Lagrangian averaged stochastic advection by Lie transport for fluids

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Abstract. We formulate a class of stochastic partial differential equations based on Kelvin’s circulation theorem for ideal fluids. In these models, the velocity field is randomly transported by white-noise vector fields, as well as by its own average over realizations of this noise. We call these systems the Lagrangian averaged stochastic advection by Lie transport (LA SALT) equations. These equations are nonlinear and non-local, in both physical and probability space. Without taking this average, the equations recover the Stochastic Advection by Lie Transport (SALT) fluid equations introduced by Holm [1]. Remarkably, the introduction of the non-locality in probability space in the form of advecting the velocity by its own mean gives rise to a closed equation for the expectation field which comprises Navier–Stokes equations with Lie–Laplacian ‘dissipation’. As such, this form of non-locality provides a regularization mechanism. The formalism we develop is closely connected to the stochastic Weber velocity framework of Constantin and Iyer [2] in the case when the noise correlates are taken to be the constant basis vectors in $\mathbb{R}^3$ and, thus, the Lie–Laplacian reduces to the usual Laplacian. We extend this class of equations to allow for advected quantities to be present and affect the flow through exchange of kinetic and potential energies. The statistics of the solutions for the LA SALT fluid equations are found to be changing dynamically due to an array of intricate correlations among the physical variables. The statistical properties of the LA SALT physical variables propagate as local evolutionary equations which when spatially integrated become dynamical equations for the variances of the fluctuations. Essentially, the LA SALT theory is a nonequilibrium stochastic linear response theory for fluctuations in SALT fluids with advected quantities.

1. Introduction – SALT and LA SALT

This paper develops a new class of fluid equations based on Stochastic Advection by Lie Transport (SALT) [1] by applying a type of Lagrangian Average (LA) which is the counterpart in probability space of the time average at fixed Lagrangian coordinate taken in the LANS-alpha turbulence model [3, 4]. As was proven previously for SALT, the new set of LA SALT fluid equations with advected quantities preserves the fundamental properties of ideal fluid dynamics. These properties include Kelvin’s circulation theorem and Lagrangian invariants such as enstrophy, helicity and potential vorticity [1]. These properties derive from the preserved Euler–Poincaré and Lie–Poisson geometric structures of the deterministic theory, which are reviewed in section 1.1. In the Kelvin circulation integral for the new theory (LA SALT), the Lagrangian Average in probability space manifests itself as advection of the Kelvin circulation loop by the expected fluid transport velocity. This is the primary modification made by applying LA to the SALT theory, [1, 5, 6, 7, 8, 9]. This modification of the SALT theory allows the dynamics of the statistical properties of the solutions for the LA SALT fluid equations to be investigated directly.

One interesting consequence is that, due to the non-local nature of the equations in probability space, the expected dynamics contain terms which regularize the (expected) solution. Thus, the introduction of the LA modification of Kelvin’s circulation integral bestows on the LA SALT version of the 3D Euler fluid similar solution properties to those of the incompressible Navier-Stokes equations. Moreover, in certain cases, the additional terms are strictly dissipative, and the LA SALT fluid equations may be regarded as non-conservative system of PDEs for the expected motion embedded into a larger conservative system which includes the fluctuation dynamics. From this viewpoint, the interaction dynamics of the two components of the full LA SALT system dissipates the Lagrangian invariant functions of the mean quantities by converting them into fluctuations, while preserving the total invariants. We will explore these properties both at the level of a general semidirect-product Hamiltonian system and more concretely for special cases of such systems where more details can be worked out.

Plan of the paper. We write the expected-quantity equations for LA SALT fluid dynamics with advected quantities in the semidirect-product Hamiltonian matrix form in Section 2. We then point out that the
fluctuations are slaved to the expected equations in a certain sense. This slaving relation enables us to
calculate the evolution equations for the local and spatially integrated variances of the fluctuations.

In section 3, we discuss the mathematical well-posedness of the LA SALT Euler equations. Theorem 1
establishes local existence for the LA SALT equations in Sobolev spaces in \( d = 2, 3 \), and global existence
provided we work in two-spatial dimensions or the noise correlates \( \{ \xi^{(k)} \} \) are sufficiently ‘large’ relative
to the data and forces in three-dimensions. To establish this result, we take advantage of the fact the dynamics
of the expectation field decouples from the fluctuations and solves a closed Navier-Stokes type equation
with Lie-Laplacian ‘dissipation’ (LL NS). Theorem 2 establishes the results above for LL NS.

Section 4 deals with several illustrative examples. Three simple examples are treated first which do not have
advected quantities, including the rigid body in 3D, Burgers equation, and the Camassa-Holm equation.
These first examples reveal the additional information made available in the LA SALT theory for investigat-
ing both the linear transport equations for the fluctuation dynamics and the equations for expected physical
variables. Finally, we demonstrate how a more advanced example fits into the LA SALT theory. For this, we
formulate the 3D LA SALT incompressible stratified magnetohydrodynamics (MHD) equations and point
out that MHD at this level also contains 3D Euler–Boussinesq, whose 2D case is treated in the paper [10].

**Main content of the paper:**

- Stochastic Euler fluid equations which are “nonlinear in the sense of McKean” [11, 2, 12, 13] are
generalised to include advected quantities.
- In certain cases, the equations for the expected values of physical variables decouple as a subsystem
  from the fluctuation dynamics. In the absence of advected quantities, a particular case yields a
  Navier–Stokes–type partial differential equation (PDE).
- The SPDE for the dynamics of the fluctuations of the momentum and advected quantities are shown
  to be transported by the PDE solutions for the expected values.
- The statistics of the fluctuations are found to be changing dynamically, driven by an intricate array
  of correlations. Specifically, the statistical properties of the LA SALT physical variables propagate
  as local equations and yield dynamical variances when spatially integrated. In certain cases for
  which the dynamics of the fluctuations occurs by linear transport, the equations for their statistical
  properties form closed evolutionary PDE systems.
- Analytical conditions are found for which the LA SALT fluid equations are well-posed.

Before developing these results, let us introduce the geometrical context of our approach by very briefly
surveying its development during the past 50 years.

**1.1. Historical background of the geometric approach to fluid mechanics.** In two papers published
in 1966, V. I. Arnold changed the way we think about fluid dynamics, forever. In the papers [14, 15], Arnold
showed that the solutions of the Euler fluid equations in a domain \( \mathcal{D} \) in fixed space, \( \mathcal{D} \subset \mathbb{R}^n \), can be mapped
by the classic Lagrange-Euler representation to a time-dependent path on the manifold of volume-preserving
diffeomorphisms acting on \( \mathbb{R}^n \) (SDiff(\( \mathbb{R}^n \))) which is geodesic with respect to the right-invariant metric on
its tangent space given by the kinetic energy of the fluid. The kinetic energy metric is right-invariant because
it is the \( L^2 \) norm of the right-invariant Eulerian velocity \( u_t \) defined by

\[
\dot{g}_t x_0 = u_t(g_t x_0),
\]

in which subscript \( t \) denotes explicit time dependence. The Lagrangian fluid parcel trajectory is given by

\[
x_t := g_t x_0 \in \mathcal{D} \quad \text{with} \quad g_0 x_0 = x_0.
\]

Upon writing \( u_t = \dot{g}_t h^{-1} \), one sees that the invariance of the Eulerian velocity corresponds to relabelling
the Lagrangian parcel label \( x_0 \rightarrow y_0 = h x_0 \) for a fixed map \( h \in \text{SDiff}(\mathbb{R}^n) \). Thus, under any fixed map \( h \)
acting from the right we have \( \dot{g}_t h(g_t h)^{-1} = \dot{g}_t h^{-1} \). Arnold’s identification of the Euler fluid solutions as
geodesics also brings in Hamilton’s variational principle and right-invariance summons Noether’s theorem for Lie group invariant variational principles.

Arnold’s idea that Euler fluid flows could be lifted to time-dependent paths on $\text{SDiff}(\mathbb{R}^n)$ has been continually fruitful. Already in 1970, Ebin and Marsden \cite{16} used this idea to prove the local in time existence and uniqueness of the Euler fluid flows in $\mathbb{R}^3$. This is still the definitive analytical result for the Euler fluid equations. By 1985, Marsden and his collaborators had used the same idea to obtain the Lie–Poisson Hamiltonian formulation for ideal fluids with advected quantities and had recognised the role of its semidirect-product structure in establishing nonlinear stability for a wide class of fluid and plasma equilibria for continuum flows with advected quantities and additional physical fields \cite{17,18}. Again, this development of nonlinear stability conditions followed Arnold’s lead in \cite{14} for the nonlinear stability of Euler fluid equilibria as critical points of a constrained variational principle for time dependent paths on the manifold $\text{SDiff}(\mathbb{R}^n)$. For further explanation of these developments in the context of momentum maps, see \cite{19,20}.

Following an observation reported in 1901 by Poincaré, \cite{21}, Holm, Marsden and Ratiu \cite{22} transferred the idea of symplectic reduction for Hamiltonian systems into the theory of reduction by Lie symmetries of the Lagrangian in Hamilton’s principle and applied the resulting Euler–Poincaré equations to derive the general dynamics of fluid flows with advected quantities. This result again followed Arnold’s lead in regarding fluid flows as curves on $\text{SDiff}(\mathbb{R}^3)$, although the dynamics discussed in \cite{22} takes place on $\text{Diff}(\mathbb{R}^3)$ in the compressible case when volume is not preserved. An interesting feature in this particular development involves the Kelvin–Noether theorem. This theorem revealed that the momentum map on the Lagrangian side for the action corresponding to the relabelling symmetry was, in fact, the circulation integral in Kelvin’s theorem for Euler’s equations. Thus, the conservation of the Kelvin circulation integral for the Euler fluid equations was found to arise via Noether theorem from the symmetry of the Eulerian velocity in (1.1) under relabelling of the Lagrangian fluid parcels.

Now, the Lagrangian fluid parcels carry advected physical variables such as mass, heat, other thermodynamic quantities, buoyancy for stratification, magnetic field lines for MHD, etc. The introduction of spatially varying initial conditions for these advected quantities breaks the symmetry of the Euler kinetic energy Lagrangian under the full set of Lie group transformations by $G = \text{Diff}(\mathbb{R}^n)$. In particular, the invariance of the Lagrangian is restricted to the “isotropy subgroups” $G_{a_0} := \text{Diff}_0(\mathbb{R}^n)$ of the full $\text{Diff}(\mathbb{R}^n)$ Lie transformations. The isotropy subgroups $G_{a_0}$ are those which leave invariant the initial conditions $a_0$ chosen for the advected quantities. The advected quantities evolve by push-forward $a_t = a_0g_t^{-1}$ by the action of the entire $\text{Diff}(\mathbb{R}^n)$. (Push-forward is pull-back by right action of $g_t^{-1}$.) Thus, symmetry breaking from $\text{Diff}(\mathbb{R}^n)$ to $\text{Diff}_0(\mathbb{R}^n)$ leads to the identification of advected quantities as order parameters $a_t \in G/G_{a_0}$, where $G/G_{a_0}$ is the corresponding coset space of the broken symmetry with $G$ by the remaining symmetry $G_{a_0}$ for $a_0$. This symmetry breaking of $\text{Diff}(\mathbb{R}^n)$ for the Euler fluid equations implies that the Euler–Poincaré equations for flows with advected quantities acquire force terms of geometric origin leading to the so-called “diamond ($\diamond$) terms” which will be discussed further below. Including these force terms arising from symmetry breaking implies that fluid solutions with advected quantities are no longer geodesic paths on $\text{Diff}$ or $\text{SDiff}$. For an application of these ideas to complex fluids, see \cite{23,24}.

This history of the development of Arnold’s $\text{SDiff}$ flow concept and its extension to include advected fluid quantities provides the context of flows on $\text{Diff}$ or $\text{SDiff}$ for the introduction of the material addressed in this paper. Because of its close connection to Lie symmetry transformations via Noether’s theorem, this material can also be addressed operationally and quite transparently from the viewpoint of Kelvin’s theorem, by using the pullbacks of the time dependent flow. This operational interpretation arises naturally because of the physical connection to Newton’s force law, since the Kelvin circulation integral is the Noether quantity for momentum distributed on closed material loops \cite{22}.

**Flow interpretation of stochastic advection by Lie transport (SALT).** In a paper in 2015, Holm \cite{1} extended the Clebsch approach of \cite{17} for deriving Euler–Poincaré equations for fluids with advected quantities to the case that the fluid variables undergo flows on $\text{Diff}$ or $\text{SDiff}$ generated via stochastic advection by
Lie transport (SALT). This step derived the SALT class of stochastic continuum equations from Hamilton’s principle with a stochastic advection constraint on Lagrangian fluid trajectories. Later, the stochastic constraint in [1] was derived from first principles by using multi-time homogenisation in [5], after first writing the fluid flow map in (1.1) as the composition of two time-dependent diffeomorphisms. Namely,

\[ g_{t,t/\epsilon} = \tilde{g}_{t/\epsilon} \circ \tilde{g}_t = (Id + \gamma_{t/\epsilon}) \circ \tilde{g}_t, \tag{1.3} \]

in which composition of maps is represented by \((\cdot)\). One of the maps in (1.1) is faster in time \((t/\epsilon)\) for \(\epsilon \ll 1\) than the other, which has time dependence, \((t)\). Upon writing \(\tilde{x}_t(x_0) = \tilde{g}_t x_0\) we have from (1.1) that

\[ \frac{d}{dt}(g_{t,t/\epsilon} x_0) = u_t(\tilde{g}_t x_0 + \gamma_{t/\epsilon} \tilde{g}_t x_0) = \hat{\xi}_t(x_0) + (\hat{\xi}_t \circ \nabla_x_t) \gamma(\tilde{x}_t(x_0), t/\epsilon) + \frac{1}{\epsilon} \frac{\partial}{\partial(t/\epsilon)} \gamma(\tilde{x}_t(x_0), t/\epsilon). \]

For certain conditions on the two-time flow, multi-time homogenisation in the limit \(\epsilon \to 0\) as the ratio of the slow rate of change to the fast rate tended to zero was used to show that the two-time map in (1.1) tends to the SALT stochastic flow map [5], denoted as

\[ \lim_{\epsilon \to 0} g_{t,t/\epsilon} = g_{t, \od W_t} := \tilde{g}_{\od W_t} \circ (Id + \gamma_{\od W_t}) \circ \tilde{g}_{t, \od W_t}, \tag{1.4} \]

where \(\od W_t\) denotes Stratonovich stochastic time dependence. The homogenisation argument of [5] shows that the corresponding spatial stochastic vector field \(dX_t\) on a given smooth manifold \(M\) which generates the stochastic flow map (1.1) is given by

\[ dx_t(x) = u_t^L(x) dt + \sum_{k=1}^{\infty} \xi^{(k)}(x) \circ dW_t^{(k)}, \quad x \in M \tag{1.5} \]

where the drift \(u_t^L\) may (and will, in many examples) be random and may depend on the noise in (1.1). In [1], the drift velocity was determined using the Clebsch variational approach. This approach ensured that the Lagrangian invariants of the deterministic flow would still hold in each realization of the process (1.1). This is the SALT approach, in which the vector field (1.1) appears in the stochastic perturbation of the Euler–Poincaré equations with Lie transport noise. In applications of the SALT approach, the stationary vector fields \(\xi^{(k)}(x)\) are expected to be determined from a data analysis procedure such as the one developed in [6, 8, 9]. Here, we will assume that these stationary vector fields are already known from the appropriate data analysis for a given application.

The present paper aims to develop the LA SALT class of stochastic continuum equations. Namely, we will consider the following stochastic flow,

\[ g_{t, \od W_t}^E = \tilde{g}_{\od W_t}, \quad \mathbb{E} [g]_t = (Id + \gamma_{\od W_t}) \circ \tilde{g}_{t, \od W_t}, \tag{1.6} \]

in which \(\mathbb{E} [g]_t\) denotes the flow generated by the expected transport velocity \(\mathbb{E} [u^L]_t\). In may be possible to interpret this choice as an additional averaging at the homogenization level, but we leave this for future work. For now, we simply adopt this framework and explore its dynamical consequences.

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\(^1\)We use the notation

\[ dx_t(x) = b_t(x) dt + \sum_{k} \xi^{(k)}(x) \circ dW_t^{(k)}, \quad x \in M, \]

to denote the **stochastic vector-field** associated with the stochastic flow \(\phi = \{\phi_{s,t}\}_{0 \leq s \leq t}\):

\[ d\phi_{s,t}(x) = b_t(\phi_{s,t}(x)) dt + \sum_{k} \xi^{(k)}(\phi_{s,t}(x)) \circ dW_t, \quad \phi_{s,s}(x) = x \in M. \]

That is to say, \(dx_t = d\phi_{0,t} \circ \phi_{0,t}^{-1}\) is the stochastic analogue of the usual Eulerian vector field. For a given tensor-field \(\tau\) on \(M\), we use the notation \(L_{dx_t} \tau\) to denote the Lie derivative of \(\tau\) along the stochastic vector field \(dX_t\). This may be defined simply as \(L_{dx_t} \tau = L_{dt} \tau + L_{\xi^{(k)}} \tau \circ dW_t^{(k)}\), or, perhaps, more satisfactorily, as \(L_{dx_t} \tau = d_t(\phi_{t,t+\epsilon}^*) \tau|_{\epsilon=0}\), which follows from the Itô-Kunita-Wentzell formula (see, e.g., [7]). Thus, our notation naturally generalizes the definition of the deterministic Lie-derivative.
For example, in the Euler fluid case the proof of the modified stochastic Kelvin theorem for SAL T 
the circulation loop velocity determines Kelvin theorem by a Stratonovich stochastic vector field whose drift velocity is the same as the circulation loop velocity \( u^L_t \) in the deterministic Kelvin theorem by a Stratonovich stochastic vector field whose drift velocity is the same as the circulation loop velocity \( u^L_t \) [1, 12]. That is, for a given smooth manifold \( M \):

\[
\int_{C(u^L_t)} u_t \cdot dx \rightarrow \int_{C(dX_t)} u_t \cdot dx,
\]

where \( dX_t \) is the SALT stochastic vector field in (1.1). In the transition to the SALT Kelvin circulation integral in (1.2), the notation \( C(u^L_t) \) (resp. \( C(dX_t) \)) is used to denote that the closed material loop at time \( t \) that is moving along the flow associated with \( u^L \) (resp. \( dx_t \)) and started at time 0. In particular, we have \( C(dX_t) = g_{t,odW_t}(C) \), which is a family of loops moving with the SALT flow \( g_{t,odW_t} \) in (1.1) at time \( t \).

For example, in the Euler fluid case the proof of the modified stochastic Kelvin theorem for SALT [7]

\[
d \int_{g_{t,odW_t}(C)} u_t \cdot dx = \int_{C_0} d(g_{t,odW_t}^*(u_0 \cdot dx))
\]

\[= \int_{C_0} g_{t,odW_t}^* [du_0 \cdot dx + L_{dx_t}(u_0 \cdot dx)]
\]

\[= \int_{g_{t,odW_t}(C)} [du_t \cdot dx + L_{dx_t}(u_t \cdot dx)]
\]

\[= \int_{C(dx_t)} [du_t \cdot dx + L_{dx_t}(u_t \cdot dx)] = 0,
\]

in which \( g_{t,odW_t}^* \) is the pull-back of the flow \( g_{t,odW_t} \) and \( L_{dx_t} \) is the Lie derivative along the spatial vector field \( dx_t \) in (1.1) which generates the flow \( g_{t,odW_t} \) in (1.1). We refer the reader to [7] for a rigorous explanation of the calculation in (1.2) and its association with Newton’s Force Law for stochastic fluids. For more discussion of the emergence of Lie derivatives in the proof of Kelvin’s circulation theorem as sketched in (1.2) for fluid flow, one may also refer to [22] in the deterministic case.

The same stochastic transport velocity \( dx_t \) in (1.1) which transports the circulation loop also advects the Lagrangian parcels in the SALT theory. The Lagrangian parcels may carry advected quantities \( a \), such as heat, mass and magnetic field lines, by Lie transport along with the flow, as

\[da + L_{dx_t} a = 0,
\]

where \( L_{dx_t} a \) is the Lie derivative of a tensor-field \( a \) with respect to the vector field \( dx_t \) in equation (1.1). That is, an advected tensor-field \( a \) satisfies

\[d(g_{t,odW_t}^*(a(t,x))) = g_{t,odW_t}^*(da(t,x) + L_{dx_t} a(t,x)) = 0, \quad \text{a.s.}
\]

where \( g_{t,odW_t}^* a = a_0 \) is the pullback of \( a \) by the map \( g_{t,odW_t} \) in (1.1). Formula (1.2) defines advection as “invariance under the SALT flow”. We refer the reader to [7] for more details about stochastic advection.

In this paper, we modify the SALT approach to stochastic fluid dynamics by replacing the SALT map in (1.1) by the LA SALT map \( g_{t,odW_t} \) in (1.1). Correspondingly, we replace the Eulerian vector field...
\[ \text{d}x_t(x) \text{ in (1.1)} \] by the vector field \( \text{d}X_t(x) \text{ in (1.1)} \) Because the drift velocity will be replaced by the expected velocity, this replacement is reminiscent of the McKean Vlasov mean field approach for finite dimensional stochastic flow, which replaces the velocities for an interacting particle system by their empirical mean \[25]. The LA SALT approach modifies the SALT Kelvin circulation in (1.2) by replacing the drift velocity in the stochastic transport loop velocity in (1.1) by its expectation, plus the same noise as in SALT. Namely, cf. equation (1.1), \[1, 12\]

\[
\int_{C(dx_t)} u_t \cdot dx \rightarrow \int_{C(dX_t)} u_t \cdot dx ,
\]

where the stochastic vector field \( \text{d}X_t \) is given in (1.1).

Since the expectation in (1.1) refers to the transport velocity \( u^L_t \) of Lagrangian loop in Kelvin’s theorem, we regard this process as a probabilistic type of Lagrangian Average (LA) which is the counterpart of the time average at fixed Lagrangian coordinate taken in the LANS-alpha turbulence model \[3, 4\].

For example, in the Euler fluid case the modified Kelvin theorem reads,

\[
\frac{d}{dt} \int_{C(dx_t)} u_t \cdot dx = \int_{C(dX_t)} \left[ du_t \cdot dx + \mathcal{L}_{dX_t}(u_t \cdot dx) \right] = 0 ,
\]

where \( \mathcal{L}_{dX_t}(u_t \cdot dx) \) denotes the Lie derivative of the 1-form \( u_t \cdot dx \) with respect to the vector field \( \text{d}X_t \) given in equation (1.1).

### 1.3. LA SALT Euler equations.

Evaluating equation (1.2) implies the following stochastic Euler fluid motion equation in Stratonovich form

\[
du_t \cdot dx + P^E \mathcal{L}_{dX_t}(u_t \cdot dx) = 0 ,
\]

where the projector \( P^E \) keeps the expectation of the transport velocity \( (\mathbb{E}[u^L_t]) \) divergence-free. The reason we choose to keep only the expected transport velocity divergence-free and allow the fluctuation field to be compressible stems from the desire to preserve the Lie–Poisson Hamiltonian structure of the equations. This point will be detailed in Section 2.

The corresponding Itô form of equation (1.3) is

\[
du_t \cdot dx + P^E \mathcal{L}_{d\hat{X}_t}(u_t \cdot dx) - \frac{1}{2} \sum_k P^E \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}(u_t \cdot dx)) = 0 ,
\]

with Itô vector field \( d\hat{X}_t \) given by

\[
d\hat{X}_t(x) = \mathbb{E}[u^L_t](x)dt + \sum_k \xi^{(k)}(x)dW^{(k)}_t .
\]

These equations are “nonlinear in the sense of McKean” \[11, 2, 13\] when the drift term involves the expected value of the flow it drives, as in the pair of stochastic differential equations for the stochastic Euler fluid in (1.3) and (1.3). This perspective is adopted by \[13\] in the special case of incompressible fluid and is used as a route towards obtaining a representation theorem for solutions of the deterministic incompressible Navier-Stokes equations.

The expectation of the Itô form of the Euler fluid motion equation (1.3) yields a motion equation with additional terms. This is the Navier–Stokes equation with Lie–Laplacian ‘dissipation’ \[12\],

\[
\partial_t \mathbb{E}[u_t] \cdot dx + P \mathbb{E}[\mathcal{L}_{u^L_t}](\mathbb{E}[u_t] \cdot dx) - \frac{1}{2} \sum_k P \mathcal{L}_{\xi^{(k)}}(\mathcal{L}_{\xi^{(k)}}(\mathbb{E}[u_t] \cdot dx)) = 0 ,
\]

where \( P \) denotes the Leray projector of the vector coefficients onto their divergence-free part. Note that, in general, the Lie–Laplacian operator is not dissipative, but provided the noise correlates satisfy some minor conditions, it is a uniformly elliptic operator which has the effect of regularizing the solution. This
LLNS equation reduces to the Navier–Stokes equation in a special choice of the noise correlates \( \xi^{(k)} = \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\} \) for \( k = 1, 2, 3 \).

The LA SALT equations (1.3) and (1.3) arose, along with additional noisy and viscous terms, in Lemma 3 of [12] where there were shown to govern the dynamics of the so-called stochastic Weber velocity. Constantin and Iyer [2] also use stochastic fluid motion equations of the type (1.3)–(1.3) with constant \( \xi^{(k)} \), where \( k = 1, 2, 3 \), as a tool to represent solutions to the incompressible Navier-Stokes equations as an average over a stochastic process. In particular, in the special case of constant \( \xi^{(k)} \)'s, they derived the following statistical Kelvin theorem,

\[
v_t = \mathbb{E} u_t, \quad \oint_C v_t \cdot dx = \mathbb{E} \oint_{A_t(C)} v_0 \cdot dx, \tag{1.17}
\]

where \( A_t = \phi^{-1}_{0,t} \) and the \( \phi_{0,t} \) is the flow (1.1) defined implicitly by the solution \( v_t \). For the more general class of equations introduced above, the statement (1.3) holding for all rectifiable loops \( C \) in fact characterizes the solution of the Lie-Laplacian Navier-Stokes equation (1.3), as discussed in Drivas & Holm [12]. See also the discussion of stochastic circulation and Hamiltonian structure of Navier-Stokes in [26].

In summary, the LA SALT Euler fluid equations (1.3) introduce a type of Lagrangian averaging, obtained by taking the expectation of the loop velocity \( u^T_\ell \) in (1.3), which is the velocity of the Lagrangian parcels. The LA SALT loop velocity in the modified Kelvin theorem in (1.2) will also be the transport velocity for advected quantities, when those quantities are included in the dynamics. Thus, we may also derive LA SALT versions of compressible, adiabatic fluid dynamics, magnetohydrodynamics (MHD), etc. As mentioned earlier, taking the expectation of the Lagrangian transport velocity in the Kelvin theorem for LA SALT is analogous to taking the temporal average at fixed Lagrangian coordinate of the transport velocity for the Navier–Stokes equation to obtain the LANS-alpha equation [27, 28, 29, 3, 4]. The corresponding expected-quantity equations produce a Lie-Laplacian version of the Navier-Stokes equation.

1.4. Notation and Setting. Before stating our class of theories, we introduce the bare minimum of geometric/probabilistic framework required to understand and interpret our equations.

Geometric Setting: We will work on a smooth \( d \)-dimensional manifold \( M \). Let \( \mathcal{X}(M) \) denote the space of vector fields on \( M \). We identify the smooth part of the dual of \( \mathcal{X} \) with \( \mathcal{X}^* = \Lambda^1(M) \otimes \text{Dens}(M) \) via the weak non-degenerate \( L^2 \)-pairing

\[
\langle m, u \rangle_{\mathcal{X}} := \int_M \alpha(u) \eta, \quad m = \alpha \otimes \eta \in \mathcal{X}^*, \quad u \in \mathcal{X},
\]

where \( \Lambda^1(M) \) is the space of one-forms on \( M \). If \( M \) has empty-boundary or if \( u, v \in \mathcal{X}(M) \) are tangential to the boundary then

\[
\mathcal{L}_v u = [u, v] = -\text{ad}_v u \quad \text{and} \quad \langle m, \text{ad}_v u \rangle_{\mathcal{X}} = \langle \mathcal{L}_v m, u \rangle_{\mathcal{X}},
\]

and hence that \( \text{ad}_v^* = \mathcal{L}_v \) with respect to the pairing \( \langle \cdot, \cdot \rangle_{\mathcal{X}} \). Let \( V^*(M) \) be a direct-sum of spaces of \( k \)-forms \( \Lambda^k(M) \), tensor-fields \( \tau^{(p,q)}(M) \), or tensor-field densities \( \tau^{(p,q)}(M) \otimes \text{Dens}(M) \). We then let \( V(M) \) be the geometric dual of \( V^*(M) \) with weak non-degenerate pairing \( \langle \cdot, \cdot \rangle_{V} : V(M) \times V^*(M) \to \mathbb{R} \). For example, if \( V^*(M) = \Lambda^k \), then we take \( V(M) = \Lambda^{d-k} \) and define

\[
\langle \alpha, \beta \rangle_{V} = \int_M \beta \wedge \alpha, \quad \alpha \in V, \beta \in V^*.
\]

Moreover, if \( V^*(M) = \tau^{(p,q)}(M) \), then we take \( V(M) = \tau^{(q,p)} \otimes \text{Dens}(M) \) and

\[
\langle \tau \otimes \eta, \tau' \rangle_{V} = \int_M \tau \cdot \tau' \mu, \quad \tau \otimes \eta \in V, \quad \tau' \in V^*.
\]
where $\tau \cdot \tau'$ denotes the contraction of tensors. It follows that there exists a bi-linear pairing $\diamond : V(M) \times V^*(M) \to \mathfrak{X}^*(M)$ such that
\[
\langle b \diamond a, v \rangle_{\mathfrak{X}(M)} := \langle b, -\mathcal{L}_v a \rangle_{V^*}, \quad \forall a \in V^*, \ b \in V, \ v \in \mathfrak{X}.
\] (1.18)

The diamond can be calculated on a case-by-case basis using the definition of the Lie-derivative (and Cartan’s formula), Stoke’s theorem, and the antiderivation properties of the exterior differential and insertion operators. See §2.2 and §4 for concrete examples including LA SALT Euler and MHD. We remark that for $\alpha \otimes \eta \in \mathfrak{X}^*$, we have $\alpha f \otimes \eta = \alpha \otimes f \eta$ for all $f \in \Lambda^0$. Thus, the diamond operation is only unique up this equivalence relation, which is not relevant for the dynamics. Since we specify (1.4) holds for all $v$, the expression for the diamond operation is unique up to the aforementioned equivalence relation.

**Probabilistic Setting:** Let $(\Omega, \mathcal{F}, \mathbb{P} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. Assume the filtered probability space supports a sequence $\{W_t^{(k)}\}_{k \in \mathbb{N}, t \geq 0}$ of $\mathbb{F}$-adapted Wiener process. Let $\{\xi(k)\}_{k \in \mathbb{N}}$ denote a collection of continuously differentiable time-independent, deterministic, and divergence-free vector fields on $M$.

2. **Reduced Euler–Poincaré Lagrangian and Lie–Poisson Hamiltonian structures**

2.1. **Euler–Poincaré and Lie–Poisson forms of the LA SALT equations.** To take these introductory remarks further, we make the replacement (1.2) in the Euler–Poincaré equation from which the SALT Kelvin circulation theorem arose in Stratonovich form, in [1]. That is, we keep the same physical class of reduced Lagrangians $\ell(u^L, a)$ as was treated for the SALT Euler–Poincaré variational principle in [1].

Let $\ell = \ell(u^L, a) : \mathfrak{X} \times V^* \to \mathbb{R}$ be a Lagrangian. We assume that the Lagrangian possesses Gateux derivatives defined in terms of the aforementioned pairings in §1.4: $\frac{\delta \ell}{\delta u^L} \in \mathfrak{X}^*$ and $\frac{\delta \ell}{\delta a} \in \mathfrak{X}^*$. The SALT equations introduced in [1] read
\[
d\frac{\delta \ell}{\delta u^L} + \mathcal{L}_d x_t \frac{\delta \ell}{\delta u^L} = \frac{\delta \ell}{\delta a} \circ a \ dt \quad \text{and} \quad d a + \mathcal{L}_d x_t a V^* = 0,
\] (2.1)

where $d x_t$ is the stochastic transport vector field (1.1). In this paper, we replace the stochastic transport vector field $d X_t$ to be given by (1.1) and consider
\[
d\frac{\delta \ell}{\delta u^L} + \mathcal{L}_d x_t \frac{\delta \ell}{\delta u^L} = \mathbb{E} \left[ \frac{\delta \ell}{\delta a} \right] \circ a \ dt \quad \text{and} \quad d a + \mathcal{L}_d x_t a V^* = 0,
\] (2.2)

The above equations form the general class of LA SALT theories that we consider.

**Legendre transform to the Hamiltonian side.**

The Legendre transform from the Lagrangian side to the Hamiltonian side for SALT in [1] is given by
\[
\mu = \delta \ell/\delta u^L \quad \text{and} \quad dh(\mu, a) = \langle \mu, d x_t \rangle - \ell(u^L, a) dt,
\]

where $d x_t$ is given in equation (1.1). By introducing $H$, the deterministic Hamiltonian, we may alternatively express the above as
\[
dh(\mu, a) = H(\mu, a) dt + \sum_k \langle \mu, \xi(k) \rangle \circ d W_t^{(k)}.
\]

Note that $\delta h/\delta u^L = 0$ and $\delta h/\delta a = \delta H/\delta a$ by definition. Taking variations of $h(\mu, a)$ yields
\[
\delta h(\mu, a) = \langle \delta \mu , \delta h/\delta \mu \rangle + \langle \delta h/\delta a , \delta a \rangle
\]
\[
= \langle \delta \mu , d x_t \rangle - \langle \delta h/\delta a , \delta a \rangle + \langle \mu - \delta \ell/\delta u^L , \delta u^L \rangle.
\]
Then, upon identifying corresponding terms, one verifies the fibre derivative $\mu = \delta \ell/\delta u^k$ and find the following variational derivatives of the SALT Hamiltonian,

$$
\frac{d\delta h}{\delta \mu} = dx_t = \frac{\delta H}{\delta \mu} dt + \sum_k \xi^{(k)} \circ dW_t^{(k)} \quad \text{and} \quad \frac{\delta H}{\delta a} = -\frac{\delta \ell}{\delta a}.
$$

(2.3)

At this point, we take expectations of the terms in (2.1) which pass from the SALT equations in (2.1) to the LA SALT equations in (2.1). Correspondingly, we set,

$$
d\frac{\delta h}{\delta \mu} = dX_t = E\left[\frac{\delta H}{\delta \mu}\right] dt + \sum_k \xi^{(k)} \circ dW_t^{(k)} \quad \text{and} \quad E\left[\frac{\delta H}{\delta a}\right] = -E\left[\frac{\delta \ell}{\delta a}\right].
$$

Taking these expectations transforms the LA SALT equations (2.1) in Euler–Poincaré into Hamiltonian form with a Lie–Poisson matrix operator. Thus, the Stratonovich version of the SDP-LPB for the SALT Hamiltonian formulation yields the LA SALT equations in (2.1) as

$$
d\begin{bmatrix} \mu \\ a \end{bmatrix} = - \left[ \begin{array}{cc} \text{ad}^*_L(\cdot) \mu & (\cdot) \circ a \\ L(\cdot) a & 0 \end{array} \right] \left[ E[\delta H/\delta \mu] dt + \sum_k \xi^{(k)} \circ dW_t^{(k)} \right] + \left[ E[\delta H/\delta a] dt \right].
$$

(2.4)

The definition of the diamond operator (\circ) (1.4) ensures that the Lie–Poisson matrix operator is skew-symmetric in $L^2$ pairing under integration by parts. The modification of the Hamiltonian form of the SALT equations to obtain the LA SALT equations in (2.1) replaces the variational derivative of the Hamiltonian with respect to momentum and the advected variable by their expected values. This modification preserves the Hamiltonian matrix operator in both the deterministic and SALT formulations and makes that operator available for exploring the solution behaviour for LA SALT, as we discuss below in a combination of theorems and illustrative examples. Essentially, the LA SALT theory is a nonequilibrium stochastic linear response theory for fluctuations in SALT fluids with advected quantities.

**Remark 1.** Note that the expectation of the system (2.1) results in a closed dynamical system, when the expected variational derivatives $\delta H/\delta \mu$ and $\delta H/\delta a$ are linear in the variables $\mu$ and $a$, after constraints (e.g. incompressibility) are accounted for. In principle, regarding SALT as the ‘mother theory’, LA SALT can be regarded as a first-order cumulant discard closure for SALT and therefore characterized as a type of linear response theory, particularly because its dynamics involves both fluctuations and dissipation. We will investigate several examples of this situation in the remainder of the paper.

**Casimirs.** The LA SALT system (2.1) undergoes stochastic coadjoint motion. That is, the LA SALT dynamics of any functional $C[\mu, a]$ is given by

$$
dC[\mu, a] = \int \left[ E[\delta H/\delta \mu] dt + \sum_k \xi^{(k)} \circ dW_t^{(k)} \right] \left[ \text{ad}^*_L(\cdot) \mu \quad (\cdot) \circ a \right] \left[ E[\delta H/\delta a] dt \right] dx.
$$

(2.5)

A functional $C[\mu, a]$ whose variational derivatives $[\delta C/\delta \mu, \delta C/\delta a]^T$ comprise a null eigenvector of the Hamiltonian matrix operator in (2.1) is called a **Casimir functional** for that Lie–Poisson system. Casimir functionals satisfy $dC[\mu, a] = 0$, so that $C[\mu_t, a_t] = C[\mu_0, a_0]$ for any Hamiltonian $H[\mu, a]$. By having preserved the Lie–Poisson structure of the deterministic Hamiltonian fluid equations in formulating the LA SALT system (2.1), one preserves the Casimir conserved quantities for the original deterministic Lie–Poisson fluid dynamics. In turn, one also preserves the expectations of the Casimirs, since for them $E[\hat{C}[\mu_t, a_t]] = E[\hat{C}[\mu_0, a_0]]$.

Thus, equation (2.1) for LA SALT encapsulates all three of the stages in the approximations of fluid dynamics which we have been discussing. The SALT formulation in [1] emerges when the expectations are not taken in the first term of the product in the integrand. The historical deterministic formulation discussed in the Introduction emerges when the $\xi^{(k)}$ also vanish in that term. Since the Casimirs are defined as null
vectors of the same Hamiltonian operator in each case, they persist in all three stages: deterministic, SALT and LA SALT fluid dynamics.

In summary, because the LA SALT modification of the SALT transport vector field preserves the form of the reduced Euler–Poincaré Lagrangian in (2.1) and the Lie–Poisson Hamiltonian operator in (2.1), one retains both the Kelvin circulation theorem and the conservation of Casimirs of the Lie–Poisson bracket in the LA SALT dynamics.

2.2. Example: LA SALT Euler. We now show that the LA SALT Euler equations introduced in the beginning of the paper are a special case of the general class of the Lie–Poisson Hamiltonian systems defined by (2.1).

Consider a $d$-dimensional smooth oriented Riemannian manifold $(M, g)$ with volume form $d\text{vol}_g$. Let $\mathfrak{X}$ denote the space of smooth vector fields on $M$. Let $V^* = \text{Dens}_g$ denote the space of smooth densities that are absolutely continuous with respect to the reference measure $d\text{vol}_g$. For a given smooth scalar function $p \in \Lambda^0$ on $M$, define $\ell : \mathfrak{X} \times V \to \mathbb{R}$ for all $u \in \mathfrak{X}$ and $D = \rho \, d\text{vol}_g$ by

$$
\ell(u, D) = \int_M \left( g(u, u) + p(\rho - 1) \right) \, d\text{vol}_g = \int_M \left( u^b(u) + p(\rho - 1) \right) \, d\text{vol}_g.
$$

We identify the smooth part of the dual of $\mathfrak{X}$ with $\mathfrak{X}^* = \Lambda^1 \otimes \text{Dens}_g$ via the pairing

$$
\langle \mu, u \rangle_{\mathfrak{X}} := \int_M \alpha(u) \rho \, d\text{vol}_g, \quad \mu = \alpha \otimes \rho \, d\text{vol}_g \in \mathfrak{X}^*, \quad u \in \mathfrak{X},
$$

where $\Lambda^1$ is the space of smooth one-forms. Let $V = \Lambda^0$ and define

$$
\langle b, D \rangle_{V} := \int_M b \rho \, d\text{vol}_g, \quad D = \rho \, d\text{vol}_g \in V^*, \quad b \in V.
$$

Then

$$
\mu = \frac{\delta \ell}{\delta u} = u^b \otimes D \in \mathfrak{X}^*, \quad \text{and} \quad \frac{\delta \ell}{\delta D} = -\frac{\delta H}{\delta D} = \frac{1}{2} u^b(u) - p \in V,
$$

and $\delta H/\delta \mu = u$. Using Cartan’s formula for the Lie derivative, Stoke’s theorem (ignoring boundary contributions by either imposing $\partial M = \emptyset$ or tangential boundary condition), and that both insertion and the exterior differential are antiderivations we obtain

$$
\langle b, -\mathcal{L}_u D \rangle_V = -\int_M b \mathcal{L}_u (\rho \, d\text{vol}_g) = -\int_M b \mathcal{L}_u (\rho \, d\text{vol}_g) = -\int_M \mathcal{L}_u (b \rho \, d\text{vol}_g) = -\int_M (\mathcal{L}_u b) \rho \, d\text{vol}_g = \langle \mathcal{L}_u \otimes D, u \rangle_{\mathfrak{X}},
$$

which implies that $b \circ D = -\mathcal{L}_u \otimes D \in \mathfrak{X}^*$. It follows that

$$
\mathbb{E} \left[ \frac{\delta \ell}{\delta D} \right] \circ D = \mathbb{E} \left[ \frac{1}{2} u^b(u) - \mathcal{L}_u \right] \otimes D.
$$

The LA SALT equations (obtained by substituting into Eq. (2.1)) in this case then read

$$
\mathcal{L}_u \otimes D + \mathcal{L}_{\mathcal{X}_i} \left( u^b \otimes D \right) = \mathbb{E} \left[ \frac{1}{2} u^b(u) - \mathcal{L}_u \right] \otimes D \; dt \quad \text{and} \quad \mathcal{L}_u + \mathcal{L}_{\mathcal{X}_i} D = 0.
$$

Recalling $D = \rho \, d\text{vol}_g \in V$ and simplifying by applying standard properties of the Lie derivative:

$$
du^b + \mathcal{L}_{\mathcal{X}_i} u^b = \frac{1}{\rho} \mathbb{E} \left[ \frac{1}{2} u^b(u) - \mathcal{L}_u \right] \; dt \quad \text{and} \quad \partial \rho + \mathcal{L}_{\mathcal{X}_i} \rho + \mathcal{L}(\mathcal{X}_i) \rho = 0.
$$
Imposing that $\rho \equiv 1$ (which arises from $\delta \ell / \delta p = (\rho - 1) \, d\text{vol}_g = 0$) and $\text{div} \, \xi^{(k)} \equiv 0$ for all $k$, and setting $\tilde{p} := \frac{1}{2} u^b(u) - p$, we arrive at the LA SALT Euler equations on $M$

\[ \text{d} u^b + \mathcal{L}_{\mathbb{E}[u]} u^b \, \text{d} t + \sum_k \mathcal{L}_{\xi^{(k)}} u^b \circ dW_t^{(k)} = \mathbb{E} [ d\tilde{p} ] \, \text{d} t \]

\[ \delta \mathbb{E} \left[ u^b \right] = 0, \tag{2.6} \]

where $\delta : \Lambda^1 \rightarrow \Lambda^0$ is the codifferential operator. Let $\Delta_H = d\delta + \delta d$ be the Hodge Laplacian and $P_H$ be the projection onto the Harmonic forms. The Hodge decomposition (e.g., Ch. 3 of [32]) for one-forms $\alpha \in \Lambda^1$ gives

\[ \alpha = P\alpha + Q\alpha + P_H\alpha \]

where $\delta P = 0$, $dQ = 0$ and $\Delta_H P_H = 0$ (abusing notation slightly in that we write the same projection operators for one-forms and vector fields). Let $P^E = I - Q \mathbb{E}$. We can then rewrite (2.2) as

\[ \text{d} u^b + P^E \mathcal{L}_{\mathbb{E}[u]} u^b \, \text{d} t + \sum_k P^E \mathcal{L}_{\xi^{(k)}} u^b \circ dW_t^{(k)} = 0. \]

These are equivalent to Eqn. (1.3) of the Introduction, where $u^b = u \cdot dx$.

### 2.3. Simplifications in the LA SALT equations for expected physical variables.

Significant simplifications occur when the drift velocity of SALT is replaced by its expectation in LA SALT. Indeed, converting (2.1) to Itô-form, we find

\[ \text{d} \mu + \mathcal{L}_{\mathbb{E} \left[ \frac{\delta H}{\delta \mu} \right]} \mu \, \text{d} t + \mathcal{L}_{\xi^{(k)}} \mu \, dW_t^{(k)} = \left( \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)}} (\mathcal{L}_{\xi^{(k)}} \mu) \, \text{d} t - \mathbb{E} \left[ \frac{\delta H}{\delta \mu} \right] \circ \mu \right) \, \text{d} t \]

\[ \text{d} a + \mathcal{L}_{\mathbb{E} \left[ \frac{\delta H}{\delta a} \right]} a \, \text{d} t + \mathcal{L}_{\xi^{(k)}} a \, dW_t^{(k)} = \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)}} (\mathcal{L}_{\xi^{(k)}} a) \, \text{d} t. \]

Applying the expectation to (2.3), we find

\[ \partial_t \mathbb{E} \left[ \mu \right] + \mathcal{L}_{\mathbb{E} \left[ \frac{\delta H}{\delta \mu} \right]} \mathbb{E} \left[ \mu \right] - \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)}} (\mathcal{L}_{\xi^{(k)}} \mathbb{E} \left[ \mu \right]) = - \mathbb{E} \left[ \frac{\delta H}{\delta \mu} \right] \circ \mathbb{E} \left[ a \right], \]

\[ \partial_t \mathbb{E} \left[ a \right] + \mathcal{L}_{\mathbb{E} \left[ \frac{\delta H}{\delta a} \right]} \mathbb{E} \left[ a \right] - \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)}} (\mathcal{L}_{\xi^{(k)}} \mathbb{E} \left[ a \right]) = 0. \]

These equations provide the history of the expectations $\mathbb{E} \left[ \mu \right]$ and $\mathbb{E} \left[ a \right]$ throughout the duration of the flow and the equations (2.1) (equivalently, (2.3)) are slaved to the expectations $\mathbb{E} \left[ \mu \right]$ and $\mathbb{E} \left[ a \right]$ as linear stochastic transport relations, for example, when the variations $\delta H / \delta \mu$ and $\delta H / \delta a$ are linear in $\mu$ and $a$. We refer the reader to the numerous examples in Section 4 for which this is the case, particularly when the flow is divergence free. Thus, introduction of nonlocality in the sense of McKean [11] in the LA SALT equations (2.1) significantly simplifies stochastic fluid dynamics in two ways. First, it preserves the differential structure and form of the nonlinear deterministic fluid motion and advection equations that results in promotion of Lagrangian conservation laws to our setting. Second, it introduces linear equations for the fluctuations (in many special cases, including incompressible Euler), which are stochastically driven while being transported by the expectation velocity and accelerated by forces involving expectations.

### 2.4. Fluctuation variance dynamics.

In this section, we will discuss the fluctuations of (2.1) about their average (i.e., expectation). We first define the fluctuation variables as

\[ \mu' := \mu - \mathbb{E} [ \mu ] \in \mathcal{X}^* , \quad a' := a - \mathbb{E} [a] \in V^*. \]
The dynamics of the fluctuations can be obtained by taking the difference between the Stratonovich-formulation (2.1) and (2.3)

\[
d\mu' + \mathcal{L}_{\mathcal{E}}[\mu'] d\mu + \mathcal{L}_{\mathcal{E}(k)}\mu \circ dW_t^{(k)} = \left( \frac{1}{2} \sum_k \mathcal{L}_{\mathcal{E}(k)}(\mathcal{L}_{\mathcal{E}(k)}[\mu]) - \mathbb{E} \left[ \frac{\delta H}{\delta a} \right] \circ a' \right) dt,
\]

or the difference between the Itô-formulation (2.3) and (2.3):

\[
d\mu' + \mathcal{L}_{\mathcal{E}}[\mu'] d\mu + \mathcal{L}_{\mathcal{E}(k)}\mu \circ dW_t^{(k)} = \left( \frac{1}{2} \sum_k \mathcal{L}_{\mathcal{E}(k)}(\mathcal{L}_{\mathcal{E}(k)}[\mu]) - \mathbb{E} \left[ \frac{\delta H}{\delta a} \right] \circ a' \right) dt.
\]

(2.9)

The Itô-formulation of the dynamics are simpler to work with for the purposes of this section.

In this section, we work with a Riemannian manifold \((M, g)\) and the associated density \(d\text{vol}_g\). The metric \(g\) induces an isomorphism between the fluctuation variables \((\mu', a') \in \mathcal{X}(M) \times V^*(M)\) and their geometric dual variables \((\hat{\mu}', \hat{a}') \in \mathcal{X}(M) \times V(M);\) we set

\[
|\mu'|_X^2 = \langle \mu', \mu' \rangle_X \quad \text{and} \quad |a'|_V^2 = \langle \hat{a}', a' \rangle_V.
\]

We note that for all \(\mu_1, \mu_2 \in \mathcal{X}(M), a_1, a_2 \in V^*(M),\)

\[
\langle \mu_1, \mu_2 \rangle_X = \langle \mu_2, \mu_1 \rangle_X \quad \text{and} \quad \langle a_1, a_2 \rangle_V = \langle a_2, a_1 \rangle_V.
\]

First, we will derive the pointwise fluctuation variances. In order to write down pointwise fluctuation variances, we need to work with duality-pairings fiberwise (at a base point \(x \in M\)). Assume that any densities involved in the definition \(\mathcal{X}(M), V^*(M)\) and \(V(M)\) are absolutely continuous with respect to \(d\text{vol}_g\). Then using the density \(d\text{vol}_g\) to reduce any tensor densities, we can define pairings for each \(x \in M\) denoted by

\[
\langle \cdot, \cdot \rangle_{X_x} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{V_x} : V(M) \times V^*(M) \to \mathbb{R}
\]

such that for all \(\mu_1, \mu_2 \in \mathcal{X}(M)\) and \(a_1, a_2 \in V^*(M),\)

\[
\langle \mu_1, \mu_2 \rangle_{X_x} = \int_M \langle \mu_1, \mu_2 \rangle_{X_x} \, d\text{vol}_g \quad \text{and} \quad \langle \hat{a}_1, a_2 \rangle_{V_x} = \int_M \langle \hat{a}_1, a_2 \rangle_{V_x} \, d\text{vol}_g.
\]

(2.11)

We emphasize that \(\langle \cdot, \cdot \rangle_{X_x}\) and \(\langle \cdot, \cdot \rangle_{V_x}\) involve i) an implicit reduction of density variables using \(d\text{vol}_g\), ii) a restriction to the fiber above \(x, \) and iii) fiberwise dual-variables, denoted also by \(\hat{\cdot}\). We do this to keep notation to a minimum so as to not have to change the operators appearing in (2.4), which of course may contain Lie derivatives of tensor-densities. Lastly, we note that for all \(\mu_1, \mu_2 \in \mathcal{X}(M), a_1, a_2 \in V^*(M)\) and \(x \in M,\)

\[
\langle \mu_1, \mu_2 \rangle_{X_x} = \langle \mu_2, \mu_1 \rangle_{X_x} \quad \text{and} \quad \langle a_1, \hat{a}_2 \rangle_{V_x} = \langle a_2, \hat{a}_1 \rangle_{V_x}.
\]

(2.12)
Applying the Itô-product rule and using (2.4), for each $x \in M$, we find

$$
\frac{1}{2} \frac{d}{dt} |\mu(t)|^2 \big|_x + \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta \mu} \right] \mu(t) , \mu(t) \right\rangle \big|_x dt + \sum_k \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta {\mu}^k} \right] \mu(t) , {\mu}^k(t) \right\rangle \big|_x dW^k_t + \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \circ a(t) , {\mu}^k(t) \right\rangle \big|_x dt
\]

$$
= \frac{1}{2} \sum_k \left( \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta {\mu}^k} \right] \mathbb{E}\left[ \frac{\delta H}{\delta {\mu}^k} \right] \right\rangle _x + \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta {\mu}^k} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle _x \right) dt,
$$

(2.13)

$$
\frac{1}{2} \frac{d}{dt} |a(t)|^2 \big|_V + \left\langle a(t) , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle \big|_V dt + \sum_k \left\langle a(t) , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle \big|_V dW^k_t
\]

$$
= \frac{1}{2} \sum_k \left( \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle _V + \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle _V \right) dt.
$$

We recall the definition of diamond:

$$
\left\langle b \circ a , v \right\rangle_X := \left\langle b , -\mathcal{L}_a v \right\rangle_X, \quad \forall a \in V^*, \ b \in V, \ v \in X.
$$

Moreover, we note the following properties concerning formal adjoints that can all be obtained by an application of the divergence theorem with appropriate boundary contributions: for all $a \in \mathcal{X}^*(M), \ u, \ v \in \mathcal{X}(M),$

$$
\left\langle \mathcal{L}_u a , v \right\rangle_X = \left\langle \text{ad}_u^* a , v \right\rangle_X = -\left\langle \alpha , \text{ad}_u v \right\rangle_X = -\left\langle \text{ad}_u^* a , v \right\rangle_X = -\left\langle \mathcal{L}_u \alpha , v \right\rangle_X,
$$

where $\text{ad}_u v = -[u, v] = -\mathcal{L}_u v$. Integrating (2.4) and using (2.4) and Stochastic Fubini (see, e.g., [33]), and the previous relations, we find

$$
\frac{1}{2} \frac{d}{dt} |\mu(t)|^2 \big|_X - \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta \mu} \right] \mu(t) \right\rangle \big|_X dt - \sum_k \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta {\mu}^k} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dW^k_t + \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \circ a(t) , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dt
\]

$$
= -\frac{1}{2} \sum_k \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dt,
$$

(2.14)

$$
\frac{1}{2} \frac{d}{dt} |a(t)|^2 \big|_V - \left\langle a(t) \circ a , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dt - \sum_k \left\langle a(t) \circ a , {\xi}^{(k)} \right\rangle_\xi dW^k_t
\]

$$
= -\frac{1}{2} \sum_k \left\langle a(t) \circ a , {\xi}^{(k)} \right\rangle_\xi dt.
$$

We remark here that one could have directly deduced (2.4) from (2.4) under weaker assumptions than classical solutions by appealing to Itô’s formula for the square of the norm (see, [34, 35, 36, 37, 33]). Finally, taking the expectation we find

$$
\frac{1}{2} \frac{d}{dt} \mathbb{E} \left[ |\mu(t)|^2 \right]_X - \mathbb{E} \left[ \frac{\delta H}{\delta \mu} \right] \big|_X + \mathbb{E} \left[ \frac{\delta H}{\delta a} \right] \big|_X \mathbb{E} \left[ \frac{\delta H}{\delta a} \right] \big|_X dt
\]

$$
= -\frac{1}{2} \sum_k \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dt,
$$

(2.15)

$$
\frac{1}{2} \frac{d}{dt} \mathbb{E} \left[ |a(t)|^2 \right]_V - \mathbb{E} \left[ \frac{\delta H}{\delta a} \right] \big|_V \frac{\delta H}{\delta a} \big|_V dt = -\frac{1}{2} \sum_k \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi \left\langle \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] , \mathbb{E}\left[ \frac{\delta H}{\delta a} \right] \right\rangle_\xi dt.
$$

Thus, we can see that the dynamics of the variances of the stochastic system (2.4) is driven by an intricate variety of correlations among the evolving fluctuation variables. Consequences of these very general equations can be more easily seen in examples. We begin with the vorticity dynamics in LA-SALT Euler.
2.5. LA SALT Euler vorticity dynamics. Let \( \omega = du^b \in \Lambda^2 \) be the vorticity two-form. Applying the exterior derivative \( d \) to (2.2) and using that it commutes with the Lie derivative, we obtain
\[
d\omega + \mathcal{L}_{\xi} \omega = 0,
\]
where, in local coordinates \( x = (x^1, \ldots, x^d) \) the Lie derivative of the two-form \( \omega = \omega_{ij} dx^i \wedge dx^j \) along a vector field \( v \) is given by
\[
\mathcal{L}_v \omega = (v^k \partial_x \omega_{ij} + (\partial_x v^k) \omega_{kj} + (\partial_x v^k) \omega_{ik}) dx^i \wedge dx^j \in \Lambda^2.
\]
Stated succinctly, the vorticity satisfies
\[
d\omega + \mathcal{L}_d \omega = 0.
\]
Thus, \( \omega \) be treated as an advected variable \( a \in \mathcal{V} = \Lambda^2 \). The geometric dual space of \( \mathcal{V} \) is given by \( \mathcal{V} = \Lambda^{d-2} \) with the duality pairing
\[
\langle \alpha, \omega \rangle_{\mathcal{V}} = \int_M \alpha \wedge \omega.
\]
In the notation of the previous section, the dual variable associated with \( \omega \) is \( \tilde{\omega} = \ast \omega \in \Lambda^{d-2} \), where \( \ast \) is the Hodge operator, and hence
\[
|\omega|^2_{\mathcal{V}} = \int_M \omega \wedge \ast \omega = \int_M \omega \wedge \ast \omega = \int_M g(\omega, \omega) \, dvol_g = \int_M \omega^2(\omega) \, dvol_g,
\]
where \( \omega^2 \) is a 2-contravariant tensor field. Moreover, adopting the notation of the previous section, we find
\[
\langle \tilde{\omega}, \omega \rangle_x := \omega^2_x(\omega_x) = g_x(\omega_x, \omega_x).
\]
Thus, setting \( a = \omega \) in (2.4), we find
\[
\frac{1}{2} \partial_t \mathbb{E} [\langle \omega^2 \rangle_{\mathcal{V}}] - \mathbb{E} [\ast \omega' \wedge \omega] - \mathbb{E} [\langle \delta H \rangle_{\mathcal{X}}] = -\frac{1}{2} \sum_k \mathbb{E} [\ast \omega' \wedge (\mathcal{L}_{\xi(k)} \omega) + \ast \mathcal{L}_{\xi} \omega \wedge \omega], \xi^{(k)}_{\mathcal{V}}.
\]
Thus, the correlates \( \xi^{(k)} \) therefore play decisive roles in balancing the spatially integrated variances of the vorticity fluctuations.

Three dimensions. In 3D, the vorticity \( \omega \) may be identified with the vector \( \vec{\omega} = \vec{\pi} \ast \omega \in \mathfrak{X} \). Moreover, using the identity \( \vec{\pi} \ast \mathcal{L}_v \vec{\omega} = 0 \) (see, e.g., appendix section A.6. of [38]), we find that \( \vec{\omega} \) is governed by the following system of stochastic partial differential equations:
\[
d\vec{\omega} + \mathcal{L}_{dX_t} \vec{\omega} = d\vec{\omega} + [dX_t, \vec{\omega}] = d\vec{\omega} - ad_{dX_t} \vec{\omega} = 0.
\]
Denote by \( \text{ad}_u^\dagger \) the formal adjoint of \( \text{ad}_u \) in the standard \( L^2 \)-inner product on \( \mathfrak{X} \) given by
\[
(u, v)_{L^2} = \int_M g(u, v) \, dvol_g, \forall u, v \in \mathfrak{X}.
\]
Note the relation \( u, v, w \in \mathfrak{X} \),
\[
(\text{ad}_u^\dagger u, v)_{L^2} = (u, \text{ad}_v^\dagger)_{L^2} = -(u, \text{ad}_u^\dagger w)_{L^2} = -(\text{ad}_u^\dagger u, w)_{L^2}.
\]
As in Section (2.4), we find
\[
\frac{1}{2} \frac{d}{dt} \mathbb{E} [\langle \omega^2 \rangle_{L^2}] + \left( \mathbb{E} [\text{ad}_{u_i}^\dagger \omega], \mathbb{E}[u_i] \right)_{L^2} + \frac{1}{2} \sum_k \left( \mathbb{E} [\text{ad}_{u_i}^\dagger \text{ad}_{\xi(k)}^\dagger \omega] + \mathbb{E} [\text{ad}_{u_i}^\dagger \text{ad}_{\xi(k)}^\dagger \omega], \xi^{(k)}_{L^2} \right)_{L^2} = 0.
\]
We note that using a similar method one could also derive a similar fluctuation relation for \( u \in \mathfrak{X} \) solving (2.2). Although more concrete, the three-dimensional vorticity dynamics is not particularly illuminating for understanding the qualitative behavior of the vorticity. In two dimensions, we can obtain much more information.
Two dimensions. The vorticity in 2D, understood as a scalar, is governed by the transport law
\[ d\omega_t + \mathbb{E}[u_t] \cdot \nabla \omega_t dt + \sum_k \xi^{(k)} \cdot \nabla \omega_t \circ dW_t^{(k)} = 0. \] (2.16)

First we remark that for an arbitrary smooth function \( \phi \), we have
\[ d\phi(\omega_t) + \mathbb{E}[u_t] \cdot \nabla \phi(\omega_t) dt + \sum_k \xi^{(k)} \cdot \nabla \phi(\omega_t) \circ dW_t^{(k)} = 0. \]

Consequently, upon noting that \( \mathbb{E}[u] \) and \( \xi^{(k)} \) are divergence-free, we find that
\[ \int \phi(\omega_t) dx = \int \phi(\omega_0) dx, \] (2.17)
for any differentiable function \( \phi \). In particular, one may choose \( \phi(x) = x^2 \) and find that all of the \( L^p \)-norms of the solution are conserved by the dynamics of equation (2.5).

We now want to investigate the fluctuations of the vorticity:
\[ \int \mathbb{E}[|\omega_t|^2] dx = \int \mathbb{E}[|\omega_0|^2] dx - \int |\mathbb{E}[\omega_t]|^2 dx. \] (2.18)

Let us start with computing the first term \( \int \mathbb{E}[|\omega_t|^2] dx \). Taking \( \phi(x) = x^2 \) in (2.5), we get
\[ \int |\omega_t|^2 dx = \int |\omega_0|^2 dx \implies \int \mathbb{E}[|\omega_t|^2] dx = \int \mathbb{E}[|\omega_0|^2] dx. \] (2.19)

To compute the term \( \int |\mathbb{E}[\omega_t]|^2 dx \), we appeal to the Itô form of the dynamics for \( \omega \), which are given by
\[ d\omega_t + \mathbb{E}[u_t] \cdot \nabla \omega_t dt + \sum_k \xi^{(k)} \cdot \nabla \omega_t dW_t^{(k)} = \frac{1}{2} \sum_k (\xi^{(k)} \cdot \nabla)(\xi^{(k)} \cdot \nabla \mathbb{E}[\omega_t]) dt. \]

Taking the expectation we get
\[ \partial_t \mathbb{E}[\omega_t] + \mathbb{E}[u_t] \cdot \nabla \mathbb{E}[\omega_t] = \frac{1}{2} \sum_k (\xi^{(k)} \cdot \nabla)(\xi^{(k)} \cdot \nabla \mathbb{E}[\omega_t]). \] (2.20)

Then appealing to the fact that \( \mathbb{E}[u] \) and \( \xi^{(k)} \) are divergence-free, we obtain
\[ \int |\mathbb{E}[\omega_t]|^2 dx = \int |\mathbb{E}[\omega_0]|^2 dx - \sum_k \int |\xi^{(k)} \cdot \nabla \mathbb{E}[\omega_t]|^2 dx, \] (2.21)
which means that magnitude \( |\mathbb{E}[\omega]| \) of the expected vorticity will decay to zero in the absence of forcing, provided that \( \{\xi^{(k)}\}_{k \in \mathbb{N}} \) span \( \mathbb{R}^3 \). Therefore, substituting (2.5) and (2.5) into (2.5), we find that fluctuations \( \omega' = \omega - \mathbb{E}[\omega] \) satisfy satisfies
\[ \int \mathbb{E}[|\omega'_t|^2] dx = \int \mathbb{E}[|\omega_0'|^2] dx + \sum_k \int |\xi^{(k)} \cdot \nabla \omega_t|^2 dx, \]
or equivalently
\[ \frac{d}{dt} \int \mathbb{E}[|\omega'_t|^2] dx = \sum_k \int |\xi^{(k)} \cdot \nabla \omega_t|^2 dx. \]

We find that the conserved total enstrophy in (2.5) transforms from the mean into the fluctuations for 2D LA SALT vorticity dynamics. The same phenomenon occurs for the magnitude of the body angular momentum in the finite-dimensional example of rigid-body dynamics, see Section 4.1. On the other hand, the total enstrophy itself is preserved because it is a Casimir for the Lie–Poisson structure of the 2D LA SALT Euler equation, given by
\[ dC(\omega) = -\int \omega \left( \frac{\delta C}{\delta \omega}, \mathbb{E} \left[ \frac{\delta H}{\delta \omega} \right] + \sum_k \xi^{(k)} \circ dW_t^{(k)} \right) dx dy, \] (2.22)
with Jacobian operator \( J(f, h) = f_x h_y - f_y h_x \) just as it is for the deterministic Euler system.

Total enstrophy is the spatial integral of the square of vorticity. It is preserved as a result of being a Casimir of the LA SALT system for the Lie–Poisson bracket expressed in (2.5), so its expectation is preserved by LA SALT. However, the integral of the square of the expectation of vorticity decays exponentially in time, while the vorticity variance increases exponentially in time.

**Remark 2.** One may regard the expected vorticity equations for 2D LA SALT in (2.5) as a dissipative system embedded into a larger conservative system (2.5). From this viewpoint, the interaction dynamics of the two components of the full LA SALT system dissipates the enstrophy of the mean vorticity by converting it into fluctuations, while preserving the mean total enstrophy. This dynamics results because the total (mean plus fluctuation) vorticity field is being linearly transported along the mean velocity in (2.5), while the mean vorticity field is decaying in 2D dissipative motion (2.5). This is the nature of stochastic coadjoint motion for LA SALT, expressed in (2.5). Namely, the Casimirs are preserved by the full LA SALT dynamics, while the equations for the expected dynamics contains dissipative terms.

### 2.6. Helicity preservation in SALT and LA SALT Euler fluid equations.

The LA SALT Euler fluid motion equation in Stratonovich form (1.3) and its spatial differential may be written together as

\[
(d + \mathcal{L}_{\xi X_i})(u_t \cdot dx) = -dp, \quad \text{and} \quad (d + \mathcal{L}_{\xi X_i})(\omega_t \cdot dS) = 0, \tag{2.23}
\]

where \( d(u_t \cdot dx) = \omega_t \cdot dS \) is the vorticity flux (a 2-form), \( \omega_t := \text{curl} u_t \) and \( p \) is a differentiable function. Since the spatial differential \( d \) commutes with the Lie derivative and satisfies \( d^2 = 0 \), and the advection operator \( (d + \mathcal{L}_{\xi X_i}) \) obeys the product rule, we have

\[
(d + \mathcal{L}_{\xi X_i})(u_t \cdot dx) \wedge (\omega_t \cdot dS) = -dp \wedge (\omega_t \cdot dS) = -d(p \omega_t \cdot dS). \tag{2.24}
\]

Here \( \wedge \) is the wedge product of differential forms, which satisfies \( dx \wedge dS = dV \), where \( dx \) is the line element, \( dS \) is the surface element and \( dV = d^3x \) is the volume element in \( \mathbb{R}^3 \). Moreover, since the wedge product is antisymmetric, one has \( (u_t \cdot dx) \wedge (\omega_t \cdot dS) = (u_t \cdot \text{curl} u_t) d^3x = : \Lambda x^3 \) and \( \mathcal{L}_{\xi X_i}(\Lambda x^3) = \text{div}(\Lambda \mathcal{X}_i) d^3x \). Hence, equation (2.6) in coordinates reads

\[
d(\Lambda x^3) = -\text{div}(\Lambda x_t + p \omega_t) d^3x.
\]

Under integration over the spatial domain of the flow, this formula becomes

\[
d \int_D u_t \cdot \text{curl} u_t d^3x = - \int_D \text{div}(\Lambda x_t + p \omega_t) d^3x = - \int_{\partial D} (\Lambda x_t + p \omega_t) \cdot dS. \tag{2.25}
\]

Consequently, for either vanishing or periodic boundary conditions on \( \partial D \), the integral at the left side of equation (2.6) is preserved by any of the deterministic, SALT, or LA SALT Euler fluid equations. This integral quantity is known as the helicity. Its topological significance as the linkage number for lines of vorticity in a volume preserving fluid flow is discussed by Arnold in [30, 31]. Its preservation for SALT and LA SALT emphasises once again the central role played by the Kelvin circulation integral in fluid dynamics.

### 3. Analytical results for LA SALT Euler

In this section, we fix \( M \) to be the flat torus \( T^d = \mathbb{R}^d/\mathbb{Z}^d \). It is possible to generalize all these results to hold on compact smooth Reimannian manifolds \((M, g)\) without boundary.

Fix a terminal time \( T > 0 \) and \( d \in \{2, 3\} \). For a given forcing term \( f : [0, T] \times T^d \to \mathbb{R}^d \) and \( \mathcal{F}_0 \)-measurable initial condition \( u_0 : \Omega \times [0, T] \times T^d \to \mathbb{R}^d \), we consider the equation for the \( \mathcal{F} \)-adapted vector field \( u : \Omega \times [0, T] \times T^d \to \mathbb{R}^d \) and scalar pressure \( p : \Omega \times [0, T] \times T^d \to \mathbb{R}^d \) satisfying

\[
\begin{align*}
\text{d}u_t + T_{E[u_t]} u_t dt + \sum_k T_{\xi(k)} u_t \circ dW_t^{(k)} &= (-E[\nabla p_t] + f_t) dt, \\
\text{div} E[u_t] &= 0, \\
u_t|_{t=0} &= u_0,
\end{align*}
\tag{3.1}
\]
where $\mathcal{L}_v^T = \text{ad}_v^\dagger = (\mathcal{L}_v u^\dagger)^\dagger = (\text{ad}_v u^\dagger)^\dagger$ is defined by

$$\mathcal{L}_v^T u_t := v \cdot \nabla u_t + (\nabla v)^T \cdot u_t,$$

or, more explicitly, as $(\mathcal{L}_v^T u_t)^j := v^i \partial_j u^i + (\partial_i v^j) u^i$. We interpret (3) via the Itô-formulation:

$$du_t + \mathcal{L}_v^T u_t dt + \sum_k \mathcal{L}_v^T \xi_{(k)}(u_t) dW_t^{(k)} = \left( \frac{1}{2} \sum_k \mathcal{L}_v^T \xi_{(k)}(\mathcal{L}_v^T u_t) - \mathbb{E}[\nabla p_t] + f_t \right) dt. \quad (3.2)$$

We note that

$$\mathcal{L}_v^T(\mathcal{L}_v^T v) = (\xi \cdot \nabla) \xi \cdot \nabla v + (\nabla \nabla) v = \partial_i (a^{ij} \partial_j u^i) + b^{i\alpha j} \partial_i u^j + c^\alpha_i u^i,$$

where

$$a^{ij} := \xi^i \xi^j, \quad b^{i\alpha j} = 2\xi^i \partial_\alpha \xi^j, \quad c^\alpha_i := (\partial_\alpha \xi^i) \partial_i \xi^\beta + \xi^i \partial_\alpha \xi^\beta,$$

where $i, j, \alpha, \beta \in \{1, \ldots, d\}$ and repeated-indices are summed-over.

**Remark 3** (Pressure). The pressure term on the right-hand-side of (3) arises from the divergence-free condition on the expectation of $u$. Indeed, let $P$ be the Leray projection onto divergence-free vector fields and $Q = I - P$ be the gradient projection. Define $P^E = I - Q^E$. Then (3) can be expressed as

$$du_t + P^E \mathcal{L}_v^T u_t dt + \sum_k P^E \mathcal{L}_v^T \xi_{(k)}(u_t) dW_t^{(k)} = \left( \frac{1}{2} \sum_k P^E \mathcal{L}_v^T \xi_{(k)}(\mathcal{L}_v^T u_t) + P^E f_t \right) dt.$$

Thus, clearly the pressure required to maintain incompressibility of $\mathbb{E}[u_t]$ is deterministic. The reason for treating the pressure to be $F$-adapted a priori rather than deterministic is to maintain the connection with the structure introduced in (2.1). However, for the dynamics, only its expectation can be recovered and it is only this that plays any role. Thus, from here on in we will simply denote $\pi_t = \mathbb{E} [p_t]$.

Taking expectation of (3) yields a closed equation for $v_t = \mathbb{E}[u_t]$ given by

$$\partial_t v + P^E \mathcal{L}_v^T v = \mathbb{P} \frac{1}{2} \sum_k \mathcal{L}_v^T \xi_{(k)}(\mathcal{L}_v^T v) + P^E f_t. \quad (3.3)$$

Equation (3) for $\mathbb{E}[u_t]$ generalizes the classical $d$-dimensional Navier-Stokes equations which appear as a special case when $\xi^{(k)} := \sqrt{2\nu} \epsilon_k$, $k = 1, 2, 3, \ldots, d$ and $\xi^{(k)} := 0$ otherwise. We term these equations (3) the Lie-Laplacian Navier-Stokes equations (LL NS).

For our well-posedness results, we will always assume a non-degeneracy condition and boundedness of the $\xi$’s, which amounts to the Lie-Laplacian being a uniformly elliptic operator:

$$\kappa |y|^2 \leq \frac{1}{2} \sum_k y^i \xi^{(k)}_{i}(x) \xi^{(k)}(x)y^j \leq C|y|^2, \quad \forall \ x, y \in \mathbb{T}^d, \quad (3.4)$$

for some $\kappa, C > 0$. This observation makes (3) essentially a ‘perturbation’ of the usual Navier-Stokes equations in terms of the existence and regularity properties of its solutions, as we will soon see.

**Remark 4.** It is clear that the spatial-mean $\bar{u}_t = \int_{\mathbb{T}^d} u_t dx$ is not conserved in (3) and (3) as a result of the zero-order term $(\nabla v)^T \cdot u_t$ in $\mathcal{L}_v^T u_t$. While we choose to account for the non-zero mean, an alternative option is to re-define the equation with a projection onto mean-free vector fields.

**Definition 1** (Solution of LA SALT). We say that $u$ is a solution of (3) on the interval $[0, T^*]$ if $u$ is a weakly continuous $H^1$-valued $\mathbb{F}$-adapted process such that $u \in L^2_0 L^2_{T}, H^1_\mathbb{F}$ and for all $\phi \in C^\infty_c (\mathbb{T}^d; \mathbb{R}^d)$, $\mathbb{P}$-a.s. for all $t \in [0, T^*]$,

$$(u_t, \phi) = (u_0, \phi) + \int_0^t \left[ -\sum_k (\mathcal{L}_v^T \xi_{(k)} u_s, \mathcal{L}_v^T \xi_{(k)} \phi) + (\mathcal{L}_v^T u_s, u_s + f_s, \phi) - \mathbb{E} [\nabla p_t] + f_t \right] ds - \sum_k \int_0^t (\mathcal{L}_v^T \xi_{(k)} u_s, \phi) dW_t^{(k)},$$
where \((\cdot, \cdot)\) denotes the usual inner product on \(L^2(\mathbb{T}^d; \mathbb{R}^d)\).

Our analytical results are as follows.

**Theorem 1** (Well-posedness of LA SALT). Let \(n \geq 1\) and \(m > \frac{d}{2} + n + 1\). Assume that \(\mathbb{E}[u_0] \in H^m\), \(u_0 \in L^2_0H^m\), \(\xi \in C^{m+2}(\mathbb{T}^d; \mathbb{R}^d)\), \(f \in L^2_0H^{m-1}\) and \(\kappa > 0\). Then there exists a time \(T^*\) depending only on \(d, n, \kappa^{-1}, |v_0|_{H^m}, |f|_{L^2_0H^{m-1}}\) and \(|\xi|_{C^{m+2}}\) and a unique solution \(u\) of LA SALT on \([0, T^*]\) satisfying \(u \in L^\infty_0L^\infty_0H^n\) for all \(q \geq 1\). Furthermore, the solution \(u\) is weakly continuous in \(H^n\) and strongly continuous in \(H^{n+1}\) \(\mathbb{P}\)-a.s. If \(d = 2\), then \(T^* = \infty\). Moreover, for \(d > 2\), there is a \(\kappa^*\) depending only on \(d, n, |v_0|_{H^m}, |f|_{L^2_0H^{m-1}}\) and \(|\xi|_{C^{m+2}}\) such that for all \(\kappa > \kappa^*\), then \(T^* = \infty\).

The proof of Theorem 1 proceeds as follows. We first solve (3) for \(v = \mathbb{E}[u]\), and then solve the linear equation (3) for \(u\). Accordingly, we need a solution theory for the deterministic LL-NS, which we state below.

**Theorem 2** (Well-posedness of LL NS). Let \(d \in \{2, 3\}\) and \(m \geq 1\). Assume \(\kappa > 0\), \(v_0 \in H^m\), \(f \in L^2_0H^{m-1}\) and \(\xi \in C^{m+2}(\mathbb{T}^d; \mathbb{R}^d)\). Then there exists a time \(T^* > 0\) depending only on \(d, m, \kappa^{-1}, |v_0|_{H^m}, |f|_{L^2_0H^{m-1}}\) and \(|\xi|_{C^{m+2}}\) and a unique strong solution of (3) on \([0, T^*]\) satisfying \(v \in C_T, H^m \cap L^2_T, H^{m+1}\). If \(d = 2\), then \(T^* = \infty\) can be taken. Moreover, if \(d > 2\), there is a \(\kappa^*\) depending only on \(d, n, |v_0|_{H^m}, |f|_{L^2_0H^{m-1}}\) and \(|\xi|_{C^{m+2}}\) such that for all \(\kappa > \kappa^*, T^* = \infty\) can be taken.

**Remark 5.** If \(\xi \in C^\infty(\mathbb{T}^d; \ell_2(\mathbb{T}^d))\) and \(f \in C^\infty C_\infty\), then \(v \in C^\infty([0, T^*] \times \mathbb{T}^d)\) for any \(\varepsilon > 0\). See Theorem 7.5 in [39]

**Remark 6** (Method of characteristics solution and representation). Let \(\alpha > 0\). If \(b \in L^\infty_0C^{2+\alpha}_x\) and \(\xi \in C^{3+\alpha}_x\), then there exists a stochastic flow of \(C^{2+\alpha}_x\)-diffeomorphisms \(\phi = \{\phi_{s,t}\}\) satisfying
\[
d\phi_{s,t}(x) = b(\phi_{s,t}(x))dt + \sum_k \xi^{(k)}(\phi_{s,t}(x)) \circ dW_t^{(k)}, \quad \phi_{0,t}(x) = x \in \mathbb{T}^d.
\]
We take \(b = \mathbb{E}[u]\), which is sufficiently regular provided we take \(u_0 \in H^m\) with \(m > d/2 + 2 + \alpha\) in Theorem 2. We denote the spatial inverse of the flow (i.e., the back-to-labels map) by \(A_t = \phi^{-1}_{0,t}\), which satisfies
\[
dA_t(x) = \mathbb{E}[u_t](x) \cdot \nabla A_t(x)dt + \sum_k \xi^{(k)}(x) \cdot \nabla A_t(x) \circ dW_t^{(k)}, \quad A_t(x) = x.
\]

It is easy to verify that \(J^{-1}_t = (\nabla \phi_{0,t})^{-1}\) satisfies
\[
dJ^{-1}_t(x) = -J^{-1}_t(x) \nabla \mathbb{E}[u_t](x)(\phi_t(x))dt - J^{-1}_t(x) \sum_k \nabla \xi^{(k)}(\phi_t(x)) \circ dW_t^{(k)}, \quad J^{-1}_0(x) = I.
\]
Noting that \(J^{-1}_t(A_t) = \nabla A_t\), we have
\[
u_t(x) = (\nabla A_t(x))^T[u_0(A_t(x)) + \Psi_t(A_t(x))], \quad \Psi_t(x) = \int_0^t J^{-1}_s(x) (f_s(\phi_s(x)) + \nabla \pi_s(\phi_s(x))) ds.
\]
In geometric notation, we may write the above as
\[
u_t^g(x) = (\phi_{0,t})^g(\nu_0^g(x) + \psi^g(x)), \quad (3.5)
\]
where \(\psi^g\) is defined in terms of \(f^g\) and \(\mathbb{E}[dp]\), where \(dp \in \Lambda^1\) is exterior differential of \(p\). This representation is closely related to the representation derived for the SALT system without a forcing term that is discussed in Remark 6 of [12]. In fact, in [12], the authors derive a stochastic representation for the solution \(\mathbb{E}[u]\) of (3) akin to the Constantin-Iyer representation [2]:
\[
\mathbb{E}[u_t] = \mathbb{E}[P[(\nabla A_t(x))^T u_0(A_t(x))]]. \quad (3.6)
\]
In fact, in those works, \(u_t\) is identified as a stochastic Weber velocity and the representation (6) can directly be obtained from (6) for the ‘Weber velocity’ by taking expectation.
We now sketch the proofs of these two results.

**Sketch of Proof of Theorem 2.** It follows, for example, from Lemmas 3.6 and 3.7, and the argument on page 3779 of [40] (see, also, Lemma 5.1 in [41]) that there exist a constant $C = C(d, m, \kappa^{-1}, |\xi|_{C^{m+2}})$ such that for all $u \in H^m$,

$$((\mathcal{L}_T^u)^2 u, u)_{H^m} \leq -\kappa |\nabla u|_{H^m}^2 + C|u|_{H^m}^2.$$

The proof then follows from a simple modification of standard arguments, see e.g., Chapters 6 and 7 of [39] or Chapter 5 of [42]. Although we do not state them, the dependence of the local existence time and requisite large viscosity for global existence can all be made explicit in terms of $T$ and the size of data and forcing.

**Sketch of Proof of Theorem 1.** Owing to Theorem 3.3 in [40] (see, also, Theorem 3.1 in [41]), if $u_0 \in H^m, \mathbb{E}u \in L^\infty_T C^{m+1}$, $\xi \in C^{m+2}(\mathbb{T}^d; \ell_2(\mathbb{T}^d))$, and $f, \nabla p \in L^2_T H^n$, there exists a unique solution $u$ of LA SALT on the interval $[0, T)$ such that $u \in L^2_T L^2 H^n$. Moreover, $u$ is weakly continuous in $H^n$ and strongly continuous in $H^{n-1}$. These results are based on the following estimate: there is a constant $C = C(d, m, |\xi|_{C^{m+2}})$ such that for all $u \in H^n$,

$$((\mathcal{L}_T^u)^2 u, u)_{H^n} + |\mathcal{L}_T^u u|_{H^n}^2 \leq C|u|_{H^n}^2. \quad (3.7)$$

By virtue of the Sobolev embedding theorem, if $\mathbb{E}u \in H^m$ for $m > d/2 + n + 1$, then $\mathbb{E}u \in C^{m+1}$. Thus, applying Theorem 2, if $u_0 \in H^m, f \in L^2_T H^{m-1}$, and $\xi \in C^{m+2}$ for $m > d/2 + n + 1$, then $\mathbb{E}u \in L^\infty_T C^{m+1}$.

By taking the divergence of both sides of (3), we obtain the following elliptic PDE for the expected pressure, denoted here as $\pi_t = \mathbb{E}p$:

$$(-\Delta)\pi_t = \text{div} \left( \mathbb{E}u \cdot \nabla \mathbb{E}u - (\mathcal{L}_T^u)(\mathcal{L}_T^{\xi(k)})(\mathbb{E}u) \right).$$

Using standard estimates of elliptic PDE in Sobolev spaces (see, e.g., Thm III 4.1 and 4.2 in [42]), we find

$$|\pi_t|_{H^{m-1}} \leq |\mathbb{E}u \otimes \mathbb{E}u|_{H^{m-1}} + \left| \text{div} \left( (\mathcal{L}_T^{\xi(k)})(\mathcal{L}_T^{\xi(k)})(\mathbb{E}u) \right) \right|_{H^{m-3}}.$$ 

Noting that $m - 1 > n + \frac{d}{2}$, by the Banach-algebra property of $H^{m-1}$ (see, e.g., Lemma 3.4 in [43]), there is a constant $C = C(d, m)$ such that $|\mathbb{E}u \otimes \mathbb{E}u|_{H^{m-1}} \leq C|\mathbb{E}u|_{H^{m-1}}^2$. Moreover, since $\text{div} \mathbb{E}u = 0$, it follows from Lemma 3.6 and the argument on page 3778 of [40] that there is a constant $C = C(d, m, |\xi|_{C^{m+2}})$ such that

$$\left| \text{div} \left( (\mathcal{L}_T^{\xi(k)})^2 \mathbb{E}u \right) \right|_{H^{m-3}} \leq C|\mathbb{E}u|_{H^{m-1}}.$$

Thus, we obtain

$$\int_0^{T^*} |\nabla \pi_t|_{H^n}^2 dt \leq \int_0^{T^*} |\pi_t|_{H^{m-1}}^2 dt \leq C \int_0^{T^*} |\mathbb{E}u|_{H^{m-1}}^2 dt < \infty,$$

which gives $\nabla \mathbb{E}p \in L^2_T H^n$. We now turn our attention to uniqueness. If $u_1, u_2$ are solutions of LA SALT, then $\mathbb{E}u_1, \mathbb{E}u_2$ are necessarily strong solutions of LL-NS. Thus, $\mathbb{E}u_1 = \mathbb{E}u_2$. It then follows from the uniqueness of linear stochastic transport equations that $u_1 = u_2$.

**Remark 7.** In fact, estimates of the form (3) hold for more general tensor fields. Lemma 3.7 along with the argument on page 3779 of [40] (see, also, Lemma 5.1 in [41]) directly imply that for all vector fields $v \in \mathcal{X}(\mathbb{T}^d)$, there is a constant $C = C(d, m, |v|_{C^{m+2}})$ such that for all $q$-contravariant, $p$-contravariant tensor fields $\tau \in \tau(q,p)(\mathbb{T}^d)$, we have

$$(\mathcal{L}_T^v \tau, \tau)_{H^m} + |\mathcal{L}_v \tau|_{H^m}^2 \leq C|\tau|_{H^m}^2. \quad (3.8)$$

Indeed, estimates of this form are derived in [40] and [41] for a first-order differential operator acting on functions $\phi : \mathbb{T}^d \to \mathbb{R}^d$ of the form

$$(\mathcal{L}\phi)^\alpha = v^j \partial_j \phi^\alpha + \lambda^{\alpha j} \phi^j,$$
where \( \xi \in C^{m+1}(T^d; \mathbb{R}^d) \) and \( \lambda \in C^{m+1}(T^d; \mathbb{R}^{d \times d}) \). That is, there is constant \( C = C(d, m, |v|_{C^{m+1}}, \lambda|_{C^{m+1}}) \) such that (7) holds with \( \mathcal{L}_v \) replaced with \( \mathcal{L} \) and \( \tau \) replaced with \( \phi \). One can see that the highest-order part (first-order) of \( \mathcal{L} \) acts diagonally, which is why the Lie-derivative is a particular case. We remark also that estimates of this type were first derived for scalar functions in Lemma 2.1 of [44] (see, also, Lemma 4.3 in [45]). In fact, estimates given in Lemma 5.1 of [41] generalize such estimates to \( L^p \)-Sobolev spaces.

4. Illustrative Examples of LA SALT Systems

4.1. LA SALT rigid body: a finite dimensional example without advection. In this example, we take \( \mathfrak{X} = \mathfrak{so}(3) \cong \mathbb{R}^3 \) and \( \mathfrak{X}^* = \mathfrak{so}^*(3) \cong \mathbb{R}^3 \) with the dot-product pairing (i.e., there is no density component \( \mathfrak{X}^* \)). Upon choosing \( \ell(\Omega) = \frac{1}{2} \Omega^T \cdot I \Omega \), where \( I = \text{diag}(I_1, I_2, I_3) \) is the moment of inertia in the body frame, we find \( \Pi := \frac{\partial \ell}{\partial \Omega} = I \Omega \). By interpreting the Lie-derivative as \( \mathcal{L}_x y = x \times y \)–which is justified from its identification with \( \text{ad}^* \) on the dual Lie algebra \( \mathfrak{so}^*(3) \) of the group \( \text{SO}(3) \) via the aforementioned identification–we obtain that the SALT formulation of the stochastic rotation of a rigid body in \( \mathbb{R}^3 \) is governed by [46]

\[
d\Pi + \left( \Omega \frac{d}{dt} + \sum_k \xi^{(k)} \circ dW_t^{(k)} \right) \times \Pi = 0 ,
\]

where \( \Pi \) is interpreted as the angular momentum vector and \( \Omega = I^{-1} \Pi \) as the angular velocity vector. If we replace \( \Omega \) by the expected angular velocity \( \mathbb{E} [\Omega] \), we get the LA SALT body dynamics

\[
d\Pi + \mathbb{E} [\Omega] \times \Pi \frac{d}{dt} + \sum_k \xi^{(k)} \times \Pi \circ dW_t^{(k)} = 0 . \tag{4.1}
\]

First note that, in either the SALT or LA SALT body dynamics, one has

\[
\frac{d}{dt} |\Pi|^2 = 0 ,
\]

which follows from (4.1) and the vector identity \( a \cdot b \times c = -c \cdot b \times a \). Therefore, the probability distribution for the SALT and LA SALT rigid body lies on a level set of \( |\Pi|^2 \). Thus, these dynamics represent stochastic coadjoint motion on a level set of the Casimir function of the Lie–Poisson bracket.

In the Itô representation, this stochastic motion equation becomes

\[
d\Pi + \left( \mathbb{E} [\Omega] \frac{d}{dt} + \sum_k \xi^{(k)} dW_t^{(k)} \right) \times \Pi = \frac{1}{2} \sum_k \xi^{(k)} \times (\xi^{(k)} \times \Pi) dt .
\]

Taking the expectation yields

\[
\frac{d}{dt} \mathbb{E} [\Pi] + \mathbb{E} [\Omega] \times \mathbb{E} [\Pi] = \frac{1}{2} \sum_k \xi^{(k)} \times (\xi^{(k)} \times \mathbb{E} [\Pi]) . \tag{4.2}
\]

Upon using the identity the same vector identity, we find

\[
\frac{d}{dt} |\mathbb{E} [\Pi]|^2 = - \sum_k |\xi^{(k)} \times \mathbb{E} [\Pi]|^2 dt .
\]

Thus, the magnitude \( |\mathbb{E} [\Pi]| \) of the expected body angular momentum vector will decay to zero in the absence of forcing. This means that \( \mathbb{E} [\Pi] \) itself and \( \mathbb{E} [\Omega] \) will also decay to zero, provided that \( \{\xi^{(k)}\}_{k \in \mathbb{N}} \) span \( \mathbb{R}^3 \). To calculate the fluctuation dynamics of \( \Pi' := \Pi - \mathbb{E} [\Pi] \), we subtract equation (4.1) from equation (4.1) to find

\[
d\Pi' + \mathbb{E} [\Omega] \times \Pi' \frac{d}{dt} + \sum_k \xi^{(k)} \times \Pi dW_t^{(k)} = \frac{1}{2} \sum_k \xi^{(k)} \times (\xi^{(k)} \times \Pi') dt . \tag{4.3}
\]
By Itô product rule we have
\[
\frac{1}{2} d|\Pi'|^2 = \Pi' \cdot d\Pi' + \frac{1}{2} d \langle \Pi', \Pi' \rangle = \Pi' \cdot d\Pi' + \frac{1}{2} \sum_k \left| \xi^{(k)} \times \Pi \right|^2 dt.
\]
Thus, upon taking the dot product of \( \Pi' \) with (4.1), applying the Itô cross-variance formula implies
\[
\frac{1}{2} d|\Pi'|^2 + \Pi' \cdot \sum_k \xi^{(k)} \times \Pi dW^{(k)}_t = \frac{1}{2} \sum_k \left( \Pi' \cdot \xi^{(k)} \times (\xi^{(k)} \times \Pi) + |\xi^{(k)} \times \Pi|^2 \right)
\]
\[
= \frac{1}{2} \sum_k \left( - |\xi^{(k)} \times \Pi'|^2 + |\xi^{(k)} \times \Pi|^2 \right)
\]
\[
= \frac{1}{2} \sum_k \left( 2(\xi^{(k)} \times \mathbb{E}[\Pi]) \cdot (\xi^{(k)} \times \Pi) - |\xi^{(k)} \times \mathbb{E}[\Pi]|^2 \right).
\]
Taking the expectation then yields
\[
\frac{d}{dt} \mathbb{E}[|\Pi'|^2] = \sum_k \left| \xi^{(k)} \times \mathbb{E}[\Pi] \right|^2,
\]
and hence the fluctuations grow as time increases.

In summary, the probability distribution for the LA SALT rigid body is constant on a level set of \( |\Pi|^2 \). This implies that the invariant measure for the motion is supported on the angular momentum sphere. This is the same result as for the SALT rigid body, treated in [46]. On the other hand, in LA SALT the variances grow monotonically and the distribution of angular velocity on the angular-momentum sphere tends to become more diffuse.

4.2. LA SALT Burgers Equation. Choosing \( \ell(u) = \frac{1}{2} \int_{S^1} |u|^2 Dx \), the one dimensional LA SALT Burgers equation reads
\[
du + \mathbb{E}[u_t] \partial_x u dt + \sum_k \xi^{(k)} \partial_x u \circ dW^{(k)}_t = 0.
\]
The SALT Burgers equations (without the expectation on the drift velocity) were studied in [47] and it was shown that shocks form almost surely. On the other hand, for LA SALT solutions stay regular. Indeed, the expectation \( v_t = \mathbb{E}[u_t] \) satisfies
\[
\partial_t v + v \partial_x v = \sum_k \xi^{(k)} \partial_x (\xi^{(k)} \partial_x v)
\]
which is a viscous Burgers equation. Thus, if the \( \xi^{(k)} \) are sufficiently smooth and non-degenerate (3), then the above equation gives rise to a global smooth solution \( v_t \). The full field is then recovered by a linear transport equation, as in the LA SALT Euler case. Thus, Burgers equation provides a clear example of regularization by ‘non-locality’ in probability space. Note also from the transport structure one has the representation
\[
u_t(x) = u_0(A_t(x)), \quad v_t = \mathbb{E}[u_0(A_t(x))]
\]
where \( A_t \) is the back-to-labels map defined in Remark 6. At the level of the mean \( v_t \), the above generalizes the stochastic method of characteristics (Feynman-Kac formula) for the usual Burgers equation [2]. This representation was used in the work [48] to study the limit of vanishing viscosity for viscous Burgers. There, it was shown the Lagrangian trajectories in the zero noise/viscosity limit become non-unique backward in time due to stochastic splitting that occurs at shock points. This phenomenon, known as spontaneous stochasticity, has many implications for understand high-Reynolds number, turbulent physics such as Richardson particle dispersion [49, 50, 51, 52], anomalous dissipation[53, 54, 55, 56], and time-irreversibility [57, 58, 59]. It would be interesting to study the effect of the \( \xi^{(k)} \) functions on the zero-noise
limit of LA-SALT Burgers, as well as their signature on the spontaneously stochastic probability measure on generalized trajectories of the entropy solutions.

4.3. LA SALT Camassa–Holm (CH) equation. Choosing $\ell(u) = \frac{1}{2} \int_{\mathbb{R}} (u^2 + \alpha^2 (\partial_x u)^2) \, dx$ and defining the Lie derivative of a one-form density $m = m(dx)^2$ by a vector field $\xi$ in 1D as

$$\mathcal{L}_\xi m = (\partial_x m + m \partial_x) \xi,$$

the LA SALT CH equation reads

$$dm = -(\partial_x m + m \partial_x) \left( \mathbb{E} \left[ \frac{\delta H}{\delta m} \right] dt + \sum_k \xi^{(k)} \circ dW_t^{(k)} \right)$$

$$= -\mathcal{L}_{K \ast \mathbb{E}[m]} m \, dt - \sum_k \mathcal{L}_{\xi^{(k)} m} \circ dW_t^{(k)}.$$  \hspace{1cm} (4.4)

where $K(x) = \frac{1}{2} \exp(-|x|/\alpha)$ is the Green’s function for the 1D Helmholtz operator $1 - \alpha^2 \partial_x^2$ and where the stochastic CH Hamiltonian is given by

$$dh(m) = H(m) dt + \sum_k \int_{\mathbb{R}} m \xi^{(k)} \circ dW_t^{(k)} \, dx$$

with deterministic part

$$H(m) = \frac{1}{2} \int_{\mathbb{R}} m K \ast m \, dx = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} m(x) K(|x - x'|) m(x') \, dx' \, dx.$$  

The expected velocity $\mathbb{E}[u] = \mathbb{E}[\delta H/\delta m]$ is given in terms of expected momentum $\mathbb{E}[m]$ by $\mathbb{E}[u] = K \ast \mathbb{E}[m]$, with $m = u - \alpha^2 u_{xx}$ and $u = K \ast m$.

As an Itô stochastic transport equation, (4.3) reads

$$dm = -\mathcal{L}_{K \ast \mathbb{E}[m]} m \, dt - \sum_k \mathcal{L}_{\xi^{(k)} m} \circ dW_t^{(k)} + \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)} \left( \mathcal{L}_{\xi^{(k)} m} \right) dt},$$

and its expectation immediately yields the dissipative equation

$$\partial_t \mathbb{E}[m] = -\mathcal{L}_{K \ast \mathbb{E}[m]} \mathbb{E}[m] + \frac{1}{2} \sum_k \mathcal{L}_{\xi^{(k)} \left( \mathcal{L}_{\xi^{(k)} \mathbb{E}[m]} \right).$$

The subsequent calculations for LA SALT CH would follow the path established in section 2 for deriving the dynamics of the fluctuations (2.4) and the spatially integrated variance (2.4) for the LA SALT CH equation. Rather then follow that path here, though, we shall consider the reduction to a finite dimensional system of SDEs which are nonlocal in probability space, arising from the singular momentum map afforded by the Lie–Poisson structure for the LA SALT equation in (4.3) [60].

LA SALT CH Peakons. The Stratonovich version of the LA SALT CH equation (4.3) admits singular solutions for the 1D momentum $m_t$. In particular, these singular solutions can be distributions of momentum on points in the real line. In previous work for the SALT version of the CH equation, the singular solutions (peakons) were found to form with positive probability [61]. The singular peakon solution Ansatz is given by [62]:

$$m_t(x) = \sum_{a=1}^N p_a(t) \delta(x - q_a(t)) \quad \text{and} \quad u_t(x) = \sum_{a=1}^N p_a(t) K(x - q_a(t)),$$  \hspace{1cm} (4.5)
with $K(x) = \frac{1}{2} \exp(-|x|/\alpha)$. Substitution of the peakon solution Ansatz into the LA SALT CH equation (4.3) yields the following closed SDEs for the time-dependent parameters $q_a(t)$ and $p_a(t)$,

$$
\begin{align*}
\frac{d q_a}{dt} &= \mathbb{E} \left[ u_t(x) \right]_{x=q_a} + \sum_k \xi^{(k)}(q_a) \circ dW^{(k)}_t, \\
\frac{d p_a}{dt} &= -p_a \mathbb{E} \left[ \frac{\partial u_t(x)}{\partial x} \right]_{x=q_a} + p_a \sum_k \frac{\partial \xi^{(k)}(x)}{\partial x} \circ dW^{(k)}_t,
\end{align*}
$$

(4.6)

with $u_t(x)$ given in (4.3).

**Remark 8.** Although peakons have been shown to emerge in the initial value problem for SALT CH with positive probability [63], the issue of whether peakons emerge for the LA SALT CH dynamics in (4.3) from confined initial conditions for velocity $u(x,0)$ remains an open question at this time. However, if the initial condition contains only peakons, then it’s clear from equation (4.3) that they persist, so long as the solution exists for their dynamics governed by the closed system of SDEs in (4.3) for any finite number of peakons. These interesting, but unfamiliar LA SALT CH peakon SDEs have yet to be studied.

### 4.4. Incompressible, vertically stratified LA SALT magnetohydrodynamics (MHD) in 3D.

To formulate a comprehensive example, we consider the LA SALT MHD equations for an incompressible stratified medium moving in three dimensions under constant acceleration of gravity, $g$. The corresponding equations for 3D LA SALT MHD are given by,

$$
\begin{align*}
\frac{d u}{dt} + (dX_t \cdot \nabla) u + u_j \nabla dX^j_t &= -\nabla \left( \mathbb{E} [p] - \frac{1}{2} \mathbb{E} [|u|^2] \right) dt - g \mathbb{E} [b] \hat{z} \, dt \\
&\quad + g \rho \nabla (b - \mathbb{E} [b]) \, dt + \mathbb{E} [J] \times B \, dt, \\
\frac{d b}{dt} + dX_t \cdot \nabla b &= 0, \quad \frac{d B}{dt} - \text{curl}(dX_t \times B) = \frac{dX_t}{dt} \cdot \text{curl} B = 0.
\end{align*}
$$

(4.7)

In these equations, the stratification is measured by buoyancy, $b$. The magnetic field $B$ is divergence free, so that $\text{div} B = 0$. The current density is given in terms of the magnetic field by $J := \text{curl} B$. Finally, the transport velocity is given by the LA SALT Stratonovich stochastic vector field,

$$
\frac{dX_t}{dt} = \mathbb{E} [u] (x) \, dt + \sum_k \xi^{(k)}(x) \circ dW^{(k)}_t.
$$

We note that the constraint $\text{div} B = 0$ is preserved, if it holds initially, which we will assume henceforth. Also, it’s clear that writing the corresponding equations in Itô form would be straightforward, given the amount of previous description above. The physical variables for 3D incompressible MHD are: momentum $\mu \in \Lambda^1 \otimes \text{Den} (\mathbb{R}^3)$, mass density $D \in \text{Den} (\mathbb{R}^3)$, buoyancy $b \in \Lambda^0 (\mathbb{R}^3)$ and magnetic flux $B \in \Lambda^2 (\mathbb{R}^3)$, with components,

$$
\mu = \mu \cdot dx \otimes d^3 x, \quad D = \rho \, d^3 x, \quad b = b, \quad \text{and} \quad B = B \cdot dS.
$$

The Hamiltonian for deterministic 3D incompressible, vertically stratified MHD in terms of the physical variables is

$$
H(\mu, D, B) = \int \frac{1}{2\rho} |\mu|^2 + g \rho b z + \frac{1}{2} |B|^2 + p(\rho - 1) \, d^3 x,
$$

whose variational derivatives are given by

$$
\delta H(\mu, D, B) = \int \frac{\mu}{\rho} \cdot \delta \mu + \left( p - \frac{|\mu|^2}{2\rho^2} + g b z \right) \delta \rho + g \rho \delta b + B \cdot \delta B \, d^3 x,
$$

so that one finds $u = \mu / \rho$. 

The entries in the Hamiltonian operator in equation (4.4) for 3D incompressible vertically stratified MHD are given in the first column by

\[
\mathcal{L}_v(\rho d^3x) = \text{div}(\rho v) d^3x, \quad \mathcal{L}_v(\mu_i dx^i \otimes d^3x) = \left((\partial_j \mu_i + \mu_i \partial_j) v^j\right) dx^i \otimes d^3x
\]

\[
\mathcal{L}_v b = v \cdot \nabla b, \quad \mathcal{L}_v (B \cdot dS) = ( - \text{curl} (v \times B) + v(\text{div} B)) \cdot dS.
\]

Then, by the definition of the diamond operator \(\diamond : V \times V^* \to \mathcal{X}^*(M)\) in terms of the Lie derivative in (1.4), the remaining entries in the first row of the Hamiltonian operator involving diamond \(\langle \cdot \rangle\) are given by

\[
\frac{\delta H}{\delta D} \diamond D = D \nabla \frac{\delta H}{\delta D}, \quad \frac{\delta H}{\delta b} \diamond b = - \frac{\delta H}{\delta b} \nabla b, \quad \frac{\delta H}{\delta B} \diamond B = B \times \text{curl} \frac{\delta H}{\delta B} - \frac{\delta H}{\delta B} \text{div} (B).
\]

These relations are sufficient to develop the dynamical equations (4.4) of the LA SALT 3D incompressible MHD example in Hamiltonian matrix form, as

\[
d \begin{bmatrix} \mu \\ D \\ b \\ B \end{bmatrix} = - \begin{bmatrix}
\mathcal{L}_v(\cdot)\mu & (\cdot) \diamond D & (\cdot) \diamond b & (\cdot) \diamond B \\
\mathcal{L}_v(\cdot)D & 0 & 0 & 0 \\
\mathcal{L}_v(\cdot)b & 0 & 0 & 0 \\
\mathcal{L}_v(\cdot)B & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} \mathbb{E}[\delta H/\delta \mu] dt + \sum_k \xi^{(k)} \diamond dW_t^{(k)} \\
\mathbb{E}[\delta H/\delta D] dt \\
\mathbb{E}[\delta H/\delta b] dt \\
\mathbb{E}[\delta H/\delta B] dt \\
\end{bmatrix}.
\]

The Casimirs for 3D vertically stratified incompressible MHD are

\[
C[b, B] = \int_D \rho \Phi(b, \rho^{-1} B \cdot \nabla b) d^3x,
\]

for any differentiable function \(\Phi\).

Equations (4.4)–(4.4) deliver the system (4.4) for 3D incompressible, stratified LA SALT MHD in the framework established in section 2 for obtaining the complete dynamics of the expected solutions in equation (2.3), the fluctuations in equation (2.4) and the variances in equation (2.4).

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