DISTRIBUTION THEORY ON P.C.F. FRACTALS

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Abstract. We construct a theory of distributions in the setting of analysis on post-critically finite self-similar fractals, and on fractafolds and products based on such fractals. The results include basic properties of test functions and distributions, a structure theorem showing that distributions are locally-finite sums of powers of the Laplacian applied to continuous functions, and an analysis of the distributions with point support. Possible future applications to the study of hypoelliptic partial differential operators are suggested.

1. Introduction

The prevalence of fractal-like objects in nature has led both physicists and mathematicians to study dynamic processes on fractals. One rigorous way to do this on post-critically finite (p.c.f.) fractals is by studying differential equations in the natural analytic structure. A brief description of this analytic structure will appear in Section 2 below, but we emphasize that it is intrinsic to the fractal, and is not necessarily related to the analysis on a space in which the fractal may be embedded. For example, the familiar Sierpinski gasket fractal SG is often visualized as a subset of $\mathbb{R}^2$, but restricting a smooth function on $\mathbb{R}^2$ to SG does not give a smooth function on the fractal [3]. Similarly, we should not expect the solutions of differential equations on fractals to be quite like the solutions of their Euclidean analogues; for example, many fractals have Laplacian eigenfunctions that vanish identically on large open sets [4], whereas eigenfunctions of the Euclidean Laplacian are analytic.

Perhaps the most important tools for studying differential equations in the Euclidean context are Fourier analysis and the theory of distributions. Since the theory of analysis on fractals relies on first constructing a Laplacian operator $\Delta$, it is unsurprising that quite a lot is known about the fractal analogue of Fourier analysis. In some interesting cases the spectrum and eigenfunctions of the Laplacian are known explicitly, and many results about Laplacian eigenfunctions have also been derived by using probability theory to study the heat diffusion on fractals. Fourier-type techniques have also been applied to treat smoothness in the fractal setting: analogues of the Sobolev, Hölder-Zygmund and Besov spaces that are so important in Euclidean analysis of differential equations were introduced and investigated in [33]. Analogues of other basic objects in Euclidean analysis are studied in [19] [4]. By contrast there has not previously been a theory of distributions on fractals, and it is the purpose of the present work to provide one.

It is relatively elementary to define distributions on fractals; as usual they are dual to the space of smooth functions with compact support, where a function $u$ is said to be smooth if $\Delta^k u$ is continuous for all $k \in \mathbb{N}$. The main theorems about distributions are then really theorems about smooth functions, and the key to proving many of them is knowing how

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to smoothly partition smooth functions. Partitions of unity are used to achieve this in the Euclidean setting, but are not useful on fractals because products of smooth functions are not smooth \cite{3}. (This latter fact also implies that products of functions and distributions are not distributions, so the distributions are not a module over the smooth functions.) Instead we rely on a partitioning theorem for smooth functions proved in \cite{27}, see Theorem 2.7 below. Using this partitioning result we are able to prove analogues of the standard structure theorems describing distributions as derivatives of continuous functions (Theorem 5.9), and identifying the positive distributions as positive measures (Theorem 5.6).

We can also characterize the distributions of point support as finite linear combinations of certain “derivatives” of Dirac masses that can be explicitly described (Corollaries 6.6 and 6.13), provided we make certain assumptions about the point in question. These assumptions are needed in order to understand the local behavior of smooth functions at the point, and are related to work done in \cite{31,38,5,26,1}. The reader should be warned that many of our proofs are quite technical in nature; we have tried to explain in advance the strategies behind the proofs, which are more conceptual.

At the end of this paper we suggest several interesting questions related to the hypoellipticity of differential operators that are natural to consider in the context of distribution theory. It should also be noted that there are a number of results on local solvability of differential equations \cite{35,24} that could be reformulated in this context. We expect that this work will provide the foundation for many subsequent investigations.

2. Setting

We begin by describing the basic elements of analysis on a post-critically finite self-similar set $X$, as laid out in the monograph of Kigami \cite{13}; in this section all unreferenced results may be found in \cite{13}, which also includes proofs and references to the original literature. The reader who prefers to have a concrete example of a p.c.f. set in mind may choose to think of $X$ as the Sierpinski Gasket, in which case an more elementary exposition of the material that follows may be found in \cite{36}.

**P.C.F. Fractals.** Let $X$ be a self-similar subset of $\mathbb{R}^d$ (or more generally a compact metric space). By this we mean that there are contractive similarities $\{F_j\}_{j=1}^N$ of $\mathbb{R}^d$, and $X$ is the unique compact set satisfying $X = \bigcup_{j=1}^N F_j(X)$. Then $X$ has a natural cell structure in which we associate to a word $w = w_1w_2\ldots w_m$ of length $m$ the map $F_w = F_{w_1} \circ \ldots \circ F_{w_m}$, and call $F_w(X)$ an $m$-cell. If $w$ is an infinite word then we let $[w]_m$ be its length $m$ truncation and note that $F_w(X) = \bigcap_m F_{[w]_{m+1}}(X)$ is a point in $X$.

We say $F_j(x)$ is a critical value of $X = \bigcup_{j=1}^N F_j(X)$ if there is $y \in X$ and $k \neq j$ such that $F_k(y) = F_j(x)$. An infinite word $w$ is critical if $F_w(X)$ is a critical value, and $\hat{w}$ is post-critical if there is $j \in \{1,\ldots,N\}$ such that $jw$ is critical. We always assume that the set of post-critical words is finite, in which case the fractal is said to be post-critically finite (p.c.f.). The boundary of $X$ is then defined to be the finite set $V_0$ consisting of all points $F_{\hat{w}}(X)$ for which $\hat{w}$ is post-critical; this set is assumed to contain at least two points. We also let $V_m = \bigcup_{j=0}^{N} F_j(V_0)$, where the union is over all words of length $m$. Points in $V_0 = \bigcup_{m=0}^{\infty} V_m$ that are not in $V_0$ are called junction points, and a key property of p.c.f. fractals is that cells intersect only at junction points.

We fix a probability measure $\mu$ on $X$ that is self-similar in the sense that there are $\mu_1,\ldots,\mu_N$ such that the cell corresponding to $w = w_1\ldots w_m$ has measure $\mu(F_w(X)) = \prod_{j=1}^N \mu_{w_j}$. The usual Bernoulli measure in which each $\mu_j = \frac{1}{N}$ is one example.
**Dirichlet Form.** Our analysis on $X$ will be constructed from a self-similar Dirichlet form. A closed quadratic form $E$ on $L^2(\mu)$ is called Dirichlet if it has the (Markov) property that if $u \in \text{dom}(E)$ then so is $\tilde{u} = u_0 + 1 + \chi_{A^c}$ and $E(\tilde{u}, \tilde{u}) \leq E(u, u)$, where $\chi_A$ is the characteristic function of $A$. Self-similarity of $E$ means that there are renormalization factors $r_1, \ldots, r_N$ such that

$$E(u, v) = \sum_{j=1}^{N} r_j^{-1} E(u \circ F_j, v \circ F_j). \quad (2.1)$$

It follows immediately that $E(u, v)$ can also be expressed as the sum over $m$-words of $r_w^{-1} E(u \circ F_w, v \circ F_w)$ where $r_w = r_{w_1} \cdots r_{w_m}$. In order to use results from [27] we assume that $0 < r_j < 1$ for all $j$, in which case $E$ is regular, meaning that that $C(X) \cap \text{dom}(E)$ is dense both in $\text{dom}(E)$ with $E$-norm and in the space of continuous functions $C(X)$ with supremum norm.

It is far from obvious that interesting fractals should support such Dirichlet forms, but in fact the conditions described so far are satisfied by many p.c.f. self-similar sets that have sufficient symmetry. In particular, if $X$ is a nested fractal in the sense of Lindstrøm [17] then a Dirichlet form of the above type may be constructed using a diffusion or a harmonic structure [15, 12, 29]. Some other approaches may be found in [20, 14, 18, 22, 21, 10, 23].

**Harmonic Functions.** Given a function on $V_0$ (usually thought of as an assignment of boundary values) there is a unique continuous function on $X$ that has these boundary values and minimizes the energy. Such functions are called harmonic, and form a finite dimensional space containing the constants. It is easy to see that there are harmonic extension matrices $A_j$, $j = 1, \ldots, N$ with the property that if $h$ is harmonic then $A_j$ maps the values of $h$ on $V_0$ to its values on $F_j(V_0)$. The largest eigenvalue of each $A_j$ is 1, corresponding to the constant functions; it is useful to know that the second eigenvalue is $r_j$, and that all other eigenvalues (which may be complex) have strictly smaller absolute value (13, Appendix A).

**The Laplacian and Normal Derivatives.** Using the energy and measure we produce a weak Laplacian by defining $f = \nabla u$ if $E(u, v) = -\int f v \, d\mu$ for all $v \in \text{dom}(E)$ that vanish on $V_0$. Our assumptions so far are sufficient to conclude that $-\Delta$ is a non-negative self-adjoint operator on $L^2(\mu)$ with compact resolvent (see Theorem 2.4.2 of [13]). We denote its eigenvalues by $\lambda_j$ and the corresponding eigenvectors by $\psi_j$. When $\lambda \in C(X)$ we write $u \in \text{dom}(\lambda)$ and think of these as the (continuously) differentiable functions on $X$. Inductively define $\text{dom}(\lambda^k)$ for each $k$ and then $\text{dom}(\lambda^\infty) = \cap_k \text{dom}(\lambda^k)$. We say $f$ is smooth if $f \in \text{dom}(\lambda^\infty)$. Harmonic functions have zero Laplacian.

By introducing a normal derivative $\partial_n$ at boundary points the defining equation for the Laplacian can be extended to functions that do not vanish on $V_0$. As a result we have the Gauss-Green formula $E(u, v) = -\int (\Delta u) v \, d\mu + \sum_{x \in V_0} \int_{V} v(x) \partial_n u(x)$ when $v \in \text{dom}(E)$, as in Theorem 3.7.8 of [13]. This formula may be localized to a cell $F_w(X)$, in which case $\partial_n u(q) = \lim_m E(u, v_m)$ at the boundary point $q = \bigcap_m F_{w_m}(X)$, where $v_m$ is the harmonic function on $F_{w_m}(X)$ with all boundary values equal to 0 other than $v_m(q) = 1$. The superscript $w$ in $\partial_n u(q)$ indicates which cell the normal derivative is taken with respect to, as there is one for each cell that intersects at $q$. In general the normal derivatives exist once $\Delta u$ exists as a measure. If $u \in \text{dom}(\lambda)$ then the normal derivatives at a point sum to zero. Conversely, if $u$ is defined piecewise by giving functions $u_j \in \text{dom}(\lambda)$ each supported on one of the cells that share the boundary point $x$, then $u \in \text{dom}(\lambda)$ if and only if all $u_j(x)$ are
equal, all $\Delta u_j(x)$ are equal, and the normal derivatives of the $u_j$ at $x$ sum to zero. We call these constraints the matching conditions for the Laplacian.

**Resistance Metric.** In addition to the Laplacian and other derivatives, the Dirichlet form also provides us with a metric intrinsic to the fractal. We define the resistance metric $R$ by

$$R(x, y) = \min(E(u)^{-1} : u \in \text{dom}(E), \ u(x) = 0, \ u(y) = 1).$$

In Sections 2.3 and 3.3 of [32] it is proven that under our assumptions this minimum exists and defines a metric, and that the $R$-topology coincides with the topology induced from the embedding of $X$ into $\mathbb{R}^d$. Of particular importance for us is the fact that continuity may be treated using the resistance metric, for which purpose the following H"older-$\frac{1}{2}$ estimate is very useful:

$$|u(x) - u(y)|^2 \leq E(u)R(x, y) \leq \|u\|_2^2\|\Delta u\|_2^2R(x, y). \quad (2.2)$$

If $u \in \text{dom}(\Delta)$ vanishes on $V_0$ then we obtain the first inequality trivially from the definition and the second by applying the Cauchy-Schwartz inequality to $E(u) = -\int u\Delta u \, d\mu$. For general $u \in \text{dom} \Delta$ we can simply subtract the harmonic function with the same boundary values and apply the same estimate. In particular, this shows that the $L^2$ domain of the Laplacian embeds in the continuous functions.

**Fractafolds.** Since the results in this paper are primarily local in nature, we will be able to work on a connected fractafold based on $X$ with a restricted cellular construction, which we denote by $\Omega$. Some results on fractafolds and their spectra may be found in [32]. As with a manifold based on Euclidean space, a fractafold based on $X$ is just a connected Hausdorff space in which each point has a neighborhood homeomorphic to a neighborhood of a point in $X$. One way to construct a fractafold is by suitably gluing together copies of $X$, for example by identifying appropriate boundary points. This leads us to the idea of a cellular construction, which is the analogue of a triangulation of a manifold. A restricted cellular construction consists of a finite or countably infinite collection of copies $X_j$ of $X$, together with an admissible identification of their boundary points. Admissibility expresses the requirement that the result of the gluing be a fractafold; more precisely, it means that if $\{x_1, \ldots, x_J\}$ are identified then there is a junction point $x \in X$ and a neighborhood $U$ of $x$ such that each of the components $U_1, \ldots, U_J$ of $U \setminus \{x\}$ is homeomorphic to a neighborhood of the corresponding point $x_j$ in $X_j$. We call any such point $x$ a gluing point, and make the obvious definition that a neighborhood of $x$ is a union of neighborhoods of $x$ in each of the cells $X_j$ that meet at $x$ in the manner previously described.

It should be noted that the above is not the most general kind of cellular construction (hence the term restricted in the definition), because some fractals have non-boundary points (called terminal points) at which cells may be glued (see [32], Section 2). Dealing with such points introduces certain technicalities that, while not insurmountable, cause complications in defining the Green’s operator (see below) that we will need for proving Theorem 5.9. It is worth noting that if $X$ has some topological rigidity then all fractafolds have restricted cellular structure. This is true, for example, for fractafolds based on the Sierpinski Gasket ([32] Theorem 2.1).

Thus far our fractafold has only topological structure; however if $\Omega$ has a restricted cellular construction then a smooth structure may be introduced in the same manner as it was on $X$ itself, specifically by defining a Dirichlet energy and a measure and thus a weak Laplacian. We can take the energy on $\Omega$ to be the sum of the energies on the cells of the cellular construction, and the measure (which is not necessarily finite, but is finite on
compacta) to be the sum of the measures on the cells:

\[ E(u, v) = \sum_j E_X(a_j|_{X_j}, v|_{X_j}) = \sum_j a_j E_X(u|_{X_j} \circ \iota_j, v|_{X_j} \circ \iota_j) \]

\[ \mu(A) = \sum_j \mu_X(A \cap X_j) = \sum_j b_j \mu_X(\iota_j^{-1}(A \cap X_j)) \]

where \( \iota_j : X \to X_j \) is the map from the cellular construction. In the same way that the angle sum at a vertex of a triangulation of a manifold determines the curvature at the vertex, the choice of the weights \( a_j \) and \( b_j \) amount to a choice of metric on \( \Omega \), with \( a_j = b_j = 1 \) for all \( j \) being the flat case (see [13], Section 6). As all of the computations made later in the paper may be made on one cell at a time, we will henceforth suppress the weights \( a_j \) and \( b_j \).

Well-known results about p.c.f. fractals imply the existence of a Green’s function (for which there is an explicit formula) on finite unions of cells in a fractal with cellular construction.

**Lemma 2.1.** Let \( K = \bigcup_j X_j \) be a connected finite union of cells in \( \Omega \) and such that \( K \neq \Omega \). Then there is a Green’s operator \( G_K \) with the property that if \( \nu \) is a Radon measure on \( K \) (i.e., a Borel measure that is finite on compacta, outer regular on Borel sets and inner regular on open sets), then \( G_K \nu \) is continuous, \( -\Delta G_K \nu = \nu \) on the interior of \( K \), and \( G_K \nu|_\partial K = 0 \). The same conclusion holds in the case \( K = \Omega \) under the additional assumption \( \int dv = 0 \).

**Remark 2.2.** It is clear that \( \partial K \) is a subset of the boundary points of the cells \( X_j \), specifically consisting of those gluing points at which not all glued cells are included in \( K \).

**Proof.** We recall from Sections 3.6 and 3.7 of [13] that our assumptions on \( X \) imply there is a Green’s operator \( G \) on \( X \) with continuous kernel \( g(x, y) \), such that \( -\Delta g \nu = \nu \) and \( G\nu|_\partial X = 0 \) for all Radon measures \( \nu \). There is an explicit formula giving \( g(x, y) \) as a series.

If \( G_j \) is the Green’s operator for the cell \( X_j \), it is easy to verify that \( -\Delta \sum G_j \nu = \nu \), except at the gluing points where the Laplacian can differ from \( \nu \) by Dirac masses, the size of which may be computed explicitly by summing the normal derivatives of the \( G_j \nu \) at the points that are glued. However it is also apparent that by assigning values at each of the gluing points and extending harmonically on the cells we obtain a continuous and piecewise harmonic function, the Laplacian of which is a sum of Dirac masses at the gluing points.

Provided the boundary \( \partial K \) is non-empty (which is obvious if \( K \neq \Omega \)), a linear algebra argument (Lemma 3.5.1 in [13]) shows that for any prescribed set of weights for Dirac masses of the Laplacian at interior gluing points, there is a unique piecewise harmonic function that vanishes on \( \partial K \) and has this Laplacian. Subtracting this piecewise harmonic function from \( \sum G_j \nu \) gives the required \( G_K \nu \).

On the other hand, if \( K = \Omega \) then \( \Omega \) is compact and the kernel of \( \Delta \) is precisely the constant functions. We can therefore invert \( -\Delta \) on the measures that annihilate constants, that is, those for which \( \int dv = 0 \). This can be done explicitly in the same manner as in the previous case, except that the linear algebra step now shows the Laplacians of piecewise harmonics span the space of mean-zero linear combinations of Dirac masses at the gluing points. In this case the choice of piecewise harmonic function is unique up to adding a constant; our convention is to choose this constant so \( \int G_K \nu(x) d\mu(x) = 0 \).

Throughout this paper we will assume that \( \Omega \) has no boundary. In some examples it is possible to deal with boundary points by passing to an appropriate cover, but relatively little
is known in terms of covering theory for general fractafolds. Elementary examples to keep in mind include non-compact cases like open subsets of $X \setminus V_0$ or the infinite Sierpinski Gasket treated in [27], and compact fractafolds like the double cover of the Sierpinski Gasket $SG$, which consists of two copies of $SG$ with each boundary point from one copy identified with exactly one boundary point of the other (see [32] for more details).

**Smooth Cutoffs and Partitioning.** As mentioned earlier, the structure theorems we shall prove for distributions rest heavily on results from [27]. In what follows we assume that $\Omega$ is a fractafold with restricted cellular structure and is based on a fractal $X$ with regular harmonic structure.

Recall that $x \in X$ is a junction point if and only if there is a neighborhood $U \ni x$ such that $U \setminus \{x\}$ is disconnected into a finite number of components $U_j$. For a smooth function $u$ the quantities $\Delta^k u(q)$ and $\partial^j_\nu \Delta^k u(q)$ exist for all $k \in \mathbb{N}$; the superscript $j$ on $\partial^j_\nu$ indicates the normal derivative with respect to the cell $U_j$. For a fixed $j$, the two sequences $\Delta^k u(q)$ and $\partial^j_\nu \Delta^k u(q)$ make up the jet of $u$ at $q$ in $U_j$. The first result we need from [27] is a Borel-type theorem on the existence of smooth functions with prescribed jets.

**Theorem 2.3 ([27], Theorem 4.3 and Equation 4.8).** Given values $\rho_0, \rho_1, \ldots$ and $\sigma_0, \sigma_1, \ldots$, there is a smooth function $f$ on $U_j$ that vanishes in a neighborhood of all boundary points except $q$, where the jet is given by $\Delta^k f(q) = \rho_k$ and $\partial^j_\nu \Delta^k f(q) = \sigma_k$ for all $k$. If we write $U_j$ as $U_j = F_w(X)$ for a word $w$, and fix a number $L$ of jet terms, then for any $\epsilon > 0$ we may construct $f$ so that for $0 \leq k \leq L$, we have the estimate

$$\|\Delta^k f\|_{\infty} \leq C(k)(r_n \mu_n)^{-k} \left( \sum_{l=0}^{L} r_n^l \mu_n^l |\rho_l| + \sum_{l=0}^{L-1} r_n^{l+1} \mu_n^l |\sigma_l| \right) + \epsilon$$  \hspace{1cm} (2.3)

where $C(k)$ depends only on $k$ and the harmonic structure on $X$.

**Remark 2.4.** Of course, it follows immediately that we can construct a smooth function with prescribed jets at each of the boundary points of a cell $K$ and an estimate like (2.3), just by applying the theorem separately to each of the boundary points and summing the result.

**Corollary 2.5.** If $K$ is a cell in $\Omega$ and $U$ is an open neighborhood of $K$, then there is a smooth function $f$ such that $f = 1$ on $K$, $f = 0$ outside $U$, and $\|f\|_{\infty} \leq C$, where $C$ is a constant that does not depend on $K$ or $U$.

**Proof.** Let $\{q_j\}$ be the boundary points of $K$ and at each $q_j$ take cells $V_{j,k} \subset U$ such that $\bigcup_k V_{j,k} \cup K$ contains a neighborhood of $q_j$. By making all of these cells sufficiently small and removing any inside $K$ we may further assume that the $V_{j,k}$ intersect $K$ only at $q_j$, intersect each other only at $q_j$ and do not intersect $V_{j,k'}$ for any $j' \neq j$.

On each $V_{j,k}$ construct the smooth function $f_{j,k}$ guaranteed by Theorem 2.3 with $f_{j,k} = 1$ at $q_j$ and all other jet terms at $q$ equal to zero, and taking $\epsilon = 1$. Then the piecewise function

$$f(x) = \begin{cases} 1 & \text{for } x \in K \\ f_{j,k} & \text{for } x \in V_{j,k} \\ 0 & \text{otherwise} \end{cases}$$

is equal 1 on $K$ and 0 off $U$ by construction. It is also smooth, simply because the pieces are smooth and the matching conditions for $\Delta^k$ apply at each of the boundary points of the $V_{j,k}$ for all $l$. The bound $\|f\|_{\infty} \leq C$ independent of $K$ and $U$ now follows from (2.3) because the scale-dependent terms are all raised to the power zero, so are constant. \qed
A more difficult task than that in Corollary 2.5 is to construct a positive bump function that is equal to 1 on $K$ and to zero outside the neighborhood $U$ of $K$. A result of this type was proven in [27] under certain assumptions on the diffusion $Y_t$ corresponding to the Laplacian. A sufficient assumption is that the heat kernel $p_t(x, y)$ (i.e. the transition density of $Y_t$) satisfies an estimate of the form

$$p_t(x, y) \leq \frac{\gamma_1}{t^\alpha} \exp(-\gamma_2 \frac{R(x, y)^\beta}{t^{1/2}}) \quad (2.4)$$

where $\alpha, \beta, \gamma_1$ and $\gamma_2$ are constants. The estimate (2.4) is known to be valid in great generality on p.c.f. fractals (Corollary 1.2 of [9]).

**Theorem 2.6** ([27] Corollary 2.9). Under the assumption (2.4), for a cell $K$ and an open neighborhood $U \supset K$, there is a smooth function $f$ such that $f = 1$ on $K$, $f = 0$ outside $U$, and $f(x) \geq 0$ for all $x$.

The final theorem from [27] that we will use extensively is concerned with the smooth partitioning of a smoothing function.

**Theorem 2.7** ([27], Theorem 5.1). Let $K \subset X$ be compact and fix $\bigcup U_\alpha \supset K$ an open cover. If $f \in \text{dom}(\Delta^n)$ then there is a decomposition $f = \sum f_j$ in which each $f_j$ is in $\text{dom}(\Delta^n)$ and has support in some $U_\alpha$.

**Remark 2.8.** Compactness of $K$ is used only to obtain finiteness of the decomposition, and may be omitted for finite covers. An analogous countable (and locally finite) decomposition is then valid in the $\sigma$-compact case; in particular it is valid on $\Omega$, because of the existence of a cellular structure.

**Remark 2.9.** The proof uses a result on the existence of smooth functions with prescribed jet at a point (Theorem 4.3 of [27]) to smoothly join cutoffs to a piece of the original function as in the proof of Theorem 2.5. This is very different from the Euclidean case where one simply multiplies the smooth function by a smooth bump. In particular, the construction of the cutoff depends explicitly on the growth rate of the jet of $f$ at the boundary points under consideration, so for a collection of sets indexed by $j$, the mapping $f \mapsto \{f_j\}$ to a sequence of smooth functions supported on these sets is nonlinear.

Although the non-linearity will make some later proofs more complicated, this method does provide good estimates. From (2.3) and standard arguments for controlling the normal derivative $\partial_n \Delta^k f$ at a point by the norms $||\Delta^j||_\infty$, $j = 0, \ldots, k + 1$, over a neighborhood of the point (like those in Section 6 below) we find that $f \mapsto f_j$ can be arranged to satisfy

$$||\Delta^j f_j||_\infty \leq C \sum_{l=0}^k ||\Delta^l f||_\infty \quad (2.5)$$

where $C$ is a constant depending only on $k$ and $K$.

### 3. Test Functions

We define test functions on $\Omega$ in the usual way, and provide notation for the space of smooth functions on $\Omega$ topologized by uniform convergence on compacta.

**Definition 3.1.** The space of test functions $\mathcal{D}(\Omega)$ consists of all $\phi \in \text{dom}(\Delta^n)$ such that $\text{Supp}(\phi)$ is compact. We endow it with the topology in which $\phi_i \rightarrow \phi$ if and only if there is a compact set $K \subset \Omega$ containing the supports of all the $\phi_i$, and $\Delta^k \phi_i \rightarrow \Delta^k \phi$ uniformly on $K$ for each $k \in \mathbb{N}$. There is a corresponding family of seminorms defined by

$$|\phi|_{lm} = \sup(|\Delta^k \phi(x)) : x \in \Omega, k \leq m| \quad (3.1)$$
though it should be noted that the topology on \( \mathcal{D}(\Omega) \) is not the usual metric topology produced by this family. For a discussion of the topology on \( \mathcal{D}(\Omega) \) and its relation to these seminorms, see Chapter 6 of [28].

**Definition 3.2.** \( \mathcal{E}(\Omega) = \text{dom}(\Delta^n) \) with the topology \( \phi_i \to \phi \) iff for every compact \( K \subset \Omega \) we have \( \Delta^k \phi_i \to \Delta^k \phi \) uniformly on \( K \) for each \( k \in \mathbb{N} \). There is a corresponding family of seminorms defined by

\[
|\phi|_{m,K} = \sup|\Delta^k(\phi(x))| : x \in K, k \leq m). \tag{3.2}
\]

The following result is immediate from Theorem 2.7 and (2.5). It will be used frequently in the results proved below.

**Lemma 3.3.** If \( \phi \in \mathcal{D}(\Omega \cup \Omega_2) \), then \( \phi = \phi_1 + \phi_2 \) for some \( \phi_j \in \mathcal{D}(\Omega_j) \). For each \( m \) there is \( C = C(M, \Omega_1, \Omega_2) \) so \( |\phi_j|_m \leq C|\phi|_m \), \( j = 1, 2 \).

One consequence is that \( \mathcal{D}(\Omega) \) is dense in \( \mathcal{E}(\Omega) \), because we may fix an increasing compact exhaustion \( \cup_j K_j = \Omega \) of our domain and for arbitrary \( \phi \in \mathcal{E}(\Omega) \) write \( \phi = \phi_j + \tilde{\phi}_j \), where \( \phi_j \) is supported in \( K_{j+1} \) and \( \tilde{\phi}_j \) is supported in \( K_j^c \), so that \( \phi_j \big|_{K_j} = \tilde{\phi}_j \big|_{K_j^c} \). The functions \( \phi_j \) are in \( \mathcal{D}(\Omega) \) and it is clear that \( \Delta^k \phi_j \to \Delta^k \phi \) uniformly on compacta, hence \( \phi_j \to \phi \) in \( \mathcal{E}(\Omega) \). Another density result that follows from Lemma 3.3 is as follows.

**Theorem 3.4.** \( \mathcal{D}(\Omega) \) is dense in \( \mathcal{C}_c(\Omega) \), the space of continuous functions with compact support, with supremum norm.

**Proof.** The dual of \( \mathcal{C}_c(\Omega) \) is the space of Radon measures, so by the Hahn-Banach Theorem, it suffices to show that if such a measure \( \nu \) satisfies

\[
\int \phi \, dv = 0, \quad \text{for all } \phi \in \mathcal{D}(\Omega), \tag{3.3}
\]

then \( \nu \equiv 0 \).

Let \( K \) be a cell and \( \{U_i\} \) a sequence of open sets containing \( K \) so that \( \nu(U_i \setminus K) \to 0 \). Using Corollary 2.5 we see that for each \( j \) we can take \( \phi_j \in \mathcal{D}(\Omega) \) with \( \phi_j \equiv 1 \) on \( K_j \), the bound \( ||\phi_j||_{\infty} \leq C \) for all \( i \), and \( \text{Sppt}(\phi_j) \subset (U_i) \). Then for \( \nu \) satisfying (3.3) we compute

\[
\nu(K) = \int \phi_j \, dv - \mu(K) \leq ||\phi_j||_{\infty} \nu(U_j \setminus K) \leq C \nu(U_j \setminus K) \to 0.
\]

As \( \nu \) vanishes on all cells it is the zero measure, and the result follows.  \( \square \)

Since \( \Omega \) is locally compact and Hausdorff, it is a standard result that \( \mathcal{C}_c(\Omega) \) is supremum-norm dense in \( \mathcal{C}_0(\Omega) \), where the latter consists of those continuous functions \( f \) for which the set \( \{x : |f(x)| \geq \epsilon \} \) is compact for all \( \epsilon > 0 \). Hence \( \mathcal{D}(\Omega) \) is also dense in \( \mathcal{C}_0(\Omega) \).

In the special case where \( \Omega \) is compact we may also characterize \( \mathcal{D}(\Omega) = \mathcal{E}(\Omega) \) by the decay of the Fourier coefficients obtained when \( \phi \) is written with respect to a basis of Laplacian eigenfunctions. This provides an alternate proof of the density of \( \mathcal{D}(\Omega) \) in \( \mathcal{C}_c(\Omega) \), which of course coincides with \( \mathcal{C}(\Omega) \) in this case.

**Theorem 3.5.** If \( \Omega \) is compact then \( \mathcal{D}(\Omega) = \mathcal{E}(\Omega) \) is the space of smooth functions with Fourier coefficients that have faster than polynomial decay, and hence is dense in \( \mathcal{C}(\Omega) \).

**Proof.** Clearly \( \phi \in \mathcal{D}(\Omega) \) is in \( L^2 \), so can be written \( \phi = \sum_{i=0}^{\infty} a_i \psi_i \), where \( \psi_i \) is the Laplacian eigenfunction with eigenvalue \(-\lambda_i\). It follows that \( (-\Delta)^k \phi = \sum_{i=0}^{\infty} a_i \lambda_i^k \psi_i \) with convergence in \( L^2 \). Since \( \Delta^k \phi \) is in \( \mathcal{C}(\Omega) \subset L^2 \) for all \( k \) we see that the sequence \( a_i \) must decay faster than any polynomial in the \( \lambda_i \). Conversely any such sequence converges to a function for which every power of the Laplacian is in \( L^2 \), whereupon the function is smooth by
iteration of (2.2). In addition, any \( u \in C(\Omega) \) can be explicitly approximated by functions from \( \mathcal{D}(\Omega) \) by taking successive truncations of the Fourier series \( u = \sum_{i=0}^{\infty} a_i \psi_i \). To see this gives convergence in \( \mathcal{D}(\Omega) \) write \((-\Delta)^k \sum_{i=0}^{\infty} a_i \psi_i = \sum_{i=0}^{\infty} a_i \lambda_i^k \psi_i \) and note this converges to zero in \( L^2 \) and therefore almost everywhere. Now from (2.2)

\[
\sup_{\Omega} \sum_{i=j}^{\infty} a_i \lambda_i^k \psi_i^2 \leq C \left\| \sum_{i=j}^{\infty} a_i \lambda_i \psi_i \right\|_2 \left\| \sum_{i=j}^{\infty} a_i \lambda_i^{k+1} \psi_i \right\|_2
\]

and both terms on the right converge to zero. \( \square \)

In [33] there is a definition of Sobolev spaces on p.c.f. fractals of the type studied here. These spaces may be defined by applying the Bessel potential \((I-\Delta)^{-s}\) (for the Dirichlet or Neumann Laplacian) or the Riesz potential \((-\Delta)^{-s}\) (for the Dirichlet Laplacian) to \( L^2 \) functions on the fractal, and adding on an appropriate space of harmonic functions. In particular, the space of \( L^2 \) functions with \( \Delta^k u \in L^2 \) for \( 0 \leq k \leq m \) may be identified with a particular \( L^2 \) Sobolev space (33 Theorem 3.7). Writing \( W^{s,2} \) for the \( L^2 \) Sobolev space arising from \((I-\Delta)^{-s}\), we have in consequence of the preceding:

**Corollary 3.6.** If \( \Omega \) is compact then \( \mathcal{D}(\Omega) = \cap_{s>0} W^{s,2} \).

4. **Distributions**

**Definition 4.1.** The space of *distributions* on \( \Omega \) is the dual space \( \mathcal{D}'(\Omega) \) of \( \mathcal{D}(\Omega) \) with the weak-star topology, so \( T_i \to T \) if and only if \( T_i \phi \to T \phi \) for all \( \phi \in \mathcal{D}(\Omega) \).

As usual, the most familiar examples of distributions are the Radon measures. If \( d\nu \) is such a measure then we define \( T_\nu \) by \( T_\nu \phi = \int \phi d\nu \). Theorem 3.4 ensures that the mapping \( \nu \mapsto T_\nu \) is injective, so we may identify \( \nu \) and \( T_\nu \). One way to obtain further examples is to take the adjoint of the Laplacian on distributions, which clearly produces another distribution.

**Definition 4.2.** If \( T \in \mathcal{D}'(\Omega) \) we define \( \Delta T \in \mathcal{D}'(\Omega) \) by \( (\Delta T) \phi = T(\Delta \phi) \) for all \( \phi \in \mathcal{D}(\Omega) \).

It is clear that powers of the Laplacian applied to the Radon measures provide a rich collection of examples of distributions. Later we prove that all distributions arise in essentially this way (Theorem 5.9), but we first need to establish some more elementary properties.

**Theorem 4.3.** A linear functional \( T \) on \( \mathcal{D}(\Omega) \) is a distribution if and only if for each compact \( K \subset \Omega \) there are \( m \) and \( M \) such that

\[
|T \phi| \leq M |\phi|_m \quad (4.1)
\]

**Proof.** It is clear that the existence of such an estimate ensures continuity of \( T \). To prove the converse we assume no such estimate exists, so there is \( K \) compact and a sequence \( \phi_j \) such that \( |T \phi_j| \geq j |\phi_j|_j \). Then the support of \( \tilde{\phi}_j = \phi_j / T \phi_j \) is in \( K \) for all \( j \) and

\[
\left\| \Delta^j \tilde{\phi}_j \right\|_\infty \leq \frac{|\phi_j|_j}{|T \phi_j|} \leq \frac{1}{j} \quad \text{once } j \geq k.
\]

Therefore \( \tilde{\phi}_j \to 0 \) in \( \mathcal{D}(\Omega) \) but has \( T \tilde{\phi}_j = 1 \) for all \( j \), contradicting the continuity of \( T \). \( \square \)

In the special case that \( \Omega \) is compact we saw in Theorem 3.5 that \( \mathcal{D}(\Omega) \) consists of smooth functions having Fourier coefficients that decay faster than polynomially. This allows us to identify its dual with coefficient sequences having at most polynomial growth.
Lemma 4.4. If $\Omega$ is compact and $T \in D'(\Omega)$ then the sequence $T\psi_i$ has at most polynomial growth. Conversely, any sequence $(b_i)$ of polynomial growth defines a distribution via $\phi = \sum_i a_i \psi_i \mapsto \sum_i a_i b_i$.

Proof. We saw in Theorem 3.5 that $\sum_i a_i \psi_i$ converges to $\phi$ in $D(\Omega)$ if and only if $(a_i)$ has faster than polynomial decay in $\lambda_i$. It follows that for any $T \in D'(\Omega)$, $\sum_i a_i T\psi_i = T \sum_i a_i \psi_i \to T\phi$, from which the sequence $T\psi_i$ has at most polynomial growth.

Conversely suppose that $(b_i)$ has polynomial growth, $|b_i| \leq C \lambda_i^n$, and consider the map $(a_i) \mapsto \sum_i a_i b_i$. This is a well defined linear map $\phi \mapsto \langle \phi, \psi_j \rangle$ which shows that $T$ is a distribution.

In particular, if we identify $D(\Omega)$ as a subset of $D'(\Omega)$ by letting $\phi' \in D(\Omega)$ act on $D(\Omega)$ via $\phi \mapsto \langle \phi, \phi' \rangle$, where $\langle \cdot, \cdot \rangle$ is the $L^2$ inner product, then this implies that the test functions are dense in the distributions when $\Omega$ is compact. To see this, define $T_j$ by

$$T_j \psi_i = \begin{cases} T \psi_i & \text{if } i \leq j \\ 0 & \text{if } i > j. \end{cases}$$

We see that $(T - T_j)\phi = \sum_i b_i \psi_i \to 0$ for any $\phi \in D(\Omega)$, so $T_j \to T$. Since $T_j$ is the inner product with the function $\sum_i b_i \psi_i$, it is in $D(\Omega)$. This is true more generally.

Theorem 4.5. $D(\Omega)$ is dense in $D'(\Omega)$.

Proof. Let $T \in D'(\Omega)$. Take an increasing exhaustion $\cup K_j$ of $\Omega$ by compact sets $K_j$ with the property that $K_j$ is contained in the interior of $K_{j+1}$, and each $K_j$ is a finite union of cells. For each $j$, the action of $T$ on $D(K_j)$ identifies it as an element of $D'(K_j)$ so by the preceding there is a sequence $\{T_{j,k}\}_{k=0}^\infty \subseteq D(K_j)$ for which $T_{j,k} \to T$ in $D'(K_j)$, and hence in $D'(\Omega)$ for all $l \leq j$.

Now consider the diagonal sequence $T_{j,j}$. For any test function $\phi$ there is some $j_0$ such that $K_{j_0}$ contains the support of $\phi$, and hence $T_{j,j_0} \phi$ is defined for $j \geq j_0$ and converges to $T\phi$. So $T_{j,j} \to T$ in $D'(\Omega)$. Of course, $T_{j,j}$ only corresponds to a test function $\phi_j$ on $K_j$, not to an element of $D(\Omega)$. To remedy this, note that for the test function $\phi_j$ corresponding to $T_{j,j}$ on $K_j$ we may apply Theorem 3.5 to each of the (finite number of) boundary points of $K_j$ and thereby continue $\phi_j$ smoothly to a function $\phi'_j$ on $\Omega$ with support in $K_{j+1}$. Denote by $T'_j$ the distribution corresponding to this new test function $\phi'_j$. Since $\phi_j$ and $\phi'_j$ coincide on $K_j$ we see that $T_{j,j} \phi = T'_j \phi$ for all $\phi$ having support in $K_j$. It follows that $T'_j$ converges to $T$ in $D'(\Omega)$, and since each $T'_j$ corresponds to a test function, the proof is complete.

Definition 4.6. If $\Omega_1 \subset \Omega$ is open, we say the distribution $T$ vanishes on $\Omega_1$ if $T\phi = 0$ for all $\phi$ supported on $\Omega_1$. This is written $T \big|_{\Omega_1} = 0$.

To make a meaningful definition of the support of a distribution we again need the partitioning property. By Lemma 3.3 we know that any $\phi \in D(\Omega_1 \cup \Omega_2)$ can be written as $\phi = \phi_1 + \phi_2$ for $\phi_j \in D(\Omega_j)$. If both $T\phi_1 = 0$ and $T\phi_2 = 0$ it follows that $T\phi = 0$. We record this as a lemma, and note that it establishes the existence of a maximal open set on which $T$ vanishes.

Lemma 4.7. If $T \big|_{\Omega_1} = 0$ and $T \big|_{\Omega_2} = 0$ then $T \big|_{\Omega_1 \cup \Omega_2} = 0$. 

**Definition 4.8.** The support of $T$ is the complement of the maximal open set on which $T$ vanishes, and is denoted $\text{Sppt}(T)$. In the special case where $\text{Sppt}(T)$ is compact we call $T$ a compactly supported distribution.

**Theorem 4.9.** The space of compactly supported distributions is (naturally isomorphic to) the dual $\mathcal{E}'(\Omega)$ of the smooth functions on $\Omega$.

**Proof.** The inclusion $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ defines a natural map from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$. We have seen (after Lemma 3.3 above) that $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$, from which it follows that the kernel of this map is trivial. For convenience we identify $\phi T$ seen (after Lemma 3.3 above) that $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}(\Omega)$, from which it follows that the kernel of this map is trivial. For convenience we identify $\mathcal{E}'(\Omega)$ with its isomorphic image in $\mathcal{D}'(\Omega)$, so we need only verify it is the space of distributions with compact support.

Fix an increasing sequence of compacta $K_j$ with $\Omega = \cup_j K_j$. If $T \in \mathcal{E}'(\Omega)$ fails to be compactly supported then for each $j$ there is $\phi_j$ supported in $\Omega \setminus K_j$ such that $T \phi_j \not= 0$, and by renormalizing we may assume $T \phi_j = 1$ for all $j$. However for any compact $K$ there is $j$ such that $K \subset K_j$ and thus $\phi_l \equiv 0$ on $K$ once $l \geq j$. This implies $\phi_j \rightarrow 0$ in $\mathcal{E}(\Omega)$ and $T \phi_j = 1$ for all $j$, a contradiction.

Conversely, let $T \in \mathcal{D}'(\Omega)$ be supported on the compact $K \subset \Omega$. Fix a strictly larger compact $K_1$ (so that $K$ is contained in the interior of $K_1$) and an open neighborhood $\Omega_1$ of $K_1$. By Remark 2.8 the conclusion of Theorem 2.7 is valid for the cover by $\Omega \setminus K_1$ and $\Omega_1$, even though $\Omega$ is noncompact. In particular we can fix a decomposition mapping in which $f \in \mathcal{E}(\Omega)$ is written as $f = f_1 + f_2$, with $f_2$ supported on $\Omega \setminus K_1$ and therefore $f_1 |_{K_1} \equiv f |_{K_1}$. Now let $T_1$ on $\mathcal{E}(\Omega)$ be given by $T_1 f = T f_1$. This is well defined, because if $f, g \in \mathcal{E}(\Omega)$ and $f_1 = g_1$, then $f - g \equiv 0$ in a neighborhood of $K$ and the support condition ensures $T f = T g$. It is also linear, even though the mapping $f \mapsto f_1$ is nonlinear (see Remark 2.9), because $(f + g)_1 = f_1 + g_1$ on $K_1$, which contains a neighborhood of $K$. Lastly, $T_1$ is continuous, as may be seen from the fact that a sequence $\{\phi_j\} \subset \mathcal{E}(\Omega)$ such that $\Delta^j \phi_j \rightarrow 0$ uniformly on compacta will have $\Delta^j (\phi_j)_1 \rightarrow 0$ on $K_1 \supset K$, or from (2.5). We conclude that every compactly supported distribution is in $\mathcal{E}'(\Omega)$.

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5. Structure Theory

**Definition 5.1.** A distribution $T$ has finite order $m$ if for each compact $K$ there is $M = M(K)$ such that $|T \phi| \leq M|\phi|_m$ for all $\phi \in \mathcal{D}(K)$.

The following theorem indicates the importance of the finite order distributions.

**Theorem 5.2.** Compactly supported distributions have finite order.

**Proof.** Let $T$ be a distribution with compact support $K$ and let $K_1$ be a compact set such that $K \subset \text{int}(K_1)$. By Lemma 3.3 we may decompose any $\phi \in \mathcal{D}(\Omega)$ as $\phi = \phi_1 + \phi_2$ where $\phi_1$ is supported on $K_1$, $\phi_2$ is supported in $K^c$, and $|\phi_j|_m \leq C_m |\phi|_m$ for $j = 1, 2$. Clearly $T \phi = T \phi_1$, but there are $M$ and $M$ so that (4.1) holds on $K_1$, from which we conclude that $|T \phi| = |T \phi_1| \leq M |\phi_1|_m \leq C_m M |\phi|_m$.

It is easy to see that the Radon measures on $\Omega$ are examples of distributions of finite order. In fact they have order zero, because the action of $\nu$ on $\mathcal{D}(\Omega)$ via $\nu \phi = \int \nu \phi$ trivially satisfies the bound $|\nu \phi| \leq ||\nu||_0 = ||\phi||_0$. The converse is also true.

**Theorem 5.3.** If $T$ is a distribution of order zero then there is a Radon measure $\nu$ such that $T \phi = \int \phi d\nu$ for all $\phi \in \mathcal{D}(\Omega)$.
Proof. Let \( K \) be compact. Since \( T \) has order zero there is \( M = M(K) \) so that \(|T\phi| \leq M||\phi||_{\infty}\) whenever \( \phi \in \mathcal{D}(\Omega) \) has support in \( K \). Theorem 5.4 shows that these functions are dense in \( C(K) \), so we may extend \( T \) to a bounded linear operator on \( C(K) \). Such operators are represented by Radon measures, so there is \( \nu_K \) with \( T\phi = \int \phi d\nu_K \) for all test functions \( \phi \) with support in \( K \). Now let \( \bigcup K_j \) be a compact exhaustion of \( \Omega \) and consider the measures \( \nu_K \). These converge weak-as \( * \) elements of the dual of \( C_r(\Omega) \) to a Radon measure \( \nu \), but by construction \( \int \phi d\nu_K \rightarrow T\phi \) on \( \mathcal{D}(\Omega) \) and the result follows. \( \Box \)

Remark 5.4. As written, the preceding proof relies on Theorem 3.4 and hence on the Hahn-Banach theorem. Since each of the \( K_j \) is compact, a constructive proof can be obtained by instead using Theorem 5.3.

A well known application of the preceding is obtaining a characterization of the distributions that have positive values on positive test functions. To prove this we need Corollary 2.6 and therefore must make the corresponding assumption (2.4) on the behavior of the heat kernel corresponding to the Laplacian.

Definition 5.5. \( T \) is a positive distribution if \( T\phi \geq 0 \) whenever \( \phi \geq 0 \) is a positive test function.

Theorem 5.6. Positive distributions have order zero. If the Laplacian on \( X \) is such that (2.4) holds and if \( T \) is a positive distribution, then there is a positive measure \( \nu \) such that \( Tf = \int f \, dv \).

Proof. Let \( K \) be compact. Using Theorem 2.5 there is \( \psi_K \in \mathcal{D}(\Omega) \) such that \( \psi \equiv 1 \) on \( K \). If \( \phi \in C^\infty \) with support in \( K \) then the functions \( ||\phi||_{\infty} \psi_K \pm \phi \) are both positive, whence

\[-||\phi||_{\infty} T\psi_K \leq T\phi \leq ||\phi||_{\infty} \psi_K.
\]

We conclude that \( T \) has order zero, so by Corollary 5.3 it is represented by integration against a measure \( \nu \). If there is a cell \( K \) for which \( \nu(K) < 0 \) then we can take \( U_j \) to be a neighborhood of \( K \) for which \( \nu(U_j \setminus K) < 1/j \) and let \( f_j \) be as in Theorem 2.6. It follows that

\[Tf_j = \int f \, dv \leq \nu(K) + \frac{1}{j} ||f||_{\infty}
\]

and for a sufficiently large \( j \) this is negative, in contradiction to the positivity of \( T \). We conclude that \( \nu(K) \geq 0 \) for all cells \( K \), and therefore that \( \nu \) is a positive measure. \( \Box \)

We noted at the beginning of Section 4 that the adjoint of the Laplacian maps \( \mathcal{D}'(\Omega) \) to itself. In particular, if \( \nu \) is a Radon measure, hence a distribution of order zero, then for each compact \( K \) there is \( M(K) \) such that

\[|\langle \Delta^n \nu \phi \rangle| = |\nu(\Delta^n \phi)| \leq M(K) |\Delta^n \phi|_0 \leq M(K) ||\phi||_m
\]

so \( \Delta^n \nu \) is a distribution of order \( m \). This result has a converse, which we prove using a modification of the Green’s function introduced in Lemma 2.1. The basic idea is to produce a Green’s operator that inverts the Laplacian on test functions, so that the adjoint of this operator lowers the order of a finite-order distribution. Iterating to produce a distribution of zero order then produces a measure by Theorem 5.3.

Lemma 5.7. Let \( K \) be a connected finite union of cells in \( \Omega \). Then \( \Delta : \mathcal{D}(K) \rightarrow \mathcal{D}(K) \) and its image consists of all test functions that are orthogonal to the harmonic functions on \( K \). Moreover there is a linear operator \( \hat{G}_K : \mathcal{D}(K) \rightarrow \mathcal{D}(K) \) such that \(-\Delta \hat{G}_K(\Delta \phi) = \Delta \phi \) for all \( \phi \in \mathcal{D}(K) \).
Theorem 5.8. If $T$ is a distribution of order $m \geq 1$ then $\tilde{G}_K T$ is a distribution of order $m - 1$, and if $T$ is a distribution of order zero then $G_K T$ is integration with respect to a continuous function on $K$.

Proof. Let $T$ be a distribution of order $m \geq 1$, so that $\tilde{G}_K T \phi$ is bounded by

$$|\tilde{G}_K \phi|_\infty = \sup \{||\Delta^k \tilde{G}_K (\phi)||_\infty : k \leq m\}.$$

Proof. For $\phi \in \mathcal{D}(K)$, the matching conditions for the Laplacian ensure that both $\phi$ and $\partial_n \phi$ vanish on $\partial K$ because $\phi$ is identically zero outside $K$. If $f \in \text{dom}(\Delta)$, then the Gauss-Green formula reduces to

$$\int_K (\Delta \phi) f \, d\mu = \int_K \phi (\Delta f) \, d\mu$$

because there are no non-zero boundary terms. We conclude that $f$ is orthogonal to the image $\Delta(\mathcal{D}(K))$ if and only if $\Delta f$ is orthogonal to $\mathcal{D}(K)$. As the latter is dense in $C(K)$ (Lemma 3.4) the first result is proven.

Let $h_1, \ldots, h_{|K|}$ be an $L^2$–orthonormal basis for the finite dimensional space of harmonic functions on $K$. As $\mathcal{D}(K)$ is dense in $C(K)$ in supremum norm and $K$ has finite measure, $\mathcal{D}(K)$ is also dense in both $C(K)$ and $L^2(K)$ in $L^2$ norm. It follows that there are $\phi_1, \ldots, \phi_{|K|}$ in $\mathcal{D}(K)$ such that $\langle \phi_i, h_j \rangle = \delta_{ij}$, where $\langle , \rangle$ is the $L^2$ inner product and $\delta_{ij}$ is Kronecker’s delta. Given $\phi \in \mathcal{D}(K)$ we let

$$\tilde{\phi} = \sum_{i=1}^{|K|} \langle \phi, h_i \rangle \phi_i$$

and define

$$G_K \phi(x) = \begin{cases} G_K (\phi - \tilde{\phi})(x) & \text{if } x \in K \\ 0 & \text{if } x \not\in K \end{cases}$$

where $G_K$ is the Green’s operator defined in Lemma 3.4. It is then clear that for $\psi \in \mathcal{D}(K)$,

$$- \Delta G_K \psi = - \Delta G_K (\psi - \tilde{\psi}) = \psi - \tilde{\psi}$$

(5.2)

except perhaps at points of $\partial K$, where we must first verify that the matching conditions for the Laplacian hold. Since $G_K \psi$ vanishes outside $K$, the matching conditions require that $\partial_n G_K \psi(q) = 0$ whenever $q \in \partial K$. One way to verify this is from the Gauss-Green formula for a harmonic function $h$, which yields

$$0 = \langle \psi - \tilde{\psi}, h \rangle = \int_K (- \Delta G_K \psi) h = - \sum_{q \in \partial K} (-\partial_n G_K \psi(q)) h(q)$$

from which we see that it suffices to know the solvability of the Dirichlet problem on $K$, that is, for every assignment of boundary values on $\partial K$ there is a harmonic function $h$ with those boundary values. This latter is true because of Lemma 3.4 for example it may be proven by taking a function that is piecewise harmonic on cells and has the desired boundary data and subtracting the result of applying $G_K$ to its Laplacian (which is simply a sum of Dirac masses at the interior gluing points). We conclude that $G_K \psi \in \mathcal{D}(K)$ and that (5.2) holds everywhere.

Finally, if $\psi = \Delta \phi$ for some $\phi \in \mathcal{D}(K)$, then $\psi$ is orthogonal to the harmonics, so $\tilde{\psi} = 0$ and $- \Delta G_K \psi = \psi$ as desired.

The adjoint of $\tilde{G}_K$ is defined on distributions by $(\tilde{G}_K T) \phi = T(G_K \phi)$. This operator is really defined on the dual of $\mathcal{D}(K)$, which is a larger space, but we will not make use of this fact.
When \( k \geq 1 \) we have \( \Delta^k \hat{G}_K(\phi) = -\Delta^{k-1}(\phi - \tilde{\phi}) \), and when \( k = 0 \) we see that \( \|\hat{G}_K(\phi)\|_\infty \leq C\|\phi - \tilde{\phi}\|_\infty \) because the operator \( G_K \) in Lemma 5.1 is clearly bounded on \( L^\infty \). Hence \( |\hat{G}_K|_\infty \leq C|\phi - \tilde{\phi}|_{l_{m-1}} \leq C(m-1,K)|\phi|_{l_{m-1}} \), where the latter inequality is from (5.1) with a constant \( C(m-1,K) \) that may depend on the set of functions \( \phi_j \). Thus \( \hat{G}_K T \) has order \( m-1 \).

If \( T \) has order zero then by Theorem 5.3 it is represented by integration against a Radon measure \( \nu \). Provided \( K \neq \Omega \) we can apply Lemma 2.1 directly to see \( \nu = \Delta f \) for some \( f \) that is continuous on \( K \) and vanishes on \( \partial K \), so can be extended continuously to be zero outside \( K \). This ensures there are no boundary terms when we compute with the Gauss-Green formula:

\[
G_K T \phi = TG_K = \int_K G_K \phi \, d\nu = \int_K (G_K \phi)(\Delta f) \, d\mu
\]

\[
= \int_K (-\Delta G_K \phi) f \, d\mu = \int_K (\phi - \tilde{\phi}) f \, d\mu
\]

\[
= \int_K (\phi - \sum_{i=1}^{i(K)} \langle \phi, h_i \rangle \phi_i) f \, d\mu
\]

\[
= \int_K \phi (f - \sum_{i=1}^{i(K)} \langle \phi, f \rangle h_i) \, d\mu
\]

and the bracketed term in the last line is continuous because \( f \) is continuous and all of the \( h_i \) are harmonic.

The argument is slightly different if \( \Omega = K \). We instead set \( t = \int d\nu/(\int d\mu) \) so \( \int d(\nu - t\mu) = 0 \), at which point Lemma 2.1 applies to show \( \nu - t\mu = \Delta f \) for a continuous \( f \), and we can compute as before

\[
\bar{G}_K T \phi = \int_K \bar{G}_K \phi \, d\nu
\]

\[
= t \int_K \bar{G}_K \phi \, d\mu + \int_K \bar{G}_K \phi (\Delta f) \, d\mu
\]

\[
= \int_K (\phi - \tilde{\phi}) f \, d\mu
\]

\[
= \int_K \phi (f - \int_K f \, d\mu) \, d\mu
\]

where we used that \( \int_K G_K \phi \, d\mu = 0 \) (from the proof of Lemma 2.1) and that the harmonic functions are constants in this case. \( \square \)

We now have all the necessary tools to prove the main structure theorem for distributions.

**Theorem 5.9.** Any distribution \( T \) may be written as a locally finite sum of the form \( T = \sum \Delta^m \nu_j \) or \( T = \sum \Delta^m f_j \), where the \( \nu_j \) are Radon measures and the \( f_j \) are continuous functions with compact support.

**Proof.** Suppose first that \( \Omega \) is non-compact and take \( K_1, K_2, \ldots \) a sequence of subsets such that each \( K_j \) is a connected finite union of cells, \( K_j \) is contained in the interior of \( K_{j+1} \), and \( \cup K_j = \Omega \). Such a sequence exists because \( \Omega \) has a restricted cellular construction. It will be convenient to also set \( K_0 = \emptyset \). For each \( j \) let \( \hat{G}_j = G_{K_j} \) be the operator from Lemma 5.7.

The key point of the proof is that for any distribution \( S \), we have \((\Delta)^m \hat{G}_j^m S = S\) as
elements of $\mathcal{D}'(K_j)$ (though not as elements of $\mathcal{D}'(\Omega)$). This may be verified by direct computation. For all $\phi \in \mathcal{D}(K_j)$,

$$-\Delta\tilde{G}_j S \phi = (-\tilde{G}_j S)(\Delta \phi) = S(-\tilde{G}_j (\Delta \phi)) = -S \phi$$

(5.3)

where the final step uses that $-\Delta\tilde{G}_j (\Delta \phi) = \Delta \phi$ from Lemma 5.7, so $\tilde{G}_j (\Delta \phi) + \phi$ is harmonic on $K$ and vanishes on $\partial K$, hence is identically zero.

Fix $T \in \mathcal{D}'(\Omega)$. Inductively suppose that for $i = 0, \ldots, j - 1$ we have found $m_i$ and a measure $\nu_i$ supported on $K_i$ such that $T - \sum_{i=0}^{j-1} \Delta^m \nu_i$ vanishes on $\mathcal{D}(K_{j-1})$. The base case $j = 0$ is trivial because $K_0 = \emptyset$. Now $T - \sum_{i=0}^{j-1} \Delta^m \nu_i$ is in $\mathcal{D}'(\Omega)$, hence its restriction to $\mathcal{D}(K_j)$ is in $\mathcal{D}'(K_j)$. We call this restriction $T_j$. As $K_j$ is compact, $T_j$ has finite order $m_j$. It satisfies $T_j = (-\Delta)^m \tilde{G}_j^m T_j$ by the argument already given, and by Theorem 5.8 there is a measure $\nu_j$ supported on $K_j$ such that $\nu_j = (-1)^{m_j} \tilde{G}_j^m T_j$. Therefore $T_j = \Delta^m \nu_j$ in $\mathcal{D}'(K_j)$, which is equivalent to saying that $T - \sum_{i=0}^{j} \Delta^m \nu_i$ vanishes on $\mathcal{D}(K_j)$.

It is immediate from the definition that $\sum_j \Delta^m \nu_j$ is a locally finite sum. If we fix $\phi \in \mathcal{D}(\Omega)$ then there is a $j$ such that $\phi \in \mathcal{D}(K_j)$, whereupon $(T - \sum_{i=1}^{j} \Delta^m \nu_i) \phi = 0$ for all $l \geq j$. This proves that $T = \sum_{j} \Delta^m \nu_j$.

The proof that $T = \sum_{j} \Delta^m \nu_j f_j$ is similar. Obviously we wish to use the latter part of Theorem 5.8 to go from the measure $\nu_j$ to a continuous function. The only technicality is that the resulting $f_j$ is continuous on $K_j$ rather than on all of $\Omega$. We fix this at each step of the induction as follows. Suppose we have determined $T_j$ as the restriction of $T - \sum_{i=0}^{j-1} \Delta^m \nu_i$ and from Theorem 5.8 a function $g_j$ continuous on $K_j$ such that $\Delta^m \nu_j = T_j$ in $\mathcal{D}'(K_j)$. Let $f_j$ be a continuous extension of $g_j$ to $\Omega$ obtained by requiring $f_j = 0$ on $\partial K_{j+1}$ and outside $K_{j+1}$, and letting it be piecewise harmonic on the cells of the cellular structure on $K_{j+1} \setminus K_j$ (here we use that $K_j$ is in the interior of $K_{j+1}$). Clearly $\Delta^m \nu_j f_j = T_j$ in $\mathcal{D}'(K_j)$, because we $f_j = g_j$ on $K_j$, so $T - \sum_{i=0}^{j} \Delta^m \nu_i f_i$ vanishes on $\mathcal{D}(K_j)$ and we may complete the proof as before.

In the case when $\Omega$ is compact the proof is somewhat more elementary because we need only a single set $K = \Omega$, but there is a small technical difference due to the fact that the final equality of (5.2) is no longer true. Indeed, $\tilde{G}_j (\Delta \phi) + \phi$ is harmonic by the same reasoning as for the non-compact case, but now it is the possibly non-zero constant $\phi = (\mu(K))^{-1} \int \phi d\mu$. The analogue of (5.3) is therefore

$$\Delta \tilde{G}_K S \phi = S \phi - \tilde{G}_K 1 \phi$$

where $1$ is the constant function.

The distribution $T$ has finite order $m$, and $T \phi = T 1 \phi + \Delta \tilde{G}_K T \phi$. Iterating, we have $\tilde{G}_K T \phi = \tilde{G}_K T 1 \phi + \Delta \tilde{G}_K T \phi$, and then

$$T \phi = \Delta^l \tilde{G}_K T \phi + \left( \sum_{i=1}^{L} a_i \right) \phi$$

where $a_i = \tilde{G}_K 1$. If $L = m$ then Theorem 5.8 implies the first term is $\Delta^m \nu$, where $\nu$ is a measure, and if $L = m + 1$ this term is $\Delta^{m+1} f$, where $f$ is a continuous function. In either case the second term is a constant multiple of the measure $\mu$, or equivalently the constant (hence continuous) function $1$, so the proof is complete. □

6. DISTRIBUTIONS SUPPORTED AT A POINT

A distribution with support a point $q$ is of finite order by Theorem 5.2 and simple modifications of the arguments in Theorem 5.9 show that it is a power of the Laplacian.
applied to a measure with support in a neighborhood of \( q \). The purpose of this section is to identify it more precisely as a finite sum of certain derivatives of the Dirac mass at \( q \); in general these derivatives are not just powers of the Laplacian, but instead reflect the local structure of harmonic functions at \( q \).

Identification of a distribution \( T \) of order \( m \) supported at \( q \) is achieved by describing a finite number of distributions \( T_j, j = 1, \ldots, J \) with the following property: if \( \phi \in \mathcal{D}(\Omega) \) has \( T_j\phi = 0 \) for all \( j \) then for any \( \epsilon > 0 \) there is a neighborhood \( U_\epsilon \) of \( q \) and a decomposition \( \phi = \phi_q + (\phi - \phi_q) \) into test functions such that \( |\phi_q|_m < \epsilon \) and \( \phi - \phi_q \) vanishes on \( U_\epsilon \). The reason is that then \( T\phi = T\phi_q \) because of the support condition, and \( |T\phi_q| \leq C\epsilon \), from which we conclude that \( T \) vanishes whenever all \( T_j \) vanish. It follows from a standard argument (for example, Lemma 3.9 of [28]) that \( T \) is a linear combination of the \( T_j \).

The argument described in the previous paragraph motivates us to find conditions on a test function \( \phi \) that ensure we can cut off outside a small neighborhood of a point \( q \) while keeping the norm \( |\phi_q|_m \) of the cutoff small. In order to proceed we will need some notation for a neighborhood base of \( q \). If \( q \) is a non-junction point then it lies in a single copy of \( X \) in the cellular structure, and within this copy there is a unique word \( w \) such that \( F_w(X) = q \). The cells containing \( q \) are then of the form \( U_i = F_{[w_1]}(X) \). For junction points the situation is different, as \( q \) can be the intersection point of several copies of \( X \), or can be in a single copy but be given by \( F_w(X) = q \) for a finite number of words \( w_1, \ldots, w_J \). We will not distinguish between these possibilities but will instead make the convention that the distinct words determining \( q \) may be used to distinguish copies of \( X \) if necessary. With this assumed, let \( U_{i,j} = F_{[w_j]}(X) \), and \( U_i = \bigcup_j U_{i,j} \).

Fix \( q \) and let \( H_{i,j} \) denote the Dirichlet Green’s operator on the cell \( U_{i,j} \), omitting the \( j \) index if \( q \) is a non-junction point. If \( \phi \in \mathcal{D}(\Omega) \) we can then decompose \( \phi \) on \( U_{i,j} \) into

\[
\phi|_{U_{i,j}} = H_{i,j}\phi + G_{i,j}\Delta\phi
\]

where \( H_{i,j}\phi \) is the (unique) harmonic function on \( U_{i,j} \) whose values on \( \partial U_{i,j} \) coincide with those of \( \phi|_{U_{i,j}} \). By induction we obtain

\[
\phi|_{U_{i,j}} = \sum_{l=0}^{m-1} G_{i,j}^l H_{i,j}^l \Delta^l \phi + G_{i,j}^m \Delta^m \phi|_{U_{i,j}} \tag{6.1}
\]

and write \( h_{i,j}^l = G_{i,j}^l H_{i,j}^l \Delta^l \phi \).

**Lemma 6.1.** In the decomposition [6.1] we have at each \( x \in U_{i,j} \) and \( p \in \partial U_{i,j} \) that \( \Delta^k h_{i,j}^l(x) = \partial_\nu \Delta^k h_{i,j}^l(p) = 0 \) if \( k > l \), while for \( k \leq l \),

\[
|\Delta^k h_{i,j}^l| \leq c(k,l) |\partial^\nu \Delta^k h_{i,j}^l|_{L^\infty(U_{i,j})}
\]

\[
|\partial_\nu \Delta^k h_{i,j}^l| \leq c(k,l) |\partial_\nu \Delta^k h_{i,j}^l|_{L^\infty(U_{i,j})}
\]

**Proof.** The conclusions for the cases \( k > l \) are immediate from the fact that \( \Delta^k h_{i,j}^l \) is harmonic, while the remaining estimates are derived from the fact that the Laplacian scales by \( r_\nu \mu_\nu \) on a cell \( F_w(X) \) while the normal derivative scales by \( r_\nu \). \( \square \)

Our purpose in making the above definitions is that estimates on the functions \( h_{i,j}^l \) are precisely what is needed to ensure we can cut off a smooth function in the manner previously described.
Theorem 6.2. If $\phi$ is such that $\Delta^m \phi(q) = 0$ and $\|H_{i,j} \Delta^k \phi\|_{L^p(U_{i,j})} = o\left(r_{w_{i,j}}^{-k} \mu_{w_{i,j}} \right)^{m-1}$ for $0 \leq k \leq m-1$ as $i \to \infty$, then for all $\epsilon > 0$ there is $\phi_q$ such that $|\phi_q|_{L^m} \leq \epsilon$ and $\phi - \phi_q$ is supported away from $q$.

Proof. We begin by constructing a neighborhood of $U_{i,j}$ by adjoining cells at each of the points $p \in \partial U_{i,j}$. At each $p$ we require finitely many such cells, and we choose them so as to intersect $U_{i,j}$ only at $p$. It will also be convenient to assume that these cells have comparable scale to the $U_{i,j}$, in the sense that they have the form $F_{\tilde{w}}(X)$ for some word with length $|\tilde{w}| \leq i + i_0$ for some constant $i_0$. Let $K$ be one of the cells adjoined at a point $p$, and let $n_p$ be the number of cells adjoined at $p$. Using Theorem 2.3 we define a smooth function $f_K$ on $K$ with jet $\Delta^k f_K(p) = \Delta^k \phi(p)$ and $\partial^k \Delta^k f_K(p) = -(1/n_p) \partial^k U_{i,j} \Delta^k \phi(p)$, and with vanishing jets at the other boundary points of $K$. Having done this for the set $\mathcal{K}$ of adjoined cells we see from the matching conditions for the Laplacian that

\[
\phi_q(x) = \begin{cases} 
\phi(x) & \text{for } x \in U_i \\
\sum_{K \in \mathcal{K}} f_K & \text{for } x \in \bigcup_{K \in \mathcal{K}} K \\
0 & \text{otherwise}
\end{cases}
\]
defines a test function with the property that $\phi - \phi_q = 0$ on $U_i$.

We must estimate $|\phi_q|_{L^m}$. There is an easy estimate for $\Delta^k \phi$ for $k \leq m$ from Lemma 6.1,

\[
|\Delta^k \phi| \leq \sum_{l=0}^{m-1} \left| \Delta^l h_{i,j}^l \right| + \left| G_{i,j}^{m-k} \Delta^m \phi \right|
\]

\[
\leq \sum_{l=0}^{m} c(k,l)r_{w_{i,j}}^{-k} \mu_{w_{i,j}}^{-k} \|H_{i,j} \Delta^l \phi\|_{L^p(U_{i,j})} + c(k,m)r_{w_{i,j}}^{-m-k} \mu_{w_{i,j}}^{-m-k} \|\Delta^m \phi\|_{L^p(U_{i,j})}
\]

\[
\leq \sum_{l=0}^{m} c(k,l) o(r_{w_{i,j}}^{-m-k} \mu_{w_{i,j}}^{-m-k})
\]

\[
= o(r_{w_{i,j}}^{-m-k} \mu_{w_{i,j}}^{-m-k})
\] (6.2)

where we used $\Delta^m \phi(q) = 0$ to obtain that $\Delta^m \phi(q) = o(1)$ when $i \to \infty$. As a result we have good control of $|\phi_q|_{L^m}$ on $U_{i,j}$.

A similar calculation allows us to estimate the size of the normal derivative $|\partial_n \Delta^k \phi(p)|$ at any of the points $p$ where pieces $f_K$ are attached. We compute

\[
|\partial_n \Delta^k \phi(p)| \leq \sum_{l=0}^{m} |\partial_n \Delta^l h_{i,j}^l| + |\partial_n G_{i,j}^{m-k} \Delta^m \phi|
\]

\[
\leq \sum_{l=0}^{m} c(k,l)r_{w_{i,j}}^{-k-1} \mu_{w_{i,j}}^{-k} \|H_{i,j} \Delta^l \phi\|_{L^p(U_{i,j})} + c(k,m)r_{w_{i,j}}^{-m-k-1} \mu_{w_{i,j}}^{-m-k} \|\Delta^m \phi\|_{L^p(U_{i,j})}
\]

\[
\leq \sum_{l=0}^{m} c(k,l) o(r_{w_{i,j}}^{-m-k-1} \mu_{w_{i,j}}^{-m-k})
\]

\[
= o(r_{w_{i,j}}^{-m-k-1} \mu_{w_{i,j}}^{-m-k})
\] (6.3)

Fix $K \in \mathcal{K}$ and examine $f_K$. By assumption $K = F_{\tilde{w}}(X)$, so by (6.3) with the fixed number of jet terms $m$ we know

\[
\|\Delta^k f_K\|_{L^\infty} \leq C(k) \left( \sum_{k^0 = 0}^{k^0} r_{\tilde{w}}^{-k^0} \mu_{\tilde{w}}^{-k^0} |\Delta^{k^0} \phi(p)| + \sum_{k^0 = 0}^{k^0} r_{\tilde{w}}^{-k^0} \mu_{\tilde{w}}^{-k^0} |\partial_n U_{i,j} \Delta^{k^0} \phi(p)| \right) + \epsilon
\] (6.4)
provided 0 ≤ k ≤ m. The terms involving |Δ^k φ(p)| may be replaced by the estimate (6.2).
For the terms involving normal derivatives we use that \( \partial_n Δ^k h^k_{ij}(p) = (1/n_p) \partial_n Δ^k φ \) and (6.3). The result is

\[
\|Δ^k f_k\|_\infty ≤ C(k) \left( \sum_{k=0}^m o(r_{\tilde{w}}^{-k-m} \mu_{\tilde{w}}^{-1} |w| \mu_{\tilde{w}}^{-1} |w|) + \sum_{k=0}^{m-1} o(r_{\tilde{w}}^{k+1-m} \mu_{\tilde{w}}^{-1} |w| \mu_{\tilde{w}}^{-1} |w|) \right) + \epsilon \]

\[
≤ o(r_{\tilde{w}}^m \mu_{\tilde{w}}^{-1} |w| \mu_{\tilde{w}}^{-1} |w|) \sum_{k=0}^m \left( \frac{r_{\tilde{w}} \mu_{\tilde{w}}}{r_{\tilde{w}} \mu_{\tilde{w}}} \right) (1 + \frac{r_{\tilde{w}}}{r_{\tilde{w}}}) + \epsilon. \tag{6.5}
\]

However, \( \tilde{w} \) and \([w_j]\) have comparable length and are adjacent, so they differ only in the final \( i_0 \) letters and therefore the ratios \( r_{\tilde{w}} r_{[w_j]}^{-1} \) and \( \mu_{\tilde{w}} \mu_{[w_j]}^{-1} \) are bounded by constants depending only on \( i_0 \) and the harmonic structure and measure. It follows that

\[
\|Δ^k f_k\|_\infty ≤ C(m, k, r, \mu) o(r_{[w_j]}^{-1} |w_j|^{-1} |w_j|^{-1}) \]

and combining this estimate for each \( K \in \mathcal{K} \) with (6.2) proves that

\[
\|Δ^k φ_q\|_{L^\infty} = o(r_{[w_j]}^{-1} |w_j|^{-1} |w_j|^{-1}) \]

as \( i \to \infty \)

for 0 ≤ k ≤ m. In particular we can make \( |φ_q|_m < \epsilon \) by making \( i \) sufficiently large. \( \Box \)

Theorem 6.2 suggests that the natural candidates for the distributions supported at \( q \) are appropriately scaled limits of the maps \( φ \mapsto H_{i,j} Δ^k φ \) as \( i \to \infty \). The question of how to take such limits has been considered by a number of authors [16, 33, 25, 26, 1], and is generally quite complicated. At the heart of this complexity is the fact that the local behavior of smooth functions in a neighborhood of a point \( q \) depends strongly (in fact almost entirely) on the point \( q \) rather than the function itself. This property – often called “geography is destiny” – contrasts sharply with the Euclidean situation where neighborhoods of points are analytically indistinguishable. Its immediate implication for the structure of distributions with point support is that the nature of these distributions must depend on the point in question. In order of increasing complexity we consider three cases: junction points, periodic points and a class of measure-theoretically generic points.

**Junction Points.** As before, the junction point \( q = q = F_{w_j}(X) \) for words \( w_1, \ldots, w_J \), each of which terminates with an infinite repetition of a single letter. The distributions corresponding to approaching \( q \) through the sequence \([w_j]\) may be understood by examining the eigenstructure of the harmonic extension matrices \( A_{ij} \), the definition of which appeared in the Harmonic Functions part of Section 2.

For notational convenience we temporarily fix one contraction \( F \), let \( A \) be the corresponding harmonic extension matrix, and suppose \( q \) is \( \cap F^1(X) \). Let \( r \) and \( \mu \) be the resistance and measure scalings of \( F \), and \( γ_1, \ldots, γ_n \) be the eigenvalues of \( A \), ordered by decreasing absolute value, with eigenspaces \( E_1, \ldots, E_n \). Of course \( γ_1 = 1 \) and \( E_1 \) is the constant functions. Let \( H_{j,\mu} \) be the harmonic function on \( F^1(X) \) that equals \( u \) on \( ∂F^1(X) \), and \( P_s \) be the projection onto \( E_s \). In what follows, \( G \) is the Dirichlet Green’s operator on \( X \) and \( G_t \) is the same on \( F^1(X) \).

**Definition 6.3.** Inductively define derivatives \( d_i \) and differentials \( D^k, k \geq 1 \) at the point \( q \) by setting \( D^0 u = u(q) \), and for each \( s \) such that \( (r\mu)^k < γ_s \) ≤ \( (r\mu)^{k+1} \)

\[
d_i u = \lim_{i \to \infty} γ_i^{-1} P_i H_i(u - GD^{k-1} \Delta u) \tag{6.6}
\]
if these limits exist. Note that $d_s$ always exists for harmonic functions as the sequence is constant in this case. Provided the necessary $d_s$ exist we then let

$$D^k u = h + GD^{k-1} \Delta u$$  \hspace{1cm} (6.7)

where $h$ is the unique harmonic function on $X$ with $d_s h = d_s u$ for those $s$ with $\gamma_s > (r\mu)^k$ and $d_s h = 0$ for all other $s$. We will also make use of $D^k u$, where $D^0 = u(0)$ and

$$D^k u = h + GD^{k-1} \Delta u$$

where $h$ is harmonic on $X$ with $d_s h = d_s u$ for those $s$ with $\gamma_s \geq (r\mu)^k$ and $d_s h = 0$ for all other $s$.

**Lemma 6.4.** For $u \in \text{dom}(\Delta^k)$ and each $s$ with $\gamma_s > (r\mu)^k$ the derivative $d_s u$ exists, and

$$|d_s u| \leq C(k) \sum_{l=0}^{k} \|\Delta^l u\|_{\infty}.$$ \hspace{1cm} (6.8)

The differential satisfies

$$\|u - D^k u\|_{L^1(F(X))} \leq C(k) \|r\mu\|^k \|\Delta^k u\|_{\infty},$$ \hspace{1cm} (6.9)

and if we further suppose that $\Delta^k u \in \text{dom}(E)$ then

$$\|u - D^k u\|_{L^1(F(X))} \leq C(k) \|r\mu\|^{k+1/2} E^{1/2}(\Delta^k u).$$ \hspace{1cm} (6.10)

**Proof.** The proof is inductive. When $k = 0$ there are no $s$ with $\gamma_s > 1 = (r\mu)^0$, so the first statement is vacuous and (6.9) is immediate. Suppose both hold up to $k - 1$.

Write $u - GD^{k-1} \Delta u$ as $H_0 u + G(\Delta u - D^{k-1} \Delta u)$, from which

$$d_s u = d_s H_0 u + \lim \gamma_s^{-i} P_s H_i(G(\Delta u - GD^{k-1} \Delta u))$$ \hspace{1cm} (6.11)

provided the latter limit exists. On the cell $F_i(X)$,

$$G(\Delta u - D^{k-1} \Delta u) = H_i G(\Delta u - D^{k-1} \Delta u) + G_i(\Delta u - D^{k-1} \Delta u)$$

thus

$$H_{i+1} G(\Delta u - D^{k-1} \Delta u) = A H_i G(\Delta u - D^{k-1} \Delta u) + H_{i+1} G_i(\Delta u - D^{k-1} \Delta u).$$

In particular, if we project onto the eigenspace $E_s$ then the action of $A$ is multiplication by $\gamma_s$. Scaling implies $G_i(\Delta u - D^{k-1} \Delta u)$ is bounded by

$$\left|G_i(\Delta u - D^{k-1} \Delta u)\right| \leq C(r\mu) \|\Delta u - D^{k-1} \Delta u\|_{L^1(F(X))} \leq CC(k - 1) \|r\mu\|^k \|\Delta^k u\|_{\infty}$$ \hspace{1cm} (6.12)

and the action of $H_{i+1}$ and $P_s$ can only improve this estimate, so

$$\gamma_s^{-(i+1)} \left|P_s H_{i+1} G(\Delta u - D^{k-1} \Delta u) - \gamma_s P_s H_i G(\Delta u - D^{k-1} \Delta u)\right|$$

$$\leq \gamma_s^{-(i+1)} \left|G_i(\Delta u - D^{k-1} \Delta u)\right|$$

$$\leq CC(k - 1) \|r\mu\|^k \|\Delta^k u\|_{\infty}$$ \hspace{1cm} (6.13)

This shows $\{\gamma_s^{-(i+1)} P_s H_i G(\Delta u - D^{k-1} \Delta u)\}$ is Cauchy when $\gamma_s > (r\mu)^k$, and that its limit is bounded by $C(k) \|\Delta^k u\|_{\infty}$. It follows from (6.11) that $d_s u$ exists for these values of $s$, and since $|d_s H_0 u| \leq \|u\|_{\infty}$ we also obtain (6.8).

Summing the tail of (6.13) establishes that

$$|d_s u - \gamma_s^{-(i+1)} P_s H_i (u - GD^{k-1} \Delta u)| \leq C(k) \left(\frac{\|r\mu\|^k}{\gamma_s}\right) \|\Delta^k u\|_{\infty}.$$
Now let \( h \) be the unique harmonic function with \( d_jh = d_ju \) for those \( s \) with \( \gamma_s > (r\mu)^k \) and \( d_jh = 0 \) otherwise. Since \( \gamma_s P_s H_jh = d_jh \) is a constant sequence, we find
\[
|P_s H_j(u - h - GD^{k-1}\Delta u)| \leq C(k)(r\mu)^k\|\Delta^k u\|_\infty 
\tag{6.14}
\]
for those \( s \) with \( \gamma_s > (r\mu)^k \). Recalling \( D^k u = h + GD^{k-1}\Delta u \) from (6.7), write
\[
(u - D^k u)\mid_{\Omega(x)} = H_j(u - D^k u) + G_j(\Delta(u - D^k u))
\]
This allows us to remove the polynomial term in \( s \) of (6.13). What remains are the terms \( P_s H_j(u - h - GD^{k-1}\Delta u) \) for \( \gamma_s \leq (r\mu)^k \). Each of these is obtained as a sum, with
\[
|P_s H_j(u - h - GD^{k-1}\Delta u)| = \left| \sum_{j=0}^{i-1} \gamma_s \gamma_s^j P_s H_j G_j(u - h - GD^{k-1}\Delta u) \right|
\]
\[
\leq CC(k - 1) \sum_{j=0}^{i-1} \gamma_s \sum_{j=0}^{i-1} j^{k-1}(r\mu)^k\|\Delta^k u\|_\infty
\]
\[
\leq CC(k - 1)(r\mu)^k\|\Delta^k u\|_\infty \sum_{j=0}^{i-1} j^{k-1} \left( \frac{\gamma_s}{(r\mu)^k} \right)^{(i-j)k}
\]
\[
\leq C(k)^k(r\mu)^k\|\Delta^k u\|_\infty 
\tag{6.16}
\]
because \( \gamma_s \leq (r\mu)^k \). This proves (6.9) for \( k \) and completes the induction.

The proof of (6.10) uses essentially the same inductive argument with \( \tilde{D} \) replacing \( D \) and the estimate from (6.10) replacing that from (6.9) throughout. Note that in (6.13) we can have \( \gamma_s \geq (r\mu)^k \) because there is an additional factor of \( r^{i/2} \) so the series still converges geometrically. Also, in (6.16) the working is simplified because for \( \tilde{D} \) we have these \( \gamma_s < (r\mu)^k \) and the \( r^{i/2} \) term is bounded, so the convergence is geometric here also. This allows us to remove the polynomial term in \( i \). The base case \( k = 0 \) is true because of the Hölder estimate (2.2).

The map \( d_s \) takes a smooth function to the eigenspace \( E_s \). We now fix orthonormal bases for each of the \( E_s \), and refer to the co-ordinates of \( d_s \) with respect to the basis for \( E_s \) as the components of \( d_s \), these components have values in \( C \).

**Corollary 6.5.** Each component \( d_{s,v} \) of \( d_s \) for which \( (r\mu)^k < \gamma_s \leq (r\mu)^{k-1} \) is a distribution supported at \( q \) and of order at most \( k \). If \( \gamma_s < (r\mu)^{k-1} \) then its order is equal to \( k \), and it is otherwise of order either \( k - 1 \) or \( k \). If \( d_{s,v} \) is a component that is a distribution of order \( k \), then \( \Delta^l d_{s,v} \) defined by \( \Delta^l d_{s,v}\phi = d_{s,v}\Delta^l \phi \) is also supported at \( q \) and has order \( k + l \).

**Proof.** It is apparent from the definition that \( d_s \) is linear on \( \mathcal{D}(\Omega) \) and that \( d_s\phi = 0 \) if \( \phi \in \mathcal{D}(\Omega) \) is identically zero in a neighborhood of \( q \), so it follows from (6.8) that the components of \( d_s \) are distributions of order at most \( k \) and are supported at \( q \).

Suppose \( \gamma_s < (r\mu)^{k-1} \) and let \( u_{s,v} \) denote the harmonic function determined by the eigenvector corresponding to \( d_{s,v} \). Then the values of \( u_{s,v} \) are \( O(\gamma_s) = o(r\mu)^{k-1} \) and \( \Delta^l u = 0 \) for \( l \leq 1 \), so Theorem 6.2 implies that for any \( \epsilon > 0 \) there is a function \( \psi \) equal to \( u_{s,v} \) in a neighborhood of \( q \) but with \( |\psi|_{k-1} < \epsilon \). Since \( d_{s,v}u_{s,v} = 1 \) and \( d_{s,v}u_{s,v} = d_{s,v}\psi \) because of the support condition, it cannot be that \( d_{s,v} \) is order \( k - 1 \) or less, so it has order \( k \).

In the case \( \gamma_s = (r\mu)^{k-1} < (r\mu)^{k-2} \) the above argument says that \( d_{s,v} \) has order at least \( k - 1 \). Both of the values \( k - 1 \) and \( k \) occur in examples. For instance, when \( k = 1 \), the
derivative $d_t u = u(q)$ corresponding to the constant harmonic functions has order $0 = k - 1$. A case where there is a $d_t$ of this type with order $k$ occurs on the Sierpinski Gasket, see Example 6.7 below. This shows that scaling alone cannot identify the order of $d_t$ when $\gamma_s = (r \mu)^{k-1}$.

The statement regarding $\Delta' d_s$ is immediate. □

We now return to using the index $j$ to distinguish the words $w_j$ for which $x = F_{w_j}(X)$, and accordingly denote by $d_j$ the derivative $d_s$ corresponding to the approach through cells $F_{[w_j]}$.

**Theorem 6.6.** Let $T$ be a distribution of order $k$ supported at the junction point $q$, where $q = F_{w_j}(X)$, $j = 1, \ldots, n$. The word $w_j$ terminates with infinite repetition of a letter which, by a suitable relabeling we assume is $j$. Then $T$ is a finite linear combination of the distributions $\Delta' d_s$, for which $\gamma_s \geq (r \mu_j)^{k-1}$. The linear combination runs over all such $s$, all basis elements $v$ for $E_s$, and all cells $j = 1, \ldots, n$ that meet at $q$.

**Proof.** Suppose that $\phi \in \mathcal{D}(\Omega)$ has the property that $\Delta' d_{s,v} \phi = 0$ for all $(r \mu_j)^{k-1} \leq \gamma_s$. It follows from Definition 6.3 that $\bar{D}^k \phi = 0$ and more generally that $\bar{D}^l \Delta \phi = 0$ for all $l \leq k$.

However, the harmonic part of $H_{i,j} \Delta \phi = H_{i,j} \Delta (\phi - \bar{D} \phi)$ on the cell $U_{i,j}$ of scale $i$ corresponding to the word $w_j$ is bounded by the maximum over the boundary vertices of this cell, so from (6.10):

$$\|H_{i,j} \Delta \phi\|_{L^\infty(U_{i,j})} = o(r \mu_j)^{(k-1)i} = o(r \mu_j)^{(k-1) \frac{2\ln w_j}{\ln r}}$$

for $0 \leq l \leq k - 1$. We also have that $\Delta^k \phi(q) = 0$ because $\Delta \phi(q) = \Delta' d_s \phi = 0$, so Theorem 6.2 shows that for any $\epsilon > 0$ there is $\psi \in \mathcal{D}(\Omega)$ that is equal to $\phi - \bar{D} \phi$ in a neighborhood of $q$ and with $|\phi| < \epsilon$.

Using the support condition and the fact that $T$ has order $k$ yields

$$T \psi = T \psi \leq M|\psi|_k < M\epsilon$$

for some fixed $M$ depending only on $T$, and all $\epsilon > 0$. Thus $T \phi = 0$, and we have shown that the kernel of $T$ contains the intersection of the kernels of the distributions described. By a standard result (e.g. Lemma 3.9 of [28]), $T$ is a linear combination of these distributions. □

**Remark 6.7.** Since $d_t$ corresponds to the eigenspace of constants, the distributions $\Delta' d_s$ are independent of $j$ and are simply powers of the Laplacian applied to the Dirac mass at $x$. It should also be noted that for each $j$ the distribution $d_j^s$ corresponds to the largest eigenvalue less than 1, so gives the normal derivative at $x$ when approaching through the cells $F_{[w_j]}$, $i \to \infty$. As a result $\sum_j d_j^s u = 0$, and not all of these distributions need appear in $T$.

It should also be noted that the linear combination in Theorem 6.6 may include distributions of the form $\Delta' d_{s,v}$ having $\gamma_s = (r \mu_j)^{k-1}$, and that it is possible for these to be of order $k + 1$. If this were to occur then we would have a non-trivial linear combination of these $(k + 1)$-order distributions such that the linear combination is of order only $k$. We do not know of an example in which this occurs, but cannot eliminate it as a possibility because our arguments rely on scaling information.

**Example 6.8.** The canonical example of a p.c.f. self-similar fractal of the type we are describing is the Sierpinski Gasket $SG$ with its usual symmetric harmonic structure (see [36] for details of all results described below). In this case $r = 3/5$ and $\mu = 1/3$, so the Laplacian scales by $r \mu = 1/5$. Each of the harmonic extension matrices $A_i$ has eigenvalues
1, 3/5 and 1/5 with one-dimensional eigenspaces. The corresponding derivatives at \( q \) are 
\[ d_1^j u = u(q) \] 
which is point evaluation, 
\[ d_0^j u(q) = \partial_q^j u(q) \] 
which is the normal derivative at \( q \) from the cell corresponding to \( j \), and 
\[ d_2^j u(q) = \partial^T_q u(q) \] 
which is the tangential derivative of \( u \) at \( q \) from this cell.

There are two cells meeting at the junction point \( q \). Without loss of generality we suppose they are indexed by \( j = 0, 1 \). The two corresponding normal derivatives \( \partial_q^j u(q) \), \( j = 0, 1 \) satisfy the single linear relation that they sum to zero, and by Corollary 6.3 they are of order 1. The two tangential derivatives \( \partial^T_q u(q) \), \( j = 0, 1 \) independent, and it is known that they cannot be controlled by \( \|u\|_\infty + \|\Delta u\|_\infty \) (see [36], page 60). They are therefore of order 2. It is also possible to see in this example that any non-trivial linear combination of the \( \partial^T_q u(q) \) has order less than 2. Writing \( \partial_q \) for the Dirac mass at \( q \), we conclude from Theorem 6.6 that any distribution \( T \) of order \( k \) at a junction point of \( S\Gamma \) can be written as a linear combination of the form

\[
T = \sum_{l=0}^{k} a_l \Delta^l \delta_q + \sum_{l=0}^{k-1} b_l \Delta^l \partial_q^0 \delta_q + \sum_{l=0}^{k-2} c_l \Delta^l \partial_q^1 \delta_q. \tag{6.17}
\]

This example also illustrates the issue described in the proof of Corollary 6.5, namely that there can be a \( d_s \) with \( \gamma_s = (r\mu)^{k-1} \) and yet \( d_s \) is order \( k \). In this case we have \( \partial_T = d_3 \) with \( \gamma_3 = 1/5 = r\mu \), so \( k = 2 \), and \( d_3 \) is of order 2.

**Periodic and Eventually Periodic Points.** Periodic points are those \( x = F_w(X) \) for which \( w \) is a periodic word, meaning that \( w \) is composed of an infinite repetition of a fixed finite word \( v \). Eventually periodic points are those for which the word \( w \) is periodic after some finite number of letters. For these points there is a theory similar to that used for junction points; we do not have to consider derivatives corresponding to multiple cells, but instead of looking at the eigenstructure of a matrix \( A_i \) we must examine that of \( A_i \), which is a finite composition of the \( A_i \) matrices. If \( \gamma \) is an eigenvalue of \( A_i \) with eigenspace \( E_i \), then we can define the derivative \( d_s \) as we did for junction points. It is easy to see that the analogues of Lemma 6.4, Corollary 6.5, and Theorem 6.6 all hold, simply by changing the notation to refer to the infinitely repeated matrix being \( A_i \), the eigenvalues \( \gamma \), being those of \( A_i \), and the Laplacian scaling factor to be \( r\mu \), instead of \( r\mu \).

**Generic Points.** We now consider a non-junction point \( x = F_w(X) \), where \( w = w_1 w_2 \ldots \) is an infinite word. The behavior of harmonic functions on the cell \( F_{w_1 w_2 \ldots}(X) \) can be understood by considering the product \( A_{w_1 w_2 \ldots} = \prod_{j=1}^n A_{w_j} \). We need to understand their scaling properties, for which we use the following approach from [38]. Define for each unit vector \( \alpha \) the corresponding Lyapunov exponent

\[
\log \gamma(\alpha) = \lim_{n \to \infty} \frac{1}{n} \log \|A_{w_1 \alpha}\| \tag{6.18}
\]

if the limit exists. In this definition we may take \( \| \cdot \| \) to be any norm on the \#\( V_0 \)-dimensional space containing \( \alpha \); all such norms are equivalent, so \( \gamma \) is unaffected by this choice.

Let us suppose that these limits exist at \( x \). It is readily seen that \( \gamma(\alpha) \neq \gamma(\alpha') \) implies \( \alpha \) and \( \alpha' \) are orthogonal, so there are at most \#\( V_0 \) distinct values \( \gamma_1 > \gamma_2 \cdots \) that occur. Corresponding to these is a direct sum decomposition \( E_1 \oplus E_2 \oplus \cdots \) with the property that writing \( \alpha = \alpha_1 + \alpha_2 + \cdots \) we have \( \gamma(\alpha) = \gamma_\alpha \) if and only if \( \alpha_1 = \cdots = \alpha_{r-1} = 0 \) and \( \alpha_r \neq 0 \). Since the constant functions are harmonic we actually know that \( \gamma_1 = 1 \) and \( E_1 \) is spanned by \( (1, 1, \ldots, 1) \). We let \( P_i \) be the orthogonal projection onto \( E_i \).

The subspaces \( E_i \) provide the natural decomposition of harmonic functions into their scaling components at \( x \). However we cannot expect to directly mimic Definition 6.3.
because the estimate (6.18) does not imply the existence of a renormalized limit of the form
\[
\lim_{t \to \infty} \gamma_t^{-1} P_t H_t (u - GD^{k-1} \Delta u)
\] (6.19)

Indeed it is easy to see that (6.18) does not even imply that \(A_{[w]} \alpha \) is \(O(\gamma(\alpha))'\).

A natural way to proceed was introduced in [16, 38] and further treated in [26]. Let \(u\) be the function we are considering, and \(H_t u\) be the harmonic function on \(F_{[w]}(X)\) with boundary values equal to \(u\) on \(\partial F_{[w]}(X)\) as usual. If we assume that the harmonic scaling matrices \(A_j\) are all invertible we can unravel the scaling structure for harmonic functions at \(x\) by applying the inverse of \(A_{[w]}\) to \(H_t u\). For later use we record an elementary result about the scaling of the adjoint of \(A_{[w]}^{\dagger}\).

Lemma 6.9. If \(\alpha \in E\), then \(\lim \frac{1}{t} \log \| (A_{[w]}^{\dagger})' \alpha \| = -\log \gamma_s\).

Proof. Writing \(\langle \cdot, \cdot \rangle\) for the usual inner product,
\[
\| (A_{[w]}^{\dagger})' \alpha \| = \sup_{\alpha'} \frac{\langle \alpha', (A_{[w]}^{\dagger})' \alpha \rangle}{\| \alpha' \|} = \sup_{\alpha'} \frac{\langle A_{[w]} \alpha', (A_{[w]}^{\dagger})' \alpha \rangle}{\| A_{[w]} \alpha' \|} = \sup_{\alpha'} \frac{\langle \alpha', \alpha \rangle}{\| A_{[w]} \alpha' \|}
\]
Since the logarithm is monotone, this implies
\[
\frac{1}{t} \log \| (A_{[w]}^{\dagger})' \alpha \| = \sup_{\alpha'} \left( \frac{1}{t} \log \frac{\langle \alpha', \alpha \rangle}{\| \alpha' \|} - \frac{1}{t} \log \frac{\| A_{[w]} \alpha' \|}{\| \alpha' \|} \right)
\]
however we know that the second term inside the supremum converges to \(-\gamma(\alpha')\), whereas the first converges to zero provided \(\langle \alpha', \alpha \rangle \neq 0\). The latter condition and \(\alpha \in E_s\) requires that \(\alpha'\) have a non-zero component in \(E_s\), from which we deduce \(\gamma(\alpha') \geq \gamma_s\), with equality provided \(P_t \alpha' = 0\) for each \(t < s\). Combining these observations it is easy to see that for each \(t\) the supremum is between \(-\gamma_s - \frac{c}{t}\) and \(-\gamma_s\) for a constant \(c\) independent of \(t\). It follows that the limit in the statement of the lemma exists and has the asserted value. \(\Box\)

In order to account for the scaling behavior of the Laplacian, we set
\[
\log \beta_w = \lim_{t \to \infty} \frac{1}{t} \log r_{[w]} H_{[w]}
\] (6.20)
provided the limit exists.

Definition 6.10. Assume that \(x = F_w(X)\) is a point at which the limits in (6.18) and (6.20) exist, and that all \(A_j\) are invertible. Inductively define derivatives \(d_s\) and differentials \(D^k\), \(k \geq 1\), at the point \(x\) by setting \(D^0 u = u(q)\), and for each \(s\) such that \(\beta_w^s < \gamma_s \leq \beta_w^{k-1}\)
\[
d_s u = \lim_{t \to \infty} P_t A_{[w]}^{\dagger} H_t (u - GD^{k-1} \Delta u)
\] (6.21)
if these limits exist. Note that \(d_s\) always exists for harmonic functions because the sequence is constant in this case. Provided the necessary \(d_s u\) exist we then let
\[
D^k u = h + GD^{k-1} \Delta u
\] (6.22)
where \(h\) is the unique harmonic function on \(X\) with \(d_s h = d_s u\) for those \(s\) with \(\gamma_s > \beta_w^s\) and \(d_s h = 0\) for all other \(s\). We will also make use of \(\tilde{D}^k u\), where \(\tilde{D}^0 = u(0)\) and
\[
\tilde{D}^k u = \tilde{h} + G\tilde{D}^{k-1} \Delta u
\]
Lemma 6.11. Assume that all $A_j$ are invertible, and that $x = F_w(X)$ is a point at which the limits in (6.18) and (6.26) exist. For $u \in \text{dom}(\Delta^k)$ and each $s$ such that $\gamma_s > \beta_w^k$, the derivative $d_s$ exists, and

$$|d_s u| \leq C(k) \sum_{j=0}^{k} \|\Delta^j u\|_{\infty}. \tag{6.23}$$

For all sufficiently small $\epsilon > 0$, the differential satisfies

$$\left\|u - D^k u\right\|_{L^\infty(F_{w_{\epsilon}}(X))} \leq C(k, \epsilon) \beta_w^{\epsilon^k} \epsilon^{|i|} \|\Delta^k u\|_{\infty}. \tag{6.24}$$

If in addition we assume that $\Delta^k u \in \text{dom}(E)$ then

$$\left\|u - D^k u\right\|_{L^\infty(F_{w_{\epsilon}}(X))} \leq C(k, \epsilon) \beta_w^{\epsilon^k} \epsilon^{\frac{1}{2}} E \left(\Delta^k u\right). \tag{6.25}$$

Proof. The proof is inductive. When $k = 0$ there are no $s$ with $\gamma_s > \beta_w^0$, so the first statement is vacuous and (6.24) is immediate. Suppose both hold up to $k - 1$.

Write $u - GD^{k-1} \Delta u = H_0 u + G(\Delta u - D^{k-1} \Delta u)$, so

$$d_s u = d_s H_0 u + \lim_{i \to \infty} P_s A_i^{-1} H_i (G \Delta u - GD^{k-1} \Delta u) \tag{6.26}$$

provided the latter limit exists. Writing $G_i$ for the Dirichlet Green’s operator on the cell $F_{w_i}(X)$, we have on that cell

$$G(\Delta u - D^{k-1} \Delta u) = H_i G(\Delta u - D^{k-1} \Delta u) + G_i (\Delta u - D^{k-1} \Delta u)$$

from which

$$H_i G(\Delta u - D^{k-1} \Delta u) = A_{w_i} H_i G(\Delta u - D^{k-1} \Delta u) + H_i G_i (\Delta u - D^{k-1} \Delta u),$$

therefore

$$A_{w_i}^{-1} H_i G(\Delta u - GD^{k-1} \Delta u) - A_{w_i}^{-1} H_i (G \Delta u - GD^{k-1} \Delta u) = A_{w_i}^{-1} H_i G_i (\Delta u - D^{k-1} \Delta u), \tag{6.27}$$

and by substitution into (6.26),

$$d_s u = d_s H_0 u + \sum_{i=0}^{\infty} P_s A_i^{-1} H_i G_i (\Delta u - D^{k-1} \Delta u) \tag{6.28}$$

provided that the series converges.
Since $G_{\alpha}$ inverts the Laplacian with Dirichlet boundary conditions on $F_{[\alpha]}(X)$, we have for any sufficiently small $\epsilon > 0$ the bound

$$
\left| G_{\alpha}(Du - D^{k-1}\Delta u) \right| \leq C r_{[\alpha]} \| \Delta u - D^{k-1}\Delta u \|_{L^\infty(F_{[\alpha]}(X))} \\
\leq C (k-1, \epsilon) r_{[\alpha]} \| \Delta u - D^{k-1}\Delta u \|_\infty \\
\leq C (k-1, \epsilon) \| \Delta^k u \|_\infty
$$

(6.29)

because of the inductive hypothesis (6.24) and the Laplacian scaling estimate (6.20). This is also applicable to $H_{[i]} G_{\alpha}(Du - D^{k-1}\Delta u)$ by the maximum principle. Using Lemma (6.29) to estimate the size of $\| (A_{[\alpha]}^{-1})^* P_{\alpha} \|$, it follows that for any sufficiently small $\epsilon > 0$, and any vector $\alpha$,

$$
\left| \langle P_{\alpha} A_{[\alpha]}^{-1} H_{[i]} G_{\alpha}(Du - D^{k-1}\Delta u), \alpha \rangle \right| = \left| \langle H_{[i]} G_{\alpha}(Du - D^{k-1}\Delta u), (A_{[\alpha]}^{-1})^* P_{\alpha} \alpha \rangle \right| \\
\leq C (k-1, \epsilon) \| \Delta_0^k \| \| \Delta u \|_\infty
$$

This and the assumption $\gamma_s > \beta_0$ imply that if $\epsilon > 0$ was chosen small enough then the series in (6.28) converges, and is bounded by $C \| \Delta u \|_\infty$. The estimate (6.23) follows because $d_{\alpha} H_{[\alpha]} u$ is bounded by $C \| u \|_\infty$.

Now $u - D^{k} u = u - h - GD^{k-1}\Delta u = H_{[0]} u - h + G(\Delta u - D^{k-1}\Delta u)$, where $h$ is the harmonic function with $d_{\alpha} u = P_{\alpha} h$ for all $s$ satisfying $\gamma_s > \beta_0$, and $P_{\alpha} h = 0$ otherwise. An expression for $h$ can be obtained by summing (6.28) over these values of $s$. Comparing it to the expression

$$
A_{[\alpha]}^{-1} H_{\alpha}(u - D^{k} u) = H_{\alpha} u - h + \sum_{i=0}^{i-1} A_{[\alpha]}^{-1} H_{[i]} G_{\alpha}(\Delta u - D^{k-1}\Delta u)
$$

from (6.27), it is apparent that for those $s$ with $\gamma_s > \beta_0$ we have

$$
P_{\alpha} A_{[\alpha]}^{-1} H_{\alpha}(u - D^{k} u) = -\sum_{i=0}^{\infty} P_{\alpha} A_{[\alpha]}^{-1} H_{[i+1]} G_{\alpha}(\Delta u - D^{k-1}\Delta u)
$$

which we note satisfies for all $\| \alpha \| \leq 1$ and sufficiently small $\epsilon > 0$

$$
\left| \langle P_{\alpha} A_{[\alpha]}^{-1} H_{\alpha}(u - D^{k} u), \alpha \rangle \right| \leq \sum_{i=0}^{\infty} \left| \langle H_{[i+1]} G_{\alpha}(\Delta u - D^{k-1}\Delta u), (A_{[\alpha]}^{-1})^* P_{\alpha} \alpha \rangle \right| \\
\leq \sum_{i=0}^{\infty} C (k-1, \epsilon) \| \Delta_0^k \| \| \Delta u \|_\infty \\
\leq C (k-1, \epsilon) \| \Delta_0^k \| \| \Delta u \|_\infty
$$

(6.30)

For those $s$ satisfying $\gamma_s \leq \beta_0$ we have instead

$$
P_{\alpha} A_{[\alpha]}^{-1} H_{\alpha}(u - D^{k} u) = \sum_{i=0}^{i} P_{\alpha} A_{[\alpha]}^{-1} H_{[i+1]} G_{\alpha}(\Delta u - D^{k-1}\Delta u),
$$

and for all vectors $\alpha$ with $\| \alpha \| \leq 1$,

$$
\left| \langle P_{\alpha} A_{[\alpha]}^{-1} H_{\alpha}(u - D^{k} u), \alpha \rangle \right| \leq \sum_{i=0}^{i} C (k-1, \epsilon) \| \Delta_0^k \| \| \Delta u \|_\infty \\
\leq C (k-1, \epsilon) \| \Delta_0^k \| \| \Delta u \|_\infty
$$

(6.31)
Equations (6.30) and (6.31) give the same estimate for each $P_r A_{[w]}^{-1} H_t(u - D^k u)$. Mapping forward again by $A_{[w]}$, increases each term by a factor at most $C(\epsilon) e^{\beta_w a/4 t}$, so summing over all $s$ we finally have

$$|H_t(u - D^k u)| \leq C \beta_w^{\delta_s} e^{\beta_u t} \|\Delta_t^s u\|_\infty$$

for some constant $C = C(k, \epsilon)$. Now the restriction of $(u - D^k u)$ to $F_{[w]}(X)$ is

$$(u - D^k u)|_{F_{[w]}(X)} = H_t(u - D^k u) + G_t(\Delta u - D^k u) = H_t(u - D^k u) + G_t(\Delta u - D^{k-1} \Delta u)$$

the second term of which is bounded by $\beta_w^{\delta_s} e^{\beta_u t} \|\Delta_t^s u\|_\infty$ from (6.29), and the first term of which we have just estimated in the same way. This establishes (6.24) and completes the induction.

The proof of (6.25) is the same, except that (6.25) is used in place of (6.24) throughout. The validity of the estimate for $k = 0$ is a consequence of the Hölder estimate (2.2). □

As previously, we fix orthonormal bases for the spaces $E_x$ and see that the components of $d_x$ are distributions.

**Corollary 6.12.** Suppose that $x$ satisfies the assumptions of Lemma 6.11 and $\beta_w \leq \gamma_s$. Any component $d_{x,v}$ of the derivative $d_x$ is a distribution of order at most $k$ supported at $x$. If also $\gamma_s < \beta_w^{k-1}$ then $d_{x,v}$ has order equal to $k$. If $d_{x,v}$ has order $k$ then defining $\Delta d_{x,v}$ by $\Delta d_{x,v} \phi = d_{x,v} \Delta \phi$ yields a distribution supported at $x$ and of order $k + l$.

**Proof.** Linearity of $d_{x,v}$ is immediate from Definition 6.10 so it is a distribution of order at most $k$ by (6.23). Again using Definition 6.10 it is apparent that $d_{x,v} \phi = 0$ if $\phi \in D(\Omega)$ vanishes in a neighborhood of $x$, so $d_{x,v}$ is supported at $x$.

To see that $d_{x,v}$ has order at least $k$, consider the harmonic function $h$ with boundary values equal to the unit vector in the $v$ direction in $E_x$. Then $H_t h = A_{[w]} H_0 h$, so the sequence in (6.21) is constant equal to $H_0 h$, and $d_{x,v} h = 1$. Now for $\epsilon > 0$ so small that $\gamma_s e^{\beta_w \epsilon} \leq \beta_w^{k-1}$ we have

$$\|H_t h\|_\infty \leq C(\epsilon) e^{\beta_u \epsilon} \|\Delta_t^s u\|_\infty \leq C(\epsilon)(r_{[w]} \mu_{[w]})^{k-1} e^{\epsilon t} = o(r_{[w]} \mu_{[w]})^{k-1}$$

and of course $\Delta h \equiv 0$ for all $l > 0$, so $\Delta h$ is a test function supported at $x$, so $d_{x,v} \phi$ is supported at $x$ and $\phi_{[w]}$ is as small as we desire.

Since $d_{x,v} h = d_{x,v} \phi$ by the support condition, $d_{x,v}$ cannot be of order less than $k - 1$.

The final statement of the lemma is obvious. □

**Theorem 6.13.** Suppose that all of the matrices $A_j$ are invertible, and that $x = F_w(X)$ is a point at which the limits in (6.18) and (6.20) exist. Then all distributions of order at most $k$ at $x$ are linear combinations of the distributions $\Delta d_{x,v}$, with $\gamma_s \geq \beta_w^{k-1}$.

**Proof.** As in the proof of Theorem 6.6, it suffices to show that $T$ vanishes whenever the distributions $\Delta d_{x,v}$, with $\gamma_s \geq \beta_w^{k-1}$ vanish.

Suppose $\phi \in D(\Omega)$ satisfies $\Delta d_{x,v} \phi = 0$ for those $\gamma_s \geq \beta_w^{k-1}$. Then the differential $D^k \phi$ (which exists by Lemma 6.11) must be zero, as must $D^{k-1} \Delta \phi$ for each $0 \leq l \leq k$.

From (6.25) we then see that for all sufficiently small $\epsilon > 0$,

$$\|H_t \Delta^s \phi\|_{L^\infty(F_{[w]}(X))} \leq \|\Delta^s \phi\|_{L^\infty(F_{[w]}(X))} \leq C(k, \epsilon)(r_{[w]} \mu_{[w]})^{k-1} e^{\epsilon t} (\Delta^k u)$$

$$\leq C(k, \epsilon)(r_{[w]} \mu_{[w]})^{k-1} e^{2\epsilon t} E^{1/2}(\Delta^k u)$$

$$= o(r_{[w]} \mu_{[w]})^{k-1}.$$
Applying Theorem 6.2 we find that for any $\delta > 0$ there is $\psi$ equal to $\phi$ in a neighborhood of $x$ and such that $|\psi| < \delta$. In particular, since $T$ is order $k$ and supported at $x$, there is $M$ independent of $\phi$ such that

$$|T\phi| = |T\psi| \leq M|\psi| \leq \delta$$

and thus $T\phi = 0$. \hfill $\square$

In concluding this section it seems appropriate to say something about the set of points $x$ satisfying the conditions in Definition 6.10. The set at which the limit $\beta_w$ exists has full $\mu$-measure by the law of large numbers, and in fact

$$\beta_w = \sum_j \mu_j \log r_j \mu_j$$

at $\mu$-a.e. point. The set on which the Lyapunov exponents exist may be treated by the theory of random matrices introduced by Furstenberg and Kesten [8]. In particular, it is possible to make certain assumptions on the matrices $A$, that guarantee that this set is also of full $\mu$-measure. This topic is discussed quite thoroughly in the paper [26] of Pelander and Teplyaev, so we will not cover it here. One consequence of their work, however, is that there are conditions that imply the spaces $E_j$ are independent of the choice of point $x$. For example, if the semigroup generated by the $A_i$ is strongly irreducible and contracting then there is a single vector $\alpha_1$ such that at $\mu$-almost every $x$, the space $E_1$ is spanned by $\alpha_1$ and has scaling $\gamma_1$. If the same strong irreducibility and contraction holds after taking the quotient to remove $E_1$ then $E_2$ is also one-dimensional and independent of $x$ on a full measure set. For a fractal where the irreducibility and contraction properties are true for the semigroup generated by the $A_i$ on each of the subspaces found by removing $E_1, E_2, \ldots, E_{i-1}$ in turn, we could conclude that all of the distributions of the form $d_s$ are independent of $x$ on a set of full $\mu$-measure. Hence in this situation any distribution of order $m$ with point support in a fixed set of full $\mu$-measure would be a finite linear combination of distributions $\Delta^l d_s$ for suitable values of $l$, where the $d_s$ are independent of $x$. This generic behavior is very different from that seen at junction points and eventually periodic points, where the structure of point-supported distributions can vary substantially from point to point.

7. DISTRIBUTIONS ON PRODUCTS

In this section we give a theory of distributions on finite products of post-critically finite self-similar fractals, using the analytic theory for such products developed in [34]. This gives genuinely new examples, because products of p.c.f. self-similar sets are not usually themselves p.c.f. Since there is no essential difference between a product $X = X' \times X''$ with two factors and a general finite product, we state our results only for the two factor case.

Following the notational conventions of [34], points are $x = (x', x'')$, functions on $X$ are called $u$ or $f$, on $X'$ they are $u'$ or $f'$, while on $X''$ they are $u''$ or $f''$. The energies on $X'$ and $X''$ are $E'$ and $E''$ and the Laplacians are $\Delta'$ and $\Delta''$. They come from a regular harmonic structure as in Section 2 and have energy and measure scaling factors $\rho'$, $\rho''$, $r'$, $r''$ and $\mu'$. The corresponding Laplacians $\Delta'$ and $\Delta''$ are defined componentwise, so $u \in \text{dom}(\Delta')$ with $\Delta' u = f$ if $u$ and $f$ are continuous on $X$ and have the property that for each fixed $x'' \in X''$ we have $\Delta' u(\cdot, x'') = f(\cdot, x'')$. A similar definition is used for $\Delta''$. By Lemma 11.2 of [34], $\Delta'$ and $\Delta''$ commute on $\text{dom}(\Delta') \cap \text{dom}(\Delta'')$.

**Definition 7.1.** A function $u$ on $X$ is smooth if for all $j, k \in \mathbb{N}$, $(\Delta')^j (\Delta'')^k u$ is a continuous function on $X$. The definition extends to a finite union of cells in the obvious manner, and $u$ is smooth on a domain in $X$ if it is smooth on every finite union of cells in the domain.
We define the test functions on a domain $\Omega$ to be the smooth functions of compact support with the usual topology and the corresponding seminorms

$$|\phi|_m = \sup \{|(\Delta^j \phi(x)| : x \in \Omega, \ j + k \leq m'\}.$$  

The distributions form the dual space with weak-star topology. A distribution $T$ has order $m$ if on any compact $K$ there is $M = M(K)$ so that $|T\phi| \leq M|\phi|_m$ for all test functions supported on $K$.

The goal of this section is to provide conditions under which analogues of our main results for distributions on p.c.f. fractals are also valid on products of these fractals. In order to avoid duplicating a great deal of work, we only give details of the proofs where they differ significantly from those for the case of a single p.c.f. fractal. In particular it is fairly easy to verify that all of the results of Section 3 (except Corollary 3.6) and Section 5 prior to Theorem 5.6 depend only on the partitioning property of Theorem 2.7 and the estimate (2.5) as well as the fact that for compact $\Omega$ there is an orthonormal basis of $L^2$ consisting of eigenfunctions. The latter is obviously true for the product $X' \times X''$ because it is true for the factors, so the original proofs transfer to the product setting once we know the partitioning property and the corresponding estimate for product spaces. These are proved in Theorem 7.7 and (7.6) below, so all of the aforementioned results are also true for products of p.c.f. fractals with regular harmonic structure, and connected fractafolds with restricted cellular structure based on such products. Only small changes are needed to obtain analogues of the remaining results from Section 5. The proof of Theorem 5.6 required that on any cell there was a positive smooth function equal to 1 on the cell and vanishing outside a specified neighborhood: such a function may be obtained on the product space as a product of functions of this type on the factors, so the theorem is true for products in which each factor has the estimate (2.4) for the heat kernel. The other results are used to prove the structure theorem (Theorem 5.9). Of these, Lemma 5.7 remains true with the same proof if it is modified to say that $\Delta^m$ maps $D(K)$ to itself with image orthogonal to those $\phi$ having $\Delta^m \phi = 0$ and there is $\tilde{G}_k$ such that $-\Delta^m \tilde{G}_k(\Delta^m \phi) = \Delta^m \phi$; there is a corresponding result for $\Delta'$. A version of Theorem 5.8 is then true with $\tilde{G}_k \tilde{G}_K$ replacing $\tilde{G}_K$ throughout. The original proof shows that for $m \geq 1$, $\tilde{G}_k \tilde{G}_K$ takes a distribution of order $m$ to one of order at most $m - 1$. To show that $\tilde{G}_k \tilde{G}_K$ takes a distribution of order zero to a continuous function it suffices to approximate the corresponding measure $\nu$ by a sequence of linear combinations of product measures. Applying $\tilde{G}_k \tilde{G}_K$ to a product measure gives a continuous function by the original proof of Theorem 5.8 so applying it to the sequence forms a uniformly convergent sequence of continuous functions whose limit represents the distribution $\tilde{G}_k \tilde{G}_K \nu$. The proof of Theorem 5.9 needs no further changes.

At this point we have essentially all of the results of Sections 3-4 and 5 in the product setting (the only exception is Corollary 3.6). In addition there are some things that can be said about distributions with point support that generalize the results of Section 6. We will return to these after giving the details of the partitioning argument, because some aspects of the procedure for cutting off a smooth function will be important for the proofs.

**Partitioning on products.** We prove analogues of the partitioning property in Theorem 2.7 and the estimate (2.5) in the product setting. As in the single variable case, the proof relies on a cell-by-cell construction of a smooth function, for which the following matching condition is essential. Note that a cell in $X$ is a product of cells from $X'$ and $X''$, so has the...
Lemma 7.2. Suppose the cells $K_1, \ldots, K_l$ all contain the face $L = \{q'\} \times F'_{w''}(X'')$, and that the union $\bigcup_j K_j$ contains a neighborhood of every point in $L$ except those of the form $(q', F'_{w''}(q''))$ with $q'' \in V_0$. If $u_j$ is smooth on $K_j$ for each $j$, then the piecewise defined function $u = u_j$ on $K_j$ is smooth on $\bigcup_j K_j$ if and only if for each $x'' \in X''$, both
(a) The functions $(\Delta^j)^m u_j(q', x'')$ are independent of $j$ for each $l$ and $m$, and
(b) For each $x''$, $\sum_j (\partial_{x''}^j)(\Delta^j)^m u_j(q', x'') = 0$, where $(\partial_{x''}^j)$ indicates the normal derivative in the $x'$ variable from within $K_j$.

Proof. For fixed $x''$, (b) is the necessary and sufficient matching condition in the first variable for $(\Delta^j)^m u(\cdot, x'')$ to exist (as a function rather than a measure with atom at $q'$).
Condition (a) is then equivalent to continuity of $(\Delta^j)^m u$.

Our construction uses an analogue of the Borel theorem from [27]. That result yields the existence of a smooth function with a prescribed jet at a junction point of a pcf fractal, whereas we need existence of a smooth function with prescribed smooth jet on the face of a cell in the product $X$.

Theorem 7.3. Fix a face $\{q'\} \times X''$ and a neighborhood $U \subset X'$ of $q'$. Given two sequences $(\rho_j(x''))_{j=1}^{m_1}$ and $(\sigma_j(x''))_{j=1}^{n_1}$ of functions that are smooth in $x''$, there is a smooth function $u$ with support in $U \times X''$ such that for each $x'' \in X''$, $(\Delta^j)^m u(q', x'') = (\Delta^j)^m \rho_j(x'')$ and $\partial_{x''}^j (\Delta^j)^m u(q', x'') = (\Delta^j)^m \sigma_j(x'')$.

Proof. The proof is almost the same as that for Theorem 4.3 of [27]. Specifically we form the series
$$u(x', x'') = \sum_l \rho_l(x') g_{l,m_1}(x') + \sigma_l(x') f_{l,n_1}(x')$$
(7.1)
where the functions $g_{l,m_1}$ and $f_{l,n_1}$ are as defined in that proof, so they satisfy
$$(\Delta^j)^m g_{l,m_1}(q) = \delta_{kl} \quad \partial_{x''}^j (\Delta^j)^m g_{l,m_1}(q) = 0$$
$$(\Delta^j)^m f_{l,n_1}(q) = 0 \quad \partial_{x''}^j (\Delta^j)^m f_{l,n_1}(q) = \delta_{kl}$$
and have supports in cells of scale $m_1$ and $n_1$ respectively. Convergence of the series (7.1) is achieved by making an appropriate choice of $m_1$ and $n_1$. In particular, it follows from the cited proof that if $|\rho_l(x'')| \leq R_l$ and $|\sigma_l(x'')| \leq S_l$ for all $x''$, then one can choose $m_1$ and $n_1$ depending only on $R_l$ and $S_l$ such that for each $x''$ the series converges to a function that is smooth in $x''$, supported in $U \times X''$, and has $(\Delta^j)^m u(q, x'') = \rho_j(x'')$ and $\partial_{x''}^j (\Delta^j)^m u(q, x'') = \sigma_j(x'')$.

Now we require convergence not only of the series for $u(x', x'')$, but also that for $(\Delta^j)^m u(x', x'')$ for each $m$, so we must diagonalize. Set
$$R_l = \max_{0 \leq j \leq l} \max_{x'' \in X''} |(\Delta^j)^m \rho_j(x'')|$$
$$S_l = \max_{0 \leq j \leq l} \max_{x'' \in X''} |(\Delta^j)^m \sigma_j(x'')|$$
which are finite by the assumed smoothness and the compactness of $X''$, and let $m_l$ and $n_l$ be chosen as described above. For fixed $x''$, all terms after the $m$-th in the partial sum
$$(\Delta^j)^m \sum_l \rho_l(x') g_{l,m_l}(x') + \sigma_l(x') f_{l,n_l}(x')$$
have coefficients bounded by $R_l$ and $S_l$, so the above reasoning implies that the partial sums converge to a function that is smooth in $x'$, and has $(\Delta')^h(\Delta'')^mu(q', x'') = (\Delta''^mu(q', x'') and

\[ \partial^\alpha_{l} (\Delta')(\Delta'')^mu(q', x'') = (\Delta''^mu(q', x'') \text{ for all } m. \]

Finally, it will be useful later to have estimated the contribution of each term to the $L^\infty$ norm of $(\Delta')^h(\Delta'')^mu$. It is convenient to write $w'(m_l)$ and $w'(n_l)$ for the words such that $F_{w'(m_l)}(X')$ is the support of $g_{l,m}$, and $F_{w'(n_l)}(X)$ is the support of $f_{l,n}$. Note that scaling then implies (see equations 4.4 and 4.5 of [27]) that

\[
\begin{align*}
\| (\Delta')^h(\Delta'')^mu(q') \|_{L^\infty} &\leq C(k, l) (r'(w'(m_l)\mu_l)(w'(n_l)))^{\frac{1}{1-k}} \| (\Delta'')^mu(q') \|_{L^\infty} \\
\| (\Delta')^h(\Delta'')^mu(q') \|_{L^\infty} &\leq C(k, l) (r'(w'(n_l)\mu_l)(w'(m_l)))^{\frac{1}{1-k}} \| (\Delta'')^mu(q') \|_{L^\infty}.
\end{align*}
\]

and in the construction in [27] it is noted that the contributions of terms with $l > k$ may be made smaller than any prescribed $\epsilon > 0$, so taking $\epsilon$ to be a small multiple of $\|u\|_\infty$ we obtain

\[
\begin{align*}
\| (\Delta')^h(\Delta'')^mu \|_{L^\infty} &\leq \sum_{l=0}^{k} C(k, l) (r'(w'(m_l)\mu_l)(w'(n_l)))^{\frac{1}{1-k}} \| (\Delta'')^mu(q') \|_{L^\infty} \\
+ \sum_{l=k+1}^{k-1} C(k, l) (r'(w'(n_l)\mu_l)(w'(m_l)))^{\frac{1}{1-k}} \| (\Delta'')^mu(q') \|_{L^\infty}.
\end{align*}
\]

(7.2)

\[\square\]

**Remark 7.4.** This result may be localized to any cell in $X$ simply by rescaling the desired jet for the cell to obtain a corresponding jet on $X$, applying the theorem, and then composing the resulting function with the inverse of the map to the cell. It may also be applied to a face in a finite union of cells, so that the face is of the form $\{ q' \} \times (\bigcup_{j=1}^{l} K_j''')$ with each $K_j''$ a cell in $X''$, because $\bigcup_{j=1}^{l} K_j'''$ is compact.

In order to make use of the preceding result we require a small lemma.

**Lemma 7.5.** If $u$ is smooth on $X$ and $q' \in V'_m$ then $\partial^\alpha_{l}u(q', x'')$ is smooth with respect to $x''$ and $(\Delta')^h\partial^\alpha_{l}u(q', x'') = \partial^\alpha_{l}(\Delta')^h\partial^\alpha_{l}u(q', x'')$. There is a bound

\[
\| \partial^\alpha_{l}(\Delta')^h\partial^\alpha_{l}u(q', x'') \|_{L^\infty} \leq C \left( \| (\Delta'')^mu \|_{L^\infty} + \| (\Delta'')^mu \|_{L^\infty} \right)
\]

(7.3)

**Proof.** For each $x''$ and each scale $m$, let $h_m(x', x'')$ be the function that is piecewise harmonic at scale $m$ in the $x'$ variable and coincides with $u$ on $V'_m \times \{ x'' \}$. Then $h_m(x', x'')$ is smooth in $x''$, because its values are obtained as uniform limits of linear combinations of the values from $V'_m \times \{ x'' \}$. Moreover, the normal derivative $\partial^\alpha_{l}h_m(q', x'')$ is a linear combination (with coefficients depending on $m$) of the differences $(h_m(p'_1, x'') - h_m(p'_2, x''))$, where $p'_1$ and $p'_2$ are neighbors of $q'$ at scale $m$. Thus $\partial^\alpha_{l}h_m(q', x'')$ is smooth in $\bar{x}'$ and

\[
(\Delta')^h\partial^\alpha_{l}h_m(q', x'') = \partial^\alpha_{l}(\Delta')^h\partial^\alpha_{l}h_m(q', x'').
\]

For each fixed $x''$, we may express $(\Delta')^h\partial^\alpha_{l}u(q', x'')$ on a cell $K'_m$ of scale $m$ containing $q'$ as the sum of $(\Delta')^h\partial^\alpha_{l}h_m(q', x'')$ and an integral involving the Dirichlet Green kernel $G'_m$ for $\Delta'$ on $K'_m$. Taking the normal derivative we obtain

\[
\begin{align*}
\partial^\alpha_{l}(\Delta')^h\partial^\alpha_{l}u(q', x'') &= \partial^\alpha_{l}(\Delta')^h\partial^\alpha_{l}h_m(q', x'') + \int (\Delta'(\Delta')^h\partial^\alpha_{l}h_m(q', x''))\partial^\alpha_{l}G'_m(q', y') \, d\mu(y').
\end{align*}
\]

(7.4)

However an easy scaling argument shows that $\partial^\alpha_{l}G'_m(q', y')$ is bounded independent of $m$ and $y'$, so the integral term is bounded by a constant multiple of $\| (\Delta')^h\partial^\alpha_{l}h_m(q', x'') \|_{L^\infty}$, independent of $m$ and $x''$. Since $\mu'(K'_m) \to 0$ as $m \to \infty$ we conclude that $(\Delta')^h\partial^\alpha_{l}h_m(q', x'')$
converges to $\partial_t^l(\Delta'')^m u(q', x'')$ uniformly in $x''$ for each $l$. Then (7.4) is obtained by using (7.3) with $m = 0$. 

We may use the preceding results to smoothly cut off a smooth function on a neighborhood of a cell.

**Theorem 7.6.** Let $u$ be smooth on a cell $K = F'_w(X') \times F''_w(X'')$, and $U \supset K$ be open. There is a function $v$ such that $v = u$ on $K$, $v = 0$ on $X \setminus U$ and $v$ is smooth on $X$. Moreover for each $k$,

$$\left\| (\Delta')^k (\Delta'')^m v \right\|_{L^p(K)} \leq C(k, U) \sum_{l=0}^k \sum_{n=0}^m \left\| (\Delta')^l (\Delta'')^n u \right\|_{L^{p'}(K)}.$$  

(7.5)

**Proof.** Let $K' = F'_w(X')$ and $K'' = F''_w(X'')$. Fix a face of $K$ having the form $\{q'\} \times K''$ and let $\rho_{q'}(x'') = (\Delta')^k u(q', x'')$ and $\sigma_{q'}(x'') = \partial_t^l(\Delta')^m u(q', x'')$. The functions $\rho_{q'}$ are smooth in $x''$ by the definition of smoothness of $u$, and the functions $\sigma_{q'}$ are smooth in $x''$ by Lemma 7.5. Now take a finite number of small cells $K'_j$ in $X'$ with the following properties: the intersection $K' \cap K'_j = \{q'\}$ for all $j$, the intersection $K'_j \cap K'_{j'} = \{q'\}$ for all $j \neq j'$, the union $K' \cup (\cup_j K'_j)$ contains a neighborhood of $q'$ in $X'$, and $(K' \cup (\cup_j K'_j)) \times K'' \subset U$. Let the number of $K'_j$ be $J$, and apply Theorem 7.3 to each $K'_j$ to obtain a smooth function $u_j$ that has jets $\rho_{q'}(x'')$ and $(-1/J)\sigma_{q'}(x'')$ at $q'$ and is supported in a neighborhood of $q'$ that is strictly contained in $K'_j$. By construction, the matching conditions of Lemma 7.2 apply to the functions $u_j$ on $K$ and $u_j$ on $K'_j \times K''$, so the piecewise defined function is smooth on the union of these cells.

Repeat the previous construction for each of the finite number of faces having the form $\{q'_i\} \times K''$. As these faces are disjoint we may choose the small cells in the construction so that those used for $q'_i$ do not intersect those for $q'_{i'}$ for $i \neq i'$. The result is a finite collection of cells $K'_j \times K'' \subset U$ and functions $u_j$ such that the piecewise function $u$ on $K$ and $u_j$ on $K'_j \times K''$ is smooth on the union of the cells, and vanishes identically in a neighborhood of any boundary face of $(K' \cup (\cup_j K'_j)) \times K''$ that has the form $\{p'\} \times K''$. We call this function $v'$.

Having treated the vertical faces $\{q'_i\} \times K''$, we then treat the horizontal faces $(K' \cup (\cup K'_j)) \times \{q''\}$ of the new function $v'$ in the same manner. All of the results we needed were valid on faces of finite unions of cells, so the same proof allows us to piecewise extend to a smooth function $v$ on a larger finite union of cells, which we call $L$, but with the additional condition that $v$ vanishes identically in a neighborhood of each horizontal face of $L$. Then $L \subset U$ and $v$ vanishes in a neighborhood of all faces of the boundary of $L$, so Lemma 7.2 ensures that extending $v$ to be identically zero outside $L$ gives a smooth function on $X$. By construction, $v = u$ on $K$.

For the estimate (7.5) we note that

$$\left\| (\Delta'')^m \rho_{q'}(x'') \right\|_{L^p(K)} \leq \left\| (\Delta')^k (\Delta'')^m u \right\|_{L^{p'}(K)}$$

by definition, while rescaling (7.3) to the cell $K$ implies that

$$\left\| (\Delta''')^m \sigma_{q'}(x'') \right\|_{L^p(K)} \leq C \left( (\rho_{q'})^{-1} \left\| (\Delta')^k (\Delta'')^m u \right\|_{L^{p'}(K)} + \mu \left\| (\Delta')^l (\Delta'')^m u \right\|_{L^{p'}(K)} \right).$$
Substituting into (7.2) and using \( r_{e''(m)}' \leq r_{e'}' \) and \( \mu_{e,v}'' < 1 \) we have

\[
\|(\Delta')^j(\Delta'')^m\|_{L^\infty} \leq \sum_{l=0}^{k} c(k,l)(r_{e'(m)}'u'_{r_{e''(m)}})' \|(\Delta')^j(\Delta'')^m u\|_{L^\infty(K)}
\]

\[
\leq C(k, U) \sum_{l=0}^{k} \|(\Delta')^j(\Delta'')^m u\|_{L^\infty(K)}.
\]

This type of estimate deals with all of the vertical faces, and an analogous argument is valid for the horizontal faces, so (7.5) holds.

**Theorem 7.7.** If \( u \) is smooth on \( X \) and \( \cup \Omega_j \) is an open cover of \( X \) then there are constants \( C(k, m) \) and smooth functions \( u_j \) such that \( u_j \) is supported on \( \Omega_j \), \( \sum_j u_j = u \), and

\[
\|(\Delta')^j(\Delta'')^m u_j\|_{L^\infty} \leq C(k, m) \sum_{l=0}^{k} \sum_{n=0}^{m} \|(\Delta')^j(\Delta'')^m u\|_{L^\infty}.
\]

**Proof.** The open cover is finite, say \( \{\Omega_j\}_1^L \) because \( X \) is compact. Moreover we may partition \( X \) into a finite number of cells \( K_l \) because each \( K_l \) is contained in some \( \Omega_j \). We proceed by induction on \( l \), with the base case being that we apply Theorem 7.6 to \( u \) on \( K_l \) to obtain a smooth function \( v_l \) with support in the open \( \Omega_j \) that contains \( K_l \). At the \( l \)-th step we apply Theorem 7.6 to \( u - \sum_{l=1}^{L} v_m \) on \( K_l \) to obtain a smooth function \( v_l \) with support in the open \( \Omega_j \) that contains \( K_l \). Note that \( u - \sum_{l=1}^{L} v_m \) vanishes on \( \cup_{m=1}^{L} K_m \) so once we have exhausted the cells we have \( \sum_j v_j = u \). By construction, each of the \( v_l \) is smooth, supported on some \( \Omega_j \) and satisfies (7.5). Setting \( u_j \) to be the sum of those \( v_l \) that are supported on \( \Omega_j \) completes the proof.

**Distributions with point support on products.** It is useful to begin with the observation that if \( T' \in \mathcal{D}'(X') \) and \( T'' \in \mathcal{D}'(X'') \) are distributions on the components of a product space \( X' \times X'' \) then there is a tensor distribution \( T' \times T'' \) which is a distribution on the product. This is not entirely immediate, but follows readily from the structure theorem for the component spaces. Specifically, the fact that \( T' \) is locally \( (-\Delta')^j f \) for a continuous \( f \) implies that for a \( \phi \in \mathcal{D}(X' \times X'') \) there are \( k \) and \( f \) such that

\[
\Delta'' T' \phi(x', x'') = \Delta'' \int_{X'} f(x')(\Delta')^k \phi(x', x'') \, d\mu'(x')
\]

\[
= \int_{X'} f(x')(\Delta')^k \Delta'' \phi(x', x'') \, d\mu'(x')
\]

\[
= T' \Delta'' \phi(x', x'')
\]

where we used that \( \Delta ' \) and \( \Delta '' \) commute. In particular \( T' \phi \) is smooth in the second variable, so \( T' \times T'' \phi = T''(T' \phi) \) is well defined. Repeating the calculation with \( T'' \) in place of \( \Delta'' \) ensures that \( T''(T' \phi) = T'(T'' \phi) \), so the order in which the distributions are applied is not important. Linearity of \( T' \times T'' \) is immediate and it is easy to check the continuity condition that ensures it is a distribution on \( X' \times X'' \).

In the special case where \( T' \) is supported at \( x' \) and \( T'' \) is supported at \( x'' \) it is apparent that \( T' \times T'' \) is supported at \( (x', x'') \), so this construction and the results of Section 6 supply a large number of distributions with point support. In fact we can show that if \( x' \) and \( x'' \) are either junction points or satisfy the conditions of Theorem 6.13 then the distributions with support at \( (x', x'') \) are of this type. As in Section 6 the key is to show that if \( \phi \in \mathcal{D}(X' \times X'') \) is annihilated by sufficiently large collection of tensor distributions at \( (x', x'') \) and if \( \epsilon > 0 \) is given, then it is possible to cut off \( \phi \) on a small neighborhood of \( (x', x'') \) such that the the
resulting function has \( \| (\Delta')^i(\Delta'')^j \psi \|_{\infty} < \epsilon \) for all \( j \) and \( k \) such that \( j + k \leq m \). It follows that all distributions of order at most \( m \) and support \( x' \times x'' \) are linear combinations of the given tensor distributions.

Our main tool is an adaptation of Theorem 6.2.

**Theorem 7.8.** Given a test function \( \psi \) and a cell \( K = K' \times K'' \) with \( K' = F_w'(X') \) and \( K'' = F_w''(X'') \), there is a test function \( \hat{\psi} \) such that \( \hat{\psi} = \phi \) on \( K \) and

\[
\| (\Delta')^i(\Delta'')^j \hat{\psi} \|_{\infty} \leq C(m, n) \sum_{i=0}^{m} \sum_{r'=0}^{n} (r'_w \mu'_w)^{j-i} (r''_w \mu''_w)^{i-j} \| (\Delta')^{j-i} \psi \|_{L^\infty(K')} + \epsilon
\]

for all \( 0 \leq j \leq m \) and \( 0 \leq k \leq n \).

**Proof.** The method for cutting-off a smooth function on a cell has already been described in the proof of Theorem 7.6. Since we cut off first in one variable and then in the other, the estimates from the proof of Theorem 6.2 may be applied directly. Suppose that we cut off in the first variable and then in the second. Taking \( \Delta'' \) for the Laplacian \( (\Delta')^k \) in the first variable on a fixed slice \( U' \times (y'') \) and substituting from the second lines of both of (6.2) and (6.3), gives

\[
\| (\Delta')^i \hat{\psi} \|_{L^\infty(U' \times (y''))} \leq C(m) \sum_{i=0}^{m} (r'_w \mu'_w)^{j-i} \| (\Delta')^{j-i} \psi \|_{L^\infty(K' \times (y''))} + \epsilon
\]

provided \( j \leq m \). In this calculation we used that the harmonic part of a function (which was denoted \( H_{i,j} \) in the proof of Theorem 6.2) is bounded by the \( L^\infty \) norm of the function because of the maximum principle, and we extracted the scaling factor \( r'_w \mu'_w \) of the Laplacian on \( K' = F_w'(X') \) using the same argument as in (6.5). The same estimate is true with the same proof when \( \psi \) is replaced by \( (\Delta'')^k \psi \) and \( \phi \) by \( (\Delta'')^j \phi \). We use this fact when we repeat the estimate in the second variable, because in this case we are cutting off the function that was modified at the first step. A little algebra then produces the desired estimate. \( \square \)

**Theorem 7.9.** Let \( T \) be a distribution supported at \( (x',x'') \in X' \times X'' \). Suppose that \( x' \) is such that either Theorem 6.6 or Theorem 6.13 may be used to identify the distributions with support at \( x' \), and make the same assumption for \( x'' \). Then \( T \) is a finite linear combination of tensor products \( T' \times T'' \) where \( T' \) is supported at \( x' \) and \( T'' \) is supported at \( x'' \).

**Proof.** In light of the preceding discussion and Theorem 7.8 it suffices to show that if the given tensor distributions vanish on a test function \( \phi \) then the right side of (7.7) may be made less than \( 2\epsilon \) by taking \( K \) sufficiently small. The proof of this estimate is elementary: we simply go from \( (x',x'') \) to \( (y',y'') \) by using two Taylor-like expansions, one in each variable.

Since \( T \) has compact support it also has finite order \( m \). It then seems reasonable that each of the terms \( T' \times T'' \) should be made up of a \( T' \) of order \( k \leq m \) and a \( T'' \) of order at most \( m-k \). Unfortunately we cannot prove this in general because our scaling estimates are insufficiently refined, as was explained in Remark 6.7. This result is true if the distributions at \( x' \) and \( x'' \) are such that none have scaling exactly equal to that of the Laplacian (meaning that if they are as in Theorem 6.5 then there is no \( \gamma_s \) equal to a power of \( r\mu \), and if they are as in Corollary 6.12 then there is no \( \gamma_s \) equal to a power of \( \beta_{m,j} \)). Given the limitations of our estimates we must instead allow the possibility that \( T' \) is order \( k+1 \) and \( T'' \) is order \( m - k + 1 \).

Suppose then that \( T' \times T'' \phi = 0 \) for all \( T' \) of order up to \( k+1 \) and \( T'' \) of order up to \( m-k+1 \). It follows that the differential \( (\tilde{D}'')^{m-k} \) vanishes on the one-variable smooth
function $T^\prime \phi(x', \cdot)$. The same reasoning as was used at the beginning of the proofs of Theorem 6.6 and Theorem 6.13 shows that then $T^\prime \phi = o(r_{w'}^m \mu_{w'}^m)^{m-k}$ on the set $\{x'\} \times K''$, so in particular at $(x', y''')$.

We now wish to repeat the argument to go from $(x', y''')$ to $(y', y''')$. Instead of having vanishing distributions in the first variable at $(x', y''')$ we have only estimates on their size, which we use to estimate the size of $(\bar{D})^m \phi$. Recall from Definitions 6.3 and 6.10 that the differential $(\bar{D})^m \phi$ for the second variable on the cell $K'$ consists of a harmonic function with coefficients obtained using distributions of order at most $m + 1$, as well as $G'(\bar{D})^{m-1} \Delta \phi$, where $G'$ is the Green’s operator for the cell $K'$. The harmonic function is itself made up of pieces (one for each $k \leq m$) with scaling bounded by $(r_{w'}^m \mu_{w'}^m)^k$ (or an equivalent quantity involving $\beta_{w'}^k$) and coefficients obtained using distributions in the first variable with order at most $k + 1$. The estimate of the previous paragraph says that these coefficients are $o((r_{w'}^m \mu_{w'}^m)^{m-k})$, so each term of the harmonic functions is $o((r_{w'}^m \mu_{w'}^m)^{(m-k)} \Delta \phi)$ on $K' \times \{y''\}$. A similar argument applies to $G'(\bar{D})^{m-1} \Delta \phi$, because the $G'$ produces an extra factor of $r_{w'}^m \mu_{w'}^m$, and the harmonic piece of $(\bar{D})^{m-1} \Delta \phi$ that has scaling $(r_{w'}^m \mu_{w'}^m)^{m-1}$ is obtained via distributions of order at most $k$ applied to $\Delta \phi$, each of which is a distribution of order $k + 1$ applied to $\phi$. We may repeat this reasoning inductively across the terms of $(\bar{D})^m \phi$ to obtain a bound of the form

$$\left| (\bar{D})^m \phi \right| = o \left( \sum_{k=0}^{m} (r_{w'}^m \mu_{w'}^m)^k (r_{w'}^m \mu_{w'}^m)^{m-k} \right).$$

(7.8)

on the set $K' \times \{y''\}$. Since we also know (from (6.10) and (6.25)) that

$$\left| \phi - (\bar{D})^l \phi \right| = o(r_{w'}^m \mu_{w'}^m)^{m}$$
on $K' \times \{y''\}$ we conclude that the estimate (7.8) is also true for $\phi$ itself. The point $y'' \in K''$ was arbitrary, so we have

$$\left\| \phi \right\|_{L^\infty(K)} = o \left( \sum_{k=0}^{m} (r_{w'}^m \mu_{w'}^m)^k (r_{w'}^m \mu_{w'}^m)^{m-k} \right).$$

(7.9)

Our working thus far has shown that if $T^\prime \times T'' \phi = 0$ for all $T'$ of order up to $k + 1$ and $T''$ of order up to $m - k + 1$, then (7.9) holds. However, this assumption obviously implies that $T^\prime \times T'' \left( \Delta^l \phi \right) = 0$ for all $T'$ of order up to $k + 1 - i$ and $T''$ of order up to $m - k + 1 - i$ if $l + i \leq m$ and $0 \leq k \leq (m - i - l)$. Thus (7.9) improves to

$$\left\| (\Delta)^l (\Delta^l \phi) \right\|_{L^\infty(K)} = o \left( \sum_{k=0}^{m-l} (r_{w'}^m \mu_{w'}^m)^k (r_{w'}^m \mu_{w'}^m)^{m-l-k} \right).$$

(7.10)

Substituting into (7.7) for the cutoff of $\phi$ yields

$$\left\| (\Delta)^l (\Delta^l \phi) \right\|_\infty \leq \epsilon + o \left( \sum_{i=0}^{m} \sum_{j=0}^{n} (r_{w'}^m \mu_{w'}^m)^{-i} (r_{w'}^m \mu_{w'}^m)^{-j-i-k} \sum_{k=0}^{m-l} (r_{w'}^m \mu_{w'}^m)^k (r_{w'}^m \mu_{w'}^m)^{m-l-k} \right).$$

The simplest way to complete the argument is to choose $K'$ and $K''$ such that $r_{w'}^m \mu_{w'}^m$ and $r_{w'}^m \mu_{w'}^m$ are comparable, at which point all terms in the sum are bounded. It follows that the sum term is $o(1)$ so can be made less than $\epsilon$ by requiring that $K'$ and $K''$ are also sufficiently small.

$\square$
8. Hypoellipticity

An important question in the analysis of PDE is to identify conditions under which a distributional solution of a PDE is actually a smooth function. In Euclidean space, an archetypal example is Weyl’s proof that a weak solution of the Laplace equation is actually $C^\infty$. In order to study these questions one uses the notion of hypoellipticity, which we may now define in the setting of fractafolds based on p.c.f. fractals and their products. We will not settle any of the questions about hypoellipticity here, but simply suggest some natural problems for which the distribution theory we have introduced is the correct setting.

We first define the singular support of a distribution, which intuitively corresponds to those points where the distribution is not locally smooth.

**Definition 8.1.** A distribution $T$ is smooth on the open set $\Omega_1 \subset \Omega$ if there is $u \in \mathcal{E}(\Omega_1)$ such that

$$T \phi = \int u \phi \, d\mu \text{ for all } \phi \in \mathcal{D}(\Omega_1)$$

Using Lemma 5.2 for the case of a single p.c.f. fractal, or the analogous result derived from Theorem 7.7 in the product setting, we see that if $T$ is smooth on $\Omega_1$ and on $\Omega_2$ then it is smooth on $\Omega_1 \cup \Omega_2$, thus there is a maximal open set on which $T$ is smooth.

**Definition 8.2.** For a distribution $T$, let $\Omega_T$ be the maximal open set on which $T$ is smooth. The singular support of $T$ is the set

$$\text{SingSppt}(T) = \text{Sppt}(T) \setminus \Omega_T$$

Let $P$ be a polynomial of order $k$ on $\mathbb{R}^m$, so $P(\xi) = \sum_{|\alpha| \leq k} a_\alpha \xi^\alpha$ where $\alpha = \alpha_1 \ldots \alpha_m$ is a multi-index, $|\alpha| = \sum \alpha_j$ is its length, and $\xi^\alpha = \prod \xi_j^{\alpha_j}$. Consider the linear differential operator $P(\Delta) = P(\Delta_1, \ldots, \Delta_m)$ on a product $\prod X_j$ of p.c.f. self-similar fractals $X_j$ with Laplacians $\Delta_j$. It is clear that for any distribution $T$ we have $\text{SingSppt}(P(\Delta)T) \subseteq \text{SingSppt}(T)$, because when $T$ is represented by $u \in \mathcal{E}(\Omega_T)$ then $P(\Delta)T$ is represented by $P(\Delta)u$. By analogy with the Euclidean case, we define a class of constant coefficient hypoelliptic linear differential operators.

**Definition 8.3.** $P(\Delta)$ is called hypoelliptic if $\text{SingSppt}(P(\Delta)T) = \text{SingSppt}(T)$ for all $T \in \mathcal{D}'(\Omega)$.

Given the importance of hypoelliptic operators in the analysis of PDE on Euclidean spaces, it is natural to seek conditions that imply hypoellipticity of an operator on a p.c.f. fractal or on products of p.c.f. fractals. We expect that if $P(\Delta)$ is elliptic then it should be hypoelliptic; it also seems possible that the celebrated hypoellipticity criterion of Hörmander might imply hypoellipticity in the fractal case, though we do not expect conditions of this type to be necessary because of examples like that motivating Conjecture 8.9.

**Definition 8.4.** For a polynomial $P(\xi) = \sum_{|\alpha| \leq k} a_\alpha \xi^\alpha$, the principal part of $P$ is $P_0 = \sum_{|\alpha| \leq k} a_\alpha \xi^\alpha$. $P$ is called elliptic if $P_0(\xi) \neq 0$ for $\xi \neq 0$; equivalently $P$ is elliptic if there is $c > 0$ so $|P_0(\xi)| \geq c|\xi|^\kappa$. We call $P(\Delta)$ elliptic if $P(\xi_1^2, \ldots, \xi_m^2)$ is elliptic.

**Remark 8.5.** The above definition is consistent with the usual one in the case that $X$ is a Euclidean interval rather than a fractal set, but they do not coincide because we are dealing with a smaller class of operators. Specifically, for such $X$ the Laplacian is $\partial^2/\partial x^2$, so our class of operators $\{P(\Delta)\}$ is smaller than the usual collection of constant coefficient linear partial differential operators $P(\partial/\partial x_1, \ldots, \partial/\partial x_m)$. Similarly our class of elliptic operators is a strict subset of the classical one.
Conjecture 8.6. If $X$ is a p.c.f. fractal and $P(Δ)$ is an elliptic operator on the product space $X^m$, then $P(Δ)$ is hypoelliptic.

In the case that $m = 1$, all operators $P(Δ)$ are elliptic, and they can all be shown to be hypoelliptic. Indeed, by factoring the polynomial we can reduce to the case of the linear polynomial $Δ + c$ for some complex constant $c$. The hypoellipticity of $Δ + c$ is readily obtained from the fact that on small cells $(Δ + c)$ has a resolvent kernel that is smooth away from the diagonal, as may be seen by representing the resolvent as an integral with respect to the heat kernel or by applying results from [12].

Conjecture 8.7. A sufficient condition for the hypoellipticity of $P(Δ)$ is that $D^x P(ξ)/P(ξ) → 0$ as $ξ → ∞$ for any partial derivative $D^x$ with $|x| > 0$ (compare to Theorems 11.1.1 and 11.1.3 of [11]).

In the Euclidean setting the above condition is necessary as well as sufficient, but we do not expect this to be the case on fractals. In essence, the idea is that hypoellipticity of $P(Δ)$ should depend only on whether the principal part $P_0(Δ)$ is hypoelliptic, and that this is equivalent (on the Fourier transform side) to estimates when inverting the algebraic equation

$$P_0(λ_1, ..., λ_m) u(λ_1, ..., λ_m) = f(λ_1, ..., λ_m)$$

for any choice of $(λ_1, ..., λ_m)$ with each $λ_j$ an eigenvalue of $Δ_j$. Since all of these $λ_j$ are negative, ellipticity of $P(Δ)$ says that $|P_0(λ_1, ..., λ_m)| ≥ c|λ_1 + ... + λ_m|$ for such $(λ_1, ..., λ_m)$, and this is sufficient to show the Fourier transform $ˆu$ has faster decay than $f$, so $u$ should be as smooth or smoother than $f$. However the ellipticity condition should only be necessary if the points $(λ_1, ..., λ_m)$ are dense in the positive orthant $\{ξ : ξ_j ≥ 0\}$. In [6] it is shown that this is not the case for the Sierpinski Gasket fractal; specifically it is shown that in the case $m = 2$, the points $(λ_1, λ_2)$ omit an open neighborhood of a ray in the positive orthant. It follows that there is $a > 0$ and $b < 0$ such that $aΔ_1 + bΔ_2$ is not elliptic but $-a/b$ lies in the omitted neighborhood, so $|a/λ_1 + b/λ_2| ≥ c|λ_1 + λ_2|$ whenever $λ_j$ is an eigenvalue of $Δ_j$.

Following [6] we call operators of this type quasielliptic. Given that quasielliptic operators satisfy elliptic-type estimates on the spectrum, it seems likely that they will have similar smoothness properties to elliptic operators; $L^p$ estimates for these operators may be found in recent work of Sikora [30].

Definition 8.8. The operator $P(Δ)$ is quasielliptic if there is $c > 0$ such that $|P_0(ξ)| ≥ c|ξ|$ for all $ξ ∈ \{λ_1, ..., λ_m : λ_j$ is an eigenvalue of $Δ_j\}$.

Conjecture 8.9. The quasielliptic operators of [6] are hypoelliptic.

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