GENERAL CONCEPTS OF GRAPHS

Dedicated to the 65th anniversary of Professor Ulrich Knauer

Sheng Bau
Center for Discrete Mathematics,
Fuzhou University, Fuzhou, China
dimacs1@fzu.edu.cn

and

School of Mathematical Sciences,
Government College University, Lahore, Pakistan

Abstract. A little general abstract combinatorial nonsense delivered in this note is a presentation of some old and basic concepts, central to discrete mathematics, in terms of new words. The treatment is from a structural and systematic point of view. This note consists essentially of definitions and summaries.

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There had been a general trend of generalization throughout 19th and the first half of the 20th centuries in mathematics. Then in the second half of the 20th century, a prevailing trend of specialization occurred. The latter trend is characteristic of almost exclusive emphasis on the immediate solutions of specific problems, especially if they were openly proposed by another mathematician, usually a famous one. This was partly encouraged by the ad hoc aim of seeking academic excellence by journals and by their authors. If a paper is on some systematic ground work which entails in proper generalization and exposure of important relations between fundamental mathematical concepts, it might easily be mistaken as “no novelty” and will be appropriated an instant rejection, sometimes even without a proper refereeing procedure.

This note consists only of definitions of basic concepts. With these, I wish to emphasize that mathematics is a unity. Individual activities and areas of activity are related in a vitally organic manner. A monopoly of dissection of the body of mathematics is clearly not always beneficial to the health and life of mathematics.

In support of the view that mathematics is an organic unity and many branches of our science are vitally related, I would like to point out the intensive and extensive interactions between algebra and combinatorics as in [20, 21] and the literature therein, between probability and combinatorics ([11] and references therein), between topology and combinatorics [22, 36], and more recently, between analysis and combinatorics [38]. Mathematics is, after all, not an exclusive instrument of some small number of executives in the community for ad hoc academic excellence.

The concepts of graphs and some of their generalizations are included in Section 2. These will be specialized to algebraic objects in Section 5. Different types of categories of graphs will be reviewed in Section 3 while the concept of graph invariants will be clarified in Section 4. Inductive sets arise naturally as graphs in Section 6. Transformation graphs arise naturally in mathematics. Generality of the concept of transformation graphs will be considered in Section 8.

2. GRAPHS AND RELATIONAL SYSTEMS

**Definition 2.1.** Let $V$ and $E$ be sets with $V \cap E = \emptyset$. Then a mapping $G : E \to V \times V$ is called a graph. This may be given by

$$E \xrightarrow{G} V \times V$$
This definition captures the essence of the concept of a graph precisely and with proper generality, as it includes the concepts of both finite and infinite, both simple and nonsimple, both directed and undirected graphs.

A special case, when $G$ is an injective mapping, of Definition 2.1 may be presented in terms of a binary relation.

**Definition 2.2.** Let $V$ be any set and $E \subseteq V \times V$ be a binary relation over $V$. Then the ordered tuple $G = (V, E)$ is called a graph.

A concept of this generality is native to mathematics. A directed graph with multiple edges that are weighted may be presented by

$$ W \xrightarrow{w} E \xrightarrow{G} V \times V $$

or

$$ W \\
\downarrow{w} \\
(V, E) $$

where $W \subseteq \mathbb{R}$ and $w$ is a nonnegative function.

If $V$ is finite then $G$ is called finite; if $E$ is irreflexive, that is, if for each $a \in V$, $(a, a) \notin E$, then $G$ is loopless; if $E$ is symmetric then the tuples in $E$ may be considered subsets of cardinality 2 and hence $G$ is undirected; if $E$ is antisymmetric then $G$ is oriented; if $E$ is antisymmetric and transitive then $G$ is a partial order, an immediate specialization.

**Proposition 2.1.** Every partially ordered set is an oriented graph.

Hence, every lattice is an oriented graph.

Denote

$$ D(V) = \{(a, a) : a \in V\} $$

and call $D(V)$ the diagonal of $\times$, as usual. Note that the diagonal is also a binary relation on $V$. Let $E$ be a symmetric binary relation on $V$. The symmetric closure $s(E)$ of $E$ is

$$ s(R) := E \cup \{(b, a) : (a, b) \in E\}. $$

Then the natural projection

$$ p_s : s[V \times V - D(V)] \to [V]^2 := \binom{V}{2} $$

of the symmetric closure forgets the order of ordered tuples and maps ordered tuples to subsets of cardinality 2. If

$$ G : E \to [V]^2 $$
is a bijection, then $G$ is called the complete graph of order $|V|$ and is denoted $G = K_{|V|}$. Any simple graph $G$ with $V(G) = V$ is a spanning subgraph of $K_{|V|}$ since

$$E(K_{|V|}) \xrightarrow{K_{|V|}^{-1}G} E(G) \xrightarrow{G} [V]^2$$

Since $K_{|V|}$ is a bijection, and $G$ is injective, hence

$$K_{|V|}^{-1} \cdot G : E(G) \rightarrow E(K_{|V|})$$

is an injective mapping (i.e., an embedding). That is, the spanning subgraph

$$G = K_{|V|} \cdot K_{|V|}^{-1} \cdot G.$$

For a graph $G$, the morphism underlying undirected graph provides a forgetful functor from the category of graphs to the category of undirected graphs, presented by the following diagram. The forgetful functor $U(G)$ forgets the directions of edges. If a graph is denoted $\overline{G}$ then its underlying undirected graph $U(\overline{G})$ may be denoted by $G$.

$$E \xrightarrow{\overline{G}} V \times V \xrightarrow{\overline{p_s}} D(V) \cup [V]^2$$

where $p'_s$ is an extension of $p_s$ over to $D(V) \cup [V]^2$. A simple graph may be presented by

$$E \xrightarrow{G} [V]^2.$$  

The following diagram presents mappings of edges of a graph to their heads ($p_2G$) and tails ($p_1G$).

$$V \xrightarrow{p_1} V \times V \xrightarrow{G} E \xrightarrow{p_2} V$$

This gives information about incidence. The incidence matrix $M_G$ of a loopless graph $G$ is given by

$$M_G : V \times E \rightarrow \{-1, 0, 1\}.$$
Thus the incidence matrix is a matrix whose rows are indexed by $V$, columns by $E$ and entries in $\{-1, 0, 1\}$ which reveals the manner of incidence of $v \in V$ and $e \in E$ as given below.

$$M_G(v, e) = \begin{cases} -1, & G(e) = (v, w), w \in V \\ 0, & v \not\in e \\ 1, & G(e) = (u, v), u \in V \end{cases}$$

Let $X \subseteq E$ where $\iota : X \to E$ is the inclusion. Then the subgraph $G|_X \subseteq G$ induced by $X$ is presented by the diagram

$$
\begin{array}{c}
X \\
\downarrow \iota \\
G|_X
\end{array} \quad \begin{array}{c}
E \\
\downarrow G \\
V \times V
\end{array}

That is, $G|_X = G \cdot \iota$. For $S \subseteq V$ with inclusion $\eta : S \to V$, the subgraph $G|_S$ induced by $S$ is presented as

$$
\begin{array}{c}
E \\
\downarrow G \\
V \times V
\end{array} \quad \begin{array}{c}
S \times S \\
\downarrow \eta \times \eta \\
G^{-1}(S \times S)
\end{array}

where $\eta_{S \times S} : S \times S \to V \times V$ is the inclusion mapping induced by $\eta$, and $\eta' : G^{-1}(S \times S) \to E$ is the natural inclusion. The diagram commutes: $\eta_{S \times S} \cdot G|_S = G \cdot \eta'$.

In this note, some further generalization will be considered before moving to the points of specializations. There are obviously two directions in which the concept of a graph as given in Definition 2.1 may be generalized, generalization at the head or at the tail of the arrow in the diagram.

**Definition 2.3.** Let $V$ be a set and let $E = (E_1, \cdots, E_s)$ be a collection of sets. If

$$G = (G_1, \cdots, G_s)$$

and

$$G_i : E_i \to V \times V, \ i = 1, \cdots, s$$

then

$$G : (E_1, \cdots, E_s) \to V \times V$$
is called a *graph system*. A graph system may be presented by the diagram

\[
\begin{array}{c}
E_1 \\
\downarrow g_1 \\
\vdots \\
V \times V \\
\downarrow g_s \\
E_s
\end{array}
\]

This seems to be an obvious generalization of the concept of a graph at its gener-
ality as given in Definition 2.1 though, as may be seen directly from the diagram,
it may be fully realized as a collection of graphs over the same set \( V(G) = V \).

The second generalization is at the head of the arrow in the diagram of Definition 2.1.

Denote

\[ V^m := V \times \cdots \times V \]

**Definition 2.4.** Let \( V \) and \( E \) be sets.

\[ G : E \to V \times \cdots \times V \]

is called a *set system* or a *hypergraph*. A set system (hypergraph) may be presented
by the diagram

\[ E \xrightarrow{G} V^m \]

This is a proper generalization. Note that this is more general than the usual a
set system or a hypergraph since at the head of the arrow is a cartesian product,
instead of

\[ \bigcup_{k=1}^{m} [V]^m := \{ S \subseteq V : |S| \leq m \} \]

For \( R \subseteq V^m \), the *symmetric closure* or \( S_m \)-*closure* \( s(R) \) is the quotient defined by
the binary relation

\[ (v_{p(1)}, v_{p(2)}, \cdots, v_{p(m)}) \sim (v_1, v_2, \cdots, v_m) \]

for all \( p \in S_m \) the symmetric group on \( \{1, \cdots, m\} \). That is, \( s(R) \) is the quotient of
the transitive action of the symmetric group \( S_m \), then the usual set system (hyper-
graph) may be presented by the diagram
The concept of a \textit{relational system} may be obtained by a generalization on both head and tail of the arrow in Definition 2.1, and insisting that the mappings concerned are injections.

**Definition 2.5.** Let $V$ be a set and let $E = (E_1, \ldots, E_s)$ be a collection of sets. If $G = (G_1, \ldots, G_s)$ and 

$$G_i : E_i \to \bigcup_{k=1}^{m}[V]^m, \quad i = 1, \ldots, s$$

then

$$G : (E_1, \ldots, E_s) \to \bigcup_{k=1}^{m}[V]^m$$

is called a \textit{relational system}. A relational system may be presented by the diagram

\[
\begin{array}{c}
E_1 \\
\downarrow G_1 \\
\vdots \\
\downarrow G_s \\
E_s
\end{array}
\quad \bigcup_{k=1}^{m}[V]^m
\]

If each $G_i$ in Definition 2.5 is injective, then we have

**Definition 2.6.** Let $V$ be any set and let $E$ be a collection of relations over $V$. Then $G = (V, E)$ is called a \textit{simple relational system}. Let $k_1 < k_2 < \cdots < k_m$ and let the number of distinct $k_i$-ary relations in $E$ be $r_i$. Then the symbol $(k_1^{r_1}, k_2^{r_2}, \ldots, k_m^{r_m})$ is called the \textit{type} of $G$. The integer $k_m$ is called the \textit{arity} of $E$ and hence of $G$.

Trivially, every hypergraph is a relational system.

Now a proper specialization of the concept of a relational system.

**Definition 2.7.** A simple relational system $G$ with arity 3 is called a \textit{ternary relational system}.
This is the definition that encompasses almost all mathematical objects. The first and the most important concept is the concept of a group. Since each binary operation is a ternary relation, every group is a ternary relational system. At the appropriate level of binary operations, algebraic objects (rings, principal ideal domains, division rings, and fields) have been studied extensively. Indeed, algebra represents one of the great successes in modern mathematics. At this point, there exists a rich possibility of specialization of the concept of relational systems, which certainly leads to an abundance of problems and questions including the investigation of graphs, groups and partially ordered sets.

The general concept of a relational system is a very recent one [25], and very little is known about relational systems, while obviously there is a rich and extensive theory in the case where these relations are operations. As a suitable generality is now at hand, this is the point where a few specializations will be considered more formally. But, before this will be dealt with in Section 5, consider a few typical categories of graphs.

3. Categories of Graphs

There are many different categories of graphs (See [27, 28, 29]). For a formal and comprehensive treatment of categories and functors, see [33]. In this section, some fundamental categories of graphs will be presented. This section is based on an excerpt from [27, 29].

**Definition 3.1.** (1) The category of all graphs with graph homomorphisms as morphisms. This category is denoted by $\mathbf{Gra}$. A homomorphism $f : G \rightarrow H$ is a mapping with $xy \in E(G) \Rightarrow f(x)f(y) \in E(H)$.

(2) The category of all graphs with egamorphisms as morphisms. This category is denoted by $\mathbf{Egra}$. An egamorphism is a mapping $f : G \rightarrow H$ such that $xy \in E(G) \Rightarrow f(x)f(y) \in E(H) \cup V(G)$.

(3) The category of all graphs with comorphisms as morphisms. This category is denoted by $\mathbf{Cgra}$. A comorphism is a mapping $f : G \rightarrow H$ such that $f(x)f(y) \in E(H) \Rightarrow xy \in E(G)$.

In each of the three categories, compositions and morphisms, respectively, obviously satisfy the categorical axioms for compositions. Note that one of many ways of defining the important concept of a contraction is that it is a preconnected egamorphism, meaning that it is an egamorphism (i.e., morphism of the category $\mathbf{Egra}$ for which the preimage of each vertex induces a connected subgraph). This will be addressed by the author in another paper.

The binary operations of graphs typically include products. Natural products in respective graph categories will now be reviewed.
Consider the category $\mathcal{S}$ of sets and mappings. Let $G_1, G_2 \in \mathcal{S}$. A pair $(G, (p_1, p_2))$ with $p_1 : G \to G_1$, $p_2 : G \to G_2$ is called (the categorical) product of $G_1, G_2$ in $\mathcal{S}$ if (1) $p_1, p_2$ are morphisms in $\mathcal{S}$; and (2) $(G, (p_1, p_2))$ solves the universal problem: for all sets $H$ and for all mappings $f_1 : H \to G_1$, $f_2 : H \to G_2$ there exists a unique mapping $f : H \to G$ such that the diagram

![Diagram](https://via.placeholder.com/150)

commutes.

**Theorem 3.1.** $(G \times G_2, (p_1, p_2))$ is the product of $G_1$ and $G_2$ in $\mathcal{S}$.

A pair $((u_1, u_2), G)$ is called the coproduct of $G_1, G_2$ in $\mathcal{S}$ if $u_1 : G_1 \to G$, $u_2 : G_2 \to G$ are mappings such that for all sets $H$ and for all mappings $f_1 : G_1 \to H$, $f_2 : G_2 \to H$ there exists exactly one mapping $f : G \to H$ such that the diagram

![Diagram](https://via.placeholder.com/150)

is commutative. Note that this diagram is obtained by reversing all arrows in the previous diagram and relabelling them.

**Theorem 3.2.** $((u_1, u_2), G_1 \cup G_2))$ is the coproduct of $G_1$ and $G_2$ in $\mathcal{S}$.

The cross product of graphs $G_1$ and $G_2$ may be defined by the requirement that for every graph $G$ and homomorphisms $f_1 : G \to G_1$ and $f_2 : G \to G_2$, there exists a unique homomorphism $f : G \to G_1 \times G_2$ so that the following diagram is commutative.

![Diagram](https://via.placeholder.com/150)

where $p_1$ and $p_2$ are natural projections (homomorphisms).

The disjunction of graphs $G_1$ and $G_2$ may be defined by the requirement that for every graph $G$ and homomorphisms $f_1 : G \to G_1$ and $f_2 : G \to G_2$, there
exists a unique homomorphism \( f : G \rightarrow G_1 \uplus G_2 \) so that the following diagram is commutative.

\[
\begin{array}{c}
\text{G} \\
\downarrow f \\
\downarrow f_1 \\
G_1 \quad \quad \quad \quad G_1 \uplus G_2 \\
\quad \downarrow p_1 \\
\quad G_2 \\
\end{array}
\]

where \( p_1 \) and \( p_2 \) are natural projections.

The strong product of graphs \( G_1 \) and \( G_2 \) is defined to be the union of their cross and cartesian products. For definitions of products, the reader may also refer to a recent monograph [26].

Many binary graph operations are interpreted categorically in [27, 29], where the following were among results established there.

(1) The cross product \( G_1 \times G_2 \) with projections is a product of \( G_1 \) and \( G_2 \) in \( \text{Gra} \).
(2) The strong product \( G_1 \boxtimes G_2 \) with projections is a product of \( G_1 \) and \( G_2 \) in \( \text{Egra} \).
(3) The disjunction with projections is a product of \( G_1 \) and \( G_2 \) in \( \text{Cgra} \).

These capture the essence of the products concerned in the respective categories.

Categories \( \text{Gra}, \text{Egra} \) and \( \text{Cgra} \) also have coproducts and tensor products [27, 29]. It was also shown in [27, 29] that products and coproducts in these three categories have right adjoints.

4. Invariants

Investigations about invariants in various fields of mathematics always concern the action of a group (usually a subgroup of the automorphism group). Combinatorics is not an exception. Let \( \mathcal{G} \) be a set of graphs and \( S \) be a set. A mapping \( f : \mathcal{G} \rightarrow S \) is called an invariant of graphs if for all \( G, H \in \mathcal{G}, G \cong H \Rightarrow f(G) \cong f(H) \). Of course, for \( S \) a set of numbers or a set of sequences of numbers, the second \( \cong \) is just \( = \). In terms of mappings, a function taking its argument as a graph \( G \) is an invariant if for each automorphism \( \varphi \) of \( G \), \( f(\varphi(G)) = f(G) \), or simply, \( f \varphi = f \), as the above is true for all graphs \( G \) in the given family. Thus, a graph invariant may
be presented by the following diagram.

\[
G \xrightarrow{f} S \\
\phi \downarrow \quad \phi \\
G 
\]

Taking into account the condition \( G \simeq H \Rightarrow f(G) \simeq f(H) \), a graph invariant for a family of graphs may be presented also by the diagram

\[
G \xrightarrow{f} S \\
\phi \downarrow \quad \phi \\
G \xrightarrow{f} S 
\]

where \( \phi \) is a graph isomorphism and \( \phi \) is an isomorphism of \( S \).

For example, (1) if \( S \) is the set of all integer sequences, then the degree sequence function \( f = d \) is an invariant, since for each automorphism \( \varphi \) of \( G \), \( d \varphi = d \); (2) if \( S = G \) is the category of all groups then the automorphism group function \( f = \text{Aut} \) is an invariant since obviously \( \text{Aut} \varphi = \text{Aut} \); (3) the determinant of the adjacency matrix is another example of an integer invariant; (4) the spectrum of a graph is an example of invariants; so also is the largest eigenvalue.

A subgraph \( H \subseteq G \) is called an invariant subgraph if \( \text{Aut}(G)(H) = H \).

If \( S \subseteq \mathbb{R} \) then the invariant \( f : G \rightarrow S \) is called a graph parameter. In particular, any integer valued invariant is an example of a graph parameter. These include, of course, the order, size, diameter, girth, circumference, connectivity, edge connectivity, independence number, covering number, chromatic number, edge chromatic number and Ramsey number. A significant part of graph theory dedicates itself to the study of graph parameters.

5. Algebraic Objects

We have already stated that graphs, groups, rings, fields, and partially ordered sets and hence lattices are instances of binary and ternary relational systems. We shall consider more formally in this section some other algebraic systems.

The specialization begins from the most abstract concept, the concept of a category. For the formal axioms for categories, the reader may see [33], where the following is explicitly stated and established.

**Proposition 5.1.** Every category is a graph.
As we have considered groups already, and category theory was essentially born out of a deep connection between groups and topological spaces, topological spaces will be addressed now.

Every topological space is a hypergraph, and hence every topological space is a relational system.

**Proposition 5.2.** Every module is a ternary relational system.

**Proof:** By definition, a module is an abelian group $M$ (a ternary relational system as seem above) together with a ring homomorphism $f : R \rightarrow \text{End}(M, M)$. Now $f$ is a ternary relation over $M$. Hence a module is a ternary relational system. \[\square\]

Thus, every vector space is a ternary relational system, and every algebra is a ternary relational system.

From category theory, topology and algebra, we now return to combinatorics and consider matroids. Since every matroid is a hypergraph, every matroid is a relational system.

Whereas the concept of a category captures mathematical concepts from algebro-axiomatic point of view (see [33] pages 10-12), relational systems capture them in an elementary combinatorial way. Having said about the generality achieved by the concept of relational systems, it needs to be pointed out that this generality is useful only as a proper generalization, as the concept of operations is considerably more special than that of a relation. Operations certainly bear more properties and these are exploited in the study of algebraic objects such as groups, rings, fields and modules.

### 6. Inductive Sets

A partially ordered set is said to be *well founded* if every descending chain is finite (this is the Jordan-Dedekind descending chain condition). A well founded partial order is also abbreviated as a *well founded order*. As a special type of partial order, every well founded order is a graph.

An *inductive class* $\mathcal{C}$ is usually understood as a set of objects such that a subset $\mathcal{B} \subseteq \mathcal{C}$ is designated and for each $X \in \mathcal{C}\setminus\mathcal{B}$ there is a well defined reduction $\rho$ such that $\rho(X) \in \mathcal{C}$. But the following definition is more essential.

**Definition 6.1.** A set $S$ is called *inductive* if there is a well founded order on $S$.

The set of minimal elements is the set $\mathcal{B}$ in the previous paragraph. Note that nothing is said about whether the set of minimal elements is *finite*. In fact, consider the set of all positive integers excepting 1, under the binary relation of divisibility:
a ≤ b ⇔ a | b. This relation is a well founded order, and the set of minimal elements (the set of all primes) is infinite. This is also an example of a well founded order that is not a well quasi order (as defined at the end of this section).

The following statement says something about the domain of usage of the important principle of mathematical induction.

**Proposition 6.1.** The mathematical induction principle is valid on a set \( S \) if and only if \( S \) has a well founded order.

This is one of the most basic statements in discrete mathematics. However, in classrooms this principle was taught in a way that gives an impression that this is a review of junior highschool mathematics. The importance of the fact that this principle should be understood here as a characterization or complete determination of the nature of a set on which mathematical induction may be used, is usually ignored or misconveyed! The true implication of this statement is that if a set \( S \) has a well founded order then the mathematical induction may be applied, and if mathematical induction may be applied to elements of a set \( S \) then \( S \) has a well founded order. Unfortunately, this is seldom done. In an inductive set \( S \), if every antichain is finite, then \( S \) is called finitely generated. Note also that each inductive set is a graph.

**Definition 6.2.** A reflexive and transitive binary relation is called a quasi-ordering. A quasi-ordering \( \leq \) on a set \( S \) is a well quasi-ordering, and the elements of \( S \) are well quasi-ordered by \( \leq \), if for every infinite sequence \( x_0, x_1, \ldots \in X \), there exist indices \( i < j \) such that \( x_i \leq x_j \).

**Proposition 6.2.** ([16] page 252) A quasi-ordering is a well quasi-ordering if and only if every antichain is finite and every descending is finite.

Since a quasi order is a binary relation, every well quasi-ordering is an oriented graph. No one can deny the importance of well quasi orders in the theory of graphs. It is, however, usual to encounter a denial of the importance of a well founded order in the theory of graphs.

Since a partially ordered set is a graph according to our definition, order preserving mappings between two partially ordered sets, and more specifically sets with well founded orders, is nothing but a graph homomorphism between the oriented graphs.

7. **Contractions and Minors**

It was mentioned in Section 3 that a contraction is a preconnected egamorphism. An equivalent formulation is by way of a connected partition of \( V(G) \).
Let $G = (V, E)$ be a graph. For $X, Y \subseteq V(G)$, denote

$$(X, Y) = \{xy : x \in X, y \in Y, xy \in E(G)\}.$$

A contraction of $G$ is defined to be a partition $\{V_1, V_2, \ldots, V_s\}$ of $V$ such that for each $i = 1, 2, \ldots, s$, the induced subgraph $G|_{V_i}$ is connected. This partition gives rise to a natural mapping (this is a preconnected egamorphism) from $G$ to a graph $H$, also called a contraction (graph) of $G$. The contraction (graph) $H$ is the graph with

$$V(H) = \{V_1, V_2, \ldots, V_s\}, \quad E(H) = \{V_iV_j : i \neq j, (V_i, V_j) \neq \emptyset\}.$$

The mapping $f$ is called a contraction (mapping) (or preconnected egamorphism) from $G$ onto $H$, and $G$ is said to be contractible to $H$.

The graph $K_1$ is a contraction of any connected graph $G$ since $\{V\}$ is a partition of $V$ and $G = G|_V$ is connected. Any automorphism of $G$ is a contraction since it is a permutation of the trivial partition of $V$ into single vertices. In particular, $1 : G \to G$ is a contraction.

Suppose that $R \subseteq G$ is a connected subgraph. Then the contraction of $R$ in $G$, denoted $G/R$, is given by the partition

$$\{V(R), \{v_1\}, \ldots, \{v_m\}\}$$

where $V(G) - V(R) = \{v_i : 1 \leq i \leq m\}$.

Let $r \geq 1$ be an integer and denote by $e^r = (uv)^r$ the presence of $r$ parallel edges between vertices $u$ and $v$ in a multigraph. A contraction $f : G \to H$ is called a faithful contraction if

$$E(H) = \{(V_iV_j)^r : |(V_i, V_j)| = r, i \neq j\}.$$

For an undirected graph, $(V_i, V_j)$ may be typed as $[V_i, V_j]$ or just $V_iV_j$.

A contraction may be understood in various ways, but this shall not be our concern here in this paper. It is only noted here that this definition is adequate for directed graphs and infinite graphs, and is equivalent to stating that $f : G \to H$ is a preconnected egamorphism.

A graph $H$ is a minor of $G$, if $G$ has a subgraph contractible to $H$. That is, there is a subgraph $K \subseteq G$ and a contraction $f : K \to H$. This is the same as saying that the following diagram commutes.

$$\begin{array}{ccc}
K & \xrightarrow{\eta} & G \\
\downarrow f & & \downarrow \mu \\
H & & 
\end{array}$$
This diagram may be used to prove some elementary properties of minor inclusions. First, since $1 : H \to H$ is a contraction, we have

\[
\begin{array}{c}
H \\ \downarrow^1 \hspace{1cm} \downarrow^\mu \\
\eta \hspace{1cm} G
\end{array}
\]

Hence,

(1) $H \subseteq G \Rightarrow H \leq G$.

As a simple consequence, $G \leq G$ (the reflexivity of the binary relation $\leq$).

If $f : G \to H$ is a contraction, then

\[
\begin{array}{c}
G \\ \downarrow^f \\
\mu \hspace{1cm} \mu \\
G \hspace{1cm} H
\end{array}
\]

Hence, we have proved

(2) If $f : G \to H$ is a contraction, then $H \leq G$.

Note that the converse is not true in general. For an example, $K_{3,3} \leq P$ where $P$ is the Petersen graph, but there is no contraction $f : P \to K_{3,3}$ (prove this!)

Denote by $\hat{G}$ a subdivision (i.e., a homeomorph) of a graph $G$. As a corollary to (2), we have

(3) If $\hat{H} \subseteq G$ then $H \leq G$.

Now, the transitivity of the binary relation $\leq$ may also be established by using the diagram defining a minor.

(4) $(J \leq H) \land (H \leq G) \Rightarrow J \leq G$;
Proof: Consider the diagram

\[ \begin{array}{ccc}
M = f^{-1}\gamma(L) & \xrightarrow{i} & K \\
\downarrow f\mid M & & \downarrow f \\
L & \xrightarrow{\gamma} & H \\
\downarrow g & & \downarrow \nu \\
J & \xrightarrow{\mu} & \end{array} \]

In this diagram, \( M = f^{-1}\gamma(L) \subseteq K \subseteq G \) is a subgraph of \( G \), \( fi = \gamma f\mid M \). \( f\mid M(M) = f\mid Mf^{-1}\gamma(L) = \gamma(L) \cong L \) and \( gf\mid M : M \to J \) is a contraction since composition of contractions is a contraction. Hence \( J \leq G \) by the definition of a minor.

We cite, without proof, two further elementary properties of minor inclusions.

(5) If \( \Delta(G) \leq 3 \), then \( H \leq G \Leftrightarrow \hat{H} \subseteq G \);

(6) If \( H \leq G \) and \( G \) is planar then \( H \) is also planar.

It will now be proved that the binary relation of minor inclusion is very close to being antisymmetric for a family of finite graphs.

**Proposition 7.1.** Let \( G \) and \( H \) be finite graphs. If \( H \leq G \) and \( G \leq H \) then \( G \cong H \).

Proof: Suppose that \( G \) and \( H \) are finite graphs and that \( H \leq G \) and \( G \leq H \). Then we have diagrams

\[ \begin{array}{ccc}
K & \xrightarrow{\eta} & G \\
\downarrow f & & \downarrow \mu \\
H & & \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
L & \xrightarrow{\gamma} & H \\
\downarrow g & & \downarrow \nu \\
G & & \\
\end{array} \]

Since \( |G| \leq |L| \leq |H| \) and \( |H| \leq |K| \leq |G| \), we have \( |G| = |H| \). Hence \( G \) is a spanning subgraph of \( H \) and \( H \) is a spanning subgraph of \( G \). (In particular, \( V(G) = V(H) \).) This means. \( E(H) \subseteq E(G) \) and \( E(G) \subseteq E(H) \). Hence, \( E(G) = E(H) \). Hence,

\[ f : G \to H, \ g : H \to G \]

are both contractions and bijections

\[ f : V(G) \to V(H), \ g : V(H) \to V(G). \]

Since \( f \) and \( g \) are contractions, hence \( uv \in E(G) \) if and only if \( f(u)f(v) \in E(H) \) and hence \( G \cong H \). \( \square \)
Thus, if isomorphic graphs are regarded as equal (which is a reasonable agreement), then the binary relation on finite graphs provided by minor inclusion is a partial order. Hence, a well quasi order in a family of graphs is essentially a well founded order. This, in addition to the principle of mathematical induction as stated in this note, is our reason why well founded order is interesting.

8. Transformation Graphs

In Definition 2.2 allow $V$ to be a specific set of mathematical objects. Then $E$ is a binary relation between mathematical objects, which may be given by a well defined set of transformations. Then Definition 2.2 itself becomes the definition of a transformation graph. The difference in the two definitions is that the set $V$ is an abstract set in the former and it is a specific set in the latter case. The binary relation $E$ is a binary relation defined on an abstract set in the former and it is one between specific mathematical objects in the latter case.

Let us now consider special examples of transformation graphs that are sufficiently general and important in mathematics.

(1) Consider the category of all finite groups and group homomorphisms. This is a transformation graph, which might have as well been called homomorphism graph of finite groups. The study of this graph comprises an essential part of the theory of groups.

(2) Consider the category of all topological spaces and continuous mappings. This is a transformation graph. The study of this graph comprises an essential part of topology. The study of the homomorphism from the graph of (1) to the graph of (2) includes homotopy and homology.

(3) Let $V$ be any finite set of positive integers, and let $E$ be the binary relation of divisibility: for $a, b \in V$, $(a, b) \in E$ if $a$ divides $b$. This transformation graph has been called the divisibility graph in [13].

(4) Let $G$ be a connected finite simple graph. Let $V$ be the set of its different spanning trees. For spanning trees $T_1, T_2 \in V$, let $(T_1, T_2) \in E$ if $||T_1|| - ||T_2|| = 2$. This is the tree transformation graph of $G$ which had been studied by Whitney as early as 1927 (I found no direct reference to this in the collection in my vicinity). The binary relation by which the edges are defined in the tree transformation graph was known in the literature as the fundamental exchange of edges.

(5) Let $G$ be any finite simple graph with a perfect matching. Let $V$ be the set of its different perfect matchings. For matchings $M_1, M_2 \in V$, let $(M_1, M_2) \in E$ if there is a unique $(M_1, M_2)$-alternating cycle $C$ in $G$. This may be called the matching transformation graph of $G$. 
(6) Let $V$ be the set of all perfect matchings in a hexagonal system (see [42, 44]), and let $E$ be the binary relation where $(M_1, M_2) \in E$ for $M_1, M_2 \in V$ if a hexagon is $(M_1, M_2)$-alternating. This gives the concept of a $Z$-transformation graph [44] of perfect matchings in a hexagonal system.

(7) Let $d$ be a graphic degree sequence, and $R(d)$ be the set of all isomorphism classes of finite simple graphs with degree sequence $d$. For $G, H \in R(d)$, $(G, H) \in E(R(d))$ if there exist $ab, cd \in E(G)$ with $ac, bd \notin E(G)$ such that

$$H = (G - \{ab, cd\}) \cup \{ac, bd\}.$$ 

This is called the realization graph of $d$. This graph has been studied for many interesting parameters in [37].

(8) This is a new concept, some special cases of which have been studied recently. Let $r$ be a fixed positive integer, $H$ be a fixed graph and $\cdot$ be a fixed binary operation. Then for a graph $G$, an $(H, r)$-transformation graph $J = T_{H,r}(G)$ may be defined by assigning

$$V(J) = \{S \subseteq E(G) : |S| = r\}, \quad E(J) = \{ST : S, T \in V(J), G|_{S,T} \supseteq H\}.$$ 

The transformation graph $T_{K_{1,2},1}(G) = L(G)$ is the usual line graph, and for $r = 2$ and $H = K_{1,2}$, the graphs $T_{H,r}(G)$ have been studied in [32]. The binary operation $\cdot$ being a natural product in an appropriate graph category seems to have not been of much attention.

These examples point to the sources of transformation graphs: transformation graphs arise from (1) a set of mathematical objects; (2) a set of subobjects of a mathematical object.

Tree transformation graphs have been studied in [14, 35] where it was established that tree transformation graphs are connected. Line graphs and super line graphs have been studied in [3, 4, 5, 8, 9, 12, 40]. $Z$-transformation graphs of hexagonal systems have been studied extensively in [15, 42, 43, 44, 45]. The study of hexagonal systems were directly motivated by organic chemistry. Matching transformation graphs have been studied in [6, 7] where it was established that these graphs are 2-connected. Euler tour graphs have been studied in [30, 41]. Divisibility graphs have been studied in [13, 18]. Transformation graphs based on some specific edge operations were studied in [19]. Switching transformation graphs or realization graphs have been investigated in [17, 23, 24, 37, 39]. Oriented transformation graphs of the quasi order arising from minor inclusion has been intensively investigated by Robertson and Seymour (see Diestel [16], Chapter 12 for a sketch).

Problems of connectivity of transformation graphs and those of traversals have been investigated. Measures of compactness and metric properties of the graphs such as diameter have also been of interest. In general, fundamental properties (combinatorial, geometric, topological or algebraic) of transformation graphs are of
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interest. A more detailed report on transformation graphs will be given in another paper by the author.

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