Roles of antinucleon degrees of freedom in the relativistic random phase approximation

Haruki Kurasawa¹,∗ and Toshio Suzuki²

1Department of Physics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan
2University of Fukui, Fukui 910-8507, Japan
∗E-mail: kurasawa@faculty.chiba-u.jp

Received August 22, 2015; Accepted September 17, 2015; Published November 16, 2015

The roles of antinucleon degrees of freedom in the relativistic random phase approximation (RPA) are investigated. The energy-weighted sum of the RPA transition strengths is expressed in terms of the double commutator between the excitation operator and the Hamiltonian, as in nonrelativistic models. The commutator, however, should not be calculated in the usual way in the local field theory, because, otherwise, the sum vanishes. The sum value obtained correctly from the commutator is infinite, owing to the Dirac sea. Most of the previous calculations take into account only some of the nucleon–antinucleon states, in order to avoid divergence problems. As a result, RPA states with negative excitation energy appear, which make the sum value vanish. Moreover, disregarding the divergence changes the sign of nuclear interactions in the RPA equation that describes the coupling of the nucleon particle–hole states with the nucleon–antinucleon states. Indeed, the excitation energies of the spurious state and giant monopole states in the no-sea approximation are dominated by these unphysical changes. The baryon current conservation can be described without touching the divergence problems. A schematic model with separable interactions is presented, which makes the structure of the relativistic RPA transparent.

1. Introduction

It has been shown by many authors [1–4] that relativistic models, assuming nuclei to be composed of Dirac particles and various mesons, work well phenomenologically to reproduce nuclear static and dynamic properties. In most of the calculations, however, antinucleon (N̄) degrees of freedom are not fully taken into account, in spite of the fact that they are one of the main differences between relativistic and nonrelativistic models. The reason why the full space is not used is because there exist divergence problems [1,5,6] that are not yet handled properly for finite nuclei.

Although not all the space was included in the previous calculations, some of the N̄ degrees of freedom were taken into account, aiming to keep some fundamental principles and to reproduce the gross properties of nuclei.

For example, in the random phase approximation (RPA), the baryon current conservation required some excitations of antinucleons [5–7]. It was necessary for the description of the center-of-mass motion to have the Landau–Migdal parameter F₁, taking into account some of the N (nucleon)–N̄ states [7,8]. In the same way, the spurious states in RPA [9,10] were described with those N–N̄ excitations in addition to nucleon particle–hole states. Furthermore, an abnormal enhancement of the isoscalar magnetic moment in the Hartree approximation demanded corrections from the N–N̄ excitations through F₁ [11].
For the above reasons, two ways to include the $N$ space, avoiding the divergence problem, were proposed. One is the no-free-term approximation (NFA), which simply neglects the divergent part of the RPA response functions. The remaining part is composed of transitions of antinucleons, which are in fact Pauli-blocked [5,6], to the Fermi sea. The other is the no-sea approximation (NSA), to assume that all the $N$ states are empty. There is no divergence problem in this case either, but transitions of nucleons in the Fermi sea to $N$ states are permitted with negative excitation energies [10]. Even though both methods look unreasonable, they have been widely used, since the experimental data are well reproduced phenomenologically [3,12–14]. In fact, as will be seen later explicitly, NFA and NSA are equivalent to each other in RPA [10].

The purpose of the present paper is to investigate the structure of the relativistic RPA in detail, and, in particular, the role of the $N$ degrees of freedom there. The relativistic RPA can be developed in almost the same way as nonrelativistic ones. It will be shown, however, that a careful treatment is required for the $N$ degrees of freedom. They cause divergence problems, but cannot simply be ignored as in NFA and NSA.

It will be shown that the energy-weighted sum of the RPA transition strengths is expressed formally in terms of the double commutator between the transition operator and the Hamiltonian. This RPA sum rule is the same as in the nonrelativistic case. The commutator, however, should not be calculated employing the usual rule in the local field theory. Otherwise, the sum vanishes, in spite of the fact that it should be positive definite. The correct calculation of the commutator gives the sum value to be infinite, because of $N-N$ excitations.

In NFA and NSA, RPA states with negative excitation energy appear, in addition to those with positive energy. This is due to the fact that the full $N-N$ excitations are not taken into account. As a result, the energy-weighted sum value of the excitation strengths vanishes. Moreover, disregarding the divergence in NSA and NFA gives rise to other unphysical results. The sign of the nuclear interactions is changed in the RPA equation relevant to the remaining $N-N$ states. Because of this fact, attractive (repulsive) forces work as repulsive (attractive) ones in the coupling of nucleon particle–hole states with the $N-N$ states considered in NFA and NSA. These effects are not negligible, but dominate the excitation energies of some low-lying states. For example, the previous numerical calculations could reproduce a spurious state [9,10] and giant monopole states [12–14], invoking those unphysical effects hidden in the RPA equation.

The above defects of NFA and NSA have not been investigated explicitly so far, since the structure of the relativistic RPA formulae may be rather complicated, compared with nonrelativistic ones. In the next section, we will briefly review the relativistic RPA, focusing on the roles of $N-N$ states in various approximations. In Sect. 3, the energy-weighted sum of the RPA excitation strengths is discussed, according to a recent new insight into the longstanding problem of the relativistic sum rule [15]. In Sects. 4 and 5, we will explore in detail why NFA and NSA seemed to describe well the continuity equation and the spurious state without the full $N-N$ excitations.

It is well known that a schematic model with separable interactions helps us to understand the structure of nonrelativistic RPA [16]. In Sect. 6, a similar model will be introduced for the relativistic RPA, in order to make clear the present discussions. The final section is devoted to a brief summary of the present paper.

2. Relativistic RPA

The relativistic RPA was formulated in various ways [1,5,6,9,10]. Taking notice of the dependence on $N-N$ states, let us briefly review it.
We begin with the Hartree polarization function for arbitrary $4 \times 4$ matrices, $A$ and $B$ [6],

$$i \Pi_H(A(1), B(2)) = \left| \left\langle T \left( \bar{\psi}(1) A \psi(1) \bar{\psi}(2) B \psi(2) \right) \right\rangle \right|,$$

(1)
defined for the Hartree ground state $. The baryon field $\psi(x)$ is written using the complete set of the eigenfunction, $\varphi_\alpha$, of the Hartree Hamiltonian,

$$\psi(x) = \sum_\alpha \varphi_\alpha(x) e^{-i E_\alpha t} a_\alpha,$$

(2)

where $E_\alpha$ denotes the eigenvalue of $\varphi_\alpha$, and $a_\alpha$ plays the role of the annihilation or creation operator of $N$ and $\bar{N}$, satisfying $\{ a_\alpha, a_\beta^\dagger \} = \delta_{\alpha\beta}$, etc. Then, together with the closure property of $\varphi_\alpha$,

$$\sum_\alpha \varphi_\alpha(x) \varphi_\alpha^\dagger(x') = \delta(x - x'),$$

(3)

the baryon field satisfies the anticommutation relation,

$$\{ \psi_a(x_0, x), \psi_b^\dagger(x_0, x') \} = \delta_{ab} \delta(x - x'),$$

(4)

where $a$ and $b$ denote the Dirac matrix indices. Since the simple interactions are rather advantageous for our purpose of investigating the details of the relativistic RPA structure, we assume, throughout the present paper, the $\sigma$–$\omega$ model [1], which provides us with the Hartree Hamiltonian as

$$h(x) = -i \alpha \cdot \nabla + \gamma_0 M + V_H(x), \quad V_H(x) = \gamma_0 \Sigma(x).$$

(5)

In the above equation, we have employed the following abbreviation for the potential:

$$\Sigma(x) = V_\sigma(x) + \gamma_\mu V_\mu(x)$$

(6)

with

$$V_\sigma(x) = g_\sigma^2 \int d^4y D_\sigma(x - y) \left| \left| \bar{\psi}(y) \psi(y) \right| \right|,$$

(7)

$$V_\mu(x) = g_\omega^2 \int d^4y D_\omega(x - y) \left| \left| \bar{\psi}(y) \gamma_\mu \psi(y) \right| \right|,$$

(8)

g$\sigma$ and $g_\omega$ being the coupling constants, and $D_\sigma$ and $D_\omega$ denoting the $\sigma$- and $\omega$-meson propagators, respectively [6]:

$$D_\sigma(x) = \int \frac{dk_0}{2\pi} e^{-ik_0 x_0} D_\sigma(k_0, x), \quad D_\sigma(k_0, x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{k_\mu k^\mu - m_\sigma^2 + i\epsilon},$$

(9)

$$D_\omega(x) = \int \frac{dk_0}{2\pi} e^{-ik_0 x_0} D_\omega(k_0, x), \quad D_\omega(k_0, x) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{k_\mu k^\mu - m_\omega^2 + i\epsilon}.$$

(10)

Here, $m_\sigma$ and $m_\omega$ represent the masses of the $\sigma$- and $\omega$-mesons.
The calculation of Eq. (1) gives [6]

\[ \Pi_{H}(A(1), B(2)) = \frac{1}{2\pi} \int d\omega e^{-i\omega(t_{1} - t_{2})}\Pi_{H}(A(x_{1}), B(x_{2}), \omega), \]  

(11)

where we have defined

\[ \Pi_{H}(A(x_{1}), B(x_{2}), \omega) = \sum_{\alpha\beta} \bar{\varphi}_{\beta}(x_{1})A\varphi_{\alpha}(x_{1})\bar{\varphi}_{\alpha}(x_{2})B\varphi_{\beta}(x_{2})\left(\Pi_{H}\right)_{\alpha\beta}(\omega) \]  

(12)

with

\[ \left(\Pi_{H}\right)_{\alpha\beta}(\omega) = \frac{N_{\alpha\beta}}{\omega - E_{\alpha\beta} + i\epsilon} - \frac{N_{\beta\alpha}}{\omega - E_{\alpha\beta} - i\epsilon}. \]

(13)

In this equation, \( E_{\alpha\beta} \) stands for the excitation energy, \( E_{\alpha} - E_{\beta} \), and \( N_{\alpha\beta} \) implies the transition from the occupied to the unoccupied state as

\[ \theta_{\alpha} = \begin{cases} 1, & \alpha : \text{occupied state in } | \rangle, \\ 0, & \alpha : \text{unoccupied state in } | \rangle. \end{cases} \]  

(14)

Defining the Fermi energy by \( E_{F} \), it is convenient to employ the following notations for \( N_{\alpha\beta} \):

\[ p_{\alpha} = \theta(E_{\alpha} - E_{F}), \quad h_{\alpha} = \theta(E_{\alpha} - E_{-\alpha}), \quad \tilde{n}_{\alpha} = \theta(-E_{\alpha}), \]  

(15)

where \( p_{\alpha}, h_{\alpha}, \) and \( \tilde{n}_{\alpha} \) indicate the particle, hole, and unoccupied states, respectively. In the full space calculation, the occupied states are expressed by \( (h + \tilde{n}) \), and unoccupied states by \( p \), so that \( N_{\alpha\beta} \) is written as \( N_{\alpha\beta} = p_{\alpha}(h_{\beta} + \tilde{n}_{\beta}) \). In NSA, where the \( N \) states are empty, we have \( N_{\alpha\beta} = \text{big}(p_{\alpha} + \tilde{n}_{\beta})h_{\beta} \). In writing \( N_{\alpha\beta} = p_{\alpha}h_{\beta} - h_{\alpha}\tilde{n}_{\beta} + (p_{\alpha} + h_{\alpha})\tilde{n}_{\beta} \) for the full space, NFA neglects the last term, which describes the vacuum polarization, to produce the divergence [5,6]. In the approximation neglecting all the \( N \) states (\( \text{No}N \)), \( N_{\alpha\beta} \) is simply given by \( p_{\alpha}h_{\beta} \). Thus, the difference between the various approximations can be represented by \( N_{\alpha\beta} \), reading it as

\[ N_{\alpha\beta} = \begin{cases} p_{\alpha}h_{\beta} + p_{\alpha}\tilde{n}_{\beta} & \text{(Full)}, \\ p_{\alpha}h_{\beta} + \tilde{n}_{\alpha}h_{\beta} & \text{(NSA)}, \\ p_{\alpha}h_{\beta} - h_{\alpha}\tilde{n}_{\beta} & \text{(NFA)}, \\ p_{\alpha}h_{\beta} & \text{(No}N \). \end{cases} \]  

(16)

We define the inverse of \( \Pi_{H} \) by \( \Pi_{H}^{-1} \) as

\[ \sum_{\alpha''\beta''} \left(\Pi_{H}^{-1}\right)_{\alpha\beta,\alpha''\beta''}\left(\Pi_{H}\right)_{\alpha''\beta'',\alpha'\beta'} = I_{\alpha\beta,\alpha'\beta'}. \]

where the following abbreviations are used:

\[ \left(\Pi_{H}\right)_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\left(\Pi_{H}\right)_{\alpha\beta}, \quad I_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\left(N_{\alpha\beta}^{2} + N_{\beta\alpha}^{2}\right). \]

For \( N_{\alpha\beta}^{2} + N_{\beta\alpha}^{2} = 1 \), we obtain

\[ \left(\Pi_{H}^{-1}(\omega)\right)_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\left(N_{\alpha\beta} - N_{\beta\alpha}\right)(\omega - E_{\alpha\beta}). \]

(17)
Because of \( p_\alpha + h_\alpha + \bar{n}_\alpha = 1 \), the above \( (N_{\alpha\beta} - N_{\beta\alpha}) \) is given for each approximation as

\[
N_{\alpha\beta} - N_{\beta\alpha} = \begin{cases} 
(h_\beta + \bar{n}_\beta) - (h_\alpha + \bar{n}_\alpha) & \text{(Full)}, \\
h_\beta - h_\alpha & \text{(NSA, NFA)}, \\
p_\alpha h_\beta - p_\beta h_\alpha & \text{(NoN)}.
\end{cases}
\]

Unlike \( \Pi_H \), the inverse \( \Pi_H^{-1} \) of NSA is the same as for NFA. This fact makes NSA and NFA equivalent to each other in RPA, as seen later.

The RPA polarization function is written in terms of \( \Pi_H \) [6]:

\[
\Pi(A(1), B(2)) = \Pi_H(A(1), B(2)) + g_\eta^2 \int d^4x d^4y \Pi_H(A(1), \Gamma^{\eta}(x)) D_\eta(x - y) \Pi(\Gamma_\eta(y), B(2)),
\]

where \( \eta = -1 \) for \( \sigma \) and, hence, \( \Gamma_\eta = 1 \) for \( \eta = -1 \), and \( \gamma_{\mu} \) for \( \eta = \mu \). As in \( \Pi_H \), we write Eq. (19) in the form

\[
\Pi(A(1), B(2)) = \frac{1}{2\pi} \int d\omega e^{-i\omega(t_1 - t_2)} \Pi(A(x_1), B(x_2), \omega),
\]

with

\[
\Pi(A(x_1), B(x_2), \omega) = \Pi_H(A(x_1), B(x_2), \omega)
\]

\[
+ g_\eta^2 \int d^3x d^3y \Pi_H(A(x_1), \Gamma^{\eta}(x), \omega) D_\eta(\omega, x - y) \Pi(\Gamma_\eta(y), B(x_2), \omega).
\]

The above \( \Pi(A(x_1), B(x_2), \omega) \) is described as

\[
\Pi(A(x_1), B(x_2), \omega) = \sum_{\alpha\beta\alpha'\beta'} \overline{\Psi}_\beta(x_1) A_{\alpha\beta}(x_1) \Pi_{\alpha\beta,\alpha'\beta'}(\omega) \overline{\Psi}_{\alpha'}(x_2) B_{\beta'}(x_2),
\]

where we have defined

\[
\Pi_{\alpha\beta,\alpha'\beta'}(\omega) = \delta_{\alpha\alpha'}\delta_{\beta\beta'}(\Pi_H)_{\alpha\beta}(\omega) + \sum_{\alpha''\beta''} (\Pi_H)_{\alpha\beta}(\omega) V_{\alpha\beta',\alpha''\beta''}(\omega) \Pi_{\alpha'',\beta''}(\omega),
\]

using the notation

\[
V_{\alpha\beta',\alpha''\beta''}(\omega) = g_\eta^2 \int d^3x_1 d^3x_2 \overline{\Psi}_\alpha(x_1) \Gamma^{\eta}\beta(x_1) D_\eta(\omega, x_1 - x_2) \overline{\Psi}_{\alpha'}(x_2) \Gamma_\eta\beta''(x_2).
\]

This is also written as

\[
V_{\alpha\beta',\alpha''}(\omega) = \langle \alpha, \beta' | V_{12}(\omega) | \beta, \alpha'' \rangle,
\]

with

\[
V_{12}(\omega) = g_\eta^2 \left( i \gamma_0 \Gamma^{\eta}(\omega) \right)_1 \left( i \gamma_0 \Gamma_\eta \right)_2 D_\eta(\omega, x_1 - x_2).
\]

In relativistic models, nuclear interactions contain \( \omega \)-dependence coming from Eqs. (9) and (10). For later discussions of the relativistic RPA, however, we have to neglect this retardation effect, as in nonrelativistic RPA. Fortunately, their contributions to the interactions have been shown to be negligible in Ref. [17], when \( m_\sigma, m_\omega > \omega \). Hence, from now on we will develop the RPA, assuming \( V_{12}(0) \).
Equation (22) is formally described as follows:

\[ \Pi(\omega) = \Pi_H(\omega) + \Pi_H(\omega)V\Pi(\omega), \quad (\Pi_H)_{\alpha\beta,\alpha'\beta'}(\omega) \equiv \delta_{\alpha\alpha'}\delta_{\beta\beta'}(\Pi_H)_{\alpha\beta}(\omega). \tag{24} \]

Then, the inverse of \( \Pi(\omega) \) can be written as \( \Pi^{-1} = \Pi_H^{-1} - V \) with

\[ (\Pi^{-1}(\omega))_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}(N_{\alpha\beta} - N_{\beta\alpha})(\omega - E_{\alpha\beta}) - V_{\alpha\beta',\beta'\alpha'}. \tag{25} \]

Now let us define the eigenvector \( C_{\alpha\beta}^{(n)} \) of the above equation [18],

\[ \Pi^{-1}(\omega_n) C_{\alpha\beta}^{(n)} = 0, \]

which is

\[ (N_{\alpha\beta} - N_{\beta\alpha}) E_{\alpha\beta} C_{\alpha\beta}^{(n)} + \sum_{\alpha'\beta'} V_{\alpha'\beta',\beta'\alpha'} C_{\alpha'\beta'}^{(n)} = (N_{\alpha\beta} - N_{\beta\alpha}) \omega_n C_{\alpha\beta}^{(n)}. \tag{26} \]

In writing

\[ C_{\alpha\beta}^{(n)} = N_{\alpha\beta} X_{\alpha\beta}^{(n)} + N_{\beta\alpha} Y_{\beta\alpha}^{(n)}, \tag{27} \]

the coupled equations are obtained:

\[ E_{\alpha\beta} X_{\alpha\beta}^{(n)} + \sum_{\alpha'\beta'} N_{\alpha'\beta'}(V_{\alpha'\beta',\beta'\alpha'} X_{\alpha'\beta'}^{(n)} + V_{\alpha'\beta',\beta'\alpha'} Y_{\alpha'\beta'}^{(n)}) = \omega_n X_{\alpha\beta}^{(n)}, \tag{28} \]

\[ E_{\alpha\beta} Y_{\alpha\beta}^{(n)} + \sum_{\alpha'\beta'} N_{\alpha'\beta'}(V_{\alpha'\beta',\beta'\alpha'} Y_{\alpha'\beta'}^{(n)} + V_{\alpha'\beta',\beta'\alpha'} X_{\alpha'\beta'}^{(n)}) = -\omega_n Y_{\alpha\beta}^{(n)}. \]

Employing the abbreviations

\[ A_{\alpha\beta,\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}N_{\alpha\beta} E_{\alpha\beta} + N_{\alpha\beta} V_{\alpha'\beta',\beta'\alpha'} N_{\alpha'\beta'}, \]

\[ B_{\alpha\beta,\alpha'\beta'} = N_{\alpha\beta} V_{\alpha'\beta',\beta'\alpha'} N_{\alpha'\beta'}. \tag{29} \]

Eq. (28) provides us with the relativistic RPA equation of the form

\[ \sum_{\alpha'\beta'} \begin{pmatrix} A_{\alpha\beta,\alpha'\beta'} & B_{\alpha\beta,\alpha'\beta'} \\ B^*_{\alpha\beta,\alpha'\beta'} & A^*_{\alpha\beta,\alpha'\beta'} \end{pmatrix} \begin{pmatrix} X_{\alpha'\beta'}^{(n)} \\ Y_{\alpha'\beta'}^{(n)} \end{pmatrix} = \omega_n N_{\alpha\beta} \begin{pmatrix} X_{\alpha\beta}^{(n)} \\ -Y_{\alpha\beta}^{(n)} \end{pmatrix}. \tag{30} \]

When the \( \tilde{N} \) degrees of freedom in \( N_{\alpha\beta} \) are neglected, the above equation reduces to the well known nonrelativistic RPA equation [16,19].

The relationship between the Full and NFA variations can be seen in Eq. (27). Comparing \( N_{\alpha\beta} \) of NFA with that of the Full case, some of the \( N-\tilde{N} \) excitations in NFA have a minus sign, as \(-h_{\alpha\beta}\), because of neglecting the divergent part, as mentioned before. This additional sign induces unphysical effects in NFA in which attractive (repulsive) forces work as repulsive (attractive) ones in the \( h_{\alpha\beta}\)-dependent part of the RPA equation, Eq. (28).
It is also seen from Eq. (27) that NSA is equivalent to NFA. In writing Eq. (27) explicitly for NSA and NFA as

\[
C_{\alpha\beta}^{\text{NSA}} = (p_\alpha h_\beta + \bar{n}_\alpha h_\beta) \chi_{\alpha\beta}^{\text{NSA}} + (p_\beta h_\alpha + \bar{n}_\beta h_\alpha) \chi_{\beta\alpha}^{\text{NSA}},
\]

\[
C_{\alpha\beta}^{\text{NFA}} = (p_\alpha h_\beta - h_\beta \bar{n}_\alpha) \chi_{\alpha\beta}^{\text{NFA}} + (p_\beta h_\alpha - h_\alpha \bar{n}_\alpha) \chi_{\beta\alpha}^{\text{NFA}},
\]

the replacements

\[X_{ph}^{\text{NSA}} = X_{ph}^{\text{NFA}}, \quad X_{nh}^{\text{NSA}} = -Y_{nh}^{\text{NFA}}, \quad Y_{ph}^{\text{NSA}} = Y_{ph}^{\text{NFA}}, \quad Y_{nh}^{\text{NSA}} = -Y_{nh}^{\text{NFA}}\]

lead to

\[C_{\alpha\beta}^{\text{NSA}} = C_{\alpha\beta}^{\text{NFA}}.\]

Hence, NSA yields the same unphysical change of sign in the interactions, and the same eigenvalues, as in NFA.

We add a few comments that we need for later discussions. First, the complex conjugate of Eq. (26) implies that \(C_{\beta\alpha}^{(n)*}\) is also its solution when the eigenvalue is \(-\omega_n\). Second, the orthogonality and normalization of the eigenvectors are written as

\[\sum_{\alpha\beta} (N_{\alpha\beta} - N_{\beta\alpha}) C_{\alpha\beta}^{(n)*} C_{\alpha\beta}^{(n')} = \delta_{nn'} (\lambda_n = \pm 1).\]  

(31)

The eigenvectors \(C_{\alpha\beta}^{(n)}\) with \(\lambda_n = 1\) describe the RPA excited states. Thus, the norm of the eigenvectors in NSA is also the same as in NFA. Third, the closure relation is given by

\[\sum_n \lambda_n C_{\alpha\beta}^{(n)} C_{\alpha'\beta'}^{(n)} = \delta_{aa'} \delta_{\beta\beta'} (N_{\alpha\beta} - N_{\beta\alpha}).\]  

(32)

Finally, we note that Eqs. (30) and (31) provide us with

\[\omega_n \lambda_n = \left(\chi^{(n)*} \gamma^{(n)*}\right) \mathcal{M} \left(\chi^{(n)} \gamma^{(n)}\right), \quad \mathcal{M} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}.\]  

(33)

In nonrelativistic models, we usually obtain \(\omega_n \geq 0\) for \(\lambda_n = 1\) [16,19]. In NSA and NFA, however, it is not necessary for \(\mathcal{M}\) to be positive definite, so there may be negative eigenvalues \(\omega_n < 0\) even for \(\lambda_n = 1\).

3. **Energy-weighted sum of the excitation strengths**

One of the reasons why RPA has been extensively used in nonrelativistic nuclear models is because the sum of the energy-weighted strengths for exciting the RPA states is constrained in a simple way by the Hartree or Hartree–Fock basis used. Once we have the double commutator of the one-body Hermitian operator \(\hat{F}\),

\[\hat{F} = \int d^3x \psi^\dagger(x) F \psi(x),\]  

(34)

with the Hamiltonian \(H\), the sum is equal to its expectation value of the Hartree or Hartree–Fock ground state [16,19],

\[\sum_n \omega_n |n|\hat{F}|0\rangle|^2 = \frac{1}{2} ||[\hat{F}, [H, \hat{F}]]||.\]  

(35)

At first glance, however, it seems that the above nonrelativistic rule is not extended to the relativistic RPA. If the operator \(F\) has the only coordinate dependence, the double commutator with the Dirac
equation vanishes,
\[ \left[ F(x), [h_0, F(x)] \right] = 0, \]
since \( h_0 \) contains the only linear derivative,
\[ h_0(x) = -i\alpha \cdot \nabla + \gamma_0 M, \]
in contrast to the nonrelativistic kinetic energy.

In fact, it was a longstanding problem for more than 50 years in relativistic local field theory that the energy-weighted sum could not be expressed with the double commutator [20–22]. Recently, this paradox has been solved by the present authors [15]. The expression with the commutator itself is correct, but the commutator should not be calculated according to the usual way in the local field theory, where the anticommutator Eq. (4) is employed. Equation (4) holds in the infinite momentum space. The commutator should be calculated first in a finite momentum space. Next, its ground-state expectation value is evaluated, and then we make the momentum space infinite to obtain a positive sum value.

In other words, in the local field theory, the commutator between the time and space components of the nuclear four-current disappears, causing the sum value to vanish. The commutator calculated in the finite momentum space, however, does not vanish, and its ground-state expectation value exists even in the infinite momentum space.

Keeping this new insight in mind, let us investigate whether or not the same relationship as in Eq. (35) holds in the relativistic RPA. We will first derive an expression for the double commutator of the one-body operator with the total Hamiltonian, and next calculate the sum of the energy-weighted strengths for exciting the RPA states, in the same way as in the nonrelativistic RPA [16,19].

The one-body operators, \( \hat{P} \) and \( \hat{Q} \), are written as
\[ \hat{P} = \int d^3 x \psi^\dagger(x) P \psi(x), \quad \hat{Q} = \int d^3 x \psi^\dagger(x) Q \psi(x). \]
Here, it is not necessary for \( P \) and \( Q \) to be a function of the only coordinate. The Hamiltonian in the \( \sigma-\omega \) model is given by
\[ H = H_0 + H_{\text{int}}, \]
where each term is described as
\[ H_0 = \int d^3 x \psi^\dagger(x) h_0 \psi(x), \quad H_{\text{int}} = \frac{1}{2} \int d^3 x_1 d^3 x_2 \psi^\dagger(x_1) \psi^\dagger(x_2) V_{12} \psi(x_2) \psi(x_1). \]

When we employ Eq. (4) as in the local field theory, the expectation value of the double commutator with the one-body Hamiltonian \( H_0 \) is expressed as
\[ \langle \left[ \left[ \hat{P}, [H_0, \hat{Q}] \right] \right] \rangle = \sum_\alpha \theta_\alpha |\alpha \rangle \left[ \left[ P, [h_0, Q] \right] \right] |\alpha \rangle. \]
As mentioned before, if the operators \( P \) and \( Q \) depend on the only coordinate, the above double commutator vanishes. For a while, however, we keep the above form, and will discuss later how the commutator should be calculated.
Finally, Eqs. (39), (41), and (42) lead to
\[
\sum\limits_{\alpha\alpha'} \theta_{\alpha}\theta_{\alpha'} \langle \alpha, \alpha' | P_1 V_{12} Q_1 | \alpha, \alpha' \rangle = \sum\limits_{\alpha\beta\beta'} \theta_{\alpha}\langle \alpha | P_1 | \beta \rangle \langle \beta | V_H | \beta \rangle \langle \beta | Q | \alpha \rangle = \sum\limits_{\alpha} \theta_{\alpha}\langle \alpha | P V_H Q | \alpha \rangle,
\]
(40)
derived from the closure property in the intermediate states, the above \(S_1\) is described in terms of the Hartree potential of Eq. (5):
\[
S_1 = \sum\limits_{\alpha} \theta_{\alpha} \langle \alpha | [P, [V_H, Q]] | \alpha \rangle.
\]
(41)

A similar calculation gives
\[
S_2 = \sum\limits_{\alpha\alpha'\beta\beta'} (\theta_{\beta} - \theta_{\alpha}) (\theta_{\beta'} - \theta_{\alpha'}) P_{\beta\alpha} Q_{\alpha'\beta'} V_{\alpha\beta',\beta\alpha'}.
\]
(42)

Finally, Eqs. (39), (41), and (42) lead to
\[
\langle [\hat{P}, [H, \hat{Q}]] \rangle = \sum\limits_{\beta} \theta_{\beta} \langle \beta | [P, [h, Q]] | \beta \rangle + S_2,
\]
(43)

where the Hartree Hamiltonian \(h\) has been given in Eq. (5). If the first term is written, employing the closure property, as
\[
\sum\limits_{\beta} \theta_{\beta} \langle \beta | [P, [h, Q]] | \beta \rangle = \sum\limits_{\alpha\beta} (1 - \theta_{\alpha}) \theta_{\beta} \left( \langle \beta | P | \alpha \rangle \langle \alpha | [h, Q] | \beta \rangle - \langle \beta | [h, Q] | \alpha \rangle \langle \alpha | P | \beta \rangle \right),
\]
(44)
we obtain the expression
\[
\langle [\hat{P}, [H, \hat{Q}]] \rangle = \sum\limits_{\alpha\beta} (\theta_{\beta} - \theta_{\alpha}) E_{\alpha\beta} P_{\beta\alpha} Q_{\alpha\beta} + \sum\limits_{\alpha\alpha'\beta\beta'} (\theta_{\beta} - \theta_{\alpha}) (\theta_{\beta'} - \theta_{\alpha'}) P_{\beta\alpha} Q_{\alpha'\beta'} V_{\alpha\beta',\beta\alpha'}.
\]
(45)

Using the notation \(N_{\alpha\beta}\), instead of the step function, it is described as
\[
\langle [\hat{P}, [H, \hat{Q}]] \rangle = \sum\limits_{\alpha\beta} (N_{\alpha\beta} - N_{\beta\alpha}) E_{\alpha\beta} P_{\beta\alpha} Q_{\alpha\beta} + \sum\limits_{\alpha\alpha'\beta\beta'} (N_{\alpha\beta} - N_{\beta\alpha}) (N_{\alpha'\beta'} - N_{\beta'\alpha'}) P_{\beta\alpha} Q_{\alpha'\beta'} V_{\alpha\beta',\beta\alpha'},
\]
(46)

which is also written with the notation Eq. (29) as
\[
\langle [\hat{P}, [H, \hat{Q}]] \rangle = \sum\limits_{\alpha\alpha'\beta\beta'} (P_{\beta\alpha} - P_{\alpha\beta}) \begin{pmatrix} A_{\alpha\beta,\alpha'\beta'} & B_{\alpha\beta,\alpha'\beta'} \\ B^*_{\alpha\beta,\alpha'\beta'} & A^*_{\alpha\beta,\alpha'\beta'} \end{pmatrix} \begin{pmatrix} Q_{\alpha'\beta'} \\ -Q_{\beta'\alpha'} \end{pmatrix}.
\]
(47)

In the No\(\overline{N}\) approximation, \(\hat{P}, \hat{Q},\) and \(H\) in the above equations should be replaced by
\[
\hat{P}_+ = \int d^3 x \, \psi^\dagger(x) \Lambda P A \psi(x), \quad \hat{Q}_+ = \int d^3 x \, \psi^\dagger(x) \Lambda Q A \psi(x).
\]
(48)
and

\[ H_+ = (H_0)_+ + (H_{\text{int}})_+, \]

\[ (H_0)_+ = \int d^3x \psi^\dagger(x) A h_0 A \psi(x), \]

\[ (H_{\text{int}})_+ = \frac{1}{2} \int d^3x_1 d^3x_2 \psi^\dagger(x_1) \psi^\dagger(x_2) A_1 A_2 V_{12} A_1 A_2 \psi(x_2) \psi(x_1), \]

with the projection operator \( A(x) = \sum_{E_{\alpha} > 0} \varphi_{\alpha}(x) \varphi_{\alpha}^\dagger(x). \)

Next, we calculate directly the energy-weighted sum:

\[ S_{\text{RPA}} = \sum_{n(\lambda_n = 1)} \omega_n \|n| \hat{F}|0\|. \]

In writing the matrix elements

\[ |n| \hat{F}|0\| = \sum_{\alpha\beta} F_{\alpha\beta} C_{\alpha\beta}^{(n)*}, \]

\[ F_{\alpha\beta} = \langle \alpha | F | \beta \rangle = \int d^3x \varphi^\dagger_{\alpha}(x) F \varphi_{\beta}(x), \]

the sum is expressed as

\[ S_{\text{RPA}} = \sum_{\alpha\beta \alpha'\beta'} F_{\alpha\beta}^{*} F_{\alpha'\beta'}^{*} \sum_{n(\lambda_n = 1)} \omega_n C_{\alpha\beta}^{(n)} C_{\alpha'\beta'}^{(n)*}. \]

From the relationship \( F_{\alpha\beta} = F_{\beta\alpha}^{*} \) and

\[ \sum_{n(\lambda_n = 1)} \omega_n C_{\alpha\beta}^{(n)} C_{\alpha'\beta'}^{(n)*} = \sum_{n(\lambda_n = -1)} (-\omega_n) C_{\beta\alpha}^{(n)} C_{\beta'\alpha'}^{(n)*}, \]

we have

\[ S_{\text{RPA}} = \frac{1}{2} \sum_{\alpha\beta \alpha'\beta'} F_{\alpha\beta}^{*} F_{\alpha'\beta'}^{*} \sum_{n} \lambda_n \omega_n C_{\alpha\beta}^{(n)} C_{\alpha'\beta'}^{(n)*}. \]

Since Eqs. (26) and (32) provide us with

\[ \sum_{n} \lambda_n \omega_n C_{\alpha\beta}^{(n)} C_{\alpha'\beta'}^{(n)*} = \sum_{n} \lambda_n C_{\alpha\beta}^{(n)} \left( E_{\alpha\beta} C_{\alpha\beta}^{(n)} + (N_{\alpha\beta} - N_{\beta\alpha}) \sum_{\alpha''\beta''} V_{\alpha''\beta''} C_{\alpha''\beta''}^{(n)} \right) \]

\[ = (N_{\alpha\beta} - N_{\beta\alpha}) E_{\alpha\beta} \delta_{\alpha\alpha'} \delta_{\beta\beta'} + (N_{\alpha\beta} - N_{\beta\alpha}) (N_{\alpha'\beta'} - N_{\beta'\alpha'}) V_{\alpha'\beta',\beta'\alpha'}, \]

finally we obtain the RPA sum value

\[ S_{\text{RPA}} = \frac{1}{2} \sum_{\alpha\beta} (N_{\alpha\beta} - N_{\beta\alpha}) E_{\alpha\beta} |F_{\alpha\beta}|^2 \]

\[ + \frac{1}{2} \sum_{\alpha\beta \alpha' \beta'} F_{\alpha\beta}^{*} F_{\alpha'\beta'}^{*} (N_{\alpha\beta} - N_{\beta\alpha}) (N_{\alpha'\beta'} - N_{\beta'\alpha'}) V_{\alpha'\beta',\beta'\alpha'}. \]

The first term of the above equation is nothing but the sum value in the Hartree approximation, \( S_{\text{H}}. \)

For each approximation, it is given by

\[ S_{\text{H}} = \sum_{p} E_{p} |F_{p}|^2 + \sum_{z} E_{z} |F_{z}|^2 \]

\[ = \sum_{\alpha\beta} \left( \sum_{\alpha''\beta''} F_{\alpha\beta}^{*} F_{\alpha''\beta''}^{*} \right) (N_{\alpha\beta} - N_{\beta\alpha}) (N_{\alpha''\beta''} - N_{\beta''\alpha''}) V_{\alpha''\beta'',\beta''\alpha''}. \]

In the Full case, \( S_{\text{H}} \) is positive, as it should be, but, in NSA and NFA, the second term on the right-hand side, coming from the \( N \) contributions, is negative owing to \( E_{\alpha\beta} = -E_{\beta\alpha} < 0. \) In Ref. [15], it
was shown that the negative contributions exactly cancel the first term from the particle excitations in nuclear matter, yielding $S_H = 0$. In writing the term in the Full approximation as

$$
\sum_{p\bar{n}} E_{p\bar{n}}|F_{p\bar{n}}|^2 = \sum_{n\bar{n}} E_{n\bar{n}}|F_{n\bar{n}}|^2 - \sum_{h\bar{n}} E_{h\bar{n}}|F_{h\bar{n}}|^2,
$$

$n$ indicating the nucleon states $(p+h)$, NSA and NFA neglect the first term on the right-hand side, which makes the left-hand side positive.

Now comparing Eq. (46) with Eq. (53), we obtain the relationship

$$S_{RPA} = \frac{1}{2} \langle [\hat{F}, [H, \hat{F}]] \rangle. \quad (56)$$

Thus, we can express formally the energy-weighted sum in terms of the double commutator, as in nonrelativistic RPA [16,19]. In the NoN approximation, $\hat{F}$ and $H$ in the above equations should be replaced as in Eqs. (48) and (49).

The above double commutator, however, should not be calculated using Eq. (4) in order to obtain the sum value, $S_{RPA}$, given by Eq. (53). Instead of Eq. (4) defined in the infinite momentum space, we must use the anticommutation relation in a finite momentum space [15],

$$\left\{ \psi_a(x), \psi^\dagger_b(y) \right\} = \delta_{ab} d(x-y), \quad (57)$$

where $d(x)$ is defined as

$$d(x) = \int \frac{d^3p}{(2\pi)^3} \Theta_p e^{ip \cdot x}, \quad \Theta_p = \theta(P_\infty - |p|). \quad (58)$$

Then, we have its ground-state expectation value, which does not vanish. By making $P_\infty$ infinite in the expectation value, we can obtain the sum value. We note that Eq. (57) is reduced to Eq. (4) in the limit $P_\infty \to \infty$, because of

$$d(x) \xrightarrow{P_\infty \to \infty} \delta(x). \quad (59)$$

The double commutator in the Hartree approximation has been calculated for nuclear matter, according to Eq. (57) in Ref. [15]. In the Full space $N_{\alpha\beta}$, the sum value of $S_H$ is divergent, being proportional to $P_\infty^2$, in contrast to $S_H = 0$ in NSA and NFA.

The energy-weighted sum for the Gamow–Teller transition strengths in the relativistic RPA has been explored in Ref. [23].

4. The continuity equation

The current conservation in the relativistic RPA has been studied in previous papers [5–7,9]. One of the reasons why NFA and NSA have been accepted so far is because they do not violate the current conservation. Generally speaking, it is important for phenomenological models to keep at least well known fundamental principles. In particular, Refs. [6,9] investigated the electron scattering, where the continuity equation must be essential.

There are various ways to show the current conservation in NFA and NSA. One of the ways is to write the Hartree polarization function in Eq. (19) in terms of the Hartree propagator, $G_H(1,2)$,

$$i \Pi(H(A(1), B(2)) = \text{Tr} \left( A G_H(1,2) B G_H(2,1) \right). \quad (60)$$
where \( G_H(1, 2) \) is given by [6]

\[
G_H(1, 2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t_1-t_2)} G_H(x_1, x_2, \omega),
\]

with

\[
G_H(x_1, x_2, \omega) = \sum_{\alpha} \phi_\alpha(x_1) \phi_\alpha(x_2) \left( \frac{1 - \theta_\alpha}{\omega - E_\alpha + i\epsilon} + \frac{\theta_\alpha}{\omega - E_\alpha - i\epsilon} \right).
\]

Owing to the closure property Eq. (3), the above Hartree propagator satisfies

\[
\left( i\gamma^\mu \partial_x^\mu - M - \Sigma(x) \right) G_H(x, y) = \delta(x - y).
\]

Then, Eq. (60) provides us with

\[
\partial_x^\mu \Pi_H(\gamma_\mu(x), B(y)) = -\text{Tr} \left[ \left( \delta(x - y) + (M + \Sigma(x)) G_H(x, y) \right) B G_H(y, x) \right] \\
+ \text{Tr} \left[ G_H(x, y) B \left( \delta(x - y) + G_H(y, x)(M + \Sigma(x)) \right) \right] = 0.
\]

The above equation and Eq. (19) imply that the RPA polarization function should satisfy

\[
\partial_x^\mu \Pi(\gamma_\mu(x), B(y)) = 0.
\]

Thus, as long as Eq. (63) holds, the relativistic RPA conserves the current. As mentioned before, Eq. (63) requires the complete set of eigenfunctions for the Hartree field. If \( \overline{N} \) states are neglected, Eq. (63) does not hold as

\[
\left( i\gamma_\mu \partial_x^\mu - M - \Sigma(x) \right) G_H(x, y) = \delta(x_0 - y_0) \sum_{\alpha} \phi_\alpha(x) \phi_\alpha^\dagger(y) (p_\alpha + h_\alpha).
\]

The above proof of Eq. (65) seems to be independent of how the Hartree ground state is occupied by \( N \) and \( \overline{N} \). Indeed, both NSA and NFA satisfy this equation. Physically, however, \( \overline{N} \) states should be occupied in the ground state, as long as the negative-energy states are required in the model framework. Hence, it may be instructive to see explicitly how the continuity equation is satisfied, when only some of \( N - \overline{N} \) excitations are included in the calculations of the excited states.

Using Eqs. (11)–(13), the Hartree polarization function can be written as

\[
i \Pi_H(\gamma_\mu(1), B(2)) = \sum_{\alpha\beta} F_{(\alpha\beta)}^{(\mu)}(x, y) T^{(\alpha\beta)}(x_0, y_0),
\]

where we have used the notations

\[
F_{(\alpha\beta)}^{(\mu)}(x, y) = \overline{\phi}_\beta(x) \gamma_\mu \phi_\alpha(x) \overline{\phi}_\alpha(y) B \phi_\beta(y)
\]

\[
T^{(\alpha\beta)}(x_0, y_0) = \left( N_{\alpha\beta} \theta(x_0 - y_0) + N_{\beta\alpha} \theta(y_0 - x_0) \right) e^{-iE_{\alpha\beta}(x_0 - y_0)}.
\]

Then, we obtain

\[
i \partial_x^\mu \Pi_H(\gamma_\mu(x), B(y)) = \delta(x_0 - y_0) \sum_{\alpha\beta} F_{(\alpha\beta)}^{(0)}(x, y) \left( N_{\alpha\beta} - N_{\beta\alpha} \right)
\]

\[
+ \sum_{\alpha\beta} \left( \partial_x^k F_{(\alpha\beta)}^{(k)}(x, y) - iE_{\alpha\beta} F_{(\alpha\beta)}^{(0)}(x, y) \right) T^{(\alpha\beta)}(x_0, y_0).
\]

Since \( \phi_\alpha(x) \) is an eigenfunction of the Hartree Hamiltonian, the second term on the right-hand side vanishes. Employing the expression of Eq. (18) for the full space, and then the closure relation Eq. (3),
the above equation can be written as
\[ i \partial_x^\mu \Pi_H(\gamma^\mu(x), B(y)) = \delta(x - y) \left( \sum_\beta \overline{\psi}_\beta(x) B \psi_\beta(x) h_\beta - \sum_\alpha \overline{\psi}_\alpha(x) B \psi_\alpha(x) h_\alpha \right) + \sum_\beta \overline{\psi}_\beta(x) B \psi_\beta(x) \bar{\eta}_\beta - \sum_\alpha \overline{\psi}_\alpha(x) B \psi_\alpha(x) \bar{\eta}_\alpha \right) = 0. \] (69)

This equation shows the first and second lines vanishing separately. As a result, NSA and NFA satisfy the continuity equation, although they have the only first line. The second line comes from the divergent terms describing the excitations of \( \overline{N} \) in the Dirac sea, \( N_{\alpha\beta} = (p_\alpha + h_\alpha) \bar{\eta}_\beta \). This fact implies that the vacuum should satisfy the continuity equation by itself, and that the current conservation is independent of the divergence problem. Thus, even if the continuity equation is described correctly, it is not assured that the same approximation is applicable to calculations of other physical quantities.

We note finally that Eq. (65) provides us with the familiar form of the continuity equation to be expressed as the transition matrix element:
\[ \sum_{\alpha\beta} \left( -i \omega_n J^0_{\beta\alpha}(x) + \nabla \cdot J_{\beta\alpha}(x) \right) C^{(n)}_{\alpha\beta} = 0, \quad J^{\mu}_{\beta\alpha}(x) = \overline{\psi}_\beta(x) \gamma^\mu \psi_\alpha(x). \]

5. The spurious state

The present relativistic RPA equation has spurious solutions in the same way as in nonrelativistic models [16,19]. If \([H, \hat{Q}] = 0\), Eq. (47) provides us with
\[ \sum_{\alpha\beta} \left( P_{\beta\alpha} - P_{\alpha\beta} \right) \left( \begin{array}{c} A_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \\ B_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \\ C_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \end{array} \right) \left( \begin{array}{c} Q_{\alpha'\beta'} \\ \bar{Q}_{\alpha'\beta'} \\ \bar{Q}_{\alpha'\beta'} \end{array} \right) = 0. \]

The above equation holds for any \( P_{\alpha\beta} \), so that we have
\[ \sum_{\alpha'\beta'} \left( \begin{array}{c} A_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \\ B_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \\ C_{\alpha\beta,\alpha'\beta'}^{\alpha\beta,\alpha'\beta'} \end{array} \right) \left( \begin{array}{c} Q_{\alpha'\beta'} \\ \bar{Q}_{\alpha'\beta'} \\ \bar{Q}_{\alpha'\beta'} \end{array} \right) = 0. \] (70)

Thus, when \( Q_{\alpha\beta} \neq 0 \), \( \hat{Q} \) is the spurious solution of Eq. (30) with \( \omega_n = 0 \). If the Full RPA has the spurious state, NSA and NFA also separate it from the other solutions. As seen in Eq. (29), \( A \) and \( B \) depend on \( N_{\alpha\beta} \). Therefore, the spurious states cannot be described without \( \overline{N} \) degrees of freedom not only in the Full calculation, but also in NSA and NFA [10]. In the NoN approximation, \([H_+, \hat{Q}_+] = 0\) would be required for \( \hat{Q}_+ \) to be the spurious solution, but not be assured by \([H, \hat{Q}] = 0\).

Up to this stage, it seems that the spurious state is well described in NSA and NFA. As discussed in Sect. 2, however, unphysical effects coming from the disregard of the divergent terms are hidden in their RPA equation. In order to see this fact in more detail, let us describe Eq. (70) explicitly. On the one hand, it is written in the Full case as
\[ E_{\alpha\beta} Q_{\alpha\beta} + \sum_{\alpha'\beta'} \left( p_{\alpha'\beta'} h_{\alpha'\beta'} + p_{\alpha'\beta'} \bar{n}_{\alpha'\beta'} \right) \left( V_{\alpha'\beta',\beta\alpha} Q_{\alpha'\beta'} - V_{\alpha'\beta',\beta\alpha} Q_{\alpha'\beta'} \right) = 0. \] (71)

On the other hand, it becomes, in NFA,
\[ E_{\alpha\beta} Q_{\alpha\beta} + \sum_{\alpha'\beta'} \left( p_{\alpha'\beta'} h_{\alpha'\beta'} - h_{\alpha'\beta'} \bar{n}_{\alpha'\beta'} \right) \left( V_{\alpha'\beta',\beta\alpha} Q_{\alpha'\beta'} - V_{\alpha'\beta',\beta\alpha} Q_{\alpha'\beta'} \right) = 0. \] (72)
and, in NSA,

\[ E_{\alpha\beta} Q_{\alpha\beta} + \sum_{\alpha'\beta'} \left( p_{\alpha'} h_{\beta'} + \bar{n}_{\alpha'} h_{\beta'} \right) \left( V_{\alpha\beta', \beta'\alpha'} Q_{\alpha'\beta'} - V_{\alpha\alpha', \beta'\beta'} Q_{\beta'\alpha'} \right) = 0. \]  

(73)

It is seen that Eq. (73) is the same as Eq. (72), by exchanging the suffixes \( \alpha' \) and \( \beta' \) in the factor with \( \bar{n}_{\alpha'} h_{\beta'} \). In writing \( p_{\alpha} h_{\beta} + p_{\alpha} \bar{n}_{\beta} = p_{\alpha} h_{\beta} - h_{\alpha} \bar{n}_{\beta} + (p_{\alpha} + h_{\alpha}) \bar{n}_{\beta} \) for the Full case, we can recognize in Eq. (72) that NFA ignores the last term \((p_{\alpha} + h_{\alpha}) \bar{n}_{\beta}\), so that the minus sign of \(-h_{\alpha} \bar{n}_{\beta}\) changes unphysically that of the interactions in NFA and NSA.

Let us make a few comments on the above discussions. Equation (70) for Full and NSA can also be derived by calculating the matrix element,

\[ \left\langle a_{\beta}^\dagger a_{\alpha} | [H, Q] \right\rangle = 0, \]  

(74)

for \([H, Q] = 0\). Remembering that the Hartree ground state \(|\rangle\) is different in Full and NSA, the above equation is expressed using \(N_{\alpha\beta}\) as

\[ N_{\alpha\beta} \sum_{\alpha'\beta'} Q_{\alpha'\beta'} \left( N_{\alpha'\beta'} \left| a_{\beta}^\dagger a_{\alpha} H a_{\alpha}^\dagger a_{\beta'}^\dagger \right\rangle - N_{\beta'\alpha'} \left| a_{\beta'}^\dagger a_{\alpha} a_{\alpha}^\dagger a_{\beta} H \right\rangle \right) = 0, \]  

(75)

which leads to, neglecting the exchange terms,

\[ \sum_{\alpha'\beta'} \left( A_{\alpha\beta', \alpha'\beta'} Q_{\alpha'\beta'} - B_{\alpha\beta, \alpha'\beta'} Q_{\beta'\alpha'} \right) = 0. \]  

(76)

Combining this with a similar equation from \([H, Q^\dagger] = 0\), we obtain Eq. (70).

Notice that \(N_{\alpha\beta}\) in Eq. (75) is not simply replaced by that of NFA. The step from Eq. (74) to Eq. (75) requires the complete set for \(N_{\alpha'\beta'}\), whereas NFA ignores the vacuum-polarization part, \((p_{\alpha} + h_{\alpha}) \bar{n}_{\beta}\), of \(p_{\alpha} h_{\beta} + p_{\alpha} \bar{n}_{\beta} = p_{\alpha} h_{\beta} - h_{\alpha} \bar{n}_{\beta} + (p_{\alpha} + h_{\alpha}) \bar{n}_{\beta}\) in the Full case. Indeed, the vacuum-polarization part of Eq. (71) does not satisfy

\[ \sum_{\alpha'\beta'} (p_{\alpha'} + h_{\alpha'}) \bar{n}_{\beta'} \left( V_{\alpha\beta', \beta'\alpha'} Q_{\alpha'\beta'} - V_{\alpha\alpha', \beta'\beta'} Q_{\beta'\alpha'} \right) = 0, \]  

(77)

and the terms corresponding to Eq. (72) in Eq. (71) also do not vanish. When Eq. (73) for NSA holds, however, Eq. (72) for NFA also does. This fact implies that neglecting the vacuum polarization in RPA is equivalent to assuming empty \(\bar{N}\) states as in NSA with the complete set for \(N_{\alpha\beta}\). In order to render Eq. (72) valid in NSA, as well as Eq. (73) in NSA, the nuclear interactions for the Full case have to be modified self-consistently together with \(E_{\alpha\beta}\). Even so, as in the case of the continuity equation, the description of the spurious state in NFA and NSA does not provide us with a justification for their approximations.

In the \(\text{No} \bar{N}\) approximation, \([H, Q]\) in Eq. (74) may be replaced by the projected one, \([H_+, Q_+] = 0\), in addition to the Hartree ground state.

6. Schematic model

It is well known in nonrelativistic RPA that a schematic model with separable interactions illustrates well the general character of the RPA solutions [16]. Let us explore the structure of NFA, NSA, and the full relativistic RPA by using a similar model.
We assume an interaction \( V \), the matrix elements of which are given by
\[
V_{\alpha \beta', \beta' a'} = \sum_a \kappa_a f^{(a)}_{\alpha} f^{(a)*}_{\alpha'}. \tag{78}
\]

Then, Eq. (26) provides us with
\[
C_{\alpha \beta} = \frac{N_{\alpha \beta} - N_{\beta \alpha}}{\omega - E_{\alpha \beta}} \sum_a f^{(a)}_{\alpha} N_a, \quad N_a = \kappa_a \sum_{\alpha \beta} f^{(a)*}_{\alpha} C_{\alpha \beta},
\]
which lead to
\[
X_{\alpha \beta} = \frac{1}{\omega - E_{\alpha \beta}} \sum_a f^{(a)}_{\alpha} N_a, \quad Y_{\alpha \beta} = -\frac{1}{\omega + E_{\alpha \beta}} \sum_a f^{(a)}_{\alpha} N_a.
\]

Since we have
\[
N_a = \kappa_a \sum_b R_{ab}(\omega) N_b, \quad R_{ab}(\omega) = \sum_{\alpha \beta} \frac{N_{\alpha \beta} - N_{\beta \alpha}}{\omega - E_{\alpha \beta}} f^{(a)*}_{\alpha} f^{(b)}_{\alpha} = R^p_{ba}(\omega), \tag{79}
\]
the eigenvalues of the RPA equation are determined by the dispersion equation
\[
\det(\delta_{ab} - \kappa_a R_{ab}(\omega)) = 0. \tag{80}
\]

The normalization equation (31) becomes
\[
\lambda = \sum_{\alpha \beta} (N_{\alpha \beta} - N_{\beta \alpha}) |C_{\alpha \beta}|^2 = -\sum_{ab} N^*_{ab} \frac{dR_{ab}}{d\omega} N_b.
\]

In the Full RPA, \( R_{ab}(\omega) \) is given by
\[
R^\text{Full}_{ab}(\omega) = \sum_{ph} \left( \frac{f^{(a)}_{ph} f^{(b)}_{ph}}{\omega - E_{ph}} - \frac{f^{(a)*}_{hp} f^{(b)}_{hp}}{\omega + E_{ph}} \right) + \sum_{pn} \left( \frac{f^{(a)*}_{pp} f^{(b)}_{pp}}{\omega - E_{pn}} - \frac{f^{(a)}_{np} f^{(b)}_{np}}{\omega + E_{np}} \right) = R^p_{ab}(\omega) + R^{\tilde{p}}_{ab}(\omega). \tag{81}
\]

The part \( R^{\tilde{p}}_{ab} \) for particle–\( \bar{N} \) excitations can be written by using Pauli-blocking terms as
\[
R^{\tilde{p}}_{ab}(\omega) = R^{\tilde{n}}_{ab}(\omega) - R^{h\tilde{n}}_{ab}(\omega), \tag{82}
\]
with
\[
R^{\tilde{n}}_{ab}(\omega) = \sum_{\tilde{n}} \left( \frac{f^{(a)}_{\tilde{n}\tilde{n}} f^{(b)}_{\tilde{n}\tilde{n}}}{\omega - E_{\tilde{n}\tilde{n}}} - \frac{f^{(a)*}_{\tilde{n}\tilde{n}} f^{(b)}_{\tilde{n}\tilde{n}}}{\omega + E_{\tilde{n}\tilde{n}}} \right), \quad R^{h\tilde{n}}_{ab}(\omega) = \sum_{\tilde{n}} \left( \frac{f^{(a)*}_{\tilde{n}\tilde{n}} f^{(b)}_{\tilde{n}\tilde{n}}}{\omega - E_{\tilde{n}\tilde{n}}} - \frac{f^{(a)}_{\tilde{n}\tilde{n}} f^{(b)}_{\tilde{n}\tilde{n}}}{\omega + E_{\tilde{n}\tilde{n}}} \right). \tag{83}
\]

Then, we may write
\[
R^\text{Full}_{ab}(\omega) = R^p_{ab}(\omega) - R^{h\tilde{n}}_{ab}(\omega) + R^{\tilde{n}}_{ab}(\omega). \tag{84}
\]
In the case of NSA and NFA, \( (N_{\alpha \beta} - N_{\beta \alpha}) \) is given by \( (h_\beta - h_\alpha) \) in \( R_{ab}(\omega) \) of Eq. (79), so that we have
\[
R^{\text{NSA,NFA}}_{ab}(\omega) = R^p_{ab}(\omega) - R^{h\tilde{n}}_{ab}(\omega). \tag{85}
\]
It is seen that NSA and NSA neglect \( R^{\tilde{n}}_{ab}(\omega) \) in the Full expression Eq. (84), leaving the Pauli-blocking term \( R^{h\tilde{n}}_{ab}(\omega) \) with the minus sign.
When the interaction has only one component $V_{\alpha\beta',\beta\alpha'} = \kappa f_{\alpha \beta} f_{\alpha' \beta'}^*$, we have simply
\[
R(\omega) = \sum_{\alpha \beta} N_{\alpha \beta} \left( \frac{|f_{\alpha \beta}|^2}{\omega - E_{\alpha \beta}} - \frac{|f_{\alpha' \beta'}|^2}{\omega + E_{\alpha' \beta'}} \right), \quad \lambda = -|N|^2 \frac{dR}{d\omega},
\] (86)
and the eigenvalues of the RPA excited states are obtained from the equation
\[
1 - \kappa R(\omega) = 0, \quad \frac{dR}{d\omega} < 0.
\] (87)
Equation (86) becomes, for the full space, and for NSA and NFA, respectively,
\[
R_{\text{Full}}(\omega) = \sum_{\text{ph}} \frac{2E_{\text{ph}}|f_{\text{ph}}|^2}{\omega^2 - E_{\text{ph}}^2} + \sum_{\text{p-n}} \frac{2E_{\text{p-n}}|f_{\text{p-n}}|^2}{\omega^2 - E_{\text{p-n}}^2} = R^{\text{ph}}(\omega) + R^{\text{p-n}}(\omega),
\] (88)
\[
R_{\text{NSA,NFA}}(\omega) = \sum_{\text{ph}} \frac{2E_{\text{ph}}|f_{\text{ph}}|^2}{\omega^2 - E_{\text{ph}}^2} - \sum_{\text{p-n}} \frac{2E_{\text{p-n}}|f_{\text{p-n}}|^2}{\omega^2 - E_{\text{p-n}}^2} = R^{\text{ph}}(\omega) - R^{\text{p-n}}(\omega).
\] (89)
As mentioned before, the sign of the second term on the right-hand side for NSA and NFA is opposite to that for the Full case. This changes the signs of $\kappa$ and the derivative in the part of the $N-\bar{N}$ excitations in Eq. (87). The former change makes the attractive (repulsive) force work as a repulsive (attractive) one, and the latter change produces RPA states with negative excitation energies.

The above unphysical effects in NSA and NFA are not independent of low-lying RPA states. When $\omega \ll 2M^*$, $M^*$ being the nucleon effective mass, on the one hand, we can set the following equation into Eq. (88) for the Full case:
\[
R^{\text{p-n}}(\omega) \approx R^{\text{p-n}}(0) = -\sum_{\text{p-n}} \frac{2|f_{\text{p-n}}|^2}{E_{\text{p-n}}} = R^{\text{p-n}}(0) + R^{\text{p-n}}(0) < 0.
\] (90)
Hence, the dispersion equation of Eq. (87) can be written as
\[
1 - \kappa_{\text{eff}}^{\text{Full}} R^{\text{ph}}(\omega) \approx 0, \quad \kappa_{\text{eff}}^{\text{Full}} = \frac{\kappa}{1 + \kappa |R^{\text{p-n}}(0)|}.
\] (91)
On the other hand, in Eq. (89) for NSA and NFA, we may use an approximation,
\[
R^{\text{p-n}}(\omega) \approx R^{\text{p-n}}(0) = -\sum_{\text{p-n}} \frac{2|f_{\text{p-n}}|^2}{E_{\text{p-n}}} < 0,
\] (92)
so that we have the dispersion equation with $\kappa_{\text{eff}}^{\text{NSA,NFA}}$, instead of $\kappa_{\text{eff}}^{\text{Full}}$ in Eq. (91):
\[
\kappa_{\text{eff}}^{\text{NSA,NFA}} = \frac{\kappa}{1 - \kappa |R^{\text{p-n}}(0)|}.
\] (93)
In the degenerate limit where all the particle–hole energies $E_{\text{ph}}$ are set equal to $\epsilon$, the dispersion equation requires all the solutions but one to be trapped at the unperturbed energy, and the one at
\[
\omega^2 = \epsilon^2 + 2\kappa_{\text{eff}} \epsilon \sum_{\text{ph}} |f_{\text{ph}}|^2,
\] (94)
where $\kappa_{\text{eff}}$ is given by $\kappa_{\text{eff}}^{\text{Full}}$ or $\kappa_{\text{eff}}^{\text{NSA,NFA}}$.

The difference between the full space RPA, and NSA and NFA appears in the denominator of $\kappa_{\text{eff}}$. The contributions from the coupling with $N-\bar{N}$ states have opposite signs to each other. If the interaction is attractive (repulsive), $\kappa < 0(> 0)$, one should have $\kappa_{\text{eff}} < \kappa$, which enhances (diminishes)
the attractive (repulsive) force. This fact is realized in Eq. (91) for the full space RPA, but not in Eq. (93) for the NSA and NFA.

In fact, these effects of the Pauli-blocking terms in NFA and NSA were recognized in the previous numerical calculations. In the response functions to quasielastic electron scattering, the effects are rather negligible [6], but not in the description of low-lying states like the spurious state [10] and giant monopole states [12–14].

The RPA spurious state to be predicted at zero energy is dominated by attractive forces. In relativistic models, the phenomenological attractive force is rather strong. Hence, in some cases, an imaginary solution of the RPA equation was obtained, when the coupling with $\mathcal{N}$–$\overline{\mathcal{N}}$ states was ignored. By taking into account the coupling, the attractive interaction is weakened effectively, reproducing the spurious state at zero energy [10].

The same thing happened in the calculation of giant monopole states that were also sensitive to attractive forces. Without the coupling, the calculated excitation energy was too low, but the coupling played the role of the repulsive force, and explained well the experimental values [12–14]. Thus, an agreement with experimental data does not always mean that models are physically appropriate for the description of phenomena.

A more intuitive understanding of the difference between the full calculation and NSA and NFA is shown in Figs. 1 and 2. Figure 1 shows schematically the dispersion relation of the full RPA, Eq. (88), in the case of $\kappa < 0$. It has a familiar structure, to be found in the literature on nonrelativistic RPA [16]. The thick and thin solid curves are calculated with and without the coupling with the $\mathcal{N}$–$\overline{\mathcal{N}}$ states, respectively. The solid circles denote the eigenvalues corresponding to $\lambda = 1(dR/d\omega < 0)$ and the open circles $\lambda = -1$. It is seen that the excitation energy of the lowest state is pushed down owing to the coupling, as expected.

Figure 2 shows the dispersion relation with Eq. (89) of NSA and NFA. In contrast to Fig. 1, the lowest positive-energy state is pushed up with the coupling, in spite of the fact that the interaction is assumed to be attractive. Moreover, except for low-lying positive-energy states around $E_{ph}$, the eigenvalues with $\lambda = 1(dR/d\omega < 0)$ appear at the negative-energy region around $-E_{h\bar{h}}$ in both NSA and NFA.

In addition to the thick and thin curves as in Fig. 1, a dotted curve is shown, which is obtained without the coupling and with a smaller number of particle–hole states. Compared with the thin curve, we see that the excitation energy of the lowest state decreases with an increased number of

---

**Fig. 1.** Graphical solution of the full RPA dispersion equation with Eq. (88). The thick and thin solid curves are calculated with and without the coupling with the $\mathcal{N}$–$\overline{\mathcal{N}}$ states, respectively. The solid circles denote the eigenvalues of the RPA excited states. The vertical broken lines indicate the position of the unperturbed energies. For details, see the text.
Fig. 2. Same as Fig. 1 for Eq. (89). The dotted curve is obtained without the $N\bar{N}$ coupling and with a smaller number of particle–hole states. For details, see the text.

particle–hole states. Therefore, when the attractive force is sufficient for the spurious state with zero energy in NSA and NFA, its eigenvalue becomes imaginary in neglecting the coupling with $N\bar{N}$ states, because of $\kappa^{-1} = R^{ph}(0) - R^{h\bar{n}}(0) > R^{ph}(0)$, as in Eq. (89). This dependence on the number of configurations was also observed numerically in Ref. [10].

Finally, it may be useful to re-examine in the present RPA framework the relativistic Landau–Migdal parameters of the $\sigma-\omega$ model developed in Refs. [7,17,24]. In discussing the response of nuclear matter at low momentum transfer $q \approx 0$, the interaction Eq. (23) of the $\sigma-\omega$ model can be expressed as a separable form. It is written in Eq. (78) as

$$f_{\alpha\beta}^{(a)} = \frac{1}{\sqrt{\Omega}} \left( i q \cdot \Gamma^a \right) \frac{1}{\sqrt{\Omega}} w^\alpha \Gamma^a w^\beta,$$

with

$$\kappa_{a} = \begin{cases} -f_{\sigma} = -g_{\sigma}^{2}/m_{\sigma}^{2}, & a = -1, \\ +f_{\omega} = g_{\omega}^{2}/m_{\omega}^{2}, & a = 0, \\ -f_{\omega}, & a = 1, 2, 3. \end{cases}$$

Here, $\Omega$ denotes the volume of the system that we need to rewrite the integral of Eqs. (9) and (10) with the summation, and $w^\alpha = w_{s}(p\sigma)$ stands for the four-component spinor, $\alpha$ representing $\{s = \pm, p, \sigma\}$,

$$w_{+}(p\sigma) = \sqrt{\frac{E_{p} + M^{*}}{2E_{p}}} \left( \begin{array}{c} 1 \\ \frac{\sigma \cdot p}{E_{p} + M^{*}} \end{array} \right) \chi_{\sigma},$$

$$w_{-}(p\sigma) = \sqrt{\frac{E_{p} + M^{*}}{2E_{p}}} \left( \begin{array}{c} -\frac{\sigma \cdot p}{E_{p} + M^{*}} \\ 1 \end{array} \right) \chi_{\sigma},$$

with $E_{p} = \sqrt{p^{2} + M^{*2}}$ and the two-component spinor, $\chi_{\sigma}$.

The matrix $R$ in Eq. (80) can be divided into longitudinal and transverse parts [6]. Taking $q = (q, 0, 0)$, the former is a $3 \times 3$ matrix depending on $a = -1, 0$ and 1, while the latter is composed of $a = 2$ and 3. The longitudinal part, which is required for the present discussions, is
calculated in NFA and NSA as

$$D_L = \det\left(\delta_{ab} - \kappa a R_{ab}(\omega)\right) = \begin{vmatrix} C_\sigma + f_\sigma R_{\sigma h}^h & f_\sigma R_{\sigma h}^a & s f_\sigma R_{\sigma h}^b \\ - f_\omega R_{\sigma h}^h & C_\omega + f_\omega R_{\sigma h}^a & - s f_\omega R_{\sigma h}^b \\ s f_\omega R_{\sigma h}^h & s f_\omega R_{\sigma h}^a & C_\omega + f_\omega R_{\sigma h}^b \end{vmatrix}, \quad (97)$$

where we have used the following abbreviations:

$$C_\sigma = 1 - f_\sigma R_{11}^{h-1}, \quad C_\omega = 1 + f_\omega R_{00}^{h-1}, \quad C_1 = 1 - f_\omega \left( R_{11}^{h-1} + \frac{N_F v_F^2}{3} \right), \quad (98)$$

together with $s = \omega / q$, $v_F = p_F / E_p$, for the relativistic Fermi velocity and $N_F = 2v_F E_p^2 / \pi^2$ for the relativistic density of states at the Fermi surface. Here $E_p$ is defined by $\sqrt{p_F^2 + M^2}$, using the Fermi momentum $p_F$. The function $R_{\sigma h}^h$ for isosymmetric nuclear matter is given by

$$R_{\sigma h}^h = N_F \left( 1 - v_F^2 \right) \Phi(x), \quad R_{\sigma h}^a = N_F \sqrt{1 - v_F^2} \Phi(x), \quad R_{\sigma h}^b = N_F \Phi(x),$$

where $\Phi(x)$ stands for the Lindhard function with $x = \omega / v_F q$:

$$\Phi(x) = -1 + \frac{x}{2} \log \left| \frac{1 + x}{1 - x} \right| - \frac{i\pi}{2} |x| \theta(1 - |x|). \quad (99)$$

By writing the determinant Eq. (97) in terms of the Landau–Migdal parameters, $F_0$ and $F_1$, as [17]

$$\frac{D_L}{C_\sigma C_\omega C_1} = 1 + \frac{f_\sigma}{C_\sigma} R_{\sigma h}^h - \frac{f_\omega}{C_\omega} R_{\omega h}^h + \frac{f_\omega}{C_1} s^2 R_{\omega h}^b = 1 - \left( F_0 + \frac{F_1}{1 + F_1/3} x^2 \right)^2 \Phi(x), \quad (100)$$

we obtain

$$F_0 = - \frac{(1 - v_F^2)}{1 - f_\sigma R_{11}^{h-1}} F_\sigma, \quad F_\omega = \frac{F_\omega}{1 + f_\omega R_{00}^{h-1}}, \quad F_1 = - \frac{v_F^2 F_\omega}{1 - f_\omega R_{11}^{h-1}}, \quad (101)$$

with $F_\omega = N_F f_\omega$ and $F_\sigma = N_F f_\sigma$. Thus, Pauli-blocking terms yield the denominators depending on each meson exchange. They are calculated as

$$R_{11}^{h-1} = -4 \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{E_p^2}, \quad R_{00}^{h-1} = 0, \quad R_{11}^{h-1} = -4 \int \frac{d^3 p}{(2\pi)^3} \frac{E_p^2 - p^2/3}{E_p^3}. \quad (102)$$

It is seen that the attractive interactions are quenched by the Pauli-blocking terms in the same way as in Eq. (93). In the present model, there is no contribution to the repulsive part, since $R_{00}^{h-1} = 0$ due to $f_\sigma^{(0) f_\sigma^{(0)}} = 0$.

The parameter $F_0$ is responsible for the nuclear incompressibility coefficient, which determines the restoring force of the giant monopole state. Reduction of the attractive part makes the value of the coefficient higher [7].

In contrast to $F_0$, the parameter $F_1$ is constrained by a more fundamental requirement that the Fermi energy $E_F$ and momentum $p_F$ are transformed as a four-vector [25]. In the $\sigma$–$\omega$ model, the parameter $F_1$ comes from the longitudinal part of the $\omega$-meson exchanges as a relativistic effect, while
the nucleon effective mass stems from the $\sigma$-meson exchange. They are, however, not independent of each other, as in nonrelativistic models, and should satisfy, according to the above requirement \[11\],

$$\frac{E_{PF}}{E_F} = 1 + \frac{F_1}{3}$$

(103)

for $E_F = E_B + M$, $E_B$ being the binding energy per nucleon. As long as Eq. (103) holds, $F_1$ describes correctly the center-of-mass motion by the Lorentz boost [8], and also restores an abnormal enhancement of magnetic moments due to the effective mass in the Hartree approximation \[11\]. Thus, although NFA and NSA seem to be consistent with the framework of the Landau–Migdal parameters, this fact does not imply that the divergence can be neglected.

We note that the last term $N_Fv_F^2/3$ in $C_1^\omega$ of Eq. (98) comes from particle–hole excitations through the space component of the $\omega$-meson exchange:

$$R_{ph}^{11} = s^2 R_{\omega}^{ph} - \frac{N_Fv_F^2}{3}.$$

(104)

Because of the last term, the continuity equation does not hold in the particle–hole space, $R_{11}^{ph} \neq s^2 R_{00}^{ph} = s^2 R_{\omega}^{ph}$. NFA and NSA, however, take into account some of the $N$–$\bar N$ states. The calculation of $R_{11}^{\omega}$ in Eq. (102) provides $-N_Fv_F^2/3$, which leads to the continuity equation, $R_{11}^{ph} - R_{11}^{\omega} = s^2 \left( R_{00}^{ph} - R_{00}^{\omega} \right)$, as discussed in Sect. 4.

7. Conclusions

The structure of the relativistic random phase approximation (RPA) has been investigated in detail. The energy-weighted sum of the RPA transition strengths is expressed formally as the Hartree ground-state expectation value of the double commutator between the excitation operator and the Hamiltonian, as in nonrelativistic models. In calculating the commutator, however, the usual anticommutation relation between the baryon fields cannot be used \[15\]. Otherwise, the sum, which should be infinite \[15\], would vanish.

The main difference between the relativistic RPA and the nonrelativistic one stems from antinucleon ($\bar N$) degrees of freedom, but they cause divergence problems. Two kinds of approximations have been proposed by previous authors in order to avoid the problems without renormalization. One \[5,6\] is the no-free-term approximation (NFA), which simply neglects the divergent terms in the RPA response function, and the other \[10\] is the no-sea approximation (NSA), where $\bar N$ states are assumed to be empty. Actually, both approximations are equivalent to each other. They have been widely employed and shown to work well for reproducing experimental data in a phenomenological way \[3,6,9,10,12–14\].

The present paper has shown that NFA and NSA have serious problems. The RPA dispersion equation yields RPA states with negative excitation energy, in addition to the low-lying positive-energy states. This fact implies that the RPA ground state is not the lowest one. Owing to these negative excitation-energy states, the energy-weighted sum of the transition strengths vanishes. These results are not avoidable for NFA and NSA, which satisfy the RPA relation of the energy-weighted strengths, since the relativistic sum-rule value stems from the excitations of the Dirac sea \[15\]. Moreover, since the only limited space of nucleon–antinucleon states is included in NFA and NSA, attractive (repulsive) forces work as repulsive (attractive) ones between their couplings. This fact also affects the couplings of particle–hole states with nucleon–$\bar N$ states. Unfortunately, these unphysical couplings played an important role in explaining the spurious state \[10\] and the giant monopole.
states [12–14] in the previous numerical calculations in NSA. These results have been shown clearly by using a schematic model.

It has been shown that there is no problem for NSA and NFA to describe the continuity equation, since it is independent of the divergence.

Thus, $\bar{N}$ degrees of freedom that provide the divergence are not ignored. As long as some of the $\bar{N}$ space is necessary, the rest of the space should also be taken into account in an appropriate way, even in phenomenological models. Indeed, it was shown in Refs. [5,7,26,27] that the renormalization of the divergence plays an important role in discussions of some physical quantities. These roles are state-dependent, and could not be incorporated into phenomenological interactions or their coupling constants. Moreover, if the divergence of the linearly energy-weighted sum is understood, we can clarify the meaning of the analyses of the distribution of transition strengths with the energy moments [4].

References

[1] B. D. Serot and J. D. Walecka, *Advances in Nuclear Physics*, eds. E. Vogt and J. Negele, (Plenum, New York, 1986) Vol. 16.
[2] P. Ring, Prog. Part. Nucl. Phys. 37, 193 (1996).
[3] H. Liang, T. Nakatsukasa, Z. Niu, and J. Meng, Phys. Rev. C 87, 054310 (2013) and references therein.
[4] W. C. Chen, J. Piekarewicz, and M. Centelles, Phys. Rev. C 88, 024319 (2013).
[5] S. A. Chin, Ann. Phys. (N.Y.) 108, 301 (1977).
[6] H. Kurasawa and T. Suzuki, Nucl. Phys. A 445, 685 (1985).
[7] H. Kurasawa and T. Suzuki, Phys. Lett. B 474, 262 (2000).
[8] S. Nishizaki, H. Kurasawa, and T. Suzuki, Nucl. Phys. A 462, 687 (1987).
[9] J. R. Shepard, E. Rost, and J. A. McNeil, Phys. Rev. C 40, 2320 (1989).
[10] J. F. Dawson and R. J. Furnstahl, Phys. Rev. C 42, 2009 (1990).
[11] H. Kurasawa and T. Suzuki, Phys. Lett. B 165, 234 (1985).
[12] J. Piekarewicz, Phys. Rev. C 64, 024307 (2001).
[13] Z. Ma, N. Van Giai, A. Wandelt, D. Vretnar, and P. Ring, Nucl. Phys. A 686, 173 (2001).
[14] Z. Ma, L. Cao, N. Van Giai, and P. Ring, Nucl. Phys. A 722, C491 (2003).
[15] H. Kurasawa and T. Suzuki, Prog. Theor. Exp. Phys. 2013, 043D04 (2013).
[16] D. J. Rowe, *Nuclear Collective Motion, Models and Theory* (World Scientific, Singapore, 2010).
[17] H. Kurasawa and T. Suzuki, Nucl. Phys. A 454, 527 (1986).
[18] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[19] D. J. Thouless, Nucl. Phys. 22, 78 (1961).
[20] T. Goto and T. Inamura, Prog. Theor. Phys. 14, 396 (1955).
[21] J. Schwinger, Phys. Rev. Lett. 3, 296 (1959).
[22] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1986), p. 530
[23] H. Kurasawa and T. Suzuki, Phys. Rev. C 69, 014306 (2004).
[24] T. Matsui, Nucl. Phys. A 370, 365 (1981).
[25] G. Baym and S. A. Chin, Nucl. Phys. A 262, 527 (1976).
[26] H. Kurasawa and T. Suzuki, Nucl. Phys. A 490, 571 (1988).
[27] H. Kurasawa and T. Suzuki, Mod. Phys. Lett. A 21, 935 (2006).