Necessary optimality conditions for the calculus of variations on time scales*

Rui A. C. Ferreira  Delfim F. M. Torres
ruiacferreira@yahoo.com  delfim@mat.ua.pt

Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract

We study more general variational problems on time scales. Previous results are generalized by proving necessary optimality conditions for (i) variational problems involving delta derivatives of more than the first order, and (ii) problems of the calculus of variations with delta-differential side conditions (Lagrange problem of the calculus of variations on time scales).

Keywords: time scales, ∆-variational calculus, higher-order ∆-derivatives, higher-order Euler-Lagrange ∆-equations, Lagrange problem on time scales, normal and abnormal ∆-extremals.

2000 Mathematics Subject Classification: 49K05, 39A12.

1 Introduction

The theory of time scales is a relatively new area, that unify and generalize difference and differential equations [5]. It was initiated by Stefan Hilger in the nineties of the XX century [7, 8], and is now subject of strong current research in many different fields in which dynamic processes can be described with discrete or continuous models [1].

The calculus of variations on time scales was introduced by Bohner [4] and by Hilscher and Zeidan [9], and appears to have many opportunities for application in economics [2]. In all those works, necessary optimality conditions are only obtained for the basic (simplest) problem of the calculus of variations on time scales: in [2, 4] for the basic problem with fixed endpoints, in [9] for the basic problem with general (jointly varying) endpoints. Having in mind the classical

*This work is part of the first author’s PhD project.
It turns out that problems (1) and (2) are equivalent: as far as we are assuming
\( C \) belonging to \( f \) boundary conditions of the type

setting (situation when the time scale \( T \) is either \( \mathbb{R} \) or \( \mathbb{Z} \) – see e.g. [6, 14] and

[10, 11], respectively), one suspects that the Euler-Lagrange equations in [2, 4, 9]

are easily generalized for problems with higher-order delta derivatives. This is

not exactly the case, even beginning with the formulation of the problem.

The basic problem of the calculus of variations on time scales is defined (cf.

[4, 9], see [2] below for the meaning of the \( \Delta \)-derivative and \( \Delta \)-integral) as

\[
\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad (y(a) = y_a), (y(b) = y_b),
\]

with \( L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (y, u) \to L(t, y, u) \) a \( C^2 \)-function for each \( t \), and where we are using parentheses around the endpoint conditions as a notation to mean that the conditions may or may not be present: the case with fixed boundary conditions \( y(a) = y_a \) and \( y(b) = y_b \) is studied in [3], for admissible functions \( y(\cdot) \) belonging to \( C^1_{prd} (\mathbb{T}; \mathbb{R}^n) \) (\( rd \)-continuously \( \Delta \)-differentiable functions); general boundary conditions of the type \( f(y(a), y(b)) = 0 \), which include the case \( y(a) \) or \( y(b) \) free, and over admissible functions in the wider class \( C^1_{prd} (\mathbb{T}; \mathbb{R}^n) \) (piecewise \( rd \)-continuously \( \Delta \)-differentiable functions), are considered in [9]. One question

immediately comes to mind. Why is the basic problem on time scales defined as (1) and not as

\[
\mathcal{L}[y(\cdot)] = \int_a^b L(t, y(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad (y(a) = y_a), (y(b) = y_b).
\]

The answer is simple: compared with [2], definition (1) simplifies the Euler-Lagrange equation, in the sense that makes it similar to the classical context.

The reader is invited to compare the Euler-Lagrange condition (6) of problem (1) and the Euler-Lagrange condition (13) of problem (2), with the classical expression (on the time scale \( \mathbb{T} = \mathbb{R} \)):

\[
\frac{d}{dt} L_y'(t, y_*(t), y'_*(t)) = L_y(t, y_*(t), y'_*(t)), \quad t \in [a, b].
\]

It turns out that problems (1) and (2) are equivalent: as far as we are assuming \( y(\cdot) \) to be \( \Delta \)-differentiable, then \( y(t) = y^\sigma(t) - \mu(t)y^\Delta(t) \) and (i) any problem (1) can be written in the form (2), (ii) any problem (2) can be written in the form (1). We claim, however, that the formulation (2) we are promoting here is more natural and convenient. An advantage of our formulation (2) with respect to (1) is that it makes clear how to generalize the basic problem on time scales to the case of a Lagrangian \( L \) containing delta derivatives of \( y(\cdot) \) up to an order \( r, r \geq 1 \). The higher-order problem will be naturally defined as

\[
\mathcal{L}[y(\cdot)] = \int_a^{\rho^{r-1}(b)} L(t, y(t), y^\Delta(t), \ldots, y^{\Delta^r}(t)) \Delta t \longrightarrow \min, \quad (y(a) = y^0_a), (y(\rho^{r-1}(b)) = y^0_b),
\]

\[
\vdots
\]

\[
(y^{\Delta^{r-1}}(a) = y^{r-1}_a), (y^{\Delta^{r-1}}(\rho^{r-1}(b)) = y^{r-1}_b),
\]

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where \( y^{\Delta^i}(t) \in \mathbb{R}^n \), \( i \in \{0, \ldots, r\} \), \( y^{\Delta^0} = y \), and \( n, r \in \mathbb{N} \) (assumptions on the data of the problem will be specified later, in Section 3). One of the new results in this paper is a necessary optimality condition in delta integral form for problem (3) (Theorem 4). It is obtained using the interplay of problems (1) and (2) in order to deal with more general optimal control problems (16).

The paper is organized as follows. In Section 2 we give a brief introduction to time scales and recall the main results of the calculus of variations on this general setting. Our contributions are found in Section 3. We start in §3.1 by proving the Euler-Lagrange equation and transversality conditions (natural boundary conditions – \( y(a) \) or/and \( y(b) \) free) for the basic problem (2) (Theorem 2). As a corollary, the Euler-Lagrange equation in [1] and [9] for (1) is obtained. Regarding the natural boundary conditions, the one which appears when \( y(a) \) is free turns out to be simpler and more close in aspect to the classical condition \( L y'(a,y_*(a),y'_*(a)) = 0 \) for problem (1) than to (2)—compare condition (4) for problem (2) with the correspondent condition (14) for problem (1); but the inverse situation happens when \( y(b) \) is free—compare condition (15) for problem (1) with the correspondent condition (10) for (2), this last being simpler and more close in aspect to the classical expression \( L y'(b,y_*(b),y'_*(b)) = 0 \) valid on the time scale \( \mathbb{T} = \mathbb{R} \). In §3.2 we formulate a more general optimal control problem (16) on time scales, proving respective necessary optimality conditions in Hamiltonian form (Theorem 3). As corollaries, we obtain a Lagrange multiplier rule on time-scales (Corollary 2), and in §3.3 the Euler-Lagrange equation for the problem of the calculus of variations with higher order delta derivatives (Theorem 4). Finally, as an illustrative example, we consider in §4 a discrete time scale and obtain the well-known Euler-Lagrange equation in delta differentiated form.

All the results obtained in this paper can be extended: (i) to nabla derivatives (see [5] §8.4] with the appropriate modifications and as done in [2] for the simplest functional; (ii) to more general classes of admissible functions and to problems with more general boundary conditions, as done in [9] for the simplest functional of the calculus of variations on time scales.

2 Time scales and previous results

We begin by recalling the main definitions and properties of time scales (cf. [1] [5] [7] [8] and references therein).

A nonempty closed subset of \( \mathbb{R} \) is called a Time Scale and is denoted by \( \mathbb{T} \).

The forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) is defined by

\[
\sigma(t) = \inf \{ s \in \mathbb{T} : s > t \}, \text{ for all } t \in \mathbb{T},
\]

while the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) is defined by

\[
\rho(t) = \sup \{ s \in \mathbb{T} : s < t \}, \text{ for all } t \in \mathbb{T},
\]

with \( \inf \emptyset = \sup \mathbb{T} \) (i.e., \( \sigma(M) = M \) if \( \mathbb{T} \) has a maximum \( M \)) and \( \sup \emptyset = \inf \mathbb{T} \) (i.e., \( \rho(m) = m \) if \( \mathbb{T} \) has a minimum \( m \)).
Lemma 1. If \( \sigma(t) = t, \sigma(t) > t, \rho(t) = t \) and \( \rho(t) < t \), respectively.

Throughout the text we let \( \mathbb{T} = [a, b] \cap \mathbb{T}_0 \) with \( a < b \) and \( \mathbb{T}_0 \) a time scale. We define \( \mathbb{T}^k = \mathbb{T} \setminus (\rho(b), b], \mathbb{T}^{k^2} = (\mathbb{T}^k)^k \) and more generally \( \mathbb{T}^{k^n} = (\mathbb{T}^{k^n-1})^k, \)
for \( n \in \mathbb{N} \). The following standard notation is used for \( \sigma \) (and \( \rho \)): \( \sigma^0(t) = t, \)
\( \sigma^n(t) = (\sigma \circ \sigma^{n-1})(t), n \in \mathbb{N} \).

The graininess function \( \mu : \mathbb{T} \to [0, \infty) \) is defined by
\[
\mu(t) = \sigma(t) - t, \quad \text{for all } t \in \mathbb{T}.
\]

We say that a function \( f : \mathbb{T} \to \mathbb{R} \) is delta differentiable at \( t \in \mathbb{T}^k \) if there is a number \( f^\Delta(t) \) such that for all \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) (i.e.,
\( U = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \)) such that
\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.
\]

We call \( f^\Delta(t) \) the delta derivative of \( f \) at \( t \).

Now, we define the \( r \)th-delta derivative \( (r \in \mathbb{N}) \) of \( f \) to be the function
\( f^{\Delta^r} : \mathbb{T}^{k^r} \to \mathbb{R} \), provided \( f^{\Delta^{r-1}} \) is delta differentiable on \( \mathbb{T}^{k^r} \).

For delta differentiable \( f \) and \( g \), the next formulas hold:
\[
f^\sigma(t) = f(t) + (t)\Delta(t)
\]
\[
(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t)
\]
\[
= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t),
\]

where we abbreviate \( f \circ \sigma \) by \( f^\sigma \).

Next, a function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by \( C_{\text{rd}} \) or \( C_{\text{rd}}[\mathbb{T}] \) and the set of all delta differentiable functions with rd-continuous derivative by \( C_{\text{rd}}^1 \) or \( C_{\text{rd}}^1[\mathbb{T}] \).

It is known that rd-continuous functions possess an antiderivative, i.e., there exists a function \( F \) with \( F^\Delta = f \), and in this case an integral is defined by
\[
\int_a^b f(t)\Delta t = F(b) - F(a). \quad \text{It satisfies}
\]
\[
\int_a^b f(\tau)\Delta \tau = \mu(t)f(t).
\]

We now present some useful properties of the delta integral:

**Lemma 1.** If \( a, b \in \mathbb{T} \) and \( f, g \in C_{\text{rd}}, \) then

1. \( \int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t. \)
2. \( \int_a^b f(t)g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t. \)
The main result of the calculus of variations on time scales is given by the following necessary optimality condition for problem (1).

**Theorem 1** ([4]). If \( y_* \) is a weak local minimizer (cf. [3]) of the problem

\[
\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \rightarrow \min
\]

\[
y(\cdot) \in C^1_{rd}[T]
\]

\[
y(a) = y_a, \quad y(b) = y_b,
\]

then the Euler-Lagrange equation

\[
L^\Delta_{y_*}(t, y_*^\sigma(t), y_*^\Delta(t)) = L_y(t, y_*^\sigma(t), y_*^\Delta(t)), \quad t \in T^k
\]  

holds.

Main ingredients to prove Theorem 1 are item 1 of Lemma 1 and the Dubois-Reymond lemma:

**Lemma 2** ([4]). Let \( g \in C^r_{rd} \), \( g : [a, b]^k \to \mathbb{R}^n \). Then,

\[
\int_a^b g(t) \cdot \eta^\Delta(t) \Delta t = 0 \quad \text{for all} \quad \eta \in C^1_{rd} \quad \text{with} \quad \eta(a) = \eta(b) = 0
\]

if and only if

\[
g(t) = c \text{ on } [a, b]^k \text{ for some } c \in \mathbb{R}^n.
\]

3 Main results

Assume that the Lagrangian \( L(t, u_0(t), u_1(t), \ldots, u_r(t)) \) \( (r \geq 1) \) is a \( C^{r+1} \) function of \( (u_0(t), u_1(t), \ldots, u_r(t)) \) for each \( t \in T \). Let \( y \in C^r_{rd}[T] \), where

\[
C^r_{rd}[T] = \left\{ y : T^k \to \mathbb{R}^n : y^\Delta^r \text{ is rd-continuous on } T^k \right\}.
\]

We want to minimize the functional \( \mathcal{L} \) of problem \( [3] \). For this, we say that \( y_* \in C^r_{rd}[T] \) is a weak local minimizer for the variational problem \( [3] \) provided there exists \( \delta > 0 \) such that \( \mathcal{L}[y_*] \leq \mathcal{L}[y] \) for all \( y \in C^r_{rd}[T] \) satisfying the constraints in \( [3] \) and \( \|y - y_*\|_{r, \infty} < \delta \), where

\[
\|y\|_{r, \infty} := \sum_{i=0}^{r} \|y^\Delta^i\|_{\infty},
\]

with \( y^\Delta^0 = y \) and \( \|y\|_{\infty} := \sup_{t \in T^k} |y(t)| \).
3.1 The basic problem on time scales

We start by proving the necessary optimality condition for the simplest variational problem \((r = 1)\):

\[
\mathcal{L}[y(\cdot)] = \int_a^b L(t, y(t), y^\Delta(t)) \Delta t \longrightarrow \min
\]

\[y(\cdot) \in C^1_{rd}[T]\]

\[(y(a) = y_a), \quad (y(b) = y_b) .\]

Remark 1. We are assuming in problem \((7)\) that the time scale \(T\) has at least 3 points. Indeed, for the delta-integral to be defined we need at least 2 points. Assume that the time scale has only two points: \(T = \{a, b\}\), with \(b = \sigma(a)\). Then,

\[
\int_a^{\sigma(a)} L(t, y(t), y^\Delta(t)) \Delta t = \mu(a)L(a, y(a), y^\Delta(a)).
\]

In the case both \(y(a)\) and \(y(\sigma(a))\) are fixed, since \(y^\Delta(a) = \frac{y(\sigma(a))-y(a)}{\mu(a)}\), then \(\mathcal{L}[y(\cdot)]\) would be a constant for every admissible function \(y(\cdot)\) (there would be nothing to minimize and problem \((7)\) would be trivial). Similarly, for \((3)\) we assume the time scale to have at least \(2r + 1\) points (see Remark 1).

Theorem 2. If \(y_\ast\) is a weak local minimizer of \((7)\) (problem \((3)\) with \(r = 1\)), then the Euler-Lagrange equation in \(\Delta\)-integral form

\[
L_y^\Delta(t, y_\ast(t), y_\ast^\Delta(t)) = \int_a^{\tau(t)} L_y(\xi, y_\ast(\xi), y_\ast^\Delta(\xi)) \Delta \xi + c
\]

holds \(\forall t \in T^k\) and some \(c \in \mathbb{R}^n\). Moreover, if the initial condition \(y(a) = y_a\) is not present \((y(a)\) is free\), then the supplementary condition

\[
L_y^\Delta(a, y_\ast(a), y_\ast^\Delta(a)) - \mu(a)L_y(a, y_\ast(a), y_\ast^\Delta(a)) = 0
\]

holds; if \(y(b) = y_b\) is not present \((y(b)\) is free\), then

\[
L_y^\Delta(\rho(b), y_\ast(\rho(b)), y_\ast^\Delta(\rho(b))) = 0 .
\]

Remark 2. For the time scale \(T = \mathbb{R}\) equalities \((9)\) and \((10)\) give, respectively, the well-known natural boundary conditions \(L_y^\Delta(a, y_\ast(a), y_\ast^\Delta(a)) = 0\) and \(L_y^\Delta(b, y_\ast(b), y_\ast^\Delta(b)) = 0\).

Proof. Suppose that \(y_\ast(\cdot)\) is a weak local minimizer of \(\mathcal{L}[\cdot]\). Let \(\eta(\cdot) \in C^1_{rd}\) and define \(\Phi : \mathbb{R} \to \mathbb{R}\) by

\[
\Phi(\varepsilon) = \mathcal{L}[y_\ast(\cdot) + \varepsilon \eta(\cdot)].
\]

This function has a minimum at \(\varepsilon = 0\), so we must have \(\Phi'(0) = 0\). Applying the delta-integral properties and the integration by parts formula 2 (second item
in Lemma \(\text{1} \), we have

\[
0 = \Phi'(0)
\]

\[
= \int_a^b [L_y(t, y_*(t), y^\Delta_*(t)) \cdot \eta(t) + L_y(\cdot, y_*(t), y^\Delta_*(t)) \cdot \eta^\Delta_*(t)] \Delta t
\]

\[
= \int_a^t L_y(t, y_*(t), y^\Delta_*(t)) \Delta t \cdot \eta(t)\bigg|_{t=a}^{t=b}
\]

\[
- \int_a^b \left[ \int_a^{\sigma(t)} L_y(\xi, y_*(\xi), y^\Delta_*(\xi)) \Delta \xi \cdot \eta^\Delta_*(t) - L_y(\cdot, y_*(t), y^\Delta_*(t)) \cdot \eta^\Delta_*(t) \right] \Delta t.
\]

(11)

Let us limit the set of all delta-differentiable functions \(\eta(\cdot)\) with rd-continuous derivatives to those which satisfy the condition \(\eta(a) = \eta(b) = 0\) (this condition is satisfied by all the admissible variations \(\eta(\cdot)\) in the case both \(y(a) = y_a\) and \(y(b) = y_b\) are fixed). For these functions we have

\[
\int_a^b \left[ L_y(\cdot, y_*(t), y^\Delta_*(t)) - \int_a^{\sigma(t)} L_y(\xi, y_*(\xi), y^\Delta_*(\xi)) \Delta \xi \right] \cdot \eta^\Delta_*(t) \Delta t = 0.
\]

Therefore, by the lemma of Dubois-Reymond (Lemma \(\text{2} \)), there exists a constant \(c \in \mathbb{R}^n\) such that \(\Phi\) holds:

\[
L_y(\cdot, y_*(t), y^\Delta_*(t)) - \int_a^{\sigma(t)} L_y(\xi, y_*(\xi), y^\Delta_*(\xi)) \Delta \xi = c,
\]

(12)

for all \(t \in T^k\). Because of (12), condition \(\text{11}\) simplifies to

\[
\int_a^t L_y(t, y_*(t), y^\Delta_*(t)) \Delta t \cdot \eta(t)\bigg|_{t=a}^{t=b} + c \cdot \eta(t)\bigg|_{t=a}^{t=b} = 0,
\]

for any admissible \(\eta(\cdot)\). If \(y(a) = y_a\) is not present in problem \(\text{7}\) (so that \(\eta(a)\) need not to be zero), taking \(\eta(t) = t - b\) we find that \(c = 0\); if \(y(b) = y_b\) is not present, taking \(\eta(t) = t - a\) we find that \(\int_a^b L_y(t, y_*(t), y^\Delta_*(t)) = 0\). Applying these two conditions to (12) and having in mind formula (5), we may state that

\[
L_y(\cdot, y_*(a), y^\Delta_*(a)) - \int_a^{\sigma(a)} L_y(\xi, y_*(\xi), y^\Delta_*(\xi)) \Delta \xi = 0
\]

\[
\Leftrightarrow L_y(\cdot, y_*(a), y^\Delta_*(a)) - \mu(a)L_y(a, y_*(a), y^\Delta_*(a)) = 0,
\]

and (note that \(\sigma(\rho(b)) = b\))

\[
L_y(\cdot, y_*(b), y^\Delta_*(b)) - \int_a^{b} L_y(\xi, y_*(\xi), y^\Delta_*(\xi)) \Delta \xi = 0
\]

\[
\Leftrightarrow L_y(\cdot, y_*(b), y^\Delta_*(b)) - \mu(b)L_y(b, y_*(b), y^\Delta_*(b)) = 0.
\]
Remark 3. Since $\sigma(t) \geq t, \forall t \in \mathbb{T}$, we must have

\[ L_{y^\Delta}(t, y_*(t), y_*^\Delta(t)) - \int_a^t L_y(\xi, y_*(\xi), y_*^\Delta(\xi)) \Delta\xi = c \]

\[ \iff L_{y^\Delta}(t, y_*(t), y_*^\Delta(t)) - \mu(t)L_y(t, y_*(t), y_*^\Delta(t)) \]

\[ = \int_a^t L_y(\xi, y_*(\xi), y_*^\Delta(\xi)) \Delta\xi + c, \]

by formula (5). Delta differentiating both sides, we obtain

\[ \left( L_{y^\Delta}(t, y_*(t), y_*^\Delta(t)) - \mu(t)L_y(t, y_*(t), y_*^\Delta(t)) \right)^\Delta \]

\[ = L_y(t, y_*(t), y_*^\Delta(t)), \quad t \in \mathbb{T}^k. \quad (13) \]

Note that we can’t expand the left hand side of this last equation, because we are not assuming that $\mu(t)$ is delta differentiable. In fact, generally $\mu(t)$ is not delta differentiable (see example 1.55, page 21 of [3]). We say that (13) is the Euler-Lagrange equation for problem (7) in the delta differentiated form.

As mentioned in the introduction, the formulations of the problems of the calculus of variations on time scales with “$(t, y^\sigma(t), y^\Delta(t))$” and with “$(t, y(t), y^\Delta(t))$” are equivalent. It is trivial to derive previous Euler-Lagrange equation (6) from our equation (13) and the other way around (one can derive (13) directly from (6)).

Corollary 1. If $y_* \in C^1_{rd}[\mathbb{T}]$ is a weak local minimizer of

\[ L[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t, \quad (y(a) = y_a), \quad (y(b) = y_b), \]

then the Euler-Lagrange equation (13) holds. If $y(a)$ is free, then the extra transversality condition (natural boundary condition)

\[ L_{y^\Delta}(a, y_*^\sigma(a), y_*^\Delta(a)) = 0 \quad (14) \]

holds; if $y(b)$ is free, then

\[ L_{y^\sigma}(\rho(b), y_*^\sigma(\rho(b)), y_*^\Delta(\rho(b)))\mu(\rho(b)) + L_{y^\Delta}(\rho(b), y_*^\sigma(\rho(b)), y_*^\Delta(\rho(b))) = 0. \quad (15) \]

Proof. Since $y(t)$ is delta differentiable, then (14) holds. This permits us to write

\[ L(t, y^\sigma(t), y^\Delta(t)) = L(t, y(t) + \mu(t)y^\Delta(t), y^\Delta(t)) = F(t, y(t), y^\Delta(t)). \]

Applying equation (13) to the functional $F$ we obtain

\[ (F_{y^\Delta}(t, y(t), y^\Delta(t)) - \mu(t)F_y(t, y(t), y^\Delta(t)))^\Delta = F_y(t, y(t), y^\Delta(t)). \]

But

\[ F_y(t, y(t), y^\Delta(t)) = L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t)), \]

\[ F_{y^\Delta}(t, y(t), y^\Delta(t)) = L_{y^\sigma}(t, y^\sigma(t), y^\Delta(t))\mu(t) + L_{y^\Delta}(t, y^\sigma(t), y^\Delta(t)), \]

and the result follows. \qed
3.2 The Lagrange problem on time scales

Now we consider a more general variational problem with delta-differential side conditions:

\[
J[y(\cdot),u(\cdot)] = \int_a^b L(t,y(t),u(t)) \Delta t \to \min,
\]
\[
y^\Delta(t) = \varphi(t,y(t),u(t)), \quad (y(a) = y_a), \quad (y(b) = y_b),
\]

where \(y(\cdot) \in C^1(T), u(\cdot) \in C(T), y(t) \in \mathbb{R}^n\) and \(u(t) \in \mathbb{R}^m\) for all \(t \in T\), and \(m \leq n\). We assume \(L : T \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) and \(\varphi : T \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n\) to be \(C^1\)-functions of \(y\) and \(u\) for each \(t\); and that for each control function \(u(\cdot) \in C(T;\mathbb{R}^m)\) there exists a correspondent \(y(\cdot) \in C^1(T;\mathbb{R}^n)\) solution of the \(\Delta\)-differential equation \(y^\Delta(t) = \varphi(t,y(t),u(t))\). We remark that conditions for existence or uniqueness are available for ODE’s from the very beginning of the theory of time scales (see [8, Theorem 8]). Roughly speaking, forward solutions exist, while existence of backward solutions needs extra assumptions (e.g., regressivity). In control theory, however, one usually needs only forward solutions, so we do not need to impose such extra assumptions [3].

We are interested to find necessary conditions for a pair \((y_*,u_*)\) to be a weak local minimizer of \(J\).

**Definition 1.** Take an admissible pair \((y_*,u_*)\). We say that \((y_*,u_*)\) is a weak local minimizer for \((16)\) if there exist \(\delta > 0\) such that \(J[y_*,u_*] \leq J[y,u]\) for all admissible pairs \((y,u)\) satisfying \(\|y - y_*\|_{1,\infty} + \|u - u_*\|_{\infty} < \delta\).

**Remark 4.** Problem \((16)\) is very general and includes: (i) problem \((7)\) (this is the particular case where \(m = n\) and \(\varphi(t,y,u) = u\)), (ii) the problem of the calculus of variations with higher-order delta derivatives \((3)\) (such problem receive special attention in Section 3.3 below), (iii) isoperimetric problems on time scales. Suppose that the isoperimetric condition

\[
I[y(\cdot),u(\cdot)] = \int_a^b g(t,y(t),u(t)) \Delta t = \beta,
\]

\(\beta\) a given constant, is prescribed. We can introduce a new state variable \(y_{n+1}\) defined by

\[
y_{n+1}(t) = \int_a^t g(\xi,y(\xi),u(\xi)) \Delta \xi, \quad t \in T,
\]

with boundary conditions \(y_{n+1}(a) = 0\) and \(y_{n+1}(b) = \beta\). Then

\[
y_{n+1}^\Delta(t) = g(t,y(t),u(t)), \quad t \in T^k,
\]

and we can always recast an isoperimetric problem as a Lagrange problem \((16)\).

To establish necessary optimality conditions for \((16)\) is more complicated than for the basic problem of the calculus of variations on time scales \((1)\) or \((2)\), owing to the possibility of existence of abnormal extremals (Definition 2). The abnormal case never occurs for the basic problem (Proposition 2).
Theorem 3 (The weak maximum principle on time scales). If \((y_\ast(\cdot), u_\ast(\cdot))\) is a weak local minimizer of problem \((10)\), then there exists a set of multipliers \((\psi_0, \psi_\ast(\cdot)) \neq 0\), where \(\psi_0\) is a nonnegative constant and \(\psi_\ast(\cdot) : \mathbb{T} \to \mathbb{R}^n\) is a delta differentiable function on \(\mathbb{T}^k\), such that \((y_\ast(\cdot), u_\ast(\cdot), \psi_0, \psi_\ast(\cdot))\) satisfy

\[
y^\Delta(t) = H\psi(t, y_\ast(t), u_\ast(t), \psi_0, \psi_\ast(t)), \quad (\Delta-\text{dynamic equation for } y) (17)
\]

\[
\psi^\Delta(t) = -H_y(t, y_\ast(t), u_\ast(t), \psi_0, \psi_\ast(t)), \quad (\Delta-\text{dynamic equation for } \psi) (18)
\]

\[H_u(t, y_\ast(t), u_\ast(t), \psi_0, \psi_\ast(t)) = 0, \quad (\Delta-\text{stationary condition}) (19)
\]

for all \(t \in \mathbb{T}^k\), where the Hamiltonian function \(H\) is defined by

\[H(t, y, u, \psi^\sigma) = \psi_0 L(t, y, u) + \psi^\sigma \cdot \varphi(t, y, u). (20)\]

If \(y(a)\) is free in \((16)\), then

\[\psi_\ast(a) = 0; (21)\]

if \(y(b)\) is free in \((16)\), then

\[\psi_\ast(b) = 0. (22)\]

Remark 5. From the definition \((20)\) of \(H\), it follows immediately that \((17)\) holds true for any admissible pair \((y(\cdot), u(\cdot))\) of problem \((10)\). Indeed, condition \((17)\) is nothing more than the control system \(y^\Delta(t) = \varphi(t, y_\ast(t), u_\ast(t))\).

Remark 6. For the time scale \(\mathbb{T} = \mathbb{Z}\), \((17)-(19)\) reduce to well-known conditions in discrete time (see e.g. \([13\text{ Ch. }8]\)): the \(\Delta\)-dynamic equation for \(y\) takes the form \(y(k + 1) - y(k) = H_\psi(k, y(k), u(k), \psi_0, \psi(k + 1))\); the \(\Delta\)-dynamic equation for \(\psi\) gives \(\psi(k + 1) - \psi(k) = -H_y(k, y(k), u(k), \psi_0, \psi(k + 1))\); and the \(\Delta\)-stationary condition reads as \(H_u(k, y(k), u(k), \psi_0, \psi(k + 1)) = 0\); with the Hamiltonian \(H = \psi_0 L(k, y(k), u(k)) + \psi(k + 1) \cdot \varphi(k, y(k), u(k))\). For \(\mathbb{T} = \mathbb{R}\), Theorem 3 is known in the literature as Hestenes necessary condition, which is a particular case of the Pontryagin Maximum Principle \([12]\).

Corollary 2 (Lagrange multiplier rule on time scales). If \((y_\ast(\cdot), u_\ast(\cdot))\) is a weak local minimizer of problem \((10)\), then there exists a collection of multipliers \((\psi_0, \psi_\ast(\cdot))\), \(\psi_0\) a nonnegative constant and \(\psi_\ast(\cdot) : \mathbb{T} \to \mathbb{R}^n\) a delta differentiable function on \(\mathbb{T}^k\), not all vanishing, such that \((y_\ast(\cdot), u_\ast(\cdot), \psi_0, \psi_\ast(\cdot))\) satisfy the Euler-Lagrange equation of the augmented functional \(J^*\):

\[
J^*[y(\cdot), u(\cdot), \psi(\cdot)] = \int_a^b L^*(t, y(t), u(t), \psi^\sigma(t), y^\Delta(t)) \Delta t
\]

\[= \int_a^b [\psi_0 L(t, y(t), u(t)) + \psi^\sigma(t) \cdot (\varphi(t, y(t), u(t)) - y^\Delta(t))] \Delta t (23)
\]

\[= \int_a^b [H(t, y(t), u(t), \psi_0, \psi^\sigma(t)) - \psi^\sigma(t) \cdot y^\Delta(t)] \Delta t.
\]

Proof. The Euler-Lagrange equations \((13)\) and \((6)\) applied to \((23)\) give

\[
\left(\frac{L_y^* \cdot \mu(t)L_y^*}{L_y^*} \right) = L_y^*, \quad (-\mu(t)L_u^*) = L_u^*, \quad L_{\psi^\sigma} = 0,
\]
that is,

\[ (\psi^\sigma(t) + \mu(t) \cdot H_y)^\Delta = -H_y, \]  
\[ (-\mu(t)H_u)^\Delta = H_u, \]  
\[ y^{\Delta}(t) = H_{\psi^\sigma}, \]

where the partial derivatives of \( H \) are evaluated at \((t, y(t), u(t), \psi_0, \psi^\sigma(t))\). Obviously, from (19) we obtain (25). It remains to prove that (18) implies (24), which is equivalent to \( \psi(t) = \psi^\sigma(t) + \mu(t)H_y \).

Remark 7. Condition (18) or (24) imply that along the minimizer

\[ \psi^\sigma(t) = -\int_a^{\sigma(t)} H_y(\xi, y(\xi), u(\xi), \psi_0, \psi^\sigma(\xi))\Delta\xi - c \]  

for some \( c \in \mathbb{R}^n \).

Remark 8. The assertion in Theorem 3 that the multipliers cannot be all zero is crucial. Indeed, without this requirement, for any admissible pair \((y(\cdot), u(\cdot))\) of (10) there would always exist a set of multipliers satisfying (18)-(19) (namely, \( \psi_0 = 0 \) and \( \psi(t) \equiv 0 \)).

Remark 9. Along all the work we consider \( \psi \) as a row-vector.

Remark 10. If the multipliers \((\psi_0, \psi(\cdot))\) satisfy the conditions of Theorem 3 then \((\gamma\psi_0, \gamma\psi(\cdot))\) also do, for any \( \gamma > 0 \). This simple observation allow us to conclude that it is enough to consider two cases: \( \psi_0 = 0 \) or \( \psi_0 = 1 \).

Definition 2. An admissible quadruple \((y(\cdot), u(\cdot), \psi_0, \psi(\cdot))\) satisfying conditions (17)-(19) (also (21) or (22) if \( y(a) \) or \( y(b) \) are, respectively, free) is called an extremal for problem (16). An extremal is said to be normal if \( \psi_0 = 1 \) and abnormal if \( \psi_0 = 0 \).

So, Theorem 3 asserts that every minimizer is an extremal.

Proposition 1. The Lagrange problem on time scales (10) has no abnormal extremals (in particular, all the minimizers are normal) when at least one of the boundary conditions \( y(a) \) or \( y(b) \) is absent (when \( y(a) \) or \( y(b) \) is free).

Proof. Without loss of generality, let us consider \( y(b) \) free. We want to prove that the nonnegative constant \( \psi_0 \) is nonzero. The fact that \( \psi_0 \neq 0 \) follows from Theorem 3. Indeed, the multipliers \( \psi_0 \) and \( \psi(t) \) cannot vanish simultaneously at any point of \( t \in T \). As far as \( y(b) \) is free, the solution to the problem must satisfy the condition \( \psi(b) = 0 \). The condition \( \psi(b) = 0 \) requires a nonzero value for \( \psi_0 \) at \( t = b \). But since \( \psi_0 \) is a nonnegative constant, we conclude that \( \psi_0 \) is positive, and we can normalize it (Remark 10) to unity.

Remark 11. In the general situation abnormal extremals may occur. More precisely (see proof of Theorem 3), abnormality is characterized by the existence of a nontrivial solution \( \psi(t) \) for the system \( \psi^\Delta(t) + \psi^\sigma(t) \cdot \varphi_y = 0 \).
Proposition 2. There are no abnormal extremals for problem (7), even in the case \( y(a) \) and \( y(b) \) are both fixed (\( y(a) = y_a, y(b) = y_b \)).

Proof. Problem (7) is the particular case of (16) with \( y^\Delta(t) = u(t) \). If \( \psi_0 = 0 \), then the Hamiltonian (20) takes the form \( H = \psi^\sigma \cdot u \). From Theorem 3, \( \psi^\Delta = 0 \) and \( \psi^\sigma = 0 \), for all \( t \in T_k \). Since \( \psi^\sigma = \psi + \mu(t)\psi^\Delta \), this means that \( \psi_0 \) and \( \psi \) would be both zero, which is not a possibility.

Corollary 3. For problem (7), Theorem 3 gives Theorem 2.

Proof. For problem (7) we have \( \varphi(t, y, u) = u \). From Proposition 2, the Hamiltonian becomes \( H(t, y, u, \psi_0, \psi_\sigma) = L(t, y, u) + \psi^\sigma \cdot u \). By the \( \Delta \)-stationary condition (19) we may write \( L_u(t, y, u) + \psi_\sigma = 0 \). Now apply (26) and the result follows.

To prove Theorem 3 we need the following result:

Lemma 3 (Fundamental lemma of the calculus of variations on time scales).

Let \( g \in C_{rd}, g : T_k \rightarrow \mathbb{R}^n \). Then,

\[
\int_a^b g(t) \cdot \eta(t) \Delta t = 0 \quad \text{for all} \ \eta \in C_{rd}
\]

if and only if

\[
g(t) = 0 \quad \text{on} \ \ T_k.
\]

Proof. If \( g(t) = 0 \) on \( T_k \), then obviously \( \int_a^b g(t) \cdot \eta(t) \Delta t = 0 \), for all \( \eta \in C_{rd} \).

Now, suppose (without loss of generality) that \( g(t_0) > 0 \) for some \( t_0 \in T_k \). We will divide the proof in two steps:

Step 1: Assume that \( t_0 \) is right scattered. Define in \( T_k \)

\[
\eta(t) = \begin{cases} 
1 & \text{if } t = t_0; \\
0 & \text{if } t \neq t_0.
\end{cases}
\]

Then \( \eta \) is rd-continuous and

\[
\int_a^b g(t) \eta(t) \Delta t = \int_{t_0}^{\sigma(t_0)} g(t) \eta(t) \Delta t = \mu(t_0) g(t_0) > 0,
\]

which is a contradiction.

Step 2: Suppose that \( t_0 \) is right dense. Since \( g \) is rd-continuous, then it is continuous at \( t_0 \). So there exist \( \delta > 0 \) such that for all \( t \in (t_0 - \delta, t_0 + \delta) \cap T_k \) we have \( g(t) > 0 \).

If \( t_0 \) is left-dense, define in \( T_k \)

\[
\eta(t) = \begin{cases} 
(t - t_0 + \delta)^2(t - t_0 - \delta)^2 & \text{if } t \in (t_0 - \delta, t_0 + \delta); \\
0 & \text{otherwise}.
\end{cases}
\]

It follows that \( \eta \) is rd-continuous and

\[
\int_a^b g(t) \eta(t) \Delta t = \int_a^{t_0-\delta} g(t) \eta(t) \Delta t + \int_{t_0-\delta}^{t_0+\delta} g(t) \eta(t) \Delta t + \int_{t_0+\delta}^b g(t) \eta(t) \Delta t > 0,
\]

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which is a contradiction.

If \( t_0 \) is left-scattered, define in \( T^k \)
\[
\eta(t) = \begin{cases} 
(t - t_0 - \delta)^2 & \text{if } t \in [t_0, t_0 + \delta); \\
0 & \text{otherwise},
\end{cases}
\]
where \( 0 < \delta < \min\{\mu(\rho(t_0)), \delta\} \). We have: \( \eta \) is rd-continuous and
\[
\int_a^b g(t) \eta(t) \Delta t = \int_{t_0}^{t_0 + \delta} g(t) \eta(t) \Delta t > 0,
\]
that again leads us to a contradiction.

Proof. (of Theorem 3) We begin by noting that \( u(t) = (u_1(t), \ldots, u_m(t)) \) in problem \([10] \), \( t \in T^k \), are arbitrarily specified functions (controls). Once fixed \( u(\cdot) \in C_{rd}[T; \mathbb{R}^m] \), then \( y(t) = (y_1(t), \ldots, y_m(t)) \), \( t \in T^k \), is determined from the system of delta-differential equations \( y^\Delta(t) = \varphi(t, y(t), u(t)) \) (and boundary conditions, if present). As far as \( u(\cdot) \) is an arbitrary function, variations \( \omega(\cdot) \in C_{rd}[T; \mathbb{R}^m] \) for \( u(\cdot) \) can also be considered arbitrary. This is not true, however, for the variations \( \eta(\cdot) \in C_{rd}[T; \mathbb{R}^n] \) of \( y(\cdot) \). Suppose that \( (y_\cdot(\cdot), u_\cdot(\cdot)) \) is a weak local minimizer of \( J[\cdot, \cdot] \). Let \( \varepsilon \in (-\delta, \delta) \) be a small real parameter and \( y_\varepsilon(t) = y_\cdot(t) + \varepsilon \eta(t) \) (with \( \eta(a) = 0 \) if \( y(a) = y_\cdot \) is given; \( \eta(b) = 0 \) if \( y(b) = y_\cdot \) is given) be the trajectory generated by the control \( u_\varepsilon(t) = u_\cdot(t) + \varepsilon \omega(t) \), \( \omega(\cdot) \in C_{rd}[T; \mathbb{R}^m] \):

\[
y_\varepsilon^\Delta(t) = \varphi(t, y_\varepsilon(t), u_\varepsilon(t)),
\]
\( t \in T^k \), \( (y_\varepsilon(a) = y_\cdot(a), (y_\varepsilon(b) = y_\cdot(b)) \). We define the following function:
\[
\Phi(\varepsilon) = J[y_\varepsilon(\cdot), u_\varepsilon(\cdot)] = J[y_\cdot(\cdot) + \varepsilon \eta(\cdot), u_\cdot(\cdot) + \varepsilon \omega(\cdot)]
\]
\[
= \int_a^b L(t, y_\cdot(t) + \varepsilon \eta(t), u_\cdot(t) + \varepsilon \omega(t)) \Delta t.
\]
It follows that \( \Phi : (-\delta, \delta) \to \mathbb{R} \) has a minimum for \( \varepsilon = 0 \), so we must have \( \Phi'(0) = 0 \). From this condition we can write that
\[
\int_a^b [\psi_0 L_y(t, y_\cdot(t), u_\cdot(t)) \cdot \eta(t) + \psi_0 L_u(t, y_\cdot(t), u_\cdot(t)) \cdot \omega(t)] \Delta t = 0 \tag{28}
\]
for any real constant \( \psi_0 \). Differentiating \((27)\) with respect to \( \varepsilon \), we get
\[
\eta^\Delta(t) = \varphi_y(t, y_\cdot(t), u_\cdot(t)) \cdot \eta(t) + \varphi_u(t, y_\cdot(t), u_\cdot(t)) \cdot \omega(t).
\]
In particular, with \( \varepsilon = 0 \),
\[
\eta^\Delta(t) = \varphi_y(t, y_\cdot(t), u_\cdot(t)) \cdot \eta(t) + \varphi_u(t, y_\cdot(t), u_\cdot(t)) \cdot \omega(t). \tag{29}
\]
Let \( \psi(\cdot) \in C_{rd}[T; \mathbb{R}^n] \) be (yet) an unspecified function. Multiplying \((29)\) by \( \psi^\sigma(t) = [\psi_1^\sigma(t), \ldots, \psi_m^\sigma(t)] \), and delta-integrating the result with respect to \( t \) from \( a \) to \( b \), we get that
\[
\int_a^b \psi^\sigma(t) \cdot \eta^\Delta(t) \Delta t = \int_a^b [\psi^\sigma(t) \cdot \varphi_y(t) \cdot \eta(t) + \psi^\sigma(t) \cdot \varphi_u(t) \cdot \omega(t)] \Delta t. \tag{30}
\]
for any $\psi(\cdot) \in C^1_{\text{ad}}[T; \mathbb{R}^n]$. Integrating by parts (see Lemma 1 formula 1),

$$
\int_a^b \psi(t) \cdot \eta^\Delta(t) \Delta t = \psi(t) \cdot \eta(t)|_a^b - \int_a^b \psi^\Delta(t) \cdot \eta(t) \Delta t, 
$$

(31)

and we can write from (28), (30) and (31) that

$$
\int_a^b \left[ (\psi \Delta(t) + \psi L_y + \psi^\sigma(t)) \cdot \eta(t) + (\psi_0 L_y + \psi^\sigma(t) \cdot \varphi_y) \cdot \omega(t) \right] \Delta t - \psi(t) \cdot \eta(t)|_a^b = 0 \quad (32)
$$

hold for any $\psi(t)$. Using the definition (20) of $H$, we can rewrite (32) as

$$
\int_a^b \left[ (\psi \Delta(t) + H_y) \cdot \eta(t) + H_u \cdot \omega(t) \right] \Delta t - \psi(t) \cdot \eta(t)|_a^b = 0. \quad (33)
$$

It is, however, not possible to employ (yet) Lemma 3 due to the fact that the variations $\eta(t)$ are not arbitrary. Now choose $\psi(t) = \psi_*(t)$ so that the coefficient of $\eta(t)$ in (33) vanishes: $\psi_*(t) = -H_y$ (and $\psi_*(a) = 0$ if $y(a)$ is free, i.e. $\eta(a) \neq 0$; $\psi_*(b) = 0$ if $y(b)$ is free, i.e. $\eta(b) \neq 0$). In the normal case $\psi_*(t)$ is determined by $(y_*(\cdot), u_*(\cdot))$, and we choose $\psi_0 = 1$. The abnormal case is characterized by the existence of a non-trivial solution $\psi_*(t)$ for the system $\psi_\Delta(t) + \psi_\sigma(t) \cdot \varphi_y = 0$: in that case we choose $\psi_0 = 0$ in order to the first coefficient of $\eta(t)$ in (32) or (33) to vanish. Given this choice of the multipliers, the necessary optimality condition (33) takes the form

$$
\int_a^b H_u \cdot \omega(t) \Delta t = 0.
$$

Since $\omega(t)$ can be arbitrarily assigned for all $t \in T^k$, it follows from Lemma 3 that $H_u = 0$. \hfill \Box

### 3.3 The higher-order problem on time scales

As a corollary of Theorem 3 we obtain the Euler-Lagrange equation for problem (3). We first introduce some notation:

$$
\begin{align*}
y^0(t) &= y(t), \\
y^1(t) &= y^{\Delta}(t), \\
\vdots \\
y^{r-1}(t) &= y^{\Delta^{r-1}}(t), \\
u(t) &= y^{\Delta^r}(t).
\end{align*}
$$
**Theorem 4.** If \( y_* \in C_{rd}[T] \) is a weak local minimizer for the higher-order problem (3), then

\[
\psi_{r-1}^r(\sigma(t)) = -L_u(t, x_*(t), u_*(t)) (34)
\]

holds for all \( t \in T^r \), where \( x_*(t) = (y_*(t), y_1^\Delta(t), \ldots, y_{r-1}^\Delta(t)) \) and \( \psi_{r-1}^r(\sigma(t)) \) is defined recursively by

\[
\psi_0^r(\sigma(t)) = -\int_\sigma^t L_y^0(\xi, x_*(\xi), u_*(\xi)) \Delta \xi + c_0 , (35)
\]

\[
\psi_i^r(\sigma(t)) = -\int_\sigma^t \left[ L_y^i(\xi, x_*(\xi), u_*(\xi)) + \psi_{i-1}^i(\sigma(\xi)) \right] \Delta \xi + c_i , \quad i = 1, \ldots, r-1 , (36)
\]

with \( c_j, j = 0, \ldots, r-1, \) constants. If \( y_1^\Delta(\alpha) \) is free in (3) for some \( i \in \{0, \ldots, r-1\}, \alpha \in \{a, \rho^r-1(b)\} \), then the correspondent condition \( \psi_i^r(\alpha) = 0 \) holds.

**Remark 12.** From (34), (35) and (36) it follows that

\[
L_u + \sum_{i=0}^{r-1} (-1)^{r-i} \int_a^\sigma L_y^i + [c_i]_{r-i-1} = 0 , (37)
\]

where \([c_i]_{r-i-1}\) means that the constant is free from the composition of the \( r-i \) integrals when \( i = r-1 \) (for simplicity, we have omitted the arguments in \( L_u \) and \( L_y^i \)).

**Remark 13.** If we delta differentiate (37) \( r \) times, we obtain the delta differentiated equation for the problem of the calculus of variations with higher order delta derivatives. However, as observed in Remark 3 one can only expand formula (37) under suitable conditions of delta differentiability of \( \mu(t) \).

**Remark 14.** For the particular case with \( \varphi(t, y, u) = u \), equation (8) is (37) with \( r = 1 \).

**Proposition 3.** The higher-order problem on time scales (3) does not admit abnormal extremals, even when the boundary conditions \( y_1^\Delta(a) \) and \( y_1^\Delta(\rho^r-1(b)) \), \( i = 0, \ldots, r-1 \), are all fixed.

**Remark 15.** We require the time scale \( T \) to have at least \( 2r + 1 \) points. Let us consider problem (3) with all the boundary conditions fixed. Due to the fact that we have \( r \) delta derivatives, the boundary conditions \( y_1^\Delta(a) = y_a^r \) and \( y_1^\Delta(\rho^r-1(b)) = y_b^r \) for all \( i \in \{0, \ldots, r-1\} \), imply that we must have at least \( 2r \) points in order to have the problem well defined. If we had only this number of
points, then the time scale could be written as $\mathbb{T} = \{a, \sigma(a), \ldots, \sigma^{2r-1}(a)\}$ and
\[
\int_{\mathbb{T}}^{\rho^{-1}(b)} L(t, y(t), y^\Delta(t), \ldots, y^\gamma(t)) \Delta t
= \sum_{i=0}^{r-1} \int_{\mathbb{T}}^{\sigma^{i+1}(a)} L(t, y(t), y^\Delta(t), \ldots, y^\gamma(t)) \Delta t
= \sum_{i=0}^{r-1} L(\sigma^i(a), y(\sigma^i(a)), y^\Delta(\sigma^i(a)), \ldots, y^\gamma(\sigma^i(a)))
\]
where we have used the fact that $\rho^{r-1}(\sigma^{2r-1}(a)) = \sigma^r(a)$. Now, having in mind the boundary conditions and the formula
\[
f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}
\]
we can conclude that the sum in (38) would be constant for every admissible function $y(\cdot)$ and we wouldn’t have nothing to minimize.

The following technical result is used in the proof of Proposition 3.

**Lemma 4.** Suppose that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is such that $f^\sigma(t) = 0$ for all $t \in \mathbb{T}$. Then, $f(t) = 0$ for all $t \in \mathbb{T} \setminus \{a\}$ if $a$ is right-scattered.

**Proof.** First note that, since $f^\sigma(t) = 0$, then $f^\sigma(t)$ is delta differentiable, hence continuous for all $t \in \mathbb{T}$. Now, if $t$ is right-dense, the result is obvious. Suppose that $t$ is right-scattered. We will analyze two cases: (i) if $t$ is left-scattered, then $t \neq a$ and by hypothesis $0 = f^\sigma(\rho(t)) = f(t)$; (ii) if $t$ is left-dense, then $f(t) = \lim_{s \to t^-} f^\sigma(s) = f^\sigma(t)$, by the continuity of $f^\sigma$. The proof is done.

**Proof.** (of Proposition 3) Suppose that $\psi_0 = 0$. With the notation introduced below, the higher order problem would have the abnormal Hamiltonian given by
\[
H(t, y_0, \ldots, y^{r-1}, u, \psi_0, \ldots, \psi^{r-1}) = \sum_{i=0}^{r-2} \psi^i(\sigma(t)) \cdot y^{i+1}(t) + \psi^{r-1}(\sigma(t)) \cdot u(t)
\]
(compare with the normal Hamiltonian). From Theorem 3 we can write the system of equations:
\[
\begin{align*}
\dot{\psi}_0(t) &= 0 \\
\dot{\psi}_1(t) &= -\psi_0(\sigma(t)) \\
&\vdots \\
\dot{\psi}_{r-1}(t) &= -\psi_{r-2}(\sigma(t)) \\
\psi_{r-1}(\sigma(t)) &= 0,
\end{align*}
\]
for all $t \in \mathbb{T}^r$, where we are using the notation $\dot{\psi}^i(t) = \psi^{i\Delta}(t)$, $i = 0, \ldots, r - 1$. From the last equation, and in view of Lemma 4 we have $\psi(t) = 0$, $\forall t \in \mathbb{T}^r$. The proof is done.
\[ y_{\psi} \text{ from Proposition 3 we can fix following form:} \]

\[ A \text{ and the matrices } I \text{ in which } c \text{ (of Theorem 4) Denoting } \hat{T}_0, \forall \text{ we pick again the first equation to point out that } \psi^0(t) = c, \forall t \in \mathbb{T}^{k_{r+1}} \text{ and some constant } c. \text{ Since the time scale has at least } 2r + 1 \text{ points (Remark 14), the set } A \text{ is nonempty and therefore } \psi^0(t) = 0, \forall t \in \mathbb{T}^{k_{r+1}}. \text{ Substituting this in the second equation, we get } \hat{T}_0(t) = 0, \forall t \in \mathbb{T}^k. \text{ As before, it follows that } \psi^0(t) = d, \forall t \in \mathbb{T}^{k_{r+1}} \text{ and some constant } d. \text{ But we have seen that there exists some } t_0 \text{ such that } \psi^0(t_0) = 0, \text{ hence } \psi^0(t) = 0, \forall t \in \mathbb{T}^{k_{r+1}}. \text{ Repeating this procedure, we conclude that for all } i \in \{0, \ldots, r-1\}, \psi^i(t) = 0 \text{ at } t \in \mathbb{T}^k. \text{ This is in contradiction with Theorem 3 and we conclude that } \psi_0 \neq 0. \]

**Proof.** (of Theorem 4) Denoting \( \hat{y}(t) = y^\Delta(t) \), then problem 3 takes the following form:

\[ \mathcal{L}[y(\cdot)] = \int_a^{t_{r-1}(b)} L(t, y^0(t), y^1(t), \ldots, y^{r-1}(t), u(t)) \Delta t \longrightarrow \min, \]

\[
\begin{cases}
\hat{y}^0 = y^1 \\
\hat{y}^1 = y^2 \\
\vdots \\
\hat{y}^{r-2} = y^{r-1} \\
\hat{y}^{r-1} = u
\end{cases}
\]

\[ (y^i(a) = y^i_a), \quad (y^i(\rho^{-1}(b)) = y^i_b), \quad i = 0, \ldots, r-1, \quad y^i_a \text{ and } y^i_b \in \mathbb{R}^n. \]

System 40 can be written in the form \( y^\Delta = Ay + Bu \), where

\[ y = (y^0, y^1, \ldots, y^{r-1}) = (y^0_1, \ldots, y^0_n, y^1_1, \ldots, y^1_n, \ldots, y^{r-1}_1, \ldots, y^{r-1}_n) \in \mathbb{R}^{nr} \]

and the matrices \( A \) (nr by nr) and \( B \) (nr by n) are

\[
A = \begin{pmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad B = \text{col}\{0, \ldots, 0, I\}
\]

in which \( I \) denotes the \( n \) by \( n \) identity matrix, and \( 0 \) the \( n \) by \( n \) zero matrix.

From Proposition 3 we can fix \( \psi_0 = 1 \): problem 40 is a particular case of 16.
with the Hamiltonian given by

\[
H(t, y^0, \ldots, y^{r-1}, u, \psi^0, \ldots, \psi^{r-1}) = L(t, y^0, \ldots, y^{r-1}, u) + \sum_{i=0}^{r-2} \psi^i(\sigma^i) \cdot y^{i+1} + \psi^{r-1}(\sigma^r) \cdot u. \tag{41}
\]

From (26) and (19), we obtain

\[
\psi^i(\sigma(t)) = -\int_a^{\sigma(t)} H_{y^i}(\xi, x(\xi), u(\xi), \psi^i(\xi)) \Delta \xi + c_i, \quad i \in \{0, \ldots, r-1\} \tag{42}
\]

\[
0 = H_u(t, x(t), u(t), \psi^\sigma(t)), \tag{43}
\]

respectively. Equation (43) is equivalent to (34), and from (42) we get (35)-(36).

### 4 An example

We end with an application of our higher-order Euler-Lagrange equation (37) to the time scale \( T = [a, b] \cap \mathbb{Z} \), that leads us to the usual and well-known discrete-time Euler-Lagrange equation (in delta differentiated form) – see e.g. [11]. Note that \( \forall t \in T \) we have \( \sigma(t) = t + 1 \) and \( \mu(t) = \sigma(t) - t = 1 \). In particular, we conclude immediately that \( \mu(t) \) is \( r \) times delta differentiable. Also for any function \( g, g^\Delta \) exists \( \forall t \in T^k \) (see Theorem 1.16 (ii) of [5]) and \( g^{\Delta^2}(t) = g(t+1) - g(t) = \Delta g \) is the usual forward difference operator (obviously \( g^{\Delta^2} \) exists \( \forall t \in T^k \)) and more generally \( g^{\Delta^r} \) exists \( \forall t \in T^k \), \( r \in \mathbb{N} \).

Now, for any function \( f : T \to \mathbb{R} \) and for any \( j \in \mathbb{N} \) we have

\[
\left[ \int_a^{\sigma(t)} \left( \int_a^{\sigma(t)} \cdots \int_a^{\sigma(t)} f \right) \Delta \tau \right]_{j-i}^\Delta = f^{\Delta^i(\sigma^{j-i}(t))}, \quad i \in \{0, \ldots, j-1\} \tag{44}
\]

where \( f^{\Delta^i(\sigma^{j-i}(t))} \) stands for \( f^{\Delta^i(\sigma^{j-i}(t))} \). To see this we proceed by induction. For \( j = 1 \)

\[
\int_a^{\sigma(t)} f(\xi) \Delta \xi = \int_a^{t+1} f(\xi) \Delta \xi = \int_a^t f(\xi) \Delta \xi + \int_t^{t+1} f(\xi) \Delta \xi = \int_a^t f(\xi) \Delta \xi + f(t),
\]

and then \( \left[ \int_a^{\sigma(t)} f(\xi) \Delta \xi \right]^\Delta = f(t) + f^\Delta(t) = f^\sigma \). Assuming that (44) is true for
all \( j = 1, \ldots, k \), then

\[
\left[ \int_a^{\sigma(t)} \left( \int_a^{\sigma} \cdots \int_a^{\sigma} f \right) \Delta \tau \right]^{\Delta^{k+1}}_{k+1-i \text{ integrals}}
\]

\[
= \left( \int_a^{t} \int_a^{\sigma} \cdots \int_a^{\sigma} f \Delta \tau + \int_a^{\sigma(t)} \int_a^{\sigma} \cdots \int_a^{\sigma} f \Delta \tau \right)^{\Delta^{k+1}}
\]

\[
= \left( \int_a^{\sigma(t)} \cdots \int_a^{\sigma} f \Delta \tau \right)^{\Delta^k} + \left[ \int_a^{\sigma(t)} \int_a^{\sigma} \cdots \int_a^{\sigma} f \Delta \tau \right]^{\Delta^k} \Delta
\]

\[
= f^{\Delta^i \sigma^{k-i}} + \left( f^{\Delta^i \sigma^{k-i}} \right)^{\Delta}
\]

\[
= f^{\Delta^i \sigma^{k+1-i}}.
\]

Delta differentiating \( r \) times both sides of equation (37) and in view of (44), we obtain the Euler-Lagrange equation in delta differentiated form (remember that \( y^0 = y, \ldots, y^{r-1} = y^{\Delta^{r-1}}, y^{\Delta^r} = u \)):

\[
L_{y^{\Delta^r}}(t, y, y^\Delta, \ldots, y^{\Delta^r}) + \sum_{i=0}^{r-1} (-1)^{r-i} L_{y^{\Delta^i \sigma^{r-i}}}^{\Delta^i \sigma^{r-i}}(t, y, y^\Delta, \ldots, y^{\Delta^r}) = 0.
\]

5 Conclusion

We introduce a new perspective to the calculus of variations on time scales. In all the previous works [2, 4, 9] on the subject, it is not mentioned the motivation for having \( y^\sigma \) (or \( y^\rho \)) in the formulation of problem (1). We claim the formulation (2) without \( \sigma \) (or \( \rho \)) to be more natural and convenient. One advantage of the approach we are promoting is that it becomes clear how to generalize the simplest functional of the calculus of variations on time scales to problems with higher-order delta derivatives. We also note that the Euler-Lagrange equation in \( \Delta \)-integral form [3], for a Lagrangean \( L \) with \( y \) instead of \( y^\sigma \), follows close the classical condition. Main results of the paper include: necessary optimality conditions for the Lagrange problem of the calculus of variations on time scales, covering both normal and abnormal minimizers; necessary optimality conditions for problems with higher-order delta derivatives. Much remains to be done in the calculus of variations and optimal control on time scales. We trust that our perspective provides interesting insights and opens new possibilities for further investigations.
Acknowledgments

This work was partially supported by the Portuguese Foundation for Science and Technology (FCT), through the Control Theory Group (cotg) of the Centre for Research on Optimization and Control (CEOC – http://ceoc.mat.ua.pt). The authors are grateful to M. Bohner and S. Hilger for useful and stimulating comments, and for them to have shared their expertise on time scales.

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