The spectrum of multi-region-relaxed magnetohydrodynamic modes in topologically toroidal geometry

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Abstract

A general formulation of the problem of calculating the spectrum of stable and unstable eigenmodes of linearized perturbations about a magnetically confined toroidal plasma is presented. The analysis is based on a new hydromagnetic dynamical model, multi-region relaxed magnetohydrodynamics, which models the plasma magnetic-field system as consisting of multiple regions, containing compressible Euler fluid and Taylor-relaxed magnetic field, separated by interfaces in the form of flexible MHD current sheets. This is illustrated using a first-principles analysis of a two-region slab geometry, with periodic boundary conditions to model the outer regions of typical tokamak or reversed-field pinch plasmas. The lowest and second-lowest eigenvalues in plasmas unstable to tearing and kink-tearing modes are calculated. Near-marginal stability, the lowest mode obtained using the incompressible approximation to the kinetic energy normalization of the present study is shown to correspond to the eigenvalues found in previous studies where all mass was artificially loaded onto the interfaces.

Supplementary material for this article is available online

Keywords: eigenvalue spectrum, relaxed MHD, MRxMHD, plasma slab, incompressible approximation

(Some figures may appear in colour only in the online journal)

1. Introduction

Despite the complex many-particle nature of plasmas in short scales, the use of fluid models has been remarkably successful in providing an understanding of the larger-scale collective behaviour of plasmas and such models are still routinely used in modelling magnetically confined fusion plasmas. The simplest such models [1] treat the plasma as a single fluid and apply them on length scales much longer than a typical ion gyroradius and timescales longer than a typical inverse ion-cyclotron frequency, and also sufficiently long for Maxwell’s displacement current to be negligible.

We may term all such theories magnetohydrodynamics (MHD), but in this paper we consider only ideal and relaxed MHD; the latter being obtained from ideal MHD by removing most of its microscopic constraints. All MHD models can be described in the Eulerian picture, which describes the fluid dynamics in terms of the space- and time-dependent fields—mass density $\rho(x, t)$, isotropic pressure $p(x, t)$, and magnetic field $B(x, t)$. Ideal MHD can also be described in the Lagrangian picture, which regards the fluid as consisting of moving fluid elements of volume $dV'$, position $r'(t)$ and mass $\rho(r', t)dV'$, with $\rho$, $p$ and $B$ holonomically (microscopically) constrained so as to be advected with the fluid elements. Relaxed MHD can be described as a hybrid theory where the Lagrangian picture can be used to describe $\rho$ as in ideal MHD, but the fields $p$ and $B$ must be treated in the Eulerian picture as they are subjected only to nonholonomic, macroscopic constraints.

1 Another hybrid case is resistive MHD, where $\rho$ may be holonomically advected, but $B$ diffuses as a free field, at least in a thin resonant reconnection layer.
Although the concept of plasma self-organization through a relaxation process, describable by a variational (energy-minimization) principle, is an old one (see, e.g. the review by Taylor [2]), the generalization to a fully fledged fluid theory, multi-region relaxed magnetohydrodynamics (MRxMHD) has only recently been promulgated [3, 4]. Unlike previous, quasi-static, multi-region generalizations of Taylor relaxation [5], this new formulation is a fully dynamical, time-dependent field theory whose self-consistency is ensured by deriving it from an action principle rather than an energy principle.

The deficiencies of ideal MHD in describing typical fusion plasmas [1] arise from its assumptions of (a) zero thermal conductivity, which implies ‘frozen-in’ entropy (i.e. an adiabatic equation of state applying in each fluid element), and (b) infinite electrical conductivity, which implies [6] frozen-in magnetic field. Assumption (a) is clearly inapplicable in high-temperature plasmas as electron mean-free-paths along magnetic-field lines are very long, making parallel heat conduction large. On the other hand, electrical conductivity is, indeed, large in fusion-relevant plasmas, so assumption (b) is (at first sight) a reasonable approximation.

However, resistivity and other non-ideal effects are enhanced in regions with short scale lengths, such as current sheets and resonances, giving rise to changes in magnetic-field-line topology (reconnection) on mesoscopic timescales, such as the growth of magnetic islands through tearing instabilities arising at resonant magnetic surfaces. Such topological changes are not allowed by the frozen-in magnetic flux property of ideal MHD, motivating the search for a simple fluid model for fusion plasmas that is more appropriate physically than ideal MHD.

The fundamental problem with ideal MHD is that conserving entropy and magnetic flux separately in an uncountable infinity of fluid elements makes it physically over-constrained. This is resolved in MRxMHD by using only a subset of ideal MHD constraints. This subset consists of entropy and magnetic helicity constraints within an arbitrary number of finite sub-volumes of the plasma, plus ideal-MHD boundary constraints on the infinity of surface elements making up the interfaces between these subregions. The parsimonious choice made in [3], to use only magnetic helicities, rather than magnetic helicities plus fluid-magnetic cross-helicities, as volume constraints decouples the magnetic field from the fluid within the subregions, giving a very simple generalization of Taylor relaxation in which the plasma behaves as an Euler fluid within each subregion.

One of the principal motivations of the development of MRxMHD has been the need for a better MHD framework than ideal MHD for numerical calculations of equilibria in fully three-dimensional (3-D) magnetic-containment devices such as stellarators, and tokamaks with resonant magnetic perturbations, where smoothly nested flux surfaces cannot be assumed. The theoretical development and physical application of this important static application is already well developed [5].

It is anticipated that the new dynamical formulation [3] will likewise provide a better framework than ideal MHD for efficient numerical calculations of stability and mode structure in realistic geometries. However, as the decoupling of fluid flow and magnetic field (except at discrete interfaces) seems, at first sight, to be a dramatic oversimplification, confidence that this novel formulation is physically reasonable needs to be built up through the analysis of simple test cases with a few interfaces, where fundamental physical effects can be isolated and analysed in detail. Then, it will need to be shown that, in the limit of an unbounded number of interfaces, MRxMHD approaches a physically applicable continuum theory that is equivalent or superior to ideal MHD (see [7]). This paper represents a first step in this programme of research.

It is the aim of the present paper to formulate a basic framework for calculating normal modes of linearized perturbations for general plasma equilibria using dynamical MRxMHD theory [3]. We also illustrate the application of this framework in a simple geometry to provide greater insight into the physics of the novel features in the MRxMHD spectrum. This insight will aid in the interpretation of the results of further studies in more realistic geometries.

For maximum simplicity we consider a simple slab geometry, in which the equilibrium quantities are independent of periodic y and z Cartesian coordinates, the system being bounded by a perfectly conducting planar wall at \( x = x_w \) and, at \( x = 0 \), a rigid, a perfectly conducting interface between the slab plasma and a notional cylindrical core plasma of radius \( a \). This is depicted in figure 1, which shows how the slab model may be associated with the plasma (and possibly vacuum) between the inner core and the outer confining wall of a large-aspect-ratio toroidal plasma, approximated by a cylinder with periodic boundary conditions in the toroidal, \( z \)-direction. The slab model ‘straightens out’ the poloidal direction, but keeps a torus topology by applying periodic boundary conditions. This toroidal confinement analogy is not meant to imply that the slab model is adequate for experimental comparisons, but rather to allow us to use terminology and notation familiar within the toroidal confinement field, and to suggest orders of magnitude for parameters relevant to illustrative tokamak and reversed-field-pinch cases.

We further simplify by assuming the plasma slab comprises only two MRxMHD regions: a Taylor-relaxed inner plasma region \( \Omega^< \) between \( x = 0 \) and an interface at \( x = x_1 < x_w \), and an outer region \( \Omega^> \) between \( x = x_1 \) and \( x_w \). (We denote parameters belonging to the inner and outer regions using superscript \(<\) and \(>\), respectively.) The outer
region can either be a vacuum or another Taylor-relaxed region. The perturbed eigenfunctions (similar to the well-known ABC solutions [see, e.g. 8]) are represented in an elementary fashion as sums of complex plane waves obeying separate local dispersion relations in the plasma(s) or vacuum. The perturbations in the inner and outer regions are coupled via a surface wave on the interface between the regions.

In the following we give a brief summary of the new MRxMHD equations in general (section 2), and linearized (section 3) form. Then, in section 4, we develop the cylindrical core + slab model and present illustrative tokamak-like and pinch-like cases. Wave perturbations of such equilibria are developed in section 5, first through calculating the modulations of the entropy and magnetic-helicity Lagrange multipliers by modulating the interface position. We then treat plane waves (including evanescent waves) within the plasma and vacuum with wave vectors in general directions compatible with the periodic boundary conditions.

These waves are superposed to find acoustic and magnetic standing waves in section 6, which are combined into the general eigenvalue problem in section 7. The general spectrum in the case of a vacuum between \( x = x_i \) and \( x = x_w \) is examined graphically, computationally, and analytically in section 8. Tearing and kink-tearing modes are discussed in section 9 and conclusions and ideas for future work are given in section 10. The online supplementary version of this article, henceforth called the supplement, provides extra discussion and detailed working.

### 2. The dynamical MRxMHD model

In [3] the equations for MRxMHD were derived as Euler–Lagrange equations from the Lagrangian

\[
L = \sum L_i - \int_{\Omega_i} \frac{B \cdot B}{2\mu_0} \, dV,
\]

where the volume integration \( \int dV \) is over a vacuum region \( \Omega_i \), with \( B \) denoting magnetic field and \( \mu_0 \) denoting the permeability of free space. The sum \( \sum \) is over Lagrangians \( L_i \) given by

\[
L_i = \int_{\Omega_i} \mathcal{L}^{\text{MHD}} \, dV + \gamma (S_i - S_0) + \mu_i (K_i - K_0).
\]

Here, \( \Omega_i \) denotes a plasma relaxation region and \( \mathcal{L}^{\text{MHD}} \) is the standard MHD Lagrangian density [9, 10], \( \rho v^2/2 - p/\gamma - B^2/2\mu_0 \), with \( \rho \) denoting mass density, \( p \) the plasma pressure, and \( \gamma \) the ratio of specific heats. The departure from ideal MHD is in constraining total entropy \( S_i \) and magnetic helicity \( 2\mu_0 K_i \) in each macroscopic subregion \( \Omega_i \), rather than in each microscopic fluid element \( dV \) (though mass is still microscopically conserved in the current formulation); the nonholonomic conservation of \( S_i \) and \( K_i \) being enforced through the Lagrange multipliers \( \gamma_i \) and \( \mu_i \) (not to be confused with magnetic permeability \( \mu_0 \)), respectively.

The entropy and magnetic helicity invariants are given explicitly by

\[
S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln \left( \frac{\rho p}{\rho} \right) \, dV,
\]

and

\[
K_i \equiv \int_{\Omega_i} \frac{A \cdot B}{2\rho_0} \, dV,
\]

where \( A \) is a vector potential for \( B \) with gauge arbitrary save for the constraint that loop integrals \( \oint A \cdot dl \) on the boundary and interface be conserved. The constant \( \gamma \) in (3), required to make the argument of \( \ln \) dimensionless, is arbitrary for our purposes, but is identified physically in appendix A of [3].

The constant reference values \( S_0 \) and \( K_0 \) are the respective initial values at \( t = t_0 \) evaluated over \( \Omega_{0i} \), making \( L_i = L_i^{\text{MHD}} \) when the \( \tau_i(t) \) and \( \mu_i(t) \) are determined so as to satisfy the conservation of \( S_i \) and \( K_i \).

Microscopic mass conservation is ensured by the continuity equation

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \mathbf{v}),
\]

and the other equations of the model are found in [3] as Euler–Lagrange equations making the action \( \int L dt \) stationary under arbitrary Eulerian variations of the free fields \( p \) and \( A \), and variations arising from infinitesimal displacements of the Lagrangian positions of the fluid elements (including geometrical variation of the boundaries \( \partial \Omega_i \)).

Varying \( p \) we find that the pressure within \( \Omega_i \) obeys an isothermal equation of state

\[
p = \tau_i \rho,
\]

with the Lagrange multiplier \( \tau_i \) thus identified as the spatially constant specific temperature \( T_i/M_i \) in \( \Omega_i \), where \( M \) is the effective ion mass \( m_i/Z_{\text{eff}} \), with \( Z_{\text{eff}} \) being the mean ionization state. An alternative physical interpretation of \( \tau_i \) is as \( C_{\text{is}} \), where \( C_{\text{is}} \) is the isothermal sound speed in \( \Omega_i \).

Variation of \( A \) gives the Beltrami equation,

\[
\nabla \times \mathbf{B} = \mu_i \mathbf{B},
\]

Variation of fluid element positions within \( \Omega_i \) gives the equation of motion

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p.
\]

No contribution from the Lorentz force appears, which is consistent with the fact that (7) describes a ‘force-free’ \( (\mathbf{j} \times \mathbf{B} = 0) \) field in region \( \Omega_i \).

Variation of fluid positions at the interface \( \partial \Omega_{ij} \equiv \partial \Omega_i \cap \partial \Omega_j \) gives the force balance condition across the current sheet on this boundary:

\[
[p + \frac{B^2}{2\mu_0}]_{ij} = 0,
\]

with the brackets \([ \cdot ]_{ij}\) denoting the jump in a quantity as the
observation point crosses the interface from the \( \Omega_i \) side to the \( \Omega_j \) side.

To complete the specification of the MRxMHD equations we give the boundary conditions on \( B \), namely tangentiality on region boundaries
\[
n_i \cdot B = 0 \quad \text{on } \partial \Omega_i,
\]
and on \( v \), with normal continuity across moving (advected) interfaces
\[
n_i \cdot [v]_{ij} = 0 \quad \text{on } \partial \Omega_{ij},
\]
where \( n_i \) is the outward unit normal at each point on \( \partial \Omega_i \). Equation (11) includes the case of a perfectly conducting confining wall, \( j = w \), moving with velocity \( v_w \), but the case of \( \Omega_j \) being a vacuum region is obvious, as velocity is not defined in a vacuum. In this case there is no constraint on \( n_i \cdot v \).

Finally, we define the normal flow velocity \( v_n(x, t) = v_n(x, t) \) where \( v_n = n_i \cdot v \) is the normal component of the full fluid velocity \( v \). Unlike \( v \), \( v_n \) is (by (11)) the same on both sides of the interface, and thus provides a suitable generator for describing the geometric evolution of the interface \( \partial \Omega_{ij} \): we define the normal flow map \( r^m_{ij}(x(t_0)) : \partial \Omega_{ij} \to \partial \Omega_{ij} \) by following the loci of points \( r^m_{ij}(x(t_0)) \) obeying \( \frac{d x_n}{d t} = (\partial_t + v_n \cdot \nabla) r^m_{ij}(x(t_0)) \equiv x \) (cf the Lagrangian flow map defined through equation (3.1) of [3]). This map is well defined provided the interface is sufficiently smooth that \( n \) is uniquely defined at each point on the interface (i.e. it must have no cusp-like behaviour).

Note that, as in ideal MHD, there is no dissipation in MRxMHD. Unlike ideal MHD, the fluid and magnetic field are decoupled, except at the interfaces, but as the number of interfaces may be arbitrarily large an arbitrarily high degree of coupling transverse to the interfaces can (in principle) be obtained. Extra physics, such as dissipation and current drive (helicity injection) can be added after the equations have been derived, but are not inherent in the theory proper.

3. Linearized equations

We now suppose the solutions to the equations above can be written as a sum of a flowless, time-independent equilibrium part and arbitrarily small perturbations, \( B = B_0 + B_1 \equiv \nabla \times (A_0 + A_1), \quad v = v_0 \), \( \rho = \rho_0 + \rho_1 \), and \( p = p_0 + p_1 \) where
\[
p_1 = \gamma_0 \rho_1 + \gamma_1 \rho_0
\]
The linearized versions of (5) and (8) are
\[
\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot v,
\]
\[
\rho_0 \frac{\partial v}{\partial t} = -\gamma_0 \nabla \rho_1.
\]
Here, \( \nabla \rho_0 \) is zero in a flowless relaxed equilibrium.

The linearized version of (7) is
\[
\nabla \times B_1 - \mu_1 \rho_1 B_1 = \mu_1 B_0
\]
Unlike \( v \), which obeys the evolution equation (14), \( B_1 \) has no evolution equation. Instead it must be found at each instant in time by solving the inhomogeneous elliptic partial differential equation (15) in each unperturbed region \( \Omega_0 \) under appropriate boundary conditions. These are the linearizations of the tangentiality constraint (10) and the loop integral constraints \( \int A \cdot dl = \text{const} \) on the disjoint components of \( \partial \Omega_{ij} \), which are succinctly captured by the vector potential constraint (see, e.g. appendix B of [3])
\[
n_0 \times (A_1 - \xi_n n_0 \times B_0 - \nabla \chi_1) = 0 \quad \text{on } \partial \Omega_0.
\]
where \( n_0 \) is the unit normal on the unperturbed interface, \( \chi_1 \) is an arbitrary single-valued gauge potential, and we have defined the (linear) normal displacements \( \xi_n \) of points on \( \partial \Omega_{ij} \) away from their positions on the equilibrium interface \( \partial \Omega_{ij}^0 \) through the normal flow map defined at the end of the previous section,
\[
\xi_n(x, t) \equiv n_0 \cdot r^m_{ij}(x) - \infty.
\]
In this equation it is assumed that a stable perturbation was switched on adiabatically from \( t = -\infty \), or an unstable perturbation has grown from a negligible amplitude in the distant past, so \( r^m_{ij} \) is essentially independent of \( t_0 \).

To solve (15) (formally) in \( \Omega_0 \), we write \( B_1 = \nabla \times A_1 \) and split the perturbed vector potential into a component driven by the normal boundary displacement \( \xi_n \) and a component driven by the perturbation (if any) in the Lagrange multiplier \( \mu_1 \):
\[
A_1 = a_1 + \frac{\mu_1}{\mu_0} A_{G0} + \nabla \chi_1,
\]
where \( \nabla \chi_1 \) is inserted to allow gauge freedom of \( A_1 \) while allowing us to restrict the driven responses to Coulomb gauge, \( \nabla \cdot a_1 = \nabla \cdot A_{G0} = 0 \). This gives the decomposition \( B_1 = \nabla \times a_1 + \frac{\mu_1}{\mu_0} G_0 \), where \( G_0(x, t) \equiv \nabla \times A_{G0} \) is the solution in \( \Omega_0 \) of the inhomogeneous equation
\[
\nabla \times G_0 - \mu_0 G_0 = \mu_0 B_0 \quad \text{such that } G_0 = 0 \quad \text{as } \mu_0 \to 0.
\]
The corresponding vector potential response satisfies
\[
\nabla^2 A_{G0} + \mu_0 \nabla \times A_{G0} = -\mu_0 B_0,
\]
with homogeneous boundary condition \( A_{G0} = 0 \) on \( \partial \Omega_0 \).

The response to boundary perturbation, \( a_1 \), is the solution to the homogeneous equation
\[
\nabla^2 a_1 + \mu_0 \nabla \times a_1 = 0
\]
driven by the inhomogeneous boundary condition from (16), \( n_0 \times (a_1 - \xi_n n_0 \times B_0) = 0 \). Equations (19) and (20) may be solved using the Green’s function for the Helmholtz equation [11], but our slab test case is sufficiently simple that we will be able to construct its solution using elementary methods.
The Eulerian linearization of the force balance condition (9) is
\[
\left[ \rho_1 + \frac{B_0 \cdot B_1}{\mu_0} \right]^{(0)}_{ij} + \xi_0 \left[ n_0 \cdot \nabla \left( \rho_0 + \frac{B_0^2}{2\mu_0} \right) \right]^{(0)}_{ij} = 0, \tag{21}
\]
where \( \left[ \cdot \right]^{(0)} \) denotes a jump evaluated on \( \partial \Omega^{(0)}_j \) (the branches of functions defined in the two adjacent regions \( \Omega_{1} \) and \( \Omega_{2} \) being assumed to be differentiable, so they may be extended, at least to linear order, into a neighbourhood on either side of \( \partial \Omega^{(0)}_j \)).

The normal component of the magnetic field \( B \) is given, (see equation (107) of [12], and the online supplement), in terms of \( \xi_0 \) as
\[
\mathbf{n}_0 \cdot \mathbf{B}_1 = \mathbf{B}_0 \cdot \nabla \xi_0 + \xi_0 \mathbf{n}_0 \cdot \nabla \times (\mathbf{n}_0 \times \mathbf{B}_0). \tag{22}
\]

The linear perturbation to the entropy, (3), is (see the supplement)
\[
S_{i1} = \rho_0 \int_{\partial \Omega_{0_i}} \xi_0 \, dS \mp \frac{\tau_1}{\tau_0} \frac{\rho_0 V_0}{\gamma - 1}, \tag{23}
\]
where we have used the mass conservation identity (see the supplement)
\[
\int_{\partial \Omega_{a0}} \rho_0 \xi_0 \, dS + \int_{\Omega_{a0}} \rho_i \, dV = 0. \tag{24}
\]
Setting \( S_{i1} = 0 \) to satisfy the entropy constraint gives
\[
\frac{\tau_1}{\tau_0} = \frac{\gamma}{\gamma - 1} \frac{V_{i1}}{V_{01}}, \tag{25}
\]
where \( V_{01} = \int_{\partial \Omega_{a0}} \mathbf{n}_0 \, dS \) and \( V_{i1} = \int_{\partial \Omega_{a0}} \xi_0 \, dS \). In the case of spatially constant \( \rho_1 \) this is seen to be consistent with the adiabatic law \( pV^\gamma = \text{const} \), but is more general as (25) does not require \( \rho_1 \) to be constant.

Decomposing \( A_1 \) as in (18), and using integration by parts (for more detail see the supplement), the linear perturbation to the magnetic helicity functional, (4), is found to be
\[
K_{i1} = \frac{1}{\mu_0} \int_{\Omega_{a0}} \left( B_0 \cdot A_\xi + \frac{\mu_{1a}}{\mu_{01}} A_0 \cdot G_0 \right) \, dV. \tag{26}
\]
Setting \( K_{i1} = 0 \) to satisfy the helicity constraint gives
\[
\frac{\mu_{1a}}{\mu_{01}} = -\frac{\int_{\partial \Omega_{a0}} B_0 \cdot a_\xi \, dV}{\int_{\partial \Omega_{a0}} A_0 \cdot G_0 \, dV}. \tag{27}
\]

Except in the special case of purely radial waves discussed in section 7.1 we shall find that \( \gamma_1 \) and \( \mu_1 \) vanish.

4. Topologically toroidal slab

4.1. Core + slab model

As sketched in figure 1 we view the plasma slab as a model, albeit imperfect, of the outer region of a toroidally confined plasma in which the two slab regions \( \Omega^\xi \) are regarded as topologically toroidal volumes (specifically, annular toroids) surrounding a cylindrical core of relaxed plasma containing the ‘magnetic axis’ at \( r = 0 \): a closed field line on which the poloidal angle \( \theta \) is singular.

In the following we constrain the core–slab boundary \( r = a \) to be rigid, so that all dynamics occur within the slab regions. However, we shall take into account the core plasma in setting up the slab equilibrium by assuming continuity of the rotational transform \( \kappa_i(r) \), or equivalently its inverse \( q_i(r) \) (see (32)), across the core–slab interface. Assuming the plasma within the cylindrical core obeys the Beltrami equation (7), with constant \( \mu_c \), the solution of (7) is well known to be obtainable in Bessel functions (see, e.g. [13, equation (27)]). The corresponding \( q \)-factors at the magnetic axis \( r = 0 \) and core edge \( r = a \), denoted by \( q_c(0) \) and \( q_c(a) \), respectively, are then found to be
\[
q_c(0) = \frac{2}{R \mu_c}, \quad q_c(a) = \frac{a J_0(\alpha \mu_c)}{R J_1(\alpha \mu_c)}. \tag{28}
\]
where the major radius \( R \) of the torus, which is approximated using the cylinder, imparts the periodicity length \( 2\pi R \) in the \( z \)-direction.

The magnetic axis cannot be included in our slab model because a slab region has no coordinate singularity. Thus, we take the origin of the slab radial coordinate \( x \) to be the core–slab boundary, so \( r = a + x \).

4.2. Slab equilibrium

The slab coordinates \( y \) and \( z \) are analogues of poloidal and toroidal coordinates: the poloidal and toroidal angles are given as \( \theta \equiv y/a \) and \( \zeta \equiv z/R \), respectively, where \( R \) is the nominal major radius and \( a \) is the minor radius of the notional core. Thus, the periodicity lengths in the \( y \) and \( z \) directions are \( L_{pol} = 2\pi a \) and \( L_{tor} = 2\pi R \), respectively. For example, the unperturbed volumes of the plasma domains \( \Omega_{0}^{\xi} \) are
\[
V_{0}^{\xi} = \int_{x_{w}^{\xi}}^{x_{a}^{\xi}} dx \int_{y_{0}^{\xi}}^{y_{a}^{\xi}} dy \int_{z}^{z + 2\pi R} dz = (2\pi)^2 a a^\xi / R, \tag{29}
\]
where, within the domains \( \Omega_{0}^{\xi} \), we have used the subscripts \( + \) and \( - \) to denote their outer and inner boundaries, respectively, and the notation \( a^\xi \) for the widths of these regions:
\[
x_{-}^{\xi} = 0, \quad x_{+}^{\xi} = x_{a}^{\xi} = x_{a}^{\xi} = x_{w}, \quad x_{+}^{\xi} = x_{w}, \quad a_{w}^{\xi} = x_{w} - x_{-}^{\xi} = x_{+}^{\xi} - x_{w} = x_{a}^{\xi} - x_{w} = -x_{0}^{\xi}. \tag{30}
\]

We consider a two-layer slab equilibrium in which the inner region \( \Omega_{0}^{\xi} : x_{a}^{\xi} = 0 \leq x < x_{a}^{\xi} = x_{a}^{\xi} = x_{a}^{\xi} = x_{a}^{\xi} = x_{w} \), may either be a vacuum region in which the unperturbed magnetic field is a spatially constant vector \( B_{a} \), or a region of relaxed plasma with magnetic field \( B^{\text{pol}}(x) \). The outer region \( \Omega_{0}^{\xi} : x_{a}^{\xi} = x_{a}^{\xi} < x < x_{a}^{\xi} = x_{a}^{\xi} = x_{w} \) may be either a vacuum region in which the standard sign function \( \text{sgn}(\cdot) \) by defining \( \text{sgn}(>0) = +1 \), \( \text{sgn}(<0) = -1 \), \( \text{sgn}(\leq 0) = \pm 1 \) and \( \text{sgn}(\leq 0) = \mp 1 \).

In the following we use the notation \( >_{+} \) to denote \( >_{+} \) and \( <_{-} \) in a similar way to how \( \pm_{+} \) denotes \( >_{+} \) or \( <_{-} \) (similarly \( \leq_{+} \) is the analogue of \( \mp_{+} \)). When we wish to associate a sign with \( >_{+} \) and a sign with \( <_{-} \) we extend the domain of the standard sign function \( \text{sgn}(\cdot) \) by defining \( \text{sgn}(>0) = +1 \), \( \text{sgn}(<0) = -1 \), \( \text{sgn}(\leq 0) = \pm 1 \) and \( \text{sgn}(\leq 0) = \mp 1 \).
In relaxed plasma domains we take the unperturbed equilibrium fields to be, consistent with (7), force-free solutions of the form

$$B_0^\parallel(x) = B_0^\parallel \left[ e_x \sin(\Theta_0 + \mu_0^\parallel x) + e_z \cos(\Theta_0 + \mu_0^\parallel x) \right].$$  

(31)

The field magnitudes $B_0^\parallel$ are spatially constant, as are the angles $\Theta_0$. Projected onto the $z, y$ plane the field lines form two families of planar magnetic surfaces parametrized by $x$, the field lines subtending angles $\Theta_0 + \mu_0^\parallel x$ with the $z$-axis. Thus, $\Theta_0$ is the angle subtended by field lines on the inner boundary, $x = x_i^\parallel = 0$, of $\Omega^\parallel$, whereas $\Theta_0$ is the extrapolation to $x = 0$ of angles subtended by field lines in $\Omega^\parallel$. As $x$ increases, the field lines rotate counterclockwise in the $z, y$ plane for $\mu_0^\parallel > 0$, clockwise for $\mu_0^\parallel < 0$, at constant shear rates determined by $|\mu_0^\parallel|$.

Continuing the analogy with tokamaks and other toroidal plasma confinement devices we define, as a measure of the winding number or helical pitch of the magnetic field lines, the ‘safety factor’ $q(x)$ given by

$$q(x) \equiv \frac{a B_{0z}(x)}{RB_{0z}(x)} = \epsilon_x B_{0x}(x) / B_{0z}(x),$$  

(32)

where $\epsilon_x \equiv a/R$ is the inverse aspect ratio. In stellarators, its inverse, the rotational transform $i(x) = 1/q(x)$, is often used instead.

Using (31) in (32) we find

$$q^\parallel(x) = \epsilon_x \cot(\Theta_0 + \mu_0^\parallel x).$$  

(33)

In the case of a vacuum field in the outer slab region $\Omega^\parallel$, $\mu_0^\parallel = 0$. Considered in the light of (33), this implies $q^\parallel(x)$ is a constant, $q_0^\parallel \equiv \epsilon_x \cot(\Theta_0)$.

While the radial profiles of density, pressure, and current are important for understanding modes and instabilities in a toroidal plasma experiment, arguably the most important radial profile is $q(r)$. Adopting the categorization of toroidal equilibria used by [14] into tokamak-like ($q \gtrsim 1$ and increasing outward) and pinch-like ($q \ll 1$ and initially decreasing outward) we give an example of each type of equilibrium in figures 2 and 3. Details on the construction of these cases are given in appendix B of the supplement.

The magnitudes of the magnetic fields on either side of the equilibrium interface $x = x_i$ separating the inner plasma from the outer plasma, or a zero-pressure vacuum, are related by equilibrium force balance. That is, from (9),

$$B_0^\parallel (1 + \beta_0^\parallel x)^{1/2} = (1 + \beta_0^\parallel x)^{1/2} B_0^\parallel,$$  

(34)

where the constants $\beta_0^\parallel \equiv 2 \mu_0^\parallel \rho_0^\parallel / B_0^2$ are the ratios of the equilibrium plasma and magnetic pressures in the two regions. Later we shall also find it useful to write $\beta$ in the form

$$\beta = 2(C_s / v_\parallel)^2,$$  

(35)

where $C_s \equiv p/\rho$ is the sound speed (see (6) ff) and $v_\parallel \equiv B/\sqrt{\mu_0 \rho}$ is the Alfvén speed.

Equation (34) implies that, in the case of a finite pressure differential across the interface, there must necessarily be a jump in $|B|$, and hence a current sheet on the interface. Even if the pressure is continuous, there may still be a current sheet if there is a tangential discontinuity in $B$, i.e. if $q_0^\parallel \neq q_0^\parallel$.

5. Plane-wave perturbations

5.1. Equilibrium variations

To model purely radial modes we can use mass, entropy, magnetic flux, and magnetic helicity conservation to calculate...
perturbations in equilibrium plasma and magnetic-field parameters under variations $\delta x_i$. Mass conservation implies $\delta p_0^z + \delta p_0^z = -\delta V_0^z$, from (29). From (30) we have

$$\delta a_z^z = -\text{sgn}(\bar{z})\delta x_i,$$

(36)

where $\text{sgn}(\bar{z}) \equiv \pm 1$ was defined in section 4.2.

Substituting (29) into (25) we find the equilibrium temperature fluctuation from entropy conservation to be $\delta T_0^z = t_0^z = (\gamma - 1)\delta a_z^z/a_z^z$, and combining these with the results above we have the equilibrium pressure fluctuation, as expected from the ideal gas law,

$$\frac{\delta p_0^z}{p_0^z} = -\bar{\gamma}\frac{\delta a_z^z}{a_z^z}. $$

(37)

To calculate magnetic helicity we need suitable vector potentials corresponding to the magnetic fields in (31),

$$A_0^z(x) = B_0^{z}(x) - B_0^{z}(x_0), \quad x < x_\parallel \leq x_\parallel^z,$$

(38)

where we have assumed gauges such that $A_0^z(x_0) = 0$, which, as $x_\parallel^z = x_\parallel$, ensures that the line integrals of $A_0$ on each side of the interface are equal. (Because $B_0$ is finite, an interface of zero width can carry no magnetic flux.)

The special case where $B_0^{z}$ is the shearless equilibrium vacuum field $B_{0\parallel} = B_0^{z}(x_0 \sin \Theta_0 + e_\parallel \cos \Theta_0)$ can be obtained by taking the limit $\mu_0^z \to 0$ in (31) in such a way that $\Theta^z \to \Theta_0$. Taking this limit in (38) gives (see the supplement) the vacuum vector potential $A_{0\parallel} = \lim_{\mu_0^z \to 0} A_0^z$ as

$$A_0(x) = B_0^{z}[e_\parallel(x - x_\parallel)\cos \Theta_0 - e_\parallel(x - x_\parallel)\sin \Theta_0],$$

(39)

The poloidal and toroidal fluxes trapped between the perfectly conducting boundaries of the two annular toroids $\Omega^z$, and thus conserved, are the differences between inner and outer line integrals $\oint_{x_\parallel^z} A_{0\parallel} \cdot dl$, specifically $\int_{x_\parallel^z_0}^{x_\parallel} [A_{0\parallel}(x_\parallel^z) - A_{0\parallel}(x_\parallel^z)]dy$ and $\int_{x_\parallel^z_0}^{x_\parallel} [A_{0\parallel}(x_\parallel^z) - A_{0\parallel}(x_\parallel^z)]dz$. As $A_0(x_0) \equiv 0$, the conservation of poloidal and toroidal fluxes thus require the boundary condition that $A_0(x_0^z)$ be a constant vector under a variation of $x_\parallel$, where we have denoted the fixed boundary in each region as $x_0^z$, defined by

$$x_0^z = x_\parallel^z = 0, \quad x_0^z = x_\parallel = x_w.$$

(40)

From (31) and (38) the toroidal and poloidal flux constraints $\delta A_0(x_0^z) = 0$ under a variation of $x_\parallel$ (and consequent variations in $B_0^{z}, \mu_0^z$ and $\Theta_0^z$) can be combined using complex exponential notation (see the supplement):

$$\delta \left\{ \frac{B_0^{z}}{\mu_0^z} \exp \left( \Theta_0^z + \frac{\mu_0^z(x_0^z + x)}{2} \right) \sin \frac{\mu_0^z a_z^z}{2} \right\} = 0,$$

(41a)

which can be broken into three independent variational constraints:

$$\frac{B_0^{z}}{\mu_0^z} \cos \left( \frac{\mu_0^z a_z^z}{2} \right) \delta \left( \frac{B_0^{z}}{\mu_0^z} a_z^z \right) = 0,$$

(41b)

$$\sin \left( \frac{\mu_0^z a_z^z}{2} \right) \delta \left( \frac{B_0^{z}}{\mu_0^z} \right) = 0,$$

(41c)

$$\left\{ \frac{B_0^{z}}{\mu_0^z} \sin \left( \frac{\mu_0^z a_z^z}{2} \right) \delta \left[ \Theta_0^z + \frac{\mu_0^z(x_0^z + x)}{2} \right] \right\} = 0.$$  

(41d)

Assuming $\mu_0^z = 0$, (41b) implies $\delta (\mu_0^z a_z^z) = 0$ (except possibly at the zeros of $\cos(\mu_0^z a_z^z)$), and (41c) and (41d) imply $\delta (B_0^{z}/\mu_0^z) = 0$ and $\delta (\Theta_0^z + \mu_0^z(x_0^z + x)/2) = 0$, respectively (except possibly at the zeros of $\sin(\mu_0^z a_z^z)$). These constraints are sufficient to completely determine how $\mu_0^z$, $B_0^{z}$ and $\Theta_0^z$ vary with $x_\parallel$.

In the vacuum-field case $\mu_0^z = 0$, we see from (39), or taking the $\mu_0^z \to 0$ limit of (41c) and (41d), that there are now only two independent variational constraints in $\Omega^z$ (see the supplement),

$$\delta (B_0^{z}(x_\parallel - x_\parallel)) = 0,$$

(42)

$$\delta \Theta_0 = 0,$$

so $\Theta_0$ is constant and $B_0^{z}$ varies as expected from elementary-flux conservation considerations.

We must now consider the constraints provided by the helicity integrals (4). Equation (38) gives (for $\mu_0^z = 0$, which is all that is required as helicity conservation does not apply for a vacuum field),

$$A_0^{z} \cdot B_0^{z}(x) = \frac{B_0^{z} - B_0^{z}(x_\parallel) \cdot B_0^{z}(x_\parallel)}{\mu_0^z}, \quad x < x_\parallel^z.$$  

(43)

Using (31) in (43), the helicity constraints become (see the supplement)

$$\int_{x_\parallel^z_0}^{x_\parallel} A_0^{z} \cdot B_0^{z} dV = \frac{(2\pi)^2 aRB_0^{z2}}{\mu_0^z} (\mu_0^z a_z^z - \sin \mu_0^z a_z^z),$$

(44)

except in the case of a vacuum field in the outer region, when only the result in $\Omega^z$ is relevant. Conservation of magnetic helicity thus implies

$$\delta \left[ \frac{B_0^{z2}}{\mu_0^z} (\mu_0^z a_z^z - \sin \mu_0^z a_z^z) \right] = 0,$$

(45)

consistent with (41b) and (41c) and showing that the zeros of $\cos(\mu_0^z a_z^z)$ and $\sin(\mu_0^z a_z^z)$ are not exceptional points.

Thus, in summary, we have shown (see the supplement)

$$\frac{\delta \mu_0^z}{\mu_0^z} = -\frac{\delta a_z^z}{a_z^z},$$

(46a)
\[
\frac{\delta O_0^\pm}{\rho_0^b \delta \theta_0} = \frac{\delta a_0^\pm}{a_0^\pm}, \quad (46b)
\]
\[
\frac{\delta B_0^\pm}{B_0^b} = -\frac{\delta a_0^\pm}{a_0^\pm}. \quad (46c)
\]

In the case of a vacuum field in the outer region, from (42) we see that (46c) remains correct in both inner and outer regions, (46a) is inapplicable and (46b) is replaced by \(\delta \theta_e = 0\).

For linearized radial perturbations, even if time-dependent (waves), (46a) allows us to bypass the complicated construction of \(\mu_1\) in (27) as relaxation is assumed to be effectively instantaneous in dynamical MRxMHD. Similarly, the expression for \(\delta \theta_0\) leading to (37) may be used to get \(\tau\) for radially propagating waves as temperature equilibration is assumed to be effectively instantaneous also. These results are used in section 7.1.

5.2. Surface, plane and standing waves

Although the lower boundary \(x = 0\) is notionally the inner face of an annular toroidal region within the plasma, we assume it is not affected by wave perturbations and may be treated as a rigid boundary. Eigenmodes in this geometry are thus somewhat analogous to water waves [15, section 4.5] in that the basic wave is a transverse wave perturbation on the two-dimensional (2-D) interface \(x = x_1\):

\[
\xi_0(y, z, t) = \tilde{\xi} \exp(i k_{xz} \cdot x - i \omega t),
\]

where the 2-D wave vector \(k_{xz} = k_x e_x + k_z e_z\) and a tilde, such as in \(\tilde{\xi}\), denotes a complex amplitude.

To ensure a correspondence with standard notations for waves in a toroidally confined plasma we write

\[
k_{xz}^{\pm \pm} = \frac{m e_y}{a} - \frac{n e_z}{R} = \frac{m e_y - n e_z e_z}{a},
\]

where \(m\) is the poloidal and \(n\) the toroidal mode number, and \(e_z\) is the inverse aspect ratio (see (32)).

Associated with this surface wave are 3-D plane waves in \(\Omega^\pm\) of the generic form

\[
w^{\alpha \pm}(x, y, z, t) = \text{Re} \; \hat{w}^{\alpha \pm} \exp(i k_{xz} \cdot x - i \omega t), \quad (49)
\]

where the scalar or vector \(w\) is a member of the set of fields \(\{\rho_0^b, p_0^b A_0^b, B_0^b, v_0^b, v_0^x\}\).

The superscripts \(\alpha \pm\) are branch labels for the wave components contributing to the total 3-D response to the 2-D surface wave. The label \(\alpha\) distinguishes two types of wave, sonic and magnetic (see sections 5.3 and 5.4 respectively), whose dispersion relations are both of the general form \(D_\alpha(\omega^2, k_z^2) = 0\). By definition, \(\omega\) is common to all components of an eigenmode, as is the 2-D wave vector \(k_{xz}\). However, the 3-D wave vector:

\[
k^{\alpha \pm} = k_{xz}^{\alpha \pm} e_x + k_{xz}^{\alpha}, \quad (50)
\]

is branch-dependent through \(k_{xz}^{\alpha \pm} \equiv \pm k_{xz}^{\alpha}\), where \(k_{xz}^{\alpha}\) is one of the two solutions of the local dispersion relation \(D_\alpha(\omega^2, k_x^{\alpha \pm} + |k_z|) = 0\). While \(k_{xz}\) is always real, \(k_{xz}^{\alpha}\) may be either real, corresponding to radial propagation, or imaginary, corresponding to radial evanescence. The details of this will be developed in section 7.

Whether propagating or evanescent, the plane waves are reflected from the inner and outer boundaries, \(x = 0\) and \(x_w\), coupling the \(\pm\) branches to form radially standing waves. At the interface, the total perturbation is thus generically of the form

\[
w^{\alpha} = \text{Re} \; \hat{w}^{\alpha}(x) \exp(i k_{xz} \cdot x - i \omega t),
\]

where

\[
\hat{w}^{\alpha}(x) = \sum_{\pm} \hat{w}^{\alpha \pm}(x).
\]

The coefficients \(\hat{w}^{\alpha \pm}\) are to be determined from the boundary conditions at the core–slab interface and the wall:

\[
\hat{w}_i^{\alpha} = 0 \quad \text{and} \quad \hat{B}_i^{\alpha} = 0 \quad \text{on} \quad x = 0,
\]

\[
\hat{w}_w^{\alpha} = 0 \quad \text{and} \quad \hat{B}_w^{\alpha} = 0 \quad \text{on} \quad x = x_w,
\]

by (10) and (11); also those at the internal interface position,

\[
v_i(x_i, y, z, t) = \delta_i \xi_n \quad \text{and} \quad B_i^{\alpha}(x_i, y, z, t) = B_0^\alpha \cdot \nabla \xi_n.
\]

The latter boundary condition follows from (22), noting that \(\mathbf{n_0} \cdot \nabla \times (\mathbf{n_0} \times \mathbf{B}_0) = 0\) in slab geometry (see the supplement). In wave representation, (54) becomes

\[
v_i(x_i, y, z, t) = \text{Re} \; -i \omega \xi \exp(i k_{xz} \cdot x - i \omega t),
\]

\[
B_i^{\alpha}(x_i, y, z, t) = \text{Re} \; i k_{xz} \cdot B_0^\alpha \xi \exp(i k_{xz} \cdot x - i \omega t). \quad (55b)
\]

In the general case \(k_{xz} = 0\) (i.e. when at least one of \(m\) and \(n\) is nonzero), the average of \(\exp(i k_{xz} \cdot x)\) over \(z\) vanishes, in which case we see from (25) that \(\eta = 0\). Similarly the volume integral of \(B_0^\alpha \cdot \mathbf{a}\) is zero; therefore, from (27), \(\mu_1\) is thus. We can take \(\tau\) and \(\mu\) to be constants when \(k_{xz} = 0\) and suppress the subscript 0. (Even for \(k_{xz} = 0\) we can easily calculate the effects associated with vanishing \(\eta\) and \(\mu_1\) separately from finite-\(\eta\) and \(\mu_1\) effects and then linearly superpose them, as will be done in section 7.1.)

5.3. Plane waves: acoustic-wave local dispersion relation

We now examine density and velocity waves propagating in the plasma, again suppressing the superscripts \(\epsilon\) in this subsection as the results apply for both inner and outer plasmas (though not a vacuum). Using (13) we find

\[
\hat{\rho}^{\pm} = \frac{\rho_0 \hat{k}_{xz} \cdot \hat{\nu}^{\pm}}{\omega}, \quad (56)
\]

From (14) we have

\[
\hat{\nu}^{\pm} = \frac{\tau_0 \hat{k}_{xz} \cdot \hat{\nu}^{\pm}}{\rho_0 \omega}, \quad (57)
\]

showing these waves are longitudinal.
Eliminating $\tilde{v}^\pm$ by substituting (57) into (56) we find the local dispersion relation for sound waves,
\[ \omega^2 = \gamma_0 k^2 \mp \frac{C_s^2}{C_i^2} \left[ |k_x|^2 + \left( k_x \frac{k_y^2}{|k_x|^2} \right)^2 \right], \]
where $C_s = \sqrt{\gamma_0}$ is the ion sound speed. We thus define
\[ k_x^\pm = \mp k_x, \]
where
\[ \frac{\omega^2}{C_i^2} = \frac{|k_x|^2}{C_s^2} - \frac{|k_x|^2}{|k_x|^2}. \]

The following expression for $\tilde{\rho}^\pm$ in terms of $\tilde{v}^\pm$, obtained from the $x$-component of (57), will be useful when deriving eigenvalue equations,
\[ \tilde{\rho}^\pm = \pm \frac{\rho_0 \omega}{\gamma_0 k_x^2} \tilde{v}_x^\pm. \]

As in ideal MHD, $\omega^2$ is real, so $\omega^2 < 0$ for all unstable modes (growth rate $\Im \omega > 0$). From (58) this implies the instability criterion that $\frac{|k_x|^2}{C_s^2} < -\frac{|k_x|^2}{|k_x|^2}$.

That is, $k_x^\pm$ must be located higher up the imaginary axis than $i|k_x|^2$, $|k_x^\pm| > |k_x|$. \hspace{1cm} (60)

5.4. Plane waves: magnetic

As well as suppressing the superscripts $\tilde{\cdot}$, in this subsection we reduce the proliferation of subscripts by suppressing the subscript 0 on equilibrium quantities and expressing the perturbed magnetic field, $B_i$, and vector potential $A_i$ as $b$ and $a$, respectively. As the case $k_x = 0$ has already been treated in section 5.1 we assume $k_{yz} = 0$, so $\mu_i = 0$ and $\mu$ is the same constant value $\mu_0$ as used to construct the equilibrium in section 4.2.

Following the general plan of section 5.2 we introduce the plane-wave ansatz $b^\pm = \tilde{b}^\pm \exp(i k^{\mu\pm} \cdot x - i \omega t)$, where (cf (50))
\[ k^{\mu\pm} \equiv k_x^{\mu\pm} e_x + i k_y^\pm. \]

Substituting this ansatz into the linearized Beltrami equation (15), $\nabla \times b = \mu b$, gives the plane-wave Beltrami equation, $i k^{\mu\pm} \times \tilde{b}^\pm = \mu \tilde{b}^\pm$. Dotting both sides of this equation with $k^{\mu\pm}$ gives $k^{\mu\pm} \cdot \tilde{b}^\pm = 0$, the plane-wave version of $\nabla \cdot b = 0$, verifying that the polarization of these magnetic perturbations is transverse. Dotting both sides with $\tilde{b}^\pm$ gives $k^{\mu\pm} \cdot \tilde{k}^{\mu\pm} = 0$, showing that these transverse perturbations are circularly polarized (see the supplement). This is in similar way, e.g. to the complex unit vector $\hat{e} \equiv \frac{(e_x + ie_y)}{\sqrt{2}}$, which has the properties $\hat{e} \cdot \hat{e} = 0, \hat{e} \times \hat{e} = 1$.

Crossing each side of the plane-wave Beltrami equation with $k^{\mu\pm}$ gives (see appendix C of the online supplement) $k^{\mu\pm \cdot \mu} = k^{\mu \pm \mu}$. Unlike more usual dispersion relations, the $\omega$-dependence of this ‘local’ dispersion relation is trivial (i.e. that of a constant) because the Beltrami equation has no time derivatives—the relaxed magnetic field adjusts instantaneously to the boundary conditions.

We now show the plane-wave Beltrami equation is solved by the ansatz (see appendix C of online supplement):
\[ \tilde{b}^\pm = \tilde{b}_x^\pm \left( e_x - i \frac{k_x^{\mu\pm} k_y^\pm}{|k_x|^2} + i \frac{k_x^{\mu\pm} \times e_x}{|k_x|^2} \right). \]

where $k_x^{\mu\pm} \equiv \pm k_x^\mu$, with $k_x^\mu$ defined by
\[ k_x^\mu = (\mu^2 - |k_x|^2)^{1/2} \quad \text{for } |\mu| > |k_x|, \]
\[ k_x^\mu = i(|k_x|^2 - \mu^2)^{1/2} \quad \text{for } |\mu| < |k_x|, \]

the latter case corresponding to radially evanescent or growing waves. It is readily verified that both forms of $k_x^{\mu\pm}$ above satisfy the local dispersion relation
\[ (k^{\mu\pm})^2 \equiv (k_x^{\mu\pm})^2 + |k_y^\pm|^2 = \mu^2. \]

Additionally, multiplying both sides of (63) by $i$ and crossing with $k^{\mu\pm}$ gives (see the supplement)
\[ i k^{\mu\pm} \times \tilde{b}^\pm = \mu \tilde{b}^\pm, \]

which, using (63) and (65), reduces to $i k^{\mu\pm} \cdot \tilde{b}^\pm = \mu \tilde{b}^\pm$, as required.

6. Standing waves

6.1. Standing sound waves

We now superepose the two ($\pm$) plane waves to give a radial standing wave (cf (51)), giving $v^\pm = \Re \{ \tilde{v}_x^\pm(x) \exp[i k_{yz} \cdot x - i \omega t] \}$ and $\rho_1^\pm = \Re \{ \tilde{\rho}_x^\pm(x) \exp[i k_{yz} \cdot x - i \omega t] \}$, where
\[ \tilde{v}_x^\pm(x) \equiv \sum \tilde{v}_x^\pm \exp(i k_{yz}^\pm x), \]
\[ \tilde{\rho}_x^\pm(x) \equiv \sum \tilde{\rho}_x^\pm \exp(i k_{yz}^\pm x). \]

To satisfy the velocity boundary conditions in (53) we thus require $e_x \cdot \tilde{v}_x^\pm(x_{w\pm}) = 0$, $\tilde{\rho}_x^\pm(x_{w\pm}) = 0$ on the innermost boundary and the wall (using the notation of (30)), which we now show are satisfied by choosing amplitudes $\tilde{v}_x^\pm = \pm \tilde{v}_{<\pm} / 2$ and $\tilde{\rho}_x^\pm = \pm \tilde{\rho}_{<\pm} \exp(\mp ik_{yz}^\pm x_{w\pm})$. From (50) and (57) the $\tilde{v}_x^\pm$ are vectors in the $k_{yz}^\pm = \pm k_{yz}^\pm e_x + k_{yz}^\pm$ direction. Normalizing to give the stated $x$-components and inserting in (67) we find
\[ \tilde{v}_x^\pm = i \tilde{v}_{<\pm} e_x \sin(k_{yz}^\pm x) + \tilde{v}_{<\pm} \frac{k_{yz}^\pm}{k_{yz}^\pm} \cos(k_{yz}^\pm x), \]
\[ \tilde{\rho}_x^\pm = -i \tilde{\rho}_{<\pm} e_x \sin(k_{yz}^\pm x) + \tilde{\rho}_{<\pm} \frac{k_{yz}^\pm}{k_{yz}^\pm} \cos(k_{yz}^\pm x), \]

which indeed satisfies $e_x \cdot \tilde{v}_x^\pm(0) = e_x \cdot \tilde{\rho}_x^\pm(x_{w\pm}) = 0$. 


Setting $x = x_i$ in (68) and comparing with the boundary condition (55a) allows us to relate $\bar{\nu}$ and $\xi$:  
\[
\bar{\nu}^< = -\frac{\omega}{\sin(k_i x \omega)} \xi, \\
\bar{\nu}^> = +\frac{\omega}{\sin k_i(x_w - x_i)} \xi,
\]
which can be summarized as $\bar{\nu}^\pm = \pm \nu^\pm/k_i x_{w0}$. Using $\bar{\nu}^<_x = \pm \nu^</2$ and $\bar{\nu}^>_x = \pm \nu^>/2$ in (60) gives, after using (69),  
\[
\bar{\nu}^\pm_x = \operatorname{sgn}(\bar{z}) \left[ \frac{\rho^\pm_{0x} a^2 \cos(k_i x \omega x - x_{\text{sgn}0} \omega)}{\tau^\pm_{0x} a^2 \sin(k_i a^2)} \right] \xi,
\]
where $\operatorname{sgn}(\bar{z}) \equiv \pm 1$ was defined in section 4.2 and the subscript $\pm$ and $a^\pm$ notations were defined in (30).

### 6.2. Standing magnetic fluctuations

Analogously to section 6.1 we form (driven) standing magnetic 'Beltrami waves' $b^\pm = \Re \{ \hat{b}^\pm(x) \exp(ik_i x - \lambda x) \}$, where  
\[
\hat{b}^\pm(x) = \sum_{\pm} \hat{b}^\pm \exp(\pm ik_i^\pm \mu x).
\]

The boundary conditions $e_x \cdot \hat{b}^\pm_x(x^\pm) = e_x \cdot \hat{b}^\pm_0(x^\pm) = 0$ are satisfied by choosing amplitudes $\hat{b}^\pm_x = \pm \hat{b}^\pm/2$ and $\hat{b}^\pm_x = \pm \hat{b}^\pm \exp(\mp i k_i^\pm \mu x_0)/2$. Using (63) we find  
\[
\hat{b}^\pm_x = \pm \hat{b}^\pm e_x \sin(\hat{k}_i^\pm \mu x - x_{\text{sgn}0} \mu), \\
\hat{b}^\pm_x = \pm \hat{b}^\pm i e_x \sin(\hat{k}_i^\pm \mu (x - x_{\text{sgn}0} \mu)),
\]
which indeed satisfies the required boundary conditions.

The remaining unknown, $\hat{b}$, is determined from the boundary condition (55b), which in the notation of this subsection is  
\[
e_x \cdot \hat{b}^\pm(x_0) = i k_i^\pm \cdot B^\pm \xi.
\]

Using (72) this gives  
\[
\hat{b}^\pm = -\operatorname{sgn}(\bar{z}) \left[ \frac{k_i^\pm \cdot B^\pm}{\sin(k_i a^2)} \right] \xi.
\]

The vacuum magnetic perturbation $b^\nu$ may be found by setting $\mu^\nu = 0$, in which case (64) gives $k_i^\nu = i k_i^\nu$ and (72) becomes (see the supplement)  
\[
\hat{b}^\nu_0(x) = \hat{b}^\nu \sinh(|k_i^\nu|(x_w - x)) e_x \\
- i \hat{b}^\nu \cosh(|k_i^\nu|(x_w - x)) k_i^\nu / |k_i^\nu|,
\]
where, from (74),  
\[
\hat{b}^\nu = \frac{i k_i^\nu \cdot B^\nu}{\sinh(|k_i^\nu|(x_w - x))} \xi.
\]

### 7. Eigenmodes

Global eigenvalue relations arise as consistency conditions for perturbed force balance to apply across the interface. Taking into account the vanishing of $\nabla p^\pm$ and $\nabla |B^\pm|^2$, (21) can be written as  
\[
(\tau_0 \rho_1^\nu - \tau^\nu_0 \rho_1^\nu) + \delta (p_0^\nu - p_0^\nu) = \frac{B_0^\nu \cdot B^\nu - B^\nu_{0} \cdot B^\nu}{\mu_0} \\
+ \frac{\delta B^\nu_{0} - B^\nu_{0}^2 - 2 \mu_0^2}{2 \mu_0^2} \bar{z},
\]
where  
\[
d \{ \cdot \} \text{ represents linear fluctuations with the same symmetry as the equilibrium, which are driven adiabatically (due to effectively instantaneous relaxation) by interface fluctuations and are thus as calculated in section 5.1. These are associated with oscillations in $\gamma$ and $\mu_i$, unlike the fluctuations denoted by $\{ \cdot \}$, which represent linear fluctuations calculated with $\tau$ and $\mu$ held fixed; defined in section 5.2. The $d \{ \cdot \}$ terms pertain only to the $m = n = 0$ modes discussed below, where they can be superposed with the $\{ \cdot \}$ terms at linear order.

In the following we use the notational simplification used in section 5.4 of supressing the subscript 0 on all equilibrium quantities.

#### 7.1. Purely radial eigenfunctions

First, consider the simple but exceptional case $k_i = 0$ (i.e., $m = n = 0$). As shown in section 5.1, in this case $\gamma(t)$ and $\mu_i(t)$ do not vanish, but are easily calculated as equilibrium variations. Additionally, $k$ is purely radial so the finite-wavelength sound-wave perturbations of section 6.1 are functions only of $x$ and $t$ through the factor $\exp(\pm ik_i^\nu \omega - \omega t)$, where, by (59) with $|k_i|^2$ set to zero,  
\[
k_i^\nu = \frac{\omega}{C_i^\nu},
\]

As in section 5.4 the subscript 0 on equilibrium quantities is implicit throughout this section. Using (37) and (70) we find (see the supplement) the LHS of (77)  
\[
\tau^\nu \rho_1^\nu - \tau^\nu_0 \rho_1^\nu + \delta (p_0^\nu - p_0^\nu) = - \Re \sum r^\nu \left[ \frac{\omega^2 \cot(k_i^\nu a_i^\nu)}{k_i^\nu} + \frac{\gamma T_i^\nu}{a_i^\nu} \right] \xi e^{-i\omega t}.
\]

There is no contribution to force balance from a vacuum or pressureless plasma in the outer region, $\Omega^\nu$, so in this case contributions from $\Omega_i^\nu$ must be deleted from (79). The issue of interpreting $k_i^\nu$ in the limit $C_i^\nu \to 0$ thus does not arise.

In MRX-MHD both relaxed and vacuum magnetic fields instantaneously adjust to the radial oscillations which means, as far as the magnetic field is concerned, the perturbed states are equivalent to the varied equilibrium states of section 4.2. Setting $B^\nu_0 = 0$ (as the magnetic response is fully accounted for by $\delta B_0$ when $k_i = 0$) and using (46c) we find the RHS of
of the Alfvén waves in favour of the Alfvén speeds in the two regions (see (35) ff).

For the two sides of (77) we use (79) (with \( \omega \) eliminated in favour of \( k_s^2 \)) (see the supplement) and \( (80) \) to find the eigenvalue condition for radial oscillations

\[
-\sum_{\xi} \left( \frac{\rho_s C_s^2}{a_i^2} \right) k_s^2 a_i^2 \cot(k_s^2 a_i^2) = \sum_{\xi} \left( \frac{\rho_s (v_s^2 + \gamma C_s^2)}{a_i^2} \right) k_s^2 a_i^2 \cot(k_s^2 a_i^2) \]  

(81)

As mentioned above, in the case of a vacuum or pressureless plasma in the outer region, sonic contributions from \( \Omega^2 \) must be deleted from (81), which can be done formally by setting \( C_s^2 \) to zero. However, as \( \rho^0 \gamma_0 k_s^2 = B_0^2/\mu_0 \) is finite as \( \rho^0 \to 0 \), the vacuum magnetic-field contribution remains well defined. In this case the eigenvalue equation is particularly simple and can be written in the form

\[
k_s^2 x_i \cot(k_s^2 x_i) = -\mathcal{V}, \]  

(82)

where the positive dimensionless parameter \( \mathcal{V} \) is defined by (see the supplement)

\[
\mathcal{V} \equiv \gamma + \left( \frac{v_s^2}{C_s^2} \right)^2 \left[ 1 + \frac{x_i}{(x_w - x_i) B_0^{-2}} \right]. \]  

(83)

Figure 4 illustrates a graphical method for solving (82), by finding the intersections of the graphs of \( \cot(k_s^2 x_i) \) and \( -\mathcal{V}/(k_s^2 x_i) \). Once solutions \( k_s^2 \) are known the spectrum of eigenvalues \( \omega \) is obtained immediately from the dispersion relation (58).

By (35) \( (v_s^2/C_s^2)^2 = 2/\beta^2 \). Thus, in a low-\( \beta \) plasma (and/or if \( x_w = x_i \to 0 \)), \( \mathcal{V} \) is a large number. In this case, the eigenvalues are given approximately by the ‘waves on a string’ spectrum (figure 4),

\[
k_s^2 x_i \approx \pi l, \ l = 1, 2, \ldots, \]  

(84)

where \( l \) is the radial-mode number. In terms of frequency (see (58)) \( \omega = \omega_l \approx \pi l C_i/x_i \), a sequence consisting of the fundamental, \( \omega_1 \), and its harmonics. Like an organ pipe open at one end, the higher-order modes are not exact harmonics—as can be seen from figure 4, \( \omega_l \approx \pi (l - 1/2) C_i/x_i \) as \( l \to \infty \).

Note that in MRxMHD the high-frequency fast magnetosonic mode within relaxation regions is eliminated because of the decoupling of velocity and magnetic-field perturbations. This should allow time steps longer than those that can be used in ideal-MHD numerical simulations using explicit methods [16, e.g. section 2.2].

To determine stability, we see from (61) that we need to find roots of (81) on the imaginary axis in the complex \( k_s^2 \) plane, where the RHS of (81) is \(-\sum_{\xi} (\rho_s C_s^2 a_i^2) k_s^2 a_i^2 \cot(k_s^2 a_i^2) \). This is negative for all \( |k_s^2 x_i| \), while the RHS is positive. Thus, there are no unstable purely radial \( (k_{\perp} = 0) \) modes.

### 7.2. Surface-wave eigenvalue problem

When \( k_{\perp} = 0 \) the only change required to the expression in (77) for the LHS of (81) is to delete the term in \( \gamma \) and to use the full expression (59) for \( k_s^2 \) rather than the simplified expression in (78).

However, the calculation of the RHS side of (77) is quite different—the \( \delta \) term vanishes whereas \( B_0 \equiv b \) does not, being given by (72). To satisfy linearized force balance we equate the RHS of (79) to \((B^0 \cdot b^0 - B^- \cdot b^-)/\mu_0 \) and multiply both sides by the non-dimensionalizing factor \( a^0 \mu_0 /B_{\perp}^0 \equiv a^0/\mu_0 C_i^2 \), where \( B^0 \) is any convenient reference magnetic-field strength, with corresponding Alfvén speed \( v_A^0 \equiv B_0^0/\sqrt{\mu_0 \rho_0} \) (see (35) ff), and the constants \( a^0 \) and \( \rho_0 \) are any convenient reference length and mass density, respectively.

This gives the eigenvalue-like equation

\[
\mathcal{K}(\lambda) \lambda = \mathcal{V}, \]  

(85)

where the factors on the LHS of (85) are the dimensionless eigenvalue

\[
\lambda \equiv \left( \frac{a^0 \omega}{v_A^0} \right)^2, \]  

(86a)

\[
= \left( \frac{C_s^2}{v_A^0} \right)^2 \left[ (k_{\perp} a^0)^2 + (k_s^2 a_i^2)^2 \right], \]  

(86b)

and the dimensionless normalization factor

\[
\mathcal{K}(\lambda) = \mathcal{K}(\lambda), \]  

(87)

and the dimensionless normalization factor

\[
\mathcal{K}(\lambda) = \mathcal{K}(\lambda), \]  

(87)
Equation (86b) makes explicit the relation between the radial sound wavenumbers \( k^2 \omega \) and \( \lambda \) using (58). By (35) the factors \( C^2 / \lambda \) can also be written as \( \beta^2 / 2 \), where \( \beta^2 \equiv 2 \mu_0 B_0^2 / B_0^2 \).

Note that we derived the form in (87b) using the identity \( \nu \cot \nu = \nu \cot \nu \), which will also be useful in the expression for \( W \) given below if \( k^2 \omega \) is imaginary. From (64) this occurs when \( |\mu^2| < |k^2| \).

Being proportional to \( \omega^2 \), the dimensionless eigenvalue \( \lambda \) is always real: \( \lambda \geq 0 \) for stable modes and \( \lambda < 0 \) for unstable modes. It is thus more convenient to use in stability studies than \( \omega \). The expression for \( K \) in (87a) is valid for all \( \lambda \), but most appropriate to cases where both \( k^2 \omega \) are real, i.e., for \( \lambda > (|k^2|a_0^0 \max x C^2 / \lambda_0^0)^2 \), by (59) and (86a).

On the other hand, the manifestly positive form in (87b) is specialized to cases where both \( k^2 \omega \) are imaginary. This includes all unstable modes, \( \lambda < 0 \), and a range of stable modes close to the instability threshold, \( 0 < \lambda < (|k^2|a_0^0 \min x C^2 / \lambda_0^0)^2 \). When \( k^2 \omega \) is imaginary, (see the supplement)

\[
[k^2 \omega] = \frac{1}{a^0} \left[ (k^2 \omega) a^0 - 2 \frac{\lambda}{b \omega^2} \right]^{1/2},
\]

from (86b). As \( K \) is positive definite in this case it can be regarded as a frequency-dependent effective mass of the artificial constant surface mass normalization used in our previous stability studies [13, 17, 18]. In fact, close enough to marginal stability (see section 8.3.1) we may approximate \( K \) by its value at \( \lambda = 0 \) to give the Rayleigh–Ritz-like approximation

\[
\lambda \approx \frac{W}{K(0)}.
\]

As \( \nabla \cdot v = 0 \) at \( \lambda = 0 \) we shall term this the 'incompressible approximation'.

The RHS of (85) is the dimensionless energy, defined, using (72) and (74) (see the supplement), as

\[
W = \sum_\xi \int \rho \xi k^2 \mu \cot(k^2 \mu a_\xi^0) |F_{m,n}(x)|^2 + \int \rho \xi m F_{m,n} G_{m,n} dx,
\]

where

\[
F_{m,n}(x) = \frac{k_{\xi} \cdot B(x)}{|k_{\xi}| B^0} = \frac{B^2 m \sin(\Theta \xi + \mu \xi x) - \mu_0 n \cos(\Theta \xi + \mu \xi x)}{(m^2 + e^2 n^2)^{1/2}},
\]

so the numerical root finding required to find eigenvalues can be done using \( k_{\xi} \) as an independent variable rather than \( \lambda \), which can then be determined from (86b).

8. Vacuum case

Provided there is no linear correction to \( \mu \), which is ensured here because \( k_{\xi} = 0 \), the case (designated by superscript \( v \)) where there is a vacuum in \( \Omega^c \) can be treated as if the vacuum were a currentless plasma. That is, by taking the limit \( \mu^v \to 0 \), a case where \( k_{\xi}^v \) is imaginary. Also, to make the region pressureless, we take \( \rho^v \to 0 \). However, without physical consequence we can keep the specific temperature \( T^v \) finite in order to avoid any complications from vanishing \( C_r^v \).

With plasma confined to \( \Omega^c \) it is natural to use the parameters of this region as reference parameters, i.e., to set \( a^0 = a_i^0 = x_i, \rho^0 = \rho^v \) and \( B^0 = B^\nu \) (so \( v_0^0 = B^\nu / \sqrt{\mu_0 \rho^v} \)). Then, the eigenvalue equation (85) simplifies to \( K^{v0} \lambda = \lambda^{v0} W^v \), where

\[
W^v = \frac{[k_{\xi} \cdot B(x)]^2 k_{\xi}^v x_i \cot(k_{\xi}^v x_i)}{|k_{\xi}| B^0} - \frac{\mu x_i k_{\xi} \cdot B(x) v_1 k_{\xi} \times B(x)}{|k_{\xi}| B^0} + \frac{B^2 [k_{\xi} x_i (k_{\xi} \cdot B(x))^2 \cot|k_{\xi}| (x - x_i)]}{|k_{\xi}| B^0} + (1 + \beta) |k_{\xi}| x_i |F_{m,n}^v|^2 \cot|k_{\xi}| (x - x_i).
\]

The superscripts \( v \) have been dropped because all plasma parameters are defined only in \( \Omega^c \) and the vacuum in \( \Omega^c \) is indicated by subscript \( v \). We have eliminated the vacuum magnetic-field strength in terms of the plasma \( \beta \) using (34): \( B_v = B(1 + \beta)^{1/2} \).

Using (86b) to eliminate \( \lambda \), the LHS of the eigenvalue equation can be put in a form reminiscent of (82):

\[
K^{v0} \lambda \equiv -\frac{\beta}{2} \left( \frac{|k_{\xi}| x_i^2 + |k_{\xi}^v|^2}{k_{\xi} x_i} \right) \cot(k_{\xi} x_i) \quad (94a)
\]

\[
= \frac{\beta}{2} \left( \frac{|k_{\xi} x_i|^2 - |k_{\xi}^v x_i|^2}{|k_{\xi}^v x_i|} \right) \cot(|k_{\xi}^v x_i|), \quad (94b)
\]

so the numerical root finding required to find eigenvalues can be done using \( k_{\xi} x_i \) as an independent variable rather than \( \lambda \), which can then be determined from (86b).

\[\text{From (57), } \mathbf{k}^+ \cdot \nu = |k_{\xi}^+|^2 - |k_{\xi}^v|^2, \text{ so } \nabla \cdot v \text{ vanishes when } \lambda \text{ vanishes.} \]
As \( \mathcal{W} \) is not positive definite, the spectrum contains unstable modes in general: the threshold between stability and instability can be found by finding where \( \mathcal{W} \) changes sign, causing \( \lambda \) to also change sign as \( \mathcal{K} \) is positive. Unlike the kinetic energy in ideal MHD, \( \mathcal{K} \) is an analytic function of \( \mathcal{W} \) through the marginal stability point \( \lambda = 0 \), allowing instability thresholds to be determined by simple interpolation methods in a scan of equilibrium parameters (a feature similar to the PEST 2 code [19, figure 4], which also has a kinetic energy depending only on displacements normal to magnetic surfaces).

From (90) we see the potentially negative factors in \( \mathcal{W} \) are the second (jump) term and the \( \cot(k_{x}^{2}x_{1}) \) factors in the first term. A simple example of the latter is given in section 9 below.

### 8.1. The spectrum in the case of radially propagating sound

When \( k_{x}^{2}x_{1} \) is real the spectrum of eigenvalues \( \omega \), or equivalently \( \lambda \), is determined by finding standing waves in the \( x \)-direction. To get an overview of this \( k_{x}^{2} \in \mathbb{R} \) subset of modes (which are all stable as \( \lambda > (\beta/2)|k_{x}^{2}x_{1}^{2}| > 0 \)), consider solutions of \( \mathcal{K}^{(v)} = \mathcal{W}^{(v)} \) on the real-\( k_{x} \) axis, corresponding to the form of \( \mathcal{K}^{(v)} \lambda \) in (94a). As seen in figure 5, solutions may be determined by finding points where the graphs of \( \cot(k_{x}^{2}x_{1}) \) and \( -Uk_{x}^{2}x_{1}/\Omega^{2} \) intersect, where \( \Omega^{2} = (|k_{x}^{2}x_{1}^{2} - |\mathcal{W}^{(v)}|x_{1}^{2} \) is now less than \( |k_{x}^{2}x_{1}^{2} \) and passes through zero (the marginal stability point, \( \lambda = 0 \)) when \( |k_{x}^{2}| = |\mathcal{W}^{(v)}| \). In contrast to the propagating case it is seen that there is only one root in the evanescent sound case, which is stable if \( \mathcal{W}^{(v)} > 0 \), but is unstable if \( \mathcal{W}^{(v)} < 0 \), consistently with the interpretation of \( \mathcal{W}^{(v)} \) as a non-dimensionalized second variation of the MHD energy [13, 17, 18] (this connection will be pursued further elsewhere).

In the strongly stable case \( U > 0 \) there is a root close to \( k_{x}^{2}x_{1} = 0 \), but in the strongly unstable case \( U < -1, |k_{x}^{2}|x_{1} \) (and hence \( -\lambda \)) are large, as shown below.

### 8.3. Asymptotics

#### 8.3.1. Close-to-marginal modes: As seen in figure 6, roots close to marginal stability occur close to the vertical asymptote at \( |k_{x}^{2}x_{1}| = |\mathcal{W}^{(v)}| \). In this case we expand coth\((|k_{x}^{2}x_{1}|)\) about \( |k_{x}^{2}x_{1}| \) in (94b), \( \mathcal{K}^{(v)} \lambda = \mathcal{W}^{(v)} = (\beta/2)U \); then giving \( |k_{x}^{2}x_{1}| \sim |\mathcal{W}^{(v)}| \sim \tanh(|k_{x}^{2}x_{1}|)U/2 + O(U^2) \). From (866) the lowest eigenvalue is found to be

\[
\lambda_{1} \sim |\mathcal{W}^{(v)}| \tanh(|\mathcal{W}^{(v)}|) \mathcal{W}^{(v)} \sim \frac{\tanh(|\mathcal{W}^{(v)}|)\mathcal{W}^{(v)}}{\beta} \frac{|\mathcal{W}^{(v)}|}{\mathcal{W}^{(v)}} + \tanh(|\mathcal{W}^{(v)}|) \mathcal{W}^{(v)} + O(V^{(v)^{2}}) \quad (95)
\]

as \( \mathcal{W}^{(v)} \to 0 \).

The striking thing about this result, at leading order, is that the lowest eigenvalue is independent of \( \beta \). This can also be seen by substituting the \( \lambda = 0 \) value \( |k_{x}^{2}x_{1}| = |\mathcal{W}^{(v)}| \) (see (88)) into (87b), giving \( \mathcal{K}^{(v)}(0) = \coth(|k_{x}^{2}x_{1}|)/|k_{x}^{2}x_{1}| \). Thus, the incompressible approximation (89) gives \( \lambda \sim |k_{x}^{2}x_{1}|\tanh(|k_{x}^{2}x_{1}|)\mathcal{W}^{(v)} \). This shows that, near marginal stability, the evanescent sound waves couple plasma inertia to the interface independent of the value of the plasma pressure. However, the range of \( \mathcal{W}^{(v)} \) over which this slow, incompressible approximation is appropriate shrinks to zero as \( \beta \to 0 \).

Thus, our previous approach [13, 17, 18] of assigning an artificial, constant mass loading to the interface gives identical stability boundaries to those found from the present dynamical MRxMHD formulation.
8.3.2. Strongly unstable modes: For large $|k_x^+x_i|$ in figure 6, $\coth |k_x^+x_i| \approx 1 + 2 \exp(-2|k_x^+x_i|)$. However, the exponential term is small to all orders in $1/|k_x^+x_i|$, so may be dropped to find an asymptotic solution of $K^{(v)} \lambda = (\beta/2)\mu$ for the strongly unstable case $W^{(v)} \to -\infty$, $k_x^+ \notin \mathbb{I}$, $|k_x^+x_i| \sim |\mu| + (k_x^+x_i)^2|\mu|^{-1} + O(|\mu|^{-3})$ (see the supplement). Thus, from (86b), the lowest eigenvalue as $W^{(v)} \to -\infty$ is

$$\lambda \sim -\frac{2}{\beta} |W^{(v)}|^2 - \frac{\beta}{2} |k_x^+x_i|^2 + O(|W^{(v)}|^2),$$

agreement with the numerical solution when the system is close to marginal stability, $\lambda_i \approx 0$.

The lower horizontal line at height $(\beta/2)|k_x^+x_i|^2$ divides the range of radially propagating sonic waves (above) from the radially evanescent lowest eigenmode range (below), and the upper line at height $(\beta/2)(|k_x^+x_i|^2 + \pi^2)$ separates the range of possible $\lambda_2$ values from the $\lambda_3$ range. The vertical line indicates the position of the first Beltrami eigenvalue (99), $\mu_1^{-1} = 3.445/x_i$, if the wall were at $x_i$.

8.3.3. Stable modes with large $|\mu'|$: As seen qualitatively in figures 5 and 6, in both the propagating and evanescent sound cases there may (depending on the sign of $W^{(v)}$) be a root where $k_x^+x_i$ is close to zero. In these cases we can expand the eigenvalue equation $K^{(v)} \lambda = (\beta/2)\mu$ in powers of $(k_x^+x_i)^2$ and solve it for $(k_x^+x_i)^2$, giving $(k_x^+x_i)^2 \sim -(|k_x^+x_i|^2|\mu|^{-1} + O(1/|\mu|^{-3})$.

Then,

$$\lambda \sim \frac{\beta}{2} |k_x^+x_i|^2 \left(1 - \frac{\beta}{2|W^{(v)}|}\right) + O(|W^{(v)}|^2),$$

for the lowest stable mode in the limit $|W^{(v)}| \to \infty$ (i.e. the lowest mode $\lambda_1$ when $W^{(v)} \to +\infty$, $k_x^+ \notin \mathbb{I}$ and the second-lowest mode $\lambda_2$ when $W^{(v)} \to -\infty$, $k_x^+ \in \mathbb{R}$).

8.4. $\mu$ scan

To illustrate these various limits we generate a family of (rather artificial) equilibria in a similar way to those depicted in section 4.2, but choosing $q'(0)$ at the core–slab interface to be the zero-radius core value $q_c(0)$ given in (28) so that $q'(0) = 2/R \mu_c$. Then (33) can be solved (see appendix B of online supplementary information) to find $\Theta = \tan^{-1}(a \mu_c/2)$. We also choose $\mu' = \mu_c$. Figure D.1 (of the supplement) plots $V^{(v)}$ versus $\mu_c x_i$, showing several stable ($V^{(v)} > 0$) and unstable ($V^{(v)} < 0$) ranges of $\mu_c$.

Figure 7 shows plots of eigenvalues $\lambda$ versus $\mu_c$, the lowest eigenvalue $\lambda_0$ (negative when the system is unstable) and the second-lowest eigenvalue $\lambda_2$. These eigenvalues were found by solving the eigenvalue equation $K^{(v)} \lambda = W^{(v)}$ numerically. The incompressible approximation to $\lambda_1$, $V^{(v)}/K^{(v)}(0)$ ((95) ff), is also plotted, showing good agreement with the numerical solution when the system is close to marginal stability, $\lambda_i \approx 0$.

9. Free- and fixed-boundary tearing instabilities

First, we consider the case where there is no vacuum region and the unperturbed inner and outer regions are simply sub-regions of the same Taylor-relaxed plasma, confined by rigid plane boundaries at $x = 0$ and $x = x_w$ and partitioned by a thin ideal interface at arbitrary $x_i$. Then, $B_y(x)$ are analytic continuations of each other, as depicted by the dashed curve in figure 2, and the jump term in (90) vanishes.

In addition to the interface at $x = x_i$ and the outer boundary at $x = x_w$, an important location in the plasma is, for given $k_x^{m,n}$ (see (48)), the mode rational surface at $x = x_w^{m,n}$, defined as the solution of the ideal-MHD resonance condition at marginal stability (i.e. $\omega^2 = 0$),

$$k_x^{m,n} \cdot B(x_w^{m,n}) = 0,$$

which, by (91), can also be written $F_{m,n}(x_w^{m,n}) = 0$.

It is well known that, for given $m$ and $n$, such an equilibrium is unstable to an $x, y$ translational symmetry-breaking ‘helical bifurcation’ when $|\mu|$ exceeds the first Beltrami eigenvalue $\mu_1^{m,n}$ (see, e.g. the review by Taylor (2)). In slab geometry this corresponds to the lowest value of $|\mu|$ for which a standing wave obeying the boundary conditions $b_y = 0$ (see (53)) at both boundaries $x = 0$ and $x = x_w$ occurs. That is, from (72), $k_x^{m,n} x_w = \pi$. From (64) this implies (see the supplement)

$$\mu_1^{m,n} = \frac{\pi}{x_w} \left[1 + \left(\frac{k_x^{m,n} x_w}{\pi}\right)^2\right]^{1/2},$$

4 Not to be confused with the linear correction to $\mu$ derived in section 5.1, which does not apply in the present case with $k_x = 0$. 

---

Figure 6. Case of radially evanescent sound, $k_x^+$ imaginary: the intersections of the graph of $\coth |k_x^+x_i|$ (solid blue curve) with the other two curves give eigenmode solutions corresponding to (94b). The vertical marginal stability line at $|k_x^+| = |k_x^0|$ separates stable modes ($\lambda > 0$, $|k_x^+| < |k_x^0|$) from unstable modes ($\lambda < 0$, $|k_x^+| > |k_x^0|$). For given $W^{(v)}$ there is only one root—stable for positive $W^{(v)}$, and unstable for negative. Illustrated for $\mu = +3$ (long-dashed orange curve) and $\mu = -3$ (short-dashed green curve).
The helical bifurcation can also be identified as being due to the tearing-mode instability [2]. In this section we see how the tearing instability manifests itself in dynamical MRxMHD.

In evaluating $\mathcal{K}$ and $\mathcal{W}$ for this equilibrium we can drop $\bar{z}$ in everything except $a_i^\omega = x_i$ and $a_i^\zeta = x_w - x_i$. From (87b), taking $a_i^\zeta = x_w$, we find

$$\mathcal{K}(\lambda, x_i) = \frac{\coth |k_x^\omega|x_i + \coth |k_x^\zeta|(x_w - x_i)}{|k_x^\omega|x_w},$$

As illustrated in figure 8, $\mathcal{K}$ is positive definite in the case relevant to stability studies, $k_x^{\zeta,\omega} = i k_x^{\zeta,\omega}$. From (48) and (90),

$$\mathcal{W}(\mu, x_i) = [F^{m,n}(x_i)]^2 (\cot k_y^m a_i^\zeta + \cot k_y^m a_i^\zeta) k_x^\omega x_w.$$  

From (85), $\omega^2$ cannot be negative unless $\mathcal{W}$ is also negative. When we are seeking unstable, or close-to-unstable, modes we can thus restrict attention to the case of real $k_x^\omega$, as it is easy to show from (101) using the identity in the previous section, that, in the case of imaginary $k_x^\omega$, $\mathcal{W}(x_i)$ is positive definite. From (64) real $k_x^\omega$ implies $|\mu| > |k_x^\omega|$, but this is not sufficient to make $\mathcal{W}$ negative—$x_i$ must lie within a range of values for which $\cot k_y^m a_i^\zeta + \cot k_y^m a_i^\zeta$ is negative, the existence of which requires $|\mu| > \mu_{1,5}^{m,n}$ as illustrated in figure 9.

Thus, we have shown that our MRxMHD stability analysis captures the onset point of the Taylor bifurcation. Beyond the bifurcation point it gives collisionless tearing-mode growth rates that can be regarded as upper bounds for tearing instabilities assisted by the mesoscopic reconnection mechanisms implicit in Taylor relaxation, such as ‘chaos-induced resistivity’ [20]. It is to be noted that placing the interface too close to either wall suppresses the instability and that the growth rate goes to infinity at the edges of the unstable region (where, though outside the scope of the present study, the nonlinearly saturated amplitude presumably goes to zero).

We confirm the predictions of this analysis in figure 10 using the same equilibrium as in figure 3, in the case of plasma filling both regions between the rigid boundaries $x = 0$ and $x_w$. As the Beltrami eigenvalue $\mu_{1,5}^{1,5} = 3.86/x_w$ in this case is less than the equilibrium value $\mu_c = 4/x_w$ we expect the plasma to be unstable to the $-1, 5$ tearing mode. The lowest eigenvalue $\lambda_1$ shown in figure 10 is indeed negative over a range of interface positions within the plasma. Also shown is the stable second eigenvalue $\lambda_2$ as well as the prediction of the incompressible approximation (89), which is seen to be accurate only over a very narrow interval of $\lambda$ for the 5% $\beta$ value used.

Moreover, note that the location $x = x_i^{m,n}$ of the mode’s rational surface does not enter into the instability threshold condition $|\mu| = \mu_{1,5}^{m,n}$, but beyond the threshold it does affect the dependence of the growth rate on the interface location $x_i$. In fact, the growth rate clearly vanishes at $x_i = x_i^{m,n}$, consistent with the picture of an MRxMHD interface as a thin
layer of ideal plasma, which does not allow reconnection [21]. This is apparent in figure 10 where the growth rate vanishes at $x = x_i^{0.3694} r_{1,5}$. In the tokamak-like case shown in figure 2, (99) gives $\mu_{i,1}^{2,1} = 5.1/x_{iw}$, which is far greater than $|\mu| = 0.16/x_{iw}$. In the case where plasma extends continuously to the wall, this tokamak-like equilibrium is, therefore, stable against the 2, 1 tearing mode (and indeed any mode, as $\mu^+ < \pi/x_{iw}$).

However, figure 11 shows that, in the case of a vacuum between $x = x_i$ and $x_w$, there is a narrow range of instability of a 2, 1 mode between $q_i = 2$ and $q_i \approx 2.23$. As the free-boundary plasma slab is known to be stable against ideal kink modes [22], this must be a kink-tearing mode, i.e. a tearing mode made unstable by the proximity of the resonant surface $x = x_i^{2,1}$ to the plasma–vacuum interface, $x_i^{2,1}$ starting at $x_i$ when $q_i = 2$ and moving inward as $q_i$ increases. Figure 11 also shows that this instability is so weak that the incompressible approximation gives excellent agreement with the exact solution even with the low 1% $\beta$ used to make the plot.

10. Conclusion

In order to make clear the physics implications of the new dynamical MRxMHD formulation, we have used as elementary an approach as possible, in particular, deriving the eigenvalue problem from first principles by linearizing the raw Euler–Lagrange equations arising from the action
principle. To develop further insights it remains to derive a quadratic Lagrangian variational principle in terms of the fluid displacement $\xi$, similar to that used in our previous energy-principle-based MRxMHD formulation [13, 17, 18].

A useful result of the analysis has been to establish that MRxMHD leads to the same stability boundaries (marginal stability points) as our previous formulation by deriving this previous formulation as the incompressible approximation to the new theory, valid near marginal stability.

We have also verified that the instability threshold, derived from linearized MRxMHD in a plasma confined between rigid boundaries, agrees with the onset of the Taylor bifurcation derived from Beltrami equilibrium theory. Given the simplicity of MRxMHD it should also be feasible, and instructive, to calculate nonlinearly saturated amplitudes of linearly unstable modes by expanding the perturbed energy up to quartic order in amplitude, assuming the bifurcation saturates when a point of minimum energy is reached. This would be much more physically useful than the rather unphysical growth rates derived in this paper (which, e.g. can become infinite).

The single-interface slab model used in the present paper is clearly inadequate for understanding mode structure in a plasma with realistic pressure and current profiles. To find more physical eigenfunctions and eigenvalues we will need to use a realistic geometry and add more interfaces so as to partition the plasma into multiple relaxed regions (as already done for MRxMHD equilibria [5]).

A feature of the dynamical formulation is that it allows treatment of background flow in a natural way, providing a further class of instability mechanisms to explore. After a few more scoping studies in simple geometries, we expect MRxMHD to provide the basis for an efficient, fully 3-D stability and stable-spectrum code.

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