Comparing Linear Width Parameters for Directed Graphs

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Abstract
In this paper we introduce the linear clique-width, linear NLC-width, neighbourhood-width, and linear rank-width for directed graphs. We compare these parameters with each other as well as with the previously defined parameters directed path-width and directed cut-width. It turns out that the parameters directed linear clique-width, directed linear NLC-width, directed neighbourhood-width, and directed linear rank-width are equivalent in that sense, that all of these parameters can be upper bounded by each of the others. For the restriction to digraphs of bounded vertex degree directed path-width and directed cut-width are equivalent. Further for the restriction to semi-complete digraphs of bounded vertex degree all six mentioned width parameters are equivalent. We also show close relations of the measures to their undirected versions of the underlying undirected graphs, which allow us to show the hardness of computing the considered linear width parameters for directed graphs. Further we give first characterizations for directed graphs defined by parameters of small width.

Keywords Graph parameters · Directed graphs · Directed path-width · Directed threshold graphs

1 Introduction
A graph parameter is a function that associates with every graph a positive integer. Examples for graph parameters are tree-width [58], clique-width [17], NLC-width [60], and rank-width [56]. Clique-width, NLC-width, and rank-width are equivalent, i.e. a graph has bounded clique-width, if and only if it has bounded NLC-width.
and that is if and only if it has bounded rank-width. The latter three parameters are more general than tree-width, since graphs of bounded tree-width also have bounded clique-width but even for dense graphs (e.g. cliques) the tree-width is unbounded while the clique-width can be small [36]. Graph classes of bounded width are interesting from an algorithmic point of view since several hard graph problems can be solved in polynomial time by dynamic programming along the tree structure of the input graph, see [2, 4, 40, 46] and [15, 16, 20, 38]. Furthermore such parameters are also interesting from a structural point of view, e.g. in the research of special graph classes [9, 11].

In this paper we study directed graph parameters which are defined by the existence of an underlying path-structure for the input graph. The parameters of our interest are obtained by generalizing path-width [57], cut-width [1], linear clique-width [37], linear NLC-width [37], neighbourhood-width [32], and linear rank-width [26] to directed graphs. With the exception of cut-width these parameters can be regarded as restrictions of the above mentioned parameters with underlying tree-structure to an underlying path-structure. The relation between these parameters corresponds to their tree-structural counterparts, since bounded path-width implies bounded linear NLC-width, linear clique-width, neighbourhood-width, and linear rank-width. Further the reverse direction is not true in general, see [32]. Such restrictions to underlying path-structures are often helpful to show results for the general parameters, see [21, 22]. These linear parameters are also interesting from a structural point of view, e.g. in the research of special graph classes [26, 31, 42].

Since several problems and applications frequently use directed graphs, during the last years, width parameters for directed graphs have received a lot of attention, see [27, 28] and the two book chapters [6, Chapter 9] and [19, Chapter 6]. Lifting the above mentioned parameters using an underlying tree-structure to directed graphs lead to directed tree-width [44], directed NLC-width [39], directed clique-width [17], and directed rank-width [45].

One of the most famous examples for a directed graph parameter defined by the existence of an underlying path-structure is the directed path-width, which was introduced by Reed, Seymour, and Thomas around 1995 (see [7]) and studied in [7, 49, 50, 59]. Further the cut-width for directed graphs was introduced by Chudnovsky et al. in [12]. Regarding the usefulness of linear width parameters for undirected graphs we introduce the directed linear NLC-width, directed linear clique-width, directed neighbourhood-width, and directed linear rank-width. In contrast to the linear width measures for undirected graphs, for directed graphs their relations turn out to be more involved. Table 1 shows some classes of digraphs demonstrating various possible combinations of the listed width measures being bounded and unbounded.

For all these linear width parameters for directed graphs we compare the directed width of a digraph and the undirected width of its underlying undirected graph, which allow us to show the hardness of computing the considered linear width parameters for directed graphs.

In order to classify graph parameters we call two graph parameters \( \alpha \) and \( \beta \) equivalent, if there are two functions \( f_1 \) and \( f_2 \) such that for every digraph \( G \) the value \( \alpha(G) \) can be upper bounded by \( f_1(\beta(G)) \) and the value \( \beta(G) \) can be upper bounded by \( f_2(\alpha(G)) \). If \( f_1 \) and \( f_2 \) are polynomials or linear functions, we call \( \alpha \) and \( \beta \)
| Measure                  | Undirected | Directed | DAG | CB | BS | OP | TT |
|--------------------------|------------|----------|-----|----|----|----|----|
| Cut-width                | cutw [1]   | d-cutw [12] | 0   | ∞  | ∞  | 0  | 0  |
| Path-width               | pw [57]    | d-pw     | 0   | ∞  | 1  | 0  | 0  |
| Linear clique-width      | lcw [37]   | d-lcw    | ∞   | 2  | 2  | 3  | 2  |
| Linear NLC-width         | lnlcw [37] | d-lnlcw  | ∞   | 1  | 1  | 3  | 1  |
| Neighbourhood-width      | nw [32]    | d-nw     | ∞   | 1  | 1  | 2  | 1  |
| Linear rank-width        | lrw [26]   | d-lrw    | ∞   | 1  | 1  | 2  | 1  |
polynomially equivalent or linearly equivalent, respectively. We show that for general digraphs we have three sets of pairwise equivalent parameters, namely \{d-cutw\}, \{d-pw\}, and \{d-nw, d-lnlcw, d-lcw, d-lrw\}. For digraphs of bounded vertex degree this reduces to two sets \{d-cutw, d-pw\} and \{d-nw, d-lnlcw, d-lcw, d-lrw\} and for semicomplete digraphs of bounded vertex degree all these six graph parameters are pairwise equivalent. With the exception of directed rank-width, the same results are even shown for polynomially and linearly equivalence.

By introducing the class of directed threshold graphs, we give characterizations for graphs defined by parameters of small width.

2 Preliminaries

We use the notations of Bang-Jensen and Gutin [5] for graphs and digraphs.

2.1 Undirected Graphs

We work with finite undirected graphs \(G = (V, E)\), where \(V\) is a finite set of vertices and \(E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}\) is a finite set of edges. For a vertex \(v \in V\) we denote by \(N_G(v)\) the set of all vertices which are adjacent to \(v\) in \(G\), i.e. \(N_G(v) = \{w \in V \mid (v, w) \in E\}\). Set \(N_G(v)\) is called the set of all neighbours of \(v\) in \(G\) or neighbourhood of \(v\) in \(G\). The degree of a vertex \(v \in V\), denoted by \(\deg_G(v)\), is the number of neighbours of vertex \(v\) in \(G\), i.e. \(\deg_G(v) = |N_G(v)|\). The maximum vertex degree is \(\Delta(G) = \max_{v \in V} \deg_G(v)\). A graph \(G' = (V', E')\) is a subgraph of graph \(G = (V, E)\) if \(V' \subseteq V\) and \(E' \subseteq E\). If every edge of \(E\) with both end vertices in \(V'\) is in \(E'\), we say that \(G'\) is an induced subgraph of digraph \(G\) and we write \(G' = G[V']\). For some graph class \(F\) we define \(\text{Free}(F)\) as the set of all graphs \(G\) such that no induced subgraph of \(G\) is isomorphic to a member of \(F\).

Special Undirected Graphs We recall some special graphs. By \(P_n = ([v_1, \ldots, v_n], \{(v_1, v_2), \ldots, (v_{n-1}, v_n)\})\), \(n \geq 2\), we denote a path on \(n\) vertices and by \(C_n = ([v_1, \ldots, v_n], \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\})\), \(n \geq 3\), we denote a cycle on \(n\) vertices. Further by \(K_n = ([v_1, \ldots, v_n], \{(v_i, v_j) \mid 1 \leq i < j \leq n\})\), \(n \geq 1\), we denote a complete graph on \(n\) vertices and by \(K_{n,m} = ([v_1, \ldots, v_n, w_1, \ldots, w_m], \{(v_i, w_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\})\) a complete bipartite graph on \(n + m\) vertices.

2.2 Directed Graphs

A directed graph or digraph is a pair \(G = (V, E)\), where \(V\) is a finite set of vertices and \(E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}\) is a finite set of ordered pairs of distinct\(^1\) vertices called arcs. For a vertex \(v \in V\), the sets \(N^+_G(v) = \{u \in V \mid (v, u) \in E\}\) and \(N^-_G(v) = \{u \in V \mid (u, v) \in E\}\) are called the set of all out-neighbours and the set of

\(^1\)Thus we do not consider directed graphs with loops.
all in-neighbours of \( v \). The outdegree of \( v \), \( \text{outdegree}_G(v) \) for short, is the number of out-neighbours of \( v \) and the indegree of \( v \), \( \text{indegree}_G(v) \) for short, is the number of in-neighbours of \( v \) in \( G \). The maximum out-degree is \( \Delta^+(G) = \max_{v \in V} \text{outdegree}_G(v) \) and the maximum in-degree is \( \Delta^-(G) = \max_{v \in V} \text{indegree}_G(v) \). The maximum vertex degree is \( \Delta(G) = \max_{v \in V} \text{outdegree}_G(v) + \text{indegree}_G(v) \). A digraph \( G' = (V', E') \) is a subdigraph of digraph \( G = (V, E) \) if \( V' \subseteq V \) and \( E' \subseteq E \). If every arc of \( E \) with both end vertices in \( V' \) is in \( E' \), we say that \( G' \) is an induced subdigraph of digraph \( G \) and we write \( G' = G[V'] \). For some digraph class \( F \) we define \( \text{Free}(F) \) as the set of all digraphs \( G \) such that no induced subdigraph of \( G \) is isomorphic to a member of \( F \).

Let \( G = (V, E) \) be a digraph.
- \( G \) is edgeless if for all \( u, v \in V, u \neq v \), none of the two pairs \( (u, v) \) and \( (v, u) \) belongs to \( E \).
- \( G \) is a tournament if for all \( u, v \in V, u \neq v \), exactly one of the two pairs \( (u, v) \) and \( (v, u) \) belongs to \( E \).
- \( G \) is semicomplete if for all \( u, v \in V, u \neq v \), at least one of the two pairs \( (u, v) \) and \( (v, u) \) belongs to \( E \).
- \( G \) is (bidirectional) complete if for all \( u, v \in V, u \neq v \), both of the two pairs \( (u, v) \) and \( (v, u) \) belong to \( E \).

**Omitting the Directions** For some given digraph \( G = (V, E) \), we define its underlying undirected graph by ignoring the directions of the edges, i.e. \( \text{und}(G) = (V, \{(u, v) \mid (u, v) \in E \text{ or } (v, u) \in E\}) \).

**Orientations** There are several ways to define a digraph \( G = (V, E) \) from an undirected graph \( G_u = (V, E_u) \). If we replace every edge \( \{u, v\} \in E_u \) by
- one of the arcs \( (u, v) \) and \( (v, u) \), we denote \( G \) as an orientation of \( G_u \). Every digraph \( G \) which can be obtained by an orientation of some undirected graph \( G_u \) is called an oriented graph.
- one or both of the arcs \( (u, v) \) and \( (v, u) \), we denote \( G \) as a biorientation of \( G_u \). Every digraph \( G \) which can be obtained by a biorientation of some undirected graph \( G_u \) is called a bioriented graph.
- both arcs \( (u, v) \) and \( (v, u) \), we denote \( G \) as a complete biorientation of \( G_u \). Since in this case \( G \) is well defined by \( G_u \) we also denote it by \( \rightarrow G_u \). Every digraph \( G \) which can be obtained by a complete biorientation of some undirected graph \( G_u \) is called a complete bioriented graph.

**Special Directed Graphs** We recall some special directed graphs. By \( \vec{P}_n = (\{v_1, \ldots, v_n\}, (\{v_1, v_2\}, \ldots, (v_{n-1}, v_n)) \), \( n \geq 2 \) we denote a directed path on \( n \) vertices and by \( \vec{C}_n = (\{v_1, \ldots, v_n\}, (\{v_1, v_2\}, \ldots, (v_{n-1}, v_n), (v_n, v_1)) \), \( n \geq 2 \) we denote a directed cycle on \( n \) vertices. The \( k \)-power graph \( G^k \) of a digraph \( G \) is a graph with the same vertex set as \( G \). There is an arc \( (u, v) \) in \( G^k \) if and only if there is a directed path from \( u \) to \( v \) of length at most \( k \) in \( G \). An oriented forest (tree) is the orientation of a forest (tree). A digraph is an out-tree (in-tree) if it is an oriented tree.
in which there is exactly one vertex of indegree (outdegree) zero. A directed acyclic digraph (DAG for short) is a digraph without any $\overrightarrow{C_n}$, $n \geq 2$ as subdigraph. Further let $\overrightarrow{K_n} = \{(v_1, \ldots, v_n), ((v_i, v_j) | 1 \leq i \neq j \leq n)\}$ be a bidirectional complete digraph on $n$ vertices.

2.3 Directed Co-graphs and Directed Threshold Graphs

2.3.1 Directed Co-graphs

Next we introduce operations in order to recall the definition of directed co-graphs from [8] and introduce an interesting and useful subclass. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex-disjoint directed graphs.

- The disjoint union of $G_1$ and $G_2$, denoted by $G_1 \oplus G_2$, is the digraph with vertex set $V_1 \cup V_2$ and arc set $E_1 \cup E_2$.
- The series composition of $G_1$ and $G_2$, denoted by $G_1 \otimes G_2$, is the digraph with vertex set $V_1 \cup V_2$ and arc set $E_1 \cup E_2 \cup \{(u, v), (v, u) | u \in V_1, v \in V_2\}$.
- The order composition of $G_1$ and $G_2$, denoted by $G_1 \oslash G_2$, is the digraph with vertex set $V_1 \cup V_2$ and arc set $E_1 \cup E_2 \cup \{(u, v) | u \in V_1, v \in V_2\}$.

**Definition 1** (Directed co-graphs, [8]) The class of directed co-graphs is recursively defined as follows.

(i) Every digraph on a single vertex ($\{v\}, \emptyset$), denoted by $\bullet$, is a directed co-graph.
(ii) If $G_1$ and $G_2$ are directed co-graphs, then (a) $G_1 \oplus G_2$, (b) $G_1 \otimes G_2$, and (c) $G_1 \oslash G_2$ are directed co-graphs.

In [18] it has been shown that directed co-graphs can be characterized by the eight forbidden induced subdigraphs shown in Table 2. In [39] the relation of directed co-graphs to the set of graphs of directed NLC-width 1 and to the set of graphs of directed clique-width 2 is analyzed.

| $D_1$ | $D_2$ | $D_3$ |
|-------|-------|-------|
| ![D1](image1) | ![D2](image2) | ![D3](image3) |
| $D_4$ | $D_5$ | $D_6$ |
| ![D4](image4) | ![D5](image5) | ![D6](image6) |
| $D_7$ | $D_8$ |
| ![D7](image7) | ![D8](image8) |

Table 2 The eight forbidden induced subdigraphs for directed co-graphs (see [18])
2.3.2 Directed Threshold Graphs

In order to characterize digraphs of directed linear NLC-width 1 and digraphs of directed neighbourhood-width 1 we introduce the following subclass of directed co-graphs.

Definition 2 (Directed threshold graphs) The class of directed threshold graphs is recursively defined as follows.

(i) Every digraph on a single vertex \( (\{v\}, \emptyset) \), denoted by \( \bullet \), is a directed threshold graph.
(ii) If \( G \) is a directed threshold graph, then (a) \( G \oplus \bullet \), (b) \( G \odot \bullet \), (c) \( \bullet \odot G \), and (d) \( G \otimes \bullet \) are directed threshold graphs.

In Theorem 11 we will show that directed threshold graphs can be characterized by the eighteen forbidden induced subdigraphs shown in Tables 2, 4, 5, and 6.

The related class oriented threshold graphs was considered by Boeckner in [10] by using all given operations except the series composition \( G \otimes \bullet \).

Observation 1 Every oriented threshold graph is a directed threshold graph and every directed threshold graph is a directed co-graph.

3 Linear Width Parameters for Directed Graphs

A layout of a graph \( G = (V, E) \) is a bijective function \( \varphi : V \to \{1, \ldots, |V|\} \). For a graph \( G \), we denote by \( \Phi(G) \) the set of all layouts for \( G \). Given a layout \( \varphi \in \Phi(G) \) we define for \( 1 \leq i \leq |V| \) the vertex sets

\[
L(i, \varphi, G) = \{u \in V \mid \varphi(u) \leq i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V \mid \varphi(u) > i\}.
\]

The reverse layout \( \varphi^R \), for \( \varphi \in \Phi(G) \), is defined by \( \varphi^R(u) = |V| - \varphi(u) + 1, u \in V \).

3.1 Directed Path-Width

The path-width (pw) for undirected graphs was introduced in [57]. The notion of directed path-width was introduced by Reed, Seymour, and Thomas around 1995 (cf. [7]) and relates to directed tree-width introduced by Johnson, Robertson, Seymour, and Thomas in [44].

Definition 3 (directed path-width) Let \( G = (V, E) \) be a digraph. A directed path-decomposition of \( G \) is a sequence \( (X_1, \ldots, X_r) \) of subsets of \( V \), called bags, such that the following three conditions hold true.

\[\text{Please note that there are some works which define the path-width of a digraph } G \text{ in a different and not equivalent way by using the path-width of } \text{und}(G), \text{ see Section 7.}\]
(1) $X_1 \cup \ldots \cup X_r = V$.
(2) For each $(u, v) \in E$ there is a pair $i \leq j$, such that $u \in X_i$ and $v \in X_j$.\(^3\)
(3) For all $i, j, \ell$ with $1 \leq i < j < \ell \leq r$, we have $X_i \cap X_\ell \subseteq X_j$.

The *width* of a directed path-decomposition $\mathcal{X} = (X_1, \ldots, X_r)$ is

$$\max_{1 \leq i \leq r} |X_i| - 1.$$ 

The directed path-width of $G$, $\text{d-pw}(G)$, for short, is the smallest integer $w$ such that there is a directed path-decomposition for $G$ of width $w$.

There are a number of results on algorithms for computing directed path-width. The directed path-width of a digraph $G = (V, E)$ can be computed in time $O(|E| \cdot |V|^{2d-pw(G)/(d-pw(G) - 1)})$ by [49] and in time $O(d\text{-pw}(G) \cdot |E| \cdot |V|^{2d-pw(G)})$ by [53]. This leads to XP-algorithms for directed path-width w.r.t. the standard parameter and implies that for each constant $w$, it is decidable in polynomial time whether a given digraph has directed path-width at most $w$. Further it is shown in [50] how to decide whether the directed path-width of an $\ell$-semicomplete digraph is at most $w$ in time $(\ell + 2w + 1)^{2w} \cdot n^{O(1)}$. Furthermore the directed path-width can be computed in time $3^{\tau(\text{und}(G))} \cdot |V|^{O(1)}$, where $\tau(\text{und}(G))$ denotes the vertex cover number of the underlying undirected graph of $G$, by [51]. For sequence digraphs with a given decomposition into $k$ sequence the directed path-width can be computed in time $O(k \cdot (1 + N)^k)$, where $N$ denotes the maximum sequence length [35]. Further the directed path-width (and also the directed tree-width) can be computed in linear time for directed co-graphs [34].

Example for digraphs of small directed path-width are given in Example 1, when considering the equivalent (cf. Lemma 6) notation of directed vertex separation number.

### 3.2 Directed Vertex Separation Number

The vertex separation number (vsn) for undirected graphs was introduced in [52]. In [61] the directed vertex separation number for a digraph $G = (V, E)$ has been introduced as follows.

**Definition 4** (directed vertex separation number, [61]) The *directed vertex separation number* of a digraph $G = (V, E)$ is defined as follows.

$$\text{d-vsn}(G) = \min_{\varphi \in \Phi(G)} \max_{1 \leq i \leq |V|} |\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : (v, u) \in E\}| \quad (1)$$

\(^3\)Please note that condition (3) in Definition 3 is the only difference to the undirected path-width [57], where $i = j$ has to be fulfilled.
Since the converse digraph has the same path-width as its original graph, we obtain an equivalent definition, which will be useful later on.

\[ d\text{-vsn}(G) = \min_{\varphi \in \Phi_1(G)} \max_{1 \leq i \leq |V|} |\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : (u, v) \in E\}| \quad (2) \]

**Example 1** (directed vertex separation number)

(a) Every directed path \( \overrightarrow{P}_n \) has directed vertex separation number 0.
(b) The \( k \)-power graph \( (\overrightarrow{P}_n)^k \) of a directed path \( \overrightarrow{P}_n \) has directed vertex separation number 0.
(c) Every directed cycle \( \overrightarrow{C}_n \) has directed vertex separation number 1.
(d) The bidirectional complete digraph \( \overrightarrow{K}_3 \) and the complete biorientation of a star \( K_{2, 2, 2} \) have directed vertex separation number 2.\(^4\)
(e) Every bidirectional complete digraph \( \overrightarrow{K}_n \) has directed vertex separation number \( n - 1 \).

### 3.3 Directed Cut-Width

The cut-width (cutw) of undirected graphs was introduced in [1]. The cut-width of digraphs was introduced by Chudnovsky, Fradkin, and Seymour in [12].

**Definition 5** (directed cut-width, [12]) The *directed cut-width* of digraph \( G = (V, E) \) is

\[ d\text{-cutw}(G) = \min_{\varphi \in \Phi_1(G)} \max_{1 \leq i \leq |V|} |\{(v, u) \in E \mid u \in L(i, \varphi, G), v \in R(i, \varphi, G)\}| \quad (3) \]

For every optimal layout \( \varphi \) we obtain the same value when we consider the arcs backwards in the reverse ordering \( \varphi^R \). Thus we obtain an equivalent definition, which will be useful later on.

\[ d\text{-cutw}(G) = \min_{\varphi \in \Phi_1(G)} \max_{1 \leq i \leq |V|} |\{(v, u) \in E \mid u \in L(i, \varphi, G), v \in R(i, \varphi, G)\}| \quad (4) \]

Subexponential parameterized algorithms for computing the directed cut-width of semicomplete digraphs are given in [25].

**Example 2** (directed cut-width)

(a) Every directed path \( \overrightarrow{P}_n \) has directed cut-width 0.
(b) The \( k \)-power graph \( (\overrightarrow{P}_n)^k \) of a directed path \( \overrightarrow{P}_n \) has directed directed cut-width 0.
(c) Every directed cycle \( \overrightarrow{C}_n \) has directed cut-width 1.
(d) The bidirectional complete digraph \( \overrightarrow{K}_3 \) has directed cut-width 2.
(e) Every bidirectional complete digraph \( \overrightarrow{K}_n \) has directed cut-width \( \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \).

\(^4\)We use the complete biorientations of the two forbidden minors for the set of all graphs of vertex separation number 1, see [48, Fig. 1].

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3.4 Directed Linear NLC-Width

The linear NLC-width (Inlcw) for undirected graphs was introduced in [37] as a parameter by restricting the NLC-width, defined in [60], to an underlying path-structure. Next we introduce the corresponding parameter for directed graphs by a modification of the edge inserting operation \( \times_S \) of the linear NLC-width, which also leads to a restriction of directed NLC-width [39]. Let \([k] = \{1, \ldots, k\}\) be the set of all integers between 1 and \(k\).

**Definition 6** (directed linear NLC-width) The directed linear NLC-width of a digraph \(G\), d-Inlcw\((G)\) for short, is the minimum number of labels needed to define \(G\) using the following three operations:

1. Creation of a new vertex with label \(a\) (denoted by \(\bullet_a\)).
2. Disjoint union of a labeled digraph \(G\) and a single vertex \(v\) labeled by \(a\) plus all arcs between label pairs from \(\rightarrow S\) directed from \(G\) to \(v\) and all arcs between label pairs from \(\leftarrow S\) directed from \(v\) to \(G\) for two relations \(\rightarrow S\) and \(\leftarrow S\) (denoted by \(G \otimes (\rightarrow S, \leftarrow S) \bullet_a\)).
3. Change every label \(a\) into label \(R(a)\) by some function \(R\) (denoted by \(\circ_R\)).

The directed linear NLC-width of an unlabeled digraph \(G = (V, E)\) is the smallest integer \(k\), such that there is a mapping \(\text{lab} : V \rightarrow [k]\) such that the labeled digraph \((V, E, \text{lab})\) has directed linear NLC-width at most \(k\). An expression \(X\) built with the operations defined above is called a directed linear NLC-width \(k\)-expression. Note that every expression defines a layout by the order in which the vertices are inserted in the corresponding digraph. The digraph defined by expression \(X\) is denoted by \(\text{val}(X)\).

**Example 3** (directed linear NLC-width) (a) Every bidirectional complete digraph \(\overrightarrow{K}_n\) has directed linear NLC-width 1.
(b) The directed paths \(\overrightarrow{P}_3\) and \(\overrightarrow{P}_4\) have directed linear NLC-width 2.
(c) Every directed path \(\overrightarrow{P}_n\) has directed linear NLC-width at most 3.
(d) Every directed cycle \(\overrightarrow{C}_n\) has directed linear NLC-width at most 4.
(e) Every \(k\)-power graph \((\overrightarrow{P}_n)^k\) of a directed path \(\overrightarrow{P}_n\) has directed linear NLC-width at most \(k + 2\).
(f) Every complete biorientation of a grid \(\overrightarrow{G}_{n}^n\), \(n \geq 3\), has directed linear NLC-width at least \(n\) and at most \(n + 2\), see [30, 33].

\(^5\)The abbreviation NLC results from the node label controlled embedding mechanism originally defined for graph grammars.
3.5 Directed Linear Clique-Width

The linear clique-width (lcw) for undirected graphs was introduced in [37] as a parameter by restricting the clique-width, defined in [17], to an underlying path-structure. Next we introduce the corresponding parameter for directed graphs by a modification of the edge inserting operation of the linear clique-width, which also leads to a restriction for directed clique-width [17].

**Definition 7** (directed linear clique-width) The directed linear clique-width of a digraph $G$, $d$-lcw($G$) for short, is the minimum number of labels needed to define $G$ using the following four operations:

1. Creation of a new vertex with label $a$ (denoted by $\bullet_a$).
2. Disjoint union of a labeled digraph $G$ and a single vertex labeled by $a$ (denoted by $G \oplus \bullet_a$).
3. Inserting an arc from every vertex with label $a$ to every vertex with label $b$ ($a \neq b$, denoted by $a_{a,b}$).
4. Change label $a$ into label $b$ (denoted by $\rho_{a \rightarrow b}$).

The linear clique-width of an unlabeled digraph $G = (V, E)$ is the smallest integer $k$, such that there is a mapping $\mathrm{lab} : V \rightarrow [k]$ such that the labeled digraph $(V, E, \mathrm{lab})$ has linear linear clique-width at most $k$. An expression $X$ built with the operations defined above is called a directed linear clique-width $k$-expression. Note that every expression defines a layout by the order in which the vertices are inserted in the corresponding digraph. The digraph defined by expression $X$ is denoted by $\mathrm{val}(X)$.

**Example 4** (directed linear clique-width) 
(a) Every edgeless digraph has directed linear clique-width 1.
(b) Every bidirectional complete digraph $\overrightarrow{K_n}$ has directed linear clique-width 2.
(c) Every directed path $\overrightarrow{P_n}$ has directed linear clique-width at most 3.
(d) Every directed cycle $\overrightarrow{C_n}$ has directed linear clique-width at most 4.
(e) Every $k$-power graph $(\overrightarrow{P_n})^k$ of a directed path $\overrightarrow{P_n}$ has directed linear clique-width at most $k + 2$. For $n \geq k(k + 1) + 2$ the given bound on the directed linear clique-width is even exact by Corollary 1.
(f) Every complete biorientation of a grid $\overleftrightarrow{G_n}$, $n \geq 3$, has directed linear clique-width at least $n$ and at most $n + 2$, see [30, 33].

3.6 Directed Neighbourhood-Width

The neighbourhood-width (nw) for undirected graphs was introduced in [32]. It differs from linear NLC-width and linear clique-width at most by one but it is independent of vertex labels.

Let $G = (V, E)$ be a digraph and $U, W \subseteq V$ two disjoint vertex sets. The set of all out-neighbours of $u$ belonging to set $W$ and the set of all in-neighbours of $u$
belonging to set $W$ are defined by $N^+_W(u) = \{ v \in W \mid (u, v) \in E \}$ and $N^-_W(u) = \{ v \in W \mid (v, u) \in E \}$. The directed neighbourhood of vertex $u$ into set $W$ is defined by $N_W(u) = (N^+_W(u), N^-_W(u))$ and the set of all directed neighbourhoods of the vertices of set $U$ into set $W$ is $N(U, W) = \{ N_W(u) \mid u \in U \}$. For some layout $\varphi \in \Phi(G)$ we define $d$-nw$(\varphi, G) = \max_{1 \leq i \leq |V|} |N(L(i, \varphi, G), R(i, \varphi, G))|$. 

**Definition 8** (directed neighbourhood-width) The directed neighbourhood-width of digraph $G$ is

$$d$-nw$(G) = \min_{\varphi \in \Phi(G)} d$-nw$(\varphi, G)$. 

**Example 5** (directed neighbourhood-width) (a) Every bidirectional complete digraph $\vec{K}_n$ has directed neighbourhood-width 1.
(b) Every directed path $\vec{P}_n$ has directed neighbourhood-width at most 2.
(c) Every directed cycle $\vec{C}_n$ has directed neighbourhood-width at most 3.
(d) Every $k$-power graph $(\vec{P}_n)^k$ of a directed path $\vec{P}_n$ has directed neighbourhood-width at most $k + 1$. For $n \geq k(k + 1) + 2$ the given bound on the directed neighbourhood-width is even exact by Corollary 1.
(e) Every complete biorientation of a grid $\vec{G}_n$, $n \geq 3$, has directed neighbourhood-width at least $n$ and at most $n + 1$, see [30, 33].

### 3.7 Directed Linear Rank-Width

The rank-width for directed graphs was introduced in Kanté in [45]. In [26] the linear rank-width (lrw) for undirected graphs was introduced by restricting the tree-structure of a rank decomposition to special caterpillars (i.e. a path in which every inner vertex may have a pendant vertex), which is also possible for the directed case as follows.

Let $G = (V, E)$ a digraph and $V_1, V_2 \subseteq V$ be a disjoint partition of the vertex set of $G$. Further let $M^{V_2}_{V_1} = (m_{ij})$ be the adjacent matrix defined over the four-element field GF(4) for partition $V_1 \cup V_2$, i.e.

$$m_{ij} = \begin{cases} 
0 & \text{if } (v_i, v_j) \notin E \text{ and } (v_j, v_i) \notin E \\
\varnothing & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \notin E \\
\varnothing^2 & \text{if } (v_i, v_j) \notin E \text{ and } (v_j, v_i) \in E \\
1 & \text{if } (v_i, v_j) \in E \text{ and } (v_j, v_i) \in E
\end{cases}$$

In GF(4) we have four elements $\{0, 1, \varnothing, \varnothing^2\}$ with the properties $1 + \varnothing + \varnothing^2 = 0$ and $\varnothing^3 = 1$.

**Definition 9** (directed linear rank-width) A directed linear rank decomposition of digraph $G = (V, E)$ is a pair $(T, f)$, where $T$ is a path in which every inner vertex has a pendant vertex and $f$ is a bijection between $V$ and the leaves of $T$. Each edge $e$ of $T$ divides the vertex set of $G$ by $f$ into two disjoint sets $A_e, B_e$. For an edge $e$
in $T$ we define the width of $e$ as $\text{rg}^{(4)}(M_{Ae})$, i.e. the matrix rank$^6$ of $M$. The width of a directed linear rank decomposition $(T, f)$ is the maximal width of all edges in $T$. The directed linear rank-width of a digraph $G$, $d\text{-lrw}(G)$ for short, is the minimum width of all directed linear rank decompositions for $G$.

**Example 6** (directed linear rank-width) (a) Every bidirectional complete digraph $\overrightarrow{K_n}$ and every directed path $\overrightarrow{P_n}$ has directed linear rank-width 1.
(b) Every directed cycle $\overrightarrow{C_n}$ has directed linear rank-width at most 2.
(c) Every complete biorientation of a grid $\overrightarrow{G_n}$, $n \geq 3$, has directed linear rank-width at least $\lceil \frac{2n}{3} \rceil$ and at most $n + 1$, see [33, 43].

### 4 Directed Width and Undirected Width

Next we compare the directed width of a digraph $G$ and the undirected width of its underlying undirected graph $\text{und}(G)$.

**Theorem 1** Let $G$ be a directed graph.

(a) $d\text{-pw}(G) \leq p\text{w}(\text{und}(G))$
(b) $d\text{-cutw}(G) \leq \text{cutw}(\text{und}(G))$
(c) $n\text{w}(\text{und}(G)) \leq d\text{-nw}(G) \leq \Delta(\text{und}(G)) \cdot n\text{w}(\text{und}(G))$
(d) $l\text{nlcw}(\text{und}(G)) \leq d\text{-lncw}(G) \leq \Delta(\text{und}(G)) \cdot l\text{nlcw}(\text{und}(G)) + 1$
(e) $l\text{cw}(\text{und}(G)) \leq d\text{-lcw}(G) \leq \Delta(\text{und}(G)) \cdot l\text{cw}(\text{und}(G)) + 1$
(f) $l\text{rw}(\text{und}(G)) \leq d\text{-lrw}(G) \leq \Delta(\text{und}(G)) \cdot 2^{\text{lrw}(\text{und}(G)) + 1} - 1$

**Proof** (a) A path-decomposition for $\text{und}(G)$ of width $k$ is also a directed path-decomposition for $G$ of width $k$.

(b) Let $G = (V, E)$ be a digraph and $\text{und}(G)$ be the underlying undirected graph of cut-width $k$. Let $\varphi$ be the corresponding ordering of the vertices, such that for every $i$, $1 \leq i \leq |V|$ there are at most $k$ edges $\{u, v\}$ such that $u \in L(i, \varphi, \text{und}(G))$ and $v \in R(i, \varphi, \text{und}(G))$. Since every undirected edge $\{u, v\}$ in $\text{und}(G)$ comes from a directed edge $(u, v)$, a directed edge $(v, u)$, or both, and the directed cut-width only counts edges directed forward, the same layout shows that the directed cut-width of $G$ is at most $k$.

(c) Let $G = (V, E)$ be a digraph of directed neighbourhood-width $k$ and $\varphi \in \Phi(G)$ a linear layout, such that for every $i \in [|V|]$, we have $|N(L(i, \varphi, G), R(i, \varphi, G))| \leq k$. Since for every pair of vertices in $G$ of the same directed neighbourhood the corresponding vertices in $\text{und}(G)$ have the same neighbourhood, it follows that for every $i \in [|V|]$, we have

$^6$We denote by $\text{rg}^{(4)}(M)$ the rank of some matrix over $\{0, 1, \emptyset, \emptyset^2\}$, i.e. the number of independent lines or rows of $M$. A set of rows $R$ (i.e. vectors) are independent, if there is no linear combination of a subset $R'$ of $R$ to define a row in $R - R'$. A linear combination for some $n$-tuple $r$ is $\sum_{i=1}^{n} a_i r_i$ for $a_i \in \{0, 1, \emptyset, \emptyset^2\}$. Springer
Let $G = (V, E)$ be a digraph and $und(G) = (V, E_u)$ be the underlying undirected graph of neighbourhood-width $k$. Then there is a layout $\varphi \in \Phi(und(G))$, such that for every $1 \leq i \leq |V|$ the vertices in $L(i, \varphi, und(G))$ can be divided into at most $k$ subsets $L_1, \ldots, L_k$, such that the vertices of set $L_j$, $1 \leq j \leq k$, have the same neighbourhood with respect to the vertices in $R(i, \varphi, und(G))$. One of these sets $L_j$ may consist of vertices having no neighbours $v \in R(i, \varphi, und(G))$. Let $1 \leq i \leq |V|$.

- If there is one set $L_j$ which consists of vertices having no neighbours $v \in R(i, \varphi, und(G))$, then there are at most $\Delta(und(G)) \cdot (k - 1)$ vertices $u \in L(i, \varphi, und(G))$, such that there is an edge $\{v, u\} \in E_u$ with $v \in R(i, \varphi, und(G))$.

- Otherwise there are at most $\Delta(und(G)) \cdot k$ vertices $u \in L(i, \varphi, und(G))$, such that there is an edge $\{v, u\} \in E_u$ with $v \in R(i, \varphi, und(G))$.

Thus for every $1 \leq i \leq |V|$ the vertices in $L(i, \varphi, G)$ can be divided into $k' \leq \Delta(und(G)) \cdot k$ subsets $L'_1, \ldots, L'_{k'}$, such that the vertices of set $L'_j$, $1 \leq j \leq k'$, have the same directed neighbourhood with respect to the vertices in $R(i, \varphi, G)$. Thus the directed neighbourhood-width of $G$ is at most $\Delta(und(G)) \cdot k$.

(d) Let $G$ be a digraph of directed linear NLC-width $k$ and $X$ be a directed linear NLC-width $k$-expression for $G$. A linear NLC-width $k$-expression $c(X)$ for $und(G)$ can recursively be defined as follows.

- Let $X = \bullet_t$ for $t \in [k]$. Then $c(X) = \bullet_t$.
- Let $X = \circ_R(X')$ for $R : [k] \to [k]$. Then $c(X) = \circ_R(c(X'))$.
- Let $X = X' \otimes (\overrightarrow{S}, \overleftarrow{S}) \bullet_t$ for $\overrightarrow{S}, \overleftarrow{S} \subseteq [k]^2$ and $t \in [k]$. Then $c(X) = c(X') \times_{\overrightarrow{S} \cup \overleftarrow{S}} \bullet_t$.

The second bound follows by

$$d-lnlcw(G) \leq d-nw(G) + 1 \leq \Delta(und(G)) \cdot nw(und(G)) + 1 \leq \Delta(und(G)) \cdot lnlcw(und(G)) + 1,$$

whereas the inequalities hold, respectively, by Lemma 2, (c), and [32].

(e) Let $G$ be a digraph of directed linear clique-width $k$ and $X$ be a directed linear clique-width $k$-expression for $G$. A linear clique-width $k$-expression $c(X)$ for $und(G)$ can recursively be defined as follows.

- Let $X = \bullet_t$ for $t \in [k]$. Then $c(X) = \bullet_t$.
- Let $X = X' \oplus \bullet_t$ for $t \in [k]$. Then $c(X) = c(X') \oplus \bullet_t$.
- Let $X = \rho_{i \to j}(X')$ for $i, j \in [k]$. Then $c(X) = \rho_{i \to j}(c(X'))$.
- Let $X = \alpha_{i,j}(X')$ for $i, j \in [k]$. Then $c(X) = \eta_{i,j}(c(X'))$. 
The second bound follows by
\[
\begin{align*}
d\text{-}lcw(G) & \leq d\text{-}nw(G) + 1 \leq \Delta(\text{und}(G)) \cdot \text{nw}(\text{und}(G)) + 1 \\
& \leq \Delta(\text{und}(G)) \cdot \text{lcw}(\text{und}(G)) + 1.
\end{align*}
\]
whereas the inequalities hold, respectively, by Lemma 2, (c), and \[32\].

(f) Let \(G\) be a digraph of directed linear rank-width \(k\) and \((T, f)\) be a directed linear rank-decomposition for \(G\) of width \(k\). Then \((T, f)\) is also a linear rank-decomposition for \(\text{und}(G)\). Let \(e\) be an edge of \(T\). Let \(N^{V_1}_{V_2} = (n_{ij})\) be the adjacent matrix defined over the two-element field GF(2) for partition \(V_1 \cup V_2\). If for \(G\) two rows in \(M^{B_e}_{A_e}\) are linearly dependent then for \(\text{und}(G)\) these two rows in \(N^{B_e}_{A_e}\) are also linearly dependent. Thus we conclude that \(\text{rg}^{(2)}(N^{B_e}_{A_e}) \leq \text{rg}^{(4)}(M^{B_e}_{A_e})\) and thus linear rank-width of \(\text{und}(G)\) \(\leq k\).

The second bound follows by
\[
\begin{align*}
d\text{-}lrw(G) & \leq d\text{-}nw(G) \leq \Delta(\text{und}(G)) \cdot \text{nw}(\text{und}(G)) \\
& \leq \Delta(\text{und}(G)) \cdot 2^{\text{lw}(\text{und}(G)) + 1} - 1,
\end{align*}
\]
whereas the inequalities hold, respectively, by Lemma 3, (c), and \[56, Proposition 6.3\].

This completes the proof.

Remark 1 In Theorem 1(a) and (b) the directed path-width of some digraph cannot be used to give an upper bound on the path-width of \(\text{und}(G)\). Any transitive tournament has directed path-width 0 but its underlying undirected graph has a path-width which corresponds to the number of vertices. Also by restricting the vertex degree this is not possible by an acyclic orientation of a grid. The same examples also show that the directed cut-width of some digraph cannot be used to give an upper bound on the cut-width of \(\text{und}(G)\).

Remark 2 Theorem 1(a) and (b) show that for path-width and cut-width the values do not grow when going to the directed variant. This changes for the other four parameters, since the set of all tournaments has unbounded directed width while the corresponding undirected width of set of all complete graphs is bounded by a small constant.

The relations shown in Theorem 1 allow to imply the following values for the directed linear clique-width and directed neighbourhood-width of a \(k\)-power graph of a path.

Corollary 1 (a) For \(n \geq k(k + 1) + 2\), we have \(d\text{-}lcw((\overrightarrow{P_n})^k) = k + 2\).

(b) For \(n \geq k(k + 1) + 2\), we have \(d\text{-}nw((\overrightarrow{P_n})^k) = k + 1\).

Proof For \(n \geq k(k + 1) + 2\) we know from \[41\] that the (undirected) linear clique-width of a \(k\)-power graph of a path on \(n\) vertices is exactly \(k + 2\).
(a) For \( n \geq k(k + 1) + 2 \) the first statement follows by
\[
  \text{lcw}(\text{und}(\vec{P}_n)^k)) \leq \text{d-lcw}(\vec{P}_n)^k) \leq k + 2,
\]
whereas these equality and inequalities hold, respectively, by [41], Theorem 1, and Example 4.

(b) For \( n \geq k(k + 1) + 2 \) the second statement follows by
\[
  \text{lcw}(\text{und}(\vec{P}_n)^k)) - 1 \leq \text{d-nw}(\vec{P}_n)^k) \leq k + 1,
\]
whereas these equality and inequalities hold, respectively, by [41], Theorem 1, Lemma 2, and Example 5.

This completes the proof.

Comparing the undirected width of a graph \( G \) and the directed width of its complete biorientation \( \vec{G} \) the following results hold.

**Theorem 2** For each width measure \( \beta \in \{ \text{pw}, \text{cutw}, \text{nw}, \text{lnlcw}, \text{lcw}, \text{lrw} \} \) and every undirected graph \( G \), we have \( \beta(G) = d-\beta(\vec{G}) \).

**Proof**
- Since \( G \) is the underlying undirected graph of \( \vec{G} \), by Theorem 1(a) it remains to show that the path-width of \( G \) is at most the directed path-width of \( \vec{G} \). Let \( (X_1, \ldots, X_r) \) be a directed path-decomposition for \( \vec{G} = (V, E) \). For every \( (u, v) \in E \), we have \( u \in X_i \) and \( v \in X_j \) for \( i \leq j \). If \( i < j \) then since in \( \vec{G} \) there is also the arc \( (v, u) \) we obtain a contradiction. Thus, we have \( i = j \) which implies that the given path-decomposition is also a path-decomposition for \( G \).
- By Theorem 1(b) it remains to show that the cut-width of \( G \) is at most the directed cut-width of \( \vec{G} \). Let \( G = (V, E) \) be a graph and \( \vec{G} \) its complete biorientation of directed cut-width \( k \). Let \( \varphi \) be the corresponding ordering of the vertices, such that for every \( i, 1 \leq i \leq |V| \) there are at most \( k \) arcs \( (u, v) \) such that \( u \in L(i, \varphi, \text{und}(\vec{G})) \) and \( v \in R(i, \varphi, \text{und}(\vec{G})) \). Since every such arc corresponds to one undirected edge \( \{u, v\} \) in \( G \), the same layout shows that the cut-width of \( G \) is at most \( k \).
- By Theorem 1(c) it remains to show that the directed neighbourhood-width of \( \vec{G} \) is at most the neighbourhood-width of \( G \). Let \( \varphi \in \Phi(G) \) a linear layout, such that for every \( i \in [|V|] \), we have \( |N(L(i, \varphi, G), R(i, \varphi, G))| \leq k \). By the definitions of \( \vec{G} \) and for neighbourhoods of directed graphs, it follows that for every \( i \in [|V|] \) for the number of directed neighbourhoods, we have \( |N(L(i, \varphi, \vec{G})), R(i, \varphi, \vec{G}))| \leq k \).
- By Theorem 1(d) it remains to show that the directed linear NLC-width of \( \vec{G} \) is at most the linear NLC-width of \( G \). Let \( X \) be an NLC-width \( k \)-expression for \( G \). A directed NLC-width \( k \)-expression \( c(X) \) for \( \vec{G} \) can recursively be defined as follows.
Let \( X = \bullet_t \) for \( t \in [k] \). Then \( c(X) = \bullet_t \).

Let \( X = \circ_R(X') \) for \( R : [k] \rightarrow [k] \). Then \( c(X) = \circ_R(c(X')) \).

Let \( X = X' \times_S X'' \) for \( S \subseteq [k]^2 \). Then \( c(X) = c(X') \otimes (S,S) c(X'') \).

By Theorem 1(e) it remains to show that the directed linear clique-width of \( \overrightarrow{G} \) is at most the linear clique-width of \( G \). Let \( X \) be a clique-width \( k \)-expression for \( G \). A directed clique-width \( k \)-expression \( c(X) \) for \( \overrightarrow{G} \) can recursively be defined as follows:

Let \( X = \bullet_t \) for \( t \in [k] \). Then \( c(X) = \bullet_t \).

Let \( X = X' \oplus X'' \). Then \( c(X) = c(X') \oplus c(X'') \).

Let \( X = \rho_{i \rightarrow j}(X') \) for \( i, j \in [k] \). Then \( c(X) = \rho_{i \rightarrow j}(c(X')) \).

Let \( X = \eta_{i,j}(X') \) for \( i, j \in [k] \). Then \( c(X) = \alpha_{j,i}(\alpha_{i,j}(c(X'))) \).

By Theorem 1(f) it remains to show that the directed linear rank-width of \( \overrightarrow{G} \) is at most the linear rank-width of \( G \). Let \((T, f)\) be a linear rank-decomposition of width \( k \) for \( G \). Then \((T, f)\) is also a linear rank-decomposition for \( \overrightarrow{G} \). Let \( N_{V_2}^{V_1} = (n_{ij}) \) be the adjacent matrix defined over the two-element field \( \text{GF}(2) \) for partition \( V_1 \cup V_2 \). Since for every bioriented graph \( N_{V_2}^{V_1} = M_{V_2}^{V_1} \) we conclude that the directed linear rank-width of \( \overrightarrow{G} \) is at most \( k \).

This completes the proof.

It is already known that recognizing path-width [3], cut-width [29], linear NLC-width [32], linear clique-width [21], neighbourhood-width [32], and linear rank-width (by [55] due [47] and [54]) are NP-hard. The results of Theorem 2 imply the same for the directed versions.

**Corollary 2** Given a digraph \( G \) and an integer \( k \), then for every width measure \( \beta \in \{d-pw, d-cutw, d-nw, d-lnlcw, d-lcw, d-lrw\} \), the problem to decide whether \( \beta(G) \leq k \) is NP-complete.

## 5 Comparing Linear Width Parameters

In order to classify graph parameters we use the following notations. Let \( \mathcal{G} \) be the set of all finite directed graphs and \( \alpha, \beta : \mathcal{G} \mapsto \mathbb{N} \) be two graph parameters. Parameters \( \alpha \) and \( \beta \) are called equivalent, if there is a function \( f_1 : \mathbb{N} \mapsto \mathbb{N} \) such that for every \( G \in \mathcal{G} \) we have \( \alpha(G) \leq f_1(\beta(G)) \) and there is a function \( f_2 : \mathbb{N} \mapsto \mathbb{N} \) such that for every \( G \in \mathcal{G} \) we have \( \beta(G) \leq f_2(\alpha(G)) \). Parameters \( \alpha \) and \( \beta \) are called polynomially equivalent, if they are equivalent and both functions \( f_1 \) and \( f_2 \) are polynomials. Parameters \( \alpha \) and \( \beta \) are called linearly equivalent, if they are equivalent and both functions \( f_1 \) and \( f_2 \) are linear.

\[ \text{Please note that two parameters are equivalent, if and only if the same families of digraphs have bounded width for them.} \]
5.1 Relations Between Linear NLC-Width, Linear Clique-Width, Neighbourhood-Width, and Linear Rank-Width

First we state the relation between the directed linear NLC-width and directed linear clique-width. The proofs can be done in the same way as for the undirected versions in [37].

**Lemma 1** For every digraph $G$, we have

$$d\text{-lnlcw}(G) \leq d\text{-lcw}(G) \leq d\text{-lnlcw}(G) + 1.$$  

Further there is also a very tight connection between the directed neighbourhood-width, directed linear NLC-width, and directed linear clique-width. The proofs of the following bounds can be done in a similar fashion as for the undirected versions in [32].

**Lemma 2** For every digraph $G$, we have

$$d\text{-nw}(G) \leq d\text{-lnlcw}(G) \leq d\text{-nw}(G) + 1$$

and

$$d\text{-nw}(G) \leq d\text{-lcw}(G) \leq d\text{-nw}(G) + 1.$$  

By the examples given in Section 3 and simple observations, we conclude that every path $\overrightarrow{P_n}$, $n \geq 3$, has directed linear clique-width 3, paths $\overrightarrow{P_3}$ and $\overrightarrow{P_4}$ have directed linear NLC-width 2, every path $\overrightarrow{P_n}$, $n \geq 5$, has directed linear NLC-width 3, and every path $\overrightarrow{P_n}$, $n \geq 3$, has directed neighbourhood-width 2, which implies that the bounds of Lemma 1 and Lemma 2 cannot be improved.

**Lemma 3** For every digraph $G$, we have

$$d\text{-lrw}(G) \leq d\text{-nw}(G).$$

**Proof** Let $G$ be a digraph with $n$ vertices of directed neighbourhood-width $k$ and $\varphi: V \to [n]$ be a layout such that $d\text{-nw}(\varphi, G) \leq k$. Using $\varphi$ we define a caterpillar $T_{\varphi}$ with consecutive pendant vertices $\varphi^{-1}(1), \ldots, \varphi^{-1}(n)$. Pair $(T_{\varphi}, \varphi)$ leads to a directed linear rank decomposition for $G$. We want to determine the width of $(T_{\varphi}, \varphi)$. Since for every $i$ the vertices in $L(i, \varphi, G)$ define at most $k$ neighbourhoods with respect to set $R(i, \varphi, G)$, every edge of $T_{\varphi}$ leads to a partition of $V$ into $L(i, \varphi, G)$ and $R(i, \varphi, G)$ for some $i$ such that $M_{L(i, \varphi, G)}^{R(i, \varphi, G)}$ has at most $k$ different rows and thus $\text{rg}(M_{L(i, \varphi, G)}^{R(i, \varphi, G)}) \leq k$.  

The following bound can be proved similarly to the case of clique-width and rank-width in [56, Proposition 6.3].

**Lemma 4** For every digraph $G$, we have

$$d\text{-lcw}(G) \leq 4^{d\text{-lrw}(G)+1} - 1.$$
The shown bounds imply the following theorem.

**Theorem 3** Any two parameters in \{\(d_{-}nw\), \(d_{-}lnlcw\), \(d_{-}lcw\), \(d_{-}lrw\)\} are equivalent.

**Theorem 4** Any two parameters in \{\(d_{-}nw\), \(d_{-}lnlcw\), \(d_{-}lcw\)\} are linearly equivalent.

Using the arguments of [23, Section 8] we obtain the next result.

**Lemma 5** There is some polynomial \(p\) such that for every digraph \(G\), we have \(d_{-}lcw(G) \leq p(\Delta(G), d_{-}lrw(G))\).

**Theorem 5** For every class of digraphs \(G\) such that, for all \(G \in G\) the value \(\Delta(G)\) is bounded, any two parameters in \{\(d_{-}nw\), \(d_{-}lnlcw\), \(d_{-}lcw\), \(d_{-}lrw\)\} are polynomially equivalent.

### 5.2 Relations Between Cut-Width and Path-Width

The directed path-width is even equal to the directed vertex separation number.

**Lemma 6** [61] For every digraph \(G\), we have

\[ d_{-}pw(G) = d_{-}vsn(G). \]

In [24] it is shown how to construct a directed path-decomposition of width twice the directed cut-width of the graph. Using the directed vertex separation number, we next show a better bound.

**Lemma 7** For every digraph \(G\), we have

\[ d_{-}pw(G) \leq d_{-}cutw(G). \]

**Proof** Let \(G = (V, E)\) be a digraph of directed cut-width \(k\). By (4) there is a layout \(\varphi \in \Phi_1(G)\), such that for every \(1 \leq i \leq |V|\) there are at most \(k\) arcs \((v, u) \in E\) such that \(v \in R(i, \varphi, G)\) and \(u \in L(i, \varphi, G)\). Thus for every \(1 \leq i \leq |V|\) there are at most \(k\) vertices \(u \in L(i, \varphi, G)\), such that there is an arc \((v, u) \in E\) with \(v \in R(i, \varphi, G)\). Thus by (1) the directed vertex separation number of \(G\) is at most \(k\) and by Lemma 6 the directed path-width of \(G\) is at most \(k\). \(\square\)

The directed path-width and directed cut-width of a digraph can differ very much, e.g. a \(\xleftrightarrow{} K_{1,n}\) has directed path-width 1 and directed cut-width \(\lceil \frac{n}{2} \rceil\).

**Lemma 8** For every digraph \(G\), we have

\[ d_{-}cutw(G) \leq \min(\Delta^{-}(G), \Delta^{+}(G)) \cdot d_{-}pw(G). \]

**Proof** Let \(G = (V, E)\) be a digraph of directed path-width \(k\). By Lemma 6 and (1) there is a layout \(\varphi \in \Phi(G)\), such that for every \(1 \leq i \leq |V|\) there are at most \(k\) arcs \((v, u) \in E\) such that \(v \in R(i, \varphi, G)\) and \(u \in L(i, \varphi, G)\). Thus for every \(1 \leq i \leq |V|\) there are at most \(k\) vertices \(u \in L(i, \varphi, G)\), such that there is an arc \((v, u) \in E\) with \(v \in R(i, \varphi, G)\). Thus by (1) the directed vertex separation number of \(G\) is at most \(k\) and by Lemma 6 the directed path-width of \(G\) is at most \(k\). \(\square\)
vertices \( u \in L(i, \varphi, G) \), such that there is an arc \((v, u) \in E \) with \( v \in R(i, \varphi, G) \). Thus for every \( 1 \leq i \leq |V| \) there are at most \( \Delta^-(G) \cdot k \) arcs \((v, u) \in E \) such that \( v \in R(i, \varphi, G) \) and \( u \in L(i, \varphi, G) \). By (4) this implies that the directed cut-width of digraph \( G \) is at most \( \Delta^-(G) \cdot k \).

The bound using \( \Delta^+ \) instead of \( \Delta^- \) can be shown in the same way using definition (2) instead of (1) and using definition (3) instead of (4).

**Theorem 6** For every class of digraphs \( \mathcal{G} \) such that for all \( G \in \mathcal{G} \) the value \( \min(\Delta^-(G), \Delta^+(G)) \) is bounded any two parameters in \( \{d\text{-cutw}, d\text{-pw}\} \) are linearly equivalent.

### 5.3 Relations Between Path-Width and Neighbourhood-Width

The directed neighbourhood-width and directed path-width of a digraph can differ very much, e.g. a \( \vec{K}_n \) has directed neighbourhood-width 1 and directed path-width \( n - 1 \).

**Lemma 9** For every digraph \( G \), we have
\[
d\text{-pw}(G) \leq \min(\Delta^- (G), \Delta^+(G)) \cdot d\text{-nw}(G).
\]

**Proof** Let \( G = (V, E) \) be a digraph of directed neighbourhood-width \( k \). Then there is a layout \( \varphi \in \Phi(G) \), such that for every \( 1 \leq i \leq |V| \) the vertices in \( L(i, \varphi, G) \) can be divided into at most \( k \) subsets \( L_1, \ldots, L_k \), such that the vertices of set \( L_j, 1 \leq j \leq k \), have the same neighbourhood with respect to the vertices in \( R(i, \varphi, G) \). Every of these sets \( L_j \) has at most \( \Delta^- (G) \) vertices \( u \) such that there is an arc \((v, u) \in E \) with \( v \in R(i, \varphi, G) \). Thus for every \( 1 \leq i \leq |V| \) there are at most \( \Delta^- (G) \cdot k \) vertices \( u \in L(i, \varphi, G) \), such that there is an arc \((v, u) \in E \) with \( v \in R(i, \varphi, G) \). Thus by (1) the directed vertex separation number of \( G \) is at most \( \Delta^- (G) \cdot k \) and by Lemma 6 the directed path-width of \( G \) is at most \( \Delta^- (G) \cdot k \).

The bound using \( \Delta^+ \) instead of \( \Delta^- \) can be shown in the same way using definition (2) instead of definition (1).

The example \( \vec{K}_n \) shows that the bound given in Lemma 9 is tight. Lemmas 9, 2, and 4 imply the following bounds.

**Corollary 3** For every digraph \( G \), we have
\[
d\text{-pw}(G) \leq \min(\Delta^- (G), \Delta^+(G)) \cdot d\text{-lnlcw}(G),
\]
\[
d\text{-pw}(G) \leq \min(\Delta^- (G), \Delta^+(G)) \cdot d\text{-lcw}(G), \text{ and}
\]
\[
d\text{-pw}(G) \leq \min(\Delta^- (G), \Delta^+(G)) \cdot (4^{d\text{-lw}(G)+1} - 1).
\]

This allows us to bound the directed path-width of directed threshold graphs (see Definition 2) as follows.
Corollary 4 The directed path-width of a directed threshold graph $G$ is at most $\min(\Delta^-(G), \Delta^+(G))$.

Proof The set of directed threshold graphs has directed linear NLC-width 1 (see Theorem 11 in Section 6). Thus the result follows by Corollary 3. \hfill \Box

Since $\Delta^-(G) \leq \Delta(G)$ and $\Delta^+(G) \leq \Delta(G)$ and thus

$$\min(\Delta^-(G), \Delta^+(G)) \leq \Delta(G)$$

the given bounds also hold for the more common measure $\Delta(G)$ instead of $\min(\Delta^-(G), \Delta^+(G))$.

After considering the maximum vertex degree, we next make a stronger restriction by excluding all possible orientations of a $K_{\ell,\ell}$ as subdigraphs.

Corollary 5 Let $G$ be a digraph where und($G$) has no $K_{\ell,\ell}$ subgraph, then we have

$$d\text{-}pw(G) \leq \text{pw(und}(G)) \leq 2 \cdot \lnlcw(\text{und}(G))(\ell - 1) \leq 2 \cdot \text{d-lnlcw}(G)(\ell - 1).$$

Proof By the results for undirected graphs in [32] we know that for every graph $G$ which has no $K_{\ell,\ell}$ subgraph, we have

$$\text{pw}(G) \leq 2 \cdot \lnlcw(G)(\ell - 1).$$

This implies for every digraph $G$, where und($G$) has no $K_{\ell,\ell}$ subgraph, we have

$$\text{pw(und}(G)) \leq 2 \cdot \lnlcw(\text{und}(G))(\ell - 1).$$

Furthermore by Theorem 1(a) and Theorem 1(d) for every digraph $G$, where und($G$) has no $K_{\ell,\ell}$ subgraph, we have

$$d\text{-}pw(G) \leq \text{pw(und}(G)) \leq 2 \cdot \lnlcw(\text{und}(G))(\ell - 1) \leq 2 \cdot \text{d-lnlcw}(G)(\ell - 1).$$

This completes the proof. \hfill \Box

This allows us to bound the directed path-width of planar directed threshold graphs (see Definition 2) as follows.

Corollary 6 Planar directed threshold graphs have directed path-width at most 4.

Proof The set of directed threshold graphs has directed linear NLC-width 1 (see Theorem 11 in Section 6) and for planar digraphs $G$ we know that und($G$) has no $K_{3,3}$ subgraph. Thus the result follows by Corollary 5. \hfill \Box

Next we want to bound the directed linear clique-width in terms of the directed path-width.

Remark 3 For general digraphs and even for digraphs of bounded vertex degree the directed linear clique-width, directed linear NLC-width, directed neighbourhood-width, and directed linear rank-width cannot be bounded by the directed path-width by the following examples.
1. Let $T'$ be an orientation of a tree, e.g., an out-tree or an in-tree. Then $d\text{-}\text{pw}(T') = 0$ by Theorem 10 in Section 6. But $d\text{-}\text{lcw}(T')$ is unbounded, since $\text{lcw}(\text{und}(T'))$ is unbounded [37] and since $\text{lcw}(\text{und}(T')) \leq d\text{-}\text{lcw}(T')$ by Theorem 1.

2. Let $G'$ be an acyclic orientation of a grid. Then $d\text{-}\text{pw}(G') = 0$ by Theorem 10 in Section 6. But $d\text{-}\text{lcw}(G')$ is unbounded, since $\text{lcw}(\text{und}(G'))$ is unbounded [30] and since $\text{lcw}(\text{und}(G')) \leq d\text{-}\text{lcw}(G')$ by Theorem 1.

3. The set of all $k$-power graphs of directed paths has directed path-width 0 (cf. Example 1) and directed linear clique-width $k + 2$ (Corollary 1).

For semicomplete digraphs the directed path-width can be used to give an upper bound on the directed clique-width, which has been shown in [24]. The main idea of the proof is to define a directed clique-width expression along a nice path-decomposition. Since the proof only uses directed linear clique-width operations we can state the next theorem.

**Lemma 10** [24] For every semicomplete digraph $S$, we have

$$d\text{-}\text{lcw}(S) \leq d\text{-}\text{pw}(S) + 2.$$ 

Lemmas 10, 1, 2, and 3 imply the following bounds.

**Corollary 7** For every semicomplete digraph $S$, we have

$$d\text{-}\text{lnlcw}(S) \leq d\text{-}\text{pw}(S) + 2,$$

$$d\text{-}\text{nw}(S) \leq d\text{-}\text{pw}(S) + 2,$$ and

$$d\text{-}\text{lrw}(S) \leq d\text{-}\text{pw}(S) + 2.$$ 

**Theorem 7** For every class of semicomplete digraphs $G$ such that for all $G \in G$ the value $\min(\Delta^-(G), \Delta^+(G))$ is bounded any two parameters in $\{d\text{-}\text{cutw}, d\text{-}\text{pw}, d\text{-}\text{nw}, d\text{-}\text{lnlcw}, d\text{-}\text{lcw}, d\text{-}\text{lrw}\}$ are equivalent.

Using the results of Theorem 1(a), [23, Section 8], and Theorem 1(f), respectively, there is some polynomial $p$ such that for every digraph $G$, we have

$$d\text{-}\text{pw}(G) \leq \text{pw}(\text{und}(G)) \leq p(\Delta(\text{und}(G)), \text{lrw}(\text{und}(G))) \leq p(\Delta(G), d\text{-}\text{lrw}(G)).$$

---

8Please note that in [24] a different notation for directed path-width was used. In Definition 3(2) the arcs are directed from bags $X_i$ to $X_j$ for $i \leq j$. The authors of [24] take arcs from bags $X_i$ to $X_j$ for $i \geq j$ into account. Since an optimal directed path-decomposition $(X_1, \ldots, X_r)$ w.r.t. Definition 3 leads to an optimal directed path-decomposition $(X_r, \ldots, X_1)$ w.r.t. the definition of [24], and vice versa, both definitions lead to the same value for the directed path-width.
Table 3  Classification of linear width parameters for directed graphs

| Digraphs          | Equivalence | d-cutw | d-pw | d-lcw | d-lnlcw | d-nw | d-lrw |
|-------------------|-------------|-------|------|-------|---------|------|-------|
| General           | equivalent  | •     | •    | •     | •       | •    | •     |
|                   | polynomially equivalent | • | • | • | • | • | • |
|                   | linearly equivalent   | • | • | • | • | • | • |
| Δ(G) bounded      | equivalent | • | • | • | • | • | • |
|                   | polynomially equivalent | • | • | • | • | • | • |
|                   | linearly equivalent | • | • | • | • | • | • |
| Semicomplete      | equivalent | • | • | • | • | • | • |
| Δ(G) bounded      | polynomially equivalent | • | • | • | • | • | • |
|                   | linearly equivalent | • | • | • | • | • | • |

The gray shades of the points represent sets of pairwise (linearly, polynomially) equivalent parameters.

This allows us to strengthen the result of Theorem 7 as follows.

**Theorem 8** For every class of semicomplete digraphs \( G \) such that for all \( G \in G \) the value \( \Delta(G) \) is bounded any two parameters in \( \{d-cutw, d-pw, d-nw, d-lnlcw, d-lcw, d-lrw\} \) are polynomially equivalent.

Except for directed linear rank-width we even have shown linear equivalence.

**Theorem 9** For every class of semicomplete digraphs \( G \) such that for all \( G \in G \) the value \( \min(\Delta^-(G), \Delta^+(G)) \) is bounded any two parameters in \( \{d-cutw, d-pw, d-nw, d-lnlcw, d-lcw\} \) are linearly equivalent.

By Lemmas 9 and 10 the restriction to semicomplete digraphs leads to the same relation between path-width an linear clique-width as for undirected graphs (see [32]).

### 5.4 Equivalent Parameters

In Table 3 we summarize our results on the equivalence of linear width parameters for directed graphs. For general digraphs we have three classes of pairwise equivalent parameters, which reduces to two or one class for \( \Delta(G) \) bounded or semicomplete \( \Delta(G) \) bounded digraphs, respectively.

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9When considering the directed path-width of almost semicomplete digraphs in [50] the class of semi-complete digraphs was suggested to be “a promising stage for pursuing digraph analogues of the splendid outcomes, direct and indirect, from the Graph Minors project”.

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6 Characterizations for Graphs Defined by Parameters of Small Width

First we summarize some quite obvious characterizations.

**Theorem 10** For every digraph $G$ the following statements are equivalent.

(a) $G$ is a DAG.
(b) $d\text{-vsn}(G) = 0$.
(c) $d\text{-pw}(G) = 0$.
(d) $d\text{-cutw}(G) = 0$.

For the next characterizations we use directed threshold graphs (see Definition 2) and the digraphs shown in Tables 2, 4, 5, and 6.

**Theorem 11** For every digraph $G$ the following statements are equivalent.

(a) $d\text{-lnlcw}(G) = 1$.
(b) $d\text{-nw}(G) = 1$.
(c) $d\text{-lcw}(G) \leq 2$ and $G \in \text{Free}([D_2, D_3, D_9, D_{10}, D_{12}, D_{13}, D_{14}])$.
(d) $G$ is a directed threshold graph.
(e) $G \in \text{Free}([D_1, \ldots, D_{15}, 2P_2, P_2 \cup \overrightarrow{P_2}, 2\overrightarrow{P_2}])$.
(f) $G \in \text{Free}([D_1, \ldots, D_6, D_{10}, D_{11}, D_{13}, D_{14}, D_{15}])$ and $\text{und}(G) \in \text{Free}([P_4, 2K_2, C_4])$.
(g) $G \in \text{Free}([D_1, \ldots, D_6, D_{10}, D_{11}, D_{13}, D_{14}, D_{15}])$ and $\text{und}(G)$ is a threshold graph.

**Proof** (a) $\Leftrightarrow$ (b) By the proof of Lemma 2 (which can be proved similarly to the case of the undirected versions in [32]) the set of all digraphs of directed linear NLC-width 1 is equal to the set of all digraphs of directed neighbourhood-width 1. 
(e) $\Rightarrow$ (d) If digraph $G$ does not contain $D_1, \ldots, D_8$ (see Table 2), then digraph $G$ is a directed co-graph by [18] and thus has a construction using disjoint union,
series composition, and order composition. By excluding $D_9$, $D_{10}$, and $D_{11}$ we know that for every series composition of $G_1$ and $G_2$ either $G_1$ or $G_2$ is bidirectional complete. Thus this subdigraph can also be added by a number of series operations with one vertex.

Further by excluding $D_{12}$, $D_{13}$, $D_{14}$, and $D_{15}$ we know that for every order composition of $G_1$ and $G_2$ either $G_1$ or $G_2$ is a tournament and since we exclude a directed cycle of length 3 by $D_5$, we know that $G_1$ or $G_2$ even is a transitive tournament. Thus this subdigraph can also be added by a number of order operations with one vertex.

By excluding $2\rightarrow P_2, P_2 \cup \leftarrow P_2, 2\leftarrow P_2$ for every disjoint union of $G_1$ and $G_2$ either $G_1$ or $G_2$ has no edge. Thus this subdigraph can also be added by a number of disjoint union operations with one vertex.

(a) $\Rightarrow$ (d) Let $G = (V, E)$ be a digraph of directed linear NLC-width 1 and $X$ be a directed linear NLC-width 1-expression for $G$. An expression $c(X)$ using directed threshold graph operations for $G$ can recursively be defined as follows.

- Let $X = \bullet_1$ for $t \in [k]$. Then $c(X) = \bullet_1$.
- Let $X = o_R(X')$ for $R : [1] \to [1]$. Then $c(X) = c(X')$.
- Let $X = X' \otimes (\vec{S}, \vec{S}) \bullet_1$ for $\vec{S}, \vec{S} \subseteq [1]^2$.
  - If $\vec{S} = \emptyset$ and $\vec{S} = \emptyset$, then $c(X)$ is the disjoint union of $c(X')$ and $\bullet_1$.
  - If $\vec{S} = \{(1, 1)\}$ and $\vec{S} = \emptyset$, then $c(X)$ is the order composition of $c(X')$ and $\bullet_1$.
  - If $\vec{S} = \emptyset$ and $\vec{S} = \{(1, 1)\}$, then $c(X)$ is the order composition of $\bullet_1$ and $c(X')$.
  - If $\vec{S} = \emptyset$ and $\vec{S} = \{(1, 1)\}$, then $c(X)$ is the series composition of $c(X')$ and $\bullet_1$.

(d) $\Rightarrow$ (a) Let $G = (V, E)$ be a directed threshold graph and $X$ be an expression using directed threshold graph operations for $G$. A directed linear NLC-width 1-expression $c(X)$ for $G$ can recursively be defined as follows.

- If $X$ defines a single vertex, then $c(X) = \bullet_1$.
- If $X$ defines the disjoint union of expression $X_1$ and $\bullet_1$, then $c(X) = c(X_1) \otimes (\emptyset, \emptyset) \bullet_1$
- If $X$ defines the order composition of expression $X_1$ and $\bullet_1$, then $c(X) = c(X_1) \otimes ((1,1), \emptyset) \bullet_1$
If $X$ defines the order composition of expression of $\bullet$ and $X_1$, then $c(X) = c(X_1) \otimes \langle (1, (1,1)) \rangle \bullet_1$

If $X$ defines the series composition of expression $X_1$ and $\bullet$, then $c(X) = c(X_1) \otimes \langle (1, (1,1)) \rangle \bullet_1$

(d) $\Rightarrow$ (c) Digraphs $D_2, D_3, D_9, D_{10}, D_{12}, D_{13}, D_{14}$ are not directed threshold graphs. Since directed threshold graphs are exactly graphs of directed linear NLC-width 1 ((a) $\Leftrightarrow$ (d)) has been shown above) by Lemma 1 we know that directed threshold graphs have directed linear clique-width at most 2.

(c) $\Rightarrow$ (e) Digraphs $D_1, D_4, D_5, D_6, D_7, D_8$ have directed clique-width greater than two and thus directed linear clique-width greater than two. $D_{11}, D_{15}$ have directed linear clique-width at least 3. Further $2\widehat{P}_2, \overrightarrow{P}_2 \cup \overrightarrow{P}_2, 2\overrightarrow{P}_2$ have an underlying $2K_2$ which has linear clique-width at least 3 and thus by Theorem 1(e) the directed linear clique-width of the three digraphs is also at least 3.

(d) $\Rightarrow$ (g) If $G$ is a directed threshold graph, then $\text{und}(G)$ is a threshold graph by the recursive definition. Further the given forbidden digraphs are no directed threshold graphs and the set of directed threshold graphs is closed under taking induced subdigraphs.

(f) $\Rightarrow$ (e) For digraphs $G$ which are excluded within (e) but not in (f), we have $\text{und}(G) \in \{P_4, C_4, 2K_2\}$.

(f) $\Leftrightarrow$ (g) Threshold graphs are exactly the set $\text{Free}(\{P_4, 2K_2, C_4\})$, see [13].

The set of digraphs of directed linear clique-width 1 is exactly the set of edgeless digraphs. While characterizing digraphs of directed linear NLC-width 1 could be done by a subclass of directed co-graphs, namely directed threshold graphs, this is not possible for digraphs of directed linear clique-width 2, since two of the forbidden induced subdigraphs for directed co-graphs ($D_2$ and $D_3$) have directed linear clique-width 2.

7 Conclusions

We reviewed the linear clique-width, linear NLC-width, neighbourhood-width, and linear rank-width for directed graphs. We compared these parameters with each other and also with the previously defined parameters directed path-width and directed cut-width. While for undirected graphs bounded cut-width implies directed path-width and directed path-width implies directed linear clique-width, linear NLC-width, neighbourhood-width, and linear rank-width (see [32]), for directed graphs the relations turn out to be more involved, see Table 1. For the restriction to semicomplete digraphs we obtain the same relation between these parameters as for undirected graphs (Lemmas 7, 10, and Corollary 7).

With the exception of directed cut-width the considered parameters can be regarded as restrictions from corresponding parameters with underlying tree-structure to an underlying path-structure. This implies that the values of the restricted parameters are greater or equal to the corresponding generalized version. This relation can be used to carry over lower bounds for parameterizations. For example
A further way to define the width of a digraph $G$ is to consider the width of the underlying undirected graph $\text{und}(G)$. This approach is used in works of Courcelle et al. [14, 15, 17], when considering the path-width and tree-width of directed graphs. We did not follow this approach, since it is less sensitive because by using the underlying undirected graph one cannot distinguish the direction of the edges. For example the existence of directed cycles in some digraph $G$ cannot be observed by the path-width of $\text{und}(G)$, while the approach of Reed, Seymour, and Thomas allows to find directed cycles by Theorem 10. Using undirected width leads to closer bounds. If we define for every digraph $G$ the value $u\text{-pw}(G) = \text{pw}(\text{und}(G))$, we can show the bound given in Lemma 10 for every arbitrary digraph $G$ as follows.

$$d\text{-nw}(G) \leq u\text{-pw}(G) + 1$$

This implies in connection with Lemma 2 the following bounds.

$$d\text{-lnlcw}(G) \leq d\text{-lcw}(G) \leq u\text{-pw}(G) + 2.$$  

The latter bound for the directed linear clique-width in terms of this approach for directed path-width was also obtained in [15, Proposition 2.114].

There are several interesting open questions. By Corollary 2 recognizing all considered linear width parameters for directed graphs is NP-hard. There are some xp-algorithms for directed path-width (cf. Section 3.1) while xp-algorithms for the other width parameters are uninvestigated up to now. Further there are fpt-algorithms for computing directed cut-width of semicomplete digraphs [25] and computing directed path-width of $\ell$-semicomplete digraphs [50]. For the other width parameters fpt-algorithms are unknown, even for semicomplete digraphs.

In Theorem 1 for several parameters we could show upper and lower bounds for the directed width of some digraph $G$ in terms of the undirected width of $\text{und}(G)$ and the maximum vertex degree of $\text{und}(G)$. For directed path-width and directed cut-width this is only possible within one direction (see Remark 1). It remains to find such bounds for special digraphs, e.g. Eulerian digraphs which were useful for directed tree-width and tree-width in (2.2) of [44].

In Lemma 4 we have shown how to bound the directed linear clique-width of a digraph exponentially in its directed linear rank-width and in Lemma 5 we used the results of [23] to show a polynomial bound for graphs of bounded vertex degree. It remains to study the existence of polynomial and linear bounds in general and linear bounds for digraphs of bounded vertex degree and semicomplete digraphs of bounded vertex degree in order to fill the four open cells in Table 3.

In order to characterize digraphs of directed linear rank-width 1 in terms of special graph operations, we propose to generalize the notation of thread graphs from [26] to digraphs.

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