TOPOLOGICAL MATTER IN FOUR DIMENSIONS

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ABSTRACT

Topological models involving matter couplings to Donaldson-Witten theory are presented. The construction is carried using both, the topological algebra and its central extension, which arise from the twisting of \( N = 2 \) supersymmetry in four dimensions. The framework in which the construction is based is constituted by the superspace associated to these algebras. The models show new features of topological quantum field theories which could provide either a mechanism for topological symmetry breaking, or the analog of two-dimensional mirror symmetry in four dimensions.

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1. Introduction

Topological matter in two dimensions [1,2,3,4] have shown to be a very interesting framework to formulate [5,6] some problems related to mirror manifolds [7-13]. Some geometrical questions which are difficult to answer for a given Calabi-Yau manifold can be stated as simpler geometrical questions in terms of the corresponding mirror manifold(s). Furthermore, when an $N=2$ Landau-Ginzburg model is known for the mirror manifold(s), those questions can be stated in terms of properties of the corresponding deformed chiral ring whose structure is fixed by the form of the Landau-Ginzburg potential. Topological matter models have the property that out of the complicated structure of $N=2$ superconformal theories they extract the information regarding the geometrical questions which arise in terms of mirror pairs. It turns out that, after twisting $N=2$ supersymmetry [14,2], there are two types of topological matter models [5,4]. These two types are described by two different forms of topological sigma models, which are called A and B models. Given a mirror pair $\mathcal{M}$ and $\mathcal{M}'$, the vacuum expectation values (vev) of observables of the A topological sigma model whose target manifold is $\mathcal{M}$ are related to the ones of the B model whose target manifold is $\mathcal{M}'$ [5]. These vev are much harder to compute for A models than for B models. Thus the simpler computations which can be done for B models translate as answers to difficult geometrical questions stated in terms of computations of vev in the A model. Furthermore, when the Landau-Ginzburg potential associated to the Calabi-Yau manifold is known, the vev for B models can be stated in terms of much simpler computations in the corresponding topological Landau-Ginzburg model [3].

Vafa has recently raised the question [6] of whether or not there exist some kind of mirror phenomena in four dimensions. For example, it would be very interesting if difficult problems as the computation of Donaldson invariants [14] could be stated in much simpler terms, as it happens in two dimensions when a Landau-Ginzburg description is available. The motivation of the work presented in this paper is to study this question from the point of view of twisting $N=2$
supersymmetry. A brief account of part of the results presented here have been reported in [15].

There are two equivalent approaches to understand the existence of two types of topological models in two dimensions. One approach consists of performing two different twists of $N = 2$ supersymmetry since in two dimensions it is possible to twist using any of the two $U(1)$ chiral currents [5]. The second approach is the result of performing one of the twists to each of the two $N = 2$ supersymmetry matter multiplets in two dimensions [4]. These multiplets are defined from chiral and twisted chiral superfields [16]. Of course, the second type of twist, when applied to these multiplets, does not lead to new theories.

In this paper we show that in four dimensions only one type of twist is possible. This suggest that one should study the second approach in order to obtain topological models, namely, one should study the twisting of different $N = 2$ multiplets which describe the same on-shell physics before the twisting. Topological Yang-Mills in four dimensions can be thought as the result of twisting an $N = 2$ supersymmetric vector multiplet [14]. Only one representation of this vector multiplet is known and therefore it seems that in this way no new topological model could be obtained. In this paper we report the results of an exhaustive analysis of other possible formulations of the vector multiplet from a superspace point of view. No new topological model with fields of spin no higher than two has been found.

The situation is rather different when one considers $N = 2$ supersymmetric matter fields. For example, several representations are known for the $N = 2$ hypermultiplet [17,18,19]. This clearly opens a line of investigation. However, this line deviates somehow from the original motivation of the present work, namely, the construction of a theory related to topological Yang-Mills theory which could allow a simpler way to compute Donaldson invariants. In order to keep ourselves within our original goal, we consider in this work topological matter coupled to topological Yang-Mills. This will lead to a generalization of Donaldson invariants which might
well be the ones that could possess features similar to mirror symmetry.

As reported in [15], topological matter coupled to topological Yang-Mills seems to possess unexpected properties. It turns out that some of the resulting models lose some of their topological features. This is an indication that the mirror-like hypothesis in four dimensions, if valid, is more complicated than in two dimensions. In fact, one of the generalizations of Donaldson invariants seems to lead to non-topological quantities. Although it shares many of the properties of Donaldson invariants, it might have a weak dependence on the metric of the four dimensional manifold. This fact connects with one of the most important physical problems in topological quantum field theory, namely, the problem of finding mechanisms to break their topological symmetry. To study a possible mechanism leading to symmetry breaking it seems natural to study couplings of topological Yang-Mills theory to matter multiplets. It is in this context where, indeed, a breaking of the topological symmetry could appear. The matter models presented in this work are topological models. However, when the coupling of these models to topological Yang-Mills is carried out such a property is lost. It turns out that the observables, although share many of the properties as the ones in topological quantum field theories, acquire a dependence on the metric of the four dimensional manifold. These results were briefly reported in [15]. In this work we present a full account of the results reported in [15] in what regards the structure of one particular representation of the $N = 2$ hypermultiplet [17,18], and we present the study of another representation.

The second representation of $N = 2$ supersymmetric matter treated in this paper is based on the relaxed hypermultiplet [19]. Only a truncated version of the theory resulting after the twisting is presented. The model is coupled to topological Yang-Mills and the theory constructed in this way turns out to be a topological quantum field theory. The observables are the same one as the ones in topological Yang-Mills but in this case the observables acquire corrections due to the presence of matter fields. The relation between the two forms of topological matter in four dimensions is discussed in sect. 8.
Let us make a brief summary of how the paper is organized. In sect. 2, the twisting of $N = 2$ supersymmetry in four dimensions is performed obtaining the resulting four dimensional topological algebra. It is shown that the twist is unique up to reversal of orientation. The corresponding topological superspace is constructed. In sect. 3, topological Yang-Mills is constructed in the framework of topological superspace and its uniqueness is discussed. In sect. 4, a central extension of the topological algebra is presented, which is needed since one of the representations of the hypermultiplet chosen in this work possesses a non-vanishing central charge. The rest of the section deals with the construction of the topological matter multiplet associated to the twisted form of the representation of the $N = 2$ hypermultiplet built in [17,18]. In sect. 5, the coupling of the resulting topological matter to topological Yang-Mills is carried out. Section 6 presents a truncated version of the model presented in sect. 5. In sect. 7 the energy-momentum tensors of the models considered in the previous sections are constructed and analyzed. Section 8 presents a topological matter model related to a twisted version of the relaxed hypermultiplet. Finally, in sect. 9 we state our final comments and remarks. An appendix describes the conventions used in this paper.
2. Topological Algebra in 4D

In this section we will analyze the possible twistings of $N = 2$ supersymmetry in four dimensions. Our conclusion is that the twisting procedure is unique (up to orientation reversal).

We will construct the 4D topological algebra by twisting the algebra of $N = 2$ supersymmetry. This is the four-dimensional analogue of the construction presented in [4]. As we will argue the twisting procedure is essentially unique. Our starting point is the algebra of $N = 2$ supersymmetry. We will denote the Poincaré generators by $P_{\alpha\dot{\beta}}, J_{\alpha\beta}, J_{\dot{\alpha}\dot{\beta}}$. For a summary of the index-convention used in this paper see the appendix. Supersymmetry generators are denoted by $Q_{a\alpha}, \overline{Q}^{a}_{\dot{\alpha}}$ while internal $SU(2)$ generators by $T^{b}_{a}$. The $N = 2$ supersymmetry algebra takes the form [20]:

\[
\begin{align*}
\{Q_{a\alpha}, \overline{Q}^{b}_{\dot{\beta}}\} &= \delta^{b}_{a}P_{\alpha\dot{\beta}}, & [J_{\alpha\beta}, J^{\gamma\delta}] &= -\frac{i}{2} \delta^{(\gamma}_{(\alpha} J^{\delta)}_{\beta)}, \\
\{Q_{a\alpha}, Q_{b\dot{\beta}}\} &= 0, & [J_{\dot{\alpha}\dot{\beta}}, J^{\dot{\gamma}\dot{\delta}}] &= -\frac{i}{2} \delta^{(\dot{\gamma}}_{(\dot{\alpha}} J^{\dot{\delta})_{\dot{\beta}}}, \\
[J_{a\beta}, Q_{c\gamma}] &= \frac{i}{2} C_{\gamma}(\alpha) Q_{c\beta}, & [J_{a\beta}, P_{\dot{\gamma}\dot{\delta}}] &= [P_{a\dot{\alpha}}, P_{\beta\dot{\beta}}] = 0, \\
[J_{a\dot{\beta}}, \overline{Q}^{c}_{\dot{\gamma}}] &= \frac{i}{2} C_{\dot{\gamma}}(\dot{\alpha}) \overline{Q}^{c}_{\dot{\beta}}, & [T^{b}_{a}, Q_{c\gamma}] &= -\frac{1}{2} (\delta^{b}_{c} Q_{c\gamma} - \frac{1}{2} \delta^{b}_{a} Q_{c\gamma}), \\
[J_{\dot{a}\dot{\beta}}, P_{\gamma\gamma}] &= \frac{i}{2} C_{\gamma}(\alpha) P_{\beta\gamma}, & [T^{b}_{a}, \overline{Q}^{c}_{\dot{\gamma}}] &= \frac{1}{2} (\delta^{b}_{c} P_{\dot{\gamma}\dot{\delta}} - \frac{1}{2} \delta^{b}_{a} \overline{Q}^{c}_{\dot{\gamma}}), \\
[J_{a\dot{\beta}}, P_{\gamma\gamma}] &= \frac{i}{2} C_{\dot{\gamma}}(\dot{\alpha}) P_{\beta\gamma}. & [T^{b}_{a}, T^{d}_{c}] &= \frac{1}{2} (\delta^{b}_{c} T^{d}_{c} - \delta^{b}_{a} T^{d}_{a}),
\end{align*}
\]

(2.1)

All other (anti)commutators vanish or are found by hermitian conjugation. Notice that we do not consider central charges. The Lorentz generator is symmetric in its two indices. The $SU(2)$ generators $T^{b}_{a}$ satisfy $T^{a}_{a} = 0$, being only three of them independent. To carry out the twisting procedure it is convenient to introduce a matrix $C_{ab}$ and its inverse $C^{ab}$ to raise and lower isospin indices. These matrices are antisymmetric and satisfy,

\[
C_{ab} C^{cd} = \delta^{c}_{a} \delta^{d}_{b} - \delta^{d}_{a} \delta^{c}_{b}.
\]

(2.2)
It allows to redefine the $SU(2)$ generators $T_a^b$ in the more convenient form,

$$T_{ab} = T_a^c C_{cb},$$

(2.3)

which turn out to be symmetric due to the condition $T_a^a = 0$. In terms of the new $SU(2)$ generators the entries of the algebra (2.1) which are modified by this redefinition become

$$[T_{ab}, Q_c] = -\frac{1}{4} C_c(b Q_a)\gamma,$$

$$[T_{ab}, \overline{Q}_c] = \frac{1}{4} \delta_c^{\gamma} (Q_a)\gamma,$$

$$[T_{ab}, T_{cd}] = -\frac{1}{4} C_c(b T_a) (d)\gamma.$$

(2.4)

The $N = 2$ supersymmetry algebra (2.1) possesses an additional $U(1)$ symmetry. This symmetry, whose generator will be denoted by $U$, is such that,

$$[U, Q_{a\alpha}] = Q_{a\alpha},$$

$$[U, \overline{Q}_{\alpha}] = -\overline{Q}_{\alpha},$$

(2.5)

while it acts trivially on the rest on the generators. This symmetry will turn out to be the ghost number symmetry of the twisted theory.

The twisting procedure consists of a redefinition of the Lorentz generators and an identification of the isospin indices as right-handed spin indices relative to the new Lorentz generator:

$$\tilde{J}_{AB} = J_{AB} - 2i T_{AB},$$

(2.6)

where capital letters will denote for the moment new spin indices. The coefficient in (2.6) is uniquely determined by the requirement that $\tilde{J}_{\alpha\beta}$ possess a commutator with itself as the one of a Lorentz generator. The choice $J_{AB} \to J_{\alpha\beta}$ is, however, a matter of convention. The opposite choice ($i.e.$, $J_{AB} \to J_{\dot{\alpha}\dot{\beta}}$) would lead to a mirror image algebra (left-handed $\leftrightarrow$ right-handed) of the one we are about to construct. If we denote the $SU(2)$ group associated to the generator $J_{\alpha\beta}$ ($J_{\dot{\alpha}\dot{\beta}}$) as $SU(2)_L$ ($SU(2)_R$), and the internal $SU(2)$ group as $SU(2)_I$, what we are doing in
(2.6) is to replace $SU(2)_L \times SU(2)_R$ by $SU(2)_L \times SU(2)'_R$, where $SU(2)'_R$ is the diagonal sum of $SU(2)_R$ and $SU(2)_L$. The choice opposite to the one taken in (2.6) would have led to $SU(2)'_L \times SU(2)_R$. Using (2.6) one finds that the following combination of components of the supersymmetry generator transforms as a scalar under $\tilde{J}_{AB}$:

$$[\tilde{J}_{AB}, Q_{++} - Q_{--}] = 0,$$  \hspace{1cm} (2.7)

which leads to the definition of the “scalar” under the new Lorentz transformations:

$$Q = -i(Q_{++} - Q_{--}) = C_{AB}Q^{AB}. \hspace{1cm} (2.8)$$

Furthermore, one finds that

$$\{Q, Q\} = 0,$$  \hspace{1cm} (2.9)

which gives a first indication of the topological structure of the resulting algebra. The choice (2.8) is unique up to a constant, i.e., (2.8) is the unique linear combination of components of $Q_{aa}$ (up to a global factor) such that it behaves as a scalar under the new Lorentz generators (2.6) and satisfies $Q^2 = 0$. The rest of the components of $Q_{aa}$ build a symmetric generator,

$$H_{AB} = Q_{(AB)}. \hspace{1cm} (2.10)$$

Finally, it is straightforward to show that the rest of the SUSY generators, $\overline{Q}_{\dot{\alpha}}^a$, build a generator,

$$G_{A\dot{A}} = C_{BA}\overline{Q}_{\dot{A}}^B, \hspace{1cm} (2.11)$$

which transforms as a vector under the new Lorentz generator (2.6):

$$[\tilde{J}_{AB}, G_{C\dot{C}}] = \frac{i}{2}C_{C(AB)\dot{C}}. \hspace{1cm} (2.12)$$

It is now simple to work out the full form of the topological algebra in terms of its defining generators $Q, H_{\alpha\beta}, G_{a\dot{\alpha}}, J_{\alpha\beta}, J_{\dot{\alpha}\dot{\beta}}, P_{\alpha\dot{\beta}}$ where we have dropped the tilde
from $\tilde{J}_{AB}$ and renamed the spin indices with capital letters by the standard Greek notation. We underline commuting vector indices:

\[ \{Q, Q\} = 0, \quad [\tilde{J}_{\alpha\beta}, H_{\gamma\delta}] = \frac{i}{2} C(\gamma(\alpha H_{\beta})\delta), \]
\[ \{Q, H_{\alpha\beta}\} = 0, \quad [\tilde{J}_{\alpha\beta}, G_{\gamma\delta}] = \frac{i}{2} C\gamma(\alpha G_{\beta}\delta), \]
\[ \{H_{\alpha\beta}, H_{\gamma\delta}\} = 0, \quad [\tilde{J}_{\alpha\beta}, J_{\gamma\delta}] = -\frac{i}{2} C(\gamma(\beta J_{\alpha})\delta), \]
\[ \{Q, G_{\alpha\dot{\beta}}\} = P_{\alpha\dot{\beta}}, \quad [\tilde{J}_{\alpha\beta}, J_{\dot{\gamma}\dot{\delta}}] = 0, \]
\[ \{H_{\alpha\beta}, G_{\gamma\dot{\delta}}\} = C(\alpha|\gamma| P_{\beta\dot{\delta}}, [H_{\dot{\gamma}\dot{\delta}}, J_{\dot{\alpha}\dot{\beta}}] = 0, \]
\[ G_{\alpha\dot{\beta}}, G_{\gamma\dot{\delta}}\} = 0, \quad [J_{\dot{\alpha}\dot{\beta}}, G_{\gamma\dot{\delta}}] = \frac{i}{2} C(\delta(\alpha G_{\gamma\beta}), \]
\[ \{P_{\alpha\dot{\beta}}, G_{\gamma\dot{\delta}}\} = C(\gamma|\beta P_{\alpha\dot{\delta}}, [J_{\dot{\alpha}\dot{\beta}}, J_{\dot{\gamma}\dot{\delta}}] = 0, \]
\[ [P_{\alpha\dot{\beta}}, P_{\gamma\dot{\delta}}] = [G_{\gamma\dot{\delta}}, P_{\alpha\dot{\beta}}] = 0, \quad [J_{\dot{\alpha}\dot{\beta}}, J_{\dot{\gamma}\dot{\delta}}] = \frac{i}{2} C(\gamma(\delta J_{\alpha})\beta), \]
\[ \{Q, J_{\alpha\beta}\} = [Q, J_{\dot{\alpha}\dot{\beta}}] = 0. \]

(2.13)

The essential feature of this algebra, which encodes its topological character, is contained in the anticommutator between $Q$ and $G_{\alpha\dot{\beta}}$. This anticommutator expresses that the translation generator $P_{\alpha\dot{\beta}}$ is $Q$-exact. This is a necessary condition for a theory to have an energy-momentum tensor which is $Q$-exact and therefore topological. Note that a nilpotent self-dual operator $H_{\alpha\beta}$ is present in this algebra. The mirror image algebra, which would have resulted from the choice $J_{AB} \rightarrow J_{\dot{A}\dot{B}}$ in (2.6) would have contained an anti-self-dual operator $H_{\dot{A}\dot{B}}$.

From (2.5), (2.8), (2.10) and (2.11), it turns out that the $U(1)$ charges of the new generators are,

\[ [U, Q] = Q, \quad [U, H_{\alpha\beta}] = H_{\alpha\beta}, \quad [U, G_{\alpha\dot{\beta}}] = -G_{\alpha\dot{\beta}}, \]

while it is zero for the rest of the generators. In the twisted algebra the charges in (2.14) are called ghost numbers.

Our next task is to construct the superspace corresponding to the topological algebra (2.13). Besides the space-time coordinates $x^{\alpha\beta}$ we introduce anticommuting coordinates $\theta^\alpha$, $\theta^{\alpha\beta}$ and $\theta^{\alpha\dot{\beta}}$, associated to the odd generators $Q$, $H_{\alpha\beta}$ and $G_{\alpha\dot{\beta}}$. 


respectively. A point in superspace is therefore labeled by 4+8 quantities \( x^{\alpha \dot{\beta}} \), \( \theta \), \( \theta^{\alpha \beta} \) and \( \theta^{\alpha \dot{\beta}} \). The representation of the operators entering (2.13) in terms of these coordinates is the following:

\[
\begin{align*}
P^{\alpha \dot{\beta}} &= i \frac{\partial}{\partial x^{\alpha \dot{\beta}}}, \\
Q &= \frac{\partial}{\partial \theta} + i \frac{\theta^{\alpha \dot{\beta}}}{2} \frac{\partial}{\partial x^{\alpha \dot{\beta}}}, \\
H^{\alpha \beta} &= \frac{\partial}{\partial \theta^{\alpha \beta}} + i \frac{1}{2} C_{(\alpha | \gamma} \theta^{\gamma \beta)} \frac{\partial}{\partial x^{\alpha \dot{\beta}}}, \\
G^{\alpha \dot{\beta}} &= \frac{\partial}{\partial \theta^{\alpha \dot{\beta}}} + i \frac{\theta}{2} \frac{\partial}{\partial x^{\alpha \dot{\beta}}} - i \frac{1}{2} C_{\alpha \delta} \theta^{\gamma \delta} \frac{\partial}{\partial x^{\gamma \dot{\beta}}}. 
\end{align*}
\]

Superspace covariant derivatives \( D, D^{\alpha \beta} \) and \( D^{\alpha \dot{\beta}} \) are introduced as operators which (anti)commute with \( P^{\alpha \dot{\beta}}, Q, H^{\alpha \beta} \) and \( G^{\alpha \dot{\beta}} \). Their representation in terms of the superspace coordinates is,

\[
\begin{align*}
D &= i \left( \frac{\partial}{\partial \theta} - \frac{i}{2} \theta^{\alpha \dot{\beta}} \frac{\partial}{\partial x^{\alpha \dot{\beta}}} \right), \\
D^{\alpha \beta} &= i \left( \frac{\partial}{\partial \theta^{\alpha \beta}} - \frac{i}{2} C_{(\alpha | \gamma} \theta^{\gamma \beta)} \frac{\partial}{\partial x^{\alpha \dot{\beta}}} \right), \\
D^{\alpha \dot{\beta}} &= i \left( \frac{\partial}{\partial \theta^{\alpha \dot{\beta}}} - \frac{i}{2} \theta \frac{\partial}{\partial x^{\alpha \dot{\beta}}} + \frac{i}{2} C_{\alpha \delta} \theta^{\gamma \delta} \frac{\partial}{\partial x^{\gamma \dot{\beta}}} \right). 
\end{align*}
\]

The prefactors are chosen for later convenience. It is simple to verify that, indeed,

\[
\begin{align*}
\{D, Q\} &= \{D, H^{\alpha \beta}\} = \{D, G^{\alpha \dot{\beta}}\} = 0, \\
\{D^{\alpha \beta}, Q\} &= \{D^{\alpha \beta}, H^{\gamma \delta}\} = \{D^{\alpha \beta}, G^{\gamma \delta}\} = 0, \\
\{D^{\alpha \dot{\beta}}, Q\} &= \{D^{\alpha \dot{\beta}}, H^{\gamma \delta}\} = \{D^{\alpha \dot{\beta}}, G^{\gamma \delta}\} = 0. 
\end{align*}
\]

The algebra of the superspace covariant derivatives turns out to be the following,

\[
\begin{align*}
\{D, D\} &= \{D^{\alpha \beta}, D^{\gamma \delta}\} = \{D, D^{\alpha \beta}\} = \{D^{\alpha \dot{\beta}}, D^{\gamma \delta}\} = 0, \\
\{D, \partial^{\alpha \dot{\beta}}\} &= \{D^{\alpha \beta}, \partial^{\gamma \delta}\} = \{D^{\alpha \dot{\beta}}, \partial^{\gamma \delta}\} = 0, \\
\{D, D^{\alpha \dot{\beta}}\} &= i \partial^{\alpha \dot{\beta}}, \\
\{D^{\alpha \beta}, D^{\gamma \delta}\} &= i C_{(\alpha | \gamma} \partial^{\beta)\delta}. 
\end{align*}
\]
The superspace formulation allows to construct theories which contain all the symmetries present in the topological algebra (2.13). The procedure is first to introduce multiplets and then suitable actions which fix their kinematics and interactions. Matter multiplets are usually constructed by imposing superspace covariant constraints on tensor superfields. These constraints involve the superspace covariant derivatives (2.16), and therefore due to the relations (2.17) their covariance under the topological algebra is guaranteed. Other multiplets as, for example, vector multiplets, are introduced by constructing gauge superspace covariant derivatives and then imposing gauge covariant constraints on their algebra. In the next section we will present in this framework a theory for the vector multiplet which is equivalent to topological Yang-Mills theory as formulated in [14], and we will study other possible sets of constraints which might lead to other models.
3. Topological vector multiplet

Let us consider a gauge group $G$ and connections $A$, $A_{\alpha\beta}$, $A_{\dot{\alpha}\dot{\beta}}$ and $A_{\dot{\alpha}\dot{\beta}}$ which allow to define gauge superspace covariant derivatives:

\[
\nabla = D - iA,
\n\nabla_{\alpha\beta} = D_{\alpha\beta} - iA_{\alpha\beta},
\n\nabla_{\dot{\alpha}\dot{\beta}} = D_{\dot{\alpha}\dot{\beta}} - iA_{\dot{\alpha}\dot{\beta}},
\n\nabla_{\dot{\alpha}\dot{\beta}} = D_{\dot{\alpha}\dot{\beta}} - iA_{\dot{\alpha}\dot{\beta}}.
\]

(3.1)

Notice that $A$, $A_{\alpha\beta}$ and $A_{\dot{\alpha}\dot{\beta}}$ are anticommuting superfields. Often we will use a condensed notation for superspace indices. We will denote by $I$ the set of indices $\ldots, \alpha\beta, \dot{\alpha}\dot{\beta}, \ldots$, by $J$ the set $\ldots, \gamma\delta, \dot{\gamma}\dot{\delta}, \ldots$, etc. Notice that $\ldots$ refers to no index, i.e., $\nabla \equiv \nabla_\ldots$ or $A \equiv A_\ldots$. Using this notation (3.1) takes the condensed form,

\[
\nabla_I = D_I - iA_I.
\]

(3.2)

These gauge superspace covariant derivatives possess the following superspace algebra,

\[
[\nabla_I, \nabla_J] = T_{IJ}^K \nabla_K - iF_{IJ},
\]

(3.3)

where $T_{IJ}^K$ is the superspace torsion in (2.18),

\[
T_{\ldots, \alpha\beta}^{\dot{\gamma}\dot{\delta}} = i\delta_\alpha^\gamma \delta_\beta^\delta, \quad T_{\alpha\beta; \gamma\delta}^{\rho\sigma} = iC_{(\alpha|\gamma} \delta_\beta^{\rho) \delta_\delta^{\sigma}},
\]

(3.4)

while all other components are zero. In (3.3), $F_{IJ}$ are superfield strengths. Constraints on the form of these superfield strengths define the vector multiplet. The
constraints leading to Donaldson-Witten theory are

\begin{align*}
F_{-,-} &= \frac{1}{2} V, \\
F_{-,\alpha\beta} &= 0, \\
F_{\alpha\beta,\gamma\delta} &= \frac{1}{2} C_{\alpha(\gamma} C_{\beta\delta)} V, \\
F_{-,\gamma\delta} &= 0, \\
F_{\alpha\beta,\gamma\delta} &= 0, \\
F_{\alpha\beta,\gamma\delta} &= C_{\alpha\gamma} C_{\beta\delta} W,
\end{align*}

where \( V \) and \( W \) are scalar superfields. From these constraints and (2.14) follow that the superfields \( V \) and \( W \) have ghost numbers 2 and -2 respectively:

\begin{align*}
[U, V] &= 2V, \\
[U, W] &= -2W.
\end{align*}

Once the constraints (3.5) are taken into account the Bianchi identities satisfied by the gauge superspace covariant derivatives (3.1) provide relations among the scalar superfields \( V \) and \( W \) and the gauge superconnections. The analysis of these identities is long and tedious and we just list here its outcome. The resulting relations can be classified in three types. There are linear constraints on the superfields \( V \) and \( W \),

\begin{align*}
\nabla V &= 0, \quad \nabla_{\alpha\beta} V = 0, \quad \nabla_{\alpha\dot{\beta}} W = 0,
\end{align*}

expressions for the superfield strengths which are not in (3.5),

\begin{align*}
F_{\alpha\beta,\gamma\delta} &= \frac{i}{4} \nabla_{\alpha\beta} V, \\
F_{\gamma\delta,\alpha\beta} &= -\frac{i}{4} C_{(\gamma|\alpha} \nabla_{\delta)\beta} V, \\
F_{\alpha\dot{\beta},\gamma\delta} &= \frac{i}{2} C_{\dot{\beta}\dot{\gamma}} [C_{\alpha\gamma} \nabla W + \nabla_{\alpha\gamma} W], \\
F_{\alpha\dot{\beta},\gamma\delta} &= \frac{1}{2} C_{\dot{\beta}\dot{\gamma}} \nabla \nabla_{\alpha\gamma} W + \frac{1}{8} [\nabla_{\alpha\dot{\beta}}, \nabla_{\gamma\delta}] V,
\end{align*}

(3.8)
and, finally, second order constraints among the fields $V$ and $W$,

$$\nabla_\gamma \nabla_\delta \sigma W = \frac{1}{2} C(\gamma|\sigma) C^{\beta\dot{\gamma}} F_{\alpha\beta\gamma\delta} - \frac{1}{8} C(\gamma|\alpha) \nabla_\sigma \nabla_{[\delta ]\dot{\beta}} V. \quad (3.9)$$

The relations (3.7), (3.8) and (3.9), are very important in the analysis of the independent component fields of the theory. They represent the consequences of the relations (3.5) which define the vector-like topological multiplet.

The theory under construction possesses the full symmetry generated by the topological algebra (2.13). Of particular importance are the odd symmetries generated by $Q$, $H_{\alpha\beta}$ and $G_{\alpha\dot{\beta}}$. Let $\Phi$ be a generic superfield. The transformations corresponding to these symmetries take the form,

$$\delta \Phi = i \epsilon Q \Phi,$$

$$\delta' \Phi = i \epsilon^{\alpha\beta} H_{\alpha\beta} \Phi,$$

$$\delta'' \Phi = i \epsilon^{\alpha\dot{\beta}} G_{\alpha\dot{\beta}} \Phi,$$

(3.10)

where $\epsilon$, $\epsilon^{\alpha\beta}$ and $\epsilon^{\alpha\dot{\beta}}$ are scalar, self-dual and vector constant anticommuting parameters. The theory is also invariant under gauge transformations. These take the form,

$$\tilde{\delta}_K A_I = D_I K,$$

(3.11)

where $A_I$ is any of the connections in (3.1) and $K$ an arbitrary scalar superfield.

Superspace actions are defined as integrations over superspace. The full superspace measure has the 8 anticommuting coordinates $\theta$, $\theta^{\alpha\beta}$ and $\theta^{\alpha\dot{\beta}}$ plus the ordinary 4 space-time commuting coordinates. The dimension of this measure is therefore 0 if one associate the standard value $1/2$ to the dimensions of $\theta$, $\theta^{\alpha\beta}$ and $\theta^{\alpha\dot{\beta}}$. This implies that it is not possible to write a suitable action unless one takes only part of the superspace measure [20]. In the untwisted analysis this would correspond to the choice of a chiral measure. In virtue of constraints (3.7) there
exist an essentially unique suitable action with zero ghost number which involves the fields $V$ and $W$ and is $Q$-exact. This action is:

$$S_0 = \int d^4x d^4\theta \text{Tr}(W^2),$$  \hspace{1cm} (3.12)

where $d^4\theta$ denotes the measure built from $\theta$ and $\theta^{\alpha\beta}$: $d\theta d\theta_{11} d\theta_{12} d\theta_{22}$. The presence of $d\theta$ in the measure assures that the action is $Q$-exact. Another suitable superspace action of zero ghost number can be built making use of the part of the full superspace measure not present in (3.12): $\hat{d}^4\theta$: $d\theta_{11} d\theta_{12} d\theta_{21} d\theta_{22}$. Taking into account (3.7) the only non-trivial choice is:

$$S_1 = \int d^4x d^4\hat{\theta} \text{Tr}(V^2).$$  \hspace{1cm} (3.13)

Both, $S_0$ and $S_1$, are invariant under the symmetries generated by the generators (2.13), and turn out to be equivalent. Their Lagrangians differ by a total derivative which is proportional to the integrand of the second Chern class.

Our next task is to formulate the theory in terms of component fields. We will do this projection in a covariant approach taking a Wess-Zumino gauge [20]. The form of the supergauge transformation (3.11) indicates that the odd connections, say $A_{\alpha\beta}$, transform as,

$$\tilde{\delta}_K A_{\alpha\beta} = \frac{\partial}{\partial \theta^{\alpha\beta}} K + ...$$  \hspace{1cm} (3.14)

and therefore the lowest component of the superfield $A_{\alpha\beta}$ can be gauged away algebraically using one of the higher components of $K$. The Wess-Zumino gauge consists of making a gauge choice while projecting into component fields of the type that we have just described for all the components of the odd connections and the higher components of the even connection $A_{\alpha\bar{\beta}}$. Some of these components are expressed via (3.5) and (3.8) in terms of the components of the fields $V$ and $W$. One is left with only the lowest component of the even connection $A_{\alpha\bar{\beta}}$, the gauge invariance corresponding to the lowest component of $K$, and the components of
the fields $V$ and $W$ which are independent after taking into account the constraints (3.7), (3.8) and (3.9). We define these independent components as,

\[
\begin{align*}
W' &= 2^{1/2} \lambda, \\
\nabla W' &= 2^{5/4} \eta, \\
\nabla_{\alpha\beta} W' &= 2^{5/4} \chi_{\alpha\beta}, \\
V' &= 2^{3/2} \phi, \\
\nabla_{\alpha\dot{\beta}} V' &= -i 2^{3/4} \psi_{\alpha\dot{\beta}}, \\
\nabla_{(\alpha\dot{\sigma}} \nabla_{\dot{\beta})} V' &= 8G_{\alpha\beta},
\end{align*}
\]  

The numerical factors introduced in (3.15) are such that the final form of the theory coincides with the one in [14]. In (3.15) | means theta-independent part. Notice that our projection procedure is covariant. All component fields appearing in (3.15) transform in the adjoint representation under gauge transformations with parameter $\kappa = K|$. On the other hand, the only component of the connections left, $A_{\alpha\dot{\beta}}|$, which will be denoted simply as $A_{\alpha\dot{\beta}}$, transforms in the standard way under gauge transformations,

\[
\delta\kappa A_{\alpha\dot{\beta}} = \nabla_{\alpha\dot{\beta}} \kappa.  
\]

Our next task is to compute the symmetry transformations of the component fields generated by $Q$, $H_{\alpha\beta}$ and $G_{\alpha\dot{\beta}}$. To carry this out one must project the transformations (3.10) taking into account that the Wess-Zumino gauge has been chosen. Let us compute as an example of the procedure the transformation of the field $\eta$ in (3.15) under the symmetry generated by $Q$. One finds,

\[
\delta\eta = i\epsilon Q \nabla W' = \epsilon \nabla \nabla W',
\]

where in the last step the fact that we work in a Wess-Zumino gauge has been
used. The first of the constraints (3.5) allows to write (3.17) as,

\[ \delta \eta = -\frac{i}{4} 2^{-5/4}\epsilon [V, W] = i2^{-5/4}\epsilon [\lambda, \phi] \equiv \epsilon' \frac{i}{2} [\lambda, \phi], \]  

(3.18)

where in the last step a redefinitions of \( \epsilon \) has been utilized. This example illustrates the general procedure. After replacing \( \epsilon' \rightarrow \epsilon \), the form of the symmetry transformation generated by \( Q \) turns out to be:

\[ \begin{align*}
\delta \lambda &= 2\epsilon \eta, \\
\delta \eta &= \frac{i}{2}\epsilon [\lambda, \phi], \\
\delta \phi &= 0, \\
\delta \chi_{\alpha\beta} &= \frac{1}{4}(F^+_{\alpha\beta} - G_{\alpha\beta}), \\
\delta \psi_{\dot{\alpha}\dot{\beta}} &= -\epsilon \nabla_{\dot{\alpha}\dot{\beta}} \phi, \\
\delta G_{\alpha\beta} &= \epsilon (\nabla_{(\dot{\alpha}\dot{\beta}} \psi_{\dot{\gamma}}) - 2i[\chi_{\alpha\beta}, \phi]), \\
\delta A_{\dot{\alpha}\dot{\beta}} &= \epsilon \psi_{\dot{\alpha}\dot{\beta}},
\end{align*} \]

(3.19)

where \( F^\pm_{\alpha\beta} \) represents the self-dual and anti-self-dual parts of the gauge field strength,

\[ F^+_{\alpha\beta} = C^\delta \dot{\gamma} F_{\dot{\alpha}\dot{\beta} \gamma}, \quad F^-_{\dot{\alpha}\dot{\beta}} = C^\sigma \rho F_{\dot{\alpha}\dot{\beta} \rho \sigma}. \]

(3.20)

The transformation of the component fields under the rest of the symmetries in (3.10) take the following form:

\[ \begin{align*}
\delta' \lambda &= 2\epsilon \eta \chi_{\alpha\beta}, \\
\delta' \eta &= \frac{1}{2}\epsilon [G_{\alpha\beta} - F^+_{\alpha\beta}], \\
\delta' \phi &= 0, \\
\delta' \chi_{\alpha\beta} &= \frac{1}{2}\epsilon [C_{\gamma(\alpha} [F^+_{\beta)\delta} + G_{\beta)\delta]] - iC_{\gamma(\alpha} C_{\beta)\delta} [\lambda, \phi]], \\
\delta' \psi_{\dot{\alpha}\dot{\beta}} &= 2\epsilon \delta C_{\alpha\gamma} \nabla_{\delta\dot{\beta}} \phi, \\
\delta' G_{\alpha\beta} &= 2\epsilon \delta C_{\gamma(\alpha} \left( \nabla_{\delta\dot{\beta}} \psi_{\dot{\gamma}} \right) - i[\eta C_{\beta)\delta} + \chi_{\beta)\delta}, \phi]], \\
\delta' A_{\dot{\alpha}\dot{\beta}} &= 2\epsilon \delta C_{\alpha\gamma} \psi_{\delta\dot{\beta}},
\end{align*} \]

(3.21)
\[ \delta'' \lambda = 0, \]
\[ \delta'' \eta = i e^{\alpha \beta} \nabla_{\alpha \beta} \lambda, \]
\[ \delta'' \chi_{\alpha \beta} = \frac{i}{2} \epsilon^{\gamma \delta} C_{(\alpha | \gamma}, \nabla_{\beta) \delta} \lambda, \]
\[ \delta'' \phi = -i e^{\alpha \beta} \psi_{\alpha \beta}, \]
\[ \delta'' \psi_{\alpha \beta} = \frac{i}{2} \epsilon^{\gamma \delta} \{ C_{\gamma \alpha} F_{\beta \delta}^\ast + C_{\beta \delta}^\ast (i C_{\gamma \alpha} [\phi, \lambda] + G_{\alpha \gamma}) \}, \]
\[ \delta'' G_{\alpha \beta} = e^{\sigma \tau} \{ i C_{(\beta | \sigma} \nabla_{\alpha) \tau} \eta + C_{(\alpha | \sigma} [\psi_{\beta \tau}, \lambda] + i \nabla_{(\alpha \tau} \chi_{\beta) \sigma} - 2 i \nabla_{\alpha \tau} \chi_{\beta \sigma} \}, \]
\[ \delta'' A_{\alpha \beta} = i e^{\gamma \delta} C_{\delta \beta} (C_{\gamma \alpha} \eta + \chi_{\alpha \gamma}). \]

These symmetries close among themselves as dictated by (2.13) up to gauge transformations. Finally, the \( U \) transformations of the component fields, or ghost numbers, can be computed easily from (2.14), (3.6) and (3.15). They are,

\[ [U, \lambda] = -2 \lambda, \]
\[ [U, \chi_{\alpha \beta}] = - \chi_{\alpha \beta}, \quad [U, \psi_{\alpha \beta}] = \psi_{\alpha \beta}, \]
\[ [U, \eta] = - \eta, \quad [U, \phi] = 2 \phi, \quad [U, G_{\alpha \beta}] = 0. \]  

(3.23)

Of course, the gauge field \( A_{\alpha \beta} \) has zero ghost number.

The action \( S_0 \) in (3.12) can be easily written in terms of component fields after taking into account that,

\[ S_0 = \frac{1}{e^2} \int d^4 x d^4 \theta \text{Tr}(W^2) = \frac{1}{6e^2} \int d^4 x D_{\alpha}^\beta D_{\beta}^\gamma D_{\gamma}^\alpha \text{Tr}(W^2) | \]
\[ = \frac{1}{6e^2} \int d^4 x D \text{Tr} \nabla_{\alpha}^\beta \nabla_{\beta}^\gamma \nabla_{\gamma}^\alpha (W^2) |, \]

(3.24)

where again use has been made of the fact that a Wess-Zumino gauge has been chosen. In (3.24) \( e \) is a dimensionless gauge coupling constant. Using the constraints of the theory and the definitions (3.15) one finds, up to irrelevant global
factors

\[ S_0 = \frac{1}{e^2} \int d^4x \text{Tr} \left\{ Q, \quad \frac{1}{2} (F_{\alpha\gamma}^+ + G_{\alpha\gamma}) \chi^{\alpha\gamma} + 2\lambda \nabla_{\alpha\dot{\gamma}} \psi^{\alpha\dot{\gamma}} + 2i\lambda [\eta, \phi] \right\} \]

\[ = \frac{1}{e^2} \int d^4x \text{Tr} \left\{ \frac{1}{8} (F_{+}^2 - G^2) - \lambda \chi^{\alpha\gamma} \nabla_{\alpha\dot{\beta}} \psi_{\dot{\beta}\gamma} - \frac{i}{2} \phi \{ \chi_{\alpha\gamma}, \chi^{\alpha\gamma} \} + 2\eta \nabla_{\alpha\dot{\gamma}} \psi^{\alpha\dot{\gamma}} 
\quad - i\lambda \{ \psi_{\alpha\dot{\gamma}}, \psi^{\alpha\dot{\gamma}} \} - \lambda \nabla_{\alpha\dot{\gamma}} \nabla^{\alpha\dot{\gamma}} \phi + 2i\phi \{ \eta, \eta \} + [\lambda, \phi]^2 \right\}. \]

(3.25)

This action is invariant under all the symmetry transformations of the theory (3.20), (3.21) and (3.22), as well as under gauge transformations. If one generalizes this action to an arbitrary curved four dimensional manifold by introducing a metric tensor it turns out that it is again invariant under all symmetries provided the parameters of the first three are covariantly constant:

\[ D_{\alpha\dot{\beta}} \epsilon = 0, \quad D_{\alpha\dot{\beta}} \epsilon_{\gamma\sigma} = 0, \quad D_{\alpha\dot{\beta}} \epsilon_{\gamma\dot{\sigma}} = 0, \]

(3.26)

where \( D_{\alpha\dot{\beta}} \) is a covariant derivative which contains gauge and Christoffel connections. Certainly, it is not guaranteed in general that a covariantly constant vector and a self-dual tensor exist for an arbitrary four-manifold. Thus, in general the \( H_{\alpha\beta} \) and \( G_{\alpha\dot{\beta}} \) symmetries do not exist. On the other hand, if one insists in having these symmetries one is led to topological gravity. Namely, if these covariantly constant vector and self-dual tensor do not exist let us gauge these global symmetries so that the parameters become arbitrary. This is the form advocated in [4] to build couplings to topological gravity in two dimensions. A similar construction should be carried out in four dimensions. It would be interesting if this approach to four dimensional topological gravity [21-24] leads to a theory of the type recently built in [25].

Let us recall here the form of the observables in Donaldson-Witten theory. These are built out of the following basic ones [14]. Let \( \gamma \) be a homology cycle on
$M$, of dimension $k_\gamma$, and $W_{k_\gamma}$ for $k = 0, 1, \ldots, 4$ the differential forms,

\[ W_0 = \frac{1}{2} \text{Tr} \phi^2, \]
\[ W_1 = \text{Tr} (\phi \wedge \psi), \]
\[ W_2 = \text{Tr} (\frac{1}{2} \psi \wedge \psi + i \phi \wedge F), \]
\[ W_3 = i \text{Tr} (\psi \wedge F), \]
\[ W_4 = -\frac{1}{2} \text{Tr} (F \wedge F), \] (3.27)

then the basic observables of the theory are

\[ \mathcal{O}(\gamma) = \int_{\gamma} W_{k_\gamma}. \] (3.28)

The superspace approach considered in this paper constitutes an excellent framework to build new topological quantum field theories involving gauge fields. The constraints (3.5) which define the theory could be properly modified to obtain new theories. The constraints chosen in (3.5) are a particular choice but certainly one could imagine many others. We have tried other possible sets of constraints and we have found that the ones in (3.5) are rather special. All our attempts to build new topological quantum field theories modifying (3.5) have led us either to trivial theories, or theories involving fields of higher spin. We have not been able to construct a non-trivial theory with fields of spin no higher than two other than the one leading to Donaldson-Witten theory. In this respect topological Yang-Mills theory seems rather unique. These results, plus the fact that there exist only one type of twisting, seem to indicate that type B models associated to topological Yang-Mills do not exist unless one introduces additional field content. In the rest of this paper we study matter couplings to Donaldson-Witten theory.
4. Topological matter multiplets

Our starting point will be the representation of the $N = 2$ hypermultiplet formulated in [17,18]. This multiplet has a non-vanishing central charge and therefore, an extension of the topological algebra (2.13) is required. The final form of the extended algebra is obvious from the form of the twisting. The only relations in (2.13) which change are,

$\{Q, Q\} = Z, \quad \{G_{\alpha\beta}, G_{\gamma\delta}\} = C_{\alpha\gamma}C_{\beta\delta}Z,$

$\{H_{\alpha\beta}, H_{\gamma\delta}\} = C_{\alpha(\gamma}C_{\beta|\delta)}Z, \quad [Z, \text{anything}] = 0,$

where $Z$ is the central charge generator. Notice that one could still have generalized the extended topological algebra introducing a dimensionless constant in the anticommutator $\{H_{\alpha\beta}, H_{\gamma\delta}\}$. We have analyzed this possibility and it seems impossible to construct invariant actions unless such a dimensionless constant is one.

As we will describe below, the presence of central charges makes the construction of invariant actions very restrictive. Notice also that the relations (4.1) break the $U(1)$ symmetry. Taking into account (2.14), the best we can do is to maintain a $Z_4$ ghost number symmetry assigning $U$-charge 2 to the central charge generator $Z$. Indeed, with this assignment, the $U(1)$ symmetry is preserved by the relations (4.1) modulo 4.

Let us introduce the extended superspace corresponding to (4.1). Let $z$ be a new real commuting coordinate corresponding to the central charge generator $Z$. We define:

$Z = i \frac{\partial}{\partial z},$

$Q = Q^{(0)} + i \frac{\theta}{2} \frac{\partial}{\partial z},$

$H_{\alpha\beta} = H^{(0)}_{\alpha\beta} + i \frac{\theta_{\alpha\beta}}{2} \frac{\partial}{\partial z},$

$G_{\alpha\dot{\beta}} = G^{(0)}_{\alpha\dot{\beta}} + i \frac{\theta_{\alpha\dot{\beta}}}{2} \frac{\partial}{\partial z},$

where the superscript $(0)$ refers to the operators without central charge. Superspace covariant derivatives are introduced as operators which (anti)commute with $P_{\alpha\dot{\beta}}$.
$Q$, $H_{\alpha\beta}$, and $G_{\alpha\dot{\beta}}$. Their representation in terms of superspace coordinates is:

\[
D = D^{(0)} + \frac{1}{2} \theta \frac{\partial}{\partial z},
\]

\[
D_{\alpha\beta} = D_{\alpha\beta}^{(0)} + \frac{1}{2} \theta_{\alpha\beta} \frac{\partial}{\partial z},
\]

\[
D_{\alpha\dot{\beta}} = D_{\alpha\dot{\beta}}^{(0)} + \frac{1}{2} \theta_{\alpha\dot{\beta}} \frac{\partial}{\partial z}.
\]

The factors are chosen for later convenience. They verify the following anticommutation relations:

\[
\{D, D\} = i \frac{\partial}{\partial z},
\]

\[
\{D_{\alpha\beta}, D_{\gamma\delta}\} = i C_{(\alpha|\gamma} C_{\beta)\delta} \frac{\partial}{\partial z},
\]

\[
\{D_{\alpha\dot{\beta}}, D_{\gamma\dot{\delta}}\} = i C_{\alpha\gamma} C_{\dot{\beta}\dot{\delta}} \frac{\partial}{\partial z},
\]

while the rest of the commutators do not change.

We are now in the position to define the massive multiplet. The superfield defining the $N = 2$ hypermultiplet is a scalar superfield in the two-dimensional representation of the internal group $SU(2)_I$. This implies that the defining superfield in the twisted theory has spin 1/2. We will denote the corresponding spinor superfield by $\Phi_\alpha$. This superfield satisfies certain superspace constraints which are easily derived from the constraints satisfied in $N = 2$ supersymmetry [18]:

\[
[C_{(\alpha|\beta} D + D_{(\alpha|\beta}] \Phi_\gamma) = 0, \quad D_{(\alpha\beta} \Phi_\gamma) = 0.
\]

Besides this superfield we will denote by $\overline{\Phi}_\alpha$ the complex conjugate spinor superfield. This superfield satisfies also the constraints (4.5),

\[
[C_{(\alpha|\beta} D + D_{(\alpha|\beta}] \overline{\Phi}_\gamma) = 0, \quad D_{(\alpha\beta} \overline{\Phi}_\gamma) = 0.
\]

Notice that the theory is certainly chiral and that the choice of one chirality over the other is dictated by the twist chosen in (2.6). In other words, the defining
superfields (which before the twisting transformed as the \( (0,0,1/2) \) representation of \( SU(2)_L \times SU(2)_R \times SU(2)_I \)) transform under the \( (0,1/2) \) representation of \( SU(2)_L \times SU(2)'_R \). The superfields \( \Phi_\alpha \) and \( \Phi_\alpha' \) have ghost number 0.

The constraints (4.5) imply that there are not component fields with spin higher than \( 1/2 \). This fact can be easily demonstrated after working out the following useful identities which are a consequence of the constraints (4.5) and the algebra (4.4):

\[
\begin{align*}
(C_{\alpha\beta}D + D_{\alpha\beta})\Phi_\gamma &= 2C_{\alpha\gamma}D\Phi_\beta, \\
(C_{\alpha\beta}D + D_{\alpha\beta})D\Phi_\gamma &= -iC_{\beta\gamma}\partial\Phi_\alpha, \\
D_{\alpha\beta}\Phi_\gamma &= \frac{1}{2}C_{\alpha\gamma}D_{\tau\beta}\Phi^{\tau}, \\
D_{\alpha\gamma}D^{\beta}\Phi^{\delta} &= -2iC^{\beta\delta}\partial\Phi_\alpha, \\
DD_{\eta}\Phi_\eta &= 2i\partial_{\bar{\eta}}\Phi_\eta, \\
D_{\alpha\gamma}D_{\eta}\Phi_\eta &= -2i\partial_{\bar{\eta}}\Phi_{\alpha\gamma}, \\
\partial\Phi_\alpha &= -\frac{1}{2}\partial_{\bar{\alpha}}\Phi_\gamma, \\
\partial^2\Phi_\gamma &= \partial_{\bar{\alpha}}\partial^{\beta}\Phi_\gamma.
\end{align*}
\]

(4.7)

One observes that all the possible higher spin components can be expressed in terms of lower ones. Furthermore, the last equation, which can be rewritten as the condition \( P^2 + Z^2 = 0 \), truncates the infinite expansion of the superfield \( \Phi_\alpha \) in powers of \( z \), remaining a finite number of component fields. Of course, a similar set of relations as the ones in (4.7) holds for \( \Phi_\alpha' \).

Component fields are defined introducing adequate superspace derivatives. From (4.7) follows that the only independent component fields are:

\[
\begin{align*}
\Phi_\alpha &= H_\alpha, \\
D\Phi_\alpha &= 2^{-1/4}u_\alpha, \\
D_{\rho\alpha}\Phi^\rho &= 2^{5/4}u_{\bar{\alpha}}, \\
\partial\Phi_\alpha &= i2^{1/2}K_\alpha,
\end{align*}
\]

(4.8)
Again, the numerical factors in these definitions are chosen for later convenience. The $Q$-transformations of the component fields are easily obtained using (4.2), (4.4) and (4.7). They turn out to be:

$$
\begin{align*}
\delta H_\alpha &= \epsilon u_\alpha, \\
\delta u_\alpha &= -\epsilon K_\alpha, \\
\delta v_\dot{\alpha} &= i\epsilon \partial_\rho H^\rho, \\
\delta K_\alpha &= i\epsilon \partial_\dot{\alpha} v^\dot{\alpha}.
\end{align*}
$$

(4.9)

In a similar way the $H_{\alpha\beta}$ and $G_{\alpha\dot{\beta}}$ transformations are worked out:

$$
\begin{align*}
\delta' H_{\alpha} &= 2\epsilon^\beta\gamma C_{\beta\alpha} u_\gamma, \\
\delta'' H_{\alpha} &= \epsilon^\beta\gamma C_{\beta\alpha} v_\gamma, \\
\delta' u_\alpha &= 2\epsilon^\beta\gamma C_{\beta\alpha} K_\gamma, \\
\delta'' u_\alpha &= i\epsilon^\beta\gamma \partial_\alpha \gamma H_\beta, \\
\delta' v_\dot{\alpha} &= -2i\epsilon^\beta\gamma \partial_\beta \dot{\alpha} H_\gamma, \\
\delta'' v_\dot{\alpha} &= \epsilon^\beta\gamma C_{\beta\dot{\alpha}} K_\beta, \\
\delta' K_\alpha &= 2i\epsilon^\beta\gamma C_{\beta\dot{\alpha}} \partial_\beta \dot{\alpha} v^\dot{\alpha}, \\
\delta'' K_\alpha &= -i\epsilon^\beta\gamma C_{\beta\dot{\alpha}} \partial_\dot{\beta} \dot{\alpha} u^\dot{\alpha}.
\end{align*}
$$

(4.10)

Finally the $z$-transformations become,

$$
\begin{align*}
\delta_z H_\alpha &= -z K_\alpha, \\
\delta_z u_\alpha &= -iz \partial_\alpha \dot{\alpha} v^\dot{\alpha}, \\
\delta_z v_\dot{\alpha} &= iz \partial_\dot{\alpha} \alpha u^\alpha, \\
\delta_z K_\alpha &= z \Box H_\alpha.
\end{align*}
$$

(4.11)

In this equation $z$ is a parameter and should not be confused with the coordinate $z$ introduced in the extended superspace as in (4.2). In these last sets of transformations the underlines of vector indices in partial derivatives have been removed since after the projection there are not anticommuting vector indices. Of course, a similar set of transformations holds for the overlined fields. The ghost numbers of the component matter fields can be obtained easily from (4.8) and the fact that the superfield $\Phi_\alpha$ has ghost number 0. It turns out that the set of matter fields $(H_\alpha, u_\alpha, v_\dot{\alpha}, K_\alpha)$ and $(\overline{H}_\alpha, \overline{u}_\alpha, \overline{v}_\dot{\alpha}, \overline{K}_\alpha)$ have both ghost numbers $(0, 1, -1, 2)$. These ghost numbers are the charges of the $Z_4$ symmetry of the extended topological algebra.
Since central charges are present in the theory there is not a natural measure to construct invariant actions. Certainly, one must require $Z$-invariance and this is not guaranteed by measures as the ones taken in (3.12) and (3.13). Actually, it is rather hard to find $Z$-invariant actions. Guided by the formulation of $N = 2$ supersymmetry in [18] there are at least two quantities invariant under all the symmetries of the extended topological algebra (4.2). These lead to the following terms entering the action for the topological hypermultiplet,

\[ \mathcal{L}_0 = \int d^4x \left[ D^2 C^\beta_\alpha - D^\sigma_\beta D_\sigma_\alpha + D_\alpha_\sigma D^\beta_\sigma \right](\Phi^\alpha \tilde{\partial} \Phi^\beta), \]

\[ \mathcal{L}_m^t = \int d^4x \left[ D^2 C^\beta_\alpha - D^\sigma_\beta D_\sigma_\alpha + D_\alpha_\sigma D^\beta_\sigma \right](\Phi^\alpha \Phi^\beta), \]  

\[ \text{(4.12)} \]

where the superscript “f” stands for free. The first quantity contains the kinetic part while the second correspond to a mass term. The action $\mathcal{L}_0$ has ghost number 0 and thus models based on this action possess a $Z_4$ symmetry. However, the action $\mathcal{L}_m^t$ has ghost number 2. This implies that in models where $m \neq 0$ the ghost number symmetry is broken to $Z_2$.

The most general form of the full action is, after writing (4.12) in terms of the component fields (4.8),

\[ \mathcal{L}^t = \mathcal{L}_0^t + m\mathcal{L}_m^t = \int d^4x \left[ \overline{H}^\alpha \Box H_\alpha + i\overline{\sigma}^\alpha \partial_\alpha \bar{v}^\hat{\alpha} - i\overline{\sigma}^\alpha \partial_\alpha \bar{u}^\alpha + \overline{K}^\alpha K_\alpha \right] + m\left( \overline{K}^\alpha H_\alpha - \overline{H}^\alpha K_\alpha \right) + m\left( \overline{\sigma}^\alpha u_\alpha + \overline{\sigma}^\alpha v_\alpha \right), \]  

\[ \text{(4.13)} \]

where $m$ is an arbitrary mass parameter.

The following redefinition of the auxiliary fields $\overline{K}$ and $K$ isolates a mass term for $\overline{H}$ and $H$:

\[ \overline{K}^\alpha = \overline{K'}^\alpha + m\overline{H}^\alpha, \]

\[ K_\alpha = K'_\alpha - mH_\alpha. \]  

\[ \text{(4.14)} \]
The final expression for the matter action is:

$$\mathcal{L}^{i} = \mathcal{L}_0^{i} + m\mathcal{L}_m^{i}$$

$$= \int d^4x \left[ \overline{\psi}^\alpha \gamma^\nu \partial_\nu \psi^\alpha - i\overline{\psi}^\alpha \gamma^\nu \partial_\nu \psi^\alpha + \overline{K}^{\alpha} K'_\alpha \right.$$ 

$$\left. + m^2 \overline{\psi}^{\alpha} H_\alpha + m \left( \overline{\psi}^{\alpha} u_\alpha + \overline{\psi}^{\dot{\alpha}} v_\dot{\alpha} \right) \right].$$ (4.15)

So far we have formulated the theory on a flat four-dimensional manifold. In order to construct the topological model we must rewrite the theory for an arbitrary curved space, in other words, we must introduce a background metric $g_{\mu\nu}$ on the manifold and covariantize the formulation. Furthermore, since there are spinorial fields, the manifold chosen must be a spin manifold. We will assume that some choice of spin structure has been made. This process presents some surprises since, as we now describe, one is forced to introduce new terms in the action which depend on the curvature associated to $g_{\mu\nu}$ in order to maintain the symmetries of the theory. Looking back to the derivation of the relations (4.7), it turns out that for an arbitrary curved space the last relation becomes,

$$\partial^2 \Phi_\gamma = 2 \left( \Box + \frac{1}{4} R \right) \Phi_\gamma,$$ (4.16)

where $\Box$ is the covariant laplacian and $R$ is the scalar curvature. This has important consequences in the new form of the $z$-transformations (4.11) and in the new form of the action (4.15). On the other hand, the rest of the transformations, namely, the ones in (4.9) and (4.10) remain the same once the partial derivatives are replaced by covariant ones. The $z$-transformations (4.11) become,

$$\delta_z H_\alpha = -zK_\alpha,$$

$$\delta_z u_\alpha = -iz\nabla_{\alpha\dot{\alpha}} v_{\dot{\alpha}},$$

$$\delta_z v_{\dot{\alpha}} = iz\nabla_{\alpha\dot{\alpha}} u_\alpha,$$

$$\delta_z K_\alpha = z(\Box + \frac{1}{4} R) H_\alpha,$$ (4.17)
where $\tilde{\nabla}_{\alpha\dot{\alpha}}$ denotes the covariant derivative. The final form of the action is,

$$L^g = L^g_0 + mL^g_m$$

$$= \int_M d^4x \sqrt{g} \left[ \tilde{H}^{\alpha} (\Box + \frac{1}{4} R) H_\alpha + i\tilde{w}^\dot{\alpha} \tilde{\nabla}_{\alpha\dot{\alpha}} \tilde{v}^\dot{\alpha} - i\tilde{v}^\dot{\alpha} \tilde{\nabla}_{\alpha\dot{\alpha}} \tilde{u}^\dot{\alpha} + K^\alpha K'^\alpha ight.$$

$$+ mL^2 \tilde{H}^{\alpha} H_\alpha + m(\tilde{w}^\alpha u_\alpha + \tilde{v}^\alpha v_\dot{\alpha}) \right].$$

(4.18)

The superscript $g$ in (4.18) indicates that a choice of metric and spin structure has been made. This action is invariant under $Q$ and $z$ transformations. However, it is not invariant under $H_{\alpha\beta}$ and $G_{\alpha\dot{\beta}}$ transformations unless their corresponding parameters satisfy,

$$\tilde{\nabla}_{\alpha\beta} \epsilon_{\gamma\sigma} = 0, \quad \tilde{\nabla}_{\alpha\dot{\beta}} \epsilon_{\gamma\dot{\sigma}} = 0.$$  

(4.19)

Certainly, not all spin manifolds admit covariantly constant vectors or self-dual tensors. Therefore, those symmetries do not exist in general. However, as discussed before, these symmetries might be useful to construct the coupling to topological gravity.

Using the covariantized version of (4.9) it is simple to verify that for $m = 0$ the action (4.18) is $Q$-exact. Notice that contrary to the case of topological Yang-Mills, this is not obvious from the form of the superspace action (4.12). In fact, it turns out that,

$$L^g_0 = \{ Q, \tilde{\Lambda}^g \},$$

(4.20)

where,

$$\tilde{\Lambda}^g = \frac{1}{2} \int_M d^4x \sqrt{g} \left[ i\tilde{H}^{\alpha} \tilde{\nabla}_{\alpha\dot{\alpha}} \tilde{v}^\dot{\alpha} + i\tilde{v}^\dot{\alpha} \tilde{\nabla}_{\alpha\dot{\alpha}} \tilde{H}^\alpha - K^\alpha u_\alpha - \tilde{u}^\alpha K_\alpha \right].$$  

(4.21)

The invariance under $Q$ and $Z$ of $L^g_0$ can be regarded simply as a consequence of (4.20) and the relations:

$$[Q, Z] = 0, \quad [Z, \tilde{\Lambda}^g] = 0.$$  

(4.22)

Equation (4.20) does not ensure that the model we have built is topological,
since from the $Q$-exactness of the whole action it does not follow that the energy-momentum tensor is also $Q$-exact. This would be so if the $Q$-transformations would not contain covariant derivatives, which is not our case. Nevertheless we show below that the non-$Q$-exact part of the energy-momentum tensor vanishes on-shell. On the other hand, the $Q$-exactness of the action makes the theory exact in the small coupling limit. These two facts suffice to render the theory topological. This implies [14] that the vacuum expectation values of operators which are invariant under $Q$-transformations lead to topological invariant quantities. A description of the observables of this theory was presented in [15]. These turn out to be a very restrictive set because of the strong conditions imposed by $z$-invariance. As we will observe in the discussion concerning the coupling of these models to topological Yang-Mills in the next section, the $z$-symmetry is so restrictive that the only observables are the ones of the form (3.28).
5. Matter coupling to Donaldson-Witten theory

We will construct the coupling of the topological hypermultiplet of the previous section to Donaldson-Witten theory covariantizing the algebra of the extended superspace derivatives, imposing the gauge constraints (3.5), and covariantizing the defining constraints on the matter superfields. Let us therefore pick a gauge group $G$ and introduce gauge connections as in (3.1). The form of the constraints (3.5) now becomes:

\[
\{\nabla, \nabla\} = i(\partial - \frac{1}{2}V), \\
\{\nabla_{\alpha\beta}, \nabla_{\gamma\delta}\} = iC_{(\alpha\beta\gamma\delta}(\partial - \frac{1}{2}V), \\
\{\nabla_{\alpha\dot{\beta}}, \nabla_{\gamma\dot{\delta}}\} = iC_{\alpha\dot{\beta}\gamma\dot{\delta}}(\partial - W),
\]

where the scalar superfields $V$ and $W$ are the same as in (3.5). The remaining fundamental (anti)commutation relations do not change. As the central charge commutes with everything, the gauge sector of the theory is as before (the Bianchi identities remain unchanged). On the other hand, the matter sector has to be reconsidered.

Let us consider a commuting spinor superfield $\Phi_\alpha$ which transforms under a representation of the gauge group $G$, and another spinor superfield $\overline{\Phi}^\dot{\alpha}$ which transform under the conjugate representation. The covariant form of the constraints (4.5), which are the defining equations of the hypermultiplet, now read,

\[
[C_{(\alpha\beta\gamma}\nabla + \nabla_{(\alpha|\beta|}\Phi_\gamma) = 0, \quad \nabla_{(\alpha\dot{\beta}\dot{\Phi}_\gamma) = 0.}
\]

The equivalent of the identities (4.7), which determine the set of independent
component fields, consist of the set of equations:

\[ (C_{\alpha\beta} \nabla + \nabla_{\alpha\beta}) \Phi_{\gamma} = 2C_{\alpha\gamma} \nabla \Phi_{\beta}, \]
\[ \nabla_{\alpha\beta} \Phi_{\gamma} = \frac{1}{2} C_{\alpha\gamma} \nabla_{\tau\beta} \Phi^\tau, \]
\[ \nabla_{\alpha\gamma} \nabla_{\delta} \Phi^\delta = 4i(W - \partial)\Phi_{\alpha}, \]
\[ \nabla \nabla_{\eta} \Phi^\eta = 2i\nabla_{\eta} \Phi_{\eta}, \]
\[ \nabla_{\alpha\gamma} \nabla_{\eta} \Phi^\eta = -2i\nabla_{(\alpha} \Phi_{\gamma)}, \]
\[ \partial \nabla \Phi_{\alpha} = -\frac{1}{2} \nabla_{\alpha\tau} \nabla_{\eta} \Phi^\eta + W \nabla \Phi_{\alpha} + \frac{1}{2} \nabla W \Phi_{\alpha} + \frac{1}{2} \nabla_{\alpha\tau} W \Phi^\tau, \] (5.3)
\[ \partial \nabla \tau_{\beta} \Phi^\tau = 4\nabla_{\tau\beta} \nabla \Phi^\tau + \frac{1}{2} V \nabla_{\tau\beta} \Phi^\tau + \nabla_{\tau\beta} V \Phi^\tau, \]
\[ \partial^2 \Phi_{\alpha} = 2\Box \Phi_{\alpha} + \frac{i}{4} \nabla_{\alpha\beta} V \nabla_{\tau} \Phi^\tau - i \nabla W \nabla \Phi_{\alpha}, \]
\[ + i \nabla_{\alpha\tau} W \nabla \Phi^\tau + W \partial \Phi_{\alpha} + \frac{1}{2} V \partial \Phi_{\alpha} \]
\[ + iG_{\alpha\tau} \Phi_{\tau} - \frac{1}{4} \{V, W\} \Phi_{\alpha}. \]

We will carry out a covariant projection into component fields. This means that component fields must be defined as in (4.8) replacing ordinary superspace derivatives by covariant ones:

\[ \Phi_{\alpha} \bigg| \rightarrow H_{\alpha}, \]
\[ \nabla \Phi_{\alpha} \bigg| = 2^{-1/4} u_{\alpha}, \]
\[ \nabla \rho_{\alpha} \Phi^\rho \bigg| = 2^{5/4} v_{\alpha}, \]
\[ \partial \Phi_{\alpha} \bigg| = i2^{1/2} K_{\alpha}, \]
\[ \nabla \Phi_{\alpha} \bigg| = \bar{H}_{\alpha}, \]
\[ \nabla \Phi_{\alpha} \bigg| = 2^{-1/4} \bar{u}_{\alpha}, \]
\[ \nabla \rho_{\alpha} \Phi^\rho \bigg| = 2^{5/4} \bar{v}_{\alpha}, \]
\[ \partial \Phi_{\alpha} \bigg| = i2^{1/2} \bar{K}_{\alpha}. \] (5.4)

The transformation laws of these component fields are derived making use of (3.10) and the relations (5.3). For Q-transformations these are:

\[ \delta H_{\alpha} = \epsilon u_{\alpha}, \]
\[ \delta u_{\alpha} = -\epsilon (K_{\alpha} + i\phi H_{\alpha}), \]
\[ \delta v_{\alpha} = i\epsilon \nabla_{aa} H_{\alpha}, \]
\[ \delta K_{\alpha} = i\epsilon (\nabla_{aa} \phi - \lambda u_{\alpha}), \]
\[ \delta H_{\alpha} = \epsilon \bar{H}_{\alpha}, \]
\[ \delta \bar{H}_{\alpha} = \epsilon \bar{u}_{\alpha}, \]
\[ \delta \bar{u}_{\alpha} = -\epsilon (K_{\alpha} - i\phi \bar{H}_{\alpha}), \]
\[ \delta \bar{v}_{\alpha} = i\epsilon \nabla_{aa} \bar{H}_{\alpha}, \]
\[ \delta \bar{K}_{\alpha} = i\epsilon (\nabla_{aa} \bar{\phi} + \lambda \bar{u}_{\alpha}), \] (5.5)
while for $H_{\alpha\beta}$-transformations,

\[
\begin{align*}
\delta' H_{\alpha} &= 2\epsilon^{\gamma\delta} C_{\gamma\alpha} u_\delta, \\
\delta' u_\alpha &= 2\epsilon^{\gamma\delta} C_{\gamma\alpha} (K_\delta + i\phi H_\delta), \\
\delta' v_\dot{\alpha} &= -2i\epsilon^{\gamma\delta} \nabla_{\gamma\dot{\alpha}} H_\delta, \\
\delta' K_\alpha &= 2i\epsilon^{\gamma\delta} C_{\gamma\alpha} (\nabla_{\delta\dot{\beta}} v^\dot{\beta} - \lambda u_\delta - \eta H_\delta - \chi_{\delta\dot{\beta}} H^{\dot{\beta}}), \\
&\quad + \eta H_\delta + \chi_{\delta\dot{\beta}} H^{\dot{\beta}}.
\end{align*}
\]

and, finally, for $G_{\alpha\dot{\beta}}$ transformations,

\[
\begin{align*}
\delta'' H_{\alpha} &= \epsilon^{\gamma\delta} C_{\gamma\alpha} v^\dot{\delta}, \\
\delta'' u_\alpha &= i\epsilon^{\gamma\delta} \nabla_{\alpha\dot{\delta}} H_\gamma, \\
\delta'' v_\dot{\alpha} &= \epsilon^{\gamma\delta} C_{\dot{\alpha}} (K_\gamma + i\lambda H_\gamma), \\
\delta'' K_\alpha &= -i\epsilon^{\gamma\delta} C_{\gamma\alpha} (\nabla_{\rho\dot{\delta}} u^\rho \\
&\quad - i\psi_{\rho\dot{\delta}} H^\rho + \phi v_\delta), \\
&\quad + i\psi_{\rho\dot{\delta}} H^\rho - \phi v_\delta.
\end{align*}
\]

On the other hand, it is also important to work out the form of the $z$-transformations, these become:

\[
\begin{align*}
\delta_z H_{\alpha} &= -z K_\alpha, \\
\delta_z u_\alpha &= -iz (\nabla_{\alpha\dot{\delta}} v_\dot{\delta} - \lambda u_\alpha - \eta H_\alpha - \chi_{\alpha\dot{\beta}} H^{\dot{\beta}}), \\
\delta_z v_\dot{\alpha} &= iz (\nabla_{\alpha\dot{\delta}} u_\alpha + \phi v_\alpha - i\psi_{\alpha\dot{\delta}} H^{\alpha}), \\
\delta_z K_\alpha &= z \left( \square H_{\alpha} + \psi_{\alpha\dot{\delta}} v^\dot{\delta} - \frac{1}{2} \{\lambda, \phi\} - i\eta u_\alpha + i\chi_{\alpha\dot{\beta}} H^{\dot{\beta}} \\
&\quad + \frac{i}{2} G_{\alpha\dot{\beta}} H^{\dot{\beta}} + i\lambda K_\alpha + i\phi K_\alpha \right), \\
\delta_z H_\alpha &= -z K_\alpha, \\
\delta_z u_\alpha &= -iz (\nabla_{\alpha\dot{\delta}} v_\dot{\delta} + \lambda u_\alpha + \eta H_\alpha + \chi_{\alpha\dot{\beta}} H^{\dot{\beta}}), \\
\delta_z v_\dot{\alpha} &= iz (\nabla_{\alpha\dot{\delta}} u_\alpha - \phi v_\alpha + i\psi_{\alpha\dot{\delta}} H^{\alpha}), \\
\delta_z K_\alpha &= z \left( \square H_{\alpha} - \psi_{\alpha\dot{\delta}} v^\dot{\delta} - \frac{1}{2} \{\lambda, \phi\} + i\eta u_\alpha - i\chi_{\alpha\dot{\beta}} H^{\dot{\beta}} \\
&\quad - \frac{i}{2} G_{\alpha\dot{\beta}} H^{\dot{\beta}} - i\lambda K_\alpha - i\phi K_\alpha \right).
\end{align*}
\]
In the transformations (5.5), (5.6), (5.7), (5.8) and (5.9) commuting vector indices have not been underlined since at the component level there is not risk to be mistaken.

The coupled matter action turns out to be made out of the covariantization of the terms in (4.12):

\[
L_{0}^{\text{tYM}} = \int d^{4}x \left[ \nabla^{2} C_{\alpha}^{\beta} - \nabla^{\sigma\beta} \nabla_{\sigma\alpha} + \nabla_{\alpha\sigma} \nabla^{\beta\sigma} \right] (\Phi^{\sigma} \delta \Phi_{\beta}) ,
\]

\[
L_{m}^{\text{tYM}} = \int d^{4}x \left[ \nabla^{2} C_{\alpha}^{\beta} - \nabla^{\sigma\beta} \nabla_{\sigma\alpha} + \nabla_{\alpha\sigma} \nabla^{\beta\sigma} \right] (\Phi^{\sigma} \Phi_{\beta}) ,
\]

where the superscript \( \text{tYM} \) indicates that the fields of topological Yang-Mills have been considered in the action (5.10) as background fields. Before writing the action in components let us analyze the form of the symmetry transformations when the theory is placed on an arbitrary spin manifold endowed with a metric \( g_{\mu\nu} \). All transformations in (5.5), (5.6) and (5.7) remain the same after the replacement of the Yang-Mills covariant derivative \( \nabla_{\alpha}^{\dot{\beta}} \) by the full covariant derivative \( D_{\alpha}^{\dot{\beta}} \) introduced in (3.26). On the other hand, since the last equation of (5.3) gets a term involving the scalar curvature,

\[
\partial^{2} \Phi_{\alpha} = 2 \left( \Box + \frac{1}{4} R \right) \Phi_{\alpha} + i D_{\dot{\alpha}} V D_{\dot{\beta}} \Phi^{\tau} - i \nabla W \nabla \Phi_{\alpha} ,
\]

\[
+ i D_{\alpha\tau} W \nabla \Phi^{\tau} + W \partial \Phi_{\alpha} + \frac{1}{2} V \partial \Phi_{\alpha} + \frac{1}{4} \{ V, W \} \Phi_{\alpha} ,
\]

the \( z \)-transformation of \( K_{\alpha} \) and \( K'_{\alpha} \) become modified in the following form:

\[
\delta_{z} K_{\alpha} = z \left[ \left( \Box + \frac{1}{4} R \right) H_{\alpha} + \psi_{\alpha\dot{a}} v^{\dot{a}} - \frac{1}{2} \{ \lambda, \phi \} - i \eta u_{\alpha} + i \chi_{\alpha\beta} H^{\beta} + i \frac{1}{2} G_{\alpha\beta} H^{\beta} + i \lambda K_{\alpha} + i \phi K_{\alpha} \right] ,
\]

\[
\delta_{z} K'_{\alpha} = z \left[ \left( \Box + \frac{1}{4} R \right) \overline{H}_{\alpha} - \psi_{\alpha\dot{a}} \overline{v}^{\dot{a}} - \frac{1}{2} \{ \lambda, \phi \} + i \eta \overline{u}_{\alpha} - i \chi_{\alpha\beta} H^{\beta} + i \frac{1}{2} G_{\alpha\beta} H^{\beta} - i \lambda K_{\alpha} - i \phi K_{\alpha} \right] .
\]
$z$-transformations in (5.8) and (5.9) have the same form once the replacement \( \nabla_{\alpha\beta} \rightarrow D_{\alpha\beta} \) is performed.

From these terms the full action is defined as in (4.13). Writing it in terms of the component fields and redefining the auxiliary fields $K$ and $\overline{K}$ as in (4.14) one finds,

\[
L_{\text{g,tYM}} = L_{\text{g,tYM}}^0 + m L_{\text{m,tYM}}^0 = \int_M d^4 x \sqrt{g} \left[ \overline{K}^\alpha K'_\alpha + \overline{H}^\alpha \left( \Box + \frac{1}{4} R \right) H_\alpha + \frac{i}{2} \overline{H}^\alpha F_{\alpha\beta}^+ H_\beta + i \overline{\alpha} D_{\alpha\beta} v^\beta \\
- i \overline{\alpha} \psi_{\alpha\beta} u^\alpha - \overline{\psi}_\alpha \dot{\psi}_\alpha H_\alpha - i \overline{H}^\alpha \eta u_\alpha - i \overline{\alpha} \eta H_\alpha + i \overline{\alpha} \chi_{\alpha\beta} u^\beta \\
- i \overline{\alpha} \chi_{\alpha\beta} H_\beta - i \overline{\alpha} \lambda u_\alpha - i \overline{\alpha} \phi u_\alpha + \frac{i}{2} \overline{\alpha} G_{\alpha\beta} H_\beta - \frac{1}{2} \overline{\alpha} \{ \phi, \lambda \} H_\alpha \\
+ m^2 \overline{\alpha} H_\alpha + m (\overline{\alpha} u_\alpha + \overline{\beta} v_\alpha) - im \overline{\alpha} (\phi + \lambda) H_\alpha \right],
\]

(5.13)

where $M$ denotes the four-dimensional spin manifold where the theory is defined. This action is invariant under the full extended topological algebra provided the parameters satisfy the relations (3.26). Certainly, in general, only the $Q$-symmetry will hold. At this moment one should ask if the action $L_{\text{g,tYM}}^0$ in (5.13) is $Q$-exact as was the case for the action with no coupling to topological Yang-Mills. Contrary to the action (4.13), $L_{\text{g,tYM}}^0$ is not $Q$-exact. In addition it turns out that the energy-momentum tensor is not $Q$-exact and therefore it is not clear if the theory is topological. Examples of topological theories whose action is not $Q$-exact but its energy momentum is are known [3,4]. However, this is not the case here.

Certainly, the mass terms of $m L_{\text{m,tYM}}^0$ in (5.13) break the topological symmetry. What is in some sense unexpected is that the action $L_{\text{g,tYM}}^0$ also might lead to a breaking of the topological symmetry. The analysis of this phenomena is carried out in the sect. 7. To finish this section let us finally write the full action $S^g$ of the topological model under consideration. This action is,

\[
S^g = L_{\text{YM}}^g + L_{\text{g,tYM}}^0,
\]

(5.14)

where $L_{\text{g,tYM}}^0$ is the action given in (5.13) and $L_{\text{YM}}^g$ is the covariantized form of the
In this action the gauge field can be taken in any representation of the gauge group, for example, one could think just in the representation chosen for the matter fields in (5.13). The difference between choosing one representation or another is just a global factor which can be reabsorbed in the coupling constant $e$. Because of the $Q$-exactness of (5.15) (see (3.25)) the observables of the theory are independent of $e$.

To show that the action $L_{g,t}^{\text{YM}}$ in (5.13) is not $Q$-exact one just has to write all possible terms quadratic in matter fields with ghost number -1. It turns out that no combination of those terms leads to $L_{g,t}^{\text{YM}}$. The $z$-symmetry present in $L_{g,t}^{\text{YM}}$ is very restrictive. If $L_{g,t}^{\text{YM}}$ were $Q$ of some quantity, presumably such a quantity should be invariant under $z$-transformations. However, it does not exist a $z$-invariant of ghost number -1 and quadratic in the matter fields. On the other hand, it is clear from the form of the action for the free case in (4.20) that part of the $L_{g,t}^{\text{YM}}$ is $Q$ exact. Indeed, one finds that

\[ L_{0}^{\text{g,tYM}} = \{Q, \Lambda^g\} + L_{0}^{\text{g,tYM}}, \quad \text{(5.16)} \]

where,

\[ \Lambda^g = \frac{1}{2} \int_{M} d^{4}x \sqrt{g} \left[ i \bar{H}^{a} D_{\alpha \dot{a}} v^{\dot{a}} \dot{\alpha} + i \bar{\pi}^{a} D_{\alpha \dot{a}} H^{\alpha} - \bar{K}^{a} u_{\alpha} - \bar{\pi}^{a} K_{\alpha} \right], \quad \text{(5.17)} \]
and,

\[
L_{0,\text{YM}} = \frac{1}{2} \int d^4 x \sqrt{g} \left[ H^\alpha \bar{\psi}_{\dot{\alpha}} \dot{\psi}^\dot{\alpha} - \bar{\psi}^\dot{\alpha} \psi_{\dot{\alpha}} H^\alpha - i \bar{H}^\alpha \eta u_\alpha - i \bar{\omega}^\alpha \eta H_\alpha \\
+ i \bar{H}^\alpha \chi_{\alpha \beta} u^\beta - i \bar{\omega}^\alpha \chi_{\alpha \beta} H^\beta - 2 i \bar{\omega}^\alpha \chi_{\alpha \beta} \phi v_\beta - \bar{H}^\alpha \{ \phi, \lambda \} H_\alpha \\
+ i \bar{H}^\alpha G_{\alpha \beta} H^\beta - i \bar{K}^\alpha \phi H_\alpha + i \bar{H}^\alpha \phi K'_\alpha \right].
\] (5.18)

This part of the action seems as complicated as the original action (5.13). Notice, however, that in (5.18) there are only interaction vertices. The form of the energy-momentum tensor of this theory will be studied in sect. 7.
6. The truncated theory

So far, the models we have presented possess all the symmetries of the topological algebra, provided that (3.26) holds. It turns out that the resulting theory can be truncated making it simpler. This truncation consist of disregarding the fields $\lambda$, $\eta$ and $\chi$, and the auxiliary field $G$ in both the coupling of matter to Donaldson-Witten theory and the matter fields transformation laws. In other words, matter is coupled to a minimal set of gauge fields, $A_\mu$, its $Q$-partner $\psi_\mu$ and the field $\phi$.

We give now the truncated $\delta$ and $\delta''$ transformations for this minimal set of fields:

$$\delta A_{\alpha\dot{\alpha}} = \epsilon \psi_{\alpha\dot{\alpha}}, \quad \delta'' A_{\alpha\dot{\alpha}} = 0,$$

$$\delta \psi_{\alpha\dot{\beta}} = -\epsilon D_{\alpha\dot{\beta}} \phi, \quad \delta'' \psi_{\alpha\dot{\alpha}} = \frac{i}{2} \epsilon \gamma^\dot{\delta} C_\gamma \alpha F^-_{\alpha\dot{\delta}},$$

$$\delta \phi = 0, \quad \delta'' \phi = -i \epsilon \beta^\gamma \psi_{\dot{\beta}\dot{\gamma}}.$$ (6.1)

It is simple to verify that the $Q$-transformations close up to gauge transformations generated by $\phi$:

$$[\delta_2, \delta_1] A_{\alpha\dot{\alpha}} = -2i \epsilon_1 \epsilon_2 D_{\alpha\dot{\alpha}} \phi$$

$$[\delta_2, \delta_1] \psi_{\alpha\dot{\alpha}} = 2i \epsilon_1 \epsilon_2 [\psi_{\alpha\dot{\alpha}}, \phi].$$ (6.2)

As for the $G$-transformations, one would expect that the commutator of two of them would give a gauge transformation generated by $\lambda$. This is indeed what happens, although in the truncated case this is zero since $\lambda$ has been set to zero,

$$[\delta''_2, \delta''_1] A_{\alpha\dot{\alpha}} = [\delta''_2, \delta''_1] \psi_{\alpha\dot{\alpha}} = 0.$$ (6.3)

Now we should calculate the commutator of a $Q$ and a $G$-transformation to make sure of having a topological algebra. It should give a derivative of the field on which the transformations act. This comes out to be true.

$$[\delta, \delta''] A_{\alpha\dot{\alpha}} = -\frac{i}{2} \epsilon \epsilon^\dot{\delta} C_\gamma \alpha F^-_{\alpha\dot{\delta}},$$

$$[\delta, \delta''] \psi_{\alpha\dot{\alpha}} = -\frac{i}{2} \epsilon \epsilon^\dot{\delta} C_\gamma \alpha D_{\rho(\dot{\alpha}\dot{\beta})} \psi_\rho,$$

$$[\delta, \delta''] \phi = -i \epsilon \epsilon^\dot{\delta} D_{\gamma\dot{\delta}} \phi.$$ (6.4)

We now give the action of the truncated $Q$ and $G$-transformations on matter fields.
and explore its consistency with the topological algebra.

\[
\delta H_\alpha = \epsilon u_\alpha, \quad \delta'' H_\alpha = \epsilon\gamma^\delta C_{\gamma\alpha} v_\delta,
\]

\[
\delta u_\alpha = -\epsilon(K_\alpha + i\phi H_\alpha), \quad \delta'' u_\alpha = i\epsilon\gamma^\delta D_{\alpha\delta} H_\gamma,
\]

\[
\delta v_\alpha = i\epsilon D_{\alpha\delta} v_\delta, \quad \delta'' v_\alpha = \epsilon\gamma^\delta C_{\delta\alpha} K_\gamma,
\]

\[
\delta K_\alpha = i\epsilon D_{\alpha\delta} v_\delta, \quad \delta'' K_\alpha = -i\epsilon\gamma^\delta C_{\gamma\alpha}(D_{\rho\delta} u_\rho - \phi v_\delta).
\]

The commutator of two $Q$-transformations is now a combined $Z$ and gauge transformation generated by $\phi$:

\[
[\delta_2, \delta_1] H_\alpha = -2\epsilon_1\epsilon_2(K_\alpha + i\phi H_\alpha)
\]

\[
[\delta_2, \delta_1] u_\alpha = -2i\epsilon_1\epsilon_2(D_{\alpha\dot{\alpha}} v^{\dot{\alpha}} + \phi u_\alpha)
\]

\[
[\delta_2, \delta_1] v_\alpha = 2\epsilon_1\epsilon_2(iD_{\alpha\dot{\alpha}} u^{\dot{\alpha}} + \psi_{\alpha\dot{\alpha}} H_\alpha)
\]

\[
[\delta_2, \delta_1] K_\alpha = 2\epsilon_1\epsilon_2\left(\Box + \frac{1}{4}R\right) H_\alpha + \frac{i}{2} F_{\alpha\beta} H_\beta + \psi_{\alpha\dot{\alpha}} v^{\dot{\alpha}}.
\]

The commutator of two $G$-transformations gives a $Z$-transformation, in analogy with the case of Donaldson-Witten fields.

\[
[\delta''_2, \delta''_1] H_\alpha = -(\epsilon_1)_{\beta_1}(\epsilon_2)_{\beta_2} K_\alpha
\]

\[
[\delta''_2, \delta''_1] u_\alpha = -i(\epsilon_1)_{\beta_1}(\epsilon_2)_{\beta_2} D_{\alpha\dot{\alpha}} v^{\dot{\alpha}}
\]

\[
[\delta''_2, \delta''_1] v_\alpha = i(\epsilon_1)_{\beta_1}(\epsilon_2)_{\beta_2}(D_{\delta\dot{\delta}} u^{\dot{\delta}} + \phi v_\delta - i\psi_{\delta\dot{\delta}} H^{\dot{\delta}})
\]

\[
[\delta''_2, \delta''_1] K_\alpha = (\epsilon_1)_{\beta_1}(\epsilon_2)_{\beta_2}\left(\Box + \frac{1}{4}R\right) H_\alpha + \frac{i}{2} F_{\alpha\beta} H_\beta + i\phi K_\alpha - \psi_{\alpha\dot{\alpha}} v^{\dot{\alpha}}.
\]

If we now work out the commutator of a $Q$ and a $G$-transformation, we find the following pattern:

\[
[\delta, \delta''] \Phi = -i\epsilon\epsilon^\alpha\dot{\beta} D_{\alpha\beta}\Phi
\]

where $\Phi$ stands for any matter field.
The action that results from (5.13) after putting $\lambda$, $\eta$, $\chi$ and $G$ to zero is invariant under the truncated transformations written above, provided that the parameters $\epsilon$ and $\epsilon_{\alpha\beta}$ are covariantly constant. This action takes the following form,

$$\mathcal{L} = \mathcal{L}^{\text{DW}} + \mathcal{L}_0 + m\mathcal{L}_m,$$  

(6.9)

where,

$$\mathcal{L}_0 = \int d^4x e \left[ \tilde{H}^\alpha \left( \Box + \frac{1}{4} R \right) H_\alpha + \frac{i}{2} \tilde{H}^\alpha \tilde{F}_{\alpha\beta} H^\beta + i\tilde{u}^\alpha D_{\alpha\dot{a}} v^\dot{a} - i\tilde{v}^{\dot{a}} D_{\dot{a}\alpha} u^\alpha 
+ \tilde{K}^\alpha K^\alpha_{\dot{a}} + \tilde{H}^\alpha \tilde{\psi}_{\alpha\beta} \tilde{v}^{\beta} - \tilde{v}^{\beta} \tilde{\psi}_{\alpha\beta} H^\alpha + i\tilde{v}^{\dot{a}} \tilde{\phi} v_{\dot{a}} \right],$$

$$m\mathcal{L}_m = \int d^4x e \left[ m^2 \tilde{H}^\alpha H_\alpha + m(\overline{u}^\alpha u_\alpha + \overline{v}^{\dot{a}} v_{\dot{a}}) - im\tilde{H}^\alpha \tilde{\phi} H_\alpha \right].$$

(6.10)

The action (6.9) represents the coupling of topological matter to a subset of the Donaldson-Witten multiplet. The question arises if this truncation can be consistently extended to (3.19), (3.21), (3.22) and (5.15) and formulate a theory of topological matter coupled to topological Yang-Mills which would be simpler than the preceding one. If we put $\lambda$, $\eta$, $\chi$ and $G$ to zero in (3.19), (3.21), (3.22) and (5.15), the resulting action contains only $F^2_+$, and is not $Q$-invariant. Then, this procedure has to be discarded. Another possibility is simply to leave all what concerns to Donaldson-Witten theory untouched, both the action and the $Q$-transformations of its fields. Then there is no problem, and we arrive to a satisfactory theory. But an important caveat should be pointed out. The truncated $Q$-transformations of the Donaldson-Witten fields coincide with the old ones, and this is why the truncation can be extended, but the $G$ transformations are different before and after the truncation, and this destroys the $G$ symmetry of (5.15).

The conclusion is that the truncation yields a $Q$-invariant theory of topological matter coupled to topological Yang-Mills, but $G$-symmetry is lost. If topological gravity is not at issue, this loose is irrelevant. Nevertheless, the inclusion of topological gravity will probably need that symmetry, and then singles out the
whole theory as the only one to which it can be consistently coupled. As for the topological character of the theory, this feature depends on the $Q$-exactness of its energy-momentum tensor. It should be fully $Q$-exact off-shell because the action is not exact and therefore no equations of motion can be used. These aspects of the theory are discussed in the next section.
7. Energy-momentum tensors

An important issue to address in every topological field theory of Witten type is the calculation of the energy-momentum tensor. These theories involve fields of integer and semi-integer spin (bosons, fermions and ghosts) living in a curved manifold $\mathcal{M}$, so we need to introduce a vierbein $e^{a\mu}$ and a spin connection $\omega_{\mu}^{ab}$ to define semi-integer spin fields and their spacetime covariant derivatives. We assume that $\mathcal{M}$ admits these structures.

From now on we will indicate vector indices on which the twisted local Lorentz transformations (2.6) act (flat or tangent indices) by using letters from the beginning of the Latin alphabet (a, b, . . . ), and vector indices on which local translations (general coordinate transformations) act (curved or world indices) by letters from the middle of the Greek alphabet ($\mu$, $\nu$, . . . ). The vierbein converts one kind of indices into the other.

It is necessary to declare which position of the indices is considered as fundamental, and which is the result of applying the metric tensor. This is important in order to keep track of dependences on the metric. Our conventions are:

$$
e^{a\mu}, \quad D_\mu \longrightarrow \text{fundamental}$$

$$e_\mu^a = g_{\mu\nu} e^{a\nu}, \quad D^\mu = g^{\mu\nu} D_\nu \longrightarrow \text{derived} \quad \quad \quad \quad (7.1)$$

Pauli matrices are always defined in a locally inertial system of reference, and according to our conventions they carry a tangent index. The action of the spacetime covariant derivative on spin-1/2 fields takes the form

$$D_\mu H_\alpha = \partial_\mu H_\alpha + \frac{1}{2} \omega_{\mu}^{ab} (\sigma_{ab})_\alpha^\beta H_\beta - i A_\mu H_\alpha,$$

$$D_\mu \overline{H}^\alpha = \partial_\mu \overline{H}^\alpha - \frac{1}{2} \omega_{\mu}^{ab} \overline{H}^\beta (\sigma_{ab})_\beta^\alpha + i \overline{H}^\beta A_\mu,$$

$$D_\mu \hat{v}^{\dot{\alpha}} = \partial_\mu \hat{v}^{\dot{\alpha}} + \frac{1}{2} \omega_{\mu}^{ab} (\tilde{\sigma}_{ab})_\dot{\alpha}^\dot{\beta} \hat{v}^{\dot{\beta}} - i A_\mu \hat{v}^{\dot{\alpha}},$$

$$D_\mu \overline{\nu}_{\dot{\alpha}} = \partial_\mu \overline{\nu}_{\dot{\alpha}} - \frac{1}{2} \omega_{\mu}^{ab} \overline{\nu}_{\dot{\beta}} (\tilde{\sigma}_{ab})_\dot{\beta}^\dot{\alpha} + i \overline{\nu}_{\dot{\alpha}} A_\mu. \quad \quad \quad \quad (7.2)$$
where $\sigma_{ab}$ are the spin matrices defined in the Appendix. The last identity needed is the variation of the spin connection under a change in the vierbein:

$$
\delta \omega^a_{\mu} = \frac{1}{2} \left[ e^{[a][\rho}] \delta e^b_{[\rho,\mu]} + e^{[a][\rho]} e^b_{\epsilon\mu} \delta e^c_{\rho,\sigma} \right].
$$

(7.3)

The energy-momentum tensor is defined as:

$$
T_{\mu\nu} = \frac{e_\alpha^\mu}{\sqrt{g}} \frac{\delta S}{\delta c_{\alpha\nu}},
$$

(7.4)

where we consider arbitrary variations of the vierbein, not only those which lead to variations of the metric tensor. The $T_{\mu\nu}$ corresponding to $\mathcal{L}_0^{\text{g.eYM}}$ in (5.14) reads,

$$
T_{\mu\nu} = \frac{1}{4} \left[ D_{\alpha\dot{a}} \bar{H}^{\dot{a}} \left( \tilde{\sigma}^{(\nu)} \right) \tilde{\alpha} \dot{\beta} D^{\mu} H_\beta - \bar{H}^{\dot{a}} \left( \sigma^{(\nu)} \right)_{\alpha\dot{a}} \tilde{D}^{\mu} \tilde{D}^{\alpha} H_\beta \\
- i\bar{u}^{\alpha} \left( \sigma^{(\nu)} \right)_{\alpha\dot{a}} D^{\mu} \dot{v}^\dot{\alpha} + i\bar{v}_\alpha \left( \tilde{\sigma}^{(\nu)} \right) \tilde{\alpha} \dot{a} D^{\mu} u_\alpha \right] \\
+ \frac{1}{2} g_{\mu\nu} \left[ \bar{H}^{\dot{a}} D_{\alpha\dot{a}} D^{\alpha} H_\beta + \left( \tilde{D}^{\alpha} D_{\alpha\dot{a}} \bar{H}^{\dot{a}} \right) H_\beta + i\bar{u}^{\alpha} \tilde{D}^{\alpha} \dot{v}^\dot{\alpha} - i\bar{v}_\alpha \tilde{D}^{\alpha} \bar{u} \right] \\
+ 2K^{\alpha} \bar{K}_\alpha - 2\bar{H}^{\dot{a}} \psi_{\alpha\dot{a}} \dot{v}^\dot{\alpha} + 2\bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} \bar{H}^\alpha + 2i\bar{H}^{\alpha} \eta_{\alpha} - i\bar{u}^{\alpha} \eta_{\alpha} - 2i\bar{H}^{\alpha} \chi_{\alpha\beta} \dot{u}^\beta \\
+ 2\sqrt{\bar{u}} \chi_{\alpha\beta} \bar{H}^\beta + 2\sqrt{\bar{u}} \chi_{\alpha\beta} \dot{u}_{\alpha} + 2\sqrt{\bar{u}} \chi_{\alpha\beta} \eta_{\alpha} + i\bar{H}^{\alpha} G_{\alpha\beta} \bar{H}^\beta + \bar{H}^{\alpha} \left\{ \phi, \lambda \right\} H_\alpha \right] \\
+ \frac{1}{2} \left[ \bar{H}^{\dot{a}} \left( \sigma^{(\nu)} \right)_\alpha \dot{\beta} D_{\alpha\dot{a}} \bar{H}^{\dot{a}} \gamma H_\gamma - \bar{D}^{\alpha\dot{a}} D_{\alpha\dot{a}} \bar{H}^{\dot{a}} \left( \sigma^{(\nu)} \right)_\beta \gamma H_\gamma - i\bar{v}_\alpha \left( \tilde{\sigma}^{(\nu)} \right)_\beta \dot{\alpha} \bar{D}^{\beta} \dot{u}_{\beta} \\
- i\bar{D}^{\dot{\alpha} \dot{a}} \bar{v}_\alpha \left( \sigma^{(\nu)} \right)_\beta \dot{u}_{\beta} + i\bar{v}_\alpha \left( \tilde{\sigma}^{(\nu)} \right)_\beta \dot{\alpha} \bar{D}^{\beta} \dot{v}^\beta + D_{\alpha\dot{a}} \bar{v}_\alpha \left( \bar{\sigma}^{(\nu)} \right)_\beta \dot{\alpha} \dot{v}^\beta \right].
$$

(7.5)

This $T_{\mu\nu}$ is not $Q$-exact, even in the free case. If we only consider matter, it can be written as a $Q$-variation of some $\Lambda^{\mu\nu}$ plus something that vanishes on shell. The general expression is

$$
T^{\mu\nu} = \left\{ Q, \Lambda^{\mu\nu}_{\text{matter}} \right\} + g^{\mu\nu} T_{\alpha\Lambda} \\
- \frac{1}{2} \left[ \bar{H}^{\dot{a}} \left( \sigma^{(\mu)} \right)_{\alpha\dot{a}} \psi^{(\nu)} \dot{v}^\dot{\alpha} + \bar{v}_\alpha \left( \sigma^{(\mu)} \right) \tilde{\alpha} \psi^{(\nu)} H_\alpha + \bar{H}^{\dot{a}} \left( \sigma^{(\mu)} \right)_\alpha \psi^{(\nu)} \psi^{(\beta)} H_\beta \right] \\
- \bar{v}_\alpha \left( \tilde{\sigma}^{(\mu)} \right) \tilde{\alpha} \beta \psi^{(\beta)} H_\beta + \bar{H}^{\dot{a}} \psi_{\alpha\dot{a}} \left( \bar{\sigma}^{(\mu)} \right) \tilde{\alpha} \psi^{(\beta)} - \bar{v}_\alpha \psi^{(\beta)} \left( \sigma^{(\mu)} \right)_\alpha \psi^{(\beta)} H_\beta \right].
$$

(7.6)
where

\[
\Lambda_{e,A} = -\frac{i}{4}\left[\overline{\Pi}^{\alpha} (\sigma^{(\nu)})_{\alpha} \overrightarrow{D}^{\mu} v^\dot{\alpha} + \Sigma_{\alpha}(\sigma^{(\nu)})_{\dot{\alpha}} \overrightarrow{D}^{\mu} H_{\alpha}\right] + \frac{i}{2}\left[\overline{\Pi}^{\alpha} (\sigma^{\mu\nu})_{\alpha} \overrightarrow{D}^\alpha v^\beta + D_{\alpha\dot{\alpha}} H^\alpha (\overline{\sigma}^{\mu\nu})_{\dot{\alpha}} v^\beta + D_{\dot{\alpha} \alpha}(\sigma^{\mu\nu})_{\alpha} H_{\beta}\right],
\]

\[
T_{e,A} = -\frac{1}{2}\left[\overline{\Pi}^{\alpha} D_{\alpha\dot{\alpha}} D^\beta H^\beta + (D^\beta_{\dot{\alpha}} D_{\alpha\dot{\alpha}} H^\alpha) H^\beta - i\overline{\psi}^\dot{\alpha} D_{\dot{\alpha} \alpha} v^\dot{\alpha} + i\overline{\psi}^\dot{\alpha} D_{\dot{\alpha} \alpha} u^\alpha - 2\overline{K}^\alpha K_{\alpha} - 2\overline{H}^{\alpha} \psi_{\alpha\dot{\alpha}} v^\dot{\alpha} + 2\overline{\psi}^\dot{\alpha} \psi_{\alpha\dot{\alpha}} H^\alpha + 2i\overline{H}^\alpha \eta u_{\alpha} - i\overline{\psi}^\dot{\alpha} \eta H_{\alpha} - 2i\overline{H}^\alpha \chi_{\alpha\beta} u^\beta + 2i\overline{\psi}^\dot{\alpha} \chi_{\alpha\beta} H^\beta + 2i\overline{\psi}^\dot{\alpha} \lambda u_{\alpha} + 2i\overline{\psi}^\dot{\alpha} \phi v_{\alpha} - i\overline{H}^\alpha G_{\alpha\beta} H^\beta + \overline{H}^\alpha \{\phi, \lambda\} H_{\alpha}\right].
\]

(7.7)

In the free case the last two sets of terms in (7.6) do not appear and therefore the energy-momentum tensor, excluding the part proportional to \(g_{\mu\nu}\), is \(Q\)-exact. On the other hand, the part proportional to \(g_{\mu\nu}\) vanishes on-shell. Since in the free case the action is \(Q\)-exact we are allowed to take the energy-momentum tensor on-shell and we therefore conclude that the theory is topological. The situation is very different for the theory coupled to topological Yang-Mills. Either the full or the truncated theory do not possess neither a \(Q\)-exact action nor a \(Q\)-exact energy-momentum tensor. The energy momentum tensor is \(Q\)-exact up two terms which vanish on-shell. However, since in this case the theory is not necessarily exact in the small coupling limit we can not conclude that the theory is topological. These facts are an indication that this theory might represent a new phenomena related to a topological symmetry breaking caused by the introduction of matter interactions.

The calculation of the energy-momentum tensor of (5.15) does not offer any new difficulty. It turns out to be \(Q\)-exact, as a consequence of the \(Q\)-exactness of the action and the independence of the transformations (3.19) on the underlying geometry. Notice that the term \(F^{\pm}_{\alpha\beta}\) is a component of a two-form. The transformation of \(\psi_{\alpha\dot{\alpha}}\) include a covariant derivative, but it acts on a scalar. Finally, the only field that transforms with the covariant derivative of a vector is \(G_{\alpha\beta}\), which is an auxiliary field that can be set to zero.

The conclusion is that neither the full nor the truncated theories are strictly
topological, since the action is $Q$-invariant, but not $Q$-exact, and the energy-momentum tensor is $Q$-exact only on-shell. As the theory does not coincide with its classical limit, the non $Q$-exact terms in $T_{\mu\nu}$ cannot be discarded.
8. Another type of topological matter in 4D

So far all our models have been built starting from an $N = 2$ theory and performing a twist. One can also think of starting directly from a set of fields which satisfy some $\delta$-transformation properties, and try to write an action invariant under that transformations. The only drawback of this approach is the difficulty to implement other symmetries (as $\delta'$ or $\delta''$). We present now an example of this procedure, which turns out to be a truncated twisted version of the relaxed hypermultiplet [19]. The analysis to build this model from the relaxed hypermultiplet will not be presented here. It goes along the same lines as the ones which led to the matter models in the previous sections. We present only the truncated model and therefore all symmetries but the one corresponding to $Q$ have been lost. A coupling of this model to topological gravity would need the full theory. Its form will presented elsewhere.

The basic set of fields of the model has the same spin content as in the previous model. We will denote these fields by $H_\alpha$, $u_\alpha$, $v_{\dot{\alpha}}$, and $L_{\dot{\alpha}}$. The reasons to make some of the choices done for the previous model will become clear below. The symmetry transformations turn out to be:

\begin{equation}
\begin{aligned}
\tilde{\delta}H_\alpha &= u_\alpha, \\
\tilde{\delta}u_\alpha &= 0, \\
\tilde{\delta}v_{\dot{\alpha}} &= L_{\dot{\alpha}} - \nabla_{\dot{\alpha}} \dot{\alpha} \bar{H}_\alpha, \\
\tilde{\delta}L_{\dot{\alpha}} &= \nabla_{\dot{\alpha}} \dot{\alpha} u_\alpha.
\end{aligned}
\end{equation}

Note that this $\tilde{\delta}$ is nilpotent, which considerably simplifies the issue of writing invariant actions; it suffices to look for $\tilde{\delta}$-exact functionals of the fields with the right dimension and ghost number. The field $L_{\dot{\alpha}}$ has the right dimension to be an auxiliary field, and it can be thought as the field needed to render the $\tilde{\delta}$-transformation nilpotent. The next task is to find out the action which is annihilated by this transformation. The choice which leads to a $\tilde{\delta}$-exact action with kinetic terms and
includes $L_{\dot{\alpha}}$ as an auxiliary field is the following:

$$L_2 = \frac{1}{2} \int_M d^4x \sqrt{g} \left[ \overline{H}^{\dot{\alpha}} \nabla_{\alpha} v^{\dot{\alpha}} + \overline{\nu}^{\dot{\alpha}} \nabla_{\alpha} H^\alpha + \overline{L}^{\dot{\alpha}} v_{\dot{\alpha}} + \overline{\nu}^{\dot{\alpha}} L_{\dot{\alpha}} \right]. \quad (8.2)$$

Its full expanded expression reads almost exactly like the one of our previous model (4.18) in the massless case:

$$L_2 = \int d^4x \sqrt{g} \left[ \nabla_{\alpha} H_{\dot{\beta}}^{\dot{\alpha}} \nabla_{\beta} H_{\dot{\beta}}^{\dot{\alpha}} + \overline{\nu}_{\alpha} \nabla_{\alpha} v^{\dot{\alpha}} - \overline{\nu}_{\dot{\alpha}} \nabla_{\dot{\alpha}} u^\alpha + \overline{L}^{\dot{\alpha}} L_{\dot{\alpha}} \right]. \quad (8.3)$$

Being the action $L_2$ manifestly $\tilde{\delta}$-invariant, the only condition for the theory being topological is the $\tilde{\delta}$-exactness of its energy-momentum tensor. This is a highly non-trivial condition since, as we have seen in the previous chapter, the exactness of the action does not imply the same property for the energy-momentum tensor because of the dependence of the transformation on the spin connection through the covariant derivatives. In our case some help to end with a topological quantum field theory comes from the fact that we can effectively consider the theory on-shell due to the exactness of the action, as we did in our previous model in the free case. We show now that this is also the case in the present model. The energy-momentum tensor can be written as follows:

$$T_{\mu\nu} = \tilde{\delta} \Lambda_{\mu\nu} + M_{\mu\nu}, \quad (8.4)$$

where,
\[ \Lambda^{\mu\nu} = \frac{1}{4} \left[ H^\alpha (\sigma^{(\nu)}_{\alpha\tilde{\alpha}} \nabla^\mu \nabla^\alpha v_{\tilde{\alpha}}) + \nabla^\alpha (\tilde{\sigma}^{(\nu)}_{\alpha\tilde{\alpha}} \nabla^\mu H_{\tilde{\alpha}}) \right] - \frac{1}{2} \left[ H^\alpha (\sigma^{\mu\nu})_{\alpha\beta} \nabla_{\beta} v^\beta \right] + \nabla_{\alpha} (\tilde{\sigma}^{\mu\nu})_{\alpha\beta} H_{\beta} + \nabla_{\alpha\dot{\alpha}} H^\alpha (\tilde{\sigma}^{\mu\nu})_{\alpha\beta} v^\beta + \nabla\dot{\alpha} \nabla_{\alpha\dot{\alpha}} (\sigma^{\mu\nu})_{\alpha\beta} H_{\beta} \]

\[ M_{\mu\nu} = \frac{1}{4} \left[ H^\alpha (\sigma_{\mu\nu})_{\alpha\beta} \nabla_{\beta} \dot{L} + \nabla_{\alpha} \dot{L} (\tilde{\sigma}_{\mu\nu})_{\alpha\beta} H^\beta \right] - \nabla_{(\mu} \bar{H}^\beta (\sigma_{\nu)}_{\alpha\beta} \nabla_{\beta} \dot{L} - \dot{L} (\tilde{\sigma}_{\mu\nu})_{\alpha\beta} \nabla_{\beta} \bar{H}^\alpha \]

\[ + \frac{1}{2} \left[ L_{\alpha} (\tilde{\sigma}_{\mu\nu})_{\alpha\beta} \nabla_{\beta} \dot{H}^\alpha + \nabla_{\gamma} L_{\alpha} (\tilde{\sigma}_{\mu\nu})_{\gamma\alpha} \dot{H}^\alpha - \nabla_{\alpha\beta} \bar{H}^\alpha (\tilde{\sigma}_{\mu\nu})_{\beta\gamma} \dot{L}^\gamma \right.

\[ - \bar{H}^\alpha (\sigma_{\mu\nu})_{\alpha\beta} \nabla_{\beta} \dot{L}^\gamma \right] - \frac{1}{2} g_{\mu\nu} \left[ \bar{H}^\alpha \nabla_{\alpha\beta} \dot{L}^\beta + \left( \nabla_{\alpha} \dot{L}_{\alpha} \right) \dot{H}^\alpha \right]. \tag{8.5} \]

The crucial observation is that, as explained before, we can set the auxiliary field to zero. In this case the energy-momentum tensor is \( \bar{\delta} \)-exact and therefore this theory is topological. Now we define a coupling of this multiplet to the Donaldson-Witten multiplet by means of the following generalized transformations:

\[ \bar{\delta} H_{\alpha} = u_{\alpha}, \quad \bar{\delta} H_{\alpha} = \bar{u}_{\alpha}, \]

\[ \bar{\delta} u_{\alpha} = -i\phi H_{\alpha}, \quad \bar{\delta} \bar{u}_{\alpha} = i\phi \bar{H}_{\alpha}, \]

\[ \bar{\delta} v_{\dot{\alpha}} = L_{\dot{\alpha}} - D_{\dot{\alpha}} \dot{H}^\alpha, \quad \bar{\delta} \bar{v}_{\dot{\alpha}} = \bar{L}_{\dot{\alpha}} - D_{\dot{\alpha}} \dot{H}^\alpha, \]

\[ \bar{\delta} L_{\dot{\alpha}} = D_{\dot{\alpha}} \dot{u}^\alpha - i\psi_{\dot{\alpha}} \dot{H}^\alpha - i\phi \dot{v}_{\dot{\alpha}}, \quad \bar{\delta} \bar{L}_{\dot{\alpha}} = D_{\dot{\alpha}} \bar{v}^\alpha + i\psi_{\dot{\alpha}} \bar{H}^\alpha + i\phi \bar{v}_{\dot{\alpha}}. \tag{8.6} \]

As in Donaldson-Witten theory, these transformations close up to a gauge transformation whose gauge parameter is \( \phi \):

\[ \bar{\delta}^2 H_{\alpha} = -i\phi H_{\alpha}, \]

\[ \bar{\delta}^2 u_{\alpha} = -i\phi u_{\alpha}, \]

\[ \bar{\delta}^2 v_{\dot{\alpha}} = -i\phi v_{\dot{\alpha}}, \]

\[ \bar{\delta}^2 L_{\dot{\alpha}} = -i\phi L_{\dot{\alpha}}. \tag{8.7} \]

This implies that we can take any suitable quantity of ghost number -1 and obtain an action using \( \bar{\delta} \). Such a quantity just must be gauge invariant. The free one does
the job, and we would obtain the same action and energy-momentum tensor as in the truncated theory except for the \( K^\alpha K_\alpha \) term, which is now \(-L^\dot{\alpha}L_{\dot{\alpha}}\):

\[ \tilde{S}^g = \mathcal{L}^g_{YM} + \tilde{\mathcal{L}}^{g_{YM}}_0, \quad (8.8) \]

where,

\[
\tilde{\mathcal{L}}^{g_{YM}}_0 = \int d^4x \left[ \mathcal{H}^\alpha (\square + \frac{1}{4}R) H_\alpha + \frac{i}{2} \mathcal{H}^\alpha F^{\alpha \beta}_{\dot{\alpha} \dot{\beta}} H^\beta + i\tilde{\mathcal{H}}^{\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} v^\alpha - i\tilde{\mathcal{D}}_{\alpha \dot{\alpha}} u^\alpha \right. \\
\left. - \mathcal{L}^\alpha L'_\alpha + \mathcal{H}^\alpha \psi_{\dot{\alpha} \dot{\beta}} v^\dot{\beta} - \tilde{\mathcal{D}}^\dot{\alpha} \psi_{\alpha \dot{\beta}} H^\beta + i\tilde{\mathcal{D}}^\dot{\alpha} \phi v^\dot{\alpha} \right]. \quad (8.9) \]

Actually, one has a result entirely analogue to the one obtained in the free case, (8.4) and (8.5). The energy-momentum tensor is then \( \tilde{\delta} \)-exact in the coupled case except for terms linear in the auxiliary field. This fact, together with the exactness of the action leads to the conclusion that this theory is topological in both the free and the coupled case.
9. Final comments and remarks

Let us first analyze the features of the possible observables of the models presented in the previous sections. Observables are $Q$-invariant quantities constructed out of the fields of the theory. Certainly, the observables (3.28) of topological Yang-Mills theory are $Q$-invariant quantities since all the fields entering in them possess the same $Q$-transformations before and after the coupling to matter fields. One would like to have another set of observables involving matter fields. We have done a thorough analysis to find observables which involve matter fields and we have not found any. This analysis goes in two steps. First, one writes all possible gauge invariant operators quadratic in matter fields of a given ghost number. Then one checks if it is possible to obtain a linear combination of them which is $Q$-invariant and it is not $Q$-exact. Our analysis shows that there are no operators of that type. Considering powers of these operators one is led the same conclusion. One is therefore left to the study of the observables (3.28) in the presence of matter. Of course, the resulting vev of the operators (3.28) are rather different than in the theory with no matter. There are relevant contributions from the matter fields in their functional integration. These contributions are very important. Indeed, for the models of sect. 6 and 7 it is not guaranteed that these observables lead to topological quantities. This follows from the fact that both, the action and energy-momentum tensor of the theory, are not $Q$-exact. For the topological matter model of sect. 8, however, since the action is $Q$-exact, the small coupling limit is exact and the observables do indeed lead to topological quantities.

The computation of these observables using the action (5.14) leads to quantities which are labeled, besides the usual homology cycles, by the matter representation chosen in the action (5.14). These quantities have properties very similar to Donaldson invariants. Let us denote by $H_*(M)$ the homology groups of the spin manifold $M$. These observables are also polynomials on $H_*(M) \times H_*(M) \times \ldots \times H_*(M)$ as Donaldson invariants are. This property is based only on the fact that the exterior differential of any of the operators (3.27) is $Q$-exact. There is no need to have
a $Q$-exact energy-momentum tensor for this to hold, just a $Q$-invariant action as it is the case. On the other hand, these observables, as in Donaldson-Witten theory, can be computed in the limit $e \to 0$. The reason for this is that all the dependence on the coupling constant $e$ in (5.14) and (8.8) is contained in a part of the action which is $Q$-exact. Again, there is no need of a $Q$-exact energy-momentum tensor for this to hold. This implies in particular that the vev of the observables are independent of $e$. Thus, the quantities associated to the vev of arbitrary products of the operators (3.28) constitute a generalization of Donaldson invariants which, however, it is not guaranteed that in general are topological invariants. The breaking of the invariance comes from the fact for an arbitrary product of operators (3.28),

$$\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \langle \prod O^{(\gamma)} \rangle = \langle \prod O^{(\gamma)} \tau_{\mu\nu} \rangle,$$

(9.1)

where $\tau_{\mu\nu}$ is the part of the energy-momentum tensor which is not $Q$-exact. For models based on the action (5.14) one does not possess an argument ensuring that (9.1) vanishes. For models based on the action (8.8), however, one can argue that the right hand side of (9.1) vanishes using the $Q$-exactness of the action.

It is also important to remark that the actions (5.14) and (8.8) of the models under considerations have a very similar structure. Their difference is very subtle since it resides in the form of the auxiliary fields. One would have to study in detail the role played by the auxiliary fields. Two possibilities could occur. If their role is trivial, the model based on the action (5.14) would be equivalent to the model based on the action (8.8) and therefore one could conclude that the model based on (5.14) is topological. If the role played by the auxiliary fields is non-trivial the model based on the action (5.14) could very well represent a situation in which the topological symmetry is broken. In either case that model is interesting and deserves further investigations.

Let us consider finally the question of the mirror-like behavior in four dimensions. The two models which have been constructed do not seem to lead to this
kind of phenomena. Their difference resides on the auxiliary fields and these possess rather simple couplings. It is likely that one has to study the coupling of the two types of matter to topological gravity to observe some kind of mirror-like phenomena. Certainly, the structure of the couplings of the auxiliary fields of the matter multiplets will be much more complicated. This study, however, requires to build the full theory resulting from the twisting of the relaxed hypermultiplet, and not the truncated one presented in sect. 8. An additional set of auxiliary fields appear in that situation which announces that the analysis of the models are rather different. Matter couplings to topological gravity will be studied in future works.

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APPENDIX

In this work we use spinor notation for all Lorentz representations, denoting spinor indices by Greek letters, dotted for $(0, \frac{1}{2})$, and undotted for $(\frac{1}{2}, 0)$. An arbitrary irreducible representation $(L, R)$ is labeled with $2L$ totally symmetrized undotted indices, and $2R$ totally symmetrized dotted indices. The symmetrization symbol of $N$ indices means the sum over all their permutations without $1/N!$, and similarly for antisymmetrization. For instance:

\[ X_{\alpha \beta} = X_{\alpha \beta} + X_{\beta \alpha}, \]
\[ X_{[\alpha \beta]} = X_{\alpha \beta} - X_{\beta \alpha}, \]  
and similarly for dotted indices. Spinor indices are raised and lowered by the second-rank antisymmetric symbol $C_{\alpha \beta}$:

\[ \psi^\alpha = \psi^\beta C_{\beta \alpha}, \quad \psi_\alpha = C^{\alpha \beta} \psi_\beta. \]

The same convention holds for dotted indices. The $C_{\alpha \beta}$ symbol is defined as follows:

\[ C_{\alpha \beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = C^{\dot{\alpha} \dot{\beta}}, \quad C^{\alpha \beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = C_{\dot{\alpha} \dot{\beta}}. \]  

Vectors belong to the $(\frac{1}{2}, \frac{1}{2})$ representation, and are labeled with one undotted and one dotted index. We have to distinguish two kinds of vectors: commuting and anticommuting. We underline the composite index in the case of commuting vectors. Examples of these two types of vectors are $A_{\alpha \beta}$ and $\psi_{\alpha \beta}$. Tangent vector indices are denoted with lower-case roman letters, and are related to the $\alpha \beta$ basis by some Clebsch-Gordan coefficients, the Pauli matrices,

\[ X^{\alpha \beta} = \sigma_a^{\alpha \beta} X^a \quad X^a = \frac{1}{2} \sigma_a^{\alpha \beta} X^{\alpha \beta}. \]

Pauli matrices with world vector indices are defined by means of the vierbein:

\[ (\sigma_{\mu})_{\alpha \dot{\alpha}} = \epsilon^a_{\mu} (\sigma_a)_{\alpha \dot{\alpha}}, \quad (\tilde{\sigma}_{\mu})^{\dot{\alpha} \alpha} = \epsilon^a_{\mu} (\tilde{\sigma}_a)^{\dot{\alpha} \alpha}. \]  

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The vierbein satisfies the usual relations

\[ e^a_\mu e^{a\nu} = g^{\mu\nu}, \quad e^a_\mu e_{b\mu} = \eta_{ab}, \tag{A.6} \]

where \( \eta_{ab} \) is the flat minkowskian metric tensor. The vierbein is required to be covariantly constant:

\[ \nabla_\mu e^{a\nu} = 0. \tag{A.7} \]

This last identity relates the vierbein to the spin connection:

\[ \omega^{ab}_\mu = \frac{1}{2} \left[ e_{[a\rho} e^{b]\rho] + e_{[a\rho} e^{b]\sigma} e_{c\mu} e^{c}_{\rho\sigma} \right]. \tag{A.8} \]

Although the spin connection does not transform as a tensor, its variation does. This property permits to calculate \( \delta \omega^{ab}_\mu \) simply going to a locally inertial system of reference and covariantizing the result. Its variation is given in (7.4).

Pauli matrices satisfy the following identities

\[ (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} = g^{\mu\nu} \delta^{\dot{\alpha}}_{\dot{\beta}} + 2(\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}, \]
\[ (\sigma^{\mu\nu})^\beta_\alpha = g^{\mu\nu} \delta^\beta_\alpha + 2(\sigma^{\mu\nu})^\beta_\alpha. \tag{A.9} \]

which also serve to define the spin matrices \( \sigma^{\mu\nu} \) and \( \tilde{\sigma}^{\mu\nu} \). These matrices are antisymmetric in their vector indices and symmetric in their spin indices. Some useful identities needed in the calculation of the energy-momentum tensor are:

\[ (\sigma^{\mu\nu})^\beta_\alpha (\sigma^{\lambda})^{\alpha\beta} = -\frac{1}{2} g^{\lambda[\mu} (\sigma^{\nu]})^{\alpha\beta} + i \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} (\sigma^{\kappa})^{\alpha\beta}, \]
\[ (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} (\tilde{\sigma}^{\lambda})^{\dot{\alpha}\beta} = -\frac{1}{2} g^{\lambda[\mu} (\tilde{\sigma}^{\nu]})^{\dot{\alpha}\beta} - i \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} (\tilde{\sigma}^{\kappa})^{\dot{\alpha}\beta}, \]
\[ (\sigma^{\lambda})^{\alpha\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} = \frac{1}{2} g^{\lambda[\mu} (\sigma^{\nu]})^{\alpha\dot{\beta}} + i \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} (\sigma^{\kappa})^{\alpha\dot{\beta}}, \]
\[ (\tilde{\sigma}^{\lambda})^{\alpha\dot{\alpha}} (\sigma^{\mu\nu})^\beta_\alpha = \frac{1}{2} g^{\lambda[\mu} (\tilde{\sigma}^{\nu]})^{\alpha\beta} - i \frac{1}{2} \epsilon^{\mu\nu\lambda\kappa} (\tilde{\sigma}^{\kappa})^{\alpha\beta}. \tag{A.10} \]

Self-dual tensors in four dimensions have two symmetric undotted indices. The self-dual part of an antisymmetric second-order tensor can be extracted using the
spin matrices, $\sigma^{ab}$:

$$X_{\alpha\beta}^+ = 2X^{ab}(\sigma_{ab})_{\alpha\beta}, \quad X_{\dot{\alpha}\dot{\beta}}^- = 2X^{ab}(\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}}.$$  \hspace{1cm} (A.11)
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