HOMOLOGICAL AND MONODROMY REPRESENTATIONS OF FRAMED BRAID GROUPS

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Abstract. In this paper, we introduce two new classes of representations of the framed braid groups. One is the homological representation constructed as the action of a mapping class group on a certain homology group. The other is the monodromy representation of the confluent KZ equation, which is a generalization of the KZ equation to have irregular singularities. We also give a conjectural equivalence between these two classes of representations.

1. Introduction

The framed braid group $FB_n$ [KS92] is the semi-direct product $FB_n \cong \mathbb{Z}^n \rtimes B_n$ where the braid group $B_n$ acts on $\mathbb{Z}^n$ as permutations of components through the projection $B_n \to \mathfrak{S}_n$ on the symmetric group of degree $n$. Similar to the diagrammatic description of the braid group $B_n$ by using $n$ strings, the framed braid group $FB_n$ can be described graphically by using $n$ ribbons. The component $B_n$ describes crossings of ribbons and the component $\mathbb{Z}^n$ describes the number of twists of each ribbon. In particular, the plat closure of an element of $FB_n$ gives a framed link. Whereas there are many studies of representations of the braid groups, representations of the framed braid groups are less known.

The purpose of this paper is to introduce two new classes of representations of the framed braid groups. (These are also new classes as representations of the braid groups.) One is the homological representation constructed as the action of a mapping class on a certain homology group. This construction generalizes the homological representation of the braid group introduced by Lawrence [Law90]. The original Lawrence representation appears as the quotient of our representation. As special cases, our representations contain natural extensions of the reduced Burau representation [Bur36] and the Lawrence-Krammer-Bigelow representation [Big01, Kra00, Kra02] to the framed braid groups.

The other is the monodromy representation of the confluent KZ equation [JNS08], which is a generalization of the KZ equation [KZ84] to have irregular singularities. The confluent KZ equation also appears in two dimensional conformal field theory as the differential equation for irregular conformal blocks of the Wess-Zumino-Witten (WZW) model [GLP]. Thus our representation also describes the monodromy of irregular conformal blocks.

In [Koh12], Kohno showed that the Lawrence homological representation of the braid group is equivalent to the monodromy representation of the KZ equation on the space of singular vectors. Following his result, we also give a conjectural equivalence between the homological representation and the monodromy representation of the framed braid group.
1.1. Homological representations. We summarize the construction of the homological representations of the framed braid groups from Section 3. For a positive integer $n > 0$, let $FB_n$ be the framed group of $n$ ribbons (see Definition 2.1). Set $\mathcal{R} = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. The homological representation

$$\rho^{(r)}_{n,m}: FB_n \to \text{Aut}_\mathcal{R} \mathcal{H}^{(r)}_{n,m},$$

is parametrized by positive integers $m, r > 0$ where $\mathcal{H}^{(r)}_{n,m}$ is a certain relative homology group constructed as follows. Let $D$ be a closed disk and take disjoint $n$ closed disks $D_1, \ldots, D_n \subset \text{Int} D$ from the interior of $D$. An open interval $A \subset \partial D_k$ is called a marked arc. Take disjoint $r$ marked arcs from each boundary $\partial D_k$ and denote by $\mathcal{A}^{(r)}$ the set of such $rn$ marked arcs (Figure 2). Define the surface $S_n^{(r)}$ (Figure 3) by

$$S_n^{(r)} = D \setminus (D_1 \cup \cdots \cup D_n) \cup \bigcup_{A \in \mathcal{A}^{(r)}} A.$$

Let $C^{(r)}_{n,m}$ be the configuration space of unordered $m$ distinct points in $S_n^{(r)}$:

$$C^{(r)}_{n,m} = \{(t_1, \ldots, t_m) \in (S_n^{(r)})^m | t_i \neq t_j \text{ if } i \neq j \}/\mathcal{S}_n.$$  

Then there is a group homomorphism from the fundamental group of $C^{(r)}_{n,m}$ to a free abelian group of rank two

$$\alpha: \pi_1(C^{(r)}_{n,m}, d) \to \langle q \rangle \oplus \langle t \rangle$$

where the generator $q$ corresponds to the loop around the cylinders $\{t_i \in D_k\}$ for $k = 1, \ldots, n$ and the generator $t$ corresponds to the loop around the hyperplanes $\{t_i = t_j\}$ for $1 \leq i < j \leq m$. Let $\pi: \tilde{C}^{(r)}_{n,m} \to C^{(r)}_{n,m}$ the covering space corresponding to $\alpha$. Introduce the subset $\mathcal{A}^{(r)} \subset C^{(r)}_{n,m}$ by

$$\mathcal{A}^{(r)} := \{ \{t_1, \ldots, t_m\} \in C^{(r)}_{n,m} | t_1 \in A \text{ for some } A \in \mathcal{A}^{(r)} \}$$

and its inverse image $\tilde{\mathcal{A}}^{(r)} = \pi^{-1}(\mathcal{A}^{(r)}) \subset \tilde{C}^{(r)}_{n,m}$. The homological representation is constructed on the relative homology group

$$\mathcal{H}^{(r)}_{n,m} = H_m(\tilde{C}^{(r)}_{n,m}, \tilde{\mathcal{A}}^{(r)}; \mathbb{Z}).$$

We note that $\mathcal{H}^{(r)}_{n,m}$ has an $\mathcal{R}$-module structure coming from the action of the deck transformation group $\langle q \rangle \oplus \langle t \rangle$. The mapping class group $\mathfrak{M}(S_n^{(r)})$ is defined to be the group of isotopy classes of orientation preserving diffeomorphisms on $S_n^{(r)}$ which fix the boundary $\partial D$ pointwise.

Then $\mathfrak{M}(S_n^{(r)})$ naturally acts on $\mathcal{H}^{(r)}_{n,m}$. In addition, there is an isomorphism $FB_n \cong \mathfrak{M}(S_n^{(r)})$ by Proposition 2.5. Our main result about the homological representation is the following.

**Theorem 1.1.** For positive integers $m, r > 0$, there is a representation of the framed braid group

$$\rho^{(r)}_{n,m}: FB_n \to \text{Aut}_\mathcal{R} \mathcal{H}^{(r)}_{n,m}.$$
which is constructed as the action of the mapping class group $\mathcal{M}(\mathcal{S}_n^{(r)})$ on the relative homology group $\mathcal{H}^{(r)}_{n,m}$. This representation has the following properties:

1. There is a subrepresentation $\mathcal{L}^{(r)}_{n,m} \subset \mathcal{H}^{(r)}_{n,m}$ which is a free $R$-module of rank
   \[
   \binom{rn + n + m - 2}{m}
   \]
   spanned by certain homology classes, called the standard multifork classes.

2. We have an equality
   \[
   \mathcal{L}^{(r)}_{n,m} \otimes_R Q = \mathcal{H}^{(r)}_{n,m} \otimes_R Q.
   \]
   over the field $Q = \mathbb{Q}(q,t)$. In particular, the standard multifork classes form a basis of $\mathcal{H}^{(r)}_{n,m}$ over $Q$.

Details are given in Section 3. The original Lawrence representation [Law90] appears as follow. In $\mathcal{L}^{(r)}_{n,m}$, there is a natural subrepresentation $\mathcal{N}^{(r)}_{n,m} \subset \mathcal{L}^{(r)}_{n,m}$ of rank
   \[
   \binom{rn + n + m - 2}{m} - \binom{n + m - 2}{m},
   \]
   and the quotient representation $\mathcal{L}_{n,m} = \mathcal{L}^{(r)}_{n,m}/\mathcal{N}^{(r)}_{n,m}$ is equivalent to the Lawrence representation [Law90] (also see a nice review of the Lawrence representation [Ito16, Section 3.1]). In this construction, the factor $\mathbb{Z}^n$ of $FB_n \cong \mathbb{Z}^n \rtimes B_n$ acts on $\mathcal{L}_{n,m}$ trivially, and hence $\mathcal{L}_{n,m}$ has only information about the representation of the braid group $B_n$. As special cases, $\mathcal{L}_{n,1}$ is the reduced Burau representation [Bur36] and $\mathcal{L}_{n,2}$ is the Lawrence-Krammer-Bigelow representation [Big01, Kra00, Kra02]. The faithfulness of the representation $\mathcal{L}_{n,m}$ of $B_n$ for $m \geq 2$ by [Big01, Kra02, Zhe] implies that the representation $\mathcal{L}^{(r)}_{n,m}$ of $FB_n$ is also faithful for $m \geq 2$ (see Corollary 3.10). The polynomial invariants of framed links associated with the reduced Burau representations will be discussed in [Ike] (see Remark 3.19).

1.2. Monodromy representations. We see the construction of the monodromy representations of the framed braid groups. Before going to the confluent KZ equation, we first recall some facts about the KZ equation and its monodromy representations. The Knizhnik-Zamolodchikov (KZ) equation was introduced in [KZ84] as the differential equation which is satisfied by correlation functions (conformal blocks) of the WZW model. Let $\mathfrak{g}$ be a semisimple Lie algebra and $M$ be a $\mathfrak{g}$-module. The KZ equation is an integrable differential equation for a $M^\otimes n$-valued function on the configuration space
   \[
   X_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j\}
   \]
   with regular singularities along the divisors $\{z_i = z_j\}$. In addition, it is invariant under the action of $\mathfrak{S}_n$. Therefore we can associate a representation of the braid group $B_n \to \text{GL}(M^\otimes n)$ as the monodromy representation. The Kohno-Drinfeld theorem [Dri89, Koh87] describes the
monodromy representation of the KZ equation as an $R$-matrix representation of the quantum group $U_q(\mathfrak{g})$. (The case $\mathfrak{g} = \mathfrak{sl}_2$ with vector representations was studied in [TK88].)

Extensions of the KZ equation to have irregular singularities were studied in [BK98, FMTV00, JNS08]. In [BK98, FMTV00], the KZ equation with an irregular singularity of Poincaré rank 1 at $\infty$ was introduced. In [JNS08], they defined the confluent KZ equation which has irregular singularities of arbitrary Poincaré rank on each divisor $\{z_i = z_j\}$ and at $\infty$ in the case $\mathfrak{g} = \mathfrak{sl}_2$. The confluent KZ equation is written by using Gaudin Hamiltonians with irregular singularities. The Gaudin model with irregular singularities was studied in [FFTL10]. In two dimensional conformal field theory, the confluent KZ equation appears as the differential equation for irregular conformal blocks of the WZW model [GLP].

We briefly review the confluent KZ equation [JNS08]. Details are given in Section 4. In the rest of this section, we assume $\mathfrak{g} = \mathfrak{sl}_2$ with the standard basis $\{E, F, H\}$. For a positive integer $r > 0$, introduce the truncated current Lie algebra $\mathfrak{g}^{(r)} := \mathfrak{g}[t]/t^{r+1} \mathfrak{g}[t]$ where $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ is a Lie algebra with the bracket $[X \otimes t^m, Y \otimes t^n] := [X, Y] \otimes t^{m+n}$. Write $X_p = X \otimes t^p$. Then we can define the confluent Verma module $M_{\lambda}(\gamma)$ from the highest weight vector

$$E_i v_{\lambda}(\gamma) = 0, \quad H_0 v_{\lambda}(\gamma) = \lambda v_{\lambda}(\gamma), \quad H_i v_{\lambda}(\gamma) = \gamma^{(i)} v_{\lambda}(\gamma) \quad (i = 1, \ldots, r)$$

where $\lambda \in \mathbb{C}$ is a weight of $H_0$ and $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(r)}) \in \mathbb{C}^r$ with $\gamma^{(r)} \neq 0$ are weights of $H_1, \ldots, H_r$. We call $\gamma^{(1)}, \ldots, \gamma^{(r)}$ movable weights. In the definition of the confluent KZ equation, movable weights $\gamma^{(1)}, \ldots, \gamma^{(r)}$ are regarded as variables of the equation. So we consider the space of movable weights $B^{(r)} = \mathbb{C}^{r-1} \times \mathbb{C}^*$. For $R = (r_1, \ldots, r_n) \in \mathbb{Z}_{>0}^n$ and $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, consider the confluent Verma module bundle

$$E^{(R)}_\Lambda \rightarrow X_n \times B^{(r_1)} \times \cdots \times B^{(r_n)}$$

whose fiber over a point $(z, \gamma_1, \ldots, \gamma_n) \in X_n \times B^{(r_1)} \times \cdots \times B^{(r_n)}$ is the tensor product of the confluent Verma modules $M_{\lambda_1}(\gamma_1) \otimes \cdots \otimes M_{\lambda_n}(\gamma_n)$. Then there is an integrable connection $\nabla^{KZ}$ on $E^{(R)}_\Lambda$ depending on a complex parameter $\kappa \in \mathbb{C}^*$, called the confluent KZ connection.

The confluent KZ connection $\nabla^{KZ}$ has irregular singularities along the divisors $\{z_i = z_j\}$ and $\{\gamma_i^{(r_i)} = 0\}$. In the case $R = (r, \ldots, r) = (r^n)$ and $\Lambda = (\lambda, \ldots, \lambda) = (\lambda^n)$, the action of the symmetric group $\mathfrak{S}_n$ on both the total space $E^{(r^n)}_\Lambda$ and the base space $X_n \times (B^{(r)})^n$ is well-defined. In addition, the confluent KZ connection is $\mathfrak{S}_n$-invariant in this setting. Hence, the confluent KZ connection $\nabla^{KZ}$ descends to an integrable connection on the quotient vector bundle

$$E^{(r^n)}_\Lambda / \mathfrak{S}_n \rightarrow (X_n \times (B^{(r)})^n) / \mathfrak{S}_n.$$
the restriction of the confluent KZ connection on some finite rank subbundles. For a positive
integer \( m \), there is a finite rank subbundle
\[
S^{(r)}[n\lambda - 2m] \subset E^{(r)}_{\lambda m}
\]
consisting of singular vectors of weight \((n\lambda - 2m)\) such that the restriction of the confluent
KZ connection \( \nabla^{KZ} \) on \( S^{(r)}[n\lambda - 2m] \) is well-defined. Our main result about the monodromy
representation is the following.

**Theorem 1.2.** For positive integers \( m, r > 0 \) and complex parameters \((\lambda, \kappa) \in \mathbb{C} \times \mathbb{C}^*\), there
is a representation of the framed braid group
\[
\theta^{(r)}_{\lambda, \kappa} : \text{FB}_n \to \text{Aut}_\mathbb{C} S^{(r)}_1[n\lambda - 2m]
\]
which is constructed as the monodromy representation of the confluent KZ connection \( \nabla^{KZ} \)
on the vector bundle \( S^{(r)}[n\lambda - 2m] \) where \( S^{(r)}_1[n\lambda - 2m] \) is a complex vector space given
by the fiber over a point \((z, \Gamma) \in (X_n \times (B^{(r)})^n)/\mathcal{S}_n\). The dimension of the representation is
given by
\[
\dim_{\mathbb{C}} S^{(r)}_1[n\lambda - 2m] = \binom{rn + n + m - 2}{m}.
\]

A new feature of this construction is that the confluent KZ connection has a non-trivial
monodromy along the weight \( \gamma^{(r)} \neq 0 \) of \( H_r \in \mathfrak{g}^{(r)} \), and this monodromy action changes
framings, i.e. it gives the action of the factor \( \mathbb{Z}^n \) of \( \text{FB}_n \cong \mathbb{Z}^n \rtimes B_n \). In Section 4, we
reformulate the confluent KZ equation [JNS08] in more geometric language and give a detailed
study of it.

1.3. Conjectural equivalences. In [Koh12], Kohno established the equivalence between the
Lawrence homological representation of the braid group and the monodromy representation
of the KZ equation. We expect the following generalization of his result to the framed braid
group.

**Conjecture 1.3.** There is an open dense subset \( U \subset \mathbb{C} \times \mathbb{C}^* \) such that if \((\lambda, \kappa) \in U \), then the
monodromy representation \( \theta^{(r)}_{\lambda, \kappa} \) of \( \text{FB}_n \) on the complex vector space \( S^{(r)}_1[n\lambda - 2m] \) is equivalent
to the homological representation \( \rho^{(r)}_{\lambda, \kappa} \) of \( \text{FB}_n \) on the complex vector space \( \mathcal{H}^{(r)}_{n,m} \otimes \mathbb{C} \) under
the specialization
\[
q = e^{\frac{2\pi \sqrt{-1} \lambda}{\kappa}}, \quad t = -e^{-\frac{2\pi \sqrt{-1}}{\kappa}}
\]
of variables of \( \mathcal{R} = \mathbb{Z}[q^\pm 1, t^\pm 1] \).

Note that our sign convention of exponent is opposite to the convention in [Koh12]. This
is due to the choice of the positive direction of \( q \) and \( t \) for our fork rules in Figure 8. Kohno’s
proof in [Koh12] is based on

- integral representations of solutions of the KZ equation by hypergeometric integrals
  over homology cycles with local system coefficients [DJMM90, SV90],
• linear independence of hypergeometric integral solutions by the determinant formula [Var91],
• identification of the homological representation with the above homology group to construct hypergeometric integral solutions [Koh12].

To show Conjecture 1.3 similarly, we need to develop integral representations of solutions of the confluent KZ equation and the determinant formula. Integral representations of solutions were studied in [JNS08, NS10]. They represented the solution of the confluent KZ equation as an integral of variables $t_1, \ldots, t_m$ of the function $\Phi(t, z, \gamma) = \Phi_1(z, \gamma)\Phi_2(t, z, \gamma)$ where

$$
\Phi_2(t, z, \gamma) = \prod_{1 \leq a < b \leq m} \left( t_a - t_b \right)^{2/\kappa} \prod_{a=1}^{m} \prod_{i=1}^{n} \left( t_a - z_i \right)^{-\lambda_i/\kappa} \exp \left( \frac{1}{\kappa} \sum_{p=1}^{r_i} \frac{1}{p} \frac{\gamma_i^{(p)}}{(t_a - z_i)^p} \right)
$$

and $\Phi_1(z, \gamma)$ is some function only depending on $z$ and $\gamma$ (for details, see [JNS08, NS10]).

Actually, the definition of the homological representation is motivated by the construction of appropriate homology cycles to integrate variables $t_1, \ldots, t_m$ of the function $\Phi_2(t, z, \gamma)$. For convergence of the integration, we need to choose the direction in which $t_a$ approaches to $z_i$ carefully. Depending on the positive integer $r_i$, we can find $r_i$ sectors in $0 < |t_a - z_i| < \epsilon$ where $\Phi_2(t, z, \gamma)$ decays exponentially as $t_a \to z_i$. These sectors essentially correspond to marked arcs in the construction of the homological representation. In two dimensional conformal field theory, the problem of the choice of this direction relates to a free field realization of irregular vertex operators (confluent primary fields) by using screening charges. For details, see [GT12, NS10]. Our construction of the surface with marked arcs also motivated by the work [HKK] which treats quadratic differentials with exponential singularities, and the name “marked arc” is taken from [HKK]. Indeed in the case $m = 1$, the function $\Phi_2$ can be identified with the (multi-valued) quadratic differential on $\mathbb{C}$ with exponential singularities at $z_1, \ldots, z_n$. Finally also in the case $m = 1$, the hypergeometric solution of the confluent KZ equation is equivalent to the hypergeometric function which was extensively studied by Haraoka in [Har97]. In this case, he constructed appropriate integration cycles and also gave the determinant formula.

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2. Framed braid groups

2.1. Definition of framed braid groups. Fix a positive integer $n > 0$. Following [KS92], we introduce the framed braid group as follows.
Definition 2.1. The framed braid group $FB_n$ is a group generated by $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1, \ldots, \tau_n$ with the relations
\[
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2 \\
\tau_i \tau_j &= \tau_j \tau_i \\
\sigma_i \tau_j &= \begin{cases} 
\tau_{i+1} \sigma_i & \text{if } j = i \\
\tau_i \sigma_i & \text{if } j = i + 1 \\
\tau_j \sigma_i & \text{if } j \neq i, i + 1.
\end{cases}
\end{align*}
\]

Graphically generators of the framed braid group can be described by using $n$ ribbons as in Figure 1. The generator $\sigma_i$ represents a crossing of the $i$-th and the $(i + 1)$-st ribbons (left of Figure 1), and the generator $\tau_i$ represents a twisting of the $i$-th ribbon (right of Figure 1). The subgroup of $FB_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$ becomes the braid group $B_n$. On the other hand, generators $\tau_1, \ldots, \tau_n$ form a free abelian group of rank $n$ given by
\[
\langle \tau_1, \ldots, \tau_n \rangle \cong \mathbb{Z}^n, \quad \tau_1^{l_1} \tau_2^{l_2} \cdots \tau_n^{l_n} \mapsto (l_1, l_2, \ldots, l_n).
\]

Let $\mathfrak{S}_n$ be the symmetric group of degree $n$ and $\pi: B_n \to \mathfrak{S}_n$ be a surjective group homomorphism sending $\sigma_i$ to the transposition $(i, i + 1)$. The kernel of $\pi$ is called the pure braid group and denoted by $P_n$. The braid group $B_n$ acts on $\{1, \ldots, n\}$ through the map $\pi$. We write this action as $\sigma(i)$ instead of $\pi(\sigma)(i)$. Then the framed braid group $FB_n$ is isomorphic to the semi-direct product $\mathbb{Z}^n \rtimes B_n$ where $B_n$ acts on $\mathbb{Z}^n$ by
\[
\sigma \cdot (l_1, l_2, \ldots, l_n) := (l_{\sigma(1)}, l_{\sigma(2)}, \ldots, l_{\sigma(n)}).
\]

The group homomorphism $\pi: B_n \to \mathfrak{S}_n$ can be extended to a group homomorphism $\pi': FB_n \to \mathfrak{S}_n$ by sending $\tau_i$ to the identity. We call the kernel of $\pi'$ the pure framed braid group and denote by $FP_n$. It is easy to see that $FP_n \cong \mathbb{Z}^n \rtimes P_n$.

2.2. Description as fundamental groups. Let $X_n$ be the configuration space of ordered distinct $n$ points in $\mathbb{C}$, i.e.
\[
X_n := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j \}.
\]
It is well-known that the fundamental group of $X_n$ is isomorphic to the pure braid group $P_n$. The quotient $X_n/\mathfrak{S}_n$ is the configuration space of unordered distinct $n$ points in $\mathbb{C}$ and its fundamental group is isomorphic to the braid group $B_n$. We give a similar description of the framed braid group. Let $T^n := (S^1)^n$ be the $n$-dimensional torus. Then it is clear that the fundamental group of the direct product $T^n \times X_n$ is isomorphic to the pure framed braid group $FP_n \cong \mathbb{Z}^n \times P_n$. Define the action of $s \in \mathfrak{S}_n$ on $T^n \times X_n$ by
\[
s \cdot (t_1, \ldots, t_n, z_1, \ldots, z_n) := (t_{s(1)}, \ldots, t_{s(n)}, z_{s(1)}, \ldots, z_{s(n)})
\]
for $(t_1, \ldots, t_n) \in T^n$ and $(z_1, \ldots, z_n) \in X_n$. In [KS92], they showed the following.

**Proposition 2.2** ([KS92], Proposition on page 2). The quotient space
\[(T^n \times X_n)/\mathfrak{S}_n\]
is a $K(FB_n, 1)$ space. Hence, its fundamental group is isomorphic to the framed braid group $FB_n$.

This result is used to construct the monodromy representations of the framed braid groups in Section 4.

2.3. **Surfaces with marked arcs.** Fix a positive integer $r$. Let $D_1, D_2, \ldots, D_n$ be disjoint closed disks in $\mathbb{C}$ with centers $1, 2, \ldots, n \in \mathbb{C}$ and the radius $1/4$, i.e.
\[
D_k := \left\{ z \in \mathbb{C} \left| |z - k| \leq \frac{1}{4} \right. \right\}.
\]
Take a sufficiently large closed disk $\mathbb{D} \subset \mathbb{C}$ which contains $D_1, D_2, \ldots, D_n$ in the interior. We define marked arcs $A_k^{(s)}$ on $\partial D_k$ by
\[
A_k^{(s)} := \left\{ k + \frac{1}{4} \exp \left( \frac{i\pi \theta}{r} \right) \in \mathbb{C} \left| \theta \in (2s - 2, 2s - 1) \right. \right\}
\]
for $k = 1, \ldots, n$ and $s = 1, \ldots, r$ (see Figure 2). Denote by $A^{(r)}$ the set of these marked arcs:

**Figure 2.** Marked arcs in $\partial D_k$. 

\[
A^{(r)} := \left\{ A_k^{(s)} \left| k = 1, \ldots, n \text{ and } s = 1, \ldots, r \right. \right\}.
\]
Definition 2.3. For positive integers $n > 0$ and $r > 0$, we define the surfaces $S_n$ and $S_n^r$ by

$$S_n := \mathbb{D} \setminus (D_1 \cup \cdots \cup D_n)$$

and

$$S_n^r := S_n \cup \bigcup_{A \in A(r)} A.$$ 

See Figure 3.

![Figure 3. Surface with marked arcs $S_n^r$.](image)

The surface $S_n^r$ is a connected oriented two dimensional smooth manifold with boundary

$$\partial S_n^r = \partial \mathbb{D} \cup \bigcup_{A \in A(r)} A,$$

and non-compact. We call $\partial \mathbb{D}$ the outer boundary.

2.4. Description as mapping class groups. Let $S_n^r$ be the surface with marked arcs constructed in the previous section. Denote by $\text{Diff}^+(S_n^r, \partial \mathbb{D})$ the group of orientation preserving diffeomorphisms on $S_n^r$ which fix the outer boundary $\partial \mathbb{D}$ pointwise. The subgroup of $\text{Diff}^+(S_n^r, \partial \mathbb{D})$ consisting of diffeomorphisms which are isotopic to the identity is denoted by $\text{Diff}_0^+(S_n^r, \partial \mathbb{D})$.

Definition 2.4. The mapping class group \( \mathcal{M}(S_n^r) \) is the quotient group defined by

$$\mathcal{M}(S_n^r) := \text{Diff}^+(S_n^r, \partial \mathbb{D}) / \text{Diff}_0^+(S_n^r, \partial \mathbb{D}).$$

In the following, we see that the mapping class group \( \mathcal{M}(S_n^r) \) is isomorphic to the framed braid group \( FB_n \). We construct two classes of elements $S_1, \ldots, S_{n-1} \in \mathcal{M}(S_n^r)$ and $T_1, \ldots, T_n \in \mathcal{M}(S_n^r)$ as follows.

(1) The element $S_i \in \mathcal{M}(S_n^r)$ is a clockwise half twist between $\partial D_i$ and $\partial D_{i+1}$ which keeps the angle of circles $\partial D_i$ and $\partial D_{i+1}$, and hence interchanges arcs $A_i^{(s)}$ and $A_{i+1}^{(s)}$ (see the left of Figure 4).


(2) The element $T_i \in \mathcal{M}(S_n^{(r)})$ is a clockwise rotation of the boundary $\partial D_i$ which maps $A_i^{(s)}$ to $A_i^{(s-1)}$ (see the right of Figure 4).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Elements $S_i$ and $T_i$.}
\end{figure}

**Proposition 2.5.** There is an isomorphism of groups

$$FB_n \cong \mathcal{M}(S_n^{(r)})$$

given by

$$\sigma_i \mapsto S_i \quad \text{and} \quad \tau_i \mapsto T_i.$$ 

**Proof.** Let $\mathcal{M}(S_n)$ be the group of isotopy classes of orientation preserving diffeomorphisms on $S_n$ which fix $\partial \mathbb{D}$ pointwise. Note that $S_i$ is also well-defined on $S_n \subset S_n^{(r)}$. It is well-known that the correspondence

$$B_n \to \mathcal{M}(S_n), \quad \sigma_i \mapsto S_i$$

gives an isomorphism between the braid group $B_n$ and the mapping class group $\mathcal{M}(S_n)$. There is a natural surjection $\varphi: \mathcal{M}(S_n^{(r)}) \to \mathcal{M}(S_n)$ and it is easy to see that $\ker \varphi$ is generated by $T_1, \ldots, T_n$. Hence we obtain a short exact sequence

$$0 \to \langle T_1, \ldots, T_n \rangle \to \mathcal{M}(S_n^{(r)}) \to \mathcal{M}(S_n) \to 0$$

and the group $\langle T_1, \ldots, T_n \rangle$ is isomorphic to $\mathbb{Z}^n$. \hfill $\Box$

This result is used to construct the homological representations of the framed braid groups in the next section.

### 3. Homological representations

**3.1. Configuration spaces and certain subsets.** Fix a positive integer $m > 0$. Let $S_n^{(r)}$ be the surface with marked arcs in Definition 2.3. Denote by $C_m(S_n^{(r)})$ the configuration space of unordered distinct $m$ points in $S_n^{(r)}$, i.e.

$$C_m(S_n^{(r)}) := \{ (t_1, \ldots, t_m) \in (S_n^{(r)})^m \mid t_i \neq t_j \text{ if } i \neq j \}/\mathfrak{S}_m$$
where $\mathfrak{S}_m$ is the symmetric group of degree $m$. Write $C_{n,m}^{(r)} := C_m(S_n^{(r)})$ for simplicity. We use the notation $\{t_1, \ldots, t_m\} \in C_{n,m}^{(r)}$ to represent unordered $m$ points.

**Definition 3.1.** The codimension one subset $A^{(r)} \subset C_{n,m}^{(r)}$ is defined as a set of unordered distinct $m$ points in $S_n^{(r)}$ such that at least one of $m$ points stays in some marked arc in $A^{(r)}$, i.e. $A^{(r)} := \{ \{t_1, \ldots, t_m\} \in C_{n,m}^{(r)} | t_1 \in A \text{ for some } A \in A^{(r)} \}$.

We introduce a specified base point in $C_{n,m}^{(r)}$ as follows.

**Definition 3.2.** Let $d_1, \ldots, d_m$ be distinct $m$ points in the outer boundary $\partial \mathbb{D}$ and assume that they lie in the lower half plane with the order $d_1, \ldots, d_m$ from left to right (see Figure 5). We define the base point $d \in C_{n,m}^{(r)}$ by $d := \{d_1, \ldots, d_m\}$.

3.2. Relative homology groups. In this section, we introduce certain relative homology groups on which our homological representations are constructed. Assume that $m \geq 2$. Then the first homology group of $C_{n,m}^{(r)}$ is given by

$$H_1(C_{n,m}^{(r)}; \mathbb{Z}) \cong \mathbb{Z}^\oplus n \oplus \mathbb{Z}$$

where the first $n$ components correspond to the loops around the cylinders $\{|t_1 - k| < (1/4)\}$ for $k = 1, \ldots, n$, and the last component corresponds to the loop around the union of the hyperplanes $\{t_i = t_j\}$ for $1 \leq i < j \leq m$. Let $\langle q \rangle \oplus \langle t \rangle$ be a free abelian group of rank two with multiplicative generators $q$ and $t$. Define the group homomorphism

$$\alpha : \pi_1(C_{n,m}^{(r)}, d) \to \langle q \rangle \oplus \langle t \rangle$$

by composing the map

$$H_1(C_{n,m}^{(r)}; \mathbb{Z}) \to \mathbb{Z}^\oplus n \oplus \mathbb{Z} \to \langle q \rangle \oplus \langle t \rangle, \quad (x_1, \ldots, x_n, y) \mapsto (q^x_1 + \cdots + x_n, t^y)$$

and the abelianization map $\pi_1(C_{n,m}^{(r)}, d) \to H_1(C_{n,m}^{(r)}; \mathbb{Z})$. Let

$$\pi : C_{n,m}^{(r)} \to \tilde{C}_{n,m}^{(r)}$$

the covering space corresponding to the normal subgroup $\text{Ker } \alpha \subset \pi_1(C_{n,m}^{(r)}, d)$. Define the subset

$$\tilde{A}^{(r)} := \pi^{-1}(A^{(r)}) \subset \tilde{C}_{n,m}^{(r)}.$$ 

The group $\langle q \rangle \oplus \langle t \rangle$ acts on $\tilde{C}_{n,m}^{(r)}$ and $\tilde{A}^{(r)}$ as deck transformations. Our homological representations are constructed on the following homology groups. Set $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$.

**Definition 3.3.** An $\mathcal{R}$-module $\mathcal{H}_{n,m}^{(r)}$ is defined to be the relative homology group

$$\mathcal{H}_{n,m}^{(r)} := H_m\left(\tilde{C}_{n,m}^{(r)}, \tilde{A}^{(r)}; \mathbb{Z}\right)$$
where the $\mathcal{R}$-module structure is induced from the action of the deck transformation group $\langle q \rangle \oplus \langle t \rangle$.

**Remark 3.4.** In the case $m = 1$, since $H_1(C_{n,1}^{(r)}; \mathbb{Z}) \cong \mathbb{Z}^n$, the above constructions are considered by using the group homomorphism $\alpha : \pi_1(C_{n,1}^{(r)}, d) \to \langle q \rangle$ instead. In the following sections, we always work over $\mathcal{R} = \mathbb{Z}[q^{\pm 1}]$ in the case $m = 1$.

### 3.3 Construction of homological representations

In this section, we define the action of the mapping class group $\mathcal{M}(S_n^{(r)})$ on $\mathcal{H}_{n,m}^{(r)}$. Recall from Section 2.4 that the group $\text{Diff}^+ (S_n^{(r)}, \partial \mathbb{D})$ consists of orientation preserving diffeomorphisms which fix $\partial \mathbb{D}$ pointwise. A diffeomorphism $f \in \text{Diff}^+ (S_n^{(r)}, \partial \mathbb{D})$ induces a new diffeomorphism $\tilde{f} : C_{n,m}^{(r)} \to C_{n,m}^{(r)}$ defined by

$$\tilde{f}(t_1, \ldots, t_m) := \{f(t_1), \ldots, f(t_m)\}$$

for $\{t_1, \ldots, t_m\} \in C_{n,m}^{(r)}$. The base point $d$ is fixed by $\tilde{f}$ since $f$ fixes $\partial \mathbb{D}$ pointwise. There is a unique lift $\tilde{f} : \mathcal{C}_{n,m}^{(r)} \to \mathcal{C}_{n,m}^{(r)}$ of $\tilde{f}$ which fixes each point in $\pi^{-1}(d)$ and commutes with the action of $\langle q \rangle \oplus \langle t \rangle$. In addition, $\tilde{f}$ preserves the subset $\mathcal{A}^{(r)} \subset C_{n,m}^{(r)}$ since $f$ preserves marked arcs $A^{(r)}$. Hence $f$ induces an $\mathcal{R}$-linear map

$$\tilde{f}_* : \mathcal{H}_{n,m}^{(r)} \to \mathcal{H}_{n,m}^{(r)}$$

on the relative homology group $\mathcal{H}_{n,m}^{(r)}$. It is clear that if $f$ is isotopic to the identity, then the induced linear map $\tilde{f}_*$ is the identity. This implies that the action of the mapping class group $\mathcal{M}(S_n^{(r)})$ on $\mathcal{H}_{n,m}^{(r)}$ is well-defined. Thus we obtain the following definition.

**Definition 3.5.** Under the identification $FB_n \cong \mathcal{M}(S_n^{(r)})$ in Proposition 2.5, the homological representation of the framed group

$$\rho_{n,m}^{(r)} : FB_n \to \text{Aut}_{\mathcal{R}} \mathcal{H}_{n,m}^{(r)}$$

is defined by $f \mapsto \tilde{f}_*$ for $f \in FB_n$.

### 3.4 Multiforks

We recall the notion of forks from [Big01, Kra00] and multiforks from [Ito16, Zhe05]. We borrow the notation from [Ito16, Section 3]. Let $d = \{d_1, \ldots, d_n\} \in C_{n,m}^{(r)}$ be the base point in Definition 3.2.

**Definition 3.6.** Let $Y$ be a graph consisting of four vertices $\{v_1, v_2, w, c\}$ and three edges $\{[v_1, c], [c, v_2], [w, c]\}$ as in the left of Figure 5. We orient edges $[v_1, c]$ and $[c, v_2]$ as in Figure 5. A *fork $F$ based on $d_i$* is the image of an embedding $\phi : Y \to S_n^{(r)}$ satisfying the followings:

- $\phi(Y \setminus \{v_1, v_2, w\})$ lies in the interior of $S_n^{(r)}$,
- $\phi(w) = d_i$,
- $\phi(v_1) \in A_1$ and $\phi(v_2) \in A_2$ for some $A_1, A_2 \in A^{(r)}$. 


for \([v_1, v_2] := [v_1, c] \cup [c, v_2]\), the closed arc \(\phi([v_1, v_2])\) is not homotopic to a closed subinterval of some \(A \in \mathcal{A}^{(r)}\).

The image \(\phi([w, c])\) is called the handle of \(F\) and denoted by \(H(F)\). The image \(\phi([v_1, v_2])\) is called the tine edge of \(F\) and denoted by \(T(F)\).

**Definition 3.7.** A multifork is a family of forks \(F := (F_1, \ldots, F_m)\) satisfying the followings:

- \(F_i\) is a fork based on \(d_i\),
- \(T(F_i) \cap T(F_j) = \emptyset\) for \(i \neq j\),
- \(H(F_i) \cap H(F_j) = \emptyset\) for \(i \neq j\).

The right of Figure 5 is an example of a multifork. For a multifork \(F = (F_1, \ldots, F_m)\), let \(\gamma_i: [0, 1] \to \mathbb{S}_n\) be the path corresponding to the handle \(H(F_i)\) with \(\gamma_i(0) = d_i\). Since all handles of \(F\) are disjoint, the path

\[
H(F): [0, 1] \to C^{(r)}_{n,m}, \quad t \mapsto \{\gamma_1(t), \ldots, \gamma_n(t)\}
\]

is well-defined. Note that \(H(F)(0) = d\). For the covering \(\pi: \tilde{C}^{(r)}_{n,m} \to C^{(r)}_{n,m}\), fix a lift of the base point \(\tilde{d} \in \pi^{-1}(d)\). Then we can take a unique lift \(\tilde{H}(F): [0, 1] \to \tilde{C}^{(r)}_{n,m}\) satisfying \(\tilde{H}(F)(0) = \tilde{d}\).

For a given multifork \(F\), define the \(m\) dimensional submanifold of \(C^{(r)}_{n,m}\) by

\[
\Sigma(F) := \{ (t_1, \ldots, t_m) \in C^{(r)}_{n,m} | t_i \in T(F_i) \}
\]

By definition, the boundary of \(\Sigma(F)\) is contained in \(\mathcal{A}^{(r)}\). Let \(\tilde{\Sigma}(F) \subset \tilde{C}^{(r)}_{n,m}\) be the connected component of \(\pi^{-1}(\Sigma(F))\) containing the point \(\tilde{H}(F)(1)\). Since the boundary of \(\tilde{\Sigma}(F)\) is contained in \(\tilde{\mathcal{A}}^{(r)}\), it defines a homology class \([\tilde{\Sigma}(F)] \in H^{(r)}_{n,m}\), called the multifork class. We denote it by \([F]\) instead of \([\tilde{\Sigma}(F)]\). Introduce the set

\[
K^{(r)}_{n,m} := \{ k = (k_1, \ldots, k_{n-1}, t_1^{(1)}, \ldots, t_1^{(r)}, \ldots, t_n^{(1)}, \ldots, t_n^{(r)}) \in \mathbb{Z}_{+}^{n-1} \times (\mathbb{Z}_{+}^{*})^n | k = m \}
\]
where

\[ |k| = \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n} \sum_{s=1}^{r} t_i^{(s)}. \]

For \( k \in K_{n,m}^{(r)} \), we define the standard multifork \( F_k \) as in Figure 6. In Figure 6, a fork with a thick handle equipped with an integer \( k_i \) or \( t_i^{(s)} \) represents parallel \( k_i \) or \( t_i^{(s)} \) forks connecting \( A_i^{(1)} \) and \( A_{i+1}^{(1)} \), or \( A_i^{(s)} \) and \( A_{i+1}^{(s)} \) as in Figure 7. By definition, the standard multifork classes \{ \([F_k]\) \( | k \in K_{n,m}^{(r)} \) \} are linearly independent in \( H_{n,m}^{(r)} \). In the relative homology group \( H_{n,m}^{(r)} \), multifork classes satisfy the diagrammatic formulas, called the fork rules as in Figure 8. In Figure 8, we use the \((-t)\)-binomial coefficient defined by

\[
\binom{k}{i}_{-t} = \frac{[k]_{-t}!}{[k-i]_{-t}! [i]_{-t}!}
\]

where

\([k]_{-t}! := [1]_{-t} [2]_{-t} \cdots [k]_{-t} \quad \text{and} \quad [n]_{-t} := \frac{(-t)^n - 1}{(-t) - 1}.\]
By using these formulas, we can compute the action of $FB_n$ on multifork classes explicitly. In particular, any multifork class can be written as a linear combination of the standard multifork classes over $\mathcal{R}$. Hence we have the following.

**Proposition 3.8.** Let $L_{n,m}^{(r)} \subset H_{n,m}^{(r)}$ be a free $\mathcal{R}$-module of rank 
$$|K_{n,m}^{(r)}| = \binom{rn + n + m - 2}{m}$$
spanned by the standard multifork classes $\{[F_k] \mid k \in K_{n,m}^{(r)}\}$. Then $L_{n,m}^{(r)}$ is a representation of $FB_n$ over $\mathcal{R}$.

We also call $L_{n,m}^{(r)}$ the homological representation. In Section 3.6, we show that $L_{n,m}^{(r)}$ coincides with $H_{n,m}^{(r)}$ over the field $\mathbb{Q}(q,t)$, and therefore the standard multifork classes form a basis of $H_{n,m}^{(r)}$ over $\mathbb{Q}(q,t)$.

![Fork rules](image)

**Figure 8.** Fork rules

### 3.5. Lawrence representations as quotients.
In this section, we see that the Lawrence representation [Law90] appears as the quotient of our homological representation. Define the set $K_{n,m}$ by

$$K_{n,m} := \left\{ k = (k_1, \ldots, k_{n-1}) \in \mathbb{Z}_+^{k-1} \mid \sum_{i=1}^{k-1} k_i = m \right\}$$
and regard $K_{n,m}$ as a subset of $K^{(r)}_{n,m}$. Let $N^{(r)}_{n,m} \subset \mathcal{L}^{(r)}_{n,m}$ be the free $\mathcal{R}$-submodule of rank
\[
\binom{rn + n + m - 2}{m} - \binom{n + m - 2}{m}
\]
spanned by the subset $\{ [F_k] \mid k \in K^{(r)}_{n,m} \setminus K_{n,m} \}$ of the standard multifork classes. Then we can check that $N^{(r)}_{n,m}$ is closed under the action of $FB_n$ by using the fork rules in Figure 8. Hence the quotient representation $\mathcal{L}_{n,m} := \mathcal{L}^{(r)}_{n,m} / N^{(r)}_{n,m}$ is well-defined.

**Proposition 3.9.** The quotient $\mathcal{L}_{n,m}$ is a free $\mathcal{R}$-module of rank
\[
\binom{n + m - 2}{m}
\]
spanned by $\{ [F_k] \mid k \in K_{n,m} \}$. The action of the factor $\mathbb{Z}^n \subset FB_n$ on the quotient representation $\mathcal{L}_{n,m}$ is trivial. As a representation of $B_n \subset FB_n$, the quotient representation $\mathcal{L}_{n,m}$ is equivalent to the Lawrence representation.

**Proof.** By the fork rules in Figure 8, it is easy to check that $\mathbb{Z}^n$ acts trivially on $\{ [F_k] \mid k \in K_{n,m} \}$ modulo $N^{(r)}_{n,m}$. The basis $\{ [F_k] \mid k \in K_{n,m} \}$ and the action of $B_n$ on it modulo $N^{(r)}_{n,m}$ precisely correspond to the description of the Lawrence representation in [Ito16, Zhe05] by using multiforks. □

**Corollary 3.10.** If $m \geq 2$, then the homological representation
\[
\rho^{(r)}_{n,m} : FB_n \to \text{Aut}_\mathcal{R} \mathcal{L}^{(r)}_{n,m}
\]
is faithful.

**Proof.** By definition, it is easy to see that the factor $\mathbb{Z}^n \subset FB_n$ acts faithfully on $\mathcal{L}^{(r)}_{n,m}$. On the other hand, the factor $B_n \subset FB_n$ acts faithfully on the quotient representation $\mathcal{L}_{n,m}$ by [Big01, Kra02] in the case $m = 2$ and by [Zhe] in the case $m \geq 3$. □

### 3.6. Dimension of homological representations

Let $Q := \mathbb{Q}(q, t)$ be the quotient field of $\mathcal{R} = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. In this section, we compute the dimension of the relative homology group $H_\ast \left( \overline{C}^{(r)}_{n,m}, \overline{A}^{(r)}; \mathbb{Z} \right)$ over $Q$. In particular, we recall the definition
\[
\mathcal{H}^{(r)}_{m,n} = H_m \left( \overline{C}^{(r)}_{n,m}, \overline{A}^{(r)}; \mathbb{Z} \right).
\]

**Theorem 3.11.** The dimension of the relative homology group is given by
\[
\dim_Q \mathcal{H}^{(r)}_{m,n} \otimes_\mathcal{R} Q = \binom{rn + n + m - 2}{m}
\]
and
\[
\dim_Q H_k \left( \overline{C}^{(r)}_{n,m}, \overline{A}^{(r)}; \mathbb{Z} \right) \otimes_\mathcal{R} Q = 0
\]
for $k \neq m$. 
In order to show Theorem 3.11, we begin by preparing some notations. For a topological space $X$, we denote by $C_m(X)$ the configuration space of unordered distinct $m$ points in $X$: 

$$C_m(X) := \{ (t_1, \ldots, t_m) \in X^m \mid t_i \neq t_j \text{ if } i \neq j \} / \mathfrak{S}_m.$$ 

Define a sequence of subsets 

$$\emptyset = B_{-1} \subset B_0 \subset B_1 \subset \cdots \subset B_{m-p} \subset \cdots \subset B_{m-1} \subset B_m \subset C_{n,m}^{(r)}$$

by 

$$B_{m-p} := \left\{ (t_1, \ldots, t_m) \in C_m(S_n^{(r)} \setminus \partial \mathbb{D}) \mid t_1 \in A_1, \ldots, t_p \in A_p \text{ for some } A_1, \ldots, A_p \in A(r) \right\}.$$ 

In other words, at least $p$ points of $(t_1, \ldots, t_m) \in C_m(S_n^{(r)} \setminus \partial \mathbb{D})$ lie in some arcs of $A(r)$.

**Remark 3.12.** The space 

$$B_m = \bigsqcup_{p=0}^{m} (B_p \setminus B_{p-1})$$

has a structure of a $2m$-dimensional smooth manifold with corners induced from the smooth structure on $S_n^{(r)} \setminus \partial \mathbb{D}$. Codimension $p$ corners of $B_m$ are given by $B_{m-p} \setminus B_{m-p-1}$, i.e. for any point $x \in B_{m-p} \setminus B_{m-p-1}$, there is an open subset $x \in U \subset B_m$ and a smooth map $f: U \to V$ on an open subset $V \subset [0, \infty)^p \times \mathbb{R}^{2m-p}$ such that $f$ is a diffeomorphism and $f(x) = 0$. Since $[0, \infty)^p \times \mathbb{R}^{2m-p}$ is homeomorphic to $[0, \infty) \times \mathbb{R}^{2m-1}$ for $p \geq 1$, if we forget the smooth structure on $B_m$ and only consider the topological structure, then $B_m$ is a $2m$-dimensional topological manifold with boundary $\partial B_m = B_{m-1}$. In the following, our computation only depends on the topological structure on $B_m$.

We note that the pair $(B_m, B_{m-1})$ is homotopy equivalent to the pair $(C_{n,m}^{(r)}, A(r))$ since $S_n^{(r)} \setminus \partial \mathbb{D}$ is homotopy equivalent to $S_n^{(r)}$.

For a subspace $U \subset C_{n,m}^{(r)}$, denote by $\tilde{U} := \pi^{-1}(U)$ its inverse image via the covering map $\pi: \tilde{C}_{n,m}^{(r)} \to C_{n,m}^{(r)}$. Then the pair $(\tilde{B}_m, \tilde{B}_{m-1})$ is also homotopy equivalent to the pair $(\tilde{C}_{n,m}^{(r)}, \tilde{A}(r))$. Therefore it is sufficient to compute the dimension of the relative homology group 

$$H_*(\tilde{B}_m, \tilde{B}_{m-1}; \mathbb{Z}) \otimes_{\mathbb{R}} \mathbb{Q}.$$ 

In the following of this section, we consider all homology groups over $\mathbb{Q}$ and write $H_*(-)$ instead of $H_*(-; \mathbb{Z}) \otimes_{\mathbb{R}} \mathbb{Q}$ for simplicity.

**Lemma 3.13.** The space $\tilde{B}_p \setminus \tilde{B}_{p-1}$ is an $(m + p)$-dimensional (smooth) manifold without boundary. The dimension of the homology group is given by 

$$\dim \mathbb{Q} H_p(\tilde{B}_p \setminus \tilde{B}_{p-1}) = \binom{rn + m - p - 1}{m - p} \binom{n + p - 2}{p}$$
and
\[ \dim_{Q} H_k(\widetilde{B}_p \setminus \widetilde{B}_{p-1}) = 0 \]
for \( k \neq p \).

**Proof.** Set
\[ O^{(r)}_n := S^{(r)}_n \setminus S_n = \bigsqcup_{A \in A^{(r)}} A. \]

The space \( O^{(r)}_n \) is an one dimensional manifold consisting of disjoint \( rn \) open intervals. By definition, there is an isomorphism
\[ B_p \setminus B_{p-1} \cong C_{m-p}(O^{(r)}_n) \times C_p(S_n \setminus \partial \mathbb{D}), \]
and hence \( B_p \setminus B_{p-1} \) is an \((m+p)\)-dimensional manifold without boundary. Since the covering map \( \pi \) is a local homeomorphism, \( \widetilde{B}_p \setminus \widetilde{B}_{p-1} = \pi^{-1}(B_p \setminus B_{p-1}) \) is also an \((m+p)\)-dimensional manifold without boundary. Let \( L^{(r)}_{n,m-p} \) be a set
\[ L^{(r)}_{n,m-p} := \left\{ 1 = \left( t_1^{(1)}, \ldots, t_1^{(r)} \right) \in (\mathbb{Z}^r_+)^n \middle| \sum_{i=1}^{n} \sum_{s=1}^{r} t_i^{(s)} = m - p \right\} \]
and for \( l \in L^{(r)}_{n,m-p} \), define a subset \( O[l] \subset C_{m-p}(O^{(r)}_n) \) by
\[ O[l] := \left\{ t_1, \ldots, t_{m-p} \in C_{m-p}(O^{(r)}_n) \middle| \begin{array}{l} t_1, \ldots, t_1^{(1)} \in A^{(1)}_1, t_1^{(s)}+1, \ldots, t_1^{(s)}+t_s^{(2)} \in A^{(2)}_1, \ldots, \\ t_i^{(s)}+t_{i-1}^{(r)}+1, \ldots, t_i^{(1)}+t_{i-1}^{(r)}+t_{i-1}^{(r)} \in A^{(r)}_1 \end{array} \right\}. \]

Namely, the space \( O[l] \) consists of configurations in \( O^{(r)}_n \) which satisfy that \( t_i^{(s)} \) points of \( \{ t_1, \ldots, t_{m-p} \} \in C_{m-p}(O^{(r)}_n) \) lie in \( A^{(s)}_1 \). Then \( C_{m-p}(O^{(r)}_n) \) is decomposed into connected components
\[ C_{m-p}(O^{(r)}_n) = \bigsqcup_{l \in L^{(r)}_{n,m-p}} O[l] \]
and each \( O[l] \) is contractible by definition. On the other hand, it is known that
\[ \dim_{Q} H_p(\widetilde{C}_p(S_n \setminus \partial \mathbb{D})) = \binom{n + p - 2}{p} \]
and \( H_k(\widetilde{C}_p(S_n \setminus \partial \mathbb{D})) = 0 \) for \( k \neq p \) since it is the dimension of the Lawrence representation (see [Koh12, Section 2 and 3]). Since
\[ \widetilde{B}_p \setminus \widetilde{B}_{p-1} \cong \bigsqcup_{l \in L^{(r)}_{n,m-p}} (O[l] \times \widetilde{C}_p(S_n \setminus \partial \mathbb{D})) \]
and
\[ | L^{(r)}_{n,m-p} | = \binom{rn + m - p - 1}{m - p}, \]
the result follows. □

In the following computation, we use the cohomology with compact support, and Poincaré and Lefschetz dualities for non-compact manifolds. For these materials, we refer to [Hat02, Section 3.3] and [Ive86, Section III]. For a (locally compact) space $X$, we denote by $H^*_c(X; \mathbb{Z})$ the singular cohomology with compact support. Since our spaces always admit the action of $\langle q \rangle \oplus \langle t \rangle$, we work over $\mathbb{Q}$ and write $H^*_c(-)$ instead of $H^*_c(-; \mathbb{Z}) \otimes \mathbb{Q}$.

**Lemma 3.14.** We have the equations of dimensions

$$\dim_{\mathbb{Q}} H^m_c(\tilde{B}_p) = \dim_{\mathbb{Q}} H^m_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) + \dim_{\mathbb{Q}} H^m_c(\tilde{B}_{p-1})$$

and

$$\dim_{\mathbb{Q}} H^k_c(\tilde{B}_p) = 0$$

for $k \neq m$.

**Proof.** Since $\tilde{B}_p \setminus \tilde{B}_{p-1}$ is an $(m + p)$-dimensional manifold without boundary by Lemma 3.13, we have an isomorphism $H^k_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) \cong H_{m+p-k}(\tilde{B}_p \setminus \tilde{B}_{p-1})$ by Poincaré duality ([Hat02, Theorem 3.35]). Again by Lemma 3.13, $H^k_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) \neq 0$ if and only if $k = m$. Since $\tilde{B}_p \setminus \tilde{B}_{p-1}$ is an open subset of $\tilde{B}_p$ and $\tilde{B}_p - (\tilde{B}_p \setminus \tilde{B}_{p-1}) = \tilde{B}_{p-1}$, we have a long exact sequence of cohomology with compact support ([Ive86, Section III.7])

$$\cdots \rightarrow H^{k-1}_c(\tilde{B}_{p-1}) \rightarrow H^k_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) \rightarrow H^k_c(\tilde{B}_p) \rightarrow H^k_c(\tilde{B}_{p-1}) \rightarrow H^{k+1}_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) \rightarrow \cdots .$$

We show the result by induction for $p$. First we consider the case $k \neq m$. By the long exact sequence, if $H^k_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) = 0$ and $H^k_c(\tilde{B}_{p-1}) = 0$, then $H^k_c(\tilde{B}_p) = 0$. By Lemma 3.13, $H^k_c(\tilde{B}_0) = H^k_c(\tilde{B}_0 \setminus \tilde{B}_-) = 0$ and $H^k_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) = 0$ for $p \geq 0$ and $k \neq m$. Inductively, we have $H^k_c(\tilde{B}_p) = 0$ for $p \geq 0$ and $k \neq m$. In the case $k = m$, since $H^{m-1}_c(\tilde{B}_{p-1}) = 0$ and $H^{m+1}_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) = 0$, the long exact sequence implies the result. □

**Proof of Theorem 3.11.** First we note that the space $\tilde{B}_m$ is a $2m$-dimensional topological manifold with boundary $\partial \tilde{B}_m = \tilde{B}_{m-1}$ (see Remark 3.12). By Lefschetz duality ([Hat02, Theorem 3.43 and Exercises 35]), we have

$$H_k(\tilde{B}_m, \tilde{B}_{m-1}) = H_k(\tilde{B}_m, \partial \tilde{B}_m) \cong H^{2m-k}(\tilde{B}_m).$$

Hence $\dim_{\mathbb{Q}} H_k(\tilde{B}_m, \tilde{B}_{m-1}) = \dim_{\mathbb{Q}} H^{2m-k}(\tilde{B}_m) = 0$ for $k \neq m$ by Lemma 3.14. In the case $k = m$, again by Lemma 3.14, we have

$$\dim_{\mathbb{Q}} H_m(\tilde{B}_m, \tilde{B}_{m-1}) = \dim_{\mathbb{Q}} H^m_c(\tilde{B}_m) = \sum_{p=0}^{m} \dim_{\mathbb{Q}} H^m_c(\tilde{B}_p \setminus \tilde{B}_{p-1}).$$

By Poincaré duality and Lemma 3.13, the dimension of $H^m_c(\tilde{B}_p \setminus \tilde{B}_{p-1})$ is given by

$$\dim_{\mathbb{Q}} H^m_c(\tilde{B}_p \setminus \tilde{B}_{p-1}) = \dim_{\mathbb{Q}} H_p(\tilde{B}_p \setminus \tilde{B}_{p-1}) = \binom{rn + m - p - 1}{m - p} \binom{n + p - 2}{p}.$$
Finally, by Vandermonde’s identity for multi-set coefficients, we have
\[
\sum_{p=0}^{m} \binom{rn + m - p - 1}{m - p} \binom{n + p - 2}{p} = \binom{rn + n + m - 2}{m}
\]
and this implies the result. \( \square \)

**Corollary 3.15.** We have an equality of \( \mathbb{Q} \)-vector spaces
\[
\mathcal{L}_{n,m}^{(r)} \otimes_{\mathcal{R}} \mathbb{Q} = \mathcal{H}_{n,m}^{(r)} \otimes_{\mathcal{R}} \mathbb{Q}.
\]
In particular, the standard multifork classes \( \{ [F_k] \mid k \in K_{n,m}^{(r)} \} \) form a basis of the homological representation \( \mathcal{H}_{n,m}^{(r)} \) over \( \mathbb{Q} \).

**Proof.** It immediately follows from Proposition 3.8 and Theorem 3.11. \( \square \)

**Remark 3.16.** In this paper, we don’t discuss the problem whether \( \mathcal{L}_{n,m}^{(r)} \) coincides with \( \mathcal{H}_{n,m}^{(r)} \) over \( \mathcal{R} \). For details of this problem, we refer to [Big03, PP02].

### 3.7. Framed Burau representations

In this section, we introduce the framed Burau representation of the framed braid group, which contains the Burau representation of the braid group as the quotient representation. We also show that the reduced framed Burau representation coincides with the homological representation in the case \( m = 1 \).

Let \( M_n^{(r)} \) be a free \( \mathbb{Z}[q^\pm] \)-module of rank \( (n + rn) \) spanned by the basis
\[
\left\{ a_i, b_i^{(s)} \mid i = 1, \ldots, n \text{ and } s = 1, \ldots, r \right\}.
\]
Define the action of \( FB_n \) on \( M_n^{(r)} \) by
\[
\begin{align*}
\sigma_i \cdot a_k &= \begin{cases} 
(1 - q)a_i + qa_{i+1} - (b_i^{(1)} + \cdots + b_i^{(r)}) & \text{if } k = i \\
q b_i^{(s-1)} & \text{if } k = i+1 \\
a_i & \text{otherwise}
\end{cases} \\
\sigma_i \cdot b_k^{(s)} &= \begin{cases} 
q b_i^{(s+1)} & \text{if } k = i \\
q b_k^{(s)} & \text{if } k = i+1 \\
b_k^{(s)} & \text{otherwise}
\end{cases} \\
\tau_i \cdot a_k &= \begin{cases} 
a_i - q b_i^{(r)} & \text{if } k = i \\
q b_i^{(s-1)} & \text{if } k = i \text{ and } s = 2, \ldots, r \\
a_k & \text{otherwise}
\end{cases} \\
\tau_i \cdot b_k^{(s)} &= \begin{cases} 
q b_i^{(r)} & \text{if } k = i \text{ and } s = 1 \\
b_k^{(s)} & \text{otherwise}.
\end{cases}
\end{align*}
\]

**Proposition 3.17.** The above action is well-defined. Hence it gives a representation of \( FB_n \) on \( M_n^{(r)} \).
Proof. By direct computation, we can easily check that the above action of $FB_n$ on $M_n^{(r)}$ satisfies the relations in Definition 2.1.

We call $M_n^{(r)}$ the framed Burau representation of the framed braid group $FB_n$. Let $N_n^{(r)} \subset M_n^{(r)}$ be the submodule spanned by $\{b_i^{(s)} | i = 1, \ldots, n$ and $s = 1, \ldots, r\}$. Then $N_n^{(r)}$ is a subrepresentation of $FB_n$. If we restrict the representation $N_n^{(r)}$ on the subgroup $B_n \subset FB_n$, then it becomes the direct sum of the standard representations of the braid group [TYM96, Sys01]. Thus our homological representation gives the homological interpretation of the standard representations of the braid groups. On the other hand, the quotient representation $M_n^{(r)}/N_n^{(r)}$ becomes the Burau representation of the braid group [Bur36] (see also [KT08, Section 3.1]).

Set $c_i := a_{i+1} - q^{-1} a_i$ for $i = 1, \ldots, n - 1$ and consider the submodule $L_n^{(r)} \subset M_n^{(r)}$ of rank $(n - 1 + rn)$ spanned by

$$\left\{ c_1, \ldots, c_{n-1}, b_1^{(1)}, \ldots, b_1^{(r)}, \ldots, b_n^{(1)}, \ldots, b_n^{(r)} \right\}.$$

Then it is easy to check that $L_n^{(r)}$ is closed under the action of $FB_n$. We call $L_n^{(r)}$ the reduced framed Burau representation of the framed braid group $FB_n$. Note that $L_n^{(r)}$ also contains $N_n^{(r)}$ as a subrepresentation. Further, the quotient representation $L_n^{(r)}/N_n^{(r)}$ is isomorphic to the reduced Burau representation of the braid group (see [KT08, Section 3.3]).

**Proposition 3.18.** The reduced framed Burau representation $L_n^{(r)}$ is equivalent to the homological representation $L_n^{(r)}$.

**Proof.** Define $k_i \in K_n^{(r)}$ and $l_i^{(s)} \in K_n^{(r)}$ by

$$k_i := (0, \ldots, 0, k_i = 1, 0, \ldots, 0)$$

$$l_i^{(s)} := (0, \ldots, 0, l_i^{(s)} = 1, 0, \ldots, 0).$$

Then we can check that the action of $FB_n$ on fork classes $[F_{k_i}]$ and $[F_{l_i^{(s)}}]$ coincides with the action on $c_i$ and $b_i^{(s)}$ respectively by using the fork rules in Figure 8. Hence the correspondence

$$[F_{k_i}] \mapsto c_i, \quad [F_{l_i^{(s)}}] \mapsto b_i^{(s)}$$

gives an isomorphism of representations $L_n^{(r)} \sim L_n^{(r)}$.

**Remark 3.19.** Let $\psi_n : B_n \to GL(n - 1, \mathbb{Z}[q^{\pm 1}])$ be the reduced Burau representation of the braid group $B_n$. Then it is well-known that the quantity

$$\Delta_\beta(q) = \frac{q - 1}{q^n - 1} \det(I_{n-1} - \psi_n(\beta))$$

is the Jones polynomial of the framed braid group $FB_n$. 

for $\beta \in B_n$ is the Alexander polynomial of the link $\hat{\beta}$ which is the plat closure of $\beta$ (see [KT08, section 3.4]). Similarly, let $\psi_n^{(r)}: FB_n \rightarrow GL(n-1+rn, \mathbb{Z}[q^{\pm 1}])$ be the reduced framed Burau representation of the framed braid group $FB_n$ and consider the quantity
\[
\Delta^{(r)}_{\beta}(q) = \frac{q-1}{q^n-1} \det(I_{n-1+rn} - \psi_n^{(r)}(\beta))
\]for $\beta \in FB_n$. Then it gives a polynomial invariant of the framed link $\hat{\beta}$. Details of this topic will be discussed in [Ike].

4. Monodromy representations

In this section, we give a detailed study of the confluent KZ equations introduced in [JNS08] and construct representations of the framed braid groups as the monodromy representations of the confluent KZ equations.

4.1. Confluent Verma module bundles. Consider the Lie algebra $g := sl_2$ over $\mathbb{C}$ with the standard basis $\{E, H, F\}$ satisfying
\[
[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.
\]

Define a truncated current Lie algebra $g^{(r)}$ by
\[
g^{(r)} := g[t]/t^{r+1}g[t]
\]where $g[t] := g \otimes \mathbb{C}[t]$ is a Lie algebra with the bracket
\[
[X \otimes t^m, Y \otimes t^n] := [X, Y] \otimes t^{m+n}.
\]Set $X_p := X \otimes t^p$ for $X \in g$ and $p = 0, 1, \ldots, r$. The Lie algebra $g^{(r)}$ has the triangular decomposition
\[
n^+_r := \bigoplus_{p=0}^r \mathbb{C}E_p, \quad h^{(r)} := \bigoplus_{p=0}^r \mathbb{C}H_p, \quad n^-_r := \bigoplus_{p=0}^r \mathbb{C}F_p.
\]Take complex numbers $\lambda, \gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(r)} \in \mathbb{C}$ with $\gamma^{(r)} \neq 0$ and set $\gamma := (\gamma^{(1)}, \ldots, \gamma^{(r)})$. A highest weight vector $v_\lambda(\gamma)$ is defined by the conditions
\[
n^+_r v_\lambda(\gamma) = 0, \quad H_0 v_\lambda(\gamma) = \lambda v_\lambda(\gamma), \quad H_p v_\lambda(\gamma) = \gamma^{(p)} v_\lambda(\gamma)
\]for $p = 1, \ldots, r$. We call $\lambda$ a weight and $\gamma^{(1)}, \ldots, \gamma^{(r)}$ movable weights. Later, movable weights $\gamma^{(1)}, \ldots, \gamma^{(r)}$ are regarded as variables of the confluent KZ equation.

**Definition 4.1.** A confluent Verma module $M_\lambda^{(r)}(\gamma)$ of a weight $\lambda \in \mathbb{C}$, movable weights $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(r)}) \in \mathbb{C}^{r-1} \times \mathbb{C}^*$ and P-rank $r > 0$ is defined to be the induced module
\[
M_\lambda^{(r)}(\gamma) := U g^{(r)} \otimes_{U b^{(r)}} \mathbb{C} v_\lambda(\gamma)
\]where $\mathbb{C} v_\lambda(\gamma)$ is the one dimensional representation of the Borel subalgebra $b^{(r)} := n^{(r)}_+ \oplus h^{(r)}$, and $U g^{(r)}$ and $U b^{(r)}$ are universal enveloping algebras of $g^{(r)}$ and $b^{(r)}$. 
Later, we will see that the P-rank $r$ controls the Poincaré ranks of irregular singularities of the confluent KZ equation. It is known that $M^{(r)}_\lambda(\gamma)$ is irreducible if $\gamma^{(r)} \neq 0$ (see [Wil11]).

Let $\mathbb{Z}_+$ be non-negative integers. For $j := (j^{(0)}, j^{(1)}, \ldots, j^{(r)}) \in \mathbb{Z}_+^{r+1}$, write

$$F^j v_\lambda(\gamma) := F_0^{j^{(0)}} F_1^{j^{(1)}} \cdots F_r^{j^{(r)}} v_\lambda(\gamma).$$

Then the set $\{F^j v_\lambda(\gamma) \mid j \in \mathbb{Z}_+^{r+1}\}$ forms a basis of $M^{(r)}_\lambda(\gamma)$, and hence we have

$$M^{(r)}_\lambda(\gamma) \cong \bigoplus_{j \in \mathbb{Z}_+^{r+1}} \mathbb{C} F^j v_\lambda(\gamma).$$

As mentioned above, to treat movable weights $\gamma^{(1)}, \ldots, \gamma^{(r)}$ as variables, we introduce the space of movable weights by

$$B^{(r)} := \{ (\gamma^{(1)}, \ldots, \gamma^{(r)}) \in \mathbb{C}^r \mid \gamma^{(r)} \neq 0 \} \cong \mathbb{C}^{r-1} \times \mathbb{C}^*. $$

Consider the family of confluent Verma modules over $B^{(r)}$ as follows.

**Definition 4.2.** A confluent Verma module bundle of a weight $\lambda \in \mathbb{C}$ and P-rank $r > 0$ is defined by

$$E^{(r)}_\lambda := \bigcup_{\gamma \in B^{(r)}} M^{(r)}_\lambda(\gamma) \to B^{(r)}$$

where a fiber over $\gamma \in B^{(r)}$ is the confluent Verma module $M^{(r)}_\lambda(\gamma)$.

**4.2. Connections on confluent Verma module bundles.** As in the previous section, we use the coordinate $(\gamma^{(1)}, \ldots, \gamma^{(r)}) \in B^{(r)}$ for the space of movable weights $B^{(r)}$. Introduce the holomorphic vector fields $D^{(0)}, D^{(1)}, \ldots, D^{(r-1)}$ on $B^{(r)}$ by

$$D^{(s)} := \sum_{p=1}^{r-s} \gamma^{(s+p)} \frac{\partial}{\partial \gamma^{(p)}}.$$

Denote by $\mathcal{O}(B^{(r)})$ the space of holomorphic functions on $B^{(r)}$.

**Lemma 4.3.** The vector fields $D^{(0)}, D^{(1)}, \ldots, D^{(r-1)}$ form a basis of the space of holomorphic vector fields on $B^{(r)}$ as an $\mathcal{O}(B^{(r)})$-module, i.e. any holomorphic vector field $V$ on $B^{(r)}$ can be uniquely written as

$$V = \sum_{s=0}^{r-1} f_s D^{(s)}$$

with $f_0, \ldots, f_{r-1} \in \mathcal{O}(B^{(r)})$.

**Proof.** First we note that $\{\partial/\partial \gamma^{(s)}\}_{s=1}^r$ is a basis of the space of holomorphic vector fields on $B^{(r)}$. Consider the base change between $\{D^{(s)}\}_{s=0}^{r-1}$ and $\{\partial/\partial \gamma^{(s)}\}_{s=1}^r$. The base change
matrix
\[
\begin{pmatrix}
\frac{\partial}{\partial \gamma_1} & \frac{\partial}{\partial \gamma_2} & \cdots & \frac{\partial}{\partial \gamma_r} \\
\end{pmatrix}
\]
\[P = (D^{(r-1)}, D^{(r-2)}, \ldots, D^{(0)})\]
is given by the upper triangular matrix
\[
P = \begin{pmatrix}
\gamma^{(r)} & \gamma^{(r-1)} & \cdots & \gamma^{(1)} \\
2\gamma^{(r)} & \cdots & 2\gamma^{(2)} \\
\vdots & \ddots & \vdots \\
\gamma^{(r)} & \cdots & \gamma^{(r)}
\end{pmatrix}
\]
Since \(\det P = r!(\gamma^{(r)})^r\) and it is nonzero on \(B^{(r)}\), we can write \(\partial/\partial \gamma^{(s)}\) as
\[
\frac{\partial}{\partial \gamma^{(s)}} = \frac{1}{(\gamma^{(r)})^r} \sum_{p=1}^{s} f_{s,p}(\gamma) D^{(r-p)}
\]
with \(f_{s,p}(\gamma) \in \mathbb{C}[\gamma^{(1)}, \ldots, \gamma^{(r)}]\). Hence the result follows.

Let \(g^{(r)} \times B^{(r)} \to B^{(r)}\) be a trivial bundle and \(S(g^{(r)} \times B^{(r)})\) be its holomorphic sections.

**Definition 4.4.** For the vector fields \(D^{(0)}, \ldots, D^{(r-1)}\), the connection
\[
\nabla_{D^{(s)}} : S(g^{(r)} \times B^{(r)}) \to S(g^{(r)} \times B^{(r)})
\]
is defined by
\[
\nabla_{D^{(s)}} (fX_p) := (D^{(s)} f) X_p + f pX_{p+s}
\]
where \(f \in \mathcal{O}(B^{(r)})\) and \(X_p = X \otimes t^p\) for \(X \in g\).

**Lemma 4.5.** The connection \(\nabla\) is integrable.

**Proof.** By direct computation, we have
\[
[D^{(s)}, D^{(t)}] = -(s-t) D^{(s+t)}
\]
and
\[
[\nabla_{D^{(s)}}, \nabla_{D^{(t)}}] X_p = -(s-t)p X_{p+s+t} = -(s-t) \nabla_{D^{(s+t)}} X_p.
\]
This implies the integrability
\[
[\nabla_{D^{(s)}}, \nabla_{D^{(t)}}] - \nabla_{[D^{(s)}, D^{(t)}]} = [\nabla_{D^{(s)}}, \nabla_{D^{(t)}}] + (s-t) \nabla_{D^{(s+t)}} = 0.
\]

The computation in the proof of Lemma 4.3 implies that the connection \(\nabla\) is a meromorphic connection on \(\mathbb{C}^r\) with a pole along \(\{\gamma^{(r)} = 0\}\). It is a regular singularity if \(r = 1\) and an irregular singularity if \(r \geq 2\). This connection can be extended to a connection on
\[
Ug^{(r)} \times B^{(r)} \to B^{(r)}
\]
by defining
\[
\nabla_{D^{(s)}}(U_1 U_2 \cdots U_k) := \sum_{l=1}^{k} \left( \nabla_{D^{(s)}} U_l \right) U_1 U_2 \cdots \hat{U}_l \cdots U_k
\]
for $U_1, \ldots, U_l \in \mathfrak{g}^{(r)}$. It is well-defined since $\nabla$ preserves the relations in $U \mathfrak{g}^{(r)}$, i.e.

$$\nabla_{D^{(s)}}(U_1U_2 - U_2U_1) = \nabla_{D^{(s)}}[U_1,U_2]$$

for $U_1, U_2 \in \mathfrak{g}^{(r)}$. As a result, we can also extend the connection on confluent Verma module bundles compatible with the action of $\mathfrak{g}^{(r)}$. Let $E^{(r)}_\lambda \to B^{(r)}$ be a confluent Verma module bundle. Denote by $v_\lambda$ the global section of $E^{(r)}_\lambda$ defined by gathering highest weight vectors $v_\lambda(\gamma)$ from each fiber $M^{(r)}_{\lambda}(\gamma)$. Since the rank of $E^{(r)}_\lambda$ is infinite, we consider the space of holomorphic sections of $E^{(r)}_\lambda$ as

$$S(E^{(r)}_\lambda) := \left\{ \sum_{\text{finite}} f_k F^{jk} v_\lambda \bigg| f_k \in \mathcal{O}(B^{(r)}) \text{ and } j_k \in \mathbb{Z}_+^{r+1} \right\}.$$ 

Then the connection $\nabla$ on $\mathfrak{g}^{(r)} \times B^{(r)}$ induces a connection on $E^{(r)}_\lambda$ as follows.

**Definition 4.6.** For the vector fields $D^{(0)}, \ldots, D^{(r-1)}$, the connection

$$\nabla_{D^{(s)}} : S(E^{(r)}_\lambda) \to S(E^{(r)}_\lambda)$$

is defined by

$$\nabla_{D^{(s)}}(f F^{j} v_\lambda) := (D^{(s)} f) F^{j} v_\lambda + f (\nabla_{D^{(s)}} F^{j}) v_\lambda$$

where $f \in \mathcal{O}(B^{(r)})$.

By Lemma 4.5, the connection $\nabla$ is integrable.

### 4.3. Spaces of singular vectors

Fix positive integers $r_1, \ldots, r_n > 0$. Let $M^{(r_i)}_{\lambda_i}(\gamma_i)$ be confluent Verma modules of weights $\lambda_i \in \mathbb{C}$ and movable weights $\gamma_i \in B^{(r_i)}$ for $i = 1, \ldots, n$. Set $\Lambda := (\lambda_1, \ldots, \lambda_n)$, $\Gamma := (\gamma_1, \ldots, \gamma_n)$, and consider the tensor product

$$M^{(R)}_{\Lambda}(\Gamma) := M^{(r_1)}_{\lambda_1}(\gamma_1) \otimes \cdots \otimes M^{(r_n)}_{\lambda_n}(\gamma_n).$$

Then the direct sum of truncated current Lie algebras $\mathfrak{g}^{(R)} := \mathfrak{g}^{(r_1)} \oplus \cdots \oplus \mathfrak{g}^{(r_n)}$ naturally acts on $M^{(R)}_{\Lambda}(\Gamma)$. The original Lie algebra $\mathfrak{g}$ acts on $M^{(R)}_{\Lambda}(\Gamma)$ through the diagonal embedding of $\mathfrak{g}$ into $\mathfrak{g}^{(R)}$. In other words, an element $X \in \mathfrak{g}$ acts on $M^{(R)}_{\Lambda}(\Gamma)$ as

$$(X_0 \otimes 1 \cdots \otimes 1) + (1 \otimes X_0 \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes X_0).$$

**Definition 4.7.** Set $|\Lambda| := \lambda_1 + \cdots + \lambda_n$. For $m \in \mathbb{Z}_{\geq 0}$, define the space of vectors of weight $(|\Lambda| - 2m)$ by

$$W_{\Gamma}^{(R)}([|\Lambda| - 2m]) := \{ w \in M^{(R)}_{\Lambda}(\Gamma) \mid Hw = (|\Lambda| - 2m)w \},$$

and the space of singular vectors of weight $(|\Lambda| - 2m)$ by

$$S_{\Gamma}^{(R)}([|\Lambda| - 2m]) := \{ w \in W_{\Gamma}^{(R)}([|\Lambda| - 2m] \mid Ew = 0 \}.$$
Let \( v_{\Lambda}(\Gamma) := v_{\lambda_1}(\gamma_1) \otimes \cdots \otimes v_{\lambda_n}(\gamma_n) \) be the tensor product of highest weight vectors. Set \( \mathbb{Z}_+^{(R+1)} := \mathbb{Z}_+^{r_1+1} \times \cdots \times \mathbb{Z}_+^{r_n+1} \). For \( J = (j_1, \ldots, j_n) \in \mathbb{Z}_+^{(R+1)} \), write

\[
F^J v_{\Lambda}(\Gamma) := F_0^{(0)} \cdots F_{r_1}^{(r_1)} v_{\lambda_1}(\gamma_1) \otimes \cdots \otimes F_0^{(0)} \cdots F_{r_n}^{(r_n)} v_{\lambda_n}(\gamma_n).
\]

Then the space \( W^{(R)}_{\Gamma}[[\Lambda] - 2m] \) has the basis

\[
\left\{ F^J v_{\Lambda}(\Gamma) \mid J \in \mathbb{Z}_+^{(R+1)} \text{ with } |J| = m \right\}
\]

where \( |J| := \sum_{i=1}^n \sum_{s=0}^{r_i} j_i^{(s)} \). Thus the dimension of \( W^{(R)}_{\Gamma}[[\Lambda] - 2m] \) is given by

\[
\dim_{\mathbb{C}} W^{(R)}_{\Gamma}[[\Lambda] - 2m] = \binom{|R| + n + m - 1}{m}
\]

where \( |R| = r_1 + \cdots + r_n \).

**Proposition 4.8.** The dimension of \( S^{(R)}_{\Gamma}[[\Lambda] - 2m] \) is given by

\[
\dim_{\mathbb{C}} S^{(R)}_{\Gamma}[[\Lambda] - 2m] = \binom{|R| + n + m - 2}{m}.
\]

**Proof.** For simplicity, we write

\[
W^{(R)} := W^{(R)}_{\Gamma}[[\Lambda] - 2m], \quad S^{(R)} := S^{(R)}_{\Gamma}[[\Lambda] - 2m]
\]

and \( x := \gamma_1^{(r_1)} \). Set \( R' := (r_1 - 1, r_2, \ldots, r_n) \) and consider the subspace \( W^{(R')} \subset W^{(R)} \) spanned by \( \{ F^J v_{\Lambda}(\Gamma) \mid J \in \mathbb{Z}_+^{(R+1)} \text{ with } |J| = m \text{ and } j_1^{(r_1)} = 0 \} \). Then

\[
\dim_{\mathbb{C}} W^{(R')} = \binom{|R| + n + m - 2}{m}.
\]

Define the operators \( f, \partial_f \) and \( e \) acting on \( M^{(R)}_{\Lambda}(\Gamma) \) by

\[
f := F_{r_1} \otimes 1 \otimes \cdots \otimes 1, \quad \partial_f := \frac{\partial}{\partial F_{r_1}} \otimes 1 \otimes \cdots \otimes 1
\]

and \( e := E - x\partial_f \). We show that the linear map \( L: W^{(R)} \to W^{(R)} \) defined by

\[
L(v) := \exp(-fe/x)v = \sum_{k=0}^{\infty} \frac{(-1)^k(ke)^k v}{k! x^k}
\]

for \( v \in W^{(R)} \) is well-defined and induces an isomorphism of subspaces \( L: W^{(R')} \cong S^{(R)} \). By direct computation, we can check that \( [f \partial_f, e] = 0 \) and \( [f \partial_f, fe] = fe \). Since

\[
(f \partial_f)(fe)^k F^J v_{\Lambda}(\Gamma) = (j_1^{(r_1)} + k)(fe)^k F^J v_{\Lambda}(\Gamma)
\]
and $j_1^{(r_1)} \leq m$, we have $(fe)^k F^J v_{\lambda}^i (\Gamma) = 0$ for $k \gg 0$. Hence $L$ is well-defined. Next we show that $E \cdot L(v) = 0$ if and only if $v \in W^{(R')}$. Since $E = e + x \partial_f$, the condition $E \cdot L(v) = 0$ implies that
\[
\partial_f \frac{(fe)^{k+1} v}{(k+1)!} = \frac{e( fe)^{k} v}{k!}
\]
for all $k \geq 0$ and $\partial_f v = 0$. We show it by induction. Note that $\partial_f v = 0$ if and only if $v \in W^{(R')}$. For $k = 0$,
\[
\partial_f (fe v) = ev + f \partial_f (ev) = ev + ef \partial_f v = ev
\]
by $[f \partial_f, e] = 0$. For general $k$,
\[
\partial_f \frac{(fe)^{k+1} v}{(k+1)!} = \frac{e( fe)^{k} v}{(k+1)!} + \frac{f \partial_f \{e( fe)^{k} v\}}{(k+1)!} = \frac{e( fe)^{k} v}{(k+1)!} + \frac{ef \partial_f \{(fe)^k v\}}{(k+1) \cdot k!}
\]
again by $[f \partial_f, e] = 0$. By induction hypothesis,
\[
\partial_f \frac{(fe)^{k+1} v}{(k+1)!} = \frac{e( fe)^{k} v}{(k+1)!} + \frac{ef}{k+1} \cdot \frac{e( fe)^{k-1} v}{(k-1)!} = \frac{e( fe)^{k} v}{k!}.
\]
Thus $L$ maps $W^{(R')}$ into $S^{(R)}$. Clearly, $L$ is invertible.  

4.4. Confluent KZ equations. Let $E_{\lambda_i}^{(r_i)} \to B^{(r_i)}$ be confluent Verma module bundles of weights $\lambda_i \in \mathbb{C}$ and P-rank $r_i > 0$ for $i = 1, \ldots, n$. Consider the external tensor product
\[
E_{\Lambda}^{(R)} := E_{\lambda_1}^{(r_1)} \otimes \cdots \otimes E_{\lambda_n}^{(r_n)} \to B^{(r_1)} \times \cdots \times B^{(r_n)}
\]
where $\Lambda := (\lambda_1, \ldots, \lambda_n)$ and $R := (r_1, \ldots, r_n)$. As the notation of coordinates on the base space $B^{(R)} := B^{(r_1)} \times \cdots \times B^{(r_n)}$, we use
\[
(\gamma_1, \ldots, \gamma_n) \in B^{(r_1)} \times \cdots \times B^{(r_n)}, \quad \gamma_i = (\gamma_i^{(1)}, \ldots, \gamma_i^{(r_i)}) \in B^{(r_i)} = \mathbb{C}^{r_i-1} \times \mathbb{C}^*.
\]
Define the elements $\Omega_{ij}^{(p,q)} \in U g^{(r_i)} \otimes U g^{(r_j)}$ by
\[
\Omega_{ij}^{(p,q)} := E_p \otimes F_q + F_p \otimes E_q + \frac{1}{2} H_p \otimes H_q
\]
for $p = 0, \ldots, r_i$ and $q = 0, \ldots, r_j$. We also denote by $\Omega_{ij}^{(p,q)} : E_{\lambda}^{(R)} \to E_{\lambda}^{(R)}$ the induced bundle map through the action of $\Omega_{ij}^{(p,q)}$ on $i$-th and $j$-th components of $E_{\lambda}^{(R)}$. Let $X_n := \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j \}$ be the configuration space. We can extend the vector bundle $E_{\lambda}^{(R)} \to B^{(R)}$ trivially on the direct product $B^{(R)} \times X_n$ by the pullback of the projection $B^{(R)} \times X_n \to B^{(R)}$. By abuse of notation, we also write the resulting vector bundle as $E_{\lambda}^{(R)} \to B^{(R)} \times X_n$. The following operators were introduced in [JNS08].
Definition 4.9. The operators
\[
\{ G_i^{(s)} \in \text{End}(E_{\Lambda}^{(R)}) \mid i = 1, \ldots, n \text{ and } s = -1, \ldots, r_i - 1 \},
\]
called the (generalized) Gaudin Hamiltonians, are defined by
\[
G_i^{(s)} := \frac{1}{2} \sum_{p=0}^{s} \Omega_{i,i}^{(p,s-p)} + \sum_{j \neq i} \sum_{p,q \geq 0}^{p+q \leq r_i + r_j - s - 1} \binom{p+q}{p} \frac{(-1)^p}{(z_i - z_j)^{p+q+1}} \Omega_{ij}^{(s+p+1,q)}.
\]

Define the space of holomorphic sections of \( E_{\Lambda}^{(R)} \) by
\[
S(E_{\Lambda}^{(R)}) := \left\{ \sum_{\text{finite}} f_k F^{J_k} v_\Lambda \mid f_k \in \mathcal{O}(B^{(R)} \times X_n) \text{ and } J_k \in \mathbb{Z}_{R+1}^n \right\}
\]
where \( v_\Lambda := v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \). Then the action of Gaudin Hamiltonians on \( S(E_{\Lambda}^{(R)}) \) is well-defined. As in Section 4.2, we introduce vector fields \( D_i^{(s)} \) for \( i = 1, \ldots, n \) and \( s = 0, \ldots, r_i - 1 \) on \( B^{(R)} \times X_n \) by
\[
D_i^{(s)} := \sum_{p=1}^{r_i-s} p \gamma_i^{(s+p)} \frac{\partial}{\partial \gamma_i^{(p)}}.
\]
Set \( \partial_i := \partial/\partial z_i \). Then
\[
\left\{ \partial_i, D_i^{(s)} \mid i = 1, \ldots, n \text{ and } s = 0, \ldots, r_i - 1 \right\}
\]
forms a basis of the space of vector fields on \( B^{(R)} \times X_n \). We extend the connection in Definition 4.6 as follows.

Definition 4.10. The connection \( \nabla \) on \( E_{\Lambda}^{(R)} \) is defined by
\[
\nabla_{D_i^{(s)}} (f F^{J} v_\Lambda) := \left( D_i^{(s)} f \right) F^J v_\Lambda + f F^{J_1} v_{\lambda_1} \otimes \cdots \otimes \left( \nabla_{D_i^{(s)}} F^{J_i} v_{\lambda_i} \right) \otimes \cdots \otimes F^{J_n} v_{\lambda_n}
\]
and
\[
\nabla_{\partial_i} (f F^{J} v_\Lambda) := \frac{\partial f}{\partial z_i} F^J v_\Lambda
\]
where \( f \in \mathcal{O}(B^{(R)} \times X_n) \).

By Lemma 4.5, \( \nabla \) is integrable. Now we introduce the confluent KZ equation as follows.

Definition 4.11. Fix a nonzero complex number \( \kappa \in \mathbb{C}^* \). The confluent Knizhnik-Zamolodchikov (KZ) equation for a holomorphic section \( \Phi \in S(E_{\Lambda}^{(R)}) \) is a system of differential equations
\[
\nabla_{\partial_i} \Phi = \frac{1}{\kappa} G_i^{(-1)} \Phi
\]
\[
\nabla_{D_i^{(s)}} \Phi = \frac{1}{\kappa} \left( G_i^{(s)} - \beta_i^{(s)} \right) \Phi
\]

for $i = 1, \ldots, n$ and $s = 0, \ldots, r_i - 1$ where $\beta_i^{(s)}$ is the function given by

$$
\beta_i^{(s)} := \frac{1}{4} \sum_{p=0}^{s} \gamma_i^{(p)} \gamma_i^{(s-p)} + \frac{s+1}{2} \gamma_i^{(s)}.
$$

By the definition of Gaudin Hamiltonians, it has poles of order $r_i + r_j + 1$ along the divisors $\{z_i = z_j\}$ for $i \neq j$. In addition, by the computation in Lemma 4.3, it has poles of order $r_i$ along the divisors $\{\gamma_i^{(r_i)} = 0\}$. Thus the confluent KZ equation has irregular singularities and their Poincaré ranks are determined by P-ranks $r_1, \ldots, r_n$. In [JNS08], they showed the integrability of the confluent KZ equation.

**Proposition 4.12** ([JNS08], Proposition 4.1). Define the confluent KZ connection $\nabla^{KZ}$ by

$$
\nabla^{KZ}_{\partial_i} := \nabla_{\partial_i} - \frac{1}{\kappa} G_i^{(-1)}
$$

$$
\nabla^{KZ}_{D_i^{(s)}} := \nabla_{D_i^{(s)}} - \frac{1}{\kappa} \left( G_i^{(s)} - \beta_i^{(s)} \right)
$$

for $i = 1, \ldots, n$ and $s = 0, \ldots, r_i - 1$. Then $\nabla^{KZ}$ is integrable.

Consider the restriction of the confluent KZ equation on finite rank subbundles of $E^{(R)}_\Lambda$. We note the following property of the confluent KZ connection.

**Lemma 4.13.** The action of the Lie algebra $g$ on $S(E^{(R)}_\Lambda)$ commutes with the action of the confluent KZ connection $\nabla^{KZ}$ on $S(E^{(R)}_\Lambda)$.

**Proof.** It is easy to check that $[\Omega_{ij}^{(p,q)}, X] = 0$ for $X \in g$. In addition, $[\nabla, X] = 0$ by the definition of $\nabla$. Hence $[\nabla^{KZ}, X] = 0$. □

Let $S^{(R)}[|\Lambda| - 2m]$ be the finite rank subbundle of $E^{(R)}_\Lambda$ whose fiber over $(\Gamma, z) \in B^{(R)} \times X_n$ is the space of singular vectors $S^{(R)}_\Gamma[|\Lambda| - 2m]$. Lemma 4.13 implies that the restriction of the confluent KZ equation on the subbundle $S^{(R)}[|\Lambda| - 2m]$ is well-defined.

### 4.5. Monodromy representations.

In this section, we construct the monodromy representations of the framed braid groups from the confluent KZ connections. We first note that the space of movable weights $B^{(r)} = \mathbb{C}^{r-1} \times \mathbb{C}^*$ is homotopic to the circle $S^1$. Hence, the space $B^{(R)} = B^{(r_1)} \times \cdots \times B^{(r_n)}$ is homotopic to the $n$-dimensional torus $T^n$ and the direct product $B^{(R)} \times X_n$ is homotopic to $T^n \times X_n$. Thus the fundamental group $\pi_1(B^{(R)} \times X_n, *)$ is isomorphic to the pure framed braid group $FP_n$ (see Section 2.2).

Recall from the previous section that the confluent KZ connection is an integrable connection on the vector bundle $E^{(R)}_\Lambda \to B^{(R)} \times X_n$. Consider the restriction of the confluent KZ connection on the finite rank subbundle $S^{(R)}[|\Lambda| - 2m] \subset E^{(R)}_\Lambda$. Then we obtain the
monodromy representation of the pure framed braid group
\[ \theta_{\Lambda, \kappa}^{(R)}: FP_n \to \text{Aut}_\mathbb{C} S^{(r_n)}_\Gamma[|\Lambda| - 2m] \]

where \( S^{(r_n)}_\Gamma[|\Lambda| - 2m] \) is the fiber over the base point \( * = (\Gamma, z) \in B^{(r_n)} \times X_n \). The dimension of the representation \( \theta_{\Lambda, \kappa}^{(R)} \) is given by Proposition 4.8.

Consider the case \( R = (r, \ldots, r) = (r^n) \) and \( \Lambda = (\lambda, \ldots, \lambda) = (\lambda^n) \). In this case, the action of the symmetric group \( \mathfrak{S}_n \) on \( B^{(r^n)} \times X_n \) is well-defined since \( B^{(r^n)} = (B^{(r)})^n \). In addition, \( \lambda_1 = \cdots = \lambda_n \) implies that the action of \( \mathfrak{S}_n \) lifts on \( E^{(r^n)}_{\lambda^n} \). Hence we obtain the quotient vector bundle
\[ E^{(r^n)}_{\lambda^n} / \mathfrak{S}_n \to (B^{(r^n)} \times X_n) / \mathfrak{S}_n. \]

Further, the confluent KZ connection \( \nabla_{KZ} \) is \( \mathfrak{S}_n \)-invariant in the case \( r_1 = \cdots = r_n \). Thus the connection \( \nabla_{KZ} \) descends to an integrable connection on the quotient vector bundle \( E^{(r^n)}_{\lambda^n} / \mathfrak{S}_n \). By Proposition 2.2, the fundamental group \( \pi_1((B^{(r^n)} \times X_n) / \mathfrak{S}_n, *) \) is isomorphic to the framed braid group \( FB_n \). Therefore we obtain the following monodromy representation.

**Proposition 4.14.** For positive integers \( r, m > 0 \) and complex parameters \( (\lambda, \kappa) \in \mathbb{C} \times \mathbb{C}^* \), there is a representation of the framed braid group
\[ \theta^{(r)}_{\Lambda, \kappa}: FB_n \to \text{Aut}_\mathbb{C} S^{(r^n)}_\Gamma[|n\lambda - 2m|] \]
constructed as the monodromy representation of the confluent KZ connection on the vector bundle
\[ S^{(r^n)}[|n\lambda - 2m|] / \mathfrak{S}_n \to (B^{(r^n)} \times X_n) / \mathfrak{S}_n. \]

The dimension of this representation is given by Proposition 4.8.

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