STABILITY FOR THE SOBOLEV INEQUALITY: EXISTENCE OF A MINIMIZER

TOBIAS KÖNIG

ABSTRACT. We prove that the stability inequality associated to Sobolev’s inequality and its set of optimizers \( \mathcal{M} \) and given by

\[
\frac{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2}{\inf_{h \in \mathcal{M}} \|\nabla (f-h)\|_{L^2(\mathbb{R}^d)}^2} \geq c_{BE} > 0 \quad \text{for every } f \in \dot{H}^1(\mathbb{R}^d),
\]

which is due to Bianchi and Egnell, admits a minimizer for every \( d \geq 3 \). Our proof consists in an appropriate refinement of a classical strategy going back to Brezis and Lieb. As a crucial ingredient, we establish the strict inequality \( c_{BE} < 2 - 2^\frac{d}{d-2} \), which means that a sequence of two asymptotically non-interacting bubbles cannot be minimizing. Our arguments cover in fact the analogous stability inequality for the fractional Sobolev inequality for arbitrary fractional exponent \( s \in (0,d/2) \) and dimension \( d \geq 2 \).

1. Introduction and main results

The famous stability inequality due to Bianchi and Egnell [2] states that, for dimension \( d \geq 3 \), there is a constant \( c_{BE} > 0 \) such that

\[
\frac{\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2}{\inf_{h \in \mathcal{M}} \|\nabla (f-h)\|_{L^2(\mathbb{R}^d)}^2} \geq c_{BE} \quad \text{for all } f \in \dot{H}^1(\mathbb{R}^d) \setminus \mathcal{M}. \tag{1.1}
\]

Here,

\[
\mathcal{M} := \left\{ x \mapsto c(a + |x-b|^2)^{-\frac{d-2}{2}} : a > 0, b \in \mathbb{R}^d, c \in \mathbb{R} \setminus \{0\} \right\} \tag{1.2}
\]

is the \((d+2)\)-dimensional manifold of Talenti bubbles, i.e., optimizers of the Sobolev inequality

\[
\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \geq S_d \|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \tag{1.3}
\]

with sharp constant \( S_d > 0 \). Indeed, \((1.1)\) makes a statement about the stability of the Sobolev inequality, in the sense that if the ‘deficit’ \( \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \) is very small for some \( f \in \dot{H}^1(\mathbb{R}^d) \), then \( f \) must be very close to \( \mathcal{M} \), in a quantitative fashion.

While the power two in the denominator of the left side of \((1.1)\) is well known to be optimal, it is a long-standing open question to determine the value of the best constant
\( c_{BE} \) in \( (1.1) \), see, e.g., the up-to-date surveys \([13, 18]\). Indeed, the proof of \( (1.1) \) in \([2]\) proceeds by compactness and therefore does not yield any positive lower bound on the value of \( c_{BE} \). Only very recently the first explicit lower bound on \( c_{BE} \) has been established in the remarkable paper \([14]\). We also refer to \([1, 8]\) for explicit constants in similar stability inequalities and to \([11]\) for a related abstract result.

A key step towards determining the value of \( c_{BE} \), which is explicitly mentioned as an open problem in \([14]\), consists in establishing the existence of an optimizer for \( (1.1) \). To achieve this, one needs to investigate the behavior of a minimizing sequence \( (f_n) \) for \( (1.1) \). Must \( (f_n) \) converge towards a minimizer, or, on the contrary, is the optimal value of \( c_{BE} \) only attained asymptotically along a certain sequence \( (f_n) \) with zero weak limit in \( \dot{H}^s(\mathbb{R}^d) \)?

We shall prove as a main result of this paper that the first alternative always holds. In particular, the Bianchi-Egnell inequality \( (1.1) \) always admits a minimizer.

1.1. Compactness vs. non-compactness of minimizing sequences. The fact that the existence of a minimizer for \( (1.1) \) cannot be proved in a straightforward way has to do with two natural scenarios for the behavior of minimizing sequences which could both prevent existence of a minimizer for \( c_{BE} \).

The simpler one of these scenarios consists of minimizing sequences \( (f_n) \) that converge towards \( \mathcal{M} \). (Since the quotient in \( (1.1) \) is ill-defined for \( f \in \mathcal{M} \), the limit of such sequences, even if non-zero, is not a minimizer.) The optimal value associated to this type of sequences can be obtained in terms of a spectral problem already analyzed in \([2]\), see also \([23]\). It is given by \( c_{BE}^{\text{spec}} := \frac{4}{d+4} \). In the recent article \([21]\), the author has excluded such behavior for minimizing sequences by showing the strict inequality \( c_{BE} < c_{BE}^{\text{spec}} \).

There is, however, another plausible scenario for non-compact minimizing sequences, namely a sequence \( (u_n) \) consisting of two Talenti bubbles which are asymptotically non-interacting by virtue of having different concentration behavior and/or being centered far apart from each other. A back-of-the-envelope calculation, which can be made rigorous as in Section 3 shows that for a configuration of two bubbles having equal mass and center, and diverging concentration rates, the quotient in \( (1.1) \) reaches the value \( c_{BE}^{2\text{-peak}} := 2 - 2\frac{d-2}{d} \) in the limit. (It can be checked that other model configurations built from non-interacting Talenti bubbles, including ones that involve more than two bubbles, do not yield a smaller value of the Bianchi-Egnell quotient.) A side remark one can make here is that, somewhat surprisingly, the question whether one or two bubbles yield a lower value in \( (1.1) \) turns out to depend on the dimension. Namely, \( c_{BE}^{\text{spec}} < c_{BE}^{2\text{-peak}} \) for \( 3 \leq d \leq 6 \), while \( c_{BE}^{\text{spec}} > c_{BE}^{2\text{-peak}} \) for \( d \geq 7 \).

Similarly to the strict inequality \( c_{BE} < c_{BE}^{\text{spec}} \) from \([21]\), we will prove in Theorem 1.1 that \( c_{BE} < c_{BE}^{2\text{-peak}} \) strictly. Thus the two-bubble configurations described above cannot
be minimizing sequences either. It turns out that the conjunction of these two strict inequalities is enough to enforce the existence of a minimizer. This is the content of Theorem 1.2 below.

Inequality (1.1) forms part of a wider class of geometric-type stability inequalities. Some celebrated quantitative stability results concern the Sobolev inequality for $W^{1,p}$ [12, 16, 17], the isoperimetric inequality [19], the logarithmic Sobolev inequality [7, 15] or the lowest eigenvalue of Schrödinger operators [9]. For most of these the existence of an optimal function is not known to date, let alone the explicit value of the stability constant corresponding to $c_{BE}$. A positive result one can mention here concerns the planar ($d = 2$) isoperimetric inequality. For this case, an optimal set for the associated stability inequality is shown to exist in [3].

1.2. Main results. Since all of our arguments work identically for any fractional order $s \in (0, d/2)$, we shall state and prove our main results in this more general situation. That is, for any $d \geq 1$ and $s \in (0, d/2)$ we consider the fractional inequality of Bianchi-Egnell-type

$$\inf_{f \in \dot{H}^s(\mathbb{R}^d) \setminus M} \mathcal{E}(f) =: c_{BE}(s) > 0,$$

for

$$\mathcal{E}(f) := \frac{\|(-\Delta)^{s/2} f\|_2^2 - S_d\|f\|_{2^*}^2}{\text{dist}(f, \mathcal{M})^2},$$

which was proved in [10]. Here and in the rest of the paper, we abbreviate $\| \cdot \|_{L^p(\mathbb{R}^d)} = \| \cdot \|_p$ and let

$$2^* := \frac{2d}{d - 2s}$$

be the critical Sobolev exponent. We denote by

$$\mathcal{M} := \left\{ x \mapsto c(a + |x - b|^2)^{-d-2s} : a > 0, b \in \mathbb{R}^d, c \in \mathbb{R} \setminus \{0\} \right\}$$

the manifold of fractional Talenti bubbles, i.e. optimizers of the Sobolev inequality

$$S(f) := \frac{\|(-\Delta)^{s/2} f\|_2^2}{\|f\|_{2^*}^2} \geq S_d > 0.$$

with sharp constant $S_d$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm $\|(-\Delta)^{s/2} f\|_2$. See, e.g., [20] for some more details about $\dot{H}^s(\mathbb{R}^d)$ and the fractional Laplacian $(-\Delta)^s$. We will always consider $\dot{H}^s(\mathbb{R}^d)$ to be equipped with that norm. Finally, in (1.4) and henceforth we employ the notation

$$\text{dist}(f, \mathcal{M}) := \inf_{h \in \mathcal{M}} \|(-\Delta)^{s/2} (f - h)\|_2$$

for the distance in $\dot{H}^s(\mathbb{R}^d)$ between $f$ and $\mathcal{M}$. (Since the value of $s \in (0, d/2)$ can be considered as fixed throughout, we choose to not include the parameter $s$ into the notation for $\mathcal{M}$, $S_d$, $2^*$ etc. in order to keep a lighter notation, with the exception of $c_{BE}(s)$.)
Our first main result is the strict inequality with respect to the constant coming from two non-interacting bubbles.

**Theorem 1.1.** For every $d \geq 1$ and $s \in (0, d/2)$, one has $c_{\text{BE}}(s) < 2 - 2^{d-2s}$. 

The proof of Theorem 1.1 consists in an asymptotic expansion of a sum $u_{\lambda}$ of two Talenti bubbles supported on length scales $1$ and $\lambda^{-1}$ respectively, as $\lambda \to 0$. It can be verified that $E(u_{\lambda}) \to 2 - 2^{d-2s}$ as $\lambda \to 0$, and that the first lower-order correction comes with a negative sign. This approach bears some similarity with that used in [21] to prove the one-bubble inequality $c_{\text{BE}}(s) < 4s^{d} + 2$ for $d \geq 2$, but the details of the required arguments and computations are entirely different. We give the proof of Theorem 1.1 in Section 3 below.

As a consequence of Theorem 1.1, together with some further analysis, we obtain that all minimizing sequences for (1.4) must converge towards a non-trivial minimizer.

**Theorem 1.2.** Let $d \geq 2$ and $s \in (0, d/2)$. Let $(u_{n})$ be a minimizing sequence for (1.4) with $\|(-\Delta)^{s/2}u_{n}\|_{2} = 1$. Then there is $u \in \dot{H}^{s}(\mathbb{R}^{d}) \setminus \mathcal{M}$ such that, up to extracting a subsequence, $u_{n} \to u$ strongly in $\dot{H}^{s}(\mathbb{R}^{d})$. Moreover, $E(u) = c_{\text{BE}}(s)$, i.e., $u$ is a minimizer for $c_{\text{BE}}(s)$.

Let us give a brief overview over the main ideas of the proof of Theorem 1.2. Its basic strategy is similar to existence proofs for simpler functionals, e.g. the Sobolev functional $S(f) = \|(-\Delta)^{s/2}f\|_{2}/\|f\|_{2}^{2}$, and goes back to Brezis’ and Lieb’s work [6]. After one has extracted a non-zero weak limit $f$ from a minimizing sequence $(f_{n})$, a suitable convexity property of the functional together with the Brezis–Lieb lemma from [6] shows that the value of $S(f_{n})$ can be strictly improved unless the weak limit has full mass. The sequence $(f_{n})$ being minimizing, a strict improvement is excluded, and hence the weak limit has full mass and is in fact a strong limit and a minimizer.

In the present situation, this idea is harder to put into practice due to the more complicated structure of the Bianchi-Egnell functional (1.4). The main difficulty stems from the term $\text{dist}(u_{n}, \mathcal{M})^{2}$, which cannot be split 'symmetrically' under a decomposition $u_{n} = f + g_{n}$. This makes it less obvious to deduce a strict improvement of $E(u_{n})$ using convexity. To overcome this, we first find by some careful arguments (including convexity) that $f$ and $g_{n}$ must both be rescaled and translated bubbles of equal mass unless $g_{n}$ vanishes asymptotically. Then we can rule out $f$ and $g_{n}$ both being bubbles by the strict inequality from Theorem 1.1. Finally, $u_{n}$ cannot converge to $\mathcal{M}$ because of the strict inequality from [21]. (Note that the result from [21] is only valid for $d \geq 2$. This is the only place in the proof where we use this assumption.) We refer to the proof of Theorem 1.2 and Remark 4.4 for more details.

This last step, which uses Theorem 1.1 and [21], reflects a widely known phenomenon in non-compact minimization problems going back to, at least, Lieb’s lemma in the
seminal paper by Brezis and Nirenberg [5, Lemma 1.2]. The decisive observation is that, for many variational problems with some loss of compactness, minimizing sequences do converge if they lie strictly below some universal energy threshold given by some limit problem. Remarkably, in the present context of the Bianchi-Egnell inequality, we find that two such compactness thresholds (i.e. the fact that $c_{BE}$ is strictly smaller than both of them) need to be taken into account to prove existence of a minimizer, namely the value $c_{BE}^{2\text{-peak}}(s) = 2 - 2^{\frac{d}{s}}$ and the value $c_{BE}^{\text{spec}}(s) = \frac{4s}{d+2+2s}$.

One can note that for this argument, at least when $s = 1$, the independent proof of Theorem 1.1 we give below is really only relevant for large dimensions $d$. Indeed, when $\frac{4s}{d+2+2s} < 2 - 2^{\frac{d+2s}{2}}$ (e.g. when $s = 1$ and $3 \leq d \leq 6$, as already observed above), then Theorem 1.1 follows immediately from the standard spectral inequality $c_{BE}(s) \leq \frac{4s}{d+2+2s}$. Conversely, when $\frac{4s}{d+2+2s} \geq 2 - 2^{\frac{d+2s}{d}}$ (e.g. when $s = 1$ and $d \geq 7$), then Theorem 1.1 implies the result from [21].

2. Preliminaries

We start by introducing some more notation. First, we denote the standard $L^{2^*}(\mathbb{R}^d)$-normalized Talenti bubble centered at zero by

$$B(x) = c_d(1 + |x|^2)^{-\frac{d-2s}{2}} (2.1)$$

with $c_d > 0$ chosen such that $\|B\|_{2^*} = 1$. For $\lambda > 0$, $x \in \mathbb{R}^d$, denote

$$B_{x,\lambda}(y) = \lambda^{\frac{d-2s}{2}} B(\lambda(x-y)).$$

If $x = 0$, we also write $B_{\lambda} = B_{0,\lambda}$. Notice that $\|B_{x,\lambda}\|_{2^*} = 1$, $\|(-\Delta)^s/2B_{x,\lambda}\|^2_2 = S_d$ and $(-\Delta)^sB_{x,\lambda} = S_dB_{x,\lambda}$ on $\mathbb{R}^d$ for all $x$ and $\lambda$.

We denote by

$$\mathcal{M}_1 = \{B_{x,\lambda}: x \in \mathbb{R}^d, \lambda > 0\} \subset \mathcal{M}$$

the submanifold of $\mathcal{M}$ consisting of normalized Talenti bubbles.

The manifolds $\mathcal{M}$ and $\mathcal{M}_1$ are invariant under conformal transformations of $\mathbb{R}^d$, i.e., dilations, translations and inversions. For later reference we collect some explicit transformations in the following lemma, which can be verified by simple computation.

**Lemma 2.1** (Conformal transformations of Talenti bubbles). (i) Let $D_\mu(x) = \mu x$ be dilation by $\mu > 0$ and set $(D_\mu u)(x) = \mu^{\frac{d-2s}{2}} u(\mu x)$. Then $D_\mu B_\lambda = B_{\mu \lambda}$.

(ii) Let $I_\tau(x) = \frac{x}{|x|^2}$ be the inversion about $\partial B(0, \tau)$ for some $\tau > 0$ and set $(I_\tau u)(x) = \left(\frac{\tau}{|x|}\right)^{-2s} u \left(\frac{\tau x}{|x|^2}\right)$. Then $I_\tau B_\lambda = B_{\tau^{-2} \lambda^{-1}}$. 

The next lemma gives a convenient reformulation of the distance $\text{dist}(f, \mathcal{M})$ in terms of a new optimization problem,

$$m(f) := \sup_{h \in \mathcal{M}_1} (f, h^{2s-1})^2, \quad (2.2)$$

which can be considered as simpler since it is over the smaller set $\mathcal{M}_1$ and involves no derivative. Here, $(f, h^{2s-1}) = \int_{\mathbb{R}^d} fh^{2s-1} \, dx$ denotes the pairing between $L^{2s}$ and its dual $(L^{2s})'$. (Be aware that $m(f)$ thus defined has nothing in common with the quantity denoted $m(\nu)$ that appears in [14].) We will mostly work with this reformulation when proving our main results below.

**Lemma 2.2.** Let $f \in \dot{H}^s(\mathbb{R}^d)$. Then

$$\text{dist}(f, \mathcal{M})^2 = \|(-\Delta)^{s/2} f\|_2^2 - S_d m(f). \quad (2.3)$$

Moreover, $\text{dist}(f, \mathcal{M})$ is achieved. The function $(f, h^{2s-1})h$ optimizes $\text{dist}(u, \mathcal{M})$ if and only if $h \in \mathcal{M}_1$ optimizes $m(f)$.

For $s = 1$, identity (2.3) is precisely the one given in [14, Lemma 3]. Its simple proof readily extends to all $s \in (0, \frac{d}{2})$.

**Proof.** Recall that $(-\Delta)^s h = S_d h^{2s-1}$. For any $c \in \mathbb{R}$ and $h \in \mathcal{M}_1$, we can thus write

$$\|(-\Delta)^{s/2} (f - ch)\|_2^2 = \|(-\Delta)^{s/2} f\|_2^2 - 2cS_d(f, h^{2s-1}) + c^2S_d$$

$$= \|(-\Delta)^{s/2} f\|_2^2 - S_d(f, h^{2s-1})^2 + S_d (c - (f, h^{2s-1}))^2$$

by completing a square. Hence

$$\text{dist}(f, \mathcal{M})^2 = \inf_{h \in \mathcal{M}_1} \inf_{c \in \mathbb{R}} \|(-\Delta)^{s/2} (f - ch)\|_2^2 = \|(-\Delta)^{s/2} f\|_2^2 - S_d \sup_{h \in \mathcal{M}_1} (f, h^{2s-1})^2$$

as claimed. The relation between the optimizers of $\text{dist}(u, \mathcal{M})$ and $m(f)$ is now immediate from the fact that $\inf_{c \in \mathbb{R}} (c - (f, h^{2s-1}))^2$ is attained uniquely at $c = (f, h^{2s-1})$.

By this relation between optimizers, it only remains to prove that $m(f)$ is always achieved. Let $(B_{x_n, \lambda_n})_n$ be a minimizing sequence for $m(f)$. This sequences converges to some $B_{x, \lambda}$, which plainly is a minimizer, unless $\lambda_n \to 0$, $\lambda_n \to \infty$ or $|x_n| \to \infty$. In all three cases it is easy to see that $(f, B_{x_n, \lambda_n}^{2s-1}) \to 0$ as $n \to \infty$. So to exclude this case, and therefore establish existence of a minimizer, it is sufficient to show that $m(f) > 0$.

We will show more, namely that in fact $m(f) > 0$ for any non-zero $f \in L^{2s}(\mathbb{R}^d) \supset \dot{H}^s(\mathbb{R}^d)$ (for which $m(f)$ is still well-defined). Indeed, if $f$ is continuous, then for every $x \in \mathbb{R}^d$ one has $(f, B_{x, \lambda}^{2s-1}) = cf(x)\lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}})$ as $\lambda \to \infty$, where $c > 0$ is some dimensional constant. Thus $m(f) > 0$ unless $f \equiv 0$ (in which case $m(0) = 0$ is trivially achieved). For the general case $f \in L^{2s}(\mathbb{R}^d)$, consider a sequence of continuous functions such that $f_k \to f$ in $L^{2s}(\mathbb{R}^d)$ and $f_k \to f$ a.e.. If $f \not\equiv 0$ a.e., we can thus find
$x \in \mathbb{R}^d$ such that $f_k(x) \to f(x) \neq 0$. Then $(f, B_{x, \lambda}^{2\epsilon-1}) = (c + o(1)) f(x) \lambda^{d-2\epsilon}$ follows by the properties of $f_k$. This completes the proof. \hfill \Box

The next two elementary lemmas will be needed naturally in the framework of the Brezis–Lieb-type argument we use to prove Theorem 1.2, as explained above.

**Lemma 2.3.** Let $p > 2$. Then the function $g(t) = (1 + t^q)^{\frac{q}{p}}$ is strictly convex on $(0, \infty)$. In particular,

$$
\eta \mapsto \frac{(1 + \eta^q)^{\frac{q}{p}} - 1}{\eta^2}
$$

is strictly increasing in $\eta \in (0, \infty)$.

**Proof.** We write $q = \frac{p}{2} > 1$. A computation shows that

$$
g''(t) = (q - 1)t^{q-2}(1 + t^q)^{\frac{q}{p} - 2}.
$$

Hence $g''(t) > 0$ for all $t > 0$, i.e. $g$ is strictly convex on $(0, \infty)$.

Now the function $rac{(1 + \eta^q)^{\frac{q}{p}} - 1}{\eta^2} = g(\eta^q) - g(0)$ is strictly increasing in $\eta$ by strict convexity of $g$. \hfill \Box

The next lemma describes how the value of a quotient changes when summands of the numerator and denominator change in a certain fashion.

**Lemma 2.4.** Let $A, B, C, D, E, F > 0$ be such that

$$
\frac{A}{B} \geq \frac{C}{D} \geq \frac{E}{F}, \quad \text{and} \quad D \leq F.
$$

Then

$$
\frac{A}{B} \geq \frac{A + C}{B + D} \geq \frac{A + E}{B + F}.
$$

Moreover, one has $\frac{A}{B} > \frac{A + C}{B + D}$ strictly if $\frac{A}{B} > \frac{C}{D}$ strictly. Likewise, $\frac{A + C}{B + D} > \frac{A + E}{B + F}$ if either $\frac{C}{D} > \frac{E}{F}$ or $D < F$.

**Proof.** Using that $\frac{DE}{F} \leq C$ by assumption, we write

$$
\frac{A + E}{B + F} = \frac{A + E + \frac{DE}{F}}{B + \frac{DE}{F}} \leq \frac{A + E + C}{B + \frac{DE}{F}D}.
$$

The inequality $\frac{A}{B} \geq \frac{C}{D}$ entails that the function $t \mapsto \frac{A + tC}{B + tD}$ is decreasing on $(0, \infty)$. Since $\frac{F}{D} \geq 1$, this yields the conclusion (with non-strict inequalities). A simple inspection of the above proof also yields all the claims about the strict inequalities. \hfill \Box
3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We will do so by considering a sequence of test functions of the form

\[ u_\lambda(x) := B(x) + B_\lambda(x), \]  

(3.1)
as \( \lambda \to 0 \). Recall that the normalized Talenti bubble \( B \) is given by (2.1), and that \( B_\lambda(x) = B_{0,\lambda}(x) = \lambda^{\frac{d-2s}{2}} B(\lambda x) \).

The following proposition contains the needed expansion of the terms appearing in \( \mathcal{E}(u_\lambda) \).

**Proposition 3.1.** Let \( c_0 := B(0) \int_{\mathbb{R}^d} B^{\frac{d+2s}{d}} \, dx \). As \( \lambda \to 0 \), the following holds.

1. \( \int_{\mathbb{R}^d} |(-\Delta)^{s/2} u_\lambda|^2 \, dx = 2S_d + 2S_d c_0 \lambda^{\frac{d+2s}{2}} + o(\lambda^{\frac{d-2s}{2}}) \).
2. \( \|u_\lambda\|_{L^2}^2 = 2^{\frac{d}{2}} + 2^{\frac{d}{2} + 1} c_0 \lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}}) \).
3. \( \text{dist}(u_\lambda, \mathcal{M})^2 = S_d + o(\lambda^{\frac{d-2s}{2}}) \).

Using these expansions, the proof of our first main result is immediate.

**Proof of Theorem 1.1.** By Proposition 3.1 as \( \lambda \to 0 \), we have

\[ \mathcal{E}(u_\lambda) = \frac{(2 - 2^{\frac{d}{2}}) S_d - 2S_d c_0 \lambda^{\frac{d+2s}{2}}}{S_d} + o(\lambda^{\frac{d-2s}{2}}) \]

\[ = 2 - 2^{\frac{d}{2}} - (2^{\frac{d}{2} + 1} c_0 \lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}})) < 2 - 2^{\frac{d}{2}} \]

for \( \lambda > 0 \) small enough, which is what we wanted to prove. \( \square \)

It remains to give the proof of Proposition 3.1.

**Proof of Proposition 3.1.** Let us first prove (i). Clearly,

\[ \|(-\Delta)^{s/2} u_\lambda\|_{L^2}^2 = \|(-\Delta)^{s/2} B\|_{L^2}^2 + \|(-\Delta)^{s/2} B_\lambda\|_{L^2}^2 + 2\langle B, B_\lambda \rangle_{H^s(\mathbb{R}^d)} = 2S_d + 2\langle B, B_\lambda \rangle_{H^s(\mathbb{R}^d)} \]

Now integrating by parts and using the equation \( (-\Delta)^s B = S_d B^{\frac{d+2s}{d}} \), we find

\[ \langle B, B_\lambda \rangle_{H^s(\mathbb{R}^d)} = S_d \int_{\mathbb{R}^d} B^{\frac{d+2s}{d}} B_\lambda \, dx = S_d c_0 \lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}}) \].

Next, let us prove (iii). Using that inversion about \( \partial B(0, \lambda^{-1/2}) \) transforms \( B + B_\lambda \) into itself by Lemma 2.1, we can write

\[ \int_{\mathbb{R}^d} (B + B_\lambda)^2 \, dx = 2 \int_{B(0, \lambda^{-1/2})} (B + B_\lambda)^2 \, dx. \]
On $B(0, \lambda^{-1/2})$, we have $0 \leq B_\lambda \leq B$. Since $(1 + a)^{2^*} = 1 + a2^* + O(a^2)$ uniformly for $a \in [0, 1]$, we have

\[(B + B_\lambda)^{2^*} = B^{2^*} \left(1 + \frac{B_\lambda}{B}\right)^{2^*} = B^{2^*} \left(1 + 2^* \frac{B_\lambda}{B} + O\left(\frac{B_\lambda^2}{B^2}\right)\right) = B^{2^*} + 2^* B^{2^*-1} B_\lambda + O(B_\lambda^2 B^{2^*-2})\]

uniformly on $B(0, \lambda^{-1/2})$. Now straightforward calculations show

\[\int_{B(0, \lambda^{-1/2})} B^{2^*} \, dx = 1 + O(\lambda^{\frac{d}{2}}) = 1 + o(\lambda^{\frac{d-2s}{2}}),\]

\[\int_{B(0, \lambda^{-1/2})} B_\lambda B^{2^*-1} \, dx = \lambda^\frac{d-2s}{2} c_0 + o(\lambda^{\frac{d-2s}{2}})\]

by dominated convergence, and

\[\int_{B(0, \lambda^{-1/2})} B^{2^*} \, dx = O(\lambda^{\frac{d}{2}}) = o(\lambda^{\frac{d-2s}{2}}).\]

Thus, in summary

\[\int_{\mathbb{R}^d} (B + B_\lambda)^{2^*} \, dx = 2 + 2 \times 2^* c_0 \lambda^\frac{d-2s}{2} + o(\lambda^{\frac{d-2s}{2}}).\]

Now (iii) follows from a first-order Taylor expansion of $t \mapsto t^{2^*}$ at $t = 2$.

We now turn to the proof of (iii), which is the most involved. To start with, by Lemma 2.2 we can write

\[\text{dist}(u_\lambda, \mathcal{M})^2 = \|(-\Delta)^{s/2} u_\lambda\|_2^2 - S_d \sup_{h \in \mathcal{M}_1} (u_\lambda, h^{2^*-1})^2. \tag{3.2}\]

Since $u_\lambda$ is positive and radially symmetric-decreasing, $\sup_{h \in \mathcal{M}_1} (u_\lambda, h^{2^*-1})^2$ can be found by optimizing over positive symmetric-decreasing functions in $\mathcal{M}_1$ only, i.e.

\[\sup_{h \in \mathcal{M}_1} (u_\lambda, h^{2^*-1})^2 = \sup_{\mu > 0} (u_\lambda, B^{2^*-1}_\mu)^2,\]

where $B_\mu(x) = \mu^\frac{d-2s}{2} B(\mu x)$. In other words we only need to find the maximum of the function of one variable $\mu \in (0, \infty)$ given by

\[H_\lambda(\mu) := (u_\lambda, B^{2^*-1}_\mu) = F(\mu) + G_\lambda(\mu)\]

with

\[F(\mu) := (B, B^{2^*-1}_\mu) \quad \text{and} \quad G_\lambda(\mu) := (B_\lambda, B^{2^*-1}_\mu). \tag{3.3}\]

Lemma 2.1 (iii) with $\tau = \lambda^{-1/2}$ implies $H_\lambda(\mu) = H_\lambda(\mu^{-1} \lambda)$. So we only need to optimize over $\mu \geq \lambda^{1/2}$.

Using the estimates

\[F(\mu) \lesssim \min\{\mu^\frac{d-2s}{2}, \mu^{-\frac{d-2s}{2}}\}, \quad G_\lambda(\mu) \lesssim \left(\frac{\lambda}{\mu}\right)^\frac{d-2s}{2}, \quad \text{uniformly for all } \mu \geq \lambda^{\frac{1}{2}}, \tag{3.4}\]
we clearly have \( \lim_{\mu \to \infty} H_\lambda(\mu) = 0 \). Hence \( \sup_{\mu \in [\lambda^{1/2}, \infty)} H_\lambda(\mu) \) is attained at some \( \mu(\lambda) \). Moreover, since \( H_\lambda(1) > F(1) = 1, \) in view of (3.4) we must have \( \mu(\lambda) \in [C^{-1}, C] \) for some \( C \). Thus there is \( \mu_0 > 0 \) such that \( \mu(\lambda) \to \mu_0 \) as \( \lambda \to 0 \).

Again by (3.4), we see that
\[
1 < H_\lambda(1) \leq H_\lambda(\mu(\lambda)) = F(\mu_0) + o(1).
\]
Passing to the limit \( \lambda \to 0 \) yields \( 1 = F(\mu_0) = (B, B^{2^* - 1}) \) and thus \( \mu_0 = 1 \) by the equality condition in Hölder’s inequality.

We have thus proved that \( \mu(\lambda) \to 1 \). By a Taylor expansion at 1, and since \( F'(1) = 0 \) by Lemma A.1, \( \mu(\lambda) \) satisfies
\[
0 = H'_\lambda(\mu(\lambda)) = F'_\lambda(\mu(\lambda)) + G'_\lambda(\mu(\lambda)) = (F'_\lambda(1) + o(1))(\mu(\lambda) - 1) + (1 + o(1))G'_\lambda(1).
\]
Still by Lemma A.1, we have \( F''(1) \neq 0 \) and \( G'_\lambda(1) \approx \lambda^{d-2s} \). Therefore
\[
|\mu(\lambda) - 1| = (1 + o(1)) \frac{|G'_\lambda(1)|}{|F'_\lambda(1) + o(1)|} \approx \lambda^{d-2s}. \tag{3.5}
\]

Now we can conclude by inserting the estimate (3.5) back into \( H_\lambda(\mu) \). We obtain, using again Lemma A.1,
\[
m(u\lambda)^{1/2} = H_\lambda(\mu(\lambda)) = F(\mu(\lambda)) + G_\lambda(\mu(\lambda)) \\
= F(1) + F'(1)(\mu(\lambda) - 1) + o(|\mu(\lambda) - 1|) + G_\lambda(1)(1 + o(1)) \\
= 1 + c_0 \lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}}).
\]
As a consequence, by (3.2) and the already established part (i) of the proposition, we obtain
\[
dist(u\lambda, \mathcal{M})^2 = \|(\Delta)^{s/2}u\lambda\|^2_2 - S_d m(u\lambda) \\
= 2S_d + 2S_d c_0 \lambda^{\frac{d-2s}{2}} - S_d (1 + c_0 \lambda^{\frac{d-2s}{2}})^2 + o(\lambda^{\frac{d-2s}{2}}) = S_d + o(\lambda^{\frac{d-2s}{2}}).
\]
This completes the proof of the proposition.

\[\square\]

4. PROOF OF THEOREM 1.2

In this section we give the proof of Theorem 1.2. We let \((u_n)\) be a normalized minimizing sequence for \( c_{BE}(s) \), i.e.
\[
\mathcal{E}(u_n) = c_{BE}(s) + o(1) \quad \text{as } n \to \infty, \quad \|u_n\|_{2^*} = 1. \tag{4.1}
\]
Then
\[
\|(\Delta)^{s/2}u_n\|^2_2 = (c_{BE}(s) + o(1)) \text{dist}(u_n, \mathcal{M})^2 + S_d \leq (c_{BE}(s) + o(1)) \|(\Delta)^{s/2}u_n\|^2_2 + S_d.
\]
Since \( c_{BE}(s) \leq \frac{4s}{d+2s+2} < 1 \) by [10], this implies that \((u_n)\) is bounded in \( \dot{H}^s(\mathbb{R}^d) \). By a theorem of Lions [22] (see also [20]), up to translating and rescaling the sequence...
The proof of Theorem 1.2 now consists in showing that (4.2). It is standard to check that the weak convergence implies \( \lim_{n \to \infty} u_n = f \) in \( \dot{H}^s(\mathbb{R}^d) \), for some non-zero \( f \). Letting \( g_n := u_n - f \), we can thus write
\[
  u_n = f + g_n, \quad \text{for some } f \in \dot{H}^s(\mathbb{R}^d) \setminus \{0\}, \quad g_n \to 0 \text{ in } \dot{H}^s(\mathbb{R}^d).
\]

We first check that if the convergences are strong, then a minimizer of \( c_{BE}(s) \) must exist.

**Proposition 4.1.** Let \((u_n)\) satisfy (4.1) and (4.2), and suppose that \( g_n \to 0 \) strongly in \( \dot{H}^s(\mathbb{R}^d) \). Then \( f \) is a minimizer for (1.4).

**Proof.** If \( u_n \to f \) strongly in \( \dot{H}^s(\mathbb{R}^d) \), then it is clear that \( \|(-\Delta)^{s/2} u_n\|^2 \to \|(-\Delta)^{s/2} f\|^2 \) and \( \text{dist}(u_n, M) \to \text{dist}(f, M) \). By Sobolev embedding, we also have \( \|u_n\|_{2^*} \to \|f\|_{2^*} \). Thus \( \mathcal{E}(u_n) \to \mathcal{E}(f) \) and \( f \) is a minimizer, provided that \( \text{dist}(f, M) \neq 0 \), i.e., that \( f \notin M \).

But for sequences \((u_n)\) such that \( \text{dist}(u_n, M) \to 0 \), it is known, e.g. from [10, Proposition 2], that \( \mathcal{E}(u_n) \geq \frac{4s}{d+2s+2} \). On the other hand, the result in [21] guarantees that \( \lim_{n \to \infty} \mathcal{E}(u_n) = c_{BE}(s) < \frac{4s}{d+2s+2} \). Hence the minimizing sequence \((u_n)\) cannot satisfy \( \text{dist}(u_n, M) \to 0 \). As explained above, this finishes the proof. \( \square \)

The proof of Theorem 1.2 now consists in showing that \( g_n \to 0 \) must in fact be the case.

To do so, let us investigate how the components of \( \mathcal{E}(u_n) \) behave under the decomposition (4.2). It is standard to check that the weak convergence implies
\[
  \|(-\Delta)^{s/2} u_n\|^2 = \|(-\Delta)^{s/2} f\|^2 + \|(-\Delta)^{s/2} g_n\|^2 + o(1),
\]
and that, using compact Sobolev embeddings and the Brezis–Lieb lemma [6],
\[
  \int_{\mathbb{R}^d} |u_n|^2 \, dx = \int_{\mathbb{R}^d} |f|^{2^*} \, dx + \int_{\mathbb{R}^d} |g_n|^2 \, dx + o(1)
\]
along a subsequence, as \( n \to \infty \). Finally, the following lemma gives the important information how the distance \( \text{dist}(u_n, M) \) decomposes. Recall that by definition \( m(u) = \sup_{h \in M_1} \langle u, h^{2^*-1} \rangle^2 \).

**Lemma 4.2.** Let \( u_n \) satisfy (4.2). As \( n \to \infty \), we have
\[
  m(u_n) = \max \{ m(f), m(g_n) \} + o(1).
\]
In particular,
\[
  \text{dist}(u_n, M)^2 = \|(-\Delta)^{s/2} f\|^2 + \|(-\Delta)^{s/2} g_n\|^2 - S_d \max \{ m(f), m(g_n) \} + o(1).
\]

Proof. By Lemma \[2.2\] \( m(u_n) \) has an optimizer \( h_n \in \mathcal{M}_1 \). We write \( h_n(x) = \mu_n B(\mu_n(x-x_n)) \) and consider two different cases.

Suppose first that \( x_n \) is bounded and \( \mu_n \) is bounded away from zero and infinity. Then up to a subsequence \( x_n \to x_\infty \in \mathbb{R}^d \) and \( \mu_n \to \mu_\infty \in (0, \infty) \), and consequently \( h_n^{2^*-1} \to B_{x_\infty, \mu_\infty}^{2^*-1} \) strongly in \( L(2^*)' \). But this implies \( (g_n, h_n^{2^*-1}) \to (0, 0) \) by weak convergence \( g_n \to 0 \).

Thus

\[
m(u_n) = \left( (f, h_n^{2^*-1}) + (g_n, h_n^{2^*-1}) \right)^2 = (f, h_n^{2^*-1})^2 + o(1) \leq m(f) + o(1). \tag{4.7}
\]

In the remaining, second case, we have \( \mu_n \to 0 \), \( \mu_n \to \infty \) or \( |x_n| \to \infty \) along a subsequence. This can be easily checked to yield \( h_n^{2^*-1} \to 0 \) in \( L(2^*)' \). Thus \( (f, h_n^{2^*-1}) \to 0 \) in that case, and we get

\[
m(u_n) = \left( (f, h_n^{2^*-1}) + (g_n, h_n^{2^*-1}) \right)^2 = (g_n, h_n^{2^*-1})^2 + o(1) \leq m(g_n) + o(1). \tag{4.8}
\]

Combining (4.7) and (4.8), we get

\[
m(u_n) \leq \max \{ m(f), m(g_n) \} + o(1), \tag{4.9}
\]

at least along some subsequence. But our argument shows that from any subsequence we can extract another subsequence such that the inequality (4.9) holds. Thus (4.9) must in fact hold for the entire sequence \( (u_n) \).

In order to establish (4.5), we will now prove the converse inequality by a similar argument. Let \( h_f \) be the optimizer for \( m(f) \). Then \( (g_n, h_f^{2^*-1}) \to 0 \) by weak convergence \( g_n \to 0 \) and thus

\[
m(u_n) \geq (u_n, h_f^{2^*-1})^2 = \left( (f, h_f^{2^*-1}) + (g_n, h_f^{2^*-1}) \right)^2 = m(f) + o(1). \tag{4.10}
\]

Now, let \( h_{g_n} \) be the optimizer for \( m(g_n) \). We write again \( h_{g_n} = \mu_n B(\mu_n(x-x_n)) \) and consider two cases. Suppose first that \( \mu_n \to 0 \), \( \mu_n \to \infty \) or \( |x_n| \to \infty \) along a subsequence. Then, as above, \( h_{g_n}^{2^*-1} \to 0 \) in \( L(2^*)' (\mathbb{R}^d) \) and thus \( (f, h_{g_n}^{2^*-1}) \to 0 \). We obtain in that case

\[
m(u_n) \geq (u_n, h_{g_n}^{2^*-1})^2 = \left( (f, h_{g_n}^{2^*-1}) + (g_n, h_{g_n}^{2^*-1}) \right)^2 = m(g_n) + o(1). \tag{4.11}
\]

If, on the other hand, \( \mu_n \), \( \mu_n^{-1} \) and \( |x_n| \) are bounded, then up to a subsequence \( \mu_n \to \mu_\infty \in (0, \infty) \) and \( x_n \to x_\infty \in \mathbb{R}^d \). But then \( m(g_n) = (g_n, h_{g_n}^{2^*-1}) \to 0 \) by weak convergence \( g_n \to 0 \), and so (4.11) holds trivially.

By the same remark as in the first part of the proof, (4.11) holds in fact along the whole sequence \( (u_n) \). Now by combining (4.10) and (4.11) with (4.9), inequality (4.5) follows.

Finally, (4.6) is immediate from Lemma \[2.2\] together with (4.3) and (4.5). \(\square\)

The next lemma serves as an important preparation for our main argument. Contrary to (4.3), (4.4) and (4.5), here the minimizing property of \( (u_n) \) comes into play.
Lemma 4.3. Let \((u_n)\) satisfy (4.1) and (4.2). Suppose that there is \(c > 0\) such that \(\|(-\Delta)^{s/2}g_n\|_2^2 \geq c\). Then \(m(f) = m(g_n) + o(1)\) as \(n \to \infty\).

Proof. Assume that, up to extracting a subsequence,

\[
\lim_{n \to \infty} m(g_n) > m(f) \quad (4.12)
\]

strictly. Multiplying by a constant, we may equivalently consider \(\tilde{u}_n = \frac{f}{\|g_n\|_2^2} + \frac{g_n}{\|g_n\|_2^2} \) := \(\tilde{f}_n + \tilde{g}_n\) with \(\|\tilde{g}_n\|_2 = 1\). Notice that \(\|g_n\|_2\) is bounded away from zero and \(\infty\), hence \(\|\tilde{f}_n\|_2\) is. Then, by (4.3), (4.4) and Lemma 4.2,

\[
\mathcal{E}(\tilde{u}_n) = \frac{\|(-\Delta)^{s/2}\tilde{g}_n\|_2^2 - S_d + \|(-\Delta)^{s/2}\tilde{f}_n\|_2^2 - S_d \left(1 + \int_{\mathbb{R}^d} |\tilde{f}_n|^{2^*} \, dx\right)^{\frac{2}{2^*}} - 1}{\|(-\Delta)^{s/2}\tilde{g}_n\|_2^2 - S_d m(\tilde{g}_n) + \|(-\Delta)^{s/2}\tilde{f}_n\|_2^2} + o(1).
\]

Our goal is now to estimate the quotient using Lemma 2.4. Suppose for the moment that \(\tilde{g}_n \notin \mathcal{M}\) for all \(n\). Then set

\[
A := \lim_{n \to \infty} \|(-\Delta)^{s/2}\tilde{g}_n\|_2^2 - S_d, \quad B := \lim_{n \to \infty} \|(-\Delta)^{s/2}\tilde{g}_n\|_2^2 - S_d m(\tilde{g}_n)^2,
\]

\[
C := \lim_{n \to \infty} \|(-\Delta)^{s/2}\tilde{f}_n\|_2^2 - S_d \left(1 + \int_{\mathbb{R}^d} |\tilde{f}_n|^{2^*} \, dx\right)^{\frac{2}{2^*}} - 1, \quad D := \lim_{n \to \infty} \|(-\Delta)^{s/2}\tilde{f}_n\|_2^2.
\]

Notice that \(A, B, C, D > 0\) because we assume \(\tilde{g}_n \notin \mathcal{M}\) and because \(\|(-\Delta)^{s/2}\tilde{f}_n\|_2^2\) is bounded away from zero. Since \(c_{BE}(s) = \lim_{n \to \infty} \mathcal{E}(\tilde{u}_n) = \frac{A + C}{B + D}\) and \(\frac{4}{B} = \lim_{n \to \infty} \mathcal{E}(g_n) \geq c_{BE}(s)\), we must have \(\frac{C}{D} \leq c_{BE}(s)\).

Now let \(F_n\) be the scalar multiple of \(\tilde{f}_n\) such that \(m(F_n) = m(\tilde{g}_n)\). Then, as a consequence of (4.12), \(\lim_{n \to \infty} \|F_n\|_2 > \lim_{n \to \infty} \|\tilde{f}_n\|_2\) strictly. By Lemma 2.3, the function

\[
\eta \mapsto \frac{(1 + \eta^2)^{\frac{2}{2^*}} - 1}{\eta^2}
\]

is strictly increasing, so that

\[
\frac{C}{D} = 1 - \lim_{n \to \infty} \frac{S_d \left(1 + \|\tilde{f}_n\|_2^2 \right)^{\frac{\frac{2}{2^*}}{2}} \, dx - 1}{S[\tilde{f}_n] \|\tilde{f}_n\|_2^2} \geq \lim_{n \to \infty} \frac{S_d \left(1 + \|F_n\|_2^2 \right)^{\frac{2}{2^*}} \, dx - 1}{S[\tilde{f}_n] \|\tilde{f}_n\|_2^2} = \frac{E}{F}.
\]

Since \(D \leq F\) and \(\frac{A}{B} \geq \frac{C}{D} > \frac{E}{F}\), Lemma 2.4 yields

\[
c_{BE}(s) = \lim_{n \to \infty} \mathcal{E}(u_n) = \lim_{n \to \infty} \mathcal{E}(\tilde{u}_n) = \frac{A + C}{B + D} > \frac{A + E}{B + F} = \lim_{n \to \infty} \mathcal{E}(F_n + \tilde{g}_n).
\]

But this contradicts the definition of \(c_{BE}(s)\). Hence (4.12) is impossible.

If, on the other hand, \(\tilde{g}_n \in \mathcal{M}\) along some subsequence, then \(A = B = 0\) in the above and we directly conclude a contradiction in the same way from \(\frac{C}{D} > \frac{E}{F}\).

The remaining case to treat is that where, up to a subsequence,

\[
m(f) > \lim_{n \to \infty} m(g_n).
\]
But here one arrives at a contradiction in a similar fashion, with the roles of \( f \) and \( g_n \) reversed and considering \( \tilde{u}_n = \frac{\tilde{f}}{\|\tilde{f}\|_{2^*}} + \frac{g_n}{\|g_n\|_{2^*}} =: \tilde{f} + \tilde{g}_n \). The fact that \( D := \lim_{n \to \infty} \|(-\Delta)^{s/2} g_n\|_2^2 > 0 \) is guaranteed here by assumption. The rest of the proof is identical to the above. \( \square \)

We are now ready to prove our second main result.

**Proof of Theorem 4.2** Let \((u_n)\) be a minimizing sequence satisfying (4.1) and (4.2). Suppose for contradiction that \( g_n = u_n - f \) does not converge strongly to zero in \( \dot{H}^s(\mathbb{R}^d) \). Then, after passing to a subsequence, we have \( \|(-\Delta)^{s/2} g_n\|_2^2 \geq c \) for some \( c > 0 \). Thus Lemma 4.3 asserts that

\[
m(f) = m(g_n) + o(1). \tag{4.13}
\]

Suppose first that \( \|g_n\|_{2^*} \leq \|f\|_{2^*} + o(1) \). As in the proof of Lemma 4.3, we may moreover assume that \( \|f\|_{2^*} = 1 \) by multiplying with a suitable scalar factor. Due to (4.13) and Lemma 4.2, we may write

\[
\text{dist}(u_n, M)^2 = \|(-\Delta)^{s/2} f\|_2^2 + \|(-\Delta)^{s/2} g_n\|_2^2 - S_d m(f) + o(1).
\]

Together with (1) and (3), we thus obtain

\[
c_{BE}(s) + o(1) = \frac{\|(-\Delta)^{s/2} f\|_2^2 - S_d + \|(-\Delta)^{s/2} g_n\|_2^2 - S_d \left( (1 + \|g_n\|_{2^*}^2)^{\frac{s}{2^*}} - 1 \right)}{\|(-\Delta)^{s/2} f\|_2^2 - S_d m(f) + \|(-\Delta)^{s/2} g_n\|_2^2}
\]

Similarly to the proof of Lemma 4.3 since by (1.4)

\[
\frac{\|(-\Delta)^{s/2} f\|_2^2 - S_d}{\|(-\Delta)^{s/2} f\|_2^2 - S_d m(f)} \geq c_{BE}(s),
\]

and since \( \|(-\Delta)^{s/2} g_n\|_2^2 \geq c \), we must have

\[
c_{BE}(s) + o(1) \geq \frac{\|(-\Delta)^{s/2} g_n\|_2^2 - S_d \left( (1 + \|g_n\|_{2^*}^2)^{\frac{s}{2^*}} - 1 \right)}{\|(-\Delta)^{s/2} g_n\|_2^2}
\]

\[
= 1 - \frac{S_d \left( (1 + \|g_n\|_{2^*}^2)^{\frac{s}{2^*}} - 1 \right)}{S(g_n)\|g_n\|_{2^*}^2} \geq 1 - \frac{S_d}{S(g_n)} (2^{\frac{s}{2^*}} - 1),
\]

where the last inequality follows from Lemma 2.3 together with \( \|g_n\|_{2^*} \leq 1 \). (Recall that \( S(g) \) is the Sobolev quotient defined in (1.6).) Since we know by Theorem 1.1 that \( c_{BE} < 2 - 2^{\frac{s}{2^*}} \) with strict inequality, we find, for \( n \) large enough, that

\[
1 - \frac{S_d}{S(g_n)} (2^{\frac{s}{2^*}} - 1) < 2 - 2^{\frac{s}{2^*}},
\]

which is equivalent to \( S(g_n) < S_d \). But this contradicts the definition of \( S_d \).
If we assume instead the reverse inequality \( \|f\|_{2^*} \leq \|g_n\|_{2^*} + o(1) \), we obtain a contradiction by writing
\[
\text{dist}(u_n, \mathcal{M})^2 = \|(-\Delta)^{s/2} f\|_2^2 + \|(-\Delta)^{s/2} g_n\|_2^2 - S_d m(g_n) + o(1)
\]
and arguing in exactly the same way with the roles of \( f \) and \( g_n \) reversed.

Thus we have shown that \( g_n \) must converge strongly to zero in \( \dot{\mathcal{H}}^s(\mathbb{R}^d) \). By Proposition 4.1, the proof of Theorem 1.2 is now complete. \( \square \)

In the previous proof, notice that it is the crucial information (4.13) from Lemma 4.3 which allows us to express \( \text{dist}(u_n, \mathcal{M})^2 \) with the help of either \( m(f) \) or \( m(g_n) \). This is what permits us to reverse the roles of \( f \) and \( g_n \), in other words to assume an inequality between \( \|f\|_{2^*} \) and \( \|g_n\|_{2^*} \) without loss of generality.

**Remark 4.4.** The following remark may help to gain some more intuition about the proof of Theorem 1.2. If we did not have Theorem 1.1 available, but only the non-strict inequality \( \text{cBE} \leq 2 - 2^\frac{d}{2s} \), then the chain of inequalities (4.14) would still imply that \( \|g_n\|_{2^*} \to 1 \) and \( S(g_n) \to S_d \), that is, \( \text{dist}(g_n, \mathcal{M}_1) \to 0 \). By \( \|f\|_{2^*} = 1 \), \( m(f) = m(g_n) \) and the equality condition in Hölder’s inequality we would then also have \( f \in \mathcal{M}_1 \).

Thus the (weaker) conclusion would be in this case that either \( (u_n) \) converges strongly or, up to rescaling and translation \( u_n = B + B_n + o(1) \) for a sequence \( (B_n) \subset \mathcal{M}_1 \) interacting weakly with \( B \).

**Appendix A. Some computations**

The following lemma was needed in the proof of Proposition 3.1.

**Lemma A.1.** Let \( F \) and \( G_\lambda \) be defined by (3.3). Then
\[
F'(1) = 0, \quad F''(1) = a_d
\]
and
\[
G_\lambda(1) = c_0 \lambda \frac{d-2s}{2} + o(\lambda \frac{d-2s}{2}), \quad G'_\lambda(1) = b_d \lambda \frac{d-2s}{2} + o(\lambda \frac{d-2s}{2}),
\]
for \( c_0 = B(0) \int_{\mathbb{R}^d} B^{2^* - 1} \) and
\[
a_d = -c_d^2 \frac{d-2s}{2(d+1)} \frac{\Gamma(d+2) \Gamma(s)}{\Gamma(d+1)}, \quad b_d = -c_d^2 \frac{d-2s}{2(d+2s)} \frac{\Gamma(d+2) \Gamma(s)}{\Gamma(d+2s)}, \quad (A.1)
\]

Here \( c_d \) is the normalization constant in (2.1).

**Proof.** We compute
\[
F'((\mu)) = \int_0^\infty r^{d-1} B(r) \left( B(\mu r)^{2^* - 1} + \frac{2}{d+2s} B(\mu r)^{2^* - 2} B'(\mu r) \mu r \right) dr.
\]
Hence
\[ c_d^{-2s} F'(1) = \int_0^\infty r^{d-1}(1 + r^2)^{-d} dr - 2 \int_0^\infty r^{d+1}(1 + r^2)^{-d-1} dr = \frac{1}{2} \Gamma \left( \frac{d}{2} \right)^2 - \frac{\Gamma \left( \frac{d+2}{2} \right) \Gamma \left( \frac{d}{2} \right)}{\Gamma(d+1)} = 0. \]

Next, we compute
\[
F''(\mu) = \frac{d}{d\mu} \left( \mu^{-d} \int_0^\infty r^{d-1} B(\mu^{-1} r) \left( B(r)^{\frac{d+2s}{d-2s}} + \frac{2}{d-2s} B^{\frac{4s}{d-2s}}(r) B'(r) r \right) dr \right)
\]
\[= -\mu^{-d-1} \int_0^\infty r^{d-1} (dB(\mu^{-1} r) + B'(\mu^{-1} r)\mu^{-1} r) \left( B(r)^{\frac{d+2s}{d-2s}} + \frac{2}{d-2s} B^{\frac{4s}{d-2s}}(r) B'(r) r \right) dr. \]

Therefore
\[c_d^{-2s} F''(1) = -dF'(1) - c_d^{-2s} \int_0^\infty r^{d-1} \left( B(r)^{\frac{d+2s}{d-2s}} B'(r) + \frac{2}{d-2s} B^{\frac{4s}{d-2s}}(r) B'(r)^2 r \right) dr
\]
\[= (d - 2s) \int_0^\infty \frac{r^{d+1}}{(1 + r^2)^{d+1}} dr - 2(d - 2s) \frac{r^{d+3}}{(1 + r^2)^{d+2}} dr
\]
\[= \frac{d - 2s}{2} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} \right) - (d - 2s) \frac{\Gamma \left( \frac{d}{2} + 2 \right) \Gamma \left( \frac{d}{2} \right)}{\Gamma(d+2)} = -\frac{d - 2s}{2(d+1)} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d}{2} \right). \]

Now let us turn to \(G_\lambda(\mu)\). First,
\[G_\lambda(1) = (B_\lambda, B^{2s-1}) = \lambda^{\frac{d-2s}{2}} B(0) \int_{\mathbb{R}^d} B^{2s-1} dx + o(\lambda^{\frac{d-2s}{2}}). \]

Finally, we have
\[G'_\lambda(\mu) = \int_0^\infty r^{d-1} B_\lambda(r) \left( B(\mu r)^{2s-1} + \frac{2}{d+2s} B(\mu r)^{2s-2} B'(\mu r) \mu r \right) dr \]

and so
\[c_d^{-2s} G'_\lambda(1)
\]
\[= c_d^{-2s} \lambda^{\frac{d-2s}{2}} \left( \int_0^\infty r^{d-1} B(\lambda r) B(r)^{\frac{d+2s}{d-2s}} dr + \frac{2}{d-2s} \int_0^\infty r^{d-1} B(\lambda r) B^{\frac{4s}{d-2s}}(r) B'(r) r \right)
\]
\[= \lambda^{\frac{d-2s}{2}} (1 + o(1)) \left( \int_0^\infty \frac{r^{d-1}}{(1 + r^2)^{d+1}} dr - 2 \int_0^\infty \frac{r^{d+1}}{(1 + r^2)^{d+2}} dr \right)
\]
\[= \lambda^{\frac{d-2s}{2}} (1 + o(1)) \left( \frac{1}{2} \frac{\Gamma \left( \frac{d}{2} \right) \Gamma(s)}{\Gamma \left( \frac{d+2s}{2} \right)} - \frac{\Gamma \left( \frac{d+2}{2} \right) \Gamma(s)}{\Gamma \left( \frac{d+2s+2}{2} \right)} \right)
\]
\[= -\frac{d - 2s}{2(d+2s)} \Gamma \left( \frac{d}{2} \right) \Gamma(s) \lambda^{\frac{d-2s}{2}} + o(\lambda^{\frac{d-2s}{2}}). \]

This completes the proof. \(\square\)
References

[1] A. Bahri, J.-M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Comm. Pure Appl. Math. 41 (1988), no. 3, 253–294.

[2] G. Bianchi, H. Egnell, A note on the Sobolev inequality. J. Funct. Anal. 100, No. 1, 18-24 (1991).

[3] C. Bianchini, G. Croce, A. Henrot. On the quantitative isoperimetric inequality in the plane. ESAIM Control Optim. Calc. Var., 23(2) (2017), 517–549.

[4] M. Bonforte, J. Dolbeault, B. Nazaret, N. Simonov, Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method. Preprint, arXiv:2007.03674.

[5] H. Brézis, L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. 36 (1983), 437–477.

[6] H. Brézis, E. H. Lieb, A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. 88 (1983), 486–490.

[7] S. G. Bobkov, N. Gozlan, C. Roberto, P.-M. Samson, Bounds on the deficit in the logarithmic Sobolev inequality. J. Funct. Anal. 267 (2014), No. 11, 4110-4138.

[8] G. Brigati, J. Dolbeault, N. Simonov, Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results. Preprint, arXiv:2211.13180.

[9] E. A. Carlen, R. L. Frank, E. H. Lieb. Stability estimates for the lowest eigenvalue of a Schrödinger operator. Geom. Funct. Anal., 24 (2014), No. 1, 63–84.

[10] S. Chen, R. L. Frank, T. Weth, Remainder terms in the fractional Sobolev inequality. Indiana Univ. Math. J. 62 (2013), No. 4, 1381–1397.

[11] L. Chen, G. Lu, H. Tang, Sharp Stability of Log-Sobolev and Moser-Onofri inequalities on the Sphere. J. Funct. Anal. 285 (2023), no. 5, 110022.

[12] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli. The sharp Sobolev inequality in quantitative form. J. Eur. Math. Soc. , 11 (2009), No. 5, 1105–1139.

[13] J. Dolbeault, M. J. Esteban, Hardy-Littlewood-Sobolev and related inequalities: stability. The physics and mathematics of Elliott Lieb. The 90th anniversary. Volume I. Berlin: European Mathematical Society (EMS). 247–268 (2022).

[14] J. Dolbeault, M. J. Esteban, A. Figalli, R. L. Frank, M. Loss, Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence. Preprint, arXiv:2209.08651.

[15] M. Fathi, E. Indrei, M. Ledoux, Quantitative logarithmic Sobolev inequalities and stability estimates. Discrete Contin. Dyn. Syst. 36 (2016), No. 12, 6835-6853.

[16] A. Figalli, R. Neumayer, Gradient stability for the Sobolev inequality: the case $p > 2$. J. Eur. Math. Soc. (JEMS) 21 (2019), No. 2, 319-354.

[17] A. Figalli, Y. R.-Y. Zhang, Sharp gradient stability for the Sobolev inequality. Duke Math. J. 171 (2022), No. 12, 2407-2459.

[18] R. L. Frank, The sharp Sobolev inequality and its stability: An introduction. Preprint, arXiv:2304.03115.

[19] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality. Ann. of Math. (2) 168 (2008), 941–980.

[20] P. Gérard, Description du défaut de compacité de l’injection de Sobolev. ESAIM: Control, Optimisation and Calculus of Variations, Volume 3 (1998), 213–233.

[21] T. König, On the sharp constant in the Bianchi-Egnell stability inequality. Bull. Lond. Math. Soc. 55 (2023), no. 4, 2070–2075.

[22] P.-L. Lions, The Concentration-Compactness Principle in the Calculus of Variations. The Limit Case, Part 2. Rev. Mat. Iberoam. 1 (1985), no. 2, 45–121.

[23] O. Rey, The role of the Green’s function in a non-linear elliptic equation involving the critical Sobolev exponent. J. Funct. Anal. 89 (1990), 1–52.
(Tobias König) Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60325 Frankfurt am Main, Germany

Email address: koenig@mathematik.uni-frankfurt.de