Switching the anomalous DC response of an AC-driven quantum many-body system

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For a class of integrable quantum many-body systems, symmetric AC driving can generically produce a steady DC response. We show how such dynamical freezing can be switched off, not by forcing the system to follow the (arbitrarily fast) driving field, but rather through a much slower but complete oscillation of each individual mode of the system at a frequency of its own, with the slowest mode exhibiting a divergent period. This switching can be controlled in detail, its sharpness depending on a particular parameter of the Hamiltonian. The phenomenon has a robust manifestation even in the few-body limit, perhaps the most promising setting for realisation within existing frameworks.

The coherent non-equilibrium dynamics of driven quantum systems offers a rich and largely uncharted field of many-body physics. While the physics of a periodically driven single quantum particle has remained a lively topic of interesting research through past decades (see, e.g., [1][5]), recent studies in periodically driven quantum many-body system have revealed surprising phenomena like dynamical many-body freezing [3][8], appearance of slow solitary oscillations [7], superfluid-insulator and localization transition due to effective renormalization of the hopping [9][11] (which often drastically fail to conform to our classical intuition). Here we add a novel phenomenon to this collection, that of regime switching: at its most extreme, simply tuning a parameter in the Hamiltonian can lead to a transition between a completely frozen and an entirely dynamic regime.

Our study contributes to the broader question of how quantum systems respond to non-adiabatic driving. A conceptually simple setting is provided by a quantum system driven periodically in time with a purely AC driving of frequency ω, with mz(t) denoting the “response” — a local quantity that varies in time under the influence of the driving. Let us assume [12] that the equilibrium value of this quantity vanishes when the driving field is zero, and that, like the driving, it is symmetric about zero over a driving period. Starting from the equilibrium state, purely AC driving must produce zero DC response over each cycle in the adiabatic limit (provided it exists, i.e. that there is no level crossing). However, if the driving is non-adiabatic, the response can exhibit an average non-zero DC contribution even in the limit of infinite driving time. In an extreme case, quantum interference may dynamically stabilize the response around a nonzero steady average, and that, like the driving, it is symmetric about zero over a driving period. Starting from the equilibrium state, purely AC driving must produce zero DC response over each cycle in the adiabatic limit (provided it exists, i.e. that there is no level crossing). However, if the driving is non-adiabatic, the response can exhibit an average non-zero DC contribution even in the limit of infinite driving time. In an extreme case, quantum interference may dynamically stabilize the response around a nonzero steady average, Q.

Here we discuss a very different but complementary phenomenon — how a non-zero Q can be made to disappear by simply tuning a coefficient in the Hamiltonian. This switching between Q finite and Q = 0 is associated with complete transfer of population between quasi-particle levels due to the driving, which surprisingly happens on very long timescales even when the driving is arbitrarily fast. Under the special condition of maximal freezing, the transition approaches a discontinuous limit. The phenomenon has a dramatic precursor already in the limit of small system-sizes (≥ 6 spins). In fact, conditions for switching are less restrictive in this limit, and hence more easily observable in experiment.

In the remainder of this paper, we first summarise the key observations for a quantum spin chain in a time-varying field. We then provide a detailed analytical theory, which agrees with a numerically exact treatment very well in the limit of fast driving. This is followed by an account of a finite-size version of the switching effect, together with a discussion of experimental realisability, and a concluding outlook.

Regime switching: Consider the Hamiltonian

\[ H_{XY} = -\frac{1}{2} \sum_{i=1}^{L} \sigma^x_i \sigma^y_{i+1} + J_y \sum_{i=1}^{L} \sigma^y_i \sigma^y_{i+1} + h_z(t) \sum_{i=1}^{L} \sigma^z_i \]

(2)

with σα Pauli matrices, h_z(t) = h_0 \cos(\omega t), and

\[ \gamma = J_x - J_y \quad \text{and} \quad \delta = J_x + J_y . \]

(3)

We follow the transverse magnetization m_z(t) = \frac{1}{L} \sum_i \sigma^z_i, taking the initial state to be fully polarized in the +z-direction (m_z = 1). Driving is strong and rapid, ω, h_0 > |γ|, |δ|.

Fig. 1 (a) shows m_z(t) for a system of L = 10^4 spins from a numerical solution of the time-dependent Schrödinger equation. After a transient, m_z oscillates about a non-zero steady average, Q. This approaches zero as δ → 0. There is a concomitant substantial enhancement of dynamics over long time-scales, visible in the increasing amplitude of oscillations about Q.

The dependence of Q on driving parameters is shown in Fig. 1 (b),(c). Besides the modulation of Q(ω) at fixed δ

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including well known integrable 1D quantum models, such as the transverse Ising chain, 1D p-wave superconductor \cite{16,17}, or quantum spin ladders \cite{18}. For instance, the Hamiltonian of a 1D p-wave superconducting wire of spin-less fermions can be described by mapping $\delta$ onto the hopping, $\gamma$ onto the pairing gap, and $h(t)$ onto the chemical potential, with $m_z = 2n_f - 1$ related \cite{16} the average fermionic occupation number, $n_f$. A similar mapping is applicable to the Kitaev spin ladder in \cite{13}.

**Analytical Theory**: We reduce Hamiltonian in \cite{2} to the direct sum of $2 \times 2$ Hamiltonians in momentum space using Jordan-Wigner and Fourier transformations \cite{13}:

$$H = \sum_k \psi_k^\dagger H_k(t) \psi_k; \quad H_k = [h(t) + \delta f_k] \sigma^z + \Delta_k \sigma^+ \Delta_k^* \sigma^-$$

where $k$ is the quasi-momentum, $\psi_k = (c_1^k(k), c_2^k(k))$ is the spinor representing the fermionic operators $c_1, c_2(k)$ in $k$-space, $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$, $\delta$ and $h(t)$ (independent of $k$) are the parameters in the many-body Hamiltonian, and $\Delta_k = \gamma \sin k$ and $\delta f_k = \delta \cos k$ are functions of $k$. In the Schrödinger picture $H_k$ represents the Hamiltonian for the time-dependent wave function, in which turn is a direct product of $k$-space wave functions:

$$|\psi(t)\rangle = \prod_k |\psi_k(t)\rangle = \prod_k |u_k(t)|_{0_k} + |v_k(t)|_{1_k}.$$  \hspace{1cm} (5)

were $\{0_k, 1_k\}$ are the basis states in which the matrix form of $H_k$ is expressed in \cite{4}.

The Hamiltonian in \cite{4} represents a class of systems including well known integrable 1D quantum models, such as the transverse Ising chain, 1D p-wave superconductor \cite{16,17}, or quantum spin ladders \cite{18}. For instance, the Hamiltonian of a 1D p-wave superconducting wire of spin-less fermions can be described by mapping $\delta$ onto the hopping, $\gamma$ onto the pairing gap, and $h(t)$ onto the chemical potential, with $m_z = 2n_f - 1$ related \cite{16} the average fermionic occupation number, $n_f$. A similar mapping is applicable to the Kitaev spin ladder in \cite{13}.

The momentum space wave-function \cite{5} satisfies the following time-dependent Schrödinger equation

$$i \frac{d |\psi(t)\rangle_k}{dt} = H_k |\psi(t)\rangle_k$$  \hspace{1cm} (6)

and the time-dependent transverse magnetization reads:

$$m_z(t) = \langle \psi(t) | \sigma^z | \psi(t) \rangle = \frac{4}{N} \sum_{k>\omega_0} (|v_k(t)|^2 - 1/2)$$  \hspace{1cm} (7)

An exact analytical solution of Eq. 6 is not available, but for large $\omega$, analytic solution is possible in the rotating wave approximation (RWA) \cite{6,22}. Starting from a fully polarized ($m_z = 1$) initial state, corresponding to $v_k(t=0) = 1$ (a good approximation to the ground state of the Hamiltonian at $t=0$ for $\hbar_0 \gg \gamma, \delta$),

$$|u_k(t)|^2 = 1 - |v_k(t)|^2 = \frac{\Delta_k^2 (2\hbar_0/\omega) \Delta_0^2}{\phi_k^2} \sin^2 (\phi_k t).$$  \hspace{1cm} (8)

where

$$\phi_k = \sqrt{\Delta_k^2 (2\hbar_0/\omega) \Delta_0^2 + \phi^2 \cos^2 k}.$$  \hspace{1cm} (9)

For periodic boundary conditions, we have wave vectors of the form $k = (2n + 1)\pi/L$, where $n$ denotes positive integers. Averaging over time and integrating over $k$ in the continuum ($L \to \infty$) limit,

$$Q = \frac{|\delta|}{|\gamma J_0(2\hbar_0/\omega)| + |\delta|}.$$  \hspace{1cm} (10)
This formula (Fig. 1b) is one of our central results. It very accurately reproduces the numerical results (Fig. 1c), in particular the switching off of $Q$ for $\delta = 0$ unless the Bessel function simultaneously vanishes.

**Disappearance of $Q$:** From Eq. (8), the maximum amplitude of excitation for any given mode is $J_0^2(2\hbar_0/\omega)\Delta_k^2$, which is almost unity, and in general smaller. This is what leads to a finite $Q$ in our case, as is apparent from the expression of $m_2$ in Eq. (7). Note here the role of the apparent “DC” part $\delta \cos k$ in the $2 \times 2$ Hamiltonian (Eq. 6), which plays the same role as would have been played by an additional DC part in the transverse field (albeit without the $k$-dependence). It is essential for the anomalous dc response: for $\delta = 0$, we have $Q = 0$ for all modes and for all $\omega$, however large!

It is important to note that $Q = 0$ emphatically does not reflect any adiabatic dynamics: eqs. (9,10) encode the drastic enhancement of dynamics of each $k$-mode at $\delta = 0$. At this point each mode has full oscillation with amplitude $|\mu_k|^2$ (but not frequency $\phi_k$ independent of both $k$ and parameters of the Hamiltonian. Thus, the characteristic time of the full population transfer for a mode of quasi-momentum $k$, $T_k = 2\pi/\phi_k$. Indeed, from Eq. (1) it follows that $T \to \infty$ for $\delta = 0$ and $\cos k \to 0$, so that some modes oscillate on diverging timescales, leading to an asymptotic $m_2(t) \sim J_0(2J_0(2\hbar_0/\omega)|t|) \sim |J_0(2\hbar_0/\omega)|t|^{-1}$. This form evidences the singular nature of the points $(\omega = \omega^P, \delta = 0)$, where $\omega^P$ are the zeroes of the Bessel function, as we discuss below.

**FIG. 2:** Short time vs long time dynamics: Discrepancy between small short-time and large long-time excursions for the dynamical case, $\delta = 0$, while, for $\delta = 0.1$, a fast initial decay is followed by long-time stability around the average $Q \neq 0$ (numerical result for $L = 100$).

By contrast, the short-time behaviour offers little clues into what happens at long times (precluding a simple picture based on naive repetition of the single sweep result) and in fact makes the dynamical curves look more static than the frozen ones, in keeping with the observation that switching involves times much longer than the sweeping period: Fig. 2 compares dynamics for $\delta = 0$ and $\delta = 0.1$ close to the sharp transition point $\omega = \omega^P + 10^{-1}$. For $\delta = 0$, while the long-time dynamics takes the system far away from the initial state, at the end of first few cycles, the system returns remarkably close to the initial state. But for $\delta = 0.1$, the system remains quite close to the initial state for all later time, while its deviation from the initial state at the end of few initial cycles is systematically much larger than that for the $\delta = 0$ case.

**Sharpness of the transition and the discontinuous limit:** As seen from Fig. 1b, (c), the sharpness of the transition varies non-monotonically with $\omega$. For discrete values of $\omega$, the limit is discontinuous: $Q = 1$ for $\delta \neq 0$ and $Q = 0$ for $\delta = 0$. As can be seen from Eq. (6), if $\delta \neq 0$, $Q = 1$ is maximal at the frequencies, where $J_0(2\hbar_0/\omega) = 0$, and, up to remarkable accuracy all the $k$-modes remain completely frozen for all time [2]. However, if $\delta = 0$, all modes undergo full oscillation. Hence the limits $J_0(2\hbar_0/\omega)\Delta_k \to 0$ and also $\cos k \to 0$ do not commute: we either obtain $Q = 0$ or $Q > 1$, switching between complete dynamics and complete freezing. The discontinuity is of course derived under RWA, whose accuracy increases with $\omega$, and it becomes exact as $\omega \to \infty$.

The tunable sharpness of the switching might be interesting from the point of view of sensitive detectors of very small static magnetic field. As discussed in the previous section, a small DC component of the transverse field plays the same role as a non-vanishing $\delta$. Now, if one sets $\delta = 0$, and tunes the system close to an $\omega^P$ then a DC transverse field will result in a jump from $Q = 0$ to $Q = 1$. Of course, the resolution time over which coherence needs to be maintained will grow as one approaches $\omega^P$. Since the transition persists in the limit of small system consisting of few spins (see below and the Supplementary Material), maintaining coherence over the period necessary for detecting even very weak fields might not be unrealistic within the framework of cold atoms in optical lattices or ion traps (see, e.g. [20, 21]).

**Experimental possibilities: finite-size versions**

Various kinds of quantum spin Hamiltonians and their coherent dynamics are realizable in cold atomic systems in optical lattice (see, e.g., [20, 21]). One important feature of the switching phenomenon that makes it particularly convenient for experimental realization is its clear signature even at very small system sizes of just a few components (with periodic boundary condition). Suppose we are away from the transition condition (i.e. $\delta \neq 0$), and drive the system with $\omega_0 \gg \gamma, \delta$ close to the large system perfect freezing condition: some small systems, $L = 3, 4, 5, 7, 8, 9$ also freeze. However, at $L = 6$ we see pronounced dynamics in the system, resulting in a strong dip in $Q$. Similar pronounced dynamics are observed at system sizes 10, 14 ... $2(2n+1)$ with gradually decreasing amplitude (Fig. 3).

The origin of such strong dynamics even for $\delta \neq 0$ can be gleaned from Eqs. (6) and (7). For $L = 2(2n+1)$ we have a pair of $k$-modes for which $\delta \cos k = 0$: again the term involving the Bessel factor responsible for the freezing cancels out for that particular pair of modes and
they execute full oscillation. For small $L$, there are only a few modes in total, and hence the contribution from the fully oscillating modes is substantial resulting in the dips in $Q$. As $L$ increases, $Q$ at the dips scales as $Q(L) = 1 - r/L$, where $r$ is a constant (inset of Fig. [3]).

Since this small system-size signature of the transition does not require $\delta \to 0$, it can also be observed even in the transverse Ising chain, which is a special of the XY chain with $\delta = \gamma = 1$. Indeed, shown in the figure are results for the Ising case where in fact no dynamics is actually expected in the large $L$ limit, since in this case we have $\gamma = \delta = 1$. We emphasize that the transverse Ising chain with time-varying transverse field already has been realized experimentally $[23, 24]$. Different initial conditions: Freezing at $\omega^P$ is maximal as there, every mode freezes individually $[9]$, while for $\delta = 0$, every mode becomes dynamical. In that sense, freezing and switching are independent of the starting state, which only affects the amplitudes $u_k(t = 0)$ and $v_k(t = 0)$, which can be accounted for by a time-independent unitary transformation. Such a transformation preserves $Q = 0$, although full population transfer would not occur in the modified basis. Thus switching and the concomitant discontinuous limit are essentially unaffected but the fully polarized initial state we have chosen provides a large $Q$ away from the transition point, making the transition most clearly visible.

Conclusion and outlook: We have presented switching between (completely) frozen and dynamical nonequilibrium regimes under driving, with regime change effected by simply tuning a parameter in the Hamiltonian. This set of phenomena is particularly intriguing on account of how fast driving yields the appearance of long timescales non-monotonically. Given the applicability of the basic Hamiltonian studied here to a wide range of systems, and the appearance of switching even for small systems, experimental realisation does not appear an unrealistic prospect in the near future.

Several further questions obviously spring to mind. Firstly, what are the necessary conditions for such behaviour to occur beyond the soluble RWA – here, e.g., the point $\delta = 0$ is special in that it exhibits an enhanced XY-like symmetry, as well as a degenerate Floquet spectrum, and appears like a critical point with a diverging timescale – but it is not clear which of these fundamentally underpin such switching mechanisms in general. As a next step, progress towards studying the long-time dynamics of non-integrable systems would be of interest to determine the stability of this phenomenology, this being hampered by the present inability to access the long-time dynamics of such systems. Finally, given the growing zoo of such nonequilibrium quantum phenomena, a systematic zoology based on some yet to be identified organising principles is clearly a most desirable goal in the long term.

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[1] D. H. Dunlap and V. M. Kenkre, Phys. Rev. B 34 3625 (1986).
[2] F. Grossmann, T. Dittrich, P. Jung, and P. Hänggi Phys. Rev. Lett. 67 516 (1991).
[3] M. Grifoni and P. Hänggi, Phys. Rep. 304 229 (1998).
[4] A. Eckardt et al., Phys. Rev. A 79 013611 (2009).
[5] E. Arimondo et al., Adv. Atomic Mol. Phys. (in press; arXiv:1203.1259v2).
[6] A. Das, Phys. Rev. B 82, 172402 (2010).
[7] S. Bhattacharyya, A. Das and S. Dasgupta, arXiv:1112.6171v1 [quant-phys] (2011).
[8] S. Mondal, D. Pekker and K. Sengupta, arXiv:1204.6331v1 (2012, unpublished).
[9] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. 95 260404 (2005).
[10] A. Eckardt and M. Holthaus, EPL 80, 550004 (2007).
[11] Y. Zou, R. Barnett and G. Refael, Phys. Rev. B 82, 224205 (2010).
[12] If the equilibrium (quasi-static) average of the response over a driving period is non-zero (say due to some equilibrium background value), one subtracts that quasi-static average from the response while defining the anomalous part of the dc response.
[13] Quantum Mechanics Vol. - II, A. Messiah, North Holland Pub. Co. (1962).
[14] V. M. Bastidas, C. Emary, G. Schaller and T. Brandes arXiv:1207.5242v1 (2012, unpublished).
[15] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).
[16] A. Kitaev, arXiv:cond-mat/0010440v2 (2000).
[17] W. DeGottardi, D. Sen and S. Vishveshwara, New J. Phys. 13 065028 (2011).
[18] D. Sen and S. Vishveshwara, EPL 91, 66009 (2010).
[19] B. Damski, W. H. Zurek, Phys.Rev. A 73 063405 (2006).
[20] I. Bloch, J. Dalibard and W. Zwerger Rev. Mod. Phys. 80 885 (2008).

[21] M. Lewenstein et al., Adv. Phys 56 243 (2007).

[22] S. Ashhab, J. R. Johansson, A. M. Zagoskin and F. Nori, Phys. Rev. A 75 063414 (2007).

[23] K. Kim et. al., New J. Phys. 13 105003 (2011); K. Kim et. al., Nature 465 590 (2010).

[24] A. Friendenauer et. al., Nat. Phys. 4, 757 (2008).