Perfect transfer of \(m\)-qubit GHZ states

M. A. Jafarizadeh\(^a,b,c,d\) *, R. Sufiani\(^a,c\), S. F. Taghavi\(^a\) and E. Barati\(^a\) †

\(^a\)Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran.

\(^b\)Center of excellence for photonic, University of Tabriz, Tabriz 51664, Iran.

\(^c\)Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

\(^d\)Research Institute for Fundamental Sciences, Tabriz 51664, Iran.

May 14, 2008

*E-mail:jafarizadeh@tabrizu.ac.ir
†E-mail:sufiani@tabrizu.ac.ir
Abstract

By using some techniques such as spectral distribution and stratification associated with the graphs, employed in [1, 2] for the purpose of Perfect state transfer (PST) of a single qubit over antipodes of distance-regular spin networks and PST of a $d$-level quantum state over antipodes of pseudo-distance regular networks, PST of an $m$-qubit GHZ state is investigated. To do so, we employ the particular distance-regular networks (called Johnson networks) $J(2m, m)$ to transfer an $m$-qubit GHZ state initially prepared in an arbitrary node of the network (called the reference node) to the corresponding antipode, perfectly.

Keywords: Perfect state transference, GHZ states, Johnson network, Stratification, Spectral distribution

PACS Index: 01.55.-b, 02.10.Yn
1 Introduction

In the interior of quantum computers good communication between different parts of the system is essential. The need is thus to transfer quantum states and generate entanglement between different regions contained within the system. In quantum information processing (QIP) protocols, quantum spin systems [3, 4, 5] (or quasi-spin [6]) systems serve as quantum channels. The idea to use quantum spin chains for short distance quantum communication was put forward by Bose [7]. He showed that an array of spins (or spin like two level systems) with isotropic Heisenberg interaction is suitable for quantum state transfer. In particular, spin chains can be used as transmission lines for quantum states without the need to have controllable coupling constants between the qubits or complicated gating schemes to achieve high transfer fidelity. After the work of Bose, the use of spin chains [8]-[23] and harmonic chains [24] as quantum wires have been proposed. In the previous work [1], the so called distance-regular graphs have been considered as spin networks (in the sense that with each vertex of a distance-regular graph a qubit or a spin was associated) and perfect state transfer (PST) of a single qubit state over antipodes of these networks has been investigated, whereas in Ref.[2] the authors have been considered the PST of a $d$-level quantum state over antipodes of pseudo-distance regular graphs. In both of these works, a procedure for finding suitable coupling constants in some particular spin Hamiltonians has been given so that perfect transfer of a quantum state between antipodes of the networks can be achieved. The present work focuses on the PST of the $m$ qubit maximally entangled GHZ states $|\psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle)$.

To this end, we will consider the Johnson networks $J(2m, m)$ (which are distance-regular) as spin networks. Then, we use the algebraic properties of these networks in order to find suitable coupling constants in some particular spin Hamiltonians so that perfect transfer of $m$-qubit GHZ states between antipodes of the networks can be achieved.

The organization of the paper is as follows: In section 2, we review some preliminary facts
about graphs, their stratifications, spectral distribution associated with them and underlying networks derived from symmetric group \( S_n \) called also Johnson networks. Section 3 is devoted to perfect transfer of \( m \)-qubit GHZ states over antipodes of the networks \( J(2m, m) \), where a method for finding suitable coupling constants in particular spin Hamiltonians so that PST be possible, is given. The paper is ended with a brief conclusion.

2 Preliminaries

In this section we recall some preliminary facts about graphs, their stratifications, spectral distribution associated with them and underlying networks derived from symmetric group \( S_n \) called also Johnson networks.

2.1 Graphs and their stratifications

A graph is a pair \( \Gamma = (V, E) \), where \( V \) is a non-empty set called the vertex set and \( E \) is a subset of \( \{(x, y) : x, y \in V, x \neq y\} \) called the edge set of the graph. Two vertices \( x, y \in V \) are called adjacent if \( (x, y) \in E \), and in that case we write \( x \sim y \). For a graph \( \Gamma = (V, E) \), the adjacency matrix \( A \) is defined as

\[
(A)_{\alpha, \beta} \begin{cases} 
1 & \text{if } \alpha \sim \beta \\
0 & \text{otherwise}
\end{cases}
\]

Conversely, for a non-empty set \( V \), a graph structure is uniquely determined by such a matrix indexed by \( V \).

The degree or valency of a vertex \( x \in V \) is defined by

\[
\kappa(x) = |\{y \in V : y \sim x\}|
\]

where, \(|\cdot|\) denotes the cardinality. The graph is called regular if the degree of all of the vertices be the same. In this paper, we will assume that graphs under discussion are regular. A finite sequence \( x_0, x_1, ..., x_n \in V \) is called a walk of length \( n \) (or of \( n \) steps) if \( x_{i-1} \sim x_i \) for all
Let \( |\beta\rangle \) denote the Hilbert space of \( \mathbb{C} \)-valued square-summable functions on \( V \). With each \( \beta \in V \) we associate a vector \( |\beta\rangle \) such that the \( \beta \)-th entry of it is 1 and all of the other entries of it are zero. Then \( \{ |\beta\rangle : \beta \in V \} \) becomes a complete orthonormal basis of \( l^2(V) \). The adjacency matrix is considered as an operator acting in \( l^2(V) \) in such a way that
\[
A|\beta\rangle = \sum_{\alpha \sim \beta} |\alpha\rangle.
\]

Now, we recall the notion of stratification for a given graph \( \Gamma \). To this end, let \( \partial(x,y) \) be the length of the shortest walk connecting \( x \) and \( y \) for \( x \neq y \). By definition \( \partial(x,x) = 0 \) for all \( x \in V \). The graph becomes a metric space with the distance function \( \partial \). Note that \( \partial(x,y) = 1 \) if and only if \( x \sim y \). We fix a vertex \( o \in V \) as an origin of the graph, called the reference vertex. Then, the graph \( \Gamma \) is stratified into a disjoint union of strata (with respect to the reference vertex \( o \)) as
\[
V = \bigcup_{i=0}^{\infty} \Gamma_i(o), \quad \Gamma_i(o) := \{ \alpha \in V : \partial(\alpha, o) = i \}
\]

Note that \( \Gamma_i(o) = \emptyset \) may occur for some \( i \geq 1 \). In that case we have \( \Gamma_i(o) = \Gamma_{i+1}(o) = \ldots = \emptyset \).

With each stratum \( \Gamma_i(o) \) we associate a unit vector in \( l^2(V) \) defined by
\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle,
\]
where, \( \kappa_i = |\Gamma_i(o)| \) is called the \( i \)-th valency of the graph (\( \kappa_i := |\{ \gamma : \partial(o, \gamma) = i \}| = |\Gamma_i(o)| \)).

One should notice that, for distance regular graphs, the above stratification is independent of the choice of reference vertex and the vectors \( |\phi_i\rangle, i = 0, 1, \ldots, d - 1 \) form an orthonormal basis for the so called Krylov subspace \( K_d(|\phi_0\rangle, A) \) defined as
\[
K_d(|\phi_0\rangle, A) = \text{span}\{ |\phi_0\rangle, A|\phi_0\rangle, \ldots, A^{d-1}|\phi_0\rangle \}.
\]

Then it can be shown that [25], the orthonormal basis \( |\phi_i\rangle \) are written as
\[
|\phi_i\rangle = P_i(A)|\phi_0\rangle,
\]
where \( P_i = a_0 + a_1 A + \ldots + a_i A^i \) is a polynomial of degree \( i \) in indeterminate \( A \) (for more details see for example [25, 26]).
2.2 Spectral distribution associated with the graphs

Now, we recall some preliminary facts about spectral techniques used in the paper, where more details have been given in Refs. [26, 27, 28, 29].

Actually the spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics [30, 31]. As an example \( \mu(dx) = |\psi(x)|^2dx \)
\( (\mu(dp) = |\tilde{\psi}(p)|^2dp) \) is a spectral distribution which is assigned to the position (momentum) operator \( \hat{X}(\hat{P}) \). Moreover, in general quasi-distributions are the assigned spectral distributions of two hermitian non-commuting operators with a prescribed ordering. For example the Wigner distribution in phase space is the assigned spectral distribution for two non-commuting operators \( \hat{X} \) (shift operator) and \( \hat{P} \) (momentum operator) with Wyle-ordering among them [32, 33]. It is well known that, for any pair \( (A, |\phi_0\rangle) \) of a matrix \( A \) and a vector \( |\phi_0\rangle \), one can assign a measure \( \mu \) as follows
\[
\mu(x) = \langle \phi_0 | E(x) | \phi_0 \rangle, \quad (2-8)
\]
where \( E(x) = \sum_i |u_i\rangle\langle u_i| \) is the operator of projection onto the eigenspace of \( A \) corresponding to eigenvalue \( x \), i.e.,
\[
A = \int x E(x) dx. \quad (2-9)
\]
Then, for any polynomial \( P(A) \) we have
\[
P(A) = \int P(x) E(x) dx, \quad (2-10)
\]
where for discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2-8) and (2-10), the expectation value of powers of adjacency matrix \( A \) over reference vector \( |\phi_0\rangle \) can be written as
\[
\langle \phi_0 | A^m | \phi_0 \rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \ldots \quad (2-11)
\]
Obviously, the relation (2-11) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure \( \mu \).
From orthonormality of the unit vectors $|\phi_i\rangle$ given in Eq. (2-5) (with $|\phi_0\rangle$ as unit vector assigned to the reference node) we have

$$\delta_{ij} = \langle \phi_i | \phi_j \rangle = \int_R P_i(x)P_j(x)\mu(dx). \quad (2-12)$$

By rescaling $P_k$ as $Q_k = \sqrt{\omega_1 \ldots \omega_k}P_k$, the spectral distribution $\mu$ under question will be characterized by the property of orthonormal polynomials $\{Q_k\}$ defined recurrently by

$$Q_0(x) = 1, \quad Q_1(x) = x,$$

$$xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \omega_k Q_{k-1}(x), \quad k \geq 1. \quad (2-13)$$

The parameters $\alpha_k$ and $\omega_k$ appearing in (2-13) are defined by

$$\alpha_0 = 0, \quad \alpha_k = \kappa - b_k - c_k, \quad \omega_k \equiv \beta_k^2 = b_{k-1}c_k, \quad k = 1, \ldots, d, \quad (2-14)$$

where, $\kappa \equiv \kappa_1$ is the degree of the networks and $b_i$’s and $c_i$’s are the corresponding intersection numbers. Following Ref. [34], we will refer to the parameters $\alpha_k$ and $\omega_k$ as QD (Quantum Decomposition) parameters (see Refs. [26, 27, 28, 34] for more details). If such a spectral distribution is unique, the spectral distribution $\mu$ is determined by the identity

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \frac{1}{x-\alpha_0 - \frac{\omega_1}{x-\alpha_1 - \frac{\omega_2}{x-\alpha_2 - \cdots}}} = \frac{Q_d^{(1)}(x)}{Q_{d+1}(x)} = \sum_{l=0}^d \frac{\gamma_l}{x-x_l}, \quad (2-15)$$

where, $x_l$ are the roots of the polynomial $Q_{d+1}(x)$. $G_\mu(x)$ is called the Stieltjes/Hilbert transform of spectral distribution $\mu$ and polynomials $\{Q_k^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1, \quad Q_1^{(1)}(x) = x - \alpha_1,$$

$$xQ_k^{(1)}(x) = Q_{k+1}^{(1)}(x) + \alpha_{k+1}Q_k^{(1)}(x) + \omega_{k+1}Q_{k-1}^{(1)}(x), \quad k \geq 1, \quad (2-16)$$

respectively. The coefficients $\gamma_l$ appearing in (2-15) are calculated as

$$\gamma_l := \lim_{x \to x_l} [(x-x_l)G_\mu(x)] \quad (2-17)$$
Now let $G_\mu(z)$ is known, then the spectral distribution $\mu$ can be determined in terms of $x_l, l = 1, 2, ...$ and Gauss quadrature constants $\gamma_l, l = 1, 2, ...$ as

$$\mu = \sum_{l=0}^{d} \gamma_l \delta(x - x_l)$$  \hspace{1cm} (2-18)

(for more details see Refs. [35, 36, 37, 38]).

### 2.3 Underlying networks derived from symmetric group $S_n$

Let $\lambda = (\lambda_1, ..., \lambda_m)$ be a partition of $n$, i.e., $\lambda_1 + ... + \lambda_m = n$. We consider the subgroup $S_m \otimes S_{n-m}$ of $S_n$ with $m \leq \lfloor \frac{n}{2} \rfloor$. Then we assume the finite set $M^\lambda = \frac{S_n}{S_m \otimes S_{n-m}}$ with $|M^\lambda| = \frac{n!}{m!(n-m)!}$ as vertex set. In fact, $M^\lambda$ is the set of $(m-1)$-faces of $(n-1)$-simplex (recall that, the graph of an $(n-1)$-simplex is the complete graph with $n$ vertices denoted by $K_n$). If we denote the vertex $i$ by $m$-tuple $(i_1, i_2, ..., i_m)$, then the adjacency matrices $A_k, k = 0, 1, ..., m$ are defined as

$$\left( A_k \right)_{i,j} = \begin{cases} 1 & \text{if } \partial(i,j) = k, \\ 0 & \text{otherwise} \end{cases} \quad (i,j \in M^\lambda), \quad k = 0, 1, ..., m. \hspace{1cm} (2-19)$$

where, we mean by $\partial(i,j)$ the number of components that $i = (i_1, ..., i_m)$ and $j = (j_1, ..., j_m)$ are different (this is the same as Hamming distance which is defined in coding theory). The network with adjacency matrices defined by (2-19) is known also as the Johnson network $J(n,m)$ and has $m+1$ strata such that

$$\kappa_0 = 1, \quad \kappa_l = \binom{m}{m-l} \binom{n-m}{l}, \quad l = 1, 2, ..., m. \hspace{1cm} (2-20)$$

One should notice that for the purpose of PST, we must have $\kappa_m = 1$ which is fulfilled if $n = 2m$, so we will consider the network $J(2m,m)$ in order to transfer $m$-qubit GHZ states (hereafter we will take $n = 2m$ so that we have $\kappa_m = 1$). If we stratify the network $J(2m,m)$ with respect to a given reference node $|\phi_0\rangle = |i_1, i_2, ..., i_m\rangle$, the unit vectors $|\phi_i\rangle, i = 1, ..., m$...
Perfect Transfer of $m$-qubit GHZ states

The quantum state transfer protocol involves two steps: initialization and evolution. First, a $m$-qubit GHZ state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|00\ldots0\rangle_A + |11\ldots1\rangle_A) \in \mathcal{H}_A$ to be transmitted is created. The state of the entire system after this step is given by

$$|\psi(t = 0)\rangle = |\psi\rangle_A|00\ldots0\rangle_B = \frac{1}{\sqrt{2}}(|00\ldots0\rangle_A|00\ldots0\rangle_B + |11\ldots1\rangle_A|00\ldots0\rangle_B).$$

3 Perfect Transfer of $m$-qubit GHZ states over antipodes of the network $J(2m,m)$

The quantum state transfer protocol involves two steps: initialization and evolution. First, a $m$-qubit GHZ state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|00\ldots0\rangle_A + |11\ldots1\rangle_A) \in \mathcal{H}_A$ to be transmitted is created. The state of the entire system after this step is given by

$$|\psi(t = 0)\rangle = |\psi\rangle_A|00\ldots0\rangle_B = \frac{1}{\sqrt{2}}(|00\ldots0\rangle_A|00\ldots0\rangle_B + |11\ldots1\rangle_A|00\ldots0\rangle_B).$$
Then, the network couplings are switched on and the whole system is allowed to evolve under $U(t) = e^{-iHt}$ for a fixed time interval, say $t_0$. The final state becomes

$$|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle_A|00\ldots0\rangle_B + \sum_{j=1}^{2m} f_{jA}(t_0)|j\rangle)$$

(3-26)

where, $|j\rangle \equiv |j_1j_2\ldots j_m\rangle = |0\ldots0\rangle_{j_1} |0\ldots0\rangle_{j_2} \ldots |0\ldots0\rangle_{j_m}$ and $|A\rangle = |11\ldots100\ldots0\rangle$ so that $f_{jA}(t_0) := \langle j|e^{-iHt_0}|A\rangle$. Any site $B$ is in a mixed state if $|f_{AB}(t_0)| < 1$, which also implies that the state transfer from site $A$ to $B$ is imperfect. In this paper, we will focus only on PST. This means that we consider the condition

$$|f_{AB}(t_0)| = 1 \text{ for some } 0 < t_0 < \infty$$

(3-27)

which can be interpreted as the signature of perfect communication (or PST) between $A$ and $B$ in time $t_0$ (with $|B\rangle = |00\ldots011\ldots1\rangle$). The effect of the modulus in (3-27) is that the state at $B$, after transmission, will no longer be $|\psi\rangle$, but will be of the form

$$\frac{1}{\sqrt{2}}(|00\ldots0\rangle + e^{i\phi}|11\ldots1\rangle).$$

(3-28)

The phase factor $e^{i\phi}$ is not a problem because $\phi$ can be corrected for with an appropriate phase gate (for more details see for example [9, 16, 39, 40]).

As regards the arguments of subsection 2.2, the evolution with the adjacency matrix $H = A \equiv A_1$ for distance-regular networks starting in $|\phi_0\rangle$, always remains in the stratification space. For distance-regular network $J(2m,m)$ for which the last stratum, i.e., $|\phi_m\rangle$ contains only one site, then PST between the antipodes $|\phi_0\rangle$ and $|\phi_m\rangle$ is allowed. Thus, we can restrict our attention to the stratification space for the purpose of PST from $|\phi_0\rangle$ to $|\phi_m\rangle$.

The model we will consider is the network $J(2m,m)$ consisting of $N = C_{2m}^m = \frac{(2m)!}{m!m!m!}$ sites labeled by $\{1,2,\ldots,N\}$ and diameter $m$. Then we stratify the network with respect to a chosen reference site, say 1. At time $t = 0$, the state in the first (input) site of the network is prepared in the $m$-qubit GHZ state $|\psi_{in}\rangle$. We wish to transfer the state to the $N$th (output) site of the network with unit efficiency after a well-defined period of time. We shall assume that initially
the network is in the state $|00...0\rangle_A|00...0\rangle_B$. Then, we consider the dynamics of the system to be governed by the quantum-mechanical Hamiltonian

$$H_G = \sum_{k=1}^{m} J_k P_k (1/2 \sum_{1 \leq i < j \leq 2m} \sigma_i \cdot \sigma_j + \frac{m}{2} I), \quad (3-29)$$

where, $\sigma_i$ is a vector with familiar Pauli matrices $\sigma^x_i, \sigma^y_i$ and $\sigma^z_i$ as its components acting on the one-site Hilbert space $H_i$, and $J_m$ is the coupling strength between the reference site 1 and all of the sites belonging to the $k$-th stratum with respect to 1.

In order to use the spectral analysis methods, we write the hamiltonian (3-29) in terms of the adjacency matrix $A$ of the network $J(2m, m)$. To do so, first we note that for the Johnson network $J(n, m)$ with adjacency matrix $A$ one can show that

$$\sum_{1 \leq i < j \leq n} P_{ij} = A + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \left( \begin{array}{c} n - m \\ 2 \end{array} \right) I, \quad (3-30)$$

where $P_{ij}$ is the permutation operator acting on sites $i$ and $j$. In fact restriction of the operator $\sum_{1 \leq i < j \leq n} P_{ij}$ on the $m$-particle space (space spanned by the states with $m$ spin down) which has dimension $C^n_m$, is written as the adjacency matrix $A$ of the network as in the Eq. (3-30). For $n = 2m$, the Eq.(3-30) is written as

$$\sum_{1 \leq i < j \leq 2m} P_{ij} = A + m(m - 1)I. \quad (3-31)$$

Then, by using the fact that

$$\sigma_i \cdot \sigma_j = 2P_{ij} - I \rightarrow \frac{1}{2} \sum_{1 \leq i < j \leq 2m} \sigma_i \cdot \sigma_j = \sum_{1 \leq i < j \leq 2m} P_{ij} - \frac{1}{2} \left( \begin{array}{c} 2m \\ 2 \end{array} \right) I = A - \frac{m}{2} I,$$

we obtain

$$H_G = \sum_{k=1}^{m} J_k P_k (A). \quad (3-32)$$

Now, for the purpose of the perfect transfer of a $m$-qubit GHZ state, we impose the constraints that the amplitudes $\langle \phi_i | e^{-iHt} | \phi_0 \rangle$ be zero for all $i = 0, 1, ..., m - 1$ and $\langle \phi_m | e^{-iHt} | \phi_0 \rangle = \frac{1}{\sqrt{2^m}}$.
$e^{i\theta}$, where $\theta$ is an arbitrary phase. Therefore, these amplitudes must be evaluated. To do so, we use the stratification and spectral distribution associated with the networks $J(2m, m)$ to write

$$
\langle \phi_i | e^{-iHt} | \phi_0 \rangle = \langle \phi_i | e^{-it} \sum_{l=0}^{m} J_l P_l(A) | \phi_0 \rangle = \frac{1}{\sqrt{\kappa_i}} \langle \phi_0 | A_i e^{-it} \sum_{l=0}^{m} J_l P_l(A) | \phi_0 \rangle
$$

Let the spectral distribution of the graph is $\mu(x) = \sum_{k=0}^{m} \gamma_k \delta(x - x_k)$ (see Eq. (2-18)). The Johnson network is a kind of network with a highly regular structure that has a nice algebraic description; For example, the eigenvalues of this network can be computed exactly (see for example the notes by Chris Godsil on association schemes [41] for the details of this calculation). Indeed, the eigenvalues of the adjacency matrix of the network $J(2m, m)$ (that is $x_k$’s in $\mu(x)$) are given by

$$
x_k = m^2 - k(2m + 1 - k), \quad k = 0, 1, \ldots, m.
$$

(3-33)

Now, from the fact that for distance-regular graphs we have $A_i = \sqrt{\kappa_i} P_i(A) [27]$, $\langle \phi_i | e^{-iHt} | \phi_0 \rangle = 0$ implies that

$$
\sum_{k=0}^{m} \gamma_k P_i(x_k) e^{-it} \sum_{l=0}^{m} J_l P_l(x_k) = 0, \quad i = 0, 1, \ldots, m - 1
$$

Denoting $e^{-it} \sum_{l=0}^{m} J_l P_l(x_k)$ by $\eta_k$, the above constraints are rewritten as follows

$$
\sum_{k=0}^{m} P_i(x_k) \eta_k \gamma_k = 0, \quad i = 0, 1, \ldots, m - 1,
$$

$$
\sum_{k=0}^{m} P_m(x_k) \eta_k \gamma_k = e^{i\theta}.
$$

(3-34)

From invertibility of the matrix $P_{ik} = P_i(x_k)$ (see Ref. [2]) one can rewrite the Eq. (3-34) as

$$
\begin{pmatrix}
\eta_0 \gamma_0 \\
\eta_1 \gamma_1 \\
\vdots \\
\eta_d \gamma_d
\end{pmatrix} = P^{-1} \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} + e^{i\theta}.
$$

(3-35)

The above equation implies that $\eta_k \gamma_k$ for $k = 0, 1, \ldots, m$ are the same as the entries in the last column of the matrix $P^{-1} = WP^t$ multiplied with the phase $e^{i\theta}$, i.e., for the purpose of PST,
the following equations must be satisfied

\[ \eta_k \gamma_k = \gamma_k e^{-2it_0 \sum_{l=0}^{m} J_l P_l(x_k)} = e^{i\theta (WP^t)_{km}} \], for \( k = 0, 1, ..., m, \) \tag{3-36} \]

with \( W := \text{diag}(\gamma_0, \gamma_1, \ldots, \gamma_m). \)

One should notice that, the Eq. (3-36) can be rewritten as

\[ (J_0, J_1, \ldots, J_D) = -\frac{1}{2t_0} [\theta + (2l_0 + f(0))\pi, \theta + (2l_1 + f(1))\pi, \ldots, \theta + (2l_D + f(D))\pi] (WP^t), \] \tag{3-37} \]

or

\[ J_k = -\frac{1}{2t_0} \sum_{j=0}^{m} [\theta + (2l_j + f(j))\pi] (WP^t)_{jk}, \] \tag{3-38} \]

where \( l_k \) for \( k = 0, 1, \ldots, m \) are integers and \( f(k) \) is equal to 0 or 1 (we have used the fact that \( \gamma_k \) and \( (WP^t)_{km} \) are real for \( k = 0, 1, \ldots, m \), and so we have \( \gamma_k = |(WP^t)_{km}| \)). The result (3-38) gives an explicit formula for suitable coupling constants so that PST between the first node \( (|\phi_0\rangle) \) and the opposite one \( (|\phi_m\rangle) \) can be achieved.

In the following we consider PST of the two qubit (the case \( m = 2 \)) GHZ state \( |\psi\rangle_A = \frac{1}{2}(|00\rangle + |11\rangle) \) in details: From Eq. (2-23), for \( m = 2 \), the QD parameters are given by

\[ \alpha_1 = 2, \quad \alpha_2 = 0; \quad \omega_1 = \omega_2 = 4, \]

Then by using the recursion relations (2-13) and (2-16), we obtain

\[ Q^{(1)}_2(x) = x^2 - 2x - 4, \quad Q_3(x) = x(x - 4)(x + 2), \]

so that the stieltjes function is given by

\[ G_\mu(x) = \frac{Q^{(1)}_2(x)}{Q_3(x)} = \frac{x^2 - 2x - 4}{x(x - 4)(x + 2)}. \]

Then the corresponding spectral distribution is given by

\[ \mu(x) = \sum_{l=0}^{2} \gamma_l \delta(x - x_l) = \frac{1}{6} \{3\delta(x) + \delta(x - 4) + 2\delta(x + 2)\}, \]
which indicates that

\[
W = \begin{pmatrix}
\gamma_0 & 0 & 0 \\
0 & \gamma_1 & 0 \\
0 & 0 & \gamma_2 \\
\end{pmatrix} = \frac{1}{6} \begin{pmatrix}
3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2 \\
\end{pmatrix}.
\]

In order to obtain the suitable coupling constants, we need also the eigenvalue matrix \(P\) with entries \(P_{ij} = P_i(x_j) = \frac{1}{\sqrt{\omega_1 \cdots \omega_i}} Q_i(x_j)\). By using the recursion relations (2-13), one can obtain \(P_0(x) = 1\), \(P_1(x) = \frac{x}{2}\) and \(P_2(x) = \frac{1}{4}(x^2 - 2x - 4)\), so that

\[
P = \begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & -1 \\
-1 & 1 & 1 \\
\end{pmatrix}.
\]

Now, from the result (3-38), we obtain

\[
J_0 = -\frac{6\theta + 6\pi}{12t_0}, \quad J_1 = -\frac{\pi}{3t_0}, \quad J_2 = \frac{\pi}{12t_0}.
\]

### 4 Conclusion

By using spectral analysis methods and employing algebraic structures of Johnson networks \(J(2m, m)\) such as distance-regularity and stratification, a method for finding a suitable set of coupling constants in some particular spin Hamiltonians associated with the networks was given so that PST of \(m\)-qubit GHZ states between antipodes of the networks can be achieved.

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