A CONSTRAINED MEAN CURVATURE FLOW AND ALEXANDROV-FENCHEL INEQUALITIES

XINQUN MEI, GUOFANG WANG, AND LIANGJUN WENG

Abstract. In this article, we study a locally constrained mean curvature flow for star-shaped hypersurfaces with capillary boundary in the half-space. We prove its long-time existence and the global convergence to a spherical cap. Furthermore, the capillary quermassintegrals defined in [29] evolve monotonically along the flow, and hence we establish a class of new Alexandrov-Fenchel inequalities for convex hypersurfaces with capillary boundary in the half-space.

1. Introduction

The mean curvature flow plays an important role in geometric analysis and differential geometry and has been extensively studied. One of the classical results proved by Huisken [16] states that it contracts a closed convex hypersurface into a round point and converges to a sphere after a suitable rescaling. Later he constructed a normalized mean curvature flow to prove the isoperimetric inequality in [17]. Inspired by the Minkowski formulas for closed hypersurfaces, Guan-Li [11] introduced a new locally constrained mean curvature flow to study the isoperimetric problem, which is defined by

$$\partial_t x = (n - \langle x, \nu \rangle H) \nu,$$

for $x : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$, (1.1)

where $M$ is a $n$-dimensional closed manifold, $H$ and $\nu$ are the mean curvature and unit outward normal vector of hypersurface $x(M, \cdot)$ respectively. Along flow (1.1), the enclosed domain has a fixed volume, while the area is monotone decreasing by the Minkowski formulas. Guan-Li obtained the long-time existence of this flow and proved that it smoothly converges to a round sphere if the initial hypersurface is star-shaped. As a result, it also yields a flow proof of classical Alexandrov-Fenchel inequalities of quermassintegrals in convex geometry. For more details see [11, Corollary 1.2]. Using similar ideas and new locally constrained curvature flows, there have been a lot of work developed in the last decades. See for instance [5, 6, 10, 12, 13, 14, 15, 26] and references therein.

In this paper, we are interested in the isoperimetric inequalities and Alexandrov-Fenchel inequalities for capillary hypersurfaces in the half-space. We first introduce a new locally constrained mean curvature type flow which is suitable for capillary hypersurfaces in the half-space. The flow is also motivated by the Minkowski formula (i.e. (1.3)) for capillary
hypersurfaces in $\mathbb{R}^{n+1}_+$. Besides, this paper is inspired by a new research on capillary hypersurfaces in [29], where the authors establish a class of new Alexandrov-Fenchel inequalities for capillary hypersurfaces in the half-space.

By abusing a little bit the terminology, a capillary hypersurface is defined in this paper as the following.

**Definition 1.1.** A hypersurface in $\mathbb{R}^{n+1}_+$ with boundary supported on $\partial \mathbb{R}^{n+1}_+$ is called capillary hypersurface if it intersects with $\partial \mathbb{R}^{n+1}_+$ at a constant angle $\theta \in (0, \pi)$.

Let $\Sigma_t$ be a family of hypersurfaces with boundary in $\mathbb{R}^{n+1}_+$ given by a family of isometric embeddings $x(\cdot, t) : M \to \mathbb{R}^{n+1}_+$ from a compact $n$-dimensional manifold $M$ with boundary $\partial M$ ($n \geq 2$) such that

$$\text{int}(\Sigma_t) = x(\text{int}(M), t) \subset \mathbb{R}^{n+1}_+, \quad \partial \Sigma_t = x(\partial M, t) \subset \partial \mathbb{R}^{n+1}_+. $$

In this paper, we study the following locally constrained mean curvature flow for capillary hypersurfaces defined by

$$\begin{cases} 
\partial_t x = \left( n + n \cos \theta \langle \nu, e \rangle - H \langle x, \nu \rangle \right) \nu, & \text{in } M \times [0, T), \\
\langle \nu, e \rangle = -\cos \theta & \text{on } \partial M \times [0, T), \\
x(\cdot, 0) = x_0(\cdot) & \text{in } M, 
\end{cases} \quad (1.2)$$

where $\nu, H$ are the unit normal, the mean curvature of $\Sigma$ respectively, $e := -E_{n+1} = (0, \cdots, 0, -1)$ is the unit outward normal of $\partial \mathbb{R}^{n+1}_+$ in $\mathbb{R}^{n+1}_+$.

Compared to flow (1.1), there is an extra term $n \cos \theta \langle \nu, e \rangle$ in the speed function $f$ of our flow (1.2), which comes naturally from the Minkowski formula, namely

$$n \int_{\Sigma} (1 + \cos \theta \langle \nu, e \rangle) \, dA = \int_{\Sigma} H \langle x, \nu \rangle \, dA, $$

for any capillary hypersurfaces. For a proof we refer to [2, 29]. There is an interesting family of capillary hypersurfaces, which are the spherical caps lying entirely in $\mathbb{R}^{n+1}_+$ and intersecting $\partial \mathbb{R}^{n+1}_+$ with a constant contact angle $\theta \in (0, \pi)$ given by

$$C_{\theta, r} := \left\{ x \in \mathbb{R}^{n+1}_+ | \| x - r \cos \theta e \| = r \right\} \subset \mathbb{R}^{n+1}_+, \quad r \in [0, \infty). $$

One can easily check that $C_{\theta, r}$ is a static solution to flow (1.2), that is

$$1 + \cos \theta \langle \nu, e \rangle - \frac{H}{n} \langle x, \nu \rangle = 0. $$

In fact, $C_{\theta, r}$ consist of all solutions of (1.5). See Proposition 3.1 below. We also denote $S^n_\theta$, $\mathbb{B}^{n+1}_\theta$ as

$$S^n_\theta := \left\{ x \in S^n | \langle x, E_{n+1} \rangle > \cos \theta \right\}, \quad \mathbb{B}^{n+1}_\theta := \left\{ x \in \mathbb{B}^{n+1} | \langle x, E_{n+1} \rangle > \cos \theta \right\},
$$

Now we state one of the main theorems in this paper.
Theorem 1.2. If the initial capillary hypersurface is star-shaped with respect to origin in $\mathbb{R}^{n+1}_+$ and the contact angle $\theta \in (0, \pi)$, then flow (1.2) exists for all time. Moreover, $x(\cdot,t)$ smoothly converges to a uniquely determined spherical cap $C_{\theta,r_0}$ around $e$, as $t \to \infty$.

It is worth noting that we could obtain the uniform a priori estimates for flow (1.2) in the whole range $\theta \in (0, \pi)$ instead of only a partial range of contact angle compared with results in [28] or [32].

It is easy to see that flow (1.2) preserves the enclosed volume

$$V_{0,\theta}(\hat{\Sigma}) := |\hat{\Sigma}|,$$

by using (1.3). A nice feature of flow (1.2) is that along the flow, the capillary area (one can refer to [8, 24] for more interpretation)

$$\mathcal{V}_{1,\theta}(\hat{\Sigma}) := \frac{1}{n+1}(|\Sigma| - \cos \theta |\partial \Sigma|),$$

is decreasing, which follows from the higher order Minkowski formulas, proved recently in [29, Proposition 2.5],

$$\int_{\Sigma} H_{k-1} (1 + \cos \theta \langle \nu, e \rangle) dA = \int_{\Sigma} H_k \langle x, \nu \rangle dA, \quad (1.6)$$

where $H_k$ ($1 \leq k \leq n$) is the normalized $k$-th mean curvature of $\Sigma \subset \mathbb{R}^{n+1}$ and $H_0 = 1$.

As a result, Theorem 1.2 yields a flow proof for capillary isoperimetric inequality for star-shaped hypersurfaces in $\mathbb{R}^{n+1}_+$.

Corollary 1.3. For $n \geq 2$, let $\Sigma \subset \mathbb{R}^{n+1}_+$ be a star-shaped capillary hypersurface with a contact angle $\theta \in (0, \pi)$, then there holds

$$\frac{|\Sigma| - \cos \theta |\partial \Sigma|}{|\Sigma_\theta| - \cos \theta |\partial \Sigma_\theta|} \geq \left( \frac{|\hat{\Sigma}|}{|\hat{\Sigma}_\theta|} \right)^{\frac{n}{n+1}}. \quad (1.7)$$

Moreover, equality holds if and only if $\Sigma$ is a spherical cap in (1.4).

The capillary isoperimetric inequality holds true in fact for any hypersurface in $\mathbb{R}^{n+1}_+$. For the proof we refer to [22, Chapter 19], using the spherical symmetrization.

Recently, as the generalization of the capillary area $\mathcal{V}_{1,\theta}(\hat{\Sigma})$, a class of new quermassintegrals $\mathcal{V}_{k,\theta}(\hat{\Sigma})$ for capillary hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}_+$ have been introduced in [29], which are, for $1 \leq k \leq n$,

$$\mathcal{V}_{k+1,\theta}(\hat{\Sigma}) := \frac{1}{n+1} \left( \int_{\Sigma} H_k dA - \frac{\cos \theta \sin k \theta}{n} \int_{\partial \Sigma} H_{k-1}^{\partial \Sigma} ds \right), \quad (1.8)$$

where $H_{k-1}^{\partial \Sigma}$ is the normalized $(k-1)$-th mean curvature of $\partial \Sigma \subset \mathbb{R}^n$. Another motivation to study flow (1.2) is to establish the following Alexandrov-Fenchel inequalities.
Theorem 1.4. For \( n \geq 2 \), let \( \Sigma \subset \mathbb{R}^{n+1}_+ \) be a convex capillary hypersurface with a contact angle \( \theta \in (0, \frac{\pi}{2}] \), then there holds
\[
\left( \frac{V_{k,\theta}(\hat{\Sigma})}{b_{\theta}} \right)^{\frac{n+1}{n+1-k}} \geq \left( \frac{V_{0,\theta}(\hat{\Sigma})}{b_{\theta}} \right)^{\frac{n+1}{n+1-k}}, \quad \forall 1 \leq k < n,
\] (1.9)
where \( b_{\theta} \) is the \((n+1)\) dimensional volume of \( B^{n+1}_{\theta} \), with equality if and only if \( \Sigma \) is a spherical cap in (1.4).

Note that (1.9) gives partially an affirmative answer to a question in [29, Conjecture 1.5], where the authors proved
\[
\frac{V_{n,\theta}(\hat{\Sigma})}{b_{\theta}} \geq \left( \frac{V_{k,\theta}(\hat{\Sigma})}{b_{\theta}} \right)^{\frac{1}{n+1-k}}, \quad \forall 1 \leq k < n,
\]
by using a locally constrained inverse curvature type flow. When \( k = 1 \) in (1.9), it is the capillary isoperimetric inequality considered above. We remark that the restriction of range \( \theta \in (0, \frac{\pi}{2}] \) is only used in showing the convexity preserving property along the flow (1.2). See Proposition 3.13 or Eq. (3.13).

As a direct consequence of Theorem 1.4, for \( k = 2 \) in (1.9), we obtain a volumetric Minkowski inequality for convex capillary hypersurfaces in \( \mathbb{R}^{n+1}_+ \).

Corollary 1.5. For \( n \geq 2 \), let \( \Sigma \subset \mathbb{R}^{n+1}_+ \) be a convex capillary hypersurface with a contact angle \( \theta \in (0, \frac{\pi}{2}] \), then there holds
\[
\int_{\Sigma} H dA - \sin \theta \cos \theta |\partial \Sigma| \geq n(n+1)b_{\theta}^{\frac{2}{n+1}}|\hat{\Sigma}|^{\frac{n+1}{n+1}},
\] (1.10)
with equality if and only if \( \Sigma \) is a spherical cap in (1.4).

It is remarkable to note that Agostiniani-Fogagnolo-Mazzieri obtained a volumetric Minkowski inequality in [1, Theorem 1.5] without any convexity assumption for closed hypersurfaces in Euclidean space, see also [4, 9, 23] for the relevant results. We expect that (1.10) holds true for general capillary hypersurfaces without any convexity assumption by replacing \( \int_{\Sigma} H dA \) with \( \int_{\Sigma} |H| dA \).

We end the introduction by mentioning some interesting results on curvature flows with free or capillary boundaries in the Euclidean space. The classical mean curvature flow with free boundary was studied by Stahl in [27], which showed that evolving a strictly convex, embedded hypersurface in the interior of a ball or half-space by mean curvature flow with free boundary produces a Type I singularity which, after rescaling, is asymptotically hemispherical. See also [3, 7] for further analysis of the behavior of singularities that develop on the free boundary of the evolving hypersurface. Lambert-Scheuer [20] proved that the inverse mean curvature flow for hypersurfaces with free boundary in a ball would drive a strictly convex hypersurfaces into a flat disk. Recently, Wang-Xia [30] studied a locally constrained mean curvature type flow for embedded hypersurface with free boundary in the unit ball and showed that it converges.
to a spherical cap, which motivated by a new Minkowski formula in [31]. See also [28] for the capillary case. Furthermore, Scheuer-Wang-Xia [25] introduced a new inverse nonlinear curvature flow for embedded hypersurfaces with free boundary in the ball and obtained a class of new Alexandrov-Fenchel inequalities after obtaining that flow converges to a spherical cap. See also [29, 32] for the very recent development of hypersurfaces with capillary boundary.

The rest of the article is structured as follows. In Section 2, some notations and known results about capillary hypersurfaces in the half space are introduced, including the relevant evolution equations. In Section 3, we first obtain the uniform estimates for flow (1.2) and prove its long-time existence and smooth convergence in Section 3.2, then we complete the proof of Theorem 1.2. In Section 3.3, we prove the convexity preserving along flow (1.2), which implies the Alexandrov-Fenchel inequalities, that is, Theorem 1.4.

2. Capillary hypersurface in half-space

Let $\Sigma \subset \mathbb{R}_{+}^{n+1}$ be a smooth, properly embedded capillary hypersurface, given by the embedding $x : M \to \mathbb{R}_{+}^{n+1}$, where $M$ is a compact, orientable smooth manifold of dimension $n$ with non-empty boundary. If there is no confusion, we do not distinguish $\Sigma$ and the embedding $x$.

Let $\mu$ be the unit outward co-normal of $\partial \Sigma$ in $\Sigma$ and $\nu$ be the unit normal to $\partial \Sigma$ in $\partial \mathbb{R}_{+}^{n+1}$ such that $\{\nu, \mu\}$ and $\{\nu, e\}$ have the same orientation in normal bundle of $\partial \Sigma \subset \mathbb{R}_{+}^{n+1}$, where $e := -E_{n+1}$. We define the contact angle $\theta$ between the hypersurface $\Sigma$ and the support $\partial \mathbb{R}_{+}^{n+1}$ by

$$\langle \nu, e \rangle = \cos(\pi - \theta).$$

It follows

$$e = \sin \theta \mu - \cos \theta \nu,$$
$$\overline{\nu} = \cos \theta \mu + \sin \theta \nu. \quad (2.1)$$

We use $D$ to denote the Levi-Civita connection of $\mathbb{R}_{+}^{n+1}$ w.r.t the Euclidean metric $\delta$, and $\nabla$ the Levi-Civita connection on $\Sigma$ w.r.t the induced metric $g$ from the immersion $x$. The operator $\text{div}, \Delta,$ and $\nabla^2$ are the divergence, Laplacian, and Hessian operator on $\Sigma$ respectively.

The second fundamental form $h$ of $x$ is defined by

$$D_X Y = \nabla_X Y - h(X, Y)\nu.$$ 

Denote $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ be the set of principal curvatures, i.e, the set of eigenvalues of $h$. For other notations, we following the paper [29].

The second fundamental form of $\partial \Sigma$ in $\mathbb{R}^n$ is given by

$$\tilde{h}(X, Y) := -\langle \nabla^\mathbb{R} X Y, \nu \rangle = -\langle D_X Y, \nu \rangle, \quad X, Y \in T(\partial \Sigma).$$

The second equality holds since $\langle \hat{\nu}, N \circ x \rangle = 0$. The second fundamental form of $\partial \Sigma$ in $\Sigma$ is given by

$$\tilde{h}(X, Y) := -\langle \nabla_X Y, \mu \rangle = -\langle D_X Y, \mu \rangle, \quad X, Y \in T(\partial \Sigma).$$
The second equality holds since \( \langle \nu, \mu \rangle = 0 \). Besides, \( \tilde{h}, \hat{h}, \) and \( h \) have some nice relationships, cf. [29, Proposition 2.3].

Let \( \Sigma_t \) be a family of smoothly embedded hypersurfaces with \( \theta \)-capillary boundary in \( \bar{\mathbb{R}}^{n+1}_+ \), given by the embeddings \( x(\cdot, t) : M \to \bar{\mathbb{R}}^{n+1}_+ \), which evolves by the general flow

\[
\partial_t x = f \nu + T, \tag{2.2}
\]

with some normal speed function \( f \) and \( T \in T_{\Sigma_t} \). In this paper, \( f \) consists mainly of mean curvature, and hence (2.2) is a mean curvature type flow, where \( T \) is added such that flow preserves the capillary condition.

Along flow (2.2), we have the following evolution equations for the induced metric \( g_{ij} \), the unit outward normal \( \nu \), the second fundamental form \( h_{ij} \), the Weingarten matrix \( h^j_i \), the mean curvature \( H \) of the hypersurfaces \( \Sigma_t \). These evolution equations have been shown in [32, Proposition 2.11] and will be used in deriving the a priori estimates.

**Proposition 2.1 ([32]).** Along flow (2.2), it holds that

1. \( \partial_t g_{ij} = 2f h_{ij} + \nabla_i T_j + \nabla_j T_i. \)
2. \( \partial_t \nu = -\nabla f + h(e_i, T) e_i. \)
3. \( \partial_t h_{ij} = -\nabla^2 f + f h_{ij} h^k_k + \nabla_T h_{ij} + h_j^k \nabla_T h^k_i + h^k_i \nabla_T h_k. \)
4. \( \partial_t h^j_i = -\nabla^i \nabla_j f - f h^k_j h^i_k + \nabla_T h^i_j. \)
5. \( \partial_t H = -\Delta f - |h|^2 f + \nabla_T H. \)

### 3. A priori estimate

In this section, we obtain the uniform a priori estimates for flow (1.2).

3.1. **Further evolution equations.** We need evolution equations for various geometric quantities. For simplicity, we introduce the linearized operator with respect to (1.2) as

\[
\mathcal{L} := \partial_t - \langle x, \nu \rangle \Delta - \langle T + H x - n \cos \theta e, \nabla \rangle.
\]

We derive the evolution equations for various geometric quantities under flow (1.2). That is, the normal speed function in (2.2) is chosen to be

\[
f := n (1 + \cos \theta \langle \nu, e \rangle) - H \langle x, \nu \rangle.
\]

**Proposition 3.1.** Along flow (1.2), the support function \( u = \langle x, \nu \rangle \) satisfies

\[
\mathcal{L} u = n + n \cos \theta \langle \nu, e \rangle - 2u H + u^2 |h|^2, \tag{3.1}
\]

and

\[
\nabla_{\mu} u = \cot \theta h(\mu, \mu) u, \quad \text{on } \partial \Sigma_t. \tag{3.2}
\]

**Proof.** Direct computation yields

\[
\nabla_i u = h_{ik} \langle x, e_k \rangle,
\]
\[
\n\nabla_{ij} u = h_{ij;k}(x, e_k) + h_{ij} - u(h^2)_{ij},
\]

then
\[
\Delta u = \langle x, \nabla H \rangle + H - u|h|^2.
\]

In view of these formulas and Proposition 2.1, we have
\[
\partial_t u = \langle \partial_t x, \nu \rangle + \langle x, \partial_t \nu \rangle = f - \langle x, \nabla f \rangle + h(x^T, T)
\]
\[
= (n + n \cos \theta \langle \nu, e \rangle - uH) - n \cos \theta \langle x, \nabla \langle \nu, e \rangle \rangle + u\langle x, \nabla H \rangle
\]
\[
+ \langle H x, \nabla u \rangle + h(x^T, T)
\]
\[
= \langle x, \nu \rangle \Delta u - n\cos \theta e, \nabla u \rangle + \langle H x, \nabla u \rangle + (n + n \cos \theta \langle \nu, e \rangle - 2uH)
\]
\[
+ u^2|h|^2 + \langle T, \nabla u \rangle,
\]

that is,
\[
\mathcal{L} u = n + n \cos \theta \langle \nu, e \rangle - 2uH + u^2|h|^2.
\]

On \(\partial \Sigma_t\), since \(\mu\) is a principal direction of \(\Sigma\) by [29, Proposition 2.3], together with (2.1), it yields
\[
\nabla_{\mu} \langle x, \nu \rangle = \langle x, h(\mu, \mu)\mu \rangle = \cos \theta h(\mu, \mu)\langle x, \bar{\nu} \rangle = \cot \theta h(\mu, \mu)\langle x, \nu \rangle.
\]

\[\square\]

For later use, now we introduce a new function.

**Definition 3.2.** We call the function defined by
\[
\bar{u} = \langle x, \nu \rangle \frac{1 + \cos \theta \langle \nu, e \rangle}{1 + \cos \theta \langle \nu, e \rangle},
\]
the capillary support function of capillary hypersurface \(\Sigma\) in \(\mathbb{R}^{n+1}_+\).

It is interesting to see that the capillary support function has nice properties.

**Proposition 3.3.** Along flow (1.2), the capillary support function \(\bar{u}\) satisfies
\[
\mathcal{L} \bar{u} = n - 2\bar{u}H + \bar{u}^2|h|^2 + 2u\left\langle \nabla \bar{u}, \frac{\nabla (1 + \cos \theta \langle \nu, e \rangle)}{1 + \cos \theta \langle \nu, e \rangle} \right\rangle,
\]
and
\[
\nabla_{\mu} \bar{u} = 0, \quad \text{on} \ \partial \Sigma_t.
\]

**Proof.** From the Codazzi formula, we have
\[
\langle \nu, e \rangle_{;kl} = h_{kl;s}(e_s, e) - (h^2)_{kl}(\nu, e),
\]
which yields
\[
\Delta \langle \nu, e \rangle = \langle \nabla H, e \rangle - |h|^2 \langle \nu, e \rangle.
\]

\[\square\]
Combining it with
\[ \langle \nabla f, e \rangle = n \cos \theta h(e^T, e^T) - \langle x, \nu \rangle \langle \nabla H, e \rangle - Hh(x^T, e^T), \]
we obtain
\[
\partial_t \langle \nu, e \rangle = -\langle \nabla f, e \rangle + h(e_i, T)\langle e_i, e \rangle
\]
\[
= -n \cos \theta h(e^T, e^T) + u\langle \nabla H, e \rangle + Hh(x^T, e^T) + h(T, e^T)
\]
Thus we conclude that
\[
\mathcal{L} \langle \nu, e \rangle = u|h|^2 \langle \nu, e \rangle.
\] (3.8)

Recall \( \bar{u} := \frac{u}{1 + \cos \theta \langle \nu, e \rangle} \).
(3.1) and (3.8) yield
\[
\mathcal{L} \bar{u} = \frac{1}{1 + \cos \theta \langle \nu, e \rangle} \mathcal{L} u - \frac{\langle x, \nu \rangle}{(1 + \cos \theta \langle \nu, e \rangle)^2} \mathcal{L} (1 + \cos \theta \langle \nu, e \rangle)
\]
\[
- \langle x, \nu \rangle \left( \frac{2u|\nabla (1 + \cos \theta \langle \nu, e \rangle)|^2}{(1 + \cos \theta \langle \nu, e \rangle)^2} - \frac{2\langle x, \nu \rangle \cos \theta \langle \nu, e \rangle}{(1 + \cos \theta \langle \nu, e \rangle)^2} \right)
\]
\[
= n - \frac{2uH}{1 + \cos \theta \langle \nu, e \rangle} + \frac{u^2|h|^2}{(1 + \cos \theta \langle \nu, e \rangle)^2} + 2u \left( \nabla \bar{u}, \frac{\nabla (1 + \cos \theta \langle \nu, e \rangle)}{1 + \cos \theta \langle \nu, e \rangle} \right) .
\]

Since \( \mu \) is a principal direction of \( \Sigma \) by [29, Proposition 2.3], we have on \( \partial \Sigma_t \)
\[
\nabla_\mu (1 + \cos \theta \langle \nu, e \rangle) = -\sin \theta h(\mu, \mu)\langle \nu, e \rangle = \cot \theta h(\mu, \mu)(1 + \cos \theta \langle \nu, e \rangle),
\] (3.9)

which, together with (3.2), implies
\[
\nabla_\mu \bar{u} = 0.
\]

□

Now we compute the evolution equation for the mean curvature \( H \).

**Proposition 3.4.** Along flow (1.2), we have
\[
\mathcal{L} H = 2\langle \nabla H, \nabla u \rangle - n|h|^2 + H^2,
\] (3.10)
and
\[
\nabla_\mu H = 0, \text{ on } \partial \Sigma_t.
\] (3.11)

**Proof.** From (3.6) it is easy to see
\[
f_{ij} = n \cos \theta h_{ij;k}\langle e_k, e \rangle - n \cos \theta (h^2)_{ij}\langle \nu, e \rangle - H_{ij}\langle x, \nu \rangle - H_i h_{jk}\langle x, e_k \rangle
\]
\[
- h_{j} h_{ik}\langle x, e_k \rangle - H h_{ij} - H h_{ij;k}\langle x, e_k \rangle + H (h^2)_{ij}\langle x, \nu \rangle.
\]
Combining it with Proposition 2.1 (6), we have
\[
\partial_t H = -\Delta f - f |h|^2 + \langle \nabla H, T \rangle
\]
\[
= -\langle n \cos \theta e, \nabla H \rangle + n \cos \theta \langle \nu, e \rangle |h|^2 + \langle x, \nu \rangle \Delta H + 2H_i \langle x, e_k \rangle
\]
\[
+ \langle H x, \nabla H \rangle - u |h|^2 - (n + n \cos \theta \langle \nu, e \rangle - Hu) |h|^2
\]
\[
+ \langle \nabla H, T \rangle + H^2
\]
\[
= \langle x, \nu \rangle \Delta H + \langle H x + T - n \cos \theta e, \nabla H \rangle + 2H_i \langle x, e_k \rangle - n |h|^2 + H^2,
\]
that is,
\[
\mathcal{L}H = 2\langle \nabla H, \nabla u \rangle - n |h|^2 + H^2.
\]
It has been shown in [29, Proposition 4.3], on \(\partial \Sigma_t\),
\[
\nabla_\mu f = \cot \theta h(\mu, \mu) f.
\]
Combining with (3.2) and (3.9), altogether implies
\[
\nabla_\mu H = \nabla_\mu \left( \frac{n(1 + \cos \theta \langle \nu, e \rangle)}{\langle x, \nu \rangle} - f \right) = 0.
\]
\[\square\]

If a hypersurface is strictly convex, we let \((b^{ij})\) denote the inverse of \((h_{ij})\), and set
\[
\bar{H} := \sum_{i=1}^{n} \frac{1}{\kappa_i} = g_{ij} b^{ji},
\]
which is the harmonic curvature. The harmonic curvature satisfies the following evolution equation, together a nice boundary inequality, provided that \(\theta \in (0, \frac{\pi}{2}]\). It will be used to prove that the flow preserves the convexity.

**Proposition 3.5.** Along flow (1.2), \(\bar{H}\) satisfies
\[
\mathcal{L}\bar{H} = -2u (b^{ij})^2 b^{ij} h_{ij, k} - 2b^{ij} H_{i,k} \langle x, e_i \rangle - u |h|^2 \bar{H} - H \bar{H} + nH \langle x, \nu \rangle + n^2. \tag{3.12}
\]
For \(\theta \in (0, \frac{\pi}{2}]\), it holds
\[
\nabla_\mu \bar{H} \leq 0, \text{ on } \partial \Sigma_t. \tag{3.13}
\]
**Proof.** Direct computation yields
\[
\nabla_k \bar{H} = -b^{is} \nabla_k h_{st} b_s^l,
\]
and
\[
\Delta \bar{H} = -b^{is} \Delta h_{st} b_s^l + 2b^{ip} \nabla^k h_{pq} b^{qs} \nabla_k h_{st} b_s^l. \tag{3.14}
\]
The Ricci equation and the Codazzi equation yield
\[
h_{kl;ii} = h_{ki;il} = h_{ki;il} + R_{di}^p h_{pk} + R_{ki}^p h_{pi}
\]
\[
= h_{ii;kl} + (h_{pl} H - h_{pi} h_{kl}) h_{pk} + (h_{pl} h_{kl} - h_{pi} h_{ki}) h_{pi}
\]
\[
= H_{,kl} + h_{pk} h_{pl} H - |h|^2 h_{kl},
\]
and hence
\[ H_{ij} = \Delta(h_{ij}) - H(h^2)_{ij} + |h|^2 h_{ij}. \]

Together with (3.14), it implies
\[ b_k^i b_k^j H_{kl} = b_k^i b_k^j (\Delta(h_{kl}) - H(h^2)_{kl} + |h|^2 h_{kl}) = -\Delta H + 2(b^{ij})^2 b^{ij} h_{ij,k}^2 - nH + |h|^2\bar{H}. \]

On the other hand,
\[ \partial_t \bar{H} = -b_k^i \partial_t b_k^i \]
\[ = -b_k^i \left( -\nabla^k \nabla^l f - f h_p^k h_p^k + \nabla_T h_k^i \right) b_i^l \]
\[ = b_k^i b_k^l \nabla^k \nabla^l f + n f - b_k^i \nabla_T h_k^i b_i^l. \]

Using again
\[ \langle x, \nu \rangle_{kl} = h_{kl} + h_{kl,s} \langle x, e_s \rangle - (h^2)_{kl} \langle x, \nu \rangle, \]
and
\[ \langle \nu, e \rangle_{kl} = h_{kl,s} \langle e_s, e \rangle - (h^2)_{kl} \langle \nu, e \rangle, \]
we obtain
\[ \nabla^k \nabla^l f = \nabla^k \nabla^l (n + n \cos \theta \langle \nu, e \rangle - H \langle x, \nu \rangle) \]
\[ = n \cos \theta (h_{kl,s} \langle e_s, e \rangle - (h^2)_{kl} \langle \nu, e \rangle) - H_{,kl} \langle x, h_{kl} e_s \rangle - H \langle x, h_{kl} e_s \rangle \]
\[ - H_{,kl} \langle x, \nu \rangle - H (h_{kl} + h_{kl,s} \langle x, e_s \rangle - (h^2)_{kl} \langle x, \nu \rangle), \]
and
\[ \partial_t \bar{H} = b_k^i b_k^l \left[ n \cos \theta \left( h_{kl,s} \langle e_s, e \rangle - (h^2)_{kl} \langle \nu, e \rangle \right) - H_{,kl} \langle x, h_{kl} e_s \rangle - H \langle x, h_{kl} e_s \rangle \right. \]
\[ - H_{,kl} \langle x, \nu \rangle - H (h_{kl} + h_{kl,s} \langle x, e_s \rangle - (h^2)_{kl} \langle x, \nu \rangle) \left] + n^2 (1 + \cos \theta \langle \nu, e \rangle) \right. \]
\[ - n H \langle x, \nu \rangle + \langle T, \nabla \bar{H} \rangle \]
\[ = -u ( - \Delta \bar{H} + 2(b^{ij})^2 b^{ij} h_{ij,k}^2 - nH + |h|^2 \bar{H} - n \cos \theta \langle e, \nabla \bar{H} \rangle \]
\[ - n^2 \cos \theta \langle \nu, e \rangle - 2 b_k^i H_{,kl} \langle x, e_i \rangle - H \bar{H} + H \langle x, \nabla \bar{H} \rangle + n H \langle x, \nu \rangle \]
\[ + n^2 (1 + \cos \theta \langle \nu, e \rangle) - n H \langle x, \nu \rangle + \langle T, \nabla \bar{H} \rangle. \]

It follows
\[ \mathcal{L} \bar{H} = -2u(b^{ij})^2 b^{ij} h_{ij,k}^2 - 2 b_k^i H_{,kl} \langle x, e_i \rangle - u |h|^2 \bar{H} - H \bar{H} + n H \langle x, \nu \rangle + n^2. \]

Along \( \partial \Sigma_t \), choosing an orthonormal frame \( \{ e^\alpha \}_{\alpha=2}^n \) of \( T \partial \Sigma_t \) such that \( \{ e_1 := \mu, (e_\alpha)_{\alpha=2}^n \} \) forms an orthonormal frame for \( T \Sigma_t \), from [29, Proposition 2.3 (3)], we know
\[ h_{\alpha \alpha, \mu} = \cot \theta h_{\alpha \alpha} (h_{11} - h_{\alpha \alpha}), \]
which, together with (3.11) and \( \theta \in (0, \frac{\pi}{2}] \), implies
\[
\nabla_\mu \bar{H} = -(b^{11})^2 h_{11;\mu} - \sum_{\alpha=2}^{n} (b^\alpha)^2 h_{\alpha\alpha;\mu}
\]
\[
= \sum_{\alpha=2}^{n} [(b^{11})^2 - (b^\alpha)^2] h_{\alpha\alpha;\mu}
\]
\[
= \cot \theta \sum_{\alpha=2}^{n} h_{\alpha\alpha}(h_{11} - h_{\alpha\alpha}) [(b^{11})^2 - (b^\alpha)^2]
\]
\[
\leq 0.
\]

\[ \square \]

### 3.2. Gradient estimates and the convergence.

One can show the short-time existence of flow (1.2) for any capillary hypersurface by using the method given in [18]. Since we consider here only for capillary hypersurfaces of star-shaped, i.e., \( \langle x, \nu \rangle > 0 \) on \( \Sigma_0 \), we can reduce flow to a scalar flow, i.e., (3.16), and hence the short-time existence follows easily. For the convenience of the reader and for our use later, we present the reduction argument here.

Assume that a capillary hypersurface \( \Sigma \subset \bar{\mathbb{R}}^{n+1}_+ \) is strictly star-shaped with respect to the origin. One can reparametrize it as a graph over \( S^n_+ \). Namely, there exists a positive smooth function \( \rho \in C^\infty \) defined on \( S^n_+ \) such that
\[
\Sigma = \{ \rho(X) X | X \in S^n_+ \},
\]
where \( X := (X_1, \ldots, X_n) \) is a local coordinate of \( S^n_+ \).

We denote \( \nabla^0 \) as the Levi-Civita connection on \( S^n_+ \) with respect to the standard round metric \( g_{ij} := g_{ij}^0 \), \( \partial_i := \partial X_i \), \( \sigma_{ij} := \sigma(\partial_i, \partial_j) \), \( \rho_i := \nabla^0_i \rho \), and \( \rho_{ij} := \nabla^0_i \nabla^0_j \rho \). Moreover, we denote \( \partial_\beta \) the unit outward normal of \( \partial S^n_+ \) in \( S^n_+ \). The induced metric \( g \) on \( \Sigma \) is given by
\[
g_{ij} = \rho^2 \sigma_{ij} + \rho_i \rho_j = e^{2\varphi} (\sigma_{ij} + \varphi_i \varphi_j),
\]
where \( \varphi(X) := \log \rho(X) \). Its inverse \( g^{-1} \) is given by
\[
g^{ij} = \frac{1}{\rho^2} \left( \sigma^{ij} - \frac{\rho^i \rho^j}{\rho^2 + |\nabla^0 \rho|^2} \right) = e^{-2\varphi} \left( \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right),
\]
where \( \rho^i := \sigma^{ij} \rho_j \), \( \varphi^i := \sigma^{ij} \varphi_j \) and \( v := \sqrt{1 + |\nabla^0 \varphi|^2} \). The unit outward normal vector field on \( \Sigma \) is given by
\[
\nu = \frac{1}{v} (\partial_\rho - \rho^{-2} \nabla^0 \rho) = \frac{1}{v} (\partial_\rho - \rho^{-1} \nabla^0 \varphi).
\]
Hence the capillary condition, i.e. \( \Sigma \) intersects with \( \partial \mathbb{R}^{n+1}_+ \) at an angle \( \theta \), is expressed by
\[
- \cos \theta = \langle \nu, \frac{1}{\rho} \partial_\beta \rangle = -\frac{1}{v} \nabla^0_{\partial_\beta} \varphi,
\]
which is equivalent to
\[
\nabla^0_{\partial_\beta} \varphi = \cos \theta \sqrt{1 + |\nabla^0 \varphi|^2}.
\]
The second fundamental form \( h \) on \( \Sigma \) is
\[
h_{ij} = \frac{e^\varphi}{v} (\sigma_{ij} + \varphi_i \varphi_j - \varphi_{ij}),
\]
and its Weingarten matrix \( (h^i_j) \) is
\[
h^i_j = g^{ik} h_{kj} = \frac{1}{e^\varphi v} \left[ \delta^i_j - (\sigma^{ik} - \frac{\varphi_i \varphi^k}{v^2}) \varphi_{kj} \right].
\]

We also have for the mean curvature,
\[
H = \frac{1}{e^\varphi v} \left[ n - \left( \sigma^{ij} - \frac{\varphi_i \varphi^j}{v^2} \right) \varphi_{ij} \right]. \tag{3.15}
\]

Now under the assumption of star-shapedness, the flow (1.2) can be reduced to a scalar evolution equation for \( \varphi(z, t) = \log \rho(X(z, t), t) \),
\[
\frac{\partial_t \varphi}{e^\varphi} = F, \quad \text{on } \partial \Sigma^+_n \times [0, T^*),
\]
\[
\varphi(\cdot, 0) = \varphi_0(\cdot), \quad \text{on } \Sigma^+_n, \tag{3.16}
\]
where \( \varphi_0 \) is the parameterization radial function of \( x_0(M) \) over \( \mathbb{S}^n_+ \), and
\[
F := \frac{1}{e^\varphi \text{div}_{\mathbb{S}^n_+}} \left( \frac{\nabla^0_\beta \varphi}{v} \right) + \frac{v}{e^\varphi} \left[ 1 - \frac{\cos \theta}{v} \left( \cos \beta + \sin \beta \nabla^0_\beta \varphi \right) - \frac{n}{v^2} \right].
\]

The short-time existence of this scalar flow follows from the standard PDE theory. It is clear that it provides the short-time existence of flow (1.2). Now assume \( T^* > 0 \) is the maximal time of existence of a solution to (1.2), in the class of star-shaped hypersurfaces.

The star-shapedness of \( \Sigma_0 \) implies that there exist some \( 0 < r_1 < r_2 < \infty \), such that
\[
\Sigma_0 \subset \widehat{C}_\theta, r_2 \setminus \widehat{C}_\theta, r_1.
\]

Following the argument in [29, Proposition 4.2], which is based on the avoidable principle, we have

**Proposition 3.6.** For any \( t \in [0, T^*) \), along flow (1.2), it holds
\[
\Sigma_t \subset \widehat{C}_\theta, r_2 \setminus \widehat{C}_\theta, r_1. \tag{3.17}
\]
where \( C_{\theta, r} \) defined by (1.4) and \( r_1, r_2, c_0 \) only depend on \( \Sigma_0 \).

Next we show the star-shapedness is preserved along flow (1.2), which follows the uniform gradient estimate of \( \varphi \) in (3.16).

**Proposition 3.7.** Let \( \Sigma_0 \) be a star-shaped hypersurface with capillary boundary in \( \mathbb{R}^{n+1}_+ \), and \( \theta \in (0, \pi) \), then there exists \( c_0 > 0 \) depending only on \( \Sigma_0 \), such that
\[
\langle x, \nu \rangle(p, t) \geq c_0. \tag{3.18}
\]
for all \( (p, t) \in M \times [0, T^*) \).
Proof. From Proposition 3.3 and \(|h|^2 \geq \frac{H^2}{n}\), we see
\[
\mathcal{L} \bar{u} = n - \frac{2uH}{1 + \cos \theta \langle \nu, e \rangle} + \frac{u^2 |h|^2}{(1 + \cos \theta \langle \nu, e \rangle)^2} \mod \nabla \bar{u}
\geq \left( \frac{u|H|}{\sqrt{n}(1 + \cos \theta \langle \nu, e \rangle)} - \sqrt{n} \right)^2 \geq 0,
\]
together with \(\nabla \mu \bar{u} = 0\) on \(\partial M\), it implies
\[
\bar{u} \geq \min_M \bar{u}(\cdot, 0).
\]
Since \(1 + \cos \theta \langle \nu, e \rangle \geq 1 - \cos \theta > 0\), we conclude
\[
\langle x, \nu \rangle = \bar{u} \cdot (1 + \cos \theta \langle \nu, e \rangle) \geq c_0,
\]
for some positive constant \(c_0 > 0\) independent of \(t\).

**Proposition 3.8.** If the initial hypersurface \(\Sigma_0 \subset \bar{\mathbb{R}}^{n+1}_+\) is star-shaped, then flow (1.2) exists for all time. Moreover, it smoothly converges to a uniquely determined spherical cap \(C_{\theta, r}(e)\) given by (1.4) with capillary boundary, as \(t \to +\infty\).

*Proof.* From (3.17) we have a uniform bound for \(\varphi\), and from (3.18), we have a uniform bound for \(v\), and hence a bound for \(\nabla^0 \varphi\). Therefore \(\varphi\) is uniformly bounded in \(C^1(S^n_+ \times [0, T^*)\) and the scalar equation in (3.16) is uniformly parabolic. Since \(|\cos \theta| < 1\), from the standard quasi-linear parabolic theory with a strictly oblique boundary condition theory (cf. [19, 21]), we conclude the uniform \(C^\infty\) estimates and the long-time existence. And the convergence can be shown similarly by using the argument as in [25, 32, 29], we omit it here. \(\square\)

### 3.3. Preserving Convexity

In this subsection, we show flow (1.2) preserves the convexity and finish the proof of Theorem 1.4. First, we show the uniform upper bound of \(H\) along flow (1.2).

**Proposition 3.9.** If \(\Sigma_t\) solves flow (1.2), then the mean curvature is uniformly bounded from above, that is,
\[
H(p, t) \leq \max_M H(\cdot, 0), \quad \forall (p, t) \in M \times [0, T^*).
\]

*Proof.* From equation (3.10) and \(n|h|^2 \geq H^2\), we see
\[
\mathcal{L} H \leq 0, \quad \mod \nabla H,
\]
and \(\nabla \mu H = 0\) on \(\partial M\), hence the conclusion follows directly from the maximum principle. \(\square\)

Next we show the uniform lower bound of mean curvature, that is, the mean convexity is preserved along flow (1.2).

**Proposition 3.10.** If \(\Sigma_t\) solves flow (1.2) with an initial hypersurface \(\Sigma_0\) being a strictly mean convex capillary hypersurface in \(\bar{\mathbb{R}}^{n+1}_+\), then
\[
H(p, t) \geq C, \quad \forall (p, t) \in M \times [0, T^*),
\]
where the positive constant \( C \) depends on the initial datum.

**Proof.** Define the function
\[
P := H\bar{u}.
\]
Using (3.5) and (3.11), it is easy to see that
\[
\nabla_{\mu} P = 0, \quad \text{on } \partial M.
\]
Using (3.4) and (3.10), we obtain
\[
\mathcal{L}P = \bar{u}\mathcal{L}H + H\mathcal{L}\bar{u} - 2u(\nabla \bar{u}, \nabla H)
= \bar{u}^2H|h|^2 - \bar{u}H^2 - n\bar{u}|h|^2 + nH + 2\bar{u}(\nabla (1 + \cos \theta \langle \nu, e \rangle), \nabla P).
\]
From (3.20) and the Hopf Lemma, if \( P \) attains the minimum value at \( t = 0 \), then the conclusion follows directly by combining with the uniform bound of \( \bar{u} \). Therefore, we assume that \( P \) attains the minimum value at some interior point, say \( p_0 \in \text{int}(M) \). At \( p_0 \), we have
\[
\nabla P = 0, \quad \mathcal{L}P \leq 0.
\]
Substituting it into (3.21) yields
\[
\left( \frac{n}{H} - \bar{u} \right) \bar{u}|h|^2 + \bar{u}H \geq n.
\]
If \( \bar{u}H \geq n \) at \( p_0 \), then we are done. Assume now that \( \bar{u}H < n \) at \( p_0 \), then using \( |h|^2 \leq nH^2 \)
in (3.22), we obtain
\[
\bar{u}H(p_0, t) \geq c,
\]
for some positive constant \( c \), which only depends on \( n \). This also yields the desired estimate.

From above discussion, we complete the proof of Proposition 3.10. \( \square \)

We now show that \( V_{k, \theta}(\Sigma_t) \) is non-increasing under flow (1.2).

**Proposition 3.11.** As long as flow (1.2) exists and \( \Sigma_t \) is strictly convex, the enclosed volume \( V_{0, \theta}(\Sigma_t) \) is preserved and \( V_{k, \theta}(\Sigma_t) \) is non-increasing for \( 1 \leq k \leq n \).

**Proof.** Using [29, Theorem 2.6] and Minkowski formula (1.3), we see
\[
\partial_t V_{0, \theta}(\Sigma_t) = \int_{\Sigma_t} f dA_t = 0,
\]
and
\[
\partial_t V_{k, \theta}(\Sigma_t) = \frac{n + 1 - k}{n + 1} \int_{\Sigma_t} f H_k dA_t
= \frac{n + 1 - k}{n + 1} \int_{\Sigma_t} [nH_k (1 + \cos \theta \langle \nu, e \rangle) - HH_k(x, \nu)] dA_t
\leq \frac{n(n + 1 - k)}{n + 1} \int_{\Sigma_t} [H_k(1 + \cos \theta \langle \nu, e \rangle) - H_{k+1}(x, \nu)] dA_t
= 0,
\]
where we have used the Newton-MacLaurin inequality $H_1 H_k \geq H_{k+1}$, Proposition 3.7 and the Minkowski formula (1.6) in the last two steps.

The proof implies the following nice property, i.e. a characterization result on the spherical cap.

**Proposition 3.12.** If a capillary hypersurface $\Sigma$ satisfies (1.5), i.e.,

$$1 + \cos \theta \langle \nu, e \rangle - \frac{H}{n} \langle x, \nu \rangle = 0,$$

(3.24)

then it is $C_{r,\theta}$ for some $r > 0$.

**Proof.** Since $|\cos \theta| < 1$, (3.24) implies that $\langle x, \nu \rangle H \geq c_0 > 0$ on $\Sigma$, which means that $H$ can never change sign. It is clear that $\Sigma$ is contained in a $C_{r,\theta}$ with $r$ large. Then we decrease $r$, there exists a first largest radius $r_0$ and a first touch point $x_0 \in \Sigma$. If $x_0$ lies in the interior of $\Sigma$, then $C_{r_0,\theta}$ touches $\Sigma$ at $x_0$ tangentially. If $x_0 \in \partial \Sigma$, then due to the assumption of the contact angle of $\Sigma$, $C_{r_0,\theta}$ touches also $\Sigma$ at $x_0$ tangentially. It follows that $H(x_0) > 0$, and hence $H > 0$ and $\langle x, \nu \rangle > 0$ on $\Sigma$. Now from (3.24) and $H^2 = n^2 H_1^2 \geq n^2 H_2$, we have

$$0 = \int_{\Sigma} \left[ nH \left( 1 + \cos \theta \langle \nu, e \rangle \right) - H^2 \langle x, \nu \rangle \right] dA$$

$$\leq n^2 \int_{\Sigma} \left[ H_1 \left( 1 + \cos \theta \langle \nu, e \rangle \right) - H_2 \langle x, \nu \rangle \right] dA$$

$$= 0,$$

the equality implies that $\Sigma$ is umbilical, which follows also the conclusion. □

Now we prove that the mean curvature type flow (1.2) preserves the convexity, if $\theta \in (0, \frac{\pi}{2}]$.

**Proposition 3.13.** Let $\Sigma_0$ be a strictly convex hypersurface, if $\Sigma_t$ solves flow (1.2) with the initial value $\Sigma_0$ and $\theta \in (0, \frac{\pi}{2}]$, then

$$\min_{1 \leq i \leq n} \kappa_i(p, t) \geq c, \quad \forall (p, t) \in M \times [0, T^*),$$

where the positive constant $c$ depends only on $\Sigma_0$.

**Proof.** It is equivalent to show the uniform upper bound for $\bar{H}$. If the maximum value of $\bar{H}$ is reached at $t = 0$, then we are done. Otherwise, from $\nabla_{\mu} \bar{H} \leq 0$ on $\partial M$ in Proposition 3.5 and the Hopf boundary Lemma, we have that $\bar{H}$ attains its maximum value at some interior point, say $p_0 \in \text{int}(\Sigma_t)$. At $p_0$, we choose an orthonormal frame $\{e_i\}_{i=1}^n$ such that $(h_{ij})$ is
diagonal, which follows that \((b_{ij})\) is also diagonal. Noticing that for each fixed \(k\), it holds
\[
\sum_{i,j} b_{ij} h_{ij;k}^2 = \sum_j \frac{1}{H} \left( 1 + \sum_{l \neq j} \frac{h_{ll}}{h_{jj}} \right) \left( h_{jj;k}^2 + \sum_{i \neq j} h_{ij;k}^2 \right) \\
\geq \frac{1}{H} \left( \sum_j h_{jj;k}^2 + \sum_{i \neq j} h_{ij;k}^2 \right) \\
= \frac{1}{H} \left[ \sum_j h_{jj;k}^2 + \sum_{i>j} \left( \frac{h_{ii}}{h_{jj}} h_{ij;k}^2 + \frac{h_{ij}}{h_{ii}} h_{ii;k}^2 \right) \right] \\
\geq \frac{H^2}{H},
\]
where the last inequality follows from the Cauchy-Schwarz inequality. Substituting this into (3.12), at \(p_0\), we have
\[
0 \leq L \bar{H} \\
\leq -2H^{-1} u(b_{ii})^2 H_i^2 - 2b_{ii} H_i \langle x, e_i \rangle - u|h|^2 \bar{H} - H \bar{H} + n^2 + nu \bar{H} \\
\leq \frac{Hu^{-1}|x|^2}{2} - u|h|^2 \bar{H} - H \bar{H} + n^2 + nu \bar{H},
\]
which together with Proposition 3.10, implies \(\bar{H} \leq C\), the desired estimate. Hence we complete the proof.

**Proof of Theorem 1.4.**

Assume that \(\Sigma\) is strictly convex, the proof of Theorem 1.4 follows from our main Theorem 1.2, the monotonicity of the relative quermassintegrals, Proposition 3.11 and Proposition 3.13.

When \(\Sigma\) is convex but not strictly convex, the inequality (1.9) follows by approximation. The equality characterization can be proved similar to [25, Section 4], by using an argument of [10]. We omit the details here.

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MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, FREIBURG IM BREISGAU, 79104, GERMANY

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, P.R.CHINA
Email address: xinquin.mei@math.uni-freiburg.de

MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, FREIBURG IM BREISGAU, 79104, GERMANY
Email address: guofang.wang@math.uni-freiburg.de

SCHOOL OF MATHEMATICAL SCIENCES, ANHUI UNIVERSITY, HEFEI, 230601, P. R. CHINA
Email address: ljweng08@mail.ustc.edu.cn