Green Currents for Meromorphic Maps of Compact Kähler Manifolds

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Received: 25 July 2011
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Abstract We study the invariance properties of positive cones under the pull back by meromorphic maps of compact Kähler manifolds. We also provide necessary and sufficient conditions for the existence of Green currents in codimension one.

Keywords Meromorphic map · Green current · Non-nef locus

Mathematics Subject Classification 37Fxx · 32Q15 · 32C30

1 Introduction

Let \( X \) be a compact Kähler manifold and \( f : X \rightarrow X \) be a dominant meromorphic map. It is known that we may define a linear pullback map \( f^* : H^{1,1}(X, \mathbb{R}) \rightarrow H^{1,1}(X, \mathbb{R}) \). In general, however, this linear action is not compatible with the dynamics of the map \( f \). We say that \( f \) is 1-regular whenever \((f^n)^* = (f^*)^n \) for \( n = 1, 2, \ldots \) on \( H^{1,1}(X, \mathbb{R}) \). In the sequel, we will assume that \( f \) is 1-regular. By a standard Perron–Frobenius type argument, there exists \( \alpha \in H^{1,1}_{psef}(X, \mathbb{R}) \) such that

\[
f^* \alpha = \lambda_1(f) \alpha,
\]

where \( \lambda_1(f) \) is defined to be the spectral radius of \( f^* \). Let

\[
H := \{ \alpha \in H^{1,1}(X, \mathbb{R}) : f^* \alpha = \lambda_1(f) \alpha \}.
\]
We also consider

\[
H_N := \left\{ \alpha \in H^{1,1}(X, \mathbb{R}) : \alpha = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{n^{m-1} \lambda_1^n} (f^n)^* \beta \right\},
\]

for some \( \beta \in H^{1,1}_{\text{nef}}(X, \mathbb{R}) \), where \( m \) denotes the size of the largest Jordan block associated with \( \lambda_1(f) \). Then it follows that \( H_N \subset H \cap H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) and \( H_N \) has a non-empty interior in \( H \).

In general, a class \( \alpha \in H_N \) is not numerically effective (nef). Boucksom [3] has defined the minimal multiplicity \( \nu(\alpha, x) \) of a class \( \alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) at a point \( x \in X \). This is a local obstruction to the numerical effectiveness of \( \alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) at \( x \). The set

\[
E_{nn}(\alpha) := \{ x \in X : \nu(\alpha, x) > 0 \}
\]

is called the non-nef locus of \( \alpha \). A property of \( E_{nn}(\alpha) \) is that if \( C \subset X \) is an irreducible algebraic curve such that \( \alpha \cdot C < 0 \), then \( C \subset E_{nn}(\alpha) \). We let \( I_{f^k} \) denote the indeterminacy locus of the iterate \( f^k \).

**Theorem 1.1** Let \( f : X \to X \) be a dominant meromorphic map. If \( f \) is 1-regular and \( \lambda_1(f) > 1 \), then \( E_{nn}(\alpha) \subset \bigcup_{k=1}^{\infty} I_{f^k} \) for every \( \alpha \in H_N \).

As a corollary, we obtain that every curve \( C \) such that \( \alpha \cdot C < 0 \) is a subset of \( \bigcup_{k=1}^{\infty} I_{f^k} \). Moreover, the non-nef locus \( E_{nn}(\alpha) \) does not contain any hypersurface of \( X \).

Many authors have constructed positive closed invariant currents to represent the invariant classes. These constructions, however, assume that the class is nef or sometimes even Kähler. Here we consider some cases where the invariant class is merely pseudo-effective (psef).

Let us fix a smooth representative \( \theta \in \alpha \). We say that an upper semi-continuous function \( \phi \in L^1(X) \) is a \( \theta \)-psh function if \( \theta + dd^c \phi \geq 0 \) in the sense of currents. Following [8], we define

\[
\nu^{\min}_\alpha := \sup\{ \phi \leq 0 : \phi \text{ is a } \theta \text{-psh function} \}.
\]

Thus, \( \theta + dd^c \nu^{\min}_\alpha \in \alpha \) is a positive closed \((1, 1)\) current with minimal singularities.

**Theorem 1.2** Let \( f : X \to X \) be a 1-regular dominant meromorphic map and \( \alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) such that \( f^* \alpha = \lambda \alpha \) for some \( \lambda > 1 \). If

\[
\frac{1}{\lambda^n} \nu^{\min}_\alpha \circ f^n \to 0 \quad \text{in } L^1(X),
\]

\((*)\)
then for every smooth form \( \theta \in \alpha \) we have the existence of the limit

\[
T_\alpha := \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* \theta,
\]
which depends only on the class \( \alpha \). \( T_\alpha \) is a positive closed \((1, 1)\) current satisfying \( f^* T_\alpha = \lambda T_\alpha \). Furthermore,

1. \( T_\alpha \) is minimally singular among the invariant currents which belong to the class \( \alpha \).
2. \( T_\alpha \) is extreme within the cone of positive closed \((1, 1)\) currents whose cohomology class belongs to \( \mathbb{R}^+ \alpha \).

We have seen that such \( \alpha \) exists for \( \lambda = \lambda_1(f) \), but we can also allow other values of \( \lambda \) as well. We also prove that \((\star)\) is a necessary condition under natural dynamical assumptions (see Proposition 4.3).

The following result provides an algebraic criterion for the existence of Green currents when \( X \) is projective.

**Theorem 1.3** Let \( X \) be a projective manifold and \( f : X \to X \) be a dominant \( 1 \)-regular rational map. Assume that \( \lambda := \lambda_1(f) > 1 \) is a simple eigenvalue of \( f^* \), with \( f^* \alpha_f = \lambda \alpha_f \). If \( \alpha_f \cdot C \geq 0 \) for every algebraic irreducible curve \( C \subset E^\perp_f := f(I_f) \), then

\[
\frac{1}{\lambda^n} \alpha_{\text{min}} \circ f^n \to 0 \quad \text{in } L^1(X).
\]

In the last part of this work, we present some examples of birational maps in higher dimensions which fall into the framework of Theorem 1.2; nevertheless, the invariant class is not nef.

Let \( f := L \circ J \), where \( J : \mathbb{P}^d \to \mathbb{P}^d \),

\[
J[x_0 : x_1 : \cdots : x_d] = [x_0^{-1} : x_1^{-1} : \cdots : x_d^{-1}],
\]

and \( L \) is a linear map given by given by a \((d + 1) \times (d + 1)\) matrix of the form

\[
L = \begin{bmatrix}
  a_0 - 1 & a_1 & a_2 & \cdots & a_d \\
  a_0 & a_1 - 1 & a_2 & \cdots & a_d \\
  a_0 & a_1 & a_2 - 1 & \cdots & a_d \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_0 & a_1 & a_2 & \cdots & a_d - 1
\end{bmatrix},
\]

with \( a_j \in \mathbb{C} \) and \( \sum_{j=0}^d a_j = 2 \). The linear map \( L \) is involutive, that is, \( L = L^{-1} \) in \( \text{PGL}(d + 1, \mathbb{C}) \). Let \( \Sigma_i := \{[x_0 : \cdots : x_d] \in \mathbb{P}^d : x_i = 0\} \); then \( p_i := f(\Sigma_i) \in \mathbb{P}^d \) is the \( i \)-th column of the matrix \( L \). We define its orbit \( O_i \) as follows: \( O_i = \{p_i, f(p_i), f^2(p_i), \ldots, f^{N_i-1}(p_i)\} \) if \( f^j(p_i) \notin I_f \) for \( 0 \leq j \leq N_i - 2 \) and \( f^{N_i-1}(p_i) \in I_f \) for some \( N_i \in \mathbb{N} \), otherwise \( O_i = \{p_i, f(p_i), f^2(p_i), \ldots\} \). If \( O_i \) is
finite (the first case above), we say that the orbit of \( \Sigma_i \) is singular of length \( N_i \). It follows from [1] that there exists a complex manifold \( X \) together with a proper modification \( \pi : X \to \mathbb{P}^d \) such that the induced map \( f_X : X \to X \) is 1-regular. Moreover, if the length of the singular orbits are long enough (see Theorem 6.1), then \( \lambda_1(f) > 1 \) is the unique simple eigenvalue of \( f^*|_{H^{1,1}(X,\mathbb{R})} \) of modulus greater than 1. We define \( S := \{i \in \{0, 1, \ldots, d\} | O_i \) is singular} and denote its cardinality by \( |S| \). If \( S \) is non-empty, by conjugating \( f \) with an involution, without loss of generality we may assume that \( S = \{0, \ldots, k\} \).

**Theorem 1.4** Let \( f_X : X \to X \) be as above with \( \lambda := \lambda_1(f_X) > 1 \) and \( \alpha_f \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) such that \( f^*\alpha_f = \lambda \alpha_f \). Then \( \alpha_f \) is nef if and only if \( |S| \leq 1 \). Moreover, if \( 2 \leq |S| \leq d \) and all singular orbits of \( f \) have the same length, then

\[
E_{\text{nn}}(\alpha_f) = \begin{cases}
\{[x_0 : x_1 : \cdots : x_d] : x_i = 0 \text{ for } k + 1 \leq i \leq d\} & \text{if } k \leq d - 2 \\
\bigcup_{i=0}^{d-1} \{[x_0 : x_1 : \cdots : x_d] : x_i = x_d = 0\} & \text{if } k = d - 1
\end{cases}
\]

We also show that these maps fall into the framework of Theorem 1.2:

**Theorem 1.5** Let \( f_X : X \to X \) be as above with \( \lambda_1(f_X) > 1 \). If \( a_i \neq 0 \) for every \( i \in S \), then condition (\( \star \)) in Theorem 1.2 holds.

The outline of the paper is as follows. In Sect. 2, we provide the basic definitions and results that we will use in the sequel. In Sect. 3, we discuss invariance properties of closed convex cones in \( H^{1,1}(X, \mathbb{R}) \) and prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. We also discuss some cases for which (\( \star \)) holds in Sect. 4. In Sect. 5, we discuss rational maps and prove Theorem 1.3. In the last section, we prove Theorems 1.4 and 1.5.

### 2 Preliminaries

#### 2.1 Positive Cones

Let \( X \) be a compact Kähler manifold of dimension \( k \), and \( \omega \) be a fixed Kähler form satisfying \( \int_X \omega^k = 1 \). All volumes will be computed with respect to the probability volume form \( dV := \omega^k \). Let \( H^{1,1}(X) \) denote the Dolbeault cohomology group, and let \( H^2(X, \mathbb{Z}) \), \( H^2(X, \mathbb{R}) \), and \( H^2(X, \mathbb{C}) \) denote the de Rham cohomology groups with coefficients in \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \). We also set

\[
H^{1,1}(X, \mathbb{R}) := H^{1,1}(X) \cap H^2(X, \mathbb{R}).
\]

**Definition 2.1** A class \( \alpha \in H^{1,1}(X, \mathbb{R}) \) is called Kähler if \( \alpha \) can be represented by a Kähler form. We denote the set of all Kähler classes by \( K \). A class \( \alpha \) is called numerically effective (nef) if it lies in the closure of the Kähler cone. The set of all nef classes will be denoted by \( H^{1,1}_{\text{nef}}(X, \mathbb{R}) \).
An upper semi-continuous function \( \varphi \in L^1(X) \) is called \textit{quasi-plurisubharmonic} (qpsh) if there exists a smooth closed form \( \theta \) such that \( \theta + dd^c \varphi \geq 0 \) in the sense of currents. Notice that a qpsh function is locally the sum of a smooth function and a psh function. A closed \((1, 1)\) current \( T \) is called \textit{almost positive} if there exists a real smooth \((1, 1)\) form \( \gamma \) such that \( T \geq \gamma \).

The \textit{Lelong number} of a positive closed \((1, 1)\) current \( T \) is defined by

\[
\nu(T, x) := \lim \inf_{z \to x} \frac{\varphi(z)}{\log |x - z|},
\]

where \( \varphi \) is a local potential for \( T \), that is, \( T = dd^c \varphi \) near \( x \). This definition is independent of the choice of the potential \( \varphi \) and the local coordinates. If \( T \) is almost positive, then the Lelong numbers are still well defined since the negative part contributes for zero. It follows from a theorem of Thie that \( \nu([D], x) = \operatorname{mult}_x D \), where \([D]\) is the current of integration along an effective divisor and \( \operatorname{mult}_x \) is the multiplicity of \( D \) at \( x \). We denote the sub-level sets by \( E_c(T) := \{ x \in X : \nu(T, x) \geq c \} \).

A theorem of Siu asserts that \( E_c(T) \) is an analytic set of codimension at least 1. We also set \( E_+ (T) := \bigcup_{c>0} E_c(T) \).

A class \( \alpha \in H^{1,1}(X, \mathbb{R}) \) is called \textit{pseudo-effective} (psef) if there exists a positive closed \((1, 1)\) current \( T \) such that \( \{ T \} = \alpha \). The set of all psef classes, \( H^{1,1}_{\text{psef}}(X, \mathbb{R}) \), is a closed convex cone containing \( H^{1,1}_{\text{nef}}(X, \mathbb{R}) \). A positive closed current \( T \) is called \textit{Kähler} if there exists small \( \epsilon > 0 \) such that \( T \geq \epsilon \omega \). A class \( \alpha \in H^{1,1}(X, \mathbb{R}) \) is said to be \textit{big} if there exists a Kähler current \( T \) such that \( \alpha = \{ T \} \). We denote the set of all big classes by \( H^{1,1}_{\text{big}}(X, \mathbb{R}) \). This is an open convex cone, and coincides with the interior of \( H^{1,1}_{\text{psef}}(X, \mathbb{R}) \). Finally, we stress that these definitions coincide with the classical ones in complex geometry [6].

**Theorem 2.2** [3, 7] A class \( \alpha \in H^{1,1}_{\text{nef}}(X, \mathbb{R}) \) is big if and only if \( \alpha^n \neq 0 \).

### 2.2 Currents with Analytic Singularities

Following [6] and [3], a closed almost positive \((1, 1)\) current \( T = \theta + dd^c \phi \) is said to have analytic singularities along a subscheme \( V(\mathcal{I}) \) defined by a coherent ideal sheaf \( \mathcal{I} \) if there exists \( c > 0 \) and locally

\[
\phi = \frac{c}{2} \log(|f_1|^2 + \cdots + |f_N|^2) + u,
\]

where \( u \) is a smooth function and \( f_1, \ldots, f_N \)'s are holomorphic functions which are local generators of \( \mathcal{I} \). Blowing up \( X \) along \( V(\mathcal{I}) \) and resolving the singularities in the sense of Hironaka, we obtain a modification \( \mu : \tilde{X} \to X \). Moreover, \( D := \mu^{-1}(V(\mathcal{I})) \) is an effective divisor in \( \tilde{X} \), and \( \mu^* T \) has analytic singularities along \( D \), thus it follows from Siu decomposition that

\[
\mu^* T = \theta + cD,
\]
where \( \theta \) is a smooth \((1,1)\) form. Furthermore, if \( T \geq \gamma \), then we have \( \theta \geq \mu^* \gamma \).
In particular, if \( T \geq 0 \), then \( \theta \geq 0 \). This decomposition is called log resolution of singularities of \( T \).

**Theorem 2.3** [6] Let \( T \geq \gamma \) be an almost positive closed \((1,1)\) current on \( X \). Then there exists a sequence of positive real numbers \( \epsilon_n \) decreasing to 0 and a sequence of almost positive closed \((1,1)\) currents \( T_n \in \{ T \} \) with analytic singularities such that \( T_n \to T \) weakly, \( T_n \geq \gamma - \epsilon_n \omega \), and \( v(T_n, x) \) increases uniformly to \( v(T, x) \) with respect to \( x \in X \).

### 2.3 Currents with Minimal Singularities

Let \( \varphi_1 \) and \( \varphi_2 \) be two qpsh functions. Following [8], we say that \( \varphi_1 \) is less singular than \( \varphi_2 \) if \( \varphi_2 \leq \varphi_1 + C \) for some constant \( C \). If \( T_1 \) and \( T_2 \) are two closed almost positive currents, we write \( T_1 = \theta_i + dd^c \varphi_i \), where \( \theta_i \in \{ T_i \} \) is a smooth closed form and \( \varphi_i \) is a qpsh function. We say that \( T_1 \) is less singular than \( T_2 \) if \( \varphi_2 \leq \varphi_1 + C \). Notice that this definition is independent of the choice of the representatives \( \theta_i \) and potentials \( \varphi_i \).

For a class \( \alpha \in H^{1,1}(X, \mathbb{R}) \) and a real smooth \((1,1)\) form \( \gamma \), we denote the set of all closed almost positive \((1,1)\) currents \( T \in \alpha \) satisfying \( T \geq \gamma \) by \( \alpha[\gamma] \). We fix a smooth representative \( \theta \in \alpha \) and define

\[
v_{\alpha, \gamma}^{\min} := \sup \{ \varphi \leq 0 | \theta + dd^c \varphi \geq \gamma \}.
\]

It follows that \( T_{\alpha, \gamma}^{\min} := \theta + dd^c v_{\alpha, \gamma}^{\min} \in \alpha[\gamma] \) and \( v(T_{\alpha, \gamma}^{\min}, x) \leq v(T, x) \) for every \( x \in X \) and for every \( T \in \alpha[\gamma] \). If, in particular, \( \gamma = 0 \), then we write \( T_{\alpha, \gamma}^{\min} = T_{\alpha}^{\min} \) and refer to it as the minimally singular current. Notice that minimally singular currents are not unique, in general. For example, if \( \alpha \in \mathcal{K} \), then every smooth positive closed form \( \theta \in \alpha \) is a minimally singular current. However, if \( S = \theta' + dd^c u_{\alpha, \gamma}^{\min} \in \alpha \) is another such current, since \( v_{\alpha, \gamma}^{\min} - u_{\alpha, \gamma}^{\min} \in L^1_{\text{loc}}(X) \) and \( dd^c (v_{\alpha, \gamma}^{\min} - u_{\alpha, \gamma}^{\min}) \) is smooth, \( v_{\alpha, \gamma}^{\min} - u_{\alpha, \gamma}^{\min} \) is also smooth and hence, bounded. Thus, \( v_{\alpha, \gamma}^{\min} \) and \( u_{\alpha, \gamma}^{\min} \) are equivalent in the sense of singularities. Therefore, for a fixed class \( \alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \), the current of minimal singularities is well defined modulo \( dd^c(C^\infty) \).

### 2.4 Minimal Multiplicities and Non-nef Locus

Following [3], for a class \( \alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R}) \) we define

\[
v(\alpha, x) := \sup_{\epsilon > 0} v(T_{\alpha, \epsilon}^{\min}, x),
\]

where \( T_{\alpha, \epsilon}^{\min} := T_{\alpha, \epsilon - \epsilon \omega}^{\min} \in \alpha[-\epsilon \omega] \). Since the right-hand side in the definition of \( v(\alpha, x) \) is increasing as \( \epsilon \) decreases, the sup coincides with the limit. This definition is independent of the choice of the Kähler form \( \omega \). We also remark that for every positive closed \((1,1)\) current \( T \in \alpha \) and \( x \in X \), we have \( 0 \leq v(\alpha, x) \leq v(T, x) \leq C \), where \( C > 0 \) is a constant depending only on the cohomology class \( \alpha \).
If $A$ is an analytic subset of $X$, we define

$$v(\alpha, A) := \inf_{x \in A} v(\alpha, x).$$

A class $\alpha \in H_{psef}^{1,1}(X, \mathbb{R})$ is called nef in codimension 1 if $v(\alpha, D) = 0$ for every prime divisor $D \subset X$. We denote the set of all such classes by $\mathcal{E}_1$. It follows from the next proposition that $\mathcal{E}_1 \subset H_{psef}^{1,1}(X, \mathbb{R})$ is also a closed convex cone.

**Proposition 2.4** [3] Let $\alpha \in H_{psef}^{1,1}(X, \mathbb{R})$ be a class

1. $\alpha$ is nef if and only if $v(\alpha, x) = 0$ for every $x \in X$.
2. $\alpha \rightarrow v(\alpha, x)$ is sub-additive and homogenous in $\alpha$ for every $x \in X$.
3. $\alpha \rightarrow v(\alpha, x)$ is lower semi-continuous on $H_{psef}^{1,1}(X, \mathbb{R})$ and continuous on $H_{big}^{1,1}(X, \mathbb{R})$ for every $x \in X$.
4. If $\alpha \in H_{big}^{1,1}(X, \mathbb{R})$, then $v(\alpha, x) = v(T^{{min}_{\alpha}} x)$ for every $x \in X$.

**Corollary 2.5** If $\alpha \in H_{nef}^{1,1}(X, \mathbb{R}) \cap H_{big}^{1,1}(X, \mathbb{R})$, then $v(T^{{min}_{\alpha}} x, x) = 0$ for every $x \in X$. Moreover, if $\alpha \in H_{psef}^{1,1}(X, \mathbb{R})$ and $v(T^{{min}_{\alpha}} x, x) = 0$ for every $x \in X$ then $\alpha$ is nef.

**Definition 2.6** [3] Let $\alpha \in H_{psef}^{1,1}(X, \mathbb{R})$ then the non-nef locus of $\alpha$ is defined by

$$E_{nn}(\alpha) := \{x \in X | v(\alpha, x) > 0\}.$$

We also have the following description of the non-nef locus:

**Proposition 2.7** Let $\alpha \in H_{psef}^{1,1}(X, \mathbb{R})$, then

$$E_{nn}(\alpha) = \bigcup_{\epsilon > 0} \bigcap_{T} \mu(|D|),$$

where $T$ runs over the set $\{T \in \alpha[-\epsilon \omega] : T$ has analytic singularities $\}$ and $\mu : \tilde{X} \rightarrow X$, $\mu^*T = \theta + |D|$ is log resolution of singularities of $T$ and $|D|$ denotes the support of the current of integration $|D|$.

**Proof** Let $x \in \bigcup_{\epsilon > 0} \bigcap_{T} \mu(|D|)$. Then there exists $\epsilon_1 > 0$ such that $x \in \mu(|D|)$ for every $T \in \alpha[-\epsilon_1 \omega]$ that has analytic singularities and log resolution $\mu : \tilde{X} \rightarrow X$, $\mu^*T = \theta + |D|$. This implies that $v(T, x) > 0$. Let $\epsilon < \epsilon_1$ and let $T^{{min}_{\alpha, \epsilon}}$ be the current of minimal singularities. By Theorem 2.3 there exists a sequence $T_k \in \alpha[-\epsilon_1 \omega]$ with analytic singularities such that $T_k$ converges weakly to $T$ and $v(T_k, x)$ increases to $v(T^{{min}_{\alpha, \epsilon}} x)$. Thus, $0 < v(T^{{min}_{\alpha, \epsilon}} x, x) \leq v(\alpha, x)$. That is, $x \in E_{nn}(\alpha)$.

To prove the reverse inclusion, let $x \in E_{nn}(\alpha)$. Then by the definition of $v(\alpha, x)$, there exists $\epsilon > 0$ such that $0 < v(T^{{min}_{\alpha, \epsilon}} x, x)$. Now, let $T \in \alpha[\epsilon]$ have analytic singularities. Resolving its singularities, we obtain $\mu : \tilde{X} \rightarrow X$ such that $\mu^*T = \theta + |D|$, where $\theta \geq -\epsilon \omega$ is smooth and $D$ is an effective divisor. Since $0 < v(T^{{min}_{\alpha, \epsilon}} x, x) \leq v(T^{{min}_{\alpha, \epsilon}} x, x)$. Thus, $0 < v(T^{{min}_{\alpha, \epsilon}} x, x) \leq v(\alpha, x)$. Therefore, $x \in E_{nn}(\alpha)$.
Let $\nu(T, x) \leq \nu(\mu^*(T), p)$ for every $p \in \tilde{X}$ with $\mu(p) = x$, we conclude that $x \in \mu(\mid D\mid)$.

**Definition 2.8** Let $\alpha \in H_{\text{psef}}^1(X, \mathbb{R})$ be a class. An irreducible algebraic curve $C$ is called $\alpha$-negative if the intersection product $\alpha \cdot C < 0$.

**Proposition 2.9** [4] For $\alpha \in H_{\text{psef}}^1(X, \mathbb{R})$, every $\alpha$-negative curve $C$ is contained in $E_{\text{nn}}(\alpha)$.

**Proof** If $C \not\subset E_{\text{nn}}(\alpha)$, then for every $\epsilon > 0$ there exists $T \in \alpha[-\epsilon \omega]$ with analytic singularities such that $C \not\subset \mu(\mid D\mid)$, where $\mu : \tilde{X} \to X$ is log resolution of $T$ and $\mu^*(T) = \theta + [D]$.

Let $\tilde{C} \subset \tilde{X}$ be the strict transform of $C$ so that $\mu_*\tilde{C} = C$, and define $S = T + \epsilon \omega \geq 0$. Then $\mu^*S = \theta_\epsilon + [D]$, where $\theta_\epsilon \geq 0$ since $S \geq 0$. Thus,

$$\alpha + \epsilon \omega \cdot C = \langle S, \mu_*\tilde{C} \rangle = \langle \mu^*S, \tilde{C} \rangle = \langle \theta_\epsilon + [D], \tilde{C} \rangle \geq 0,$$

where the last inequality follows from $\theta_\epsilon \geq 0$ and $\tilde{C} \not\subset D$. Since $\epsilon > 0$ is arbitrary, we obtain $\alpha \cdot C \geq 0$. □

**Remark 2.10** Thus, the non-nef locus contains the union of the $\alpha$-negative curves. However, in general, the non-nef locus of a pseudo-effective class $\alpha$ is not equal to union of $\alpha$-negative curves (see [4]).

3 Dynamics of Meromorphic Maps

Let $X$ be a compact Kähler manifold of dimension $k$ and $f : X \to X$ be a meromorphic map that is $f$ is holomorphic on the set $X \setminus I_f$ such that the closure of the graph $\Gamma_f$ of $f : X \setminus I_f \to X$ in $X \times X$ is an irreducible analytic set of dimension $k$.

Let $\pi_i : X \times X \to X$ denote the canonical projections. Then $I_f$ coincides with the set of points $z$ for which $\pi_i^{-1}(z) \cap \Gamma_f$ contains more than one point. The set $I_f$, called the indeterminacy set of $f$, is also an analytic set of codimension at least 2. In fact, for every $z \in I_f$, the set $\pi_i^{-1}(z) \cap \Gamma_f$ has positive dimension. Moreover, $X \setminus I_f$ is the largest open set where $f$ is holomorphic. We also set

$$I_\infty := \bigcup_{n \geq 1} I_f^n.$$

For a subset $Z \subset X$, we define the total transform of $Z$ under $f$ by

$$f(Z) := \pi_2(\pi_1^{-1}(Z) \cap \Gamma_f).$$

With the above convention, we define $E_f^- := f(I_f)$.

We say that $f$ is dominant if the projection $\pi_2$ restricted to $\Gamma_f$ is surjective. This is equivalent to saying that the Jacobian determinant of $f$ does not vanish identically
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in any coordinate chart. We refer the reader to the surveys [28] and [21] for the basic properties of meromorphic maps.

It is well known that $f$ induces a linear action on $H^{p,p}(X, \mathbb{R})$ as follows. Let $\Gamma_f$ denote a desingularization of $\Gamma_f$. Then the diagram

$$
\begin{array}{ccc}
\Gamma_f & \xrightarrow{\pi_1} & X \\
\downarrow & & \downarrow f \\
\Gamma_f & \xrightarrow{\pi_2} & X
\end{array}
$$

commutes. Let $\theta$ be a smooth closed $(p, p)$ form on $X$. Then we set $f^*\theta = (\pi_1)_*(\pi_2)^*\theta$, where $(\pi_2)^*\theta$ is a smooth form and the latter is push-forward as a current. Since pull-back and push-forward commute with the $d$ operator, $f^*\theta$ is a $d$-closed $(p, p)$ current. Then we set

$$f^*\{\theta\} = \{f^*\theta\},$$

where $\{\theta\}$ denotes the de Rham cohomology class of $\theta$ and $\{f^*\theta\}$ is the de Rham cohomology class of $f^*\theta$.

Similarly, one can define push-forward by $f_*\theta = (\pi_2)_*(\pi_1)^*\theta$. It follows that the action of pull-back on $H^{p,p}(X, \mathbb{R})$ is dual to that of push-forward on $H^{k-p,k-p}(X, \mathbb{R})$ with respect to the intersection product.

We say that $f$ is $p$-regular whenever $(f^n)^* = (f^*)^n$ on $H^{p,p}(X, \mathbb{R})$ as linear maps for $n = 1, 2, \ldots$.

We also denote

$$\delta_p(f) := \int_{X \setminus I_f} f^*\omega^p \wedge \omega^{k-p},$$

and the $p$-th dynamical degree of $f$ by

$$\lambda_p(f) := \limsup_{n \to \infty} \left[ \delta_p(f^n) \right]^{\frac{1}{n}}.$$

In particular, if $f$ is $p$-regular, then $\lambda_p(f)$ coincides with the spectral radius of $f^*: H^{p,p}(X, \mathbb{R})$.

3.1 Invariant Classes and Singularities

We denote the set of all positive closed $(1, 1)$ currents by $T(X)$. Following [19], we define the pull-back of $T$ as follows. We fix a point $x_0 \in X \setminus I_f$; then locally we can write $T = dd^c u$ for a psh function $u$ near $f(x_0)$ and define $f^*T = dd^c (u \circ f)$ near $x_0$ that is independent of the choice of the local potential $u$. Therefore, we obtain a well-defined positive closed $(1, 1)$ current on $X \setminus I_f$. Now, since $I_f$ is an analytic set of codimension at least 2, it follows from [22] that it extends trivially to a unique positive closed $(1, 1)$ current $f^*T$ on $X$. Furthermore, since $f$ is dominant, the action $T \to f^*T$ is continuous with respect to weak topology on positive closed $(1, 1)$
currents [26]. Moreover, \(\{f^* T\} \in H^{1,1}(X, \mathbb{R})\) is independent of the choice of \(T \in \alpha\) for \(\alpha \in H^{1,1}_{\text{psf}}(X, \mathbb{R})\) and \(f^*\alpha = \{f^* T\} \in H^{1,1}_{\text{psf}}(X, \mathbb{R})\). Notice that \(H^{1,1}_{\text{psf}}(X, \mathbb{R})\) is a closed, convex cone which is strict (i.e., \(H^{1,1}_{\text{psf}}(X, \mathbb{R}) \cap -H^{1,1}_{\text{psf}}(X, \mathbb{R}) = \{0\}\)). Since \(H^{1,1}_{\text{psf}}(X, \mathbb{R})\) is invariant under the linear action \(f^* : H^{1,1}(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R})\), it follows from a Perron–Frobenius type argument [10] that there exists a class \(\alpha \in H^{1,1}_{\text{psf}}(X, \mathbb{R})\) such that \(f^* \alpha = r_1(f)\alpha\), where \(r_1(f)\) is the spectral radius of \(f^*|_{H^{1,1}(X, \mathbb{R})}\). In particular, if \(f\) is 1-regular, then \(r_1(f) = \lambda_1(f)\).

The following argument is adapted from [13]. We choose a basis for \(H^{1,1}(X, \mathbb{C}) = H^{1,1}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}\) so that the associated matrix of \(f^*\) is in Jordan form. Let \(J_{\lambda_j, m_j}\) denote its Jordan blocks for \(1 \leq j \leq r\). In other words, we can decompose \(H^{1,1}(X, \mathbb{C})\) into a direct sum of complex subspaces \(E_j\)

\[
H^{1,1}(X, \mathbb{C}) = \bigoplus_{1 \leq j \leq r} E_j \quad \text{with} \quad \dim E_j = m_j \quad \text{and} \quad \sum_{j=1}^{r} m_j = H^{1,1}
\]

such that the restriction of \(f^*\) to \(E_j\) is given by the Jordan block \(J_{\lambda_j, m_j}\). Since \(f^*\) preserves the psf cone that is a proper cone, we may assume that \(\lambda_1 = r_1(f)\) and \(m := m_1\) is the index of the spectral radius. Moreover, we can also assume that \((|\lambda_j|, m_j)\) is ordered, so that either \(|\lambda_j| > |\lambda_{j+1}|\) or \(|\lambda_j| = |\lambda_{j+1}|\) and \(m_j \geq m_{j+1}\) for \(1 \leq j \leq r\). Let \(v\) be the integer such that \(|\lambda_j| = \lambda_1\) for \(1 \leq j \leq v\). Let \(E_{\lambda_j, m_j}\) denote the hyperplane generated by the first \(m_j - 1\) vectors of the basis of \(E_j\) associated with the Jordan form. Then we have \(||(f^*)^n v|| \sim n^{m-1}\lambda_1^n\) for every \(v \notin E_1 \oplus \cdots \oplus E_v \oplus E_{v+1} \oplus \cdots \oplus E_r\). Notice that this property holds for every \(v \in K\) because given any \(v' \in H^{1,1}(X, \mathbb{R})\), we can find \(v'' \in K\) and \(\sigma \geq 0\) such that \(v'' = \sigma v - v''\). We let \(F_j\) denote the eigenspace of \(f^*|_{E_j}\), which is a complex line. We define

\[
F_{\lambda_1} := F_1 \oplus \cdots \oplus F_v \quad \text{and} \quad H_{\lambda_1} := \bigoplus_{\lambda_j = \lambda_1, m_j = m} F_j.
\]

We also set \(F := F_{\lambda_1} \cap H^{1,1}(X, \mathbb{R})\) and \(H := H_{\lambda_1} \cap H^{1,1}(X, \mathbb{R})\). Notice that for any \(2 \leq j \leq v\), there exists a unique \(\theta_j \in S := \mathbb{R}/2\pi \mathbb{Z}\) such that \(\lambda_j = \lambda_1 \exp(i\theta_j)\). Let \(\Theta := (\theta_2, \ldots, \theta_v) \in S^{v-1}\). We let \(\Theta\) denote the closed subgroup of \(S^{v-1}\) generated by \(\Theta\). This is a finite union of real tori. The orbit of each point \(\theta' \in \Theta\) under the translation \(\theta' \to \theta' + \theta\) is dense in \(\Theta\). If \(\lambda_j = \lambda_1\) for every \(2 \leq j \leq v\), then \(F = H\) and \(\Theta = \{0\}\). We also set

\[
\Lambda_N := \frac{1}{N} \sum_{n=1}^{N} \frac{(f^*)^n}{n^{m-1}\lambda_1^n}.
\]

For the proof of the following proposition, we refer the reader to [13].

**Proposition 3.1** Assume that \(\lambda_1 > 1\). Then the sequence \((\Lambda_N)\) converges to a surjective real linear map \(\Lambda_\infty : H^{1,1}(X, \mathbb{R}) \to H\). Let \(n_i\) be an increasing sequence
of positive integers. Then \((n_1^{1-m}\lambda_1^{-n_i}(f^*)^{n_i})\) converges if and only if \((n_i\theta)\) converges. Moreover, any limit \(L_\infty\) of \((n_1^{1-m}\lambda_1^{-n_i}(f^*)^{n_i})\) is a surjective real linear map \(L_\infty : H^{1,1}(X, \mathbb{R}) \to F\).

We also define
\[
H_{\text{psef}} := H \cap H^{1,1}_{\text{psef}}(X, \mathbb{R})
\]
and
\[
H_N := \Lambda_\infty(H^{1,1}_{\text{nef}}(X, \mathbb{R})).
\]
It is clear that \(H_N \subset H_{\text{psef}}\). Moreover, it follows from Proposition 3.1 that \(H_N\) has a non-empty interior in \(H\). Indeed, since \(\Lambda_\infty\) is surjective, it is open. The Kähler cone, \(K \subset H^{1,1}(X, \mathbb{R})\), is also open, and \(\Lambda_\infty(K)\) is contained in \(H_N\).

**Theorem 3.2** [15, 24] Let \(f : X \to X\) be a dominant meromorphic map and \(T\) be a positive closed \((1, 1)\) current on \(X\). Then for every \(x \in X \setminus I_f\),
\[
\nu(T, f(x)) \leq \nu(f^*T, x) \leq C_f \nu(T, f(x)),
\]
where \(C_f > 0\) is a constant that does not depend on \(T\).

**Theorem 3.3** Let \(f : X \to X\) be a dominant meromorphic map.

(1) For every \(\alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R})\) and \(p \in X \setminus I_f\),
\[
\nu(f^*\alpha, p) \leq C_f \nu(\alpha, f(p)),
\]
where \(C_f\) is independent of \(\alpha\). In particular, \(f(E_{mn}(f^*\alpha) \setminus I_f) \subset E_{mn}(\alpha)\).

(2) Assume that \(f\) is 1-regular and \(\lambda := \lambda_1(f) > 1\). Then \(E_{mn}(\alpha) \subset I_\infty\) for every \(\alpha \in H_N\). In particular, \(H_N \subset E_1\).

**Proof** (1) We assume that \(\alpha \in H^{1,1}_{\text{big}}(X, \mathbb{R})\). Let \(p \in X \setminus I_f\) and \(T^\text{min}_\alpha \in \alpha\) be a positive closed \((1, 1)\) current with minimal singularities. Since \(f^*T^\text{min}_\alpha \in f^*\alpha\), by the definition of \(\nu(f^*\alpha, p)\) and by Theorem 3.2 we have
\[
\nu(f^*\alpha, p) \leq \nu(f^*(T^\text{min}_\alpha), p) \leq C_f \nu(T^\text{min}_\alpha, f(p)) = C_f \nu(\alpha, f(p)),
\]
where the last equality follows from \(\alpha \in H^{1,1}_{\text{big}}(X, \mathbb{R})\).

If \(\alpha\) is merely psef, then we consider \(\alpha_\delta := \alpha + \delta(\omega) \in H^{1,1}_{\text{big}}(X, \mathbb{R})\) for \(\delta > 0\). Now, by the lower semi-continuity of \(\nu(\cdot, p)\) and the continuity of \(f^*\), we get
\[
\nu(f^*\alpha, p) \leq \liminf_{\delta \to 0} \nu(f^*\alpha_\delta, p).
\]
Since \(\alpha_\delta\) is big, by the above argument we also have
\[
\nu(f^*\alpha_\delta, p) \leq C_f \nu(\alpha_\delta, f(p))
\]
for $\delta > 0$. Then by sub-additivity and homogeneity of $v(\cdot, f(p))$, we get

$$v(\alpha_{\delta}, f(p)) \leq v(\alpha, f(p)) + \delta v([\omega], f(p)) = v(\alpha, f(p)),$$

since $[\omega]$ is Kähler. Therefore, the assertion follows.

(2) Let $\alpha \in H_N$. Then $\alpha = \lim_{N \to \infty} \Lambda_N \beta$ for some $\beta \in H^{1,1}_{\text{nef}}(X, \mathbb{R})$ and $f^* \alpha = \lambda \alpha$. By part (1) we have

$$v\left(\frac{1}{n^{m-1} \lambda^n} (f^n)^* \beta, p\right) = 0$$

for $p \notin I_f$. It follows from sub-additivity, lower semi-continuity of $v(\cdot, p)$, and continuity of $f^*$ that

$$v(\alpha, p) \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{n^{m-1} \lambda^n} v((f^n)^* \beta, p).$$

Thus, $v(\alpha, p) = 0$ for $p \notin I_{\infty}$. Finally, since $f$ is 1-regular, $I_{\infty}$ does not contain any divisors, and hence, $\alpha \in \mathcal{E}_1$. □

Without the assumption $\alpha \in H_N$, the assertion of Theorem 3.3(2) is not true in general. The following example was communicated by V. Guedj.

**Example 3.4** Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree $\lambda \geq 2$ with a totally invariant point $p$, i.e., $f^{-1}(p) = p$. We define $\pi : X \to \mathbb{P}^2$ to be the blow-up of $\mathbb{P}^2$ at $p$. Let $f_X$ denote the induced map and $E := \pi^{-1}(p)$ denote the exceptional fiber. Then $f_X^* \{E\} = \lambda \{E\}$. Thus, $f_X^* : H^{1,1}(X, \mathbb{R}) \to H^{1,1}(X, \mathbb{R})$ is given by

$$f_X^* = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

Notice that the class $\{E\} \in H^{1,1}_{\text{psef}}(X, \mathbb{R})$, but $E \cdot E = -1$, hence, $\{E\} \notin H_N = H^{1,1}_{\text{nef}}(X, \mathbb{R})$.

We also remark that in the above case $\lambda^2(f) = \lambda_1(f)^2$. It follows from [10] that if $\dim_{\mathbb{C}}(X) = 2$ and $r_1(f)^2 > \lambda_2(f)$, then $r_1(f)$ is a simple root of the characteristic polynomial of $f^*$.

The next corollary follows from Proposition 2.9.

**Corollary 3.5** Let $f$ and $\alpha \in H_N$ be as in Theorem 3.3(2). Then every $\alpha$-negative curve $C \subset I_{\infty}$.

**Corollary 3.6** Let $f : X \to X$ be a dominant meromorphic map such that $\dim(I_f) = 0$. Then $f^*(H^{1,1}_{\text{nef}}(X, \mathbb{R})) \subset H^{1,1}_{\text{nef}}(X, \mathbb{R})$. In particular, if $X$ is a compact Kähler surface, $f^*$ preserves the nef cone.
Proof Let $\alpha \in H^{1,1}_{\text{nef}}(X, \mathbb{R})$. Then it follows from Theorem 3.3 that $E_{\text{nn}}(f^*\alpha) \subset I_f$, and since $\dim(I_f) = 0$, $E_{\text{nn}}(\alpha)$ is a finite set. Thus, the assertion follows from the regularization argument of [6, Lemma 6.3].

If $X$ is a compact Kähler surface, then the cone $E_1$ coincides with $H^{1,1}_{\text{nef}}(X, \mathbb{R})$ [3]. Thus, $f^*(E_1) \subset E_1$ when $\dim \mathcal{C} X = 2$.

However, if $\dim \mathcal{C} X \geq 3$, then this is no longer true, as the following example shows.

Example Let $J : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$

$$J[x_0 : x_1 : x_2 : x_3] = [x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2]$$

and $L \in \text{Aut}(\mathbb{P}^3)$ given by the matrix

$$L = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}.$$

We consider the birational map $f = L \circ J : \mathbb{P}^k \rightarrow \mathbb{P}^k$

$$f[x_0 : x_1 : x_2 : x_3] = [(x_0 + x_3)x_1x_2 : (x_2 + x_3)x_0x_1 : (x_1 + x_3)x_0x_2 : (x_1 + x_2)x_0x_3].$$

Notice that $f$ has four exceptional hypersurfaces: $\{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3\}$, where $\Sigma_i := \{x_i = 0\}$ with the following orbit data:

$$\Sigma_0 \rightarrow e_0 := [1 : 0 : 0 : 0] \sim \Sigma_\beta$$
$$\Sigma_1 \rightarrow e_{23} := [0 : 0 : 1 : 1] \sim l_1 \rightarrow e_{23}$$
$$\Sigma_2 \rightarrow e_{13} := [0 : 1 : 0 : 1] \sim l_2 \rightarrow e_{13}$$
$$\Sigma_3 \rightarrow [1 : 1 : 1 : 0],$$

where “$\sim$” indicates the total transform under $f$ and $\Sigma_\beta = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : 2x_0 - x_1 - x_2 + x_3 = 0\}$, $l_1 \subset \Sigma_1$ is the line passing through $e_0$ and $e_{23}$, and $l_2 \subset \Sigma_2$ is the line passing through $e_0$ and $e_{13}$.

We define the complex manifold $X$ to be $\mathbb{P}^3$ blown up at $e_0$, $e_{23}$, and $e_{13}$ successively. We denote the exceptional fibers on $e_0$, $e_{23}$, and $e_{13}$ by $E_0$, $E_{23}$, and $E_{13}$, respectively, and the induced map by $f_X : X \rightarrow X$. Then we have the following orbit data:

$$\Sigma_0 \rightarrow E_0 \rightarrow \Sigma_\beta$$
$$\Sigma_1 \rightarrow E_{23} \rightarrow l_1 \rightarrow \gamma \rightarrow l_1$$
\[ \Sigma_2 \to E_{13} \to l_2 \to \sigma \to l_1 \]
\[ \Sigma_3 \to [1 : 1 : 1 : 0] \cap, \]

where \( \gamma \subset E_{23} \) and \( \sigma \subset E_{13} \) are lines that are regular. It follows that no exceptional hypersurface of \( f_X \) is mapped into the indeterminacy locus \( I_{f_X} \); thus, \( f_X \) is 1-regular. Now, \( \langle H_X, E_0, E_{23}, E_{13} \rangle \) forms a basis for \( H^{1,1}(X) \), and the action \( f^*_X : H^{1,1}(X) \to H^{1,1}(X) \) with respect to this ordered basis is given by the integer coefficient matrix

\[
\begin{pmatrix}
3 & 1 & 1 & 1 \\
-2 & 0 & -1 & -1 \\
-1 & -1 & -1 & 0 \\
-1 & -1 & 0 & -1
\end{pmatrix}
\]

the characteristic polynomial of \( f^*_X \) is \( \chi(x) = x^4 - x^3 - 3x^2 + x + 2 \), and the first dynamical degree of \( f \), the largest root of \( \chi(x) \), \( \lambda_1(f) = 2 \) is a simple eigenvalue.

Let \( \tilde{\Sigma}_1 \subset X \) denote the strict transform of \( \Sigma_1 \). Since \( e_0, e_{23} \in \Sigma_1 \subset \mathbb{P}^3 \), the class \( \alpha := \{ \tilde{\Sigma}_1 \} \) is nef in codimension 1. Indeed, for any hyperplane \( H \subset \mathbb{P}^3 \) containing \( e_0 \) and \( e_{23} \) that does not contain \( e_{13} \), \( \tilde{H} \) is cohomologous to \( \tilde{\Sigma}_1 \), but this is a 1-parameter family of hyperplanes and \( [\tilde{H}] \in \alpha \) defines a positive closed \((1,1)\) current. Thus, we infer that \( v(T^\alpha_{\min}, x) = 0 \) for every \( x \not\in l_1 \).

Hence, by Proposition 2.4, \( E_{\text{nn}}(\{ \tilde{\Sigma}_1 \}) \subset l_1 \). Now, since \( \tilde{\Sigma}_1 \cdot l_1 = -1 \), by Proposition 2.9, \( l_1 \subset E_{\text{nn}} \). Therefore, \( E_{\text{nn}}(\{ \tilde{\Sigma}_1 \}) = l_1 \). On the other hand,

\[
f^*(\alpha) = f^*_X(H_X - E_0 - E_{23}) = H_X - E_0 + E_{23} = \{ \tilde{H}_0 \} + E_{23},
\]

where \( H_0 \) is any hyperplane in \( \mathbb{P}^3 \) containing \( e_0 \) that does not contain \( e_{23} \) or \( e_{13} \). Since this is a 2-parameter family, by the same argument above we see that \( E_{\text{nn}}(H_X - E_0 + E_{23}) \subset E_{23} \). Moreover, for a generic line \( \sigma \subset E_{23} \) we have \( E_{23} \cdot \sigma = -1 \). Therefore, \( E_{\text{nn}}(H_X - E_0 + E_{23}) = E_{23} \), hence \( f^*(\alpha) \not\in \mathcal{E}_1 \).

4 Green Currents

**Proof of Theorem 1.2** We fix a smooth representative \( \theta \in \alpha \) and let

\[
T^\alpha_{\min} = \theta + dd^c v^\alpha_{\min}
\tag{4.1}
\]

denote the current of minimal singularities. Since \( \alpha \) is invariant by the \( dd^c \)-lemma [18, p. 149], we can write

\[
\frac{1}{\lambda} f^* T^\alpha_{\min} = \theta + dd^c \phi_1,
\tag{4.2}
\]

where \( \phi_1 \) is a qpsh function; thus, we can assume that \( \phi_1 \leq 0 \). Then by the definition of \( v^\alpha_{\min} \), we have \( \phi_1 \leq v^\alpha_{\min} \).

Now, by using the invariance of \( \alpha \) again, we write

\[
\frac{1}{\lambda^2} (f^2)^* T^\alpha_{\min} = \theta + dd^c \phi_2
\tag{4.3}
\]
since \( f \) is 1-regular by (4.2) we obtain
\[
\frac{1}{\lambda^2} (f^2)^* T^\text{min}_\alpha = \frac{1}{\lambda} f^* \left( \frac{1}{\lambda} f^* T^\text{min}_\alpha \right) = \frac{1}{\lambda} f^* (\theta) + \frac{1}{\lambda} dd^c \phi_1 \circ f
\] (4.4)
and using (4.1) and (4.2) we have
\[
\frac{1}{\lambda} f^* (\theta) = \theta + dd^c \left( \phi_1 - \frac{1}{\lambda} v^\text{min}_\alpha \circ f \right)
\] (4.5)
and substituting (4.5) in (4.4) we obtain
\[
\frac{1}{\lambda^2} (f^2)^* T^\text{min}_\alpha = \theta + dd^c \left( \phi_1 + \frac{1}{\lambda} (\phi_1 - v^\text{min}_\alpha) \circ f \right).
\]
Therefore, by adding a constant, if necessary, we can choose
\[
\phi_2 = \phi_1 + \frac{1}{\lambda} (\phi_1 - v^\text{min}_\alpha) \circ f.
\]
Since \( \phi_1 \leq v^\text{min}_\alpha \), we get \( \phi_2 \leq \phi_1 \).
Iterating this argument we obtain
\[
\frac{1}{\lambda^n} (f^n)^* T^\text{min}_\alpha = \theta + dd^c \phi_n,
\]
where
\[
\phi_n = \phi_1 + \sum_{j=1}^{n-1} \frac{1}{\lambda^j} (\phi_1 - v^\text{min}_\alpha) \circ f^j
\]
for \( n \geq 2 \) and \( \{\phi_n\} \) is a decreasing sequence of negative qpsf functions. Thus, by Hartogs’ lemma, either \( \phi_n \) converges uniformly to \(-\infty\) or \( \phi_n \) converges to some qpsf function \( g \). We will show that the former case is not possible by using a trick due to Sibony [28].

Let \( R \in \alpha \) be a positive closed \((1, 1)\) current. We consider the Cesaro means of the form
\[
R_N = \frac{1}{N} \sum_{i=0}^{N-1} \frac{1}{\lambda^i} (f^i)^* R.
\]
Notice that \( R_N \)'s are positive closed \((1, 1)\) currents and \( \|R_N\| = \|R\| \), where \( \|R\| = \int_X R \wedge w^{k-1} \). Therefore, we can extract a subsequence \( R_{N_k} \to S \) for some positive closed current \( S \in \alpha \) such that \( f^* S = \lambda S \), and we have
\[
S = \theta + dd^c u,
\]
where \( u \) is a qpsf function. Then, by the invariance of \( S \), we get
\[
\frac{1}{\lambda} (f^* \theta + dd^c u \circ f) = \theta + dd^c u
\]
and by (4.5) we infer
\[ \theta + dd^c \left( \phi_1 - \frac{1}{\lambda} (v_{\alpha}^{\min} + u) \circ f \right) = \theta + dd^c u. \] (4.6)

Thus, by adding a constant to \( u \), we can assume that
\[ \phi_1 - \frac{1}{\lambda} v_{\alpha}^{\min} \circ f = u - \frac{1}{\lambda} u \circ f. \] (4.7)

Pulling back (4.7) by \( \frac{1}{\lambda} f \) and adding the result to (4.7) again, we obtain
\[ \phi_n - \frac{1}{\lambda^n} v_{\alpha}^{\min} \circ f^n = u - \frac{1}{\lambda^n} u \circ f^n. \]

Since \( u \) is qpsh, it is bounded from above \( u \leq C \) and we have
\[ \phi_n \geq u + \frac{1}{\lambda^n} v_{\alpha}^{\min} \circ f^n - \frac{1}{\lambda^n} C. \]

Thus, we infer that \( \phi_n \) converges to \( g_\theta \) for some qpsh function \( g_\theta \). We denote the limit current by \( T_\alpha = \theta + dd^c g_\theta \). Since it is a limit of positive closed currents belonging to \( \alpha \), \( T_\alpha \in \alpha \) is a positive closed current. Moreover, by the continuity of \( f^* \), we have \( f^* T_\alpha = \lambda T_\alpha \). Now, we will show that \( g_\theta \) depends only on the class \( \alpha \).

First of all, since \( \frac{1}{\lambda^n} v_{\alpha}^{\min} \circ f^n \to 0 \) in \( L^1(X) \), we get
\[ T_\alpha = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* T_{\alpha}^{\min} = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* \theta. \]

Now, if \( \theta' \) is another smooth form representing the class \( \alpha \), then by the \( dd^c \)-lemma we can write \( \theta' = \theta + dd^c \varphi \), where \( \varphi \) is a smooth function and since \( X \) is compact \( \varphi \) is bounded. Therefore, the current \( T_\alpha = \theta + dd^c g_\alpha \) is independent of the choice of the representative form. So far, we have proved the first part.

To prove (1): let \( \sigma \in \alpha \) be an invariant current, i.e., \( f^* \sigma = \lambda \sigma \) and \( \sigma = \theta + dd^c \psi \) for some qpsh function \( \psi \leq 0 \). Then by the same argument as above, we obtain
\[ \phi_n - \frac{1}{\lambda^n} v_{\alpha}^{\min} \circ f^n = \psi - \frac{1}{\lambda^n} \psi \circ f^n + C \sum_{j=0}^{n-1} \frac{1}{\lambda^j}. \]

Thus, \( g_\alpha + C_1 \geq \psi \).

The proof of (2) appears in the literature; see [20, 28].

Remark 4.1 Notice that in the proof of Theorem 1.2, to get the convergence \( \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* T_{\alpha}^{\min} = T_\alpha \), we only need that \( \{ \frac{1}{\lambda^n} v_{\alpha}^{\min} \circ f^n \} \) is locally bounded near some point \( x \in X \).

Theorem 1.2 was proved by Fornaess and Sibony when \( X = \mathbb{P}^k \) (see [17] and [28]). More recently, it was proved in [20] under a cohomological assumption that we replace here by a weaker dynamical assumption. See also [9–11, 16, 19] for similar
constructions and [27] for the case of non-1-regular meromorphic self maps of $\mathbb{P}^k$.

Finally, in the case of holomorphic maps of $\mathbb{C}\mathbb{P}^k$, [14] provides an optimal condition on the test forms for the equidistribution towards the Green current.

It seems that the first two assumptions (1-regularity and $\lambda_1 > 1$) in Theorem 1.2 are quite natural. Indeed, if $\lambda_1(f) = 1$, then it follows from the concavity of the function $j \to \log(\lambda_j)$ and the upper bound for the entropy [12] that $h_{\text{top}}(f) = 0$.

What about the condition ($\star$)? If $\alpha$ is a Kähler class, then $v_{\alpha}^{\min} \equiv 0$; thus, ($\star$) holds. More generally, if $\alpha$ can be represented by a semi-positive form, then $v_{\alpha}^{\min}$ is bounded; hence, ($\star$) holds. This is the case for complex homogenous manifolds (i.e., when the group of automorphisms $\text{Aut}(X)$ acts transitively on $X$). Indeed, if $X$ is a complex homogeneous manifold, then every positive closed current $T$ can be approximated by positive smooth forms $\theta_\epsilon \in \{T\}$ (see [23]). Therefore, any psef class can be represented by a semi-positive form. If $X$ is a compact Kähler surface and $\lambda_1(f)^2 > \lambda_2(f)$, it is well known that $\lambda_1$ is a simple eigenvalue (see [10]). Furthermore, the corresponding eigenvector $\alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R})$ can be represented by a positive closed $(1, 1)$ current with bounded potentials (see [9]). Thus, in this case ($\star$) holds.

However, there are some examples for which the condition ($\star$) does not hold.

**Example 4.2** Let $f : \mathbb{P}^2 \to \mathbb{P}^2$

$$f[x_0 : x_1 : x_2] = [x_0^2 : x_1^2 : x_2^2].$$

Then $f$ is a holomorphic map with the totally invariant point $p = [1 : 0 : 0]$, i.e., $f^{-1}(p) = p$. Let $\pi : X \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at $p$. We let $f_X$ denote the induced map and $E := \pi^{-1}(p)$ denote the exceptional fiber. Then $f_X^* [E] = 2[E]$, where $[E]$ denotes the current of integration along $E$. Notice that $\alpha := [E]$ contains only one positive closed $(1, 1)$ current, namely, $[E]$. Thus, $T_{\alpha}^{\min} = [E]$ and $E = \{v_{\alpha}^{\min} = -\infty\}$. We will show that

$$\text{Vol}\left(\frac{1}{2^n} v_{\alpha}^{\min} \circ f_X^n < -1\right) \not\to 0.$$

Indeed, we choose the local coordinates $(s, \eta)$ on $X$ such that $\pi(s, \eta) = [1 : s : s\eta]$. Then in these coordinates, $E = \{s = 0\}$, $v_{\alpha}^{\min}(s, \eta) = \log |s|$, and

$$\frac{1}{2^n} v_{\alpha}^{\min} \circ f_X^n = \log |s|.$$

Thus,

$$\{|s| < e^{-1}\} \subset \left\{\frac{1}{2^n} v_{\alpha}^{\min} \circ f_X^n < -1\right\}$$

for every $n \in \mathbb{N}$, and the claim follows.

**Proposition 4.3** Let $f : X \dashrightarrow X$ be a dominant 1-regular meromorphic map, and $f^* \alpha = \lambda \alpha$ for some $\alpha \in H^{1,1}_{\text{psef}}(X, \mathbb{R})$ with $\lambda > 1$. Assume that there exists a positive
closed (1, 1) current $T := \theta + dd^c \phi$ such that

$$T = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* \theta$$

for some (equivalently, for every) smooth form $\theta \in \alpha$. Then (★) holds.

Proof Since $\theta \in \alpha$ is a smooth form, we have $T = \theta + dd^c \phi$, where $\phi$ is a qpsh function and $\phi \leq v_{\alpha}^{\min} + O(1)$. It is enough to show that $\frac{1}{\lambda} f^* \phi \to 0$ in $L^1(X)$. Let

$$\frac{1}{\lambda} f^* \theta = \theta + dd^c \gamma$$

for some $\gamma \in L^1(X)$. By the continuity of $f^*$ we have $f^* T = \lambda T$. Thus, by adding a constant to $\gamma$, if necessary, we may assume that

$$\gamma + \frac{1}{\lambda} \phi \circ f = \phi.$$ 

Then we have

$$\phi = \sum_{i=0}^{n-1} \frac{1}{\lambda^i} \gamma \circ f^i + \frac{1}{\lambda^n} \phi \circ f^n,$$

and by assumption $\sum_{i=0}^{n-1} \frac{1}{\lambda^i} \gamma \circ f^i \to \phi$ in $L^1(X)$; hence, $\frac{1}{\lambda} \phi \circ f^n \to 0$ in $L^1(X)$. □

Thus, it follows from the next theorem that if $\lambda_1(f) > 1$ is simple and $\alpha$ is nef, then condition (★) holds.

Theorem 4.4 [11] Let $f : X \dashrightarrow X$ be a dominant meromorphic map. Assume that $f$ is $1$-regular and $\lambda := \lambda_1(f) > 1$ is the unique simple eigenvalue and $\alpha \in H^{1,1}(X, \mathbb{R})$ is the corresponding eigenvector. If $\alpha$ is nef, then for every smooth form $\theta \in \alpha$ we have the limit $T_\alpha := \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* \theta$ that depends only on the class $\alpha$. Moreover, $T_\alpha$ is a positive closed $(1, 1)$ current satisfying $f^* T_\alpha = \lambda T_\alpha$.

5 An Algebraic Criterion

The following result is a consequence of the Hodge index theorem [25, Lemma 3.39].

Lemma 5.1 (Negativity Lemma) Let $\pi : Z \dashrightarrow Y$ be a proper birational morphism between normal projective varieties $Z$ and $Y$. Let $-E$ be a $\pi$-nef $\mathbb{R}$-divisor on $Z$ (that is, $(-E) \cdot C \geq 0$ for every $\pi$-exceptional curve $C$). Then

1. $E$ is effective if and only if $\pi_* E$ is effective;
2. Assume that $E$ is effective. Then for every $y \in Y$, either $\pi^{-1}(y) \subset E$ or $\supp(E) \cap \pi^{-1}(y) = \emptyset$.

Proposition 5.2 Let $X$ be a projective manifold, $f : X \dashrightarrow X$ be a dominant rational map, and $\omega$ be a Kähler form. If $p \in I_f$, then $\nu(f^* \omega, p) > 0$. 
Proof We consider the pull-backs
\[(\pi_1)^* f^* \omega = (\pi_1)^* (\pi_1)_* (\pi_2)^* \omega = (\pi_2)^* \omega + E, \tag{5.1}\]
where \(E\) is a (possibly trivial) \(\pi_1\)-exceptional divisor. We claim that \(E\) is a non-trivial effective divisor. Indeed, for any \(\pi_1\)-exceptional curve \(C\), we have
\[(\pi_1)^* f^* \omega \cdot C = 0, \]
and since \(\omega\) is a Kähler form, we get
\[0 \leq \langle \omega, \pi_2(C) \rangle = \langle -E, C \rangle.
\]
Thus, by the Negativity Lemma we conclude that \(E\) is effective.

Let us fix \(p \in I_f\). Since \(\dim(f(p)) \geq 1\), there exists a curve \(C \subset \pi_1^{-1}(p)\) such that \(C \not\subseteq \mathcal{E}(\pi_2)\), and by the above argument \(E \cdot C < 0\); hence, \(E\) is non-trivial. Moreover, \(\pi_1^{-1}(p) \subseteq E\).

Since the left-hand side of (5.1) defines a positive closed \((1, 1)\) current, we infer that \(\nu((\pi_1)^* f^* \omega, q) = \text{mult}_q(E)\) for any \(q \in \pi_1^{-1}(p)\) and
\[0 < \nu((\pi_1)^* f^* \omega, q).
\]
Now, it follows from Theorem 3.2 that
\[0 < \nu((\pi_1)^* f^* \omega, q) \leq C \nu(f^* \omega, p),\]
where \(C > 0\) is a constant that does not depend on \(f^* \omega\).

The following result is well known when \(X = \mathbb{P}^k\) [17] or \(X\) is a compact Kähler surface [10]. To our knowledge, it is new in this generality.

**Theorem 5.3** Let \(X\) be a projective manifold and \(f : X \dasharrow X\) be a dominant rational map. Then the following are equivalent:

(i) \((f^n)^* T = (f^*)^n T\) for every \(T \in T(X)\) and \(n = 1, 2, \ldots;\)
(ii) \((f^*)^n \omega = (f^n)^* \omega\) for every Kähler form \(\omega\) on \(X\) and \(n = 1, 2, \ldots;\)
(iii) \(f\) is 1-regular;
(iv) there is no exceptional hypersurface \(H\) and \(n \in \mathbb{N}\) such that \(f^n(H - I_f) \subseteq I_f\).

**Proof** (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) is clear.

(iii) \(\Rightarrow\) (iv) Let be \(\omega\) be a Kähler form. Assume that there exists an exceptional hypersurface \(H\) such that \(f^n(H - I_f) \subseteq I_f\) for some \(n\). By replacing \(f\) with \(f^n\), we may assume that \(f(H - I_f) \subseteq I_f\). Then by Proposition 5.2, for each \(p \in f(V - I_f) \subseteq I_f\) we have \(\nu(f^* \omega, p) > 0\), but this implies that \(\nu((f^*)^2 \omega, q) > 0\) for every \(q \in H - I_f\). However, \((f^*)^2 \omega\) is an \(L^1_{\text{loc}}\) coefficient form and does not charge \(H\), and hence the cohomology classes of \((f^*)^2 \omega\) and \((f^2)^* \omega\) are different.

(iv) \(\Rightarrow\) (i) Let \(T \in T(X)\). Notice that \(f^{n-1}\) and \(f^n\) are both holomorphic on \(X - (I_f \cup f^{-1}(I_f)) \cup \cdots \cup f^{n-1}(I_f))\). Thus, \((f^n)^* T = (f^*)(f^{n-1})^* T\) on this set. Since there is no hypersurface contained in \(X - (I_f \cup f^{-1}(I_f) \cup \cdots \cup f^{n-1}(I_f))\), we get the equality on \(X\). \(\square\)
Lemma 5.4 Let \( \pi : Z \to Y \) be a proper modification between smooth projective varieties. Let \( \eta \) be a smooth closed real \((1,1)\) form on \( Z \) such that \( \langle \eta, C \rangle \geq 0 \) for every \( \pi \)-exceptional curve \( C \). Then \( \pi_* \eta \) has potentials bounded from above.

Proof Let \( \pi_* \eta = \eta' + dd^c u \) for some smooth form \( \eta' \) and \( u \in L^1(X) \). We claim that \( u \) is bounded from above. Indeed,

\[
\pi^* \pi_* \eta = \pi^* \eta' + dd^c (u \circ \pi) = \eta + E,
\]

where \( E \) is an \( \mathbb{R} \) divisor supported in \( \mathcal{E}(\pi) \). Since

\[
0 \leq \langle \eta, C \rangle = \langle -E, C \rangle
\]

for every \( \pi \)-exceptional curve \( C \), by the Negativity Lemma \( E \) is an effective divisor. Hence, \( u \circ \pi \) is qpsh on \( Z \) and bounded from above. Thus, so is \( u \). \( \square \)

Proposition 5.5 Let \( X \) be a projective manifold and \( f : X \dashrightarrow X \) be a dominant rational map. Let \( \theta \) be a smooth closed real \((1,1)\) form on \( X \) such that \( \langle \theta, C \rangle \geq 0 \) for every curve \( C \subset E_f := f(I_f) \). Then the potentials of \( f^* \theta \) are bounded from above.

Proof We write \( f^* \theta = (\pi_1)_* (\pi_2)^* \theta \), where \( (\pi_2)^* \theta \) is a smooth form on the desingularization of the graph of \( f \), \( \tilde{\Gamma} \subset X \times X \). Notice that for any \( \pi_1 \)-exceptional curve \( C \subset \tilde{\Gamma} \), \( \pi_2(C) \) is either a point in \( X \) or a curve in \( E_f \). Thus, we have

\[
\langle \pi_2^* \theta, C \rangle = \langle \theta, \pi_2(C) \rangle \geq 0.
\]

Then, applying Lemma 5.4 with \( \eta = (\pi_2)^* \theta \), the assertion follows. \( \square \)

For a convex cone \( \mathcal{C} \) in a finite dimensional vector space \( V \), we define the dual cone \( \mathcal{C}^\vee \) to be the set of linear forms in \( V^* \) that have non-negative values on \( \mathcal{C} \). By the Hahn–Banach theorem, we have \( \mathcal{C}^{\vee \vee} = \overline{\mathcal{C}} \).

Theorem 5.6 Let \( X \) be a projective manifold, and \( f : X \dashrightarrow X \) be a 1-regular dominant rational map. We assume that \( \lambda_1(f) > 1 \) is a simple eigenvalue of \( f^* \) and \( \alpha \) denotes the corresponding eigenvector. If \( \alpha \cdot C \geq 0 \) for every curve \( C \subset E_f \), then for every smooth representative \( \theta \in \alpha \), we have

\[
T_\alpha = \lim_{n \to \infty} \frac{1}{\lambda^n} (f^n)^* \theta
\]

exists. Moreover, \( T_\alpha \) is a positive closed \((1,1)\) current such that \( f^* T_\alpha = \lambda T_\alpha \).

Proof We will sketch the proof: let \( \theta \in \alpha \) be a smooth representative. Then by the \( dd^c \)-lemma, we have

\[
\frac{1}{\lambda} f^* \theta = \theta + dd^c \gamma.
\]
and by Proposition 5.5, \( \gamma \in L^1(X) \) is bounded from above. Thus, we may assume that \( \gamma \leq 0 \). Iterating this equation, we get

\[
\frac{1}{\lambda^n}(f^n)^* \theta = \theta + dd^c \gamma_n,
\]

where

\[
\gamma_n = \sum_{j=0}^{n-1} \frac{1}{\lambda^j} \gamma \circ f^j.
\]

Then \( \{\gamma_n\} \) is a decreasing sequence in \( L^1(X) \). It follows from Sibony’s argument [28] that \( \{\gamma_n\} \geq \phi \) for some qpsh function \( \phi \). Thus, \( \gamma_n \to \gamma_\infty \) for some \( \gamma_\infty \in L^1(X) \). Therefore,

\[
T_\alpha := \theta + dd^c \gamma_\infty
\]
defines a closed \((1, 1)\) current. It follows from the continuity of \( f^* \) that \( f^* T_\alpha = \lambda T_\alpha \).

It remains to show that \( T_\alpha \) is positive. We will follow the arguments from [2] and [11]. It is enough to show that for every smooth cutoff function \( \chi \) supported in a coordinate chart \( U \subset X \) and positive \((k-1, k-1)\) form \( \sigma \) that is constant relative to the coordinates on \( U \),

\[
\langle T, \chi \sigma \rangle \geq 0.
\]

Since \( \lambda \) is simple, it follows from [2, Lemma 1.3] that the sequence \( \{\frac{1}{\lambda^n}(f^n)_*(\chi \sigma)\} \) has weak limit points that are positive and closed. Moreover, since \( (f_*)|_{H^{k-1,k-1}(X, \mathbb{R})} \) preserves classes, these limit points belong to the dual of the psef cone. Thus,

\[
\langle T_\alpha, \chi \sigma \rangle = \lim_{n_k \to \infty} \langle \frac{1}{\lambda^{n_k}}(f^{n_k})^* \theta, \chi \sigma \rangle = \langle \theta, S \rangle \geq 0,
\]

where \( S = \lim_{n_k \to \infty} \frac{1}{\lambda^{n_k}}(f^{n_k})_*(\chi \sigma) \). \( \square \)

Note that if \( \alpha \) is nef, then \( \alpha \cdot C \geq 0 \) for every curve \( C \). This is the case when \( X \) is a compact Kähler surface and \( \lambda_1(f) > \lambda_2(f) \), and the corresponding results were obtained in [9] as a consequence of the so-called “push-pull formula” [10].

If there exists an irreducible curve \( C \subset E_f^- \) such that \( \alpha \cdot C < 0 \), then \( C \subset E_{\text{in}}(\alpha) \).

Thus, if \( \dim(E_{\text{in}}(\alpha) \cap E_f^-) = 0 \), then \( \alpha \cdot C \geq 0 \) for every curve \( C \subset E_f^- \).

The next result follows from Proposition 4.3.

**Corollary 5.7** Let \( f : X \to X \) and \( \alpha \) be as in Theorem 5.6. Then

\[
\frac{1}{\lambda^n} v_{\alpha}^{\text{min}} \circ f^n \to 0 \quad \text{in} \quad L^1(X).
\]
Let \( P^d \) denote the complex projective space of dimension \( d \), and for a point \( x \in P^d \)
\[
x = [x_0 : x_1 : \cdots : x_d]
\]
denotes the homogenous coordinates on \( P^d \). For a subset \( I \subset \{0, 1, \ldots, d\} \), we denote its complement by \( \hat{I} := \{0, 1, \ldots, d\} - I \) and its cardinality by \( |I| \). We also define the sets
\[
D_I := \{ [x_0 : \cdots : x_d] \in \mathbb{P}^d : x_i = x_i' \text{ for every } i, i' \in \hat{I} \}.
\]
In particular, if \( I = \{i\} \), then we set \( D_i := D_{\{i\}} \), which is a complex line. We also denote
\[
\Sigma_I := \{ [x_0 : \cdots : x_d] \in \mathbb{P}^d : x_i = 0 \text{ for } i \in I \}.
\]
In this section, we consider the maps of the form \( f = L \circ J : \mathbb{P}^d \rightarrow \mathbb{P}^d \), where \( J : \mathbb{P}^d \rightarrow \mathbb{P}^d \) is the involution defined by
\[
J[x_0 : x_1 : \cdots : x_d] = [x_0^{-1} : x_1^{-1} : \cdots : x_d^{-1}] = [x_0 : x_1 : \cdots : x_d]
\]
with \( x_j := \prod_{i \neq j} x_i \) and \( L \) is a linear map given by a \((d + 1) \times (d + 1)\) matrix of the form
\[
L = \begin{bmatrix}
  a_0 - 1 & a_1 & a_2 & \cdots & a_d \\
  a_0 & a_1 - 1 & a_2 & \cdots & a_d \\
  a_0 & a_1 & a_2 - 1 & \cdots & a_d \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_0 & a_1 & a_2 & \cdots & a_d - 1
\end{bmatrix},
\]
with \( a_j \in \mathbb{C} \) and \( \sum_{j=0}^d a_j = 2 \). It follows that \( \det(L) = (-1)^d \) and \( L \) is involutive, that is, \( L = L^{-1} \) in \( \text{PGL}(d + 1, \mathbb{C}) \). A map of this form is called a Noetherian mapping in [5]. Notice that \( f \) is a birational mapping with \( f^{-1} = J \circ L \). Moreover, the indeterminacy locus is given by
\[
I_f = \bigcup_{|I| \geq 2} \Sigma_I.
\]
For a point \( p \in \mathbb{P}^d \), we define its orbit \( \mathcal{O}(p) \) as follows: \( \mathcal{O}(p) = \{p\} \) if \( p \in I_f \) and \( \mathcal{O}(p) = \{p, f(p), f^2(p), \ldots, f^{N-1}(p)\} \) if \( f^j(p) \notin I_f \) for \( 0 \leq j \leq N - 2 \) and \( f^{N-1}(p) \in I_f \) for some \( N \in \mathbb{N} \); otherwise, \( \mathcal{O}(p) = \{p, f(p), f^2(p), \ldots\} \). If \( \mathcal{O}(p) \) is finite with \( f^{N-1}(p) \in I_f \), then we say that \( p \) has a singular orbit of length \( N \); otherwise, we say that it has a non-singular orbit.

A hypersurface \( H \) is called exceptional if \( \dim f(H - I_f) < d - 1 \). The only exceptional hypersurfaces of \( f \) are of the form
\[
\Sigma_i := \{ [x_0 : \cdots : x_d] \in \mathbb{P}^d : x_i = 0 \}.
\]
In fact, \( p_i := f(\Sigma_i - I_f) \) is the \( i \)-th column of the matrix \( L \). It follows from Theorem 5.3 that \( f \) is 1-regular if and only if \( f^n(\Sigma_i - I_f) \not\subset I_f \) for every \( n \geq 1 \) and \( i \in \{0, 1, \ldots, d\} \). We denote the orbit of \( \Sigma_i \) by \( O_i := O(p_i) \). Then it is easy to see that the orbit \( O_i \) is given by \( p_i, j = [1 : \cdots : 1 : \frac{f(a_i - 1)}{j_{a_i - (j-1)}} : 1 : \cdots : 1] \) for \( j = 1, 2, \ldots \) and \( O_i \) is contained in the complex line \( D_i \). Thus, \( O_i \cap O_j = \emptyset \) for \( i \neq j \). In particular, \( O_i \) is singular if and only if \( a_i = \frac{N_i - 1}{N_i} \) for some \( N_i \in \mathbb{N}_+ \), and in this case \( f^{N_i - 1}(p_i) = e_i := [0 : \cdots : 1 : 0 : \cdots : 1] \) where 1 is in the \( i \)-th component. Let \( O_i = \{ p_{i,j} \}_{j=1}^{N_i} \) be a singular orbit; then we denote its length by \( |O_i| := N_i \).

We define the set \( S := \{ i : O_i \text{ is singular} \} \) and we also set \( O_S := \bigcup_{i \in S} O_i \). By conjugating \( f \) with an involution, without lost of generality we may assume that \( S = \{0, 1, \ldots, k \} \) with \( N_0 \leq N_1 \leq \cdots \leq N_k \), where \( 0 \leq k \leq d + 1 \), and we define \( l \) by \( l := \# \{ i \in S : a_i = 0 \} \) if the latter set is non-empty; otherwise, we set \( l = 0 \).

Let \( \pi : X \rightarrow \mathbb{P}^d \) be the complex manifold obtained by blowing up the points in the set \( O_S \) successively. Then \( f \) induces a birational map \( f_X : X \dashrightarrow X \).

We denote the exceptional fiber over the point \( p_{i,j} \in O_S \) by \( P_{i,j} := \pi^{-1}(p_{i,j}) \).

We also define the class \( H_X := \pi^* H \), where \( H \subset \mathbb{P}^d \) is the class of a generic hyperplane and let \( P_{i,j} \) denote the class of exceptional divisors over \( p_{i,j} \). Then \( \{ H_X, P_{0,1}, P_{0,2}, \ldots, P_{k,N_k} \} \) forms a basis for \( H^{1,1}(X, \mathbb{R}) \), and the action of \( f_X^* \) on \( H^{1,1}(X, \mathbb{R}) \) is given by

\[
\begin{align*}
    f_X^*(H_X) &= dH_X - (d - 1) \sum_{i \in S} P_{i,N_i} \quad \text{(6.2)} \\
    f_X^*(P_{i,j+1}) &= P_{i,j} \quad \text{for } 1 \leq j \leq N_i - 1 \quad \text{(6.3)} \\
    f_X^*(P_{i,1}) &= \{ \widetilde{\Sigma}_j \} = H_X - \sum_{j \in S, j \neq i} P_{j,N_j} \quad \text{for } 1 \leq j \leq l \quad \text{and } l \leq k \quad \text{(6.4)}
\end{align*}
\]

where \( \widetilde{\Sigma}_j \subset X \) denotes the strict transform of \( \Sigma_j \subset \mathbb{P}^d \) (see [1, Sect. 3] for details).

**Theorem 6.1** [1] Let \( f_X : X \dashrightarrow X \) be as above. Then \( f_X \) is 1-regular and the characteristic polynomial of \( f_X^* \) is given by

\[
\chi(x) = (x - 1)^l \left[ (x - (d - l)) \prod_{j=l}^{k} (x^{N_j} - 1) + (x - 1) \sum_{j=l}^{k} \prod_{i \neq j}^{k} (x^{N_i} - 1) \right].
\]

Moreover, if \( S \neq \emptyset \) and

\[
d - l - 3 \geq 0, \quad \text{(6.5)}
\]

then \( d - l - 1 \leq \lambda := \lambda_1(f) \leq d \) is the unique eigenvalue of \( f_X^* \) of modulus greater than 1 and is a simple root of \( \chi(x) \).

In the sequel, we will assume that \( d \geq 3 \) and \( f_X \) is as in Theorem 6.1, so that (6.5) holds. We denote the corresponding eigenvector by \( \alpha_f \in H^{1,1}(X, \mathbb{R}) \) with \( f_X^* \alpha_f =
\( \lambda \alpha_f \) and we normalize it so that

\[
\alpha_f = H_X - c \cdot E,
\]

where \( c = (c_{0,1}, c_{0,2}, \ldots, c_{k,N_k}) \) and \( E = (P_{0,1}, P_{0,2}, \ldots, P_{k,N_k}) \).

**Lemma 6.2** Let \( \alpha_f = H_X - c \cdot E \) be as above. Then for every \( 0 \leq i \leq k \),

1. \( c_{i,j+1} = \lambda c_{i,j} \) for \( 1 \leq j \leq N_i - 1 \)
2. \( \sum_{i=0}^{k} c_{i,1} = d - \lambda \)
3. \( c_{i,1} = \frac{\lambda - 1}{\lambda N_i - 1} > 0 \)
4. \( \sum_{j=1}^{N_i} c_{i,j} = 1 \).

**Proof** (1) and (2) follow from the invariance of \( \alpha_f \) and (6.2)–(6.4).
(3) For fixed \( i \), we compare the coefficient of \( P_{i,N_i} \) on both sides of \( f^* \alpha_f = \lambda \alpha_f \) and obtain

\[
(d - 1) - \sum_{j=0 \atop j \neq i}^{k} c_{j,1} = \lambda c_{i,N_i}.
\]

Then by (1) and (2) we get

\[
c_{i,1} = \frac{\lambda - 1}{\lambda N_i - 1}.
\]

Item (4) follows from (1) and (3). \( \square \)

**Proposition 6.3** The class \( \alpha_f \in H^{1,1}_{\text{nef}}(X, \mathbb{R}) \) if and only if \( |S| \leq 1 \).

**Proof** If \( S = \emptyset \), then \( X = \mathbb{P}^d \) and \( \alpha_f = \{\omega_{FS}\} \), which is Kähler.

Assume that \( |S| = 1 \). Then \( \mathcal{O}_0 \) is singular, and the orbit is

\[
\Sigma_0 \to p_1 \to \cdots \to p_N = e_0.
\]

Let \( H_i \subset \mathbb{P}^d \) denote a hyperplane such that \( p_i \in H_i \) and \( p_j \notin H_i \) for \( j \neq i \). Notice that \( H_i \)'s form a \((d - 1)\)-parameter family of hyperplanes. Since \( \{\widehat{H}_i\} = H_X - P_i \), by Lemma 6.2 we can represent the class \( \alpha_f \) as the class of effective divisors \( \sum_{i=1}^{N} c_i \widehat{H}_i \), where \( \widehat{H}_i \) is the strict transform of \( H_i \). Since \( \sum_{i=1}^{N} c_i [\widehat{H}_i] \in \alpha_f \) defines a positive closed \((1, 1)\) current, we infer that \( \nu(T^\text{min}_\alpha, x) = 0 \) for every \( x \in X \). Thus, it follows from Proposition 2.4 that \( \alpha_f \) is nef.

Now, we will prove that if \( |S| \geq 2 \), then \( \alpha_f \) is not nef. Indeed, let \( \mathcal{O}_{l_1} \) and \( \mathcal{O}_{l_2} \) be two singular orbits; then by Lemma 6.2 and Theorem 6.1,

\[
c_{ij, N_j} > 1 - \frac{1}{\lambda} \geq 1 - \frac{1}{d - l - 1} \geq \frac{1}{2} \quad \text{for } j = 1, 2.
\]
Let $\ell$ denote the complex line passing through the points $e_{i_1}$ and $e_{i_2}$, and let $\tilde{\ell}$ be its strict transform in $X$. Then

$$\alpha_f \cdot \tilde{\ell} = 1 - c_{i_1,N_{i_1}} - c_{i_2,N_{i_2}} < 0.$$ 

Hence, by Proposition 2.9 we get $\tilde{\ell} \subset E_{\text{nn}}(\alpha_f)$.

Let $\Sigma_l \subset \mathbb{P}^d$ be as above; we also write $\Sigma_l$ for its strict transform inside $X$.

**Proposition 6.4** If $1 \leq k \leq d - 1$ and $2 \leq N := N_i$ for every $0 \leq i \leq k$, then

$$E_{\text{nn}}(\alpha_f) = \left\{ \begin{array}{ll}
\Sigma_{(k+1,\ldots,d)} & \text{if } k \leq d - 2, \\
\bigcup_{i=0}^k \Sigma_{[i,d]} & \text{if } k = d - 1.
\end{array} \right.$$ 

In particular, $1 \leq \dim C E_{\text{nn}}(\alpha_f) \leq (d - 2)$ and $E_{\text{nn}}(\alpha_f) \subset I_{f_X}$ is algebraic.

**Proof** It follows from Lemma 6.2 that $c_{i_1,l} = c_{i_2,l}$ for all $i_1, i_2 \in S = \{0, \ldots, k\}$ and $1 \leq l \leq N$. We denote $c_i := c_{i,l}$ for $i \in \{0, \ldots, k\}$.

If $|S| = 2$, then $\Sigma_{[2,\ldots,d]}$ is a complex line, and in the proof of Proposition 6.3 we have already showed that $\Sigma_{[2,\ldots,d]} \subset E_{\text{nn}}(\alpha_f)$.

Assume that $3 \leq |S| = k + 1 \leq (d - 1)$ and let $p \in \Sigma_{[k+1,\ldots,d]} \cong \mathbb{P}^k$ be a point. Let $\gamma \subset \Sigma_{[k+1,\ldots,d]}$ be an algebraic irreducible curve of degree $k$ such that $p, \gamma \in \gamma$ for every $0 \leq i \leq k$. Then by Lemma 6.2 and by (6.5) we have $c_N > 1 - \frac{1}{d-1}$ and

$$\alpha_f \cdot \gamma = k - (k+1)c_N < \frac{k+1}{d-1} - 1 \leq 0.$$ 

Thus, by Proposition 2.9 we get $\Sigma_{[k+1,\ldots,d]} \subset E_{\text{nn}}(\alpha_f)$.

If $k = d - 1$, we can apply the same argument to $\{0, \ldots, d - 1\} - \{i\}$ for $0 \leq i \leq d - 1$.

To prove the reverse inclusion, we will represent the class $\alpha_f$ by effective divisors: Notice that each $p_{l,i} = [1 : \cdots : 1 : \frac{l(a_i-1)}{l(a_i-1)}} : 1 : \cdots : 1] \in D_i$, which is a complex line. Let $H_l \subset \mathbb{P}^d$ be a hyperplane such that $p_{l,i} \in H_l$ for every $0 \leq i \leq k$ and $p_{l,m} \notin H_l$ for $m \neq l$. This is a $d - k - 1$ parameter family of hyperplanes for each $l$. Then, the class $\{\widetilde{H_l}\} = H_X - \sum_{i=0}^k P_{l,i}$, where $\widetilde{H_l}$ denotes the strict transform of $H_l$. Hence, by Lemma 6.2 we can represent $\alpha_f$ by

$$\alpha_f = \sum_{l=1}^N c_l \{\widetilde{H_l}\}. \quad (6.6)$$

Next, we assume that $k \leq d - 2$. We consider the hyperplanes of the form $D_i = \{x \in \mathbb{P}^d : 2x_i - x_{d-1} - x_d = 0\}$, where $0 \leq i \leq k$ is fixed. Then the complex line $D_j \subset D_i$ for $0 \leq j \neq i \leq k$ and $O_i \cap D_i = \emptyset$. Thus, $\{\widetilde{D_i}\} = H_X - \sum_{0 \leq j \neq i \leq k} P_{j,i}$.

We also denote $H_{\Sigma_{(k+1,\ldots,d)}} \subset \mathbb{P}^d$ to be a hyperplane containing $\Sigma_{[k+1,\ldots,d]}$ such that
\{H_{\Sigma_{[k+1,\ldots,d]}}\} = H_X - \sum_{i \in S} P_{i,N} \text{ (e.g., } H_{\Sigma_{[k+1,\ldots,d]}} = \Sigma_j \text{ for some } k + 1 \leq j \leq d).\) Then by Lemma 6.2,

\begin{equation}
\alpha_f = \sigma \left( \sum_{i=0}^{k} \{ \tilde{D}_i \} + (1 - \sigma (k + 1)) \{ \tilde{H}_{\Sigma_{[k+1,\ldots,d]}} \} \right) + \mathcal{E}, \tag{6.7}
\end{equation}

where \(\sigma = 1 - c_N\) and \(\mathcal{E}\) is an effective divisor supported on \(\bigcup_{i \in [0,\ldots,k]} P_{i,l}\). Indeed, it follows from Lemma 6.2 that

\[1 - \sigma (k + 1) = \frac{d - k - 1}{\lambda} > 0.\]

On the other hand, we can also represent \(\alpha_f\) as follows: let \(i_1, i_2 \in S\); then

\begin{equation}
\alpha_f = \sigma \left( \{ \tilde{D}_{i_1} \} + \{ \tilde{D}_{i_2} \} \right) + (1 - 2\sigma) \{ \tilde{H}_{\Sigma_{[k+1,\ldots,d]}} \} + \mathcal{E}', \tag{6.8}
\end{equation}

where \(\mathcal{E}'\) is an effective divisor supported on \(\bigcup_{i \in [0,\ldots,k]} P_{i,l}\) and \(\sigma\) is as above. Since the non-nef locus is contained in the intersection of the supports of the effective divisors in (6.6), (6.7), and (6.8), we conclude that

\[E_{en}(\alpha_f) \subset \Sigma_{[k+1,\ldots,d]}.\]

If \(k = d - 1\), then we claim that \(c_N = \frac{d-1}{d}\). Indeed, by Lemma 6.2(2) \(c_1 = \frac{(d-\lambda)}{d}\), and by using Lemma 6.2(3) we get \(\lambda^{N-1} = \frac{d-1}{d-\lambda}\). Then by Lemma 6.2(1) we get \(c_N = \lambda^{N-1} c_1 = \frac{d-1}{d}\). This implies that we can represent \(\alpha_f\) as

\begin{equation}
\alpha_f = \frac{1}{d} \sum_{i=0}^{d-1} \{ \tilde{L}_i \} + \mathcal{E}, \tag{6.9}
\end{equation}

where \(\tilde{L}_i = \{ x \in \mathbb{P}^d : x_i - x_d = 0 \}\) and \(\mathcal{E}\) is an effective divisor supported on \(\bigcup_{0 \leq i \leq d-1} P_{i,l}\). Now, for fixed \(0 \leq j \leq d - 1\) we also have

\begin{equation}
\alpha_f = \frac{1}{d} \sum_{i=0}^{d-1} \{ \tilde{L}_i \} + \frac{1}{d} \{ \tilde{F}_j \} + \mathcal{E}', \tag{6.10}
\end{equation}

where \(\tilde{F}_j = \{ x \in \mathbb{P}^d : x_j - ax_d = 0 \}\) is a 1-parameter family of hyperplanes and \(\mathcal{E}'\) is an effective divisor supported on \(\bigcup_{0 \leq i \leq d-1} P_{i,l}\). Hence, by (6.6), (6.9), and (6.10) we get

\[E_{en}(\alpha_f) \subset \bigcup_{i=0}^{d-1} \Sigma_{[i,d]}.\]

Now, we prove that a generic mapping of the form \(f = L \circ J\) falls into framework of Theorem 1.2.
Proof of Theorem 1.5 If $|S| \leq 1$, then the assertion follows from Proposition 6.3 and Theorem 4.4.

We assume that $|S| \geq 2$ and set $S = \{0, \ldots, k\}$. By Theorem 5.6 and Corollary 5.7, it is enough to show that $\alpha_f \cdot C \geq 0$ for every algebraic irreducible curve $C \subset E_{fX}^-$. Since $f_X$ is biholomorphic near the exceptional fibers $P_{i,j}$’s, the indeterminacy locus is given by

$$I_{fX} = \bigcup_{|I| \geq 2} \Sigma_I.$$ 

This implies that $f_X(I_{fX}) \subset \bigcup_{i=0}^{d} L(\Sigma_i)$, where $L(\Sigma_i) = \{x \in \mathbb{P}^d : a \cdot x - x_i = 0\}$ and $a = [a_0 : \cdots : a_d]$. Since $p_{i,j} = [1 : \cdots : j(a_i-1) : \cdots : 1]$, we infer that

$$\bigcup_{i=0}^{d} L(\Sigma_i) \cap O_S = \{p_{0,1}, p_{1,1}, \ldots, p_{k,1}\}.$$ 

By Lemma 6.2, for any algebraic irreducible curve $C \subset E_{fX}^-$,

$$\alpha \cdot C \geq \deg C - \sum_{i=0}^{k} c_{i,1}(\text{mult}_{p_{i,1}} C) \geq \deg C(1 - (d - \lambda)) \geq 0,$$

where the last inequality follows from Theorem 6.1.

\[\square\]

Theorem 6.5 Let $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be as above. If $|S| \leq 3$ and $2 \leq N := N_i$ for every $i \in S$ then there exists a birational model $\mu : Y \to \mathbb{P}^3$ such that $f_Y : Y \dashrightarrow Y$ is a dominant 1-regular map with $\lambda := \lambda_1(f_Y)$ is the unique simple eigenvalue of modulus greater than 1 with the corresponding normalized eigenvector $\widetilde{\alpha_f} \in H_{\text{nef}}^{1,1}(Y, \mathbb{R})$.

Proof If $|S| \leq 1$, then the assertion follows from Theorem 6.1 and Proposition 6.3.

We assume that $S = \{0, 1\}$; then we define the complex manifold $Y$ to be $X$ blown up along $E_{\text{nn}}(\alpha_f) = \Sigma_{23}$, which is a complex line. We denote the projection by $\mu : Y \to X$ and the exceptional divisor by $\mathcal{F} := \mu^{-1}(E_{\text{nn}}(\alpha_f))$.

We first show that the induced map $f_Y : Y \dashrightarrow Y$ is 1-regular. Notice that the only exceptional hypersurfaces of $f_Y$ are $\Sigma_i$ for $i \not\in S$ and $\mathcal{F}$. Since $f_Y^n(\Sigma_i - f_Y^n) \not\subset I_{f_Y}$ for $n \geq 1$ and $i \not\in S$, by Theorem 5.3 it is enough to check that $f_Y^n(\mathcal{F}) \not\subset I_{f_Y}$ for every $n \in \mathbb{N}$.

We claim that $f_Y(\mathcal{F} \setminus I_{f_Y}) = \widetilde{\Sigma_S}$. Indeed, we write $f_Y$ in the local coordinates: $(\eta_1, \eta_2, s) \in Y$, where $\mathcal{F} = \{s = 0\}$ and

$$\pi_Y : Y \to \mathbb{P}^3$$

$$\mu(\eta_1, \eta_2, s) = [1 : \eta_1 : \eta_2 s : s].$$
Then, we may identify
\[ f_Y(\eta_1, \eta_2, 0) = \eta_1[a_2 : a_2 : a_2 - 1 : a_2] + \eta_1\eta_2[a_3 : a_3 : a_3 : a_3 - 1], \]

which proves the claim. Since the points \([a_2 : a_2 : a_2 - 1 : a_2]\) and \([a_3 : a_3 : a_3 - 1]\) have non-singular orbits, we conclude that \(f_Y\) is 1-regular. Similarly, one can show that \(f_Y^{-1}(\mathcal{F}\setminus I_{f_Y}) = J(\text{span}([a_0 - 1 : a_0 : a_0], [a_1 - 1 : a_1 : a_1]))\), where the latter set has codimension 2.

Now, \(\{H_Y, P_{0,1}, P_{0,2}, \ldots, P_{1,N}, \mathcal{F}\}\) forms an ordered basis for \(H^{1,1}(X, \mathbb{R})\), where \(H_Y := \mu^*(H_X)\) and \(P_{i,l} := \mu^*(P_{i,l})\) for each \(i, 1 \leq l \leq N\), and the action of \(f_Y^* : H^{1,1}(Y) \to H^{1,1}(Y)\) is given by
\[
\begin{align*}
    f_Y^*(H_Y) &= 3H_X - 2P_{0,N} - 2P_{1,N} - \mathcal{F} \\
    f_Y^*(P_{i,l+1}) &= P_{i,l} \quad \text{for } 1 \leq l \leq N - 1 \text{ and } i \in S \\
    f_Y^*(P_{i,1}) &= \Sigma_i \quad \text{for } i \in S \\
    f_Y^*(\mathcal{F}) &= 0,
\end{align*}
\]

where \(\Sigma_i \subset Y\) denotes the strict transform of \(\Sigma_i \subset \mathbb{P}^3\). Thus, the characteristic polynomial of \(f_Y^*\) is given by \(p(x) = x\chi(x)\), where \(\chi(x)\) is as in Theorem 6.1. This implies that \(\lambda = \lambda_1(f_Y)\) is a simple eigenvalue. Moreover, the corresponding eigenvector \(\tilde{\alpha}_f\) is of the form
\[
    \tilde{\alpha}_f = H_Y - c \cdot E - \frac{1}{\lambda} \mathcal{F},
\]

where \(c\) and \(E\) are as in Lemma 6.2.

Now, we claim that \(\tilde{\alpha}_f\) is nef. Indeed, it follows from Lemma 6.2 that
\[
    1 - 2\sigma = \frac{1}{\lambda},
\]
and by the representations (6.6), (6.7), and (6.8), we infer that \(\nu(T_{\tilde{\alpha}_f}^{\min}, y) = 0\) for every \(y \in Y\). Hence, the claim follows.

If \(|S| = 3\) then by Proposition 6.4 \(E_{nm}(\alpha_f)\) has 3 components that are pairwise disjoint complex lines in \(X\). In this case, we define the complex manifold \(Y\) to be \(X\) blown up along each component of \(E_{nm}(\alpha_f)\) successively, and apply the above analysis to drive the assertion. We omit the details of this part. \(\square\)

**Acknowledgements**

I would like to express my sincere thanks to my advisor E. Bedford for his guidance and interest in this work. I am also grateful to J. Diller and V. Guedj for their comments and suggestions on an earlier draft. I also thank the referee for his suggestions.

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