Central Measures of Continuous Graded Graphs: the Case of Distinct Frequencies

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Abstract

We define a class of continuous graded graphs similar to the graph of Gelfand–Tsetlin patterns, and describe the set of all ergodic central measures of discrete type on the path spaces of such graphs. The main observation is that an ergodic central measure on a subgraph of a Pascal-type graph can often be obtained as the restriction of the standard Bernoulli measure to the path space of the subgraph. This observation dramatically changes the approach to finding central measures also on discrete graphs, such as the famous Young graph.

The simplest example of this type is given by the theorem on the weak limits of normalized Lebesgue measures on simplices; these are the so-called Cesàro measures, which are concentrated on the sequences with prescribed Cesàro limits (this limit parametrizes the corresponding measure).

More complicated examples are the graphs of continuous Young diagrams with fixed number of rows and the graphs of spectra of infinite Hermitian matrices of finite rank. We prove existence and uniqueness theorems for ergodic central measures and describe their structure. In particular, our results 1) give a new spectral description of the so-called infinite-dimensional Wishart measures [15] — ergodic unitarily invariant measures of discrete type on the set of infinite Hermitian matrices; 2) describe the structure of continuous analogs of measures on discrete graded graphs.

New problems and connections which appear are to be considered in new publications.

1 Introduction

Locally finite $\mathbb{N}$-graded graphs, often called Bratteli diagrams, proved to be a useful instrument both for algebraic and stochastic problems. It is important to stress that the geometry of such graphs is related to the theory of discrete topological Markov chains, although the usual questions about Markov chains differ from the questions of the theory of Bratteli diagrams. Combining these two theories may yield interesting results. It is well known in the theory of Markov chains that it is fruitful to consider

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not finitely or countably many states, but continuously many states. This approach is well developed in probability theory, although some generalizations of notions and results are nontrivial. But in the theory of graded graphs, it is not yet developed. In this paper we want to consider first examples of results concerning so-called continuous graded graphs. We recall that the theory of continuous graphs is a new and active branch [6], but the theory of graded graphs has its own problems which are somehow different from the usual graph theory problems.

A basic notion for a graph is the path space, possibly structured. For a graded graph, we consider “graded” paths which begin at the first level and go down. In this paper we consider graphs in which both the levels and the sets of edges between consecutive levels correspond to convex cones in Euclidean spaces. Such graphs are said to be of Gelfand–Tsetlin type.

The simplest such graph has the half-line \( \mathbb{R}_+ \) as the vertex set at each level, and there is an edge from \( x \) to \( y \) iff \( x \leq y \). This makes it possible to identify a path in this graph with a series with nonnegative terms. We call it the Cesàro graph. We are especially interested in the following two generalizations of the Cesàro graph. The first one is the continuous Gelfand–Tsetlin graph: the vertices at level \( n \) are the increasing sequences \( x_1 \leq x_2 \leq \ldots \leq x_n \) of real numbers, and there is an edge between two sequences \( x_1 \leq x_2 \leq \ldots \leq x_n \) and \( y_1 \leq y_2 \leq \ldots \leq y_{n+1} \) iff they interlace:

\[
  y_1 \leq x_1 \leq y_2 \leq x_2 \leq \ldots \leq x_n \leq y_{n+1}.
\]

This graph has the subgraph consisting of nonnegative sequences (“positive Gelfand–Tsetlin graph”). An important intimately related graph is the so-called rank \( d \) Gelfand–Tsetlin graph, in which the vertices at level \( n \) are the increasing sequences \( x_1 \leq x_2 \leq \ldots \leq x_d \) of \( d \) nonnegative numbers, and an edge corresponds to the interlacing condition

\[
  x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_d \leq y_d.
\]

This graph may be regarded as a subgraph of the positive Gelfand–Tsetlin graph by considering only the levels \( \geq d \) and sequences of the form \( (0,0,\ldots,0,x_1,\ldots,x_d) \) (with \( n-d \) initial zeros). These graphs correspond to the spectra of corners of Hermitian matrices (arbitrary, or nonnegative definite, or nonnegative definite of rank at most \( d \)).

For an infinite nonnegative definite Hermitian matrix \( A \) of rank at most \( d \), we may define \( A_n \) for \( n \geq d \) as the principal \( n \times n \) minor of \( A \), and let \( x_1 \leq \ldots \leq x_d \) be the \( d \) largest eigenvalues of \( A_n \) (all the other eigenvalues are equal to 0). Thus we get a path in the Gelfand–Tsetlin graph of rank \( d \). The pushforward of measures on nonnegative definite infinite Hermitian matrices of rank \( d \) are, therefore, measures on the path space of the Gelfand–Tsetlin graph of rank \( d \), and \( U(\infty) \)-invariant measures on matrices correspond to central measures on the path space. For matrices, we get the so-called Wishart measures. The Wishart measure of rank \( d \) with frequencies \( \lambda_1, \ldots, \lambda_d \) is defined as follows: take \( d \) i.i.d. standard complex Gaussian vectors \( \xi_1, \xi_2, \ldots, \xi_d \in \mathbb{C}^n \) and consider the matrix \( \sum \lambda_i \xi_i \cdot \xi_i^\dagger \). Without rank restrictions, the \( U(\infty) \)-invariant measures include, besides Wishart measures, which now may have infinitely many frequencies, the so-called Gaussian unitary ensemble (GUE), corresponding to Gaussian matrix entries (real on the main diagonal and complex otherwise; the entries on the diagonal and above the diagonal are independent).
Possibly the first paper in which the Gaussian unitary ensemble is considered in the framework of central measures on graded graphs is [2]. The complete list of invariant measures for infinite Hermitian matrices (and other similar series) is known for a while and was obtained by different methods by Pickrell [9] and Olshanski and Vershik [8] (the latter paper also uses the ergodic method but in a very different way from what we mean here).

This problem was considered in an old paper [11] of the first author about the ergodic method, but the list of Wishart measures was missed there. The invariant measures on positive definite Hermitian matrices are easier to find, this problem was addressed in [5, 1].

Gaussian measures (GUE, GOE) on matrices are especially popular. But we study these measures in terms of the graph of spectra of matrices rather than the graph of matrices themselves (for the GUE, this is done in [2]). The spectral approach is important not only in itself, but also because it allows us to discover more subtle properties of invariant measures. Moreover, this point of view is also possible for other fields (see, for example, [7]): for self-adjoint and quaternionic matrices. For more on the relation between graphs of matrices and graphs of spectra, see [14].

Another important generalization is the graph of increasing sequences \( x_1 \leq \ldots \leq x_d \) of nonnegative real numbers in which there is an edge between two sequences iff \( x_i \leq y_i \) for all \( i = 1, \ldots, d \). In other words, vertices are continuous Young diagrams with at most \( d \) rows, and an edge corresponds to one diagram being contained in the other.

A probabilistic Markov chain is determined by transition probabilities, but in the theory of Bratteli diagrams, like in many other problems in mathematics and physics [12], the basic notion is that of cotransition probability. Cotransition, or cocycle, is the main additional structure on a graded graph, and here we chose the cotransition probability measures to be the normalized Lebesgue measures on the phase spaces of cotransitions. These cotransitions arise in the theory of invariant measures on Hermitian matrices [2]. The Lebesgue cocycle is a continuous analog of the uniform cocycle for discrete graded graphs. A measure on the path space of a graded graph with Lebesgue cocycle is called a central measure.

Next, a remarkable problem arises of describing all central measures for a given graph. We discuss it for specific cases. For many countable graphs (the Pascal, Young, Schur graphs, dynamical graphs on groups and random walks), this problem has been considered in the recent decades.

A natural method for solving this problem is the ergodic method [11, 12], see the details below. Applying it to find central measures on continuous graphs is especially spectacular because of the geometric nature of the cocycle.

In many cases, the list of ergodic central measures is conjecturally known (or even already found by other methods). For example, in our case, each such measure is given by a set of finite or countable frequencies of natural objects. The problem is to explain why the frequencies uniquely determine an ergodic central measure, why they are determined by it, and, most importantly, how the frequency description of the measure is related to the corresponding product measure. Up to now, there was no such description not only for continuous graphs, but even for the Young graph. This is what is done below for the chosen class of graded graphs and for some discrete graphs.

Now we outline the contents of the paper. We begin with the fundamental example of the continuous Cesàro graph. The cocycle here is the set of normalized Lebesgue measures on simplices.
\{x_i \geq 0, \sum x_i = S\} of growing dimension with the sum \(S\) growing linearly with the dimension. It is easy to understand that a central measure on this graph is the weak limit of Lebesgue measures on simplices, but it is more difficult to prove that there exists a unique central measure for each growth rate, i.e., for each frequency. The structure of the measure is clear: it is a measure on the set of Cesàro sequences, which explains the name. Cesàro measures play here the same role as Bernoulli measures for discrete graphs.

For graphs more general than the Cesàro graph, that is, for the finite rank Gelfand–Tsetlin graphs and for the graph of continuous Young diagrams, the central measures with distinct frequencies are the restrictions of direct products of Cesàro measures to the corresponding sets of paths. In the case where some frequencies are equal, some regularization of the weak limits is needed. But, quite unexpectedly, it turns out that exactly the same, and even simpler, answer can be obtained for the problem of describing the central measures of discrete type for many countable graphs, e.g., for the Young graph.

Note that in [13] all central measures for the Young graph were realized by means of a generalized RSK algorithm as homomorphic images of Bernoulli measures, and in the subsequent paper [10] it was proved that this homomorphism is an isomorphism. However, this approach, while important, does not prove the completeness of the list of central measures. It turns out that the completeness of the list in this and other cases follows from an entirely different relation between Bernoulli measures and central measures: the latter are not only factors of Bernoulli measures, but also restrictions to their proper subsets. To the authors’ knowledge, for some reason this fact has been so far overlooked by mathematicians.

The class of continuous graphs under consideration is wide enough. It is convenient to regard infinite paths in these graphs as positive vector series. The dimension of the vector being equal to 1 corresponds to the case of the Cesàro graph.

We emphasize that the main conclusion from our considerations is that the ergodic central measures for a rather wide class of continuous graphs are determined by their frequencies, i.e., growth rates of specific coordinates. More precisely, a central measure is a Markov measure on positive vector series, and the behavior of these random series reflects the asymptotic behavior of frequencies.

It is important to stress that for the Gaussian unitary ensemble (for the Plancherel measure in the Young graph case), the picture is quite different, since all the limiting frequencies are zero; GUE (as well as the Plancherel measure) owes its existence to a very nontrivial interaction between the positive and negative parts of the spectra, each of which has a sublinear (rather than linear) asymptotics.

A remarkable parallel between these examples will be discussed in another paper. We are grateful to a referee who drew our attention to the interesting paper [4] where the connection between these two measures is established via the values of moments. But we mean a different type of parallelism.

We emphasize that the calculations involved in the ergodic method are sometimes quite nontrivial. In this paper we observe the very fact that central measures are determined by frequencies and find them explicitly in the simplest case of the so-called Cesàro graph. Further calculations are the subject of a forthcoming publication.
2 Setup

The vertex set of an \(N\)-graded graph is

\[
\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n,
\]

where \(\Gamma_n\) is the set of vertices of level \(n\) and every edge joins two vertices of consecutive levels. Here we assume that the following two conditions are satisfied.

1. Euclideness. All levels \(\Gamma_n\), \(n = 0, 1, 2, \ldots\), are identified with closed finite-dimensional convex polyhedral cones \(\Gamma_n \subset \mathbb{R}^{d_n}\) with nonempty interior (but the levels in a graph are disjoint, so, rigorously speaking, we must write something like \(\Gamma_n \subset \mathbb{R}^{d_n} \times \{n\}\)). The dimensions \(d_n\) may or may not depend on \(n\).

2. Convex incidence. Two vertices \(x \in \Gamma_n\), \(y \in \Gamma_{n+1}\) of consecutive levels are joined by an edge iff the pair \(x, y\) belongs to a certain closed convex polyhedral cone

\[
D_n \subset \Gamma_n \times \Gamma_{n+1}, \quad n = 0, 1, \ldots
\]

We require the natural projections \(D_n \rightarrow \Gamma_n\) and \(D_n \rightarrow \Gamma_{n+1}\) to be surjective; in other words, every vertex in \(\Gamma_0\) has a neighbor in \(\Gamma_1\), and every vertex in \(\Gamma_n\), \(n > 0\), has neighbors in both \(\Gamma_{n-1}\) and \(\Gamma_{n+1}\).

An infinite path is defined as a sequence of vertices \((v_0, v_1, \ldots)\), \(v_i \in V_i\), such that \((v_i, v_{i+1}) \in D_i\) for all \(i = 0, 1, \ldots\). The path space is the set of all infinite paths endowed with a natural (weak) topology.

3. Nondegeneracy. For all interior points \(x \in \Gamma_n\), \(z \in \Gamma_{n+1}\), the section \(D_n^+(x) = \{y \in \Gamma_{n+1} : (x, y) \in D_n\}\) (the set of second endpoints of the edges from \(x\) to \(\Gamma_{n+1}\)) is a nondegenerate convex polyhedral set that does not contain lines; and the section \(D_n^-(z) = \{y \in \Gamma_n : (y, z) \in D_n\}\) (the set of starting points of the edges from \(\Gamma_n\) to \(z\)) is a nondegenerate convex polytope. (The nondegeneracy here means the maximum dimension). This property can be weakened, but it determines an important special class of continuous graphs.

A certain class of continuous graphs is already defined. We proceed to specify the class of graphs under consideration by imposing new conditions on the graph structure.

4. Homogeneity. This condition concerns the case where all levels are isomorphic: a graph is said to be homogeneous if the incidence cones \(D_n\) coincide for all \(n\).

Graphs satisfying the above properties 1–3 are called continuous graphs of Gelfand–Tsetlin type, and if all levels of such a graph coincide and property 4 holds, we call it a homogeneous continuous graph of Gelfand–Tsetlin type.

3 Cocycles

We proceed to the definition of cotransition probabilities and cocycle for graded graphs. They are introduced in the same (and even simpler) manner as for discrete graded graphs.
For every interior vertex \( x \in \Gamma_n, n > 0 \), the paths from \( \Gamma_0 \) to \( x \) form a compact set \( \mathcal{P}(x) \) of dimension \( d_0 + d_1 + \ldots + d_{n-1} \), which is stratified in a natural way into sets \( \mathcal{P}(y) \) for \( y \in D_n^-(x) \subset \Gamma_{n-1} \). Assume that we have already fixed a probability measure \( \mu_y \) on each \( \mathcal{P}(y) \) for \( y \in D_n^-(x) \subset \Gamma_{n-1} \). Then, if we fix a probability measure on \( D_n^-(x) \) (so-called cotransition probability measure), it defines a probability measure \( \mu_x \) on \( \mathcal{P}(x) \). So, by an obvious induction, a system of cotransition measures defines a measure on the paths to \( x \) for every vertex \( x \) of \( \Gamma \). More generally, it defines a probability measure on the paths from \( y \) to \( x \) for any two vertices \( x \) and \( y \) such that there exists a path from \( y \) to \( x \). Hereafter, we assume that the measures are absolutely continuous with respect to the Lebesgue measures in the corresponding Euclidean spaces. Then, for tail-equivalent paths \( P_1, P_2 \) (that is, infinite paths that coincide from some point on) we can define a cocycle \( c(P_1, P_2) \) as follows. Take a large number \( n \) such that \( P_1, P_2 \) coincide after level \( n \) and cut the paths \( P_1, P_2 \) at this level. We get two paths \( P_1(n), P_2(n) \) belonging to the same set \( \mathcal{P}(x) \) for certain \( x \in \Gamma_n \). The ratio of the densities of \( \mu_x \) at \( P_2(n) \) and \( P_1(n) \) is denoted by \( c(P_1, P_2) \). It is straightforward that this number does not depend on the choice of \( n \).

In the discrete case, it is natural to consider the uniform distributions on the sets \( \mathcal{P}(x) \), which correspond to the central cocycle \( c(P_1, P_2) \equiv 1 \) (instead of considering densities, here we may simply divide probabilities). Borel measures on infinite paths with such a cocycle are called central measures. As follows from the definition, they are Markov measures (with time corresponding to the grading), but not vice versa. The set of ergodic (indecomposable) central measures is called the absolute of the graph. Finding the absolute is the most important problem in the theory of graded graphs.

For the class of continuous Gelfand–Tsetlin type graphs introduced above, we introduce a natural analog of the central cocycle, which we call the Lebesgue cocycle. It corresponds to considering the normalized Lebesgue measure on each \( \mathcal{P}(x) \), in other words, again \( c \equiv 1 \). Measures with such a cocycle are again called central measures, and the main problem is again to describe all ergodic central measures, i.e., the absolute of Gelfand–Tsetlin type graphs.

More general cocycles are to be considered separately.

## 4 Ergodic method

The so-called ergodic method for describing central and invariant measures was suggested in [II]. It essentially follows from the individual ergodic theorem or the martingale convergence theorem. To state the method, we introduce the necessary notion of weak convergence of measures on the path space. A sequence of measures on the path space of a graph is weakly convergent if for any cylinder set the values of the measures of this set converge. In our situation, the ergodic method can be described as follows (cf. [II, II2]).

**Theorem 1.** For every ergodic central measure \( \mu \) on the path space of a Gelfand–Tsetlin type graph \( \Gamma \) there exists a path, i.e., a sequence of vertices \( \{x_n\}_n, x_n \in D_n^-(x_{n+1}), n = 1, 2, \ldots, \) for which the sequence of the Lebesgue measures on the sets \( \mathcal{P}(x_n) \) weakly converges to \( \mu \).

Although the continuous setting is formally different from what has been considered earlier, the usual proof holds verbatim.
This theorem reduces the problem of describing the absolute to concrete calculations, which may be laborious in practice. On the other hand, the conditions of weak convergence in concrete cases may already be restrictive enough and allow one to describe the absolute explicitly.

The ergodic method was used in [13] to describe the characters of the infinite symmetric group $S_\infty$, in [8] to describe the invariant measures on infinite Hermitian matrices, and in many other problems about invariant or central measures.

5  Frequencies

Here we restate the main problem of describing the absolute of a Gelfand–Tsetlin type graph in concrete coordinate terms.

Assume that each level has dimension $d$ and coincides with the cone

$$C := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_d\}.$$

A convex cone $D_n$ (which is also the same for all $n$) is defined by several linear inequalities satisfied by two vertices $x = (x_1, \ldots, x_n) \in \Gamma_n$ and $y = (y_1, \ldots, y_n) \in \Gamma_{n+1}$.

Our two main examples are as follows.

(i) The rank $d$ Gelfand–Tsetlin graph is defined by the interlacing inequalities

$$D_n = D_{GT} := \{x, y : 0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_d \leq y_d\}.$$

This graph arises in random matrix theory. Namely, consider an $n \times n$ Hermitian matrix with eigenvalues $0$ (of multiplicity $n-d$), $y_1, \ldots, y_d$, where $0 \leq y_1 \leq \ldots \leq y_d$. Its principal minor of order $n-1$ has eigenvalues $0$ (of multiplicity $n-d-1$) and $x_1, \ldots, x_d$, where $0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \ldots \leq x_d \leq y_d$. Moreover, a unitarily invariant (in the natural sense) measure on the orbit of matrices with fixed eigenvalues and fixed eigenvalues of the minor of order $k$, $d \leq k \leq n$, is the Lebesgue measure on the corresponding polytope. This is proved by Baryshnikov [2] in greater generality.

So, along with the Gelfand–Tsetlin graph, we may consider a “covering” graph of matrices: at level $k$, the vertices are the $k \times k$ Hermitian matrices of rank at most $d$, and an edge joins a $k \times k$ matrix with its principal $(k-1) \times (k-1)$ minor.

The ordinary Gelfand–Tsetlin graph is defined similarly, but without the rank restriction.

Note that the definition of our graphs uses only inequalities between elements, and thus it can be automatically considered over an arbitrary linearly ordered set (for example, over a segment instead of the half-line) and even a poset.

In the discrete case of the linearly ordered set $\mathbb{Z}$, it is related to the representation theory of unitary groups, see, for example, [3].

(ii) The Young jumps graph. Here $D_n$ is the following cone:

$$D_Y := \{x, y \in C : x_1 \leq y_1, x_2 \leq y_2, \ldots, x_d \leq y_d\}.$$
If we identify a point \((x_1, \ldots, x_d)\) with the Young diagram with \(d\) rows of (not necessarily integer) lengths \(x_1, \ldots, x_d\), then the condition \((x, y) \in \mathcal{D}_Y\) means that one diagram is contained in the other. In the ordinary discrete Young graph, an edge corresponds to the condition “one diagram is contained in the other and differs from it by exactly one box.” Here we may remove (or add, if we consider moving along a forward path) arbitrarily many “boxes.”

Let \(x(n) = (x_1(n), x_2(n), \ldots, x_d(n)) \in \Gamma_n, n = 1, 2, \ldots\), be a path defining a central probability measure \(\mu\) on a homogeneous Gelfand–Tsetlin type graph \(\Gamma\) according to Theorem 1. If \(\lim \frac{x_i(n)}{n} = \lambda_i \in [0, \infty]\) for all \(i\), we say that the \(\lambda_i\)’s are the frequencies of the corresponding measure. We always may pass to a subsequence of \(\{x(n)\}\) for which the frequencies exist. The importance of this notion is due to the fact that in many cases (and we conjecture that for homogeneous Gelfand–Tsetlin type graphs, this is always the case) the frequencies uniquely determine an ergodic central measure.

6 Restricting and recovering

Let \(\Gamma\) be a graded graph with levels \(\Gamma_n\) and \(\tilde{D}_n \subset D_n\) be (measurable) subsets of the edge sets of \(\Gamma\). Denote by \(\tilde{\Gamma}\) the subgraph induced on these subsets.

Let \(\mu\) be a central probability measure on the path space of \(\Gamma\). Assume that with positive \(\mu\)-probability a random path is a path in \(\tilde{\Gamma}\), that is, it goes only along the edges in \(\bigcup \tilde{D}_n\).

Our crucial observation is the following almost obvious theorem.

Theorem 2. The restriction of \(\mu\) to the path space of \(\tilde{\Gamma}\) is also a central measure.

Proof. Fix a level \(n_0\). Like every central measure, \(\mu\) enjoys the Markov property, that is, it is characterized by the distribution at the level \(\Gamma_{n_0}\) and the conditional measures on the tails after this level. The initial segments before \(\Gamma_{n_0}\) are distributed (Lebesgue) uniformly on the corresponding polytopes \(\mathcal{P}(x), x \in \Gamma_{n_0}\), and do not depend on the tails. All this is preserved under restriction, since the restriction of a Lebesgue measure is again a Lebesgue measure.

Note that the theorem does not give a recipe for obtaining specific formulas for the restricted measure, and this task can be quite nontrivial. For example, consider the ordinary (discrete) Young graph consisting of Young diagrams with at most \(k\) rows. It may be viewed as the subgraph consisting of decreasing integer sequences \(n_1 \geq n_2 \geq \ldots \geq n_k \geq 0\) in the Pascal graph \(\mathbb{Z}_{\geq 0}^k\). Consider the Bernoulli measure with frequencies \(p_1 > p_2 > \ldots > p_k\) on \(\mathbb{Z}_{\geq 0}^k\) (which means that at each step we increase the \(i\)th coordinate by 1 with probability \(p_i\) and do not change the other coordinates). Its restriction to the Young graph is the Thoma measure with parameters \(p_1, \ldots, p_k\): the probability of a diagram with row lengths \(n_1 \geq n_2 \geq \ldots \geq n_k \geq 0\) is the value of the corresponding Schur function at the point \((p_1, \ldots, p_k)\).

A counterpart of Theorem 2 states that a finite measure \(\nu\) on the path space of the subgraph \(\tilde{\Gamma}\) can be extended to a measure \(\mu\) on the path space of the whole graph \(\Gamma\) such that the restriction of \(\mu\) to the path space of \(\tilde{\Gamma}\) coincides with \(\nu\) (but this \(\mu\) is not always finite, this is a delicate issue).
Indeed, the centrality property allows us to define $\mu$ on the images of the path space of $\tilde{\Gamma}$ under transformations that preserve central measures. For example, we may consider transformations that fix the tail after level $n_0$ and change the initial segments accordingly (preserving the Lebesgue measure). This defines $\mu$ on the set of paths that are eventually in $\tilde{\Gamma}$. It is not hard to see that this definition is consistent, and we get a measure on the path space of $\Gamma$ supported on the paths that are eventually in $\tilde{\Gamma}$.

If the resulting measure is finite, then the original measure on the path space of $\tilde{\Gamma}$ can be obtained by restriction (and normalization). For homogeneous Gelfand–Tsetlin type graphs, as well as for the discrete Young graph, this corresponds to the case of distinct frequencies.

### 7 One-dimensional case: Cesàro measures

Here we deal with the $d = 1$ case of the Gelfand–Tsetlin graph (the Young jumps graph coincides with the Gelfand–Tsetlin graph for $d = 1$) and compute the central measures of this graph.

A path in our graph is simply a sequence $0 \leq x_0 \leq x_1 \leq \ldots$ of nonnegative numbers, and the path space $P(a)$ for a vertex $a \in \Gamma_n = [0, \infty)$ is the $n$-dimensional simplex

$$\Delta_n(a) := \{0 \leq x_0 \leq x_1 \leq \ldots \leq x_n = a\}.$$  

We must study the distribution of finitely many coordinates $x_0, \ldots, x_{k-1}$ (where $k$ is fixed and $n$ is large) with respect to the normalized Lebesgue measure $\mu_a$ on this simplex.

Consider an ergodic central measure on the one-dimensional Gelfand–Tsetlin graph. By Theorem 1, it corresponds to a certain sequence of vertices (even to a path, but it is more convenient to consider a sequence: the difference is that in a sequence some levels may be skipped). Passing to a subsequence, we may assume that the vertices $a_i \in \mathbb{R}_+$ at levels $n_i$ satisfy the relation $\lim a_i/n_i = \lambda$, where $\lambda \in [0, \infty]$.

Thus, the following theorem implies that every ergodic central measure on the one-dimensional Gelfand–Tsetlin graph is an exponential random walk over the levels of the graph.

**Theorem 3.** Let $\lambda > 0$ be constant and $a = \lambda n + o(n) \in \Gamma_n$ be a vertex of level $n$ of the one-dimensional Gelfand–Tsetlin graph. Then for every fixed positive integer $k$, the measure induced by $\mu_a$ on the sequences $x_0 < x_1 < \ldots < x_{k-1}$ converges to the exponential random walk with mean $\lambda$: $x_0, x_1 - x_0, \ldots, x_{k-1} - x_{k-2}$ are i.i.d. with distribution $\text{Exp}(\lambda)$.

If $a = o(n)$, then the corresponding distributions weakly converge to the $\delta$-distribution at the zero sequence $(0, 0, \ldots)$; if $a/n \to \infty$, then the corresponding distributions weakly converge to the zero measure.

Because of the importance of this theorem and its relation to generalizations, we provide three different proofs. We focus on the case $0 < \lambda < \infty$, the proofs for $\lambda = 0$ and $\lambda = \infty$ are similar (and even simpler).

**Proof 1.** Denote $y_0 = x_0$, $y_i = x_i - x_{i-1}$ for $i = 1, \ldots, n - 1$. Then our simplex is defined as $\{y_i \geq 0, \sum y_i \leq a\}$. Its volume equals $a^n/n!$. If we restrict the distribution to the first $k$ coordinates
Of course, this vector has the same distribution as $x \cdot (n \Phi)$. Proof 3. This proof, based on the law of large numbers, is ideologically close to the proof of the Maxwell–Poincaré lemma, which states that the distribution of coordinates of a Lebesgue-random point on the $n$-dimensional sphere of radius $\sqrt{n}$ is asymptotically standard normal.

We start with the distribution $\eta_n$ of a vector $y \in (0, \infty)^n$ defined as the product of the $\text{Exp}(\lambda)$ distributions over all $n$ coordinates, and prove that the distribution of the first $k$ coordinates of this vector is close to that of a random point in our simplex $\Delta_n(a)$. The density of $\eta_n$ at a point $y = (y_1, \ldots, y_n)$ equals

$$p(y) = \prod_{i=1}^{d} \lambda^{-1} e^{-y_i/\lambda} = \lambda^{-n} e^{-(y_1 + \ldots + y_n)/\lambda}.$$
Thus, we see that \( p(y) \) depends only on the sum \( y_1 + \ldots + y_n \) of the coordinates. In other words, the random point \( y \) can be obtained by choosing the value of \( x = y_1 + \ldots + y_n \) at random (according to a certain distribution, which is in fact a gamma distribution, but we do not use this) and then choosing a uniformly distributed random point in the simplex \( S_x := \{ y_i \geq 0, \sum y_i = x \} \). Now fix a set \( \Omega \subset \mathbb{R}^k \) that is a product of intervals. The probability that \( (y_1, \ldots, y_k) \in \Omega \) is equal to the expectation against \( x \) of the probability that \( (y_1, \ldots, y_k) \in \Omega \) for a random point \( (y_1, \ldots, y_n) \in S_x \). Note that \( (y_1, \ldots, y_k) \in \Omega \) if and only if \( \frac{\lambda x}{x} (y_1, \ldots, y_k) \in \lambda \Omega \), and the point \( \frac{\lambda x}{x} (y_1, \ldots, y_n) \) is uniformly distributed in \( S_{\lambda x} \). Now observe that, by the law of large numbers, for any \( \varepsilon > 0 \) the probability that \( \frac{\lambda x}{x} \notin [1 - \varepsilon, 1 + \varepsilon] \) tends to 0. When \( \tau \in [1 - \varepsilon, 1 + \varepsilon] \), the sets \( \tau \Omega \) vary a little. In particular, their intersection contains a set \( \Omega_- (\varepsilon) \) and their union is contained in a set \( \Omega_+ (\varepsilon) \) such that the \( \eta_k \)-measures of both sets are close to each other. This yields the required weak convergence.

Thus, any ergodic measure on the one-dimensional Gelfand–Tsetlin graph is concentrated on the series \( x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots \) that are Cesàro summable to a certain number \( \lambda > 0 \). That is why we suggest to call it a Cesàro measure.

## 8 Multi-dimensional case

Here we explain how combining the one-dimensional Cesàro case and Theorem 2 allows us to handle the case of a finite rank Gelfand–Tsetlin graph and the Young jumps graph.

Note that the rank \( d \) Gelfand–Tsetlin graph is a subgraph of \( \text{Ces}^d \), where \( \text{Ces} \) is the Cesàro graph. On \( \text{Ces}^d \) we can consider a Bernoulli measure that is the product of Cesàro measures with frequencies \( p_1 < p_2 < \ldots < p_d \). With positive probability, any vertex \( (x_1, \ldots, x_d) \) is an increasing sequence and any two consecutive vertices \( (x_1, \ldots, x_d) \) and \( (y_1, \ldots, y_d) \) interlace (by the law of large numbers, with probability 1 this happens eventually, and it can easily be seen that with positive probability this “eventually” is from the very beginning). Thus, by our recovering procedure we get a unique ergodic central measure on the path space of the Gelfand–Tsetlin graph with given distinct frequencies.

A measure for which some frequencies are equal can be obtained as a weak limit of measures with distinct frequencies, but we skip the details here. All the same holds for the Young jumps graph: the ergodic central measures are in one-to-one correspondence with arrays of frequencies, and for distinct frequencies the corresponding measures are obtained by restricting Cesàro–Bernoulli measures.

To summarize, we provided a new way of studying central measures, which is based not on computations and estimates, but on the intrinsic properties of these measures. Of course, it can also be applied to discrete graphs, but, quite surprisingly, it was realized for continuous graphs. Applications to discrete graphs are to be considered in a separate paper. The calculations of transition probabilities also get a new interpretation.

For the Gelfand–Tsetlin graph of growing rank (the graph of interlacing sequences), we have countably many frequencies with a finite sum of absolute values, and if all these frequencies are distinct, the same restriction argument works.

This method reduces an explicit evaluation of central measures with given frequencies to a probabilistic problem of finding the “Cesàro–Bernoulli” measures of the corresponding cones (in order to
compute the restriction). An alternative, more standard, way is to apply a continuous version of the Lindström–Gessel–Viennot lemma.

As to general Gelfand–Tsetlin type graphs, we conjecture that for them central measures are also uniquely determined by limiting frequencies.

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