Generalized Fractional Integrals and Their Commutators over Non-homogeneous Metric Measure Spaces

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Abstract Let \((X, d, \mu)\) be a metric measure space satisfying both the upper doubling and the geometrically doubling conditions. In this paper, the authors establish some equivalent characterizations for the boundedness of fractional integrals over \((X, d, \mu)\). The authors also prove that multilinear commutators of fractional integrals with RBMO(\mu) functions are bounded on Orlicz spaces over \((X, d, \mu)\), which include Lebesgue spaces as special cases. The weak type endpoint estimates for multilinear commutators of fractional integrals with functions in the Orlicz-type space \(\text{Osc}_{\text{exp}L^r}(\mu)\), where \(r \in [1, \infty)\), are also presented. Finally, all these results are applied to a specific example of fractional integrals over non-homogeneous metric measure spaces.

1 Introduction

During the past ten to fifteen years, considerable attention has been paid to the study of the classical theory of harmonic analysis on Euclidean spaces with non-doubling measures only satisfying the polynomial growth condition (see, for example, \([11, 10, 35, 37, 38, 39, 40, 5, 27, 14, 15, 16, 17, 4, 42]\)). Recall that a Radon measure \(\mu\) on \(\mathbb{R}^d\) is said to only satisfy the polynomial growth condition, if there exists a positive constant \(C\) such that,

\[
\mu(B(x, r)) \leq Cr^\kappa,
\]

where \(\kappa\) is some fixed number in \((0, d]\) and \(B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}\). The analysis associated with such non-doubling measures \(\mu\) as in (1.1) has proved to play a striking role in solving the long-standing open Painlevé’s problem and Vitushkin’s conjecture by Tolsa \([38, 39, 40]\).

Obviously, the non-doubling measure \(\mu\) as in (1.1) may not satisfy the well-known doubling condition, which is a key assumption in harmonic analysis on spaces of homogeneous type in the sense of Coifman and Weiss \([6, 7]\). To unify both spaces of homogeneous type...
type and the metric spaces endowed with measures only satisfying the polynomial growth condition, Hytönen [18] introduced a new class of metric measure spaces satisfying both the so-called geometrically doubling and the upper doubling conditions (see also, respectively, Definitions 1.1 and 1.3 below), which are called non-homogeneous metric measure spaces. Recently, many classical results have been proved still valid if the underlying spaces are replaced by the non-homogeneous metric measure spaces (see, for example, [18, 22, 2, 19, 20, 21, 25, 8, 24]). It is now also known that the theory of the singular integral operators on non-homogeneous metric measure spaces arises naturally in the study of complex and harmonic analysis questions in several complex variables (see [41, 20]). More progresses on the Hardy space $H^1$ and boundedness of operators on non-homogeneous metric measure spaces can be found in the survey [43] and the monograph [44].

Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [18]. In this paper, we establish some equivalent characterizations for the boundedness of fractional integrals over $(\mathcal{X}, d, \mu)$. We also prove that multilinear commutators of fractional integrals with RBMO($\mu$) functions are bounded on Orlicz spaces over $(\mathcal{X}, d, \mu)$, which include Lebesgue spaces as special cases. The weak type endpoint estimates for multilinear commutators of fractional integrals with functions in the Orlicz-type space $\text{Osc}_{\exp L^r} (\mu)$, where $r \in [1, \infty)$, are also presented. Finally, all these results are applied to a specific example of fractional integrals over non-homogeneous metric measure spaces. The results of this paper round out the picture on fractional integrals and their commutators over non-homogeneous metric measure spaces.

Recall that the well-known Hardy-Littlewood-Sobolev theorem (see, for example, [32, pp. 119-120, Theorem 1]) states that the classical fractional integral $I_\alpha$, with $\alpha \in (0, d)$, is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$, for all $p \in (1, d/\alpha)$ and $1/q = 1/p - \alpha/d$, and bounded from $L^1(\mathbb{R}^d)$ to weak $L^{d/(d-\alpha)}(\mathbb{R}^d)$. Chanillo [3] further showed that the commutator $[b, I_\alpha]$, generated by $b \in \text{BMO}(\mathbb{R}^d)$ and $I_\alpha$, which is defined by

$$[b, I_\alpha](f)(x) := b(x)I_\alpha(f)(x) - I_\alpha(bf)(x), \quad x \in \mathbb{R}^d,$$

is bounded from $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ for all $\alpha \in (0, d)$, $p \in (1, d/\alpha)$ and $1/q = 1/p - \alpha/d$. These results, when the $d$-dimensional Lebesgue measure is replaced by the non-doubling measure $\mu$ as in (1.1), were obtained by García-Cuerva and Martell [11] and by Chen and Sawyer [5], respectively. Moreover, also in this setting with the non-doubling measure $\mu$ as in (1.1), some equivalent characterizations for the boundedness of fractional integrals were established in [17] and the boundedness for the multilinear commutators of fractional integrals with RBMO($\mu$) or $\text{Osc}_{\exp L^r} (\mu)$ functions was presented in [14].

On the other hand, due to the request of applications, as a natural extension of Lebesgue spaces, the Orlicz space was introduced by Birnbaum-Orlicz in [1] and Orlicz in [28]. Since then, the theory of Orlicz spaces and its applications have been well developed (see, for example, [30, 31, 26]).

To state the main results of this paper, we first recall some necessary notions.

The following notion of the geometrically doubling is well known in analysis on metric spaces, which was originally introduced by Coifman and Weiss in [6, pp.66-67] and is also known as metrically doubling (see, for example, [13, p.81]).
Definition 1.1. A metric space \((X,d)\) is said to be \textit{geometrically doubling} if there exists some \(N_0 \in \mathbb{N}\) such that, for any ball \(B(x,r) \subset X\), there exists a finite ball covering \(\{B(x_i,r/2)\}_{i}\) of \(B(x,r)\) such that the cardinality of this covering is at most \(N_0\).

Remark 1.2. Let \((X,d)\) be a metric space. In [18], Hytönen showed that the following statements are mutually equivalent:

(i) \((X,d)\) is geometrically doubling.

(ii) For any \(\epsilon \in (0,1)\) and any ball \(B(x,r) \subset X\), there exists a finite ball covering \(\{B(x_i,\epsilon r)\}_i\) of \(B(x,r)\) such that the cardinality of this covering is at most \(N_0\%\), here and in what follows, \(N_0\) is as in Definition 1.1 and \(n:= \log_2 N_0\).

(iii) For every \(\epsilon \in (0,1)\), any ball \(B(x,r) \subset X\) contains at most \(N_0\%\) centers of disjoint balls \(\{B(x_i,\epsilon r)\}_i\).

(iv) There exists \(M \in \mathbb{N}\) such that any ball \(B(x,r) \subset X\) contains at most \(M\) centers \(\{x_i\}_i\) of disjoint balls \(\{B(x_i,r/4)\}_{i=1}^M\).

Recall that spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [6, pp.66-68].

The following notion of upper doubling metric measure spaces was originally introduced by Hytönen [18] (see also [19, 25]).

Definition 1.3. A metric measure space \((X,d,\mu)\) is said to be upper doubling if \(\mu\) is a Borel measure on \(X\) and there exist a dominating function \(\lambda : X \times (0,\infty) \rightarrow (0,\infty)\) and a positive constant \(C_\lambda\), depending on \(\lambda\), such that, for each \(x \in X\), \(r \rightarrow \lambda(x,r)\) is non-decreasing and, for all \(x \in X\) and \(r \in (0,\infty)\),

\[
\mu(B(x,r)) \leq \lambda(x,r) \leq C_\lambda \lambda(x,r/2).
\]

A metric measure space \((X,d,\mu)\) is called a non-homogeneous metric measure space if \((X,d)\) is geometrically doubling and \((X,d,\mu)\) upper doubling.

Remark 1.4. (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function \(\lambda(x,r) := \mu(B(x,r))\). On the other hand, the Euclidean space \(\mathbb{R}^d\) with any Radon measure \(\mu\) as in (1.1) is also an upper doubling space by taking the dominating function \(\lambda(x,r) := C_0 r^n\).

(ii) Let \((X,d,\mu)\) be upper doubling with \(\lambda\) being the dominating function on \(X \times (0,\infty)\) as in Definition 1.3. It was proved in [21] that there exists another dominating function \(\overline{\lambda}\) such that \(\overline{\lambda} \leq \lambda, C_{\overline{\lambda}} \leq C_\lambda\) and, for all \(x, y \in X\) with \(d(x,y) \leq r\),

\[
\overline{\lambda}(x,r) \leq C_{\overline{\lambda}} \overline{\lambda}(y,r).
\]

(iii) It was shown in [33] that the upper doubling condition is equivalent to the \textit{weak growth condition}: there exist a dominating function \(\lambda : X \times (0,\infty) \rightarrow (0,\infty)\), with \(r \rightarrow \lambda(x,r)\) non-decreasing, positive constants \(C_\lambda\), depending on \(\lambda\), and \(\epsilon\) such that

(a) for all \(r \in (0,\infty), t \in [0,r], x, y \in X\) and \(d(x,y) \in [0,r]\),

\[
|\lambda(y,r + t) - \lambda(x,r)| \leq C_\lambda \left[\frac{d(x,y) + t}{r}\right]^\epsilon \lambda(x,r);
\]
(b) for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$
\mu(B(x, r)) \leq \lambda(x, r).
$$

Based on Remark 1.4(ii), from now on, we always assume that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space with the dominating function $\lambda$ satisfying (1.3).

We now recall the notion of the coefficient $K_{B,S}$ introduced by Hytönen [18], which is analogous to the quantity $K_{Q,R}$ introduced by Tolsa [36, 37]. It is well known that $K_{B,S}$ well characterizes the geometrical properties of balls $B$ and $S$.

**Definition 1.5.** For any two balls $B \subset S$, define

$$
K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x),
$$

where $c_B$ is the center of the ball $B$.

**Remark 1.6.** The following discrete version, $\tilde{K}_{B,S}$, of $K_{B,S}$ defined in Definition 1.5, was first introduced by Bui and Duong [2] in non-homogeneous metric measure spaces, which is more close to the quantity $K_{Q,R}$ introduced by Tolsa [35] in the setting of non-doubling measures. For any two balls $B \subset S$, let $\tilde{K}_{B,S}$ be defined by

$$
\tilde{K}_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(6^k B)}{\lambda(c_B, 6^k r_B)},
$$

where $r_B$ and $r_S$ respectively denote the radii of the balls $B$ and $S$, and $N_{B,S}$ the smallest integer satisfying $6^{N_{B,S}} r_B \geq r_S$. Obviously, $K_{B,S} \lesssim \tilde{K}_{B,S}$. As was pointed by Bui and Duong [2], in general, it is not true that $K_{B,S} \sim \tilde{K}_{B,S}$.

Though the measure doubling condition is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [18] that there exist still many balls which have the following $(\eta, \beta)$-doubling property.

**Definition 1.7.** Let $\eta, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be $(\eta, \beta)$-doubling if $\mu(\eta B) \leq \beta \mu(B)$.

To be precise, it was proved in [18, Lemma 3.2] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\eta, \beta \in (1, \infty)$ satisfying $\beta > C_\lambda^\log_2 \eta =: \eta'$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ such that $\eta^j B$ is $(\eta, \beta)$-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \eta^n$ with $n := \log_2 N_0$ and $\mu$ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [18, Lemma 3.3] also showed that, for $\mu$-almost every $x \in \mathcal{X}$, there exist arbitrary small $(\eta, \beta)$-doubling balls centered at $x$. Furthermore, the radii of these balls may be chosen to be the form $\eta^j B$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this paper, for any $\eta \in (1, \infty)$ and ball $B$, the smallest $(\eta, \beta_\eta)$-doubling ball of the form $\eta^j B$ with $j \in \mathbb{N}$ is denoted by $B^n$, where

$$
\beta_\eta := \max\{\eta^{3\eta}, \eta^{3\eta'}\} + 30^n + 30^{\eta'} = \eta^{3(\max\{n, \eta\})} + 30^n + 30^{\eta'}.
$$

(1.4)
In what follows, by a doubling ball we mean a $(6, \beta_0)$-doubling ball and $\tilde{B}^6$ is simply denoted by $\tilde{B}$.

Now we recall the following notion of $\text{RBMO}(\mu)$ from [18].

**Definition 1.8.** Let $\rho \in (1, \infty)$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space $\text{RBMO}(\mu)$ if there exist a positive constant $C$ and, for any ball $B \subset \mathcal{X}$, a number $f_B$ such that

\[
\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \, d\mu(x) \leq C
\]

and, for any two balls $B$ and $B_1$ such that $B \subset B_1$,

\[
|f_B - f_{B_1}| \leq CK_{B,B_1}.
\]

The infimum of the positive constants $C$ satisfying both (1.5) and (1.6) is defined to be the $\text{RBMO}(\mu)$ norm of $f$ and denoted by $\|f\|_{\text{RBMO}(\mu)}$.

From [18, Lemma 4.6], it follows that the space $\text{RBMO}(\mu)$ is independent of the choice of $\rho \in (1, \infty)$.

In this paper, we consider a variant of the generalized fractional integrals from [10, Definition 4.1] (see also [17, (1.4)]).

**Definition 1.9.** Let $\alpha \in (0, 1)$. A function $K_\alpha \in L^1_{\text{loc}}(\mathcal{X} \times \mathcal{X} \setminus \{(x, y) : x = y\})$ is called a generalized fractional integral kernel if there exists a positive constant $C_{K_\alpha}$, depending on $K_\alpha$, such that

(i) for all $x, y \in \mathcal{X}$ with $x \neq y$,

\[
|K_\alpha(x, y)| \leq C_{K_\alpha} \frac{1}{\lambda(x, d(x, y))^{1-\alpha}};
\]

(ii) there exist positive constants $\delta \in (0, 1]$ and $c_{K_\alpha} \in (0, \infty)$ such that, for all $x, \tilde{x}, y \in \mathcal{X}$ with $d(x, y) \geq c_{K_\alpha}d(x, \tilde{x})$,

\[
|K_\alpha(x, y) - K_\alpha(\tilde{x}, y)| + |K_\alpha(y, x) - K_\alpha(y, \tilde{x})| \leq C_{K_\alpha} \frac{[d(x, \tilde{x})]^{\delta}}{[d(x, y)]^{\delta}[\lambda(x, d(x, y))]^{1-\alpha}}.
\]

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A linear operator $T_\alpha$ is called a generalized fractional integral with kernel $K_\alpha$ satisfying (1.7) and (1.8) if, for all $f \in L^\infty_b(\mu)$ and $x \notin \text{supp } f$,

\[
T_\alpha f(x) := \int_{\mathcal{X}} K_\alpha(x, y) f(y) \, d\mu(y).
\]

**Remark 1.10.** (i) Without loss of generality, for the simplicity, we may assume in (1.8) that $c_{K_\alpha} \equiv 2$.

(ii) If a kernel $K_\alpha$ satisfies (1.7) and (1.8) with $\alpha = 0$, then $K_\alpha$ is called a standard kernel and the associated operator $T_\alpha$ as in (1.9) is called a Calderón-Zygmund operator on non-homogeneous metric measure spaces (see [20, Subsection 2.3]).
We give a specific example of the generalized fractional integrals, which is a natural variant of the so-called “Bergman-type” operators from [41, Section 2.1] (see also [20, Section 12] and [34, Section 2.2]). Let $\mathcal{X} := \mathbb{B}_{2d}$ be the open unit ball in $\mathbb{C}^d$. Suppose that the measure $\mu$ satisfies the upper power bound $\mu(B(x, r)) \leq r^m$ with $m \in (0, 2d]$ except the case when $B(x, r) \subseteq \mathbb{B}_{2d}$. However, in the exceptional case it holds true that $r \leq \widetilde{d}(x) := d(x, \mathbb{C}^d \setminus \mathbb{B}_{2d})$, where $d(x, y) := ||x - y|| + |1 - x \cdot y/|y||$ for all $x, y \in \mathbb{B}_{2d} \subset \mathbb{C}^d$, and hence $\mu(B(x, r)) \leq \max\{\{d(x)\}^m, r^m\} =: \lambda(x, r)$. By similar arguments to those used in the proofs of [34, Proposition 2.13] and [20, Section 2], we conclude that, if $\alpha \in (0, 1)$, then the kernel $K_{m, \alpha}(x, y) := (1 - x \cdot y)^{-m(1 - \alpha)}$, $x, y \in \mathbb{B}_{2d} \subset \mathbb{C}^d$, satisfies the conditions (1.7) and (1.8). So, when $\alpha \in (0, 1)$, the fractional integral $T_{m, \alpha}$, associated with $K_{m, \alpha}$, is an example of the generalized fractional integrals as in Definition 1.9. Recall that, when $\alpha = 0$, the operator $T_{m, 0}$, associated with $K_{m, 0}$, is just the so-called “Bergman-type” operator (see [34, 41, 20]).

Now we recall the notion of the atomic Hardy space from [21].

**Definition 1.11.** Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(p, 1)_\lambda$-atomic block if

(i) there exists a ball $B$ such that $\text{supp} b \subset B$;

(ii) $\int_{\mathbb{X}} b(x) \, d\mu(x) = 0$;

(iii) for any $j \in \{1, 2\}$, there exist a function $a_j$ supported on ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and $||a_j||_{L^p(\mu)} \leq [\mu(B_j)]^{1/p - 1} K_{B_j, B}^{-1}$. Moreover, let $|b|_{H^1_{\text{at}}(\mu)} := |\lambda_1| + |\lambda_2|$

A function $f \in L^1(\mu)$ is said to belong to the atomic Hardy space $H^1_{\text{at}}(\mu)$ if there exist $(p, 1)_\lambda$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and $\sum_{i=1}^\infty |b_i|_{H^1_{\text{at}}(\mu)} < \infty$. The $H^1_{\text{at}}(\mu)$ norm of $f$ is defined by $\|f\|_{H^1_{\text{at}}(\mu)} := \inf\{\sum_{i=1}^\infty |b_i|_{H^1_{\text{at}}(\mu)}\}$, where the infimum is taken over all the possible decompositions of $f$ as above.

**Remark 1.12.** (i) It was proved in [21] that, for each $p \in (1, \infty]$, the atomic Hardy space $H^1_{\text{at}}(\mu)$ is independent of the choice of $\rho$ and that, for all $p \in (1, \infty]$, the spaces $H^1_{\text{at}}(\mu)$ and $H^1_{\text{at}}(\mu)$ coincide with equivalent norms. Thus, in what follows, we denote $H^1_{\text{at}}(\mu)$ simply by $H^1(\mu)$ and, unless explicitly pointed out, we always assume that $\rho = 2$ in Definition 1.11.

(ii) It was proved in [25, Remark 1.3(ii)] that the atomic Hardy space introduced by Bui and Duong [2] and the atomic Hardy space in Definition 1.11 coincide with equivalent norms.

Then we state the first main theorem of this paper.

**Theorem 1.13.** Let $\alpha \in (0, 1)$ and $T_\alpha$ be as in (1.9) with kernel $K_\alpha$ satisfying (1.7) and (1.8). Then the following statements are equivalent:

(I) $T_\alpha$ is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$;

(II) $T_\alpha$ is bounded from $L^1(\mu)$ into $L^{1/(1-\alpha)}(\mu)$;

(III) There exists a positive constant $C$ such that, for all $f \in L^{1/\alpha}(\mu)$ with $T_\alpha f$ being finite almost everywhere, $\|T_\alpha f\|_{\text{RBMO}(\mu)} \leq C \|f\|_{L^{1/\alpha}(\mu)}$.
(IV) \( T_\alpha \) is bounded from \( H^1(\mu) \) into \( L^{1/(1-\alpha)}(\mu) \);
(V) \( T_\alpha \) is bounded from \( H^1(\mu) \) into \( L^{1/(1-\alpha)\infty}(\mu) \).

Remark 1.14. Theorem 1.13 covers [17, Theorem 1.1] by taking \( X := \mathbb{R}^d \), \( d \) being the usual Euclidean metric and \( \mu \) as in (1.1). The difference between Theorem 1.13 and [17, Theorem 1.1] exists in that no conclusion of Theorem 1.13 is known to be true, while all conclusions of [17, Theorem 1.1] are true.

Let \( \Phi \) be a convex Orlicz function on \([0, \infty)\), namely, a convex increasing function satisfying \( \Phi(0) = 0, \Phi(t) > 0 \) for all \( t \in (0, \infty) \) and \( \Phi(t) \to \infty \) as \( t \to \infty \). Let

\[
(1.10) \quad a_\Phi := \inf_{t \in (0, \infty)} \frac{t \Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi := \sup_{t \in (0, \infty)} \frac{t \Phi'(t)}{\Phi(t)}.
\]

We refer to [26] for more properties of \( a_\Phi \) and \( b_\Phi \).

The Orlicz space \( L^\Phi(\mu) \) is defined to be the space of all measurable functions \( f \) on \((X, d, \mu)\) such that \( \int_X \Phi(|f(x)|) \, d\mu(x) < \infty \); moreover, for any \( f \in L^\Phi(\mu) \), its Luxemburg norm in \( L^\Phi(\mu) \) is defined by

\[
\|f\|_{L^\Phi(\mu)} := \inf \left\{ t \in (0, \infty) : \int_X \Phi(|f(x)|/t) \, d\mu(x) \leq 1 \right\}.
\]

For any sequence \( \vec{b} := (b_1, \ldots, b_k) \) of functions, the multilinear commutator \( T_{\alpha, \vec{b}} \) of the generalized fractional integral \( T_\alpha \) with \( \vec{b} \) is defined by setting, for all suitable functions \( f \),

\[
(1.11) \quad T_{\alpha, \vec{b}} f := \left[ b_k, \ldots, [b_1, T_\alpha] \right] f,
\]

where

\[
(1.12) \quad [b_1, T_\alpha] f := b_1 T_\alpha f - T_\alpha (b_1 f).
\]

The second main result of this paper is the following boundedness of the multilinear commutator \( T_{\alpha, \vec{b}} \) on Orlicz spaces.

Theorem 1.15. Let \( \alpha \in (0, 1) \), \( k \in \mathbb{N} \) and \( b_j \in \text{RBMO}(\mu) \) for all \( j \in \{1, \ldots, k\} \). Let \( \Phi \) be a convex Orlicz function and \( \Psi \) defined, via its inverse, by setting, for all \( t \in (0, \infty) \), \( \Psi^{-1}(t) := \Phi^{-1}(t)^{1-\alpha} \), where \( \Phi^{-1}(t) := \inf\{s \in (0, \infty) : \Phi(s) > t\} \). Suppose that \( T_\alpha \) is as in (1.9), with kernel \( K_\alpha \) satisfying (1.7) and (1.8), which is bounded from \( L^p(\mu) \) into \( L^q(\mu) \) for all \( p \in (1, 1/\alpha) \) and \( 1/q = 1/p - \alpha \). If \( 1 < a_\Phi \leq b_\Phi < \infty \) and \( 1 < a_\Psi \leq b_\Psi < \infty \), then the multilinear commutator \( T_{\alpha, \vec{b}} \) as in (1.11) is bounded from \( L^\Phi(\mu) \) to \( L^\Psi(\mu) \), namely, there exists a positive constant \( C \) such that, for all \( f \in L^\Phi(\mu) \),

\[
\|T_{\alpha, \vec{b}} f\|_{L^\Psi(\mu)} \leq C \prod_{j=1}^k \|b_j\|_{\text{RBMO}(\mu)} \|f\|_{L^\Phi(\mu)}.
\]
Remark 1.16. (i) Let all the notation be the same as in Theorem 1.15. By Theorem 1.13, we can, in Theorem 1.15, replace the assumption that $T_\alpha$ is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$ by any one of the statements (II)-(V) in Theorem 1.13.

(ii) In Theorem 1.15, if $p \in (1, 1/\alpha)$ and $\Phi(t) := t^p$ for all $t \in (0, \infty)$, then $\Psi(t) = t^q$ and $1/q = 1/p - \alpha$. In this case, $a_\Phi = b_\Phi = p \in (1, \infty)$, $a_\Psi = b_\Psi = q \in (1, \infty)$, $L_\Phi^q(\mu) = L^p(\mu)$ and $L_\Psi^q(\mu) = L^q(\mu)$. Thus, Theorem 1.15, even when $\mathcal{X} := \mathbb{R}^d$, $d$ being the usual Euclidean metric and $\mu$ as in (1.1), also contains [14, Theorem 1.1] as a special case. In the non-homogenous setting, Theorem 1.15, even when $k = 1$, is also new.

The end point counterpart of Theorem 1.15 is also considered in this paper. To this end, we first recall the following Orlicz type space $\text{Osc}_\exp L^r(\mu)$ of functions (see, for example, Pérez and Trujillo-González [29] for Euclidean spaces and [14] for non-doubling measures).

In what follows, let $L^1_{\text{loc}}(\mu)$ be the space of all locally $\mu$-integrable functions on $\mathcal{X}$. For all balls $B$ and $f \in L^1_{\text{loc}}(\mu)$, $m_B(f)$ denotes the mean value of $f$ on ball $B$, namely,

$$(1.13) \quad m_B(f) := \frac{1}{\mu(B)} \int_B f(y) \, d\mu(y).$$

Definition 1.17. Let $r \in [1, \infty)$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to belong to the space $\text{Osc}_\exp L^r(\mu)$ if there exists a positive constant $C_1$ such that

(i) for all balls $B$,

$$\|f - m_B(f)\|_{\exp L^r, B, \mu/\mu(2B)} := \inf \left\{ \lambda \in (0, \infty) : \frac{1}{\mu(2B)} \int_B \exp \left( \frac{|f - m_B(f)|}{\lambda} \right)^r \, d\mu \leq 2 \right\} \leq C_1;$$

(ii) for all doubling balls $Q \subset R$, $|m_Q(f) - m_R(f)| \leq C_1 K_{Q,R}.$

The $\text{Osc}_\exp L^r(\mu)$ norm of $f$, $\|f\|_{\text{Osc}_\exp L^r(\mu)}$, is then defined to be the infimum of all positive constants $C_1$ satisfying (i) and (ii).

Remark 1.18. Obviously, for any $r \in [1, \infty)$, $\text{Osc}_\exp L^r(\mu) \subset \text{RBMO}(\mu)$. Moreover, from [18, Corollary 6.3], it follows that $\text{Osc}_\exp L^1(\mu) = \text{RBMO}(\mu)$.

We recall some notation from [15]. For $i \in \{1, \ldots, k\}$, the family of all finite subsets $\sigma := \{\sigma(1), \ldots, \sigma(i)\}$ of $\{1, \ldots, k\}$ with $i$ different elements is denoted by $C_i^k$. For any $\sigma \in C_i^k$, the complementary sequence $\sigma'$ is defined by $\sigma' := \{1, \ldots, k\} \setminus \sigma$. For any $\sigma := \{\sigma(1), \ldots, \sigma(i)\} \subset C_i^k$ and $k$-tuple $r := (r_1, \ldots, r_k)$, we write that $1/r_\sigma := 1/r_{\sigma(1)} + \cdots + 1/r_{\sigma(i)}$ and $1/r_{\sigma'} := 1/r - 1/r_\sigma$, where $1/r := 1/r_1 + \cdots + 1/r_k$.

Now we state the third main result of this paper.

Theorem 1.19. Let $\alpha \in (0, 1)$, $k \in \mathbb{N}$, $r_i \in [1, \infty)$ and $b_i \in \text{Osc}_\exp L^{r_i}(\mu)$ for $i \in \{1, \ldots, k\}$. Let $T_\alpha$ and $T_{\alpha,b}$ be, respectively, as in (1.9) and (1.11) with kernel $K_\alpha$ satisfying (1.7) and (1.8). Suppose that $T_\alpha$ is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. Then, there exists a positive constant $C$ such that, for all $\lambda \in (0, \infty)$ and $f \in L^p(\mu)$,

$$
\mu(\{x \in \mathcal{X} : |T_{\alpha,b}f(x)| > \lambda\})
$$
there exists \( m \) (see Section 4 below), which is weaker than the assumption introduced by Bui and Duong [8, Lemma 4.1] and the Calderón-Zygmund decomposition mentioned above. 

we need to use the generalized Hölder inequality over the non-homogeneous setting from To obtain the weak type endpoint estimates of multilinear commutators in Theorem 1.19, theorem in [8] on Orlicz spaces and borrowing some ideas from the proof of [15, Theorem 2].

(see Remark 1.4(iii)) introduced by Tan and Li [33], which is equivalent to the upper doubling condition. 

doubling condition. 

satisfying all the assumptions of this article. The key tool is the weak growth condition for a positive constant which is independent of the main parameters, but it may vary from

\[
\leq C \left[ \Phi_{1/r} \left( \prod_{j=1}^{k} \|b_j\|_{\text{osc}_{\exp L^t_j}(\mu)} \right) \right] \left[ \sum_{j=0}^{k} \sum_{\sigma \in C_j^b} \Phi_{1/r_{\sigma}} \left( \|\Phi_{1/r_{\sigma}}(\lambda^{-1}|f|)\|_{L^1(\mu)} \right) \right],
\]

where \( \Phi_s(t) := \log^s(2 + t) \) for all \( t \in (0, \infty) \) and \( s \in (0, \infty) \).

Remark 1.20. Theorem 1.19 covers [17, Theorem 1.1] by taking \( \mathcal{X} := \mathbb{R}^d \), \( d \) being the usual Euclidean metric and \( \mu \) as in (1.1).

The organization of this paper is as follows.

In Section 2, we show Theorem 1.13 by first establishing a new interpolation theorem (see Theorem 2.7 below), which, when \( p_0 = \infty \), is just [23, Theorem 1.1] and whose version on the linear operators over the non-doubling setting is just [17, Lemma 2.3]. Moreover, we prove Theorem 2.7 by borrowing some ideas from the proof of [23, Theorem 1.1], which seals some gaps existing in the proof of [17, Lemma 2.3]. The key tool for the proof of Theorem 2.7 is the Calderón-Zygmund decomposition in the non-homogeneous setting obtained by Bui and Duong [2] (see also Lemma 2.6 below). Again, using the Calderón-Zygmund decomposition (Lemma 2.6) and the interpolation theorem (Theorem 2.7), together with the full applications of the geometrical properties of \( K_{B,S} \) and the underlying space \( (\mathcal{X}, d, \mu) \), we then complete the proof of Theorem 1.13.

Section 3 is devoted to proving Theorems 1.15 and 1.19. We first prove, in Theorem 3.9 below, that, if the generalized fractional integral \( T_\alpha \) \((\alpha \in (0,1))\) is bounded from \( L^p(\mu) \) into \( L^q(\mu) \) for some \( p \in (1,1/\alpha) \) and \( 1/q = 1/p - \alpha \), then so is its commutator with any RBMO(\( \mu \)) function, by borrowing some ideas of [5, Theorem 1]. The main new ingredient appearing in our approach used for the proof of Theorem 3.9 is that we introduce a quantity \( \tilde{K}_{B,S}^{(\alpha)} \), which is a fractional variant of \( \tilde{K}_{B,S} \) and, in the setting of non-doubling measures, was introduced by Chen and Sawyer in [5, Section 1]. As the case \( \tilde{K}_{B,S} \), \( \tilde{K}_{B,S}^{(\alpha)} \) also well characterizes the geometrical properties of balls \( B \) and \( S \) and, moreover, it preserves all the properties of \( K_{Q,R}^{(\beta)} \) in [5, Lemma 3]. To prove Theorem 3.9, we also need to introduce the maximal operator \( \tilde{M}^{#,\alpha} \), associated with \( \tilde{K}_{B,S}^{(\alpha)} \), adapted from the maximal operator \( M^{#,\alpha} \) in [5, Section 2]. Then we complete the proof of Theorem 1.15 by the interpolation theorem in [8] on Orlicz spaces and borrowing some ideas from the proof of [15, Theorem 2]. To obtain the weak type endpoint estimates of multilinear commutators in Theorem 1.19, we need to use the generalized Hölder inequality over the non-homogeneous setting from [8, Lemma 4.1] and the Calderón-Zygmund decomposition mentioned above.

In Section 4, under some weak reverse doubling condition of the dominating function \( \lambda \) (see Section 4 below), which is weaker than the assumption introduced by Bui and Duong in [2, Subsection 7.3]: there exists \( m \in (0, \infty) \) such that, for all \( x \in \mathcal{X} \) and \( a, r \in (0, \infty) \), \( \lambda(x, ar) = a^m \lambda(x, r) \), we construct a non-trivial example of generalized fractional integrals satisfying all the assumptions of this article. The key tool is the weak growth condition (see Remark 1.4(iii)) introduced by Tan and Li [33], which is equivalent to the upper doubling condition.

Finally, we make some conventions on notation. Throughout the whole paper, \( C \) stands for a positive constant which is independent of the main parameters, but it may vary from
line to line. Moreover, we use $C_{\rho, \gamma, \ldots}$ or $C_{(\rho, \gamma, \ldots)}$ to denote a positive constant depending on the parameter $\rho, \gamma, \ldots$. For any ball $B$ and $f \in L^1_{\text{loc}}(\mu)$, $m_B(f)$ denotes the mean value of $f$ over $B$ as in (1.13); the center and the radius of $B$ are denoted, respectively, by $c_B$ and $r_B$. If $f \leq Cg$, we then write $f \lesssim g$; if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any subset $E$ of $X$, we use $\chi_E$ to denote its characteristic function.

2 Proof of Theorem 1.13

In this section, we prove Theorem 1.13. We begin with recalling some useful properties of $\delta$ in Definition 1.9 (see, for example, [18, Lemmas 5.1 and 5.2] and [21, Lemma 2.2]).

**Lemma 2.1.** (i) For all balls $B \subset R \subset S$, $K_{B,R} \leq K_{B,S}$.

(ii) For any $\rho \in [1, \infty)$, there exists a positive constant $C(\rho)$, depending on $\rho$, such that, for all balls $B \subset S$ with $r_S \leq \rho r_B$, $K_{B,S} \leq C(\rho)$.

(iii) For any $\alpha \in (1, \infty)$, there exists a positive constant $C(\alpha)$, depending on $\alpha$, such that, for all balls $B$, $K_{B,B^\alpha} \leq C(\alpha)$.

(iv) There exists a positive constant $c$ such that, for all balls $B \subset R \subset S$,

$$K_{B,S} \leq K_{B,R} + cK_{R,S}.$$  

In particular, if $B$ and $R$ are concentric, then $c = 1$.

(v) There exists a positive constant $\bar{c}$ such that, for all balls $B \subset R \subset S$, $K_{R,S} \leq \bar{c}K_{B,S}$; moreover, if $B$ and $R$ are concentric, then $K_{R,S} \leq K_{B,S}$.

Now we recall the following equivalent characterizations of $\text{RBMO}(\mu)$ established in [21, Proposition 2.10].

**Lemma 2.2.** Let $\rho \in (1, \infty)$ and $f \in L^1_{\text{loc}}(\mu)$. The following statements are equivalent:

(a) $f \in \text{RBMO}(\mu)$;

(b) there exists a positive constant $C$ such that, for all balls $B$,

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - m_B f| \, d\mu(x) \leq C$$

and, for all doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \leq C K_{B,S}. \tag{2.1}$$

Moreover, let $\|f\|_*$ be the infimum of all admissible constants $C$ in (b). Then there exists a constant $\bar{C} \in [1, \infty)$ such that, for all $f \in \text{RBMO}(\mu)$, $\|f\|_* / \bar{C} \leq \|f\|_{\text{RBMO}(\mu)} \leq \bar{C} \|f\|_*$.  

We also need the following conclusion, which is just [8, Corollary 3.3].

**Corollary 2.3.** If $f \in \text{RBMO}(\mu)$, then there exists a positive constant $C$ such that, for any ball $B$, $\rho \in (1, \infty)$ and $r \in [1, \infty)$,

$$\left\{ \frac{1}{\mu(\rho B)} \int_B |f(x) - m_B f|^r \, d\mu(x) \right\}^{1/r} \leq C \|f\|_{\text{RBMO}(\mu)}. \tag{2.2}$$

Moreover, the infimum of the positive constants $C$ satisfying both (2.2) and (2.1) is an equivalent $\text{RBMO}(\mu)$-norm of $f$.  

The following interpolation result is from [8, Theorem 2.2].

**Lemma 2.4.** Let $\alpha \in [0, 1)$, $p_i, q_i \in (0, \infty)$ satisfy $1/q_i = 1/p_i - \alpha$ for $i \in \{1, 2\}$, $p_1 < p_2$ and $T$ be a sublinear operator of weak type $(p_i, q_i)$ for $i \in \{1, 2\}$. Then $T$ is bounded from $L^{p_i}(\mu)$ to $L^{q_i}(\mu)$, where $\Phi$ and $\Psi$ are convex Orlicz functions satisfying the following conditions: $1 < p_1 < a_\Phi \leq b_\Phi < p_2 < \infty$, $1 < q_1 < a_\Psi \leq b_\Psi < q_2 < \infty$ and, for all $t \in (0, \infty)$, $\Psi^{-1}(t) = \Phi^{-1}(t)t^{-\alpha}$.

We also recall some results in [2, Subsection 4.1] and [18, Corollary 3.6].

**Lemma 2.5.** (i) Let $p \in (1, \infty)$, $r \in (1, p)$ and $\rho \in (0, \infty)$. The following maximal operators defined, respectively, by setting, for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$,

$$M_{r,\rho}f(x) := \sup_{Q \ni x} \left[ \frac{1}{\mu(\rho Q)} \int_Q |f(y)|^r \, d\mu(y) \right]^{\frac{1}{r}},$$

$$Nf(x) := \sup_{Q \ni x, Q \text{ doubling}} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y)$$

and

$$M(\rho)f(x) := \sup_{Q \ni x} \frac{1}{\mu(\rho Q)} \int_Q |f(y)| \, d\mu(y),$$

are bounded on $L^p(\mu)$ and also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

(ii) For all $f \in L^1_{\text{loc}}(\mu)$, it holds true that $|f(x)| \leq Nf(x)$ for $\mu$-almost every $x \in \mathcal{X}$.

Before we prove Theorem 1.13, we establish a new interpolation theorem, which is adapted from [23, Theorem 1.1]. To this end, we first recall the following Calderón-Zygmund decomposition theorem obtained by Bui and Duong [2, Theorem 6.3]. Let $\gamma_0$ be a fixed positive constant satisfying that $\gamma_0 > \max \{C^3_\lambda, 6\}$, where $C_\lambda$ is as in (1.2) and $n$ as in Remark 1.2(ii).

**Lemma 2.6.** Let $p \in [1, \infty)$, $f \in L^p(\mu)$ and $t \in (0, \infty)$ ($t > \gamma_0^{1/p} \|f\|_{L^p(\mu)}/|\mu(\mathcal{X})|^{1/p}$ when $\mu(\mathcal{X}) < \infty$). Then

(i) there exists a family of finite overlapping balls $\{6B_j\}_j$ such that $\{B_j\}_j$ is pairwise disjoint,

$$\frac{1}{\mu(6^2B_j)} \int_{B_j} |f(x)|^p \, d\mu(x) > \frac{t^p}{\gamma_0} \text{ for all } j,$$

$$\frac{1}{\mu(6^2\eta B_j)} \int_{\eta B_j} |f(x)|^p \, d\mu(x) \leq \frac{t^p}{\gamma_0} \text{ for all } j \text{ and all } \eta \in (2, \infty)$$

and

$$|f(x)| \leq t \text{ for } \mu-\text{almost every } x \in \mathcal{X} \setminus (\bigcup_j 6B_j);$$
(ii) for each $j$, let $R_j$ be a $(3 \times 6^2, C^{\log_2(3 \times 6^2)+1}_\lambda)$-doubling ball of the family $\{(3 \times 6^2)^k B_j\}_{k \in \mathbb{N}}$, and $\omega_j := \chi_{6B_j}/(\sum_k \chi_{6B_k})$. Then there exists a family $\{\varphi_j\}_j$ of functions such that, for each $j$, $\text{supp}(\varphi_j) \subset R_j$, $\varphi_j$ has a constant sign on $R_j$.

(2.5) \[ \int_{\mathcal{X}} \varphi_j(x) \, d\mu(x) = \int_{6B_j} f(x) \omega_j(x) \, d\mu(x) \]

and

(2.6) \[ \sum_j |\varphi_j(x)| \leq \gamma t \text{ for } \mu\text{-almost } x \in \mathcal{X}, \]

where $\gamma$ is a positive constant depending only on $(\mathcal{X}, \mu)$ and there exists a positive constant $C$, independent of $f$, $t$ and $j$, such that, if $p = 1$, then

(2.7) \[ \|\varphi_j\|_{L^\infty(\mu)} \mu(R_j) \leq C \int_{\mathcal{X}} |f(x)\omega_j(x)| \, d\mu(x) \]

and, if $p \in (1, \infty)$, then

(2.8) \[ \left\{ \int_{R_j} |\varphi_j(x)|^p \, d\mu(x) \right\}^{1/p} \left[ \mu(R_j) \right]^{1/p'} \leq \frac{C}{t^{p-1}} \int_{\mathcal{X}} |f(x)\omega_j(x)|^p \, d\mu(x); \]

(iii) when $p \in (1, \infty)$, if, for any $j$, choosing $R_j$ to be the smallest $(3 \times 6^2, C^{\log_2(3 \times 6^2)+1}_\lambda)$-doubling ball of the family $\{(3 \times 6^2)^k B_j\}_{k \in \mathbb{N}}$, then $h := \sum_j (f\omega_j - \varphi_j) \in H^1(\mu)$ and there exists a positive constant $C$, independent of $f$ and $t$, such that

(2.9) \[ \|h\|_{H^1(\mu)} \leq \frac{C}{t^{p-1}} \|f\|_{L^p(\mu)^p}. \]

Recall that the \textit{sharp maximal operator} $M^\#$ in [2] is defined by setting, for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$,

\[ M^\#f(x) := \sup_{B \ni x } \frac{1}{\mu(6B)} \int_B |f(x) - m_B f| \, d\mu(x) + \sup_{(Q,R) \in \Delta_x} \frac{|m_Q f - m_R f|}{K_{Q,R}}, \]

where $\Delta_x := \{(Q,R) : x \in Q \subset R \text{ and } Q,R \text{ are doubling balls}\}$.

\textbf{Theorem 2.7.} Let $T$ be a bounded sublinear operator from $L^{p_0}(\mu)$ into $\text{RBMO}(\mu)$ and from $H^1(\mu)$ into $L^{p_0,\infty}(\mu)$, where $p_0 \in (1, \infty]$ and $1/p_0 + 1/p_0' = 1$. Then $T$ extends to a bounded linear operator from $L^p(\mu)$ into $L^q(\mu)$, where $p \in (1, p_0)$ and $1/q = 1/p - 1/p_0$.

\textit{Proof.} By the Marcinkiewicz interpolation theorem, it suffices to prove that

(2.10) \[ \mu(\{x \in \mathcal{X} : |Tf(x)| > t\}) \lesssim [t^{-1} \|f\|_{L^p(\mu)}]^q \]

for all $p \in (1, p_0)$ and $1/q = 1/p - 1/p_0$. We consider the following two cases.
Case i) \( \mu(\mathcal{X}) = \infty \). Let \( L^\infty_{b,0}(\mu) := \{ f \in L^\infty_b(\mu) : \int_{\mathcal{X}} f(x) \, d\mu(x) = 0 \} \). Then, by a standard argument, we know that \( L^\infty_{b,0}(\mu) \) is dense in \( L^p(\mu) \) for all \( p \in (1,p_0) \). Let \( r \in (0,1) \). Define \( N_r(g) := \{ N(|g|^r) \}^{1/r} \) for all \( g \in L^\infty_{b,0}(\mu) \). By Lemma 2.5(ii) and a standard density argument, to prove (2.10), it suffices to prove that, for any \( f \in L^\infty_{b,0}(\mu) \), \( p \in (1,p_0) \) and \( 1/q = 1/p - 1/p_0 \),

\[
\sup_{t \in (0,\infty)} t^q \mu(\{ x \in \mathcal{X} : |N_r(Tf)(x)| > t \}) \lesssim \| f \|^q_{L^p(\mu)}.
\]

To this end, for any given \( f \in L^\infty_{b,0}(\mu) \), applying Lemma 2.6 to \( t \) with \( t \) replaced by \( t^{q/p} \), and letting \( R_j \) be as in Lemma 2.6(iii), we see that \( f = g + h \), where \( g := f \chi_{\mathcal{X} \setminus \bigcup B_j} + \sum_j \varphi_j \) and \( h := \sum_j (\omega_j f - \varphi_j) \). By Minkowski's inequality, Hölder's inequality and \( 1/p_0 \), together with (2.4), (2.6) and (2.8) with \( t \) replaced by \( t^{q/p} \), we conclude that

\[
\| g \|_{L^{p_0}(\mu)} \leq \left\| f \chi_{\mathcal{X} \setminus \bigcup B_j} \right\|_{L^{p_0}(\mu)} + \left\| \sum_j \varphi_j \right\|_{L^{p_0}(\mu)}
\]

\[
\lesssim t^{q(\frac{1}{p} - \frac{1}{p_0})} \| f \|^{p/p_0}_{L^{p}(\mu)} + t^{(q/p)/p_0} \left[ \sum_j \| \varphi_j \|_{L^1(\mu)} \right]^{1/p_0}
\]

\[
\lesssim t \| f \|^{p/p_0}_{L^{p}(\mu)} + t^{(q/p)/p_0} \left( \sum_j \| \varphi_j \|_{L^{p}(\mu)} [\mu(R_j)]^{1/p'} \right)^{1/p_0}
\]

\[
\lesssim t \| f \|^{p/p_0}_{L^{p}(\mu)} + t^{(q/p)/p_0} \left( \sum_j \int_{\mathcal{X}} |\omega_j(x)| f(x) \, d\mu(x) \right)^{1/p_0}
\]

For each \( r \in (0,1) \), define \( M_r^\# g := \{ M_r^\#(|g|^r) \}^{1/r} \). Then, from [23, Lemma 3.1], together with the boundedness of \( T \) from \( L^{p_0}(\mu) \) into \( \text{RBMO}(\mu) \) and (2.12), we deduce that

\[
\| M_r^\# Tg \|_{L^\infty(\mu)} \lesssim \| Tg \|_{\text{RBMO}(\mu)} \lesssim \| g \|_{L^{p_0}(\mu)} \lesssim t \| f \|^{p/p_0}_{L^{p}(\mu)}.
\]

Hence, if \( C_0 \) is chosen to be a sufficiently large positive constant, we then see that

\[
\mu \left( \left\{ x \in \mathcal{X} : M_r^\# (Tg)(x) > C_0 t \| f \|^{p/p_0}_{L^{p}(\mu)} \right\} \right) = 0.
\]

On the other hand, since both \( f \) and \( h \) belong to \( H^1(\mu) \), by (2.9) with \( t \) replaced by \( t^{q/p} \), we conclude that \( g \in H^1(\mu) \) and

\[
\| g \|_{H^1(\mu)} \leq \| f \|_{H^1(\mu)} + \| h \|_{H^1(\mu)} \lesssim \| f \|_{H^1(\mu)} + \frac{1}{t^{(p-1)/q/p}} \| f \|^{p/p_0}_{L^{p_0}(\mu)}.
\]

From this, together with the boundedness of \( T \) from \( H^1(\mu) \) into \( L^{p_0}(\mu) \) and [23, Lemma 3.3], we deduce that, for any \( q \) satisfying \( 1/q = 1/p - 1/p_0 \) and \( R \in (0,\infty) \),

\[
\sup_{t \in (0,R)} t^q \mu(\{ x \in \mathcal{X} : N_r(Tg)(x) > t \})
\]
\[ \lesssim \sup_{t \in (0, R)} t^{q/p'} \sup_{\tau \in [t, \infty)} \tau^{p_0} \mu \left( \{ x \in X : |Tg(x)| > \tau \} \right) \]
\[ \lesssim R^{q/p_0} \| Tg \|_{L_{p_0}^q(\mu)} \lesssim R^{q/p_0} \| g \|_{H^1(\mu)} < \infty. \]

From the fact that \( N_r \circ T \) is quasi-linear, (2.14), [23, Lemma 3.2] and (2.13), we deduce that there exists a positive constant \( \tilde{C} \) such that, for all \( f \in L_{p_0}^\infty(\mu) \),

\[
\begin{align*}
(2.15) \quad & \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : N_r(Tf)(x) > \tilde{C}C_0 \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
& \lesssim \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : N_r(Tg)(x) > C_0 \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
& \quad + \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : N_r(Th)(x) > C_0 \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
& \lesssim \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : M_r(Tg)(x) > C_0 \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
& \quad + \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : N_r(Th)(x) > C_0 \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
& \sim \sup_{t \in (0, \infty)} t^q \mu \left( \{ x \in X : N_r(Th)(x) > t \| f \|_{L_{p_0}^p(\mu)} \} \right).
\end{align*}
\]

By the boundedness of \( N \) from \( L^1(\mu) \) into \( L^{1, \infty}(\mu) \) (see Lemma 2.5(i)), the layer cake representation, the boundedness of \( T \) from \( H^1(\mu) \) into \( L_{p_0}^{\infty}(\mu) \) and (2.9) with \( t \) replaced by \( t^{q/p} \), we conclude that

\[
(2.16) \quad \mu \left( \{ x \in X : N_r(Th)(x) > t \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
= \mu \left( \{ x \in X : N(|Th|^r)(x) > t^r \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
\leq \mu \left( \{ x \in X : N(|Th|^r \chi_{x \in X : |Th(x)| > 2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)}})(x) > t^r \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
\lesssim t^{-r} \| f \|_{L_{p_0}^p(\mu)} \int_X |Th|^r \chi \{ x \in X : |Th(x)| > 2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)} \}(x) d\mu(x) \\
\sim t^{-r} \| f \|_{L_{p_0}^p(\mu)} \int_0^{2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)}} s^{r-1} \\
\times \mu \left( \{ x \in X : |Th(x)| > 2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)} \} \right) ds \\
+ \int_{2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)}}^\infty s^{r-1} \mu \left( \{ x \in X : |Th(x)| > s \} \right) ds \\
\lesssim \mu \left( \{ x \in X : |Th(x)| > 2^{-1/r}t \| f \|_{L_{p_0}^p(\mu)} \} \right) \\
+ [|t \| f \|_{L_{p_0}^p(\mu)}|^{-p_0'} \sup_{s \in (0, \infty)} s^{p_0} \mu \left( \{ x \in X : |Th(x)| > s \} \right) \\
\lesssim \| h \|_{H^1(\mu)}^{p_0'} \left[ |t \| f \|_{L_{p_0}^p(\mu)}|^{-p_0} \right] \lesssim t^{-q} \| f \|_{L_{p_0}^p(\mu)}.
\]
which, together with (2.15), completes the proof of (2.11).

Case ii) $\mu(\mathcal{X}) < \infty$. In this case, assume that $f \in L_b^\infty(\mu)$. Notice that, if $t \in (0, t_0]$, where $t_0^\beta := \beta_0 \|f\|^p_{L_p(\mu)} / \mu(\mathcal{X})$, then (2.10) holds true trivially. Thus, we only need to consider the case when $t \in (t_0, \infty)$. Let $N_t$ and $M_t$ be as in Case i). For each $t \in (t_0, \infty)$, applying Lemma 2.6 to $f$ with $t$ replaced by $t^{p/p}$, we then see that $f = g + h$ with $g$ and $h$ as in Case i), which, together with the boundedness of $T$ from $L^{p_0}(\mu)$ into $\text{RBMO}(\mu)$ and [23, Lemma 3.1], shows that (2.13) still holds true for $M_t^*(Tg)$.

We now claim that, for any $r \in (0, 1),\quad (2.17)\quad F := \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |Tg(x)|^r \, d\mu(x) \lesssim \|f\|_{L_p(\mu)}^{rp/p_0},$

where the implicit positive constant only depends on $\mu(\mathcal{X})$ and $r$. To see this, since $\mu(\mathcal{X}) < \infty$, we may regard $\mathcal{X}$ as a ball, then $g_0 := g - \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(x) \, d\mu(x) \in H^1(\mu)$. Precisely, by (2.12), we see that

$\quad (2.18)\quad \|g_0\|_{H^1(\mu)} \lesssim \|f\|_{L_p(\mu)}^{rp/p_0}.$

On the other hand, by Hölder’s inequality, the fact that $T1 \in \text{RBMO}(\mu)$ and the locally integrability of $\text{RBMO}(\mu)$ functions, we conclude that

$\int_{\mathcal{X}} |T1(x)|^r \, d\mu(x) \leq \left( \int_{\mathcal{X}} |T1(x)| \, d\mu(x) \right)^r \mu(\mathcal{X})^{1-r} < \infty.$

From this and the layer cake representation, together with $r \in (0, 1)$, Hölder’s inequality, (2.12), the boundedness of $T$ from $H^1(\mu)$ into $L^{p_0,\infty}(\mu)$ and (2.18), we deduce that

$\int_{\mathcal{X}} |Tg(x)|^r \, d\mu(x) \lesssim \|g_0\|_{H^1(\mu)} + \|g\|_{\text{RBMO}(\mu)} \lesssim \|f\|_{L_\mu(\mu)}^{rp/p_0} \lesssim \|f\|_{L_p(\mu)}^{rp/p_0},$

which implies (2.17).

Observe that $\int_{\mathcal{X}} |Tg(x)|^r - F| \, d\mu(x) = 0$ and, for any $R \in (0, \infty),
\sup_{t \in (0, R)} t^q \mu(\mathcal{X} : N(|Tg|^r - F)(x) > t) \leq R^q \mu(\mathcal{X}) < \infty.$

From this and (2.17), together with [23, Lemma 3.2], $M_t^*(F) = 0$, (2.13) and some arguments similar to those used in the estimates for (2.15) and (2.16), we deduce that
there exists a positive constant $c$ such that

$$
\sup_{t \in (t_0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : N_r(Tf)(x) > cC_0 t \| f \|_{L^p(\mu)}^{p/p_0} \right\} \right)
\lesssim \sup_{t \in (t_0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : N_r(Tg - F)(x) > (C_0 t)^r \| f \|_{L^p(\mu)}^{r/p_0} \right\} \right)
+ \sup_{t \in (t_0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : N_r(Th)(x) > C_0 t \| f \|_{L^p(\mu)}^{p/p_0} \right\} \right)
\lesssim \sup_{t \in (0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : M_{\#}(Tg)(x) > C_0 t \| f \|_{L^p(\mu)}^{p/p_0} \right\} \right)
+ \sup_{t \in (0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : N_r(Th)(x) > C_0 t \| f \|_{L^p(\mu)}^{p/p_0} \right\} \right)
\sim \sup_{t \in (0, \infty)} t^q \mu \left( \left\{ x \in \mathcal{X} : N_r(Th)(x) > t \| f \|_{L^p(\mu)}^{p/p_0} \right\} \right)
\lesssim t^{-q} \| f \|_{L^p(\mu)}^p,
$$

where $C_0$ is chosen to be a sufficiently large positive constant, which completes the proof of Theorem 2.7.

Proof of Theorem 1.13. (I)⇒(II) Let $f \in L^1(\mu)$. Without loss of generality, we may assume that $\| f \|_{L^1(\mu)} = 1$. We denote $1/(1 - \alpha)$ by $q_0$. Applying Lemma 2.6 to $f$ with $p = 1$ and $t$ replaced by $t^{q_0}$, and letting $R_j$ be as in Lemma 2.6(iii), we see that $f = g + h$, where $g := f \chi_{X \setminus \bigcup_{j} 6B_j} + \sum_{j} \varphi_j$ and $h := \sum_{j}(\omega_j f - \varphi_j)$. By (2.7) and the assumption $\| f \|_{L^1(\mu)} = 1$, we easily see that

$$
\| g \|_{L^1(\mu)} \lesssim \| f \|_{L^1(\mu)} \sim 1. \tag{2.19}
$$

From (2.4) and (2.6) with $t$ replaced by $t^{q_0}$, it follows that, for $\mu$-almost every $x \in \mathcal{X},$

$$
|g(x)| \lesssim t^{q_0}. \tag{2.20}
$$

Since $T_\alpha$ is bounded from $L^{p_1}(\mu)$ into $L^{q_1}(\mu)$ for any $p_1 \in (1, 1/\alpha)$ and $1/q_1 = 1/p_1 - \alpha$, by (2.20) and (2.19), we conclude that

$$
\mu(\{ x \in \mathcal{X} : |T_\alpha g(x)| > t \}) \lesssim t^{-q_1} \| T_\alpha g \|_{L^{q_1}(\mu)}^{q_1} \lesssim t^{-q_1} \| g \|_{L^{p_1}(\mu)}^{q_1}
\lesssim t^{-q_1} (t^{q_0})^{(p_1 - 1)q_1/p_1} \lesssim t^{-q_0}. \tag{2.21}
$$

On the other hand, from (2.3) with $p = 1$ and $t$ replaced by $t^{q_0}$, and the fact that $\{B_j\}_j$ is a sequence of pairwise disjoint balls, we deduce that

$$
\mu(\cup_{j} 6^2B_j) \lesssim t^{-q_0} \int_{\mathcal{X}} \| f(y) \| d\mu(y) \lesssim t^{-q_0}. \tag{2.22}
$$

Therefore, to show (II), by $f = g + h$, (2.21) and (2.22), it suffices to prove that

$$
\mu \left( \left\{ x \in \mathcal{X} \setminus (\cup_{j} 6^2B_j) : |T_\alpha h(x)| > t \right\} \right) \lesssim t^{-q_0}. \tag{2.23}
$$
To this end, denote the center of $B_j$ by $x_j$, and let $N_1$ be the positive integer satisfying $R_j = (3 \times 6^2)^{N_1} B_j$. Let $\theta$ be a bounded function with $\|\theta\|_{L^0(\mu)} \leq 1$ whose support is contained in $\mathcal{X} \setminus (\cup_j 6^2 B_j)$. Then

$$
\int_{\mathcal{X} \setminus (\cup_j 6^2 B_j)} |T_\alpha h(x)\theta(x)| \, d\mu(x) \leq \sum_j \int_{\mathcal{X} \setminus 6R_j} |T_\alpha h_j(x)\theta(x)| \, d\mu(x) + \sum_j \int_{6R_j \setminus 6^2 B_j} \cdots
$$

$$=: F_1 + F_2,$$

where $h_j := \omega_j f - \varphi_j$. By (2.5), we see that $\int_{\mathcal{X}} h_j(x) \, d\mu(x) = 0$, which, together with (1.8), Hölder’s inequality and (2.7), further implies that

$$F_1 \leq \sum_j \int_{\mathcal{X} \setminus 6R_j} \int_{\mathcal{X}} |\theta(x)||K_\alpha(x,y) - K_\alpha(x,x_j)||h_j(y)| \, d\mu(y) \, d\mu(x)
$$

$$\lesssim \sum_j \int_{\mathcal{X}} \sum_{i=1}^\infty \sum_{j_{i+1}} \int_{6^{i+1} B_j \setminus 6^i B_j} (\frac{q_j}{2})^\delta |\lambda(x_j, 6^i B_j)|^{-1-\alpha} |\theta(x)| \, d\mu(x) \, |h_j(y)| \, d\mu(y)
$$

$$\lesssim \sum_j \int_{\mathcal{X}} |f(y)\omega_j(y)| \, d\mu(y) \, \sum_{i=1}^\infty 6^{-i\delta} \|\theta\|_{L^0(\mu)} \lesssim 1.$$

For $F_2$, by $h_j := \omega_j f - \varphi_j$, (1.7), Hölder’s inequality and an argument similar to that used in the proof of [8, Lemma 3.5(iii)], together with the boundedness of $T_\alpha$ from $L^{p_2}(\mu)$ into $L^{q_2}(\mu)$ with $p_2 \in (1, 1/\alpha)$ and $1/q_2 = 1/p_2 - \alpha$, we have

$$F_2 \leq \sum_j \int_{6R_j \setminus 6^2 B_j} |\theta(x)||T_\alpha(\omega_j f)(x)| \, d\mu(x) + \sum_j \int_{6R_j} |\theta(x)||T_\alpha \varphi_j(x)| \, d\mu(x)
$$

$$\lesssim \sum_j \int_{6R_j \setminus 6^2 B_j} \frac{|\theta(x)|}{|\lambda(x_j, d(x,x_j))|^{1-\alpha}} \, d\mu(x) \int_{\mathcal{X}} |f(y)\omega_j(y)| \, d\mu(y)
$$

$$+ \sum_j \left[ \int_{6R_j} |T_\alpha \varphi_j(x)|^{q_0} \, d\mu(x) \right]^{1/q_0} \|\theta\|_{L^0(\mu)}
$$

$$\lesssim \sum_j \int_{\mathcal{X}} |f(y)\omega_j(y)| \, d\mu(y) \left[ \sum_{k=1}^{N_1+1} \frac{\mu((3 \times 6^2)^k B_j)}{\lambda(x_j, (3 \times 6^2)^k B_j)} \right]^{1/q_0} \|\theta\|_{L^0(\mu)}
$$

$$+ \sum_j \left[ \int_{6R_j} |T_\alpha \varphi_j(x)|^{q_2} \, d\mu(x) \right]^{1/q_2} [\mu(6R_j)]^{1/q_0 - 1/q_2} \lesssim 1,$$

where we chose $p_2$ and $q_2$ such that $p_2 \in (1, 1/\alpha)$ and $1/q_2 = 1/p_2 - \alpha$. The estimates for $F_1$ and $F_2$ give (2.23), and hence complete the proof of (I)⇒(II).

(II)⇒(III) Indeed, for any $f \in L^{1/\alpha}(\mu)$, to show $T_\alpha f \in RBMO(\mu)$, by the assumption that $T_\alpha f$ is finite almost everywhere, it suffices to show that, for any ball $Q$ and $h_Q = m_Q(T_\alpha(f \chi_{\mathcal{X} \setminus (6/5)Q})),

$$\frac{1}{\mu(6Q)} \int_Q |T_\alpha f(x) - h_Q| \, d\mu(x) \lesssim \|f\|_{L^{1/\alpha}(\mu)}$$

(2.24)
and, for any two balls $Q \subset R$, where $R$ is doubling,

\begin{equation}
|h_Q - h_R| \lesssim K_{Q,R} \|f\|_{L^{1/\alpha} (\mu)}.
\end{equation}

Now we first show (2.24). Write

\[
\frac{1}{\mu(6Q)} \int_Q |T_\alpha f(x) - h_Q| \, d\mu(x) \leq \frac{1}{\mu(6Q)} \int_Q |T_\alpha (f \chi_{(6/5)Q})(x)| \, d\mu(x) + \frac{1}{\mu(6Q)} \int_Q |T_\alpha (f \chi_{(6/5)Q})(x) - h_Q| \, d\mu(x) =: H + I.
\]

Notice that Kolmogorov’s inequality (see, for example, [12, p. 485, Lemma 2.8]) also holds true in the non-homogeneous setting. By Kolmogorov’s inequality, namely, for $0 < p < q < \infty$ and any function $f$,

\[
\|f\|_{L^{q,\infty} (\mu)} \leq \sup_E \|f \chi_E\|_{L^{p}(\mu)}/\|\chi_E\|_{L^{q} (\mu)} \lesssim \|f\|_{L^{q,\infty} (\mu)},
\]

where $1/s = 1/p - 1/q$ and the supremum is taken over all measurable sets $E$ with $0 < \mu(E) < \infty$, together with (II) of Theorem 1.13 and Hölder’s inequality, we know that

\[
H \lesssim \frac{1}{\mu(6Q)} \|\chi_Q\|_{L^{1/\alpha} (\mu)} \|T_\alpha (f \chi_{(6/5)Q})\|_{L^{q,\infty}(\mu)} \lesssim \frac{[\mu(Q)]^\alpha}{\mu(6Q)} \|f \chi_{(6/5)Q}\|_{L^1 (\mu)} \lesssim \|f\|_{L^{1/\alpha} (\mu)}.
\]

To estimate $I$, we write

\[
|T_\alpha (f \chi_{(6/5)Q})(x) - T_\alpha (f \chi_{(6/5)Q})(y)| \\
\leq \int_{6Q \setminus (6/5)Q} |K_\alpha(x,z) - K_\alpha(y,z)||f(z)| \, d\mu(z) \\
= \int_{\chi \setminus 6Q} |K_\alpha(x,z) - K_\alpha(y,z)||f(z)| \, d\mu(z) + \int_{\chi \setminus (6/5)Q} \cdot \cdot \cdot =: I_1 + I_2.
\]

Let $c_Q$ and $r_Q$ be the center and the radius of $Q$, respectively. To estimate $I_1$, from (1.7) and Hölder’s inequality, together with (1.2) and (1.3), it follows that

\[
I_1 \lesssim \int_{6Q \setminus (6/5)Q} \left( \frac{1}{[\lambda(x,d(x,z))]^{1-\alpha}} + \frac{1}{[\lambda(y,d(y,z))]^{1-\alpha}} \right) |f(z)| \, d\mu(z) \\
\lesssim \frac{1}{[\lambda(c_Q,r_Q)]^{1-\alpha}} \int_{6Q} |f(z)| \, d\mu(z) \lesssim \|f\|_{L^{1/\alpha} (\mu)}.
\]

To estimate $I_2$, by (1.8), (1.2), Hölder’s inequality and (1.3), we see that, for any $x, y \in Q$,

\[
I_2 \lesssim \sum_{i=1}^\infty \int_{2^i(6Q) \setminus 2^{i-1}(6Q)} \frac{[d(x,y)]^\delta}{[d(z,y)]^\delta[\lambda(y,d(y,z))]^{1-\alpha}} |f(z)| \, d\mu(z) \\
\lesssim \sum_{i=1}^\infty \int_{2^i(6Q) \setminus 2^{i-1}(6Q)} \frac{d_Q^\delta}{[2^{(i-1)}(6r_Q)]^\delta[\lambda(y,2^{(i-1)}6r_Q)]^{1-\alpha}} |f(z)| \, d\mu(z)
\]

where $d_Q = \max\{r_Q, d(x,y)\}$. Now, notice that $\lambda(2r)$ holds true in the non-homogeneous setting.
\[ \|f\|_{L_1} \lesssim \sum_{i=1}^{\infty} 2^{-(i-1)\alpha} \left[ \frac{\mu(2^i(6Q))}{\lambda(C_4, 2^i(6R))} \right]^{1-\alpha} \|f\|_{L_1} \lesssim \|f\|_{L_1}^{\alpha}. \]

Therefore, \( I \lesssim \|f\|_{L_1}^{\alpha} \).

Combining the estimates for \( H \) and \( I \), we obtain (2.24).

Now we show (2.25) for the chosen \( \{h_Q\}_Q \). Denote \( N_{Q,R} + 1 \) simply by \( N_2 \). Write

\[ |h_Q - h_R| = |m_Q(T_a(f\chi\setminus(6/5)Q)) - m_R(T_a(f\chi\setminus(6/5)R))| \]
\[ \leq |m_Q(T_a(f\chi\setminus(6/5)Q))| + |m_Q(T_a(f\chi\setminus(6/5)Q))| \]
\[ + |m_Q(T_a(f\chi\setminus(6/5)Q)) - m_R(T_a(f\chi\setminus(6/5)Q))| + m_R(T_a(f\chi\setminus(6/5)Q))| \]
\[ := J_1 + J_2 + J_3 + J_4. \]

An argument similar to that used in the estimate for \( H \) shows that \( J_4 \lesssim \|f\|_{L_1}^{\alpha} \).

Also, an argument similar to that used in the estimate for \( I \) gives us that \( J_3 \lesssim \|f\|_{L_1}^{\alpha} \).

Next we estimate \( J_2 \). For any \( x \in Q \), by Hölder’s inequality, the fact that \( 6^{N_2}Q \subset 72R \) and (ii) and (iv) of Lemma 2.1, we have

\[ \left| T_a(f\chi_{6^{N_2}Q\setminus6Q})(x) \right| \lesssim \int_{6^{N_2}Q\setminus6Q} \frac{1}{\lambda(x, d(x, z))} d\mu(z) \]|f|_{L_1}^{\alpha} \lesssim K_{Q,R}\|f\|_{L_1}^{\alpha} \lesssim K_{Q,R}\|f\|_{L_1}^{\alpha}. \]

This implies that \( J_2 \lesssim K_{Q,R}\|f\|_{L_1}^{\alpha} \). Similarly, we have

\[ J_1 \lesssim K_{Q,R}\|f\|_{L_1}^{\alpha} \lesssim K_{Q,R}\|f\|_{L_1}^{\alpha}. \]

Combining the estimates for \( J_1, J_2, J_3 \) and \( J_4 \), we obtain (2.25) and hence complete the proof of (II)⇒(III).

(III)⇒(IV) We first show that, for any ball \( B \), bounded function \( a \) supported on \( B \) and \( q_0 := 1/(1 - \alpha) \),

\[ \int_B |T_a(x)|^{q_0} d\mu(x) \lesssim [\mu(2B)]^{q_0} \|a\|_{L_{-\infty}}^{q_0}. \]  

To prove this, we borrow some ideas from the proof of [25, Lemma 3.1] by considering the following two cases for \( r_B \).

Case (i) \( r_B \leq \text{diam}(\text{supp}\mu)/40 \), where \( \text{diam}(\text{supp}\mu) \) denotes the diameter of the set \( \text{supp}\mu \). By Corollary 2.3 and (III) of Theorem 1.13, we have

\[ \int_B |T_a(x)|^{q_0} d\mu(x) \lesssim \mu(2B)\|a\|_{L_1}^{q_0} \lesssim [\mu(2B)]^{q_0} \|a\|_{L_{-\infty}}^{q_0}. \]

Thus, by (2.27), to prove (2.26), it suffices to show that

\[ |m_{B}(T_a)| \lesssim [\mu(2B)]^{q_0} \|a\|_{L_{-\infty}}. \]

We first claim that there exists \( j_0 \in \mathbb{N} \) such that

\[ \mu(6^{j_0}B \setminus 2B) > 0. \]
Indeed, if, for all \( j \in \mathbb{N} \), \( \mu(6^j B \setminus 2B) = 0 \), then we see that \( \mu(X \setminus 2B) = 0 \), which implies that \( \text{supp} \mu \subset \overline{2B} \), the closure of \( 2B \). This contradicts to that \( r_B \leq \text{diam(supp} \mu)/40 \) and thus \( (2.29) \) holds true. Now assume that \( S \) is the smallest ball of the form \( 6^j B \) such that \( \mu(S \setminus 2B) > 0 \). We then know that \( \mu(6^{-1}S \setminus 2B) = 0 \) and \( \mu(S \setminus 2B) > 0 \). Thus, \( \mu(S \setminus (6^{-1}S \cup 2B)) > 0 \). By this and [18, Lemma 3.3], we choose \( x_0 \in S \setminus (6^{-1}S \cup 2B) \) such that the ball centered at \( x_0 \) with the radius \( 6^{-k}r_S \) for some \( k \geq 2 \) is doubling. Let \( B_0 \) be the biggest ball of this form. Then we see that \( B_0 \subset 2S \) and \( \text{dist}(B_0, B) \geq r_B \). We now claim that

\[
(2.30) \quad K_{B, 2S} \lesssim 1.
\]

Indeed, if \( S = 6B \), then by Lemma 2.1(ii), we have \( (2.30) \). If \( S \supset 6^2B \), then \((1/12)S \supset 2B \). Notice that, in this case, \( \mu(6^{-1}S \setminus 2B) = 0 \) implies that \( K_{2B,(1/12)S} = 1 \). By this, together with (iv) and (ii) of Lemma 2.1, we further have

\[
K_{B, 2S} \lesssim K_{B, 2B} + K_{2B,(1/12)S} + K_{(1/12)S, 2S} \lesssim K_{B, 2B} + K_{(1/12)S, 2S} \lesssim 1.
\]

Thus, \( (2.30) \) also holds true in this case, which shows \( (2.30) \). Moreover, assume that \( r_{B_0} = 6^{-k_0}r_S \), where \( k_0 \geq 2 \), and there exists \( N \in \mathbb{N} \) such that \( 6B_0 = 6^{N+1}B_0 \). By the definition of \( B_0 \), we see that \( N - k_0 + 1 \geq -1 \), hence \( r_{6(6B_0)} \geq r_S \) and \( 2S \subset 24(6B_0) \). Therefore, by (i) through (iv) of Lemma 2.1, we see that

\[
(2.31) \quad K_{B_0, 2S} \lesssim K_{B_0, 24(6B_0)} \lesssim K_{B_0, 6B_0} + K_{6B_0, 24(6B_0)} \lesssim 1.
\]

By (2.1), (2.31), Lemma 2.1(iii) and Theorem 1.13(III), we know that

\[
(2.32) \quad |m_{B_0}(T_\alpha a) - m_{\hat{B}}(T_\alpha a)| \\
\leq |m_{B_0}(T_\alpha a) - m_{2S}(T_\alpha a)| + |m_{2S}(T_\alpha a) - m_B(T_\alpha a)| + |m_B(T_\alpha a) - m_{\hat{B}}(T_\alpha a)| \\
\lesssim (K_{B_0, 2S} + K_{B, 2S} + K_{B, \hat{B}}) ||T_\alpha a||_{\text{RBMO}(\mu)} \\
\lesssim ||a||_{L^{1/\alpha}(\mu)} \lesssim [\mu(2B)]^{\alpha} ||a||_{L^\infty(\mu)},
\]

Moreover, by (1.7), \( \text{dist}(B_0, B) \geq r_B \), (1.2) and (1.3), we conclude that, for all \( y \in B_0 \),

\[
(2.33) \quad |T_\alpha a(y)| \lesssim \frac{\mu(B)}{[\lambda(c_B, r_B)]^{1-\alpha}} ||a||_{L^\infty(\mu)} \lesssim [\mu(2B)]^{\alpha} ||a||_{L^\infty(\mu)}.
\]

The estimate (2.28) follows from (2.32) and (2.33), which completes the proof of (2.26) in this case.

Case (ii) \( r_B > \text{diam(supp} \mu)/40 \). In this case, without loss of generality, we may assume that \( r_B \leq 8\text{diam(supp} \mu) \). Then, by Remark 1.2(ii), we see that \( B \cap \text{supp} \mu \) is covered by finite number balls \( \{B_j \}_{j=1}^J \) with radius \( r_B/800 \), where \( J \in \mathbb{N} \) is independent of \( r_B \). For any \( j \in \{1, \ldots, J\} \), we define \( a_j := \frac{x_B}{\sum_{k=1}^J x_B} a \). Since (2.26) is true if we replace \( B \) by \( 2B \), which contains the support of \( a_j \) by (1.7), (2.26), (1.3), (1.2) and the fact that, if \( B \cap B_j \neq \emptyset \), then \( 4B_j \subset 2B \), we have

\[
\int_B |T_\alpha a(x)|^{q_0} d\mu(x)
\]
\[
\lesssim \sum_{j=1}^{J} \int_{B_{2,j}} |T_{\alpha}a(x)|^{q_0} \, d\mu(x) + \sum_{j=1}^{J} \int_{2B_{j}} \cdots \\
\lesssim \sum_{j=1}^{J} \int_{B_{2,j}} \left[ \int_{B_{j}} \frac{|a_j(y)|}{\lambda(x,d(x,y))^{1-\alpha}} \, d\mu(y) \right]^{q_0} \, d\mu(x) + \sum_{j=1}^{J} \|a_j\|_{L^{q_0}(\mu_b)}^{q_0} \|\mu(4B_j)\|^{q_0} \\
\lesssim \sum_{j=1}^{J} \|a_j\|_{L^{q_0}(\mu)}^{q_0} \left\{ \left[ \frac{\mu(B_j)}{(\lambda(c_{B_j},r_{B_j}))^{1-\alpha}} \right]^{q_0} \mu(B) + \|\mu(4B_j)\|^{q_0} \right\} \\
\lesssim \sum_{j=1}^{J} \|a_j\|_{L^{q_0}(\mu)}^{q_0} \{ \|\mu(2B)\|^{q_0} \mu(B) + \|\mu(4B_j)\|^{q_0} \} \lesssim \|a\|_{L^{q_0}(\mu)}^{q_0} \|\mu(2B)\|^{q_0}.
\]

Thus, (2.26) also holds true in this case.

Now we turn to prove (IV). By a standard argument (see [21, Theorem 4.1] for the details), it suffices to show that, for any \((\infty, 1)\)\_{\chi}-atomic block \(b\),

\[(2.34) \quad \|T_{\alpha}b\|_{L^{q_0}(\mu)} \lesssim \|b\|_{H^{1,\infty}_{\text{atb}}(\mu)}.
\]

Assume that \(\text{supp} \ b \subset R\) and \(b = \sum_{j=1}^{2} \lambda_j a_j\), where, for \(j \in \{1, 2\}\), \(a_j\) is a function supported in \(B_j \subset R\) such that \(\|a_j\|_{L^{q_0}(\mu)} \lesssim \|\mu(4B_j)\|^{-1} K_{B_j,R}^{-1}\) and \(|\lambda_1| + |\lambda_2| \sim \|b\|_{H^{1,\infty}_{\text{atb}}(\mu)}\).

Write
\[
\int_{\mathcal{X}} |T_{\alpha}b(x)|^{q_0} \, d\mu(x) = \int_{2R} |T_{\alpha}b(x)|^{q_0} \, d\mu(x) + \int_{\mathcal{X} \setminus 2R} \cdots =: L_1 + L_2.
\]

For \(L_1\), we see that
\[
L_1 \lesssim \sum_{j=1}^{2} \int_{2B_j} |T_{\alpha}a_j(x)|^{q_0} \, d\mu(x) + \sum_{j=1}^{2} \int_{2R \setminus 2B_j} \cdots =: L_{1,1} + L_{1,2}.
\]

From (2.26), \(\|a_j\|_{L^{q_0}(\mu)} \lesssim \|\mu(B_j)\|^{-1} K_{B_j,R}^{-1}\) for \(j \in \{1, 2\}\), and Definition 1.11(iii), it follows that
\[
L_{1,1} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \|a_j\|_{L^{q_0}(\mu)}^{q_0} \|\mu(4B_j)\|^{q_0} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \lesssim \|b\|_{H^{1,\infty}_{\text{atb}}(\mu)}^{q_0}.
\]

For \(L_{1,2}\), by (1.7), Minkowski’s inequality, (1.2), (1.3), (ii) and (iv) of Lemma 2.1, the fact that \(\|a_j\|_{L^{q_0}(\mu)} \lesssim \|\mu(B_j)\|^{-1} K_{B_j,R}^{-1}\) and Definition 1.11(iii), we see that
\[
L_{1,2} \lesssim \sum_{j=1}^{2} \int_{2R \setminus 2B_j} \left\{ \int_{B_j} \frac{|a_j(y)|}{\lambda(x,d(x,y))^{1-\alpha}} \, d\mu(y) \right\}^{q_0} \, d\mu(x) \\
\lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \left\{ \int_{B_j} |a_j(y)| \left[ \int_{2R \setminus 2B_j} \frac{1}{\lambda(x,d(x,y))} \, d\mu(x) \right]^{1/q_0} \, d\mu(y) \right\}^{q_0}.
\]
\[ \lesssim \sum_{j=1}^{2} |\lambda_j|^{90} \|\mu(B_j)\|^{90} a_j \|f\|_{L^\infty(\mu)} \int_{2R \setminus 2B_j} \frac{1}{\lambda(c_{B_j}, d(x, c_{B_j}))} d\mu(x) \]
\[ \lesssim \sum_{j=1}^{2} |\lambda_j|^{90} \|\mu(B_j)\|^{90} a_j \|f\|_{L^\infty(\mu)} K_{B_j, R} \lesssim \sum_{j=1}^{2} |\lambda_j|^{90} \lesssim |b|_{H^{1, \infty}_a(\mu)}^{90}. \]

Therefore, \( L_1 \lesssim |b|_{H^{1, \infty}_a(\mu)}^{90}. \)

On the other hand, from the fact that \( \int_X b(y) d\mu(y) = 0, \) (1.8) and Definition 1.11(iii), we deduce that
\[ L_2 \leq \int_{X \setminus 2R} \left[ \int_R |K_\alpha(x, y) - K_\alpha(x, c_R)| |b(y)| d\mu(y) \right] d\mu(x) \]
\[ \lesssim \left[ \int_R |b(y)| d\mu(y) \right] \sum_{i=1}^{\infty} \int_{2^{i+1}R \setminus 2^i R} \frac{r_{\delta q_0}}{\lambda(c_R, d(x, c_R)) |d(x, c_R)|^{\delta q_0}} d\mu(x) \]
\[ \lesssim \left( |\lambda_1| + |\lambda_2| \right) \sum_{i=1}^{\infty} 2^{-iq_0} \lesssim |b|_{H^{1, \infty}_a(\mu)}^{q_0}, \]

which, together with the estimate for \( L_1, \) implies (2.34) and hence completes the proof of (III) \( \Rightarrow \) (IV).

(IV) \( \Rightarrow \) (V) is obvious, the details being omitted.

(V) \( \Rightarrow \) (I) We first claim that, for any ball \( B \) and \( f \in L^1(\mu) \) with bounded support in \((6/5)B, \)
\[ \frac{1}{\mu(6B)} \int_B |T_\alpha f(y)| d\mu(y) \lesssim \|f\|_{L^{1/\alpha}(\mu)}. \]

Assume first that \( r_B \leq \text{diam(supp } \mu)/40. \) We consider the same construction in the proof of (III) \( \Rightarrow \) (IV). Let \( B, B_0 \) and \( S \) be the same as there. We know that \( B, B_0 \subset 2S, B_0 \) is doubling, \( K_{B, 2S} \lesssim 1, K_{B_0, 2S} \lesssim 1 \) and \( \text{dist}(B_0, B) \gtrsim r_B. \) Let \( g = f + C_{B_0} \chi_{B_0}, \) where \( C_{B_0} \) is a constant such that \( \int_X g(x) d\mu(x) = 0. \) Then \( g \) is an \((\infty, 1)\) \( \lambda \)-atomic block supported in \( R. \) It is easy to show that
\[ \|g\|_{H^{1}(\mu)} \lesssim [\mu(6B)]^{1/q_0} \|f\|_{L^{1/\alpha}(\mu)}, \]

where \( q_0 := 1/(1 - \alpha). \) For \( y \in B, \) by (1.7), the fact that \( \text{dist}(B_0, B) \gtrsim r_B, \) (1.3), \( \int_X g(x) d\mu(x) = 0, \) Hölder’s inequality and (1.2), we have
\[ \|T_\alpha(C_{B_0} \chi_{B_0})(y)| \]
\[ \lesssim |C_{B_0}| \int_{B_0} \frac{1}{\lambda(y, d(x, y))} d\mu(x) \lesssim \frac{|C_{B_0}| \mu(B_0)}{\lambda(c_B, r_B)} \]
\[ \lesssim \|f\|_{L^{1}(\mu)} \frac{1}{\lambda(c_B, r_B)} \lesssim \frac{\mu((6/5)B)}{\lambda(c_B, r_B)} \|f\|_{L^{1/\alpha}(\mu)} \lesssim \|f\|_{L^{1/\alpha}(\mu)}. \]

Denote \( \|g\|_{H^{1}(\mu)} [\mu(B)]^{-1/q_0} \) simply by \( E. \) Then by (V) of Theorem 1.13 and (2.36), we conclude that
\[ \int_B |T_\alpha g(y)| d\mu(y) = \int_0^E \mu(\{y \in B : |T_\alpha g(y)| > t\}) dt + \int_E^\infty \cdots \]
We first prove that

\[ \text{Proof.} \]

Hence, by the proceeding arguments, we see that

\[ \mu \]

Lemma 3.1.

Let \( E \)

By [13, Theorem 1.2] and [18, Lemma 2.5], there exist countable disjoint subsets \( \{Q_j\} \) of \( \{Q_x : x \in E\} \) such that \( E \subset \cup_j \rho Q_j \). Let \( q := p/(1 - \alpha p) \). Then \( p/q \leq 1 \). Hence, by (3.2) and \( p/q = 1 - \alpha p \), we see that

\[ \mu(E)^{p/q} \leq [\mu(\cup_j \rho Q_j)]^{p/q} \leq \sum_j [\mu(\rho Q_j)]^{p/q} \leq \sum_j \frac{1}{t^p} \int_{Q_j} |f(y)|^p \, d\mu(y) \leq \frac{\|f\|_{L^p(\mu)}^p}{t^p}. \]

Hence \( \mu(E) \lesssim t^{-q} \|f\|_{L^p(\mu)}^q \), namely, (3.1) holds true.

Notice that, if \( p < s < 1/\alpha \), by using Hölder’s inequality, we have \( M_{s,\rho} f \leq M_{s,\rho} f \). Hence, by the proceeding arguments, we see that \( \mu(E) \lesssim \frac{1}{t^p} \|f\|_{L^s(\mu)}^s/(1-\alpha s) \), which, together with (3.1) and the Marcinkiewicz interpolation theorem, further implies the desired result and hence completes the proof of Lemma 3.1. \( \square \)
Remark 3.2. Let $\alpha \in (0, 1)$. By Lemma 3.1, the maximal operators $M^{(\alpha)}_{r,p}$ ($r < p < 1/\alpha$) and $M^{(\alpha)}_{\rho} := M^{(\alpha)}_{L,\rho}$ are bounded from $L^p(\mu)$ to $L^q(\mu)$ for $p \in (r, 1/\alpha)$ and $1/q = 1/p - \alpha$.

Now we introduce the fractional coefficient $\tilde{K}^{(\alpha)}_{B,S}$ adapted from [5].

Definition 3.3. For any two balls $B \subset S$, $\tilde{K}^{(\alpha)}_{B,S}$ is defined by

$$\tilde{K}^{(\alpha)}_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \left[ \frac{\mu(6^kB)}{\lambda(x_B, 6^kB)} \right]^{1-\alpha},$$

where $\alpha \in [0, 1)$ and $N_{B,S}$ is defined as in Remark 1.6.

Now we give out some simple properties of $\tilde{K}^{(\alpha)}_{B,S}$, which are completely analogous to [5, Lemma 3]. We omit the details; see [8, Lemma 3.5] for the proofs of the case that $\alpha = 0$.

Lemma 3.4. Let $\alpha \in [0, 1)$.

(i) For all balls $B \subset R \subset S$, $\tilde{K}^{(\alpha)}_{B,R} \leq 2\tilde{K}^{(\alpha)}_{B,S}$.

(ii) For any $\rho \in [1, \infty)$, there exists a positive constant $C(\rho)$, depending only on $\rho$, such that, for all balls $B \subset S$ with $r_S \leq \rho r_B$, $\tilde{K}^{(\alpha)}_{B,S} \leq C(\rho)$.

(iii) There exists a positive constant $C(\alpha)$, depending on $\alpha$, such that, for all balls $B$, $\tilde{K}^{(\alpha)}_{B,B} \leq C(\alpha)$.

(iv) There exists a positive constant $c$, depending on $C_\lambda$ and $\alpha$, such that, for all balls $B \subset R \subset S$, $\tilde{K}^{(\alpha)}_{B,S} \leq \tilde{K}^{(\alpha)}_{B,R} + c\tilde{K}^{(\alpha)}_{R,S}$.

(v) There exists a positive constant $\tilde{c}$, depending on $C_\lambda$ and $\alpha$, such that, for all balls $B \subset R \subset S$, $\tilde{K}^{(\alpha)}_{R,S} \leq \tilde{c}\tilde{K}^{(\alpha)}_{B,S}$.

Now we introduce the sharp maximal operator $\tilde{M}^{#,\alpha}$ associated with $\tilde{K}^{(\alpha)}_{B,S}$.

Definition 3.5. Let $\alpha \in [0, 1)$. For all $f \in L^1_{\text{loc}}(\mu)$ and $x \in \mathcal{X}$, the sharp maximal function $\tilde{M}^{#,\alpha} f(x)$ of $f$ is defined by

$$\tilde{M}^{#,\alpha} f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(x) - m_B f| \, d\mu(x) + \sup_{(Q,R) \in \Delta_x} \frac{|m_Q f - m_R f|}{\tilde{K}^{(\alpha)}_{Q,R}},$$

where $\Delta_x := \{(Q, R) : x \in Q \subset R$ and $Q, R$ are doubling balls$\}$.

Similar to [2, Theorem 4.2], we have the following lemma.

Lemma 3.6. Let $f \in L^1_{\text{loc}}(\mu)$ satisfy that $\int_{\mathcal{X}} |f(x)| \, d\mu(x) = 0$ when $\|\mu\| := \mu(\mathcal{X}) < \infty$. Assume that, for some $p \in (1, \infty)$, $\inf \{1, Nf\} \in L^p(\mu)$. Then there exists a positive constant $C$, independent of $f$, such that $\|Nf\|_{L^p(\mu)} \leq C\|\tilde{M}^{#,\alpha} f\|_{L^p(\mu)}$.

The following two lemmas are completely analogous to [5, Lemmas 5 and 6], the details being omitted.
Lemma 3.7. For any $\alpha \in [0,1)$, there exists some positive constant $P_\alpha$ (big enough), depending only on $C_\lambda$ in (1.2) and $\alpha$, such that, if $m \in \mathbb{N}$, $B_1 \subset \cdots \subset B_m$ are concentric balls with $K_{B_i,B_{i+1}}^{(\alpha)} > P_\alpha$ for $i \in \{1, \ldots, m-1\}$, then there exists a positive constant $C$, depending only on $C_\lambda$ and $\alpha$, such that $\sum_{i=1}^{m-1} K_{B_i,B_{i+1}}^{(\alpha)} \leq C K_{B_1,B_m}^{(\alpha)}$.

Lemma 3.8. For any $\alpha \in [0,1)$, there exists some positive constant $\widetilde{P}_\alpha$ (large enough), depending on $C_\lambda$, $\beta_6$ as in (1.2) with $\eta = 6$ and $\alpha$, such that, if $x \in X$ is some fixed point and $\{f_B\}_{B \ni x}$ is a collection of numbers such that $|f_B - f_S| \leq K_{B,S}^{(\alpha)} C_\alpha$ for all doubling balls $B \subset S$ with $x \in B$ satisfying $K_{B,S}^{(\alpha)} \leq \widetilde{P}_\alpha$, then there exists a positive constant $C$, depending on $C_\lambda$, $\beta_6$, $\alpha$ and $\widetilde{P}_\alpha$, such that $|f_B - f_S| \leq C K_{B,S}^{(\alpha)} C_\alpha$ for all doubling balls $B \subset S$ with $x \in B$, where $C_\alpha$ is a positive constant, depending on $x$, and $C_\alpha$ a positive constant depending only on $C_\lambda$, $\beta_6$ and $\alpha$.

The following theorem is adapted from [5, Theorem 1].

Theorem 3.9. Let $b \in \text{RBMO}(\mu)$ and $T_\alpha$ for $\alpha \in (0,1)$ be as in (1.9) with kernel $K_\alpha$ satisfying (1.7) and (1.8), which is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1/\alpha)$ and $1/q = 1/p - \alpha$. Then the commutator $[b, T_\alpha]$ satisfies that there exists a positive constant $C$ such that, for all $f \in L^p(\mu)$, $\|b, T_\alpha f\|_{L^q(\mu)} \leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)}$.

Proof. The case that $\mu(X) < \infty$ can be proved by a way similar to the proof of [8, Theorem 3.10]. Thus, without loss of generality, we may assume that $\mu(X) = \infty$. Let $p \in (1,1/\alpha)$.

We first claim that, for all $r \in (1, \infty)$, $f \in L^p(\mu)$ and $x \in X$,

$$\mathcal{M}^{#,\alpha}([b, T_\alpha]) f(x) \lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ \|M_{r,5}^{(\alpha)} f(x)\| + \|M_{r,6} T_\alpha f(x)\| + \|T_\alpha(|f|)(x)\| \right\}. \quad (3.3)$$

Once (3.3) is proved, taking $1 < r < p < 1/\alpha$, by Lemma 2.5(ii), Lemma 3.6, an argument similar to that used in the proof of [8, Theorem 3.10], and Remark 3.2, we conclude that

$$\|b, T_\alpha f\|_{L^q(\mu)} \leq \|N([b, T_\alpha] f)\|_{L^q(\mu)} \lesssim \|\mathcal{M}^{#,\alpha}([b, T_\alpha] f)\|_{L^q(\mu)} \lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ \|M_{r,5}^{(\alpha)} f\|_{L^q(\mu)} + \|M_{r,6} T_\alpha f\|_{L^q(\mu)} + \|T_\alpha(|f|)(x)\| \right\} \lesssim \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)},$$

which is just the desired conclusion.

To show (3.3), by Definition 1.9, there exists a family of numbers, $\{b_Q\}_Q$, such that, for any ball $Q$,

$$\int_Q |b(y) - b_Q| \, d\mu(y) \lesssim \mu(6Q) \|b\|_{\text{RBMO}(\mu)}$$

and, for all balls $Q, R$ with $Q \subset R$, $|b_Q - b_R| \lesssim K_{Q,R} \|b\|_{\text{RBMO}(\mu)}$. For any ball $Q$, let

$$h_Q := m_Q(T_\alpha([b - b_Q]) f \chi_{X \setminus (6/5) Q}).$$

Next we show that, for all $x$ and $Q$ with $x \ni Q$,

$$\frac{1}{\mu(6Q)} \int_Q \|b, T_\alpha f(y) - h_Q| \, d\mu(y) \lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ \|M_{p,5}^{(\alpha)} f(x)\| + \|M_{p,6} T_\alpha f(x)\| \right\}. \quad (3.4)$$
and, for all balls $Q$, $R$ with $Q \subset R$ and $Q \ni x$,

\[
(3.5) \quad |h_Q - h_R| \lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ M_{p,5}^{(\alpha)} f(x) + T_\alpha(|f|)(x) \right\} K_{Q,R}K_{Q,R}^{(\alpha)}.
\]

To prove (3.4), for a fixed ball $Q$ and $x$ with $x \in Q$, we write $[b,T_\alpha]f$ as

\[
(3.6) \quad [b,T_\alpha]f = [b-b_Q]T_\alpha f - T_\alpha([b-b_Q]f_1) - T_\alpha([b-b_Q]f_2),
\]

where $f_1 := f\chi_{(6/5)Q}$ and $f_2 := f - f_1$.

Let us first estimate the term $[b-b_Q]T_\alpha f$. By Hölder’s inequality and [18, Corollary 6.3], we see that

\[
(3.7) \quad \frac{1}{\mu(6Q)} \int_Q |(b(y)-b_Q)T_\alpha f(y)| \, d\mu(y)
\leq \left[ \frac{1}{\mu(6Q)} \int_Q |b(y)-b_Q|^{p'} \, d\mu(y) \right]^{1/p'} \left[ \frac{1}{\mu(6Q)} \int_Q |T_\alpha f(y)|^p \, d\mu(y) \right]^{1/p}
\lesssim \|b\|_{\text{RBMO}(\mu)} M_{p,6}^{(\alpha)}(T_\alpha f)(x),
\]

which is desired.

To estimate $T_\alpha([b-b_Q]f_1)$, take $s := \sqrt{p}$ and $1/r := 1/s - \alpha$. From Hölder’s inequality, the $(L^s(\mu), L^r(\mu))$-boundedness of $T_\alpha$ and [18, Corollary 6.3], it follows that

\[
(3.8) \quad \frac{1}{\mu(6Q)} \int_Q |T_\alpha([b-b_Q]f_1)(y)| \, d\mu(y)
\leq \left[ \frac{\mu(Q)}{\mu(6Q)} \right]^{1-1/r} \|T_\alpha([b-b_Q]f_1)\|_{L^r(\mu)} \lesssim \left[ \frac{\|Q\|}{\mu(6Q)} \right]^{1-1/r} \|(b-b_Q)f_1\|_{L^r(\mu)}
\lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ \int_{(6/5)Q} |b(y)-b_Q|^{ss'} \, d\mu(y) \right\} \left[ \int_{(6/5)Q} |f(y)|^p \, d\mu(y) \right]^{1/p}
\lesssim \|b\|_{\text{RBMO}(\mu)} M_{p,5}^{(\alpha)} f(x),
\]

which is desired.

By (3.6), (3.7) and (3.8), to obtain (3.4), we still need to estimate the difference $|T_\alpha([b-b_Q]f_2) - h_Q|$ by writing that, for all $y_1, y_2 \in Q$,

\[
|T_\alpha([b-b_Q]f_2)(y_1) - T_\alpha([b-b_Q]f_2)(y_2)|
\lesssim \int_{6Q\setminus(6/5)Q} |K_\alpha(y_1,z) - K_\alpha(y_2,z)||b(z)-b_Q||f(z)| \, d\mu(z) \, d\mu(z) + \int_{X\setminus 6Q} \cdots
=: I_1 + I_2.
\]

Let $c_Q$ and $r_Q$ be the center and the radius of $Q$, respectively. To estimate $I_1$, from (1.7) and Hölder’s inequality, together with (1.2) and (1.3), it follows that

\[
I_1 \lesssim \int_{6Q\setminus(6/5)Q} \left( \frac{1}{|\lambda(y_1,d(y_1,z))|^{1-\alpha}} + \frac{1}{|\lambda(y_2,d(y_2,z))|^{1-\alpha}} \right) |f(z)||b(z)-b_Q| \, d\mu(z)
\]
Thus, we used the fact that\( b \in RBMO(\mu)M^{(\alpha)}_{p,5}f(x) \), which is desired.

For any \( y_1, y_2 \in Q \), by (1.8), (1.3), (1.2), Hölder’s inequality and \([18, Corollary 6.3]\), we know that

\[
I_2 \lesssim \int_{x\in Q} \left[ \frac{1}{\mu(30Q)} \int_{6Q} [d(y_1, y_2)]^\delta |b(z) - b_Q||f(z)| \, d\mu(z) \right]^{1/p'}
\]

\[
\lesssim \sum_{k=1}^{\infty} \int_{2^{k+1}(6Q)} \left[ \frac{1}{\mu(3k \times 30Q)} \int_{2^{k}(6Q)} |b(z) - b_{2^k(6Q)}||f(z)| \, d\mu(z) \right]^{1/p'}
\]

\[
+ k \|b\|_{RBMO(\mu)} \left( \int_{2^{k}(6Q)} |f(z)| \, d\mu(z) \right)^{1/p'}
\]

\[
\lesssim \sum_{k=1}^{\infty} 2^{-k\delta} \left( \int_{2^{k}(6Q)} |b(z) - b_{2^k(6Q)}|^{p'} \, d\mu(z) \right)^{1/p'}
\]

\[
\times \left( \int_{2^{k}(6Q)} |f(z)|^p \, d\mu(z) \right)^{1/p'}
\]

\[
+ k \|b\|_{RBMO(\mu)} \left( \int_{2^{k}(6Q)} |f(z)|^p \, d\mu(z) \right)^{1/p'}
\]

\[
\lesssim \sum_{k=1}^{\infty} (k + 1) 2^{-k\delta} \|b\|_{RBMO(\mu)} M^{(\alpha)}_{p,5}f(x) \lesssim \|b\|_{RBMO(\mu)} M^{(\alpha)}_{p,5}f(x),
\]

where we used the fact that

\[
|b - b_{2^k(6/5)Q}| \lesssim K_{Q,2^k(6Q)} \|b\|_{RBMO(\mu)} \lesssim k \|b\|_{RBMO(\mu)}.
\]

Combining the estimates for \( I_1 \) and \( I_2 \), we see that, for all \( y \in Q \),

\[
|T_\alpha([b - b_Q]f_2)(y) - h_Q| \lesssim \|b\|_{RBMO(\mu)} M^{(\alpha)}_{p,5}f(x).
\]

Thus,

\[
\frac{1}{\mu(6Q)} \int_{Q} |T_\alpha([b - b_Q]f_2)(y) - h_Q| \, d\mu(y) \lesssim \|b\|_{RBMO(\mu)} M^{(\alpha)}_{p,5}f(x),
\]

which, together with (3.6), (3.7) and (3.8), implies (3.4).

Now we show the regularity condition (3.5) for the numbers \( \{h_Q\}_Q \). Consider two balls \( Q \subset R \) with \( x \in Q \) and let \( N := N_{Q,R} + 1 \). Write \(|h_Q - h_R|\) as

\[
|m_Q(T_\alpha([b - b_Q]f_\chi_{X\setminus(6/5)Q})) - m_R(T_\alpha([b - b_Q]f_\chi_{X\setminus(6/5)R}))|
\]
Thus, by (1.7), (1.3) and (1.2), we conclude that

\[
    \frac{\mu}{|m_Q(T_a([b - b_Q] f \chi_{6Q \setminus (6/5)Q})| + |m_Q(T_a([b_Q - b_R] f \chi_{6Q \setminus 6Q})|) + |m_Q(T_a([b - b_R] f \chi_{6Q \setminus 6Q})|) - m_R(T_a([b - b_R] f \chi_{6Q \setminus 6Q}))| + |m_R(T_a([b - b_R] f \chi_{6Q \setminus (6/5)R}))|)
    =: U_1 + U_2 + U_3 + U_4 + U_5.
\]

Following the proof of [5, Theorem 1], it is easy to see that

\[
    U_1 + U_4 + U_5 \lesssim \|b\|_{\text{RBMO}(\mu)} M_p^{(\alpha)} f(x)
\]

and \( U_2 \lesssim Q,R ||b||_{\text{RBMO}(\mu)} [T_a(|f|)(x) + M_p^{(\alpha)} f(x)] \).

Now we turn to the estimate for \( U_3 \). For \( y \in Q \), by (1.7) and Hölder’s inequality, we conclude that

\[
    |T_a([b - b_R] f \chi_{6Q \setminus 6Q})(y)| \leq \sum_{k=0}^{N-1} \frac{1}{\lambda(x_Q, 6^k r_Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b(y) - b_R| |f(y)| d\mu(y)
\]

\[
    \lesssim \sum_{k=0}^{N-1} \frac{1}{\lambda(x_Q, 6^k r_Q)} \left[ \int_{6^{k+1}Q} |b(y) - b_{6^{k+1}Q}|^{p'} d\mu(y) \right]^{1/p'} \left[ \int_{6^{k+1}Q} |f(y)|^p d\mu(y) \right]^{1/p}.
\]

Notice that, by Minkowski’s inequality and Lemma 2.1(i), we see that

\[
    \left[ \int_{6^{k+1}Q} |b(y) - b_{6^{k+1}Q}|^{p'} d\mu(y) \right]^{1/p'} \leq \left[ \int_{6^{k+1}Q} |b(y) - b_{6^{k+1}Q}|^{p'} d\mu(y) \right]^{1/p'} + \left[ \mu(6^{k+1}Q) \right]^{1/p'} |b_{6^{k+1}Q} - b_R| \lesssim K_{Q,R} ||b||_{\text{RBMO}(\mu)} \left[ \mu(5 \times 6^{k+1}Q) \right]^{1/p'}.
\]

Thus, by (1.7), (1.3) and (1.2), we conclude that

\[
    |T_a([b - b_R] f \chi_{6Q \setminus 6Q})(y)| \lesssim K_{Q,R} ||b||_{\text{RBMO}(\mu)} \sum_{k=0}^{N-1} \left[ \mu(5 \times 6^{k+1}Q) \right]^{1/p} \left[ \lambda(x_Q, 6^k r_Q) \right]^{1/\alpha} \left[ \int_{6^{k+1}Q} |f(y)|^p d\mu(y) \right]^{1/p}
\]

\[
    \lesssim K_{Q,R} ||b||_{\text{RBMO}(\mu)} \sum_{k=0}^{N_Q,R} \left[ \mu(6^{k+2}Q) \right]^{1/\alpha} \left[ \lambda(x_Q, 6^k r_Q) \right]^{1-\alpha} \left[ \frac{1}{\mu(5 \times 6^{k+1}Q)} \right]^{1-\alpha p} \int_{6^{k+1}Q} |f(y)|^p d\mu(y) \right]^{1/p} \lesssim K_{Q,R} \tilde{K}_{Q,R}^{(\alpha)} ||b||_{\text{RBMO}(\mu)} M_p^{(\alpha)} f(x).
\]

Taking the mean over \( Q \), we obtain \( U_3 \lesssim K_{Q,R} \tilde{K}_{Q,R}^{(\alpha)} ||b||_{\text{RBMO}(\mu)} M_p^{(\alpha)} f(x) \), which, together with the estimates \( U_1, U_2, U_4 \) and \( U_5 \), further implies (3.5).
By (3.4), if $Q$ is a doubling ball and $x \in Q$, we have

\begin{equation}
|m_Q([b, T_\alpha]f) - h_Q| \lesssim \|b\|_{\text{RBMO}(\mu)} \left[ M_{p,5}^{(\alpha)} f(x) + M_{p,6}(T_\alpha f)(x) \right].
\end{equation}

Since, for any ball $Q$ with $x \in Q$, $K_{Q,\tilde{Q}} \leq C$ and $\tilde{K}_{Q,\tilde{Q}}^{(\alpha)} \leq C$, by (3.4), (3.5) and (3.9), we see that

\begin{equation}
\frac{1}{\mu(6Q)} \int_Q \| [b, T_\alpha]f(y) - m_Q([b, T_\alpha]f) \| \, d\mu(y)
\leq \frac{1}{\mu(6Q)} \int_Q \| [b, T_\alpha]f(y) - h_Q \| \, d\mu(y) + |h_Q - m_Q([b, T_\alpha]f)|
\lesssim \|b\|_{\text{RBMO}(\mu)} \left\{ M_{p,5}^{(\alpha)} f(x) + M_{p,6}(T_\alpha f)(x) + T_\alpha(|f|)(x) \right\}.
\end{equation}

On the other hand, for all doubling balls $Q \subset R$ with $x \in Q$ such that $\tilde{K}_{Q,R}^{(\alpha)} \leq \tilde{P}_\alpha$, where $\tilde{P}_\alpha$ is the constant as in Lemma 3.8, by (3.5), we have

\begin{equation}
|h_Q - h_R| \lesssim K_{Q,R} \|b\|_{\text{RBMO}(\mu)} \left[ M_{p,5}^{(\alpha)} f(x) + T_\alpha(|f|)(x) \right] \tilde{P}_\alpha.
\end{equation}

Hence, by Lemma 3.8, we know that, for all doubling balls $Q \subset R$ with $x \in Q$,

\begin{equation}
|h_Q - h_R| \lesssim \tilde{K}_{Q,R}^{(\alpha)} \|b\|_{\text{RBMO}(\mu)} \left[ M_{p,5}^{(\alpha)} f(x) + T_\alpha(|f|)(x) \right]
\end{equation}

and, using (3.9), we further obtain

\begin{equation}
|m_Q([b, T_\alpha]f) - m_R([b, T_\alpha]f)|
\lesssim \tilde{K}_{Q,R}^{(\alpha)} \|b\|_{\text{RBMO}(\mu)} \left\{ M_{p,5}^{(\alpha)} f(x) + M_{p,6}(T_\alpha f)(x) + T_\alpha(|f|)(x) \right\},
\end{equation}

which, together with (3.10), induces (3.3) and hence completes the proof of Theorem 3.9.

To prove Theorem 1.15, we need to recall some notation from [14]. Let $C_i^k$ be as in Section 1. For any sequence $\tilde{b} := (b_1, \ldots, b_k)$ of functions and all $i$-tuples $\sigma := \{\sigma(1), \ldots, \sigma(i)\} \in C_i^k$, let $\tilde{b}_\sigma := (b_{\sigma(1)}, \ldots, b_{\sigma(i)})$ and

\[ \|\tilde{b}_\sigma\|_{\text{RBMO}(\mu)} := \prod_{j=1}^i \|b_{\sigma(j)}\|_{\text{RBMO}(\mu)}.
\]

For any $\sigma \in C_i^k$ and $z \in X$, let

\[ \left[ m_B(\tilde{b}) - \tilde{b}(z) \right]_\sigma := \prod_{j=1}^i \left[ m_B(b_{\sigma(j)}) - b_{\sigma(j)}(z) \right]
\]

and $T_{\alpha,\tilde{b}_\sigma} := \{b_{\sigma(i)}, [b_{\sigma(i-1)}, \ldots, [b_{\sigma(1)}, T_\alpha] \cdots] \}$. In particular, when $\sigma := \{1, \ldots, k\}$, $T_{\alpha,\tilde{b}_\sigma}$ coincides with $T_{\alpha,\tilde{b}}$ as in (1.11).

Now we are ready to prove Theorem 1.15.
Proof of Theorem 1.15. By Lemma 2.4, it suffices to prove that $T_{\alpha,\tilde{b}}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$ for all $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. We show this by induction on $k$.

By Theorem 3.9, the conclusion is valid for $k = 1$. Now assume that $k \geq 2$ is an integer and, for any $i \in \{1, \ldots, k-1\}$ and any subset $\sigma = \{\sigma(1), \ldots, \sigma(i)\}$ of $\{1, \ldots, k-1\}$, $T_{\alpha,\tilde{b}_\sigma}$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for the same $p, q$ as those such that $T_\alpha$ is bounded from $L^p(\mu)$ to $L^q(\mu)$.

The case that $\mu(\mathcal{X}) < \infty$ can be proved by a way similar to that used in the proof of [8, Theorem 3.10], the details being omitted. Thus, without loss of generality, we may assume that $\mu(\mathcal{X}) = \infty$. We first claim that, for any $r \in (1, \infty)$, $f \in L^p(\mu)$ and $x \in \mathcal{X}$,

$$
(3.11) \quad \tilde{M}^{\#, \alpha}(T_{\alpha,\tilde{b}}f)(x) \lesssim \|\tilde{b}\|_{\text{RBMO}(\mu)} \left[ M_{r,6}T_\alpha f(x) + M_{r,5}^{(\alpha)} f(x) \right] + \sum_{i=1}^{k-1} \sum_{\sigma \in C^i_k} \|\tilde{b}_\sigma\|_{\text{RBMO}(\mu)} M_{r,6}(T_{\alpha,\tilde{b}_\sigma} f)(x).
$$

Once (3.11) is proved, by Lemmas 2.5 and 2.6, an argument similar to that used in the proof of Theorem 3.9, and Remark 3.2, we conclude that, for all $p \in (1, 1/\alpha)$, $1/q = 1/p - \alpha$ and $f \in L^p(\mu)$,

$$
\|T_{\alpha,\tilde{b}} f\|_{L^q(\mu)} \leq \|N(T_{\alpha,\tilde{b}} f)\|_{L^q(\mu)} \lesssim \left\| \tilde{M}^{\#}(T_{\alpha,\tilde{b}} f) \right\|_{L^q(\mu)}
\lesssim \|\tilde{b}\|_{\text{RBMO}(\mu)} \left[ \|M_{r,6}(T f)\|_{L^q(\mu)} + \|M_{r,5}(f)\|_{L^q(\mu)} \right]
+ \sum_{i=1}^{k-1} \sum_{\sigma \in C^i_k} \|\tilde{b}_\sigma\|_{\text{RBMO}(\mu)} \|M_{r,6}(T_{\alpha,\tilde{b}_\sigma} f)\|_{L^q(\mu)}
\lesssim \|\tilde{b}\|_{\text{RBMO}(\mu)} \left[ \|T f\|_{L^q(\mu)} + \|f\|_{L^p(\mu)} + \sum_{i=1}^{k-1} \sum_{\sigma \in C^i_k} \|T_{\alpha,\tilde{b}_\sigma} f\|_{L^q(\mu)} \right]
\lesssim \|\tilde{b}\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)},
$$

which is desired.

As in the proof of [14, Theorem 2], to prove (3.11), it suffices to show that, for all $x$ and $B$ with $B \ni x$,

$$
(3.12) \quad \frac{1}{\mu(6B)} \int_B |T_{\alpha,\tilde{b}} f(y) - h_B| \, d\mu(y) \lesssim \|\tilde{b}\|_{\text{RBMO}(\mu)} \left[ M_{r,6}(T_\alpha f)(x) + M_{r,5}^{(\alpha)} f(x) \right]
+ \sum_{i=1}^{k-1} \sum_{\sigma \in C^i_k} \|\tilde{b}_\sigma\|_{\text{RBMO}(\mu)} M_{r,6}(T_{\alpha,\tilde{b}_\sigma} f)(x)
$$

and, for an arbitrary ball $Q$, a doubling ball $R$ with $Q \subset R$ and $x \in Q$,

$$
(3.13) \quad |h_Q - h_R| \lesssim \left[ \tilde{K}_{Q,R} \right]^k \tilde{K}_{Q,R}^{(\alpha)} \|\tilde{b}\|_{\text{RBMO}(\mu)} \left\{ M_{r,6}T_\alpha f(x) + M_{r,5}^{(\alpha)} f(x) \right\}
$$
\[
+ \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \| \tilde{b}_\sigma \|_{\text{RBMO}(\mu)} M_{r,6}(T_{\alpha,\tilde{b}_\sigma} f)(x) \right),
\]

where

\[
h_Q := m_Q \left( T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f \chi_\frac{Q}{2} \right) \right)
\]

and

\[
h_R := m_R \left( T_\alpha \left( \prod_{i=1}^{k} [m_R(b_i) - b_i] f \chi_\frac{Q}{2} \right) \right).
\]

Let us first prove (3.12). With the aid of the formula that, for all \( y, z \in \mathcal{X} \),

\[
\prod_{i=1}^{k} [m_Q(b_i) - b_i(z)] = \sum_{i=0}^{k} \sum_{\sigma \in C_i^k} [b(y) - b(z)]_{\sigma'} [m_Q(b) - b(y)]_{\sigma},
\]

where, if \( i = 0 \), then \( \sigma' = \{1, \ldots, k\} \) and \( \sigma = \emptyset \), \( [m_Q(b) - b(y)]_{\emptyset} = 1 \), it is easy to see that, for all \( y \in \mathcal{X} \),

\[
T_{\alpha, \tilde{b}} f(y) = T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f \right)(y) - \sum_{i=1}^{k} \sum_{\sigma \in C_i^k} [m_Q(b) - b(y)]_{\sigma} T_{\alpha, \tilde{b}_\sigma} f(y),
\]

where, if \( i = k \), \( T_{\alpha, \tilde{b}_\sigma} f(y) := T_\alpha(|f|)(y) \). Therefore, for all balls \( Q \ni x \), we have

\[
\frac{1}{\mu(6Q)} \int_Q |T_{\alpha, \tilde{b}} f(y) - h_Q| \, d\mu(y)
\]

\[
\leq \frac{1}{\mu(6Q)} \int_Q \left| T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f \chi_\frac{Q}{2} \right) \right| (y) \, d\mu(y)
\]

\[
+ \sum_{i=1}^{k} \sum_{\sigma \in C_i^k} \frac{1}{\mu(6Q)} \int_Q \left| [m_Q(b) - b(y)]_{\sigma} \right| \left| T_{\alpha, \tilde{b}_\sigma} f(y) \right| \, d\mu(y)
\]

\[
+ \frac{1}{\mu(6Q)} \int_Q \left| T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f \chi_\frac{Q}{2} \right) \right| (y) - h_Q \, d\mu(y) =: I_1 + I_2 + I_3.
\]

Take \( 1/s^2 = 1/r - \alpha \). Using the boundedness of \( T_\alpha \) from \( L^{s/(1+\alpha)}(\mu) \) into \( L^s(\mu) \) for \( s \in (1, \infty) \) and some arguments similar to those used in the proofs of [14, Theorem 1.1] and [8, Theorem 1.9], we conclude that, for all \( x \in \mathcal{X} \),

\[
I_1 \lesssim \| \tilde{b} \|_{\text{RBMO}(\mu)} M_{r,5}^{(\alpha)} f(x),
\]

and

\[
I_2 \lesssim \sum_{i=1}^{k} \sum_{\sigma \in C_i^k} \| \tilde{b}_\sigma \|_{\text{RBMO}(\mu)} M_{r,6} \left( T_{\alpha, \tilde{b}_\sigma} f \right)(x)
\]

which imply (3.12).
Now we turn to prove (3.13). Let $Q$ be an arbitrary ball and $R$ a doubling ball in $\mathcal{X}$ such that $x \in Q \subset R$. Denote $N_{Q,R} + 1$ simply by $N$. Write

$$
|h_Q - h_R|
\leq m_R \left[ T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right] - m_Q \left[ T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right]
+ m_R \left[ T_\alpha \left( \prod_{i=1}^{k} [m_Q(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right] - m_R \left[ T_\alpha \left( \prod_{i=1}^{k} [m_R(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right]
+ m_Q \left[ T_\alpha \left( \prod_{i=1}^{k} [m_R(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right] + m_R \left[ T_\alpha \left( \prod_{i=1}^{k} [m_R(b_i) - b_i] f_{\mathcal{X}\setminus \delta N Q} \right) \right] =: L_1 + L_2 + L_3 + L_4.
$$

An estimate similar to that for $I_3$, together with $K_{Q,R} \lesssim \tilde{K}_{Q,R}$, we see that, for all $x \in \mathcal{X}$, $L_1 \lesssim \lbrack \tilde{K}_{Q,R} \rbrack^{k} \|\tilde{b}\|_{RBMO(\mu)}M_{r,5}^{(\alpha)} f(x)$. By some arguments similar to those used in the proofs of [14, Theorem 1.1] and [8, Theorem 1.9], we easily see that, for all $x \in \mathcal{X}$,

$$
L_2 \lesssim \lbrack \tilde{K}_{Q,R} \rbrack^{k} \left\{ \sum_{i=1}^{k-1} \sum_{\sigma \in C_i^k} \|\tilde{b}_\sigma\|_{RBMO(\mu)}M_{r,6}(T_\alpha f)(x) \right\}
+ \|\tilde{b}\|_{RBMO(\mu)}M_{r,6}(T_\alpha f)(x) + \|\tilde{b}\|_{RBMO(\mu)}M_{r,5}^{(\alpha)} f(x)
$$

$L_3 \lesssim \lbrack \tilde{K}_{Q,R} \rbrack^{k} \|\tilde{b}\|_{RBMO(\mu)}M_{r,5}^{(\alpha)} f(x)$ and $L_4 \lesssim \|\tilde{b}\|_{RBMO(\mu)}M_{r,5}^{(\alpha)} f(x)$.

Combining the estimates for $L_1$, $L_2$, $L_3$ and $L_4$, we then obtain (3.13) and hence complete the proof of Theorem 1.13.

Now we are ready to prove Theorem 1.19. In what follows, for any $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$, let $C_i^k$ be as in the introduction. For all sequences of numbers, $r := (r_1, \ldots, r_k)$, and $i$-tuples $\sigma := \{\sigma(1), \ldots, \sigma(i)\} \in C_i^k$, let $\tilde{b}$ and $\tilde{b}_\sigma$ be as in Theorem 1.15,

$$
\|\tilde{b}_\sigma\|_{Osc_{\exp L^{r,\sigma}}(\mu)} := \prod_{j=1}^{i} \|b_{\sigma(j)}\|_{Osc_{\exp L^{r,\sigma(j)}}(\mu)}
$$

and, in particular,

$$
\|\tilde{b}\|_{Osc_{\exp L^{r,\sigma}}(\mu)} := \prod_{j=1}^{k} \|b_{j}\|_{Osc_{\exp L^{r,\sigma(j)}}(\mu)}.
$$

Then we prove Theorem 1.19.
Proof of Theorem 1.19. Without loss of generality, by homogeneity, we may assume that \( \|f\|_{L^1(\mu)} = 1 \) and \( \|b_i\|_{\text{osc}}(L^r(\mu)) = 1 \) for all \( i \in \{1, \ldots, k\} \).

We prove the theorem by two steps: \( k = 1 \) and \( k > 1 \).

Step i) \( k = 1 \). It is easy to see that the conclusion of Theorem 1.19 automatically holds true if \( t \leq \beta_0 \|f\|_{L^1(\mu)} / \mu(\mathcal{X}) \) when \( \mu(\mathcal{X}) < \infty \). Thus, we only need to deal with the case that \( t > \beta_0 \|f\|_{L^1(\mu)} / \mu(\mathcal{X}) \). For any given bounded function \( f \) with bounded support, \( q_0 := 1/(1 - \alpha) \) and any \( t > \beta_0 \|f\|_{L^1(\mu)} / \mu(\mathcal{X}) \), applying Lemma 2.6 to \( f \) with \( t \) replaced by \( t^{q_0} \), and letting \( R_j \) be as in Lemma 2.6(iii), we see that \( f = g + h \), where

\[
\begin{align*}
g := f\chi_{\mathcal{X}\backslash \cup_j B_j} + \sum_j \varphi_j \quad \text{and} \quad h := \sum_j (\omega_j f - \varphi_j) =: \sum_j h_j.
\end{align*}
\]

Let \( p_1 \in (1, 1/\alpha) \) and \( 1/q_1 := 1/p_1 - \alpha \). By (2.7), we easily know that \( \|g\|_{L^{q_1}(\mu)} \lesssim t^{q_0} \). From this, the boundedness of \( T_\alpha \) from \( L^{p_1}(\mu) \) to \( L^{q_1}(\mu) \) and (2.19), it follows that

\[
\mu(\{x \in \mathcal{X} : |T_{\alpha,b} f(x)| > t\}) \lesssim t^{-q_1} \int_{\mathcal{X}} |T_{\alpha,b} g(y)|^{q_1} d\mu(y) \lesssim t^{-q_1} \|g\|_{L^{q_1}(\mu)}^{q_1}
\]

\[
\lesssim t^{-q_1} t^{q_0(p_1 - 1)q_1/p_1} \|f\|_{L^{p_1}(\mu)}^{q_1} \lesssim t^{-q_0},
\]

where \( T_{\alpha,b} := T_{\alpha,b_1} \). On the other hand, by (2.3) with \( p = 1 \) and \( t \) replaced by \( t^{q_0} \), and the fact that the sequence of balls, \( \{B_j\}_j \), is pairwise disjoint, we see that \( \mu\left(\bigcup_j B_j\right) \lesssim t^{-q_0} \int_{\mathcal{X}} |f(y)|\,d\mu(y) \lesssim t^{-q_0} \), and hence the proof of Step i) can be reduced to proving

\[
\mu \left( \left\{ x \in \mathcal{X} : \left| \sum_j b_j(x) h_j(x) \right| > t \right\} \right) \lesssim \left[ \|\Phi_{1/q}(t^{-1}|f|)\|_{L^1(\mu)} + \Phi_{1/q}(t^{-1}\|f\|_{L^1(\mu)}) \right]^{q_0}.
\]  

(3.14)

For each fixed \( j \) and all \( x \in \mathcal{X} \), let \( b_j(x) := b(x) - m_{\tilde{B}_j}(b) \) and write

\[
T_{\alpha,b} h(x) = \sum_j b_j(x) T_{\alpha} h_j(x) - \sum_j T_{\alpha}(b_j h_j)(x) =: I(x) + \Pi(x).
\]

For the term \( \Pi(x) \), by the boundedness of \( T_{\alpha} \) from \( L^1(\mu) \) to \( L^{q_0,\infty}(\mu) \), we conclude that

\[
\mu (\{x \in \mathcal{X} : |\Pi(x)| > t\}) \lesssim t^{-q_0} \left[ \sum_j \int_{\mathcal{X}} |b_j(y) h_j(y)|\,d\mu(y) \right]^{q_0}
\]

\[
\lesssim t^{-q_0} \left[ \sum_j \int_{\mathcal{X}} |b(y) - m_{\tilde{B}_j}(b)| f(y) |\omega_j(y)|\,d\mu(y) \right]^{q_0}
\]

\[
+ t^{-q_0} \left[ \sum_j \|\varphi_j\|_{L^{q_0}(\mu)} \int_{R_j} |b(y) - m_{\tilde{B}_j}(b)|\,d\mu(y) \right]^{q_0} =: U + V.
\]

By Lemma 2.6(iii), we easily know that \( R_j \) is also \((6, \beta_0)\)-doubling and \( R_j = \tilde{R}_j \). Thus, from Lemmas 2.2 and 2.1, an argument similar to that used in the proof of [14, Theorem
1.2], (2.5) and the fact that \(\{6B_j\}_j\) is a sequence of finite overlapping balls, we deduce that

\[
(3.15) \quad V \lesssim t^{-q_0} \left[ \sum_j \|\varphi_j\|_{L^\infty(\mu)} \mu(R_j) \right]^{q_0} \lesssim t^{-q_0} \left[ \int_{\mathcal{X}} |f(y)| \, d\mu(y) \right]^{q_0}
\]

On the other hand, by the generalized Hölder inequality ([8, Lemma 4.1]), Lemma 2.2 and an argument similar to that used in the proof of [14, Theorem 1.2], we have

\[
(3.16) \quad U \lesssim \left[ \|\Phi_1/(t^{-1}f(\mu))\|_{L^1(\mu)} + \Phi_1/(t^{-1}f(\mu)) \right]^{q_0}.
\]

Combining (3.15) and (3.16), we know that

\[
(3.17) \quad \mu(\{x \in \mathcal{X} : |I(x)| > t\}) \lesssim \left[ \|\Phi_1/(t^{-1}f(\mu))\|_{L^1(\mu)} + \Phi_1/(t^{-1}f(\mu)) \right]^{q_0},
\]

which is desired.

Now we turn our attention to \(I(x)\). Let \(x_j\) be the center of \(B_j\). Let \(\theta\) be a bounded function with \(\|\theta\|_{L^q(\mu)} \leq 1\) and the support contained in \(\mathcal{X} \setminus (\cup_j 6^2B_j)\). By the vanishing moment of \(h_j\) and (1.8), we see that

\[
\int_{\mathcal{X} \setminus (\cup_j 6^2B_j)} |I(x)\theta(x)| \, d\mu(x) \\
\lesssim \sum_j \int_{\mathcal{X} \setminus 2R_j} |b_j(x)\theta(x)| \left| \int_{\mathcal{X}} h_j(y) [K_\alpha(x,y) - K_\alpha(x,x_j)] \, d\mu(y) \right| \, d\mu(x) \\
+ \sum_j \int_{2R_j \setminus 6^2B_j} |b_j(x)\theta(x)||T_\alpha h_j(x)| \, d\mu(x) \\
\lesssim \sum_j \int_{2R_j \setminus 6^2B_j} |h_j(y)| \, d\mu(y) \int_{\mathcal{X} \setminus 2R_j} \frac{|b_j(x)\theta(x)|}{d(x,x_j)^q [\lambda(x_j,d(x,x_j))]^{1-\alpha}} \, d\mu(x) \\
+ \sum_j \int_{2R_j \setminus 6^2B_j} |b_j(x)\theta(x)||T_\alpha (\varphi_j f)(x)| \, d\mu(x) \\
+ \sum_j \int_{2R_j} |b_j(x)\theta(x)||T_\alpha (\varphi_j f)(x)| \, d\mu(x) =: G + H + J.
\]

From (1.2), Hölder’s inequality, Corollary 2.3, (2.1), (i) through (iv) of Lemma 2.1, we deduce that

\[
\int_{\mathcal{X} \setminus 2R_j} \frac{|b_j(x)\theta(x)|}{d(x,x_j)^q [\lambda(x_j,d(x,x_j))]^{1-\alpha}} \, d\mu(x) \\
\lesssim \sum_{k=1}^\infty \left(2^k R_j\right)^{-\delta} \frac{1}{[\lambda(x_j,2^k R_j)]^{1-\alpha}} \int_{2^{k+1} R_j} |b(x) - m_{2^{k+1} R_j} (b)||\theta(x)| \, d\mu(x) \\
+ \sum_{k=1}^\infty \left(2^k R_j\right)^{-\delta} \frac{1}{[\lambda(x_j,2^k R_j)]^{1-\alpha}} |m_{B_j} (b) - m_{2^{k+1} R_j} (b)| \int_{2^{k+1} R_j} |\theta(x)| \, d\mu(x)
\]
where we used the fact that

\[ K_{B_j, 2^{k+1}R_j} \lesssim K_{\tilde{B}_j, R_j} + K_{R_j, 2^{k+1}R_j} + K_{2^{k+1}R_j, 2^{k+1}R_j} \lesssim K_{R_j, 2^{k+1}R_j} \lesssim k. \]

Since \( \|h_j\|_{L^1(\mu)} \lesssim \int \chi |f(y)| \omega_j(y) \, d\mu(y) \), we further see that \( G \lesssim \|f\|_{L^1(\mu)} \).

On the other hand, applying Hölder’s inequality, Corollary 2.3, (2.1), (iv), (i) and (iii) of Lemma 2.1, the boundedness of \( T_\alpha \) from \( L^{p_1}(\mu) \) to \( L^{q_1}(\mu) \) with \( p_1 \in (p_0, 1/\alpha) \) and \( 1/q_1 = 1/p_1 - \alpha, (2.7) \), and the fact that \( \{6Q_j\}_j \) is a sequence of finite overlapping balls, we obtain

\[
J \lesssim \sum_j \int_{2R_j} \left[ |b(x) - m_{2R_j}(b)| + |m_{B_j}(b) - m_{2R_j}(b)| \right] |T_\alpha(\phi_j)(x)\theta(x)| \, d\mu(x) \\
\lesssim \|\theta\|_{L^{q_0}(\mu)} \sum_j \left\{ \left[ \int_{2R_j} |b(x) - m_{2R_j}(b)|^{q_0}|T_\alpha|\phi_j(x)|^{q_0} \, d\mu(x) \right]^{1/q_0} \\
+ \left[ \int_{2R_j} |T_\alpha|\phi_j(x)|^{q_0} \, d\mu(x) \right]^{1/q_0} \left[ |m_{B_j}(b) - m_{2R_j}(b)| \right] \right\}^{1/q_0 - 1/q_1} \\
\lesssim \sum_j \left[ |T_\alpha|\phi_j|_{L^1(\mu)} \left[ \int_{2R_j} |b(x) - m_{2R_j}(b)|^{q_0(q_1/0)} \, d\mu(x) \right]^{1/q_0 - 1/q_1} \\
+ |(4R_j)|^{1/q_0 - 1/q_1} |m_{B_j}(b) - m_{2R_j}(b)| \right\} \lesssim \sum_j |(4R_j)|^{1/q_0 - 1/q_1} \|\phi_j\|_{L^{p_1}(\mu)}^{1/p_1} \lesssim \int \chi |f(x)| \, d\mu(x),
\]

where we used the fact that

\[ |m_{B_j}(b) - m_{2R_j}(b)| \leq |m_{B_j}(b) - m_{R_j}(b)| + |m_{R_j}(b) - m_{2R_j}(b)| \lesssim 1. \]

To estimate \( H \), by (1.7), (1.2) and (1.3), we see that, for all \( x \in 2R_j \setminus 6^2 B_j \),

\[ |T_\alpha(\omega_j f)(x)| \lesssim \frac{1}{[\lambda(x_j, d(x, x_j))]^{1-\alpha}} \int_{6B_j} |f(y)| \omega_j(y) \, d\mu(y), \]

which further implies that

\[
H \lesssim \sum_j \left\{ \int_{2R_j \setminus R_j} \frac{|b_j(x)\theta(x)|}{[\lambda(x_j, d(x, x_j))]^{1-\alpha}} \, d\mu(x) + \int_{R_j \setminus 6^2 B_j} \ldots \right\} \int \chi |f(y)| \omega_j(y) \, d\mu(y)
\]
We see that in the case that which, together with (3.17), implies (3.14) and hence completes the proof of Theorem 1.19. Consequently, by the fact that if, for all \( \mu \in (3 \times 6^2)^N \), and an argument similar to that used in the proof of Lemma 3.4(iii), we see that

\[
H \lesssim \sum_j \left( 1 + \sum_{k=0}^{N-1} \left[ \frac{\mu((3 \times 6^2)^k B_j)}{\lambda(x_j, (3 \times 6^2)^k r_{B_j})} \right]^{-1-\alpha} \right) \int_X |f(y)| \omega_j(y) \, d\mu(y) \lesssim \int_X |f(y)| \, d\mu(y).
\]

Combining the estimates for \( G, H \) and \( J \), we then conclude that

\[
\int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |I(x)\theta(x)| \, d\mu(x) \lesssim \|f\|_{L^1(\mu)}.
\]

Thus, we have

\[
\mu \left( \left\{ x \in X : |I(x)| > t \right\} \cup \bigcup_j (3 \times 6^2)^2 B_j \right) \lesssim t^{-q_0} \int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |I(x)|^{q_0} \, d\mu(x) \lesssim t^{-1} \int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |f(x)| \, d\mu(x)^{q_0}
\]

which, together with (3.17), implies (3.14) and hence completes the proof of Theorem 1.19 in the case that \( k = 1 \).

Step ii) \( k > 1 \). The proof of this case is completely analogous to that of [14, Theorem 1.2], the details being omitted, which completes the proof of Theorem 1.19.

\[ \square \]

4 Some applications

In this section, we apply all the results of Theorems 1.13, 1.15 and 1.19 to a specific example of fractional integrals to obtain some interesting conclusions.

We first need the following notion.

**Definition 4.1.** Let \( \epsilon \in (0, \infty) \). A dominating function \( \lambda \) is said to satisfy the \( \epsilon \)-weak reverse doubling condition if, for all \( r \in (0, 2 \text{diam}(X)) \) and \( a \in (1, 2 \text{diam}(X)/r) \), there exists a number \( C(a) \in [1, \infty) \), depending only on \( a \) and \( X \), such that, for all \( x \in X \),

\[
(4.1) \quad \lambda(x, ar) \geq C(a) \lambda(x, r)
\]

where \( N \in \mathbb{N} \) satisfies that \( R_j = (3 \times 6^2)^N B_j \). Obviously, for each \( k \in \{0, \ldots, N-1\} \), \((3 \times 6^2)^k B_j \subset R_j \) and hence

\[
|m_{B_j}(b) - m_{(3 \times 6^2)^{k+1} B_j}(b)| \lesssim K_{B_j, (3 \times 6^2)^{k+1} B_j} \lesssim K_{B_j, R_j} \lesssim 1.
\]

Consequently, by the fact that \( R_j \) is the smallest \((3 \times 6^2, C(3 \times 6^2)^N \text{-doubling ball of the family } \{(3 \times 6^2)^k B_j\}_{k \in \mathbb{N}} \) \text{ and an argument similar to that used in the proof of Lemma 3.4(iii), we see that}

\[
H \lesssim \sum_j \left( 1 + \sum_{k=0}^{N-1} \left[ \frac{\mu((3 \times 6^2)^k B_j)}{\lambda(x_j, (3 \times 6^2)^k r_{B_j})} \right]^{-1-\alpha} \right) \int_X |f(y)| \omega_j(y) \, d\mu(y) \lesssim \int_X |f(y)| \, d\mu(y).
\]

Combining the estimates for \( G, H \) and \( J \), we then conclude that

\[
\int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |I(x)\theta(x)| \, d\mu(x) \lesssim \|f\|_{L^1(\mu)}.
\]

Thus, we have

\[
\mu \left( \left\{ x \in X : |I(x)| > t \right\} \cup \bigcup_j (3 \times 6^2)^2 B_j \right) \lesssim t^{-q_0} \int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |I(x)|^{q_0} \, d\mu(x) \lesssim t^{-1} \int_{X \setminus (\bigcup_j (3 \times 6^2)^2 B_j)} |f(x)| \, d\mu(x)^{q_0}
\]

which, together with (3.17), implies (3.14) and hence completes the proof of Theorem 1.19 in the case that \( k = 1 \).

Step ii) \( k > 1 \). The proof of this case is completely analogous to that of [14, Theorem 1.2], the details being omitted, which completes the proof of Theorem 1.19.

\[ \square \]
and, moreover,

\[
\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty.
\]

**Remark 4.2.** (i) We remark that the 1-weak reverse doubling condition is just the weak reverse doubling condition introduced in [9, Definition 3.1]. Moreover, it is easy to see that, if \( \epsilon_1 < \epsilon_2 \) and \( \lambda \) satisfies the \( \epsilon_1 \)-weak reverse doubling condition, then \( \lambda \) also satisfies the \( \epsilon_2 \)-weak reverse doubling condition.

(ii) Assume that \( \text{diam}(\mathcal{X}) = \infty \). Let \( a = 2^k \) and \( r = 2^{-k} \) in (4.1). Then, by (4.2), we see that, for any fixed \( x \in \mathcal{X} \),

\[
\lim_{k \to \infty} \lambda(x, 2^{-k}) \leq \lim_{k \to \infty} \frac{1}{C(2^k)} \lambda(x, 1) = 0.
\]

Thus, by the fact that \( r \to \lambda(x, r) \) is non-decreasing for any fixed \( x \in \mathcal{X} \), we further know that \( \lim_{r \to 0} \lambda(x, r) = 0 \).

On the other hand, by (4.2), we see that \( \lim_{k \to \infty} C(2^k) = \infty \). Letting \( a = 2^k \) and \( r = 1 \) in (4.1), by an argument similar to the case \( r \to 0 \), we know that, for any fixed \( x \in \mathcal{X} \), \( \lim_{r \to \infty} \lambda(x, r) = \infty \).

(iii) By Remark 1.4(i), the dominating function in the Euclidean space \( \mathbb{R}^d \) with a Radon measure \( \mu \) as in (1.1) is \( \lambda(x, r) := C_0 r^\kappa \), which satisfies the \( \epsilon \)-weak reverse doubling condition for any \( \epsilon \in (0, \infty) \).

(iv) If \( (\mathcal{X}, d, \mu) \) is an RD-space, namely, a space of homogeneous type in the sense of Coifman and Weiss with a measure \( \mu \) satisfying both the doubling and the reverse doubling conditions, then \( \lambda(x, r) := \mu(B(x, r)) \) is the dominating function satisfying the \( \epsilon \)-weak reverse doubling condition for any \( \epsilon \in (0, \infty) \). It is known that a connected space of homogeneous type in the sense of Coifman and Weiss is always an RD-space (see [45, p. 65] and [9, Remark 3.4(ii)]).

(v) We remark that the \( \epsilon \)-weak reverse doubling condition is much weaker than the assumption introduced by Bui and Duong in [2, Subsection 7.3]: there exists \( m \in (0, \infty) \) such that, for all \( x \in \mathcal{X} \) and \( a, r \in (0, \infty) \), \( \lambda(x, ar) = a^m \lambda(x, r) \).

Before we give an example, we first establish a technical lemma adapted from [10, Lemma 2.1]. It turns out that the integral kernel \( 1/[\lambda(y, d(x, y))]^{1-\alpha} \) for \( \alpha \in (0, 1) \) is locally integrable.

**Lemma 4.3.** Let \( \alpha \in (0, 1) \) and \( \lambda \) satisfy the \( \alpha \)-weak reverse doubling condition. Then there exists a positive constant \( C \), depending on \( \alpha \) and \( m \), such that, for all \( x \in \mathcal{X} \) and \( r \in (0, 2 \text{diam}(\mathcal{X})) \),

\[
\int_{B(x, r)} \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}} d\mu(y) \leq C[\lambda(x, r)]^\alpha.
\]

**Proof.** From (1.3), (1.2), (4.1) and (4.2), we deduce that

\[
\int_{B(x, r)} \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}} d\mu(y)
\]
Proof. For all \( \lambda \in (0, 1) \), \( f \in L^\infty_b(\mu) \) and \( x \in \mathcal{X} \), the fractional integral \( I_\alpha f(x) \) is defined by

\[
I_\alpha f(x) := \int_{\mathcal{X}} \frac{f(y)}{[\lambda(y, d(x, y))]^{1-\alpha}} \, d\mu(y).
\]

Notice that, if \( (\mathcal{X}, d, \mu) = (\mathbb{R}^d, |\cdot|, \mu) \), \( \lambda(x, r) = r^\kappa \) with \( \kappa \in (0, d) \) and the measure \( \mu \) is as in (1.1), then \( I_\alpha \) is just the classical fractional integral in the non-doubling space \( (\mathbb{R}^d, |\cdot|, \mu) \).

We now show that the kernel of \( I_\alpha \) satisfies all the assumptions of this article. By (1.3), we know that the integral kernel \( K_\alpha(x, y) := \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}} \) satisfies (1.7). By Remark 1.4(iii), without loss of generality, we may assume that \( \lambda \) satisfies that there exist \( \epsilon, C \in (0, \infty) \) such that, for all \( x \in \mathcal{X} \), \( r \in (0, \infty) \) and \( t \in [0, r] \),

\[
|\lambda(x, r + t) - \lambda(x, r)| \leq C \frac{t^\epsilon}{r^\epsilon} \lambda(x, r).
\]

Remark 4.4. By (4.4), we see that, for a fixed \( x \in \mathcal{X} \), \( r \to \lambda(x, r) \) is continuous on \( (0, \infty) \).

Now we show that the integral kernel \( K_\alpha \) of \( I_\alpha \) also satisfies (1.8).

Proposition 4.5. Assume that \( \lambda \) satisfies (4.4). Then the integral kernel \( K_\alpha \) of \( I_\alpha \) in (4.3) satisfies (1.8).

Proof. For all \( x, \, \tilde{x}, \, y \in \mathcal{X} \) with \( d(x, y) \geq 2d(x, \tilde{x}) \), we consider the following two cases.

Case i) \( d(x, y) \leq d(\tilde{x}, y) \). Let \( t = d(\tilde{x}, y) - d(x, y) \) and \( r = d(x, y) \). Then, by \( 0 \leq t \leq d(\tilde{x}, \tilde{x}) \leq \frac{1}{2}d(x, y) \leq d(x, y) = r \) and (4.4), we see that

\[
|\lambda(y, d(\tilde{x}, y)) - \lambda(y, d(x, y))| \leq \frac{d(\tilde{x}, y) - d(x, y)}{d(x, y)} \lambda(y, d(x, y)) \lesssim \left(\frac{d(\tilde{x}, \tilde{x})}{d(x, y)}\right)^\epsilon \lambda(y, d(x, y)).
\]

From this, \( d(x, y) \leq d(\tilde{x}, y) \), Definition 1.3 and (1.3), we further deduce that

\[
|K_\alpha(x, y) - K_\alpha(\tilde{x}, y)| \lesssim \frac{1}{[d(x, y)]^{(1-\alpha)}} \lesssim \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}} \lesssim \frac{1}{[\lambda(y, d(\tilde{x}, y))]^{1-\alpha}} \lesssim \frac{1}{[\lambda(y, d(x, y))]^{1-\alpha}}.
\]

This finishes the proof of (1.8) in this case.
Case ii) $d(\tilde{x}, y) \leq d(x, y)$. In this case, since $d(x, y) \geq 2d(x, \tilde{x})$, it follows that
\[
d(x, \tilde{x}) \leq \frac{1}{2}d(x, y) \leq \frac{1}{2}[d(x, \tilde{x}) + d(\tilde{x}, y)],
\]
and hence $d(x, \tilde{x}) \leq d(\tilde{x}, y)$. Then, by an argument similar to that used in the proof of Case i), we see that
\[
|K_\alpha(x, y) - K_\alpha(\tilde{x}, y)| \lesssim \frac{|d(x, \tilde{x})|^{(1-\alpha)}}{|d(\tilde{x}, y)|^{(1-\alpha)}|\lambda(x, d(x, y))|^{1-\alpha}},
\]
which, together with $d(x, y) \leq d(x, \tilde{x}) + d(\tilde{x}, y) \leq 2d(\tilde{x}, y)$, further implies that (1.8) holds true in this case. This finishes the proof of Proposition 4.5. \qed

To consider the boundedness of $I_\alpha$ on Lebesgue spaces, we need the following Welland inequality in the present setting, which is a variant of [11, Theorem 6.4].

**Lemma 4.6.** Assume that $\text{diam}(X) = \infty$. Let $\alpha \in (0, 1), \epsilon \in (0, \min\{\alpha, 1 - \alpha\})$ and $\lambda$ satisfy the $\epsilon$-weak reverse doubling condition. Then there exists a positive constant $C$, independent of $f$ and $x$, such that, for all $x \in X$ and $f \in L^\infty_d(\mu),$
\[
|I_\alpha f(x)| \leq C \left[ M_{1, 6}^{(\alpha+\epsilon)} f(x) M_{1, 6}^{(\alpha-\epsilon)} f(x) \right]^{1/2},
\]
where $M_{1, 6}^{(\alpha)}$ for $\alpha \in (0, 1)$ is defined as in Lemma 3.1.

**Proof.** Without loss of generality, we may assume that the right-hand side of the desired inequality is finite. Let $s \in (0, \infty)$. We write
\[
|I_\alpha f(x)| \leq \int_{B(x,s)} \frac{|f(y)|}{\lambda(y, d(x, y))^{1-\alpha}} \cdot \mu(y) + \int_{X \setminus B(x,s)} \cdots =: I + II.
\]
By (1.3), (1.2), (4.1) and (4.2), we see that,
\[
I \lesssim \int_{B(x,s)} \frac{|f(y)|}{\lambda(x, d(x, y))^{1-\alpha}} \cdot \mu(y) \leq \sum_{j=0}^{\infty} \frac{1}{\lambda(x, 2^{-j-1}s)^{1-\alpha}} \int_{B(x,2^{-j}s)} |f(y)| \cdot \mu(y) \\
\lesssim \sum_{j=0}^{\infty} \frac{1}{\lambda(x, 2^{-j-1}s)^{1-\alpha}} \int_{B(x,2^{-j}s)} |f(y)| \cdot \mu(y) \\
\lesssim |\lambda(x, s)|^{\epsilon} \sum_{j=1}^{\infty} \frac{1}{|\lambda(x, s)|^{\epsilon - \alpha}} M_{1, 6}^{(\alpha-\epsilon)} f(x) \lesssim |\lambda(x, s)|^{\epsilon} M_{1, 6}^{(\alpha-\epsilon)} f(x).
\]

Similarly, we also see that $II \lesssim |\lambda(x, s)|^{-\epsilon} M_{1, 6}^{(\alpha+\epsilon)} f(x)$. Thus,
\[
|I_\alpha f(x)| \lesssim |\lambda(x, s)|^{\epsilon} M_{1, 6}^{(\alpha-\epsilon)} f(x) + |\lambda(x, s)|^{-\epsilon} M_{1, 6}^{(\alpha+\epsilon)} f(x).
\]
By Remark 4.2(ii) and Remark 4.4, we can choose $s \in (0, \infty)$ such that
\[
|\lambda(x, s)|^{\epsilon} := \left[ \frac{M_{1, 6}^{(\alpha+\epsilon)} f(x)}{M_{1, 6}^{(\alpha-\epsilon)} f(x)} \right]^{1/2}.
\]
Then we obtain the desired conclusion and hence complete the proof of Lemma 4.6. \qed
Now we are ready to state the main theorem of this section.

**Theorem 4.7.** Assume that $\text{diam}(X) = \infty$. Let $\alpha \in (0, 1)$, $p \in (1, 1/\alpha)$ and $1/q = 1/p - \alpha$. If $\lambda$ satisfies the $\epsilon$-weak reverse doubling condition for some $\epsilon \in (0, \min\{\alpha, 1 - \alpha, 1/q\})$, then $I_{\alpha}$ is bounded from $L^p(\mu)$ into $L^q(\mu)$.

**Proof.** Let $q_{\epsilon}^{-} := \frac{1}{q} - \epsilon$, $q_{\epsilon}^{+} := \frac{1}{q} + \epsilon$, $q := 2^{q_{\epsilon}^{+}}$ and $q_{\epsilon}^{-} := 2^{q_{\epsilon}^{-}}$. Then we have $1 < p < q_{\epsilon}^{-} < q < q_{\epsilon}^{+} < \infty$, $1 < q_{\epsilon}^{-} < q_{\epsilon}^{+} < \infty$ and $1/q_{\epsilon}^{-} + 1/q_{\epsilon}^{+} = 1$. From Lemma 4.6, Hölder’s inequality and Lemma 3.1, it follows that

$$
\|I_{\alpha}f\|_{L^q(\mu)} \lesssim \left\|M_{1/(\alpha+\epsilon)}^{1/2} f\right\|_{L^q(\mu)}^{1/2} \left\|M_{1/(\alpha-\epsilon)}^{1/2} f\right\|_{L^q(\mu)}^{1/2} \lesssim \|f\|_{L^p(\mu)}^{1/2} \|f\|_{L^p(\mu)}^{1/2} \sim \|f\|_{L^p(\mu)},
$$

which completes the proof of Theorem 4.7. \qed

From Theorems 4.7, 1.13, 1.15 and 1.19, we immediately deduce the following interesting conclusions, the details being omitted.

**Corollary 4.8.** Under the same assumption as that of Theorem 4.7, all the conclusions of Theorems 1.13, 1.15 and 1.19 hold true, if $T_{\alpha}$ therein is replaced by $I_{\alpha}$ as in (4.3).

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