Kinematics of semiclassical spin and spin fiber bundle associated with so(n) Lie-Poisson manifold

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Abstract. We describe geometric construction underlying the Lagrangian actions for non-Grassmann spinning particles proposed in our recent works. If we discard the spatial variables (the case of frozen spin), the problem reduces to formulation of a variational problem for Hamiltonian system on a manifold with so(n) Lie-Poisson bracket. To achieve this, we identify dynamical variables of the problem with coordinates of the base of a properly constructed fiber bundle. In turn, the fiber bundle is embedded as a surface into the phase space equipped with canonical Poisson bracket. This allows us to formulate the variational problem using the standard methods of Dirac theory for constrained systems.

Consider \( \frac{1}{2} n(n-1) \)-dimensional Euclidean space equipped with the coordinates \( J^{ij} = -J^{ji}, \ i, j = 1, 2, \ldots, n \) and with the Lie-Poisson bracket

\[
\mathbb{R}^{\frac{n(n-1)}{2}} = \left\{ J^{ij}, \{ J^{ij}, J^{kl} \}_{LPB} = 2(\delta^{ik} J^{jl} - \delta^{il} J^{jk} - \delta^{jk} J^{il} + \delta^{jl} J^{ik}) \right\}. \tag{1}
\]

This is the Lie-Poisson manifold associated with so(n) algebra \([1, 2, 3]\). The construction presented below works for the case of Minkowski space with so(m, n) algebra as well.

We discuss the Hamiltonian flow

\[
\dot{J} = \{ J, H_0 \}_{LPB}, \tag{2}
\]

generated by given Hamiltonian \( H_0(J) \). Our aim is to formulate variational problem for the Hamiltonian flow on the submanifold \( \mathbb{S} \) which will be specified below. We call \( \mathbb{S} \) spin surface, as the canonical quantization of the submanifold gives quantum mechanics of the spin one-half particle. In \([8]\) it has been demonstrated that SO(3) spin surface leads to a reasonable model of non-relativistic spin. SO(1, 3) spin surface can be used to construct variational problem for unified description of both the Frenkel \([6]\) and BMT \([7]\) theories of relativistic spin \([9]\). SO(2, 3) spin surface implies the model of Dirac electron \([11, 10]\), and represents an example of pseudoclassical mechanics \([12]\).

Roughly speaking, our procedure is as follows. The formulation of a variational problem in closed form is known for the equations constructed on the phase space equipped with canonical Poisson bracket, \( \{ \omega_i, \pi_j \} = \delta_{ij} \). In this case the Hamiltonian equations can be obtained by variation of the action \( \int \pi \dot{\omega} - H(\omega, \pi) \), where \( H(\omega, \pi) \) stands for Hamiltonian of the system. To apply this prescription, we reformulate our problem (1), (2) in terms of the properly constructed phase space with constraints.
CONSTRUCTION 1.
1.1. Take $2n$-dimensional phase space equipped with the Poisson bracket

$$\mathbb{R}^{2n} = \{ \omega^i, \pi^j, \{ \omega^i, \pi^j \}_{PB} = \delta^{ij} \}.$$  \hspace{1cm} (3)

1.2. Define the map from the phase to angular-momentum space (1)

$$f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}},$$

$$f : (\omega^i, \pi^j) \rightarrow J^{ij} = 2(\omega^i \pi^j - \omega^j \pi^i).$$  \hspace{1cm} (4)

We have, for $n > 2$,

$$\text{rank } \frac{\partial (J^{ij})}{\partial (\omega^k, \pi^l)} = 2n - 3,$$  \hspace{1cm} (5)

so an image of the map is $(2n - 3)$-dimensional surface $\mathbb{M}$

$$f(\mathbb{R}^{2n}) = \mathbb{M}^{2n-3} \subseteq \mathbb{R}^{\frac{n(n-1)}{2}}.$$  \hspace{1cm} (6)

Poisson bracket of the functions $J^{ij}(\omega, \pi)$ coincides with the Lie-Poisson bracket (1). More generally, for any functions $A(J), B(J),$

$$\{ A(J), B(J) \}_{PB} = \{ A(J), B(J) \}_{LPB}|_{J \rightarrow J(\omega, \pi)}.$$  \hspace{1cm} (7)

1.3. Look for the surface $T = \{ \omega, \pi | T_a(\omega^i, \pi^j) = 0 \} \subseteq \mathbb{R}^{2n}$ which is invariant under action of $SO(n)$, that is

$$\{T_a, J^{ij}\}_{PB} = 0.$$  \hspace{1cm} (8)

This is essentially unique surface of $2n - 3$ dimensions

$$T^{2n-3} = \{T_3 = \pi^2 + a_3 = 0, \ T_4 = \omega^2 + a_4 = 0, \ T_5 = \omega \pi = 0\}.$$  \hspace{1cm} (9)

where $a_3, a_4 \in \mathbb{R}$, and it has been denoted $\pi^2 = \pi^i \pi^i$, and so on.

COMMENTS. A. Any trajectory of $H_0(J)$ which starts on $\mathbb{M}^{2n-3}$ lies entirely on $\mathbb{M}^{2n-3}$ (the proof is similar to those of Proposition 3 below).

B. $SO(3)$ is the exceptional case, when $\mathbb{M} = \mathbb{R}^{\frac{n(n-1)}{2}}$, and the vector representation of $SO(3)$ coincides with the adjoint one. Besides the surface $T^{2n-3}$ can be identified with the group manifold $SO(3)$, see [13].

C. The invariance condition (8) guarantees the validity of important Propositions 2, 3, see below.

D. Casimir operators of $SO(n)$ group are scalar functions of the generators, $C(J^{ij})$. On the surface (9) they will have fixed values determined by the constants $a_3$ and $a_4$: $C(J^{ij}) = C(\omega^2, \pi^2, \omega \pi) = C(a_3, a_4)$. In particular, the first Casimir operator is $J^2 = 8a_3a_4$, see Eq. (22) below.

Denote $S$ image of $T^{2n-3}$ under the map $f$ (this is called the spin surface, see Figure 1)

$$S = f(T^{2n-3}) \subseteq \mathbb{R}^{\frac{n(n-1)}{2}}.$$  \hspace{1cm} (10)

Denote $F_J \in T^{2n-3}$ preimage of a point $J \in S$, $F_J = f^{-1}(J)$.

The manifold $T^{2n-3}$ has natural structure of fiber bundle

$$T^{2n-3} = (S, F, f),$$  \hspace{1cm} (11)
with the base $S$, the projection map $f$, the standard fiber $F$ and structure group of the fiber $SO(2)$. The standard fiber can be described as follows. Let us identify the vector $(\omega, \pi) \in T^{2n-3}$ with the pair of orthogonal vectors of $\mathbb{R}^n$ $(\omega, \pi) \sim \vec{\omega} \perp \vec{\pi}$, $\vec{\omega}^2 = -a_4$, $\vec{\pi}^2 = -a_3$. \hspace{1cm} (12)

Then $F$ is composed by all the pairs obtained from $(\vec{\omega}, \vec{\pi})$ by rotations in the plane of these vectors. So the structure group is composed of rotations

$$\vec{\omega}' = \vec{\omega} \cos \beta + \sqrt{\frac{a_4}{a_3}} \vec{\pi} \sin \beta, \hspace{0.5cm} \vec{\pi}' = -\sqrt{\frac{a_3}{a_4}} \vec{\omega} \sin \beta + \vec{\pi} \cos \beta.$$ \hspace{1cm} (13)

By construction, $(\omega', \pi') \in T^{2n-3}$. Infinitesimal form of the symmetry is

$$\delta \vec{\omega} = \sqrt{\frac{a_4}{a_3}} \beta \vec{\pi}, \hspace{0.5cm} \delta \vec{\pi} = -\sqrt{\frac{a_3}{a_4}} \beta \vec{\omega}.$$ \hspace{1cm} (14)

As it must be, the transformations leave inert the points of base, $\delta J^{ij} = 0$. In the dynamical realization, the structure group acts independently at each instance of time and turn into the local (gauge) symmetry of a Lagrangian. The spin-plane symmetry determines physical sector of the theory, and hence play the fundamental role in our construction. Indeed, according to Eq. (4), we consider the spin $J^{ij}$ as angular-momentum of an "inner-space particle" $\vec{\omega}$. The crucial difference with the usual (spacial) angular momentum is the presence of the plane-spin symmetry, which acts on the basic variables $\omega, \pi$, while leaves invariant the spin variables $J$. According to the general theory [4, 5], the gauge non-invariant coordinates $\omega$ of the inner-space are not physical (observable) quantities. The only observable quantities are the gauge-invariant variables $J$. So our geometric construction realizes, in a systematic form, the oldest idea about spin as the "intrinsic angular momentum".
LOCAL COORDINATES ON $M^{2n-3}$ AND EQUATIONS OF THIS SURFACE. They can be obtained solving Eq. (4). Namely, the subset $J'$ of $2n-3$ independent functions among $J(\omega, \pi)$, represents the local coordinates.

Let us discuss this in some details. In accordance with the rank condition (5), we can separate $J^j$ on two groups, $J = (J', J'')$, in such a way that the number of $J'$ is equal to $2n - 3$, and

$$\text{rank} \frac{\partial J'}{\partial (\omega, \pi)} = 2n - 3. \quad (15)$$

Then equations for $J'$ from the system (4) can be resolved with respect some of $2n - 3$ variables among $\omega, \pi$. Substitute these expressions into the remaining equations from (4). By construction, the result does not depend on $\omega$ and $\pi$, and we obtain the expressions for $J''$ through $J'$

$$J'' = g(J') \equiv J''(J'). \quad (16)$$

In the result, the image $f(\mathbb{R}^{2n})$ is the surface with equations (16) and with local coordinates $J'$

$$\mathbb{M}^{2n-3} = \left\{ J = (J', J'') \mid J'' = J''(J') \right\}. \quad (17)$$

So, points of $\mathbb{R}^{2n}$ are mapped into

$$f(\omega, \pi) = (J', J''(J')) \in \mathbb{M}^{2n-3}. \quad (18)$$

For example, for $SO(2, 3)$ case, the local coordinates are $J^5\mu$, $J^0\nu$ while as the equations of the surface we can take [12]

$$\epsilon^{\mu\nu\alpha\beta} J^5_\nu J_{\alpha\beta} = 0, \iff J^{ij} = (J^{50})^{-1}(J^{5i} J^{0j} - J^{5j} J^{0i}). \quad (19)$$

Among the four relativistic-covariant equations on the l.h.s., there are only three independent.

LOCAL COORDINATES ON $T^{2n-3}$ ADJUSTED WITH THE STRUCTURE OF FIBRATION. Since

$$\text{rank} \frac{\partial (J', \omega^n, T_4, T_5)}{\partial (\omega^k, \pi^l)} = 2n, \quad (20)$$

we can make the change of coordinates on $\mathbb{R}^{2n}$

$$(\omega^j, \pi^j) \leftrightarrow (J', \omega^n, T_4, T_5). \quad (21)$$

In the new coordinates the function $T_3$ does not depend on $\omega^n$. Indeed, $J^{ij} J_{ij}$ can be identically rewritten through $T_a$ as follows

$$J^2 = 8 \left[ (T_4 - a_4)T_3 - T^2_5 - a_3T_4 + a_3a_4 \right], \quad (22)$$

then

$$T_3 = \frac{J^2 + 8T^2_2 + 8a_3T_4 - 8a_3a_4}{8(T_4 - a_4)}. \quad (23)$$

On the other hand, substitute the new coordinates into the expression $(J^{ij}(\omega, \pi))^2$. By construction, this gives $J^2 = (J')^2 + (J''(J'))^2$. Using this in Eq. (23), we obtain

$$T_3(\omega, \pi)|_{(J', \omega^n, T_4, T_5)} = T_3(J', T_4, T_5). \quad (24)$$
Hence in the new coordinates the surface looks as

\[ T^{2n-3} = \left\{ J', \omega^n, T_4, T_5 \mid T_4 = T_5 = 0, \ T_3(J') = 0 \right\}. \tag{25} \]

**Proposition 1.** \( \dim S = 2n - 4 \). (then \( \dim F = 1 \)).

Indeed, take restriction of the map (18) on \( T^{2n-3} \). Since any point on the surface obeys the condition \( T_3(J') = 0 \), we have

\[ f|_T: (\omega, \pi) \to (J', J''(J')), \text{ where } T_3(J') = 0. \tag{26} \]

Hence \( T_3(J') = 0 \) is equation of the base in space \( \mathbb{M}^{2n-3} \), and \( \dim S = 2n - 4 \).

Let as take some \( 2n - 4 \) variables \( J \) among \( J' \) which form the coordinate system of the base \( S^{2n-4} \). Then (21) and (25) imply that \( (J, \omega^n) \) can be taken as local coordinates of the fibration \( T^{2n-3} \).

**Construction 2.**

**2.1.** Let \( H_0(J) \) is some Hamiltonian on \( \mathbb{R}^{\frac{n(n-1)}{2}} \). The map \( f \) can be used to induce the Hamiltonian \( H_0(\omega, \pi) \) on the phase space \( \mathbb{R}^{2n} \)

\[ H_0(\omega, \pi) \equiv H_0(J(\omega, \pi)). \tag{27} \]

Let us confirm that Hamiltonian flows of \( H_0(\omega, \pi) \) and \( H_0(J) \) are adjusted with the surfaces \( T^{2n-3} \) and \( S \).

**Proposition 2.** Any trajectory of \( H_0(\omega, \pi) \) which starts on \( T^{2n-3} \) lies entirely on \( T^{2n-3} \).

Indeed, let \( T_a(\omega(\tau_0), \pi(\tau_0)) = 0 \) for some trajectory of \( H_0(\omega, \pi) \). We have

\[ \dot{T}_a(\omega(\tau), \pi(\tau)) = \{T_a, H_0(J(\omega, \pi))\}_{PB} = \{T_a, J\}_{PB} \frac{\partial H_0}{\partial J} = 0, \tag{28} \]

due to the invariance condition \( \{T_a, J\} = 0 \). Hence \( T_a(\tau) = T_a(\tau_0) = 0 \) for any \( \tau \), that is \( (\omega(\tau), \pi(\tau)) \) belong to \( T^{2n-3} \) at each \( \tau \).

**Proposition 3.** Any trajectory of \( H_0(J) \) which starts on \( S \) lies entirely on \( S \).

Indeed, the problem

\[ \dot{J} = \{J, H_0(J)\}_{LPB} = 0, \quad J(\tau_0) = J_0 \in S, \tag{29} \]

has unique solution, we denote it \( J(\tau) \). Take any point \( (\omega_0, \pi_0) \) which belong to preimage of \( J_0, f(\omega_0, \pi_0) = J_0 \). Construct the (unique) solution to the problem \( \dot{\omega} = \{\omega, H_0(\omega, \pi)\}_{PB} = 0, \dot{\pi} = \{\pi, H_0(\omega, \pi)\}_{PB} = 0, (\omega(\tau_0), \pi(\tau_0)) = (\omega_0, \pi_0) \). According the Proposition 2, it lies in \( T^{2n-3} \), then \( f(\omega(\tau), \pi(\tau)) \) lies in \( S \). Besides, this obeys the problem (29). Since solution to the problem is unique, we conclude \( J(\tau) = f(\omega(\tau), \pi(\tau)) \) \( \in S \) for each \( \tau \).

**2.2.** Consider variational problem on the extended phase space \( (\omega^i, \pi^j, e_a, \pi_a, \lambda_{ea}), \ a = 3, 4 \), with the functional

\[ S_H = \int d\tau \pi \dot{\omega} - \left[ H_0(\omega, \pi) + \frac{e_3^3}{2}T_3 + \frac{e_4^4}{2}T_4 + \pi_{ea}(\lambda_{ea} - \dot{e}_a) \right]. \tag{30} \]

Variation of the functional leads to the equations

\[ \pi_{ea} = 0, \quad \dot{e}_a = \lambda_{ea}, \tag{31} \]
\[ \pi^2 + a_3 = 0, \quad \omega^2 + a_4 = 0, \]  
\[ \dot{\omega}^i = \frac{\partial H_0}{\partial \pi^i} - e_3 \pi^i, \quad \dot{\pi}^i = -\frac{\partial H_0}{\partial \omega^i} + e_4 \omega^i. \]  

Eq. (31) has been used in obtaining (32). Compute derivative of the first equation from (32)

\[ 0 = (\pi^2 + a_3) = 2(-\pi^i \frac{\partial H_0}{\partial \omega^i} + e_4 (\omega \pi)) = 2e_4 (\omega \pi) \Rightarrow (\omega \pi) = 0, \]  
where we have used \( \pi^i \frac{\partial H_0(j(\omega \pi))}{\partial \omega^i} = 4 \frac{\partial H_0}{\partial J^i} \pi^i \pi^j = 0. \) We continue the process, obtaining the following algebraic consequences of the system (31)-(33)

\[ (\omega \pi) = 0, \quad e_4 = \frac{a_3}{a_4}, \quad \lambda_4 = \frac{a_3}{a_4} \lambda_3. \]  
The auxiliary variables \( \lambda_3, e_4, \pi_4 \) and \( \pi_3 \) are fixed by the algebraic equations in terms of \( \lambda_3 \). For the remaining variables we have the differential equations

\[ \dot{\epsilon}_3 = \lambda_3, \]  
\[ \dot{\omega}^i = \frac{\partial H_0}{\partial \pi^i} - e_3 \pi^i, \quad \dot{\pi}^i = -\frac{\partial H_0}{\partial \omega^i} + \frac{a_3}{a_4} e_3 \omega^i, \]  
as well as the algebraic constraints

\[ T_3 = \pi^2 + a_3 = 0, \quad T_4 = \omega^2 + a_4 = 0, \quad T_5 = (\omega \pi) = 0. \]  

Equations (38) and (9) imply the following

**Proposition 4.** All the trajectories \((\omega(\tau), \pi(\tau))\) of the problem (30) live on the fiber bundle \( \mathbb{T}^{2n-3} \).

Let us consider the projection map (4), \( J^{ij}(\tau) = f(\omega(\tau), \pi(\tau)) \). According the Proposition 4 and Eq. (10), \( J(\tau) \) lies on \( S \). Besides, \( J(\tau) \) represents a solution to the problem (2)

\[ J^{ij} = \frac{\partial J^{ij}}{\partial \omega^a} \dot{\omega}^a + \frac{\partial J^{ij}}{\partial \pi_a} \dot{\pi}_a = \]  
\[ \frac{\partial J^{ij}}{\partial \omega^a} \left( \frac{\partial H_0}{\partial \pi_a} - e_3 \pi^a \right) + \frac{\partial J^{ij}}{\partial \pi_a} \left( -\frac{\partial H_0}{\partial \omega^a} + \frac{a_3}{a_4} e_3 \omega^a \right) = \]  
\[ \frac{\partial J^{ij}}{\partial \omega^a} \frac{\partial H_0}{\partial \pi_a} - \frac{\partial J^{ij}}{\partial \pi_a} \frac{\partial H_0}{\partial \omega^a} + \frac{\partial J^{ij}}{\partial \omega^a} \frac{\partial H_0}{\partial \omega^a} \frac{\partial J^{kl}}{\partial \pi_a} - \frac{\partial J^{ij}}{\partial \pi_a} \frac{\partial J^{kl}}{\partial \omega^a} \frac{\partial H_0}{\partial J^{kl}} = \]  
\[ \{J^{ij}, J^{kl}\}_{PB} \frac{\partial H_0}{\partial J^{kl}} = \{J^{ij}, J^{kl}\}_{LPB} \frac{\partial H_0}{\partial J^{kl}} = \{J^{ij}, H_0(J)\}_{LPB}. \]  

Hence we have obtained

**Proposition 5.** Any trajectory of the Hamiltonian flow of \( H_0(J) \) on \( S, J(\tau)|_S \), is a projection of some trajectory \((\omega(\tau), \pi(\tau))\) of the variational problem (30), \( J(\tau)|_S = f(\omega(\tau), \pi(\tau)) \).

In this sense, trajectories of the Lie-Poisson system (2) lying on \( S \in \mathbb{R}^{n(n-1)} \), are obtained starting from the variational problem (30) formulated on \( \mathbb{R}^{2n} \).

For the physically interesting cases \( SO(3), SO(1,3) \) and \( SO(2,3) \), the Lagrangian actions can be restored from the Hamiltonian one, (30), see [8-13].
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