On the analytic spread and the reduction number of the ideal of maximal minors

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Abstract: Let \( m, n, a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r \) be integers with \( 1 \leq a_1 < \cdots < a_r \leq m \) and \( 1 \leq b_1 < \cdots < b_r \leq n \). And let \( x \) be the universal \( m \times n \) matrix with the property that \( i \)-minors of first \( a_i - 1 \) rows and first \( b_i - 1 \) columns are all zero, for \( i = 1, \ldots, r + 1 \) \((a_{r+1} := m + 1 \text{ and } b_{r+1} := n + 1)\). For an integer \( u \) with \( 1 \leq u \leq m \), we denote by \( U \) the \( u \times n \) matrix consisting of the first \( u \) rows of \( x \). In this paper, we consider the analytic spread and the reduction number of the ideal of maximal minors of \( U \).

Key words: ideal of maximal minors, analytic spread, reduction number, ASL, distributive lattice

1 Introduction

In this paper all rings and algebras are assumed to be commutative with identity element. For an \( m \times n \) matrix \( U \) with entries in a ring \( R \), we denote by \( I_t(U) \) the ideal of \( R \) generated by all the \( t \)-minors of \( U \), where we put \( I_0(U) = R \) and \( I_t(U) = (0) \) for \( t \) with \( t > \min\{m, n\} \). And if \( I_k(U) \neq (0) \) and \( I_{k+1}(U) = (0) \), we call \( I_k(U) \) the ideal of maximal minors of \( U \). In this paper, we consider the analytic spread and the reduction number of the ideal of maximal minors of the matrix defined below.

Let \( K \) be an infinite field, \( m, n, r \) be integers with \( 1 \leq r \leq \min\{m, n\} \), \( a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r \) be integers with \( 1 \leq a_1 < \cdots < a_r \leq m \) and \( 1 \leq b_1 < \cdots < b_r \leq n \). In this situation, there is a universal \( m \times n \) matrix \( x \) with the condition

\[
\begin{align*}
I_i(\text{first } a_i - 1 \text{ rows}) &= (0) \\
I_i(\text{first } b_i - 1 \text{ columns}) &= (0)
\end{align*}
\]

for \( i = 1, \ldots, r + 1 \), where we set \( a_{r+1} := m + 1 \) and \( b_{r+1} := n + 1 \). That is, \( x \) satisfies (1.1) and if \( U \) is an \( m \times n \) matrix with entries in a \( K \)-algebra \( S \) satisfying (1.1), there is a unique \( K \)-algebra homomorphism \( K[x] \to S \) mapping \( x \) to \( U \), where \( K[x] \) is the \( K \)-algebra generated by the entries of \( x \).

There are two ways to construct such a matrix \( x \). One is to define \( x \) as a homomorphic image of the generic \( m \times n \) matrix (i.e. an \( m \times n \) matrix whose
entries are independent indeterminates) $X$ in the quotient ring of $K[X]$. The
other is to define $x$ as the array of products of generic matrices, see [HE, 
HR, BV]. In this paper, we follow the first way.

Let $X$ be the generic $m \times n$ matrix, $K[X]$ the polynomial ring generated
by the entries of $X$. Then it is known that $K[X]$ is a graded algebra with
straightening laws (ASL for short) over $K$ generated by $\Delta(X)$, where

$$\Delta(X) := \{[c_1, c_2, \ldots, c_s|d_1, d_2, \ldots, d_s] \mid s \leq \min\{m, n\},
1 \leq c_1 < \cdots < c_s \leq m, 1 \leq d_1 < \cdots < d_s \leq n\}$$

and the partial order of $\Delta(X)$ is defined by

$$[c_1, c_2, \ldots, c_s|d_1, d_2, \ldots, d_s] \leq [c'_1, c'_2, \ldots, c'_s|d'_1, d'_2, \ldots, d'_s]$$
$$\iff s \geq s', c_1 \leq c'_1, \cdots, c'_s \leq c_s, d_1 \leq d'_1, \cdots, d'_s \leq d_s.$$ 

And $\Delta(X)$ is embedded in $K[X]$ by corresponding $[c_1, c_2, \ldots, c_s|d_1, d_2, \ldots, d_s]$
to $\det(X_{i,j})$. See [DEP1], [BV].

Set

$$\delta := [a_1, a_2, \ldots, a_r|b_1, b_2, \ldots, b_r] \in \Delta(X)$$
$$\Delta(X; \delta) := \{\gamma \in \Delta(X) \mid \gamma \geq \delta\}$$
$$\Omega := \Delta(X) \setminus \Delta(X; \delta)$$
$$A := K[X]/\Omega K[X].$$

Then, since $\Omega$ is a poset ideal of $\Delta(X)$, $A$ is a graded ASL over $K$
generated by $\Delta(X; \delta)$. If we denote the image of $X$ in $A$ by $x$, then, by the Laplace
expansion, we see that $x$ is the universal $m \times n$ matrix satisfying (1.1).

Let $u$ be an integer with $1 \leq u \leq m$, and $U$ be the $u \times n$ matrix consisting
of the first $u$ rows of $x$. In this paper, we consider the analytic spread and
the reduction number of the ideal of maximal minors $I$ of $U$. If we take $k$
such that $a_k \leq u < a_{k+1}$, then by the definition of $A$, we see that $I_k(U) \neq (0)$
and $I_{k+1}(U) = (0)$. Therefore $I = I_k(U)$.

## 2 Analytic spread

We denote the irrelevant maximal ideal of $A$ by $m$.

Northcott-Rees [NR] defined for an ideal $a$ of a local ring $(R, m)$ with
infinite residue field, the analytic spread $\ell(a)$ of $a$ to be the dimension of

$$R/m \otimes \text{Gr}_a(R) = R/m \oplus a/m \oplus a^2/m^2 \oplus \cdots$$

and showed that the analytic spread of $a$ is the number of minimal generators
of any minimal reduction of $a$. They also showed, essentially, that for an ideal
b contained in a,

\[(2.1) \quad ba^n = a^{n+1} \iff \bar{b} \supseteq (R/n \otimes \text{Gr}_a(R))^{n+1},\]

where \(\bar{b}\) is the ideal of \(R/n \otimes \text{Gr}_a(R)\) generated by \((b + na)/na(\subseteq a/na)\).

Now set
\[
\Theta := \{ \gamma \in \Delta(X) \mid [a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k] \leq \gamma \\
\leq [u-k+1, \ldots, u|n-k+1, \ldots, n]\}.
\]

If \(\alpha\) and \(\beta\) are incomparable elements in \(\Delta(X)\), then the standard representation of \(\alpha\beta\) in the ASL \(K[X]\) is of the form
\[
\alpha\beta = \sum_i b_i \gamma_{i1} \gamma_{i2} + \sum_j b'_j \delta_j
\]
and for each \(i\) and \(j\), the union of row (column) numbers of \(\gamma_{i1}\) and \(\gamma_{i2}\) (\(\delta_j\)) as a multi-set is the same as that of \(\alpha\) and \(\beta\) \([\text{DEP1}], [\text{BV}]\). Therefore, if \(\theta_1\) and \(\theta_2\) are incomparable elements of \(\Theta\), then, since minors of \(U\) size greater than \(k\) are zero, the standard representation of \(\theta_1\theta_2\) in the ASL \(A\) is of the form
\[
\theta_1\theta_2 = \sum_i b_i \gamma_{i1} \gamma_{i2}, \quad \gamma_{ij} \in \Theta.
\]

It follows from \([\text{DEP2}]\) Proposition 1.1 that \(K[\Theta]\) is a sub-ASL of \(A\) and

\[(2.2) \quad A/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots \simeq K[\Theta].\]

Therefore
\[
\ell(I) = \dim K[\Theta] = \text{rank}\Theta + 1.
\]

By counting the rank of \(\Theta\), we see the following

**Theorem 2.1** The analytic spread \(\ell(I)\) of \(I\) is 

\[k(u + n - k + 1) - \sum_{i=1}^{k} (a_i + b_i) + 1.\]

### 3 Reduction number

In the following, we multiply the degree of elements in \(K[\Theta]\) by \(1/k\) and adjust the degree of the right hand side of \([2.2]\) to the left hand side. We also denote the irrelevant maximal ideal of \(K[\Theta]\) by \(n\), and the analytic spread \(\ell(I)\) of \(I\) by \(l\). By \([2.1]\) if \(J\) is a homogeneous ideal of \(A\) generated by elements of degree \(k\) (if we consider the degree in \(K[\Theta]\), then degree 1 by the convention above) and is a minimal reduction of \(I\), then the minimal
generating system of \( J \) is a homogeneous system of parameters of degree 1 in \( K[\Theta] \). Conversely, any homogeneous system of parameters of degree 1 in \( K[\Theta] \) generates a minimal reduction of \( I \) in \( A \) (see the proof of [NR §2 Theorem 1]).

If \( J \) is a homogeneous ideal of \( A \), and a minimal reduction of \( I \), we denote by \( r_J(I) \) the reduction number \( \min\{ n \midJI^n = I^{n+1}\} \) of \( I \) with respect to \( J \). If \( v_1, v_2, \ldots, v_l \) is a minimal system of generators of \( J \) of degree 1, then by (2.1),

\[
\begin{align*}
r_J(I) &= \min\{ n \mid (v_1, v_2, \ldots, v_l) \supseteq n^{n+1}\} \\
&= \max\{ n \mid (K[\Theta]/(v_1, v_2, \ldots, v_l))_n \neq 0 \} \\
&= a(K[\Theta]/(v_1, v_2, \ldots, v_l)),
\end{align*}
\]

where \( a(\cdot) \) is the \( a \)-invariant defined by Goto-Watanabe (see [GW Definition (3.1.4)]). Since \( \Theta \) is a distributive lattice, we see that \( K[\Theta] \) is a Cohen-Macaulay ring. Therefore by [GW, Remark (3.1.6)], we see that

\[
r_J(I) = a(K[\Theta]) + l.
\]

On the other hand by the proof of [Sta2, 4.4 Theorem], we see that

\[
\text{Hilb}(K_R, \lambda) = (-1)^{\dim R} \text{Hilb}(R, \lambda^{-1})
\]

for a Cohen-Macaulay standard graded ring \( R \) over a field, where \( \text{Hilb}(\cdot, \cdot) \) denotes the Hilbert series, \( K_R \) denotes the canonical module of \( R \). So in order to calculate the \( a \)-invariant, we may replace the ring with a Cohen-Macaulay ring with the same Hilbert series. Since the two ASL's generated by the same poset has the same Hilbert series, we compute the \( a \)-invariant of \( K[\Theta] \) by computing the \( a \)-invariant of the Hibi ring \( R_K(\Theta) \).

In general, for a distributive lattice \( D \), if we denote the set of all the join irreducible elements of \( D \) (i.e. elements \( x \) of \( D \) such that there is exactly one \( y \in D \) such that \( y < x \)) by \( P \), it is known that

\[
D \simeq J(P) := \{ J \mid J \text{ is a poset ideal of } P \}.
\]

And if one takes a family \( \{X_\alpha\}_{\alpha \in P \cup \{-\infty\}} \) of indeterminates and set \( \varphi(I) := X_\infty \prod_{\alpha \in I} X_\alpha \) for \( I \in J(P) \), then Hibi [Hib] showed that

\[
\mathcal{R}_K(D) := K[\varphi(I) \mid I \in J(P)] \subseteq K[X_\alpha \mid \alpha \in P \cup \{-\infty\}]
\]

is a homogeneous ASL over \( K \) generated by \( D \). Where we set \( \deg X_\infty = 1 \) and \( \deg X_\alpha = 0 \) for any \( \alpha \in P \).

Set

\[
M := \{(n_\alpha)_{\alpha \in P \cup \{-\infty\}} \in \mathbb{N}^{#P+1} \mid \alpha \leq \beta \implies n_\alpha \geq n_\beta \}.
\]
Then $M$ is a submonoid of $N^{#P+1}$ and

$$R_{K}(D) = K[M] := K[X^{\omega} \mid \omega \in M],$$

where $X^{\omega}$ is the multi-index.

Since $R_{\geq 0} M \cap N^{#P+1} = M$, we see by [Sta1, Theorem 4.1],

$$K_{K[M]} = \bigoplus_{\omega \in \text{int}(R_{\geq 0}M) \cap M} KX^{\omega}.$$

Because $\text{int}(R_{\geq 0}M) \cap M = \{(n_{\alpha})_{\alpha \in P U \{-\infty\}} \in N^{#P+1} \mid n_{\alpha} > 0, \alpha < \beta \Rightarrow n_{\alpha} > n_{\beta}\}$, we see that

$$a(K[M]) = -\min\{\deg X^{\omega} \mid \omega \in \text{int}(R_{\geq 0}M) \cap M\}
= - (\text{rank}P + 2).$$

In particular, by taking $D$ to our $\Theta$, we see that

$$r_{J}(I) = a(K[\Theta]) + l = a(R_{K}(\Theta)) + l = l - (\text{rank}P + 2),$$

where $P$ is the set of join irreducible elements of $\Theta$.

By considering row numbers and column numbers separately, we see that $\Theta$ is the poset product of two distributive lattices, say $D_{1}$ and $D_{2}$. $(x_{1}, x_{2}) \in D_{1} \times D_{2}$ is join irreducible if and only if $x_{1}$ is a join irreducible element of $D_{1}$ and $x_{2}$ is the minimal element of $D_{2}$ or $x_{1}$ is the minimal element of $D_{1}$ and $x_{2}$ is a join irreducible element of $D_{2}$. So if we denote the set of all the join irreducible elements of $D_{i}$ by $P_{i}$ for $i = 1, 2$, the set of join irreducible elements of $\Theta$ is isomorphic to the disjoint union of $P_{1}$ and $P_{2}$. Therefore

$$\text{rank}P = \max\{\text{rank}P_{1}, \text{rank}P_{2}\}.$$ 

Since $D_{1}$ and $D_{2}$ are of the same form, we consider $D_{1}$ in the following.

Since

$$D_{1} = \{[c_{1}, c_{2}, \ldots, c_{k}] \mid 1 \leq c_{1} < \cdots < c_{k} \leq u, a_{i} \leq c_{i} (i = 1, \ldots, k)\},$$

$$[c_{1}, c_{2}, \ldots, c_{k}] \leq [d_{1}, d_{2}, \ldots, d_{k}] \iff \forall i; c_{i} \leq d_{i},$$

$$[c_{1}, c_{2}, \ldots, c_{k}] < [d_{1}, d_{2}, \ldots, d_{k}] \iff \exists i; d_{i} = c_{i} + 1, d_{j} = c_{j} (j \neq i),$$

$[c_{1}, c_{2}, \ldots, c_{k}]$ is a join irreducible element of $D_{1}$ if and only if there is unique $i$ such that

$$c_{i} > a_{i}, \quad c_{i} > c_{i-1} + 1,$$

(3.2)
where we assume that \( c_1 > c_0 + 1 \) is always valid. For a join irreducible element \([c_1, c_2, \ldots, c_k]\), we take \( i \) satisfying (3.2) and set
\[
p := u - c_i - (k - i), \quad q := i - 1.
\]

We denote the map which send \([c_1, c_2, \ldots, c_k]\) to \((p, q)\) by \(\varphi\).

It is easy to construct a join irreducible element \([c_1, c_2, \ldots, c_k]\) such that \(\varphi([c_1, c_2, \ldots, c_k]) = (p, q)\), if \((p, q)\) is in the image of \(\varphi\). And it also easy to verify that if \((p, q)\) is in the image of \(\varphi\) and \(0 \leq p' \leq p, 0 \leq q' \leq q\), then \((p', q')\) is also in the image of \(\varphi\). Moreover, if \([c_1, c_2, \ldots, c_k]\) and \([d_1, d_2, \ldots, d_k]\) are join irreducible elements of \(D_1\), and \(\varphi([c_1, c_2, \ldots, c_k]) = (p, q), \varphi([d_1, d_2, \ldots, d_k]) = (p', q')\), then
\[
[c_1, c_2, \ldots, c_k] \leq [d_1, d_2, \ldots, d_k] \iff p \geq p', q \geq q'.
\]

In particular, the coheight of \([c_1, c_2, \ldots, c_k]\) in \(P_1\) is \(p + q\). Therefore, if we set
\[
\{i \mid a_i + 1 < a_{i+1}, a_i < u, i \leq k\} = \{l_1, \ldots, l_v\}, \quad l_1 < \cdots < l_v,
\]
then the minimal elements of \(P_1\) are \([a_1, a_2, \ldots, a_{l_1-1}, a_l + 1, a_{l+1}, \ldots, a_k]\), \(\ldots, [a_1, a_2, \ldots, a_{l_v-1}, a_{l_v} + 1, a_{l_v+1}, \ldots, a_k]\) and their coheights are \(u - k - a_{l_1} + 2l_1 - 2, \ldots, u - k - a_{l_v} + 2l_v - 2\) respectively.

**Example 3.1** If \(u = 13, k = 8, [a_1, a_2, \ldots, a_k] = [1, 2, 3, 7, 8, 10, 11, 12]\), then \(v = 3, l_1 = 3, l_2 = 5\) and \(l_3 = 8\) and the minimal elements of \(P_1\) are
\[
\gamma_1 = [1, 2, 4, 7, 8, 10, 11, 12]
\gamma_2 = [1, 2, 3, 7, 9, 10, 11, 12]
\gamma_3 = [1, 2, 3, 7, 8, 10, 11, 13].
\]

And the Hasse diagram of \(P_1\) is the following.

![Hasse diagram](attachment:image.png)

Summing up, we obtain the following
Theorem 3.2 If we set

\[
\begin{align*}
\{ i \mid a_i + 1 < a_{i+1}, \ a_i < u, \ i \leq k \} &= \{ l_1, \ldots, l_v \}, \ l_1 < \cdots < l_v \\
\{ i \mid b_i + 1 < b_{i+1}, \ b_i < n, \ i \leq k \} &= \{ l'_1, \ldots, l'_{v'} \}, \ l'_1 < \cdots < l'_{v'},
\end{align*}
\]

then for any minimal reduction J of I, the reduction number \( r_J(I) \) of I with respect to J is equal to

\[
\ell(I) - \max \{ u - k - a_{l_1} + 2l_1, \ldots, u - k - a_{l_v} + 2l_v, \\
n - k - b_{l'_1} + 2l'_1, \ldots, n - k - b_{l'_{v'}} + 2l'_{v'} \}.
\]

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