FROBENIUS MANIFOLDS AND A NEW CLASS OF EXTENDED AFFINE WEYL GROUPS $\tilde{\mathcal{W}}^{(k,k+1)}(A_l)$

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Abstract. We present a new class of extended affine Weyl groups $\tilde{\mathcal{W}}^{(k,k+1)}(A_l)$ for $1 \leq k < l$ and obtain an analogue of Chevalley-type theorem for their invariants. We further show the existence of Frobenius manifold structures on the orbit spaces of $\tilde{\mathcal{W}}^{(k,k+1)}(A_l)$ and also construct Landau–Ginzburg superpotentials for these Frobenius manifold structures.

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1. Introduction

E.Witten, R.Dijkgraaf, E.Verlinde and H.Verlinde ([7, 8]) in 1990’s introduced a remarkable system of partial differential equations, i.e., WDVV equations of associativity, on two dimensional topological field theory (briefly 2dTFT). In order to understand a geometrical foundation of 2dTFT on the bases of WDVV equations, B.Dubrovin ([9, 10]) extended the Atiyah’s axioms of 2dTFT ([6]) and invented a nice geometrical object, that is, Frobenius manifold designed as a coordinate-free formulation of WDVV equations.

For an arbitrary $n$-dimensional Frobenius manifold, B.Dubrovin ([10]) defined a monodromy group, which acts on $n$-dimensional linear space and can be regarded as (an extension of) a group generated by reflections. The Frobenius manifold itself can be identified with the orbit space of the group in the sense to be specified for each class of monodromy groups, which gives a clue to understanding of Frobenius manifold structure on the orbit space. It was shown by B.Dubrovin ([10, 11]) that any finite Coxeter group can serve as a monodromy group of a polynomial Frobenius manifold, that is to say, the potential is a polynomial with respect to the flat coordinates $t^1, \ldots, t^l$. Furthermore, he put forward the following conjecture, “Any massive polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group”, which was proved by C.Hertling ([16]). Besides these, in [13, 18] it has been shown that there are $l$ different Frobenius manifold structures on the orbit spaces of the Coxeter groups $B_l$ and $D_l$. Especially, we also proved in [18] that the corresponding potentials are meromorphic along the divisors $t^k = 0$ for a given $1 \leq k \leq l - 2$, where $t^1, \ldots, t^l$ are the flat coordinates of the Frobenius manifold.

Let $R$ be an irreducible reduced root system defined in an $l$-dimensional Euclidean space $V$ with Euclidean inner product $(\ ,\ )$, we fix a basis of simple roots $\alpha_1, \ldots, \alpha_l$ and denote by $\alpha_j^\vee$, $j = 1, 2, \cdots, l$ the corresponding coroots. The Weyl group $W(R)$ is generated by the reflections $x \mapsto x - (\alpha_j^\vee, x)\alpha_j$, $\forall x \in V$, $j = 1, \ldots, l$. The semi-direct product of $W(R)$ by the lattice of coroots yields the affine Weyl group $W_a(R)$ that acts on $V$ by the affine transformations

$$x \mapsto w(x) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W, \ m_j \in \mathbb{Z}.$$ 

We denote by $\omega_1, \ldots, \omega_l$ the fundamental weights defined by the relations

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \ldots, l.$$
B.Dubrovin and Y.Zhang in [12] (also [19]) defined an extended affine Weyl group $\tilde{W}^{(k)}(R)$, which acts on the extended space $V \oplus \mathbb{R}$ and is generated by the transformations

$$x = (x, x_{l+1}) \mapsto (w(x) + \sum_{j=1}^{l} m_j \alpha_j^\vee, x_{l+1}), \quad w \in W, \ m_j \in \mathbb{Z},$$

and

$$x = (x, x_{l+1}) \mapsto (x + \gamma \omega_k, x_{l+1} - \gamma).$$

Here $1 \leq k \leq l$, $\gamma = 1$ except for the cases when $R = B_l, k = l$ and $R = F_4, k = 3$ or $k = 4$, in these three cases $\gamma = 2$. For a particular choice of a simple root $\alpha_k$, they proved an analogue of Chevalley theorem for their invariants. On the orbit space of $\tilde{W}^{(k)}(R)$, they constructed a Frobenius manifold structure whose potential is a weighted homogeneous polynomial of $t^1, \ldots, t^{l+1}, e^{tl+1}$, where $t^1, \ldots, t^{l+1}$ are the flat coordinates of the Frobenius manifold.

Observe that for the root system of type $A_l$, there is in fact no restrictions on the choice of $\alpha_k$. However, for the root systems of type $B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ there is only one choice for each. In [14] Slodowy pointed out that the Chevalley type theorem is a consequence of the results of Looijenga and Wirtheimüller [2, 3, 5], and in fact it holds true for any choice of the base element $\alpha_k$. A natural question is that “Whether the geometric structures that were revealed by Dubrovin-Zhang’s construction also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of $\alpha_k$?” Our recent work in [19] is to give an affirmative answer to this question for the root systems of type $B_l, C_l$ and also for $D_l$. We show, by fixing another integer $0 \leq m \leq l - k$, that on the corresponding orbit spaces there also exist Frobenius manifold structures with potentials $F(t)$ that are weighted homogeneous polynomials w.r.t $t^1, \ldots, t^{l+1}, \frac{1}{t^{l+1}}, \frac{1}{t^m}$, $e^{tl+1}$. We also construct Landau–Ginzburg (briefly LG) superpotentials for these Frobenius manifold structures.

In this paper we will present a new extension of affine Weyl groups denoted by $\tilde{W}^{(k,k+1)}(R)$, which is different from those in [12, 19], and study Frobenius manifold structures on the corresponding orbit spaces $\mathcal{M}^{(k,k+1)}(R)$, where the new extended affine Weyl groups $\tilde{W}^{(k,k+1)}(R)$ act on the extended space $V \oplus \mathbb{R}^2$ generated by the transformations

$$x = (x, x_{l+1}, x_{l+2}) \mapsto (w(x) + \sum_{j=1}^{l} m_j \alpha_j^\vee, x_{l+1}, x_{l+2}), \quad w \in W, \ m_j \in \mathbb{Z},$$

and

$$x = (x, x_{l+1}, x_{l+2}) \mapsto (x + \gamma \omega_k, x_{l+1} - \gamma, x_{l+2}).$$
and 

\[ x = (x, \ x_{l+1}, \ x_{l+2}) \mapsto (x + \gamma \omega_{k+1}, \ x_{l+1}, \ x_{l+2} - \gamma) \]

Here \( 1 \leq k \leq l - 1, \gamma = 1 \) except for the cases when \( R = B_t, k = l \) and \( R = F_t, k = 3 \) or \( k = 4 \), in these three cases \( \gamma = 2 \).

By a direct verification, we could not obtain any flat pencil of metrics and Frobenius manifold structures on the orbit spaces \( \mathcal{M}^{(1,2)}(B_2), \mathcal{M}^{(1,2)}(C_2) \) and \( \mathcal{M}^{(1,2)}(G_2) \) etc. We thus have to restrict our study to the type \( A_l \) case, i.e. \( \mathcal{M}^{(k,k+1)}(A_l) \) and will show that (see Theorem 4.5)

**Main Theorem 1.** For any fixed integer \( 1 \leq k < l \), there exists a unique Frobenius manifold structure of charge \( d = 1 \) on the orbit space \( \mathcal{M}^{(k,k+1)}(A_l) \setminus \{ \tilde{y}_{l+1} = 0 \} \cup \{ \tilde{y}_{l+2} = 0 \} \) of \( \tilde{W}^{(k,k+1)}(A_l) \) such that

1. the invariant flat metric and the intersection form of the Frobenius manifold structure coincide with the metrics \((\eta^{ij}(y))\) in (3.18) and \((g^{ij}(y))\) in (3.5) respectively;
2. the unity and the Euler vector fields have the form
   \[
   e = \frac{\partial}{\partial y^k} + \frac{\partial}{\partial y^{k+1}} \tag{1.1}
   \]
   and
   \[
   E = \sum_{\alpha=1}^{l} d_{\alpha} y^{\alpha} \frac{\partial}{\partial y^{\alpha}} + \frac{1}{k} \frac{\partial}{\partial y^{l+1}} + \frac{1}{l-k} \frac{\partial}{\partial y^{l+2}}, \tag{1.2}
   \]
   where \( d_1, \ldots, d_l \) are defined in (2.13);
3. in the flat coordinates \( t^1, \ldots, t^{l+2} \) of the metric (3.18) defined on certain covering of \( \mathcal{M}^{(k,k+1)}(A_l) \) the potential of the Frobenius manifold structure is of the form
   \[
   F(t) = \bar{F}(t) + \frac{1}{2} (t^{k+1})^2 \log(t^{k+1}), \text{ where } \bar{F}(t) \text{ is a weighted homogeneous polynomial in } t^1, t^2, \ldots, t^{l+2}, e^{t^{l+1}}, e^{t^{l+2}-t^{l+1}}.
   \]

On the orbit space of the extended affine Weyl group \( \tilde{W}^{(k)}(A_l) \), an alternative construction of the Frobenius manifold structure was given in [12]. This structure was given in terms of a LG superpotential construction. In particular, it was shown that \( \tilde{W}^{(k)}(A_l) \) describes the monodromy of roots of trigonometric polynomials - the superpotential - with a given bidegree being of the form

\[
\lambda(\varphi) = e^{ik\varphi} + a_1 e^{i(k-1)\varphi} + \cdots + a_{l+1} e^{i(k-l-1)\varphi}, \quad a_{l+1} \neq 0.
\]

A natural question is that

"Whether a similar construction about the Frobenius manifold structure exists on the orbit space \( \mathcal{M}^{(k,k+1)}(A_l) \setminus \{ \tilde{y}_{l+1} = 0 \} \cup \{ \tilde{y}_{l+2} = 0 \} \) of the extended affine Weyl group \( \tilde{W}^{(k,k+1)}(A_l) \)?"
Let $M_{k,l-k+1,1}$ be the space of a particular class of LG superpotentials consisting of trigonometric-Laurent series of one variable with tri-degree $(k+1, l-k, 1)$, these being functions of the form

$$\lambda(\varphi) = (e^{ik\varphi} - a_l)^{-1}(e^{i(k+1)\varphi} + a_1e^{ik\varphi} + \cdots + a_{l+1}e^{i(k-l)\varphi}), \quad a_{l+1}a_l \neq 0,$$

where $a_j \in \mathbb{C}$ for $j = 1, \ldots, l+2$. The space $M_{k,l-k+1,1}$ carries a natural structure of Frobenius manifold. Its invariant inner product $\eta$ and the intersection form $g$ of two vectors $\partial', \partial''$ tangent to $M_{k,l-k+1,1}$ at a point $\lambda(\varphi)$ can be defined by the formulae (5.2) and (5.3). We will show that (see Theorem 5.1)

**Main Theorem 2.** The Frobenius manifolds $M^{(k,k+1)}(A_l)$ and $M_{k,l-k+1,1}$ are locally isomorphic.

2. A new class of extended affine Weyl groups $\tilde{W}^{(k,k+1)}(A_l)$

To keep self-contained, we recall some known facts about Weyl groups of type $A_l$, see [4] for details. Let $\mathbb{R}^{l+1}$ be a $(l+1)$-dimensional Euclidean space with Euclidean inner product $(\ , \ )$ and an orthonormal basis $\epsilon_1, \cdots, \epsilon_{l+1}$. Let $A_l$ be an irreducible reduced root system in the hyperplane $V = \left\{ \sum_{s=1}^{l+1} v_s \epsilon_s \in \mathbb{R}^{l+1} \bigg| \sum_{s=1}^{l+1} v_s = 0 \right\}$.

We fix a basis

$$\alpha_1 = \epsilon_1 - \epsilon_2, \cdots, \alpha_l = \epsilon_l - \epsilon_{l+1}$$

of simple roots. The corresponding coroots are $\alpha_j^\vee = \alpha_j$ for $j = 1, \cdots, l$. The Weyl group $W = W(A_l)$ is generated by the reflections

$$x \mapsto x - (\alpha_j^\vee, x)\alpha_j, \quad \forall x \in V, \quad j = 1, \ldots, l. \quad (2.1)$$

$W$ acts on $V$ by permutations of the coordinates $v_1, \cdots, v_{l+1}$. The basic $W$-invariant Fourier polynomials coincide with the elementary symmetric functions

$$y_j(x) = \sigma_j(e^{2\pi iv_1}, \cdots, e^{2\pi iv_{l+1}}), \quad j = 1, \cdots, l. \quad (2.2)$$

**Definition 2.1.** For any fixed integer $1 \leq k < l$, we call $\tilde{W} = \tilde{W}^{(k,k+1)}(A_l)$ to be an extended affine Weyl group of type $A$ if it acts on

$$\tilde{V} = V \oplus \mathbb{R}^2$$

generated by the transformations

$$x = (x, x_{l+1}, x_{l+2}) \mapsto (w(x) + \sum_{j=1}^{l} m_j \alpha_j^\vee, x_{l+1}, x_{l+2}), \quad w \in W, \quad m_j \in \mathbb{Z},$$

(2.3)
where \( x = (x, x_{l+1}, x_{l+2}) \mapsto (x + \omega_k, x_{l+1} - 1, x_{l+2}), \) \( (2.4) \)

and

\[ x = (x, x_{l+1}, x_{l+2}) \mapsto (x + \omega_{k+1}, x_{l+1}, x_{l+2} - 1). \] \( (2.5) \)

Coordinates \( x_1, \ldots, x_l \) may be introduced on the space \( V \) via the expression

\[ x = x_1 \alpha_1^y + \cdots + x_l \alpha_l^y. \] \( (2.6) \)

That is to say,

\[ v_1 = x_1, \quad v_i = x_j - x_{j-1}, \quad v_{l+1} = -x_l, \quad j = 2, \ldots, l. \] \( (2.7) \)

**Definition 2.2.** \( \mathcal{A} = \mathcal{A}^{(k,k+1)}(\hat{A}_l) \) is the ring of all \( \hat{W} \)-invariant Fourier polynomials of \( x_1, \ldots, x_l, \frac{1}{l+1}x_{l+1}, \frac{1}{l+1}x_{l+2} \) that are bounded in the following limit conditions

\[ x = x^0 - i\omega_k \tau, \quad x_{l+1} = x^0_{l+1} + i\tau, \quad x_{l+2} = x^0_{l+2}, \quad \tau \rightarrow +\infty \] \( (2.8) \)

and

\[ x = x^0 - i\omega_{k+1} \tau, \quad x_{l+1} = x^0_{l+1}, \quad x_{l+2} = x^0_{l+2} + i\tau, \quad \tau \rightarrow +\infty \] \( (2.9) \)

for any \( x^0 = (x^0, x^0_{l+1}, x^0_{l+2}) \).

Conditions \( (2.8) \) and \( (2.9) \) are essential for this construction as did in \( [12, 19] \).

For simplicity, we introduce a set of numbers

\[ d_{j,k} := (\omega_j, \omega_k) = \begin{cases} \frac{j(l-k+1)}{l+1}, & j = 1, \ldots, k, \\ \frac{k(l-j+1)}{l+1}, & j = k + 1, \ldots, l \end{cases} \] \( (2.10) \)

and define the following Fourier polynomials

\[ \tilde{y}_j(x) = e^{2\pi i(d_{j,k} x_{l+1} + d_{j,k+1} x_{l+2})} y_j(x), \quad j = 1, \ldots, l, \]

\[ \tilde{y}_{l+1}(x) = e^{2\pi i x_{l+1}}, \quad \tilde{y}_{l+2}(x) = e^{2\pi i x_{l+2}}. \] \( (2.11) \)

**Lemma 2.3.** \( (12) \) For any fixed integer \( 1 \leq k_r \leq l \), we have

\[ y_j(x) = e^{2\pi d_{j,k_r} \tau} \left[ y_j^{0,r}(x^0) + o(e^{-2\pi \tau}) \right], \quad \tau \rightarrow +\infty, \quad \text{for } j = 1, \ldots, l, \]

where \( x = x^0 - i\omega_{k_r} \tau \) and

\[ y_j^{0,r}(x^0) = \frac{1}{n_j} \sum_{w \in W, (w(\omega_j), x^0) = 0} e^{2\pi i (w(\omega_j), x^0)}, \]

where \( n_j = \# \{ w \in W | e^{2\pi i (\omega_j, w(x^0))} = e^{2\pi i (\omega_j, x^0)} \} \). Moreover, the Fourier polynomials \( y_j^{0,r}(x^0), \ldots, y_j^{0,r}(x^0) \) are algebraically independent.

From these explicit expressions in \( (2.11) \) and Lemma \( 2.3 \) it is not difficult to see that \( \tilde{y}_j(x) \in \mathcal{A} \) for \( j = 1, \ldots, l + 2 \). Furthermore, we have
Theorem 2.4. (Chevalley-type theorem) The ring $A$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x), \ldots, \tilde{y}_{t+2}(x)$.

Proof. Observe that $\tilde{y}_1(x), \ldots, \tilde{y}_{t+2}(x)$ are algebraically independent. So in order to prove the theorem, we only need to show that any element $f(x)$ of the ring $A$ can be represented as a polynomial of $\tilde{y}_1(x), \ldots, \tilde{y}_{t+2}(x)$. From the invariance with respect to $\tilde{W}$, it follows that $f(x)$ can be represented as a polynomial of $\tilde{y}_1(x), \ldots, \tilde{y}_{t+1}(x), \tilde{y}_{t+2}(x)^{-1}, \tilde{y}_{t+1}(x)^{-1}$. It suffices to show that in $f(x)$ there are no negative powers of $\tilde{y}_{t+1}(x)$ and $\tilde{y}_{t+2}(x)$.

Assume that

$$f(x) = \sum_{s \geq -S} \tilde{y}_1^s \sum_{t \in \Lambda} \tilde{y}_{t+2}^t Q_{s,t}(\tilde{y}_1(x), \ldots, \tilde{y}_l(x))$$

for a positive integer $S$ and the polynomial $Q_{-S,t_0}(\tilde{y}_1(x), \ldots, \tilde{y}_l(x))$ does not vanish identically, where $t_0 = \min\{t \in \Lambda \mid Q_{-S,t} \text{ does not vanish identically}\}$ and $\Lambda$ is a finite subset of $\mathbb{Z}$. With the use of Lemma 2.3 in the limit (2.8) the function $f(x)$ behaves as

$$f(x) = e^{2\pi S\tau + 2\pi Sx_0^0} \sum_{s \in \Lambda} e^{2\pi it\alpha_0^0} [Q_{s,t}(\tilde{y}_1^{0,1}(x^0), \ldots, \tilde{y}_l^{0,1}(x^0)) + O(e^{-2\pi \tau})],$$

where $\tilde{y}_j^{0,1}(x^0) = e^{2\pi i(d_jx_0^{0,l} + d_j,k+1x_0^{l}+1)}y_j^{0,1}(x^0)$ for $j = 1, \ldots, l$. In order to assure the function $f(x)$ bounded for $\tau \mapsto +\infty$, it is necessary to have

$$Q_{-S,t_0}(\tilde{y}_1^{0,1}(x^0), \ldots, \tilde{y}_l^{0,1}(x^0)) \equiv 0,$$

which is a contradiction with the algebraic independence of $\tilde{y}_1^{0,1}(x^0), \ldots, \tilde{y}_l^{0,1}(x^0)$. This means that there are no negative powers of $\tilde{y}_{t+1}(x)$. Similarly one can show that there are no negative powers of $\tilde{y}_{t+2}(x)$. This completes the proof of the theorem.

\[\square\]

Corollary 2.5. The function $\text{deg}$ defined as

$$\text{deg} \tilde{y}_1 = \frac{1}{k}, \quad \text{deg} \tilde{y}_{t+2} = \frac{1}{l-k},$$

$$d_j := \text{deg} \tilde{y}_j = \frac{d_{j,k}}{k} + \frac{d_{j,k+1}}{l-k} = \begin{cases} \frac{j}{k}, & j = 1, \ldots, k, \\ \frac{l-j+1}{l-k}, & j = k+1, \ldots, l \end{cases}$$

determines on $A$ a structure of graded polynomial ring. Especially,

$$d_k = d_{k+1} = 1 > d_s, \quad s \neq k, k+1.$$ 

The numbers $d_1, \ldots, d_{t+2}$ with $d_{t+1} = d_{t+2} = 0$ satisfy a duality relation. For any given integer $k$, we denote $A_1 \setminus \{\alpha_k, \alpha_{k+1}\} = R_1 \cup R_2$, where $R_1 = \{\alpha_1, \ldots, \alpha_{k-1}\}$.

and \( R_2 = \{ \alpha_{k+2}, \cdots, \alpha_l \} \). On each component we have an involution \( j \mapsto j^* \) given by the reflection with respect to the center of the component. Let us define

\[
k^* = l + 1, \quad (k + 1)^* = l + 2, \quad (l + 2)^* = k + 1, \quad (l + 1)^* = k, \tag{2.15}\]

then

\[
d_j + d_{j^*} = 1, \quad j = 1, \ldots, l + 2. \tag{2.16}\]

3. A Flat Pencil of Metrics on the Orbit Space \( \mathcal{M} \)

Let us denote

\[ \mathcal{M}^{(k,k+1)} := \widetilde{V} \otimes \mathbb{C}/\tilde{W}, \]

called the orbit space of the extended Weyl group \( \tilde{W} \). We define an indefinite flat metric \((dx_i, dx_j)^\sim\) on \( \tilde{V}_\mathbb{C} = \tilde{V} \otimes_{\mathbb{R}} \mathbb{C} \) where \( \tilde{V} \) is the orthogonal direct sum of \( V \) and \( \mathbb{R}^2 \). Here \( V \) is endowed with the \( W \)-invariant Euclidean metric

\[
(dx_a, dx_b)^\sim = \frac{1}{4\pi^2} (\omega_a, \omega_b), \quad 1 \leq a, b \leq l \tag{3.1}
\]

and \( \mathbb{R}^2 \) is endowed with the metric

\[
(dx_{l+1}, dx_{l+1})^\sim = -\frac{\tau_{11}}{4\pi^2}, \quad (dx_{l+1}, dx_{l+2})^\sim = -\frac{\tau_{12}}{4\pi^2}, \quad (dx_{l+2}, dx_{l+2})^\sim = -\frac{\tau_{22}}{4\pi^2}, \tag{3.2}
\]

where

\[
\begin{pmatrix}
\tau_{11} & \tau_{12} \\
\tau_{12} & \tau_{22}
\end{pmatrix} = \begin{pmatrix}
d_{k,k} & d_{k,k+1} \\
d_{k+1,k} & d_{k+1,k+1}
\end{pmatrix}^{-1} = \begin{pmatrix}
\frac{k+1}{k} & -1 \\
-1 & \frac{l-k+1}{l-k}
\end{pmatrix}.
\]

The set of generators for the ring \( \mathcal{A} \) are defined by (2.11). They form a system of global coordinates on \( \mathcal{M}^{(k,k+1)} \). We now introduce a system of local coordinates on \( \mathcal{M}^{(k,k+1)} \) as follows

\[
y_1 = \tilde{y}_1, \ldots, y^l = \tilde{y}_l, \quad y^{l+1} = \log \tilde{y}_{l+1} = 2\pi ix_{l+1}, \quad y^{l+2} = \log \tilde{y}_{l+2} = 2\pi ix_{l+2}. \tag{3.3}
\]

They live on the universal covering \( \tilde{\mathcal{M}} \) of \( \mathcal{M} \), where \( \mathcal{M} := \mathcal{M}^{(k,k+1)} \setminus \{ \tilde{y}_{l+1} = 0 \} \cup \{ \tilde{y}_{l+2} = 0 \} \). The projection

\[
\text{Pr} : \tilde{V} \rightarrow \tilde{\mathcal{M}}, \quad (x_1, \ldots, x_{l+2}) \mapsto (y_1, \ldots, y^{l+2}) \tag{3.4}
\]

induces a symmetric bilinear form on \( T^*\tilde{\mathcal{M}} \)

\[
(dy^i, dy^j)^\sim = g^{ij}(y) := \sum_{a,b=1}^{l+2} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_b} (dx_a, dx_b)^\sim. \tag{3.5}
\]

---

\(^1\) As is common in the Frobenius manifold literature, we use the word metric to denote a complex-valued, symmetric, non-degenerate, bilinear form.
Lemma 3.1. The matrix entries \( g^{ij}(y) \) of (3.5) are weighted homogeneous polynomials in \( y^1, \ldots, y^l, e^{y^1}, e^{y^2} \) of the degree \( \deg g^{ij}(y) = \deg y^i + \deg y^j \), here \( \deg y^{i+1} = 0 \) for \( \nu = 0,1 \). The matrix \( (g^{ij}(y)) \) does not degenerate outside the \( Pr \)-images of the hyperplanes

\[
\{(x,x_{l+1},x_{l+2})| (\beta,x) = m \in \mathbb{Z}, \forall x_{l+1}, \forall x_{l+2} \}, \quad \beta \in \Phi^+,
\]

where \( \Phi^+ \) is the set of the all positive roots.

Proof. With the use of (3.5), (2.10), and (2.13), we obtain

\[
g^{i,l+1}(y) = \zeta_j d_j y^j, \quad g^{i,l+2}(y) = (1 - \zeta_j) d_j y^j, \quad j = 1, \ldots, l, \quad g^{i+1,l+1}(y) = \frac{k+1}{k}, \quad g^{i+1,l+2}(y) = -1, \quad g^{i+2,l+2}(y) = \frac{l-k}{l-k},
\]

(3.6)

where \( \zeta_j = \begin{cases} 1, & 1 \leq j \leq k, \\ 0, & k+1 \leq j \leq l. \end{cases} \)

Also, for \( 1 \leq i, j \leq l \) we have

\[
g^{ij}(y) = c_{ij} y^i y^j + \frac{1}{4\pi^2} \sum_{p,q=1}^l \frac{\partial y^i}{\partial x_p} \frac{\partial y^j}{\partial x_q} (\omega_p, \omega_q), \quad c_{ij} = (\zeta_j d_{i,k} + (1 - \zeta_j) d_{i,k+1}) d_j,
\]

(3.7)

which are Fourier polynomials invariant with respect to \( \bar{W} \) and bounded in the limits (2.8) and (2.9). It follows from Theorem 2.4 and (3.5) that \( g^{ij}(y) \) are weighted homogeneous polynomials in \( y^1, \ldots, y^l, e^{y^1}, e^{y^2} \) of the degree \( \deg g^{ij}(y) = \deg y^i + \deg y^j \).

Observe that the Jacobian of the projection map (3.4) is given by

\[
\det \left( \frac{\partial y^j}{\partial x_a} \right) = -4\pi^2 e^{2\pi i \sum_{j=1}^l (d_{j,k} x_{l+1} + d_{j,k+1} x_{l+2})} \det \left( \frac{\partial y^j(x)}{\partial x_p} \right) = c \prod_{p=1}^l \left( e^{2\pi i (\beta, x)} - 1 \right) J(x),
\]

(3.8)

where \( J(x) = e^{-\sum_{a \in \Phi^+} \pi i (\alpha, x)} \prod_{\beta \in \Phi^+} (e^{2\pi i (\beta, x)} - 1) \) and \( c \) is a nonzero constant (4). So the projection map (3.4) is a local diffeomorphism outside the above hyperplanes, which assures the nondegeneracy of \( (g^{ij}(y)) \). \( \square \)

Lemma 3.2. For \( k \leq i, j \leq k + 1 \), the term \( y^i y^j \) only possibly appears in \( g^{ij}(y) \) and \( g^{ji}(y) \) with the coefficient \( c_{ij} - d_{i,j} \), where \( c_{ij} = (\zeta_j d_{i,k} + (1 - \zeta_j) d_{i,k+1}) d_j \) and \( d_{i,j} = (\omega_i, \omega_j) \).

Proof. From (3.7), we have for \( k \leq i, j \leq k + 1 \)

\[
g^{ij}(y) = e^{2\pi i (d_{i,k} + d_{j,k}) x_{l+1} + (d_{i,k+1} + d_{j,k+1}) x_{l+2}} (c_{ij} y_i(x) y_j(x) + \beta_{ij}(x)),
\]

(3.9)
where

\[
\beta_{ij}(x) = \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial y_k(x)}{\partial x_p} \frac{\partial y_j(x)}{\partial x_q} (\omega_p, \omega_q)
\]

\[
= -\frac{1}{n_i n_j} \sum_{w,w'\in W} e^{2\pi i (w(\omega_i)+w'(\omega_j),x)} (w(\omega_i), w'(\omega_j)).
\]

Now we use the standard partial ordering of the weights (see the page 69 in [1])

\[
\omega \succ \omega' \iff \omega - \omega' = \sum_{m=1}^{l} c_m \alpha_m
\]

for some nonnegative integers \(c_1, \ldots, c_l\). In this case, we will write \(e^{2\pi i (w,x)} \succ e^{2\pi i (w',x)}\). All the terms in the \(W\)-invariant Fourier polynomials \(\beta_{ij}(x)\) are strictly less than \(e^{2\pi i (\omega_i+\omega_j,x)}\), except the terms \(-d_{i,j} e^{2\pi i (\omega_i+\omega_j,x)}\). So the term \(y_j y^j\) possibly appears in \(g_{ij}(y)\) and \(g_{ji}(y)\) with the coefficient \(c_{ij} - d_{i,j}\).

Observe that

\[
g_{kk}^{kk}(y) = e^{2\pi i \left(\frac{1}{l+1}x_{l+1} - \frac{1}{l+1}x_{l+1} \right)} \left( \frac{c_{kk} y_k(x)}{y_{k+1}(x)} + \frac{\beta_{kk}(x)}{y_k(x) y_{k+1}(x)} \right)
\]

and

\[
g_{kk}^{kk}(y) = e^{2\pi i \left(\frac{1}{l+1}x_{l+1} - \frac{1}{l+1}x_{l+1} \right)} \frac{c_{kk} (y_k(x))^2 + \beta_{kk}(x)}{(y_{k+1}(x))^2}.
\]

So \(y_k y_{k+1}^{k+1}\) and \((y_{k+1}^{k+1})^2\) do not appear in \(g_{kk}^{kk}(y)\). Similarly, we could prove the other cases.

\[\square\]

**Lemma 3.3.** Denote

\[
e = s_1 \frac{\partial}{\partial y^k} + s_2 \frac{\partial}{\partial y^{k+1}}, \quad s_1, s_2 \in \mathbb{R},
\]

then for \(1 \leq i, j \leq l + 2\),

\[
\mathcal{L}_e (\mathcal{L}_e g^{ij}(y)) = 0,
\]

where \(\mathcal{L}_e\) is the Lie derivative along the vector field \(e\).

**Proof.** According to the weighted homogeneity and (2.14) and (3.6), it suffices to show that \(k \leq i, j \leq k + 1\),

\[
\frac{\partial^2}{\partial y^k \partial y^k} g^{ij}(y) = \frac{\partial^2}{\partial y^{k+1} \partial y^{k+1}} g^{ij}(y) = \frac{\partial^2}{\partial y^k \partial y^{k+1}} g^{ij}(y) = 0.
\]

It follows from

\[
c_{k(k+1)} = d_{k,k+1}, \quad c_{(k+\nu)(k+\nu)} = d_{k+\nu,k+\nu}
\]
that \(g^{k+\nu,k+\nu}(y)\) does not contain \((y^{k+\nu})^2\) for \(\nu = 0, 1\), and \(g^{k,k+1}(y) = g^{k+1,k}(y)\) does not contain \(y^{k}y^{k+1}\). Combining with Lemma 3.2 we obtain the desired (3.12) and complete the proof of this lemma.

\[\square\]

**Corollary 3.4.** For \(1 \leq i, j \leq l + 2\),

\[g^{ij}(\cdot \cdot \cdot , y^k + \zeta_1 \lambda, y^{k+1} + \zeta_2 \lambda, \cdot \cdot \cdot )\]

are linear in the parameter \(\lambda\).

Suppose \(\Sigma\) is the discriminant of \(\mathcal{M}^{(k,k+1)}\), i.e., \(\Sigma = \{y \mid \det(g^{ij}(y)) = 0\}\), then on \(\mathcal{M}^{(k,k+1)} \setminus \Sigma\) the matrix \((g^{ij}(y))\) is invertible. The inverse matrix \((g_{ij}(y))\) determines a flat metric on \(\mathcal{M}^{(k,k+1)} \setminus \Sigma\). Let us now compute the coefficients of the correspondent Levi-Civita connection \(\nabla\) for the metric \(g_{ij}(y)\). It is convenient to consider the contravariant components of the connection

\[\Gamma^m_{ij}(y) = (dy^i, \nabla_m dy^j),\]

which are related to the standard Christoffel coefficients by the formula

\[\Gamma^m_{ij}(y) = -g^{is}(y)\Gamma^j_{sm}(y)\]

For the contravariant components, we have the following formulae

\[\Gamma^m_{ij}(y)dy^m = \frac{\partial y^i}{\partial x_a} \frac{\partial^2 y^j}{\partial x_b \partial x_r} (dx_a, dx_b)^\sim dx_r \quad (3.13)\]

and

\[2g^{sm}(y)\Gamma^m_{ij}(y) = g^{im}(y) \frac{\partial g^{js}(y)}{\partial y^m} + g^{em}(y) \frac{\partial g^{ji}(y)}{\partial y^m} - g^{jm}(y) \frac{\partial g^{is}(y)}{\partial y^m} \quad (3.14)\]

and

\[\frac{\partial g^{ij}(y)}{\partial y^m} = \Gamma^m_{ij}(y) + \Gamma^m_{ji}(y). \quad (3.15)\]

**Lemma 3.5.** \(\Gamma^m_{ij}(y)\) are weighted homogeneous polynomials in \(y^1, \cdot \cdot \cdot , y^l, e^{y^{l+1}}, e^{y^{l+2}}\) of the degree \(\deg \Gamma^m_{ij}(y) = \deg y^i + \deg y^j - \deg y^m\).

**Proof.** By using (3.13) and (3.15), we can represent

\[(\Gamma^1_{ij}(y), \cdot \cdot \cdot , \Gamma^l_{ij}(y), \Gamma_{i+2}^m(y)) \left(\frac{\partial y^r}{\partial x_a}\right) = \left(\frac{\partial y^i}{\partial x_a} \frac{\partial^2 y^j}{\partial x_b \partial x_1} (dx_a, dx_b)^\sim, \cdot \cdot \cdot , \frac{\partial y^i}{\partial x_a} \frac{\partial^2 y^j}{\partial x_b \partial x_{l+2}} (dx_a, dx_b)^\sim\right),\]

and

\[\Gamma^m_{ij}(y) = e^{2\pi i \sum_{\nu} d_{i,k+\nu} + d_{j,k+\nu} - d_{m,k+\nu}} x_{i+1+\nu} P^m_{ij}(x) \frac{\text{J}(x)}{J(x)}, \quad (3.16)\]

where \(d_{i+1+\nu,k+\nu} = 0\) for \(\nu = 0, 1\) and \(P^m_{ij}(x)\) is certain Fourier polynomial in \(x_1, \cdot \cdot \cdot , x_l\). As discussed the Lemma 2.2 in [12], \(P^m_{ij}(x)\) is anti-invariant with
respect to Weyl group $W$ and divisible by $J(x)$. We thus knows $\Gamma_{m}^{ij}(y) \in \mathcal{A}$, whose homogeneity property is obvious. □

**Lemma 3.6.** For $1 \leq i, j \leq l + 2$, we have

$$\mathcal{L}_e(\mathcal{L}_e \Gamma_{m}^{ij}(y)) = 0.$$  (3.17)

Equivalently, $\Gamma_{m}^{ij}(\cdots, y^k + \zeta_1 \lambda, y^{k+1} + \zeta_2 \lambda, \cdots)$ are linear in the parameter $\lambda$.

**Proof.** By the degrees, it suffices to show that

$$\mathcal{L}_e(\mathcal{L}_e \Gamma_{k+\nu, k+2}^{l+i}(y)) = 0.$$  (3.17)

Observe that

$$\Gamma_{k+\nu, k+2}^{l+i}(y) = \frac{\partial x_r}{\partial y^{l+i}} \frac{\partial y^{k+\nu}}{\partial x_p} \frac{\partial^2 y^{k+\nu}}{\partial x_q \partial x_r} (dx_p, dx_q)^{\sim}$$

and

$$= \frac{\partial y^{k+\nu}}{\partial x_p} \frac{\partial}{\partial y^{l+i}} \left( \frac{\partial y^{k+\nu}}{\partial x_q} \right) (dx_p, dx_q)^{\sim}$$

$$= 1 \frac{\partial}{\partial y^{l+i}} g^{k+\nu, k+\nu}(y),$$

and using (3.11), then

$$\mathcal{L}_e(\mathcal{L}_e \Gamma_{k+\nu, k+2}^{l+i}(y)) = 0$$

for $\nu = 0, 1$ and $i = 1, 2$.

By choosing $s = k, i = k, j = k + 1$ in (3.14) and using (3.6), we get

$$y^k \Gamma_{l+1}^{k,k+1}^{l+1}(y) = g^{k,l+1}(y) \Gamma_{l+1}^{k,k+1}^{l+1}(y) + g^{k,l+2}(y) \Gamma_{l+1}^{k,k+1}^{l+2}(y)$$

$$= g^{km}(y) \frac{\partial g^{k,k+1}(y)}{\partial y^m} - \frac{1}{2} g^{k+1,m}(y) \frac{\partial g^{k+1}(y)}{\partial y^m} - \sum_{j=1}^{l} g^{kj}(y) \Gamma_{j}^{k,k+1}(y).$$

Repeat using the degrees and Lemma 3.2 and Lemma 3.3, we thus conclude

$$\mathcal{L}_e(\mathcal{L}_e \Gamma_{k+1}^{k,k+1}^{l+1}(y)) = 0.$$  (3.17)

Furthermore, with the help of (3.15), we have $\mathcal{L}_e(\mathcal{L}_e \Gamma_{k+1}^{k+1, k}(y)) = 0$.

Similarly, by choosing $s = k + 1, i = k, j = k + 1$ in (3.14) and using (3.6), (3.16) and (3.15), we have

$$y^{k+1} \Gamma_{l+2}^{k,k+1}(y) = g^{k,l+1}(y) \Gamma_{l+1}^{k,k+1}(y) + g^{k+1,l+2}(y) \Gamma_{l+2}^{k,k+1}(y)$$

$$= \frac{1}{2} g^{km}(y) \frac{\partial g^{k+1,k+1}(y)}{\partial y^m} - \sum_{j=1}^{l} g^{k+1,j}(y) \Gamma_{j}^{k,k+1}(y)$$

and

$$\Gamma_{l+2}^{k,k+1}(y) = \frac{\partial g^{k+1,k+1}(y)}{\partial y^{l+2}} - \Gamma_{l+2}^{k,k+1}(y).$$
So

\[ \mathcal{L}_e(\mathcal{L}_e \Gamma_{l+2}^{k,k+1}(y)) = \mathcal{L}_e(\mathcal{L}_e \Gamma_{l+2}^{k+1,k}(y)) = 0. \]

This completes the proof of the lemma. □

**Lemma 3.7.** Setting

\[ \eta^{ij}(y) = \mathcal{L}_e g^{ij}(y) \] (3.18)

and denoting \( R_{k,k+1} = R_1 \cup R_2 \), then we have

1. If \( \alpha_i \) and \( \alpha_j \) belong to different components of \( R_{k,k+1} \), then \( \eta^{ij}(y) = 0 \);
2. The block \( \eta_{t} = (\eta^{ij}(y))|_{\alpha_i,\alpha_j \in R_t} \) of the matrix \( (\eta^{ij}(y)) \) corresponding to any branch \( R_t \) has triangular form. The antidiagonal elements of \( \eta_{t} \) consists of the constant numbers \( \eta^{ii} \) for \( \alpha_i \in R_t \), where \( t = 1, 2 \);
3. \( \eta^{a,l+1}(y) = \varsigma_1 \delta_{a,k}, \ \eta^{a,l+2}(y) = \varsigma_2 \delta_{a,k+1} \) for \( a = 1, \ldots, l+2 \).

**Proof.** (1) Let \( \alpha_i \in R_1 \) and \( \alpha_j \in R_2 \), i.e. \( 1 \leq i < k \) and \( k + 1 < j \leq l \). As discussed above, if \( \beta_{ij}(x) \) as a polynomial in \( y_1(x), \ldots, y_l(x) \) contains a monomial \( y_1^{p_1} \cdots y_l^{p_l} \) with \( p_k = 1 \), then

\[ \omega_i + \omega_j = p_1 \omega_1 + \cdots + p_l \omega_l + \sum_{s=1}^{l} q_s \alpha_s \] (3.19)

for some nonnegative integers \( q_1, \ldots, q_l \). We multiply \( (3.19) \) by \( \omega_1 \) and obtain

\[ l + 1 + k - i - j - \sum_{s \neq k} p_s(l - s - 1) = (l + 1)q_1. \]

Since \( k - i - j < 0 \), then we have \( q_1 = 0 \) and

\[ l - i + 1 - (j - k) = \sum_{s \neq k} p_s(l - s - 1) \]

which yields that \( p_s = 0 \) for \( s = 1, \ldots, i \). So \( (3.19) \) becomes

\[ \omega_i + \omega_j = p_{i+1} \omega_{i+1} + \cdots + p_l \omega_l + \sum_{s=2}^{l} q_s \alpha_s. \] (3.20)

We multiply \( (3.20) \) by \( \alpha_1, \ldots, \alpha_i \) and get \( q_2 = \cdots = q_i = 0, q_{i+1} = -1 \), which contradicts nonnegativity of \( q \)'s. So in this case \( \frac{\partial}{\partial y^k} g^{ij}(y) = 0 \). Similarly, one can show that

\[ \frac{\partial}{\partial y^{k+1}} g^{ij}(y) = 0. \]

We thus complete the proof of the first statement.

(2) Observe that in any component of \( R_{k,k+1} \), the numbers \( d_i \) are distinct and ordered monotonically and \( \deg \eta^{ij}(y) = d_i + d_j - d_k \). We thus conclude that \( \eta^{ij}(y) = 0 \) when \( d_i + d_j = d_k \), and \( \eta^{ij}(y) = \text{constant} \) when \( d_i + d_j = d_k \) which happens if the labels \( i \) and \( j \) are dual to each other in the sense of \( (2.16) \).

(3) Obviously, the third statement follows from \( (3.6) \). □
**Proposition 3.8.** If $\varsigma_1 \varsigma_2 \neq 0$, then the determinant of $(\eta^{ij}(y))$ is a non-zero constant.

**Proof.** By using Lemma 3.7 we know that

$$
\det(\eta^{ij}(y)) = (-1)^{\frac{k(k+1)+l(l-k)(l-k+1)}{2}} \prod_{i=1}^{l} \eta^{ii*}. \tag{3.21}
$$

It suffices to show that $\eta^{ii*}$ are non-zero constants for $i = 1, \cdots l$.

For a fixed $1 \leq i < k$, with the use of (3.9) we obtain

$$
\eta^{ii*} = \eta^{i(k-i)}(y) = L_e g^{i(k-i)}(y)
$$

$$
= e^{2\pi i(d_k k x_{i+1} + d_k k x_{i+2})} L_e (c_i(i-k) y_i y_{(i-k)}(x) + \beta_{i(k-i)}(x))
$$

$$
= e^{2\pi i(d_k k x_{i+1} + d_k k x_{i+2})} L_e \beta_{i(k-i)}(x)
$$

$$
= \frac{\partial}{\partial y_k(x)} \beta_{i(k-i)}(x) + e^{2\pi i(k-1)x_{i+1} - \frac{l-k}{l+i+2}} \frac{\partial}{\partial y_{k+1}(x)} \beta_{i(k-i)}(x).
$$

Since $\eta^{ii*}$ is a constant, we thus have

$$
\frac{1}{\varsigma_1} \eta^{ii*} = \frac{\partial}{\partial y_k(x)} \beta_{i(k-i)}(x) = \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial y_i(x)}{\partial x_p} \frac{\partial y_{k-i}(x)}{\partial x_q} (\omega_p, \omega_q)
$$

which coincides with the non-zero constant $\eta^{i(k-i)}$ used in the case $\tilde{W}^{(k)}(A_l)$ (e.g., please see the Corollary 2.3 in [12]).

Similarly, for a fixed $k+1 < i \leq l$, we have

$$
\frac{1}{\varsigma_2} \eta^{ii*} = \eta^{i(l+k+2-i)}(y) = L_e g^{i(l+k+2-i)}(y)
$$

$$
= e^{2\pi i(d_k k x_{i+1} + d_k k x_{i+2})} L_e (c_i(i(l+k+2-i) y_i y_{k-i}(x) + \beta_{i(k-i)}(x))
$$

$$
= e^{2\pi i(d_k k x_{i+1} + d_k k x_{i+2})} L_e \beta_{i(l+k+2-i)}(x)
$$

$$
= e^{2\pi i(k-1)x_{i+1} - \frac{k}{l+i+1}} \frac{\partial}{\partial y_k(x)} \beta_{i(k-i)}(x) + \frac{\partial}{\partial y_{k+1}(x)} \beta_{i(l+k+2-i)}(x)
$$

$$
= \frac{\partial}{\partial y_{k+1}(x)} \beta_{i(l+k+2-i)}(x) = \frac{1}{4\pi^2} \sum_{p,q=1}^{l} \frac{\partial y_i(x)}{\partial x_p} \frac{\partial y_{l+k+2-i}(x)}{\partial x_q} (\omega_p, \omega_q)
$$

which is exactly the non-zero constant $\eta^{i(l+k+2-i)}$ used in the case $\tilde{W}^{(k+1)}(A_l)$ ([12]).

Observe that $\eta^{kk*} = \varsigma_1$ and $\eta^{(k+1)(k+1)*} = \varsigma_2$ in Lemma 3.7, we thus complete the proof of this proposition.

According to Lemma D.1 in [10] (or see Lemma 3.3 in [19]) and using Lemma 3.7, Proposition 3.8 and Lemma 3.6 we have
Theorem 3.9. \( g^{ij}(y) \) and \( \eta^{ij}(y) \) form a flat pencil of metrics, i.e., the metric

\[ g^{ij}(y) + \lambda \eta^{ij}(y) \]

is flat for arbitrary \( \lambda \) and the Levi-Civit\`a connection for this metric has the form

\[ \Gamma^{ij}_m(y) + \lambda \gamma^{ij}_m(y) \].

Here \( \gamma^{ij}_m(y) := \mathcal{L}_c \Gamma^{ij}_m(y) \) are the contravariant components of the Levi-Civita connection for the metric \( (\eta^{ij}(y)) \).

Without loss of generality, in what follows we take \( \varsigma_1 = \varsigma_2 = 1 \) unless otherwise stated.

4. Frobenius manifold structures on the orbit space \( \mathcal{M} \)

In this section we want to describe Frobenius manifold structures on the orbit space \( \mathcal{M} \) of \( \widetilde{W}^{(k,k+1)}(A_l) \) for \( 1 \leq k < l \).

4.1. The change of coordinates. In order to do this, we firstly make the change of coordinates

\[
\begin{align*}
  z^j &= y^j, \quad j = 1, \cdots, k - 1, k + 2, \cdots, l, \\
  z^k &= y^k, \quad z^{k+1} = y^{k+1} - y^k, \\
  z^{l+1} &= y^{l+2}, \quad z^{l+2} = y^{l+1} + y^{l+2},
\end{align*}
\]

such that

\[ e = \frac{\partial}{\partial y^k} + \frac{\partial}{\partial y^{k+1}} = \frac{\partial}{\partial z^k}. \]

Let us denote

\[
\begin{align*}
  g^{ij}(z) &= \sum_{a,b=1}^{l+2} \frac{\partial z^i}{\partial y^a} \frac{\partial z^j}{\partial y^b} (dy^a, dy^b)^\sim, \\
  \eta^{ij}(z) &= \sum_{a,b=1}^{l+2} \frac{\partial z^i}{\partial y^a} \frac{\partial z^j}{\partial y^b} \eta^{ab}(y),
\end{align*}
\]

and \( \Gamma^{ij}_m(z) \) and \( \gamma^{ij}_m(z) \) are the contravariant components of the Levi-Civita connection for the metric \( g^{ij}(z) \) and \( \eta^{ij}(z) \). Under the simple change of coordinates (4.1), it is easy to know that

1. \( g^{ij}(z) \) and \( \Gamma^{ij}_m(z) \) are weighted homogeneous polynomials in \( z^1, \cdots, z^{l+2}, z^{l+1}, e^{z^{l+2} - z^{l+1}} \) of the degrees \( \deg g^{ij}(z) = \deg z^i + \deg z^j \) and

\[
\deg \Gamma^{ij}_m(z) = \deg z^i + \deg z^j - \deg z^m,
\]

(4.4)
where \( \deg z^i = d_i \). Moreover, \( g^{ij}(z) \) and \( \Gamma^{ij}_{km}(z) \) are at most linear in \( z^k \) and
\[
\begin{align*}
g^{k,l+1}(z) &= 0, & g^{k+1,l+1}(z) &= z^{k+1} + z^k, \\
g^{k,l+2}(z) &= z^k, & g^{k+1,l+2}(z) &= z^{k+1}, \quad (4.5) \\
g^{l+1,l+1}(z) &= \frac{l-k+1}{l-k}, & g^{l+1,l+2}(z) &= \frac{1}{l-k}, \\
g^{l+2,l+2}(z) &= \frac{l}{k(l-k)}.
\end{align*}
\]

(2) \( \eta^{ij}(z) = \frac{\partial g^{ij}(z)}{\partial z^k} \) and \( \gamma^{ij}_{km}(z) = \frac{\partial \Gamma^{ij}_{km}(z)}{\partial z^k} \). Especially,
\[
\eta^{i,l+1}(z) = \delta^k_{i+1}, \quad \eta^{i,l+2}(z) = \delta^k_i, \quad i = 1, \cdots, l + 2. \quad (4.6)
\]

So we rename \( k^* = l + 2 \) and \((k+1)^* = l + 1\), and also have
\[
\eta^{ii^*}(z) = \eta^{ii^*}, \quad i \neq k, k + 1, \quad \eta^{kk^*}(z) = \eta^{(k+1)(k+1)^*}(z) = 1.
\]

(3) \( g^{ij}(z) \) and \( \eta^{ij}(z) \) form a flat pencil of metrics.
(4) \( \eta_{ij}(z) \) are weighted homogeneous polynomials in \( z^1, \cdots, z^l, e^{z^1+2-z^l+1}, e^{z^{l+1}} \),
where \( (\eta^{ij}(z)) \) is the inverse matrix of \( (\eta^{ij}(z)) \).

4.2. Flat coordinates of the metric \( \eta^{ij}(z) \). In this subsection, we want to describe flat coordinates of the metric \( \eta^{ij}(z) \).

**Lemma 4.1.** For \( \nu, s, t = 0, 1 \), we have
\[
\frac{\partial}{\partial z^{k+\nu}} \eta^{k+s, k+t}(z) = 0. \quad (4.7)
\]

**Proof.** Observe that
\[
\begin{align*}
\frac{\partial}{\partial z^{k+1}} \eta^{k,k+1}(z) &= \frac{\partial}{\partial z^{k+1}} \sum_{a,b=1}^{l+2} \frac{\partial z^k}{\partial y^a} \frac{\partial z^{k+1}}{\partial y^b} g^{ab}(y) \\
&= \frac{\partial}{\partial y^{k+1}} \left( \frac{\partial}{\partial y^k} + \frac{\partial}{\partial y^{k+1}} \right) \left( g^{k,k+1}(y) - g^{kk}(y) \right) = 0
\end{align*}
\]
which follows from (3.12). The other cases are similar. \( \square \)

**Lemma 4.2.** For \( \nu = 0, 1 \), we have
\[
\gamma^{i, l+1+\nu}(z) = 0, \quad i, j = 1, \cdots, l + 2. \quad (4.8)
\]

**Proof.** With the use of (4.4) and \( \gamma^{i, l+1+\nu}(z) = \frac{\partial}{\partial z^k} \Gamma^{i}_{j, l+1+\nu}(z) \), we obtain
\[
\deg \gamma^{i, l+1+\nu}(z) = d_i - d_j - d_k < 0, \quad i.e., \quad \gamma^{i, l+1+\nu}(z) = 0 \quad (4.9)
\]
except the cases \( \gamma_{l+1+\mu}^{k+\sigma} \) for \( \nu, \mu, \sigma = 0, 1 \). So it suffices to show that
\[
\gamma_{l+1+\mu}^{k+\sigma} = 0, \quad \nu, \mu, \sigma = 0, 1. \tag{4.10}
\]

Since \( \gamma_{ij}^m(z) \) are the contravariant components of the Levi-Civita connection for the metric \( g_{ij}^nz \), then
\[
2\eta^m(z)\gamma_{ij}^m(z) = \eta^m(z) \frac{\partial \eta_{is}^j(z)}{\partial z^m} + \eta^m(z) \frac{\partial \eta_{is}^j(z)}{\partial z^m} - \eta^m(z) \frac{\partial \eta_{is}^j(z)}{\partial z^m}. \tag{4.11}
\]
By choosing \( i = k + \sigma \) and \( j = l + 1 + \nu \) in (4.11), it follows from (4.9) that
\[
2\eta^s(z)\gamma_{l+1+\nu}^{k+\sigma}(z) + 2\eta^s(z)\gamma_{l+2}(z) = \eta^s(z)\frac{\partial \gamma_{l+1+\nu}^{k+\sigma}(z)}{\partial z^m} + \eta^s(z)\frac{\partial \gamma_{k+\sigma}^{l+1+\nu}(z)}{\partial z^m}.
\tag{4.12}
\]
Taking \( s = k \) and \( s = k + 1 \) respectively in (4.12) and with the help of (4.6) and (4.7), we get the desired identities (4.10).

**Theorem 4.3.** There exist flat coordinates of the metric \( (\eta_{ij}^m(z)) \) in the form
\[
t^\alpha = z^\alpha + h^\alpha(z^1, \ldots, z^\alpha, \ldots, z^l, e^{z^i-1}, e^{z^i-2}, \ldots), \quad t^{l+1} = z^{l+1}, \quad t^{l+2} = z^{l+2}, \quad \alpha = 1, \ldots, l,
\tag{4.13}
\]
where \( h^\alpha \) are weighted homogeneous polynomials in \( z^1, \ldots, z^\alpha-1, z^\alpha+1, \ldots, z^l, e^{z^i+1}, e^{z^i+2} \) of degree \( d_\alpha \) defined in (2.13).

**Proof.** Local existence of the coordinates \( t^1, \ldots, t^{l+2} \) follows from flatness of the metric \( (\eta_{ij}^m(z)) \). The flat coordinates \( t = t(z) \) are to be found from the following system
\[
\frac{\partial^2 t}{\partial z^i \partial z^j} + \eta_{is}^j(z)\gamma_{ij}^m(z) \frac{\partial t}{\partial z^m} = 0, \quad i, j = 1, \ldots, l + 2. \tag{4.14}
\]
The system (4.14) can be written as linear differential equations
\[
\frac{\partial w_i}{\partial z^i} - \gamma_{ij}^m(z)w_m = 0, \quad w_i = \frac{\partial t}{\partial z^i}.
\]
This is an overdetermined holonomic system. So the space of solutions has dimension \( l + 2 \). Observe that those coefficients in (4.14) are weighted homogeneous polynomials in \( z^1, \ldots, z^l, e^{z^i+1}, e^{z^i+2} \). From (4.8), it follows that
\[
t^{l+1} = z^{l+1}, \quad t^{l+2} = z^{l+2}
\]
are two solutions of (4.14). We choose remaining solutions
\[
t^\alpha = z^\alpha + h^\alpha(z^1, \ldots, z^\alpha, \ldots, z^l, e^{z^i+1}, e^{z^i+2}), \quad \alpha = 1, \ldots, l
\]
in such a way that
\[ \frac{\partial t^\alpha}{\partial z^\beta}(0,\ldots,0,0,0) = \delta_\beta^\alpha, \quad \alpha, \beta = 1, \ldots, l. \] (4.15)

These solutions \( t^\alpha \) are power series in \( z^1, \ldots, z^l, e^{z^l+1}, e^{z^l+2-z^l+1} \). The system (4.14) is invariant with respect to the transformation
\[ z^\alpha \mapsto c_0^d a^\alpha, z^{l+1} \mapsto z^{l+1} + \frac{1}{l-k} \log(c_0), z^{l+2} \mapsto z^{l+2} + \frac{l}{k(l-k)} \log(c_0) \]
for any positive constant \( c_0 \). This yields that \( t^\alpha \) are weighted homogeneous in \( z^1, \ldots, z^l, e^{z^l+1}, e^{z^l+2-z^l+1} \) of the same degree \( d_\alpha > 0 \). Thus \( h^\alpha \) are weighted homogeneous polynomials in \( z^1, \ldots, z^l, e^{z^l}, e^{z^l+1}, e^{z^l+2} - z^l+1 \) of degree \( d_\alpha \).

**Corollary 4.4.** In the flat coordinates \( t^1, \ldots, t^{l+2} \), the entries of the metric \( (\eta^{ij}(t)) \) have the form
\[ \eta^{ij} := \eta^{ij}(t) = \begin{cases} \eta^{ii}(y), & j = i^*, \\ 0, & j \neq i^*, \end{cases} \]
for \( i, j = 1, \ldots, l+2 \). Especially,
\[ \eta^{i,l+1}(t) = \delta_i^{l+1}, \quad \eta^{i,l+2}(t) = \delta_i^k, \quad i = 1, \ldots, l+2. \]

The entries of the matrix \( (g^{ij}(t)) \) and the Christoffel symbols \( \Gamma^{ij}_{kl}(t) \) are weighted homogeneous polynomials of \( t^1, \ldots, t^l, e^{t^{l+1}}, e^{t^{l+2}-t^l+1} \) of degrees \( d_i + d_j \) and \( d_i + d_j - d_k \) respectively. In particular,
\[ g^{\alpha,l+2}(t) = d_\alpha t^\alpha, \quad 1 \leq \alpha \leq l, \quad g^{l+1,l+1}(t) = \frac{l-k+1}{l-k}, \]
\[ g^{l+1,l+2}(t) = \frac{1}{l-k}, \quad g^{l+2,l+2}(t) = \frac{l}{k(l-k)}, \] (4.16)
and
\[ \Gamma^{l+2,i}_{j}(t) = d_j \delta^i_j, \quad 1 \leq i, j \leq l+2 \] (4.17)
and
\[ g^{k+1,l+1}(t) = t^k + t^{k+1} + \eta^k(t) \] (4.18)
for certain weighted homogenous polynomial \( \eta^k(t) \) in \( t^1, \ldots, t^{k-1}, t^{k+2}, \ldots, t^l, e^{t^{l+1}}, e^{t^{l+2}-t^l+1} \) of degree 1.

**Proof.** In the flat coordinates \( t^1, \ldots, t^{l+2} \), using (4.15) and Theorem 4.3, we have
\[ \frac{\partial}{\partial t^k} = \frac{\partial}{\partial z^k}, \quad \eta^{ij}(t) = \frac{\partial g^{ij}(t)}{\partial t^k}. \]
Thus the first statement of this corollary follows from the fact that the linear part of $t^\alpha$ is $z^\alpha$.

By definition, we easily obtain (4.16). The identity (4.18) follows from (4.5) and (4.13). It remains to prove (4.17). Notice that

$$t^{l+2} = 2\pi i (x_{l+1} + x_{l+2})$$

and

$$l^{l+2} \sum_{j=1}^{l+2} \Gamma_{j}^{l+2, i}(t) dt^j
= \sum_{p, q, r=1}^{l+2} \partial t^{l+2} \frac{\partial^2 t^i}{\partial x_p \partial x_q \partial x_r} (dx_p, dx_q)^{i} dx_r$$

$$= \sum_{p, q=1}^{l+2} \partial t^{l+2} \frac{\partial^2 t^i}{\partial x_p \partial x_q \partial x_r} (dx_p, dx_q)^{i} dx_r$$

$$= -4\pi^2 \sum_{p=1}^{l+2} [d_{i,k}(dx_p, dx_{l+1})^{i} + d_{i,k+1}(dx_p, dx_{l+2})^{i}] dt^i$$

$$= \left( \frac{d_{i,k}}{k} + \frac{d_{i,k+1}}{l-k} \right) dt^i = d_i dt^i.$$ 

This completes the proof of the corollary. \(\Box\)

4.3. Frobenius manifold structures on the orbit space \(\mathcal{M}\). Now we are ready to describe the Frobenius manifold structures on the orbit space of the extended affine Weyl group \(\tilde{W}^{(k,k+1)}(A_l)\).

**Theorem 4.5.** For any fixed integer \(1 \leq k < l\), there exists a unique Frobenius manifold structure of charge \(d = 1\) on the orbit space \(\mathcal{M}\) of \(\tilde{W}^{(k,k+1)}(A_l)\) such that the potential \(F(t) = \hat{F}(t) + \frac{1}{2}(t^{k+1})^2 \log(t^{k+1})\), where \(\hat{F}(t)\) is a weighted homogeneous polynomial in \(t^1, t^2, \ldots, t^{l+2}, e^{t^{l+1}}, e^{t^{l+2}-t^{l+1}}\), satisfying

1. the unity vector field \(e\) coincides with \(\frac{\partial}{\partial t^k}\);
2. the Euler vector field has the form

\[
E = \sum_{\alpha=1}^{l} d^\alpha_t t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{l-k} \frac{\partial}{\partial t^{l+1}} + \frac{l}{k(l-k)} \frac{\partial}{\partial t^{l+2}},
\]

where \(d_1, \ldots, d_l\) are defined in (2.13);
3. the invariant flat metric and the intersection form of the Frobenius manifold structure coincide respectively with the metrics \(\eta^{ij}\) and \(g^{ij}(t)\) on \(\mathcal{M}\).

**Proof.** The idea of the proof is similar to that of [12], i.e., using the theory of flat pencils of metrics ([10]).

Let \(\Gamma^{\alpha\beta}_\gamma(t)\) be the coefficients of the Levi-Civita connection for the metric \((\ , \ )^{\sim}\) in the coordinates \(t^1, \ldots, t^{l+2}\). According to Proposition D.1 of [10] one can represent these functions as

\[
\Gamma^{\alpha\beta}_\gamma(t) = \eta^{\alpha\epsilon} \partial_\epsilon \partial_\gamma f^\beta(t)
\]

(4.20)
for some functions $f^\beta(t)$. From the weighted homogeneity of $\Gamma^{\alpha\beta}_\gamma(t)$ and Corollary 4.4, one has

$$\partial_\alpha \partial_\gamma \left( L_E f^\beta - (1 + d_\beta) f^\beta \right) = 0$$

for any $\alpha, \beta$. So

$$L_E f^\beta(t) = (d_\beta + 1) f^\beta(t) + A_\beta^\gamma t^\gamma + B^\beta$$

(4.21)

for some constants $A_\beta^\gamma, B^\beta$. Doing a transformation

$$f^\beta(t) \mapsto \tilde{f}^\beta(t) = f^\beta(t) + R_\gamma^\beta t^\gamma + Q^\beta$$

all the coefficients $A_\beta^\gamma, B^\beta$ in (4.21) can be killed except $A_{k+\nu}^l + \tau$, for $\nu, \tau = 0, 1$. Indeed, after the transformation,

$$L_E \tilde{f}^\beta(t) = (d_\beta + 1) \tilde{f}^\beta(t) + \sum_{\gamma=1}^{l} \left[ R_\gamma^\beta(d_\gamma - 1 - d_\beta) + A_\gamma^\beta \right] t^\gamma$$

$$+ \left[ A_{l+1}^\beta - (1 + d_\beta) R_{l+1}^\beta \right] t^{l+1} + \left[ A_{l+2}^\beta - (1 + d_\beta) R_{l+2}^\beta \right] t^{l+2}$$

$$+ \frac{1}{l-k} R_{l+1}^\beta + \frac{l}{k(l-k)} R_{l+2}^\beta + B^\beta - (1 + d_\beta) Q^\beta$$

The function $\tilde{f}^\beta(t)$ does still satisfy (4.20). Choosing

$$R_{l+1}^\beta = \frac{1}{1 + d_\beta} A_{l+1}^\beta, \quad R_{l+2}^\beta = \frac{1}{1 + d_\beta} A_{l+2}^\beta, \quad Q^\beta = \frac{1}{1 + d_\beta} \left[ B^\beta + \frac{1}{l-k} R_{l+1}^\beta + \frac{l}{k(l-k)} R_{l+2}^\beta \right],$$

one kills the constant term in the r.h.s. of (4.21) and the term linear in $t^{l+1}$ and $t^{l+2}$. In order to kill other linear terms, putting

$$R_\gamma^\beta = \frac{1}{d_\beta + d_{\gamma^*}} A_\gamma^\beta,$$

where $\gamma^*$ is the index dual to $\gamma$ in the sense of duality defined in the above subsection. We can do this unless $d_\beta = d_{\gamma^*} = 0$. The last equation holds only for

$$\beta = l + 1 + \tau, \quad \gamma = k + \nu, \quad \nu, \tau = 0, 1.$$

So all linear terms can be killed except $A_{k+\nu}^{l+1+\tau}$ for $\nu, \tau = 0, 1$ in (4.21). Thus for $\beta \neq l + 1, l + 2$ the polynomials $f^\beta(t)$ can be assumed to be homogeneous of the degree $d_\beta + 1$.

Next we want to show that for $1 \leq \beta \leq l$ the functions $f^\beta(t)$ are polynomials in $t^1, \ldots, t^l, e^{l+1}, e^{l+2-l+1}$. We already know that this is true for the Christoffel
coefficients $\Gamma^{\alpha\beta}_{\gamma}(t)$. Denoting

$$\eta_{\alpha i} \Gamma^{\alpha\beta}_{\gamma}(t) = \sum_{m=0}^{M_i} \sum_{n=0}^{N_i} C^{\beta,i}_{\alpha,m,n} e^{m t^{l+1} + n (t^{l+2} - t^{l+1})} \equiv \partial_{\alpha} \partial_{t+i} f^{\beta}(t), \quad i = 1, 2,$$

where the coefficients $C^{\beta,i}_{\alpha,m,n}$ are polynomials in $t^1, \ldots, t^l$ and $M_i, N_i$ are certain positive integers. From the compatibility condition

$$\partial_{t+i} \left( \partial_{\alpha} \partial_{t+j} f^{\beta}(t) \right) = \partial_{\alpha} \left( \partial_{t+i} \partial_{t+j} f^{\beta}(t) \right), \quad i, j = 1, 2,$$

one obtains

$$\partial_{\alpha} C^{\beta,j}_{t+i,0,0} = 0, \quad \alpha = 1, \ldots, l, \quad i, j = 1, 2.$$

So $C^{\beta,j}_{t+i,0,0}$ are constants. But $\partial_{t+i} \partial_{t+j} f^{\beta}(t)$ must be a weighted homogeneous polynomial in $t^1, \ldots, t^l$, $e^{t^{l+1}}, e^{t^{l+2} - t^{l+1}}$ of the positive degree $1 + d_{\beta}$. Thus

$$C^{\beta,j}_{t+i,0,0} = 0, \quad i, j = 1, 2.$$

and

$$\partial_{\alpha} \partial_{t+i} f^{\beta}(t) = \sum_{m=1}^{M_i} \sum_{n=1}^{N_i} C^{\beta,i}_{\alpha,m,n} e^{m t^{l+1} + n (t^{l+2} - t^{l+1})}, \quad i = 1, 2. \quad (4.22)$$

From the compatibility condition

$$\partial_{t+i} \left( \partial_{t+j} f^{\beta}(t) \right) = \partial_{t+j} \left( \partial_{t+i} f^{\beta}(t) \right) = \partial_{t+j} \partial_{t+i} f^{\beta}(t),$$

for $i, j = 1, 2$, one gets $M := M_1 = M_2$ and $N := N_1 = N_2$ and

$$C^{\beta,1}_{l+1,m,n} = \frac{m-n}{m} C^{\beta,1}_{l+2,m,n} = \frac{m-n}{m} C^{\beta,2}_{l+1,m,n} = \frac{(m-n)^2}{m^2} C^{\beta,2}_{l+2,m,n}. \quad (4.23)$$

It follows from (4.22) and (4.23) that

$$f^{\beta}(t) = \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{m^2} C^{\beta,2}_{l+2,m,n} e^{m t^{l+1} + n (t^{l+2} - t^{l+1})} + t^{l+1} D^{\beta}_1 + t^{l+2} D^{\beta}_2 + D^{\beta}_0$$

for some new polynomials $D^{\beta}_i = D^{\beta}_i(t^1, \ldots, t^l), \ i = 0, 1, 2$. Since $\partial_{\alpha} \partial_{\gamma} f^{\beta}(t)$ must not contain terms linear in $t^l$ and $t^{l+2}$, these polynomials $D^{\beta}_i$ are at most linear in $t^1, \ldots, t^l$. Using the homogeneity of $f^{\beta}(t)$, so $D^{\beta}_i = 0$, $i = 1, 2$ and

$$f^{\beta}(t) = \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{m^2} C^{\beta,2}_{l+2,m,n} e^{m t^{l+1} + n (t^{l+2} - t^{l+1})} + D^{\beta}_0, \quad \beta = 1, \ldots, l.$$

The coefficients $\Gamma^{\alpha\beta}_{\gamma}(t)$ must also satisfy the conditions

$$g^{\alpha\sigma} \Gamma^{\beta\gamma}_{\sigma} = g^{\beta\sigma} \Gamma^{\alpha\gamma}_{\sigma}. \quad (4.24)$$

For $\alpha = l + 2$, it follows from (4.20), (4.24) and (4.17) that

$$\mathcal{L}_E(\eta^{\beta\gamma} \partial_{\epsilon} f^{\gamma}) = d_{\gamma} g^{\beta\gamma}. \quad (4.25)$$
Notice that the right side has no summation w.r.t the index $\gamma$. Because of $\deg f^\gamma = d_\gamma + 1$, then
\[
\deg(\eta^{\beta \gamma} \partial_\gamma f^\gamma) = d_\beta + d_\gamma, \quad \gamma \neq l + 1, l + 2.
\]
Thus,
\[
(d_\gamma + d_\beta)\eta^{\beta \gamma} \partial_\gamma f^\gamma = d_\gamma g^{\beta \gamma}, \quad \gamma \neq l + 1, l + 2. \tag{4.26}
\]
Putting
\[
F^\gamma = \frac{1}{d_\gamma} f^\gamma, \quad \gamma \neq l + 1, l + 2
\]
and using (4.26), one has
\[
\eta^{\beta \gamma} \partial_\gamma F^\gamma = \eta^{\gamma \beta} \partial_\beta F^\gamma, \quad 1 \leq \gamma, \beta \leq l. \tag{4.27}
\]
From (4.27) it follows that a function $F = F(t)$ exists such that
\[
F^\gamma = \eta^{\gamma \beta} \partial_\beta F, \quad 1 \leq \gamma \leq l. \tag{4.28}
\]
The dependence of $F$ on $t^k$ is not determined from (4.28). However, putting $\beta = l + 2$ in (4.26), one obtains
\[
\partial_k F^\gamma = t^\gamma, \quad 1 \leq \gamma \leq l. \tag{4.29}
\]
By using (4.28) and (4.29), one gets
\[
\partial_{l+2} (\partial_k F) = t^k, \quad \partial_{l+1} (\partial_k F) = t^{k+1} = \sum_{\alpha=1}^{l} \eta_{l+1, \alpha} t^\alpha,
\]
\[
\partial_\gamma (\partial_k F) = \sum_{\alpha=1}^{l} \eta_{\gamma \alpha} t^\alpha, \quad \gamma \neq k, k + 1, l + 1, l + 2.
\]
Notice that $(l + 1)^* = k + 1$, hence
\[
\partial_k F = t^k t^{l+2} + \frac{1}{2} \sum_{\alpha, \beta \neq k, l+2} \eta_{\alpha \beta} t^\alpha t^\beta + \mu(t^k)
\]
for some function $\mu(t^k)$. Shifting $F(t) \mapsto F(t) + \int \mu(t^k) dt^k$ one can kill this function, and the equations in (4.28) still hold true due to $\eta^{ik} = \delta_{i,l+2}$. So one has the representation
\[
F(t) = \frac{1}{2} (t^k)^2 t^{l+2} + \frac{1}{2} t^k \sum_{\alpha, \beta \neq k, l+2} \eta_{\alpha \beta} t^\alpha t^\beta + G(t) \tag{4.30}
\]
with some $G(t)$ independent on $t^k$.

From the definition (4.28) of $F$ and the weighted homogeneity of $f^\gamma, \gamma \neq l + 1, l + 2$, it follows that
\[
\mathcal{L}_E F(t) = 2F(t) + \rho(t^k, t^{k+1}) \tag{4.31}
\]
for certain unknown function \( \rho(t^k, t^{k+1}) \). By using (4.30) and the duality condition of the degrees, one has
\[
\mathcal{L}_EF(t) = (t^k)^2 t^{l+2} + t^k \sum_{\alpha, \beta \neq k, l+2} \eta_{\alpha \beta} t^\alpha t^\beta \\
+ \mathcal{L}_EG(t) + \frac{l}{2k(l-k)} (t^k)^2 + \frac{1}{l-k} t^k t^{k+1}.
\] (4.32)

From (4.31) and (4.32), one gets
\[
\mathcal{L}_EG(t) = 2G(t) + \rho(t^k, t^{k+1}) - \frac{l}{2k(l-k)} (t^k)^2 - \frac{1}{l-k} t^k t^{k+1}.
\]

But \( \mathcal{L}_EG(t) \) does not depend on \( t^k \), thus there exists an unknown function \( \varphi(t^{k+1}) \) with \( \deg \varphi(t^{k+1}) = 2 \), which does not depend on \( t^k \), such that
\[
\varphi(t^{k+1}) = s\rho(t^k, t^{k+1}) - \frac{l}{2k(l-k)} (t^k)^2 - \frac{1}{l-k} t^k t^{k+1}
\]
and
\[
\mathcal{L}_EG(t) = 2G(t) + \varphi(t^{k+1}) + c_0,
\]
for a constant \( c_0 \). Killing the constant by a shift, it follows that
\[
\mathcal{L}_EG(t) = 2G(t) + \varphi(t^{k+1}),
\] (4.33)
and \( G(t) = G(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^{l+1}, e^{l+1}, e^{l+2}-t^{l+1}) \) is a weighted homogeneous function of the degree 2. The above conditions determine this function uniquely. By integrating polynomials, it yields that \( G \) is a polynomial in \( t^1, \ldots, t^{k-1}, t^{k+2}, \ldots, t^{l+1}, e^{l+1}, e^{l+2}-t^{l+1} \). But because of the existence of \( \varphi(t^{k+1}) \), \( G \) may have some terms which are not polynomials in \( t^{k+1} \).

Substituting \( F(t) \) into (4.25), one gets
\[
\eta_{\alpha \beta} = \mathcal{L}_EF_{\alpha \beta}, \quad F_{\alpha \beta} = \eta_{\alpha \epsilon} \eta_{\delta \beta} \partial_\epsilon \partial_\delta F.
\] (4.34)

For \( \alpha = l + 1, \beta = l + 2 \), the equation (4.33) reads
\[
\mathcal{L}_E \frac{\partial^2 F(t)}{\partial t^k \partial t^{k+1}} = \frac{1}{l-k}.
\] (4.35)

For \( \alpha = l + 2, \beta = l + 2 \) the equation, (4.33) reads
\[
\mathcal{L}_E \frac{\partial^2 F(t)}{\partial t^k \partial t^k} = \frac{l}{k(l-k)}.
\] (4.36)

For \( \alpha = \beta = l + 1 \), the equation (4.33) reads
\[
\mathcal{L}_E \frac{\partial^2 F(t)}{\partial t^{k+1} \partial t^{k+1}} = \frac{l - k + 1}{l - k}.
\] (4.37)
By using (4.32) and (4.33), we get
\[ L_E \frac{\partial^2 G(t)}{\partial t^{k+1} \partial t^{l+1}} = \frac{l - k + 1}{l - k} \]
and
\[ \partial^2_{k+1} \varphi(t^{k+1}) = \partial^2_{k+1} L_E G(t) - 2 \partial^2_{k+1} G(t) = \frac{l - k + 1}{l - k}. \tag{4.38} \]
Notice that \( \deg \varphi = 2 \) and integrating (4.38) to obtain
\[ \varphi(t^{k+1}) = \frac{l - k + 1}{2(l - k)} (t^{k+1})^2. \tag{4.39} \]

By using (4.28) and (4.30), one can write
\[ G(t) = H_0(t) + \psi(t^{k+1}) + a_0 t^{l+1} (t^{k+1})^2, \quad \deg \psi(t^{k+1}) = 2. \tag{4.40} \]
Here \( a_0 \) is a constant and \( H_0(t) \) is a weighted homogeneous polynomial in \( t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^{l+1}, e^{t^{k+1}}, e^{t^{l+1}} \) of degree 2 with \( L_E H_0(t) = 2 H_0(t) \). In particular, \( H_0(t) \) is at most linear in \( t^{k+1} \). By substituting (4.39) and (4.40) into (4.33), one has
\[ t^{k+1} \frac{\partial \psi(t^{k+1})}{\partial t^{k+1}} = 2 \psi(t^{k+1}) + \left( \frac{k + 1}{2k} - \frac{a_0}{k} \right) (t^{k+1})^2. \tag{4.41} \]
The solutions of (4.41) are
\[ \psi(t^{k+1}) = c_1 (t^{k+1})^2 + \left( \frac{l - k + 1}{2(l - k)} - \frac{a_0}{l - k} \right) (t^{k+1})^2 \log(t^{k+1}), \tag{4.42} \]
where \( c_1 \) is an integral constant.

For \( \alpha = k + 1, \beta = l + 1 \), the equation (4.34) reads
\[ g^{k+1,l+1}(t) = L_E \frac{\partial^2 F(t)}{\partial t^{k+1} \partial t^{l+1}} = t^k + 2a_0 t^{k+1} + L_E \frac{\partial^2 H_0(t)}{\partial t^{k+1} \partial t^{l+1}}. \tag{4.43} \]
By comparing (4.18) with (4.43), one gets
\[ a_0 = \frac{1}{2}, \quad L_E \frac{\partial^2 H_0(t)}{\partial t^{k+1} \partial t^{l+1}} = g_0(t). \]
Thus,
\[ G(t) = H(t) + \frac{1}{2} (t^{k+1})^2 \log(t^{k+1}), \]
where \( H(t) = H_0(t) + c_1 (t^{k+1})^2 + \frac{1}{2} (t^{k+1})^2 t^{l+1} \).

In a word, we show that the existence of a unique weighted homogeneous polynomial
\[ H(t) := H(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^{l+1}, e^{t^{k+1}}, e^{t^{l+1}}, e^{t^{k+1} - t^{l+1}}) \]
of degree 2 such that the function
\[ F = \frac{1}{2} (t^k)^2 t^{l+2} + \frac{1}{2} \sum_{\alpha, \beta \neq k, l+2} \eta_{\alpha, \beta} t^\alpha t^\beta + \frac{1}{2} (t^{k+1})^2 \log(t^{k+1}) + H(t) \tag{4.44} \]
satisfies the equations
\[ g^{\alpha\beta} = \mathcal{L}_E F^{\alpha\beta}, \quad \Gamma_{\gamma}^{\alpha\beta} = d_\beta c_\gamma^{\alpha\beta}, \quad \alpha, \beta, \gamma = 1, \ldots, l + 2, \tag{4.45} \]
where \( c_\gamma^{\alpha\beta} = \frac{\partial F^{\alpha\beta}}{\partial t^\gamma}. \)

Obviously, the function \( F \) satisfies the equations
\[ \frac{\partial^3 F}{\partial t^k \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \ldots, l + 2 \tag{4.46} \]
and the quasi-homogeneity condition
\[ \mathcal{L}_E F = 2F + \frac{l}{2k(l - k)} (t^k)^2 + \frac{1}{l - k} t^k t^{k+1} + \frac{l - k + 1}{2(l - k)} (t^{k+1})^2. \tag{4.47} \]

From the properties of a flat pencil of metrics [10], it follows that \( F \) also satisfies associativity equations of WDVV
\[ c_{mp}^{ij} c_{q}^{mj} = c_{m}^{ip} c_{q}^{mj} \tag{4.48} \]
for any set of fixed indices \( i, j, p, q \). Now the theorem follows from above properties of the function \( F \) and the simple identity \( \mathcal{L}_E e = -e \). This completes the proof of the theorem. \( \square \)

4.4. Examples. We end this section by giving some examples to illustrate the above construction. For the brevity, instead of \( t^1, \ldots, t^{l+2} \) we will denote the flat coordinates of the metric \( \eta^{ij} \) by \( t_1, \ldots, t_{l+2} \), and also denote \( \partial_i = \frac{\partial}{\partial t_i} \) in this subsection.

**Example 4.6.** Let \( \tilde{W} \) be the extended affine Weyl group \( \tilde{W}^{(1,2)}(A_2) \), then
\[ d_{1,1} = \frac{2}{3}, \quad d_{1,2} = d_{2,1} = \frac{1}{3}, \quad d_{2,2} = \frac{2}{3} \]
and
\[ y^1 = e^{\frac{2\pi i}{3}} (2x_3 + x_4) (e^{2\pi i x_1} + e^{2\pi i x_2} + e^{-2\pi i (x_1 + x_2)}), \]
\[ y^2 = e^{\frac{2\pi i}{3}} (x_3 + 2x_4) (e^{-2\pi i x_1} + e^{-2\pi i x_2} + e^{2\pi i (x_1 + x_2)}), \]
\[ y^3 = 2i\pi x_3, \quad y^4 = 2i\pi x_4. \]
The metric \( (\ , \ )^- \) has the form
\[
(dx_a, dx_b)^- = \frac{1}{4\pi^2} \begin{pmatrix}
\frac{2}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{2}{3} & 0 & 0 \\
0 & 0 & -2 & 1 \\
0 & 0 & 1 & -2
\end{pmatrix}.
\]
We thus have
\[
(g^{ij}(y)) = \left( \sum_{a,b=1}^{4} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_b} (dx_a, dx_b)^* \right) = \begin{pmatrix}
2y^2e^{y^3} & 3e^{y^3+y^4} & y^1 & 0 \\
3e^{y^3+y^4} & 2y^1e^{y^4} & 0 & y^2 \\
y^1 & 0 & 2 & -1 \\
0 & y^2 & -1 & 2 \\
\end{pmatrix}
\]
and
\[
(\eta^{ij}(y)) = (L_e g^{ij}(y)) \begin{pmatrix}
2e^{y^3} & 0 & 1 & 0 \\
0 & 2e^{y^4} & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad e = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}.
\]

The flat coordinates for the metric \((\eta^{ij}(y))\) are
\[
t_1 = y^1 - e^{y^3}, \quad t_2 = -y^1 + y^2 + e^{y^3} - e^{y^4}, \quad t_3 = y^4, \quad t_4 = y^3 + y^4.
\]
The potential has the expression
\[
F = \frac{1}{2} t_1^2 t_4 + t_1 t_2 t_3 + \frac{1}{2} t_2^2 t_3 + e^{t_4} - t_2 e^{t_3} + t_2 e^{t_3-t_4} + \frac{1}{2} t_2^2 \log t_2
\]
and the unit vector field is
\[
e = \partial_1
\]
and the Euler vector field is given by
\[
E = t_1 \partial_1 + t_2 \partial_2 + \partial_3 + 2 \partial_4.
\]

Example 4.7. Let \(\widetilde{W}\) be the extended affine Weyl group \(\widetilde{W}^{(1,2)}(A_3)\), then
\[
d_{1,1} = \frac{3}{4}, \quad d_{1,2} = \frac{1}{2}, \quad d_{1,3} = \frac{1}{4}, \quad d_{2,2} = 1, \quad d_{2,3} = \frac{1}{2}, \quad d_{3,3} = \frac{3}{4}
\]
and
\[
y^1 = e^{\frac{\pi}{4} (3x_4 + 2x_5)} \sum_{j=1}^{4} \xi_a, \quad y^2 = e^{\pi i (x_4 + 2x_5)} \sum_{1 \leq a < b \leq 4} \xi_a \xi_b, \\
y^3 = e^{\frac{\pi}{4} (x_4 + 2x_5)} \sum_{1 \leq a < b < c \leq 4} \xi_a \xi_b \xi_c, \quad y^4 = 2i\pi x_4, \quad y^5 = 2i\pi x_5,
\]
where \( \xi_j = e^{2\pi i (x_j - x_{j-1})} \) for \( j = 1, 2, 3 \) and \( x_0 = 0 \), \( \xi_4 = e^{-2\pi i x_3} \). The metric \((\cdot, \cdot)^\sim\)
has the form

\[
((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2}
\begin{pmatrix}
\frac{3}{4} & \frac{1}{2} & \frac{3}{4} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & 0 \\
0 & 0 & 0 & -2 & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & -\frac{3}{2}
\end{pmatrix}.
\]

We thus have

\[
(g^{ij}(y)) =
\begin{pmatrix}
2y^2 e^{y^4} & 3y^3 e^{y^4+y^5} & 4 e^{y^4+y^5} & y^1 & 0 \\
3y^3 e^{y^4+y^5} & 2y^1 y^3 e^{y^5} + 4 e^{2y^5+y^4} & 3y^1 e^{y^5} & 0 & y^2 \\
4 e^{y^4+y^5} & 3y^1 e^{y^5} & -\frac{1}{2} (y^3)^2 + 2y^2 & 0 & \frac{1}{2} y^3 \\
y^1 & 0 & 0 & 2 & -1 \\
0 & y^2 & \frac{1}{2} y^3 & -1 & \frac{3}{2}
\end{pmatrix}
\]

and

\[
(\eta^{ij}(y)) = \left( \mathcal{L}_e g^{ij}(y) \right) =
\begin{pmatrix}
2 e^{y^4} & 0 & 0 & 0 & 1 \\
0 & 2y^3 e^{y^5} & 3 e^{y^5} & 0 & 1 \\
0 & 3 e^{y^5} & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad e = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}.
\]

To write down the flat coordinates, we first introduce the following variables

\[
z^1 = y^1, \quad z^2 = y^2 - y^1, \quad z^3 = y^3, \quad z^4 = y^5, \quad z^5 = y^4 + y^5,
\]

and obtain

\[
(\eta^{ij}(z)) =
\begin{pmatrix}
2 e^{z^5-z^4} & -2 e^{z^5-z^4} & 0 & 0 & 1 \\
-2 e^{z^5-z^4} & 2 e^{z^5-z^4} + 2 z^3 e^{z^4} & 3 e^{z^4} & 1 & 0 \\
0 & 3 e^{z^4} & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

then the flat coordinates for the metric \((\eta^{ij}(z))\) are given by

\[
t_1 = z^1 - e^{z^5-z^4}, \quad t_2 = z^2 - z^3 e^{z^4} + e^{z^5-z^4} + e^{2z^4},
\]
\[
t_3 = z^3 - e^{z^4}, \quad t_4 = z^4, \quad t_5 = z^5.
\]
The potential has the expression

\[ F = \frac{1}{2} t_1^2 t_5 + \frac{1}{2} t_1 t_2 t_4 + \frac{1}{2} t_2^2 t_4 + \frac{1}{4} t_3^2 t_2 + \frac{1}{4} t_4^2 t_1 - \frac{1}{96} t_3^4 + t_3 e^{t_5} - t_2 t_3 e^{t_4} + t_2 e^{t_5-t_4} + \frac{1}{2} t_2 e^{2 t_4} + \frac{1}{2} t_2^2 \log(t_2) \]

and the unit vector field is

\[ e = \partial_1 \]

and the Euler vector field is given by

\[ E = t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2} t_3 \partial_3 + \frac{1}{2} \partial_4 + \frac{3}{2} \partial_5. \]

**Example 4.8.** Let \( \widetilde{W} \) be the extended affine Weyl group \( \widetilde{W}^{(2,3)}(A_3) \), then

\[ d_{1,1} = \frac{3}{4}, \quad d_{1,2} = \frac{1}{2}, \quad d_{1,3} = \frac{1}{4}, \quad d_{2,2} = 1, \quad d_{2,3} = \frac{1}{2}, \quad d_{3,3} = \frac{3}{4} \]

and

\[ y_1 = e^{\pi i (2x_1 + x_5)} \sum_{j=1}^{4} \xi_j, \quad y_2 = e^{\pi i (2x_4 + x_5)} \sum_{1 \leq a < b \leq 4} \xi_a \xi_b, \]

\[ y_3 = e^{\pi i (2x_1 + 3x_5)} \sum_{1 \leq a < b < c \leq 4} \xi_a \xi_b \xi_c, \quad y_4 = 2 i \pi x_4, \quad y_5 = 2 i \pi x_5, \]

where \( \xi_j = e^{2 \pi i (x_j - x_{j-1})} \) for \( j = 1, 2, 3 \) and \( x_0 = 0 \), \( \xi_4 = e^{-2 \pi i x_3} \). The metric \(( , )^{-}\) has the form

\[ (d x_i, d x_j)^{-} = \frac{1}{4 \pi^2} \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}. \]

We thus have

\[ (g^{ij}(y)) = \begin{pmatrix} -\frac{1}{2} (y_1)^2 + 2 y_2^2 & 3 y_3 e^{y_4} & 4 e^{y_4} + y_5 & \frac{1}{2} y_1 & 0 \\ 3 y_3 e^{y_4} & 2 y_1 y_3 e^{y_4} + 4 e^{y_4} + 2 y_4^2 & 3 y_1 e^{y_4} + y_5 & y_2 & 0 \\ 4 e^{y_4} + y_5 & 3 y_1 e^{y_4} + y_5 & 2 y_2 e^{y_5} & 0 & y_3 \\ \frac{1}{2} y_1 & y_2 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & y_3 & -1 & 2 \end{pmatrix} \]
and

\[
(\eta^{ij}(y)) = (L_e g^{ij}(y)) = \begin{pmatrix}
2 & 3 e^{y^4} & 0 & 0 & 0 \\
3 e^{y^4} & 2 y^1 e^{y^4} & 0 & 1 & 0 \\
0 & 0 & 2 e^{y^5} & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
e = \frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^3}.
\]

To write down the flat coordinates, we first introduce the following variables

\[
z^1 = y^1, \quad z^2 = y^2, \quad z^3 = y^3 - y^2, \quad z^4 = y^5, \quad z^5 = y^4 + y^5,
\]

and obtain

\[
(\eta^{ij}(z)) = \begin{pmatrix}
2 & 3 e^{z^5-z^4} & -3 e^{z^5-z^4} & 0 & 0 \\
3 e^{z^5-z^4} & 2 e^{z^5-z^4} z^1 & -2 z^1 e^{z^5-z^4} & 0 & 1 \\
-3 e^{z^5-z^4} & -2 z^1 e^{z^5-z^4} & 2 z^1 e^{z^5-z^4} + 2 e^{z^4} & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

then the flat coordinates for the metric \((\eta^{ij}(z))\) are given by

\[
t_1 = z^1 - e^{z^5-z^4}, \quad t_2 = z^2 - z^1 e^{z^5-z^4} + e^{2z^5-2z^4},
\]
\[
t_3 = z^3 - e^{z^4} + z^1 e^{z^5-z^4} - e^{2(z^5-z^4)}, \quad t_4 = z^4, \quad t_5 = z^5.
\]

The potential has the expression

\[
F = \frac{1}{2} t_2^2 t_5 + t_2 t_3 t_4 + \frac{1}{2} t_3^2 t_4 + \frac{1}{4} t_1^2 t_2 - \frac{1}{96} t_1^4
\]
\[
+ t_1 e^{t_5} - t_3 e^{t_4} + t_1 t_3 e^{t_5-t_4} - \frac{1}{2} e^{2t_5-2t_4} t_3 + \frac{1}{2} t_3^2 \log(t_3)
\]

and the unit vector field is

\[
e = \partial_1
\]

and the Euler vector field is given by

\[
E = \frac{1}{2} t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3 + \partial_4 + \frac{3}{2} \partial_5.
\]
5. The group $\tilde{W}^{(k,k+1)}(A_l)$ and the Hurwitz space $\mathbb{M}_{k,l-k+1,1}$

In this section we want to show that the space $\mathbb{M}_{k,l-k+1,1}$ as a Frobenius manifold is isomorphic to the orbit space $\mathcal{M}$ of the extended affine Weyl group $\tilde{W}^{(k,k+1)}(A_l)$, where $\mathcal{M} = \mathcal{M}^{(k,k+1)} \setminus \{ \tilde{g}_{l+1} = 0 \} \cup \{ \tilde{g}_{l+2} = 0 \}$ for $1 \leq k < l$.

Let $\mathbb{M}_{k,l-k+1,1}$ be the space of a particular class of LG superpotentials consisting of trigonometric-Laurent series of one variable with tri-degree $(k+1,l-k,1)$, these being functions of the form

$$\lambda(\varphi) = (e^{i\varphi} - a_{l+2})^{-1}(e^{i(k+1)\varphi} + a_1 e^{ik\varphi} + \cdots + a_{l+1} e^{i(l-k)\varphi}), \quad a_{l+1}a_{l+2} \neq 0, \quad (5.1)$$

where $a_j \in \mathbb{C}$ for $j = 1, \ldots, l+2$. For brevity, we denote $m = l - k$ in this section. According to [10], the space $\mathbb{M}_{k,m+1,1}$ is a simple Hurwitz space and carries a natural structure of Frobenius manifold. The invariant inner product $\eta$ and the intersection form $g$ of two vectors $\partial'$, $\partial''$ tangent to $\mathbb{M}_{k,m+1,1}$ at a point $\lambda(\varphi)$ can be defined by the following formulae

$$\tilde{\eta}(\partial', \partial'') = (-1)^{k+1} \sum_{|\lambda| < \infty} \res_{d\lambda = 0} \partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi) \frac{d\lambda(\varphi)}{d\lambda(\varphi)}, \quad (5.2)$$

and

$$\tilde{g}(\partial', \partial'') = - \sum_{|\lambda| < \infty} \res_{d\lambda = 0} \partial'(\log(\lambda(\varphi)d\varphi))\partial''(\log(\lambda(\varphi)d\varphi)) \frac{d\log(\lambda(\varphi))}{d\lambda(\varphi)}. \quad (5.3)$$

In these formulae, the derivatives $\partial'(\lambda(\varphi)d\varphi)$ etc. are to be calculated keeping $\varphi$ fixed. The formulae (5.2) and (5.3) uniquely determine multiplication of tangent vectors on $\mathbb{M}_{k,m+1,1}$ assuming that the Euler vector field $E$ has the form

$$E = \sum_{j=1}^{l+1} \frac{j}{k}a_j \frac{\partial}{\partial a_j} + \frac{1}{k}a_{l+2} \frac{\partial}{\partial a_{l+2}}. \quad (5.4)$$

For any tangent vectors $\partial'$, $\partial''$ and $\partial'''$ to $\mathbb{M}_{k,m+1,1}$, one has

$$c(\partial', \partial'', \partial''') = \tilde{\eta}(\partial' \cdot \partial'', \partial''') = - \sum_{|\lambda| < \infty} \res_{d\lambda = 0} \partial'(\lambda(\varphi)d\varphi)\partial''(\lambda(\varphi)d\varphi)\partial'''(\lambda(\varphi)d\varphi) \frac{d\lambda(\varphi)d\varphi}{d\lambda(\varphi)}. \quad (5.5)$$

The canonical coordinates $u_1, \cdots, u_{l+2}$ for this multiplication are the critical values of $\lambda(\varphi)$ and

$$\partial_{u_\alpha} \cdot \partial_{u_\beta} = \delta_{\alpha\beta} \partial_{u_\alpha}, \quad \text{where} \quad \partial_{u_\alpha} = \frac{\partial}{\partial u_\alpha}. \quad (5.6)$$

We start with factorizing

$$\lambda(\varphi) = (e^{i\varphi} - e^{i\varphi_{l+2}})^{-1} e^{-i m \varphi} \prod_{b=1}^{l+1} (e^{i\varphi} - e^{i\varphi_b}). \quad (5.7)$$
Theorem 5.1. Let $\bar{f} : \mathcal{M} \to \mathbb{M}_{k,m+1,1}$ be induced by the map

\[
(x_1, \ldots, x_{l+2}) \mapsto (\varphi_1, \ldots, \varphi_{l+2})
\]

(5.8)

with

\[
\varphi_1 = 2\pi(\rho + x_1), \quad \varphi_j = 2\pi(\rho + x_j - x_{j+1}), \quad j = 2, \ldots, l, \\
\varphi_{l+1} = 2\pi(\rho - x_l), \quad \varphi_{l+2} = 2\pi x_{l+1},
\]

(5.9)

where $\rho = \frac{m+1}{l+1} x_{l+1} + \frac{m}{l+1} y_{l+2}$. Then $\bar{f}$ is an $m$-fold covering map, which is also a local isomorphism between the Frobenius manifolds $\mathcal{M}$ and $\mathbb{M}_{k,m+1,1}$.

**Proof.** With the use of (5.11) and (5.7), one obtains

\[
a_j = (-1)^j \sigma_j (e^{i\varphi_1}, \ldots, e^{i\varphi_{l+1}}), \quad a_{l+2} = e^{i\varphi_{l+2}}, \quad j = 1, \ldots, l + 1.
\]

From the formulae (2.2), (2.11) and (5.9), it follows

\[
a_j = \begin{cases}
(-1)^j y^j, & j = 1, \ldots, k, \\
(-1)^j y^j e^{(j-k)y^{j+1}+(j-k-1)y^{j+2}}, & j = k + 1, \ldots, l, \\
(-1)^{l+1} e^{(m+1)y^{l+1}+my^{l+2}}, & j = l + 1, \\
e^{y^{l+1}}, & j = l + 2.
\end{cases}
\]

(5.10)

Then the map $f : (\bar{y}_1, \ldots, \bar{y}_{l+2}) \mapsto (a_1, \ldots, a_{l+2})$ is given by

\[
a_j = (-1)^j \bar{y}_j, \quad j = 1, \ldots, k, \\
a_{k+s} = (-1)^k \bar{y}_{k+s-1} \bar{y}_{l+2}, \quad s = 1, \ldots, l - k, \\
a_{l+1} = (-1)^{l+1} \bar{y}_{l+1} \bar{y}_{l+2}, \quad a_{l+2} = \bar{y}_{l+1}
\]

(5.11)

and the Jacobian of $f$ is proportional to $\bar{y}_{l+1}^{-1} \bar{y}_{l+2}^{-1}$. So $f$ is an $m$-fold covering map.

Now let us proceed to prove that $f$ is a local isomorphism between the two Frobenius manifolds. By using (5.11), it is easy to check that the Euler vector fields (5.4) and (4.19) coincide. So it suffices to prove that the intersection form (5.3) coincides with the intersection form of the orbit space, and the metric (5.2) coincides with the metric (5.18).

Let us denote the roots of $\lambda'(\varphi)$ by $\psi_\gamma$, $\gamma = 1, \ldots, l + 2$, then

\[
\lambda'(\varphi) = (k + 1)i(e^{i\varphi} - e^{i\varphi_{l+2}})^{-2} e^{-i\psi_\varphi} \prod_{\gamma=1}^{l+2} (e^{i\varphi} - e^{i\psi_\gamma}).
\]

(5.12)

We define $u_\alpha = \lambda(\psi_\alpha)$, $\alpha = 1, \ldots, l + 2$, then

\[
\partial_{u_\alpha} \lambda(\varphi)|_{\varphi = \psi_\beta} = \delta_{\alpha\beta}.
\]

(5.13)
Using (5.7), (5.13) and the Lagrange interpolation formula one obtains
\[ \partial_{u_{\alpha}} \lambda(\varphi) = \frac{i e^{i\psi_{\alpha}} \lambda'(\varphi)}{(e^{i\varphi} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})}. \] (5.14)

It follows from (5.7) and (5.14) that
\[ i e^{i\psi_{\alpha}} \lambda'(\varphi) = \lambda(\varphi) \frac{i \partial_{u_{\alpha}} \varphi_{l+2} e^{i\varphi_{l+2}}}{e^{i\varphi} - e^{i\varphi_{l+2}}} - \sum_{s=1}^{l+1} \frac{i \partial_{u_{\alpha}} \varphi_{s} e^{i\psi_{s}} \lambda(\varphi)}{e^{i\varphi} - e^{i\varphi_{s}}}. \] (5.15)

Putting \( \varphi = \varphi_{b} \) in (5.15) for \( b = 1, \ldots, l + 1 \), then
\[ \partial_{u_{\alpha}} \varphi_{b} = - \frac{i e^{i\psi_{\alpha}}}{(e^{i\varphi_{b}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})}, \quad b = 1, \ldots, l + 1. \] (5.16)

Let us rewrite (5.15) as
\[ i e^{i\psi_{\alpha}} \lambda'(\varphi) = \frac{i \partial_{u_{\alpha}} \varphi_{l+2} e^{i\varphi_{l+2}}}{(e^{i\varphi} - e^{i\varphi_{l+2}}) \lambda(\varphi)} - \sum_{b=1}^{l+1} \frac{i \partial_{u_{\alpha}} \varphi_{b} e^{i\varphi_{b}}}{(e^{i\varphi} - e^{i\varphi_{b}}) \lambda(\varphi)}. \] (5.17)

Also, putting \( \varphi = \varphi_{l+2} \) in (5.17) one gets
\[ \partial_{u_{\alpha}} \varphi_{l+2} = - \frac{i e^{i\psi_{\alpha}}}{(e^{i\varphi_{l+2}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})}. \] (5.18)

We denote
\[ \varpi_{1} = x_{1}, \quad \varpi_{j} = x_{j} - x_{j-1}, \quad \varpi_{l+1} = x_{l+1}, \quad \varpi_{l+2} = x_{l+2} \] (5.19)
for \( j = 2, \ldots, l \). With the help of (5.9), (5.10) and (5.18), one has
\[ \partial_{u_{\alpha}} \varpi_{\beta} = \frac{e^{i\psi_{\alpha}}}{2\pi i (e^{i\varphi_{\beta}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})} \left[ \frac{m + 1}{l + 1} \partial_{u_{\alpha}} \varpi_{l+1} - \frac{m}{l + 1} \partial_{u_{\alpha}} \varpi_{l+2} \right], \] (5.20)
\[ \partial_{u_{\alpha}} \varpi_{l+1} = \frac{e^{i\psi_{\alpha}}}{2\pi i (e^{i\varphi_{l+2}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})}, \] (5.21)
\[ \partial_{u_{\alpha}} \varpi_{l+2} = \sum_{b=1}^{l+1} \frac{2\pi i (e^{i\varphi_{b}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})}{2\pi i (e^{i\varphi_{l+2}} - e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})} \left[ \frac{m + 1}{2\pi i (e^{i\psi_{\alpha}}) \lambda''(\psi_{\alpha})} \right]. \] (5.22)

From (5.2), (5.3) and (5.13) one gets
\[ \tilde{\eta}_{\alpha\beta} = \tilde{\eta}(\partial_{u_{\alpha}} \partial_{u_{\beta}}) = (-1)^{k+1} \frac{\delta_{\alpha\beta}}{\lambda''(\psi_{\alpha})}, \] (5.23)
and
\[ \tilde{g}_{\alpha\beta} = \tilde{g}(\partial_{u_{\alpha}} \partial_{u_{\beta}}) = - \frac{\delta_{\alpha\beta}}{u_{\alpha} \lambda''(\psi_{\alpha})}. \] (5.24)
It follows from (5.10) that the vector field \( e = \frac{\partial}{\partial y^k} + \frac{\partial}{\partial y^{k+1}} \) in the coordinates \( a_1, \ldots, a_{l+2} \) coincides with \( e = (-1)^k \left( \frac{\partial}{\partial a_k} - a_{l+2} \frac{\partial}{\partial a_{k+1}} \right) \). We shift
\[
a_k \mapsto a_k + c, \quad a_{k+1} \mapsto a_{k+1} - ca_{l+2}
\]
which produces the corresponding shift
\[
u_i \mapsto \nu_i + c, \quad i = 1, \ldots, l + 2
\]
of the critical values. This shift does not change the critical points \( \psi_\alpha \) neither the values of the second derivative \( \lambda''(\psi_\alpha) \). So
\[
\mathcal{L}_e \tilde{g}^{\alpha\beta} = \mathcal{L}_e (-u_\alpha \lambda''(\psi_\alpha) \delta_{\alpha\beta}) = (-1)^{k+1} \lambda''(\psi_\alpha) \delta_{\alpha\beta} = \tilde{\eta}^{\alpha\beta}.
\]

Finally we want to show that the bilinear form (5.3) in the coordinates \( x_1, \ldots, x_{l+2} \) coincides with the form defined in (3.19) and (3.2). We shall use the following identity
\[
\sum_{\alpha=1}^{l+2} u_\alpha e^{2i\psi_\alpha} \left( e^{i\varphi_a} - e^{i\varphi_b} \right) \lambda''(\psi_\alpha) = \sum_{\alpha=1}^{l+2} \frac{\lambda(\varphi) u_\alpha e^{2i\varphi}}{\text{res}_{\varphi=\psi_\alpha} (e^{i\varphi} - e^{i\varphi_a})(e^{i\varphi} - e^{i\varphi_b})\lambda'(\varphi)} = \begin{cases} 
\delta_{ab} - \frac{1}{k}, & 1 \leq a, b \leq l + 1, \\
-\frac{1}{k}, & 1 \leq a \leq l + 1, \ b = l + 2, \\
-1 - \frac{1}{k}, & a = b = l + 2,
\end{cases}
\quad (5.23)
\]
which follows from the explicit form of \( \lambda(\varphi) \). With the use of (5.20), (5.22) and (5.23), then
\[
(d\omega_{l+1}, d\omega_{l+1}) = \sum_{\alpha=1}^{l+2} \frac{1}{\tilde{g}_{\alpha\alpha}(u)} \partial_{u_\alpha} \omega_{l+1} \partial_{u_\alpha} \omega_{l+1} = \frac{1}{4\pi^2} \sum_{\alpha=1}^{l+2} u_\alpha e^{2i\psi_\alpha} \left( e^{i\varphi_{l+2}} - e^{i\psi_\alpha} \right)^2 \lambda''(\psi_\alpha) = -\frac{1}{4\pi^2} \frac{k + 1}{k}
\]
and
\[(d\omega_{l+1}, d\omega_{l+2}) = \sum_{\alpha=1}^{l+2} \frac{1}{g_{\alpha\alpha}(u)} \partial_{u\alpha} \omega_{l+1} \partial_{u\alpha} \omega_{l+2} \]
\[= \frac{1}{4\pi^2 m} \sum_{\alpha=1}^{l+2} \sum_{b=1}^{l+1} \frac{u_{\alpha} e^{{2i\psi}_{\alpha}}}{(e^{i\varphi_{b}} - e^{i\psi_{\alpha}})(e^{i\varphi_{l+2}} - e^{i\psi_{\alpha}})} \lambda''(\psi_{\alpha}) \]
\[- \frac{m+1}{4\pi^2 m} \sum_{\alpha=1}^{l+2} \frac{u_{\alpha} e^{{2i\psi}_{\alpha}}}{(e^{i\varphi_{l+2}} - e^{i\psi_{\alpha}})^2} \lambda''(\psi_{\alpha}) \]
\[= \frac{1}{4\pi^2 m} \sum_{b=1}^{l+1} \left( -\frac{1}{k} \right) + \frac{(m+1)(k+1)}{4\pi^2 mk} = \frac{1}{4\pi^2} \]

and
\[(d\omega_{l+2}, d\omega_{l+2}) = \sum_{\alpha=1}^{l+2} \frac{1}{g_{\alpha\alpha}(u)} \partial_{u\alpha} \omega_{l+2} \partial_{u\alpha} \omega_{l+2} \]
\[= \frac{1}{4\pi^2 m^2} \sum_{a,b=1}^{l+1} (\delta_{ab} - \frac{1}{k}) - \frac{2(m+1)}{4\pi^2 m^2} \sum_{a=1}^{l+1} \left( -\frac{1}{k} \right) + \frac{(m+1)^2}{4\pi^2 m^2} \left( -1 - \frac{1}{k} \right) \]
\[= -\frac{1}{4\pi^2} \frac{m+1}{m} \]

and for \(b = 1, \ldots, l\),
\[(d\omega_b, d\omega_{l+1}) = (d\omega_b, d\omega_{l+2}) = 0 \]

and for \(1 \leq s, b \leq l\),
\[(d\omega_a, d\omega_b) = \sum_{\alpha=1}^{l+2} \frac{1}{g_{\alpha\alpha}(u)} \partial_{u\alpha} \omega_a \partial_{u\alpha} \omega_b = \frac{1}{4\pi^2} (\delta_{ab} - \frac{1}{l+1}). \]

By using (5.19) and the above explicit forms, it is easy to verify that the intersection form (5.3) coincides with (3.1) and (3.2). This completes the proof of the theorem. \(\square\)

6. Conclusions

We have presented a new class of extended affine Weyl groups \(\tilde{W}^{(k,k+1)}(A_l)\) for \(1 \leq k < l\). On the orbit spaces \(\mathcal{M}^{(k,k+1)}(A_l) \setminus \{\tilde{y}_{l+1} = 0\} \cup \{\tilde{y}_{l+2} = 0\}\), we have shown the existence of Frobenius manifold structures and constructed LG superpotentials for these Frobenius manifold structures. Besides these, there are still some open problems deserved further study.
• Is it possible to obtain an explicit realization of the integrable hierarchies associated with the Frobenius manifolds on the orbit space of \( \tilde{W}^{(k,k+1)}(A_l) \)? Perhaps this problem is related to the works in [23] or [24] about rational reductions of the 2D-Toda hierarchy, or in [25, 26] about the finite Toda lattice of CKP type when \( k = 1 \) and \( l = 2 \).

• How about the almost dual structure of the resulting Frobenius manifold structures? ([17, 27])

• Whether the resulting Frobenius manifolds could be regarded as Frobenius submanifolds in Strachan’s sense ([15]) of certain infinite-dimensional Frobenius manifolds ([20, 21, 22]) or not?

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