**IMEX ERROR INHIBITING SCHEMES WITH POST-PROCESSING**

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**Abstract.** High order implicit-explicit (IMEX) methods are often desired when evolving the solution of an ordinary differential equation that has a stiff part that is linear and a non-stiff part that is nonlinear. This situation often arises in semi-discretization of partial differential equations and many such IMEX schemes have been considered in the literature. The methods considered usually have a global error that is of the same order as the local truncation error. More recently, methods with global errors that are one order higher than predicted by the local truncation error have been devised [18, 28, 5]. In prior work [6, 7] we investigated the interplay between the local truncation error and the global error to construct explicit and implicit error inhibiting schemes that control the accumulation of the local truncation error over time, resulting in a global error that is one order higher than expected from the local truncation error, and which can be post-processed to obtain a solution which is two orders higher than expected. In this work we extend our error inhibiting with post-processing framework introduced in [6] and further in [7] to a class of additive general linear methods with multiple steps and stages. We provide sufficient conditions under which these methods with local truncation error of order $p$ will produce solutions of order $p + 1$, which can be post-processed to order $p + 2$, and describe the construction of one such post-processor. We apply this approach to obtain implicit-explicit (IMEX) methods with multiple steps and stages. We present some of our new IMEX methods and show their linear stability properties, and investigate how these methods perform in practice on some numerical test cases.

1. **Introduction.** Efficient high order numerical methods for evolving the solution of an ordinary differential equation (ODE) are frequently desired, especially for the time-evolution of PDEs that are semi-discretized in space. Higher order methods are desirable as they enable more accurate solutions for larger step-sizes. In this paper we consider numerical solvers for ordinary differential equations (ODEs)

\[
\begin{align*}
\frac{du}{dt} &= F(t,u), \quad t \geq 0 \\
u(t_0) &= u_0.
\end{align*}
\]

(1)

(where $F(t,u)$ is sufficiently smooth). Any non-autonomous system of this form can be converted to an autonomous system (see [15]) so that without loss of generality, we restrict ourselves to the autonomous case:

\[
\begin{align*}
\frac{du}{dt} &= F(u), \quad t \geq 0 \\
u(t_0) &= u_0.
\end{align*}
\]

(2)

The forward Euler method is the simplest numerical method for evolving this problem forward in time with time-steps $\Delta t$:

\[
v_{n+1} = v_n + \Delta t F(v_n),
\]

where $v_n$ approximates the exact solution at time $t^n$ and we let $v_0 = u_0$. For this method, we have

\[
\tau^n = \Delta t L T E^n = u(t_{n-1}) + \Delta t F(t_{n-1}, u(t_{n-1})) - u(t_n) \approx O(\Delta t^2)
\]
where $LTE^n$ is called the local truncation error, and $\tau^n$ the approximation error, at any given time $t_n$. At any such time $t_n$ we are interested in is the \textit{global error} which is the difference between the numerical and exact solution. For the forward Euler method the global error is first order accurate:

$$E_n = v_n - u(t_n) \approx O(\Delta t).$$

Observe that the global error $E^n$ and the local truncation error $LTE^n$ are of the same order, $O(\Delta t)$. This is the common experience that one step methods and linear multi-step methods that are typically used produce solutions that have a global error that is of the same order as the local truncation error.

To generate methods with higher local truncation errors, we can generalize the forward Euler scheme by adding steps (e.g. linear multistep methods), stages (e.g. Runge–Kutta methods), or derivatives (e.g. Taylor series methods). The Dahlquist Equivalence Theorem \cite{dahlquist} states that any zero-stable, consistent linear multistep method with truncation error $O(\Delta t^p)$ will have global error $O(\Delta t^p)$, provided that the solution $u$ has at least $p + 1$ smooth derivatives. This behavior is usually seen in other types of schemes, and is the general expectation: so much so that the order of a numerical method is typically defined solely by the order conditions derived by Taylor series analysis of the local truncation error. However, the Lax-Richtmeyer equivalence theorem (see e.g. \cite{lax}, \cite{richtmeyer}) states that if the numerical scheme is stable then its global error is \textit{at least} of the same order as its local truncation error. Recent work \cite{dikowski1, weiner, ditkowski2, ditkowski3, ditkowski4} highlights that while for a stable scheme the global error must \textit{at least} of the same order as its local truncation error, it may in fact be of higher order.

Schemes that have global errors that are \textit{higher order} than predicted by the local truncation errors were devised by Kulikov \cite{kulikov} and by Weiner and colleagues \cite{weiner} following the quasi-consistency theory first introduced by Skeel in 1978 \cite{skeel1}. These methods have truncation errors that are of order $p$ but have global errors of order $p + 1$. In \cite{ditkowski3} Ditkowski and Gottlieb derived sufficient conditions for a class of general linear methods (GLMs) under which one can control the accumulation of the local truncation error over time evolution and produce methods that are one order higher than expected from the truncation error alone.

Additional conditions were derived in \cite{ditkowski2, ditkowski3, ditkowski4}, that allow the computation of the the precise form of the first surviving term in the global error (the vector multiplying $\Delta t^{p+1}$). In these works, this form was leveraged for error estimation. However, the conditions in \cite{ditkowski2, ditkowski3, ditkowski4}, were overly restrictive and made it difficult to find methods. In \cite{ditkowski5} Ditkowski, Gottlieb, and Grant showed that under less restrictive conditions the form of the first surviving term in the global error can be computed explicitly and they showed how the solution can be post-processed to obtain accuracy of order $p + 2$. In this work, they produced implicit method and explicit methods that have favorable stability properties. The coefficients of those methods can be downloaded from \cite{ditkowski6}. In \cite{ditkowski7} Ditkowski, Gottlieb, and Grant derived sufficient conditions under which two-derivative GLMs of a certain form can produce solutions of order $p + 1$ and can be post-processed to obtain solutions of order $p + 2$. The coefficients of those methods can be downloaded from \cite{ditkowski8}.

In many problems, there is a component that is stiff and easy to invert (e.g. linear) and a component that is non-stiff but difficult to invert (e.g. nonlinear). Such examples include advection-diffusion PDEs, where the diffusion term requires a small time-step when used with an explicit time-stepping method but is linear, and the advection term is nonlinear but allows a relatively large time-step when used with an
explicit time-stepping method. In such cases it is convenient to distinguish these two components and write the resulting autonomous ODE as

\[ u_t = F(u) + G(u), \quad t \geq 0 \]
\[ u(t_0) = u_0. \]

(where we assume, as before, the required smoothness on \( F(u) \) and \( G(u) \)). Here, \( G(u) \) is a stiff but 'simple to invert' operator while \( F(u) \) is non-stiff but complicated (or costly) to invert and typically non-linear.

Implicit-explicit (IMEX) methods were developed to treat the \( F \) component explicitly and the \( G \) component implicitly, thus alleviating the linear stability constraint on the stiff component \( G \) while treating the difficult to invert component \( F \) explicitly. IMEX methods were first introduced by Crouziex in 1980 [4] for evolving parabolic equations. For time dependent PDEs (notably (convection-diffusion equations) Ascher, Ruuth, and Wetton [1] introduced IMEX multi-step methods and Ascher, Ruuth and Spiteri [2] IMEX RungeKutta schemes. Implicit methods are often particularly desirable when applied to a linear component, in which case the order conditions may simplify: such methods were considered by Calvo, Frutos, and Novo in [3]. In 2003, Kennedy and Carpenter [16] derived IMEX RungeKutta methods based on singly diagonally implicit RungeKutta (SDIRK) methods. This work introduced sophisticated IMEX methods with good accuracy and stability properties, as well as high quality embedded methods for error control and other features that make these methods usable in complicated applications. Recently, there has been more interest in IMEX methods with multiple steps and stages, including [14], [26]. Optimized stability regions for IMEX methods with multiple steps and stages have also been of recent interest, as in [30] and [20].

In recent work, Schneider and colleagues [23, 24] produced super-convergent IMEX Peer methods (a Peer method is a GLM where each stage is of the same order). These methods satisfy error inhibiting conditions similar to those in [5] and so produce order \( p + 1 \) although their truncation error is of order \( p \), so we shall refer to them here as IMEX-EIS schemes. In this work, we extend the error inhibiting with post-processing theory in [6] and [7] to IMEX methods. We proceed to devise error inhibiting IMEX methods of up to five steps and order six that have truncation errors or order \( p \) but which can be post-processed to attain a solution of order \( p + 2 \). We refer to these as IMEX-EIS+ methods. We test these methods on a number of numerical examples to demonstrate their enhanced accuracy properties. This paper is structured as follows: in Section 2 we present our notation and some information about our IMEX methods which have multiple steps and stages. In Section 3.1 we provide sufficient conditions under which these methods with local truncation error of order \( p \) will produce solutions of order \( p + 1 \), which can be post-processed to order \( p + 2 \). We provide the construction of one such post-processor in Section 3.2. In Section 4 we present some of our new methods and show their linear stability properties, and in Section 5 we show how these methods perform in practice on some numerical test cases.

2. Multistep multi-stage additive methods. In this work we consider the class of additive general linear methods (GLMs) of the form

\[ V^{n+1} = DV^n + \Delta t \left[ A_F F(V^n) + R_F F(V^{n+1}) + A_G G(V^n) + R_G G(V^{n+1}) \right], \]
where \( V^n \) is a vector of length \( s \) that contains the numerical solution at times \((t_n + c_j \Delta t)\) for \( j = 1, \ldots, s \):

\[
V^n = (v(t_n + c_1 \Delta t)), v(t_n + c_2 \Delta t), \ldots, v(t_n + c_s \Delta t))^T.
\]

The functions \( F(V^n) \) and \( G(V^n) \) are defined as the component-wise function evaluation on the vector \( V^n \):

\[
F(V^n) = (F(v(t_n + c_1 \Delta t)), F(v(t_n + c_2 \Delta t)), \ldots, F(v(t_n + c_s \Delta t))^T.
\]

and

\[
G(V^n) = (FGv(t_n + c_1 \Delta t)), G(v(t_n + c_2 \Delta t)), \ldots, G(v(t_n + c_s \Delta t))^T.
\]

For convenience, we select \( c_1 = 0 \) so that the first element in the vector \( V^n \) approximates the solution at time \( t_n \), and the abscissas are non-decreasing \( c_1 \leq c_2 \leq \ldots \leq c_s \).

To initialize these methods, we define the first element in the initial solution vector \( V_1^0 = u(t_0) \) and the remaining elements \( v(t_0 + c_j \Delta t) \) are computed using a sufficiently accurate method.

Similarly, we define the projection of the exact solution of the ODE (3) onto the temporal grid by:

\[
U^n = (u(t_n + c_1 \Delta t), u(t_n + c_2 \Delta t), \ldots, u(t_n + c_s \Delta t))^T,
\]

with \( F(U^n) \) and \( G(U^n) \) are the component-wise function evaluation on the vector \( U^n \).

We define the global error as the difference between the vectors of the exact and the numerical solutions at some time \( t^n \)

\[
E^n = V^n - U^n.
\]

**Remark 1.** Note that the additive form (4) includes both explicit and implicit schemes, as \( V^{n+1} \) appears on both sides of the equation. However, if \( R_F \) and \( R_G \) are both strictly lower triangular the scheme is explicit. A special case of additive methods is that the method is explicit in \( R_F \) and implicit in \( R_G \): such methods are called implicit-explicit or IMEX methods. In this work we are primarily interested in IMEX schemes so we require that in the form (4) \( R_F \) is strictly lower triangular, and that \( R_G \) contains some diagonal or super-diagonal elements. However, in this section and in Section (3.1) we do not assume this is the case, and so our error inhibiting with post-processing theory is true for all additive methods of the form (4), which is a much more general class than IMEX methods.

### 2.1. Truncation errors.

A method of the form (4) has an approximation error \( \tau^n \) at time \( t^n \)

\[
\tau^n = [DU^n + \Delta t (A_F F(U^{n-1}) + R_F F(U^n)) + A_G G(U^{n-1}) + R_G G(U^n))] - U^n
\]

where

\[
\tau^n = \sum_{j=0}^{\infty} \tau^n_j \Delta t^j = \tau_0 u(t_n) + \sum_{j=1}^{\infty} \Delta t^j \left( \tau_j \frac{d^j u}{dt^j} \bigg|_{t=t_n} + \hat{\tau}_j \frac{d^{j-1} G(u)}{dt^{j-1}} \bigg|_{t=t_n} \right)
\]

\[
F(U^n) = (F(v(t_n + c_1 \Delta t)), F(v(t_n + c_2 \Delta t)), \ldots, F(v(t_n + c_s \Delta t))^T.
\]
where

\[(12a) \quad \tau_0 = (D - I) 1\]
\[(12b) \quad \tau_j = -\frac{1}{(j-1)!} \left( \frac{\partial^j}{\partial t^j} u(t) \bigg|_{t=n} + \hat{\tau}_j \frac{\partial^{j-1} G(u)}{\partial t^{j-1}} \bigg|_{t=t_n} \right) \quad \text{for } j > 0\]
\[(12c) \quad \hat{\tau}_j = \frac{1}{(j-1)!} \left( [A_G - A_F] (c - 1)^{j-1} + (R_G - R_F) c^{j-1} \right) \quad \text{for } j=1,2, ...,\]

We denote the vector of abscissas by \(c = (c_1, c_2, ..., c_s)^T\), and the vector of ones by \(1 = (1,1, ..., 1)^T\). Any terms of the form \(c^j\) are to be understood component-wise \(c^j = (c_1^j, c_2^j, ..., c_s^j)^T\). Note that this notation matches the one used in [20].

Alternatively, observing that \(u_t = F + G\) we can write this as:

\[
\tau^n = \tau_0 u(t_n) + \sum_{j=1}^{\infty} \Delta t^j \left( \tau_j \frac{d^j u}{d t^j} \bigg|_{t=t_n} + \hat{\tau}_j \frac{d^{j-1} G(u)}{d t^{j-1}} \bigg|_{t=t_n} \right)
\]

\[
= \tau_0 u(t_n) + \sum_{j=1}^{\infty} \Delta t^j \left( \tau_j \frac{d^{j-1} u}{d t^{j-1}} \bigg|_{t=t_n} + \hat{\tau}_j \frac{d^{j-1} G(u)}{d t^{j-1}} \bigg|_{t=t_n} \right)
\]

\[
= \tau_0 u(t_n) + \sum_{j=1}^{\infty} \Delta t^j \left( \tau_j \frac{d^{j-1} F(u)}{d t^{j-1}} \bigg|_{t=t_n} + (\tau_j + \hat{\tau}_j) \frac{d^{j-1} G(u)}{d t^{j-1}} \bigg|_{t=t_n} \right)
\]

\[
= \tau_0 u(t_n) + \sum_{j=1}^{\infty} \Delta t^j \left( \tau_j^F \frac{d^{j-1} F(u)}{d t^{j-1}} \bigg|_{t=t_n} + \tau_j^G \frac{d^{j-1} G(u)}{d t^{j-1}} \bigg|_{t=t_n} \right)
\]

where

\[(13a) \quad \tau_0 = (D - I) 1\]
\[(13b) \quad \tau_j^F = -\frac{1}{(j-1)!} \left( \frac{\partial^j}{\partial t^j} u(t) \bigg|_{t=n} + \hat{\tau}_j \frac{\partial^{j-1} G(u)}{\partial t^{j-1}} \bigg|_{t=t_n} \right) \quad \text{for } j > 0\]
\[(13c) \quad \tau_j^G = \frac{1}{(j-1)!} \left( \frac{\partial^j}{\partial t^j} u(t) \bigg|_{t=n} + \hat{\tau}_j \frac{\partial^{j-1} G(u)}{\partial t^{j-1}} \bigg|_{t=t_n} \right) \quad \text{for } j > 0\]

This notation is the same as used in [26].

Regardless of whether we use the notation (12) or (13), for a method to have truncation error of order \(p\) it must satisfy the order conditions

\[(14) \quad \tau_j^F = 0 \quad j \leq p.\]

As this must be true for all \(F\) and \(G\) these become

\[\tau_0 = 0, \quad \text{and} \quad \tau_j = \hat{\tau}_j = 0 \quad \forall \ j = 1, ..., p,\]

or, equivalently,

\[\tau_0 = 0, \quad \text{and} \quad \tau_j^F = \tau_j^G = 0 \quad \forall \ j = 1, ..., p.\]

We are only interested in zero-stable methods. A sufficient condition for this is that the coefficient matrix \(D\) is a rank one matrix that has row sum one: this satisfies the consistency condition

\[\tau_0 = (D - I) 1 = 0.\]

For simplicity we assume this to be the case in this the remainder of this work.
2.2. Preliminaries. In this subsection we make an observation that will be useful in the remainder of the paper.

**Observation 1** Given the smoothness of \( F \) and \( G \), and the assumption that \( \| E^n \| < < 1 \), we observe that

\[
F(U^n + E^n) = F(U^n) + F_y^n E^n + O(\Delta t)O (\| E^n \|),
\]

and

\[
G(U^n + E^n) = G(U^n) + G_y^n E^n + O(\Delta t)O (\| E^n \|),
\]

where \( F_y^n = F_y(u(t_n)) \) and \( G_y^n = G_y(u(t_n)) \).

**Proof.** This is simply due to the fact that \( F \) is smooth enough and \( E^n \) is small enough that we can expand:

\[
F(U^n + E^n) = F(U^n) + \begin{pmatrix}
F_y(u(t_n + c_1 \Delta t))e_{n+c_1} \\
F_y(u(t_n + c_2 \Delta t))e_{n+c_2} \\
\vdots \\
F_y(u(t_n + c_r \Delta t))e_{n+c_r}
\end{pmatrix} + O (\| E^n \|^2),
\]

\[
= F(U^n) + \begin{pmatrix}
F^n_{y+c_1} & 0 & \ldots & 0 \\
0 & F^n_{y+c_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & F^n_{y+c_r}
\end{pmatrix} E^n + O (\| E^n \|^2),
\]

where the error vector is \( E^n = (e_{n+c_1}, e_{n+c_2}, \ldots, e_{n+c_r})^T \), and we use the notation \( F_y^{n+c_j} = F_y(u(t_n + c_j \Delta t)) \). Each term can be expanded as

\[
F_y^{n+c_j} = F_y(u(t_n + c_j \Delta t)) = F_y(u(t_n)) + c_j \Delta t F_y'(u(t_n)) + O(\Delta t^2),
\]

so that, using the assumption that \( \| E^n \| < < 1 \) we have

\[
F(U^n + E^n) = F(U^n) + (F_y^n I + O(\Delta t)) E^n + O (\| E^n \|^2)
\]

\[
= F(U^n) + F_y^n E^n + O(\Delta t)O (\| E^n \|).
\]

The proof for \( G \) is identical. \( \square \)

We now describe the growth of the error:

**Lemma 1.** Given a zero-stable method of the form (4) which satisfies the order conditions

\[
\tau^p_j = 0 \quad \text{for} \quad j = 0, \ldots, p
\]

and where the functions \( F \) and \( G \) are smooth, the evolution of the error can be described by:

\[
(I - \Delta t \left( R_F F_y^n + R_G G_y^n \right) + O(\Delta t^2)) E^{n+1} =
\]

\[
(D + \Delta t \left( A_F F_y^n + A_G G_y^n \right)) E^n + \tau^{n+1}.
\]

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Proof. Subtracting (10) from (4), we obtain
\[ V^{n+1} - U^{n+1} = D V^n - DU^n + \Delta t A_F (F(V^n) - F(U^n)) \\
+ \Delta t A_G (G(V^n) - G(U^n)) + \Delta t R_F (F(V^{n+1}) - F(U^{n+1})) \\
+ \Delta t R_G (G(V^{n+1}) - G(U^{n+1})) + \tau^{n+1} \]
using Observation 1 we have
\[ E^{n+1} = DE^n + \Delta t (A_F F^n y E^n + R_F F^{n+1} y E^{n+1} + A_G G^n y E^n + R_G G^{n+1} y E^{n+1}) \\
+ \tau^{n+1} + O(\Delta t)O(\|E^n\|^2, \|E^{n+1}\|^2). \]
The term final term is very small: \( O(\Delta t)O(\|E^n\|^2, \|E^{n+1}\|^2) \ll \tau^{n+1} \) so we can neglect it. We then have an equation for the error which is essentially linear because the terms \( F^n, G^n, F^{n+1}, \) and \( G^{n+1} \) are all constant scalars, which depend only on the solution of the ODE.

Now move the \( E^{n+1} \) terms to the left side:
\[ (I - \Delta t R_F F_y^{n+1} - \Delta t R_G G_y^{n+1}) E^{n+1} = DE^n + \Delta t A_F F^n y E^n + \Delta A_G G^n y E^n + \tau^{n+1}, \]
Noting that \( F^{n+1} = F^n + O(\Delta t) \) and \( G^{n+1} = G^n + O(\Delta t) \) we have the desired result
\[ (I - \Delta t (R_F F^n y + R_G G^n y) + O(\Delta t^2)) E^{n+1} = \\
(\mathbf{D} + \Delta t (A_F F^n y + A_G G^n y)) E^n + \tau^{n+1}. \]

3. Constructing IMEX-EIS+ methods. Schneider, Lang, and Hundsdorfer [23] showed how to construct IMEX methods of the form (4) which satisfy the order conditions to order \( p \) but are error inhibiting and so produce a solution of order \( p+1 \). In this section we show that under additional conditions we can express the exact form of the next term in error and define an associated post-processor that allow us to recover order \( p+2 \) from a scheme that would otherwise be only \( p \)-th-order accurate. We note that also our main interest is in IMEX methods, and we will focus on these later in the paper, the theory presented in this section is general enough for all additive methods.

3.1. IMEX error inhibiting schemes that can be post-processed. In this section we consider an additive method of the form (4) where we assume that \( \mathbf{D} \) is a rank one matrix that satisfies the consistency condition \( \mathbf{D} \mathbf{1} = \mathbf{1} \) so that this scheme is zero-stable. Furthermore, we assume that the coefficient matrices \( \mathbf{D}, \mathbf{A}_F, \mathbf{R}_F, \mathbf{A}_G, \mathbf{R}_G \) are such that the order conditions are satisfied to order \( p \), so that the method will give us a numerical solution that has error that is guaranteed of order \( p \).

In [23] it was shown that if the truncation error vector \( \tau^{p+1}_n \) lives in the null-space of the operator \( \mathbf{D} \) then the order of the error is of order \( p+1 \). In the following theorem we establish additional conditions on the coefficient matrices \( \mathbf{D}, \mathbf{A}_F, \mathbf{R}_F, \mathbf{A}_G, \mathbf{R}_G \), which allow us to determine precisely what the leading term of this error will look like and therefore remove it by post-processing.

**Theorem 1.** Consider a zero-stable additive general linear method of the form
\[ V^{n+1} = DV^n + \Delta t \left[ A_F F(V^n) + R_F F(V^{n+1}) + A_G G(V^n) + R_G G(V^{n+1}) \right], \]
where \( \mathbf{D} \) is a rank one matrix that satisfies the consistency condition \( \mathbf{D} \mathbf{1} = \mathbf{1} \), the coefficient matrices \( \mathbf{D}, \mathbf{A}_F, \mathbf{R}_F, \mathbf{A}_G, \mathbf{R}_G \), satisfy the order conditions
\[ \tau^s_j = 0 \quad \text{for} \quad j = 1, ..., p, \quad \text{and} \quad n \geq 0 \]
As we showed in Lemma 3 in [72x-308] that we can expand the first term the fact 

\[ F(21) \]

are satisfied, then error vector will have the more precise form:

\[ E(20) \]

large time interval \((0, T)\) is satisfied (so that numerical solution produced by this method will have error \(E^k = O(\Delta t^p+1)\)). If the conditions

\[(19a) \quad \mathbf{D} \tau_{p+1}^n = 0 \]

are satisfied, then error vector will have the more precise form:

\[(19b) \quad \mathbf{D} \tau_{p+2}^n = 0 \]

\[(19c) \quad \mathbf{D}(\mathbf{A}_F + \mathbf{R}_F) \tau_{p+1}^n = 0 \]

\[(19d) \quad \mathbf{D}(\mathbf{A}_G + \mathbf{R}_G) \tau_{p+1}^n = 0 \]

are satisfied, then error vector will have the more precise form:

\[(19e) \quad E^n = \Delta t^{p+1} \tau_{p+1}^n + O(\Delta t^{p+2}). \]

Proof. Using Lemma 1 we obtain the equation for the evolution of the error

\[(20) \quad E^{n+1} = \left[ I - \Delta t \left( \mathbf{R}_F F^n_y + \mathbf{R}_G G^n_y \right) + O(\Delta t^2) \right]^{-1} \left[ \left( \mathbf{D} + \Delta t \left( \mathbf{A}_F F^n_y + \mathbf{A}_G G^n_y \right) \right) E^n + \tau^{n+1} \right]. \]

the fact \(F\) and \(G\) are smooth assures us that \(\|\Delta t (\mathbf{R}_F F^n_y + \mathbf{R}_G G^n_y)\| << O(1)\) so that we can expand the first term

\[
E^{n+1} = \left[ I + \Delta t \left( \mathbf{R}_F F^n_y + \mathbf{R}_G G^n_y \right) + O(\Delta t^2) \right] \left[ \left( \mathbf{D} + \Delta t \left( \mathbf{A}_F F^n_y + \mathbf{A}_G G^n_y \right) \right) E^n + \tau^{n+1} \right] \\
= \left( \mathbf{D} + \Delta t F^n_y (\mathbf{R}_F D + \mathbf{A}_F) + \Delta t G^n_y (\mathbf{R}_G D + \mathbf{A}_G) + O(\Delta t^2) \right) E^n \\
+ \left[ \Delta t^{p+1} \tau_{p+1}^{n+1} + \Delta t^{p+2} (\tau_{p+2}^{n+1} + (\mathbf{R}_F F^n_y + \mathbf{R}_G G^n_y) \tau_{p+1}^{n+1}) + O(\Delta t^2) \right] \\
= Q^n E^n + \Delta t T^n_e. 
\]

As we showed in Lemma 3 in [6], we can assume that \(\|E^n\| << 1\) for some reasonably large time interval \((0, T)\).

The discrete Duhamel principle (given as Lemma 5.1.1 in [12]) states that given an iterative process of this form where \(Q^n\) is a linear operator, we have

\[(22) \quad E^n = \prod_{\mu=0}^{n-1} Q^n E^0 + \Delta t \sum_{\nu=0}^{n-1} \left( \prod_{\mu=\nu+1}^{n-1} Q^n \right) T^n_e. \]

To analyze the error \((22)\), we separate it into four parts:

\[
E^n = \prod_{\mu=0}^{n-1} Q^n E^0 + \Delta t T^{n-1}_I + \Delta t Q^{n-1} T^{n-2}_I + \Delta t \sum_{\nu=0}^{n-1} \left( \prod_{\mu=\nu+1}^{n-3} Q^n \right) T^n_e
\]

and discuss each part separately:

I. The method is initialized so that the numerical solution vector \(V^0\) is accurate enough to ensure that the initial error \(E^0\) is negligible and we can ignore the first term.
II. The final term in the summation is $\Delta t T_{e}^{n-1}$. Recall that
\[ T_{e}^{n-1} = \Delta t^p \tau_{p+1}^n + \Delta t^{p+1} \left( \tau_{p+2}^n + (R_F F_y^{n-1} + R_G G_y^{n-1}) \tau_{p+1}^n \right) + O(\Delta t^{p+2}) \]
so that this term contributes to the final time error the term
\[ \Delta t T_{e}^{n-1} = \Delta t^{p+1} \tau_{p+1}^n + O(\Delta t^{p+2}). \]

III. The term $\Delta t Q^{n-1}T_{e}^{n-2}$ is a product of the operator
\[ Q^{n-1} = (D + \Delta t F_y^{n-1}(R_F D + A_F) + \Delta t G_y^{n-1}(R_G D + A_G) + O(\Delta t^2)) \]
and the approximation error
\[ T_{e}^{n-2} = \Delta t^p \tau_{p+1}^n + \Delta t^{p+1} \left( \tau_{p+2}^n + (R_F F_y^{n-2} + R_G G_y^{n-2}) \tau_{p+1}^n \right) + O(\Delta t^{p+2}) \]
so we obtain
\[ Q^{n-1}T_{e}^{n-2} = \Delta t^p D \tau_{p+1}^n + O(\Delta t^{p+1}) = O(\Delta t^{p+1}), \]
due to the condition (19a) that states that $DT_{e}^{n-1} = 0$.
The result in this theorem requires a closer look at the $O(\Delta t^{p+1})$ terms in this product
\[
(23) \quad D \left[ \tau_{p+1}^{n-1} + (R_F F_y^{n-2} + R_G G_y^{n-2}) \tau_{p+1}^{n-1} \right] \\
+ \left[ F_y^{n-1}(R_F D + A_F) + G_y^{n-1}(R_G D + A_G) \right] \tau_{p+1}^{n-1} + O(\Delta t) \\
= D \left[ \tau_{p+2}^{n-1} + (R_F F_y^{n-2} + R_G G_y^{n-2}) \tau_{p+1}^{n-1} \right] \\
+ \left[ F_y^{n-1} A_F + G_y^{n-1} A_G \right] \tau_{p+1}^{n-1} + O(\Delta t) \\
= D \left[ \tau_{p+2}^{n-1} + (R_F F_y^{n-1} + R_G G_y^{n-1}) \tau_{p+1}^{n-1} \right] \\
+ \left[ F_y^{n-1} A_F + G_y^{n-1} A_G \right] \tau_{p+1}^{n-1} + O(\Delta t) \\
= DT_{n+1}^{n-1} + \left[ F_y^{n-1}(D R_F + A_F) + G_y^{n-1}(D R_G + A_G) \right] \tau_{p+1}^{n-1} + O(\Delta t) \\
\]
where we applied (19a) and used the observation that $F_y^{n-2} = F_y^{n-1} + O(\Delta t)$ and $G_y^{n-2} = G_y^{n-1} + O(\Delta t)$.
We now apply the conditions of the theorem (19b) - (19d), which make the the $O(\Delta t)^p$ terms in (23) vanish and allow us to conclude that:
\[ Q^{n-1}T_{e}^{n-2} = O(\Delta t^{p+2}). \]

IV. Finally we look at the rest of the sum and use the boundedness of the operator $Q^n$ to observe
\[
\left\| \Delta t \sum_{\nu=0}^{n-3} \left( \prod_{\mu=\nu+1}^{n-1} Q^\mu \right) T_{e}^{\nu} \right\| = \left\| \Delta t \sum_{\nu=0}^{n-3} \left( \prod_{\mu=\nu+3}^{n-1} \hat{Q}^\mu \right) \left( \hat{Q}^{\nu+2} \hat{Q}^{\nu+1} T_{e}^{\nu} \right) \right\| \\
\leq \Delta t \sum_{\nu=0}^{n-3} \left\| \prod_{\mu=\nu+3}^{n-1} Q^\mu \right\| \left\| \hat{Q}^{\nu+2} \hat{Q}^{\nu+1} T_{e}^{\nu} \right\| \\
\leq \Delta t \sum_{\nu=0}^{n-3} \left( 1 + c \Delta t \right)^{n-\nu-3} \left\| \hat{Q}^{\nu+2} \hat{Q}^{\nu+1} T_{e}^{\nu} \right\| \\
= \exp \left( \frac{c t_n - 1}{c} \right) \max_{\nu=0,\ldots,n-3} \left\| \hat{Q}^{\nu+2} \hat{Q}^{\nu+1} T_{e}^{\nu} \right\|. 
\]
Clearly, the first term here is a constant that depends only on the final time, so it is the product $Q^{\nu+2}Q^{\nu+1}T_\nu$ we need to bound. Using the definition of the operators $Q^\mu$

$$Q^{\nu+2}Q^{\nu+1} = D^2 + \Delta t \left[ F_y^{\nu+1} (R_F D + A_F) + G_y^{\nu+1} (R_G D + A_G) + F_y^{\nu+2} (R_F D + A_F) D + G_y^{\nu+2} (R_G D + A_G) D \right] + O(\Delta t^2)$$

and the Truncation error

$$T_\nu = [\Delta t \tau_{p+1}^{\nu+1} + \Delta t^{p+1} (\tau_{p+1}^{\nu+1} + (R_F F_y^{\nu} + R_G G_y^{\nu}) \tau_{p+1}^{\nu+1}) + O(\Delta t^{p+2})]$$

we have

$$Q^{\nu+2}Q^{\nu+1}T_\nu = \Delta t^p \left[ D + \Delta t F_y^{\nu+2} (R_F D + A_F) + \Delta t G_y^{\nu+2} (R_G D + A_G) \right] D \tau_{p+1}^{\nu+1}$$

We use the facts that: (1) the form of $D$ means that $D^2 = D$; (2) $F_y^{\nu+2} = F_y^{\nu+1} + O(\Delta t)$, $G_y^{\nu+2} = G_y^{\nu+1} + O(\Delta t)$; and (3) $D \tau_{p+1}^{n} = 0$ for any integer $n$, to obtain

$$Q^{\nu+2}Q^{\nu+1}T_\nu = \Delta t^{p+1} \left[ F_y^{\nu+1} (R_F D + A_F) + G_y^{\nu+1} (R_G D + A_G) \right] \tau_{p+1}^{\nu+1}$$

The last terms disappear because of (19b), $D \tau_{p+1}^{\nu+1} = 0$, $n = 0, 1, \ldots$. The first term is eliminated by (19c) and (19d), namely.

$$D (R_F + A_F) \tau_{p+1}^{\nu+1} = 0, \ n = 0, 1, \ldots$$

and

$$D (R_G + A_G) \tau_{p+1}^{n} = 0, \ n = 0, 1, \ldots$$

So that

$$Q^{\nu+2}Q^{\nu+1}T_\nu = O(\Delta t^{p+2}).$$

All these parts together give us

$$E_n = \prod_{\mu=0}^{n-1} Q^\mu E^0 \prod_{\nu=0}^{n-1} \left[ \frac{\Delta t Q^{\nu-1} T_\nu}{I I} \right] \prod_{\mu=0}^{n-1} \left[ \sum_{\nu=0}^{n-1} \frac{Q^\mu}{I V} \right] T_\nu$$

$$= 0 + \prod_{\nu=0}^{n-1} \frac{\Delta t^{p+1} \tau_{p+1}^{n}}{I I} + O(\Delta t^{p+2}) + O(\Delta t^{p+2})$$

$$= \Delta t^{p+1} \tau_{p+1}^{n} + O(\Delta t^{p+2}).$$

(24)

### 3.1.1. Implementation of the EIS conditions

Unlike the previous cases [6, 7], the terms $\tau_j^n$ here are the sum of two quantities, each of which must be zero. As we saw in Section 2 above, there is some flexibility in these terms, for example we may have

$$\tau_j^n = \left( \tau_j \frac{d^j u}{dt} \bigg|_{t=t_n} + \hat{\tau}_j \frac{d^{j-1} G(u)}{dt^{j-1}} \bigg|_{t=t_n} \right).$$
or
\[ \tau_j^n = \left( \tau_j^F \frac{d^{j-1}F}{dt^{j-1}} \bigg|_{t=t_n} + \tau_j^G \frac{d^{j-1}G(u)}{dt^{j-1}} \bigg|_{t=t_n} \right). \]

The order conditions above
\[ \tau_j^n = 0 \quad \forall j \leq p, \]
become
\[ \tau_j = 0 \quad \text{and} \quad \hat{\tau}_j = 0 \quad \forall j \leq p, \]
or
\[ \tau_j^F = 0 \quad \text{and} \quad \tau_j^G = 0 \quad \forall j \leq p, \]
respectively. It does not matter which formulation one chooses, as these two sets of conditions are equivalent.

However, once we also consider the conditions (19) which use the truncation error vectors \( \tau_{p+1}^n \) and \( \tau_{p+2}^n \), we see a variety of different ways to write these conditions. The straightforward way to write these conditions would be
\[ \tau_j = 0 \quad \text{and} \quad \hat{\tau}_j = 0, \quad \text{for} \quad j = 0, \ldots, p \]
(25)
\[ \begin{align*}
\text{D} \tau_{p+1} = 0, & \quad \text{D} \hat{\tau}_{p+1} = 0, \\
\text{D} \tau_{p+2} = 0, & \quad \text{D} \hat{\tau}_{p+2} = 0, \\
\text{D}(A_F + R_F)\tau_{p+1} = 0, & \quad \text{D}(A_F + R_F)\hat{\tau}_{p+1} = 0, \\
\text{D}(A_G + R_G)\tau_{p+1} = 0, & \quad \text{D}(A_G + R_G)\hat{\tau}_{p+1} = 0.
\end{align*} \]

or equivalently,
\[ \tau_j^F = 0 \quad \text{and} \quad \tau_j^G = 0 \quad \text{for} \quad j = 0, \ldots, p \]
(26)
\[ \begin{align*}
\text{D} \tau_{p+1}^F = 0, & \quad \text{D} \tau_{p+1}^G = 0, \\
\text{D} \tau_{p+2}^F = 0, & \quad \text{D} \tau_{p+2}^G = 0, \\
\text{D}(A_F + R_F)\tau_{p+1}^F = 0, & \quad \text{D}(A_F + R_F)\tau_{p+1}^G = 0, \\
\text{D}(A_G + R_G)\tau_{p+1}^F = 0, & \quad \text{D}(A_G + R_G)\tau_{p+1}^G = 0.
\end{align*} \]

Again, these are mathematically equivalent.

However, we can create conditions that are more constrained than (25) or (26) by requiring higher order conditions to be satisfied for the second component. For example, conditions (25) can be satisfied by requiring that \( \hat{\tau}_{p+1} = 0 \), which reduces the number of conditions:
\[ \tau_j = 0 \quad \text{for} \quad j = 0, \ldots, p, \quad \text{and} \quad \hat{\tau}_j = 0 \quad \text{for} \quad j = 0, \ldots, p + 1 \]
(27)
\[ \begin{align*}
\text{D} \tau_{p+1} = 0, & \quad \text{D} \tau_{p+2} = 0, \quad \text{and} \quad \text{D} \tau_{p+2} = 0, \\
\text{D}(A_F + R_F)\tau_{p+1} = 0, & \quad \text{D}(A_F + R_F)\tau_{p+1} = 0, \\
\text{D}(A_G + R_G)\tau_{p+1} = 0, & \quad \text{D}(A_G + R_G)\tau_{p+1} = 0.
\end{align*} \]

Or even requiring that \( \hat{\tau}_{p+1} = \hat{\tau}_{p+2} = 0 \), which further reduces the number of conditions:
\[ \tau_j = 0 \quad \text{for} \quad j = 0, \ldots, p, \quad \text{and} \quad \hat{\tau}_j = 0 \quad \text{for} \quad j = 0, \ldots, p + 2 \]
(28)
\[ \begin{align*}
\text{D} \tau_{p+1} = 0, & \quad \text{D} \tau_{p+2} = 0, \\
\text{D}(A_F + R_F)\tau_{p+1} = 0, & \quad \text{D}(A_F + R_F)\tau_{p+1} = 0.
\end{align*} \]

We can make other reduced sets of conditions by placing additional requirements on \( \tau_{p+1} \) instead of \( \tau_{p+1} \), etc. Alternatively, we can add similar constraints to the
conditions (26) by requiring \( \tau_{p+1}^G = 0 \) or even \( \tau_{p+1}^G = \tau_{p+2}^G = 0 \) and reducing the number of conditions. However, we stress that these although these forms seem to give simpler conditions, in fact they constrain us further than (25) (or (26)) and actually reduce the size of the possible solution space. This means that while methods that satisfy (27) or (28) will always satisfy (25) (or equivalently (26)), but that methods satisfying (25) (and therefore (26)) may not satisfy (27) or (28). Simply put, the set of conditions (25), or equivalently (26)), allows for the broadest selection of methods.

3.2. Designing a post-processor to recover \( p + 2 \) order. The form of the error vector (20)

\[
E^n = \Delta t^{p+1} \tau_p^{n+1} + O(\Delta t^{p+2}),
\]

at any time-step \( t_n \), allows us to remove the leading order term \( \Delta t^{p+1} \tau_p^{n+1} \) at the end of the computation. This produces a final time solution of order \( p + 2 \).

We observe that the form of the error for IMEX method (4) is exactly the same as that of the error of the EIS methods presented in [6] and [7], the only difference being the definition of the truncation error vector, and the number of truncation error vectors. Thus, we refer the reader to [6] for a theoretical discussion of the post-processor. In this subsection, we review one possible construction of a post-processor.

1. Select the number of computation steps \( m \) that will be used. For additive methods we typically have two truncation error vectors and so require that

\[
ms \geq p + 4
\]

where \( s \) is the number of steps and \( p \) the order of the time-stepping method. However, in some cases the two truncation error vectors are linearly dependent and so we only use one of them. In this case we require that

\[
ms \geq p + 3.
\]

2. Define the time vector of all the temporal grid points in the last \( m \) computation steps:

\[
\tilde{t} = (t_{n-m+1} + c_1 \Delta t, ..., t_{n-m+1} + c_s \Delta t, ..., t_n + c_1 \Delta t, ..., t_n + c_s \Delta t)^T
\]

3. Correspondingly, concatenate the final \( m \) solution vectors

\[
\tilde{V}^n = \begin{pmatrix}
V_{n-m+1} \\
\vdots \\
V^n
\end{pmatrix}, \quad \text{and} \quad \tilde{U}^n = \begin{pmatrix}
U_{n-m+1} \\
\vdots \\
U^n
\end{pmatrix}.
\]

4. Stack \( m \) copies of the final truncation error vectors \( \tau_{p+1}^F \) and \( \tau_{p+1}^G \)

\[
\tilde{\tau}^F = \left( \tau_{p+1}^F \right)_{m}^{T}, ..., \left( \tau_{p+1}^F \right)_{m}^{T} \quad \text{and} \quad \tilde{\tau}^G = \left( \tau_{p+1}^G \right)_{m}^{T}, ..., \left( \tau_{p+1}^G \right)_{m}^{T}
\]

5. Define the vertically flipped Vandermonde interpolation matrix \( T \) on the vector \( \tilde{t} \), and replace the first two columns (the ones corresponding to the highest polynomial term) by \( \tilde{\tau}^F \) and \( \tilde{\tau}^G \):

\[
T = (\tilde{\tau}^F, \tilde{\tau}^G, \tilde{\tau}_{ms-2}, \tilde{\tau}_{ms-3}, ..., \tilde{\tau}^2, \tilde{t}, \tilde{\tau})
\]
where terms of the form $\tilde{t}^q$ are understood as component-wise exponentiation. If the two truncation error vectors are linearly dependent, we use only one of them, so the matrix $T$ would be defined as

$$T = (\tilde{\tau}^F, \tilde{t}^{ms-1}, \tilde{t}^{ms-2}, ..., \tilde{t}^2, \tilde{t}, 1).$$

Note that in either case $T$ is a square matrix of dimension $ms \times ms$.

6. Define the post-processing filter

$$\Phi = T \text{diag} \left( 0, 0, 1, ..., 1 \right) T^{-1}.$$  

If two truncation error vectors are linearly dependent, we use instead

$$\Phi = T \text{diag} \left( 0, 1, ..., 1 \right) T^{-1}.$$  

Finally, left-multiply the solution vector $\tilde{V}^n$ by the post-processing filter $\Phi$ to obtain the post-processed solution

$$\hat{V}^n = \Phi \tilde{V}^n,$$

which will be of order $p+2$. As in [6] we remark that this process may break down if the matrix $T$ is not invertible, and that numerical instabilities may result if $\|\Phi\|$ is large. This should be verified while building these matrices. In the examples we produced, we pre-computed the post-processing matrix and present the post-processing procedure along with the numerical methods. The post-processor can be downloaded along with the methods from our GitHub repository [10].

4. Error inhibiting IMEX schemes with post-processing. Using the truncation error conditions and the error inhibiting conditions in Section 3.1, we construct new additive methods that are error inhibiting and admit postprocessing to raise the order (after post-processing) to $P = p + 2$ where $p$ is the order predicted by truncation error analysis. We are interested in the stability regions of these new methods: to study these we look at the explicit and implicit methods separately. In other words, we consider separately the linear stability regions when $G$ is zero (the explicit case) and when $F$ is zero (the implicit case).

We denote s-step methods of the form (4) that satisfy the order conditions

$$\tau^n_j = 0 \quad \text{for} \quad j = 0, ..., p,$$

and the EIS conditions (19) by the notation IMEX-EIS+(s,P) where $P = p + 2$. We focus on IMEX methods, so we require $R_F$ to be strictly lower triangular. We also restrict ourselves to the diagonally implicit cases so we require $R_G$ to be lower triangular. If the methods have $R_G$ and $R_F$ that have only diagonal elements (i.e. $R_F$ is the zero matrix and $R_G$ is a diagonal matrix) then the method can be implemented efficiently in parallel, and to highlight this we denote the method pIMEX-EIS+(s,P) (where $P = p + 2$).

All of our methods have an $A$-stable implicit part (i.e. when $F = 0$). For the explicit part (i.e. when $G = 0$), we measure the radius $R_{\text{stab}}$ of the semicircle in the
Table 1
The linear stability radius of the explicit component of our IMEX methods.

| Method            | \( R_{stab} \) |
|-------------------|----------------|
| pIMEX-EIS(2,2)    | 0.73           |
| pIMEX-EIS(2,3)    | 0.63           |
| pIMEX-EIS+(3,3)   | 0.70           |
| pIMEX-EIS+(3,4)   | 0.41           |
| pIMEX-EIS+(4,5)   | 0.45           |
| IMEX-EIS+(2,2)    | 1.42           |
| IMEX-EIS(2,3)     | 1.42           |
| IMEX-EIS+(3,3)    | 2.26           |
| IMEX-EIS+(3,4)    | 2.12           |
| IMEX-EIS+(4,5)    | 1.62           |
| IMEX-EIS+(5,6)    | 1.46           |

left half plane in which this method is linearly stable. Table 1 lists the methods we found and shows the linear stability radius for which the explicit part is stable. We note that some of the methods converged to the case where \( \tau_n^j = 0 \) for \( j = 0, \ldots, p+1 \), in which case the conditions (19) reduce to \( D \tau_{p+2}^n = 0 \). This is a method which is in the previous class considered by [23] and does not require post-processing. We denote such methods by IMEX-EIS\((s,p+1)\), and if they are also parallel efficient we denote them by pIMEX-EIS\((s,p+1)\). As shown in Table 1, we found IMEX-EIS+ methods of up to five stages and order \( P = p + 2 = 6 \). Note that the methods that do not require the parallel (i.e. diagonal \( R_F \) and \( R_G \)) structure allow a significantly larger linear stability radius for the explicit part.

We are interested in comparing the linear stability regions of the IMEX-EIS+ methods we found here to the IMEX methods in [23]. The comparison is limited by several factors: first, the methods in [23] are not only diagonally implicit, but singly diagonally implicit (i.e. the lower triangular matrix \( R_G \) has a diagonal that is all the same number), which makes them different from ours; second, we were not able to find an IMEX-EIS+(2,3) method, so we compare our IMEX-EIS(2,3) to the corresponding (but singly diagonally implicit) method in [23]; finally, the (4,5) method in [23] was obtained by loosening the conditions on the matrix \( D \) and requiring only that \( D^k \tau_n^5 = 0 \) for some value \( k \), so this does not exactly satisfy the condition (19a), but still gives a fifth order results. Subject to all these caveats in the comparison of the methods, we consider the stability regions in Figure 1 and observe that our methods have larger imaginary axis stability for the same value of \( s, P \).

![Fig. 1](image)

Due to space constraints we do not list all the coefficients of all the methods. However, we list the coefficients of three methods of interest (IMEX-EIS+(3,4), pIMEX-EIS+(4,5) and IMEX-EIS+(5,6)) in the appendix, and all the coefficients can be downloaded from [10].
5. Numerical Results. In this section we test the IMEX methods developed above for convergence on a well-studied nonlinear system of ODEs, and on a nonlinear ODE system coming from the semi-discretization of a PDE. The numerical results confirm that we observe the predicted order (or better) from our methods, both before and after post-processing.

Example 1: Van der Pol oscillator problem The nonlinear system of ODEs is given by

\[
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix}' = \begin{pmatrix}
  y_2 \\
  a(1 - y_1^2)y_2 - y_1
\end{pmatrix}
\]

with \( a = 2 \) and initial condition \( y(0) = (2, 0)^T \). We split the right-hand-side into

\[
y' = F(y) + G(y)
\]

where:

\[
F = \begin{pmatrix}
  0 \\
  a(1 - y_1^2)y_2
\end{pmatrix}, \quad \text{and} \quad G = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix}.
\]

In the following, \( F \) is treated explicitly and \( G \) is implicit. We use a selection of parallel-efficient and non-parallel-efficient IMEX-EIS+ methods with a range of stepsizes \( \Delta t = \frac{T_f}{N} \) for different values of \( N \) to evolve this problem to the final time \( T_f = 3.0 \). We then postprocess the solution at the final time as described in Section 3.2. We compute the reference solution at the final time by Matlab’s \texttt{ode45} routine, and subtract them from our numerical results at this final time, the errors are then the sum of squares of the two errors, divided by \( \sqrt{2} \).

In Table 2 we show the convergence rates, calculated. For the methods that are third through fifth order we use a value of \( N \) between 400 and 1200; for the sixth order method, we must use smaller values of \( N \) as larger values give machine-accuracy solutions. For the IMEX-EIS+(3,3), IMEX-EIS+(3,4), and IMEX-EIS+(4,5) and pIMEX-EIS+(3,3), pIMEX-EIS+(3,4), and pIMEX-EIS+(4,5) methods we see the expected order of accuracy before \( (p + 1) \) and after \( (p + 2) \) post-processing. The IMEX-EIS+(5,6) method, on the other hand, shows sixth order both before and after post-processing for both the numerical examples we ran, so that the expected order before post-processing is higher than expected but the overall order is as expected.

In Figure 2 we show the errors for different values of \( \Delta t \), with \( \log_{10}(\text{errors}) \) graphed against \( \log_{10}(\Delta t) \). We show the the non-parallelizable methods on the right and the parallelizable methods on the left. The solid lines represent the errors before post-processing and the dashed ones are the errors after post-processing. This figure shows that these solutions attain the expected order of convergence before and after post-processing. We omit the results from the IMEX-EIS+(5,6) method from this graph due to the different values of \( N \) used for this example.

| \( N \)       | parallelizable methods | non-parallelizable methods |
|--------------|------------------------|---------------------------|
|              | method before after    | method before after       |
| 400:1200     | pIMEX-EIS+(3,3) 2.20    | IMEX-EIS+(3,3) 2.08        |
| 400:1200     | pIMEX-EIS+(3,4) 3.05    | IMEX-EIS+(3,4) 3.05        |
| 400:1200     | pIMEX-EIS+(4,5) 3.90    | IMEX-EIS+(4,5) 3.82        |
| 50:800       |                        | IMEX-EIS+(5,6) 6.02        |
Fig. 2. Example 1: Convergence of a variety of methods for the Van der Pol oscillator problem. On the x-axis is $\log_{10}(\Delta t)$ and on the y-axis $\log_{10}(\text{error})$. Solid line is before post-processing and dotted line after post-processing. The slopes of the lines are given in Table 2. Left: parallelizable methods pIMEX-EIS+(3,3) in blue, pIMEX-EIS+(3,4) in red, and pIMEX-EIS+(4,5) in green. Right: non-parallelizable methods IMEX-EIS+(3,3) in blue, IMEX-EIS+(3,4) in red, and IMEX-EIS+(4,5) in green.

Example 2: Viscous Burgers’ equation We solve the PDE

$$u_t + \left( \frac{1}{2} u^2 \right)_x = \frac{1}{10} u_{xx}$$

on the domain $x \in [0, 2\pi]$ with initial condition $u(0) = \sin(5x) + \cos(2x)$ and periodic boundary conditions. We semidiscretize this problem using a Fourier spectral collocation method with 41 points in space. We treat the Burgers’ term $\left( \frac{1}{2} u^2 \right)_x$ explicitly and the viscous term $\frac{1}{10} u_{xx}$ implicitly. Using a selection of the IMEX-EIS+ methods we developed, we evolve this in time to $T_f = 0.5$, for $\Delta t = \frac{T_f}{N}$ for various values of $N$.

In Table 3 we show the convergence rates of the IMEX-EIS+ methods and the pIMEX-EIS+ methods (3,3), (3,4), and (4,5) for values of $N$ from 210 to 1440, and the IMEX-EIS+(5,6) method for $N$ from 60 to 280 (due to the high accuracy of the solutions). Convergence plots shown in Figure 3. For most of the third through fifth order methods we observe a convergence rate of $p + 1$ before post-processing and $P = p + 2$ after post-processing. For the IMEX-EIS+(4,5) the order before post-processing is higher than expected, at fifth order (4.67), while the order after post-processing is improved but also, as expected, fifth order (4.90). For the sixth order method we observe a higher-than-expected convergence rate of sixth order (5.69) before post-processing, and the expected convergence rate of sixth order (also 5.69) after post-processing. Table 3 confirms that our methods perform as expected or better than expected for the numerical tests.

**Table 3**

Example 2: Convergence slopes before and after post-processing

| $N$    | parallelizable methods | non-parallelizable methods |
|--------|------------------------|----------------------------|
|        | method                 | before | after  | method                  | before | after  |
| 210:1440 | pIMEX-EIS+(3,3)        | 1.90   | 2.96   | IMEX-EIS+(3,3)          | 1.97   | 2.92   |
| 210:1440 | pIMEX-EIS+(3,4)        | 3.21   | 3.97   | IMEX-EIS+(3,4)          | 2.99   | 4.00   |
| 210:1440 | pIMEX-EIS+(4,5)        | 4.04   | 4.86   | IMEX-EIS+(4,5)          | 4.67   | 4.90   |
| 60:280  | IMEX-EIS+(5,6)         | 5.69   | 5.69   | IMEX-EIS+(5,6)          | 5.69   | 5.69   |
6. Conclusions. It has been previously shown in [23] that IMEX methods of the form (4) with truncation error of order \( p \) can attain order \( p + 1 \) if the coefficients satisfy the condition (19a). In this work we showed how, under additional conditions on the coefficients of (4), it is possible to devise a post-processor that can extract a solution of order \( p + 2 \).

In Section 3.1 we proved that if a method of the form (4) with truncation error of order \( p \) satisfies the set of conditions (19) then the growth of the errors will be inhibited so that at any time \( t^n \) the error has the form

\[
E^n = \tau^n \Delta t^{p+1} + O(\Delta t^{p+2}).
\]

In [6] we showed that given an error of this form we can construct a post-processor that allows us to remove this first term of the error and obtain a solution that has error

\[
\hat{e}^n = O(\Delta t^{p+2}).
\]

In Section 3.2 we detail the construction of one such post-processor. We proceed to find method that satisfy the order conditions and error inhibiting conditions and the post-processor for that method. We list the coefficients of two such methods and their post-processor in Section 4 and provide coefficients of other methods in our GitHub repository [10]. Our methods are designed to be A-stable for the implicit part \( G = 0 \) and have a good stability regions for the explicit part \( F \), and we show the radius of the semicircle in the complex plane that includes the linear stability region of the explicit part. Finally, in Section 5 we test some of our methods on a nonlinear system of ODEs and on a nonlinear PDE and show that the order of convergence before and after post-processing is as expected.

Appendix A. The coefficients of few IMEX-EIS+ methods. In this appendix we provide the coefficients of three EIS+ methods: the non-parallelizable IMEX-EIS+(3,4) and IMEX-EIS+(5,6) methods and the pIMEX-EIS+(4,5) method which is parallelizable. We also provide the details of a post-processor for each of these, and their linear stability regions for the explicit case \( G = 0 \). Note that these methods are all A-stable for the implicit case \( F = 0 \).
Non-parallelizable IMEX-EIS+(3,4) This method has the form for each stage $i$

\[
v_{n+1+c_i} = \sum_{j=1}^{i} D_{ij} v_{n+c_j} + \Delta t \left( (A_F)_{ij} F(v_{n+c_j}) + (A_G)_{ij} G(v_{n+c_j}) \right)
+ \Delta t \left[ \sum_{j=1}^{i-1} (R_F)_{ij} F(v_{n+1+c_j}) + \sum_{j=1}^{i} (R_G)_{ij} G(v_{n+1+c_j}) \right],
\]

where $c_1 = 0$, $c_2 = 0.726140175537503$, $c_3 = 0.673358282778651$.

The coefficients in $D$ are

\[
D_{ij} = \begin{cases}
0.669589009596231, & i = j = 1 \\
0.324903855316460, & i = j = 2 \\
0.113098975835751, & i = j = 3
\end{cases}
\]

The coefficient matrices $A_F$ and $A_G$ are

\[
A_F = \begin{pmatrix}
0.114204309138172 & -0.00390083432031 & 1.07955287341509 \\
0.46138154216379 & 1.84520907440007 & -2.681606546815293 \\
0.354966311057433 & 1.044611661302771 & -1.341592157784282
\end{pmatrix}
\]

and

\[
A_G = \begin{pmatrix}
0.284198645406530 & -0.015257351367544 & 0.236227411970908 \\
0.324903855316460 & -0.362534474009427 & 0.207162116346608 \\
0.095855552702204 & 0.715560227998031 & 0.177838308334027
\end{pmatrix}
\]

The final coefficients are

\[
R_F = \begin{pmatrix}
0 & 0 & 0 \\
1.891771006717059 & 0 & 0 \\
1.309753253604631 & 0.099260727618746 & 0
\end{pmatrix}
\]

and

\[
R_G = \begin{pmatrix}
0.288202807010756 & 0 & 0 \\
1.07901350783908 & 0.27507840122604 & 0 \\
0.113098975835751 & -0.492120079122587 & 0.856527688304053
\end{pmatrix}
\]

The truncation error vectors for post-processing are

\[
\tau_F^3 = \begin{pmatrix}
-0.029109337573875 \\
-0.039680299841934 \\
0.012001277545145
\end{pmatrix},
\tau_G^3 = \begin{pmatrix}
0.079790724801134 \\
0.108766469751468 \\
-0.032896338895454
\end{pmatrix}
\]

We note that $\frac{\tau_F^3}{\tau_G^3} = -0.36482106969733$ so that the two are linearly dependent. Using this form we construct the post-processor and find that to obtain a solution $\hat{v}^n$ that is fifth order we use

\[
\hat{v}^n = \sum_{j=1}^{3} w_j v_{n-1+c_j} + \sum_{j=1}^{3} w_{3+j} v_{n+c_j},
\]

where

\[
w_1 = -0.005813528106374, \quad w_2 = -0.825824388871650, \quad w_3 = 0.671784878748904, \quad w_4 = 1.187717516309380, \quad w_5 = 0.117883101641288, \quad w_6 = -0.145747579721548.
\]
Parallelizable pIMEX-EIS+(4,5) Each stage $i$ of this method can be written as

$$v_{n+1+c_i} = \sum_{j=1}^{i} [D_{ij}v_{n+c_j} + \Delta t ((A_F)_{ij} F(v_{n+c_j}) + (A_G)_{ij} G(v_{n+c_j}))] + \Delta t (R_G)_{ij}G(v_{n+1+c_i}),$$

where $c_1 = 0$ and

$$c_2 = 0.168033239597551, \quad c_3 = 1.757182407781971, \quad c_4 = 1.859454471327513.$$

The matrix $D$ has coefficients

$$D_{1j} = -0.318365990733397, \quad D_{2j} = 1.304472100371239,$$

$$D_{3j} = 0.549931869327788, \quad D_{4j} = -0.536037978965630.$$

The coefficients in $A_F$ are:

$$
\begin{pmatrix}
-1.664522119422666 & 2.437573230962123 & -0.769668590642686 & 0.807830422310789 \\
-0.78168985324564 & 1.397193436278877 & 1.659473775700052 & -1.295731181519254 \\
1.321744800130581 & -1.02276965721561 & 1.83547792707761 & 0.433936718202950 \\
1.792224287686993 & -1.556690154187516 & 1.162924903269568 & 1.272208371916028
\end{pmatrix}
$$

and the coefficients in $A_G$ are:

$$
\begin{pmatrix}
5.130504311291359 & -6.868827443719447 & -6.722550006878589 & 4.94972109038540 \\
1.365036148735676 & -1.731952546490924 & -8.799998237141496 & 6.717460091357383 \\
-4.040734278322292 & 5.102367660896668 & 8.373021332707967 & -8.044233252055056 \\
-4.19539468061772 & 8.899796721907132 & 8.79999823663552 & -8.48672201934611
\end{pmatrix}
$$

The matrix $R_F$ has all zeros, and the matrix $R_G$ has only diagonal elements:

$$(R_G)_{11} = 4.322293969405709, \quad (R_G)_{22} = 3.428700720653071,$$

$$(R_G)_{33} = 1.177973876898242, \quad (R_G)_{44} = 1.217134341860772.$$

For this method, the truncation error vectors are

$$\tau_4^F = \begin{pmatrix} 0.488267196647527 \\ -0.076569016719893 \\ -1.995223807311692 \\ -2.523266318943553 \end{pmatrix}, \quad \tau_4^G = \begin{pmatrix} -0.902269383509413 \\ 0.141491953557654 \\ 3.686974503536061 \\ 4.662746057189559 \end{pmatrix},$$

and we observe that $\tau_4^F = (-0.541154565999338)\tau_4^G$. These truncation error vectors are linearly dependent so we only need to use one in the post-processing.

Using this form we construct the post-processor and find that to obtain a solution $\hat{v}^n$ that is fifth order we use

$$\hat{v}^n = \sum_{j=1}^{4} w_j v_{n-1+c_j} + \sum_{j=1}^{4} w_{4+j} v_{n+c_j},$$

where

$$w_1 = -0.039322995751032, \quad w_2 = 0.075926208780666, \quad w_3 = -1.415777364482847, \quad w_4 = 1.158626364485013,$$

$$w_5 = 0.331161725962668, \quad w_6 = 0.925152344959055, \quad w_7 = -0.108628113639943, \quad w_8 = 0.072861829686421.$$
These methods are both A-stable for the implicit case $F = 0$, and in Figure 4 we show the linear stability regions for the explicit case $G = 0$. Clearly, the non-parallelizable IMEX-EIS+($3,4$) has a much larger explicit linear stability region than the parallelizable IMEX-EIS+($4,5$) method.

**Non-parallelizable IMEX-EIS+($5,6$)** The coefficient matrices for this method are only given to the first four decimal places due to space constrains: please download the correct values from [10] when using these in practice.

$$D_{1j} = 0.0659, \quad D_{2j} = -0.1352, \quad D_{3j} = 0.2825, \quad D_{4j} = 0.5511, \quad D_{5j} = 0.2355.$$
The truncation error vectors are linearly independent:

\[
\begin{pmatrix}
-0.013938778257283 \\
-0.02923291698827 \\
-0.030789029777504 \\
0.012120485455712 \\
-0.004300675470610
\end{pmatrix},
\quad
\begin{pmatrix}
-0.005170695208015 \\
0.158776476412505 \\
0.157237715751941 \\
-0.023750671362163 \\
-0.040451707588424
\end{pmatrix}.
\]

Using this form we construct the post-processor and find that to obtain a solution \( \hat{v}^n \) that is sixth order we use

\[
\hat{v}^n = \sum_{j=1}^{5} w_j v_{n-1+c_j} + \sum_{j=1}^{5} w_{5+j} v_{n+c_j},
\]

where

\[
\begin{align*}
    w_1 &= 0.199430122528852, &
    w_2 &= 0.527176733379556, &
    w_3 &= 1.849342116119140, \\
    w_4 &= -1.990042072366473, &
    w_5 &= -0.236068245820308, &
    w_6 &= -8.50034072029671, \\
    w_7 &= -0.444089994252464, &
    w_8 &= -0.10579856591708, &
    w_9 &= 0.318463049906237, \\
    w_{10} &= 9.381927579036839.
\end{align*}
\]

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