Helicity invariants of force-free field for a rectangular box
Solar Physics

G. V. Rudenko · I. I. Myshyakov

Abstract An algorithm for calculating three gauge-invariant helicities (self-, mutual- and Berger relative helicity) for a magnetic field specified in a rectangular box is described. The algorithm is tested on a well-known force-free model (Low and Lou, 1990) presented in vector-potential form.

Keywords: Magnetic fields, Corona; Force-free fields; Helicity invariants

1. Introduction

Gauge-invariant helicities are meaningful measures of nonpotentiality of a magnetic field in active regions. These measures in combination with energetic estimates and their time dependence may be used for interpretation and prediction of various forms of solar activity (see Brown and Priest (1999); Amari et al. (2003a); Amari et al. (2003b); Regnier et al. (2005)).

Self and mutual helicities correspond to the twist and writhe of confined flux bundles, and the crossing of field lines in the magnetic configuration respectively (Regnier et al., 2005). Time evolution of Berger relative helicity measures the transport of magnetic helicity through the surface and the effect of boundary transverse motions (Berger and Field, 1984). These invariants represent magnetic topology of nonpotential magnetic configurations in active regions and its transformation caused by eruption.

This paper considers a problem of exact calculation of the helicity invariants within a rectangular box. It is supposed that the full magnetic field vector is prescribed everywhere in the rectangular box. The helicity calculation problem stated in this form as opposed to that stated for a half-space (Longcope and Malanushenko, 2008) is best suited for fixing physical characteristics of an active region. At present, it is possible to calculate force-free spatial magnetic field distribution in a bounded volume (see Rudenko and Myshyakov (2009); Rudenko et al. (2010)). Magnetic field extrapolation and calculation of helicity invariants and free energy may be used for physical analysis of solar activity. In this paper, we examine the algorithm for calculating helicity invariants and test it on a force-free model (Low and Lou, 1990).
presented in vector-potential form. A key feature of the algorithm is the numerical solution to the boundary problem for a potential magnetic field $\mathbf{B}$ in the rectangular box $V$ in terms of the vector potential satisfying conditions $\nabla \cdot \mathbf{A} (r) = 0 |_{r \in \partial V}$ and $\mathbf{A} (r) \cdot \hat{n} (r) = 0 |_{r \in \partial V}$.

2. Basic formulations and tools description

2.1. Definitions and general formulation of problems

Magnetic helicity as a gauge invariant measure may be strictly defined only for infinite space or finite volume bounded by magnetic surfaces. The helicity has a property to remain constant even when magnetic reconnection dissipates energy. The term of magnetic helicity and the condition of constant magnetic topology can not be applied to any bounded volume because the vector potential can not be affected by an arbitrary additional gradient function (gauge transformation). Choosing potential field as a reference field $\mathbf{B}_{ref} = \mathbf{B}_{pot}$,

$$\mathbf{B}_{pot} = \nabla \times \mathbf{A}_{pot},$$  

one can obtain three gauge-invariant helicity measures (see Berger (1999); Priest (1999) and Berger and Field (1984)) for the bounded volume $V$:

$$H_{self} = \int_{V} \mathbf{A}_{cl} \cdot \mathbf{B}_{cl} dv,$$  

$$H_{mut} = 2 \int_{V} \mathbf{A}_{pot} \cdot \mathbf{B}_{cl} dv,$$  

$$\Delta H_{BF} = \int_{V} \mathbf{A}_{cl} \cdot (\mathbf{B}_{cl} + \mathbf{B}_{pot}) dv - \oint_{S} \zeta (\mathbf{B}_{cl} + \mathbf{B}_{pot}) \cdot \hat{n} ds.$$  

Here $\mathbf{B}_{cl} = \mathbf{B} - \mathbf{B}_{pot}$; $\mathbf{A}_{cl} = \mathbf{A} - \mathbf{A}_{pot}$; $\hat{n}$ is the outer normal to $V$ on $S$. $\zeta$ – the scalar potential of the gradient field beyond $V$ whose normal component on the surface $S = \partial V$ corresponds to that of $\mathbf{A}_{cl}$

$$\left(\nabla \zeta \cdot \hat{n}\right)|_{S} = \left(\mathbf{A}_{cl} \cdot \hat{n}\right)|_{S},$$  

where $\mathbf{A}_{cl}$ for $\mathbf{B}_{cl}$ in $V$ is defined as

$$\mathbf{A}_{cl} (r) = -\frac{1}{4\pi} \int_{V} \frac{r - r'}{|r - r'|} \times \mathbf{B}_{cl} (r') dv'.$$

Using gauge invariance, one can choose the following conditions on the vector potential

$$\nabla \cdot \mathbf{A}_{pot} = 0,$$
Helicity invariants of force-free field for a rectangular box

\[(\mathbf{A}_{\text{pot}} \cdot \mathbf{n})|_S = 0. \tag{9}\]

Condition (9) is useful for defining reference helicity (5). In this case, the time evolution of \(\Delta H_{BF}\) describes helicity transport through the surface (see Berger and Field (1984)),

\[\frac{d}{dt}\Delta H_{BF} = -2 \oint_S (\mathbf{B} \cdot \mathbf{A}_{\text{pot}}) \mathbf{v}_{\text{pot}} \cdot \mathbf{n} \, ds + 2 \oint_S (\mathbf{v} \cdot \mathbf{A}_{\text{pot}}) \mathbf{B}_{\text{pot}} \cdot \mathbf{n} \, ds. \tag{10}\]

The sum of two gauge measures (3) and (4) is the reference helicity defined by Finn and Antonsen (1985):

\[\Delta H_{FA}(\mathbf{B}) = \int_V (\mathbf{A} + \mathbf{A}_{\text{pot}}) \cdot (\mathbf{B} - \mathbf{B}_{\text{pot}}) \, dv = H_{self} + H_{mut}. \tag{11}\]

Reference helicities (5) and (11) coincide only in special cases of \(V\) - the half space or infinite space above the sphere.

Given only a magnetic field vector in a bounded volume \(V\), it is necessary to do the following steps to obtain helicity invariants (3)-(5):

a) to solve boundary problems (1), (2) for the vector potential \(\mathbf{A}_{\text{pot}}\) (if \(\mathbf{A}_{\text{pot}}\) is known, Eq. (2) gives a solution to \(\mathbf{B}_{\text{pot}}\) for a given \(g\));

b) to obtain the vector potential of the confined magnetic field \(\mathbf{A}_{cl}\) using Eq. (7);

c) to obtain the scalar potential \(\zeta\) on \(\partial V\) corresponding to (6).

In this paper, we solve problems a) and c) for the rectangular domain: \(V = (0, L_x) \times (0, L_y) \times (0, L_z)\). As mentioned above, problem b) has already its analytical solution (7).

2.2. Problem a)

Let

\[g \in L^2(S). \tag{12}\]

Consider the problem of finding the vector potential \(\mathbf{A}^0 \in C^\infty(\overline{V})\) satisfying equations:

\[\nabla \times \nabla \times \mathbf{A}^0 = \mathbf{0}, \quad \nabla \cdot \mathbf{A}^0 = 0 \quad \text{in} \quad \overline{V}. \tag{13}\]

and the boundary condition

\[\|(\nabla \times \mathbf{A}^0) \cdot \mathbf{n} - g\|_{L^2(S)} = 0. \tag{14}\]

If it is possible to solve the problem (12)-(14), the vector-potential equation corresponding to (9) can be written as

\[\mathbf{A}_{\text{pot}} = \mathbf{A}^0 - \nabla \times \mathbf{A}^1, \tag{15}\]

where \(\mathbf{A}^1\) is the solution to (13) with boundary conditions

\[\|(\nabla \times \mathbf{A}^1) \cdot \mathbf{n} - (\mathbf{A}^0 \cdot \mathbf{n})|_S\|_{L^2(S)} = 0. \tag{16}\]
The solution to (12)-(14) can be written as a sum:

\[ A^0 = \sum_{i=1}^{6} A^{g_i} + A^m. \]  \hspace{1cm} (17)

Here \( A^{g_i} \) is the solution to (13) with the boundary function in (14)

\[ g_i(r) = \begin{cases} g(r) - G_i; & r \in S_i \\ 0; & r \in S_j, j \neq i \end{cases}, \]  \hspace{1cm} (18)

\[ G_i = \int_{S_i} g ds = \int_{S_i} \nabla \times A^m \cdot \hat{n}_i ds, \]  \hspace{1cm} (19)

where \( S_i \) - the \( i \)-th side of the rectangular box \( V \); \( A^m \) is the superposition of vector potentials given by the system of 5 magnetic monopoles located beyond the box \( V \)

\[ A^m = \sum_{i=1}^{5} \frac{m_i \hat{z}_i \times (r - r_i)}{|r - r_i| |(r - r_i) + (r - r_i) \cdot \hat{z}_i|}. \]  \hspace{1cm} (20)

This system of monopoles satisfying Equations (19) can be easily constructed by fixing their \( r_i \) locations and \( \hat{z}_i \) orientations in space and solution of linear algebraic system of equations for unknown magnitudes \( m_i \) (orts \( \hat{z}_i \) should be chosen so that tails of monopoles do not cross the volume \( V \)). To each problem (13),(18) on \( A^{g_i} \), there is a solution giving a unique field \( \nabla \times A^{g_i} \), and thus there is a solution to (12)-(14) on \( A_0 \), giving a unique \( \mathbf{B}_{pot} \).

Due to the freedom to choose gradient normalization, we can reformulate problem (13),(18) on \( A^{g_i} \) in terms given by Amari et al. (1999):

\[ \Delta A^{g_i} = 0 \quad \text{in} \quad \nabla, \]  \hspace{1cm} (21)

\[ \nabla \cdot A^{g_i} = 0 \quad \text{in} \quad \nabla, \]  \hspace{1cm} (22)

\[ \nabla_t \cdot A^{g_i} = 0 \quad \text{on} \quad S, \]  \hspace{1cm} (23)

\[ A^{g_i}_t = \nabla \perp \chi = \nabla_t \chi \times \hat{n}_i \quad \text{on} \quad S_i, \]  \hspace{1cm} (24)

\[ \partial_n A^{g_i} = 0 \quad \text{on} \quad S, \]  \hspace{1cm} (25)

\[ \hat{n}_i \cdot A^{g_i} = 0 \quad \text{in} \quad \nabla, \]  \hspace{1cm} (26)

\[ -\nabla^2 \chi = g_i \quad \text{on} \quad S_i, \]  \hspace{1cm} (27)

\[ \partial_n \chi = 0 \quad \text{on} \quad \partial S_i. \]  \hspace{1cm} (28)

Here the subscript \( t \) stands for the trace of the operator or the field on the boundary; the derivative \( \partial_n \chi \) in Equation (28) means the derivative of the normal to the surface \( \partial S_i \) in the plane of \( S_i \).


2.2.1. BVP for the vector-potential function given on one side of the rectangular box

Further, without loss of generality, is assumed that index $i$ corresponds to the side lying in the plane ($z = 0$). Then $A_{yi}^0 = 0$ from Equation (26). If $\chi$ is known, tangential components $A_x$ and $A_y$ are BVP solutions to (21)-(26) and satisfy the properties for $\partial S_i$, which follow from Equations (24) and (28):

$$
\begin{align*}
A_{yi}^0(x, 0, 0) &= \partial_y \chi(x, 0, 0) = 0, \\
A_{yi}^0(x, L_y, 0) &= \partial_y \chi(x, L_y, 0) = 0, \\
\partial_x A_{yi}^0(0, y, 0) &= \partial_x \partial_y \chi(0, y, 0) = 0, \\
\partial_x A_{yi}^0(L_x, y, 0) &= \partial_x \partial_y \chi(L_x, y, 0) = 0;
\end{align*}
$$

$$
A_{yi}^0(0, y, 0) = -\partial_x \chi(0, y, 0) = 0, \\
A_{yi}^0(L_x, y, 0) = -\partial_x \chi(L_x, y, 0) = 0, \\
\partial_y A_{yi}^0(x, 0, 0) = -\partial_x \partial_y \chi(x, 0, 0) = 0, \\
\partial_y A_{yi}^0(x, L_y, 0) = -\partial_x \partial_y \chi(x, L_y, 0) = 0.
$$

(29) and (30) allow us to choose suitable orthonormal bases for $A_{yi}^0$ and $A_{yi}^0$ on $S_i$ in $L^2$

$$
h_{x}^{mn} = \sqrt{\frac{2\pi}{L_x}} \sqrt{\frac{2}{L_y}} \cos(\pi m x / L_x) \sin(\pi n y / L_y),
$$

$$
h_{y}^{mn} = \sqrt{\frac{2\pi}{L_x}} \sqrt{\frac{2}{L_y}} \sin(\pi m x / L_x) \cos(\pi n y / L_y),
$$

(31) and (32).

Here $L_x$, $L_y$ are the scales of $S_i$. Basis functions (31) and (32) satisfy (29) and (30), therefore expansions $A_{xi}^0$ and $A_{yi}^0$ in these bases converge according to the norm of $W^1$ in the neighborhood of $\partial S_i$ (i.e., their expansions converge to them smoothly on $\partial S_i$).

Using Equations (31) and (32), one can find a harmonical solution in $V$ to problem (21)-(28):

$$
A_{yi}^0 = \begin{pmatrix}
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{x}^{mn} h_{x}^{mn}(x, y) p^{mn}(z) \\
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{y}^{mn} h_{y}^{mn}(x, y) p^{mn}(z)
\end{pmatrix}.
$$

(33)

Here

$$
p^{mn}(z) = e^{-q^{mn}z} - e^{-q^{mn}(2L_z - z)}, \\
q^{mn} = \sqrt{(\pi m / L_x)^2 + (\pi n / L_y)^2},
$$

(34)

$$
a_{x}^{mn} = \int_0^{L_x} \int_0^{L_y} \partial_y \chi(x, y) h_{x}^{mn}(x, y) dx dy
$$

(35)

$$
a_{y}^{mn} = -\int_0^{L_x} \int_0^{L_y} \partial_x \chi(x, y) h_{y}^{mn}(x, y) dx dy
$$

(36)
Solution (33) satisfies (24) in $L^2$ and strictly satisfies (21)-(23), (25) and (26). Choosing suitable bases (31) and (32) provides smooth solutions in the neighborhood of $\partial S_i$. The magnetic components expressed through partial derivatives $A^g_i$ can be easily represented by analytical expressions of (33). The same method is applied to all sides of $V$. In numerical implementation of this scheme, expansions (33) may be limited by the number of terms corresponding to dimensions of the grid $\chi(x,y)$. In our implementation, we show integrals of (35) as a sum of analytical integrals in grid cells, representing functions $\partial_y \chi$ and $\partial_x \chi$ as a 2D linear interpolation in the cell of their grid values.

2.2.2. BVP-$\chi$

Let us consider the same side ($z = 0$) as in previous item. The solution to (27), (28) will be presented as follows:

$$\chi(r) = \chi^* - \chi^0 - \sum_{j=1}^{4} \chi^{\tau_j},$$

where

$$\chi^* = \frac{1}{2\pi} \int_{S_i} g_i(r') \ln \frac{1}{|r - r'|} dx' dy',$$

is the inhomogeneous solution to (27). The second and third components are potential functions satisfying the two-dimensional homogeneous Laplace equation. $\chi^0$ will be defined as superposition of the potentials

$$\chi^0 = m_1 x + m_2 y + m_3 xy + m_4 (x^2 - y^2) + m_5 (3y x^2 - y^3) + m_6 (3y^2 - x^3) + m_7 (x^3 y - xy^3) + m_8 (6x^2 y^2 - x^4 - y^4) + m_9 (5x y^4 - 10x^3 y^2 + x^5) + m_{10} (5x^4 - 10x^2 y^3 + y^5) + m_{11} (y^6 - x^6 - 15y^4 x^2 + 15x^4 y^2),$$

where the vector of coefficients $m$ is selected to satisfy eight conditions of equality between tangential derivatives $\chi^0$ and $\chi^*$ at four vertices of $S_i$ and three conditions of equality between $\chi^0$ and $\chi^*$ integrals on any three edges of $S_i$. Thus, given that the surface integral of $g_i$ is 0, the difference of $\chi^* - \chi^0$ have zero tangential derivatives at vertices of $S_i$ and zero means on all edges of $S_i$. Let us label $S_i$ edges by $j$ and define boundary univariate functions $\tau_j$ equal to values of the normal derivative of $\chi^* - \chi^0$ on corresponding edges $j$. $\chi^{\tau_j}$ will be defined as potential functions satisfying the Neumann problem and the boundary condition:

$$\|\partial_n \chi^{\tau_k} - \tau_k\|_{L^2} = 0 \text{ on } (\partial S_i)_k, \quad k = j \text{ and } \partial_n = 0 \text{ on } (\partial S_i)_k, \quad k \neq j.$$
\[ b^m = \int_0^{L_x} \tau_j(x, y) \sqrt{\frac{2}{L_x}} \cos(\pi mx/L_x) dx. \]  

(42)

Solution (41) satisfies (40) in \( L^2 \) and strictly satisfies (27), (28) in \( S_i \). Solution (41) is smooth at the vertices of \( S_i \). Components \( A_{il}^x \) and \( A_{il}^y \) expressed through partial derivatives \( \chi^\tau_i \) can be described by the analytical expressions of (41). The same method is applied to every edge of \( S_i \). In numerical implementation of this scheme, expansions (41) can be limited by the number of terms corresponding to grid dimensions \( \tau_k(x, y) \). In our implementation, we show integrals of (42) as a sum of analytical integrals in grid cells, representing functions \( \tau_k \) as a 2D linear interpolation in the cell of its grid values.

2.3. Problem c)

To obtain potential \( \zeta \) satisfying Equation (6) on the surface \( S \), we use (7). Equation (7) yields \( A_{cl} \) on the surface \( S' \) limiting the volume \( V' \subset V \)

\[ V' = (-\Delta x, L_x + \Delta x) \times (-\Delta y, L_y + \Delta y) \times (-\Delta z, L_z + \Delta z), \]

(43)

where \( \Delta x, \Delta y, \Delta z \), are scales of the elementary cell of the discrete grid defined on \( V \). To calculate \( \zeta \), we first estimate its values on the grid \( S' \). To do this, we specify its zero value at an arbitrary vertex of \( V' \) \( \zeta_0(\mathbf{r'}_0) = 0 \) and then calculate integrals on the broken curves \( \Gamma_i \) jointing vertices \( \mathbf{r'}_i \) on \( S' \) and \( \mathbf{r'}_0 \):

\[ \zeta'(\mathbf{r'}_i) = \int_{\mathbf{r'}_0}^{\mathbf{r'}_i} \mathbf{A}_{cl} \cdot \mathbf{\tau}_i dl, \]

(44)

where \( \mathbf{\tau}_i \) is the unit local tangential vector to \( \Gamma_i \). To calculate \( \zeta \) on \( S \), we add integrals

\[ \zeta(\mathbf{r}_i) = \zeta'(\mathbf{r'}_i) - \int_{\mathbf{r}_i}^{\mathbf{r'}_i} \mathbf{A}_{cl} \cdot \mathbf{n} dl. \]

(45)

Integrals (44), (45) are approximated by simple summation. Integrals (44) do not depend on \( \Gamma_i \) curve shape. Therefore in their discrete implementation the choice of different configurations of \( \Gamma_i \) should result in small differences between results of integration of (44), (45).

3. Results of model calculations of helicity invariants

Let us take the calculation of helicity invariants (3)-(5), (11) as an example for the analytical force-free model (Low and Lou, 1990). The expression for the magnetic field will be written in terms of vector-potential for \( n = 1 \):

\[ \mathbf{A}_{LL} = -\frac{1}{r^2} W \widehat{\mathbf{r}} + U \frac{1}{r\sqrt{1 - \mu^2}} \widehat{\varphi}, \]

(46)

\[ W = -a \int_0^\mu \frac{P^2(\mu')}{\sqrt{1 - \mu'^2}} d\mu', \]

(47)
where $\mu = \cos \theta$,

$$U = \frac{P(\mu)}{r},$$

(48)

$P(\mu)$ satisfies nonlinear differential equation

$$(1 - \mu^2)\frac{d^2 P}{d\mu^2} + 2P + 2a^2 P^3 = 0$$

(49)

with boundary values

$$P(-1) = P(1) = 0.$$  

(50)

Applying operator $\nabla \times$ to (46) yields

$$\mathbf{B}_{LL} = \nabla \times \mathbf{A}_{LL} = \frac{1}{r \sqrt{1 - \mu^2}} \left( \frac{1}{r} \partial_\theta \mathbf{r} \cdot \partial_r U \hat{\mathbf{r}} + \partial_r U \hat{\theta} + aU^2 \varphi \right)$$

(51)

that coincides with Equation (3) from Low and Lou (1990).

For the tests we employed the force-free model calculated in the rectangular box $V$ ($L_x = L_y = 1$, $L_z = 0.8$; $N_x = N_y = 100$, $N_z = 80$) with the following parameters: $a^2 = a^2_{1,1} = 0.425$; $r_c = [0.5, 0.5, -0.25]$ are coordinates of the source; the axis of dipole lies in the plane ($x, z$) and has an angle $\Phi = 45^o$ to the z-axis.

Next, the notations in (3)-(5) and (11) for helicity invariants are used to denote their calculated values. To estimate the accuracy of the algorithm, we have also calculated helicity invariant (11) in terms of the known vector potential $\mathbf{A}_{LL}$:

$$\Delta H_{FA, LL}(\mathbf{B}) = \int_V (\mathbf{A}_{LL} + \mathbf{A}_{pot}) \cdot (\mathbf{B} - \mathbf{B}_{pot}) \, dv.$$ 

(52)

Relative helicity (5) was calculated for two different sets $\{\Gamma_i\}$ (44) and marked with indices $\{1, 2\}$. All helicity invariants were normalized to the true (not invariant!) helicity

$$H_{LL} = \int_V \mathbf{A}_{LL} \cdot \mathbf{B}_{LL} \, dv.$$ 

(53)

Table 1 lists the results. Numbers in Table 1 are helicities of the selected model characteristics. The relative errors indicating the level of completion of gauge invariance of Finn & Antonsen reference helicity (3 and 4 columns) and the degree of accuracy of Berger reference helicity (5 and 6 columns) are 0.0025 and 0.0134 respectively. Such sufficiently small errors justify the application of the algorithm presented here to calculate helicity invariants of real active regions.

| $H_{sel}/H_{LL}$ | $H_{mut}/H_{LL}$ | $\Delta H_{FA}/H_{LL}$ | $\Delta H_{FA, LL}/H_{LL}$ | $\Delta H_{BF,1}/H_{LL}$ | $\Delta H_{BF,2}/H_{LL}$ |
|------------------|------------------|-----------------------|--------------------------|------------------------|------------------------|
| 0.1166           | 0.6009           | 0.7175                | 0.7157                   | 0.7916                 | 0.7810                 |

G.V. Rudenko and I.I. Myshyakov

Table 1.
References

Amari, T., Luciani, J. F., Aly, J. J., Mikic, Z., and Linker, J.: 2003a, *Astrophys. J.* **585**, 1073.
Amari, T., Luciani, J. F., Aly, J. J., Mikic, Z., and Linker, J.: 2003a, *Astrophys. J.* **595**, 1231.
Amari, T., Boulmezaoud, T. Z., and Mikic, Z.: 1999, *Astron. Astrophys.* **350**, 1051.
Berger, M. A.: 1999, in *Magnetic Helicity in Space and Laboratory Plasmas*, ed. M. R. Brown, R. C. Canfield, and A. A. Pevtsov, 1.
Berger, M. A., and Field, G. B.: 1984, *J. Fluid Mech.* **147**, 133.
Brown, D. S., and Priest, E. R.: 1999, *Solar Phys.* **190**, 25.
Finn, J. M., and Antonsen, T. M.: 1985, *Comments Plasma Phys. Controlled Fusion* **9**, 111.
Longcope, D. W., Malanushenko, A.: 2008, *Astrophys. J.* **674**, 1130.
Low, B. C., Lou, Y. Q.: 1990, *Astrophys. J.* **352**, 343.
Priest, E. R.: 1999, in *Magnetic Helicity in Space and Laboratory Plasmas*, ed. M. R. Brown, R. C. Canfield, and A. A. Pevtsov, 141.
Regnier, S., Amari, T., and Canfield, R. C.: 2005, *Astron. Astrophys.* **442**, 345.
Rudenko, G. V., and Myshyakov, I. I.: 2009, *Solar Phys.* **257**, 287.
Rudenko, G. V., Myshyakov, I. I., and Anfinogentov, S. A.: 2011, eprint arXiv:1007.0298, (*Solar Phys.*, in press)
