Simulation of the stress-strain state of shells under internal pressure using the mixed finite element method, taking into account physical nonlinearity

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Abstract. In this article, the finite element method (FEM) is used to model the stress-strain state (SSS) of shells experiencing internal pressure with elastic-plastic state being taken into account. Main geometric equations determining the location of an arbitrary point at load step are given. Mixed functionality used in implementation of FEM which allows to determine stresses and displacements simultaneously is obtained based on equality of possible and actual work of external and internal forces. Examples of stress-strain state of rotation shells with spherical and elliptical bottoms under internal pressure are provided. Static check of resultant external forces arising from applied pressure and resultant internal forces arising from stresses equality to zero is performed. Achieved results show correctness of the developed algorithm with elastic-plastic properties of the material being taken into account, ensuring that main requirements for the mathematical model are secured, namely adequacy, versatility and cost-effectiveness. Costs of resources computing are reduced by adopting a number of hypotheses and determination of stresses and strains at once ensuring cost-effectiveness and simplicity. Using developed scheme of the mixed functionality application in implementation of FEM the mathematical model acquires one of its main properties - "Potentiality".

Keywords: modeling, stress-strain state, finite element, displacements, stresses, functional, shell structures

1. Introduction
Different shells are among most common elements of engineering constructions and due to their curvilinear shape shells work as spatial elements. One of the most frequent numerical methods of stress-strain state of shells study is Finite Element Method (FEM). It allows to achieve fairly accurate solutions[1], therefore in [2] spherical and cylindrical geometry of the shell is studied for two different material configuration which are single-layer structures and multi-layer structures with an internal core embedded in them. Three-dimensional (3D) exact shell model and various two-dimensional (2D) computational models are compared by frequency and oscillation mode. Proposed numerical solutions are typical two-dimensional finite elements (FE) as well as classical and improved generalized two-dimensional differential quadrature (TDDQ) solutions. Problems regarding 2D Generalized Differential Quadrature solutions for FGM plates and shells analyses are discussed in [2-4]. 2D static analysis of FGM plates and shells is being proposed in [2-3].
In [5] the dynamic Koiter model is being viewed for hyperbolic parabolic shell, more precisely the saddle surface as its mean surface. Numerical calculation for this type of shell is the most complex due to its complex differential geometry.

Using a simplified model that reflects individual features of the object being studied allows us to see the relations of causes and effects, inputs and outputs more clearly and make necessary conclusions faster, and make correct decisions when modeling shells of different configurations [5-7, 10]. In [8] we consider a variant of Koiter shell model based on internal geometry methods by Michael Delfour and Jean-Paul Zolesio.

2. Informal statement of the problem
Analyzing the stress-strain state of complex elements and structures used in agro-industrial production and manufacturing processes in mechanical engineering and other industrial sectors it's clear that usage of material modeling is difficult since it is necessary to operate with large number of criteria and restrictions that may be incompatible thus it's often impossible. Therefore, in [9], physicomathematical interpretation of alternative shear strain concept is proposed in order to get rid of intuitive aspect of its basic premise that total deviation \( w \) can be taken as sum of bending and transverse shear deviations. One of solutions to these problems is to use numerical solving procedures with a computational experiment [7-9, 11].

Among many schemes of FEM in calculations of the stress-strain state of shells special space is occupied by mixed methods, which proved to be effective here. Usage of methods based on Reissner variational principle for linear and geometrically nonlinear problems is repeatedly reported in literature, results of numerical calculations and their analysis are presented in [13, 14].

Browsing literature sources for calculation of plate and shell structures shows that modeling and analysis of the stress-strain state based on FE is of interest to researchers and engineers worldwide [2-9, 15-17]. There are practically no works devoted to the mixed formulation analysis of SSS with physical nonlinearity of material taken into account. This method allows us to determine displacements and stresses without additional operations.

3. Formalization and analysis of the model
In Cartesian coordinate system when rotation shell is deformed an arbitrary point \( M \) of median surface of the shell is determined by radius vector:

\[
\mathbf{R} = x\mathbf{i} + r\sin\theta\mathbf{j} + r\cos\theta\mathbf{k},
\]

where \( x \) is axis coordinate; \( r \) is radius of rotation; \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are unit vectors of Cartesian system; \( \theta \) is angular coordinate being calculated counterclockwise from vertical.

Basis vectors of arbitrary point tangent to median surface are determined by differentiating the radius vector (1):

\[
\begin{align*}
\mathbf{a}_1 &= \mathbf{R}_t = \mathbf{i} + r\sin\theta\mathbf{j} + r\cos\theta\mathbf{k} ; \\
\mathbf{a}_2 &= \mathbf{R}_\theta = r\cos\theta\mathbf{j} - r\sin\theta\mathbf{k}.
\end{align*}
\]

The normal to the median surface is defined by next equation:

\[
\mathbf{a}_3 = \frac{A}{r} = -\mathbf{i}r\theta + x\sin\theta\mathbf{j} + x\cos\theta\mathbf{k}.
\]

Relations (2) and (3) can be written in matrix form:

\[
\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} M \\ 3x3 \\ 3x3 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \end{bmatrix}
\]

Derivatives of basis vectors of point \( M \) of the median surface are defined by differentiation (2), (3), expressed in terms of orts of Cartesian coordinate system in matrix form it will look like following:

\[
\begin{bmatrix} \mathbf{a}_1 \end{bmatrix} = \begin{bmatrix} N_1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \end{bmatrix} ; \quad \begin{bmatrix} \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} N_2 \end{bmatrix} \begin{bmatrix} \mathbf{j} \end{bmatrix} ; \quad \begin{bmatrix} \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} N_3 \end{bmatrix} \begin{bmatrix} \mathbf{k} \end{bmatrix}
\]
Arbitrary point M' receives displacement $\tilde{V}$ when shell is deformed [18]. This vector can be represented by components in basis of corresponding point M of the median surface:

$$\tilde{V} = v^i \tilde{a}_i + v^j \tilde{a}_j + v^k \tilde{a}_k = \{\tilde{a}\}^T \{v\}, \quad (6)$$

where $\{v\}^T = \{v^1, v^2, v^3\}$.

At (j+1)-th loading step displacement of the point M' is determined by a vector, which in terms of vectors of local basis of the point M of the median surface and the corresponding components is expressed as

$$\tilde{w} = w^i \tilde{a}_i + w^j \tilde{a}_j + w^k \tilde{a}_k = \{\tilde{a}\}^T \{w\}, \quad (7)$$

where $\{w\}^T = \{w^1, w^2, w^3\}$.

Movement of the point M after the (j+1)-th loading step is determined by the total vector $\tilde{V} + \tilde{w}$. Position of the point M' in deformed state of the shell is determined by radius vector

$$\tilde{R}^* = \tilde{R} + \tilde{V}. \quad (8)$$

Deformations at the point M' are defined as differences between the components of metric tensors of deformed and initial states and determined by relations that can be represented in matrix form:

$$\{\varepsilon\} = [L] \{v\}, \quad (9)$$

where $\{\Delta \varepsilon\}^T = \{\Delta \varepsilon_{11}, \Delta \varepsilon_{22}, \Delta \varepsilon_{33}, 2\Delta \varepsilon_{12}, 2\Delta \varepsilon_{13}, 2\Delta \varepsilon_{23}\}$ is vector-string of strain tensor components at the point M'; $[L]$ is matrix of algebraic and differential operators.

In calculations for geometrically linear deformation between the increments of deformations and the increments of displacements at the (j+1)-th step of loading relations could be written in matrix form [18]:

$$\{\Delta \varepsilon\} = [C^p_{\varepsilon}] [\Delta \sigma], \quad (10)$$

Arbitrary eight-node finite element with nodes i, j, k, l, m, n, p, h is accepted as finite element. Nodal unknowns take stress increments at nodal points and movements of nodal points at the loading step. To perform numerical integration, an arbitrary octagon is mapped to a cube whose local coordinates change within $-1 \leq \xi, \eta, \zeta \leq 1$.

Displacement vector components of the finite element inner point are expressed in terms of the nodal values of the displacement components using the matrix relation

$$\{\tilde{w}\} = [A] \{w_j\}, \quad (11)$$

where $[A]_{3 \times 24} = \begin{bmatrix} [\varphi(\xi, \eta, \zeta)]^T & \{0\}^T & \{0\}^T \\ \{0\}^T & [\varphi(\xi, \eta, \zeta)]^T & \{0\}^T \\ \{0\}^T & \{0\}^T & [\varphi(\xi, \eta, \zeta)]^T \end{bmatrix}$.

Stress increment tensor components of the finite element inner point are expressed in terms of the nodal values of the stress increment tensor components by following matrix relation

$$\{\Delta \sigma\} = [G] \{\Delta \sigma'\}, \quad (12)$$

Replacing the internal forces increments actual workload at the loading step with the difference between the possible and additional internal forces workload at the loading step and taking into account the matrix relations (10), (11) and (12), the function at the loading step is written as:
\[ \Pi_{LNS} = \left\{ \Delta \sigma_{y} \right\}_{1x48}^{T} \int \left[ G^{T} \right] \left[ B \right] dV \left\{ w_{y} \right\}_{24x1} - \frac{1}{2} \left\{ \Delta \sigma_{y} \right\}_{1x48}^{T} \int \left[ G^{T} \right] \left[ D \right] \left[ G \right] dV \left\{ \Delta \sigma_{y} \right\}_{48x1} - \frac{1}{2} \left\{ \Delta \sigma_{y} \right\}_{1x24}^{T} \int \left[ A \right] \left\{ \Delta p \right\} dS - \left\{ \Delta \sigma_{y} \right\}_{1x24}^{T} \int \left[ A \right] \left\{ p \right\} dS + \left\{ \Delta \sigma_{y} \right\}_{1x24}^{T} \int \left[ B \right] \left\{ \sigma \right\} dV = 0. \] 

Minimizing the function (13) by nodal unknowns \( \left\{ \Delta \sigma_{y} \right\}_{1x24}^{T} \) and \( \left\{ w_{y} \right\}_{1x24}^{T} \) we get system of equations:

\[ \frac{\partial \Pi_{R}}{\partial \left\{ \Delta \sigma_{y} \right\}_{24x1}^{T}} = \left[ H \right] \left\{ \Delta \sigma_{y} \right\}_{24x1} + \left[ Q \right] \left\{ w_{y} \right\}_{24x1} = 0; \]

\[ \frac{\partial \Pi_{R}}{\partial \left\{ w_{y} \right\}_{24x1}^{T}} = \left[ Q \right] \left\{ \Delta \sigma_{y} \right\}_{24x1} - \left\{ f_{p} \right\}_{24x1} - \left\{ f_{\sigma} \right\}_{24x1} = 0. \] 

The system of equations (14) can be represented in traditional form for the finite element method

\[ \left[ k \right] \left\{ \varepsilon_{y} \right\} = \left\{ F \right\}, \] 

where

\[ \left[ k \right]_{72x72} = \left[ \begin{array}{cc} - \left[ H \right]_{24x48}^{T} & \left[ Q \right]_{24x48}^{T} \\ \left[ Q^{T} \right]_{24x48} & \left[ 0 \right]_{24x24} \end{array} \right] \] is matrix of deformation of FE at the loading step;

\[ \left\{ \varepsilon_{y} \right\}_{1x72}^{T} = \left\{ \Delta \sigma_{y} \right\}_{1x48}^{T} \left\{ w_{y} \right\}_{1x24}^{T} \] is vector of nodal unknowns of FE;

\[ \left\{ F \right\}_{1x72}^{T} = \left\{ 0 \right\}_{1x64}^{T} \left\{ \Delta f_{p} \right\}_{1x8}^{T} + \left\{ R \right\}_{1x8}^{T} \] is nodal forces vector of FE;

\[ \left\{ R \right\}_{1x8}^{T} = \left\{ f_{p} \right\}_{1x8} - \left\{ f_{\sigma} \right\}_{1x8} \] is discrepancy.

4. Results

This paragraph shows the results of solving test problems, in which adequacy of the results obtained through described algorithm basis was checked.

Results obtained with numerical experiments and results obtained with equations of structural mechanics solution basis have been compared. Purpose of these tests was to check both the capabilities of developed algorithm and the convergence of results.

Test example 1. The stress-strain state of a rotation shell with spherical bottom under internal pressure is considered (the diagram is shown in figure 1). The following initial data are accepted: \( q = 2.39 \text{ MPa} \), \( E=2\times10^4 \text{ MPa} \); \( \mu=0.3 \). The shell has been divided along Meridian into 75 elements, with a thickness of is 3, 4, 5, 8, 10 and 11 elements. Results of calculations at step 30 are shown in sheet 1. First line shows the number of nodes along thickness of the shell, second one shows normal stresses in the inner fiber of section 2-2 and the third one shows the stress intensity \( \sigma \) in the same fiber of section 2-2.

It can be seen that even with 5 elements in thickness of the shell sufficient computational process convergence has been achieved.
Figure 1 shows zones of plastic deformations in the longitudinal section of the shell. When loading the first plastic deformations occur in the section 3-3 of shell's inner surface.

Table 1. Calculation results.

| Number of elements | 3   | 4   | 5   | 8   | 10  | 11  |
|-------------------|-----|-----|-----|-----|-----|-----|
| \( \sigma_{xx} \) normal stress | 1062,34 | 1074,64 | 1126,23 | 1126,84 | 1125,35 | 1126,74 |
| \( \sigma_i \) stress intensity | 2195,21 | 2205,44 | 2214,62 | 2214,113 | 2213,73 | 2214,51 |

The statics equation of zero equal sum of external and internal forces projections on axis of the shell has the form

\[
\sum_{i=1}^{n} \sigma_{xx} \Delta h_i \cdot 2\pi r_i = q \pi \left( R^2 - r_k^2 \right),
\]

\( N_q = q \pi \left( R^2 - r_k^2 \right) \) is resultant of external forces arising from applied pressure;

\( N_{in} = \sum_{i=1}^{n} \sigma_{xx} \Delta h_i \cdot 2\pi r_i \) is resultant of internal forces arising from stresses in the cross-section 2-2;

\( N_q = 3680,16 \kappa H ; \ N_{in} = 3614,02 \kappa H \) the difference is \( \delta = 1,87\% \).

Result shows the correctness of the developed algorithm for the elastic–plastic properties of material taken into account.

Test example 2. The stress-strain state of the rotation shell with an elliptical bottom under internal pressure is considered (scheme is shown in figure 2). The following input data is accepted: \( q=1,74 \) MPa, \( E=2\times10^4 \) MPa; \( \mu=0,3 \).

Figure 2. Scheme of load distribution through the thickness of the cross section
Static check: sum of the projections of external and internal forces on the shell axis is zero, was performed for section 2-2.

\[ N'_q = 2822.17 \text{kN} \; ; \; N'^{in} = 2884.24 \text{kN} \] the difference is \( \delta = 2.24\% \).

Figure 2 shows the zones of plastic deformations in longitudinal section of the shell. When loading first plastic deformations occur in the section 3-3 of the shell’s outer surface.

**Test example 3.** The stress-strain state of the rotation shell with an elliptical bottom under internal pressure is considered (scheme is shown in figure 3). The following input data is accepted:

\[ q=2.91 \text{ MPa}, \; E =2 \times 10^4 \text{ MPa}; \; \mu =0.3. \]

![Figure 3. Load distribution diagram for cross-section thickness](image)

In this example the resultant in section 2-2 of internal stresses \( N'^{in} =3782.37 \text{kN} \) and resultant from external pressure \( N_q=3841.17 \text{kN} \). The statics equation of sum of external and internal forces projections on the axis of the shell in section 2-2 equality to zero was performed with accuracy \( \delta = 1.56\% \).

In all cases the convergence of numerical calculation results was checked depending on the number of elements in the thickness of the shell. Results showed that a sufficient number of elements in the thickness is equal to 5.

### 5. Conclusion

Results show correctness of the developed algorithm with elastic–plastic properties of the material being taken into account.

It is known that the main requirements for mathematical models are those of adequacy, universality and economy. Developed finite element allows us to meet mentioned above requirements. By adopting a number of hypotheses and determining both stresses and deformations, resources computing is reduced ensuring cost-effectiveness and simplicity. Using developed scheme of mixed functional application in implementation of FEM the mathematical model acquires one of its main properties called "Potentiality".

### References

[1] Arkov D P 2012 Application of the mixed functional finite element method to the calculation of plates and shells taking into account physical nonlinearity (Cand. techn. Sci. diss. Volgograd, p 156)

[2] Alibeigloo A and Nouri V 2010 Static analysis of functionally graded cylindrical shell with piezoelectric layers using differential quadrature method (Composite Structures) 92(8), pp 1775-1785

[3] Akbari R and Alashi M, Khorsand M H, Tarahomi 2013 Thermo-elastic analysis of a functionally graded spherical shell with piezoelectric layers by differential quadrature method (Scientia Iranica) Volume 20(1), pp 109-119
[4] Asanjarani A and Satouri S, Alizadeh A, Kargarnovin M H 2015 Free vibration analysis of 2D-FGM truncated conical shell resting on Winkler-Pasternak foundations based on FSDT (Proceedings of the Institution of Mechanical Engineers) Part C: Journal of Mechanical Engineering Science 229(5), pp 818-839

[5] Xiaojin Shen and Qian Yang, Linjin Li, Zhiming Gao, Tiantian Wang 2020 Numerical approximation of the dynamic Koiter's model for the hyperbolic parabolic shell (Applied Numerical Mathematics) vol 150, pp 194-205

[6] Hernandez E and Naranjo J C 2020 Velloj in Modelling of thin viscoelastic shell structures under Reissner–Mindlin kinematic assumption (Applied Mathematical Modelling) vol 79, pp 180-199

[7] Semenov A 2019 Mathematical model of deformation of orthotropic shell structures under dynamic loading with transverse shears (Computers & Structures) vol 221, pp 65-73

[8] Lebiedzik C 2007 Exact boundary controllability of a shallow intrinsic shell model (Journal of Mathematical Analysis and Applications) 335(1), pp 584-614

[9] Mitsuru E 2016 An alternative first-order shear deformation concept and its application to beam, plate and cylindrical shell models (Composite Structures) 146(20), pp 50-61

[10] He X and Du H, Ying Z, Wang L, Melnik R. 2019 Modeling static microstructure of shape memory alloy via (Legendre wavelets collocation method Journal of Physics: Conference Series) 1419(1), 012003

[11] Cheng Q and Sun W, Cheng Y, Chen L, Li T 2019 Nonlinear Control Strategy Based on Lyapunov Function for MMC Under Grid Voltage Unbalance Condition | [电网电压不平衡条件下MMC的基于Lyapunov函数非线性控制策略] Gaodianya Jishu/High Voltage Engineering 45(12), pp 3984-3992

[12] Errico F and Tufano G, Robin O, Ichchou M, Atalla N 2019 Simulating the sound transmission loss of complex curved panels with attached noise control materials using periodic cell wavemodes (Applied Acoustics) 156, pp 21-28

[13] Tyukalov Y Y 2019 Calculation of circular plates with assuming shear deformations (Conference Series: Materials Science and Engineering) 687(3), 033004

[14] Kress G and Filipovic D 2019 Exact model for the response of moderately thick laminates to transverse forces (Composite Structures) 227, 111261

[15] Morse L and Sharif Khodaei Z, Aliabadi M H 2019 A dual boundary element based implicit differentiation method for determining stress intensity factor sensitivities for plate bending problems (Engineering Analysis with Boundary Elements) 106, pp 412-426

[16] Klochkov Yu V and Nikolaev A P, Ishchanov T R, Andreev A S, Klochkov M Yu 2020 Consideration of geometric nonlinearity in finite element strength calculations of thin-walled structures of the shell type (Construction mechanics of engineering structures and structures) vol 16 (1), pp 31-37

[17] Klochkov Yu V and Nikolaev A P, Vakhmina O V, Klochkov M Yu 2019 Variants of defining relations of the deformation theory of plasticity in the calculation of the rotation shell based on the finite element method (Construction mechanics of engineering structures and structures) vol 15(4) pp 315-322

[18] Arkov D P and Gureeva N A 2011 Application of the mixed finite element method for strength calculations of silos intended for grain storage (Proceedings of the lower Volga agrodiversity complex: Science and higher professional education) vol 1 (21), pp 189-197