Approaches to general field theory
(The method of skew-symmetric differential forms)

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The basis for the field theory are properties of the closed exterior differential forms (skew-symmetric differential forms defined on manifolds with the closed metric forms), which reflect properties of the conservation laws for physical fields. It is possible to classify physical fields and interactions. So, the (0-form) corresponds to the strong interaction, the (1-form) corresponds to the weak interaction, the (2-form) corresponds to the electromagnetic interaction, and the (3-form) corresponds to the gravitational interaction. This is the basis of unified field theory.

As a general field theory it can be a theory, which not only describes possible physical fields and relation between them, but also discloses a mechanism of forming physical fields and the causality of such processes. It occurs that as the basis of such a theory it can become the theory of skew-symmetric differential forms defined on manifolds with unclosed metric forms. These differential forms, which were named the evolutionary ones, reflect the properties of the conservation laws for material media (the balance conservation laws for energy, linear and angular momentum, and mass) and disclose a mechanism of the evolutionary processes in material media. It is in such processes the physical structures that form physical fields originate. The theory of exterior and evolutionary skew-symmetric differential forms discloses the causality of physical processes, establishes a relation between physical fields and material media and allows to introduce a classification of physical fields and interactions.

1 The role of exterior differential forms in invariant field theories

The analysis of operators and equations of existing invariant field theories shows that the mathematical principles of the theory of closed exterior differential forms lie at the basis of existing field theories.

A connection of field theory with the exterior differential forms is explained by the fact that the closed exterior differential forms describe physical structures, which constitute physical fields. This is connected with the conservation laws [1-3].

Properties of closed exterior differential forms, which reflect properties of the conservation laws

The exterior differential form of degree \(p\) (\(p\)-form) can be written as [4-6]

\[
\theta^p = \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \quad 0 \leq p \leq n
\]  

(1.1)
Here $a_{i_1 \ldots i_p}$ are the functions of the variables $x^{i_1}, x^{i_2}, \ldots, x^{i_p}$, $n$ is the dimension of space, $\wedge$ is the operator of exterior multiplication, $dx^i$, $dx^i \wedge dx^j$, $dx^i \wedge dx^j \wedge dx^k$, $\ldots$ is the local basis which satisfies the condition of exterior multiplication:

\[
\begin{align*}
    dx^i \wedge dx^i &= 0 \\
    dx^i \wedge dx^j &= -dx^j \wedge dx^i & \text{if } i \neq j
\end{align*}
\] (1.2)

[From here on the symbol $\sum$ will be omitted and it will be implied that the summation is performed over double subscripts. Besides, the symbol of exterior multiplication will be also omitted for the sake of presentation convenience].

The differential of the (exterior) form $\theta^p$ is expressed as

\[
d\theta^p = \sum_{i_1 \ldots i_p} da_{i_1 \ldots i_p} dx^{i_1} dx^{i_2} \ldots dx^{i_p}
\] (1.3)

From a definition of differential one can see that, firstly, the differential of the exterior form is also the exterior form (but with the degree $(p+1)$), and, secondly, he can see that the components of the differential form commutator are the coefficients of the form differential. Thus, the differential of the first-degree form $\omega = a_i dx^i$ can be written as $d\omega = K_{ij} dx^i dx^j$ where $K_{ij}$ are the components of the commutator for the form $\omega$ that are defined as $K_{ij} = (\partial a_j / \partial x^i - \partial a_i / \partial x^j)$.

The theory of exterior differential forms was developed just for differentiable manifolds and manifolds with structures of any types. They may be the Hausdorff manifolds, fiber spaces, the comological, characteristic, configuration manifolds, and so on. These manifolds and their properties are treated in [6-8] and in some other works. Since all these manifolds possess structures of any types, they have one common property, namely, locally they admit one-to-one mapping into the Euclidean subspaces and into other manifolds or submanifolds of the same dimension [8]. What they have in common is that the metric forms of such manifolds are closed. Below we will consider differential forms, which are defined on manifolds with metric forms that are unclosed. Differential of such forms will include an additional term that contains a differential of unclosed metric form. Such skew-symmetric differential forms, which were named the evolutionary ones, possess new unique possibilities that disclose properties of field theories.

If $\theta^p$ be the exterior differential form of degree $p$ ($p$-form), the closure conditions of the exterior differential form (vanishing the form differential) can be written as

\[
d\theta^p = 0
\] (1.4)

From this equation one can see that the closed form is a conservative quantity. This means that it can correspond to the conservation law, namely, to some conservative physical quantity.

In relation (1.4) the exterior differential form is an exact one. If the exterior differential form is closed only on pseudostructure, that is, this form is a closed inexact differential form, the closure condition is written as

\[
d_* \theta^p = 0
\] (1.5)
And the pseudostructure $\pi$ obeys the condition

$$d_\pi^* \theta^p = 0 \quad (1.6)$$

where $^*\theta^p$ is the dual form.

From conditions (1.5) and (1.6) one can see that the exterior differential form closed on pseudostructure is a conservative object, namely, this quantity conserves on pseudostructure. This can also correspond to some conservation law, i.e. to conservative object.

The closure conditions for the exterior differential form ($d_\pi \theta^p = 0$) and the dual form ($d_\pi^* \theta^p = 0$) are mathematical expressions of the conservation law. Such conservation laws that state the existence of conservative physical quantities or objects can be named the exact ones.

The pseudostructure and the closed exterior form defined on the pseudostructure make up a binary differential and geometrical structure. Such a binary object can be named a Bi-Structure. (This is an example of the differential and geometrical structure (G-Structure).) It is evident that such a structure does correspond to the conservation law.

The physical structures, from which physical fields are formed, are precisely structures that correspond to the exact conservation law.

Relations that define the physical structures ($d_\pi \theta^p = 0, \ d_\pi^* \theta^p = 0$) turn out to be coincident with the mathematical expression for the exact conservation law.

The mathematical expression for the exact conservation law and its connection with physical fields can be schematically written in the following way

$$\left\{ \begin{array}{l}
\ d_\pi \theta^p = 0 \\
\ d_\pi^* \theta^p = 0
\end{array} \right. \implies \left\{ \begin{array}{l}
\ ^*\theta^p = d\theta^{p-1}
\end{array} \right. \text{physical structures} \implies \text{physical fields}$$

It is obvious that the exact conservation law is that for physical fields.

**Characteristic properties of the closed exterior forms and their relation to properties of existing field theories**

Since the relations for exact conservation laws and corresponding physical structures (which form physical fields) are expressed in terms of closed and dual forms, it is obvious that at the basis of all existing field theories (which describe physical fields) there lie properties of the closed exterior differential and dual forms. The properties and the mathematical apparatus of exterior differential forms allow to disclose specific features peculiar to all existing field theories.

1) **Invariance of closed exterior forms**

From the closure condition of exterior differential it follows a property of exterior differential forms, which has a physical meaning, namely, any closed exterior form is a differential of the form of lower degree: the total one if the form is exact

$$\theta^p = d\theta^{p-1} \quad (1.7)$$
or the interior one on pseudostructure if the form is inexact

\[ \theta^p = d_x \theta^{p-1} \]  

(1.8)

Since the closed exterior form is a differential then it is obvious that the closed form proves to be invariant under all transformations that conserve the differential. The unitary transformations (0-form), the tangent and canonical transformations (1-form), the gradient and gauge transformations (2-form) and so on are examples of such transformations. These are gauge transformations for spinor, scalar, vector, tensor (3-form) fields.

It is well known that these are transformations typical for existing field theories. The equations of existing field theories remain invariant under such transformations.

At this point it should be emphasized that the relation between the closed exterior form and the form of lower degree shows that the form of lower degree can correspond to the potential, and the closed form by itself can correspond to the potential force.

2) Conjugacy of the closed exterior forms

The closure of the exterior differential forms and hence their invariance result from the conjugacy of elements of the exterior or dual forms. On the other hand, the concept of conjugacy may imply something that leads to the closure of exterior or dual forms, obeys the closure condition, or establishes a relation between closed forms.

From the definition of the exterior differential form one can see that the exterior differential forms have complex structure. The specific features of the exterior form structure are a homogeneity with respect to the basis, skew-symmetry, the integration of terms each consisting of two objects of different nature (the algebraic nature for the form coefficients, and the geometric nature for the base components). Besides, the exterior form depends on the space dimension and on the manifold topology. The closure property of the exterior form means that any objects, namely, elements of the exterior form, components of elements, elements of the form differential, exterior and dual forms, and others, turn out to be conjugated. The variety of objects of conjugacy leads to the fact that the closed forms can describe a great number of different physical and spatial structures. It is the conjugacy that leads to realization of the invariant and covariant properties of the exterior and dual forms. These properties of exterior differential forms lie just at the basis of field theories.

2) Identical relations of the closed exterior forms

Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that relations can be used to express conjugacy mathematically. Just such relations, which are the identical one, constitute the basis of the mathematical apparatus of the exterior differential forms.

The identical relations for exterior differential forms reflect the closure conditions of the differential forms, namely, vanishing the form differential (see formulas (1.4), (1.5), (1.6)) and the conditions connecting the forms of consequent degrees (see formulas (1.7), (1.8)). Hence they are a mathematical expression of
the conservation laws (which correspond to physical structures forming physical fields) and a mathematical expression of the invariance and covariance. And this lies at the basis of existing field theories.

One can assure himself that all existing field theories contain the identical relations, which are the identical relations of the exterior differential forms, or their differential or integral representations.

Examples of such relations are canonical relations in the Schrödinger equations, gauge invariance in electromagnetic theory, commutator relations in the Heisenberg theory, symmetric connectedness, identity relations by Bianchi in the Einstein theory, cotangent bundles in the Yang-Mills theory, the covariance conditions in the tensor methods, the characteristic relations (integrability conditions) in equations of mathematical physics, etc.

**Characteristical properties and peculiarities of existing field theories**

A connection between the exterior differential forms and existing field theories allow to disclose peculiarities of the field theory equations, their common functional properties and their interconnection.

Practically all field theory operators are expressed in terms of following operators of the exterior differential forms: \(d\) (exterior differential), \(\delta\) (the operator of transforming the form of degree \(p+1\) into the form of degree \(p\)), \(\delta'\) (the operator of cotangent transformations), \(\Delta\) (that of the transformation \(d\delta - \delta d\)), \(\Delta'\) (the operator of the transformation \(d\delta' - \delta' d\)). In terms of these operators that act onto exterior forms one can write down the operators by Green, d’Alembert, Laplace and the operator of canonical transformations [9,10]. Eigenvalues of these operators reveal themselves as conjugacy conditions for the differential form elements.

The equations, that are equations of the existing field theories, are those obtained on the basis of the properties of the exterior differential form theory. To the equations of quantum mechanics (equations by Shrödinger, Heisenberg, Dirac) there correspond the closed exterior forms of zero degree or appropriate dual forms. The closed exterior form of zero degree corresponds to the Schrödinger equation, the close dual form corresponds to the Heisenberg equation. It can be pointed out that, whereas the equations by Shrödinger and Heisenberg describe a behavior of potential obtained from the zero degree closed form, Dirac’s bra- and ket- vectors constitute the zero degree closed exterior form itself as the result of conjugacy (vanishing the scalar product).

The Hamilton formalism is based on the properties of closed exterior and dual forms of the first degree. The closed exterior differential form \(ds = -H dt + p_j dq_j\) (the Poincare invariant) corresponds to field equation [10].

The properties of closed exterior and dual forms of the second degree lie at the basis of the electromagnetic field equations. The Maxwell equations may be written as \(d\theta^2 = 0\), \(d^* \theta^2 = 0\) [9], where \(\theta^2 = \frac{1}{4} F_{\mu\nu} dx^\mu dx^\nu\) (here \(F_{\mu\nu}\) is the strength tensor).
Closed exterior and dual forms of the third degree correspond to the gravitational field.

The connection between field theory and closed exterior differential forms supports the invariance of field theory.

The invariance of field theories is an invariance under transformations that conserve the differential. These are transformations under which the invariance of closed exterior forms is conserved. As it was already pointed out, these are the unitary transformations (0-form), the tangent and canonical transformations (1-form), the gradient and gauge transformations (2-form) are gauge transformations for tensor fields (3-form).

The covariance of the dual form is directly connected with the invariance of the exterior closed inexact form. The covariance of the dual form play an important role in describing physical structures and manifolds.

And here it should underline that the field theories are based on the properties of closed inexact forms. This is explained by the fact that only inexact exterior forms can correspond to the physical structures that form physical fields. The condition that the closed exterior forms, which constitute the basis of field theory equations, are inexact ones reveals in the fact that essentially all existing field theories include a certain elements of noninvariance, i.e. they are based either on functionals that are not identical invariants (such as Lagrangian, action functional, entropy) or on equations (differential, integral, tensor, spinor, matrix and so on) that have no identical invariance (integrability or covariance).

Such elements of noninvariance are, for example, nonzero value of the curvature tensor in Einstein’s theory [11], the indeterminacy principle in Heisenberg’s theory, the torsion in the theory by Weyl [11], the Lorentz force in electromagnetic theory [12], an absence of general integrability of the Schrödinger equations, the Lagrange function in the variational methods, an absence of the identical integrability of the mathematical physics equations and an absence of identical covariance of the tensor equations, and so on. Only if we assume elements of noncovariance, we can obtain closed inexact forms that correspond to physical structures.

And yet, the existing field theories are invariant ones because they are provided with additional conditions under which the invariance or covariance requirements have to be satisfied. These conditions are the closure conditions of exterior or dual forms. Examples of such conditions are the above pointed identity relations: canonical, gauge, commutator relations, symmetric connectednesses, identity relations by Bianchi etc.

From the aforesaid one can see that both the field theory transformations and the field theory equations (identical relations) as well are characterized by a degree of the closed form. This discloses a relation between them and shows that it is possible to introduce a classification of physical fields according to the degree of exterior differential form.

As it will be shown below, such a classification is true also for physical interactions. If to denote the degree of closed exterior form by $k$, the case $k = 0$ will correspond to the strong interaction, $k = 1$ will do to the weak interaction, $k = 2$ will correspond to the electromagnetic interaction, and $k = 3$
will correspond to the gravitational interaction.

But within the framework of only exterior differential forms one cannot understand how this classification is explained. This can be elucidated only by application of skew-symmetric differential forms of another type, which possess not invariant properties but evolutionary ones. Such differential forms are just skew-symmetric differential forms, which are defined on manifolds with non-closed metric forms and were named the evolutionary differential forms.

2 A role of evolutionary differential forms in field theory

In paper [2] it has been noted that one must distinguish two types of differential equations of mathematical physics:

1) differential equations that describe physical processes, and
2) the invariant equations of the field theory that describe physical structures forming physical fields.

As it has been shown above the field theories are based on the conservation laws. At the basis of the field theory equations there lie properties of the skew-symmetric differential forms.

It turns out that differential equations, which describe physical processes, are also based on the conservation laws. And at the basis of these equations there also lie properties of the skew-symmetric differential forms.

A difference between two types of equations of mathematical physics consists in the following.

The conservation laws, on which field theories are based, are those for physical fields. The skew-symmetric differential forms correspond to the closed exterior differential forms (skew-symmetric differential forms defined on manifolds with closed metric forms).

In contrast to this, the conservation laws, on which differential equations that describe physical processes are based, are the conservation laws for material media (material systems). And skew-symmetric differential forms correspond to evolutionary differential forms (skew-symmetric differential forms defined on manifolds with unclosed metric forms).

The connection between the mathematical physics equations and the skew-symmetric differential forms enables one to see a connection of the field theory equations with equations that describe physical processes. And this, in turn, allows to see an internal connection of existing invariant field theories and a validity of these theories.

Here it should be noted some functional peculiarities of differential equations.

In differential equations of mathematical physics, which describe physical processes, the functions are found by means of integration of derivatives obtained from the differential equation. Whereas in the field theory equations the functions are obtained not from derivatives, but from differentials, and they are exterior forms (potentials or the state functions). And differentials themselves
are closed forms, i.e. they are invariants.

Differential equations of mathematical physics, which describe physical processes in material media, in addition to operators (derivatives) of the functions desired, involve the terms that are connected with an external action on the system under consideration. Such terms cannot be invariant ones. Hence, these differential equations cannot be invariant equations.

A peculiarity of the invariant equations consists in that they involve only functions or operators on the functions desired. Due to this fact they can be reduced to identical relations or are identical relations.

To such identical relations it can be reduced the field theory equations, for example, the Maxwell equations, Einstein’s equations, the Schrödinger equation, Dirac’s equation, and so on.

So, the Maxwell equations are reduced to the forms $\theta^2 = 0$ and $\ast\theta^2 = 0$, where $\theta^2$ is the second degree form. Field equation [2] is reduced to the canonical relations that corresponds to the closure condition of the dual form and the first degree exterior form. The Schrödinger equation is an analog of the field equation for zero degree form. Einstein’s equation is an identical relation. This equation connects a differential of the first degree form and the closed form of the second degree, namely, the energy-momentum tensor. (It would we noted that, though Einstein’s equation connects the closed forms of the second degree, this equation follows from the third degree differential forms [13]).

Thus, we obtain that differential equations are connected with relations.

It appears that noninvariant differential equations are also connected with relations. However, in contrast to invariant equations, which are connected with identical relations, noninvariant differential equations are connected with nonidentical relations.

Relations, with which differential equations are connected, are expressed in terms skew-symmetric differential forms. In this case identical relations are expressed in terms of closed exterior forms (as it has been shown above), and the nonidentical relations involve the unclosed form.

As it will be shown below, differential equations that describe physical processes are convolved into nonidentical relations. From such nonidentical relations it can be obtained identical relations of the closed exterior forms that lie at the basis of invariant equations of field theory.

Nonidentical relations are those that involve skew-symmetric differential forms defined on manifolds with metric forms - evolutionary differential forms.

**Some properties of evolutionary differential forms**

As it was already mentioned, the evolutionary differential forms are skew-symmetric differential forms defined on manifolds with metric forms that are unclosed. The evolutionary differential form of degree $p$ ($p$-form), as well as the exterior differential form, can be written down as

$$\omega^p = \sum_{\alpha_1 \ldots \alpha_p} a_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p} \quad 0 \leq p \leq n$$

(2.1)
where the local basis obeys the condition of exterior multiplication

\[ dx^\alpha \wedge dx^\alpha = 0 \]

\[ dx^\alpha \wedge dx^\beta = -dx^\beta \wedge dx^\alpha \quad \alpha \neq \beta \]

(summation over repeated subscripts is implied).

But the evolutionary form differential cannot be written similarly to that presented for exterior differential forms (see formula (1.3)). In the evolutionary form differential there appears an additional term connected with the fact that the basis of the form changes. For the differential forms defined on the manifold with unclosed metric form one has \( d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p}) \neq 0 \) (it should be noted that for differentiable manifold the following is valid: \( d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p}) = 0 \)). For this reason a differential of the evolutionary form \( \omega^p \) can be written as

\[
d\omega^p = \sum_{\alpha_1 \ldots \alpha_p} da_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p} + \sum_{\alpha_1 \ldots \alpha_p} a_{\alpha_1 \ldots \alpha_p} d(dx^{\alpha_1} dx^{\alpha_2} \ldots dx^{\alpha_p})
\]

(2.2)

where the second term is connected with a differential of the basis. That is expressed in terms of the metric form commutator[2]. For manifold with closed metric form this term vanishes.

Every evolutionary form is unclosed form, since its commutator, and, consequently, a differential of this form are nonzero (the evolutionary form commutator involves a commutator of unclosed metric form, which is nonzero).

In more detail about properties of the evolutionary forms and peculiarities of their mathematical apparatus it was written in work [2]. Here we shall call attention only to properties of the evolutionary forms that correspond to the conservation laws.

The evolutionary differential forms, as well as the exterior differential forms, can reflect properties of the conservation laws. However, in contrast to exterior differential forms, which reflect properties of the conservation laws for physical fields, the evolutionary differential forms reflect properties of the conservation laws for material systems (material media).

{Material system is a variety of elements that have internal structure and interact to one another. As examples of material systems it may be thermodynamic, gas dynamical, cosmic systems, systems of elementary particles and others. Examples of elements that constitute a material system are electrons, protons, neutrons, atoms, fluid particles, cosmic objects, and othersoppel.

The conservation laws for material systems are the conservation laws for energy, linear momentum, angular momentum, and mass. These are conservation laws that can be named as balance conservation laws. In contrast to the conservation laws for physical fields, which state an existence of conservative physical quantities or objects, the conservation laws for material systems establish a balance between a variation of physical quantity and the corresponding external action.

In works [2,3] it has been shown that the balance conservation laws play a controlling role in the evolutionary processes, which lead to origination of physical structures. Mathematical apparatus of the evolutionary differential
forms, that describe properties of the balance conservation laws is significant for understanding foundations of the general field theory.

**Properties of evolutionary differential forms, which reflect properties of the balance conservation laws**

Let us analyze the equations that describe the balance conservation laws for energy and linear momentum.

We introduce two frames of reference: the first is an inertial one (this frame of reference is not connected with material system), and the second is an accompanying one (this system is connected with manifold constructed of trajectories of material system elements). The energy equation in the inertial frame of reference can be reduced to the form:

\[
\frac{D\psi}{Dt} = A_1
\]

(2.3)

where \(D/Dt\) is the total derivative with respect to time (or another evolutionary variable), \(\psi\) is the functional of the state that specifies a material system, \(A_1\) is the quantity that depends on specific features of the system and on external energy actions onto the system. (The action functional, entropy, wave function can be regarded as examples of the functional \(\psi\). Thus, the equation for energy presented in terms of the action functional \(S\) has a similar form: \(DS/Dt = L\), where \(\psi = S, A_1 = L\) is the Lagrange function. In mechanics of continuous media the equation for energy of ideal gas can be presented in the form [14]: \(Ds/Dt = 0\), where \(s\) is entropy. In this case \(\psi = s, A_1 = 0\). It is worth noting that the examples presented show that the action functional and entropy play the same role.)

In the accompanying frame of reference the total derivative with respect to time is transformed into the derivative along trajectory. Equation (2.3) is now written in the form

\[
\frac{\partial\psi}{\partial\xi^1} = A_1
\]

(2.4)

here \(\xi^1\) is the coordinate along trajectory.

In a similar manner, in the accompanying frame of reference the equation for linear momentum appears to be reduced to the equation of the form

\[
\frac{\partial\psi}{\partial\xi^\nu} = A_\nu, \quad \nu = 2, \ldots
\]

(2.5)

where \(\xi^\nu\) are the coordinates in the direction normal to trajectory, \(A_\nu\) are the quantities that depend on the specific features of the system and external force actions.

Eqs. (2.4) and (2.5) can be convolved into the relation

\[
d\psi = A_\mu d\xi^\mu, \quad (\mu = 1, \nu)
\]

(2.6)

where \(d\psi\) is the differential expression \(d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu\).
Relation (2.6) can be written as

\[ d\psi = \omega \] (2.7)

Here \( \omega = A_\mu \, d\xi^\mu \) is the differential form of the first degree.

Relation (2.7) was obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form \( \omega \) is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form in the evolutionary relation will be the form of the second degree. And in combination with the equation of the balance conservation law of mass this form will be the form of degree 3.

Thus, in the general case the evolutionary relation can be written as

\[ d\psi = \omega^p \] (2.8)

where the form degree \( p \) takes the values \( p = 0, 1, 2, 3 \ldots \) (The evolutionary relation for \( p = 0 \) is similar to that in the differential forms, and it was obtained from the interaction of energy and time.)

Since the equation of the balance conservation laws are the evolutionary ones, the relation obtained is also an evolutionary relation.

The evolutionary relation is a nonidentical one as it involves the unclosed differential form.

Let us consider the commutator of the form \( \omega = A_\mu \, d\xi^\mu \). Components of the commutator of such a form can be written as follows:

\[ K_{\alpha\beta} = \left( \frac{\partial A_\beta}{\partial \xi^\alpha} - \frac{\partial A_\alpha}{\partial \xi^\beta} \right) \] (2.9)

(here the term connected with a nondifferentiability of the manifold has not yet been taken into account). The coefficients \( A_\mu \) of the form \( \omega \) must be obtained either from the equation of the balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. A commutator of the form \( \omega \) constructed of derivatives of such coefficients is nonzero. This means that a differential of the form \( \omega \) is nonzero as well. Thus, the form \( \omega \) proves to be unclosed. This means that the evolutionary relation cannot be an identical one. In the left-hand side of this relation it stands a differential whereas in the right-hand side it stands an unclosed form that is not a differential.

Since the evolutionary relation is not identical, from this relation one cannot get the state differential \( d\psi \) that may point to the equilibrium state of a material system. This means that the material system state is nonequilibrium. (The nonequilibrium state means that there is an internal force in the material system. It is evident that the internal force originates at the expense of some quantity described by the evolutionary form commutator). The nonequilibrium
state of material system induced by the action of internal forces leads to that the accompanying manifolds turns out to be a deforming manifold. The metric forms of such manifold cannot be closed. The metric form commutator, which describes a deformation of the manifold and is nonzero, enters into the commutator of the differential form $\omega$ defined on the accompanying manifold. That is, in formula (2.9) it will arise the second term connected with the metric form commutator with nonzero value. In this case the second term will correlate with the first term, and this term cannot make the differential form $\omega$ commutator to be zero. That is, the differential form, which enters into the evolutionary equation, cannot become closed. And this means that the evolutionary relation cannot become the identical relation. Unclosed differential form $\omega$, which enters into this relation, is an example of the evolutionary differential form.

In such a way it can be shown that the evolutionary differential form $\omega^p$, involved into this evolutionary relation (2.8), is an unclosed one for real processes. Evolutionary relation (2.8) is nonidentical one.

**Obtaining an identical relation from a nonidentical one**

A role of the nonidentical evolutionary relation in field theory consists in that it discloses a connection of the balance conservation law equations (equations, which describe physical processes in material media) and the field theory equations. Identical relations, which correspond to equations of existing field theories, are obtained from the evolutionary nonidentical relations, which correspond to the equations describing physical processes in material media.

The nonidentical relation includes an unclosed differential form. A differential of such a form is nonzero. The identical relation includes a closed differential form. A differential of such a form equals zero. Hence one can see that a transition from the nonidentical relation to the identical one can proceed only as a degenerate transformation.

Let us consider nonidentical evolutionary relation (2.8).

As it has been already mentioned, the evolutionary differential form $\omega^p$, involved into this relation is an unclosed one for real processes. The commutator, and hence the differential, of this form is nonzero. That is,

$$d\omega^p \neq 0$$

(2.10)

If the transformation is degenerate, from the unclosed evolutionary form it can be obtained a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$$d\omega^p \neq 0 \rightarrow (\text{degenerate transform}) \rightarrow d_x\omega^p = 0, \ d_{\pi}\omega^p = 0$$

The degenerate transformation is realized as a transition from the accompanying noninertial coordinate system to the locally inertial that.

To the degenerate transformation it must correspond a vanishing of some functional expressions. Such functional expressions may be Jacobians, determinants, the Poisson brackets, residues and others. A vanishing of these functional expressions is the closure condition for a dual form. An equality to zero of such
functional expressions is an identical relation written in terms of derivatives (like the Cauchy-Riemann conditions, canonical relations, the Bianchi identities and so on). The conditions of degenerate transformation are connected with symmetries that can be obtained from the coefficients of evolutionary and dual forms and their derivatives. Since the evolutionary relation has been obtained from the equations for material system, it is obvious that the conditions of degenerate transformation are specified by properties of the material system. The degrees of freedom of material system can correspond to such conditions. Translational, rotational, oscillatory degrees of freedom are examples.

On the pseudostructure $\pi$ evolutionary relation (2.8) transforms into the relation

$$d_\pi \psi = \omega^p_\pi$$  \hspace{1cm} (2.11)

which proves to be the identical relation. Indeed, since the form $\omega^p_\pi$ is a closed one, on the pseudostructure it turns out to be a differential of some differential form. In other words, this form can be written as $\omega^p_\pi = d_\pi \theta$. Relation (2.11) is now written as

$$d_\pi \psi = d_\pi \theta$$

There are the differentials in the left-hand and right-hand sides of this relation. This means that the relation is an identical one.

Transition from nonidentical relation (2.8) obtained from the balance conservation laws to identical relation (2.11) means the following. Firstly, it is from such a relation that one can find the state differential $d_\pi \psi$. Existence of the state differential (left-hand side of relation (2.11)) points to a transition of material system to the locally-equilibrium state. And, secondly, an emergence of the closed (on pseudostructure) inexact exterior form $\omega^p_\pi$ (right-hand side of relation (2.11)) points to an origination of the physical structure, namely, the conservative object. This object is a conservative physical quantity (the closed exterior form $\omega^p_\pi$ on the pseudostructure (the dual form $\ast \omega^p$, which defines the pseudostructure).

From a nonidentical evolutionary relation of degree $p$ (evolutionary relation that contains a differential form of degree $p$) one can obtain an identical relation of degree $k$, where $k$ ranges from $p$ to 0. Under degenerate transformation from a nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. It is just a relation that one can integrate and obtain a relation with differential forms of less by one degree. The relation obtained after integration proves to be nonidentical as well. The obtained nonidentical relation of degree $(p - 1)$ can be integrated once again if the corresponding degenerate transformation is realized and the identical relation is formed. By sequential integrating the evolutionary relation of degree $p$ (in the case of realization of the corresponding degenerate transformations and forming the identical relation), one can get closed (on the pseudostructure) exterior forms of degree $k$, where $k$ ranges from $p$ to 0.

An emergency of identical relation with closed inexact form of degree $k$ points to origination of corresponding physical structure.
Thus, a transition from the nonidentical evolutionary relation to the identical one elucidates the mechanism of origination of physical structures. Such structures form physical fields.

Since the evolutionary relation is obtained from equations for material media, it is evident that physical fields are produced by material systems (material media). The mechanism of evolutionary processes, which proceed in material media and lead to origination of physical structures, has been detailed in works [2,3]. In that works it has been shown a connection between characteristics of the physical structures originated with characteristics of the evolutionary forms, of the evolutionary form commutators and of the material system producing these structures. In the present work we shall not focus our attention on these problems.

Besides, in papers [2,3] it has been shown that parameters, which enter into the evolutionary relation, and the identical relations obtained allow to classify physical structures and physical fields.

Here it should emphasize the following.

The evolutionary relation is obtained from not a single, but from several equations of the balance conservation laws. A nonidentity of the evolutionary relation obtained from equations of the balance conservation laws means that the equations of balance conservation laws turn out to be not consistent. And this points to a noncommutativity of the balance conservation laws and the nonequilibrium material system state produced as a result. A quantity described by the evolutionary differential form commutator serves as the internal force. A noncommutativity of the balance conservation laws is a moving force of evolutionary processes in material media. An interaction of the noncommutative balance conservation laws causes the evolutionary processes in material media, which lead to origination of physical structures. The noncommutativity of the balance conservation laws and their controlling role in the evolutionary processes, that are accompanied by emerging physical structures, practically have not been taken into account in the explicit form anywhere. The mathematical apparatus of evolutionary differential forms enables one to take into account and describe these points. An account for the noncommutativity of the balance conservation laws in material systems enables one to unveil the causality of physical processes and phenomena and to understand a meaning of postulates that lie at the basis of existing field theories.

3 Mathematical apparatus of exterior and evolutionary skew-symmetric differential forms as the basis of the general field theory

In section 1 it has been shown that at the basis of the invariant field theories there lie the mathematical apparatus of closed exterior differential forms, which reflect properties of the conservation laws.

A connection of field theory with the exterior differential forms allow to dis-
close peculiarities of the field theory equations, their common functional properties. From properties of the closed exterior differential forms one can see that the field theory equations, the field theory transformations and physical interactions are characterized by a degree of the closed form. This discloses a relation between them and shows that it is possible to introduce a classification of physical fields according to a degree of the exterior differential form. Such classification shows that the theory of closed exterior differential forms can lie at the basis of the unified field theory.

The field theories that are based on exact conservation laws allow to describe the physical fields. However, because these theories are invariant ones they cannot answer the question about the mechanism of originating physical structures that form physical fields. The origination of physical structures and forming physical fields are evolutionary processes, and hence they cannot be described by invariant field theories. Only evolutionary theory can do this.

As the basis of such evolutionary theory it can be the theory of evolutionary skew-symmetric differential forms. It has been shown above that the theory of evolutionary skew-symmetric differential forms elucidates a mechanism of originating physical structures and forming physical fields and indicates that the physical structures, which form physical fields, are produced by material systems (material media). The connection of physical fields and material media elucidates the causality of physical phenomena and allows to understand what specifies the characteristics of physical structures and physical fields. Here it should be pointed out that (as the present study shows) the emergence of physical structures in the evolutionary process proceeds spontaneously and is manifested as an emergence of certain observable formations of material system. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, creating massless particles, and others.

The evolutionary theory has to be based on properties of the balance conservation laws for material systems, because just the interaction of the noncommutative balance conservation laws leads to creation of physical structures that are generated by material system.

For developing the evolutionary theory one must know the following.
Firstly, it is necessary to know which material system (medium) generates the given physical field. Further, one needs to have an equation that describes the balance conservation laws (of energy, linear momentum, angular momentum, and mass) for material system [14-16]. After this, it is necessary to get the nonidentical evolutionary relation from these equations and to develop the method of studying such evolutionary relation by using the balance conservation law equations themselves and properties of material system (being connected with degrees of freedom).

The basic mathematical foundations of the theory of evolutionary differential forms that describe the evolutionary process in material systems, and the mechanism of originating physical structures evidently must be included into the evolutionary field theory. However, to realize all these mathematical foundations is rather difficult and in many cases this turns out to be impossible. However, a knowledge of the basic mathematical principles of the theory of
evolutionary differential forms may be helpful while studying a mechanism of originating physical fields.

The results of qualitative investigations of evolutionary processes on the basis of the mathematical apparatus of evolutionary differential forms enables one to see the common properties that unify all physical fields. The physical fields are generated by material media, and at the basis of this it lies the interaction of the noncommutative conservation laws of energy, linear momentum, angular momentum, and mass for material media. This explains the causality of physical phenomena and clarifies the essence of postulates that lie at the basis of existing field theories. The postulates, which lie at the basis of the existing field theories, correspond to the closure conditions for exterior and dual form, which correspond to the conservation laws.

These results allow to classify the physical structures and hence to see internal connections between various physical fields. The properties of physical structures depend primarily on which material systems (media) generate physical structures (but the physical structures generated by different material media possess common properties as well).

In section 2 there were presented parameters according to which one can classify physical structures and physical fields.

One of these parameters is, firstly, the evolutionary form degree that enters into the evolutionary relation. This is the parameter $p$ that ranges from 0 to 3 (the case $p = 1$ corresponds to interaction of the balance conservation laws of energy and linear momentum, the case $p = 2$ does to that of energy, linear momentum, and angular momenta, the case $p = 3$ corresponds to interaction of the balance conservation laws of energy, linear and angular momenta, and mass, and to $p = 0$ it corresponds an interaction between time and the balance conservation law of energy or an interaction between the coordinate and the momentum). This parameter specifies a type of physical fields. So, the electromagnetic field is obtained from interaction between the balance conservation laws of energy and linear and angular momenta. The gravitational field is obtained as the result of interactions between the balance conservation laws of energy, linear momentum, angular momentum, and mass.

The other parameter is the degree of closed differential forms that were realized from given evolutionary relation. The values of these parameters designated by $\kappa$ range from $p$ to 0. This parameter, which corresponds to physical structures realized, characterizes a connection between physical structures and exact conservation laws. A parameter that classifies the equations of invariant field theories is such a parameter.

One more parameter is the dimension of space in which the physical structures are generated. This parameter points to the fact that the physical structures, which belong to common type of the exact conservation laws, can be distinguished by their space structure.

The classification with respect to these parameters may be traced in the Table of interactions presented below. In the Table some specific features of classification of physical structures were considered. It will be shown that the classification with respect to these parameters not only elucidates connections
between physical fields generated by material media, but explains a mechanism of creating elements of material media themselves and demonstrates connections between material media as well.

In work [2] examples of using the methods developed are presented.

The thermodynamic system has been inspected, and the analysis of the principles of thermodynamics has been carried out. It was shown that the first principle of thermodynamics is a nonidentical evolutionary relation for thermodynamic system, and the second principle of thermodynamics is an example of identical relation that is obtained from the nonidentical evolutionary relation (the first principle of thermodynamics) under realization of the additional condition, namely, under realization of the integrating factor that turns out to be the inverse temperature. In this case as the closed exterior form it serves a differential of entropy.

Derivation of the evolutionary relation for gas dynamic system is presented. The evolutionary relation for gas dynamic system is written for the entropy differential as well. But whereas the thermodynamic evolutionary relation involves entropy that depends on the thermodynamic parameters, the gas dynamic evolutionary relation involves entropy that depends on space-time coordinates. It was carried out the investigation of the evolutionary form commutator that enters into the gas dynamic evolutionary relation. This investigation has shown that the external actions, which give contributions into the evolutionary form commutator, effect on development of instability and origination of physical structures. This analysis allows to understand a mechanism of turbulence.

It was carried out an analysis of the equations of electromagnetic field. It was shown that there are two equations for the Pointing vector from which the nonidentical evolutionary relation can be obtained. It was shown under which conditions the identical relation follows from that, and this corresponds to origination of electromagnetic wave.

These examples show that the evolutionary approach to field theory enables one to get radically new results and to explain the causality of physical phenomena.

By comparison of the invariant and evolutionary approaches to field theory one can state the following. Physical fields are described by invariant field theory that is based on exact conservation laws. Properties of closed exterior differential forms lie at the basis of mathematical apparatus of the invariant theory. A mechanism of forming physical fields can be described only by evolutionary theory. The evolutionary theory that is based on the balance conservation laws for material systems is just such a theory. As the mathematical apparatus of such a theory it can be the mathematical apparatus of evolutionary differential forms. It is evident that as the common field theory it must serve a theory that involves the basic mathematical foundations of the evolutionary and invariant field theories.

In conclusion we present the Table of data, which can be obtained within the framework of the skew-symmetrical differential form theory. The Table shows that this theory can be regarded as an approach to the general field theory.
Certain classification of physical structures

As it was shown above, the type of physical structures (and accordingly of physical fields) generated by the evolutionary relation depends on the degrees of differential forms $p$ and $k$ and on the dimension of initial inertial space $n$. Here $p$ is the degree of the evolutionary form in the evolutionary relation, which is connected with a number of interacting balance conservation laws, and $k$ is the degree of a closed form generated by the evolutionary relation. By introducing a classification by numbers $p$, $k$, $n$ one can understand the internal connection between various physical fields. Since the physical fields are carriers of interactions, such classification enables one to see a connection between interactions. This is reflected in the Table presented below. This Table corresponds to elementary particles.

It should be emphasized the following. Here the concept of “interaction” is used in a twofold meaning: an interaction of the balance conservation laws that relates to material systems, and the physical concept of “interaction” that relates to physical fields and reflects the interactions of physical structures, namely, it is connected with the exact conservation laws.

Recall that the interaction of balance conservation laws for energy and linear momentum corresponds to the value $p = 1$, with the balance conservation law for angular momentum in addition this corresponds to the value $p = 2$, and with the balance conservation law for mass in addition it corresponds to the value $p = 3$. The value $p = 0$ corresponds to interaction between time and energy or an interaction between coordinate and momentum.

In the Table the names of particles created are given. Numbers placed near particle names correspond to the space dimension. In braces {} the sources of interactions are presented. In the next to the last row we present the massive particles (elements of the material system) formed by interactions (the exact forms of zero degree obtained by sequential integrating the evolutionary relations with the evolutionary forms of degree $p$ correspond to these particles). In the bottom row the dimension of the metric structure created is presented.

From the Table one can see a correspondence between the degree $k$ of the closed forms realized and the type of interactions. Thus, $k = 0$ corresponds to the strong interaction, $k = 1$ corresponds to the weak interaction, $k = 2$ corresponds to the electromagnetic interaction, and $k = 3$ corresponds to the gravitational interaction. The degree $k$ of the closed forms realized and the number of interacting balance conservation laws determine a type of interactions and a type of particles created. The properties of particles are governed by the space dimension. The last property is connected with the fact that closed forms of equal degrees $k$, but obtained from the evolutionary relations acting in spaces of different dimensions $n$, are distinctive because they are defined on pseudostructures of different dimensions (the dimension of pseudostructure $(n + 1 - k)$ depends on the dimension of initial space $n$). For this reason the realized physical structures with closed forms of equal degrees $k$ are distinctive in their properties.
### Table

| interaction          | 0 \(k\), 1 \(p\), 2 \(n\) | 1   | 2   | 3   |
|----------------------|-------------------------------|-----|-----|-----|
| gravitation          | 3                             |     |     |     |
| electron             | photon                        |     |     |     |
| proton               | neutron                       |     |     |     |
| photon2              |                               |     |     |     |
| electron             | photon                        |     |     |     |
| proton               | neutrino                      |     |     |     |
| photon3              |                               |     |     |     |
| electron             | neutrino2                     |     |     |     |
| neutrino2            | neutrino3                     |     |     |     |
| weak                 | 1                             |     |     |     |
| electron             | quanta1                       |     |     |     |
| neutrino2            | neutrino3                     |     |     |     |
| strong               | 0                             |     |     |     |
| quanta0              | quanta1                       |     |     |     |
| quanta2              | quanta3                       |     |     |     |
| particles            | exact forms                   | electron | proton | neutron | deuteron? |
| material nucleons?   | N 1                           | 2    | 3    | 4    |
| time                 | time+                         | time+ | time+ |
| 1 coord.             | 2 coord.                      | 3 coord. |

(For the value \(k = 0\) the commutative relations \(\hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar\) correspond to such quantities. The left-hand side of the commutative relations is analog of the commutator value of the nonintegrable form of zero degree, and the right-hand side is equal to its value at the instant of realization of the closed zero degree form, the imaginary unit points to the direction transverse to the pseudostructure).

The parameters \(p\), \(k\), \(n\) can range from 0 to 3. This determine some completed cycle. In the Table a single cycle of forming physical structures is presented. This cycle is related to material system. Each material system has its own completed cycle. This distinguishes one material system from another system. One completed cycle can serve as the beginning of another cycle (the structures formed in the preceding cycle serve as the sources of interactions for beginning a new cycle). This may mean that one material system (medium) proves to be imbedded into the other material system (medium). The sequential cycles reflect properties of sequentially imbedded material systems.

In each cycle one can determine the levels and stages. In the Table presented rows correspond to the levels and columns correspond to the stages.

From the Table one can see that the cycle level (to which in the Table it corresponds the row) points to a type of interaction. This relates to the degree
A stage of the cycle (to which in the Table there corresponds a column) is connected with a total number of the balance conservation laws interacting in the given space, namely, with the evolutionary form degree $p$, and with space dimension $n$. Each cycle involves four stages, to every of which there corresponds its own value $p$ ($p = 0, 1, 2, 3$) and the space dimension $n$.

At each stage of given cycle the transitions from the closed exterior form of degree $k = p$ to the closed exterior form of degree $k = 0$ are the transitions from one type of interaction to another. Such transitions execute the connection between different types of interactions.

At each stage the transition from the closed inexact form of zero degree $k = 0$ to the exact exterior form of the same degree corresponds to the transition from relevant physical structure to the element of material system. To each type there corresponds its own appropriate coupling constant. This means that from the physical structure it can be obtained the appropriate elements of material system. In every cycle four types of elements that are distinguished by the dimensions of their metric structure are created. In the Table presented electron, proton, neutron, and deuteron(?) are such elements.

Each stage has the specific features that are inherent to the same stages in other cycles. {One can see this, for example, by comparison of the cycle described with the other cycle, where to the exact form there correspond sequentially conductors, semiconductors, dielectrics, and neutral elements. The properties of elements of the third stage, namely, neutrons in one cycle and dielectrics in the other coincide with those of the so-called "magnetic monopole" \cite{17,18}.} Physical structures that have the same parameters exhibit common properties. And yet the physical structures that have the same parameters $p, k, n$ will be distinctive according to in what cycle they are located. That is, which material system generates these structures. (As it was already pointed out, thermodynamic, gas dynamic, cosmic systems, the system of elementary particles and so on can serve as examples of material system. The physical vacuum in its properties may be regarded as an analog of a material system that generates some physical fields.).

The Table presented provides the idea about the dimension of pseudostructures and metric structures.

It was shown that the evolutionary relation of the degree $p$ can generate (with the availability of degenerate transformations) closed forms of degrees $0 \leq k \leq p$ on the pseudostructures. Under generation of the forms of sequential degrees $k = p$, $k = p - 1$, $\ldots$, $k = 0$ the pseudostructures of the dimensions $(n + 1 - k)$: 1, $\ldots$, $n + 1$, where $n$ is the dimension of initial inertial space, are obtained. While going to the exact exterior form of zero degree the metric structure of the dimension $N = n + 1$ is obtained. With a knowledge of the values $n$ and $k$ for each physical structure presented in the Table one can find the dimension of relevant pseudostructure. In the bottom row of the Table the dimension $N$ of the metric structure formed is presented. From initial space of the dimension 0 the metric space of the dimension 1 (it can occurs to be time) can be realized. From space of the dimension 1 the metric space of dimension 2
(time and coordinate) can appear and so on. From initial space of the dimension 3 it can be formed the metric space of dimension 4 (time and 3 coordinates). Such space is convolved and a new dimension cannot already be realized. This corresponds to ending the cycle. (Such metric space with corresponding physical quantity that is defined by the exact exterior form is the element of new material system.)

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