Metals in high magnetic field: a new universality class of Fermi liquids

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Parquet equations, describing the competition between superconducting and density-wave instabilities, are solved for a three-dimensional isotropic metal in a high magnetic field when only the lowest Landau level is filled. In the case of a repulsive interaction between electrons, a phase transition to the density-wave state is found at finite temperature. In the opposite case of attractive interaction, no phase transition is found. With decreasing temperature $T$, the effective vertex of interaction between electrons renormalizes toward a one-dimensional limit in a self-similar way with the characteristic length (transverse to the magnetic field) decreasing as $\ln^{-1/6}(\omega_c/T)$ ($\omega_c$ is a cutoff). Correlation functions have new forms, previously unknown for conventional one-dimensional or three-dimensional Fermi-liquids.

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The behavior of an isotropic three-dimensional (3D) metal in a high magnetic field has attracted attention of physicists for a long time. In this system, the energy of an electron depends only on the momentum along the magnetic field. Thus, the system exhibits effects characteristic of one-dimensional (1D) metals, while intrinsically it is 3D. As an immediate consequence of this fact, it was suggested that the system should be unstable with respect to charge- or spin-density-wave (DW) formation [1]. Another suggestion was that the system could remain superconducting (SC) in an arbitrarily high magnetic field, since the 1D dispersion law still allows for a SC instability [2]. It was pointed out already in Ref. [1] that both instabilities must be taken into account simultaneously in the so-called parquet approximation. The parquet equations were written correctly in Ref. [3], where it was found that the DW solution is indeed an asymptotic solution of the equations. However, the equations were not solved numerically, thus the question of when the asymptotic solution develops remained open. The present author previously solved numerically similar equations for a quasi-one-dimensional conductor in a high magnetic field and found phase transitions to the DW state in both cases of repulsive and attractive interaction between electrons [4]. Even if initially the interaction was attractive (favorable for superconductivity), in the course of renormalization the sign of the interaction was effectively changed giving rise to a non-trivial DW state exhibiting the quantum Hall effect [5]. In present paper, parquet equations for an isotropic 3D metal in a very high magnetic field, with only the lowest spin-polarized Landau level filled, are solved numerically. It is found that in the repulsive case a phase transition to the DW state occurs, in agreement with the analysis given in Ref. [4]. In the case of attraction, no phase transition occurs and the system remains a non-trivially correlated metal to arbitrarily low temperatures. With decreasing temperature $T$, the system becomes progressively more one-dimensional, with the characteristic interaction length perpendicular to the magnetic field decreasing as $\zeta^{-1/6}$, where $\zeta = \ln(\omega_c/T)$, and $\omega_c$ is a cutoff of the order of the cyclotron frequency. However, the system never becomes strictly 1D. By this reason, the SC susceptibility grows as $\zeta^{1/3} \exp(\text{const}\zeta^{2/3})$. This dependence is neither characteristic of 1D, nor of conventional 3D metals. Thus, the system is a new non-trivial example of an
unconventional Fermi-liquid, neither of the Landau, nor of the Luttinger type \[6\]. It should be emphasized also that there exists a model where the competition between SC and DW channels is suppressed and, thus superconductivity indeed may survive in an arbitrarily high magnetic field \[7\]. However, this is not the case for the model considered below.

Let us consider an isotropic 3D metal with an effective mass \(m\) and a parabolic dispersion law. We assume that magnetic field is so high that only the lowest Landau level is filled and all electrons are spin-polarized. Dispersion law has the form \(\varepsilon = p_z^2 / 2m\), where \(p_z\) is the momentum along the field. The Fermi surface consists of the two Fermi points \(p_z = \mp p_F = \pm 2\pi^2 \hbar n l_H^2\), where \(n\) is the volume concentration of electrons and \(l_H = (\hbar c / eH)^{1/2}\) is the magnetic length. Coordinates \(X, Y\) in the plane perpendicular to the field will be measured in units of \(l_H\). The wave functions of the lowest Landau level are \[8\]:

\[
\psi_x(X,Y) = \pi^{-1/4} l_H^{-1/2} \exp(iX - (x - X)^2 / 2).
\] (1)

Another representation can be obtained making a superposition of states \(\|1\rangle\) and \(\|2\rangle\):

\[
\psi_y(X,Y) = (2\pi)^{-1/2} \int dx \exp(-ixy) \psi_x(X,Y)
= \pi^{-1/4} l_H^{-1/2} \exp(iX(Y - y) - (y - Y)^2 / 2).
\] (2)

To be distinguished from the running variables \(X, Y\), the quantum numbers \(x, y\), which label the wave functions \(\|1\rangle\) and \(\|2\rangle\), are denoted everywhere by the lowercase letters.

Let us introduce now the operators \(\hat{a}^+(x,p_z)\) and \(\hat{b}^+(y,p_z)\) which create electrons in the eigenstates \(\|1\rangle\) and \(\|2\rangle\) with \(p_z\) close to \(+p_F\) and \(-p_F\), respectively. Near the Fermi energy, the electron spectrum can be linearized in \(p_z\) and the Hamiltonian of the model can be written in the form \(\hat{H} = \hat{H}_0 + \hat{H}_1\), where

\[
\hat{H}_0 = 2\pi l_H \int \frac{dp_z}{2\pi \hbar} v_F [(p_z - p_F) \int dx \hat{a}^+(x,p_z)\hat{a}(x,p_z)
-(p_z + p_F) \int dy \hat{b}^+(y,p_z)\hat{b}(y,p_z)], \quad v_F = p_F / m,
\] (3)

\[
\hat{H}_1 = \int g \frac{dp_1 dp_2 dp_3}{(2\pi \hbar)^3} dx dy dx' dy' \exp(-ixy' + iyx') \gamma_0(x - x', y - y')
\hat{a}^+(x,p_1)\hat{b}^+(y,p_2)\hat{b}(y',p_3)\hat{a}(x',p_1 + p_2 - p_3),
\] (4)
\( \gamma_0(\mathbf{r} - \mathbf{r}') = \exp(i r \wedge r') \int dX dY \psi_\mathbf{x}^\ast(X, Y) \psi_\mathbf{y}^\ast(X, Y) \psi_\mathbf{y}'(X, Y) \psi_\mathbf{x}'(X, Y) \)
\[ = \exp(- (\mathbf{r} - \mathbf{r}')^2 / 2). \]

In Eq. (5), \( \mathbf{r} = (x, y) \), \( \mathbf{r}' = (x', y') \) and \( r \wedge r' = xy' - yx' \). The function \( \gamma_0 \) is a form-factor of the interaction between electrons in the representation given in (1) and (2). The interaction amplitude is \( g = g_2 - g_1 \), where \( g_2 \) and \( g_1 \) are the amplitudes of forward and backward scattering. We assume that the original interaction is not completely local, so that \( g_1 \neq g_2 \) and \( g \neq 0 \); however, its spatial range is much shorter than \( l_H \).

The so-called parquet diagrams, which consist of the electron-hole and electron-electron loops inserted into each other in all possible ways, are the most important many-body corrections to the interaction vertex \( \gamma_0(\mathbf{r}) [1], [3] \). The corrections form a series in powers of \( \xi = (|g|/(2\pi)^3 v_F L_H^2) \ln[\omega_c/\max(T, \omega, v_F|p_z - p_F|)] \), where \( \omega \) is a frequency, since both of the one-loop diagrams are logarithmically divergent.

The renormalized vertex of interaction \( \gamma(\mathbf{r}, \xi) \) obeys the following equation, shown graphically in Fig.1:
\[ \partial \gamma(\mathbf{r}, \xi)/\partial \xi = \int d^2 \mathbf{r}' \gamma(\mathbf{r}', \xi) \gamma(\mathbf{r} - \mathbf{r}', \xi)(1 - e^{i r \wedge r'}), \]
\[ \gamma(\mathbf{r}, 0) = \text{sign}(g) \gamma_0(\mathbf{r}). \] (5)

Eq. (3) can be obtained from Eq. (2) of Ref. 3 via a Fourier transformation over the variable \( k_x \).

The r.h.s. of Eq. (3) is a difference of two terms. The first term is the contribution of the electron-hole loop, the second term — of the electron-electron loop. If only one of these terms is retained in the r.h.s., that corresponds to a ladder approximation instead of a parquet one, then Eq. (3) can be solved analytically. If one neglects the second term, then the equations are diagonalized via two-dimensional (2D) Fourier transformation \( \gamma(\mathbf{r}, \xi) \rightarrow \Gamma(\mathbf{k}, \xi) [3] \):
\[ \Gamma(\mathbf{k}, \xi) = (\text{sign}(g) \Gamma_0^{-1}(\mathbf{k}) - \xi)^{-1}, \]
(8)
where $\Gamma_0(k)$ is the 2D Fourier transform of $\gamma_0(r)$ \((3)\). In the case $g > 0$ (repulsion), solution \((8)\) is called a “moving pole” \([9]\), because the position $\xi_p(k) = \Gamma_0^{-1}(k)$ of the pole singularity in $\xi$ depends on the value of $k$. It follows from \((5)\) that the minimum value of $\xi_p(k)$, which is equal to $\xi_c = 1/2\pi$, is attained at $k_c = 0$. As was shown by a calculation of the appropriate susceptibility \([3]\), the moving pole singularity \((8)\) indicates a phase transition to a DW state, where the densities of the electron charge and spin are modulated along the magnetic field with a wave vector $2p_F/h$, and are homogeneous in the perpendicular plane (because $k_c = 0$).

Once the moving pole \((8)\) develops, the second term in Eq. \((6)\) indeed can be neglected, because this term contains an integration over $k$ which makes it less singular than the first term \([3]\).

Solving numerically the full equation \((6)\) at $g > 0$, we find the moving pole singularity occurring at $k_c = 0$ and $\xi_c = 1.3/2\pi$. The value of $\xi_c$ is related to the transition temperature by the formula $T_c = \omega_c \exp(-(2\pi)^2 \xi_c v_F l_H / |g|)$. Thus, the only effect of the second term in Eq. \((6)\) in the repulsive case is a certain decrease of the transition temperature. Otherwise, the ladder approximation gives qualitatively correct results.

If the first term in the r.h.s. of Eq. \((6)\) is neglected, then the resulting equation can be solved by performing the Fourier transformation $\gamma(x_1, y, \xi) \rightarrow \lambda(x_1, x_2, \xi)$ over the variable $y$ and introducing the function $h(x_1, x_2, \xi) = \lambda(x_1 - x_2, x_1 + x_2, \xi)$ obeying the following equation \([4]\):

$$
\partial h(x_1, x_2, \xi) / \partial \xi = -\int dx h(x_1, x, \xi) h(x, x_2, \xi).
$$

With initial conditions \((7)\), Eq. \((6)\) has a solution:

$$
h(x_1, x_2, \xi) = \sqrt{2\pi} \exp(-x_1^2 - x_2^2) / (\text{sign}(g) + \pi \xi).
$$

In the case $g < 0$ (attraction), Eq. \((10)\) has a pole singularity at $\xi_c = 1/\pi$ that indicates a phase transition to a SC state. Solution \((10)\) is called a “standing pole” \([4]\), because the position of the pole in $\xi$ does not depend on any continuous variable. For this reason, when expression \((10)\) is substituted into the full Eq. \((6)\), the first term has the same singularity as
the second, thus the SC ladder approximation can never be justified \[9\]. This fact explains why it is important to solve numerically the full Eq. \((\text{I})\) in the case \(g < 0\).

Since the initial vertex \(\gamma_0(\mathbf{r})\) \[9\] depends only on \(r\), which is the absolute value of \(\mathbf{r}\), then the same holds for \(\gamma(\mathbf{r}, \xi)\). Eq. \((\text{I})\) can be rewritten for a new function \(\tilde{\gamma}(r, \xi) = \gamma(\mathbf{r}, \xi)\), which depends on one spatial argument:

\[
\frac{\partial \tilde{\gamma}(r, \xi)}{\partial \xi} = 8 \int_0^\infty r_1 dr_1 \int_{|r-r_1|}^{r+r_1} r_2 dr_2 \tilde{\gamma}(r_1, \xi) \tilde{\gamma}(r_2, \xi) \sin^2([4(r_1 r_2)^2 - (r_1^2 + r_2^2 - r^2)^2]^{1/2}/4) \over [4(r_1 r_2)^2 - (r_1^2 + r_2^2 - r^2)^2]^{1/2}, \tag{11}
\]

\[
\tilde{\gamma}(r, 0) = \text{sign}(g) \exp(-r^2/2). \tag{12}
\]

The numerical solution of Eq. \((\text{I})\)\((\text{II})\)\((\text{III})\) for the case \(g < 0\) is shown in Fig. 2 for several values of “time” \(\xi\). After a short initial evolution, the function \(\tilde{\gamma}(r, \xi)\) attains the form \(\tilde{\gamma}_c(w(\xi)r)\), where \(w(\xi)\) is a monotonically growing function of \(\xi\). The ansatz \(\tilde{\gamma}(r, \xi) = \tilde{\gamma}_c(w(\xi)r)\) is consistent with Eq. \((\text{I})\) provided \(w\) is sufficiently large. In this case, the sine in Eq. \((\text{I})\) can be replaced by its argument, and Eq. \((\text{I})\) decouples into two equations:

\[
dw(\xi)/d\xi = Aw^{-5}(\xi), \tag{13}
\]

\[
2A \rho \partial \tilde{\gamma}_c(\rho)/\partial \rho = \int_0^\infty \rho_1 d\rho_1 \int_{|\rho-\rho_1|}^{\rho+\rho_1} \rho_2 d\rho_2 \tilde{\gamma}_c(\rho_1) \tilde{\gamma}_c(\rho_2) \over \big[4(\rho_1 \rho_2)^2 - (\rho_1^2 + \rho_2^2 - \rho^2)^2\big]^{1/2}, \tag{14}
\]

\(A\) is a constant. It follows from \((\text{I})\)\((\text{III})\) that \(w(\xi) = [A(\xi - \xi_0)]^{1/6}\). This dependence indeed was found numerically with \(\xi_0 = 0\). The function \(\tilde{\gamma}_c(\rho)\) is also known numerically: with the convention \(A = 1\), \(\tilde{\gamma}_c(\rho) = \tilde{\gamma}(\rho^{1/6}, \xi)\), where for the latter function one can take any of the plots in Fig. 2, except one corresponding to \(\xi = 0\). In summary, the solution of the parquet equations for the case \(g < 0\) has the self-similar form

\[
\gamma(\mathbf{r}, \xi) = \gamma_c(\mathbf{r}\xi^{1/6}), \tag{15}
\]

where we introduced a function of two variables \(\gamma_c(\mathbf{r}) = \tilde{\gamma}_c(\mathbf{r})\). Eq. \((\text{I})\)\((\text{III})\) is neither a moving, nor a standing pole; it is rather a “squeezing” solution: the effective range of
interaction varies as \( \ln^{-1/6}(\omega_c/T) \), thereby making the system increasingly one-dimensional as temperature is reduced.

In order to calculate susceptibilities, let us add to the Hamiltonian (3)–(4) the fictitious external fields, \( f_{SC} \) and \( f_{DW} \), which create electron-electron and electron-hole pairs:

\[
\hat{H}_2 = \int \frac{dp_z}{2\pi} dx dy \left[ f_{SC}(x, y) \hat{a}^+(x, p_z) \hat{b}^+(y, -p_z) + f_{DW}(x, y) \exp(-i p_z y) \hat{a}^+(x, p_z) \hat{b}(y, p_z - 2p_F) \right] + \text{h.c.}
\]

(16)

Let us start with the SC susceptibility. According to the parquet rules \[3\], it is necessary to calculate first a vertex \( \Phi(x, y, \xi) \), which is determined by the graphical equation shown in Fig. 3a:

\[
\frac{\partial \Phi(r, \xi)}{\partial \xi} = - \int dr' \gamma(r - r', \xi) \Phi(r', \xi) e^{-ir\lambda r'},
\]

(17)

\[
\Phi(r, 0) = f_{SC}(r).
\]

(18)

To solve this equation let us make a Fourier transformation over the variable \( y \): \( \Phi(x_1, y, \xi) \rightarrow \Lambda(x_1, x_2, \xi) \) and \( f_{SC}(x_1, y) \rightarrow F_{SC}(x_1, x_2) \); then, introduce the new variables: \( \Xi(\bar{x}, x, \xi) = \Lambda(\bar{x} - x, \bar{x} + x, \xi) \) and \( \Phi_{SC}(\bar{x}, x) = F_{SC}(\bar{x} - x, \bar{x} + x) \). In the new variables, Eq. (17) reads:

\[
\frac{\partial \Xi(\bar{x}, x, \xi)}{\partial \xi} = - \int dx' \Xi(\bar{x}, x', \xi) h(x', x, \xi),
\]

(19)

\[
\Xi(\bar{x}, x, 0) = \Phi_{SC}(\bar{x}, x),
\]

(20)

where the function \( h \) was introduced earlier. Eq. (19) is diagonal and degenerate with respect to \( \bar{x} \), the center-of-mass coordinate of a Cooper pair, so this variable can be omitted. Taking into account Eq. (15), one can rewrite Eq. (19) in the form:

\[
\frac{\partial \Xi(x, \xi)}{\partial \xi} = - \int dx' \lambda_c(x', 2x' \xi^{1/3} + x' \xi^{1/3}) \Xi(x + x', \xi),
\]

where \( \lambda_c(x_1, x_2) \) is the Fourier transform of \( \gamma_c(x_1, y) \) over the variable \( y \). An approximate solution is:

\[
\Xi(x, \xi) = \Phi_{SC}(x) \exp\left(\frac{3\Gamma_0 \xi^{2/3}}{2} - \frac{2\Gamma_1 x^2 \xi^{1/3}}{3}\right),
\]

(21)

\[
\Gamma_0 = - \int_0^\infty 2\pi r \tilde{\gamma}_c(r) dr = 3.1,
\]

\[
\Gamma_1 = - \int_0^\infty \pi r^3 \tilde{\gamma}_c(r) dr = 1.1.
\]
The contribution to the free energy $\delta_1 F$ due to the external field $f_{SC}$ is shown graphically in Fig. 3b:

$$\delta_1 F = 2 \int_0^\xi d\zeta \int dx |\Xi(x, \zeta)|^2 / |g|.$$  

(22)

Substituting Eq. (21) into Eq. (22), one finds the susceptibility with respect to creation of an electron pair with the relative distance between electrons equal to $x$:

$$\chi(x, \xi) = \xi^{1/3} \exp(3\Gamma_0 \xi^{2/3} - \Gamma_1 x^2 \xi^{1/3}/3)/|g|\Gamma_0.$$  

(23)

When $\xi \to \infty$, the susceptibility diverges following an unusual law $\xi^{1/3} \exp(\text{const} \xi^{2/3})$. This behavior can be understood qualitatively in the following way. The total amplitude of scattering to all possible channels, $\gamma_t(\xi) = \int d^2 r \gamma(r, \xi)$, goes to zero with increasing $\xi$. However, the function $-2 \int_0^\xi \gamma_t(\zeta) d\zeta$, which appears in the exponent of (23), increases as $\xi^{2/3}$. At the same time, the characteristic distance between electrons in a Cooper pair in (23) squeezes as $\xi^{-1/6}$, which indicates the increasingly one-dimensional character of the system.

Following an analogous procedure, one can find a contribution to the free energy due to $f_{DW}$:

$$\delta_2 F = \int_0^\xi \frac{d\zeta}{|g|} d^2 k |F_{DW}(k)|^2 \exp \left(2 \int_0^\xi \Gamma_c \left(\frac{k}{\eta^{1/6}}\right) \frac{d\eta}{\eta^{1/3}}\right),$$

where $F_{DW}(k)$ and $\Gamma_c(k)$ are the 2D Fourier transforms of $f_{DW}(r)$ and $\gamma_c(r)$. For all values of $k$ the DW susceptibilities have finite limits at zero temperature.

The lowest order diagram, which gives a logarithmic correction to the one-particle Green function, is shown in Fig. 4. Renormalized Green functions $G_{\pm}$, where the indices $\pm$ refer to the $\hat{a}$ and $\hat{b}$ fermions respectively, can be found in the framework of a scaling hypothesis [10]:

$$G_{\pm}(p_z, \omega) = \frac{\exp(-3\Gamma_2 \xi^{2/3} |g| / 4(2\pi)^2 v_F l_H^2)}{\omega \mp v_F (p_z \mp p_F)},$$

(24)

$$\Gamma_2 = \int_0^{\infty} 2\pi r \gamma_c^2(r) dr = 1.9.$$
Eq. (24) differs from expressions known for the 3D Landau and the 1D Luttinger Fermi-liquids [6].

In conclusion, the self-similar “squeezing” solution (15) of the parquet equations has never appeared previously in condensed matter physics. The correlation functions (23) and (24) represent a new universality class of Fermi-liquid behavior. All of this enriches our intuition beyond the Luttinger and the Landau schemes in consideration of what happens when many 1D channels interact [11].

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FIGURES

FIG. 1. Parquet equations for the vertex of interaction $\gamma(x_2 - x_1, y_2 - y_1, \xi)$ shown as a filled circle. The bare vertex $\gamma_0(x_2 - x_1, y_2 - y_1)$ is denoted as a dot. The Green functions of electrons with momenta $p_z$ close to $\pm p_F$ are shown as solid and dotted lines.

FIG. 2. The vertex of interaction $\bar{\gamma}(r, \xi)$ as a function of $r$ at $2\pi \xi = 0$ (curve a), 5 (b), 500 (c) and 3200 (d).

FIG. 3. a) Parquet equations for the vertex $\Psi(x, y, \xi)$ shown as a filled triangle. The external field $f_{SC}(x, y)$ is denoted as a square. b) Contribution to the free energy $\delta_1 F$ due to the field $f_{SC}$.

FIG. 4. The lowest order logarithmic correction to a Green function.