Exact Relativistic Two-Body Motion
in Lineal Gravity

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PACS numbers: 13.15.-f, 14.60.Gh, 04.80.+z

March 24, 2022

Abstract

The N-body problem for one-dimensional self-gravitating systems
has been often studied to test theories of galactic evolution and sta-
tistical mechanics. We consider the general relativistic version of this
system and obtain the first exact solution to the 2-body problem in
which spacetime is not flat. In the equal mass case we obtain an
explicit expression for the proper separation of the two masses as a
function of their mutual proper time.
$N$-body self-gravitating systems have a long history in physics and are of interest in studying both star systems ($N \geq 2$) and galactic evolution ($N$). One-dimensional (or lineal) models of such systems have been of particular interest in that they avoid some difficulties due to three dimensions, including evaporation, singularities, and energy dissipation in the form of gravitational radiation, as well as admitting a level of computational and analytic analysis which is dramatically simpler.

We consider in this paper the $N$-body problem for a relativistic self-gravitating lineal system and formulate a canonical theory determining its Hamiltonian. In the 2-body case we obtain an exact solution (valid to all orders in the gravitational coupling $\kappa$) in which spacetime outside the moving matter sources is not flat. In the equal mass case we obtain an explicit expression for the proper separation of the two point masses as a function of their mutual proper time. This is the first non-perturbative relativistic curved-spacetime treatment of this problem, providing new avenues for investigation of one-dimensional self-gravitating systems.

For our lineal self-gravitating system we choose a modification of Jackiw-Teitelboim lineal gravity, in which the scalar curvature is equated to a cosmological constant

$$R - \Lambda = 0$$

in the absence of other matter fields. This model has been of considerable interest as a model theory of quantum gravity. Here we couple $N$ point masses to this theory so that we have a generally covariant self-gravitating system with non-zero curvature outside the point sources. We do not include collisional terms, so that the bodies pass through each other.

Since the Einstein action is a topological invariant in 2 spacetime dimensions, we must incorporate a scalar (dilaton) field into our action, which we take to be

$$I = \int d^2x \left[ \frac{1}{2\kappa} \sqrt{-g} \left\{ \Psi R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi + \Lambda \right\} \right] + I_P$$

where $\Psi$ is the dilaton field, $g_{\mu\nu}$ and $g$ are the metric and its determinant and $R$ is the Ricci scalar, with $\kappa = 8\pi G/c^4$. $I_P$ is the action of $N$ point masses minimally coupled to gravity

$$I_P = -\int d^2x \left[ \sum_{a=1}^{N} m_a \int d\tau_a \left\{ -g_{\mu\nu}(x) \frac{dz_a^{\mu}}{d\tau_a} \frac{dz_a^{\nu}}{d\tau_a} \right\}^{1/2} \delta^2(x - z_a(\tau_a)) \right]$$
where $\tau_a$ is the proper time of $a$-th particle. Variation of the action (2) with respect to the metric, dilaton field, and particle coordinates yields the field equations

$$R - \Lambda = \kappa T^\mu_\mu \frac{d}{d\tau_a} \left\{ \frac{dz^\nu_a}{d\tau_a} \right\} + \Gamma^\nu_{\alpha\beta}(z_a) \frac{dz^\alpha_a}{d\tau_a} \frac{dz^\beta_a}{d\tau_a} = 0$$

(3)

$$\frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi - g_{\mu\nu} \left( \frac{1}{4} \nabla^\lambda \Psi \nabla_\lambda \Psi - \nabla^2 \Psi \right) - \nabla_\mu \nabla_\nu \Psi = \kappa T^\mu_\nu + \frac{\Lambda}{2} g_{\mu\nu}$$

(4)

where the stress-energy due to the point masses is

$$T^\mu_\nu = \sum_{a=1}^{N} m_a \int d\tau_a \frac{1}{\sqrt{-g}} g_{\mu\sigma} g_{\nu\rho} \frac{dz^\sigma_a}{d\tau_a} \frac{dz^\rho_a}{d\tau_a} \delta^2(x - z_a(\tau_a))$$

(5)

and is conserved. We observe that (3) is a closed system of $N + 1$ equations for which one can solve for the single metric degree of freedom and the $N$ degrees of freedom of the point masses; it reduces to (4) if all masses vanish. The evolution of the dilaton field is governed by the evolution of the point-masses via (4). The left-hand side of (4) is divergenceless (consistent with the conservation of $T^\mu_\nu$), yielding only one independent equation to determine the single degree of freedom of the dilaton.

Working in the canonical formalism we make use of the decomposition $\sqrt{-g}R = -2\partial_0(\sqrt{\gamma}K) + 2\partial_1(\sqrt{\gamma}N^1K - \gamma^{-1}\partial_1N_0)$ where the extrinsic curvature $K = (2N_0\gamma)^{-1}(2\partial_1N_1 - \gamma^{-1}N_1\partial_1\gamma - \partial_0\gamma)$, and rewrite the action (2) in the form [4]

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a(x^0)) + \pi \ddot{\Psi} + \Pi \dot{\Psi} + N_0 R^0 + N_1 R^1 \right\}$$

(6)

where $\gamma = g_{11}, N_0 = (-g^{00})^{-1/2}, N_1 = g_{10}, \pi$ and $\Pi$ are conjugate momenta to $\gamma$ and $\Psi$ respectively. The quantities $N_0$ and $N_1$ are Lagrange multipliers which enforce the constraints $R^0 = 0 = R^1$, where

$$R^0 = -\kappa \sqrt{\gamma} \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \left( \frac{\Psi'}{\kappa \sqrt{\gamma}} \right)' + \frac{\Lambda}{2\kappa \sqrt{\gamma}} - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(x^0))$$

$$R^1 = \frac{\gamma'}{\gamma} - \frac{1}{\gamma} \Pi \Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(x^0))$$

(7)
with the symbols (\(\dot{\cdot}\)) and (\(\cdot^\prime\)) denoting \(\partial_0\) and \(\partial_1\), respectively. We identify the dynamical and gauge (i.e. co-ordinate) degrees of freedom by writing the generator arising from the variation of the action at the boundaries in terms of \((\Psi^\prime/\sqrt{\gamma})^\prime\) and \(\pi^\prime\), which we can easily solve for since these are the only linear terms in the constraints. We then fix the frame of the physical space-time coordinates in a manner similar to the \((3 + 1)\)-dimensional case [5].

Carrying out this procedure, we find that we can consistently choose the coordinate conditions \(\gamma = 1\) and \(\Pi = 0\). Eliminating the constraints, the action (6) then reduces to

\[
I = \int d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} .
\]

where the reduced Hamiltonian is \(\mathcal{H} = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \Delta \Psi\), where \(\Delta \equiv \partial^2/\partial x^2\), and \(\Psi = \Psi(x, z_a, p_a)\) and is understood to be determined from the constraint equations which are now

\[
\Delta \Psi - \frac{(\Psi^\prime)^2}{4} + \kappa^2 \pi^2 - \frac{\Lambda}{2} + \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) = 0
\]

\[
2\pi^\prime + \sum_a p_a \delta(x - z_a) = 0 .
\]

When \(\Lambda = 0\) the Hamiltonian reduces to that considered in ref. [1] in the non-relativistic limit [4].

For \(N = 2\) we solve these equations exactly following the method given in ref. [6]. First, for \(z_2 < z_1\), we divide spacetime into three regions: \(z_1 < x\) (\((+)\) region), \(z_2 < x < z_1\) (\((0)\) region) and \(x < z_2\) (\((-)\) region) and set \(\Psi = -4\log|\phi|\) and \(\pi = \chi^\prime\). In each region \(\chi\) is a sum of terms linear in \(x\) and \(\phi\) obeys a harmonic oscillator equation, whose solutions are linear combinations of exponential functions of \(x\) in the \(\pm\) regions and of either trigonometric or exponential functions of \(x\) in the 0 region depending on the size of \(\Lambda\) relative to the other integration constants. Matching these solutions at the boundaries \(x = z_1\) and \(x = z_2\) of each region allows a determination of the coefficients of these linear combinations in the + and − regions in terms of those in the 0 region.

The magnitudes of both \(\phi\) and \(\chi\) increase with increasing \(|x|\) and so we must employ a boundary condition which guarantees that the surface
terms obtained in passing from (2) to (6) vanish and the Hamiltonian remain finite. A perturbative analysis indicates that this condition must be $Ψ^2 - 4\kappa^2\chi^2 + 2\Lambda x^2 = C_\pm x$ in the $\pm$ regions. Incorporating this into the matching conditions determines the coefficients of the linear combination in the 0 region in terms of the momenta and positions of the bodies.

This procedure is sufficient to solve for $N_0$, $N_1$ and all other field variables exactly. Repeating the analysis for $z_1 < z_2$ yields a similar solution with $p_i \rightarrow -p_i$, and so the full solution is a function of $(z_i, \tilde{p}_i)$, where $\tilde{p}_i = p_i \text{sgn}(z_1 - z_2)$. The resultant expressions are rather long and cumbersome. However it is straightforward to show that the canonical equations imply conservation of the total momenta $p_1 + p_2$. Choosing a center of inertia frame with $p_1 = -p_2 = p$, yields considerable simplification of the exact solution, which we write employing the notation $K = \sqrt{(\kappa X)^2 - \frac{A}{2}}$, $K_0 = \sqrt{Y_0^2 - \frac{A}{2}}$ and

\[
K_{1,2} \equiv 2K_0 + 2K - \kappa \sqrt{p^2 + m^2_{1,2}} \quad \tilde{K} \equiv K_0 - K + \frac{\kappa \epsilon}{2} \tilde{p} \\
\mathcal{M}_{1,2} \equiv \kappa \sqrt{p^2 + m^2_{1,2}} + 2K_0 - 2K \quad Y_0 \equiv \kappa \left[ X - \frac{\epsilon}{2} \tilde{p} \right] .
\]

We obtain

\[
\phi_\pm(x) = A_\pm e^{\pm \frac{1}{2}Kx} \quad \phi_0(x) = A_0 e^{\frac{\kappa \epsilon}{2}K_0} + B_0 e^{-\frac{\kappa \epsilon}{2}K_0} \quad (10)
\]

in the (+), (−) and (0) regions respectively, where

\[
A_\pm = \left( \frac{K_{1,2}}{\mathcal{M}_{1,2}} \right)^{1/2} e^{-\tilde{K}(z_1 - z_2) + \frac{1}{2}Kz_{1,2}} \quad (11)
\]

\[
\{A_0, B_0\} = \frac{(K_{1,1} \mathcal{M}_{2,1})^{1/2}}{4K_0} e^{-\tilde{K}(z_1 - z_2) + \frac{1}{2}K_{1,1} z_{2,1}} .
\]

The metric components are

\[
N_{0(\pm)} = A\phi_\pm^2 \quad N_{0(0)} = A\phi_0^2 \\
N_{1(\pm)} = \pm \epsilon \frac{\kappa X}{K} \left( A\phi_\pm^2 - 1 \right) \quad (12)
\]

\[
N_{1(0)} = \epsilon \left\{ AY_0 \left[ 2A_0B_0x + \frac{A_0^2 e^{K_0x} - B_0^2 e^{-K_0x}}{K_0} \right] + D_0 \right\}
\]
where \( A = \frac{16\kappa K_0 X}{J_+ K} e^{\frac{1}{2}K(z_1 - z_2)} \), \( D_0 = \frac{J_+ X}{J_+ K} \) and

\[
J_\pm = 2 \left( \frac{Y_0}{K_0} + \frac{\kappa X}{K} \right) (K_1 \pm K_2) - 2K_1 K_2 \left( \frac{Y_0}{K_0} - \frac{\kappa X}{K} \right) \left( \frac{1}{\mathcal{M}_1} \pm \frac{1}{\mathcal{M}_2} \right) - \frac{Y_0}{K_0} K_1 K_2(z_1 \mp z_2). 
\]

The solution for \( z_1 < z_2 \) is obtained by interchanging the suffices 1 and 2 in the preceding solution.

The matching conditions at \((z_1, z_2)\) force the relation

\[
K_1 K_2 = \mathcal{M}_1 \mathcal{M}_2 e^{\kappa J_0 |z_1 - z_2|} 
\]

which determines \( X \). From this the Hamiltonian

\[
H = -\frac{1}{\kappa} \int dx \triangle \Psi = -\frac{1}{\kappa} [\Psi']_\infty = \frac{4K}{\kappa}. 
\]

may be explicitly determined as a function of the coordinates and momenta of the particles.

The parameter \( \epsilon = \pm 1 \), and is a constant of integration associated with the metric degree of freedom. Under time reversal, solutions with \( \epsilon = 1 \) transform into those with \( \epsilon = -1 \), ensuring invariance of the whole theory under this symmetry. It is straightforward to show from this solution that the Ricci scalar is equal to the cosmological constant everywhere except at the locations \((z_1(t), z_2(t))\) of the point masses. Hamilton’s equations imply that

\[
\dot{p} = -\frac{4X K_0 K_1 K_2}{K} \frac{J_+}{J_+} 
\]

\[
\dot{z}_i = (-1)^{i+1} \frac{\kappa X}{K} \left( \epsilon + \frac{16}{J_+} \frac{K_0 K_i}{\mathcal{M}_i} \left\{ \frac{p}{\sqrt{p^2 + m_i^2}} - \epsilon \frac{\kappa X}{K} \right\} \right) 
\]

are the dynamical equations for the 2-body system coupled to gravity, where \( i = 1, 2 \).

The defining equation for the Hamiltonian is, using eqs. (14) and (13),

\[
\tanh(\frac{\kappa \mathcal{J}}{8} |r|) = \frac{\mathcal{J}(B_1 + B_2)}{\mathcal{J}^2 + B_1 B_2} 
\]

where \( B_{1,2} = H - 2\sqrt{p^2 + m_{1,2}^2} \), \( \mathcal{J}^2 = (\sqrt{H^2 + 8\Lambda/\kappa^2} - 2\epsilon \bar{p})^2 - 8\Lambda/\kappa^2 \) and \( r \equiv z_1 - z_2 \). For a given \( \Lambda \geq -(\kappa H)^2/8 \), equation (14) describes the surface in
(r, p, H) space of all allowed phase-space trajectories. Since H is a constant of the motion, (a fact easily verified by differentiation of (17) with respect to t) a given trajectory in the (r, p) plane is uniquely determined by setting \( H = H_0 \) in (17).

In the equal mass case the canonical equations of motion (16) can be solved exactly. The proper time of each particle is

\[
d\tau = dt N_0(z_0) \frac{m}{\sqrt{p^2 + m^2}} = dt \frac{16\kappa K_0 K_1 X}{J+K'M_1} \frac{m}{\sqrt{p^2 + m^2}}
\]

yielding upon insertion in (16) the solution

\[
p = \frac{\epsilon m^2}{\gamma_m (1 + \sqrt{\gamma_m^2 - \gamma_m \tan \frac{\epsilon m^2}{\sqrt{H^2 - 4m^2}}})} \quad (18)
\]

where \( \gamma_m \equiv 1 + \frac{8\Lambda}{\kappa^2 m^4}, \sigma = (1 + \sqrt{\gamma_m})(\sqrt{p_0^2 + m^2} - \epsilon p_0) - \frac{m^2}{H}, \) and \( \eta = \frac{\sigma - \frac{m^2}{H} \sqrt{\gamma_m}}{\sigma + \frac{m^2}{H} \sqrt{\gamma_m}}, \) with \( p_0 \) the initial momentum at \( \tau = \tau_0. \) It is then straightforward to obtain an exact expression for \( r \) as a function of \( \tau, \) either directly from the second of eqs. (16) or by inserting the expression for \( p(\tau) \) into (17) and solving for \( r. \)

Analysis of these solutions shows that two types of motion are possible, depending on the amount of energy and the value of the cosmological constant. For \( \Lambda < \frac{\kappa^2 m^4}{2(H^2 - 4m^2)} \equiv \Lambda_c \) the two particles will always execute periodic motion, with period

\[
T = \begin{cases} 
\frac{16}{\sqrt{\kappa^2 m^2 + 8\Lambda}} \tanh^{-1} \left( \frac{\sqrt{\kappa^2 m^2 + 8\Lambda \sqrt{H^2 - 4m^2}}}{\kappa H m} \right) & \gamma_m > 0 \\
\frac{16\sqrt{H^2 - 4m^2}}{\kappa H m} & \gamma_m = 0 \\
\frac{16}{\sqrt{-\kappa^2 m^2 - 8\Lambda}} \tan^{-1} \left( \frac{\sqrt{-\kappa^2 m^2 - 8\Lambda \sqrt{H^2 - 4m^2}}}{\kappa H m} \right) & \gamma_m < 0
\end{cases}
\]

which can be obtained from the initial and final values of \( p \) when \( r = 0 \) from (17). If \( \Lambda < 0 \) this is the only state of motion possible. However if \( \Lambda > 0 \) there also exist a countably infinite set of unbound states of motion, in which
the particles begin with some (sufficiently large) separation, approach one another at some minimal value of $|r|$, and then move apart from one another toward infinite separation.

For $\Lambda > \Lambda_c$, only unbounded motion is possible. If the initial momentum is sufficiently small, the particles will cross one another before receding toward infinity. Otherwise they simply approach one another at some minimal value of $|r|$ and then reverse direction toward infinity as previously mentioned. One peculiar feature in this regime is that the two particles diverge to infinite separation at finite $\tau$! Specifically, if the above condition is satisfied, then the determining equation requires that $r \to \infty$ as $p$ approaches the value $\frac{1}{2\kappa} \left( \sqrt{\kappa^2 H^2 + 8\Lambda} + \sqrt{8\Lambda} \right)$. The corresponding value of $\tau$ at which this will occur can be found from the $\gamma_m > 0$ solution above. The explicit expression for this $\tau$ is cumbersome and is omitted.

In figure 1 we plot $r$ vs. $\tau$ for two bodies initially at $r = 0$ for fixed $\Lambda = -1.5$ and $H = 16$ for several different values of $m$. As the motion becomes more relativistic (i.e. $m$ gets smaller) we find that a second maximum develops in the curve, which then vanishes for very small $m$. We find this behaviour for a broad range of values of $\Lambda < 0$ and $H$. Figure 2 contains an analogous plot for $\Lambda = 1.5$, showing the transition from bounded to unbounded motion. As $\Lambda \to \Lambda_c$ (i.e. as $m$ becomes small) the particles rapidly separate, remaining nearly stationary for an increasingly large period of proper time before coming together again. At $\Lambda = \Lambda_c$ this separation time becomes infinite, and for $\Lambda > \Lambda_c$, the separation diverges at finite $\tau$.

To summarize, we have obtained an exact solution to the 2-body problem in relativistic lineal gravity in which spacetime is not flat, in contrast to other lower-dimensional solutions to this problem in which spacetime is either flat outside matter \cite{3, 4, 5} or the matter sources are stationary \cite{1}. The system is described by a conservative Hamiltonian in the canonical formalism. Many interesting features of this problem remain to be explored, including an extension to the unequal-mass case, the transition from bounded to unbounded motion for $\Lambda > 0$, the appearance/disappearance of a second maximum for $\Lambda < 0$, and an investigation of the statistical properties of the model for large $N$. In this last case many-body forces play an important role – as shown in ref. \cite{4}, in the post-Newtonian approximation $n$-body forces appear at order $\kappa^{n-1}$. We intend to make these the subjects of future investigation.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.
Figure 1: A sequence of equal mass curves for $\Lambda = -1.5, H = 16$. Note the presence of the second maximum for $m = 0.05$. 

Equal Mass -- $H=16$ Lambda=-1.5

- $m=7$
- $m=3$
- $m=1$
- $m=0.05$
- $m=0.001$
Figure 2: A sequence of equal mass curves for $\Lambda = 1.5$, $H = 16$. The motion becomes unbounded between $m = 4.72$ and $m = 4.73$. 

Equal Mass -- $H=16$  $\Lambda=1.5$
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