Incentivizing efficient use of shared infrastructure:
Optimal tolls in congestion games

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Throughout modern society, human users interact with large-scale engineered systems, e.g., road-traffic networks, electric power grids, wireless communication networks. As the performance of such systems greatly depends on the decisions made by the individual users - often leading to undesirable system behaviour - a natural question arises: How can we design incentives to promote efficient use of the existing infrastructure? Here, we answer this question in relation to the well-studied class of congestion games, used to model a variety of problems arising in theory and practice including traffic routing. In this context, a methodology for designing efficient mechanisms is so far missing, in spite of the vast scientific interest. In this manuscript, we resolve this problem by means of an elegant and computationally tractable approach, recovering and generalizing many results in the literature. Surprisingly, optimal mechanisms designed using local information perform closely to those designed using global information. Additionally, we show how mechanisms that perform optimally in the continuous-flow approximation (marginal cost tolls), worsen the performance when applied to the original discrete setup.

Modern society is based on large-scale engineered systems, often at the service of human end-users, e.g., transportation, power and communication networks. While their design and operation is typically grounded on purely engineering principles, their performance heavily depends on the interaction between the “human element” and the underlying technological infrastructure (1). The rapid penetration of smartphones and the decrease in mobile communication costs has resulted in an even tighter integration between these two components, further blurring the boundaries between engineered systems and the social fabric. As a result, the operation of such systems - often referred to as sociotechnical systems (2) - requires interdisciplinary considerations at the confluence between economics, engineering, and social sciences, thus advocating for concepts and tools pertaining to the field theory (3). Sociotechnical systems emerge in connection to numerous applications including ride-sharing platforms, where both drivers and customers utilize the road-traffic infrastructure (4), electric power grids, in which consumers’ requirements determine the instantaneous energy production (5), internet routing (6), supply chain management (7), bike-sharing (8), cloud computing (9), road-traffic routing (10), and countless others.

A common issue arising in these settings is the performance degradation incurred when the users’ individual objectives are not aligned to the “greater good”. Within the environmental sciences this phenomenon has been extensively studied, and, in its extreme form, is referred to as the “tragedy of the commons” (11). A prime example of how users’ behaviour degrades the performance of an engineered system is provided by road-traffic routing: when drivers choose routes that minimize their individual travel time, the aggregate congestion could be much higher compared to that of a centrally-imposed routing policy. A fruitful paradigm to tackle this issue is to influence users’ behaviour through appropriately designed incentives, as widely acknowledged in the economic and computer science literature (12–14). Leveraging this approach in the context of sociotechnical systems (Fig. 1) gives rise to two fundamental questions:

\begin{itemize}
  \item How can we design behaviour-influencing mechanisms to incentivize efficient use of the existing infrastructure?
  \item How does the performance of a behaviour-influencing mechanism depend on the available information?
\end{itemize}

In this paper we answer these questions in relation to a well-studied class of problems known as congestion games, most notably utilized to model problems in which drivers need to be routed through a congestion-sensitive network (15). Congestion games were introduced by Rosenthal in 1973 (16), and since then have found applications in diverse fields such as animal dispersal (17), energy markets (18), machine scheduling (19), wireless data networks (20), sensor allocation (21), network design (22), and many more. While our results can be applied to a variety of problems, in this paper we consider routing problems as our application of interest. Therefore, we introduce congestion games in this context purely for clarity of exposition.

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As users are assumed to be self-interested, the emergent behaviour is well captured by the notion of Nash equilibrium, i.e., a system cost of 3. Hence, we say that the equilibrium configuration on the left has efficiency of 4/3 ≈ 1.33.

In a congestion game, we are given a set of edges $E$ representing the road-network, and a set of users $N = \{1, \ldots, n\}$ wishing to travel on the network (see Fig. 2 for a simplified example). To reach her destination, each user $i \in N$ can choose between different subsets of $E$. We list all the options of user $i \in N$ in the set $A_i \subseteq 2^E$. The time required to transit through edge $e \in E$ depends only on the number of users traveling through that edge, and is captured by the latency function $\ell_e : N \rightarrow \mathbb{R}_{\geq 0}$. The function $\ell_e$ takes into consideration geometric properties such as the length, number of lanes and speed limit of edge $e$ (1). Once all users have chosen their route $a_i \in A_i$, each user experiences a travel time equal to the sum of the latencies over the edges she selected. Finally, the system cost describes the time spent on the network by all users, i.e.,

$$SC(a) = \sum_{e \in E} |a|_e \ell_e(|a|_e),$$

where $|a|_e$ is the number of users selecting edge $e$ in allocation $a = (a_1, \ldots, a_n)$. Since the overall congestion can be much higher when drivers act selfishly (23), there has been considerable interest in the use of road tolls as a behaviour-influencing mechanisms (24–29). Towards this goal, each edge $e$ is associated to a congestion-dependent tolling function $\tau_e : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$. As a result, user $i \in N$ incurs a cost factoring both the travel time, as well as the tolls levied along her route, i.e.,

$$C_i(a) = \sum_{e \in a_i} \ell_e(|a|_e) + \tau_e(|a|_e).$$

The performance of a given set of tolls is typically measured by the ratio between the system cost resulting from the worst-performing emergent behaviour and the minimum system cost. Such a performance metric is referred to as the price-of-anarchy. As users are assumed to be self-interested, the emergent behaviour is well captured by the notion of Nash equilibrium, i.e., a configuration where no single user can reduce her travel time by modifying only her route choice. The transportation and computer science communities have long been asking questions of the form: "How do we design tolls that optimize the system efficiency?" (23–30). Fig. 3 illustrates the advantages and challenges associated with the use of behaviour-influencing tolls.

Designing tolls that utilize all the data defining a routing problem is not possible, as that requires the central planner to access private information such as the origin/destination of all users, in addition to the associated computational burden. Therefore, a large portion of the literature has focused on designing tolls that exploit only local information (local tolls), i.e., for which the toll on edge $e$ is constructed only using information on that edge. Since the central planner does not have perfect knowledge about the specific routing instance at hand (e.g., the demand could shift, or the latency functions could be mischaracterized), the tolls need to be designed robustly: while a set of tolls could perform particularly well on one instance, they might be harmful on another one, see Fig. 3. In light of these challenges, and despite the vast scientific interest, a methodology for designing local tolls that robustly optimize the efficiency is so far missing (see the next section for further details). In this article we show how to resolve this question by means of an elegant and computationally tractable approach.

Our Main Contributions.

- In this manuscript we resolve the problem of designing local tolls that yield the best achievable price-of-anarchy through a tractable linear program.
- Surprisingly, optimal tolls designed using local information perform closely to optimal tolls designed using global information (≤ 1% difference for affine latency functions).
- The efficiency of optimal congestion-dependent and fixed tolls is comparable for polynomial latencies of low degree.
- The efficiency of marginal cost tolls, known to be optimal in the continuous-flow approximation model, is unexpectedly lower than that encountered levying no toll, when applied to the original discrete setup.
Review of Tolling in Congestion Games. The scientific interest in the design of tolls dates back to the early 1900s (23). Since then, a large body of literature in the areas of transportation and computer science has investigated this approach (24–27).

Designing tolling mechanisms that optimize the efficiency is particularly challenging in the context of congestion games, as observed, e.g., by (31), in part due to the multiplicity of the equilibria. While most of the research (32–35) has focused on providing efficiency bounds for given schemes (or in the un-tolled case), little is known regarding the design question, except that no tolling scheme can incentivize optimal routing in every instance (30). While tolling schemes utilizing global information have recently been proposed for the case of polynomial latency functions (29), they require complete network and user information - an often impractical scenario. As a result, the proposed tolls lack the desired robustness property (see Fig. 3).

Despite the vast scientific interest, there is so far no means to derive a set of optimal local tolls, let alone a computationally tractable approach to do so. The only available answer is confined to problems where all latencies are affine functions, and under the restrictive requirement that each toll is congestion-independent (28). Our work resolves this problem and shows how to derive local tolls that maximize the system efficiency through the solution of a tractable linear program.

The continuous-flow approximation of the congestion game model is better understood, as uniqueness of the Nash equilibrium is guaranteed, greatly simplifying the analysis. In this setting, marginal cost tolls produce an emergent behaviour which is always optimal (23, 36). Here, we demonstrate that utilizing marginal cost tolls in the original setup does not improve - and instead significantly deteriorates - the resulting efficiency.

Results

We begin by introducing the formal notion of price-of-anarchy, before presenting the main result. Towards this goal, we consider congestion games where all latencies belong to a common set of functions $\mathcal{L}$, which we assume is generated by a linear combination of given bases $b_j : \mathbb{N} \to \mathbb{R}_{\geq 0}$, $j = 1, \ldots, m$, with nonnegative coefficients. This framework is extremely rich, and accommodates well-studied classes of problems, e.g., polynomial congestion games (33). Affine congestion games are obtained, for example, setting $b_1(x) = 1$, $b_2(x) = x$. We denote with $\mathcal{G}$ the set of all congestion games that can be constructed with latencies as in the above, and a maximum of $n$ players. Our objective is that of associating each latency $\ell$ to a corresponding toll, which we denote with $\tau = T(\ell)$. Observe that $\tau : \{1, \ldots, n\} \to \mathbb{R}_{\geq 0}$ is congestion-dependent (i.e., it associates a real number to every integer in $\{1, \ldots, n\}$). The inefficiency of a tolling mechanism $T$ is commonly measured by the price-of-anarchy (37), defined as

$$\text{PoA}(T) = \sup_{G \in \mathcal{G}} \frac{\text{NashCost}(G, T)}{\text{MinCost}(G)},$$

where MinCost($G$) and NashCost($G$, $T$) are the minimum social cost, and the highest social cost at a Nash equilibrium when employing the mechanism $T$ on the game $G$. By definition $\text{PoA}(T) \geq 1$, and a lower value of the price-of-anarchy corresponds to more efficient tolling. Within this context, the central planner is interested in designing a mechanism $T$ that minimizes the price-of-anarchy. Unexpectedly, this problem can be solved through a tractable linear program.

Optimal Local Tolls. Let $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$ be a latency function. A local taxation mechanism minimizing the price-of-anarchy defined in Eq. (2) is given by

$$T^{opt}(\ell) = \sum_{j=1}^{m} \alpha_j \cdot \tau_j^{opt}, \quad \text{where} \quad \tau_j^{opt}(x) = f_j^{opt}(x) - b_j(x),$$

and $f_j^{opt} : \{1, \ldots, n\} \to \mathbb{R}$ is a solution to the linear program

$$\max_{f_j \in \mathbb{R}^n, \rho \in \mathbb{R}} \rho \quad \text{s.t.} \quad b_j(x+z)(x+z) - \rho b_j(x+y)(x+y) + f_j(x+y)y - f_j(x+y+1)z \geq 0$$

$$\forall (x, y, z) \in \mathbb{N}^3 \text{ with } 1 \leq x + y + z \leq n,$$

$$f_j(x) \geq b_j(x) \quad \forall x \in \{1, \ldots, n\},$$

where we define $f_j(0) = f_j(n+1) = 0$ to ease the notation. The resulting optimal price-of-anarchy is $\text{PoA}(1/\rho_j^{opt})$, where $\rho_j^{opt}$ is the value of the linear program in Eq. (3).

The above statement contains two fundamental results. The first part of the statement shows that an optimal taxation mechanism is a linear map. As a consequence, the toll applied to any latency function $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$ can be obtained as the linear combination of $\tau_j^{opt}(x)$, with the same coefficients $\alpha_j$ used to define $\ell(x)$. Complementary to this, the second part of the statement provides a practical technique to compute $\tau_j^{opt}(x)$ for each of the basis $b_j(x)$, as the solution of a simple linear program. This is particularly valuable, as linear programs can be solved extremely efficiently through widely available software packages, e.g., (38, 39). Furthermore, the approach presented allows to precompute and store in a library the values of $\tau_j^{opt}(x)$ for different basis functions (e.g., polynomials). Having done so, the only operations required to compute an optimal toll $T^{opt}(\ell)$ are mere addition and multiplications of $\tau_j^{opt}(x)$ - stored in the library - with $\alpha_j$. A graphical representation of this process is included in Fig. 4.

![Fig. 4. Graphical representation of the main result on the design of optimal tolls. The input consists of a given latency $\ell(x)$ expressed as a combination of basis $b_j(x)$ with coefficients $\alpha_j$. For each basis, we compute the associated optimal toll $\tau_j^{opt}(x)$ by solving the linear program (LP) appearing in Eq. (3). The resulting optimal toll is obtained as the linear combination of $\tau_j^{opt}(x)$ with the same coefficients $\alpha_j$. The quantities $\tau_j^{opt}(x)$ can be precomputed and stored in a library, therefore offloading the solution of the linear programs.](image-url)
We finally observe that a different linear program can be derived in order to compute (and not optimize) the efficiency of given tolls (see the supplementary material). As the un-tolled case corresponds to setting the tolls to be identically zero, our linear programming approach recovers many well-known results on the efficiency of un-tolled systems, e.g., (19, 32–35), reported in the second column of Table 1. In addition to that, it allows to automatically discover novel efficiency bounds: for any given choice of basis functions, the associated linear program returns the exact value of the price-of-anarchy, avoiding the need for ad hoc constructions as those appearing in the above-cited works.

Local tolling vs Global tolling. Leveraging the framework presented above, we have computed optimal local tolls for a variety of basis functions, including the widely-studied polynomial case. The resulting values of the price-of-anarchy are presented in the fourth column of Table 1, while the corresponding tolls are depicted in Fig. 5. We compare these results with those derived in (28, 29), which instead make use of global information (for example, by letting the tolling function on edge \( e \) depend on the latency functions on all the other edges).

Surprisingly, the efficiency of optimal tolls designed using only local information is almost identical to that of optimal tolls designed using global information (compare the fourth and third columns in Table 1). In addition to providing similar performances by means of less information, local tolls offer two fundamental advantages. First, local tolls are robust against uncertain scenarios, e.g., modification of the origin/destination pairs, modification in the network topology, etc. Second, local tolls can be computed very efficiently, as discussed in the previous section. These observations are extremely valuable to the central planner and to the practitioners, in that they suggest - through principled analysis - to focus on the design of mechanisms that utilize only local information.

![Fig. 5. Optimal local tolls for an edge with latency \( \ell(x) = x^d \) resulting as a solution of the linear program with \( n = 20 \).](image)

**Table 1. Comparison of the price-of-anarchy values for congestion games with polynomial latency functions of degree at most \( d \).** The second column features the price-of-anarchy in the un-tolled case, recovering the results of (32–35). The third column describes the values of the price-of-anarchy for optimal tolls utilizing global information (28, 29). The fourth column contains the prices-of-anarchy obtained utilizing optimal local tolls, designed as described earlier. The fifth column contains the prices-of-anarchy resulting from constant (i.e., congestion-independent) optimal tolls. These two columns are composed of entirely novel results, except for the case of constant tolls and \( d = 1 \), which appeared in (28). While such results have been obtained with \( n = 100 \), identical efficiency values are found for larger \( n \), see the supplementary material. Note that i) optimal tolls relying only on local information perform very closely to optimal tolls designed using global information, with a difference in performance below \( 1\% \) for \( d = 1 \), see last column; ii) congestion-independent tolls result in a price-of-anarchy that is comparable to that obtained using congestion-dependent local tolls only for polynomials of low degree.

**Congestion-independent vs congestion-aware local tolls.** In this section we compare the performance of congestion-independent optimal local tolls (i.e., tolls that are constant for any value of the congestion), with that of congestion-aware optimal local tolls, designed as per the above-procedure. Congestion-independent local tolls are attractive from the standpoint of the central planner because of their simplicity. Indeed, congestion-independent tolls do not require any physical infrastructure devoted to measuring the congestion in real-time, and to broadcasting the corresponding tolls. In this respect, many of the existing congestion-pricing schemes are of this form (e.g., London Congestion Charge (40), or the proposed Congestion Princing Plan for New York (41)).

Congestion-independent optimal local tolls can be computed through a simple modification of the linear program presented in Eq. (3), further showcasing the breadth of this approach. In further details, Eq. (3) needs to be modified so as to include the constraints imposing \( \tau_j^{\text{opt}} \) to be congestion-independent, i.e., \( \tau_j^{\text{opt}}(1) = \cdots = \tau_j^{\text{opt}}(n) \), see supplementary material. The values of the price-of-anarchy associated to congestion-independent optimal local tolls are presented in the fifth column of Table 1. Observe that the corresponding performance is comparable with that of congestion-aware tolls for polynomial latency functions of low degree (\( d \leq 3 \)). This suggests that, whenever the travel time can be modeled as a low order polynomial, congestion-independent tolls are not only robust and simple to implement, but also efficient.

**Marginal cost tolls are worse than no tolls.** In the continuous-flow approximation model studied in, e.g., (42–44), users are thought of as a continuous mass flowing on a road network. Within this setup, any Nash equilibrium resulting from the application of marginal contribution tolls (known also as Pigovian tolls) is optimal, i.e., it has a price-of-anarchy equal to one. This is a consequence of two features: i) utilizing marginal cost tolls ensures that the optimal routing is a Nash equilibrium; ii) the Nash equilibrium is guaranteed to be unique.

We have utilized the newly developed approach presented earlier to compute the efficiency of marginal cost tolls \( T^p(\ell) = \tau^p \) in the original setup of (16), where they take the form

\[
\tau^p(x) = (x - 1)(\ell(x) - \ell(x - 1)),
\]

with the convention that \( \ell(0) = 0 \). The resulting values of the price-of-anarchy are presented in the third column of Table 2. Surprisingly, utilizing marginal cost tolls on the original atomic model yields a system performance that is lower than

\[\text{(32–35)}\]
We believe that the linear programming formulation developed here coupled with recent results in robust optimization can be further developed to probe this question.

While we considered incentives that vary depending on the traffic congestion, it is important to understand whether such mechanisms can be easily implemented utilizing the current infrastructure. As technology in vehicles and mobile devices further integrates into society’s means of transportation, the ability to reach users with incentives improves, and the prospect of adjusting prices in real-time becomes increasingly realistic (45). Certainly, less sophisticated tolling schemes can be considered, such as fixed-price tolls (also featured in this work) or pricing only certain roads or lanes. We remark that in all these scenarios one should expect a complex and nuanced compromise between the simplicity of the mechanism, its robustness, and the loss in performance. While this work gives insight into such a trade-off by analyzing both fixed and non-fixed tolls, characterizing this trade-off more broadly remains an open question.

Finally, our analysis is based on the assumption that users act rationally, purely interested in minimizing their own travel time. This assumption has found partial empirical evidence on a macroscopic scale (46) and it’s further justified by the increase in the utilization of routing maps that provide the users with “selfish” routing options (47). Nevertheless, the rationality assumption should be thought of as a first order approximation that does not fully capture the users response.

Table 2. Comparison between the price-of-anarchy for congestion games with polynomial latency functions of degree at most d. The second column describes the price of anarchy in the un-tolled case; the third column contains the price-of-anarchy resulting from the use of marginal cost tolls (Pigouvian tolls). Unlike in the continuous-flow approximation, marginal cost tolls result in a lower equilibrium time than that encountered if no mechanisms was used (see Fig. 6 for a worst-case instance achieving PoA = 3 for d = 1). While such results have been obtained with n = 100, identical efficiency values are found for larger n, see the supplementary material.

| d | No toll | Marginal cost toll (this work) |
|---|---|---|
| 1 | 2.500 | 3.000 |
| 2 | 9.583 | 13.000 |
| 3 | 41.536 | 57.364 |
| 4 | 267.643 | 391.000 |
| 5 | 1513.570 | 2124.205 |
| 6 | 12345.198 | 21337.000 |

Fig. 6. This example demonstrates that marginal cost tolls (Pigouvian tolls) induce equilibria whose performance is worse than that encountered by introducing a marginal cost toll at all. Each link has a latency equal to the number of users on that link, while the red/blue arrows denote a Nash equilibrium. (A) and (B) feature two users willing to travel from O₁/O₂ to D. In the untolled case (A) the system cost at the worst Nash-equilibrium (coinciding with the optimum for this instance) is 2. In this setting, Table 1 guarantees that the price of anarchy is always lower than 2.5. The situation worsens when using marginal cost tolls (B), as the worst Nash equilibrium gives a system cost of 6, thus corresponding to a price of anarchy of 3. Introducing marginal cost tolls has worsened the price-of-anarchy from 2.5 to 3.

Discussion and Conclusions

In this article, we studied the problem of designing behaviour-influencing mechanisms for sociotechnical systems (see Fig. 1), where the infrastructure is represented by a road-traffic network, the users behaviour is captured by the notion of Nash equilibrium, and the mechanisms consists in the use of congestion-pricing tolls. Surprisingly, designing tolls that maximize the system-level efficiency is equivalent to solving a simple linear program, thus significantly improving upon the existing work in the literature. As a consequence, we have shown that mechanisms designed utilizing only local information are i) robust to modifications in the problem instance (origin/destination demand, network topology, etc); ii) simple to compute; and iii) perform closely to mechanisms designed utilizing global information, which lack properties i) and ii).

Though our results are encouraging, they prompt as many questions as they answer. First, throughout this paper, we focused on the design of congestion-pricing schemes that are efficient in face of the worst case realization both with regards to the network topology, and the traffic demand. This design paradigm is motivated by the need for mechanisms that are robust against modification in the problem instance. As the network topology is often partially known, and past realization of the traffic demand are available, the following question naturally arises: “Can we incorporate this information to reduce the set of allowable instances and thus provide improved performance while retaining the required robustness properties?” We believe that the linear programming formulation developed here coupled with recent results in robust optimization can be further developed to probe this question.

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Lemma 1. Consider the class of congestion games $G$. For any linear tolling mechanism $T$, it holds that

$$\text{PoA}(T) = \sup_{G \in \mathcal{G}(Q)} \frac{\text{NashCost}(G,T)}{\text{MinCost}(G)} =: \text{PoA}_0(T),$$

where $G(Q) \subset \mathcal{G}$ is the subclass of games with $\alpha_j \in Q$ for all $j \in \{1, \ldots, m\}$, for all edges in $\mathcal{E}$.

Proof. Note that the price-of-anarchy is defined as a supremum, and that $\text{PoA}(T) \leq \text{PoA}(T)$, in general. According to (49, Thm. 1), we know that the price-of-anarchy of a given class of cost-minimization games is the solution of a linear program and that the game $G$ exists. Note that the decision variables entering in (49, Thm. 1) are represented by $\theta(x,y,z) \in \mathbb{R}_{\geq 0}$, where $x,y,z \in \mathbb{N}$ and $1 \leq x + y + z \leq n$, and that a given set of tuples $\theta(x,y,z)$ encodes the action sets of the agents as well as the coefficients of the latency functions corresponding to any game $G$. Let $\theta'((x,y,z)$ represent a tuple solving the linear program in (49, Thm. 1). If all the values $\theta'(x,y,z)$ are rational, then we are done, as the corresponding game $G' \in \mathcal{G}(Q)$. Otherwise, we can show that there exists a sequence of rational-valued games $G_k$ such that the supremum over their prices-of-anarchy is equal to $\text{PoA}(T)$. This follows because the feasible set is a nonempty compact polytope, so that there exists a series of games $G_k$ with $b_k(x,y,z) \in Q$ converging to $\theta'(x,y,z)$ that satisfies the constraints of the linear program, and whose corresponding worst-case efficiency converges to $\text{PoA}(T)$ from below.

Lemma 2. Consider the class of congestion games $G$. For any tolling mechanism $T$, the following equality holds,

$$\text{PoA}(T) = \text{PoA}_0(T) = \sup_{G \in \mathcal{G}(Q)} \frac{\text{NashCost}(G,T)}{\text{MinCost}(G)},$$

where $G(Q) \subset \mathcal{G}$ is the subclass of games with $\alpha_j \in Q$ for all $j \in \{1, \ldots, m\}$, for all edges in $\mathcal{E}$.

Proof. Observe that the left-most equality holds by Lemma 1. For a given game $G \in \mathcal{G}(Q)$, we denote by $d_G$ the lowest common denominator among the latency function coefficients $\alpha_j$, across all the edges of the game. Define $\alpha^*_j = \alpha_j / d_G \in \mathbb{N}$ for all $j \in \{1, \ldots, m\}$, for all edges in $\mathcal{E}$. Since the equilibrium conditions are independent to uniform scaling of the latency functions and tolls, any game $G \in \mathcal{G}(Q)$ with latency coefficients $(\alpha^*_j)^{j=1}_{j=m}$ has the same worst-case equilibrium scaling as a game $G$ which is identical to $G$ except that it has latency coefficients $(\alpha^*_j)^{j=1}_{j=m}$. Observing that $G$ belongs to both $\mathcal{G}(Q)$ and $\mathcal{G}(N)$ concludes the proof.

Characterizing the price-of-anarchy. The authors of (48) show how to compute the price-of-anarchy for given linear tolls as the solution of a linear program. In this section we recall that result, which we will exploit throughout the remainder of the manuscript.

Consider the class of congestion games where the latency functions are generated by the linear combination of the bases $b_1(x), \ldots, b_n(x)$ with non-negative coefficients $(\alpha^*_j)^{j=1}_{j=m}$. Given a linear taxation mechanism $T(\ell) = \sum_{j=1}^{m} \alpha_j \tau_j$, define $f_j(x) = b_j(x) + \tau_j$ for all $1 \leq j \leq n$, and all $j \in \{1, \ldots, m\}$. The authors of (48) show that the price-of-anarchy of the mechanisms $T$ is given by $\text{PoA}(T) = 1/C^*$ where

$$C^* \underset{\text{max}}{=} \max_{\rho, \mu \in \mathbb{R}_{>0}} \rho \sum_{j=1}^{m} b_j(x) \left( x^2 + x + \mu \delta_j(x+y+y) + \nu f_j(x+y) y \right) \left( f_j^*(x+y) + z \right) \geq 0$$

$s.t. b_j(x+y) \in \mathbb{N}$, $1 \leq j \leq m$, $\forall j \in \{1, \ldots, m\}$.

In the special case where all the functions $(f_j^*)^{n=1}_{n=1}$ are non-decreasing in $x$, i.e., $f_j(x+1) \geq f_j(x)$ for all $1 \leq x \leq n-1$ and all $j \in \{1, \ldots, m\}$, it is sufficient to only consider the constraints with $x = \max(0, y + z - n)$, as demonstrated in (49, Cor. 1). Then, the linear program simplifies to

$$C^* \underset{\text{max}}{=} \max_{\rho \in \mathbb{R}_{>0}} \rho \sum_{j=1}^{m}$$

$s.t. b_j(x+y) \in \mathbb{N}$, $1 \leq j \leq m$, $\forall j \in \{1, \ldots, m\}.$

where $u, v \in \{0, \ldots, n\}$.

2. Optimal local tolls for arbitrary number of agents

While the linear programming formulation introduced in Eq. (3) provides an optimal tolling mechanism and a corresponding price-of-anarchy when the number of agents is upper-bounded by $n$, in this section we show how to design tolling mechanisms that provide price-of-anarchy values identical to those in Table 1 and apply to any $n$ (possibly infinite).

For clarity and conciseness of exposition, we consider the case of $d = 1$ as a similar approach can be applied to any $d$. For $d = 1$, Table 1 shows that the optimal price-of-anarchy is 2.012 (up to three decimal digits), obtained with a choice of $n = 100$. Here we intend to derive a mechanism that will give the same performance for any arbitrarily large $n$. When $d = 1$ (affine congestion games) any latency function of the form $f(x) = \alpha_1 + \alpha_2 x$ (with non-negative $\alpha_1$, $\alpha_2$) is allowed, thus corresponding to bases $b_1(x) = 1$ and $b_2(x) = x$. A linear tolling mechanism is obtained as $T(\ell) = \alpha_1 \tau_1 + \alpha_2 \tau_2$, where we let $\tau_1(x) = f_1(x) - b_1(x)$ and $\tau_2(x) = (f_2(x) - b_2(x)$). We are therefore left to define $f_1(x)$, $f_2(x)$ for any $x \in \mathbb{N}$ (Part 1) and to quantify the resulting performance (Part 2). We begin with an overview of these two parts, followed by a detailed proof.

- Part 1. We define $f_1(x) = 1$ for any $x \in \mathbb{N}$. The definition of $f_2$ is instead more involved, and is described in the following. We utilize a modified version of the linear program of Eq. (3) where we consider only the basis $b_2(x) = x$, and compute a solution $f_2^{\text{opt}}$ for a given $n = \overline{n}$. We then construct $f_2(x)$ for any $x \in \mathbb{N}$, matching $f_2^{\text{opt}}$ for $x \leq \overline{n}/2$, while we provide an analytical expression for $f_2(x)$ when $x > \overline{n}/2$.

- Part 2. We show that for some choice of $\overline{n}$ finite (in fact we will choose $\overline{n} = 100$) the price-of-anarchy associated to the pair $f_1, f_2$ matches 2.012 up to three decimal digits, regardless of how large $n$ is.

Part 1. As anticipated, we let $f_1(x) = 1$ for all $x \in \mathbb{N}$, while the definition of $f_2(x)$ is described in the following. Towards this goal, for any given choice of $n$, even, we solve the following linear program corresponding to the basis $b_1(x) = x$

$$\max_{\rho, \mu \in \mathbb{R}_{>0}} \rho \sum_{j=1}^{m}$$

$s.t. (x+y)^2 - \rho (x+y)^2 + f(x+y) y \geq (f(x+y)+1) i \geq 0$\n
$v \geq 0, \forall x \in \{1, \ldots, n\}$, \n
$f(x) \geq x$\n
$f(x+1) \geq f(x)$ \n
$\forall x \in \{1, \ldots, n\}$. 

with the usual convention that $f(0) = f(n+1) = 0$. Note that the previous program is identical to that in Eq. (3) except that i) we are no longer requiring positivity of the tolls (we will fix this at the end); and ii) we have included two additional sets of constraints, i.e., $f(x) \leq x$ for $x \in \{1, \ldots, n\}$ and $f(x+1) \geq f(x)$ for $x \in \{1, \ldots, n-1\}$. We let $(\rho^{\text{opt}}, \mu^{\text{opt}})$ be a solution of this program and utilize it to define $f_2 : \mathbb{N} \rightarrow \mathbb{R}$ as follows

$$f_2(x) = \begin{cases} f_2^{\text{opt}}(x) & \text{for } x \leq \overline{n}/2, \\
\beta \cdot x & \text{for } x > \overline{n}/2, \end{cases} \text{ where } \beta = \frac{f_2^{\text{opt}}(\overline{n}/2)}{\overline{n}/2}. \ \ \ [9]$$

Note that $f_2$ is defined for any integer $x \in \mathbb{N}$, while $\beta \geq 0$ was chosen so that the two expressions defining $f_2$ match for $x = 1^6$ Observe that $f_2(1) \geq 0$ since having $f_2(1) < 0$ would always result in a lower performance, as shown in (50). Therefore $f_2(\overline{n}/2) \geq 0$ as it is feasible for Eq. (8), which includes the constraint $f(x+1) \geq (x)$. Hence, $\beta \geq 0$.
\( n / 2 \). While the expression of \( f_2 \) and all forthcoming quantities depends on the choice of \( n \), we do not make this explicit to ease the notation. When \( n = 10 \), one has, for example, \( \rho^{\text{opt}} = 0.4971, \beta = 0.6364 \), where the first \( n / 2 \) components of \( f^{\text{opt}} \) are \( f^{\text{opt}} = [1, 1.5029, 2.0176, 2.5793, 3.1822] \), thus fully defining \( f_2 \).

**Part 2.** We intend to show that the pair \( f_1, f_2 \) results in a price-of-anarchy of 2.012 when \( n \) is chosen to be sufficiently large (but finite), regardless of the value of \( n \). Towards this goal, we observe that \( f_1 \) is non-decreasing by definition, while \( f_2 \) is non-decreasing because it is a solution of Eq. (8) for \( x \leq n/2 \) and because of its definition in Eq. (9) for \( x > n/2 \). Since both \( f_1 \) and \( f_2 \) are non-decreasing, we utilize the linear program in Eq. (7), to characterize the performance of \( f_1, f_2 \), which reads as

\[
\begin{align*}
\max_{\rho \in [0, \infty), \nu \in [0, \infty)} & \quad \rho \\
\text{s.t.} & \quad v - pu + \nu[u f_1(u) - v f_2(u + 1)] \geq 0 \quad u + v \leq n, \quad u + v > n, \\
& \quad v - pu + \nu[(n + v) f_1(u) - (n + u) f_1(u + 1)] \geq 0 \quad u + v \leq n, \\
& \quad v^2 - pu^2 + \nu[u f_2(u) - v f_2(u + 1)] \geq 0 \quad u + v \leq n, \\
& \quad v^2 - pu^2 + \nu[(n - v) f_2(u) - (n - u) f_2(u + 1)] \geq 0 \quad u + v > n,
\end{align*}
\]

where the indices \( u, v \) belong to \( \{0, \ldots, n\} \). Characterizing the performance of the given mechanism amounts to finding the largest possible \( \rho \) that is feasible for the latter program. The challenging task is that we intend to do so for \( n \) arbitrary large.

It is immediate to observe, that the constraints involving \( f_1 \) reduce to \( v - pu + \nu u f_1(u) \geq 0 \), both in the case of \( u + v \leq n \) and \( u + v > n \). These inequalities are satisfied for any choice of \( \rho \leq 1 \), and for any \( n \), once we set \( \nu = 1 \). As it will appear clear later, these constraints will not play any role in defining the resulting price-of-anarchy, which will be purely determined by \( f_2 \).

We now turn our attention to the constraints involving \( f_2 \), and divide the discussion in two parts, as the expressions are different for \( u + v \leq n \) and \( u + v > n \). Before doing so, we study the degenerate case of \( u = 0 \), for which the constraints in Eq. (10) reduce to \( f_2(1) \leq 1 \), which holds as \( f_2 \) is feasible for the linear program in Eq. (8) (which already includes this constraint).

- Case of \( u + v \leq n, \ u \geq 1 \). The constraints reduce to

\[
v^2 - pu^2 + \nu u f_2(u) - v f_2(u + 1) \geq 0
\]

which we want to hold for any integers \( u \geq 1, v \geq 0 \) (the bound on the indices \( u + v \leq n \) can be dropped as we are interested in the case of arbitrary large \( n \)).

- We are left with the region where \( u > n/2, v \geq 0 \) (region C in Fig. 7, left). In this case, \( f_2(u) = \beta - u \) by definition. With the choice of \( \nu = 1 \), the constraints in Eq. (11) read

\[
v^2 - pu^2 + \beta u^2 - \beta u(u + 1) \geq 0.
\]

One observes (see Fig. 7, right) that \( v^2 - pu^2 + \beta u^2 - \beta u(u + 1) = 0 \) represents an ellipse in the plane \( (u, v) \) (when allowing \( u, v \) to be real numbers). Additionally, one notes that the ellipse lies entirely on the left of the vertical line \( u = u_1 \), where \( u_1 \) is given by

\[
u_1 = \frac{-\beta - 2\sqrt{\beta - \rho}}{\beta^2 - 4\beta + 4\rho}.
\]

One further observes that the pairs \( (u, v) \) for which Eq. (12) holds are all those lying “outside” of the region enclosed by the ellipse (shaded region in Fig. 7, right). Therefore, in order for Eq. (11) to be satisfied for any \( u \geq n/2 \), it suffices to select \( \rho \) so that \( u_1 \leq n/2 \) in Eq. (13). This results in the following constraint on \( \rho \)

\[
\rho \leq \rho_1 = \frac{\beta(n/2)^2(4 - \beta) - 2(\beta/2)^2\beta - \beta}{4(n/2)^2}.
\]

When \( n = 10 \), \( u_1 \leq 5 \) is satisfied with \( \rho \leq \rho_1 = 0.4906 \).
increasing in \( n \geq \bar{n} \). It follows that the tightest constraint on \( \rho \) is achieved when \( n = \bar{n} \), resulting in
\[
\rho \leq \rho_2 = 1 + 2\beta + \frac{(\beta - \bar{n} + 1)^2}{4\beta}.
\]
With \( \bar{n} = 10 \), this reduces to \( \rho \leq 0.5056 \).

In conclusion, we have shown that \( f_1, f_2, \rho = \min(\rho^{opt}, \rho_1, \rho_2) \), \( \nu = 1 \), is a feasible solution for the linear program in Eq. (10).

3. Congestion-Independent Tolls

As mentioned in the main body of this manuscript, congestion-independent tolls (often referred to as fixed tolls) can be computed using a modification of the linear program in Eq. (3) where additional constraints are introduced. A fixed toll requires that for each basis latency function \( b_j \), the tolling function \( T(b_j) \) is constant with the amount of traffic, i.e., \( \tau(1) = \tau(2) = \ldots = \tau(n) = \tau \).

In conclusion, we have shown that \( f_1, f_2, \rho = \min(\rho^{opt}, \rho_1, \rho_2) \), \( \nu = 1 \), is a feasible solution for the linear program in Eq. (10).

The most binding constraint of interest reduces to
\[
\max_{\rho \in R, \nu \geq 0} \sigma \in R_{\geq 0}^n \geq (1, 1, \ldots, 1) \rho^{opt} \nu \sigma^{opt}
\]
resulting linear program reads as
\[
\begin{align*}
\text{s.t. } & b_j(v) = \rho b_j(u) + \nu b_j(u) + b_j(u) - b_j(u + 1) + \nu \sigma_j(u - v) \geq 0 \\
& u + v < n, \quad \forall j \in \{1, \ldots, n\}, \\
& b_j(v) - \rho b_j(u) + \nu b_j(u) - b_j(u + 1)(1 - \nu \sigma_j(u - v)) \geq 0 \\
& u + v < n, \quad \forall j \in \{1, \ldots, n\},
\end{align*}
\]
where \( u, v \in \{0, \ldots, n\} \). If \( (\rho^{opt}, \nu^{opt}, \sigma^{opt}) \) is the solution to Eq. (17), then the resulting price-of-anarchy is \( 1/\rho^{opt} \), and the optimal fixed toll for resources with latency \( b_j \) is
\[
T(b_j) = \tau = \frac{\rho^{opt}}{\rho^{opt}}.
\]
The price-of-anarchy bounds from the solutions of Eq. (17) for polynomials of order at most \( d = 1, 2, \ldots, 6 \) are shown in Table 1.

To illustrate that these are indeed the true price-of-anarchy values, and that these numbers hold for any arbitrarily large \( n \), we will independently prove the bound for congestion games where each edge is associated to a quadratic latency function, i.e., a latency function of the form \( \ell(x) = ax^2 \), \( a \geq 0 \) (the case of \( d = 1 \) has been shown in (28) resulting in an optimal price-of-anarchy of \( 1 + 2/\sqrt{3} \approx 2.557 \)). In this case, we will also provide an analytic expression for the optimal fixed toll. A similar argument can be applied to the case when \( d \geq 2 \).

Theorem 1. Consider any congestion game where the latency function on each edge is of the form \( \ell(x) = ax^2 \) for some \( a \geq 0 \). An optimal constant toll minimizing the price-of-anarchy is \( T^{opt}(\ell) = a\sigma^{opt} = 3a \), resulting in a price-of-anarchy of \( 16/3 \approx 5.33 \).

Proof. We prove the claim in two steps. First, we show that the price-of-anarchy for any fixed toll is larger than \( 16/3 \). Second, we show that the price-of-anarchy obtained with \( T = a\sigma^{opt} \) is lower-bounded by \( 16/3 \). Towards this goal, observe that any constant tolling mechanisms applied to a quadratic latency \( \ell(x) = ax^2 \) takes the form \( T(\ell) = a\sigma \), for some \( \sigma \geq 0 \).

For any \( \tau \geq 3 \) we consider the following problem instance: there are 8 agents each with two actions \( a_1^{i*} \) and \( a_2^{i*} \). In action \( a_1^{i*} \), user \( i \) selects 6 of the available 8 edges, which are associated to a latency \( \gamma a_1^{i*} \); in action \( a_2^{i*} \) user \( i \) selects the remaining two edges with latency \( \gamma a_2^{i*} \), as well as one edge with latency \( \beta a_2^{i*} \). Each player has a similar pair of actions, but each subsequent agent is offset by one from the prior user, as depicted in Fig. 9.

In this game, the system and user costs read as follows
\[
\begin{align*}
\text{SC}(a^{i*}) &= 6(\gamma b(6)) = 216\gamma \\
\text{SC}(a^{opt}) &= 2b(2) + \beta b(1) = 8\gamma + \beta \\
C_i(a^{i*}) &= 6\gamma(b(6) + \tau) = 216\gamma + 6\tau \\
C_i(a^{opt}, a^{i*}, a^{opt}) &= 2(49 + \tau) + \beta(1 + \tau).
\end{align*}
\]
We normalize the costs in the game setting \( \text{SC}(a^{i*}) = 1 \), which results in \( \gamma = 1/216 \) from Eq. (17). To ensure that the joint action \( a^{i*} \) is a Nash equilibrium (at least weakly), we impose that
\[
C_i(a^{i*}) = C_i(a^{opt}, a^{i*}) \quad \text{for any player } i.
\]
This condition is satisfied when
\[
\beta = \frac{2\tau + 59}{108(1 + \tau)}.
\]
Finally, for the fixed toll $\tau = 3$, we upper-bound the price-of-anarchy at $16/3$. Let $a^{ne}$ be an equilibrium in a congestion game $G$, with $n$ users, edges in $E$, where all latencies take the form $\ell(x) = c x^2$, for some $c \geq 0$. Then the cost at equilibrium satisfies

$$SC(a^{ne}) \leq \sum_{i=1}^{n} C_i(a^{opt}, a^{ne}_i) - \sum_{i=1}^{n} C_i(a^{ne}) + SC(a^{ne})$$

and by Eq. (20) follows from the parameterization introduced in (30) (also recalled above), and substituting $h_s(x) = x^2$. Eq. (21) follows by replacing $\tau = 3$ and by $x_e \geq 0$. To see that Eq. (22) holds for all integers $x_e, y_e \geq 0$, we define $u = x_e + y_e \geq 0$, $v = x_e + y_e \geq 0$, and divide the argument in two parts depending on whether the integer tuple $(u, v) \in \{u \geq 22$ or $v \geq 7\}$ or not. For the case of $(u, v) \in \{u \geq 22$ or $v \geq 7\}$, we observe that

$$4v^3 + \frac{1}{4}v^3 - v(u + 2)^2 \geq 0.$$ 

Differentiating with respect to $v$ shows that $v = (u + 2)/\sqrt{12}$ are the only minima in $v$ that occur in the positive orthant and the left hand side of Eq. (24) is convex in $v$ in this space. For any $u > 22$, this minimum satisfies Eq. (24), thus for any $v > 0$ and $u > 22$ Eq. (24) is satisfied. Additionally, when $v = 7$, for each $u \in \{0, \ldots, 22\}$, Eq. (24) is satisfied and the minimum in $v$ occurs at $v^* < v$, implying that Eq. (24) holds for every $v \geq 7$ as well. Therefore Eq. (24) (and consequently Eq. (22)) is satisfied for all $(u, v) \in \{u \geq 22$ or $v \geq 7\}$. One can enumerate the finitely-many non-negative integers $(u, v)$ with $u < 22$, $v < 7$ and verify that Eq. (22) holds. The inequality in Eq. (23) implies that the price-of-anarchy is upper bounded by $16/3$. Observe that this bound holds for arbitrarily large $n$ and matches the solution of the linear program, stated in Table 1.

### 4. Price-of-anarchy of marginal cost tolls

In this section we derive the exact value of the price-of-anarchy when Pigouian tolls are utilized. We do so for the case of $d = 1$ as similar arguments carry over for the case of $d \geq 1$.

We begin by observing that Fig. 6 is an example of an affine congestion game under discretized Pigouian tolls with $n = 2$, and that it has a price-of-anarchy of 3. Thus, it must be that $\text{PoA}(TP) \geq 3$ for congestion games with affine latency functions. We now show that $\text{PoA}(TP) \leq 3$ for the linear latency basis, $b(x) = x$, via a smoothness argument. We note that $TP(x) = x - 1$ for the linear latency basis. For $a^{new} \in A$ satisfying the Nash equilibrium
constraints, and $a^{\text{opt}} \in \mathcal{A}$, the optimal allocation, we get that

$$SC(a^{\text{ne}}) \leq \sum_{i=1}^{n} C_i(a_i^{\text{opt}}, a_i^{\text{ne}}) - \sum_{i=1}^{n} C_i(a_i^{\text{ne}}) + SC(a^{\text{ne}}) \quad [25]$$

$$= \sum_{e \in \mathcal{E}} \alpha_e \left[ x_e(2x_e + 2y_e + 1) - y_e(2x_e + 2y_e - 1) + (x_e + y_e)^2 \right]$$

$$+ \beta_e [x_e - y_e + (x_e + y_e)]$$

$$\leq \sum_{e \in \mathcal{E}} \alpha_e \left[ (x_e+y_e)-(x_e+y_e)^2+2(x_e+y_e)(x_e+z_e)+(x_e+z_e) \right]$$

$$+ \beta_e [x_e+z_e] \quad [26]$$

$$\leq \sum_{e \in \mathcal{E}} \alpha_e \left[ 3(x_e + z_e)^2 \right] + \beta_e [3(x_e + z_e)] = 3 \cdot SC(a^{\text{opt}}), \quad [27]$$

where again we utilize the notation $y_e = |a_i^{\text{ne}}|_e - x_e$, $x_e = |a^{\text{opt}}|_e - x_e$, and $x_e = |\{i \in N \text{ s.t. } e \in a_i^{\text{ne}} \cap a_i^{\text{opt}}\}|$. Eq. (25) holds by the definition of the Nash equilibrium, and Eq. (26) holds due to nonnegativity of $x_e$ and $\alpha_e$. One can verify that Eq. (27) holds, using $u = x_e + y_e \geq 0$, $v = x_e + z_e \geq 0$, and observing that the region

$$3v^2 + u^2 - u - 2uv - v \geq 0$$

is the exterior of an ellipse, and contains all $(u, v) \in \mathbb{N}^2$. Rearranging the above inequality, we get $\text{PoA}(T_p) \leq SC(a^{\text{ne}})/SC(a^{\text{opt}}) \leq 3$. 

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