Shannon–Kotel’nikov Mappings for Analog Point-to-Point Communications

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Abstract—In this paper an approach to joint source-channel coding (JSCC) named Shannon-Kotel’nikov mappings (S-K) mappings is discussed. S-K mappings are continuous, or piecewise smooth direct source-to-channel mappings operating on amplitude-continuous and discrete-time signals, and they encompass several existing JSCC schemes as special cases. Many existing approaches to analog- or hybrid discrete-analog JSCC provide both excellent performance as well as robustness to variable noise level at both low and arbitrary complexity and delay. However, a general theory explaining their performance and behaviour, as well as guidelines on how to construct well-performing mappings, do not exist. Therefore, such mappings are often based on educated guesses inspired by configurations that are known in advance to produce good solutions through numerical optimization methods. The objective of this paper is to develop a theoretical framework for analysis of analog- or hybrid discrete-analog S-K mappings which enables calculation of distortion when applying them on point-to-point links, reveal more about their fundamental nature, and provide guidelines for their construction at low as well as arbitrary complexity and delay. Such guidelines will likely help constrain solutions to numerical approaches and help explain why deep learning approaches obtain the solutions they do. The overall task is difficult and we do not provide a complete framework at this stage: We focus on high SNR and memoryless sources with an arbitrary continuous unimodal density function and memoryless Gaussian channels. We also provide example mappings based on surfaces which are chosen, or constructed, based on the provided theory.

Index Terms—Joint source channel coding, analog mappings, distortion analysis, differential geometry, OPTA.

I. INTRODUCTION

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VER the last decades attention has been directed towards miniature devices, for example in-body sensors and miniature electronic modules replacing neural network function in the brain. For this reason, and several others, it has become important to study communication systems with low complexity and delay with the highest possible performance.

In this paper we investigate a broad set of analog or hybrid discrete-analog (HDA) joint source-channel coding (JSCC) schemes named Shannon-Kotel’nikov mappings (S-K mappings). S-K mappings operate directly on analog information sources and are known to perform well at low complexity and delay when carefully constructed [4], [5], [6], [7], [8].

Shannons’ separation theorem or information transmission theorem (see [9, pp. 224–227]) for communication of a single source over a point-to-point link states that source-and channel coding can be performed separately, without any loss compared to a joint technique. To prove that separation is optimal, arbitrary complexity and delay is assumed. With a constraint on complexity and delay, separate source and channel coding (SSCC) does not necessarily result in the best possible performance, as some examples illustrate: It was shown in [10] and [11] that for an independent and identically distributed (i.i.d.) source and an additive white Gaussian noise (AWGN) channel, both of the same bandwidth, the information theoretical bound,1 optimal performance theoretically attainable (OPTA), is achieved by a simple linear source-channel mapping operating on a symbol-by-symbol basis. This result was generalized in [11] and [12] to special combinations of correlated sources and channels with memory. Furthermore, it was shown in [13], that with an ideal feedback channel, OPTA is achieved when the channel-source bandwidth ratio is an integer. This was extended to simple sensor networks in [14] as well as noisy feedback in [15]. However, with limited or no feedback, the asymptotic bounds cannot be obtained at finite complexity and delay when source and channel are of different bandwidth or dimension, or in general, when the source and channel are not probabilistically matched [16]. An open question is what the best possible performance is under complexity and delay constraints. Efforts dealing with this issue are Kostina and Verdú [17] and Merhav [18], [19].

Several analog and semi-analog JSCC schemes for the bandwidth mismatch case, operating at low and arbitrary complexity and delay, have been suggested in the literature: The analog matching scheme in [20] is a structured semi-analog approach built on lattices that achieves OPTA in the limit of infinite complexity and delay for any colored Gaussian source transmitted on any colored Gaussian channel. However, the performance of analog matching scheme in the finite

1By information theoretical bound we refer to a bound derived assuming no restriction on complexity and delay.
complexity and delay regime is, to our knowledge, unknown at present. Efforts dealing with this problem using HDA schemes are \cite{21} and \cite{22}, the latter considering the broadcast problem. Schemes that are known to perform well at low complexity and delay are the HDA approaches in \cite{4}, \cite{23}, \cite{24}, and \cite{25}, certain analog mappings like the Archimedes Spiral \cite{26}, \cite{27}, \cite{28} and mappings found by deep learning in \cite{8} and \cite{29}.

The approach to JSCC studied in this paper, namely S-K mappings, is inspired by many earlier efforts: First of all, Shannon suggested using continuous mappings through space-curves as a way of getting close to the information theoretical bounds \cite{[30]}. Simultaneously, Kotel’nikov developed a theory for analyzing distortion of amplitude-continuous and time-discrete systems realized as parametric curves in $N$ dimensions in \cite{[31]}. These are basically bandwidth- or dimension expanding systems with pulse-position modulation as a special case. The efforts of Goblick \cite{10}, Berger and Tufts \cite{11}, Ziv \cite{32}, and Vaishampayan \cite{33}, \cite{34} are pioneering works on the subject. The effort by Gastpar et. al. \cite{16} is another important contribution and Merhav’s efforts \cite{18} provides insight into the workings of such schemes through analysis based on statistical mechanics. Other important works include power-constrained channel-optimized vector quantizers (PCCOVQ) \cite{33}, \cite{35}, \cite{36}, the HDA schemes in \cite{4}, \cite{23}, and \cite{24}, the linear block pulse-amplitude modulation (BPAM) scheme in \cite{37} and the use of parametric curves for both bandwidth expansion \cite{34} and compression \cite{26}, \cite{27}. Other efforts dedicated to analog or semi-analog mappings ‘are [5], [6], \cite{7}, \cite{8}, \cite{38}, \cite{39}, \cite{40}, \cite{41}, \cite{42}. These efforts illustrate that such schemes perform well at low complexity and delay, some providing excellent performance not matched by any other known scheme.

Besides Goblick’s \cite{10}, Ziv’s \cite{32}, Gastpar’s \cite{16} and Merhav’s approaches \cite{18}, \cite{19}, there are, as far as we know, no theory to analyze such mappings nor guide their construction on a general basis. The objective of this paper is to introduce a theoretical framework based on differential geometry, encompassing most analog and HDA schemes. This approach seeks to complement the above mentioned approaches. The proposed framework facilitate calculation and analysis of the overall distortion in order to reveal the fundamentals of S-K mappings, as well as guide their construction towards well performing mappings. The reason for developing a theory is to gain knowledge on how to optimally construct such mappings, not having to rely solely on educated guesses, numerical optimization sensitive to initial conditions, or deep learning approaches in which little is known about why certain results are produced.

Treating nonlinear mappings on a general basis is difficult, and we do not present a complete framework at this point, rather introduce a set of necessary tools providing insights into the construction of S-K mappings. We limit the treatment to memoryless, independent analog sources drawn from an arbitrary unimodal density function. The sources are transmitted on memoryless, independent Gaussian point-to-point channels. Most of the results provided are proven under the assumption of high signal-to-noise ratio (SNR). Generally, S-K mappings apply when the channel-source dimension (or bandwidth) ratio is a positive rational number. We focus on low complexity and delay but also consider how S-K mappings perform when their dimensionality increase. That is, what gains may be obtained if we increase the mappings dimension to code blocks of samples, ultimately by letting mapping dimensionality go to infinity. When analyzing distortion behaviour of such mappings, we also consider how mapping curvature affects their performance. To keep the treatment as general as possible, all results are provided without reference to a specific mapping realization. Finally, we consider mapping construction providing specific examples focusing on surface-based mappings chosen or constructed based on the provided theory. We also illustrate the gain obtained by increasing mapping dimensionality for a particular type of piecewise smooth (HDA) mapping for dimension expansion.

The paper is organized as follows: In Section II the problem is formulated, OPTA is introduced, S-K mappings are defined and key concepts from differential geometry are presented. In Section III a distortion framework for S-K mappings based on concepts from differential geometry is developed and guidelines for their construction are given. In Section IV asymptotic analysis is considered and it is shown under which conditions S-K mappings may achieve OPTA for Gaussian sources. Section V deals with construction of S-K mappings to illustrate some of the theory developed in preceding sections. In Section VI a discussion is given.

II. PROBLEM FORMULATION AND PRELIMINARIES

Fig. 1 shows the communication systems under consideration. The terminology will be explained throughout this section and in Section III.

Assume a source $x \in \mathcal{D}_s \subseteq \mathbb{R}^M$, drawn from a continuous unimodal probability density function (pdf) $f_x(x)$, with i.i.d. components $x_i$. $x$ is encoded to a vector $z \in \mathcal{D}_c \subseteq \mathbb{R}^N$, which is transmitted on a memoryless channel with average power $P$, i.e., $1/N \sum_{i=1}^{N} E(z_i^2) \leq P$, and additive Gaussian noise $n \in \mathbb{R}^N$ with pdf $f_n(n)$ and i.i.d. components $n_i \sim \mathcal{N}(0, \sigma_n^2)$. The operation where a source of dimension $M$ is mapped onto a channel of dimension $N$ is an $M:N$ mapping. The channel output, $\hat{z} = z + n$, is mapped through the decoder to reconstruct $x$. As the focus is high SNR, maximum likelihood (ML) decoding is assumed, producing an estimate $\hat{x}$ of $x$. Ideally, MMSE decoding should be considered, but
as we seek analytical results, ML decoding is more convenient, and as high SNR is considered, the loss is small [43, pp. 216–219]. Both encoder and decoder contain an S-K mapping, S, defined in Section II-B. How S is applied for dimension expansion (M < N) and dimension reduction (M > N) will be explained in Section III.

Definition 1 (Source- and Channel Domains): The source domain, \( D_s \), is a simply connected, \( M \)-dimensional subset of \( \mathbb{R}^M \), \( D_s \subseteq \mathbb{R}^M \). The channel domain, \( D_c \), is a simply connected, \( N \)-dimensional subset of \( \mathbb{R}^N \), \( D_c \subseteq \mathbb{R}^N \).

Whenever \( M < N \), we consider the sub-division of \( D_c \) into two orthogonal subsets, an \( M \)-dimensional simply connected domain \( D_{cc} \) and an \( N - M \) dimensional simply connected domain \( D_{cd} \), so that \( D_c = D_{cc} \times D_{cd} \). Let \( D_{sc} \) and \( D_{sd} \) denote subsets of these two domains.

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Remark 1: Subscripts \( c \) and \( d \) are short for continuous and discrete (as clarified later). Simply connected domains are considered as unimodal pdf's are assumed. \( D_c \) is usually unbounded if \( f_c(x) \) does not have compact support, but can be bounded if \( f_c(x) \) has compact support, or if \( x \) is limited for the purpose of code design. Typically, \( D_s \) is limited to contain the significant probability mass, referring to all \( x \) except those with very low probability. For Gaussian pdf one typically limit \( ||x|| \) at four standard deviations (like “\( \sigma \) loading” assumed when constructing scalar quantizers [44, pp. 124–125]). Similarly, \( D_c \) may be bounded due to code design or a power constraint.

As a measure of performance, the end-to-end mean squared error per source-sample between the input- and reconstructed vector, \( D_t = (1/M)E[||x - \hat{x}||^2] \), is considered and compared to the optimal performance theoretically attainable (OPTA) [11].

A. OPTA

OPTA in the i.i.d. case is obtained by equating the rate-distortion function for the relevant source with the relevant channel capacity. The equation is solved with respect to the signal-to-distortion ratio (SDR), which becomes a function of the channel signal-to-noise ratio (SNR). For the case of Gaussian sources and channels, OPTA is explicitly given by

\[
\frac{\sigma^2}{D_t} = \left(1 + \frac{P}{\sigma_n^2}\right)^{N/M},
\]

where \( \sigma^2 \) is the source variance, \( \sigma^2/D_t \) is the SDR and \( P/\sigma_n^2 \) is the channel SNR. Assuming Nyquist-sampling and an ideal Nyquist-channel, with equal duration of both source- and channel signal, the ratio between channel signalling rate \( f_c \), and source sampling rate \( f_s \), can be obtained by combining \( M \) source samples with \( N \) channel samples. That is, \( f_c/f_s = M/N = r \), where \( r \) is a positive rational number, \( r \in \mathbb{Q}_+ \), named dimension change factor. If \( r > 1 \), the channels dimension is higher than that of the source and this can be utilized for noise reduction. If \( r \in [0,1) \), the source dimension, and hence its information, has to be reduced before transmission in general. Note that (1) is an asymptotic bound, which is generally obtained as \( M, N \to \infty \) while \( r = N/M \in \mathbb{Q}_+ \) is kept constant.

B. Shannon-Kotel'nikov Mappings

S-K mappings operate directly on amplitude-continuous, discrete-time signals. Let \( S \) denote a general S-K mapping and \( S \) a specific realization. Generally, \( S \) is an \( M \)- or \( N \)-dimensional smooth (hyper) surface, or a set of \( M \)- or \( N \)-dimensional smooth surface patches, \( S_i \). In the latter case, \( S \) may provide a mixture of discrete and continuous representations (a HDA system). Let \( N_i(S_i) \) denote set of indices for all nearest neighboring patches of \( S_i \) in \( D_c \) or \( D_s \).

Definition 2 (Piecewise Smooth Source-to-Channel Mapping:) Consider \( k \) (\( M \)- or \( N \)-dimensional) smooth surface patches, \( S_i \), so that \( S_i = \bigcup_{i=1}^{k} S_i \), and where \( S_i \cap S_j = \emptyset \), \( \forall j \neq i \), to assure uniqueness for all points on \( S \).

For \( M < N \): Consider a sub-division of source domain, \( D_s \), into \( k \) \( M \)-dimensional subsets, \( D_{si} \), so that \( \cup_{i=1}^{k} D_{si} = D_s \), and \( D_{si} \cap D_{sj} = \emptyset \), \( \forall j \neq i \). To each \( D_{si} \), there is a corresponding \( M \)-dimensional \( S_i \subseteq D_c \subseteq \mathbb{R}^N \), giving every point in \( D_{si} \) a unique representation in \( D_c \). Consider the sub-division of \( D_c \) in Definition 1, and let \( D_{cc} \) and \( D_{cd} \) be sets of (vector) values \( S_i \) takes in these subsets. Then \( S_i \) should be chosen so that \( \cup_{i=1}^{k} D_{cc} = D_{cc} \), and \( D_{cc} \cap D_{cc'} \neq \emptyset \), \( \forall i, j \in \{1, \cdots, k\} \). Further, \( D_{cd} \) may contain disjoint subsets so that \( \cup_{i=1}^{k} D_{cd} \subseteq D_{cd} \).

For \( M > N \): Consider a sub-division of channel domain, \( D_c \), into \( k \) \( N \)-dimensional subsets, \( D_{ci} \), so that \( \cup_{i=1}^{k} D_{ci} = D_c \), and \( D_{ci} \cap D_{cj} = \emptyset \), \( \forall j \neq i \). To each \( D_{ci} \), a corresponding \( N \)-dimensional \( S_i \subseteq D_s \subseteq \mathbb{R}^M \), giving every point of \( S \) a unique representation in \( D_c \). Consider the sub-division of \( D_s \) in Definition 1, and let \( D_{sc} \) and \( D_{sd} \) be the set of (vector) values \( S_i \) takes in these subsets. Then \( S_i \) should be chosen so that \( \cup_{i=1}^{k} D_{sc} = D_{sc} \), and \( D_{sc} \cap D_{sc'} \neq \emptyset \), \( \forall i, j \in \{1, \cdots, k\} \). \( D_{cd} \) may contain disjoint subsets so that \( \cup_{i=1}^{k} D_{sd} \subseteq D_{sd} \).

In both cases the neighbours to \( S_i, S_j, j \in N_i(S_i) \), should, as far as possible, correspond to neighbouring subsets in \( D_s \) (\( M < N \)) or \( D_c \) (\( M > N \)), to avoid large decoding errors.

Remark 2: For \( M < N \), the condition \( D_{cc} \cap D_{cc'} \neq \emptyset \), \( \forall i, j \in \{1, \cdots, k\} \) ensures a continuous representation in \( D_{cc} \) and that the channel space is properly utilized. Ideally, at high SNR, each \( D_{cc} \) should cover \( D_{cc} \). In low SNR range, or if there is significant correlation among the source variables [45], it may be convenient to relax this to \( D_{cc} \cap D_{cc'} \neq \emptyset \), \( \forall j \in N_i(S_i) \). This condition, together with \( S_i \cap S_j = \emptyset \), \( \forall j \neq i \), ensures a nonzero distance among the \( S_i \)'s. The same goes for the \( M > N \) case relating to \( D_c \). In \( M > N \) case, \( \cup_{i=1}^{k} D_{ci} \subseteq D_c \), to allow some space between subsets in \( D_c \), avoiding large decoding errors.

Remark 3: Generally, the number of \( S_i \)'s, \( k \), increases with SNR. There is some flexibility in placement of the \( S_i \)'s in \( D_c \) (or \( D_s \)) though index assignment. To avoid large decoding errors, one should choose an assignment where neighbours to \( S_i \) in \( D_c \) (or \( D_s \)) corresponds to neighboring subsets in \( D_s \) (or \( D_c \)). However, if quantized values are transmitted directly, this issue is usually resolved.
In the bottom middle figure, \( D \), where each of Definition II-B, but the first satisfies a stricter condition \( D \).

In the upper right, where \( S \subset D \), unbounded as the structures can be extended indefinitely. The map \( D_s \to S \) is generally not bijective. In the continuous case, a particular \( S \in S \) can be represented by the parametrization

\[
S(x) = [S_1(x), S_2(x), \ldots, S_N(x)].
\]

In the piecewise smooth case, \( S \) is as in Definition II-B.

Dimension reduction \( M > N \), \( S \subset D_s \), an \( N \)-dimensional subset of the source space \( \mathbb{R}^M \), realized by a continuous or piecewise smooth \( M \)-dimensional surface, so that each source vector \( x \in D_s \) has a unique representation on \( S \) in \( D_c \). In the continuous case, a particular \( S \in S \) can be represented by the parametrization

\[
S(z) = [S_1(z), S_2(z), \ldots, S_M(z)].
\]

In the piecewise smooth case, \( S \) is as in Definition II-B.

Remark 4: The parametrization is of mathematical convenience, not a restriction on the set of mappings. I.e., \( S \) does not need to have a parametrization. Note that when \( M > N \) with continuous \( S \), large decoding errors are completely avoided (see Section III-B). Then it is not crucial that \( S \) does not intersect itself. I.e., this condition may be relaxed for that case.

We are concerned with cases 2 and 3 as case 1 was dealt with in [6] and [10]. We will in the rest of the paper refer to \( S \) as either \( S \)-K mapping, signal surface or simply \( S \).

Example 1: Throughout the paper we give several examples of \( S \)-K mappings satisfying Definition 2 above: The 3:2 mappings in Figs. 9(a) and 10(a) are continuous mappings with parametric equations where both \( D_s \) and \( D_c \) can be unbounded as the structures can be extended indefinitely. The 1:2 mappings in Fig. 8 (also potential 2:1 mappings), are valid mappings, one satisfying the conditions for a piecewise smooth \( S \)-K mapping. In Fig. 2, we give more examples of piecewise smooth source-to-channel mappings for \( M < N \), i.e., with \( S \subset D_c \). The same examples are valid for \( M > N \) mappings in which \( S \subset D_s \). To the upper left we have a 2:3 (or 3:2) \( S \)-K mapping where \( S \) is parallel planes, a mapping we will exemplify throughout. In the upper middle figure is a 1:2 (or 2:1) \( S \)-K mapping where \( S \) is line segments. Both satisfy the condition \( D_{cc_i} \cap D_{cc_j} \neq \emptyset, \forall i, j \in \{1, \ldots, k\} \) of Definition II-B, but the first satisfies a stricter condition where each \( D_{cc_i} \) covers all of \( D_{cc} \). Both take on discrete values in \( D_{cd} \) (or \( D_{ed} \)). The 1:2 (or 2:1) \( S \)-K mapping to the upper right, where \( S_i \) is a curve segment, takes on continuous values in \( D_{ed} \) (or \( D_{cd} \)), but each \( S_i \) represents a disjoint set in \( D_{cd_i} \subset D_{cd} \). Note that all these \( S \)-K mappings have bounded \( D_{cc} \) (or \( D_{cc_i} \)). The three mappings in bottom row does not comply with Definition II-B: To the bottom left \( S_i \cap S_j \neq \emptyset \). In the bottom middle figure, \( \bigcup_{i=1}^k D_{cc_i} = D_{cc} \) is violated.

To the bottom left, \( D_{cc_i} \cap D_{cc_j} \neq \emptyset, \forall i, j \in \{1, \ldots, k\} \) is violated.

C. Relevant Concepts From Differential Geometry

The theory of \( S \)-K mappings is based on concepts from differential geometry. A brief introduction of the concepts we apply are given here with more details provided in Appendix A and in [46] (available online). All concepts are taken from Kreyszig’s book [47]. We use variables \( u \in \mathbb{R} \) and \( u_i \in \mathbb{R} \) here to keep the discussion general, avoiding reference to source- or channel variables.

Consider a curve, \( C : S(u) = [S_1(u), S_2(u), \ldots, S_M(u)] \in \mathbb{R}^M \). We will denote the derivatives with respect to \( (w.r.t.) \) a general parameter, \( u \), as \( S', S'' \) etc. In the special case of the parameter being arc-length of \( C \), we denote the derivatives \( S, S' \) etc. That is, when we parameterize by

\[
\ell(u) = \int_{u_0}^u \sqrt{S' \cdot S'} \, du = \int_{u_0}^u ||S'|| \, du.
\]

Then \( ||S|| = ||t|| = 1, \forall u \), with \( t \) the curves tangent vector(s) (See Appendix A-A).

For a curve, \( S(u) \), one can define the principal curvature \( \alpha \) at \( u_0 \) such that \( \alpha = ||S'(u_0)|| \) [47, p. 34]. Consider \( \kappa = ||S'(u)||/||S''(u)|| \) arc-length parametrization. Then \( \kappa = \alpha/||S'(u)|| \). Now, and as \( S \perp \alpha \), then

\[
\kappa = \sqrt{S''_0(u_0)}/||S'_0(u_0)||^2 \text{ according to (63) in Appendix A-A.}
\]

For surfaces, \( S \), with parametric representation as in (2), (3), we denote partial derivatives as

\[
S_\alpha = \frac{\partial S}{\partial u_\alpha}, \quad S_{\alpha \beta} = \frac{\partial^2 S}{\partial u_\alpha \partial u_\beta}.
\]

The use of sub- and superscripts here and in the following relate to Einstein’s summation convention [46, p. 2].

The curvature of a surface, \( S \), depends on the choice of coordinate curves on \( S \). For simplicity, consider a 2-dimensional surface in \( \mathbb{R}^3 \) in the following. A curve, \( C \), on surface \( S(u_1, u_2) \in S \) is represented by the parametrization \( C : u_1 = u_1(t), u_2 = u_2(t) \), which is in \( C^1 \) (the set of differentiable functions), where \( t \in \mathbb{R} \). The coordinate curves, \( u_1 = \text{constant} \) and \( u_2 = \text{constant} \), corresponds to parallel curves in the \( u_1, u_2 \)-plane. One must always choose allowable
coordinates which conditions are provided in [46, p. 2]. The normal curvature, \( \kappa_n \), of \( S \) at point \( P \) is given by \( \kappa_n = b_{\alpha\beta} \partial u^\alpha \partial u^\beta / g_{\alpha\beta} d\alpha d\beta \) (see Appendix A-B.2, Eqn. (67)), where \( g_{ij} = S_i \cdot S_j \) are the components of the metric tensor \( G \), or first fundamental form (FFF) of \( S \), and \( b_{\alpha\beta} = S_{\alpha\beta} \cdot N \) are components of the second fundamental form (SFF) of \( S \), with \( N \), the unit normal to \( S \) at \( P \) (see Appendix A-B.1). A special case of particular interest is the extremal values of \( \kappa_n \), named lines of curvature (LoC). If one chooses LoC as coordinate curves then the curvature of \( S \) in those directions, the principal curvatures, are given by \( \kappa_1 = b_{ii} / g_{ii}, \forall i \) (see Appendix A-B.2 for details). For general coordinate curves, \( \kappa_1 \) are the roots of (69) in Appendix A-B. The normal curvature, \( \kappa_n \), for any (tangent) direction can be represented in terms of \( \kappa_1 \) and \( \kappa_2 \) according to theorem of Euler [47, p. 132] as \( \kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha \), with \( \alpha \) the angle between an arbitrary direction at \( P \) and the direction corresponding to \( \kappa_1 \). For m-dimensional surfaces in \( \mathbb{R}^n \), the same concepts exist. However, the definition of \( b_{\alpha\beta} \) is more elaborate (see for example [48]).

III. DISTORTION ANALYSIS FOR S-K MAPPINGS

In this section we quantify distortion for S-K mappings.

A. Dimension Expanding S-K Mappings

In this section Kotelnikov’s theory from [31, pp. 62–99] on 1:N mappings is generalized to include vector sources, enabling analysis of more general mappings. The results presented in this section are extensions of [1].

Fig. Fig. 1(a) depicts the block diagram for a dimension expanding communication system. Consider source vector \( \mathbf{x} \in D \subseteq \mathbb{R}^M \), which is represented on the channel as \( \mathbf{x} \rightarrow \mathbf{S}(\mathbf{x}) \in S \subseteq D \subseteq \mathbb{R}^N \), (see Definition 2). Applying a specific realization \( \mathbf{S} \in S \), the likelihood-function of the received vector \( \mathbf{z} = \mathbf{S}(\mathbf{x}) + \mathbf{n} \) is

\[
f_{\mathbf{z}|\mathbf{x}}(\mathbf{z}|\mathbf{x}) = \frac{1}{2\pi \sigma_n^2} \frac{N^2/2}{e^{-\frac{||\mathbf{z} - \mathbf{S}(\mathbf{x})||^2}{2\sigma_n^2}}}. \quad (6)
\]

The ML-estimate is then defined as [49]

\[
\hat{\mathbf{x}} = \max_{\mathbf{x} \in \mathbb{R}^M} f_{\mathbf{z}|\mathbf{x}}(\mathbf{z}|\mathbf{x}), \quad (7)
\]

and is maximized by the \( \mathbf{x} \) minimizing \( ||\hat{\mathbf{z}} - \mathbf{S}(\mathbf{x})|| \). I.e., the ML-estimate is the \( \mathbf{x} \) in which \( \mathbf{S}(\mathbf{x}) \) is closest to \( \hat{\mathbf{z}} \) in Euclidean distance. That is, a projection of \( \hat{\mathbf{z}} \) onto \( S \).

Ideally one could formulate the exact distortion for any such scheme once a specific \( \mathbf{S} \in S \) is chosen. This is inconvenient when it comes to analysis of the general behavior of such mappings as it is usually very hard, if at all possible, to find closed form solutions. Therefore we use the approach by Kotelnikov [31, pp. 62–99], which reasoned that there are two main contributions to the total distortion using such mappings: low-intensity noise and strong noise. Low-intensity noise refers to errors in the reconstruction varying gradually with the magnitude of \( \mathbf{n} \). Distortion due to low-intensity noise can be analyzed without reference to a specific \( \mathbf{S} \) whenever the noise can be considered weak. The resulting distortion is named weak noise distortion, denoted by \( \varepsilon^2_{wn} \), as defined in Definition 3, section III-A.1. Strong noise is known as anomalous errors in the literature, and results from a threshold effect [19], [30]. The resulting distortion is named anomalous distortion and denoted by \( \varepsilon^2_{an} \).

1) Weak Noise Distortion: To analyze non-linear mappings without reference to a chosen structure, the Taylor expansion and the concepts introduced in Section II-C apply. Let \( \mathbf{S}_{Lin}(\mathbf{x}) \) denote 1st order Taylor approximation of \( \mathbf{S}(\mathbf{x}) \in C^1 \) at \( \mathbf{x}_0 \)

\[
\mathbf{S}_{Lin}(\mathbf{x}) = \mathbf{S}(\mathbf{x}_0) + J(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad (8)
\]

where \( J(\mathbf{x}_0) \) denotes the Jacobian (see Appendix A-B) of \( \mathbf{S} \) evaluated at \( \mathbf{x}_0 \). Under (8) one can, for mathematical convenience, approximate the ML-decoder as a projection onto the tangent (hyper) plane of \( S \) at \( \mathbf{S}(\mathbf{x}_0) \), as depicted in Fig. 3(a) for the 2:3 case. We have the following proposition providing the exact distortion under linear approximation:

**Proposition 1 (Minimum Weak Noise Distortion):** For any continuous i.i.d. source \( \mathbf{x} \in \mathbb{R}^M \) with unimodal pdf \( f_{\mathbf{x}}(\mathbf{x}) \) sent on an i.i.d. Gaussian channel of dimension \( N \) using a continuous dimension expanding \( S \) mapping \( \mathbf{S} \) as in (2) where \( \mathbf{S}_i \in C^r(\mathbb{R}^M), r \geq 1, i = 1, \ldots, N \), the minimum distortion under the approximation in (8) is

\[
\varepsilon^2_{wn} = \sigma_n^2 \int \cdots \int_{D_1} \cdots \int_{D_N} \frac{1}{g_{ii}(x)} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}, \quad (9)
\]

obtained when the metric tensor \( G \) (or FFF) of \( S \) is diagonal with entries \( g_{ii} = ||\partial S(\mathbf{x}) / \partial x_i||^2 \). I.e., the squared norm of the tangent vector along the \( i \)th coordinate curve.

**Proof:** See Appendix B-A.1. □

Remark 5: Proposition 1 is directly extendable to piecewise smooth \( S \) as one can compute (9) for each \( \mathbf{S}_i \), then sum all contributions afterwards (like in [23] for 1:N mappings).

Eqn. (9) states that weak noise distortion is lowered when the \( g_{ii} \)’s are increased (i.e., when the length of the tangent vectors of \( S \) are increased), which is obtained when \( S \) is stretched like a rubber-sheet. Bending or cutting of \( S \) into pieces does not affect weak noise distortion.

The concept is illustrated in Fig. 3(b) for the 1:N case when \( S \) is a curve. Stretching of the curve makes the distance between points in source space \( (x_1 \) and \( x_2) \) appear larger relative to a given noise vector. Equivalently, the more the source is stretched at the transmitter trough \( S \), the more the noise will be attenuated at the receiver, lowering distortion. This result implies that the source space should be stretched indefinitely. However, as will be seen in Section III-A.2,
under a channel power constraint this cannot be done without introducing large errors.

The following corollary is a special case of Proposition 1:

**Corollary 1 (Shape-Preserving Mapping):** When $S$ has a diagonal metric $G$ with $g_{ii}(x) = g_{jj}(x) = \alpha, \forall x, i, j$, and with $\alpha$ a constant, then

\[
\varepsilon_{wn}^2 = \frac{\sigma_n^2}{\alpha^2}. \tag{10}
\]

That is, all source vectors are equally scaled when mapped through $S$, and so the noise will affect all $x$ equally.

**Proof:** Insert $g_{ii}(x) = g_{jj}(x) = \alpha$ in (9).

A shape-preserving mapping can be seen as an amplification factor $\alpha$ from source to channel. Although a shape-preserving mapping leads to simple analysis, its not necessarily optimal in general, as shown in [50, 294–297].

In order to determine the error made in the distortion estimate under linear approximation, we need to consider 2nd order Taylor expansion. We have the following proposition:

**Proposition 2 (Weak Noise Error Under 2nd Order Taylor Approximation):** Under 2nd order Taylor approximation, the special case of 1:$N$ mappings (curves) has an error in the absence of anomalies given by

\[
\varepsilon_{wn}^2 \approx \frac{\sigma_n^2}{\|S_i^p\|^2} \left(1 + \frac{1 + \sigma_n^2 \|S_i^b\|^2}{4 g_{ii}(x_0)} \right), \tag{11}
\]

valid for any S-K mapping $S(x) \in C^n, n \geq 2$. The last equality is true under scaled arc-length parametrization. Further, for any continuous dimension expanding S-K mapping as defined in Definition 2, with LoC coordinate curves, the error is given by

\[
\varepsilon_{wn}^2(x_0) \approx \frac{\sigma_n^2}{M} \sum_{i=1}^M \left\{ \frac{1}{g_{ii}(x_0)} \left(1 + \frac{\sigma_n^2 b_{ii}^2(x_0)}{4 g_{ii}(x_0)} \right) \right\}, \tag{12}
\]

Here, $\kappa_i = b_{ii}/g_{ii}$ is the curvature along coordinate curve $i$ with $b_{ii}$ the diagonal components of the SFF (see Appendix A-B).

**Proof:** See Appendix B-A.2.

**Remark 6:** We only treat 2nd order Taylor approximation here to simplify the analysis. It will be seen in Section III-B that higher order terms are even less influential as $\sigma_n$ is raised to a power twice that of the order, which at high SNR ($\sigma_n << 1$) leads to a negligible contribution. For piecewise smooth $S$, one evaluate (12) for each $S_i$, then sum all contributions.

In the absence of anomalies, one can characterize distortion for S-K mappings in general without choosing a specific $S$ in advance, as it is expressed solely w.r.t. FFF and SFF components, making it easier to evaluate the distortion analytically for such mappings.

From (12) alone a linear mapping seems convenient as $\kappa_i = 0, \forall i$. However, at high SNR linear mappings perform poorly, and with (12) in mind, one would seek nonlinear mappings with the smallest possible $\kappa_i(x)$. Therefore the weak noise regime, as defined next, is a good approximation for any reasonably chosen mapping at high SNR.

**Definition 3 (Weak Noise Regime, Dimension Expansion):** With $x_0$ the transmitted vector and $S(x_0)$ its representation on the channel, we say that we are in the weak noise regime whenever the 2nd order term in (12) (the term containing $\kappa_i$), is negligible compared to the 1st order term. That is, (8) is a close approximation to $S$ at any point and (9) is an accurate approximation to the actual distortion in the absence of anomalies.

**Example 2:** When is Definition 3 satisfied? There are at least three cases:

i) $\text{SNR} \rightarrow \infty (\sigma_n \rightarrow 0)$: The linear approximation in (8) is exact as $S$ is locally Euclidean.

ii) When $S$ is linear then $\kappa_i = 0, \forall i$. The same goes for HDA schemes composed of piecewise line- or (hyper) plane patches (as the HVQLC in Section V).

iii) Small maximal principal curvature $\kappa_{\text{max}}$: The smaller $\kappa_{\text{max}}$ is, the larger $\sigma_n$ can be before (9) becomes inaccurate. This is inline with mappings obtained from numerical optimization algorithms, which tend to bend less and less the lower the SNR is (36) and [51].

2) **Anomalous Distortion:** With a channel power constraint, $S$ must be constrained to lie within some $N−1$ sphere, $S \subseteq \mathbb{R}^{N−1}$. To make weak noise distortion small, $S$ must be stretched, bent and twisted to “fit” within this sphere. As an example take the 1:2 case in Fig. 4. Take a decomposition of the noise $n$ into a tangential component to $S$, $n_{\text{tan}} = n_{\text{t}}$, and a normal component $n_{\text{nor}} = n_{\text{n}}$, as depicted in Fig. 4(a). $n_{\text{nor}}$ contributes to weak noise, whilst $n_{\text{tan}}$ contributes to anomalous errors, which occur whenever $\|n_{\text{tan}}\|$ crosses a certain boundary. Then the transmitted vector $S(x_0)$, representing $x_0$, will be detected as the vector $S(x_{err})$ on another arm (or fold) of the spiral in Fig. 4(b). This happens if the distance, $\Delta$, between the spiral arms is chosen too small w.r.t. $\sigma_n$. Although $S(x_{err})$ is close to $S(x_0)$ in the channel space, $x_{err}$ is far away from $x_0$ in source space, leading to large decoding errors.

To quantify anomalous distortion in general, the concept of a **canal surface** is needed:

The definition of an $N$-sphere is $S^N = \{y \in \mathbb{R}^{N+1}|d(y,0) = \text{constant}\}$ [52, p. 7], where $d$ is the distance from any point $y$ on $S^N$ to the origin of $\mathbb{R}^{N+1}$; E.g. the sphere embedded in $\mathbb{R}^3$ is denoted $S^2$. 
Definition 4: Canal surface With \( S(x) \), a curve in \( \mathbb{R}^N \), the corresponding canal surface, \( S_c \), is the envelope to the family, \( F \), of congruent \( N-1 \) spheres, \( \mathbb{S}^{N-1} \), of radius \( r \) with center on \( S(x) \), given by the equation

\[
H(z, x) = (z - S(x)) \cdot (z - S(x)) - r^2 = 0.
\]

I.e., (13) defines a hyper-surface in \( \mathbb{R}^N \), with \( z \in \mathbb{R}^N \), the channel variables.

The following definition is an extension of [47, p. 263].

Definition 5 (Characteristic and Characteristic Intersection): Consider the equations:

\[
H(z, x) = 0, \quad \frac{\partial H(z, x)}{\partial x} = -2\dot{S}(x) \cdot (z - S(x)) = 0, \quad \frac{\partial^2 H(z, x)}{\partial x^2} = -2\ddot{S}(x) \cdot (z - S(x)) + 2 = 0.
\]

The two first equations define the characteristic of \( S_c \), an \( N-2 \) sphere lying in the space normal to \( S \). All three equations defines the characteristic intersection of \( S_c \), an \( N-3 \) sphere.

If a solution to (14) does not exists for any \( x \), the canal surface will not intersect itself at any point, and it can be used to bound the probability for anomalous errors by relating the noise variance \( \sigma_n^2 \) to the radius \( r \) (see Example 4 below).

Example 3: In \( \mathbb{R}^3 \), with \( S(x) \) a space curve, the canal surface is a tube as shown in Fig. 5. The characteristic is a circle lying in the normal plane of \( S \) (see Appendix A-A), and the characteristic intersection is two points corresponding to the intersection of two circles (the characteristic at two points of \( S \)). In \( \mathbb{R}^4 \), the characteristic is a sphere, and the characteristic intersection is a circle. In \( \mathbb{R}^2 \) the characteristic consists of two points and the characteristic intersection consist of one point corresponding to the value of \( S(x) \) where the points of the characteristic coincide.

A generalization for \( M \times N \) mappings is straightforward:

Definition 6 (Extended Canal Surface): The extended canal surface, \( S_c \), of an \( M \)-dimensional \( S \) embedded in \( \mathbb{R}^N \) is the envelope of the congruent hyper-spheres \( \mathbb{S}^{N-M} \) with radius \( r \) and center on \( S(x) \). We refer to an extended canal surface simply as “canal surface” in the following.

We also need the following definition to properly define anomalous distortion:

Definition 7 (Folds and Minimum Distance of \( S \)): Let \( S \) have a canal surface with no characteristic intersection. Then any two points on \( S \) will reside on different folds of \( S \) whenever the straight line between them intersects the canal surface in one- or several points.

Given that \( x_0 \) is transmitted, then \( \Delta_{\text{min}}(x_0) \) is the minimum distance from \( S(x_0) \) to the closest point on another fold in Euclidean distance, denoted \( S(x_{\text{err}}) \).

To bound the anomalous error probability one can consider the coordinate curve as being the LoC along the maximal principal curvature \( \kappa_{\text{max}} \). Then the following Lemma provides conditions for a non-intersecting canal surface:

Lemma 1: Consider \( M \times N \) dimension expanding \( S \) with \( \rho_{\text{min}} = 1/\kappa_{\text{max}} \) and \( \kappa_{\text{max}} \) the maximal principal curvature of \( S \) at any point \( x_0 \). With \( r \), the radius of the hyper-sphere \( \mathbb{S}^{N-M} \) from Definition 6, and \( \Delta_{\text{min}} \) the minimum distance in Definition 7, then the canal surface, \( S_c \), in Definition 6 will not intersect itself at any point if and only if i) \( \Delta_{\text{min}} > 2r \) and ii) \( \rho_{\text{min}} > r \) for all points of \( S \). That is, the characteristic intersection does not exist at any point on \( S \).

Proof: See Appendix B-A.3.

Remark 7: Note that condition ii) is incorporated into condition i). The reason we explicitly state ii) is to constrain the curvature of \( S \) so that it can be removed from later analysis.

Example 4 (1:3 Mapping): Fig. 5 depicts the canal surface surrounding a curve, \( S(x) \), in \( \mathbb{R}^3 \). The radius of the canal surface is linked to the noise vector \( n_{\text{an}} = n_\perp \). Bending of the tube can increase the probability for anomalous errors, implying that straight lines have the lowest probability for such errors. However, according to Lemma 1, one can circumvent this if the radius of curvature of \( S \) is small enough. The significant probability mass of the normalized noise vector, \( n_{\text{an}} \), is located within a circle of radius \( \rho_n = \sqrt{2\rho_n^2 \sigma_n^4/3} \) (see Section IV-A), with \( \rho_n \) related to the variance of \( n_{\text{an}} \) (with \( \rho_n > 4 \) about 99.99% of the probability mass is included when the dimension of \( n_{\text{an}} \) is small). Therefore, if i) is satisfied, and if \( \rho_{\text{min}} > r = \rho_n \geq \sqrt{2\rho_n^2 \sigma_n^4/3} \), no characteristic intersection exist and the tube in Fig. 5 will not intersect itself.

We provide a definition of anomalous distortion valid in the vicinity of the optimal operational SNR. That is, we only consider jumps to the nearest point on another fold, \( S(x_{\text{err}}) \), from a given transmitted point, \( S(x_0) \) (jumps across several folds may happen as \( \sigma_n \) grows, but this is far from optimal). Fig. 4 shows the terminology used in the following definition.

Definition 8 (Anomalous Distortion): Let \( x_0 \) denote the transmitted vector and \( S(x_0) \) its representation in the channel space. Let \( n_{\text{an}} \) denote the \( K \) (\( \leq N \)) dimensional component of a decomposition of the noise \( n \) pointing in the direction of the closest point, \( S(x_{\text{err}}) \), on any other fold of \( S \) from \( S(x_0) \) (see Fig. 4(b)). \( x_{\text{err}}(x_0) \) is reconstruction in the case of this anomaly and \( \Delta_{\text{min}}(x_0) \) is the distance between \( S(x_0) \) and \( S(x_{\text{err}}) \). Let \( \rho_{\text{an}} = \|n_{\text{an}}\| \) with \( f_{\rho_{\text{an}}}(\rho_{\text{an}}) \) its pdf (see (15) in Section IV-A). The probability that \( x_0 \) is detected as \( x_{\text{err}} \) is then

\[
P_{\text{an}}(x_0) = \int_{\Delta_{\text{min}}(x_0)/2}^{\infty} f_{\rho_{\text{an}}}(\rho_{\text{an}}) d\rho_{\text{an}}.
\]

The anomalous distortion close to the optimal operational SNR is then defined as

\[
\varepsilon_{\text{an}}^2 = E_x \{ P_{\text{an}}(x) \| x - x_{\text{err}}(x) \|^2 \}.
\]

Remark 8: To lower \( P_{\text{an}}(x_0) \), \( \Delta_{\text{min}} \) should be chosen larger. There is thus a tradeoff between reducing weak noise.
distortion, obtained by decreasing $\Delta_{\min}$, and anomalous distortion. This tradeoff results in one optimal point for a fixed $\Delta_{\min}$. If $\sigma_n$ changes, then $\Delta_{\min}$ must be adapted to maintain optimality. This is inline with Ziv’s results in [32].

### B. M:N Dimension Reducing S-K Mappings

Results presented in this section are extensions of [2]. Fig. 1(b) shows a block diagram for the dimension reducing communication system under consideration. From Definition 2, a continuous dimension reducing $S$ is an $N$ dimensional subset of the source space $D_s \subseteq \mathbb{R}^M$, that can be realized by the parametrization $S(z)$ in (3), with the channel signal $z$ acting as parameters. In that sense, $S$ is a representation of the channel in source space.

To reduce the dimension of a source under a channel power constraint in general, its information content must be reduced. The source vectors $x$ are approximated by their projection onto $S$, a lossy operation denoted $q(x) \in S \subseteq D_s \subseteq \mathbb{R}^M$. The dimension is subsequently changed from $M$ to $N$ by a lossless operator $d_c : S \rightarrow D_c \subseteq \mathbb{R}^N$, with $D_c$ as in Definition 1. The total operation is named projection operation, and denoted $p = d_c \circ q \in D_s \mapsto p(x) \in D_c$. The vector $\tilde{x} = p(x)$ is transmitted over the AWGN channel. The ML-decoder will now produce the estimate $\hat{x} = S(z+n)$. The noise $n$ leads to displacements of the projected source vector along $S$. With a continuous $S$, the distortion due to channel noise will be gradually increasing with $\sigma_n^2$, i.e. no anomalous errors will occur. The concept is illustrated for a 2:1 mapping in Fig. 6(b).

There are two contributions to the total distortion for continuous $S$: approximation distortion from the projection operation, and channel distortion resulting from channel noise mapped through $S$ at the receiver. However, anomalous errors may also occur if $S$ is piecewise continuous.

1) Channel Distortion: The received vector $\tilde{x} = x + n$ is mapped through $S$ to reconstruct $x$. When noise is sufficiently small, distortion can be modelled by considering a linear approximation $S_{lin}(z_0)$ of $S(z) \in C^1$ at $z_0$

$$S_{lin}(z_0 + n) = S(z_0) + J(z_0)n.$$(17)

The following proposition provides the exact distortion under linear approximation:

**Proposition 3 (Minimum Weak Channel Distortion):** For any continuous i.i.d. Gaussian channel of dimension $N$ and any continuous dimension reducing S-K mapping $S \in \mathcal{S}$, where $S_i \in C^r, r \geq 1$, $i = 1, \ldots, M$, the distortion due to channel noise under the linear approximation in (17) is given by

$$\varepsilon_{ch}^2 = \frac{\sigma_n^2}{M} \int \cdots \int \sum_{i=1}^N g_{ii}(z) f_{\varepsilon}(\varepsilon) d\varepsilon,$$(18)

where $f_{\varepsilon}(\varepsilon)$ is the channel pdf, and $g_{ii}$ the diagonal components of the metric tensor of $S$.

**Proof:** See Appendix B-B.1.

**Remark 9:** For piecewise smooth $S$, one applies (18) over each $S_i$, then sum all contributions.

The name weak channel distortion is due to Definition 9 given below. Proposition 3 states that weak channel distortion increases in magnitude when $S$ is stretched as the $g_{ii}$’s increases. To keep the channel distortion small, $S$ should be stretched minimally. The opposite is sought in the dimension expansion case as an increase of $g_{ii}$ leads to larger attenuation of noise at the receiver side. The following corollary is a special case of Proposition 3:

**Corollary 2 (Shape-Preserving Mapping):** When $S$ has a diagonal metric with $g_{ii}(z) = g_{jj}(z) = \alpha, \forall z,i,j$, and $\alpha$ constant, then

$$\varepsilon_{ch}^2 = \frac{N\sigma_n^2}{M} \alpha^2.$$(19)

i.e. all channel vectors are equally scaled when mapped through $S$, and so noise will affect all source vectors $x$ equally.

**Proof:** Insert $g_{ii}(z) = g_{jj}(z) = \alpha$ in (18).

Under Corollary 2, $S$ can be seen as an amplification $\alpha$ from channel to source at the receiver.

As the channel noise becomes larger, (18) becomes inaccurate as illustrated in Fig. 6(b). To determine the error under linear approximation, we consider 2nd order Taylor expansion:

**Proposition 4 (Error Under 2nd Order Taylor Approximation):** Under 2nd order Taylor approximation, in the special case of $M:1$ mappings, the error due to channel noise is given by

$$\varepsilon_{ch}^2(z_0) = \sigma_n^2 \|S_0\|^2 + \frac{3\sigma_n^2}{4} \frac{\|S'_0\|^2}{\|S_0\|^4},$$

valid for any S-K mapping $S(z) \in C^n, n \geq 2$. The last equality is true under scaled arc-length parametrization. For any continuous dimension reducing S-K mapping in Definition 2, with LoC coordinate curves, the error is given by

$$\varepsilon_{ch}^2(z_0) \approx \sigma_n^2 \sum_{i=1}^N \left( g_{ii}(z_0) + \frac{3\sigma_n^2}{4} \kappa_i^2(z_0) \right)$$

$$= \sigma_n^2 \sum_{i=1}^N \left( g_{ii}(z_0) + \frac{3\sigma_n^2}{4} \frac{b_{ii}^2(z_0)}{g_{ii}(z_0)} \right).$$

**Proof:** See Appendix B-B.2.

**Remark 10:** Comparing with dimension expansion in Proposition III-A.1 we see that distortion is scaled by the components of the SFF (or curvature), $b_{ii}$, in a similar manner. However, the scaling w.r.t. $g_{ii}$ is opposite, corresponding to the results in (9) and (18).
Remark 11: From the proof of Proposition III-B.1, Appendix B-B.2, Eq. (88), we have

$$
\varepsilon_{ch}(x_0) = \sigma_n^2 \| \tilde{S}(x_0) \|^2 + \frac{3\sigma_n^2}{4} \| \tilde{S}(x_0) \|^2 + \frac{5\sigma_n^6}{12} \| \tilde{S}(x_0) \|^2
$$

for the channel error under 3rd order Taylor expansion. The last equality is a canonical representation [47, p. 48], valid for any curve $S \in C^r$, $r \geq 3$, where $\tau_0$ the torsion of $S$ (see Appendix A-A). This shows that higher order terms become smaller as $\sigma_n$ decreases, at least for curves with small $\kappa_0$ and $\tau_0$. Referring to Section III-A, this is the reason why we did not consider Taylor expansion beyond 2nd order there.

Definition 9 (Weak Noise Regime, Dimension Reduction): Let $z_0$ denote the transmitted vector and $S(z_0)$ its representation in the source space. We are in the weak noise regime whenever the second order term in (21) is negligible compared to the 1st order term. That is, (17) is a close approximation to $S$ and (18) is an accurate approximation to the actual distortion due to channel noise. □

Remark 12: Generally the error in the ML-estimate increases with $\kappa_{max}$. However, for continuous mappings, $\kappa_{max}$ must be non-zero in order to cover the source space and thereby keep the approximation distortion low. One should therefore choose a mapping that fills the source space with the smallest possible $\kappa$. Alternatively, one may consider a scheme consisting of parallel planes or (hyper) planes where $\kappa_\parallel = 0$, at the expense of introducing anomalous errors. Therefore the weak noise regime is a good approximation for any reasonably chosen mappings at high SNR.

2) Approximation Distortion: Approximation distortion results from the lossy operation $p$. Its magnitude depends on the average distance source vectors have to $S$. In order to make the approximation distortion as small as possible, $S$ should cover the source space so that any $x$ is as close to it as possible. Covering of the source space is obtained by stretching, bending and twisting the transformed channel space $S$ inside the subset of the source space with significant probability mass (an example for the 2:1 case is provided in Fig. 6(a)). This is in conflict with the requirement of reducing channel distortion in which the stretching of $S$ should be minimized. This tradeoff leads to a single optimal point, inline with [32].

Since approximation distortion is structure dependent, one cannot find a closed form expression for it in general. However, one can find a general expression valid for certain special cases that becomes exact as the dimension of the mapping becomes large.

Definition 10 (Uniform S-K Mapping): For an S-K mapping where at each point, $P \in S$, there is a fixed distance $\Delta$ to the closest point on another fold of $S$, is a uniform S-K mapping. The maximal approximation error from $x$ to $S$ will then be $\Delta/2$ for any $x$ to any point of $P \in S$.

Remark 13: For a uniform $S$, any vector being approximated to any point of $S$ will be confined within a canal surface, $S_c$, as in Definition 6, but now with radius $r = \Delta/2$ and $S(z)$ its center, and with the roles of $z$ and $x$ exchanged in equations (13) and (14).

The 2:1 S-K mapping shown in Fig. 6(a) is a uniform mapping (except close to the origin). For uniform S-K mappings, a similar distortion lower bound as that found for vector quantizers in [53] can be derived for small $\Delta$, i.e., a sphere bound [54]. We have the following proposition:

Proposition 5 (Sphere Bound for Approximation Distortion): For a uniform $S$ with distance $\Delta$ between closest points on neighboring folds, the approximation distortion for sufficiently small $\Delta$ is bounded by

$$
\varepsilon_q^2 \geq \frac{M - N}{4M(M - N + 2)} \Delta^2.
$$

Equality is obtained when $M = N + 1$, and generally as $M, N \to \infty$, with $N/M = r$ constant.

Proof: See Appendix B-B.3.

Remark 14: Uniform mappings with $\varepsilon_q^2$ bounded by (23) exist. For $M = N + 1$, where (23) is exact, an example is parallel N-dimensional planes with distance $\Delta$ embedded in $\mathbb{R}^{N+1}$. The 3:2 HVQLC in Section V-A.3, depicted in Fig. 2 to the upper left, is an example of this construction. Another is concentric N-spheres with radii $r = n/\Delta$, $n \in \mathbb{N}$ (the 2:1 spiral in Fig. 6(a) is similar to this). For 3:1 case, place parallel lines in $\mathbb{R}^3$ such that each line have distance $\Delta$ to each of its nearest neighbors. With the lines oriented along $x_1$-axis, one would obtain a lattice in the $x_2$-axis, and (23) serve as a lower bound. This construction, generalized to $\mathbb{R}^M$ for $M > 3$ was treated in [45] for Gaussian multiple access channel. There, this mappings was also generalized to higher dimensions using hyper planes, analyzing its asymptotic performance.

IV. ASYMPTOTIC ANALYSIS FOR S-K MAPPINGS

We investigate how S-K mappings perform as their dimensionality (or block-length) increases. I.e., letting $M, N$ increase while $r = N/M \in \mathbb{Q}^+$ is kept constant. Ultimately, can S-K mappings obtain OPTA as $M, N \to \infty$, and what conditions on $S$ are needed to do so? We consider Gaussian sources here.

When constructing mappings based on (hyper) surfaces, one method is to construct several parallel and independent systems based on curves (i.e., $1:N$ or $M:1$ mappings), each one representing a coordinate curve on the resulting surface. This approach was taken in [7] and is a simple way of constructing higher dimensional mappings. However, one cannot obtain optimal performance at high SNR in this way: Take a $(m + n):2$ mapping when $M > N$ and a $2:(m + n)$ mapping when $M < N$, both realized as two parametric curve-based systems in parallel. A $m:1$ and $n:1$ mapping when $M > N$ and a $1:m$ and $1:n$ mapping when $M < N$. Then we have

Proposition 6 (Sub-Optimality of Decomposable Mappings): Any $(m + n):2$ or $2:(m + n)$ mapping composed of lower dimensional sub-mappings will always follow $\text{SNR} \sim \text{SNR}^\ell$ as $\text{SNR} \to \infty$, with $\ell$, the dimension change factor of the sub-system with highest distortion. Therefore, such mappings will diverge from OPTA at high SNR.

Proof: See Appendix C-1.

Remark 15: A result for $M:N$ mappings follows from Proposition IV using several such systems in parallel.
with power allocation over all sub-systems (water filling [55, p. 277]).

Decomposable mappings should be avoided if one seeks a scheme approaching OPTA at high SNR. However, as will be seen in Section V, such mappings can be applicable in lower SNR regions.

The asymptotic result provided in the following do not refer to a particular $S$, as such the results are non-constructive. However, the results provide conditions for $S$ to potentially obtain OPTA in the limit $M,N \to \infty$. The results presented are extensions of [3].

We start with a proposition concerning anomalous distortion as $M,N \to \infty$:

**Proposition 6 (Asymptotic Anomalous Distortion):** Take normalized noise vectors $\hat{n} = n/\sqrt{N}$. With $\Delta_{\text{min}}$ as in Definition 7, let $n_{an}$ be the $K(N)\leq N$ dimensional sub-vector of $\hat{n}$ pointing in the direction of the closest point on another fold of $S$ along which $\Delta_{\text{min}}$ is measured. Then $\varepsilon_{an}^2 \to 0$ as $K,N \to \infty$ if $\Delta_{\text{min}} > 2\sqrt{K/N}/\sigma_n$, $\forall n$. 

**Proof:** By definition, $\hat{n}$ has mean length $\sigma_n$. It is shown in [43, pp. 324–325] that the variance of $\|\hat{n}\|$ decreases as $N$ increases and that $\lim_{N \to \infty} \|\hat{n}\| = \sigma_n$ with probability one. For $n_{an}$, a $K(\leq N)$ dimensional subset of $\hat{n}$, we get $\|n_{an}\| = \sqrt{K/N}\sigma_n$ with probability one. $\square$

**Remark 6:** Proposition 6 is sufficient to have a non-intersecting canal surface as in Lemma 1. I.e., $1/k_{\text{max}} \geq \rho_n \geq \sqrt{\frac{2^2}{N}\sigma_n^2(N-M)/N}$, with $K = N-M$, where $b_n \to 1$ as $M,N \to \infty$.

Proposition 6 is the key to performance gain through increased mapping dimensionality: Consider Definition 8. The distribution of $\rho = \|n_{an}\|$, $n_{an} \in \mathbb{R}^K$, is given by [56, p. 237]

$$f_\rho(\rho) = \frac{2(\frac{K}{2})^{\frac{K}{2}} \rho^{K-1}}{\Gamma(\frac{K}{2})\sigma_n^K} e^{-\frac{\rho}{2}\sigma_n^2}, \quad K \geq 1,$$  

(24)

where $\Gamma(\cdot)$ is the Gamma function [57]. Fig. 7(a) shows (24) for selected values of $K$. Note that the probability mass of $\rho$ becomes more located around $\sigma_n$ as $N$ increases. For $S$-K mappings, a gain can potentially be obtained when increasing dimensionality as $\Delta_{\text{min}}$ can be reduced: Consider $r = 2$, which can be accomplished by both 1:2 and 2:4 mappings. Take a 2:4 mapping with diagonal $G$ with $g_{11} = g_{22}$, both chosen optimally. The 2:4 mapping can then be "packed" more densely in the channel space as $f_\rho(\rho)$ narrows. That is, $\Delta_{2:4} < \Delta_{1:2}$ as Fig. 7(b) illustrates (note that the intersected 2:4 mapping in the figure is just an illustration, not an actual 2:4 mapping.). The $g_{ii}$'s can therefore be made larger with the 2:4 mapping, effectively reducing $\varepsilon_{an}^2$ and thereby the gap to OPTA. We provide a mapping obtaining such a gain for the $r = 2$ case in Section V-B-3.

**Remark 17:** Linear mappings do not introduce anomalous errors, so they cannot benefit from increased dimensionality.

According to Proposition 6, anomalous errors can be avoided as $M,N \to \infty$ if $\Delta_{\text{min}} \geq 2\sqrt{K/N}/\sigma_n$, which is the same as satisfying Lemma 1. To determine the optimal $S$, we must determine the smallest obtainable $\varepsilon_{an}^2$ under this condition without violating a channel power constraint.

As proving the existence of general hyper surfaces satisfying a distortion criterion is hard, if at all possible, we use a geometrical argument and consider how large volume the transformed source will occupy in the channel space, a generalization of results presented in [43, pp. 666–674]. In order to determine the volume $S$ occupies in the channel space it must be enclosed within an entity of $N$-dimensional volume. Arguments in [43, pp. 670–672] state that for $1$ mappings this is an $N - 1$ dimensional tube, $S \times \mathbb{R}^{N-2}$, with constant radius $\rho_{MN} \geq \|n_{an}\| > \sqrt{\frac{2}{N}\sigma_n^2(N-M)/N}$ ($b_{NM} \to 1$ as $N \to \infty$), with the signal curve at its center. This is a canal surface after Definition 4 with characteristic $\mathbb{S}^{N-2}$, a $N - 2$ sphere with radius $\rho_{MN}$. Locally this canal surface can be approximated by $L \times \mathbb{S}^{N-2}$, with $L$ a line-segment. Referring back to Example 4, we locally have $\mathbb{L} \times \mathbb{S}^{N-2}$ when we are in the weak noise regime of Definition 3. Then, for $M : N$ mappings, one can consider extended canal surfaces in Definition 6. Thus, $S$ should be enclosed within $S \times \mathbb{S}^{N-1}$, which can locally be approximated by $B^M \times \mathbb{S}^{N-1}$ if we are in the weak noise regime of Definition 3. Here, $\mathbb{S}^{N-1}$ is a hyper sphere with radius $\rho_{MN} \geq \sqrt{\frac{2}{N}\sigma_n^2(N-M)/N} = \sqrt{1 - \frac{r^2}{\gamma - 1}b_{NM}\sigma_n}$ and $B^M$ is an $M$-dimensional ball of radius $\rho_{MN}$, which size corresponds to the stretching of $D_r$ through $S$.

**Definition 11 (Local $B^M \times \mathbb{S}^{N-M-1}$ Regime):** We say we are in the local $B^M \times \mathbb{S}^{N-M-1}$ regime whenever: i) Definition 3 is satisfied. ii) $\Delta_{\text{min}} = 2\rho_{MN} > 2\sqrt{1 - r^2}b_{NM}\sigma_n$ at every point of $S$. iii) Lemma 1 is satisfied, i.e., the canal surface $S \times \mathbb{S}^{N-1}$ has no characteristic intersection. $\square$

**Example 5:** Definition 11 is satisfied for the 1:3 mapping in Fig. 5 when the cylinder is a valid model locally along the whole curve.

**Remark 18:** For fixed SNR there is an optimal $\rho_{MN}$, inline with [32]: If $\sigma_n$ increases the performance will deteriorate due to anomalous errors, while if $\sigma_n$ decreases there will be non-utilized space available to stretch $S$, implying sub-optimal

Fig. 7. (a) Pdf of $\|n_{an}\|$ when $\sigma_n = 0.1$. (b) Performance improvement from increased mapping dimensionality for $r = 2$: The dashed curve is an intersection of a surface in $\mathbb{R}^4$. As $\Delta_{2:4} < \Delta_{1:2}$, it is possible to obtain a gain for the 2:4 mapping.
Assume that $\Delta_{\text{min}} = 2\rho_{MN}$ should decrease with increasing SNR, for any $r = N/M \in \mathbb{Q}([1, \infty))$, have anomalous distortion $\tilde{\varepsilon}^2_{\text{wn}} \to 0$ and potentially obtain weak noise distortion given by

$$\tilde{\varepsilon}^2_{\text{wn}} = \sigma_x^2 \left( 1 + \frac{P_N}{\sigma_n^2} \right)^{-r}.$$  \hspace{1cm} (25)

**Proof:** See Appendix C-.2.

Remark 19: [Summary of Conditions That $S$ Must Satisfy to Obtain the Distortion in (25)]

1. Definition 3 should be satisfied: $S$ should be nearly flat within a hyper-sphere of radius $\sigma_n$. The larger $\sigma_n$ is, the smaller $\kappa_{\text{max}}$ should be, so that the first term in (11) dominates.

2. At any point of $S$, $\Delta_{\text{min}} > 2\sqrt{(1-r^{-1})} \sigma_n$ to avoid anomalous errors. That is, the canal surface $S \times S^{N-M-1}$ should satisfy Lemma 1.

3. $S$ should fill the channel space as densely as possible while satisfying 1 and 2 above for a given power constraint in order to stretch (amplify) the source as much as possible and thereby minimize $\tilde{\varepsilon}^2_{\text{wn}}$, i.e., a $S$ with $\rho_{MN} = \Delta_{\text{min}} = \text{constant}$ to optimally utilize the channel space.

4. $S$ can be shape preserving (Corollary 1). This condition was chosen to simplify the proof of Proposition IV-A. It may therefore be that more general $S$ can obtain OPTA.

**Example 6:** What S-K mapping would satisfy the conditions in Remark 19? The parallel planes mapping illustrated in Fig. 2 and discussed in Remark 14 (but now residing in $D_r$) is a potential candidate. The special case of 1:2 is depicted in Fig 8. The mapping in Fig. 8(a) potentially fulfill all condition as $\kappa = 0$ and it fills the channel space properly with constant $\Delta_{\text{min}} = 2\rho_{MN}$. We will consider this mapping in Section V-B.3, where we show that it can achieve OPTA as $M, N \to \infty$. For a fully continuous mapping, consider the spiral in Fig. 8(b): It satisfies 2-4, but also 1 as long as $\kappa << \sigma_n^2$. That is, the spiral must have smaller curvature as the SNR drops, obtained by choosing $\Delta_{\text{min}}$ larger, inline with earlier efforts [5]. The question is if higher dimensional generalizations satisfying the above conditions can be constructed.

B. Dimension Reducing S-K Mappings

We determine under which conditions dimension reducing S-K mappings may achieve OPTA for $\forall r \in \mathbb{Q}(0,1)$ as $M, N \to \infty$.

Again we need to enclose $S$ within a canal surface. By similar reasoning as in Section IV-A, we obtain the canal surface $S \times S^{M-N-1}$, now residing in the source space. This canal surface can locally be approximated as $\mathbb{B}^N \times S^{M-N-1}$, with $\mathbb{B}^N$ a ball with radius $\rho_x$, a local representation of the transformed channel space in source space, and $S^{M-N-1}$, a hyper-sphere with radius $\rho_{MN}$, corresponding to the decision borders for approximation to a uniform $S$ (Definition 10).

**Definition 12** (Local $\mathbb{B}^N \times S^{M-N-1}$ Regime): We say we are in the local $\mathbb{B}^N \times S^{M-N-1}$ regime whenever: i) Definition 9 is satisfied ii) Definition 10 is satisfied with $\Delta = 2\rho_{MN}$, where $\rho_{MN}$ is the radius of $S^{M-N-1}$.

Condition i) states that $S$ must be approximately flat inside a sphere of radius $\alpha \sqrt{B_N} \sigma_n$ at any point, where $\alpha$ is the amplification factor in (19). Condition ii) ensures uniformity (Definition 10). Note that both i) and ii) will be satisfied iff $S \times S^{M-N-1}$ has no characteristic intersection, which limits the maximal principal curvature $\kappa_{\text{max}}$. Take the 3:1 case: then we have the canal surface in Fig. 5, but where $n_{\text{wn}}$ now corresponds to the approximation error and $n_{\text{wn}}$ corresponds to the channel error.

To determine optimal performance, the free parameter $\Delta$ must be determined. Different from the expansion case, $\Delta$ is not directly related to the noise variance and has to be found through a minimization process determining the optimal balance between the two distortion terms $\tilde{\varepsilon}^2_{\text{q}}$ and $\tilde{\varepsilon}^2_{\text{ch}}$ (see Eq.(114), Appendix C-.3). We have the Proposition.

**Proposition 8** (Minimum Asymptotic Distortion for Dimension Reducing S-K Mappings): Assume that $f_x(x)$ is Gaussian. Any non-decomposable shape preserving dimension reducing S-K mapping satisfying Definition 12, will in the limit $M, N \to \infty$, for any $r = N/M \in \mathbb{Q}([0,1])$, potentially obtain the distortion

$$D_{\text{min}} = \tilde{\varepsilon}^2_q + \tilde{\varepsilon}^2_{\text{ch}} = \sigma_x^2 \left( 1 + \frac{P_N}{\sigma_n^2} \right)^{-r}.$$ \hspace{1cm} (26)

**Proof:** See Appendix C-.3.

Remark 20: [Summary of conditions that $S$ should satisfy to obtain the distortion in (26)]

1. Definitions 9 and 12 must be satisfied: $S$ should be approximately flat within a sphere of radius $\alpha \sigma_n$, implying that larger $\sigma_n$ necessitates smaller curvature $\kappa_{\text{max}}$.

2. For fixed approximation distortion, the canal surface of $S$ should cover the source space with the least possible stretching and curvature to minimize channel distortion.

3. $S$ can be uniform (Definition 10) and shape-preserving (Corollary 2). These were chosen to simplify the proof of Proposition IV-B. More general mappings may therefore obtain OPTA.

**Example 7:** What $S$ satisfies these conditions? The mapping in Fig 6(a) satisfies 2-3 in the finite dimensional case. However, as in the expansion case, $\kappa << \sigma_n^2$, if point 1 should be satisfied. A mapping similar to the one shown in Fig. 8(a), now residing in the source space, satisfies all these conditions.
By making sure that that the ends of each line/plane are close in the channel space, anomalous errors are avoided. We proved that the generalization of this approach to an $M:1$ mapping of arbitrary dimensionality, i.e., the parallel planes mapping mentioned in Remark 14, can achieve OPTA when $M, N \to \infty$ at high SNR (see [58], Proposition 1).

Lastly, we state a condition on $\Delta$ following from the proof of Proposition IV-B, which will be important for mapping construction in Section V:

Corollary 3: For any uniform dimension reducing $S$, then $\varepsilon^2_{ch} \sim \Delta^{-2(M-N)/N}$ to obtain the same slope as OPTA as SNR $\to \infty$.

Proof: Follows from the proof of Proposition IV-B in Appendix C, Eqn (113).

Example 8: Take a uniform 3:2 mapping where $\varepsilon^2_{ch} \sim \Delta^2$ according to Proposition III-B.2. Then we need $\varepsilon^2_{ch} \sim 1/\Delta$, in order to obtain SDR $\sim$ SNR$^{2/3}$ as SNR $\to \infty$.

V. MAPPING CONSTRUCTION

Construction of 1:N or M:1 mappings follow more or less directly from results and conditions derived for curves throughout this paper as illustrated in [5]. However, when it comes to (hyper) surfaces some more care needs to be taken.

We mainly consider surfaces in $\mathbb{R}^3$, if not otherwise stated, in order to obtain simple and explicit results.

Earlier investigations [40, pp. 88–89] indicated that a diagonal $G$ with $g_{ii}(x_i) =$constant $\forall i$, is convenient as it avoids nonlinear distortion, providing a shape preserving mapping3 (see Corollary 1 and 2). Further, for general (source) distributions it can be convenient to choose

$$G(x_1, x_2) = \text{diag}[g_{11}(x_1), g_{22}(x_2)]$$

(27)

where $g_{ii}(x_i)$ can be optimized for the source pdf for each coordinate curve on $S$ (see [50, pp. 296–297] for 1:N mappings). However, Proposition IV and Corollary 3 reveal that the metric in (27) is not sufficient for a mapping to follow the same slope as OPTA curve as SNR $\to \infty$.

Coordinate curves on $S$ where $g_{ii}$ only depends on $x_i$ are possible for certain sub-families of surfaces: An isometric mapping between two surfaces $S$ and $S^*$ is length preserving, i.e., $g_{i\beta} = g^*_{i\beta}$ under the same choice of coordinates [47, pp. 176–177]. Any $S$ that has a metric like (27) can be mapped isometrically to the Euclidean plane, and Theorem 59.3 in [47, p. 189] states that this has to be a developable surface:

Definition 13: Developable surface in $\mathbb{R}^3$:

Consider a ruled surface (RS), obtained by a set of straight lines, $z(\ell)$, named generators, interrelated through a space curve $y(\ell)$, named indicatrix [47, p. 181], as $S(\ell, t) = y(\ell) + tz(\ell)$. $z$ is a unit vector linearly independent of the tangent vector $y$, i.e., $y \times z \neq 0$. $y(\ell)$ acts like the trajectory for a straight line through space, and both $z$ and $y$ are coordinate curves on $S$. The RS is a developable surface (DS) $\iff [y \times z] = 0$ according to Theorem 58.1 in [47, p. 182]. □

For any DS, $g_{ii}$ can be made constant equal to 1 $\forall x_i$, by arc-length parametrization of $y$. An example of DS is shown in Fig. 9(a) in Section V-A.1.

Remark 21: As all DS can be seen as a straight line (a 1:1 system) moved along a curve $y$ (a $M:1$ or $1:N$ system), any DS will diverge from OPTA as SNR grows large, including the higher dimensional mappings proposed in [7].

To avoid the problem of non-optimal slope one has to widen the set of mappings beyond DS, keeping a similar type of $G$ as in (27). The most direct generalization are surfaces that can be mapped in an angle preserving way (or conformally), to the Euclidean plane. The metric between two surfaces $S$ and $S^*$ are then proportional, i.e. $g^*_{i\beta} = \eta(u^1, u^2)g_{i\beta}$ [47, pp. 193–194]. A subset of such surfaces are shape preserving, for example when $\eta$ is a constant.

Remark 22: Another condition for an S-K mapping to minimize channel power and reduce the effect of noise is to construct the mapping so that source vectors with highest probability are allocated to channel representations with low power [59, p. 103]. Also, the condition in Corollary 3 is important to consider for dimension reducing mappings.

In the following we illustrate the results of this paper with focus on 3:2 and 2:3 mappings. We also provide an example on the effect of dimensionality increase in the $r = 2$ case.

A. Examples on 3:2 Mappings

Three 3:2 mappings selected based on intuition obtained through previous sections are evaluated: The first two are selected from the Encyclopedia of Analytical Surfaces [60] as likely candidates for well-performing mapping based on the criteria of this paper: One DS-based mapping as well as a non-decomposable mapping. The third is a piecewise smooth (HDA) mapping specifically constructed to satisfy all requirements for obtaining the slope of OPTA at high SNR.

To evaluate performance of the example mappings we compare them with OPTA and block pulse amplitude modulation (BPAM) [37], the optimal linear mapping. Obviously, any meaningful choice of nonlinear mapping should rise well above BPAM as the SNR increases. At the end of the section all suggested schemes are compared to existing mappings.

1) Right Cylinder With Archimedes Spiral Directrix (RCASD): Fig. 9(a) depicts the RCASD in the source space. Its parametric equation is given by [60, p. 51]

$$S(z) = [\pm a \varphi(z_1) \cos(\varphi(z_1)), \pm a \varphi(z_1) \sin(\varphi(z_1)), \alpha_2 z_2],$$

(28)

with $\alpha_2$ some amplification factor and $a = \Delta/\pi$, where $\pm$ refers to positive and negative channel values, $\Delta$ is the smallest distance between the two “spiral surfaces”, or folds, seen in Fig. 9(a). The RCASD is a DS with Archimedes spiral as directrix (a 2:1 sub-mapping treated in [5]).

The components of FFF are

$$g_{11} = \left(\frac{\Delta}{\pi}\right)^2 (\varphi'(z_1))^2 (1 + \varphi^2(z_1)),$$

$$g_{22} = a_2^2, g_{12} = g_{21} = 0,$$

(29)
The components in (29) are computed as $g_{αβ} = S_{α} \cdot S_{β}$, and the components of the SFF are computed using Eq. (33) in [46, p. 4]. With (29) and (30), one can from Theorem 2 in Appendix A-B, conclude that the coordinate curves are LoC as $g_{12} = b_{12} = 0$. Therefore (20) describes the 2nd order behavior of this mapping.

**Evaluation of Curvature**: With LoC coordinates, the principal curvatures are found from the fundamental forms as:

$$κ_1 = \frac{b_{11}}{g_{11}} = -a(ϕ'(z_1))^2 \frac{2 + ϕ^2(z_1)}{\sqrt{1 + ϕ^2(z_1)}}, \quad κ_2 = \frac{b_{22}}{g_{22}} = 0. \quad (31)$$

By choosing

$$ϕ(z_1) = \pm \sqrt{α_1 z_1 / (η Δ)}, \quad (32)$$

with $α_1$ some amplification factor, one approximates arc-length parametrization along the directrix as shown in [5]. Evaluation of $κ_1$ as function of the free parameters $Δ$ and $α_1$ is provided in [46, p. 23], Fig. 18(a). Generally, the curvature is relatively small for this mapping. By inserting optimized parameters for 30dB SNR found by the optimization procedure below, i.e., $Δ' = 0.608$ and $α_1 = 3.33$, one obtains $|κ_1| < 1$ averaged over the relevant range of $z_1$. By considering the distortion terms in (12), with total transmission power 1, then $σ_n^2 = 0.001$, and one can see that the 1st order term is in the order of about $10^{-3}/(10^{-3})^2 = 1000$ over the 2nd order term. Therefore, RCASD is a mapping following Definition 3 at high SNR.

**Optimization of RCASD as 3:2 Mapping**: The RCASD’s performance is made scalable with SNR through the factor $a = Δ/π$ where $Δ$ is adapted to the noise variance $σ_n^2$.

**Distortion**: With $z_i = z_i + n_i$ mapped through (28), the channel distortion is computed from (18). With $ϕ$ as in (32), $g_{11} ≈ α_1^2$, $∀ z_1, z_2$. Similarly, since $x_3 = α_2 z_2$, then $g_{22} = α_2^2$, and so $G$ is diagonal with constant $g_{αi}$’s, which was one of the criteria sought. Therefore

$$ε_{ch}^2 = \frac{σ_n^2}{3} \int \sum_{i=1}^{2} g_{ii}(z) f_{x_i}(z) dz = \frac{σ_n^2}{3} (α_1^2 + α_2^2) f_{x_2}(z) dz = \frac{σ_n^2}{3} (α_1^2 + α_2^2). \quad (33)$$

From Fig. 9(a) one can see that we have a uniform S-K mapping (Definition 10), implying that Eq. (23) with $N = 2$ and $M = 3$ applies exactly as $M = N + 1$,

$$ε_q^2 = Δ^2/36. \quad (34)$$

**Power**: The Archimedes’ spiral directrix was applied as 2:1 mapping in [5]. With $ϕ$ as in (32) it was shown that a Laplace distribution over $z_1$ is obtained with variance $σ_q^2 = 2(2nσ^2Δ/Δα_1)^2$, $η = 0.16$, at high SNR. As $z_2 = x_3/α_3$, $z_2$ has a Gaussian distribution with variance $σ_{z_2}^2 = σ_x^2/α_2^2$. Therefore, the total channel power becomes

$$P_t = \frac{1}{2} \left( 2 \frac{2nσ^2Δ}{Δα_1} + \frac{σ_q^2}{α_2^2} \right). \quad (35)$$

**Optimization**: With constraint $C_t = P_{max} - P_t(Δ, α_1, α_2) \geq 0$, the following objective function is obtained,

$$L(Δ, α_1, α_2) = ε_q^2(Δ) + ε_{ch}^2(α_1, α_2) - λC_t(Δ, α_1, α_2). \quad (36)$$

The optimal parameters are found by a numerical approach, as in [40, pp. 87].

The performance of the optimized RCASD is shown in Fig. 9(b) (red curve). The RCASD improves with SNR, rising well above BPAM as SNR increases, and is also robust to varying SNR, having both graceful improvement and degradation for a fixed set of parameters (red dashed curve). The calculated performance is also shown (green curve) in order to demonstrate the accuracy of the theory in Section III-B. The distortion contributions in Section III-B is observed from the robustness graphs: $ε_q^2$ dominates above the optimal SNR, whereas below, $ε_{ch}^2$ dominates. Simulated and calculated performance correspond well, confirming that RCASD follows Definition 9, inline with the curvature evaluation above. However, the slope at high SNR follows that of 2:1 OPTA (black dashed curve) which is expected from Proposition IV as the RCASD is decomposable into a 2:1- and 1:1 system.
2) Snail Surface: The snail surface is non-decomposable and covers a spherical subset of the source space properly. Its parametrization has components [60, p. 280]

\[
\begin{align*}
S_1(z_1, z_2) &= a\varphi(z_1) \sin(\varphi(z_1)) \cos(\alpha_2 z_2 + \phi), \\
S_2(z_1, z_2) &= b\varphi(z_1) \cos(\varphi(z_1)) \cos(\alpha_2 z_2 + \phi), \\
S_3(z_1, z_2) &= -c\varphi(z_1) \sin(\alpha_2 z_2 + \phi),
\end{align*}
\]

which are valid for \(0 \leq z_1 \leq k\pi, -\pi \leq z_2 \leq \pi\). To include negative values of \(z_1\), i.e., \(-k\pi \leq z_1 \leq 0\), one flips the sign of all components in (37), and obtain a double snail surface (DSS), as depicted in Fig. 10(a). By choosing \(\psi = \pi/2\) and \(a = b = c = 2\Delta/\pi\) one obtains a spherical symmetry which leads to a (close to) uniform S-K mapping (Definition 10), and so (34) approximates \(f_{\rho}^2\), \(\phi\) will be decided later. For a general \(\varphi(z_1)\), the metric tensor is found to be [46]

\[
\begin{align*}
g_{11} &= (a\varphi'(z_1))^2(1 + \varphi^2(z_1) \cos^2(\alpha_2 z_2 + \phi)), \\
g_{22} &= a^2\alpha_2^2 \varphi^2(z_1), \\
g_{12} &= g_{21} = 0.
\end{align*}
\]

By inserting \(\varphi(z_1) = \alpha_1 z_1\) in (38) one observes that \(g_{ii} \sim z_1^2\), \(i = 1, 2\), and so \(\varepsilon_{ch}^2\) increases with \(z_1^2\). One can compensate this for both \(g_{ii}\) components simultaneously by choosing \(\varphi \sim \sqrt{z_1}\). As the RCASD also has \(g_{11} \sim z_1^2\) when \(\varphi(z_1) = \alpha_1 z_1\), and that DSS scales with \(\Delta\) like RCASD, it makes sense to use (32) for DSS as well, the choice of \(\eta\) being arbitrary.

a) Evaluation of curvature: The components of the SFF, as derived in [46, p. 23], are

\[
\begin{align*}
b_{11} &= -\frac{a(\alpha_1^2 z_1^2 \varphi^2(z_1) \cos^2 \theta)}{\sqrt{1 + \varphi^2(z_1) \cos^2 \theta}}, \\
b_{12} &= \frac{a\alpha_1 z_1 \varphi^2(z_1) \sin \theta}{\sqrt{1 + \varphi^2(z_1) \cos^2 \theta}}, \\
b_{22} &= -\frac{a\alpha_2 z_2 \varphi^2(z_1) \sin \theta}{\sqrt{1 + \varphi^2(z_1) \cos^2 \theta}},
\end{align*}
\]

As the coordinates are not LoC, the principal curvatures are the roots of (69) in Appendix A-B

\[
\begin{align*}
\kappa_{1/2} &= \frac{1}{2} \left( \frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \pm \sqrt{\left( \frac{b_{11}}{g_{11}} + \frac{b_{22}}{g_{22}} \right)^2 - \frac{4b_{12}b_{22} - b_{11}b_{22}^2}{g_{11}g_{22}}} \right),
\end{align*}
\]

Evaluation of \(\kappa_i\) as function of the free parameters \(\Delta\), \(\alpha_1\) and \(\alpha_2\) is provided in [46, p. 23], Fig. 18(b). Not surprisingly, the curvature is larger than for RCASD in general, particularly when \(\Delta\) is large and \(\alpha_1\) is small (corresponding to low SNR case). However, when \(\Delta\) is small and \(\alpha_1\) is large, corresponding to high SNR case, the curvature is relatively small: By inserting optimized parameters for 30dB SNR found by the optimization procedure below (\(\Delta^* = 0.539, \alpha_1^* = 4.76, \alpha_2^* = 2.57\)) one obtains maximal curvature \(\kappa_{21} < 1\) averaged over the relevant range of \(z_1\) [46, p. 23]. Similar to the RCASD, the DSS can be considered to follow Definition 3 at high SNR.

b) Optimization of DSS as 3:2 mapping: Channel Power and Density Function: To evaluate the channel input from DSS it is convenient to analyze variation over each channel separately (see [46] p. 21–22 for more details).

Take channel 1 first: It was argued in [46, p. 21] that the mapping from DSS to \(z_1\) can be approximated as the radius, \(\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}\), tracing out points inside a sphere as \(z_1\) and \(z_2\) vary over their domains. Then, with \(\Delta\) small, one can approximate the mapping \(x \rightarrow z_1\) as a continuous function \(h : \mathbb{R}^3 \rightarrow \mathbb{R}\). This assumption becomes more accurate as SNR grows, i.e., as \(\Delta\) decreases. By choosing \(\varphi = (\gamma z_1)^n, n \in \mathbb{Q}^+\), then \(z_1 = h(x_1, x_2, x_3) = 1 - \gamma a^n(x_1^2 + x_2^2 + x_3^2)^{n/2} = \pm \gamma \rho^n \rho^n = \ell(\rho)\), and we have:

\[\text{Lemma 2: At high SNR, with } \varphi = (\gamma z_1)^n, \text{the pdf for } z_1 \text{ when } S \text{ is a DSS, is approximated by} \]

\[
f_{z_1}(z_1) = \frac{mn^3\gamma z_1^{3n-1}}{\sqrt{2n\sigma_3^2}^{3/2}} e^{-\frac{a^2 z_1^2}{2\gamma}\frac{1}{\sigma_3}}.
\]

\[\text{Proof: See Appendix D.1.}\]

Now assume that \(\varphi(z_1)\), then \(\gamma = \sqrt{\alpha_1/(\eta \Delta)}\), and thus

\[
f_{z_1}(z_1) = \frac{\alpha_1^{3/2} z_1 \sqrt{\eta \Delta}}{2\sqrt{2n\sigma_3^2}^{3/2}} e^{-\frac{a^2 z_1^2}{2\gamma}\frac{1}{\sigma_3}}.
\]

According to [61, p. 87 and 154] a Gamma distribution has the form \(f_{\Gamma}(x) = u(x) x^{c-1} e^{-x/\Gamma(c)b^2}\) with second moment \(E[x^2] = (c+1)b^2\). Therefore (42) is a double Gamma distribution with \(c = 3/2\) and \(b = (2\eta \Delta a_2^2)/(a^2 \alpha_1)\). Since (42) has zero mean, the power on channel 1 becomes

\[
P_1 = \text{Var}[z_1] = \frac{15(\eta \Delta a_2^2)^2}{a^4 \alpha_1^4} = \frac{15(\eta \Delta a_2^2)^2}{16a_1^4 \Delta^2}.
\]

Now take channel 2: \(z_2\) makes out a circle which “rotates” about the \(x_3\) axis when \(z_1\) varies (see [46], Fig. 15(b) for illustration). The circle’s radius is given by \(2\Delta z_1/\pi\). If \(\phi = 0\),
the radius of the rotating circle does not affect the pdf of \( z_2 \) as \( z_2 \in [-\pi, \pi] \), independent of \( z_1 \). From the perspective of \( z_2 \), as the joint pdf of \( x \) is spherically symmetric, we have a uniform mass distribution over a virtual spherical shell of arbitrary radius, \( r_0 \), as depicted in Fig. 10(b).

To find the probability mass associated with different values of \( z_2 \), one considers the sum of all points along circles resulting from the intersections of this virtual sphere by planes perpendicular to the rotation axis (the green circle in Fig. 10(b)). The radius of this circle is \( r_1 = r_0 \cos(\upsilon) \), where \( \upsilon = z_2 \) is the angle from the equatorial plane, \( \upsilon = \pi/2 - \phi \), and \( \phi \) the polar angle. The circumference as a function of \( \upsilon \), the angle from the equatorial plane, is then

\[
f_r(x) = 2(\pi r_0 \cos(z_2)) \cos(\upsilon)
\]

implies that \( f_{z_2}(z_2) \sim |\cos(z_2)| \), \( z_2 \in [-\pi, \pi] \). To avoid high probability for the largest channel amplitude values, one can set \( \phi = \pi/2 \) to obtain zero probability there. Then a sine distribution results. To normalize, as \( \int_0^\pi \sin(z_2)dz_2 = 2 \), then

\[
f_{z_2}(z_2) = \frac{\alpha_2}{4} |\sin(\alpha_2 z_2)|.
\]

Since \( f_{z_2}(z_2) \) is proportional to Gilbert’s sine distribution \( f_{x}(x) = \sin(2x) \), which according to [62] has variance \( E\{x^2\} = (\pi^2/4 - 1)/2 \), the power for channel 2 becomes

\[
P_2 = \text{Var}\{z_2\} = \frac{2}{\alpha_2^2} \left( \frac{\pi^2}{4} - 1 \right).
\]

**Distortion:** Using (38), we obtain

\[
I_2 = \int \int g_{z_2}(z)f_{z}(z)dz = 2(\alpha_2^2 \frac{\alpha_1}{\eta_2^2}) \int_{-\infty}^{\infty} z_1 f_{z_1}(z_1)dz_1 = 3\alpha_2^2 \sigma_x^2.
\]

The last equality comes from the fact that \( z_1 \) is gamma distributed, therefore the integral in (46) becomes [61, p. 154] \( \frac{bc}{2} = 3\eta_2 \Delta \sigma_x^2/(a^2 \alpha_1) \). Further, we have (see [46] for details),

\[
I_1 = \int \int g_{z_1}(z)f_{z}(z)dz = \frac{\alpha_1 \Delta}{\eta_1^2} E\{z_1^{-1}\} + \frac{2\alpha_1^2}{3\pi^2 \eta_1^2}.
\]

Through power series expansion one can show that \( E\{z_1^{-1}\} \approx 4(1 + \text{Var}\{z_1\}) \) up to 3rd order. Inserting this into (47), the channel distortion results from (18),

\[
\tilde{\varepsilon}_{ch}^2 = \frac{\sigma_x^2}{3} \left( I_1 + I_2 \right) = \frac{\sigma_x^2}{3} \left( \frac{4\alpha_1 \Delta}{\eta_1^2} (1 + 16\eta_1^2 \alpha_1^2 \Delta^2) + \frac{2\alpha_1^2}{3\pi^2 \eta_1^2} \right) + \frac{2\alpha_1^2}{3\pi^2 \eta_1^2}
\]

\[
\approx \frac{\sigma_x^2}{3} \left( \frac{4\alpha_1 \Delta}{\eta_1^2} (1 + 16\alpha_1^2 \Delta^2) + \frac{2\alpha_1^2}{3\pi^2 \eta_1^2} + 3\alpha_2^2 \sigma_x^2 \right).
\]

**Optimization:** With constraint \( C_t = P_{\text{max}} - P_{\text{leak}}(\Delta, \alpha_1, \alpha_2) \geq 0 \), we get a similar objective function as in (36).

The performance of the optimized DSS is plotted in Fig. 10(c). Comparing with Fig. 9(b) it’s clear that DSS has better performance than the RCASD at high SNR, which is expected from Proposition IV. The DSS is also noise robust (green dashed curve), and the magenta line confirms that the theoretical model derived for DSS above is quite accurate. However, one can see that the gap to OPTA increases somewhat above 40dB, the reason being that \( \tilde{\varepsilon}_{ch}^2 \sim 1/\Delta^2 \) instead of \( \tilde{\varepsilon}_{ch}^2 \sim 1/\Delta \), which is required according to Corollary 3. Therefore, the DSS will eventually diverge from OPTA (albeit at a higher SNR than a decomposable mapping). One option that has not yet been investigated that may lead to the right slope is a change of coordinates curves on DSS.

3) 3:2 Hybrid Vector Quantizer Linear Coder (HVQLC): We construct a mapping satisfying all criteria for obtaining \( \text{SDR} \sim \text{SNR}^2/3 \) as \( \text{SNR} \rightarrow \infty \). A piecewise smooth (HDA) approach is taken: Consider approximating \( x \) to planes parallel to \( x_1, x_2 \)-plane in \( \mathbb{R}^3 \) with distance \( \Delta \) between them, like the parallel planes mapping suggested in Remark 14, depicted in Fig. 2, but now residing in \( D_s \). This is a uniform \( S \), and so \( \tilde{\varepsilon}_{ch}^2 \sim \Delta^2 \) from Proposition III-B.2. With parallel planes \( G = \alpha I \), with \( \alpha \) some scaling factor. To obtain a non-decomposable mapping, we map each plane onto the channel (\( D_s \)) with its center placed on Archimedes spiral, thereby obtaining a mix of several sources on each channel. A spiral is chosen for two reasons: i) The condition \( \tilde{\varepsilon}_{ch}^2 \sim 1/\Delta^2 \) in Corollary 3 is obtained, as shown in Proposition 9, ii) To easily scale the mapping with SNR: By choosing \( \varphi \) as in (32), an equal distance, \( \Delta \), between the spiral arms as well as between the centroids along each arm results. I.e., a uniform VQ on a disc, as illustrated in [40, p. 90]. The block diagram for the 3:2 HVQLC is depicted in Fig. 11(a). \( Q_\Delta \) is a uniform scalar quantizer producing an index, \( i \), mapped to two values through the 1:2 spiral, then scaled before transmission. The received vector \( \tilde{z} \) is projected to the closest discrete point on the spiral.
to reproduce the two index values. \(x_1\) and \(x_2\) are scaled, then transmitted.

**Optimization of 3:2 HVQLC:**

**Distortion:** A drawback of this mapping is that it introduces anomalous errors when centroids are mis-detected. This happens with probability \(Pr\{\|y_{12} + n\| \geq \Delta/(2\alpha_3)\}\), where \(y_{12} = [x_1 \ x_2]/\alpha\). The error is bounded by \(2b_x\sigma_x\), with \(b_x\) depending on the limiting of \(x_1\) and \(x_2\) described below. The pdf of \(y_{12} + n\) is the product of the distributions of \(x_i + n_i \sim \mathcal{N}(0, \sigma_y^2 + \sigma_n^2)\), \(i = 1, 2\) [61, pp. 181–182]. The variable \(w = \|y_{12} + n\|\) is then Rayleigh distributed [61, pp. 202–203], \(f_w(w) = \frac{w}{(\sigma_y^2 + \sigma_n^2)} \exp(-w^2/(2(\sigma_y^2 + \sigma_n^2)))\). Therefore the anomalous distortion becomes

\[
\varepsilon_{an}^2 = 4b_x^2\sigma_x^2 Pr\left\{w \geq \frac{\Delta}{2\alpha_3}\right\} = 4b_x^2\sigma_x^2 \int_{\Delta/(2\alpha_3)}^{\infty} f_w(w)dw = 4b_x^2\sigma_x^2 e^{-\frac{\Delta^2}{8\sigma_y^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)}}.
\]

(49)

In order to obtain a mapping where anomalous errors happen with low probability one will either condition \(\Delta/\alpha_3\) to be small, or limit \(x_1\) and \(x_2\) at some value by the green blocks in Fig. 11(a). By limiting \(x_1\) at the value \(b_x\sigma_x\), one introduces a distortion [63]

\[
\varepsilon_{an}^2 = \frac{4}{3} \int_{b_x\sigma_x}^{\infty} (x_i - b_x\sigma_x)^2 f_x(x_i)dx_i, \ i = 1, 2.
\]

(50)

By limiting each source separately, the HVQLC will result in parallel planes in \(D_s \subseteq \mathbb{R}^3\), whereas by limiting \(\sqrt{x_1^2 + x_2^2}\), parallel discs will be obtained.

To make the probability of anomalous errors small, the following constraint is needed

\[
\frac{\Delta}{\alpha_3} > \frac{2b_x\sigma_x}{\alpha_1} + 2b_n\sigma_n.
\]

(51)

The probability is adjusted with \(b_x\). With \(b_x > 4\) then 99.99% of all source values are included.

As mentioned above, \(g_{ii} = \alpha^2\) and so the channel distortion becomes \(\varepsilon_{ch}^2 = 2\sigma_x^2\alpha^2/3\). We also have a uniform \(S\). Therefore, the total distortion becomes

\[
D_t = \frac{\Delta^2}{36} + \frac{2\sigma_x^2\alpha^2}{3} + 4b_x^2\sigma_x^2 e^{-\frac{\Delta^2}{8\sigma_y^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)}}.
\]

(52)

**Power:** Since \(x_1\) and \(x_2\) are scaled Gaussians, their transmission power becomes \(P_1 + P_2 = 2\sigma_x^2/\alpha^2\). As \(x_3\) is mapped through a discretized version of the 1:2 mapping in [5], the same power expression applies for small \(\Delta\), or high SNR (a term in [5] is removed as \(x_3\) takes values over \(\mathbb{R}\)). In [5] the source was in the range \([-1, 1]: P_3 \approx 2\Delta\sigma_x/(\eta\sqrt{2\pi^2\alpha_3^2})\). The fact that \(P_3 \sim \Delta/\alpha_3^2\) makes \(\varepsilon_{ch}^2 \sim 1/\Delta\) as required by Corollary 3. The total power is then

\[
P_t = \frac{2\sigma_x^2}{\alpha^2} + \frac{2\Delta\sigma_x}{\eta\sqrt{2\pi^2\alpha_3^2}}.
\]

(53)

**Optimization:** To determine optimal performance we consider the Lagrangian

\[
L(\Delta, \alpha, \alpha_3) = D_t(\Delta, \alpha) - \lambda_1 C_1(\Delta, \alpha, \alpha_3) - \lambda_2 C_2(\Delta, \alpha, \alpha_2),
\]

where \(C_1 = P_{\text{max}} - (P_1 + P_2 + P_3)\) and \(C_2(\Delta, \alpha, \alpha_2) = \Delta - 2b_x\sigma_x\alpha_3/\alpha_1 - 2b_n\sigma_n\alpha_3\). The slight difference from the constraint in (51) is to obtain better numerical stability when solving (54).

The optimized performance of HVQLC (with no limitation of \(x\)), is shown in Fig. 11(b), magenta curve. The HVQLC seem to follow the OPTA slope at high SNR and it is noise robust (magenta dashed curve) despite of the anomalous errors. But anomalies are likely the reason why HVQLC has a larger distance to OPTA than DSS for SNR < 40 dB.

**High SNR analysis:** We prove that 3:2 HVQLC has the same slope as OPTA as \(\text{SNR} \rightarrow \infty\).

**Proposition 9:** 3:2 HVQLC at high SNR. At high SNR the SDR of 3:2 HVQLC follows

\[
\text{SDR} = \left(\frac{9\eta\sqrt{\pi^3}}{80b_x^2}\right)\text{SNR}^{\frac{1}{2}}.
\]

(55)

**Proof:** See Appendix D-2. \(\Box\)

From (55), with \(\sigma_x = 1, b_x = 4\) and \(\eta = 0.16\), the loss from OPTA becomes \(\text{SDR}_{\text{loss}} = -10\log(9 \cdot 0.16\sqrt{\pi^3}/(8 \cdot 16))^{2/3} \approx 4.7\text{dB}\), corresponding to the gap seen in Fig. 11(b).

4) **Comparison of Different 3:2 Schemes:** The performance of all 3:2 mappings proposed are compared in Fig. 11(b).

We also include the performance of the 3:2 mapping of Saidutta et. al. [8] found by deep learning, i.e., variational auto encoders (VAE), as well as power constrained channel optimized vector quantizer (PCCOVQ) [36], [64]. PCCOVQ is a numerically optimized discrete mapping replicating continuous (or piecewise smooth) mappings when the number of source- and channel symbols in the mapping is large. Approaching the VAE or PCCOVQ is a good indication of a well performing mapping as these are properly optimized structures.

Not surprisingly, Saidutta’s VAE mapping (gray dash-dot curve) and PCCOVQ (black dashed curve) have superior performance in the SNR range they have been optimized for. However, the proposed mappings are only about 1dB away from the reference systems. The RCASD is best at low SNR, the DSS is best between 20 and 45 dB, while the HVQLC is best from 45dB and above. The HVQLC is the only system that does not diverge from OPTA at high SNR. Although DS-based mappings, like RCASD, diverge from OPTA at high SNR, decomposable mappings have their virtue as a simple alternative that perform well at lower SNR.

Although the mappings proposed are inferior to the two optimized schemes, the loss is small, and they have the advantage of having a fixed geometrical structure that only needs to be scaled in order to adapt to varying SNR, thus lowering complexity. Its is also observed that widely different configurations provide well performing mappings. However, any such configuration will need to comply with the conditions presented in Sections III and IV.

**B. Examples on 2:3 Mappings and the Gain From Dimensionality Increase When \(r = 2\)**

First we analyze two 2:3 mappings: i) A piecewise smooth scheme, hybrid vector quantizer linear coders (HVQLC)

4The reason why the PCCOVQ declines above 22dB is because 4096 symbols are used during optimization, a too small a number for high SNR.
suggested in [40, pp. 89–93]. ii) The RCASD treated in Section V-A.1. Then we illustrate performance gain obtained by increasing mapping dimensionality when \( r = 2 \).

1) 2:3 HVQLC. This is a generalization of the hybrid scalar quantizer linear coder (HSQLC) proposed in [24]. The block diagram is depicted in Fig. 12.

Here VQ centroid indices are denoted by \( i \), \( e_1 \) and \( e_2 \) denote the two error components from the VQ, and \( \alpha_1, \alpha_2 \) are scaling factors to adjust channel power. To make the VQ adaptable to varying SNR, its centroids are placed on Archimedes’ spiral with distance \( \delta \) between them, see [40, p. 90] for illustration. Arc-length parametrization is chosen along the spiral for the same reason as for the 3:2 HVQLC. The scaled VQ indices are transmitted as PAM symbols on channel 1, while the scaled error components are transmitted on channels 2 and 3, leading to a “mix” of both sources on all three channels. Geometrically, the 2:3 HVQLC consists of planes parallel to the \( z_1, z_2 \)-plane in channel domain \( \delta_c \), as illustrated in Fig. 2 to the upper left, (see also [40, p. 90]), making it similar to 3:2 HVQLC (parallel planes in source domain \( \delta_s \)).

Distortion: \( \varepsilon_{\text{wn}}^2 \) can be found from (10), as the HVQLC is shape preserving. Only the error components \( e_1, e_2 \) contribute, and so \( \varepsilon_{\text{wn}}^2 = \sigma_{\text{e}}^2 / \alpha_1^2 \).

As the VQ indices are placed on the Archimedes’ spiral in an equidistant manner, the pdf of \( z_1 \) will be a discretized version of the pdf for RCASD directrix. i.e., a discretized Laplace pdf. For small \( \delta \), its variance can be approximated by the variance of a Laplace pdf. Therefore, the power on channel 1 is approximated as

\[
P_1 = \text{Var}(z_1) \approx 2\alpha_1^2 \left(2\eta \frac{\pi^2}{\delta^2} \sigma_{\text{e}}^2 \right)^2.
\]

Generally, the right hand side will be somewhat smaller than the real power, but the smaller \( \delta \) is (higher SNR) the better they coincide. Note particularly that \( \sigma_{\text{e}}^2 \sim 1/\delta^4 \), different from the 3:2 RCASD directrix where \( \sigma_{\text{e}}^2 \sim 1/\delta^2 \). The reason is that indices are sent on the channel and so the length measured along the spiral is independent of \( \Delta \). This difference in exponent is crucial to make HVQLC obtain the same slope as 2:3 OPTA.

For channels 2 and 3, assuming that \( \delta \) is small, \( e_1 \) and \( e_2 \) are uniformly distributed over \( (\delta/2) \times (\delta/2) \). Therefore, the power on channel 2 and 3 can be approximated by \( P_2 = P_3 \approx \alpha_2^2 \delta^2 / 12 \). The total channel power is then

\[
P_t = \frac{2}{3} \left(2\alpha_1 \eta \frac{\pi^2}{\delta^2} \sigma_{\text{e}}^2 \right)^2 + \frac{\alpha_2^2 \delta^2}{12}.
\]

Optimization: The Lagrangian is

\[
\mathcal{L}(\delta, \alpha_1, \alpha_2, \lambda) = \varepsilon_{\text{wn}}^2 (\alpha_2) + \varepsilon_{\text{wn}}^2 (\delta, \alpha_1) - \lambda c_t(\delta, \alpha_1, \alpha_2),
\]

where \( c_t(\delta, \alpha_1, \alpha_2) = P_{\text{max}} - P_t(\delta, \alpha_1, \alpha_2) \geq 0 \), with \( P_t(\delta, \alpha_1, \alpha_2) \) as in (58), and \( P_{\text{max}} \) the maximum power per channel. Numerical optimization is applied (see [40, p. 92]).

The performance of the optimized 2:3 HVQLC system is shown in Fig. 13(a) (green curve). The HVQLC is about 5dB from OPTA above 20dB SNR. It also seem to follow the slope of OPTA at high SNR, as will be shown in Proposition 10. Both simulated (green dashed curve) and calculated (magenta curve) robustness performance are shown, where the two distortion contributions can be seen: \( \varepsilon_{\text{wn}}^2 \) dominates above
the optimal SNR point and behaves like a linear scheme (having the same slope as BPAM) which is to be expected from Definition 3 and Proposition 1. Below the optimal SNR, $\varepsilon_{an}^2$ dominates, and is observed to diverge faster from OPTA than $\varepsilon_{wn}^2$. The theoretical model coincides well with simulations at high SNR.

**High SNR analysis:** We prove that 2:3 HVQLC has the same slope as OPTA as SNR $\to \infty$. To simplify one can eliminate anomalous errors by choosing $\alpha_1$ sufficiently large. By letting $\alpha_1 \geq 2 b_n \sigma_n$, with $b_n > 4$, 99.99% of all noise events are included, and the total distortion can be approximated as $D_t \approx \sigma_n^2/\alpha_1^2$.

**Proposition 10** (2:3 HVQLC at High SNR): At high SNR, the SDR of 2:3 HVQLC follows

$$\text{SDR} = \frac{7\sqrt{3}}{6\sqrt{2} b_n} \text{SNR}^2.$$

**Proof:** See Appendix D-3. $\Box$

With $b_n = 4$, $\eta = 0.16$, the loss from OPTA is $\text{SDR}_{\text{loss}} = -10 \log(7\sqrt{3}/(6\sqrt{2} b_n)) \approx 4.95$dB, corresponding to the performance gap seen in Fig.13(a).

2) 2:3 RCASD: The parametrization is the same as in (32), but now as a function of $x$. The distortion and power is easily derived using results from preceding sections and existing papers:

To compute $\varepsilon_{wn}^2$ we assume arc-length parametrization and obtain the same $G$ as for the 3:2 RCASD in Section V-A.1. Then (9) is reduced to $\varepsilon_{wn}^2 = 0.5\sigma_n^2 (\alpha_1^2 + \alpha_2^2 - 2)$. Furthermore, $\varepsilon_{wn}^2$ is the same as for the 1:2 mapping in [5], Eqn.(25), scaled by 0.5.

The power on channels 1 and 2 are also the same as for the 1:2 mapping in [5], and is given by $P_1 + P_2 = 2\Delta_2 \sigma_2^2(\eta \sqrt{2\pi})$. As $x_2$ is scaled with $\alpha_3$ and sent on channel 3, the power is $P_3 = \sigma_3^2 \alpha_3^2$. Then $P_t = (P_1 + P_2 + P_3)/3$.

The performance of the optimized RCASD is shown in Fig. 13(a). As expected from Proposition IV, the RCASD diverges from OPTA, following the slope of a 1:1 system at high SNR. Between 10-22 dB SNR, the RCASD outperforms the HVQLC, and so a DS-based mapping is advantageous at low to medium SNR. The correspondence between the calculated and simulated robustness curves indicate that the theoretical model fits well with reality at high SNR.

3) $M:N$ HVQLC for $r = 2$: The construction we consider here was briefly discussed in Example 6 as a potential candidate for obtaining OPTA in $r = 2$ case, satisfying all necessary conditions in Proposition IV-A. The approach is a generalization of the HSQCL treated in [24], using an $M$-dimensional uniform VQ instead of a scalar quantizer. We use a similar construction as for the 2:3 HVQLC in Fig. 12, except that the $M$ VQ centroid values (not the indices) are sent directly on the first $N/2$ channels after scaling by $\alpha_1$. The error from this quantization process is sent on the remaining $N/2$ channels after scaling by $\alpha_2$.

As this is a shape preserving mapping, the weak noise distortion is given by (10), $\varepsilon_{wn}^2 = \sigma_n^2/\alpha_1^2$. An anomalous error is, as for 2:3 HVQLC, equal to $\delta$, but now scaled with $1/M$. The probability for this event is found by integrating (24) from $\alpha_1 \delta/2$ to $\infty$, substituting $M = 2M$ so that $N = M$. From the integration rules of exponential functions, one can find a closed-form expression for the integrand $x^n e^{-x}$ as a series expansion. This expansion can be simplified by some straightforward rearrangements. Then the distortion is

$$D_t = \varepsilon_{wn}^2 + \varepsilon_{wn}^2 = \frac{\sigma_n^2}{\alpha_2^2} + \frac{\delta^2}{M} \left( \frac{M^2}{2} \frac{2^{-M/2}}{e^{\delta^2/2}} \sum_{i=0}^{M-1} \delta^{M-1-i} \right)$$

The pdf on the first $N/2$ channels can be approximated by a Gaussian distribution as $\delta \to 0$ (at high SNR). As the error of the VQ is sent on the remaining $N/2$ channels, the power is lower bounded by the sphere bound distortion for a $M$-dimensional VQ as stated in (90), Appendix B-B.3. The power per channel therefore becomes

$$P_t \approx \frac{1}{2} \left( \alpha_1^2 \sigma_n^2 + \alpha_2^2 \frac{M \delta^2}{4(M + 2)} \right).$$

A more realistic power expression for small block length is the error of a scalar quantizer on each channel. This is obtained by replacing the last term in (61) by $\alpha_2^2 \delta^2/12$.

By setting up a similar Lagrangian as in Section V-B.1, one can numerically determine the optimal parameters. We plot the optimized performance for HVQLC for selected values of $M, N$ in Fig. 13(b). Its clear that the HVQLC benefits from increased dimensionality.

**High SNR analysis:** We provide a sphere bound for HVQLC valid at high SNR. To simplify, we eliminate anomalous errors by choosing $\alpha_1 \geq 2 b_N \sigma_N$, with $b_N$ determined by integrating (24) from $b_N \sigma_N$ to $\infty$ so a very small probability results. Then one can approximate $D_t \approx \sigma_n^2/\alpha_1^2$.

**Proposition 11** (Sphere Bound for $M:N$ HVQLC With $r = 2$ at High SNR): At high SNR, the SDR of $r = 2$ HVQLC has the sphere bound

$$\text{SDR} = \frac{1 + 2/M}{b_N^2} \text{SNR}^2,$$

and so, as $b_N \to 1$ when $M, N \to \infty$, the HVQLC obtains OPTA.

**Proof:** See Appendix D-4. $\Box$

C. Remarks for Both 3:2 and 2:3 Mappings

From the analysis and simulations in Sections V-A and V-B it may appear that fully continuous mappings based on surfaces cannot obtain the slope of OPTA at high SNR. This is in contrast to S-K mappings realized by continuous surfaces is due to sub-optimal choice of coordinate system on non-decomposable mappings like DSS. Just as curve-based mappings will diverge from OPTA if $\varphi$ is chosen non-wisely [40], it may be that the divergence observed for continuous surfaces is due to sub-optimal choice of coordinates on $S$. Its worth noting the VAE-based 3:2 mapping in [8], a structure quite similar to the DSS, is continuous and seems to follow OPTA at high SNR. However its performance is only evaluated up to 30dB, where also DSS follows the slope of OPTA, so this is inconclusive. An intuitive choice of coordinate curves for dimension reducing mappings are

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geodesics (see [46, p. 11] or [47, pp. 162–168]) as they represent the path of minimum length between points on \(S\), and thereby reduces the \(g_{ii}\)’s. Determining the optimal coordinate system in general is difficult even for 2D surfaces, and should be followed up in future effort(s).

VI. SUMMARY, DISCUSSION AND EXTENSIONS

In this paper a theoretical framework for analyzing and constructing continuous or piecewise smooth mappings used for joint source-channel coding, named named Shannon-Kotel’nikov (S-K) mappings, has been proposed for the case of memoryless sources and channels.

The distortion framework introduced describes S-K mapping behaviour in general, that is, without reference to a specific mapping realization. Also, the framework provides guidelines for construction of well-performing mappings for both low and arbitrary complexity and delay.

We showed that any S-K mapping which can be decomposed into combinations of lower dimensional sub-mappings cannot obtain the same slope as OPTA at high SNR. We also showed that the performance of S-K mappings has the potential to improve under certain criteria when their dimensionality (or block length) is increased, potentially obtaining OPTA when their dimensionality goes to infinity.

Further, the provided theory was applied to construct mappings for 2:3 and 3:2 cases as well as \(M:N\) case for \(r = 2\). The mappings display decent performance. albeit some of them are inferior to mappings found by deep learning, the loss is small (about 1dB), and the mappings found can easily be adapted to varying channel conditions simply by scaling one given structure, thereby reducing complexity. The \(M:N\) mapping was shown to obtain OPTA as \(M, N \to \infty\).

The conditions stated throughout this paper can provide constraints on numerical approaches like [6] and [65] that may provide mappings closer to the global optimum without having to input a pre-determined, close to optimal mapping. The conditions presented may also provide deeper understanding of why certain configurations are favored by deep learning approaches [8], [29].

Future extensions should deal with optimal global structure, low SNR case and optimal choice of coordinate curves for S-K mappings. The theory could also be extended to multimodal source distributions, correlated sources and channels, as well as simple network structures. The case of a bivariate source transmitted on two channels has been treated in [66] and the case of multivariate sources communicated on a Gaussian multiple access channel has been treated in [45].

APPENDIX A

CONCEPTS FROM DIFFERENTIAL GEOMETRY

A. Arc-Length Parametrization, Differential Geometry of Curves and Formula of Frenet

Let \(S : u \in [a, b] \subseteq \mathbb{R} \to S(u) \subseteq \mathbb{R}^N\) with \(S(u) \in C^1\) be a parametrization for the curve \(C\), w.r.t. \(\ell(u)\), where \(\ell(u)\) is the arc-length of \(S\) as defined in (4), and \(\phi\) its inverse.

**Theorem 1:** Let \(\gamma(\ell)\) be a parametrization of \(C\). Then \(\gamma(\ell)\) and \(S(\phi(u))\) will have the same image, and \(\|\gamma'()\| = \|S'(\phi(\ell))\| \equiv 1, \forall \ell.\)

**Proof:** See [67, pp. 115–116].

There are \(n\) orthonormal vectors connected to any curve \(C : S(u) \in \mathbb{R}^n\), and related to these are the generalized curvatures [68]. In this paper we will only need three of them. The unit tangent vector \(t = S' / \|S'\|\), the unit principal normal vector \(p_n = \hat{t} / \|\hat{t}\| = S / \|S(x_0)\|\), and the unit binormal vector which in \(\mathbb{R}^n\) is defined as \(b = tx \times p_n\). In \(\mathbb{R}^n\), \(b\) is found by a Gram-Schmidt orthogonalization process [69, pp. 172–173] involving the vectors \(t, p_n, \hat{S}, t, p_n\) and \(b\) define three mutually orthogonal planes. We consider two of them: i) Osculating plane spanned by \(t\) and \(p_n\), ii) Normal plane spanned by \(p_n\) and \(b\).

For a curve, \(S(u)\), the principal curvature \(\kappa\) w.r.t. arc-length is defined as \(\kappa_0 = \|S(u_0)\|\) [47, p. 34]. Then we also have \(p_n = (1/\kappa)S = \rho S\). The torsion [47, p. 37–40] is defined as \(\tau(x) = - \rho \cdot b\). For a general parametrization we have [47, pp. 35, 39]

\[
\kappa = \sqrt{\|S'\|^2 \|S''\|^2 - (S' \cdot S'')^2 \|S'\|^2}.
\]

The principal curvature can locally be interpreted as a circle of radius \(\rho = 1/\kappa\), named radius of curvature, lying in the osculating plane of \(S\). This is the osculating circle and its center is the centre of curvature. i.e., the curvature in a neighborhood of \(u_0\) is equivalent to that of a circle with radius \(\rho\) (illustration is given in [46], Fig. 3). When \(\tau = 0, \forall u\), \(S\) will be confined to a plane, otherwise it will twist up into space.

The Formula of Frenet (FoF) [47, p. 41] relates the derivatives \(t, p_n\) and \(b\) to linear combinations of \(t, p_n\), and \(b\) of curve \(C\) as follows:

\[
t = sp_n, \quad p_n = -\kappa t + \tau b, \quad b = -\tau p_n.
\]

B. Surfaces, Fundamental Forms and Curvature

To keep the introduction to curvature of surfaces short and informative, we mainly introduce concepts for 2D surfaces in \(\mathbb{R}^3\). All relevant concepts are extendable to hyper surfaces in \(\mathbb{R}^n\). However, as we do not need to explicitly evaluate particular components or values for the sake of this paper (with the exception of the metric tensor), we do not need the more elaborate and subtle generalized definitions. The reader may consult [48] for details.

1) Fundamental Forms: **First fundamental form (FFF)***

Consider a hyper surface, \(S\), realized by (2) or (3). In order to measure lengths, angles and areas on \(S\), a metric is needed. A length differential of a curve \(C \in S\) is given by [47, p. 82]

\[
d^2 = (S_1(du^1 + S_2(du^2)) \cdot (S_1(du^1 + S_2(du^2)) = S_1 \cdot S_1(du^1)^2 + 2S_1 \cdot S_2(du^1) \cdot S_2(du^2) + S_2 \cdot S_2(du^2)^2.
\]

A generalization to M-dimensional surfaces is obtained by adding \(M\) “\(S_\alpha(du^\alpha)\)-terms” in the above expression. The quantities \(g_{\alpha\beta} = S_\alpha \cdot S_\beta, \alpha, \beta = 1, \cdots, m\), are components of a 2nd order covariant tensor (see [47], pp. 88–105) or [46] for definition) named metric tensor. By Einstein’s summation convention, \(d^2 = g_{\alpha\beta}(du^\alpha)^2(du^\beta)^2\), which is the first fundamental form (FFF). For a smooth embedding \(S \in \mathbb{R}^N\) (\(M \leq N\) the
metric tensor is a symmetric, positive definite \( M \times M \) matrix \( G = J^T J \) [52, pp. 301–304], with \( J \) the Jacobian of \( S \), a \( N \times M \) matrix with entries \( J_{ij} = \partial S_i / \partial u_j \), \( i = 1, \ldots, N \), \( j = 1, \ldots, M \). I.e., \( g_{ij} \) is the squared norm of the tangent vector along the \( i \)'th coordinate curve of \( S \). All cross terms, \( g_{ij} \), are inner products of tangent vectors along the \( i \)'th and \( j \)'th coordinate curve of \( S \).

Second fundamental form (SFF): For any point, \( P \), of a curve \( C \) on \( S \subset \mathbb{R}^3 \), the corresponding unit normal to \( S \), \( N = S_1 \times S_2 /\| S_1 \times S_2 \| \), lies in the normal plane \( C \) of \( S \) which also contains its principal normal \( p_n \). The angle between \( N \) and \( p_n \), denoted \( \gamma \), depends on the geometry of both \( C \) and \( S \) in a neighborhood of \( P \). Its shown in [44, pp. 118–119] that \( \kappa \cos \gamma = (S_{\alpha \beta} \cdot N) u^\alpha u^\beta \), with \( \kappa \), the principal curvature of \( C \). The term in the parentheses,

\[
b_{\alpha \beta} = S_{\alpha \beta} \cdot N, \quad \alpha, \beta = 1, \ldots, N, \tag{66}
\]

depends on \( S \) only (independent of \( C \)), and \( b_{\alpha \beta} = b_{\beta \alpha} \). The \( b_{\alpha \beta} \)'s are components of a 2nd order covariant tensor, where the quadratic form \( b_{\alpha \beta} u^\alpha u^\beta \) is the second fundamental form (SFF). The SFF exist for hyper surfaces as well, but its definition is more elaborate.

2) Normal Curvature, Principal Curvature, Lines of Curvature: Let \( t \) be any allowable parameter for curve \( C \). Then \( \dot{u}^\alpha = (du^\alpha / dt)(dt/d\ell) = u^{\alpha'} / \ell' \), and therefore

\[
\kappa \cos \gamma = b_{\alpha \beta} \dot{u}^\alpha \dot{u}^\beta = b_{\alpha \beta} u^{\alpha'} u^{\beta'} = \frac{\ell'}{g_{\alpha \beta} u^{\alpha'} u^{\beta'}} = g_{\alpha \beta} b_{\alpha \beta} u^{\alpha'} u^{\beta'}. \tag{67}
\]

It is shown in [44, pp. 121–124] that \( \kappa_n = \kappa \cos \gamma \) is the curvature at point \( P \in S \) of the normal section \( C \subset S \), an intersection of \( S \) by a plane passing through both \( t \) of \( C \) and \( N \) of \( S \), named normal curvature at \( P \). The Theorem of Meusnier [44, p. 122] states that one can restrict treatment of curvature at any point \( P \in S \) to that of normal sections without loss of generality.

The directions where \( \kappa_n \) has extremal values can be determined (except when \( b_{\alpha \beta} \sim g_{\alpha \beta} \)). From (67) one can derive [44, pp. 128–129],

\[
(b_{\alpha \gamma} - \kappa_n g_{\alpha \gamma}) u^\alpha = 0, \quad \gamma = 1, 2. \tag{68}
\]

The roots of (68) are directions for which \( \kappa_n \) is extreme, named principal directions of normal curvature at \( P \). The corresponding curvatures, \( \kappa_1 \) and \( \kappa_2 \), are the principal curvatures of \( S \), corresponding to the maximal and minimal curvature of \( S \) at \( P \), respectively. It is proven in [44, p. 129] that the roots of (68) are real and the principal directions are orthogonal at every point. Further, a curve on \( S \) whose direction at every point is a principal direction is a line of curvature (LoC) of \( S \). One may always choose coordinates \( u^1, u^2 \) on \( S \) so that the LoCs are allowable (see [46, p. 2]). Then [44, p. 130]:

**Theorem 2:** The coordinate curves of any allowable coordinate system on \( S \) coincide with the LoC \( g_{12} = 0 \) and \( b_{12} = 0 \), at any point where those coordinates are allowable.

\[
\text{Proof: See [44, p. 130].} \quad \Box
\]

Under LoC coordinates, (68) leads to \( \kappa_i = b_i/g_{ii} \). For general coordinates, with \( B \) the matrix of \( b_{\alpha \beta} \), \( \kappa_i \) are the roots of [47, p. 130]

\[
\kappa_i^2 - b_{\alpha \beta} g^{\alpha \beta} \kappa_i + \det(B)/\det(G) = 0. \tag{69}
\]

Here \( g^{\alpha \beta} \) is the contravariant metric tensor (see [46, pp. 3–4]) defined so that \( g_{\alpha \beta} g^{\alpha \beta} = \delta_\alpha^\beta \). Note that the \( \kappa_i \)'s as defined above are also valid for \( M \)-dimensional surfaces, i.e., \( i = 1, \ldots, M \).

**APPENDIX B**

**PROOFS FOR SECTION III**

**A. Proofs for Section III-A**

1) **Proof, Proposition 1:** For \( S \subset S \), assume that \( S_i \in C^1(\mathbb{R}^M) \), \( i = 1, \ldots, N \). The tangent space at \( x_0 \) is given by (8). Applying ML detection, then \( S(x_{ML}) = S(x_0) + P_{proj} \) (see Fig. 3(a)), where \( P_{proj} \) is a projection matrix given by [69, p. 158]

\[
P_{proj} = J(x_0)(J(x_0)^T J(x_0))^{-1} J(x_0)^T. \tag{70}
\]

Setting \( S_{lim}(x) = S(x_{ML}) \), with \( S_{lim} \) as in (8) and from (70), we get \( J(x_0)(x_{ML} - x_0) = J(x_0)(G(x_0))^{-1} J(x_0)^T n \). Multiplying both sides from the left with \( J^T \), with \( G \) is invertible, then \( (x_{ML} - x_0) = G(x_0)^{-1} J(x_0)^T n \). The MSE given that \( x_0 \) was transmitted is then

\[
\varepsilon_{wn}^2 = \frac{1}{M} E\{(x_{ML} - x_0)^T (x_{ML} - x_0)\}. \tag{71}
\]

**Lemma 3:** With ML detection, the minimum MSE in (71) is achieved with a diagonal \( G \)

\[
\varepsilon_{wn}^2 = \frac{\sigma_n^2}{M} \sum_{i=1}^{M} 1 / g_{ii}, \tag{72}
\]

where \( g_{ii} \) are the diagonal components of \( G \) at \( x_0 \).

**Proof, Lemma 3:** Consider the MSE in (71). To avoid matrix multiplication, the \( N \)-dimensional noise vector \( n \) is, without loss of generality, replaced by its \( M \) dimensional projection \( n_P \), which is also Gaussian since \( P_{proj} \) is a linear transformation [69, p. 117]. Let \( J = J(x_0) \). Assume that a hypothetical inverse \( B = J^{-1} \) exists. Let \( S_i \) denote the \( (M \)-dimensional) tangent space of \( S \) at \( x_0 \). Under Definition 3 the linear approximation to \( S^{-1} \) can be applied, and so \( \hat{x} = S^{-1}(S(x_0) + n_P) \approx S^{-1}(S(x_0) + Bn_P) = x_0 + Bn_P. \) Then, Eqn. (71) becomes \( \varepsilon_{wn}^2 = E\{n_P^T B^T Bn_P\} / M = b_i^T b_j E\{n'^j n'i\} / M, \) using Einstein summation convention in the last equality, where \( b_i \) column vector no. \( i \) of \( B \). With i.i.d. noise, \( E\{n_i n_j\} = \sigma_n^2 \delta_{ij} \), then

\[
\varepsilon_{wn}^2 = \frac{1}{M} E\{n_P^T B^T Bn_P\} = \frac{\sigma_n^2}{M} \sum_{i=1}^{M} 1 / g_{ii} = \frac{\sigma_n^2}{M} \sum_{i=1}^{M} \|b_i\|^2. \tag{73}
\]
Since $B = J^{-1}$, and since any orthogonal matrix has an inverse, the above result implies that the basis of the tangent space of $S$ can be chosen orthogonal without any loss. An orthogonal $J$ results in a diagonal $G$ (see Appendix A-B.1). Therefore $G^{-1}$, as well as $G^{-n}$, are also diagonal with elements $1/g_{ii}$. With $G^{-2}$ diagonal, $E\{n_{i}n_{j}\} = \sigma_{n}^{2}\delta_{ij}$, and with (73) in mind, (71) leads to
\[
\varepsilon_{\text{wn}}^{2} = \frac{1}{M}E\{(G^{-1}JT)^{T}(G^{-1}JT)\} = \frac{1}{M}E\{(JT)^{T}G^{-2}(JT)\} = \frac{\sigma_{n}^{2}}{M} \sum_{i=1}^{M} \frac{1}{g_{ii}} ||J_{i}||^{2},
\]
where $J_{i}$ is column vector no. $i$ of $J$ and $||J_{i}||^{2} = g_{ii}$.

2) Proof. Proposition III-A.1: The ML-estimate of this problem, using 2nd order Taylor approximation, has no simple solution [46]. However, we can apply the analysis for pulse-modulation positioning (PPM) in [43, pp. 703–704]. Geometrically, PPM is a curve on a hyper sphere where the arc between any two coordinate axes is like the circle segment depicted in Fig. 14. As the curvature of any curve can be described locally by the osculating circle, the analysis done for PPM is also valid locally for any 1:N mapping under arc-length parameterization. In the following, the curve segment in Fig. 14 is named circle approximation.

We divide the noise into the components Proj$(\mathbf{n}) = \mathbf{n}_{i}$, i.e., the projection onto the closest point on the circle in Fig. 14, and its normal, $\mathbf{n}_{i}$. For the circle approximation $R(x) = \rho(x) = 1/\kappa(x) = 1/||S_{0}||$ if $t = ||S_{0}|| = 1$. In polar coordinates, $S(x) \approx [R(x), \theta(x)] = [\rho, \theta]$. Then $R(x) = R = \rho$, $\forall x$ and we have $S = [R\cos(\theta(x)), R\sin(\theta(x))]$. Then
\[
dS = \left[ R\sin(\theta(x)) \frac{d\theta(x)}{dx}, -R\cos(\theta(x)) \frac{d\theta(x)}{dx} \right].
\]
By taking the norm we find that $dS/dx = 2R^2(d\theta(x)/dx)^2$. From this we get
\[
d\theta(x)/dx = \frac{1}{R} \frac{dS}{dx} = \frac{||S_{0}||}{\rho} = \alpha\kappa \approx \alpha ||S_{0}||.
\]
That is, $d\theta(x)/dx \sim \kappa$. The two last equalities in (76) are valid under scaled arc-length parameterization. To determine the ML estimate for the circle approximation, rewrite (76) as $dx = d\theta(x)/(||S_{0}'||||S_{0}||) = \theta(x)\rho(x) / ||S_{0}||$. Then
\[
x_{0} = \hat{x}_{\text{ML}} = \rho(x_{0}) ||S_{0}'|| / \theta = \frac{\theta}{\kappa(x_{0}) ||S_{0}'||}.
\]
\[
\theta \text{ must be expressed in terms of } n_{i} \text{ and } \rho \text{ (or } \kappa). \text{ In Fig. 14 we have a right-angled triangle where } \phi = (\pi - \theta)/2, \text{ and with } b = \rho \sin(\theta) \text{ its right normal. Therefore sin}(\phi) = b/||n_{i}|| = b/\sin(\phi) = \rho \sin(\theta)/\sin(\phi). \text{ Furthermore, } \sin(\phi) = \sin(\pi/2 - \theta/2) = \cos(\theta/2). \text{ Since } \sin(2y) = 2\sin y \cos y, \text{ with } y = \theta/2, \text{ then } \sin(\theta) = 2\sin(\theta/2)\cos(\theta/2), \text{ implying that } ||n_{i}|| = 2\rho \sin(\theta/2). \text{ Therefore } \theta = 2\sin^{-1}((||n_{i}||/(2\rho(x_{0}))). \text{ By the expansion [70, p. 117]}\sin^{-1}(x) = x + x^3/(2 \cdot 3) + x^5 \cdot 3/2 \cdot 4 \cdot 5 + \ldots, \text{ we get, up to two terms}\]
\[
\theta \approx ||n_{i}|| (1 + \frac{||n_{i}||^2}{2\rho^2(x_{0})}),
\]
One can then compute the error up to second order from (77) and (78).

In general, $\kappa$ is given by (63). Under scaled arc-length parameterization, $||S'(x_{0})|| = \alpha ||S_{0}|| = \alpha, \forall x$, then (63) reduces to $\kappa(x_{0}) = ||S_{0}'(x_{0})||/||S_{0}||^2$. Then the error can be expressed in terms of the derivatives of $S$ as
\[
\varepsilon_{\text{wn}}^{2} = \frac{\sigma_{n}^2}{||S_{0}'||^2} \left(1 + \frac{\sigma_{n}^2}{4\rho^2(x_{0})} + \frac{5\sigma_{n}^4}{48\rho^4(x_{0})} \right).
\]
As will be shown in Lemma 1, one should let $\sigma_{n}^2 << \rho^2(x_{0})$ to avoid larger errors (at least high SNR). At high SNR ($\sigma_{n}^2 << 1$) one can therefore make the approximation
\[
\varepsilon_{\text{wn}}^{2} \approx \frac{\sigma_{n}^2}{||S_{0}'||^2} \left(1 + \frac{\sigma_{n}^2}{4\sigma_{n}^2} \right).
\]
To see that the circle approximation above is valid locally for any 1:N mapping, consider a general curve in polar coordinates. The product rule gives
\[
\frac{dS}{dx} = \left[ R'(x) \cos(\theta(x)) + R(x) \sin(\theta(x)) \right] \left[ R'(x) \sin(\theta(x)) - R(x) \cos(\theta(x)) \right].
\]
(83)
Locally, \( R'(x) \) is small if the curvature of \( S \) changes slowly with \( x \). Therefore, the smaller the derivative \( \kappa'(x) \), the more accurate the circle approximation is. As will become clear in Lemma 1, it is not convenient to use a curve (or surface in general) with a rapidly changing curvature.

For generalization to surfaces, consider first 2:N mappings. Assume that \( x_1 \) and \( x_2 \) are parameters in a LoC coordinate representation. Then, according to Theorem 2 in Section A-B.2, \( g_{12} = b_{12} = 0 \), therefore the curvature is \( \kappa_i(x_0) = b_{1i}(x_0)/g_{1i}(x_0) \), \( i = 1, 2 \). Since \( x_1, x_2, n_1 \) and \( n_2 \) are independent and i.i.d., and we have two orthogonal coordinate curves,
\[
\varepsilon^2_{w_0}(x_0) \approx \frac{\sigma_n^2}{2} \sum_{i=1}^{2} \left\{ \frac{1}{g_{1i}(x_0)} \left( 1 + \frac{\sigma_n^2}{4} \kappa_i^2(x_0) \right) \right\} = \frac{\sigma_n^2}{2} \sum_{i=1}^{2} \left\{ \frac{1}{g_{1i}(x_0)} \left( 1 + \frac{\sigma_n^2}{4} \frac{b_{1i}^2(x_0)}{g_{1i}(x_0)} \right) \right\},
\]
(84)
which is a straightforward generalization of the result for 1:N mappings. For \( M:N \) mappings, the generalization follows directly as we have \( M \) orthogonal curves forming a coordinate grid, and the result is obtained by letting the sum in (84) run from 1 to \( M \).

3) Proof, Lemma 1: Condition i) is obvious, and can be seen directly from Fig. 4(b).

Condition ii): We begin with 1:N mappings. From (14) we have \( \partial H/\partial x = -2S_{\mathbf{u}}(z-S) = 0 \), implying that \( z-S \perp \mathbf{S} \), and \( \partial^2 H/\partial x^2 = -2S_{\mathbf{u}}(z-S(x)) = 2 - 2\kappa_s p_a \cdot (z-S) + 2 = 0 \). The last equality is due to the FoF (64). Also, \( \kappa_s = 1/p_s \). From (14), \( \|z-S\| = r \).

\[\Rightarrow: \text{Since } \|p_a \cdot (z-S)\| = \|z-S\| = r, \text{ this condition follows from } -2p_a^s \cdot (z-S) + 2 = 0.\]

\[\Leftarrow: \text{If } p_s > r \forall x, \text{ the last equation in (14) will not have a real solution.}\]

With the definition of principal curvature in Appendix A-B.2 it is straightforward to extend the proof to \( M:N \) mappings (canal hyper surfaces): The result follows directly from 1:N case by letting a curve \( C \) be LoC with maximal principal curvature for all points of \( S \). That is, \( C \) is always in the direction of the maximal curvature on \( S \).

B. Proofs Section III-B

1) Proof Proposition 3: Under Definition 9, the received signal \( \hat{x} = S(\hat{z}) \) can be approximated by (17), where \( J(z_0)n \) contributes to the distortion. The MSE per source component given that \( z_0 \) was transmitted is then
\[
\varepsilon^2_{ch} = \frac{1}{M} E \left\{ (J(z_0)n)^T (J(z_0)n) \right\} = \left\{ \frac{\partial S_1}{\partial z_1} n_1 + \cdots + \frac{\partial S_1}{\partial z_N} n_N \right\}^2 + \cdots + \left\{ \frac{\partial S_N}{\partial z_1} n_1 + \cdots + \frac{\partial S_N}{\partial z_N} n_N \right\}^2 \right\}.
\]
(85)
Since noise on each sub-channel is independent, \( E\{n_in_j\} = \sigma^2_n b_{ij} \). After some rearrangement,
\[
\varepsilon^2_{ch} = \frac{\sigma_n^2}{M} \left\{ \left( \frac{\partial S_1}{\partial z_1} \right)^2 + \cdots + \left( \frac{\partial S_N}{\partial z_N} \right)^2 \right\} + \cdots + \left( \frac{\partial S_N}{\partial z_1} \right)^2 + \cdots + \left( \frac{\partial S_N}{\partial z_N} \right)^2 \right\}.
\]
(86)
Expectation w.r.t. \( z \) gives the wanted result.

2) Proof, Proposition III-B.1: With \( z_0 \) transmitted, and noise \( n \), we have:
\[
S(z_0 + n) \approx S(z_0) + nS'(z_0) + \frac{n^2}{2} S''(z_0) + \frac{n^3}{3!} S'''(z_0),
\]
(87)
from the 3rd order Taylor expansion. From this we can derive the channel distortion as
\[
\varepsilon^2_{ch}(z_0) = E \left\{ \left\| nS'(z_0) + \frac{n^2}{2} S''(z_0) + \frac{n^3}{3!} S'''(z_0) \right\|^2 \right\}.
\]
(88)
To expand this expression and take the expectation, it is advantageous to use arc-length parametrization (see Appendix A-A). Then, \( S S = 0, S S = 0 \) and \( \hat{S} S = 0 \) [47, pp. 36–37]. With this in mind, using the fact that \( E\{n^a\} = 1 \cdot 3 \cdots (a-1) \sigma^a_n, a \) even, and zero otherwise [61, p. 148], the expectation in (88) can be found from straight forward calculations
\[
\varepsilon^2_{ch}(z_0) = \sigma^2_n \| S(z_0) \|^2 + \frac{3\sigma^4_n}{4} \| S(z_0) \|^2 + \frac{5\sigma^6_n}{12} \| S(z_0) \|^2,
\]
(89)
where \( \| S(x_0) \|^2 = 1 \) according to Theorem 1 in Appendix A-A. The first term in (89) dominates when \( \sigma_n \) is small if \( \kappa \) and \( \tau \) are sufficiently small (see Eq. (22)).

Consider scaled arc-length parametrization, i.e., \( \| S'(z_0) \| = \alpha \| S(z_0) \| = \alpha, \forall z_0 \). Then (63) reduces to \( \kappa(z_0) = \| S'(z_0) \| / \| S''(z_0) \|^2 \) (since \( S''_0 \perp S'''_0 \), still) and the channel error can be expressed up to 2nd order in terms of the signal curves derivatives as in (20).

For general hyper surfaces the Taylor expansion is a complicated expression, making it hard to draw conclusions. It is more convenient to consider LoC: By choosing LoC as coordinates on a \( M:2 \) mapping \( S \), then a direct generalization of the \( M:1 \) case results. Assume that \( z_1 \) and \( z_2 \) are along the LoC. Then, according to Theorem 2, \( g_{12} = b_{12} = 0 \). Therefore \( \kappa_0 = b_{0i}(z_0)/g_{0i}(z_0) \) with \( g_i \) and \( b_{0i} \) evaluated w.r.t. the channel variables \( z_i \). Then (21) follows.
3) Proof, Proposition III-B.2: Consider an \( m \)-dimensional vector quantizer (VQ) with equal distance among each neighboring centroid, \( \Delta \), i.e., a uniform VQ. The distortion is then lower bounded by assuming \( m - 1 \)-spheres as Voronoi regions [53], [54], i.e., a sphere bound. Let the radius of these spheres be \( \rho_m \). Then we have

\[
\varepsilon_a^2 \geq E\{\|x - q(x)\|^2\} = \frac{m}{4(m+2)}\Delta^2,
\]

when \( \Delta \) is small, with \( q(x) \), the centroids of the VQ. \( \square \)

As the decision borders of a uniform VQ become spherical as \( m \to \infty \) [54], [72], equality is obtained in (90) as \( m \to \infty \) when \( \Delta \) is sufficiently small and when \( m = 1 \).

The bound (90) must be modified to entail S-K mappings. Assuming a uniform \( S \), then for each point \( P \) on \( S \), all points in the \( M - N \) dimensional space orthogonal to \( S \) at \( P \), which will be approximated onto \( P \), will for large \( M, N \), lie within an \( M - N - 1 \)-sphere, \( S^{M-N-1}_\Delta \), with radius \( \Delta/2 \). This sphere corresponds to the characteristic of the canal surface for \( S \), acting as decision borders for the approximation from \( \mathbb{R}^M \) to \( S \). From (14), the characteristic is in the space normal to \( S \) at any point. As the decision borders for any point, \( P \), is an \( M - N - 1 \)-sphere, the approximation to an \( N \)-dimensional uniform S-K mapping results in the same distortion as that of an \( M - N \) dimensional VQ. By substituting \( m = M - N \) in (90), dividing by \( M \) to obtain distortion per source component, the source result is obtained. Further, as \( M = N + 1 \) corresponds to \( m = 1 \) in (90), equality is obtained. \( \square \)

Proof, Lemma 4: Consider first the special case of high dimensional VQ with spherical Voronoi regions of unit radius \( \rho_m = 1 \), and with one centroid at the origin. If the source variance \( \sigma_s \gg \rho_m \), then the pdf of \( \|x - q(x)\| \), \( f_q \), will be approximately uniform [53]. Then one may consider an arbitrary centroid to quantify \( \varepsilon_a^2 \). Let \( q(x) = 0 \) for simplicity, and let \( B_m \) denote the volume contained within the relevant Voronoi region (see (97) for analytical expression). Then \( f_q = 1/B_m \), \( q \in [0, \rho_m] \), and \( 0 \) otherwise. Then

\[
\varepsilon_a^2 = \int \cdots \int_{B_m} \|x\|^2 f_q \, dx = \frac{1}{B_m} \int \cdots \int_{B_m} \|x\|^2 \, dx.
\]

From [53, p. 375], we have that the moment of inertia

\[
\int \cdots \int_{B_m} \|x\|^2 \, dx = \frac{m}{m+1} B_m.
\]

Therefore \( \varepsilon_a^2 = m/(m+1) \). As the quantization error scales with the radius of the Voronoi regions, \( \rho_m \), the distortion \( \varepsilon_a^2 \), will scale with \( \rho_m^2 = \Delta^2/4 \).

\textbf{APPENDIX C}

\textbf{PROOFS FOR SECTION IV}

1) Proof, Proposition IV: Take the dimension reduction case: The minimal distortion for an \( n:1 \) system is found by solving (1) w.r.t. \( D_t \), setting \( N = 1, M = n \),

\[
D_{n:1} = \left( \frac{\sigma_s^2}{1 + P_{n:1}/\sigma_n^2} \right)^{\frac{1}{n}}.
\]

For an \( m:1 \) system, simply substitute \( n \) with \( m \) in (93). With \( P_t \), the total power of the \( (m + n):2 \) system, one can allocate power to the two sub-systems through a factor \( \kappa \in [0,1] \) so that \( P_{n:1} = \kappa P_t \). Let \( \text{SNR} = P_t/\sigma_n^2 \) and use the fact that \( 1 + x \approx x \) as \( x \) becomes large. Then,

\[
\lim_{\text{SNR} \to \infty} D_{t(m+n:2)} = \lim_{\text{SNR} \to \infty} \frac{\sigma_s^2}{m+n} \left[ \frac{\left(1 + \kappa\text{SNR}\right)^{\frac{1}{n}}}{\left(1 + (1-\kappa)\text{SNR}\right)^{\frac{1}{n}}} \right] = \frac{\sigma_s^2}{m+n} \lim_{\text{SNR} \to \infty} \left[ \frac{1}{\kappa^{1/n}\text{SNR}^{1/m}} + \frac{1}{(1-\kappa)\text{SNR}^{1/m}} \right].
\]

According to the laws of limits \( \lim_{x \to \infty} \kappa x = \kappa \lim_{x \to \infty} x \), and so the power allocation factor(s) can be moved outside the limit. Then, for \( m > n \) since \( \text{SNR}^{1/m} \) grows more slowly than \( \text{SNR}^{1/n} \), \( D_{t(m+n:2)} \) will be dominated by the second term in (94) as \( \text{SNR} \to \infty \).

In the expansion case, a similar derivation leads to

\[
\lim_{\text{SNR} \to \infty} D_{t(2m+n)} = \frac{\sigma_s^2}{m+n} \lim_{\text{SNR} \to \infty} \left[ \frac{1}{\kappa^{1/n}\text{SNR}^{1/m}} + \frac{1}{(1-\kappa)\text{SNR}^{1/m}} \right].
\]

If \( m > n \), the first term in (95) dominates as \( \text{SNR} \to \infty \). \( \square \)

2) Proof, Proposition IV-A: Assume the channel- and noise vectors are normalized with \( N \). With power constrained Gaussian channel, the received vector will lie within an \( N-1 \) sphere of radius

\[
\rho_N = \sqrt{P_N + b_N^2\sigma_n^2},
\]

with high probability. \( P_N \) and \( \sigma_n^2 \) denote channel power and noise variance per dimension, respectively. With \( b_N \) one takes into consideration that \( \rho_N \) exceeds \( \sqrt{P_N + \sigma_n^2} \) for finite \( N \), so \( b_N \to 1 \) as \( N \to \infty \). Let \( B_n \) denote the volume inside an \( (n-1) \)-sphere of unit radius [73]

\[
B_n = \frac{\pi^n}{\Gamma(n/2 + 1)}.
\]

To satisfy the power constraint then \( \text{Vol}(S^M \times S^{N-M-1}) \leq B_M \rho_N^M \). Under Definition 11 we have \( \mathbb{B}^M \times S^{N-M-1} \) locally for all points on \( S \). With Gaussian pdf, \( x \) is distributed over \( D_x \) like an \( M \)-ball. Further, under Definition 3, in the absence of anomalies, the act of \( S \) is simply a stretching of \( D_s \) (as bending, or cutting, of \( S \) does not affect \( \varepsilon_a^2 \)), and thereby the stretching of an \( M \)-ball. If \( S \) is shaped preserving (Corollary 1), the stretching is the same for all \( x \in D_s \), and so, equivalent to an amplification factor. Under Definition 11, we can identify this amplification with the radius \( \rho_m \) of \( \mathbb{B}^M \). Then \( \text{Vol}(S^M \times S^{N-M-1}) \leq B_M \rho_N^M \) becomes identical to

\[
B_M \rho_M^M B_{N-M} \rho_{MN}^{N-M} \leq B_{NP}^N,
\]

where \( \rho_{MN} \) is the canal hyper surface radius and \( \rho_M \) is the radius of the stretched source space \( D_s \). Further, assume the
same decomposition of \( n \) as in Section III-A.2 (see Fig. 4(a)), the \( M \)-dimensional tangent to \( S \), \( \mathbf{n}_{wn} \) and the \( N - M \) dimensional normal \( \mathbf{n}_{n} \). To avoid anomalous errors, Proposition 6 states that \( \rho_{MN} \geq \| \mathbf{n}_{wn} \| = \sqrt{((N - M)/N)B_{MN}^2} \), where \( b_{NM} \to 1 \) as \( M, N \to \infty \). When \( M, N \) is large enough, (98) can be written (\( b_N = b_{NM} = 1 \) as \( M, N \to \infty \))

\[
B_M \rho_{MN}^M B_{N-M} \left( \frac{N - M}{N} - \frac{2}{N^2} \right)^{N-M} \leq B_N(P + \frac{2}{\alpha_n^2})^{1/2}. \tag{99}
\]

As explained above, with shape preserving \( S \), \( \hat{\varepsilon}_{wn}^2 \) is determined by \( \rho_M \). Solving (99) w.r.t. \( \rho_M \),

\[
\rho_M \leq \frac{\sqrt{E} \sigma_n}{\left( \frac{1}{1 - M/N} \right)^{N-M}} \left( 1 + P_N \frac{N}{\sigma_n^2} \right)^{1/2}, \tag{100}
\]

where from (97)

\[
\tilde{B} = \frac{B_N}{B_M B_{N-M}} = \frac{\Gamma\left( N - M + 1 \right)}{\Gamma\left( M + 1 \right)} \left( \frac{M + 1}{N} \right)^{N-M} \left( \frac{M}{N} \right)^{N-M} \tag{101}
\]

Eqn. (101) can be expressed through the Beta function using [57, p. 9]

\[
\mathcal{B}(\varphi, \varsigma) = \int_0^1 (1 - t)^{\varphi - 1} t^{\varsigma - 1} dt = \frac{\varphi^{\varphi}(\varphi)}{\varphi^{\varphi} + \varphi^{\varsigma}}. \tag{102}
\]

and the Functional relation \( \Gamma(a + 1) = a \Gamma(a) \) [57, p. 3]. Letting \( a = (N - M)/2 + 1 \) and \( \varsigma = M/2 + 1 \), using the above relations, we obtain

\[
\tilde{B} = \left( \frac{N}{M} + 1 \right) \left( \frac{M}{N} + 1 \right) = \left( \frac{N}{M} + 1 \right) B_{(N,M)}. \tag{103}
\]

As \( M \) of \( N \) noise components (\( \mathbf{n}_{wn} \)) contribute to weak noise distortion, we get

\[
\hat{\varepsilon}_{wn}^2 = \frac{E\{ \mathbf{n}_{wn}^2 \}}{\rho_M^2} = \frac{M \sigma_n^2}{N \rho_M^2}. \tag{104}
\]

from (10). With \( \rho_{MN} > \| \mathbf{n}_{wn} \|, \hat{\varepsilon}_{an}^2 = 0 \) from Proposition 6. Then \( \hat{\varepsilon}_{wn}^2 \) is the total distortion \( D_t \). Assume a fixed \( r = N/M \). Substituting \( M = N/r \) and inserting (100) into (104) then

\[
D_t = \frac{1}{r} \left( 1 - \frac{1}{r} \right)^{r-1} \frac{N}{M + 1} \left( \frac{N}{M} + 1 \right)^{-\frac{2}{2}} \mathcal{B}_{(N,r)}^{-\frac{\mathcal{B}_{(N,r)}}{N}} \left( 1 + P_N \frac{N}{\sigma_n^2} \right)^{-r}, \tag{105}
\]

where

\[
\mathcal{B}_{(N,r)} = \left( \frac{N}{M} + 1 \right) \left( \frac{M}{N} + 1 \right) = \left( \frac{N}{M} + 1 \right) \mathcal{B}_{(N,M)}. \tag{106}
\]

What is left to show is then

\[
\lim_{N \to \infty} \left( \frac{N}{M} + 1 \right) \left( \frac{N}{M} + 1 \right) \mathcal{B}_{(N,r)}^{-\frac{\mathcal{B}_{(N,r)}}{N}} = r \left( 1 - \frac{1}{r} \right)^{1-r}. \tag{107}
\]

Using the product rule for limits [74, p. 68], the first term on the left in (107) is eliminated since its limit equals 1. Further, using Hölders inequality [75, pp. 135–136], we get

\[
\mathcal{B}_{(N,r)} \leq \left\| t \right\|_{\infty} \left\| \frac{2}{\alpha_n^2} \right\|_{\infty}, \tag{108}
\]

with equality as \( N \to \infty \). By differentiation one find that \( t_{max} = 1 - 1/r \) maximizes the norm in (108), and so, as \( N \to \infty \),

\[
\mathcal{B}_{(N,r)} = \left( 1 - \frac{1}{r} \right)^{\frac{N}{M}} \left( \frac{1}{r} \right)^{\frac{2}{N}}. \tag{109}
\]

Raising both sides of (109) to the power \(-2r/N\) gives the wanted result. To include \( \sigma_x \), sets \( \rho_M = \alpha \sigma_x \), solve (100) with respect to \( \alpha \) and substitute \( \alpha \) for \( \rho_M \) in (104).

3) Proof, Proposition IV-B: To make \( \hat{\varepsilon}_{ch}^2 \) small, the canal surface \( \mathbf{S} \times \mathbb{S}^{M-N-1} \) should cover the source space. Under Definition 11, then locally we have \( \mathbb{B}^N \times \mathbb{S}^{M-N-1} \) for all points on \( S \). Under the same reasoning as in the proof of proposition IV-A, we have

\[
B_N \rho_{MN}^N B_{N-M} \rho_{MN}^{M-N} \geq B_M \rho_M. \tag{110}
\]

\( \rho_{MN} = \| x \| = \sqrt{MB_{MN}^2} \) is the radius of the source-space, \( \rho_N = \alpha \sqrt{N(P_N + b_N^2 \sigma_n^2)} \) is the radius of the stretched channel space (these are not normalized here), where \( \alpha \) is an amplification factor, \( \rho_{MN} = \Delta/2 \) is the canal surface radius, and \( b_M, b_N \to 1 \) as \( M, N \to \infty \). As in Appendix C.2 these are set to one in what follows. Inserting into (110) and solving w.r.t. \( \alpha \), we obtain

\[
\alpha \geq \sqrt{\frac{M}{N} \frac{N}{M} \Delta} \frac{1}{\sigma_x} \left( \frac{1}{N} \right)^{\frac{M-N}{2}} \left( \frac{M}{N} \right)^{\frac{M-N}{2}} \frac{1}{\sigma_n^2} \left( 1 + P_N \sigma_n^2 \right)^{-1/2}, \tag{111}
\]

where

\[
\Delta = \left( \frac{M}{N} + 1 \right) \left( \frac{M}{N} + 1, N + 1 \right) = \left( \frac{M}{N} + 1 \right) \mathcal{B}_{(M,N)}, \tag{112}
\]

derived in a similar way as in Appendix C.2. Assuming a shape preserving mapping and inserting (111) into (19), an expression for \( \hat{\varepsilon}_{ch}^2 \) is found. Furthermore, \( \hat{\varepsilon}_{ch}^2 \) and \( \hat{\varepsilon}_{an}^2 \) can be considered independent under Definition 12 as they are perpendicular, thus

\[
D_t = \hat{\varepsilon}_{an}^2 + \hat{\varepsilon}_{ch}^2 = \frac{M - N}{4(M - N + 2)} \Delta^2 + M \frac{N}{N} \Delta \left( \frac{\Delta}{2} \right) \frac{2}{\sigma_x} \sigma_n^2 \left( 1 + P_N \frac{N}{\sigma_n^2} \right)^{-1}. \tag{113}
\]

This is a convex function in \( \Delta \), and so a unique minimum exists. Differentiating (113) with respect to \( \Delta \), equating to zero and solving for \( \Delta \), we obtain

\[
\Delta_{opt} = \frac{M + N}{4M} \left( \frac{M + N + 2}{M} \right)^{\frac{N}{M}} \left( \frac{M + N}{N} \right)^{\frac{N}{M}} \left( \frac{M}{N} + 1 \right) \left( \frac{M}{N} + 1 \right) \frac{2}{\sigma_x} \sigma_n^2 \left( 1 + P_N \frac{N}{\sigma_n^2} \right)^{-1}. \tag{114}
\]

Inserting (114) and (112) into (113) and using the relation \( N = Mr, \) with \( r \in Q[0,1] \), we get

\[
D_t = \left( \frac{1}{r} \right)^{\frac{N}{M}} \left( \frac{1}{N} \right)^{\frac{1}{N}} \left( \frac{M}{r} + 1 \right)^{\frac{2}{N}} \mathcal{B}_{(M,r)}^2 \sigma_n^2 \left( 1 + P_N \frac{N}{\sigma_n^2} \right)^{-r}. \tag{115}
\]
where

\[ B_{(M,r)} = \int_0^1 t^{\frac{M}{2}(1-r)}(1-t)^{-\frac{M}{2}} dt. \]  

(116)

We get rid of two terms as

\[ \lim_{M \to \infty} \left[ \frac{M}{2} + 1 \right]^{\frac{M}{2}} \left( \frac{1-r}{1-r+2/M} \right)^{1-r} = [1, 1], \]

(117)

using the product rule for limits [74, p. 68]. From Hölders inequality [75, pp. 135–136], \( B_{(M,r)} \leq (1-r)^{\frac{M}{2}} r^{\frac{M}{2}}, \) with equality when \( M \to \infty, \) and so

\[ \lim_{M \to \infty} \left( 1 + \frac{r}{1-r} \right)^{\frac{M}{2}} B_{(M,r)}^{\frac{M}{2}} = \left( 1 + \frac{r}{1-r} \right)^{r} (1-r)^{(1-r)r} = 1 \]  

(118)

\[ \Box \]

APPENDIX D

PROOFS FOR SECTION V

1) Proof, Lemma 2: The cumulative distribution is given by a straightforward generalization of the \( h : \mathbb{R}^2 \to \mathbb{R} \) case in [61, pp. 180–181]. With \( x = (x_1, \cdots, x_3) \)

\[ F_{z_1}(z_1) = p_r \{ x \in D_{z_1}^+ \cup D_{z_1}^- \} = \int \int \int_{D_{z_1}^+ \cup D_{z_1}^-} f_{X}(x) dx, \]  

(119)

where \( f_X(x) \) is the 3 D i.i.d. Gaussian distribution and

\[ D_{z_1}^+ = \{ x ||x||^n \leq \rho^n, z_1 \geq 0 \}, \]

\[ D_{z_1}^- = \{ x ||x||^n \geq -\rho^n, z_1 < 0 \}. \]  

(120)

Then

\[ f_{z_1}(z_1) = \frac{d}{dz_1} F_{z_1}(z_1) = \frac{d}{dz_1} \int_0^{2\pi} \int_0^{r} \int_0^{a \varphi(z_1)} f_{x}(x) d\rho d\theta d\phi, \]  

(121)

where \( f_{x}(x) = \exp(-\rho^2/(2\sigma^2_x))/(\sqrt{2\pi} \sigma^2_x) \). The integrals over \( \theta, \phi \) become \( I_{(\theta, \phi)} = \pi, \) and

\[ \frac{d}{dz_1} \int_0^{a \varphi(z_1)} f_{x}(x) \rho^2 d\rho = \frac{1}{(2\pi)^{3/2} \sigma^2_x} \frac{d}{dz_1} \int_0^{a \varphi(z_1)} e^{-\frac{x^2}{2\sigma^2_x}} \rho^2 d\rho \]

\[ = \frac{1}{(2\pi)^{3/2} \sigma^2_x} \frac{e^{-\frac{x^2}{2\sigma^2_x}}}{a \varphi(z_1) - \frac{2e^{a \varphi(z_1)}}{2\sigma^2_x} - \frac{2e^{a \varphi(z_1)}}{2\sigma^2_x}}. \]  

(122)

Using absolute value to include negative values, the wanted result is obtained.

2) Proof, Proposition 9: With \( b_x \) sufficiently large, the probability for anomalies becomes small due to the constraint in (51), and the last term in (52) becomes negligible. A constant gap to OPTA at high SNR then implies that

\[ D_t = \Delta^2/36 + 2\sigma^2_x \alpha^2/3 = C \cdot \text{SNR}^{-\frac{4}{3}}, \]  

(123)

with \( C \) some constant. We show that such a constant exists, complying with the optimization problem in (54) with \( D_t \) as in (123). Let \( \beta = \eta \sqrt{2\pi} \). As \( \alpha_3 \) does not occur explicitly in (123), we eliminate it by equating the constraints in (54) to zero and solving w.r.t. \( \alpha_3 \). Then

\[ \alpha_3^2 = \frac{\Delta \sigma_x \alpha^2}{\beta(P_\alpha^2)} = \frac{\Delta^2 \alpha^2}{(2b_x \sigma_x + 2b_n \sigma_n \alpha)^2}. \]  

(124)

From this an equation for \( \alpha \) results, \((4b_n^2 \sigma_n^2 - P_t \Delta \beta) \alpha^2 + 8b_x b_n \sigma_n \alpha + (4b_x^2 + \Delta \beta) = 0\) (assuming \( \sigma_x = 1 \)). With SNR = \( P_t/\sigma_n^2 \), the solution is

\[ \alpha = \frac{-4b_x \pm \sqrt{\Delta \beta(4SNR \sigma_n^2 + \Delta \beta 3SNR^2 - 4)}}{8b_n \sigma_n(4 - \Delta \beta \sigma_n^2 \text{SNR})}, \]

(125)

where we used \( x + \) constant \( \to x \) for large \( x \) in the last step. Only the positive solution is viable

\[ \varepsilon_{e_{ch}}^2 = \frac{2\sigma^2}{3} \alpha^2 = \frac{4b_x^2 + \Delta \beta}{3\Delta \beta \text{SNR}}. \]  

(126)

The distortion contributions should balance at high SNR [77], i.e., \( \varepsilon_{e_{ch}}^2 = \varepsilon_{e_{q}}^2 \), and so \( \Delta^2/36 = (C/2)\text{SNR}^{-\frac{4}{3}} \). Therefore \( \Delta = 3\sqrt{2}\text{SNR}^{-\frac{4}{3}} \), and from (123) and (126),

\[ D_t = 2\varepsilon_{e_{ch}}^2 = \frac{4}{3} \frac{4b_x^2 + 3\beta \sqrt{2\text{SNR}}}{3\sqrt{2\text{SNR}}^4} \]

\[ \approx \frac{3C}{4} \text{SNR}^{-\frac{4}{3}} - \beta \approx \frac{3C}{4} \text{SNR}^{-\frac{4}{3}}. \]  

(127)

Therefore

\[ \frac{4b_x^2}{3\sqrt{2\text{SNR}}^4} = \frac{3C}{4} \text{SNR}^{-\frac{4}{3}} - \beta \approx \frac{3C}{4} \text{SNR}^{-\frac{4}{3}}. \]  

(128)

Solving (128) w.r.t. \( C \), then \( C = (16b_x^2/((9\sqrt{2})^2)^{2/3}. \)

3) Proof, Proposition 10: The Lagrangian for the problem is now

\[ \mathcal{L}(\delta, \alpha_2, \lambda) = \frac{\sigma^2_n}{\alpha^2_2} + \lambda \left( \frac{\zeta^2 \alpha^2_2}{\delta^4} + \frac{\alpha_2^2 \delta^2}{12} - P_t \right) \]

\[ + \lambda_2(2b_n \sigma_n - \alpha_1), \]  

(129)

where \( \zeta = 2\eta \alpha^2_x \). Equality constraints are assumed, \( P_t = 0.5(P_1 + P_2) \) and \( \alpha_1 = 2b_n \sigma_n \), as all the power should be used, and HVQLC should fill the channel space as properly as possible under the given constraints. Solving (58) w.r.t. \( \alpha_2 \) we get \( \alpha_2 = 12(3P_t/2 - \zeta^2 \alpha^2_1/\delta^4)/\delta^2 \). Then

\[ \text{SDR} = \frac{\sigma^2_n}{\alpha^2_2} \frac{\sigma^2_n}{\alpha^2_2} = \frac{12\sigma^2_n}{\alpha^2_2} \frac{3P_t - \zeta^2 \alpha^2_1}{\delta^4}. \]  

(130)

The constrained problem over \( \delta, \alpha_1, \alpha_2 \) is then converted to an unconstrained problem over \( \delta \). By solving \( \partial \text{SDR}/\partial \delta = 0 \) we get \( \delta^* = \sqrt{12\zeta^2 b_n^2}/\text{SNR} \), with SNR = \( P_t/\sigma_n^2 \). By inserting this into (130), we get the wanted result. \[ \Box \]
4) **Proof, Proposition V-B.3:** The Lagrangian for the problem is now

\[
\mathcal{L}(\delta, \alpha_2, \lambda) = \frac{\sigma_n^2}{\alpha_2^2} + \lambda \left( \frac{1}{2} (\alpha_1 \sigma_x^2 + \alpha_2^2 \frac{M \sigma^2}{(M + 2)}) - P_t \right) + \lambda(2b_{NM} \sigma_n - \alpha_1 - 1).
\]

Equality constraints are assumed for the same reason as for 2:3 HVQLC. Solving (61) w.r.t. \( \alpha_2 \) and inserting into \( D_t = \frac{\sigma_n^2}{\alpha_2^2} \),

\[
D_t = \frac{\sigma_n^2}{\alpha_2^2} = \frac{\sigma_n^2 \delta^4}{8(1 + 2/M)(P_t \delta^2 - 2b_{NM} \sigma_n \sigma_x^2)}.
\]

The constrained problem over \( \delta, \alpha_1, \alpha_2 \) is then converted to an unconstrained problem over \( \delta \). By solving \( \partial D_t/\partial \delta = 0 \) we get \((\delta^*)^2 = 4b_{NM} \sigma_n^2 \sigma_x^2 / P_t\). This implies that \( \alpha_1 = 1 \), and so \( \Delta_{\min} = \delta \). By inserting this into (132), with SNR = \( P_t/\sigma_n^2 \), we get the wanted result. That \( b_{NM} \to 1 \) in the limit follows from Proposition 6. \( \square \)

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