On the Quantumness of a Hilbert Space

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Abstract

We derive an exact expression for the quantumness of a Hilbert space (defined in C. A. Fuchs and M. Sasaki, Quant. Info. Comp. 3, 377 (2003)), and show that in composite Hilbert spaces the signal states must contain at least some entangled states in order to achieve such a sensitivity. Furthermore, we establish that the accessible fidelity for symmetric informationally complete signal ensembles is equal to the quantumness. Though spelling the most trouble for an eavesdropper because of this, it turns out that the accessible fidelity is nevertheless easy for her to achieve in this case: Any measurement consisting of rank-one POVM elements is an optimal measurement, and the simple procedure of reproducing the projector associated with the measurement outcome is an optimal output strategy.

1 Introduction

Memorable experiences sometimes happen in elevators. I have had two in my life: This paper has to do with both.

The setting of the second was QIC ’96 in Fuji-Hakone, Japan. It was the first time I met Alexander Holevo, to whom this paper is dedicated. As we entered an elevator, Richard Jozsa gave Prof. Holevo a brief description of the then recent quantum channel-capacity superadditivity result of Ref. [1]. Holevo—apparently not quite absorbing what he had just heard—said something like, “The issue of quantum channel capacity is very tricky. For instance, collective measurements on individual signals can increase capacity. There is no classical analogue to this. You can read about the phenomenon in my 1979 paper; I will give you the reference.” When Holevo left the elevator, Jozsa and I looked at each other in awe! Had he really known this effect so long before Ref. [1] and even before Ref. [2]? Sure enough [3]. And like so many of Holevo’s great early contributions to quantum information theory, it went essentially unnoticed for many years. He has always been a man ahead of his time.

In contrast, ten years before that second experience, I met an odd fellow in an elevator at the physics department of the University of Texas. All alone, with the doors shut, he looked at me with a crazed look in his eyes and asked, “What is energy?” Feeling uncomfortable, I turned my own eyes to the ground and was happy that the doors soon opened. I slipped away, but regaining composure just before the doors shut again, I replied, “I don’t know, that which gravitates?” A cheap answer! But at least I had something to say. Looking back over the years I thank my lucky stars he didn’t ask a tough question. He could have asked, “What is Hilbert space?”

Associated with each quantum system is a Hilbert space. In the case of finite dimensional ones, it is commonly said that the dimension corresponds to the number of distinguishable states a system can “have.” But what are these distinguishable states? Are they potential properties
a system can possess in and of itself, much like a cat’s possessing the binary value of whether it is alive or dead? If the Bell-Kochen-Specker theorem\(^4\) has taught us anything, it has taught us that these distinguishable states should not be thought of in that way.

In this paper, I present some results that take their motivation (though not necessarily their interpretation) in a different point of view about the meaning of a system’s dimensionality. From this view, dimensionality may be the raw, irreducible concept—the single property of a quantum system—from which other consequences are derived (for instance, the maximum number of distinguishable preparations which can be imparted to a system in a communication setting)\(^5\)\(^6\). The best I can put my finger on it is that dimensionality should have something to do with a quantum system’s “sensitivity to the touch,”\(^7\)\(^8\) its ability to be modified with respect to the external world due to the interventions of that world upon its natural course. Thus, for instance, in quantum computing each little push or computational step has the chance of counting for more than in the classical world.

Various aspects of quantum eavesdropping seem to be perfect for sussing out and quantifying such ideas. One is the setting introduced in Ref.\(^9\) and explored further in Ref.\(^10\). Here I show that, of the definitions spelled out there, the quantumness of a Hilbert space \(H_d\) of dimension \(d\) can be calculated explicitly. Moreover, if a certain class of symmetric signal ensembles exists, such a sensitivity to eavesdropping can by achieved by using signals drawn from an ensemble of no more than \(d^2\) elements. Interestingly, for composite systems, entangled states are a necessary ingredient for achieving the quantumness: In particular, by the measure of quantumness, systems’ “sensitivity to the touch” is strictly superrmultiplicative.

The plan of the paper is as follows. In Section 2 I reacquaint the reader with the definitions of Ref.\(^9\). In Section 3 I derive an expression for the quantumness of \(H_d\). I also point out that on composite Hilbert spaces \(H_{d_1} \otimes H_{d_2}\), ensembles containing entangled states are necessary for achieving the quantumness. In Section 4 I introduce the idea of a symmetric informationally complete signal ensemble\(^11\)\(^12\) and show that—if it exists—it achieves the quantumness of the Hilbert space.\(^1\) In Section 5 I show that when a complete set of mutually unbiased bases\(^1\)\(^4\)\(^5\) exists for \(H_d\), signals drawn from such an ensemble of \(d(d + 1)\) elements also achieve the quantumness. Furthermore, I look into the question of the minimal number of elements in an ensemble required for it to achieve the quantumness of \(H_d\). In Section 6 I reemphasize the open question, and finally in Section 7 I conclude with the hint of another elevator story.

### 2 Preliminary Notations

Recall the main definitions from Ref.\(^9\). Given a signal ensemble \(\mathcal{P}\) (i.e., a collection \(\mathcal{P} = \{\Pi_i, \pi_i\}\) of pure quantum states \(\Pi_i = |\psi_i\rangle\langle\psi_i|\) along with associated probabilities \(\pi_i\)), a measurement \(\mathcal{E} = \{E_b\}\) (i.e., a positive operator-valued measure or POVM\(^16\)) and a state-reproduction strategy \(\mathcal{M}: b \rightarrow \sigma_b\) (i.e., a map taking measurement outcomes to new quantum states), we can define an average fidelity for \(\mathcal{E}\) and \(\mathcal{M}\) according to

\[
F_P(\mathcal{E}, \mathcal{M}) = \sum_{b,i} \pi_i \text{tr}(\Pi_i E_b) \text{tr}(\Pi_i \sigma_b),
\]

The average fidelity represents an eavesdropper’s probability of going unnoticed after performing an “intercept-resend” strategy of this type.

The **achievable fidelity** for a given measurement \(\mathcal{E}\) is the average fidelity optimized over all reconstruction strategies \(\mathcal{M}\):

\[
F_P(\mathcal{E}) = \sup_{\mathcal{M}} F_P(\mathcal{E}, \mathcal{M}).
\]

The achievable fidelity, it turns out, can be explicitly calculated\(^9\) in terms of the trace-nonincreasing completely positive linear map—the **ensemble map**—\(\Psi : \mathcal{L}(H_d) \rightarrow \mathcal{L}(H_d)\) defined

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\(^4\)In another connection, the use of such ensembles for quantum cryptography has also been considered in Refs.\(^12\)\(^13\).
by
\[ \Psi(X) = \sum_i \pi_i \Pi_i X \Pi_i. \tag{3} \]

With it,
\[ F_P(\mathcal{E}) = \sum_b \lambda_1(\Psi(\mathcal{E}_b)) , \tag{4} \]

where \( \lambda_1(X) \) denotes the largest eigenvalue of a Hermitian operator \( X \).

The accessible fidelity of the ensemble \( P \) is the best possible average fidelity an eavesdropper can attain over all measurements and all reproduction strategies:
\[ F_P = \sup_{\mathcal{E}, M} F_P(\mathcal{E}, M) = \sup_{\mathcal{E}} \sum_b \lambda_1(\Psi(\mathcal{E}_b)) . \tag{5} \]

Thus, the accessible fidelity is a natural measure of an ensemble’s intrinsic sensitivity to eavesdropping.

Finally, the quantumness of a Hilbert space \( \mathcal{H}_d \) is smallest possible accessible fidelity the space can support
\[ Q_d = \inf_P F_P . \tag{6} \]

The quantumness, it should be noted, is an inverted measure: The smaller the quantumness of a Hilbert space, the greater a system’s ultimate sensitivity to intercept-resend style quantum eavesdropping.

### 3 Quantumness of a Hilbert Space

Of the various ensembles explored in Ref. [9], the one with the smallest accessible fidelity was the (unique) unitarily invariant ensemble on \( \mathcal{H}_d \). Its accessible fidelity was shown to be \( 2/(d + 1) \). This establishes that
\[ Q_d \leq \frac{2}{d + 1}. \tag{7} \]

Since it is hard to imagine a more difficult ensemble for an eavesdropper to successfully eavesdrop upon than this one, it was speculated in Ref. [9] that
\[ Q_d = \frac{2}{d + 1}. \tag{8} \]

To prove this is the case, we establish the following theorem.

**Theorem 1** For any ensemble \( P \), the accessible fidelity for that ensemble satisfies
\[ F_P \geq \frac{2}{d + 1} . \tag{9} \]

To see this, we use a trick similar to the one used in Ref. [17]. For any (discrete) ensemble \( P = \{\Pi_i, \pi_i\} \), imagine an eavesdropper partaking in the following strategy. She performs a standard, but random, von Neumann measurement \( \mathcal{G} = \{G_b\}_{b=1}^d \) consisting of one-dimensional projection operators, and then uses a strategy \( M \) that simply reproduces the state \( G_b \) corresponding to the outcome she finds. By definition,
\[ F_P \geq F_P(\mathcal{G}, M) = \sum_{b,i} \pi_i (\text{tr}\Pi_i G_b)^2 . \tag{10} \]

However, also,
\[ F_P \geq F_P(\mathcal{G}, M) , \tag{11} \]

3
where the overline represents an average over all such measurements, for instance with respect to the unitarily invariant measure.

Thus, all we need to do is resurrect the measure $d\Omega$ of Eq. (111) in Ref. [9], along with the result of Eq. (114) there, to get:

$$F_P(G,M) = \sum_i \sum_b \pi_i \int (\text{tr}\Pi_i G_b)^2 d\Omega_G = d \sum_i \pi_i \int (\text{tr}\Pi_i G)^2 d\Omega_G$$

$$= d \sum_i \pi_i \int |\langle \psi_i | \phi \rangle|^4 d\Omega_\phi = d \sum_i \pi_i \frac{\Gamma(d) \Gamma(3)}{\Gamma(1) \Gamma(d+2)}$$

$$= d \frac{(d-1)!}{(d+1)!} = \frac{2}{d+1} \quad (12)$$

Here, $d\Omega_G$, $d\Omega_G$, and $d\Omega_\phi$ represent the various incarnations of the unitarily invariant measure as it is translated from being about complete von Neumann measurements to projection operators on $\mathcal{H}_d$ to normalized vectors in $\mathcal{H}_d$, respectively, and $G = |\phi \rangle \langle \phi|$ is a dummy one-dimensional projector.

It is interesting to couple Eq. (8) with the findings of Ref. [10]. By the result here, if we consider a composite Hilbert space $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$, with components of dimension $d_1$ and $d_2$, its quantumness is given by

$$Q_{\text{comp}} = \frac{2}{d_1 d_2 + 1} \quad (13)$$

On the other hand, in Ref. [10] a general multiplicativity result was proven for sets of product states. That result along with Eq. (8) shows that the smallest accessible fidelity that can be achieved with signal ensembles of product states is

$$F_{P_1 \otimes P_2} = \left(\frac{2}{d_1+1}\right) \left(\frac{2}{d_2+1}\right) \quad (14)$$

The implication of this is that ensembles $\tilde{P}$ optimal for achieving the quantumness of $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ must contain entangled states. Indeed the quantumness of a composite Hilbert space is not simply multiplicative in the quantumnesses of its components.

4 Symmetric Informationally Complete Ensembles

Suppose there exists $d^2$ unit vectors $|\psi_i \rangle \in \mathcal{H}_d$ such that

$$|\langle \psi_i | \psi_j \rangle|^2 = \frac{1}{d+1} \quad \forall i \neq j \quad (15)$$

If such a set exists, it follows that the $d^2$ projection operators $\Pi_i = |\psi_i \rangle \langle \psi_i|$ form a linearly independent set [21]. To see this, suppose there exist real numbers $\alpha_i$ such that

$$\sum_i \alpha_i \Pi_i = 0 \quad (16)$$

Multiplying by $\Pi_k$ and taking the trace of both sides we get

$$\alpha_k + \frac{1}{d+1} \sum_{i \neq k} \alpha_i = \left(1 - \frac{1}{d+1}\right) \alpha_k + \frac{1}{d+1} \sum_i \alpha_i = 0 \quad (17)$$

\footnote{Be careful to note that this is no trivial supposition. The problem of the existence of such a set, in fact, has existed in the mathematical literature since the early 1970s [18]. To date, such sets have only been proven to exist in dimensions $d = 2, 3, 4, 8, 11, 20, 24$. The remainder of the evidence for their existence (in dimensions up to $d = 45$) comes through numerical work [11]. Thus, it does seem likely that such sets exist, but it cannot be taken for granted.}
which implies that, for all $\alpha_k$,

$$\alpha_k = -\frac{1}{d} \sum_i \alpha_i .$$  \hspace{1cm} (18)

On the other hand, taking the trace of Eq. (16) reveals that $\sum_i \alpha_i = 0$. It follows that all $\alpha_k = 0$ for all $k$. The $\Pi_i$ are thus linearly independent.

Because of this latter property, the projectors $\Pi_i$ form a complete basis on $\mathcal{L}(\mathcal{H}_d)$, the vector space of linear operators over $\mathcal{H}_d$. It follows that for any operator $X \in \mathcal{L}(\mathcal{H}_d)$, the $d^2$ numbers $\text{tr} X \Pi_i$ generated by the Hilbert-Schmidt inner product $(A, B) = \text{tr} A^\dagger B$ uniquely specify the operator $X$.

It is convenient to use the projectors $\Pi_i$ to form a completely positive linear map $\Phi : \mathcal{L}(\mathcal{H}_d) \to \mathcal{L}(\mathcal{H}_d)$ in the following way:

$$\Phi(X) = \sum_i \frac{1}{d^2} \Pi_i X \Pi_i .$$  \hspace{1cm} (19)

**Theorem 2** An alternative representation of $\Phi$ is this:

$$\Phi(X) = \frac{1}{d(d+1)} \left( (\text{tr}X)(I + X) \right) ,$$  \hspace{1cm} (20)

where $I$ denotes the identity operator.

To see this, note that $\Phi(I)$ is a density operator and that

$$\text{tr} \left( \Phi(I)^2 \right) = \frac{1}{d} .$$  \hspace{1cm} (21)

Thus,

$$\Phi(I) = \frac{1}{d} I .$$  \hspace{1cm} (22)

Now, for any $X \in \mathcal{L}(\mathcal{H}_d)$ there exists an expansion

$$X = \sum_i c_i \Pi_i ,$$  \hspace{1cm} (23)

where

$$\text{tr}X = \sum_i c_i .$$  \hspace{1cm} (24)

With these ingredients, simply follow the action of $\Phi$ on $X$:

$$\Phi(X) = \frac{1}{d^2} \sum_{ij} c_j \Pi_i \Pi_j \Pi_i$$

$$= \frac{1}{d^2} \sum_{ij} c_j \text{tr}(\Pi_i \Pi_j) \Pi_i$$

$$= \frac{1}{d^2} \sum_i c_i \Pi_i + \frac{1}{d^2(d+1)} \sum_{i \neq j} c_j \Pi_i$$

$$= \frac{1}{d^2} X + \frac{1}{d^2(d+1)} \left( \sum_{ij} c_j \Pi_i - \sum_i c_i \Pi_i \right)$$

$$= \frac{1}{d^2} X + \frac{1}{d^2(d+1)} \left( \sum_j c_j \left( \sum_i \Pi_i \right) - X \right)$$

$$= \frac{1}{d^2} X + \frac{1}{d^2(d+1)} \left( d(\text{tr}X)I - X \right)$$

$$= \frac{1}{d(d+1)} \left( (\text{tr}X)I + X \right) .$$  \hspace{1cm} (25)
This proves the theorem.

When \( \Phi \) acts on a density operator \( \rho \), its action is (up to a scaling factor) that of a “just barely” entanglement-breaking depolarizing channel. In the language of Ref. [22],

\[
\Phi(\rho) = \frac{1}{d} \Delta_\lambda(\rho), \quad \text{with} \quad \lambda = \frac{1}{d+1},
\]

(26)

Also note that by acting \( \Phi \) on the identity, we obtained that \( \sum_\frac{1}{d} \Pi_i = I \), which means the operators

\[
E_i = \frac{1}{d} \Pi_i
\]

(27)

form a positive operator-valued measure. Such a POVM is known as a symmetric informationally complete POVM, or SIC-POVM for short [11]. The appellation ‘informationally complete’ is used because for any density operator \( \rho \), if one knows the probabilities

\[
p(i) = \text{tr} \rho E_i
\]

(28)

for the outcomes of such a measurement, then one knows the operator \( \rho \) itself [23, 24]. In fact, using Eq. (20), one sees immediately that for any density operator \( \rho \)

\[
\rho = (d + 1) \sum_i p(i) \Pi_i - I.
\]

(29)

For all these reasons, we will call any ensemble

\[
\mathcal{P} = \left\{ \Pi_i, \frac{1}{d^2} \right\}
\]

(30)

satisfying Eq. (15), a symmetric informationally complete ensemble (or SIC ensemble for short) in analogy to the SIC-POVMs studied in Ref. [11].

Another useful quantity to know in these terms is the purity of \( \rho \):

\[
\text{tr} \rho^2 = d^2(d+1)^2 \sum_{gh} p(g)p(h) \text{tr} E_g E_h - 2d(d+1) \sum_h p(h) \text{tr} E_h + \text{tr} I
\]

\[
= d^2(d+1)^2 \sum_{gh} p(g)p(h) \text{tr} E_g E_h - 2(\text{tr} \rho + \text{tr} I) + \text{tr} I
\]

\[
= (d+1)^2 \sum_h p(h)^2 + (d+1) \sum_{g \neq h} p(g)p(h) - d - 2
\]

\[
= ((d+1)^2 - (d+1)) \sum_h p(h)^2 + (d+1) \sum_{g,h} p(g)p(h) - d - 2
\]

\[
= d(d+1) \sum_h p(h)^2 - 1
\]

(31)

Thus all pure states give rise to a probability distribution \( p(h) \) for the outcomes of a SIC-POVM such that

\[
\sum_h p(h)^2 = \frac{2}{d(d+1)}.
\]

(32)

With these preliminary remarks, we are ready to explore the accessible fidelity for SIC-ensembles.

**Theorem 3** The accessible fidelity for any SIC ensemble \( \mathcal{P} \) is given by

\[
F_P = \frac{2}{d+1}.
\]

(33)

and so achieves the quantumness of the Hilbert space. Moreover, any POVM \( \{G_b\} \) consisting of rank-1 elements \( G_b = g_b |\phi_b \rangle \langle \phi_b| \) can be used for an optimal eavesdropping strategy.
To prove this, we simply fix any POVM $\mathcal{G} = \{G_b\}$ consisting of rank-1 elements and use the general formula derived in for the achievable fidelity in Eq. (4):

$$F_P(\mathcal{G}) = \sum_b \lambda_1 (\Phi(G_b)) .$$

With this,

$$F_P(\mathcal{G}) = \sum_b g_b \lambda_1 \left( \frac{1}{d(d+1)} (I + |\phi_b\rangle \langle \phi_b|) \right)$$

$$= \frac{2}{d(d+1)} \sum_b g_b .$$

(35)

Finally, because $I = \sum_b G_b$, it follows that $\sum_b g_b = d$. Thus,

$$F_P(\mathcal{G}) = \frac{2}{d+1} .$$

(36)

regardless of the measurement $\mathcal{G}$ (so long as it consists of rank-1 elements). In particular,

$$F_P = \frac{2}{d+1} .$$

(37)

Furthermore notice that $F_P$ can be achieved through a very simple reconstruction strategy $\mathcal{M}$. In particular, we do not need to use the more difficult-to-express measurement derived in Ref. [9] which gives rise to Eq. (4).

**Theorem 4** For any measurement consisting of rank-1 elements $G_b = g_b |\phi_b\rangle \langle \phi_b| \equiv g_b \sigma_b$, the accessible fidelity $F_P$ can be achieved via the simple reconstruction strategy $\mathcal{M}$: $b \rightarrow \sigma_b$.  

To prove this, define the conditional probabilities $p(i|b)$ by

$$p(i|b) = \frac{1}{d} \text{tr} (\Pi_i |\phi_b\rangle \langle \phi_b|) = \text{tr} (|\phi_b\rangle \langle \phi_b| E_i) ,$$

(38)

Now, simply write out the expression for the average fidelity of such a strategy.

$$F_P(\mathcal{G}, \mathcal{M}) = \sum_{b,i} \frac{1}{d^2} \text{tr} (\Pi_i G_b) \text{tr} (\Pi_i \sigma_b)$$

$$= \sum_{b,i} g_b (\text{tr} (E_i \sigma_b))^2$$

(39)

Using the conditional probabilities in Eq. (38), this becomes

$$F_P(\mathcal{G}, \mathcal{M}) = \sum_b g_b \sum_i p(i|b)^2 .$$

(40)

Noting that the $\sigma_b$ are pure states, so that Eq. (32) is satisfied for the conditional probabilities, we have finally

$$F_P(\mathcal{G}, \mathcal{M}) = \frac{2}{d(d+1)} \sum_b g_b$$

$$= \frac{2}{d+1} ,$$

(41)

which is just the accessible fidelity.

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3 A theorem like this was first shown for the case of the unitarily invariant signal ensemble by Barnum in Ref. [26].
5 Other Ensembles Achieving Quantumness

It is clear that for any other ensemble \( \mathcal{P} = \{ \Pi_i, \pi_i \} \), if its ensemble map \( \Psi \), Eq. (3), happens to coincide with \( \Phi \) in Eq. (20), then that ensemble too will have an accessible fidelity that achieves the quantumness of the Hilbert space. Here is another example.

Suppose \( \mathcal{H}_d \) can be equipped with a complete set of mutually unbiased bases. That is, suppose one can find \( d(d+1) \) one dimensional projectors \( \Pi^j_i \), with \( j = 1, \ldots, d+1 \) and \( i = 1, \ldots, d \), such that

\[
\text{tr}(\Pi^j_i \Pi^k_l) = \delta_{ik} \quad (42)
\]

\[
\text{tr}(\Pi^j_i \Pi^l_k) = \frac{1}{d} \quad \text{when } j \neq l. \quad (43)
\]

It is known that such sets always exist when \( d \) is an integer power of a prime number [14, 15]. (Though it is speculated that they do not exist for general \( d \), for instance, for \( d = 6 \) [25].) When such a set exists, it provides an (overcomplete) basis for \( \mathcal{L}(\mathcal{H}_d) \). Thus one can write any operator \( X \) in the form

\[
X = \sum_{ij} \alpha^j_i \Pi^j_i, \quad (44)
\]

where the \( \alpha^j_i \) are \( d(d+1) \) complex numbers. Performing now a calculation similar to the one in Eq. (25), one obtains that

\[
\sum_{ij} \frac{1}{d(d+1)} \Pi^j_i X \Pi^j_i = \Phi(X). \quad (45)
\]

Hence, an ensemble consisting of elements drawn from a complete set of mutually unbiased bases (all equally weighted) achieves the quantumness of the Hilbert space.

One can also ask the question of whether there are any ensembles with strictly less than \( d^2 \) elements that still achieve the quantumness of the Hilbert space. If there are such ensembles, then it will have to be for a reason more subtle than that the ensemble map \( \Psi \) in Eq. (3) coincides with \( \Phi \). For, proving \( \Psi = \Phi \) is a sufficient condition achieving the quantumness, but it is not a priori a necessary condition.

However, one can show the following about this sufficient condition:

**Theorem 5** For each one-dimensional projector \( \Pi \in \mathcal{L}(\mathcal{H}_d) \), define the completely positive linear map \( \Phi_\Pi : \mathcal{L}(\mathcal{H}_d) \to \mathcal{L}(\mathcal{H}_d) \) by

\[
\Phi_\Pi(X) = \Pi X \Pi. \quad (46)
\]

Denote by \( \mathcal{Q} \) the set of all such maps, and let \( \mathcal{B} \) be the convex hull of \( \mathcal{Q} \).

If the map \( \Phi \) in Eq. (20) can be written as a convex combination of \( d^2 \) or less extremal maps \( \Phi_\Pi \) of \( \mathcal{B} \), then the projectors \( \Pi \) in such a decomposition of \( \Phi \) must correspond to a SIC ensemble.

Here is how to see this. Let \( \{ \Pi_i \} \) be the set projectors in such a decomposition. Note that there must be \( d^2 \) of them and that they must be linearly independent. This follows for the simple reason that the range of \( \Phi \) spans \( \mathcal{L}(\mathcal{H}_d) \). If some of the \( \Pi_i \) were linearly dependent or there were less than \( d^2 \) of them, then, for any probability distribution \( \pi_i \), the operators

\[
\sum_i \pi_i \Pi_i X \Pi_i = \sum_i \alpha_i \Pi_i \quad \forall X, \quad (47)
\]

where \( \alpha_i = \pi_i \langle \psi_i | X | \psi_i \rangle \), will not be able to span \( \mathcal{L}(\mathcal{H}_d) \).

Now, working with the fact that the \( \Pi_i \) are linearly independent, try to satisfy the two equations:

\[
\sum_i \pi_i \Pi_i \Pi_i = \Phi(I) = \frac{1}{d} I \quad (48)
\]
and
\[ \sum_i \pi_i \Pi_i \Pi_k = \Phi(\Pi_k) = \frac{1}{d(d+1)}(I + \Pi_k). \] (49)

Putting these two equations together, one obtains
\[ \left( d\pi_k - \frac{1}{d} \right) \Pi_k + (d+1) \sum_{i \neq k} \pi_i \left( \text{tr}(\Pi_i \Pi_k) - \frac{1}{d+1} \right) \Pi_i = 0. \] (50)

By the linear independence of the \( \Pi_i \), the only way to satisfy this is to have
\[ \pi_i = \frac{1}{d^2} \quad \forall i, \] (51)

and
\[ \text{tr}(\Pi_i \Pi_j) = \frac{1}{d+1}, \quad \forall i \neq j. \] (52)

That completes the proof.

6 Open Question

The previous section still leaves the open question: Are there any ensembles \( \mathcal{P} = \{ \Pi_i, \pi_i \} \) with strictly less than \( d^2 \) elements such that
\[ F_\mathcal{P} = Q_d? \] (53)

If there are, what are the minimal number of elements required of an ensemble so that it achieves the quantumness of the Hilbert space?

Whatever the answer—whether it be \( d^2 \) or strictly less—what is the essential structure of such sets? The suspicion here is that this structure will have more to do with the intrinsic defining characteristics of a quantum system than anything based on the imagery of “the number of distinguishable states a system can have.”

7 Concluding Remarks

In several pieces of recent literature much to-do has been made of the fungibility of quantum information [27]. Hilbert spaces of the same dimension are said to be fungible: What can be done in one can be done in the others. Thus, for instance, it would not matter if a quantum cryptographer decided to build a quantum key distribution scheme based on \( d \)-dimensional subspaces gotten from pellets of platinum, or \( d \)-dimensional subspaces gotten from pellets of magnalium. The ultimate security he can achieve—at least in principle—will be the same in either case.

What is the meaning of this? One might say that this is the very reason quantum information is quantum information! If a protocol like quantum key distribution depended upon the kinds of matter used for its implementation, one would hardly be justified in thinking of it as a pure protocol solely of (quantum) information-theoretic origin. That is a useful and fruitful point of view.

However, another point of view is that this may be a call to reexamine physics, much like the miraculous equivalence between gravitational and inertial mass (as revealed by the Eötvös experiment [28]) was once a call to reexamine the origin of gravitation. From the perspective of gravity, the “implementation” of the mass is inconsequential—platinum and magnalium fall with the same acceleration. And that, in the hands of Einstein, led ultimately to the realization that gravity is a manifestation of spacetime curvature.

What is Hilbert space? Who knows! But in quantum information we are learning how to make use of it as a raw resource, basking in the good fortune that it is fungible—that the implementation of a Hilbert space dimension \( d \) is inconsequential. There could be some very deep physics in that, but it might take another elevator story.
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