A perturbative analysis of interacting scalar field cosmologies

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Abstract
Scalar field cosmologies with a generalized harmonic potential are investigated in flat and negatively curved Friedmann–Lemaître–Robertson–Walker and Bianchi I metrics. An interaction between the scalar field and matter is considered. Asymptotic methods and averaging theory are used to obtain relevant information about the solution space. In this approach, the Hubble parameter plays the role of a time-dependent perturbation parameter which controls the magnitude of the error between full-system and time-averaged solutions as it decreases. Our approach is used to show that full and time-averaged systems have the same asymptotic behavior. Numerical simulations are presented as evidence of such behavior. Relevant results show that the asymptotic behavior of the solutions is independent of the coupling function.

Keywords: scalar field cosmologies, asymptotic analysis, averaging methods, generalized harmonic potentials

(Some figures may appear in colour only in the online journal)

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1. **Introduction**

There are a number of gravitational theories, some of them including scalar fields, that can be studied using local and global variables, providing a qualitative description of the space of solutions. In addition, it is possible to provide precise schemes to find analytical approximations of the solutions, as well as exact solutions or solutions in quadrature by choosing various approaches, e.g. [1–1]. In particular, relevant information about the properties of the flow associated with an autonomous system of ordinary differential equations can be obtained by using qualitative techniques of dynamical systems. See textbooks related to qualitative theory of differential equations [90–99] and with some applications in cosmology [100–105]. The tools of averaging theory and qualitative techniques of dynamical systems have been applied successfully in recent years to cosmological models, say in [106–116].

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In this paper, methods of perturbation theory and averaging theory are applied to differential equations that arise from cosmological models with a scalar field that evolves according to the Klein–Gordon (KG) equation under the influence of a self-interacting potential. Models with and without interaction between the scalar field and the matter (described by an ideal gas with a barotropic equation of state (EoS)) are investigated. That is, we are interested in the study of models where the matter of the universe is described by a scalar field \( \phi \), which is assumed to be homogeneous, with an energy–momentum tensor given by \( [T^b_b] = \text{diag}(-\rho_\phi, p_\phi, p_\phi, p_\phi) \), where \( \rho_\phi = \frac{1}{2}(\dot{\phi}^2 + 2V(\phi)) \) and \( p_\phi = \frac{1}{2}(\dot{\phi}^2 - 2V(\phi)) \) are the energy density and isotropic pressure of the scalar field, and \( V(\phi) \) is the self-interacting potential, e.g. the generalized harmonic potential \( V(\phi) = \mu^2 \phi^2 + \text{cosine corrections} \); and by an ideal gas described by the tensor \( [T^b_b^\text{matter}] = \text{diag}(-\rho_m, p_m, p_m, p_m) \), where \( \rho_m \geq 0 \) and \( p_m = (\gamma - 1)\rho_m \), where \( \gamma \in [0, 2] \) is the barotropic index.

The natural generalization of the models examined in [114–116], where a universe dominated by a scalar field and a barotropic fluid was studied for the Bianchi I, Bianchi III, Friedmann–Lemaître–Robertson–Walker (FLRW) (open, flat and closed) and Kantowski–Sachs metrics in minimally coupled scalar field theories, is to consider spatially homogeneous and isotropic matter-scalar field interactive schemes. Interactive matter-scalar field schemes refer to models where the conservation equations have the structure

\[
\dot{\rho}_m + 3H(\rho_m + p_m) = -\dot{Q}, \quad \dot{\phi} + 3H\phi + V(\phi) = Q, \tag{1}
\]

where a dot means derivative with respect to cosmic time \( t \), comma derivative with respect to \( \phi \), \( \rho_m \) is the energy density of matter, \( \phi \) is the scalar field, \( V(\phi) \) its potential, \( Q \) is the interaction term, and \( H = \dot{a}/a \) stands for the Hubble parameter (which is a general measure of the isotropic rate of spatial expansion), where \( a \) denotes the scale factor of the universe.

When considering models with interaction, which have various physical implications, different results would be expected from the case without interaction. An interesting research program is to investigate the dynamics and asymptotic behavior of the solutions of the equations of the gravitational field for various interacting functions \( Q = Q(H, \rho_m, \rho_\phi) \). As a first step towards generalization we can consider an interaction of the form \( Q = \lambda/2\rho_m \dot{\phi} \), arising from the coupling function (8) within the interacting scheme (1). Then, the complexity can be increased by incorporating interaction terms \( Q = 3\alpha H\rho_m, Q = 3\beta H\rho_\phi \) and \( Q = 3H(\alpha\rho_m + \beta\rho_\phi) \) [117, 118].

Our methodology consists of using perturbation theory, in particular multi-scale methods as well as averaging theory and qualitative analysis to describe oscillating solutions in a wide class of cosmological models going beyond the usual linear stability analysis. The first sections are devoted to showing that asymptotic methods and the averaging theory are powerful tools for investigating scalar field models, so we will start with examples from low to high complexity.

In this work we have considered the action for a general class of scalar–tensor theories (STT), written in the so-called Einstein frame (EF), which is given by [119]

\[
S_{\text{EF}} = \int_{M^4} d^4x \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) + \chi(\phi)^{-2} \mathcal{L}_{\text{matter}}(\mu, \nabla \mu, \chi(\phi)^{-1} g_{\alpha\beta}) \right\}, \tag{2}
\]

where \( R \) is the curvature scalar, \( \phi \) is the scalar field, \( \nabla_\alpha \) is the covariant derivative, \( V(\phi) \) is the quintessence self-interacting potential, \( \chi(\phi)^{-2} \) is the coupling function, \( \mathcal{L}_{\text{matter}} \) is the matter Lagrangian, and \( \mu \) is a collective name for the matter degrees of freedom, repeated indexes
mean sum over them. The energy–momentum tensor of matter is defined by

\[ T_{\alpha\beta} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\alpha\beta}} \left\{ \sqrt{|g|} \chi^{-2}(\phi) \mathcal{L}(\mu, \nabla \mu, \chi^{-1}(\phi) g_{\alpha\beta}) \right\}. \tag{3} \]

By considering the conformal transformation \( \bar{g}_{\alpha\beta} = \chi(\phi)^{-1} g_{\alpha\beta} \), defining the Brans–Dicke (BD) coupling ‘constant’ \( \omega(\chi) \) in such way that \( d\phi = \pm \sqrt{\omega(\chi)} + 3/2 \chi^{-1} d\chi \) and recalling \( \nabla(\chi) = \chi^2 V(\phi, \chi) \), the action (2) can be written in the Jordan frame (JF) as [101]

\[ S_{\text{JF}} = \int d^4x \sqrt{|\bar{g}|} \left\{ \frac{1}{2} \chi \mathcal{R} - \frac{1}{2} \chi (\nabla \chi)^2 - \nabla(\chi) + \mathcal{L}_{\text{matter}}(\mu, \nabla \mu, \bar{g}_{\alpha\beta}) \right\}. \tag{4} \]

Here the bar is used to denote geometrical objects defined with respect to the metric \( \bar{g}_{\alpha\beta} \). In the next sections a bar or an over-line will be referring to averaged quantities. In the STT given by (4), the energy–momentum of the matter fields,

\[ \bar{T}_{\alpha\beta} = -\frac{2}{\sqrt{|\bar{g}|}} \frac{\delta}{\delta \bar{g}^{\alpha\beta}} \left\{ \sqrt{|\bar{g}|} \mathcal{L}(\mu, \nabla \mu, \bar{g}_{\alpha\beta}) \right\}, \tag{5} \]

is separately conserved. That is \( \nabla^\alpha \bar{T}_{\alpha\beta} = 0 \). However, when is written in the EF (2), with a matter energy–momentum tensor given by (3), this is no longer the case (although the overall energy density is conserved). In fact in the EF we find that

\[ \bar{Q}_{\beta} \equiv \nabla^\alpha \bar{T}_{\alpha\beta} = -\frac{1}{2} T(\phi)^{-1} \frac{d\chi(\phi)}{d\phi} \nabla_{\beta} \phi, \quad \bar{T} = T^\alpha_{\alpha}. \tag{6} \]

By making use of the above ‘formal’ conformal equivalence between the EF and JF we can find, for example, that the theory formulated in the EF with the coupling function \( \chi(\phi) = \chi_0 \exp((\phi - \phi_0)/\tau) \), \( \tau \equiv \pm \sqrt{\omega_0 + 3/2} \) and potential \( V(\phi) = \beta \exp((\alpha - 2)\tau(\phi - \phi_0)) \), corresponds to the BD theory with a power-law potential, i.e. \( \omega(\chi) = \omega_0 \) and \( \nabla(\chi) = \beta \chi^\alpha \).

Exact solutions with exponential couplings and exponential potentials (in the EF) were investigated in [25]. Quintessential DE models [120–122], for instance, are described by an ordinary scalar field minimally coupled to gravity. A particular choice of the scalar field self-interacting potentials can drive the past and current accelerated expansion.

The natural generalizations to quintessence models evolving independently from the matter are models that exhibit non-minimal coupling between both components. Several physical theories predict the presence of a scalar field coupled to matter. For example, in string theory, the dilaton field is generally coupled to matter [123]. Non-minimally coupling occurs also in STT of gravity [124, 125], in higher order gravity theories [126] and in models of chameleon gravity [127]. Coupled quintessence was investigated also in [128–130] by using dynamical systems techniques. The cosmological dynamics of scalar–tensor gravity have been investigated in [131, 132]. Phenomenological coupling functions were studied for instance in [133] which can describe either the decay of dark matter into radiation, the decay of the curvaton field into radiation or the decay of dark matter into dark energy [133]. In the reference [132], the authors constructed a family of viable scalar–tensor models of dark energy, which includes pure \( F(R) \) theories and quintessence. There is the possibility of a universal coupling of dark energy to all sorts of matter, including baryons, but excluding radiation [134]. For action (2), the strength of the coupling between the perfect fluid and the scalar field is \( Q = \frac{4}{3} (4 - 3\gamma) \rho_m \phi \frac{d\ln \chi}{d\phi} \), where \( \chi(\phi) \) is an input function. In reference [130] the interaction terms (in the flat FLRW geometry) \( Q = \alpha \phi \rho_m \) and \( Q = \alpha \rho_m H \) were investigated, here \( \alpha \) is a constant, \( \phi \) is the scalar field, \( \rho \) is the...
energy density of background matter and $H$ is the Hubble parameter. The first choice corresponds to an exponential coupling function $\chi(\phi) = \chi_0 \exp \left( \frac{2\alpha \phi}{(4 - 3\gamma)} \right)$. The second case corresponds to the choice $\chi = \chi_0 a^{-2\alpha/(4 - 3\gamma)}$ (and then, $\rho \propto a^{-\alpha - 3\gamma}$).

In this paper, some perturbation problems in scalar field cosmologies in a vacuum and including matter will be studied. Relevant information about the solution’s space for scalar field cosmologies in FLRW and Bianchi I metrics is expected to be obtained using qualitative techniques, asymptotic methods, and averaging theory. In this regard, this paper is a continuation of [109, 135]. There, some well-known results were reviewed and new theorems in the context of scalar field cosmologies with arbitrary potential (and with an arbitrary coupling to matter) were proved. In particular, cosine-like corrections with small phases were incorporated to the harmonic potential for FLRW metric and Bianchi I metrics inspired in [136]. Following this line, we select a self-interacting potential

$$V(\phi) = \frac{\phi^2}{2} + f \left[ 1 - \cos \left( \frac{\phi}{f} \right) \right] = \frac{(f + 1)\phi^2}{2f} + \mathcal{O}(\phi^3), \quad f > 0,$$

(7)

and the coupling function

$$\chi(\phi) = \chi_0 e^{\lambda \phi / (4 - 3\gamma)}, \quad \lambda \text{ is a constant and } \gamma \neq \frac{4}{3}.$$

(8)

We must emphasize that there is a close relationship between the KG equation

$$\ddot{\phi} + \phi = H[-3\dot{\phi}],$$

(9)

and that of a harmonic oscillator with non-linear damping, where the damping depends on time through the coupling of the Einstein equations with the KG equation through the Hubble parameter $H$. Motivated by the works [109, 135] and based on the previous analogy, an amplitude-phase transformation $(\dot{\phi}, \phi) \to (r, \varphi)$ (chapter 11 of [137]; p 22, 24–27, 42, 54, 361 of [138]), which is defined as

$$\dot{\varphi}(t) = r(t) \cos(t - \varphi(t)), \quad \dot{r}(t) = r(t) \sin(t - \varphi(t)),$$

(10)

such that

$$r(t) = \sqrt{\dot{\varphi}(t)^2 + \dot{r}(t)^2}, \quad \varphi(t) = t - \arctan \left( \frac{\dot{r}(t)}{\dot{\varphi}(t)} \right),$$

(11)

will be used [137]; which allows obtaining new equations which will be averaged in time to obtain new systems. With this approach, the oscillations present in the non-linear systems, which enter/modify the dynamics through the KG equation, can be controlled and smoothed as long as the Hubble parameter $H$, which acts as a time-dependent perturbation parameter, decreases monotonically. We will use the methods of the averaging theory of systems of nonlinear differential equations to prove that the original time-dependent systems and their corresponding averaged versions have the same late dynamics. Therefore, to determine the future asymptotic behavior, the simpler averaged systems are investigated. Numerical simulations will be carried out to show the oscillatory behavior of the solutions. These simulations will also show how the averaged solutions has the same asymptotic behavior as compared to the original ones. These results will allow making conjectures about the dynamics of the universe at local or cosmological scales and will establish demonstration schemes to prove them.
The paper is organized as follows. In section 2 we discuss some asymptotic expansion techniques, in particular the two-timing method. In section 3 we present a review on averaging techniques, with special emphasis on applications in cosmology. In section 4 some applications of perturbation and averaging methods in cosmology are presented. In particular, in section 4.1 is studied a scalar field with generalized harmonic potential (7) non-minimally coupled to matter with coupling (8). Sections 4.2 and 4.3 are devoted to the minimally coupled and vacuum cases, respectively. We are focused on studying the imprint of coupling function, as well as the influence of the metric on the dynamics of the averaged problem. In section 5 we present numerical simulations as evidence that the solutions of the full system for each model follow the track of the solutions of their corresponding averaged version when $H$ is monotonically decreasing. Section 6 is devoted to results and conclusions.

2. Perturbation problems

Perturbation problems focus on the study of the phase portrait of the differential system

$$\dot{x} = X(x; \varepsilon), \quad x \in \mathbb{R}^k, \quad \varepsilon \sim 0,$$

near the zero of $X(x; 0)$ [137–144]. In general, perturbation problems (12) are expressed in Fenichel’s normal form, i.e. given $x = (x, y) \in \mathbb{R}^{k+m}$, where $k = n + m$, and $f, g$ smooth functions, the equations can be written as

$$\dot{x} = f(x, y; \varepsilon), \quad \dot{y} = \varepsilon g(x, y; \varepsilon), \quad x = x(t), \quad y = y(t).$$

The system (13) is called ‘fast system’, unlike the system

$$\varepsilon \dot{x} = f(x, y; \varepsilon), \quad \dot{y} = g(x, y; \varepsilon), \quad x = x(\tau), \quad y = y(\tau),$$

obtained after the re-scaling $\tau = \varepsilon t$, that is called the ‘slow system’. Notice that for $\varepsilon > 0$, the phase portraits of (13) and (14) coincide. However, this two problems manifestly depend on two scales: (i) the problem in terms of the ‘slow time’ variable, whose solution is analogous to the outer solution in a boundary layer problem; and (ii) the fast system, a change of scale on the system which describes the rapid evolution that occurs in shorter times, analogous to the inner solution of a boundary layer problem. The solution of each subsystem will be sought in the form of a regular perturbation expansion. For singularly perturbed problems the subsystems will have simpler structures than the full problem, allowing the characterization of the slow and fast dynamics in terms of a reduced phase line or phase plane dynamics.

For $\varepsilon > 0$, let $\mathcal{S}$ denotes the singular points of (13). Equation (14) define a dynamical system on $\mathcal{S}$ called the reduced problem. The implicit equation $f(x, y; 0) = 0$ is called the slow manifold or ‘slow solution curve’. Very often the solution is pushed out of the slow manifold at which point the solution is no longer described by the dynamics of the slow system; all out the slow manifold in the phase plane is part of the fast problem. Combining the results of these two limiting problems, some information of the dynamics for small values of $\varepsilon$ is obtained. This technique is used to construct uniformly valid approximations of the solutions of perturbation problems using as seed solutions those which satisfy the original equations in the limit of $\varepsilon \to 0$. One approach used to construct that asymptotic expansions is to introduce the two time scales $t_1 = t$ and $t_2 = \varepsilon t$. For this reason, the method is sometimes called two-timing, and $t_1$ is said to be the fast time scale and $t_2$ the slow scale. The list of possible scales includes the following [143]:
(a) Several time scales like \( t_1 = t/\varepsilon, t_2 = t, t_3 = t\varepsilon, t_4 = t\varepsilon^2 \ldots \) may be needed.
(b) More complex dependence on \( \varepsilon \), for example, \( t_1 = t \left( 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots \right) \) and \( t_2 = t\varepsilon \) where the \( \omega_n \) are determined while solving the problem (Poincaré–Lindstedt’s method).
(c) The correct scaling may not be immediately apparent, and one starts off with something like \( t_1 = t\varepsilon^\alpha \) and \( t_2 = t\varepsilon^\beta \), where \( \alpha < \beta \).
(d) Nonlinear time dependence, for example, one may have to assume \( t_1 = f(t, \varepsilon) \) and \( t_2 = \varepsilon t_1 \), where the function \( f(t, \varepsilon) \) is determined from the problem.

Perturbations methods and averaging methods were used, for example, in [26], in investigations of the oscillating behavior in scalar field cosmologies with harmonic potential using amplitude–phase variables of the form (10). In [106], these techniques were used to prove statements about how the relationship between the EoS of the fluid and the monomial exponent of the scalar field affects the asymptotic source dominance and asymptotic late time behavior. Slow-fast methods were used for example in GUP theories, say in [108]. In [109] averaging over an angle \( \phi \) by using an amplitude-angle transformation (p 358 [138]) of the form \( \dot{\phi}(t) = r(t)\sin\varphi(t) \) and \( \ddot{\phi}(t) = r(t)\cos\varphi(t) \) was used to study oscillations of the scalar field driven by generalized harmonic potentials. In the reference [110], was applied the averaging theory of first-order to study the periodic orbits of Hamiltonian systems describing a universe filled with a scalar field. There were provided sufficient conditions on the parameters of these cosmological models which guarantee that at any positive or negative Hamiltonian level, the Hamiltonian system has periodic orbits. Additionally, it was shown the non-integrability of these cosmological systems in the sense of Liouville–Arnold, proving that there cannot exist any second first integral of class \( C^1 \). These techniques can be applied to Hamiltonian systems with an arbitrary number of degrees of freedom. In reference [111], the method of multiple scales was applied to the analysis of cosmological dynamics. This method was used to construct solutions to the governing equations of the universe filled with a scalar field in the FLRW metric. A general scheme is described for choosing small dimensionless parameters of the expansion of model functions and applying the multiple scales method to the cosmological equations for two different types of a small parameter, a small field value, and a small slow-roll parameter.

In general, the regular asymptotic expansion fails in presence of resonant (secular) terms, which are those terms in the solution proportional to the time variable. The alternative Poincaré–Lindstedt’s method would determine solutions of perturbed oscillators by suppressing resonant forcing terms that would yield spurious secular terms in the asymptotic expansions. The \( t_1 \) and \( t_2 \) time variables are introduced to keep a well ordered expansion, where \( t_1 \) is the regular (or ‘fast’) time variable and \( t_2 \) is a new variable describing the ‘slow-time’ dependence of the solution. The idea is to use any freedom that is in the \( t \)-dependence of \( t_1 \) and \( t_2 \) to minimize the approximation’s error, and whenever is possible to remove unbounded or secular terms. To our knowledge, Poincaré–Lindstedt’s method has not been implemented yet in the cosmological setup. However, basic examples of oscillators show that by implementing a time-averaged version of the model instead of multiple scales, the issue of secular terms is overcome; getting the same accuracy as in the two-timing method. Alternatively, the method of multiple time scales makes a less restrictive assumption on the form of the solution than those employed by Poincaré–Lindstedt’s method. It assumes that the solution can be expressed as a function of multiple (just two for our purposes) time variables, which are introduced to keep

\[ We elaborate more on averaging techniques in subsection 3. \]
Figure 1. Numerical solution of (16) (solid line) vs asymptotic expansion (19) (dashed line).

a well-ordered expansion,

\[ x(t) = X(t, \tau), \]  

where \( t \) is the regular (or ‘fast’) time variable and \( \tau \) is a new variable describing the ‘slow-time’ dependence of the solution. As commented before, the idea is to use any freedom that is in the \( \tau \)-dependence to minimize the approximation’s error, and whenever is possible to remove unbounded or secular terms. Some examples to illustrate the use of perturbation methods are the following.

2.1. Example 1

Considering the following initial value problem with \( t > 0 \)

\[
\frac{d^2 y}{d\tau^2} = -\frac{1}{(1 + \varepsilon y)^2}, \quad y(0) = 0, \quad y'(0) = 1.
\]  

(16)

Assuming the solution has an asymptotic expansion of the form

\[ y(t) \sim y_0(t) + \varepsilon y_1(t) + \cdots, \]

(17)

and considering a very small \( \varepsilon \), \( (1 + \varepsilon)^{-2} \sim 1 - 2\varepsilon \), the original problem becomes

\[
y_0''(t) + \varepsilon y_1''(t) + \cdots = -\frac{1}{[1 + \varepsilon(y_0(t) + \cdots)]^2} \sim -1 + 2\varepsilon y_0(t) + \cdots,
\]

(18)

with initial conditions

\[ y_0(0) + \varepsilon y_1(0) + \cdots = 0, \quad \text{and} \quad y_0'(0) + \varepsilon y_1'(0) + \cdots = 1.\]

Collecting terms the following systems are obtained:

To order \( O(1) \): \( y_0''(t) = -1, \ y_0'(0) = 1, \ y_0(0) = 0 \) has solution \( y_0(t) = -\frac{1}{2}t^2 + t \).

To order \( O(\varepsilon) \): \( y_1''(t) = 2y_0(t), \ y_1'(0) = 0, \ y_1(0) = 0 \) has solution \( y_1(t) = \frac{t^3}{4} - \frac{1}{12}t^4 \).

Finally, the solution is given by

\[
y(t) \sim t \left( 1 - \frac{1}{2}t^2 \right) + \frac{1}{3} \varepsilon t^3 \left( 1 - \frac{1}{4}t^2 \right).
\]

(19)

This example illustrate how the regular asymptotic expansion method works. As shown in figure 1 as \( \varepsilon \) becomes small the numerical solution of (16) (solid line) and the asymptotic expansion (19) coincide.

The next example shows the failure of the regular asymptotic expansion due to the appearance of spurious secular terms in the asymptotic expansions.
2.2. Example 2

Considering the classical example [143], given by the ordinary differential equation
\[ y'' + \varepsilon y' + y = 0, \quad t > 0, \quad y(0) = 0, \quad y'(0) = 1. \]  
Equation (20) admits an exact solution of the form
\[ y(t) = 2e^{-\frac{t}{2}} \sin \left( \frac{1}{2} t \sqrt{4 - \varepsilon^2} \right) \frac{1}{\sqrt{4 - \varepsilon^2}}. \]  
Using regular asymptotic expansions to solve (20) would yield spurious secular terms, for instance, the solution by regular expansion is
\[ x(t, \varepsilon) = \sin(t) - \varepsilon t \sin(t) + O(\varepsilon^2), \]  
notice that the ‘next to leading term’ \( \varepsilon t \sin(t) \) is dominant on scales \( \varepsilon t = O(1) \), that is, it holds for \( 0 \leq \varepsilon t \leq T \) where \( T \) is fixed. Therefore, it becomes larger than the zeroth-order terms as the time increases as shown in figure 2. This is the typical behavior of secular/resonant terms.

Observe that solution (21) has an oscillatory component running on the scale of order \( O(1) \), as well as a slow variation of order \( O(\varepsilon^{-1}) \). Therefore, two time scales \( t, \tau = \varepsilon t \) are introduced and treated as independent variables. Using the chain rule
\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \varepsilon \frac{\partial f}{\partial \tau}, \quad \frac{d^2 f}{dt^2} = \frac{\partial^2 f}{\partial t^2} + 2\varepsilon \frac{\partial^2 f}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2 f}{\partial \tau^2}, \]  
the initial value problem of a scalar differential equation
\[ y_{tt} + 2\varepsilon y_{t \tau} + \varepsilon^2 y_{\tau \tau} + \varepsilon (y_t + \varepsilon y_{\tau}) + y = 0, \]  
\[ y = 0, \quad y_t + \varepsilon y_{\tau} = 1 \quad \text{for} \quad t = 0 = \tau, \]  
is obtained, where the subscripts \( y_t, y_{t \tau}, \ldots \), denote the partial derivatives. Now, using a series expansion of the form
\[ y \sim y_0(t, \tau) + \varepsilon y_1(t, \tau) + \cdots, \]  
the following equation
\[ y_{0tt} + y_0 + \varepsilon (y_{1tt} + y_1 + 2y_{0t \tau} + y_0) + O(\varepsilon^2) = 0, \]  
is obtained. Collecting terms of order 1 and \( \varepsilon \) leads to
Figure 3. Exact solution (21) of equation (20) (thin blue line) vs two-timing expansion (22) (thick dashed red line).

\[ O(1): \]
\[ y_{0\tau} + y_0 = 0, \tag{28} \]
with a general solution
\[ y_0(t, \tau) = A(\tau) \sin(t) + B(\tau) \cos(t), \tag{29} \]
and
\[ O(\epsilon): \]
\[ y_{1\tau} + y_1 = -(2y_{0\tau} + y_0) \]
\[ = \sin(t) \left( 2B'(\tau) + B(\tau) \right) - \cos(t) \left( 2A'(\tau) + A(\tau) \right). \tag{30} \]

Then, secular terms \( \propto \epsilon t \sin(t), \propto \epsilon t \cos(t) \) are removed by setting
\[ \left( 2B'(\tau) + B(\tau) \right) = 0, \quad \left( 2A'(\tau) + A(\tau) \right) = 0. \tag{31} \]

After imposing the initial conditions, it follows that \( B(\tau) = 0 \) and \( A(\tau) = e^{-\tau} \) and the solution
\[ y \sim e^{-\tau} \sin(t) = e^{-\epsilon t} \sin(t), \tag{32} \]
valid up to the first order of \( \epsilon \) is obtained, which gives a good approximation to the solution of the problem. Indeed, the previous approximation holds up to \( \epsilon t = O(1) \). Therefore, this procedure alleviates the failure of the regular asymptotic expansion (22) that yielded spurious secular terms \( \propto \epsilon t \sin(t) \) in the asymptotic expansion. A comparison between figures 2 and 3 illustrates the benefit of the two-timing procedure over the regular asymptotic expansion to avoid secular/resonant terms.

2.3. Example 3

The so-called induced gravity model has the action [145, 146]
\[ S_{IG} = \int \sqrt{-g} \left( \frac{\sigma^2}{8\omega_0} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{\gamma^2 U_0 \sigma^2}{4 - 6\gamma^2} \right), \tag{33} \]
where \( \omega_0 > 0 \) and \( \gamma \geq 0 \). A massless scalar field is added to the action in [147] of the form
\[ S_{IG(\phi)} = S_{IG} + \int \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \tag{34} \]
The equation of motion for a massless scalar field is given by
\[ \dddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} = 0, \] (35)
and admits the solution \( \dot{\phi} = \varepsilon a^{-3} \), where \( \varepsilon \) is an integration constant. Using the parametrization [145]
\[ a = \sigma^{-1} \exp(u + v), \] (36a)
\[ \sigma = \exp(A(u - v)), \] (36b)
with \( A = \sqrt{\frac{3}{2}} \gamma \), the Raychaudhuri equation and the equation of motion for \( \sigma \) lead to
\[ \frac{3}{\gamma^2} (\sqrt{6} \gamma + 2) (3 \gamma^2 - 2) \varepsilon^2 \exp \left( 2 \sqrt{6} \gamma (u - v) - 6u - 6v \right) + 12 \left( \sqrt{6} \gamma - 6 \right) \dddot{u} + \left( \sqrt{6} \gamma + 6 \right) U_0 - 24 \dddot{u} = 0, \] (37a)
\[ - \frac{3}{\gamma^2} (\sqrt{6} \gamma - 2) (3 \gamma^2 - 2) \varepsilon^2 \exp \left( 2 \sqrt{6} \gamma (u - v) - 6u - 6v \right) - 12 \left( \sqrt{6} \gamma + 6 \right) \dddot{v} + \left( 6 - \sqrt{6} \gamma \right) U_0 - 24 \dddot{v} = 0. \] (37b)
where the Friedmann equation
\[ \dot{u} \dot{v} = \frac{1}{12} \left( \frac{2 - 3 \gamma^2}{\gamma^2} \varepsilon^2 \exp \left( 2 \sqrt{6} \gamma (u - v) - 6u - 6v \right) + U_0 \right), \] (38)
is used to eliminate the mixed terms \( \propto \dddot{u} \dddot{v} \).
Using series expansion of the form
\[ u \sim u_0(t, \tau) + \varepsilon u_1(t, \tau) + \mathcal{O}(\varepsilon^2), \quad v \sim v_0(t, \tau) + \varepsilon v_1(t, \tau) + \mathcal{O}(\varepsilon^2), \] (39)
where the time variables \( t, \tau = \varepsilon t \), are introduced and treated as independent variables. Collecting terms of order \( \varepsilon \) (see reference [147]) the following problems are found: \( \mathcal{O}(1) \):
\[ \begin{align*}
12(\sqrt{6} \gamma - 6)u_0(t, \tau) + 24u_0u_1(t, \tau) + (\sqrt{6} \gamma + 6)U_0 &= 0 \\
-12(\sqrt{6} \gamma + 6)v_0(t, \tau) - 24v_0v_1(t, \tau) + (6 - \sqrt{6} \gamma)U_0 &= 0 \\
u_0(t, \tau) - \frac{U_0}{12} &= 0
\end{align*} \] (40)
(see, e.g., similar equation (28) in [147], (2.24) in [148]).
\[ O(\varepsilon): \]
\[
\begin{aligned}
-2w_{0\tau}(t, \tau) + \left( \sqrt{6} \gamma - 6 \right) u_0(t, \tau) (u_0(t, \tau) + u_1(t, \tau)) - u_{1\tau}(t, \tau) &= 0 \\
2v_{0\tau}(t, \tau) + \left( \sqrt{6} \gamma + 6 \right) v_0(t, \tau) (v_0(t, \tau) + v_1(t, \tau)) + v_{1\tau}(t, \tau) &= 0 . \\
v_0(t, \tau) (u_0(t, \tau) + u_1(t, \tau)) + u_0(t, \tau) (v_0(t, \tau) + v_1(t, \tau)) &= 0 \\
\end{aligned}
\]
\hspace{2cm} (42)
\]

Solving up to order \( O(1) \), the following systems are obtained

\[
\begin{aligned}
u_0(t, \tau) &= \begin{cases}
c_2(\tau) - \frac{2 \ln(\cosh(\Delta_0))}{\sqrt{6} - 6}, & \gamma^2 < 6 \\
c_2(\tau) - \frac{2 \ln(\cos(\Delta))}{\sqrt{6} \gamma - 6}, & \gamma^2 \geq 6
\end{cases}, \\
v_0(t, \tau) &= \begin{cases}
c_3(\tau) + \frac{2 \ln(\sin(\Delta))}{\sqrt{6} \gamma + 6}, & \gamma^2 < 6 \\
c_3(\tau) + \frac{2 \ln(\sin(\Delta))}{\sqrt{6} \gamma + 6}, & \gamma^2 \geq 6
\end{cases}
\end{aligned}
\]
\hspace{2cm} (42a)
\]

where \( c_1(\tau), c_2(\tau) \) and \( c_3(\tau) \) are integration functions, and

\[
\Delta := \Delta(t, \tau) = \frac{\sqrt{|\gamma^2 - 6|} \sqrt{U_0} (24c_1(\tau) + f)}{2 \sqrt{2}}.
\]
\hspace{2cm} (43)

Substituting (42a) into the equations at order \( O(\varepsilon) \), the following is obtained

\[
\begin{aligned}
u_{1\tau} &= \begin{cases}
\sqrt{U_0} \left( 12 \sqrt{U_0} (\gamma^2 - 6) \cosh^4(\Delta) (u_0 + c_2(\tau)) + \left( \sqrt{3} \gamma - 3 \sqrt{2} \right) \sqrt{6 - \gamma^2} \tanh(\Delta) \cosh(\Delta_0) (c_0(\tau) + u_0) \right), & \gamma^2 < 6 \\
\sqrt{U_0} \left( \left( \sqrt{3} \gamma - 3 \sqrt{2} \right) \sqrt{6 - \gamma^2} \tan(\Delta) (c_0(\tau) + u_0) - 12 \sqrt{U_0} (\gamma^2 - 6) \cosh(\Delta_0) \cosh(\Delta) \right), & \gamma^2 \geq 6
\end{cases},
\end{aligned}
\]
\hspace{2cm} (44)
\]

\[
\begin{aligned}
v_{1\tau} &= \begin{cases}
\sqrt{U_0} \cosh(\Delta) \left( 4 \sqrt{6 - \gamma^2} \cosh(\Delta_0) (u_0 + c_2(\tau)) + 12 \sqrt{2} (\gamma^2 - 6) \sqrt{6 + \gamma^2} \sin(\Delta_0) + \sin(\Delta) \sin(\Delta_0) \right), & \gamma^2 < 6 \\
\sqrt{U_0} \cosh(\Delta) \left( 4 \sqrt{6 - \gamma^2} \cosh(\Delta_0) (u_0 + c_2(\tau)) + 12 \sqrt{2} (\gamma^2 - 6) \sqrt{6 + \gamma^2} \sin(\Delta_0) + \sin(\Delta) \sin(\Delta_0) \right), & \gamma^2 \geq 6
\end{cases},
\end{aligned}
\]
\hspace{2cm} (45)
\]

\[
\begin{aligned}
u_{1\tau} &= \begin{cases}
\left( \sqrt{6 - \gamma^2} \cosh(\Delta_0) + \sqrt{6} u_0 \cosh(\Delta_0) - 6u_0 \cosh(\Delta_0) - 24 \sqrt{2} \sqrt{6 - \gamma^2} \gamma (c_0(\tau) \cosh(\Delta_0) - \sqrt{6} \gamma (c_0(\tau) - 6u_0(\tau)) \right), & \gamma^2 < 6 \\
\left( \sqrt{6 - \gamma^2} \cosh(\Delta_0) + \sqrt{6} u_0 \cosh(\Delta_0) - 6u_0 \cosh(\Delta_0) + 24 \sqrt{2} \sqrt{6 - \gamma^2} \gamma (c_0(\tau) \cosh(\Delta_0) + \sqrt{6} \gamma (c_0(\tau) + 6u_0(\tau)) \right), & \gamma^2 \geq 6
\end{cases},
\end{aligned}
\]
\hspace{2cm} (46)
\]
Integration of (44) leads to
\[
u_1 = c_4(\tau) + \begin{cases} 
\frac{2\tanh(\Delta)}{\sqrt{\gamma^2 - 6}} & \gamma^2 < 6 \\
\frac{2\tan(\Delta)}{\sqrt{\gamma^2 - 6}} & \gamma^2 \geq 6
\end{cases}
\]
\[\frac{2\tanh(\Delta)}{\sqrt{\gamma^2 - 6}} \frac{2c_2\tanh(\Delta)}{\sqrt{\gamma^2 - 6}} c_3(\rho) c_3(\rho) - \epsilon c_3(\rho) c_5(\rho), \gamma^2 < 6
\]
\[\frac{2\tan(\Delta)}{\sqrt{\gamma^2 - 6}} \frac{2c_2\tan(\Delta)}{\sqrt{\gamma^2 - 6}} c_3(\rho) c_3(\rho) - \epsilon c_3(\rho) c_5(\rho), \gamma^2 \geq 6
\]
To avoid the two secular terms \(\propto t\), conditions \(c_1' = c_2' = 0\) are imposed, i.e. \(c_1\) and \(c_2\) are constants. Hence,
\[
u_1 = c_4(\tau) + \begin{cases} 
\frac{2\sqrt{2}c_2\tanh(\Delta)}{\sqrt{U_0(6 - \gamma^2)}} & \gamma^2 < 6 \\
\frac{2\sqrt{2}c_2\tan(\Delta)}{\sqrt{U_0(6 - \gamma^2)}} & \gamma^2 \geq 6
\end{cases}
\]
where \(\Delta := \Delta(t) = \frac{(2c_1 + 1)\sqrt{c_0 - 6}}{4\sqrt{3}}\). Then,
\[
v_1 = \begin{cases} 
\frac{(6 - \sqrt{6})\sqrt{U_0(6 - \gamma^2)}c_2\coth(\Delta)\csch^2(\Delta)}{\sqrt{2}(\sqrt{6} + 6)} & \gamma^2 < 6 \\
\frac{(\sqrt{6} - 6)\sqrt{U_0(6 - \gamma^2)}c_2\coth(\Delta)\csch^2(\Delta)}{\sqrt{2}(\sqrt{6} + 6)} & \gamma^2 \geq 6
\end{cases}
\]
\[
v_1 = \begin{cases} 
\frac{(\sqrt{6} - 6)c_2\csch^2(\Delta)}{\sqrt{6} + 6} - c_3(\tau), \gamma^2 < 6 \\
\frac{(\sqrt{6} - 6)c_2\csch^2(\Delta)}{\sqrt{6} + 6} - c_3(\tau), \gamma^2 \geq 6
\end{cases}
\]
Solving the second equation the following is obtained
\[
v_1 = \begin{cases} 
\frac{-4(\sqrt{3} - 3)\sqrt{2}c_2\coth(\Delta)}{(\sqrt{6} + 6)\sqrt{U_0(6 - \gamma^2)}} & -\epsilon c_3(\tau) + c_5(\tau), \gamma^2 < 6 \\
\frac{4(\sqrt{3} - 3)\sqrt{2}c_2\coth(\Delta)}{(\sqrt{6} + 6)\sqrt{U_0(6 - \gamma^2)}} & -\epsilon c_3(\tau) + c_5(\tau), \gamma^2 \geq 6
\end{cases}
\]
such that both differential equations for \(v_1\) are identically satisfied. To avoid the secular terms \(\propto t\), the condition \(c_3(\tau) = 0\) is imposed, i.e. \(c_3\) is a constant. For simplicity, we set \(c_4 = c_5 = 0\). Therefore, it follows that
\[
u(t; \varepsilon) = c_2 - \begin{cases} 
\frac{2\coth(\Delta)}{2\ln(\cos(\Delta))} & \gamma^2 < 6 \\
\frac{2\tan(\Delta)}{2\ln(\cos(\Delta))} & \gamma^2 \geq 6
\end{cases}
\]
\[ v(t; \varepsilon) = c_3 + \begin{cases} 2 \ln(\sinh(\Delta)) \over \sqrt{6\gamma + 6}, & \gamma^2 < 6 \\ 2 \ln(\sin(\Delta)) \over \sqrt{6\gamma + 6}, & \gamma^2 \geq 6 \end{cases} + \varepsilon \begin{cases} 4 \left(\sqrt{3\gamma - 3\sqrt{2}}\right) c_2 \coth(\Delta), & \gamma^2 < 6 \\ -4 \left(\sqrt{6\gamma + 6}\right) U_0 (6 - \gamma^2), & \gamma^2 < 6 \\ 4 \left(\sqrt{3\gamma - 3\sqrt{2}}\right) c_2 \cot(\Delta), & \gamma^2 \geq 6 \end{cases} + O(\varepsilon^2). \] (52b)

The relative errors in the approximation of (52) by \( u = u(t; 0), v = v(t; 0) \) are

\[ E_v(u) := u(t; \varepsilon) - u(t; 0) \over u(t; \varepsilon) = \begin{cases} 2 \ln(\cosh(\Delta)) \over \sqrt{6\gamma + 6}, & \gamma^2 < 6 \\ 2 \sqrt{2} c_2 \tanh(\Delta) \over \sqrt{6\gamma + 6}, & \gamma^2 \geq 6 \end{cases} + O(\varepsilon^2), \] (53a)

\[ E_v(v) := v(t; \varepsilon) - v(t; 0) \over v(t; \varepsilon) = \begin{cases} 2 \ln(\sin(\Delta)) \over \sqrt{6\gamma + 6}, & \gamma^2 < 6 \\ 4 \left(\sqrt{3\gamma - 3\sqrt{2}}\right) c_2 \coth(\Delta), & \gamma^2 < 6 \\ -4 \left(\sqrt{6\gamma + 6}\right) U_0 (6 - \gamma^2), & \gamma^2 < 6 \\ 4 \left(\sqrt{3\gamma - 3\sqrt{2}}\right) c_2 \cot(\Delta), & \gamma^2 \geq 6 \end{cases} + O(\varepsilon^2). \] (53b)

Taking the limit \( t \to +\infty \) it follows that the above relative errors tend to zero. Thus, the linear terms in \( \varepsilon \) in the equation (52) can be made a small percent of the contribution of the zeroth-solutions by taking \( \tau \) large enough. Henceforth, this shows that the behavior of the solutions for the induced gravity model does not change abruptly when a massless scalar field \( \phi \) with a small kinetic term is added to the setup.

3. Review on averaging techniques

The averaging methods were applied extensively in [106–110, 112–116] to single field scalar field cosmologies and was extended to scalar field cosmologies of two fields in [148]. New variables and dimensionless time variables were adopted, which have not been used to analyze these cosmological dynamics. The main difficulties that arise when using standard dynamical systems approaches are due to the oscillations that enter the nonlinear system through the KG equations. This motivates the analysis of the oscillations using averaging techniques.
The theory of averaging studies initial value problems of the general form
\[ \dot{x} = f(x, t, \varepsilon), \quad x(0) = x_0, \]
with \( x, f(x, t, \varepsilon) \in \mathbb{R}^n \), where \( \varepsilon \) plays the role of a, usually small, perturbation parameter. Typically one would then perform a Taylor expansion of \( f \) in \( \varepsilon \) around \( \varepsilon = 0 \). For the simplest form of averaging, \textit{periodic averaging}, the zeroth order term usually vanishes, and one is typically looking at problems of the standard form
\[ \dot{x} = \varepsilon f^1(x, t), \quad x(0) = x_0, \]
(54)
with \( f^1 \) \( T \)-periodic in \( t \). The exponents correspond to the respective perturbative order, and the square bracket marks the remainder of the series (notation 1.5.2, p 13 [138]).

To first order, the theory is then concerned with the question to what degree solutions of (54) can be approximated by the solutions of an associated \textit{averaged system}
\[ \dot{y} = \varepsilon \bar{f}^1(y), \quad y(0) = x_0, \]
(55)
with
\[ \bar{f}^1(y) = \frac{1}{T} \int_0^T f^1(y, s) \, ds. \]
(56)

Take the following two definitions from [138]:

**Definition 1. (p 31 [138]).** \( D \subset \mathbb{R}^n \) is a connected, bounded open set (with compact closure) containing the initial value \( x_0 \), and constants \( L > 0, \varepsilon_0 > 0 \), such that the solutions \( x(t, \varepsilon) \) and \( y(t, \varepsilon) \) with \( 0 \leq \varepsilon \leq \varepsilon_0 \) remain in \( D \) for \( 0 \leq t \leq L/\varepsilon \).

**Definition 2. (Definition 4.2.4 of [138]).** Consider the vector field \( f(x, t) \) with \( f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \). Let \( f \) be Lipschitz continuous in \( x \) on \( D \subset \mathbb{R}^n, t \geq 0 \). Let further \( f \) be continuous in \( t \) and \( x \) on \( \mathbb{R}_+ \times D \). If the average
\[ \bar{f}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x, s) \, ds, \]
exists and the limit is uniform in \( x \) on compact subsets of \( D \), then \( f \) is called a \textbf{KBM-vector field} (from the initials Krylov, Bogoliubov and Mitropolsky). If the vector field \( f(x, t) \) contains parameters, we assume that the parameters and the initial conditions are independent of \( \varepsilon \) and that the limit is uniform in the parameters.

The basic result is given by the following theorem:

**Lemma 3.1. (Theorem 11.1 of [137]).** Let be the \( n \)-dimensional system (54). Supposing that \( f^1(t, x) \) is \( T \)-periodic in \( t \), with \( T > 0 \) a constant independent of \( \varepsilon \). Performing the averaging process (56) where \( y \) is considered as a parameter that is kept constant during integration. Let be the associated initial value problem
\[ \dot{y} = \varepsilon \bar{f}^1(y), \quad y(0) = x_0. \]
(58)
Then, we have \( y(t) = x(t) + \mathcal{O}(\varepsilon) \) on the time scale \( 1/\varepsilon \), under fairly general conditions:
(a) The vector functions \( f^1 \) and \( f^{[2]} \) are continuously differentiable in a bounded \( n \)-dimensional domain \( D \), with \( x_0 \) an interior point, on the time scale \( 1/\varepsilon \).
(b) \( y(t) \) remains interior to the domain \( D \) on the time scale \( 1/\varepsilon \) to avoid boundary effects.
A similar result is:

**Lemma 3.2. (Theorem 2.8.1, p 31 [138]).** Let $f^1$ be Lipschitz continuous, let $f^{(2)}$ be continuous, and let $\varepsilon_0, D, L$ be as in definition 1. Then there exists a constant $c > 0$ such that

$$\|x(t, \varepsilon) - y(t, \varepsilon)\| < c\varepsilon,$$

for $0 \leq \varepsilon \leq \varepsilon_0$ and $0 \leq t \leq L/\varepsilon$, and where $\|\cdot\|$ denotes the norm $\|u\| := \sum_{i=1}^n |u_i|$ for $u \in \mathbb{R}^n$.

Now, supposing that the slowly varying system $\dot{x} = \varepsilon f^1(t, x)$ is such that $f^1(t, x)$ is not periodic, nor a finite sum of periodic vector fields, we have the following result:

**Lemma 3.3. (Theorem 11.3 of [137]).** Let $b$ be the $n$-dimensional system (54). Supposing that $f^1(t, x)$ can be averaged over $t$ in the sense that the limit (57) exists. Let $b$ be the associated initial value problem

$$\dot{y} = \varepsilon \overline{f}(y), \quad y(0) = x_0,$$

where $y$ is again considered a parameter that is kept constant during integration. Then, we have

$$y(t) = x(t) + \mathcal{O}(\delta(\varepsilon)), \quad (60)$$

on the timescale $1/\varepsilon$ under fairly general conditions:

(a) The vector functions $f^1$ and $f^{(2)}$ are continuously differentiable in a bounded $n$-dimensional domain $D$ with $x_0$ an interior point on the timescale $1/\varepsilon$.

(b) $y(t)$ remains interior to the domain $D$ on the timescale $1/\varepsilon$ to avoid boundary effects.

For the error $\delta(\varepsilon)$, we have the explicit estimate

$$\delta(\varepsilon) = \sup_{x \in D} \sup_{0 \leq \varepsilon \leq C} \left\| \int_0^t \left( f^1(s, x) - \overline{f}(x) \right) ds \right\|,$$

with $C$ a constant independent of $\varepsilon$.

In other words, the error made when approximating the entire system (54) by the averaged system (55) will be of the order $\delta(\varepsilon)$ on timescales of the order $\varepsilon^{-1}$. When the solutions of the complete or averaged system are attracted by an asymptotically stable critical point, the approximation domain can be extended to all times (see chapter 5 of [138]). For instance:

**Lemma 3.4. (Theorem 5.5.1 by Eckhaus/Sanchez–Palencia of p 101 [138]).** Consider the initial value problem

$$\dot{x} = \varepsilon f^1(x, t), \quad x(0) = x_0,$$

with $x_0, x \in D \subset \mathbb{R}^n$. Suppose $f^1$ is a KBM-vector field (definition 2) producing the averaged equation

$$\dot{y} = \varepsilon \overline{f}(y), \quad y(0) = x_0,$$

where $y = 0$ is an asymptotically stable critical point in the linear approximation, $\overline{f}$ is continuously differentiable with respect to $y$ in $D$ and has a domain of attraction $D' \subset D$. Then for any compact $K \subset D'$ there exists a $\delta(\varepsilon) > 0$ such that for all $x_0 \in K$

$$x(t) - y(t) = \mathcal{O}(\delta(\varepsilon)), \quad 0 \leq t < \infty,$$
with \( \delta(\varepsilon) = O(1) \) in the general case and \( O(\varepsilon) \) in the periodic case.

For periodic solutions we have the following:

**Lemma 3.5. (Theorem 11.4 of [137]).** \( \dot{x} = \varepsilon f(t, x) \) is such that \( f(t, x) \) is \( T \)-periodic and the averaged equations

\[
\dot{y} = \varepsilon \mathbf{\bar{f}}(y),
\]

with

\[
\mathbf{\bar{f}}(y) = \frac{1}{T} \int_0^T f(t, y) dt,
\]

where \( y_0 \) is a stationary solution (equilibrium point) of the averaged equation \( \mathbf{\bar{f}}(y_0) = 0 \). If (a) \( f(t, x) \) is a smooth vector field, (b) for the Jacobian in \( y_0 \) we have

\[
\left| \frac{\partial \mathbf{\bar{f}}}{\partial y} \right|_{y=y_0} \neq 0,
\]

then a \( T \)-periodic solution of the equation \( \dot{x} = \varepsilon f(t, x) \) exists in an \( \varepsilon \)-neighborhood of \( x = y_0 \). We can establish the stability of the periodic solution as it matches exactly the stability of the stationary solution of the averaged equation. This reduces the stability problem of the periodic solution to determine the eigenvalues of a matrix.

To summarize, methods from the theory of averaging nonlinear dynamical systems allow us to prove that time-dependent systems and their corresponding time-averaged versions have the same late-time dynamics. Therefore, simple time-averaged systems determine the future asymptotic behavior.

### 3.1 Example 4: harmonic oscillator

Giving a differential equation \( \dot{x} = f(x, t, \varepsilon) \) with \( f \) periodic in \( t \). An approximation scheme that can be used consists of solving the problem for \( \varepsilon = 0 \) (unperturbed problem). Then, use this approximated unperturbed solution to formulate variational equations in standard form which can be averaged.

Take the simple equation

\[
\ddot{\phi} + \phi = \varepsilon(-2\dot{\phi}),
\]

with \( \phi(0), \dot{\phi}(0) \) given. The unperturbed problem:

\[
\ddot{\phi} + \phi = 0,
\]

have as solution

\[
\dot{\phi}(t) = r_0 \cos(t - \varphi_0), \quad \phi(t) = r_0 \sin(t - \varphi_0),
\]

where \( r_0 \) and \( \varphi_0 \) are constants depending on the initial conditions. Using the amplitude-phase variables defined as (10) with inverse transformation (11), and under the coordinate transformation \((\dot{\phi}, \phi) \rightarrow (r, \varphi)\), equation (65) leads to

\[
\dot{r} = -2\varepsilon\cos^2(t - \varphi), \quad \dot{\varphi} = -\varepsilon\sin(2(t - \varphi)).
\]
These equations mean that $r$ and $\varphi$ are slow-varying with time, and the system is in the form $\dot{y} = \varepsilon f(y)$. The idea is to consider only the nonzero average of the right-hand-sides, keeping $r$ and $\varphi$ fixed, and leave out the terms with average zero ignoring the slow-varying dependence of $r$ and $\varphi$ on $t$ in the averaging process. Now, replacing $r$, $\varphi$ by their averaged approximations $\overline{r}$, $\overline{\varphi}$, is obtained

\begin{align}
\dot{\overline{\varphi}} &= -\varepsilon \frac{1}{2\pi} \int_0^{2\pi} 2r \cos^2(t - \varphi) \, dt = -\varepsilon \overline{r}, \\
\dot{\overline{\varphi}} &= -\varepsilon \frac{1}{2\pi} \int_0^{2\pi} \sin(2(t - \varphi)) \, dt = 0,
\end{align}

(69)

where, by lemma 3.2, we know that the error between $[r, \varphi]^T$ and $[\overline{r}, \overline{\varphi}]^T$ will be of order $\varepsilon$ on timescales $O(\varepsilon^{-1})$.

Solving (69) with initial conditions $\overline{r}(0) = r_0$ and $\overline{\varphi}(0) = \varphi_0$, the approximation takes the form

$$
\overline{\varphi} = r_0 e^{-\varepsilon t} \sin(t - \varphi_0),
$$

(70)

which coincides with the result that would be obtained using the two-timing expansion procedure. These two procedures alleviate the failure of the regular asymptotic expansion that would yield spurious secular terms in the asymptotic expansions, say, on the regular asymptotic expansion (22), the ‘next to leading term’ $\varepsilon t \sin t$ is dominant on scales $\varepsilon t = O(1)$.

These techniques can be extended to homogeneous cosmologies when $H$, the Hubble parameter, is considered as a time-dependent perturbation parameter. One example is the model in [112] for LRS Bianchi III Einstein-KG system. This system is analogous to an harmonic oscillator with nonlinear damping, and where the time dependence of the latter is governed by the coupling of the Einstein equations with the KG equation (9) via $H$. In [112] the state vector $x = [\Sigma, \Omega, \varphi]^T$, $\varphi = r^2/(6H^2)$, and $(r, \varphi)$ defined by (11) is introduced, and the system takes the form

$$
\begin{bmatrix}
\dot{H} \\
\dot{x}
\end{bmatrix} = H F^1(x, t) + H^2 F^{[2]}(x, t) = H \begin{bmatrix}
0 \\
F^1(x, t)
\end{bmatrix} + H^2 \begin{bmatrix}
F^{[2]}(x, t) \\
0
\end{bmatrix},
$$

(71)

where $F^1, F^{[2]}$ are independent of $H$. One can see that (71) is resembling the standard form (54) with $H(t)$ playing the role of the perturbation parameter $\varepsilon$. The resulting system was studied in [112] using averaging techniques.

Let $\overline{y}(t)$ denote the solution of the corresponding averaged system. Then from lemma 3.2 one knows that $y(t) - \overline{y}(t) = O(H_\ast)$ on time scales of $O(H_\ast^{-1})$, where $H_\ast$ is the value of $H$ at a large truncation time $t^*$, $H(t^*)$. Furthermore, one have a case of averaging with attraction and, one can extend the validity of this error estimate for all times for the $x$-components. In [113], a more general result was proved, where the long-term behavior of solutions of a general class of systems in standard form (71) was studied; where $H > 0$ is strictly decreasing in $t$ and $\lim_{t \to \infty} H(t) = 0$. Theorem by [113], gives local-in-time asymptotics for system (71).

Let the norm $\| \cdot \|$ denotes the standard discrete $l^1$-norm $\| u \| := \sum_{i \leq \infty} |u_i|$ for $u \in \mathbb{R}^n$. Let also $L^\infty_{\mathbb{R}}$ denotes the standard $L^\infty$ space in both $t$ and $x$ variables with norm defined as $\| f \|_{L^\infty_{\mathbb{R}}} := \sup_{x \in \mathbb{R}} |f(x, t)|$.

**Theorem 3.1. (Theorem 3.1 of [113]).** Suppose $H(t) > 0$ is strictly decreasing in $t$ and $\lim_{t \to \infty} H(t) = 0$. Fix any $\varepsilon > 0$ with $\varepsilon < H(0)$ and define $t_\ast > 0$ such that $\varepsilon = H(t_\ast)$. Suppose that $\| F^1 \|_{L^\infty} \leq \| F^{[2]} \|_{L^\infty} \leq \infty$ and that $F^1(x, t)$ is Lipschitz continuous and $F^{[2]}$ is continuous with
respect to $x$ for all $t \geq t_c$. Also, assume that $f^1$ and $f^{[2]}$ are $T$-periodic for some $T > 0$. Then for all $t > t_c$ with $t = t_c + \mathcal{O}\left(H(t_c)^{\delta}\right)$ for any given $\delta \in (0,1)$ we have

$$x(t) - y(t) = \mathcal{O}\left(H(t_c)^{\min\{1.2-2\delta\}}\right),$$

where $x$ is the solution of system (71) with initial condition $x(0) = x_0$ and $y(t)$ is the solution of the time-averaged system

$$\dot{y} = H(t_c)\bar{F}(y), \quad \text{for } t > t_c,$$

with initial condition $y(t_c) = x(t_c)$ where the time-averaged vector $\bar{F}$ is defined as

$$\bar{F}(y) = \frac{1}{T}\int_{t_c}^{t_c+T} f^1(y,s)ds.$$

In references [114, 115], systems which are not in the standard form (71), but can be expressed as a series with center in $H = 0$ according to the equation

$$\begin{bmatrix} \dot{H} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & H^0(x, t) \\ H^0(x, t) & \bar{f}(x, t) \end{bmatrix} + \mathcal{O}\left(H^2, \mu^2\right),$$

were studied. These systems depend on a parameter $\omega$ which is a free angular frequency that can be tuned to make $\bar{F}^1(x, t) = 0$. Therefore, systems can be expressed in the standard form (71). The examples worked in reference [114] correspond to generalized scalar-field cosmologies with matter in LRS Bianchi III and open FLRW model with generalized harmonic potential

$$V(\phi) = \mu^2\phi^2 + f^2(\omega^2 - 2\mu^2 \left(1 - \frac{\sin(\frac{x}{T})}{x}\right)).$$

(73)

The asymptotic features of potential (73) are the following. Near the global minimum $\phi = 0$, we have $V(\phi) \sim \frac{\omega^2\phi^2}{2} + \mathcal{O}\left(\phi^3\right)$, as $\phi \to 0$. That is, $\omega^2$ can be related to the mass of the scalar field near its global minimum. As $\phi \to \pm \infty$ the cosine-correction is bounded, then $V(\phi) \sim \mu^2\phi^2 + \mathcal{O}(1)$. This makes it suitable to describe oscillatory behavior in cosmology.

The state vector is $x = (\Omega, \Sigma, \Omega_b, \Phi)\top$, the system can be symbolically written as a Taylor series of the form (72). The term $\bar{F}^1(t, x)$ in expression (72) is eliminated imposing the condition $b\mu^2 + 2f\mu^2 - f\omega^2 = 0$, which defines an angular frequency $\omega \in \mathbb{R}$. Then, order zero terms in the series expansion around $H = 0$ are eliminated assuming $\omega^2 > 2\mu^2$ and setting $f = \frac{b\mu}{2\omega^2}$, which is equivalent to tune $\omega$. In theorem 2 of [114] it was proved that if $\Omega, \Sigma, \Omega_b, \Phi$ and $H$ are the solutions of averaged equations. Then, there exist continuously differentiable functions $g_1, g_2, g_3$ and $g_4$, such that $\Omega, \Sigma, \Omega_b$ and $\Phi$ are locally given by [106, 107]

$$x_0 := (\Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0)\top \mapsto x := (\Omega, \Sigma, \Omega_b, \Phi)\top,$$

$$x = x_0 + Hg(H, x_0, t), \quad g(H, x_0, t) = \begin{bmatrix} g_1(H, \Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0, t) \\ g_2(H, \Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0, t) \\ g_3(H, \Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0, t) \\ g_4(H, \Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0, t) \end{bmatrix},$$

(74)

where $\Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0$ are order zero approximations of them as $H \to 0$. Then, functions $\Omega_0, \Sigma_0, \Omega_{b0}, \Phi_0$ and averaged solution $\bar{\Omega}, \bar{\Sigma}, \bar{\Omega}_b, \bar{\Phi}$ have the same limit as $t \to \infty$. Setting $\Sigma = \Sigma_0 = 0$ are derived the analogous results for the negatively curved FLRW model.
Theorem 3 of [114] shows that the late time attractors of the full system and averaged system for Bianchi III line element are the same. The results from the linear stability analysis combined with theorem 2 of [114] (for $\Sigma = 0$, open FLRW) lead to theorem 4 in [114], which shows that the late time attractors of the full system and the averaged system are the same.

The examples worked in reference [115] correspond to generalized scalar-field cosmologies with the matter in LRS Bianchi I and flat FLRW model. Denoting $x = (\Omega, \Sigma, \Phi)^T$ and using the condition $b\mu^3 + 2g\mu^2 - f\omega^2 = 0$, to obtain a system can be expressed in the standard form (71). Proceeding in analogous way as in references [106, 107] but for three dimensional systems instead of a one-dimensional one, it was implemented a local nonlinear transformation

$$x_0 := (\Omega_0, \Sigma_0, \Phi_0)^T \mapsto x := (\Omega, \Sigma, \Phi)^T,$$

$$x = x_0 + H g(H, x_0, t), g(H, x_0, t) = \begin{bmatrix} g_1(H, \Omega_0, \Sigma_0, \Phi_0, t) \\ g_2(H, \Omega_0, \Sigma_0, \Phi_0, t) \\ g_3(H, \Omega_0, \Sigma_0, \Phi_0, t) \end{bmatrix}. \quad (75)$$

Theorem 1 of [115], states that, given the functions $\Omega, \Sigma, \Phi$ such that $\Omega, \Sigma, \Phi$ are locally given by (75) where $\Omega_0, \Sigma_0, \Phi_0$ are zero order approximations of $\Omega, \Sigma, \Phi$ as $H \to 0$. Then, functions $\Omega_0, \Sigma_0, \Phi_0$ and averaged solution $\Omega, \Sigma, \Phi$ have the same limit as $t \to \infty$. Setting $\Sigma = \Sigma_0 = 0$ analogous results for flat FLRW model are derived. Results from the linear stability analysis which are combined with theorem 1 of [115], lead to theorem 2 of [115], where the late-time attractors of the full system and time-averaged system for LRS Bianchi I line element are proved to be the same. For flat FLRW metric, theorem 3 of [115] shows that the late-time attractors of the full system and averaged system with $\Sigma = 0$ are the same too.

The core of these examples is to show how methods from the theory of averaging in non-linear dynamical systems can be used to prove that time-dependent systems and their corresponding time-averaged versions have the same late-time dynamics. Therefore, the simplest time-averaged system determines the future asymptotic behavior. Depending on the values of free parameters, we can find the late-time attractors of physical interests. With this approach, the oscillations entering the system through the KG equation can be controlled and smoothed out as the Hubble parameter $H$—acting as time-dependent perturbation parameter—tends monotonically to zero. In other words, these results show that one can ‘average out’ the oscillations arising due to the harmonic functions, thus simplifying the problem.

4. Perturbation and averaging methods applied to interacting scalar field cosmology

It is worth noticing that when Hubble-normalized quantities are used more often the evolution equation for $H$, which is given by the Raychaudhuri equation, decouples. The asymptotic of the remaining reduced system is then typically given by the equilibrium points and often it can be determined by a dynamical system analysis [102, 151]. In particular, this is always the case for a scalar field with exponential potential. This is due to the fact the exponential potential has symmetry such that its derivative is also an exponential function. For other potentials that do not satisfy the above symmetry, like the generalized harmonic potential $V(\phi) = \mu^2 \phi^2 + \text{cosine corrections}$, the Raychaudhuri equation fails to decouple [55]. Hubble-normalized equations often are very difficult to be analyzed using the standard dynamical systems approach due to oscillations entering the system via the KG equation [112, 114, 115].
The preliminary analysis of oscillations in scalar-field cosmologies with generalized harmonic potentials of type \( V(\phi) = \mu^2 \phi^2 + \text{cosine corrections} \) is extended here using averaging techniques similar to those used in \([112, 114, 115]\) for a family of generalized harmonic potentials when \( H \) is monotonically decreasing. In this approach, the Hubble scalar plays a role of a time-dependent perturbation parameter which controls the magnitude of the error between full-system and time-averaged solutions. These oscillations can be viewed as perturbations that can be smoothed out with the benefit that the averaged Raychaudhuri equation decouples in the averaged system. In the end, the analysis of the system is reduced to the study of corresponding averaged equations.

In this section, we investigate a cosmological model obtained by varying the action (2) for FLRW and Bianchi I geometries. An auxiliary function is used to include them, defined by

\[
G_0(a) = \begin{cases} 
-\frac{3}{a^2}, & k = 0, \pm 1, \text{ spatial curvature of FLRW metrics} \\
\sigma_0^2, & \text{anisotropies of Bianchi I metric}
\end{cases}
\]

We assume that the energy–momentum tensor (3) is in the form of a perfect fluid

\[
T^\alpha_\beta = \text{diag}(-\rho_m, p_m, p_m, p_m),
\]

where \( \rho_m \) and \( p_m \) are respectively the isotropic energy density and the isotropic pressure (consistently with FLRW metric, pressure is necessarily isotropic \([149]\)). For simplicity we will assume a barotropic EoS

\[
p_m = (\gamma - 1) \rho_m,\]

Also we consider a quintessence scalar field, \( \phi \), interacting in the action with the perfect fluid. In this case, the equations for FLRW and Bianchi I metrics are \([28, 48]\):

\[
\begin{align*}
\ddot{\phi} + 3H \dot{\phi} + \frac{dV(\phi)}{d\phi} &= \frac{1}{2} (4 - 3\gamma) \rho_m \frac{d \ln \chi(\phi)}{d\phi}, \\
\dot{\rho}_m + 3\gamma H \rho_m &= -\frac{1}{2} (4 - 3\gamma) \rho_m \frac{d \ln \chi(\phi)}{d\phi}, \\
\dot{a} &= aH, \\
\dot{H} &= -\frac{1}{2} \left( \gamma \rho_m + \dot{\phi}^2 \right) + \frac{1}{6} aG_0(a), \\
3H^2 &= \rho_m + \frac{1}{2} \dot{\phi}^2 + V(\phi) + G_0(a),
\end{align*}
\]

where \( a(t) \) denotes the scale factor of the universe, \( H = \frac{\dot{a}}{a} \) denotes the Hubble parameter, a dot accounts for the derivative with respect to \( t \), \( \phi \) is the scalar field, \( V(\phi) \) the scalar field self-interacting potential which is assumed to be of class \( C^2 \), \( \chi(\phi)^{-2} \) is the coupling function, \( \rho_m \) corresponds to the energy density of matter with EoS parameter \( w_m = \frac{p_m}{\rho_m} \equiv \gamma - 1 \), where \( 0 \leq \gamma \leq 2 \) denotes the barotropic index. The integration of (77b) leads to

\[
\rho_m = \frac{\rho_{m0}}{a^{3\gamma}} \chi(\phi(a))^{-2+\frac{3\gamma}{2}}.
\]

As in \([25]\), here the baryons (a subdominant component at present, but important in the past of the cosmic evolution) are included in the background of dark matter. We assume a generalized harmonic potential (7) non-minimally coupled to matter with coupling (8). Potential (7) belongs to the class of potentials studied by \([26]\). In the figure 4, it is presented this the generalized harmonic potential and its derivative for \( f = 0.1, f = 0.3 \) and \( f = 0.9 \). In
first case the potential has three local minimums and two local maximums. In other two cases the origin is the unique stationary point and the global minimum of the potential. Harmonic potentials plus cosine corrections were introduced in the context of inflation in loop-quantum cosmology in \[136\]. In \[135\], some theorems related to the asymptotic behavior of a very general cosmological model given by system \((77)\) were presented. Using the Hubble-normalized formulation for a scalar field non-minimally coupled to matter with generalized harmonic potential \((7)\) and with coupling function \((8)\) where \(\lambda\) is a constant and \(0 \leq \gamma \leq 2, \gamma \neq \frac{4}{3}\) the late time attractors corresponding to the non zero local minimums of the potential for FLRW metrics and for the Bianchi I metric were found. These equilibrium points are related to de Sitter solutions. The global minimum of \(V(\phi)\) at \(\phi = 0\) is unstable to curvature perturbations for \(\gamma > \frac{2}{3}\) in the case of a negatively curved FLRW model. This confirms the result in \[89\], that in a non-degenerated minimum with zero critical value, the curvature will eventually dominate both the perfect fluid and the scalar field densities on the late evolution of the universe for \(\gamma > 2/3\). For the Bianchi I model the global minimum \(V(0) = 0\) is unstable to shear perturbations. Equations for a scalar field cosmology minimally coupled to matter for FLRW metrics and for Bianchi I metrics are obtained by setting \(\chi(\phi) = 1\) in \((77)\) with \(G_0(a)\) given by \((76)\) \[102, 150\]. Equation \((78)\) reduces to \(\rho_m = \frac{\rho_{m0}}{a^3}\). The field equations of a scalar field with self-interacting potential \(V(\phi)\) in a vacuum for flat FLRW metric are obtained by setting \(\chi(\phi) = 1, \rho_m = 0\) in \((77)\) with \(G_0(a) = 0\). In \[109\], a local dynamical systems analysis for arbitrary \(V(\phi)\) and \(\chi(\phi)\) using Hubble normalized equations was provided. The analysis relies on two arbitrary functions \(f(\lambda)\) and \(g(\lambda)\) which encode a potential and a coupling function through a quadrature. Afterward, a global dynamical systems formulation using the Alho and Uggla’s approach \[55\] was implemented. The equilibrium points that represent some solutions of cosmological interest were obtained. In particular, several scaling solutions are found, as well as
stiff solutions, and a solution dominated by the effective energy density of the geometric term $G_0(a)$, a quintessence scalar field dominated solution, the vacuum de Sitter solution associated to the minimum of the potential and a non-interacting matter-dominated solution. All of which reveals a very rich cosmological behavior.

4.1. Scalar field with generalized harmonic potential non-minimally coupled to matter

In this section the averaging methods are applied for FLRW and Bianchi I metrics for the generalized harmonic potential (7) coupled to matter with coupling function (8). In the following sections the FLRW and Bianchi I models will be studied separately.

4.1.1. FLRW metric. In this case the field equations are:

\begin{equation}
\ddot{\phi} + 3H\dot{\phi} + \phi + \sin\left(\frac{\phi}{f}\right) = \frac{\lambda}{2} \rho_m,
\end{equation}

\begin{equation}
\dot{\rho}_m + 3\gamma H \rho_m = -\frac{\lambda}{2} \rho_m \dot{\phi},
\end{equation}

\begin{equation}
\dot{a} = aH,
\end{equation}

\begin{equation}
H = -\frac{1}{2} \left(\gamma \rho_m + \phi^2\right) + \frac{k}{a^2},
\end{equation}

\begin{equation}
3H^2 = \rho_m + \frac{1}{2} \dot{\phi}^2 + \frac{\phi^2}{2} + f \left[1 - \cos\left(\frac{\phi}{f}\right)\right] - \frac{3k}{a^2}.
\end{equation}

Using the amplitude-phase variables (10) with inverse transformation (11), it follows

\begin{equation}
\dot{r} = \frac{\phi}{r} \left[\ddot{\phi} + \phi + \sin\left(\frac{\phi}{f}\right)\right] = \frac{\phi}{r} \left[-3H\dot{\phi} - \sin\left(\frac{\phi}{f}\right) + \frac{\lambda}{2} \rho_m\right]
\end{equation}

and

\begin{equation}
\dot{\phi} = \frac{\phi}{r^2} \left[\ddot{\phi} + \phi + \sin\left(\frac{\phi}{f}\right)\right] = \frac{\phi}{r^2} \left[-3H\dot{\phi} - \sin\left(\frac{\phi}{f}\right) + \frac{\lambda}{2} \rho_m\right]
\end{equation}

\begin{equation}
= -3H \sin(t - \varphi) \cos(t - \varphi) - \frac{\sin(t - \varphi) \sin\left(\frac{r \sin(t - \varphi)}{f}\right)}{r} + \frac{\lambda}{2} \rho_m \sin(t - \varphi). 
\end{equation}

Defining

\begin{equation}
\Omega = \frac{r^2}{6H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_k = -\frac{k}{a^2 H^2},
\end{equation}

such that

\begin{equation}
f \cos\left(\frac{\sqrt{6\sqrt{\Omega}} H \sin(t - \varphi)}{f}\right) = f - 3H^2 (1 - \Omega - \Omega_k - \Omega_m),
\end{equation}

23
the following dynamical system is obtained

\[
\begin{align*}
\dot{H} &= -\frac{1}{2}H^2 (3\gamma \Omega_m + 6\Omega \cos^2(t - \varphi) + 2\Omega_k) \\
\dot{\Omega} &= \frac{1}{2}H (2\Omega (3\gamma \Omega_m + 3\Omega - 1) \cos(2(t - \varphi)) + 3\Omega + 2\Omega_k - 3) \\
&\quad + \sqrt{6\lambda \Omega \Omega_m} \cos(t - \varphi) - \sqrt{\frac{3}{2}\Omega \sin(t - \varphi)} \\
\dot{\Omega}_m &= \frac{1}{2} \Omega_m H \left(6\gamma (\Omega_m - 1) - \sqrt{6\lambda \Omega \Omega_m} \cos(t - \varphi) + 6\Omega \cos(2(t - \varphi)) + 6\Omega_m + 4\Omega_k\right), \\
\dot{\Omega}_k &= \Omega_k H (3\gamma \Omega_m + 6\Omega \cos^2(t - \varphi) + 2\Omega_k - 2) \\
\dot{\varphi} &= \frac{1}{4}H \left(\sqrt{6\lambda \Omega_m} \sin(t - \varphi) - 6 \sin(2(t - \varphi))\right) - \frac{\sin(t - \varphi) \sin\left(\sqrt{\frac{3}{2}\Omega \sin(t - \varphi)}\right)}{\sqrt{6\Omega H}}.
\end{align*}
\]

For the problem (84), using the techniques of section 3.1, we obtain the averaged system

\[
\begin{align*}
\ddot{\Omega} &= -\frac{1}{2}H^2 (3\gamma \Omega_m + 3\Omega + 2\Omega_k) \\
\ddot{\Omega}_m &= \Omega_m H (3\gamma \Omega_m - 1) + 3\Omega + 2\Omega_k, \\
\ddot{\Omega}_k &= \Omega_k H (3\gamma \Omega_m + 3\Omega + 2\Omega_k - 2)
\end{align*}
\]

where the angular equation is decoupled. Defining the new temporary variable \(\tau = \ln a\), the following guiding system is obtained:

\[
\begin{align*}
\partial_t \ddot{\Omega} &= \ddot{\Omega}_m (3\gamma \Omega_m + 3\Omega + 2\Omega_k - 3), \\
\partial_t \ddot{\Omega}_m &= \Omega_m H (3\gamma \Omega_m - 1) + 3\Omega + 2\Omega_k, \\
\partial_t \ddot{\Omega}_k &= \Omega_k H (3\gamma \Omega_m + 3\Omega + 2\Omega_k - 2).
\end{align*}
\]

The equilibrium points for system (86) are \(P_1 = (1, 0, 0), P_2 = (0, 1, 0), P_3 = (0, 0, 1)\) and \(P_4 = (0, 0, 0)\). By evaluating the linearization matrix of system (86) on each of the equilibrium points and calculating its eigenvalues, we obtain the stability of each point depending on \(\gamma\), these results are summarized in the table 1.

Point \(P_1\) corresponds to a flat FLRW scalar field dominated solution, that is a saddle point for \(1 < \gamma \leq 2\) (i.e. if the perfect fluid EoS is in the matter domain), or a source for \(0 \leq \gamma < 1\) (i.e. if the perfect fluid has a negative pressure). Point \(P_2\) corresponds to the flat FLRW matter-dominated solution and it is unstable to matter perturbations. It is a source for \(1 < \gamma \leq 2\) (i.e. if the perfect fluid EoS is in the matter domain) or a saddle if \(0 < \gamma < 2/3\) or \(2/3 < \gamma < 1\). It corresponds to a transient epoch in cosmological history. Point \(P_3\) is a curvature-dominated solution with positive curvature (Misner solution). The energy density of the scalar field scales as \(a^{-2}\) and is a saddle (unstable to curvature perturbations). Finally, point \(P_4\) corresponds to a vacuum Minkowski solution, that is a sink. The Minkowski solution represents an empty universe. Physically, Minkowski space-time can be used as a local approximation of space-time in reasonably small regions and the presence of matter, as long as it does not self-gravitate.
This example would be possible in the context of inflation from an isotropic initial state, and we see that isotropization is a transient state in the universe, before reaching the Minkowski solution (that is flat, isotropic, and empty of matter). This approach, in which the scalar field oscillates in the minimum, before reaching a Minkowski solution, is useful for describing the oscillations of the inflaton around the potential minimum during reheating after inflation. This behavior was described in models like the $N$-field inflation model [152] as well as in axion-like matter [148]. In figure 5 is shown that the origin is a sink as indicated in table 1.

4.1.2. Bianchi I metric. In this case, the field equations are:

\[
\ddot{\phi} + 3H\dot{\phi} + \phi + \sin\left(\frac{\phi}{f}\right) = \frac{\lambda}{\pi} \rho_m, \tag{87a}
\]

\[
\dot{\rho}_m + 3\gamma H \rho_m = -\frac{\lambda}{2} \rho_m \dot{\phi}, \tag{87b}
\]

\[
\dot{a} = aH, \tag{87c}
\]

\[
\dot{H} = -\frac{1}{2} \left(\gamma \rho_m + \dot{\phi}^2\right) - \frac{\sigma_0^2}{a^6}, \tag{87d}
\]

\[
3H^2 = \rho_m + \frac{1}{2} \dot{\phi}^2 + \frac{\phi^2}{2} + f \left[1 - \cos\left(\frac{\phi}{f}\right)\right] + \frac{\sigma_0^2}{a^6}. \tag{87e}
\]

Using the amplitude-phase transformation (10) with (11), and defining

\[
\Omega = \frac{r^2}{6H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2}, \quad \Sigma = \frac{\sigma_0}{a^6 H}, \tag{88}
\]

such that

\[
f \cos\left(\frac{\sqrt{6\sqrt{3}t} \sin(t - \varphi)}{f}\right) = f - H^2 \left(3(1 - \Omega - \Omega_m) - \Sigma^2\right), \tag{89}
\]
Figure 5. Phase portrait of the system (86) for $\gamma = 0, 1/3, 2$.

the following dynamical system is obtained

$$
\begin{align*}
\dot{H} &= -\frac{1}{2}H^2 \left(3\gamma \Omega_m + 2\Sigma^2 + 6\Omega \cos^2(t - \varphi)\right) \\
\dot{\Omega} &= \frac{1}{2}H \left(2\Omega \left(3\gamma \Omega_m + 2\Sigma^2 + 3(\Omega - 1)\cos(2(t - \varphi)) + 3\Omega - 3\right)
+ \sqrt{6}\lambda \Omega_m \cos(t - \varphi) - \frac{\sqrt{2}\sqrt{\Omega} \cos(t - \varphi) \sin \left(\frac{\sqrt{6}\lambda \Omega_m \sin(t - \varphi)}{\sqrt{\Omega}}\right)}{\sqrt{6} \sqrt{\Omega} H} \right) \\
\dot{\Omega}_m &= \frac{1}{2} \Omega_m H \left(6\gamma (\Omega_m - 1) + 4\Sigma^2 - \sqrt{6}\lambda \Omega_m \cos(t - \varphi) + 6\Omega \cos(2(t - \varphi)) + 6\Omega\right) \\
\dot{\Sigma} &= \frac{1}{2} \Sigma H \left(3\gamma \Omega_m + 2\Sigma^2 + 6\Omega \cos^2(t - \varphi) - 6\right) \\
\dot{\varphi} &= \frac{1}{4} \left(\frac{\sqrt{6}\lambda \Omega_m \sin(t - \varphi)}{\sqrt{\Omega}} - 6 \sin(2(t - \varphi))\right) - \frac{\sin(t - \varphi) \sin \left(\frac{\sqrt{6}\lambda \Omega_m \sin(t - \varphi)}{\sqrt{\Omega}}\right)}{\sqrt{6} \sqrt{\Omega} H}
\end{align*}
$$

(90)
solution (that is flat, isotropic, and empty of matter). As commented before, this model can be investigated in only one part of the phase portrait.

Point $P_3$ corresponds to the flat FLRW matter-dominated solution, that is a saddle point, unstable to scalar field perturbations. Point $P_4$ corresponds to the flat FLRW matter-dominated solution and it is a saddle as expected. It is always unstable to matter perturbations. It corresponds to a transient epoch in cosmological history. The sources are the points $P_3^\pm$ which are Bianchi I solutions dominated by anisotropies. The total energy density at points $P_3^\pm$ scales as $a^{-6}$. That is, they mimic early-time stiff fluid solutions (assuming an FLRW cosmology). Finally, $P_4$ corresponds to a vacuum Minkowski solution, that is a sink. This example would be possible in the context of inflation from an anisotropic initial state, with transient isotropization (domination of matter or the scalar field) before reaching Minkowski solution (that is flat, isotropic, and empty of matter). As commented before, this model can

Table 2. Stability criteria for the equilibrium points of the system (92).

| Label | $(\Omega, \bar{\Omega}_m, \Sigma)$ | Eigenvalues | Stability |
|-------|----------------------------------|-------------|-----------|
| $P_1$ | $(1,0,0)$                         | $\{3, -\frac{1}{\gamma}, -3(\gamma - 1)\}$ | Saddle for $0 \leq \gamma < 1$ or $1 < \gamma \leq 2$ Nonhyperbolic saddle for $\gamma = 1$ |
| $P_2$ | $(0,1,0)$                         | $\{\frac{3(\gamma - 2)}{\gamma}, \gamma - 1, 3\gamma\}$ | Saddle for $0 < \gamma < 1$ or $1 < \gamma < 2$ Nonhyperbolic for $\gamma = 0, 1, 2$ |
| $P_3^\pm$ | $(0,0,\pm \sqrt{3})$ | $\{6, -3(\gamma - 2)\}$ | Source for $0 \leq \gamma < 2$ Nonhyperbolic for $\gamma = 2$ Sink for $0 < \gamma \leq 2$ Nonhyperbolic for $\gamma = 0$ |
| $P_4$ | $(0,0,0)$                         | $\{-3, -3, -3\gamma\}$ | |

For the problem (90), using the techniques of section 3.1, we obtain the averaged system

$$
\begin{align*}
\dot{H} &= -\frac{1}{2}H^2 \left( 3(\gamma \bar{\Omega}_m + \Omega) + 2\Sigma^2 \right), \\
\dot{\Omega} &= \bar{\Omega} H \left( 3(\gamma \bar{\Omega}_m + \Omega - 1) + 2\Sigma^2 \right), \\
\dot{\bar{\Omega}}_m &= \bar{\Omega}_m H \left( 3\gamma (\bar{\Omega}_m - 1) + 2\Sigma^2 + 3\bar{\Omega} \right), \\
\dot{\Sigma} &= \frac{1}{2} \Sigma H \left( 3(\gamma \bar{\Omega}_m + \bar{\Omega} - 2) + 2\Sigma^2 \right), \\
\dot{\bar{\nu}} &= -\frac{1}{2} \bar{\nu}
\end{align*}
$$

(91)

where the angular equation is decoupled. Introducing the new variable $\tau = \ln a$, the following guiding system is obtained:

$$
\begin{align*}
\dot{\bar{\Omega}} &= \bar{\Omega} \left( 3(\gamma \bar{\Omega}_m + \Omega - 1) + 2\Sigma^2 \right), \\
\dot{\bar{\Omega}}_m &= \bar{\Omega}_m \left( 3\gamma (\bar{\Omega}_m - 1) + 2\Sigma^2 + 3\bar{\Omega} \right), \\
\dot{\Sigma} &= \frac{1}{2} \Sigma \left( 3(\gamma \bar{\Omega}_m + \bar{\Omega} - 2) + 2\Sigma^2 \right).
\end{align*}
$$

(92a, 92b, 92c)

Observe that the system (92) is invariant under the change of coordinates $\Sigma \to -\Sigma$, therefore it can be investigated in only one part of the phase portrait.

The equilibrium points of the system (92) are $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3^\pm = (0,0,\pm \sqrt{3})$ and $P_4 = (0,0,0)$. The stability criteria for each of them is summarized in table 2.

Point $P_1$ corresponds to a flat FLRW scalar field dominated solution, that is a saddle point, unstable to scalar field perturbations. Point $P_2$ corresponds to the flat FLRW matter-dominated solution and it is a saddle as expected. It is always unstable to matter perturbations. It corresponds to a transient epoch in cosmological history. The sources are the points $P_3^\pm$ which are Bianchi I solutions dominated by anisotropies. The total energy density at points $P_3^\pm$ scales as $a^{-6}$. That is, they mimic early-time stiff fluid solutions (assuming an FLRW cosmology). Finally, $P_4$ corresponds to a vacuum Minkowski solution, that is a sink. This example would be possible in the context of inflation from an anisotropic initial state, with transient isotropization (domination of matter or the scalar field) before reaching Minkowski solution (that is flat, isotropic, and empty of matter).
describe the oscillations of the inflaton around the potential minimum during reheating after inflation, like it was described in the $N$-field inflation model [152] as well as in axion-like matter [148].

In figure 6, it can be corroborated that the origin is a sink as it was indicated in table 2.

4.2. Scalar field with generalized harmonic potential minimally coupled to matter

In this section, a scalar field cosmology is investigated in the presence of matter for FLRW metrics and Bianchi I metrics. The averaging methods are applied for a generalized harmonic potential of the type (7). In every case, the stability criteria of their equilibrium points are obtained.

4.2.1. FLRW metric. For the minimally coupled case of the FLRW metric, the field equations are given by setting $\lambda = 0$ in (79). Using the amplitude-phase variables (10) with (11) and
defining (82), which satisfy (83), we obtain the following dynamical system:

\[
\begin{align*}
\dot{H} &= -\frac{1}{2}H^2 \left( 3\gamma \Omega_m + 6\Omega \cos^2(t - \varphi) + 2\Omega_k \right) \\
\dot{\Omega} &= \Omega H \left( 3\gamma \Omega_m + 3(\Omega - 1) \cos(2(t - \varphi)) + 3\Omega + 2\Omega_k - 3 \right) \\
&\quad - \sqrt{\frac{3}{5}} \sqrt{\Omega} \cos(t - \varphi) \sin \left( \frac{\sqrt{3} \Omega H \sin(t - \varphi)}{f} \right) \\
\dot{\Omega}_m &= \Omega_m H \left( 3\gamma (\Omega_m - 1) + 6\Omega \cos^2(t - \varphi) + 2\Omega_k \right) \\
\dot{\Omega}_k &= \Omega_k H \left( 3\gamma \Omega_m + 6\Omega \cos^2(t - \varphi) + 2\Omega_k - 2 \right) \\
\dot{\varphi} &= -\frac{3}{2}H \sin(2(t - \varphi)) - \frac{\sin(t - \varphi) \sin \left( \frac{\sqrt{3} \Omega H \sin(t - \varphi)}{f} \right)}{\sqrt{6\sqrt{\Omega}H}}
\end{align*}
\]

For the problem (93), the corresponding averaged system is again (85). Introducing the time variable \( \tau = \ln a \), we obtain once again the guiding system (86). Therefore, we find the same equilibrium points of the system (92); they are \( P_1 = (1, 0, 0) \), \( P_2 = (0, 1, 0) \), \( P_3 = (0, 0, 1) \) and \( P_4 = (0, 0, 0) \). The physical interpretation is the same as in section 4.1.1. Therefore, the asymptotic behavior of the model on average is independent of the coupling function.

4.2.2. Bianchi I metric. For the minimally coupled case of the Bianchi I metric the field equations are obtained from (87) by setting \( \lambda = 0 \). Using the amplitude-phase transformation (10) with (11), and defining (88), which satisfies (89), it is derived the dynamical system:

\[
\begin{align*}
\dot{H} &= -\frac{1}{2}H^2 \left( 3\gamma \Omega_m + 2\Sigma^2 + 6\Omega \cos^2(t - \varphi) \right) \\
\dot{\Omega} &= \Omega H \left( 3\gamma \Omega_m + \Omega - 1 \right) + 2\Sigma^2 + 3(\Omega - 1) \cos(2(t - \varphi)) \\
&\quad - \sqrt{\frac{3}{5}} \sqrt{\Omega} \cos(t - \varphi) \sin \left( \frac{\sqrt{3} \Omega H \sin(t - \varphi)}{f} \right) \\
\dot{\Omega}_m &= \Omega_m H \left( 3\gamma (\Omega_m - 1) + 2\Sigma^2 + 6\Omega \cos^2(t - \varphi) \right) \\
\dot{\Omega}_k &= \Omega_k H \left( 3\gamma \Omega_m + 2\Sigma^2 + 6\Omega \cos^2(t - \varphi) - 6 \right) \\
\dot{\varphi} &= -\frac{3}{2}H \sin(2(t - \varphi)) - \frac{\sin(t - \varphi) \sin \left( \frac{\sqrt{3} \Omega H \sin(t - \varphi)}{f} \right)}{\sqrt{6\sqrt{\Omega}H}}
\end{align*}
\]

For the problem (94), the corresponding averaged system is again (91). Introducing the time variable \( \tau = \ln a \), we obtain again the guiding system (92). Therefore, the equilibrium points are the same: \( P_1 = (1, 0, 0) \), \( P_2 = (0, 1, 0) \), \( P_3 = (0, 0, \pm \sqrt{5}) \), \( P_4 = (0, 0, 0) \). The physical interpretation of these equilibrium points is the same as in section 4.1.2, but for the minimally coupled set up. The stability criteria of the equilibrium points for system (92) are summarized in table 2. Therefore, the asymptotic behavior of the model on average is independent of the coupling function.

4.3. A scalar field in a vacuum with generalized harmonic potential

In this section, the perturbation methods are applied for analyzing the dynamics of a scalar field in a vacuum with generalized harmonic potential (7). The amplitude-phase variables (10)
leads to the system:

\[
\dot{r} = -\cos(t - \phi) \sin \left( \frac{r \sin(t - \phi)}{f} \right) - 3rH \cos^2(t - \phi), \quad (94a)
\]

\[
\dot{\phi} = -\sin(t - \phi) \sin \left( \frac{r \sin(t - \phi)}{f} \right) - 3H \sin(t - \phi) \cos(t - \phi), \quad (94b)
\]

with restriction

\[
f \cos \left( \frac{r \sin(t - \phi)}{f} \right) - f - \frac{r^2}{2} + 3H^2 = 0. \quad (94c)
\]

Defining the transformation \( r \to \Omega = \frac{r^2}{6H^2} \), it follows:

\[
\begin{aligned}
\dot{H} &= -3\Omega H^2 \cos^2(t - \phi) \\
\dot{\Omega} &= 6(\Omega - 1)\Omega H \cos^2(t - \phi) - \frac{\sqrt{2} \sqrt{6} \sqrt{\Omega} \cos(t - \phi) \sin \left( \frac{\sqrt{6} \sqrt{\Omega} \sin(t - \phi)}{f} \right)}{H}, \\
\dot{\phi} &= -\frac{\sin(t - \phi) \sin \left( \frac{\sqrt{6} \sqrt{\Omega} \sin(t - \phi)}{f} \right)}{\sqrt{6} \sqrt{\Omega} H} - \frac{3}{2}H \sin(2(t - \phi))
\end{aligned}
\]

(95)

where

\[
f \cos \left( \frac{\sqrt{6} \sqrt{\Omega} \sin(t - \phi)}{f} \right) - f - 3(\Omega - 1)H^2 = 0. \quad (96)
\]

**Proposition 1.** System (95) admits the approximated solution as \( H \to 0 \):

\[
\begin{aligned}
\Omega_0(t) &= c_2 \left( \cos \left( \frac{2(2c_1 + t)}{\sqrt{f + t}} \right) + 2f + 1 \right), \\
\varphi_0(t) &= t - \arctan \left( \frac{f}{f + 1} \tan \left( \frac{2c_1 + t}{\sqrt{f + t}} \right) \right)
\end{aligned}
\]

(97a, 97b)

where \( c_1 \) and \( c_2 \) are integration constants.

**Proof.** The sketch of the proof is given in appendix A.

From (11), using the approximation \( \varphi \approx \varphi_0 \), given by (97b), and restricting the domain where the \( \arctan(x) \) is a one-to-one function, we have for large \( t \) (and as \( H \to 0 \)),

\[
\dot{\phi} \approx \phi \left( \frac{f + 1}{f} \cot \left( \frac{2c_1 + t}{\sqrt{f + t}} \right) \right). \quad (98)
\]
Solving the system (98) and (99) we obtain
\begin{equation}
\dot{a} \approx a \frac{\csc \left( \frac{\sqrt{t+1(2c_1+1)}}{\sqrt{t}} \right)}{2 \sqrt{3 \sqrt{c_2^{32}}}}.
\end{equation}

Substituting (98) in (82) and using \( H(t) = \dot{a}(t)/a(t), \) \( \dot{H}(t) = r(t)/\sqrt{6\Omega(t)}, \) \( r(t) = \sqrt{(1 + \Phi(t)^2)\phi(t)}, \) where \( \Phi(t) \) is defined in (98), and using the approximation \( \Omega \approx \Omega_0 \) given by (97a), we have for large \( t \) (and as \( H \to 0 \)),

\begin{equation}
\dot{\phi}(t) \approx c_3 \sin \left( \sqrt{\frac{1}{f} + 1(t + 2c_1)} \right), \quad a(t) = c_4 e^{\frac{1}{2}(\sqrt{3} \sqrt{c_2^{32}})}.
\end{equation}

That is, asymptotically we have a de Sitter solution with ‘small’ \( H \approx \frac{c_3}{2 \sqrt{3} \sqrt{c_2^{32}}} \).

Now, continuing with the applications of the perturbation theory tools it is proved the following:

**Proposition 2.** System (95) admits the expansion

\begin{equation}
\Omega \equiv \Omega(t) = \Omega_0(t) + H(t)\Omega_1(t) + \mathcal{O}(H^2),
\end{equation}

\begin{equation}
\varphi \equiv \varphi(t) = \varphi_0(t) + H(t)\varphi_1(t) + \mathcal{O}(H^2),
\end{equation}

where \( \Omega_0(t) \) and \( \varphi_0(t) \) are the solutions (97) of the unperturbed problem \( P(0) \),

\begin{equation}
\varphi_1(t) = \frac{1}{2} \left( \frac{2c_3 - 3f(2f + 1)}{\cos \left( \frac{\sqrt{1/3}}{2} \left( t + 2c_1 \right) \right) + 2f + 1} + 3f \right),
\end{equation}

and \( \Omega_1 \) is given in quadratures

\begin{equation}
\Omega_1(t) = \exp \left( -\int_{s_1}^{t} \frac{\sin(2s - \varphi_0(s))}{f} \, ds \right) \times \left( \int_{s_1}^{t} g(s) \exp \left( \int_{s_1}^{s} \frac{\sin(2(s - \varphi_0(u)))}{f} \, du \right) \, ds \right) + c_1,
\end{equation}

where

\begin{equation}
g(t) = c_3 \left[ \cos \left( \frac{2(2c_1 + t)}{\sqrt{1/t + 1}} \right) + 2f + 1 \right]
\times \left[ 3f \cos \left( \frac{2(2c_1 + t)}{\sqrt{1/t + 1}} \right) + 2c_2 \right] \cos \left( 2 \arctan \left( \frac{2c_1 + t}{\sqrt{1/t + 1}} \right) \right)
\times \cos \left( \frac{2(2c_1 + t)}{\sqrt{1/t + 1}} \right) + 2f + 1
+ f(f + 1) \left( c_3 \cos \left( \frac{2(2c_1 + t)}{\sqrt{1/t + 1}} \right) + 2c_3f + c_3 - 1 \right) \right]
\times \left( 2(2c_1 + t) \sqrt{1/t + 1} + 2c_3f + c_3 - 1 \right) \right]
+ \frac{f \tan^2 \left( \frac{2c_1 + t}{\sqrt{1/t + 1}} \right) + f + 1}{f \tan^2 \left( \frac{2c_1 + t}{\sqrt{1/t + 1}} \right) + f + 1} \right) + f(f + 1).
\end{equation}
where \( c_1, c_2 \) and \( c_3 \) are integration constants.

**Proof.** The sketch of the proof is given in appendix B.

The system (95) can be expressed as

\[
\frac{dY}{d\eta} = HG(Y, t, H), \quad \frac{dt}{d\eta} = H,
\]

where

\[
G(Y, t, H) = \begin{bmatrix}
-3\Omega H^2 \cos^2(t - \varphi) \\
6(\Omega - 1)\Omega H \cos^2(t - \varphi) - \frac{3}{2} H \sin(2(t - \varphi)) \\
-\frac{3}{2} \Omega \sin(2(t - \varphi)) - \frac{H}{\sqrt{6\Omega H}} \sin(t - \varphi) \sin\left(\frac{H}{\sqrt{6\Omega H}} \sin(t - \varphi)\right)
\end{bmatrix},
\]

where \( Y \) denotes the phase vector \((H, \Omega, \varphi)^T\).

For the problem (95) the following averaged system is deduced:

\[
\begin{cases}
\dot{H} = -\frac{3\Omega H^2}{2} \\
\dot{\Omega} = -3(1 - \Omega)\Omega H, \\
\varphi = \frac{1}{2f}
\end{cases}
\]

where the angular equation is decoupled. Introducing the new variable \( \tau = \ln a \), the following guiding equation is obtained

\[
\partial_t \Omega = -3(1 - \Omega)\Omega,
\]

for which \( \Omega = 0 \) is a sink and \( \Omega = 1 \) is a source.

Starting with the averaged equation (107), it is proved that \( \Omega, \varphi \) evolve at first order according to the averaged equations for \( \Omega, \varphi \).

**Proposition 3.** Given \((H, \Omega, \varphi, t)\) solutions of (105), there exists a transformation

\[
\begin{aligned}
t &= t_0 + H\alpha_1(t_0, \varphi_0), \\
\Omega &= \Omega_0 + H \left[ \alpha_2(t_0, \varphi_0) - \frac{\eta}{f} \sin(2(t_0 - \varphi_0))\Omega_0 \right], \\
\varphi &= \varphi_0 + H \left[ \alpha_3(t_0, \varphi_0) + \frac{\eta}{2f} \cos(2(t_0 - \varphi_0)) \right],
\end{aligned}
\]

where \( \alpha_i(t_0, \varphi_0), i = 1, 2, 3 \) are differentiable, such that the functions \( t_0, \Omega_0, \varphi_0 \) have the same asymptotic of the averaged solutions \( \Omega, \varphi \) of (107) as \( H \to 0 \) and \( \eta \to \infty \).

**Proof.** The sketch of the proof is given in appendix C.
Figure 7. Some solutions of the full system \((84)\) (blue) and time-averaged system \((85)\) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when \(\lambda = 0.1, f = 0.1\) and \(\gamma = 0.0\). We have used for both systems the initial data sets presented in table 3.
Figure 8. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.1$ and $\gamma = \frac{1}{2}$. We have used for both systems the initial data sets presented in table 3.
Figure 9. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1$, $f = 0.1$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 3.
Figure 10. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.1$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 3.
Figure 11. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.3$ and $\gamma = 0$. We have used for both systems the initial data sets presented in table 3.
Figure 12. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.3$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 3.
Figure 13. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 3.
Figure 14. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 3.
Figure 15. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.9$ and $\gamma = 0$. We have used for both systems the initial data sets presented in table 3.
Figure 16. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1$, $f = 0.9$ and $\gamma = \frac{1}{2}$. We have used for both systems the initial data sets presented in table 3.
Figure 17. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.9$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 3.
Figure 18. Some solutions of the full system (84) (blue) and time-averaged system (85) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric when $\lambda = 0.1, f = 0.9$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 3.
Table 3. Seven initial data set for the simulations of the full and time averaged system for the FLRW metric in the non-minimally and minimally coupled cases. The initial conditions were chosen in order to fulfill the inequality $\Omega(0) + \Omega_m(0) + \Omega_k(0) < 1$.

| Sol. | $H(0)$ | $\Omega(0)$ | $\Omega_m(0)$ | $\Omega_k(0)$ | $\varphi(0)$ | $n(0)$ |
|------|--------|-------------|--------------|--------------|-------------|--------|
| i    | 0.1    | 0.8         | 0.01         | 0.09         | 0           | 0      |
| ii   | 0.1    | 0.1         | 0.16         | 0.64         | 0           | 0      |
| iii  | 0.1    | 0.1         | 0.36         | 0.44         | 0           | 0      |
| iv   | 0.02   | 0.02        | 0.2304       | 0.6496       | 0           | 0      |
| v    | 0.1    | 0.02        | 0.2304       | 0.6496       | 0           | 0      |
| vi   | 0.1    | 0.01        | 0.59         | 0.3          | 0           | 0      |
| vii  | 0.1    | 0.584       | 0.315        | 0.001        | 0           | 0      |

5. Numerical simulations

In this section we present the numerical results obtained from the integration of the full system and its corresponding averaged version of the scalar field with a generalized harmonic potential model in the non-minimally coupled, for FLRW and Bianchi I metrics, and vacuum cases, as evidence that the full and averaged systems have the same late-time dynamics. To that end, we elaborated an algorithm in the programming language Python, where the systems of differential equations were numerically integrated using the `solve_ivp` code provided by the Scipy open-source Python-based ecosystem. As an integration method we use Radau, which is an implicit Runge–Kutta method of the Radau Ia family of order 5, with relative and absolute tolerances of $10^{-3}$ and $10^{-6}$, respectively. In the numerical integration we use as a time variable $\tau$, which is related to the cosmic time $t$ through the expression $dt/d\tau = 1/H$, in an integration range of $-40 \leq \tau \leq 3$ for the full systems and $-40 \leq \tau \leq 40$ for the averaged system, all of them partitioned in 20000 and 60000 data points for the non-minimal coupling and vacuum cases, respectively. Furthermore, the full and time-averaged systems were solved for a value of $\gamma = 0$ (CC), $2/3$, $1$ (dust) and $2$ (stiff fluid); all of them for a value of $f = 0.1$, $0.3$ and $0.9$, for the non-minimally coupling case. It is worth noticing that in the case of the scalar field with generalized harmonic potential minimally coupled to matter ($\lambda = 0$), for FLRW and Bianchi I metrics, the numerical results are very similar to their respective non-minimally coupled cases ($\lambda \neq 0$). Observe that the interaction appears in the equations explicitly in the form $\lambda \sin(t - \varphi)$, $\lambda \cos(t - \varphi)$, which are zero in average.

5.1. Scalar field with generalized harmonic potential non-minimally coupled to matter

5.1.1. FLRW metric. In figures 7–18 we present the numerical results obtained from the integration of the full system (84) (blue lines) and time-averaged system (85) (orange lines) for the non-minimally coupled case in the FLRW metric, using for both systems the seven initial data set presented in the table 3.

In figures 7, 11 and 15 we depict the results obtained for $\gamma = 0$ when $f = 0.1$, $0.3$ and $0.9$, respectively. Figures 7(a), 11(a) and 15(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 7(b), 11(b) and 15(b) shows the projections in the space $(\Omega_k, H, \Omega)$, and figures 7(c), 11(c) and 15(c) shows the projections in the space $(\Omega_m, \Omega_k, \Omega)$.

In figures 8, 12 and 16 we depict the results obtained for $\gamma = 2/3$ when $f = 0.1$, $0.3$ and $0.9$, respectively. Figures 8(a), 12(a) and 16(a) shows the projections in the space $(\Omega_m, H, \Omega)$. 

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Figure 19. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.1$ and $\gamma = 0$. We have used for both systems the initial data sets presented in table 4.
Figure 20. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.1$ and $\gamma = \frac{3}{4}$. We have used for both systems the initial data sets presented in table 4.
Figure 21. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.1$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 4.
Figure 22. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.1$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 4.
Figure 23. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = 0$. We have used for both systems the initial data sets presented in table 4.
Figure 24. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = \frac{3}{4}$. We have used for both systems the initial data sets presented in table 4.
Figure 25. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 4.
Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.3$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 4.

Figure 26.
Figure 27. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1, f = 0.9$ and $\gamma = 0$. We have used for both systems the initial data sets presented in table 4.
Figure 28. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.9$ and $\gamma = \frac{2}{3}$. We have used for both systems the initial data sets presented in table 4.
Figure 29. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.9$ and $\gamma = 1$. We have used for both systems the initial data sets presented in table 4.
Figure 30. Some solutions of the full system (90) (blue) and time-averaged system (91) (orange) for a scalar field with generalized harmonic potential non-minimally coupled to matter in the Bianchi I metric when $\lambda = 0.1$, $f = 0.9$ and $\gamma = 2$. We have used for both systems the initial data sets presented in table 4.
Table 4. Seven initial data set for the simulations of the full and time averaged system for the Bianchi I metric in the non-minimally and minimally coupled cases. The initial conditions were chosen in order to fulfill the inequality $\Omega(0) + \Omega_m(0) + \Sigma(0)^2/3 < 1$.

| Sol. | $H(0)$ | $\Omega(0)$ | $\Omega_m(0)$ | $\Sigma(0)$ | $\phi(0)$ | $n(0)$ |
|------|--------|-------------|---------------|-------------|-----------|--------|
| i    | 0.1    | 0.8         | 0.01          | 0.52        | 0         | 0      |
| ii   | 0.1    | 0.1         | 0.16          | 1.39        | 0         | 0      |
| iii  | 0.1    | 0.1         | 0.36          | 1.15        | 0         | 0      |
| iv   | 0.02   | 0.02        | 0.2304        | 1.3959      | 0         | 0      |
| v    | 0.1    | 0.02        | 0.2304        | 1.3959      | 0         | 0      |
| vi   | 0.1    | 0.01        | 0.59          | 0.9         | 0         | 0      |
| vii  | 0.1    | 0.584       | 0.315         | 0.055       | 0         | 0      |

Figures 8(b), 12(b) and 16(b) shows the projections in the space $(\Omega_k, H, \Omega)$, and figures 8(c), 12(c) and 16(c) shows the projections in the space $(\Omega_m, \Omega_k, \Omega)$.

In figures 9, 13 and 17 we depict the results obtained for $\gamma = 1$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 9(a), 13(a) and 17(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 9(b), 13(b) and 17(b) shows the projections in the space $(\Omega_k, H, \Omega)$, and figures 9(c), 13(c) and 17(c) shows the projections in the space $(\Omega_m, \Omega_k, \Omega)$.

In figures 10, 14 and 18 we depict the results obtained for $\gamma = 2$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 10(a), 14(a) and 18(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 10(b), 14(b) and 18(b) shows the projections in the space $(\Omega_k, H, \Omega)$, and figures 10(c), 14(c) and 18(c) shows the projections in the space $(\Omega_m, \Omega_k, \Omega)$.

These figures show evidence that the solutions of the full system (blue lines), obtained for a scalar field with generalized harmonic potential non-minimally coupled to matter in the FLRW metric, follow the track of the solutions of the averaged system (orange lines). Therefore, they have the same asymptotic behavior. Furthermore, we can see that the amplitude of oscillations decreases when the value of $f$ increases.

5.1.2. Bianchi I metric. In figures 19–30 we present the numerical results obtained from the integration of the full system (90) (blue lines) and time-averaged system (91) (orange lines) for the non-minimally coupled case in the Bianchi I metric, using for both systems the seven initial data set presented in the table 4. The integration range of the full system used in the $\gamma = 2$ case was $-35 \leq \tau \leq 3$.

In figures 19, 23 and 27 we depict the results obtained for $\gamma = 0$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 19(a), 23(a) and 27(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 19(b), 23(b) and 27(b) shows the projections in the space $(\Sigma, H, \Omega)$, and figures 19(c), 23(c) and 27(c) shows the projections in the space $(\Omega_m, \Sigma, \Omega)$.

In figures 20, 24 and 28 we depict the results obtained for $\gamma = 2/3$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 20(a), 24(a) and 28(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 20(b), 24(b) and 28(b) shows the projections in the space $(\Sigma, H, \Omega)$, and figures 20(c), 24(c) and 28(c) shows the projections in the space $(\Omega_m, \Sigma, \Omega)$.

In figures 21, 25 and 29 we depict the results obtained for $\gamma = 1$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 21(a), 25(a) and 29(a) shows the projections in the space $(\Omega_m, H, \Omega)$. Figures 21(b), 25(b) and 29(b) shows the projections in the space $(\Sigma, H, \Omega)$, and figures 21(c), 25(c) and 29(c) shows the projections in the space $(\Omega_m, \Sigma, \Omega)$.

In figures 22, 26 and 30 we depict the results obtained for $\gamma = 2$ when $f = 0.1, 0.3$ and 0.9, respectively. Figures 22(a), 26(a) and 30(a) shows the projections in the space $(\Omega_m, H, \Omega)$. 

Some solutions of the full system (105) (blue) and time-averaged system (107) (orange) for a scalar field with generalized harmonic potential in a vacuum. We have used for both systems the seven initial data sets presented in table 5.

Figures 22(b), 26(b) and 30(b) show the projections in the space $\left( \Sigma, H, \Omega \right)$, and figures 22(c), 26(c) and 30(c) show the projections in the space $\left( \Omega_m, \Sigma, \Omega \right)$.

These figures are evidence that the solutions of the full system (blue lines), obtained for a scalar field with generalized harmonic potential in a vacuum, follow the track of the solutions of the averaged system (orange lines) as $H$ monotonically decreases. Therefore, they have the same asymptotic behavior. Furthermore, we can see that the amplitude of oscillations decreases when the value of $f$ increases.

5.2. Scalar field with generalized harmonic potential in a vacuum

In figure 31 we present the numerical results obtained from the integration of the full system (105) (blue lines) and time-averaged system (107) (orange lines) for the vacuum case, using for both systems the seven initial data sets presented in the table 5.

In figures 31(a)–(c) we depict the results obtained for $f = 0.1, 0.3$ and $0.9$, respectively, in the $(H, \Omega)$ projection.

These figures are evidence that the solutions of the full system (blue lines), obtained for a scalar field with generalized harmonic potential in a vacuum, follow the track of the solutions of the averaged system (orange lines) as $H$ monotonically decreases. Therefore, they have the same asymptotic behavior. Furthermore, we can see that the amplitude of oscillations decreases when the value of $f$ increases.
Table 5. Seven initial data set for the simulations of the full and time averaged system for the scalar field with generalized harmonic potential in a vacuum. The initial conditions were chosen in order to fulfill the inequality $\Omega(0) < 1$.

| Sol. | $H(0)$ | $\Omega(0)$ | $\varphi(0)$ | $n(0)$ |
|------|--------|-------------|--------------|--------|
| i    | 0.1    | 0.4         | 0            | 0      |
| ii   | 0.1    | 0.5         | 0            | 0      |
| iii  | 0.1    | 0.6         | 0            | 0      |
| iv   | 0.02   | 0.6         | 0            | 0      |
| v    | 0.1    | 0.7         | 0            | 0      |
| vi   | 0.1    | 0.8         | 0            | 0      |
| vii  | 0.1    | 0.9         | 0            | 0      |

It is important to mention that these figures confirm the result of proposition 1. Therefore, asymptotically we have a de Sitter solution with "small" $H \approx \frac{c_1}{\sqrt{3}}$, where $c_1$ and $c_2$ are integration constants that depend on the initial conditions.

6. Results and conclusions

This paper was devoted to the study of perturbation problems in scalar field cosmologies in the FLRW metric with $k = -1, 0$, and Bianchi I metric, in a vacuum and with matter. In the last case, considering minimal and non-minimal couplings between matter and the scalar field. Qualitative techniques, asymptotic methods, and averaging theory were used to obtain relevant information about the solution’s space of the aforementioned cosmologies. Variables that lead to regular equations in a bounded state space were chosen. This allows us to give a global description of the dynamics, in particular, the behavior in early and late times and the evolution in intermediate stages that can be of physical interest. Furthermore, differential equations, suitable for performing systematic numerical simulations, were derived. Averaged versions of original systems were constructed where the oscillations of the solutions are smoothed out. The analysis is then reduced to studying the late dynamics of a simpler averaged system where the oscillations entering the full system through the KG equation can be controlled.

The tools of the averaging theory and the qualitative techniques of dynamical systems have been applied successfully in recent years in similar cosmological models, say in [114–116]. As the natural generalization of these models, we considered spatially homogeneous and isotropic dark energy (scalar field)-matter interactive schemes. To illustrate the relevance of these tools, we have discussed some basic examples of applications of perturbation techniques in sections 2 and 3. Regarding the cosmological applications of these techniques (which are the core of the present research), in section 4 some applications of perturbation and averaging methods in cosmology were presented. In particular, in section 4.1 it was studied a scalar field with generalized harmonic potential (7) non-minimally coupled to matter with coupling (8). Sections 4.2 and 4.3 were devoted to the minimally coupled and vacuum cases, respectively. The focus was to study the imprint of coupling function, as well as the influence of the metric on the dynamics of the averaged problem. As a first step towards generalization, we have considered an interaction with the background matter with strength of type $Q = \lambda/2\rho_m\dot{\varphi}$ arising from the coupling function (8) within the interacting scheme (1). It is worth noting that in the case of the scalar field with generalized harmonic potential minimally coupled to the matter ($\lambda = 0$), for the FLRW metrics, the numerical results are very similar to their respective non-minimally coupled cases ($\lambda \neq 0$), and the same happens for the Bianchi I models. Note that the interaction
appears in the equations explicitly in the form $\lambda \sin(t - \varphi), \lambda \cos(t - \varphi)$, expressions that have zero average. Using averaging methods for periodic functions of a given period $T$, it can be concluded that, regardless of whether the scalar field is minimally or non-minimally coupled to the matter field, there is no difference in dynamics when performing the averaging process at least for interactions of the type $Q = \lambda/2\rho_m \dot{\phi}$. This indicates that the asymptotic results are independent of this coupling function.

For the FLRW metric, we have found the following results. Point $P_1$ corresponds to a flat FLRW scalar field dominated solution, that is a saddle point for $1 < \gamma \leq 2$, or a source for $0 \leq \gamma < 1$. Point $P_2$ corresponds to the flat FLRW matter-dominated solution and it is unstable to matter perturbations. It is a source for $1 < \gamma < 2$ or a saddle if $0 < \gamma < 2/3$ or $2/3 < \gamma < 1$. It corresponds to a transient epoch in cosmological history. Point $P_3$ is a curvature-dominated solution with positive curvature (Misner solution). The energy density of the scalar field scales as $a^{-2}$ and is a saddle (unstable to curvature perturbations). For LRS Bianchi I metric we found that point $P_1$ corresponds to a flat FLRW scalar field dominated solution, that is a saddle point, unstable to scalar field perturbations. Point $P_2$ corresponds to the flat FLRW matter-dominated solution and it is a saddle as expected. It is always unstable to matter perturbations. It corresponds to a transient epoch in cosmological history. The sources are the points $P_3^\pm$ which are Bianchi I solutions dominated by anisotropies. The total energy density at points $P_3^\pm$ scales as $a^{-6}$. That is, they mimic early-time stiff fluid solutions (assuming an FLRW cosmology).

Finally, for each metric (LRS Bianchi I or FLRW) point $P_4$ corresponds to a vacuum Minkowski solution, that is a sink. The Minkowski solution represents an empty universe. It can be used as a local approximation of space-time in reasonably small regions and in the presence of matter, as long as it does not self-gravitate. This example would be possible in the context of inflation from an isotropic initial state, and we see that isotropization is a transient state in the universe, before reaching the Minkowski solution (that is flat, isotropic, and empty of matter). This approach, in which the scalar field oscillates in the minimum, before reaching a Minkowski solution, is useful for describing the oscillations of the inflaton around the potential minimum during reheating after inflation. This behavior was described in models like the $N$-field inflation model [152] as well as in axion-like matter [148]. The relevant calculations depend on the shape of the potential and, in particular, are quite complicated for harmonic potentials. The result presented here shows that the oscillations arising due to harmonic functions can be ‘averaged’, thus simplifying the problem. For non-zero $H$, this gives rise to time-dependent oscillatory dynamics. This is maybe responsible for the production of particles through quantum field theory. Using some inverse transformations, one can find from the averaged version of the scalar field variables, the approximate temporal dependence of the original fields. This approach is also suitable in the context of linear cosmological perturbations. In cosmological perturbation theory, cosmological perturbations at the linear level are governed by equations whose coefficients are made up of background quantities. Therefore, adequate knowledge of the background dynamics is necessary to perform further perturbation analysis.

Summarizing,

(a) We have obtained relevant information about the solution space of scalar field cosmologies with generalized harmonic potential for the FLRW metrics, in a vacuum, and in the presence of matter (within minimal or non-minimal interacting schemes) and for the locally and rotationally symmetric Bianchi I metric.
(b) We have incorporated asymptotic expansion with multiple timescales, averaging theory, and qualitative analysis of dynamical systems to describe oscillatory solutions to a wide class of perturbation problems for these models.

(c) We built averaged versions of the original systems where oscillations are smoothed out. The analysis can then be reduced to studying the late dynamics of a simpler averaged system where oscillations entering the full system can be controlled through the KG equation.

(d) We have constructed regular equations defined in bounded state spaces that allow giving a global description of the dynamics. In particular, the behavior at early and late-time and the evolution at intermediate stages may be of physical interest. In addition to proposing suitable differential equations to carry out systematic numerical simulations.

There are several issues to be discussed within a new research, but it is worth noting that the success in the implementation of mathematical techniques during this research allows an immediate implementation of these to the case with more general interaction terms, so that new results can be achieved as a continuation of this project.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Proof of proposition 1

Now is given the proof of proposition 1

Using Taylor series in a neighborhood of \( H = 0 \) of (95) the following holds:

\[
\dot{H} = -3\Omega H^2 \cos^2(t - \varphi) + O(H)^3, \tag{A.1a}
\]

\[
\dot{\Omega} = -\frac{\Omega \sin(2(t - \varphi))}{f} + 6(\Omega - 1)\Omega H \cos^2(t - \varphi) + O(H)^2, \tag{A.1b}
\]

\[
\dot{\varphi} = -\frac{\sin^2(t - \varphi)}{f} - \frac{3}{2}H \sin(2(t - \varphi)) + O(H)^2. \tag{A.1c}
\]
Then, as $H \to 0$, it follows the unperturbed problem:

$$P(0): \begin{cases} \dot{H} = 0 \\ \dot{\Omega} = -\frac{\Omega \sin(2(t - \varphi))}{f} \\ \dot{\varphi} = -\frac{\sin^2(t - \varphi)}{f} \end{cases} \quad (A.2)$$

whose solution is given by (97).

Appendix B. Proof of proposition 2

Now is given the proof of proposition 2.

Continuing with the applications of the perturbation theory tools is proposed an expansion of type (101), where $\Omega_0(t)$ and $\varphi_0(t)$ are the solutions of the unperturbed problem $P(0)$. Applying the chain rule and using the fact that $\Omega_1 \frac{dH}{dt} = O(H^2)$ according to (A.1a), it follows:

$$\frac{d\Omega}{dt} = \frac{d\Omega_0}{dt} + H \frac{d\Omega_1}{dt} + \frac{\Omega_1 dH}{dt} + O(H^2) = \frac{d\Omega_0}{dt} + H \frac{d\Omega_1}{dt} + O(H^2), \quad (B.1a)$$

$$\frac{d\varphi}{dt} = \frac{d\varphi_0}{dt} + H \frac{d\varphi_1}{dt} + \frac{\varphi_1 dH}{dt} + O(H^2) = \frac{d\varphi_0}{dt} + H \frac{d\varphi_1}{dt} + O(H^2). \quad (B.1b)$$

Hence,

$$H \frac{d\Omega_1}{dt} = \frac{d\Omega}{dt} - \frac{d\Omega_0}{dt}$$

$$= 6((\Omega_0 + H\Omega_1) - 1)(\Omega_0 + H\Omega_1)H \cos^2(t - (\varphi_0 + H\varphi_1))$$

$$- (\Omega_0 + H\Omega_1) \sin(2(t - (\varphi_0 + H\varphi_1))) + \frac{\Omega_1 \sin^2((t - \varphi_0))}{f} + O(H^2)$$

$$= H \left(6f(\Omega_0 - 1)\Omega_0 \cos^2(t - \varphi_0) + 2\varphi_1\Omega_0 \cos(2(t - \varphi_0)) - \Omega_1 \sin(2(t - \varphi_0))\right)$$

$$+ O(H^2), \quad (B.2a)$$

$$H \frac{d\varphi_1}{dt} = \frac{d\varphi}{dt} - \frac{d\varphi_0}{dt}$$

$$= \frac{3}{2} H \sin(2(t - (\varphi_0 + H\varphi_1))) - \frac{\sin^3(t - (\varphi_0 + H\varphi_1))}{f}$$

$$+ \frac{\sin^2(t - \varphi_0)}{f} + O(H^2)$$

$$= \frac{H(2\varphi_1 - 3f) \sin(2(t - \varphi_0))}{2f} + O(H^2). \quad (B.2b)$$
Therefore, to find analytically the functions $\Omega_1$ and $\varphi_1$, equations
\[
\frac{d\Omega_1}{dt} = \left(6f(\Omega_0 - 1)\Omega_0 \cos^2(t - \varphi_0) + 2\varphi_1\Omega_0 \cos(2(t - \varphi_0)) - \Omega_1 \sin(2(t - \varphi_0))\right),
\]
and
\[
\frac{d\varphi_1}{dt} = \frac{(2\varphi_1 - 3f) \sin(2(t - \varphi_0))}{2f},
\]
have to be solved with the substitution of $\Omega_0$ and $\varphi_0$ in (B.3). Integrating for $\varphi_1$, it follows equation (102). For $\Omega_1$ the following quadrature (103) with $f(t)$ defined by (104) is obtained.

The next result is useful in the following proof.

**Lemma B.1. (Gronwall’s lemma. Integral form).** Let be $\xi(t)$ a nonnegative function, summable over $[0, T]$ which satisfies almost everywhere the integral inequality
\[
\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2, \quad C_1, C_2 \geq 0.
\]
Then
\[
\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}),
\]
almost everywhere for $t$ in $0 \leq t \leq T$. In particular, if
\[
\xi(t) \leq C_1 \int_0^t \xi(s)ds, \quad C_1 \geq 0,
\]
almost everywhere for $t$ in $0 \leq t \leq T$, then
\[
\eta \equiv 0,
\]
almost everywhere for $t$ in $0 \leq t \leq T$.

**Appendix C. Proof of proposition 3**

Now is given the proof of proposition 3.

It is easy to see that the system (A.1) can be conveniently written as:
\[
\frac{d\Omega}{d\eta} = -\frac{H\Omega \sin(2(t - \varphi))}{f} + 6(\Omega - 1)\Omega H^2 \cos^2(t - \varphi) + \mathcal{O}(H)^3,
\]
\[
\frac{d\varphi}{d\eta} = -\frac{H\sin^2(t - \varphi)}{f} - \frac{3}{2}H^2 \sin(2(t - \varphi)) + \mathcal{O}(H)^3.
\]
and the averaged problem is:
\[
\frac{d\Omega}{d\eta} = H, \quad \frac{d\bar{\Omega}}{d\eta} = -3(1 - \bar{\Omega})H^2, \quad \frac{d\bar{\varphi}}{d\eta} = -\frac{H}{2f}.
\]
Now, the expansion (109) is proposed. Next, it is proved that the equations for $t_0, \Omega_0, \varphi_0$ have the same asymptotic that the averaged equations for $\mathcal{T}, \mathcal{P}, \varphi$.

After some algebraic manipulations and recalling that

$$\frac{dH}{d\eta} = -3\Omega'H^3 \cos^2(t - \varphi) = \mathcal{O}(H)^3,$$

(C.3)

it follows:

$$\frac{d\alpha_0}{d\eta} = H + H^2 \left( \frac{\alpha_{1,0}}{2f} - \alpha_{1,0} \right) + \mathcal{O} \left( H^3 \right),$$

(C.4a)

$$\frac{d\Omega_0}{d\eta} = H^2 \left\{ \frac{\Omega_0 \left( 3f^2(\Omega_0 - 1) + 2f\eta + \eta \cos(2(t_0 - \varphi_0)) + 3f^2(\Omega_0 - 1) + \eta \right)}{f^2} + \frac{\Omega_0\alpha_{2,0}}{2f} - \frac{\alpha_3\alpha_{2,0}}{2f} \right\} + \mathcal{O} \left( H^3 \right),$$

(C.4b)

$$\frac{d\varphi_0}{d\eta} = -H + H^2 \left( \frac{\sin(2(t_0 - \varphi_0)) \left( -3f^2 - 2\alpha_1 f + 2\alpha_3 f + 2f\eta + \eta \cos(2(t_0 - \varphi_0)) \right)}{2f^2} + \frac{\alpha_{3,0} - \alpha_{2,0}}{2f} \right) + \mathcal{O} \left( H^3 \right).$$

(C.4c)

Imposing the conditions

$$\frac{\alpha_{1,0}}{2f} - \alpha_{1,0} = 0 \Rightarrow \alpha_1(t_0, \varphi_0) = c_1 \left( \frac{t_0}{2f} + \varphi_0 \right),$$

(C.5)

$$\alpha_{2,0} - 2f\alpha_{2,0} - 2\alpha_2 \sin(2(t_0 - \varphi_0)) = 0 \Rightarrow \alpha_2(t_0, \varphi_0) = e^{\frac{\cos(t_0 - \varphi_0)}{2f + \varphi_0}} c_2 \left( \frac{t_0}{2f} + \varphi_0 \right),$$

(C.6)

and assuming $\alpha_3 = \alpha_1 + g(t_0, \varphi_0)$, the following equations are deduced:

$$\frac{d\alpha_0}{d\eta} = H + \mathcal{O} \left( H^3 \right),$$

(C.7a)

$$\frac{d\Omega_0}{d\eta} = H^2 \left\{ \frac{\Omega_0 \cos(2(t_0 - \varphi_0)) \left( 2fg + 3f^2(\Omega_0 - 1) + 2f\eta + \eta \right)}{f^2} + \frac{\Omega_0 \left( \frac{\eta}{f^2} + 3\Omega_0 - 3 \right)}{f^2} \right\} + \mathcal{O} \left( H^3 \right),$$

(C.7b)

$$\frac{d\varphi_0}{d\eta} = -H + H^2 \left\{ \frac{\eta \sin(2(t_0 - \varphi_0))(2f + \cos(2(t_0 - \varphi_0)) + 1)}{2f^2} + \frac{-2fg_{1,0} - 3f \sin(2(t_0 - \varphi_0)) + 2g \sin(2(t_0 - \varphi_0)) + g_{2,0}}{2f} \right\} + \mathcal{O} \left( H^3 \right).$$

(C.7c)
The condition
\[-2fg_0 - 3f \sin(2(t_0 - \varphi_0)) + 2g \sin(2(t_0 - \varphi_0)) + g_0 = 0\]
leads to
\[g(t_0, \varphi_0) = \frac{3f}{2} + e^{-\frac{\cos(2(t_0 - \varphi_0))}{2f}} c_3 \left( \frac{t_0}{2f} + \varphi_0 \right), \quad (C.8)\]

Equation (C.7b) becomes
\[
\frac{d\Omega_0}{d\eta} = H^2 \left\{ 6(\Omega_0 - 1)\Omega_0 \cos^2(t_0 - \varphi_0) + \frac{\eta(2f + 1)\Omega_0 \cos(2(t_0 - \varphi_0)) + \Omega_0}{f^2} \right.

\left. + \Omega_0 \cos(2(t_0 - \varphi_0)) \left( 3 + \frac{2e^{-\frac{\cos(2(t_0 - \varphi_0))}{2f}} c_3 \left( \frac{t_0}{2f} + \varphi_0 \right)}{f} \right) \right\}. \quad (C.12)\]

Equation (C.7c) becomes
\[
\frac{d\varphi_0}{d\eta} = -\frac{H}{2f} + \frac{\eta H^2 \sin(2(t_0 - \varphi_0))(f + \cos^2(t_0 - \varphi_0))}{f^2}. \quad (C.13)\]

From the equation
\[\dot{H} = -3\Omega H^2 \cos^2(t - \varphi) + O(H^3), \quad (C.14)\]

or its averaged version, it follows $H$ is a monotonic decreasing function of $t$ due to $0 \leq \Omega, \bar{\Omega} \leq 1$. This allows to define recursively the sequences:

\[
\left\{ \begin{array}{l}
\eta_0 = 0 \\
H_0 = H(\eta_0)
\end{array} \right\}, \quad \left\{ \begin{array}{l}
\eta_{n+1}^2 = \eta_n^2 + \frac{1}{H_n} \\
H_{n+1} = H(\eta_{n+1})
\end{array} \right\}, \quad (C.15)

such that $\lim_{n \to \infty} H_n = 0$ and $\lim_{n \to \infty} \eta_n = \infty$. 
Defining $\Delta \varphi(\eta) = \varphi_0(\eta) - \overline{\varphi}(\eta)$ and taking the same initial conditions at $\eta = \eta_n$, $\varphi_0(\eta_n) = \overline{\varphi}(\eta_n)$, it follows:

$$
|\Delta \varphi(\eta)| = \left| \int_{\eta_n}^{\eta} [\varphi'_0(s) - \overline{\varphi}'(s)] ds \right|
$$

$$
= \left| \int_{\eta_n}^{\eta} 2s \left[ \frac{H^2 \sin(2(t_0 - \varphi_0))(f + \cos^2(t_0 - \varphi_0))}{2f^2} + \mathcal{O}(H^2) \right] ds \right|
$$

$$
\leq M_1 H_n^2 \int_{\eta_n}^{\eta} 2s \left| ds \right| + \mathcal{O}(H_n^2) + \mathcal{O}(H_n)
$$

where $M_1$ is a constant, for all $\eta \geq \eta_n$. Then, for $\eta \in [\eta_n, \eta_{n+1}]$, it follows the inequality

$$
|\Delta \varphi(\eta)| \leq M_1 H_n.
$$

Finally, taking the limit as $n \to \infty$, it follows $H_n \to 0, \eta_n \to \infty$, then, it follows $\lim_{\eta \to \infty}|\Delta \varphi(\eta)| = 0$. This means that $\varphi_0$ and $\overline{\varphi}$ have the same limit as $\eta \to \infty$.

Without losing generality, $c_3(\varphi_0) \equiv 0$ is chosen in (C.12). Therefore, it follows

$$
\frac{d\Omega_0}{d\eta} = H^2 \left\{ -3\Omega_0(1 - \Omega_0) + \eta \Omega_0 (1 + (2f + 1) \cos(2(t_0 - \varphi_0))) + 3\Omega_0^2 \cos(2(t_0 - \varphi_0)) \right\} + \mathcal{O}(H^2). \tag{C.17}
$$

Defining $\Delta \Omega = \Omega_0 - \overline{\Omega}$, it follows

$$
\Delta \Omega'(s) = -3 \Delta \Omega H^2 (1 - \overline{\Omega} - \Omega_0) + 2H^2 \eta \Omega_0 \left[ (1 + (2f + 1) \cos(2(t_0 - \varphi_0))) \right]
$$

$$
+ 3H^2 \Omega_0^2 \cos(2(t_0 - \varphi_0)). \tag{C.18}
$$

Choosing the same initial conditions $\Omega_0(\eta_n) = \overline{\Omega}(\eta_n)$ at $\eta = \eta_n$, it follows

$$
|\Delta \Omega(\eta)| = \left| \int_{\eta_n}^{\eta} [\Omega'_0(s) - \overline{\Omega}'(s)] ds \right|
$$

$$
= \left| \int_{\eta_n}^{\eta} -3 \Delta \Omega H^2 (1 - \overline{\Omega} - \Omega_0) + 2H^2 \eta \Omega_0 \left[ (1 + (2f + 1) \cos(2(t_0 - \varphi_0))) \right]
$$

$$
+ 3H^2 \Omega_0^2 \cos(2(t_0 - \varphi_0)) + \mathcal{O}(H^2) \right| ds \right|
$$

$$
\leq 3H_n^2 \int_{\eta_n}^{\eta} |\Delta \Omega(s)| ds + MM_2 H_n^2 |\eta + \eta_n| + 3M_2 H_n^2 |\eta - \eta_n|, \quad \eta \geq \eta_n, \tag{C.19}
$$

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where $M_2$ is a constant, and
\[
M = \max_{s \in [\eta_n, \eta]} |\Omega_0(s)|,
\]
which exists due to the continuity of $\Omega_0$ on the compact set $[\eta_n, \eta]$. Applying Gronwall’s lemma B.1, it follows:
\[
|\Delta \Omega(\eta)| \leq \left[ M M_2 H_n^2 |\eta + \eta_n| - |\eta - \eta_n| \right] \left[ 1 + H_n^2 |\eta - \eta_n| \right] + O(H_n^3)
\]
\[
= \left[ M M_2 H_n^2 |\eta + \eta_n| - |\eta - \eta_n| \right] \left[ 1 + H_n^2 |\eta - \eta_n| \right] + O(H_n^3)
\]
\[
= M M_2 H_n^2 |\eta + \eta_n| - |\eta - \eta_n| + 3 M_2^2 H_n^2 |\eta - \eta_n| + O(H_n^3).
\]
(C.20)

Then, for $\eta \in [\eta_n, \eta_{n+1}]$ and for $n$ large enough such that $|\eta + \eta_n| \geq 1$, it follows
\[
M M_2 |\eta + \eta_n| - |\eta - \eta_n| + 3 M_2^2 |\eta - \eta_n| \leq (M M_2 + 3 M_2^2) |\eta^2 - \eta_n^2|
\]
\[
\leq (M M_2 + 3 M_2^2) |\eta_{n+1} - \eta_n^2| = (M M_2 + 3 M_2^2) H_n^{-1}.
\]

Therefore, it follows the inequality $|\Delta \Omega(\eta)| \leq KH_n$, for a positive constant $K \geq (M M_2 + 3 M_2^2)$. Finally, taking the limit as $n \to \infty$, it follows $H_n \to 0$, $\eta_n \to \infty$. Then, it follows $\lim_{\eta \to \infty} |\Delta \Omega(\eta)| = 0$. This means that $\Omega_0$ and $\Omega$ have the same limit as $\eta \to \infty$. □

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