ON \( L^1 \)-ESTIMATES FOR PROBABILITY SOLUTIONS TO FOKKER–PLANCK–KOLMOGOROV EQUATIONS

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Abstract

We prove two new results connected with elliptic Fokker–Planck–Kolmogorov equations with drifts integrable with respect to solutions. The first result answers negatively a long-standing question and shows that a density of a probability measure satisfying the Fokker–Planck–Kolmogorov equation with a drift integrable with respect to this density can fail to belong to the Sobolev class \( W^{1,1}(\mathbb{R}^d) \). There is also a version of this result for densities with respect to Gaussian measures. The second new result gives some positive information about properties of such solutions: the solution density is proved to belong to certain fractional Sobolev classes.

Keywords: Fokker–Planck–Kolmogorov equation, \( L^1 \)-estimate

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1. Introduction

In the recent years, there has been a growing interest to various \( L^1 \)-estimates for second order partial differential operators, see, e.g., [1], [11], [12], [18], [21], [24], [27], and [29]–[32], where additional references can be found. Their main feature is that classical \( L^p \)-estimates for solutions to second order elliptic equations valid for \( p > 1 \) do not extend directly to the case \( p = 1 \) (see [26], [15], [23], [14], [11], and [19, Example 7.5]). Some concrete examples are mentioned below. In particular, for the solution \( f \) to the Poisson equation

\[ \Delta f = g, \quad f|_{\partial B} = 0 \]

on the unit ball \( B \) in \( \mathbb{R}^d \) one has

\[ \|f\|_{W^{2,2}(B)} \leq C(p, d)\|g\|_{L^p(B)} \]

with some number \( C(p, d) \) provided that \( p > 1 \), but there is no such estimate for \( p = 1 \) if \( d > 1 \). Here and throughout we use the symbol \( W^{p,k}(\Omega) \) to denote the Sobolev space of functions on a domain \( \Omega \subset \mathbb{R}^d \) that belong to \( L^p(\Omega) \) along with their partial derivatives up to order \( k \); the Sobolev norm \( \|f\|_{W^{p,k}} \) is the sum of the \( L^p \)-norms of the function \( f \) and its partial derivatives up to order \( k \). By \( W^{p,k}_{loc} \) we denote the class of functions \( f \) such that \( \zeta f \in W^{p,k}(\mathbb{R}^d) \) for all functions \( \zeta \) from the class \( C_0^{\infty}(\mathbb{R}^d) \) of smooth compactly supported functions.

The failure of an \( L^1 \)-estimate of this kind is connected with the fact that a solution \( f \) for some \( g \in L^1(B) \) does not belong to the second Sobolev class \( W^{1,2}(B) \). As a consequence, a solution to the equation

\[ \Delta u = \text{div } v \]

with a vector field \( v \) of class \( L^1 \) can fail to belong to the first Sobolev class \( W^{1,1}(B) \) and in case of smooth solutions there is no estimate of \( \|\nabla u\|_{L^1} \) through \( \|v\|_{L^1} \). For example, one can consider \( u = \partial_{x_1} f \) and \( v = (\partial_{x_1} g, 0, \ldots) \) for \( f \) and \( g \) satisfying the first equation.
Questions of this type arise also for solutions to Fokker–Planck–Kolmogorov equations on the whole space, which is the subject of this paper and which has not been studied so far. A bounded Borel measure $\mu$ on $\mathbb{R}^d$ is said to satisfy the Fokker–Planck–Kolmogorov equation

$$L_0 \mu = 0$$

(1.1)

with a Borel vector field $b$ locally integrable with respect to $\mu$ if for the operator

$$L_0 \varphi = \Delta \varphi + \langle b, \nabla \varphi \rangle$$

we have the identity

$$\int_{\mathbb{R}^d} L_0 \varphi(x) \mu(dx) = 0 \quad \forall \varphi \in C^\infty_0(\mathbb{R}^d).$$

It is known (see [4] or [7]) that in this case the measure $\mu$ has a density $\varrho$ with respect to Lebesgue measure and

$$\Delta \varrho - \text{div} (\varrho b) = 0$$

in the sense of the integral identity

$$\int_{\mathbb{R}^d} [\Delta \varphi + \langle b, \nabla \varphi \rangle] \varrho \, dx = 0 \quad \forall \varphi \in C^\infty_0(\mathbb{R}^d).$$

Moreover, if $|b|$ is locally integrable to some power $p > d$ with respect to Lebesgue measure or with respect to $\mu$, then $\varrho \in W^{p,1}_{\text{loc}}(\mathbb{R}^d)$ (although $\varrho$ can fail to be in the second Sobolev class $W^{p,2}_{\text{loc}}(\mathbb{R}^d)$ unlike the case of non-divergence form equations). This is not true for $p < d$, but in case of the global integrability the following fact holds for $p = 2$ (see [7, 8], and [2]). Suppose that $\mu = \varrho \, dx$ is a probability measure on $\mathbb{R}^d$ satisfying equation (1.1), where, in addition,

$$\int_{\mathbb{R}^d} |b|^2 \, d\mu = \int_{\mathbb{R}^d} |b|^2 \varrho \, dx < \infty.$$

Then $\varrho \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ and the logarithmic gradient $\nabla \varrho / \varrho$ belongs to the weighted space $L^2(\varrho \, dx)$ and one has a dimension-free estimate

$$\left\| \frac{\nabla \varrho}{\varrho} \right\|^2_{L^2(\mu)} = \int_{\mathbb{R}^d} \left| \frac{\nabla \varrho}{\varrho} \right|^2 \varrho \, dx \leq \int_{\mathbb{R}^d} |b|^2 \varrho \, dx = \left\| |b| \right\|^2_{L^2(\mu)}.$$  

(1.3)

To be more precise, $\nabla \varrho / \varrho$ is the orthogonal projection of $b$ onto the closure of gradients of smooth compactly supported functions in the Hilbert space $L^2(\varrho \, dx, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued mappings. Note that a local version of this result fails: see [7, Example 1.6.10]. It is obvious that it is not valid for signed solutions.

There are also some sufficient conditions for membership of $|\nabla \varrho / \varrho|$ in $L^p(\varrho \, dx)$ with $p > 2$. However, such conditions are not of the same form as in case $p = 2$ and require additional assumptions such as a certain rate of convergence of $\langle b(x), x \rangle$ to $-\infty$ as $|x| \to +\infty$ (see [25], [5], [6], and [7]).

It is still unknown whether there are $L^p$-analogs of the above estimate for $p \neq 2$. One goal of this paper is to show that there is no such estimate for $p = 1$. We actually show that there is a sequence of smooth probability densities $\varrho_n$ satisfying the equations

$$\Delta \varrho_n - \text{div} (\varrho_n b_n) = 0$$

on $\mathbb{R}^2$ with smooth mappings $b_n$ such that

$$\int_{\mathbb{R}^2} |\nabla \varrho_n| \, dx \geq n \int_{\mathbb{R}^2} |b_n| \, \varrho_n \, dx.$$

We also show that there is a smooth probability solution $\varrho$ to the equation with smooth $b$ such that $|b| \varrho \in L^1(\mathbb{R}^2)$ and $|\nabla \varrho|$ is not in $L^1(\mathbb{R}^2)$, so that $\varrho \notin W^{1,1}(\mathbb{R}^2)$. 


We emphasize that the difficulty concerned probability solutions, there was no problem with signed solutions.

The assumption that $|b|$ is integrable with the weight $\varrho$ rather than with respect to Lebesgue measure is quite natural. For example, any probability measure with a density $\varrho \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ satisfies the equation $L^*\mu = 0$ with $b = \nabla \varrho/\varrho$, where we let $\nabla \varrho/\varrho := 0$ on the set $\{\varrho = 0\}$. Obviously, such $b$ can be very singular with respect to Lebesgue measure, but with weight $\varrho$ it is locally integrable, and if $\varrho \in W^{1,1}(\mathbb{R}^d)$, then $|b|\varrho \in L^1(\mathbb{R}^d)$.

The situation is similar with $L^1$-estimates with respect to Gaussian measures. It is known (see [8] or [2]) that if a Borel probability measure $\mu$ on $\mathbb{R}^d$ satisfies equation (1.1) with a drift $b$ such that

$$b(x) = -x + v(x),$$

where $x_i, |v| \in L^2(\mu)$ then $\mu$ has a density $f$ with respect to the standard Gaussian measure $\gamma$ on $\mathbb{R}^d$, this density is in $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, and the mapping $\nabla f/f$ is the orthogonal projection of $v$ to the closure of the gradients of smooth compactly supported functions taken in the Hilbert space $L^2(\mu, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued mappings. Hence

$$\int_{\mathbb{R}^d} |\nabla f|^2 f^2 d\mu = \int_{\mathbb{R}^d} |\nabla f|^2 f d\gamma \leq \int_{\mathbb{R}^d} |v|^2 d\mu.$$

Our next result says that there is no such estimate for the $L^1$-norm of $|\nabla f|$, namely, there is a sequence of smooth vector fields $v_n$ on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} |\nabla f_n| d\gamma \geq n \int_{\mathbb{R}^2} |v_n| f_n d\gamma.$$

It is instructive to consider the following formal manipulations. For locally Sobolev $\varrho$ one can write (1.2) as

$$\int_{\mathbb{R}^d} \langle \nabla \varrho, \nabla \varphi \rangle dx = \int_{\mathbb{R}^d} \langle \varrho b, \nabla \varphi \rangle dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d). \quad (1.4)$$

If we substitute $\varphi = \log \varrho$, then we obtain

$$\int_{\mathbb{R}^d} \frac{|\nabla \varrho|^2}{\varrho} dx = \int_{\mathbb{R}^d} \langle b, \nabla \varrho \rangle dx \leq \left( \int_{\mathbb{R}^d} \frac{|\nabla \varrho|^2}{\varrho} dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |b|^2 \varrho dx \right)^{1/2},$$

which yields (1.3). It turns out that this manipulation can be justified in case of $\mathbb{R}^d$ and in case of connected Riemannian manifolds with certain curvature conditions (see [9]), but not in case of arbitrary connected manifolds. The latter is indeed impossible even in case $b = 0$ because that would mean the absence of nonzero integrable nonnegative harmonic functions while examples of such functions are known from [13], [20], and [22]. Let now $\varphi$ be such that $\nabla \varphi = \nabla \varrho/|\nabla \varrho|$. This would give the bound

$$\int_{\mathbb{R}^d} |\nabla \varrho| dx \leq \int_{\mathbb{R}^d} |b| \varrho dx,$$

which, as we show below, is false even on $\mathbb{R}^2$. Certainly, both substitutions are illegal, but the first one leads to a correct conclusion. It would be interesting to find conditions under which the second one can be also justified.

Our construction is based on a thorough study and certain modification of the known old result of Ornstein [20], who showed that there are smooth functions $g_n$ with support in a square in $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} \left| \partial_x \partial_y g_n(x, y) \right| dx dy \geq n \int_{\mathbb{R}^2} \left[ |\partial_x^2 g_n(x, y)| + |\partial_y^2 g_n(x, y)| \right] dx dy.$$
This result shows that the $L^1$-norm of the mixed derivative is not controlled by the $L^1$-norm of the Laplacian. In particular, the Sobolev norm in the second class $W^{1,2}$ is not controlled by the $L^1$-norm of the Laplacian. The latter effect is much easier seen by example of radial functions, as noted, e.g., in [15] and [23]. One can show that the function $f$ that is log log $r$ near zero in $\mathbb{R}^2$ is not in the class $W^{1,2}$, but $\Delta f$ in the sense of distributions is given by the usual pointwise expression for the Laplacian outside of the origin and is integrable. However, for our purposes this elementary example (actually, any radial function) is not enough, as explained in Remark [2,4].

This is why we need a modification of Ornstein’s result with a Lipschitz function and some other bound (see (2.1)). Moreover, taking into account that in the original example in [26] some important technical details of justification are omitted, we have to reproduce the whole example from that paper with all details and verification of some additional properties. This is done in the last section. Moreover, we explain there how the desired modification can be derived from Ornstein’s result (but this reasoning does not provide the missing details in Ornstein’s construction).

Our positive result presented in Section 4 says that the Sobolev class $W^{1,1}$ to which solutions can fail to belong is actually the border line and that the integrability of $|b|$ with respect to the measure $\rho dx$ yields that $\rho$ belongs to fractional Sobolev classes of order of differentiability as close to 1 as we wish.

2. A modification of Ornstein’s example

Here we present a modification of Ornstein’s result that differs from his original result by extra terms in the inequality. These extra terms are needed in the case of the Fokker–Planck–Kolmogorov equation. Its justification is postponed until the last section, since it is rather involved technically, although the construction follows Ornstein’s method. In this and the last sections vectors in $\mathbb{R}^2$ are denoted by $(x, y)$ unlike the rest of the paper where single letters like $x$ are used to denote vectors.

**Theorem 2.1.** For each $\delta \in (0, 1)$ there is a function $g_\delta \in C_0^\infty([-1, 1]^2)$ such that

$$\|\partial_x \partial_y g_\delta\|_1 \geq \frac{1}{\delta} \left( \|\partial_x^2 g_\delta\|_1 + \|\partial_y^2 g_\delta\|_1 + \|\partial_x g_\delta\|_\infty + \|\partial_y g_\delta\|_\infty \right).$$

(2.1)

In addition, there is a Lipschitz (even of class $C^1$) function $f$ that vanishes outside of $[-1, 1]^2$ such that there exist repeated Sobolev derivatives with

$$\partial_x^2 f, \partial_y^2 f \in L^1, \text{ but } \partial_x \partial_y f \notin L^1,$$

where $\partial_x \partial_y$ is taken in the sense of distributions.

**Corollary 2.2.** There exist a probability density $\rho \in C^\infty(\mathbb{R}^2)$ and a $C^\infty$-mapping $v : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|v| \in L^1(\mathbb{R}^2)$ and $\Delta \rho - \text{div } v = 0$, but $|\nabla \rho|$ does not belong to $L^1(\mathbb{R}^2)$.

**Proof.** It is clear from the theorem that for every $n$ one can find a function $g_n \in C_0^\infty([-1, 1]^2)$ such that

$$\|\partial_x \partial_y g_n\|_1 = 1, \|\partial_x^2 g_n\|_1 + \|\partial_y^2 g_n\|_1 \leq 1/n, \|\partial_x g_n\|_\infty \leq 1/n, \|\partial_y g_n\|_\infty \leq 1/n.$$

For the function $f_n = \partial_y g_n$ we have $\|\partial_x f_n\|_1 = 1, \|f_n\|_\infty \leq 1/n, \text{ and } \Delta f_n = \text{div } v_n$, where $v_n = (0, \Delta g_n)$, so $\|v_n\|_1 \leq 1/n$. The function $f_n$ need not be nonnegative, but $|f_n| \leq 1/n$. We now consider $f_n$ on the square $[-2, 2]^2$ and find a bump function $u_n \in C_0^\infty([-2, 2]^2)$ such that $0 \leq u_n \leq 1/n, u_n = 1/n$ on $[-1, 1]^2$ and $|\nabla u_n| \leq 2/n$. The function $w_n = f_n + u_n \in C_0^\infty([-2, 2]^2)$ is nonnegative, bounded by $2/n$ and $\Delta w_n = \text{div } (v_n + \nabla u_n)$, where $\|v_n + \nabla u_n\|_1 \leq 3/n$. 

For the function $f_n = \partial_y g_n$ we have $\|\partial_x f_n\|_1 = 1, \|f_n\|_\infty \leq 1/n, \text{ and } \Delta f_n = \text{div } v_n$, 

where $v_n = (0, \Delta g_n)$, so $\|v_n\|_1 \leq 1/n$. The function $f_n$ need not be nonnegative, but $|f_n| \leq 1/n$. We now consider $f_n$ on the square $[-2, 2]^2$ and find a bump function $u_n \in C_0^\infty([-2, 2]^2)$ such that $0 \leq u_n \leq 1/n, u_n = 1/n$ on $[-1, 1]^2$ and $|\nabla u_n| \leq 2/n$. The function $w_n = f_n + u_n \in C_0^\infty([-2, 2]^2)$ is nonnegative, bounded by $2/n$ and $\Delta w_n = \text{div } (v_n + \nabla u_n)$, where $\|v_n + \nabla u_n\|_1 \leq 3/n$. 

By using shifts we can find such functions \( w_n \) with supports in disjoint squares. Then the function \( w = \sum_{n=1}^{\infty} n^{-1} w_n \) is infinitely differentiable, nonnegative, and \( \Delta w = \text{div} v \), where \( v = \sum_{n=1}^{\infty} n^{-1} (v_n + \nabla u_n) \), \( \|v\|_1 \leq 3 \sum_{n=1}^{\infty} n^{-2} < \infty \), but \( \partial_x w \) does not belong to \( L^1(\mathbb{R}^2) \). Finally, \( \|w\|_1 \leq 2 \sum_{n=1}^{\infty} n^{-2} \), so multiplying \( w \) by a constant we obtain a probability density.

\[ \square \]

**Remark 2.4.** Let us explain why we could not use a much simpler example of the function \( f(x, y) = \log \log r \) on \( \mathbb{R}^2 \) not belonging to the second Sobolev class on the unit disc and satisfying the equation \( \Delta f = g \) with \( g \) integrable near the origin and also leading to the equation \( \Delta \partial_x f = \text{div} (g, 0) \) whose solution is not in \( W^{1,1} \). The point is that we need a probability solution for the latter equation, but if \( f(x) = V(r) \) is an integrable radial function on the unit disc with integrable \( V''(r) \) near zero on the real line such that \( \Delta f \) is integrable near zero in the plane, then, recalling that \( \Delta f \) in polar coordinates is \( \Delta f = V''(r) + r^{-1} V'(r) \), we see that \( V''(r) \) must be integrable near zero in the plane (i.e., \( V''(r) r \) is integrable near zero on the real line). Hence all second order partial derivatives of \( f \) are integrable as well, so \( f \) is in the second Sobolev class.

3. **The Fokker–Planck–Kolmogorov equation and the Gaussian case**

We now apply the example described above to constructing some examples with the Fokker–Planck-Kolmogorov equation. Let us explain at once why such examples are impossible on the real line. The point is that in the one-dimensional case we have the equation \( \varphi'' - (\varphi b)' = 0 \), hence \( \varphi' - \varphi b = C \) for some constant \( C \). It follows that \( \varphi \) has a locally absolutely continuous version. Since \( \varphi b \) is integrable and \( \varphi \) cannot be separated from zero as \( |x| \to \infty \), the constant \( C \) must be zero, hence \( \varphi' \) is integrable as well.

**Theorem 3.1.** There exist a continuous probability density \( g \) with compact support and a Borel vector field \( b \) with compact support on \( \mathbb{R}^2 \) such that \( |b|g \in L^1(\mathbb{R}^2) \) and \( \Delta g - \text{div} (gb) = 0 \), but \( g \) does not belong to the Sobolev class \( W_{1,1}^{1} \).

There exist also a probability density \( g \in C^\infty(\mathbb{R}^2) \) and a \( C^\infty \)-vector field \( b \) such that \( \Delta g - \text{div} (gb) = 0 \) and \( |b|g \in L^1(\mathbb{R}^2) \), but \( |\nabla g| \) does not belong to \( L^1(\mathbb{R}^2) \).

**Proof.** We know that there is a continuous probability density \( w \) with compact support in \( \mathbb{R}^2 \) satisfying the equation \( \Delta w = \text{div} v \) with a Borel vector field \( v \) with compact support such that \( |v| \in L^1(\mathbb{R}^2) \) and \( w \) does not belong to \( W_{1,1}^{1}(\mathbb{R}^2) \). We now write the same equation as

\[ \Delta w = \text{div} (wb), \quad b := \frac{v}{w}, \]

where on the set \( \{w = 0\} \) we define \( b \) by the zero value. Obviously, \( |b| \in L^1(\text{w} \, dx) \), although now we can loose the Lebesgue integrability of \( b \), of course.

We now construct an example of a smooth probability density \( g \) satisfying the equation \( L^*_b (g \, dx) = 0 \) with smooth \( b \), but still not belonging to \( W_{1,1}^{1}(\mathbb{R}^2) \). To this end, we return to the examples of the previous section and using also Remark 2.3 find smooth nonnegative functions \( g_n \) with support exactly \( [0, 1]^2 \) and smooth vector fields \( v_n \) with support in \( [0, 1]^2 \) such that \( \Delta g_n = \text{div} v_n \), \( \|\nabla g_n\|_1 = n^{-1}, \|g_n\|_\infty \leq n^{-2}, \|v_n\|_1 \leq n^{-2} \). It is obvious from our construction that we can ensure the bound
\[ \| \nabla g_n \|_{L^1(D)} > (2n)^{-1} \] on the twice smaller square \( D \) with the same center. Next we cover the whole plane by squares of unit length with vertices at the integer points and slightly increase the obtained squares in order to produce overlapping squares \( B_n \) such that every point is contained in the interior of some of these larger squares. Now each \( B_n \) has intersections with eight other squares. Translating our functions \( g_n \) we can construct smooth nonnegative functions \( f_n \) with supports exactly \( B_n \) and vector fields \( u_n \) of class \( C_0^\infty(B_n) \) such that \( \Delta f_n = \text{div} \, u_n \), \( \| u_n \|_{L^1(D_n)} > (2n)^{-1} \), where \( D_n \) is the square of edge length 1/2 with the same center as \( B_n \), \( \| f_n \|_\infty \leq n^{-2} \), and \( \| u_n \|_1 \leq 2n^{-2} \).

The purpose of making \( B_n \) overlapping is that now the function \( f = \sum_{n=1}^\infty f_n \) is positive (simple translations of \( g_n \) would give a function vanishing on the edges). Clearly, this function is infinitely differentiable and satisfies the equation \( \Delta f = \text{div} \, u \) with \( u = \sum_{n=1}^\infty u_n \), where \( |u|_1 \leq \sum_{n=1}^\infty \| u_n \|_1 \leq 2 \sum_{n=1}^\infty n^{-2} \). It is also obvious that \( |\nabla f| \) is not integrable over the plane, since already the integral over the union of \( D_n \) diverges. Taking \( b = v/f \) as above, we obtain a smooth vector field with \( |b|f \in L^1(\mathbb{R}^2) \) such that \( \Delta f = \text{div} \, (fb) \). It remains to normalize \( f \) to obtain a probability density.

We now consider the connection between the two cases mentioned above, where densities are taken with respect to Lebesgue measure and with respect to the standard Gaussian measure \( \gamma \) on the plane with density \( \varrho_2(x) = (2\pi)^{-1} \exp(-x^2/2) \). Suppose that a probability measure \( \mu \) with a density \( \varrho \) satisfies the equation \( L^*_{\varrho} \mu = 0 \) with a drift \( b \). Let us set

\[ f(x) = \varrho(x)/\varrho_2(x). \]

Certainly, the same measure \( \mu = f \cdot \gamma \) satisfies the equation with the same drift written as \( -x + v(x) \), where \( v(x) := b(x) + x \). Therefore, once we use the aforementioned field \( b \) that coincides with \( -x \) outside of a compact set, we obtain \( v \) with compact support, so that its integrability with respect to Lebesgue measure is the same as the integrability with respect to the Gaussian measure.

**Theorem 3.2.** There exist a vector field \( v \) on \( \mathbb{R}^2 \) with compact support such that \( |v| \) is integrable with respect to Lebesgue measure, hence with respect to \( \gamma \), and a continuous probability density \( \varrho \) proportional to \( \varrho_2 \) outside of a ball such that the measure \( \mu \) with density \( \varrho \) satisfies the equation \( L^*_{\varrho} \mu = 0 \) with \( b(x) = -x + v(x) \), where \( |x|, |v| \in L^1(\mu) \), but \( \varrho \) does not belong \( W_{1,1}^1(\mathbb{R}^2) \).

**Proof.** Let us take the function \( w \geq 0 \) and the vector field \( v \) with compact support considered in the proof of Theorem 3.1, where \( \Delta w = \text{div} \, v \), \( |v| \in L^1(\mathbb{R}^2) \), and \( w \notin W_{1,1}^1(\mathbb{R}^2) \). We take the density \( w + \varrho_2 \), which satisfies the equation

\[ \Delta(w + \varrho_2) = \text{div} \, (v - x \varrho_2) = \text{div} \, ((w + \varrho_2)b) \]

with the drift

\[ b = \frac{v - x \varrho_2}{w + \varrho_2}, \]

which is locally Lebesgue integrable and \( b(x) = -x \) outside of the support of \( w \), so \( |b|(w + \varrho_2) \in L^1(\mathbb{R}^2) \). Again, \( w + \varrho_2 \notin W_{1,1}^1(\mathbb{R}^2) \).

It is worth noting that we have constructed above examples of two types in which solutions to Fokker–Planck–Kolmogorov equations have no Sobolev regularity. One example gives a density \( \varrho \) with compact support and a drift \( b \) with compact support such that \( |b| \varrho \) is integrable, but \( |b| \) is not locally Lebesgue integrable. The other one gives a positive density \( \varrho \) and a locally Lebesgue integrable drift \( b \) such that \( |b| \varrho \) is integrable on the plane. We have no examples in which the probability density \( \varrho \).
and the drift $b$ have compact support and $|b|$ and $|b|^2 \rho$ are both integrable. If in the two-dimensional case $|b|$ is locally integrable to power larger than 2, then $\rho$ not only belongs to $W^{2,1}_{\text{loc}}$, but also has a positive continuous version by Harnack’s inequality (see [4], [6] or [7]), so that it is impossible to make its support compact.

4. A positive result in the $L^1$-setting

Let us prove a positive result on fractional differentiability of solutions. Although this result actually follows from the facts presented in the recent book [7 Chapter 1], it is not explicitly formulated there for the whole space in case $p \leq d$. For the definition of the Sobolev space $H^{p,s}(\mathbb{R}^d)$, see [7 §1.8.1], [10] or [28]; for example, one can set

$$H^{p,s}(\mathbb{R}^d) = (I - \Delta)^{-s/2}(L^p(\mathbb{R}^d)),$$

where the operator $(I - \Delta)^{-s/2}$ is applied in the sense of distributions.

Theorem 4.1. (i) Suppose that $\mu$ is a bounded Borel measure on $\mathbb{R}^d$ satisfying the equation $L^s \mu = 0$ with $|b| \in L^1(\mu)$. Then $\mu$ has a density $\rho$ belonging to the fractional Sobolev class $H^{r,\alpha}(\mathbb{R}^d)$ for each $r > 1$ and $\alpha < 1 - d(r - 1)/r$, where $1 - d(r - 1)/r > 0$ whenever $1 < r < d/(d - 1)$. In particular, $\rho \in L^s(\mathbb{R}^d)$ for each exponent $s \in [1, d/(d - 1))$.

(ii) If $|b| \in L^p(\mu)$ with some $p \in (1, d]$, then $\rho \in W^{q,-1}(\mathbb{R}^d)$ for each exponent $q < d/(d + 1 - p)$, hence $\rho \in L^s(\mathbb{R}^d)$ for all $s < d/(d - p)$.

Proof. (i) We know that $\mu$ has a density $\varrho \in L^1(\mathbb{R}^d)$. Then

$$\Delta \varrho - \varrho = g + \text{div} F, \quad F = gb,$$

where $g := -\varrho \in L^1(\mathbb{R}^d)$, $|F| \in L^1(\mathbb{R}^d)$. By the Sobolev embedding theorem, the function $g$ and the components of $F$ belong to the negative Sobolev class $W^{s,-1}(\mathbb{R}^d)$ with any $s < d' = d/(d - 1)$. Therefore, $g, \text{div} F \in W^{s,-2}(\mathbb{R}^d)$, which yields that $\varrho \in L^s(\mathbb{R}^d)$. Moreover, we also have

$$\Delta \varrho - \varrho \in H^{r,-1-d(r-1)/r-\varepsilon}(\mathbb{R}^d) \quad \text{for all} \quad r > 1.$$

Therefore,

$$\varrho \in H^{r,1-d(r-1)/r-\varepsilon}(\mathbb{R}^d).$$

(ii) In case $p > 1$, by assertion (i) we have $\varrho \in L^s(\mathbb{R}^d)$ for any $s \in [1, d')$. Let us take $\delta > 0$ so small that

$$1 + \delta + \delta^2 < d', \quad 1 + \delta + \delta^2 + \delta^3 \leq p.$$

Set

$$r := 1 + \delta, \quad p_1 := 1 + \delta^2.$$

By Hölder’s inequality

$$\int_{\mathbb{R}^d} |b|^{p_1} \rho^{p_1} \, dx \leq \left( \int_{\mathbb{R}^d} |b|^{p_1 r} \rho \, dx \right)^{1/r} \left( \int_{\mathbb{R}^d} \rho^{1+(p_1-1)r/(r-1)} \, dx \right)^{r/(r-1)}.$$

By our choice $p_1 r \leq p$ and $1 + (p - 1)r/(r - 1) = 1 + \delta(1 + \delta) < d'$. Hence the right-hand side of the previous estimate is finite. This yields the inclusion $\varrho \in W^{p_1,1}(\mathbb{R}^d)$. Therefore, $\rho \in L^{p_1,d/(d-p_1)}(\mathbb{R}^d)$ by the Sobolev embedding theorem. Now the same reasoning with iterations as in [7 Theorem 1.8.2] in the local case shows that we can raise the order of the Sobolev class for $\rho$ as close to $d/(d + 1 - p)$ as we wish. □
In case $p > d$ we have $q \in W^{p,1}(\mathbb{R}^d)$ (see [7] Chapter 1), hence $q$ has a bounded continuous version.

However, even weaker assumptions are sufficient to increase the global order of integrability of $q$. Namely, suppose that $q$ is the density of a bounded measure $\mu$ satisfying the equation $L_\sigma^* \mu = 0$ with $b$ such that

$$
\sup_{a \in \mathbb{R}^d} \int_{U(a)} |b(x)| \omega(x) \, dx \leq M,
$$

where $U(a)$ is the ball of radius 1 centered at $a$. Then $q \in L^p(\mathbb{R}^d)$ for every exponent $p \in [1, d/(d - 1))$ and

$$
\|q\|_{L^p} \leq C(M, d, p)\|\omega\|_{L^1},
$$

where $C(M, d, p)$ is a number depending only on $d, p, M$. Indeed, we can assume that $\|\omega\|_{L^1} \leq 1$. It follows from the local estimates established in [7] \S 1.5 that there is a number $C_1(p, d, M)$ independent of $q$ such that

$$
\|q\|_{L^p(B(a))} \leq C_1(p, d, M)\|q\|_{L^1(U(a))}
$$

for every ball $B(a)$ of radius 1/2 centered at $a$. Since $\|q\|_{L^1(U(a))} \leq 1$, this yields the bound

$$
\|q\|^p_{L^p(B(a))} \leq C_1(p, d, M)^p \|q\|_{L^1(U(a))}.
$$

Hence the integral of $|q|^p$ over the whole space is estimated by $C(d)C_1(p, d, M)^d$ with some number $C(d)$ depending only on $d$.

Note that a local version of the previous theorem is proved in [3] (see also [7] Chapter 1) for nonconstant infinitely differentiable matrix $A$. The previous theorem can be also generalized to nonconstant $A$ provided that the second order elliptic operator $a^{ij}\partial_{x_i} \partial_{x_j} - 1$ has the same properties as the Laplacian in the scale of Sobolev spaces. For example, this is true if $A(x) = A_0 + A_1(x)$, where $A_0$ is a constant positive definite matrix and $A_1$ has entries in $C_0^\infty(\mathbb{R}^d)$ and $\|A\|_\infty$ is sufficiently small.

Once the density $q$ belongs to $H^{p,s}(\mathbb{R}^d)$, one can use known embedding theorems for fractional Sobolev spaces (see, e.g., [16]), in particular, there is a continuous embedding into $L^{p^*}(\mathbb{R}^d)$ with $p^* = dp/(d - sp)$ if $sp < d$. There are also fractional Hardy inequalities estimating integrals of functions like $|f(x)|^{q}/\text{dist}(x, \partial\Omega)^\alpha$ over a domain $\Omega$ via a suitable fractional Sobolev norm (see [17]).

For additional results on regularity of solutions in case of $A$ of low regularity, see the recent paper [10].

5. Justification of Theorem 2.1

Here we present a detailed justification of Theorem 2.1, which is needed not only because some details have been omitted in [26], but also because we need a bound with additional terms. So it does not come as a surprise that our justification is twice longer than in [26], although we essentially follow Ornstein’s construction and partly use the same notation. On the other hand, we show below how a similar result (which can be also used for our purposes) can be deduced from Ornstein’s example (if we do not intend to provide all details for the latter).

Proof of Theorem 2.1. For any function $\psi$, let us set $\psi_x = \frac{\partial \psi}{\partial x}$,

$$
\psi^x(x_0, y_0) = \int_{x_0}^{x_0} \psi(x, y_0) \, dx,
$$

and let $Var_x \psi(y_0)$ be the variation of the function $s \mapsto \psi(s, y_0)$ on $[-1, 1]$ for fixed $y_0$. So if $\psi$ is smooth (but actually at this stage we construct piecewise constant
functions), \( Var_x \psi(y_0) \) is the integral of \( |\partial_x \psi(s, y_0)| \) in \( s \) over \([-1, 1]\). Similarly we define \( \psi^y(x_0, y_0) \) and \( Var_y \psi(x_0) \).

Let \( \delta \in (0, 1) \) be a rational number. We shall find \( g_\delta \) in the form of the integral over \([-1, x] \times [-1, y]\) of a certain smoothing of a suitable function \( p_n \) described below. We construct a sequence of functions \( p_n(x, y) \) on \([-1, 1]^2\) with the following properties (all double integrals are taken over \([-1, 1]^2\)):

1. For each \( n \) there exist a partition of \([-1, 1]\) on the \( y \)-axis into intervals \( J_{n,i} = \{a_{n,i} \leq y \leq a_{n,i+1}\}\) and a partition of the interval \([-1, 1]\) on the \( x \)-axis into intervals \( L_{n,j} = \{b_{n,j} \leq x \leq b_{n,j+1}\}\). The function \( p_n \) is defined and constant on every open rectangle \((b_{n,j}, b_{n,j+1}) \times (a_{n,i}, a_{n,i+1})\), but is not defined on the boundaries of the rectangles.
2. \( p_n(x, y) = 0 \) outside of \([-1, 1]^2\).
3. \( \int_{-1}^1 p_n(x_0, y) \, dy = 0 \, \forall \, x_0 \in [-1, 1], \int_{-1}^1 p_n(x, y_0) \, dx = 0 \, \forall \, y_0 \in [-1, 1]. \)
4. \( \int_{-1}^1 |p_n(x, y)| \, dx \, dy > C_1 \delta n, \) where \( C_1 \) is a constant independent of \( n \) and \( \delta \).
5. \( \int_{-1}^1 Var_x p_n^x(y) \, dy < C_2 \delta \int_{-1}^1 |p_n(x, y)| \, dx \, dy, \) where \( C_2 \) is a constant independent of \( n \) and \( \delta \).
6. \( \int_{-1}^1 Var_y p_n^y(x) \, dx = C_3, \) where \( C_3 \) is a constant independent of \( n \) and \( \delta \). Note that here and in the previous item we would have \( \|\partial_y p_n^x\|_1 \) and \( \|\partial_x p_n^y\|_1 \) for smooth functions.
7. For every \( x_0 \in [-1, 1] \), the function \( p_n^x(x_0, y) \) is nonincreasing piecewise constant on \((a_{n,2}, a_{n,N_n-1})\), and whenever \( 2 < i < N_n \) one has

\[
|p_n^x(x_0, y_i) - p_n^x(x_0, y_{i-1})| \leq 2^{-(n-1)}, \quad y_i \in J_{n,i}, y_{i-1} \in J_{n,i-1}.
\]

8. \( |p_n^x(x, y)| \leq 1, \) \( |p_n^y(x, y)| \leq \delta(2 - 2^{-2(n-1)}) \) for all \( x, y \in [-1, 1] \).
9. \( p_n(x, y) = -p_n(x, -y) \).

We now define the function \( p_1 \):

\[
p_1 = 1 \quad \text{if} \quad (x, y) \in (-1/2, 0) \times (-\delta, -\delta/4) \cup (0, 1/2) \times (\delta/4, \delta),
\]

\[
p_1 = -1 \quad \text{if} \quad (x, y) \in (0, 1/2) \times (-\delta, -\delta/4) \cup (-1/2, 0) \times (\delta/4, \delta),
\]

and \( p_1 = 0 \) else.

Then

\[
\int_{-1}^1 |p_1(x, y)| \, dx \, dy = 3\delta/2,
\]

\[
\int_{-1}^1 Var_x p_1^x(y) \, dy \leq \tilde{C}_1 \delta^2;
\]

\[
\int_{-1}^1 Var_y p_1^y(x) \, dx =: C_3,
\]

where \( C_3 \) is independent of \( \delta \), because

\[
p_1^x(x, y) = 0 \quad \text{outside of} \quad [-1/2, 1/2] \times ([-\delta, -\delta/4] \cup [\delta/4, \delta])
\]

and \( p_1^y(x, y) = (1 - 2|x|) \text{sign} \, y \) else, so \( Var_y p_1^y(x) = 0 \) if \( x \in [-1, -1/2] \cup [1/2, 1] \), \( Var_y p_1^y(x) = -2|x| \) if \( x \in (-1/2, 1/2) \). Hence \( p_1 \) satisfies 1)–9).

Suppose that \( p_n \) is defined and show how to define \( p_{n+1} \). It suffices to define \( p_{n+1} \) for \( y < 0 \) and use 9) to extend to \( y > 0 \). For every \( i > 2 \) we take the interval \( J_{n,i} = (a_{n,i} - \alpha/2, a_{n,i} + \alpha/2) \) of length \( \alpha \) (where a rational number \( \alpha \) will be chosen
later). Outside of \((-1,1) \times (\cup_i \tilde{J}_{n,i})\) we let \(p_{n+1} = p_n\). On \((-1,1) \times \tilde{J}_{n,i}\) we define \(p_{n+1}\) as the sum of two functions \(r_1^i\) and \(r_2^i\) (we omit \(n\) in their notation), where

\[
r_1^i(x,y) = \frac{p_n(x,y_{i-1}) + p_n(x,y_i)}{2}, \quad y \in \tilde{J}_{n,i}, \quad y_{i-1} \in J_{n,i-1}, \quad y_i \in J_{n,i},
\]

outside of the strips \((-1,1) \times \tilde{J}_{n,i}\) we let \(r_1^i = 0\), and \(r_2^i\) is defined as follows. We partition the strip \((-1,1) \times \tilde{J}_{n,i}\) into rectangles \(K_k\) of height \(\alpha\) and width \(\alpha/\delta\) (again we suppress \(n\) in this notation). Next, each rectangle \(K_k\) is partitioned into four rectangles of height \(\alpha/2\) and width \(\alpha/(2\delta)\) each. Take \(\beta_k^i\) such that

\[
\frac{\alpha}{2\delta} \beta_k^i = \frac{p_n^x(x_k,y_{i-1}) - p_n^x(x_k,y_i)}{4}, \quad y_{i-1} \in J_{n,i-1}, \quad y_i \in J_{n,i},
\]

where \(x_k\) is the \(x\)-coordinate of the center of the rectangle \(K_k\). Define \(r_2^i\) to equal \(\beta_k^i\) on the lower left and upper right rectangle of \(K_k\), and let \(r_2^i\) equal \(-\beta_k^i\) on the remainder of \(K_k\). Outside of the strips \((-1,1) \times \tilde{J}_{n,i}\) we let \(r_2^i = 0\).

Let us verify that \(p_{n+1}\) satisfies 1)–9). Properties 1), 2), 9) are obvious. Property 3) follows from 9) and the fact that

\[
\int_{-1}^{1} r_1^i(x,y_0) \, dx = 0, \quad \int_{-1}^{1} r_2^i(x,y_0) \, dx = 0 \quad \forall \, y_0 \in [-1,1].
\]

Let us prove that \(p_{n+1}\) satisfies 7). It suffices to show that for all \(i > 2\) and all \(x\)

\[
p_{n+1}^x(x,y_{i-1}) - p_{n+1}^x(x,y'), p_{n+1}^x(x,y') - p_{n+1}^x(x,y''), p_{n+1}^x(x,y'') - p_{n+1}^x(x,y_i)
\]

\[
\in \left[0, \frac{p_n^x(x,y_i) - p_n^x(x,y_{i-1})}{2}\right],
\]

where \(y_{i-1} \in (a_{n,i-1},a_{n,i} - \alpha/2), \, y' \in (a_{n,i} - \alpha/2,a_{n,i}), \, y'' \in (a_{n,i},a_{n,i} + \alpha/2), \, y_i \in (a_{n,i} + \alpha/2,a_{n,i+1})\). Since for every \(y_0\) the function \(p_{n+1}^x(x,y_0)\) is linear on the intervals \([0,\alpha/(2\delta)], [\alpha/(2\delta),\alpha/\delta], \ldots\), it suffices to verify (5.1) for the endpoints \(x \in \{\alpha m/(2\delta)\} : m \in \mathbb{N}\). First we consider the endpoints of the form \(x = \alpha m/\delta, \, m \in \mathbb{N}\). In this case in our calculation of \(p_{n+1}^x\) the \(\beta\)-terms mutually cancel, hence

\[
p_{n+1}^x(x,y') = p_{n+1}^x(x,y'') = \frac{p_n^x(x,y_i) + p_n^x(x,y_{i-1})}{2},
\]

so (5.1) is fulfilled. The remaining endpoints are the \(x\)-coordinates \(x_k\) of the centers of the rectangles \(K_k\) (i.e., equal \(\alpha m/(2\delta)\) with odd \(m\)). For them we have

\[
p_{n+1}^x(x_k,y') = \frac{p_n^x(x_k,y_i) + p_n^x(x_k,y_{i-1})}{2} + \frac{\alpha}{2\delta} \beta_k^i,
\]

\[
p_{n+1}^x(x_k,y'') = \frac{p_n^x(x_k,y_i) + p_n^x(x_k,y_{i-1})}{2} - \frac{\alpha}{2\delta} \beta_k^i,
\]

and the definition of \(\beta_k^i\) yields that

\[
p_{n+1}^x(x_k,y_{i-1}) - p_{n+1}^x(x_k,y') = \frac{p_n^x(x_k,y_{i-1}) - p_n^x(x_k,y_i)}{4},
\]

\[
p_{n+1}^x(x_k,y') - p_{n+1}^x(x_k,y'') = \frac{p_n^x(x_k,y_{i-1}) - p_n^x(x_k,y_i)}{2},
\]

\[
p_{n+1}^x(x_k,y'') - p_{n+1}^x(x_k,y_i) = \frac{p_n^x(x_k,y_{i-1}) - p_n^x(x_k,y_i)}{4}.
\]

Hence 7) is fulfilled.

We observe that 7) obviously yields 6): since \(p_{n+1}^x(x,y) = p_1^x(x,y)\) whenever \((x,y) \in (0,1) \times (a_{n,2},a_{n,N_n})\) and \(p_{n+1}^x(x,y)\) is a nonincreasing function of \(y\) on the interval \((a_{n,2},a_{n,N_n})\), one has \(\text{Var}_y p_{n+1}^x(x) = \text{Var}_y p_1^x(x)\), whence we obtain 6).
Let us show that \( p_{n+1} \) satisfies 8). Since \( p_{n+1}^x(x, y) = p_1^x(x, y) \) whenever \((x, y) \in (0, 1) \times (a_n, a_{n,N_n}) \) and \( p_{n+1}^x(x, y) \) is a nonincreasing function of \( y \) on \((a_n, a_{n,N_n-1})\), one has
\[
\max_{x,y} |p_{n+1}^x(x, y)| \leq \max_{x,y} |p_1^x(x, y)| \leq 1.
\]

Let us estimate \( \max_{x,y} |p_{n+1}^y(x, y)| \). If \( y \notin J_{n,i} \), then \( p_{n+1}^y(x, y) = p_n^y(x, y) \), since
\[
\int_{J_{n,i}} [r_1^i(x, y) + r_2^i(x, y)] dy = \int_{\tilde{J}_{n,i}} p_n(x, y) dy.
\]

If \( y \in \tilde{J}_{n,i} \), then
\[
|p_{n+1}^y(x, y)| \leq \frac{\alpha}{2} \max_k \beta_k^i + \frac{\alpha}{2} \max_k \beta_k^i
\]
\[
= |p_n^y(x, y)| + \frac{\delta}{2} \max_k (p_n^x(x, y_i) - p_n^x(x_k, y_i))
\]
\[
\leq \delta(2 - 2^{-(n-1)}) + \delta 2^{-n} = \delta(2 - 2^n),
\]
where the last inequality follows from the fact that \( p_n \) satisfies 8) and 7).

Let us show that \( p_{n+1} \) satisfies 5) for sufficiently small \( \alpha \). We observe that for every \( i \) we have
\[
\int_{-1}^1 Var_x(r_2^i(y)) dy = \tilde{C}_2 \alpha^2 \beta_k^i,
\]
where \( \tilde{C}_2 \) is a constant \( \text{independent of } n \) and \( \delta \) and
\[
\int \|r_2^i\| dx dy = \frac{\alpha^2}{\delta} \sum_k \beta_k^i.
\]

Hence
\[
\int_{-1}^1 Var_x(r_2^i(y)) dy = \tilde{C}_2 \int \|r_2^i\| dx dy.
\]

Note that
\[
\lim_{\alpha \to 0} \int_{-1}^1 Var_x(r_2^i(y)) dy = 0, \quad \lim_{\alpha \to 0} \int \|r_1\| dx dy = 0, \quad \text{where} \quad r_1 = \sum_i r_1^i.
\]

Therefore, for sufficiently small \( \alpha \) there holds the inequality
\[
\int_{\cup J_{n,i}} Var_x p_{n+1}^y(y) dy < ((\tilde{C}_2 + 1)\delta \int_{\cup J_{n,i}} \int_{-1}^1 |p_{n+1}(x, y)| dx dy.
\]

We have
\[
Var_x p_{n+1}^y(y) = Var_x p_n^y(y) \quad \text{if} \quad y \notin J_{n,i},
\]
because \( p_{n+1}^y(x, y) = p_n^y(x, y) \) if \( y \notin \tilde{J}_{n,i} \). Since \( p_n \) satisfies 5), one has
\[
\int_{-1}^1 Var_x p_{n+1}^y(x, y) dy < \max(C_2, \tilde{C}_2 + 1)\delta \int \|p_{n+1}(x, y)\| dx dy,
\]
whence we obtain that \( p_{n+1} \) satisfies 5) (from the very beginning we take \( C_2 > \tilde{C}_2 + 1 \), which is possible, since \( \tilde{C}_2 \) is a universal constant independent of \( n \) and \( \delta \)).

Let us show that \( p_{n+1} \) satisfies 4) for sufficiently small \( \alpha \). We have
\[
\lim_{\alpha \to 0} \int \int |p_{n+1}| dx dy = \int \int |p_n| dx dy + \lim_{\alpha \to 0} \sum_i \int_{\tilde{J}_{n,i}} \int_{-1}^1 |r_1^i(x, y)| dx dy,
\]
\[
\lim_{\alpha \to 0} \int_{J_{\alpha}} \int_{-1}^{1} |\varphi_{\alpha}^{2}(x, y)| \, dx \, dy = \lim_{\alpha \to 0} \frac{\alpha^2}{\delta} \sum_{k} \beta_{k}^{3} = \frac{\alpha}{2} \sum_{k} \left(p_{n}(x_{k}, y_{i-1}) - p_{n}(x_{k}, y_{i})\right) = \frac{\delta}{2} \int_{-1}^{1} \left(p_{n}(x, y_{i-1}) - p_{n}(x, y_{i})\right) \, dx, \quad \text{where} \quad y_{i-1} \in J_{n,i-1}, y_{i} \in J_{n,i}.
\]

The last equality is just the limit of the Riemann sums with partitions of length \(\alpha/\delta\). Hence

\[
\lim_{\alpha \to 0} \int_{-1}^{1} \int |p_{n+1}| \, dx \, dy = \int_{-1}^{1} |p_{n}| \, dx \, dy + \frac{1}{2} \delta \int_{-1}^{1} Var_{y}p_{n}(x) \, dx > C_{1}\delta n + \frac{C_{3}}{2},
\]

since \(p_{n}\) satisfies 4) and

\[
\int_{-1}^{1} Var_{y}p_{n}(x) \, dx = C_{3}.
\]

We now take \(C_{1} < \min(C_{3}/2, 3/2)\).

For \(p_{n}\) we have

\[
\int_{-1}^{1} Var_{x}p_{n}^{y}(y) \, dy + \int_{-1}^{1} Var_{y}p_{n}^{x}(x) \, dx + \|p_{n}^{x}\|_{\infty} + \|p_{n}^{y}\|_{\infty} < C_{2}\delta \|p_{n}\|_{1} + C_{3} + 1 + 2\delta,
\]

\[
\|p_{n}\|_{1} > C_{1}\delta n.
\]

Hence for sufficiently large \(n\) we obtain

\[
\int_{-1}^{1} Var_{x}p_{n}^{y}(y) \, dy + \int_{-1}^{1} Var_{y}p_{n}^{x}(x) \, dx + \|p_{n}^{x}\|_{\infty} + \|p_{n}^{y}\|_{\infty} < C_{2}\delta \|p_{n}\|_{1}. \tag{5.2}
\]

For each \(n\) we can smooth \(p_{n}\) in the variable \(x\) as follows. Let \(\varrho \in C_{\infty}(\mathbb{R})\) be a probability density with support in \([-1, 1]\). Let

\[
q_{n}(x, y) = \int_{-1}^{1} p_{n}(x - t, y) \varrho_{\varepsilon}(t) \, dt, \quad \text{where} \quad \varrho_{\varepsilon}(t) = \frac{1}{\varepsilon} \varrho\left(\frac{t}{\varepsilon}\right), \quad \varepsilon > 0.
\]

We do not indicate dependence of \(q_{n}\) on \(\varepsilon\) that will be taken sufficiently small. Then \(\|q_{n}\|_{1} \to \|p_{n}\|_{1}\) as \(\varepsilon \to 0\). The functions \(q_{n}^{x}\) and \(q_{n}^{y}\) satisfy the equalities

\[
q_{n}^{x}(x, y) = \int_{-1}^{1} p_{n}(x - t, y) \varrho_{\varepsilon}(t) \, dt, \quad q_{n}^{y}(x, y) = \int_{-1}^{1} p_{n}(x - t, y) \varrho_{\varepsilon}(t) \, dt.
\]

It follows that

\[
\|q_{n}^{x}\|_{\infty} \leq \|p_{n}^{x}\|_{\infty}, \quad \|q_{n}^{y}\|_{\infty} \leq \|p_{n}^{y}\|_{\infty}.
\]

Let us estimate

\[
\int_{-1}^{1} Var_{x}q_{n}^{y}(y) \, dy \quad \text{and} \quad \int_{-1}^{1} Var_{y}q_{n}^{x}(x) \, dx
\]

from above. We have

\[
Var_{x}q_{n}^{y}(y) = \sup \left\{ \sum_{i} \left| q_{n}^{y}(c_{i}, y) - q_{n}^{y}(c_{i-1}, y) \right| : -1 \leq c_{1} \leq \cdots \leq c_{n} \leq 1 \right\}
\]

\[
= \sup \left\{ \sum_{i} \left| \int_{-1}^{1} p_{n}^{y}(c_{i} - t, y) - p_{n}^{y}(c_{i-1} - t, y) \varrho_{\varepsilon}(t) \, dt \right| \right\}
\]

\[
\leq \sup \left\{ \int_{-1}^{1} \sum_{i} \left| p_{n}^{y}(c_{i} - t, y) - p_{n}^{y}(c_{i-1} - t, y) \varrho_{\varepsilon}(t) \, dt \right| \right\} \leq Var_{x}p_{n}^{y}(y).
\]
Therefore,
\[ \int_{-1}^{1} Var_x q_n^y(y) \, dy \leq \int_{-1}^{1} Var_x p_n^y(y) \, dy, \]

\[ Var_y q_n^x(x) = \sup \left\{ \sum_i |q_n^x(x, c_i) - q_n^x(x, c_{i-1})| : -1 \leq c_1 \leq \cdots \leq c_n \leq 1 \right\} \]

= \sup \left\{ \sum_i \left| \int_{-1}^{1} p_n^x(x-t, c_i) - p_n^x(x-t, c_{i-1}) \varphi(t) \, dt \right| \right\}

\leq \sup \left\{ \int_{-1}^{1} \sum_i |p_n^x(x-t, c_i) - p_n^x(x-t, c_{i-1})| \varphi(t) \, dt \right\} \leq \int_{-1}^{1} Var_y p_n^x(x-t) \varphi(t) \, dt, \]

which yields that
\[ \int_{-1}^{1} Var_y q_n^x(x) \, dx \leq \int_{-1}^{1} Var_y p_n^x(x) \, dx. \]

Therefore, for sufficiently small \( \varepsilon \), for \( q_n \) we have inequality (5.2).

Let us show that \( q_n \) has property 3) from the list for \( p_n \), i.e., we have to show that \( q_n^y(x, 1) = 0 \) for all \( x \) and \( q_n^x(1, y) = 0 \) for all \( y \). This is needed in order to ensure that \( q_n^x \) and \( q_n^y \) vanish outside of \([-1, 1]^2\). The equality \( q_n^y(x, 1) = 0 \) follows from property 3) for \( p_n \) and the fact that \( p_n^y(x, 1) = 0 \) for all \( x \). In addition,

\[ q_n^x(1, y) = \int_{-1}^{1} q_n(x, y) \, dx = \int \int p_n(t, y) \varphi(x-t) \, dt \, dx \]

\[ = \int_{-1}^{1} p_n(t, y) \int_{-1}^{1} \varphi(x-t) \, dx \, dt = \int_{-1}^{1} p_n(t, y) \, dt = 0. \]

Similarly, smoothing the constructed function in the variable \( y \), we obtain a function of class \( C_0^\infty \), again denoted by \( q_n \), satisfying 3) from the list for \( p_n \) and inequality (5.2).

Let \( g_\delta = (q_n^x)^\prime \). Then \( g_\delta \in C_0^\infty \) and

\[ \| \partial_x^2 g_\delta \|_1 + \| \partial_y^2 g_\delta \|_1 + \| \partial_x g_\delta \|_\infty + \| \partial_y g_\delta \|_\infty = \| (q_n^y)_{x} \|_1 + \| (q_n^x)_{y} \|_1 + \| q_n^x \|_\infty + \| q_n^y \|_\infty \]

\[ = \int_{-1}^{1} Var_x q_n^y(y) \, dy + \int_{-1}^{1} Var_y q_n^x(x) \, dx + \| q_n^x \|_\infty + \| q_n^y \|_\infty, \]

because

\[ Var_x q_n^y(y) = \int_{-1}^{1} |(q_n^y)_{x}(x, y)| \, dx, \]

\[ Var_y q_n^x(x) = \int_{-1}^{1} |(q_n^x)_{y}(x, y)| \, dy. \]

Hence

\[ \| \partial_x \partial_y g_\delta \|_1 > \frac{1}{C_2\delta} \left( \| \partial_x^2 g_\delta \|_1 + \| \partial_y^2 g_\delta \|_1 + \| \partial_x g_\delta \|_\infty + \| \partial_y g_\delta \|_\infty \right), \]

which completes the justification of the first claim. Now the second one follows by the closed graph theorem. Indeed, if there is no function with the desired properties, then we obtain a linear operator \( T \) from the space \( E \) of Lipschitz functions \( f \) on the square \([-1, 1]^2\) vanishing on the boundary and having Sobolev repeated derivatives \( \partial_x^2 f \) and \( \partial_y^2 f \) in \( L^1([-1, 1]^2) \) to the space \( L^1([-1, 1]^2) \) defined by \( Tf = \partial_x \partial_y f \), where \( \partial_x \partial_y \) is taken in the sense of distributions. The space \( E \) is Banach with respect to the natural norm

\[ \| f \|_E = \| f \|_{\text{Lip}} + \| \partial_x^2 f \|_1 + \| \partial_y^2 f \|_1. \]
where the Lipschitz norm \(\|f\|_{\text{Lip}}\) is defined by
\[
\|f\|_{\text{Lip}} = \max_{[-1,1]^2} |f(x, y)| + L(f),
\]
and \(L(f)\) is the minimal Lipschitz constant for \(f\). The graph of the operator \(T\) is closed, which is seen, for example, from the fact that \(T\) is continuous on \(E\) with values in the space of distributions (or in the negative Sobolev space \(W^{2,-2}([-1,1]^2)\)) and \(L^1([-1,1]^2)\) is continuously embedded into the space of distributions (respectively, into \(W^{2,-2}([-1,1]^2)\)). Similarly one can obtain a function \(f\) of class \(C^1\); in the definition of \(E\) we replace the class of Lipschitz functions by the space \(C^1([-1,1]^2)\) with its natural norm. \(\square\)

We now show how a similar result can be deduced from Ornstein’s example. We are grateful to A.V. Shaposhnikov for suggesting the following lemma.

**Lemma 5.1.** Let \(B = B(0,1)\) be the open unit ball in \(\mathbb{R}^2\). There is no number \(C\) such that for every function \(f \in C_0^\infty(B)\) one has
\[
\|\partial^2_x f\|_1 \leq C(\|f\|_\infty + \|\nabla f\|_\infty + \|\Delta f\|_1).
\]

**Proof.** Let us assume that such \(C\) exists. Then for all \(f \in C_0^\infty(B)\) we have
\[
\|\partial^2_x f\|_1 \leq C(\|f\|_\infty + \|\nabla f\|_\infty + \|\Delta f\|_1).
\] (5.3)

Let us fix \(f \in C_0^\infty(B)\). For any point \(P = (P_x, P_y) \in B(0,1)\) and any number \(N \in \mathbb{N}\) we define \(g_{P,N}\) as follows:
\[
g_{P,N}(x,y) := f(N(x - P_x), N(y - P_y)).
\]

It is easy to see that
\[
\text{supp}(g_{P,N}) \subset B(P,1/N),
\]
\[
\|g_{P,N}\|_\infty = \|f\|_\infty, \quad \|\nabla g_{P,N}\|_\infty = N\|f\|_\infty,
\]
\[
\|\partial_x^2 g_{P,N}\|_1 = \|\partial_x^2 f\|_1, \quad \|\Delta g_{P,N}\|_1 = \|\Delta f\|_1.
\]

Let us take \(M = N^2/100\) disjoint balls \(\{B(P,1/N)\}_{i=1}^M\) in \(B(0,1)\). Let us define \(g_N\) by the following formula:
\[
g_N := \sum_{i=1}^M g_{P_i,N}.
\]

Then
\[
\|g_N\|_\infty = \|f\|_\infty, \quad \|\nabla g_N\|_\infty = N\|f\|_\infty,
\]
\[
\|\partial_x^2 g_N\|_1 = M\|\partial_x^2 f\|_1, \quad \|\Delta g_N\|_1 = M\|\Delta f\|_1.
\]

Next we apply (5.3) to the function \(g_N\):
\[
M\|\partial_x^2 f\|_1 \leq C(\|f\|_\infty + N\|\nabla f\|_\infty + M\|\Delta f\|_1),
\]
which is
\[
\|\partial_x^2 f\|_1 \leq C\left(\frac{1}{M}\|f\|_\infty + \frac{N}{M}\|\nabla f\|_\infty + \|\Delta f\|_1\right).
\]

Letting \(N \to \infty\) we obtain
\[
\|\partial_x^2 f\|_1 \leq C\|\Delta f\|_1.
\]

Now it is easy to see that since \(f\) was an arbitrary function in \(C_0^\infty(B)\), this inequality holds for every function \(f \in C_0^\infty(\mathbb{R}^2)\). This contradicts the result of Ornstein. \(\square\)

We now prove an analog of Theorem 2.1 (with the repeated derivative in place of the mixed derivative).
Theorem 5.2. Let $B = B(0, 1)$ be the open unit ball in $\mathbb{R}^2$. There exists a Lipschitz function $f$ on $B$ such that

$$\Delta f \in L^1(B), \; \partial^2_x f \notin L^1(B).$$

Proof. Let $X$ be the completion of $C_0^\infty(B)$ with respect to the norm

$$\|f\|_X := \|f\|_\infty + \|\nabla f\|_\infty + \|\Delta f\|_1.$$ 

Let us assume that for each $f \in X$ we have $\partial^2_x f \in L^1(B)$, where $\partial^2_x f$ is understood in the sense of distributions. Then by the closed graph theorem the operator $f \mapsto \partial^2_x f$ from $X$ to $L^1(B)$ is bounded. This contradicts the previous lemma. \hfill $\square$

This result can be used in our main construction.

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