Time evolution and decoherence of entangled states realized in coupled superconducting flux qubits

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We study theoretically how decoherence affects superposition states composed of entangled states in inductively coupled two superconducting flux-qubits. We discover that the quantum fluctuation of an observable in a coupled flux-qubit system plays a crucial role in decoherence when the expectation value of the observable is zero. We also find that there exists a decoherence free subspace for the environment coupled via a charge degree of freedom of the qubit system.

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I. INTRODUCTION

Josephson-junction circuits behave quantum mechanically, if they are sufficiently decoupled from their environment. However, in reality, since a macroscopic quantum state is difficult to be decoupled from their environment completely, a quantum mechanical superposition state suffers decoherence. This is one of the central problems that must be solved before these circuits can be used for quantum information processing[1]. In order to minimize decoherence on quantum states, it is very important to search what is the dominant source of decoherence and how sensitively qubits feel the effect of environment. In this paper, we theoretically present that the quantum fluctuation of an observable in a coupled flux-qubit plays a crucial role in decoherence, even if the expectation value of the observable is always zero. We also report that we can extract a decoherence-free single-qubit basis from an inductively coupled flux-qubit system.

II. INDUCTIVELY COUPLED TWO FLUX QUBIT SYSTEM

Among qubits based on Josephson-junction circuits, a flux qubit was realized as a superconducting ring with three Josephson-junctions[2,3] and coherent oscillations were demonstrated[4,5,6]. We discuss decoherence appearing in the dynamics of an inductively coupled two superconducting flux-qubits in Fig. 1(a)[7,8]. Four of Josephson-junctions have the Josephson energy $E_J$ and capacitance $C$. The remaining two have $\alpha E_J$ and $\alpha C$, where $\alpha = 0.75$ is a constant. The Hamiltonian of the system is given by

$$H = \frac{C}{2} \left( \frac{\hbar}{2e} \right)^2 \left( \gamma_L^2 + \gamma_R^2 + \gamma_L^2 + \gamma_R^2 + \alpha (\gamma_L^2 + \gamma_R^2) \right) - E_J (\cos[\gamma_L] + \cos[\gamma_R] + \cos[\gamma_L] + \cos[\gamma_R]) + \frac{\alpha (\cos[\gamma_L] + \cos[\gamma_R])}{2M} \left( \frac{\hbar}{2e} \right)^2 \gamma_M^2,$$

where $\gamma_i (i = L1, R1, L2, R2, L3, R3, M)$ is the phase difference across each junction or the inductance. By calculating the Hamiltonian with parameters $E_J = 100$GHz, $E_C = \frac{(2e)^2}{C} = 8$GHz, $\alpha = 0.75$, $M = \frac{200 J}{(2e)^2}$, we can obtain the energy dispersion shown in Fig. 1(b) and we call the point at $f = 0.5$ as the “degeneracy point”.

![FIG. 1: (a)Effective circuit of inductively coupled two flux qubits. Two superconducting closed loops correspond to two flux qubits. A qubit has three Josephson junctions. An external flux $\Phi_1$ or $\Phi_2$ is penetrating through each qubit loop. Qubits are coupled to each other through the mutual inductance $M$. (b)Energy dispersions obtained when identical external fluxes are applied to the qubits, $f = \Phi_1/\Phi_0 = \Phi_2/\Phi_0 = \Phi_0/h/2e$.](image)

We can approximate the Hamiltonian for the two-qubit system using Pauli matrices $\sigma_i^x, \sigma_i^y, \sigma_i^z$.

$$H_S = h_1 \sigma_1^x + \Delta_1 \sigma_1^z + h_2 \sigma_2^x + \Delta_2 \sigma_2^z + j \sigma_1^y \sigma_2^y.$$

Here, subscripts 1 and 2 indicate the left and right qubits, respectively. The basis of the two-qubit state are $|\downarrow\downarrow, \downarrow\downarrow, \downarrow\uparrow, \uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow\rangle$, $|\uparrow\uparrow\rangle$ corresponds to the state where the persistent current flows counterclockwise and $|\downarrow\downarrow\rangle$ to the clockwise state around the loop of the qubit $i(i = 1, 2)$. $h_1(i = 1, 2)$ is the parameter that depends on the external flux $\Phi_i$ for the qubit $i$, and $h_i = I_{ps}(\Phi_i - \frac{2\Phi_0}{h}) = I_{ps} \Phi_0 (f_i - \frac{1}{2})$, where $I_{ps}(i = 1, 2)$ is the absolute value of the persistent current in each qubit determined by $E_J$ and $\alpha$. $\Phi_0 = h/2e$ is the flux quantum. $\Delta_i(i = 1, 2)$ is the tunneling matrix element each qubit has, which is the order of $\sqrt{E_C}/E_J e^{\sqrt{|E_J|/E_C}}$. 

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$j$ is the coupling constant between two qubits, which is determined approximately by $MI_{11}I_{p2}$. If the external flux is equally applied ($h_1 = h_2 = h$) and the two qubits are assumed to be identical ($\Delta_1 = \Delta_2 = \Delta$), we can simplify the Hamiltonian as

$$H_S = \hbar \sigma_1^x + \Delta \sigma_1^z + h \sigma_2^z + \Delta \sigma_2^z + j \sigma_1^x \sigma_2^x.$$ 

By comparing energy eigenstates of the $H_S$ and Eq. 1(b) near the degeneracy point, we find that this approximated Hamiltonian $H_S$ reproduces well the result of the original two flux-qubit Hamiltonian with $j = 1.78$ GHz and $\Delta = 2.07$ GHz. Therefore, hereafter we analyze the decoherence in our coupled flux-qubit system with this $H_S$. In terms of the original flux-qubit system, $\sigma_i^z$ corresponds to the flux the qubit $i$ has, and $\sigma_i^x$ to the charge polarization in the junction of the qubit. At the degeneracy point $h = 0$, we can obtain eigenvalues and eigenstates[7],

$$E_0 = -\sqrt{j^2 + 4\Delta^2}, \quad |\psi_0\rangle = |\uparrow_1\downarrow_2\rangle + \downarrow_1\uparrow_2\rangle,$$

$$-(j + \sqrt{j^2 + 4\Delta^2})(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle),$$

$$E_1 = -j, \quad |\psi_1\rangle = |\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle,$$

$$E_2 = j, \quad |\psi_2\rangle = |\uparrow_1\uparrow_2\rangle - |\downarrow_1\downarrow_2\rangle,$$

$$E_3 = \sqrt{j^2 + 4\Delta^2}, \quad |\psi_3\rangle = |\uparrow_1\uparrow_2\rangle + |\downarrow_1\downarrow_2\rangle$$

(2) Here, a remarkable fact is that all the eigenstates of the two-qubit Hamiltonian at the degeneracy point are strongly entangled ones. Adopting a state composed of these entangled states as an initial state, we will discuss the time evolution of the density operator and some quantities.

III. PHYSICAL MODEL OF DECOHERENCE AND QUANTUM MASTER EQUATION

In order to take into account decoherence, we consider the total system as the sum of the qubit system and the environment. The total Hamiltonian is $H_{TOT} = H_S + H_E + H_{INT}$, where $H_S$, $H_E$, and $H_{INT}$ are the Hamiltonian of the qubit system, the environment, and the interaction between them, respectively. We showed already $H_S$ and the components of environment and interaction are described as follows; 

$$H_E = \sum_k \hbar \omega_k a_k^\dagger a_k, \quad H_{INT} = x \sum_k \lambda_k (a_k + a_k^\dagger).$$

We regard the environment as a boson bath. $a_k$, $a_k^\dagger$ are the annihilation and the creation operators of the environmental mode $k$. $\lambda_k$ is the strength of the coupling in each mode. $x$ means a physical quantity of the qubit system that is coupled to the environment. Assuming that the system-environment coupling is small enough, we can treat $H_{INT}$ as a perturbative term so that we acquire a quantum master equation[9],

$$\dot{\rho}_S = \frac{1}{\hbar} \int_0^t dt \{ \nu(t)|x, [x(-t), \rho_S]\} - i\eta(t)|x, \{x(-t), \rho_S\}$$

$$= -i\nu(t)|x, \{x(-t), \rho_S\} \rangle \langle x + i\eta(t)|x, \{x(-t), \rho_S\}.$$ 

where

$$\nu(t) = \int_0^\infty d\omega \cos[\omega t] \coth\left[\frac{\omega}{2kB}\right]J(\omega),$$

$$\eta(t) = \int_0^\infty d\omega \sin[\omega t]J(\omega).$$

$k_B$ is the Boltzman constant and $T$ is the absolute temperature. We adopt the spectral density $J(\omega) = g^2 \omega^2 / (\pi^2 + \omega^2)$. Here $a$ is the cut off frequency and we set $a = 5$.

IV. RESULTS AND DISCUSSION

A. decoherence via flux degree

First, we assume that the interaction between the system and the environment is caused only through the total flux of qubits; then we take $x = \sigma_1^x + \sigma_2^x$. This is a reasonable situation because we can expect that the system suffers decoherence due to the coupling to the detector like a SQUID and the fluctuation of the external flux, both of which are related to small flux generated by the persistent currents flowing in qubits. Now, we consider a superposition state composed of the first and second excited states $|\psi_{12}(0)\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ at the degeneracy point ($h = 0$) as the initial state. We can analytically calculate the time evolutions of expectation values $\langle \sigma_1^x \rangle, \langle \sigma_2^x \rangle, \langle \sigma_1^z \rangle$ and $\langle \sigma_2^z \rangle$ when there is no decoherence. Provided that it is possible to set independent probes which can detect both flux and charge on each of qubits, we can measure expectation values $\langle \sigma_1^x \rangle, \langle \sigma_2^x \rangle, \langle \sigma_1^z \rangle$ and $\langle \sigma_2^z \rangle$ independently. $|\psi_{12}(t)\rangle$ and the time evolutions of observables at $h = 0$ are obtained as follows;

$$|\psi_{12}(t)\rangle = \cos[\nu t]|\uparrow_1\downarrow_2\rangle + \sin[\nu t]|\downarrow_1\uparrow_2\rangle + \nu[\sin[\nu t]|\uparrow_1\downarrow_2\rangle + \sin[\nu t]|\downarrow_1\uparrow_2\rangle.$$

(3) Here $|\uparrow_i\rangle = |\uparrow_i\rangle + |\downarrow_i\rangle, \quad |\downarrow_i\rangle = |\uparrow_i\rangle + |\downarrow_i\rangle$ 

$(i = 1, 2)$. This shows that this coherent oscillation is impossible to detect through the flux, but there is a possibility to detect through the charge. We will show the result of the numerical calculation in the presence of decoherence, which is obtained by solving the quantum master equation with the above initial state. As we have expected in the analytical calculation, we realize that expectation values $\langle \sigma_1^x \rangle$ and $\langle \sigma_2^x \rangle$ obtained in numerical
calculations are always zero. The most interesting feature of this example is that the expectation value of the physical quantity which interacts with the environment \( \langle \sigma^z_1 + \sigma^z_2 \rangle \) is zero. Although we would speculate that there is no coupling between qubits and the environment, this turns out not true according to the calculation results. Figure 2(a) shows the oscillations of \( \langle \sigma^z_1 \rangle \) and \( \langle \sigma^z_2 \rangle \). We find that thier amplitudes are obviously suffering decoherence. Time evolution of eigenvalues of the density operator in Fig. 2(b) helps us realize that a pure state is rapidly changing into a mixed state due to decoherence because the emergence of the finite second and later eigenvalues means the system becomes a mixed state. Therefore, we can make sure that, even if the expectation value of the interacting quantity between system and environment is always zero, there exists certain effect of decoherence.

Next, we discuss when the initial state is the superposition state \( |\psi_{12}(0)\rangle \equiv |\psi_0\rangle + |\psi_1\rangle \) of the ground and the first excited states. Figure 3(a) shows oscillations of the flux induced at the qubit1 and qubit2, and these oscillations suggest that the possibility to detect the fluxes \( \langle \sigma^y_1 \rangle \), \( \langle \sigma^y_2 \rangle \) separately by setting an independent SQUIDs over each of the qubits. Figure 3(a) also shows that amplitudes of oscillations hardly decay and it seems that the oscillations are hardly affected by decoherence. This suggests that it could be possible to realize coherent oscillation which is robust against the decoherence via \( x = \sigma^z_1 + \sigma^z_2 \). We can again observe the expectation value of the coupling between the qubit system and the environment is always zero \( \langle \sigma^z_1 + \sigma^z_2 \rangle = 0 \). From the above two examples, surely we can make coherent oscillations where \( \langle H_{INT} \rangle = \langle x \rangle \sum \lambda_k (a_k + a_k\dagger) = 0 \) by using initial states which is composed by entangled energy eigenstates. However, one is strongly affected by decoherence and the other is not. Then, we are interested in considering how quantum fluctuations of the coupling \( x \) may become the source of decoherence[8]. In fact, the calculations obtaining Figs. 2 and 3 also give the finite quantum fluctuations

\[
\sqrt{\langle \psi_{12}(t)\rangle \langle \Delta x^2 \rangle \langle \psi_{12}(t)\rangle} \sim 1.4, \\
\sqrt{\langle \psi_{01}(t)\rangle \langle \Delta x^2 \rangle \langle \psi_{01}(t)\rangle} \sim 0.75,
\]

where \( \Delta x^2 \equiv x^2 - \langle x \rangle^2 \). It looks like that the quantum fluctuations of the coupling \( x \) and resulting magnitude of decoherence are determined by the curvatures of energy dispersions in Fig. 1(b) around the degeneracy point \( f = 0.5 \). The quantum fluctuation of \( x = \sigma^z_1 + \sigma^z_2 \) qualitatively corresponds to the quantum fluctuation of an applied flux in the horizontal axis of Fig. 1(b). In fact, by a perturbation calculation under the condition \( |h| \ll |\Delta_i| \), the energy dispersion of an eigenstate \( |i\rangle \) in the vicinity of the degeneracy point is given by

\[
E_i - E_i^0 = \hbar^2 \sum_{j \neq i} \langle i| \sigma^z_1 + \sigma^z_2 |j\rangle \langle j| \sigma^z_1 + \sigma^z_2 |i\rangle / (E_j - E_i).
\]

where \( E_i^0 \) and \( |i\rangle \) are the eigenenergy and eigenstate at the degeneracy point. So, the curvature \( \sigma^2 E_i / \sigma \omega^2 \) (\( i = 0, 1, 2 \)) is approximately proportional to

\[
\sum_j \langle i| \sigma^z_1 + \sigma^z_2 |j\rangle \langle j| \sigma^z_1 + \sigma^z_2 |i\rangle,
\]

that is the quantum fluctuation of \( \sigma^z_1 + \sigma^z_2 \). As a result, we have reached a conclusion that the decoherence at the degeneracy point depends on the curvature of the energy dispersion around the degeneracy point.

Now we discuss the concurrence which is defined as

\[
C(t) = \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4}
\]

where \( \xi_i (i = 1, 2, 3, 4) \) is the \( i \) th eigenvalue of \( \Lambda(t) \equiv \sum |p_i|^2 \langle \psi_i | \sigma^y_1 \sigma^y_2 | \psi_i \rangle^2 \). This is a very important measure because the concurrence tells us how the two qubits keep entanglement which provide the possibility to implement quantum parallelism calculations. Figure 4 shows the time evolution of the concurrence. When compared with the decays of the corresponding coherent oscillations in Figs. 2(a) and 4(a), we can find that the entanglement decays more rapidly than the observables \( \langle \sigma^y_1 \rangle \) and \( \langle \sigma^y_2 \rangle \) in the presence of decoherence.

**B. decoherence via charge degree**

Next, we suppose that the interaction between the system and the environment is only through the total charge...
can suggest that it is possible to design a single

states is kept perfectly. We find this is because the two basis amplitudes, and we have confirmed that the pure state amazingly keep oscillating without any reduction of the time evolution gives the behavior of the expectation values, shown in Fig. 5(a). We can see \( \langle \sigma_1^z \rangle \) and \( \langle \sigma_2^z \rangle \) oscillate sinusoidally. The numerical calculation of the position state of the quantum suppression of decoherence by introducing a redundant type of interaction might appear in the presence of residual charges near the qubit junctions. When the initial state is composing the initial state construct a Decoherence Free Subspace (DFS) for the particle-junctions. When the initial state of the qubit junctions. When the initial state is an example of a suppression of decoherence by introducing a redundant qubit and a qubit-qubit interaction, like quantum error correction codes. Figure 5(b) represents time evolution of the concurrence and we can make sure that the entanglement never deteriorates because of the suppression of decoherence in the DFS.

V. SUMMARY AND CONCLUSION

In summary, we analyzed the time evolution of an inductively coupled flux qubit system in the presence of environment. We confirmed the importance of quantum fluctuations of the observable coupled to the environment. We also found two basis states constructing a DFS for charge fluctuations.

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