SUBHARMONIC SOLUTIONS AND MINIMAL PERIODIC SOLUTIONS OF FIRST-ORDER HAMILTONIAN SYSTEMS WITH ANISOTROPIC GROWTH

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Abstract. Using a homologically link theorem in variational theory and iteration inequalities of Maslov-type index, we show the existence of a sequence of subharmonic solutions of non-autonomous Hamiltonian systems with the Hamiltonian functions satisfying some anisotropic growth conditions, i.e., the Hamiltonian functions may have simultaneously, in different components, superquadratic, subquadratic and quadratic behaviors. Moreover, we also consider the minimal period problem of some autonomous Hamiltonian systems with anisotropic growth.

1. Introduction. In this paper, we first consider subharmonic solutions of the following Hamiltonian system

\[
\begin{aligned}
-J \dot{z} &= H_z'(t, z), \\
z(k\tau) &= z(0), \quad k \in \mathbb{N},
\end{aligned}
\]

where \(H_z'\) is the gradient of \(H\) with respect to \(z = (p_1, \cdots, p_n, q_1, \cdots, q_n) \in \mathbb{R}^{2n}\) and \(J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\) with \(I_n\) being the \(n \times n\) identity matrix.

Denote any principal diagonal matrix \(\text{diag}\{a_1, \cdots, a_n, b_1, \cdots, b_n\} \in \mathbb{R}^{2n}\) by \(V(a, b)\) with \(a = (a_1, \cdots, a_n)\) and \(b = (b_1, \cdots, b_n)\), then

\[V(a, b)(z) = (a_1 p_1, \cdots, a_n p_n, b_1 q_1, \cdots, b_n q_n).\]

Now we suppose the Hamiltonian function \(H\) satisfying the following conditions as in [32] with a bit difference.

(H1) \(H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})\) is nonnegative and \(\tau\)-periodic with respect to \(t\).

(H2) There exist \(\beta > 1\) and \(c_1, c_2, \alpha_i, \beta_i > 0\) with \(\alpha_i + \beta_i = 1\) (\(i = 1, 2, \cdots, n\)) such that

\[H_z'(t, z) \cdot V_1(z) - H(t, z) \geq c_1 |z|^{\beta} - c_2, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n},\]

where \(V_1 = V(\alpha, \beta)\).
Theorem 1.1. Suppose $H$ satisfies (H1), (H2), (H4), (H5) and (H3) there exist constants $\xi_i, \eta_i > 0$ with $\xi_i + \eta_i = 1$ \((i = 1, 2, \cdots, n)\) such that

$$
H'_i(t, z) \cdot V_2(z) - H(t, z) \geq c_1 |H'_i(t, z)| - c_2, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n},
$$

where $V_2 = V(\xi, \eta)$ and $\max_{1 \leq i \leq n} \left\{ \frac{\xi_i}{\eta_i}, \frac{\eta_i}{\xi_i}, \frac{\xi_i}{\eta_i}, \frac{\eta_i}{\xi_i} \right\} < 1 + \beta$.

Then for each integer $k \geq 1$, the system (1) possesses a $k\tau$-periodic nonconstant solution $z_k$ such that $z_k$ and $z_{pk}$ are geometrically distinct provided $p > 2n + 1$. If all $z_k$ are non-degenerate, then $z_k$ and $z_{pk}$ \((p > 1)\) are geometrically distinct.

Note that (H3)' is weaker than (H3), so we have a similar result stated as a corollary of Theorem [1,1].

Corollary 1. Suppose $H$ satisfies (H1)-(H5), then we have the same results as in Theorem [1,1].

Theorem 1.2. The conclusions of Theorem [1,1] still hold if $H$ satisfies the conditions

\begin{enumerate}
  \item [(C1)] $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ is nonnegative and $\tau$-periodic with respect to $t$,
  \item [(C2)] there exist constants $0 < \theta < 1$, $R, \varphi_i, \psi_i > 0$ with $\frac{1}{\varphi_i} + \frac{1}{\psi_i} = 1$ \((i = 1, 2, \cdots, n)\) such that, by setting $V_3 = V(\varphi^{-1}, \psi^{-1})$, we have
    $$
    \theta H'_i(t, z) \cdot V_3(z) \geq H(t, z) > 0, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq R,
    $$
  \item [(C3)] there exist constants $b_1 > 0$ and $b_2 > 0$ such that
    $$
    |H'_i(t, z)| \leq b_1 H'_i(t, z) \cdot V_3(z) + b_2, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n},
    $$
  \item [(C4)] $\sum_{i=1}^n \left( |p_i| \psi_i + |q_i| \varphi_i \right) \to 0$ as $|z| \to 0$ uniformly in $t$.
\end{enumerate}

The above conditions (C1)-(C4) are similar to that of [2] with minor difference.
Remark 1. In the case where $H(t,z) = \frac{1}{2}(\dot{B}(t)z, z) + \dot{H}(t,z)$ with $\dot{B}(t)$ being a $\tau$-periodic, continuous symmetric matrix function and $\dot{H}$ satisfying the conditions as stated in Theorems 1.1 1.2 and Corollary 1, we also obtain the similar results with some restrictive conditions on $\hat{B}$ as stated in Theorems 1.1, 1.2 and Corollary 1. We refer to [1], [21] and [24] for details. In Section 4, we consider the existence of subharmonic solutions of Hamiltonian systems we refer to [22]. We note that all the results obtained in the references mentioned here are related with the Hamiltonian functions with superquadratic growth or subquadratic growth. This paper is organized as follows, in Section 2, as preliminary we recall some notions about the Maslov-type index theory and some iteration inequalities developed by Y. Long and the first author of this paper in [21]. In this section we also recall the homologically link theorem in [1] from which we can find a critical point of the corresponding functional together with index information. Under the conditions as in Theorems 1.1 1.2 and Corollary 1, we show that there is a homologically link structure for the functional. In Section 3, we give a proof of Theorems 1.1, 1.2 and Corollary 1. In Section 4, we consider the existence of subharmonic solutions of the Hamiltonian systems in the case where $H$ may contain a quadratic term as $H(t,z) = \frac{1}{2}(\dot{B}(t)z, z) + \dot{H}(t,z)$. We consider minimal period problem for the autonomous Hamiltonian systems in Section 5.

2. Preliminaries. We first recall the notion of Maslov-type index and some iteration estimates. We refer to [1], [21] and [24] for details.

Denote by $Sp(2n) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^tJM = J\}$ the symplectic group, where $M^t$ denotes the transpose of $M$. Define $P(2n) = \{\gamma \mid \gamma \in C([0, \tau], Sp(2n)), \gamma(0) = I_{2n}\}$. For $\gamma \in P(2n)$, according to [1] and [21], there is a Maslov-type index theory which assigns to $\gamma$ a pair of integers

$$(\iota_\tau, \nu_\tau) := (i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\},$$

where $i_\tau$ is the index part of $\gamma$ and $\nu_\tau$ is the nullity.

For $\gamma \in P(2n)$, define $\gamma^k : [0, k\tau] \to Sp(2n)$ by

$$\gamma^k(t) = \gamma(t - j\tau)\gamma(j\tau)^l, \quad j\tau \leq t \leq (j + 1)\tau, \quad 0 \leq j \leq k - 1.$$ 

We denote the Maslov index of $\gamma^k$ on $[0, k\tau]$ by $(i_{k\tau}, \nu_{k\tau}) := (i_{k\tau}(\gamma^k), \nu_{k\tau}(\gamma^k))$.

In the case of linear Hamiltonian systems

$$\begin{cases} -J\dot{z} = B(t)z, \\ z(0) = z(\tau), \end{cases}$$

where $B(t)$ is a $\tau$-periodic, symmetric and continuous matrix function, its fundamental solution is denoted by $\gamma_B \in C([0, \tau], Sp(2n))$ with $\gamma(0) = I_{2n}$. The Maslov-type

This paper is organized as follows, in Section 2, as preliminary we recall some notions about the Maslov-type index theory and the iteration inequalities developed by Y. Long and the first author of this paper in [21]. In this section we also recall the homologically link theorem in [1] from which we can find a critical point of the corresponding functional together with index information. Under the conditions as in Theorems 1.1 1.2 and Corollary 1, we show that there is a homologically link structure for the functional. In Section 3, we give a proof of Theorems 1.1, 1.2 and Corollary 1. In Section 4, we consider the existence of subharmonic solutions of the Hamiltonian systems in the case where $H$ may contain a quadratic term as $H(t,z) = \frac{1}{2}(\dot{B}(t)z, z) + \dot{H}(t,z)$. We consider minimal period problem for the autonomous Hamiltonian systems in Section 5.
obtained by extending the bilinear form \((i_\tau(B), \nu_\tau(B))\) is also called the Maslov-type index of the matrix function \(B(t)\).

If \(z\) is a \(\tau\)-periodic solution of the system \([1]\), we denote by \((i_\tau(z), \nu_\tau(z)) := (i_\tau(B), \nu_\tau(B))\) with \(B(t) = H''_zz (t, z(t))\). The solution \(z\) is non-degenerate if \(\nu_\tau(z) = 0\).

**Proposition 1.** \([15]\) If \(z\) is a \(k\tau\)-periodic solution of the system \([1]\), then \(i_{k\tau}(j \ast z) = i_k \tau(z)\) and \(\nu_{k\tau}(j \ast z) = \nu_k \tau(z)\) hold for \(0 \leq j \leq k\).

**Proposition 2.** \([15]\) For \(k \in \mathbb{N}\), there holds
\[
k(i_\tau + \nu_\tau - n) - n \leq i_{k\tau} \leq k(i_\tau + n) + n - \nu_{k\tau}.
\]

**Proposition 3.** \([21]\) For \(m \in \mathbb{N}\), there holds
\[
m(i_\tau + \nu_\tau - n) + n - \nu_\tau \leq i_{m\tau} \leq m(i_\tau + n) - n - (\nu_{m\tau} - \nu_\tau).
\]

**Lemma 2.1.** \([1]\) Let \(B(t)\) be a \(\tau\)-periodic, symmetric and continuous matrix function. Assume \(B(t)\) are positive for \(t \in [0, \tau]\) and \(B(t_0)\) is strictly positive for some \(t_0 \in [0, \tau]\). Then \(i_{\tau}(B) \geq n\).

**Lemma 2.2.** \([1]\) Let \(B(t)\) be a \(\tau\)-periodic, symmetric and continuous matrix function. Assume for some \(k \in \mathbb{N}\), there hold \(i_{k\tau}(B) \leq n+1\), \(i_{\tau}(B) \geq n\) and \(\nu_{\tau}(B) \geq 1\). Then \(k = 1\).

Now we introduce some concepts and results of Sobolev space theory.

Let \(E = W^{\frac{1}{2}, 2}(S_\tau, \mathbb{R}^{2n}) = \left\{ z \in L^2(S_\tau, \mathbb{R}^{2n}) \left| \sum_{j \in \mathbb{Z}} |j||a_j|^2 < +\infty \right. \right\}\), where \(S_\tau := \mathbb{R}/\tau\mathbb{Z}\), \(z(t) = \sum_{k \in \mathbb{Z}} \exp\left(\frac{2k\pi i}{\tau} J\right) a_k, a_k \in \mathbb{R}^{2n}\).

For \(\zeta \in E\), \(\zeta(t) = \sum_{k \in \mathbb{Z}} \exp\left(\frac{2k\pi i}{\tau} J\right) b_k, b_k \in \mathbb{R}^{2n}\), the inner product on \(E\) is
\[
\langle z, \zeta \rangle = \tau\sum_{k} |a_k| \cdot b_k.
\]

**Lemma 2.3.** \([18]\) (Embedding Theorem) The space \(E\) compactly embeds into \(L^s(S_\tau, \mathbb{R}^{2n})\) \((s \geq 1)\), in particular, there exists a constant \(C_s > 0\) such that \(\|z\|_{L^s} \leq C_s \|z\|\) holds for \(z \in E\), where \(\|\cdot\|\) denotes the norm on \(E\).

There exists a linear bounded self-adjoint operator \(A\) such that
\[
\langle Az, \zeta \rangle = 2\pi \sum_{k \in \mathbb{Z}} k a_k \cdot b_k
\]

obtained by extending the bilinear form \(\langle Az, \zeta \rangle = \int_0^{2\pi} (-J \dot{z}, \zeta) dt, z, \zeta \in W^{1, 2}(S_\tau, \mathbb{R}^{2n})\).

Set \(E^\pm = \left\{ z \in E | z(t) = \sum_{k > 0} \exp\left(\frac{2k\pi i}{\tau} J\right) a_k, a_k \in \mathbb{R}^{2n}\right\}\) and \(E^0 = \mathbb{R}^{2n}\), then \(Az^\pm = \pm \mathbb{Z}^\pm z^\pm, z^\pm \in E^\pm\) (see \([18]\)). Moreover, we set \(E_m = \{ z \in E | z(t) = \sum_{k=-m}^{m} \exp\left(\frac{2k\pi i}{\tau} J\right) a_k, a_k \in \mathbb{R}^{2n}\}\) and \(E_m^\pm = E^\pm \cap E_m\), and let \(P_m\) be the corresponding orthogonal projection.

For \(d > 0\), we denote by \(M^+_d(C), M^-_d(C)\) and \(M^0_d(C)\) the eigenspaces of any linear bounded self-adjoint Fredholm operator \(C\) corresponding to the eigenvalue \(\lambda\) belonging to \((d, +\infty)\), \((-\infty, d)\) and \([-d, d]\) respectively.

Given \(B(t)\) a \(\tau\)-periodic, symmetric and continuous matrix function with Maslov-type index \((i_\tau(B), \nu_\tau(B))\), define \(\langle Bz, \zeta \rangle = \int_0^\tau (B(t)z, \zeta) dt, z, \zeta \in E\). Set \((A-B)\) := \((A-B)|_{\{t \in \mathbb{R} | t \neq s, s \in \mathbb{R}\}}\)^{-1}, then we have the following theorem.
Lemma 2.4. ([10]) Suppose $0 < d < \frac{1}{4}\|(A - B)^2\|^{-1}$, then for $m$ large enough, there hold
\begin{align*}
\dim M^+_d(P_m(A - B)P_m) &= \frac{1}{2}\dim(P_mE) - i_v(B) - \nu_v(B), \\
\dim M^-_d(P_m(A - B)P_m) &= \frac{1}{2}\dim(P_mE) + i_v(B), \\
\dim M^0_d(P_m(A - B)P_m) &= \nu_v(B).
\end{align*}

For $e \in E_1 \cap E^+$ with $\|e\| = 1$, set $W = \{z \in \text{span}\{e\} \oplus E^- \oplus E^0|1 \leq \|z\| \leq 2, \|z^-\| \leq \|z^+ + z^0\|\}$, then we have

Lemma 2.5. ([1]) There exists a constant $\varepsilon_1 > 0$ such that
\[\text{measure } \{t \in [0, \tau]|z(t)| \geq \varepsilon_1\} \geq \varepsilon_1, \quad z \in W.\]

Finally, we recall the homologically link theorem in [1].

Definition 2.6. ([3]) Let $Q$ be a topologically embedded closed $q$-dimensional ball on a Hilbert manifold $M$ and let $S \subset M$ be a closed subset such that $\partial Q \cap S = \emptyset$. We say that $\partial Q$ and $S$ homotopically link if $\varphi(Q) \cap S \neq \emptyset$ for $\varphi \in C(Q, M)$ with $\varphi|_{\partial Q} = \text{id}|_{\partial Q}$.

Definition 2.7. ([1]) Let $Q$ be a topologically embedded closed $q$-dimensional ball on a Hilbert manifold $M$ and let $S \subset M$ be a closed subset such that $\partial Q \cap S = \emptyset$. We say that $\partial Q$ and $S$ homologically link if $\partial Q$ is the support of a non-vanishing homology class in $H_{q-1}(M \setminus S)$.

Lemma 2.8. Let $M = M_1 \bigoplus M_2$ be a Hilbert space with $\dim M_2 = q - 1$, $S = \partial B_v \cap M_1$ and $Q = (B_v \cap M_2) \bigoplus [0, \nu]e$, where $e \in M_1$ with $\|e\| = 1$ and $\nu > \mu > 0$. Let $B_v$, $B_\nu$ be two bounded linear invertible operators on $M$ such that $\nu > \mu\|B^{-1}_vB_\nu\|$ and $P_{B^{-1}_\nu}B_\nu: M_2 \to M_2$ is invertible, where $P : M \to M_2$ is the orthogonal projection. Then $B_v(\partial Q)$ and $B_\nu(S)$ homologically link.

Proof. It is easy to prove $B_v(\partial Q)$ and $B_\nu(S)$ homotopically link (see [13]).

Indeed, to show $B_v(Q) \cap B_\nu(S) \neq \emptyset$, it is equivalent to proving $\psi_0(t, v) = (\mu, 0)$ has a solution in $[0, \nu] \times (\bar{B}_v \cap M_2)$, where
\[\psi_0(t, v) = (\|B^{-1}_vB_v(te + v)\|, \|PB^{-1}_\nu B_v(te + v)\|), \quad (t, v) \in [0, \nu] \times (\bar{B}_v \cap M_2).\]

Note that $t = \mu\|B^{-1}_vB_v - B^{-1}_\nu B_\nu B_0\|^{-1}$, $v = -tB^{-1}_0B_0e \in B_v \cap M_2$, $\mu > 0$, $\nu > \mu\|B^{-1}_vB_\nu\|$ and $B^{-1}_\nu B_\nu$ denotes the inverse of $B_\nu$. Thus, $(\mu, 0) \notin \psi_0(\partial([0, \nu] \times (B_v \cap M_2)))$.

For $\varphi \in C(B_v(Q), M)$ with $\varphi|_{B_v(\partial Q)} = \text{id}|_{B_v(\partial Q)}$, define $\psi : [0, \nu] \times (B_v \cap M_2) \to R \times M_2$ as
\[\psi(t, v) = (\|B^{-1}_v \varphi B_v(te + v)\|, \|PB^{-1}_\nu \varphi B_v(te + v)\|).\]

In order to show $\varphi(B_v(Q)) \cap B_\mu(S) \neq \emptyset$, it is equivalent to proving $\psi(t, v) = (\mu, 0)$ has a solution in $[0, \nu] \times (B_v \cap M_2)$.

Since $\psi = \psi_0$ on $\partial([0, \nu] \times (B_v \cap M_2))$, by Brouwer degree theory, $\deg(\psi, (0, \nu) \times (B_v \cap M_2), (\mu, 0)) = \pm 1$, then $\psi(t, v) = (\mu, 0)$ has a solution in $[0, \nu] \times (\bar{B}_v \cap M_2)$. Hence, $B_v(\partial Q)$ and $B_\mu(S)$ homologically link. From Theorem II.1.2 in [3], we see $B_v(\partial Q)$ and $B_\mu(S)$ homologically link. \qed
Lemma 2.9. (1) Let $f$ be a $C^2$ functional on a Hilbert manifold $M$. Recall that the Morse index $m(x)$ of $f$ at a critical point $x$ is the dimension of a maximal subspace on which $D^2f(x)$ is strictly negative definite, while the large Morse index $m^*$ of $f$ is $m(x) + \dim \ker D^2f(x)$.

**Lemma 2.9.** Let $f$ be a $C^2$ functional on a Hilbert manifold $M$ with Fredholm gradient. Let $Q \subset M$ be a topologically embedded closed $q$-dimensional ball and let $S \subset M$ be a closed subset such that $\partial Q \cap S = \emptyset$. Assume that $\partial Q$ and $S$ homologically link. Moreover, assume $\Gamma$ denotes the set of all $q$-chains in $M$ whose boundary has support $\partial Q$, the number $c = \inf_{\xi \in \Gamma} \sup_{|\xi|} f \in [\inf_f f, \sup_f f]$ is a critical value of $f$, where $|\xi|$ denotes the support of the chain $\xi$. Moreover, $f$ has a critical point $\bar{x}$ such that $f(\bar{x}) = c$ and $m(\bar{x}) \leq q \leq m^*(\bar{x})$.

**Remark 2.** If $M$ is a finite dimensional Hilbert space, and $f$ satisfies (C) condition instead of (PS) condition, the above theorem still holds, the proof is the same as that of Theorem 4.1.7 in [1] (see [27] for results obtained under (C) condition).

Recall that the functional $f$ satisfies the so-called Cerami condition ((C) condition for short) on $J \subset \mathbb{R} \cup \{-\infty\}$ if $\{z_m\} \subset M$ such that $f(z_m) \to c \in J$ and $(1 + \|z_m\|)\|\nabla f(z_m)\| \to 0$ as $m \to +\infty$ has a convergent subsequence.

3. **Proofs of the main results.** For simplicity, we first prove Corollary 1. Define $f(z) = \frac{1}{2} \langle Az, z \rangle - \int_0^\tau H(t, z) dt, \quad z \in E,$ by (H3), we have $f \in C^2(E, \mathbb{R})$. As usual, finding periodic solutions of the system (1) converging to looking for critical points of $f$.

Let $f_m = f|_{E_m}, X_m = E_m \bigoplus E^0$ and $Y_m = E^+_m$. Now we check the conditions in Lemma 2.9 for $f_m$ when $H$ satisfies (H1)-(H5). The proofs are similar to those in [32, 34].

**Lemma 3.1.** The functional $f$ satisfies (C)* condition with respect to $\{E_m|m = 1, 2, \cdots\}$, that is, any sequence $\{z_m\} \subset E_m$ such that $z_m \in E_m, \{f_m(z_m)\}$ is bounded and $(1 + \|z_m\|)\|\nabla f_m(z_m)\| \to 0$ as $m \to +\infty$ has a convergent subsequence.

**Proof.** Let $\{z_m\}$ be such a sequence, we only need to prove $\{z_m\}$ is bounded. Otherwise, we may suppose $\|z_m\| \to +\infty$ as $m \to +\infty$.

Note that for $z \in E_m$, we have $\nabla f_m(z) = P_m \nabla f(z)$. By using $\alpha_i + \beta_i = 1$ and the integration by parts, there hold

\[
\left( f_m(z) - \langle \nabla f_m(z), V_1(z) \rangle \right) = f(z) - \langle \nabla f(z), V_1(z) \rangle = \int_0^\tau \left( H'_1(t, z) \cdot V_1(z) - H(t, z) \right) dt,
\]

where $V_1(z)$ is defined in (H2). By (H2), we have

\[
f_m(z_m) - \langle \nabla f_m(z_m), V_1(z_m) \rangle = \int_0^\tau \left( H'_1(t, z_m) \cdot V_1(z_m) - H(t, z_m) \right) dt \\
\geq c_1 \|z_m\|_L^{\beta} - \tau c_2.
\]

Hence $\{\|z_m\|_{L^\beta}\}$ is bounded.
We note that on $E$ from $\parallel z \parallel$ we can suppose that there are only finitely many $p > 2$. It is easy to see $\lambda - \frac{\beta}{p} = \frac{\lambda + \beta}{2p + 1} < 1$ and $2q(\lambda - \frac{\beta}{p}) = \frac{\lambda + \beta}{\beta + 1} > 1$. By (H3), Lemma 2.3 and (3), we have

$$\int_0^\tau |H_z^1(t, z_m) \cdot \frac{z_m^+}{|z_m^+|}| \, dt \leq c_3 \int_0^\tau |z_m^\beta| |z_m^+| \, dt + c_4 \|z_m^+\|$$

$$\leq c_3 \int_0^\tau |z_m^\beta| |z|^{\lambda - \frac{\beta}{p}} |z_m^+| \, dt + c_4 \|z_m^+\|$$

$$\leq c_3 \left( \int_0^\tau |z_m^\beta| \, dt \right)^{\frac{1}{\beta}} \left( \int_0^\tau |z_m^{(\lambda - \frac{\beta}{p})q}| |z_m^+| \, dt \right)^{\frac{1}{q}} + c_4 \|z_m^+\|$$

$$\leq c_5 \left( \int_0^\tau |z_m^{(\lambda - \frac{\beta}{p})2q}| \, dt \right)^{\frac{1}{2q}} \left( \int_0^\tau |z_m^+| \, dt \right)^{\frac{1}{2q}} + c_4 \|z_m^+\|$$

$$\leq c_6 \|z_m\|^{\lambda - \frac{\beta}{p}} \|z_m^+\| + c_6 \|z_m^+\|, \quad (4)$$

where $c_i > 0$ are suitable constants.

By (4), we have

$$\|\nabla f_m(z_m)\| \cdot \|z_m^+\| \geq \pm \langle \nabla f_m(z_m), z_m^+ \rangle$$

$$= \pm \langle Az_m, z_m^+ \rangle \mp \int_0^\tau H_z^1(t, z_m) \cdot z_m^+ \, dt$$

$$\geq \frac{2\pi}{\tau} \|z_m^+\|^2 - c_6 \|z_m\|^{\lambda - \frac{\beta}{p}} \|z_m^+\| - c_6 \|z_m^+\|. \quad (5)$$

We can suppose that there are only finitely many $z_m^+ = 0$. Dividing the two sides by $\|z_m\| \cdot \|z_m^+\|$, it implies $\frac{z_m^+}{|z_m^+|} \to 0$ as $m \to +\infty$. By (3), we see $\|z_m^+\| \to 0$, $m \to +\infty$.

From $\frac{\|z_m^+\|^2 + \|z_m\|^2 + \|z_m^+\|^2}{\|z_m\|^2} = 1$, we obtain a contradiction. \qed

We note that if $f$ satisfies (C)* condition on $E$, then $f_m$ satisfies (C) condition on $E_m$.

There exists a constant $\eta > 0$ such that $\bar{\sigma} = \frac{n \eta}{\sigma + \tau}, \bar{\xi} = \frac{n \eta}{\sigma + \tau}, \geq 1$ for $\rho > 0$ and $z = (p_1, \cdots, p_n, q_1, \cdots, q_n) \in E$, we set

$$B_\rho(z) = (\rho^{\bar{\sigma} - 1} p_1, \cdots, \rho^{\bar{\xi} - 1} p_n, \rho^{\bar{\sigma} - 1} q_1, \cdots, \rho^{\bar{\xi} - 1} q_n).$$

We note that $B_\rho$ is a linear bounded and invertible operator and $\|B_\rho\| \leq 1$, if $\rho \leq 1$.

For $z = z^+ + z^0 + z^- \in E$, we have

$$\langle AB_\rho z, B_\rho z \rangle = \rho^{\bar{\sigma} - 2} \langle Az, z \rangle = \frac{2\pi}{\tau} \rho^{\bar{\sigma} - 2} (\|z^+\|^2 - \|z^-\|^2).$$

(6)

**Lemma 3.2.** There exist constants $\mu \in (0, 1)$ and $\delta > 0$ independent of $m$ such that $\inf_{B_\mu(S_m)} f_m \geq \delta$, where $S_m = S \cap Y_m$ and $S = \{ z \in E^+ \|z\| = \mu \}$.

**Proof.** It suffices to show $\inf_{B_\mu(S)} f \geq \delta$.

By (H3) and (H4), for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$H(t, z) \leq \varepsilon \sum_{i=1}^n (|p_i |^{\lambda + \frac{\beta}{p}} + |q_i |^{\lambda + \frac{\beta}{p}}) + M_\varepsilon \sum_{i=1}^n (|p_i |^{\lambda + \beta} + |q_i |^{\lambda + \beta}), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^n. \quad (7)$$
By (7), for \( z = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in E, \| z \| = \mu \), we have
\[
\int_0^\tau H(t, B_\mu z) dt \\
\leq \varepsilon \sum_{i=1}^n \int_0^\tau \left( |\hat{\mu} \hat{\tau}^{-1} | p_i \| + | \hat{\mu} \hat{\tau}^{-1} | q_i \| \right) dt \\
+ M_\varepsilon \sum_{i=1}^n \int_0^\tau \left( |\hat{\mu} \hat{\tau}^{-1} | p_i \| + | \hat{\mu} \hat{\tau}^{-1} | q_i \| \right) dt \\
\leq \varepsilon \sum_{i=1}^n \int_0^\tau \left( \left| \hat{\mu} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \right| \| z \| + \mu (\hat{\sigma} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \| z \| + \mu (\hat{\sigma} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \right) dt \\
+ M_\varepsilon \sum_{i=1}^n \int_0^\tau \left( \left| \hat{\mu} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \right| \| z \| + \mu (\hat{\sigma} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \right) dt \\
\leq 2\varepsilon \mu^n \sum_{i=1}^n C(\sigma_i, \tau_i) + M_\varepsilon \mu^n \sum_{i=1}^n C(\lambda) \left( \hat{\mu} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \| z \| + \mu (\hat{\sigma} \hat{\tau}^{-1} (1+ \frac{\eta}{\mu}) \right) \right)
\] (8)
where \( C(\sigma_i, \tau_i) \), \( C(\lambda) > 0 \) are the embedding constants.

By (6) and (8), for \( \varepsilon > 0 \) and \( \mu > 0 \) so small that \( f(B_\mu z) \geq \delta := \frac{\pi}{\mu_1} \mu_0 \) for \( z \in E^+ \) and \( \| z \| = \mu \). Thus \( \inf_{B_\mu(S)} f \geq 0 \). \( \square \)

**Lemma 3.3.** Set \( Q = [\beta_\nu \cap (E^{-} \oplus E^0)] \oplus [0, \nu] \) and \( Q_m = Q \cap [X_m \oplus \text{span}\{e\}] \), where \( B_\nu = \{ z \in E \| | z | \| \leq \nu \} \). For any \( \nu \) with \( \nu > \mu > 0 \), we have \( B_\nu(\partial Q_m) \) and \( B_\mu(S_m) \) homologically link.

**Proof.** Since \( \nu > \mu > 0 \), then \( \nu > \mu \| B_\nu^{-1} B_\mu \| = \mu \| B_\mu \| \) and \( PB_\nu^{-1} B_\nu : E^{-} \oplus E^0 \to E^{-} \oplus E^0 \) is linear bounded and invertible (see [2][32]), where \( P : E \to E^{-} \oplus E^0 \) denotes the orthogonal projection. Furthermore, by noting \( B_i(E_m) \subset E_m \) and \( B_i|_{E_m} : E_m \to E_m \) is linear bounded and invertible, then \( \bar{P} \bar{m}(B_\mu|_{E_m})^{-1} B_\nu|_{E_m} : X_m \to X_m \) is linear bounded and invertible, where \( \bar{P} : E_m \to X_m \) is the orthogonal projection. From Lemma 2.8 we complete the proof. \( \square \)

For \( \varepsilon_1 > 0 \) as in Lemma 2.5 we set
\[
\frac{2\pi}{\tau} \cdot \sqrt{2n}A_1^{-1} = \varepsilon_1 \min_{1 \leq i \leq n} \left\{ \left( \frac{\varepsilon_1}{\sqrt{2n}} \right)^{1+\frac{\eta}{\mu}} \right\},
\]
by (H5), there exists a constant \( A_2 > 0 \) such that
\[
H(t, z) \geq A_2 \sum_{i=1}^n \left( |p_i|^{1+\frac{\eta}{\mu}} + |q_i|^{1+\frac{\eta}{\mu}} \right), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq A_2. \quad (9)
\]
Lemma 3.4. Choose $\nu > \frac{A^2}{\varepsilon^2} + 1$, then $f_m|_{B_\nu(\partial Q_m)} \leq 0$.

Proof. Since $\partial Q_m \subset \partial Q$, we show $f|_{B_\nu(\partial Q)} \leq 0$.

For $z \in \partial Q$, $z = se + z^- + z^0$, then $f(B_\nu z) \leq 0$. We show this in two cases.

Case 1. If $s = 0$, by (H1) and (10), we have $f(B_\nu z) \leq 0$.

Case 2. If $s \neq 0$, then $s = \nu$ and $\|z^- + z^0\| \leq \nu$ or $0 \leq s \leq \nu$ and $\|z^- + z^0\| = \nu$. In the two situations, we always have $\|z\| \leq 2\nu$. We now consider two subcases.

Subcase 1. If $\|se + z^0\| < \|z^-\|$, so $\|se\| < \|z^-\|$, by (H1) and (6), then $f(B_\nu z) \leq 0$.

Subcase 2. If $\|se + z^0\| \geq \|z^-\|$, set $\Omega_z = \{t \in [0, \tau] | z(t) \geq \nu \varepsilon\}$, Lemma 2.5 shows that measure $\Omega_z \geq \varepsilon_1$. From definition, we have

$$\frac{\sqrt{2n}}{\nu \varepsilon_1} |z(t)| \geq \sqrt{2n}, \quad t \in \Omega_z$$

(10) and

$$|B_\nu z(t)| \geq |z(t)| \geq \nu \varepsilon_1 > A_2, \quad t \in \Omega_z.$$  

(11)

From (10) and Remark 1.4 of [32], there hold

$$\sum_{i=1}^{n} \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} p_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} + \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} q_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} \geq \frac{1}{2n} \sum_{i=1}^{n} \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} p_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} + \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} q_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} \geq \frac{1}{2n} \frac{\sqrt{2n}}{\nu \varepsilon_1} |z(t)|, \quad t \in \Omega_z.$$  

(12)

By (9), (11) and (12), we have

$$H(t, B_\nu z(t)) \geq A_1 \sum_{i=1}^{n} \left( |\nu^{-1} p_i(t)|^{1+\frac{\varepsilon_1}{\tau}} + |\nu^{-1} q_i(t)|^{1+\frac{\varepsilon_1}{\tau}} \right)$$

$$\geq A_1 \nu^n \sum_{i=1}^{n} \left( |\nu^{-1} p_i(t)|^{1+\frac{\varepsilon_1}{\tau}} + |\nu^{-1} q_i(t)|^{1+\frac{\varepsilon_1}{\tau}} \right)$$

$$\geq A_1 \nu^n \min_{1 \leq i \leq n} \left\{ \left( \frac{\varepsilon_1}{\sqrt{2n}} \right)^{1+\frac{\varepsilon_1}{\tau}}, \left( \frac{\varepsilon_1}{\sqrt{2n}} \right)^{1+\frac{\varepsilon_1}{\tau}} \right\},$$

$$\sum_{i=1}^{n} \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} p_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} + \left( \frac{\sqrt{2n}}{\nu \varepsilon_1} q_i(t) \right)^{1+\frac{\varepsilon_1}{\tau}} \geq A_1 \nu^n \min_{1 \leq i \leq n} \left\{ \left( \frac{\varepsilon_1}{\sqrt{2n}} \right)^{1+\frac{\varepsilon_1}{\tau}}, \left( \frac{\varepsilon_1}{\sqrt{2n}} \right)^{1+\frac{\varepsilon_1}{\tau}} \right\},$$

$$\geq \frac{2\pi}{\nu \varepsilon_1} \nu^n, \quad t \in \Omega_z.$$  

(13)

By (H1), (6) and (13), we have $f(B_\nu z) \leq \frac{2\pi}{\nu} \nu^n - \int_{\Omega_z} H(t, B_\nu z(t)) dt \leq 0$. \hfill \Box

Theorem 3.5. If $H$ satisfies (H1)-(H5), then there exists a nonconstant solution $z$ of the system (7) satisfying

$$i_\tau(z) \leq n + 1 \leq i_\tau(z) + \nu_\tau(z).$$  

(14)
Proof. We follow the ideas of [18]. Now Lemmas 3.1-3.4 show that all conditions of Lemma 2.9 are satisfied for $f_m$ (see Remark 2). By (6) and (H1), we have $f|_{B,(Q)} \leq \frac{2}{2}v^0$. So $f_m$ has a critical point $z_m$ satisfying

$$\delta \leq f_m(z_m) \leq \frac{2}{2}v^0 \quad \text{and} \quad m(z_m) \leq \dim X_m + 1 \leq m^*(z_m). \quad (15)$$

By Lemma 3.1, we may assume $z_m \to z \in E$ with $\delta \leq f(z) \leq \frac{2}{2}v^0$ and $\nabla f(z) = 0$. By (H1), we see $z$ is a nonconstant solution of the system (11). Now we show that the critical point $z$ satisfies (14).

Let $B$ be the operator for $B(t) = H^2_{zz}(t, z(t))$ defined in Section 2, then we have

$$\|f''(x) - f''(z)\| = \|f''(x) - (A - B)\| \to 0, \quad \|x - z\| \to 0. \quad (16)$$

Let $0 < d < ||(A - B)||^{-1}$, by (16), there exists a constant $\kappa > 0$ such that

$$\|f''(x) - (A - B)\| < \frac{d}{3}, \quad x \in \tilde{B}(z, \kappa) := \{z \in E ; \|x - z\| \leq \kappa\}.

Then for $m$ large enough, we have

$$\|f''_m(x) - P_m(A - B)P_m\| < \frac{d}{2}, \quad x \in \tilde{B}(z, \kappa) \cap E_m. \quad (17)$$

For $x \in \tilde{B}(z, \kappa) \cap E_m$, Eq. (17) implies that

$$\langle f''_m(x)u, u \rangle \leq \langle P_m(A - B)P_mu, u \rangle + \|f''_m(x) - P_m(A - B)P_m\|\|u\|^2 \leq \frac{d}{2}\|u\|^2 < 0, \quad u \in \mathcal{M}^{-}(P_m(A - B)P_m) \setminus \{0\}.

Thus,

$$\dim M^{-}(f''_m(x)) \geq \dim M^{-}(P_m(A - B)P_m), \quad x \in \tilde{B}(z, \kappa) \cap E_m. \quad (18)$$

Similarly, we have

$$\dim M^{+}(f''_m(x)) \geq \dim M^{+}(P_m(A - B)P_m), \quad x \in \tilde{B}(z, \kappa) \cap E_m. \quad (19)$$

By (15), (18) and (19), for $m$ large enough, Lemma 2.4 shows that

$$\frac{1}{2}\dim E_m + n + 1 = \dim X_m + 1 \geq m(z_m) \geq \dim M^{-}(P_m(A - B)P_m) = \frac{1}{2}\dim E_m + i_\tau(z)$$

and

$$\frac{1}{2}\dim E_m + n + 1 = \dim X_m + 1 \leq m^*(z_m) \leq \dim M^{-}(P_m(A - B)P_m) + M_{\tau}(P_m(A - B)P_m) = \frac{1}{2}\dim E_m + i_\tau(z) + v_\tau(z).

The above two estimates show that (14) holds. \(\square\)

Remark 3. Under either the conditions of Theorem 1.1 or Theorem 1.2, the conclusion of Theorem 3.5 still holds (see Remarks below).
**Proof of Corollary 1.** The proof is the same as that in [18]. For readers’ convenience we give the details here.

Since $H$ is $k\tau$-periodic, by Theorem 3.5 the system (1) possesses a nonconstant $k\tau$-periodic solution $z_k$ satisfying

$$i_{k\tau}(z_k) \leq n + 1 \leq i_{k\tau}(z_k) + \nu_{k\tau}(z_k).$$

If $z_k$ and $z_{pk}$ are not geometrically distinct, by definition, there exist integers $l$ and $m$ such that $l \ast z_k = m \ast z_{pk}$. By Proposition 4 we have $i_{k\tau}(l \ast z_k) = i_{k\tau}(z_k)$, $\nu_{k\tau}(l \ast z_k) = \nu_{k\tau}(z_k)$ and $i_{pk\tau}(m \ast z_{pk}) = i_{pk\tau}(z_{pk})$, $\nu_{pk\tau}(m \ast z_{pk}) = \nu_{pk\tau}(z_{pk})$.

Eq. (20) shows that $p - n \leq n + 1$ and $i_{k\tau}(z_k) + \nu_{k\tau}(z_k) \geq n + 1$. Proposition 2 shows that $p - n \leq n + 1$ contradicting with the assumption $p > 2n + 1$. Hence if $p > 2n + 1$, then $z_k$ and $z_{pk}$ are geometrically distinct.

If all $z_k$ are non-degenerate, then $\nu_{k\tau}(z_k) = 0$ and $i_{k\tau}(z_k) = n + 1$ for $k \in \mathbb{N}$. Proposition 3 shows that $p + n \leq n + 1$, so we get $p = 1$. Hence $z_k$ and $z_{pk}$ are geometrically distinct when $p > 1$. We complete the proof of Theorem 1.1.

From Remark 4, we see the proofs of Theorem 1.1 and Theorem 1.2 are similar to the proof of Corollary 1.

**Remark 4.** Under the conditions of Theorem 1.1, the conclusion of Theorem 3.5 still holds.

Indeed, for any $K > 0$, we take a cut-off function defined by

$$\chi(s) = \begin{cases} 1, & 0 \leq s \leq K, \\ 0, & s \geq K + 1, \end{cases} \quad \text{and} \quad \chi'(K,K+1) < 0.$$ 

We set $\gamma = \max_{1 \leq i \leq n} \frac{\alpha_i}{\gamma_i}, \frac{\beta_i}{\gamma_i}, \frac{\xi_i}{\gamma_i}, \frac{\eta_i}{\gamma_i}, \frac{\sigma_i}{\gamma_i}, \frac{\tau_i}{\gamma_i}, \beta - 1$. Choose $\lambda_0 \in (\gamma, 1 + \beta)$ and

$$C_K \geq \max_{K < |z| \leq K + 1} \left\{ \max_{|z| \geq K} \frac{H(t,z)}{|z|^{|\lambda_0+1|}}, \frac{c_1}{\min_{1 \leq i \leq n} (|\alpha_i\lambda_0 - \beta_i|, |\beta_i\lambda_0 - \alpha_i|)}, A_1 \right\},$$

where $A_1$ is defined in (0), $c_1, \alpha_i, \beta_i$ are defined in (H2). For $(t,z) \in \mathbb{R} \times \mathbb{R}^{2n}$, we set

$$H_K(t, z) = \chi(|z|)H(t, z) + (1 - \chi(|z|))C_K|z|^{|\lambda_0+1|}.$$

If $K > 0$ is large enough, it is easy to show that $H_K$ satisfies (H2) and (H3)' with the constants independent of $K$ (see [34]). The modified function $H_K$ also satisfies (H1), (H3)-(H5).

Let $f_K(z) = \frac{1}{K}(A_{f_K}, z) - \int_0^T H_K(t, z)dt$, then $f_K \in C^2(E, \mathbb{R})$.

By the choice of $\lambda_0$, there exists a constant $A_2 > 0$ such that

$$|z|^{\lambda_0+1} \geq \sum_{i=1}^n \left( |p_i|^{|\xi_i|} + |q_i|^{|\tau_i|} \right), \quad |z| \geq A_2,$$

then we have

$$H_K(t, z) \geq A_1 \sum_{i=1}^n \left( |p_i|^{|\xi_i|} + |q_i|^{|\tau_i|} \right), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq A_2.$$

In all the arguments before, we replace $H$, $\lambda$ and $f$ by $H_K$, $\lambda_0$ and $f_K$ respectively, we see $f_K$ possesses a critical point $z_K$ satisfying $0 < \delta_K < f_K(z_K) \leq \frac{\tau_0}{2} \nu^0$ and $i_{*}(z_K) \leq n + 1 \leq i_{*}(z_K) + \nu_{*}(z_K)$.

By (H3)', it is easy to prove that $z = z_K$ is independent of $K$ and a $\tau$-periodic nonconstant solution of the system (1) for $K$ large enough (see [34]).
Remark 5. Under the conditions of Theorem 1.2, the conclusion of Theorem 3.5 still holds.

In fact, we also take the cut-off function $\chi \in C^\infty([0, +\infty), R)$ as before. For $R$ defined in (C3), we define

$$H_K(t, z) = \chi(|z|)H(t, z) + (1 - \chi(|z|))R(K)\sum_{i=1}^n \left(|p_i|\theta^{-1}\varphi_i + |q_i|\theta^{-1}\psi_i\right), \quad (t, z) \in R \times R^{2n},$$

where $R(K) = \max_{t \in R} \frac{H(t, z)}{K \leq |z| \leq K + 1}$ for $K > R$. Then $H_K$ satisfies (C2) and (C3) with the constants independent of $K$ (see [2]) if $R(K)$ and $R$ are large enough.

Let $f_K(z) = \frac{1}{2}(Az, z) - \int_0^t H_K(t, z)\, dt$, then $f_K \in C^2(E, R)$. It is easy to show that $f_K$ satisfies (PS) condition (see [2]). By the definition of $H_K$, we can choose $\lambda_1 > \max_{1 \leq i \leq n} \left\{\frac{\varphi_i}{\psi_i}, \frac{\psi_i}{\varphi_i}\right\}$ such that $H_K$ satisfies (H4), then $f_K|_{B_{\mu_i(K)(S)}} \geq \delta_K > 0$ (see Lemma 3.2).

From [2], we know that there exist constants $d_1 > 0$ and $d_2 > 0$ such that

$$H_K(t, z) \geq d_1 \sum_{i=1}^n \left(|p_i|\theta^{-1}\varphi_i + |q_i|\theta^{-1}\psi_i\right) - d_2 = d_1 \sum_{i=1}^n \left(|p_i|\theta^{-1}(1 + \frac{\varphi_i}{\psi_i}) + |q_i|\theta^{-1}(1 + \frac{\psi_i}{\varphi_i})\right) - d_2, \quad (t, z) \in R \times R^{2n},$$

which indicates an inequality similar to [9].

4. The case: $H$ contains a quadratic term. Now we consider the case where $H(t, z) = \frac{1}{2}(A(t)z, z) + H(t, z)$. The proof of the following results is similar to that of Theorems 1.1, 1.2 and Corollary 1.1, we only state the results.

We set $\omega = \max_{t \in R} |\hat{B}(t)|$ and suppose $H(t, z) \geq 0, \quad (t, z) \in R \times R^{2n}$.

Theorem 4.1. Suppose $\hat{H}$ satisfies (H1), (H2), (H3), (H4), (H5) and $\hat{B}(t)$ satisfies

(H6) $\hat{B}(t)$ is a $\tau$-periodic, symmetric and continuous matrix function and satisfies

$$\hat{B}(t)z = 2(\hat{B}(t)z, V_k(z)), \quad (t, z) \in R \times R^{2n} \quad (k = 1, 2).$$

We also require there exists an unbounded sequence $\{q_m\} \subset (0, +\infty)$ with $\inf_m q_m = 0$ such that

$$\hat{B}(t)B_{\theta^2}z, B_{\theta^2}z = \theta^{q-2}(\hat{B}(t)z, z), \quad (t, z) \in R \times R^{2n}$$

holds for $\theta \in \{q_m\}$, where $B_{\theta^2} = (\theta^{\tau_1-1}p_1, \cdots, \theta^{\tau_n-1}p_n, \theta^{\tau_1-1}q_1, \cdots, \theta^{\tau_n-1}q_n)$, $\theta > 0$ with $\eta, \eta_1, \tau_n$ defined as in Section 3.

Then for each integer $k \geq 1$ and $k < \frac{2\pi}{\omega}$, the system [1] possesses a $k\tau$-periodic nonconstant solution $z_k$ such that $z_k$ and $z_{pk}$ are geometrically distinct provided $p > 2n + 1$ and $pk \leq \frac{2\pi}{\omega}$. If all $z_k$ are non-degenerate, then $z_k$ and $z_{pk}$ ($p > 1$) are geometrically distinct.

Note that (H6) is satisfied if $b_{ij}(t) = 0$ whenever $|i - j| \neq n$. If $a_i = b_i = 2, \sigma_i = \tau_i = 1$ ($i = 1, 2, \cdots, n$), then $\hat{B}(t)$ is just a $\tau$-periodic, symmetric and continuous matrix function.

Similarly we have the following results.
Corollary 2. Replace (H3)' with (H3), then we have the same results as in Theorem 4.1.

Theorem 4.2. Suppose \( \hat{H} \) satisfies (C1)-(C4) and \( \hat{B}(t) \) satisfies
\[
(C5) \quad \hat{B}(t) \text{ is a } \tau\text{-periodic, symmetric and continuous matrix function and satisfies}
\]
\[
(\hat{B}(t)z, z) = 2(\hat{B}(t)z, V_3(z)), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}.
\]
Moreover, set \( \tilde{\eta} = \max_{1 \leq i \leq n} \{ \varphi_i + \psi_i \} \), \( \tilde{\delta}_i = \frac{\eta}{\varphi_i + \psi_i} \varphi_i \), and \( \tilde{\tau}_i = \frac{\eta}{\varphi_i + \psi_i} \psi_i \), we require that there exists an unbounded sequence \( \{ \varrho_m \} \subset (0, +\infty) \) with \( \inf \varrho_m = 0 \) such that
\[
(\hat{B}(t)B_0z, B_0z) = \varrho^{n-2}(\hat{B}(t)z, z), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}
\]
holds for \( \varrho \in \{ \varrho_m \} \), where \( B_0z = (\varrho^{\tau_1-1}p_1, \ldots, \varrho^{\tau_n-1}p_n, \varrho^{\eta_1-1}q_1, \ldots, \varrho^{\eta_n-1}q_n) \), \( \varrho > 0 \). Then we have the same results as in Theorem 4.1.

Remark 6. If \( \varphi_i = \psi_i \) (\( i = 1, 2, \ldots, n \)), then (C1)-(C5) are the conditions in [18] with the difference that we require \( H(t, z) = \frac{1}{2}(\hat{B}(t)z, z) + \hat{H}(t, z) \geq 0 \) instead of \( \hat{B}(t) \) being semi-positive-definite. Thus Theorem 4.2 generalizes the theorems in [18] in the semi-positive-definite case.

For \( z, \zeta \in E \), define \( \langle Bz, \zeta \rangle = \int_0^\tau (\hat{B}(t)z, \zeta)dt \), then \( B \) is a linear bounded and self-adjoint operator on \( E \) and \( \| Bz, z \| \leq \omega \| z \| \).

Remark 7. The key point of the proof of Corollary 2 is that if (H6) holds, then we have \( f(z) - f'(z)V_1(z) = \int_0^\tau (H_\tau^*(z) - \hat{H}(t, z))dt \) and \( (BB_\rho z, B_\rho z) = \varrho^{n-2}(Bz, z) \), where \( z \in E \), and \( f, \eta \) and \( B_\rho (\rho > 0) \) are defined in Section 3. Note that \( H \) satisfies (H3) if \( \hat{H} \) does. Then the proof of (C)' condition is the same as that of Lemma [3, 1]. We can define \( B_\mu \) for small \( \mu \in \{ \varrho_m \} \) and \( B_\nu \) for large \( \nu \in \{ \varrho_m \} \) as in Section 3. So the arguments can be applied to the current case.

The first equation in (H6) implies that \( \hat{B}(t) = B(t)V_1 + V_1 \hat{B}(t), t \in \mathbb{R} \). For \( \beta \) in (H2), we require that \( \beta \geq 2 \), so there exists \( c_\beta > 0 \) such that \( \| z \|_{L^\beta} \geq c_\beta \| z \|_{L^2} \), \( z \in L^\beta(S_\tau, \mathbb{R}^{2n}) \). So similarly we have the following result.

Theorem 4.3. Suppose \( \hat{H} \) satisfies (H1)-(H5) and \( \hat{B}(t) \) satisfies
\[(H6)' \hat{B}(t) \text{ is a } \tau\text{-periodic, symmetric and continuous matrix valued function with max}_{t \in \mathbb{R}} |\hat{B}(t) - B(t)V_1 - V_1 \hat{B}(t)| < c_1 \| z \|_{L^{\gamma}} \text{ and satisfies}
\]
\[
\lim_{\varrho \to 0^+} (\hat{B}(t)B_\varrho z, B_\varrho z) \leq \omega_1 \varrho^{n-2} \quad \text{and} \quad \lim_{\varrho \to +\infty} (\hat{B}(t)B_\varrho z, B_\varrho z) \geq \omega_2 \varrho^{n-2}
\]
holds uniformly for \( (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \) and \( |z| = 1 \), where \( c_1 \) is as in (H2) and \( \omega_1, \omega_2 \geq 0 \).

Then we have the same results as in Theorem 4.1.

For \( \theta, \varphi_i, \psi_i \) (\( i = 1, 2, \ldots, n \)) in (C2), we require that \( \gamma \geq 2\theta \), so there exists \( c_\gamma > 0 \) such that \( \| z \|_{L^\gamma} \geq c_\gamma \| z \|_{L^2} \), \( z \in L^\gamma(S_\tau, \mathbb{R}^{2n}) \), where \( \gamma = \varphi_i, \psi_i \) (\( i = 1, 2, \ldots, n \)). So similarly we also have the following result.

Theorem 4.4. Suppose \( H \) satisfies a condition similar to (C3), \( \hat{H} \) satisfies (C1)-(C5) and \( \hat{B}(t) \) satisfies
\[
(C5)' \hat{B}(t) \text{ is a } \tau\text{-periodic, symmetric and continuous matrix valued function with max}_{t \in \mathbb{R}} |\hat{B}(t) - B(t)V_3 - V_3 \hat{B}(t)| < c_1 \min_{1 \leq i \leq n} \{ c_{\varphi_i}, c_{\psi_i} \} \text{ and satisfies}
\]
\[
\lim_{\varrho \to 0^+} (\hat{B}(t)B_\varrho z, B_\varrho z) \leq \omega_3 \varrho^{n-2} \quad \text{and} \quad \lim_{\varrho \to +\infty} (\hat{B}(t)B_\varrho z, B_\varrho z) \geq \omega_4 \varrho^{n-2}
\]
hold uniformly for \((t, z) \in \mathbb{R} \times \mathbb{R}^{2n}\) and \(|z| = 1\), where \(\hat{\eta}\) and \(B_\varrho\) are as in Theorem 4.2 and \(\omega_3, \omega_4 \geq 0\).

Then we have the same results as in Theorem 4.1.

5. Minimal periodic solutions for the autonomous Hamiltonian systems.

In this section, we consider the minimal period problem of the following autonomous Hamiltonian systems

\[
\begin{align*}
\dot{z} &= JH'_z(z), \quad z \in \mathbb{R}^{2n}, \\
\tau(z) &= z(0).
\end{align*}
\]

Then we have the same results as in Theorem 4.1.

We say that \((z, \tau)\) is a minimal periodic solution of (21) if \(z\) solves the problem (21) with \(\tau\) being the minimal period of \(z\).

As shown in [21], we can also obtain minimal periodic solutions for the autonomous Hamiltonian systems (21).

From Theorem 3.5 and Remark 3, we see that for any \(\tau > 0\) the Hamiltonian system (21) possesses a nontrivial \(\tau\)-periodic solution \((z, \tau)\) satisfying \(i_\tau(z) \leq n + 1\) provided the function \(H\) satisfying the conditions in one of the Theorems 1.1, 1.2 and Corollary 1 with some necessary modifications in an obvious way (since the function \(H\) does not depend on time \(t\)). In the following result we understand the statements of the conditions in this sense.

**Theorem 5.1.** Suppose the autonomous Hamiltonian function \(H(z)\) satisfies the conditions in one of the Theorems 1.1, 1.2 and Corollary 1 and

\(\text{(H7)} \ H''_{zz}(z)\) is strictly positive for every \(z \in \mathbb{R}^{2n} \setminus \{0\}\).

Then \((z, \tau)\) is a minimal periodic solution of the nonlinear Hamiltonian system (21).

Note that the Hamiltonian function \(H\) of (2) satisfies (H7).

**Proof of Theorem 5.1.** The proof is almost the same as that in [21]. For readers’ convenience, we estimate the iteration number of the solution \((z, \tau)\) now.

Assume \((z, \tau)\) has minimal period \(\bar{\tau}_1\), i.e., its iteration number is \(k \in \mathbb{Z}\). Since the nonlinear Hamiltonian system in (21) is autonomous and (H7) holds, we have \(\nu^\tau_k(z) \geq 1\) and \(i^\tau_k(z) \geq n\) by Lemma 2.1. From Lemma 2.2 we see \(k = 1\), that is, the solution \((z, \tau)\) has minimal period \(\tau\).

In his pioneer work [28], P. Rabinowitz proposed a conjecture on whether a superquadratic Hamiltonian system possesses a periodic solution with a prescribed minimal period. This conjecture has been deeply studied by many mathematicians. We refer to [5, 6, 8, 11, 12, 21, 24] for the original Rabinowitz’s conjecture under some further conditions (for example the convex case). For the minimal period problem of brake solution of Hamiltonian systems, we refer to [15, 19, 33]. For the minimal period problem of \(P\)-symmetric solution of Hamiltonian systems, we refer to [20, 23]. Up to our knowledge, Theorem 5.1 is the first result on the minimal period problem of nonlinear Hamiltonian systems with anisotropic growth.

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**REFERENCES**

[1] A. Abbondandolo, *Morse Theory for Hamiltonian Systems*, Chapman, Hall, London, 2001.

[2] T. An and Z. Wang, Periodic solutions of Hamiltonian systems with anisotropic growth *Commun. Pure Appl. Anal.*, 9 (2010), 1069–1082.

[3] K. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.
[4] S. Chen and C. Tang, Periodic and subharmonic solutions of a class of superquadratic Hamiltonian systems, *J. Math. Anal. Appl.*, **297** (2004), 267–284.

[5] D. Dong and Y. Long, The iteration formula of the Maslov-type index theory with applications to nonlinear Hamiltonian systems, *Trans. Amer. Math. Soc.*, **349** (1997), 2619–2661.

[6] I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*, Springer, Berlin, 1990.

[7] I. Ekeland and H. Hofer, Subharmonics of convex nonautonomous Hamiltonian systems, *Comm. Pure Appl. Math.*, **40** (1987), 1–36.

[8] G. Fei, Relative morse index and its application to Hamiltonian systems in the presence of symmetries, *J. Differential Equations*, **122** (1995), 302–315.

[9] G. Fei, On periodic solutions of superquadratic Hamiltonian systems, *Electron. J. Differential Equations*, **8** (2002), 1–12.

[10] G. Fei and Q. Qiu, Periodic solutions of asymptotically linear Hamiltonian systems, *Chinese Ann. Math. Ser. B*, **18** (1997), 359–372.

[11] G. Fei and Q. Qiu, Minimal period solutions of nonlinear Hamiltonian systems, *Nonlinear Anal.*, **27** (1996), 821–839.

[12] G. Fei, S. Kim and T. Wang, Minimal period estimates of periodic solutions for superquadratic Hamiltonian systems, *J. Math. Anal. Appl.*, **238** (1999), 216–233.

[13] P. L. Felmer, Periodic solutions of “superquadratic” Hamiltonian systems, *J. Differential Equations*, **102** (1993), 188–207.

[14] C. Li, Brake subharmonic solutions of subquadratic Hamiltonian systems, *Acta Math. Sin. (Engl. Ser.)*, **31** (2015), 215–234.

[15] C. Liu, Subharmonic solutions of first order Hamiltonian systems, *Sci. China Math.*, **53** (2010), 2719–2732.

[16] C. Li, Z. Ou and C. Tang, Periodic and subharmonic solutions for a class of non-autonomous Hamiltonian systems, *Acta Math. Sin. (Engl. Ser.)*, **31** (2015), 1645–1658.

[17] C. Li, Brake subharmonic solutions of first order Hamiltonian systems, *Sci. China Math.*, **53** (2010), 2719–2732.

[18] C. Liu, Subharmonic solutions of Hamiltonian systems, *Nonlinear Anal.*, **42** (2000), 185–198.

[19] C. Liu, Minimal period estimates for brake orbits of nonlinear symmetric Hamiltonian systems, *Discrete Contin. Dyn. Syst.*, **27** (2010), 337–355.

[20] C. Liu, Relative index theories and applications, preprint.

[21] C. Liu and Y. Long, Iteration inequalities of the Maslov-type index theory with applications, *J. Differential Equations*, **165** (2000), 355–376.

[22] C. Liu and S. Tang, Subharmonic P-solutions of first order Hamiltonian systems, preprint.

[23] C. Liu and S. Tang, Iteration inequalities of the Maslov P-index theorem with applications, *Nonlinear Anal.*, **127** (2015), 215–234.

[24] Y. Long, *Index Theory for Symplectic Paths with Applications*, Birkhauser Verlag Basel · Boston · Berlin, 2002.

[25] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York, 1989.

[26] R. Michalek and G. Tarantello, Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems, *J. Differential Equations*, **72** (1988), 28–55.

[27] K. Perera and M. Schechter, *Topics in Critical Point Theory*, Cambridge University Press, 2013.

[28] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.*, **31** (1978), 157–184.

[29] P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.*, **33** (1980), 609–633.

[30] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.

[31] E. Silva, Subharmonic solutions for subquadratic Hamiltonian systems, *J. Differential Equations*, **115** (1995), 120–145.

[32] Q. Xing, F. Guo and X. Zhang, One generalized critical point theorem and its applications on super-quadratic Hamiltonian systems, *Taiwanese J. Math.*, **20** (2016), 1093–1116.

[33] D. Zhang, Symmetric period solutions with prescribed minimal period for even autonomous semipositive Hamiltonian systems, *Sci. China Math.*, **57** (2014), 81–96.

[34] X. Zhang and F. Guo, Existence of periodic solutions of a particular type of super-quadratic Hamiltonian systems, *J. Math. Anal. Appl.*, **421** (2015), 1587–1602.
X. Zhang and F. Guo, Multiplicity of Subharmonic Solutions and Periodic Solutions of a Particular Type of Super-quadratic Hamiltonian Systems, *Commun. Pure Appl. Anal.*, **15** (2016), 1625–1642.

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