THE PERTURBATION OF THE DE RHAM HODGE OPERATOR AND
THE KASTLER-KALAU-WALZE TYPE THEOREM FOR MANIFOLDS
WITH BOUNDARY

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Abstract. In this paper, we give Lichnerowicz type formulas for the perturbation
of the de Rham Hodge operator. We prove the Kastler-Kalau-Walze type theorems for the
perturbation of the de Rham Hodge operator on 4-dimensional and 6-dimensional com-
pact manifolds with or without boundary. Some concrete examples of the perturbation
of the de Rham Hodge operator are provided for our main theorems.

1. Introduction

The noncommutative residue was found in [1, 2]. Since noncommutative residues are of
great importance to the study of noncommutative geometry, more and more attention has
been attached to the study of noncommutative residues. By using the noncommutative
residue, Connes derived a conformal 4-dimensional Polyakov action analogy [3]. Connes
showed us that the noncommutative residue on a compact manifold $M$ coincided with
the Dixmiers trace on pseudodifferential operators of order $-\dim M$ in [4]. Connes put
forward that the noncommutative residue of the square of the inverse of the Dirac operator
was proportioned to the Einstein-Hilbert action, which is called the Kastler-Kalau-Walze
theorem now. Kastler gave a brute-force proof of this theorem [5]. In the same time,
Kalau and Walze proved this theorem in the normal coordinates system [6]. Ackermann
proved this theorem by using the heat kernel expansion, in [7]. The result of Connes was
extended to the higher dimensional case [8]. Fedosov et al. gave the definition about the
noncommutative residues on Boutet de Monvel algebra [9].

On the other hand, Wang generalized the Connes’ results to the case of manifolds with
boundary in [10, 11], and proved the Kastler-Kalau-Walze type theorem for the Dirac
operator and the signature operator on lower-dimensional manifolds with boundary. In
[12, 13], for the Dirac operator and the signature operator, Wang computed $\widetilde{\text{Wres}}[\pi + D^{-1} \circ \pi^+ D^{-1}]$, in these cases the boundary term vanished. But for $\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}]$, authors got a nonvanishing boundary term [14], and gave a theoretical explanation for

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gravitational action on boundary. In other words, Wang provided a kind of method to study the Kastler-Kalau-Walze type theorem for manifolds with boundary. In [15], Wei and Wang proved the Kastler-Kalau-Walze type theorem for modified Novikov operators on compact manifolds. In [16], Wu-Wang-Wang obtained two Lichnerowicz type formulas for the Dirac-witten operators, and gave the proof of Kastler-Kalau-Walze type theorems for the Dirac-witten operators on 4-dimensional and 6-dimensional compact manifolds with (resp.without) boundary. In [17], Wang proved a Kastler-Kalau-Walze type theorem for perturbations of the Dirac operators on compact manifolds with or without boundary.

The motivation of this paper is to prove the Kastler-Kalau-Walze type theorems for the perturbation of the de Rham Hodge operator. Specifically, we calculate \( \tilde{W}_{\text{res}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] \), \( \tilde{W}_{\text{res}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*D_A)^{-1}] \) and \( \tilde{W}_{\text{res}}[\pi^+D_A^{-1} \circ \pi^+D_A^{-3}] \), for \( D_A \) see (2.5).

A brief description of the organization of this paper is as follows. In Section 2, this paper will firstly introduce the basic notions of the perturbation of the de Rham Hodge operator, by means of which we can compute the Lichnerowicz formulas for the perturbation of the de Rham Hodge operator. And then we present the Kastler-Kalau-Walze type theorems for the perturbation of the de Rham Hodge operator on \( n \)-dimensional compact manifolds without boundary. In the next section, we calculate \( \tilde{W}_{\text{res}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] \) and \( \tilde{W}_{\text{res}}[\pi^+D_A^{-1} \circ \pi^+D_A^{-1}] \) for 4-dimensional compact manifolds with boundary. In Section 4, we prove the Kastler-Kalau-Walze type theorems on 6-dimensional compact manifolds with boundary for the perturbation of the de Rham Hodge operator.

2. The perturbation of the de Rham Hodge operator and its Lichnerowicz formulas

We give some definitions and basic notions which we will use in this paper. Let \( M \) be a \( n \)-dimensional \((n \geq 3)\) oriented compact Riemannian manifold with a Riemannian metric \( g^M \). And let \( \nabla^L \) be the Levi-Civita connection about \( g^M \). In the local coordinates \( \{x_i; 1 \leq i \leq n\} \) and the fixed orthonormal frame \( \{\tilde{e}_1, \cdots, \tilde{e}_n\} \), the connection matrix \((\omega_{s,t})\) is defined by

\[
\nabla^L(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(\omega_{s,t}).
\]

Let \( \epsilon(\tilde{e}_j^*) \), \( \iota(\tilde{e}_j^*) \) be the exterior and interior multiplications respectively, \( \tilde{e}_j^* \) be the dual base of \( \tilde{e}_j \) and \( c(\tilde{e}_j) \) be the Clifford action. Suppose that \( \partial_i \) is a natural local frame on \( TM \) and \((g^{ij})_{1 \leq i,j \leq n}\) is the inverse matrix associated to the metric matrix \((g_{ij})_{1 \leq i,j \leq n}\) on \( M \). Write

\[
c(\tilde{e}_j) = \epsilon(\tilde{e}_j^*) - \iota(\tilde{e}_j^*); \quad \overline{c}(\tilde{e}_j) = \epsilon(\tilde{e}_j^*) + \iota(\tilde{e}_j^*),
\]
which satisfies
\[
\begin{align*}
(2.3) & \quad c(\tilde{e}_i)c(\tilde{e}_j) + c(\tilde{e}_j)c(\tilde{e}_i) = -2\delta^j_i; \\
& \quad \tau(\tilde{e}_i)c(\tilde{e}_j) + c(\tilde{e}_j)\tau(\tilde{e}_i) = 0; \\
& \quad \tau(\tilde{e}_i)\tau(\tilde{e}_j) + \tau(\tilde{e}_j)\tau(\tilde{e}_i) = 2\delta^j_i.
\end{align*}
\]

By [12], we have the signature operator
\[
(2.4) \quad D = d + \delta = \sum_{i=1}^n c(\tilde{e}_i) \left[ \tilde{e}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) [\tau(\tilde{e}_s)\tau(\tilde{e}_t) - c(\tilde{e}_s)c(\tilde{e}_t)] \right] + A,
\]

We define the perturbation of the de Rham Hodge operator $D_A$ as follows:
\[
(2.5) \quad D_A = d + \delta + A
\]
\[
= \sum_{i=1}^n c(\tilde{e}_i) \left[ \tilde{e}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) [\tau(\tilde{e}_s)\tau(\tilde{e}_t) - c(\tilde{e}_s)c(\tilde{e}_t)] \right] + A,
\]

then we have
\[
(2.6) \quad D^*_A = d + \delta + A^*
\]
\[
= \sum_{i=1}^n c(\tilde{e}_i) \left[ \tilde{e}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) [\tau(\tilde{e}_s)\tau(\tilde{e}_t) - c(\tilde{e}_s)c(\tilde{e}_t)] \right] + A^*.
\]

where
\[
(2.7) \quad A = \tau(X_1) \cdots \tau(X_k)c(X_{k+1}) \cdots c(X_{k+t});
\]
\[
(2.8) \quad A^* = (-1)^t c(X_{k+t}) \cdots c(X_{k+1})\tau(X_k) \cdots \tau(X_1),
\]

where $X_1, \ldots, X_{k+t}$ are the smooth vector fields on $M$.

By some simple calculations, we get Lichnerowicz formulas.

**Theorem 2.1.** The following equalities hold:
\[
(2.9) \quad D_A^2 = -[g^{ij}(\nabla^1_{\partial_i} \nabla^1_{\partial_j} - \nabla^1_{\nabla^1_{\partial_i}}\partial_j)] - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tau(\tilde{e}_i)\tau(\tilde{e}_j)c(\tilde{e}_k)c(\tilde{e}_l) + \frac{1}{4} s + A^2
\]
\[
+ \frac{1}{4} \sum_{i=1}^n [c(\tilde{e}_i) A + Ac(\tilde{e}_i)]^2 - \frac{1}{2} \sum_{j=1}^n [\nabla^\Lambda_{\tilde{e}_j} T^M(A)c(\tilde{e}_j) - c(\tilde{e}_j)\nabla^\Lambda_{\tilde{e}_j} T^M(A)],
\]
\[
(2.10) \quad D^*_A D_A = -[g^{ij}(\tilde{\nabla}^2_{\tilde{\partial}_i} \tilde{\nabla}^2_{\tilde{\partial}_j} - \nabla^2_{\nabla^2_{\tilde{\partial}_i}}\tilde{\partial}_j)] - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tau(\tilde{e}_i)\tau(\tilde{e}_j)c(\tilde{e}_k)c(\tilde{e}_l) + \frac{1}{4} s + A^* A
\]
\[
+ \frac{1}{4} \sum_{i=1}^n [c(\tilde{e}_i) A + A^* c(\tilde{e}_i)]^2 - \frac{1}{2} \sum_{j=1}^n [\nabla^\Lambda_{\tilde{e}_j} T^M(A^*)c(\tilde{e}_j) - c(\tilde{e}_j)\nabla^\Lambda_{\tilde{e}_j} T^M(A)],
\]

where $s$ is the scalar curvature.
Now we prove the Theorem 2.1. Let \( M \) be a smooth compact oriented Riemannian \( n \)-dimensional manifolds without boundary and \( N \) be a vector bundle on \( M \). We say that \( P \) is a differential operator of Laplace type, if it has locally the form
\[
P = -(g^{ij}\partial_i\partial_j + A^i\partial_i + B),
\]
where \( \partial_i \) is a natural local frame on \( TM \) and \((g^{ij})_{1\leq i,j\leq n}\) is the inverse matrix associated to the metric matrix \((g_{ij})_{1\leq i,j\leq n}\) on \( M \), and \( A^i \) and \( B \) are smooth sections of \( \text{End}(N) \) on \( M \) (endomorphism). If \( P \) satisfies the form (2.11), then there is a unique connection \( \nabla \) on \( N \) and a unique endomorphism \( E \) such that
\[
P = -[g^{ij}((\nabla_{\partial_i} - \nabla_{\partial_j}) + \nabla_{\partial_j}) - \nabla_{\partial_i} + E],
\]
where \( \nabla^L \) is the Levi-Civita connection on \( M \). Moreover (with local frames of \( T^*M \) and \( N \)), \( \nabla_{\partial_i} = \partial_i + \omega_i \) and \( E \) are related to \( g^{ij} \), \( A^i \) and \( B \) through
\[
\omega_i = \frac{1}{2}g_{ij}(A^i + g^{kl}\Gamma^j_{kl}\text{id}),
\]
\[
E = B - g^{ij}(\partial_i(\omega_j) + \omega_j\omega_j - \omega_k\Gamma^k_{ij}),
\]
where \( \Gamma^j_{kl} \) is the Christoffel coefficient of \( \nabla^L \).

Let \( g^{ij} = g(dx^i, dx^j) \), \( \xi = \sum_k \xi_k dx_k \) and \( \nabla^L_k \partial_j = \sum_k \Gamma^j_{ij}\partial_k \), we denote that
\[
\begin{align*}
\sigma_i &= -\frac{1}{4}\sum_{s,t} \omega_{s,t}(\bar{e}_i)c(\bar{e}_s)c(\bar{e}_t); \quad a_i = \frac{1}{4}\sum_{s,t} \omega_{s,t}(\bar{e}_i)c(\bar{e}_s)c(\bar{e}_t); \\
\xi^j &= g^{ij}\xi_i; \quad \Gamma^k = g^{ij}\Gamma^k_{ij}; \quad \sigma^j = g^{ij}\sigma_i; \quad a^j = g^{ij}a_i.
\end{align*}
\]

Then, the perturbation of the de Rham Hodge operator \( D_A \) and \( D_A^* \) can be written as
\[
D_A = \sum_{i=1}^n c(\bar{e}_i)[\bar{e}_i + \sigma_i + a_i] + A;
\]
\[
D_A^* = \sum_{i=1}^n c(\bar{e}_i)[\bar{e}_i + \sigma_i + a_i] + A^*.
\]

By [18], the local expression of \((d + \delta)^2\) is
\[
(d + \delta)^2 = -\Delta_0 - \frac{1}{8}\sum_{i,j,k,l} R_{ijkl}c(\bar{e}_i)c(\bar{e}_j)c(\bar{e}_k)c(\bar{e}_l) + \frac{1}{4}s.
\]

By [7] and [18], we have
\[
-\Delta_0 = -g^{ij}(\nabla^L_i \nabla^L_j - \Gamma^k_{ij}\nabla^L_k).
\]

We note that
\[
D_A^2 = (d + \delta)^2 + (d + \delta)A + A(d + \delta) + A^2,
\]
\[(d + \delta)A + A(d + \delta) = \sum_{i,j} g^{ij}[c(\partial_i)A + Ac(\partial_i)]\partial_j + \sum_{i,j} g^{ij}[c(\partial_i)\partial_j(A) + c(\partial_i)a_jA + c(\partial_i)\sigma_jA + Ac(\partial_i)a_j + Ac(\partial_i)\sigma_j],\]

then we obtain

\[(2.22)\]
\[D_A^2 = -\sum_{i,j} g^{ij}[\partial_i\partial_j + 2\sigma_i\partial_j + 2a_i\partial_j - \Gamma_k^{ij}\partial_k] + (\partial_i\sigma_j) + (\partial_i a_j) + \sigma_i\sigma_j + \sigma_j a_j + a_i\sigma_j + a_i a_j - \Gamma_k^{ij}\sigma_k - \Gamma_k^{ij}a_k] + \sum_{i,j} g^{ij}[c(\partial_i)\partial_j(A) + c(\partial_i)a_jA + c(\partial_i)\sigma_jA + Ac(\partial_i)a_j + Ac(\partial_i)\sigma_j],\]

\[\frac{1}{s} + A^2.\]

Similarly, we have

\[(2.23)\]
\[D_A^*D_A = -\sum_{i,j} g^{ij}[\partial_i\partial_j + 2\sigma_i\partial_j + 2a_i\partial_j - \Gamma_k^{ij}\partial_k] + (\partial_i\sigma_j) + (\partial_i a_j) + \sigma_i\sigma_j + \sigma_j a_j + a_i\sigma_j + a_i a_j - \Gamma_k^{ij}\sigma_k - \Gamma_k^{ij}a_k] + \sum_{i,j} g^{ij}[c(\partial_i)\partial_j(A) + c(\partial_i)a_jA + c(\partial_i)\sigma_jA + A^*c(\partial_i)a_j + A^*c(\partial_i)\sigma_j].\]

By (2.13), (2.14) and (2.22), we have

\[(2.24)\]
\[(\omega_i)D_A^2 = \sigma_i + a_i - \frac{1}{2}[c(\partial_i)A + Ac(\partial_i)].\]
\begin{equation}
E_{D_A^2} = \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tilde{\partial}_i (\tilde{e}_j) \tilde{c}(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i=1}^{n} c(\partial_i) \partial^i (A) - \sum_{i=1}^{n} c(\partial_i) a^i A - \sum_{i=1}^{n} c(\partial_i) \sigma^i A
+ \frac{1}{2} \sum_{j=1}^{n} \partial^j [c(\partial_j) A + Ac(\partial_j)] - \frac{1}{2} \sum_{k=1}^{n} \Gamma^k [c(\partial_k) A + Ac(\partial_k)] - \frac{1}{4} s - A^2
+ \sum_{i,j} g^{ij} [c(\partial_i) A + Ac(\partial_i)] (\sigma_j + a_j) + \sum_{i,j} \frac{g^{ij}}{2} (\sigma_i + a_i) [c(\partial_j) A + Ac(\partial_j)]
- \sum_{i,j} \frac{g^{ij}}{4} [c(\partial_i) A + Ac(\partial_i)] [c(\partial_j) A + Ac(\partial_j)] - \sum_{i=1}^{n} Ac(\partial_i) a^i - \sum_{i=1}^{n} Ac(\partial_i) \sigma^i.
\end{equation}

Since \( E \) is globally defined on \( M \), taking normal coordinates at \( x_0 \), we have \( \sigma^i (x_0) = 0 \), \( a^i (x_0) = 0 \), \( \partial^j [c(\partial_j)] (x_0) = 0 \), \( \Gamma^k (x_0) = 0 \), \( g^{ij} (x_0) = \delta^j_1 \), so that

\begin{equation}
E_{D_A^2} (x_0) = \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tilde{\partial}_i (\tilde{e}_j) \tilde{c}(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i=1}^{n} [c(\partial_i) A + Ac(\partial_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\partial^j (A) c(\partial_j) - c(\partial_j) \partial^j (A)] - \frac{1}{4} s - A^2
= \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tilde{\partial}_i (\tilde{e}_j) \tilde{c}(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i=1}^{n} [c(\tilde{e}_i) A + Ac(\tilde{e}_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\tilde{c}_j (A) c(\tilde{e}_j) - c(\tilde{e}_j) \tilde{c}_j (A)] - \frac{1}{4} s - A^2
= \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tilde{\partial}_i (\tilde{e}_j) \tilde{c}(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i=1}^{n} [c(\tilde{e}_i) A + Ac(\tilde{e}_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\nabla_{\tilde{e}_j} A^* T^M (A) c(\tilde{e}_j) - c(\tilde{e}_j) \nabla_{\tilde{e}_j} A^* T^M (A)] - \frac{1}{4} s - A^2.
\end{equation}

Similarly, we have

\begin{equation}
E_{D_A^* D_A} (x_0) = \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} \tilde{\partial}_i (\tilde{e}_j) \tilde{c}(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i=1}^{n} [c(\tilde{e}_i) A + A^* c(\tilde{e}_i)]^2
+ \frac{1}{2} \sum_{j=1}^{n} [\nabla_{\tilde{e}_j} A^* T^M (A^*) c(\tilde{e}_j) - c(\tilde{e}_j) \nabla_{\tilde{e}_j} A^* T^M (A^*)] - \frac{1}{4} s - A^* A.
\end{equation}

By (2.12), we get Theorem 2.1.

According to the detailed descriptions in [7], we know that the noncommutative residue
of a generalized laplacian \( \tilde{\Delta} \) is expressed as

\[
(n - 2)\Phi_2(\tilde{\Delta}) = (4\pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \tilde{\text{res}}(\tilde{\Delta}^{\frac{n}{2}} - 1),
\]

where \( \Phi_2(\tilde{\Delta}) \) denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of \( \tilde{\Delta} \). Now let \( \tilde{\Delta} = D_A^2 \). Since \( D_A^2 \) is a generalized laplacian, we can suppose \( D_A^2 = \Delta - E \), then, we have

\[
W_{\text{res}}(D_A^2) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}\left( -\frac{1}{12} s + \frac{n}{2} A^2 - \frac{1}{2} \sum_{j=1}^{n} A c(\tilde{e}_j) A c(\tilde{e}_j) \right) d\text{Vol}_M,
\]

\[
W_{\text{res}}(A^* D_A D_A) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}\left( -\frac{1}{12} s + \frac{n}{2} A^* A - \frac{1}{4} \sum_{j=1}^{n} A c(\tilde{e}_j) A c(\tilde{e}_j) \right) d\text{Vol}_M,
\]

where \( W_{\text{res}} \) denote the noncommutative residue. By applying the formulae shown in (2.26), (2.27), (2.29) and (2.30), we get:

**Theorem 2.2.** If \( M \) is a \( n \)-dimensional compact oriented manifolds without boundary, we have the following:

\[
W_{\text{res}}(D_A^2) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}\left( -\frac{1}{12} s + \frac{n}{2} A^2 - \frac{1}{2} \sum_{j=1}^{n} A c(\tilde{e}_j) A c(\tilde{e}_j) \right) d\text{Vol}_M,
\]

\[
W_{\text{res}}(A^* D_A D_A) = \frac{(n - 2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 1)!} \int_M \text{tr}\left( -\frac{1}{12} s + \frac{n}{2} A^* A - \frac{1}{4} \sum_{j=1}^{n} A c(\tilde{e}_j) A c(\tilde{e}_j) \right) d\text{Vol}_M,
\]

where \( s \) is the scalar curvature.

### 3. A Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary

Firstly, we explain the basic notions of Boutet de Monvel’s calculus and the definition of the noncommutative residue for manifolds with boundary that will be used throughout the paper. For the details, see Ref.[12].

Let \( U \subset M \) be a collar neighborhood of \( \partial M \) which is diffeomorphic with \( \partial M \times [0, 1) \). By the definition of \( h(x_n) \in C^\infty([0, 1)) \) and \( h(x_n) > 0 \), there exists \( \tilde{h} \in C^\infty((0, 1)) \) such that \( \tilde{h}|_{0,1} = h \) and \( \tilde{h} > 0 \) for some sufficiently small \( \varepsilon > 0 \). Then there exists a metric \( g' \)
on $\widetilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$g' = \frac{1}{h(x_n)}g^{\partial M} + dx^2_n;$$

such that $g'|_M = g$. We fix a metric $g'$ on the $\widetilde{M}$ such that $g'|_M = g$.

We define the Fourier transformation $F'$ by

$$F': L^2(\mathbb{R}_d) \to L^2(\mathbb{R}_v); \quad F'(v) = \int e^{-iuv}u(t)dt$$

and let

$$r^+: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^+); \quad f \to f|_{\mathbb{R}^+}; \quad \mathbb{R}^+ = \{x \geq 0; x \in \mathbb{R}\}$$

where $\Phi(\mathbb{R})$ denotes the Schwartz space and $\Phi(\mathbb{R}^+) = r^+\Phi(\mathbb{R}), \Phi(\mathbb{R}^-) = r^-\Phi(\mathbb{R})$.

We define $H^+ = F'(\Phi(\mathbb{R}^+))$; $H^-_0 = F'(\Phi(\mathbb{R}^-))$ which satisfies $H^+ \cap H^-_0$. We have the following property: $h \in H^+(H^-_0)$ if and only if $h \in C^\infty(\mathbb{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\Im \xi < 0\}$ ($\{\Im \xi > 0\}$) such that for all nonnegative integer $l$,

$$\frac{d^l h}{d \xi^l}(\xi) \sim \sum_{k=1}^\infty \frac{d^l}{d \xi^l}\left(\frac{c_k}{\xi^k}\right),$$

as $|\xi| \to +\infty, \Im \xi \leq 0 \ (\Im \xi \geq 0)$.

Let $H'$ be the space of all polynomials and $H^- = H^+_0 \bigoplus H'$; $H = H^+ \bigoplus H^-$. Denote by $\pi^+ (\pi^-)$ respectively the projection on $H^+ (H^-)$. For calculations, we take $H = \widetilde{H} = \{\text{rational functions having no poles on the real axis}\}$ ($\widetilde{H}$ is a dense set in the topology of $H$). Then on $\widetilde{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} \, d\xi,$$

where $\Gamma^+$ is a Jordan close curve included $\Im(\xi) > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. Similarly, define $\pi'$ on $\widetilde{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) \, d\xi.$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(\mathbb{R}), \pi' h = \frac{1}{2\pi} \int_{\mathbb{R}} h(v)dv$ and for $h \in H^+ \cap L^1(\mathbb{R}), \pi' h = 0$.

Let $M$ be a $n$-dimensional compact oriented manifold with boundary $\partial M$. Denote by $B$ Boutet de Monvel’s algebra, we recall the main theorem in [9, 12].

**Theorem 3.1.** [Fedosov-Golse-Leichtnam-Schrohe] Let $X$ and $\partial X$ be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi + P & G \\ T & S \end{pmatrix} \in B$, and denote by $p$, $b$ and $s$ the local symbols
of $P, G$ and $S$ respectively. Define:

\begin{equation}
\widetilde{\text{Wres}}(A) = \int_X \int_S \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx + 2\pi \int_{\partial X} \int_{S'} \{ \text{tr}_E [(\text{tr}b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx',
\end{equation}

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

**Definition 3.2.** [12] Lower dimensional volumes of spin manifolds with boundary are defined by

\begin{equation}
\text{Vol}^{(p_1, p_2)}_n M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}],
\end{equation}

By [12], we get

\begin{equation}
\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi| = 1} \text{trace}_{\Lambda^* T^* M} [\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi\end{equation}

and

\begin{equation}
\Phi = \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \sum_{j, k=0}^\infty \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^* T^* M} \left[ \partial_{x_n}^j \partial_{\xi_n}^k \sigma_r^+(D^{-p_1})(x', 0, \xi', \xi_n) \right] \partial_{\xi_n}^r \sigma(\xi') dx',
\end{equation}

where the sum is taken over $r + l - k - |\alpha| - j - 1 = -n$, $r \leq -p_1, l \leq -p_2$.

Since $[\sigma_{-n}(D^{-p_1-p_2})]|_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without boundary, so locally we can compute the first term by [5], [6], [12], [19].

For any fixed point $x_0 \in \partial M$, we choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Phi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1) \subset M$ and the metric $\frac{1}{h(x_n)} g^{\partial M} + dx_n^2$. The dual metric of $g^M$ on $\tilde{U}$ is $h(x_n) g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $g_{ij}^\partial = g^M(dx_i, dx_j)$, then

\begin{equation}
[g_{ij}^M] = \begin{bmatrix} h(x_n) g_{ij}^{\partial M} & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_{ij}^\partial] = \begin{bmatrix} h(x_n) g_{ij}^{\partial M} & 0 \\ 0 & 1 \end{bmatrix}
\end{equation}

and

\begin{equation}
\partial_{x_i} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n-1; \quad g_{ij}^M(x_0) = \delta_{ij}.
\end{equation}

We review the following three lemmas.

**Lemma 3.3.** [12] With the metric $g^M$ on $M$ near the boundary

\begin{equation}
\partial_{x_j}(|\xi'|^2 g^M)(x_0) = \begin{cases} 0, & \text{if } j < n, \\
h'(0)|\xi'|^2 g^{\partial M}, & \text{if } j = n; \end{cases}
\end{equation}

\begin{equation}
\partial_{x_j} [c(\xi')](x_0) = \begin{cases} 0, & \text{if } j < n, \\
\partial x_n(c(\xi'))(x_0), & \text{if } j = n, \end{cases}
\end{equation}
where $\xi = \xi' + \xi_n dx_n$.

**Lemma 3.4.** [12] With the metric $g^M$ on $M$ near the boundary

$$\omega_{s,t}(\tilde{e}_i)(x_0) = \begin{cases} 
\omega_{n,i}(\tilde{e}_i)(x_0) = \frac{1}{2} h'(0), & \text{if } s = n, t = i, i < n; \\
\omega_{i,n}(\tilde{e}_i)(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, i < n; \\
\omega_{s,t}(\tilde{e}_i)(x_0) = 0, & \text{other cases},
\end{cases}$$

where $(\omega_{s,t})$ denotes the connection matrix of Levi-Civita connection $\nabla^L$.

**Lemma 3.5.** [12]

$$\Gamma^k_{st}(x_0) = \begin{cases} 
\Gamma^n_{ii}(x_0) = \frac{1}{2} h'(0), & \text{if } s = t = i, k = n, i < n; \\
\Gamma^n_{ni}(x_0) = -\frac{1}{2} h'(0), & \text{if } s = n, t = i, k = i, i < n; \\
\Gamma^n_{in}(x_0) = -\frac{1}{2} h'(0), & \text{if } s = i, t = n, k = i, i < n, \\
\Gamma^i_{st}(x_0) = 0, & \text{other cases}.
\end{cases}$$

Similar to (3.9) and (3.10), we firstly compute

$$\int_M \int_{[\xi]=1} \text{trace}_{\Lambda^* T^* M}[\sigma_{-4}((D^*_A D_A)^{-1})] \sigma(\xi) dx + \int_{\partial M} \Psi,$$

where

$$\Psi = \int_{[\xi']=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(\xi')^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^* T^* M}[\partial_{\xi'}^\alpha \partial_{\xi_n}^j \partial_{\xi_n}^k \sigma_{(D^*_A)^{-1}}(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx',$$

the sum is taken over $r + l - k - j - |\alpha| - 1 = -4, r \leq -1, l \leq -1$.

Then we can compute the interior of $\int_M \int_{[\xi]=1} \text{trace}_{\Lambda^* T^* M}[\sigma_{-4}((D^*_A D_A)^{-1})] \sigma(\xi) dx = 32\pi^2 \int_M \text{tr} \left(-\frac{1}{12}s + A^* A - \frac{1}{4} \sum_{j=1}^{n} A c(\tilde{e}_j) A c(\tilde{e}_j) - \frac{1}{4} \sum_{j=1}^{n} A^* c(\tilde{e}_j) A^* c(\tilde{e}_j) + \frac{1}{2} \sum_{j=1}^{n} \nabla_{\tilde{e}_j}^{\Lambda^*T^* M}(A^*) c(\tilde{e}_j) \right) d\text{Vol}_M$.

Now we need to compute $\int_{\partial M} \Psi$. Since, some operators have the following symbols.
Lemma 3.6. The following identities hold:

\[ (3.20) \quad \sigma_1(D_A) = \sigma_1(D_A^*) = ic(\xi); \]

\[ \sigma_0(D_A) = \frac{1}{4} \sum_{i,s,t} \omega_{i,s,t}(\overline{\xi}_i) c(\overline{\xi}_i) c(\overline{\xi}_s) c(\overline{\xi}_t) - \frac{1}{4} \sum_{i,s,t} \omega_{i,s,t}(\overline{\xi}_i) c(\overline{\xi}_i) c(\overline{\xi}_s) + A; \]

\[ \sigma_0(D_A^*) = \frac{1}{4} \sum_{i,s,t} \omega_{i,s,t}(\overline{\xi}_i) c(\overline{\xi}_i) c(\overline{\xi}_s) c(\overline{\xi}_t) - \frac{1}{4} \sum_{i,s,t} \omega_{i,s,t}(\overline{\xi}_i) c(\overline{\xi}_i) c(\overline{\xi}_s) + A^*. \]

Write

\[ (3.21) \quad D_x^\alpha = (-i)^{\alpha_i} \partial_x^\alpha; \quad \sigma(D_A) = p_1 + p_0; \quad \sigma(D_A^{-1}) = \sum_{j=1}^{\infty} q_j. \]

By the composition formula of pseudodifferential operators, we have

\[ (3.22) \quad 1 = \sigma(D_A \circ D_A^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_x^\alpha [\sigma(D_A)] D_x^\alpha [\sigma(D_A^{-1})] \]

\[ = (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) \]

\[ + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \cdots) \]

\[ = p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \cdots, \]

so

\[ (3.23) \quad q_{-1} = p_1^{-1}; \quad q_{-2} = -p_1^{-1} [p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1})]. \]

Lemma 3.7. The following identities hold:

\[ (3.24) \quad \sigma_{-1}(D_A^{-1}) = \sigma_{-1}((D_A^*)^{-1}) = \frac{ic(\xi)}{|\xi|^2}; \]

\[ \sigma_{-2}(D_A^{-1}) = \frac{c(\xi) \sigma_0(D_A) c(\xi)}{|\xi|^4} \quad \text{and} \quad \sigma_{-2}(D_A^*)^{-1} = \frac{c(\xi) \sigma_0(D_A^*) c(\xi)}{|\xi|^4} \]

\[ \quad \frac{c(\xi) \sigma_0(D_A) c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j} (c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right]; \]

\[ \quad \frac{c(\xi) \sigma_0(D_A^*) c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j} (c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} (|\xi|^2) \right]. \]

We denote tr as shorthand of trace. When \( n = 4, \) then \( \text{tr}_{\Lambda \ast T \cdot M} [\text{id}] = \text{dim}(\Lambda^*(4)) = 16, \)

since the sum is taken over \( r + l - k - j - |\alpha| - 1 = -4, \quad r \leq -1, \quad l \leq -1, \) then we have the following five cases:

case a) \( I) \quad r = -1, \quad l = -1, \quad k = j = 0, \quad |\alpha| = 1 \)
By applying the formula shown in (3.18), we can calculate (3.25)

\[ \Psi_1 = -\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}\left[ \partial^\alpha_{\xi \eta} \sigma_{-1} \left( (D_A^{-1}) \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \]

**case a) II)** \( r = -1, \ l = -1, \ k = |\alpha| = 0, \ j = 1 \)

By (3.18), we get

(3.26) \[ \Psi_2 = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[ \partial_{x_n} \pi_{\xi_n} \sigma_{-1} \left( (D_A^{-1}) \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \]

**case a) III)** \( r = -1, \ l = -1, \ j = |\alpha| = 0, \ k = 1 \)

By (3.18), we calculate that

(3.27) \[ \Psi_3 = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[ \partial_{\xi_n} \pi_{\xi_n} \sigma_{-1} \left( (D_A^{-1}) \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \]

Similar to the formulae (2.17)-(2.31) in [12], we have

(3.28) \[ \Psi_1 + \Psi_2 + \Psi_3 = 0. \]

**case b)** \( r = -2, \ l = -1, \ k = j = |\alpha| = 0 \)

Similarly, we get

(3.29) \[ \Psi_4 = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\left[ \pi_{\xi_n} \sigma_{-2} \left( (D_A^{-1}) \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \]

We first compute

(3.30) \[ \sigma_{-2}(D_A^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_A)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n} \left[ c(\xi') \right] (x_0) |\xi|^2 - c(\xi)h'(0) |\xi|^2 \left[ \partial_{\xi} \right] \right], \]

where

(3.31) \[ \sigma_0(D_A)(x_0) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\xi_i)(x_0)c(\xi_i)c(\xi_s)c(\xi_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\xi_i)(x_0)c(\xi_i)c(\xi_s)c(\xi_t) + A. \]
We denote
\begin{equation}
\tag{3.32}
b_0^1(x_0) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\overline{\tau}(\tilde{e}_s)\overline{\tau}(\tilde{e}_t);
\end{equation}
\begin{equation}
\tag{3.33}
b_0^2(x_0) = -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t).
\end{equation}

Means that
\begin{align*}
\pi_{\xi_n}^+ \sigma_{-2}(D_A^{-1})(x_0)|_{\xi'|=1} &= \pi_{\xi_n}^+ \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right) + \pi_{\xi_n}^+ \left( \frac{c(\xi)Ac(\xi)}{(1 + \xi_n^2)^2} \right) \\
&+ \pi_{\xi_n}^+ \left( \frac{c(\xi)b_0^2(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2} - h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right).
\end{align*}

Since
\begin{equation}
\tag{3.34}
b_0^1(x_0)c(dx_n) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\tilde{e}_i)\overline{\tau}(\tilde{e}_i)c(\tilde{e}_n)\overline{\tau}(\tilde{e}_n),
\end{equation}
then by the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \), we have the equalities:
\begin{equation*}
\text{tr}[c(\tilde{e}_i)\overline{\tau}(\tilde{e}_i)c(\tilde{e}_n)\overline{\tau}(\tilde{e}_n)] = 0 \quad (i < n); \quad \text{tr}[b_0^1(x_0)c(dx_n)] = 0.
\end{equation*}

Therefore
\begin{equation}
\tag{3.35}
\partial_{\xi_n} \sigma_{-1}((D_A^*)^{-1})(x_0)|_{\xi'|=1} = \partial_{\xi_n} q_{-1}(x_0)|_{\xi'|=1} = i \left( \frac{c(dx_n)}{1 + \xi_n^2} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1 + \xi_n^2)^2} \right).
\end{equation}

Hence, we have
\begin{align*}
\tag{3.36}
\text{tr} \left[ \pi_{\xi_n}^+ \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-1}((D_A^*)^{-1})(x_0) \right]|_{\xi'|=1} &= \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[b_0^1(x_0)c(\xi')] + \frac{i}{2(1 + \xi_n^2)^2} \text{tr}[b_0^1(x_0)c(dx_n)] \\
&= \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[b_0^1(x_0)c(\xi')].
\end{align*}

We note that \( i < n \), \( \int_{\xi'|=1} \{ \xi_{i_1} \cdots \xi_{i_{2d+1}} \} \sigma(\xi') = 0 \), so \( \text{tr}[c(\xi')c(dx_n)] \) has no contribution for computing case b),

\begin{equation}
\tag{3.37}
- i \int_{\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-1}((D_A^*)^{-1}) \right](x_0)d\xi_n\sigma(\xi')dx' = 0.
\end{equation}
Since
\[
(3.38) \quad \text{tr} \left[ \pi^+_{\xi_n} \left( \frac{c(\xi) Ac(\xi)}{1 + \xi_n^2} \right) \right] \times \partial_{\xi_n} \sigma_1 ((D^*_A)^{-1})(x_0) \bigg|_{\xi'_n = 1} = \frac{1}{2(1 + \xi_n^2)} \text{tr} [Ac(\xi')] + \frac{i}{2(1 + \xi_n^2)} \text{tr} [Ac(dx_n)],
\]
then, we have
\[
(3.39) \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \left( \frac{c(\xi) Ac(\xi)}{1 + \xi_n^2} \right) \right] \times \partial_{\xi_n} \sigma_1 ((D^*_A)^{-1}) (x_0) d\xi_n \sigma(\xi') dx',
\]
\[
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2(1 + \xi_n^2)} \text{tr} [Ac(\xi')] d\xi_n \sigma(\xi') dx' \\
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{i}{2(1 + \xi_n^2)} \text{tr} [Ac(dx_n)] d\xi_n \sigma(\xi') dx' \\
= \frac{\Omega_3}{2} \text{tr} [Ac(dx_n)] \int_{\Gamma^+} \frac{1}{(\xi_n - i)^2(\xi_n + i)^2} dx' \\
= \frac{\Omega_3}{2} \text{tr} [Ac(dx_n)] \frac{2\pi i}{(\xi_n + i)^2} \bigg|_{\xi_n = i} dx' \\
= \frac{\pi}{4} \Omega_3 \text{tr} [Ac(dx_n)] dx'.
\]
We have
\[
(3.40) \quad \pi^+_{\xi_n} \left( \frac{c(\xi) b^2_0(x_0) c(\xi) + c(\xi) c(dx_n) \sigma_{\xi_n} [c(\xi')]}{(1 + \xi_n^2)^2} \right) - h'(0) \pi^+_{\xi_n} \left( \frac{c(\xi) c(dx_n) c(\xi)}{(1 + \xi_n^2)^3} \right) := B_1 - B_2,
\]
where
\[
(3.41) \quad B_1 = \frac{-1}{4(\xi_n - i)^2} [(2 + i\xi_n) c(\xi') b^2_0(x_0) c(\xi') + i\xi_n c(dx_n) b^2_0(x_0) c(dx_n) - i\sigma_{\xi_n} c(\xi') + (2 + i\xi_n) c(\xi') c(dx_n) \sigma_{\xi_n} c(\xi') + ic(dx_n) b^2_0(x_0) c(\xi') + ic(\xi') b^2_0(x_0) c(dx_n)]
\]
and
\[
(3.42) \quad B_2 = \frac{h'(0)}{2} \left( \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - i\xi_n c(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} (ic(\xi') - c(dx_n)) \right).
\]
A simple calculation shows that
\[
(3.43) \quad \text{tr} [B_2 \times \partial_{\xi_n} \sigma_1 ((D^*_A)^{-1})(x_0)] \bigg|_{\xi'_n = 1} = \frac{i}{2} h'(0) \frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2} \text{tr} [id] \\
= 8ih'(0) \frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}.
\]
Similarly, we have

\[
\text{(3.44)} \quad \text{tr}[B_1 \times \partial_{\xi_n} \sigma_{-1}((D_A^*)^{-1})(x_0)]|_{\xi'|=1} = \frac{-8ic_0}{(1 + \xi_n^2)} + 2h'(0) \frac{\xi_n^2 - i\xi_n - 2}{(\xi_n - i)(1 + \xi_n^2)},
\]

where \( b^2_0 = c_0 c(dx_n) \) and \( c_0 = -\frac{3}{4} h'(0) \).

By (3.44) and (3.45), we have

\[
\text{(3.45)} \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[(B_1 - B_2) \times \partial_{\xi_n} \sigma_{-1}((D_A^*)^{-1})](x_0)d\xi_n \sigma(\xi')dx' = -\Omega_3 \int_{\Gamma^+} \frac{8c_0(\xi_n - i) + 4ih'(0)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_n dx'
\]

\[
= \frac{9}{2} \pi h'(0) \Omega_3 dx'.
\]

Hence, we have

\[
\text{(3.46)} \quad \Psi_4 = \frac{9}{2} \pi h'(0) \Omega_3 dx' + \frac{\pi}{4} \Omega_3 \text{tr}[Ac(dx_n)]dx'.
\]

**case c) \( r = -1, \quad l = -2, \quad k = j = |\alpha| = 0 \)**

Using (3.18), we get

\[
\text{(3.47)} \quad \Psi_5 = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi^+_{\xi_n} \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \sigma_{-2}((D_A^*)^{-1})](x_0)d\xi_n \sigma(\xi')dx'.
\]

Considering (3.5) and (3.6), we have

\[
\pi^+_{\xi_n} \sigma_{-1}(D_A^{-1})(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\]

Since

\[
\text{(3.49)} \quad \sigma_{-2}((D_A^*)^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_A^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) [\partial_{x_n}[c(\xi')(x_0)]|\xi|^2 - c(\xi)h'(0)|\xi|^2_{\partial_{x_n}}],
\]

where

\[
\text{(3.50)} \quad \sigma_0(D_A^*)(x_0) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\bar{e}_i)(x_0)c(\bar{e}_i)c(\bar{e}_s)c(\bar{e}_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\bar{e}_i)(x_0)c(\bar{e}_i)c(\bar{e}_s)c(\bar{e}_t) + A^*
\]

\[
= b^1_0(x_0) + b^2_0(x_0) + A^*,
\]
then

\[(3.51)\]

\[
\partial_n \sigma_{-2}((D^*_n)^{-1})(x_0)|_{\xi'|=1} = \partial_n \left( \frac{c(\xi)(b_0^1(x_0) + b_0^2(x_0) + A^*)c(\xi)}{|\xi|^4} \right)
+ \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n} [c(\xi')] (x_0)|\xi|^2 - c(\xi)h'(0)|\xi|^2 \right]
\]

\[
= \partial_n \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^4} \right) + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n} [c(\xi')] (x_0)|\xi|^2 - c(\xi)h'(0) \right]
+ \partial_n \left( \frac{c(\xi)A^*c(\xi)}{|\xi|^4} \right).
\]

By computation, we have

\[(3.52)\]

\[
\partial_n \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^4} \right) = \frac{c(dx_n)b_0^1(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)b_0^1(x_0)c(dx_n)}{|\xi|^4} - 4\xi_n c(\xi)b_0^1(x_0)c(\xi),
\]

\[(3.53)\]

\[
\partial_n \left( \frac{c(\xi)A^*c(\xi)}{|\xi|^4} \right) = \frac{c(dx_n)A^*c(\xi)}{|\xi|^4} + \frac{c(\xi)A^*c(dx_n)}{|\xi|^4} - 4\xi_n c(\xi)A^*c(\xi).
\]

For the sake of convenience in writing, we denote

\[(3.54)\]

\[
q_{-2}^1 = \frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n} [c(\xi')] (x_0)|\xi|^2 - c(\xi)h'(0) \right],
\]

then

\[(3.55)\]

\[
\partial_n (q_{-2}^1) = \frac{1}{(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3) c(dx_n)b_0^2(x_0) c(dx_n) + (1 - 3\xi_n^2) c(dx_n)b_0^2(x_0)c(\xi') + (1 - 3\xi_n^2)c(\xi')b_0^2(x_0)c(dx_n) - 4\xi_n c(\xi')b_0^2(x_0) c(\xi') + (3\xi_n^2 - 1) \partial_{x_n} c(\xi') - 4\xi_n c(\xi')c(dx_n) \partial_{x_n} c(\xi') + 2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n) \right]
+ 6\xi_n h'(0) \frac{c(dx_n)c(\xi)}{|\xi|^4}.
\]

By (3.48) and (3.52), we have

\[(3.56)\]

\[
\text{tr} \left[ \pi_+^\dagger \sigma_{-1} (D^*_n)^{-1})(x_0) \times \partial_n \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^4} \right) \right] |_{\xi'|=1}
= \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[b_0^1(x_0)c(\xi')] + \frac{i}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[b_0^1(x_0)c(dx_n)]
= \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[b_0^1(x_0)c(\xi')] ,
\]
it is shown that

\[(3.57) \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^{+} \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) b_0^c(\xi)}{\xi} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \]

\[= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr} [b_0^l(x_0) c(\xi')] d\xi_n \sigma(\xi') dx' \]

\[= 0. \]

Similarly to (3.56), we have

\[(3.58) \quad \text{tr} \left[ \pi_{\xi_n}^{+} \sigma_{-1}(D_A^{-1})(x_0) \times \partial_{\xi_n} \left( \frac{c(\xi) A^* c(\xi)}{\xi} \right) \right] \bigg|_{|\xi'|=1} \]

\[= \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr} [A^* c(\xi')] + \frac{i}{2(\xi_n - i)(\xi_n + i)^3} \text{tr} [A^* c(dx_n)], \]

then

\[(3.59) \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^{+} \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) A^* c(\xi)}{\xi} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \]

\[= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr} [A^* c(\xi')] d\xi_n \sigma(\xi') dx' \]

\[= \frac{\Omega_3}{2} \text{tr} [A^* c(dx_n)] \int_{\Gamma} \left( \frac{1}{(\xi_n + i)^3} \right) d\xi_n dx' \]

\[= \frac{\Omega_3}{2} \text{tr} [A^* c(dx_n)] 2\pi i \left[ \frac{1}{(\xi_n + i)^3} \right]^{(1)} |_{\xi_n = i} dx' \]

\[= -\frac{\pi}{4} \Omega_3 \text{tr} [A^* c(dx_n)] dx'. \]

Observing (3.48) and (3.55), we have

\[(3.60) \quad \text{tr} [\pi_{\xi_n}^{+} \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} (q_{-2}^1)] (x_0) \bigg|_{|\xi'|=1} = \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4}. \]
By \( \int_{|\xi'|=1} \{\xi_{i_1}, \ldots, \xi_{i_{2d+1}}\} \sigma(\xi') = 0 \) and (3.60), we have

\[
(3.61) \quad -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\pi^{+}_{\xi_n} \sigma_{-1}(D_{A}^{-1}) \times \partial_{\xi_n}(q_{-2})](\xi_0) d\xi_n \sigma(\xi') dx' \\
= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4} d\xi_n \sigma(\xi') dx' \\
= -i\Omega_3 \int_{\Gamma^*} \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4} d\xi_n dx' \\
= -i\Omega_3 \frac{2\pi i}{2!} \left[ \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi + i)^3} \right]^{(2)} |_{\xi_n=\xi} dx' - i\Omega_3 \frac{2\pi i}{2!} \left[ \frac{48h'(0)i\xi_n}{(\xi + i)^4} \right]^{(2)} |_{\xi_n=\xi} dx' \\
= -\frac{9}{2} \pi h'(0)\Omega_3 dx'.
\]

Then,

\[
(3.62) \quad \Psi_5 = -\frac{9}{2} \pi h'(0)\Omega_3 dx' - \frac{\pi}{4} \Omega_3 \text{tr}[A^*c(dx_n)] dx'.
\]

In summary,

\[
(3.63) \quad \Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 = \frac{\pi}{4} \Omega_3 \text{tr}[Ac(dx_n)] dx' - \frac{\pi}{4} \Omega_3 \text{tr}[A^*c(dx_n)] dx'.
\]

Applying (3.17), (3.19) and (3.63), we can assert that:

**Theorem 3.8.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^{M} \) as above, \( D_{A} \) be the perturbation of the de Rham Hodge operator on \( \mathring{M} \), then

\[
(3.64) \quad \widetilde{\text{Wres}}[\pi^+D_{A}^{-1} \circ \pi^+(D_{A}^*)^{-1}] = 32\pi^2 \int_{M} \text{tr} \left( -\frac{1}{12} s + A^*A - \frac{1}{4} \sum_{j=1}^{n} Ac(e_j)Ac(e_j) \right. \\
- \frac{1}{4} \sum_{j=1}^{n} A^*c(e_j)Ac(e_j) + \frac{1}{2} \sum_{j=1}^{n} \nabla_{e_j}^* T^{*M}(A^*)c(e_j) - \frac{1}{2} \sum_{j=1}^{n} c(e_j) \nabla_{e_j}^* T^{*M}(A) \big) dV_{\mathring{M}} \\
+ \int_{\partial M} \frac{\pi}{4} \Omega_3 \text{tr}[Ac(dx_n)] dV_{\partial M} - \int_{\partial M} \frac{\pi}{4} \Omega_3 \text{tr}[A^*c(dx_n)] dV_{\partial M},
\]

where \( s \) is the scalar curvature.

When \( A = c(X) \), then we have

\[
(3.65) \quad \int_{M} \int_{|\xi'|=1} \text{trace}_{\lambda^* T^{*M}}[\sigma_{-4}((D_{A}^*D_{A})^{-1})] \sigma(\xi) dx \equiv 512\pi^2 \int_{M} \left( -\frac{1}{12} s + 2|X|^2 + \sum_{j=1}^{n} g(\nabla_{e_j}^* T^{*M}X, e_j) \right) dV_{\mathring{M}}.
\]
and

\[(3.66) \quad \int_{\partial M} \Psi = - \int_{\partial M} 8\pi \Omega_3 g(\partial x_n, X)d\text{Vol}_{\partial M}.\]

We can immediately state the following corollary:

**Corollary 3.9.** Let \(M\) be a 4-dimensional oriented compact manifolds with the boundary \(\partial M\) and the metric \(g^M\) as above, and let \(A = c(X)\), then

\[(3.67) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12} s + 2|X|^2 + \sum_{j=1}^n g(\nabla_{e_j}^T X, e_j) \right) d\text{Vol}_M

- \int_{\partial M} 8\pi \Omega_3 g(\partial x_n, X)d\text{Vol}_{\partial M},

where \(s\) is the scalar curvature.

When \(A = \pi(X)\), we can get

\[(3.68) \quad \int_M \int_{|\xi|=1} \text{trace}_{\pi^*T^*M}[\sigma_A((D_A^* D_A)^{-1})] \sigma(\xi) dx = 512\pi^2 \int_M \left( -\frac{1}{12} s - |X|^2 \right) d\text{Vol}_M,

and

\[(3.69) \quad \int_{\partial M} \Psi = 0.

Now, we compute \(\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^*)^{-1}]\).

**Corollary 3.10.** Let \(M\) be a 4-dimensional oriented compact manifolds with the boundary \(\partial M\) and the metric \(g^M\) as above, and let \(A = \pi(X)\), then

\[(3.70) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12} s - |X|^2 \right) d\text{Vol}_M,

where \(s\) is the scalar curvature.

When \(A = c(X)c(Y)\), then \(A^* = c(Y)c(X)\).

By computation, we have

\[(3.71) \quad \text{tr}[A^* A] = \text{tr}[(-|X|^2)(-|Y|^2)] = |X|^2 |Y|^2 \text{tr}[\text{id}],

\text{tr}[c(dx_n)A] = 0, \text{tr}[c(dx_n)A^*] = 0,\]
\[
\sum_{j=1}^{n} \text{tr}[Ac(\bar{e}_j)Ac(\bar{e}_j)] = \sum_{j=1}^{n} \text{tr}[c(X)c(Y)c(\bar{e}_j)c(X)c(\bar{e}_j)] \\
= - \sum_{j=1}^{n} \text{tr}[c(X)c(Y)c(\bar{e}_j)c(Y)c(\bar{e}_j)] - 2 \sum_{j=1}^{n} g(\bar{e}_j, X) \text{tr}[c(X)c(Y)c(\bar{e}_j)] \\
= - \sum_{j=1}^{n} \text{tr}[c(X)c(Y)c(\bar{e}_j)c(Y)c(\bar{e}_j)] - 2|X|^2|Y|^2 \text{tr}[id] \\
= (n - 4)|X|^2|Y|^2 \text{tr}[id] + (4 - 2n)g(X, Y)^2 \text{tr}[id]
\]

(3.72)

\[
\sum_{j=1}^{n} \text{tr}[A^*c(\bar{e}_j)A^*c(\bar{e}_j)] = \sum_{j=1}^{n} \text{tr}[c(Y)c(X)c(\bar{e}_j)c(Y)c(\bar{e}_j)] \\
= - \sum_{j=1}^{n} \text{tr}[c(Y)c(X)c(\bar{e}_j)c(X)c(\bar{e}_j)] - 2 \sum_{j=1}^{n} g(\bar{e}_j, Y) \text{tr}[c(Y)c(X)c(\bar{e}_j)] \\
= - \sum_{j=1}^{n} \text{tr}[c(Y)c(X)c(\bar{e}_j)c(X)c(\bar{e}_j)] - 2|X|^2|Y|^2 \text{tr}[id] \\
= (n - 4)|X|^2|Y|^2 \text{tr}[id] + (4 - 2n)g(X, Y)^2 \text{tr}[id]
\]

(3.73)

\[
\sum_{j=1}^{n} \text{tr}[\nabla_{\bar{e}_j}^\Lambda^* T^M (A^*)c(\bar{e}_j)] = \sum_{j=1}^{n} \text{tr}[\nabla_{\bar{e}_j}^\Lambda^* T^M (c(Y)c(X))c(\bar{e}_j)] \\
= \sum_{j=1}^{n} \text{tr}[\nabla_{\bar{e}_j}^\Lambda^* T^M (c(Y))c(X)c(\bar{e}_j) + c(Y)\nabla_{\bar{e}_j}^\Lambda^* T^M (c(X))c(\bar{e}_j)] \\
= \sum_{j=1}^{n} \text{tr}[c(\nabla_{\bar{e}_j} T^M Y)c(X)c(\bar{e}_j) + c(Y)c(\nabla_{\bar{e}_j} T^M X)c(\bar{e}_j)] = 0,
\]

(3.74)

\[
\sum_{j=1}^{n} \text{tr}[c(\bar{e}_j)\nabla_{\bar{e}_j}^\Lambda^* T^M (A)] = \sum_{j=1}^{n} \text{tr}[c(\bar{e}_j)\nabla_{\bar{e}_j}^\Lambda^* T^M (c(X)c(Y)))] \\
= \sum_{j=1}^{n} \text{tr}[c(\bar{e}_j)\nabla_{\bar{e}_j}^\Lambda^* T^M (c(Y))c(X)c(Y) + c(\bar{e}_j)c(X)c(\nabla_{\bar{e}_j}^\Lambda^* T^M (c(Y)))] \\
= \sum_{j=1}^{n} \text{tr}[c(\bar{e}_j)c(\nabla_{\bar{e}_j} T^M X)c(Y) + c(\bar{e}_j)c(X)c(\nabla_{\bar{e}_j} T^M Y)] = 0.
\]

(3.75)
By applying the formulae shown in (3.19), we can calculate

\[(3.76) \quad \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^*M} [\sigma_{-4}((D_A^* D_A)^{-1})] \sigma(\xi) dx \]

\[= 32\pi^2 \int_M \text{tr} \left( -\frac{1}{12}s + A^* A - \frac{1}{4} \sum_{j=1}^n A c(\tilde{e}_j) A c(\tilde{e}_j) - \frac{1}{4} \sum_{j=1}^n A^* c(\tilde{e}_j) A^* c(\tilde{e}_j) \right) \]

\[+ \frac{1}{2} \sum_{j=1}^n \nabla_{\tilde{e}_j}^{A^* T^* M} (A^*) c(\tilde{e}_j) - \frac{1}{2} \sum_{j=1}^n c(\tilde{e}_j) \nabla_{\tilde{e}_j}^{A^* T^* M} (A) d\text{Vol}_M \]

\[= 32\pi^2 \int_M \left( -\frac{1}{12}s + |X|^2 |Y|^2 - \frac{1}{4}(-4)g(X,Y)^2 - \frac{1}{4}(-4)g(X,Y)^2 \right) \text{tr}[1d] d\text{Vol}_M \]

\[= 512\pi^2 \int_M \left( -\frac{1}{12}s + |X|^2 |Y|^2 + 2g(X,Y)^2 \right) d\text{Vol}_M, \]

and

\[(3.77) \quad \int_{\partial M} \Psi = 0. \]

We can claim the following corollary:

**Corollary 3.11.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = c(X)c(Y) \), then

\[(3.78) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s + |X|^2 |Y|^2 + 2g(X,Y)^2 \right) d\text{Vol}_M, \]

where \( s \) is the scalar curvature.

When \( A = c(X)c(Y) \), similar to (3.78), we can get:

**Corollary 3.12.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = c(X)c(Y) \), then

\[(3.79) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s + 2|X|^2 |Y|^2 \right) d\text{Vol}_M, \]

where \( s \) is the scalar curvature.

When \( A = c(X)c(Y) \), similar to (3.78), we can get the following corollary:

**Corollary 3.13.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = c(X)c(Y) \), then

\[(3.80) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s - |X|^2 |Y|^2 + 4g(X,Y)^2 \right) d\text{Vol}_M, \]

where \( s \) is the scalar curvature.
When \( A = c(X)c(Y)c(Z) \), then \( A^* = -c(Z)c(Y)c(X) \).

By computation, we have

\[(3.81)\]
\[
\text{tr}[A^*A] = -\text{tr}[(-|X|^2)(-|Y|^2)(-|Z|^2)] = |X|^2|Y|^2|Z|^2\text{tr}[\text{id}],
\]

\[(3.82)\]
\[
\sum_{j=1}^{n} \text{tr}[Ac(\tilde{e}_j)Ac(\tilde{e}_j)] = \sum_{j=1}^{n} \text{tr}[c(X)c(Y)c(Z)c(\tilde{e}_j)c(X)c(Y)c(Z)c(\tilde{e}_j)]
\]
\[
= (n-6)|X|^2|Y|^2|Z|^2\text{tr}[\text{id}] + (8-2n)|X|^2g(Y,Z)^2\text{tr}[\text{id}] + (8-2n)|Y|^2g(X,Z)^2\text{tr}[\text{id}]
\]
\[
+ (8-2n)|Z|^2g(X,Y)^2\text{tr}[\text{id}] + (4n-16)g(X,Y)g(X,Z)g(Y,Z)\text{tr}[\text{id}]
\]

\[(3.83)\]
\[
\sum_{j=1}^{n} \text{tr}[A^*c(\tilde{e}_j)A^*c(\tilde{e}_j)] = \sum_{j=1}^{n} \text{tr}[(-c(Z)c(Y)c(X))c(\tilde{e}_j)(-c(Z)c(Y)c(X))c(\tilde{e}_j)]
\]
\[
= (n-6)|X|^2|Y|^2|Z|^2\text{tr}[\text{id}] + (8-2n)|X|^2g(Y,Z)^2\text{tr}[\text{id}] + (8-2n)|Y|^2g(X,Z)^2\text{tr}[\text{id}]
\]
\[
+ (8-2n)|Z|^2g(X,Y)^2\text{tr}[\text{id}] + (4n-16)g(X,Y)g(X,Z)g(Y,Z)\text{tr}[\text{id}]
\]

\[(3.84)\]
\[
\sum_{j=1}^{n} \text{tr}[\nabla_{\tilde{e}_j}^{\Lambda^*T^M}(A^*)c(\tilde{e}_j)] = \sum_{j=1}^{n} \text{tr}[\nabla_{\tilde{e}_j}^{\Lambda^*T^M}(-c(Z)c(Y)c(X))c(\tilde{e}_j)]
\]
\[
= \left[ -\sum_{j=1}^{n} g(\nabla_{\tilde{e}_j}^{T^M}Z, \tilde{e}_j)g(X, Y) + \sum_{j=1}^{n} g(Y, \tilde{e}_j)g(X, \nabla_{\tilde{e}_j}^{T^M}Z) - \sum_{j=1}^{n} g(X, \tilde{e}_j)g(Y, \nabla_{\tilde{e}_j}^{T^M}Z) \right]
\]
\[
- \sum_{j=1}^{n} g(Z, \tilde{e}_j)g(X, \nabla_{\tilde{e}_j}^{T^M}Y) + \sum_{j=1}^{n} g(\nabla_{\tilde{e}_j}^{T^M}Y, \tilde{e}_j)g(X, Z) - \sum_{j=1}^{n} g(X, \tilde{e}_j)g(\nabla_{\tilde{e}_j}^{T^M}Y, Z)
\]
\[
- \sum_{j=1}^{n} g(Z, \tilde{e}_j)g(\nabla_{\tilde{e}_j}^{T^M}X, Y) + \sum_{j=1}^{n} g(Y, \tilde{e}_j)g(\nabla_{\tilde{e}_j}^{T^M}X, Z) - \sum_{j=1}^{n} g(\nabla_{\tilde{e}_j}^{T^M}X, \tilde{e}_j)g(Y, Z) \right] \text{tr}[\text{id}],
\]

\[(3.85)\]
\[
\sum_{j=1}^{n} \text{tr}[c(\tilde{e}_j)\nabla_{\tilde{e}_j}^{\Lambda^*T^M}(A)] = \sum_{j=1}^{n} \text{tr}[c(\tilde{e}_j)\nabla_{\tilde{e}_j}^{\Lambda^*T^M}(c(X)c(Y)c(Z))]
\]
\[
= \left[ \sum_{j=1}^{n} g(\tilde{e}_j, Z)g(\nabla_{\tilde{e}_j}^{T^M}X, Y) - \sum_{j=1}^{n} g(\nabla_{\tilde{e}_j}^{T^M}X, Z)g(\tilde{e}_j, Y) + \sum_{j=1}^{n} g(Y, Z)g(\tilde{e}_j, \nabla_{\tilde{e}_j}^{T^M}X) \right]
\]
\[
+ \sum_{j=1}^{n} g(\tilde{e}_j, Z)g(X, \nabla_{\tilde{e}_j}^{T^M}Y) - \sum_{j=1}^{n} g(X, Z)g(\tilde{e}_j, \nabla_{\tilde{e}_j}^{T^M}Y) + \sum_{j=1}^{n} g(\nabla_{\tilde{e}_j}^{T^M}Y, Z)g(\tilde{e}_j, X)
\]
\[
+ \sum_{j=1}^{n} g(\tilde{e}_j, \nabla_{\tilde{e}_j}^{T^M}Z)g(X, Y) - \sum_{j=1}^{n} g(X, \nabla_{\tilde{e}_j}^{T^M}Z)g(\tilde{e}_j, Y) + \sum_{j=1}^{n} g(Y, \nabla_{\tilde{e}_j}^{T^M}Z)g(\tilde{e}_j, X) \right] \text{tr}[\text{id}],
\]
\[\text{Corollary 3.14.}\]

Let \( M \) be an \( n \)-dimensional oriented compact manifold with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = c(X)c(Y)c(Z) \), then

\[\sum_{j=1}^{4} g(\nabla^T_{\xi_j} X, \tilde{\xi}_j)Y - \sum_{j=1}^{4} g(\nabla^T_{\xi_j} Y, \tilde{\xi}_j)X + \sum_{j=1}^{4} g(\nabla^T_{\xi_j} Y, \tilde{\xi}_j)Z - \sum_{j=1}^{4} g(\nabla^T_{\xi_j} Z, \tilde{\xi}_j)Y - \sum_{j=1}^{4} g(\nabla^T_{\xi_j} Z, \tilde{\xi}_j)X + \sum_{j=1}^{4} g(\nabla^T_{\xi_j} X, \tilde{\xi}_j)Z - \sum_{j=1}^{4} g(\nabla^T_{\xi_j} Z, \tilde{\xi}_j)X\]
where $s$ is the scalar curvature.

When $A = \pi(X)c(Y)c(Z)$, similar to Corollary 3.14, we have:

**Corollary 3.15.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)c(Y)c(Z)$, then

$$
\tilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s + |X|^2|Y|^2|Z|^2 - 2|X|^2g(Y, Z)^2 \right) d\text{Vol}_M,
$$

where $s$ is the scalar curvature.

When $A = \pi(X)e(Y)c(Z)$, we can get the following corollary:

**Corollary 3.16.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)e(Y)c(Z)$, then

$$
\tilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s + 2|Z|^2g(X, Y)^2 + 4 \sum_{j=1}^4 g(\nabla^{T_M}_{\tilde{e}_j}X, Y)g(Z, \tilde{e}_j) 
+ \sum_{j=1}^4 g(X, \nabla^{T_M}_{\tilde{e}_j}Y)g(Z, \tilde{e}_j) + \sum_{j=1}^4 g(X, Y)g(\nabla^{T_M}_{\tilde{e}_j}Z, \tilde{e}_j) \right) d\text{Vol}_M
\quad - \int_{\partial M} 8\pi\Omega_3 g(\partial_n, Z)g(X, Y) d\text{Vol}_{\partial M},
$$

where $s$ is the scalar curvature.

When $A = \pi(X)e(Y)e(Z)$, we get:

**Corollary 3.17.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)e(Y)e(Z)$, then

$$
\tilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s + 3|X|^2|Y|^2|Z|^2 - 4|X|^2g(Y, Z)^2 
- 4|Y|^2g(X, Z)^2 - 4|Z|^2g(X, Y)^2 + 8g(X, Y)g(X, Z)g(Y, Z) \right) d\text{Vol}_M,
$$

where $s$ is the scalar curvature.

Next, we also prove the Kastler-Kalau-Walze type theorem for 4-dimensional manifolds with boundary associated to $D_A^2$. By (3.9) and (3.10), we will compute

$$
\tilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\Lambda^*T^*M}[\sigma_4(D_A^{-2})] \sigma(\xi) dx + \int_{\partial M} \Phi,
$$
where

\[ \Phi = \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{\alpha|j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\mathcal{L} \cdot \mathcal{T} \cdot \mathcal{M}} \left[ \partial_{\xi}^j \partial_{\xi}^k \sigma_r(D_A^{-1})(x', 0, \xi', \xi_n) \right] \]

\[ \times \partial_{\xi'}^j \partial_{\xi'}^k \sigma_l(D_A^{-1})(x', 0, \xi', \xi_n) \]dξ_nσ(ξ')dx',

and the sum is taken over \( r + l - k - j - |\alpha| = -3 \), \( r \leq -1, l \leq -1 \).

By Theorem 2.2, we compute the interior of \( \text{Wres}[^+\pi D_A^{-1} \circ ^+\pi D_A^{-1}] \), then

\[ \int_M \int_{|\xi|=1} \text{trace}_{\mathcal{L} \cdot \mathcal{T} \cdot \mathcal{M}}[\sigma_{-4}(D_A^{-2})]\sigma(\xi)dx \]

\[ = 32\pi^2 \int_M tr \left( -\frac{1}{12} s + A^2 - \frac{1}{2} \sum_{j=1}^{n} Ac(\tilde{e}_j) Ac(\tilde{e}_j) \right) dVol_M. \]

When \( n = 4 \), then \( \text{tr}_{\mathcal{L} \cdot \mathcal{T} \cdot \mathcal{M}}[\text{id}] = \dim(\wedge^r(4)) = 16 \), where \( \text{tr} \) as shorthand of trace, the sum is taken over \( r + l - k - j - |\alpha| = -3 \), \( r \leq -1, l \leq -1 \), then we have the following five cases:

case a) I) \( r = -1, l = -1, k = j = 0, |\alpha| = 1 \)

By (3.94), we get

\[ \Phi_1 = -\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial_{\xi}^{\alpha} \pi_{\xi}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi}^{\alpha} \partial_{\xi'} \sigma_{-1}(D_A^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \]

case a) II) \( r = -1, l = -1, k = |\alpha| = 0, j = 1 \)

Likewise, we get

\[ \Phi_2 = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi} \pi_{\xi}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi} \partial_{\xi'} \sigma_{-1}(D_A^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \]

case a) III) \( r = -1, l = -1, j = |\alpha| = 0, k = 1 \)

Observing (3.94), we get

\[ \Phi_3 = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi} \pi_{\xi}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi} \partial_{\xi'} \sigma_{-1}(D_A^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \]

By Lemma 3.7, we have \( \sigma_{-1}(D_A^{-1}) = \sigma_{-1}((D_A^*)^{-1}) \).

In combination with the calculation,

\[ \Phi_1 + \Phi_2 + \Phi_3 = 0. \]
case b) \( r = -2, \ l = -1, \ k = j = |\alpha| = 0 \)

By applying the formulae shown in (3.94), we get

\[
\Phi_4 = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-2}(D_A^{-1}) \times \partial_{\xi_n} \sigma_{-1}(D_A^{-1})](x_0)d\xi_n \sigma(\xi')dx'.
\]

By Lemma 3.7, we have \( \sigma_{-1}(D_A^{-1}) = \sigma_{-1}((D_A^*)^{-1}) \). Then, we have

\[
\Phi_4 = \frac{9}{2} \pi h'(0)\Omega_3 dx' + \frac{\pi}{4} \Omega_3 \text{tr}[Ac(dx_n)]dx',
\]

where \( \Omega_4 \) is the canonical volume of \( S^4 \).

case c) \( r = -1, \ l = -2, \ k = j = |\alpha| = 0 \)

By (3.94), we can calculate

\[
\Phi_5 = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D_A^{-1})](x_0)d\xi_n \sigma(\xi')dx'.
\]

By (3.5) and (3.6), we have

\[
\pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1})(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\]

Since

\[
\sigma_{-2}(D_A^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_A)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)|\xi|^2_{\partial_{x'}} \right],
\]

where

\[
\sigma_0(D_A)(x_0) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\overline{c(\tilde{e}_s)}\overline{c(\tilde{e}_t)} - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) + A
\]

\[
= b_0^1(x_0) + b_0^2(x_0) + A,
\]
hence

\[(3.106)\]
\[
\frac{\partial_{\xi_n} \sigma_{-2}(D_A^{-1})(x_0)|_{|\xi'|=1}}{|\xi'|^4} = \partial_{\xi_n} \left( \frac{c(\xi)(b_0^1(x_0) + b_0^2(x_0) + A)c(\xi)}{|\xi'|^4} \right)
\]
\[
+ \frac{c(\xi)}{|\xi'|^6} c(dx_n) \left( \partial_{\xi_n} [c(\xi')(\xi_0)](\xi_0)|\xi'|^2 - c(\xi)h'(0)|\xi'|^2 \right) \right)
\]
\[
= \partial_{\xi_n} \left( \frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi'|^4} \right) + \partial_{\xi_n} \left( \frac{c(\xi)Ac(\xi)}{|\xi'|^4} \right). \]

By calculation, we have

\[(3.107)\]
\[
\frac{\partial_{\xi_n} \left( \frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi'|^4} \right)}{|\xi'|^4} = \frac{c(dx_n)b_0^1(x_0)c(\xi)}{|\xi'|^4} + \frac{c(dx_n)b_0^1(x_0)c(dx_n)}{|\xi'|^4} - \frac{4\xi_n c(\xi)b_0^1(x_0)c(\xi)}{|\xi'|^6};
\]
\[(3.108)\]
\[
\frac{\partial_{\xi_n} \left( \frac{c(\xi)Ac(\xi)}{|\xi'|^4} \right)}{|\xi'|^4} = \frac{c(dx_n)Ac(\xi)}{|\xi'|^4} + \frac{c(dx_n)Ac(dx_n)}{|\xi'|^4} - \frac{4\xi_n c(\xi)Ac(\xi)}{|\xi'|^6}. \]

For brevity, we denote

\[(3.109)\]
\[
q_{-2}^1 = \frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi'|^4} + \frac{c(\xi)}{|\xi'|^6} c(dx_n) \left( \partial_{\xi_n} [c(\xi')(\xi_0)](\xi_0)|\xi'|^2 - c(\xi)h'(0) \right), \]

then

\[(3.110)\]
\[
\frac{\partial_{\xi_n} (q_{-2}^1)}{\xi_n} = \frac{1}{(1 + \xi_n^2)^3} \left( [2\xi_n - 2\xi_n^3] c(dx_n)b_0^2(x_0)c(dx_n) + (1 - 3\xi_n^2)c(dx_n)b_0^2(x_0)c(\xi') \right.
\]
\[
+ (1 - 3\xi_n^2) c(\xi')b_0^2(x_0)c(dx_n) - 4\xi_n c(\xi')b_0^2(x_0)c(\xi') + (3\xi_n^2 - 1) \partial_{\xi_n} c(\xi')
\]
\[
- 4\xi_n c(\xi')\partial_{\xi_n} c(\xi') + 2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n) \right)
\]
\[
+ 6\xi_n h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^4}. \]

We calculate

\[(3.111)\]
\[
\frac{\mathrm{tr} \left( \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1})(x_0) \times \partial_{\xi_n} \left( \frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi'|^4} \right) \right) \}|_{|\xi'|=1} \]
\[
= \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \mathrm{tr}[b_0^1(x_0)c(\xi')] + \frac{i}{2(\xi_n - i)(\xi_n + i)^3} \mathrm{tr}[b_0^1(x_0)c(dx_n)]
\]
\[
= \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \mathrm{tr}[b_0^1(x_0)c(\xi')], \]
then

\[
\begin{align*}
\text{(3.112)} & \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) b_0'(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[b_0'(x_0)c(\xi')] d\xi_n \sigma(\xi') dx' \\
& = 0.
\end{align*}
\]

Likewise, we have

\[
\begin{align*}
\text{(3.113)} & \quad \text{tr} \left[ \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1})(x_0) \times \partial_{\xi_n} \left( \frac{c(\xi) A_c(\xi)}{|\xi|^4} \right) \right] \bigg|_{|\xi'|=1} \\
& = \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[A_c(\xi')] + \frac{i}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[A_c(dx_n)];
\end{align*}
\]

then by \( \int_{|\xi'|=1} \{\xi_{i_1} \cdots \xi_{i_{2d+1}}\} \sigma(\xi') = 0 \), we have

\[
\begin{align*}
\text{(3.114)} & \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \left( \frac{c(\xi) A_c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
& = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{-1}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[A_c(\xi')] d\xi_n \sigma(\xi') dx' \\
& \quad - i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{i}{2(\xi_n - i)(\xi_n + i)^3} \text{tr}[A_c(dx_n)] d\xi_n \sigma(\xi') dx' \\
& \quad = \frac{\Omega_3}{2} \text{tr}[A_c(dx_n)] \int_{\Gamma^+} \frac{1}{(\xi_n + i)^3} |_{\xi_n = i} dx' \\
& \quad = \frac{\Omega_3}{2} \text{tr}[A_c(dx_n)] 2\pi i \left[ \frac{1}{(\xi_n + i)^3} \right]^{(1)} |_{\xi_n = i} dx' \\
& \quad = -\frac{\pi}{4} \Omega_3 \text{tr}[A_c(dx_n)] dx'.
\end{align*}
\]

By (3.103) and (3.109), we have

\[
\begin{align*}
\text{(3.115)} & \quad \text{tr} [\pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} (q_1^2_2)] (x_0)] |_{|\xi'|=1} = \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4}.
\end{align*}
\]
We compute that

\[
(3.116) \quad -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} [\pi^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n}(q_{-2})](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4} d\xi_n \sigma(\xi') dx'
\]

\[
= -i\Omega_3 \int_{\Gamma^+} \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^3(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^3(\xi + i)^4} d\xi_n dx'
\]

\[
= -i\Omega_3 \frac{2\pi i}{2!} \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi + i)^3} |_{\xi_n=i} dx' - i\Omega_3 \frac{2\pi i}{2!} \frac{48h'(0)i\xi_n}{(\xi + i)^4} |_{\xi_n=i} dx'
\]

\[
= -\frac{9}{2} \pi h'(0)\Omega_3 dx'.
\]

Then,

\[
(3.117) \quad \Phi_5 = -\frac{9}{2} \pi h'(0)\Omega_3 dx' - \frac{\pi}{4} \Omega_3 \text{tr}[Ac(dx_n)] dx'.
\]

So

\[
(3.118) \quad \Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 = 0.
\]

**Theorem 3.18.** Let \( M \) be a 4-dimensional oriented compact manifold with the boundary \( \partial M \) and the metric \( g^M \) as above, \( D_A \) be the perturbation of the de Rham Hodge operator on \( \widetilde{M} \), then

\[
\widetilde{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}]
\]

\[
= 32\pi^2 \int_M \text{tr} \left( -\frac{1}{12} s + A^2 - \frac{1}{2} \sum_{j=1}^{n} Ac(\tilde{e}_j) Ac(\tilde{e}_j) \right) d\text{Vol}_M.
\]

where \( s \) is the scalar curvature.

We can directly state the following facts as a corollary of Theorem 3.18.

**Corollary 3.19.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = c(X) \), then

\[
(3.119) \quad \widetilde{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12} s \right) d\text{Vol}_M,
\]

where \( s \) is the scalar curvature.

When \( A = \pi(X) \), we can get the following corollary:

**Corollary 3.20.** Let \( M \) be a 4-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric \( g^M \) as above, and let \( A = \pi(X) \), then

\[
(3.120) \quad \widetilde{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12} s - |X|^2 \right) d\text{Vol}_M,
\]

where \( s \) is the scalar curvature.
When $A = c(X)c(Y)$, similar to Corollary 3.20, we have:

**Corollary 3.21.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)$, then

$$\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( - \frac{1}{12} s + |X|^2 |Y|^2 + 2g(X,Y)^2 \right) d\text{Vol}_M,$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)$, we compute $\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}]$.

**Corollary 3.22.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)$, then

$$\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( - \frac{1}{12} s \right) d\text{Vol}_M,$$

where $s$ is the scalar curvature.

A simple calculation shows that:

**Corollary 3.23.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)$, then

$$\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( - \frac{1}{12} s - |X|^2 |Y|^2 + 4g(X,Y)^2 \right) d\text{Vol}_M,$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)c(Z)$, we have:

**Corollary 3.24.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)c(Z)$, then

$$\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( - \frac{1}{12} s \right) d\text{Vol}_M,$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)c(Z)$, we notice that:

**Corollary 3.25.** Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)c(Z)$, then

$$\widehat{\text{res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( - \frac{1}{12} s + |X|^2 |Y|^2 |Z|^2 - 2|X|^2 g(Y,Z)^2 \right) d\text{Vol}_M,$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)c(Z)$, we have:
Corollary 3.26. Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \overline{\varphi}(X, \overline{\varphi}(Y) c(Z)$, then

\begin{equation}
\tag{3.126}
\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ D_A^{-1}] = 512\pi^2 \int_M \left( -\frac{1}{12}s - 2|X|^2|Y|^2|Z|^2 + 2|Z|^2g(X,Y)^2 \right) d\text{Vol}_M,
\end{equation}

where $s$ is the scalar curvature.

When $A = \overline{\varphi}(X, \overline{\varphi}(Y) c(Z)$, then $A^* = \overline{\varphi}(Z, \overline{\varphi}(Y) c(X)$.

By computation, we have

\begin{equation}
\tag{3.127}
\text{tr}[A^2] = \text{tr}[|X|^2|Y|^2|Z|^2] = |X|^2|Y|^2|Z|^2 \text{tr}[\text{id}],
\end{equation}

\begin{equation}
\tag{3.128}
\text{tr}[c(dx_n)A^*] = 0, \text{tr}[c(dx_n)A] = 0,
\end{equation}

\begin{equation}
\sum_{j=1}^n \text{tr}[Ac(e_j)Ac(e_j)] = \sum_{j=1}^n \text{tr}[\overline{\varphi}(X) \overline{\varphi}(Y) c(Z) \overline{\varphi}(X) \overline{\varphi}(Y) c(Z)]
\end{equation}

\begin{align*}
&= -\sum_{j=1}^n \text{tr}[\overline{\varphi}(Y) \overline{\varphi}(X) \overline{\varphi}(Z) \overline{\varphi}(X) \overline{\varphi}(Y) c(Z)] + 2 \sum_{j=1}^n g(X,Y) \text{tr}[\overline{\varphi}(Z) \overline{\varphi}(X) \overline{\varphi}(Y) c(Z)] \\
&= -n|X|^2|Y|^2|Z|^2 \text{tr}[\text{id}] + 2n|X|^2|Y|^2g(Y,Z)^2 \text{tr}[\text{id}] + 2n|Y|^2g(X,Z)^2 \text{tr}[\text{id}] \\
&+ 2n|Z|^2g(X,Y)^2 \text{tr}[\text{id}] - 4ng(X,Y)g(X,Z)g(Y,Z) \text{tr}[\text{id}]
\end{align*}

We can calculate

\begin{equation}
\tag{3.129}
\int_M \int_{|\xi|=1} \text{trace}_{A^*T^*M}[\sigma_{-4}(D_A^{-2})] \sigma(\xi) dx
\end{equation}

\begin{align*}
&= 32\pi^2 \int_M \left( -\frac{1}{12}s + A^2 - \frac{1}{2} \sum_{j=1}^n Ac(e_j)Ac(e_j) \right) d\text{Vol}_M \\
&= 32\pi^2 \int_M \left( -\frac{1}{12}s + |X|^2|Y|^2|Z|^2 + 2|X|^2|Y|^2|Z|^2 - 4|X|^2g(Y,Z)^2 - 4|Y|^2g(X,Z)^2 \\
&- 4|Z|^2g(X,Y)^2 + 8g(X,Y)g(X,Z)(Y,Z) \right) \text{tr}[\text{id}] d\text{Vol}_M \\
&= 512\pi^2 \int_M \left( -\frac{1}{12}s + 3|X|^2|Y|^2|Z|^2 - 4|X|^2g(Y,Z)^2 - 4|Y|^2g(X,Z)^2 \\
&- 4|Z|^2g(X,Y)^2 + 8g(X,Y)g(X,Z)(Y,Z) \right) d\text{Vol}_M,
\end{align*}

and

\begin{equation}
\tag{3.130}
\int_{\partial M} \Psi = 0.
\end{equation}
Corollary 3.27. Let $M$ be a 4-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)\pi(Z)$, then

\begin{align}
(3.131) \quad \widetilde{\text{Res}}[\pi^+D_A^{-1} \circ \pi^+D_A^{-1}] &= 512\pi^2 \int_M \left( -\frac{1}{12} s + 3|X|^2|Y|^2|Z|^2 - 4|X|^2g(Y, Z)^2 \\
&- 4|Y|^2g(X, Z)^2 - 4|Z|^2g(X, Y)^2 + 8g(X, Y)g(X, Z)g(Y, Z) \right) d\text{Vol}_M, \end{align}

where $s$ is the scalar curvature.

4. A Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary

Firstly, we prove the Kastler-Kalau-Walze type theorems for 6-dimensional manifolds with boundary. From [11], we know that

\begin{align}
(4.1) \quad \widetilde{\text{Res}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*D_A D_A^*)^{-1}] &= \int_M \int_{|\xi|=1} \text{trace}_{\lambda^*T^*M}[\sigma_{-4}(D_A^*D_A)^{-2}]\sigma(\xi)dx + \int_{\partial M} \Psi, \end{align}

where

\begin{align}
(4.2) \quad \Psi &= \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{+\infty} \sum_{\alpha} \frac{(-i)^{\alpha}j+k+1}{\alpha!(j+k+1)!} \times \text{trace}_{\lambda^*T^*M}[\partial_{x_n}^j \partial_{\xi_n}^k \sigma_r(D_A^{-1})(x', 0, \xi', \xi_n) \\
&\times \partial_{x_n}^j \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((D_A^*D_A D_A^*)^{-1})(x', 0, \xi', \xi_n)]d\xi_n \sigma(\xi')dx', \end{align}

and the sum is taken over $r + \ell - k - j - |\alpha| - 1 = -6, \ r \leq -1, \ \ell \leq -3$. By Theorem 2.2, we compute the interior term of (4.1), then

\begin{align}
(4.3) \quad \int_M \int_{|\xi|=1} \text{trace}_{\lambda^*T^*M}[\sigma_{-4}(D_A^*D_A)^{-2}]\sigma(\xi)dx \\
= 128\pi^3 \int_M \text{tr} \left( -\frac{1}{12} s + 2A^*A - \frac{1}{4} \sum_{j=1}^n A\pi(e_j)A\pi(e_j) - \frac{1}{4} \sum_{j=1}^n A^*c(e_j)A^*c(e_j) \\
+ \frac{1}{2} \sum_{j=1}^n \nabla^A_{\pi(e_j)}(A^*)c(e_j) - \frac{1}{2} \sum_{j=1}^n c(e_j)\nabla^A_{\pi(e_j)}(A) \right) d\text{Vol}_M. \end{align}
Next, we compute $\int_{\partial M} \Psi$. By computation, we get

(4.4)\[D^*_A D_A D^*_A = \sum_{i=1}^n c(\tilde{e}_i) \langle \tilde{e}_i, dx_i \rangle (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(\tilde{e}_i) \langle \tilde{e}_i, dx_i \rangle \left\{ - (\partial_i g^{ij}) \partial_i \partial_j - g^{ij} \left( \begin{array}{l} 4(\sigma_i + a_i) \partial_j - 2\Gamma^k_{ij} \partial_k \\ \end{array} \right) \right\} + \sum_{i=1}^n c(\tilde{e}_i) \langle \tilde{e}_i, dx_i \rangle \left\{ - 2(\partial_i g^{ij})(\sigma_i + a_i) \partial_j + g^{ij}(\partial_i \Gamma_{ij}^k) \partial_k \\ - 2g^{ij}[(\partial_i \sigma_i) + (\partial_i a_i)] \partial_j + (\partial_i g^{ij}) \Gamma_{ij}^k \partial_k + \sum_{j,k} \left( c(\tilde{e}_j) A + A^* c(\tilde{e}_j) \right) \right\} \langle \tilde{e}_j, dx_k \rangle \partial_k \right\} + \sum_{j,k} \left( c(\tilde{e}_j) A + A^* c(\tilde{e}_j) \right) \left\{ - g^{ij} \left( \begin{array}{l} (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \end{array} \right) \right\} + \sum_{i,j} g^{ij} \left( c(\partial_i) \partial_j (A) \right) + c(\partial_i) \sigma_j A + c(\partial_i) a_j A + A^* c(\partial_i) \sigma_j + A^* c(\partial_i) a_j \right\} - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} c(\tilde{e}_i) c(\tilde{e}_j) c(\tilde{e}_k) c(\tilde{e}_l) \right\} + \frac{1}{4} s + A^* A \right\} + \sum_{i=1}^n c(\tilde{e}_i) (\sigma_i + a_i) + A^* \right\} \left\{ - g^{ij} \left( \begin{array}{l} (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \end{array} \right) \right\} + \sum_{i,j} g^{ij} \left( c(\partial_i) A + A^* c(\partial_i) \right) \partial_j + \sum_{i,j} g^{ij} \left( c(\partial_i) \partial_j (A) + c(\partial_i) \sigma_j A + c(\partial_i) a_j A + A^* c(\partial_i) \sigma_j + A^* c(\partial_i) a_j \right\} - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl} c(\tilde{e}_i) c(\tilde{e}_j) c(\tilde{e}_k) c(\tilde{e}_l) \right\} + \frac{1}{4} s + A^* A \right\}.\]

Then, we obtain

**Lemma 4.1.** The following identities hold:

(4.5)\[\sigma_2(D^*_A D_A D^*_A) = \sum_{i,j,l} c(dx_i) \partial_l (g^{ij}) \xi_l \xi_j + c(\xi) (4a^k + 4\sigma^k - 2\Gamma^k) \xi_k + 2[|\xi|^2 A - c(\xi) A^* c(\xi)] + \frac{1}{4} |\xi|^2 \sum_{s,t,l} \omega_{s,t} (\tilde{e}_i) [c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t) - c(\tilde{e}_i) c(\tilde{e}_s) c(\tilde{e}_t)] + |\xi|^2 A^*;\]

\[\sigma_3(D^*_A D_A D^*_A) = i c(\xi)|\xi|^2.\]
Write

\begin{equation}
(4.6) \quad \sigma(D_A^*D_AD_A^*) = p_3 + p_2 + p_1 + p_0; \quad \sigma((D_A^*D_AD_A^*)^{-1}) = \sum_{j=3}^{\infty} q_j.
\end{equation}

By the composition formula of pseudodifferential operators, we have
\begin{equation}
(4.7) \quad 1 = \sigma((D_A^*D_A^*D_A^*) \circ (D_A^*D_AD_A^*)^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^n \left[ \sigma(D_A^*D_AD_A^*) \right] D_\xi^n \left[ (D_A^*D_AD_A^*)^{-1} \right]
\end{equation}

\begin{align*}
&= (p_3 + p_2 + p_1 + p_0) (q_{-3} + q_{-4} + q_{-5} + \cdots) \\
&+ \sum_j (\partial_\xi p_3 + \partial_\xi p_2 + \partial_\xi p_1 + \partial_\xi p_0) \\
&\quad \left( D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \cdots \right) \\
&= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_\xi p_3 D_{x_j} q_{-3}) + \cdots,
\end{align*}

by (4.7), we have
\begin{equation}
(4.8) \quad q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial_\xi p_3 D_{x_j} (p_3^{-1})].
\end{equation}

By Lemma 4.1, we have some symbols of operators.

**Lemma 4.2.** The following identities hold:

\begin{align}
(4.9) \quad &\sigma_{-3}((D_A^*D_A^*D_A^*)^{-1}) = \frac{ic(\xi)}{(|\xi|^4} \\
&\sigma_{-4}((D_A^*D_A^*D_A^*)^{-1}) = \frac{c(\xi) \sigma_2(D_A^*D_A^*D_A^*) c(\xi)}{|\xi|^8} + \frac{c(\xi)}{|\xi|^8} \left( |\xi|^2 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) \\
&\quad + 2\xi_n c(\xi) \partial_{x_n} c(\xi') + 4\xi_n h'(0) \right).
\end{align}

When \( n = 6 \), then \( \text{tr}_{\Lambda^{3}T^*M}[\text{id}] = 64 \), where \( \text{tr} \) as shorthand of trace. Since the sum is taken over \( r + \ell + k + j - |\alpha| - 1 = -6 \), \( r \leq -1, \ell \leq -3 \), then we have the \( \int_{\partial M} \Psi \) is the sum of the following five cases:

**case (a) (I)** \( r = -1, l = -3, j = k = 0, |\alpha| = 1 \).

By (4.2), we obtain
\begin{equation}
(4.10) \quad \Psi_1 = -\int_{|\xi'|=1} \int_{|\alpha|=1} \sum_{|\alpha|=1} \text{trace} \left[ \partial_\xi^\alpha \pi_\alpha \sigma_{-1}(D_A^{-1}) \partial_{\xi'}^\beta \partial_{\xi_n} \sigma_{-3}(D_A^*D_AD_A^*)^{-1} \right] (x_0) d\xi_n c(\xi') dx'.
\end{equation}
case (a) (II) \( r = -1, l = -3, |\alpha| = k = 0, j = 1. \)

By (4.2), we have
\[
(4.11) \quad \Psi_2 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n}^2 \sigma_{-3}((D_A^* D_A D_A^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'.
\]

case (a) (III) \( r = -1, l = -3, |\alpha| = j = 0, k = 1. \)

It is easy to check that
\[
(4.12) \quad \Psi_3 = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_{-3}((D_A^* D_A D_A^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'.
\]

For [13], we have
\[
(4.13) \quad \Psi_1 + \Psi_2 + \Psi_3 = \frac{10}{2} \pi h'(0) \Omega_4 dx'.
\]

case (b) \( r = -1, l = -4, |\alpha| = j = k = 0. \)

By observing (4.2), we have
\[
(4.14) \quad \Psi_4 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi_n} \sigma_{-4}((D_A^* D_A D_A^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'.
\]

In the normal coordinate, \( g^{ij}(x_0) = \delta^j_i \) and \( \partial_{x_j}(g^{\alpha\beta})(x_0) = 0, \) if \( j < n; \partial_{x_j}(g^{\alpha\beta})(x_0) = h'(0)\delta^j_3, \) if \( j = n. \) So by [12], when \( k < n, \) we have \( \Gamma_k(x_0) = \frac{5}{2} h'(0), \Gamma^k(x_0) = 0, \delta^k(x_0) = 0 \) and \( \delta^k = \frac{1}{2} h'(0)c(e_k)c(e_n). \) Then, we obtain
\[
(4.15) \quad \sigma_{-4}((D_A^* D_A D_A^*)^{-1})(x_0) = \frac{1}{|\xi|^8} c(\xi) \left( h'(0)c(\xi) \sum_{k<n} \xi_k c(\bar{e}_k)c(\bar{e}_n) - h'(0)c(\xi) \sum_{k<n} \xi_k \bar{c}(\bar{e}_k)\bar{c}(\bar{e}_n) \right.
\]

\[
- 5h'(0)\xi_n c(\xi) + 2[|\xi|^2 A - c(\xi) A^* c(\xi)] + \frac{5}{4} |\xi|^2 h'(0)c(\bar{e}_n)\bar{c}(dx_n)\bar{c}(\bar{e}_i)
\]

\[
- \frac{1}{4} |\xi|^2 h'(0) c(dx_n) + |\xi|^2 A^* c(\xi) + \frac{c(\xi)}{|\xi|^8} \left( |\xi|^2 c(dx_n) \partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi) \partial_{x_n} c(\xi') + 4\xi_n h'(0) \right).
\]

\[
(4.16) \quad \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D_A^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}.
\]
By (4.15) and (4.16), we have

\[
\text{(4.17)} \quad \text{tr}[\partial_{\xi_n} \pi_n^+ \sigma_{-1}(D_A^{-1}) \times \sigma_{-4}(D_A^* D_A D_A^*)^{-1}] (x_0) |_{\xi' = 1} = -\frac{64h'(0)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} \left( -\frac{15}{4} i + 16 \xi_n + \frac{19}{2} i \xi_n^2 + 8 \xi_n^3 + \frac{21}{4} i \xi_n^4 \right) + \frac{2 + 4 i \xi_n - 2 \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A c(\xi')] - \frac{-2 i + 4 \xi_n + 2 i \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A c(dx_n)] + \frac{3 + 2 i \xi_n + \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A^* c(\xi')] + \frac{i + 2 \xi_n + 3 i \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A^* c(dx_n)].
\]

Consequently,

\[
\text{(4.18)} \quad \Psi_4 = i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} -\frac{64h'(0)(-\frac{15}{4} i + 16 \xi_n + \frac{19}{2} i \xi_n^2 + 8 \xi_n^3 + \frac{21}{4} i \xi_n^4)}{2(\xi_n - i)^2(1 + \xi_n^2)^4} d\xi_n \sigma(\xi') d\xi' + i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} -\frac{2 i + 4 \xi_n + 2 i \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A c(dx_n)] d\xi_n \sigma(\xi') d\xi' + i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \frac{i + 2 \xi_n + 3 i \xi_n^2}{2(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[A^* c(dx_n)] d\xi_n \sigma(\xi') d\xi'
\]

\[
= \frac{111}{2} \pi h'(0) \Omega_4 d\xi' - \frac{3}{16} \pi \Omega_4 \text{tr}[A c(dx_n)] d\xi' + \frac{1}{16} \pi \Omega_4 \text{tr}[A^* c(dx_n)] d\xi'.
\]

case (c) \( r = -2, l = -3, |\alpha| = j = k = 0. \)

By (4.2), we obtain

\[
\text{(4.19)} \quad \Psi_5 = -i \int_{|\xi'| = 1} \int_{-\infty}^{+\infty} \text{trace}[\pi_n^+ \sigma_{-2}(D_A^{-1}) \times \partial_{\xi_n} \sigma_{-3}((D_A^* D_A D_A^*)^{-1})](x_0) d\xi_n \sigma(\xi') d\xi'.
\]

By Lemma 4.1 and Lemma 4.2, we have

\[
\text{(4.20)} \quad \sigma_{-2}(D_A^{-1})(x_0) = \frac{c(\xi)\sigma_0(D_A)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi)) |\xi|^2 - c(\xi) \partial_{x_j} |\xi|^2 \right](x_0),
\]

where

\[
\text{(4.21)} \quad \sigma_0(D_A)(x_0) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\overline{\sigma}_s(\overline{e}_i) + \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\overline{c}(\overline{e}_i)c(\overline{e}_i) + A.
\]

On the other hand,

\[
\text{(4.22)} \quad \partial_{\xi_n} \sigma_{-3}((D_A^* D_A D_A^*)^{-1})(x_0) |_{\xi' = 1} = -\frac{8 i e_n c(\xi')}{(1 + \xi_n^2)^5} + \frac{i(1 - 7 \xi_n^2)}{(1 + \xi_n^2)^5} c(dx_n).
\]
It is easy to obtain that

\begin{equation}
\pi^+_{\xi_n}(\sigma_{-2}(D_A^{-1}))(x_0)|_{\xi'=1} = \pi^+_{\xi_n}\left(\frac{(\xi)c\sigma_0(D_A)(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2}\right. \\
- h'(0)\pi^+_{\xi_n}\left(\frac{(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3}\right).
\end{equation}

We denote

\begin{equation}
\sigma_0(D_A)(x_0) = b_0^1(x_0) + b_0^2(x_0) + A.
\end{equation}

Then, we obtain

\begin{equation}
\pi^+_{\xi_n}(\sigma_{-2}(D_A^{-1}))(x_0)|_{\xi'=1} = \pi^+_{\xi_n}\left(\frac{(\xi)c\sigma_0(D_A)(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2}\right. \\
- h'(0)\pi^+_{\xi_n}\left(\frac{(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3}\right) + \pi^+_{\xi_n}\left(\frac{(\xi)c(b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2}\right) \\
+ \pi^+_{\xi_n}\left(\frac{(\xi)Ac(\xi)}{(1 + \xi_n^2)^2}\right).
\end{equation}

Furthermore,

\begin{equation}
\pi^+_{\xi_n}\left(\frac{(\xi)c(b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2}\right) = \pi^+_{\xi_n}\left(\frac{(\xi)c(b_0^1(x_0)c(\xi)}{(1 + \xi_n^2)^2}\right) + \pi^+_{\xi_n}\left(\frac{(\xi)c(b_0^1(x_0)c(dx_n)}{(1 + \xi_n^2)^2}\right) \\
+ \pi^+_{\xi_n}\left(\frac{(\xi)c(dx_n)b_0^1(x_0)c(\xi')}{(1 + \xi_n^2)^2}\right) + \pi^+_{\xi_n}\left(\frac{(\xi)c(dx_n)b_0^1(x_0)c(dx_n)}{(1 + \xi_n^2)^2}\right) \\
= - \frac{(\xi)c(b_0^1(x_0)c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} - \frac{ic(\xi')b_0^1(x_0)c(dx_n)}{4(\xi_n - i)^2} \\
- \frac{ic(dx_n)b_0^1(x_0)c(\xi')}{{4(\xi_n - i)^2}} - \frac{ic_n(dx_n)\xi_n b_0^1(x_0)c(dx_n)}{4(\xi_n - i)^2}.
\end{equation}

For the sake of convenience in writing,

\begin{equation}
\pi^+_{\xi_n}\left(\frac{(\xi)c(b_0^2(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1 + \xi_n^2)^2}\right) - h'(0)\pi^+_{\xi_n}\left(\frac{(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3}\right) := B_1 - B_2,
\end{equation}

where

\begin{equation}
B_1 = \frac{-1}{4(\xi_n - i)^2}\left[(2 + i\xi_n)c(\xi')b_0^2(x_0)c(\xi) + i\xi_n c(dx_n)b_0^2(x_0)c(dx_n) \\
+ (2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)b_0^2(x_0)c(\xi') + ic(\xi')b_0^2(x_0)c(dx_n) - i\partial_{x_n}c(\xi')\right]
\end{equation}

and

\begin{equation}
B_2 = \frac{h'(0)}{2}\left(\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^2}(ic(\xi') - c(dx_n))\right).
\end{equation}
Similarly, we have

\begin{equation}
\text{tr}[B_1 \times \partial_{\xi_n} \sigma_{-3}((D_A^* A D_A^*)^{-1})(x_0)]|_{\xi'|=1} = \text{tr} \left[ \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') b_0^2(x_0) c(\xi') + i\xi_n c(dx_n) b_0^2(x_0) c(dx_n) + ic(dx_n) b_0^2(x_0) c(dx_n) + (2 + i\xi_n) c(dx_n) \partial_{x_n} c(\xi') + ic(\xi') b_0^2(x_0) c(dx_n) - i\partial_{x_n} c(\xi') \right] \right] \\
= -32h'(0) \frac{-2i + 9\xi_n + 14i\xi_n^2 - 7\xi_n^3}{4(\xi_n - i)^2(1 + \xi_n^2)^5} - 64h'(0) \frac{-\frac{3}{2}i + 12\xi_n + \frac{21}{2}i\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^5} \\
= 8h'(0) \frac{-5 - 28i\xi_n + 7\xi_n^2}{(\xi_n - i)^6(\xi_n + i)^5},
\end{equation}

thus

\begin{equation}
\text{tr}[B_2 \times \partial_{\xi_n} \sigma_{-3}((D_A^* A D_A^*)^{-1})(x_0)]|_{\xi'|=1} = \text{tr} \left[ \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} \right] \\
+ \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [ic(\xi') - c(dx_n)] \times \frac{-8i\xi_n c(\xi') + (i - 7i\xi_n^2)c(dx_n)}{(1 + \xi_n^2)^5} \right] \\
= -8h'(0) \frac{4i - 23\xi_n - 14i\xi_n^2 + 7\xi_n^3}{(\xi_n - i)^7(\xi_n + i)^5},
\end{equation}

We calculate that

\begin{equation}
\text{tr} \left[ \pi_+^{\xi_n} \left( \frac{c(\xi) b_0^1(x_0) c(\xi')}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((D_A^* A D_A^*)^{-1})(x_0) \right]|_{\xi'|=1} = \frac{-2 - 16i\xi_n + 14\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[b_0^1(x_0) c(\xi')].
\end{equation}

Similar calculations to (4.33), it is shown that

\begin{equation}
\text{tr} \left[ \pi_+^{\xi_n} \left( \frac{c(\xi) Ac(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((D_A^* A D_A^*)^{-1})(x_0) \right]|_{\xi'|=1} = \frac{-2 - 16i\xi_n + 14\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[Ac(\xi')] + \frac{-2i + 16\xi_n + 14i\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[Ac(dx_n)].
\end{equation}
By \( \int_{|\xi'|=1} \xi_1 \cdots \xi_{2q+1} \sigma(\xi') = 0 \), we have

\[
(4.35) \quad \overline{\Psi}_5 = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} 8h'(0) \frac{9i - 56\xi_n - 49i\xi_n^2 + 14\xi_n^3}{(\xi_n - i)^7(\xi_n + i)^5} d\xi_n \sigma(\xi') dx' \\
- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-2 - 16i\xi_n + 14\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[b_0^1(x_0)c(\xi')] d\xi_n \sigma(\xi') dx' \\
- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-2 - 16i\xi_n + 14\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[Ac(\xi')] d\xi_n \sigma(\xi') dx' \\
- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-2i + 16\xi_n + 14i\xi_n^2}{4(\xi_n - i)^7(\xi_n + i)^5} \text{tr}[Ac(dx_n)] d\xi_n \sigma(\xi') dx' \\
= -i 8h'(0) \Omega_4 \left[ \frac{9i - 56\xi_n - 49i\xi_n^2 + 14\xi_n^3}{(\xi_n + i)^5} \right] \bigg|_{\xi_n=i} dx' \\
- i\Omega_4 \text{tr}[Ac(dx_n)] \frac{2\pi i}{6!} \left[ \frac{-2i + 16\xi_n + 14i\xi_n^2}{4(\xi_n + i)^5} \right] \bigg|_{\xi_n=i} dx' \\
= -\frac{105}{4} \pi h'(0) \Omega_4 dx' + \frac{161}{512} \pi\Omega_4 \text{tr}[Ac(dx_n)] dx'.
\]

Now \( \overline{\Psi} \) is the sum of the cases (a), (b) and (c), then

\[
(4.36) \quad \overline{\Psi} = \frac{137}{4} \pi h'(0) \Omega_4 dx' - \frac{31}{512} \pi\Omega_4 \text{tr}[Ac(dx_n)] dx' + \frac{1}{16} \pi\Omega_4 \text{tr}[A^*c(dx_n)] dx'.
\]

**Theorem 4.3.** Let \( M \) be a 6-dimensional compact oriented manifold with the boundary \( \partial M \) and the metric \( g^M \) as above, \( D_A \) be the perturbation of the de Rham Hodge operator on \( \widetilde{M} \), then

\[
(4.37) \quad \overline{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 128\pi^3 \int_M \left( -\frac{1}{12} s + 2A^*A - \frac{1}{4} \sum_{j=1}^n Ac(e_j) Ac(e_j) \right) \\
- \frac{1}{4} \sum_{j=1}^n A^*c(e_j) A^*c(e_j) + \frac{1}{2} \sum_{j=1}^n \nabla^\Lambda_{e_j} T^* M(A^*) c(e_j) - \frac{1}{2} \sum_{j=1}^n c(e_j) \nabla^\Lambda_{e_j} T^* M(A) \right) d\text{Vol}_M \\
+ \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\text{Vol}_{\partial M} - \int_{\partial M} \frac{31}{512} \pi\Omega_4 \text{tr}[Ac(dx_n)] d\text{Vol}_{\partial M} \\
+ \int_{\partial M} \frac{1}{16} \pi\Omega_4 \text{tr}[A^*c(dx_n)] d\text{Vol}_{\partial M},
\]

where \( s \) is the scalar curvature.

We can state the following facts as a corollary of Theorem 4.3.
Corollary 4.4. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \bar{c}(X)$, then

$$
\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 4|X|^2 + \sum_{j=1}^n g(\nabla_{\vec{e}_j}^T X, \vec{e}_j) \right) d\text{Vol}_M
$$

$$
+ \int_{\partial M} \frac{137}{4}\pi h'(0)\Omega_4 d\text{Vol}_{\partial M} + \int_{\partial M} \frac{63}{8}\pi \Omega_4 g(\partial x_n, X) d\text{Vol}_{\partial M},
$$

where $s$ is the scalar curvature.

When $A = \bar{c}(X)$, we can get the following corollary:

**Corollary 4.5.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \bar{c}(X)$, then

$$
\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s - |X|^2 \right) d\text{Vol}_M
$$

$$
+ \int_{\partial M} \frac{137}{4}\pi h'(0)\Omega_4 d\text{Vol}_{\partial M},
$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)$, we compute $\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}]$.

**Corollary 4.6.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \bar{c}(X)c(Y)$, then

$$
\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + |X|^2|Y|^2 + 4g(X, Y)^2 \right) d\text{Vol}_M
$$

$$
+ \int_{\partial M} \frac{137}{4}\pi h'(0)\Omega_4 d\text{Vol}_{\partial M},
$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)$, we obtain:

**Corollary 4.7.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)$, then

$$
\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 4|X|^2|Y|^2 \right) d\text{Vol}_M
$$

$$
+ \int_{\partial M} \frac{137}{4}\pi h'(0)\Omega_4 d\text{Vol}_{\partial M},
$$

where $s$ is the scalar curvature.

When $A = \bar{c}(X)c(Y)$, similar to Corollary 4.7, we can get:
Corollary 4.8. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \tau(X)\tau(Y)$, then

(4.41)
\[
\tilde{\operatorname{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s - |X|^2|Y|^2 + 6g(X, Y)^2 \right) d\operatorname{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\operatorname{Vol}_{\partial M},
\]

where $s$ is the scalar curvature.

When $A = c(X)c(Y)c(Z)$, we compute $\tilde{\operatorname{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}]$.

Corollary 4.9. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)c(Z)$, then

(4.42)
\[
\tilde{\operatorname{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 2|X|^2|Y|^2|Z|^2 - 2|X|^2g(Y, Z)^2 + 2|Y|^2g(X, Z)^2 - 2|Z|^2g(X, Y)^2 - 4g(X, Y)g(X, Z)g(Y, Z) \right.
\]
\[
- \sum_{j=1}^{6} g(\nabla_{e_j}^TM Z, \tilde{e}_j)g(X, Y) + \sum_{j=1}^{6} g(Y, \tilde{e}_j)g(X, \nabla_{e_j}^TM Z) - \sum_{j=1}^{6} g(X, \tilde{e}_j)g(Y, \nabla_{e_j}^TM Z)
\]
\[
- \sum_{j=1}^{6} g(Z, \tilde{e}_j)g(X, \nabla_{e_j}^TM Y) + \sum_{j=1}^{6} g(\nabla_{e_j}^TM Y, \tilde{e}_j)g(X, Z) - \sum_{j=1}^{6} g(X, \tilde{e}_j)g(\nabla_{e_j}^TM Y, Z)
\]
\[
- \sum_{j=1}^{6} g(Z, \tilde{e}_j)g(\nabla_{e_j}^TM X, Y) + \sum_{j=1}^{6} g(Y, \tilde{e}_j)g(\nabla_{e_j}^TM X, Z) - \sum_{j=1}^{6} g(\nabla_{e_j}^TM X, \tilde{e}_j)g(Y, Z) \bigg) d\operatorname{Vol}_M
\]
\[
+ \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\operatorname{Vol}_{\partial M} + \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, X)g(Y, Z) d\operatorname{Vol}_{\partial M}
\]
\[
- \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, Y)g(X, Z) d\operatorname{Vol}_{\partial M} + \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, Z)g(X, Y) d\operatorname{Vol}_{\partial M},
\]

where $s$ is the scalar curvature.

When $A = \tau(X)c(Y)c(Z)$, we get:

Corollary 4.10. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \tau(X)c(Y)c(Z)$, then

(4.43)
\[
\tilde{\operatorname{Res}}[\pi^+ D_A^{-1} \circ \pi^+ (D_A^* D_A D_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 3|X|^2|Y|^2|Z|^2 - 4|X|^2g(Y, Z)^2 \right) d\operatorname{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\operatorname{Vol}_{\partial M},
\]
where $s$ is the scalar curvature.

When $A = \pi(X)\pi(Y)c(Z)$, we conclude that:

**Corollary 4.11.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)c(Z)$, then

(4.44)

$$\widetilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*D_AD_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 4|\xi|^2g(X,Y)^2 \right.$$ 

$$+ \sum_{j=1}^6 g(\nabla^{TM}_{e_j}X,Y)g(Z,\tilde{e}_j) + \sum_{j=1}^6 g(X,\nabla^{TM}_{\xi_j}Y)g(Z,\tilde{\xi}_j) + \sum_{j=1}^6 g(X,Y)g(\nabla^{TM}_{e_j}Z,\tilde{e}_j) \bigg)d\text{Vol}_M$$

$$+ \int_{\partial M} \frac{137}{4}\pi\hbar'(0)\Omega_4d\text{Vol}_{\partial M} + \int_{\partial M} \frac{63}{8}\pi\Omega_4g(\partial_{r},Z)g(X,Y)d\text{Vol}_{\partial M},$$

where $s$ is the scalar curvature.

When $A = \pi(X)\pi(Y)c(Z)$, we find that:

**Corollary 4.12.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)c(Z)$, then

(4.45)

$$\widetilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+(D_A^*D_AD_A^*)^{-1}] = 8192\pi^3 \int_M \left( -\frac{1}{12}s + 5|X|^2|Y|^2|Z|^2 - 6|X|^2g(Y,Z)^2 \

- 6|Y|^2g(X,Z)^2 - 6|Z|^2g(X,Y)^2 + 12g(X,Y)g(X,Z)g(Y,Z) \right)d\text{Vol}_M$$

$$+ \int_{\partial M} \frac{137}{4}\pi\hbar'(0)\Omega_4d\text{Vol}_{\partial M},$$

where $s$ is the scalar curvature.

Next, we prove the Kastler-Kalau-Walze type theorem for 6-dimensional manifold with boundary associated to $D_A^3$. From [14], we know that

(4.46)  \( \widetilde{\text{Wres}}[\pi^+D_A^{-1} \circ \pi^+D_A^{-3}] = \int_M \int_{|\xi|=1} \text{trace}_{\Lambda^+T^*M}[\sigma^{-4}(D_A^{-4})]\sigma(\xi)d\xi + \int_{\partial M} \Phi, \)

where $\widetilde{\text{Wres}}$ denote noncommutative residue on minifolds with boundary,

(4.47)

$$\Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{\alpha+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^+T^*M}[\partial_{x'}^j\partial_\xi^\alpha\partial_\xi^k\sigma_r(D_A^{-1})(x',0,\xi',\xi_n) \times \partial_{x'}^\alpha\partial_\xi^{j+1}\partial_{x_n}^k\sigma_r(D_A^{-3})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',$$

and the sum is taken over $r + \ell - k - j - |\alpha| - 1 = -6$, $r \leq -1$, $\ell \leq -3$. 

\[\]
By Theorem 2.2, we compute the interior term of (4.46)

\begin{equation}
\int_M \int_{\xi=1} \text{trace}_{\lambda_t \cdot T(M)}[\sigma_{-4}(DA^{-1})]\sigma(\xi) dx
\end{equation}

\[= 128\pi^3 \int_M \text{tr} \left( -\frac{1}{12}s + 2A^2 - \frac{1}{2} \sum_{j=1}^n Ac(\tilde{e}_j)Ac(\tilde{e}_j) \right) d\text{Vol}_M.\]

So we only need to compute \(\int_{\partial M} \Phi\). Let us now turn to compute the specification of \(DA^3\).

\begin{equation}
DA^3 = \sum_{i=1}^n c(\tilde{c}_i)(\tilde{\epsilon}_i, dx_i)(-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(\tilde{c}_i)(\tilde{\epsilon}_i, dx_i) \left\{ - (\partial_i g^{ij}) \partial_i \partial_j - g^{ij} \left( 4(\sigma_i + a_i) \partial_i \partial_j \right) \right\}
\end{equation}

\[+ \sum_{i=1}^n c(\tilde{c}_i)(\tilde{\epsilon}_i, dx_i) \left\{ - 2(\partial_i g^{ij}) (\sigma_i + a_i) \partial_j + g^{ij} (\partial_i \Gamma_k^{ij}) \partial_k \right\} + \sum_{i,j} g^{ij} \left( (\partial_i \sigma_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_k^{ij} \sigma_k - \Gamma_k^{ij} a_k \right) + \sum_{i,j} g^{ij} \left( c(\partial_i) \partial_j (A) + c(\partial_i) a_j A + Ac(\partial_i) \sigma_j + Ac(\partial_i) a_j \right) - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl}(\tilde{c}_i)\tilde{c}(\tilde{c}_j) c(\tilde{c}_k) c(\tilde{c}_l)
\]

\[+ \frac{1}{4}s + A^2 \right\} + \left[ \sum_{i=1}^n c(\tilde{c}_i)(\sigma_i + a_i) + A \right] (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(\tilde{c}_i)(\tilde{\epsilon}_i, dx_i) \left\{ 2 \sum_{i,j} g^{ij} (2(\sigma_i + a_i) \partial_i \partial_j - \Gamma_k^{ij} \partial_k + (\partial_i \Gamma_k^{ij}) \partial_j + \sum_{i,j} g^{ij} \left( c(\partial_i) \partial_j (A) + c(\partial_i) a_j A + c(\partial_i) \sigma_j + Ac(\partial_i) a_j \right) - \frac{1}{8} \sum_{i,j,k,l} R_{ijkl}(\tilde{c}_i)\tilde{c}(\tilde{c}_j) c(\tilde{c}_k) c(\tilde{c}_l) + \frac{1}{4}s + A^2 \right\}.
\]

Then, we obtain
Lemma 4.13. The following identities hold:

\[
\sigma_2(D_A^3) = \sum_{i,j,l} c(dx_i)\partial_i(g^{ij})\xi_i \xi_j + c(\xi)(4\sigma^k + 4a^k - 2\Gamma^k)\xi_k + 2[|\xi|^2 A - c(\xi)Ac(\xi)] \\
+ \frac{1}{4}|\xi|^2 \sum_{s,t,l} \omega_{s,t}(\tilde{e}_i)[c(\tilde{e}_i)\tilde{\sigma}(\tilde{e}_s)\tilde{\sigma}(\tilde{e}_t) - c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)] + |\xi|^2 A;
\]

\[
\sigma_3(D_A^3) = ic(\xi)|\xi|^2.
\]

Write

\[
\sigma(D_A^3) = p_3 + p_2 + p_1 + p_0; \quad \sigma(D_A^{-3}) = \sum_{j=3}^{\infty} q_{-j}.
\]

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D_A^3 \circ D_A^{-3}) = \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha_\xi [\sigma(D_A^3)]D^\alpha_\xi [\sigma(D_A^{-3})] \\
= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots) \\
+ \sum_j (\partial^j_\xi p_3 + \partial^j_\xi p_2 + \partial^j_\xi p_1 + \partial^j_\xi p_0)(D_{x,j} q_{-3} + D_{x,j} q_{-4} + D_{x,j} q_{-5} + \cdots) \\
= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial^j_\xi p_3 D_{x,j} q_{-3}) + \cdots,
\]

by (4.52), we have

\[
q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1}[p_2 p_3^{-1} + \sum_j \partial^j_\xi p_3 D_{x,j} (p_3^{-1})].
\]

By (4.49)-(4.53), we have some symbols of operators.

Lemma 4.14. The following identities hold:

\[
\sigma_{-3}(D_A^{-3}) = \frac{ic(\xi)}{|\xi|^4};
\]

\[
\sigma_{-4}(D_A^{-3}) = \frac{c(\xi)c(\xi)}{|\xi|^8} + \frac{c(\xi)}{|\xi|^8} \left( |\xi|^2 c(dx_n)\partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) \\
+ 2\xi_n c(\xi)\partial_{x_n} c(\xi') + 4\xi_n h'(0) \right).
\]

When \( n = 6 \), then \( \text{tr}_{\lambda \ast T_{-\lambda}}[\text{id}] = 64 \), where \( \text{tr} \) as shorthand of trace. Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -6, r \leq -1, \ell \leq -3 \), then we have the \( \int_{\partial M} \Phi \) is the sum of the following five cases:

**case (a) (I)** \( r = -1, l = -3, j = k = 0, |\alpha| = 1. \)
By (4.47), we get
\[(4.55)\]
\[\Phi_1 = - \int_{|\xi'|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi',\xi,n}^\alpha \sigma_{-1}(D_A^{-1}) \times \partial_{\xi,n}^\alpha \sigma_{-3}(D_A^{-3})](x_0) d\xi_n \sigma(\xi') dx'.\]

**case (a) (II) \(r = -1, l = -3, |\alpha| = k = 0, j = 1\).**

We notice that
\[(4.56)\]
\[\Phi_2 = - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \text{trace}[\partial_{x,n} \pi_{\xi,n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi,n} \sigma_{-3}(D_A^{-3})](x_0) d\xi_n \sigma(\xi') dx'.\]

**case (a) (III) \(r = -1, l = -3, |\alpha| = j = 0, k = 1\).**

By (4.47), we compute that
\[(4.57)\]
\[\Phi_3 = - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \text{trace}[\partial_{\xi,n} \pi_{\xi,n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi,n} \partial_{x,n} \sigma_{-3}(D_A^{-3})](x_0) d\xi_n \sigma(\xi') dx'.\]

By Lemma 4.2 and Lemma 4.5, we have \(\sigma_{-3}((D_A^*D_A)^{-1}) = \sigma_{-3}(D_A^{-3})\), then we obtain
\[(4.58)\]
\[\Phi_1 + \Phi_2 + \Phi_3 = \frac{10}{2} \pi h'(0) \Omega_4 dx',\]

where \(\Omega_4\) is the canonical volume of \(S^4\).

**case (b) \(r = -1, l = -4, |\alpha| = j = k = 0\).**

By (4.47), we notice that
\[(4.59)\]
\[\Phi_4 = -i \int_{|\xi'|=1}^{+\infty} \text{trace}[\pi_{\xi,n}^+ \sigma_{-1}(D_A^{-1}) \times \partial_{\xi,n} \sigma_{-4}(D_A^{-3})](x_0) d\xi_n \sigma(\xi') dx'.\]

In the normal coordinate, \(g^{\alpha\beta}(x_0) = \delta^\alpha_\beta\) and \(\partial_{x}^{(g^{\alpha\beta})}(x_0) = 0\), if \(j < n\); \(\partial_{x}^{(g^{\alpha\beta})}(x_0) = h'(0)\delta^\alpha_\beta\), if \(j = n\). So by [12], when \(k < n\), we have \(\Gamma^\alpha(x_0) = \frac{1}{2} h'(0), \Gamma^k(x_0) = 0, \delta^\alpha_n(x_0) = 0\) and \(\delta^k = \frac{1}{4} h'(0)c(\tilde{e}_k)c(\tilde{e}_n)\). Then, we obtain
\( \Phi = \sum \text{(a), (b) and (c)}, \text{hence} \) 
\( \Phi = 137 \) 

By calculation, we have 
\( (4.61) \)

By (4.16) and (4.60), we have 
\( (4.63) \)

By applying the formula shown in (4.59), we can calculate 
\( (4.62) \)

case (c) \( r = -2, l = -3, |\alpha| = j = k = 0. \)

We calculate 
\( (4.63) \)

By calculation, we have 
\( (4.64) \)

Now \( \overline{\Phi} \) is the sum of the cases (a), (b) and (c), hence 
\( (4.65) \)
Theorem 4.15. Let $M$ be a 6-dimensional compact oriented manifold with the boundary $\partial M$ and the metric $g^M$ as above, $D_A$ be the perturbation of the de Rham Hodge operator on $\tilde{M}$, then

$$\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 128\pi^3 \int_M \text{tr} \left( -\frac{1}{12} s + 2A^2 - \frac{1}{2} \sum_{j=1}^{n} Ac(\tilde{e}_j)Ac(\tilde{e}_j) \right) \text{dVol}_M$$

$$+ \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 \text{dVol}_{\partial M} + \int_{\partial M} \frac{1}{512} \pi \Omega_4 \text{tr} [Ac(dx_n)] \text{dVol}_{\partial M},$$

where $s$ is the scalar curvature.

When $A = c(X)$, we can directly state the subsequent corollary:

**Corollary 4.16.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)$, then

$$(4.66) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s \right) \text{dVol}_M$$

$$+ \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 \text{dVol}_{\partial M} - \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, X) \text{dVol}_{\partial M},$$

where $s$ is the scalar curvature.

When $A = \pi(X)$, we can get the following corollary:

**Corollary 4.17.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)$, then

$$(4.67) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s - |X|^2 \right) \text{dVol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 \text{dVol}_{\partial M},$$

where $s$ is the scalar curvature.

When $A = c(X)c(Y)$, we compute $\widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})]$.

**Corollary 4.18.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)$, then

$$(4.68) \quad \widetilde{\text{Res}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s + |X|^2|Y|^2 + 4g(X, Y)^2 \right) \text{dVol}_M$$

$$+ \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 \text{dVol}_{\partial M},$$

where $s$ is the scalar curvature.

When $A = c(X)\pi(Y)$, we notice that:
Corollary 4.19. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)\pi(Y)$, then

(4.69)\[
\text{Wres}[\pi^+D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s - \frac{1}{12} |X|^2 |Y|^2 + 6g(X, Y)^2 \right) d\text{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\text{Vol}_{\partial M},
\]
where $s$ is the scalar curvature.

When $A = \pi(X)\pi(Y)$, similar to (4.68), we can get the following corollary:

Corollary 4.20. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)$, then

(4.70)\[
\text{Wres}[\pi^+D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s - \frac{1}{12} |X|^2 |Y|^2 + 6g(X, Y)^2 \right) d\text{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\text{Vol}_{\partial M},
\]
where $s$ is the scalar curvature.

When $A = c(X)c(Y)c(Z)$, we compute that:

Corollary 4.21. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = c(X)c(Y)c(Z)$, then

(4.71)\[
\text{Wres}[\pi^+D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s - \frac{1}{12} |X|^2 |Y|^2 |Z|^2 + 2|X|^2 g(Y, Z)^2
\]
\[
+ 2|Y|^2 g(X, Z)^2 + 2|Z|^2 g(X, Y)^2 - 4g(X, Y)g(X, Z)g(Y, Z) \right) d\text{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\text{Vol}_{\partial M} + \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, X)g(Y, Z) d\text{Vol}_{\partial M}
\]
\[
- \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, Y)g(X, Z) \text{Vol}_{\partial M} + \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, Z)g(X, Y) \text{Vol}_{\partial M},
\]
where $s$ is the scalar curvature.

When $A = \pi(X)c(Y)c(Z)$, we can conclude the following facts:

Corollary 4.22. Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)c(Y)c(Z)$, then

(4.72)\[
\text{Wres}[\pi^+D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s + 3|X|^2 |Y|^2 |Z|^2 - 4|X|^2 g(Y, Z)^2 \right) d\text{Vol}_M + \int_{\partial M} \frac{137}{4} \pi h'(0) \Omega_4 d\text{Vol}_{\partial M}.
\]
where $s$ is the scalar curvature.

Computations show that:

**Corollary 4.23.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)c(Z)$, then

$$ \widetilde{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s - 4|X|^2|Y|^2|Z|^2 + 4|Z|^2g(X,Y)^2 \right)d\text{Vol}_M $$

$$ + \int_{\partial M} \frac{137}{4} \pi h'(0)\Omega_4 d\text{Vol}_{\partial M} - \int_{\partial M} \frac{1}{8} \pi \Omega_4 g(\partial x_n, Z)g(X,Y)d\text{Vol}_M, $$

where $s$ is the scalar curvature.

When $A = \pi(X)\pi(Y)\pi(Z)$, we can get the following corollary:

**Corollary 4.24.** Let $M$ be a 6-dimensional oriented compact manifolds with the boundary $\partial M$ and the metric $g^M$ as above, and let $A = \pi(X)\pi(Y)\pi(Z)$, then

$$ \widetilde{\text{Wres}}[\pi^+ D_A^{-1} \circ \pi^+(D_A^{-3})] = 8192\pi^3 \int_M \left( -\frac{1}{12} s + 5|X|^2|Y|^2|Z|^2 - 6|X|^2g(Y,Z)^2 $$

$$ - 6|Y|^2g(X,Z)^2 - 6|Z|^2g(X,Y)^2 + 12g(X,Y)g(X,Z)g(Y,Z) \right)d\text{Vol}_M $$

$$ + \int_{\partial M} \frac{137}{4} \pi h'(0)\Omega_4 d\text{Vol}_{\partial M}, $$

where $s$ is the scalar curvature.

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**References**

[1] Guillemin V W.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).

[2] Wodzicki M.: Local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1995).

[3] Connes A.: Quantized calculus and applications. 11th International Congress of Mathematical Physics (Paris, 1994), Internat Press, Cambridge, MA, 15-36, (1995).

[4] Connes A.: The action functional in noncommutative geometry. Comm. Math. Phys. 117, 673-683, (1998).

[5] Kastler D.: The Dirac operator and gravitation. Comm. Math. Phys. 166, 633-643, (1995).

[6] Kalau W, Walze M.: Gravity, noncommutative geometry and the Wodzicki residue. J. Geom. Phys. 16, 327-344, (1995).
[7] Ackermann T.: A note on the Wodzicki residue. J. Geom. Phys. 20, 404-406, (1996).
[8] Ugalde W J.: Differential forms and the Wodzicki residue. J. Geom. Phys. 58, 1739-1751, (2008).
[9] Fedosov B V, Golse F, Leichtnam E, Schrohe E.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).
[10] Wang Y.: Differential forms and the Wodzicki residue for manifolds with boundary. J. Geom. Phys. 56, 731-753, (2006).
[11] Wang Y.: Differential forms the noncommutative residue for manifolds with boundary in the non-product case. Lett. Math. Phys. 77, 41-51, (2006).
[12] Wang Y.: Gravity and the noncommutative residue for manifolds with boundary. Lett. Math. Phys. 80, 37-56, (2007).
[13] Wang Y.: Lower-dimensional volumes and Kastler-Kalau-Walze type theorem for manifolds with boundary. Commun. Theor. Phys. Vol 54, 38-42, (2010).
[14] Wang J, Wang Y.: The Kastler-Kalau-Walze type theorem for six-dimensional manifolds with boundary. J. Math. Phys. 56, 052501, (2015).
[15] Wei S, Wang Y.: Modified Novikov operators and the Kastler-Kalau-Walze type theorem for manifolds with boundary. Adv. Math. Phys. 1-28, (2020).
[16] Wu T, Wang J, Wang Y.: Dirac-witten operators and the Kastler-Kalau-Walze type theorem for manifolds with boundary. J. Nonliear Math. Phys. arXiv:2103.11842.
[17] Wang Y.: A Kastler-Kalau-Walze type theorem and the spectral action for perturbations of Dirac operators on manifolds with boundary. Abstr. Appl. Anal. 17, 286-299, (2014).
[18] Yu Y.: The index theorem and the heat equation method, nankai tracts in mathematics-Vol.2. World Scientific Publishing, (2001).
[19] Ponge R.: Noncommutative geometry and lower dimensional volumes in Riemannian geometry. Lett. Math. Phys. 83, no.1, 19-32, (2008).

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