Factorization Theorem for the Transfer Function Associated with an Unbounded Non-Self-Adjoint $2 \times 2$ Operator Matrix

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We construct operators which factorize the transfer function associated with a non-self-adjoint $2 \times 2$ operator matrix whose diagonal entries can have overlapping spectra and whose off-diagonal entries are unbounded operators.

1. Introduction

In the present work we extend some of the results of Refs. [HMM1, HMM2] obtained for a class of unbounded self-adjoint block operator matrices to the case where these matrices are unbounded non-self-adjoint operators.

Throughout the paper we consider a $2 \times 2$ block operator matrix having the form

$$H_0 = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

and acting in the orthogonal sum $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_C$ of separable Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_C$. The entry $C$ is a possibly unbounded self-adjoint operator in $\mathcal{H}_C$ on the domain $\text{dom}(C)$. This operator is assumed to be semibounded from below, that is,

$$C \geq \lambda_C \quad \text{for some} \quad \lambda_C \in \mathbb{R},$$

In the following we assume without loss of generality that the lower bound $\lambda_C$ for the entry $C$ in the assumption (1.2) is positive,

$$\lambda_C > 0$$

(otherwise one could simply shift the origin of the spectral parameter axis). The entry $A$ is supposed to be a bounded and not necessarily self-adjoint operator in $\mathcal{H}_A$, that is, $A \in \mathcal{B}(\mathcal{H}_A)$.

Regarding the couplings $B$ and $D$ we assume the following hypotheses:

(i) $D$ is a densely defined closable operator from $\text{dom}(D) \subset \mathcal{H}_A$ to $\mathcal{H}_C$ such that the product $C^{-1/2}D$ admits extension from $\text{dom}(D)$ to the whole space $\mathcal{H}_A$ as a bounded operator in $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_C)$,

(ii) $B$ is a densely defined closable operator from $\text{dom}(B) \subset \mathcal{H}_C$ to $\mathcal{H}_A$ such that
(1.3) \( \text{dom}(B) \supset \text{dom}(C^{1/2}). \)

Since \( \text{dom}(C^{1/2}) \supset \text{dom}(C) \), these assumptions imply that the matrix \( H_0 \) is a densely defined closable operator on the domain \( \text{dom}(H_0) = \text{dom}(D) \oplus \text{dom}(C) \). We denote its closure by \( H = \overline{H}_0. \)

From the assumption (i) one infers that \( \tilde{D} := C^{1/2}D \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_C). \)

Also one concludes from (i) that

\[
\text{ran} \left( \left. \tilde{D} \right|_{\text{dom}(D)} \right) \subset \text{dom}(C^{1/2})
\]

and

\[
D = C^{1/2} \left. \tilde{D} \right|_{\text{dom}(D)}.
\]

At the same time the assumption (1.3) yields that the product \( \tilde{B} := BC^{-1/2} \) is a bounded linear operator between \( \mathcal{H}_C \) and \( \mathcal{H}_A \), i.e. \( \tilde{B} \in \mathcal{B}(\mathcal{H}_C, \mathcal{H}_A) \), and \( B|_{\text{dom}(C^{1/2})} = BC^{1/2}. \)

In addition, the hypotheses (i) and (ii) imply that for \( z \) in the resolvent set \( \varrho(C) \) of \( C \) the operator-valued function \( A - z - B(C - z)^{-1}D \) is densely defined on \( \text{dom}(D) \) and admits a bounded extension onto the whole space \( \mathcal{H}_A \). We denote this extension by \( M(z) \),

\[
M(z) := A - z - B(C - z)^{-1}D,
\]

and call it the transfer function associated with the operator \( H \). By definition

\[
M(z)|_{\text{dom}(D)} = A - z - B(C - z)^{-1}D = A - z - BC^{1/2}(C - z)^{-1}C^{1/2} \tilde{D} = \tilde{A} - z + V(z),
\]

where

\[
(1.5) \quad \tilde{A} := A - \tilde{B} \tilde{D} \quad \text{and} \quad V(z) := z \tilde{B}(z - C)^{-1} \tilde{D}.
\]

Meanwhile \( \tilde{A} \) is a bounded operator on \( \mathcal{H}_A \) while \( V(z), z \in \varrho(C) \), is an operator-valued function with values in \( \mathcal{B}(\mathcal{H}_A) \). Then it follows that

\[
(1.6) \quad M(z) = \tilde{A} - z + V(z).
\]

Before going into more details we mention the recent paper [LMMT] studying by means of the new concept of the quadratic numerical range, introduced in [LT], in particular the factorization properties of the transfer functions associated with bounded non-self-adjoint \( 2 \times 2 \) block operator matrices. As a matter of fact, the
operators factorizing the transfer functions for bounded non-self-adjoint operator matrices have been also constructed in [ALT] and for unbounded non-self-adjoint operator matrices with bounded entries $B$ and $D$ in [LT]. The results of [ALT, LMMT, LT] are obtained under the assumption that the spectra of $A$ and $C$ do not overlap.

We are studying the transfer function (1.6), associated with the unbounded non-self-adjoint operator matrix (1.1) satisfying the assumptions (i) and (ii), in the case where the numerical range $\nu(\tilde{A})$ of the operator $\tilde{A}$ (and in particular its spectrum $\sigma(\tilde{A}))$ may have a non-empty intersection with the spectrum $\sigma(C)$ of $C$. Notice that, since the resolvent of the operator $H$ can be expressed explicitly in terms of $[M(z)]^{-1}$ (see, e.g., [ALMSa, MS, MM]), in studying the spectral properties of the transfer function $M$ one studies at the same time the spectral properties of the operator matrix $H$.

Obviously, the transfer function (1.6), considered for $z \in \rho(C)$, represents a holomorphic operator-valued function. (We refer to the standard definition of holomorphy of an operator-valued function with respect to the operator norm topology, see, e.g., [ALMSa].) We study the transfer function $M(z)$ under the assumption that it admits analytic continuation through the absolutely continuous spectrum of the entry $C$ at least in a neighborhood of the numerical range $\nu(\tilde{A})$ of the operator $\tilde{A}$. In order to present our main ideas in a more transparent form, we suppose, for the sake of simplicity, that the entry $C$ only has the absolutely continuous spectrum, that is, $\sigma(C) = \sigma_{\text{ac}}(C)$.

Section 2 includes a description of the conditions making the analytic continuation of $M(z)$ through the set $\sigma_{\text{ac}}(C)$ to be possible. In Section 3 we introduce the nonlinear transformation equations (3.4) making a rigorous sense to the formal operator equation $M(Z) = 0, Z \in \mathcal{B}(\mathcal{H}_A)$. We explicitly show that eigenvalues and accompanying eigenvectors of a solution $Z$ to the equation (3.4) are eigenvalues and eigenvectors of the analytically continued transfer function $M$. Further, we prove the solvability of (3.4) under smallness conditions concerning the operator $\tilde{B}$ and $\tilde{D}$, see (3.8). In Section 4 a factorization theorem (Theorem 4.1) is proven for the analytically continued transfer function. This theorem implies in particular that there exist certain domains in $\mathbb{C}$ lying partly on the unphysical sheet(s) where the spectrum of the analytically continued transfer function is represented by the spectrum of the corresponding solutions to the transformation equations (3.4). Finally, in Section 5 we present an example.

### 2. Analytic continuation

Throughout this paper we assume that the spectrum $\sigma(C)$ of the entry $C$ is absolutely continuous and it fills the semiaxis $[\lambda_C, +\infty)$. By $\nu(\tilde{A})$ we denote the
numerical range of the (bounded) operator $\tilde{A}$,

$$\nu(\tilde{A}) = \left\{ \langle \tilde{A}x, x \rangle : x \in \mathcal{H}_A, \|x\| = 1 \right\}.$$  

We suppose that the set $\nu(\tilde{A})$ lies inside a neighborhood $O_\eta([\alpha_1, \alpha_2]) := \{ z \in \mathbb{C} : \text{dist}(z, [\alpha_1, \alpha_2]) \leq \eta \}$, $\eta > 0$, of a finite real interval $[\alpha_1, \alpha_2] \subset \mathbb{R}$, $\alpha_1 < \alpha_2$, and $\alpha_1 - \eta > \lambda_C$. Notice that the numerical range $\nu(\tilde{A})$ is a convex set containing the spectrum $\sigma(\tilde{A})$ (see, e.g., [GK], § V.6) and, moreover,

$$(2.1) \quad \| (\tilde{A} - zI_A)^{-1} \| \leq \frac{1}{\text{dist}(z, \nu(\tilde{A}))}, \quad z \in \mathbb{C} \setminus \nu(\tilde{A}),$$

where $I_A$ stands for the identity operator in $\mathcal{H}_A$ (see Lemma V.6.1 in [GK]).

Let $\{E_C(\mu)\}_{\mu \in \mathbb{R}}$ be the spectral family for the entry $C$, $C = \int_{\sigma(C)} \mu \, dE_C(\mu)$. Then the function $V(z)$ can be written

$$V(z) = \int_{\lambda_C}^{\infty} \frac{dK(\mu)}{z - \mu}$$

with

$$K(\mu) := \tilde{B}\hat{E}_C(\mu)\tilde{D}$$

We assume that the function $K(\mu)$ is continuously differentiable in $\mu \in (\lambda_C, +\infty)$ in the operator norm topology and, moreover, that it admits analytic continuation from the interval $(\lambda_C, \beta)$, $\beta > \alpha_2 + \eta$, to a simply connected domain $\mathcal{D} \subset \mathbb{C}$, $\mathcal{D} \supset O_\eta([\alpha_1, \alpha_2])$. For the continuation we keep the same notation $K(\mu)$ and by $K'(\mu)$ denote the derivative of $K$. We suppose that at the points $\lambda = \lambda_C$ and $\lambda = \beta$ the function $K'(\mu)$ satisfies the condition

$$\|K'(\mu)\| \leq c|\mu - \lambda|^\gamma, \quad \mu \in \mathcal{D},$$

with some $c > 0$ and $\gamma \in (-1, 0]$. We also assume that the operators $\tilde{B}$ and $\tilde{D}$ are such that

$$(2.2) \quad \int_\beta^{\infty} |d\mu| (1 + |\mu|)^{-1} \|K'(\mu)\| < \infty.$$  

In the following we use the notation

$$\mathcal{D}^{(l+1)} := \mathcal{D} \cap \mathbb{C}^+ \quad \text{and} \quad \mathcal{D}^{(-1)} := \mathcal{D} \cap \mathbb{C}^-.$$  

**Lemma 2.1.** Let $\Gamma_l$ ($l = \pm 1$) be a rectifiable Jordan curve in $\mathcal{D}^{(l)}$ resulting from continuous deformation of the interval $(\lambda_C, \beta)$, the end points of this interval being fixed, and let $\Gamma_l = \Gamma_l \cup [\beta, +\infty)$. Then the analytic continuation of the transfer
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function \( M(z), z \in \mathbb{C} \setminus [\lambda_C, +\infty) \), through the spectral interval \((\lambda_C, \beta)\) into the subdomain \( D(\Gamma_l) \subset D^{(l)} \) \((l = \pm 1)\) bounded by \((\lambda_C, \beta)\) and \( \Gamma_l \) is given by

\[
M_{\Gamma_l}(z) := \overline{A} - z + V_{\Gamma_l}(z),
\]

(2.3)

where

\[
V_{\Gamma_l}(z) := \int_{\Gamma_l} d\mu K'(\mu) \frac{z}{z - \mu}.
\]

(2.4)

For \( z \in D(\Gamma_l) \) the function \( M_{\Gamma_l}(z) \) may be written as

\[
M_{\Gamma_l}(z) = M(z) + 2\pi i lzK'(z).
\]

(2.5)

Proof. The function (2.4) is well defined for \( z \notin \Gamma_l \) since (2.2) holds and for any \( z \in \mathbb{C} \setminus \Gamma_l \) there exist a number \( c(z) > 0 \) such that the estimate \(|(z - \mu)^{-1}| < c(z)(1 + |\mu|)^{-1} \) \((\mu \in \Gamma_l)\) is valid. Then the proof of this lemma is reduced to the observation that the function \( M_{\Gamma_l}(z) \) is holomorphic for \( z \in \mathbb{C} \setminus \Gamma_l \) and coincides with \( M(z) \) for \( z \in \mathbb{C} \setminus D(\Gamma_l) \). The equation (2.5) is obtained from (2.4) using the Residue Theorem.

\( \square \)

Remark 2.2. From formula (2.5) one concludes that the transfer function \( M(z) \) has a Riemann surface larger than a single sheet of the spectral parameter plane. The sheet of the complex plane where the transfer function \( M(z) \) together with the resolvent \((H - z)^{-1}\) is initially considered is said to be the physical sheet. Hence, the formula (2.5) implies that the domains \( D^{(+1)} \) and \( D^{(-1)} \) should be placed on additional sheet(s) of the complex plane different from the physical sheet. Recall that these additional sheets are usually called unphysical sheets (see, e.g., [RS]).

Remark 2.3. For \( z \in \mathbb{C} \setminus \Gamma_l \), the formula (2.4) defines values of the function \( V_{\Gamma_l}(\cdot) \) in \( B(H_A) \). The inverse transfer function \([M(z)]^{-1}\) coincides with the compressed resolvent \( P_{H_A} (H - z)^{-1} \) where \( P_{H_A} \) stands for the orthogonal projection on \( H_A \). Thus, \([M(z)]^{-1}\) is holomorphic in \( \mathbb{C} \setminus \sigma(H) \). Since \( M_{\Gamma_l}(z) \) coincides with \( M(z) \) for all \( z \in \mathbb{C} \setminus D(\Gamma_l) \), one concludes that \([M_{\Gamma_l}(z)]^{-1}\) exists as a bounded operator and is holomorphic in \( z \) at least for \( z \in (\mathbb{C} \setminus \sigma(H) \cup D(\Gamma_l)) \).

3. The transformation equations

Let \( \Gamma_l \subset D^{(l)} \) \((l = \pm 1)\) be a contour described in hypothesis of Lemma 2.1. Assume that \( Z \in B(H_A) \) is a bounded operator such that its spectrum \( \sigma(Z) \) is separated from \( \Gamma_l \),

\[ \text{dist} (\sigma(Z), \Gamma_l) > 0. \]
Following to [MM, M1, M2] (cf. [ALT]), for such \( Z \) we introduce the “right”, \( V^\top_l(Z) \), and “left”, \( V^\downarrow_l(Z) \), transformations respectively

\[(3.1)\]
\[
V^\top_l(Z) := \int_{\Gamma_l} d\mu K'(\mu) Z(Z - \mu)^{-1}.
\]
and

\[(3.2)\]
\[
V^\downarrow_l(Z) := \int_{\Gamma_l} Z(Z - \mu)^{-1}K'(\mu) d\mu.
\]

Obviously, for both “right” and “left” symbols \( \kappa = \top \) and \( \kappa = \downarrow \)

\[
\|V^\kappa_l(Z)\| \leq \|Z\| \sup_{\mu \in \Gamma_l} [(1 + |\mu|)(|Z - \mu|^{-1})] \\
\times \int_{\Gamma_l} |d\mu| (1 + |\mu|)^{-1}\|K'(\mu)\| < \infty.
\]

(3.3)

In what follows we consider the operator transformation equation on \( B(\mathcal{H}_A) \) (cf. [ALT, MM, M1, M2])

\[(3.4)\]
\[
Z = \tilde{A} + V^\kappa_l(Z), \quad \kappa = \top \quad \text{or} \quad \kappa = \downarrow.
\]

In particular, for \( \kappa = \top \) the equation (3.4) possesses the following characteristic property: If an operator \( Z_\circ \) is a solution of (3.4) and \( u \in \mathcal{H}_A \) is an eigenvector of \( Z_\circ \), i.e., \( Z_\circ u = zu \) for some \( z \in \sigma(Z_\circ) \), then

\[
zu = \tilde{A}u + V^\top_l(Z_\circ)u = \tilde{A}u + \int_{\Gamma_l} d\mu K'(\mu)Z_\circ(Z_\circ - \mu)^{-1}u
\]

\[
= \tilde{A}u + \int_{\Gamma_l} d\mu K'(\mu)\frac{z}{z - \mu}u = \tilde{A}u + V^\top_l(z)u.
\]

Hence, any eigenvalue \( z \) of such an operator \( Z_\circ \) is automatically an eigenvalue for the analytically continued transfer function \( M_{\Gamma_l}(\cdot) \) and \( u \) is a corresponding eigenvector. One can easily see that a similar relation holds between the operator \( Z^*_{\circ} \), adjoint of a solution \( Z_\circ \) to the transformation equation (3.4) for \( \kappa = \downarrow \), and the adjoint transfer function \([M_{\Gamma_l}(\cdot)]^*\). This means that, having found the solutions of the equations (3.4) for \( \kappa = \top \) and/or \( \kappa = \downarrow \), one obtains an effective means of studying the spectral properties of the transfer function \( M_{\Gamma_l}(z) \), referring to well known facts of operator theory [GR, KR].

Let \( \Gamma_l (l = \pm 1) \) be a contour described in the hypothesis of Lemma 2.1 and let the numerical range of the operator \( \tilde{A} \) be separated from \( \Gamma_l \), that is,

\[(3.5)\]
\[
d(\Gamma_l) := \text{dist}(\nu(\tilde{A}), \Gamma_l) > 0.
\]

Then it is obvious that the following quantity

\[(3.6)\]
\[
\text{Var}_{\tilde{A}}(K, \Gamma_l) := \int_{\Gamma_l} |d\mu| \frac{\|K'(\mu)\|}{\text{dist}(\mu, \nu(\tilde{A}))}
\]
is finite,

\[
\text{Var}_{\tilde{A}}(K, \Gamma_l) \leq \sup_{\mu \in \Gamma_l} \frac{1 + |\mu|}{\text{dist}(\mu, \nu(\tilde{A}))} \times \int_{\Gamma_l} |d\mu| (1 + |\mu|)^{-1} \|K'(\mu)\| < \infty.
\]

**Hypothesis 3.1** Assume all the assumptions of Section 2 concerning the operators \(\tilde{A}\) and \(C\), and the operator-valued function \(K\). Assume, in addition, that for both \(l = +1\) and \(l = -1\) there are contours \(\Gamma_l = \tilde{\Gamma}_l \cup [\beta, +\infty)\), described in the hypothesis of Lemma 2.1, such that the following estimates hold true:

\[
\text{Var}_{\tilde{A}}(K, \Gamma_l) < 1, \quad \text{Var}_{\tilde{A}}(K, \Gamma_l) \|\tilde{A}\| < \frac{1}{4} d(\Gamma_l) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_l)]^2.
\]

In the following such contours \(\Gamma_l (\lambda = \pm 1)\) are called admissible contours.

It is convenient to rewrite the equations (3.4), \(\varkappa = \prec\) or \(\varkappa = \succ\), in the equivalent form

\[
X = \text{V}_{\Gamma_l}^{\varkappa}(\tilde{A} + X),
\]

where \(X := Z - \tilde{A}\). We have the following statement regarding the solvability of the transformation equation (3.9).

**Theorem 3.2.** Assume Hypothesis 3.1 and let \(\Gamma_l (l = \pm 1)\) be an admissible contour. Then for each \(\varkappa = \succ\) and \(\varkappa = \prec\) the equation (3.9) has a solution \(X_{\varkappa} \in \mathcal{B}(\mathcal{H}_A)\) satisfying the estimate

\[
\|X_{\varkappa}\| \leq r_{\min}(\Gamma_l),
\]

where

\[
r_{\min}(\Gamma_l) := \frac{1}{2} d(\Gamma_l) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_l)]
\]

\[
- \sqrt{\frac{1}{4} d^2(\Gamma_l) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_l)]^2 - d(\Gamma_l) \text{Var}_{\tilde{A}}(K, \Gamma_l) \|\tilde{A}\|}.
\]

Moreover, for given \(\varkappa = \succ\) or \(\varkappa = \prec\) this solution is a unique solution to the equation (3.9) in the open ball

\[
\{Y \in \mathcal{B}(\mathcal{H}_A) : \|Y\| < r_{\max}(\Gamma_l)\},
\]

where

\[
r_{\max}(\Gamma_l) := d(\Gamma_l) - \sqrt{\text{Var}_{\tilde{A}}(K, \Gamma_l) d(\Gamma_l) [d(\Gamma_l) + \|\tilde{A}\|]}.
\]
Proof. One can prove this theorem making use of Banach’s Fixed Point Theorem. The proof is very similar to the proof of Theorem 3.1 in [HMM2] (also cf. [MM]). Thus we omit it.

Remark 3.3. Conditions (3.8) imply that the following inequalities hold true:

\[ r_{\text{min}}(\Gamma_l) < \frac{1}{2} d(\Gamma_l) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_l)] < r_{\text{max}}(\Gamma_l). \]

Lemma 3.4. Assume Hypothesis 3.1. Then for given \( l = \pm 1 \) and \( \varkappa = \triangleright \) or \( \varkappa = \triangleleft \) the solution \( X_{\varkappa} \) of the equation (3.9) guaranteed by Theorem 3.2 is the same for any admissible contour \( \Gamma_l \). Moreover, this solution satisfies the estimate

\[ \|X\| \leq r_0^{(l)}(K) \]

where

\[ r_0^{(l)}(K) := \inf \{ r_{\text{min}}(\Gamma_l) : \text{Var}_{\tilde{A}}(K, \Gamma_l) < 1, \omega(K, \Gamma_l) > 0 \} \]

with \( r_{\text{min}}(\Gamma_l) \) given by (3.10) and

\[ \omega(K, \Gamma_l) := d(\Gamma_l) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_l)]^2 - 4\|\tilde{A}\|\text{Var}_{\tilde{A}}(K, \Gamma_l). \]

Proof. This statement can be proven essentially in the same way as Theorem 3.3 in [MM].

Therefore, for a given holomorphy domain \( D_l \) \((l = \pm 1)\) and fixed \( \varkappa = \triangleright \) or \( \varkappa = \triangleleft \) the solution \( X_{\varkappa} \) to the equation (3.4) and the solution \( Z_{\varkappa} = \tilde{A} + X_{\varkappa} \) to the equation (3.9) do not depend on the admissible contours \( \Gamma_l \). But when the index \( l \) changes, \( X_{\varkappa} \) and \( Z_{\varkappa} \) can also change. For this reason we shall supply them in the following, when it is necessary, with the index \( l \) writing \( X_{\varkappa}^{(l)} \) and \( Z_{\varkappa}^{(l)} = \tilde{A} + X_{\varkappa}^{(l)} \). Surely, the equations (3.4) and (3.9) are nonlinear equations and, outside the balls \( \|X\| < r_{\text{max}}(\Gamma_l) \), they may, in principle, have other solutions, different from the \( X_{\varkappa}^{(l)} \) or \( Z_{\varkappa}^{(l)} \) the existence of which is guaranteed by Theorem 3.2. In the following we only deal with the solutions \( X_{\varkappa}^{(l)} \) or \( Z_{\varkappa}^{(l)} \) guaranteed by Theorem 3.2.

4. Factorization

In this section we prove a factorization theorem for the transfer function \( M_{\Gamma_l}(z) \). Note that this theorem resembles the corresponding statements from [MrMt, VM]. It is an extension of Theorem 4.1 in [HMM2].

Theorem 4.1. Assume Hypothesis 3.1 and let \( \Gamma_l \) \((l = \pm 1)\) be an admissible contour. Let for \( \varkappa = \triangleright \) and \( \varkappa = \triangleleft \) the operators \( X_{\varkappa}^{(l)} \) be the solutions of the
transformation equations $\{3.9\}$, $\|X_\infty^{(l)}\| \leq r_0^{(l)}(K)$, and $Z_\infty^{(l)} = \bar{A} + X_\infty^{(l)}$. Then, for $z \in \mathbb{C} \setminus \Gamma_1$, the transfer function $M_{\Gamma_1}(z)$ admits the factorizations
\begin{equation}
M_{\Gamma_1}(z) = W_{\Gamma_1}^\varphi(z) (Z_\varphi^{(l)} - z) = (Z_\varphi^{(l)} - z) W_{\Gamma_1}^\varphi(z),
\end{equation}
where $W_{\Gamma_1}^\varphi(z)$, $\varphi = \triangleright$ or $\varphi = \triangleleft$, are bounded operators in $\mathcal{H}_A$ which read
\begin{align}
W_{\Gamma_1}^\varphi(z) &= I_A - \int_{\Gamma_1} d\mu K'(\mu) (Z_\varphi^{(l)} - \mu)^{-1} \\
&\quad + z \int_{\Gamma_1} d\mu K'(\mu)(z - \mu)^{-1}(Z_\varphi^{(l)} - \mu)^{-1}
\end{align}
and
\begin{align}
W_{\Gamma_1}^\varphi(z) &= I_A - \int_{\Gamma_1} (Z_\varphi^{(l)} - \mu)^{-1} K'(\mu) d\mu \\
&\quad + z \int_{\Gamma_1} (Z_\varphi^{(l)} - \mu)^{-1} K'(\mu) d\mu
\end{align}

If, in addition,
\begin{equation}
\text{dist} \left( z, \nu(\bar{A}) \right) \leq d(\Gamma_1) \left[ 1 - \text{Var}_A(K, \Gamma_1) \right]/2,
\end{equation}
then
\begin{equation}
\|W_{\Gamma_1}^\varphi(z) - I_A\| < 1 \quad \varphi = \triangleright \text{ or } \varphi = \triangleleft
\end{equation}
and the operator $W_{\Gamma_1}^\varphi(z)$ is boundedly invertible, that is, $\|W_{\Gamma_1}^\varphi(z)^{-1}\| < \infty$.

Proof. For both $\varphi = \triangleright$ and $\varphi = \triangleleft$ this statement is proven in the same way. Thus, we only present the proof for the “right” case $\varphi = \triangleright$.

First we prove the factorization (4.1). Note that, according to (3.1) and (3.9),
\begin{equation}
\bar{A} = Z_\varphi^{(l)} - V_{\Gamma_1}(\bar{A} + X^{(l)}) = Z_\varphi^{(l)} - \int_{\Gamma_1} d\mu K'(\mu) Z_\varphi^{(l)} (Z_\varphi^{(l)} - \mu)^{-1}.
\end{equation}
Thus, in view of (2.3) and (2.4), the value of $M_{\Gamma_1}(z)$ can be written as
\begin{align}
M_{\Gamma_1}(z) &= \bar{A} - z + \int_{\Gamma_1} d\mu K'(\mu) \frac{z}{z - \mu} \\
&= Z_\varphi^{(l)} - z - \int_{\Gamma_1} d\mu K'(\mu) (Z_\varphi^{(l)} - \mu)^{-1} (Z_\varphi^{(l)} - z) \\
&\quad + z \int_{\Gamma_1} d\mu K'(\mu) \left[ \frac{1}{z - \mu} - (Z_\varphi^{(l)} - \mu)^{-1} \right] \\
&= (Z_\varphi^{(l)} - z) - \int_{\Gamma_1} d\mu K'(\mu) (Z_\varphi^{(l)} - \mu)^{-1} (Z_\varphi^{(l)} - z) \\
&\quad + z \int_{\Gamma_1} d\mu K'(\mu)(z - \mu)^{-1} (Z_\varphi^{(l)} - \mu)^{-1} (Z_\varphi^{(l)} - z).
which proves the equation (1.3). The boundedness of the operator $W_{\Gamma_1}(z)$ for $z \in \mathbb{C} \setminus \Gamma_l$ is obvious.

Further, assume that the condition (4.4) holds true. Using the triangle inequality, it is easy to see that this condition yields

$$\sup_{\mu \in \Gamma_l} |z - \mu|^{-1} \leq \frac{2}{d(\Gamma_l)[1 + \Var_{\tilde{A}}(K, \Gamma_l)].}$$

For any $x \in \mathcal{H}_A$, $\|x\| = 1$, we have

$$z = \langle \tilde{A}x, x \rangle + \langle (z - \tilde{A})x, x \rangle$$

and, hence,

$$|z| \leq \|\tilde{A}\| + \inf_{x = 1} \|((z - \tilde{A})x, x)\| = \|\tilde{A}\| + \inf_{\zeta \in \nu(\tilde{A})} |z - \zeta| = \|\tilde{A}\| + \text{dist} \,(z, \nu(\tilde{A})).$$

Then it follows from (4.4) that

$$|z| \leq \|\tilde{A}\| + \frac{1}{2}d(\Gamma_l)[1 - \Var_{\tilde{A}}(K, \Gamma_l)].$$

Meanwhile the estimate (2.4) implies

$$\| (\tilde{A} + X^{(l)} - \mu)^{-1} \| = \| [I_A + (\tilde{A} - \mu)^{-1}X^{(l)}(\tilde{A} - \mu)^{-1}] \| \leq \frac{1}{\text{dist} \,(\mu, \nu(\tilde{A})) - \|X^{(l)}\|}. \tag{4.7}$$

Therefore, taking in to account the definitions of $d(\Gamma_l)$ and $r_{\min}(\Gamma_l)$, and Remark 3.3 we find

$$\left\| \int_{\Gamma_l} d\mu \, K'(|\mu|) (Z^{(l)} - \mu)^{-1} \right\| \leq \frac{2 \Var_{\tilde{A}}(K, \Gamma_l)}{1 + \Var_{\tilde{A}}(K, \Gamma_l)}. \tag{4.8}$$

Using the inequality (4.7) and Remark 3.3 one also finds

$$\left\| \int_{\Gamma_l} d\mu \, K'(|\mu|) (Z^{(l)} - \mu)^{-1} (z - \mu)^{-1} \right\| \leq |z| \frac{2 \Var_{\tilde{A}}(K, \Gamma_l)}{1 + \Var_{\tilde{A}}(K, \Gamma_l)} \sup_{\mu \in \Gamma_l} |z - \mu|^{-1}.$$ 

Finally, taking into account the second assumption in (3.3), we obtain

$$\| W_{\Gamma_1}(z) - I_A \| \leq \frac{2 \Var_{\tilde{A}}(K, \Gamma_l)}{1 + \Var_{\tilde{A}}(K, \Gamma_l)} + \frac{4 \Var_{\tilde{A}}(K, \Gamma_l)}{d(\Gamma_l)[1 + \Var_{\tilde{A}}(K, \Gamma_l)]} \left\{ \|\tilde{A}\| + \frac{1}{2}d(\Gamma_l)[1 - \Var_{\tilde{A}}(K, \Gamma_l)] \right\}$$

$$= \frac{4 \Var_{\tilde{A}}(K, \Gamma_l)}{d(\Gamma_l)[1 + \Var_{\tilde{A}}(K, \Gamma_l)]} < 1.$$
and, thus, if $z$ satisfies (4.11), then the operator $W_{\Gamma}^\kappa(z)$ has a bounded inverse.

The proof is complete. \(\square\)

**Corollary 4.2.** The spectra $\sigma(Z_{\kappa}^{(l)})$ and $\sigma(Z_{\kappa}^{(l)})$ of the operators $Z_{\kappa}^{(l)} = \tilde{A} + X_{\kappa}^{(l)}$, $\kappa = \nu$ or $\kappa = \theta$, coincide and belong to the closed $r_{0}^{(l)}(K)$-neighbourhood

$$O_{r_{0}^{(l)}(K)}(\tilde{A}) := \{z \in \mathbb{C} : \text{dist} (z, \nu(\tilde{A})) \leq r_{0}^{(l)}(K)\}$$

of the numerical range $\nu(\tilde{A})$. Moreover, for any admissible contour $\Gamma_{l}$ these spectra coincide with a subset of the spectrum of the transfer function $M_{\Gamma_{l}}(\cdot)$. More precisely,

$$\sigma(M_{\Gamma_{l}}(\cdot)) \cap O(\tilde{A}, \Gamma_{l}) = \sigma(Z_{\kappa}^{(l)}) = \sigma(Z_{\kappa}^{(l)}),$$

where

$$O(\tilde{A}, \Gamma_{l}) := \{z \in \mathbb{C} : \text{dist} (z, \nu(\tilde{A})) \leq d(\Gamma_{l}) [1 - \text{Var}_{\tilde{A}}(K, \Gamma_{l})]/2\}.$$  

**Proof.** This statement is an immediate consequence of the factorizations (4.1) and bounded invertibility of the operators $W_{\Gamma}^\kappa(z)$ and $W_{\Gamma}^\kappa(z)$ whenever (4.4) holds. \(\square\)

Let us introduce the operator

$$\Omega^{(l)} := \int_{\Gamma_{l}} d\mu \mu \left( Z_{\kappa}^{(l)} - \mu \right)^{-1} K''(\mu) \left( Z_{\kappa}^{(l)} - \mu \right)^{-1}, \quad l = \pm 1,$$

where $\Gamma_{l}$ stands for an admissible contour and, as before, $Z_{\kappa}^{(l)} = \tilde{A} + X_{\kappa}^{(l)}$ where $X_{\kappa}^{(l)}$, $\kappa = \nu$ or $\kappa = \theta$, are solutions of the transformation equations (3.9). It is obvious that $\Omega^{(l)}$ does not depend on the choice of the admissible contour $\Gamma_{l}$.

**Theorem 4.3.** The operators $\Omega^{(l)}$ ($l = \pm 1$) possess the following properties (cf. [HMM1], [MrM1], [MM], [MS], [VM]):

$$\|\Omega^{(l)}\| < 1,$$

$$- \frac{1}{2\pi i} \int_{\gamma} dz [M_{\Gamma_{l}}(z)]^{-1} = (I_{A} + \Omega^{(l)})^{-1},$$

$$- \frac{1}{2\pi i} \int_{\gamma} dz z [M_{\Gamma_{l}}(z)]^{-1} = (I_{A} + \Omega^{(l)})^{-1} Z_{\kappa}^{(l)} = Z_{\kappa}^{(l)} (I_{A} + \Omega^{(l)})^{-1},$$

where $\gamma$ stands for an arbitrary rectifiable closed contour encircling the spectrum $\sigma(Z_{\kappa}^{(l)}) = \sigma(Z_{\kappa}^{(l)})$ inside the set $O(\tilde{A}, \Gamma_{l})$ in the anticlockwise direction. The integration along $\gamma$ is understood in the sense of the operator norm topology.
Thus the term \( F \) and (4.16) where (4.15) can be written as where (4.14)

Further, the formula (4.14) yields (4.17)

By the resolvent equation and the definition (4.10) the product \( \Omega^{(l)}(Z^{(l)}_{\phi} - z)^{-1} \) can be written as

(4.15)

\[
\Omega^{(l)}(Z^{(l)}_{\phi} - z)^{-1} = F_1(z) + F_2(z),
\]

where (4.16)

\[
F_1(z) := \int_{\Gamma_l} d\mu \mu (Z^{(l)}_{\phi} - \mu)^{-1} K'(-\mu) (Z^{(l)}_{\phi} - \mu)^{-1} (\mu - z)^{-1}
\]

and

\[
F_2(z) := \left(- \int_{\Gamma_l} d\mu \frac{\mu}{\mu - z} (Z^{(l)}_{\phi} - \mu)^{-1} K'(-\mu) \right) (Z^{(l)}_{\phi} - z)^{-1}
\]

\[
= (W^{d}_{\Gamma_l}(z) - I_A)(Z^{(l)}_{\phi} - z)^{-1}.
\]

Further, the formula (4.14) yields

\[
(I_A + \Omega^{(l)}) [M_{\Gamma_l}(z)^{-1} = F_1(z) [W^{d}_{\Gamma_l}(z)]^{-1} + (Z^{(l)}_{\phi} - z)^{-1}.
\]

The function \( F_1(z) \) is holomorphic inside the contour \( \gamma \subset \mathcal{O}(\tilde{A}, \Gamma_l) \) since the argument \( \mu \) of the integrand in (4.16) belongs to \( \Gamma_l \) and thereby

\[
|z - \mu| \geq |d(\Gamma_l) + \text{Var}_{\tilde{A}}(K, \Gamma_l)|/2 > 0.
\]

Thus the term \( F_1(z) [W^{d}_{\Gamma_l}(z)]^{-1} \) does not contribute to the integral

\[
-\frac{1}{2\pi i} \int_{\gamma} dz (I_A + \Omega^{(l)}) [M_{\Gamma_l}(z)^{-1}
\]

while the resolvent \( (Z^{(l)}_{\phi} - z)^{-1} \) gives the identity \( I_A \) which proves (4.12).

Similarly, to prove the first equality in (4.13) we calculate

\[
-\frac{1}{2\pi i} \int_{\gamma} dz (I_A + \Omega^{(l)}) z [M_{\Gamma_l}(z)]^{-1} =
\]

\[
= -\frac{1}{2\pi i} \int_{\gamma} dz z F_1(z) [W^{d}_{\Gamma_l}(z)]^{-1} - \frac{1}{2\pi i} \int_{\gamma} dz z (Z^{(l)}_{\phi} - z)^{-1}.
\]
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The first integral vanishes whereas the second integral equals \( Z_d^{(l)} \). The second equality of (4.13) can be checked in the same way. \( \square \)

**Corollary 4.4.** Given \( l = \pm 1 \) the formula (4.13) implies that the operators \( Z_d^{(l)} \) and \( Z_\neq^{(l)} \) are similar to each other:

\[
Z_d^{(l)} = (I_A + \Omega^{(l)}) Z_\neq^{(l)} (I_A + \Omega^{(l)})^{-1}.
\]

**Remark 4.5.** The formulae (4.12) and (4.13) allow, in principle, to construct the operators \( Z_d^{(l)} \) and, thus, to resolve the equations (3.9) by a contour integration of the inverse of the transfer function \( M_{\Gamma_l}(z) \).

**Theorem 4.6.** Let \( \lambda \) be an isolated eigenvalue of \( Z_d^{(l)} \) and, hence, of \( Z_\neq^{(l)} \) and \( M_{\Gamma_l}(\cdot) \) where \( \Gamma_l \) is an admissible contour. Denote by \( P_{\neq,\lambda} \) and \( P_{\neq,\lambda} \) the eigenprojections of the operators \( Z_d^{(l)} \) and \( Z_\neq^{(l)} \), respectively, and by \( P_{M,\lambda} \) the residue of \( M_{\Gamma_l}(z) \) at \( z = \lambda \),

\[
P_{\neq,\lambda}(\gamma) := \frac{1}{2\pi i} \int_\gamma dz \left( Z_\neq^{(l)} - z \right)^{-1} \quad (\neq = \triangleleft \text{ or } \neq = \triangleright),
\]

and

\[
P_{\lambda}(\gamma) := -\frac{1}{2\pi i} \int_\gamma dz \left[ M_{\Gamma_l}(z) \right]^{-1},
\]

where \( \gamma \) stands for an arbitrary rectifiable closed contour going around \( \lambda \) in the positive direction in a sufficiently close neighbourhood such that \( \gamma \cap \Gamma_l = \emptyset \) and no points of the spectrum of \( M_{\Gamma_l}(\cdot) \), except the eigenvalue \( \lambda \), lie inside \( \gamma \). Then the following relations hold:

\[
P_{M,\lambda} = P_{\neq,\lambda} (I_A + \Omega^{(l)})^{-1} = (I_A + \Omega^{(l)})^{-1} P_{\neq,\lambda}.
\]

**Proof.** Proof of this statement can be done in the same way as the proof of the relation (4.14), only the path of integration is changed. \( \square \)

5. **An example**

Let \( \mathcal{H}_A = \mathcal{H}_C = L^2(\mathbb{R}) \) and \( C = P^2 + \lambda_C I_C \) where \( P = i \frac{d}{dx}, \lambda_C \) is some positive number, and \( I_C \) denotes the identity operator in \( \mathcal{H}_C \). It is assumed that the domain \( \text{dom}(P) \) is the Sobolev space \( W^2_2(\mathbb{R}) \) and the domain \( \text{dom}(C) \) is the Sobolev space \( W^2_2(\mathbb{R}) \). The spectrum of \( C \) is absolutely continuous and fills the semi-axis \( [\lambda_C, +\infty) \). By the operator \( A \) we understand the multiplication by a bounded complex-valued function \( a, Af = af, f \in \mathcal{H}_A \).
The operators $B$ and $D$ are defined on $\text{dom}(B) = \text{dom}(D) = W^1_2(\mathbb{R})$ by

$$B = SP \quad \text{and} \quad D = PQ,$$

where $S$ and $Q$ are the multiplications by bounded functions $s \in L^2(\mathbb{R})$ and $q \in W^1_2(\mathbb{R})$, that is, $Qf = qf$ and $Sg = sg$ where $f, g \in L^2(\mathbb{R})$. Both $S$ and $D$ are densely defined closable operators.

Notice that $\text{dom}(C^{1/2}) = W^1_2(\mathbb{R})$. The proof of this statement is based on the second representation theorem for quadratic forms, see Theorem VI.2.23 in [K]. It is similar to the proof of Proposition 2.4 in [FMM].

Further, we assume that the functions $s$ and $q$ are exponentially decreasing at infinity, so that the estimates

$$(5.1) \quad |s(x)| \leq c \exp(-\alpha_0|x|) \quad \text{and} \quad |q(x)| \leq c \exp(-\alpha_0|x|) \quad (x \in \mathbb{R})$$

hold with some $c \geq 0$ and $\alpha_0 > 0$.

For this example the operators $\tilde{B}$ and $\tilde{D}$ are given by

$$\tilde{B} = SP (P^2 + \lambda_C I_C)^{-1/2} = S \int_{\mathbb{R}} \frac{\mu}{(\mu^2 + \lambda_C)^{1/2}} \, dE_p(\mu),$$

$$\tilde{D} = (P^2 + \lambda_C I_C)^{-1/2} PQ = \int_{\mathbb{R}} \frac{\mu}{(\mu^2 + \lambda_C)^{1/2}} \, dE_p(\mu) Q,$$

where \(\{E_p(\mu)\}_{\mu \in \mathbb{R}}\) denotes the spectral family of the selfadjoint operator $P$. Thus

$$\tilde{A} = A - \tilde{B} \tilde{D}$$

$$= A - S \int_{\mathbb{R}} \frac{\mu}{(\mu^2 + \lambda_C)^{1/2}} \, dE_p(\mu) \int_{\mathbb{R}} \frac{\bar{\mu}}{(\bar{\mu}^2 + \lambda_C)^{1/2}} \, dE_p(\bar{\mu}) Q$$

$$= A - S \int_{\mathbb{R}} \frac{\mu^2}{\mu^2 + \lambda_C} \, dE_p(\mu) Q$$

$$= A - SQ + \lambda_C S(P^2 + \lambda_C I_C)^{-1}Q.$$

The operator $A - SQ$ is the multiplication by the function

$$\tilde{a}(x) = a(x) - s(x)q(x)$$

while the term $S(P^2 + \lambda_C I_C)^{-1}Q$ is a compact (even Hilbert-Schmidt) operator in $L^2(\mathbb{R})$. Indeed, the inverse operator $C^{-1} = (P^2 + \lambda_C I_C)^{-1}$ is the integral operator whose kernel reads

$$C^{-1}(x, x') = \frac{1}{2\sqrt{\lambda_C}} \exp(-\sqrt{\lambda_C} |x - x'|).$$

Thus, the double integral $\int_{\mathbb{R}} \int_{\mathbb{R}} |(SC^{-1}Q)(x, x')|^2 \, dx \, dx'$ is convergent. Obviously,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |(SC^{-1}Q)(x, x')|^2 \, dx \, dx' \leq \frac{1}{4\lambda_C} \|s\|_{L^2(\mathbb{R})}^2 \|q\|_{L^2(\mathbb{R})}^2.$$
Therefore, the essential spectrum of $\tilde{A}$ coincides with the range of the function $\tilde{a}$.

In the following we assume that there are an interval $[\alpha_1, \alpha_2] \subset (\lambda_C, +\infty)$ with $\alpha_1 < \alpha_2$ and a number $\eta > 0$ such that all the numerical range $\nu(\tilde{A})$ of $\tilde{A}$ lies inside the domain

$$\mathcal{O}_\eta([\alpha_1, \alpha_2]) := \{ z \in \mathbb{C} : \text{dist}(z, [\alpha_1, \alpha_2]) \leq \eta \}, \quad \eta > 0,$$

of a finite real interval $[\alpha_1, \alpha_2] \subset \mathbb{R}$, and, moreover, $\alpha_1 - \eta > \lambda_C$.

It is easy to check that the spectral projections $E_C(\mu)$ of the operator $C = P^2 + \lambda_C I$ are given by the integral operator whose kernel reads

$$E_C(\mu; x, x') = \begin{cases} 0 & \text{if } \mu < \lambda_C, \\ \frac{1}{\sqrt{2\pi}} \int_{\lambda_C}^{\mu} \frac{\cos[(\mu' - \lambda_C)^{1/2}(x - x')] d\mu'}{(\mu' - \lambda_C)^{1/2}} & \text{if } \mu \geq \lambda_C. \end{cases}$$

Thus, the derivative $K'(\mu)$ is also an integral operator in $L_2(\mathbb{R})$. Its kernel $K'(\mu; x, x')$ is only nontrivial for $\mu > \lambda_C$ and, moreover, for these $\mu$

$$K'(\mu; x, x') = \frac{(\mu - \lambda_C)^{1/2}}{2\sqrt{2\pi} \mu} \cos[(\mu - \lambda_C)^{1/2}(x - x')] s(x) q(x').$$

Obviously, this kernel is degenerate for $\mu > \lambda_C$,

\begin{equation}
K'(\mu; x, x') = \frac{(\mu - \lambda_C)^{1/2}}{2\sqrt{2\pi} \mu} \left[ s_+(\mu, x) q_+(-\mu, x') + s_-(\mu, x) q_-(\mu, x') \right],
\end{equation}

where $s_\pm(\mu, x) = e^{\pm i\sqrt{\mu - \lambda_C}} q(x)$ and $q_\pm(\mu, x) = e^{\pm i\sqrt{\mu - \lambda_C}} q(x)$. From the assumptions \((5.1)\) on $s$ and $q$ we conclude that in the domain $\pm \Im \sqrt{\mu - \lambda_C} < \alpha_0$, i.e., inside the parabola

\begin{equation}
D = \left\{ \mu \in \mathbb{C} : \Re \mu > \lambda_C - \alpha_0^2 + \frac{1}{4\alpha_0^2} (\Im \mu)^2 \right\},
\end{equation}

the functions $s_\pm(\mu, \cdot)$ and $q_\pm(\mu, \cdot)$ are elements of $L_2(\mathbb{R})$. The function $K'(\mu)$ admits an analytic continuation into this domain (cut along the interval $\lambda_C - \alpha_0^2 < \mu \leq \lambda_C$) as a holomorphic function with values in $\mathcal{B}(\mathcal{H}_A)$ and the equation \((5.2)\) implies that

$$\|K'(\mu)\| \leq \frac{|\mu - \lambda_C|^{1/2}}{2\sqrt{2\pi} |\mu|} \left[ \|s_-(\mu, \cdot)\| \|q_-(\mu, \cdot)\| + \|s_+(\mu, \cdot)\| \|q_+(\mu, \cdot)\| \right].$$

Obviously, for $\mu \geq \lambda_C$ we have $\|s_\pm(\mu, \cdot)\| = \|s\|$ and $\|q_\pm(\mu, \cdot)\| = \|q\|$.

Let us make our final assumption that $D \supset \mathcal{O}_\eta([\alpha_1, \alpha_2])$. In this case one can always choose a contour $\Gamma = \Gamma \cup [\beta, +\infty)$ where $\beta > \alpha_2 + \eta$ and the rectifiable Jordan curve $\Gamma \subset D \setminus \mathcal{O}_\eta([\alpha_1, \alpha_2])$ results from continuous deformation of the
interval \((\lambda_C, \beta)\), the end points being fixed. Assume, in addition, that the functions 
\(s\) and \(q\) are sufficiently small in the sense that the conditions \((3.8)\) hold. In such a case the contour \(\Gamma\) is an admissible contour (see Hypothesis \([3.1]\)) and, thus, one can apply all the statements of the Sections 3 and 4 to the corresponding transfer function \(M_\Gamma(z)\).

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