Backward bifurcation of a disease-severity-structured epidemic model with treatment

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Abstract

This paper presents a disease-severity-structured epidemic model with treatment necessary only to severe infective individuals to discuss the effect of the treatment capacity on the disease transmission. It is shown that a backward bifurcation occurs in the basic reproduction number $R_0$, where a stable endemic equilibrium co-exists with a stable disease-free equilibrium when $R_0 < 1$, if the capacity is relatively small. This epidemiological implication is that, when there is not enough capacity for treatment, the requirement $R_0 < 1$ is not sufficient for effective disease control and disease outbreak can happen to a high endemic level even though $R_0 < 1$.

Keywords: epidemic; disease-severity structure, bifurcation, treatment

1 Introduction

Treatment is an important method to decrease the spread of diseases such as measles, tuberculosis and flu (see, for example [3]). In classical disease transmission models, the treatment rate of infective individuals is assumed to be proportional to their number. However, it is natural to think that there is some capacity for the treatment, including limited beds in hospitals, or an insufficient supply of medicine. Such a limited capacity, in [3], was considered as

$$\frac{dS}{dt} = A - \sigma SI - \mu S$$
$$\frac{dI}{dt} = \sigma SI - (\mu + \rho + \varepsilon)I - T(I)$$
$$\frac{dR}{dt} = T(I) + \rho I - \mu R$$

where $T(\cdot)$ is the treatment rate defined as the function of $I$:

$$T(I) = \begin{cases} 
  rI, & I < C_I \\
  rC_I, & I \geq C_I 
\end{cases}$$

with the per capita treatment rate $r$ and the capacity $C_I$. Here, $S(t)$, $I(t)$, and $R(t)$ denote the numbers of susceptible, infective, and recovered individuals at time $t$, respectively. $A$ is
the recruitment rate of the population, $\mu$ the per capita natural death rate of the population, $\varepsilon$ the per capita disease-related death rate, $\rho$ the per capita natural recovery rate of infective individuals, $\sigma$ the disease transmission coefficients of infective individuals. Characterizing the dynamics of disease transmission models often requires the basic reproduction number $R_0$, the average number of new cases that would be generated by a typical infected individual introduced into a completely susceptible population. In general, the phenomenon forward bifurcation is observed, where the disease-free equilibrium loses its stability and a stable endemic equilibrium appears as $R_0$ increases through one. [5] figured out, however, backward bifurcations occur, where a stable endemic equilibrium co-exists with a stable disease-free equilibrium when $R_0 < 1$ (as illustrated in Fig. 3 in [5]), due to the low treatment capacity. The fact implies that the requirement $R_0 < 1$ is not sufficient for effective disease control and disease outbreak can happen to a high endemic level even though $R_0 < 1$.

In case of less-lethal diseases (not to mention non-lethal diseases), we can assume $\varepsilon = 0$ when the disease-related death rate is negligibly small compared with the natural death rate. Besides, for such diseases, infective individuals do not always need treatment in that they can recover by themselves in mild case. To discuss the effect of the treatment capacity on the transmission of non-lethal or less-lethal diseases, in this paper, we consider the following disease-severity-structured epidemic model, which is more realistic than the above mentioned model if $\varepsilon = 0$:

$$
\frac{dS}{dt} = A - \sigma_m SI_m - \sigma_s S[I_s]^+_{C_I} - \mu S \\
\frac{dI_m}{dt} = \sigma_m SI_m + \sigma_s S[I_s]^+_{C_I} - (\mu + \rho + \beta)I_m \\
\frac{dI_s}{dt} = \beta I_m - T(I_s) - \mu I_s \\
\frac{dR}{dt} = T(I_s) + \rho I_m - \mu R.
$$

Here, new unknown functions $I_s(t)$ and $I_m(t)$ denote the number of severe infective individuals who need treatment, and the number of non-severe (that is, mild) infective individuals who do not need it, respectively. Also, new parameters $\sigma_m$ and $\sigma_s$ are the disease transmission coefficients of mild and severe infective individuals, respectively, $\beta$ the severity coefficient of mild infective individuals. Furthermore,$$
[I_s]_{C_I}^+ = \max\{0, I_s - C_I\},
$$
by which $\sigma_s S[I_s]^+_{C_I}$ means the transmission rate of severe infective individuals exceeding the capacity. We assume all the parameters to be positive constants and the initial data given as

$$S(0) > 0, \quad I_m(0) \geq 0, \quad I_s(0) \geq 0, \quad I_m(0) + I_s(0) > 0, \quad R(0) \geq 0.$$

As a result, backward bifurcation occurs for (1.1), which leads to the same scenario as mentioned above, if the treatment capacity $C_I$ is relatively small. Details of our results and their proof are found in the next section and Section 3. We summarize our findings in Section 4.
2 Basic reproduction number and equilibria

To analyze (1.1), we only focus on the three dimensional ODEs:

\[
\begin{align*}
\frac{dS}{dt} &= A - \sigma_m S I_m - \sigma_s S I_s + \mu S, \\
\frac{dI_m}{dt} &= \sigma_m S I_m + \sigma_s S I_s - (\mu + \rho + \beta) I_m, \\
\frac{dI_s}{dt} &= \beta I_m - T(I_s) - \mu I_s
\end{align*}
\]  

(2.1)

since the first three equations in (1.1) are independent of the variable \( R \). For (2.1), uniqueness of the solutions is ensured by the Lipschitz continuity of its right-hand side although the right-hand side is not differentiable because of \([I_s]_{C_l}^+\) and \( T(I_s) \).

Disease-free equilibrium is required to derive the basic reproduction number \( R_0 \). Obviously, the model (2.1) has always a disease-free equilibrium \( E_0 \) expressed as

\[(S, I_m, I_s) = \left( \frac{A}{\mu}, 0, 0 \right).\]

According to the concept of next generation matrix, let \( x = (I_m, I_s, S) \) and write (2.1) as

\[
\frac{dx}{dt} = \mathcal{F}(x) - \mathcal{V}(x),
\]

where \( \mathcal{F} \) is the rate of production term of new infection and \( -\mathcal{V} \) otherwise. From the viewpoint of local behavior in completely susceptible population, the first principal matrices (defined as \( \mathcal{F}, \mathcal{V} \) below) of the Jacobian matrix of \( \mathcal{F} \) and \( \mathcal{V} \) at disease-free equilibrium play a key role of defining \( R_0 \) (see [2], [4]). Then the disease-free equilibrium is as \((I_m, I_s, S) = (0, 0, A/\mu)\) and the first principal matrices \( \mathcal{F}, \mathcal{V} \) are given as

\[
\mathcal{F} = \begin{pmatrix}
\frac{A\sigma_m}{\mu} & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix}
\mu + \beta + \rho & 0 \\
-\beta & \mu + r
\end{pmatrix}.
\]

Hence, the basic reproduction number for (2.1) is

\[
R_0 = \|\mathcal{F}^{-1}\mathcal{V}\| = \frac{A\sigma_m}{\mu(\mu + \beta + \rho)},
\]

where \( \|\mathcal{M}\| \) represents the spectral radius of the matrix \( \mathcal{M} \).

An endemic equilibrium of (2.1) satisfies

\[
\begin{align*}
A - \sigma_m S I_m - \sigma_s S I_s + \mu S &= 0, \\
\sigma_m S I_m + \sigma_s S I_s - (\mu + \rho + \beta) I_m &= 0, \\
\beta I_m - T(I_s) - \mu I_s &= 0,
\end{align*}
\]
which becomes
\[
A - \sigma_m S I_m - \mu S = 0 \\
\sigma_m S I_m - (\mu + r + \beta) I_m = 0 \\
\beta I_m - (\mu + r) I_s = 0
\]
if \(0 < I_s \leq C_I\), while
\[
A - \sigma_m S I_m - \sigma_s S (I_s - C_I) - \mu S = 0 \\
\sigma_m S I_m + \sigma_s S (I_s - C_I) - (\mu + r + \beta) I_m = 0 \\
\beta I_m - \mu I_s - r C_I = 0
\]
if \(C_I < I_s\). When \(0 < I_s \leq C_I\) and \(R_0 > 1\), (2.2) admits a unique positive solution \(E^* = (S^*, I^*_m, I^*_s)\): 
\[
S^* = \frac{\mu + \beta + \rho}{\sigma_m}, \quad I^*_m = \frac{\mu (R_0 - 1)}{\sigma_m}, \quad I^*_s = \frac{\mu \beta (R_0 - 1)}{\sigma_m (\mu + r)}.
\]
Clearly, \(E^*\) is an endemic equilibrium of (2.1) if and only if
\[
1 < R_0 \leq 1 + \frac{\sigma_m (\mu + r) C_I}{\mu \beta C_I}.
\]

In order to obtain positive solutions of (2.3), solving (2.3) in terms of \(I_m, I_s\) with \(S\) and substituting the result into the first equation, we have
\[
S^2 - \frac{\mu + \beta + \rho}{\sigma_m} (R_0 - p + q) S + \frac{(\mu + \beta + \rho)^2}{\sigma_m^2} q R_0 = 0,
\]
where
\[
p = \frac{(\mu + r) \sigma_m \sigma_s C_I}{\mu (\mu \sigma_m + \beta \sigma_s)}, \quad q = \frac{\mu \sigma_m}{\mu \sigma_m + \beta \sigma_s}.
\]
Note that \(p > 0\) and \(0 < q < 1\). When \(C_I < I_s\), (2.3) admits possible two positive solutions
\[
E_1^* = (S_1^*, I^*_{m1}, I^*_{s1}), \quad E_2^* = (S_2^*, I^*_{m2}, I^*_{s2}) \quad \text{where}
\]
\[
S_1^* = \frac{\mu + \beta + \rho}{2 \sigma_m} \left\{ R_0 - p + q - \sqrt{(R_0 - p - q)^2 - 4pq} \right\}, \\
S_2^* = \frac{\mu + \beta + \rho}{2 \sigma_m} \left\{ R_0 - p + q + \sqrt{(R_0 - p - q)^2 - 4pq} \right\}, \\
I^*_{m1} = \frac{\mu R_0}{\sigma_m} - \frac{\mu S_1^*}{\mu + \beta + \rho}, \quad I^*_{s1} = \frac{\beta R_0}{\sigma_m} - \frac{\beta S_1^*}{\mu + \beta + \rho} - \frac{r C_I}{\mu}, \quad i = 1, 2.
\]
We see that \(I^*_{m1} > 0\) and \(I^*_{s1} > C_I\) are equivalent to
\[
S < \frac{\mu + \beta + \rho}{\sigma_m} \left( R_0 - \frac{\sigma_m (\mu + r) C_I}{\mu \beta} \right),
\]
which implies that (2.1) does not have any endemic equilibria if \(R_0 \leq \sigma_m (\mu + r) C_I / (\mu \beta)\). From the above, we have the following:
Lemma 2.1. \(2.1\) has always a unique disease-free equilibrium \(E_0\), and has an endemic equilibrium \(E^*\) if and only if
\[
1 < \mathcal{R}_0 \leq 1 + \frac{\sigma_m(\mu + r)}{\mu \beta} C_I.
\]
Moreover, \(2.1\) does not have any endemic equilibria in the region \(C_I < I_s\) if
\[
C_I \geq \frac{\beta A}{(\mu + r)(\mu + r + \rho)}.
\]

For convenience, let
\[
a = \frac{\mu + \beta + \rho}{\sigma_m} \left( \mathcal{R}_0 - \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \right).
\]
In order to find endemic equilibria in the region \(C_I < I_s\), by Lemma 2.1 we should consider the case
\[
C_I < \frac{\beta A}{(\mu + r)(\mu + r + \rho)},
\]
which is equivalent to \(\mathcal{R}_0 > \sigma_m(\mu + r)C_I / (\mu \beta)\). To figure out \(S_1^*\), \(S_2^*\) feasible for endemic equilibria, define
\[
f(S) = S^2 - \frac{\mu + \beta + \rho}{\sigma_m} (\mathcal{R}_0 - p + q) S + \frac{(\mu + \beta + \rho)^2}{\sigma_m^2} q \mathcal{R}_0.
\]
When \(2.4\) holds, this quadratic function axis is positive since
\[
\frac{\sigma_m(\mu + r)C_I}{\mu \beta} > p > p - q.
\]
Then we see that \(E_1^*\) is an endemic equilibrium of \(2.1\) but not \(E_2^*\) if and only if \(f(a) \leq 0\) holds as the axis of \(f(S)\) is less than \(a\), or \(f(a) < 0\) holds as the axis of \(f(S)\) is greater than or equal to \(a\), which equivalently implies
\[
\mathcal{R}_0 \geq 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \quad \text{and} \quad \mathcal{R}_0 > -p + q + \frac{2\sigma_m(\mu + r)C_I}{\mu \beta}
\]
or
\[
\mathcal{R}_0 > 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \quad \text{and} \quad \mathcal{R}_0 \leq -p + q + \frac{2\sigma_m(\mu + r)C_I}{\mu \beta}
\]
holds, respectively. Since we obtain
\[
1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} - \left( -p + q + \frac{2\sigma_m(\mu + r)C_I}{\mu \beta} \right) = -\frac{(\mu + r)\sigma_m^2}{\beta (\mu \sigma_m + \beta \sigma_s)} \left( C_I - \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m^2} \right),
\]
\(2.7\) is equivalent to
\[
\mathcal{R}_0 > 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \quad \text{if} \quad C_I \geq \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m^2}
\]
while
\[
\mathcal{R}_0 \geq 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \quad \text{else.}
\]
Similarly, both of $E_1^*$ and $E_2^*$ are endemic equilibria of (2.1) if and only if

\[ f(a) > 0, \]
\[ (\mathcal{R}_0 - p - q)^2 - 4pq \geq 0, \]
\[ \frac{\mu + \beta + \rho}{2\sigma_m} (\mathcal{R}_0 - p + q) < a. \]

Clearly, the first and third conditions of (2.9) are equivalent to

\[ \mathcal{R}_0 < 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \]  
and
\[ \mathcal{R}_0 > -p + q + \frac{2\sigma_m(\mu + r)C_I}{\mu \beta}, \]

respectively. Also, by (2.6) and (2.11), the second one of (2.9) is replaced with

\[ \mathcal{R}_0 \geq (\sqrt{p} + \sqrt{q})^2. \]

Thus, it follows from the facts (2.8) and (2.13) that a common range where (2.4), (2.10), (2.11), and (2.12) hold is

\[ (\sqrt{p} + \sqrt{q})^2 \leq \mathcal{R}_0 < 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \]

if $C_I < \beta^2 \sigma_s / \{(\mu + r)\sigma_m^2\}$ while no the common range exists else. Furthermore, by some tedious calculation, we see that $\sqrt{p} + \sqrt{q} < 1$, $\sqrt{p} + \sqrt{q} = 1$ are equivalent to

\[ C_I < \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{1}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{1 \over \sigma_s} \right)^2, \quad C_I = \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{1}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{1 \over \sigma_s} \right)^2, \]

respectively. Here,

\[ \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{1}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{1 \over \sigma_s} \right)^2 = \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m^2} \left( \sqrt{\frac{\beta}{\mu \sigma_m} + \frac{1}{\sigma_s}} + \sqrt{1 \over \sigma_s} \right)^2 < \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m^2}. \]

Hence, we have the following:

**Theorem 2.1.** Suppose (2.4) and

\[ \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m^2} \leq C_I \]

hold. Then $E^*$ is a unique endemic equilibrium of (2.1) if $1 < \mathcal{R}_0 \leq 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta}$, and $E_1^*$ is a unique endemic equilibrium if $\mathcal{R}_0 > 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta}$. 

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Theorem 2.2. Suppose (2.4) and
\[
\frac{\mu^2}{\mu + r} \left( \sqrt{\frac{\beta}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{\frac{1}{\sigma_s}} \right)^2 < C_I < \frac{\beta^2 \sigma_s}{(\mu + r) \sigma_m^2}
\]
hold. Then $E^*$ is a unique endemic equilibrium of (2.1) if $1 < R_0 < (\sqrt{p} + \sqrt{q})^2$, all endemic equilibria $E^*$, $E_1^*$, and $E_2^*$ exist if $(\sqrt{p} + \sqrt{q})^2 \leq R_0 < 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$, $E_2^*$ does not exist but $E^*$ and $E_1^*$ exist if $R_0 = 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$, and $E_1^*$ is a unique endemic equilibrium if $R_0 > 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$.

Theorem 2.3. Suppose (2.4) and
\[
C_I \leq \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{\beta}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{\frac{1}{\sigma_s}} \right)^2
\]
hold. Then $E^*$ does not exist as endemic equilibrium for (2.1) but $E_1^*$ and $E_2^*$ exist if $(\sqrt{p} + \sqrt{q})^2 \leq R_0 \leq 1$, all endemic equilibria $E^*$, $E_1^*$, and $E_2^*$ exist if $1 < R_0 < 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$, $E_2^*$ does not exist but $E^*$ and $E_1^*$ exist if $R_0 = 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$, and $E_1^*$ is a unique endemic equilibrium if $R_0 > 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$.

Note that a backward bifurcation with endemic equilibria when $R_0 < 1$ is very interesting in applications. We present the following corollary, a consequence of Theorems 2.1–2.3, to give a necessary and sufficient condition for such a backward bifurcation to occur.

Corollary 2.1. (2.1) has a backward bifurcation with endemic equilibria when $R_0 < 1$ if and only if (2.4) and
\[
C_I \leq \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{\beta}{\mu \sigma_m} + \frac{1}{\sigma_s}} - \sqrt{\frac{1}{\sigma_s}} \right)^2
\]

3 Stability of the equilibria

We have the following results on stability for all the equilibria of (2.1).

Theorem 3.1. $E_0$ is asymptotically stable if $R_0 < 1$, but unstable if $R_0 > 1$.

Proof. This theorem is a simple consequence of Theorem 2 of [3].

Theorem 3.2. $E^*$ is asymptotically stable if $1 < R_0 < 1 + \frac{\sigma_m(\mu + r)}{\mu^3} C_I$. $E_1^*$ is asymptotically stable whenever it exists and does not shrink to $E_2^*$, while $E_2^*$ is unstable whenever it exists and does not shrink to $E_1^*$.

Proof. We analyze the eigenvalues of the Jacobian matrices of (2.1) at the equilibria, to which Lemma A.28 in [3] and Rough-Hurwiz criteria (see, for example [1]) are applied. First, we have the Jacobian matrix at $E^*$:
\[
\mathcal{J}(E^*) = \begin{pmatrix}
-\frac{A}{s} & -\sigma_m S^* & 0 \\
\frac{A}{s} - \mu & 0 & 0 \\
0 & \beta & -\mu - r
\end{pmatrix}
\]
and then consider the characteristic equation \( \det(\lambda I - J(E^*)) = 0 \) with an identity matrix \( I \), which is given as

\[
(\lambda + \mu + r) \left\{ \lambda^2 + \frac{A}{S^*} \lambda + \mu(\mu + \beta + \rho) (R_0 - 1) \right\} = 0.
\]

It is clear that all eigenvalues of \( J(E^*) \) have negative real parts when \( 1 < R_0 < 1 + \frac{\sigma_m(\mu + r)}{\mu \beta} C_I \), which implies that \( E^* \) is asymptotically stable.

For the Jacobian matrix at \( E_2^* \):

\[
J(E_2^*) = \begin{pmatrix}
-\frac{A}{S_2^*} & -\sigma_m S_2^* & -\sigma_s S_2^* \\
\frac{A}{S_2^*} - \mu & \sigma_m S_2^* - \mu - \beta - \rho & \sigma_s S_2^* \\
0 & \beta & -\mu
\end{pmatrix},
\]

the characteristic equation \( \det(\lambda I - J(E_2^*)) = 0 \) is given as

\[
(\lambda + \mu) \left\{ \lambda^2 + \left( \frac{A}{S_2^*} - \sigma_m S_2^* + \mu + \beta + \rho \right) \lambda - \frac{(\mu + \beta + \rho)(\mu \sigma_m + \beta \sigma_s) \sqrt{D_1}}{\sigma_m} \right\} = 0,
\]

where \( D_1 = (R_0 - p - q)^2 - 4pq \) and we used relations between roots and coefficients for \( S_1^* \) and \( S_2^* \). Note that \( D > 0 \) since we now consider the case \( S_1^* \neq S_2^* \). Then it is clear that one eigenvalue of \( J(E_2^*) \) is a positive real number, which implies that \( E_2^* \) is unstable.

In order to consider the stability of \( E_1^* \), we similarly have the Jacobian matrix at \( E_1^* \):

\[
J(E_1^*) = \begin{pmatrix}
-\frac{A}{S_1^*} & -\sigma_m S_1^* & -\sigma_s S_1^* \\
\frac{A}{S_1^*} - \mu & \sigma_m S_1^* - \mu - \beta - \rho & \sigma_s S_1^* \\
0 & \beta & -\mu
\end{pmatrix},
\]

and the characteristic equation

\[
(\lambda + \mu) \left\{ \lambda^2 + \left( \frac{A}{S_1^*} - \sigma_m S_1^* + \mu + \beta + \rho \right) \lambda - \frac{(\mu + \beta + \rho)(\mu \sigma_m + \beta \sigma_s) \sqrt{D_1}}{\sigma_m} \right\} = 0.
\]

To conclude \( E_1^* \) is asymptotically stable, we only have to show that

\[
\frac{A}{S_1^*} - \sigma_m S_1^* + \mu + \beta + \rho > 0,
\]

which is proven separately in the following two cases. Recall that

\[
0 < S_1^* < \frac{\mu + \beta + \rho}{\sigma_m} \left( R_0 - \frac{\sigma_m (\mu + r) C_I}{\mu \beta} \right) = a. \tag{3.1}
\]

(i) The case where \( R_0 < 1 + \sigma_m(\mu + r)C_I/(\mu \beta) \) holds. By (3.1) we easily see that

\[
\frac{A}{S_1^*} - \sigma_m S_1^* + \mu + \beta + \rho > \frac{A}{S_1^*} - (\mu + \beta + \rho) \left( R_0 - \frac{\sigma_m (\mu + r) C_I}{\mu \beta} \right) + \mu + \beta + \rho
\]

\[
= \frac{A}{S_1^*} - (\mu + \beta + \rho) \left( R_0 - 1 - \frac{\sigma_m (\mu + r) C_I}{\mu \beta} \right) > 0.
\]
(ii) When $\mathcal{R}_0 \geq 1 + \sigma_m(\mu + r)C_I/(\mu \beta)$ holds, it follows that

$$f(a) = -\frac{(\mu + \beta + \rho)^2 q(\mu + r)C_I}{\sigma_m \mu \beta} \left( \mathcal{R}_0 - 1 - \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \right) \leq 0,$$

where $f$ is defined by (2.5). We obtain

$$\frac{A}{S_1^*} - \sigma_m S_1^* + \mu + \beta + \rho
= \frac{\mu + \beta + \rho}{S_1^*} \left\{ - (\mathcal{R}_0 - 1 - p + q) S_1^* + \frac{(\mu + \beta + \rho) \mathcal{R}_0}{\sigma_m} \left( q + \frac{\mu}{\mu + \beta + \rho} \right) \right\}$$

since $f(S_1^*) = 0$. Define

$$g(S) = - (\mathcal{R}_0 - 1 - p + q) S + \frac{(\mu + \beta + \rho) \mathcal{R}_0}{\sigma_m} \left( q + \frac{\mu}{\mu + \beta + \rho} \right).$$

Then the proof will be completed if $g(S_1^*) > 0$, equivalent to

$$0 < S_1^* < \frac{(\mu + \beta + \rho) \mathcal{R}_0}{\sigma_m (\mathcal{R}_0 - 1 - p + q)} \left( q + \frac{\mu}{\mu + \beta + \rho} \right)$$

(3.3)

since $\mathcal{R}_0 - 1 - p + q > 0$ by (2.6).

Now we consider a quadratic function $h$ given as

$$h(x) = x^2 - \left( 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} + p + \frac{\mu}{\mu + \beta + \rho} \right) x + (1 + p - q) \frac{\sigma_m(\mu + r)C_I}{\mu \beta}.$$

By some tedious calculation, we see that $h(\mathcal{R}_0) \leq 0$ if and only if

$$a \leq \frac{(\mu + \beta + \rho) \mathcal{R}_0}{\sigma_m (\mathcal{R}_0 - 1 - p + q)} \left( q + \frac{\mu}{\mu + \beta + \rho} \right),$$

which, together with (3.2), implies that (3.3) holds. Clearly, $h(x)$ has the two zeros $\alpha_1, \alpha_2$:

$$\alpha_1 = \frac{1}{2} \left( 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} + p + \frac{\mu}{\mu + \beta + \rho} - \sqrt{D_2} \right),$$

$$\alpha_2 = \frac{1}{2} \left( 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} + p + \frac{\mu}{\mu + \beta + \rho} + \sqrt{D_2} \right),$$

where

$$D_2 = \left( p + \frac{\mu}{\mu + \beta + \rho} - 1 - \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \right)^2 + \frac{(1 + p - q)\sigma_m(\mu + r)C_I}{\beta p (\mu + \beta + \rho)} > 0.$$

Note that $\alpha_1 < 1 + \sigma_m(\mu + r)C_I/(\mu \beta) < \alpha_2$. Hence, only consideration in the case

$$\mathcal{R}_0 > \alpha_2$$

(3.4)

remains to complete the proof. Let

$$b = \frac{(\mu + \beta + \rho) \mathcal{R}_0 q}{\sigma_m (\mathcal{R}_0 - 1 - p + q)}.$$
Then we have
\[ b < \frac{(\mu + \beta + \rho)R_0}{\sigma_m (R_0 - 1 - p + q)} \left( q + \frac{\mu}{\mu + \beta + \rho} \right) < a, \] (3.5)
where the first inequality is clearly verified and the second one is by the fact that the case (3.4) ensures \( h(R_0) > 0 \) as discussed above. Furthermore, we obtain
\[ f(b) = \frac{q(q - 1)}{(R_0 - 1 - p + q)^2} \left( R_0 - 1 - \frac{\sigma_m(\mu + r)C_I}{\mu \beta} \right) < 0 \] (3.6)
since \( q < 1 \) and (3.4). It follows from (3.2), (3.5), (3.6), and the convexity of the function \( f \) that
\[ f \left( \frac{(\mu + \beta + \rho)R_0}{\sigma_m (R_0 - 1 - p + q)} \left( q + \frac{\mu}{\mu + \beta + \rho} \right) \right) < 0, \]
which implies that (3.3) holds. The proof is then complete.

Figure 1: A bifurcation diagram with \( C_I = 40 \) that satisfies \( \frac{\beta^2 \sigma_s}{(\mu + r)\sigma_m} \leq C_I \), where the vertical axis shows \( R_0 \) and the horizontal one the value \( I_s \) in equilibrium. The bifurcation from the disease-free equilibrium at \( R_0 = 1 \) is forward and (2.1) has a unique endemic equilibrium for \( R_0 > 1 \).

\( E^* \) exists even if \( R_0 = 1 + \sigma_m(\mu + r)C_I/(\mu \beta) \) as mentioned in the previous section. For the case, the stability of \( E^* \) is not determined by the same method as in the proof of Theorem 3.2 since the right-hand side of (2.1) is not differentiable at \( E^* \). Such situations where the equal sign just holds, however, cannot be expected to be found in nature and can be assumed to be neglected without loss of biological generality. Theorems 3.1 and 3.2 then completely give local properties of all equilibrium solutions for (2.1). Typical bifurcation diagrams are
Figure 2: A bifurcation diagram with $C_I = 10$ that satisfies
\[ \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{\beta}{\mu \sigma_m}} + \frac{1}{\sigma_s} - \sqrt{\frac{1}{\sigma_s}} \right)^2 < C_I < \frac{\beta^2 \sigma_s}{(\mu + r) \sigma_m}, \]
where the vertical and horizontal axes are the same as Fig. 1. Dashed line represents the unstable equilibrium $E^*_2$. The bifurcation at $R_0 = 1$ is forward and there is a backward bifurcation from an endemic equilibrium at $R_0 = 1 + \frac{\sigma_m(\mu + r)C_I}{\mu \beta} = 1.286$, which leads to the existence of multiple endemic equilibria.

Illustrated in Figs. 1–3 in correspondence with three sizes of $C_I$, where other parameters are given as $\sigma_m = \sigma_s = 0.01, \mu = 0.8, \rho = 1.0, \sigma = 0.8$, and $\beta = 0.7$. In particular, we present the following corollary to clarify the stability of endemic equilibria for the backward-bifurcation case mentioned in Corollary 2.1.

**Corollary 3.1.** Suppose (2.4) and
\[ C_I \leq \frac{\mu^2}{\mu + r} \left( \sqrt{\frac{\beta}{\mu \sigma_m}} + \frac{1}{\sigma_s} - \sqrt{\frac{1}{\sigma_s}} \right)^2 \] (3.7)
hold. Then $E^*_1$ is asymptotically stable while $E^*_2$ is unstable if $(\sqrt{\rho} + \sqrt{q})^2 \leq R_0 \leq 1$, both $E^*_1$ and $E^*_2$ are asymptotically stable while $E^*_2$ is unstable if $1 < R_0 < 1 + \frac{\sigma_m(\mu + r)}{\mu \beta} C_I$, and $E^*_1$ is asymptotically stable if $R_0 > 1 + \frac{\sigma_m(\mu + r)}{\mu \beta} C_I$.

**4 Concluding remarks**

In this paper, we have proposed a disease-severity-structured epidemic model with treatment necessary only to severe infective individuals, which is higher dimensional and also more realistic.
Figure 3: A bifurcation diagram with $C_I = 4$ that satisfies $C_I \leq \frac{\mu^2}{\mu+\tau} \left( \sqrt{\frac{\beta}{\mu \sigma_m}} + \frac{1}{\sigma_s} - \sqrt{\frac{1}{\sigma_s}} \right)$, where the vertical and horizontal axes are the same as Fig. 1. Dashed line represents the unstable equilibrium $E_2^*$. The graph shows a backward bifurcation with endemic equilibria when $R_0 < 1$.

than [5] for non-lethal or less-lethal diseases, to discuss the effect of the treatment capacity on the disease transmission. Our bifurcation analysis reveals local properties of all the equilibrium solutions to fully mathematically obtain bifurcation diagrams for any situation. We have shown in Corollary 2.1 that backward bifurcations occur because of the insufficient capacity for treatment, which generalizes [5] in the non-lethal disease case or in the less-lethal disease case where the disease-related death rate is too small to be neglected. Once a backward bifurcation occurs, as shown in Corollary 3.1, a stable endemic equilibrium co-exists with a stable disease-free equilibrium when $R_0 < 1$. This leads to the same scenario as [5], that is, the requirement $R_0 < 1$ is not sufficient for effective disease control and disease outbreak can happen to a high endemic level even though $R_0 < 1$.

By Corollary 3.1, no backward bifurcations occur if $C_I$ is so large that (2.4) does not hold. When $C_I$ is not so large and satisfy (2.4), however, it follows from Theorems 2.1-2.3 that a backward bifurcation can occur depending on either or both of $\sigma_m$ and $\sigma_s$. A novel aspect of our results is a clear formulation representing the effect of the treatment capacity by the necessary and sufficient condition (3.7) derived for the occurrence of backward bifurcations. This means that we understand in an explicit way how each parameter, as well as $C_I$, plays a role of preventing or promoting the backward-bifurcation scenario. Let us fix $C_I$ such that (3.7) does not hold. Then no backward bifurcations occur. Even if $\sigma_m$ increases, (3.7) has not held
forever since its right-hand side is a decreasing function in $\sigma_m$, which implies that $\sigma_m$ plays a role of preventing backward bifurcation from occurring. On the other hand, from a similarly simple observation, we can see that $\sigma_s$ plays a role of promoting the backward-bifurcation scenario.

We have not shown global properties of all solutions for (2.1). This is because our model has a higher dimension than that of [5] and then cannot have the same method as in [5], that is, using the theory of planar dynamical system, applied to it. In order to carry out global analysis for (2.1), appropriate Lyapunov functions should be constructed when an endemic equilibrium uniquely exists, while much more sophisticated mathematics may be required when multiple endemic equilibria coexists, which will be left for future work. Besides that, as with treatment, vaccination is important to decrease the spread of diseases, and the number of vaccines is also practically limited. Based on the work of this paper, we can consider an epidemic model with capacities of vaccination as well as treatment to discuss the effect of both these capacities on the disease transmission, which will be presented on another occasion.

Acknowledgments

We thank Professor Hiromi Seno, Tohoku University, for his helpful comments on mathematical modeling for the infection force of severe infective individuals. This work is partially supported by JSPS KAKENHI Grant Numbers 20K03750 (the second author).

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