Bifurcation Results for a Fractional Quasilinear System with Critical Nonlinearities

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Authors’ contributions

This work was carried out in collaboration between both authors. Author ZY designed the study and guided the research. Author ZZ performed the analysis and wrote the first draft of the manuscript. Authors ZY and ZZ managed the analyses of the study. Both authors read and approved the final manuscript.

Abstract

In this paper, we prove bifurcation results for a fractional quasilinear system with critical nonlinearities. Under different assumptions on the exponents of nonlinearities, we give existence results respectively.

Keywords: Fractional p-q Laplacian; critical nonlinearities; variational methods.

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Symbol list: $(-\Delta)^s_p$ fractional $p$-Laplacian operator; $\bigoplus$ direct sum operator.

1 Introduction

In this paper, we investigate bifurcation results of non-trivial solutions ($u \neq 0$) for the following fractional elliptic system,

$$
\begin{align*}
(-\Delta)^s_p u + (-\Delta)^s_q u &= \lambda V|u|^{r-2}u + \frac{2q}{\alpha+\beta} |u|^{\alpha-2}|v|^\beta v, \quad x \in \Omega \\
(-\Delta)^s_p v + (-\Delta)^s_q v &= \mu V|v|^{r-2}v + \frac{2r}{\alpha+\beta} |v|^{\alpha-2}|u|^\beta u, \quad x \in \Omega \\
u = v = 0 & \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

(1.1)

where $\Omega$ is bounded in $\mathbb{R}^N$, $0 < s < 1$, $1 < q < p < p^*_s$, $\lambda$ and $\mu$ are positive constants, $\alpha + \beta = p^*_s = pN/(N - sp)$ is the fractional critical exponent and $(-\Delta)^s_p$ defined as

$$
(-\Delta)^s_p u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} \, dy
$$

(1.2)

where P.V. for principle value, see [1] for details.

In recent years, there has been growing interest in study of fractional elliptic equations, see [2]-[9]. Let $u = v$, $s = 1$, and $V$ is constant, problem (1.1) becomes an equation of the form

$$
-\Delta_p u - \Delta_q u = \lambda |u|^{r-2}u + |u|^{p^*_s - 2}u
$$

(1.3)

which has been studied in [10] and [11]. When $1 < r < q < p$, Li and Zhang [10] proved by Lusternik-Schnirelmann theory that there exits a $\lambda_0$ such that problem (1.3) has infinitely many solutions in $W^{1,p}_0(\Omega)$ for any $\lambda \in (0, \lambda_0)$. While for $1 < q < p < r < p^*$, Yin and Yang [11] proved by mountain pass theorem that there exists a $\lambda^* > 0$ such that for any $\lambda > \lambda^*$, problem (1.3) has a nontrivial solution in $W^{1,p}_0(\Omega)$. Later on similar results are extended by variational methods to p-q Laplacian systems in [12] and [13], and [14] considered the problem (1.1) and by mountain pass theorem prove the existence of a nontrivial solution. Thus our result in this paper can be viewed as extension of [12] to fractional sense and bifurcation results are given depending on the exponent $r$ of subcritical nonlinearities, compared with the result in [14]. Moreover, Ho and Sim [15] studied the existence result of (1.3) involving sandwich-type growth, i.e., $q < r < p$, by concentration compactness principle and Ekeland variational principle they obtained a nonnegative solution.

Recently, Figueriedo et al. [16] concerned with multiplicity results of the following equaiton

$$
(-\Delta)^s_p u = \mu |u|^{q-2}u + |u|^{p^*_s - 2}u
$$

(1.4)

where $\mu$ is a positive number, by Lusternik-Schnirelman theory, they obtain at least $cat(\Omega)$ nontrivial solutions. Very recently, Chen et al. [17] investigate the following fractional p-Laplacian equation

$$
(-\Delta)^s_p u + V(x)|u|^{p-2}u = h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N
$$

(1.5)

where $1 < q < p < r < p^*_s$, and the potential function $h_1$ and $h_2$ change sign in $\mathbb{R}^N$. By variational methods they proved (1.4) admits infinitely many solutions. Moreover, Bhakta and Mukherjee [18] considered the equation

$$
(-\Delta)^s_p u + (-\Delta)^s_q u = \theta |u|^{r-2}u + |u|^{p^*_s - 2}u
$$

(1.6)

where $2 \leq q < N(p-1)/(N-s_1) < p \leq \max\{p, p^*_s - \frac{s}{s-1}\} < r < p^*_s$, and they obtain $cat(\Omega)$ nontrivial nonnegative solutions. For more recent results on fractional equations, see ([19],[20],[21]) and references therein.
We first consider the Banach space $H = W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$, where fractional Sobolev space $W_0^{s,p}(\Omega) = \{ u \in W^{s,p}(\Omega) | u = 0 \in \mathbb{R}^N \setminus \Omega \}$ defined as follows:

$$W^{s,p}(\Omega) = \{ u \in L^p(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \}$$

equipped with the norm

$$\|u\|_p^p = \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy,$$

where $Q = \mathbb{R}^N \setminus (C \Omega \times C \Omega)$ with $C \Omega = \mathbb{R}^N \setminus \Omega$. By results of [1], the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $s \in [1, p_*]$ and compact for $s \in [1, p^*_0)$. For the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$, we denote the best Sobolev constant by $S$,

$$S := \inf_{u \in W_0^{s,p}} \frac{\|u\|_p}{\|u\|_{p^*_0}^p}$$

and

$$S_{\alpha, \beta} := \inf_{u \in H_0^1(\Omega)} \frac{\|u\|_p}{\left( \int_{\Omega} |u|^\alpha dx \right)^{\frac{\beta}{\alpha}}}$$

From Lemma 3.4 of [22], we know

$$S_{\alpha, \beta} = \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{\alpha}{\beta}} + \left( \frac{\beta}{\alpha} \right)^{\frac{\beta}{\alpha}} \right] S$$

Moreover, we assume the following

(V0) $V(x) \in L^{\frac{2p}{p+q}}(\Omega)$, and $V(x)$ is bounded below by $\sigma > 0$.

Our approach to study problem (1.1) is variational, including application of dual fountain theorem, Lusternik-Schnirelman category and Kajikiya’s critical point theorem, which investigate the energy functional of the problem in topological and analytical sense. Generally, we check the geometric structure of the functional and prove the compactness results of the functional to meet the demands of the critical point theorems. Since $r$ varies, combined with the presence of the critical nonlinearity, the energy functional no longer satisfies global compactness conditions but on different ranges respectively, thus we apply different variational theorems to the functional for multiplicity results. We improve the multiplicity result of Yin and Yang [12] for infinitely many small energy solutions and extend the result of Figueiredo et al. [16] to fractional p-q Laplacian systems. We have the following results.

**Theorem 1.1.** Assume $1 < r < q < p < p^*_0$ and $V_0$ hold. Then there exists $\Lambda^* > 0$ such that problem (1.1) admits infinitely many small energy solutions $(u_k, v_k) \in H$ satisfying $I(u_k, v_k) \to 0$ as $k \to \infty$, provided $0 < (\lambda + \mu) \leq \Lambda^*$.

**Theorem 1.2.** Assume $1 < q < p < r < p^*_0$ and $V_0$ hold with $N > p^2 s, 1 < q < p, \max\{ p_0, \frac{Nq}{N-p} : p \} < r < p^*_0$, then there exist $\lambda^*, \mu^* > 0$ such that problem (1.1) has at least $\text{cat}(\Omega)$ nontrivial solutions, provided $\lambda \in (0, \lambda^*)$ and $\mu \in (0, \mu^*)$.

**Theorem 1.3** Assume $1 < q < r < p < q^*_0$ and $V_0$ hold with $\alpha + \beta = q^*_0$. Then there exists $\Lambda_* > 0$ such that for any $\lambda, \mu > 0$ problem (1.1) has a sequence of solutions $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset H$ satisfying $(u_n, v_n) \to (0, 0)$ as $n \to \infty$, provided $0 < (\lambda + \mu) \leq \Lambda_*$. 

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2 Preliminaries

We prepare to give some auxiliary results on the energy functional. We say \((u,v)\) is a weak solution if it appears to be a critical point of the following functional

\[
I(u,v) := \frac{1}{p} \|(u,v)\|_p^p + \frac{1}{q} \|(u,v)\|_q^q - \frac{1}{r} \int_{\Omega} (\lambda V|u|^\gamma + \mu V|v|^\gamma) \, dx - \frac{2}{p^2} \int_{\Omega} |u|^\alpha |v|^\beta \, dx
\]

It is easy to see that \(I(u,v) \in C^1(X,\mathbb{R})\) and the weak solution satisfies the following

\[
\int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{N+\eta}} (\phi(x) - \phi(y)) + \int_Q \frac{|u(x) - u(y)|^q}{|x-y|^{N+\eta}} (\psi(x) - \psi(y)) + \int_Q |v(x) - v(y)|^r \, dx + \int_Q |w(x) - w(y)|^s \, dx = 0
\]

for all \( (\phi, \psi) \in X \), i.e., \((I'(u,v), (\phi, \psi)) = 0\). Next we define the Palais-Smale condition.

**Definition 2.1** We call \((u_k)\) a \((PS)_c\) sequence in \(X\) if \(I(u_k) = c + o(1)\) and \(I'(u_k) = o(1)\) as \(n \to \infty\).

The functional \(I\) satisfies \((PS)_c\) condition if every \((PS)_c\) sequence admits convergent subsequence.

Then we have the following local compactness result.

**Lemma 2.2** There exists a positive constant \(C_0\) such that \(I\) satisfies the \((PS)_c\) condition when \(c\) satisfies

\[
c \leq \frac{2}{N} \left( \frac{S_{N, \beta}}{2} \right)^{\frac{N}{p}} - C_0 (\lambda + \mu)^{\frac{p}{p-\eta}}
\]

**Proof** Let \(z_k = (u_k, v_k)\) be a \((PS)_c\) sequence,

\[
I(z_k) = c + o(1); \quad \langle I'(z_k), z_k \rangle = o(1) \|z_k\|. \tag{2.1}
\]

Argue by contradiction, it is easy to see that \(z_k\) is bounded. Since \(H\) is reflexive Banach space, there exists \(z \in H\) such that

\[
\begin{align*}
z_k & \rightharpoonup z \quad \text{in} \quad H \\
u_k & \to u \quad \text{in} \quad L^s(\Omega), v_k \to v \quad \text{in} \quad L^s(\Omega), 1 \leq s < p^* \\
(u_k, v_k) & \to (u, v) \quad \text{a.e. on} \quad \Omega
\end{align*}
\]

Note that

\[
\left\{ \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+p}} \right\}_{k \in \mathbb{N}}, \left\{ \frac{|u_k(x) - u_k(y)|^q}{|x-y|^{N+q}} \right\}_{k \in \mathbb{N}}
\]

are bounded in \(L^r(\Omega)\) and \(L^s(\Omega)\) and by the pointwise convergence \(u_k \to u\) we have,

\[
\begin{align*}
\frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+p}} & \rightharpoonup_{L^r(\Omega)} \frac{|u(x) - u(y)|^p}{|x-y|^{N+p}} \\
\frac{|u_k(x) - u_k(y)|^q}{|x-y|^{N+q}} & \rightharpoonup_{L^s(\Omega)} \frac{|u(x) - u(y)|^q}{|x-y|^{N+q}}
\end{align*}
\]

Thus we have for any \((\phi, \psi) \in H\), there holds

\[
\lim_{n \to \infty} \langle I'(u_k, v_k), (\phi, \psi) \rangle = \langle I'(u, v), (\phi, \psi) \rangle = 0 \tag{2.3}
\]
It then suffices to check that \((u_k, v_k) \to (u, v)\) in \(H\). We first show that there exists \(C_0 > 0\) satisfying

\[
I(z) \geq -C_0(\lambda + \mu)^{-\frac{p^*}{p^* - r}}
\]  

(2.4)

By (2.3) we have

\[
2 \int_\Omega |u|^\alpha |v|^\beta \, dx = \|u, v\|_p^p + \|(u, v)\|_q^q - \int_\Omega \Lambda V|u|^r + \mu V|v|^r \, dx
\]

Thus

\[
I(z) = \left( \frac{1}{p} - \frac{1}{p'} \right) \frac{1}{p'} \|z\|_p^p + \left( \frac{1}{q} - \frac{1}{q'} \right) \|z\|_q^q - \left( \frac{1}{r} - \frac{1}{r'} \right) \int_\Omega \Lambda V|u|^r + \mu V|v|^r \, dx
\]

(2.5)

By Hölder inequality and Young inequality, we derive

\[
I(z) = \left( \frac{1}{p} - \frac{1}{p'} \right) \frac{1}{p'} \|z\|_p^p + \left( \frac{1}{q} - \frac{1}{q'} \right) \|z\|_q^q - \left( \frac{1}{r} - \frac{1}{r'} \right) \int_\Omega \Lambda V|u|^r + \mu V|v|^r \, dx
\]

\[
\geq \frac{s}{N} \|z\|_p^p - \frac{p - r}{rp^*} \|V\|^\frac{p^*}{p^* - r} (\Lambda V|u|^r + \mu V|v|^r)
\]

\[
\geq \frac{s}{N} \|z\|_p^p - \frac{s}{N} \|z\|_p^p - C_0(\lambda V^\frac{p^*}{p^* - r} + \mu V^\frac{p^*}{p^* - r})
\]

\[
= -C_0(\lambda V^\frac{p^*}{p^* - r} + \mu V^\frac{p^*}{p^* - r})
\]

where \(C_0\) is a positive constant depending on \(p, r, S\) and \(V\).

Let \(\bar{u}_k = u_k - u, \bar{v}_k = v_k - v\), then by variants of Brezis-Lieb Lemma (see [3, 23]) we have

\[
\|\bar{z}_k\|_p^p = \|z_k\|_p^p - \|z\|_p^p + o(1), \|\bar{z}_k\|_q^q = \|z_k\|_q^q - \|z\|_q^q + o(1)
\]  

(2.6)

By Lemma 2.1 of Han [24], combined with (2.1),(2.6),(2.7), we have

\[
\frac{1}{p} \|\bar{z}_k\|_p^p + \frac{1}{q} \|\bar{z}_k\|_q^q - \frac{2}{p} \int_\Omega |u_k|^\alpha |v_k|^\beta \, dx = c - I(z) + o(1)
\]  

(2.7)

and

\[
\|\bar{z}_k\|_p^p + \|\bar{z}_k\|_q^q - 2 \int_\Omega |u_k|^\alpha |v_k|^\beta \, dx = o(1)
\]  

(2.8)

Thus we have

\[
\|\bar{z}_k\|_p^p \to a, \|\bar{z}_k\|_q^q \to b, \int_\Omega |u_k|^\alpha |v_k|^\beta \, dx \to l,
\]

If \(a = 0\), the proof is complete. Assume the contrary, \(a > 0\), then we have

\[
a \leq l \leq 2(S_{\alpha, \beta})^{-\frac{p^*}{p^* - r}} a^\frac{p^*}{p^* - r}
\]  

(2.9)

which implies that \(a \geq 2(S_{\alpha, \beta})^{-\frac{p^*}{p^* - r}}.\) Note that by (2.8),

\[
c = a \frac{p}{p^*} + b \frac{q}{q^*} - \frac{1}{p} + I(z)
\]

\[
> 2a \left( \frac{S_{\alpha, \beta}}{2} \right)^\frac{p^*}{p^* - r} - C_0(\lambda + \mu)^{-\frac{p^*}{p^* - r}}
\]

which contradicts the assumption, thus \(a = 0\).
3 Proof of Theorem 1.1

To prove Theorem 1.1, we need the dual fountain theorem established in [25]. Let
\[ X = \bigoplus_{j=1}^{\infty} X_j \text{ with } \dim X_j < \infty, \]
define
\[ Y_k = \bigoplus_{j=1}^{k} X_j \quad Z_k = \bigoplus_{j=k}^{\infty} X_j \] (3.1)

Lemma 3.1 (Dual fountain theorem) Let \( I \) be an even functional on Banach space \( X \), and for every \( k \geq k_0 \), there exists \( \tau_k > \delta_k > 0 \) such that
\[
\begin{align*}
(A_1) \quad & a_k = \inf \{ I(u) | u \in Z_k \cap \partial B_{\tau_k} \} \geq 0; \\
(A_2) \quad & b_k = \max \{ I(u) | u \in Y_k \cap \partial B_{\delta_k} \} < 0; \\
(A_3) \quad & d_k = \inf \{ I(u) | u \in Z_k \cap B_{\delta_k} \} \to 0 \text{ as } k \to \infty; \\
(A_4) \quad & I \text{ satisfies the } (PS)_c \text{ condition for all } c \in [d_k, 0]; \\
\end{align*}
\]
Then \( I \) has a sequence of negative critical values converging to 0.

Since \( H \) is Banach space, there exists \( \{ e_i \}_{i=1}^{\infty} \subset H \) and \( \{ e_i^* \}_{i=1}^{\infty} \subset H^* \) satisfying
\[
E = \text{span} \{ e_1, e_2, \ldots \}, \quad \text{span} \{ e_1^*, e_2^*, \ldots \} \] (3.2)
and
\[
\langle e_i^*, e_j \rangle = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases} \] (3.3)

Let \( X_i = \text{Re} e_i \), and \( Y_k \) and \( Z_k \) defined as in (3.1). We first need the following result.

Lemma 3.2 Assume \((V_0)\) holds, and define
\[
\zeta_k := \sup_{u \in Z_k : \|u\| = 1} (\int_{\Omega} V(x)|u|^r dx)^{\frac{1}{r}} \] (3.4)
then \( \zeta_k \to 0 \) as \( k \to \infty \).

Proof It is easy to see that \( 0 < \zeta_{k+1} < \zeta_k \), thus \( \zeta_k \) converges. Let \( u_k \in Z_k \) such that
\[
(\int_{\Omega} V|u_k|^r dx)^{\frac{1}{r}} > \frac{1}{2} \zeta_k > 0. \] (3.5)
Since \( H \) is reflexive, there exists a subsequence such that \( u_k \rightharpoonup u \) in \( H \), and
\[
\langle e_i^*, u \rangle = \lim_{k \to \infty} \langle e_i^*, u_k \rangle = 0, \] (3.6)
which implies that \( u_k \to 0 \) in \( H \). Then by compact embedding \( H \hookrightarrow L^r(\Omega) \) we may assume that \( u_k \to 0 \) a.e., and we have
\[
\int_{\Omega} V|u_k|^r dx \leq |V|_{\infty} \|u_k\|_{r^*}^r \leq C. \] (3.7)
By Lebesgue convergence theorem we have
\[
\lim_{k \to \infty} \int_{\Omega} V|u_k|^r dx = 0 \] (3.8)
Then from (3.5) and (3.7), we conclude that \( \zeta_k \to 0 \) as \( k \to \infty \).

Proof of Theorem 1.1 We try to check the conditions \( A(1)-A(4) \).
We will denote by $I$ satisfying (3) verified. Then from Lemma 3.1 we see that

$$I(z) = 1_p \|z\|_p^p - \frac{1}{q} \|v\|_q^q - C\|z\|_p^q - C(\lambda + \mu)\|z\|_p^q$$

where $C_i > 0$ for $i = 1, 2, 3$.

We then introduce function

$$\theta(t) = C_1 t^p - C_2(\lambda + \mu) t^r - C_3 t^r^*, \quad t > 0$$

By hypothesis $1 < r < p < p^*$, there exists $\Lambda^* > 0$ such that

(a) $\theta(t)$ attains its positive maximum between two positive solutions $t_1$ and $t_2$ of $\theta(t) = 0$.

(b) $\frac{\lambda}{\mu} (S_{r})\frac{\lambda}{\mu} - C_0(\lambda + \mu)\frac{\lambda}{\mu} \geq 0$, where $C_0$ is given in Lemma 2.2.

provided $0 < (\lambda + \mu) \leq \Lambda^*$, thus (A1) and (A4) is verified.

Verification of A(2). For all $z \in Y_k$, by equivalence of norms in finite dimensional space, we have

$$I(z) \leq \frac{1}{p} \|z\|_p^p + \frac{1}{q} \|z\|_q^q - C\|z\|_p^q - C(\lambda + \mu)\|z\|_p^q$$

By hypothesis $1 < r < p < p^*$, we can choose $\delta_k$ sufficiently small that $\delta_k < \tau_k$ and

$$b_k = \max \{ I(\mu) | u \in Y_k \cap \partial B_{\theta_k} \} < 0$$

Verification of A(3). By Lemma 3.2, we have

$$\int_{\Omega} V(x)|u|\,dx \leq (a_k \|u\|)^r$$

and notice $z_k \to 0$ a.e. as $k \to \infty$, we have

$$I(z) \geq -\frac{1}{r} a_k \rho_k^r + o(1) \to 0 \quad \text{as} \quad k \to \infty$$

Thus we have A(3) verified. Then from Lemma 3.1 we see that $I$ possesses infinitely many solutions satisfying $I(z_k) \to 0$ as $k \to \infty$ Thus completes the proof.

4 Proof of Theorem 1.2

Hereafter we assume

$$N > p^2 s, \quad 1 < q < p, \quad \max \left\{ \frac{Nq}{N - ps}, p \right\} < r < p^*$$

We introduce the Nehari manifold $N_{\lambda, \mu}$ defined as

$$N_{\lambda, \mu} = \{ z \in H \setminus \{0\} | \langle I_{\lambda, \mu}(z), z \rangle = 0 \}$$

and

$$c_N = \inf_{z \in N_{\lambda, \mu}} \{ I_{\lambda, \mu}(z) : z \in H \}$$

We will denote by $c_{\lambda, \mu}$ the mountain pass level:

$$c_{\lambda, \mu} := \inf_{t \geq 0} \sup_{z \in H \setminus \{0\}} I(tz), \quad z \in H \setminus \{0\}$$
Let condition (4.1) holds. Then there exists \( \tau \) and 
and the function

Hereafter, we define the set

Assume the contrary, there exist 

Proof The proof is standard and we refer to [14].

We then consider the barycenter function \( \tau : N_\Omega \to \mathbb{R}^N \),

and the following sets homotopically equivalent to \( \Omega \),

provided \( r > 0 \) sufficiently small. Moreover, we introduce the functional space

where \( B_r = B(0, r) \) is a ball centered at 0 with radius \( r \), and the functional \( I_{B_r} : W^{s,p}_{0,rad}(B_r) \to \mathbb{R} \) defined as

which satisfies results in Lemma 4.1. We then set

where

By standard arguments, it is known that \( m_r = c_r \), where \( c_r \) is the mountain pass level of \( I_{B_r} \). Thus we can find by Lemma 4.1 of [14] a radial function \( z_r \in N_{B_r} \) satisfying

Hereafter, we define the set

and the function \( \psi : \Omega_r^- \to \Gamma_{\lambda, \mu} \),

for all \( y \in \Omega_r^- \), where \( B(y, r) \) is the ball centered at \( y \) with radius \( r \). It is trivial that \( \psi(y) \in \Gamma_{\lambda, \mu} \), and \( \tau(\psi(y)) = y \), for all \( y \in \Omega_r^- \).

Lemma 4.2 Let condition (4.1) holds. Then there exists \( \lambda^*, \mu^* > 0 \) such that for all \( \lambda \in (0, \lambda^*) \) and \( \mu \in (0, \mu^*) \), we have \( \tau(z) \in \Omega_r^+ \) if \( z \in \Gamma_{\lambda, \mu} \).

Proof Assume the contrary, there exist \( \lambda_n, \mu_n \to 0 \) in \( \mathbb{R} \), and \( \{z_n\}_{n \in \mathbb{N}} \) in \( \Gamma_{\lambda_n, \mu_n} \) such that

Since \( z_n \in N_{\lambda_n, \mu_n} \), it is easy to see that \( I_{\lambda_n, \mu_n}(z_n) = \max_{t \geq 0}(tz_n) \) and

\[
\|z_n\|_p^p + \|z_n\|_q^q = (\lambda_n + \mu_n) \int_\Omega V(x)|z_n|^p dx + 2 \int_\Omega |u_n|^p|v_n|^q dx \tag{4.5}
\]
Multiplying both sides by $1/\|z_n\|^p$, we derive
\[
1 \leq (\lambda_n + \mu_n)S_p^{r/p}V(x)\frac{\|z_n\|^p + S_{\alpha,\beta}^{-r/p}\|z_n\|^{p^* - p}}{\|z_n\|^p + S_{\alpha,\beta}^{-r/p}\|z_n\|^{p^* - p}}
\]
for $n$ sufficiently large, where $C$ is a positive constant. Thus we know that $\|z_n\|^p$ is bounded from below by a positive constant $C'$ for $n$ sufficiently large.

Then by definition of $\Gamma_{\lambda_n, \mu_n}$ it is easy to see that $\{I_{\lambda_n, \mu_n}\}_{n \in \mathbb{N}}$ is a bounded sequence, which implies immediately $\{\|z_n\|^p\}_{n \in \mathbb{N}}, \{\int_\Omega |u_n|^\alpha|v_n|^\beta dx\}_{n \in \mathbb{N}}$, and $\{(\lambda_n + \mu_n)|z_n|^r\}_{n \in \mathbb{N}}$ are bounded. Thus there exits $l > 0$ such that, up to a subsequence,
\[
\int_\Omega |u_n|^\alpha|v_n|^\beta dx \to l \ (n \to \infty) \tag{4.6}
\]
Note that
\[
2 \int_\Omega |u_n|^\alpha|v_n|^\beta dx = \|z_n\|^p + \|z_n\|^q - (\lambda_n + \mu_n) \int_\Omega V(x)|z_n|^r dx
\]
\[
\geq \|z_n\|^p - (\lambda_n + \mu_n)S_p^{r/p}V(x)\frac{\|z_n\|^p + S_{\alpha,\beta}^{-r/p}\|z_n\|^{p^* - p}}{\|z_n\|^p + S_{\alpha,\beta}^{-r/p}\|z_n\|^{p^* - p}}
\]
\[
\geq C' - (\lambda_n + \mu_n)\tilde{C}
\]
Since $\lambda_n, \mu_n \to 0$, we get $l > 0$. We then check that there exits $t_n > 0$ satisfying
\[
\|t_n z_n\|^p = \int_\Omega t_n^{\alpha + \beta}|u_n|^\alpha|v_n|^\beta dx \tag{4.7}
\]
and up to a subsequence, $\{t_n\}_{n \in \mathbb{N}}$ is bounded. We take
\[
t_n = \left(\frac{\|z_n\|^p}{\int_\Omega |u_n|^\alpha|v_n|^\beta dx}\right)^{1/(p^* - p)}
\]

Since $\{\|z_n\|^p\}_{n \in \mathbb{N}}$ is bounded, by (4.6) we have that $\{t_n\}_{n \in \mathbb{N}}$ is bounded, up to a subsequence. Then we have
\[
\frac{S}{N}\|t_n z_n\|^p \leq I_{\lambda_n, \mu_n}(t_n z_n) + \frac{(\lambda_n + \mu_n)t_n^r}{\rho} \int_\Omega V(x)|z_n|^r dx
\]
\[
\leq I_{\lambda_n, \mu_n}(z_n) + o(1)
\]
\[
\leq 2\frac{S_{\alpha,\beta}}{N}z_n^{N/p} + o(1)
\]

Let $\tilde{z}_n = t_n z_n$, and consider $\tilde{z}_n = (\widetilde{u}_n/(\int_\Omega |\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx)^{1/p^*}, \widetilde{v}_n/(\int_\Omega |\widetilde{v}_n|^\alpha|\widetilde{v}_n|^\beta dx)^{1/p^*})$. Then it is easy to verify that
\[
\int_\Omega |\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx = 1, \quad \|\tilde{z}_n\|^p \leq S_{\alpha,\beta} \tag{4.8}
\]
Moreover, by (1.8) and (4.8), we have
\[
\|\tilde{z}_n\|^p \geq S_{\alpha,\beta}(\int_\Omega |\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta dx)^{p/p^*} = S_{\alpha,\beta} \tag{4.9}
\]
which combined with (4.8) implies
\[
\|\tilde{z}_n\|^p \to S_{\alpha,\beta} \ (n \to \infty) \tag{4.10}
\]
By \([18],\) Theorem 2.1, there exists a sequence \(\{y_n, \nu_n\}_{n \in \mathbb{N}}\) in \(\mathbb{R}^N \times \mathbb{R}^+\) such that
\[
(z_n(x), \eta_n(x)) = (\nu_n^{(n-p)/p} \tilde{a}_n(\nu_n x + y_n), \nu_n^{(n-p)/p} \tilde{a}_n(\nu_n x + y_n))
\]
has a convergent subsequence, still denoted by \(z_n, \eta_n\), with
\[
(z_n, \eta_n) \rightarrow (z, \eta) \neq 0 \text{ in } W^{s,p}(\mathbb{R}^N), \quad \nu_n \rightarrow 0, \quad y_n \rightarrow y \quad (4.11)
\]
as \(n \rightarrow +\infty\).

Assume \(\omega \in C_0(\mathbb{R}^N)\) with \(\omega(x) = x\) for all \(x \in \overline{\Omega}\), then we have
\[
\tau(z_n) = \int_{\Omega} x |u_n|^p |v_n|^p \, dx
\]
\[
= \int_{\Omega} x |u_n|^p |\tilde{a}_n|^p \, dx
\]
\[
= \int_{\mathbb{R}^N} \omega(v_n x + y_n) |\zeta_n|^p |\eta_n|^p \, dx
\]
Combined with (4.11) and Lebesgue convergence theorem we have
\[
\lim_{n \rightarrow \infty} \tau(z_n) = y \int_{\mathbb{R}^N} |\zeta|^p |\eta|^p \, dx = y \in \overline{\Omega} \quad (4.12)
\]
which contradicts with (4.4). Thus completes the proof.

We now give an estimate on the category \(\text{cat}(\Omega)\).

**Lemma 4.3** Assume condition (4.1) holds and \(\lambda \in (0, \lambda^+), \mu \in (0, \mu^+)\) as in Lemma 4.3, then
\[\text{cat}(\Gamma_{\lambda, \mu}) \geq \text{cat}(\Omega)\]

**Proof** Let \(\text{cat}(\Gamma_{\lambda, \mu}) = n\), i.e.,
\[\Gamma_{\lambda, \mu} = A_1 \cup A_2 \cup \cdots \cup A_n, \quad (4.13)\]
where \(A_i, i = 1, \ldots, n\) is closed and contractible in \(\Gamma_{\lambda, \mu}\), that is, there exist \(h_i \in C([0, 1] \times A_i, \Gamma_{\lambda, \mu})\) and \(u_i \in A_i\) satisfying
\[h_i(0, u) = u, \quad h_i(1, u) = u_i \text{ for any } u \in A_i\]
By (4.13) we have
\[B_i = \psi_{\lambda, \mu}^{-1}(A_i), \quad i = 1, \ldots, n\]
where \(B_i\) are closed sets in \(\Omega^+_\mu\) satisfying
\[\Omega^+_\mu = B_1 \cup \cdots \cup B_n\]
Consider the deformation \(\sigma_i : [0, 1] \times B_i \rightarrow \Omega^+_\mu\) defined as
\[\sigma_i(t, y) = \tau(h_i(t, \psi_{\lambda, \mu}(y))\]
By Lemma 4.3 the maps are contractions of \(B_i\) in \(\Omega^+_\mu\). Thus we have
\[\text{cat}(\Omega) = \text{cat}_{\Omega^+_\mu}(\Omega^+_\mu) \leq n\]
This completes the proof.

**Lemma 4.4** Assume \(z\) is a constrained critical point of \(I_{\lambda, \mu}\) on \(N_{\lambda, \mu}\), then it is also a critical point of \(I_{\lambda, \mu}\) on \(H\).
Proof Since $z$ is a constrained critical point of $I_{\lambda,\mu}$ on $N_{\lambda,\mu}$, it holds
\begin{equation}
I'_{\lambda,\mu}(z) = \theta J'_{\lambda,\mu}(z)
\end{equation}
for some $\theta \in \mathbb{R}$, where
\begin{align*}
J_{\lambda,\mu} &= \|z\|^p_p + \|z\|^q_q - \int_{\Omega} V(x)(\lambda|u|^r + \mu|v|^s)\,dx - 2\int_{\Omega} |u|^\alpha|v|^\beta\,dx
\end{align*}
Note that $J_{\lambda,\mu} \in C^1(H)$ and
\begin{align*}
\langle J'_{\lambda,\mu}(z), z \rangle &= p\|z\|^p_p + q\|z\|^q_q - r\int_{\Omega} V(x)(\lambda|u|^r + \mu|v|^s)\,dx - 2(\alpha + \beta)\int_{\Omega} |u|^\alpha|v|^\beta\,dx \\
&= (p - r)\|z\|^p_p + (q - r)\|z\|^q_q - 2(\alpha + \beta)\int_{\Omega} |u|^\alpha|v|^\beta\,dx < 0
\end{align*}
By (4.10) and (4.26) we have
\begin{equation}
\langle I'_{\lambda,\mu}(z), z \rangle = \theta \langle J'_{\lambda,\mu}(z), z \rangle = 0
\end{equation}
which implies $\theta = 0$. Again by (4.26) we derive $I_{\lambda,\mu}(z) = 0$, hence completes the proof.

Lemma 4.5 Assume condition (4.1) holds. Then the functional $I_{\lambda,\mu}$ constrained on $N_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for all $c \in (0, \frac{1}{2^*}(\frac{S_{\mu,\nu}}{2})^{N/p^*})$.

Proof By hypothesis, there exits a sequence $\{\theta_n\}_{n \in \mathbb{N}}$ such that
\begin{equation}
\|I_{\lambda,\mu}(z_n) - \theta_n J_{\lambda,\mu}(z_n)\| \to 0
\end{equation}
as $n \to \infty$. Thus we have
\begin{equation}
I'_{\lambda,\mu}(z_n) = \theta_n J'_{\lambda,\mu}(z_n) + o(1)
\end{equation}
Note that $\{z_n\}_{n \in \mathbb{N}}$ is bounded in $H$ and $\langle J'_{\lambda,\mu}(z_n), z_n \rangle < 0$, we may assume up to a subsequence,
\begin{equation}
\langle J'_{\lambda,\mu}(z_n), z_n \rangle \to d < 0
\end{equation}
Since $\langle I'_{\lambda,\mu}(z_n), z_n \rangle = 0$, we conclude that $\theta_n \to 0$ as $n \to \infty$, which implies $I'_{\lambda,\mu}(z_n) \to 0$. Thus the assertion follows from Lemma 4.2.

Proof of Theorem 1.2 By Lemma 4.6, $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition on $N_{\lambda,\mu}$ for all $c \in (0, \frac{1}{2^*}(\frac{S_{\mu,\nu}}{2})^{N/p^*})$. Moreover, by Lemma 4.2, $c_{\lambda,\mu}, c_N \in (0, \frac{1}{2^*}(\frac{S_{\mu,\nu}}{2})^{N/p^*})$. Then we can apply Lusternik-Schnirelman theory and Lemma 4.4 to obtain $cat(\Gamma_{\lambda,\mu}) \geq cat(\Omega)$ critical points of $I_{\lambda,\mu}$ restricted to $N_{\lambda,\mu}$. Hence Lemma 4.5 implies that $I_{\lambda,\mu}$ has at least $cat(\Omega)$ critical points. Thus completes the proof.

5 Proof of Theorem 1.3

In this section we give multiplicity results for problem (1.1) with sandwich-type and critical growth, i.e., $q < r < p < q^*_s = \alpha + \beta$. We denote the best Sobolev constant for the embedding of $W^{s,q}_0(\Omega) \hookrightarrow L^r(\Omega)$ by $S'$,
\begin{equation}
S' := \inf_{u \in W^{s,q}_0} \|u\|^r_r / |u|^{s\alpha}_{\Sigma^r}
\end{equation}
and
\begin{equation}
S'_{\alpha,\beta} := \inf_{u \in H((0,0) \cap W^{s,q}_0)} \|u\|_{\Sigma^r}^r / (\int_{\Omega} |u|^{\alpha}|v|^\beta\,dx)^{\frac{r}{\alpha + \beta}}
\end{equation}
We first introduce Kransnoselski's genus. Let $E$ be a real Banach space. A closed subset $A$ of $E$ is called symmetric if $x \in A$ implies $-x \in A$. Denote by $\Sigma$ the family of all symmetric closed sets of
E. The genus of $A$ is defined to be the least integer $n$ if there is an odd map $\varphi \in C(A, R^n \setminus \{0\})$. If $n$ does not exist, then $\gamma(A) = \infty$. Typically, $\gamma(\hat{B}) = 0$.

Following the same argument in Lemma 2.2, there exists $\Lambda_* > 0$ such that the Euler-Lagrange functional $I$ satisfies local compactness condition for some $d^* > 0$, when $d^* \leq \frac{\epsilon}{4} (\frac{S^*_\alpha}{S^*_{\alpha,\beta}})^{\frac{r}{r-q}} - C_0(\lambda + \mu)^{\frac{r}{r-q}}$, provided $\lambda + \mu < \Lambda_*$.

**Lemma 5.1** For any $k \in \mathbb{N}$, there exists $\delta(k) > 0$ such that $\gamma(\{(u, v) \in H : I(u, v) \leq -\delta(k)\} \setminus \{(0, 0)\}) \geq k$.

**Proof** Given $k \in \mathbb{N}$ and let $E_k$ be a $k$-dimensional subspace of $H$, since all norms in $E_k$ are equivalent, we define

$$\alpha_k = \sup_{\|u\|_q = 1} \|u\|_p^p, \quad \beta_k = \inf_{\|u\|_q = 1} \int_\Omega |u|^r \, dx$$

then for any $\|(u, v)\|_q = 1, (u, v) \in E_k$,

$$I(tu, tv) = \frac{p^p}{p} \|(u, v)\|_p^p + \frac{q^q}{q} \|(u, v)\|_q^q - \frac{t^r}{r} \int_\Omega (\lambda V|u|^r + \mu V|v|^r) \, dx - \frac{2q^r}{q^r} \int_\Omega |u|^r |v|^r \, dx$$

$$\leq \frac{\alpha_k}{p} p^p + \frac{1}{q} q^q - \frac{\sigma(\lambda + \mu)\beta_k}{r} t^{r-q}$$

Let

$$\vartheta(t) = \frac{\alpha_k}{p} t^{p-q} + \frac{1}{q} q^q - \frac{\sigma(\lambda + \mu)\beta_k}{r} t^{r-q}$$

Since $q < r < p$, it is easy to see that $\vartheta(t)$ attains minimum at $t_0 = \left[\frac{2q^r}{q} \int_\Omega |u|^r |v|^r \, dx\right]^{1/(p-r)}$, and there exists $t' > t_0$ such that $\vartheta(t') = 0$. Note that

$$\vartheta(t_0) = \frac{1}{q} q^q - \frac{(\lambda + \mu)\beta_k}{r} \left[\frac{2q^r}{q} \int_\Omega |u|^r |v|^r \, dx\right]^{1/(p-r)} \left[\frac{r}{q} - \frac{q}{p} - \frac{r}{q} \left(\frac{p}{q} - 1\right)\right] < 0$$

if $\sigma$ sufficiently large. Thus there exists $\delta_k > 0$ such that

$$\{(u, v) \in E_k : \|(u, v)\| = t_0\} \subset \{(u, v) \in H : I(u) \leq -\delta(k)\} \setminus \{0\}.$$  

This completes the proof.

Since $I(u)$ is not bounded below, we use truncation argument. First, by Hölder inequality,

$$I(u, v) = \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} q^q \|(u, v)\|_q^q - \frac{1}{r} \int_\Omega (\lambda V|u|^r + \mu V|v|^r) \, dx - \frac{2q^r}{q^r} \int_\Omega |u|^r |v|^r \, dx$$

$$\geq \frac{1}{q} q^q - \frac{(\lambda + \mu)S^*_\alpha^{r/q}}{r} |V(x)| \frac{2^q}{2^q} \|z\|_q^q - \frac{2}{q^r S^*_\alpha^{r/q}} \|z\|_q^q$$

$$= A\|z\| + B\|z\|^r - C\|z\|^{r_q}$$

where $A = \frac{1}{q} q^q$, $B = \frac{(\lambda + \mu)S^*_\alpha^{r/q}}{r} |V(x)| \frac{2^q}{2^q}$, $C = \frac{2}{q^r S^*_\alpha^{r/q}}$.

Consider

$$\varphi(t) := At^q - Bt' - Ct^{q'}$$

since $q < r < q'$, $\varphi(t)$ attains its maximum at some point $t_1$, which satisfies

$$(\lambda + \mu)S^*_\alpha^{r/q} |V(x)| \frac{2^q}{2^q} t_1^{q'-q} + 2S^*_\alpha^{r/q} t_1^{q'-q} = 1$$

(5.6)
Our results extend the result of solutions \( \{ u \} \) for \( q < r < p < q \). Consider
\[
\text{Lemma 5.2}
\]
Let \( \text{Lemma 5.2} \). The authors declare that they have no competing interests.

Competing Interests

The authors declare that they have no competing interests.

6 Conclusions
By our proofs on the multiplicity results of the solutions to problem (1.1), we can see that the exponent \( r \) of the subcritical nonlinearities contributes to the bifurcation result of the solutions.

When \( 1 < q < r < p < p^* \), there exists \( \Lambda^* > 0 \) such that problem (1.1) admits infinitely many small energy solutions \( \{ u_k, v_k \} \in H \) satisfying \( I( u_k, v_k ) \to 0 \) as \( k \to \infty \), provided \( 0 < ( \lambda + \mu ) \leq \Lambda^* \). When \( 1 < q < p < r < p^* \) and \( N > p^2 s, 1 < q < p, \max \{ \frac{N}{N-2}, p \} < r < p^* \), there exist \( \lambda^*, \mu^* > 0 \) such that problem (1.1) has as \( \Lambda^* \) nontrivial solutions, provided \( \lambda \in (0, \lambda^*) \) and \( \mu \in (0, \mu^*) \). When \( 1 < q < r < p < p^* \) and \( \alpha + \beta = q^* \), there exists \( \Lambda^* > 0 \) such that problem (1.1) has a sequence of solutions \( \{ ( u_n, v_n ) \}_{n \in \mathbb{N}} \subset H \) satisfying \( ( u_n, v_n ) \to (0,0) \) as \( n \to \infty \), provided \( 0 < ( \lambda + \mu ) \leq \Lambda^* \). Our results extend the result of [12] to fractional sense and improve the result of [14] to bifurcation results.

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References

[1] Di Nezza E, Palatucci G, Valdinoci E. Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 2012;136:521-573.
[2] Servadei R, Valdinoci E. Variational methods for non-local operators of elliptic type. Discret. Contin. Dyn. Syst. 2013;33(5):2105-2137.
[3] Brasco L, Squassina M, Yang Y. Global compactness results for nonlocal problems. Discrete Contin. Dyn. Syst. Ser. S. 2018;11(3):391-424.
[4] Fiscella A, Bisci G, Servadei R. Multiplicity results for fractional Laplace problems with critical growth, Manuscripta Math. 2018;155:369-388.
[5] Barrios B, Colorado E, Servadei R, Soria F. A critical fractional equation with concave-convex power nonlinearities. Ann. Inst. H. Poincare Anal. Non Lineaire. 2015;32:875-900.
[6] Fiscella A, Pucci P. On certain nonlocal Hardy-Sobolev critical elliptic Dirichlet problems. Adv. Differ. Equ. 2016;21:571-599.
[7] Servadei R, Valdinoci E. The Brezis-Nirenberg result for the fractional Laplacian. Trans. Amer. Math. Soc. 2015;367(1):67-102.
[8] Servadei R, Valdinoci E. A Brezis-Nirenberg result for non-local critical equations in low dimensions. Commun. Pure Appl. Anal. 2013;12:2445-2464.
[9] Mosconi S, Perera K, Squassina M, Yang Y. The Brezis-Nirenberg problem for the fractional p-Laplacian. Calc. Var. Partial Differ. Equ. 2016;55:55-105.
[10] Li G, Zhang G. Multiple solutions for the p-q Laplacian problem with critical exponent. Acta Math. Sci. Ser. B Engl. Ed. 2009;29(4):903-918.
[11] Yin H, Yang Z. Multiplicity of positive solutions to a p-q-Laplacian equation involving critical nonlinearity. Nonlinear Anal. 2012;75:3021-3035.
[12] Yin H, Yang Z. Existence of positive solutions for a class of quasilinear elliptic system with concave-convex nonlinearities. J. Appl. Math. Informatics. 2011;29:921-936.
[13] Yin H. Existence of multiple positive solutions for a p-q Laplacian system with critical nonlinearities. J. Math. Anal. Appl. 2013;403:200-214.
[14] Chen W. Existence of solutions for critical fractional p,q Laplacian system. Complex Variables and Elliptic Equations; 2020.
[15] Ho K, Sim I. An existence result for (p,q)-Laplace equations involving sandwich-type and critical growth. Appl. Math. Letters. 2021;111.
[16] Figueiredo F, Bisci G, Servadei R. The effect of the domain topology on the number of solutions of fractional Laplace problems. Calc. Var. 2018;103.
[17] Chen Q, Chen C, Shi Y. Multiple solutions for fractional p-Laplace equation with concave-convex nonlinearities. Bound. Value Prob. 2020;63.
[18] Bhakta M, Mukherjee D. Multiplicity results for (p, q) fractional elliptic equations involving critical nonlinearities. Adv. Differential Equations. 2019;24:185-228.
[19] Zhi Z, Yang Z. On a fractional p-q Laplacian equation with critical nonlinearity. J. Ineq. Appl. 2020;183.
[20] Chen C, Bao J. Existence, nonexistence, and multiplicity of solutions for the fractional p-q-Laplacian equation in R^N. Bound. Value Prob. 2016;153.
[21] Behboudi F, Razani A, Oveisah M. Existence of a mountain pass solution for a nonlocal fractional (p, q)-Laplacian problem. Bound. Value Prob. 2020;149.
[22] Chen W, Squassina M. Critical nonlocal systems with concave-convex towers. Adv. Nonlinear Stud. 2016;16:821-842.

[23] Brezis H, Lieb E. A relation between pointwise convergence of functions and convergence of functional. Proc. Amer. Math. Soc. 1983;88:486-490.

[24] Han P. The effect of the domain topology on the number of positive solutions of elliptic systems involving critical Sobolev exponents. Houston J. Math. 2006;32:1241-1257.

[25] Bartsch T, Willem M. On an elliptic equation with concave and convex nonlinearities. Proc. Amer. Math. Soc. 1995;123:3555-3561.

[26] Kajikiya R. A critical-point theorem related to the symmetric mountain-pass lemma and its applications to elliptic equations. J. Funct. Anal. 2005;225:352-370.

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