Influence on Observation from IR Divergence during Inflation. II
— Multi-Field Inflation —

Yuko URAKAWA¹,*) and Takahiro TANAKA²,**)

¹Department of Physics, Waseda University, Tokyo 169-8555, Japan
²Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

(Received May 24, 2009; Revised July 27, 2009)

We propose a way to regularize the fluctuations generated during inflation, whose infrared (IR) corrections diverge logarithmically. In the case of a single-field inflation model, recently, we have proposed a solution to the IR divergence problem. There, we introduced new perturbative variables that better mimic actual observable fluctuations, and proved the regularity of correlation functions with respect to these variables. In this paper, we extend our previous discussions to a multi-field inflation model. We show that, as long as we consider the case that the nonlinear interaction acts for a finite duration, observable fluctuations are also free from IR divergences in the multi-field model. In contrast to the single-field model, to discuss observables, we need to take into account the effects of quantum decoherence, which pick up a unique history of the universe from various possibilities contained in an initial quantum state set naturally in the early stage of the universe.

Subject Index: 440, 442

§1. Introduction

It is widely known that, on the computation of the nonlinear perturbations generated during inflation we encounter the divergence coming from infrared (IR) corrections.¹)–¹²) As the possibility of detecting the non-linear primordial perturbations is increasing,¹³)–³⁷) it becomes more important to solve the IR divergence problem for the primordial perturbations and to predict their finite amplitude that we observe.³⁸)–⁴²) In our previous work,⁴²) we proposed one way to solve this IR divergence problem in the single-field inflation model. The key observation is that the variables that are commonly used in describing fluctuations are affected by what we cannot observe. This is because we can observe only the fluctuations within the region causally connected to us. We usually define the fluctuation by the deviation from the background value which is the spatial average over the whole universe. However, since we can observe only a finite volume of the universe, the fluctuations evaluated in such a way are inevitably influenced by the information contained in the unobservable region. In general, the deviation from the global average is much larger than the deviation from the local average, which leads to the overestimation of the fluctuations owing to the contribution from long-wavelength fluctuations. In addition to that, to discuss the so-called observable quantities in the framework of

*) E-mail: yuko@gravity.phys.waseda.ac.jp
**) E-mail: tanaka@yukawa.kyoto-u.ac.jp
the standard cosmological perturbation (even though people call it gauge-invariant formulation), in practice, it is necessary to fix the gauge, say, the flat gauge. As long as the gauge is determined by solving elliptic-type equations on each constant time slice, the gauge choice is inevitably affected by the information in the region causally disconnected to us. The gauge dependence of the perturbation variables itself is not a problem since the “true observables” such as the statistical property of the sky map of the temperature fluctuation of the cosmic microwave background are not affected by the gauge choice. However, if divergences appear in the quantities computed at the intermediate steps, such as \( n \)-point functions of the perturbation fields, it becomes almost impossible to extract information about the “true observables” from them.

To shut off the influence from the unobservable region of the universe, we focused on the presence of residual gauge degrees of freedom in the flat gauge. To remove the harmful part in the residual gauge degrees of freedom, we imposed a further gauge condition, which insists that the local average of the inflaton vanishes. Then, the fluctuation is not affected by the information from the causally disconnected region. We gave a proof that IR divergences are absent in this new scheme. In this paper, we extend our discussion to multi-field models. Since the adjustment of the average value is possible only for one field, even if we adopt the local gauge, the fluctuations of the other components of the scalar fields \( \varphi^I \) are still influenced by the causally disconnected region. Here, we denote the index for the \( D \)-component scalar fields by \( I = 1, \cdots, D \). Reflecting this fact, when plural fields are associated with the IR divergences, the prescription presented in our previous paper\(^{42} \) is not enough to regularize IR divergences. To remove the influence from what we cannot observe, we introduce new perturbation variables from which we can compute the “true observables”. As the new variables, we consider the local perturbations of the fields defined by the deviation from their local average values. Then, we prove the regularity of \( n \)-point functions of these new variables.

We should note that \( n \)-point functions for the local perturbations are still affected by what we cannot observe. This is due to the difference between the quantum states of the universe, which we set as a natural initial condition and the one which we observe in our real universe. The natural wave function of the inflationary universe is not peaked at a specific point in the space of the local average values of \( \varphi^I \). At the observation time, this “natural” state of the universe can be decomposed into a superposition of wave packets that have a peak at a certain point. As the universe evolves, constituent wave packets lose their correlations to each other. Through this so-called decoherence, the coherent superposition of the wave packets starts to behave as a statistical ensemble of many different worlds, where each world means the universe described by a decohered wave packet.\(^{43}–45 \) Our observed world is just a representative one expressed using a wave packet randomly chosen from various possibilities. Once we select one wave packet after the decoherence occurs, the evolution of our world will not be affected by other parallel worlds. However, the initial quantum state does include the contributions from all the wave packets. This implies that a naive computation of \( n \)-point functions is contaminated by the other worlds uncorrelated to ours. In this paper, taking into account the decoherence of
the quantum state of the universe, we propose a way to define “observables” and show that they are actually finite without suffering from IR divergence.

However, to be honest, the “observables” that we introduce do not correspond exactly to what we measure in the actual observations. Not to mislead the reader, we should stress here that our definition of “observables” does not respect the aspect explained in the preceding paragraph in a completely satisfactory manner. They are not the expectation values for a single decohered wave packet. What we define is not completely free from the contamination of the other worlds decohered from ours. However, since we define our “observables” so as to overestimate the amplitude of fluctuations, the proof of their finiteness ensures the finiteness of what we really observe. Recently, the stochastic approach\textsuperscript{45), 46}) has been employed in order to solve the IR divergence problem.\textsuperscript{39)–41}) This is in harmony with our claim. In the stochastic approach, we assume that the modes that exceed a certain length scale are automatically decohered.\textsuperscript{45)–52}) Namely, long-wavelength fluctuations are treated as the statistical variance. Since our unique “world” is one realization in this statistical ensemble, there is no contribution to the observed quantum fluctuations from long-wavelength modes by definition. This is a practical way to take into account the quantum decoherence in the inflationary universe, but this scheme cannot completely remove the artificial IR cutoff scale from the discussion. Moreover, it is difficult to deny the spiteful criticism that the reason why the problem of IR divergence does not appear in the stochastic approach might be simply because quantum fluctuations in the IR limit are neglected by hand. In contrast, in our approach, to avoid underestimating the amplitude of fluctuations, we accepted a small contamination from other parallel worlds, which will make the amplitude of fluctuations larger. Namely, sacrificing the accuracy of the estimate of the amplitude of fluctuations, we choose to show the IR regularity of the observables by overestimating the amplitude of fluctuations.

This paper is organised as follows. In §2, we give the set up of our problems. Following it, we propose a prescription to define “observables”. The basic idea of our proposal that assures the IR regularity is stated in this section. In §3 we explain the details of the proof of IR regularity. In §4, we summarize our results. On the computation of the non-linear corrections, both integrations over the temporal and spatial coordinates can make IR corrections singular. In this paper, maintaining the initial time $t_i$ at a finite past, we restrict ourselves to the time evolution of perturbations for a finite period of time during inflation. In §4, we add the comments on the case in which we send $t_i$ to distant past.

\section*{§2. A solution to IR problem}

\subsection*{2.1. Setup of the problem}

We first define the setup of the problem that we study in this paper. We consider the multi-component inflation model with the conventional kinetic term. The total action is given by
\[ S = \frac{1}{2} \int \sqrt{-g} \left[ M_{\text{pl}}^2 R - G_{IJ} g^{\mu\nu} \Phi_I^{,\mu} \Phi_J^{,\nu} - 2U(\Phi) \right] d^4x, \]  

(2.1)

where \( M_{\text{pl}} \) is the Planck mass. We denote the field-space metric by \( G_{IJ} \). For simplicity, we assume that \( G_{IJ} \) is a constant matrix. We perform the following change of variables:

\[ \phi_I \equiv \Phi_I / M_{\text{pl}}, \quad V(\phi) \equiv U(\Phi) / M_{\text{pl}}^2, \]  

(2.2)

to factorize \( M_{\text{pl}}^2 \) from the action as

\[ S = \frac{M_{\text{pl}}^2}{2} \int \sqrt{-g} \left[ R - g^{\mu\nu} \phi_I^{,\mu} \phi_J^{,\nu} - 2V(\phi) \right] d^4x, \]  

(2.3)

where \( \phi_I \equiv G_{IJ} \phi^J \). Since the Planck mass is completely factored out, the equations of motion do not depend on it. The Planck mass appears only in the amplitude of quantum fluctuation. Namely, the typical amplitude of fluctuation of \( \Phi^I \) is \( H \), and hence, that of \( \phi_I \) is \( H / M_{\text{pl}} \).

To discuss the nonlinearity, it is convenient to use the ADM formalism, where the line element is expressed in terms of the lapse function \( N \), the shift vector \( N^a \), and the spatial metric \( h_{ab} \):

\[ ds^2 = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \]  

(2.4)

Substituting this metric form, we can denote the action as

\[ S = M_{\text{pl}}^2 \int \sqrt{h} \left[ (3)R - 2NV(\phi) + \frac{1}{N}(E_{ab}E^{ab} - E^2) \right. \]
\[ + \frac{1}{N}(\dot{\phi}_I - N^a \partial_a \phi_I)(\dot{\phi}^I - N^b \partial_b \phi^I) - Nh^{\mu\nu} \partial_{\mu} \phi_I \partial_{\nu} \phi^I \]  

\[ \left. d^4x \right), \]  

(2.5)

where

\[ E_{ab} = \frac{1}{2}(\dot{h}_{ab} - D_a N_b - D_b N_a), \]  

(2.6)

\[ E = h_{ab} E_{ab}, \]  

(2.7)

and \( D \) is the covariant differentiation associated with \( h_{ab} \). A dot “\( \cdot \)” represents a differentiation with respect to the time coordinate. In the ADM formalism, we can obtain the constraint equations easily by varying the action with respect to \( N \) and \( N^a \), which play the role of Lagrange multipliers. We obtain the Hamiltonian constraint equation and the momentum constraint equations as

\[ (3)R - 2V - N^{-2}(E_{ab}E^{ab} - E^2) \]
\[ - N^{-2}(\dot{\phi}_I - N^a \partial_a \phi_I)(\dot{\phi}^I - N^b \partial_b \phi^I) - nh_{\mu\nu} \partial_{\mu} \phi_I \partial_{\nu} \phi^I = 0, \]  

(2.8)

\[ D_b [N^{-1}(E_{ab} - \delta_{ab} E)] - N^{-1} \partial_a \phi_I (\dot{\phi}^I - N^b \partial_b \phi^I) = 0. \]  

(2.9)

Hereafter, neglecting the vector perturbation, we denote the shift vector as \( N^a = h_{ab} \delta^b_{\chi} \). In this paper, we work in the flat gauge, defined as

\[ h_{ab} = e^{2\rho} \delta_{ab}, \]  

(2.10)
where $a \equiv e^\rho$ is the background scale factor. Here we have also neglected the tensor perturbation, focusing only on the scalar perturbation, in which the IR divergence of our interest arises.\cite{4,12}

In this gauge, using $N$, $\chi$, and the fluctuation of the scalar fields $\varphi^I$, the total action is written as

$$S = \frac{M_{\text{pl}}^2}{2} \int dt d^3x e^{3\rho} \left[ -2N \sum_{n=0}^{\infty} \frac{1}{n!} V_{I_1 \ldots I_n}(\phi) \varphi^{I_1} \ldots \varphi^{I_n} 
+ N^{-1} \{ -6\dot{\rho}^2 + 4\dot{\rho} \Delta \chi + (\nabla^a \nabla^b \chi \nabla_a \nabla_b \chi - (\Delta \chi)^2) \} 
+ N^{-1} (\dot{\varphi}^I + \dot{\varphi}^I - \nabla^a \chi \nabla_a \varphi^I)(\dot{\phi}_I + \dot{\phi}_I - \nabla^b \chi \nabla_b \varphi^I) - N \nabla_a \varphi^I \nabla^a \varphi^I \right],$$

(2.11)

and two constraint equations are

$$2N^2 \sum_{n=0}^{\infty} \frac{1}{n!} V_{I_1 \ldots I_n}(\phi) \varphi^{I_1} \ldots \varphi^{I_n} - 6\dot{\rho}^2 + 4\dot{\rho} \Delta \chi + \{ \nabla^a \nabla^b \chi \nabla_a \nabla_b \chi - (\Delta \chi)^2 \} 
+ (\dot{\varphi}_I + \dot{\varphi}_I - \nabla^a \chi \nabla_a \varphi_I)(\dot{\phi}_I + \dot{\phi}_I - \nabla^b \chi \nabla_b \varphi^I) + N^2 \nabla_a \varphi^I \nabla^a \varphi^I = 0,$$

(2.12)

$$\nabla_b N \{ 2\dot{\rho} \delta^b_a + (\nabla^b \nabla_a \chi - \delta^b_a \Delta \chi) \} - (\nabla_a \varphi^I)N(\dot{\phi}_I + \dot{\phi}_I - \nabla^b \chi \nabla_b \varphi^I) = 0,$$

(2.13)

where

$$\nabla_a \equiv e^{-\rho} \partial_a,$$

represents the three-dimensional partial differentiation with respect to the proper length coordinates $X \equiv e^\rho x$ and

$$\Delta \equiv \delta^{ab} \nabla_a \nabla_b.$$

Spatial indices, $a, b, \cdots$, are raised by $\delta^{ab}$. We use this notation, which respects the proper distance, because it eliminates all the complicated scale factor dependences from the action. We define the derivatives of the potential as

$$V_{I_1 I_2 \ldots I_n}(\phi) \equiv \frac{\partial^n V(\phi)}{\partial \phi^{I_1} \partial \phi^{I_2} \ldots \partial \phi^{I_n}}.$$  

(2.14)

The background quantities $\rho$ and $\phi$ satisfy the following equations:

$$3\dot{\rho}^2 = \frac{1}{2} \dot{\phi}_I^2 + V(\phi),$$

(2.15)

$$\ddot{\phi}_I + 3\dot{\rho} \dot{\phi}_I + V^I = 0,$$

(2.16)

$$\ddot{\rho} = -\frac{1}{2} \dot{\phi}_I^2 \dot{\phi}_I.$$

(2.17)

Expanding the variables as

$$N = 1 + \delta N_1 + \frac{1}{2} \delta N_2 + \cdots,$$
we can solve Eq. (2.21) formally as

\[ \chi = \chi_1 + \frac{1}{2} \chi_2 + \cdots , \]
\[ \varphi^I = \varphi_1^I + \frac{1}{2} \varphi_2^I + \cdots , \]

we find that the first-order constraint equations are

\[
V_I \varphi_1^I + 2V \delta N_1 + 2\dot{\rho} \Delta \chi + 3\dot{\rho} \Delta \varphi_1^I - \dot{\varphi}_1^I = 0 ,
\]
\[
\nabla_a \left( 2\dot{\rho} \delta N_1 - \dot{\varphi}_1^I \right) = 0 .
\]

The first order perturbation \( \varphi_1^I \) is identified with the field perturbation in the interaction picture. Taking the deviation of the action with respect to \( \varphi_1^I \), we can derive the equation of motion for \( \varphi_1^I \), which includes the Lagrange multipliers \( \delta N \) and \( \chi \). For example, from the third-order action, we can derive the equation of motion with quadratic interaction terms as,

\[
\dot{\varphi}_1^I + 3\dot{\rho} \dot{\varphi}_1^I - \Delta \varphi_1^I + V_I^I \varphi_1^I - \dot{\varphi}_1^I \Delta \chi + \delta N V_I^I - 3\dot{\rho} \dot{\varphi}_1^I \delta N - \partial_t (\delta N \dot{\varphi}_1^I) + \frac{1}{2} V_{JK}^I \varphi_1^J \varphi^K - \nabla_a (\varphi_1^I \dot{\delta N}) \nabla^a \chi - (\varphi_1^I \dot{\varphi}_1^I \Delta N) \Delta^2 \chi - \rho \nabla^a \chi \nabla_a \varphi_1^I - \partial_t (\nabla^a \chi \nabla_a \varphi_1^I) - 3\dot{\rho} \dot{\varphi}_1^I \delta N - \partial_t (\delta N \varphi_1^I) - \nabla_\mu (\delta N \nabla^\mu \varphi_1^I) + V_{IK}^J \varphi_1^J \delta N + 3\dot{\rho} \dot{\varphi}_1^I \delta N^2 + \partial_t (\dot{\varphi}_1^I \delta N^2) = 0 .
\]

Solving the constraint equations at each order, we can express the lapse function and the shift vector as functions of \( \varphi_1^I \) at lower order. By substituting these expressions into the equation of motion for \( \varphi_1^I \), which is up to the third-order given by Eq. (2.20), the equation is written solely in terms of the dynamical degree of freedom, \( \varphi_1^I \).

2.2. Tree-shaped graphs

In this subsection, as a preparation for computing \( n \)-point functions of \( \varphi_I(x) \), we consider an expansion of the Heisenberg field \( \varphi_I(x) \) in terms of the interaction picture field, using the retarded Green function \( G_{R}^I_J(x, x') \), which is causal. Since the retarded Green function \( G_{R}^I_J(x, x') \) has a finite non-vanishing support for fixed \( x \) and \( t' \), its three dimensional Fourier transform with respect to \( x' \) becomes regular in the IR limit.

Let us denote the equation of motion for \( \varphi^I \) schematically by

\[
\mathcal{L}_{IJ}^I \varphi^I = -\Gamma_I^I[\varphi] ,
\]

where \( \mathcal{L}_{IJ}^I \) is a second order differential operator corresponding to the linearized equation for \( \varphi^I \) (Eq. (2.32)) and \( \Gamma_I^I \) stands for all the nonlinear interaction terms. Using the retarded Green function \( G_{R}^I_K(x, x') \) that satisfies

\[
\mathcal{L}_{IJ}^I G_{R}^I_K(x, x') = -a^{-3} \delta^I(x - x') \delta_K^I ,
\]

we can solve Eq. (2.21) formally as

\[
\varphi^I(x) = \varphi_1^I(x) + \int d^4x' G_{R}^I_J(x, x') a^3(t') \Gamma_I^I[\varphi](x') ,
\]
where the first order perturbation $\varphi_1^I$ satisfies
\[
\mathcal{L}_I^I\varphi_1^I(x) = 0.
\] (2.24)

Here, the factor $a^3$ originates from the background value of $\sqrt{-g}$. Substituting the expression (2.23) for $\varphi^I(x)$ iteratively into $\Gamma_I[\varphi]$ on its r.h.s., we obtain the Heisenberg field $\varphi^I(x)$ expanded in terms of $\varphi_1^I(x)$ to an arbitrary high order using the retarded Green function $G_{RI}^I(x, x')$. As we have shown in Ref. 42), to expand $\varphi^I$, a diagrammatic illustration will be useful. The Heisenberg field can be expressed as a summation of tree-shaped graphs in which all the retarded Green functions $G_{RI}^I(x, x')$ are followed by two or more $\varphi_1^I(x')$ or $G_{RI}^I(x', x'')$ with some integro-differential operators and all the interaction picture fields $\varphi_1^I(x)$ are located at the rightmost ends of the graphs.

When we compute the expectation value for $n$-point functions of the Heisenberg field, the interaction picture fields $\varphi_I$ in the tree-shaped graphs are contracted with each other to make pairs, which are replaced with Wightman functions, $G_{IJ}^+(x, x') \equiv \langle \varphi_I^+(x) \varphi_J^+(x') \rangle$ or $G_{IJ}^-(x, x')(= G_{JI}^{+}(x', x))$. These propagators are IR singular ($\propto 1/k^3$), which is the possible origin of IR divergences in momentum integrations, while, the retarded Green function,
\[
G_{RI}^I(x, x') = i\theta(t - t')M_{pl}^2\{G_{+IJ}^I(x, x') - G_{-IJ}^I(x, x')\},
\] (2.25)
is regular in the IR limit.

2.3. Iteration scheme and local gauge conditions

In our previous paper,\textsuperscript{42}) we have shown that the flat gauge still has residual gauge degrees of freedom. For instance, we can introduce an arbitrary function $f_n(t)$ to the $n$-th order lapse function and the shift vector as
\[
\delta N_n \rightarrow \delta N_n + f_n(t), \quad \chi_n \rightarrow \chi_n - \frac{V}{\rho_0} X_{n}X^{a} f_n(t).
\]
This gauge degree of freedom corresponds to the scale transformation. As mentioned in §1, our final goal is to define finite observable quantities in place of the naively divergent quantum correlation functions. For this purpose, we need to define gauge-invariant variables without the information contained in the region far outside $\mathcal{O}$. Then, we have to fix the residual gauge only using the information near the observable region $\mathcal{O}$.

In the multi-field model, it is convenient to decompose the perturbation into the adiabatic one, which is tangential to the background trajectory, and the entropy one, which is orthogonal to the background trajectory.\textsuperscript{53}) Using the residual gauge degrees of freedom, we fix the homogeneous mode in the direction of the background trajectory $e_I \equiv \dot{\phi}_I / (\dot{\phi}_J \dot{\phi}^J)^{\frac{1}{2}}$ as
\[
\hat{W}_I e_I \varphi^I(t) \equiv \frac{1}{L_i^3} \int d^3 x \ W_t(x) \ e_I \varphi^I(t, x) = 0,
\] (2.26)
where $W_t(x)$ is a window function, which is unity in the finite region $\mathcal{O}_t \equiv \mathcal{O} \cap \Sigma_t$ with a rapidly vanishing halo in the surrounding region, where $\Sigma_t$ represents a $t = \text{const}$
Y. Urakawa and T. Tanaka

dimensional, we introduce \( \mathcal{O}_t \supset \mathcal{O}_t \) and define \( \mathcal{O} \) as the causal past of \( \mathcal{O}_t \). We require \( W_t(x) \) to vanish in the region outside \( \mathcal{O} \). In addition, \( W_t(x) \) is assumed to be a sufficiently smooth function so that an artificial UV contribution is not induced by a sharp cutoff. \( L_t \), an approximate radius of the region \( \mathcal{O}_t \), is defined such that the normalization condition

\[ \hat{W}_t = 1 \]

is satisfied.

We associated ‘\( \sim \)’ with the variables in the particular gauge satisfying Eq. (2.26), in order to clearly distinguish them from the variables for which the additional gauge condition is not imposed. The difference between the variables with and without ‘\( \sim \)’ is only in the boundary conditions. Hence, they obey the same differential equations, (2.19) and (2.20).

To fix the arbitrary functions \( f_n(t) \) \( (n = 1, 2, 3, 4, \ldots) \) so as to satisfy the gauge condition (2.26), we need to obtain a formal solution for \( \tilde{\phi} \). The higher order lapse functions are determined by solving the momentum constraint given in the form

\[ \nabla_a \left( \delta \bar{N}_n - \frac{1}{2\bar{\rho}} \dot{\varphi}^I_n \right) = \Xi_a^{(n)}, \quad (n = 1, 2, 3, \ldots) \tag{2.27} \]

where the r.h.s. is the \( n \)-th order nonlinear term expressed in terms of the lower order lapse functions, shift vectors, and \( \tilde{\phi} \). As we neglect the vector perturbation, we consider only the scalar part of these equations, i.e., its divergence, which is formally solved as

\[ \delta \bar{N}_n = \delta \hat{N}_n + f_n, \tag{2.28} \]

with

\[ \delta \hat{N}_n = \frac{1}{2\bar{\rho}} \dot{\varphi}^I_n + \nabla_a \Xi_a^{(n)}. \]

We define the operation \( \triangle^{\sim -1} \) by

\[ \triangle^{\sim -1} F(x) = -\frac{1}{4\pi} \int \frac{W_t(e^{-\bar{\rho}Y})d^3Y}{|X - Y|} F(t, e^{-\bar{\rho}Y}), \tag{2.29} \]

so that it is completely determined using the local information in the neighborhood of \( \mathcal{O}_t \). Similarly, the higher order shift vectors satisfy the Hamiltonian constraint in the form

\[ \triangle \dot{\bar{\chi}}_n = \frac{1}{2} \left[ \dot{\varphi}^I_n - \frac{\dot{\phi}^I_n}{\bar{\rho}^2} \partial_t (\dot{\varphi}^I_n) \right] - \frac{V}{\bar{\rho}} \left( f_n + \nabla^a \Xi_a^{(n)} \right) + C_n, \]

where \( C_n \) on the r.h.s. is a function expressed in terms of the lower order lapse functions, shift vector and \( \tilde{\varphi}^I \). A formal solution for \( \bar{\chi}_n \) is given by

\[ \bar{\chi}_n = \hat{\chi}_n - \frac{\bar{\rho}^2 V}{6\bar{\rho}} f_n, \tag{2.30} \]
with
\[ \dot{\chi}_n = \triangle^{-1} \left( \frac{1}{2} \left[ \ddot{\phi}_I \phi_n^I - \frac{\dot{\phi}_I}{\rho} \partial_t (\dot{\rho} \phi_n^I) \right] - \frac{V}{\dot{\rho}} \triangle^{-1} \left( \nabla^a \Xi^{(n)}_a + C_n \right) \right). \]

Substituting the expressions for the lapse function (2.28) and the shift vector (2.30) into the equation of motion for \( \tilde{\phi} \) truncated at the \( n \)-th order, we obtain an equation
\[ \mathcal{L}^I_J \tilde{\phi}^J_n - \dot{\phi}_I^I \dot{f}_n + \left( \frac{V \dot{\phi}_I}{\dot{\rho}} + 2V^I \right) f_n = -W_t(x) \Gamma^I_n, \tag{2.31} \]
where, for later convenience, we have inserted a window function \( W_t(x) \) on the r.h.s., which does not alter the evolution in \( \mathcal{O} \). The explicit form of \( \mathcal{L}^I_J \) is given by
\[ \mathcal{L}^I_J \equiv (\partial_t^2 + 3\dot{\rho} \partial_t - \triangle) \delta^I_J + \left( V^I_J - e^{-3\rho} \dot{A}^I_J \right), \tag{2.32} \]
with
\[ \dot{A}^I_J(t) \equiv e^{3\rho} \dot{\phi}_I \dot{\phi}_J / \dot{\rho}, \]
and \( \Gamma^I_n \) on the r.h.s. of Eq. (2.31) represents all the \( n \)-th order nonlinear terms expressed in terms of lower order variables.

By acting the operator \( 1 - \hat{W}_t \) on Eq. (2.31), the equation for the inhomogeneous part of \( \tilde{\phi}^I_n \) is obtained as
\[ (1 - \hat{W}_t) \mathcal{L}^I_J \tilde{\phi}^J_n = -(1 - \hat{W}_t)W_t(x) \Gamma^I_n[\tilde{\varphi}] \tag{2.33} \]
Then, we find that \( \tilde{\phi}^I_n(x) \) is obtained as
\[ \tilde{\phi}^I_n(x) = \hat{W}^I_n x \tilde{\phi}^J_n(x) + B^I_{\perp n}(t), \tag{2.34} \]
where \( \hat{W}^I_n x \tilde{\phi}^J_n(x) \) satisfies
\[ \mathcal{L}^I_J \tilde{\phi}^J_n(x) = -W_t(x) \Gamma^I_n[\tilde{\varphi}] \tag{2.35} \]
\( B^I_{\perp n}(t) \) is a homogeneous field perpendicular to \( e^I \). The solution (2.34) satisfies the gauge condition \( \hat{W}^I_n e^I \tilde{\varphi}^I_n = 0 \).

The remaining unknowns are \( f_n(t) \) and \( B^I_{\perp n}(t) \), which has \( D - 1 \) components. These \( D \) unknown components are determined using the homogeneous part of the equations of motion obtained by substituting (2.34) into Eq. (2.31),
\[ \mathcal{L}^I_J B^I_{\perp n} - \dot{\phi}^I \dot{f}_n + \left( \frac{V \dot{\phi}_I}{\dot{\rho}} + 2V^I \right) f_n = \mathcal{L}^I_J e^I e^K \hat{W}_t \tilde{\varphi}^K_n. \tag{2.36} \]

2.4. Projection to one decohered wave packet

When plural fields have scale-invariant or even redder spectrum, the entropy perturbation can give divergences. However, in this subsection, we show that a naive computation of the correlation functions does not give the correlation functions that we actually observe.
When there is no isocurvature mode related to IR divergence, making use of the
gauge degree of freedom, we can arrange that the adiabatic perturbation variable
\( \tilde{A} \equiv \epsilon_I \tilde{\phi}^I \) be the deviation from the local average value. In contrast, there is no such
gauge degree of freedom for the isocurvature perturbation
\( S_I \equiv \tilde{\phi}^I - \epsilon^I J \epsilon^J \tilde{\phi}^J \).

Hence, we have to use the isocurvature perturbation variables defined as the devi-
ation from the average values on a whole time slice, which contains information on
the causally disconnected region. As an observable isocurvature perturbation, we
introduce the local fields,
\[
\tilde{S}_I(x) \equiv S_I(x) - \hat{W}_t S_I(x).
\]  

(2.37)

However, even if we restrict our attention to the local quantity \( \tilde{S}_I(x) \) on the final
surface, the variables \( S_I(x) \) that contain the information outside the causal region
appear in describing the time evolution of the field. Although in our previous work
we have stressed that the locality of the observables is the key issue in order to
 assure the IR regularity, the locality is inevitably violated under the presence of IR
divergence originating from isocurvature perturbation.

Here, we need to raise another key issue, i.e., the quantum decoherence. The
primordial perturbations are expected to decohere through the cosmic expansion
and/or through various interactions.\(^{43)}\)–\(^{45)}\) This decoherence process transmutes the
quantum fluctuations at a long wavelength to statistical variances.\(^{45)}\)–\(^{51)}\),\(^{56)}\),\(^{57)}\) At the
initial time when the wavelength of relevant modes is short, the adiabatic vacuum
state will be a natural vacuum state. However, the adiabatic vacuum state is not a
wave packet sharply peaked around a specific value of the homogeneous part of the
scalar field \( \hat{W}_t \tilde{\phi}^I \). Instead, it is infinitely broad and can be interpreted as a coherent
superposition of peaked wave packets. (A detailed explanation will be given in \( \S \) B.1
below.) In the early stage of inflation, these wave packets correlate with each other,
but the quantum coherence is gradually lost in the course of time evolution. Thus,
at the observation time \( (t = t_f) \), the coherence will remain only between adjacent
overlapping wave packets. Our observed world corresponds to one decohered wave
packet picked up from this superposition. For the later time evolution of our world,
we can completely neglect the other wave packets whose peak is located very far
from ours in the space of isocurvature components of the local average values of
fields \( \tilde{S}_I \equiv (\hat{W}_t \tilde{\phi}^I)_\perp \). Hence, keeping the superposition of all wave packets as the
wave function of the universe gives rather misleading results, i.e., huge overestimates
of quantum fluctuations. We should therefore remove the contributions from the
other worlds.

It is standard to discuss the decoherence by coarse-graining some degrees of
freedom in the quantum interacting system, by which the reduced density matrix
evolves from its initial pure state to a mixed state. This process is interpreted as the
transition from the initial coherent superposition of many different worlds to their
final statistical ensemble. In Refs. 39)–41), the decoherence of the long-wavelength
modes is treated using the stochastic approach to inflation, in which all the quantum
fluctuations of long-wavelength modes are assumed to turn into the variance of classical ensemble at each time step. This assumption of complete classicalization can be justified to some extent by coarse-graining the short wavelength modes. On physical grounds, we consider that this approach gives a good approximate description of the dynamics of inflation. However, here, we take a different approach because in the stochastic approach, by assumption, the quantum fluctuations of long-wavelength modes, which we focus on in this paper, cannot enter into the quantum loop corrections from the beginning. In this regard, we consider that stochastic approach is not much more satisfactory than a naive prescription of introducing a cutoff length scale by hand.

The accurate evaluation of what we really observe requires the elucidation of the decoherence of the primordial perturbations, which is a long-lasting and unsettled issue. Here, instead, we aim at proving the IR regularity of our “observables” independently of the details of the decoherence process, which is the heart of this paper. Although it is difficult to understand how the decoherence proceeds until the observation time \( t_f \), it is natural to expect that the wave function of the universe has already been decohered at \( t = t_f \) to a large extent. The observation picks up one world from the superposition of many decohered worlds. The wave function corresponding to each decohered world will have a rather sharp peak in the coordinate space of

\[
\bar{S}^\bar{\alpha}(t) \equiv e_I^\bar{\alpha} \hat{W}_I \hat{S}^I(x), \quad (\bar{\alpha} = 2, 3, \cdots, D)
\]  

where \( \{e^\alpha_I\} \) with \( e^1_I = e_I \) is a set of orthonormal bases in field space. Hence, we insert a projection operator \( \mathcal{P}_{\{\alpha\}} \), which restricts the values of \( \bar{S}^\bar{\alpha}(t_f) \) to a small range near \( \bar{S}^\bar{\alpha}(t_f) = \alpha^{\bar{\alpha}} \) without having any significant effect on each decohered wave packet. Then, the insertion of \( \mathcal{P} \) is expected to reduce the contamination from the other worlds significantly.

For simplicity, we choose the Gaussian projection operator

\[
\mathcal{P}_{\{\alpha\}} \equiv \exp \left[ -\frac{C^{-1}_{\bar{\alpha}\bar{\beta}}(\bar{S}_{\bar{\alpha}}(t_f) - \alpha^{\bar{\alpha}})(\bar{S}_{\bar{\beta}}(t_f) - \alpha^{\bar{\beta}})}{2} \right],
\]

where \( \alpha^{\bar{\alpha}} \) denotes \( D - 1 \) real C-numbers. The dispersion should be sufficiently large compared with the width of one decohered wave packet to guarantee that the evaluated amplitude of fluctuations is always larger than that for a single wave packet. Inserting an identity

\[
\frac{1}{\sqrt{\det C}} \left[ \prod_{\bar{\alpha} = 2}^{D} \int_{-\infty}^{\infty} \frac{d\alpha^{\bar{\alpha}}}{\sqrt{2\pi}} \right] \mathcal{P}_{\{\alpha\}} = 1,
\]

we can expand the \( n \)-point function of the variables whose local average values are subtracted. \( \{\bar{A}, \bar{S}^I\} \) are schematically denoted by \( \bar{O} \). We can expand the \( n \)-point

\[\text{footnote}^1\] Here, we used the word “projection operator”, but this operator does not satisfy the relation \( \mathcal{P}_{\{\alpha\}} = \mathcal{P}^2_{\{\alpha\}} \) expected from its name. However, this kind of property is unnecessary for our present discussion.
function of $\hat{O}(x)$ for the adiabatic vacuum $|0\rangle_a$ as

$$a\langle 0|\hat{O}(t_f,x_1)\hat{O}(t_f,x_2)\cdots\hat{O}(t_f,x_n)|0\rangle_a = \frac{1}{\sqrt{\det C}} \left[ \prod_{\alpha=2}^D \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi}} \right] a\langle 0|P_{\{\alpha\}}\hat{O}(t_f,x_1)\cdots\hat{O}(t_f,x_n)|0\rangle_a . \quad (2.41)$$

Then, we regard

$$\langle P\hat{O}(t_f,x_1)\hat{O}(t_f,x_2)\cdots\hat{O}(t_f,x_n) \rangle = \frac{a\langle 0|P\hat{O}(t_f,x_1)\hat{O}(t_f,x_2)\cdots\hat{O}(t_f,x_n)|0\rangle_a}{a\langle 0|P|0\rangle_a} \quad (2.42)$$

in the integrand on the right-hand side of Eq. (2.41) as the observable $n$-point function of $\hat{O}$s after the selection of a single decohered world. Here, we set $\alpha^\alpha = 0$ and denote $P_{\{\alpha^\alpha=0\}}$ by $P$. Setting $\alpha^\alpha = 0$ does not lose generality because the classical average values of isocurvature perturbation $S^\alpha$ can be changed by choosing the background trajectory. We will prove the IR regularity of this $n$-point function in the succeeding section.

Now, the question is how to determine the width of the projection operator, $\sigma$. (Here, we are assuming that $C^\alpha\bar{\alpha} \approx \sigma^2 \delta^\alpha\bar{\alpha}$.) On one hand, $\sigma$ must be sufficiently large to exceed the width of a decohered wave packet. Naively, there is a minimum size of the wave packet, since a very narrow wave packet cannot maintain its width for a long period of time. Later, we find that the minimum size of a wave packet that we can choose is determined using the typical amplitude of quantum fluctuations generated during inflation, which is characterized by the Hubble scale for a nearly massless scalar field. This amplitude is $H/M_{pl}$ in terms of the fluctuation of $\hat{O}(x)$. Therefore, we need to set $\sigma$ to be larger than $H/M_{pl}$. On the other hand, to suppress the influence from other wave packets, $\sigma$ should not be very large. Later, we find that the condition that the higher order contributions are more suppressed requires $\sigma$ to be much less than unity. These two conditions are compatible by choosing $\sigma$ to satisfy $H/M_{pl} \ll \sigma \ll 1$.

Owing to the inaccurate evaluation of the decohered wave packet, the effect of insertion of the Gaussian projection is not equivalent to selecting our world through the actual decoherence process. Hence, we cannot claim that the $n$-point function given by Eq. (2.42) is the true observable $n$-point function. However, the former amplitude is larger than the latter one. Thus, if the $n$-point function given by Eq. (2.42) is proved to be finite, we can conclude that the $n$-point function of $\hat{O}(x)$ evaluated for the actual decohered wave packet is also finite.

In the above discussion, we assumed that the average values $S^{\alpha}(t)$ of all entropy modes have accomplished the decoherence process successfully before we measure $n$-point functions of $\hat{O}(x)$ at $t = t_f$. Here, we want to stress that whether the superposition of decohered wave packets come to be statistical ensemble has nothing to do with whether the mode is measurable for us. Let us consider a hidden variable $x$ that interacts extremely weakly with our visible sector. Even in that case, if $x$ represents an average of a field over a large volume, the quantum coherence between
two wave packets $|1\rangle$ and $|2\rangle$ that peaked at largely different values of $x$ will be lost (at least after integrating out the other degrees of freedom in the hidden sector). Assuming that $x$ takes the two discrete states $|1\rangle$ and $|2\rangle$ with an equal weight for simplicity, the evolved density matrix will be schematically written as $\rho = (|1\rangle\langle 1|\rho_1 + |2\rangle\langle 2|\rho_2)/2$, after integrating out the other degrees of freedom in the hidden sector. Here, $\rho_1$ and $\rho_2$ are the density matrices of our visible sector. (If the interaction between the hidden and visible sectors is extremely weak, $\rho_1$ and $\rho_2$ are identical.) Then, for any operator $\mathcal{O}$ in our visible sector, $\text{tr}\rho \mathcal{O} = (\text{tr}\rho_1 \mathcal{O} + \text{tr}\rho_2 \mathcal{O})/2$. This indicates that, as far as the variables measurable for us are concerned, the state can be understood as a statistical ensemble composed of $\rho_1$ and $\rho_2$. Therefore what we actually observe is the expectation value for either $\rho_1$ or $\rho_2$. Therefore, irrespective of whether the isocurvature perturbation is in the visible sector or in the hidden sector, we can insert a projection operator $\mathcal{P}$ to take into account the influence of decoherence. In the succeeding section, we discuss the regularity of the “observed” $n$-point function $\langle \mathcal{P} \tilde{O}(t_f, x_1)\tilde{O}(t_f, x_2) \cdots \tilde{O}(t_f, x_n) \rangle$.

§3. Proof of IR regularity

In this section, we prove the IR regularity of the $n$-point function (2.42). In this paper, we discuss the evolution of perturbation during a finite period of inflation. First we describe the method of the quantization in §3.1. Before starting the detailed discussion, in §3.2, we briefly explain the basic idea of the proof of the IR regularity. In this subsection, we clarify the difference between the regularization in multi-field models and that in single field models. After that, in §3.3, we adapt the basis transformation. In the new basis, it becomes easier to understand the regularization in the multi-field models. On the basis of these preparations, in §3.4, we show that IR suppression due to the projection operator regularizes the IR divergence when the initial conditions are set at a finite past.

In this section, we discuss the IR regularity after we remove the influence of the unobservable quantities. For technical reasons, it is better to avoid treating the divergent quantities directly. Therefore, first, we assume that the total volume of the universe $V_c = L_c^3$ is finite. Then, the normal modes take a discrete spectrum, and as a result, the IR divergence is concentrated on the spatially homogeneous mode with $p = 0$, as long as $L_c$ is kept finite. Even with a finite volume, the quantum fluctuation of the homogeneous mode with $p = 0$ in adiabatic vacuum is still divergent in contrast to the other IR modes. In §3.1 we introduce a parameter $s_\alpha$ that represents the deviation from the adiabatic one for the $p = 0$ mode. At the end of the calculations, we take the limit $V_c \to \infty$ and $s_\alpha \to 0$.

3.1. Quantization

In the previous section, we described how we can expand the Heisenberg field $\tilde{\varphi}_I^\dagger$ in terms of $\tilde{\varphi}_I^\dagger(x)$. The interaction picture field $\tilde{\varphi}_I^\dagger(x)^\ast$ satisfies the equation of

\footnote{The leading term of the Heisenberg picture field $\tilde{\varphi}_I^\dagger(x)$ agrees with the interaction picture field $\tilde{\varphi}_I^\dagger_{\text{int}}(x)$. Thus, we denote the interaction picture field as $\tilde{\varphi}_I^\dagger(x)$.}
motion $\mathcal{L}_{IJ} \bar{\phi}_1^I(x) = 0$. Using a set of mode function \{ $\phi_{\alpha,p}^I(\mathbf{x}) \equiv u_{\alpha,p}^I(t)e^{i\mathbf{p} \cdot \mathbf{x}}$ \} that satisfies

$$0 = e^{-i\mathbf{p} \cdot \mathbf{x}} \mathcal{L}_{IJ} \phi_{\alpha,p}^J = [(\delta^2_t + 3\dot{\rho} \partial_t + \mathbf{p}^2)\delta^I_J + (V^I_J - e^{-3\dot{\rho}}\dot{A}^I_J)]u_{\alpha,p}^J(t), \quad (3.1)$$

we expand $\bar{\phi}_1^I(x)$ as

$$\bar{\phi}_1^I(x) = \frac{1}{V^2_c} \sum\sum D \sum_{\alpha=1} \left\{ \frac{u_{\alpha,p}^I(t)}{M_{pl}} e^{i\mathbf{p} \cdot \mathbf{x}} u_{\alpha,p}^I(t) + \text{h.c.} \right\}, \quad (3.2)$$

where the index $\alpha = 1 \cdots D$ is the label of the orthonormal basis. By making use of the Gram-Schmidt orthogonalization, the mode functions $\phi_{\alpha,p}^I \equiv e^{i\mathbf{p} \cdot \mathbf{x}} u_{\alpha,p}^I/M_{pl}V^{1/2}$ are orthonormalized such that

$$(\phi_{\alpha,p}, \phi_{\beta,p'}) = V_c \delta_{\alpha\beta} \delta_{pp'} \quad (3.3)$$

is satisfied, where the Klein-Gordon inner product is defined as

$$(\phi, \psi) = -ia^3 \int_\Sigma \left\{ \phi^I \partial_\alpha \psi^*_I - (\partial_\alpha \phi^I) \psi^*_I \right\} d\Sigma^\alpha. \quad (3.4)$$

In Eq. (3.3), the factor $V_c$ is necessary in order that the same functional form of the mode functions satisfies the natural orthonormal conditions in the continuum limit, $V_c \to \infty$. (See Appendix A.) Using the normalization conditions (3.3), we find that the creation and annihilation operators $a_{\alpha,p}^\dagger$ and $a_{\alpha,p}$ satisfy the following commutation relations,

$$[a_{\alpha,p}, a_{\beta,p'}^\dagger] = \delta_{\alpha\beta} \delta_{pp'} \quad (3.5)$$

The initial vacuum state $|0\rangle_a$ is annihilated by the operation of any annihilation operator:

$$a_{\alpha,p}|0\rangle_a = 0, \quad \text{for } \forall \alpha \text{ and } \forall p.$$

The mode function $u_{\alpha,p}^I(t)$ is normalized by

$$u_{\alpha,p}^I(t)u_{\alpha,p}^{J*}(t) - u_{\alpha,p}^I(t)u_{\alpha,p}^{J*}(t) = \frac{i}{a^3(t)} \delta_{\alpha\beta}. \quad (3.6)$$

In the long-wavelength limit, we obtain two real independent growing and decaying solutions as

$$g_{\alpha,p}^I(t) = g_{\alpha}^I(t) \left[ 1 + O((p/aH)^2) \right], \quad (3.7)$$
$$d_{\alpha,p}^I(t) = d_{\alpha}^I(t) \left[ 1 + O((p/aH)^2) \right], \quad (3.8)$$

and $g_{\alpha}^I(t)$ and $d_{\alpha}^I(t)$ satisfy the normalization condition

$$\dot{g}_{\alpha}^I(t)d_{\alpha}^I(t) - g_{\alpha}^I(t)\dot{d}_{\alpha}^I(t) = a^{-3}(t) \delta_{\alpha\beta}. \quad (3.9)$$
After we define appropriate observables, we take the limit $s \to 0$ and $V_c \to \infty$. Combining these two solutions, we can construct a mode function as

$$u^{I}_{\alpha,k}(t) = \frac{1}{c_{\alpha}(k)} g^{I}_{\alpha,k}(t) + ic^{*}_{\alpha}(k) d^{I}_{\alpha,k}(t),$$

with an arbitrary parameter $c_{\alpha}(k)$. The squared amplitude of $u^{I}_{\alpha,k}(t)$ gives the amplitude of the primordial perturbations. It is common to set the initial state to the adiabatic vacuum, which is a natural state in the inflationary universe. At the horizon crossing, where $k \approx aH$, the growing and decaying solutions should contribute to the positive frequency function $u^{I}_{\alpha,k}(t)$ to the same order unless the initial quantum state is very different from the adiabatic vacuum one. Assuming that the time variations of $g^{I}_{\alpha,k}$ and $a^{3}Hd^{I}_{\alpha,k}$ are not very fast after the horizon crossing time, i.e., $|g^{I}_{\alpha,k}| / |d^{I}_{\alpha,k}| \approx Ha^{3+\delta_{\alpha}}$ with $\delta_{\alpha} \ll 1$, this requirement determines the order of magnitude of $c_{\alpha}(k)$ as

$$|c_{\alpha}(k)| = O \left( \sqrt{\frac{k^{3+\delta_{\alpha}}}{H^{2+\delta_{\alpha}}}} \right).$$

(3.11)

Thanks to the local gauge conditions, as in the single field case discussed in our previous paper,\(^{42}\) we can prove the regularity of the IR fluctuations initially in the adiabatic direction, i.e., the tangential direction to the background trajectory. Looking at Eq. (3.1), we find that $\ddot{\phi}^{I} / \rho = d\phi^{I} / d\rho$ satisfies the mode equation for the homogeneous mode $u^{I}_{\alpha,0}$. We choose one of the bases $u^{I}_{\alpha,k} \approx g^{I}_{\alpha,k} / c_{1}(k)$ so as to approach $d\phi^{I} / d\rho$ in the homogeneous limit $k \to 0$. Then, as we will show in §3.2, the modes with $\alpha = 1$ no longer cause IR divergences. We give the other modes $u^{I}_{\alpha,k} (\alpha = 2, \cdots D)$ so as to be orthogonal to each other.

Because we are considering the universe in a finite box, wave numbers $k$ are discrete. Hence, unless we take the infinite volume limit, $V_c \to \infty$, the divergence is concentrated on the spatially homogeneous mode with $k = 0$ in the above expression for the mode functions. To deal with this divergence in the $k = 0$ mode, we regularize $c_{\alpha}(0)$, introducing a small parameter $s_{\alpha},$ as

$$c_{\alpha}(0) \equiv s_{\alpha} / V_{c}^{1/2}.$$

Then, we obtain

$$u^{I}_{\alpha,0}(\tau) = \frac{V_{c}^{1/2}}{s_{\alpha}} g^{I}_{\alpha,0}(t) + i \frac{s_{\alpha}}{V_{c}^{1/2}} d^{I}_{\alpha,0}(t),$$

(3.12)

After we define appropriate observable, we take the limit $s_{\alpha} \to 0$ and $V_{c} \to \infty$.

Giving the Wightman function $G^{I}_{IJ}(x,x')$ in Eq. (2.25) as $G^{I}_{IJ}(x,x') = \langle 0 | \hat{\phi}_{I,1}(x) \hat{\phi}_{J,1}(x') | 0 \rangle_{a}$, we can expand the retarded Green function in terms of mode functions $u^{I}_{\alpha}$ as

$$G_{RJ}^{I}(x,x') = -i(t - t') \frac{1}{V_{c}} \sum_{k} e^{ik \cdot (x-x')} R^{I}_{J,k}(t,t'),$$

(3.13)
where

\[ R_{J,k}^I(t, t') \equiv \sum_{\alpha=1}^{D} \{ u_{\alpha,k}^I(t) u_{\alpha,k}^* J(t') - \text{c.c.} \}. \]  (3.14)

Then, using the expressions in Eq. (3.10), we find that \( R_{J,k}^I \) is regular even in the IR limit \( k \to 0 \).

3.2. IR vanishing smooth function

In this paper, we do not consider the secular growth of the amplitude of perturbation due to the integration for a long period of time. Namely, we consider the case that \( t_i \) is set at a finite past from \( t_f \). Deferring the detailed explanation to our succeeding paper, we give a brief comment on the regularization of the secular growth in the multi-field model in §4. In this paper we concentrate on the IR divergences originating from the momentum integration.

The first part of our proof of IR regularity in the multi-field model goes in parallel with the single field case.\(^{42}\) In the single-field model, the proof of IR regularity was quite simple if we do not care about the long time integration. However, multi-field extension turns out to be nontrivial even for this restricted case. To maintain the simplicity of notation, we suppress the field index \( I \) and the labels of mode \( \alpha \) for a moment. As \( \tilde{\varphi}(x) \) is composed of \( \tilde{\varphi}_n(x) \) (\( n = 1, 2, 3, \ldots \)), we make use of the mathematical induction to show the regularity of all \( \tilde{\varphi}_n(x) \).\(^{42}\) Formally, we define \( C(\tilde{\varphi}_n)(x; p_1, \cdots, p_n) \) by expanding \( \tilde{\varphi}_n(x) \) as

\[
\tilde{\varphi}_n(x) = \left[ \prod_{j=1}^{n} \frac{1}{V_c^2} \sum_{p_j \neq 0} \frac{a_{p_j}}{p_j^{3/2}} \right] C^{(0)}[\tilde{\varphi}_n](x; p_1, \cdots, p_n)
+ a_0 \left[ \prod_{j=1}^{n-1} \frac{1}{V_c^2} \sum_{p_j \neq 0} \frac{a_{p_j}}{p_j^{3/2}} \right] C^{(1)}[\tilde{\varphi}_n](x; p_1, \cdots, p_{n-1}) + \cdots, (3.15)
\]

where we have suppressed the terms containing creation operators. \( C^{(j)} \) represents the coefficient of the term that contains \( j \)-th order product of 0-mode operators \( a_0 \). We also suppress this superscript \( (j) \) for simplicity. The above expression is the result that we obtain after conducting all the integrations over the intermediate vertexes. The momenta \( \{ p_j \} \) in the argument of \( C[\tilde{\varphi}_n] \) are those associated with the rightmost ends of the corresponding tree-shaped graph.

A key ingredient of the first part of our proof is to show that \( C[\tilde{\varphi}_n](x; p_1, \cdots, p_n) \) has the following properties,

- It is a smooth function with respect to \( x \) for \( \forall p_j \equiv |p_j| < a(t)\Lambda \), where \( \Lambda \) is an UV momentum cutoff scale.
- It vanishes when the long-wavelength limit \( p_j \to 0 \) is taken for any momentum in its arguments.

If \( C[\tilde{\varphi}_n] \) satisfies the properties mentioned above, (then we say \( C[\tilde{\varphi}_n] \) is an IR vanishing smooth function (IRVSF)), one can easily show that the \( n \)-point functions
In the continuum limit. Here, we note that the interaction picture field appears only in the projected form \( \hat{\varphi} \). One of the momentum integrations over \( p_i \) and \( p_j \) is performed to obtain an expression in the form

\[
\int \frac{d^3p_j}{(2\pi p_j)^3} C[\hat{\varphi}_{n1}](x_1; \ldots, p_j, \ldots)C[\hat{\varphi}_{n2}](x_2; \ldots, p_j, \ldots)
\]

in the continuum limit. Here, we note that \( V_c^{-\frac{3}{2}} \sum_{j=1} a_{p_j} \) should be replaced with \((2\pi)^{-3} \int d^3p_j a_{p_j} \) in the continuous limit. (See Appendix A.) The resulting momentum integration does not have IR divergences owing to the second property of \( C[\hat{\varphi}_n] \), i.e., \( \lim_{p \to 0} C[\hat{\varphi}_n](x; \ldots, p, \ldots) = 0 \).

Before we start the mathematical induction, let us note the following properties of IRVSFs:

**Lemma** If \( C(x; \{ p_j \}) \) and \( C(x; \{ q_j \}) \) are IRVSFs and there is no overlap between the list of momenta \( \{ p_j \} \) and \( \{ q_j \} \), then \( \nabla_a C_1(x; \{ p_j \}), x C_1(x; \{ p_j \}), \tilde{C}_1(x; \{ p_j \}), \Delta^{-1} C_1(x; \{ p_j \}), \hat{W}_t C_1(x; \{ p_j \}), \int dt C_1(x; \{ p_j \}) \), and \( C_1(x; \{ p_j \}) \times C_2(x; \{ q_j \}) \) are all IRVSFs.

Now, let us prove that \( C[\hat{\varphi}_n] \) is IRVSF by induction if \( C[\hat{\varphi}_1] \) is so. The \( n \)-th order perturbation is obtained by

\[
\hat{\varphi}_n = \hat{W}_t \int dt' \int d^3x' a_3(t') G_R(x, x') W'_{t'}(x') \Gamma_n(x') .
\]

\( W'_{t'}(x') \Gamma_n(x') \) is constructed from lower order perturbations \( \delta\tilde{N}_j, \tilde{\chi}_j, f_j, B_{\perp j}, \) and \( \tilde{\varphi}_j \) with \( j < n \) using the operations listed in the above Lemma. Furthermore, from Eqs. (2-28), (2-30), and (2-36), we find that \( \delta\tilde{N}_j, \tilde{\chi}_j \) and \( f_j \) are all constructed from \( \tilde{\varphi}_j \) with \( l \leq j \) by the operations listed there, too. Hence, \( C[W'_{t'} \Gamma_n] \), the expansion coefficient of \( W'_{t'}(x') \Gamma_n(x') \) analogous to \( C[\tilde{\varphi}_n] \) in Eq. (3-15), is also an IRVSF. Since the Fourier mode of the retarded Green function described using Eq. (3-14) is regular in the IR limit, its Fourier transform \( G_R(x, x') \) should also be regular. (Regularity in UV is assumed to be guaranteed by an appropriate UV renormalization.) Since the integration volume of \( x' \) is finite, the integral of a product of regular functions \( \int d^3x' a_3(t') G_R(x, x') W'_{t'}(x') \Gamma_n(x') \) should be finite, and hence, it is IRVSF. Since the operation \( \hat{W}_t \) preserves the properties of IRVSF, \( \hat{\varphi}_n = \hat{W}_t G_R(W'_{t'} \Gamma_n) \) is also found to be IRVSF.

Now, our concern is whether the first step of induction is true. Namely, we examine if \( C[\hat{\varphi}_1] \) is IRVSF. Utilizing the residual gauge degrees of freedom, we fix the local average of the adiabatic mode \( \tilde{A} = \epsilon_I \tilde{\varphi}^I \). Then, IR modes in this direction are controlled to be free from divergences, but the modes in the other directions are not. The interaction picture field appears only in the projected form \( \hat{W}'_{fj} \hat{\varphi}^j \), which can be expanded using the mode function \( u^I_{\alpha, k} \) as

\[
\hat{W}'_{fj} \hat{\varphi}^j(x) = \frac{1}{V_c} \sum_{\alpha, p} \left[ e^{ip \cdot x} G^I_{fj} - \frac{W_{t,-} - p e^I e_j}{W_{t,0} M_{pl}} a_{\alpha, p} + \{h.c.\} \right] u^I_{\alpha, p}(t) M_{pl} + \{h.c.\}. \quad (3.17)
\]
where
\[ W_{t,-p} \equiv \int d^3x e^{ip \cdot x} W_t(x), \quad (3.18) \]
and we note that \( W_{t,0} = \int d^3x W_t(x) = L_i^3 \). To make it easy to take the limit \( V_c \to \infty \), we define the Fourier mode of the window function in a different manner from those of fluctuations. (See Appendix A.) Hence, we have the coefficient for \( a_{\alpha,p} \) as
\[ C_\alpha^{(0)}[\hat{W}^I J \hat{\varphi}^I_1](x,p) = \left[ e^{ip \cdot x} \delta_J^I - \frac{W_{t,-p} e^{I e_J}}{W_{t,0}} \right] \frac{p^2 u_{\alpha,p}^J(t)}{M_{pl}}. \quad (3.19) \]
We have chosen the adiabatic mode (\( \alpha = 1 \)) so as to be tangential to the background trajectory in \( p \to 0 \) limit, i.e., \( g_1^J(t) \propto e^t \). Then, multiplying \( p^{\frac{3}{2}} u_{1,p}^J(t) \approx p^{-\frac{1}{2}} \) by the factor \( [e^{ip \cdot x} - W_{t,-p}/W_0] \), \( C_1^{(0)}[\hat{W}^I J \hat{\varphi}^I_1](x,p) \) vanishes in this limit. Therefore \( C_\alpha^{(0)}[\hat{W}^I J \hat{\varphi}^I_1](x,p) \) vanishes in the limit \( p \to 0 \). Thus, we find that \( C_\alpha^{(0)}[\hat{W}^I J \hat{\varphi}^I_1](x,p) \) is an IRVSF. However, the factor \( [e^{ip \cdot x} \delta_J^I - W_{t,-p}/W_{t,0} e^{I e_J}] \) does not suppress isocurvature fluctuations \( \tilde{S}^I \), which is pointing in the orthogonal direction to the background trajectory. When one of the basis with \( \bar{\alpha} \neq 1 \) has a nonnegative value of \( \delta_{\bar{\alpha}} \), the IR contribution of such a mode diverges and \( C_\alpha^{(0)}[\hat{W}^I J \hat{\varphi}^I_1](x,p) \) is not IRVSF. In this case, \( n \)-point functions of \( \tilde{O}(x) = \{, \tilde{A}, \tilde{S}^I \} \) actually diverge.

The case with \( p = 0 \) goes in a similar manner. The coefficient for \( a_{\alpha,0} \) of \( \hat{\varphi}^I_1 \) is given by
\[ C_\alpha^{(1)}[\hat{W}^I J \hat{\varphi}^I_1](x) = [\delta_J^I - e^{I e_J}] \frac{g_J^0(t)}{s_{\alpha} M_{pl}}, \quad (3.20) \]
where we take the limit \( V_c \to \infty \) using Eq. (3.12). This expression also vanishes for the adiabatic perturbation, but it does not for the isocurvature perturbation. Then, the contribution from the isocurvature perturbation diverges when the limit \( s_{\alpha} \to 0 \) is taken.

The above divergences arise only in the multi-field model. In the rest of this section, we discuss the regularization of this divergence. Even when \( \delta_{\bar{\alpha}} \) is negative, the following discussion is still relevant. For \( \delta_{\bar{\alpha}} < 0 \) there is no IR divergence, but the IR contribution can be large if \( |\delta_{\bar{\alpha}}| \ll 1 \). If one can show that \( n \)-point functions for \( \tilde{O}(x) \) are free from the IR divergence for \( \delta_{\bar{\alpha}} > 0 \), it also implies the absence of an enhanced IR contribution for \( \delta_{\bar{\alpha}} < 0 \).

3.3. Squeezed wave packet

For later use, we transform the mode functions \( \{u_{\alpha,p}^I e^{ip \cdot x}\} \) to other functions suitable for discussing the effect of inserting projection operator. Transformation proceeds in two steps. Deferring the detailed explanations to Appendix B, here, we just provide a brief sketch of the transformations. At the first step, the new basis mode functions \( \{v_{\alpha,p}^I\} \) for \( p \neq 0 \) are arranged so that the leading term in the long wavelength limit is cancelled. While, the IR divergent contributions are localized to a single mode with \( p = 0 \). At the second step of transformation, without changing
the mode function for \( p \neq 0 \), we introduce another mode function for \( p = 0 \) mode, which naturally defines wave packets with a finite width even in the limit \( s_\alpha \to 0 \) and \( V_c \to \infty \). In this limit, \( \tilde{\varphi}_1^I \) can be expanded as

\[
\tilde{\varphi}_1^I(x) = \sum_\alpha \left\{ \frac{\tilde{v}_{\alpha,0}^I}{M_{pl}} a_{\alpha,0} + \int_{p \neq 0} \frac{d^3p}{(2\pi)^3} \frac{\tilde{v}_{\alpha,p}^I(t)}{M_{pl}} \right\} + (\text{h.c.}) ,
\]

where

\[
\tilde{v}_{\alpha,0}^I(x) = H_f g_{\alpha,0}^I(t) + \frac{i}{H_f} \int_{p \neq 0} \frac{d^3p}{(2\pi)^3} \frac{W_p g_{\alpha,p}^I(t) e^{ip \cdot x}}{W_0 c_\alpha(p)} ,
\]

\[
\tilde{v}_{\alpha,p}^I(x) = u_{\alpha,p}^I(t) e^{ip \cdot x} - \frac{W_p g_{\alpha,0}^I(t)}{W_0 c_\alpha(p)} .
\]

Now, we introduce the coefficients \( \tilde{C}^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I] \) analogous to \( C^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I] \) for the creation and annihilation operators \( \tilde{a}_{\alpha,p}^I \) and \( \tilde{a}_{\alpha,p}^I \). Then, in the continuum limit, \( \tilde{\varphi}_n^I \) is expanded as

\[
\tilde{\varphi}_n^I(x) = \sum_{k=0}^n \prod_{l=1}^k \tilde{a}_{\alpha_{n-k+l},0}^I \prod_{j=1}^{n-k} \int \frac{d^3p_j}{(2\pi)^3} \tilde{a}_{\alpha_j,p_j}^I C^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I](x; p_1, \cdots, p_{n-k}) + \cdots .
\]

Since the induction with respect to \( n \) proceeds as before, we can say that \( \tilde{C}^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I] \) is IRVSF if the coefficients of the first-order variables \( \hat{W}_j^I \tilde{\varphi}_1^I(x) \) are IRVSFs.

In the same way as Eq. (3.19), the coefficient for \( \tilde{a}_{\alpha,0}^I \) is obtained as

\[
C^{(0)}_{\alpha}[\hat{W}_j^I \tilde{\varphi}_1^I](x, p) = \left[ e^{ip \cdot x} \delta^I_{j,1} - \frac{W_{l_0-p} e^{I_{l_0}}}{W_{l_0}} \right] \frac{p^2 \tilde{v}_{\alpha,0}^I(t)}{M_{pl}} .
\]

From Eq. (3.23), \( p^{3/2} \tilde{v}_{\alpha,p}^I \) vanishes in the limit \( p \to 0 \). The adiabatic component also vanishes owing to the projection \( \hat{W}_j^I \). Hence, these coefficients are IRVSFs. While, the coefficient for \( \tilde{a}_{\alpha,0}^I \) is given by

\[
C^{(1)}_{\alpha}[\hat{W}_j^I \tilde{\varphi}_1^I](x) = \left[ \delta^I_{j,1} - e^{I_{l_0}} e_{j} \right] \frac{\tilde{v}_{\alpha,0}^I(t)}{M_{pl}} ,
\]

which is finite from Eq. (3.22) in contrast to the previous case in Eq. (3.20). Therefore, it is also IRVSF. As we find that all the coefficients of \( \hat{W}_j^I \tilde{\varphi}_1^I \) are IRVSFs or just a regular function independent of \( p \), all higher order coefficients \( \tilde{C}^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I] \) are proven to be IRVSFs by induction.

We should emphasize that even if the coefficients \( C^{(k)}_{\{\alpha\}}[\tilde{\varphi}_n^I] \) are IRVSFs, this does not imply the regularity of \( n \)-point functions of \( \tilde{O} \) for the initial adiabatic vacuum state \( |0\rangle \). In the following discussion, we use an expression for the adiabatic vacuum.
state $|0\rangle_a$ expanded in terms of the coherent states $|\beta\rangle_{\tilde{a}}$ associated with $\tilde{a}_{\alpha,0}$. The coherent state satisfies

$$\tilde{a}_{\alpha,p}|\beta\rangle_{\tilde{a}} = \beta_{\tilde{a}}|\beta\rangle_{\tilde{a}}, \quad \tilde{a}_{1,p}|\beta\rangle_{\tilde{a}} = 0.$$  

As shown in Appendix B, the original vacuum state $|0\rangle_a$ can be expressed as

$$|0\rangle_a = \prod_{\tilde{a}=2} D \int_{-\infty}^{\infty} d\beta_{\tilde{a}} E_{\tilde{a}}(\beta_{\tilde{a}}) |\beta\rangle_{\tilde{a}},$$  

(3.27)

where the coefficient $E_{\tilde{a}}(\beta_{\tilde{a}})$ approaches

$$E_{\tilde{a}}(\beta_{\tilde{a}}) \to \sqrt{\frac{s_{\tilde{a}} H_f}{\pi}} e^{-(s_{\tilde{a}} H_f \beta_{\tilde{a}})^2},$$  

(3.28)

in the limit of $s_{\tilde{a}} \to 0$. The nonvanishing support for $E_{\tilde{a}}$ extends to infinitely large $|\beta_{\tilde{a}}|$ in this limit. This is a consequence of the fact that the wave function in the adiabatic vacuum is highly squeezed in the direction corresponding to $\tilde{a}_{\alpha,0}$.

Using Eq. (3.27), we can expand the $n$-point function of the “observables” $\tilde{O}(t_f, x)$ as

$$\langle \mathcal{P}\tilde{O}(t_f, x_1)\tilde{O}(t_f, x_2) \cdots \tilde{O}(t_f, x_n) \rangle$$

$$= \prod_{\tilde{a}=2} D \int_{-\infty}^{\infty} d\beta_{\tilde{a}} \int_{-\infty}^{\infty} d\gamma_{\tilde{a}} E_{\tilde{a}}(\beta_{\tilde{a}}) E_{\tilde{a}}(\gamma_{\tilde{a}}) \tilde{a}_{\beta} \mathcal{P} \tilde{O}(t_f, x_1) \cdots \tilde{O}(t_f, x_n) |\gamma\rangle_{\tilde{a}}^{\text{conn}},$$  

(3.29)

where, to clarify that we should sum up only the connected graphs, we have added the suffix “conn”.

One remark is in order in computing the expectation value of the product of $\{\tilde{a}_{\alpha,p}\}$ and $\{\tilde{a}_{\alpha,p}^\dagger\}$. Basically, pairs between $\tilde{a}_{\alpha,p}$ and $\tilde{a}_{\alpha',p'}^\dagger$ are replaced with $\delta(p - p')\delta_{\alpha,\alpha'}$ except for the case with $p = 0$. After the replacement, only the operators $\{\tilde{a}_{\alpha,0}\}$ and $\{\tilde{a}_{\alpha,0}^\dagger\}$ are left on the right-hand side of Eq. (3.29). The expectation value of the product of these operators becomes the summation of the commutators and their normal ordered products. Since the operators $\tilde{a}_{\alpha,0}$ and $\tilde{a}_{\alpha,0}^\dagger$ are not annihilated by the coherent state, the expectation values of the normal ordered products composed of these zero-mode operators are nonvanishing. The annihilation operators $\tilde{a}_{\alpha,0}$ in the normal ordered parts acting on the coherent state $|\gamma\rangle_{\tilde{a}}$ produces the factor $\gamma_{\alpha}$, while the creation operator $\tilde{a}_{\alpha,0}^\dagger$ acting on $\tilde{a}_{\beta}\rangle$ produces $\beta_{\alpha}$. In other words, $\tilde{a}_{\alpha,0}$ and $\tilde{a}_{\alpha,0}^\dagger$ are replaced either with a commutator by making a pair or with $\gamma_{\alpha}$ and $\beta_{\alpha}$, respectively. These two exclusive possibilities can be concisely expressed by the replacement

$$\tilde{a}_{\alpha,0} \to \gamma_{\alpha} + \tilde{a}_{\alpha,0}^{(q)}, \quad \tilde{a}_{\alpha,0}^\dagger \to \beta_{\alpha} + \tilde{a}_{\alpha,0}^{(q)^\dagger},$$  

(3.30)

where $\tilde{a}_{\alpha,0}^{(q)}$ and $\tilde{a}_{\alpha,0}^{(q)^\dagger}$ satisfy the same commutation relation as $\tilde{a}_{\alpha,0}$ and $\tilde{a}_{\alpha,0}^\dagger$, and they annihilate the coherent state $|\gamma\rangle_{\tilde{a}}$ and $\tilde{a}_{\beta}\rangle$, respectively.
Now, it will be obvious that the $n$-point function evaluated for the coherent states,
\[ \hat{a} (\beta | \tilde{O}(t_f, x_1) \cdots \tilde{O}(t_f, x_n) | \gamma)_{\tilde{a}}^{\text{conn}} \]
is finite. To show its regularity, the insertion of the projection operator $\mathcal{P}$ is unnecessary. We were focusing on $\bar{\hat{a}}(\beta)$ to be IRV SFs in the same manner. The effect of the coherent state is taken care of using the relation
\[ W_{t, p} \equiv \frac{W_{t, -p}}{W_{t, 0}} = \left( \frac{W_{p, -p} W_{t, 0}}{W_{t, -p} W_{t, 0}} \right) \]
which is sourced by $\bar{\hat{a}}(\beta)$ \[ e \int d^{3}p \left( \frac{W_{p} W_{p, -p}}{W_{t, 0} W_{t, 0}} \right) d_{\alpha, p} = (\text{c.c.}) \]
We focus on the coefficient of $\bar{\hat{a}}(\beta)$ to be IRV SFs in the same manner. The effect of the coherent state is taken care of using the relation
\[ W_{t, p} = W_{t, -p} \]
which is derived from the reality condition of $W_t(x)$, the second term on the right-hand side vanishes. Contraction with $\epsilon_{I}$ also erases the first term. Since the source term of the equation for $B_{-11}^{I}$ does not contain $(\bar{a}_{\alpha, 0} + a^{\dagger}_{\alpha, 0})$, $B_{-11}^{I}$ does not, either.

3.4. Role of projection operator

Without the projection operator $\mathcal{P}$, the $n$-point functions of $\tilde{O}(x)$ for the adiabatic vacuum diverge, although the expectation values for the coherent states were proven to be finite in the preceding subsection. The divergences for the adiabatic vacuum appear in the integration over $\beta$ and $\gamma$. The limiting behavior of $E_{\alpha}(\beta)$ for $s_{\alpha} \to 0$ given in Eq. (3.28) tells that this factor does not restrict the effective range of these integrations in the limit $s_{\alpha} \to 0$. Therefore, integrating over $\beta$ and $\gamma$ without the insertion of $\mathcal{P}$, the $n$-point function for the adiabatic vacuum diverges. This result is as expected since the basis transformation does not change the final result for the $n$-point function.

To remedy these divergences, we need the insertion of the projection operator. The projection operator $\mathcal{P}$ takes care of the effect of quantum decoherence, removing the contamination from the other parallel worlds. We will see that the insertion of $\mathcal{P}$ makes the effective range of integration for $\beta$ and $\gamma$ finite.

In the same way as $\tilde{O}(t_f, x)$, we expand $\mathcal{P}$, which is the functional of $\{\tilde{S}^{\alpha}(t_f)\}$, in terms of $\{\bar{\alpha}_{\alpha, p}\}$ and $\{\bar{a}^{\dagger}_{\alpha, p}\}$. Expanding $\tilde{S}^{\alpha}(t_f)$ as
\[ \tilde{S}^{\alpha}(t_f) = S_{1}^{\alpha}(t_f) + \frac{1}{2} S_{2}^{\alpha}(t_f) + \cdots, \] (3.31)
we focus on the coefficient of $\bar{\alpha}_{\alpha, 0} + \bar{a}^{\dagger}_{\alpha, 0}$ in the leading term $S_{1}^{\alpha}(t_f) \equiv e_{I}^{\alpha}(\hat{W}_{t, \bar{\alpha}}^{I} + B_{-11}^{I})$. We decompose the terms that contain $\bar{\alpha}_{\alpha, 0}$ and $\bar{a}^{\dagger}_{\alpha, 0}$ into two pieces proportional to $(\bar{a}_{\alpha, 0} + \bar{a}^{\dagger}_{\alpha, 0})$ and $(\bar{a}_{\alpha, 0} - \bar{a}^{\dagger}_{\alpha, 0})$. Then, we can see that the former does not contain a contribution from $B_{-11}^{I}$ as follows. $B_{-11}^{I}$ is determined by solving Eq. (2.36), which is sourced by $e_{I} \hat{W}_{t, \bar{\alpha}}^{I}$ on the right-hand side. The coefficient of $\bar{a}_{\alpha, 0} + \bar{a}^{\dagger}_{\alpha, 0}$ in $\hat{W}_{t, \bar{\alpha}}^{I}$ is given by
\[ \frac{\hat{W}_{t}}{2} (\bar{a}_{\alpha, 0} + \bar{a}^{\dagger}_{\alpha, 0}) = H_{f} g_{\alpha, 0} + \frac{i}{2 H_{f}} \int_{p \neq 0} \frac{d^{3}p}{(2 \pi)^{3}} \left( \frac{W_{p} W_{t, -p}}{W_{t, 0} W_{t, 0}} d_{\alpha, p} - (\text{c.c.}) \right). \] (3.32)
Using the replacement (3.30), we divide \((\tilde{a}_{\alpha,0} + \tilde{a}_{\alpha,0}^\dagger)\) into \((\beta_\alpha + \gamma_\alpha)\) and \((\tilde{a}_{\alpha,0}^{(q)} + \tilde{a}_{\alpha,0}^{(q)\dagger})\). Separating the coefficient of \((\beta_\alpha + \gamma_\alpha)\), \(\mathbf{S}_1^{\tilde{a}}(t_f)\) is expressed as

\[
\mathbf{S}_1^{\tilde{a}}(t_f) = \Delta^{\tilde{a}}_{\beta}(\beta^{\tilde{a}} + \gamma^{\tilde{a}}) + \delta\mathbf{S}_1^{\tilde{a}}(t_f),
\]

where

\[
\Delta^{\tilde{a}}_{\beta} \equiv \frac{H_f}{M_{pl}} e^{\tilde{a}}_f g_{\alpha,0}(t_f), \tag{3.34}
\]

and the summation over repeated index \(\tilde{\beta}\) is understood.

Inserting the above expression for \(\mathbf{S}_1^{\tilde{a}}(t_f)\) into \(\mathcal{P}\) given in (2.39), the observable \(n\)-point function (3.29) is recast into

\[
\langle \mathcal{P} \tilde{O}(t_f, x_1)\tilde{O}(t_f, x_2) \cdots \tilde{O}(t_f, x_n) \rangle \\
= \prod_{\epsilon=2}^{D} \left[ \int_{-\infty}^{\infty} d\beta^\epsilon \int_{-\infty}^{\infty} d\gamma^\epsilon \ E(\beta^\epsilon) E(\gamma^\epsilon) \right] \exp \left[ -\frac{C_{\alpha\tilde{\alpha}}^{-1} \Delta^{\tilde{a}}_{\gamma}(\gamma^{\tilde{a}} + \beta^{\tilde{a}}) \Delta^{\tilde{a}}_{\beta}(\gamma^{\tilde{a}} + \beta^{\tilde{a}})}{2} \right] \\
\times \tilde{a}(\beta) \exp \left[ -\frac{C_{\alpha\tilde{\alpha}}^{-1} \left( 2\Delta^{\tilde{a}}_{\gamma}(\gamma^{\tilde{a}} + \beta^{\tilde{a}}) \delta\mathbf{S}^{\tilde{a}} + \delta\mathbf{S}^{\tilde{a}} \delta\mathbf{S}^{\tilde{a}} \right)}{2} \right] \\
\times \tilde{O}(t_f, x_1)\tilde{O}(t_f, x_2) \cdots \tilde{O}(t_f, x_n) |\gamma^{\tilde{a}}_{\text{conn}} \rangle, \tag{3.35}
\]

where

\[
\delta\mathbf{S}^{\tilde{a}} \equiv \delta\mathbf{S}_1^{\tilde{a}}(t_f) + \frac{1}{2} \mathbf{S}_2^{\tilde{a}}(t_f) + \cdots.
\]

Owing to the first exponential factor, the contribution from the integration region with \(|\Delta^{\tilde{a}}_{\beta}(\gamma^{\tilde{a}} + \beta^{\tilde{a}})| \gg \sigma\) is exponentially suppressed. Since the inner product between the coherent states gives

\[
\tilde{a}(\beta) |\gamma^{\tilde{a}}_{\text{conn}} \rangle = \prod_{\alpha=2}^{D} \exp \left[ -\frac{(\beta^{\tilde{a}} - \gamma^{\tilde{a}})^2}{2} \right],
\]

(see Eq. (B.13)), the contribution from the region with \(|\beta^{\tilde{a}} - \gamma^{\tilde{a}}| \gg 1\) is also exponentially suppressed. The directions of these two suppressions are orthogonal, and hence the effective integration area is restricted to a finite region

\[
|\beta^{\tilde{a}}|, \ |\gamma^{\tilde{a}}| \lesssim \max \left( 1, \frac{\sigma}{\Delta} \right), \tag{3.36}
\]

where \(\Delta\) and \(\sigma\) are the typical amplitudes of the eigenvalues of \(\Delta^{\tilde{a}}_{\beta}\) and the square of the eigenvalues of \(C^{\tilde{a}}_{\beta}\), respectively. Our discussion up to here ensures the finiteness of the effective range of the Gaussian integrations over \(\beta^{\tilde{a}}\) and \(\gamma^{\tilde{a}}\). This proves the IR regularity of the \(n\)-point function of the local fluctuation \(\{\tilde{O}(x)\}\) with the projection \(\mathcal{P}\) at each order of loop expansion even if the initial state is set to the adiabatic vacuum state.
Now, the remaining task is to examine if the perturbative expansion is still reliable after all the changes that we made. For a sufficiently wide projection operator $P$, the expected amplitude of $|\tilde{\varphi}|$ is dominated by the contribution from $\beta\bar{\alpha}$ and $\gamma\bar{\alpha}$, which is $O(\sigma)$. Hence, the validity of the perturbative expansion requires

$$\sigma \ll 1.$$  (3.37)

The next question is whether one can safely expand the second exponential factor in Eq. (3.35),

$$e^{-\frac{c^{-1}}{\alpha\beta}\left\{2\Delta(\gamma+\beta)\delta S^\beta + 2\delta S^\alpha\delta S^\beta\right\}},$$

coming from $P$. Using the conditions (3.36), we find that this expansion converges if

$$|\delta S^\alpha| \ll \sigma$$

is satisfied. On the other hand, the amplitude of $|\delta S^\alpha|$ is estimated to be given by the linear order contribution $\Delta$. Therefore the necessary condition is

$$\Delta \ll \sigma.$$  (3.38)

Since $\Delta \ll 1$, we can choose $\sigma$ such that satisfies (3.37) and (3.38) simultaneously.

§4. Conclusion

In this paper, we have proposed one solution to the IR divergence problem in multi-field models. We discuss $n$-point functions for the local perturbed variables, $\tilde{O}(x)$, defined by the deviations from the local average values with an additional gauge condition that fixes one of the residual gauge degrees of freedom remaining in the usual flat gauge. Even if we consider these local perturbative variables, when plural fields have a scale-invariant or red spectrum, we encounter IR divergences. This is because the effects of quantum decoherence are not taken into account yet; the $n$-point function for $\tilde{O}(x)$ is affected by the contaminations from other uncorrelated worlds. To remove the contaminations, we have inserted the projection operator $P$, which projects the final quantum state to a wave packet with a sharp peak in the space of the local average values of the fields, $\{\hat{W}_I S_I\}$. Here, we give an intuitive way to understand how the insertion of the projection operator regularizes the IR corrections. When plural fields contribute to IR divergences, the wave function corresponding to the initial adiabatic vacuum is highly squeezed in the corresponding directions in the space of $\{\hat{W}_I S_I\}$. However, only a part of the squeezed wave function does contribute to a decohered wave packet, which represents our world. Introducing the projection operator, we have taken into account the restriction of the wave function to the nonvanishing support of the decohered wave packet. This restriction is recast into the exponential factor in Eq. (3.35), by which the nonvanishing support of $(\beta, \gamma)$ becomes a finite region. This assures the regularity of the observable $n$-point functions.
The “observable” $n$-point function (2.42) depends on the parameter $\sigma$ that we introduced to incorporate the decoherence effect without discussing the detailed process. It also depends on the size of the observable region. These dependences may disappear when we compute the actual observables, taking into account the decoherence process appropriately. Definitely, to predict the accurate value of observables, further study is necessary. We leave this issue for our future work.

Here, fixing the temporal coordinate on each vertex, we showed the regularity of the integration over the spatial coordinates. Hence, we cannot deny the possibility that the temporal integral makes the $n$-point function diverge when we send the initial time $t_i$ to a distant past. In single-field models, we showed, by considering the local fluctuations, that we can suppress the secular growth that appears from the temporal integration, unless a very high order perturbation is considered. We expect that in the multi-field model, a similar suppression appears from the inserted projection operator. The reason is as follows. If the nonvanishing contributions from the distant past make the amplitude of $\tilde{O}(t_f, \mathbf{x})$ significantly large, they must increase also the amplitude of the local average $\{\tilde{W}_{t_f}S_I\}$ to a large extent. However, the decohered wave packet that we pick up has the bounded average values of $\{\tilde{W}_{t_f}S_I\}$. Therefore, we expect that as long as we compute the observable $n$-point function evaluated for a decohered wave packet, the contributions from the distant past are suppressed and the temporal integration converges. We also defer the examination of our optimistic expectation to a future work.

In general, the IR divergences originate from massless (or quasi-massless) fields with nonlinear interactions in the (quasi-) de Sitter universe.\textsuperscript{5)–7), 59)–62}) In Refs. 5)–7), by using the prescription with an IR cutoff, the logarithmic amplification is discussed for a massless test scalar field $\phi$ with a quadratic interaction term. We can discuss the regularization of the IR corrections from a test field in a similar manner to the entropy perturbations discussed in this paper. In Refs. 59) and 60), the effects of IR gravitons that grow logarithmically are argued to screen the cosmological constant. However, the Hubble parameter defined in Refs. 59) and 60) is gauge-dependent and in another definition, we do not encounter the screening.\textsuperscript{61)} Thus, this problem is still controversial.\textsuperscript{61), 62)} We would also like to apply our prescription to these issues in our future work.

Acknowledgements

YU would like to thank Kei-ichi Maeda for his continuous encouragement. YU is supported by JSPS under Contact No. 19-720. TT is supported by Monbukagakusho Grants-in-Aid for Scientific Research Nos. 19540285 and 21244033. This work is also supported in part by the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. The authors thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the GCOE/YITP workshop YITP-W-09-01 on “Nonlinear cosmological perturbations” were useful for completing this work.
Appendix A

Correspondence in the Continuum Limit

In this paper, to tame the divergence in the IR limit, we begin with the model with a finite volume of the universe $V_c \equiv L_c^3$. After we define appropriate observables, we take the infinite volume limit $V_c \to \infty$. At this step, the discrete label of the comoving wave number changes to the continuum one. We use two different notations for the Fourier components between the perturbation variable like $\varphi^I(x)$ and the window function $W_{t,f}(x)$. Because of that, these two classes of quantities should be treated differently when we take the continuum limit. Hereafter, we use variables $Q$ and $W$ to represent the variables of the first and second classes, respectively. We abbreviate the suffixes, $I$ and $\alpha$.

A.1. Quantized variables

When we consider the first class of variables $Q(x)$, which is to be quantized like $\varphi(x)$, the corresponding mode functions $\{q_k\}$ play the more important role than the Fourier components of $Q(x)$ themselves. Therefore, we adopt a convention so that $q_k$ remains unchanged in the continuum limit $V_c \to \infty$.

When we quantize $Q(x)$, we expand the Fourier mode $Q_k$ in terms of the creation and annihilation operator like

$$ Q_k = (2\pi)^3 \{a_k q_k + a_k^\dagger q_k^*\}. \quad (A.1) $$

We require that the mode functions $q_k$ remain unchanged in the continuum limit. On the other hand, the commutation relation for the creation and annihilation operator

$$ \left[ a_k^{(d)} , a_{k'}^{(d)\dagger} \right] = \delta_{k,k'} $$

changes in the continuum limit to

$$ \left[ a_k^{(c)} , a_{k'}^{(c)\dagger} \right] = \delta^3(k - k'). $$

From the above commutation relation, trivial relations

$$ 1 = \sum_{k'} \left[ a_k^{(d)} , a_{k'}^{(d)\dagger} \right], \quad 1 = \int d^3k' \left[ a_k^{(c)} , a_{k'}^{(c)\dagger} \right], $$

follow. Since the wave number $k$ is discrete like $k_i = \Delta k_{ji} \equiv 2\pi j_i/L_c$ with $j_i$ being an integer, $\sum_{k'}$ is to be replaced with $(2\pi)^{-3}V_c \int d^3k$ in the continuum limit. This requires the correspondence like

$$ a_k^{(d)} \leftrightarrow \bar{V}_c^{-\frac{1}{2}} a_k^{(c)} , $$

where $\bar{V}_c \equiv V_c/(2\pi)^3$ and hence we find

$$ Q_k^{(d)} \leftrightarrow \bar{V}_c^{-\frac{1}{2}} Q_k^{(c)}. \quad (A.2) $$
To realize the above correspondence, we define the Fourier components \( Q^{(d)}_k \) as
\[
Q^{(d)}_k = \frac{1}{V^2} \int_{V} d^3x Q(x)e^{-ik \cdot x}.
\] (A.3)

Then the definition of the Fourier components in the continuum limit becomes a normal one:
\[
Q^{(c)}_k = \int d^3x Q(x)e^{-ik \cdot x}.
\] (A.4)

The inverse transform is given by
\[
Q(x) = \frac{1}{V^2 (2\pi)^2} \sum_k Q^{(d)}_k e^{i k \cdot x},
\] (A.5)

and its continuum limit consistently recovers
\[
Q(x) = \frac{1}{(2\pi)^3} \sum_k (\Delta k)^3 V^2 \frac{1}{2} Q^{(d)}_k e^{i k \cdot x} \rightarrow \frac{1}{(2\pi)^3} \int d^3k Q^{(c)}_k e^{i k \cdot x}.
\] (A.6)

In §B.1, we treat the \( k = 0 \) mode separately. Even in the limit \( V \rightarrow \infty \), this mode is treated as a discrete spectrum. Hence, the corresponding commutators keep the normalization in terms of Kronecker \( \delta \), and we have
\[
\left[ a^{(c)}_{\alpha,0} , a^{(c) \dagger}_{\alpha,0} \right] = 1.
\]

This means that the correspondence of the creation and annihilation operators should be like \( a^{(c)}_{\alpha,0} \leftrightarrow a^{(d)}_{\alpha,0} \). Taking into account the exception for the \( k = 0 \) mode, Eq. (A.6) is to be modified to
\[
Q^{(c)}_{\alpha,0} \rightarrow V^\frac{1}{2} Q^{(d)}_{\alpha,0}.
\]

Correspondingly, we have
\[
q^{(c)}_{\alpha,0} \rightarrow V^\frac{1}{2} q^{(d)}_{\alpha,0}.
\]

Finally, we mention the normalization conditions of mode functions. The normalization conditions for \( q^{(d)}_k \) given in Eq. (3.3) are understood as
\[
(q^{(d)}_k , q^{(d)}_{k'}) = V\delta_{k,k'} = \int d^3x e^{i(k-k') \cdot x}.
\] (A.8)
Hence, in the limit \( V_c \to \infty \), the right-hand side is to be understood as \((2\pi)^3 \delta^3(k-k')\). Therefore, the normalization conditions of mode functions in the continuum limit should be

\[
(q_k^{(c)}, q_{k'}^{(c)}) = (2\pi)^3 \delta^3(k-k') .
\] (A.9)

To summarize, the correspondence between the discrete Fourier components and the Fourier components in the continuum limit is given by

\[
Q_k^{(c)} \leftrightarrow \tilde{V}_c^{\frac{1}{2}} Q_k^{(d)}, \quad q_k^{(c)} \leftrightarrow q_k^{(d)}, \quad a_k^{(c)} \leftrightarrow \tilde{V}_c^{\frac{1}{2}} a_k^{(d)},
\] (A.10)

for all modes \( k \) with \( \alpha \neq \alpha \) and \( k \neq 0 \) with \( \alpha = \alpha \) and

\[
Q_{\alpha,0}^{(c)} \leftrightarrow V_c^{-\frac{1}{2}} Q_{\alpha,0}^{(d)}, \quad q_{\alpha,0}^{(c)} \leftrightarrow V_c^{-\frac{1}{2}} q_{\alpha,0}^{(d)}, \quad a_{\alpha,0}^{(c)} \leftrightarrow a_{\alpha,0}^{(d)}.
\] (A.11)

A.2. Unquantized variables

The quantities of the second class, \( W(x) \), are not supposed to be quantized. In this case, it is more convenient to consider the Fourier components \( W_k \) that remain unchanged in the continuum limit:

\[
W_k^{(c)} \leftrightarrow W_k^{(d)}.
\]

For this purpose, we simply define the Fourier components in the usual manner as

\[
W_k^{(d)} = \int_{V_c} d^3 x W(x) e^{-ik \cdot x}.
\] (A.12)

Then, the inverse transform is given by

\[
W(x) = \frac{1}{V_c} \sum_k W_k^{(d)} e^{ik \cdot x},
\] (A.13)

and its continuum limit \( V_c \to \infty \) becomes

\[
W(x) = \frac{1}{(2\pi)^3} \sum_k (\Delta k)^3 W_k^{(d)} e^{ik \cdot x} \to \frac{1}{(2\pi)^3} \int d^3 k W_k^{(c)} e^{ik \cdot x},
\] (A.14)

as is expected.

Appendix B

Bogoliubov Transformation

B.1. Another set of mode functions

In §3.3, we transformed the mode functions to more suitable ones to understand the role of the projection operator. In this subsection, we give the detailed explanations for the two transformations. On the first transformation, the new basis
mode functions \(\{v^I_{\alpha,p}\}\) are constructed so that the leading amplitude in the long-wavelength limit is canceled for \(p \neq 0\). While, the IR divergent contributions are localized to a single mode with \(p = 0\). Hence, when we expand \(\tilde{\varphi}^I_n\) in terms of the creation and annihilation operators associated with the new set of mode functions, the expansion coefficients \(C[\tilde{\varphi}^I_n]\) are IRVSFs, except for the case with \(p_j = 0\). Such transformation can be achieved by taking

\[
v^I_{\alpha,0}(x) \equiv \frac{1}{\mathcal{N}_\alpha}\left\{ u^I_{\alpha,0}(t) + \sum_{p \neq 0} \frac{W_{-p}^* c^*_\alpha(0)}{W_{0}^* c^*_\alpha(p)} u^I_{\alpha,p}(t) e^{ipx} \right\},
\]

(B.1)

\[
v^I_{\alpha,p}(x) \equiv u^I_{\alpha,p}(t)e^{ipx} - \frac{W_{-p} c^*_\alpha(0)}{W_0 c^*_\alpha(p)} u^I_{\alpha,0}(t),
\]

(B.2)

where we denote \(W_{t,p}\) by \(W_p\). The normalization constant \(\mathcal{N}_\alpha\) is chosen as

\[
\mathcal{N}^2_\alpha = \sum_p \left| \frac{W_{-p} c^*_\alpha(0)}{W_0 c^*_\alpha(p)} \right|^2 = 1 + O(s_\alpha),
\]

(B.3)

where the second equality immediately follows from the observation that only the term with \(p = 0\) remains in the limit \(s_\alpha \to 0\).

The orthonormal relation for \(\{v^I_{\alpha,k}\}\) is given by

\[
(v_{\alpha,0},v_{\beta,0}) = \delta_{\alpha\beta},
\]

\[
(v_{\alpha,0},v_{\beta,p}) = 0,
\]

\[
(v_{\alpha,p},v_{\beta,p'}) = \delta_{\alpha\beta} \left\{ \delta_{p,p'} + \frac{W_{-p} W_{-p'} c^*_\alpha(0)^2}{|W_0|^2 c^*_\alpha(p)c^*_\alpha(p')} \right\} = \mathcal{V}_\alpha \delta_{\alpha\beta} \left\{ \delta_{p,p'} + O(s^2_\alpha/V_c) \right\},
\]

(B.4)

\[
(v_{\alpha,0},v_{\beta,0}^*) = (v_{\alpha,0},v_{\beta,p}^*) = (v_{\alpha,p},v_{\beta,0}^*) = 0.
\]

The mode functions \(v^I_{\alpha,p \neq 0}\) are not mutually orthogonal before we take the limit \(s_{\alpha} \to 0\) in Eq. (B.4). After taking the limit \(s_{\alpha} \to 0\), however, \(\{v^I_{\alpha,p}\}\) becomes a set of orthonormal bases. We denote the creation and annihilation operators associated with \(\{v^I_{\alpha,p}\}\) by \(b^I_{\alpha,p}\) and \(a^I_{\alpha,p}\), respectively.

After this transformation, the IR divergent contribution is confined to \(v^I_{\alpha,0}(x)\). Indeed, the problematic term in \(u^I_{\alpha,p}\) which scales as \(p^{-(3+\delta_\alpha)/2}\) in the IR limit is cancelled by the second term in Eq. (B.2). Then, even when \(\delta_\alpha > 0\), as long as \(\delta_\alpha < 1\), we find that \(p^{\delta_\alpha/2} v^I_{\alpha,p \neq 0}\) vanishes in the limit \(p \to 0\). This is the necessary and sufficient condition for the coefficients of the creation and annihilation operators to be IRVSFs. In contrast, \(v^I_{\alpha,0}\) has a diverging amplitude in the limit \(s_{\alpha} \to 0\) and \(V_c \to \infty\). Notice that this transformation does not mix the positive frequency modes with the negative frequency ones. Therefore, in the limit \(s_{\alpha} \to 0\), the vacuum annihilated by \(b_{\alpha,p}\) is identical to the vacuum annihilated by \(a^I_{\alpha,p}\).

Now, we move on to the second step of the transformation. Here, we introduce another mode function for \(p = 0\) mode, which naturally defines wave packets with a finite width even in the limit \(s_{\alpha} \to 0\) and \(V_c \to \infty\). In Fig. 1, we depicted the changes in the wave packets under these two Bogoliubov transformations. For brevity, we
have suppressed the indices $I$ and $\alpha$. We denote a new set of mode functions by $\{\bar{v}_{\alpha,p}^I\}$, which is defined as

\begin{align}
\bar{v}_{\alpha,0}^I(x) &= \cosh r_\alpha \, v_{\alpha,0}^I(x) - \sinh r_\alpha \, v_{\alpha,0}^I(x), \\
\bar{v}_{\alpha,k}^I(x) &= v_{\alpha,k}^I(x),
\end{align}

with the squeeze parameter $r_\alpha$ such that

\[ e^{r_\alpha} = \frac{1}{N_\alpha A_\alpha s_\alpha}, \]

where $A_\alpha$ is a real constant. The mode functions for $p \neq 0$ are unchanged:

\[ \bar{v}_{\alpha,p}^I = v_{\alpha,p}^I, \quad \text{for } p \neq 0. \]

We denote the creation and annihilation operators associated with $\{\bar{v}_{\alpha,p}^I\}$ by $\bar{a}_{\alpha,p}^\dagger$ and $\bar{a}_{\alpha,p}$.

In the limit $s_\alpha \to 0$, we obtain

\[ \bar{v}_{\alpha,0}^I(x) = V_c^2 \left[ A_\alpha g_{\alpha,0}^I(t) + \frac{i}{A_\alpha} \frac{1}{V_c} d_{\alpha,0}^I(t) + \frac{i}{A_\alpha} \frac{1}{V_c} \sum_{p \neq 0} \frac{W_p}{W_0} d_{\alpha,p}^I(t)e^{ip\cdot x} \right]. \]

The first and second terms in Eq. (B-9) originate from the homogeneous mode, $u_{\alpha,0}^I$, and its complex conjugate. The terms that diverge in the limit $s_\alpha \to 0$ are common in $u_{\alpha,0}^I$ and $u_{\alpha,0}^{I*}$, which cancel with each other in the definition of $\bar{v}_{\alpha,0}^I$ in Eq. (B-5).
The third term in Eq. (B.9) originates from the inhomogeneous modes, $u^I_{\alpha,p \neq 0}$. The contribution from $u^I_{\alpha,p \neq 0}$ is also large in $v^I_{\alpha,0}$ in the IR limit, but it is not in $\bar{v}^I_{\alpha,0}$. The leading terms of $d^I_{\alpha,p}(t)$ in $p \to 0$ limit behaves as

$$d^I_{\alpha,p}(t) \approx d^I_{\alpha}(t).$$

Hence, the momentum integral $\int d^3p W_p d^I_{\alpha,p}(t) e^{i p \cdot x}$, which arises in the continuum limit, remains finite for $x < \infty$.

Here, we choose the unspecified constant $A_\alpha$ such that the mode functions $\bar{v}^I_{\alpha,0}$ take the minimum amplitude on average. To do so, we compare the amplitude of the first term with that of the last term in Eq. (B.9). Near the final time $t_f$ the last term is negligibly small, but it is larger in the past. Assuming $\dot{\phi}$ and $H$ are almost constant, the relative amplitude between these two terms is roughly estimated as $|A_\alpha^{-2} \int_0^\infty d^3 p d^I_{\alpha,p}| \approx H^2/A_\alpha^2$, where we used $d^I_{\alpha,p} \approx 1/(6H a^3) \delta^I_{\alpha}$. Hence, we find that it is appropriate to choose $A_\alpha = H$ to minimize the amplitude of the mode functions. (We choose $A_1 = 1/s_1 = \infty$ so that the mode function for the adiabatic mode is unchanged: $\bar{v}^I_{1,0} = v^I_{1,0}$.)

Now, sending $V_c \to \infty$, we take the continuum limit. We have summarized the correspondence between the discrete description and the continuous one in Appendix A. The overall factor $V_c^{1/2}$ in Eq. (B.9) is cancelled by $V_c^{-1/2}$ in Eq. (3.2). In this limit, $\sum_k$ is replaced with $(2\pi)^{-3} V_c \int d^3k$. Since the expression (B.9) has a factor $1/V_c$ in front of the summation $\sum_k$, we find that $\bar{v}^I_{\alpha,0}$ is free from the explicit divergence in the limit $V_c \to \infty$. In the continuum limit, $\bar{v}^I_1$ can be expanded as

$$\bar{v}^I_1(x) = \sum_\alpha \left\{ \frac{\bar{v}^I_{\alpha,0}}{M_{pl}} \bar{a}_{\alpha,0} + \int_{p \neq 0} \frac{d^3p}{(2\pi)^3} \frac{\bar{v}^I_{\alpha,p}(t)}{M_{pl}} \bar{a}_{\alpha,p} \right\} + \text{(h.c.)}, \quad (B.10)$$

where

$$\bar{v}^I_{\alpha,0}(x) = H_f g^I_{\alpha,0}(t) + \frac{i}{H_f} \int_{p \neq 0} \frac{d^3p}{(2\pi)^3} \frac{W_p}{W_0} q^I_{\alpha,p}(t) e^{i p \cdot x}, \quad (B.11)$$

$$\bar{v}^I_{\alpha,p}(x) = u^I_{\alpha,p}(t) e^{i p \cdot x} - \frac{W_p g^I_{\alpha,0}(t)}{W_0} \frac{1}{c_{\alpha}(p)}. \quad (B.12)$$

**B.2. Expansion by coherent states**

In this subsection, we show that the adiabatic vacuum state $|0\rangle_\alpha$ can be expanded in terms of the coherent states $|\beta\rangle_\bar{a}$ associated with $\bar{a}_{\alpha,0}$. The coherent state $|\beta\rangle_\bar{a}$ is defined as

$$|\beta\rangle_\bar{a} \equiv \prod_{\bar{a}=2}^D e^{\beta_{\bar{a}}(\bar{a}_{\bar{a},0} - \bar{a}_{\bar{a},0})} |0\rangle_\bar{a} = \prod_{\bar{a}=2}^D e^{-\frac{\beta_{\bar{a}}^2}{2}} e^{\beta_{\bar{a}} \bar{a}_{\bar{a},0}} |0\rangle_\bar{a}, \quad (B.13)$$

where $|0\rangle_\bar{a}$ is the vacuum state annihilated by $\bar{a}_{\bar{a},k}$. This new vacuum state $|0\rangle_\bar{a}$ is related to $|0\rangle_\alpha$ by

$$|0\rangle_\alpha = \prod_{\bar{a}=2}^D (\cosh r_{\bar{a}})^{-1/2} e^{\frac{1}{2} \tanh r_{\bar{a}} (\bar{a}_{\bar{a},0}^2)} |0\rangle_\bar{a}. \quad (B.14)$$
It is easy to verify that the left-hand side is annihilated by 
\( b_{\bar{\alpha},0} = (\cosh r_{\bar{\alpha}}) \bar{a}_{\bar{\alpha},0} - (\sinh r_{\bar{\alpha}}) \bar{a}_{\bar{\alpha},0}^\dagger \). The coherent state satisfies

\[
\bar{a}_{\bar{\alpha},p}|\beta\rangle_{\bar{\alpha}} = \beta_{\bar{\alpha}}|\beta\rangle_{\bar{\alpha}}, \quad \bar{a}_{1,p}|\beta\rangle_{\bar{\alpha}} = 0.
\]

The original vacuum state

\[
|0\rangle_a = \prod_{\bar{\alpha}=2}^D \int_{-\infty}^{\infty} d\beta_{\bar{\alpha}} E_{\bar{\alpha}}(\beta_{\bar{\alpha}}) |\beta\rangle_{\bar{\alpha}}, \quad (B.15)
\]

where the coefficient \( E_{\bar{\alpha}}(\beta_{\bar{\alpha}}) \) is given by

\[
E_{\bar{\alpha}}(\beta_{\bar{\alpha}}) \equiv (2\pi \sinh r_{\bar{\alpha}})^{-1/2} e^{-\beta_{\bar{\alpha}}^2 (e^{2r_{\bar{\alpha}}} - 1)^{-1}} \rightarrow \sqrt{\frac{s_{\bar{\alpha}} H_f}{\pi}} e^{-(s_{\bar{\alpha}} H_f \beta_{\bar{\alpha}})^2}, \quad (s_{\bar{\alpha}} \rightarrow 0).
\]

(B.16)

The formula (B.15) can be verified simply by performing Gaussian integral about \( \{\beta_{\bar{\alpha}}\} \).

References

1) D. Boyanovsky and H. J. de Vega, Phys. Rev. D 70 (2004), 063508, astro-ph/0406287.
2) D. Boyanovsky, H. J. de Vega and N. G. Sanchez, Phys. Rev. D 71 (2005), 023509, astro-ph/0409406.
3) D. Boyanovsky, H. J. de Vega and N. G. Sanchez, Nucl. Phys. B 747 (2006), 25, astro-ph/0503669.
4) D. Boyanovsky, H. J. de Vega and N. G. Sanchez, Phys. Rev. D 72 (2005), 103006, astro-ph/0507596.
5) V. K. Onemli and R. P. Woodard, Class. Quantum Grav. 19 (2002), 4607, gr-qc/0204065.
6) T. Brunier, V. K. Onemli and R. P. Woodard, Class. Quantum Grav. 22 (2005), 59, gr-qc/0408080.
7) T. Prokopec, N. C. Tsamis and R. P. Woodard, arXiv:0707.0847.
8) M. S. Sloth, Nucl. Phys. B 748 (2006), 149, astro-ph/0604488.
9) M. S. Sloth, Nucl. Phys. B 775 (2007), 78, hep-th/0612138.
10) D. Seery, J. Cosmol. Astropart. Phys. 11 (2007), 025, arXiv:0707.3377.
11) D. Seery, J. Cosmol. Astropart. Phys. 02 (2008), 006, arXiv:0707.3787.
12) Y. Urakawa and K. i. Maeda, Phys. Rev. D 78 (2008), 064004, arXiv:0801.0126.
13) E. Komatsu et al. (WMAP Collaboration), arXiv:0803.0547.
14) N. Bartolo, S. Matarrese and A. Riotto, Phys. Rev. D 65 (2002), 103505, hep-ph/0112261.
15) N. Bartolo, E. Komatsu, S. Matarrese and A. Riotto, Phys. Rep. 402 (2004), 103, astro-ph/0406398.
16) J. Maldacena, J. High Energy Phys. 05 (2003), 013, astro-ph/0210603.
17) S. A. Kim and A. R. Liddle, Phys. Rev. D 74 (2006), 063522, astro-ph/0608186.
18) D. Babich, P. Creminelli and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 08 (2004), 009, astro-ph/0405356.
19) D. Seery and J. E. Lidsey, J. Cosmol. Astropart. Phys. 06 (2005), 003, astro-ph/0503692.
20) D. Seery and J. E. Lidsey, J. Cosmol. Astropart. Phys. 09 (2005), 011, astro-ph/0506056.
21) S. Weinberg, Phys. Rev. D 72 (2005), 043514, hep-th/0506236.
22) S. Weinberg, Phys. Rev. D 74 (2006), 023508, hep-th/0605244.
23) G. I. Rigopoulos, E. P. S. Shellard and B. J. W. van Tent, Phys. Rev. D 73 (2006), 083521, astro-ph/0504508.
24) G. I. Rigopoulos, E. P. S. Shellard and B. J. W. van Tent, Phys. Rev. D 73 (2006), 083522, astro-ph/0506704.
25) G. I. Rigopoulos, E. P. S. Shellard and B. J. W. van Tent, Phys. Rev. D 76 (2007), 083512, astro-ph/0511041.
26) F. Vernizzi and D. Wands, J. Cosmol. Astropart. Phys. 05 (2006), 019, astro-ph/0603799.
27) X. Chen, M. x. Huang, S. Kachru and G. Shiu, J. Cosmol. Astropart. Phys. 01 (2007), 002, hep-th/0605045.
28) T. Battefeld and R. Easther, J. Cosmol. Astropart. Phys. 03 (2007), 020, astro-ph/0610296.
29) S. Yokoyama, T. Suyama and T. Tanaka, Phys. Rev. D 77 (2008), 083511, arXiv:0711.2920.
30) S. Yokoyama, T. Suyama and T. Tanaka, J. Cosmol. Astropart. Phys. 02 (2009), 012, arXiv:0810.3053.
31) D. Seery, M. S. Sloth and F. Vernizzi, arXiv:0811.3934.
32) A. Naruko and M. Sasaki, Prog. Theor. Phys. 121 (2009), 193, arXiv:0807.0180.
33) S. Weinberg, arXiv:0805.3781.
34) S. Weinberg, Phys. Rev. D 78 (2008), 123521, arXiv:0808.2909.
35) S. Weinberg, arXiv:0810.2831.
36) H. R. S. Cogollo, Y. Rodriguez and C. A. Valenzuela-Toledo, J. Cosmol. Astropart. Phys. 08 (2008), 029, arXiv:0806.1546.
37) Y. Rodriguez and C. A. Valenzuela-Toledo, arXiv:0811.4092.
38) D. H. Lyth, J. Cosmol. Astropart. Phys. 12 (2007), 016, arXiv:0707.0361.
39) N. Bartolo, S. Matarrese, M. Pietroni, A. Riotto and D. Seery, J. Cosmol. Astropart. Phys. 01 (2008), 015, arXiv:0711.4263.
40) A. Riotto and M. S. Sloth, J. Cosmol. Astropart. Phys. 04 (2008), 030, arXiv:0801.1845.
41) K. Enqvist, S. Nurmi, D. Podolsky and G. I. Rigopoulos, J. Cosmol. Astropart. Phys. 04 (2008), 025, arXiv:0802.0395.
42) Y. Urakawa and T. Tanaka, Prog. Theor. Phys. 122 (2009), 779, arXiv:0902.3209.
43) D. Polarski and A. A. Starobinsky, Class. Quantum Grav. 13 (1996), 377, gr-qc/9504030.
44) C. Kiefer, I. Lohmar, D. Polarski and A. A. Starobinsky, Class. Quantum Grav. 24 (2007), 1699, astro-ph/0610700.
45) A. A. Starobinsky, Lect. Notes Phys. 246 (1986), 107.
46) A. A. Starobinsky and J. Yokoyama, Phys. Rev. D 50 (1994), 6357, astro-ph/9407016.
47) K. i. Nakao, Y. Nambu and M. Sasaki, Prog. Theor. Phys. 80 (1988), 1041.
48) Y. Nambu and M. Sasaki, Phys. Lett. B 219 (1989), 240.
49) M. Morikawa, Phys. Rev. D 42 (1990), 1027.
50) M. Morikawa, Prog. Theor. Phys. 77 (1987), 1163.
51) T. Tanaka and M. a. Sakagami, Prog. Theor. Phys. 100 (1998), 547, gr-qc/9705054.
52) D. Seery, arXiv:0903.2788.
53) C. Gordon, D. Wands, B. A. Bassett and R. Maartens, Phys. Rev. D 63 (2001), 023506, astro-ph/0009131.
54) D. H. Lyth and Y. Rodriguez, Phys. Rev. D 71 (2005), 123508, astro-ph/0502578.
55) P. R. Jarnhus and M. S. Sloth, J. Cosmol. Astropart. Phys. 02 (2008), 013, arXiv:0709.2708.
56) A. Roura and E. Verdaguer, arXiv:0709.1940.
57) Y. Urakawa and K. i. Maeda, Phys. Rev. D 77 (2008), 024013, arXiv:0710.5342.
58) F. Finelli, G. Marozzi, A. A. Starobinsky, G. P. Vacca and G. Venturi, Phys. Rev. D 79 (2009), 044007, arXiv:0808.1786.
59) N. C. Tsamis and R. P. Woodard, Ann. of Phys. 253 (1997), 1, hep-ph/9602316.
60) N. C. Tsamis and R. P. Woodard, Nucl. Phys. B 474 (1996), 235, hep-ph/9602315.
61) J. Garriga and T. Tanaka, Phys. Rev. D 77 (2008), 024021, arXiv:0706.0295.
62) N. C. Tsamis and R. P. Woodard, arXiv:0708.2004.