GRADIENT INEQUALITY AND CONVERGENCE OF THE NORMALIZED RICCI FLOW

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Dedicated to Professor Nicholas Alikakos on the occasion of his retirement

Abstract. We study the problem of convergence of the normalized Ricci flow evolving on a compact manifold \( \Omega \) without boundary. In [11, 12] we derived, via PDE techniques, global-in-time existence of the classical solution and pre-compactness of the orbit. In this work we show its convergence to steady-states, using a gradient inequality of Lojasiewicz type. We have thus an alternative proof of [7], but for general manifold \( \Omega \) and not only for unit sphere. As a byproduct of that approach we also derive the rate of convergence according to this steady-state being either degenerate or non-degenerate as a critical point of a related energy functional.

1. Introduction

In the current work we revisit the following problem of logarithmic diffusion

\[
\begin{align*}
\frac{u_t}{u} &= \Delta \log u + u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx, & x \in \Omega, \ t > 0 \\
u(x,0) &= u_0(x) > 0, & x \in \Omega
\end{align*}
\]

with

\[
\int_{\Omega} u_0(x) \, dx = \lambda,
\]

where \( \lambda > 0 \) is a constant and \( \Omega \) is a compact Riemannian surface without boundary. When \( \lambda = 8\pi \) and \( \Omega \) equals to the unit sphere \( S^2 = \{ x \in \mathbb{R}^2 : ||x|| < 1 \} \), problem (1.1) - (1.3) describes an evolution of the metric \( g = g(t) \) on \( \Omega \), that is, the normalized Ricci flow introduced by [7] as

\[
\frac{\partial g}{\partial t} = (r - R)g,
\]

where \( R \) is the scalar curvature, \( r \) is the volume mean

\[
r = \frac{\int_{\Omega} R \, \mu}{\int_{\Omega} \mu},
\]

and \( \mu = \mu(t) \) is the area element, cf. [1, 13, 19, 20].

The standard parabolic theory, cf. [14], assures that for smooth initial data \( u_0(x) > 0 \) problem (1.1) - (1.2) has a unique classical solution local in time, denoted by \( u(x,t) > 0 \) in \( \Omega \times (0,T) \) with \( T = T_{\text{max}} > 0 \), there also holds that

\[
\int_{\Omega} u(x,t) \, dx = \int_{\Omega} u_0(x) \, dx \equiv \lambda > 0
\]

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by (1.2). The main result in [7] reads as: if 

$$\lambda = 8\pi$$  

then there holds that 

$$T = +\infty$$ and 

$$u(\cdot, t) \to u^*, t \to \infty$$ in $$C^\infty$$ topology,  

where $$u^* = u^*(x) > 0$$ is a stationary solution to (1.1) - (1.3):

$$- \Delta \log u^* = u^* - 1 \frac{1}{|\Omega|} \int_{\Omega} u^* \, dx, \quad x \in \Omega, \quad \int_{\Omega} u^* \, dx = \lambda. \quad (1.6)$$

Remarkably, Hamilton’s approach in [7], uses the geometric structure of problem (1.1)-(1.3) valid only for the special case (1.4), which results in the control of several key geometric quantities related to the evolution of the metric $$g = g(t)$$.

On the other hand, via a PDE approach, which works to any $$0 < \lambda \leq 8\pi$$ and any compact Riemannian surface $$\Omega$$ without boundary, we could derive convergence (1.5), cf. [11, 12]. Specifically, in spite of the lack of geometric structure of this general case, the following holds.

**Theorem 1.1** ([11, 12]). Assume that $$\Omega$$ is a compact Riemannian surface without boundary and $$0 < \lambda \leq 8\pi$$. Then the solution $$u = u(x, t)$$ to (1.1) - (1.3) exists global in time and satisfies the uniform estimates

$$\sup_{t \geq 0} \{ ||u(\cdot, t)||_\infty + ||u^{-1}(\cdot, t)||_\infty \} < \infty. \quad (1.7)$$

We note that Theorem 1.1 reproduces the convergence (1.5) for the case (1.4) with the aid of dynamical and elliptic theories.

The first step to confirm this fact is to notice that system (1.1)-(1.3) is provided with a Lyapunov function and $$L^1$$ conservation. In fact, problem (1.1) is written as the parabolic-elliptic system

$$u_t = \Delta (\log u - v), \quad (1.8)$$

$$-\Delta v = u - 1 \frac{1}{|\Omega|} \int_{\Omega} u \, dx, \quad \int_{\Omega} v \, dx = 0, \quad x \in \Omega, \quad t > 0. \quad (1.9)$$

Notably (1.8)-(1.9) takes the form of model (B) equation studied by [13, 19], that is

$$u_t = \Delta \delta F(u), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad t > 0,$$

where $$\delta F$$ stands for the first variation of the functional

$$F(u) = \int_{\Omega} u (\log u - 1) \, dx + \frac{1}{2} \int_{\Omega} u \cdot \Delta^{-1} u \, dx$$

whilst by

$$v = -\Delta^{-1} u$$

we mean that $$v$$ satisfies (1.9) for given $$u$$. As a consequence it holds that

$$\frac{d}{dt} \int_{\Omega} u = 0, \quad \frac{d}{dt} F(u) = \langle u_t, \delta F(u) \rangle = -\|\nabla \delta F(u)\|_2^2 \leq 0 \quad (1.10)$$

for the solution $$u = u(\cdot, t)$$.

By this variational structure of (1.10), the steady-state $$u^*$$ of (1.8)-(1.9) is defined by

$$\delta F(u^*) = \log u^* + \Delta^{-1} u^* = \text{constant}, \quad \int_{\Omega} u^* \, dx = \lambda, \quad u^* = u^*(x) > 0. \quad (1.11)$$

For more details regarding the above formulation of the steady-state as a model (B) equation see [19].
Since Theorem 1.1 guarantees the pre-comapcness in $C^\infty$ topology of the orbit $\{u(\cdot, t)\}$ of the global-in-time solution to (1.1)-(1.3), the following fact arises by the theory of dynamical systems, that is, the LaSalle principle [10, 19], where
\[ \omega(u_0) = \{u_* = u_*(x) > 0 \mid \text{there exists } t_k \to \infty \text{ such that } u(\cdot, t_k) \to u_* \text{ in } C^\infty \text{ topology} \} \]
denotes the $\omega$-limit set.

**Theorem 1.2.** Under the assumption of Theorem 1.1, the $\omega$-limit set $\omega(u_0)$ is non-empty, connected, and compact, contained in the set of stationary solutions denoted by
\[ F_\lambda = \{u^* = u^*(x) > 0 \mid \text{classical solution to (1.11)} \}. \]

**Remark 1.3.** To show the equivalence of (1.6) and (1.11), first, put $v^* = -\Delta -1 u^*$ in (1.11). Then (1.11) implies
\[ -\Delta v^* = u^* - \frac{1}{|\Omega|} \int_\Omega u^* \, dx, \quad \int_\Omega v^* = 0 \tag{1.12} \]
\[ \log u^* = v^* + \text{constant}, \quad \int_\Omega u^* = \lambda \tag{1.13} \]
and hence
\[ -\Delta v^* = \lambda \left( \frac{e^{v^*}}{\int_\Omega e^{v^*} \, dx} - \frac{1}{|\Omega|} \right), \quad \int_\Omega v^* \, dx = 0 \tag{1.14} \]
\[ u^* = \frac{\lambda e^{v^*}}{\int_\Omega e^{v^*} \, dx}, \quad \int_\Omega e^{v^*} \, dx, \tag{1.15} \]
which implies (1.6). If $u^* = u^*(x) > 0$ solves (1.6), then (1.14) arises for
\[ v^* = w^* - \frac{1}{|\Omega|} \int_\Omega w^* \, dx, \quad w^* = \log u^*, \tag{1.16} \]
and hence (1.11) holds true. Thus (1.6) is equivalent to (1.11).

Theorem 1.2 implies (1.5) if $F_\lambda$ is discrete, particularly, a singleton:
\[ F_\lambda = \{\lambda/|\Omega|\}. \tag{1.17} \]

Under the transformation (1.13), furthermore, property (1.17) is equivalent to
\[ E_\lambda = \{0\}, \tag{1.18} \]
where
\[ E_\lambda = \{v^* \mid \text{solution to (1.14)} \}. \]
If (1.18) holds for $0 < \lambda \leq 8\pi$, then arises (1.5) with
\[ u^* = \frac{\lambda}{|\Omega|} \tag{1.19} \]
in (1.1)-(1.3) by Theorem 1.2.

We are ready to begin the second step of deriving (1.5) for (1.4) by Theorem 1.2 that is, the confirmation of (1.18) for (1.4). First of all, we note that this fact follows from a geometric property, that is, the classification of the closed surface with constant Gaussian curvature. Direct proof in the context of the elliptic theory, however, is also available [2, 3, 15]. The analytic proof of (1.5) for (1.4) is thus complete.

**Remark 1.4.** Another case when (1.18) is valid is for
\[ \lambda = 8\pi, \quad \Omega = \mathbb{T} \equiv \mathbb{R}^2/a\mathbb{Z} \times b\mathbb{Z}, \quad \frac{b}{a} \geq \frac{\pi}{4} \tag{1.20} \]
by the elliptic theory [16]. Hence, (1.5) arises with (1.19) and for (1.20).
The convergence (1.5), however, is valid even if $E_\lambda$ forms a continuum. Furthermore, we can even determine the rate of convergence and thus Theorem 1.2 is improved as follows.

**Theorem 1.5.** Under the assumption of Theorem 1.2 there is a solution $u^* = u^*(x) > 0$ to (1.6) so that (1.5) is valid. The rate of this convergence is at least algebraic.

**Theorem 1.6.** If $u^*$ is non-degenerate in the previous theorem, the rate of convergence in (1.5) is exponential.

To define the non-degeneracy of the steady-state $u^*$, we use the fact that $v^*$ defined by (1.16) is a solution to (1.14), which is the Euler-Lagrange equation of the energy functional

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|^2_2 - \lambda \log \int v^2 \, dx$$

(1.21)

defined for $v \in V_0$, where

$$V_0 = \{v \in H^1(\Omega) \mid \int \Omega v \, dx = 0\}.$$  

(1.22)

Thus we say that $u^* = u^*(x) > 0$ is non-degenerate in Theorem 1.6 if $v^* \in V_0$ defined by (1.16) is non-degenerate as a critical point of $J_\lambda$ on $V_0$. Later in Lemma 4.1 we show that this non-degeneracy of $u^* = u^*(x) > 0$ means that

$$\psi \in H^2(\Omega), \quad -\Delta \psi = u^* \psi \text{ in } \Omega, \quad \int \Omega \psi u^* \, dx = 0 \implies \psi = 0.$$  

(1.23)

**Remark 1.7.** We can regard (1.14) as a nonlinear eigenvalue problem of finding $(\lambda, v^*)$ simultaneously. Then, if $\Omega = \mathbb{R}^2$, non-trivial solutions bifurcate at $\lambda = 8\pi$ from the branch of trivial solutions \{$(\lambda, v^*) \mid \lambda \in \mathbb{R}, v^* = 0$\}. Hence we cannot apply Theorem 1.6 for this case, but still have the rate at least of algebraic order in (1.5) with (1.19) for $\lambda = 8\pi$. In the case of $\Omega = \mathbb{R}^2/a\mathbb{Z} \times b\mathbb{Z}$ with $\frac{2}{a} \geq \frac{2}{b}$, on the other hand, $\lambda = 8\pi$ is a bifurcation point of the non-trivial solution, and $v^* = 0$ is still non-degenerate at this value of $\lambda = 8\pi$. Hence in the case of a torus given by (1.20), there holds (1.5), with (1.19) for $\lambda = 8\pi$, in the exponential rate.

Theorem 1.1 without geometric structure for problem (1.1)-(1.3) is proven as follows. First, the range $0 < \lambda < 8\pi$ of this problem is sub-critical in accordance with the Trudinger-Moser-Fontana inequality [6]

$$v \in V_0, \quad \|v\|_2 \leq 1 \implies \int \Omega e^{4\pi v^2} \, dx \leq C$$

(1.24)

which entails

$$\inf\{J_{8\pi}(v) \mid v \in V_0\} > -\infty$$

as in [19]. Hence Moser’s iteration ensures $T = +\infty$ and (1.7) for $0 < \lambda < 8\pi$, under

$$\|u(\cdot, t)\|_1 = \lambda, \quad F(u(\cdot, t)) \leq F(u_0)$$

derived from (1.10), cf. [11]. On the other hand, Benilan-Crandall’s inequality

$$\frac{u_t(x, t)}{u(x, t)} \leq \frac{e^t}{e^t - 1}$$

is used to confirm $T = +\infty$ for $\lambda = 8\pi$ ([11]). To derive (1.7) for $\lambda = 8\pi$, we finally appeal to a concentration compactness argument, cf. [12].

**Remark 1.8.** An immediate consequence of (1.24) is that any $K > 0$ admits $C(K) > 0$ such that

$$v \in V = H^1(\Omega), \quad \|v\|_V \leq K \implies \|e^{|v|}\|_1 \leq C(K),$$

where $\|v\|_V = (\|v\|^2_2 + \|\nabla v\|^2_2)^{1/2}$.

The main aim of the current work for is to provide an analytic proof of Theorems 1.5, 1.6, by using a gradient inequality which takes the following classical form in the finite dimensional case.

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Lemma 1.9 ([17]). Let $E = E(x) : \mathbb{R}^n \to \mathbb{R}$ be real-analytic at $x = 0$, satisfying $E(0) = 0$ and $\delta E(0) = 0$. Then there is $0 < \theta \leq \frac{1}{2}$ such that
\[ |E(x)|^{1-\theta} \leq C|\delta E(x)|, \quad |x| \ll 1. \]

We thus provide an alternative proof of Hamilton’s convergence result in [7] which is entirely based on the parabolic theory; that is no use of any geometric or elliptic structures of (1.1)-(1.3) is made. Furthermore, our proof assures (1.5) for any $0 < \lambda \leq 8\pi$ and a general compact manifold $\Omega$ without boundary and it also shows that the convergence rate is at least algebraic and exponential, provided that $u^* = u^*(x) > 0$ is degenerate and non-degenerate steady-state respectively.

This paper is composed of five sections. Section 2 is devoted to the key concept of critical manifold developed in [4, 5]. Then, Theorem 1.5 is proven in Section 3 employing the method presented in [8, 18]. Section 4 is devoted to the study of non-degenerate steady-state solution, and finally Theorem 1.6 is proven in Section 5.

Notations. In the sequel $|| \cdot ||_p$ denotes the $L^p(\Omega)$-norm for $1 \leq p \leq \infty$. The letter $C$ denotes inessential constants which may vary from line to line. The dependence of $C$ upon parameters is indicated explicitly.

2. Theory of Critical Manifolds

Under the change of variables $u = e^w$, problem (1.1)-(1.3) is reduced to
\[ \frac{\partial e^w}{\partial t} = \Delta w + \lambda \left( \int_{\Omega} e^w dx - \frac{1}{|\Omega|} \right), \quad x \in \Omega, \quad t > 0, \quad (2.1) \]
\[ w(x, 0) = w_0(x), \quad x \in \Omega. \quad (2.2) \]

Integrating equation (2.1) over $\Omega$, taking also into account that $\partial \Omega = \emptyset$, we obtain the total mass conservation
\[ \int_\Omega e^w dx = \int_\Omega e^{w_0} dx = \lambda. \]

Hence it holds that
\[ \frac{\partial e^w}{\partial t} = \Delta w + e^w - \frac{\lambda}{|\Omega|}, \quad x \in \Omega, \quad t > 0 \quad (2.3) \]
\[ w(x, 0) = w_0(x), \quad x \in \Omega, \quad \int_\Omega e^{w_0} dx = \lambda, \quad (2.4) \]

and a related variational functional is
\[ \mathcal{E}(w) = \int_\Omega \frac{1}{2} |\nabla w|^2 - e^w + \frac{\lambda}{|\Omega|} w \ dx, \quad w \in H^1(\Omega) = V. \quad (2.5) \]

In relation to the Gel’fand triple
\[ V = H^1(\Omega) \hookrightarrow X = L^2(\Omega) \cong X^* \hookrightarrow V^*, \]

the first variation of $\mathcal{E}(w)$ is given by
\[ \delta \mathcal{E}(w) = -\Delta w - e^w + \frac{\lambda}{|\Omega|} \quad (2.6) \]

and thus (2.3)-(2.4) is reduced to
\[ \frac{\partial e^w}{\partial t} = -\delta \mathcal{E}(w), \quad x \in \Omega, \quad t > 0 \quad (2.7) \]
\[ w(x, 0) = w_0(x), \quad x \in \Omega, \quad \int_\Omega e^{w_0} dx = \lambda. \quad (2.8) \]
Each steady-state $u^* \in F_\lambda$ to (1.1)-(1.3) is a solution to (1.6), and hence $w^* = \log u^*$ satisfies
\[ \Delta w^* + e^{w^*} - \frac{\lambda}{|\Omega|} = 0, \quad x \in \Omega, \]
or equivalently,
\[ \delta \mathcal{E}(w^*) = 0. \]

This section is devoted to the proof of the following inequality, which casts a basis for proving Theorem 2.1 as in the standard theory of gradient inequality, cf. [18].

**Theorem 2.1.** Given $w^* \in V$ satisfying $\delta \mathcal{E}(w^*) = 0$, there exist $0 < \theta \leq \frac{1}{2}$ and $\varepsilon_0 > 0$ such that
\[ w \in V, \quad ||w - w^*||_V < \varepsilon_0 \Rightarrow \mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C||\delta \mathcal{E}(w)||_V^\theta. \quad (2.9) \]

To prove this result we decompose $\mathcal{E}(w)$ as in
\[ \mathcal{E}(w) = \mathcal{E}_1(w) - \mathcal{E}_2(w) \]
for
\[ \mathcal{E}_1(w) = \int_\Omega \frac{1}{2} \nabla |w|^2 + \frac{\lambda}{|\Omega|} w \, dx, \quad \mathcal{E}_2(w) = \int_\Omega e^w \, dx. \]
The first functional $\mathcal{E}_1 : V \to \mathbb{R}$ is analytic, and it holds that
\[ \delta \mathcal{E}_1(w^*)[w] = \int_\Omega \nabla w \cdot \nabla w^* + \frac{\lambda}{|\Omega|} w \, dx \]
\[ \delta^2 \mathcal{E}_1(w^*)[w,w] = \int_\Omega |\nabla w|^2 \, dx = (\nabla w, \nabla w) \]
\[ \delta^k \mathcal{E}_1(w^*)[w,w,\ldots,w] = 0, \quad k \geq 3, \quad w, w^* \in V, \]
where $( , , )$ denotes the $L^2$-inner product. The second functional $\mathcal{E}_2 : V \to \mathbb{R}$ is also analytic by the Trudinger-Moser-Fontana inequality (1.24), which assures
\[ \sum_{k=0}^{\infty} \frac{1}{k!} \int_\Omega e^{w^*} |w|^k \, dx = \int_\Omega e^{w^*} |w| \, dx < +\infty, \quad w, w^* \in V. \]

Then
\[ \mathcal{E}_2(w + w^*) - \mathcal{E}_2(w^*) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_\Omega e^{w^*} w^k \, dx \]
and hence
\[ \delta^k \mathcal{E}_2(w^*)[w,w,\ldots,w] = \int_\Omega e^{w^*} w^k \, dx, \quad k \geq 1. \]

Given $w^* \in V$ satisfying $\delta \mathcal{E}(w^*) = 0$, the linearized operator
\[ \mathcal{L} \equiv \delta^2 \mathcal{E}(w^*) = -\Delta - e^{w^*} : V \to V^* \]
is realized as a self-adjoint operator in $X = L^2(\Omega)$ with domain $D(\mathcal{L}) = H^2(\Omega)$. To develop the theory of critical manifold, cf. [23], we first introduce
\[ X_1 \equiv \text{Ker } \mathcal{L} = \{ v \in D(\mathcal{L}) | \mathcal{L}v = 0 \} \subset V = H^1(\Omega). \]

Let $\text{dim } X_1 = n < \infty$ and $\{ \phi_1, \ldots, \phi_n \}$ be an orthonormal basis of $X_1$, and define the orthogonal projection $\mathcal{P} : X \to X_1$, which can be extended to $\mathcal{P} : V^* \to X_1$, by
\[ \mathcal{P}v = \sum_{i=1}^{n} (v, \phi_i) \phi_i = \sum_{i=1}^{n} \langle \phi_i, v \rangle_{V,V^*} \phi_i. \]

We next recall the following theorem, derived from the implicit function theorem applied to
\[ (I - \mathcal{P}) \delta \mathcal{E}(v) = 0. \]
The local manifold $S$ defined by (2.10) below is analytic because $E : V \to \mathbb{R}$ is so.

**Theorem 2.2** ([5]). Each $w^* \in V$ with $\delta E(w^*) = 0$ admits a neighbourhood $U \subset V$ of $w^*$ such that

$$S = \{ w \in U | (I - \mathcal{P}) \delta E(w) = 0 \}$$

(2.10)

is a local analytic manifold around $w^*$ with dimension equal to $n$.

More precisely, we have the analytic mapping

$$g : U_1 = U \cap X_1 \to U_2 = (I - \mathcal{P}) U$$

such that $g(w_1^*) = w_2^*$ for $w^* = w_1^* + w_2^* \in U_1 \oplus U_2$, and define

$$S = \{ w_1 + g(w_1) | w_1 \in U_1 \}.$$ (2.11)

Then the following decomposition is valid

$$w = w_1 + w_2 \in S = U_1 \oplus U_2, \quad w_1 = \mathcal{P} w, \quad w_2 = g(w_1),$$

and then, the analytic mapping $Q : U \to S$ is defined by

$$Qw = w_1 + g(w_1) \in S, \quad w = w_1 + w_2 \in U_1 \oplus U_2.$$ (2.12)

We then obtain

$$w - Qw = w_2 - g(w_1) \in U_2$$

(2.13)

and also

$$Qw = w, \quad w \in S$$

(2.14)

by (2.11)-(2.12).

In the sequel we confirm several lemmas derived from the above structure.

**Lemma 2.3.** It holds that

$$|E(w) - E(Qw)| \leq C ||w - Qw||_V^2, \quad w \in U.$$

**Proof.** First, we have

$$E(w) - E(Qw) = \langle w - Qw, \delta E(Qw) \rangle + \frac{1}{2} \delta^2 E(Qw)[w - Qw, w - Qw] + o \left( ||w - Qw||_V^2 \right).$$

(2.15)

Second, there arises

$$(I - \mathcal{P})(w - Qw) = w - Qw$$

by (2.13), and therefore, $Qw \in S$ implies

$$\langle w - Qw, \delta E(Qw) \rangle = \langle (I - \mathcal{P})(w - Qw), \delta E(Qw) \rangle$$

$$= \langle w - Qw, (I - \mathcal{P})\delta E(Qw) \rangle = 0.$$

Then (2.15) entails the desired estimate

$$|E(w) - E(Qw)| \leq C ||w - Qw||_V^2.$$

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**Lemma 2.4.** Any $\varepsilon > 0$ admits $\delta > 0$ such that

$$w \in V, \quad ||w - w^*||_V < \delta \quad \Rightarrow \quad ||w - Qw||_V < \varepsilon.$$

**Proof.** We may assume $w \in U$. Since $w^* \in S$ it holds that $Qw^* = w^*$ by (2.14), which implies

$$||w - Qw||_V \leq ||w - w^*||_V + ||Qw^* - Qw||_V.$$

The result is now obvious because $Q : U \to S$ is analytic.

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**Lemma 2.5.** There is $\varepsilon_0 > 0$ such that

$$w \in V, \quad ||w - w^*||_V < \varepsilon_0 \quad \Rightarrow \quad ||w - Qw||_V \leq C ||\delta E(w)||_V.$$
Proof. First, we have
\[ \delta \mathcal{E}(w) - \delta \mathcal{E}(Qw) = \delta^2 \mathcal{E}(Qw)(w - Qw) + o(||w - Qw||_V). \] (2.16)

Since
\[ (I - \mathcal{P})\delta \mathcal{E}(Qw) = 0 \]
by \( Qw \in S \), it follows that
\[ (I - \mathcal{P})\delta \mathcal{E}(w) = (I - \mathcal{P})\delta^2 \mathcal{E}(Qw)(w - Qw) + o(||w - Qw||_V). \] (2.17)

Let \( V_2 = (I - \mathcal{P})(V) \) and recall \( L = \delta^2 \mathcal{E}(w^*). \) Then,
\[ (I - \mathcal{P})L : V_2 \to V^*_2 \]
is an isomorphism. By Lemma 2.4, therefore, there is \( \varepsilon_1 > 0 \) such that
\[ (I - \mathcal{P})\delta^2 \mathcal{E}(Qw) : V_2 \to V^*_2 \]
is an isomorphism, provided that \( ||w - w^*||_V < \varepsilon_1 \) for \( w \in V \).

More precisely, we have \( C_1 > 0 \) such that
\[ ||z||_V \leq C_1 \left( ||(I - \mathcal{P})\delta^2 \mathcal{E}(Qw)z||_{V^*}, z \in V_2 \right) \] (2.18)
for any \( w \in V \) in \( ||w - w^*||_V < \varepsilon_1 \). Putting
\[ z = w - Qw \in V_2 = (I - \mathcal{P})V \]
in (2.18), we deduce
\[ ||w - Qw||_V \leq C_1 \left( ||(I - \mathcal{P})\delta^2 \mathcal{E}(Qw)(w - Qw)||_{V^*} + o(||w - Qw||_V) \right) \]
by (2.17). Hence there is \( \varepsilon_0 > 0 \) such that
\[ w \in V, \ ||w - w^*||_V < \varepsilon_0 \quad \Rightarrow \quad ||w - Qw||_V \leq C_2 \left( ||(I - \mathcal{P})\delta \mathcal{E}(w)||_{V^*} \leq C_3 ||\delta \mathcal{E}(w)||_{V^*} \right) \]
by Lemma 2.4.

Lemma 2.6. There is \( \varepsilon_0 > 0 \) such that
\[ w \in V, \ ||w - w^*||_V < \varepsilon_0 \quad \Rightarrow \quad ||\delta \mathcal{E}(Qw)||_{V^*} \leq C||\delta \mathcal{E}(w)||_{V^*}. \]

Proof. Equality (2.16), combined with \( Qw \in S \), implies
\[ \delta \mathcal{E}(Qw) = \mathcal{P}\delta \mathcal{E}(Qw) = \mathcal{P}\delta \mathcal{E}(w) - \mathcal{P}\delta^2 \mathcal{E}(Qw)(w - Qw) + o(||w - Qw||_V). \]

Hence it follows that
\[ ||\delta \mathcal{E}(Qw)||_{V^*} \leq ||\delta \mathcal{E}(w)||_{V^*} + C_4 ||w - Qw||_V + o(||w - Qw||_V). \] (2.19)

Then Lemma 2.4 assures there exists \( \varepsilon_1 > 0 \) such that
\[ w \in V, \ ||w - w^*||_V < \varepsilon_1 \quad \Rightarrow \quad ||\delta \mathcal{E}(Qw)||_{V^*} \leq ||\delta \mathcal{E}(w)||_{V^*} + C_5 ||w - Qw||_V. \]

We finally obtain \( \varepsilon_0 > 0 \) such that
\[ w \in V, \ ||w - w^*||_V < \varepsilon_0 \quad \Rightarrow \quad ||\delta \mathcal{E}(Qw)||_{V^*} \leq C_5 ||\delta \mathcal{E}(w)||_{V^*} \]
by Lemma 2.5.

Lemma 2.7. There exist \( 0 < \theta \leq \frac{1}{2} \) and \( \varepsilon_0 > 0 \) such that
\[ w \in S, \ ||w - w^*||_V < \varepsilon_0 \quad \Rightarrow \quad |\mathcal{E}(w) - \mathcal{E}(w^*)|^{1-\theta} \leq C||\delta \mathcal{E}(w)||_{V^*}. \] (2.20)

Proof. Since \( S \) is a finite dimensional analytic manifold and \( \mathcal{E} : S \to \mathbb{R} \) is analytic, the result follows from Lemma 1.9.

We are ready to give the proof of Theorem 2.1.
Proof of Theorem 2.1. Given \( w \in V \) in \( ||w - w^*||_V \ll 1 \), we have
\[
|E(w) - E(w^*)| \leq |E(w) - E(Qw)| + |E(Qw) - E(w^*)| \\
\leq C(||w - Qw||^2_V + ||\delta E(Qw)||_{V^*}^{1/2})
\]
by Lemma 2.3, Lemma 2.7, and \( Qw \in \mathcal{S} \), where \( 0 < \theta \leq \frac{1}{2} \). Then Lemmas 2.5 and 2.6 imply
\[
|E(w) - E(w^*)| \leq C(||\delta E(w)||^2_{V^*} + ||\delta E(w)||_{V^*}^{1/2})
\]
and hence the desired property (2.20).

3. Proof of Theorem 1.5

Note that assuming \( 0 < \lambda \leq 8\pi \) in (2.3)-(2.4), we have readily obtained that \( T = +\infty \) and the orbit \( \mathcal{O} = \{w(\cdot, t)\} \) is pre-compact in \( C(\Omega) \) by Theorem 1.1. To apply Theorem 2.1 we use the parabolic regularity in the following form.

Lemma 3.1. Given \( w^* \in V \) with \( \delta E(w^*) = 0 \), we obtain
\[
\sup_{t_0 \leq t < t_0 + T} ||w(\cdot, t) - w^*||_V \leq C(||w(\cdot, t_0) - w^*||_V + \sup_{t_0 \leq t < t_0 + T} ||w(\cdot, t) - w^*||_2), \tag{3.1}
\]
for any \( t_0 \geq 0 \) and \( T > 0 \).

Proof. Since
\[
w_t = e^{-w} \Delta w + 1 - \frac{\lambda}{|\Omega|} e^{-w}, \quad 0 = e^{-w^*} \Delta w^* + 1 - \frac{\lambda}{|\Omega|} e^{-w^*}
\]
the function \( z = w - w^* \) solves
\[
\begin{align*}
z_t &= e^{-w} \Delta z + \left(e^{-w} - e^{-w^*}\right) \Delta w^* - \frac{\lambda}{|\Omega|} \left(e^{-w} - e^{-w^*}\right) \\
&= \left(e^{-w} \Delta - 1\right) z + bz
\end{align*}
\]
with \( b = b(x, t) \) uniformly bounded. Here, \( e^{-w} \Delta \) generates an evolution operator, denoted by \( \{U(t, s)\} \), satisfying
\[
||U(t, s)z_0||_V \leq C||z_0||_V, \quad ||U(t, s)z_0||_V \leq C(t - s)^{-1/2}||z_0||_2, \quad 0 \leq s < t < \infty. \tag{3.2}
\]
Hence \( z(t) = U(t, z_0) \) is the solution to
\[
z_t = e^w \Delta z \quad \text{in} \ \Omega \times (s, +\infty), \quad z|_{t=s} = z_0.
\]
If \( \tilde{U}(t, s) \) denotes the evolution operator associated with \( e^{-w} \Delta - 1 \), therefore, it holds that
\[
||\tilde{U}(t, s)||_V \leq C e^{-\langle t-s\rangle}, \quad ||\tilde{U}(t, s)||_{X \rightarrow V} \leq C(t - s)^{-1/2}e^{-\langle t-s\rangle}, \quad 0 \leq s < t < \infty,
\]
and furthermore,
\[
z(t) = \tilde{U}(t, t_0)z(t_0) + \int_{t_0}^{t} \tilde{U}(t, r)(b)(r) dr, \quad t \geq t_0.
\]
Thus we obtain
\[
||z(t)||_V \leq C \left(||z(t_0)||_V + \int_{t_0}^{t} (t - r)^{-1/2}e^{-\langle t-r\rangle} dr \sup_{t_0 \leq r < t} \||z(r)||_2\right)
\]
which finally entails
\[
\sup_{t_0 \leq t < t_0 + T} ||z(t)||_V \leq C \left(||z(t_0)||_V + \int_{0}^{\infty} s^{-1/2}e^{-s} ds \sup_{t_0 \leq t < t_0 + T} ||z(t)||_2\right)
\]
\[
\leq C(||z(t_0)||_V + \sup_{t_0 \leq t < t_0 + T} ||z(t)||_2).
\]
\[\square\]
Remark 3.2. The second inequality of (3.2) implies
$$\|w(\cdot, t+1) - w^*\|_V \leq C \sup_{t \leq s < t+1} \|w(\cdot, s) - w^*\|_2$$
for any $t \geq 0$ by
$$z(t+1) = \Bar{U}(t+1, t) z(t) + \int_t^{t+1} \Bar{U}(t+1, r)(b z)(r) \, dr.$$ 

Now we are ready to give the proof of the main result in the current section.

Proof of Theorem 1.5. We prescribe the constant $C$ in (3.1) as $C = C_1 \geq 1$ and thus:
$$\sup_{t_0 \leq t < t_0 + T} \|w(\cdot, t) - w^*\|_V \leq C_1 (\|w(\cdot, t_0) - w^*\|_V + \sup_{t_0 \leq t < t_0 + T} \|w(\cdot, t) - w^*\|_2). \quad (3.3)$$
Let the $\omega$-limit set of (2.3) be
$$\omega(w_0) = \{w^* \in V \mid \text{there exists } t_k \to \infty \text{ such that } w(t_k) \to w^* \text{ in } C^\infty \text{ topology}\}.$$ 
By Theorem 1.2 this $\omega(w_0)$ is non-empty, compact, connected, and satisfies
$$\omega(w_0) \subset \{w^* \in V \mid \delta \mathcal{E}(w^*) = 0\}.$$ 
Hence we have $w^* \in V$ with $\delta \mathcal{E}(w^*) = 0$ and $t_k \to \infty$ such that
$$w(\cdot, t_k) \to w^* \text{ in } C^\infty \text{ topology} \quad (3.4)$$
and in particular,
$$\|w(\cdot, t_k) - w^*\|_V \leq \frac{\varepsilon_0}{4C_1}, \quad \text{for } k \gg 1, \quad (3.5)$$
where $\varepsilon_0 > 0$ and $C_1 \geq 1$ are constants prescribed in Theorem 2.2 and (3.3), respectively. We have
$$\frac{d}{dt} \mathcal{E}(w) = -\langle w_t, \delta \mathcal{E}(w) \rangle_{V, V^*} = -(w_t, e^w w_t) \leq 0,$$
by (2.3), and hence the existence of
$$\lim_{t \to \infty} \mathcal{E}(w(\cdot, t)) = \mathcal{E}_\infty = \mathcal{E}(w^*), \quad (3.6)$$
where the second equality follows from $w^* \in \omega(w_0)$. In particular,
$$\mathcal{H}(t) = (\mathcal{E}(w(\cdot, t)) - \mathcal{E}(w^*))^\theta \geq 0$$
is well-defined, and it holds that
$$\lim_{t \to \infty} \mathcal{H}(t) = 0. \quad (3.7)$$
Since
$$C_2^{-1} \leq e^w \leq C_2 \quad \text{in } \Omega \times (0, \infty)$$
is valid with $C_2 \geq 1$, we obtain
$$-\frac{d\mathcal{H}}{dt} = -\theta (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \langle w_t, \delta \mathcal{E}(w) \rangle_{V, V^*}$$
$$= \theta (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} (e^w, w_t^2)$$
$$\geq \theta C_2^{-1} (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|w_t\|^2_2$$
$$\geq \theta C_2^{-3/2} (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|w_t\|_2 \left(\int_\Omega e^w w_t^2 \, dx\right)^{1/2}$$
$$= \theta C_2^{-3/2} (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|w_t\|_2 \|\delta \mathcal{E}(w)\|_2$$
again by (2.3). Therefore, there is $C_3 > 0$ such that
$$-\frac{d\mathcal{H}}{dt} \geq \frac{1}{C_3} (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|w_t\|_2 \|\delta \mathcal{E}(w)\|_{V^*}. \quad (3.8)$$
To apply Theorem 2.1, assume the existence of \( t_0 > t_k \) such that
\[
\|w(\cdot,t) - w^*\|_V < \varepsilon_0, \quad t_k \leq t \leq t_0.
\] (3.9)
Then inequality (3.8) implies
\[
\|w_t\|_2 \leq -C_4 \frac{d\mathcal{H}}{dt}, \quad t_k \leq t \leq t_0,
\] (3.10)
where \( C_4 > 0 \) is a constant. It follows that
\[
\|w(\cdot,t) - w^*(\cdot,t_k)\|_2 \leq C_4 \mathcal{H}(t_k), \quad t_k \leq t \leq t_0,
\] and thus we obtain
\[
\|w(\cdot,t) - w^*(\cdot,t_k)\|_V \leq \frac{\varepsilon_0}{4} + C_1 C_4 \mathcal{H}(t_k) \quad t_k \leq t \leq t_0
\] (3.11)
by (3.3) and (3.5) with \( C_1 \geq 1 \).
Equality (3.7) assures \( k \gg 1 \) satisfying
\[
\mathcal{H}(t_k) < \frac{\varepsilon_0}{4 C_1 C_4}.
\] (3.12)
Fix such \( k \). By the above argument, if there is \( t_0 > t_k \) provided with (3.9), it holds that (3.11) and hence
\[
\|w(\cdot,t) - w^*(\cdot,t_k)\|_V < \varepsilon_0/2, \quad t_k \leq t \leq t_0
\] (3.13)
by (3.12). Since we have readily assumed (3.5) with \( C_1 \geq 1 \), inequality (3.13) implies
\[
\|w(\cdot,t) - w^*\|_V < 3\varepsilon_0/4, \quad t_k \leq t \leq t_0.
\] (3.14)
We have thus observed that (3.9) implies (3.14). Regarding (3.5) with \( C_1 \geq 1 \) again, we conclude
\[
\|w(\cdot,t) - w^*\|_V < \varepsilon_0, \quad t \geq t_k.
\] (3.15)
Consequently, by (3.15) inequality (3.10) is improved as
\[
\|w_t\|_2 \leq -C_4 \frac{d\mathcal{H}}{dt}, \quad t \geq t_k,
\] (3.16)
which implies
\[
\int_0^{\infty} \|w_t\|_2 dt < \infty.
\]
Then we obtain
\[
\lim_{t \to \infty} \|w(\cdot,t) - w^*\|_2 = 0
\] by (3.2), and hence \( \omega(w_0) = \{w^*\} \) from the uniqueness of the limit. It thus follow that
\[
w(\cdot,t) \to w^*, \quad t \to \infty \quad \text{in } C^\infty \text{ topology.}
\] (3.17)
Turning to the rate of convergence, we use
\[
|\mathcal{E}(w(\cdot,t)) - \mathcal{E}(w^*)|^{1-\theta} \leq C_5 \|\delta \mathcal{E}(w(\cdot,t))\|_{V^*}, \quad t \geq t_k
\] derived from Theorem 2.1 and (3.15). Using (2.3) again we derive
\[
-\frac{d\mathcal{H}}{dt} = \theta (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \langle w_t, -\delta \mathcal{E}(w) \rangle
\]
\[
= \theta (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|\delta \mathcal{E}(w)\|_2
\geq \frac{1}{C_6} (\mathcal{E}(w) - \mathcal{E}(w^*))^{\theta-1} \|\delta \mathcal{E}(w)\|_{V^*}
\geq \frac{1}{C_6 C_5^2} (\mathcal{E}(w(t)) - \mathcal{E}(w^*))^{1-\theta}
\geq \gamma \mathcal{H}^{-1}, \quad \gamma = \frac{1}{C_6 C_5^2}, \quad t \geq t_k.
\]
We thus obtain
\[ \mathcal{H}(t) \leq C\Phi(t), \quad t \geq t_k, \]
where
\[ \Phi(t) = \left\{ \begin{array}{ll} t^{-\frac{\theta}{1-2\theta}}e^{-\gamma t}, & 0 < \theta < \frac{1}{2} \\ e^{-\gamma t}, & \theta = \frac{1}{2}. \end{array} \right. \]

Inequality (3.16) now implies
\[ ||w(\cdot, t) - w(\cdot, s)||_2 \leq C\Phi(s), \quad t \geq s \geq t_k, \]
and sending \( t \to \infty \), we get
\[ ||w^{\ast}(\cdot, t) - w^{\ast}(\cdot, s)||_2 \leq C\Phi(t), \quad s \geq t_k, \]
or
\[ ||w(\cdot, t) - w^{\ast}(\cdot, t)||_2 \leq C\Phi(t), \quad t \geq t_k, \quad (3.18) \]
Then, Remark 3.2 entails
\[ ||w(\cdot, t) - w^{\ast}(\cdot, t)||_V \leq C\Phi(t), \quad t \to \infty. \]

Given multi-index \( \alpha \), we can derive an equation of
\[ z^{\alpha} = D^{\alpha}(w - w^{\ast}), \]
where the second estimate of (3.2) is applicable. Then an iteration ensures the rate of convergence \( \Phi(t) \) in (3.17).

4. Non-degenerate Steady-States

Recall that \( u^{\ast} = u^{\ast}(x) > 0 \) is called a steady-state to (1.1)-(1.3) when it solves (1.6). Then \( v^{\ast} = v^{\ast}(x) \) defined by (1.16) satisfies (1.14), which is the Euler-Lagrange equation for the functional \( J_\lambda = J_\lambda(v) \) of \( v \in V_0 \), defined by (1.21)-(1.22). We say that \( u^{\ast} \) is non-degenerate if this \( v^{\ast} \in V_0 \) is a non-degenerate critical point of \( J_\lambda \) on \( V_0 \). Here, \( u^{\ast} \) is reproduced by \( v^{\ast} \) through (1.15).

More precisely, first, we notice
\[
\delta J_\lambda(v)[\phi] = \frac{d}{ds} J_\lambda(v + s\phi) \bigg|_{s=0} = (\nabla \phi, \nabla \phi) - \frac{\lambda}{\int_\Omega e^v dx} \int_\Omega e^v \phi^2 dx, \quad \phi \in V_0,
\]
to identify
\[
\delta J_\lambda(v) = -\Delta v - \lambda \left( \frac{e^v}{\int_\Omega e^v dx} - \frac{1}{|\Omega|} \right) \in V_0^*, \quad v \in V_0.
\]
Hence the above \( v^{\ast} \), realized as a solution to (1.14), belongs to \( V_0 \) and is a critical point of \( J_\lambda \) on \( V_0 \).

Second, the quadratic form \( Q : V_0 \times V_0 \to \mathbb{R} \) defined by
\[
Q(\phi, \phi) = \frac{d^2}{ds^2} J_\lambda(v^{\ast} + s\phi) \bigg|_{s=0} = (\nabla \phi, \nabla \phi) - \frac{\lambda}{\int_\Omega e^{v^{\ast}} dx} \int_\Omega e^{v^{\ast}} \phi^2 dx + \lambda \left( \frac{\int_\Omega e^{v^{\ast}} \phi dx}{\int_\Omega e^{v^{\ast}} dx} \right)^2
\]
is associated with the linearized operator \( \delta^2 J_\lambda(v^{\ast}) : V_0 \to V_0^* \) through
\[
Q(\phi, \phi) = \langle \phi, \delta^2 J_\lambda(v^{\ast}) \phi \rangle_{V^*, V}.
\]
This \( \delta^2 J_\lambda(v^{\ast}) \) is realized as a self-adjoint operator in \( X_0 = L^2(\Omega) \cap V_0 \), denoted by \( B \), with the domain \( D(B) = H^2(\Omega) \cap V_0 \), satisfying
\[
(B\phi, \psi) = Q(\phi, \psi), \quad \phi \in D(B) \subset V_0, \quad \psi \in V_0.
\]
Hence it holds that

\[
\mathcal{B}\phi = -\Delta \phi - \frac{\lambda e^{v^*}}{\int_\Omega e^{v^*} \phi \, dx} \phi + \frac{\lambda}{\int_\Omega e^{v^*} \phi \, dx} e^{v^*} = -\Delta \phi - u^* \phi + \frac{1}{\lambda} (\phi, u^*) u^*, \quad \phi \in D(\mathcal{B}) = H^2(\Omega) \cap V_0
\]

by (1.15).

Now we show the following lemma stated in Section 1.

**Lemma 4.1.** The stationary solution \( u^* = u^*(x) > 0 \) to (1.1)-(1.3) is non-degenerate if and only if the property (1.23) holds.

**Proof.** By the definition, the non-degeneracy of \( u^* \) means the non-degeneracy of \( \mathcal{B} \) in \( X_0 = L^2(\Omega) \cap V_0 \), which is equivalent to

\[
\phi \in D(\mathcal{B}), \quad \mathcal{B}\phi = 0 \implies \phi = 0.
\]

Assume, first, \( \phi \in D(\mathcal{B}) \setminus \{0\} \) with \( \mathcal{B}\phi = 0 \), and let

\[
\psi = \phi - \frac{1}{\lambda} \int_\Omega u^* \phi \, dx \in H^2(\Omega).
\]

Then we have

\[
-\Delta \psi = u^* \psi, \quad \int_\Omega \psi u^* \, dx = 0.
\]

It also holds that \( \psi \neq 0 \) by \( \phi \in V_0 \setminus \{0\} \). Hence if \( u^* \) is degenerate there is \( \psi \in V \setminus \{0\} \) satisfying (1.23).

If problem (1.23) admits \( \psi \in H^2(\Omega) \setminus \{0\} \), second, we take

\[
\phi = \psi - \frac{1}{|\Omega|} \int_\Omega \psi \, dx \in H^2(\Omega) \cap V_0 = D(\mathcal{B}).
\]

It holds that

\[
(\phi, u^*) = \frac{1}{|\Omega|} \int_\Omega \psi \, dx,
\]

and hence

\[
\mathcal{B}\phi = -\Delta \phi - u^* \phi + \frac{1}{\lambda} (\phi, u^*) u^* = -\Delta \psi - u^* \psi + \frac{1}{\lambda} (\phi, u^*) u^* = -\Delta \psi - u^* \psi = 0
\]

by (1.23). If \( \phi = 0 \), then it holds that \( \psi = \text{constant} \) and by virtue of (1.23), there arises \( \psi = 0 \), a contradiction. Thus \( \mathcal{B} \) has the eigenvalue 0, and hence this operator is degenerate. \( \square \)

**Lemma 4.2.** Let \( u^* = u^*(x) > 0 \) be a steady-state to (1.1)-(1.3), and define \( w^* \in V = H^1(\Omega) \) by (1.16), i.e.,

\[
w^* = \log u^*.
\]

Set

\[
\mathcal{M} = -\Delta - u^*: V \to V^*.
\]

Then the following statements are equivalent.

(i) There exists \( C > 0 \) such that

\[
\phi \in V, \quad \int_\Omega u^* \phi \, dx = 0 \implies \|\phi\|_V \leq C \|\mathcal{M}\phi\|_{V^*}.
\]

(ii) There exists \( \varepsilon_0 > 0 \) and \( C > 0 \) such that

\[
w \in V, \quad \int_\Omega e^w \, dx = \lambda, \quad \|w - w^*\|_V < \varepsilon_0 \implies \|w - w^*\|_V \leq C \|\mathcal{M}(w - w^*)\|_{V^*}.
\]
Proof. (i) $\implies$ (ii): Assume (i), take
\[
w \in V; \int_{\Omega} e^w \, dx = \lambda, \|w - w^*\|_V < \varepsilon_0,\] (4.7)
and let
\[
\phi^* = \frac{u^*}{\|u^*\|_2}, \quad z = w - w^*, \quad Pz = z - (\phi^*, z)\phi^*.
\]
It holds that \((Pz, u^*) = 0\), and hence,
\[
\|Pz\|_V \leq C \|M(z)\|_V + \|\phi^*\|_V \|u^*\|_2^{-1}.
\] (4.9)

Here we have
\[
\int_{\Omega} e^w \, dx = \int_{\Omega} e^{w^*} \, dx = \lambda
\]
by (4.3), \(\|u^*\|_1 = \lambda\), and (4.7). Hence it holds that
\[
0 = \int_{0}^{1} \int_{\Omega} e^{sw+(1-s)w^*} (w - w^*) \, dx \, ds
\] (4.10)
by
\[
e^w - e^{w^*} = \int_{0}^{1} \frac{d}{ds} e^{sw+(1-s)w^*} \, ds = \int_{0}^{1} e^{sw+(1-s)w^*} (w - w^*) \, ds.
\]
Then (4.10) implies
\[
(u^*, z) = \int_{\Omega} e^{w^*} (w - w^*) \, dx
\]
\[
= \int_{0}^{1} \int_{\Omega} (e^{w^*} - e^{sw+(1-s)w^*}) (w - w^*) \, dx \, ds.
\]
\[
= \int_{0}^{1} \int_{\Omega} (e^{w^*} - e^{sw+(1-s)w^*}) z \, dx \, ds.
\] (4.11)

In (4.11) we have
\[
e^{w^*} - e^{sw+(1-s)w^*} = \int_{0}^{1} \frac{d}{dr} e^{r w^*+(1-r)(sw+(1-s)w^*)} \, dr
\]
\[
= \int_{0}^{1} e^{r w^*+(1-r)(sw+(1-s)w^*)} (-s)z \, dr,
\]
and therefore,
\[
|(u^*, z)| \leq \int_{0}^{1} \int_{\Omega} e^{r w^*+(1-r)(sw+(1-s)w^*)} s z^2 \, dx \, dr \, ds.
\]
By the Trudinger-Moser-Fontana inequality, any \(K > 0\) admits \(C_1(K)\) such that
\[
\|w\|_V \leq K \implies \|e^{rw^*+(1-r)(sw+(1-s)w^*)}\|_2 \leq C_1(K), \quad 0 \leq r, s \leq 1,
\]
and hence we find \(C_2(K) > 0\) such that
\[
|(u^*, z)| \leq C_1(K) \|z\|_2^2 \leq C_2(K) \|z\|_V^2.
\] (4.12)

Combining (4.9) and (4.12), we reach to
\[
\|z\|_V \leq C_3(K) (\|Mz\|_V + \|z\|_V^2)
\]
for \( \|z\|_V \leq K \). Then (4.6) follows for \( \varepsilon_0 = \frac{1}{2C_0(K)} \), because then we have
\[
\|z\|_V = \|w - w^*\|_V < \varepsilon_0 \quad \Rightarrow \quad C_3(K)\|z\|^2_V \leq \frac{1}{2}\|z\|_V
\]
and hence (4.6) with \( C = 2C_3(K) \).

\((ii) \implies (i)\): Given
\[
\phi \in V, \int_{\Omega} u^* \phi \, dx = 0, \quad (4.13)
\]
we show the conclusion of (4.5):
\[
\|\phi\|_V \leq C\|M\phi\|_{V^*}. \quad (4.14)
\]
For this purpose, it suffices to assume \( \phi \neq 0 \).

Define
\[
\Phi(s, z) = \begin{cases} \frac{1}{s} \int_{\Omega} e^{s\phi + s^2z + w^*} - e^{w^*} \, dx, & s \neq 0 \\ 0, & s = 0 \end{cases}, \quad (s, z) \in \mathbb{R} \times V.
\]
Note that
\[
e^{s\phi + s^2z + w^*} - e^{w^*} = e^{w^*}(e^{s\phi + s^2z} - 1) = e^{w^*}\{(s\phi + s^2z) + \frac{1}{2}(s\phi + s^2z)^2 + o(s^2)\}
\]
\[
= \{s\phi + s^2(z + \frac{1}{2}\phi^2)\}e^{w^*} + o(s^2), \quad s \to 0,
\]
to deduce
\[
\Phi(s, z) = \int_{\Omega} \{\phi + s(z + \frac{1}{2}\phi^2)\}e^{w^*} \, dx + o(s), \quad s \to 0.
\]
First, this \( \Phi = \Phi(s, z) \) is continuous in \( (s, z) \in \mathbb{R} \times V \) because
\[
\lim_{s \to 0} \Phi(s, z) = 0
\]
follows from (4.3) and (4.13):
\[
\int_{\Omega} e^{w^*} \phi \, dx = 0. \quad (4.16)
\]
Second, the following limit arises
\[
\lim_{s \to 0} \Phi_s(s, z) = \int_{\Omega} (z + \frac{1}{2}\phi^2)e^{w^*} \, dx
\]
and hence \( \Phi \) is \( C^1 \) in \( \mathbb{R} \times V \). It holds, in particular, that
\[
\Phi_s(0, 0) = \frac{1}{2} \int_{\Omega} e^{w^*}\phi^2 \, dx \neq 0
\]
by (4.15), and therefore, the implicit function theorem guarantees the existence of a \( C^1 \) function \( z = z(s) \) of \( s \) such that
\[
z(0) = 0, \quad \Phi(s, z(s)) = 0, \quad |s| \ll 1.
\]
Accordingly,
\[
w(s) = s\phi + s^2z(s) + w^*
\]
satisfies
\[
w(0) = w^*, \quad \dot{w}(0) = \phi, \quad \int_{\Omega} e^{w(s)} \, dx = \int_{\Omega} e^{w^*} \, dx = \lambda, \quad |s| \ll 1
\]
and hence
\[
\|w(s) - w^*\|_V \leq C\|M(w(s) - w^*)\|_{V^*}, \quad |s| \ll 1
\]
by (4.6). Then, (4.14) follows from (4.17)-(4.18). \( \square \)
5. PROOF OF THEOREM 1.6

Given a non-degenerate steady-state \( u^* = u^*(x) > 0 \) of \((1.1)-(1.3)\), define \( w^* \in V \) by \((1.3)\). Then it holds that

\[
\delta E(w^*) = 0, \quad \int_\Omega e^{w^*} \, dx = \lambda. \tag{5.1}
\]

By Lemma 4.4 the operator \( M : V \to V^* \) defined by \((4.3)\) is provided with the property \((4.5)\). Then we obtain \( \varepsilon_0 > 0 \) satisfying \((4.6)\) by Lemma 4.2.

Having these properties, we see that Theorem 1.6 is reduced to the following lemma by the proof of Theorem 1.5.

**Lemma 5.1.** Let \( w^* \in V \) satisfy \((5.7)\), and assume the property \((4.7)\). Then, there arises that \( \theta = \frac{1}{2} \) in the conclusion of \((2.9)\) for \( w \) satisfying

\[
w \in V, \quad \|w - w^*\|_V < \varepsilon_1, \quad \int_\Omega e^w \, dx = \lambda \tag{5.2}
\]

for \( \varepsilon_1 > 0 \) sufficiently small.

For the proof of this lemma, we first verify several facts derived from the Trudinger-Moser-Fontana inequality.

**Lemma 5.2.** Any \( K > 0 \) admits \( C(K) > 0 \) such that

\[
w_1, w_2 \in V, \quad \|w_1\|_V, \|w_2\|_V \leq K \quad \Rightarrow \quad \|\delta E(w_1) - \delta E(w_2)\|_V \leq C(K)\|w_1 - w_2\|_V. \tag{5.3}
\]

**Proof.** Given \( w \in V = H^1(\Omega) \), let

\[
\overline{w} = \frac{1}{|\Omega|} \int_\Omega w \, dx, \quad [w] = w - \overline{w} \in V_0.
\]

Take \( z \in V \) then we have

\[
\langle z, \delta E(w_1) - \delta E(w_2) \rangle_{V', V} = \int_\Omega \nabla z \cdot \nabla (w_1 - w_2) - z(e^{w_1} - e^{w_2}) \, dx \tag{5.4}
\]

by \((2.6)\), where

\[
e^{w_1} - e^{w_2} = \int_0^1 \frac{d}{ds} e^{sw_1+(1-s)w_2} \, ds = \int_0^1 e^{sw_1+(1-s)w_2} \, ds \cdot (w_1 - w_2)
\]

\[
= \int_0^1 e^{sw_1+(1-s)\overline{w}} \cdot e^{[sw_1+(1-s)w_2]} \, ds \cdot (w_1 - w_2).
\]

Hence it follows that

\[
|e^{w_1} - e^{w_2}| \leq e^{\overline{w}_1+w_1} \cdot \int_0^t e^{[sw_1+(1-2)w_2]} \, ds \cdot |w_1 - w_2|. \tag{5.5}
\]

Letting \( w \in V \setminus \mathbb{R} \), on the other hand, we use

\[
[w] \leq \frac{4\pi|w|^2}{||\nabla[w]||_2^2} + \frac{1}{\pi}||\nabla[w]||_2^2
\]

to deduce

\[
\int_\Omega e^{[w]} \, dx \leq C \cdot \exp \left( \frac{1}{\pi}||\nabla[w]||_2^2 \right), \quad w \in V, \tag{5.6}
\]

by \((1.24)\).

Inequalities \((5.5)-(5.6)\) imply

\[
\left| \int_\Omega z \left( e^{w_1} - e^{w_2} \right) \, dx \right| \leq \|z\|_4 \exp \left( |\overline{w}_1| + |\overline{w}_2| \right) \left| \int_0^1 e^{[sw_1+(1-s)w_2]} \, ds \right|_4 |w_1 - w_2|_2
\]

\[
\leq C(K)|z|_V |w_1 - w_2|_V, \quad \|w_1\|_V, \|w_2\|_V \leq K,
\]

and hence \((5.3)\) is valid due to \((5.4)\). \qed
Lemma 5.3. Given $w^* \in V$ with $\delta E(w^*) = 0$, any $K > 0$ admits $C = C(K) > 0$ such that
\[ w \in V, \ |w|_V \leq K \implies |E(w) - E(w^*)| \leq C|w - w^*|^2_V. \]

Proof. Since
\[
E(w) - E(w^*) = \int_0^1 \frac{d}{ds} E(sw + (1-s)w^*) \, ds
\]
\[
= \int_0^1 \langle w - w^*, \delta E(sw + (1-s)w^*) \rangle_{V,V^*} \, ds
\]
\[
= \int_0^1 \langle w - w^*, \delta E(sw + (1-s)w^*) - \delta E(w^*) \rangle_{V,V^*} \, ds
\]
we obtain
\[
|E(w) - E(w^*)| \leq \|w - w^*\|_V \int_0^1 \|\delta E(sw + (1-s)w^*) - \delta E(w^*)\|_{V,V^*} \, ds
\]
\[
\leq C(K)\|w - w^*\|_V \int_0^1 s ds = \frac{C(K)}{2} \|w - w^*\|^2_V
\]
by the previous lemma. \(\square\)

We are ready to prove the key result in the current section.

Proof of Lemma [5.7] We take $w$ as in [5.2]. Recall $\delta E(w^*) = 0$, and deduce from [2.6] that
\[
-\delta E(w) = -\delta E(w) + \delta E(w^*)
\]
\[
= \Delta (w - w^*) + (e^w - e^{w^*})
\]
\[
= \Delta (w - w^*) + \int_0^1 \frac{d}{ds} e^{sw+(1-s)w^*} \, ds
\]
\[
= \Delta (w - w^*) + \int_0^1 e^{sw+(1-s)w^*} (w - w^*) \, ds
\]
\[
= \Delta (w - w^*) + e^{w^*} (w - w^*) + \int_0^1 (e^{sw+(1-s)w^*} - e^{w^*}) (w - w^*) \, d\zeta
\]
\[
= -\mathcal{M}(w - w^*) + z,
\]
where
\[
z = \int_0^1 (e^{sw+(1-s)w^*} - e^{w^*}) (w - w^*) \, ds.
\]

Here we use
\[
e^{sw+(1-s)w^*} - e^{w^*} = \int_0^1 \frac{d}{d\zeta} e^{\zeta(sw+(1-s)w^*)+(1-\zeta)w^*} \, d\zeta
\]
\[
= \int_0^1 e^{\zeta(sw+(1-s)w^*)+(1-\zeta)w^*} s(w - w^*) \, d\zeta,
\]
to derive
\[
|z| \leq |w - w^*|^2 e^{|w|+|w^*|}.
\]

Hence it holds that
\[
\|z\|_2 \leq \|\exp(|w|)\|_4 \cdot \|\exp(|w^*|)\|_4 \cdot |w - w^*|^2,
\]
and therefore, the assumption [5.2] ensures
\[
\|z\|_{V^*} \leq C_1 \|z\|_2 \leq C_2 \|w - w^*\|^2 \leq C_3 \|w - w^*\|^2_V\tag{5.8}
\]
by the Trudinger-Moser-Fontana inequality.
Since $\mathcal{M}: V \to V^*$ is provided with (4.6), it follows that
\[ \|w - w^*\|_V \leq C_1 \|\mathcal{M}(w - w^*)\|_{V^*}. \]
from (5.2) if $\varepsilon_1 \leq \varepsilon_0$. By (5.7)-(5.8), therefore, we obtain
\[ \|w - w^*\|_V \leq C_2(\|\delta \mathcal{E}(w)\|_{V^*} + \|w - w^*\|_{V^*}^2). \]
Choosing $0 < \varepsilon_1 \ll 1$ in (5.2), then we reach to
\[ \|w - w^*\|_{V^*} \leq C_3\|\delta \mathcal{E}(w)\|_{V^*}. \] (5.9)

Lemma 5.3 now guarantees
\[ |\mathcal{E}(w) - \mathcal{E}(w^*)| \leq C_4\|w - w^*\|_V^2 \leq C_5\|\delta \mathcal{E}(w)\|_{V^*}^2, \]
which entails the conclusion of (2.9) for $\theta = 1/2$ under the presence of (5.2).

\[ \square \]

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References

[1] J. BARTZ, M. STRUWE & R. YE, A new approach to the Ricci flow on $S^2$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV 21 (1994) 475-482.
[2] S. CHANILLO & M. KIESSLING, Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry, Comm. Math. Phys. 160 (1994) 217-238.
[3] K.-S. CHENG & C.-S. LIN, On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in $\mathbb{R}^2$, Math. Ann. 308 (1997) 119-139.
[4] R. CHILL, On the Lojasiewicz-Simon gradient inequality, J. Funct. Analysis, 201 (2003), 572-601.
[5] R. CHILL, On the Lojasiewicz-Simon gradient inequality on Hilbert spaces, Proceedings of 5th European-Magrekbean Workshop on Semigroup Theory, Evolution Equations and Applications, M.A. Jendoubi ed. (2006), 25-36.
[6] L. FONTANA, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helvetici 68 (1993) 415-454.
[7] R. HAMILTON, The Ricci flow on surfaces, Contem. Math., 71 (1988) 237-262.
[8] A. HARAUX & M.A. JENDOUBLI, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asym. Analysis 26 (2001), 21–36.
[9] A. HARAUX & M.A. JENDOUBLI, The Lojasiewicz gradient inequality in the infinite-dimensional Hilbert space framework, J. Funct. Analysis, 260 (2011), 2826–2842.
[10] D. HENRY, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. 840, Springer-Verlag, Berlin, 1981.
[11] N.I. KAVALLARIS & T. SUZUKI, An analytic approach to the normalized Ricci flow-like equation, Nonl. Analysis, 72 (2010), 2300–2317.
[12] N.I. KAVALLARIS & T. SUZUKI, An analytic approach to the normalized Ricci flow-like equation: revisited, Appl. Math. Letters, 44 (2015), 30-33.
[13] N.I. KAVALLARIS & T. SUZUKI, Non-Local Partial Differential Equations for Engineering and Biology: Mathematical Modeling and Analysis, Mathematics for Industry Vol. 31 Springer Nature 2018.
[14] O. LADYZHENSKAJA, V.A. SOLONNIKOV & N.N. URAIČEVA, Linear and Quasi-Linear Equations of Parabolic Type, Amer. Math. Soc. Providence, R.I. 1968.
[15] C.-S. LIN, Uniqueness of solutions to the mean field equations for the spherical Onsager vortex, Arch. Rational Mech. Anal. 153 (2000) 153–176.
[16] C.-S. LIN & M. LUCIA, Uniqueness of solutions to a mean field equation on torus, J. Differential Equations 229 (2006) 172–185.
[17] S. LOJASIEWICZ, Une propriété topologique des sous-ensembles analytiques réels, Colloques internationaux du C.N.R.S #117, Les équations aux dérivées partielles, 1963.
[18] L. SIMON, Asymptotics for a class of non-linear evolution equations, with applications to geometric problems, Ann. of Math. 118 (1983), 525-571.
[19] T. SUZUKI, Mean Field Theories and Dual Variation, 2nd edition, Atlantis Press, 2015.
[20] T. SUZUKI, Semilinear Elliptic Equations: Classical and Modern Theories, De Gruyter, Berlin, 2020.
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