C^0 \text{ FINITE ELEMENT APPROXIMATIONS OF LINEAR ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM AND HAMILTON-JACOBI-BELLMAN EQUATIONS WITH CORDES COEFFICIENTS}^{*}

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Abstract. This paper is concerned with C^0 (non-Lagrange) finite element approximations of the linear elliptic equations in non-divergence form and the Hamilton-Jacobi-Bellman (HJB) equations with Cordes coefficients. Motivated by the Miranda-Talenti estimate, a discrete analog is proved once the finite element space is C^0 on the (n - 1)-dimensional subsimplex (face) and C^1 on (n - 2)-dimensional subsimplex. The main novelty of the non-standard finite element methods is to introduce an interior penalty term to argument the PDE-induced variational form of the linear elliptic equations in non-divergence form or the HJB equations. As a distinctive feature of the proposed methods, no penalization or stabilization parameter is involved in the variational forms. As a consequence, the coercivity constant (resp. monotonicity constant) for the linear elliptic equations in non-divergence form (resp. the HJB equations) at discrete level is exactly the same as that from PDE theory. The quasi-optimal order error estimates as well as the convergence of the semismooth Newton method are established. Numerical experiments are provided to validate the convergence theory and to illustrate the accuracy and computational efficiency of the proposed methods.

Key words. Elliptic PDEs in non-divergence form, Hamilton-Jacobi-Bellman equations, Cordes condition, C^0 (non-Lagrange) finite element methods

AMS subject classifications. 65N30, 65N12, 65N15, 35D35, 35J15, 35J66

1. Introduction. In this paper, we study the C^0 (non-Lagrange) finite element approximations of the linear elliptic equations in non-divergence form

\begin{equation}
Lu := A : D^2 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\end{equation}

and the Hamilton-Jacobi-Bellman (HJB) equations

\begin{equation}
\sup_{\alpha \in \Lambda} (L^\alpha u - f^\alpha) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\end{equation}

where \( \Lambda \) is a compact metric space, and

\[ L^\alpha v := A^\alpha : D^2 v + b^\alpha \cdot \nabla v - c^\alpha v. \]

Here, \( D^2 u \) and \( \nabla u \) denote the Hessian and gradient of real-valued function \( u \), respectively. \( \Omega \) is a bounded, open, convex polytope in \( \mathbb{R}^n \) (\( n = 2, 3 \)). Problems (1.1) and (1.2) arise in many applications from areas such as probability and stochastic processes. In particular, the HJB equations (1.2), which are of the class of fully nonlinear second order partial differential equations (PDEs), play a fundamental role in the field of stochastic optimal control [13]. In the study of fully nonlinear second order PDEs, linearization techniques naturally lead to problems such as (1.1) [4].

In contrast to the PDEs in divergence form, the theory of linear elliptic equations in non-divergence form and, more generally, fully nonlinear PDEs, hinges on different solution concepts such as strong solutions, viscosity solutions, or \( H^2 \) solutions. For these different solution concepts, numerical methods for (1.1) and (1.2) have

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experienced some rapid developments in recent years. In [10, 12], the authors proposed non-standard primal finite element methods for (1.1) with the coefficient matrix $A \in C^0(\Omega; \mathbb{R}^{n \times n})$. They showed the convergence in the sense of $W^{2,p}$ strong solutions by establishing the discrete Calderon-Zygmund estimates. In [26], the authors studied a two-scale method based on the integro-differential formulation of (1.1), where the monotonicity under the weakly acute mesh condition and discrete Alexandroff-Bakelman-Pucci estimates were established to obtain the pointwise error estimates. In the Barles-Souganidis framework [1], the monotonicity is a key concept of numerical schemes for the convergence to the viscosity solutions of fully nonlinear PDEs, which is also applicable to the monotone finite difference methods (FDM) for the HJB equations [2, 13]. A monotone finite element like scheme for the HJB equations was proposed in [5]. The convergent monotone finite element methods (FEM) for the viscosity solutions of parabolic HJB equations were proposed and analyzed in [18, 17]. For a general overview, we refer the reader to the survey articles [9, 24]. Recently a narrow-stencil FDM for the HJB equations based on the generalized monotonicity was proposed in [11], where the numerical solution was proved to be convergent to the viscosity solution.

For the PDE theory of $H^2$ solutions, it is allowable to have the discontinuous coefficient matrix $A$ in (1.1). As a compensation, the coefficients are required to satisfy the Cordes condition stated below in Definition 2.1 for (1.1) and Definition 2.3 for (1.2), respectively. We refer the reader to [22] for the analysis of PDEs with discontinuous coefficients under the Cordes condition. In addition, the analysis of the problems (1.1) and (1.2) hinges on the following Miranda-Talenti estimate.

**Lemma 1.1 (Miranda-Talenti estimate).** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain. Then, for any $v \in H^2(\Omega) \cap H^1_0(\Omega)$,

\begin{equation}
|v|_{H^2(\Omega)} \leq \|\Delta v\|_{L^2(\Omega)}.
\end{equation}

For the finite element approximations of $H^2$ solutions, the most straightforward way is to apply the $C^1$-conforming finite elements [7], which are sometimes considered impractical on unstructured meshes (at least $P_5$ in 2D and $P_9$ in 3D). It is worth mentioning that the $H^2$ nonconforming elements would fail to mimic the Miranda-Talenti estimate at the discrete level. For example, a direct calculation shows that three basis functions of the Morley element [7] are harmonic.

Instead of the $C^1$-conforming finite element methods, the first discontinuous Galerkin (DG) method for (1.1) in the case of discontinuous coefficients with Cordes condition was proposed in [29], which has been extended to the elliptic and parabolic HJB equations in [30, 31]. In [25], the authors developed the $C^0$-interior penalty DG methods to both (1.1) and (1.2). The above DG methods are applicable when choosing the penalization parameters suitably large. A mixed method for (1.1) based on the stable finite element Stokes spaces was proposed in [14], which has been extended to the HJB equations in [15]. Other numerical methods for (1.1) include the discrete Hessian method [20], weak Galerkin method [33], and least square methods [21, 27].

The primary goal of this paper is to develop and analyze the $C^0$ primal finite element approximations of (1.1) and (1.2) without introducing any penalization parameter. In view of the proof of Miranda-Talenti estimate [22], the difference between $\|\Delta u\|_{L^2(\Omega)}^2$ and $|u|_{H^2(\Omega)}^2$ is a positive term that involves the mean curvature of $\partial \Omega$. The key idea in this paper is to adopt the $C^0$ finite element with the enhanced regularity on some subsimplex so as to have the ability to detect the information of mean curvature. More precisely, we show in Section 3 that, a feasible condition pertaining
to the finite element is the $C^1$-continuity on the $(n-2)$-dimensional subsimplex. In 2D case, a typical family of finite elements that meets this requirement is the family of $P_k$-Hermite finite elements ($k \geq 3$) depicted in Figure 1a. In 3D case, the family of Argyris finite elements with local space $P_k$ ($k \geq 5$) [23, 6] satisfies the condition.

Having the families of 2D Hermite finite elements and 3D Argyris finite elements, we prove a discrete analog of the Miranda-Talenti estimate in Lemma 3.3, on any conforming triangulation of convex polytope $\Omega$. The jump terms in the discrete Miranda-Talenti-type estimate (3.3) naturally induce the interior penalty in the variational form of the linear elliptic equations in non-divergence form (4.1) or the HJB equations (5.2). However, in the same spirit of [10], the proposed methods are not the DG methods per se since no penalization parameter is used. The convergence analysis mimics the analysis of $H^2$ solutions to a great extent. As a striking feature of the proposed methods, the coercivity constant (resp. monotonicity constant) for the linear elliptic equations in non-divergence form (resp. the HJB equations) at discrete level is exactly the same as that from PDE theory. Since the methods are consistent, the coercivity or monotonicity naturally leads to the energy norm error estimates.

The rest of the paper is organized as follows. In Section 2, we establish the notation and state some preliminaries results. In Section 3, we state the finite element spaces on which the discrete Miranda-Talenti-type estimate holds. We then give the applications to the discretizations of the linear elliptic equations in non-divergence form (1.1) and the HJB equations (1.2), respectively, in Section 4 and Section 5. Numerical experiments are presented in Section 6.

For convenience, we use $C$ to denote a generic positive constant that may stand for different values at its different occurrences but is independent of the mesh size $h$.

The notation $X \lesssim Y$ means $X \leq CY$.

2. Preliminaries. In this section, we first review the $H^2$ solutions to the linear elliptic equations in non-divergence form and the HJB equations under the Cordes conditions. Let $\Omega$ be a bounded, open, convex domain in $\mathbb{R}^n$ ($n = 2, 3$), in which the Miranda-Talenti estimate (1.3) holds. We shall use $U$ to denote a generic subdomain of $\Omega$ and $\partial U$ denotes its boundary. Given an integer $k \geq 0$, let $H^k(U)$ and $H^k_0(U)$ denote the usual Sobolev spaces. We also denote $V := H^2(\Omega) \cap H^1_0(\Omega)$.

2.1. Review of strong solutions to the linear elliptic equations in non-divergence form. For the problem (1.1), it is assumed that $A \in L^\infty(\Omega; \mathbb{R}^{n \times n})$, and that $L$ is uniformly elliptic, i.e., there exists $\nu, \bar{\nu} > 0$ such that

\begin{equation}
\nu |\xi|^2 \leq A(x)\xi \leq \bar{\nu} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \ a.e. \ in \ \Omega.
\end{equation}

It is well-known that, the above assumptions can not guarantee the well-posedness of (1.1); See, for instance, the example in [16, p. 185]. In addition, we assume the coefficients satisfy the Cordes condition below.

**Definition 2.1 (Cordes condition for (1.1)).** The coefficient satisfies that there is an $\varepsilon \in (0, 1]$ such that

\begin{equation}
\frac{|A|^2}{(\text{tr} A)^2} \leq \frac{1}{n - 1 + \varepsilon} \quad a.e. \ in \ \Omega.
\end{equation}

In two dimensions, it is not hard to show that uniform ellipticity implies the Cordes condition (2.2) with $\varepsilon = 2\nu/(\nu + \bar{\nu})$, see [30]. Define the strictly positive function $\gamma \in L^\infty(\Omega)$ by

\begin{equation}
\gamma := \frac{\text{tr} A}{|A|^2}.
\end{equation}
Hence, the variational formulation of (1.1) reads

\[
B_0(u, v) = \int_{\Omega} \gamma f \Delta v \, dx \quad \forall v \in V,
\]

where the bilinear form \( B_0 : V \times V \to \mathbb{R} \) is defined by

\[
B_0(w, v) := \int_{\Omega} \gamma Lw \Delta v \, dx \quad \forall w, v \in V.
\]

The well-posedness hinges on the following lemma; See [29, Lemma 1] for a proof.

**Lemma 2.2.** Under the Cordes condition (2.2), for any \( v \in H^2(\Omega) \) and open set \( U \subset \Omega \), the following inequality holds a.e. in \( U \):

\[
|\gammaLv - \Delta v| \leq \sqrt{1 - \varepsilon} |D^2v|.
\]

Using the Miranda-Talenti estimate in Lemma 1.1, Lemma 2.2 and the Lax-Milgram Lemma, it is readily seen that there exists a unique strong solution to (2.4) under the Cordes condition (2.2). The proof is given in [22, p. 256], see also [29, 25].

### 2.2. Review of strong solutions to the HJB equation.

For the HJB equations (1.2), the coefficient \( A^\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R}^{n \times n}) \) is assumed uniformly elliptic, i.e., there exists \( \nu, \bar{\nu} > 0 \) such that

\[
\nu |\xi|^2 \leq A^\alpha(x) \xi \cdot \xi \leq \bar{\nu} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. in } \Omega, \forall \alpha \in \Lambda.
\]

The corresponding Cordes condition for \( A^\alpha, b^\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R}^n) \) and \( c^\alpha \in C(\overline{\Omega} \times \Lambda) \) can be stated as follows.

**Definition 2.3 (Cordes condition for (1.2)).** The coefficients satisfy

1. whenever \( b^\alpha \not\equiv 0 \) or \( c^\alpha \not\equiv 0 \) for some \( \alpha \in \Lambda \), there exist \( \lambda > 0 \) and \( \varepsilon \in (0, 1] \) such that

\[
\frac{|A^\alpha|^2 + |b^\alpha|^2/(2\lambda) + (c^\alpha/\lambda)^2}{(\text{tr}A^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{n + \varepsilon} \quad \text{a.e. in } \Omega, \forall \alpha \in \Lambda.
\]

2. whenever \( b^\alpha \equiv 0 \) and \( c^\alpha \equiv 0 \) for all \( \alpha \in \Lambda \), there is an \( \varepsilon \in (0, 1] \) such that

\[
\frac{|A^\alpha|^2}{(\text{tr}A^\alpha)^2} \leq \frac{1}{n - 1 + \varepsilon} \quad \text{a.e. in } \Omega.
\]

Similar to (2.3) for the linear elliptic equations in non-divergence form, for each \( \alpha \in \Lambda \), we define

\[
\gamma^\alpha := \begin{cases} 
\frac{\text{tr}A^\alpha + c^\alpha/\lambda}{|A^\alpha|^2 + |b^\alpha|^2/(2\lambda) + (c^\alpha/\lambda)^2} & b^\alpha \not\equiv 0 \text{ or } c^\alpha \not\equiv 0 \text{ for some } \alpha \in \Lambda, \\
\frac{\text{tr}A^\alpha + c^\alpha/\lambda}{|A^\alpha|^2} & b^\alpha \equiv 0 \text{ and } c^\alpha \equiv 0 \text{ for all } \alpha \in \Lambda.
\end{cases}
\]

Note here that the continuity of data implies \( \gamma^\alpha \in C(\overline{\Omega} \times \Lambda) \). Define the operator \( F_\gamma : H^2(\Omega) \to L^2(\Omega) \) by

\[
F_\gamma[v] := \sup_{\alpha \in \Lambda} \gamma^\alpha(L^\alpha v - f^\alpha).
\]

It is readily seen that the HJB equations (1.2) is in fact equivalent to the problem \( F_\gamma[u] = 0 \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). The Cordes condition (2.8) leads to the following lemma; See [30, Lemma 1] for a proof.
LEMMA 2.4. Under the Cordes condition (2.8), for any open set $U \subset \Omega$ and $w, v \in H^2(U)$, $z := w - v$, the following inequality holds a.e. in $U$:

$$\|F_\gamma[w] - F_\gamma[v] - L_\Lambda z\| \leq \sqrt{1 - \varepsilon^2}\|D^2z\|^2 + 2\lambda|\nabla z|^2 + \lambda^2|z|^2. \tag{2.11}$$

For $\lambda$ as in (2.8a), we define a linear operator $L_\Lambda$ by

$$L_\Lambda v := \Delta v - \lambda v \quad v \in V. \tag{2.12}$$

Let the operator $M : V \rightarrow V^*$ be

$$\langle M[w], v \rangle := \int_{\Omega} F_\gamma[w]L_\Lambda v \, dx \quad \forall w, v \in V. \tag{2.13}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $V^*$ and $V$. Under the condition of Miranda-Talenti estimate in Lemma 1.1, it is straightforward to show that

$$\|L_\Lambda v\|_{L^2(\Omega)}^2 \geq \int_{\Omega} |D^2v|^2 + 2\lambda|\nabla v|^2 + \lambda^2|v|^2 \, dx \quad \forall v \in V. \tag{2.14}$$

By using the Cordes condition (2.8), one can show the strong monotonicity of $M$. Together with the Lipschitz continuity of $M$ by the compactness of $\Lambda$ and the Browder-Minty Theorem [28, Theorem 10.49], one can show the existence and uniqueness of the following problem: Find $u \in V$ such that

$$\langle M[u], v \rangle = 0 \quad \forall v \in V. \tag{2.15}$$

We refer to [30, Theorem 3] for a detailed proof.

3. Finite element spaces and discrete Miranda-Talenti-type estimate.

Let $T_h$ be a conforming and shape-regular simplicial triangulation of $\Omega$ and $F_h$ be the set of all faces of $T_h$. Let $F_h^i := F_h \setminus \partial \Omega$ and $F_h^0 := F_h \cap \partial \Omega$. Here, $h := \max_{K \in T_h} h_K$, and $h_K$ is the diameter of $K$ (cf. [7, 3]). Since each element has piecewise flat boundary, the faces may also be chosen to be flat.

We define the jump of a vector function $w$ on an interior face $F = \partial K^+ \cap \partial K^-$ as follows:

$$\|w\|_F := |w^+ \cdot n^+|_F + |w^- \cdot n^-|_F,$$

where $w^\pm = w|_{K^\pm}$ and $n^\pm$ is the outward unit normal vector of $K^\pm$, respectively. We also denote $\omega_F := K^+ \cup K^-$ for any $F \in F_h^i$. For an element $K \in T_h$, $(\cdot, \cdot)_K$ denotes the standard inner product on $L^2(K)$. The standard inner products $(\cdot, \cdot)_\partial K$ and $(\cdot, \cdot)_F$, are defined in a similar way.

For $F \in F_h$, following [29, 30], we define the tangential gradient $\nabla_T : H^s(F) \rightarrow H^{s-1}_T(F)^n$ and the tangential divergence $\text{div}_T : H^s_T(F) \rightarrow H^{s-1}(F)$, where $s \geq 1$. Here, $H^s_T(F) := \{v \in H^s(F) : v \cdot n_F = 0 \text{ on } F\}$. Let $\{t_i\}_{i=1}^{n-1}$ be an orthogonal coordinate system on $F$. Then, for $w \in H^s(F)$ and $v = \sum_{i=1}^{n-1} v_i t_i$ with $v_i \in H^s(F)$ for $i = 1, \cdots, n-1$, define

$$\nabla_T w := \sum_{i=1}^{n-1} t_i \frac{\partial w}{\partial t_i}, \quad \text{div}_T v := \sum_{i=1}^{n-1} \frac{\partial v_i}{\partial t_i}. \tag{3.1}$$

We also define $\Delta_T w := \text{div}_T \nabla_T w$ for $w \in H^s(F)$, where $s \geq 2$. 


3.1. The families of 2D Hermite finite elements and 3D Argyris finite elements. In this subsection, we shall describe the finite elements that will be used to solve the linear elliptic equations in non-divergence form (1.1) and the HJB equations (1.2). More precisely, we adopt the Hermite elements in 2D and the Argyris elements in 3D [23, 6], both of which have the $C^0$-continuity on the face (namely, $(n-1)$-dimensional subsimplex) and $C^1$-continuity on the $(n-2)$-dimensional subsimplex.

The family of 2D Hermite finite elements. Following the description of [7, 3], the geometric shape of Hermite elements is triangle $K$. The shape function space is given as $P^k(K)$ ($k \geq 3$), where $P^k(K)$ denotes the set of polynomials with total degree not exceeding $k$ on $K$. In $K$ the degrees of freedom are defined as follows (cf. Figure 1):

- Function value $v(a)$ and first order derivatives $\partial_i v(a)$, $i = 1, 2$ at each vertex;
- Moments $\int_e vq \, ds$, $\forall q \in P_{k-4}(e)$ on each edge $e$;
- Moments $\int_K vq \, dx$, $\forall q \in P_{k-3}(K)$ on element $K$.

It is simple to check that the degrees of freedom given above form a unisolvent set of $P_k(K)$ for $k \geq 3$ [7].

![Fig. 1: Degrees of freedom of 2D $P_k$ Hermite elements, in the case of $k = 3$ and $k = 4$](image)

The family of 3D Argyris finite elements. In 3D case, the finite elements are required to be $C^0$ on face and $C^1$ on edge. The typical elements that meet the requirement in 3D are the Argyris elements [6], which coincide with each component of the velocity finite elements in the 3D smooth de Rham complex [23]. Given a tetrahedron $K$, the shape function space is given as $P_k(K)$, for $k \geq 5$. In $K$ the degrees of freedom are defined as follows (cf. Figure 2):

- One function value and (nine) derivatives up to second order at each vertex;
- Moments $\int_e vq \, ds$, $\forall q \in P_{k-6}(e)$ on each edge $e$;
- Moments $\int_e \frac{\partial v}{\partial n_{e,i}} q \, ds$, $\forall q \in P_{k-5}(e)$, $i = 1, 2$ on each edge $e$;
- Moments $\int_F vq \, ds$, $\forall q \in P_{k-6}(F)$ on each face $F$;
- Moments $\int_F vq \, dx$, $\forall q \in P_{k-4}(K)$ on element $K$.

Here, $n_{e,i}$ ($i = 1, 2$) are two unit orthogonal normal vectors that are orthogonal to the edge $e$.

We sketch the main argument of the unisolvent property. We first note that the number of degrees of freedom above is

$$4 \times 10 + 6 \times (k - 5) + 6 \times 2(k - 4) + 4 \times \frac{(k - 4)(k - 3)}{2} + \frac{(k - 1)(k - 2)(k - 3)}{6},$$

which is exactly the dimension of $P_k(K)$. Therefore, it suffices to show that $v \in P_k(K)$ vanishes if it vanishes at all the degrees of freedom. It is readily seen that the trace of $v|_F \in P_k(F)$, which has to be zero since the degrees of freedom on the face are...
that of 2D Argyris elements [7] (this also shows the \( C^0 \)-continuity on face). Therefore \( v = b_K p \) for some \( p \in \mathcal{P}_{k-4}(K) \), where \( b_K \) is the quartic volume bubble function. By the set of degrees of freedom on element \( K \), we deduce \( v \equiv 0 \). The \( C^1 \)-continuity on the edge follows from the \( C^1 \)-continuity of the Argyris elements in 2D.

### 3.2. Finite element spaces.

For every triangulation \( T_h \) of the polytope \( \Omega \), we are now ready to define the finite element spaces \( V_h \) as

1. For \( n = 2 \), with \( k \geq 3 \),

\[
V_h := \{ v \in H^1_0(\Omega) : v|_K \in \mathcal{P}_k(K), \forall K \in T_h, \ v \text{ is } C^1 \text{ at all vertices} \}.
\]

2. For \( n = 3 \), with \( k \geq 5 \),

\[
V_h := \{ v \in H^1_0(\Omega) : v|_K \in \mathcal{P}_k(K), \forall K \in T_h, \ v \text{ is } C^1 \text{ on all edges,} \ v \text{ is } C^2 \text{ at all vertices} \}.
\]

A unisolvent set of degrees of freedom of \( V_h \) is given locally by that of 2D Hermite elements or 3D Argyris elements. Since the finite elements may have extra continuity on sub-simplex, we briefly explain the implementation of the boundary conditions.

**Boundary condition of 2D Hermite FEM space.** Since the first order derivatives are imposed at vertices, for any \( v_h \in V_h \), the tangential derivatives along the boundary should also be zero due to the consistency. In this case, we follow the terminology in [8, 6] to introduce the definition of *corner vertices*.

**Definition 3.1.** A boundary vertex is called a corner vertex if the two adjacent boundary edges sharing this vertex do not lie on a straight line.

At each corner boundary vertex, since the two tangential directions along the boundary form a basis of \( \mathbb{R}^2 \), we should impose both the first order derivatives to be zero. At a non-corner boundary vertex, the two tangential derivatives along its two adjacent edges coincide up to a sign. In this case, we only specify the degree of freedom value of this tangential derivative.

**Boundary condition of 3D Argyris FEM space.** The enhanced continuities are imposed on the edge and vertex for the 3D Argris finite element. Therefore, we follow the terminology in [6] to introduce the corner vertices and corner edges.

**Definition 3.2.** A boundary vertex is called corner vertex in 3D if the adjacent boundary edges sharing this vertex are not coplanar. A boundary edge is called a corner edge in 3D if the two adjacent faces on the boundary sharing this edge are not coplanar.
For a corner boundary vertex in 3D, there are three linearly independent boundary edges connected to it. Hence, all the degrees of freedom at the corner boundary vertex should be set to zero. Similarly, on a corner edge, derivatives of a function along two normal directions can be determined by the function value on the boundary. Hence, all the degrees of freedom at the corner boundary vertex should be set to zero.

On the other hand, there are only two independent directions, namely $t_1$ and $t_2$ along the boundary at a non-corner boundary vertex or on a non-corner edge (cf. Figure 3b). Therefore at a non-corner vertex, function value, two tangential first order derivatives (i.e. $\partial_{t_1}$, $\partial_{t_2}$) and three tangential second order derivatives (i.e. $\partial_{t_1,t_1}$, $\partial_{t_2,t_2}$, $\partial_{t_1,t_2}$) are set to be zero. The degrees of freedom corresponding to the normal first order derivative (i.e. $\partial_n$) and three second order derivatives (i.e. $\partial_{nn}$, $\partial_{t_1,n}$, $\partial_{t_2,n}$) should be treated as knowns. The treatment of degree of freedoms on non-corner edge follows a similar way.

### 3.3. Discrete Miranda-Talenti-type estimate.

The following lemma is crucial in the design and analysis of $C^0$ (non-Lagrange) finite element approximations of the linear elliptic equations in non-divergence form (1.1) and the HJB equations (1.2).

**Lemma 3.3 (Discrete Miranda-Talenti-type estimate).** Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be a bounded Lipschitz polytopal domain and $\mathcal{T}_h$ be a conforming triangulation. For $v \in V_h$, it holds that

\[
\sum_{K \in \mathcal{T}_h} \|\Delta v_h\|^2_{L^2(K)} = \sum_{K \in \mathcal{T}_h} \|D^2 v_h\|^2_{L^2(K)} + 2 \sum_{F \in \mathcal{F}_h} \langle \|\nabla v_h\|, \Delta_T v_h \rangle_F.
\]

**Proof.** For any simplicial element $K \in \mathcal{T}_h$, the outward unit normal vector of $\partial K$ is piecewise constant. Using integration by parts, we obtain (see e.g. [29, Equ. (3.7)])

\[
\|\Delta v_h\|^2_{L^2(K)} = \|D^2 v_h\|^2_{L^2(K)} + \langle \Delta v_h, \frac{\partial v_h}{\partial n} \rangle_{\partial K} - \langle \nabla \frac{\partial v_h}{\partial n}, \nabla v_h \rangle_{\partial K}
\]

\[
= \|D^2 v_h\|^2_{L^2(K)} + \langle \Delta_T v_h, \frac{\partial v_h}{\partial n} \rangle_{\partial K} - \langle \nabla_T \frac{\partial v_h}{\partial n}, \nabla_T v_h \rangle_{\partial K}.
\]
Here, the common term \(<\partial^2 v_h/\partial n^2, \partial v_h/\partial n>\)_{\partial K}\) is cancelled in the last step. We use \(n_{t_F}\) to denote the outward unit normal vector of \(\partial F\) coplanar to \(F\). Using integration by parts on \(F \subset \partial K\), we have
\[
\langle \nabla_T \frac{\partial v_h}{\partial n}, \nabla_T v_h \rangle_{\partial K} = \sum_{F \subset \partial K} \int_F \nabla_T \frac{\partial v_h}{\partial n} \cdot \nabla_T v_h \, ds
\]
\[
= \sum_{F \subset \partial K} \left( -\int_F \frac{\partial v_h}{\partial n} \Delta_T v_h \, ds + \int_{\partial F} \frac{\partial v_h}{\partial n} \frac{\partial v_h}{\partial n_{t_F}} \, ds \right).
\]
Hence, (3.4) can be reformulated as
\[
\|\Delta v_h\|^2_{L^2(K)} = \|D^2 v_h\|^2_{L^2(K)} + 2\langle \Delta_T v_h, \frac{\partial v_h}{\partial n} \rangle_{\partial K} - \sum_{F \subset \partial K} \langle \frac{\partial v_h}{\partial n}, \frac{\partial v_h}{\partial n_{t_F}} \rangle_{\partial F}.
\]
Thanks to the \(C^0\)-continuity, it is readily seen that \(\Delta_T v_h\) is continuous across the face and vanishes on the boundary. Summing over all elements yields
\[
\sum_{K \in T_h} \|\Delta v_h\|^2_{L^2(K)} = \sum_{K \in T_h} \|D^2 v_h\|^2_{L^2(K)} + 2 \sum_{F \in F_h} \langle [\nabla v_h], \Delta_T v_h \rangle_F
\]
\[
- \sum_{K \in T_h, F \subset \partial K} \langle \frac{\partial v_h}{\partial n}, \frac{\partial v_h}{\partial n_{t_F}} \rangle_{\partial F}.
\]
For any \(F \in F_h^0\), the boundary condition implies that \(\frac{\partial v_h}{\partial n_{t_F}} = 0\). For any interior face \(F = \partial K^+ \cap \partial K^-\), the \(C^1\)-continuity on the \((n-2)\)-dimensional subsimplex implies that
\[
\frac{\partial v^+}{\partial n^+} \bigg|_{\partial F} = -\frac{\partial v^-}{\partial n^-} \bigg|_{\partial F}, \quad \frac{\partial v^+_h}{\partial n_{t_F}} \bigg|_{\partial F} = \frac{\partial v^-_h}{\partial n_{t_F}} \bigg|_{\partial F}.
\]
Then, we deduce that the last term in (3.5) vanishes, which gives the desired result (3.3).

4. Applications to the linear elliptic equations in non-divergence form.
In this section, we apply the \(C^0\) (non-Lagrange) finite element method to solve the linear elliptic equations in non-divergence form (1.1).

4.1. Numerical scheme. Define the broken bilinear form for \(w \in V + V_h\) and \(v_h \in V_h\):
\[
B_{0,h}(w, v_h) := \sum_{K \in T_h} (\gamma L w, \Delta v_h)_K - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^0} \langle [\nabla w], \Delta_T v_h \rangle_F.
\]
We propose the following finite element scheme to approximate the solution to linear elliptic equations in non-divergence form (1.1): Find \(u_h \in V_h\) such that
\[
B_{0,h}(u_h, v_h) = \sum_{K \in T_h} (\gamma f, \Delta u_h)_K \quad \forall v_h \in V_h.
\]
We emphasize that no penalty or stabilization parameter is involved in the scheme above. The broken norm is introduced on \(V + V_h\):
\[
\|v\|^2_{0,h} := \sum_{K \in T_h} \|D^2 v\|^2_{L^2(K)} \quad \forall v \in V + V_h.
\]
Lemma 3.3. The coercivity result follows directly from the discrete Miranda-Talenti-type estimate in (3.3) where the discrete Miranda-Talenti-type estimate (3.3) is used in the last step.

A simple modification of (4.1) reads

\[ \|v\|_{0,h} = 0 \] implies that \( D^2 v|_K = 0 \) on each element \( K \). Together with the \( C^1 \)-continuity on \((n-2)\)-dimensional subsimplex, we immediately have that \( v \) is a linear polynomial on \( \Omega \), which means \( v \equiv 0 \) since \( v \) vanishes on \( \partial \Omega \). The following coercivity result follows directly from the discrete Miranda-Talenti-type estimate in Lemma 3.3.

**Lemma 4.1.** There holds that

\[ B_{0,h}(v_h, v_h) \geq (1 - \sqrt{1 - \varepsilon}) \|v_h\|_{0,h}^2 \quad \forall v_h \in V_h. \]  

**Proof.** By using Lemma 2.2 and Cauchy-Schwarz inequality, we have

\[ B_{0,h}(v_h, v_h) = \sum_{K \in T_h} (\gamma L v_h - \Delta v_h, \Delta v_h)_K + \sum_{K \in T_h} \|\Delta v_h\|_{L^2(K)}^2 \]

\[ - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^I} \langle |\nabla v_h|, \Delta_T v_h \rangle_F \]

(by (2.6)) \[ \geq \sum_{K \in T_h} \|\Delta v_h\|_{L^2(K)}^2 - \sqrt{1 - \varepsilon} \sum_{K \in T_h} \|D^2 v_h\|_{L^2(K)} \|\Delta v_h\|_{L^2(K)} \]

\[ - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^I} \langle |\nabla v_h|, \Delta_T v_h \rangle_F \]

\[ \geq \sum_{K \in T_h} \|\Delta v_h\|_{L^2(K)}^2 - \frac{\sqrt{1 - \varepsilon}}{2} \sum_{K \in T_h} \left( \|D^2 v_h\|_{L^2(K)}^2 + \|\Delta v_h\|_{L^2(K)}^2 \right) \]

\[ - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^I} \langle |\nabla v_h|, \Delta_T v_h \rangle_F \]

\[ = (1 - \sqrt{1 - \varepsilon}) \left( \sum_{K \in T_h} \|\Delta v_h\|_{L^2(K)}^2 - 2 \sum_{F \in F_h^I} \langle |\nabla v_h|, \Delta_T v_h \rangle_F \right) \]

\[ - \frac{\sqrt{1 - \varepsilon}}{2} \sum_{K \in T_h} \|D^2 v_h\|_{L^2(K)}^2 \]

(recall (4.3)) \[ = (1 - \sqrt{1 - \varepsilon}) \|v_h\|_{0,h}^2, \]

where the discrete Miranda-Talenti-type estimate (3.3) is used in the last step.

We note that the coercivity constant (namely, \( 1 - \sqrt{1 - \varepsilon} \)) under the broken norm \( \|\cdot\|_{0,h} \) is exactly the same as that for the PDE theory. As a corollary, the uniqueness of the finite element method (4.2) implies the existence, since equation (1.1) is linear. Further, assume that the solution \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), a straightforward argument shows the consistency, namely

\[ B_{0,h}(u, v_h) = \sum_{K \in T_h} (\gamma Lu, \Delta v_h)_K = \sum_{K \in T_h} (\gamma f, \Delta v_h)_K \quad \forall v_h \in V_h. \]

**Remark 4.2.** We note that the bilinear form (4.1) explicitly uses the constant \( \varepsilon \) in Cordes condition (2.2). In case that the optimal value of \( \varepsilon \) is not easy to compute, a simple modification of (4.1) reads

\[ \tilde{B}_{0,h}(w, v_h) := \sum_{K \in T_h} (\gamma L w, \Delta v_h)_K - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^I} \langle |\nabla w|, \Delta_T v_h \rangle_F, \]
where \( \hat{\varepsilon} \) is an approximation of \( \varepsilon \) that satisfies \( \sqrt{1 - \hat{\varepsilon}} + \frac{1 - \varepsilon}{2\sqrt{1 - \hat{\varepsilon}}} < 2 \). Using the inequality

\[
2\|D^2v_h\|_{L^2(\Omega)}\|\Delta v_h\|_{L^2(\Omega)} 
\leq \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 - \hat{\varepsilon}}} \|D^2v_h\|_{L^2(K)}^2 + \frac{\sqrt{1 - \hat{\varepsilon}}}{\sqrt{1 - \varepsilon}} \|\Delta v_h\|_{L^2(K)}^2 \quad \forall K \in \mathcal{T}_h,
\]

we have the coercivity result

\[
\tilde{B}_{0,h}(v_h, v_h) \geq (1 - \frac{\sqrt{1 - \varepsilon}}{2} - \frac{1 - \varepsilon}{2\sqrt{1 - \hat{\varepsilon}}}) \|v_h\|_{0,h}^2 \quad \forall v_h \in V_h,
\]

following a similar argument as Lemma 4.1. Clearly, the optimal coercivity constant is attained at \( \hat{\varepsilon} = \varepsilon \). Even if there is no any a priori estimate of \( \varepsilon \), one may simply take \( \hat{\varepsilon} = 0 \), which leads to the coercivity constant \( \frac{1}{2} \).

4.2. Error estimate. Thanks to the coercivity result (4.4) and the consistency (4.6), we then arrive at the quasi-optimal error estimate.

**Theorem 4.3.** Let \( \Omega \) be a bounded, convex polytope in \( \mathbb{R}^n \), and let \( \mathcal{T}_h \) be a simplicial, conforming, shape-regular mesh. Suppose that the coefficients satisfy the Cordes condition (2.2) and the solution to (1.1) \( u \in H^s(\Omega) \cap H_0^1(\Omega) \) for some \( s \geq 2 \). Then, there holds

\[
(4.7) \quad \|u - u_h\|_{0,h}^2 := \sum_{K \in \mathcal{T}_h} \|D^2(u - u_h)\|_{K}^2 \leq C \sum_{K \in \mathcal{T}_h} h_{K}^{2t-4} \|u\|_{H^s(K)}^2,
\]

where \( t = \min\{s, k + 1\} \).

**Proof.** Since the sequence of meshes is shape regular, it follows from the standard polynomial approximation theory [3] that, there exists a \( z_h \in V_h \), such that

\[
(4.8a) \quad \|u - z_h\|_{H^s(K)} \leq Ch_{K}^{1-q} \|u\|_{H^s(\omega_K)}, \quad 0 \leq q \leq 2,
\]

\[
(4.8b) \quad \|D^\beta(u - z_h)\|_{L^2(\partial K)} \leq Ch_{K}^{1-q-1/2} \|u\|_{H^s(\omega_K)} \quad \forall |\beta| = q, \quad 0 \leq q \leq 1,
\]

where \( \omega_K \) represents the union of the local neighborhood of element \( K \). Let \( \psi_h = z_h - u_h \). Then, by the coercivity result (4.4), we obtain

\[
(4.9) \quad \|z_h - u_h\|_{0,h}^2 \leq B_{0,h}(z_h - u_h, \psi_h) = B_{0,h}(z_h, \psi_h) - \sum_{K \in \mathcal{T}_h} (\gamma_f, \Delta \psi_h)_K
\]

\[
= \sum_{K \in \mathcal{T}_h} (\gamma L(z_h - u), \Delta \psi_h)_K - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in \mathcal{F}_h} \langle \nabla z_h, \Delta_T \psi_h \rangle_F.
\]

By the boundedness of the data, the fact that \( \|\Delta \psi_h\|_{L^2(\Omega)} \leq \sqrt{n} \|D^2 \psi_h\|_{L^2(\Omega)} \) and the approximation result (4.8a), we have

\[
|E_1| \leq \sum_{K \in \mathcal{T}_h} \|\gamma L(u - z_h)\|_{L^2(\Omega)} \|\Delta \psi_h\|_{L^2(\Omega)}
\]

\[
\leq \sum_{K \in \mathcal{T}_h} \sqrt{n} \|\gamma\|_{L^\infty(\Omega)} \|A\|_{L^\infty(\Omega)} \|D^2(u - z_h)\|_{L^2(\Omega)} \|D^2 \psi_h\|_{L^2(\Omega)}
\]

\[
\leq \left( \sum_{K \in \mathcal{T}_h} h_{K}^{2t-4} \|u\|_{H^s(K)}^2 \right)^{1/2} \|\psi_h\|_{0,h}.
\]
Further, the local trace inequality implies that \( \| \Delta_T \psi_h \|_{L^2(\partial K)} \lesssim h_K^{-1/2} \| D^2 \psi_h \|_{L^2(K)} \), together with the approximation result (4.8b), we have
\[
|E_2| = \left| \sum_{F \in F^1_h} \left\langle \left\| \nabla u - \nabla z_h \right\|, \Delta_T \psi_h \right\rangle_F \right| \leq \sum_{F \in F^1_h} \left\| \nabla (u - z_h) \right\|_{L^2(F)} \| \Delta_T \psi_h \|_{L^2(F)} \\
\lesssim \left( \sum_{K \in T_h} h_K^{2l-4} \| u \|_{H^s(K)} \right)^{1/2} \| \psi_h \|_{0,h}.
\]
The above inequalities and (4.9) give rise to
\[
\| z_h - u_h \|_{0,h} \leq C \left( \sum_{K \in T_h} h_K^{2l-4} \| u \|_{H^s(K)} \right)^{1/2},
\]
which implies the desired result by triangle inequality.

5. Applications to the Hamilton-Jacobi-Bellman equations. In this section, we apply the \( C^0 \) (non-Lagrange) finite element method to solve the HJB equations (1.2), which can be viewed as a natural extension of the numerical scheme for the linear elliptic equations in non-divergence form. Since an additional parameter \( \lambda > 0 \) is introduced in the Cordes condition (2.8a), the broken norm is defined as
\[
\| v \|_{\lambda,h}^2 := \sum_{K \in T_h} \| v \|_{\lambda,h,K}^2 := \sum_{K \in T_h} \left( \| D^2 v \|_{L^2(K)}^2 + 2 \lambda \| \nabla v \|_{L^2(K)}^2 + \lambda^2 \| v \|_{L^2(K)}^2 \right).
\]
Thanks to the discussion of \( \| \cdot \|_{0,h} \) in (4.3), it is readily seen that \( \| \cdot \|_{\lambda,h} \) is indeed a norm on \( V + V_h \) for all \( \lambda \geq 0 \).

5.1. Numerical scheme. We describe the finite element method. In light of (2.13), we define the operator \( M_h : V + V_h \mapsto V_h^* \) by
\[
\langle M_h[w], v_h \rangle := \sum_{K \in T_h} \left( F_v[w], L_{\lambda,v} \right)_K - \left( 2 - \sqrt{1 - \varepsilon} \right) \sum_{F \in F^1_h} \left\langle \left\| \nabla w \right\|, \Delta_T v_h - \lambda v_h \right\rangle_F,
\]
where we recall that \( L_{\lambda,v} = \Delta v - \lambda v \) in (2.12). The following finite element method is proposed to approximate the solution to the HJB equations (1.2): Find \( u_h \in V_h \) such that
\[
\langle M_h[u_h], v_h \rangle = 0 \quad \forall v_h \in V_h.
\]

**Lemma 5.1.** For every \( w_h, v_h \in V_h \), we have
\[
\langle M_h[w_h] - M_h[v_h], w_h - v_h \rangle \geq (1 - \sqrt{1 - \varepsilon}) \| w_h - v_h \|_{\lambda,h}^2.
\]

**Proof.** Set \( z_h = w_h - v_h \). Using the discrete Miranda-Talenti-type estimate (3.3) and integration by parts, we obtain
\[
\sum_{K \in T_h} \| L_{\lambda,z_h} \|_{L^2(K)}^2 \\
= \sum_{K \in T_h} \| \Delta z_h \|_{L^2(K)}^2 - 2 \lambda \sum_{K \in T_h} \left( z_h, \Delta z_h \right)_K + \lambda^2 \| z_h \|_{L^2(\Omega)}^2 \\
= \sum_{K \in T_h} \| \Delta z_h \|_{L^2(K)}^2 + 2 \lambda \| \nabla z_h \|_{L^2}^2 + \lambda^2 \| z_h \|_{L^2(\Omega)}^2 - 2 \lambda \sum_{K \in T_h} \int_{\partial K} z_h \frac{\partial z_h}{\partial n} \, ds \\
= \| z_h \|_{\lambda,h}^2 + \sum_{F \in F^1_h} \left\langle \left\| \nabla z_h \right\|, \Delta_T z_h - \lambda z_h \right\rangle_F,\]
where we use the definition of \(\| \cdot \|_{\lambda,h} (5.1)\) in the last step. Further, by Lemma 2.4, we have

\[
(M_h[w_h] - M_h[v_h], z_h)
\]

\[
= \sum_{K \in T_h} (F_\gamma[w_h] - F_\gamma[v_h] - L_\lambda z_h, L_\lambda z_h)_K + \sum_{K \in T_h} \|L_\lambda z_h\|_{L^2(K)}^2
\]

\[
- (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^i} \langle \|\nabla z_h\|, \Delta_T z_h - \lambda z_h \rangle_F
\]

\[
\geq \sum_{K \in T_h} \|L_\lambda z_h\|_{L^2(K)}^2 - \sqrt{1 - \varepsilon} \sum_{K \in T_h} \|z_h\|_{\lambda, h, K} \|L_\lambda z_h\|_{L^2(K)}
\]

\[
- (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^i} \langle \|\nabla z_h\|, \Delta_T z_h - \lambda z_h \rangle_F
\]

\[
\geq \frac{2 - \sqrt{1 - \varepsilon}}{2} \sum_{K \in T_h} \|L_\lambda z_h\|_{L^2(K)}^2 - 2 \sum_{F \in F_h^i} \langle \|\nabla z_h\|, \Delta_T z_h - \lambda z_h \rangle_F
\]

\[
- \frac{\sqrt{1 - \varepsilon}}{2} \|z_h\|_{\lambda,h}^2.
\]

Applying (5.5), we conclude the strong monotonicity of \(M_h\) in (5.4).

Again, the monotonicity constant under the broken norm is exactly the same as that for the PDE theory. Similar to the Remark 4.2, the monotonicity constant becomes \(1 - \frac{\sqrt{1 - \varepsilon}}{2 - 2\sqrt{1 - \varepsilon}}\) if \(\varepsilon\) is replaced by its approximation \(\tilde{\varepsilon}\). Next, we show that \(M_h\) is Lipschitz continuous on \(V_h\) with respect to \(\cdot \|_{\lambda,h}\).

**Lemma 5.2.** For any \(v_h, w_h, z_h \in V_h\),

\[
\langle (M_h[w_h] - M_h[v_h], z_h) \rangle \leq C \|w_h - v_h\|_{\lambda, h} \|z_h\|_{\lambda,h}.
\]

**Proof.** In light of the definition of \(M_h\) in (5.2), using Cauchy-Schwarz inequality, we have

\[
\|M_h[w_h] - M_h[v_h]\|
\]

\[
\leq \sum_{K \in T_h} \|F_\gamma[w_h] - F_\gamma[v_h] - L_\lambda (w_h - v_h)\|_{L^2(K)} \|L_\lambda z_h\|_{L^2(K)}
\]

\[
+ \sum_{K \in T_h} \|L_\lambda (w_h - v_h)\|_{L^2(K)} \|L_\lambda z_h\|_{L^2(K)}
\]

\[
+ (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h^i} \|\nabla (w_h - v_h)\|_{L^2(F)} \left( \|\Delta_T z_h\|_{L^2(F)} + \|\lambda z_h\|_{L^2(F)} \right).
\]

Revolving Lemma 2.4, and the fact that \(\sum_{K \in T_h} \|L_\lambda \|_{L^2(K)}^2 \leq 2n \|v\|_{\lambda,h}^2\) for any \(v \in V + V_h\), we have

\[
I_1 \leq \sqrt{2n(1 - \varepsilon)} \|w_h - v_h\|_{\lambda,h} \|z_h\|_{\lambda,h}, \quad I_2 \leq 2n \|w_h - v_h\|_{\lambda,h} \|z_h\|_{\lambda,h}.
\]

For any interior face \(F = \partial K^+ \cap \partial K^-\), the standard scaling argument [7, 3] gives

\[
\|\nabla (w_h - v_h)\|_{L^2(F)}^2 \lesssim h_F \sum_{K \in \{K^+, K^-, K\}} \|\Delta^2 (w_h - v_h)\|_{L^2(K)}^2.
\]
where the $C^0$-continuity at face and $C^1$-continuity at $(n-2)$-dimensional subsimplex guarantee that the piecewise linear function on $\omega_F = K^+ \cup K^-$ has to be a linear function on the $\omega_F$. Further, By the local trace inequality, we have that for $F \subset \partial K$

$$\| \Delta_T z_h \|^2_{L^2(F)} \lesssim h_F^{-1} \| D^2 z_h \|_{L^2(K)}, \quad \| \lambda z_h \|^2_{L^2(F)} \lesssim h_F^{-1} \lambda^2 \| z_h \|^2_{L^2(K)}.$$ 

Hence, we have $I_3 \lesssim \| w_h - v_h \|_{\lambda,h} \| z_h \|_{\lambda,h}$. The bound (5.6) is obtained from the above estimates of $I_i \ (i = 1, 2, 3)$.

Having the strong monotonicity and the Lipschitz continuity, by the Browder-Minty Theorem, there exists a unique solution $u_h \in V_h$ to (5.3).

5.2. Error estimate. The consistency of (5.3) follows naturally since the term $\sum_{F \in F_h} (\| \nabla u \|, \Delta_T v_h - \lambda v_h)_F$ vanishes for $u \in H^2(\Omega) \cap H_1^0(\Omega)$. Finally, we arrive at the quasi-optimal error estimate.

**Theorem 5.3.** Let $\Omega$ be a bounded, convex polytope in $\mathbb{R}^n$, and let $T_h$ be a simplicial, conforming, shape-regular mesh. Let $\Lambda$ be a compact metric space. Suppose that the coefficients satisfy the Cordes condition (2.8). Then, there exists a unique solution $u_h \in V_h$, satisfying (5.3). Moreover, there holds that

$$(5.7) \quad \| u - u_h \|^2_{\lambda,h} \leq C \sum_{K \in T_h} h_K^{2t-4} \| u \|_{H^t(K)}^2,$$

where $t = \min\{s, k+1\}$ provided that $u \in H^s(\Omega) \cap H_1^0(\Omega)$ for some $s \geq 2$.

**Proof.** Since the sequence of meshes is shape regular, it follows from the standard polynomial approximation theory [3] that, there exists a $z_h \in V_h$, such that

$$\| u - z_h \|^2_{H^t(K)} \leq C h_K^{t-q} \| u \|_{H^{t}(\omega_K)}, \quad 0 \leq q \leq 2,$$

$$\| D^\beta(z_h - u) \|^2_{L^2(\partial K)} \leq C h_K^{t-q-1/2} \| u \|_{H^{t}(\omega_K)} \quad \forall |\beta| = q, \quad 0 \leq q \leq 1.$$

Let $\psi_h = z_h - u_h$. In light of the consistency, the strong monotonicity of $M_h$ on $V_h$, as shown in Lemma 5.1, yields

$$(5.9) \quad \| \psi_h \|^2_{\lambda,h} \lesssim \langle M_h[z_h] - M_h[u_h], \psi_h \rangle = \langle M_h[z_h] - M[u], \psi_h \rangle \underbrace{= \sum_{K \in T_h} (F_\gamma[z_h] - F_\gamma[u] - L\lambda(z_h - u), L\lambda\psi_h)_K}_{E_1} + \sum_{K \in T_h} (L\lambda(z_h - u), L\lambda\psi_h)_K_{E_2} - (2 - \sqrt{1-\varepsilon}) \sum_{F \in F_h} (\| \nabla z_h \|, \Delta_T \psi_h - \lambda \psi_h)_F_{E_3}.$$ 

Similar to the proof of Lemma 5.2, we obtain

$$|E_1| \leq \sqrt{2n(1-\varepsilon)} \| u - z_h \|_{\lambda,h} \| \psi_h \|_{\lambda,h} \lesssim \left( \sum_{K \in T_h} h_K^{2t-4} \| u \|_{H^t(K)}^2 \right)^{1/2} \| \psi_h \|_{\lambda,h},$$

$$|E_2| \leq 2n \| u - z_h \|_{\lambda,h} \| \psi_h \|_{\lambda,h} \lesssim \left( \sum_{K \in T_h} h_K^{2t-4} \| u \|_{H^t(K)}^2 \right)^{1/2} \| \psi_h \|_{\lambda,h}.$$
The above estimates of $E_i$ (i = 1, 2, 3) and (5.9) yield

$$
\|z_h - u_h\|_{\lambda,h} \leq C\left(\sum_{K \in T_h} h^{2l-4}_K \|u\|_{H^s(K)}^2\right)^{1/2},
$$

which implies the desired result by triangle inequality.

5.3. Semismooth Newton method. We use the semismooth Newton method [32] to solve the discrete problem (5.3). We follow a similar argument as [30] in this subsection. Since transferring the proofs in [30] to our setting is straightforward, we only describe the algorithm and the convergence result.

Following the discussion in [30], we define the set of admissible maximizers for any $v \in V + V_h$,

$$
\Lambda[v] := \left\{ g : \Omega \to \Lambda \mid g(x) \in \arg \max_{\alpha \in \Lambda} (A^\alpha : D^2_h v + b^\alpha \cdot \nabla v - c^\alpha v - f^\alpha) \right\},
$$

where $D^2_h v$ denotes the broken Hessian of $v$. As shown in [30, Lemma 9, Theorem 10], the set $\Lambda[v]$ is nonempty for any $v \in V + V_h$, where a selection theorem in [19] is applied. For any measurable $g(x) : \Omega \to \Lambda$, thanks to the uniform continuity of $\gamma^\alpha$ defined in (2.9) on $\Omega \times \Lambda$, $\gamma^g := \gamma^\alpha \mid_{\alpha=g(x)}$ satisfies $\gamma^g \in L^\infty(\Omega)$ and $\|\gamma^g\|_{L^\infty(\Omega)} \leq \|\gamma^\alpha\|_{C(\bar{\Omega} \times \Lambda)}$. The functions $A^g$, $b^g$, $c^g$ and $f^g$ and the operator $L^g$ are defined in a similar way and are likewise bounded.

The semismooth Newton algorithm for solving (5.3) is described as follows.

**Input:** Given initial guess $u^0_h \in V_h$ and a stopping criterion.

**for** $j = 0, 1, 2, \cdots$ **until** termination **do**

Choose any $\alpha_j \in \Lambda[u^j_h]$ and compute $u^{j+1}_h \in V_h$ as the solution to the linear problem

$$
B^j_{\lambda,h}(u^{j+1}_h, v_h) = \sum_{K \in T_h} (\gamma^{\alpha_j} f^{\alpha_j}, L_\lambda v_h)_K \quad \forall v_h \in V_h,
$$

where the bilinear form $B^j_{\lambda,h} : V_h \times V_h \to \mathbb{R}$ is defined by

$$
B^j_{\lambda,h}(w_h, v_h) := \sum_{K \in T_h} (\gamma^{\alpha_j} L^\alpha, w_h, L_\lambda v_h)_K
$$

$$
- (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h} \langle \|\nabla w_h\|, \Delta_T v_h - \lambda v_h \rangle_F.
$$

**end do**
Remark 5.4. We note here that (5.11) is indeed a finite element scheme for solving
the linear elliptic equations in non-divergence form with lower-order terms:
\[ L^{\alpha_j}u^{j+1} := A^{\alpha_j} : D^2u^{j+1} + b^{\alpha_j} \cdot \nabla u^{j+1} - c^{\alpha_j}u^{j+1} = f^{\alpha_j} \quad \text{in } \Omega, \quad u^{j+1} = 0 \quad \text{on } \partial \Omega, \]
where the coefficients, which is allowed to be discontinuous, satisfy a similar Cordes
condition as (2.8) with \( \alpha = \alpha_j \). Using very similar arguments as Lemma 5.1 and
Lemma 5.2, the coercivity and boundedness of \( B_j^{\lambda,h} \) regarding to \( \| \cdot \|_{\lambda,h} \)
can be proved. A quasi-optimal error estimate then follows directly, which is also confirmed
numerically in subsection 6.2.

We state the main result as follows.

Theorem 5.5. Under the hypotheses of Theorem 5.3, there exists a constant \( R > 0 \)
that may depend on \( h \) as well as on the polynomial degree, such that if
\( \| u_h - u_0^h \|_{\lambda,h} < R \), then the sequence \( \{ u_j^h \}_{j=1}^\infty \) generated by the semismooth Newton algorithm
converges to \( u_h \) with a superlinear convergence rate.

Proof. The proof is similar to [30, Theorem 11] and is therefore omitted here. \( \square \)

6. Numerical experiments. In this section we present some numerical experi-
ments of the \( C^0 \) (non-Lagrange) finite element methods for the linear elliptic equations
in non-divergence form (1.1) and the HJB equations (1.2). For all the convergence
order experiments, the convergence history plots are logarithmically scaled.

6.1. First experiment. In the first experiment, we consider the problem (1.1)
in two dimensions on the domain \( \Omega = (-1,1)^2 \). The coefficient matrix is set to be
\[
A = \begin{pmatrix}
2 & \frac{x_1x_2}{|x_1|x_2} \\
\frac{x_1x_2}{|x_1|x_2} & 2
\end{pmatrix},
\]
A straightforward calculation shows that, for the coefficient matrix in (6.1), the Cordes
condition (2.2) is satisfied with \( \varepsilon = 3/5 \). We note here that the coefficient matrix is
discontinuous across the set \( \{(x_1,x_2) \in \Omega : x_1 = 0 \text{ or } x_2 = 0\} \). In order to test the
convergence order, the smooth solution
\[
u(x) = (x_1e^{1-|x_1|} - x_1)(x_2e^{1-|x_2|} - x_2)
\]
is considered from many works (e.g., [29, 14]). The right hand side \( f := A : D^2u \) is
directly calculated from the coefficient matrix and solution.

On a sequence of uniform triangulations \( \{ T_h \}_{0<h<1} \), we apply the numerical
scheme (4.2) to the problem with 2D Hermite finite element spaces for polynomial
degrees \( k = 3 \) and \( k = 4 \). After computing (4.2) for various \( h \), we report the errors in
Figure 4. The expected optimal convergence rate \( \| D^2u - D^2u_h \|_{L^2(T_h)} = O(h^{k-1}) \) is
observed, which is in agreement with Theorem 4.3. Further, the experiments indicate
that the scheme converges with (sub-optimal) second order convergence in both \( H^1 \)
and \( L^2 \) when \( k = 3 \). As for \( k = 4 \), the \( H^1 \) error converges with (optimal) fourth order,
and the \( L^2 \) error converges with (sub-optimal) fourth order.

6.2. Second experiment. In this experiment, we consider the linear elliptic
equations in non-divergence form with lower-order terms on \( \Omega = (-1,1)^2 \):
\[
A : D^2u + b \cdot \nabla u - cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]
Here, $A$ is taken the same as (6.1), $b = (x_1, x_2)^T$, $c = 3$. By choosing $\lambda = 1$, we have
\[
\frac{|A|^2 + |b|^2/(2\lambda) + (c/\lambda)^2}{(\text{tr} A + c/\lambda)^2} = \frac{19 + (x_1^2 + x_2^2)/2}{49} \leq \frac{20}{49},
\]
which means that the Cordes condition holds for $\varepsilon = 9/20$ (see Remark 5.4). The right hand side $f$ is chosen so that the exact solution is (6.2). The scheme converges with the optimal order $h^{k-1}$ in the broken $H^2$ norm, as shown in Table 1. The convergence orders in $H^1$ and $L^2$ norms are similar to the Experiment 1.

| $h$   | $\|u - u_h\|_{L^2(\Omega)}$ | Order | $\|u - u_h\|_{H^1(\Omega)}$ | Order | $\|D^2u - D^2u_h\|_{L^2(T_h)}$ | Order |
|-------|----------------------------|-------|----------------------------|-------|-------------------------------|-------|
| $k = 3$ |
| $2^{-2}$ | 1.72705E-03 | — | 1.17301E-02 | — | 1.41359E-01 | — |
| $2^{-3}$ | 4.10255E-04 | 2.07 | 2.33656E-03 | 2.33 | 3.58960E-02 | 1.98 |
| $2^{-4}$ | 1.00457E-04 | 2.03 | 5.42544E-04 | 2.10 | 9.5321E-03 | 1.99 |

| $k = 4$ |
| $2^{-2}$ | 1.20953E-04 | 3.87 | 4.62102E-04 | 3.84 | 8.63084E-03 | 2.98 |
| $2^{-3}$ | 7.9999E-06 | 3.96 | 2.96137E-05 | 3.96 | 1.06983E-03 | 3.01 |
| $2^{-4}$ | 4.8888E-07 | 4.00 | 1.85296E-06 | 4.00 | 1.32677E-04 | 3.01 |

Table 1: Errors and observed convergence orders for Experiment 2.

### 6.3. Third experiment.
In this experiment, we solve the nonlinear HJB equations (1.2) in two dimensions on the domain $\Omega = (0, 1)^2$. Following [30], we take $\Lambda = [0, \pi/3] \times \text{SO}(2)$, where $\text{SO}(2)$ is the set of $2 \times 2$ rotation matrices. The coefficients are given by $b^T = 0$, $c^\alpha = \pi^2$, and
\[
A^\alpha = \frac{1}{2} \sigma(\sigma^\alpha)^T, \quad \sigma^\alpha = R^T \begin{bmatrix} 1 & \sin \theta \\ 0 & \cos \theta \end{bmatrix}, \quad \alpha = (\theta, R) \in \Lambda.
\]
Since $\text{tr} A^\alpha = 1$ and $|A^\alpha|^2 = (1 + \sin^2 \theta)/2 \leq 7/8$, the Cordes condition (2.8a) holds with $\varepsilon = 1/7$ by taking $\lambda = 8\pi^2/7$. We choose $f^\alpha = \sqrt{3}\sin^2 \theta/\pi^2 + g$, $g$ inde-
pendent of $\alpha$ so that the exact solution of the HJB equations (1.2) is $u(x_1, x_2) = \exp(x_1 x_2) \sin(\pi x_1) \sin(\pi x_2)$.

On a sequence of uniform triangulations, we apply the numerical scheme (5.3) to the HJB equations (1.2). The finite element spaces are defined by employing the 2D Hermite finite elements for polynomial degrees $k = 3$ and $k = 4$. The plots of the errors given in Figure 5a and 5c show that the scheme converges with $\|u - u_h\|_{L^2} = O(h^{k-1})$, which is in agreement with Theorem 5.3. The convergence orders in $H^1$ and $L^2$ norms are similar to the Experiment 1 and Experiment 2.

In the semismooth Newton algorithm, the initial guess is $u_0^h = 0$, and the stopping criterion is set to be $\|u_j^h - u_{j-1}^h\|_{L^2} < 10^{-8}$. The convergence histories shown in Figure 5b and 5b demonstrate the fast convergence of the algorithm.

![Convergence rate, $k = 3$](image1.png)

![Semismooth Newton, $k = 3$](image2.png)

![Convergence rate, $k = 4$](image3.png)

![Semismooth Newton, $k = 4$](image4.png)

Fig. 5: Numerical scheme (5.3) applied to the HJB equations (1.2) for Experiment 3: Convergence rate and convergence histories of the semismooth Newton method.

### 6.4. Fourth experiment

In the last example, we consider a test case of the HJB equations (1.2) from [30, 15] with near degenerate diffusion and a boundary layer in the solution. Let $\Omega = (0, 1)^2$, $b^\alpha = (0, 1)^T$, $c^\alpha = 10$, and define

$$A^\alpha = a^T \begin{pmatrix} 20 & 1 \\ 1 & 1/10 \end{pmatrix} a, \quad \forall \alpha \in \Lambda := \text{SO}(2).$$
For this choice of parameters and $\lambda = 1/2$, the Cordes condition (2.8a) is satisfied for $\varepsilon = 0.0024$ (cf. [30]). Let $\delta = 0.01$ and $f^\alpha = A^\alpha : D^2 u + b^\alpha \cdot \nabla - c^\alpha u$, where the exact solution is

$$u(x) = (2x_1 - 1)(\exp(1 - |2x_1 - 1|) - 1)\left(x_2 + \frac{1 - \exp(x_2/\delta)}{\exp(1/\delta) - 1}\right).$$

This solution exhibits a sharp boundary layer near the line $\bar{\Omega} \cap \{x_2 = 1\}$. Following [30], we use a sequence of graded bisection meshes $\{T_\ell\}_{\ell \in \mathbb{N}_0}$ with grading factor $1/2$, see Figure 6a for example. More precisely, we mark the element $T$ whose $|T| > C(2x_2_T - 1)^2/\#T$, where $x_{2,T}$ is the second component of barycenter of the element $T$, $\#T$ is the number of elements in $T$. In the experiment, we set $C = 120$, and use the 2D Hermite finite element spaces for polynomial degrees $k = 3$. In Figure 6b, we plot the errors in broken $H^2$-seminorm, $H^1$-seminorm and $L^2$ on the graded bisection meshes, against the number of degrees of freedom (ndof). We observe a reduction in the order of error $O(\text{ndof}^{-1})$, which confirms the second-order convergence as shown in Theorem 5.3.

![Graded mesh at level 13: 3,587 vertices, 17,629 degrees of freedom](image1)

![Convergence rate, $k = 3$](image2)

**Fig. 6:** Convergence rate for the numerical scheme (5.3) applied to the HJB equations (1.2) for Experiment 4.

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