Asymptotic analysis of a Cucker-Smale system with leadership and distributed delay

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Abstract

We extend the analysis developed in [33] in order to prove convergence results for a Cucker-Smale type model with hierarchical leadership and distributed delay. Flocking estimates are obtained for a general interaction potential with divergent tail. We analyze also the model when the ultimate leader can change its velocity. In this case we give a flocking result under suitable conditions on the leader’s acceleration.

1 Introduction

The celebrated Cucker-Smale model has been introduced in [14, 15] as a model for flocking, namely for phenomena where autonomous agents reach a consensus based on limited environmental information. Let us consider \( N \in \mathbb{N} \) agents and let \((x_i(t), v_i(t)) \in \mathbb{R}^{2d}, i = 1, \ldots, N,)\) be their phase-space coordinates. As usual \( x_i(t) \) denotes the position of the \( i \)th agent and \( v_i(t) \) the velocity. The Cucker-Smale model reads, for \( t > 0 \),

\[
\dot{x}_i(t) = v_i(t), \\
\dot{v}_i(t) = \sum_{j=1}^{N} \psi_{ij}(t)(v_j(t) - v_i(t)), \quad i = 1, \ldots, N, \tag{1.1}
\]

where the communication rates \( \psi_{ij}(t) \) are of the form

\[
\psi_{ij}(t) = \psi(|x_i(t) - x_j(t)|), \tag{1.2}
\]

being \( \psi : [0, +\infty) \to (0, +\infty) \) a suitable non-increasing potential functional.

Definition 1.1. We say that a solution of (1.1) converges to consensus (or flocking) if

\[
\sup_{t>0} |x_i(t) - x_j(t)| < +\infty \quad \text{and} \quad \lim_{t \to +\infty} |v_i(t) - v_j(t)| = 0, \quad \forall \ i, j = 1, \ldots, N. \tag{1.3}
\]

The potential function considered by Cucker and Smale in [14, 15] is \( \psi(s) = \frac{1}{(1+s^2)^\beta} \) with \( \beta \geq 0 \). They proved that there is unconditional convergence to flocking whenever \( \beta < 1/2 \). In the

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case $\beta \geq 1/2$, they obtained a conditional flocking result, namely convergence to flocking under appropriate assumptions on the initial data. Actually, unconditional flocking can be obtained also for $\beta = 1/2$ (see e.g. [20]).

The extension of the flocking result to cover the case of non symmetric communication rates is due to Motsch and Tadmor [30]. Other variants and generalizations have been proposed, e.g. more general interaction potentials, cone-vision constraints, leadership (see e.g. [10, 12, 21, 29, 31, 35, 37, 39]), stochastic terms (see [13, 18, 19]), pedestrian crowds (see [11, 23]), infinito-dimensional kinetic models (see [1, 2, 4, 7, 17, 22, 36]) and control models (see [3, 5, 6, 32, 38]).

Here, we consider the Cucker-Smale system with hierarchical leadership introduced by Shen [35]. In this model the agents are ordered in a specific way, depending on which other agents they are leaders of or led by. This reflects natural situations, e.g. in animals groups, where some agents are more influential than the others. We also add a distributed delay term, namely we assume that the agent $i$ adjusts its velocity depending on the information received from other agents on a time interval $[t-\tau, t]$. Indeed, it is natural to assume that there is a time delay in the information’s transmission from an agent to the others. The case of CS-model with hierarchical leadership and a pointwise time delay has been recently studied by the authors ([33]). Other models with (pointwise) time delay, without leadership, have been considered in [8, 9, 28, 34], while for other extensions of Shen’s results, without delay, we refer to [16, 21, 27, 26, 25].

In order to present our model, we first recall some definitions from [35].

**Definition 1.2.** The leader set $\mathcal{L}(i)$ of an agent $i$ in a flock $[1, 2, \ldots, N]$ is the subgroup of agents that directly influence agent $i$, i.e. $\mathcal{L}(i) = \{j \mid \psi_{ij} > 0\}$.

The Cucker-Smale system considered by Shen is then, for all $i \in \{1, 2, \ldots, N\}$ and $t > 0$,

$$
\frac{dx_i}{dt} = v_i, \\
\frac{dv_i}{dt} = \sum_{j \in \mathcal{L}(i)} \psi_{ij}(t)(v_j - v_i).
$$

The interaction potential was analogous to the one of Cucker and Smale’s papers and Shen proved convergence to consensus for $\beta < 1/2$.

**Definition 1.3.** A flock $[1, 2, \ldots, N]$ is an **HL-flock**, namely a flock under hierarchical leadership, if the agents can be ordered in such a way that:

1. if $\psi_{ij} \neq 0$ then $j < i$, and
2. for all $i > 1$, $\mathcal{L}(i) \neq \emptyset$.

**Definition 1.4.** For each agent $i = 1, \ldots, N$, we define the $m$-th level leaders of $i$ as

$$
\mathcal{L}^0(i) = \{i\}, \quad \mathcal{L}^1(i) = \mathcal{L}(i), \quad \mathcal{L}^2(i) = \mathcal{L}(\mathcal{L}(i)), \quad \ldots, \quad \mathcal{L}^m = \mathcal{L}(\mathcal{L}^{m-1}(i)), \quad \ldots
$$

for $m \in \mathbb{N}$, and denote the set of all leaders of the agent $i$, direct or indirect, as

$$
[\mathcal{L}](i) = \mathcal{L}^0(i) \cup \mathcal{L}^1(i) \cup \ldots
$$

For a fixed positive time $\tau$ and for every $t > 0$, our system is the following:
\[
\frac{dx_i(t)}{dt} = v_i(t), \\
\frac{dv_i(t)}{dt} = \sum_{j \in \mathcal{L}(i)} \mu(t-s) \psi_{ij}(s)[v_j(s) - v_i(t)] ds,
\]
for all \(i \in \{1, \ldots, N\}\), with initial conditions, for \(s \in [-\tau, 0]\),
\[
\begin{align*}
  x_i(s) &= x_i^0(s), \\
  v_i(s) &= v_i^0(s),
\end{align*}
\]
for some continuous functions \(x_i^0\) and \(v_i^0\), \(i = 1, \ldots, N\). The communication rates are
\[
\psi_{ij}(t) = \psi(|x_i(t) - x_j(t)|)
\]
for some non-increasing, nonnegative, continuous interaction potential \(\psi\). The weight function \(\mu : [0, \tau] \to \mathbb{R}\) is assumed to be bounded and nonnegative, with
\[
\int_0^\tau \mu(s) ds = \mu_0 > 0.
\]

We will prove a flocking result under the assumption
\[
\int_0^{+\infty} \psi(s) ds = +\infty.
\]

Then, our result extends and generalizes the one of Shen. Note that in [33] we have proved a flocking result in the case of a pointwise time delay. We can formally obtain the model studied in [33] if the weight \(\mu(\cdot)\) is a Dirac delta function centered at \(t = \tau\).

The paper is organized as follows. In section 2 we give some preliminary properties of system (1.5), in particular we prove the positivity and boundedness properties for the velocities. In section 3 we will prove the flocking result for the system (1.5). Finally, in section 4 we will consider the model under hierarchical leadership and a free–will leader and we will prove flocking estimates under suitable growth assumptions on the acceleration of the free–will leader.

## 2 Preliminary properties

Before proving our main result, namely the convergence to consensus theorem, we need some general properties of the Cucker-Smale model (1.5), such as the positivity property and the boundedness of the velocities. The following propositions extend analogous results of [35].

**Proposition 2.1.** Let us consider the system of scalar equations
\[
\frac{du_i(t)}{dt} = \sum_{j \in \mathcal{L}(i)} \int_{t-\tau}^t \mu(t-s) \psi_{ij}(s)[u_j(s) - u_i(t)] ds, \quad i = 1, \ldots, N, \quad t > 0,
\]
\[
u_i(s) = u_i^0(s), \quad i = 1, \ldots, N, \quad s \in [-\tau, 0],
\]
where \(u_i^0(\cdot), \ i = 1, \ldots, N\), are continuous functions. If \(u_i^0(s) \geq 0\) for all \(i = 1, \ldots, N\), and all \(s \in [-\tau, 0]\), then \(u_i(t) \geq 0\) for all \(i\) and \(t > 0\).
Using (2.2), the equation for the agent 2 becomes

\[ \frac{du_2}{dt} = 0 \] and so \[ u_2(t) = u_2(0) = u^0_2(0) \geq 0, \quad \forall t \geq 0. \] (2.2)

Using (2.2), the equation for the agent 2 becomes

\[ \frac{du_2}{dt}(t) = \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s)[u_1(s) - u_2(t)] ds = (u_1(0) - u_2(t)) \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s) ds. \]

Arguing by contradiction, we assume that \( u_2(\bar{t}) < 0 \) for some \( \bar{t} > 0 \). Then, let us denote

\[ t^* = \inf\{ t > 0 \mid u_2(s) < 0 \quad \text{for} \quad s \in (t, \bar{t}) \}. \]

Hence, by definition of \( t^* \), \( u_2(t^*) = 0 \) and \( u_2(s) < 0 \) for \( s \in (t^*, \bar{t}) \). So, using again (2.2),

\[ \frac{du_2}{dt}(t) = (u_1(0) - u_2(t)) \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s) ds \geq 0, \quad t \in [t^*, \bar{t}), \]

which is in contradiction with \( u_2(t) < 0 \) for \( t \in (t^*, \bar{t}) \) and \( u_2(t^*) = 0 \). This ensures that \( u_2(t) \geq 0 \) for all \( t \geq 0 \).

Now, as the induction hypothesis, assume that \( u_i(t) \geq 0 \) for all \( t > 0 \) and for all \( i \in \{1, \ldots, k-1\} \).

The equation for agent \( k \) is

\[ \frac{du_k}{dt}(t) = \sum_{j \in \mathcal{L}(k)} \int_{t-\tau}^{t} \mu(t-s)\psi_{kj}(s)[u_j(s) - u_k(t)] ds, \quad t > 0. \]

As in the first step, let us assume by contradiction that \( u_k(\bar{t}) < 0 \) for some \( \bar{t} > 0 \) and let us denote

\[ t^* = \inf\{ t > 0 \mid u_k(s) < 0 \quad \text{for} \quad s \in (t, \bar{t}) \}. \]

Then, \( u_k(t^*) = 0 \) and \( u_k(s) < 0 \) for \( s \in (t^*, \bar{t}) \). We can use the induction hypothesis on the agents \( j \in \mathcal{L}(k) \subseteq \{1, \ldots, k-1\} \), so

\[ \frac{du_k}{dt}(t) = \sum_{j \in \mathcal{L}(k)} \int_{t-\tau}^{t} \mu(t-s)\psi_{kj}(s)[u_j(s) - u_k(t)] ds \geq 0, \quad t \in [t^*, \bar{t}), \]

which gives a contradiction.

Therefore, we have proved that \( u_i(t) \geq 0 \) for all \( i \in \{1, \ldots, N\} \).}

As in the undelayed case (see Th. 4.2 of [35]) we can now deduce from the previous proposition the boundedness result for the velocities.
Proposition 2.2. Let $\Omega$ be a convex and compact domain in $\mathbb{R}^d$ and let $(x_i, v_i)$ be a solution of system (1.5). If $v_i(s) \in \Omega$ for all $i = 1, \ldots, N$ and $s \in [\tau, 0]$, then $v_i(t) \in \Omega$ for all $i = 1, \ldots, N$ and $t > 0$. In particular, if $\Omega$ is the ball with center 0 and radius $D_0 = \max_{1 \leq i \leq N} \max_{s \in [\tau, 0]} |v_i(s)|$, then $|v_i(t)| \leq D_0$ for all $t > 0$ and $i = 1, \ldots, N$.

3 Convergence to consensus

Here we will prove the announced flocking result for the CS-model under hierarchical leadership with distributed delay (1.5). Our proof extends to the model at hand the one in [33], with pointwise delay. We need a preliminary lemma.

Lemma 3.1. Let $(x, v)$ be a trajectory in the phase-space, namely $\frac{dx}{dt}(t) = v(t)$ for $t \geq 0$. Assume that

$$\frac{d|v|}{dt}(t) \leq -d_0|\psi(|x(t)| + M)|v(t)| + ce^{-bt} \quad \forall \; t \geq t_0, \tag{3.1}$$

for some nonnegative constants $M, c, t_0$ and $b, d_0 > 0$, where $\psi : [0, +\infty) \rightarrow (0, +\infty)$ is a continuous function satisfying (1.5). Then, there exists a suitable positive constant $C$ such that

$$|x(t)| \leq C, \quad t \geq 0.$$

Proof. Let us consider the functionals (cfr. [20, 33])

$$F_\pm(t) = |v(t)| \pm d_0\phi(|x(t)| + M), \tag{3.2}$$

where $\phi$ is a primitive of $\psi$, namely $\phi'(s) = \psi(s), s \in (0, +\infty)$.

From (3.1) we deduce

$$\frac{dF_\pm}{dt}(t) = \frac{d|v|}{dt}(t) \pm d_0\psi(|x(t)| + M)\frac{d|x|}{dt}(t)
\leq -d_0\psi(|x(t)| + M)|v(t)| \pm d_0\psi(|x(t)| + M)\frac{d|x|}{dt}(t) + ce^{-bt}$$
$$= d_0\psi(|x(t)| + M)\left(\frac{d|x|}{dt}(t) - |v(t)|\right) + ce^{-bt} \leq ce^{-bt}, \quad t \geq t_0, \tag{3.3}$$

where we have used

$$\left|\frac{d|x|}{dt}\right| \leq |v(t)|. \tag{3.4}$$

Now, integrating (3.3) on the time interval $[t_0, t]$, we obtain

$$F_\pm(t) - F_\pm(t_0) \leq c \int_{t_0}^{t} e^{-bs} ds = \frac{c}{b}(e^{-bt_0} - e^{-bt}) \leq \frac{c}{b},$$

which implies

$$|v(t)| - |v(t_0)| \leq \pm d_0 (\phi(|x(t_0)| + M) - \phi(|x(t)| + M)) + \frac{c}{b},$$

$$\left|\frac{d|x|}{dt}\right| \leq |v(t)|.$$
namely

\[ |v(t)| - |v(t_0)| \leq -d_0 \int_{|x(t_0)|+M}^{x(t)+M} \psi(s) \, ds + \frac{c}{b}. \quad (3.5) \]

In particular, from (3.5), we deduce

\[ |v(t_0)| + \frac{c}{b} \geq d_0 \int_{|x(t_0)|+M}^{x(t)+M} \psi(s) \, ds. \quad (3.6) \]

Then, assumption (1.8) ensures the existence of a constant \( x_M > 0 \) such that

\[ |v(t_0)| + \frac{c}{b} = d_0 \int_{|x(t_0)|+M}^{x(t)+M} \psi(s) \, ds, \]

which, together with (3.6), implies

\[ |x(t)| \leq C, \quad \forall \ t \geq 0, \]

being \( \psi \) is a nonnegative function. □

**Theorem 3.2.** Let \((x_i, v_i), i = 1, \ldots, N,\) be a solution of the Cucker-Smale system under hierarchical leadership with distributed delay (1.5) with initial conditions (1.6). Assume that the potential function \( \psi \) satisfies (1.8). Then,

\[ |v_i(t) - v_j(t)| = O(e^{-Bt}), \quad \forall \ i, j = 1, \ldots, N, \]

for a suitable constant \( B > 0 \) depending only on the initial configuration and the parameters of the system.

**Proof.** We will use induction on the number of agents in the flock. Consider first a flock of 2 agents \([1, 2]\). Recall that, by definition of an HL-flock, \( \mathcal{L}(2) \neq \emptyset \), i.e. \( \psi_{21} > 0 \). Moreover, \( \psi_{12} = 0 \). Then,

\[ \frac{dv_1}{dt} = 0 \Rightarrow v_1(t) = v_1(0), \quad \forall \ t > 0, \quad (3.8) \]

and

\[ \frac{dv_2}{dt} = \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s)[v_1(s) - v_2(s)]ds = (v_1(0) - v_2(t)) \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s)ds, \quad t \geq \tau. \quad (3.9) \]

We now denote

\[ y_2(t) = x_2(t) - x_1(t) \quad \text{and} \quad w_2(t) = v_2(t) - v_1(t). \quad (3.10) \]

Then, from (3.9), we obtain

\[ \frac{dw_2}{dt}(t) = \frac{dv_2}{dt}(t) - \frac{dv_1}{dt}(t) = \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s)[v_1(s) - v_2(s)]ds, \quad t \geq \tau, \quad (3.11) \]

and thus, using also (3.8),

\[ \frac{1}{2} \frac{d|w_2|^2}{dt}(t) = -|w_2(t)|^2 \int_{t-\tau}^{t} \mu(t-s)\psi_{21}(s)ds, \]
which implies
\[
\frac{d|w_2|}{dt}(t) \leq -|w_2(t)| \int_{t-\tau}^{t} \mu(t-s)\psi(|x_2(s) - x_1(s)|) \, ds, \quad t \geq \tau. \tag{3.12}
\]

Therefore, from \eqref{3.12}, we deduce that \(|w_2(t)|\) is decreasing in time for \(t \geq \tau\). Now, observe that for \(t > \tau\) and \(s \in [t - \tau, t]\), we have
\[
x_1(s) - x_2(s) = x_1(t) - x_2(t) + \int_{t}^{s} (x_1 - x_2)'(\sigma) \, d\sigma
= x_1(t) - x_2(t) + \int_{t}^{s} w_2(\sigma) \, d\sigma,
\]
which gives, recalling Lemma 2.2
\[
|x_1(s) - x_2(s)| \leq |x_1(t) - x_2(t)| + 2D_0 \tau = |y_2(t)| + 2D_0 \tau, \quad t \geq \tau, \tag{3.13}
\]
with \(y_2(t), w_2(t)\) defined in \eqref{3.10} and \(D_0\) the bound on the initial velocities defined in \eqref{2.3}.

Using this inequality in \eqref{3.12} and recalling that the potential function \(\psi\) is not increasing, we obtain
\[
\frac{d|w_2|}{dt}(t) \leq -|w_2(t)| \int_{t-\tau}^{t} \mu(t-s)\psi(|y_2(t)| + 2\tau D_0) \, ds = -\mu_0|w_2(t)|\psi(|y_2(t)| + 2\tau D_0) \quad t \geq \tau, \tag{3.14}
\]
where \(\mu_0\) is the positive constant in \eqref{1.7}. Then, the pair state-velocity \((y_2, w_2)\) satisfies the inequality \eqref{3.1} with \(t_0 = \tau, d = \mu_0, M = 2\tau D_0\) and \(c = 0\). Therefore, we can apply Lemma 3.1 obtaining \(|y_2(t)| \leq C_2\) for some positive constant \(C_2\). So, for a suitable constant \(y_M^2\),
\[
|y_2(t)| + 2\tau D_0 \leq y_M^2, \quad t \geq \tau. \tag{3.15}
\]
Now, from \eqref{3.13} and \eqref{3.15} we deduce
\[
\frac{d|w_2(t)|}{dt} \leq -\mu_0\psi(y_M^2)|w_2(t)|, \quad t \geq \tau,
\]
and the Gronwall inequality implies
\[
|w_2(t)| \leq e^{-\mu_0\psi(y_M^2)(t-\tau)}|w_2(\tau)|, \quad t \geq \tau. \tag{3.16}
\]

In order to complete our inductive step we will need also estimates on the distances \(|v_1(s) - v_2(j)(t)|\) and \(|v_1(s) - v_2(j)(t)|\) for \(j = 1, 2\) and \(s \in [t - \tau, t]\).

Now, since \(v_1(t)\) is constant for \(t \geq \tau\), we easily deduce
\[
|v_1(s) - v_2(t)| = |v_1(t) - v_2(t)| = O(e^{-\psi(y_M^2)\mu}). \tag{3.17}
\]
Observe also that, for \(s \in [t - \tau, t]\),
\[
|v_2(s) - v_2(t)| = \left| \int_{t}^{s} v_2'(r) \, dr \right|
\leq c \int_{t}^{s} e^{-\psi(y_M^2)\sigma} \, d\sigma
\leq c\tau e^{-\psi(y_M^2)(t-\tau)} = c\tau e^{\psi(y_M^2)\tau} e^{-\psi(y_M^2)\mu} = O(e^{-\psi(y_M^2)\mu}). \tag{3.18}
\]
Since
\[ |v_2(s) - v_1(t)| \leq |v_2(s) - v_2(t)| + |v_2(t) - v_1(t)|, \quad (3.19) \]
from previous estimates we thus obtain
\[ |v_2(s) - v_1(t)| = O(e^{-\psi(y_M^2)t}), \quad t > \tau, \ s \in [t - \tau, t]. \quad (3.20) \]

Moreover, of course, \( |v_1(s) - v_1(t)| = O(e^{-\psi(y_M^2)t}) \), being \( v_1(t) \) constant for \( t \geq \tau \).

We assume now, by induction, that analogous exponential estimates are satisfied for a flock of \( l - 1 \) agents \([1, \ldots, l - 1]\) with \( l > 2 \), i.e. there exists some constant \( b > 0 \) such that, \( \forall i, j \in \{1, \ldots, l - 1\} \),

\[
|v_i(t) - v_j(t)| = O(e^{-bt}), \quad (3.21)
\]
\[
|v_i(s) - v_j(t)| = O(e^{-bt}), \quad t > \tau, \ s \in [t - \tau, t]. \quad (3.22)
\]

Then, we want to prove that such estimates hold true also for a flock with \( l > 2 \) agents \([1, \ldots, l]\). This will complete the proof. For this aim, define the average position and velocity of the leaders of agent \( l \),

\[
\hat{x}_l = \frac{1}{d_l} \sum_{i \in \mathcal{L}(l)} x_i(t) \quad \text{and} \quad \hat{v}_l = \frac{1}{d_l} \sum_{i \in \mathcal{L}(l)} v_i(t), \quad d_l = \#\mathcal{L}(l). \quad (3.23)
\]

Also, define
\[
g_l(t) = x_l(t) - \hat{x}_l(t) \quad \text{and} \quad w_l(t) = v_l(t) - \hat{v}_l(t). \quad (3.24)
\]

Then,
\[
\frac{dw_l}{dt}(t) = \frac{dv_l}{dt}(t) - \frac{d\hat{v}_l}{dt}(t) = \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)[v_j(s) - v_l(t)]ds - \frac{d\hat{v}_l}{dt}(t). \quad (3.25)
\]

By adding and subtracting \( \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)ds \) \( \hat{v}_l(t) \) in \((3.25)\) we get
\[
\frac{dw_l}{dt} = -w_l(t) \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)ds + \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)[v_j(s) - \hat{v}_l(t)]ds - \frac{d\hat{v}_l}{dt} \quad (3.26)
\]

Using the induction hypothesis \((3.22)\), since \( \mathcal{L}(i), \mathcal{L}(l) \subseteq [1, \ldots, l - 1] \),

\[
\frac{d\hat{v}_l}{dt} = \frac{1}{d_l} \sum_{i \in \mathcal{L}(l)} \frac{dv_i}{dt} = \frac{1}{d_l} \sum_{i \in \mathcal{L}(l)} \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)[v_j(s) - v_i(t)]ds = O(e^{-bt}). \quad (3.27)
\]

Using again the induction hypothesis \((3.22)\),

\[
\sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)[v_j(s) - \hat{v}_l(t)]ds = \frac{1}{d_l} \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{ij}(s)\left( \sum_{i \in \mathcal{L}(l)} [v_j(s) - v_i(t)] \right)ds = O(e^{-bt}). \quad (3.28)
\]
So, identity (3.26) can be rewritten as

\[
\frac{dw_i}{dt}(t) = -w_i(t) \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s) \psi_{ij}(s) \, ds + O(e^{-bt}), \quad t \geq \tau. \tag{3.29}
\]

with

\[
\psi_{ij}(s) = \psi(|x_i(s) - x_j(s)|).
\]

Observe that for every \( j \in \mathcal{L}(l) \) it results

\[
|x_i(s) - x_j(s)| \leq |x_i(s) - \hat{x}_i(s)| + |x_j(s) - \hat{x}_i(s)| \leq |y_i(s)| + M_l, \tag{3.30}
\]

for some positive \( M_l \), due to the induction’s assumption. Then, (3.29) gives

\[
\frac{d|w_i|}{dt}(t) \leq -d_i |w_i(t)| \int_{t-\tau}^{t} \mu(t-s) \psi(|y_i(s)| + M_l) \, ds + ce^{-bt}, \quad t \geq \tau. \tag{3.31}
\]

Now, note that from Proposition 2.2, \(|v_i(t)| \leq D_0\) for all \( i \) and for all \( t > 0 \), which implies

\[
|w_i(t)| \leq \frac{1}{d_i} \sum_{j \in \mathcal{L}(l)} |v_j(t) - v_i(t)| \leq \frac{1}{d_i} \sum_{j \in \mathcal{L}(l)} 2D_0 = 2D_0.
\]

Then,

\[
|y_i(s)| \leq |y_i(t)| + 2\tau D_0, \quad t \geq \tau, \quad s \in [t-\tau, t], \tag{3.32}
\]

which used in (3.31), recalling that \( \psi \) in not increasing, yields

\[
\frac{d|w_i|}{dt}(t) \leq -d_i \mu_0 \psi(|y_i(t)| + 2\tau D_0 + M_l) |w_i(t)| + ce^{-bt}. \tag{3.33}
\]

We can then apply Lemma 3.1 to the pair state-velocity \((y_i, w_i)\) to conclude that \(|y_i(t)| \leq C_l\) for some positive constant \( C_l \). So, for a suitable constant \( y^l_{M} \),

\[
|y_i(t)| + 2\tau D_0 + M_l \leq y^l_{M}, \quad t \geq \tau.
\]

Using the above estimate in (3.32) we then obtain

\[
\frac{d|w_i|}{dt} \leq -d_i \mu_0 \psi(y^l_{M}) |w_i(t)| + ce^{-bt},
\]

and therefore, from the Gronwall’s inequality we deduce,

\[
|w_i(t)| \leq Ce^{-B^l t}, \tag{3.34}
\]

for suitable positive constants \( C, B^l \).

Thus, from (4.5) and the induction hypothesis (3.21), for every \( j \in \mathcal{L}(l) \), we have

\[
|v_i(t) - v_j(t)| \leq |v_i(t) - \hat{v}_i(t)| + |\hat{v}_i(t) - v_j(t)| = O(e^{-Bt}). \tag{3.35}
\]

Now, to complete the induction argument, we only have to prove that, for all \( t > 0 \) and \( i, j \in \{1, \ldots, l\} \),

\[
|v_i(s) - v_j(s)| = O(e^{-Bt}), \tag{3.36}
\]
for a suitable positive constant $B$.

If $i, j \in \{1, \ldots, l-1\}$, then (4.14) is true by (3.22). Let us consider the case $i \in \{1, \ldots, l-1\}$ and $j = l$. Then,

$$|v_i(s) - v_l(t)| \leq |v_i(s) - v_i(t)| + |v_i(t) - v_l(t)| = O(e^{-Bt}),$$

by (3.22) and (4.13), for suitable $B$.

Consider now $i = j = l$. Then, using previous estimates we see that

$$|v_l(s) - v_l(t)| \leq c \int_s^t e^{-B\sigma} d\sigma \leq c\tau e^{-B(t-\tau)} = O(e^{-Bt}).$$

Also for the last case, where $j \in \{1, \ldots, l-1\}$ and $i = l$, using (4.15) we have

$$|v_l(s) - v_j(t)| \leq |v_l(s) - v_l(t)| + |v_l(t) - v_j(t)| = O(e^{-Bt}),$$

by the previous case and (4.13). Then, we have proved that (4.14) is satisfied for all $i, j \in \{1, \ldots, l\}$ and this concludes the proof of the theorem.

4 The case of free-will leader

It may happen that the leader of the flock, instead of moving at a constant velocity, takes off or changes its rate in order to avoid a danger, for instance due to the presence of predator species. Thus, it is important to consider this situation in the mathematical model.

The Cucker-Smale model with a free-will leader is, then,

$$\frac{dx_1}{dt}(t) = v_1(t),$$
$$\frac{dv_1}{dt}(t) = f(t),$$

where $f : [0, +\infty) \rightarrow \mathbb{R}^d$ is a continuous integrable function, that is,

$$\|f\|_1 = \int_0^{+\infty} |f(t)| \, dt < +\infty,$$

for the motion of the free-will leader, and the Cucker-Smale model under hierarchical leadership and distributed delay, as in the previous sections, for the other agents, namely

$$\frac{dx_i}{dt}(t) = v_i(t),$$
$$\frac{dv_i}{dt}(t) = \sum_{j \in \mathcal{L}(i)} \int_{t-\tau}^t \mu(t-s)\psi_{ij}(s)[v_j(s) - v_i(t)] \, ds,$$

for all $i \in \{2, \ldots, N\}$. The initial data are assigned, as usual, on the time interval $[-\tau, 0]$, i.e.

$$x_i(s) = x_i^0(s),$$
$$v_i(s) = v_i^0(s),$$

(4.4)
for some continuous functions $x_i^0$ and $v_i^0$, for $i = 1, \ldots, N$.

The flocking result below extends the one proved by Shen \[35\] for the undelayed case. The case with pointwise delay has been studied in \[33\]. Here, we consider a more general acceleration function with respect to \[35, 33\], for the free-will leader. Indeed we assume

$$|f(t)| = o((1 + t)^{1-N}) \quad \text{and} \quad t^{N-2}|f(t)| \in L^1(0, +\infty) \quad (4.5)$$

instead of

$$|f(t)| = O((1 + t)^{-\mu}), \quad \mu > N - 1. \quad (4.6)$$

Then, for instance, $f$ can be in the form

$$f(t) = \frac{C}{(1 + t)^{N-1}} \ln^2(2 + t).$$

Note that, from (4.5) it results

$$t^k|f(t)| = o((1 + t)^{1-N+k}), \quad \forall \ k = 1, \ldots, N - 1. \quad (4.7)$$

In order to prove our flocking result, we will need the following lemma, which is a generalization of Lemma 3.1 above.

**Lemma 4.1.** Let $(x, v)$ be a trajectory in the phase–space, namely $\frac{dx}{dt}(t) = v(t)$ for $t \geq 0$. Assume that

$$\frac{d|v|}{dt}(t) \leq -d_0\psi(|x(t)| + M)|v(t)| + g(t) \quad \forall \ t \geq t_0, \quad (4.8)$$

for some nonnegative constants $M, t_0$, a constant $d_0 > 0$ and a continuous and integrable function $g : [t_0, +\infty) \to (0, +\infty)$, where $\psi : [0, +\infty) \to (0, +\infty)$ is a continuous function satisfying (1.8). Then, there exists a suitable positive constant $C$ such that

$$|x(t)| \leq C, \quad t \geq 0.$$

**Proof.** Let us consider the functionals $F_{\pm}$ introduced in (3.2) with $d_0, M, \psi$ as in the statement. From (4.8) we deduce

$$\frac{dF_{\pm}}{dt}(t) = \frac{d|v|}{dt}(t) \pm d_0\psi(|x(t)| + M)\frac{d|x|}{dt}(t)$$

$$\leq -d_0\psi(|x(t)| + M)|v(t)| \pm d_0\psi(|x(t)| + M)\frac{d|x|}{dt}(t) + g(t)$$

$$= d_0\psi(|x(t)| + M)\left(\pm \frac{d|x|}{dt}(t) - |v(t)|\right) + g(t) \leq g(t), \quad t \geq t_0, \quad (4.9)$$

where we have used inequality (3.4).

Now, we integrate (4.9) on the time interval $[t_0, t]$, obtaining

$$F_{\pm}(t) - F_{\pm}(t_0) \leq \|g\|_{L^1(t_0, +\infty)},$$

which gives
\[ |v(t)| \leq \pm d_0 (\phi (|x(t_0)| + M) - \phi (|x(t)| + M)) + |v(t_0)| + \|g\|_{L^1(t_0, +\infty)}, \]

namely
\[ |v(t)| \leq -d_0 \int_{|x(t)| + M}^{\phi (|x(t_0)| + M)} \psi(s) \, ds + |v(t_0)| + \|g\|_{L^1(t_0, +\infty)}. \tag{4.10} \]

Therefore, from (4.10), we have
\[ |v(t_0)| + \|g\|_{L^1(t_0, +\infty)} \geq d_0 \int_{|x(t)| + M}^{\phi (|x(t_0)| + M)} \psi(s) \, ds. \tag{4.11} \]

The assumption (1.8) ensures then the existence of a constant \( x_M > 0 \) such that
\[ |v(t)| + \|g\|_{L^1(t_0, +\infty)} = d_0 \int_{|x(t)| + M}^{x_M} \psi(s) \, ds, \]
which, together with (4.11), implies \( |x(t)| \leq C, \quad \forall \ t \geq 0. \]

**Theorem 4.2.** Let \((x_i, v_i), i = 1, \ldots, N\), be a solution of the Cucker-Smale system under hierarchical leadership with delay (4.1) - (4.3) with initial conditions (4.4). Assume that (1.8) is satisfied and that the acceleration of the free-will leader satisfies (4.5). Then, it results
\[ |v_i(t) - v_j(t)| \to 0, \quad \text{for} \quad t \to +\infty, \quad \forall \ i, j = 1, \ldots, N. \tag{4.12} \]

**Proof.** As in the previous convergence to consensus result, we argue by induction. First, we look at the first agent, i.e. the free-will leader. Equation (4.1) gives
\[ v_1(t) = v_1(0) + \int_0^t f(s) \, ds, \]
and so, from (4.2),
\[ |v_1(t)| \leq |v_1(0)| + \|f\|_1 = C_1, \quad \forall \ t \geq 0. \tag{4.13} \]

Now, let us consider the 2-flock. As before, let us denote
\[ w_2(t) = v_2(t) - v_1(t) \quad \text{and} \quad y_2(t) = x_2(t) - x_1(t), \quad t \geq 0. \]

From (4.1) and (4.3)
\[
\frac{dw_2}{dt}(t) = \frac{dv_2}{dt}(t) - \frac{dv_1}{dt}(t) = \int_{t-\tau}^t \mu(t-s) \psi_1(s)(v_1(s) - v_2(t)) \, ds - f(t) \\
= (v_1(t) - v_2(t)) \int_{t-\tau}^t \mu(t-s) \psi_1(s) \, ds - \int_{t-\tau}^t \mu(t-s) \psi_1(s)(v_1(t) - v_1(s)) \, ds - f(t) \\
= -w_2(t) \int_{t-\tau}^t \mu(t-s) \psi_1(s) \, ds - \int_{t-\tau}^t \mu(t-s) \psi_1(s) \int_s^t f(\sigma) \, d\sigma \, ds - f(t), \quad t \geq \tau. \tag{4.14} \]

Now, from (4.5), it results
\[
\left| \int_{t-\tau}^t \mu(t-s) \psi_1(s) \int_s^t f(\sigma) \, d\sigma \, ds \right| + |f(t)| \\
\leq \tau \mu_0 \max_{s \in [0, +\infty)} \psi(s) \int_{t-\tau}^t |f(s)| \, ds + |f(t)| = O(|f|). \tag{4.15} \]
Then, from (4.14) and (4.15) we obtain

\[
\frac{d}{dt}|w_2(t)| \leq -|w_2(t)| \int_{t-\tau}^{t} \mu(t-s)\psi_2(s)ds + \tilde{f}(t), \quad t \geq \tau.
\]  

(4.16)

where

\[
\tilde{f}(t) := \tau\mu_0 \max_{s \in [0, +\infty)} \psi(s) \int_{t-\tau}^{t} |f(s)|ds + |f(t)| = O(|f|).
\]  

(4.17)

Therefore,

\[
|w_2(t)| \leq |w_2(\tau)| + \int_{\tau}^{+\infty} \tilde{f}(t) dt \leq D_2, \quad \forall \ t \geq \tau,
\]  

(4.18)

for some constant \( D_2 > 0 \). Since

\[
y_2(s) = y_2(t) + \int_{t}^{s} w_2(\sigma)d\sigma,
\]

from (4.18) we have

\[
|y_2(s)| \leq |y_2(t)| + \tau D_2, \quad \forall \ s \in [t - \tau, t].
\]  

(4.19)

From (4.16) and (4.19), we then deduce

\[
\frac{d}{dt}|w_2(t)| \leq -\mu_0\psi(|y_2(t)| + \tau D_2)|w_2(t)| + \tilde{f}(t), \quad t \geq \tau.
\]  

(4.20)

Then, we can apply Lemma 4.1 to the pair \((y_2, w_2)\) with \( d = \mu_0, M = \tau D_2 \) and \( g = \tilde{f} \), obtaining that

\[
|y_2(t)| + \tau D_2 \leq y_2^2, \quad t \geq 0,
\]  

(4.21)

for a suitable positive constant \( y_2^2 \). So, from (4.20) and (4.21) we have

\[
\frac{d}{dt}|w_2(t)| \leq -\psi(y_2^2)|w_2(t)| + \tilde{f}(t), \quad t \geq \tau,
\]

and thus, for every \( T > \tau \), applying Gronwall’s lemma we deduce

\[
|w_2(T)| \leq e^{-\psi(y_2^2)\frac{T}{2}}|w_2(T/2)| + \int_{\tau}^{T} e^{-\psi(y_2^2)(T-t)} \tilde{f}(t) dt
\]

\[
\leq e^{-\psi(y_2^2)\frac{T}{2}} D_2 + \int_{\tau}^{T} \tilde{f}(t) dt \leq e^{-\psi(x_2^2)\frac{T}{2}} D_2 + \tilde{f}_2(T),
\]  

(4.22)

where, recalling (4.5), \( \tilde{f}_2 \), is a suitable function satisfying

\[
\tilde{f}_2(t) = O(t|f|) = o((1 + t)^{2-N}).
\]  

(4.23)

Thus,

\[
|v_2(t) - v_1(t)| = o((1 + t)^{2-N}).
\]  

(4.24)

Note also that

\[
|v_1(t - \tau) - v_1(t)| \leq \int_{t-\tau}^{t} |f(t)| dt = O(|f|),
\]  

(4.25)
and then
\[ |v_2(t - \tau) - v_2(t)| \leq |v_2(t - \tau) - v_1(t - \tau)| + |v_1(t - \tau) - v_1(t)| + |v_1(t) - v_2(t)| = o((1 + t)^{2-N}). \] (4.26)

Therefore, (4.24)–(4.26) imply
\[ |v_i(t - \tau) - v_j(t)| = O(\tilde{f}_2) = o((1 + t)^{2-N}), \quad \text{for } i, j \in \{1, 2\}. \] (4.27)

Now, as induction hypothesis, assume that for a flock of \( l - 1 \) agents \([1, \ldots, l - 1]\) with \( 2 < l \leq N \), we have
\[ |v_i(t) - v_j(t)| = O(t^{l-2}|f|) = o((1 + t)^{l-1-N}), \] (4.28)
\[ |v_i(t - \tau) - v_j(t)| = O(t^{l-2}|f|) = o((1 + t)^{l-1-N}), \] (4.29)
for all \( i, j \in \{1, \ldots, l - 1\} \).

Then, we want to prove the same kind of estimates for a flock with \( l \) agents. This will complete our theorem.

As before, we will use the average position and velocity of the leaders of agent \( l \), introduced in (3.23) and let \( y_l, w_l \) be defined as in (3.24). Then, as before we can write
\[
\frac{dw_l}{dt} = -w_l(t) \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{lj}(s)ds + \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{lj}(s)[v_j(s) - \hat{v}_l(t)]ds - \frac{dv_l}{dt}. \tag{4.30}
\]

Using the induction hypothesis (4.29), since \( \mathcal{L}(i), \mathcal{L}(l) \subseteq [1, \ldots, l - 1] \),
\[
\frac{d\hat{v}_l}{dt} = \frac{1}{d_l} \sum_{i \in \mathcal{L}(l)} \frac{dv_i}{dt} = \chi_{1 \in \mathcal{L}(l)} \frac{1}{dt} f(t) + \frac{1}{d_l} \sum_{i \in \mathcal{L}(l) \setminus \{1\}} \frac{dv_i}{dt} = O(t^{l-2}|f|) = o((1 + t)^{l-1-N}). \tag{4.31}
\]

From the induction hypotheses (4.29) we deduce also
\[
\sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{lj}(s)[v_j(s) - \hat{v}_l(t)] ds = \frac{1}{d_l} \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{lj}(s) \left( \sum_{i \in \mathcal{L}(l)} [v_j(s) - v_i(t)] \right) ds = O(t^{l-2}|f|) = o((1 + t)^{l-1-N}). \tag{4.32}
\]

Then, identity (4.30) can be rewritten as
\[
\frac{dw_l}{dt}(t) = -w_l(t) \sum_{j \in \mathcal{L}(l)} \int_{t-\tau}^{t} \mu(t-s)\psi_{lj}(s) ds + O(t^{l-2}|f|), \quad t \geq \tau. \tag{4.33}
\]

As before one can now observe that for every \( j \in \mathcal{L}(l) \) it results
\[
|x_l(s) - x_j(s)| \leq |x_l(s) - \hat{x}_l(s)| + |x_j(s) - \hat{x}_l(s)| \leq |y_l(s)| + R_l, \tag{4.34}
\]
for some positive \( R_l \), due to the induction’s assumption. Thus, (4.33) implies
Thus, we can apply the Gronwall’s lemma analogously to the 2−by (4.29) and (4.41).

Then, from (4.40) and the induction hypothesis (4.28), for every

\[ |v_i(t) - v_j(t)| \leq |v_i(t) - \hat{v}_i(t)| + |\hat{v}_i(t) - v_j(t)| = O(t'^{-1}|f|) = o(t'^{-N}). \]  

Note that (4.35) implies

\[ \frac{d|w_i|}{dt}(t) \leq -d_l|w_i(t)| \int_{t-\tau}^{t} \mu(t-s)\psi (|y_i(s)| + R_l) \ ds + O(t'^{-2}|f|), \quad t \geq \tau. \]  

(4.35)

So, recalling the assumptions (4.5) on the acceleration \( f \) of the free–will leader, we deduce

\[ |w_i(t)| \leq |w_i(\tau)| + \int_{\tau}^{t} O(t'^{-2}|f|) \ dt \leq C_l. \]  

(4.37)

Then,

\[ |x'(t-\tau)| \leq |x'(t)| + \int_{t-\tau}^{t} |v'(s)| \ ds \leq |x'(t)| + C_l \tau, \quad t \geq \tau, \]  

(4.38)

which, used in (4.35), gives

\[ \frac{d|w_i|}{dt}(t) \leq -d_l\mu_0 \psi (|y_i(t)| + 2\tau C_l + R_l) |w_i(t)| + O(t'^{-2}|f|). \]  

(4.39)

We can then apply Lemma (4.1) to the pair state–velocity \((y_i, w_i)\) and conclude that \( |y_i(t)| \leq C_l \) for some positive constant \( C_l \). So, for a suitable constant \( y^f_M \),

\[ |y_i(t)| + 2\tau C_l + R_l \leq y^f_M, \quad t \geq \tau. \]

Using the above estimate in (4.39) we then obtain

\[ \frac{d|w_i|}{dt}(t) \leq -d_l\mu_0 \psi (y^f_M |w_i(t)|) + O(t'^{-2}|f|). \]

Thus, we can apply the Gronwall’s lemma analogously to the 2–flock case obtaining

\[ |v^f(t)| = O(t'^{-1}|f|) = o(t'^{-N}). \]  

(4.40)

Then, from (4.40) and the induction hypothesis (4.28), for every \( j \in L(l) \), we have

\[ |v_i(t) - v_j(t)| \leq |v_i(t) - \hat{v}_i(t)| + |\hat{v}_i(t) - v_j(t)| = O(t'^{-1}|f|) = o(t'^{-N}). \]  

(4.41)

Now, it remains to prove that, for all \( i, j \in \{1, \ldots, l\} \),

\[ |v_i(t-\tau) - v_j(t)| = O(t'^{-1}|f|) = o(|f|^{l'-N}). \]  

(4.42)

If \( i, j \in \{1, \ldots, l-1\} \), then (4.42) is true by (4.29). Consider the case \( i \in \{1, \ldots, l-1\} \) and \( j = l \). Then,

\[ |v_i(t-\tau) - v_i(t)| \leq |v_i(t-\tau) - v_i(t)| + |v_j(t) - v_i(t)| = O(t'^{-1}|f|) = o(|f|^{l'-N}), \]

by (4.29) and (4.41).

For the case \( i = j = l \), using the previous estimates, we obtain
$$|v_l(t) - v_l(t)| = \left| \int_t^s v_l'(\sigma) \, d\sigma \right| = \left| \int_t^s \sum_{k \in L(l)} \int_{\sigma - \tau}^\sigma \mu(\sigma - r) \psi_{k(r)}(r) (v_k(r) - v_l(\sigma)) \, dr \, d\sigma \right|$$

$$\leq C \int_t^s O(\sigma^{l-1} |f(\sigma)|) \, d\sigma = O(t^{l-1} |f|), \quad s \in [t - \tau, t].$$

(4.43)

Also for the last case, where $j \in \{1, \ldots, l - 1\}$ and $i = l$, using (4.41) and (4.43) we obtain

$$|v_l(t - \tau) - v_j(t)| \leq |v_l(t - \tau) - v_l(t)| + |v_l(t) - v_j(t)| = O(t^{l-1} |f|) = o(|f|^{l-N}).$$

Therefore, (4.42) is satisfied for all $i, j \in \{1, \ldots, l\}$ and so the theorem is proved. □

**Remark 4.3.** Note that our generalization concerning the acceleration function $f$ of the free–will leader is suitable also for the problem without delay considered by Shen [35] and for the problem with pointwise delay studied by the authors [33]. Therefore, our flocking estimates (4.13) could be obtained, under the same assumptions on $f$, for the problem with free–will leader studied in [35] and the more general one considered in [33].

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