Bifurcation avoidance control of stable periodic points using the maximum local Lyapunov exponent

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Abstract: We propose a novel controlling methodology for avoiding local bifurcations of stable, hyperbolic fixed and periodic points in nonlinear discrete-time dynamical systems with parameter variation. Dynamical systems may not work as expected owing to bifurcations of desired behavior caused by parameter variation. Controlling parameters to avoid bifurcations enables us to construct robust dynamical systems against unexpected parameter variation. Assuming that desired behavior is a hyperbolic fixed or periodic point, as a control strategy, optimizing the degree of its stability (i.e. the maximum modulus of its characteristic multipliers) is considerable. However, it cannot be optimized by simple gradient methods, and off-line calculation to find the exact positions of fixed and periodic points and their characteristic multipliers is also needed. In contrast, the method we propose is to control the maximum local Lyapunov exponent (MLLE) that is defined in finite time and is closely related to the degree of stability on hyperbolic fixed and periodic points. This method can predict occurrence of local bifurcations for parameter variation by monitoring the MLLE along the passage of time and adjust parameter values on line to control the MLLE. This leads that occurrence of local bifurcations can be avoided without directly analyzing bifurcations in advance. The parameter regulation to avoid local bifurcations is theoretically derived from the minimization of an objective function with respect to the MLLE. Our experimental results applied to the Kawakami and Hénon maps demonstrate that the proposed controller could be used to avoid local bifurcations.

Key Words: bifurcation control, fixed and periodic points, maximum local Lyapunov exponent, discrete-time dynamical system
1. Introduction

Nonlinear dynamical systems expressed by parameterised difference equations are widely used for mathematical modeling of physical systems [1, 2]. The values of system parameters are determined so that a desired behavior can appear in a steady state through bifurcation analysis for the desired behavior [3–5]. However, when parameter values are changed from appropriate values for any reason, the systems may not work as expected owing to occurrence of bifurcations which mean that a desired behavior vanishes. For example, we know that cardiac alternans [6] is deadly to humans and can be understood to be caused by bifurcations in mathematical models [7–9].

Suppressing an undesirable state and preventing its occurrence are important for not only diseases but also a variety of physical systems. We now assume that a desired behavior of a given discrete-time dynamical system is a hyperbolic fixed or periodic point. The OGY method [10] is known as the original way of controlling chaos [11, 12] and can stabilize unstable fixed and periodic points by rearranging characteristic multipliers that correspond to the eigenvalues of the Jacobian matrix. This means that the OGY method can be used to recover desired behavior destabilized by bifurcations from an undesirable state. For example, by using ideas of controlling chaos, we previously proposed a recovery system [13] for a dynamic image-segmentation system consisting of a discrete-time oscillator network [14, 15].

On the other hand, in terms of robust control, developing robust dynamical systems that can hold on a desired behavior even if the values of system parameters are changed from appropriate values for any reason is also important. Since desired behavior vanishes by its bifurcation, controlling parameters to avoid bifurcations becomes the key to construct robust systems, and moreover, it is a new challenge in bifurcation control [16, 17].

Local bifurcations of stable, hyperbolic fixed and periodic points occur when the maximum modulus of characteristic multipliers becomes one. Kitajima et al. [18] defined the degree of stability (the stability index) on hyperbolic fixed and periodic points using the characteristic multiplier of the maximum modulus and proposed updating parameter values to optimize the stability index. It is expected that this method can be used to construct robust dynamical systems that are controlled so as not to make bifurcations. However, since the stability index (i.e. the maximum characteristic multiplier) is not differentiable with respect to system parameters in general, simple gradient methods cannot be used to optimize it directly. In addition, this method [18] needs off-line calculation with a high computational cost to find the exact positions of fixed and periodic points and their characteristic multipliers whenever parameter values change. This means that the parametric control cannot be realized along the passage of time.

In this paper, we instead formulate the problem of bifurcation avoidance on the basis of the maximum Lyapunov exponent (MLE) [19–23] that is closely related to the maximum characteristic multiplier [24–26], and then, we theoretically derive parameter regulation to optimize the MLE using a simple gradient method from the differentiability of the MLE with respect to system parameters. This parametric controller can be used to avoid local bifurcations of stable, hyperbolic fixed and periodic points. In parameter updating, we practically use the maximum local Lyapunov exponent (MLLE) [27, 28] that can be computed in finite time to relieve computational difficulty of the MLE defined in infinite time. The MLLE and parameter variation to optimize the MLLE can be calculated along the passage of time. Therefore, the proposed method predicts occurrence of bifurcations by monitoring the MLLE along the passage of time and adjusts parameter values on line so as to avoid bifurcations without directly analyzing bifurcations in advance. We here note that a control system we propose is to adjust parameter values and is different from a state-feedback control system of LEs [29, 30]. To evaluate whether the proposed controller works effectively, we also carry out several experiments for stable fixed and periodic points in the Kawakami [31] and Hénon [32] maps.

2. Preliminaries

2.1 Bifurcation of fixed and periodic points

We consider a differentiable \( N \)-dimensional map \( f \) or a discrete-time dynamical system described by
where \( t \) denotes the discrete time; \( x \in \mathbb{R}^N \) and \( p \in \mathbb{R}^M \) correspond to the vectors consisting of \( N \) state variables and \( M \) system parameters in which \( \mathbb{R} \) denotes the set of real numbers. Here, we assume that the dynamics of \( f \) is known and all state variables are always observable.

Fixed and periodic points of \( f \) are defined as follows. If a point \( \mathbf{x}^* \in \mathbb{R}^N \) satisfies
\[
\mathbf{x}^* - f(\mathbf{x}^*, p) = 0,
\]
then \( \mathbf{x}^* \) is a fixed point of \( f \). In the same way, a periodic point with period \( n \), i.e., an \( n \)-periodic point, of \( f \) is defined as a point \( \mathbf{x}^* \) such that
\[
\mathbf{x}^* - f^n(\mathbf{x}^*, p) = 0 \quad \text{and} \quad \mathbf{x}^* - f^k(\mathbf{x}^*, p) \neq 0 \quad \text{for} \quad k < n,
\]
where \( f^n \) denotes the \( n \)th iterate of \( f \).

We now describe the Jacobian matrix of \( f \) as
\[
Df(x(t), p) = \frac{\partial}{\partial x} f(x(t), p),
\]
where all entries of the Jacobian matrix are explicitly described. Using the Jacobian matrix of \( f \), we introduce the characteristic equation of the fixed point \( \mathbf{x}^* \) as
\[
\chi(x^*, p, \mu) = \det(\mu I - Df(x^*, p)) = 0,
\]
where \( I \) denotes the \( N \times N \) identity matrix. A complex number \( \mu \) is an eigenvalue of \( Df(x^*, p) \) and is called a characteristic multiplier of \( x^* \). By replacing \( Df(x^*, p) \) to \( Df^n(\mathbf{x}^*, p) \) in Eq. (5), the characteristic equation of \( \mathbf{x}^* \) can also be defined and the eigenvalues of \( Df^n(\mathbf{x}^*, p) \) correspond to the characteristic multipliers of \( \mathbf{x}^* \). Here, assuming that \( \mathbf{x}^* \) corresponds to \( x(t) \) at \( t = \tau \), the matrix \( Df^n(\mathbf{x}(\tau), p) \) consists of the product of \( n \) Jacobian matrices as
\[
Df^n(\mathbf{x}^*, p) = Df^n(\mathbf{x}(\tau), p)
= Df(\mathbf{x}(\tau - 1), p) \cdot Df(\mathbf{x}(\tau - 2), p) \cdots Df(\mathbf{x}(\tau + 1), p) \cdot Df(\mathbf{x}(\tau), p).
\]

We call \( x^* \) (resp. \( \mathbf{x}^* \)) hyperbolic fixed point (resp. periodic point) if the absolute values of all eigenvalues of \( Df(x^*, p) \) (resp. \( Df^n(\mathbf{x}^*, p) \)) are not one. In the following, we consider only hyperbolic fixed and periodic points.

The arrangement of characteristic multipliers in the complex plane determines the local stability of fixed and periodic points. When all characteristic multipliers of a fixed point are inside the unit circle in the complex plane, the fixed point is stable. On the other hand, the fixed point is unstable if one or more characteristic multipliers are outside the unit circle. The local stability of a periodic point is also determined by the same way. Thus, local bifurcations of stable fixed and periodic points occur when the characteristic multiplier of the maximum modulus is on the circumference of the unit circle.

Kitajima et al. [18] defined the stability index on fixed and periodic points using the characteristic multiplier of the maximum modulus.

General bifurcations of codimension-one are tangent bifurcations (or saddle-node bifurcations or fold bifurcations), period-doubling bifurcations (or flip bifurcations), and the Neimark–Sacker bifurcations (or the Hopf bifurcations in a discrete-time system) [4]. The conditions in which the respective local bifurcations of \( \mathbf{x}^* \) occur correspond to \( \chi(x^*, p^*, 1) = 0, \chi(x^*, p^*, -1) = 0, \) and \( \chi(x^*, p^*, e^{i\theta}) \mid_{\theta \neq 0, \pi} = 0 \) where the superscript \( i \) expresses the imaginary unit and \( \theta \in [0, 2\pi) \) denotes the argument of a characteristic multiplier \( \mu = e^{i\theta} \). Moreover, \( p^* \) represents system parameters at which the local bifurcation occurs, and the values of \( p^* \) can be numerically found by using a powerful computing method [3, 4].
2.2 Maximum local Lyapunov exponent

We assume that the limit of the trajectory \( x(t) \) starting with an initial point \( x(0) \) exists, and moreover, the limit set is a stable fixed or periodic point. We here consider two nearby points \( x(0) + v(0) \) and \( x(0) \) in the state space where \( v(0) \) is a small perturbation at \( x(0) \). After \( T \) time steps, these points will evolve to \( f^T(x(0) + v(0)) \) and \( f^T(x(0)) \). Assuming that \( v(t) \) is evolved in accordance with the linearized dynamics described by

\[
v(t + 1) = Df(x(t), p) \cdot v(t),
\]

we obtain

\[
v(T) = f^T(x(0) + v(0), p) - f^T(x(0), p) = Df^T(x(0), p) \cdot v(0).
\]

Therefore, we define the average exponential rate of divergence or convergence between the two trajectories emanating from \( x(0) + v(0) \) and \( x(0) \) as follows [22]:

\[
\Lambda(x(0), p) = \lim_{T \to \infty} \frac{1}{T} \ln \frac{\|v(T)\|}{\|v(0)\|} = \lim_{T \to \infty} \frac{1}{T} \ln \frac{\|Df^T(x(0), p) \cdot v(0)\|}{\|v(0)\|},
\]

where \( \|\cdot\| \) denotes the Euclidean norm of a vector. From the consequence of Oseledec’s theorem [24], the limit of \( \Lambda(x(0), p) \) exists and is equal to the MLE for almost all \( v(0) \) [22, 25]. Hence, regardless of a given \( v(0) \), the MLE takes the same value for almost all initial points in the basin of a stable fixed or periodic point.

The MLE is closely related to the maximum modulus of characteristic multipliers of a fixed or periodic point. From the definition of the operator norm for a vector, \( \|Df^T(x(0), p) \cdot v(0)\|/\|v(0)\| \) in Eq. (9) is equal to the spectral norm of \( Df^T(x(0), p) \) if \( \Lambda(x(0), p) \) gives the MLE. That is, for a stable fixed point \( x^* \), we obtain the relation described by

\[
\Lambda(x(0), p) = \ln \rho(Df(x^*, p)),
\]

according to the spectral radius formula [33]. \( \rho(\cdot) \) represents the spectral radius of a matrix. Therefore, the MLE corresponds to the logarithm of the maximum modulus of the characteristic multipliers, \( \mu_{\text{max}} = \arg\max_j |\mu_j|, (j = 1, 2, \ldots, N) \), of \( x^* \). In the same way, for a stable \( n \)-periodic point \( \hat{x}^* \), we also obtain the relation as

\[
\Lambda(x(0), p) = \frac{1}{n} \ln \rho(Df^n(\hat{x}^*, p)).
\]

From the two relations and the assumption that the limit set of a trajectory is a stable fixed or periodic point, the MLE gives us local information whether a stable fixed or periodic point bifurcates at the values of \( p \). For example, a stable fixed or periodic point is present at the values of \( p \) if \( \Lambda(x(0), p) \) takes a negative value; when the value of \( \Lambda(x(0), p) \) approaches zero, the values of \( p \) are close to a bifurcation point of the stable fixed or periodic point.

Since the MLE in Eq. (9) is defined in infinite time, computing the MLE is numerically difficult. To relieve the difficulty, we here introduce the MLLE that is defined in finite time [27, 28]. The MLLE on an initial value \( x(0) \) is computed as

\[
\lambda(x(0), p, T) = \frac{1}{T} \sum_{t=0}^{T-1} \ln \frac{\|Df(x(t), p) \cdot v(t)\|}{\|v(0)\|},
\]

where the interval \( T \) is set so that \( T \gg n \), i.e., this gives us an approximate value of \( \Lambda(x(0), p) \).

Moreover, we normalize \( v(t) \) so that \( \|v(t)\| = 1 \) at every discrete time as

\[
v(t) = \frac{w(t)}{\|w(t)\|},
\]

and

\[
w(t + 1) = Df(x(t), p) \cdot v(t),
\]

where \( w(0) \in \mathbb{R}^N \) is given as a perturbation at \( x(0) \). This normalization is due to relieve the difficulty in the numerical computation of Eq. (12) for chaotic trajectories.
3. Problem statement and proposed controlling method

Let us rewrite the discrete-time dynamical system \( f \) in Eq. (1) as

\[
x(t+1) = f(x(t), q(t), r(t)),
\]

where \( q \in \mathbb{R}^K \) and \( r \in \mathbb{R}^L \) with \( K + L = M \) are time-variant system parameters. We here assume that the values of \( q \) can be forcibly changed for any reason and are out of control, but \( r \) are able to be handled. To simplify the problem, we also assume that these parameter values can be changed only at \( t = mT \) \( (m = 0, 1, 2, \ldots) \), and besides, for \( x^* \) and \( r(mT) \) such that \( \lambda^* < \lambda(x^*, q(mT), r(mT), T) < 0 \), \( x^* \) and \( r^* \) such that \( \lambda(x^*, q((m+1)T), r^*, T) \approx \lambda^* \) exist and, in the parameter space \( r \in \mathbb{R}^L \) containing the point \( q((m+1)T) \), there exists at least one path from \( r(mT) \) to \( r^* \) without a bifurcation point of \( x^* \). Here, \( x^* \) represents a stable fixed or periodic point corresponding to desired behavior and \( \lambda^* \) is a user-defined parameter that is set to a negative value to predict occurrence of local bifurcations and to avoid it.

We now consider the situation in which bifurcations of desired behavior (\( x^* \)) may occur owing to forcible variation of \( q \). This means that the desired behavior vanishes by its bifurcation, and consequently, the dynamical system does not work as expected. To construct robust dynamical systems that withstand parameter variation, we design a parametric controller that predicts the occurrence of local bifurcations for variation of \( q \) and can avoid it by adjusting the values of \( r \).

The considered problem resembles optimization of the stability index (the maximum characteristic multiplier) that Kitajima et al. [18] treated. By using the parameter updating that the authors have proposed, we can construct robust dynamical systems that are controlled so as not to make bifurcations. However, since the stability index is not differentiable with respect to system parameters in general, the stability index cannot be directly optimized by using simple gradient methods. Moreover, their updating requires calculation with a high computational cost to find inverse matrices and the exact position of a stable fixed or periodic point.

By using the MLLE instead, we formulate the problem of bifurcation avoidance. This is because the MLLE is closely related to the maximum characteristic multiplier as seen in Eqs. (10) and (11), the MLLE is differentiable with respect to system parameters, simple gradient methods can be applied to its optimization, and calculation to find inverse matrices and the exact position of a stable fixed or periodic point in optimization of the MLLE is not needed as described below.

We here simplify the notation of \( \lambda(x(mT), q, r, T) \) as \( \lambda \) and describe a design of a parametric controller that can be used to avoid bifurcation. Consider the minimization problem of an objective function defined by

\[
J(\lambda) = \frac{1}{2} (\lambda - P(\lambda))^2,
\]

where the map \( P: \mathbb{R} \rightarrow \mathbb{R} \) is defined as

\[
P(\lambda) = \left\{ \begin{array}{ll}
\lambda & \text{if } \lambda \leq \lambda^*, \\
\lambda^* & \text{otherwise}.
\end{array} \right.
\]

As described above, \( \lambda^* \) is used to detect the approach of parameter values to a bifurcation point and becomes a set point to control \( \lambda \) when \( \lambda > \lambda^* \).

Since \( \lambda \) is differentiable with respect to \( r \), by using the method of steepest descent, we can derive a gradient system of Eq. (15), i.e. the parameter regulation of \( r \), as

\[
r((m+1)T) - r(mT) = -\gamma \frac{\partial J(\lambda)}{\partial r} = -\gamma(\lambda - \lambda^*) \frac{\partial \lambda}{\partial r},
\]

where \( \gamma \) is a user-defined positive parameter. Let \( r_{\ell} \) be the \( \ell \)th entry of \( r \) \( (\ell = 1, 2, \ldots, L) \). The \( \ell \)th entry of \( \partial \lambda / \partial r \), i.e. \( \partial \lambda / \partial r_{\ell} \), is derived as follows. From Eqs. (12) and (13), \( \lambda \) on \( x(mT) \) can be calculated as

\[
\lambda = \frac{1}{T} \sum_{t=mT}^{(m+1)T-1} \ln \|w(t+1)\|.
\]
By calculating the partial differentiation of Eq. (18) by $r_\ell$, we obtain

$$\frac{\partial \lambda}{\partial r_\ell} = \frac{1}{T} \sum_{t=mT}^{(m+1)T-1} \frac{w(t+1)^T}{\|w(t+1)\|^2} \frac{\partial w(t+1)}{\partial r_\ell},$$

(19)

$$\frac{\partial \|w(t+1)\|}{\partial r_\ell} = \frac{w(t+1)^T}{\|w(t+1)\|} \cdot \frac{\partial w(t+1)}{\partial r_\ell},$$

(20)

where $\top$ denotes the transpose of a vector.

Since we assumed that the values of $q(t)$ and $r(t)$ are changed only at $t = mT$, these values are constant for the duration of $t \in [mT, (m+1)T - 1]$. Let $q_0$ and $r_0$ be the values of $q(t)$ and $r(t)$ at $t = mT$. The values of $\partial w(t+1)/\partial r_\ell$ in Eqs. (19) and (20) are calculated as

$$\frac{\partial w(t+1)}{\partial r_\ell} = \frac{\partial Df(x(t), q_0, r_0)}{\partial r_\ell} \cdot w(t) + Df(x(t), q_0, r_0) \cdot \frac{\partial w(t)}{\partial r_\ell}.$$  

(21)

Here, the $(i,j)$th entry of $\partial Df(x(t), q_0, r_0)/\partial r_\ell$ $(i = 1, 2, \ldots, N, j = 1, 2, \ldots, N)$ in Eq. (21) is expanded as

$$\frac{\partial Df(x(t), q_0, r_0)}{\partial r_\ell}_{ij} = \sum_{k=1}^{N} \left( \frac{\partial^2 f_i(x(t), q_0, r_0)}{\partial x_k(t) \partial r_\ell} \cdot \frac{\partial x_k(t)}{\partial r_\ell} \right) + \frac{\partial f_i(x(t), q_0, r_0)}{\partial x_j}.$$  

(22)

where we can obtain $\partial x_k(t+1)/\partial r_\ell$ $(k = 1, 2, \ldots, N)$ from the first variational equation of solution $(x)$ with respect to $r_\ell$ as

$$\frac{\partial x(t+1)}{\partial r_\ell} = Df(x(t), q_0, r_0) \cdot \frac{\partial x(t)}{\partial r_\ell} + \frac{\partial f(x(t), q_0, r_0)}{\partial r_\ell}.$$  

(23)

Moreover, $\partial w(t+1)/\partial r_\ell$ in Eq. (21) becomes

$$\frac{\partial w(t+1)}{\partial r_\ell} = \frac{1}{\|w(t+1)\|} \cdot \frac{\partial w(t+1)}{\partial r_\ell} - \frac{1}{\|w(t+1)\|^2} \cdot \frac{\partial \|w(t+1)\|}{\partial r_\ell} \cdot w(t+1),$$  

(24)

where $\partial w(t+1)/\partial r_\ell$ and $\partial \|w(t+1)\|/\partial r_\ell$ can be assigned with the values calculated from Eqs. (21) and (20), respectively. In Eqs. (19)–(24), the initial conditions are set to $\partial x(mT)/\partial r_\ell = 0$, $\partial w(mT)/\partial r_\ell = 0$, and $\partial w(mT)/\partial r_\ell = 0$ because we arbitrarily give the values of $x(mT)$ and $w(mT)$ to Eqs. (14) and (18) regardless of the values of $r$. The procedure to find $\partial \lambda/\partial r_\ell$ for $t = (m+1)T$ is summarized in Algorithm 1. We note that while Honma [34] derived the differential of the MLLE for a specific nonlinear discrete-time dynamical system, we here derived its general form that can be used in a variety of nonlinear discrete-time dynamical systems.

4. Experimental results

To clarify whether the proposed method is effective, we carried out numerical experiments on avoiding local bifurcations of fixed and periodic points. We here set the parameters of the proposed controller to $\lambda^* = -0.1$ and $T = 500$ in Eqs. (17) and (18). The value of $\gamma$ was individually given in each experiment. The upper limit of control input corresponding to $\|r((m+1)T) - r(mT)\|$ in Eq. (17) was also set in all our experiments so as not to make bifurcations owing to the control input itself. Note that we need to set these parameters to appropriate values in accordance with experimental situations.

4.1 Avoiding bifurcation of fixed point

We considered a fixed point observed in the Kawakami map [31] described by

$$x_1(t+1) = q(t) \cdot x_1(t) + x_2(t),$$  

(25a)

$$x_2(t+1) = x_2(t) + r(t),$$  

(25b)

Although $x(mT)$ depends on $x(mT-1)$ except when $m = 0$, we may suppose that $x(mT)$, $(m = 1, 2, \ldots)$ is newly set regardless of $x(mT-1)$ because the values of system parameters are changed at $t = mT$. 

7
Algorithm 1 Computation $\partial \lambda / \partial r_\ell$ at $t = (m+1)T$

$t \leftarrow mT$
Set $w(t)$ around $x(t)$
Compute $v(t)$ according to Eq. (13a)
\[
\frac{\partial x(t)}{\partial r_\ell} \leftarrow 0
\]
\[
\frac{\partial w(t)}{\partial r_\ell} \leftarrow 0
\]
\[
\frac{\partial v(t)}{\partial r_\ell} \leftarrow 0
\]
$q_0 \leftarrow q(t)$
$r_0 \leftarrow r(t)$
$s \leftarrow 0$

for $t \leftarrow mT$ to $(m+1)T - 1$ do

Compute $\frac{\partial w(t+1)}{\partial r_\ell}$ according to Eqs. (21) and (22)

Compute $\|w(t+1)\|$ according to Eq. (13b)

Compute $\frac{\partial \|w(t+1)\|}{\partial r_\ell}$ according to Eq. (20)

\[
s \leftarrow s + \left( \frac{w(t+1)^T}{\|w(t+1)\|^2} \cdot \frac{\partial w(t+1)}{\partial r_\ell} \right) \quad \text{(see Eq. (19))}
\]

Compute $\frac{\partial x(t+1)}{\partial r_\ell}$ according to Eq. (14)

Compute $v(t+1)$ according to Eq. (13a)

Compute $\frac{\partial v(t+1)}{\partial r_\ell}$ according to Eq. (24)

end for

$\frac{\partial \lambda}{\partial r_\ell} \leftarrow s/T$ \quad \text{(see Eq. (19))}

where $x_1$ and $x_2$ are state variables, $t$ denotes the discrete time, and $q$ and $r$ represent the parameter not to be handled and that to be handled, respectively. Namely, we assumed the situation that $q$ was forcibly changed for any reason.

When we set to $q = 0$ and $r = 0$, we found a stable fixed point $(x_1^*, x_2^*) = (0, 0)$ with $(\mu_1, \mu_2) = (0, 0)$. The bifurcation curves of the fixed point were drawn in Fig. 1 using the method of bifurcation analysis [3, 4]. The solid curve indicated by $G^1$, the dotted curve with $I^1$, and the broken curve with $NS^1$ correspond to the sets of tangent, period-doubling, and the Neimark–Sacker bifurcation points where the superscript number represents the period of periodic points. The stable fixed point exists in the colored parameter region that is surrounded by the three bifurcation curves. The colors indicate

Fig. 1. Bifurcation curves of a fixed point in the Kawakami map. MLLE in the region that a stable fixed point exists is represented by colors in the right bar graph. Blue and red curves correspond to trajectories of parameters without and with control.
the MLLE for the stable fixed point and the correspondence between colors and values of the MLLE are shown in the bar graph on the right.

Let us demonstrate that the proposed controller can be used to avoid these bifurcations. We set to $q(0) = -0.2$ and $r(0) = 0.2$ corresponding to the point “a” in Fig. 1 and assumed that $q(t)$ is forcibly increased along the blue line with increments of 0.005 every interval of $T$. From the fact that the blue line crossed the curve $G^1$, the stable fixed point bifurcated on this curve without the proposed controller. On the other hand, the red curve branching from the blue line represents the trajectory of the parameter values controlled by using the controller we designed. This red trajectory indicated that our controller with $\gamma = 0.1$ could be used to avoid the tangent bifurcation at which the stable

![Diagram](image)

(a) Tangent bifurcation  
(b) Period-doubling bifurcation

(c) Neimark-Sacker bifurcation

Fig. 2. Experimental results of bifurcation avoidance for a fixed point in the Kawakami map displayed as time series. Sequences of blue and red points correspond to trajectories without and with control.
fixed point vanishes.

The detailed time series are shown in Fig. 2(a). The abscissa axes in the four graphs are the discrete time. The four graphs from the top to the bottom show the time evolution of \( x_1, \lambda, q, \) and \( r \) where the blue and red sequences correspond to the trajectories both without and with control. As shown on the graph of \( x_1, \) we plotted 100 transient points at \( t = mT \) \((m = 0, 1, 2, \ldots, 100)\). The blue points with \( 0 \leq t \leq 61T \) show the stable fixed point. However, the value of \( x_1 \) in the blue points suddenly diverged at \( t = 62T \) without control because the parameter values passed through the tangent bifurcation point. This can be confirmed that the value of \( \lambda \) also diverged at the time. In contrast, the red locus of \( x_1 \) indicates that the stable fixed point remained for the duration of \( 62T \leq t \leq 100T \) because of the designed controller with adjusting the value of \( r(t) \) after \( t = 59T \) so that \( \lambda = \lambda^* \).

We then considered the parameter variation passing through the curve \( I^1 \) from the point "b" in Fig. 1. The parameter values were \( q(0) = -1.0 \) and \( r(0) = -0.5 \) at the point "b". When the parameter \( q(t) \) was forcibly changed to \( q((m+1)T) = q(mT) - 0.005 \), the stable fixed point bifurcated on the curve \( I^1 \) and then a two-periodic point appeared at \( t = 78T \) as shown in Fig. 2(b). On the other hand, as illustrated by the red curve branching from the blue line in Fig. 1, the designed controller with \( \gamma = 0.5 \) avoided the period-doubling bifurcation, and the stable fixed point remained after \( t = 78T \).

The designed controller with \( \gamma = 0.5 \) could also avoid the Neimark–Sacker bifurcation. We set \( q(0) = -1.2 \) and \( r(0) = -1.0 \) at the point “c” in Fig. 1. When we changed parameter \( q(t) \) such that \( q((m+1)T) = q(mT) + 0.01 \), as shown in Fig. 2(c), the Neimark–Sacker bifurcation occurred without our controller at \( t \approx 70T \), and consequently, a quasi-periodic solution was observed after the bifurcation. However, the Neimark–Sacker bifurcation could be avoided by using our controller as shown by the red curve in Figs. 1 and 2(c).

4.2 Avoiding bifurcation of periodic point
To show that our controller is also effective for avoiding local bifurcations of periodic points, we consider the Hénon map [32] defined by

\[
x_1(t+1) = 1 + x_2(t) - q(t) \cdot x_1(t)^2, \quad (26a)
\]

\[
x_2(t+1) = r(t) \cdot x_1(t), \quad (26b)
\]

where \( x_1 \) and \( x_2 \) are state variables, and \( t \) is the discrete time. As is the case in the Kawakami map, we here assumed that \( r \) is able to be handled, and \( q \) is uncontrollable and is forcibly changed for any reason.

As shown in Fig. 3, we found several period-doubling bifurcation curves with regard to fixed and periodic points. A stable fixed point exists in the left-hand-side parameter region of the curve \( I^1 \) and a stable \( n \)-periodic one \((n = 2, 4, 8)\) exists in the parameter region surrounded by the curves of \( I^2 \) and \( I^n \). We also computed the MLLE on the grid points of \( 301 \times 301 \) in the parameter plane of Fig. 3 and showed them in color. For example, the color in the parameter region surrounded by the curves of \( I^1 \) and \( I^2 \) shows the MLLE on the stable two-periodic point. The relationship between colors and values of the MLLE is illustrated in the right-hand-side bar graph in Fig. 3.

Let us demonstrate avoiding control for the period-doubling bifurcations of the stable two- and four-periodic points. We now set initial values to \( q(0) = 0.6, r(0) = 0.2, \) and \((x_1(0), x_2(0)) = (1.244, 0.017)\) so that the two-periodic point is present in a steady state. These initial parameter values correspond to the point “a” in Fig. 3. We also set the variation of \( q(t) \) every \( T \) to be \(-0.0015\) so that it could cross the curve \( I^1 \) as the blue locus starting from the point “a”. As the red points of \( x_1, \lambda, \) and \( r \) in Fig. 4(a) show that the designed controller with \( \gamma = 0.1 \) kept the value of \( \lambda \) so that \( \lambda \leq \lambda^* \), and the adjustment of \( r(t) \) began at \( t = 53T \) when \( \lambda > \lambda^* \) without control. The locus of the adjusted \( r(t) \) is also illustrated as the red curve running parallel to the curve \( I^1 \) in Fig. 3. This resulted in the persistence of the two-periodic point for the duration of \( 0 \leq t \leq 100T \) without bifurcating into the fixed point at \( t = 86T \) as shown in the blue points of Fig. 4(a).

We also experimented to find a way to avoid the period-doubling bifurcation of the stable four-periodic point. The initial values were set to \( q(0) = 0.9, r(0) = 0.35, \) and \((x_1(0), x_2(0)) = (1.01, -0.22)\)
so that the four-periodic point is observed in a steady state. This parameter setting corresponds to the point “b” in Fig. 3. The increment of $q(t)$ every $T$ was set to be 0.0015. Since we assumed the parameter variation that $q(t)$ crosses the bifurcation curves of $I^4$ and $I^8$ as the blue line starting the point “b”, in the case of without control, the chaotic state was observed after $t = 82T$ as shown in Fig. 4(b). In contrast, using the designed controller with $\gamma = 0.1$ enabled us to avoid the bifurcation curve of $I^4$ as seen in Figs. 3 and 4(b).

5. Discussion
Since the MLE is closely related to the maximum characteristic multiplier (the stability index) of stable fixed and periodic points, the MLLE that gives an approximate value of the MLE tells us information
on their local bifurcations. From this, we formulated the problem of bifurcation avoidance on stable fixed and periodic points as an optimization problem of an objective function with regard to the MLLE. In this formulation, using the MLLE has an advantage that simple gradient methods can be used to optimize the objective function because the MLLE is differentiable with respect to system parameters. In practice, as shown in Eq. (17), the parameter regulation we derived is based on the method of steepest descent, and moreover, calculation to find inverse matrices and the exact position of stable fixed and periodic points is not needed in the optimization. In addition, from generality of the derivation, it is expected that this can be used for any nonlinear discrete-time dynamical systems described by differentiable difference equations.

Our experiments with the Kawakami and Hénon maps showed that we could know information whether stable fixed and periodic points bifurcate by monitoring the MLLE along the passage of time. This is because the MLLE, which can be computed from finite-time trajectories of state variables and a perturbation, can be given without finding the exact positions of stable fixed and periodic points. Moreover, by superimposing the trajectories of controlled parameters on bifurcation diagrams we drew, we confirmed that the designed controllers could be used to avoid local bifurcations of stable fixed and periodic points. That is, using the proposed method of parametric control means that bifurcations can be avoided without using the results of bifurcation analysis.

We here note that our method can be used only if dynamical systems are known and all states are always observable. For example, in the case that only some of all states in a system (14) are always observable, according to the embedding theorem [35, 36], we can reconstruct trajectories of stable fixed and periodic points to be controlled objects in certain time-delay coordinates. Further, for a class of dynamical systems such as the Kawakami and Hénon maps, we can explicitly derive the Jacobian matrix of dynamical systems reconstructed in the time-delay coordinates, and besides, other formulas can be derived according to the procedure described in Secs. 2 and 3. However, in Eqs. (20) and (23) derived for the reconstructed dynamical systems, a problem on undetermined initial values remains. This is because initial states in the time-delay coordinates and a small perturbation around the initial states cannot be set arbitrarily, i.e., these should be determined by a time series of observable state values in the original system (14) and depend on the parameter values of the system (14).

6. Conclusion
We have considered the problem of bifurcation avoidance on stable, hyperbolic fixed and periodic points in nonlinear discrete-time dynamical systems with parameter variation. To solve the problem under assumptions that the dynamics of system is known and all state variables are always observable, we have proposed a novel methodology of parametric control derived theoretically from the optimization of an objective function with respect to the MLLE. This is because the MLLE not only tells us information on local bifurcations of stable, hyperbolic fixed and periodic points but also it is differentiable with respect to system parameters unlike the stability index [18]. In addition, it is also remarkable to calculate the MLLE along the passage of time without finding the exact positions of fixed and periodic points. Compared with a method to optimize the stability index, these lead to the advantages that the proposed method can predict occurrence of local bifurcations by monitoring the MLLE along the passage of time and adjust parameter values on line to avoid local bifurcations. Our experiments for stable fixed and periodic points in the Kawakami and Hénon maps demonstrated that the proposed controller effectively worked to avoid local bifurcations without directly analyzing the bifurcations in advance.

We note that developing practical methods to avoid local bifurcations in the cases that only some of all states are always observable and dynamics of systems are known or unknown should be considered from the viewpoints of dynamical systems theory and time series analysis.

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