Linear Convergent Decentralized Optimization with Compression

Xiaorui Liu\textsuperscript{1}, Yao Li\textsuperscript{2,3}, Rongrong Wang\textsuperscript{3,2}, Jiliang Tang\textsuperscript{1}, and Ming Yan\textsuperscript{3,2}

\textsuperscript{1}Department of Computer Science and Engineering
\textsuperscript{2}Department of Mathematics
\textsuperscript{3}Department of Computational Mathematics, Science and Engineering
Michigan State University
\{xiaorui, liyao6, wangron6, tangjili, myan\}@msu.edu

Abstract

Communication compression has been extensively adopted to speed up large-scale distributed optimization. However, most existing decentralized algorithms with compression are unsatisfactory in terms of convergence rate and stability. In this paper, we delineate two key obstacles in the algorithm design – data heterogeneity and compression error. Our attempt to explicitly overcome these obstacles leads to a novel decentralized algorithm named LEAD. This algorithm is the first \textit{LinEAr} convergent \textit{Decentralized} algorithm with communication compression. Our theory describes the coupled dynamics of the inaccurate model propagation and optimization process. We also provide the first consensus error bound without assuming bounded gradients. Empirical experiments validate our theoretical analysis and show that the proposed algorithm achieves state-of-the-art computation and communication efficiency.

1. Introduction

Distributed optimization solves the following large-scale optimization problem

$$\mathbf{x}^* := \arg \min_{\mathbf{x} \in \mathbb{R}^d} \left[ f(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x}) \right]$$

with a distributed computing system composed with \( n \) computing agents and a communication network. Each \( f_i(x) : \mathbb{R}^d \rightarrow \mathbb{R} \) is a local objective function of agent \( i \) and typically defined on the data \( \mathcal{D}_i \) settled at the agent. The variable \( \mathbf{x} \in \mathbb{R}^d \) often represents model parameters in machine learning. A distributed optimization algorithm seeks optimal solutions that minimize the overall objective function \( f(\mathbf{x}) \) with computation and communication. According to the communication strategy, existing algorithms can be conceptually categorized into centralized and decentralized ones. Specifically, the former allows global communication between agents such that gradient communication is sufficient for synchronization with parameter-server or all-reduce architectures. While the latter only permits local communication between connected agents such that variable exchange is required to achieve a consensual solution,
e.g., gossip-type algorithms. In both paradigms, the computation can be relatively fast with powerful computing devices; efficient communication is the key to improve algorithm efficiency and system scalability, especially when the network bandwidth is limited.

In recent years, various communication compression techniques, such as quantization and sparsification, have been developed to reduce the communication cost. Notably, extensive studies (Seide et al., 2014; Alistarh et al., 2017; Bernstein et al., 2018; Stich et al., 2018; Karimireddy et al., 2019; Mishchenko et al., 2019; Tang et al., 2019b; Liu et al., 2020) have utilized gradient compression to significantly boost communication efficiency for centralized optimization. They enable efficient large-scale optimization while maintaining comparable convergence rates and practical performance with their non-compressed counterparts. This great success has suggested the potential and significance of communication compression in decentralized algorithms.

While most attention has been paid to centralized optimization, communication compression is crucial for decentralized optimization because of its broad applications such as decision making in sensor networks (Olfati-Saber and Shamma, 2005), power system control (Yang et al., 2015), and distributed inference and learning (Yan et al. 2012; Lian et al., 2017). Due to the limited (compressed) and local neighboring communication restrictions, the coupled dynamics of inaccurate model propagation and optimization process should be carefully considered. Therefore, the algorithm design and convergence analysis are nontrivial, and dedicated efforts are desired.

Recent efforts (Tang et al. 2018a; Koloskova et al., 2019; Koloskova* et al., 2020; Tang et al., 2019a) try to push this research direction. For instance, DCD-SGD and ECD-SGD (Tang et al. 2018a) introduce difference compression and extrapolation compression to reduce model compression error. DeepSqueeze (Tang et al. 2019a; Koloskova* et al., 2020) compensates compression error to the next iteration. CHOCO-SGD (Koloskova et al. 2019; Koloskova* et al. 2020) presents a novel quantized gossip algorithm that preserves the model average and achieves state-of-the-art performance. Nevertheless, these algorithms are still unsatisfactory in terms of convergence rate and stability. First, to the best of our knowledge, there is no linear convergent decentralized optimization algorithm with communication compression even for strongly convex objective functions and with full gradient. Second, the convergence rates of these algorithms are analyzed on the average model, and a consensus error bound is largely missing. This explains why a consensus solution is not always reliably obtained, and sometimes instability happens. Besides, a bounded-gradient assumption is mostly needed for the convergence analysis, though it is strong and might not hold in some cases. For example, this assumption is often violated in the strongly convex case (Nguyen et al., 2018; Gower et al., 2019), but the analysis of CHOCO-SGD in (Koloskova et al., 2019) requires both strong convexity and bounded gradient. Thus, there is a pressing need to bridge this gap.

In this paper, we study the decentralized optimization problem with communication compression. In particular, we aim to achieve a linear convergence rate and better stability by addressing the aforementioned challenges. Our key contributions can be summarized as:

- We delineate two key obstacles in the algorithm design, namely data heterogeneity and compression error, for communication compression in decentralized optimization, and our attempt to explicitly overcome these obstacles leads to a novel algorithm, LEAD.
• We prove that for LEAD, a constant stepsize in the range $(0, 2/(\mu + L)]$ is sufficient to achieve linear convergence for strongly convex and smooth objective functions.\(^1\) To the best of our knowledge, LEAD is the first linear convergent decentralized algorithm with communication compression. Moreover, it provably works with arbitrary compression precision.

• Our theory provides the first consensus error bound for this problem without using bounded gradient and shows that LEAD can reliably converge to the consensual solution, which stabilizes the algorithm.

• Extensive experiments on regularized linear regression and logistic regression problems validate our theoretical analysis. All the experiments demonstrate that the proposed algorithm achieves state-of-art computation and communication efficiency.

2. Related Works

Communication efficiency has been widely concerned for large-scale distributed optimization. Here we review related works on communication compression in centralized and decentralized settings.

Centralized optimization. Centralized distributed optimization algorithms such as parallel SGD (Zinkevich et al., 2010) are widely used in PyTorch, Tensorflow, MXNet, and CNTK. Recent works such as 1bit SGD Seide et al. (2014), QSGD (Alistarh et al., 2017), SignSGD (Bernstein et al., 2018), and Terngrad (Wen et al., 2017) show that the gradient information can be compressed via quantization or sparsification to reduce the number of bits being transmitted during the distributed training. Error compensation and gradient difference compression have been introduced to reduce the impact of compression error in ECQ-SGD (Wu et al., 2018), MEM-SGD (Stich et al., 2018), EF-SGD (Karimireddy et al., 2019), DIANA (Mishchenko et al., 2019), DoubleSqueeze (Tang et al., 2019b), and DORE Liu et al. (2020). The great success in centralized optimization suggests the potential and significance of communication compression in decentralized optimization.

Decentralized optimization. Decentralized optimization can be traced back to the seminal work (Tsitsiklis et al. 1986) in the 1980s. Compared to centralized optimization, decentralized algorithms only require local neighboring communication such that it greatly expands the application scenarios of distributed optimization with scalability, robustness, dynamic topologies, and balanced communication. Classical algorithms include distributed subgradient algorithm (Nedic and Ozdaglar, 2009) and decentralized gradient descent (DGD) (Yuan et al., 2016). These algorithms are intuitive and simple but usually slow due to the diminishing step size that is needed to obtain a consensual and optimal solution. The stochastic version D-PSGD (Lian et al., 2017) has been shown effective for training nonconvex deep learning models. Among those works attempting to improve the convergence rate, EXTRA (Shi et al., 2015) and NIDS (Li et al., 2019) use a second-order difference structure to cancel the steady-state error in DGD (Nedic and Ozdaglar, 2009) (Yuan

\(^1\) This work considers the non-stochastic setting only, where the full-batch gradient is used. For fairness, when comparing with other algorithms, we assume the availability of the full gradient, i.e., gradient variance is zero. Note the stochastic gradient can be used in LEAD, and it converges to the neighborhood of optimal solutions. We leave the stochastic version with variance reduction as future works.
et al., 2016) and achieve a sublinear (linear) convergence rate when the objective function is convex (strongly convex).

Recently, communication compression is applied to decentralized settings in (Tang et al., 2018a). It proposes two algorithms, i.e., DCD-SGD and ECD-SGD, which require compression of high accuracy and are not stable. DeepSqueeze (Tang et al., 2019a) compensates the compression error to the next iteration. Motivated by the quantized average consensus algorithms such as (Carli et al., 2010), the quantized gossip algorithm CHOCO-Gossip (Koloskova et al., 2019) converges linearly to the consensus solution. Combining CHOCO-Gossip and D-PSGD (Lian et al., 2017) leads to a decentralized algorithm with compression, CHOCO-SGD, which only converges sublinearly under the strong convexity and bounded gradient assumptions. Its nonconvex variant is further provided in (Koloskova et al., 2020). A new compression scheme using the modulo operation is introduced in (Lu and De Sa, 2020), which is complementary to our work. Note that most of existing works with compression are analyzed only for nonconvex functions, so their convergence is not directly comparable to our proposed LEAD. Among them, only CHOCO-SGD has convergence rates for convex functions so we will majorly compare with it in this work.

3. Algorithm

In this section, we first introduce the proposed algorithm LEAD. Then we provide an equivalent algorithm for efficient implementation. Finally, its connections to existing works are discussed. Before details, we introduce notation and definitions used in this work.

We use bold upper-case letters such as $X$ to define matrices and bold lower-case letters such as $x$ to define vectors. Let $1$ and $0$ be vectors with all ones and zeros, respectively. Their dimensions will be provided when necessary. Given two matrices $X, Y \in \mathbb{R}^{n \times d}$, we define their inner product as $\langle X, Y \rangle = \text{tr}(X^\top Y)$ and the norm as $\|X\| = \sqrt{\langle X, X \rangle}$. We further define $\langle X, Y \rangle_P = \text{tr}(X^\top PY)$ and $\|X\|_P = \sqrt{\langle X, X \rangle}_P$ for any given symmetric positive semidefinite matrix $P \in \mathbb{R}^{n \times n}$. For simplicity, we will majorly use the matrix notation in this work. For instance, each agent $i$ holds an individual estimate $x_i \in \mathbb{R}^d$ of the global variable $x \in \mathbb{R}^d$. Let $X^k$ and $\nabla F(X^k)$ be the collections of $\{x_i^k\}_{i=1}^n$ and $\{\nabla f_i(x_i^k)\}_{i=1}^n$, which are defined below:

$$X^k = [x_1^k, \ldots, x_n^k]^\top \in \mathbb{R}^{n \times d}, \quad \nabla F(X^k) = [\nabla f_1(x_1^k), \ldots, \nabla f_n(x_n^k)]^\top \in \mathbb{R}^{n \times d}.$$ (2)

With this notation, the update $X^{k+1} = X^k - \eta \nabla F(X^k)$ means that $x_i^{k+1} = x_i^k - \eta \nabla f_i(x_i^k)$ for all $i$. In this paper, we need the average of all rows in $X^k$ and $\nabla F(X^k)$, so we define $\bar{X}^k = (1^\top X^k)/n$ and $\bar{\nabla F}(X^k) = (1^\top \nabla F(X^k))/n$. They are row vectors, and we will take a transpose if we need a column vector. The pseudoinverse of a matrix $M$ is denoted as $M^\dagger$. The largest, ith-largest, and smallest nonzero eigenvalues of a symmetric matrix $M$ are $\lambda_{\text{max}}(M)$, $\lambda_i(M)$, and $\lambda_{\text{min}}(M)$, respectively.

**Assumption 1** (Mixing matrix). The connected network $G = \{V, E\}$ consists of a node set $V = \{1, 2, \ldots, n\}$ and an undirected edge set $E$. The primitive symmetric doubly-stochastic matrix $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ encodes the network structure such that $w_{ij} = 0$ if nodes $i$ and $j$ are not connected and cannot exchange information.
Assumption 1 implies that $-1 < \lambda_n(W) \leq \lambda_{n-1}(W) \leq \cdots \leq \lambda_2(W) \leq \lambda_1(W) = 1$ and $W1 = 1$. The matrix multiplication $X^{k+1} = WX^k$ describes that agent $i$ takes a weighted sum from its neighbors and itself, i.e., $x_i^{k+1} = \sum_{j \in N_i \cup \{i\}} w_{ij} x_j^k$, where $N_i$ denotes the neighbors of agent $i$.

### 3.1 The Proposed Algorithm

The proposed algorithm LEAD to solve (1) is shown in Alg. 1 for the purpose of analysis. We will refer to the line number in the analysis. An equivalent but communication efficient algorithm for implementation will be discussed in Section 3.2. For the initialization, we require $D^1 = (I - W)Z$ for some $Z \in \mathbb{R}^{n \times d}$. One simple way is to set $D^1 = 0^{n \times d}$. The function $\text{Compress}$ is the compression operator that compresses the variable for each agent independently. The motivation behind Alg. 1 is to achieve two goals – (a) consensus ($x_i^k - (X^k)^\top \to 0$) and (b) convergence ($(X^k)^\top \to x^*$). Next we discuss how the goal (a) leads to the goal (b) and explain how LEAD fulfills the goal (a).

#### Algorithm 1 The Proposed Algorithm: LEAD

**Input:** stepsize $\eta$, parameters $(\alpha, \gamma)$, $X^0$, $H^1$, $D^1 = (I - W)Z$ for any $Z$

**Output:** $X^K$

1. $X^1 = X^0 - \eta \nabla F(X^0)$
2. for $k = 1, 2, \ldots, K - 1$ do
3. $Y^k = X^k - \eta \nabla F(X^k) - \eta D^k$
4. $Q^k = \text{Compress}(Y^k - H^k)$
5. $\hat{Y}^k = H^k + Q^k$
6. $H^{k+1} = (1 - \alpha)H^k + \alpha \hat{Y}^k$
7. $D^{k+1} = D^k + \frac{\gamma}{2n}(I - W)\hat{Y}^k$
8. $X^{k+1} = X^k - \eta \nabla F(X^k) - \eta D^{k+1}$
9. end for

In essence, LEAD runs the approximate gradient descent globally and reduces to the exact gradient descent under consensus. One key property for LEAD is $1_n^\top D^k = 0$, regardless of the compression error in $\hat{Y}^k$. It holds because $D^k \in \text{Range}(I - W)$ for all $k$ and $1_n^\top (I - W) = 0$. Therefore, multiplying $(1/n)1_n^\top$ on both sides of Line 8 leads to a global average view of Alg. 1

$$X^{k+1} = \bar{X}^k - \eta \nabla F(X^k),$$ (3)

which doesn’t contain the compression error. Note that this is an approximate gradient descent step because, as shown in (2), the gradient $\nabla F(X^k)$ is not evaluated on a global synchronized model $\bar{X}^k$. If the solution converges to the consensus solution, i.e., $x_i^k - (\bar{X}^k)^\top \to 0$, then $(\nabla F(X^k))^\top - \nabla f((\bar{X}^k)^\top) \to 0$ and (3) gradually reduces to the exact gradient descent.

With the establishment on how consensus leads to convergence, the obstacle becomes how to achieve consensus under local communication and compression error. It requires
to address two issues, i.e., data heterogeneity and compression error. Existing algorithms, such as DCD-SGD, ECD-SGD, DeepSqueeze, and CHOCO-SGD, need diminishing or a constant but small step size depending on the total number of iterations to deal with these issues. These choices unavoidably cause much slower convergence and bring in the difficulty of step size tuning. In contrary, one distinctive advantage of LEAD is that a constant step size is sufficient to ensure the consensual solution and achieve linear convergence as proved in Section 4. This is achieved by a novel way of resolving the data heterogeneity and compression error issues.

**Data heterogeneity.** It is common in distributed settings that there exists data heterogeneity among agents. In other words, there exist two agents \(i\) and \(j\) such that \(f_i(x) \neq f_j(x)\). The optimality condition of problem (1) gives \(1_{n \times 1}^\top \nabla F(X^*) = 0\), where \(X^*\) is a consensual and optimal solution. The data heterogeneity and optimality condition imply that there may exist two agents \(i\) and \(j\) such that \(\nabla f_i(x^*) \neq 0\) and \(\nabla f_j(x^*) \neq 0\). As a result, a simple decentralized gradient descent algorithm cannot converge to the consensual solution as \(X^* \neq X^* - \eta \nabla F(X^*)\).

In LEAD, \(D^k\) plays the role as gradient correction. As \(k \to \infty\), we expect \(D^k \to -\nabla F(X^*)\) and \(X^k\) will converge to \(X^*\) via the update \(X^{k+1} = X^k - \eta \nabla F(X^k) - \eta D^{k+1}\) (Line 8), since \(D^{k+1}\) corrects the nonzero gradient \(\nabla F(X^k)\) asymptotically. The update of \(D^k\) is inspired by the design of NIDS (Li et al., 2019)(Li and Yan, 2019) and the corresponding connection is demonstrated in Proposition 1.

**Compression error.** LEAD reduces the impact of compression error by difference compression and implicit error compensation. On the one hand, we compress the difference between \(Y^k\) and a state variable \(H^k\) (Line 4). We recover an estimate \(\hat{Y}^k\) by adding the state back (Line 5). The state is updated as a running average of \(H^k\) and \(\hat{Y}^k\), i.e., \(H^{k+1} = (1 - \alpha)H^k + \alpha \hat{Y}^k\) (Line 6). We expect that both \(Y^k\) and \(H^k\) converge to \(X^*\), such that the compression error vanishes asymptotically. On the other hand, even the compression error exists, LEAD essentially compensates the error in an implicit way. To illustrate how it works, let \(E^k = \hat{Y}^k - Y^k\) denote the compression error where \(e_i^k\) is its \(i\)-th row. The update of \(D^k\) gives

\[
D^{k+1} = D^k + \frac{\gamma}{2\eta} (I - W) \hat{Y}^k = D^k + \frac{\gamma}{2\eta} (I - W)Y^k + \frac{\gamma}{2\eta} (I - W)E^k
\]

where \((I-W)E^k\) indicates that each agent \(i\) spreads total compression error \(-\sum_{j \in \mathcal{N} \cup \{i\}} w_{ji} e_j^k = -e_i^k\) to other agents and compensates this error locally by adding \(e_i^k\). The implicit error compensation also explains why the global view in (3) doesn’t involve compression error.

### 3.2 Efficient Implementation

The communication process \((I - W)\hat{Y}^k\) in Alg. 1 involves the full-precision variable \(\hat{Y}^k\) such that Alg. 1 does not enjoy the benefits of compression. Thus, we introduce an equivalent algorithm in Alg. 2, which allows a communication and memory efficient implementation.

The main idea behind Alg. 2 is the introduction of two new variables \(H_w\) and \(\hat{Y}_w\). With the initialization \(H_0^w = WH^1\), we have \(\hat{Y}_w^1 = H_0^w + WQ^1 = WH^1 + Q^1 = WH^1\) and \(H_w^2 = (1 - \alpha)H_w^1 + \alpha \hat{Y}_w^1 = WH^1 + \alpha \hat{Y}_w^1 = WH^2\). Therefore, by recursion, we only need the communication of the compressed variable \(Q^k\) to ensure \(H_w^k = WH^k\) and \(\hat{Y}_w^k = WH^k\) for all \(k\). Let’s describe the procedure \(\text{COMM}(Y, H, H_w)\) in details:
\textbf{Algorithm 2} An Efficient Implementation of LEAD

\textbf{Input:} Stepsize $\eta$, parameter $(\alpha, \gamma)$, $X^0$, $H^1$, $D^1 = (I - W)Z$ for any $Z$

\textbf{Output:} $X^K$

\begin{align*}
H^1_w &= WH^1 \\
X^1 &= X^0 - \eta \nabla F(X^0)
\end{align*}

for $k = 1, 2, \cdots, K - 1$ do

\begin{align*}
Y^k &= X^k - \eta \nabla F(X^k) - \eta D^k \\
\hat{Y}^k, \hat{Y}^k_w, H^{k+1}, H^{k+1}_w &= \text{COMM}(Y^k, H^k, H^k_w) \\
D^{k+1} &= D^k + \frac{\gamma}{2\eta}(\hat{Y}^k - \hat{Y}^k_w) \\
X^{k+1} &= X^k - \eta \nabla F(X^k) - \eta D^{k+1}
\end{align*}

end for

\textbf{procedure} COMM($Y, H, H_w$)

\begin{align*}
Q &= \text{COMPRESS}(Y - H) \quad \triangleright \text{Compression} \\
\hat{Y} &= H + Q \\
\hat{Y}_w &= H_w + WQ \quad \triangleright \text{Communication} \\
H &= (1 - \alpha)H + \alpha \hat{Y} \\
H_w &= (1 - \alpha)H_w + \alpha \hat{Y}_w \\
\text{Return: } \hat{Y}, \hat{Y}_w, H, H_w
\end{align*}

\textbf{end procedure}

- Compress the difference between $Y$ and the state variable $H$ as $Q$.

- $Q$ is encoded into the low-bit representation, which enables the efficient local communication step $\hat{Y}_w = H_w + WQ$. It is the only communication step in each iteration.

- Each agent recovers its estimate $\hat{Y}$ by $\hat{Y} = H + Q$ and we have $\hat{Y}_w = W\hat{Y}$.

- States $H$ and $H_w$ are updated based on $\hat{Y}$ and $\hat{Y}_w$, respectively. We have $H_w = WH$.

After the COMM procedure, the update on the gradient correction variable $D^{k+1}$ is given by

\begin{equation}
D^{k+1} = D^k + \frac{\gamma}{2\eta}(\hat{Y}^k - \hat{Y}^k_w) = D^k + \frac{\gamma}{2\eta}(I - W)\hat{Y}^k, \quad (4)
\end{equation}

which demonstrates the equivalence between Alg. 1 and Alg. 2. With this reformulation, we compute $\hat{Y}_w^k = W\hat{Y}^k$ without the explicit transmission of $\hat{Y}^k$ in full-precision. In addition to communication efficiency, we also avoid storing all neighbor’s variables ($h^k_j$ and $\hat{y}^k_j, j \in N_i$) because they are summarized in the weighted sums ($h^k_w$ and $\hat{y}^k_w$). Thus, it also enables memory efficiency. Both Algs. 1 and 2 use the matrix notations while we also present an algorithm from the agent’s perspective in Appendix A.
3.3 Connections to exiting works

The non-compressed variant of Alg. 1 recovers recently proposed NIDS (Li et al., 2019) and $D^2$ (Tang et al., 2018b) as shown in Proposition 1 and the detailed proof is provided in Appendix D.2

Proposition 1 (Connection to NIDS and $D^2$). When there is no communication compression (i.e., $Y^k = Y^k$) and $\gamma = 1$, Alg. 1 recovers NIDS:

$$X^{k+1} = \frac{I + W}{2} \left(2X^k - X^{k-1} - \eta \nabla F(X^k) + \eta \nabla F(X^{k-1})\right).$$  

(5)

Furthermore, if the stochastic estimator of the gradient $F(X^k; \xi^k)$ is applied, it recovers $D^2$.

4. Theoretical Analysis

In this section, we show the convergence rate for our proposed algorithm LEAD. Before showing the main theorem, we make some assumptions which are commonly used for the analysis of the decentralized optimization algorithm. All proofs are provided in Appendix D.

Assumption 2 (Unbiased and $C$-contracted operator). The compression operator $Q : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is unbiased, i.e., $E Q(x) = x$, and there exists $C \geq 0$ such that $E \|x - Q(x)\|_2^2 \leq C \|x\|_2^2$ for all $x \in \mathbb{R}^d$.

Assumption 3. Each $f_i$ is $L$-smooth and $\mu$-strongly convex with $L \geq \mu > 0$, i.e., for $i = 1, 2, \ldots, n$ and $\forall x, y \in \mathbb{R}^d$, we have

$$f_i(y) + \langle \nabla f_i(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2 \leq f_i(x) \leq f_i(y) + \langle \nabla f_i(y), x - y \rangle + \frac{L}{2} \|x - y\|^2.$$  

(6)

Theorem 1 (Linear convergence). Let $\{X^k, H^k, D^k\}$ be the sequence generated from Alg. 1 and $X^*$ is the optimal solution with $D^* = -\nabla F(X^*)$. Under Assumptions 1-3, for any constant stepsize $\eta \in (0, 2/(\mu + L)]$, if the compression parameters $\alpha$ and $\gamma$ satisfy

$$\gamma \in \left(0, \min \left\{\frac{(6C + 2) - \sqrt{(6C + 2)^2 - 16C}}{2C\beta}, \frac{2\mu\eta(2 - \mu\eta)}{2 - \mu\eta(2 - \mu\eta)}\right\}\right),$$  

(7)

$$\alpha \in \left[\frac{C\beta\gamma}{2(1 + C)}, \min \left\{\frac{2 - \beta\gamma}{4 - \beta\gamma 4(1 + C)}, \frac{\mu\eta(2 - \mu\eta)C\beta\gamma}{4(1 + C)}\right\}\right],$$  

(8)

with $\beta := \lambda_{\text{max}}(I - W)$. Then, in total expectation we have

$$E \mathcal{L}^{k+1} \leq \rho E \mathcal{L}^k,$$  

(9)

where

$$\mathcal{L}^k := (1 - a_1\alpha)\|X^k - X^*\|^2 + (2\eta^2/\gamma)E\|D^k + 1 - (I - W)^\dagger + a_1\|H^k - X^*\|^2,$$

$$\rho := \max \left\{1 - \frac{\mu\eta(2 - \mu\eta)}{1 - a_1\alpha}, 1 - \frac{\gamma}{2\lambda_{\text{max}}(I - W)^\dagger}, 1 - \alpha\right\} < 1, \quad a_1 := \frac{4(1 + C)}{C\beta\gamma + 2}.$$  

The result holds for $C = 0$, if we let $\frac{(6C + 2) - \sqrt{(6C + 2)^2 - 16C}}{2C\beta} = \frac{2}{3}$, which is the limit when $C \rightarrow 0$.  

8
The linear convergence of LEAD holds when \( \eta < \frac{2}{L} \), but it is omitted because of space limit.

**Remark 1.** LEAD converges linearly, which is significantly faster than the sublinear convergence of CHOCO-SGD (Koloskova et al., 2019) for the full-gradient case. Moreover, LEAD only makes mild assumptions. For instance, CHOCO-SGD has extra assumptions \( \|\nabla f_i(x)\|^2 \leq G \) and \( C < 1 \). The relaxed requirement \( 0 < C \) in LEAD also suggests that LEAD is able to tolerate more aggressive compression, i.e., \( C > 1 \). However, LEAD assumes unbiased compression operator and the convergence under biased compression method is not investigated yet. In the theorem of CHOCO-SGD (Koloskova et al., 2019), it requires a specific point set of \( \gamma \) while LEAD only requires \( \gamma \) to be within a rather large range. This explains the advantages of LEAD over CHOCO-SGD in terms of stability and ease of parameter tuning.

**Remark 2.** Pick any \( \eta \in (0, \frac{2}{\mu + L}) \), based on the compression-related constant \( C \) and the network-related constant \( \beta \), we can find \( \gamma \) with a lower bound 0 and \( \alpha \) in certain range to achieve the exact linear convergence. It also suggests that LEAD supports compression method with arbitrary precision, i.e., any \( C > 0 \).

**Corollary 1** (Consensus error). Let \( x^k = \frac{1}{n} \sum_{i=1}^{n} x^k_i \) be the averaged model parameter, then all agents achieve consensus at the exponential rate with
\[
\frac{1}{n} \sum_{i=1}^{n} \left\| x^k_i - x^k \right\|_2^2 \leq \frac{2L^1}{\rho} \rho^{k-1}.
\] (10)

**Corollary 2** (Consistency with NIDS). When there is no communication compression, i.e. \( C = 0 \), and \( \gamma = 1 \), the generated sequence is deterministic, and we have the convergence for NIDS with \( \eta \in (0, \frac{2}{\mu + L}) \):
\[
L^{k+1} \leq \max \left\{ 1 - \mu(2\eta - \mu \eta^2), 1 - \frac{1}{2\lambda_{\max}((I - W)^i)} \right\} L^k.
\] (11)

5. Numerical Experiments

We consider two machine learning problems – \( \ell_2 \)-regularized linear regression and logistic regression. The proposed algorithm LEAD is compared with the state-of-the-art baseline CHOCO-SGD and two non-compressed algorithms, i.e., NIDS (Li et al. 2019) and DGD (Yuan et al., 2016). We do not compare LEAD with DCD-SGD, ECD-SGD, and DeepSqueeze, because their convergence results are shown under nonconvex settings. Moreover, it has been shown that CHOCO-SGD outperforms these algorithms and is more stable in (Koloskova et al., 2019; Koloskova* et al., 2020). Note that the full-batch gradient is applied for all algorithms in the experiments to exclude the impact of gradient variance. We evaluate the algorithms using the distance to the optimal solution \( \|X^k - X^*\|_F \) (or the loss on the average model for logistic regression\(^2 \) \( f((X^k)^\top)) \), consensus error \( \|X^k - 1_{n \times 1} X^k\|_F \), and compression error.

\(^2\) For algorithms achieving consensual solutions such as NIDS and LEAD, the final loss on the average model is the same as that on the individual model. However, for algorithms without achieving consensual solutions such as DGD and CHOCO-SGD, the true performance has to be evaluated on the average model.
Setup. We consider eight machines connected in a ring topology network. Each agent can only exchange information with its two 1-hop neighbors. The mixing weight is simply set as 1/3. For compression, we use the unbiased $b$-bits quantization method with $\infty$-norm.  

\[
Q_{\infty}(x) := \left( \|x\|_\infty 2^{-(b-1)} \text{sign}(x) \right) \cdot \left[ \frac{2^{(b-1)}|x|}{\|x\|_\infty} + u \right], \tag{12}
\]

where $\cdot$ is the Hadamard product, $|x|$ is the elementwise absolute value of $x$, and $u$ is a random vector uniformly distributed in $[0,1]^d$. Only $\text{sign}(x)$, norm $\|x\|_\infty$, and integers in the bracket need to be transmitted. Theoretical justification and the advantage of this quantization method are fully discussed in Appendix B. When using 2-bit quantization, we quantize the data blockwise (block size = 512) to reduce the compression error. Due to the space limit, we only present the results with 2-bit compression here, while the experiments with 3-bit compression are in Appendix C.1.

The stepsize $\eta$ is tuned from $\{0.01, 0.05, 0.1, 0.5\}$. We fix $\alpha = 0.5$ and $\gamma = 1.0$ for LEAD. For CHOCO-SGD, $\gamma$ is tuned from $\{0.2, 0.4, 0.6, 0.8, 1.0\}$. Note that $\gamma = 1.0$ works well for LEAD. However, a large $\gamma$ often makes CHOCO-SGD diverge and unstable, while a small $\gamma$ slows down the model average and convergence because it changes the effective mixing matrix. Detailed parameter settings are summarized in Appendix C.2.

**Linear regression.** We consider the problem: $f(x) = \sum_{i=1}^{n} (\|A_i x - b_i\|^2 + \lambda \|x\|^2)$.

Data matrices $A_i \in \mathbb{R}^{200 \times 200}$ and the true solution $x^*$ are randomly synthesized. The values $b_i$ are generated by adding Gaussian noise to $A_i x^*$. We let $\lambda = 0.1$ and the optimal solution of the linear regression problem be $x^*$. The performance is showed in Fig. 1. The distance to $x^*$ and consensus error in Fig. 1a and 1c verify that LEAD converges exponentially to the optimal consensual solution. It significantly outperforms DGD and CHOCO-SGD while matching NIDS. Fig. 1b demonstrates the benefit of compression. Fig. 1d shows that the compression error vanishes for both LEAD and CHOCO-SGD.

**Logistic regression** We further consider a logistic regression problem on the MNIST dataset. The regularization parameter is $1 \times 10^{-4}$. We consider two settings: shuffled and unshuffled data splitting. In the shuffled case, the data samples are randomly shuffled before uniformly partitioned among all agents. In the unshuffled case, the samples are first sorted by their labels and then partitioned among agents. The unshuffled case is to enlarge the data heterogeneity and better evaluate how it impacts the performance of different algorithms. The results in both settings are shown in Fig. 2 and Fig. 3.

In the shuffled case, the data in different agents are very similar and the data heterogeneity is negligible. Therefore, all algorithms almost overlap with each other as shown in Fig. 2a. In Fig. 2b, CHOCO-SGD overlaps with LEAD and DGD overlaps with NIDS. But Fig. 2c still can show that LEAD achieves much smaller consensus error than CHOCO-SGD and DGD. Compression error vanishes for both algorithms as shown in Fig. 2d. In the unshuffled case, the data heterogeneity is considerable. LEAD converges to the optimal and consensual solution.
Figure 1: Performance comparison for the linear regression problem.

Figure 2: Performance comparison on logistic regression in the shuffled case.
solution much faster than DGD and CHOCO-SGD while matching NIDS as shown in Fig. 3a and 3c. Fig. 3b shows LEAD significantly outperforms all baselines in terms of communication efficiency. Fig. 3d shows the decrease of compression error for both LEAD and CHOCO-SGD.

In summary, all experiments support that LEAD converges to the exact consensual and optimal solution much faster than CHOCO-SGD and DGD. It also significantly outperforms NIDS in terms of communication efficiency. Both LEAD and CHOCO-SGD have vanishing compression error provided that they converge. These results are consistent with our theoretical analysis and suggest that LEAD achieves the state-of-the-art computation and communication efficiency.

6. Conclusion

We study the decentralized optimization problem with communication compression and propose a novel algorithm LEAD, the first linear convergent algorithm in such setting. The theoretical behavior of LEAD is demonstrated by an analysis on the coupled dynamics of inaccurate model propagation and optimization process. Our theory suggests that LEAD achieves linear convergence to the consensual and optimal solution of the optimization problem under mild assumptions. Experiments demonstrate state-of-the-art efficiency of LEAD. Stochastic extension and nonconvex analysis are not considered in this paper. They are important directions and left for future work.
References

Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. QSGD: Communication-efficient sgd via gradient quantization and encoding. In Advances in Neural Information Processing Systems, pages 1709–1720. 2017.

Jeremy Bernstein, Yu-Xiang Wang, Kamyar Azizzadenesheli, and Animashree Anandkumar. SIGNSGD: compressed optimisation for non-convex problems. In Proceedings of the 35th International Conference on Machine Learning, pages 559–568, 2018.

Ruggero Carli, Fabio Fagnani, Paolo Frasca, and Sandro Zampieri. Gossip consensus algorithms via quantized communication. Automatica, 46(1):70–80, 2010.

Robert Mansel Gower, Nicolas Loizou, Xun Qian, Alibek Sailanbayev, Egor Shulgin, and Peter Richtárik. SGD: General analysis and improved rates. arXiv preprint arXiv:1901.09401, 2019.

Sai Praneeth Karimireddy, Quentin Rebjock, Sebastian Urban Stich, and Martin Jaggi. Error feedback fixes SignSGD and other gradient compression schemes. In Proceedings of the 36th International Conference on Machine Learning, pages 3252–3261. PMLR, 2019. URL https://arxiv.org/abs/1901.09847.

Anastasia Koloskova, Sebastian U. Stich, and Martin Jaggi. Decentralized stochastic optimization and gossip algorithms with compressed communication. In Proceedings of the 36th International Conference on Machine Learning, pages 3479–3487. PMLR, 2019. URL http://proceedings.mlr.press/v97/koloskova19a.html.

Anastasia Koloskova*, Tao Lin*, Sebastian U Stich, and Martin Jaggi. Decentralized deep learning with arbitrary communication compression. In International Conference on Learning Representations, 2020. URL https://openreview.net/forum?id=SkgGCkrKvH.

Yao Li and Ming Yan. On linear convergence of two decentralized algorithms. arXiv preprint arXiv:1906.07225, 2019.

Zhi Li, Wei Shi, and Ming Yan. A decentralized proximal-gradient method with network independent step-sizes and separated convergence rates. IEEE Transactions on Signal Processing, 67(17):4494–4506, 2019.

Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. In Advances in Neural Information Processing Systems, pages 5330–5340, 2017.

Xiaorui Liu, Yao Li, Jiliang Tang, and Ming Yan. A double residual compression algorithm for efficient distributed learning. The 23rd International Conference on Artificial Intelligence and Statistics, 2020.

Yucheng Lu and Christopher De Sa. Moniqua: Modulo quantized communication in decentralized sgd. arXiv preprint arXiv:2002.11787, 2020.
Konstantin Mishchenko, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. Distributed learning with compressed gradient differences. arXiv preprint arXiv:1901.09269, 2019.

Angelia Nedic and Asuman Ozdaglar. Distributed subgradient methods for multi-agent optimization. IEEE Transactions on Automatic Control, 54(1):48–61, 2009.

Lam Nguyen, Phuong Ha Nguyen, Marten van Dijk, Peter Richtarik, Katya Scheinberg, and Martin Takac. SGD and Hogwild! Convergence without the bounded gradients assumption. In Proceedings of the 35th International Conference on Machine Learning, pages 3750–3758. PMLR, 2018. URL http://proceedings.mlr.press/v80/nguyen18c.html.

Reza Olfati-Saber and Jeff S Shamma. Consensus filters for sensor networks and distributed sensor fusion. In Proceedings of the 44th IEEE Conference on Decision and Control, pages 6698–6703. IEEE, 2005.

Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and application to data-parallel distributed training of speech DNNs. In Interspeech 2014, September 2014.

Wei Shi, Qing Ling, Gang Wu, and Wotao Yin. Extra: An exact first-order algorithm for decentralized consensus optimization. SIAM Journal on Optimization, 25(2):944–966, 2015. doi: 10.1137/14096668X. URL https://doi.org/10.1137/14096668X.

Sebastian U. Stich, Jean-Baptiste Cordonnier, and Martin Jaggi. Sparsified SGD with memory. In Proceedings of the 32Nd International Conference on Neural Information Processing Systems, pages 4452–4463, 2018.

Hanlin Tang, Shaoduo Gan, Ce Zhang, Tong Zhang, and Ji Liu. Communication compression for decentralized training. In Advances in Neural Information Processing Systems, pages 7652–7662. 2018a. URL http://papers.nips.cc/paper/7992-communication-compression-for-decentralized-training.pdf.

Hanlin Tang, Xiangru Lian, Ming Yan, Ce Zhang, and Ji Liu. $D^2$: Decentralized training over decentralized data. In Proceedings of the 35th International Conference on Machine Learning, pages 4848–4856. PMLR, 2018b. URL http://proceedings.mlr.press/v80/tang18a.html.

Hanlin Tang, Xiangru Lian, Shuang Qiu, Lei Yuan, Ce Zhang, Tong Zhang, and Ji Liu. Deepsqueeze: Parallel stochastic gradient descent with double-pass error-compensated compression. CoRR, abs/1907.07346, 2019a. URL http://arxiv.org/abs/1907.07346.

Hanlin Tang, Chen Yu, Xiangru Lian, Tong Zhang, and Ji Liu. DoubleSqueeze: Parallel stochastic gradient descent with double-pass error-compensated compression. In Proceedings of the 36th International Conference on Machine Learning, pages 6155–6165, 2019b.

John Tsitsiklis, Dimitri Bertsekas, and Michael Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. IEEE transactions on automatic control, 31(9):803–812, 1986.
Wei Wen, Cong Xu, Feng Yan, Chunjpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. Terngrad: Ternary gradients to reduce communication in distributed deep learning. In *Advances in neural information processing systems*, pages 1509–1519, 2017.

Jiaxiang Wu, Weidong Huang, Junzhou Huang, and Tong Zhang. Error compensated quantized SGD and its applications to large-scale distributed optimization. In *Proceedings of the 35th International Conference on Machine Learning*, pages 5325–5333, 2018.

Feng Yan, Shreyas Sundaram, SVN Vishwanathan, and Yuan Qi. Distributed autonomous online learning: Regrets and intrinsic privacy-preserving properties. *IEEE Transactions on Knowledge and Data Engineering*, 25(11):2483–2493, 2012.

Tao Yang, Di Wu, Yannan Sun, and Jianming Lian. Minimum-time consensus-based approach for power system applications. *IEEE Transactions on Industrial Electronics*, 63(2):1318–1328, 2015.

Kun Yuan, Qing Ling, and Wotao Yin. On the convergence of decentralized gradient descent. *SIAM Journal on Optimization*, 26(3):1835–1854, 2016.

Martin Zinkevich, Markus Weimer, Lihong Li, and Alex J. Smola. Parallelized stochastic gradient descent. In *Advances in Neural Information Processing Systems 23*, pages 2595–2603. Curran Associates, Inc., 2010.
Appendix

Appendix A. Algorithm LEAD

In the main paper, we described the algorithm with matrix notations for concision. Here we further provide a complete algorithm description from the agents’ perspective.

Algorithm 3 LEAD

| input: stepsize $\eta$, compression parameters $(\alpha, \gamma)$, initial values $x_i^0$, $h_i^1$, $z_i$, $\forall i \in \{1, 2, \ldots, n\}$ |
| output: $x_i^K$, $\forall i \in \{1, 2, \ldots, n\}$ or $\sum_{i=1}^{n} x_i^K$ |

1: for each agent $i \in \{1, 2, \ldots, n\}$ do
2: \hspace{1em} $d_i^1 = z_i - \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} z_j$
3: \hspace{1em} $(h_w)_i^1 = \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} (h_w)_j^1$
4: \hspace{1em} $x_i^1 = x_i^0 - \eta \nabla f_i(x_i^0)$
end for
6: for $k = 1, 2, \ldots, K - 1$ do in parallel for all agents $i \in \{1, 2, \ldots, n\}$
7: \hspace{1em} compute $\nabla f_i(x_i^k)$ \hspace{1em} $\triangleright$ Gradient computation
8: \hspace{1em} $y_i^k = x_i^k - \eta \nabla f_i(x_i^k) - \eta d_i^k$
9: \hspace{1em} $q_i^k = \text{Compress}(y_i^k - h_i^k)$ \hspace{1em} $\triangleright$ Compression
10: \hspace{1em} $\hat{y}_i^k = h_i^k + q_i^k$
11: \hspace{1em} for neighbors $j \in N_i$ do
12: \hspace{2em} Send $q_i^k$ and receive $q_j^k$ \hspace{1em} $\triangleright$ Communication
end for
14: \hspace{1em} $(\hat{y}_w)_i^k = (h_w)_i^k + \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} q_j^k$
15: \hspace{1em} $h_i^{k+1} = (1 - \alpha) h_i^k + \alpha \hat{y}_i^k$
16: \hspace{1em} $(h_w)_i^{k+1} = (1 - \alpha) (h_w)_i^k + \alpha (\hat{y}_w)_i^k$
17: \hspace{1em} $d_i^{k+1} = d_i^k + \frac{\gamma}{2\eta} (\hat{y}_i^k - (\hat{y}_w)_i^k)$
18: \hspace{1em} $x_i^{k+1} = x_i^k - \eta \nabla f_i(x_i^k) - \eta d_i^{k+1}$ \hspace{1em} $\triangleright$ Model update
end for

Appendix B. Compression method

B.1 p-norm b-bits quantization

Theorem 2 (p-norm b-bit quantization). Let us define the quantization operator as

$$Q_p(x) := \left( \|x\|_p \text{sign}(x) 2^{-(b-1)} \right) \cdot \left[ \frac{2^{b-1} |x|}{\|x\|_p} + u \right]$$

(13)

where $\cdot$ is the Hadamard product, $|x|$ is the elementwise absolute value and $u$ is a random dither vector uniformly distributed in $[0, 1]^d$. $Q_p(x)$ is unbiased, i.e., $\mathbb{E} Q_p(x) = x$, and the
compression variance is upper bounded by
\[
\mathbb{E}\|x - Q_p(x)\|^2 \leq \frac{1}{4}\|\text{sign}(x)2^{-(b-1)}\|^2\|x\|^2_p,
\] (14)
which suggests that \(\infty\)-norm provides the smallest upper bound for the compression variance due to \(\|x\|_p \leq \|x\|_q\), \(\forall x\) if \(1 \leq q \leq p \leq \infty\).

**Remark 3.** For the compressor defined in (13), we have the following compression constant
\[
C = \sup_x \frac{\|\text{sign}(x)2^{-(b-1)}\|^2\|x\|^2_p}{4\|x\|^2}.
\]

**Proof.** Let denote \(v = \|x\|_p\text{sign}(x)2^{-(b-1)}\), \(s = \frac{2^{b-1}|x|}{\|x\|_p}\), \(s_1 = \left\lfloor \frac{2^{b-1}|x|}{\|x\|_p} \right\rfloor\) and \(s_2 = \left\lceil \frac{2^{b-1}|x|}{\|x\|_p} \right\rceil\).

We can rewrite \(x\) as \(x = s \cdot v\).

For any coordinate \(i\) such that \(s_i = (s_1)_i\), we have \(Q_p(x_i) = (s_1)_i v_i\) with probability 1. Hence \(\mathbb{E}Q_p(x_i) = s_i v_i = x_i\) and
\[
\mathbb{E}(x_i - Q_p(x_i))^2 = (x_i - s_i v_i)^2 = 0.
\]

For any coordinate \(i\) such that \(s_i \neq (s_1)_i\), we have \((s_2)_i - (s_1)_i = 1\) and \(Q_p(x)_i\) satisfies
\[
Q_p(x)_i = \begin{cases} (s_1)_i v_i, & \text{w.p. } (s_2)_i - s_i, \\ (s_2)_i v_i, & \text{w.p. } s_i - (s_1)_i. \end{cases}
\]

Thus, we derive
\[
\mathbb{E}Q_p(x)_i = v_i((s_1)_i(s_2) + v_i((s_2)_i(s_2) - 1) = v_i(s_2 - s_1)_i = v_i s_i = x_i,
\]
and
\[
\mathbb{E}[x_i - Q_p(x)_i]^2 = (x_i - v_i(s_1)_i)^2(s_2 - s_1) + (x_i - v_i(s_2)_i)^2(s_2 - s_1)
= (s_2 - s_1)x_i^2 + ((s_1)_i(s_2)_i(s_2 - s_1) + s_i((s_2)_i^2 - (s_1)_i^2))v_i^2 - 2s_i(s_2 - s_1)x_i v_i
= x_i^2 + (1 - (s_1)_i(s_2)_i + s_i(s_2 + s_1))v_i^2 - 2s_i x_i v_i
= (x_i - s_i v_i)^2 + ((s_2)_i(s_2)_i + s_i(s_2 + s_1)_i - s_i^2)v_i^2
= (x_i - s_i v_i)^2 + (s_2 - s_1)_i v_i^2
= (s_2 - s_1)_i(s_2 - s_1)_i v_i^2.
\]

Considering both cases, we have \(\mathbb{E}Q(x) = x\) and
\[
\mathbb{E}\|x - Q_p(x)\|^2 = \sum_{\{s_i = (s_1)_i\}} \mathbb{E}[x_i - Q_p(x)_i]^2 + \sum_{\{s_i \neq (s_1)_i\}} \mathbb{E}[x_i - Q_p(x)_i]^2
\leq 0 + \frac{1}{4} \sum_{\{s_i \neq (s_1)_i\}} v_i^2
\leq \frac{1}{4}\|v\|^2
= \frac{1}{4}\|\text{sign}(x)2^{-(b-1)}\|^2\|x\|^2_p.
\]
\[\square\]
B.2 Compression error

To verify Theorem 2, we compare the compression error of the quantization method defined in (13) with different norms ($p = 2, 3, \ldots, 7, \infty$). Specifically, we uniformly generate 100 random vectors in $\mathbb{R}^{10000}$ and compute the average compression error. The result shown in Figure 4 verifies our proof in Theorem 2 that the compression error decreases when $p$ increases. This suggests that $\infty$-norm provides the best compression precision under the same bit constraint.

Under similar setting, we also compare the compression error with other popular compression methods, such as top-k and random-k sparsification. The x-axes represents the average bits needed to represent each element of the vector. The result is showed in Fig. 5. Note that intuitively top-k methods should perform better than random-k method, but the
top-k method needs extra bits to transmitted the index while random-k method can avoid this by using the same random seed. Therefore, top-k method doesn’t outperform random-k too much under the same communication budget. The result in Fig. 5 suggests that $\infty$-norm $b$-bits quantization provides significantly better compression precision than others under the same bit constraint.

Appendix C. Experiments

C.1 Additional Experiments

We provide additional experiments on all settings with 3-bit compression as shown in Figures 6-8. These results support similar conclusions as the 2-bit quantization setting.

Figure 6: Linear regression. (a) and (b) show distance between $X^k$ and the optimal solution $X^*$ w.r.t. epochs and communication cost (bits transmitted). (c) and (d) show the consensus error and compression error.
Figure 7: Logistic regression (shuffled case). (a) and (b) show the training loss w.r.t. epoch and communication cost (bits transmitted). (c) and (d) show the consensus error and compression error.
Figure 8: Logistic regression (unshuffled case). (a) and (b) show the training loss w.r.t. epoch and communication cost (bits transmitted). (c) and (d) show the consensus error and compression error.
C.2 Parameter settings

The stepsize $\eta$ is tuned from $\{0.01, 0.05, 0.1, 0.5\}$. In all experiments, we fix $\alpha = 0.5$ and $\gamma = 1.0$ for LEAD. For CHOCO-SGD, $\gamma$ is tuned from $\{0.2, 0.4, 0.6, 0.8, 1.0\}$. Note that LEAD works well with $\gamma = 1.0$ for all experiments but a large $\gamma$ makes CHOCO-SGD diverge when the compression error is large, e.g., 2-bit quantization. This is consistent with the experiments in (Koloskova et al., 2019), which typically chooses a very small $\gamma$. However, a small $\gamma$ will slow down the model averaging as it changes the effective mixing matrix $W$ in CHOCO-SGD. The detailed parameter settings are summarized in Tables 1 and 2.

| Algorithm   | $\eta$ | $\gamma$ | $\alpha$ |
|-------------|--------|-----------|----------|
| DGD         | 0.1    | -         | -        |
| NIDS        | 0.1    | -         | -        |
| CHOCO-SGD   | 0.1    | 0.8       | -        |
| LEAD        | 0.1    | 1.0       | 0.5      |

2 bit quantization

Table 1: Parameter settings for the linear regression problem.

| Algorithm   | $\eta$ | $\gamma$ | $\alpha$ |
|-------------|--------|-----------|----------|
| DGD         | 0.05   | -         | -        |
| NIDS        | 0.05   | -         | -        |
| CHOCO-SGD   | 0.05   | 0.6       | -        |
| LEAD        | 0.05   | 1.0       | 0.5      |

2 bit quantization

Table 2: Parameter settings for the logistic regression problem (shuffled and unshuffled cases).
Appendix D. Proofs of the theorems

D.1 Illustrative flow

The following flow graph depicts the relation between iterative variables and clarifies the range of conditional expectation.

\[(X_1, D_1, H_1) \rightarrow (X_2, D_2, H_2) \rightarrow (X_3, D_3, H_3) \rightarrow \cdots \rightarrow (X_k, D_k, H_k) \rightarrow \cdots \]

\[Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{k-1} \rightarrow Y_k\]

\[F_0 \subset \cdots \subset F_{k-2} \subset F_{k-1}\]

The solid and dotted arrows in the top flow illustrate the dynamics of the algorithm, while in the bottom, the arrows stand for the relation between successive \(\sigma\)-algebras. The downward arrows determine the range of \(\sigma\)-algebras. E.g., up to \(E_k\), all random variables are in \(F_{k-1}\).

The conditional expectation of \(E^k\) in the \(k\)th iteration is \(E[E^k|F_{k-1}]\). Throughout the appendix, without specification, \(E\) is the expectation conditioned on the corresponding iteration as defined.

D.2 Proof of Proposition 1

Proof of Proposition 1. Let \(\gamma = 1\) and \(\hat{Y}^k = Y^k\). Combing Lines 3 and 7 of Alg. 1 gives

\[D^{k+1} = D^k + I - W \frac{1}{2\eta} (X^k - \eta \nabla F(X^k) - \eta D^k).\]  \hspace{1cm} (15)

Based on line 8, we can represent \(\eta D^k\) from the previous iteration as

\[\eta D^k = X^{k-1} - X^k - \eta \nabla F(X^{k-1}).\]  \hspace{1cm} (16)

Eliminating both \(D^k\) and \(D^{k+1}\) by substituting (15)-(16) into Line 8, we obtain

\[X^{k+1} = X^k - \eta \nabla F(X^k) - \left(\eta D^k + I - W \frac{1}{2} (X^k - \eta \nabla F(X^k) - \eta D^k)\right)\]  \hspace{1cm} (from (15))

\[= \frac{I + W}{2} (X^k - \eta \nabla F(X^k)) - \frac{I + W}{2} \eta D^k\]

\[= \frac{I + W}{2} (X^k - \eta \nabla F(X^k)) - \frac{I + W}{2} (X^{k-1} - X^k - \eta \nabla F(X^{k-1}))\]  \hspace{1cm} (from (16))

\[= \frac{I + W}{2} (2X^k - X^{k-1} - \eta \nabla F(X^k) + \eta \nabla F(X^{k-1})),\]  \hspace{1cm} (17)

which is exactly NIDS. The connection to \(D^2\) is completed by replacing the full gradient \(\nabla F(X^k)\) with its stochastic estimation \(\nabla F(X^k; \xi^k)\).

D.3 Two central Lemmas

**Lemma 1** (Fundamental equality). Let \(X^*\) be the optimal solution, \(D^* := -\nabla F(X^*)\) and \(E^k\) denote the compression error in the \(k\)th iteration, that is \(E^k = Q^k - (Y^k - H^k) = \hat{Y}^k - Y^k\).
From Alg. 1, we have

\[
\|X^{k+1} - X^*\|^2 + (\eta^2/\gamma)\|D^{k+1} - D^*\|_M^2 \\
= \|X^k - X^*\|^2 + (\eta^2/\gamma)\|D^k - D^*\|_M^2 - (\eta^2/\gamma)\|D^{k+1} - D^k\|_M^2 - \eta^2\|D^{k+1} - D^*\|^2 \\
- 2\eta(X^k - X^*, \nabla F(X^k) - \nabla F(X^*)) + \eta^2\|\nabla F(X^k) - \nabla F(X^*)\|^2 + 2\eta(E^k, D^{k+1} - D^*),
\]

where \( M := 2(I - W)^\dagger - \gamma I \) and \( \gamma < 2/\lambda_{\max}(I - W) \) ensures the positive definiteness of \( M \) over \( \text{range}(I - W) \).

**Lemma 2** (State inequality). Let the same assumptions in Lemma 1 hold. From Alg. 1, if we take the expectation over the compression operator conditioned on the \( k \)-th iteration, we have

\[
E\|H^{k+1} - X^*\|^2 \leq (1 - \alpha)\|H^k - X^*\|^2 + \alpha E\|X^{k+1} - X^*\|^2 + \alpha \eta^2 E\|D^{k+1} - D^k\|^2 \\
+ \frac{2\alpha \eta^2}{\gamma} E\|D^{k+1} - D^k\|_M^2 + \alpha^2 E\|E^k\|^2 - \alpha \gamma E\|E^k\|^2_{I - W} - \alpha(1 - \alpha)\|Y^k - H^k\|^2.
\]

**D.4 Proof of Lemma 1**

Before proving Lemma 1, we let \( E^k = \hat{Y}^k - Y^k \) and introduce the following three Lemmas.

**Lemma 3.** Let \( X^* \) be the consensus solution. Then, from Line 3-8 of Alg. 1, we obtain

\[
\frac{I - W}{2\eta}(X^{k+1} - X^*) = \left( I - \frac{I - W}{2} \right) (D^{k+1} - D^k) - \frac{I - W}{2\eta}E^k.
\]

**Proof.** From the iterations in Alg. 1, we have

\[
D^{k+1} = D^k + \frac{\gamma}{2\eta}(I - W)\hat{Y}^k \quad \text{(from Line 7)} \\
= D^k + \frac{\gamma}{2\eta}(I - W)(Y^k + E^k) \\
= D^k + \frac{\gamma}{2\eta}(I - W)(X^k - \eta \nabla F(X^k) - \eta D^k + E^k) \quad \text{(from Line 3)} \\
= D^k + \frac{\gamma}{2\eta}(I - W)(X^k - \eta \nabla F(X^k) - \eta D^{k+1} - X^* + \eta(D^{k+1} - D^k) + E^k) \\
= D^k + \frac{\gamma}{2\eta}(I - W)(X^{k+1} - X^*) + \frac{\gamma}{2}(I - W)(D^{k+1} - D^k) + \frac{\gamma}{2\eta}(I - W)E^k,
\]

where the fourth equality holds due to \((I - W)X^* = 0\) and the last equality comes from Line 8 of Alg. 1. Rewriting this equality, and we obtain (18). \( \square \)

**Lemma 4.** Let \( D^* = -\nabla F(X^*) \in \text{span}\{I - W\} \), we have

\[
\langle X^{k+1} - X^*, D^{k+1} - D^k \rangle = \frac{\eta}{\gamma} \|D^{k+1} - D^k\|_M^2 - \langle E^k, D^{k+1} - D^k \rangle, \quad (19) \\
\langle X^{k+1} - X^*, D^{k+1} - D^* \rangle = \frac{\eta}{\gamma} \|D^{k+1} - D^k, D^{k+1} - D^*\|_M - \langle E^k, D^{k+1} - D^* \rangle, \quad (20)
\]

where \( M = 2(I - W)^\dagger - \gamma I \) and \( \gamma < 2/\lambda_{\max}(I - W) \) ensures the positive definiteness of \( M \) over \( \text{span}\{I - W\} \).
Proof. Since $D^{k+1} \in \text{span}\{I - W\}$ for any $k$, we have

$$\langle X^{k+1} - X^*, D^{k+1} - D^* \rangle$$
$$= \langle (I - W)(X^{k+1} - X^*), (I - W)^\dagger(D^{k+1} - D^*) \rangle$$
$$= \left\langle \frac{\eta}{\gamma} (2(I - \gamma(I - W))(D^{k+1} - D^k) - (I - W)E^k, (I - W)^\dagger(D^{k+1} - D^*) \right\rangle \quad \text{(from (18))}$$
$$= \left\langle \frac{\eta}{\gamma} (2(I - W)^\dagger - \gamma I)(D^{k+1} - D^k) - E^k, D^{k+1} - D^k \right\rangle$$
$$= \frac{\eta}{\gamma} \|D^{k+1} - D^k\|_M^2 - \langle E^k, D^{k+1} - D^k \rangle.$$

Similarly, we have

$$\langle X^{k+1} - X^*, D^{k+1} - D^* \rangle$$
$$= \langle (I - W)(X^{k+1} - X^*), (I - W)^\dagger(D^{k+1} - D^*) \rangle$$
$$= \left\langle \frac{\eta}{\gamma} (2(I - \gamma(I - W))(D^{k+1} - D^k) - (I - W)E^k, (I - W)^\dagger(D^{k+1} - D^*) \right\rangle$$
$$= \left\langle \frac{\eta}{\gamma} (2(I - W)^\dagger - \gamma I)(D^{k+1} - D^k) - E^k, D^{k+1} - D^* \right\rangle$$
$$= \frac{\eta}{\gamma} \langle D^{k+1} - D^k, D^{k+1} - D^* \rangle_M - \langle E^k, D^{k+1} - D^* \rangle.$$

To make sure that $M$ is positive definite over $\text{span}\{I - W\}$, we need $\gamma < 2/\lambda_{\max}(I - W)$. □

**Lemma 5.** Taking the expectation conditioned on the compression in the $k$th iteration, we have

$$2\eta \mathbb{E}\langle E^k, D^{k+1} - D^* \rangle = 2\eta \mathbb{E}\left\langle E^k, D^k + \frac{\gamma}{2\eta}(I - W)Y^k + \frac{\gamma}{2\eta}(I - W)E^k - D^* \right\rangle$$
$$= \gamma \mathbb{E}\langle E^k, (I - W)E^k \rangle = \gamma \mathbb{E}\|E^k\|_{I - W}^2.$$

$$2\eta \mathbb{E}\langle E^k, D^{k+1} - D^k \rangle = 2\eta \mathbb{E}\left\langle E^k, \frac{\gamma}{2\eta}(I - W)Y^k + \frac{\gamma}{2\eta}(I - W)E^k \right\rangle$$
$$= \gamma \mathbb{E}\langle E^k, (I - W)E^k \rangle = \gamma \mathbb{E}\|E^k\|_{I - W}^2.$$

Proof. The proof is straightforward and omitted here. □

25
Proof of Lemma 1. From Alg. 1, we have

\begin{align*}
2\eta(X^k - X^* , \nabla F(X^k) - \nabla F(X^*)) \\
= 2 \langle X^k - X^* , \eta \nabla F(X^k) - \eta \nabla F(X^*) \rangle \\
= 2 \langle X^k - X^* , X^k - X^{k+1} - \eta(D^{k+1} - D^*) \rangle \quad \text{(from Line 8)} \\
= 2 \langle X^k - X^* , X^k - X^{k+1} \rangle - 2\eta \langle X^k - X^* , D^{k+1} - D^* \rangle \\
= 2 \langle X^k - X^* - \eta(D^{k+1} - D^*) , X^k - X^{k+1} \rangle - 2\eta \langle X^k - X^* , D^{k+1} - D^* \rangle \\
= 2 \langle X^{k+1} - X^* + \eta(\nabla F(X^k) - \nabla F(X^*)) , X^k - X^{k+1} \rangle - 2\eta \langle X^{k+1} - X^* , D^{k+1} - D^* \rangle \quad \text{(from Line 8)} \\
= 2 \langle X^{k+1} - X^* , X^k - X^{k+1} \rangle + 2\eta \langle \nabla F(X^k) - \nabla F(X^*) , X^k - X^{k+1} \rangle \\
- 2\eta \langle X^{k+1} - X^* , D^{k+1} - D^* \rangle. \quad (21)
\end{align*}

Then we consider the terms on the right hand side of (21) separately. Using $2 \langle A - B , B - C \rangle = \|A - C\|^2 - \|B - C\|^2 - \|A - B\|^2$, we have

\begin{align*}
2 \langle X^{k+1} - X^* , X^k - X^{k+1} \rangle &= 2 \langle X^* - X^{k+1} , X^{k+1} - X^k \rangle \\
&= \|X^k - X^*\|^2 - \|X^{k+1} - X^k\|^2 - \|X^{k+1} - X^*\|^2. \quad (22)
\end{align*}

Using $2 \langle A , B \rangle = \|A\|^2 + \|B\|^2 - \|A - B\|^2$, we have

\begin{align*}
2\eta \langle \nabla F(X^k) - \nabla F(X^*) , X^k - X^{k+1} \rangle \\
= \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + \|X^k - X^{k+1}\|^2 - \|X^k - X^{k+1} - \eta(\nabla F(X^k) - \nabla F(X^*))\|^2 \\
= \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + \|X^k - X^{k+1}\|^2 - \eta^2 \|D^{k+1} - D^*\|^2. \quad \text{(from Line 8)} \quad (23)
\end{align*}

Combining (21), (22), (23), and (19), we obtain

\begin{align*}
2\eta \langle X^k - X^* , \nabla F(X^k) - \nabla F(X^*) \rangle \\
= \|X^k - X^*\|^2 - \|X^{k+1} - X^k\|^2 - \|X^{k+1} - X^*\|^2 \\
- \frac{\eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + \|X^k - X^{k+1}\|^2 - \eta^2 \|D^{k+1} - D^*\|^2}{2\eta \langle \nabla F(X^k) - \nabla F(X^*) , X^k - X^{k+1} \rangle} \\
- \frac{\langle 2\eta^2 \langle D^{k+1} - D^* , D^{k+1} - D^* \rangle , M - 2\eta \langle E^k , D^{k+1} - D^* \rangle \rangle}{2\eta \langle X^{k+1} - X^* , D^{k+1} - D^* \rangle} \\
= \|X^k - X^*\|^2 - \|X^{k+1} - X^k\|^2 - \|X^{k+1} - X^*\|^2 \\
+ \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + \|X^k - X^{k+1}\|^2 - \eta^2 \|D^{k+1} - D^*\|^2 \\
+ \frac{\eta^2 \|D^k - D^*\|^2 - \|D^{k+1} - D^*\|^2 + \|D^{k+1} - D^k\|^2}{2\eta \langle E^k , D^{k+1} - D^* \rangle M - 2\eta \langle E^k , D^{k+1} - D^* \rangle M} \right) + 2\eta \langle E^k , D^{k+1} - D^* \rangle,
\end{align*}

26
where the last equality holds because
\[ 2(D^k - D^{k+1}, D^{k+1} - D^*)_M = ||D^k - D^*||_M^2 - ||D^{k+1} - D^*||_M^2 - ||D^{k+1} - D^k||_M^2. \]
Thus, we reformulate it as
\[ \|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} \|D^{k+1} - D^*\|^2_M \]
\[ = \|X^k - X^*\|^2 + \frac{\eta^2}{\gamma} \|D^k - D^*\|^2_M + \frac{\eta^2}{\gamma} \|D^{k+1} - D^k\|^2_M - \eta^2 \|D^{k+1} - D^*\|^2 \]
\[ - 2\eta \langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle + \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + 2\eta \langle E^k, D^{k+1} - D^* \rangle, \]
which completes the proof.
\[ \square \]

D.5 Proof of Lemma 2

Proof of Lemma 2. From Alg. 1, we take the expectation conditioned on \( k \)th compression and obtain
\[ \mathbb{E}||H^{k+1} - X^*||^2 \]
\[ = \mathbb{E}||(1 - \alpha)(H^k - X^*) + \alpha(Y^k - X^*) + \alpha E^k||^2 \] (from Line 6)
\[ = ||(1 - \alpha)(H^k - X^*) + \alpha(Y^k - X^*)||^2 + \alpha^2 \mathbb{E}||E^k||^2 \]
\[ = (1 - \alpha)||H^k - X^*||^2 + \alpha||Y^k - X^*||^2 + \alpha(1 - \alpha)||H^k - Y^k||^2 + \alpha^2 \mathbb{E}||E^k||^2. \] (24)

In the second equality, we used the unbiasedness of the compression, i.e., \( \mathbb{E}E^k = 0 \). The last equality holds because of
\[ \|(1 - \alpha)A + \alpha B\|^2 = (1 - \alpha)\|A\|^2 + \alpha\|B\|^2 - \alpha(1 - \alpha)\|A - B\|^2. \]

In addition, we have
\[ ||Y^k - X^*||^2 = ||X^k - \eta \nabla F(X^k) - \eta D^k - X^*||^2 \] (from Line 1)
\[ = \mathbb{E}||X^{k+1} + \eta D^{k+1} - \eta D^k - X^*||^2 \] (from Line 8)
\[ = \mathbb{E}||X^{k+1} - X^*||^2 + \eta^2 \mathbb{E}||D^{k+1} - D^k||^2 + 2\eta \mathbb{E}<X^{k+1} - X^*, D^{k+1} - D^k> \]
\[ = \mathbb{E}||X^{k+1} - X^*||^2 + \eta^2 \mathbb{E}||D^{k+1} - D^k||^2 \]
\[ + \frac{2\eta^2}{\gamma} \mathbb{E}||D^{k+1} - D^k||^2_M - 2\eta \mathbb{E}<E^k, D^{k+1} - D^k>. \] (from (19))
\[ = \mathbb{E}||X^{k+1} - X^*||^2 + \eta^2 \mathbb{E}||D^{k+1} - D^k||^2 \]
\[ + \frac{2\eta^2}{\gamma} \mathbb{E}||D^{k+1} - D^k||^2_M - \gamma \mathbb{E}||E^k||^2_{\mathbb{I} - \mathbb{W}}. \] (from Line 7) (25)

Combing the above two equations (24) and (25) together, we have
\[ \mathbb{E}||H^{k+1} - X^*||^2 \]
\[ \leq (1 - \alpha)||H^k - X^*||^2 + \alpha \mathbb{E}||X^{k+1} - X^*||^2 + \alpha \eta^2 \mathbb{E}||D^{k+1} - D^k||^2 + \frac{2\alpha \eta^2}{\gamma} \mathbb{E}||D^{k+1} - D^k||^2_M \]
\[ - \alpha \gamma \mathbb{E}||E^k||^2_{\mathbb{I} - \mathbb{W}} + \alpha^2 \mathbb{E}||E^k||^2 - \alpha(1 - \alpha)||Y^k - H^k||^2, \] (26)
which completes the proof.
\[ \square \]
D.6 Proof of Theorem 1

Proof of Theorem 1. Combining Lemmas 1, 2, and 5, we have

\[
\begin{align*}
\mathbb{E}[\|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} \mathbb{E}[\|D^{k+1} - D^*\|^2_M + a_1 \mathbb{E}[\|H^{k+1} - X^*\|^2 ] \\
\leq \|X^k - X^*\|^2 + \frac{\eta^2}{\gamma} \mathbb{E}[\|D^k - D^*\|^2_M + \frac{\eta^2}{\gamma} \mathbb{E}[\|D^{k+1} - D^k\|^2_M - \eta^2 \mathbb{E}[\|D^{k+1} - D^*\|^2 ] ] \\
- 2\eta \langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle + \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 + \gamma \mathbb{E}[\|k\|^2_{\lambda - W} ] \\
+ a_1 (1 - \alpha) \|H^k - X^*\|^2 + a_1 \alpha \mathbb{E}[\|X^{k+1} - X^*\|^2 ] + a_1 \alpha \eta^2 \mathbb{E}[\|D^{k+1} - D^k\|^2 ] \\
+ \frac{2a_1 \alpha \eta^2}{\gamma} \mathbb{E}[\|D^{k+1} - D^k\|^2_M + a_1 \alpha^2 \mathbb{E}[\|E^k\|^2 ] - a_1 \alpha \gamma \mathbb{E}[\|E^k\|^2_{\lambda - W} - a_1 (1 - \alpha) \|Y^k - H^k\|^2 ] \\
= \|X^k - X^*\|^2 - 2\eta \langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle + \eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 ] \\
+ a_1 \alpha \mathbb{E}[\|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} \mathbb{E}[\|D^k - D^*\|^2_M - \eta^2 \mathbb{E}[\|D^{k+1} - D^*\|^2 ] ] \\
+ a_1 (1 - \alpha) \|H^k - X^*\|^2 - (1 - 2a_1 \alpha) \frac{\eta^2}{\gamma} \mathbb{E}[\|D^{k+1} - D^k\|^2_M + a_1 \alpha \eta^2 \mathbb{E}[\|D^{k+1} - D^k\|^2 ] ] \\
+ a_1 \alpha^2 \mathbb{E}[\|E^k\|^2 ] + (1 - a_1 \alpha) \gamma \mathbb{E}[\|E^k\|^2_{\lambda - W} - a_1 (1 - \alpha) \|Y^k - H^k\|^2 ] ,
\end{align*}
\]

(27)

where \(a_1\) is a non-negative number to be determined. Then we deal with the three terms on the right hand side separately. We want the terms \(\mathcal{B}\) and \(\mathcal{C}\) to be nonpositive. First, we consider \(\mathcal{B}\). Note that \(D^k \in \text{Range}(I - W)\). If we want \(\mathcal{B} \leq 0\), then, we need \(1 - 2a_1 \alpha > 0\), i.e., \(a_1 \alpha < 1/2\). Therefore we have

\[
\begin{align*}
\mathcal{B} &= - (1 - 2a_1 \alpha) \frac{\eta^2}{\gamma} \mathbb{E}[\|D^{k+1} - D^k\|^2_M + a_1 \alpha \eta^2 \mathbb{E}[\|D^{k+1} - D^k\|^2 ] ] \\
&\leq \left( a_1 \alpha - \frac{(1 - 2a_1 \alpha) \lambda_{n-1}(M)}{\gamma} \right) \eta^2 \mathbb{E}[\|D^{k+1} - D^k\|^2 ] ,
\end{align*}
\]

where \(\lambda_{n-1}(M) > 0\) is the second smallest eigenvalue of \(M\). It means that we also need

\[
a_1 \alpha + \frac{(2a_1 \alpha - 1) \lambda_{n-1}(M)}{\gamma} \leq 0 ,
\]

which is equivalent to

\[
a_1 \alpha \leq \frac{\lambda_{n-1}(M)}{\gamma + 2\lambda_{n-1}(M)} < 1/2 .
\]

(28)

Then we look at \(\mathcal{C}\). We have

\[
\begin{align*}
\mathcal{C} &= a_1 \alpha^2 \mathbb{E}[\|E^k\|^2 ] + (1 - a_1 \alpha) \gamma \mathbb{E}[\|E^k\|^2_{\lambda - W} - a_1 \alpha (1 - \alpha) \|Y^k - H^k\|^2 ] \\
&\leq ((1 - a_1 \alpha) \beta \gamma + a_1 \alpha^2) \mathbb{E}[\|E^k\|^2 ] - a_1 \alpha (1 - \alpha) \|Y^k - H^k\|^2 ] \\
&\leq C ((1 - a_1 \alpha) \beta \gamma + a_1 \alpha^2) \|Y^k - H^k\|^2 - a_1 \alpha (1 - \alpha) \|Y^k - H^k\|^2 ]
\end{align*}
\]

28
Because we have $1 - a_1 \alpha > 1/2$, so we need

$$C((1 - a_1 \alpha)\beta \gamma + a_1 \alpha^2) - a_1 \alpha (1 - \alpha) = (1 + C)a_1 \alpha^2 - a_1 (C \beta \gamma + 1)\alpha + C \beta \gamma \leq 0.$$ (29)

That is

$$\alpha \geq \frac{a_1 (C \beta \gamma + 1) - \sqrt{a_1^2 (C \beta \gamma + 1)^2 - 4(1 + C)Ca_1 \beta \gamma}}{2(1 + C)a_1} =: \alpha_0,$$ (30)

$$\alpha \leq \frac{a_1 (C \beta \gamma + 1) + \sqrt{a_1^2 (C \beta \gamma + 1)^2 - 4(1 + C)Ca_1 \beta \gamma}}{2(1 + C)a_1} =: \alpha_1.$$ (31)

Next, we look at $A$. With the smoothness and strong convexity from Assumptions 3, we have the co-coercivity of $\nabla g_i(x)$ with $g_i(x) := f_i(x) - \frac{\mu}{2}\|x\|^2_2$, which gives

$$\langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle \geq \frac{\mu L}{\mu + L} \|X^k - X^*\|^2 + \frac{1}{\mu + L} \|\nabla F(X^k) - \nabla F(X^*)\|^2.$$ 

When $\eta \leq 2/(\mu + L)$, we have

$$\langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle = \left(1 - \frac{\eta(\mu + L)}{2}\right) \langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle + \frac{\eta(\mu + L)}{2} (X^k - X^* \langle \nabla F(X^k) - \nabla F(X^*) \rangle$$

$$\geq \left(\mu - \frac{\eta \mu (\mu + L)}{2} + \frac{\eta \mu L}{2}\right) \|X^k - X^*\|^2 + \frac{\eta}{2} \|\nabla F(X^k) - \nabla F(X^*)\|^2$$

$$= \mu \left(1 - \frac{\eta \mu}{2}\right) \|X^k - X^*\|^2 + \frac{\eta}{2} \|\nabla F(X^k) - \nabla F(X^*)\|^2.$$ 

Therefore, we obtain

$$-2\eta \langle X^k - X^*, \nabla F(X^k) - \nabla F(X^*) \rangle$$

$$\leq -\eta^2 \|\nabla F(X^k) - \nabla F(X^*)\|^2 - \mu (2\eta - \mu \eta^2) \|X^k - X^*\|^2.$$ (32)

So the inequality (27) becomes

$$E\|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} E\|D^{k+1} - D^*\|^2_M + a_1 E\|H^{k+1} - X^*\|^2$$

$$\leq (1 - \mu(2\eta - \mu \eta^2)) \|X^k - X^*\|^2 + a_1 \alpha E\|X^{k+1} - X^*\|^2$$

$$+ \frac{\eta^2}{\gamma} \|D^k - D^*\|^2_M - \frac{\eta \mu}{\gamma} \|D^{k+1} - D^*\|^2 + a_1 (1 - \alpha) \|H^k - X^*\|^2,$$ (33)

if the step size satisfies $\eta \leq \frac{2}{\mu + L}$. Rewriting (33), we have

$$(1 - a_1 \alpha)E\|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} E\|D^{k+1} - D^*\|^2_M + \eta^2 E\|D^{k+1} - D^*\|^2 + a_1 E\|H^{k+1} - X^*\|^2$$

$$\leq (1 - \mu(2\eta - \mu \eta^2)) \|X^k - X^*\|^2 + \frac{\eta^2}{\gamma} \|D^k - D^*\|^2_M + a_1 (1 - \alpha) \|H^k - X^*\|^2,$$ (34)
and thus
\[
(1 - a_1 \alpha) \mathbb{E} \|X^{k+1} - X^*\|^2 + \frac{\eta^2}{\gamma} \mathbb{E} \|D^{k+1} - D^*\|^2_{M+\gamma I} + a_1 \mathbb{E} \|H^{k+1} - X^*\|^2 \\
\leq (1 - \mu(2\eta - \mu\eta^2)) \|X^k - X^*\|^2 + \frac{\eta^2}{\gamma} \|D^k - D^*\|^2_M + a_1 (1 - \alpha) \|H^k - X^*\|^2.
\]

(35)

With the definition of $\mathcal{L}^k$ in (10), we have
\[
\mathbb{E} \mathcal{L}^{k+1} \leq \rho \mathcal{L}^k,
\]
with
\[
\rho = \max \left\{ \frac{1 - \mu(2\eta - \mu\eta^2)}{1 - a_1 \alpha}, \frac{\lambda_{\max}(M)}{\gamma + \lambda_{\max}(M)}, 1 - \alpha \right\},
\]
where
\[
\lambda_{\max}(M) = 2\lambda_{\max}((I - W)^\dagger) - \gamma.
\]

Recall all the conditions on the parameters $a_1$, $\alpha$, and $\gamma$ to make sure that $\rho < 1$:

\[
a_1 \alpha \leq \frac{\lambda_{n-1}(M)}{\gamma + 2\lambda_{n-1}(M)},
\]
(37)

\[
a_1 \alpha \leq \mu(2\eta - \mu\eta^2),
\]
(38)

\[
\alpha \geq \frac{a_1(C\beta \gamma + 1) - \sqrt{a_1^2(C\beta \gamma + 1)^2 - 4(1+C)Ca_1\beta \gamma}}{2(1+C)a_1} =: \alpha_0,
\]
(39)

\[
\alpha \leq \frac{a_1(C\beta \gamma + 1) + \sqrt{a_1^2(C\beta \gamma + 1)^2 - 4(1+C)Ca_1\beta \gamma}}{2(1+C)a_1} =: \alpha_1.
\]
(40)

In the following, we show that there exist parameters that satisfy these conditions.

Since we can choose any $a_1$, we let
\[
a_1 = \frac{4(1+C)}{C\beta \gamma + 2},
\]
such that
\[
a_1^2(C\beta \gamma + 1)^2 - 4(1+C)Ca_1\beta \gamma = a_1^2.
\]

Then we have
\[
\alpha_0 = \frac{C\beta \gamma}{2(1+C)} \to 0, \quad \text{as } \gamma \to 0,
\]
\[
\alpha_1 = \frac{C\beta \gamma + 2}{2(1+C)} \to \frac{1}{1+C}, \quad \text{as } \gamma \to 0.
\]

Conditions (39) and (40) show
\[
a_1 \alpha \in \left[ \frac{2C\beta \gamma}{C\beta \gamma + 2}, 2 \right] \to [0, 2], \text{ if } C = 0 \text{ or } \gamma \to 0.
\]
Hence in order to make (37) and (38) satisfied, it’s sufficient to make

\[
\frac{2C\beta\gamma}{C\beta\gamma + 2} \leq \min \left\{ \frac{\lambda_{n-1}(M)}{\gamma + 2\lambda_{n-1}(M)}, \mu(2\eta - \mu\eta^2) \right\} = \min \left\{ \frac{2 - \gamma}{\beta - \gamma}, \mu(2\eta - \mu\eta^2) \right\}.
\]

(41)

where we use \( \lambda_{n-1}(M) = \frac{2}{\lambda_{\max}(1-W)} - \gamma = \frac{2}{\beta} - \gamma. \)

When \( C > 0, \) the condition (41) is equivalent to

\[
\gamma \leq \min \left\{ \frac{(6C + 2) - \sqrt{(6C + 2)^2 - 16C}}{2C\beta}, \frac{2\mu\eta(2 - \mu\eta)}{|2 - \mu\eta(2 - \mu\eta)| C\beta} \right\}.
\]

(42)

For a given stepsize \( \eta, \) if we choose

\[
\gamma \in \left( 0, \min \left\{ \frac{(6C + 2) - \sqrt{(6C + 2)^2 - 16C}}{2C\beta}, \frac{2\mu\eta(2 - \mu\eta)}{|2 - \mu\eta(2 - \mu\eta)| C\beta} \right\} \right)
\]

and

\[
\alpha \in \left[ \frac{C\beta\gamma}{2(1 + C)}, \min \left\{ \frac{C\beta\gamma + 2}{2(1 + C)} \cdot \frac{2 - \beta\gamma}{2 - \beta\gamma 4(1 + C)}, \frac{\mu\eta(2 - \mu\eta)}{4(1 + C)} \right\} \right],
\]

then, all conditions (37)-(40) hold. Note that \( \gamma < \frac{2}{\beta} \) is to ensure the positive definiteness of \( M \) over \( \text{span}\{I - W\} \) in Lemma 4.

Note that \( \eta \leq \frac{2}{\mu + L} \) ensures

\[
\mu\eta(2 - \mu\eta) \frac{C\beta\gamma + 2}{4(1 + C)} \leq \frac{C\beta\gamma + 2}{2(1 + C)}.
\]

(43)

So, we can simplify the bound for \( \alpha \) as

\[
\alpha \in \left[ \frac{C\beta\gamma}{2(1 + C)}, \min \left\{ \frac{2 - \beta\gamma}{4 - \beta\gamma 4(1 + C)}, \frac{\mu\eta(2 - \mu\eta)}{4(1 + C)} \right\} \right].
\]

Lastly, taking the total expectation on both sides of (36) and using tower property, we complete the proof for \( C > 0. \)

When \( C = 0, \) the condition (41) is satisfied when \( \gamma < \frac{2}{\beta} \) and \( \eta \leq \frac{2}{(\mu + L)}. \) So, the upper bound of \( \gamma \) in (42) holds if we let \( \frac{(6C + 2) - \sqrt{(6C + 2)^2 - 16C}}{2C\beta} = \frac{2}{\beta}, \) which is the limit when \( C \to 0. \) Because \( \beta < 2, \) we can choose \( \gamma = 1, \) and no tuning parameter is needed to control the communication. \( \square \)
Proof of Corollary 1. Note that $(\mathbf{x}^k)^\top = \mathbf{X}^k$ and $\mathbf{1}_{n \times 1} \mathbf{X}^* = \mathbf{X}^*$, then
\[
\sum_{i=1}^{n} \|\mathbf{x}_i^k - \mathbf{x}^k\|^2 = \left\| \mathbf{X}^k - \mathbf{1}_{n \times 1} \mathbf{X}^\top \right\|^2
\]
\[
= \left\| \mathbf{X}^k - \mathbf{X}^* - \mathbf{X}^* - \mathbf{1}_{n \times 1} \mathbf{1}_{n \times 1} \mathbf{X}^\top \right\|^2
\]
\[
= \left\| \mathbf{X}^k - \mathbf{X}^* - \frac{1}{n} \mathbf{1}_{n \times 1} \mathbf{1}_{n \times 1} \mathbf{X}^\top \mathbf{X}^* \right\|^2
\]
\[
\leq \|\mathbf{X}^k - \mathbf{X}^*\|^2
\]
\[
\leq \frac{\rho L^{k-1}}{1 - a_1 \alpha}
\]
\[
\leq 2 \rho^{k-1} L^1.
\]
Proof of Corollary 2. From the proof of Theorem 1, when $C = 0$, we can set $\gamma = 1$, $\alpha = 1$, and $a_1 = 0$. Plug those values into $\rho$, and we obtain the convergence rate for NIDS.