On the Isomorphism Problem for Helly Circular-Arc Graphs

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Abstract

The isomorphism problem is known to be efficiently solvable for interval graphs, while for the larger class of circular-arc graphs its complexity status stays open. We consider the intermediate class of intersection graphs for families of circular arcs that satisfy the Helly property. We solve the isomorphism problem for this class in logarithmic space. If an input graph has a Helly circular-arc model, our algorithm constructs it canonically, which means that the models constructed for isomorphic graphs are equal.

1 Introduction

An intersection representation of a graph $G$ is a mapping $\alpha$ of the vertex set $V(G)$ onto a family $\mathcal{A}$ of sets such that vertices $u$ and $v$ of $G$ are adjacent if and only if the sets $\alpha(u)$ and $\alpha(v)$ have a non-empty intersection. The family $\mathcal{A}$ is called an intersection model of $G$. $G$ is an interval graph if it admits an intersection model consisting of intervals of reals (or, equivalently, intervals of consecutive integers). The larger class of circular-arc (CA) graphs arises if we consider intersection models consisting of arcs on a circle. These two archetypal classes of intersection graphs have important applications, most noticeably in computational genomics, and have been intensively studied for decades in graph theory and algorithmics; for an overview see e.g. [17]. In general, fixing a class of admissible intersection models, we obtain the corresponding class of intersection graphs.

In the canonical representation problem for a class $\mathcal{C}$ of intersection graphs, we are given a graph $G \in \mathcal{C}$ and have to compute its intersection representation $\alpha$ so that isomorphic graphs receive equal intersection models. This subsumes both recognition of $\mathcal{C}$ and isomorphism testing for graphs in $\mathcal{C}$. In their seminal work [1]...
Booth and Lueker solve both the representation and the isomorphism problems for interval graphs in linear time. Together with Laubner, we designed a canonical representation algorithm for interval graphs that takes logarithmic space [9].

The case of CA graphs remains a challenge up to now. While a circular-arc intersection model can be constructed in linear time (McConnell [15]), no polynomial-time isomorphism test for CA graphs is currently known (though some approaches [7] have appeared in the literature; see the discussion in [4]). A few natural subclasses of CA graphs have received special attention among researchers. In particular, for proper CA graphs both the recognition and the isomorphism problems are solved in linear time, respectively, in [13] and in [4], and in logarithmic space in [10]. The history of the isomorphism problem for circular-arc graphs is surveyed in more detail by Uehara [20].

Here we are interested in the class of Helly circular-arc (HCA) graphs. Those are graphs that admit circular-arc models having the Helly property, which requires that every family of arcs with non-empty pairwise intersections has a non-empty overall intersection. Obeying this property is assumed in the representation problem for HCA graphs. Since any family of intervals has the Helly property, the canonical representation problem for HCA graphs generalizes the canonical representation problem for interval graphs. On the other hand, not every CA model is Helly; see Fig. 1 for examples. Joeris et al. characterize HCA graphs among CA graphs by a family of forbidden induced subgraphs [8].

HCA graphs were introduced by Gavril under the name of \( \Theta \) circular-arc graphs [6]. Gavril gave an \( O(n^3) \) time representation algorithm for HCA graphs. Hsu improved this to \( O(nm) \) [7]. Recently, Joeris et al. gave a linear time representation algorithm [8]. The fastest known isomorphism algorithm for HCA graphs is due to Curtis et al. and works in linear time [4]. Chen gave a parallel AC\(^2\) algorithm [2].

We aim at designing space efficient algorithms. In [12] we already presented a logspace canonical representation algorithm for HCA graphs. Our approach in [12] uses techniques developed by McConnell in [15], and the algorithm is rather intricate. Now we suggest an alternative approach that is independent of [15]. The new algorithm admits a much simpler analysis and exploits some new ideas that may be of independent interest.

**Theorem 1.1.** The canonical representation problem for the class of Helly circular-arc graphs is solvable in logspace.

Figure 1: Two non-Helly CA models and their intersection graphs. The graph in (a) admits an HCA model (even an interval model), while the graph in (b) does not.
Note that solvability in logspace implies solvability in logarithmic time by a CRCW PRAM with polynomially many parallel processors, i.e., in $\mathbf{AC}^1$. Prior to our work, no $\mathbf{AC}^1$ algorithm was known for recognition and isomorphism testing of HCA graphs.

In general, solvability of the isomorphism problem for a non-trivial class of graphs in logarithmic space is an interesting result because the general graph isomorphism problem is known to be $\mathbf{DET}$-hard \[^{18}\] and, therefore, $\mathbf{NL}$-hard. It is also interesting that for some classes of intersection graphs, the isomorphism problem is as hard as in general. For example, Uehara \[^{19}\] shows this for intersection graphs of axis-parallel rectangles in the plane. Note that any family of such rectangles has the Helly property.

Our strategy. Recall that a hypergraph $\mathcal{H}$ is interval (resp. circular-arc) if it is isomorphic to a hypergraph whose hyperedges are intervals of integers (resp. arcs of a discrete circle). Such an isomorphism is called an interval (resp. CA) representation of $\mathcal{H}$. The overall idea of our algorithm is, like in our approach to interval graphs in \[^{9}\], to exploit the relationship between an input graph $G$ and the dual of its maxclique hypergraph, which will be denoted by $\mathcal{B}(G)$. Fulkerson and Gross \[^{5}\] established that $G$ is an interval graph iff $\mathcal{B}(G)$ is an interval hypergraph. Moreover, represented as an interval system, $\mathcal{B}(G)$ can serve as an intersection model of $G$. Our approach in \[^{9}\] consists, therefore, of two steps: first, construct $\mathcal{B}(G)$ (or, equivalently, find all maxcliques in $G$) and, second, design a canonical representation algorithm for interval hypergraphs and apply it to $\mathcal{B}(G)$. The first step is implementable in logspace because all connected interval graphs are maximal clique irreducible, which means that every maxclique $C$ contains an edge $uv$ that is contained in no other maxclique and, therefore, $C$ is equal to the common neighborhood of $u$ and $v$.

The Fulkerson-Gross theorem is extended to the class of HCA graphs by Gavril \[^{6}\]: $G$ is a HCA graph iff $\mathcal{B}(G)$ is a CA hypergraph. Also in this case, $\mathcal{B}(G)$ can serve as an isomorphic image of an intersection model for $G$. The canonical representation problem for CA hypergraphs is solved in logspace in \[^{10}\]. However, the similarity between interval and HCA graphs ends there because HCA graphs are in general not maximal clique irreducible.

Though we are not able to find all maxcliques of an HCA graph $G$ directly, the discussion above shows that the canonical representation problem for HCA graphs is logspace reducible to the representation problem, where we need just to construct an HCA representation and do not need to take care of canonicity. Indeed, once we have an arbitrary HCA model of an input graph $G$, we get all maxcliques of $G$ by inspection of the sets of arcs sharing a common point. After all the maxcliques are found, we form the hypergraph $\mathcal{B}(G)$ and compute its canonical representation according to \[^{10}\] (the details are given in Section \[^{4}\]).

It remains to explain how we compute an HCA representation $\alpha$ of $G$. It is handy to assume that, if $G$ has $n$ vertices, then its HCA model $\alpha(G)$ has $2n$ points, and that no arc in $\alpha(G)$ shares extreme points with others. Given $C \subset V(G)$, let $\alpha^C(G)$ denote the arc system obtained from $\alpha(G)$ by flipping the arc $\alpha(v)$ for all $v \in C$,
that is, by replacing \( \alpha(v) \) with the other arc on the same circle that has the same extreme points. We make use of a simple consequence of the Helly property: If \( C \) is a maxclique, then \( \alpha^C(G) \) becomes an interval system. As was said, we cannot find all maxcliques of \( G \) at once. However, we are able to find one of them, which will be used for the flipping operation. Our next goal is to compute the interval system \( \alpha^C(G) \) up to isomorphism. Once this is done, we obtain the desired \( \alpha \) (or its isomorphic version) by performing the \( C \)-flipping for \( \alpha^C(G) \) (note that \( (\alpha^C)^C = \alpha \)). The flipping operation is considered in detail in Section 6.

The interval system \( \alpha^C(G) \) is constructed as follows. In Section 5 we argue that we always can suppose that \( \alpha \) has an additional property: If two arcs intersect and cover the whole circle, then each of the arcs contains both extreme points of the other. Under this assumption we are able to compute the pairwise-intersection matrix \( M_\alpha = (m_{uv}) \), defined by \( m_{uv} = |\alpha(u) \cap \alpha(v)| \), and then also the pairwise-intersection matrix \( M_{\alpha^C} \) for the interval system \( \alpha^C(G) \). Afterwards we use another result of Fulkerson and Gross saying that an interval system is determined by its pairwise-intersection matrix up to isomorphism. Moreover, it can be reconstructed from the pairwise-intersection matrix in logspace by an algorithm worked out in [11]; see Section 7.

The pairwise-intersection matrix \( M_\alpha \) is computed in Section 8. The computation is based on the fact that any arc model \( \alpha(G) \) is, in a sense, close to \( B(G) \) and on some generic relations between \( B(G) \) and the closed neighborhood hypergraph of \( G \), that we explore in Section 3.

## 2 Formal definitions

**Hypergraphs.** Recall that a hypergraph is a pair \( (X, \mathcal{H}) \), where \( X = V(\mathcal{H}) \) is a set of vertices and \( \mathcal{H} \) is a family of subsets of \( X \), called hyperedges. We will use the same notation \( \mathcal{H} \) to denote a hypergraph and its hyperedge set. A hypergraph has the Helly property if every set of pairwise intersecting hyperedges has a common vertex. An isomorphism from a hypergraph \( \mathcal{H} \) to a hypergraph \( \mathcal{K} \) is a bijection \( \phi: V(\mathcal{H}) \to V(\mathcal{K}) \) such that \( H \in \mathcal{H} \) iff \( \phi(H) \in \mathcal{K} \) for every \( H \subseteq V(\mathcal{H}) \).

**Arc systems.** For \( n \geq 3 \), consider the directed cycle on the vertex set \( \{1, \ldots, n\} \) with arrows from \( i \) to \( i + 1 \) and from \( n \) to 1. An arc \( A = [a, b] \) consists of the points appearing in the directed path from \( a \) to \( b \). The arc \( A = \{1, \ldots, n\} \) is called complete. If \( A = [a, b] \) is not complete, \( a \) and \( b \) are referred to as extreme points of \( A \), the start point and the end point respectively. An arc system \( \mathcal{A} \) is a hypergraph on the vertex set \( \{1, \ldots, n\} \) whose hyperedges are arcs.

**Arc representations of hypergraphs.** An arc representation of a hypergraph \( \mathcal{H} \) is an isomorphism \( \rho \) from \( \mathcal{H} \) to an arc system \( \mathcal{A} \). It can be thought of as a circular ordering of \( V(\mathcal{H}) \) where every hyperedge is a segment of consecutive vertices. The arc system \( \mathcal{A} \) is referred to as an arc model of \( \mathcal{H} \). The notions of an interval representation and an interval model of a hypergraph are introduced similarly, where
interval means an interval of integers. Hypergraphs having arc representations are called circular-arc (CA) hypergraphs, and those having interval representations are called interval hypergraphs.

A representation scheme is a function defined on CA hypergraphs that on input $H$ outputs an arc representation $\rho_H$ of $H$. A representation scheme is called canonical if isomorphic hypergraphs $H \cong K$ always receive equal arc models $\rho_H(H) = \rho_K(K)$. In [10] we designed a canonical representation scheme for CA hypergraphs computable in logarithmic space. Moreover, our algorithm works for hypergraphs with multiple hyperedges (the multiplicity $c(H)$ of a hyperedge $H$ has to be preserved under isomorphisms).

Graphs. The vertex set of a graph $G$ is denoted by $V(G)$. The closed neighborhood $N[v]$ of a vertex $v$ consists of $v$ itself and all vertices adjacent to it. A vertex $u$ is universal if $N[u] = V(G)$. Two vertices $u$ and $v$ are twins if $N[u] = N[v]$. Note that twins are always adjacent. The twin class $[v]$ of a vertex $v$ consists of $v$ itself along with all its twins. Between two different twin classes there are either none or all possible edges. This allows us to consider the quotient graph $G'$ on the vertex set $V(G') = \{[v] \}_{v \in V(G)}$ where two distinct twin classes $[v]$ and $[u]$ are adjacent if $v$ and $u$ are adjacent in $G$. The map $v \mapsto [v]$, that is a homomorphism from $G$ to $G'$, will be referred to as the quotient map.

The intersection graph $\mathbb{I}(H)$ of a hypergraph $H$ has the hyperedges of $H$ as vertices, and two such vertices $A, B \in H$ are adjacent if $A \cap B \neq \emptyset$. If $H$ has multiple hyperedges, they appear in $\mathbb{I}(H)$ as twins.

Arc representations of a graph. An intersection representation of a graph $G$ is an isomorphism $\alpha: V(G) \to A$ from $G$ to the intersection graph $\mathbb{I}(A)$ of a hypergraph $A$. The hypergraph $A$ is then called an intersection model of $G$. If $A$ is an arc system, we speak of arc representation and arc model of $G$. Graphs having arc representations are called circular-arc (CA) graphs. In other words, those are graphs isomorphic to the intersection graphs of CA hypergraphs. Helly circular-arc (HCA) graphs are graphs having Helly arc representations, i.e., representations providing arc models that obey the Helly property.

It is practical to allow an arc model $A$ to have multiple arcs and to require that an arc representation of a graph $G$ map twins in $G$ to the same arc in $A$ (of multiplicity more than 1). Also, one can require that universal vertices of $G$ are mapped to the complete arc. Unless stated otherwise, we will consider arc representations of this kind. This causes no loss of generality as any such representation can be made injective in logarithmic space.

A representation scheme for a class $\mathcal{C}$ of CA graphs is a function that on input $G \in \mathcal{C}$ outputs an arc representation $\alpha_G$ of $G$. A representation scheme for HCA graphs must produce Helly arc representations. If a representation scheme produces equal arc models for isomorphic input graphs, it is called canonical.
3 The maxclique bundle hypergraph

An inclusion-maximal clique in a graph $G$ will be called maxclique. The maxclique hypergraph $C(G)$ of a graph $G$ has the same vertex set as $G$ (i.e., $V(C(G)) = V(G)$) and the maxcliques of $G$ as its hyperedges. We now define the bundle hypergraph $B(G)$, which is the dual of $C(G)$. The hypergraph $B(G)$ has maxcliques of $G$ as vertices (i.e., $V(B(G)) = C(G)$) and a hyperedge $B_v$ for each vertex $v$ of $G$, where $B_v$ consists of all maxcliques that contain $v$. We call $B_v$ the (maxclique) bundle of $v$.

We begin with general properties of the bundle hypergraph that are true for any graph $G$.

**Lemma 3.1.** Define the map $\beta_G : V(G) \rightarrow B(G)$ by $\beta_G(v) = B_v$. Then the correspondence $G \mapsto \beta_G$ is an intersection representation scheme for the class of all graphs.

**Proof.** Note that, for any two distinct vertices $u$ and $v$,

$$B_u \cap B_v \neq \emptyset \text{ iff } u \text{ and } v \text{ are adjacent.} \quad (1)$$

Indeed, if $C \in B_u \cap B_v$, then both $u$ and $v$ are in the clique $C$ and hence adjacent. On the other hand, if $u$ and $v$ are adjacent, extend the set $\{u, v\}$ to a maxclique $C$ and notice that $C \in B_u \cap B_v$. Thus, $\beta_G$ is an intersection representation of $G$. ■

We now notice that the map $\beta_G$ in Lemma 3.1 is, in fact, a Helly representation of the graph $G$.

**Lemma 3.2.** $B(G)$ is a Helly hypergraph.

**Proof.** Suppose that $\{B_x\}_{x \in X}$ is a family of bundles with nonempty pairwise intersections. By [1], $X$ is a clique. Extend $X$ to a maxclique $C$. Then $C \in B_x$ for every $x \in X$. ■

Moreover, $\beta_G$ is the smallest possible among all Helly representations of $G$ in the following sense. Given a map $\alpha : V(G) \rightarrow \mathcal{H}$ and a set $X \subseteq V(\mathcal{H})$, we consider the hypergraph $\mathcal{H}|_X = \{H \cap X : H \in \mathcal{H}\}$ on the vertex set $X$ and define the map $\alpha|_X : V(G) \rightarrow \mathcal{H}|_X$ by $\alpha|_X(v) = \alpha(v) \cap X$.

**Lemma 3.3.** For every Helly intersection representation $\alpha : V(G) \rightarrow \mathcal{H}$ of a graph $G$ there is a set $X \subseteq V(\mathcal{H})$ such that $\alpha|_X$ is an intersection representation of $G$ equivalent with $\beta_G$: there is a hypergraph isomorphism $\psi$ from $B(G)$ to $\mathcal{H}|_X$ such that $\alpha|_X = \psi \circ \beta_G$; see Fig. [2].

**Proof.** For each $C \in C(G)$, consider the family of hyperedges $\mathcal{H}_C = \{\alpha(v)\}_{v \in C}$. Since $\alpha$ is an intersection representation of $G$, all pairwise intersections of the family members are non-empty. Since $\mathcal{H}$ is a Helly hypergraph, the overall intersection $\bigcap \mathcal{H}_C$ is non-empty. We fix a point $x_C \in \mathcal{H}_C$ and let $X = \{x_C : C \in C(G)\}$. Note that $x_C \neq x_{C'}$ if $C \neq C'$ (indeed, the equality $x_C = x_{C'}$ implies that the family $\mathcal{H}_C \cup \mathcal{H}_{C'}$ has non-empty overall intersection; therefore, the union $C \cup C'$ of two maxcliques must be a clique, which is possible only when $C = C'$). For every $v \in V(G)$, we have $\alpha(v) \cap X = \{x_C : C \ni v \text{ (or } C \in B_v)\}$. Hence, $\psi(C) = x_C$ is an isomorphism from $B(G)$ to $\mathcal{H}|_X$ with the desired property. ■
Figure 2: Lemma 3.3: the intersection representations $\alpha|_X$ and $\beta_G$ of $G$ are equivalent up to an isomorphism $\psi$ between the intersection models.

The following classical result provides a link between HCA graphs and CA hypergraphs; it is exemplified in Fig. 3.

**Lemma 3.4 (Gavril [6]).** $G$ is an HCA graph iff $B(G)$ is a CA hypergraph.

**Proof.** If $B(G)$ is a CA hypergraph, consider an arc representation $\rho$ of $B(G)$. By Lemmas 3.1 and 3.2, $\beta_G$ is a Helly intersection representation of $G$. It remains to notice that $\rho \circ \beta_G$ is a Helly arc representation of this graph.

Conversely, assume that $G$ is an HCA graph and consider a Helly arc representation $\alpha: V(G) \rightarrow A$ of $G$. By Lemma 3.3, $B(G)$ is isomorphic to the hypergraph $A|_X$ for some set of points $X$. For any arc system $A$ and for any set of points $X \subseteq V(A)$, the hypergraph $A|_X$ is CA. Therefore, $B(G) \cong A|_X$ is CA as well.

The last lemma of the section describes local similarity between the bundle hypergraph $B(G)$ and the closed neighborhood hypergraph $\{N[v]\}_{v \in V(G)}$. It shows that the set-theoretic relations between maxclique bundles can be understood in terms of the adjacency relation of the graph.

**Lemma 3.5.** Let $u$ and $v$ be arbitrary vertices of a graph $G$.

1. $B_u \cap B_v \neq \emptyset$ iff $u \in N[v]$ and iff $v \in N[u]$.
2. $B_u \subseteq B_v$ iff $N[u] \subseteq N[v]$.

Figure 3: The graph $G$ contains the maxcliques $C_1 = \{a,b,c\}$, $C_2 = \{b,c,d,e\}$, $C_3 = \{c,d,e,f\}$, $C_4 = \{e,f,g\}$, $C_5 = \{f,g,h\}$, and $C_6 = \{a,h\}$. Its bundle hypergraph $B_G$ admits the HCA model $A$ via the representation $\rho: C(G) \rightarrow \{1,2,3,4,5,6\}$ that maps each maxclique $C_i$ to the point $i$, and thus $\rho(B_v) = A_v$ for each $v \in V(G)$. The function $\alpha: V(G) \rightarrow A$ that maps each vertex $v$ to the arc $A_v$ is an HCA representation of $G$. 

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3. Suppose that \( u \) and \( v \) are adjacent. Then \( B_u \cup B_v = \mathcal{C}(G) \) iff the following three conditions are met:

   (a) \( N[u] \cup N[v] = V(G) \);
   (b) \( w \in N[u] \setminus N[v] \) implies \( N[w] \subseteq N[u] \);
   (c) \( w \in N[v] \setminus N[u] \) implies \( N[w] \subseteq N[v] \).

4. Suppose that \( u \) and \( v \) are adjacent. Then \( B_u \cap B_v \subseteq B_w \) iff \( N[u] \cap N[v] \subseteq N[w] \).

**Proof.** 1 readily follows from (1).

2. \((\implies)\) In this direction, the claim readily follows from Part 1. Indeed, if \( x \in N[u] \), then \( B_x \) intersects \( B_u \) and, hence, also \( B_v \). Therefore, \( x \in N[v] \).

   \((\impliedby)\) Suppose that \( C \in B_u \), that is, \( u \in C \). It follows that \( C \subseteq N[u] \) and, by assumption, also \( C \subseteq N[v] \). This implies that \( C \cup \{v\} \) is a clique. Since the clique \( C \) is maximal, \( v \in C \), that is, \( C \in B_v \).

3. \((\implies)\) Again, this direction follows from Part 1, even without the assumption that \( u \) and \( v \) are adjacent.

   (a) For any \( x \), the bundle \( B_x \) intersects at least one of the bundles \( B_u \) and \( B_v \). Therefore, \( x \) belongs to one of the neighborhoods \( N[u] \) or \( N[v] \).

   (b) Assume that \( w \in N[u] \setminus N[v] \). This implies, in particular, that \( B_w \) is disjoint from \( B_v \). If follows from \( B_u \cup B_v = \mathcal{C}(G) \) that \( B_w \subseteq B_u \). Now, if \( x \in N[w] \), then \( B_x \) intersects \( B_w \) and, hence, also \( B_u \). Therefore, \( x \in N[u] \) as well.

   (c) is symmetric to (b).

   \((\impliedby)\) For this direction, the assumption that \( u \) and \( v \) are adjacent is essential. Assuming that \( B_u \cup B_v \neq \mathcal{C}(G) \), we will infer that at least one of the conditions (a) and (b) is false. Indeed, let \( C \) be a maxclique that does not belong to \( B_u \cup B_v \), that is, \( u \notin C \) and \( v \notin C \). Since \( C \) is inclusion-maximal, it contains a vertex \( x \) non-adjacent to \( u \) and a vertex \( y \) non-adjacent to \( u \). Suppose that (a) is true. Then \( x \) must be adjacent to \( u \) and, similarly, \( y \) must be adjacent to \( v \). Thus, \( vuxy \) is an induced cycle of length 4 in \( G \). Now, (b) is refuted by taking \( w = x \) because \( x \in N[u] \setminus N[v] \) while \( y \in N[x] \setminus N[u] \).

4. \((\implies)\) Once again, this direction follows from Part 1. Indeed, let \( x \in N[u] \cap N[v] \). It follows that \( B_x \) intersects both \( B_u \) and \( B_v \). Since \( B_u \) and \( B_v \) intersect (because \( u \) and \( v \) are adjacent), Lemma 3.2 implies that \( B_x \) intersects even the intersection \( B_u \cap B_v \). Since \( B_u \cap B_v \subseteq B_w \), \( B_x \) intersects also \( B_w \) and, therefore, \( x \in N[w] \).

   \((\impliedby)\) For this direction, the assumption that \( u \) and \( v \) are adjacent is not needed (as then \( B_u \cap B_v = \emptyset \)). Let \( C \in B_u \cap B_v \), that is, \( u \in C \) and \( v \in C \). It follows that the clique \( C \) is contained in both \( N[u] \) and \( N[v] \). Since \( C \subseteq N[u] \cap N[v] \subseteq N[w] \), the set \( C \cup \{w\} \) is a clique. Since \( C \) is inclusion-maximal, \( w \in C \) and \( C \in B_w \) as well.
4 Getting canonicity for free

Lemma 4.1. The canonical representation problem for HCA graphs is logspace reducible to the (not necessarily canonical) representation problem for HCA graphs with no twins and no universal vertices.

Proof. We first show that the canonical representation problem for HCA graphs reduces in logspace to the problem of computing \( C(G) \), that is, to finding all maxcliques in a given HCA graph \( G \). Indeed, given \( C(G) \), we can easily construct the bundle hypergraph \( B(G) \) and the mapping \( \beta_G \). As shown in the proof of Lemma 3.3, we can combine \( \beta_G \) with an arc representation \( \rho_{C(G)} \) of the CA hypergraph \( B(G) \) and obtain an arc representation \( \alpha_G = \rho \circ \beta_G \). If \( \rho = \rho_{C(G)} \) is chosen according to the logspace-computable canonical representation scheme for CA hypergraphs designed in [10], then \( G \mapsto \alpha_G \) will be a canonical representation scheme for HCA graphs.

Indeed, if \( G \cong H \), then \( C(G) \cong C(H) \), which implies that \( \alpha_G(G) = \rho_{C(G)}(C(G)) \) is equal to \( \alpha_H(H) = \rho_{C(H)}(C(H)) \).

Note now that the problem of computing \( C(G) \) is equivalent to its restriction to graphs with no twins and no universal vertices. Indeed, let \( G' \) be obtained from \( G \) by computing its quotient-graph with respect to the twin-relation and removing the universal vertex \([u]\) from it (if \( G \) contains a universal vertex \( u\)). Given \( C(G') \), we easily obtain \( C(G) \) by inserting \([u]\) in each maxclique of \( G' \) and by converting each maxclique \( \{[v_1], \ldots, [v_k]\} \) of the quotient-graph to the maxclique \( [v_1] \cup \ldots \cup [v_k] \) of the original graph \( G \).

It remains to show that finding \( C(G) \) reduces to computing an arbitrary Helly arc representation \( \alpha \) of \( G \). Given the arc model \( \alpha(G) \), for each point \( x \) of the circle we can compute the set \( C_x = \{v \in V(G) : x \in \alpha(v)\} \). Obviously, \( C_x \) is a clique in \( G \).

By Lemma 3.3 among these cliques there are all maxcliques of \( G \). Since maximality of a given clique is easy to detect, this allows us to compute all \( C(G) \). \( \blacksquare \)

5 A sharpening of an arc system

If two sets \( A \) and \( B \) intersect but neither of them includes the other, we say that they overlap and write \( A \not\subset B \). Suppose now that \( A \) and \( B \) are arcs on a circle \( \mathbb{C} \). If \( A \not\subset B \) and \( A \cup B \neq \mathbb{C} \), we say that \( A \) and \( B \) strictly overlap and write \( A \not\subset^{*} B \). If, moreover, \( A = [a^-, a^+] \) and \( a^+ \in B \), we say that \( A \) overlaps \( B \) on the left (or that \( B \) overlaps \( A \) on the right) and write \( A \preceq^{*} B \) in this case.

A system \( \mathcal{A} \) of \( m \) arcs on the 2\( m \)-point circle will be called sharp if all extreme points of the arcs in \( \mathcal{A} \) are pairwise distinct; in other words, every point of the circle is either start or end point of exactly one arc. Furthermore, let \( A, B \in \mathcal{A} \), \( A = [a^-, a^+] \), and \( B = [b^-, b^+] \). If the extreme points of these arcs occur in the circular order \( a^- b^+ b^- a^+ \), we say that \( A \) and \( B \) form a circle cover and write \( A \succ B \).

Note that \( A \succ B \) exactly when \( A \cup B = \mathbb{C} \) and \( A \) contains both \( b^- \) and \( b^+ \) (hence, \( B \) contains both \( a^- \) and \( a^+ \)).
Definition 5.1. Let $\mathcal{A}$ be an arc system on a circle $C$ with no multiple and no complete arcs. Let $\mathcal{A}'$ be another arc system on a circle $C'$. A bijection $\sigma: \mathcal{A} \to \mathcal{A}'$ is a sharpening of $\mathcal{A}$ if $\mathcal{A}'$ is sharp and the following conditions are met for every $A,B \in \mathcal{A}$:

1. $A \cap B = \emptyset$ iff $\sigma(A) \cap \sigma(B) = \emptyset$;
2. $A \subseteq B$ iff $\sigma(A) \subseteq \sigma(B)$;
3. $A \preceq B$ iff $\sigma(A) \preceq \sigma(B)$;
4. Let $A,B \neq C$ and $A \cap B \neq \emptyset$. Then $A \cup B = C$ iff $\sigma(A) \bowtie \sigma(B)$.

Condition 4 means that if $A \cup B = C$ and $A$ contains one extreme point of $B$, then $A$ must contain both extreme points of $B$.

Lemma 5.2. Every arc system $\mathcal{A}$ can be sharpened to an arc system $\mathcal{A}'$ so that, if $\mathcal{A}$ has the Helly property, this is true also for $\mathcal{A}'$.

Proof. For each pair of successive points $x$ and $y$ on $C$ we do the following. Suppose that $y$ is the successor of $x$. Suppose that $x$ serves as the end point for the arcs $A_1, \ldots, A_k$ and $y$ serves as the start point for the arcs $B_1, \ldots, B_l$. W.l.o.g., assume that $A_i \subset A_{i+1}$ and $B_j \subset B_{j+1}$. The arcs $B_1, \ldots, B_l$ will get new pairwise distinct start points $b_1^-, \ldots, b_l^-$ that will be inserted between $x$ and $y$ in this order. The arcs $A_1, \ldots, A_k$ will get new pairwise distinct end points $a_1^+, \ldots, a_k^+$ that will be inserted between $x$ and $y$ in this order. Our goal is to ensure Condition 4 in Definition 5.1 for each pair $A_i, B_j$. For this purpose, the sequences $a_1^+, \ldots, a_k^+$ and $b_1^-, \ldots, b_l^-$ will interlace as follows. Note that, if $A_i$ intersects $B_j$, then $A_i$ intersects also the longer arc $B_{j+1}$. This suggests that we put $a_i^+$ after $b_j^-$, where $j_i$ is the minimum index $j$ such that $A_i$ intersects $B_j$. This rule ensures that

- disjoint $A_i$ and $B_j$ remain disjoint and that
- if intersecting $A_i$ and $B_j$ cover all points of the circle, then their modified versions will form a circle cover.

Furthermore, this rule is compatible with the the order $a_1^+, \ldots, a_k^+$ because, if $A_i$ intersects $B_j$, then the longer arc $A_{i+1}$ also intersects $B_j$. All the other arcs retain their old extreme points. Finally, all points that are not extreme for any arc are removed.

Let us analyze the outcome of performing this transformation for a pair $x, y$.

- The arcs with end point at $x$ get now pairwise distinct end points and do not share these new extreme points with any other arc.
- The arcs with start point at $y$ get pairwise distinct start points.
- Disjoint arcs remain disjoint.
• Any set of arcs with nonempty overall intersection still has nonempty overall intersection (because every arc either stays the same or becomes longer in one direction). Therefore,
  - intersecting arcs remain intersecting, and
  - the Helly property is preserved.
• The inclusion relation is respected.
• The left/right overlapping relations are respected.
• All pairs of arcs \( A \) and \( B \) such that \( A \cap B \neq \emptyset \) and \( A \cup B = C \) are “extended” so that they form a circle cover.

Thus, when the transformation is performed for all pairs \( x, y \), the resulting arc system \( \mathcal{A}' \) is as required; in particular it satisfies all the conditions in Definition 5.1.

**Remark 5.3.** Our notion of sharpening is related to the concepts of a stable arc system introduced in [8] and of a normalized arc representation of a graph introduced in [7]. A sharp arc system is stable if no additional circle-cover pair can be introduced by moving an extreme point, unless the intersection graph is also changed by this modification. In particular, a stable arc system cannot contain any pair of arcs \( A = [a^-, a^+] \), and \( B = [b^-, b^+] \) such that \( a^+ \) and \( b^- \) are consecutive points of the circle and \( b^+ \in A \). Due to Condition 4 in Definition 5.1 the latter is true also for any sharpened arc system.

In a normalized arc representation \( \alpha : V(G) \to \mathcal{A} \), the resulting arc system \( \mathcal{A} \) must be stable, and the containment between arcs must reflect the containments between neighborhoods, i.e., \( \alpha(u) \subseteq \alpha(v) \) if and only if \( N[u] \subseteq N[v] \). By Lemma 3.4, every HCA graph \( G \) has an arc representation \( \alpha : V(G) \to \mathcal{A} \) of the form \( \alpha = \rho \circ \beta_G \), where \( \rho \) is an arc representation of the bundle hypergraph \( \mathcal{B}(G) \). Let us modify \( \alpha \) to another Helly arc representation \( \alpha' = \sigma \circ \alpha \) of \( G \), where \( \sigma : \mathcal{A} \to \mathcal{A}' \) is a sharpening of \( \mathcal{A} \). Lemmas 3.5 and 5.2 imply that \( \alpha' \) is normalized.

### 6 Flipping in a sharp arc system

For the complete arc \( C \) the notion of extreme points becomes ambiguous, as then \( C = [a, a - 1] \) for any \( a > 1 \) and also \( C = [1, n] \). We will call \([1, n]\) and \([a, a - 1]\) a complete arc with designated extreme points. Suppose that an arc \( A = [a, b] \) contains more than one point, that is, \( a \neq b \). In this case, we will say that the arc \( A = [b, a] \) is obtained from \( A \) by flipping. This operation applies, in particular, to complete arcs with designated extreme points, producing two-point arcs \([a - 1, a]\) or \([n, 1]\). If applied to two-point arcs, it produces complete arcs with designated extreme points. Note that flipping preserves sharpness.

Suppose that an arc system \( \mathcal{A} \) contains no one-point arc but possibly contains complete arcs with designated extreme points. Let \( X \subseteq \mathcal{A} \). The \( X\)-flipped system
Lemma 6.1. Let $\mathcal{I}$ be an arc system containing no one-point arc but possibly complete arcs with designated extreme points. Let $\psi$ be a hypergraph isomorphism from $\mathcal{I}$ to another arc system $\mathcal{J}$ that takes the extreme points of each arc $A \in \mathcal{I}$ to the endpoints of the arc $\psi(A) \in \mathcal{J}$. Consider mappings $\lambda: V \to \mathcal{I}$ and $\mu: V \to \mathcal{J}$ such that $\mu = \psi \circ \lambda$; see Fig. 4. Let $C \subseteq V$. Then $\psi$ is an isomorphism from $\mathcal{I}^\lambda(C)$ to $\mathcal{J}^\mu(C)$ and $\mu^C = \psi \circ \lambda^C$.

Proof. For every $v \in V$, the isomorphism $\psi$ maps the arc $\lambda(v)$ in $\mathcal{I}$ onto the arc $\mu(v)$ in $\mathcal{J}$. If $\lambda(v) = [a^-, a^+]$ and $\mu(v) = [b^-, b^+]$, it is also known that $\psi([a^-, a^+]) = [b^-, b^+]$. This implies that $\psi$ maps $\lambda(v)$ onto $\mu(v)$. Therefore, $\psi$ maps $\lambda^C(v)$ onto $\mu^C(v)$, which means exactly that it is an isomorphism from $\mathcal{I}^\lambda(C)$ to $\mathcal{J}^\mu(C)$ and $\mu^C = \psi \circ \lambda^C$.

Lemma 6.1 is true for isomorphisms between arc systems that respect extreme points. The last condition is not always met. For example, the transposition (23), while being an automorphism of the interval system $\{(1, 3), (2, 4)\}$, exchanges extreme points of two different intervals. However, two isomorphic sharp interval systems always admit an isomorphism that does respect extreme points. Before we prove this below in Lemma 6.3, we need to recall some general notions and facts about interval systems.

A slot of a hypergraph $\mathcal{H}$ is an inclusion-maximal subset $S$ of $V(\mathcal{H})$ such that each hyperedge of $\mathcal{H}$ contains either all of $S$ or none of it. Recall that hyperedges $A$ and $B$ overlap, which is denoted as $A \nparallel B$, if they intersect but neither of them includes the other. With respect to the relation $\nparallel$, any hypergraph $\mathcal{H}$ is either connected or is split into overlap-connected components. If $\mathcal{O}$ and $\mathcal{O}'$ are different overlap-connected components, then either they are vertex-disjoint or all hyperedges of one of the two components are contained in a single slot of the other component. If $\mathcal{H}$ is connected, this containment relation determines a tree-like decomposition of $\mathcal{H}$ into its overlap-connected components. The root in this tree will be referred to as the top component; the other components will be called inner. The following fact is due to [3, Theorem 2]; see also [9, Section 2.2].

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1This follows from a simple observation that the conditions $B \subset A$, $B \nparallel B'$, and $\neg(B' \nparallel A)$ imply that $B' \subset A$.

2If $\mathcal{H}$ is an interval system, this decomposition gives rise to the concept of a PQ-tree.
Lemma 6.2 (Chen and Yesha [3]). Suppose that $\mathcal{I}$ and $\mathcal{J}$ are isomorphic overlap-connected interval systems. Let $I_1, \ldots, I_k$ be all slots of $\mathcal{I}$ listed in the order as they appear in the line. Similarly, let $J_1, \ldots, J_k$ be the sequence of slots of $\mathcal{J}$. Then any isomorphism from $\mathcal{I}$ to $\mathcal{J}$ maps either each $I_s$ onto $J_s$ or each $I_s$ onto $J_{k+1-s}$.

Lemma 6.3. Let $\mathcal{I}$ and $\mathcal{J}$ be isomorphic sharp interval systems. For every hypergraph isomorphism $\psi$ from $\mathcal{I}$ to $\mathcal{J}$ there is a hypergraph isomorphism $\psi'$ from $\mathcal{I}$ to $\mathcal{J}$ such that $\psi'(A) = \psi(A)$ for all $A \in \mathcal{I}$ and, moreover, $\psi'$ respects extreme points, that is, takes the extreme points of each arc $A \in \mathcal{I}$ to the endpoints of the arc $\psi(A) \in \mathcal{J}$.

Proof. We proceed by induction on the number of overlap-connected components of $\mathcal{I}$. In the base case, $\mathcal{I}$ and $\mathcal{J}$ are overlap-connected. Using Lemma 6.2 we can assume that an isomorphism $\psi$ from $\mathcal{I}$ to $\mathcal{J}$ maps each $I_s$ onto $J_s$; the other case is symmetric.

We show that, for each $A \in \mathcal{I}$, the isomorphism $\psi$ either respects the extreme points of $A$ or can be locally modified to respect them. Let $A = [a^-, a^+]$ and $A = \bigcup_{s=p}^q I_s$. It follows that $\psi(A) = \bigcup_{s=p}^q J_s$, $a^- \in I_p$, and $a^+ \in I_q$. Moreover, if $\psi(A) = [b^-, b^+]$, then $b^- \in J_p$ and $b^+ \in J_q$.

Notice now that, since $\mathcal{I}$ is sharp, every slot contains at most two points. Moreover, every two-point slot $[c^-, d^+]$ consists of the start point of some interval $C$ and the end point of another interval $D$. The transposition of the points $c^-$ and $d^+$ does violate neither $C$ nor $D$, nor any other interval.

If $I_p$ is a one-point slot, we immediately conclude that $\psi(a^-) = b^-$. Suppose that $I_p = [a^-, x^+]$ is a two-point slot. Let $J_p = [b^-, y^+]$. If $\psi(a^-) = b^-$, we are done. Otherwise we can ensure $\psi'(a^-) = b^-$ by changing $\psi$ only on $I_p$.

In order to ensure that $\psi'(a^+) = b^+$, we may need to modify $\psi$ on $I_q$. In fact, we just need to inspect all two-point slots; if such a slot needs modification, this will simultaneously fix inconsistency between a pair of start points and a pair of end points. The analysis of the overlap-connected case is complete.

Suppose now that $\mathcal{I}$ and $\mathcal{J}$ have more than one overlap-connected component, that is, are not overlap-connected. If $\mathcal{I}$ and $\mathcal{J}$ are disconnected, the claim readily follows by applying the induction assumption to the corresponding connected components of $\mathcal{I}$ and $\mathcal{J}$.

It remains to consider the case when $\mathcal{I}$ and $\mathcal{J}$ are connected but not overlap-connected. Assume that an interval $A \in \mathcal{I}$ contains an inner overlap-connected component $S \subset \mathcal{I}$, then $\psi(V(S)) \subset \psi(A)$ for any isomorphism $\psi$ from $\mathcal{I}$ to $\mathcal{J}$. If we remove all points in $V(S)$ from $\mathcal{I}$ and all points in $\psi(V(S))$ from $\mathcal{J}$, the resulting interval systems $\mathcal{I}'$ and $\mathcal{J}'$ will still contain the extreme points of $A$ and $\psi(A)$ respectively, and $\psi$ will induce an isomorphism from $\mathcal{I}'$ to $\mathcal{J}'$. By the induction assumption, there are isomorphisms from $\mathcal{I}'$ to $\mathcal{J}'$ and from $S$ to $\psi(S)$ that agree with $\psi$ on hyperedges and respect extreme points. Merging them, we get the desired isomorphism $\psi'$ from $\mathcal{I}$ to $\mathcal{J}$. 

When we want to apply Lemmas 6.1 and 6.3 the interval systems under consideration need to be sharp. It may happen that we deal an isomorphic copy of a sharp
interval system that itself is not sharp; consider for example, \{[1, 4], [2, 3]\} that is isomorphic to \{[1, 4], [1, 2]\}. In such cases the following fact will be helpful.

**Lemma 6.4.** Suppose that for an interval system \(J\) there is an isomorphic sharp interval system \(J'\). Then such \(J'\) can be computed in logspace along with an isomorphism \(\phi\) from \(J\) to \(J'\).

**Proof.** Suppose that \(J\) is isomorphic to a sharp interval system \(S\) and \(\phi\) is an isomorphism from \(J\) to \(S\). Since \(S\) cannot contain any 1-point interval, the same holds true for any isomorphic system, in particular, for \(J\). Furthermore, \(J\) cannot contain any point that serves simultaneously as the start point of an interval \(A\) and the end point of another interval \(B\); otherwise the intervals \(\phi(A)\) and \(\phi(B)\) in \(S\) would also intersect at only one point and thus share an extreme point.

Given \(J\), we construct an interval system \(J'\) in three steps, each doable in logspace.

1. Remove all interior points from \(J\), that is, those points that are not extreme for any interval.

2. For each point \(x\) that is the start point of more than two intervals \(A_1, \ldots, A_k\), do the following. W.l.o.g., assume that \(A_1 \supset A_2 \supset \cdots \supset A_k\). Let \(y \in A_1\) be the point next to \(x\). We provide the arcs \(A_1, \ldots, A_k\) with new pairwise distinct start points \(x = a_1, \ldots, a_k\) that will be inserted between \(x\) and \(y\) in this order.

3. Do similarly with the shared end points.

Being removed in the first step, interior points never appear later. The 2nd and the 3rd steps ensure that no two intervals in \(J'\) share an extreme point. Thus, \(J'\) is sharp. The main efforts are needed to show that \(J'\) is isomorphic to \(J\).

To prove this, we use induction on the number of overlap-connected components of \(J\). In the base case, \(J\) is overlap-connected. Note that Lemma 6.2 has the following interpretation.

**Claim A.** Let \(I\) and \(I'\) be interval systems isomorphic as hypergraphs. If they are overlap-connected, then either \(I = I'\) or \(I'\) is obtained from \(I\) by a reflection of the line.

Claim A implies that an overlap-connected \(J\) is geometrically congruent to \(S\) and is, therefore, sharp. Thus, the algorithm just returns \(J' = J\) in this case.

Suppose now that \(J\) is disconnected. Note that we can obtain \(J'\) by applying the algorithm to each connected component of \(J\) and merging the results. The isomorphism \(J' \cong J\) readily follows by the induction assumption.

It remains to consider the case when \(J\) is connected but not overlap-connected. Let \(T\) denote the top overlap-connected component of \(J\). By Claim A, \(T\) is congruent to the top overlap-connected component of \(S\). In particular, no extreme point is shared by two intervals in \(T\) (but \(T\) has interior points). Assume first that no extreme point of an interval in \(T\) is shared with any interval in \(J \setminus T\). Then the output \(J'\) is obtainable by leaving \(T\) as it is and by applying the algorithm to the
children-components within each slot of \( T \). Note that \( \phi \) maps every inner overlap-connected component of \( J \) to an inner component of \( S \), which is sharp. Using the induction assumption for each child-component within each slot of \( T \), we conclude that \( J' \cong J \) also in this case.

Assume now that there is an extreme point \( x \) of an interval \( T \) that is also an extreme point of some interval \( A \in J \setminus T \). Fix \( A \) to be the longest of such intervals. Denote the overlap-connected component of \( J \) containing \( A \) by \( A' \). Let \( T \) denote the slot of \( T \) containing \( x \). Note that \( A \) is one of the children-components located in \( T \).

Looking at the image \( \phi(T) \) in \( S \), we see that \( T \) must contain a point \( z \) not included in any inner overlap-connected component (namely \( z = \phi^{-1}(z') \) for \( z' \in \phi(T) \) being an extreme point of an interval in the top component of \( S \)). Moving \( z \) to any other place in \( T \) outside the children-components results in an interval system isomorphic to \( J \). In particular, we can make \( z \) a new extreme point of an interval in \( T \) instead of \( x \). Denote the resulting interval system by \( \tilde{J} \). We can, therefore, obtain the same outcome \( J' \) as follows.

- Remove \( V(A) \) from \( \tilde{J} \) and denote the result by \( K \). Looking at \( \phi \) on \( V(A) \) and on \( V(J) \setminus V(A) \), we see that both \( A \) and \( K \) are isomorphic to sharp interval systems.

- Apply the algorithm to \( A \) and \( K \) and denote the outputs by \( A' \) and \( K' \), respectively.

- Reinsert \( A' \) in \( K' \) within the corresponding slot.

Since \( A' \cong A \) and \( K' \cong K \) by the induction assumption, we conclude that \( J' \cong J \) as claimed.

In general, if the algorithm is run on an arbitrary \( J \), after computing \( J' \) we invoke the algorithm of \([9]\) to find a hypergraph isomorphism \( \psi \) from \( J \) to \( J' \). In the case of failure, we conclude that the input system \( J \) is not isomorphic to any sharp interval system.

\[ \square \]

7 Pairwise intersections as a complete isomorphism invariant for interval hypergraphs

Given a hypergraph \( \mathcal{H} \) and a bijection \( \nu : V \rightarrow \mathcal{H} \), we define the pairwise-intersection matrix \( M_\nu = (m_{uv})_{u,v \in V} \) by \( m_{uv} = |\nu(u) \cap \nu(v)| \). If \( \psi \) is an isomorphism from \( \mathcal{H} \) to \( \mathcal{K} \) and the bijection \( \mu : V \rightarrow \mathcal{K} \) is defined by \( \mu = \psi \circ \lambda \), then obviously \( M_\lambda = M_\mu \).

It turns out that the converse is also true if \( \mathcal{H} \) is an interval hypergraph.

Lemma 7.1 (Fulkerson and Gross \([5]\)). Let \( I \) be an interval system and \( J \) be an arbitrary hypergraph. Suppose that \( M_\lambda = M_\mu \) for bijections \( \lambda : V \rightarrow I \) and \( \mu : V \rightarrow J \). Then there is a hypergraph isomorphism \( \psi \) such that \( \mu = \psi \circ \lambda \); see Fig. 5.
Figure 5: Lemma [7.1] If $M_\lambda = M_\mu$ and $I$ is an interval hypergraph, then $I \cong J$.

We will use the fact that $I$ and $\lambda$ are efficiently reconstructible from a given $M = M_\lambda$.

**Lemma 7.2 (Köbler, Kuhnert, and Watanabe [11])**. There is a logspace algorithm that, given an integer matrix $M = (m_{uv})_{u,v \in V}$, constructs an interval system $I$ and a bijection $\lambda: V \to I$ such that $M = M_\lambda$ or detects that such an interval system does not exist.

## 8 A representation scheme for HCA graphs in logspace

We are now prepared to prove Theorem [1.1]. By Lemma 4.1, it suffices to design a (not necessarily canonical) representation scheme for HCA graphs that have no twins and no universal vertices and to show that this scheme is computable in logspace.

Let $G$ be an input graph on $n$ vertices. We assume that $G$ is HCA and has neither twins nor universal vertices. Note that then its bundle hypergraph $B(G)$ has no multiple hyperedges $B_u = B_v$ and no complete hyperedge $B_u = C(G)$.

Let $\beta_G: V(G) \to B(G)$ be the Helly intersection representation of $G$ as defined in Lemma 3.1. By Lemma 3.4, $B(G)$ is a CA hypergraph. Consider its arbitrary arc representation $\rho: B(G) \to B$. As it will be beneficial to deal with sharp arc models, consider an arbitrary sharpening $\sigma: B \to A$ of $B$ to a sharp Helly arc system $A$, which exists because $B$ contains no multiple and no complete arcs. Define

$$\alpha = \sigma \circ \rho \circ \beta_G. \tag{2}$$

Thus, $\alpha: V(G) \to A$ is a Helly arc representation of $G$ by a sharp arc model $A$.

**Lemma 8.1.** For $\alpha$ defined by (2), the pairwise-intersection matrix $M_\alpha$ depends on $G$ only (and neither on $\rho$ nor on $\sigma$) and can be computed in logspace.

**Proof.** Consider first $m_{uv} = |\alpha(v)|$. The arc $\alpha(v)$ contains two its own extreme points and, furthermore, every vertex $u$ adjacent to $v$ contributes one or two extreme points of $\alpha(u)$ into $\alpha(v)$. More precisely, the following configurations are possible.

$$\alpha(u) \subset \alpha(v) \rightharpoonup 2 \text{ contributed points:}$$ By the definition of sharpening, this happens exactly when $B_u \subset B_v$, which is equivalent to the logspace-verifiable condition $N[u] \subset N[v]$ by Lemma 3.5.2.
\(\alpha(u) \gg \alpha(v)\) — 2 contributed points: By the definition of sharpening, this happens exactly when \(B_u \cup B_v = C(G)\), which is equivalent to the logspace-verifiable conditions (a) and (c) in Lemma 3.5.3.

\(\alpha(u) \nleq \alpha(v)\) — 1 contributed point: the remaining case.

Consider now \(m_{uv} = |\alpha(u) \cap \alpha(v)|\) for \(u \neq v\). In the simplest case of non-adjacent \(u\) and \(v\) we have \(m_{uv} = 0\). Also, \(m_{uv} = m_{uu}\) if \(\alpha(u) \subset \alpha(v)\) or, equivalently, \(N[u] \subset N[v]\). Similarly, \(m_{uv} = m_{vv}\) if \(N[v] \subset N[u]\). Furthermore, \(m_{uv} = m_{uu} + m_{vv} - 2n\) if \(\alpha(u) \gg \alpha(v)\), which is verifiable by Lemma 3.5.3.

It remains to compute \(m_{uv}\) if \(\alpha(u) \nleq \alpha(v)\). The intersection contains one extreme point of \(\alpha(u)\) and one of \(\alpha(v)\). Any other vertex \(w\) contributes 0, 1, or 2 extreme points of \(\alpha(w)\). The contribution is 0 when \(\alpha(w)\) is disjoint from \(\alpha(u)\) or \(\alpha(v)\) or when it contains at least one of these arcs. Let us analyze the remaining cases (some cases symmetric up to swapping \(u\) and \(v\) are omitted). The first four conditions are verifiable in logspace similarly to the above by Lemma 3.5.3.

\(\alpha(w) \subset \alpha(u)\) and \(\alpha(w) \subset \alpha(v)\) — 2 contributed points,

\(\alpha(w) \subset \alpha(u)\) and \(\alpha(w) \nleq \alpha(v)\) — 1 contributed point,

\(\alpha(w) \gg \alpha(u)\) and \(\alpha(w) \nleq \alpha(v)\) — 2 contributed points,

\(\alpha(w) \gg \alpha(u)\) and \(\alpha(w) \nleq \alpha(v)\) — 1 contributed point,

\(\alpha(w) \nleq \alpha(u)\) and \(\alpha(w) \nleq \alpha(v)\): This case is more complicated. W.l.o.g., suppose that \(\alpha(u) \nleq \alpha(v)\) and, hence, \(\rho(B_u) \nleq \rho(B_v)\). We split our analysis into two subcases.

Note first that the arc configuration \(\alpha(v) \nleq \alpha(w) \nleq \alpha(u)\) is non-Helly and, hence, cannot occur. There remain two subcases.

\(\alpha(u) \nleq \alpha(w) \nleq \alpha(v)\) — 0 contributed points: By the definition of sharpening, this happens exactly when \(\rho(B_u) \nleq \rho(B_v) \nleq \rho(B_w)\), which is equivalent \(\rho(B_u) \cap \rho(B_v) \subset \rho(B_w)\). Since \(\rho\) is a hypergraph isomorphism, the last condition reads \(B_u \cap B_v \subset B_w\), which is equivalent to the logspace-verifiable condition \(N[u] \cap N[v] \subseteq N[w]\) by Lemma 3.5.4.

\(\alpha(w) \nleq \alpha(u)\) or \(\alpha(u) \nleq \alpha(v)\) or \(\alpha(u) \nleq \alpha(v)\) — 1 contributed point: This is the complementary subcase.

The analysis is complete. The matrix entry \(m_{uv}\) is obtained by summing up the contributions of \(\alpha(w)\) over all \(w\).

Next, we need to find an arbitrary maxclique \(C \in C(G)\). We have to argue that this is doable in logspace. An edge \(uv\) in a graph \(G\) is called essential if it is contained in a unique maxclique \(C\). The following lemma implies that, for each \(uv\), we can check in logspace if it is essential. If so, the corresponding maxclique \(C\) can be computed also in logspace as \(C = N[u] \cap N[v]\).
Lemma 8.2. An edge $uv$ is essential if and only if the intersection $N = N[u] \cap N[v]$ is a clique.

Proof. Note first that any clique containing $uv$ is included in $N$. If $N$ is a clique, this implies that $N$ is actually a maxclique and, moreover, it is the only maxclique containing $uv$.

Suppose now that $N$ contains non-adjacent vertices $x$ and $y$. Then two triangles \{u, v, x\} and \{u, v, y\} can be extended to two different maxcliques both containing $uv$. ■

It is known [16] that if $G$ is a connected interval graph, then every maxclique in $G$ contains an essential edge. This allows to compute $\mathcal{B}(G)$ in logspace, which was an important ingredient of our canonical representation scheme for interval graphs in [9]. However, connected HCA graphs do not enjoy this property, for instance, the Hajós (or 3-sun) graph on the vertex set \{u_1, u_2, u_3, v_1, v_2, v_3\} whose maxcliques are \{u_1, u_2, u_3\}, \{v_1, v_2, u_3\}, \{u_1, v_2, u_3\}, and \{u_1, u_2, v_3\}. Fortunately, every nonempty HCA graph has at least one maxclique that can be efficiently found due to the fact that it contains an essential edge.

Lemma 8.3. Every nonempty HCA graph $G$ contains an essential edge $uv$.

Proof. It is enough to prove the lemma for $G$ with no twins and no universal vertices. Consider the Helly arc representation $\beta = \rho \circ \beta_G$ of $G$ where $\rho$ is an arc representation of the CA hypergraph $\mathcal{B}(G)$. Fix $v$ to be a vertex whose maxclique bundle $B_v$ is minimal under inclusion. Note that $B_v \cup B_w = \mathcal{C}(G)$ for no vertex $w$ for else $w$ would be universal. Thus, for every $w$ either $B_v \subseteq B_w$ or $B_v \not\subseteq B_w$. If all $w$ satisfy the former condition, $N[v]$ is a clique and we are done (we can choose $u$ arbitrarily from $N[v]$). Otherwise fix $u$ to be a vertex with $|B_v \cap B_u|$ as small as possible. Note that $B_v \not\subseteq B_u$.

It remains to argue that $uv$ is an essential edge. By Lemma 8.2, we have to show that the intersection $N = N[u] \cap N[v]$ is a clique. Assume, to the contrary, that $N$ contains non-adjacent $x$ and $y$. Looking at the arc representation $\beta$, we see that the arcs $\beta(x)$ and $\beta(y)$ must intersect the arc $\beta(v) \cap \beta(u)$ from different sides. One of $\beta(x)$ and $\beta(y)$ must, therefore, contain the extreme point of $\beta(v)$ contained in $\beta(u)$. Without loss of generality, suppose that this is $\beta(x)$. It follows that $|\beta(v) \cap \beta(x)| < |\beta(v) \cap \beta(u)|$, giving a contradiction with the assumption that $|B_v \cap B_u|$ is the smallest possible. ■

Lemma 8.4. Let $\alpha$ be a sharp Helly representation of a graph $G$ without universal vertices. Let $C \in \mathcal{C}(G)$ be a maxclique in $G$. Consider the $C$-flipped mapping $\alpha^C : \mathcal{V}(G) \rightarrow \mathcal{A}^{\alpha(C)}$. Then $\mathcal{I} = \mathcal{A}^{\alpha(C)}$ is an interval system, that is, there are two consecutive points $x$ and $y$ on the circle such that no interval $I \in \mathcal{I}$ contains both $x$ and $y$ unless $I = [x, y]$ is the complete arc with designated extreme points $x$ and $y$ (obtained by flipping the arc $\{x, y\}$).

Proof. Since $\alpha$ is a Helly representation of $G$, the arcs in the set $\alpha(C)$ have a common point $x$. Suppose that $x$ is an extreme point of an arc $A \in \alpha(C)$. Choosing $y$ to be the point of $A$ next to $x$, we obtain the claimed pair $x, y$. ■
We remark that the sharpness condition in Lemma 8.4 is crucial. Indeed, consider the graph \( G \) and its HCA representation \( \alpha \) given in Fig. 3. The \( C_6 \)-flipped mapping \( \alpha^{C_6} \) results in a non-interval arc system.

**Lemma 8.5.** Let \( \lambda = \alpha^C \) where \( \alpha \) and \( C \) are as in Lemma 8.4. Then \( M_\lambda \) can be computed in logspace from \( M_\alpha \) and \( C \).

**Proof.** Let \( M_\lambda = (m^\lambda_{uv}) \) and \( M_\alpha = (m^\alpha_{uv}) \). We have \( m^\lambda_{uv} = m^\alpha_{uv} \) if \( v \notin C \) and \( m^\lambda_{uv} = 2n + 2 - m^\alpha_{uv} \) if \( v \in C \). For different \( u \) and \( v \), \( m^\lambda_{uv} \) is computed by inspection of several cases. If \( u \notin C \) and \( v \notin C \), then \( m^\lambda_{uv} = m^\alpha_{uv} \). If \( u \in C \) and \( v \notin C \), then

\[
\begin{align*}
\alpha(u) \cap \alpha(v) &= \emptyset \quad \Rightarrow \quad m^\lambda_{uv} = m^\alpha_{uv}; \\
\alpha(u) \subset \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = m^\alpha_{uv} - m^\alpha_{uu} + 2; \\
\alpha(u) \supset \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 0; \\
\alpha(u) \bowtie \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 2n + 2 - m^\alpha_{uu}; \\
\alpha(u) \triangleright \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = m^\alpha_{uv} - m^\alpha_{uv} + 1.
\end{align*}
\]

The case of \( u \notin C \) and \( v \in C \) is symmetric. If \( u \in C \) and \( v \in C \), then

\[
\begin{align*}
\alpha(u) \cap \alpha(v) &= \emptyset \quad \Rightarrow \quad m^\lambda_{uv} = 2n + 4 - m^\alpha_{uu} - m^\alpha_{vv}; \\
\alpha(u) \subset \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 2n + 2 - m^\alpha_{vv}; \\
\alpha(u) \supset \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 2n + 2 - m^\alpha_{uv}; \\
\alpha(u) \bowtie \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 0; \\
\alpha(u) \triangleright \alpha(v) \quad &\Rightarrow \quad m^\lambda_{uv} = 2n + 2 + m^\alpha_{uv} - m^\alpha_{uv} - m^\alpha_{vv}.
\end{align*}
\]

Recall that the relationship between \( \alpha(u) \) and \( \alpha(v) \) is recognizable by Lemma 8.5 and Definition 5.1.

Let us continue description of the algorithm. Lemma 8.1 allows us to compute the intersection matrix \( M_\alpha \). According to Lemmas 8.2 and 8.3, we are able to compute a maxclique \( C \) of \( G \) in logspace. Denote \( \lambda = \alpha^C \). By Lemma 8.4, the flipped arc system \( I = A^{(\alpha^C)} \) is actually an interval system and can be treated as such. Now we invoke the algorithm of Lemma 7.2 and compute an interval system \( J \) and a mapping \( \mu: V(G) \to J \) such that \( M_\mu = M^*_G \). By Lemma 7.1, \( J \) and \( I \) are isomorphic hypergraphs. Since \( I \) is a sharp interval system, Lemma 6.4 allows us to make \( J \) also sharp if it is not such from the very beginning. Lemma 7.1 ensures, moreover, that there is a hypergraph isomorphism \( \psi \) from \( I \) to \( J \) such that

\[
\mu = \psi \circ \lambda.
\]

By Lemma 6.3, we can assume that \( \psi \) respects extreme points. Now, we “close” the interval \( 1, \ldots, 2n \) to the circle where \( 1 \) succeeds \( 2n \) and regard both \( I \) and \( J \) as arc systems, that possibly have complete arcs with designated extreme points. The map \( \psi \) stays a hypergraph isomorphism respecting extreme points of all arcs. The last step of the algorithm is to compute the \( C \)-flipped mapping \( \mu^C: V(G) \to J^{\mu(C)} \).

By Lemma 6.1,

\[
\mu^C = \psi \circ \lambda^C = \psi \circ \alpha
\]

and \( \psi \) is a hypergraph isomorphism from \( I^{\lambda(C)} = A \) to \( J^{\mu(C)} \). It follows that, like \( \alpha \), \( \mu^C \) is a Helly arc representation of \( G \).

The proof of Theorem 1.1 is complete.
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