GROUPOID EQUIVALENCE AND THE ASSOCIATED ITERATED CROSSED PRODUCT

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Abstract. Given groupoids $G$ and $H$ and a $(G, H)$-equivalence $X$ we may form the transformation groupoid $G \rtimes X \rtimes H$. Given a separable groupoid dynamical system $(A, G \rtimes X \rtimes H, \omega)$ we may restrict $\omega$ to an action of $G \rtimes X$ on $A$ and form the crossed product $A \rtimes (G \rtimes X \rtimes H)$. We show that there is an action of $H$ on $A \rtimes (G \rtimes X \rtimes H)$ and that the iterated crossed product $(A \rtimes (G \rtimes X \rtimes H)) \rtimes H$ is naturally isomorphic to the crossed product $A \rtimes (G \rtimes X \rtimes H)$.

1. Introduction

If $\alpha : G \to \text{Aut} A$ and $\beta : H \to \text{Aut} A$ are commuting actions of locally compact groups $G$ and $H$ on a $C^*$-algebra $A$, then we trivially obtain an action $\alpha \times \beta : G \times H \to \text{Aut} A$. Furthermore, every action of $G \times H$ arises in this way. It is straightforward to check that the crossed product $A \rtimes (G \times H)$ decomposes (up to isomorphism) as $(A \rtimes_G G) \rtimes_H H$, where $\beta := \beta \times 1$ is the associated action of $H$ on $A \rtimes_G G$.

Recently, we discovered we needed a version of this iterated crossed product result for groupoid dynamical systems. However, it is far from clear just what form such a general result would take. For example, groupoids act on fibred objects, so $A$ would need to be fibred over the unit spaces of both groupoids. Even then, it is not so obvious what it should mean for the actions to commute.

Rather than sort out the most general possible theorem, we opted to let our applications dictate the form of our result. In particular, we want to consider locally compact Hausdorff groupoids $G$ and $H$ that are equivalent via a $(G, H)$-equivalence $X$. Then we can form the (bi)transformation groupoid $G \rtimes X \rtimes H$ which naturally contains the transformation groupoids $G \rtimes X$ and $X \rtimes H$ as subgroupoids. (To see how this set-up relates to commuting actions of groups, see Remark 3.2.) Given a groupoid dynamical system $(A, G \rtimes X \rtimes H, \omega)$, we can get actions $\alpha$ of $G \rtimes X$ and $\beta$ of $X \rtimes H$ via restriction. It turns out that the crossed product $A \rtimes (G \rtimes X)$ naturally fibres over the orbit space $(G \rtimes X) \setminus X$. Since the latter equals $G \setminus X$ and since $G \setminus X$ is homeomorphic to $H(0)$ because $X$ is a $(G, H)$-equivalence, it is not surprising that $\beta$ induces an action $\tilde{\beta}$ of $H$ on $A \rtimes (G \rtimes X)$. Then our main result (Theorem 4.1) states that the iterated crossed product

$$(A \rtimes (G \rtimes X)) \rtimes \tilde{\beta} H$$

is naturally isomorphic to the crossed product $A \rtimes (G \rtimes X \rtimes H)$.

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Although such a result might be expected, especially in analogy with the group case, it is nontrivial to prove. Some of the subtleties are foreshadowed by the proof of the “standard result” in the group case. Even in that setting, one needs to work with covariant representations. Of course covariant representations of groupoid crossed products are considerably more subtle than their classical counterparts, and come with an (un-)healthy dose of measure theory.

Before we plunge into the details, we feel obligated to say a bit about our main application. In [2], the first two authors introduce the Brauer semigroup $S(G)$ of a locally compact groupoid $G$. The Brauer semigroup is a natural outgrowth of the Brauer group $\text{Br}(G)$ introduced in [9] and consists of Morita equivalence classes $[A, \alpha]$ of groupoid dynamical $G$-systems with a suitable multiplication. For example, if $G = G \times X$ is the transformation groupoid associated to the transformation group $(G, X)$, then $S(G)$ is (isomorphic to) the equivariant Brauer semigroup $S_G(X)$ of $G$. The main theorem in [2] is the full groupoid dynamical system analogue of [8, Theorem 5.2] and asserts that if $G$ and $H$ are equivalent groupoids, then there is a semigroup isomorphism $\Theta$ from $S(H)$ onto $S(G)$. A critical additional ingredient from [8] is that $\Theta$ is constructed such that if $\Theta([B, \beta]) = [A, \alpha]$, then the crossed products $B \rtimes_{\beta} H$ and $A \rtimes_{\alpha} G$ are Morita equivalent.

The proof in [2] is modeled on the proof in [8] and it goes as follows. Suppose that $\alpha$ is a class in $S(H)$. Then there is an associated dynamical system $(A, G \times X \rtimes H, \omega)$ such that the following statements can be verified. The action $\alpha = \omega|_{G \times X}$ is proper and saturated as defined in [9]. Consequently, we can form the generalized fixed point algebra $A^\alpha$ which is Morita equivalent to $A \rtimes_{\alpha} G$. Furthermore, $A^\alpha$ is fibred over $G \times X \cong H^{(0)}$, and $\beta$ induces an action $\hat{\beta}$ of $H$ on $A^\alpha$. The point is that the construction of $(A, G \times X \rtimes H, \omega)$ is such that our class $\alpha$ must be of the form $[A^\alpha, \hat{\beta}]$, and such that there is a well-defined map $\Theta$ sending $\alpha$ to the class $[A^\beta, \hat{\alpha}]$ (with $A^\beta$ and $\hat{\alpha}$ defined analogously to $A^\alpha$ and $\hat{\beta}$). An application of the Equivalence Theorem [13, Theorem 5.5] (as in [13, §9.1]) shows that $A^\alpha \rtimes_{\beta} H$ is Morita equivalent to $(A \rtimes_{\alpha} (G \times X)) \rtimes_{\beta} H$. Employing symmetric arguments, we also have $A^\beta \rtimes_{\hat{\alpha}} G$ Morita equivalent to $(A \rtimes_{\beta} (X \rtimes H)) \rtimes_{\hat{\alpha}} G$. The main result from this paper (and a little symmetry) implies that both $(A \rtimes_{\alpha} (G \times X)) \rtimes_{\beta} H$ and $(A \rtimes_{\beta} (X \rtimes H)) \rtimes_{\hat{\alpha}} G$ are isomorphic to $A \rtimes_{\omega} (G \times X \rtimes H)$. This implies that $A^\alpha \rtimes_{\beta} H$ and $A^\beta \rtimes_{\hat{\alpha}} G$ are Morita equivalent, and provides the last bit of the main result in [2].

The structure of this paper is roughly as follows. We start by briefly reviewing the groupoid crossed product construction in Section 2. In Section 5 we introduce the iterated product and build the outer action. Section 4 contains the main result, as described above, although significant portions of the proof are postponed until Section 5. As is usual, we assume that homomorphisms between $C^*$-algebras are $*$-preserving. Furthermore, representations of $C^*$-algebras are assumed to be non-degenerate. Since we use the Equivalence and Disintegration Theorems, we require separability in a nontrivial way (see Remark 2.3). Hence virtually all the topological spaces that appear here are second countable, and with the exception of $B(H)$ and other multiplier algebras, our $C^*$-algebras and Banach spaces are assumed to be separable. In particular, our groupoids are all second countable and Hausdorff, and will be assumed to have a Haar system.
2. Groupoid Crossed Products

For background on groupoid crossed products, we refer to the exposition in [13]. For the correspondence between $C_0(X)$-algebras and upper semicontinuous $C^*$-bundles we refer to [19] Appendix C. For convenience, we review some of the basics here.

An upper semicontinuous $C^*$-bundle $\mathcal{A}$ over $X$ is a continuous open surjection $p : \mathcal{A} \to X$ such that $A(x) := p^{-1}(x)$ is a $C^*$-algebra for each $x \in X$ and which satisfies some additional continuity conditions [13] Definition C.16. In particular, the algebra $A := \Gamma_0(X; \mathcal{A})$ of continuous sections vanishing at infinity is a $C^*$-algebra with respect to the supremum norm. On the other hand, we say that a $C^*$-algebra $A$ is a $C_0(X)$-algebra if it comes equipped with a nondegenerate homomorphism of $C_0(X)$ into the center of the multiplier algebra $M(A)$. There is a correspondence between $C_0(X)$-algebras and upper semicontinuous $C^*$-bundles over $X$: given a $C_0(X)$-algebra $A$, there is a bundle $\mathcal{A}$ such that $A$ is $C_0(X)$-isomorphic to $\Gamma_0(X; \mathcal{A})$ (with its natural $C_0(X)$-action) [19] Theorem C.26.

Let $G$ be a locally compact Hausdorff groupoid, with unit space $G^{(0)}$, Haar system $\{\lambda^\gamma\}_{\gamma \in G^{(0)}}$, and range and source maps $r_G$ and $s_G$ respectively [17]. We omit the subscripts on $r_G$ and $s_G$ when the domain is clear from context. Let $A$ be a $C_0(G^{(0)})$-algebra and $p : \mathcal{A} \to G^{(0)}$ be the associated upper semicontinuous $C^*$-bundle. An action $\alpha$ of $G$ on $A$ is a set of isomorphisms $\{\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))\}_{\gamma \in G}$ such that $\alpha_\gamma \circ \alpha_\eta = \alpha_{\gamma \eta}$ and the map $(\gamma, a) \mapsto \alpha_\gamma(a)$ is jointly continuous [13] Definition 4.1. We refer to the triple $(\mathcal{A}, G, \alpha)$ as a groupoid dynamical system. Let $r^* \mathcal{A} = \{(\gamma, a) \in G \times \mathcal{A} : r(\gamma) = p(a)\}$ be the pull back bundle of $\mathcal{A}$. Let $\Gamma_c(G; r^* \mathcal{A})$ be the continuous compactly supported sections of $r^* \mathcal{A}$.

Proposition 2.1 ([13] Proposition 4.4]). The space of compactly supported sections, $\Gamma_c(G; r^* \mathcal{A})$, is a $\ast$-algebra with respect to the operations

$$f \ast g(\gamma) := \int f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^\gamma(\eta) \quad \text{and} \quad f^*(\gamma) = \alpha_\gamma(f(\gamma^{-1}))^*.$$ 

If $f \in \Gamma_c(G; r^* \mathcal{A})$, we define

$$\|f\|_I = \max\left\{\sup_{\gamma \in G^{(0)}} \int \|f(\gamma)\| \, d\lambda^\gamma(\gamma), \sup_{\gamma \in G^{(0)}} \int \|f^*(\gamma)\| \, d\lambda^\gamma(\gamma)\},$$

and say that $\pi : \Gamma_c(G; r^* \mathcal{A}) \to B(H)$ is in $\text{Rep}(G, A)$ if $\|\pi(f)\| \leq \|f\|_I$ for all $f \in \Gamma_c(G; r^* \mathcal{A})$. Then

$$\|f\| := \sup\{\|\pi(f)\| : \pi \in \text{Rep}(G, A)\}$$

defines a norm on $\Gamma_c(G; r^* \mathcal{A})$. The crossed product, $\mathcal{A} \rtimes_\alpha G$, is the completion of $\Gamma_c(G; r^* \mathcal{A})$ with respect to $\|\cdot\|$.

Remark 2.2 (Notation). When working with groupoid dynamical systems $(\mathcal{A}, G, \alpha)$ it is a matter of taste whether to emphasize the bundle $\mathcal{A}$ or the $C^*$-algebra $A = \Gamma_0(G^{(0)}; \mathcal{A})$. Hence many authors write $A \rtimes_\alpha G$ in place of $\mathcal{A} \rtimes_\alpha G$. We did this in the introduction so that our main theorem looks like an associative law: $A \rtimes (G \times X \rtimes G) \cong (A \rtimes (G \times X)) \rtimes H$. However, in the remainder of this paper we will stick with the bundle notation. This has a number of advantages — for example, see Remark 3.3 where the same $C^*$-algebra is the section algebra of different bundles.
We say that a uniformly convergent net in \( \Gamma_c(G; r^* \mathcal{A}) \) is eventually compactly supported if there is an \( i_0 \) and a compact set \( K \) such that \( \supp f_i \subset K \) for all \( i \geq i_0 \). There exists a topology on \( \Gamma_c(G; r^* \mathcal{A}) \) such that \( F: \Gamma_c(G; r^* \mathcal{A}) \to V \) is continuous into a locally convex linear space \( V \) if and only if \( F \) maps eventually uniformly convergent nets to convergent nets \[15\] Lemma D.10. We call this the \textit{inductive limit topology} on \( \Gamma_c(G; r^* \mathcal{A}) \). Note that convergence in the inductive limit topology implies convergence in the \( \| \cdot \|_I \)-norm, and hence in the \( C^* \)-norm \( \| \cdot \| \).

\textbf{Remark 2.3 (Separability Assumptions).} In the sequel, we will be exclusively interested in \textit{separable} dynamical systems \((\mathcal{A}, G, \alpha)\). By this we mean that \( G \) is second countable and that \( A := \Gamma_0(G^{(0)}; A) \) is separable. This not only allows us to use the Disintegration Theorem and other results from \[13\], but has a number of other important consequences:

(a) The total space, \( \mathcal{A} \), is a second countable topological space.

(b) The \(*\)-algebra \( \Gamma_c(G; r^* \mathcal{A}) \) is separable in the inductive limit topology. (That is, there is a countable dense set \( D \) which is dense in the inductive limit topology.)

(c) If \((\mathcal{A}, G, \alpha)\) is separable, then the crossed product, \( \mathcal{A} \rtimes_\alpha G \) is a separable \( C^* \)-algebra.

(d) Any nondegenerate \(*\)-homomorphism \( \pi: \Gamma_c(G; r^* \mathcal{A}) \to B(\mathcal{H}) \) which is continuous with respect to the inductive limit topology on \( \Gamma_c(G; r^* \mathcal{A}) \) and the weak-\(*\) topology on \( B(\mathcal{H}) \) is in \( \text{Rep}(G, A) \). (Since the converse is automatic, we can view \( \text{Rep}(G, A) \) as the set of inductive limit continuous representations in the separable case.)

In the case that \( \mathcal{A} \) is a (continuous) \( C^* \)-bundle, assertion (a) follows from \[4\] Proposition II.13.21. The proof in general carries over easily using \[19\] Theorem C.25 to describe a basis for the topology on \( \mathcal{A} \). Assertion (b) is just \[4\] Proposition II.14.10 in the Banach bundle case. The proof in the general case follows \textit{mutatis mutandis}. Assertion (c) follows from assertion (b). Assertion (d) is a consequence of the Disintegration Theorem \[13\] Theorem 7.12.

3. The Iterated Crossed Product

Throughout, \( G \) and \( H \) will be second countable, locally compact Hausdorff groupoids with Haar systems \( \{\lambda^u\}_{u \in G^{(0)}} \) and \( \{\tau^v\}_{v \in H^{(0)}} \), respectively. We also fix a \((G, H)\)-equivalence \( X \) as in \[10\] Definition 2.1. Thus there are maps \( r_X: X \to G^{(0)} \) and \( s_X: X \to H^{(0)} \) and commuting free and proper actions of \( G \) and \( H \), respectively. (We will quickly drop the subscript ‘\( X \)’ from \( r_X \) and \( s_X \) since the domain should be clear from context.) Then we can define the following groupoid.

\textbf{Definition 3.1.} We set \( E = G \times X \rtimes H := \{ (\gamma, x, \eta) \in G \times X \times H : r(\gamma) = r(x) \text{ and } s(x) = r(\eta) \} \), and give \( E \) the subspace topology inherited from the product topology on \( G \times X \times H \). The groupoid operations are given by

\[ (\gamma, x, \eta)(\zeta, x^{-1} \cdot x \cdot \eta, \zeta) = (\gamma \zeta, x, \eta \zeta) \quad \text{and} \quad (\gamma, x, \eta)^{-1} = (\gamma^{-1}, x^{-1} \cdot x \cdot \eta, \eta^{-1}). \]

Then we can identify \( E^{(0)} \) with \( X \) so that the range and source maps are given by

\[ s(\gamma, x, \eta) = \gamma^{-1} \cdot x \cdot \eta \quad \text{and} \quad r(\gamma, x, \eta) = x. \]
We define a Haar system on $E$ by $\lambda^E_x = \lambda^r(x) \times \delta_x \times \sigma^s(x)$ where $\delta_x$ is the Dirac $\delta$-measure at $x$.

It is routine to verify that with these operations, $E$ is a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^E_x\}_{x \in X}$.

**Remark 3.2** (The Group Case). To see that $E$ is a natural iterated construct in the groupoid realm, consider the situation where $G$ and $H$ are groups acting freely and properly on the left and right, respectively, of $X$. Then we can form the transformation groupoids $G := G \times X/H$ and $H := G\backslash X \rtimes H$, and let them act on $X$ in the natural way so that $X$ becomes a $(G,H)$-equivalence. Suppose that $A = \Gamma_0(X$; $\mathcal{A}$) is a $C_0(X)$-algebra with commuting $G$ and $H$ actions $\alpha$ and $\beta$, respectively, which induce the given actions on $X$ as in [13, Example 4.8]. Then, just as in [13, Example 4.8], the crossed product $A \rtimes_{\alpha \times \beta} (G \times H)$ is isomorphic to the groupoid crossed product $\mathcal{A} \rtimes_{\alpha \times \beta}$ $((G \times H) \rtimes X)$ for an appropriate action $(\alpha \times \beta)$. If we let $E = (G \times X \rtimes H)$, then the map $(s,x,H), x, (G \times x,t) \mapsto (s,t,x)$ is an groupoid isomorphism of $E = (G \rtimes X/H) \times X \rtimes (G\backslash X \times H)$ onto $(G \times H) \rtimes X$ which intertwines an action $\omega$ with $(\alpha \times \beta)$. Hence the groupoid crossed product $\mathcal{A} \rtimes_{\alpha \omega} E$ is (isomorphic to) the iterated crossed product $A \rtimes_{\alpha \times \beta} (G \times H)$.

We will identify the transformation groupoid $G := G \times X/H$ with the closed subgroupoid $\{(\gamma,x,s(x)) \in E : r(\gamma) = r(x)\}$ of $E$. Thus we will often write $(\gamma,x) \in G$ in place of $(\gamma,x,s(x))$. We equip $G$ with the Haar system $\lambda_G = \{\lambda^r(x) \times \delta_x\}_{x \in X}$. Similar statements hold for the transformation groupoid $H = X \rtimes H$. Notice that $E$, $G$, and $H$ all have unit spaces identified with $X$.

If $E$ acts continuously on an upper semicontinuous $C^*$-bundle $\mathcal{A}$ by isomorphisms, then the same is true of any closed subgroupoid. Hence we get the following proposition.

**Proposition 3.3.** Let $(\mathcal{A}, E, \omega)$ be a groupoid dynamical system. Then the restrictions

$$\alpha(\gamma,x) := \omega(\gamma,x,s(x)) \quad \text{and} \quad \alpha'(x,\eta) = \omega(r(x),x,\eta)$$

are continuous actions of $G$ and $H$, respectively, by isomorphisms on $\mathcal{A}$.

To obtain the inner portion of our iterated crossed product, we form the crossed product $B := \mathcal{A} \rtimes_{\alpha} G$.

**Remark 3.4.** We can also view $B$ as a crossed product by $G$. Since $r_X : X \to G^{(0)}$ is continuous, there is an upper semicontinuous $C^*$-bundle $\mathcal{A}'$ over $G^{(0)}$ such that $A := \Gamma_0(X; \mathcal{A}) \cong \Gamma_0(G^{(0)}; \mathcal{A}')$. Furthermore there is an induced action $\hat{\alpha}$ of $G$ on $\mathcal{A}'$ such that $\mathcal{A}' \rtimes_{\hat{\alpha}} G$ is isomorphic to $\mathcal{A} \rtimes G$. (See [3, Theorem 2] for the details in the case where $A$ has continuous trace.)

Since $X$ is a $(G,H)$-equivalence, the source map $s_X$ factors through a homeomorphism of $G\backslash X$ with $H^{(0)}$. In particular, $G\backslash X$ is Hausdorff and we can identify $v \in H^{(0)}$ with the orbit $s_X^{-1}(v)$. Thus the following proposition follows immediately from [3, Proposition 4.2].

**Proposition 3.5.** Let $H$ and $B$ be as above. Then $B$ is a $C_0(H^{(0)})$-algebra with respect to the action

$$\phi \cdot f(\gamma,x) = \phi(s(x))f(\gamma,x)$$
for $\phi \in C_c(H(0))$ and $f \in \Gamma_c(G; r^*\mathcal{A})$. Furthermore, the restriction map
\[\Gamma_c(G; r^*\mathcal{A}) \to \Gamma_c(G|_{s^{-1}(v)}; r^*\mathcal{A})\] factors to an isomorphism of the fibre $B(v)$ with $\mathcal{A}^a|_{s^{-1}(v)} \times G|_{s^{-1}(v)}$.

**Remark 3.6.** Since $B$ is separable by Remark 2.3(c) and since $H(0)$ is second countable, $\mathcal{B}$ must be second countable as in Remark 2.3(a).

We will write $B_0(v)$ for the dense $*$-subalgebra $\Gamma_c(G|_{s^{-1}(v)}; r^*\mathcal{A})$ of $B(v)$. If $f \in \Gamma_c(G; r^*\mathcal{A})$ and $v \in H(0)$, then we’ll write $f_v$ for the element of $B_0(v)$ obtained by restriction. We use the set of such sections to define a topology on $\mathcal{B} := \bigsqcup_{v \in H(0)} B(v)$ as in [19] Theorem C.25 making $\mathcal{B}$ an upper semicontinuous $C^*$-bundle. Note that $B \cong \Gamma_0(H(0); \mathcal{B})$ and $v \mapsto f_v$ is a prototypical section in $\Gamma_c(H(0); \mathcal{B})$. We can now build the outer action of our iterated crossed product.

**Proposition 3.7.** Let $(\mathcal{A}, E, \omega)$ be a separable groupoid dynamical system with $H$ and $\mathcal{B}$ as above. Then there is a groupoid dynamical system $(\mathcal{B}, H, \beta)$ where, for $f \in B_0(s(\eta))$,

\[\beta_\eta(f)(\gamma, x) = \omega_{(r(\gamma), x, \eta)}(f(\gamma, x, \eta)).\]

**Proof.** Since multiplication by $\eta$ is a homeomorphism of $s^{-1}(s_H(\eta))$ onto $s^{-1}(r_H(\eta))$, it is not hard to check that $f$ defines a $*$-homomorphism of $B_0(s(\eta))$ into $B_0(r(\eta))$. Since $\beta_\eta$ is $\|\cdot\|_t$-isometric, it extends to all of $B(s(\eta))$. Elementary calculations show that $\beta_\eta = \id$ if $v \in H(0)$, and that for composable $\eta$ and $\zeta$ we have $\beta_\zeta = \beta_\eta \circ \beta_\zeta$. Thus $\eta \mapsto \beta_\eta$ is an action of $H$ on $\mathcal{B}$ by isomorphisms. It only remains to show that the action is continuous.

Since $\mathcal{B}$ is second countable by Remark 3.6 we can work with sequences. Thus we assume that $\eta_i \to \eta_0$ in $H$ and $b_i \to b_0$ in $\mathcal{B}$ with $b_i \in B(s(\eta_i))$ for all $i$. We need to verify that $\beta_{\eta_i}(b_i) \to \beta_{\eta_0}(b_0)$ in $\mathcal{B}$.

Fix $\epsilon > 0$. Let $u_i = s(\eta_i)$ and $v_i = r(\eta_i)$ for all $i \geq 0$. Choose $b \in B := \mathcal{A} \rtimes_{\omega} G$ such that $b(u_0) = b_0$, and let $F \in \Gamma_c(G; r^*\mathcal{A})$ be such that $\|F - b\| < \epsilon/2$. Observe that

\[\|F_u - b(u)\| < \epsilon/2 \quad \text{for all } u \in H(0)\]

where $F_u$ denotes the restriction of $F$ to $G|_{s^{-1}(u)}$. We first show that it suffices to prove the following claim:

**Claim.** If $F \in \Gamma_c(G; r^*\mathcal{A})$, $\eta_i \to \eta_0$ and $u_i, v_i$ are as above, then $\beta_{\eta_i}(F_{u_i}) \to \beta_{\eta_0}(F_{u_0})$ in $\mathcal{B}$.

Suppose that claim is valid. By (2) we have

\[\|\beta_{\eta_i}(F_{u_i}) - \beta_{\eta_0}(b_0)\| < \epsilon/2 < \epsilon.\]

Since both $b_i \to b_0$ and $b(u_i) \to b(u_0)$ we have $\|b(u_i) - b_i\| \to 0$. Consequently it follows that for large $i$

\[\|\beta_{\eta_i}(F_{u_i}) - \beta_{\eta_0}(b_i)\| \leq \|F_{u_i} - b(u_i)\| + \|b(u_i) - b_i\| \to \epsilon.\]

It follows from [19] Proposition C.20 that $\beta_{\eta_i}(b_i) \to \beta_{\eta_0}(b_0)$ in $\mathcal{B}$ as required.

Thus it will suffice to prove the claim. Since it will suffice to see that every subsequence of $\{\beta_{\eta_i}(F_{u_i})\}$ has a subsequence converging to $\beta_{\eta_0}(F_{u_0})$, we can replace $\{\beta_{\eta_i}(F_{u_i})\}$ by a subsequence, relabel, and find a convergent subsequence. If $v_i = v_0$ infinitely often, then we can pass to another subsequence and assume that $v_i = v_0$ for all $i \geq 0$. Then, since the relative topology of $B(v_0)$ in $\mathcal{B}$ is the norm
with respect to the topology on $B$. Let $eta$ be the restriction of $\omega$ to $G = G \ltimes X$ and \( B = \mathcal{A} \ltimes_{\alpha} G = \Gamma_0(H; \mathcal{B}) \). Let $\beta$ be the action of $H$ on $\mathcal{B}$ given by

$$\beta_0(f)(\gamma, x) = \omega(r(x), x, \eta)(f(\gamma, x\eta)).$$

Then there is an isomorphism $\Upsilon: \mathcal{A} \ltimes_{\omega} E \to \mathcal{B} \ltimes_{\beta} H$ characterized by

$$\Upsilon(f)(\eta)(\gamma, x) := f(\gamma, x, \eta)$$

for all $f \in \Gamma_c(E, r^*\mathcal{A})$.  

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1The proof is the same as that for (continuous) Banach bundles in [12] Proposition A.5.
The proof of Theorem 7.12 is involved, and we divide it up into a series of propositions. We begin by showing that \( \Upsilon \) is a \(*\)-homomorphism. Although this assertion is considerably easier than the assertion that the map is injective, even this part of the result is technical and far from immediate. Showing that \( \Upsilon \) is isometric is subtle and requires delicacies with unitary groupoid representations that we address in a separate section.

**Proposition 4.2.** The map \( \Upsilon \) defined in Theorem 4.1 extends to a \(*\)-homomorphism from \( \mathcal{A} \times_\omega \mathcal{E} \) onto \( \mathcal{B} \times_\beta H \).

*Proof.* If \( f \in \Gamma_c(\mathcal{G}; r^* \mathcal{A}) \), then \( \Upsilon(f)(\eta) \in B_0(r(\eta)) \). Furthermore \( \Upsilon(f) \) is a compactly supported section of \( r^* \mathcal{B} \). Thus to see that \( \Upsilon \) maps into \( \mathcal{B} \times_\beta H \) we need to show that \( \eta \mapsto \Upsilon(f)(\eta) \) is continuous. The topology on \( \mathcal{B} \) is determined by the sections coming from \( \Gamma_c(\mathcal{G}; r^* \mathcal{A}) \), and is second countable (Remark 3.6). Hence we can proceed as in Proposition 3.7. We just sketch the details.

Suppose \( \eta_i \to \eta_0 \) in \( H \). Replacing \( \{ \Upsilon(f)(\eta_i) \} \) by a subsequence, it suffices to see that it has a subsequence converging to \( \Upsilon(f)(\eta_0) \). If \( r(\eta_i) = r(\eta_0) \) infinitely often, then after passing to a subsequence, we may assume \( \Upsilon(f)(\eta_i) \in B(r(\eta_0)) \) for all \( i \). Since we are dealing with a fixed fibre it suffices to show that \( \| \Upsilon(f)(\eta_i) - \Upsilon(f)(\eta_0) \| \to 0 \) and this follows from a standard argument. Alternatively, if we eventually have \( r(\eta_i) \neq r(\eta_j) \) then we may pass to a subsequence and assume \( r(\eta_i) \neq r(\eta_j) \) for all \( i \neq j \). Next, as in the proof of Proposition 3.7, we build a continuous function \( F_0 \in \Gamma_c(\mathcal{G}; r^* \mathcal{A}) \) with the property that the restriction of \( F_0 \) to \( s^{-1}(r(\eta_i)) \) is equal to \( \Upsilon(f)(\eta_i) \) for all \( i \geq 0 \). Since \( F_0 \) defines a continuous section \( v \mapsto F_0|_{s^{-1}(x)} \), we have \( \Upsilon(f)(\eta_i) = F_0|_{s^{-1}(r(\eta_i))} \to F_0|_{s^{-1}(r(\eta_0))} = \Upsilon(f)(\eta_0) \). In either case \( \Upsilon(f) \) is a continuous, compactly supported section.

The next step is to see that \( \Upsilon \) is a \(*\)-homomorphism on \( \Gamma_c(\mathcal{G}; r^* \mathcal{A}) \). This is mostly routine, but some care is required to see that it is multiplicative: \( \Upsilon(f * g) = \Upsilon(f) \ast \Upsilon(g) \). The issue is that \textit{a priori} \( \Upsilon(f) \ast \Upsilon(g)(\eta) \) is the element of \( B(r(\eta)) = \mathcal{A}|_{s^{-1}(r(\eta))} \times_\alpha \mathcal{G}|_{s^{-1}(r(\eta))} \) given by the \( B(r(\eta)) \)-valued integral

\[
\int_H \Upsilon(f)(\zeta) * \beta_\zeta(\Upsilon(g)(\zeta^{-1}\eta)) \, d\sigma^{r(\eta)}(\zeta).
\]

We claim that first, \( \Upsilon(f) \ast \Upsilon(g)(\eta) \) belongs to \( B_0(r(\eta)) \), and second that

\[
\int_H \Upsilon(f)(\zeta) * \beta_\zeta(\Upsilon(g)(\zeta^{-1}\eta)) \, d\sigma^{r(\eta)}(\zeta) = \int_H \Upsilon(f)(\zeta) * \beta_\zeta(\Upsilon(g)(\zeta^{-1}\eta)) \, d\sigma^{r(\eta)}(\zeta).
\]

This will suffice since a routine computation shows that the \( A(x) \)-valued integral on the right-hand side of (4) simplifies to

\[
f * g(\gamma, x, \eta) = \Upsilon(f * g)(\eta)(\gamma, x).
\]

However both claims follow just as in [19, Lemma 1.108], and \( \Upsilon \) is a \(*\)-homomorphism as claimed.

To show that \( \Upsilon \) is bounded with respect to the \( C^* \)-norms, we note that it suffices to prove that \( \Upsilon \) is continuous with respect to the inductive limit topologies. Then if \( L \) is a faithful representation of \( \mathcal{B} \times_\beta H \), \( L \circ \Upsilon \) is a bounded representation of \( \mathcal{A} \times_\omega \mathcal{E} \) by [13, Theorem 7.12], and we have

\[
\|f\| \leq \|L \circ \Upsilon(f)\| = \|\Upsilon(f)\|.
\]
To see that $\Upsilon$ is continuous in the inductive limit topologies, suppose that $f_i \to f_0$ with respect to the inductive limit topology in $\Gamma_c(E, r^*\mathcal{A})$ and let $K$ be the compact set which eventually contains the supports of the $f_i$. Pick $\epsilon > 0$. Let $K_G$ be the restriction of $K$ to $G$ and observe that $K_G$ is compact so that the set 
\[
\{ \lambda_u(K_G) : u \in G(0) \},
\]
is bounded by some $M$. Pick $i_0$ so that
\[
\|f_i - f_0\|_\infty < \frac{\epsilon}{M}
\]
for all $i > i_0$. Then given any $\eta \in H$ and $x \in X$ such that $s(x) = r(\eta)$ we have
\[
\int \|f_i(\gamma, x, \eta) - f_0(\gamma, x, \eta)\| d\lambda_r(x) \leq \|f_i - f_0\|_\infty \lambda_r(x)(K_G) < \epsilon,
\]
and
\[
\int \|f_i(\gamma, x, \eta) - f_0(\gamma, x, \eta)\| d\lambda_r(x)(\gamma) \leq \|f_i - f_0\|_\infty \lambda_r(x)(K_G) < \epsilon.
\]
It follows that for all $i > i_0$ we have
\[
\|\Upsilon(f_i)(\eta) - \Upsilon(f_0)(\eta)\| \leq \|\Upsilon(f_i)(\eta) - \Upsilon(f_0)(\eta)\|_I < \epsilon.
\]
Consequently, $\Upsilon(f_i) \to \Upsilon(f_0)$ uniformly and, since the supports of the $\Upsilon(f_i)$ are eventually contained in the restriction of $K$ to $H$, this convergence occurs with respect to the inductive limit topology.

It only remains to see that $\Upsilon$ is surjective. But \{ $\Upsilon(f)(\eta) : f \in \Gamma_c(E, r^*\mathcal{A})$ \} is dense in $B(r(\eta))$ and it follows from [19, Proposition C.24] that $\Upsilon(\Gamma_c(E, r^*\mathcal{A}))$ is dense in $\Gamma_c(H, r^*\mathcal{B})$ in the supremum norm. But then a compactness argument implies the image of $\Upsilon$ is dense in the inductive limit topology. Hence $\Upsilon$ has dense image and is necessarily onto. $\square$

Thus to prove Theorem 4.1, we just need to see that $\Upsilon$ is isometric. The idea of the proof is elementary: if $R$ is a representation of $\mathcal{A} \rtimes_\omega E$, then we want to show that it factors through $\mathcal{B} \rtimes_\beta H$. More precisely, we will prove the following.

**Proposition 4.3.** If $R$ is a representation of $\mathcal{A} \rtimes_\omega E$, then there is a representation $L$ of $\mathcal{B} \rtimes_\beta H$ such that $L \circ \Upsilon$ is equivalent to $R$.

Of course, once we’ve proved Proposition 4.3, it follows easily that $\Upsilon$ is isometric: If $R$ is a faithful representation of $\mathcal{A} \rtimes_\omega E$, then
\[
\|f\| = \|R(f)\| = \|L \circ \Upsilon(f)\| \leq \|\Upsilon(f)\|.
\]
Since Proposition 4.2 implies $\|\Upsilon(f)\| \leq \|f\|$, this completes the proof of Theorem 4.1.

However our proof of Proposition 4.3 requires that we work with covariant representations of groupoid dynamical systems, and unitary groupoid representations in particular. While the corresponding details in the group case are straightforward, working out the niceties for groupoids is extremely subtle. We do this in the next sections after briefly reviewing the necessary definitions.

5. **Proof of Proposition 4.3**

We refer to [19, Appendix F] for background on Borel Hilbert bundles and [13, §7] for background on covariant representations of groupoid dynamical systems. For convenience, we review some of the basic concepts here.
5.1. Covariant Representations. Let $\mathcal{H} = \{ \mathcal{H}(y) \}_{y \in Y}$ be a collection of Hilbert spaces indexed by an analytic Borel space $Y$. The disjoint union $Y \ast \mathcal{H}$, viewed as a bundle $p : Y \ast \mathcal{H} \to Y$ in the obvious way, is called a Borel Hilbert bundle if it has a natural Borel structure respecting the Hilbert space structure on the $\mathcal{H}(y)$. (For a precise statement, see [19, Definition F.1].) By [19, Proposition F.6] there exists a sequence of sections $e_i : Y \to Y \ast \mathcal{H}$ called a special orthogonal fundamental sequence such that $\{ e_i(x) : e_i(x) \neq 0 \}$ is an orthonormal basis for $\mathcal{H}(x)$ and such that $h : Y \to Y \ast \mathcal{H}$ is Borel if and only if the maps $x \mapsto (h(x) | e_i(x))_{\mathcal{H}(x)}$ are Borel for all $i$. If $\mu$ is a measure on $X$ then $(h, k) \mapsto \int_Y (h(x) | k(x))_{\mathcal{H}(x)} d\mu(x)$ defines an inner product on the bounded Borel sections of $Y \ast \mathcal{H}$. We denote the Hilbert space completion of these sections by $L^2(Y \ast \mathcal{H}, \mu)$. Given a Borel Hilbert bundle $Y \ast \mathcal{H}$, its isomorphism groupoid $\text{Iso}(Y \ast \mathcal{H})$ is the set $\{(x, V, y) : V$ a unitary from $\mathcal{H}(y)$ to $\mathcal{H}(x)\}$ with multiplication $(x, V, y)(y, U, z) = (x, VU, z)$ and the weakest Borel structure such that $(x, V, y) \mapsto (Vh(y) | k(x))$ is Borel for all bounded Borel sections $h$ and $k$.

Let $S$ be a locally compact Hausdorff groupoid with Haar system $(\kappa^u)_{u \in S(0)}$. Let $\mu$ be a Radon measure on $S(0)$ and define $\nu = \mu \circ \kappa := \int_{S(0)} \kappa^u d\mu(u)$. We say $\mu$ is quasi-invariant if $\nu$ is equivalent to its image under inversion [15, Definition 3.2]. We define the modular function $\Delta := \Delta^S_\mu$ to be the Radon-Nikodym derivative of $\nu$ with respect to its image under inversion. We will use the fact that the modular function can be taken to be multiplicative (see [13, Remark 7.1]). If $\mu$ is an arbitrary Radon measure on $S(0)$ and $\nu_0$ is a finite measure on $S$ equivalent to $\nu = \mu \circ \kappa$, then we define the saturation of $\mu$ to be the push-forward $[\mu] = s_\kappa \nu_0$. It is shown in [15, Proposition 3.6] that the saturation $[\mu]$ is quasi-invariant, and if $\mu$ is quasi-invariant to begin with, then $\mu$ is equivalent to $[\mu]$.

Let $(\mathcal{C}, S, \vartheta)$ be a groupoid dynamical system. Following [13, Definition 7.9], a covariant representation $(\pi, U, S(0) \ast \mathcal{H}, \mu)$ of $(\mathcal{C}, S, \vartheta)$ consists of a Borel Hilbert bundle $S(0) \ast \mathcal{H}$ over $S(0)$, a quasi invariant measure $\mu$, a Borel field of representations $\pi_u : C(u) \to H(u)$ and a Borel homomorphism $U : S \to \text{Iso}(S(0) \ast \mathcal{H})$ that satisfies the covariance condition: there is a $\nu$-null set $N$ such that for all $\gamma \in N$
\begin{equation}
U_{\gamma} \pi_s(\gamma)(b) = \pi_{s(\gamma)}(\vartheta_{\gamma}(b)) U_{\gamma} \quad \text{for all } b \in C(s(\gamma)).
\end{equation}
It will be convenient to recall that if $(\pi, U, S(0) \ast \mathcal{H}, \mu)$ is covariant, then there is a $\mu$-conull set $V \subset S(0)$ such that the covariance condition [15] holds for all $\gamma \in S \setminus V$ [13, Remark 7.10]. (Notice that $S \setminus V$ is $\nu$-conull.)

By [13, Proposition 7.11], each covariant representation $(\pi, U, S(0) \ast \mathcal{H}, \mu)$ determines a representation $\pi \times U$ of $\mathcal{C} \rtimes_\vartheta S$ on $L^2(S(0) \ast \mathcal{H}, \mu)$ (called the integrated form) such that
\begin{equation}
\pi \times U(f)h(u) = \int_G \pi_u(f(\gamma))U_{\gamma}h(s(\gamma))\Delta^S_\mu(\gamma)^{-1/2}d\kappa^u(\gamma)
\end{equation}
for $f \in \Gamma_c(S; r^*\mathcal{C})$ and $h \in L^2(S(0) \ast \mathcal{H}, \mu)$. It suffices to define $U$ only on a restriction of the form $S \setminus V$ where $V$ is a $\mu$-conull set in $S(0)$. By [13, Theorem 7.12] every representation of $\mathcal{C} \rtimes_\vartheta S$ is equivalent to the integrated form of a covariant representation.

\footnote{Note that the saturation depends on our choice of $\nu_0$ and so is well-defined only up to equivalence of measures.}
5.2. A Representation of $\mathcal{B}$. Let $(\pi, U, X \ast \mathcal{H}, \mu)$ be a covariant representation of $(A, E, \omega)$. The first step will be to build a covariant representation of $(\mathcal{A}, G, \alpha)$. We already have the Borel Hilbert bundle $X \ast \mathcal{H}$ and $\pi$ is already a Borel field of representations of $\mathcal{A}$. The restriction of $U$ from $E$ to $G$, which we shall denote $U_G$, is still a Borel homomorphism into $\text{Iso}(X \ast \mathcal{H})$. Furthermore, since $U_G$ and $\alpha$ are the restrictions of $U$ and $\omega$ to $G$, respectively, there is a $\mu$-conull set $V \subset X$ such that the covariance relation between $U_G$ and $\pi$ holds on $G \setminus V$. Thus it will follow that $(\pi, U_G, X \ast \mathcal{H}, \mu)$ is a covariant representation of $(\mathcal{A}, G, \alpha)$ provided we can show that $\mu$ is quasi-invariant with respect to $G$.

**Proposition 5.1.** Let $\mu$ be a quasi-invariant measure on $X$ with respect to $E$.

(a) Then $\mu$ is quasi-invariant with respect to $G$ and $H$.
(b) The push forward measure $\tau = s_* \mu$ on $H^{(0)}$ is quasi-invariant with respect to $H$.

**Proof.** Since $\mu$ is quasi-invariant on $X$ with respect to $E$, it is equivalent to its saturation $[\mu]$ with respect to $E$. Proposition 3.6. So it will suffice to show that $[\mu]$ is quasi-invariant with respect to $G$. Let $\nu = \mu \circ \lambda_E$ be the measure induced on $E$ by $\mu$, $\nu_0$ a finite measure equivalent to $\nu$, and $d\nu/d\nu_0$ the (strictly positive) Radon-Nikodym derivative of $\nu$ and $\nu_0$. By definition, $[\mu] = s_* \nu_0$. To see that $[\mu]$ is quasi-invariant with respect to $G$, it will suffice to see that if $f \in C^*_\alpha(G)$ is such that $\int f(\gamma, x) d\lambda^{C^*_\alpha}(\gamma) d[\mu](x) = 0$, then $\int f(\xi^{-1}, \xi^{-1} \cdot x) d\lambda^{C^*_\alpha}(\xi) d[\mu](x) = 0$.

Now,

$$0 = \int \int f(\gamma, x) d\lambda^{C^*_\alpha}(\gamma) d[\mu](x)$$

$$= \int \int f(\gamma, \xi^{-1} \cdot x \cdot \eta) d\lambda^C(\gamma) d\nu_0(\xi, x, \eta)$$

$$= \int \int \int \int f(\gamma, \xi^{-1} \cdot x \cdot \eta) \frac{d\nu}{d\nu_0}(\xi, x, \eta) d\lambda^{C}(\gamma) d\lambda^{C^*_\alpha}(\xi) d\sigma^C(\eta) d\mu(x)$$

$$= \int \int \int \int f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta) \frac{d\nu}{d\nu_0}(\gamma, x, \eta) d\lambda^{C^*_\alpha}(\xi) d\lambda^{C^*_\alpha}(\gamma) d\sigma^C(\eta) d\mu(x)$$

$$= \int \int \int f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta) \frac{d\nu}{d\nu_0}(\gamma, x, \eta) d\lambda^{C^*_\alpha}(\xi) d\nu_0(\gamma, x, \eta).$$

So off a $\nu_0$-null set we have

$$0 = \int f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta) \frac{d\nu}{d\nu_0}(\gamma, x, \eta) d\lambda^{C^*_\alpha}(\xi).$$

However, because $d\nu/d\nu_0$ is strictly positive, the supports of

$$\xi \mapsto f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta) \frac{d\nu}{d\nu_0}(\gamma, x, \eta)$$

and

$$\xi \mapsto f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta)$$

are the same. Thus, since $f \geq 0$, (6) holds if and only if

$$0 = \int f(\xi^{-1}, \xi^{-1} \gamma^{-1} \cdot x \cdot \eta) d\lambda^{C^*_\alpha}(\xi).$$
As a result
\[
0 = \int \int f(\xi, \xi^{-1} \cdot x \cdot \eta) \, d\lambda^x(\gamma) \, d\nu_0(\gamma, x, \eta) \\
= \int \int f(\xi^{-1}, \xi^{-1} \cdot x) \, d\lambda^x(\gamma) \, d\nu_0(\gamma, x, \eta).
\]

It follows that $[\mu]$ is quasi-invariant with respect to $G$. The corresponding assertion for $H$ follows by symmetry.

The proof of (5.1) is similar but easier.

Now that we have a covariant representation $(\pi, U_G, X, \mathcal{H}, \mu)$ of $(\mathcal{A}, G, \alpha)$ we may form the integrated representation $R = \pi \times U_G$ of $B$ on $L^2(X \ast \mathcal{H}, \mu)$. This will make up the $C^*$-algebraic portion of a covariant representation of $(\mathcal{B}, H, \beta)$.

5.3. A Representation of $H$. Next we must build a unitary representation of $H$. We use the fact that $\mu$ is quasi-invariant with respect to $H$ (Proposition 5.1) to restrict $U$ to a unitary representation $(U_H, X \ast \mathcal{H}, \mu)$ of the groupoid $H$. This yields a representation of the transformation groupoid $C^*$-algebra $C^*(H)$. However, it is routine to check that $C^*(H)$ is naturally isomorphic to the groupoid crossed product $C_0(X) \rtimes_r H$ (for example, see [7, Remark 2.7]). Next, we recall from the proof of the Disintegration Theorem for crossed products [13, Theorem 7.12], that there is a nondegenerate map from $C^*(H)$ into the multiplier algebra $M(C_0(X) \rtimes_r H)$ defined for $\phi \in C_c(H)$ and $f \in \Gamma_c(H, r^* C_0(X))$ by

\[
\phi \cdot f(\eta) = \int \phi(\zeta) r_\zeta(f(\zeta^{-1} \eta)) \, d\sigma^r(\eta)(\zeta).
\]

Since the latter is isomorphic to $M(C^*(H))$, we obtain a map $m$ of $C^*(H)$ into $M(C^*(H))$ which is given on $\phi \in C_c(H)$ and $f \in C_c(H)$ by

\[
(7) \quad m(\phi) f(x, \eta) = \int \phi(\zeta) f(x \cdot \zeta, \zeta^{-1} \eta) \, d\sigma^r(\eta)(\zeta).
\]

This multiplier action is important because we can use it to form a representation $W$ of $H$ on $L^2(X \ast \mathcal{H}, \mu)$.

Proposition 5.2. Let $m$ be the action given in (7) and let $U_H$ be the extension of (the integrated form of) $U_H$ to the multipliers of $C^*(H)$. Then $W = U_H \circ m$ is a representation of $C^*(H)$ which, for $\phi \in C_c(H)$ and $f \in L^2(X \ast \mathcal{H}, \mu)$, is given by

\[
(8) \quad W(\phi) h(x) = \int \phi(\eta) U_{(r(x), x, \eta)} h(x) \, d\sigma^r(\eta)(\zeta).
\]

Proof. Since $U_H$ is a representation of $M(C^*(H))$ by [13, Proposition 3.5(ii)] and since $m$ is a nondegenerate $*$-homomorphisms, $W$ is a representation. That $W$ has the form given in (8) follows from the following computation. By nondegeneracy it suffices to verify (8) for vectors of the form $U_H(f) h$ for $f \in C_c(H)$ and $h \in L^2(X \ast \mathcal{H}, \mu)$. Let $\phi \in C_c(H)$. Then

\[
W(\phi) U_H(f) h(x) = U_H(m(\phi) f) h(x) \\
= \int m(\phi) f(x, \zeta) U_{(r(x), x, \zeta)} h(x) \, d\sigma^r(x)(\zeta) \\
= \int \int \phi(\eta) f(x, \eta^{-1} \zeta) U_{(r(x), x, \zeta)} h(x) \, d\sigma^r(x)(\eta) \, d\sigma^r(x)(\zeta).
\]
where \( \theta \) is an isomorphism between the two spaces. However, we have

\[
H(\tau(x),x,\eta)h(x,\eta)\Delta_H(x,\eta)\gamma \frac{1}{2} \sigma^*(\eta) \frac{1}{2} \sigma^*(x)(\eta)
\]

\[
= \int \phi(x)U(\tau(x),x,\eta)h(x,\eta)\Delta_H(x,\eta)\gamma \frac{1}{2} \sigma^*(\eta) \frac{1}{2} \sigma^*(x)(\eta)
\]

\[
= \int \phi(x)U(\tau(x),x,\eta)U_H(f)h(x,\eta)\Delta_H(x,\eta)\gamma \frac{1}{2} \sigma^*(\eta) \frac{1}{2} \sigma^*(x)(\eta).
\]

\[\Box\]

In order to proceed any further we need to examine the measure \( \mu \) and the relationship between the various modular functions more closely. But first we need to recall some issues concerning disintegration of measures from \([19, \text{Appendices F and I}]\). Let \( \mu \) be a measure on \( X \) and \( \tau = s_*\mu \) be the push forward of \( \mu \) to \( H(0) \).

We now use the Disintegration Theorem for measures \([19, \text{Theorem I.5}]\) to generate Radon measures \( \{\mu_u\}_{u \in H(0)} \) on \( X \) such that:

(a) Off a \( \tau \)-null set \( N \), each \( \mu_u \) is a probability measure supported on \( s^{-1}(u) \subset X \) and \( \mu_u = 0 \) for \( u \in N \).

(b) For all bounded Borel functions \( h \), the map

\[
u \mapsto \int h(x)d\mu_u(x)
\]

is bounded and Borel.

(c) For all bounded Borel functions \( h \),

\[
\int_X h(x)d\mu(x) = \int \int h(x)d\mu_u(x)d\tau(u).
\]

Let \( \{e_i\} \) be the special orthogonal fundamental sequence for a Borel Hilbert bundle \( X * H \) over \( X \). By \([19, \text{Example F.19}]\) there exists a Borel Hilbert bundle \( H(0) \) with fibres \( K(u) = L^2(s^{-1}(u) * H, \mu_u) \) for all \( u \) and so that

\[
ed_i(u)(x) = e_i(x)
\]

defines a special orthogonal fundamental sequence for \( H(0) \). Furthermore, the map \( V : L^2(X * H, \mu) \to L^2(H(0) \to H, \tau) \) given by \( V(h)(u)(x) = h(x) \) is a natural isomorphism between the two spaces.

By Proposition \([\text{5.1}]\) \( \tau \) is quasi-invariant with respect to \( H \). Thus we may form the modular function on \( H \), \( \Delta_H(\eta) \), as well as the usual one, \( \Delta_H(\eta) \), on \( \overline{H} \).

Our next proposition connects the decomposition of \( \mu \) with respect to \( \tau \) with the action of \( H \) on \( X \). (As a special case, it simply says that \( H \) and \( \tau \) give us a measured groupoid as in \([11, \text{Chap. 4}]\).

**Proposition 5.3.** Let \( \{\mu_u\}_{u \in H(0)} \) be the decomposition of \( \mu \) as above. There is a \( \tau \)-null set \( V \subset H(0) \) such that for all \( \eta \in H|_V \) and all bounded Borel functions \( \phi \) we have

\[
\int \phi(x)d\mu_{r(\eta)}(x) = \int \phi(x,\eta)\theta(x,\eta)d\mu_u(\eta)(x)
\]

where \( \theta(x,\eta) = \Delta_{H}(x,\eta)/\Delta_{H}(\eta) \).

Proving Proposition \([\text{5.3}]\) will take some work. We start by using the groupoid structure to generate a very special “invariant” section of \( X * H \).
Lemma 5.4. There exists $e \in \mathcal{B}(X \ast \mathscr{K})$ such that $e(x)$ is a unit vector if $\mathcal{H}(x) \neq 0$ and for all $(x, \eta) \in H$

$$e(x \cdot \eta) = U^*_{(r(x),c(q(x))}e(q(x)).$$

Proof. Since second countable, locally compact Hausdorff spaces are Polish — see [19, Lemma 6.5] — and since the orbit space $X/H$ is Hausdorff, it follows from the Corollary following [4, 3.4.1] that there is a Borel cross section $c$ for the quotient map $q : X \to X/H$. Using $c$ we may construct a Borel map $\varsigma : X \to H$ with the property that

$$c(q(x))\varsigma(x) = x \text{ for all } x \in X.$$

Now let $e_i$ be a special orthogonal fundamental sequence for $X \ast \mathscr{K}$ as in [19, Remark F.7]. By definition, $e_1(x)$ is a unit vector whenever $\mathcal{H}(x)$ is non-trivial. We define

$$e(x) = U^*_{(r(x),c(q(x)),c(x))}e_1(c(q(x))).$$

Since $U$ is a unitary representation, $e(x)$ is a unit vector if $\mathcal{H}(x) \neq 0$. All that remains is to show that $e$ is Borel and invariant. According to the definition of a fundamental sequence, to show that $e$ is Borel it suffices to show that $x \mapsto (e(x) \mid e_i(x))$ is Borel for all $i$ ([19, Proposition F.6]). For this note that

$$(\xi, x, \eta) \mapsto \left( U^*_{(\xi,x,\eta)}e_1(x) \mid e_i((\xi^1 \cdot x \cdot \eta)) \right)$$

is Borel on $E$ (since $U$ is a representation). Therefore, since $x \mapsto (r(x), c(q(x)), \varsigma(x))$ is Borel, so is

$$x \mapsto (e(x) \mid e_i(x)) = \left( U^*_{(r(x),c(q(x)),c(x))}e_1(c(q(x)))) \mid e_i(x) \right).$$

The “invariance” portion of the lemma now follows from a brief computation and the observation that $\varsigma(x \cdot \eta) = \varsigma(x)\eta$. \hfill \Box

This special invariant vector, combined with the representation $W$, is the key to showing that the measure decomposition respects the groupoid action.

Proof of Proposition [5.3] Let $h, k \in L^2(X \ast \mathscr{K}, \mu)$ and $\psi \in C_c(H)$. Using (8), we have that

$$(W(\psi)h \mid k) = \int (W(\psi)h(x) \mid k(x)) \, d\mu(x)$$

$$= \iint \psi(\eta)(U(r(x),x,\eta)h(x \cdot \eta) \mid k(x)) \Delta_H^H(x,\eta)^{-\frac{1}{2}} \, d\sigma^\mu(x)(\eta) \, d\mu_u(x) \, d\tau(u)$$

$$= \iint \psi(\eta) \int (U(r(x),x,\eta)h(x \cdot \eta) \mid k(x)) \Delta_H^H(x,\eta)^{-\frac{1}{2}} \, d\mu_u(x) \, d\sigma^\mu(\eta) \, d\tau(u).$$

Given $\eta \in H$ define the Radon measure $\eta \cdot \mu_s(\eta)$ supported in $r_X^{-1}(r(\eta))$ by

$$\int \varrho(x \cdot \eta) d(\eta \cdot \mu_s(\eta))(x) := \int \varrho(x) d\mu_s(\eta)(x) \text{ for all } \varrho \in C_c(X).$$

Using the fact that $W$ is a representation, we may also write

$$(W(\psi)h \mid k) = \frac{(W(\psi^*)k \mid h)}{\phi}$$

$$= \iint \psi^*(\eta)(U(r(x),x,\eta)k(x \cdot \eta) \mid h(x)) \Delta_H^H(x,\eta)^{-\frac{1}{2}} \, d\sigma^\mu(x)(\eta) \, d\mu_u(x) \, d\tau(u)$$

$$= \iint \psi(\eta^{-1}) (h(x) \mid U(r(x),x,\eta)k(x \cdot \eta)) \Delta_H^H(x,\eta)^{-\frac{1}{2}} \, d\mu_{r(\eta)}(x) \, d\sigma^\mu(\eta) \, d\tau(u).$$
\[
\int \int \psi(\eta) (h(x) | U(r(x), x, \eta^{-1}) k(x \cdot \eta^{-1})) \Delta^H_F(x, \eta^{-1})^{-\frac{1}{2}} \Delta^H_W(x, \eta) \, d\mu_{s(\eta)}(x) \, d\sigma^a(\eta) \, d\tau(u) = \int \int \psi(\eta) \int \left( U(r(x), x, \eta) h(x \cdot \eta) | k(x) \right) \Delta^H_F(x, \eta)^{-\frac{1}{2}} \vartheta(x, \eta) \, d(\eta \cdot \mu_{s(\eta)})(x) \, d\sigma^a(\eta) \, d\tau(u),
\]

where we used the groupoid homomorphism properties of \( U \) and \( \Delta^H_F \) to get the last equality. Since these two forms of (12) are equal for all \( \psi \in C_c(H) \), we may conclude that given \( h \) and \( k \) there is a \( \tau \circ \sigma \)-null set \( N_{h,k} \) such that \( \eta \in N_{h,k} \).

(To be precise, we are choosing \( N_{h,k} \) so that both sides of (10) are well-defined, finite and equal to each other.)

Let \( e \) be the invariant vector from Lemma 5.4. Then, by construction, we have

\[
\int \left( U(r(x), x, \eta) e(x) | \varrho e(x) \right) = (e(x) | \varrho(x) e(x)) = \varrho(x)
\]

for all \( x \) such that \( H(x) \) is nontrivial. Let \( \{ \varrho_i \} \) be an inductive limit dense set in \( C_c(X) \) (see Remark 2.3(b)). If we set \( h = e \) and \( k = \varrho_i e \), then we can conclude from (10) and (14) that for all \( \eta \in N_{e,\varrho_i e} \)

\[
\int \varrho_i(x) \Delta^H_F(x, \eta)^{-\frac{1}{2}} \, d\mu_{\tau(\eta)}(x) = \int \varrho_i(x) \Delta^H_F(x, \eta)^{-\frac{1}{2}} \vartheta(x, \eta) \, d(\eta \cdot \mu_{s(\eta)})(x)
\]

We may take the countable union \( N = \bigcup_i N_{e,\varrho_i e} \) to obtain a null set \( N \) such that (12) holds for all \( i \) provided \( \eta \notin N \). We can assume that if \( K \subset X \) is compact, then there is an \( i_0 \) such that \( \varrho_{i_0} \geq 0 \) and equal to 1 on \( K \). This implies that the measures \( \Delta^H_F(x, \eta)^{-\frac{1}{2}} \, d\mu_{\tau(\eta)}(x) \) and \( \Delta^H_F(x, \eta)^{-\frac{1}{2}} \vartheta(x, \eta) \, d(\eta \cdot \mu_{s(\eta)})(x) \) are finite on compact subsets of \( X \). Thus they are Radon measures by \([14\text{, Theorem 2.18}]\). In particular, they are determined on \( C_c(X) \). Since the \( \varrho_i \) are dense in the inductive limit topology, it follows that the two measures are equal. Thus (12) holds for all nonnegative Borel functions. Replacing an arbitrary nonnegative Borel function \( \phi(x) \) by \( \phi(x) \Delta(x, \eta)^{\frac{1}{2}} \) we conclude that for \( \eta \notin N \) equation (11) holds for all nonnegative Borel functions. Since the \( \mu_{s(\eta)} \) are probability measures, (10) holds for all bounded Borel functions as claimed.

It is clear that the set \( \Sigma = \{ \eta \in H : (19) \text{ holds} \} \) is conull. Since the modular functions are all homomorphisms it is straightforward to show that if \( \eta, \zeta \in \Sigma \) such that \( s(\eta) = r(\zeta) \) then \( \eta \zeta \in \Sigma \). It follows from a result of Ramsay’s (see [14\text{, Lemma 5.2}] or [14\text{, Lemma 4.9}]) that there is a \( \tau \)-conull set \( V \) such that \( H|_V \subset \Sigma \). This completes the proof.

The proof of the following lemma is a brief computation and has been omitted.

**Lemma 5.5.** Let \( \nu = \mu \circ \lambda_\nu \). Then, \( \nu \)-almost everywhere, we have

\[
\Delta^F_{\nu}(\gamma, x) \Delta^H_F(\gamma^{-1} \cdot x, \eta) = \Delta^F_{\nu}(\gamma, x, \eta) = \Delta^F_{\nu}(\gamma, x \cdot \eta) \Delta^H_F(x, \eta).
\]
5.4. Back to the Proof of Proposition 4.3. Now that we have dealt with the major measure theoretic issues, we can turn to Proposition 4.3.

Proof of Proposition 4.3. Let \( R = \pi \times U \) be the representation of \( B \) given by the integrated form of \((\pi, U, X * K, \mu)\). We construct a groupoid representation of \( H \) which is covariant with \( R \). From the discussion before the statement of Proposition 5.3, we can use [19, Example F.19] to obtain a Borel Hilbert bundle \( H \) for \( \mathcal{K} = \mathcal{H} \) and \( \mathcal{K} = \mathcal{H} \) with fibres \( \mathcal{K}(u) = L^2(s^{-1}(u) * \mathcal{H}, \mu_u) \) so that \( V : L^2(X * \mathcal{H}, \mu) \rightarrow L^2(H \otimes \mathcal{K}, \tau) \) given by \( V(h)(x) = h(x) \) is a natural isomorphism between the two spaces. Note that if \( \{e_i\} \) is a special orthogonal fundamental sequence for \( X * \mathcal{H} \), then \( \tilde{e}(u) = e_i(u) \) is a special orthogonal fundamental sequence for \( H \).

Define a representation \( Q \) of \( B \) on \( L^2(H \otimes \mathcal{K}, \tau) \) by \( Q = VRV^* \). Simple computations show that \( Q \) is a \( C_0(H) \)-linear homomorphism and that for \( f \in B_0(u) \) and \( x \in s^{-1}(u) \)

\[
(Q_u(f)h)(u)(x) = \int \pi_x(f(\gamma,x)h(u)(\gamma^{-1} \cdot x)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

Next we define \( W_\eta : \mathcal{K}(s(\eta)) \rightarrow \mathcal{K}(r(\eta)) \) by

\[
(W_\eta h)(r(\eta))(x) = U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

Straightforward computations using Proposition 5.3 show that there is a \( \tau \)-null set \( V \subset H \) such that for all \( \eta \in H \setminus V \), \( W_\eta \) is a unitary and that \( W_\eta W_\zeta = W_{\eta \zeta} \) when \( \eta \) and \( \zeta \) are composable in \( H \setminus V \). The last thing we need to check is that \( W \) is Borel. Given \( e_j \) and \( e_k \) we have

\[
(W_\eta \hat{e}_j(s(\eta)) | \hat{e}_k(r(\eta))) = \int (U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

The integrand is Borel with respect to \( \eta \) because the \( e_j \) are a fundamental sequence and \( U \) is a Borel representation. Since \( \mu_u \) is a Borel field of measures it follows from standard, albeit lengthy, arguments that

\[
\eta \mapsto (W_\eta \hat{e}_j(s(\eta)) | \hat{e}_k(r(\eta)))
\]

is Borel. Thus we have all of the components to form a Borel representation \((W, H \otimes \mathcal{K}, \tau)\) of \( H \).

Remark 5.6. Although we won’t make use of this fact, the integrated form of \( W_\eta \) is the push forward of the representation \( W \) defined in Proposition 4.3 from \( L^2(X * \mathcal{H}) \) to \( L^2(H \otimes \mathcal{K}, \tau) \) by \( V \).

We now show that \((Q, W, H \otimes \mathcal{K}, \tau)\) satisfies the covariance condition. Using Lemma 5.5.

\[
(W_\eta Q_{\pi}(f)h)(r(\eta))(x) = \int U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

Next we define \( W_\eta : \mathcal{K}(s(\eta)) \rightarrow \mathcal{K}(r(\eta)) \) by

\[
(W_\eta h)(r(\eta))(x) = U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

Straightforward computations using Proposition 5.3 show that there is a \( \tau \)-null set \( V \subset H \) such that for all \( \eta \in H \setminus V \), \( W_\eta \) is a unitary and that \( W_\eta W_\zeta = W_{\eta \zeta} \) when \( \eta \) and \( \zeta \) are composable in \( H \setminus V \). The last thing we need to check is that \( W \) is Borel. Given \( e_j \) and \( e_k \) we have

\[
(W_\eta \hat{e}_j(s(\eta)) | \hat{e}_k(r(\eta))) = \int (U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]

The integrand is Borel with respect to \( \eta \) because the \( e_j \) are a fundamental sequence and \( U \) is a Borel representation. Since \( \mu_u \) is a Borel field of measures it follows from standard, albeit lengthy, arguments that

\[
\eta \mapsto (W_\eta \hat{e}_j(s(\eta)) | \hat{e}_k(r(\eta)))
\]

is Borel. Thus we have all of the components to form a Borel representation \((W, H \otimes \mathcal{K}, \tau)\) of \( H \).

We now show that \((Q, W, H \otimes \mathcal{K}, \tau)\) satisfies the covariance condition. Using Lemma 5.5.

\[
(W_\eta Q_{\pi}(f)h)(r(\eta))(x) = \int U(r(x),x,\eta)\Lambda_{\mathcal{H}^\gamma}(\gamma)\Lambda_{\mathcal{H}^\gamma}(\gamma^{-1} \cdot x) \, d\lambda^x(\gamma).
\]
\[
\left( \frac{\Delta_G^f(\gamma, x \cdot \eta) \Delta_H(\gamma, x)}{\Delta_H(\eta)} \right)^{-\frac{1}{2}} d\lambda^r(x)(\gamma)
\]
\[
= \int \pi_x(\beta_\eta(f)(\gamma, x)) U(\gamma, x, \eta) h(\gamma^{-1} \cdot x \cdot \eta) \left( \frac{\Delta_G^f(\gamma, x) \Delta_H(\gamma^{-1} \cdot x, \eta)}{\Delta_H(\eta)} \right)^{-\frac{1}{2}} d\lambda^r(x)(\gamma)
\]
\[
= \int \pi_x(\beta_\eta(f)(\gamma, x)) U(\gamma, x, s(x)) W_\eta h(r(\eta))(\gamma^{-1} \cdot x) \Delta_H^f(\gamma, x)^{-\frac{1}{2}} d\lambda^r(x)(\gamma)
\]
\[
= Q_r(\eta)(\beta_\eta(f)) W_\eta h(r(\eta))(x).
\]

This shows that \((Q, W, H^{(0)} \ast \mathcal{K}, \tau)\) is a covariant representation of \((\mathcal{B}, H, \beta)\). It remains to show that \(V\) intertwines \(\pi \circ U\) and \((Q \ast W) \circ \Upsilon\). Given \(h, k \in L^2(X \ast \mathcal{K})\) and \(f \in \Gamma_\epsilon(E, r^* \mathcal{A})\),
\[
(V^* Q \ast W(\Upsilon(f)) V h | k) = (Q \ast W(\Upsilon(f)) V h | V k)
\]
\[
= \int \int \int \left( Q(\Upsilon(f)(\eta)) W_\eta V h(s(\eta))(x) | V k(r(\eta))(x) \right) \Delta_H^f(\eta)^{-\frac{1}{2}} d\mu_u(x) d\sigma^u(\eta) d\tau(u)
\]
\[
= \int \int \int \pi_x(f(\gamma, \eta)) U(\gamma, x, s(x)) W_\eta V h(s(\eta))(\gamma^{-1} \cdot x) | k(x) \Delta_H^f(\eta)^{-\frac{1}{2}} \Delta_H^f(\gamma, x)^{-\frac{1}{2}}
\]
\[
\int \lambda^r(x)(\gamma) d\sigma^s(x)(\eta) d\mu(x)
\]
\[
= \int \int \int \pi_x(f(\gamma, \eta)) U(\gamma, x, \eta) h(\gamma^{-1} \cdot x \cdot \eta) | k(x) \Theta(\gamma^{-1} \cdot x, \eta)^{-\frac{1}{2}} \Delta_H^f(\gamma, x)^{-\frac{1}{2}}
\]
\[
\Delta_H^f(\eta)^{-\frac{1}{2}} d\lambda^r(x)(\gamma) d\sigma^s(x)(\eta) d\mu(x)
\]
\[
= \int \int \int \pi_x(f(\gamma, \eta)) U(\gamma, x, \eta) h(\gamma^{-1} \cdot x \cdot \eta) | k(x) \Delta_H^f(\gamma, x, \eta)^{-\frac{1}{2}}
\]
\[
\lambda^r(x)(\gamma) d\sigma^s(x)(\eta) d\mu(x)
\]

Since this holds on a dense subset it holds everywhere and we get the result. \(\square\)

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