A GENERALIZED QUOT SCHEME AND MEROMORPHIC VORTICES

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Abstract. Let $X$ be a compact connected Riemann surface. Fix a positive integer $r$
and two nonnegative integers $d_p$ and $d_z$. Consider all pairs of the form $(\mathcal{F}, f)$, where $\mathcal{F}$
is a holomorphic vector bundle on $X$ of rank $r$ and degree $d_z - d_p$, and

$$f : \mathcal{O}_X^{\oplus r} \rightarrow \mathcal{F}$$

is a meromorphic homomorphism which an isomorphism outside a finite subset of $X$ and
has pole (respectively, zero) of total degree $d_p$ (respectively, $d_z$). Two such pairs
$(\mathcal{F}_1, f_1)$ and $(\mathcal{F}_2, f_2)$ are called isomorphic if there is a holomorphic isomorphism of $\mathcal{F}_1$
with $\mathcal{F}_2$ over $X$ that takes $f_1$ to $f_2$. We construct a natural compactification of the
moduli space equivalence classes pairs of the above type. The Poincaré polynomial of
this compactification is computed.

1. Introduction

Take a compact connected Riemann surface $X$. Fix positive integers $r$ and $d$. Consider
pairs of the form $(E, f)$, where $E$ is a holomorphic vector bundle on $X$ of rank $r$ and
degree $d$, and

$$f : \mathcal{O}_X^{\oplus r} \rightarrow E$$

is an $\mathcal{O}_X$–linear homomorphism which is an isomorphism outside a finite subset of $X$. This implies that the total degree of zeros of $f$ is $d$. Two such pairs $(E_1, f_1)$ and $(E_2, f_2)$
are called equivalent if there is a holomorphic isomorphism

$$\phi : E_1 \rightarrow E_2$$

such that $\phi \circ f_1 = f_2$. Such pairs are examples of vortices [BDW], [Br], [BR], [Ba],
[EINOS].

For any pair $(E, f)$ of the above type, consider the dual homomorphism

$$f^* : E^* \rightarrow (\mathcal{O}_X^{\oplus r})^* = \mathcal{O}_X^{\oplus r}.$$ The quotient $\mathcal{O}_X^{\oplus r}/\text{image}(f^*)$ is an element of the Quot scheme $\text{Quot}(r, d)$ that parametrizes
all torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree $d$. Conversely, given any torsion quotient

$$\mathcal{O}_X^{\oplus r} \rightarrow T$$
of degree $d$, consider the homomorphism

$$\mathcal{O}_X^{\oplus r} \rightarrow (\mathcal{O}_X^{\oplus r})^* \rightarrow \text{kernel}(\psi)^*$$

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induced by the inclusion kernel(ψ) ↦ O_X^{pr}. The pair (kernel(ψ)*, ψ') is clearly of the above type. Therefore, the moduli space of equivalence classes of pairs \((E, f)\) is identified with the Quot scheme Quot\((r, d)\).

Here we consider pairs of the form \((E, f)\), where \(E\) is a holomorphic vector bundle on \(X\) of rank \(r\) and degree \(d\), and

\[ f : O_X^{pr} \longrightarrow E \]

is an \(O_X\)-linear meromorphic homomorphism which is an isomorphism outside a finite subset of \(X\). We assume that the total degree of the poles of the meromorphic homomorphism is \(d_p\). This implies that the total degree of the zeros of the meromorphic homomorphism is \(d + d_p\). As before, two such pairs \((E_1, f_1)\) and \((E_2, f_2)\) will be called equivalent if there is a holomorphic isomorphism

\[ \phi : E_1 \longrightarrow E_2 \]

such that \(\phi \circ f_1 = f_2\). The equivalence classes of pairs can be considered as examples of meromorphic vortices.

We construct a natural compactification of the moduli space of these meromorphic vortices. We compute the Poincaré polynomial of this compactification.

2. Preliminaries

Let \(S\) be a scheme and \(Y \longrightarrow S\) a smooth projective morphism. Given a coherent sheaf \(F\) on \(Y\) flat over \(S\) and a numerical polynomial \(r(t)\), we denote by Quot\((F/S, r(t))\) the Grothendieck Quot scheme over \(S\) parametrizing quotients of \(F\) with Hilbert polynomial \(r(t)\) [Gr]. There is a universal exact sequence on Quot\((F/S, r(t)) \times_S Y\)

\[ 0 \longrightarrow K_{\text{Quot}(F/S, r(t))}^{\text{univ}} \longrightarrow \pi_Y^*F \longrightarrow Q_{\text{Quot}(F/S, r(t))}^{\text{univ}} \longrightarrow 0, \]

where \(\pi_Y : \text{Quot}(F/S, r(t)) \times_S Y \longrightarrow Y\) is the natural projection. Often we will just drop the subscripts and write \(K^{\text{univ}}\) or \(Q^{\text{univ}}\) instead. This construction is well behaved with respect to pull-backs, so let us record the following:

**Lemma 2.1.** For any morphism \(g : T \longrightarrow S\), the base change

\[ \text{Quot}(g^*F/T, r(t)) \cong \text{Quot}(F/S, r(t)) \times_S T \]

holds.

**Proof.** This follows by examining the corresponding representable functors. \qed

We will mostly be interested in the case where \(Y \longrightarrow S\) is a smooth, connected and of relative dimension one, that is a relative curve, and \(F\) is locally free of rank \(r\). Further, we will only consider torsion quotients of rank zero and degree \(d\). This Quot scheme will be denoted by Quot\((F/S, d)\). When \(r = 1\) and \(S\) is a point, then

\[ \text{Quot}(O, d) = \text{Sym}^d(Y), \]

the \(d\)-th symmetric power of \(Y\).
Given an positive integer $d$ by a partition of length $k > 0$ of $d$ we mean a sequence $P = (p_1, p_2, \ldots, p_k)$ of non-negative integers with $\sum_{i=1}^{k} p_i = d$. For such a partition define $d(P) := \sum_{i=1}^{k} (i-1)p_i$. We will write

$$\text{Sym}^P(Y) = \text{Sym}^{p_1}(Y) \times \cdots \times \text{Sym}^{p_r}(Y).$$

3. A relative Quot scheme

Let $X$ be a compact connected Riemann surface. Let $\mathcal{E}$ and $\mathcal{F}$ be two holomorphic vector bundles on $X$ of common rank $r$. Take a dense open subset $U \subset X$, such that the complement $S := X \setminus U$ is a finite set, and take an isomorphism of coherent analytic sheaves

$$f : \mathcal{E}|_U \longrightarrow \mathcal{F}|_U$$

over $U$. This homomorphism $f$ will be called meromorphic if there is a positive integer $n$ such that $f$ extends to a homomorphism of coherent analytic sheaves

$$\widehat{f} : \mathcal{E} \longrightarrow \mathcal{F} \otimes \mathcal{O}_X(ns) \supset \mathcal{F}$$

over $X$, where $S$ is the reduced divisor defined by the finite subset $S$. Note that since the divisor $S$ is effective, we have $\mathcal{F} \subset \mathcal{F} \otimes \mathcal{O}_X(ns)$. Therefore, $f$ is meromorphic if and only if the homomorphism $\widehat{f}$ is algebraic with respect to the algebraic structures on $\mathcal{E}|_U$ and $\mathcal{F}|_U$ given by the algebraic structures on $\mathcal{E}$ and $\mathcal{F}$ respectively.

Take a meromorphic homomorphism $f$ as above. We note that the extension $\widehat{f}$ is uniquely determined by $f$ because $f$ and $\widehat{f}$ coincide over $U$. The inverse image

$$\mathcal{E}(f) := \widehat{f}^{-1}(\mathcal{F}) \subset \mathcal{E}$$

(recall that $\mathcal{F} \subset \mathcal{F} \otimes \mathcal{O}_X(ns)$) is clearly independent of the choice of $n$. We note that both $\mathcal{E}(f)$ and $\widehat{f}(\mathcal{E}(f))$ are holomorphic vector bundles on $X$ because they are coherent analytic subsheaves of holomorphic vector bundles. Both of then are of rank $r$, and the restriction

$$\widehat{f}|_{\mathcal{E}(f)} : \mathcal{E}(f) \longrightarrow \widehat{f}(\mathcal{E}(f))$$

is an isomorphism of holomorphic vector bundles. Define

$$Q_p(f) := \mathcal{E}/\mathcal{E}(f) \quad \text{and} \quad Q_z(f) := \mathcal{F}/(\widehat{f}(\mathcal{E}(f)))$$

(the subscripts “$p$” and “$z$” stand for “pole” and “zero” respectively). We note that both $Q_p(f)$ and $Q_z(f)$ are torsion coherent analytic sheaves on $X$. In particular, their supports are finite subsets of $X$. From \((3.2)\) it follows that

$$\text{degree}(Q_p(f)) = \text{degree}(\mathcal{E}) - \text{degree}(\mathcal{E}(f)) \quad \text{and} \quad \text{degree}(Q_z(f)) = \text{degree}(\mathcal{F}) - \text{degree}(\widehat{f}(\mathcal{E}(f))).$$

Fix positive integers $r, d_p$ and $d_z$. Set the domain $\mathcal{E}$ to be the trivial vector bundle $\mathcal{O}_X^{\oplus r}$ of rank $r$. Consider all triples of the form $(\mathcal{F}, U, f)$, where

- $\mathcal{F}$ is a holomorphic vector bundle on $X$ of rank $r$,
- $U$ is the complement of a finite subset of $X$, and
- $f : \mathcal{O}_X^{\oplus r}|_U = \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U$ is a meromorphic homomorphism such that

$$\text{degree}(Q_p(f)) = d_p \quad \text{and} \quad \text{degree}(Q_z(f)) = d_z.$$
Since $\tilde{f}|_{\mathcal{E}(f)}$ in (3.1) is an isomorphism, from (3.3) we conclude that
\begin{equation}
\text{degree}(\mathcal{F}) = d_z - d_p + \text{degree}(\mathcal{O}_X^{\oplus r}) = d_z - d_p.
\end{equation}

Two such triples $(\mathcal{F}_1, U_1, f_1)$ and $(\mathcal{F}_2, U_2, f_2)$ will be called equivalent if there is a holomorphic isomorphism of vector bundles over $X$
\[\beta : \mathcal{F}_1 \longrightarrow \mathcal{F}_2\]
such that\[\beta \circ (f_1|_{U_1 \cap U_2}) = f_2|_{U_1 \cap U_2}.
\]

Therefore, the equivalence class of $(\mathcal{F}, U, f)$ depends only on $(\mathcal{F}, f)$ and it is independent of $U$. More precisely, $(\mathcal{F}, U, f)$ is equivalent to $(\mathcal{F}, W, f|_W)$ for every $W \subset U$ such that the complement $U \setminus W$ is a finite set.

Let
\begin{equation}
Q^0 = Q^0_X(r, d_p, d_z)
\end{equation}
be the space of all equivalence classes of triples of the above form. We will embed $Q^0$ as a Zariski open subset of a smooth complex projective variety.

Take any triple $(\mathcal{F}, U, f)$ as above that is represented by a point of $Q^0$. Consider the short exact sequence
\begin{equation}
0 \longrightarrow \mathcal{E}(f) := \text{kernel}(q_p) \longrightarrow \mathcal{E} = \mathcal{O}_X^{\oplus r} \longrightarrow Q_p(f) \longrightarrow 0,
\end{equation}
where $q_p$ denotes the projection to the quotient in (3.2). We also have
\[\mathcal{E}(f) = \tilde{f}(\mathcal{E}(f)) \hookrightarrow \mathcal{F}\]
(recall that $\tilde{f}|_{\mathcal{E}(f)}$ in (3.1) is an isomorphism). Let
\begin{equation}
0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{E}(f)^*
\end{equation}
be the dual of the above inclusion of $\mathcal{E}(f)$ in $\mathcal{F}$. From (3.6) we have $\text{degree}(\mathcal{E}(f)^*) = \text{degree}(Q_p(f)) = d_p$. Therefore, from (3.4) it follows that
\[\text{degree}(\mathcal{E}(f)^*/\mathcal{F}^*) = \text{degree}(\mathcal{E}(f)^*) - \text{degree}(\mathcal{F}^*) = d_p + d_z - d_p = d_z\]
as $\text{degree}(\mathcal{F}^*) = -\text{degree}(\mathcal{F})$. These imply that we can recover the equivalence class of $(\mathcal{F}, f)$ once we know the following two:
\begin{itemize}
  \item the torsion quotient $Q_p(f)$ of $\mathcal{O}_X^{\oplus r}$ of degree $d_p$, and
  \item the torsion quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ of $\mathcal{E}(f)^*$ of degree $d_z$.
\end{itemize}

(It should be clarified that “knowing the torsion quotient $Q_p(f)$” means knowing the sheaf $Q_p(f)$ along with the surjective homomorphism $\mathcal{O}_X^{\oplus r} \longrightarrow Q_p(f)$; similarly “knowing the torsion quotient $\mathcal{E}(f)^*/\mathcal{F}^*$” means knowing the sheaf $\mathcal{E}(f)^*/\mathcal{F}^*$ along with the surjective homomorphism from $\mathcal{E}(f)^*$ to it.) Indeed, once we know $Q_p(f)$, we know the kernel $\mathcal{E}(f)$ and hence know $\mathcal{E}(f)^*$; if we know the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$, then we know the subsheaf $\mathcal{F}^*$ of $\mathcal{E}(f)^*$. The dual of this inclusion $\mathcal{F}^* \hookrightarrow \mathcal{E}(f)^*$, namely the homomorphism
\[\mathcal{E}(f) \longrightarrow \mathcal{F},\]
gives the meromorphic homomorphism $f$. In other words, we have the diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
0 & \rightarrow & \mathcal{E}(f) & \rightarrow & \mathcal{O}^{\oplus r} & \rightarrow & \mathcal{Q}_p & \rightarrow & 0 \\
\downarrow^f \\
\mathcal{F} \\
\downarrow \\
\mathcal{Q}_z \\
\downarrow \\
0
\end{array}
$$

Let Quot$(r, d_p)$ be the Quot scheme parametrizing the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree $d_p$. We have the tautological short exact sequence of coherent analytic sheaves on $X \times \text{Quot}(r, d_p)$

$$(3.8) \quad 0 \rightarrow \mathcal{K}^{\text{univ}} \rightarrow p_X^* \mathcal{O}_X^{\oplus r} \rightarrow \mathcal{Q}^{\text{univ}} \rightarrow 0,$$

where $p_X$ is the projection of $X \times \text{Quot}(r, d_p)$ to $X$. We write $\mathcal{K} = \mathcal{K}^{\text{univ}}$. Now consider the dual vector bundle

$$
\mathcal{K}^* \rightarrow X \times \text{Quot}(r, d_p) \xrightarrow{p_Q} \text{Quot}(r, d_p),
$$

where $p_Q$ is the natural projection. Using $p_Q$, we will consider $\mathcal{K}^*$ as a family of vector bundles on $X$ parametrized by $\text{Quot}(r, d_p)$. For any point $y \in \text{Quot}(r, d_p)$, the vector bundle $\mathcal{K}^*|_{X \times \{y\}}$ over $X$ will be denoted by $\mathcal{K}^*|_{y}$. Let

$$(3.9) \quad \varphi : \text{Quot}(r, d_p, d_z) := \text{Quot}(\mathcal{K}^*/\text{Quot}(r, d_p), d_z) \rightarrow \text{Quot}(r, d_p)$$

be the relative Quot scheme over $\text{Quot}(r, d_p)$, for the family $\mathcal{K}^*$, parametrizing the torsion quotients of degree $d_z$. Therefore, for any point $y \in \text{Quot}(r, d_p)$, the fiber $\varphi^{-1}(y)$ is the Quot scheme parametrizing the torsion quotients of degree $d_z$ of the vector bundle $\mathcal{K}^*|_{y}$.

Both $\text{Quot}(r, d_p)$ and the fibers of $\varphi$ are irreducible smooth projective varieties. The morphism $\varphi$ is smooth. Therefore, the projective variety $\text{Quot}(r, d_p, d_z)$ is irreducible and smooth.

Consider $Q^0$ defined in (3.5). We have a map

$$
\eta' : Q^0 \rightarrow \text{Quot}(r, d_p)
$$

that sends any triple $(\mathcal{F}, U, f) \in Q_0$ to the point representing the quotient $Q_p(f)$ in (3.6). Let

$$(3.10) \quad \eta : Q^0 \rightarrow \text{Quot}(\mathcal{K}^*, d_z) = \text{Quot}(r, d_p, d_z)$$

be the map that sends any point $\alpha = (\mathcal{F}, U, f) \in Q_0$ to the point of $\varphi^{-1}(\eta'(\alpha))$ that represents the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ in (3.7). This map $\eta$ is injective because, as observed earlier, the equivalence class of the pair $(\mathcal{F}, f)$ can be recovered from the quotient $Q_p(f)$ of $\mathcal{O}_X^{\oplus r}$ and the quotient $\mathcal{E}(f)^*/\mathcal{F}^*$ of $\mathcal{E}(f)^*$. The image of $\eta$ is clearly a Zariski open subset of $\text{Quot}(r, d_p, d_z)$. 
Now define the morphism sheaf $\delta_1$ where

$$\delta_1 : \text{Quot}(r, d_p) \rightarrow \text{Quot}(1, d_p) = \text{Sym}^{d_p}(X).$$

Let $y$ be the morphism that sends any $x \in X$ to the scheme theoretic support of the quotient sheaf

$$\mathcal{S} \hookrightarrow (\text{Id}_X \times \varphi)^* \mathcal{K}^*$$
on $X \times Q$. Let $\Lambda^r \mathcal{S} \hookrightarrow \Lambda^r (\text{Id}_X \times \varphi)^* \mathcal{K}^*$ be the $r$-th exterior power of the above inclusion. Let

$$\delta_2 : Q \rightarrow \text{Sym}^{d_z}(X)$$

be the morphism that sends any $y \in Q$ to the scheme theoretic support of the quotient sheaf

$$((\Lambda^r (\text{Id}_X \times \varphi)^* \mathcal{K}^*)_{|_{\varphi^{-1}(y)}}) / ((\Lambda^r \mathcal{S})_{|_{X \times \{y\}}}) \rightarrow X.$$ 

Now define the morphism

$$\delta := (\delta_1, \delta_2) : \text{Quot}(r, d_p, d_q) = Q \rightarrow \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X),$$

where $\delta_1$ and $\delta_2$ are constructed in (3.11) and (3.12) respectively. It can be shown that $\delta$ is surjective. In fact, in Section 4 we will construct, and use, a section of $\delta$.

**Remark 3.1.** Let $\mathcal{M}_X(r, d_z - d_p)$ denote the moduli stack of vector bundles on $X$ of rank $r$ and degree $d_z - d_p$. Since there is a universal bundle over $X \times \text{Quot}(r, d_p, d_q)$, we get a morphism

$$\text{Quot}(r, d_p, d_q) \rightarrow \mathcal{M}_X(r, d_z - d_p).$$

*4. Fundamental group of Quot$(r, d_p, d_q)$*

**Proposition 4.1.** The homomorphism between fundamental groups induced by the morphism $\delta$ in (3.13) is an isomorphism.

**Proof.** We will first construct a section of $\delta$. Let

$$D(d_p) \subset X \times \text{Sym}^{d_p}(X)$$

be the divisor consisting of all $(x, \{y_1, \cdots, y_{d_p} \})$ such that $x \in \{y_1, \cdots, y_{d_p} \}$. Then the subsheaf

$$\mathcal{O}_{X \times \text{Sym}^{d_p}(X)}(-D(d_p)) \oplus \mathcal{O}_{X \times \text{Sym}^{d_p}(X)}^{\oplus r-1} \subset \mathcal{O}_{X \times \text{Sym}^{d_p}(X)}^{\oplus r}$$

produces a classifying morphism

$$\theta_1 : \text{Sym}^{d_p}(X) \rightarrow \text{Quot}(r, d_p).$$

Let $\xi_1$ (respectively, $\xi_2$) denote the projection of $\text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X)$ to $\text{Sym}^{d_p}(X)$ (respectively, $\text{Sym}^{d_z}(X)$). Like before, $D(d_z) \subset X \times \text{Sym}^{d_z}(X)$ be the divisor consisting of all $(x, \{y_1, \cdots, y_{d_z} \})$ such that $x \in \{y_1, \cdots, y_{d_z} \}$. The subsheaf

$$((\text{Id}_X \times \xi_1)^* (\mathcal{O}_{X \times \text{Sym}^{d_p}(X)}(D(d_p))) \otimes (\text{Id}_X \times \xi_2)^* (\mathcal{O}_{X \times \text{Sym}^{d_z}(X)}(-D(d_z))))$$


The homomorphism isomorphism, this implies that the homomorphism sequence associated to \( \delta \) injective (see (4.3)) we conclude that \( \theta \) induced in \( \iota \) induced by the inclusion \( Z \) Zariski open subset of it. Therefore, the homomorphism induced in \( \theta \) produces a classifying morphism

\[
\theta : \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X) \rightarrow \text{Quot}(r, d_p, d_q).
\]

We note that \( \varphi \circ \theta = \theta_1 \), where \( \varphi \) and \( \theta_1 \) are the morphisms constructed in (3.9) and (4.1) respectively.

It is straightforward to check that

\[
\delta \circ \theta = \text{Id}_{\text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X)},
\]

where \( \delta \) is constructed in (3.13). In view of this section \( \theta \), we conclude that the induced homomorphism between fundamental groups

\[
\delta_* : \pi_1(\text{Quot}(r, d_p, d_q)) \rightarrow \pi_1(\text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X))
\]

is surjective (the base points of fundamental groups are suppressed in the notation).

Let

\[
U \subset \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X)
\]

be the Zariski open subset consisting of all

\[
(x, y) = (\{x_1, \ldots, x_{d_p}\}, \{y_1, \ldots, y_{d_z}\}) \in \text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X)
\]

such that the \( d_p + d_z \) points \( \{x_1, \ldots, x_{d_p}, y_1, \ldots, y_{d_z}\} \) are all distinct, equivalently, the effective divisor \( x + y \) is reduced. Let

\[
\theta_0 := \theta|_U : U \rightarrow \text{Quot}(r, d_p, d_z)
\]

be the restriction of the map \( \theta \) in (4.2). Also, consider the restriction

\[
\delta_0 := \delta|_{\delta^{-1}(U)} : \delta^{-1}(U) \rightarrow U.
\]

Every fiber of \( \delta_0 \) is identified with \((\mathbb{P}_{\mathbb{C}}^{r-1})^{d_p} \times (\mathbb{P}_{\mathbb{C}}^{r-1})^{d_z} \), where \( \mathbb{P}_{\mathbb{C}}^{r-1} \) is the projective space parametrizing the hyperplanes in \( \mathbb{C}^r \) and \( \mathbb{P}_{\mathbb{C}}^{r-1} \) is the projective space parametrizing the lines in \( \mathbb{C}^r \) (so \( \mathbb{P}_{\mathbb{C}}^{r-1} \) parametrizes the hyperplanes in \( (\mathbb{C}^r)^* \)). From the homotopy exact sequence associated to \( \delta_0 \) it follows that the induced homomorphism of fundamental groups

\[
\delta_{0,*} : \pi_1(\delta^{-1}(U)) \rightarrow \pi_1(U)
\]

is an isomorphism. The variety \( \text{Quot}(r, d_p, d_z) \) is smooth, and \( \delta^{-1}(U) \) is a nonempty Zariski open subset of it. Therefore, the homomorphism

\[
\iota_* : \pi_1(\delta^{-1}(U)) \rightarrow \pi_1(\text{Quot}(r, d_p, d_z))
\]

induced by the inclusion \( \iota : \varphi^{-1}(U) \hookrightarrow \text{Quot}(r, d_p, d_z) \) is surjective. Since \( \delta_{0,*} \) is an isomorphism, this implies that the homomorphism

\[
\theta_{0,*} : \pi_1(U) \rightarrow \pi_1(\text{Quot}(r, d_p, d_z))
\]

induced in \( \theta_0 \) in (4.5) is surjective. Since \( \theta_0 \) extends to \( \theta \), this immediately implies that the homomorphism

\[
\theta_* : \pi_1(\text{Sym}^{d_p}(X) \times \text{Sym}^{d_z}(X)) \rightarrow \pi_1(\text{Quot}(r, d_p, d_z))
\]

induced in \( \theta \) in (4.2) is surjective. Since \( \theta_* \) is surjective, and the composition \( \delta_* \circ \theta_* \) is injective (see (4.3)) we conclude that \( \delta_* \) is also injective.
5. Cohomology of Quot($r, d_p, d_z$)

5.1. Generalization of a theorem of Bifet. Let $S_1, S_2, \ldots, S_k$ be a smooth connected projective varieties over $\mathbb{C}$. Fix some line bundles $\mathcal{L}_i$ on $S_i \times X$ of relative degree $d_i$ over $S_i$. In other words
\[ \deg(\mathcal{L}_i|_{s \times X}) = d_i \]
for each point $s \in S_i$. Set $S = S_1 \times \cdots \times S_k$. Let
\[ \pi_{S \times X} : S \times X \rightarrow S_i \times X \]
be the natural projection. Define
\[ \tilde{\mathcal{L}}_i := \pi_{S \times X}^* \mathcal{L}_i. \]

Let
\[ \phi : \text{Quot}(i \tilde{\mathcal{L}}_i/S, d) \rightarrow S \]
be the relative Quot scheme that parametrizes the torsion quotients of degree $d$. So for any $s = (s_1, \cdots, s_k) \in S$, the fiber $\phi^{-1}(s)$ parametrizes the torsion quotients of $i \mathcal{L}_i|_{s \times X}$ of degree $d_i$. By deformation theory, $\phi$ is a smooth morphism of relative dimension $kd$, so Quot($i \tilde{\mathcal{L}}_i/S, d$) is smooth of dimension $\dim(S) + kd$. The torus $\mathbb{G}_m^k$ acts on Quot($i \tilde{\mathcal{L}}_i/S, d$) via its action on $\bigoplus_{i=1}^k \tilde{\mathcal{L}}_i$.

For any positive integer $p$, let Quot($\mathcal{L}_i/S, p$) $\rightarrow S_i$ denote the relative Quot scheme parametrizing the torsion quotients of $\mathcal{L}_i|_{S_i}$ of degree $p$. So the fiber of Quot($\mathcal{L}_i/S, p$) over any $s \in S_i$ parametrizes the torsion quotients of $\mathcal{L}_i|_{s \times X}$ of degree $p$.

Proposition 5.1. There is a bijection between the partitions $P = (p_1, p_2, \cdots, p_k)$ of $d$ of length $k$ and the connected components of the fixed point loci of the $\mathbb{G}_m^k$ action on Quot($i \tilde{\mathcal{L}}_i/S, d$). The component corresponding to the partition $\sum_{i=1}^k p_i = d$ is the product of Quot schemes
\[ \text{Quot}(\mathcal{L}_i/S, P) := \text{Quot}(\mathcal{L}_1/S_1, p_1) \times \text{Quot}(\mathcal{L}_2/S_2, p_2) \times \cdots \times \text{Quot}(\mathcal{L}_k/S_k, p_k) \]
with the obvious structure morphism to $S$.

Proof. On applies the argument used to prove Lemme 1 in [Bif].

As all schemes and morphisms are assumed to be projective it is possible to choose a one parameter subgroup
\[ \mathbb{G}_m \hookrightarrow \mathbb{G}_m^k \]
so that
\[ \text{Quot}(i \tilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m} = \text{Quot}(i \tilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m^k}. \]

Further, the above one-parameter subgroup can be chosen to be given by an increasing sequence of weights $\lambda_1 < \lambda_2 < \cdots < \lambda_k$.

There is an induced action of $\mathbb{G}_m$ on the tangent space at a fixed point $x$. The action preserves the normal space to the fixed point locus and we wish to describe the subspace of positive weights.

Take a partition $P = (p_1, \cdots, p_k)$ of $D$. As before, let
\[ \text{Quot}(\mathcal{L}_i/S, P) \subset \text{Quot}(i \tilde{\mathcal{L}}_i/S, d)^{\mathbb{G}_m^k} \]
be the connected component corresponding to $P$. For a point $x \in \text{Quot}(\mathcal{L}/S, P)$, its image in $S_i$ will be denoted by $x_i$. The line bundle $\mathcal{L}_i|_{x_i \times X}$ on $X$ will be denoted by $\mathcal{L}_i^x$. The point $x_i$ is given by the exact sequence
\begin{equation}
0 \to \mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i) \to \mathcal{L}_i^x \to \mathcal{O}_{D_i} \to 0
\end{equation}
where $D_i$ is an effective divisor on $X$ with $\deg D_i = p_i$. The relative tangent bundle for the projection $\phi$ is
\begin{equation}
T_{x, \text{Quot}(\mathcal{E}/S, d)/S} = \bigoplus_{i=1}^k \text{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i), \mathcal{O}_{D_i}).
\end{equation}
On the other hand, the relative tangent space to the fixed point locus Quot$(\mathcal{L}/S, P)$ is
\begin{equation}
T_{x, \text{Quot}(\mathcal{L}/S, P)/S} = \bigoplus_{i=1}^k \text{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i), \mathcal{O}_{D_i}).
\end{equation}
Consequently, the normal bundle $N$ to Quot$(\mathcal{L}/S, P)$ is
\begin{equation}
N_x = \bigoplus_{i \neq j} \text{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i), \mathcal{O}_{D_j}).
\end{equation}
Also, the subspace of positive weights is
\begin{equation}
N_x^+ = \bigoplus_{i < j} \text{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i), \mathcal{O}_{D_j})
\end{equation}
because the torus acts on $\text{Hom}(\mathcal{L}_i^x \otimes \mathcal{O}_X(-D_i), \mathcal{O}_{D_j})$ with weight $\lambda_j - \lambda_i$. It follows that
\begin{equation}
d(P) := \dim N_x^+ = \sum_{i=1}^k (i-1)p_i.
\end{equation}

**Proposition 5.2.** The Quot schemes for line bundles
\begin{equation}
\text{Quot}(\mathcal{L}_i/S_i, p_i) = \text{Quot}(\mathcal{O}/S_i, p_i) = \text{Sym}^{p_i}(X) \times S_i.
\end{equation}
The Poincaré polynomial of Quot$(\oplus_i \mathcal{E}_i/S, d)$ is given by
\begin{equation}
P(\text{Quot}(\oplus_i \mathcal{E}_i/S, d), t) = \sum_{(P)} t^{2d(P)} \text{P}(\text{Quot}(\mathcal{L}_i/S_i, p_i), t) = \sum_{(P)} t^{2d(P)} \text{P}(S_i, t) \text{P}(\text{Sym}^{p_i}(X), t),
\end{equation}
where the sum is over all partitions of $d$ of length $k$.

**Proof.** The isomorphism $\text{Quot}(\mathcal{O}/S_i, p_i) \xrightarrow{\sim} \text{Quot}(\mathcal{L}_i/S_i, p_i)$ is by tensoring exact sequences with $\mathcal{L}_i$. The second equality is by (2.1).

We need to recall the theorems of [Bia] and [Ki] in our present context. The torus action determines two stratifications of the variety Quot$(\oplus_i \mathcal{E}_i/S, d)$. The strata are in bijection with connected components of the fixed point locus which are in turn in bijection with partitions of $d$ of length $k$. Given such a partition $P$, its corresponding strata are
\begin{equation}
\text{Quot}(\mathcal{L}_x/S, P)^+ := \{ x \mid \lim_{t \to 0} t.x \in \text{Quot}(\mathcal{L}_x/S, P) \}.
\end{equation}
and
\[ \text{Quot}(\mathcal{L}_* / S, P)^- := \{ x \mid \lim_{t \to \infty} t \cdot x \in \text{Quot}(\mathcal{L}_* / S, P) \}. \]

Both of these stratifications are known to be perfect. There are affine fibrations
\[ \text{Quot}(\mathcal{L}_* / S, P)^+ \to \text{Quot}(\mathcal{L}_* / S, P) \quad \text{and} \quad \text{Quot}(\mathcal{L}_* / S, P)^- \to \text{Quot}(\mathcal{L}_* / S, P) \]
of relative dimensions \( \dim N^+_x \) and \( \dim N^-_x \) respectively, where \( x \in \text{Quot}(\mathcal{L}_* / S, P) \) is an arbitrary closed point. It follows that the codimension of \( \text{Quot}(\mathcal{L}_* / S, P)^- \) is \( \dim N^+_x \) which gives the above formula for the Poincaré polynomial.

5.2. The cohomology of \( \text{Quot}(r, d_p, d_z) \). In this subsection we describe the Poincaré polynomial of \( \text{Quot}(r, d_p, d_z) \). Consider the morphism \( \varphi \) in (3.9). There is a natural action of the torus \( \mathbb{G}_m^r \) on the target \( \text{Quot}(O^r, d_p) = O(r, d_p) \). This action clearly lifts to the domain \( \text{Quot}(r, d_p, d_z) \) for \( \varphi \).

The previous subsection provides us with a decomposition and an induced formula for the Poincaré polynomial of \( \text{Quot}(r, d_p, d_z) \). Let us recall it quickly in the present context. There is a bijection between connected components of fixed point loci and partitions of \( d_p \) of length \( r \). Given a partition \( P = (p_1, p_2, \cdots, p_r) \), the corresponding component of \( \text{Quot}(O^r, d_p)_{\mathbb{G}_m^r} \) is
\[ \text{Quot}(O, p_1) \times \cdots \times \text{Quot}(O, p_r) = \text{Sym}^{p_1}(X) \times \cdots \times \text{Sym}^{p_r}(X) = \text{Sym}^P X. \]

There are universal divisors \( D^\text{univ}_{p_i} \) inside \( \text{Sym}^{p_i}(X) \times X \). The component of \( \text{Quot}(r, d_p, d_z)_{\mathbb{G}_m^r} \) corresponding to \( P \), that is
\[ \phi^{-1}(\text{Sym}^{p_1}(X) \times \text{Sym}^{p_2}(X) \times \cdots \times \text{Sym}^{p_r}(X)) \]
is then identified with \( \text{Quot}(\oplus_i O_{\text{Sym}^{p_i}(X) \times X}(D^\text{univ}_{p_i}) / \text{Sym}^{p_i}(X), d_z) \). As the morphism \( \varphi \) in (3.9) is smooth, and smooth morphisms preserve codimension, we obtain the following formula for the Poincaré polynomial:
\[ P(\text{Quot}(r, d_p, d_z), t) = \sum P(\oplus_i O_{\text{Sym}^{p_i}(X) \times X}(D^\text{univ}_{p_i}) / \text{Sym}^{p_i}(X), d_z), t). \]

To complete the calculation we need to compute the Poincaré polynomials of
\[ \text{Quot}(\oplus_i O_{\text{Sym}^{p_i}(X) \times X}(D^\text{univ}_{p_i}) / \text{Sym}^P(X), d_z). \]

Once again Proposition 5.2 applies. The connected components of the fixed point loci are in bijection with partitions of \( d_z \) of length \( r \). Given a partition \( Q = (q_1, \cdots, q_r) \), the corresponding connected component is
\[ \text{Quot}(O_{\text{Sym}^{p_1}(X) \times X}(-D_{p_i}) / \text{Sym}^{p_i}(X), q_1) \times \cdots \times \text{Quot}(O_{\text{Sym}^{p_r}(X) \times X}(-D_{p_r}) / \text{Sym}^{p_r}(X), q_r) \]
which is canonically isomorphic to
\[ \text{Sym}^P Q X := \text{Sym}^{p_1}(X) \times \cdots \times \text{Sym}^{p_r}(X) \times \text{Sym}^{q_1}(X) \times \cdots \times \text{Sym}^{q_r}(X). \]

We obtain the following formula:
\[ P(\oplus_i O_{\text{Sym}^{p_i}(X) \times X}(D^\text{univ}_{p_i}) / \text{Sym}^P(X), d_z)) = \sum P(\text{Sym}^P Q(X), t). \]

Putting this all together we obtain the following:
Theorem 5.3. The Poincaré polynomial for $\text{Quot}(r, d_p, d_z)$ is

$$P(\text{Quot}(r, d_p, d_z), t) = \sum_{P} \sum_{Q} t^{2[d(P)+d(Q)]} P(\text{Sym}^P(X), t)P(\text{Sym}^Q(X), t),$$

where $P$ varies over all partitions of $d_p$ of length $r$ and $Q$ varies over all partitions of $d_z$ of length $r$.

Poincaré polynomial of $\text{Sym}^n(X)$ is the coefficient of $t^n$ in

$$\frac{(1 + tx)^{2g_X}}{(1 - t)(1 - tx^2)},$$

where $g_X$ is the genus of $X$ [Ma, p. 322, (4.3)]. Using this and Theorem 5.3 we get an explicit expression for $P(\text{Quot}(r, d_p, d_z), t)$.

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