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Keywords
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A note on the boundary behavior for a modified Green function in the upper-half space

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Abstract
Motivated by (Xu et al. in Bound. Value Probl. 2013:262, 2013) and (Yang and Ren in Proc. Indian Acad. Sci. Math. Sci. 124(2):175-178, 2014), in this paper we aim to construct a modified Green function in the upper-half space of the n-dimensional Euclidean space, which generalizes the boundary property of general Green potential.

Keywords: modified Green function; capacity; upper-half space

1 Introduction and main results
Let $\mathbb{R}^n$ ($n \geq 2$) denote the n-dimensional Euclidean space. The upper half-space $H$ is the set $H = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}$, whose boundary and closure are $\partial H$ and $\bar{H}$ respectively.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball with center at $x$ and radius $r$.

Set $E_\alpha(x) = \begin{cases} -\log |x| & \text{if } \alpha = n = 2, \\ |x|^{\alpha-n} & \text{if } 0 < \alpha < n. \end{cases}$

Let $G_\alpha$ be the Green function of order $\alpha$ for $H$, that is,

$G_\alpha(x, y) = E_\alpha(x - y) - E_\alpha(x - y^*)$, \quad $x, y \in \bar{H}, x \neq y, 0 < \alpha \leq n,$

where $*$ denotes reflection in the boundary plane $\partial H$ just as $y^* = (y_1, y_2, \ldots, -y_n)$.

In case $\alpha = n = 2$, we consider the modified kernel function, which is defined by

$E_{n,m}(x - y) = \begin{cases} E_\alpha(x - y) & \text{if } |y| < 1, \\ E_\alpha(x - y) + 3i(\log y - \sum_{k=1}^{m-1} \left( \frac{x}{ky} \right) ) & \text{if } |y| \geq 1. \end{cases}$

In case $0 < \alpha < n$, we define

$E_{\alpha,m}(x - y) = \begin{cases} E_\alpha(x - y) & \text{if } |y| < 1, \\ E_\alpha(x - y) - \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{\alpha-n}} \frac{n-\alpha}{\alpha} C_k \left( \frac{x}{|y|} \right) & \text{if } |y| \geq 1. \end{cases}$
where $m$ is a non-negative integer, $C^\omega_k(t)$ ($\omega = \frac{2-n}{2}$) is the ultraspherical (or Gegenbauer) polynomial (see [1]). The expression arises from the generating function for Gegenbauer polynomials

$$(1 - 2tr + r^2)^{-\omega} = \sum_{k=0}^{\infty} C^\omega_k(t)r^k,$$  \hspace{1cm} (1.1)

where $|r| < 1$, $|t| \leq 1$ and $\omega > 0$. The coefficient $C^\omega_k(t)$ is called the ultraspherical (or Gegenbauer) polynomial of degree $k$ associated with $\omega$, the function $C^\omega_k(t)$ is a polynomial of degree $k$ in $t$.

Then we define the modified Green function $G_{\alpha,m}(x,y)$ by

$$G_{\alpha,m}(x,y) = \begin{cases} E_{n,m+1}(x-y) - E_{n,m+1}(x-y^*) & \text{if } \alpha = n = 2, \\ E_{a,m+1}(x-y) - E_{a,m+1}(x-y^*) & \text{if } 0 < \alpha < n, \end{cases}$$

where $x,y \in \overline{H}$ and $x \neq y$. We remark that this modified Green function is also used to give unique solutions of the Neumann and Dirichlet problem in the upper-half space [2–4].

Write

$$G_{\alpha,m}(x,\mu) = \int_H G_{\alpha,m}(x,y) d\mu(y),$$

where $\mu$ is a non-negative measure on $H$. Here note that $G_{2,0}(x,\mu)$ is nothing but the general Green potential.

Let $k$ be a non-negative Borel measurable function on $\mathbb{R}^n \times \mathbb{R}^n$, and set

$$k(y,\mu) = \int_E k(y,x) d\mu(x) \quad \text{and} \quad k(\mu,x) = \int_E k(y,x) d\mu(y)$$

for a non-negative measure $\mu$ on a Borel set $E \subset \mathbb{R}^n$. We define a capacity $C_k$ by

$$C_k(E) = \sup \mu(\mathbb{R}^n), \quad E \subset H,$$

where the supremum is taken over all non-negative measures $\mu$ such that $S_\mu$ (the support of $\mu$) is contained in $E$ and $k(y,\mu) \leq 1$ for every $y \in H$.

For $\beta \leq 0$, $\delta \leq 0$ and $\beta \leq \delta$, we consider the kernel function

$$k_{\alpha,\beta,\delta}(y,x) = x^{\beta} y^{\delta} G_\alpha(x,y).$$

Now we prove the following result. For related results in a smooth cone and tube, we refer the reader to the papers by Qiao (see [5, 6]) and Liao-Su (see [7]), respectively. The readers may also find some related interesting results with respect to the Schrödinger operator in the papers by Su (see [8]), by Polidoro and Ragusa (see [9]) and the references therein.

Theorem Let $n + m - \alpha + \delta + 2 \geq 0$. If $\mu$ is a non-negative measure on $H$ satisfying

$$\int_H \frac{y^{\delta+1}}{n^{\alpha+m-\alpha+\delta+2}} d\mu(y) < \infty,$$  \hspace{1cm} (1.2)
then there exists a Borel set $E \subset H$ with properties:

\[
\begin{align*}
(1) \quad & \lim_{{x_n \to 0, x_n \in H - E}} x_n^{n-\alpha - \beta + \delta + 1} (1 + |x|)^{\alpha + \gamma - \delta - \beta + 2} G_{\alpha, \mu}(x, \mu) = 0; \\
(2) \quad & \sum_{i=1}^{\infty} 2^i (n-\alpha + \beta + \delta) C_{\mu, \mu}(E_i) < \infty,
\end{align*}
\]

where $E_i = \{x \in E : 2^{-i} \leq x_n < 2^{-i+1}\}$.

**Remark** By using Lemma 4 below, condition (2) in Theorem with $\alpha = 2$, $\beta = 0$, $\delta = 0$ means that $E$ is $2$-thin at $\partial H$ in the sense of [10].

## 2 Some lemmas

Throughout this paper, let $M$ denote various constants independent of the variables in questions, which may be different from line to line.

**Lemma 1** There exists a positive constant $M$ such that $G_\alpha(x, y) \leq \frac{M}{|x-y|^{n-\alpha}}$, where $0 < \alpha \leq n$, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $H$.

This can be proved by a simple calculation.

**Lemma 2** Gegenbauer polynomials have the following properties:

\[
\begin{align*}
(1) \quad & |C_k^n(t)| \leq C_k^n(1) = \frac{(2n+1)!}{(2n)!} |t| \leq 1; \\
(2) \quad & \frac{d}{dt} C_k^n(t) = 2n C_{k-1}^{n+1}(t), k \geq 1; \\
(3) \quad & \sum_{k=0}^{\infty} C_k^n(t)^k = (1 - r^{-2})\omega; \\
(4) \quad & |C_k^n(t) - C_k^{*n}(t)| \leq (n-\alpha)C_k^{n-1}(1)|t - t^*|, |t| \leq 1, |t^*| \leq 1.
\end{align*}
\]

**Proof** (1) and (2) can be derived from [1], p.232. Equality (3) follows from expression (1.1) by taking $t = 1$; property (4) is an easy consequence of the mean value theorem, (1) and also (2).  

**Lemma 3** For $x, y \in \mathbb{R}^n (\alpha = n = 2)$, we have the following properties:

\[
\begin{align*}
(1) \quad & |\sum_{k=0}^{m} \frac{x^k}{y^k}| \leq \sum_{k=0}^{m} \frac{|x^k|}{|y^k|}; \\
(2) \quad & |\sum_{k=0}^{\infty} \frac{x^{k+1}}{y^{k+1}}| \leq M \sum_{k=0}^{\infty} \frac{|x|^{k+1}}{|y|^{k+1}}; \\
(3) \quad & |G_{\alpha, \mu}(x, y) - G_\alpha(x, y)| \leq M \sum_{k+1}^{m} \frac{|x^k y^{k-1}|}{|y|^{k+1}}; \\
(4) \quad & |G_{\alpha, \mu}(x, y)| \leq M \sum_{k=m+1}^{\infty} \frac{|x^k y^{k-1}|}{|y|^{k+1}}.
\end{align*}
\]

The following lemma can be proved using Fuglede (see [11], Théorèm 7.8).

**Lemma 4** For any Borel set $E$ in $H$, we have $C_{k_\alpha}(E) = \hat{C}_{k_\alpha}(E)$, where $\hat{C}_{k_\alpha}(E) = \inf \lambda(H)$, $k_\alpha = k_{\alpha, 0, 0}$, the infimum being taken over all non-negative measures $\lambda$ on $H$ such that $k_\alpha(\lambda, x) \geq 1$ for every $x \in E$.

Following [10], we say that a set $E \subset H$ is $\alpha$-thin at the boundary $\partial H$ if

\[
\sum_{i=1}^{\infty} 2^i (n-\alpha) C_{k_\alpha}(E_i) < \infty,
\]

where $E_i = \{x \in E : 2^{-i} \leq x_n < 2^{-i+1}\}$.  

3 Proof of Theorem

We write

\[ G_{\alpha,m}(x, \mu) = \int_{G_1} G_{\alpha}(x, y) \, d\mu(y) + \int_{G_2} G_{\alpha}(x, y) \, d\mu(y) + \int_{G_3} \left[ G_{\alpha,m}(x, y) - G_{\alpha}(x, y) \right] \, d\mu(y) \]

\[ + \int_{G_4} G_{\alpha,m}(x, y) \, d\mu(y) + \int_{G_5} G_{\alpha,m}(x, y) \, d\mu(y) \]

\[ = U_1(x) + U_2(x) + U_3(x) + U_4(x) + U_5(x), \]

where

\[ G_1 = \left\{ y \in H : |y - x| \leq \frac{x_n}{2} \right\}, \quad G_2 = \left\{ y \in H : |y| \geq 1, \frac{x_n}{2} < |y - x| \leq 3|x| \right\}, \]

\[ G_3 = \left\{ y \in H : |y| \geq 1, |y - x| \leq 3|x| \right\}, \quad G_4 = \left\{ y \in H : |y| \geq 1, |y - x| > 3|x| \right\}, \]

\[ G_5 = \left\{ y \in H : |y| < 1, |y - x| > \frac{x_n}{2} \right\}. \]

We distinguish the following two cases.

Case 1. 0 < \alpha < n.

By assumption (1.2) we can find a sequence \{a_i\} of positive numbers such that \(\lim_{i \to \infty} a_i = \infty\) and \(\sum_{i=1}^{\infty} a_i b_i < \infty\), where

\[ b_i = \int_{|y|<2^{-i-1},|y|>2^{-i+2}} \frac{y^{d+1}}{(1 + |y|)^{n+m-\alpha+d+2}} \, d\mu(y). \]

Consider the sets

\[ E_i = \left\{ x \in H : 2^{-i} \leq x_n < 2^{-i+1}, \frac{x^{d+1}}{(1 + |x|)^{n+m-\alpha+d+2}} U_1(x) \geq a_i^{-1} 2^{-i(\beta-1)} \right\} \]

for \(i = 1, 2, \ldots\). Set

\[ G = \bigcup_{x \in E_i} B \left( x, \frac{x_n}{2} \right). \]

Then \( G \subseteq \{ y \in H : 2^{-i-1} < y_n < 2^{-i+2} \} \). Let \( \nu \) be a non-negative measure on \( H \) such that \( S_\nu \subseteq E_i \), where \( S_\nu \) is the support of \( \nu \). Then we have \( k_{\alpha, \beta}(y, \nu) \leq 1 \) for \( y \in H \) and

\[ \int_H d\nu \leq a_1 2^{1-i(\beta-1)} \int_H \frac{x^{d+1}}{(1 + |x|)^{n+m-\alpha+d+2}} U_1(x) \, d\nu(x) \]

\[ \leq Ma_1 2^{1-i(\beta-1)} 2^{(i+1)(n-\alpha+d+1)} \int_G k_{\alpha, \beta}(y, \nu) \frac{x^{d}}{(1 + |y|)^{n+m-\alpha+d+2}} \, d\mu(y) \]

\[ \leq Ma_1 2^{1-i(\beta-1)} 2^{(i+1)(n-\alpha+d+1)} 2^{i+1} \int_{|y|<2^{-i-1},|y|>2^{-i+2}} \frac{x^{d+1}}{(1 + |y|)^{n+m-\alpha+d+2}} \, d\mu(y) \]

\[ \leq M2^{-\alpha+\beta+d+2} 2^{-i(n-\alpha+\beta+d)} a_i b_i. \]
So that

\[ C_{\alpha,\beta}(E_i) \leq M2^{-i(n-\alpha+\beta+1)} \beta_i, \]

which yields

\[ \sum_{i=1}^{\infty} 2^{i(n-\alpha+\beta+1)} C_{\alpha,\beta}(E_i) < \infty. \]

Setting \( E = \bigcup_{i=1}^{\infty} E_i \), we see that (2) in Theorem is satisfied and

\[ \lim_{x_n \to 0, x \in H - E} \sum_{k=0}^{\infty} x_n^{n-\alpha-\beta-\delta+1} \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) = 0. \] (3.1)

For \( U_2(x) \), by Lemma 1 we have

\[ |U_2(x)| \leq Mx_n \int_{G_2} \frac{y_n}{|x - y|^{n-\alpha+\beta+1}} d\mu(y) \]
\[ \leq Mx_n^{n-\alpha-1} |x|^{n+m-\alpha+\beta+2} \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) \]
\[ \leq Mx_n^{n-\alpha-1} |x|^{n+m-\alpha+\beta+2} \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y). \] (3.2)

Note that \( C_0(t) \equiv 1 \). By (3) and (4) in Lemma 2, we take \( t = \frac{\xi^*}{|x|} \), \( \xi^* = \frac{x\mu^*}{|x|} \) in Lemma 2(4) and obtain

\[ |U_3(x)| \leq \int_{G_2} \sum_{k=1}^{m} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n - \alpha) C_{k-1}^{n-\alpha+2} \left(1 \times y_n \right) |y|^{n+m-\alpha+\beta+2} \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) \]
\[ \leq Mx_n |x|^m \sum_{k=1}^{m} \frac{1}{2^{k-1}} C_{k-1}^{n-\alpha+2} \left(1 \right) \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) \]
\[ \leq Mx_n |x|^m. \] (3.3)

Similarly, we have by (3) and (4) in Lemma 2

\[ |U_4(x)| \leq \int_{G_4} \sum_{k=m+1}^{\infty} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n - \alpha) C_{k-1}^{n-\alpha+2} \left(1 \times y_n \right) |y|^{n+m-\alpha+\beta+2} \int_{G_2} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) \]
\[ \leq Mx_n |x|^m \sum_{k=m+1}^{\infty} \frac{1}{2^{k-1}} C_{k-1}^{n-\alpha+2} \left(1 \right) \int_{G_4} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y) \]
\[ \leq Mx_n |x|^m. \] (3.4)

Finally, by Lemma 1, we have

\[ |U_5(x)| \leq Mx_n^{n-\alpha-1} \int_{G_5} \frac{y_n^{\delta+1}}{\left(1 + |y|\right)^{n+s+m-\alpha+\beta+2}} d\mu(y). \] (3.5)
Combining (3.1), (3.2), (3.3), (3.4) and (3.5), by Lebesgue’s dominated convergence theorem, we prove Case 1.

Case 2. \( \alpha = n = 2 \).

In this case, \( U_1(x) \), \( U_2(x) \) and \( U_5(x) \) can be proved similarly as in Case 1. Here we omit the details and state the following facts:

\[
\lim_{x_n \to 0, x_n \in H} x_n^{\beta+1} U_1(x) = 0,
\]

where \( E = \bigcup_{i=1}^{\infty} E_i \), and \( \sum_{i=1}^{\infty} 2^{(\beta+i)} C_{\alpha,\beta,i}(E_i) < \infty \),

\[
\lim_{x_n \to 0, x_n \in H} x_n^{\beta+1} \left[ U_2(x) + U_5(x) \right] = 0.
\]

By Lemma (3), we obtain

\[
\left| U_3(x) \right| \leq \int_{G_3} \sum_{k=1}^{m} k x_n y_n |x|^{k-1} \frac{2 |y|^{m+\delta+2}}{y_n^{m+\delta+2}} \frac{y_n^{\beta+1}}{(1 + |y|)^{m+\delta+2}} d\mu(y)
\]

\[
\leq M x_n |x|^m \sum_{k=1}^{m} k \int_{G_3} \frac{y_n^{\beta+1}}{(1 + |y|)^{m+\delta+2}} d\mu(y)
\]

\[
\leq M x_n |x|^m.
\]

By Lemma (4), we have

\[
\left| U_4(x) \right| \leq \int_{G_4} \sum_{k=m+1}^{\infty} k x_n y_n |x|^{k-1} \frac{2 |y|^{m+\delta+2}}{y_n^{m+\delta+2}} \frac{y_n^{\beta+1}}{(1 + |y|)^{m+\delta+2}} d\mu(y)
\]

\[
\leq M x_n |x|^m \sum_{k=m+1}^{\infty} k \int_{G_4} \frac{y_n^{\beta+1}}{(1 + |y|)^{m+\delta+2}} d\mu(y)
\]

\[
\leq M x_n |x|^m.
\]

Combining (3.6), (3.7), (3.8) and (3.9), we prove Case 2.

Hence the proof of the theorem is completed.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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