Branes and Calibrated Geometries

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Abstract

The fivebrane worldvolume theory in eleven dimensions is known to contain BPS threebrane solitons which can also be interpreted as a fivebrane whose worldvolume is wrapped around a Riemann surface. By considering configurations of intersecting fivebranes and hence intersecting threebrane solitons, we determine the Bogomol’nyi equations for more general BPS configurations. We obtain differential equations, generalising Cauchy-Riemann equations, which imply that the worldvolume of the fivebrane is wrapped around a calibrated geometry.
1 Introduction

The dynamics of branes have played an important role in elucidating the structure of M-theory (for a review see [1]). In particular the fivebrane has received substantial interest recently due to its intricate worldvolume theory. This theory has been shown to contain supersymmetric threebrane \[ \mathcal{4} \] and self-dual string \[ \mathcal{3} \] solitons. A remarkable feature of these solitons, and closely related solitons on the worldvolumes of D-branes, is that they incorporate their spacetime interpretation \[ \mathcal{2}, \mathcal{3}, \mathcal{4}, \mathcal{7}, \mathcal{8}, \mathcal{9} \]. For example, the self-dual string corresponds to a membrane ending on the fivebrane. Similarly, the simplest threebrane soliton solution can be interpreted as the orthogonal intersection of two fivebranes lying along flat hyperplanes. In fact, for this case the Bogomol’nyi equations are precisely the Cauchy-Riemann equations. Thus there are more general solutions corresponding to desingular deformations of this configuration which can be interpreted as a single fivebrane with its worldvolume wrapped around an arbitrary Riemann surface.

There are solutions of the supergravity equations of motion corresponding to orthogonal intersections of branes, but the BPS solutions that are known at present are not fully localised \[ \mathcal{8}, \mathcal{9}, \mathcal{10} \] (for a review see [11]). The description of intersecting branes given by examining the worldsheet theory thus provides a useful avenue of obtaining more insights into the properties of M-branes. Moreover, the existence of branes with non-trivial worldvolumes has important applications in relation to the low energy dynamics of quantum Yang-Mills theories, e.g. the derivation of the Seiberg-Witten curve \[ \mathcal{12} \] (see also \[ \mathcal{14} \]) and indeed all of the Seiberg-Witten dynamics \[ \mathcal{13} \] from the fivebrane.

It is natural to enquire if there are other BPS solutions of the worldvolume that correspond to intersecting threebranes and self-dual strings. From the supergravity point of view this seems rather natural: supersymmetric configurations of orthogonal intersecting membranes and fivebranes are known and we might expect to see analogous
configurations in the worldvolume theory. For example, a supersymmetric configuration is given by a fivebrane in \( \{x^1, x^2, x^3, x^4, x^5\} \) plane orthogonally intersecting another fivebrane in the \( \{x^3, x^4, x^5, x^6, x^7\} \) plane, with a membrane in the \( \{x^3, x^6\} \) plane, a configuration that we will denote

\[
M5 : \begin{array}{c}
1 \ 2 \ 3 \ 4 \ 5 \\
3 \ 4 \ 5 \ 6 \ 7 \\
2 \ 6
\end{array}
\] (1)

Considering the first fivebrane’s worldvolume theory we expect this configuration to correspond, in the simplest setting, to a BPS solution consisting of a threebrane soliton in the \( x^3, x^4, x^5 \) directions orthogonally intersecting a self-dual string in the \( x^2 \) direction. This self-dual string then acts as a source for the three form field \( h \) on the fivebrane worldvolume. More general solutions should correspond to BPS solitons in the fivebrane approach to N=2 superYang-Mills theory \([12, 13]\).

As a first step towards studying all supersymmetric configurations of branes, in this paper we will consider configurations with only fivebranes. In the simplest setting these should correspond to intersecting configurations of threebranes on the worldvolume, but more generally they can be interpreted as the worldvolume of a single fivebrane with a non-trivial worldvolume, i.e. these BPS states may simply be viewed as a single fivebrane wrapped on a non-trivial submanifold embedded in eleven dimensions. Since there are no membranes and we are considering solitons with only scalars active, our discussion is universal to all types of branes by dimensional reduction and T-duality. The fivebrane in eleven dimensions is particularly useful in this sense because it has both a large worldvolume and transverse space. We will address the issue of configurations involving fivebranes, membranes and momentum modes in a future paper. In our analysis we will choose the target space to be flat space throughout, although the generalisation to a curved space should be straightforward and will be briefly discussed in the conclusion.

The supersymmetry of (Euclidean) membranes wrapped on three cycles of a Calabi-Yau manifold and threebranes wrapped around three cycles and four cycles of exceptional
holonomy manifolds has been studied in [16, 17]. From those results we expect the supersymmetric configurations of fivebranes to correspond to calibrated surfaces. In this work we shall focus on a full description of the non-linear worldvolume theory of the fivebrane and its supersymmetry. In this way we hope to obtain a more detailed picture. In particular our derivation shows that such surfaces satisfy elegant differential equations, generalising Cauchy-Riemann equations, which appear in the work of Harvey and Lawson [18] as necessary and sufficient conditions for the manifold to be calibrated. In addition, since we will directly show that the surfaces must be calibrated using similar ideas to [16, 17], our results can be viewed as a supersymmetric proof of some of the results in [18].

The plan of the rest of the paper is as follows. In the next section we obtain a list of orthogonally intersecting fivebranes which preserve some fraction of eleven-dimensional spacetime supersymmetry. The purpose of this section is to characterise some features of potential supersymmetric solutions on the fivebrane. In particular we will identify which transverse scalars we expect to be active in the solutions and determine sets of projection operators acting on the supersymmetry parameters that will be useful in later sections. Following that we turn our attention to the non-linear worldvolume theory of the fivebrane in section three. For the reader who is not interested in all the details of this section, we point them to equation (42), which is the condition for the fivebrane to preserve some supersymmetry in cases where the self-dual three form vanishes. Following this equation we present the argument that the fivebranes must be wrapped along calibrated geometries. In the section four we combine the results of sections two and three to derive the Bogomol’nyi equations for supersymmetric fivebrane configurations.
2 Intersecting Fivebranes

In this section we construct a number of orthogonally intersecting fivebrane configurations which preserve some fraction of eleven-dimensional spacetime supersymmetry (see also [19]) and list the corresponding supersymmetry projectors. This will provide a guide in our search for Bogomol’nyi conditions for supersymmetric solutions in the fivebrane worldvolume theory. We first note that a fivebrane in the \{x^0, x^1, x^2, x^3, x^4, x^5\} plane preserves the supersymmetries \(\epsilon \Gamma^{012345} = \epsilon\), where \(\Gamma\) are the eleven-dimensional \(\Gamma\)-matrices. The addition of other fivebranes will therefore imply further projections on \(\epsilon\). We shall list the various configurations in the order of the amount of supersymmetry that they preserve.

It turns out that in many configurations the supersymmetry conditions allow for additional fivebranes to be included, without breaking more supersymmetries. Thus the number of fivebranes can be rather large and does not immediately reflect the amount of supersymmetry preserved. We follow the practice of always including these extra fivebranes, which make the configurations more symmetric. However we only list an independent set of projectors for each configuration.

The reader will note in the following that there is clearly some choice between adding fivebranes or anti-fivebranes, although only for those fivebranes corresponding to independent projectors. Once these fivebranes are fixed, there is no choice for the others. In this section however, we merely wish to motivate the choice of projections used the in worldvolume analysis in the following sections. Clearly one could find other solitons by changing fivebranes to anti-fivebranes and visa-versa. However this would only lead to trivial changes in our analysis and correspond to changing the signs of the coordinates.
2.1 1/4 Supersymmetry

\begin{align*}
M5 : & \quad 1 \ 2 \ 3 \ 4 \ 5 \\
M5 : & \quad 3 \ 4 \ 5 \ 6 \ 7 \\
\epsilon \Gamma^{012345} = & \epsilon , \quad \epsilon \Gamma^{1267} = -\epsilon .
\end{align*}

This spacetime configuration should manifest itself as two active scalars \((X^6, X^7)\) depending on two worldvolume coordinates \((x^1, x^2)\), i.e. a two-dimensional surface embedded in four dimensions. As mentioned above the differential equation that the scalars satisfy in BPS solutions are simply Cauchy-Riemann equations, and hence this situation corresponds to a fivebrane wrapped around a Riemann surface.

2.2 1/8 Supersymmetry

\begin{align*}
M5 : & \quad 1 \ 2 \ 3 \ 4 \ 5 \\
M5 : & \quad 3 \ 4 \ 5 \ 6 \ 7 \\
M5 : & \quad 3 \ 4 \ 5 \ 8 \ 9 \\
\epsilon \Gamma^{012345} = & \epsilon , \quad \epsilon \Gamma^{1267} = -\epsilon , \quad \epsilon \Gamma^{1289} = -\epsilon .
\end{align*}

BPS worldvolume solutions corresponding to this configuration should have four active scalars depending on two worldvolume coordinates. Thus it should appear as a two-dimensional surface embedded in six dimensions (and moreover it must not be possible to embed the surface in four dimensions). In fact it corresponds to a Riemann surface but this time embedded in a six dimensional space. We note that one also has \(\epsilon \Gamma^{10} = -\epsilon\).
\[\epsilon\Gamma^{012345} = \epsilon, \quad \epsilon\Gamma^{1267} = -\epsilon, \quad \epsilon\Gamma^{3467} = -\epsilon.\]  

(7)

For this case we expect two active scalars depending on four worldsurface coordinates. BPS solutions should appear as a four surface embedded in six dimensions and in fact corresponds to a complex manifold. Note that \(\epsilon\Gamma^{05} = -\epsilon\) so that we could add a pp-wave in the \(x^5\) direction without breaking any more supersymmetries.

\[M5: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\]
\[\overline{M5}: \quad 3 \quad 4 \quad 5 \quad 6 \quad 7\]
\[M5: \quad 2 \quad 4 \quad 5 \quad 6 \quad 8\]
\[\overline{M5}: \quad 1 \quad 4 \quad 5 \quad 7 \quad 8\]

(8)

\[\epsilon\Gamma^{012345} = \epsilon, \quad \epsilon\Gamma^{1267} = \epsilon, \quad \epsilon\Gamma^{1368} = \epsilon.\]  

(9)

This configuration should correspond to solutions with three active scalars depending on three worldvolume coordinates. We will see that this corresponds to a three-dimensional special Lagrangian manifold embedded in six dimensions.

2.3 1/16 Supersymmetry

\[M5: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\]
\[M5: \quad 1 \quad 4 \quad 5 \quad 6 \quad 9\]
\[\overline{M5}: \quad 1 \quad 4 \quad 5 \quad 7 \quad 8\]
\[\overline{M5}: \quad 1 \quad 2 \quad 5 \quad 8 \quad 9\]
\[M5: \quad 1 \quad 2 \quad 5 \quad 6 \quad 7\]
\[M5: \quad 1 \quad 3 \quad 5 \quad 6 \quad 8\]
\[M5: \quad 1 \quad 3 \quad 5 \quad 7 \quad 9\]

(10)

\[\epsilon\Gamma^{012345} = \epsilon, \quad \epsilon\Gamma^{2369} = -\epsilon, \quad \epsilon\Gamma^{2378} = \epsilon, \quad \epsilon\Gamma^{3489} = \epsilon.\]  

(11)
For this configuration we should have four scalars depending on three worldvolume coordinates. We will see below that it describes an associative three surface in seven dimensions. Note that we also have $\epsilon \Gamma^{10} = \epsilon$ for this configuration.

\begin{align*}
M5 : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
M5 : & \quad 3 \quad 4 \quad 5 \quad 8 \quad 9 \\
M5 : & \quad 2 \quad 4 \quad 5 \quad 7 \quad 9 \\
\overline{M5} : & \quad 1 \quad 4 \quad 5 \quad 7 \quad 8 \\
\overline{M5} : & \quad 1 \quad 2 \quad 5 \quad 8 \quad 9 \\
M5 : & \quad 2 \quad 3 \quad 5 \quad 7 \quad 8 \\
M5 : & \quad 1 \quad 3 \quad 5 \quad 7 \quad 9 \\
\end{align*}

(12)

\begin{align*}
\epsilon \Gamma^{012345} = \epsilon, \quad \epsilon \Gamma^{1289} = -\epsilon, \quad \Gamma^{1379} = \epsilon, \quad \epsilon \Gamma^{2378} = \epsilon.
\end{align*}

(13)

Here we should look for solutions with three scalars depending on four worldvolume coordinates. We will see below that this corresponds to a coassociative four surface in seven-dimensions. Note that we have $\epsilon \Gamma^{05} = \epsilon$ so that we could add a pp-wave in the $x^5$ direction without breaking any more supersymmetries.

\begin{align*}
M5 : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
M5 : & \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
M5 : & \quad 1 \quad 2 \quad 5 \quad 8 \quad 9 \\
M5 : & \quad 3 \quad 4 \quad 5 \quad 8 \quad 9 \\
M5 : & \quad 1 \quad 2 \quad 5 \quad 6 \quad 7 \\
M5 : & \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\
\end{align*}

(14)

\begin{align*}
\epsilon \Gamma^{012345} = \epsilon, \quad \epsilon \Gamma^{1267} = -\epsilon, \quad \epsilon \Gamma^{3489} = -\epsilon, \quad \epsilon \Gamma^{1289} = -\epsilon.
\end{align*}

(15)

This configuration corresponds to four scalars depending on four worldvolume coordinates. We will see that it corresponds to a complex four dimensional surface embedded
in eight dimensions. Note that $\epsilon \Gamma^{05} = -\epsilon$, $\epsilon \Gamma^{10} = -\epsilon$ and $\epsilon \Gamma^{0510} = \epsilon$ so we could add a

pp-wave in the $x^5$ direction and a membrane in the \(\{x^0, x^5, x^{10}\}\) plane. The presence of

the membrane is related to the fact that the second and third fivebranes intersect over

a string, rather than a threebrane. We have not considered this string soliton by itself

because there is no known worldvolume solution to describe it. Such configurations will

appear again but unlike this case, where the orthogonal intersection is necessary to ob-

tain the corresponding projections, the fivebranes which contribute string intersections
could be discarded.

\[
\begin{align*}
M5 & : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\overline{M5} & : \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\
M5 & : \quad 2 \quad 4 \quad 5 \quad 6 \quad 8 \\
\overline{M5} & : \quad 2 \quad 3 \quad 5 \quad 6 \quad 9 \\
M5 & : \quad 1 \quad 4 \quad 5 \quad 7 \quad 8 \\
M5 & : \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\
\overline{M5} & : \quad 1 \quad 3 \quad 5 \quad 7 \quad 9 \\
\overline{M5} & : \quad 1 \quad 2 \quad 5 \quad 8 \quad 9
\end{align*}
\]

(16)

$\epsilon \Gamma^{012345} = \epsilon$, $\epsilon \Gamma^{1267} = \epsilon$, $\epsilon \Gamma^{1368} = \epsilon$, $\epsilon \Gamma^{1469} = \epsilon$ . (17)

Here we again have four scalars depending on four worldvolume coordinates. We will see

below that this corresponds to a four-dimensional special Lagrangian surface embedded

in eight dimensions. Note that we also have $\epsilon \Gamma^{0510} = \epsilon$ so again we could add a membrane

in the \(\{x^0, x^5, x^{10}\}\) plane.
2.4 1/32 Supersymmetry

\[ M5 : \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array} \]
\[ \overline{M5} : \begin{array}{ccccc}
3 & 4 & 5 & 6 & 7 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
2 & 4 & 5 & 6 & 8 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
1 & 2 & 5 & 6 & 7 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
1 & 3 & 5 & 6 & 8 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
2 & 3 & 5 & 7 & 8 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
1 & 4 & 5 & 7 & 8 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
5 & 6 & 7 & 8 & 9 \\
\end{array} \]
\[ \overline{M5} : \begin{array}{ccccc}
1 & 4 & 5 & 7 & 8 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
5 & 6 & 7 & 8 & 9 \\
\end{array} \]
\[ \overline{M5} : \begin{array}{ccccc}
2 & 3 & 5 & 6 & 9 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
3 & 4 & 5 & 8 & 9 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
2 & 4 & 5 & 7 & 9 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
1 & 3 & 5 & 7 & 9 \\
\end{array} \]
\[ M5 : \begin{array}{ccccc}
1 & 4 & 5 & 6 & 9 \\
\end{array} \]
\[ \overline{M5} : \begin{array}{ccccc}
1 & 2 & 5 & 8 & 9 \\
\end{array} \]

\[ \epsilon \Gamma^{012345} = \epsilon, \quad \epsilon \Gamma^{1267} = \epsilon, \quad \epsilon \Gamma^{1308} = \epsilon, \quad \Gamma^{1469} = \epsilon, \quad \Gamma^{1289} = -\epsilon. \]  

In this configuration we expect four scalars depending on four worldvolume coordinates. We will see below that this solution is described by a Cayley four surface in eight dimensions. Note that here we have \( \epsilon \Gamma^{0510} = -\epsilon, \epsilon \Gamma^{065} = -\epsilon \) and \( \epsilon \Gamma^{10} = \epsilon \). Thus we could add membranes in the \( \{x^0, x^5, x^{10}\} \) plane and pp-waves in the \( x^5 \) direction without breaking any additional supersymmetry.
\( M5 : \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  \bar{M5} : & 3 & 4 & 5 & 6 & 7 \\
  M5 : & 2 & 4 & 5 & 6 & 8 \\
  \bar{M5} : & 2 & 3 & 5 & 6 & 9 \\
  M5 : & 2 & 3 & 4 & 6 & 10 \\
  \bar{M5} : & 1 & 4 & 5 & 7 & 8 \\
  M5 : & 1 & 3 & 5 & 7 & 9 \\
  \bar{M5} : & 1 & 2 & 5 & 8 & 9 \\
  \bar{M5} : & 1 & 3 & 4 & 7 & 10 \\
  M5 : & 1 & 2 & 4 & 8 & 10 \\
  \bar{M5} : & 1 & 2 & 3 & 9 & 10 \\
  M5 : & 5 & 6 & 7 & 8 & 9 \\
  \bar{M5} : & 4 & 6 & 7 & 8 & 10 \\
  M5 : & 3 & 6 & 7 & 9 & 10 \\
  \bar{M5} : & 2 & 6 & 8 & 9 & 10 \\
  M5 : & 1 & 7 & 8 & 9 & 10 \\
\end{array}\) (20)

\( \epsilon_{\Gamma^{012345}} = \epsilon, \quad \epsilon_{\Gamma^{1267}} = \epsilon, \quad \epsilon_{\Gamma^{1368}} = \epsilon, \quad \epsilon_{\Gamma^{1469}} = \epsilon, \quad \epsilon_{\Gamma^{15610}} = \epsilon. \) (21)

In this configuration all five scalars are active and depend on all five worldvolume coordinates. We will see that it manifests itself as a five-dimensional special Lagrangian surface in ten dimensions. Again there are fivebranes intersecting over strings and \( \epsilon_{\Gamma^{0510}} = \epsilon, \epsilon_{\Gamma^{049}} = \epsilon, \epsilon_{\Gamma^{038}} = \epsilon, \epsilon_{\Gamma^{027}} = \epsilon \) and \( \epsilon_{\Gamma^{016}} = \epsilon \) so that we can add membranes in the \( \{x^0, x^1, x^6\}, \{x^0, x^2, x^7\}, \{x^0, x^3, x^8\}, \{x^0, x^4, x^9\} \) and \( \{x^0, x^5, x^{10}\} \) planes.
3 Supersymmetry and the Fivebrane

In this paper we are interested in bosonic solutions of the fivebrane equations of motion that preserve some supersymmetry. This will be the case if there exists constant spinors such that the variation of the spinor field of the fivebrane theory vanishes: the resulting condition is the Bogomol’nyi equation for the bosonic fields. We will see that the Bogomol’nyi condition will determine the geometry of the fivebrane configuration. In this section we derive an explicit expression for the supersymmetric variation of the spinor field of the fivebrane for the case of vanishing self-dual three form, generalising and refining the discussion found in [3].

We use the fivebrane dynamics and conventions of [20]. In our paper the fivebrane is embedded in flat eleven-dimensional Minkowski superspace. We must distinguish between world and tangent indices, fermionic and bosonic indices and indices associated with the target space $\underline{M}$ and the fivebrane worldvolume $M$. On the fivebrane worldvolume the bosonic tangent space indices are denoted by $a, b, ... = 0, 1, 2, ..., 5$ and bosonic world indices by $m, n, ... = 0, 1, 2, ..., 5$. For example, the inverse vielbein of the bosonic sector of the fivebrane worldvolume is denoted by $E^m_a$. The bosonic indices of the tangent space of the target space $\underline{M}$ are denoted by the same symbols, but underlined, i.e. the inverse vielbein in the bosonic sector is given by $\underline{E}^\underline{m}_a$. The fermionic indices follow the same pattern, those in the tangent space are denoted by $\alpha$ and $\underline{\alpha}$ for worldvolume $M$ and target space $\underline{M}$ respectively, while the world spinor indices are denoted by $\mu$ and $\underline{\mu}$.

The fivebrane sweeps out a superspace $\underline{M}$ in the target superspace $\underline{M}$ which is specified in local coordinates $Z^\underline{M} = (X^m, \Theta^\underline{\mu})$, $m = 0, 1, \ldots, 10$, $\underline{\mu} = 1, \ldots, 32$. These coordinates are functions of the worldvolume superspace parameterised by $z^M = (x^m, \theta^\mu)$, $m = 0, 1, \ldots, 5$; $\mu = 1, \ldots, 16$. The $\theta^\mu$ expansion of the $Z^\underline{M}$ contains $x^m$ dependent fields of which the only independent ones are their $\theta^\mu = 0$ components, also denoted $X^m$ and $\Theta^\underline{\mu}$,
and a self-dual tensor $h_{abc}$ which occurs at level $\theta^\mu$ in $\Theta^\mu$. Despite the redundancy of notation it will be clear from the context when we are discussing the component fields and the superfields.

The bosonic target space indices tangent to $M$ may be decomposed as those that lie in the fivebrane worldvolume and those that lie in the space transverse to the fivebrane; we denote these indices by $a$ and $a'$ respectively (i.e. $a = (a, a'), a = 0,1,\ldots, 5; a' = 1',\ldots, 5'$) with a similar convention for world indices. The initially thirty-two component spinor indices $\alpha$ are split into a pair of sixteen component spinor indices (i.e. $\alpha = (\alpha, \alpha'), \alpha = 1,\ldots, 16; \alpha' = 1',\ldots, 16'$) corresponding to the breaking of half of the supersymmetries by the fivebrane.

We will use the super-reparameterisations of the worldvolume to choose the so-called static gauge. In this gauge we identify the bosonic coordinates in the worldvolume with the bosonic coordinates on the worldvolume (i.e. $X^a = x^n$, $n = 0,1,\ldots, 5$) and set the fermionic fields $\Theta^\alpha = 0$, $\alpha = 1,\ldots, 16$. For a flat background $\Theta^\mu = \Theta^\mu \delta^\mu_5$. The component field content of the fivebrane is $X^a'$ ($a' = 1',\ldots, 5'$), $\Theta^\alpha'$ ($\alpha' = 1',\ldots, 16'$) and the self-dual field strength $h_{abc}$.

We recall some of the salient points of the super-embedding formalism. The frame vector fields on the target manifold $M$ and the fivebrane worldvolume submanifold $M$ are given by $E_\mathcal{A} = E_\mathcal{A}^M \partial_M$ and $E_A = E_A^M \partial_M$ respectively. The coefficients $E_\mathcal{A}^A$ encode the relationship between the vector fields $E_\mathcal{A}$ and $E_\mathcal{A}$, i.e. $E_A = E_\mathcal{A}^A E_\mathcal{A}$. Applying this relationship to the coordinate $Z^M$ we find the equation

$$E_\mathcal{A}^\Lambda = E_A^N \partial_N Z^M E_\mathcal{A}^M .$$

(22)

In this paper we will be primarily interested in fivebranes whose worldvolumes have $h_{abc} = 0$. In this case the geometry of the fivebrane simplifies considerably. The vector

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*We will also use $a' = 6,7,8,9,10$.
fields $E^\beta_\alpha \equiv (E^\beta_\alpha, E^\beta_{\alpha'})$ and $E^\beta_a \equiv (E^b_a, E^b_{a'})$ on the fivebrane can be chosen to be equal to the $Spin(1,10)$ and $SO(1,10)$ matrices $u^\beta_\alpha$ and $u^\beta_a$ respectively. For example

$$E^\beta_\alpha = u^\beta_\alpha, \quad E^\beta_{\alpha'} = u^\beta_{\alpha'}, \quad E^\beta_a = u^\beta_a.$$ (23)

The matrix $u^b_a \equiv (u^b_a, u^b_{a'})$ is an element of $SO(1,10)$ and the matrix $u^\beta_{\alpha'} \equiv (u^\beta_{\alpha}, u^\beta_{\alpha'})$ forms an element of $Spin(1,10)$. As is clear from the notation, the indices with an overbar take the same range as those with an underline. We recall that the connection between the Lorentz and spin groups is given by

$$u^\beta_\alpha u^\delta_\beta (\Gamma_a^\gamma)_{\gamma\delta} = (\Gamma^b_a)_{\beta\alpha}.$$ (24)

For a flat target superspace the super-reparameterisation invariance reduces to translations and rigid supersymmetry transformations. The latter take the form

$$\delta x^\underline{\mu} = \frac{i}{2} \Theta \Gamma^{\underline{\mu}} \epsilon, \quad \delta \Theta^\underline{\mu} = \epsilon^\underline{\mu}.$$ (25)

Unlike other formulations, the super-embedding approach of [20, 21] is invariant under super-reparameterisations of the worldvolume, that is, invariant under

$$\delta z^M = -v^M,$$ (26)

where $v^M$ is a supervector field on the fivebrane worldvolume. The corresponding motion induced on the target space $\underline{M}$ is given by

$$\delta \underline{Z}^B = v^A E^B_A,$$ (27)

where $v^M = v^A E^M_A$ and rather than use the embedding coordinates $Z^A$ we referred the variation to the background tangent space, i.e. $\delta \underline{Z}^B \equiv \delta Z^M E^B_M$. We are interested in supersymmetry transformations and so consider $v^a = 0$, $v^\alpha \neq 0$; with this choice and including the rigid supersymmetry transformation of the target space of equation (25) the transformation of $\Theta^\underline{\mu}$ is given by [20]

$$\delta \Theta^\underline{\mu} = v^\beta E^\underline{\mu}_\beta + \epsilon^\underline{\mu}.$$ (28)
The local supersymmetry transformations $v^\alpha$ are used to set $\Theta^\alpha = 0$ which is part of the static gauge choice. However, by combining these transformations with those of the rigid supersymmetry of the target space $\epsilon^\alpha$ we find a residual rigid worldvolume supersymmetry which is determined by the requirement that the gauge choice $\Theta^\alpha = 0$ is preserved. Consequently, we require $v^\beta E_\beta^\alpha = -\epsilon^\alpha$. Following the discussion in [3] the variation of the remaining spinor is given by

$$
\delta \Theta^\alpha' = v^\beta E_\beta^\alpha (E^{-1})_{\alpha' \alpha} \delta E^\alpha',
$$

where we have set the non-linearly realized symmetry parameterized by $\epsilon^\alpha'$ to zero. Introducing the projectors [20]

$$
(E^{-1})_{\alpha' \beta} E_\beta^\gamma = \frac{1}{2} (1 + \Gamma)_{\alpha' \gamma}, \quad (E^{-1})_{\beta' \alpha} E_\beta^\gamma = \frac{1}{2} (1 - \Gamma)_{\beta' \gamma},
$$

we then find that the supersymmetry transformation for the fermions is given by

$$
\delta \Theta^\alpha' = -\frac{1}{2} \epsilon^\gamma (1 + \Gamma)_{\gamma'} + \frac{1}{2} \delta \Theta^\gamma (1 + \Gamma)_{\gamma'} \alpha'.
$$

Hence we may write the variation of the spinor as

$$
\delta \Theta^\gamma \left( \frac{1 - \Gamma}{2} \right)_{\gamma'} = -\frac{1}{2} \epsilon^\gamma (\Gamma)_{\gamma'} \alpha'.
$$

Note that since only primed indices occur, the matrix $\frac{1}{2} (1 - \Gamma)_{\gamma'}$ is invertible. Therefore by multiplying by its inverse we find the variation of $\delta \Theta^\gamma$.

Bosonic configurations will preserve some supersymmetry if there exist spinors $\epsilon$ such that $\delta \Theta^\gamma$ vanishes in the limit $\Theta^\alpha = 0$. It will actually be more convenient to look for the conditions required for the vanishing of the right hand side of (32). We thus write (32) as

$$
\hat{\delta} \Theta^\alpha' = -\frac{1}{2} \epsilon^\gamma (\Gamma)_{\gamma'} \alpha',
$$

where we have absorbed the factor of $\frac{1}{2} (1 - \Gamma)$ into the definition of $\hat{\delta} \Theta^\alpha'$. To further analyse this expression we are required to find $E^A_{\alpha'}$, or equivalently the $u$'s of $SO(1,10)$.
and $\text{Spin}(1,10)$, in terms of the component fields in the limit $\Theta^\alpha = 0$. Using equation (22), the Lorentz condition $u_a^a \eta_{ab} u_b^b = \eta_{cd}$ and the static gauge choice $X^n = x^n$ we find that

$$(u_a^b, u_a'^{b'}) = (e_a^n \delta^b_n, e_a^n \partial_n x^{b'}) ,$$

where $g_{nm} = e_a^n e_m^b \eta_{ab} = \eta_{nm} + \partial_n X^a \partial_m X^{b'} \delta_{a'b'}$. Using the remaining Lorentz conditions we find, up to a local $\text{SO}(5)$ rotation, that the full Lorentz matrix $u_n^{\alpha \beta} \in \text{Spin}(1,10)$ is given by

$$u = \begin{pmatrix} e^{-1} & e^{-1} \partial X \\ -d^{-1}(\partial X)^T (\eta_1)^T & d^{-1} \end{pmatrix} ,$$

where the matrix $d$ is defined by the condition $dd^T = I + (\partial X)^T \eta_1 (\partial X)$, $(\partial X)^T$ is the transpose of the matrix $(\partial_n X^a')$ and $\eta_1$ is the Minkowski metric on the fivebrane and is given by $\eta_1 = \text{diag}(-1,1,1,1,1,1)$. The $u_n^{\beta} \in \text{Spin}(1,10)$ corresponding to the above $u_n^{\alpha \beta} \in \text{SO}(1,10)$ are found using equation (24).

We now consider in more detail the decomposition of the spinor indices. We recall that the bosonic indices of the fields on the fivebrane can be decomposed into longitudinal and transverse indices i.e. $a = (a, a')$ according to the decomposition of the Lorentz group $\text{SO}(1,10)$ into $\text{SO}(1,5) \times \text{SO}(5)$. The corresponding decomposition of the spin group is $\text{Spin}(1,10) \rightarrow \text{Spin}(1,5) \times \text{USp}(4)$. The spinor indices of the groups $\text{Spin}(1,5)$ and $\text{USp}(4)$ are denoted by $\alpha, \beta, ... = 1, ..., 4$ and $i, j, ... = 1, ..., 4$ respectively. Six-dimensional Dirac spinor indices normally take eight values, however the spinor indices we use for $\text{Spin}(1,5)$ correspond to Weyl spinors. Although we began with spinor indices $\alpha$ that took thirty-two dimensional values and were broken into two pairs of indices each taking sixteen values $\alpha = (\alpha, \alpha')$, in the final six-dimensional expressions the spinor indices are further decomposed according to the above decomposition of the spin groups and we take $\alpha \rightarrow \alpha i$ and $\alpha' \rightarrow i$ when appearing as superscripts and $\alpha \rightarrow \alpha i$ and $\alpha' \rightarrow i$ when appear as subscripts [20]. It should be clear whether we mean $\alpha$ to be sixteen or four dimensional depending on the absence or presence of $i, j, ...$ indices respectively. For example, we will write $\Theta^{\alpha'} \rightarrow \Theta^{\alpha i}$.
Using the corresponding decomposition of the spinor indices, the eleven dimensional Γ-matrices can be written as

\[ (\Gamma^a)_{\alpha}^\beta = (\gamma^a)_{ij} \delta_{\beta}^\alpha \]

\[ (\Gamma^\alpha)_{\alpha}^\beta = \delta_{ij} (\tilde{\gamma}^\alpha)_{\alpha}^\beta \]

where \( \gamma^a = \gamma^0 \tilde{\gamma}^a \). Using this equation the eleven dimensional Γ-matrices with several indices can be expressed as

\[ (\Gamma^{a_1 \ldots a_{2n}})_{\alpha}^\beta = (\gamma^{a_1 \ldots a_{2n}})_{ij} \delta_{\beta}^\alpha \]

\[ (\Gamma^{a_1 \ldots a_{2n+1}})_{\alpha}^\beta = (\gamma^{a_1 \ldots a_{2n+1}})_{ij} \delta_{\beta}^\alpha \]

where, for example, \( \gamma^{a_1 \ldots a_{2n}} \equiv \gamma^{a_1 \tilde{\gamma}^a \ldots \tilde{\gamma}^{a_{2n}}} \).

We will need the relationship

\[ (\gamma^{a_1 \ldots a_n}) = -\frac{1}{(6-n)!} (-1)^{n(n+1)/2} \epsilon^{a_1 \ldots a_2 a_{2n+1} a_{2n+2} \ldots a_6} \gamma^{a_1 \ldots a_{2n+1} a_{2n+2} \ldots a_6} \]

for the chiral six dimensional \( \gamma \)-matrices. The other chiral six dimensional \( \tilde{\gamma} \)-matrices satisfy an identical condition except for an additional minus sign on the right hand side.

Using the expressions for the supervielbeins of equation (23) in terms of the \( SO(1,10) \) matrices, the variation of the spinor can be written as

\[ \hat{\delta} \Theta' = -\epsilon^\gamma (u^{-1})^\gamma_{\gamma} u^\gamma_{\beta} = -\frac{1}{2} \epsilon^\gamma (u^{-1})^\gamma_{\gamma} (1 - \frac{1}{6!} \epsilon^{a_1 a_2 a_3 a_4 a_5 a_6} \Gamma_{a_1 a_2 a_3 a_4 a_5 a_6} \delta_{\beta}^\gamma u^\gamma_{\beta} \]

The last step in the above equation used the relation

\[ -\frac{1}{6!} \epsilon^{a_1 a_2 a_3 a_4 a_5 a_6} (\Gamma_{a_1 a_2 a_3 a_4 a_5 a_6})_{\alpha}^\beta = \delta_{ij} (\delta_{\alpha}^\beta 0)

(40)

Using equation (24) we find that

\[ \hat{\delta} \Theta' = \frac{1}{2 \cdot 6!} \epsilon^{a_1 a_2 a_3 a_4 a_5 a_6} u^\beta_{a_1} u^\beta_{a_2} u^\beta_{a_3} u^\beta_{a_4} u^\beta_{a_5} u^\beta_{a_6} \epsilon^\alpha (\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6})_{\alpha}^\gamma \].

(41)
Equation (41) however contains an eleven dimensional $\Gamma$-matrix that involves the upper off diagonal block and as such it vanishes unless the $b_i$ indices take values in the longitudinal direction an odd number of times. Substituting in this matrix we find that

$$\hat{\delta}\Theta^{j} = -\frac{1}{2}\det(e^{-1})e^{\alpha i}\left\{ \partial_{\alpha}X^{c'}(\gamma^{a})_{\alpha\beta}(\gamma_{c'})^{j}_{i} - \frac{1}{3!}\partial_{\alpha_{1}}X^{c'_{1}}\partial_{\alpha_{2}}X^{c'_{2}}\partial_{\alpha_{3}}X^{c'_{3}}(\gamma^{a_{1}a_{2}a_{3}})_{\alpha\beta}(\gamma_{c'_{1}c'_{2}c'_{3}})^{j}_{i} + \frac{1}{5!}\partial_{\alpha_{1}}X^{c'_{1}}\cdots\partial_{\alpha_{5}}X^{c'_{5}}(\gamma^{a_{1}\cdots a_{5}})_{\alpha\beta}(\gamma_{c'_{1}\cdots c'_{5}})^{j}_{i} \right\}. \tag{42}$$

When deriving this equation we have used equation (38) and equation (35) for the $u$'s. In the next section we will derive Bogomol'nyi equations for bosonic configurations with vanishing self-dual three form which preserve some worldvolume supersymmetry, i.e. configurations associated with the vanishing of (42). We will do this by further manipulating (42) by imposing the projections on the spinor $\epsilon$ that we obtained in the last section from considerations of orthogonally intersecting branes.

Before proceeding to that analysis, it is interesting to consider the conditions for the preservation of supersymmetry without using static gauge. Clearly $\delta\Theta^{\underline{a}} = 0$ implies that $v^{\beta}E^{\underline{a}}_{\beta} = -\epsilon_{\underline{a}}$. Multiplying by the inverse of the embedding matrix this condition is equivalent to the two conditions $v^{\beta} = -\epsilon_{\underline{a}}(E^{-1})_{\underline{a}}^{\beta}$ and $\epsilon_{\underline{a}}(E^{-1})_{\underline{a}}^{\beta'} = 0$. Since $v^{\beta}$ is an arbitrary function the first of these equations is automatically satisfied. The second condition is equivalent to $\epsilon_{\underline{a}}(E^{-1})_{\underline{a}}^{\beta'}E_{\gamma'\beta'} = 0$ which using the projectors of equation (30) we may rewrite as

$$\epsilon_{\underline{a}}(1 - \Gamma)_{\underline{a}}^{\gamma} = 0. \tag{43}$$

Hence this is the necessary and sufficient condition for the preservation of supersymmetry.

We can now make contact with the work of [16, 17]. For the static configurations
which are studied in this paper the matrix $\Gamma$ takes the form
\[
\Gamma = -\frac{1}{5!} \text{det}(e^{-1}) \epsilon^{m_1 m_2 m_3 m_4 m_5} \partial_{m_1} X \frac{b_1}{m_1} \partial_{m_2} X \frac{b_2}{m_2} \partial_{m_3} X \frac{b_3}{m_3} \partial_{m_4} X \frac{b_4}{m_4} \partial_{m_5} X \frac{b_5}{m_5} \Gamma_0 \Gamma_0 \Gamma_0 \Gamma_0 \Gamma_0 .
\]
where the sums exclude the value 0. Although the matrix $\Gamma$ is in general not a hermitian matrix it is for the case of static configurations. One can also verify that it is symmetric in its spinor indices.

Following similar arguments to those of [16] for the case of the Euclidean two brane we conclude that
\[
\eta^\dagger (1 - \Gamma)(1 - \Gamma) \eta = \eta^\dagger (1 - \Gamma) \eta \geq 0 ,
\]
where $\eta = \epsilon^\dagger$. The transverse coordinates will not depend on all the longitudinal coordinates of the brane. Let us suppose that they depend on $q$ spatial coordinates leaving $p = 5 - q$ spatial coordinates upon which there is no dependence. In static gauge the matrix $\Gamma$ then further simplifies
\[
\Gamma = -\frac{1}{q!} \text{det}(e^{-1}) \epsilon^{m_1 \ldots m_q} \partial_{m_1} X \frac{b_1}{m_1} \ldots \partial_{m_q} X \frac{b_q}{m_q} \Gamma_0 \ldots \Gamma_0 \Gamma_0 \ldots \Gamma_0 \Gamma_0 \ldots \Gamma_0 \eta ,
\]
where $\text{det} e$ is the determinant of the vielbein induced on the embedded surface. Integrating equation (46) over the $q$ longitudinal coordinates of the brane we find that
\[
\int d^q x (\text{det} e) \eta^\dagger \eta \geq \int d^q x (\text{det} e) \eta^\dagger \Gamma_0 \ldots \Gamma_0 \Gamma_0 \ldots \Gamma_0 \eta
\]
\[
= -\int d^q x \frac{1}{q!} \epsilon^{m_1 m_2 m_3 \ldots m_q} \partial_{m_1} X \frac{b_1}{m_1} \ldots \partial_{m_q} X \frac{b_q}{m_q} \eta^\dagger \Gamma_0 \ldots \Gamma_0 \Gamma_0 \ldots \Gamma_0 \eta .
\]
Hence we find that the volume of the volume of the embedded surface is greater than or equal to the integral of the form $-\frac{1}{q!} \epsilon^{m_1 m_2 m_3 \ldots m_q} \partial_{m_1} X \frac{b_1}{m_1} \ldots \partial_{m_q} X \frac{b_q}{m_q} \eta^\dagger \Gamma_0 \ldots \Gamma_0 \Gamma_0 \ldots \Gamma_0 \eta$. This is precisely the statement [18] that the form calibrates the embedded surface. Furthermore there is equality if and only if supersymmetry is preserved.

To illustrate how this works in detail let us consider the particular example of (18). In this case four of the transverse fields of the fivebrane are active and they depend
on only four of the longitudinal coordinates of the brane (i.e. \( q = 4 \)). Thus we have a four dimensional space embedded in eight dimensions which are made up of the four longitudinal coordinates of the brane and the four active coordinates of the fivebrane.

In this case the form of the right hand side of (47) has the components

\[
- \partial_{m_1} X^{b_1} \ldots \partial_{m_4} X^{b_4} \eta^i \gamma^{05} \Gamma_{b_1 \ldots b_4} \eta,
\]

where the sum over the \( b_i \) excludes the values 0, 5, 10. Since \( \gamma^{05} \epsilon = -\epsilon \), this is just the pull back to the fivebrane world surface of the four form \( \eta^i \gamma^{05} \Gamma_{b_1 \ldots b_4} \eta \). This form lives on the eight-dimensional space and, given the projections in (19), is none other than the \( \text{Spin}(7) \) invariant self-dual four form \( \Omega \) which lives on this eight-dimensional space (see for example [22]). One can work out the calibrating form for all the spaces considered in this paper in a similar manner.

Finally, it is interesting to compare the worldsurface supersymmetry of the spinor with that of \( \kappa \)-supersymmetry. In fact \( \kappa \)-supersymmetry is just a consequence of world-volume supersymmetry which is found by taking \( \kappa \) transformation \( \delta \Theta^\alpha = \frac{1}{2} \kappa (1 + \Gamma) \alpha + \epsilon^\alpha \). Making this replacement in equation (28) and using the projector of equation (30) we find the standard result for the \( \kappa \) transformation

\[
\delta \Theta^\alpha = \frac{1}{2} \kappa (1 + \Gamma) \alpha + \epsilon^\alpha.
\]

In addition setting \( \Theta^\alpha = 0 \) in static gauge requires \( \frac{1}{2} \kappa (1 + \Gamma) \alpha + \epsilon^\alpha = 0 \) and following the same argument as before we find the variation of the remaining spinor is given by

\[
\delta \Theta^{\beta'} (1 - \Gamma)_{\beta'} = \frac{1}{2} \kappa (1 + \Gamma)_{\beta'} (1 + \Gamma) \alpha' = -\epsilon^\beta (1 + \Gamma) \alpha',
\]

again setting \( \epsilon^{\alpha'} = 0 \), which is the same as (32). Thus one can find the conditions for supersymmetry preservation by studying either worldvolume or \( \kappa \)-supersymmetry. Given that the origin of \( \kappa \)-supersymmetry is worldsurface supersymmetry this is to be expected.
4 Geometry and Calibrations

In section two above we wrote down static intersecting brane configurations which preserve some fraction of spacetime supersymmetry. Let us now examine these configurations from the point of view of the worldvolume of the first fivebrane. In particular we shall further manipulate the full non-linear supersymmetry conditions on the worldvolume theory \cite{[12]} using the projection operators associated with each of the configurations in section two. We will obtain differential equations for the coordinates of all the manifolds constructed above which correspond precisely to the necessary and sufficient conditions of Harvey and Lawson for these to be calibrated manifolds. We will see that all of these configurations correspond to the standard Kähler, Special Lagrangian and exceptional calibrations of the mathematical literature. As calibrated manifolds they all have minimal area in their homology class \cite{[18]}. Thus they all solve the field equations of the fivebrane with the three form set to zero.

4.1 Kähler Submanifolds

Let us consider the case of an \( n \) complex dimensional manifold embedded in \( \mathbb{C}^m \cong \mathbb{R}^{2m} \) with \( m > n \). It is helpful to introduce the complex coordinates

\[
\begin{align*}
  z^\mu &= x^{2\mu-1} + ix^{2\mu}, \quad \mu = 1, 2, 3, \ldots, n \\
  Z^\alpha &= X^{2\alpha+4} + iX^{2\alpha+5}, \quad \alpha = 1, 2, 3, \ldots, m - n
\end{align*}
\]

(51)

and their complex conjugates \( \bar{z}^\mu \) and \( \bar{Z}^\bar{\alpha} \). Let us denote the corresponding \( \gamma \)-matrices by \( \gamma^\mu \) and \( \gamma^\alpha_\alpha = \frac{1}{2} \gamma^\mu_{\bar{\mu}} \). Here and in the rest of this paper we denote the transverse \( \gamma \)-matrices with primes to distinguish them from the worldvolume \( \gamma \)-matrices. These furnish commuting representations of the 2\( n \)-dimensional and 2\( m \)-dimensional Clifford algebras respectively;

\[
\begin{align*}
  \{ \gamma^\mu, \gamma^\nu \} &= \{ \gamma^\bar{\mu}, \gamma^\bar{\nu} \} = 0, \quad \{ \gamma^\mu, \gamma^\bar{\nu} \} = 2\delta^{\mu\bar{\nu}},
\end{align*}
\]
\begin{align}
\{ \gamma'_{\alpha}, \gamma'_{\beta} \} &= \{ \gamma'_{\bar{\alpha}}, \gamma'_{\bar{\beta}} \} = 0 , \quad \{ \gamma'_{\alpha}, \gamma'_{\bar{\beta}} \} = 2 \delta_{\alpha \bar{\beta}} . \tag{52}
\end{align}

We then consider the projections

\begin{equation}
\epsilon \gamma^\mu \gamma'_\alpha = 0 . \tag{53}
\end{equation}

One can easily check that these form a commuting set of \( n(m - n) \) projectors, although they are not always independent. Indeed for \( (n, m) = (1, 2), (1, 3), (2, 3) \) one finds the configurations \( \{2\}, \{4\}, \{3\} \) which preserve \( 1/2, 1/4, 1/4 \) of worldsheet supersymmetry respectively. The only other case occurring on the fivebrane (i.e. with \( n \leq 2 \) and \( m - n \leq 2 \)) is the configuration \( \{14\} \) where \( (n, m) = (2, 4) \) and this preserves \( 1/8 \) of worldsheet supersymmetry (i.e. only three of the four projections are independent).

We now consider the linear term in \( [12] \)

\begin{equation}
0 = \epsilon \left[ \gamma^\mu \partial_\mu Z^\alpha \gamma'_\alpha + \gamma^\mu \partial_\mu Z^\alpha \gamma'_\alpha + c.c. \right] . \tag{54}
\end{equation}

Clearly the first term is zero as a result of the projections and the equation is satisfied if and only if the scalars are holomorphic functions; \( \partial_\mu Z^\alpha = 0 \). For all the above cases with the exception of \( n = 2, m = 4 \), the higher order terms vanish automatically. Thus the only supersymmetric configurations correspond to holomorphic embeddings. For the \( n = 2, m = 4 \) case one finds a non-trivial third order term coming from \( [12] \). Vanishing of the full non-linear supersymmetry then yields the equation

\begin{equation}
0 = \epsilon \left\{ \gamma^\mu \gamma'_\alpha \partial_\mu Z^\beta \left[ \delta^\alpha_{\bar{\beta}} \delta^\nu_{\bar{\mu}} - \frac{3}{2} \left( \partial_\mu Z^\gamma \partial^\nu Z_\gamma \delta^\alpha_{\bar{\beta}} - \partial_\mu Z_\beta \partial^\nu Z^{\bar{\alpha}} \right.ight.ight.
\end{equation}

\begin{equation}
\left. - \delta^\nu_{\bar{\mu}} \partial^\rho Z^\gamma \partial_\rho Z_\gamma \delta^\alpha_{\bar{\beta}} + \delta^\nu_{\bar{\mu}} \partial^\rho Z_\beta \partial_\rho Z^{\bar{\alpha}} \right) + c.c. \right\} . \tag{55}
\end{equation}

Clearly \( \partial_\mu Z^{\bar{\alpha}} = 0 \) is a solution however we have not checked that it is the only solution. Note that the corresponding complex submanifolds are calibrated by powers of the Kähler form \( \omega, \frac{1}{m!} \omega^n \).
4.2 Special Lagrangian Submanifolds

Here we consider the case of an $n$-dimensional manifold embedded into $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Let $i = 1, 2, 3, \ldots, n$ and introduce the notation

$$\gamma'^i = \gamma'_i + 5, \quad X_i = X^{i+5},$$

and again the two Clifford algebras $\gamma^i$ and $\gamma'^i$ commute. We now consider the projections

$$\epsilon_1 \gamma^i \gamma'^1 \gamma'^i = \epsilon,$$

where there is no sum over $i$. These projections in turn imply that

$$\epsilon_1 \gamma^i \gamma'^j = -\epsilon_1 \gamma^j \gamma'^i, \quad i \neq j.$$

It is easy to see that these form a set of $n - 1$ independent commuting projectors which correspond to the preservation of $2^{-(n-1)}$ of the worldvolume supersymmetry. Clearly the $n = 1$ case is trivial and the $n = 2$ case corresponds to the $n = 1, m = 2$ complex case above.

Let us now consider the supersymmetry condition. First take $n = 3$, corresponding to the configuration (8) preserving $1/4$ of worldvolume supersymmetry. A little algebra shows that (42) may be written as

$$0 = \sum_{i< j} \epsilon_1 \gamma^i \gamma'^j (\partial_i X_j - \partial_j X_i) + \epsilon_1 \sum_i \partial_i X_i - \det(\partial X).$$

(59)

Therefore we find from the first term that

$$\partial_i X_j = \partial_j X_i,$$

(60)

and so we take $X_j = \partial_j F$ for some $F$. The second term then gives

$$\partial^2 F = \det(\text{Hess} F),$$

(61)

where $(\text{Hess} F)_{ij} = \partial_i \partial_j F$. 

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Now consider $n = 4$ describing the configuration (16) preserving $1/8$ of worldvolume supersymmetry. Here we find

\[ 0 = \sum_{i<j} \epsilon^i \epsilon^j \left[ \partial_i X_j - \partial_j X_i - 3\partial_{[k} X^k \partial_{l]} \partial_{i} X_{j} + 3\partial_{[k} X^k \partial_{l]} \partial_{j} X_{i} \right] + \epsilon^1 \epsilon^1 \sum_i \left[ \partial_i X_i - \det_{i|i}(\partial X) \right], \tag{62} \]

where $\det_{i|i}(\partial X)$ is the determinant of the matrix found by deleting the $i$th row and $j$th column of the matrix $\partial X$. The simple condition $\partial_i X_j - \partial_j X_i = 0$ has now become non-linear. Some work shows that it can be written as

\[ 0 = (\partial_m X_n - \partial_n X_m) \left( \delta^m_i \delta^n_j - \delta^m_j \delta^n_i - \frac{1}{2} \delta^m_i \delta^n_j [ (\partial \cdot X)^2 - \partial_i X_k \partial^k X^i ] + \partial^m X_j \partial^n X_i \right) - (\partial \cdot X) [ \delta^m_i \partial^n X_j + \delta^n_i \partial^m X_j ] - \delta^m_i \partial^n X_k \partial^k X_j - \delta^n_i \partial^m X_k \partial^k X_j \right). \tag{63} \]

From this one readily sees that $\partial_j X_i - \partial_i X_j = 0$ is still a solution (although we have not checked that it is the only solution). Again write $X_j = \partial_j F$ so that the first line in (62) vanishes. The second line then yields the equation

\[ \partial^2 F = \sum_i \det_{i|i}(\text{Hess} F). \tag{64} \]

Finally we consider $n = 5$ This describes the configuration (20) preserving $1/16$ of the worldvolume supersymmetry. Here we find

\[ 0 = \sum_{i<j} \epsilon^i \epsilon^j \left[ \partial_i X_j - \partial_j X_i - 3\partial_{[k} X^k \partial_{l]} \partial_{i} X_{j} + 3\partial_{[k} X^k \partial_{l]} \partial_{j} X_{i} \right] + \epsilon^1 \epsilon^1 \sum_i \left[ \partial_i X_i - \sum_{i \neq j} \det_{i|ij}(\partial X) + \det(\partial X) \right], \tag{65} \]

where $\det_{i|ij|(\partial X}$ is the determinant of the matrix found by deleting the $i$th and $j$th rows and $k$th and $l$th columns of the matrix $\partial X$. Again equation (63) appears and so we write $X_i = \partial_i F$ and we arrive at the equation

\[ \partial^2 F = \sum_{i \neq j} \det_{i|ij}(\text{Hess} F) - \det(\text{Hess} F). \tag{66} \]
Equations (61), (64) and (66) above are precisely the necessary and sufficient conditions derived by Harvey and Lawson [18] for the embedded manifold in $\mathbb{C}^n$ to be Special Lagrangian. By definition such manifolds are calibrated by the form $\text{Re}(dz^1 \wedge \ldots \wedge dz^n)$, where the $z^\mu$ are complex coordinates of $\mathbb{C}^n$.

### 4.3 Exceptional Submanifolds

We are now left with only a few of the configurations in section two left to analysis. As we will see, these cases correspond to the exceptional calibrated submanifolds discussed in [18]. For these cases it will be convenient to work with an explicit representation of gamma matrices (36) using quaternions. Specifically we choose

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

(67)

and

$$\gamma^6 = \begin{pmatrix} 0 & 1' \\ 1 & 0' \end{pmatrix}, \quad \gamma^7 = \begin{pmatrix} 0 & i' \\ -i' & 0 \end{pmatrix}, \quad \gamma^8 = \begin{pmatrix} 0 & j' \\ -j' & 0 \end{pmatrix},$$

$$\gamma^9 = \begin{pmatrix} 0 & k' \\ -k' & 0 \end{pmatrix}, \quad \gamma_{10} = \begin{pmatrix} -1 & 0' \\ 0 & 1 \end{pmatrix},$$

(68)

where $(i, j, k)$ and $(i', j', k')$ are two commuting sets of quaternions that can be realised as Pauli matrices.
4.3.1 Cayley Submanifolds

As before, the aim is now to reinterpret the spacetime configuration (18) as a supersymmetric configuration on the first fivebrane. For this case four transverse scalars are excited and they should be functions of four coordinates on the fivebrane, i.e., the configurations should correspond to a four surface in eight dimensions. We will now show that the conditions for preserved supersymmetry after imposing the projections lead to the Cayley differential equation in [18] corresponding to Cayley submanifolds i.e. submanifolds that are calibrated by the $Spin(7)$ invariant self-dual four-form $\Omega$.

Before we present the derivation, we first note that the projections (19) can be rewritten in the elegant form

$$\frac{3}{4}(\gamma_{ij} + \frac{1}{6} \Omega_{ijkl} \gamma^{kl}) = 0,$$

where we have taken the only non-zero components of $\Omega$ to be

$$+1 = \Omega_{1234} = \Omega_{6789} = \Omega_{3489} = \Omega_{2479} = \Omega_{1379} = \Omega_{1267} = \Omega_{1368} = \Omega_{1469} = \Omega_{2468},$$

$$-1 = \Omega_{1289} = \Omega_{1478} = \Omega_{3467} = \Omega_{2369},$$

which are the same as those in [18] after the redefinition 6789 $\rightarrow$ 5678. Thinking of $\epsilon$ as an $SO(8)$ spinor, (69) says that under the decomposition $SO(8) \rightarrow Spin(7)$, it is in fact a $Spin(7)$ singlet. To see this note that the adjoint of $SO(8)$ decomposes as $28 \rightarrow 21 + 7$, where $21$ is the adjoint of $Spin(7)$, and that the matrix that appears in (69) is precisely the operator that projects onto the $21$ (see, for example [23, 24]). We thus conclude that the projection operators (19) that we obtained from considerations of orthogonally intersecting branes are equivalent to the more abstract statement that we are working with a spinor that is a $Spin(7)$ singlet. One implication of this observation is that we expect the same projections to appear for just two fivebranes rotated by a $Spin(7)$ rotation [24].
Let us now begin the derivation of the Cayley equation. We first rewrite the projections using the explicit basis (67)-(68). We conclude the following:

\[
\begin{align*}
\epsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 0, \\
\epsilon \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & i' \end{pmatrix}, \\
\epsilon \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & j' \end{pmatrix}, \\
\epsilon \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & k' \end{pmatrix},
\end{align*}
\]

which allows one to trade $\text{Spin}(5,1)$ matrices for $\text{Spin}(5)$ matrices when acting on the spinor $\epsilon$. The signs are necessary and essentially arise from the fact that the $\text{Spin}(5,1)$ $\gamma$-matrices commute with the $\text{Spin}(5)$ matrices.

We look for configurations with $\partial_0 = \partial_5 = 0$ and all transverse scalars excited except $X^{10}$. It is convenient to introduce the quaternion valued fields and derivatives

\[
X' = X^6 + iX^7 + jX^8 + kX^9,
\]
\[
\partial = \partial_1 + i\partial_2 + j\partial_3 + k\partial_4,
\]
\[
\overline{\partial} = \partial_1 - i\partial_2 - j\partial_3 - k\partial_4.
\]  

We first consider the terms in the supersymmetry variation that are linear in $X$:

\[
\epsilon X'^\gamma \gamma^\nu \bar{\partial}_a \gamma^a = \epsilon \begin{pmatrix} 0 & X' \\ X & 0 \end{pmatrix}' \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix},
\]
\[
= \epsilon \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}' \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
= \epsilon \begin{pmatrix} 0 & 0 \\ 0 & X\overline{\partial} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}'.
\]  

(73)
where \( X = X^6 + iX^7 + jX^8 + kX^9 \), \( X\overline{\partial} \equiv \partial_1 X - \partial_2 X i - \partial_3 X j - \partial_4 X k \) and we have used (71).

Next we turn to the terms in the supersymmetry variation that are cubic in \( X \). By performing similar steps we obtain

\[
- \frac{1}{3!}\epsilon \left[ \partial_{a_1} X^{b_1} \partial_{a_2} X^{b_2} \partial_{a_3} X^{b_3} \gamma_{a_1 a_2 a_3} \right],
\]

where we have introduced the triple \( \times \) product of quaternions defined by

\[
x \times y \times z = \frac{1}{2}(x\overline{y}z - z\overline{y}x),
\]

and we have used the fact that it is alternating. Next we let the indices \( a_1 a_2 a_3 \) run over the values 1, 2, 3, 4 and substitute their explicit form using (67), (68). Combining with the terms linear in \( X \) we conclude that the condition for preserved supersymmetry is encapsulated by the differential equation

\[
\partial_1 X - \partial_2 X i - \partial_3 X j - \partial_4 X k = \partial_2 X \times \partial_3 X \times \partial_1 X + \partial_4 X \times \partial_1 X \times \partial_3 X - \partial_1 X \times \partial_2 X \times \partial_4 X i
\]

\[
- \partial_1 X \times \partial_2 X \times \partial_3 X j + \partial_1 X \times \partial_3 X \times \partial_2 X \times \partial_4 X k.
\]

This is the Cayley equation derived in [18] for submanifolds that are calibrated by the Cayley calibration.
4.3.2 Associative Submanifolds

Next consider the configuration (10) preserving 1/16 of the spacetime supersymmetry. For this case four transverse scalars are excited and they should be functions of three coordinates on the fivebrane. i.e. the configurations should correspond to a three surface in seven dimensions. We will now show that the conditions for preserved supersymmetry after imposing the projections (11) lead to the associator equation in [18] for associative submanifolds i.e. submanifolds calibrated by the $G_2$ invariant three form $\varphi$. The non-zero components of $\varphi$ can taken to be

$$
+1 = \varphi_{234} = \varphi_{267} = \varphi_{469} = \varphi_{379} = \varphi_{368},
$$

$$
-1 = \varphi_{289} = \varphi_{478}.
$$

(77)

As in the Cayley case, we first note that our projections (11) can be recast in the form

$$
\frac{2}{3}(\Gamma_{ij} + \frac{1}{4}\psi_{ijkl}\Gamma^{kl}) = 0,
$$

(78)

where the four-form $\psi$ is the Hodge-dual of $\varphi$ in the directions $\{2,3,4,6,7,8,9\}$. Specifically the non-zero components of $\psi$ are

$$
+1 = \psi_{6789} = \psi_{3489} = \psi_{2479} = \psi_{2378} = \psi_{2468},
$$

$$
-1 = \psi_{3467} = \psi_{2369},
$$

(79)

which are simply the components of $\Omega$ in (70) without a 1 component. Thinking of $\epsilon$ as a $Spin(7)$ spinor, (78) says that it is actually a $G_2$ singlet under the decomposition $Spin(7) \rightarrow G_2$. This is because the adjoint of $Spin(7)$ decomposes as $21 \rightarrow 14 + 7$ and the matrix appearing in (78) projects onto the $14$, the adjoint of $G_2$.

For this case, after we rewrite the projections (11) using our explicit basis (67),(68),
we conclude the following:

\[
\begin{align*}
\epsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}' &= 0 , \\
\epsilon \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & i' \end{pmatrix}' , \\
\epsilon \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & j' \end{pmatrix}' , \\
\epsilon \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & k' \end{pmatrix}'.
\end{align*}
\]

(80)

Now we turn to the supersymmetry variation. We now define \[X' = X^6 + i'X^7 + j'X^8 + k'X^9\] and \[\partial = +i\partial_2 + j\partial_3 + k\partial_4 = \overline{\mathcal{D}}.\] The terms linear in \(X\) can now be processed as follows:

\[
\epsilon X''^ \gamma \gamma \delta^a \gamma^a = \epsilon \begin{pmatrix} \overline{\mathcal{D}} & 0 \\ 0 & \overline{\mathcal{D}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \overline{X} \end{pmatrix}' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' ,
\]

\[
= \epsilon \begin{pmatrix} 0 & 0 \\ 0 & \partial \overline{X} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}'.
\]

(81)

Similarly, the cubic terms can be rewritten

\[
-\epsilon \begin{pmatrix} 0 & 0 \\ 0 & \partial_2 \overline{X} \partial_3 X' \partial_4 \overline{X} \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' \gamma^{234} \]

\[
= \epsilon \begin{pmatrix} 0 & 0 \\ 0 & \partial_2 \overline{X}' \times \partial_3 \overline{X} \times \partial_4 \overline{X} \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(82)

After taking the hermitian conjugate the condition for unbroken supersymmetry is given by the differential equation (after dropping the primes)

\[-\partial_2 X i - \partial_3 X j - \partial_4 X k = \partial_2 X \times \partial_3 X \times \partial_4 X ,\]

(83)
which is the associator equation that appears in [18]. Recently solutions to this equation have been studied in relation to domain walls in MQCD [27].

### 4.3.3 Coassociative Submanifolds

Next consider the configuration (12) preserving 1/16 of the spacetime supersymmetry. For this case three transverse scalars are excited and they should be functions of four coordinates on the fivebrane. i.e. the configurations should correspond to a four surface in seven dimensions. We will now show that the conditions for preserved supersymmetry after imposing the projections (13) lead to the coassociator differential equation in [18] for coassociative submanifolds i.e. manifolds calibrated by the $G_2$ invariant four-form $\psi$ that is Hodge dual to the three-form $\varphi$ in the associative case. For this case the projectors can be recast in the form (78) where the components of $\psi$ are now given by (79) after the relabelling $\{2346789\} \rightarrow \{7891234\}$. Thus, the projections for this case also imply that the spinor is a $G_2$ singlet.

To obtain the corresponding differential equation we begin by rewriting the projections using the explicit basis (67), (68), to conclude:

\[
\begin{align*}
\epsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= 0, \\
\epsilon \begin{pmatrix} i' & 0 \\ 0 & i' \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \\
\epsilon \begin{pmatrix} j' & 0 \\ 0 & j' \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \\
\epsilon \begin{pmatrix} k' & 0 \\ 0 & k' \end{pmatrix}' &= -\epsilon \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.
\end{align*}
\]

(84) (85)

We now have $X = i'X^7 + j'X^8 + k'X^9 = -\overline{X}$ and $\partial = \partial_1 + \partial_2i + \partial_3j + \partial_4k$. The
terms in the supersymmetry variation that are linear in $X$ can be reexpressed

$$\epsilon X' \gamma_{\nu} \partial_\nu \gamma^a = \epsilon \begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix}' \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Similarly, the cubic terms can be rewritten

$$-\epsilon \begin{pmatrix} 0 & 0 \\ 0 & \overline{\partial} X [7 \partial X^8 \partial X^9] \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^{789} \gamma^{i j k},$$

After taking the hermitian conjugate the condition for unbroken supersymmetry is then given by the differential equation

$$-\partial X^7 i - \partial X^8 j - \partial X^9 k = \partial X^7 \times \partial X^8 \times \partial X^9,$$

which is the coassosiator equation that appears in [18].

5 Conclusion

In this paper we have analysed the conditions necessary for the fivebrane worldvolume theory to preserve some supersymmetry, when the self-dual three form is set to zero. Our approach was to first consider spacetime configurations of orthogonally intersecting fivebranes in order to derive a set of projection operators acting on the worldvolume supersymmetry parameters. By manipulating the supersymmetry variation using these projections we derived a set of differential equations for the transverse scalar fields of the worldvolume. These Bogomol’nyi equations are none other than the equations resulting from calibrated geometries [18]. It would be interesting to find explicit solutions to these equations, i.e. calibrated geometries, for simple cases such as the orthogonal brane configurations of section two, or branes in flat space rotated by elements of groups associated
with special holonomy \[23, 26, 24\] (since we might expect that these configurations are associated with the same projection operators as for the orthogonal configurations that we explicitly considered in section two).

It will be very interesting to extend the analysis of this paper to include membranes by allowing for a non-zero self-dual three form. We expect that the resulting differential equations will be associated with a generalised notion of calibrated geometries \[28\]. We pointed out in section two some configurations allow for pp-waves and membranes to be introduced without breaking any addition supersymmetry. One may expect that this has a simple interpretation in the resulting generalised calibrations.

In this paper we have studied fivebranes in a flat target space. It is well known that calibrated geometries can be defined in manifolds with special holonomy. For example, in eight dimensions it is well known that in a curved manifold with reduced \(\text{Spin}(7)\) holonomy the self-dual four form \(\Omega\) is globally defined and this allows one to have submanifolds calibrated by \(\Omega\). With a flat target space we saw that the preserved supersymmetries for the calibrated geometries are \(\text{Spin}(7)\) invariant spinors. This also has a natural generalisation since manifolds with \(\text{Spin}(7)\) holonomy contain parallel spinors. In a similar manner the associative and coassociative cases will generalise to seven dimensional manifolds with \(G_2\) holonomy while the Kähler and special Lagrangian cases can be generalised to manifolds with \(SU(n)\) holonomy.

It would interesting to generalise our analysis to find the analogues of the differential equations of \[18\] in a curved manifold of special holonomy. We leave this to future work, but we would like to mention how some of the analysis of section three could be generalised to a curved target space. Examining equation (22) we find that now

\[ u^b_a = e^m_a \partial_m X^N \bar{E}^b_N , \tag{89} \]

where \(e^m_a g_{nm} e^m_d = \eta_{ad}\) and \(g_{nm} = \partial_n X^N \bar{E}^b_N \eta_{ad} \partial_m X^R \bar{E}^d_R\). The equation for the matrix \(\Gamma\)
is still given by
\[ -\frac{1}{6!} \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6} u_{a_1} b_{a_2} u_{a_3} b_{a_4} u_{a_5} b_{a_6} \epsilon^\alpha (\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6})_{\alpha} \gamma' . \] (90)

But if we define \( v_c^k \) by \( u_{b}^k = e^m_a (f^{-1})^c_m v_c^k \) where \( (f^{-1})^c_m \) is the matrix \( (f^{-1})^c_m = e^b_m u_b^c \) then we may write \( \Gamma \) as
\[ -\frac{1}{6!} (\det(e^{-1}f)) \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6} v_{a_1} b_{a_2} v_{a_3} b_{a_4} v_{a_5} b_{a_6} \epsilon^\alpha (\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6})_{\alpha} \gamma' . \] (91)

The net effect of these changes is just to make the replacement
\[ \delta_c^n \partial_n X^{b'} \rightarrow ((f^{-1})^c_m (E_{m}^{b'} + \partial_M X^{n'} E_{n'}^{b'}) , \] (92)
in all formulae for the supervariation of the spinor.

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While this paper was being prepared we learnt of the work [29] which has some overlap with this work.

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