LINEAR LOCALLY NILPOTENT DERIVATIONS AND THE CLASSICAL INARIANT THEORY, I: THE POINCARÉ SERIES

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Abstract. By using classical invariant theory approach a formulas for computation of the Poincaré series of the kernel of linear locally nilpotent derivations is found.

1. Introduction

Let \( K \) be a field of characteristic 0. A derivation \( D \) of the polynomial algebra \( K[z_n] \), \( X_n = \{z_1, z_2, \ldots, z_n\} \) is called a linear derivation if

\[
D(z_i) = \sum_{j=1}^{n} a_{i,j} z_j, \quad a_{i,j} \in K, \quad i = 1, \ldots, n.
\]

If the matrix \( A_D := \{a_{i,j}\} \), \( i, j = 1, \ldots, n \) is nilpotent one then the linear derivation is called a Weitzenböck derivation. The Weitzenböck derivation is a locally nilpotent derivation of \( K[z_1, z_2, \ldots, z_n] \). Any Weitzenböck derivation \( D \) is completely determined by Jordan normal form of the matrix \( A_D \).

Denote by \( D_d := (d_1, d_2, \ldots, d_s) \) a Weitzenböck derivation with Jordan normal form of \( A_D \) which consists of \( s \) Jordan blocks of sizes \( d_1 + 1, d_2 + 1, \ldots, d_s + 1 \). Weitzenböck derivation which determined by unique Jordan block of size \( d + 1 \) is called the basic Weitzenböck derivation and denoted by \( \Delta_d \).

The algebra

\[
\ker D_d = \{ f \in K[z_n] | D_d(f) = 0 \},
\]

is called the kernel of the derivation \( D_d \). It is well known that the kernel \( \ker D_d \) is a finitely generated algebra, see [1]–[3]. However, it remained an open problem to find a minimal system of homogeneous generators (or even the cardinality of such a system) of the algebra \( \ker D_d \) even for small sets \( d \).

On the other hand, the problem to describe of the kernel \( \ker D_d \) can be reduced to an old problem of the classical invariant theory, namely to the problem to describe of the algebra of joint covariants of several binary forms.

It is well known that there is a one-to-one correspondence between \( G_a \)-actions on an affine algebraic variety \( V \) and locally nilpotent \( K \)-derivations on its algebra of polynomial functions. Let us identify the algebra \( K[z_n] \) with the algebra \( O[\mathbb{K}^n] \) of polynomial functions of the algebraic variety \( \mathbb{K}^n \). Then, the kernel of the derivation \( D_d \) coincides with the invariant ring of the induced via \( \exp(t D_d) \) action:

\[
\ker D_d = K[z_n]^{G_a} \cong O(\mathbb{K}^n)^{G_a}.
\]

Now, let \( B_{d_1}, B_{d_2}, \ldots, B_{d_s} \) be the vector \( K \)-spaces of binary forms of degrees \( d_1, d_2, \ldots, d_s \) endowed with the natural action of the group \( SL_2 \).

Consider the induced action of the group \( SL_2 \) on the algebra of polynomial functions \( O[B_d \oplus \mathbb{K}^2] \) on the vector space \( B_d \oplus \mathbb{K}^2 \), where

\[
B_d := B_{d_1} \oplus B_{d_2} \oplus \ldots \oplus B_{d_s}, \quad \dim(B_d) = d_1 + d_2 + \ldots + d_s + s.
\]

Let \( U_2 \) be the maximal unipotent subgroup of the group \( SL_2 \). The application of the Grosshans principle, see [4], [5] gives
Thus
\[ \mathcal{O}[B_d \oplus \mathbb{K}^2]^{SL_2} \cong \mathcal{O}[B_d]^{U_2}. \]

Since \( U_2 \cong (\mathbb{K}, +) \) and \( \mathbb{K} \mathbb{Z}_1 \oplus \mathbb{K} \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{K} \mathbb{Z}_n \cong B_d \) it follows
\[ \ker \mathcal{D}_d \cong \mathcal{O}[B_d \oplus \mathbb{K}^2]^{sl_2}. \]

In the language of classical invariant theory the algebra \( \mathcal{C}_d := \mathcal{O}[B_d \oplus \mathbb{K}^2]^{sl_2} \) is called the algebra of joint covariants for \( s \) binary forms, the algebra \( \mathcal{S}_d := \mathcal{O}[B_d]^{u_1} \) is called the algebra of joint semi-invariants for binary forms and the algebra \( \mathcal{I}_d := \mathcal{O}[B_d]^{sl_2} \) is called the algebra of invariants for binary forms of degrees \( d_1, d_2, \ldots, d_s \). The algebras of joint covariants of the binary forms were an object of research in the classical invariant theory of the 19th century.

The reductivity of \( SL_2 \) implies that the algebras \( \mathcal{I}_d, \mathcal{S}_d \cong \ker \mathcal{D}_d \), are finitely generated \( \mathbb{Z} \)-graded algebras. The formal power series \( \mathcal{P} \mathcal{I}_d, \mathcal{P} \mathcal{D}_d = \mathcal{P} \mathcal{S}_d \in \mathbb{Z}[[z]] \),
\[ \mathcal{P} \mathcal{I}_d(z) = \sum_{i=0}^{\infty} \dim((\mathcal{I}_d)_i)z^i, \quad \mathcal{P} \mathcal{S}_d(z) = \sum_{i=0}^{\infty} \dim((\mathcal{S}_d)_i)z^i, \]
are called the Poincaré series of the algebras of joint invariants and semi-invariants. The finitely generation of the algebras \( \mathcal{I}_d \) and \( \mathcal{S}_d \) implies that their Poincaré series are expansions of certain rational functions. We consider here the problem of computing efficiently these rational functions. It can be the first step in describing these algebras.

Let us recall that the Poincaré series of the algebra of covariants for binary form of degree \( d \) equals the Poincaré series of kernel of the basic Weitzenböck derivation \( \Delta_d \). For the cases \( d \leq 10, d = 12 \) the Poincaré series of the algebra of invariants and covariants for the binary \( d \)-form were calculated by Sylvester and Franklin, see \([14], [15]\). To do so, they used the Sylvester-Cayley formula for the dimension of graded subspaces. In \([16]\) the Poincaré series for \( \Delta_5 \) was rediscovered. Springer \([7]\) derived the formula for computing the Poincaré series of the algebras of invariants of the binary \( d \)-forms. This formula has been used by Brouwer and Cohen \([8]\) for the Poincaré series calculations in the cases \( d \leq 17 \) and also by Littelmann and Procesi \([12]\) for even \( d \leq 36 \). For the case \( d \leq 30 \) in \([9]\) the explicit form of the Poincaré series is given.

In \([10], [11]\) we have found Sylvester-Cayley type and Springer type formulas for the basic derivation \( \Delta_d \) and for the derivation \( \mathcal{D}_d \) for \( d = (d_1, d_2) \). Also, for those derivations the Poincaré series was found for \( d, d_1, d_2 \leq 30 \). Relatively recently, in \([13]\) the formula for computing the Poincare series of the Weitzenböck derivation \( \mathcal{D}_d \) for arbitrary \( d \) was announced.

In this paper we have given Sylvester-Cayley type formulas for calculation of \( \dim((\mathcal{I}_d)_i), \dim(\ker \mathcal{D}_d) \) and, Springer-type formulas for calculation of \( \mathcal{P} \mathcal{I}_d(z), \mathcal{P} \mathcal{D}_d(z) = \mathcal{P} \mathcal{S}_d(z) \) for arbitrary \( d \). Also, for the cases \( d = (1, 1, \ldots, 1), d = (2, 2, \ldots, 2) \) the explicit formulas for \( \mathcal{P} \mathcal{I}_d(z), \mathcal{P} \mathcal{D}_d(z) \) are given.

2. Sylvester-Cayley type formula for the kernel

To begin with, we give a proof of the Sylvester-Cayley type formula for the kernel of the \( \Delta_d \), \( d := (d_1, d_2, \ldots, d_s) \).

Let us consider the polynomial algebra \( \mathbb{K}[X_d] \) generated by the set of variables
\[ X_d := \left\{ x_0^{(1)}, x_1^{(1)}, \ldots, x_d^{(1)}, x_0^{(2)}, x_1^{(2)}, \ldots, x_d^{(2)}, \ldots, x_0^{(s)}, x_1^{(s)}, \ldots, x_d^{(s)} \right\}. \]

Define on \( \mathbb{K}[X_d] \) the linear nilpotent derivation \( \mathcal{D}_d, d := (d_1, d_2, \ldots, d_s) \) by
\[ \mathcal{D}_d(x_i^{(k)}) = i x_i^{(k)}, k = 1, \ldots, s. \]
Also, define on \( \mathbb{K}[X_d] \) two linear derivations \( \mathcal{D}_d^* \) and \( \mathcal{E}_d \), by

\[
\mathcal{D}_d^*(x_i^{(k)}) = (d_k - i) x_i^{(k)}, \quad \mathcal{E}_d(x_i^{(k)}) = (d_k - 2i) x_i^{(k)}, \quad k = 1, \ldots, s.
\]

The linear locally nilpotent derivation \( \mathcal{D}_d^* \) is said to be the dual derivation with respect to the derivation \( \mathcal{D}_d \).

By direct calculation we get

\[
[\mathcal{D}_d, \mathcal{D}_d^*](x_i^{(k)}) = \mathcal{D}_d (\mathcal{D}_d^* (x_i^{(k)})) - \mathcal{D}_d^* (\mathcal{D}_d (x_i^{(k)})) = (d_k - 2i) x_i^{(k)} = \mathcal{E}_d(x_i^{(k)}).
\]

In the same way we get \( [\mathcal{D}_d, \mathcal{E}_d] = -2\mathcal{D}_d \) and \( [\mathcal{D}_d^*, \mathcal{E}_d] = 2\mathcal{D}_d^* \). Therefore, the polynomial algebra \( \mathbb{K}[X_d] \) considered as a vector space becomes a \( \mathfrak{sl}_2 \)-module. The basis elements \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) of the algebra \( \mathfrak{sl}_2 \) act on \( \mathbb{K}[X_d] \) as follows:

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.f = \mathcal{D}_d (f), \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.f = \mathcal{D}_d^* (f), \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.f = \mathcal{E}_d (f),
\]

for any \( f \in \mathbb{K}[X_d] \).

Let \( u_2 = \mathbb{K}[X_d] \) be the maximal unipotent subalgebra of \( \mathfrak{sl}_2 \). As above, let us identify the algebras \( \mathcal{I}_d, \mathcal{S}_d \),

\[
\mathcal{I}_d := \mathbb{K}[X_d]^{\mathfrak{sl}_2} = \{ v \in \mathbb{K}[X_d] | \mathcal{D}_d(v) = \mathcal{D}_d^*(v) = 0 \},
\]

\[
\mathcal{S}_d := \ker \mathcal{D}_d = \mathbb{K}[X_d]^{u_2} = \{ v \in \mathbb{K}[X_d] | \mathcal{D}_d(v) = 0 \},
\]

with the algebras of joint invariants and joint semi-invariants of the binary forms of the degrees \( d_1, d_2, \ldots, d_s \). For any element \( v \in \mathcal{S}_d \) a natural number \( m \) is called the order of the element \( v \) if the number \( r \) is the smallest natural number such that

\[
(\mathcal{D}_d^*)^r(v) \neq 0, (\mathcal{D}_d)^{r+1}(v) = 0.
\]

It is clear that any semi-invariant of order \( r \) is the highest weight vector for an irreducible \( \mathfrak{sl}_2 \)-module of the dimension \( r + 1 \) in \( \mathbb{K}[X_d] \).

The algebra simultaneous covariants is isomorphic to the algebra of simultaneous semi-invariants. Therefore, it is enough to compute the Poincaré series of the algebra \( \mathcal{S}_d \).

The algebras \( \mathbb{K}[X_d], \mathcal{I}_d, \mathcal{S}_d \) are graded algebras:

\[
\mathbb{K}[X_d] = (\mathbb{K}[X_d])_0 + (\mathbb{K}[X_d])_1 + \cdots + (\mathbb{K}[X_d])_m + \cdots,
\]

\[
\mathcal{I}_d = (\mathcal{I}_d)_0 + (\mathcal{I}_d)_1 + \cdots + (\mathcal{I}_d)_m + \cdots,
\]

\[
\mathcal{S}_d = (\mathcal{S}_d)_0 + (\mathcal{S}_d)_1 + \cdots + (\mathcal{S}_d)_m + \cdots.
\]

and each \((\mathbb{K}[X_d])_m\) is the complete reducible representation of the Lie algebra \( \mathfrak{sl}_2 \).

Let \( V_k \) be the standard irreducible \( \mathfrak{sl}_2 \)-module, \( \dim V_k = k + 1 \). Then, the following primary decomposition holds

\[
(\mathbb{K}[X_d])_m \cong \gamma_m(d; 0)V_0 + \gamma_m(d; 1)V_1 + \cdots + \gamma_m(d; m \cdot d^*)V_{m \cdot d^*},
\]

(1)

here \( d^* := \max(d_1, d_2, \ldots, d_s) \) and \( \gamma_m(d; k) \) is the multiplicity of the representation \( V_k \) in the decomposition of \((\mathbb{K}[X_d])_m\). On the other hand, the multiplicity \( \gamma_m(d; k) \) of the representation \( V_k \) is equal to the number of linearly independent homogeneous simultaneous semi-invariants of the degree \( m \) and the order \( k \). In particular, the number of linearly independent simultaneous invariants of degree \( m \) is equal to \( \gamma_m(d; 0) \). These arguments prove

Lema 2.1.

(i) \( \dim(\mathcal{I}_d)_m = \gamma_m(d; 0) \),

(ii) \( \dim(\mathcal{S}_d)_m = \gamma_m(d; 0) + \gamma_m(d; 1) + \cdots + \gamma_m(d; m \cdot d^*) \).
Let us recall some general facts about the representation theory of the Lie algebra $\mathfrak{sl}_d$.

The set of weights of a representation $W$ denote by $\Lambda_W$, in particular, $\Lambda_{V_d} = \{-d, -d + 2, \ldots, d\}$. A formal sum

$$\text{Char}(W) = \sum_{\lambda \in \Lambda_W} n_W(\lambda)q^\lambda,$$

is called the character of a representation $W$, here $n_W(\lambda)$ denotes the multiplicity of the weight $\lambda \in \Lambda_W$. Since, a multiplicity of any weight of the irreducible representation $V_d$ is equal to 1, we have

$$\text{Char}(V_d) = q^{-d} + q^{-d+2} + \ldots + q^d.$$

Let us consider the $s$ sets of variables $x^{(1)}_0, x^{(1)}_1, \ldots, x^{(1)}_{d_1}, x^{(2)}_0, x^{(2)}_1, \ldots, x^{(2)}_{d_2}, \ldots, x^{(s)}_0, x^{(s)}_1, \ldots, x^{(s)}_{d_s}$. The character $\text{Char}((\mathbb{K}[X_d])_m)$ of the representation $(\mathbb{K}[X_d])_m$, see [17], equals

$$H_m(q^{-d_1}, q^{-d_1+2}, \ldots, q^{-d_2}, q^{-d_2+2}, \ldots, q^{-d_s}, q^{-d_s+2}, \ldots, q^d),$$

where $H_m(x^{(1)}_0, x^{(1)}_1, \ldots, x^{(1)}_{d_1}, x^{(2)}_0, x^{(2)}_1, \ldots, x^{(2)}_{d_2}, \ldots, x^{(s)}_0, x^{(s)}_1, \ldots, x^{(s)}_{d_s})$ is the complete symmetrical function

$$H_m(x^{(1)}_0, x^{(1)}_1, \ldots, x^{(1)}_{d_1}, x^{(2)}_0, x^{(2)}_1, \ldots, x^{(2)}_{d_2}, \ldots, x^{(s)}_0, x^{(s)}_1, \ldots, x^{(s)}_{d_s}) =
= \sum_{|\alpha(1)| + \ldots + |\alpha(s)| = m} (x^{(1)}_0)^{\alpha^{(1)}_0} (x^{(1)}_1)^{\alpha^{(1)}_1} \ldots (x^{(1)}_{d_1})^{\alpha^{(1)}_{d_1}} \ldots (x^{(s)}_0)^{\alpha^{(s)}_0} (x^{(s)}_1)^{\alpha^{(s)}_1} \ldots (x^{(s)}_{d_s})^{\alpha^{(s)}_{d_s}},$$

where $|\alpha(k)| := \sum_{i=0}^{d_k} \alpha^{(k)}_i$.

By replacing $x^{(k)}_i = q^{d_k - 2i}$, we obtain the specialized expression for the character $(\mathbb{K}[X_d])_m$, namely

$$\text{Char}((\mathbb{K}[X_d])_m) =
= \sum_{|\alpha(1)| + \ldots + |\alpha(s)| = n} (q^{d_1})^{\alpha^{(1)}_0} (q^{d_1-2})^{\alpha^{(1)}_1} \ldots (q^{-d_1})^{\alpha^{(1)}_{d_1}} \ldots (q^{d_s})^{\alpha^{(s)}_0} (q^{d_s-2-1})^{\alpha^{(s)}_1} \ldots (q^{-d_s})^{\alpha^{(s)}_{d_s}} =
= \sum_{|\alpha(1)| + \ldots + |\alpha(s)| = n} q^{d_1|\alpha(1)| + \ldots + d_s|\alpha(s)| + (\alpha^{(1)}_1 + 2\alpha^{(1)}_2 + \ldots + d_1 \alpha^{(1)}_{d_1}) + \ldots + (\alpha^{(s)}_1 + 2\alpha^{(s)}_2 + \ldots + d_s \alpha^{(s)}_{d_s})} =
= \sum_{i=-md^*}^{md^*} \omega_m(d; i)q^i,$$

here $\omega_m(d; i)$ is the number of nonnegative integer solutions of the following system of equations:

$$\begin{align*}
&d_1|\alpha^{(1)}| + \ldots + d_s|\alpha^{(s)}| + (\alpha^{(1)}_1 + 2\alpha^{(1)}_2 + \ldots + d_1 \alpha^{(1)}_{d_1}) + \\
&\ldots + (\alpha^{(s)}_1 + 2\alpha^{(s)}_2 + \ldots + d_s \alpha^{(s)}_{d_s}) = i \\
&|\alpha^{(1)}| + \ldots + |\alpha^{(s)}| = m.
\end{align*}$$

(2)

We can summarize what we have shown so far in
Theorem 2.1.  

(i) \( \dim(I_d)_m = \omega_m(d; 0) - \omega_m(d; 2) \).

(ii) \( \dim(S_d)_m = \omega_m(d; 0) + \omega_m(d; 1) \).

Proof. (i) The zero weight appears once in any representation \( V \), for even \( k \), therefore
\[
\omega_m(d; 0) = \gamma_m(d; 0) + \gamma_m(d; 2) + \gamma_m(d; 4) + \ldots
\]
The weight 2 appears once in any representation \( V \), for even \( k > 0 \), therefore
\[
\omega_m(d; 2) = \gamma_m(d; 2) + \gamma_m(d; 4) + \gamma_m(d; 6) + \ldots
\]
Taking into account Lemma 2.1, we obtain
\[
\omega_m(d; 0) - \omega_m(d; 2) = \gamma_m(d; 0) = \dim(I_d)_m.
\]
(ii) The weight 1 appears once in any representation \( V \), for odd \( k \), therefore
\[
\omega_m(d; 1) = \gamma_m(d; 1) + \gamma_m(d; 3) + \gamma_m(d; 5) + \ldots
\]
Thus,
\[
\omega_m(d; 0) + \omega_m(d; 1) =
\]
\[
= \gamma_m(d; 0) + \gamma_m(d; 1) + \gamma_m(d; 2) + \ldots + \gamma_m(d; n d) =
\]
\[
= \dim(S_d)_m.
\]

\[ \square \]

Simplify the system (2) to
\[
\begin{align*}
(d_1 - 2)\alpha_0^{(1)} + (d_1 - 4)\alpha_2^{(1)} + \cdots + (d_1)\alpha_d^{(1)} + \cdots + \\
+ d_s\alpha_0^{(s)} + (d_s - 2)\alpha_1^{(s)} + (d_s - 4)\alpha_2^{(s)} + \cdots + (d_s)\alpha_d^{(s)} = i,
\end{align*}
\]
\[
\alpha_0^{(1)} + \alpha_1^{(1)} + \cdots + \alpha_d^{(1)} + \cdots + \alpha_0^{(s)} + \alpha_1^{(s)} + \cdots + \alpha_d^{(s)} = n.
\]

It well-known that the number \( \omega_m(d; i) \) of non-negative integer solutions of the above system is equal to the coefficient of \( t^m z^i \) of the expansion of the series
\[
f_d(t, z) =
\]
\[
= \frac{1}{(1 - tz^{d_1})(1 - t z^{d_2}) \cdots (1 - t z^{d_s})(1 - t z^{d_2-2}) \cdots (1 - t z^{d_s-2}) \cdots (1 - t z^{d_s})}.
\]
Denote it in such a way: \( \omega_m(d; i) := \lfloor t^m z^i \rfloor (f_d(t, z)) \). Observe that \( f_d(t, z) = f_d(t, z^{-1}) \).

The following statement holds

Theorem 2.2.

(i) \( \dim(I_d)_m = [t^m](1 - z^2)f_d(t, z) \),

(ii) \( \dim(S_d)_m = [t^m](1 + z)f_d(t, z) \).
Proof. Taking into account the formal property \([x^{i-k}] f(x) = [x^i](x^k f(x))\), we get
\[
\dim(I_d)_m = \omega_m(d; 0) - \omega_m(d; 2) = [t^m] f_d(t, z) - [t^m z^2] f_d(t, z) = [t^m] f_d(t, z) - [t^m z^2] f_d(t, z^-1) = [t^m] (1 - z^2) f_d(t, z).
\]
In the same way
\[
\dim(S_d)_m = \omega_m(d; 0) + \omega_m(d; 1) = [t^m] f_d(t, z) + [t^m z] f_d(t, z) = [t^m] f_d(t, z) + [t^m z^{-1}] f_d(t, z) = [t^m] (1 + z) f_d(t, z).
\]
\[
\square
\]
It is easy to see that the dimensions \(\dim(I_d)_m\), \(\dim(S_d)_m\) allow the following representations:
\[
\dim(I_d)_m = [t^m] \frac{1}{2\pi i} \oint_{|z|=1} (1 - z^2) f_d(t, z) \frac{dz}{z},
\]
\[
\dim(S_d)_m = [t^m] \frac{1}{2\pi i} \oint_{|z|=1} (1 + z) f_d(t, z) \frac{dz}{z}.
\]

3. Springer type formulas for the Poincaré series

Let us prove a Springer type formulas for the Poincaré series \(PL_d(z), PS_d(z) = PD_d(z)\) of the algebras simultaneous invariants and semi-invariants of two binary forms.

Consider the \(C\)-algebra \(C[[t, z]]\) of the formal power series. For an arbitrary \(m, n \in Z^+\) define \(C\)-linear function
\[
\Psi_{m, n} : C[[t, z]] \rightarrow C[[z]],
\]
in the following way:
\[
\Psi_{m, n} \left( \sum_{i,j=0}^{\infty} a_{i, j} t^i z^j \right) = \sum_{i=0}^{\infty} a_{im, in} z^i.
\]
Denote by \(\varphi_n\) the restriction of \(\Psi_{m, n}\) to \(C[[z]]\), namely
\[
\varphi_n \left( \sum_{i=0}^{\infty} a_i z^i \right) = \sum_{i=0}^{\infty} a_{in} z^i.
\]
There is an effective algorithm of calculation for the function \(\varphi_n\), see [10]. In some cases calculation of the functions \(\Psi\) can be reduced to calculation of the functions \(\varphi\). The following statements hold:

Lema 3.1. For \(R(z) \in C[[z]]\) and for \(m, n, k \in \mathbb{N}\) we have:
\[
\Psi_{1, n} \left( \frac{R(z)}{(1 - tz^k)^m} \right) = \begin{cases} 
\frac{1}{(m - 1)!} \frac{d^{m-1} \varphi_{n-k}(R(z))}{dz^{m-1}}, & n > k; \\
\frac{R(0)}{(1 - z)^m}, & n = k; \\
R(0), & \text{if } k > n.
\end{cases}
\]
Proof. Let $R(z) = \sum_{j=0}^{\infty} r_j z^j$. Observe, that

$$\frac{1}{(1-x)^m} = \frac{1}{(m-1)!} \left[ \frac{1}{1-x} \right]^{(m-1)} = \sum_{i=0}^{\infty} \binom{s+m-1}{m-1} x^s.$$ 

Then for $n > k$ we have

$$\Psi_{1,n} \left( \frac{R(z)}{(1-tz^k)^m} \right) = \Psi_{1,n} \left( \sum_{j,s \geq 0} \binom{s+m-1}{m-1} r_j (tz^k)^s \right) = \sum_{s \geq 0} \binom{s+m-1}{m-1} r_{s(n-k)} z^s.$$ 

On other hand

$$\frac{1}{(m-1)!} (z^{m-1} \varphi_{n-k}(R(z)))^{(m-1)} = \frac{1}{(m-1)!} \left( \sum_{s=0}^{\infty} r_{s(n-k)} z^{m+s-1} \right)^{(m-1)} = $$

$$= \frac{1}{(m-1)!} \sum_{s \geq 0} (s+m-1)(s+m-2) \ldots (s+1)r_{s(n-k)} z^s = \sum_{s \geq 0} \binom{s+m-1}{m-1} r_{s(n-k)} z^s.$$ 

This proves the case $n > k$.

Taking into account the formal property

$$\Psi_{1,n}(F(tz^n) H(t, z)) = F(z) \Psi_{1,n}(H(t, z)), F(z), H(t, n) \in \mathbb{C}[[t, z]],$$

for the case $n = k$ we have

$$\Psi_{1,n} \left( \frac{R(z)}{(1-tz^k)^m} \right) = \frac{1}{(1-z)^m} \Psi_{1,n} (R(z)) = \frac{R(0)}{(1-z)^m}.$$ 

To prove the case $n < k$, observe that, the equation $ks + j = ns$ for $n < k$ and $j, s \geq 0$ has only one trivial solution $j = s = 0$. We have

$$\Psi_{1,n} \left( \frac{R(z)}{1-tz^k} \right) = \Psi_{1,n} \left( \sum_{j,s \geq 0} r_j z^j \right) = \Psi_{1,n} \left( \sum_{j,s \geq 0} t^j z^s \right) = r_0 = R(0).$$

The main idea of the calculations of the paper is that the Poincaré series $\mathcal{P}d(z), \mathcal{P}S_d(z)$ can be expressed in terms of functions $\Psi$. The following simple but important statement holds:

Lema 3.2. Let $d^* := \max(d)$. Then

$$(i) \quad \mathcal{P}d(z) = \Psi_{1,d^*} \left( (1-z^2) f_d(tz^{d^*}, z) \right),$$

$$(ii) \quad \mathcal{P}S_d(z) = \Psi_{1,d^*} \left( (1+z) f_d(tz^{d^*}, z) \right).$$

Proof. Theorem 2 states that $\dim(I_d) = [t^n] (1-z^2) f_d(t, z)$. Then

$$\mathcal{P}d(z) = \sum_{n=0}^{\infty} \dim(I_d)_n \cdot z^n = \sum_{n=0}^{\infty} [t^n] (1-z^2) f_d(t, z) \cdot z^n =$$

$$= \sum_{n=0}^{\infty} \left( (tz^{d^*})^n \right) (1-z^2) f_d(tz^{d^*}, z) \cdot z^n = \Psi_{1,d^*} \left( (1-z^2) f_d(tz^d, z) \right).$$
Similarly, we prove the statement (ii).

We replaced \( t \) with \( tz^{d^*} \) to avoid of a negative powers of \( z \) in the denominator of the function \( f_d(t, z) \).

Write the function \( f_d(t, z) \) in the following way

\[
f_d(t, z) = \frac{1}{\prod_{k=1}^n (tz^{-d_k}, z^2)^{d_k+1}},
\]

here \((a, q)_n = (1 - a)(1 - a q) \cdots (1 - a q^{n-1})\) denotes the \( q \)-shifted factorial.

The above lemma implies the following representations of the Poincaré series via the contour integrals:

**Lemma 3.3.**

(i) \( \mathcal{P} \mathcal{I}_d(t) = \frac{1}{2 \pi i} \oint_{|z|=1} \frac{1 - z^2}{\prod_{k=1}^n (tz^{-d_k}, z^2)^{d_k+1}} \frac{dz}{z} \),

(ii) \( \mathcal{P} \mathcal{S}_d(t) = \frac{1}{2 \pi i} \oint_{|z|=1} \frac{1 + z}{\prod_{k=1}^n (tz^{-d_k}, z^2)^{d_k+1}} \frac{dz}{z}. \)

**Proof.** We have

\[
\mathcal{P} \mathcal{S}_d(t) = \sum_{n=0}^\infty \dim(I_d)_n t^n = \sum_{n=0}^\infty (\lfloor t^n \rfloor (1 + z)f_d(t, z)) t^n = \sum_{n=0}^\infty \left( \lfloor t^n \rfloor \frac{1}{2 \pi i} \oint_{|z|=1} (1 + z)f_d(t, z) \frac{dz}{z} \right) t^n = \frac{1}{2 \pi i} \oint_{|z|=1} (1 + z)f_d(t, z) \frac{dz}{z}.
\]

Similarly we get the Poincaré series \( \mathcal{P} \mathcal{I}_d(t) \). \( \square \)

Note that the Molien-Weyl integral formula for the Poincaré series \( \mathcal{P}_d(t) \) of the algebra of invariants of binary \( d \)-form can be reduced to the following formula

\[
\mathcal{P}_d(t) = \frac{1}{2 \pi i} \oint_{|z|=1} \frac{1 - z^2}{\prod_{k=1}^n (1 - tz^d)(1 - tz^{d-2}) \cdots (1 - tz^{-d})} \frac{dz}{z} = \frac{1}{2 \pi i} \oint_{|z|=1} \frac{1 - z^2}{\prod_{k=1}^n (tz^{-d}, z^2)^{d_k+1}} \frac{dz}{z}.
\]

see [19], p. 183. An ingenious way to calculate such integrals proposed in [20].

After simplification we can write \( f_d(tz^{d^*}, z) \) in the following way

\[
f_d(tz^{d^*}, z) = ((1 - t)^{\beta_0}(1 - tz)^{\beta_1}(1 - tz^2)^{\beta_2} \cdots (1 - tz^{2d^*})^{\beta_{d^*}})^{-1},
\]

for some integer \( \beta_0, \ldots, \beta_{d^*} \). For example

\[
f_{(1, 2, 4)}(tz^4, z) = \frac{1}{(1 - t)(1 - tz^2)(1 - tz^3)(1 - tz^4)^2(1 - tz^5)(1 - tz^6)^2(1 - tz^8)}.
\]

It implies the following partial fraction decomposition of \( f_d(tz^{d^*}, z) \):

\[
f_d(tz^{d^*}, z) = \sum_{i=0}^{2d^*} \sum_{k=1}^{\beta_i} \frac{A_{i, k}(z)}{(1 - tz^i)^k},
\]

for some polynomials \( A_{i, k}(z) \).

By direct calculations we obtain

\[
A_{i, k}(z) = \frac{(-1)^{\beta_i - k}}{\beta_i - k)! (z^{\beta_i - k}) \lim_{t \to z^{-i}} \frac{\partial t^{\beta_i - k}}{\partial t^{\beta_i - k}} (f_d(tz^{d^*}, z)(1 - tz^i)^{\beta_i}).
\]

Now we can present Springer type formulas for the Poincaré series \( \mathcal{P} \mathcal{I}_d(z) \) and \( \mathcal{P} \mathcal{S}_d(z) \).
Theorem 3.1.
\[
\mathcal{P}_I d(z) = \sum_{i=0}^{d^*} \sum_{k=1}^{\beta_i} \frac{1}{(k-1)!} z^{k-1} (z^{k-1} \varphi_{d^*-k} ((1 - z^2) A_{i,k}(z)))
\]
\[
\mathcal{P}_S d(z) = \sum_{i=0}^{d^*} \sum_{k=1}^{\beta_i} \frac{1}{(k-1)!} z^{k-1} (z^{k-1} \varphi_{d^*-k} ((1 + z) A_{i,k}(z)))
\]

Proof. Taking into account Lemma 3.1 and linearity of the map \( \Psi \) we get
\[
\mathcal{P}_S d(z) = \Psi_{1,d^*} \left((1 + z) f_d(tz^{d^*}, z)\right) = \Psi_{1,d^*} \left(\sum_{i=0}^{2d^*} \sum_{k=1}^{\beta_i} \frac{(1 + z) A_{i,k}(z)}{(1 - tz^2)^k}\right)
\]
\[
= \sum_{i=0}^{d^*} \sum_{k=1}^{\beta_i} \frac{1}{(k-1)!} z^{k-1} (z^{k-1} \varphi_{d^*-k} ((1 + z) A_{i,k}(z)))
\]

The case \( \mathcal{P}_I d(z) \) can be considered similarly. \( \square \)

Note, the Poincaré series \( \mathcal{P}_I d(z) \) and \( \mathcal{P}_C d(z) \) of the algebras of invariants and covariants of binary \( d \)-form equal
\[
\mathcal{P}_I d(z) = \sum_{0 \leq k < d/2} \varphi_{d-2k} \left(\frac{(-1)^k z^{k(k+1)} (1 - z^2)}{(z^2, z^2)_k (z^2, z^2)_{d-k}}\right)
\]
\[
\mathcal{P}_C d(z) = \sum_{0 \leq k < d/2} \varphi_{d-2k} \left(\frac{(-1)^k z^{k(k+1)} (1 + z)}{(z^2, z^2)_k (z^2, z^2)_{d-k}}\right)
\]
see \([7]\) and \([10]\) for details.

4. Explicit formulas for small \( d \)

The formulas of Theorem 3.1 allow the simplification for some small values \( d \).

Theorem 4.1. Let \( s = n \) and \( d_1 = d_2 = \ldots = d_n = 1 \), i.e. \( d = (1, 1, \ldots, 1) \). Then
\[
\mathcal{P}_I d(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)! (n-k)!} z^{k-1} \left(\frac{z}{1 - z^2}\right)^{2n-k-1}
\]
\[
\mathcal{P}_S d(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k} (n)_{n-k}}{(k-1)! (n-k)!} z^{k-1} \left(\frac{(1 + z) z^{2n-k-1}}{(1 - z^2)^{2n-k}}\right)
\]
where \((n)_m := n(n + 1) \cdots (n + m - 1), (n)_0 := 1\) denotes the shifted factorial.

Proof. For \( d = (1, 1, \ldots, 1) \in \mathbb{Z}^n\) we have \( d^* = 1 \) and
\[
f_d(tz^{d^*}, z) = \frac{1}{((1 - t)(1 - t z^2))^n} = \frac{A_{0,1}(z)}{1 - t} + \cdots + \frac{A_{0,n}(z)}{(1 - t)^n} + R(z), \Psi_{1,1} (R(z)) = 0,
\]
where
\[
A_{0,k} = \frac{(-1)^{n-k} (n)_{n-k}}{(n-k)!} \lim_{t \to 1} \frac{\partial^{n-k}}{\partial t^{n-k}} \left(\frac{1}{(1 - t z^2)^n}\right).
\]
By induction we get
$$\lim_{t \to 1} \frac{\partial^m}{\partial t^m} \left( \frac{1}{(1-tz^2)^m} \right) = (n)_m \frac{(z^2)^m}{(1-z^2)^{n+m}}.$$ Thus,
$$A_{0,k} = \frac{(-1)^{n-k} (n)_{n-k} (z^2)^{n-k}}{(n-k)! (1-z^2)^{2n-k}}.$$ Now, using Theorem 3.1 and the property \(\varphi_1(F(z)) = F(z)\), for any \(F(z) \in \mathbb{Z}[[z]]\) we have
$$\mathcal{P}S_d(z) = \Psi_{1,1} \left( \sum_{k=1}^{s} \frac{(1+z) A_{0,k}}{(1-t)^k} \right) = \sum_{k=1}^{s} \Psi_{1,1} \left( \frac{(1+z) A_{0,k}}{(1-t)^k} \right) =$$
$$= \sum_{k=1}^{n} \frac{1}{(m-1)!} d_z^{k-1} \left( z^{k-1} \varphi_1((1+z) A_{0,k}) \right) = \sum_{k=1}^{n} \frac{1}{(m-1)!} d_z^{k-1} \left( (1+z) z^{2k-1} A_{0,k} \right) =$$
$$= \sum_{k=1}^{n} \frac{(-1)^{n-k} (n)_{n-k} d_z^{k-1}}{(k-1)! (n-k)!} \left( (1+z) z^{2k-1} \right).$$

The case \(\mathcal{P}I_d(z)\) can be considered similarly. 

**Theorem 4.2.** Let \(d_1 = d_2 = \ldots = d_n = 2\), \(d = (2, 2, \ldots, 2)\), then
$$\mathcal{P}I_d(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)(k-1)!} d_z^{k-1} \left( \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n-k)_i (1-z) z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right),$$
$$\mathcal{P}S_d(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)! (k-1)!} d_z^{k-1} \left( \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n-k)_i (1-z) z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}} \right).$$

**Proof.** It is easy to check that in this case we have
$$f_d(tz^2, z) = \frac{1}{(1-t)(1-tz^2)(1-tz^4)^n}.$$ The decomposition \(f_d(tz^2, z)\) into partial fractions yields
$$f_d(tz^2, z) = \sum_{k=1}^{n} \left( \frac{A_k(z)}{(1-t)^k} + \frac{B_k(z)}{(1-tz^2)^k} + \frac{C_k(z)}{(1-tz^4)^k} \right),$$
for some rational functions \(A_k(z), B_k(z), C_k(z)\). Then
$$\mathcal{P}S_d(z) = \Psi_{1,2} \left( (1+z) f_d(tz^2, z) \right) =$$
$$= \sum_{k=1}^{n} \left( \Psi_{1,2} \left( \frac{(1+z) A_k(z)}{(1-t)^k} \right) + \Psi_{1,2} \left( \frac{(1+z) B_k(z)}{(1-tz^2)^k} \right) + \Psi_{1,2} \left( \frac{(1+z) C_k(z)}{(1-tz^4)^k} \right) \right).$$

Lemma 3.1 implies that
$$\Psi_{1,2} \left( \frac{(1+z) C_k(z)}{(1-tz^4)^k} \right) = 0,$$
and
$$\Psi_{1,2} \left( \frac{(1+z) B_k(z)}{(1-tz^2)^k} \right) = \frac{B_k(0)}{(1-z)^k}, k = 1, \ldots, n.$$
But
\[ B_k(z) = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} \frac{\partial^{n-k}}{\partial t^{n-k}} \left( \frac{1}{(1-t)^{n}(1-tz^4)^n} \right). \]

It is easy to see that this partial derivatives has the following form
\[ \frac{\partial^{n-k}}{\partial t^{n-k}} \left( \frac{1}{(1-t)^{n}(1-tz^4)^n} \right) = \frac{B_k(t, z)}{((1-t)(1-tz^4))^{2n-k}}. \]

for some polynomial \( B_k(t, z) \). Moreover, \( \deg_s(B_k(t, z)) = n - k \). Then
\[ B_k(z) = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} B_k(t, z) \cdot \frac{1}{((1-t)(1-tz^4))^{2n-k}} = \]
\[ = \frac{(-1)^{n-k}z^{2n}B_k(1/z^2, z)}{(n-k)!((z^2 - 1)(1-tz^4))^{2n-k}}. \]

It follows that \( B_k(z) \) has the factor \( z^{2k} \) and then \( B_k(0) = 0 \). Thus
\[ \Psi_{1,2} \left( \frac{(1+z)B_k(z)}{(1-tz^2)^k} \right) = 0, k = 1, \ldots, n. \]

Therefore
\[ \mathcal{PS}_d(z) = \sum_{k=1}^{n} \Psi_{1,2} \left( \frac{(1+z)A_k(z)}{(1-t)^k} \right) = \sum_{k=1}^{n} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left( z^{k-1} \varphi_2((1+z)A_k(z)) \right). \]

Let us to calculate \( A_k(z) \). We have
\[ A_k(z) = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} \frac{\partial^{n-k}}{\partial t^{n-k}} \left( f_a(tz^2, z)(1-t)^n \right) = \]
\[ = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} \frac{\partial^{n-k}}{\partial t^{n-k}} \left( \frac{1}{(1-tz^2)^n(1-tz^4)^n} \right) = \]
\[ = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) \left( \frac{1}{(1-tz^2)^n} \right)^{i} \left( \frac{1}{(1-tz^4)^n} \right)^{n-k-i} = \]
\[ = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to z} \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (n)_i(n)_{n-k-i} \frac{z^{2i}}{(1-tz^2)^{n+i}} \frac{z^{4(n-k-i)}}{(1-tz^4)^{2n-k-i}} = \]
\[ = \frac{(-1)^{n-k}}{(n-k)!} \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (n)_i(n)_{n-k-i} \frac{(z^2)^{2(n-k)-i}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}. \]

Taking into account that \( \varphi_2(F(z^2)) = F(z) \), and \( \varphi_2(zF(z^2)) = 0 \) we obtain
\[ \varphi_2((1+z)A_k(z)) = \varphi_2(A_k(z)) = \]
\[ = \frac{(-1)^{n-k}}{(n-k)!} \sum_{i=0}^{n-k} \left( \begin{array}{c} n-k \\ i \end{array} \right) (n)_i(n)_{n-k-i} \frac{(z)^{2(n-k)-i}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}. \]
Thus,

\[
\mathcal{PS}_d(z) = \sum_{k=1}^{n} \Psi_{1,2} \left( \frac{(1 + z)A_k(z)}{(1 - t)^k} \right) = \sum_{k=1}^{n} \frac{1}{(k-1)!} d^{k-1} \varphi_2((1 + z)A_k(z)) = 
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^{n-k} d^{k-1}}{(n - k)! (k-1)!} \frac{d^{k-1}}{dz^{-1}} \left( \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(n)_i (n-k-i)_i}{(1 - z)^{n+i} (1 - z^2)^{2{n-k-i}-1}} \right).
\]

By replacing the factor 1 + \( z \) with 1 - \( z^2 \) in \( \mathcal{PS}_d(z) \) and taking into account that

\[
\varphi_2((1 - z^2)A_k(z)) = (1 - z)\varphi_2(A_k(z)),
\]

get the Poincaré series \( \mathcal{PD}_d(z) \).

\[\square\]

5. Examples

For direct computations of the function \( \varphi_n \) we use the following technical lemma, see [10]:

**Lemma 5.1.** Let \( R(z) \) be some polynomial of \( z \). Then

\[
\varphi_n \left( \frac{R(z)}{(1 - z^k_1)(1 - z^k_2) \cdots (1 - z^k_m)} \right) = \varphi_n \left( \frac{R(z)Q_n(z^k_1)Q_n(z^k_2)Q_n(z^k_m)}{(1 - z^k_1)(1 - z^k_2) \cdots (1 - z^k_m)} \right),
\]

here \( Q_n(z) = 1 + z + z^2 + \ldots + z^{n-1} \), and \( k_i \) are natural numbers.

As example, let us calculate the Poincaré series \( \mathcal{PD}_{(1,2,3)} \). We have \( d^* = 3 \) and

\[
f_{(1,2,3)}(t, z) = \frac{1}{(1 - tz^4)^2 (1 - tz^2)^2 (1 - tz^3)(1 - t)(1 - tz^2)^2 (1 - t^2 z^2)}. \]

The decomposition \( f_{(1,2,3)}(t, z) \) into partial fractions yields:

\[
f_{(1,2,3)}(t, z) = \frac{A_{0,1}(z)}{1 - t} + \frac{A_{1,1}(z)}{1 - tz} + \frac{A_{2,1}(z)}{1 - tz^2} + \frac{A_{2,2}(z)}{(1 - tz^2)^2} + \frac{A_{3,1}(z)}{1 - t z^3} + \frac{A_{4,1}(z)}{1 - t z^4} + \\
+ \frac{A_{4,2}(z)}{(1 - t z^4)^2} + \frac{A_{5,1}(z)}{1 - t z^5} + \frac{A_{6,1}(z)}{1 - t z^6}.
\]

By using Lemma 3.1 we have

\[
\mathcal{PD}_{(1,2,3)}(z) = \Psi_{1,3} \left( (1 + z) f_{(1,2,3)}(t, z) \right) = 
\]

\[
= \Psi_{1,3} \left( \frac{(1 + z)A_{0,1}(z)}{1 - t} \right) + \Psi_{1,3} \left( \frac{(1 + z)A_{1,1}(z)}{1 - t z} \right) + \\
+ \Psi_{1,3} \left( \frac{(1 + z)A_{2,1}(z)}{1 - t z^2} \right) + \Psi_{1,3} \left( \frac{(1 + z)A_{2,2}(z)}{(1 - t z^2)^2} \right) + \Psi_{1,3} \left( \frac{(1 + z)A_{3,1}(z)}{1 - t z^3} \right) = \\
= \varphi_3 ((1 + z)A_{0,1}(z)) + \varphi_2 ((1 + z)A_{1,1}(z)) + \varphi_1 ((1 + z)A_{2,1}(z)) + \\
+ (z \varphi_1 ((1 + z)A_{2,2}(z)))' + A_{3,1}(0).
\]
Now

\[ A_{0,1}(z) = \lim_{t \to 1} \left( f_{(1,2,3)}(t, z)(1 - t) \right) = \]

\[ = \frac{1}{(1 - z^4)^2 (1 - z^2)^2 (1 - z^5) (1 - z^3) (1 - z) (1 - z^6)}. \]

and

\[ \varphi_3((1 + z)A_{0,1}(z)) = \]

\[ = \frac{2 z^{11} + 7 z^{10} + 14 z^9 + 29 z^8 + 34 z^7 + 42 z^6 + 42 z^5 + 33 z^4 + 21 z^3 + 14 z^2 + 4 z + 1}{(1 - z^5) (1 - z)^3 (1 - z^4)^2 (1 - z^2)^2}. \]

As above we obtain

\[ A_{1,1}(z) = \lim_{t \to z^{-1}} \left( f_{(1,2,3)}(t, z)(1 - tz) \right) = \]

\[ = \frac{z}{(1 - z^3)^2 (1 - z)^2 (1 - z^4) (1 - z^5) (z - 1)}, \]

\[ \varphi_2((1 + z)A_{1,1}(z)) = -\frac{z(4 + 13 z^2 + 6 z + 6 z^6 + 7 + 13 z^4 + 9 z^5 + 12 z^3)}{(1 - z^2) (1 - z^5) (1 - z^3)^2 (1 - z)^4}. \]

\[ A_{2,1}(z) = -\frac{1}{z^2} \lim_{t \to z^{-2}} \left( f_{(1,2,3)}(t, z)(1 - tz^2)^2 \right)'_t = \]

\[ = -\frac{z^3 (5 z^6 + 5 z^5 + 6 z^4 + 2 z^3 - z^2 - 2 z - 2)}{(1 - z)^2 (1 - z^2)^2 (1 - z^3)^2 (1 - z^4)^2}, \]

\[ \varphi_1((1 + z)A_{2,1}(z)) = -\frac{z^3 (5 z^6 + 5 z^5 + 6 z^4 + 2 z^3 - z^2 - 2 z - 2)}{(1 - z^4) (1 - z)^2 (1 - z^3)^2 (1 - z^2)^2}. \]

\[ A_{2,2}(z) = \lim_{t \to z^{-2}} \left( f_{(1,2,3)}(t, z)(1 - tz^2)^2 \right) = \frac{z^3}{(1 - z^4) (1 - z^3)^2 (1 - z^3)^2}, \]

\[ (z \varphi_1 ((1 + z)A_{2,2}(z)))'_z = (z(1 + z)A_{2,2}(z))'_z = \]

\[ = \frac{z^3 (10 z^6 + 13 z^5 + 20 z^4 + 16 z^3 + 14 z^2 + 7 z + 4)}{(1 - z^2)^2 (1 - z^3)^2 (1 - z)^2 (1 - z^4)^2}. \]

At last

\[ A_{3,1}(z) = \lim_{t \to z^{-3}} \left( f_{(1,2,3)}(t, z)(1 - tz^3) \right) = \frac{z^7}{(1 - z^3)^2 (1 - z)^b (1 - z^2)}. \]

Thus \(A_{3,1}(0) = 0.\)

After summation and simplification we obtain the explicit expression for the Poincaré series

\[ \mathcal{P}D_{(1,2,3)}(z) = \frac{p_{(1,2,3)}(z)}{(1 - z^4)^2 (1 - z)^2 (1 - z^2)^2 (1 - z^3)^2 (1 - z^5)}, \]

where

\[ p_{(1,2,3)}(z) = z^{14} + z^{13} + 6 z^{12} + 12 z^{11} + 20 z^{10} + 29 z^9 + 35 z^8 + 39 z^7 + 35 z^6 + 29 z^5 + \]

\[ + 20 z^4 + 12 z^3 + 6 z^2 + z + 1. \]
Theorem 3.3

The following Poincaré series obtained by using the explicit formulas of Theorem 3.2 and Theorem 3.3

\[
\mathcal{P}D_{(1,1)}(z) = \frac{1}{(1 - z)^2 (1 - z^2)}, \quad \mathcal{P}D_{(1,1)}(z) = \frac{1 - z^3}{(1 - z)^2 (1 - z^2)^3},
\]

\[
\mathcal{P}D_{(1,1,1)}(z) = \frac{z^4 + 2 z^3 + 4 z^2 + 2 z + 1}{(1 - z)^2 (1 - z^2)^5},
\]

\[
\mathcal{P}D_{(1,1,1,1)}(z) = \frac{z^6 + 3 z^5 + 9 z^4 + 9 z^3 + 9 z^2 + 3 z + 1}{(1 - z)^2 (1 - z^2)^7},
\]

\[
\mathcal{P}D_{(1,1,1,1,1)}(z) = \frac{p_7(z)}{(1 - z)^2 (1 - z^2)^{11}},
\]

\[
p_7(z) = z^{10} + 5 z^9 + 25 z^8 + 50 z^7 + 100 z^6 + 100 z^5 + 100 z^4 + 50 z^3 + 25 z^2 + 5 z + 1.
\]

\[
\mathcal{P}D_{(2,2)}(z) = \frac{1 + 4 z^2 + z^4}{(1 - z)^3 (1 - z^2)^3}, \quad \mathcal{P}D_{(2,2)}(z) = \frac{1 + 9 z^2 + 9 z^4 + z^6}{(1 - z)^4 (1 - z^2)^4},
\]

\[
\mathcal{P}D_{(2,2,2,2)}(z) = \frac{1 + 16 z^2 + 36 z^4 + 16 z^6 + z^8}{(1 - z)^3 (1 - z^2)^9},
\]

\[
\mathcal{P}D_{(2,2,2,2,2)}(z) = \frac{z^{10} + 25 z^8 + 100 z^6 + 100 z^4 + 25 z^2 + 1}{(1 - z)^6 (1 - z^2)^{11}},
\]

\[
\mathcal{P}D_{(2,2,2,2,2,2)}(z) = \frac{z^{12} + 36 z^{10} + 225 z^8 + 400 z^6 + 225 z^4 + 36 z^2 + 1}{(z - 1)^7 (1 - z^2)^{13}}.
\]

By using Maple we computed the Poincaré series up to \( n = 30 \). The case \( n = 2 \) agrees to the results of the paper [11].

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