A THOM ISOMORPHISM IN FOLIATED DE RHAM THEORY

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In memory of Hans Duistermaat

ABSTRACT. We prove a Thom isomorphism theorem for differential forms in the setting of transverse Lie algebra actions on foliated manifolds and foliated vector bundles.

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1. INTRODUCTION

Any multiplicative cohomology theory has a Thom isomorphism theorem, which relates the cohomology of the Thom space of an oriented vector bundle to the cohomology of its base. The Thom isomorphism theorem for de Rham cohomology theory is well known and can be found for instance in the textbooks [4, § 6] and [11, Ch. LIX]. We present an alternative proof of the Thom isomorphism for de Rham cohomology, which as far as we know was first given by Paradan and Vergne in their unpublished preprint [20, § 4]. Their proof, which is an adaptation of Atiyah’s elegant proof of Bott periodicity [2], offers some advantages: it is quite short, and it leads to an explicit homotopy equivalence between de Rham complexes, which is useful in extending the Thom isomorphism theorem in new directions.

As an application we establish a Thom isomorphism theorem for the “equivariant basic” differential forms on a foliated vector bundle (Theorem 4.6.1). These differential forms are basic with respect to foliations on the base manifold and the total space of a vector bundle, and they are equivariant with respect to a “transverse” action of a Lie algebra on these foliations. This theorem is motivated by results of Töben [23] and Goertsches et al. [9], and finds an application in our paper [14] on cohomological localization formulas for transverse Lie algebra actions on Riemannian foliations. (A Thom isomorphism is also available for the equivariant version of the Crainic-Moerdijk Čech-de Rham complex [17, § 7], [7, § 2], but we have not pursued this here.)
Constructing a Thom form in this setting turns out to be not always possible, and necessitates a rather long excursion into the Cartan-Chern-Weil theory of equivariant characteristic forms, which we have placed in Appendix A. In the geometric cases of interest to us, where the vector bundle is the normal bundle of a submanifold of a manifold equipped with a Riemannian foliation, an equivariant basic Thom form exists, as we show in §5. A consequence of this fact is the existence of “wrong-way” homomorphisms and a long exact Thom-Gysin sequence in equivariant basic de Rham theory, as we also discuss in §5.

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2. Preliminaries

Throughout this paper $M$ denotes a paracompact smooth manifold and $\pi: E \to M$ a real vector bundle of rank $r$ with zero section $\zeta: M \to E$. For simplicity we assume the vector bundle $E$ to be oriented, noting only that the non-orientable case of the Thom isomorphism can be handled by using forms with coefficients in the orientation bundle as in [4, §7]. A subset $A$ of $E$ is vertically compact if the restriction $\pi|_A: A \to M$ is a proper map. We denote the family of all vertically compact subsets by “cv” and say that a differential form on $E$ is vertically compactly supported if its support is a member of cv.

We denote the translation functor on cochain complexes by $[k]$. So if $(C, d)$ is a cochain complex we have $C[k]^i = C^{i+k}$ and $d[k] = (-1)^kd$. The algebraic mapping cone of a morphism of complexes $\phi: C \to D$ is the complex $C(\phi)$ with $C(\phi)^i = C^i \oplus D^{i-1}$ and differential $d(x, y) = (dx, \phi(x) - dy)$. We have an obvious short exact sequence $D[-1] \to C(\phi) \to C$; the connecting homomorphism in the associated long exact sequence in cohomology is the map $H^i(C) \to H^i(D)$ induced by $\phi$. We say $\phi$ is a quasi-isomorphism if it induces an isomorphism in cohomology; in other words if its mapping cone is acyclic. A smooth map $f: X \to Y$ induces a map of de Rham complexes $f^*: \Omega(Y) \to \Omega(X)$, the mapping cone of which we denote by $\Omega(f)$. If $f$ is the inclusion of a smooth submanifold, we call $\Omega(f)$ the relative de Rham complex and denote it by $\Omega(Y, X)$.

3. The Thom isomorphism

In this expository section we review the proof of the Thom isomorphism theorem for differential forms on vector bundles over manifolds given by Paradan and Vergne [20, §4], which is a translation into de Rham theory of Atiyah’s proof of Bott periodicity [2]. The trick is to take the direct sum of two copies of the vector bundle and to exploit the additional symmetries (the interchange map and the rotation) produced in this way. Given the existence of a Thom form, the Thom isomorphism (Theorem 3.1.1) then follows quickly from standard properties of differential forms. The proof yields an explicit, though rather long formula for a homotopy equivalence between the two de Rham complexes, which will be of use in the next section. Other uses of the Paradan-Vergne idea can be found in [8].

It was Chern [6] who first noticed that the Thom form has a natural primitive defined on the complement of the zero section, which is now called a transgression form; see also Mathai and Quillen [15, §7]. The Thom isomorphism can be expressed either in terms of vertically compactly supported forms on the vector bundle $E$, or in terms of relative differential forms on $E$ modulo the complement of the zero section. It is well known, but appears not to be recorded in the literature, that these two pictures are isomorphic. We give a proof of this fact (Proposition 3.2.2) that is based on the notion of transgression.
3.1. **The Thom isomorphism.** The vertically compactly supported differential forms on $E$ constitute a subcomplex $\Omega_{cv}(E)$ of the de Rham complex $\Omega(E)$, and we write its cohomology as $H_{cv}(E)$. By assumption $E$ is orientable, so every $k$-form $\beta$ in $\Omega_{cv}(E)$ can be integrated over the fibres of $E$ (see Appendix B) and the result is a $k-r$-form on $M$ denoted by $\pi_\ast \beta$. The projection formula (B.1) shows that the fibre integration map $\pi_\ast : \Omega_{cv}(E)[r] \to \Omega(M)$ is a morphism of graded left $\Omega(M)$-modules. The map in cohomology induced by $\pi_\ast$ is also denoted by $\pi_\ast$. A Thom form of $E$ is an $r$-form $\tau \in \Omega_r^c(E)$ which satisfies $\pi_\ast \tau = 1$ and $d\tau = 0$. It is well known that Thom forms exist. An explicit construction is given in [15]; see also §4 below.

3.1.1. **Theorem.** Let $E$ be an oriented vector bundle of rank $r$ over $M$.

(i) Fibre integration $\pi_\ast : \Omega_{cv}(E)[r] \to \Omega(M)$ is a homotopy inverse of $\pi_\ast$ is the Thom map

$$\zeta : \Omega(M) \to \Omega_{cv}(E)[r]$$

defined by $\zeta_*(\alpha) = \tau \wedge \pi^* \alpha$, where $\tau \in \Omega_r^c(E)$ is a Thom form of $E$. A homotopy $\zeta : \pi_\ast \simeq \id$ is given by (3.1.5) below.

(ii) All Thom forms of $E$ are cohomologous. Their cohomology class $\Th(E) \in H_{cv}^r(E)$ is uniquely determined by the property that $\pi_\ast(\Th(E)) = 1$.

(iii) $H_{cv}(E)$ is a free $\Omega(M)$-module of rank 1 generated by the Thom class $\Th(E)$.

**Proof.** (i) Let $\tau$ be a Thom form of $E$. Then $\pi_\ast \tau = 1 \in \Omega^0(M)$, so by the projection formula

$$\pi_\ast \zeta_*(\alpha) = \pi_\ast (\tau \wedge \pi^* \alpha) = \pi_\ast \tau \wedge \alpha = \alpha$$

for all $\alpha \in \Omega(M)$. This shows that $\pi_\ast \zeta_* = \id$. Our next task is to find a cochain homotopy $\kappa$ of the complex $\Omega_{cv}(E)$ with the property

$$\zeta_* \pi_\ast = \id = \partial \kappa + \kappa \partial.$$

Let $\beta \in \Omega_{cv}(E)$. Then $\zeta_\ast \pi_\ast (\beta) = \tau \wedge \pi^* \beta_\ast$. To rewrite this expression we introduce the direct sum bundle $E \oplus E$, which has two projection maps $p_1, p_2 : E \oplus E \to E$ that make up a pullback diagram

$$E \oplus E \xrightarrow{p_2} E \xleftarrow{p_1} E \xrightarrow{\pi} M.$$

The pullback property (Lemma B.2(iii)) and the projection formula give

$$\zeta_\ast \pi_\ast (\beta) = \tau \wedge \pi^* \beta_\ast = \tau \wedge p_1 \ast (p_2^* \beta) = (-1)^r p_1 \ast (p_2^* \tau \wedge p_2^* \beta).$$

Write elements of $E \oplus E$ as pairs $(a, b)$ and let $\phi(a, b) = (b, a)$ be the automorphism that switches the two copies of $E$. Then $p_1 \phi = p_2$ and $p_2 \phi = p_1$, so

$$\zeta_\ast \pi_\ast (\beta) = (-1)^r p_1 \ast (p_2^* \tau \wedge p_2^* \beta) = (-1)^r p_1 \ast \phi^* (p_2^* \tau \wedge p_2^* \beta).$$

The form $p_2^* \beta \in \Omega(E \oplus E)$ is not vertically compactly supported with respect to the bundle projection $E \oplus E \to M$, but the product $p_2^* \tau \wedge p_2^* \beta$ is, and therefore

$$p_2^* \tau \wedge p_2^* \beta \in \Omega_{cv}(E \oplus E).$$

The automorphism $\phi$ of $E \oplus E$ is the composition $\phi = \varphi_1 \circ \sigma$ of the quarter-turn map $\varphi_1(a, b) = (-b, a)$ and the reflection $\sigma(a, b) = (a, -b)$. The map $\varphi_1$ is homotopic to the identity through the family of rotations $\varphi : [0, 1] \times (E \oplus E) \to E \oplus E$ defined by

$$\varphi(t, a, b) = (\cos(\frac{1}{4} \pi t)a - \sin(\frac{1}{4} \pi t)b, \sin(\frac{1}{4} \pi t)a + \cos(\frac{1}{4} \pi t)b).$$
By Corollary B.5 this homotopy induces a cochain homotopy $p_\ast \varrho^\ast$ of the complex $\Omega(E \oplus E)$,

\[
\varrho^\ast_1 - \text{id} = dp_\ast \varrho^\ast + p_\ast \varrho^\ast d,
\]

where $p : [0, 1] \times (E \oplus E) \to E \oplus E$ is the projection. The homotopy $p_\ast \varrho^\ast$ preserves forms that are vertically compactly supported with respect to the bundle projection $E \oplus E \to M$, so (3.1.4) is valid as a homotopy of the complex $\Omega_{cv}(E \oplus E)$. Since $\phi = \varrho_1 \circ \varrho$, this yields

\[
\varrho^\ast = \varrho^\ast_1 \circ \varrho = (\text{id} + dp_\ast \varrho^\ast + p_\ast \varrho^\ast d).
\]

We substitute this formula into (3.1.3),

\[
\begin{align*}
\zeta_\ast \pi_\ast(\beta) &= (-1)^r p_{1,\ast} \varrho^\ast_1 \circ \varrho (\text{id} + dp_\ast \varrho^\ast + p_\ast \varrho^\ast d)(p_2^\ast \tau \wedge p_1^\ast \beta) \\
&= p_{1,\ast} \circ \varrho (\text{id} + dp_\ast \varrho^\ast + p_\ast \varrho^\ast d)(p_2^\ast \tau \wedge p_1^\ast \beta) \\
&= \pi_\ast(\tau) \wedge \beta + (p_{1,\ast} \circ \varrho^\ast_1)(p_2^\ast \tau \wedge p_1^\ast \beta) \\
&= \beta + (\varrho^\ast_1)(p_2^\ast \tau \wedge p_1^\ast \beta) \\
&= \beta + (\gamma \varrho^\ast_1)(p_2^\ast \tau \wedge p_1^\ast \beta),
\end{align*}
\]

and so we see that (3.1.2) holds. Here we used the fact that $p_{1,\ast} \varrho^\ast_1 = (-1)^r p_{1,\ast}$ (since $\varrho$ reverses the fibres of $p_1$), the projection formula, the fact that $[p_{1,\ast}, d] = 0$ (Lemma B.2(ii)), and the fact that $d \tau = 0$. The notation $l(\gamma)$ means left multiplication by a form $\gamma$, and the formula for the homotopy is

\[
\kappa = (-1)^r p_{1,\ast} \circ \varrho^\ast_1 \circ l(p_2^\ast \tau) \circ p_1^\ast.
\]

(ii) This assertion follows immediately from (i).

(iii) It follows from (i) that the Thom map in cohomology $\zeta_\ast : H(M) \to H_{cv}(E)[r]$ is an isomorphism of $H^\ast(M)$-modules. Hence $H_{cv}(E)[r]$ is free on the single generator $\zeta_\ast(1) = [\tau]$. QED

3.2. Transgression and relative forms. The punctured vector bundle is $E^\times = E \setminus \zeta(M)$. Define a projection $p$ and a homotopy $h$

\[
E^\times \xleftarrow{p} [1, \infty) \times E^\times \xrightarrow{h} E
\]

by $p(t, v) = t$ and $h(t, v) = tv$. For every vertically compact subset $A$ of $E$ the map $p : h^{-1}(A) \to E^\times$ is proper; in other words the subset $h^{-1}(A)$ of $[1, \infty) \times E^\times$ is vertically compact for $p$. Therefore we have a well-defined map

\[
\psi = p \circ h : \Omega_{cv}(E) \to \Omega(E^\times)[{-1}],
\]

which we call the transgression map. Lemma B.2 (ii) shows that $[d, \phi] = h_1^\ast$. Since $h_1$ is the inclusion $E^\times \to E$, we obtain the transgression formula

\[
\beta|_{E^\times} = d\phi(\beta) + \phi(d\beta)
\]

for all $\beta \in \Omega_{cv}(E)$. In particular, if $\beta$ is closed (for instance the Thom form), its restriction to $E^\times$ has a primitive $\phi(\beta)$. (If we regard the map $h$ as a homotopy $[0, \infty) \times E^\times \to E$, as in [15, §7], we obtain a similar formula for $h_0^\ast \beta = (\pi^\ast \zeta^\ast \beta)|_{E^\times}$.)

The following proposition is a folklore result, which provides several alternative models for vertically compactly supported cohomology. Recall (see §2) that the relative de Rham complex $\Omega(E, E^\times)$ is the mapping cone of the restriction map $\Omega(E) \to \Omega(E^\times)$. The transgression formula (3.2.1) can be interpreted as saying that the map

\[
\psi : \Omega_{cv}(E) \to \Omega(E, E^\times)
\]

defined by $\psi(\beta) = (\beta, \phi(\beta))$ is a morphism of complexes. The result asserts that $\psi$ induces an isomorphism in cohomology. There are two other complexes quasi-isomorphic
to $\Omega_{cv}(E)$, the definition of which involves the choice of a Riemannian fibre metric on $E$. Let $BE$ be the unit disc bundle and $SE = \partial (BE)$ the unit sphere bundle with respect to this metric. In order not to overload the notation, we will use $j$ for any of the four inclusion maps

$$BE \hookrightarrow E, \quad SE \hookrightarrow E^\times, \quad (BE, SE) \hookrightarrow (E, E^\times), \quad SE \hookrightarrow \overline{E \setminus BE},$$

where $\overline{E \setminus BE}$ denotes the closure of $E \setminus BE$ in $E$. Then we have a restriction map $j^*: \Omega(E, E^\times) \to \Omega(BE, SE)$. Let $\Omega_{BE}(E)$ be the collection of all differential forms on $E$ which have support contained in $BE$. Then $\Omega_{BE}(E)$ is a subcomplex of $\Omega_{cv}(E)$ and we denote the inclusion map by $\iota: \Omega_{BE}(E) \hookrightarrow \Omega_{cv}(E)$.

### 3.2.2. Proposition. The morphisms

$$\Omega_{BE}(E) \xrightarrow{j^*} \Omega_{cv}(E) \xrightarrow{\psi} \Omega(E, E^\times) \xrightarrow{j^*} \Omega(BE, SE)$$

are quasi-isomorphisms.

**Proof.** There exists a Thom form $\tau$ which is supported on $BE$, as shown for instance in §4 below. With such a choice of $\tau$ we have $\zeta_\alpha(\alpha) = \tau \wedge \pi^*\alpha \in \Omega_{BE}(E)$ for all $\alpha \in \Omega(M)$. Let us write $\zeta^B_\alpha$ for $\zeta_\alpha$ viewed as a map from $\Omega(M)$ into $\Omega_{BE}(E)$. Then $\zeta_\alpha = j_* \circ \zeta^B_\alpha$. Theorem 3.1.1 says that $\zeta_\alpha$ is a quasi-isomorphism, so to prove the proposition it suffices to show that the maps

$$(3.2.3) \quad \zeta^B: \Omega(M)[-r] \to \Omega_{BE}(E),$$

$$(3.2.4) \quad j^* \circ \psi \circ j_*: \Omega_{BE}(E) \to \Omega(BE, SE),$$

$$(3.2.5) \quad j^*: \Omega(E, E^\times) \to \Omega(BE, SE)$$

are quasi-isomorphisms. The map $\zeta^B$ has a left inverse $\pi^B_\alpha$, defined by integration over the fibres of the projection $\pi^B: BE \to M$. We also have the formula

$$(3.2.6) \quad \zeta^B \pi^B_\alpha = (dk + kd)(\beta)$$

for all $\beta \in \Omega_{BE}(E)$, where $k$ is as in (3.1.5). But the support of $\kappa(\beta) \in \Omega(E)$ may not be contained in $BE$, so (3.2.6) is not valid as a homotopy of the complex $\Omega_{BE}(E)$. Indeed, the support of $p^*_1 \tau$ and $p^*_2 \beta$ is contained in the “bi-disc” bundle $BE \times_E BE$, so we see from (3.1.5) that the support of $\kappa(\beta)$ is a subset of $p_1^*p_2^* \Omega(\overline{BE} \times_E BE)$, which is contained in the disc bundle $B_RE$ of radius $R = \sqrt{2}$. (Here the maps $p_1, p_2, \rho$, and $p$ are as in the proof of Theorem 3.1.1.) We fix this by retracting the disc bundle $B_RE$ into the disc bundle $BE$: we have a homotopy $\tilde{h}$ and a projection $\tilde{p}$

$$E \xrightarrow{\tilde{p}} [1, R] \times E \xrightarrow{\tilde{h}} E$$

defined by $h(t, v) = tv$ and $p(t, v) = t$. By Lemma B.2 (ii) this homotopy gives rise to a homotopy formula on the de Rham complex $\Omega(E)$, namely $h^*_R - id^*_E = [\lambda, d\lambda]$, where $\lambda = \tilde{p}^* \circ \tilde{h}^*$. Combining this with (3.2.6) gives

$$(3.2.7) \quad \zeta^B \pi^B_\alpha = (d\mu + \mu d)(\beta)$$

for all $\beta \in \Omega_{BE}(E)$, where $\mu = h^*_R \circ \kappa + \lambda \circ (\zeta^B \pi^B_\alpha - id^*_E)$. From $\text{supp}(\beta) \subseteq BE$ we get $\text{supp}(h^*_R \kappa(\beta)) \subseteq \overline{h^*_R(B_R E) = BE}$ and $\text{supp}(\lambda(\beta)) \subseteq \overline{\tilde{p} h^{-1}(BE) = BE}$, so $\text{supp}(\mu(\beta)) \subseteq BE$. So (3.2.7) shows that $\pi^B_\alpha$ is a homotopy inverse of $\zeta^B_\alpha$, and hence (3.2.3) is a quasi-isomorphism. Next we show (3.2.4) is a quasi-isomorphism. Let $\beta \in \Omega_{cv}(E)$ and let $A$ be the support of $\beta$. Then the support of $\phi(\beta) = p_* h^*(\beta)$ is contained in $p(h^{-1}(A)) = \bigcup_{t \geq 1} \frac{1}{t} A$. It follows that the transgression $\phi$ maps $\Omega_{BE}(E)$ to $\Omega_{BE}(E)$. In particular, if
$\beta \in \Omega_{BE}(E)$, then $\phi(\beta) = 0$ on $SE$ and hence $j^* \psi j_*(\beta) = j^*(\beta, \phi(\beta)) = (\beta, 0)$. In other words the map (3.2.4) is just the map induced by the inclusion $j: BE \to E$. The complexes $\Omega_{BE}(E)$ and $\Omega(BE, SE)$ fit into two short exact sequences

$$
\Omega_{BE}(E) \to \Omega(E) \to \Omega(BE) \to \Omega(\Omega(E)) \to \Omega(\Omega(BE)).
$$

We have two corresponding long exact cohomology sequences and a map between them,

$$
\cdots \to H^k_{BE}(E) \to H^k(BE) \to H^k(BE, SE) \to H^{k+1}_{BE}(E) \to \cdots
$$

The inclusions $BE \to E$ and $SE \to E\backslash BE$ are homotopy equivalences, so by the homotopy lemma, Corollary B.5, they induce isomorphisms $H^k(E) \cong H^k(BE)$ and $H^k(E\backslash BE) \cong H^k(SE)$. Hence $j^*: H^k_{BE}(E) \to H^k(BE, SE)$ is also an isomorphism, i.e. (3.2.4) is a quasi-isomorphism. The proof that (3.2.5) is a quasi-isomorphism is similar but easier, and uses the fact that the inclusions $BE \to E$ and $SE \to E^\times$ are homotopy equivalences. QED

3.3. The Thom-Gysin sequences. The Thom isomorphism theorem leads to two exact sequences, Propositions 3.3.1 and 3.3.4 below, each of which is known as the Thom-Gysin sequence. If $\tau \in \Omega_c^*(E)$ is a Thom form on the bundle $E$, then $\eta = \xi^* \tau = \xi^* \xi \zeta, 1 \in \Omega^*(M)$ is called the Euler form corresponding to $\tau$. Its class $\text{Eu}(E) = [\eta] \in H^r(M)$ is the Euler class of $E$. Let $SE$ be the sphere bundle of $E$ with respect to some Riemannian fibre metric on $E$ and let $\pi_{SE} = \pi|SE: SE \to M$ be the projection.

3.3.1. Proposition. The sequence

$$
\cdots \to H^{k-r}(M) \xrightarrow{\text{Eu}(E) \cup} H^k(M) \xrightarrow{\pi_{BE}^*} H^k(SE) \xrightarrow{\pi_{SE}^*} H^{k+r}(M) \to \cdots
$$

is exact.

Proof. Make the following substitutions into the long exact sequence for the pair $(BE, SE)$:

$$
\cdots \to H^k(BE, SE) \to H^k(SE) \xrightarrow{\delta} H^k(BE, SE) \to \cdots
$$

Here $\pi_{BE}: BE \to M$ is the projection and $i_{SE}: SE \to BE$ is the inclusion. The isomorphism $H^k(BE, SE) \cong H^{k-r}(M)$ follows from Theorem 3.1.1 and Proposition 3.2.2 and the isomorphism $H^k(BE) \cong H^k(M)$ follows from the fact that $\pi_{BE}: BE \to M$ is a deformation retraction. The map $H^{k-r}(M) \to H^k(M)$ is given by

$$
\alpha \mapsto \xi^* \zeta \alpha = \xi^* (\tau \wedge \pi_{BE}^* \alpha) = \eta \wedge \alpha,
$$

and the map $H^k(M) \to H^k(SE)$ is $i^*_SE \circ \pi_{BE}^* = \pi_{SE}^*$. The map $\pi_{BE}^* \circ \delta: H^k(SE) \to H^{k+r}(M)$ can be described as follows. Let $\lambda$ be a closed $k$-form on $SE$, extend it to a form $\mu$ on $BE$; then $v = (d\mu, \lambda)$ is a cocycle in $\Omega^{k+1}(BE, SE)$ and $\delta([\alpha]) = [v]$. Also

$$
\pi_{BE}^* v = \pi_{BE}^* d\mu = \pi_{SE}^* \lambda + (-1)^r d\pi_{BE}^* \lambda,
$$

where the second equality follows from Lemma B.2(ii). We conclude that $\pi_{BE}^* \delta[\lambda] = \pi_{SE}^* [\lambda]$. QED
The de Rham complex has the following excision property.

3.3.2. Lemma. Let $X$ be a manifold and $i: Y \to X$ a closed submanifold with normal bundle $N = \iota^*TX/\iota^*TY$. Choose a tubular neighborhood embedding $i_N: N \to X$. Then $i_N^*: \Omega(X, X\setminus Y) \to \Omega(N, N^\alpha)$ is a quasi-isomorphism.

Proof. Put $Z = X\setminus Y$ and let $\{U, V\}$ be an open cover of $X$. We have two short exact Mayer-Vietoris sequences of complexes

$$\Omega(X) \longrightarrow \Omega(U) \oplus \Omega(V) \longrightarrow \Omega(U \cap V),$$

$$\Omega(Z) \longrightarrow \Omega(Z \cap U) \oplus \Omega(Z \cap V) \longrightarrow \Omega(Z \cap U \cap V).$$

Forming mapping cones we obtain an exact sequence of relative de Rham complexes

$$\Omega(X, Z) \longrightarrow \Omega(U, Z \cap U) \oplus \Omega(V, Z \cap V) \longrightarrow \Omega(U \cap V, Z \cap U \cap V).$$

Taking $U = i_{N}(N)$ and $V = Z$ yields the exact sequence

$$\Omega(X, Z) \longrightarrow \Omega(N, N^\alpha) \oplus \Omega(Z, Z) \longrightarrow \Omega(N^\alpha, N^\alpha).$$

Writing the corresponding long exact cohomology sequence gives the result. QED

In the setting of Lemma 3.3.2 suppose that $Y$ is co-oriented, let $r$ be the codimension of $Y$, and choose a Thom form $\tau(N) \in \Omega^r(N)$ of $N$. For each $\beta \in \Omega^r(N)$ the form $(i_N^*)^r\beta$ is supported near $Y$, so extends by zero to a form on $X$, which we denote by $i_{N,*}\beta$. The wrong-way homomorphism is the degree $r$ morphism of complexes

$$(3.3.3) \quad i_* = i_{N,*} \circ \zeta_{N,*}: \Omega(Y)[-r] \longrightarrow \Omega(X)$$

defined by $i_*\alpha = i_{N,*}(\tau(N) \wedge \pi_N^*\alpha)$.

3.3.4. Proposition. Let $X$ be a manifold, let $Y$ be a closed co-oriented submanifold of codimension $r$, and let $i: Y \to X$ and $j: X\setminus Y \to X$ be the inclusions. We have a long exact sequence

$$\cdots \longrightarrow H^{k-r}(Y) \overset{i_*}{\longrightarrow} H^k(X) \overset{f^*}{\longrightarrow} H^k(X\setminus Y) \longrightarrow H^{k-r+1}(Y) \longrightarrow \cdots$$

Proof. In the long exact sequence for the pair $(X, X\setminus Y)$ the term $H^k(X, X\setminus Y)$ is isomorphic to $H^k(N, N^\alpha)$ by Lemma 3.3.2, which is isomorphic to $H^{k-r}(Y)$ by Theorem 3.1.1 and Proposition 3.2.2.

QED

4. THE EQUIVARIANT BASIC THOM ISOMORPHISM

As in the previous section, $\pi: E \to M$ denotes a smooth oriented real vector bundle over a manifold $M$. The treatment of the Thom isomorphism theorem given in §3 can be readily modified to apply to various subcomplexes and extensions of the de Rham complex. For instance, if the vector bundle is equivariant with respect to a compact Lie group $G$, it leads to a new proof of the Thom isomorphism in $G$-equivariant de Rham theory. This proof has the advantage over proofs such as the one given in [12, §10.6] that it offers an explicit homotopy equivalence between the $G$-equivariant de Rham complexes $\Omega_{G, cv}(E)[r]$ and $\Omega_G(M)$.

In this section, for the purpose of our work [14], we will consider a generalization of this $G$-equivariant case, namely a situation where both $M$ and $E$ are equipped with foliations and where a finite-dimensional Lie algebra $\mathfrak{g}$ acts transversely on the foliated manifolds $M$ and $E$. We obtain an “equivariant basic” Thom isomorphism, Theorem 4.6.1, for differential forms which are equivariant with respect to the action of $\mathfrak{g}$ and basic with respect to the foliations. This theorem can be regarded as a substitute for a Thom isomorphism for
A Thom class does not necessarily exist in this context. In § 4.7 we offer a simple example where it does not exist and in § 4.8 we state a sufficient condition for when it does exist. For our purposes the most important case where an equivariant basic Thom class exists is when $M$ is a submanifold of a Riemannian foliated manifold and $E$ is the normal bundle of $M$, as we will discuss in § 5.

We start with a review of some topics in foliation theory (§§ 4.1–4.5), standard references for which include [13], [18], [19], and [24].

4.1. Foliations. By a foliation of $M$ we mean a smooth and regular (i.e. constant rank) foliation. We denote the 0-dimensional foliation of $M$ by $* = *_M$. The tangent bundle of a foliation $\mathcal{F}$ is written as $T\mathcal{F}$, and the leaf of a point $x \in M$ as $\mathcal{F}(x)$. We call the dimension of the leaves $\mathcal{F}(x)$ the rank or dimension of the foliation. If $\mathcal{F}$ is a foliation of $M$ and $U$ an open subset of $M$, the restricted foliation $\mathcal{F}|_U$ is the foliation of $U$ whose leaves are the connected components of the intersections $U \cap \mathcal{F}(x)$. We have $T(\mathcal{F}|_U) = (T\mathcal{F})|_U$. We say that a smooth map $f: M \to M'$ of foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ is foliate if $f$ maps each leaf $\mathcal{F}(x)$ of $\mathcal{F}$ to the leaf $\mathcal{F}'(f(x))$ of $\mathcal{F}'$, and that $f$ is a foliate isomorphism if $f$ is a diffeomorphism and $f$ and $f^{-1}$ are foliate.

The space of sections of the tangent bundle of a foliation $\mathcal{F}$ is a Lie subalgebra $\mathfrak{X}(\mathcal{F})$ of the Lie algebra of vector fields $\mathfrak{X}(M)$. Let $\mathfrak{R}(\mathcal{F}) = N_{\mathfrak{X}(M)}(\mathfrak{X}(\mathcal{F}))$ be the normalizer of this subalgebra. Elements of $\mathfrak{R}(\mathcal{F})$ are foliate vector fields, i.e. vector fields whose flow consists of foliate maps. The quotient $\mathfrak{R}(\mathcal{F})/\mathfrak{X}(\mathcal{F})$ is a Lie algebra, which we denote by $\mathfrak{X}(M, \mathcal{F})$, and the elements of which we call transverse vector fields. Thus a transverse vector field is not a vector field, but an equivalence class of foliate vector fields modulo $\mathfrak{X}(M, \mathcal{F})$. The flow of a transverse vector field is well-defined only up to a flow along the leaves of $\mathcal{F}$.

The notion of a transverse vector field is a substitute for the ill-defined notion of a vector field on the leaf space $M/\mathcal{F}$. We say that the leaf space is a manifold if $M/\mathcal{F}$ is equipped with a smooth structure with respect to which the quotient map $M \to M/\mathcal{F}$ is a submersion. If $M/\mathcal{F}$ is a manifold, transverse vector fields on $M$ can be identified naturally with vector fields on $M/\mathcal{F}$.

4.2. Basic differential forms. Let $\mathcal{F}$ be a foliation of the manifold $M$. A differential form $\alpha$ on $M$ is $\mathcal{F}$-basic if its Lie derivatives $L(u)\alpha$ and contractions $\iota(u)\alpha$ vanish for all vector fields $u$ on $M$ that are tangent to $\mathcal{F}$. The set of $\mathcal{F}$-basic forms is a differential graded subalgebra of the de Rham complex $\Omega(M)$, which we denote by $\Omega(M, \mathcal{F})$. Its cohomology is a graded commutative algebra called the $\mathcal{F}$-basic de Rham cohomology and denoted by $H(M, \mathcal{F})$. If the leaf space $M/\mathcal{F}$ is a manifold, then the basic de Rham complex of $(M, \mathcal{F})$ is isomorphic to the de Rham complex of $M/\mathcal{F}$. A foliate map $f: (M, \mathcal{F}) \to (M', \mathcal{F}')$ induces a pullback morphism of differential graded algebras $f^*: \Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})$ and hence a morphism of graded algebras $f^*: H(M', \mathcal{F}') \to H(M, \mathcal{F})$.

4.3. Foliated bundles. Let $(M, \mathcal{F} = \mathcal{F}_M)$ and $(P, \mathcal{F}_P)$ be foliated manifolds. Let $\pi: P \to M$ be a smooth map and let $F$ be a third manifold. By a foliated fibre bundle chart on $P$ with fibre $F$ we mean a pair $(U, \phi)$, where $U$ is an open subset of $M$ and $\phi$ is a diffeomorphism $\phi: \pi^{-1}(U) \to U \times F$ which satisfies $\text{pr}_1 \circ \phi = \pi$ (with $\text{pr}_1: U \times F \to U$ being the projection onto the first factor) and which is a foliate isomorphism with respect to
the restricted foliation $\mathcal{T}_p|_{\pi^{-1}(U)}$ of $\pi^{-1}(U)$ and the product foliation $\mathcal{T}_U \times_F U \times F$.

We say that $P$ is a foliated fibre bundle with fibre $F$ if $P$ is equipped with a foliated fibre bundle atlas with fibre $F$, i.e., a collection of foliated fibre bundle charts $(U_i, \phi_i)$ with fibre $F$ whose domains $U_i$ form an open cover of $M$.

We single out some special classes of foliated fibre bundles. If the fibre is a Lie group $K$ and the foliated fibre bundle charts $(U_i, \phi_i)$ are principal $K$-bundle charts, we say $P$ is a foliated principal $K$-bundle. If the fibre is a (real) vector space and the foliated fibre bundle charts $(U_i, \phi_i)$ are vector bundle charts, we say $P$ is a foliated vector bundle. If $P$ is a foliated principal $K$-bundle over $M$ and $F$ is any manifold on which $K$ acts smoothly, then the quotient $Q = (P \times F)/K$ by the diagonal $K$-action has a natural structure of a foliated fibre bundle over $P/K = M$ with fibre $F$. We call a bundle $Q$ that arises in this way a foliated fibre bundle with structure group $K$. In particular a foliated vector bundle is a foliated fibre bundle with fibre $F = \mathbb{R}^r$ and structure group $K = \text{GL}(r, \mathbb{R})$.

Let $\pi : (P, \mathcal{T}_P) \to (M, \mathcal{T}_M)$ be a foliated fibre bundle with structure group a finite-dimensional Lie group $K$. The bundle projection $\pi$ is a foliate map and the foliations $\mathcal{T}_M$ and $\mathcal{T}_P$ have the same rank. Indeed, for each $p \in P$ the tangent map $T_p \pi : T_p P \to T_{\pi(p)} M$ maps the tangent space $T_p \mathcal{T}_P$ to the leaf $\mathcal{T}_P(p)$ isomorphically onto the tangent space $T_{\pi(p)} \mathcal{T}_M$ of the leaf $\mathcal{T}_M(\pi(p))$. Hence for each $p \in P$ the restriction of $\pi$ to the leaf $\mathcal{T}_P(p)$ is a local diffeomorphism $\mathcal{T}_P(p) \to \mathcal{T}_M(\pi(p))$, and $\pi$ induces a vector bundle isomorphism $\mathcal{T}_P \cong \pi^* T \mathcal{T}_M$. The inverse of this isomorphism is a map

$$\pi^* T \mathcal{T}_M \to TP,$$

known as the partial connection of the foliated bundle $P$, whose image is equal to $T \mathcal{T}_P$. The partial connection gives us, for each vector field $v$ tangent to $\mathcal{T}_M$, a unique vector field $v_P$ tangent to $\mathcal{T}_P$ which is $\pi$-related to $v$. The map $v \mapsto v_P$ is a Lie algebra homomorphism

$$\mathfrak{X}(\mathcal{T}_M) \to \mathfrak{X}(\mathcal{T}_P)$$

called the horizontal lifting homomorphism of the partial connection. It follows that, for each leaf $L$ of $\mathcal{T}_M$, the partial connection restricts to a genuine ( Ehresmann) connection $\pi^*(TL) \to T(P|_L)$ on $P|_L$ which is flat. Hence for each $p \in P$ the map $\mathcal{T}_P(p) \to \mathcal{T}_M(\pi(p))$ is a Galois covering map, whose covering group is the holonomy of the connection.

4.3.2. Remark. Suppose that the leaf space $\tilde{M} = M / \mathcal{T}_M$ is a manifold and that the holonomy of the partial connection on $P$ is trivial for all leaves of $M$. Then, by [19, § 2.6, Lemma 2.5], there exists a pair $(\tilde{P}, q_P)$, unique up to isomorphism, consisting of a fibre bundle $\tilde{P} \to \tilde{M}$ with structure group $K$ and an isomorphism $q_P : P \cong q^* \tilde{P}$.

Every foliated fibre bundle over $M$ with structure group $K$ is locally, in a foliation chart on $M$, isomorphic to a pullback of this kind.

An instance of a foliated principal bundle over $M$ that we shall frequently return to is the transverse frame bundle $P$ of the foliation $\mathcal{F}$, which is defined as the frame bundle of the normal bundle $N \mathcal{F} = TM / T \mathcal{F}$ of the foliation, and which is a principal $K = \text{GL}(q, \mathbb{R})$-bundle, where $q$ is the codimension of the foliation $\mathcal{F}$. We explain briefly how the foliation of $P$ comes about; see [19, § 2.4] for details. The flow $\phi_t$ of a foliate vector field $v \in \mathfrak{X}(\mathcal{F})$ is a local 1-parameter group of foliate diffeomorphisms of $M$, and so the tangent flow $T_t \phi_t$ induces a local 1-parameter group of vector bundle automorphisms of $N \mathcal{F}$, which lifts naturally to a $K$-equivariant flow $\phi_{t, P}$ of bundle automorphisms of $P$. The infinitesimal generator $\pi^* (v)$ of $\phi_{P, t}$ is a $K$-invariant vector field on $P$, and the map $v \mapsto \pi^* (v)$ is a homomorphism of Lie algebras $\pi^* : \mathfrak{X}(\mathcal{F}) \to \mathfrak{X}(P)^K$. The restriction of $\pi^*$ to $\mathfrak{X}(\mathcal{F})$
is a homomorphism $\mathfrak{X}(\mathcal{F}) \to \mathfrak{X}(P)$, which is the horizontal lifting map for a partial connection $\pi^*\mathcal{F} \to TP$ on $P$. The image of the map $\pi^*\mathcal{F} \to TP$ is the tangent bundle to the foliation $\mathcal{F}_P$ that makes $P$ a foliated principal $K$-bundle. We call the Lie algebra map
\begin{equation}
\pi^\#: \mathfrak{X}(M, \mathcal{F}) \to \mathfrak{X}(P, \mathcal{F}_P)^K
\end{equation}
induced by $\pi^\#$ the natural lifting homomorphism of the transverse frame bundle. The normal bundle $N\mathcal{F}$ is isomorphic to the associated bundle $(P \times \mathbb{R}^d)/K$ and so inherits the structure of a foliated vector bundle from $P$.

4.4. Transverse Lie algebra actions and equivariant basic differential forms. Let $(M, \mathcal{F})$ be a foliated manifold and let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. A transverse action of $\mathfrak{g}$ on $(M, \mathcal{F})$ is a Lie algebra homomorphism $\alpha$ from $\mathfrak{g}$ to the Lie algebra of transverse vector fields $\mathfrak{X}(M, \mathcal{F})$. If $\alpha$ is a transverse $\mathfrak{g}$-action on $M$, we call the triple $(M, \mathcal{F}, \alpha)$ a foliated $\mathfrak{g}$-manifold. The notion of a transverse Lie algebra action on a foliated manifold is a substitute for the ill-defined notion of a Lie algebra action on the leaf space. If the leaf space $M/\mathcal{F}$ is a manifold, a transverse $\mathfrak{g}$-action on $M$ amounts to a $\mathfrak{g}$-action on $M/\mathcal{F}$.

Let $(M, \mathcal{F}, \alpha)$ be a foliated $\mathfrak{g}$-manifold. For $\xi \in \mathfrak{g}$ let $\xi_M = \alpha(\xi) \in \mathfrak{X}(M, \mathcal{F})$ denote the transverse vector field on $M$ defined by the $\mathfrak{g}$-action. For $\alpha \in \Omega(M, \mathcal{F})$ define
\[ t(\xi)\alpha = t(\xi_M)\alpha, \quad L(\xi)\alpha = L(\xi_M)\alpha, \]
where $\xi_M \in \mathfrak{R}(\mathcal{F})$ is a foliate vector field that represents $\xi_M$. Since $\alpha$ is $\mathcal{F}$-basic, these contractions and derivatives are independent of the choice of the representative $\xi_M$ of $\xi_M$. Goertsches and Töben [10, Proposition 3.12] observed that they obey the usual rules of É. Cartan’s differential calculus, namely $[L(\xi), L(\eta)] = L([\xi, \eta])$ etc. In other words the transverse $\mathfrak{g}$-action makes the basic de Rham complex $\Omega(M, \mathcal{F})$ a $\mathfrak{g}$-differential graded algebra in the sense of [22, Appendice] or [1]. (These objects are called $\mathfrak{g}^*$-algebras in [12, Ch. 2]). See §§A.1–A.3 for a detailed definition of $\mathfrak{g}$-differential graded modules and algebras. Like any other $\mathfrak{g}$-differential graded module, the $\mathcal{F}$-basic de Rham complex $\Omega(M, \mathcal{F})$ has a Weil complex
\[ \Omega^\#(M, \mathcal{F}) = (\mathfrak{W}\mathfrak{g} \otimes \Omega(M, \mathcal{F}))^{\text{q-bas}}. \]
Here $\mathfrak{W}\mathfrak{g}$ denotes the Weil algebra of $\mathfrak{g}$, which is a differential graded commutative algebra isomorphic to $\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^*$ as an algebra, and “$\text{q-bas}$” means “$\mathfrak{g}$-basic subcomplex”. (See §§A.4–A.5.) We refer to elements of $\Omega^\#(M, \mathcal{F})$ as $\mathfrak{g}$-equivariant $\mathcal{F}$-basic differential forms. The cohomology of the Weil complex is the $\mathfrak{g}$-equivariant $\mathcal{F}$-basic de Rham cohomology $H^\#_q(M, \mathcal{F})$ of the foliated manifold with respect to the transverse action. We will frequently abbreviate the cumbersome phrase “$\mathfrak{g}$-equivariant $\mathcal{F}$-basic” to “equivariant basic”.

Let $(M', \mathcal{F}', \alpha')$ be another foliated $\mathfrak{g}$-manifold. We say that a foliate map $f : M \to M'$ is $\mathfrak{g}$-equivariant if the transverse vector fields $\xi_M$ and $\xi_{M'}$ are $\alpha$-related for all $\xi \in \mathfrak{g}$. A $\mathfrak{g}$-equivariant foliate map $f$ induces a pullback map of $\mathfrak{g}$-differential graded algebras $\Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})$. We say that a smooth map $f : [0, 1] \times M \to M'$ is a $\mathfrak{g}$-equivariant foliate homotopy if the map $f_t : M \to M'$ defined by $f_t(x) = f(t, x)$ is $\mathfrak{g}$-equivariant foliate for all $t \in [0, 1]$.

The following statement, the non-equivariant version of which is due to Töben [23, §2], is a variant of the standard homotopy lemma in de Rham theory. A morphism of $\mathfrak{g}$-differential graded modules $\phi : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a degree 0 linear map $\phi$ satisfying $[d, \phi] = [t(\xi), \kappa] = [L(\xi), \kappa] = 0$ for all $\xi \in \mathfrak{g}$. Let $\phi_0, \phi_1 : \mathfrak{M}_1 \to \mathfrak{M}_2$ be two morphisms
of $\mathfrak{g}$-differential graded modules. A homotopy of $\mathfrak{g}$-differential graded modules between two morphisms $\phi_0$ and $\phi_1$ is a degree $-1$ map $\kappa : \mathfrak{M}_1 \to \mathfrak{M}_2$ satisfying $\phi_1 - \phi_0 = [d, \kappa] = dk + k\delta$ and $[i(\xi), \kappa] = [L(\xi), \kappa] = 0$ for all $\xi \in \mathfrak{g}$.

**4.4.1. Lemma.** Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds equipped with transverse actions of a Lie algebra $\mathfrak{g}$. Let $f : [0, 1] \times M \to M'$ be a $\mathfrak{g}$-equivariant foliate homotopy. Then the pullback morphisms $f_0$ and $f_1 : \Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})$ are homotopic as morphisms of $\mathfrak{g}$-differential graded algebras. In particular they induce the same homomorphisms in equivariant basic cohomology: $f_0^* = f_1^* : H_0(M', \mathcal{F}') \to H_0(M, \mathcal{F})$.

**Proof.** As reviewed in Appendix B, on the ordinary de Rham complexes we have a homotopy $f_0^* - f_1^* = [d, \kappa]$ given by the homotopy operator $\kappa = \pi \circ f^* : \Omega(M') \to \Omega(M)[-1]$. Here $\pi : [0, 1] \times M \to M$ is the projection and $\pi_*$ denotes integration over the fibre $[0, 1]$. We assert that the operator $\kappa$ preserves the basic subcomplexes and is a homotopy of $\mathfrak{g}$-differential graded algebras when restricted to those subcomplexes. To show this let us furnish the cylinder $[0, 1] \times M$ with the foliation $\ast \times \mathcal{F}$, whose leaves are of the form $\{t\} \times \mathcal{F}(\pi)$ for $t \in [0, 1]$ and $\pi \in M$. Then the homotopy $f$ is a foliate map. Let $u$ be a vector field on $M$ tangent to $\mathcal{F}$. Then $(0, u)$ is a vector field on $[0, 1] \times M$ tangent to $\ast \times \mathcal{F}$. The vector fields $u$ and $(0, u)$ are $\pi$-related, so Lemma B.2(i) tells us that

$$i(u) \circ \pi_* = -\pi_* \circ i((0, u)), \quad L(u) \circ \pi_* = \pi_* \circ L((0, u)).$$

If $u' \in \mathfrak{X}(\mathcal{F}')$ is a vector field on $M'$ which is $f$-related to $(0, u)$, then

$$i((0, u)) \circ f^* = f^* \circ i((0, u)), \quad L((0, u)) \circ f^* = f^* \circ L((0, u)).$$

Combining these identities gives

$$i(u) \circ \kappa = -\kappa \circ i(u'), \quad L(u) \circ \kappa = \kappa \circ L(u').$$

It follows that $\kappa$ maps $\Omega(M', \mathcal{F}')$ to $\Omega(M, \mathcal{F})$. The transverse $\mathfrak{g}$-action on $(M, \mathcal{F})$ extends to a transverse $\mathfrak{g}$-action on $([0, 1] \times M, \ast \times \mathcal{F})$ by letting $\mathfrak{g}$ act trivially in the $t$-direction. Then the maps $\pi$ and $f$ are $\mathfrak{g}$-equivariant, so the transverse vector fields $\xi_M, \xi_{[0, 1] \times M}$ and $\xi_{M'}$ are $\pi$- and $f$-related. Again by Lemma B.2(i), $\pi_*$ commutes with the operations $i(\xi)$ and $L(\xi)$, and so does $f^*$. This shows that $\kappa : \Omega(M', \mathcal{F}') \to \Omega(M, \mathcal{F})[-1]$ is a homotopy of $\mathfrak{g}$-differential graded modules.

**QED**

### 4.5. Equivariant basic characteristic forms.

Let $K$ be a Lie group with Lie algebra $\mathfrak{k}$ and let $\pi : (P, \mathcal{F}_P) \to (M, \mathcal{F} = \mathcal{F}_M)$ be a foliated principal $K$-bundle as in § 4.3. For the purpose of constructing Thom classes in § 4.8 we recall how under appropriate hypotheses one can define characteristic classes of $P$ in the basic cohomology of $M$ and how, in the presence of a transverse action of a Lie algebra, these classes lift to equivariant basic classes.

A connection on the principal bundle $P$ can be viewed either as a bundle map $\pi^* TM \to TP$ or as a 1-form $\theta \in \Omega^1(P, \mathfrak{k})$. A connection on $P$ is called basic (or projectable in [19]) if the associated 1-form is $\mathcal{F}_P$-basic. According to [19, § 2.6] a connection 1-form $\theta$ is $\mathcal{F}_P$-basic if and only if for every foliation chart $U$ of $M$ with quotient manifold $\bar{U} = U/(\mathcal{F}|U)$ the restriction of $\theta$ to $P|U$ is the pullback of a connection 1-form $\theta_U \in \Omega^1(U, P_U)$ on the quotient principal bundle $P_U = P|U/(\mathcal{F}^P|\pi^{-1}(U))$ over $U$. In particular a basic connection on $P$, viewed as a bundle map $\pi^* TM \to TP$, is an extension of the partial connection $\pi^* T\mathcal{F}_M \to T\mathcal{F}_P$ defined in (4.3.1).

Now suppose $P$ is equipped with a transverse action of a finite-dimensional Lie algebra $\mathfrak{g}$ which commutes with the $K$-action. This transverse action descends to a unique transverse action on the base manifold $M$ with the property that the bundle projection $\pi : P \to M$ is $\mathfrak{g}$-equivariant. We say that $P$ is a $\mathfrak{g}$-equivariant foliated principal $K$-bundle.
A natural instance of such a bundle is the transverse frame bundle $P$ of the foliation $\mathcal{F}$. If $a: g \to \mathfrak{X}(M, \mathcal{F})$ is any transverse $g$-action on $M$, the homomorphism $\pi^T \circ a: g \to \mathfrak{X}(P, \mathcal{F}_P)^K$, where $\pi^T$ is the lifting homomorphism (4.3.3), makes $P$ a $g$-equivariant foliated principal $K$-bundle.

4.5.1. **Lemma.** Suppose that the $g$-equivariant foliated principal $K$-bundle $P$ admits a principal connection $\pi^T \nu M \to TP$ which is $g$-invariant and $\mathcal{F}_P$-basic. Then the associated connection 1-form $\theta \in \Omega^1(P, \mathcal{F}_P; \mathfrak{t})$, viewed as a map $\mathfrak{t}^* \to \Omega^1(P, \mathcal{F}_P)$ defines a $g$-invariant connection on the $\mathfrak{t}$-differential graded algebra $\Omega(P, \mathcal{F}_P)$.

**Proof.** This is a restatement of the definition of $g$-invariant connections on $\mathfrak{t}$-differential graded algebras; see §§ A.3 and A.6.

Under the hypothesis of Lemma 4.5.1 we have the $g$-equivariant characteristic homomorphism defined in (A.6.5),

$$c_{g, \theta}: S(\mathfrak{t}[2]^*\mathfrak{t}) \to (\Omega_{t\text{-bas}}(P, \mathcal{F}_P))_g.$$

Here the left-hand side denotes the algebra of $\mathfrak{t}$-invariant polynomials on $\mathfrak{t}^*$ with the generators placed in degree 2. The right-hand side is the $g$-Weil complex $(\mathcal{W}_g \otimes \Omega_{t\text{-bas}}(P, \mathcal{F}_P))_g\text{-bas}$ of the $\mathfrak{t}$-basic $\mathcal{F}_P$-basic de Rham complex of $P$. If moreover the structure group $K$ is connected, then $\Omega_{t\text{-bas}}(P, \mathcal{F}_P) = \Omega_{K\text{-bas}}(P, \mathcal{F}_P) = \Omega(M, \mathcal{F})$ as a $\mathfrak{g}$-differential graded algebra, so $(\Omega_{t\text{-bas}}(P, \mathcal{F}_P))_g = \Omega_g(M, \mathcal{F})$. So if $K$ is connected, the $g$-equivariant characteristic homomorphism is an algebra homomorphism

$$c_{g, \theta}: S(\mathfrak{t}[2]^*\mathfrak{t}) \to \Omega_g(M, \mathcal{F}).$$

Elements in the image of (4.5.2) are $g$-equivariant $\mathcal{F}$-basic characteristic forms; their cohomology classes are $g$-equivariant $\mathcal{F}$-basic characteristic classes.

If $g = 0$, the characteristic map is an algebra homomorphism $c_\theta: S(\mathfrak{t}[2]^*\mathfrak{t}) \to \Omega(M, \mathcal{F})$ and we speak of $\mathcal{F}$-basic characteristic forms and classes.

If the leaf space $\bar{M} = M/\mathcal{F}$ is a manifold and the principal bundle $P$ descends to a principal bundle $\bar{P}$ on $\bar{M}$ as in Remark 4.3.2, then the $g$-equivariant $\mathcal{F}$-basic characteristic forms of $P$ are the same as the $g$-equivariant characteristic forms of the quotient bundle $\bar{P}$.

An obstruction to the existence of basic connections on foliated principal bundles known as the Atiyah class, and examples of bundles whose Atiyah class does not vanish, can be found in [13, Ch. 8] and [19, Ch. 2]. We are not aware of any references for a $g$-equivariant version of this obstruction.

4.6. **The equivariant basic Thom isomorphism.** Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and let $M$ be a foliated $\mathfrak{g}$-manifold. Let $E$ be an oriented $\mathfrak{g}$-equivariant foliated vector bundle over $M$. An **equivariant basic Thom form** of $E$ is an $r$-form $\tau_\mathfrak{g} \in \Omega^r_{\mathfrak{g}, cv}(E, \mathcal{F}_E)$ which satisfies $\pi_\mathfrak{g}^*\tau_\mathfrak{g} = 1$ and $d_{\mathfrak{g}}\tau_\mathfrak{g} = 0$. Such a form does not necessarily exist. An example where it does not exist is given in § 4.7, and sufficient conditions for it to exist are stated in Proposition 4.8.1. If it exists we have the following $\mathfrak{g}$-equivariant $\mathcal{F}$-basic Thom isomorphism theorem.

4.6.1. **Theorem.** Suppose that $E$ admits an equivariant basic Thom form $\tau_\mathfrak{g}$. Then the following conclusions hold.

(i) **Fibre integration**

$$\pi_\mathfrak{g}: \Omega_{\mathfrak{g}, cv}(E, \mathcal{F}_E)[r] \to \Omega_\mathfrak{g}(M, \mathcal{F})$$
Moreover, the maps defined by $\zeta'(\alpha) = \tau_\theta \wedge \pi'^* \alpha$. A homotopy $\zeta \circ \pi_\theta \simeq \text{id}$ is induced by the homotopy of $g$-differential graded modules given in (4.6.4) below.

(ii) All equivariant basic Thom forms of $E$ are cohomologous. Their cohomology class $\text{Th}_\theta(E/F_E) \in H^r_{g,cv}(E, F_E)$ is uniquely determined by the property $\pi_\theta(\text{Th}_\theta(E, F_E)) = 1$.

(iii) $H_{g,cv}(E, F_E)$ is a free $H_g(M, F)$-module of rank 1 generated by the Thom class $\text{Th}_\theta(E, F_E)$.

**Proof.** We will verify that each step in the proof of Theorem 3.1.1 is valid in the present context. Only the proof of (i) requires comment. We start by showing that the fibre integral of an equivariant basic form is an equivariant basic form. The usual fibre integration map $\pi_\theta : \Omega_{g,cv}(E)[r] \to \Omega(M)$ has the properties

$$d\pi_\theta = (-1)^r \pi_* d, \quad L(v)\pi_\theta = \pi_* L(w), \quad i(v)\pi_\theta = (-1)^r \pi_* i(w)$$

for every pair of $\pi_\theta$-related vector fields $v \in \mathfrak{x}(M)$ and $w \in \mathfrak{x}(E)$. (See Appendix B.) It follows from these properties that $\pi_\theta$ maps $F_E$-basic forms to $F$-basic forms and restricts to a degree $-r$ morphism of $g$-differential graded modules

$$\pi_\theta : \Omega_{g,cv}(E, F_E)[r] \to \Omega(M, F),$$

where $\Omega_{g,cv}(E, F_E) = \Omega_{g,cv}(E) \cap \Omega(M, F)$ denotes the $g$-differential graded module of vertically compactly supported basic forms on $E$. The morphism (4.6.2) extends uniquely to an $(S^g)^b$-linear degree $-r$ morphism of Weil complexes

$$\pi_\theta : \Omega_{g,cv}(E, F_E)[r] \to \Omega_g(M, F),$$

and the projection formula (B.1) shows that this map is a degree $-r$ morphism of graded left $\Omega_g(M, F)$-modules. Now let $\zeta_\theta$ be the Thom map defined by the equivariant basic Thom form $\tau_\theta$. The identity $\pi_\theta \zeta_\theta = \text{id}$ holds as in the proof of Theorem 3.1.1. Next we must find a cochain homotopy $\kappa_\theta$ of the complex $(\Omega_{g,cv}(E, F_E), d_{\theta})$ satisfying

$$\zeta_\theta \pi_\theta - \text{id} = d_{\theta} \kappa_\theta + \kappa_{\theta} d_{\theta}.$$

By definition $\Omega_{g,cv}(E, F_E)$ is the Weil complex $M_\theta = (W_\theta \otimes M)_{g,bas}$ of the $g$-differential graded module $M = \Omega_{g,cv}(E, F_E)$, so it is enough to find a homotopy $\kappa$ of the $g$-differential graded module $W_\theta \otimes M$ with the property

$$\zeta_\theta \pi_\theta - \text{id} = d\kappa + \kappa d.$$

Replacing $\tau$ with $\tau_\theta$ in (3.1.5) we put

$$\kappa = (-1)^r p_{1,*} \circ p_* \circ \phi^* \circ l(p_2^* \tau_\theta) \circ p_{1,*}$$

where the maps

$$p_1, p_2 : E \oplus E \to E, \quad p : [0, 1] \times (E \oplus E) \to E \oplus E,$$

$$\phi : [0, 1] \times (E \oplus E) \to E \oplus E$$

are as in the proof of Theorem 3.1.1. Then $\kappa$ satisfies (4.6.3) and maps vertically compactly supported forms to vertically compactly supported forms just as before. The direct sum $E \oplus E$ is a foliated vector bundle with foliation $F_{E\oplus E}$, and the cylinder $[0, 1] \times (E \oplus E)$ carries the foliation $\ast \times F_{E\oplus E}$. With respect to these foliations the maps $p_1, p_2, \phi$ and $p$ are foliate, and the Thom form $\tau_\theta$ is $F_E$-basic, so $\kappa$ maps basic forms to basic forms. Moreover, the maps $p_1, p_2, \phi$ and $p$ are $g$-equivariant and the Thom form $\tau_\theta$, regarded an
element of \( M_\xi \subset W_g \otimes M \), is \( g \)-basic, so the map \( \kappa \) commutes with the contractions \( i(\xi) \) and the derivations \( L(\xi) \) for all \( \xi \in g \). This shows that \( \kappa \) is a homotopy of the \( g \)-differential graded module \( W_g \otimes M \).

4.7. A foliated vector bundle without basic Thom form. Not all foliated vector bundles have basic Thom forms. As an example consider the 2-torus, i.e. the product \( M = T_1 \times T_2 \) of two copies \( T_1, T_2 \) of the circle \( T = \mathbb{R}/\mathbb{Z} \). Let \( \mathcal{F} = \mathcal{F}_M \) be a foliation of \( M \) with the following properties:

(a) the \( T \)-action \( t \cdot (t_1, t_2) = (t_1, t_2t) \) on \( M \) is foliate, i.e. maps leaves to leaves;
(b) the \( T \)-orbit \( \mathcal{L}_0 = \{0\} \times T_2 \) is a leaf;
(c) the leaf \( \mathcal{L}_0 \) is in the closure of every other leaf;
(d) every leaf other than \( \mathcal{L}_0 \) is transverse to the \( T \)-orbits.

An example of such a foliation is produced at the end of [19, § 1.4].

![Figure 1. Foliation of 2-torus. Compact leaf \( \mathcal{L}_0 \) shown in bright green, noncompact leaves in blue. Orbits of \( T \)-action shown in green. Lifted foliation of unit square shown on right.](image)

The normal bundle \( E = N\mathcal{F} = TM/T\mathcal{F} \) of the foliation \( \mathcal{F} \) is a foliated vector bundle over \( M \) of rank 1. It is also an equivariant vector bundle with respect to the \( T \)-action on \( M \). The foliate vector field \( \tilde{v} = \partial/\partial t_2 \) is the generator of the \( T \)-action on \( M \). Let \( v = \tilde{v} \mod \mathfrak{X}(\mathcal{F}) \in \Gamma(E) \) be the transverse vector field determined by \( \tilde{v} \). The actions of \( v \) on \( M \) and \( E \) make \( E \) a \( g \)-equivariant foliated vector bundle, where \( g = \mathbb{R} \).

4.7.1. Lemma. (i) \( \Omega^0(M, \mathcal{F}) = \mathbb{R} \) and \( \Omega^1(M, \mathcal{F}) = 0 \).
(ii) \( \Omega^0(E, \mathcal{F}_E) = \mathbb{R} \).

Proof. (i) This is verified as in [19, pp. 17, 40].
(ii) Let \( f \) be an \( \mathcal{F}_E \)-basic function on \( E \). Viewed as a map \( M \to E \) the section \( v \) of \( E \) is foliate, and therefore the function \( f \circ v \) is \( \mathcal{F} \)-basic on \( M \). More generally, define a map \( u : \mathbb{R} \times M \to E \) by

\[
u(\lambda, x) = u_A(x) = \lambda v(x).
\]

For each \( \lambda \in \mathbb{R} \) the scalar multiple \( u_A = \lambda v \) is a foliate map \( M \to E \), and therefore for each \( \lambda \) the function \( f \circ u_A \) is \( \mathcal{F} \)-basic on \( M \). Thus \( f \circ u_A \) is constant by (i). Since \( \tilde{v} \) is tangent to the compact leaf \( \mathcal{L}_0 \) (by property (b) above), we have \( v = 0 \) on \( \mathcal{L}_0 \) and so \( u_A(x_0) = \lambda v(x_0) = 0 \) for all \( x_0 \in \mathcal{L}_0 \). Hence \( f(u_A(x)) = f(u_A(x_0)) = f(x_0) = c \) for all \( x \in M \), where \( c \) is the (constant) value of \( f \) on \( \mathcal{L}_0 \). (Here we have for simplicity identified \( M \) with the zero section of \( E \).) In other words, \( f \) is constant on the image of the map \( u \).

The transverse vector field \( v \) vanishes nowhere on \( M \setminus \mathcal{L}_0 \) (by property (d) above) and \( E \) is
of rank 1, so the image of the map \( \alpha \) contains \( E \setminus \mathcal{L}_0 \), which is dense in \( E \). We conclude that \( f = c \) on \( E \). QED

4.7.2. Proposition. If \( \alpha \in \Omega^1(E, \mathcal{F}_E) \) is closed, then \( \alpha = 0 \). It follows that \( H^1_{\mathcal{F}_E}(E, \mathcal{F}_E) = 0 \). Therefore the foliated vector bundle \( E \) has no basic Thom form, nor does \( E \) have an equivariant basic Thom form with respect to any transverse Lie algebra action on \( E \) (such as the action generated by the transverse vector field \( v \)).

Proof. The retraction along the fibres is a foliate homotopy equivalence between \( E \) and \( M \), so \( H^1(E, \mathcal{F}_E) \cong H^1(M, \mathcal{F}) \) by Lemma 4.4.1. Hence \( H^1(E, \mathcal{F}_E) = 0 \) by Lemma 4.7.1(i). Therefore \( \alpha = df \) for some basic function \( f \in \Omega^0(E, \mathcal{F}_E) \). But \( f \) is constant by Lemma 4.7.1(ii), so \( \alpha = 0 \). In particular \( H^1_{\mathcal{F}_E}(E, \mathcal{F}_E) = 0 \). On the other hand \( H^0(M, \mathcal{F}) = \mathbb{R} \neq H^1_{\mathcal{F}_E}(E, \mathcal{F}_E) \) by Lemma 4.7.1(i), so by Theorem 4.6.1 there cannot exist a basic Thom form for \( E \). If a Lie algebra \( \mathfrak{g} \) acts transversely on \( E \), then the image of an equivariant basic Thom form \( \tau_\mathfrak{g} \) under the forgetful homomorphism \( \Omega_\mathfrak{g}(E, \mathcal{F}_E) \to \Omega(E, \mathcal{F}_E) \) would be an ordinary basic Thom form, so \( \tau_\mathfrak{g} \) does not exist either. QED

4.8. Existence of Thom forms. The main result of this section is Proposition 4.8.1, which extends results of [9, Appendix A], and which gives a sufficient condition for an oriented \( \mathfrak{g} \)-equivariant foliated vector bundle \( E \) to possess an equivariant basic Thom form. The condition is that the structure group of the foliated bundle \( E \) should admit a Riemannian metric compatible with the foliation, that the Lie algebra should act isometrically, and that \( E \) should admit a \( \mathfrak{g} \)-invariant basic metric connection.

Let \((M, \mathcal{F})\) be a foliated manifold and \((E, \mathcal{F}_E)\) a oriented foliated vector bundle over \( M \). A Riemannian metric on \( E \) is a Riemannian fibre metric \( g_E \) which satisfies \( \nabla_E v = 0 \) for all \( v \in \mathfrak{X}(\mathcal{F}) \), where \( \nabla_E \) is the partial connection of \((E, \mathcal{F}_E)\). If \( \mathfrak{g} \) is a finite-dimensional Lie algebra acting transversely on \( M \), the vector bundle \( E \) is \( \mathfrak{g} \)-equivariant, and the metric \( g_E \) is \( \mathfrak{g} \)-invariant, we say \((E, \mathcal{F}_E, g_E)\) is a \( \mathfrak{g} \)-equivariant Riemannian foliated vector bundle.

4.8.1. Proposition. Let \((M, \mathcal{F}, a)\) be foliated \( \mathfrak{g} \)-manifold and \((E, \mathcal{F}_E, g_E)\) an oriented \( \mathfrak{g} \)-equivariant Riemannian foliated vector bundle over \( M \). Suppose that the oriented orthogonal frame bundle \( P \) of \( E \) admits a connection that is \( \mathfrak{g} \)-invariant and \( \mathcal{F}_P \)-basic as in Lemma 4.5.1. Then there exists an equivariant basic Thom form on \( E \), and hence the Thom isomorphism theorem, Theorem 4.6.1, applies to \( E \).

A very special case where the hypotheses of this proposition are satisfied is when the foliations \( \mathcal{F}_M \) and \( \mathcal{F}_E \) are 0-dimensional and the \( \mathfrak{g} \)-actions on \( M \) and \( E \) are induced by actions of a compact Lie group \( G \) with Lie algebra \( \mathfrak{g} \). In this case the existence of the invariant metric on \( E \) is automatic and the proposition gives the existence of \( G \)-equivariant Thom forms, which is well known and can be found e.g. in [20, §4.5]. Another case where the hypotheses are satisfied is when \( E \) is the normal bundle of \( M \) in an ambient Riemannian foliated \( \mathfrak{g} \)-manifold; we will investigate this case in §5. Proposition 4.8.1 fails for the foliation of §4.7 because that foliation is not Riemannian.

Proposition 4.8.1 is an immediate consequence of Lemma 4.8.2 below, which exhibits a specific equivariant basic Thom form on \( E \), called universal. Suppose that the conditions of Proposition 4.8.1 are satisfied. Let \( K \) be the special orthogonal group \( \text{SO}(r) \) and \( \mathfrak{f} = o(r) \) its Lie algebra. Let \( g_E \) be a fibre metric as in Proposition 4.8.1 and let \( \theta \in \Omega^1(P, \mathfrak{f}) \) be a \( \mathfrak{g} \)-invariant \( \mathcal{F}_P \)-basic connection on the oriented orthogonal frame bundle \( P \) of the \( \mathfrak{g} \)-equivariant foliated vector bundle \( E \to M \). Let \( \tau_0 \in \Omega^r_{\mathcal{F}_E}(\mathbb{R}^r) \) be a compactly supported modification of the Mathai-Quillen-Thom form on \( \mathbb{R}^r \) as defined in [15, p.98]; cf. also [12, §10.3] or [16, §8]. The subscript “c” refers to compact supports; we regard \( \tau_0 \) as a \( \mathfrak{f} \)-basic
element of the $\mathfrak{t}$-differential graded module $W_\mathfrak{t} \otimes \Omega_\mathfrak{t}(\mathbb{R}^r)$. We view this module as a submodule of the $\mathfrak{h}$-differential graded module $W_\mathfrak{h} \otimes \Omega_\mathfrak{h}(\mathbb{R}^r)$, where $\mathfrak{h}$ is the product Lie algebra $\mathfrak{t} \times \mathfrak{q}$ and where we let $\mathfrak{q}$ act trivially on $\mathbb{R}^r$. The frame bundle $P$ is a $\mathfrak{g}$-equivariant foliated principal $K$-bundle over $M$ with foliation $\mathcal{F}_P$. The product $P \times \mathbb{R}^r$ has a foliation $\mathcal{F}_P \times *$, where $*$ denotes the 0-dimensional foliation of $\mathbb{R}^r$. The $\mathfrak{g}$-action and the transverse $\mathfrak{g}$-action provide the complex $M = \Omega_{\mathfrak{g},v}(P \times \mathbb{R}^r, \mathcal{F}_P \times *)$ with the structure of an $\mathfrak{h}$-differential graded module. Let $pr_2$ be the projection onto the second factor $P \times \mathbb{R}^r \to \mathbb{R}^r$. Since $\tau_0$ is $\mathfrak{t}$-basic, the pullback $pr_2^* (\tau_0)$ is an $\mathfrak{h}$-basic element of $W_\mathfrak{h} \otimes M$. Let $C_{\mathfrak{g},0}$ be the $\mathfrak{g}$-equivariant Cartan-Chern-Weil homomorphism associated with the $\mathfrak{g}$-invariant connection $\theta$, as defined in § A.6. Then $C_{\mathfrak{g},0}(pr_2^* (\tau_0))$ is an $\mathfrak{h}$-basic element of $W_\mathfrak{g} \otimes M$. We summarize the situation with the diagram

$$
\tau_0 \in W_\mathfrak{t} \otimes \Omega_\mathfrak{t}(\mathbb{R}^r) \leftarrow W_\mathfrak{h} \otimes \Omega_\mathfrak{h}(\mathbb{R}^r) \sim pr_2^* W_\mathfrak{h} \otimes M \sim C_{\mathfrak{g},0} W_\mathfrak{g} \otimes M.
$$

The foliated vector bundle $E$ is the quotient of $P \times \mathbb{R}^r$ by the free diagonal $K$-action, so the $\mathfrak{t}$-basic subcomplex of $M$ is

$$
M_{1\text{-bas}} = \Omega_{1\text{-bas},cv}(P \times \mathbb{R}^r, \mathcal{F}_P \times *) = \Omega_{K\text{-bas},cv}(P \times \mathbb{R}^r, \mathcal{F}_P \times *) = \Omega_{cv}(E, \mathcal{F}_E).
$$

Therefore the $\mathfrak{h}$-basic submodule of $W_\mathfrak{g} \otimes M$ is

$$
(W_\mathfrak{g} \otimes M)_{\mathfrak{h}\text{-bas}} = (W_\mathfrak{g} \otimes M_{1\text{-bas}})_{\mathfrak{g}\text{-bas}} = \Omega_{\mathfrak{g},cv}(E, \mathcal{F}_E).
$$

We define the universal equivariant basic Thom form of $(E, \theta)$ to be the image

$$
\tau_{\mathfrak{g},0}(E, \mathcal{F}_E) = C_{\mathfrak{g},0}(pr_2^* (\tau_0)) \in \Omega_{\mathfrak{g},cv}(E, \mathcal{F}_E).
$$

In part (iii) of the next result, $Pf \in S^{r/2}(\pi_*^*)^1$ denotes the Pfaffian (which is defined when $r$ is even), and the map $c_{\mathfrak{g},0} : \Sigma((\pi_*)^1) \to \Omega^*_\mathfrak{g}(M, \mathcal{F})$ denotes the $\mathfrak{g}$-equivariant characteristic homomorphism (4.5.2) of the foliated bundle $E$ with respect to the invariant basic orthogonal connection $\theta$.

4.8.2. Lemma. Let $(M, \mathcal{F}, a)$ be foliated $\mathfrak{g}$-manifold and $(E, \mathcal{F}_E, g_E)$ an oriented $\mathfrak{g}$-equivariant Riemannian foliated vector bundle over $M$. Let $\theta \in \Omega^1(P, \mathfrak{t})$ be an invariant basic connection on the oriented orthogonal frame bundle $P$ of $E$. The form $\tau_{\mathfrak{g},0}(E, \mathcal{F}_E)$ has the following properties.

(i) $\tau_{\mathfrak{g},0}(E, \mathcal{F}_E)$ is an equivariant basic Thom form.
(ii) $\tau_{\mathfrak{g},0}(E, \mathcal{F}_E)$ is universal in the sense that $\tau_{\mathfrak{g},0}(f^* \mathcal{F}_E) = \tau_{\mathfrak{g},0}(f^* E, \mathcal{F}_E)$ for all $\mathfrak{g}$-equivariant foliate maps $f : (M', \mathcal{F}') \to (M, \mathcal{F})$, where $f_\mathcal{F} : f^* \mathcal{F}_E \to \mathcal{F}'$ is the natural lift of $f$.
(iii) $\zeta^* \tau_{\mathfrak{g},0}(E, \mathcal{F}_E) = 0$ if $r$ is odd and $\zeta^* \tau_{\mathfrak{g},0}(E, \mathcal{F}_E) = (-2\pi)^{-r/2} c_{\mathfrak{g},0}(Pf)$ if $r$ is even.
(iv) Let $p : P \times \mathbb{R}^r \to E$ be the quotient map for the $K$-action. Let $\mathcal{E}$ be any (possibly singular) foliation of $E$ with the property that every vector field tangent to $\mathcal{E}$ is $p$-related to a vector field on $P \times \mathbb{R}^r$ which is tangent to the fibres of $pr_2$. Then $\tau_{\mathfrak{g},0}(E, \mathcal{F}_E)$ is basic with respect to $\mathcal{E}$.

Proof. (i) Put $\tau = \tau_{\mathfrak{g},0}(E, \mathcal{F}_E)$. We have $\int_{K_{\mathfrak{a}}^r} \tau = \int_{\mathbb{R}^r} \tau_0 = 1$ for all $x \in M$, so $\pi_* \tau = 1$. Also $d_{\mathfrak{g}} \tau = 0$ because $d_{\mathfrak{t}} \tau_0 = 0$ and the Cartan-Chern-Weil map is a cochain map.
(ii) This follows from the naturality of the Cartan-Chern-Weil map with respect to maps.
(iii) We have $\zeta^* \tau = j^* \tau_0$, where $j : 0 \to \mathbb{R}^r$ is the inclusion of the origin. The assertion now follows from the fact that $j^* \tau_0 = 0$ if $r$ is odd and $j^* \tau_0 = (-2\pi)^{-r/2} c_{\mathfrak{g},0}(Pf)$ if $r$ is even; see [15, §7] or [12, (7.20)].
4.9. **Euler forms.** Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra, \((M, \mathcal{F}, a)\) foliated \( \mathfrak{g} \)-manifold, and \((E, \mathcal{F}_E, g_E)\) an oriented \( \mathfrak{g} \)-equivariant Riemannian foliated vector bundle of rank \( r \) over \( M \). Suppose that the oriented orthogonal frame bundle of \( E \) admits an invariant basic connection \( \theta \) as in Proposition 4.8.1. Then we have a \( \mathfrak{g} \)-equivariant characteristic homomorphism \( c_{\mathfrak{g}, \theta} \) as in (4.5.2). The universal \( \mathfrak{g} \)-equivariant \( \mathcal{F} \)-basic Euler form of the foliated Riemannian vector bundle \( E \) is the element \( \eta_{\mathfrak{g}, \theta}(E, \mathcal{F}_E) \in \Omega^r_\mathfrak{g}(M, \mathcal{F}) \) given by

\[
\eta_{\mathfrak{g}, \theta}(E, \mathcal{F}_E) = \begin{cases} 
0 & \text{if } r \text{ is odd} \\
(-2\pi)^{-r/2}c_{\mathfrak{g}, \theta}(\text{Pf}) & \text{if } r \text{ is even}.
\end{cases}
\]

The next statement follows immediately from Theorem 4.6.1 and Lemma 4.8.2.

4.9.1. **Proposition.** Let \((M, \mathcal{F}, a)\) be foliated \( \mathfrak{g} \)-manifold and \((E, \mathcal{F}_E, g_E)\) an oriented \( \mathfrak{g} \)-equivariant Riemannian foliated vector bundle of rank \( r \) over \( M \). Suppose that \( E \) admits an invariant basic metric connection. The universal equivariant basic Euler form satisfies

\[
\eta_{\mathfrak{g}, \theta}(E, \mathcal{F}_E) = \xi^*(\tau_{\mathfrak{g}, \theta}(E, \mathcal{F}_E)) = \xi^*(\xi^r_\mathfrak{g}(1),
\]

where \( \tau_{\mathfrak{g}, \theta}(E, \mathcal{F}_E) \) is the universal equivariant basic Thom form of \( E \), and

\[
\eta_{\mathfrak{g}, f^* \theta}(f^* E, f^* \mathcal{F}_E) = f^* \eta_{\mathfrak{g}, \theta}(E, \mathcal{F}_E)
\]

for all \( \mathfrak{g} \)-equivariant foliate maps \( f: (M', \mathcal{F}') \to (M, \mathcal{F}) \).

5. **Riemannian foliations and normal bundles**

In this section \( \mathfrak{g} \) denotes a finite-dimensional real Lie algebra and \((X, \mathcal{F})\) a foliated \( \mathfrak{g} \)-manifold as defined in § 4. We let \( i: Y \to X \) be a co-oriented closed submanifold which is preserved by the \( \mathfrak{g} \)-action. The normal bundle \( NY = i^* TX / TY \) is an oriented \( \mathfrak{g} \)-equivariant foliated vector bundle over \( Y \). The first goal of this section is to establish Proposition 5.2.1, which says that \( NY \) admits an equivariant basic Thom form, provided that the foliation \( \mathcal{F} \) is Riemannian and the \( \mathfrak{g} \)-action is isometric (the definition of which we recall below). This fact enables us to define a wrong-way homomorphism \( i_*: \Omega_Y(\mathcal{F}_Y) \to \Omega_X(\mathcal{F}) \) and to obtain a long exact Thom-Gysin sequence, which relates the equivariant basic cohomology of \( X \) to that of \( Y \) and the complement \( X \setminus Y \) (Proposition 5.3.3).

5.1. **Extending and reducing connections.** We state without proof two elementary facts regarding principal connections. Suppose we are given a Lie group homomorphism \( f = f_G: G_1 \to G_2 \), a principal \( G_1 \)-bundle \( P_1 \to X_1 \), a principal \( G_2 \)-bundle \( P_2 \to X_2 \), and a smooth map \( f = f_P: P_1 \to P_2 \) with the equivariance property \( f_P(gp) = f_G(g)f_P(p) \) for all \( g \in G_1 \) and \( p \in P_1 \). Then \( f_G \) induces a Lie algebra homomorphism \( f = f_{\mathfrak{g}}: \mathfrak{g}_1 \to \mathfrak{g}_2 \) and \( f_P \) descends to a smooth map \( f = f_X: X_1 \to X_2 \) as in the commutative diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{f_P} & P_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{f_X} & X_2.
\end{array}
\]

5.1.1. **Lemma.** (i) Let \( X_1 = X_2 \) and \( f_X = \text{id}_{X_1} \). For every connection \( \theta_1 \in \Omega^1(P_1, \mathfrak{g}_1)^{G_1} \) on \( P_1 \) there is a unique connection \( \theta_2 \in \Omega^1(P_2, \mathfrak{g}_2)^{G_2} \) on \( P_2 \) with the property that \( Tf_P \) maps \( \theta_1 \)-horizontal subspaces to \( \theta_2 \)-horizontal subspaces.
(ii) Suppose there exists an \( \text{Ad}(G) \)-equivariant linear map \( \varphi : g_2 \to g_1 \) satisfying \( \varphi \circ f_2 = \text{id}_{g_1} \). For every connection \( \theta_2 \) on \( P_2 \) the formula \( \theta_1 = \varphi \circ \theta_2 \circ T f_P \) defines a connection \( \theta_1 \) on \( P_1 \).

5.2. Thom form of the normal bundle. Recall that a Riemannian structure on the foliated manifold \((X, \mathcal{F})\) is a Riemannian metric \( g \) on the normal bundle of the foliation \( N\mathcal{F} = TM / T\mathcal{F} \) that satisfies \( \nabla g = 0 \), where \( \nabla \) is the partial connection on \( N\mathcal{F} \). A Killing vector field on \((X, \mathcal{F})\) is a vector field \( v \in \mathfrak{X}(X) \) that satisfies \( L(v)g = 0 \). By [19, Lemma 3.3] a Killing vector field \( v \in \mathfrak{X}(X) \) is automatically foliate and therefore the Killing vector fields form a Lie subalgebra \( \mathcal{R}(\mathcal{F}, g) \) of the Lie algebra of foliate vector fields \( \mathfrak{R}(\mathcal{F}) \). Vector fields in \( \mathfrak{X}(\mathcal{F}) \) are by definition Killing, so \( \mathfrak{X}(\mathcal{F}) \) is an ideal of \( \mathfrak{R}(\mathcal{F}, g) \). The quotient \( \mathfrak{X}(X, \mathcal{F}, g) = \mathfrak{R}(\mathcal{F}, g) / \mathfrak{X}(\mathcal{F}) \subseteq \mathfrak{X}(X, \mathcal{F}) \) is a Lie algebra, whose elements we call transverse Killing vector fields. A transverse action \( a : \mathfrak{g} \to \mathfrak{X}(X, \mathcal{F}) \) is isometric if \( a(\xi) \) is transverse Killing for all \( \xi \in \mathfrak{g} \).

Fix a Riemannian structure \( g \) on \((X, \mathcal{F})\) and an isometric transverse \( \mathfrak{g} \)-action \( a : \mathfrak{g} \to \mathfrak{X}(X, \mathcal{F}, g) \) on \( X \). We write \( a(\xi) = \xi_X \) for \( \xi \in \mathfrak{g} \). Let \( n \) be the codimension of \( \mathcal{F} \) and let \( P \to X \) be the orthonormal frame bundle of \( N\mathcal{F} \), which has structure group \( K = O(n) \). By [13, § 2.35] or [19, § 3.3], \( P \) is a foliated bundle with foliation \( \mathcal{F}_P \), whose partial connection extends to a unique torsion-free basic connection \( \theta \in \Omega^1(P, \mathfrak{f})^K \) on \( P \), called the transverse Levi Civita connection. Killing vector fields on \( X \) lift naturally to foliate vector fields on \( P \), which preserve \( \theta \). Thus \( \theta \) is a \( \mathfrak{g} \)-invariant \( \mathcal{F}_P \)-basic connection.

A submanifold \( Y \) of \( X \) is \( \mathfrak{g} \)-invariant if for every \( \xi \in \mathfrak{g} \) and every foliate representative \( \xi_X \in \mathfrak{R}(\mathcal{F}, g) \) of \( \xi_X \), the flow of \( \xi_X \) preserves \( Y \). Let \( Y \) be \( \mathfrak{g} \)-invariant. Then in particular for each \( x \in Y \) the leaf \( \mathcal{F}_y(x) \) is contained in \( Y \). Let \( \mathcal{F}_Y \) be the induced foliation of \( Y \). The restriction of the normal bundle \( N\mathcal{F} \) to \( Y \) is an orthogonal direct sum \( N\mathcal{F} = N\mathcal{F}_Y \oplus N\mathcal{Y} \), where \( N\mathcal{Y} = TX|_Y / TY \) is the normal bundle of \( Y \) in \( X \). Let \( p \) be the codimension of \( \mathcal{F}_Y \) in \( Y \) and \( q \) the codimension of \( Y \) in \( X \). Then \( p + q = n \). We form the orthonormal frame bundle \( P_1 \) of \( N\mathcal{F}_Y \) and, assuming \( Y \) to be co-orientable, the oriented orthonormal frame bundle \( P_2 \) of \( N\mathcal{Y} \). The structure group of \( P_1 \) is \( K_1 = O(p) \) and the structure group of \( P_2 \) is \( K_2 = SO(q) \). For every \( x \in Y \) a pair consisting of a frame of \( N_x \mathcal{F}_Y \) and a frame of \( N_x \mathcal{Y} \) gives a frame of \( N_x \mathcal{F} \), so we have an embedding \( j \) of the fibred product \( P' = P_1 \times_Y P_2 \) into \( P \), which is equivariant with respect to the embedding \( K' = K_1 \times K_2 \to K \). We view \( P' \) as a principal \( K' \)-bundle over \( Y \). Choose a \( K' \)-invariant projection \( pr : \mathfrak{f} \to \mathfrak{f}' \); then by Lemma 5.1.1(ii) the form \( \theta' = pr \circ \theta \circ Tj \) is a connection on \( P' \). The bundle \( P_2 \) is the quotient of \( P' \) with respect to the \( K_1 \)-action, so by Lemma 5.1.1(i) the form \( \theta' \) descends uniquely to a connection \( \theta_Y \) on \( N\mathcal{Y} \). Since \( \theta \) is \( \mathfrak{g} \)-invariant and \( \mathcal{F}_P \)-basic, \( \theta' \) and \( \theta_Y \) are \( \mathfrak{g} \)-invariant and \( \mathcal{F}_Y \)-basic. The following statement now follows from Proposition 4.8.1.

5.2.1. Proposition. Let \((X, \mathcal{F}, g)\) be a Riemannian foliated manifold equipped with an isometric transverse action of a Lie algebra \( \mathfrak{g} \). Let \( Y \) be a co-orientable \( \mathfrak{g} \)-invariant submanifold of \( X \). Then the normal bundle \( N\mathcal{Y} \) possesses an invariant basic metric connection. Hence \( N\mathcal{Y} \) has an equivariant basic Thom form, and hence the Thom isomorphism theorem, Theorem 4.6.1, applies to \( N\mathcal{Y} \).

5.3. The Thom-Gysin sequences. We now have all the ingredients for the equivariant basic version of the Thom-Gysin theorem, Propositions 3.3.1 and 3.3.4.

5.3.1. Proposition. Let \((M, \mathcal{F}, a)\) be foliated \( \mathfrak{g} \)-manifold and \((E, \mathcal{F}_E, g_E)\) an oriented \( \mathfrak{g} \)-equivariant Riemannian foliated vector bundle over \( M \). Suppose that \( E \) admits an invariant basic metric connection. Let \( \tau_E \in \Omega^r_{g_E}(E, \mathcal{F}_E) \) be an equivariant basic Thom form of \( E \). \( \eta_E = \zeta^* \tau_E \in \Omega^r_{g}(M, \mathcal{F}) \) the associated equivariant basic Euler form, and
\( \textbf{Eu}_g(E) = [\eta_\mathfrak{g}] \) the equivariant basic Euler class. Let \( SE \) be the sphere bundle of \( E \) and let \( \pi_{SE} = \pi|_{SE} : SE \to M \) be the projection. Then we have a long exact sequence

\[
\ldots \to H^{k-r}_\mathfrak{g}(Y, \mathcal{F}_Y) \xrightarrow{i_*} H^k_\mathfrak{g}(X, \mathcal{F}) \xrightarrow{j^*} H^k_\mathfrak{g}(X \setminus Y, \mathcal{F}_{X \setminus Y}) \xrightarrow{\epsilon} H^{k-r+1}_\mathfrak{g}(Y, \mathcal{F}_Y) \to \ldots
\]

**Sketch of proof.** The proof of Proposition 3.3.1 works in the present context, relying on the equivariant basic Thom isomorphism, Theorem 4.6.1, and on the equivariant basic version of Proposition 3.2.2.

For the submanifold version of this sequence we consider a manifold \( X \) equipped with a Riemannian foliation \((\mathcal{F}, g)\) and an isometric transverse \( \mathfrak{g}\)-action, and a closed \( \mathfrak{g}\)-invariant submanifold \( i: Y \to X \) with normal bundle \( N = i^*TX/TY \). By [14, Proposition 3.3.4] the existence of a \( \mathfrak{g}\)-equivariant foliate tubular neighbourhood embedding \( i_N : N \to X \) is guaranteed if \( X \) is complete. Suppose this to be the case, and also that \( Y \) is co-oriented. Let \( r \) be the codimension of \( Y \) and choose an equivariant basic Thom form \( \tau_g(N) \in \Omega^{\xi_{g,v},N}(N, \mathcal{F}_N) \) of \( N \), the existence of which follows from Proposition 5.2.1. For each \( \beta \in \Omega^{\xi_{g,v},N}(N, \mathcal{F}_N) \) the form \((i_N^*)\beta \) extends by zero to an equivariant basic form on \( X \), which we denote by \( i_N.\beta \). In the same way as (3.3.3) we define the wrong-way homomorphism to be the degree \( r \) morphism of complexes

\[
\text{(5.3.2)} \quad i_* = i_{N,*} \circ \xi_{N,*} : \Omega_g(Y, \mathcal{F}_Y)[-r] \to \Omega_g(X, \mathcal{F}),
\]

given by \( i_* = i_{N,*}(\tau_g(N) \wedge \xi_{N,*}) \).

5.3.3. **Proposition.** Let \((X, \mathcal{F}, g)\) be a complete Riemannian foliated manifold equipped with an isometric transverse action of a Lie algebra \( \mathfrak{g} \). Let \( Y \) be a closed co-orientable \( \mathfrak{g}\)-invariant submanifold of codimension \( r \). Let \( i : Y \times X \) and \( j : X \setminus Y \to X \) be the inclusions. We have a long exact sequence

\[
\ldots \to H^{k-r}_\mathfrak{g}(Y, \mathcal{F}_Y) \xrightarrow{i_*} H^k_\mathfrak{g}(X, \mathcal{F}) \xrightarrow{j^*} H^k_\mathfrak{g}(X \setminus Y, \mathcal{F}_{X \setminus Y}) \xrightarrow{\epsilon} H^{k-r+1}_\mathfrak{g}(Y, \mathcal{F}_Y) \to \ldots
\]

**Sketch of proof.** We follow the proof of Proposition 3.3.4, using the equivariant basic Thom isomorphism, Theorem 4.6.1, the equivariant basic version of Proposition 3.2.2, and the equivariant basic version of the excision lemma, Lemma 3.3.2. (The latter relies on the Mayer-Vietoris principle for equivariant basic de Rham theory, for which see [14, Proposition 3.3.7].)

**QED**

**Appendix A. Cartan-Chern-Weil theory**

This appendix is a summary of the algebraic principles of H. Cartan’s equivariant de Rham theory [5]. We follow the exposition of [12], [1] and [10]. This material is mostly standard, but we state a few results, notably Theorems A.5.1 and A.6.3, under weaker hypotheses than our references.

The action of a Lie algebra \( \mathfrak{g} \) on a manifold induces two actions on differential forms, namely through Lie derivatives and through contractions, and these actions intertwine in a distinctive way with each other and with the exterior derivative. These features are abstracted in the notion of a \( \mathfrak{g}\)-differential graded module. An example of a \( \mathfrak{g}\)-differential graded module is the Weil algebra \( \mathsf{Wg} \), which models the de Rham complex of a universal bundle \( E_G \), where \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \). A key feature of the theory is a convenient criterion for the equivariant cohomology of a \( \mathfrak{g}\)-differential graded module to be isomorphic to its basic cohomology, namely Theorem A.5.1 and its equivariant analogue Theorem A.6.3. This criterion asks whether the \( \mathfrak{g}\)-differential graded module structure extends to a compatible \( \mathsf{Wg}\)-module structure. Such a \( \mathsf{Wg}\)-module structure enables an
algebraic version of Chern and Weil’s connection-curvature construction of characteristic forms.

In this section we place ourselves in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbf{F}$ of characteristic 0. An ungraded object is considered as a graded object concentrated in degree 0. “Module” means “graded module”, “dual” means “graded dual”, “commutative” means “graded commutative”, “derivation” means “graded derivation”, etc. Tensor products are taken over $\mathbf{F}$ and equipped with the total grading. Our complexes will be cochain complexes. We denote the translation functor on graded objects by $[r]$. So if $(C, d)$ is a cochain complex we have $C[k]^l = C^{l+k}$ and $d[k] = (-1)^k d$. We will abbreviate “differential graded module” to “dgm” and “differential graded algebra” to “dga”.

A.1. The Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{F}$ (placed in degree 0). Define the (graded) vector space $\mathfrak{a}$ by

$$\mathfrak{a} = \mathfrak{g}[1] \oplus \mathfrak{g} \oplus \mathbf{F}[-1],$$

i.e. place copies of $\mathfrak{g}$ in degrees $-1$ and 0 and a copy of $\mathbf{F}$ in degree 1. For $\xi \in \mathfrak{g}$ denote the corresponding element of $\mathfrak{g}[1]$ by $\iota(\xi)$ and the corresponding element of $\mathfrak{g}[0]$ by $L(\xi)$. Denote the basis element of $\mathbf{F}[-1]$ corresponding to $1 \in \mathbf{F}$ by $d$. The conditions

$$[\iota(\xi), \iota(\eta)] = 0, \quad [L(\xi), L(\eta)] = L([\xi, \eta]), \quad [d, d] = 0,$$

(A.1.1)

$$[L(\xi), d] = 0, \quad [\iota(\xi), d] = L(\xi), \quad [L(\xi), \iota(\eta)] = \iota([\xi, \eta])$$

for all $\xi, \eta \in \mathfrak{g}$ determine a bilinear bracket on $\mathfrak{a}$ which makes $\mathfrak{a}$ a (graded) Lie algebra. As for any (graded) Lie algebra, we can talk about (graded) modules and algebras over $\mathfrak{a}$.

A.2. $\mathfrak{g}$-differential graded modules. A $\mathfrak{g}$-differential graded module (abbreviated to $\mathfrak{g}$-dgm) is a graded $\mathfrak{a}$-module, in other words a graded vector space $M$ equipped with an endomorphism $d$ of degree 1 and, for each $\xi \in \mathfrak{g}$, endomorphisms $\iota(\xi)$ of degree $-1$ and $L(\xi)$ of degree 0, which depend linearly on $\xi$ and satisfy the commutation rules (A.1.1). In particular, the rule $[L(\xi), L(\eta)] = L([\xi, \eta])$ means that a $\mathfrak{g}$-dgm $M$ is a $\mathfrak{g}$-module, and the rule $d^2 = \frac{1}{2}[d, d] = 0$ means that $M$ is a cochain complex of $\mathfrak{g}$-representations.

Let $M$ be a $\mathfrak{g}$-dgm. An element $m$ of $M$ is $\mathfrak{g}$-invariant if $L(\xi)m = 0$ for all $\xi \in \mathfrak{g}$, $\mathfrak{g}$-horizontal if $\iota(\xi)m = 0$ for all $\xi \in \mathfrak{g}$, and $\mathfrak{g}$-basic if it is $\mathfrak{g}$-invariant and $\mathfrak{g}$-horizontal. We denote by

$$M^0, \quad M_{\mathfrak{g} \text{-hor}} = M_{\text{hor}}, \quad M_{\mathfrak{g} \text{-bas}} = M_{\text{bas}} = M^0 \cap M_{\text{hor}}$$

the subspaces of $M$ consisting of invariant, horizontal and basic elements, respectively. The subspaces $M^0$ and $M_{\mathfrak{g} \text{-bas}}$ are $\mathfrak{a}$-submodules of $M$. The $\mathfrak{g}$-basic cohomology of $M$ is the vector space $H_{\mathfrak{g} \text{-bas}}(M) = H(M_{\mathfrak{g} \text{-bas}}, d)$. A morphism of $\mathfrak{g}$-dgm $f : M \to M'$ is degree 0 morphism $f$ of $\mathfrak{g}$-modules. A homotopy of $\mathfrak{g}$-dgm between two morphisms $f_0, f_1 : M \to M'$ is a degree $-1$ linear map $F : M \to M'[-1]$ satisfying

$$[\iota(\xi), F] = 0, \quad [L(\xi), F] = 0, \quad [d, F] = f_1 - f_0$$

for all $\xi \in \mathfrak{g}$. Two homotopic morphisms $f_0$ and $f_1$ of $\mathfrak{g}$-dgm induce the same maps in cohomology $H(f_0) = H(f_1) : H(M) \to H(M')$ as well as in basic cohomology $H_{\text{bas}}(f_0) = H_{\text{bas}}(f_1) : H_{\text{bas}}(M) \to H_{\text{bas}}(M')$.

A.3. $\mathfrak{g}$-differential graded algebras. A $\mathfrak{g}$-differential graded algebra (abbreviated to $\mathfrak{g}$-dga) is a graded $\mathfrak{a}$-algebra, i.e. a $\mathfrak{a}$-module which is also an algebra, always assumed to be unitary, associative and graded, on which the operators $d$, $L(\xi)$ and $\iota(\xi)$ act as graded derivations. The basic complex of a $\mathfrak{g}$-dga $A$ is a differential subalgebra of $A$, so the basic cohomology $A$ is an $\mathbf{F}$-algebra.
The algebra of $\mathbf{F}$-linear endomorphisms $E = \text{End}(M)$ of a $g$-dgm $M$ is a $g$-dga. The basic subalgebra of $E$ is $E_{\text{bas}} = \text{End}_g(M)$, the algebra of $g$-dgm endomorphisms of $M$.

Let $A$ be a $g$-dga. By an $A$-module we mean an $A$-module $M$ in the category of $g$-dgm. So $M$ is also equipped with a $\mathfrak{g}$-module structure with the property that the multiplication map $A \otimes M \to M$ is $\mathfrak{g}$-equivariant, i.e. $\gamma(am) = \gamma(a)m + \gamma(\gamma(m))a\gamma(m)$ for all homogeneous $\gamma \in \mathfrak{g}, a \in A$ and $m \in M$.

We say that $A$ is locally free if it admits a connection, i.e. a linear map $\theta : g^* \to A^1$ satisfying
\[
i(\xi)\theta(x) = \langle \xi, x \rangle \in A^0 \quad \text{and} \quad L(\xi)(\theta(x)) = -\theta(\text{ad}^*(\xi)x)
\]
for all $\xi \in g$ and $x \in g^*$. It is useful to reformulate this notion as follows. Let $V_g$ be the vector space $\mathfrak{g}[1]^* = g[2]^* \oplus g[1]^* \oplus \mathbf{F}[0]$. For $x \in g^*$ denote the corresponding element of $\mathfrak{g}[1]^*$ by $\delta(x)$ and the corresponding element of $g[2]^*$ by $\hat{\delta}(x)$. Denote the degree 0 element corresponding to $1 \in \mathbf{F}$ by $z$. The rules
\[
(A, 3.1) \quad i(\xi)\delta(x) = \langle \xi, x \rangle, \quad L(\xi)\delta(x) = -\delta(\text{ad}^*(\xi)x), \quad d\delta(x) = \hat{\delta}(x),
\]
\[
(A, 3.2) \quad i(\xi)\hat{\delta}(x) = -\delta(\text{ad}^*(\xi)x), \quad L(\xi)\hat{\delta}(x) = -\hat{\delta}(\text{ad}^*(\xi)x), \quad d\hat{\delta}(x) = 0,
\]
\[
(A, 3.3) \quad i(\xi)z = 0, \quad L(\xi)z = 0, \quad dz = 0
\]
for all $\xi \in g$ and $x \in g^*$ determine a structure of $g$-dgm on $V_g$. Here $\langle \xi, x \rangle \in \mathbf{F} \subseteq A^0$ denotes the dual pairing. The $\delta(x)$ are the connection elements of $V_g$. These rules ensure that a connection on $A$ is equivalent to a degree 0 homomorphism of $g$-dgm $\theta : V_g \to A$ satisfying $\theta(z) = 1$. We call a connection $\theta$ commutative if the image $\theta(V_g)$ generates a commutative subalgebra of $A$.

A typical example of a locally free $g$-dga is $A = \Omega(P)$, the de Rham complex of a principal $G$-bundle $P$, where $G$ is any Lie group with Lie algebra $g$. Typical examples of $A$-modules are $M = \Omega_\infty(P)$, the compactly supported de Rham complex of $P$, and $M = \Omega(P, E)$, the de Rham complex with coefficients in an equivariant flat vector bundle $E$ over $P$.

A.4. The Weil algebra. Let $S(V_g) = S(\mathfrak{g}[1]^*)$ be the (graded) symmetric algebra of $V_g$, equipped with the commutative $g$-dga structure induced by the $g$-dgm structure on $V_g$. The Weil algebra of $\mathfrak{g}$ is the commutative $g$-dga $W_\mathfrak{g} = S(V_g)/(z - 1)$, where $(z - 1)$ is the ideal generated by $z - 1$. The inclusion $\theta_{\text{uni}} : V_g \to W_\mathfrak{g}$ is a connection on $W_\mathfrak{g}$ called the universal or tautological connection. The Weil algebra has the following universal property: every commutative connection $\theta$ on a $g$-dga $A$ is of the form $\theta = c_\theta \circ \theta_{\text{uni}}$ for a unique $g$-dga homomorphism $c_\theta : W_\mathfrak{g} \to A$, as in the diagram
\[
\begin{array}{c}
V_g \xrightarrow{\theta} A \\
\downarrow \theta_{\text{uni}} \\
W_\mathfrak{g} \xrightarrow{c_\theta} A
\end{array}
\]
(We have no need here of the noncommutative connections and noncommutative Weil algebra of [1].) We call $c_\theta$ the characteristic homomorphism of the connection. Any two connections on a principal bundle are homotopic. See [1, Proposition 3.1] for the following algebraic counterpart of this fact. We review the proof (which is formally identical to the construction of homotopies in de Rham theory; see Corollary B.5), because we will need the formula for the homotopy.
A.4.1. Proposition. Let \( \theta_0 \) and \( \theta_1 \) be commutative connections on a \( \mathfrak{g} \)-dga \( A \). Suppose that \( \theta_0 \) and \( \theta_1 \) commute in the sense that \( [\theta_0(v_0), \theta_1(v_1)] = 0 \) for all \( v_0, v_1 \in \mathfrak{V}_g \). Then the characteristic homomorphisms \( c_{\theta_0} \) and \( c_{\theta_1} \) are homotopic.

Proof. The graded line is the Koszul differential algebra \( S = S(F[-2] \oplus F[-1]) \) of \( F \). Let \( s \in F[-2] \) and \( \delta \in F[-1] \) be the two elements corresponding to the identity \( 1 \in F \). The Koszul differential is given on generators by \( ds = \delta \) and \( d\delta = 0 \). For each \( a \in F \) the evaluation map is the \( F \)-linear functional \( ev_a : S \to F \) defined on basis elements by \( ev_a(s^k) = a^k \) and \( ev(s^k \delta) = 0 \). We often write \( f(a) \) instead of \( ev_a(f) \). The integral is the \( F \)-linear functional \( J = \int_0^1 : S \to F \) defined by \( J(s^k) = 0 \) and \( J(s^k \delta) = 1/(k+1) \). These functionals are not homogeneous with respect to the \( \mathbb{Z} \)-grading, but with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading: \( ev_a \) is even and \( J \) is odd. They satisfy the “fundamental theorem of calculus” \( Jdf = f(1) - f(0) \) for \( f \in S \). They extend \( A \)-linearly to functionals 

\[
    ev_a = ev_a \otimes \text{id}_A, \quad J = J \otimes \text{id}_A: S \otimes A \to A.
\]

The functionals \( ev_a \) are morphisms of \( \mathfrak{g} \)-dgm and the functional \( J \) satisfies \( [\iota(\xi), J] = [L(\xi), J] = 0 \) for all \( \xi \in \mathfrak{g} \). The fundamental theorem of calculus for \( \alpha \in S \otimes A \) says that \( Jd\alpha = \alpha(1) - \alpha(0) - dJ\alpha \), i.e. \( [d, J] = ev_1 - ev_0 \). Thus (the degree \( -1 \) component of) \( J \) is a homotopy from \( ev_0 \) to \( ev_1 \). The \( \mathfrak{g} \)-dga \( S \otimes A \) has a commutative connection \( \theta \) defined by \( \theta(x) = (1 - s) \otimes \theta_0 + s \otimes \theta_1 \) with corresponding characteristic homomorphism \( c_\theta : W_\mathfrak{g} \to S \otimes A \). The operator \( J \circ c_\theta : W_\mathfrak{g} \to A \) is then a homotopy from \( c_{\theta_0} \) to \( c_{\theta_1} \). QED

A few words about the structure of the Weil algebra. The connection elements \( \theta(x) \in \mathfrak{g}[1]^* \subseteq W_\mathfrak{g} \) are of degree 1. The map \( \theta : \mathfrak{g}^* \leftarrow W_\mathfrak{g} \) defined by \( x \mapsto \theta(x) \) extends to an algebra homomorphism \( \tilde{\theta} : \Lambda \mathfrak{g}^* \equiv \mathfrak{S}(\mathfrak{g}[1]^*) \hookrightarrow W_\mathfrak{g} \). Similarly the map \( \tilde{\theta} : \mathfrak{g}^* \leftarrow W_\mathfrak{g} \) defined by \( x \mapsto \tilde{\theta}(x) \) extends to an algebra homomorphism \( \hat{\theta} : \mathfrak{S}(\mathfrak{g}^*[2]) \hookrightarrow W_\mathfrak{g} \). The degree 2 elements \( \theta(x) \in \mathfrak{g}[2]^* \subseteq W_\mathfrak{g} \) are not horizontal (see (A.3.2)), but the curvature elements \( \mu(x) = \theta(x) + \frac{1}{2} \hat{\theta}(\lambda(x)) \in W_\mathfrak{g} \) are, where \( \lambda : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \) is the map dual to the Lie bracket. The map \( \mu : \mathfrak{g}^* \to W_\mathfrak{g} \) which sends \( x \) to \( \mu(x) \) extends to an algebra morphism \( \mu : \mathfrak{S}(\mathfrak{g}^*[2]) \to W_\mathfrak{g} \). See e.g. [12, Ch. 3] or [1, §3] for the following statement.

A.4.2. Proposition. (i) The homomorphism \( \theta \otimes \hat{\theta} : \mathfrak{S}(\mathfrak{g}[2] \oplus \mathfrak{g}[1])^* \to W_\mathfrak{g} \) is an isomorphism of algebras. In terms of the generators \( \theta(x) \) and \( \hat{\theta}(x) \) the \( \tilde{\theta} \)-structure equations of \( W_\mathfrak{g} \) are given by (A.3.1)–(A.3.2). In particular, as a complex \( W_\mathfrak{g} \) is isomorphic to the Koszul complex of \( \tilde{\theta} \) and is therefore non-homotopic.

(ii) The homomorphism \( \theta \otimes \mu : \mathfrak{S}(\mathfrak{g}[2] \oplus \mathfrak{g}[1])^* \to W_\mathfrak{g} \) is also an isomorphism of algebras. The image of the homomorphism \( \mu : \mathfrak{S}(\mathfrak{g}[2]^*) \to W_\mathfrak{g} \) is the horizontal subalgebra \( (W_\mathfrak{g})_{\text{hor}} \). In terms of the generators \( \theta(x) \) and \( \mu(x) \) the \( \hat{\theta} \)-structure equations of \( W_\mathfrak{g} \) read as follows:

\[
    \iota(\xi)(\theta(x)) = (\xi, x), \quad L(\xi)\theta(x) = -\theta(\text{ad}^*(\xi)x), \quad d\theta(x) = \frac{1}{2}d_{\text{CE}}\theta(x) + \mu(x),
\]

\[
    \iota(\xi)(\mu(x)) = 0, \quad L(\xi)\mu(x) = -\mu(\text{ad}^*(\xi)x), \quad d\mu(x) = d_{\text{CE}}\mu(x).
\]

(iii) The isomorphism \( \mu : \mathfrak{S}(\mathfrak{g}[2]^*) \to (W_\mathfrak{g})_{\text{hor}} \text{ induces an isomorphism } \mathfrak{S}(\mathfrak{g}[2]^*) \equiv (W_\mathfrak{g})_{\text{bas}} \). The restriction of the differential \( d_{W_\mathfrak{g}} \) to \( (W_\mathfrak{g})_{\text{bas}} \) is 0, so \( H_{\text{bas}}(W_\mathfrak{g}) \equiv \mathfrak{S}(\mathfrak{g}[2]^*) \).

Here \( d_{\text{CE}} \) denotes the differential on \( W_\mathfrak{g} \) viewed as the Cartan-Eilenberg complex \( \Lambda^* \otimes \mathfrak{S} \) of the \( \mathfrak{g} \)-module \( \mathfrak{S} \). This differential sends \( x \in \Lambda^1 \mathfrak{g}^* \) to \( -\lambda(x) \in \Lambda^2 \mathfrak{g}^* \) and \( x \in \Lambda^1 \mathfrak{g}^* \) to
the element of \( \Lambda^1 \mathfrak{g}^* \otimes S^1 \mathfrak{g}^* \equiv \text{Hom}(\mathfrak{g}, S^1 \mathfrak{g}^*) \) given by \( d_\mathfrak{g}(x)(\xi) = L(\xi)(x) \). The formulas for \( d\theta \) and \( d\mu \) are the Cartan-Bianchi identities.

Let \( M \) be a \( \mathfrak{g} \)-dgm. Suppose that the algebra of \( \mathbb{F} \)-linear endomorphisms \( E = \text{End}(M) \) admits a commutative connection \( \theta \). Then the characteristic map \( \theta: W_\mathfrak{g} \to E \) defines a \( \mathfrak{g} \)-linear multiplication law \( W_\mathfrak{g} \otimes M \to M \) on \( M \). In other words a \( W_\mathfrak{g} \)-module structure on \( M \) is nothing but a commutative connection \( \theta \) on its endomorphism algebra. With this in mind we denote the multiplication law of a \( W_\mathfrak{g} \)-module \( M \) by

\[
\theta: W_\mathfrak{g} \otimes M \to M: w \otimes m \mapsto c_\theta(w)m
\]

and call it the Cartan-Chern-Weil homomorphism of \( M \). It is a morphism of \( \mathfrak{g} \)-dgm and therefore induces a morphism of complexes

\[
(W_\mathfrak{g} \otimes M)_{\text{bas}} \longrightarrow M_{\text{bas}}
\]

and of graded vector spaces \( H_{\text{bas}}(W_\mathfrak{g} \otimes M) \to H_{\text{bas}}(M) \).

A.5. Equivariant cohomology. Let \( M \) be a \( \mathfrak{g} \)-dgm. The Weil complex of \( M \) is \( M_\mathfrak{g} = (W_\mathfrak{g} \otimes M)_{\text{bas}} \), the basic complex of the \( \mathfrak{g} \)-dgm \( W_\mathfrak{g} \otimes M \), the differential of which we denote by \( d_\mathfrak{g} \). The cohomology of the Weil complex \( H_\mathfrak{g}(M) = H(M_\mathfrak{g}) \) is the equivariant cohomology of \( M \). A morphism of \( \mathfrak{g} \)-dgm \( f: M \to M' \) induces a morphism in equivariant cohomology \( f_\mathfrak{g}: H_\mathfrak{g}(M) \to H_\mathfrak{g}(M') \). Two homotopic morphisms induce the same map in equivariant cohomology.

For the trivial 1-dimensional \( \mathfrak{g} \)-dgm \( M = \mathbb{F} \) we have \( (W_\mathfrak{g} \otimes \mathbb{F})_{\text{bas}} = (W_\mathfrak{g})_{\text{bas}} = S(\mathfrak{g}[2]^*)^\theta \), so \( H_\mathfrak{g}(\mathbb{F}) = S(\mathfrak{g}[2]^*)^\theta \). For an arbitrary trivial \( \mathfrak{g} \)-dgm \( M \) we have \( (W_\mathfrak{g} \otimes M)_{\text{bas}} = (W_\mathfrak{g})_{\text{bas}} \otimes M \) with \( d_\mathfrak{g} = \text{id} \otimes d_M \), so \( H_\mathfrak{g}(M) = S(\mathfrak{g}[2]^*)^\theta \otimes H(M) \).

The map \( M \to W_\mathfrak{g} \otimes M \) sending \( m \to 1 \otimes m \) is a morphism of \( \mathfrak{g} \)-dgm and therefore restricts to an injective morphism of complexes \( M_{\text{bas}} \to (W_\mathfrak{g} \otimes M)_{\text{bas}} \), which induces a map \( H_{\text{bas}}(M) \to H_\mathfrak{g}(M) \). The latter map is in general neither injective nor surjective.

Suppose however that the \( \mathfrak{g} \)-dgm structure on \( M \) extends to a \( W_\mathfrak{g} \)-module structure with multiplication map \( C_\theta: W_\mathfrak{g} \otimes M \to M \) (where \( \theta \) is the commutative connection on the endomorphism algebra \( \text{End}(M) \) that defines the \( W_\mathfrak{g} \)-module structure on \( M \)). For our purposes the following statement, which is a variant of \([12, \text{Theorem 5.2.1}] \) and \([1, \text{Proposition 3.2}] \), is the main point of Cartan-Chern-Weil theory. Note the absence of any hypotheses on the Lie algebra \( \mathfrak{g} \). In particular \( \mathfrak{g} \) need not be reductive, nor is \( M \) required to be semisimple as a \( \mathfrak{g} \)-module.

A.5.1. Theorem. Let \( M \) be a \( W_\mathfrak{g} \)-module. The Cartan-Chern-Weil map \( C_\theta: W_\mathfrak{g} \otimes M \to M \) is a homotopy inverse of the inclusion \( j: M \to W_\mathfrak{g} \otimes M \). Therefore \( j \) restricts to a homotopy equivalence \( M_{\text{bas}} \xrightarrow{\sim} M_\mathfrak{g} \) and induces an isomorphism \( H_{\text{bas}}(M) \xrightarrow{\sim} H_\mathfrak{g}(M) \).

Proof. We have \( C_\theta(j(m)) = C_\theta(1 \otimes m) = m \) for \( m \in M \), so \( C_\theta \circ j = \text{id}_M \). Next we show that the map \( h_1 = j \circ C_\theta \) is homotopic to the identity map \( h_0 = \text{id}_{W_\mathfrak{g} \otimes M} \). We have

\[
h_1(w \otimes m) = j(C_\theta(w \otimes m)) = j(c_\theta(w)m) = 1 \otimes c_\theta(w)m
\]

for \( w \in W_\mathfrak{g} \) and \( m \in M \). Let \( D \) be the algebra \( \text{End}(W_\mathfrak{g} \otimes M) \) and define \( \mathfrak{g} \)-dga homomorphisms \( f_0, f_1: W_\mathfrak{g} \to D \) by

\[
f_0(v)(w \otimes m) = vw \otimes m, \quad f_1(v)(w \otimes m) = (-1)^{|v||w|}w \otimes c_\theta(v)m
\]

for \( v, w \in W_\mathfrak{g} \) and \( m \in M \). Then

\[
f_0(v)(1 \otimes m) = v \otimes m = h_0(v \otimes m),
\]

\[
f_1(v)(1 \otimes m) = 1 \otimes c_\theta(v)m = h_1(v \otimes m).
\]
We have \([f_0(v_0), f_1(v_1)] = 0\) for all \(v_0, v_1 \in W g\), so the maps \(f_0\) and \(f_1\) are the characteristic homomorphisms of a commutating pair of connections \(\theta_0\) and \(\theta_1\) on \(D\). Proposition A.4.1 gives us a homotopy \(F: W g \to D[-1]\) satisfying \(f_1 - f_0 = [d, F]\). This homotopy is given by \(F(v) = \int_0^1 c_\Theta(v)\) for \(v \in W g\), and \(c_\Theta: W g \to S \otimes D\) is the characteristic homomorphism of the connection \(\Theta(x) = (1 - s) \otimes \theta_0 + s \otimes \theta_1\) on \(S \otimes D\). Using (A.5.2) we obtain that \(h_1 - h_0 = [d, H]\), where \(H: W g \otimes M \to (W g \otimes M)[-1]\) is the homotopy given by

\[
H(w \otimes m) = F(w)(1 \otimes m) - \left(\int_0^1 c_\Theta(w)\right)(1 \otimes m)
\]

for \(w \in W g\) and \(m \in M\). QED

A.6. Change of Lie algebra. A Lie algebra homomorphism \(h \to g\) induces a homomorphism \(h^* \to g^*\) and hence a pullback functor from \(g\)-dgm to \(h\)-dgm. If \(M\) is a \(g\)-dgm we have natural maps

\[
M^h \to M^h, \quad M_{g, \text{hor}} \to M_{h, \text{hor}}, \quad M_{g, \text{bas}} \to M_{h, \text{bas}}, \quad M_g \to M_h,
\]

and hence maps in basic cohomology \(H_{g, \text{bas}}(M) \to H_{h, \text{bas}}(M)\) and in equivariant cohomology \(H_g(M) \to H_h(M)\). If \(M\) is an \(h\)-dgm and the morphism \(h \to g\) is surjective with kernel \(f\), then on the subcomplex \(M_{f, \text{bas}}\) the operations \(L(\eta)\) and \(i(\eta)\) for \(\eta \in f\) are trivial, so \(M_{f, \text{bas}}\) is in a natural way a \(g\)-dgm, and we have

\[(A.6.1) \quad M_{h, \text{bas}} = (M_{f, \text{bas}})_g,\]

This implies \(H_{h, \text{bas}}(M) = H_{g, \text{bas}}(M_{f, \text{bas}})\). In the case of a product of Lie algebras we have the following simple statement about the Weil complex of \(M\).

A.6.2. Lemma. Let \(h = f \times g\) be the product of two Lie algebras \(f\) and \(g\).

(i) \(W h\) is isomorphic to \(W f \otimes W g\) as an \(h\)-dga.

(ii) Let \(M\) be an \(h\)-dgm. The \(f\)-Weil complex \(M_f\) is a \(g\)-dgm and the \(f\)-Weil complex \(M_{f, \text{bas}}\) is isomorphic to \((M_{f, \text{bas}})_{h, \text{bas}}\) as an \(h\)-dgm. It follows that \(H_g(M) = H_{f, \text{bas}}(M_{f, \text{bas}})\).

Proof. (i) The product of the universal connections \(f^* \to W f\) and \(g^* \to W g\) is a connection \(h^* \to W f \otimes W g\) and therefore induces an \(h\)-dga homomorphism \(\psi: W h \to W f \otimes W g\). The projections \(h \to f\) and \(h \to g\) induce two connections \(f^* \to W h\) and \(g^* \to W h\), and hence algebra maps \(\psi_f: W f \to W h\) and \(\psi_g W g \to W h\). The map \(\psi: W f \otimes W g \to W h\) given by \(\psi(v \otimes w) = \psi_f(v)\psi_g(w)\) is the inverse of \(\phi\).

(ii) The algebra \(W f\) is trivial as a \(g\)-dgm and the \(g\)-operations on \(M\) commute with the \(f\)-operations, so \(M_f = (W f \otimes M)_{f, \text{bas}}\) is a \(g\)-dgm. It follows from (A.6.1) and from (i) that we have an isomorphism of complexes

\[
M_g = (W h \otimes M)_{h, \text{bas}} = ((W g \otimes W f \otimes M)_{f, \text{bas}})_{g, \text{bas}} = (M_f)_g.
\]

Therefore \(H_g(M)\) is isomorphic to \(H_{f, \text{bas}}(M_f)\). QED

Let \(M\) be an \(h\)-dgm and \(E = \text{End}(M)\) its algebra of endomorphisms. Suppose that \(E\) is locally free as a \(f\)-dga, so that it has a \(f\)-connection \(\theta: f^* \to E^1\). We say that the connection \(\theta\) is \(g\)-invariant if \(\theta(y) \in E^1\) is \(g\)-invariant for all \(y \in f^*\). Similarly, \(\theta\) is \(g\)-horizontal if \(\theta(y)\) is \(g\)-horizontal for all \(y \in f^*\); and \(\theta\) is \(g\)-basic if \(\theta(y)\) is \(g\)-basic for all \(y \in f^*\). Define

\[
\iota_f: f^* \to g[1]^* \otimes E^0 \cong \text{Hom}(g[1], E^0)
\]

by \(\iota_f(y)(\xi) = -\iota(\xi)\theta(y)\) for \(\xi \in g\) and \(y \in f^*\). Berline and Vergne’s [3] \(g\)-equivariant extension of the connection \(\theta\) is the entity

\[
\theta_g: f^* \to (W g \otimes E)^1 = F \otimes E^1 \oplus g[1]^* \otimes E^0
\]
defined by \(\theta_\mathfrak{g} = \theta + \iota_\mathfrak{g}\). This terminology is justified by the next theorem, which is a \(\mathfrak{g}\)-equivariant version of Theorem A.5.1. This theorem refines the earlier results [12, §4.6] and [10, Proposition 3.9] in two ways: it is valid for arbitrary Lie algebras \(\mathfrak{f}\) and \(\mathfrak{g}\), and the cohomology isomorphism is given by an explicit homotopy equivalence.

A.6.3. Theorem. Let \(\mathfrak{h} = \mathfrak{f} \times \mathfrak{g}\) be the product of two Lie algebras \(\mathfrak{f}\) and \(\mathfrak{g}\). Let \(M\) be an \(\mathfrak{h}\)-dga. Suppose that there exists a \(\mathfrak{g}\)-invariant connection \(\theta : \mathfrak{f}^* \to E_1\) on the \(\mathfrak{f}\)-dga \(E = \text{End}(M)\).

(i) The \(\mathfrak{g}\)-equivariant extension \(\theta_\mathfrak{g} = \theta + \iota_\mathfrak{g}\) is a \(\mathfrak{g}\)-basic connection on the \(\mathfrak{f}\)-dga \(E_\mathfrak{g} = W_\mathfrak{g} \otimes E\).

(ii) Suppose that the connection \(\theta_\mathfrak{g}\) is commutative. Let \(c_{\mathfrak{g}, \theta} : W\mathfrak{f} \to E_\mathfrak{g}\) be the characteristic homomorphism and

\[C_{\mathfrak{g}, \theta} : W\mathfrak{h} \otimes M \cong W\mathfrak{f} \otimes W\mathfrak{g} \otimes M \to W\mathfrak{g} \otimes M\]

the Cartan-Chern-Weil map associated with \(\theta_\mathfrak{g}\). Then \(C_{\mathfrak{g}, \theta}\) is a homotopy inverse of the inclusion \(j : W\mathfrak{g} \otimes M \to W\mathfrak{h} \otimes M\). It follows that \(j\) induces a homotopy equivalence \(M_{\mathfrak{f}-\text{bas}}(\mathfrak{g}) \cong M_{\mathfrak{h}}\) and hence an isomorphism \(H_\mathfrak{g}(M_{\mathfrak{f}-\text{bas}}) \cong H_\mathfrak{h}(M)\).

Proof. (i) Let \(\eta, \xi, \xi' \in \mathfrak{g}\). Then
\[
i(\eta)\iota_\theta(y)(\xi) = -\iota_\theta(\eta)(\xi', \eta, y) = 0,
\]
because the derivation \(\iota(\xi)\) of \(E\) kills the scalar \((\eta, y)\). It follows that \(\iota(\eta)\theta_\mathfrak{g}(y) = \iota(\eta)\theta(y) = (\eta, y)\). Similarly,
\[
\theta L(\eta)\iota_\theta(\xi) = -\theta L(\eta)(\iota_\theta(y)) = -\iota(\theta)(\eta)(\iota_\theta(y)) = \eta(\theta)(\eta)(\eta) = \iota_\theta(\eta)(\eta)(\eta) = 0,
\]
because \([\xi, \eta] = 0\). Therefore
\[
\theta L(\eta)\theta_\mathfrak{g}(y) = \theta L(\eta)\theta(y) + \theta L(\eta)\iota_\theta(y) = -\theta(\eta)(\theta)(\eta)(\eta) = -\theta(\eta)(\theta)(\eta)(\eta).
\]
This proves that \(\theta_\mathfrak{g}\) is a connection. Next we have
\[
(L(\xi)\iota_\theta(y))(\xi') = L(\xi)(\iota_\theta(y)(\xi')) - \iota_\theta(\xi)(L(\xi)(\xi'))
\]
\[
-\iota(\xi', \xi)\theta(y) = \iota(\xi', \xi')\theta(y) + \iota(\xi', \xi')\theta(y)
\]
from which it follows that \(L(\xi)\theta_\mathfrak{g}(y) = L(\xi)\theta_\mathfrak{g}(y) + L(\xi)\iota_\theta(y) = 0\). So \(\theta_\mathfrak{g}\) is \(\mathfrak{g}\)-invariant.

The identity \(\iota(\xi)\theta(y) = \theta(y) - \iota(\xi)\theta(y) = 0\) shows that \(\theta_\mathfrak{g}\) is \(\mathfrak{g}\)-horizontal.

(ii) Because \(\theta_\mathfrak{g}\) is a \(\mathfrak{f}\)-connection, the maps \(c_{\mathfrak{g}, \theta}\) and \(C_{\mathfrak{g}, \theta}\) are \(\mathfrak{f}\)-dga homomorphisms. To see that they are \(\mathfrak{h}\)-dga homomorphisms we must show that for all \(v \in W\mathfrak{f}\) the element \(c_{\mathfrak{g}, \theta}(v) \in E_\mathfrak{g}\) is annihilated by \(L(\xi)\) and \(\iota(\xi)\) for all \(\xi \in \mathfrak{g}\). It is enough to check this for the generators \(v = \theta(y)\) and \(v = \dot{\theta}(y)\) of \(W\mathfrak{f}\) for \(y \in \mathfrak{f}\). But the elements \(\dot{\theta}(y)\) and \(\dot{\theta}(y)\) act on \(E_\mathfrak{g}\) through multiplication by \(\theta_\mathfrak{g}(y)\) and \(d\theta_\mathfrak{g}(y)\). By (i) the latter elements are \(\mathfrak{g}\)-basic, so they are annihilated by \(L(\xi)\) and \(\iota(\xi)\). Thus \(c_{\mathfrak{g}, \theta}\) and \(C_{\mathfrak{g}, \theta}\) are \(\mathfrak{h}\)-dga homomorphisms. It follows from Theorem A.5.1 that \(C_{\mathfrak{g}, \theta}\) is a \(\mathfrak{f}\)-homotopy inverse of the inclusion \(j\). According to the proof of that theorem a \(\mathfrak{f}\)-homotopy from the identity map of \(W\mathfrak{h} \otimes M\) to \(j \circ C_{\mathfrak{g}, \theta}\) is given by

\[(A.6.4) \quad H(v \otimes w \otimes m) = \left(\int_0^1 c_\theta(v)\right)(1 \otimes w \otimes m)\]
for \( v \in \mathcal{W}t, w \in \mathcal{W}g \) and \( m \in \mathcal{M} \). The quantity \( c_{g,\theta} : \mathcal{W}t \rightarrow S \otimes D \) in this formula is the characteristic homomorphism of the connection \( \Theta(x) = (1 - s) \otimes \theta_0 + s \otimes \theta_1 \) on the algebra \( S \otimes D \), where \( D = \text{End}(\mathcal{W}g \otimes \mathcal{M}) \). Here \( \theta_0 \) and \( \theta_1 \) are the \( t \)-connections on \( D \) determined by the following algebra homomorphisms \( f_0, f_1 : \mathcal{W}t \rightarrow D \):

\[
f_0(u)(v \otimes w \otimes m) = uv \otimes w \otimes m,
\]

\[
f_1(u)(v \otimes w \otimes m) = (-1)^{|u||v|}v \otimes c_{g,\theta}(u)(w \otimes m)
\]

for \( u, v \in \mathcal{W}t, w \in \mathcal{W}g \) and \( m \in \mathcal{M} \). The connection \( \theta_0 \) is the universal \( t \)-connection \( t^* \rightarrow \mathcal{W}t \) pulled back to \( \mathcal{W}g \otimes \mathcal{M} = \mathcal{W}t \otimes \mathcal{W}g \otimes \mathcal{M} \) and so is \( g \)-basic. The connection \( \theta_1 \) is \( g \)-basic by (i). Therefore \( \Theta \) is \( g \)-basic. The graded line \( S \) is a trivial \( g \)-dga and the functional \( \int: S \otimes D \rightarrow D \) commutes with the operations \( t(\xi) \) and \( L(\xi) \) for \( \xi \in \mathfrak{g} \). It follows that the map \( H \) defined in (A.6.4) commutes with \( t(\xi) \) and \( L(\xi) \). We conclude that the \( t \)-homotopy \( H \) is an \( \mathfrak{h} \)-homotopy. QED

Under the assumptions of the theorem we call the map \( C_{g,\theta} \) the \( g \)-equivariant Cartan-Chern-Weil homomorphism defined by the connection \( \Theta \). The hypothesis that \( \theta_0 \) should be commutative is satisfied in all our applications, but one can extend Theorem A.6.3 to noncommutative connections by resorting to the noncommutative Weil algebra of \([1]\).

In the special case where \( \mathcal{M} = \mathcal{A} \) is a commutative \( \mathfrak{h} \)-dga equipped with a \( g \)-invariant \( t \)-connection \( \theta \), we have a \( g \)-equivariant characteristic homomorphism

\[
c_{g,\theta} : \mathcal{W}t \rightarrow \mathcal{W}g \otimes \mathcal{A}.
\]

Taking \( \mathfrak{h} \)-basics and applying Lemma A.6.2(ii) gives an algebra homomorphism

\[
(A.6.5) \quad c_{g,\theta} : S(t[2]^t)^t \rightarrow (\mathcal{A}_{t-bas})_g.
\]

Taking cohomology then gives an algebra homomorphism that is independent of the connection,

\[
(A.6.6) \quad c_g : S(t[2]^t)^t \rightarrow H_0(\mathcal{A}_{t-bas}).
\]

The maps (A.6.5) and (A.6.6) are also referred to as \( g \)-equivariant characteristic homomorphisms.

A.7. Group versus algebra. Let \( \mathcal{F} = \mathcal{R} \) or \( \mathcal{C} \) and let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Let \( \mathcal{M} \) be a \( g \)-dgm and suppose that \( G \) acts linearly on \( \mathcal{M} \) in a way which is compatible with the \( \mathfrak{g} \)-module structure in the sense of \([12, \S \, 3.2.1]\). (For this to make sense we must assume either that the \( G \)-module \( \mathcal{M} \) is locally finite, or that \( \mathcal{M} \) has a complete Hausdorff locally convex topology in which the function \( G \rightarrow M \) defined by \( g \mapsto gm \) is smooth for every \( m \in \mathcal{M} \).) The \( G \)-basic complex of \( \mathcal{M} \) is the set of \( g \)-horizontal \( G \)-fixed vectors, \( \mathcal{M}_{G-bas} = \mathcal{M}_{g-bas} \cap M^G \). The \( G \)-basic cohomology is \( H_{G-bas}(\mathcal{M}) = H(\mathcal{M}_{G-bas}) \). The \( G \)-equivariant cohomology is \( H_0(\mathcal{M}) = H((\mathcal{W}g \otimes \mathcal{M})_{G-bas}) \).

A.7.1. Lemma. Suppose that \( G \) has finitely many components. Let \( \Pi = \pi_0(G) \) be the component group of \( G \) and let \( \mathcal{M} \) be a \( g \)-dgm with a compatible \( G \)-action. Then \( \mathcal{M}_{G-bas} = (\mathcal{M}_{g-bas})^\Pi, H_{G-bas}(\mathcal{M}) \cong H_{g-bas}(\mathcal{M})^\Pi, \) and \( H_G(\mathcal{M}) \cong H_g(\mathcal{M})^\Pi \).

Proof. The equality \( \mathcal{M}_{G-bas} = (\mathcal{M}_{g-bas})^\Pi \) follows from the compatibility of the \( G \)-action. The group \( \Pi \) being finite, a simple averaging argument shows that \( H((\mathcal{M}_{g-bas})^\Pi) \cong H((\mathcal{M}_{g-bas})^\Pi) \), which proves the second assertion. The last assertion follows from the second applied to the module \( \mathcal{W}g \otimes \mathcal{M} \). QED
APPENDIX B. fibre integration

By integrating a differential form along the fibres of a submersion one obtains a form of lower degree on the base manifold. We review a few properties of this useful process. Let \( \pi: E \to B \) be a smooth oriented locally trivial fibre bundle with fibre \( F \). The base \( B \) and the fibre \( F \) are allowed to have a boundary. Let \( r \) be the dimension of \( F \). Fibre integration or pushforward is a map

\[
\pi_*: \Omega_{cv}(E)[r] \to \Omega(B),
\]

which is uniquely determined by the requirement that it satisfy the projection formula

\[
\pi_*(\beta \wedge \pi^* \alpha) = \pi_* \beta \wedge \alpha
\]

for all \( \alpha \in \Omega(B) \) and \( \beta \in \Omega_{cv}(E) \). The subscript “cv” indicates vertically compactly supported forms; see §2. For existence and uniqueness of \( \pi_* \), see e.g. [21, §13], [4, §6], or [12, §10.1]. We adopt the convention (B.1), which agrees with [21], but differs by a sign from [4] and [12], so as to comply with the Koszul sign rule. The projection formula is equivalent to \( \pi_*(\pi^* \alpha \wedge \beta) = (-1)^{\text{deg} \beta} \alpha \wedge \pi_* \beta \), where \( k \) is the degree of \( \alpha \). In other words

\[
[\pi_*, l(\alpha)] = 0,
\]

where \( l(\alpha) \) denotes left multiplication by \( \alpha \) on \( \Omega(B) \) and left multiplication by \( \pi^* \alpha \) on \( \Omega_{cv}(E) \), and \([\pi_*, l(\alpha)]\) denotes the (graded) commutator of \( \pi_* \) and \( l(\alpha) \). Thus \( \pi_* \) is a degree \(-r\) morphism of left \( \Omega(B)\)-modules.

The proof of the next lemma is a routine verification based on the projection formula.

In part (ii) we denote by \( E^\partial \) the manifold \( E^\partial = \bigcup_{b \in B} \partial E_b \), which is a bundle over \( B \) with fibre \( \partial F \) and projection \( \pi^\partial = \pi|_{E^\partial} \).

B.2. Lemma.  
(i) For every pair of \( \pi \)-related vector fields \( \nu \in \mathfrak{X}(B) \) and \( \omega \in \mathfrak{X}(E) \) we have \( L(\nu) \circ \pi_* = \pi_* \circ L(\omega) \) and \( l(\nu) \circ \pi_* = (-1)^{\text{deg} \nu} \pi_* \circ l(\omega) \).

(ii) Let \( \pi^\partial_*: \Omega_{cv}(E^\partial)[r-1] \to \Omega(B) \) be the fibre integral for \( \pi^\partial: E^\partial \to B \). Then

\[
[\pi^\partial_*, d] = \pi^\partial_.
\]

(iii) A pullback diagram of oriented fibre bundles

\[
\begin{array}{ccc}
E_2 & \xrightarrow{f_\nu} & E_1 \\
\pi_2 & \downarrow & \pi_1 \\
B_2 & \xrightarrow{f_\omega} & B_1
\end{array}
\]

induces a commutative diagram

\[
\begin{array}{ccc}
\Omega_{cv}(E_2)[r] & \xrightarrow{f^*_\nu} & \Omega_{cv}(E_1)[r] \\
\pi^\partial_2 & \downarrow & \pi^\partial_1 \\
\Omega(B_2) & \xrightarrow{f^*_\omega} & \Omega(B_1)
\end{array}
\]

We spell out some consequences of part (ii). First, if the fibre has no boundary, fibre integration commutes with the differential. See §A.2 for the definition of \( g \)-differential graded modules, and their morphisms and homotopies.

B.3. Corollary. If \( \partial F = \emptyset \), then \( \pi_*: \Omega_{cv}(E)[r] \to \Omega^*(B) \) is a morphism of cochain complexes of degree \(-r\). If in addition the bundle \( E \to B \) is equivariant with respect to the action of a Lie algebra \( \mathfrak{g} \) on \( E \) and \( B \), then \( \pi_* \) is a degree \(-r\) morphism of \( \mathfrak{g} \)-differential graded modules.
If on the other hand $E$ is the cylinder $[0, 1] \times B$, then the bundle $E^\partial$ consists of two copies $E_0$ and $E_1$ of $E$, and $\pi^\partial_\beta = \beta|_{E_1} - \beta|_{E_0}$.

**B.4. Corollary.** Let $E = [0, 1] \times B$. Let $i_t : B \to E$ be the embedding $i_t(b) = (t, b)$ and $i_t^\partial : \Omega(E) \to \Omega(B)$ the induced morphism. Then we have the cylinder formula $i_1^\partial - i_0^\partial = d\pi^\partial + \pi^\partial d$. If a Lie algebra $\mathfrak{g}$ acts on $B$, then $\pi^\partial$ is a homotopy of $\mathfrak{g}$-differential graded modules.

Substituting a homotopy of maps into the cylinder formula gives the homotopy formula.

**B.5. Corollary.** Let $h : [0, 1] \times B_1 \to B_2$ be a smooth homotopy. Let $h_t(b) = h(t, b)$ and let $\pi : [0, 1] \times B_1 \to B_1$ be the projection. Then $\pi^\partial h^\partial : \Omega(B_2)[1] \to \Omega(B_1)$ is a cochain homotopy: $h_1^\partial - h_0^\partial = d\pi^\partial + \pi^\partial d$. If $h$ is equivariant with respect to a Lie algebra $\mathfrak{g}$ acting on $B_1$ and $B_2$, then $\pi^\partial h^\partial$ is a homotopy of $\mathfrak{g}$-differential graded modules.

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