VARIETIES WITH ONE APPARENT DOUBLE POINT

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Abstract. The number of apparent double points of a smooth, irreducible projective variety \( X \) of dimension \( n \) in \( \mathbb{P}^{2n+1} \) is the number of secant lines to \( X \) passing through the general point of \( \mathbb{P}^{2n+1} \). This classical notion dates back to Severi. In the present paper we classify smooth varieties of dimension at most three having one apparent double point. The techniques developed for this purpose allow to treat a wider class of projective varieties.

Introduction

The number \( \nu(X) \) of apparent double points of a smooth, irreducible projective variety \( X \) of dimension \( n \) in \( \mathbb{P}^{2n+1} \) is the number of secant lines to \( X \) passing through the general point of \( \mathbb{P}^{2n+1} \). The variety \( X \) is called a variety with one apparent double point, or \( OADP \)-variety, if \( \nu(X) = 1 \). Hence \( OADP \)-varieties of dimension \( n \) are not defective, and actually they can be regarded as the simplest non defective varieties of dimension \( n \) in \( \mathbb{P}^{2n+1} \). Probably this is the reason why \( OADP \)-varieties attracted since a long time the attention of algebraic geometers. There are other reasons however. Among which we will mention here the fact that the known \( OADP \)-varieties are rather interesting objects, as we will indicate later on in this paper, and the unexpected relation of \( OADP \)-varieties with an important subject like Cremona transformations (see \([AR]\)). Unpublished work of F. Zak also suggests connections with the classification of self-dual varieties.

The first instance of an \( OADP \)-variety is the one of a rational normal cubic in \( \mathbb{P}^3 \), which actually is the only \( OADP \)-curve. It goes back to Severi the first attempt of classifying \( OADP \)-surfaces. According to him the only \( OADP \)-surfaces in \( \mathbb{P}^5 \) are quartic rational normal scrolls and del Pezzo quintics (see Theorem 4.10 below). Severi’s nice geometric argument contained, however, a gap, recently fixed by the third author of the present paper \([Ru]\). Meanwhile, inspired by Severi’s result, Edge produced in \([Ed]\) two infinite series of \( OADP \)-varieties of any dimension (see Example 2.4). In dimension 2 the two types of \( OADP \)-surfaces are either Edge varieties or deformations of them (see again Example 2.4). In more recent times E. Sernesi called the attention on Edge’s paper, asking whether, like in the surface case, all \( OADP \)-varieties can be somehow recovered by Edge varieties. F. Zak has been the first who produced, in dimension 4 or more a few very interesting sporadic examples of \( OADP \)-varieties (see \([Zak]\) below). Unfortunately however, there is no other infinite list of examples but Edge’s varieties and rational normal scrolls.

The challenge of classifying \( OADP \)-varieties seems to be quite hard. A first remark is that one may associate to an \( OADP \)-variety \( X \) of dimension \( n \) another variety \( F(X) \) in \( \mathbb{P}^{2n+1} \) which is called the focal variety of \( X \). Basically \( F(X) \) is the

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locus of points in $\mathbb{P}^{2n+1}$ which are contained in more than one, hence in infinitely many, secant lines to $X$ (see §1 below). Of course $X$ is contained in $F(X)$ and one proves that $F(X)$ has at most dimension $2n-1$. One is therefore tempted to classify $OADP$-varieties according to the dimension of their focal loci. This essentially works in the case of surfaces, but falls short for higher dimensions, where one needs some new idea.

In the present paper we deal with the case of $OADP$-threefolds, and we are able to give a full classification theorem for them (see Theorem 7.1). The result is that, like in the case of surfaces, the degree $d$ of such varieties is bounded and precisely one has $5 \leq d \leq 8$. The case $d = 5$ corresponds to rational normal scrolls, the cases $d = 6, 7$ correspond to Edge’s varieties, whereas the case $d = 8$ corresponds to an interesting scroll in lines with rational hyperplane section and sectional genus 3, which naturally arises in the context of Cremona transformations (see [AR], [HKS]).

The new idea on which we base our analysis, and which we believe could be useful also in other contexts, is to study the degeneration of the singular locus of a general projection of a variety when the center of projection tends to meet the variety itself. Using this, we are able to prove a few interesting general facts about $OADP$-varieties. What is really relevant for us here, is that any irreducible 1-dimensional component of the intersection of an $OADP$-variety of dimension $n$ with $n-1$ general tangent hyperplanes at a general point is a rational curve. Applying this to surfaces one recovers Severi’s classification theorem right away. In the case of threefolds the situation is much more involved and requires a case by case rather delicate analysis. Nonetheless it works, and at the end it provides the bound $d \leq 9$ for the degree. Once one has this, the conclusion of the classification can be done in several ways: we used in a rather efficient way adjunction theory. This enables us to exclude the case $d = 9$ and to conclude that the other cases $5 \leq d \leq 8$ correspond to the aforementioned varieties.

It should be noticed that M. G. Violo, in her Ph.D. thesis [V], also studied the same classification problem. She tried to adapt to higher dimensions Severi’s argument for surfaces. However, in order to do so, she had to make a rather unnatural assumption and her list of $OADP$-threefolds does not include the degree 8 scroll.

Our approach could in principle be applied also in higher dimensions. However the analysis becomes then much heavier, to the extent of being in fact discouraging. What it suggests is that it should be possible to bound the degree of an $OADP$-variety of dimension $n$ by a function depending on $n$. In other words, it should still be the case that, in any dimension, one has finitely many families of $OADP$-varieties. The bound that our approach suggests for the degree is exponential in $n$, and this is one of the reasons why the classification, in this way, becomes rather unfeasible. It could happen, however, that the cases which are really possible are much more restricted: after all, even in dimension 3, the case $d = 9$, a priori possible according to our approach, does not in fact exist.

One more information which we are able to give is that both an $OADP$-variety and its general hyperplane section have Kodaira dimension $-\infty$ (see Proposition 4.6). This suggests an alternative approach to the classification which we briefly outline in Remark 4.11. It also relates to an interesting conjecture of Bronowski [Br] to the effect that a variety $X \subset \mathbb{P}^{2n+1}$ of dimension $n$ has one apparent double point if and only if the projection of $X$ to $\mathbb{P}^n$ from the projective tangent space
to $X$ at its general point is birational (see Remark 1.3). Here we prove that, in general, the degree of aforementioned projection is bounded above by the number of apparent double points of $X$, which yields one implication of Bronowski's conjecture (see Proposition 1.1 and Corollary 1.2). In particular we have the important consequence that $OADP$-varieties are rational.

Finally, as an application of our classification of $OADP$-threefolds, we are able to classify $OADP$-varieties which are Mukai varieties, i.e. such that their general curve sections are canonical curves (see Theorem 5.1). Related classification results for $OADP$-varieties of small degree can be found in [AR].

The paper is organized as follows. In section 1 we give the main definitions and prove a few basic facts about $OADP$-varieties. In §2 we present all the known examples of $OADP$-varieties. In §3 we make our analysis of degenerations of projections. In §4 we apply it to the study of a special class of varieties, which we call varieties of full rational type, which include the $OADP$-varieties. As a first result we give a very easy proof of Severi’s classification theorem of $OADP$-surfaces. In §5 we use the previous analysis for the study of the tangential behavior of $OADP$-threefolds, i.e. we study the intersection of such a threefold with its general tangent space. As a result we are able to prove the bound $5 \leq d \leq 9$ for the degree $d$ of a $OADP$-threefold. On the way, we need a few technical propositions, some of them of independent interest, which we collected in §6. In §7 we prove our classification theorem for $OADP$-threefolds. Finally in §8 we prove the classification of $OADP$-Mukai varieties.

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1. Preliminaries

Let $X \subseteq \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension $n \geq 1$ over $\mathbb{C}$. If $d$ is the degree of $X$ we will sometimes denote $X$ by $X_{n,d}$ or by $X_n$.

Let $X(2)$ be the twofold symmetric product of $X$. One considers $S(X)$ the abstract secant variety of $X$, i.e. the Zariski closure of the scheme

$$I(X) := \{([x,y],p) \in X(2) \times \mathbb{P}^r : x \neq y, p \not\in \langle x,y \rangle \}$$

inside $X(2) \times \mathbb{P}^r$. The image of the projection of $S(X)$ to $\mathbb{P}^r$ is the secant variety $Sec(X)$ of $X$. Thus one has the projection map $p_X : S(X) \to Sec(X)$. Since $\dim(S(X)) = 2n + 1$, then $\dim(Sec(X)) \leq \min\{2n + 1, r\}$, and the variety $X$ is said to be defective if the strict inequality holds. If $x, y \in X$ are distinct smooth points, the secant line $L = \langle x, y \rangle$ is contained in $Sec(X)$. As limit of secant lines, one has also the tangent lines, i.e. the lines contained in the tangent space $T_xX$ and passing through $x$, where $x \in X$ is a smooth point. The Zariski closure of the union of the tangent lines is the so-called tangential variety $Tan(X)$ of $X$, which is contained in $Sec(X)$. The tangent lines are considered to be improper secant lines, whereas the usual secant lines are sometimes called proper secants.

Let us assume now that $X$ is smooth and that $r \geq 2n + 1$. Let $\epsilon$ be an integer such that $0 \leq \epsilon \leq n - 1$ and let $\Pi$ be a general $\mathbb{P}^s$ in $\mathbb{P}^r$, with $s = r - n - \epsilon - 2$. 
We denote by \( \pi_L : X \to \mathbb{P}^{n+1} \) the projection of \( X \) from II, by \( X^{(c)} \) its image and by \( \Gamma^{(c)} \) the singular scheme of \( X^{(c)} \). We will assume from now on that \( X \) is not defective, i.e. that its secant variety \( \text{Sec}(X) \) has dimension \( 2n + 1 \). Then \( \Gamma^{(c)} \) has pure dimension \( n - \epsilon - 1 \).

The projective invariants of the varieties \( \Gamma^{(c)} \), which are projective invariants of the variety \( X \) itself, are relevant in various classification problems. For instance, if \( \epsilon = n - 1 \), then \( \Gamma^{(n-1)} \) consists of finitely many points, which are improper double points of \( X^{(n-1)} \subset \mathbb{P}^{2n} \), i.e. double points whose tangent cone consists of two subspaces of dimension \( n \) meeting transversely at the points in question. Following a classical terminology, we call the points of \( \Gamma^{(n-1)} \) the apparent double points of \( X \). Their number, which is not 0 since \( X \) is not defective, will be denoted by \( \nu(X) \). Usually one sets \( \nu(X) = 0 \) for defective varieties \( X \).

It is useful to point out right away the following basic facts:

**Proposition 1.1.** Let \( X \subset \mathbb{P}^r \) be a smooth, irreducible, non-degenerate, non-defective, projective variety of dimension \( n \), with \( r \geq 2n + 1 \). One has:

1. If \( r = 2n + 1 \) then \( \nu(X) \) is the number of secant lines to \( X \) passing through a general point of \( \mathbb{P}^{2n+1} \).
2. If \( r > 2n + 1 \) then \( \nu(X) = \deg(\text{Sec}(X)) \cdot \deg(p_X) \) and \( r \leq 2n + \nu(X) - 1 \), hence \( \nu(X) \geq 3 \).
3. If \( \nu(X) \leq 2 \) then \( r = 2n + 1 \) and \( X \) is linearly normal.

**Proof.** Assertion (i) is trivial, and (iii) is an immediate consequence of (ii), which we are now going to prove. The first assertion of (ii) is also trivial and one has:

\[
\nu(X) = \deg(\text{Sec}(X)) \cdot \deg(p_X) \geq \deg(\text{Sec}(X)) \geq r - \dim(\text{Sec}(X)) + 1 = r - 2n
\]

Assume the equality holds. Then \( \text{Sec}(X) \) would be a variety of minimal degree (see [EH] and Example 2.1 below). Such a variety, if singular, is a cone. Actually \( \text{Sec}(X) \) is singular along \( X \), hence \( \text{Sec}(X) \) would be a cone, whose vertex would contain the span of \( X \), which is the whole \( \mathbb{P}^r \), a contradiction. \( \square \)

A smooth, irreducible, non-degenerate, non-defective, projective variety \( X \subset \mathbb{P}^r \) of dimension \( n \geq 1 \) is called a variety with one apparent double point, or a OADP-variety, if \( \nu(X) = 1 \). Note that, by part (iii) of Proposition 1.1, if \( X \) is an OADP-variety, one has \( r = 2n + 1 \) and \( X \) is linearly normal in \( \mathbb{P}^{2n+1} \) (see also [Sc] and [Re]).

To say that \( X \) is an OADP-variety is the same as saying that the projection \( p_X : S(X) \to \mathbb{P}^{2n+1} \) is dominant and birational. This fact has some important consequences.

Let us consider, in general, \( X_n \subset \mathbb{P}^{2n+1} \) an irreducible, non-degenerate, non-defective, projective variety. We can see \( S(X) \) as a family of dimension \( 2n \) of lines in \( \mathbb{P}^{2n+1} \). Let \( L = < x, y >, x, y \in X \), be a proper secant line. In this situation there is focal square matrix \( F_L \) arising, whose rows are given by the sections of the normal bundle \( N_{L, \mathbb{P}^{2n+1}} \) corresponding to the \( 2n \) independent infinitesimal deformations of \( L \) determined by \( 2n \) independent vectors in \( T_{x,y}(X(2)) \) (see [CS] for the general theory of foci, which we are going to apply in this paper). Since \( X \) is not defective, hence \( p_X : S(X) \to \mathbb{P}^{2n+1} \) is dominant, then \( \det(F_L) \) is not identically zero for \( L \) general. The scheme \( Z_L \) of degree \( 2n \) defined by the equation \( \det(F_L) = 0 \), if \( \det(F_L) \) is not identically zero, coincides with \( nx + ny \). The same considerations
apply if \( L \) is an improper secant line. In this case \( L \) is tangent to \( X \) at a smooth point \( x \in X \) and the focal matrix \( F_L \) can still be considered. If \( \det(F_L) \) is not identically zero, the focal scheme \( Z_L \) of degree \( 2n \) defined by \( \det(F_L) = 0 \) is \( 2nz \). A proper or improper secant line \( L \) is called a focal line if \( \det(F_L) \equiv 0 \). The union of focal lines is a Zariski closed subset \( F(X) \) of \( \mathbb{P}^{2n+1} \) called the focal locus of \( X \) and its points are called foci.

Assume now \( X \) is an OADP-variety. By the general theory of foci, \( F(X) \) coincides with the indeterminacy locus of the inverse map \( p_X^{-1} : \mathbb{P}^{2n+1} \rightarrow S(X) \).

Hence:

i) \( z \) is a focus if and only if there are two distinct, hence infinitely many, secant lines through \( z \),

ii) the secant line \( L \) is a focal line if and only if for some \( z \in L - L \cap X \) there is some other secant line, different from \( L \), containing \( z \). This happens if and only if for every \( z \in L \), there are infinitely many secant lines containing \( z \).

Remark 1.2. The definition of OADP-varieties can be extended also to singular varieties. A non-degenerate, projective variety \( X \) of pure dimension \( n \) in \( \mathbb{P}^{2n+1} \) has one apparent double point if there is a unique secant to \( X \) passing through the general point of \( \mathbb{P}^{2n+1} \). These varieties share with their smooth relatives, several properties. For example they also turn out to be linearly normal (see the proof of Proposition 1.1). Though the subject of singular OADP-variety is very interesting, we will not consider it in this paper.

Going back to the loci \( \Gamma^{(\epsilon)} \) for general varieties, let us consider the next one, corresponding to the case \( \epsilon = n - 2 \), thus we have to assume \( n \geq 2 \). Then \( \Gamma^{(n-2)} \) is a curve whose geometric genus we denote by \( PG(X) \) and we will call it the projective genus of \( X \). Then \( PG(X) \geq 0 \), and we will say that \( X \) is of rational projective type, or a RPT-variety, if \( PG(X) = 0 \) and, in addition, the curve \( \Gamma^{(n-2)} \) is irreducible. More generally, we will say that \( X \) is of rational projective \( i \)-type, for some \( i = 1, \ldots, n-1 \), or a RPT-variety, if the variety \( \Gamma^{(n-i-1)} \) is an irreducible, rational, \( i \)-dimensional variety. We will say that \( X \) is of full rational projective type, or a FRPT-variety, if it is of rational projective \( i \)-type, for all \( i = 1, \ldots, n-1 \).

As the following proposition shows, examples of FRPT-varieties are provided by OADP-varieties.

**Proposition 1.3.** Every OADP-variety is a FRPT-variety.

**Proof.** For every \( 0 \leq \epsilon \leq n-1 \) we consider \( \Pi \) a general \( \mathbb{P}^{n-\epsilon-1} \) and \( \pi_\epsilon : X \to \mathbb{P}^{n+1+\epsilon} \) the projection of \( X \) from \( \Pi \). Since through the general point \( p \in \Pi \) there is a unique secant line to \( X \) corresponding to a point of \( \Gamma^{(\epsilon)} \), we have a rational map \( f_\epsilon : \Pi \to \Gamma^{(\epsilon)} \), which is birational. \( \square \)

**Remark 1.4.** If \( X \subset \mathbb{P}^{2n+1} \) is an OADP-variety, then its symmetric product \( X(2) \) is rational. Indeed, if \( H \) is a general hyperplane of \( \mathbb{P}^{2n+1} \), then, referring to the map \( \pi : S(X) \to \mathbb{P}^{2n+1} \), one has that \( H' = \pi^*(H) \) is rational and the projection map \( H' \to X(2) \) is clearly birational. Thus the Kodaira dimension of \( X \) is \( \kappa(X) = -\infty \) and \( h^i(X, \mathcal{O}_X) = 0 \), \( 1 \leq i \leq n \). This can be proved as in \([Ma]\).

We prove now a lemma indicated to us by F. Zak, which will be useful later on. First we recall a definition. Let \( X \subset \mathbb{P}^r \) be an irreducible, projective variety of dimension \( n \). For a point \( p \in (\text{Sec}(X) \setminus X) \) we consider the Zariski closure \( C_p(X) \) of
the union of (proper or improper) secant lines to $X$ passing through $p$. We denote by $S_p(X)$ the scheme theoretical intersection of $C_p(X)$ and $X$, and we call $S_p(X)$ the entry locus of $p$ with respect to $X$. Intuitively, this is the Zariski closure of the set of all points $x \in X$ for which there is a point $y \in X$ such that $p \in \langle x, y \rangle$.

Assume $Sec(X) = \mathbb{P}^r$, which yields $r \leq 2n + 1$. A count of parameters shows that for $p \in \mathbb{P}^r$ a general point, $S_p(X)$ has dimension $2n + 1 - r$.

Notice that $X_n \subset \mathbb{P}^{2n+1}$ is an OADP-variety if and only if the entry locus $S_p(X)$ of a general point $p \in \mathbb{P}^{2n+1}$ is a pair of points.

**Lemma 1.5.** Let $Y \subset \mathbb{P}^N$ be an irreducible, non-degenerate, projective variety of dimension $m$, such that $Sec(Y) = \mathbb{P}^N$. Set $n = N - m - 1$ and $r = 2n + 1$. Consider $\Pi$ a $\mathbb{P}^r$ in $\mathbb{P}^N$ intersecting $Y$ along a smooth variety $X$ of dimension $n$. Suppose that for $p \in \Pi$ general the entry locus $S_p(Y)$ is a quadric of dimension $2m + 1 - N$. Then $X$ is an OADP-variety.

**Proof.** Let $p$ be a general point of $\Pi$. The entry locus $S_p(X)$ is the intersection of the quadric $S_p(Y)$ with $\Pi$. By a dimension count, this intersection is not empty. This proves that $Sec(X) = \Pi$ and therefore $S_p(X)$ is finite. Since, as we saw, it is the intersection of the quadric $S_p(Y)$ with $\Pi$, it consists of two points. This implies the assertion about $X$. 

To put the previous lemma in perspective the reader should perhaps take into account the relations between OADP-varieties and Cremona transformations, as indicated in [AR]. We will not dwell on this here but we will only mention the following result (see [AR]):

**Proposition 1.6.** Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, non-degenerate, projective variety, which is not defective. Suppose that the linear system of quadrics $|I_X|^{2n+1}(2)$, restricted to a general hyperplane of $\mathbb{P}^{2n+1}$ defines a rational map which is birational onto its image. Then $X$ is an OADP-variety.

**Proof.** Let $\phi$ be the map associated to $|I_X|^{2n+1}(2)$. The map $\phi$ contracts any secant line to $X$ to a point. Hence the general fiber of $\phi$ has positive dimension $l$ and degree $d$. Since the restriction of $\phi$ to a general hyperplane has to be birational onto its image, we must have $l = d = 1$. This implies the assertion. 

2. Examples of OADP-varieties

In this section we collect some examples, essentially the only ones known to us, of smooth OADP-varieties. We refer the reader to [AR] for more details.

**Example 2.1.** (Scrolls). Let $0 \leq a_0 \leq a_1 \leq \ldots \leq a_k$ be integers and set $r = a_0 + \ldots + a_k + k$. Recall that a rational normal scroll $S(a_0, \ldots, a_k)$ in $\mathbb{P}^r$ is the image of the projective bundle $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_k) := \mathbb{P}(O_{\mathbb{P}}(a_0) \oplus \ldots \oplus O_{\mathbb{P}}(a_k))$ via the linear system $|O_{\mathbb{P}}(1)|$. The dimension of $S(a_0, \ldots, a_k)$ is $k + 1$, its degree is $a_0 + \ldots + a_k = r - k$ and $S(a_0, \ldots, a_k)$ is smooth if and only if $a_0 > 0$. Otherwise, if $0 = a_0 = \ldots = a_i < a_{i+1}$, it is the cone over $S(a_{i+1}, \ldots, a_k)$ with vertex a $\mathbb{P}^i$.

One uses the simplified notation $S(a_0^{h_0}, \ldots, a_m^{h_m})$ if $a_i$ is repeated $h_i$ times, $i = 1, \ldots, m$.

Recall that rational normal scrolls, the (cones over) Veronese surface in $\mathbb{P}^5$, and quadrics, can be characterized as those non degenerate irreducible varieties in a projective space having minimal degree (see [DI]).
Given positive integers $0 < m_1 \leq \ldots \leq m_h$ we will denote by $Seg(m_1, \ldots, m_h)$ the Segre embedding of $\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_h}$ in $\mathbb{P}^N$, $N = (m_1 + 1) \ldots (m_h + 1) - 1$. We use the shorter notation $Seg(m_1^k, \ldots, m_h^k)$ if $m_i$ is repeated $k_i$ times, $i = 1, \ldots, s$. Recall that $Pic(Seg(m_1, \ldots, m_h)) \cong \mathbb{Z}^h$, generated by the line bundles $\xi_i = pr_i^* (\mathcal{O}_{\mathbb{P}^{m_i}}(1))$, $i = 1, \ldots, h$. A divisor $D$ on $Seg(m_1, \ldots, m_h)$ is said to be of type $(\ell_1, \ldots, \ell_h)$ if $\mathcal{O}_{Seg(m_1, \ldots, m_h)}(D) \cong \mathcal{O}_1 \otimes \ldots \otimes \mathcal{O}_h$. The hyperplane divisor of $Seg(m_1, \ldots, m_h)$ is of type $(1, \ldots, 1)$.

Notice now that $S(1^n) \subset \mathbb{P}^{2n-1}$ coincides with $Seg(1, n - 1)$. Let us point out the following easy fact.

**Lemma 2.2.** Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, linearly normal, regular variety of dimension $n$. Then $X$ is a scroll over a curve if and only if $X$ is isomorphic either to $S(1^{n-2}, 2^2)$, $n \geq 2$, or $S(1^{n-1}, 3)$, $n \geq 1$. Moreover for $n \geq 3$ these scrolls can be realized as divisors of type $(2, 1)$ on $Seg(1, n)$, the general one being $S(1^{n-2}, 2^2)$, while the scroll $S(2^2)$ can be also realized as a divisor of type $(0, 2)$ on $Seg(1, 2)$.

Furthermore for all $n \geq 2$ the scrolls $S(1^{n-2}, 2^2)$ and $S(1^{n-1}, 3)$ can also be obtained by intersecting $Seg(1, n + 1)$ with a suitable $\mathbb{P}^{2n+1}$.

**Proof.** By definition $X$ is isomorphic to $\mathbb{P}(\mathcal{E})$, with $\mathcal{E}$ a locally free sheaf of rank $n$ over a smooth curve $C$. Since $X$ is regular, $C \cong \mathbb{P}^1$ and hence $\mathcal{E} \cong \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $0 < a_1 \leq a_2 \leq \cdots a_n$. We know that $X$ is linearly normal and hence that $2n + 2 = h^0(\mathbb{P}^1, \mathcal{E}) = \sum_{i=1}^n (a_i + 1)$, from which it follows that $X$ is either $S(1^{n-2}, 2^2)$, $n \geq 2$, or $S(1^{n-1}, 3)$, $n \geq 1$. The description as divisors in $Seg(1, n)$ as well as sections of $Seg(1, n + 1)$ is well known. We leave the easy proof to the reader. \(\square\)

In view of Proposition [1,3] and of Remark [1,3] (see also Proposition [4,6] below), an OADP-variety $X \subset \mathbb{P}^{2n+1}$ can be a scroll over a curve only if it is a smooth rational normal scroll, i.e. only if $X$ is either of type $S(1^{n-1}, 3)$ or of type $S(1^{n-2}, 2^2)$.

We will now prove, using an argument of F. Zak, that $S(1^{n-1}, 3)$ and $S(1^{n-2}, 2^2)$ are indeed OADP-varieties for all $n \geq 2$. Notice that $S(1^{n-1}, 3)$ makes sense even for $n = 1$, i.e. one has $S(3)$, a rational normal cubic, which is indeed an OADP-variety.

**Proposition 2.3.** For all $n \geq 2$ the scrolls $S(1^{n-1}, 3)$ and $S(1^{n-2}, 2^2)$ are OADP-varieties. These are the only scrolls over a curve which are OADP-varieties.

**Proof.** By Lemma 2.2 the scrolls in question can be obtained by intersecting $Y := Seg(1, n + 1)$ with a $\mathbb{P}_p^{2n+3}$.

Let now $p \in \mathbb{P}_p^{2n+3}$ be any point not on $Y$. It is well known (see for instance [Ed]) that the entry locus $S_p(Y)$ has the form $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}_p^{2n+3}$, hence it is a 2-dimensional quadric in a linear $\mathbb{P}_p^3$. We are then in position to apply Lemma 1.5 to conclude with the first assertion.

The final assertion is a consequence of Lemma 2.2 and of Proposition 1.3 part iii) and Remark [1,3]. \(\square\)

**Example 2.4.** (Edge varieties). Consider $Seg(1, n) = S(1^{n+1}) \subset \mathbb{P}^{2n+1}$ which has degree $n + 1$, $n \geq 1$. Recall that $Pic(\mathbb{P}(1, n)) \cong \mathbb{Z} < \Pi > \oplus \mathbb{Z} < \Sigma >$, where $\Pi$ is one of the $\mathbb{P}^n$s of the rulings of $S(1^{n+1})$ and $\Sigma$ is $Seg(1, n - 1)$ naturally embedded in $Seg(1, n)$. Hence, an effective divisor $D$ of type $(a, b)$ has degree $a + nb$ in $\mathbb{P}^{2n+1}$. The hyperplane divisor of $Seg(1, n)$ is of type $(1, 1)$. 

Let $Q$ be a general quadric of $\mathbb{P}^{2n+1}$ containing $\Pi$. Then the residual intersection of $Q$ with $\text{Seg}(1, n)$ is an $n$-dimensional smooth variety $E_{n, 2n+1}$ of degree $2n + 1$ in $\mathbb{P}^{2n+1}$ which is a general divisor of type $(1, 2)$ on $\text{Seg}(1, n)$. Its degree is $2n + 1$.

Let $\Pi_1$ and $\Pi_2$ be two fixed $\mathbb{P}^n$'s of the ruling of $\text{Seg}(1, n)$. If $Q$ is a general quadric through $\Pi_1$ and $\Pi_2$, then the residual intersection of $Q$ with $\text{Seg}(1, n)$ is an $n$-dimensional smooth variety $E_{n, 2n}$ of degree $2n$ which is a general divisor of type $(0, 2)$ on $\text{Seg}(1, n)$. Notice that $E_{n, 2n} \cong \mathbb{P}^1 \times Q'$, where $Q'$ is a general quadric of $\mathbb{P}^n$. In particular $E_{3,6}$ coincides with $\text{Seg}(1^3)$.

Varieties of the type $E_{n,2n+\epsilon}$, with $\epsilon = 0, 1$, have been considered in [4]. Edge proved that they are $OADP$-varieties as soon as $n \geq 2$. Therefore they are called $Edge$ varieties. If $n = 1$, then $E_{1,3}$ is the rational normal cubic, which is an $OADP$-variety, whereas $E_{1,2}$ is a pair of skew lines which, strictly speaking, cannot be considered an $OADP$-variety because it is not irreducible. However it still has one apparent double point.

We remark that, by degree reasons, the Edge varieties and the scrolls are different examples of $OADP$-varieties, as soon as $n \geq 3$. If $n = 2$ instead, $E_{2,4}$ is a rational normal scroll of type $S(2^2)$. Notice that, instead, the other scroll $S(1, 3)$, which is also an $OADP$-surface, is not an Edge variety.

We now sketch Edge’s argument from [4] to the effect that smooth irreducible divisors of type $(0, 2)$, $(1, 2)$ and $(2, 1)$ on the Segre varieties $\text{Seg}(1, n)$, $n \geq 2$, have one apparent double point. This generalizes the trivial fact that the only smooth curves on a smooth quadric in $\mathbb{P}^3$ having one apparent double point are of the above types.

**Proposition 2.5.** Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, $n$-dimensional variety contained as a divisor of type $(a, b)$ in $\text{Seg}(1, n)$, $n \geq 2$. Then $X$ has one apparent double point if and only if $(a, b) \in \{(2, 1), (0, 2), (1, 2)\}$, i.e. if and only if $X$ is either a rational normal scroll or one of the two Edge’s varieties as in examples 2.4 and 2.5.

**Sketch of the proof.** As we know for, $p \notin Y := \text{Seg}(1, n)$ the entry locus $S_p(Y)$ has the form $\text{Seg}(1^2) \cong \mathbb{P}^1 \times \mathbb{P}^1_p$ for some $\mathbb{P}^1_p \subset \mathbb{P}^n$ and spans a linear $\mathbb{P}^1_p$. So if $X$ is a divisor of type $(a, b)$ of $Y$, the secant lines of $X$ passing through $p$ are exactly the secant lines of $X \cap \mathbb{P}^1_p$ passing through $p$. For a general $p \in \mathbb{P}^{2n+1}$, $X \cap \mathbb{P}^1_p$ is a reduced, not necessarily irreducible curve and it is a divisor of type $(a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1_p$. Hence $X$ has one apparent double point if and only if $(a, b) \in \{(1, 2), (2, 1), (2, 0), (0, 2)\}$. If $(a, b) = (2, 0)$, then $X = \mathbb{P}^n \setminus \Pi \mathbb{P}^n$ is reducible. \[\square\]

In relation with Edge’s varieties we notice the following useful generalization of [Ru, Lemma 3 and Proposition 8], the main ingredients of the proof of Severi’ s classification Theorem of $OADP$-surfaces given in [Ru].

**Proposition 2.6.** Let $X \subset \mathbb{P}^{2n+1}$ be an $OADP$-variety of dimension $n \geq 2$. Suppose $X$ contains a 1-dimensional family $D$ of divisors $D$ of degree $\alpha$, whose general element spans a $\mathbb{P}^n$. Then $\alpha = 2$, $X$ is an Edge variety, hence $X$ is either a divisor of type $(0, 2)$ or $(1, 2)$ on $\text{Seg}(1, n)$, and $D$ is cut out on $X$ by divisors of type $(1, 0)$.

**Proof.** The assertion is true if $n = 2$ (see [Ru, 1.c.]), so we assume $n \geq 3$ from now on. By Zak’s Theorem on Tangencies, [3, Corollary 1.8], every divisor $D \in D$ has at most a finite number of singular points, so that it is irreducible.
First we will prove that $D$ is a base point free pencil, which is rational because $X$ is regular (see Remark 2.4). In order to see this, it suffices to prove that two general divisors $D_1, D_2$ of $D$ do not intersect. Suppose this is not the case. Then $D_1$ and $D_2$ intersect along a variety $D_{1,2}$ of dimension $n - 2$. Let $x$ be a general point in a component of $D_{1,2}$ and let $p_i$ be a general point of $D_i$, $i = 1, 2$. Of course $p_1, p_2$ are general points of $X$. Notice that $\mathbb{P}^{n-2} \simeq T_x D_{1,2} \subset \langle D_i \rangle \simeq \mathbb{P}^n$, $i = 1, 2$. Since $T_{p_i} D_i \subset \langle D_i \rangle$, then $\dim(T_{p_i} D_i \cap T_x D_{1,2}) \geq n - 3$, $i = 1, 2$, hence $\dim(T_{p_i} D_i \cap T_{p_2} D_2) \geq n - 4$ and therefore also $\dim(T_{p_i} X \cap T_{p_2} X) \geq n - 4$. Since $X$ is not defective, by Terracini’s Lemma we have a contradiction as soon as $n \geq 4$. If $n = 3$, by arguing as above we see that $\langle D_1 \rangle \cap \langle D_2 \rangle$ cannot be a plane. Thus $D_{1,2}$ has to be a line along which $D_1$ and $D_2$ intersect transversely. We claim that $\langle x, p_i \rangle \cap X = \{x, p_i\}$. Indeed, if $\alpha \geq 3$, the line $\langle x, p_i \rangle$ would intersect the divisor $D_1$ in a point $p'_i$ different from $x$ and $p_i$ and the line $\langle p'_1, p'_2 \rangle$ would cut the general secant line $\langle p_1, p_2 \rangle$ in a point different from $p_1$ and $p_2$ and $X$ would not be an OADP-variety. The claim proves that $\alpha = 2$ and that $D$ is a smooth quadric being smooth along a line contained in it. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0,$$

together with $h^1(X, \mathcal{O}_X) = 0$ and $h^0(D, \mathcal{O}_D(D)) = 2$, yields $h^0(X, \mathcal{O}_X(D)) = 3$.

The general hyperplane section $H$ of $X$ would be a smooth surface containing a 2-dimensional family of conics, and therefore $H$ would either be the Veronese surface in $\mathbb{P}^5$ or a projection of it (e.g. see [Mez]), which is not possible.

Now we consider the scroll $Y$ swept out by $\langle D \rangle$ as $D$ varies in $\mathcal{D}$. This is a scroll over a rational curve since there is a unique $\mathbb{P}^n$ through the general point of $X$ and since $X$ is regular. Furthermore $X$ is linearly normal, therefore $Y$ is a rational normal scroll, i.e. a cone over a smooth rational normal scroll or $\text{Seg}(1, n)$ (see for instance [EH, Theorem 2]). Let us prove that $Y$ cannot be a cone. Let $V$ be the vertex of $Y$ and let $s \in V$ be any point. Keeping the above notation, we remark that $s \notin D_i$ and the lines $\langle s, p_1 \rangle$ and $\langle s, p_2 \rangle$ span a plane $\Pi$, which contains the general secant line $\langle p_1, p_2 \rangle$ to $X$. The line $\langle s, p_i \rangle$ intersects the divisor $D_i$ at another point $p'_i$ different from $p_i$, because it is a general line through $s \notin D_i$ in the linear space $\langle D_i \rangle$, $i = 1, 2$. Then the secant line $\langle p'_1, p'_2 \rangle$ is contained in $\Pi$ and intersects the general secant line $\langle p_1, p_2 \rangle$ outside $X$, implying that $Y$ is not an OADP-variety.

In conclusion, $Y$ is smooth, then $Y = \text{Seg}(1, n)$ and we conclude by Proposition 2.5.

**Example 2.7.** (OADP 3-folds of degree 8 in $\mathbb{P}^7$). Let $X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}^7$ be the scroll in lines over $\mathbb{P}^2$ associated to the very ample vector bundle $\mathcal{E}$ of rank 2, given as an extension by the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_{\{p_1, \ldots, p_8\}, \mathbb{P}^2}(4) \to 0,$$

where $p_1, \ldots, p_8$ are distinct points in $\mathbb{P}^2$, no 7 of them lying on a conic and no 4 of them collinear. Since $h^0(\mathbb{P}^2, \mathcal{E}(-2)) = 0$ such a vector bundle is stable, it has Chern classes $c_1(\mathcal{E}) = 4$, $c_2(\mathcal{E}) = 8$ and it exists (see [FP] and [Io5]). A general hyperplane section of $X$ corresponds to a section as in sequence (2.1) and it is a smooth octic rational surface $Y$, the embedding of the blow-up $\mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_i$ given by the quartics through them. The surface $Y$ is arithmetically Cohen-Macaulay and
it is cut out by 7 quadrics which define a Cremona transformation \( \varphi : \mathbb{P}^6 \rightarrow \mathbb{P}^6 \) (see [ST], [HKS], [MR] and also [AR], §6). Hence \( X \) is an arithmetically Cohen-Macaulay 3-fold cut out by 7 quadrics and it is an \( OADP \)-variety by Proposition 14.

We sketch now another description of \( X \), from which we will deduce another proof of the fact that \( X \) is an \( OADP \)-variety. Notice that \( h^0(\mathbb{P}^2, I_{(p_1, \ldots, p_8)} \mathbb{P}^2(3)) = 2 \) and call \( f_i := f_i(x_0, x_1, x_2) \) two independent cubics through \( p_1, \ldots, p_8 \). Then \( H^0(\mathbb{P}^2, I_{(p_1, \ldots, p_8)} \mathbb{P}^2(4)) \) is spanned by \( f_i x_j, i = 0, 1, j = 0, 1, 2 \), and an irreducible quartic, \( \phi := \phi(x_0, x_1, x_2) \), vanishing at \( p_1, \ldots, p_8 \). Then \( H^0(\mathbb{P}^2, \mathcal{E}) \) is spanned by sections \( s_{i,j} \) and \( s \) respectively mapping to \( f_i x_j, i = 0, 1, j = 0, 1, 2 \), and to \( \phi \) via the surjective map \( H^0(\mathbb{P}^2, \mathcal{E}) \rightarrow H^0(\mathbb{P}^2, I_{(p_1, \ldots, p_8)} \mathbb{P}^2(4)) \) deduced by sequence (2.1), and by the section \( \sigma \) given by the inclusion \( O_{\mathbb{P}^2} \rightarrow \mathcal{E} \) also deduced from (2.1). Introduce homogeneous coordinates \( z_{i,j}, i = 0, 1, j = 0, 1, 2, z_0, z_1 \) in \( \mathbb{P}^7 \). Then the inclusion \( X = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^7 \) is defined by

\[
z_0 = \sigma, z_1 = s, z_{i,j} = s_{i,j}, i = 0, 1, j = 0, 1, 2.
\]

Whence we see that \( X \) verifies the equations

\[
z_{i,j} z_{h,k} = z_{i,k} z_{h,j}, i, h = 0, 1, j, k = 0, 1, 2.
\]

which, in the \( \mathbb{P}^5 \) with coordinates \( z_{i,j}, i = 0, 1, j = 0, 1, 2 \), define \( \text{Seg}(1,2) \). Thus the line \( L \) with equations \( z_{i,j} = 0, i = 0, 1, j = 0, 1, 2 \), sits on \( X \) and by projecting \( X \) from \( L \) to \( \mathbb{P}^5 \) we have \( \text{Seg}(1,2) \). In other words \( X \) sits on the cone \( F \) with vertex \( L \) over \( \text{Seg}(1,2) \). The variety \( X \) is the complete intersection of two divisors of type \((1,2)\) on \( F \) (see [HKS]). Notice that a divisor of type \((1,2)\) on \( F \) is the intersection of \( F \) with a quadric containing a \( \mathbb{P}^4 \) of the ruling.

This description allows us to see in another way that \( X \) is an \( OADP \)-variety. Indeed, let \( p \) be a general point of \( \mathbb{P}^7 \). Let \( p' \) be its projection to \( \mathbb{P}^5 \) from \( L \). The entry locus \( S_{p'}(\text{Seg}(1,2)) \) is a 2-dimensional smooth quadric \( Q \). Consider the \( \mathbb{P}^5 \) spanned by \( Q \) and \( L \). Any secant to \( X \) through \( p \) sits in it. The intersection of \( X \) with this \( \mathbb{P}^5 \) is the intersection of the rank 4 quadric cone with vertex \( L \) over \( Q \) with two divisors of type \((1,2)\). This is a rational normal scroll of degree 4, which is an \( OADP \)-surface. This yields that there is a unique secant line to \( X \) through \( p \).

The focal locus of \( X \) contains the cone \( F \).

The following examples have been first pointed out by F. Zak.

**Example 2.8.** (A smooth 4-fold of degree 12). For every \( k \geq 1 \), we denote by \( S^{(k)} \subset \mathbb{P}^{2k-1} \) the \( k \)-th spinor variety of dimension \( k(k+1)/2 \), parameterizing the subspaces of dimension \( k-1 \) contained in a smooth quadric \( Q_{2k-1} \subset \mathbb{P}^{2k} \).

Consider \( Y := S^{(4)} \subset \mathbb{P}^{15} \). It is known that for \( p \in \mathbb{P}^{15} \) general, the entry locus \( S_p(Y) \) is a smooth 6-dimensional quadric (see [FS]). By lemma 14, the intersection \( X \) of \( Y \) with a general \( \mathbb{P}^9 \) is an \( OADP \)-variety of dimension 4.

A different, but related, proof that \( X \) is an \( OADP \)-variety, can be obtained by applying Proposition 14 and by using the fact that the linear system \([I_{Y^*} \mathbb{P}^{15}(2)]\) maps \( \mathbb{P}^{15} \) to a smooth quadric in \( \mathbb{P}^9 \), with fibers the \( \mathbb{P}^7 \)'s spanned by the entry loci \( S_p(Y) \) (see [AR]).

Notice that \( Y \) is a Mukai variety, i.e. its general curve section is a canonical curve (see [Mul]). The same happens for \( X \).
**Example 2.9.** (Lagrangian Grassmannians). Let us consider the Lagrangian Grassmannians $G^{\text{lag}}_k(2, 5)$, over the four composition algebras $k = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, embedded via their Plücker embedding.

These varieties can be described also as twisted cubics over the four cubic Jordan algebras $\text{Sym}^3 \mathbb{C}, M_3 \mathbb{C}, \text{Alt}_6 \mathbb{C}$, respectively $H_3 \mathbb{O}$, see [Mu2]. Note that also Edge varieties $E_n, 2n$ are of this type for split Jordan algebras, see [Mu3]. Another possible description is as follows, see [Za2]:

i) $G^{\text{lag}}_\mathbb{R}(2, 5)$ is the symplectic Grassmannians $G^{\text{sym}F}(2, 5)$ of the 2-planes in $\mathbb{P}^5$ on which a non-degenerate null-polarity is trivial, i.e. which are isotropic with respect to a non-degenerate antisymmetric form. This is a variety of dimension 6 which turns out to be embedded via the Plücker embedding in $\mathbb{P}^{13}$,

ii) $G^{\text{lag}}_\mathbb{C}(2, 5)$ is the usual Grassmannians $G(2, 5)$ of 2-planes in $\mathbb{P}^5$ which has dimension 9 and sits in $\mathbb{P}^{19}$ via the Plücker embedding,

iii) $G^{\text{lag}}_{\text{spin}}(2, 5)$ is the spinor variety $S(5)$ of dimension 15 in $\mathbb{P}^{31}$,

iv) $G^{\text{lag}}_\mathbb{O}(2, 5)$ is the $E_7$-variety of dimension 27 in $\mathbb{P}^{55}$.

These are OADP-varieties. For the proof, we refer the reader to [AR]. We notice that $G^{\text{lag}}_\mathbb{R}(2, 5)$ is a Mukai variety (see [Mu1]).

3. Degenerations of projections

One of our tools for studying OADP-varieties, or, more generally, RPTV-varieties, is the analysis of certain degenerations of projections, which, we believe, can be useful also in other contexts. This is what we present in this section.

Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate, non-defective, projective variety of dimension $n$. Let $\Pi \subset \mathbb{P}^r$ be a general projective subspace of dimension $s$ with $r - 2n - 1 \leq s \leq r - n - 2$. Consider the projection morphism $X \to \mathbb{P}^{r-s-1}$ of $X$ from $\Pi$. Let $x$ be a general point on $X$ and let $T_x X$ be the projective tangent space to $X$ at $x$. Roughly speaking, in this section we will study how the projection $X \to \mathbb{P}^{r-s-1}$ degenerates when its center $\Pi$ tends to a general $s$-dimensional subspace $\Pi_0$ containing $x$, i.e. such that $\Pi_0 \cap X = \Pi_0 \cap T_x X = \{x\}$.

To be more precise we want to describe the limit of the double point scheme of $X \to \mathbb{P}^{r-s-1}$ in such a degeneration.

Let us describe in more details the set up in which we work. We let $T$ be a general $\mathbb{P}^{n+s+1}$ which is tangent to $X$ at $x$, i.e. $T$ is a general $\mathbb{P}^{n+s+1}$ containing $T_x X$. Then we choose a general line $L$ inside $T$ containing $x$, and we also choose $\Sigma$ a general $\mathbb{P}^{s-1}$ inside $T$. For every $t \in L$, we let $\Pi_t$ be the span of $t$ and $\Sigma$. For $t \in L$ a general point, $\Pi_t$ is a general $\mathbb{P}^s$ in $\mathbb{P}^r$. For a general $t \in L$, we denote by $\pi_t : X \to \mathbb{P}^{r-s-1}$ the projection morphism of $X$ from $\Pi_t$. We want to study the limit of $\pi_t$ when $t$ tends to $x$.

In order to do this, consider a neighborhood $U$ of $x$ in $L$ such that $\pi_t$ is a morphism for all $t \in U - \{x\}$. We will fix coordinates on $L$ so that $x$ has the coordinate 0, thus we may identify $U$ with a disk around $x = 0$ in $\mathbb{C}$. Consider the products $\mathcal{X} = X \times U, \mathbb{P}_U = \mathbb{P}^{r-s-1} \times U$. The projections $\pi_t$, for $t \in U$, fit together to give a rational map $\pi : \mathcal{X} \dasharrow \mathbb{P}_U$, which is defined everywhere except in the pair $(x, x) = (x, 0)$. In order to extend it, we have to blow up $\mathcal{X}$ at $(x, 0)$. Let $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be this blow up and let $Z \simeq \mathbb{P}^n$ be the exceptional divisor. Looking at the obvious morphism $\phi : \tilde{\mathcal{X}} \to U$, we see that this is a flat family of varieties
over $U$. The fiber $X_t$ over a point $t \in U \setminus \{0\}$ is isomorphic to $X$, whereas the fiber $X_0$ over $t = 0$ is of the form $X_0 = \tilde{X} \cup Z$, where $\tilde{X} \to X$ is the blow-up of $X$ at $x$, and $\tilde{X} \cap Z = E$ is the exceptional divisor of this blow up, the intersection being transverse. On $\tilde{X}$ the projections $\pi_t$, for $t \in U$, fit together to give a morphism $\tilde{\pi} : \tilde{X} \to \mathbb{P}_U$. If we set $H = p^*(\pi^*(\mathcal{O}_{\mathbb{P}_U}(1)))$, then the map $\tilde{\pi}$ is determined by the line bundle $H \otimes \mathcal{O}_{\mathbb{P}_U}(-Z)$. The restriction of $\tilde{\pi}$ to a fiber $X_t$, with $t \in U \setminus \{0\}$, is the projection $\pi_t$. By abusing notation, we will denote by $\pi_0$ the restriction of $\tilde{\pi}$ to $X_0$.

We want to understand how $\pi_0$ acts on $X_0$. The restriction of $\pi_0$ to $\tilde{X}$ is the projection of $X$ from the subspace $\Pi_0$. Notice that, since $\Pi_0 \cap X = \Pi_0 \cap T_xX = \{x\}$, this projection is not defined on $X$ but it is well defined on $\tilde{X}$.

The main point of our analysis is to describe the action of $\pi_0$ on $Z$. This is the content of the following lemma:

**Lemma 3.1.** In the above setting, $\pi_0$ maps isomorphically $Z$ to the $n$-dimensional linear space $S$ which is the projection of $T$ from $\Pi_0$.

**Proof.** The restriction of $\pi_0$ to $Z \cong \mathbb{P}^n$ is determined by the line bundle $\mathcal{O}_Z(-Z) \cong \mathcal{O}_Z(E) \cong \mathcal{O}_{\mathbb{P}^n}(1)$. Since $\pi_0$ is a morphism, the image of $Z$ via $\pi_0$ is a $\mathbb{P}^n$.

The projection $S$ of $T$ in $\mathbb{P}^{r-s-1}$ from $\Pi_0$ has also dimension $n$. Notice that $S$ contains the image $\Omega$, under the same projection, of $T_xX$. This is a subspace of dimension $n - 1$ which is also the image of the exceptional divisor $E$ of $\tilde{X}$ via $\pi_0$, and therefore it is contained in $\pi_0(Z)$ too.

Consider the smooth curve $C$ in $\tilde{X}$, whose general point is $(x,t)$, with $t \in U$ general, and let $\tilde{C}$ be its proper transform on $\tilde{X}$. We let $\gamma$ be the point $\tilde{C} \cap Z$.

This curve is contracted by the obvious projection map $\tilde{X} \to \mathbb{P}^{r-s-1}$, to a point $p \in \mathbb{P}^{r-s-1}$. Of course $p = \pi_0(\gamma) \in \pi_0(Z)$. Furthermore $p \in S$. This follows from the fact that $p \in S_t$, for all $t \in U \setminus \{x\}$, where $S_t$ is the projection of $T$ via $\pi_t$.

We claim that $p \notin \Omega$. Otherwise the $\mathbb{P}^{s+1}$ joining $L$ and $\Sigma$ would cut $T_xX$ along a line, against the generality assumptions about $L$ and $\Sigma$. Finally our assertion follows: indeed $\pi_0(Z)$ and $S$ are two $\mathbb{P}^n$’s, meeting along $\Omega$ which is a $\mathbb{P}^{n-1}$, and also meeting at a point $p \notin \Omega$. \hfill $\square$

The double point scheme of the map $\tilde{\pi}$ is the scheme $\tilde{X} \times_{\mathbb{P}_U} \tilde{X}$. Consider the obvious map $\psi : \tilde{X} \times_{\mathbb{P}_U} \tilde{X} \to U$. We will denote by $\Delta_t$ the fiber of $\psi$ over $t \in U$. We will assume, up to shrinking $U$, that $\dim(\Delta_t)$ is the minimum possible, i.e.

$$\dim(\Delta_t) = 2n - r + s + 1 \geq 0$$

for all $t \in U \setminus \{0\}$. The family $\psi : \tilde{X} \times_{\mathbb{P}_U} \tilde{X} \to U$ may very well be not flat over $x = 0$. However, there is a unique flat limit of $\Delta_t$, $t \neq 0$, sitting inside $\Delta_0$. We will denote by $\tilde{\Delta}_0$ this flat limit, and we will call it the limit double point scheme of the projection $\pi_t$, $t \neq 0$. We want to give some information about it. We notice the following lemma:

**Lemma 3.2.** In the above setting, every irreducible component of $\Delta_0$ of dimension $2n - r + s + 1$ sits in the limit double point scheme $\tilde{\Delta}_0$.

**Proof.** Every irreducible component of $\tilde{X} \times_{\mathbb{P}_U} \tilde{X}$ has dimension at least $2n - r + s + 2$. Hence every irreducible component of $\Delta_0$ of dimension $2n - r + s + 1$ sits in a component of $\tilde{X} \times_{\mathbb{P}_U} \tilde{X}$ of dimension exactly $2n - r + s + 2$ dominating $U$. The assertion immediately follows. \hfill $\square$
Let us introduce some more notation. We will denote by:

i) $X_T$ the scheme cut out by $T$ on $X$. $X_T$ is cut out on $X$ by $r-n-s-1$ general hyperplanes tangents at $x$. We call $X_T$ a general $(r-n-s-1)$-tangent section. Remark that each component of $X_T$ has dimension at least $2n-r+s+1$.

ii) $\tilde{X}_T$ the proper transform of $X_T$ on $\tilde{X}$.

iii) $Y_T$ the image of $\tilde{X}_T$ via the restriction of $\pi_0$ to $\tilde{X}$. By Lemma 3.1, $Y_T$ sits in $S = \pi_0(Z)$, which is naturally isomorphic to $Z$. Hence we may consider $Y_T$ as a subscheme of $Z$.

iv) $\Delta'_0$ the double point scheme of the restriction of $\pi_0$ to $\tilde{X}$.

With this notation in mind, the next lemma follows right away:

**Lemma 3.3.** In the above setting, $\Delta_0$ is the union of $\Delta'_0$ and $\tilde{X}_T$ on $\tilde{X}$ and of $Y_T$ on $Z$.

As an immediate consequence of Lemma 3.2 and Lemma 3.3, we have the following proposition, which will play a crucial role in what follows:

**Proposition 3.4.** In the above setting, every irreducible component of $X_T$ of dimension $2n-r+s+1$ gives rise to an irreducible component of $\tilde{X}_T$ which is contained in the limit double point scheme $\Delta_0$. In particular if $X_T$ has dimension $2n-r+s+1$, then $\tilde{X}_T$ is contained in $\Delta_0$.

4. APPLICATIONS TO RPT AND OADP-VARIETIES

One may apply the results from the previous section to the study of $n$-dimensional RPT and OADP-varieties. We start with the following:

**Theorem 4.1.** Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, projective variety of dimension $n$ whose number of apparent double points is $\nu(X) > 0$. Let $\delta(X)$ be the degree of the projection of $X$ to $\mathbb{P}^n$ from the tangent $\mathbb{P}^n$ to $X$ at its general point. Then $0 < \delta(X) \leq \nu(X)$.

**Proof.** Notice that $X$ is not defective, because $\nu(X) > 0$. Hence by Terracini’s lemma $\delta(X) > 0$ (see [CC2] or [AR], Proposition 3). Now we apply Proposition 3.3 to $X$ with $s = 0$. Notice that, by assumption, $X_T$ has $\delta(X)$ isolated points, which give rise to $\delta(X)$ points in the limit double point scheme. Since this has degree $\nu(X)$, we have the assertion. \qed

As a consequence we have the following.

**Corollary 4.2.** Let $X \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, projective OADP-variety of dimension $n$. Then the projection of $X$ to $\mathbb{P}^n$ from the tangent $\mathbb{P}^n$ to $X$ at its general point is birational. In particular $X$ is rational.

**Remark 4.3.** J. Bronowski claims in [Br] that $X_n \subset \mathbb{P}^{2n+1}$ is an OADP-variety if and only if the projection of $X$ to $\mathbb{P}^n$ from a general tangent $\mathbb{P}^n$ to $X$ is a birational map. Unfortunately Bronowski’s argument is very obscure and, at the best of our knowledge, there is no convincing proof in the present literature for this statement as it is. It is therefore sensible to consider it as a conjecture. The above corollary proves one implication of Bronowski’s conjecture.

For more information about this conjecture see [AR].
Before stating and proving the next result, we need to recall a basic fact concerning degenerations of curves (see also [H]).

**Proposition 4.4.** Let $p : C \to \Delta$ be a proper, flat family of curves parametrized by a disk $\Delta$. Suppose that the general curve $C_t, t \in \Delta \setminus \{0\}$, of the family is irreducible of geometric genus $g$. Then every irreducible component of the central fiber $C_0$ has geometric genus $g' \leq g$.

**Proof.** Let $f : C' \to C$ be the normalization map and set $p' = p \circ f$. Then $p' : C' \to \Delta$ is again a flat family and its general curve $C'_t, t \in \Delta \setminus \{0\}$, is smooth of genus $g$. Moreover the central fiber $C'_0$ has no embedded points and it is a partial normalization of $C_0$.

Let $p'' : C'' \to \Delta$ be a semistable reduction of $p' : C' \to \Delta$ such that all the irreducible components of the central fiber $C''_0$ are smooth. The curve $C''_0$ is reduced, connected (see [Ha, Exercise 11.4, pg. 281]), of arithmetic genus $g$. Hence all its irreducible components have geometric genus bounded above by $g$. Every irreducible component of $C_0$ is birational to some irreducible component of $C''_0$, which, in turn, is dominated by some irreducible component of $C''_0$. This proves the assertion. \qed

**Corollary 4.5.** Let $p : F \to \Delta$ be a proper, flat family of varieties over a disk $\Delta$ such that the general fiber is smooth, irreducible and uniruled. Then every irreducible component of the central fiber is also uniruled.

**Proof.** There is no lack of generality in assuming that $F$ is smooth. Take an irreducible component of the central fiber and let $x$ be a general point on it. We can consider a section of the family, i.e. a morphism $\gamma : U \to F$ such that $p(\gamma(t)) = t$, for all $t \in U$, and such that $\gamma(0) = x$. As a consequence of the uniruledness assumption, there is, up to a harmless base change, a subvariety $C$ of $F$ such that the restriction $p : C \to \Delta \setminus \{0\}$ is a flat family of rational curves such that for all $t \in \Delta \setminus \{0\}$ the curve $C_t$ of the family contains $\gamma(t)$. The flat limit $C_0$ of this family of curves contains $x$ and all of its components are rational by Proposition 4.4. This proves the assertion. \qed

Now we are ready for the announced result about RPT-varieties.

**Proposition 4.6.** Let $X \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate, projective variety of dimension $n$.

i) Suppose $X$ is a RPT-variety and let $C$ be its general $(n-1)$-tangent section. Then every 1-dimensional irreducible component of $C$ is rational.

ii) Suppose $X$ is a RPT$_{n-1}$-variety of dimension $n \geq 3$, and let $D$ be its general tangent hyperplane section. Assume $X$ is not a scroll over a curve. Then $D$ is irreducible and uniruled and the general hyperplane section $H$ of $X$ has $k(H) = -\infty$. Furthermore $X$ is also uniruled, hence $\kappa(X) = -\infty$. Finally $h^i(X, \mathcal{O}_X) = 0, i = n - 1, n$.

**Proof.** The first assertion of (i) is an immediate consequence of Proposition 4.4 and Proposition 4.4.

As for (ii), if $X$ is not a scroll, its general tangent hyperplane section $D$ has double points along a subspace $\Pi_D$ of dimension $s$ such that $0 \leq s \leq n - 3$ and it is smooth elsewhere (see [K] and [E] Theorems 2.1 and 3.2]). This implies that $D$, which is connected, is also irreducible (see also Proposition 3.1 below). By
Proposition 3.4 and Corollary 4.5, $\kappa$ and $X$ are similar if $X$ is a scroll. If $X$ is smooth elsewhere, if $X$ is not a scroll, $C$ is also irreducible (cfr. Proposition 3.2). If $X$ is a scroll it consists of a general line $L$ of the ruling plus a unisecant $C'$ meeting $L$ at the contact point.

By Proposition 4.4 and Proposition 4.5, every irreducible component of $C$ has genus $\gamma \leq PG(X)$. Hence, if $X$ is not a scroll, $C$ is irreducible of geometric genus $\gamma \leq PG(X)$. Since $C$ has a node, its arithmetic genus is $\gamma + 1 \leq PG(X) + 1$. On the other hand, the arithmetic genus of $C$ is the sectional genus $\pi$ of $X$. The argument is similar if $X$ is a scroll.

As a consequence, by taking into account the results from [AB], [Io1] and [Io4] (see also [3]), one may classify all surfaces $X$ with $PG(X) \leq 4$. We will not dwell on this now, and we will simply classify here the RPT-surfaces:

Proposition 4.8. Let $X \subset \mathbb{P}^r$, $r \geq 4$, be a smooth, irreducible, non-degenerate surface which is different from the Veronese surface in $\mathbb{P}^r$, $r = 4, 5$. Then $PG(X)$ is defined. Furthermore the sectional genus $\pi$ of $S$ satisfies $\pi \leq PG(X) + \mu$, where $\mu = 0$ if $X$ is a scroll, and $\mu = 1$ otherwise.

Proof. The Veronese surface in $\mathbb{P}^5$ is the only smooth defective surface and the same surface, and its general projection in $\mathbb{P}^4$, are the only surfaces such that the double curve of their general projection in $\mathbb{P}^3$ is reducible (see [4], [F1], [F2]).

The general tangent curve section $C$ of $X$ has a node at the contact point and it is smooth elsewhere. If $X$ is not a scroll, $C$ is also irreducible (cfr. Proposition 4.2). If $X$ is a scroll it consists of a general line $L$ of the ruling plus a unisecant $C'$ meeting $L$ at the contact point.

By Proposition 3.4 and Proposition 4.4, every irreducible component of $C$ has genus $\gamma \leq PG(X)$. Hence, if $X$ is not a scroll, $C$ is irreducible of geometric genus $\gamma \leq PG(X)$. Since $C$ has a node, its arithmetic genus is $\gamma + 1 \leq PG(X) + 1$. On the other hand, the arithmetic genus of $C$ is the sectional genus $\pi$ of $X$. The argument is similar if $X$ is a scroll.

As a consequence, by taking into account the results from [AB], [Io1] and [Io4] (see also [4]), one may classify all surfaces $X$ with $PG(X) \leq 4$. We will not dwell on this now, and we will simply classify here the RPT-surfaces:

Proposition 4.8. Let $X \subset \mathbb{P}^r$, $r \geq 4$, be a smooth, irreducible, non-degenerate surface which is different from the Veronese surface in $\mathbb{P}^r$, $r = 4, 5$. Then $X$ is a RPT-surface if and only if it is one of the following:

i) a rational normal scroll of degree 3 in $\mathbb{P}^4$,

ii) a rational normal scroll of degree 4 in $\mathbb{P}^5$,

iii) a del Pezzo surface of degree $n = 4, 5$ in $\mathbb{P}^n$.

Proof. By Proposition 4.7, the sectional genus of $X$ is $\pi \leq 1$ and $X$ cannot be an elliptic scroll. Hence $X$ is either a rational scroll or a del Pezzo surface. By checking the various possible cases and using the classical formula for the geometric genus of the double curve of the general projection of a surface (see [Ex], pg. 176] and also [P2]), one proves the assertion.

□
Remark 4.9. The hypothesis of smoothness for $X$ in the previous proposition can be relaxed. It suffices to assume that $X$ has at worst improper double points.

By Proposition 1.3 all OADP-surfaces are RPT-surfaces, i.e. they are listed in the previous proposition. Actually all surfaces there lying in $\mathbb{P}^5$ are OADP-surfaces, and one finds a new, rather cheap, proof of the classification of OADP-surfaces (see [St], [Ru]):

Theorem 4.10. Let $X \subset \mathbb{P}^5$ be an OADP-surface. Then $X$ is one of the following:

(i) a rational normal scroll of degree 4;

(ii) a del Pezzo surface of degree 5.

Remark 4.11. We notice that an equally easy proof of Theorem 4.10 could be deduced from Corollary 4.2. In particular for smooth non degenerate surfaces in $\mathbb{P}^5$, OADP is equivalent to FRPT. It is reasonable to suspect that this relation holds also in higher dimensions, see Remark 5.6.

Turning to the classification of OADP-threefolds $X$, which is our main aim, we notice that Proposition 4.6 and Remark 1.4 imply that the general hyperplane section of such an $X$ is rational. A proof of the classification Theorem 7.1 below could be based on this remark. Indeed by applying Mori’s program (see [Mc]) as in [Io2, Io3] or [CF], one sees that $X$ is of one of the following types (see [Io2, Theorem II]):

i) a scroll over a curve or over a surface,

ii) a quadric fibration,

iii) a del Pezzo threefold,

iv) a quadric in $\mathbb{P}^4$,

v) (a projection of) a Veronese surface fibration,

vi) a projection of the Veronese embedding of $\mathbb{P}^3$ via the cubics or of the Veronese embedding of a quadric in $\mathbb{P}^4$ via the quadrics.

A case by case analysis leads to the classification. Most of the tools needed for this have already been introduced so far or will be in §. However, the approach we have chosen to follow here is slightly different and, we think, somewhat conceptually easier. It is, in a sense, dual to the one outlined above. In fact one first bounds the degree of $X$, drastically reducing the a priori possible cases (see next §, this is not more expensive than the analysis required in the discussion of the cases (i), ..., (vi) above). Then a bit of Mori’s theory, i.e. adjunction theory, is used to determine, for each possible degree, the varieties that really occur.

5. Tangential behavior of OADP 3-folds

Let $X \subset \mathbb{P}^7$ be an OADP-variety of dimension 3. The main result of this section is an upper bound for the degree $d$ of $X$.

Theorem 5.1. Let $X \subset \mathbb{P}^7$ be a OADP 3-fold of degree $d$. Then $d \leq 9$. Furthermore either $X$ is a scroll over a curve and $d = 5$ or $d = 9 - k$, where $k \leq 3$ is the number of lines passing through a general point of $X$. 
The proof is based on the analysis of the tangential behavior of \( X \), i.e., on the study of the intersection scheme \( Z := Z_x \) of \( X \) with \( T_x X \), where \( x \in X \) is a general point. We need a few technical results, some of them of independent interest. We state them at the first occurrence, but we will prove them separately in the next section, in order not to break the main line of the argument. We refer to them freely here.

First we remark that \( 0 \leq \dim(Z_x) \leq 2 \). The case of \( \dim(Z_x) = 2 \) is classically known.

**Proposition 5.2.** Let \( X \subset \mathbb{P}^r \) be an irreducible, projective, non-degenerate variety of dimension \( n \) with \( 1 < n < r \). Let \( x \) be a general point of \( X \). The intersection scheme of \( X \) with \( T_x X \) has an irreducible component of dimension \( n - 1 \) if and only if either \( X \) is a hypersurface in \( \mathbb{P}^{n+1} \) or \( X \) is a scroll over a curve.

We will prove this Proposition in the next section for the lack of a modern reference. This allows to restrict our attention to the case of an OADP 3-fold \( X \subset \mathbb{P}^7 \) such that the scheme \( Z_x \), with \( x \in X \) a general point, has dimension at most 1. Equivalently we may and will assume from now on that \( X \) is not a scroll.

The main difficulty will be to treat the one-dimensional case. A great part of this section is indeed devoted to prove the following Lemma

**Lemma 5.3.** If \( \dim(Z_x) = 1 \), then \((Z_x)_{\text{red}} \) is a bunch of \( k \) lines through \( x \).

Let us now fix some notations which we are going to use throughout this section. Let \( f : Y \to X \) be the blow up of \( X \) at \( x \) with exceptional divisor \( E \). We let \( H' \) be the pull back on \( Y \) via \( f \) of the general hyperplane section \( H \) of \( X \). We consider the proper transform \( \Lambda \) on \( Y \) of the linear system of tangent hyperplanes to \( X \) at \( x \), i.e., \( \Lambda = |H' - 2E| \). Let \( S \) be a general element in \( \Lambda \), i.e., \( S \) is the proper transform on \( Y \) of a general tangent hyperplane section \( \Sigma \) of \( X \). The surface \( \Sigma \) has an ordinary double point at \( x \) and no other singularity, see for instance [Ei, Theorem 2.1]. Thus \( S \) is a smooth surface and the exceptional curve \( F = E|_S \) is an irreducible \((-2)\)-curve.

We will consider the characteristic system \( \mathcal{C} \) of \( \Lambda \) on \( S \), i.e., the linear system cut out by \( \Lambda \) on \( S \). We may write \( \mathcal{C} = \mathcal{Z} + aF + \mathcal{L} \) where \( \mathcal{L} \) is the movable part of \( \mathcal{C} \) and \( \mathcal{Z} + aF \) is the fixed part, with \( \mathcal{Z} \) not containing \( F \). Since \( X \) has non-degenerate tangential variety, by a classical result of Terracini’s, one has \( a = 0 \) (see [H], or [GH, pg. 417], or also [La, Theorem 7.3 i]). Moreover the image of \( \mathcal{Z} \) via \( f \) is the support of the 1-dimensional part of \( Z \).

By abusing notation, we often denote by \( \mathcal{C} \) and \( \mathcal{L} \) also a curve of the linear systems in question. Notice then the basic linear equivalence relation

\[
H'|_S \equiv \mathcal{C} + 2F = \mathcal{Z} + 2F + \mathcal{L}
\]

which we frequently use later on.

Next we are going to study the linear system \( \mathcal{C} \), by first studying its movable part \( \mathcal{L} \) and then its fixed part \( \mathcal{Z} \).

**Proposition 5.4.** In the above setting, \( \mathcal{L} \) is a complete linear system which is base point free and not composed with a pencil. Moreover \( \mathcal{L}^2 = 1 \) and \( h^1(S, \mathcal{O}_S(\mathcal{L})) = 0 \). In particular either \( Z_x \) has dimension 0 supported at \( x \) or it is equidimensional of dimension 1 off \( x \). Moreover the morphism \( \phi_\mathcal{C} : S \to \mathbb{P}^2 \) is a sequence of blowing-ups.
Proof: The threefold $X$ is linearly normal and, as we saw in remark 4.4, $h^1(X,\mathcal{O}_X) = 0$. Thus any hyperplane section of $X$ is linearly normal. This means that the linear system $\mathcal{H}'$ cut out by $|\mathcal{H}'|$ on $S$ is complete. Notice that $\mathcal{C} = \mathcal{H}'(-2\mathcal{F})$, which proves that $\mathcal{C}$, and therefore also $\mathcal{L}$, is complete.

Since $X$ is non-defective, the projection of $X$ to $\mathbb{P}^3$ from the tangent space $T_xX$ is dominant (see [CC2] or [AR, Proposition 3]). This implies that the rational map $\phi_{\mathcal{C}} : S \to \mathbb{P}^2$ determined by the linear system $\mathcal{L}$ is dominant, namely $\mathcal{L}$ is not composed with a pencil.

By Proposition 4.6, the general curve in $\mathcal{L}$, which is irreducible, is also rational. Here we need the following auxiliary result, whose proof is postponed to the next section.

Proposition 5.5. Let $S$ be a smooth, irreducible, projective, regular surface and let $\mathcal{M}$ be a line bundle on $S$. Suppose that the general curve $C \in |\mathcal{M}|$ is irreducible with geometric genus $g$ and that $h^0(S,\mathcal{M}) \geq g+2$. Then the complete linear system $|\mathcal{M}|$ is base point free. Furthermore $S$ is rational.

By Proposition 5.5, $\mathcal{L}$ is base point free. Therefore the general element in $\mathcal{L}$ is a smooth rational curve. Hence $K_S \cdot \mathcal{L} = -2 - \mathcal{L}^2$, and, by Riemann–Roch formula, $\chi(\mathcal{O}_S(\mathcal{L})) = 3 + \mathcal{L}^2$. On the other hand one has $h^0(S,\mathcal{O}_S(\mathcal{L})) = 3$ and $h^2(S,\mathcal{O}_S(\mathcal{L})) = h^0(S,\mathcal{O}_S(K_S - \mathcal{L})) = 0$, because $\mathcal{L}$ is effective and $S$ is rational. Since $\mathcal{L}^2 > 0$ we finally obtain $h^1(S,\mathcal{O}_S(\mathcal{L})) = 0$ and $\mathcal{L}^2 = 1$.

The rest of the assertion is trivial. □

Remark 5.6. Notice that, as a consequence of the above proposition, one finds a new proof, in the threefold case, of Corollary 4.2. This also proves the same statement for linearly normal FRPT $3$-folds embedded in $\mathbb{P}^7$. Recall that Bronowski, [Br], claimed that OADP varieties are characterized by this property. Summing all one can conjecture that OADP and FRPT are equivalent also, keep in mind Remark 1.11 for smooth non degenerate $3$-folds in $\mathbb{P}^7$ and maybe a similar statement is true in any dimension.

The next step toward the proof of Theorem 5.1 is to study the fixed part $Z$ of the system $\mathcal{C}$, i.e. the scheme $Z := Z_s$. We will denote by $Z_s$ the Zariski closure of the support of the scheme $Z_s \setminus \{x\}$. By Proposition 5.4, either $Z_s = \emptyset$ or every irreducible component of $Z_s$ has dimension $1$. We will consider the latter case next, for this we use the notion of $k$-filling curve.

Definition 5.7. Let $C \subset X$ be a reduced irreducible curve on $X$. We say that $C$ is a $k$-filling curve for $X$ if $C$ is the general member of a $k$-dimensional family of curves such that there is a curve of the family passing through the general point of $X$.

Lemma 5.8. If $\dim(Z) = 1$, let $D \subset Z_s$ be any irreducible component. Then $D$ is a $k$-filling curve for $X$ for some $k \geq 2$. In particular if $Z_s$ contains a line then $x$ sits on some line contained in $Z_s$.

Proof: Consider the family $\mathcal{D}$ in which $D$ varies as $x$ varies on $X$. By abusing notation, we denote by $\mathcal{D}$ also the parameter space of the family.

First we claim that $\dim(\mathcal{D}) := k \geq 2$. Otherwise there would be a surface $\Delta$ on $X$ such that $D \subset T_xX$ for the general point $x \in \Delta$. Then $1 \leq \dim(<D>) \leq 2$. Suppose $D$ is a line. By generically projecting to $\mathbb{P}^4$ we would find a hypersurface
$X'$, the projection of $X$, containing a line $D'$, the projection of $D$, such that the general hyperplane through $D'$ is tangent to $X'$ at some smooth point $x' \notin D'$. This contradicts Bertini’s theorem. Also if $D$ spans a plane a similar argument, which we can leave to the reader, works to reach a contradiction to Bertini’s theorem. This proves our assertion on $k$.

Now assume that all the curves $D$ of $\mathcal{D}$ are contained in a surface $B$. Then through the general point of $B$ there is a $(k - 1)$-dimensional family of curves of $\mathcal{D}$. In particular two general curves of $\mathcal{D}$ have to intersect. This would imply that $T_xX \cap T_yX \neq \emptyset$ for $x$ and $y$ general in $X$, contradicting Terracini’s lemma.

Lemma 5.9. If $\dim(Z) = 1$, then either $Z_s$ consists of a bunch of lines passing through $x$ and spanning $T_xX$ or it is a plane curve. In the latter case the plane spanned by $Z_s$ contains $x$.

Proof. Recall that the focal locus $F(X)$ of $X$ has dimension at most 5 (see [3]). Thus $F(X)$ cannot contain the tangential variety $Tan(X)$, which has dimension 6. Therefore for the general point of $z \in Tan(X)$, with $z \in T_xX$, $x \in X$, there is a unique secant to $X$ through it, i.e. the tangent line $< x, z >$. Since $Z_s$ is a subvariety of $T_xX \simeq \mathbb{P}^3$, we conclude that $Sec(Z_s)$ is properly contained in $T_xX$. In particular every irreducible component of $Z_s$ is planar. Moreover the classical trisecant lemma (see, for instance, [CC2]) yields that either $Z_s$ itself is planar or $Z_s$ consists of a bunch of lines spanning $T_xX$ and passing through a point $y \in T_xX$. In this latter case, since the general tangent hyperplane section $\Sigma$ of $X$ contains $Z$, we conclude that $x = y$, otherwise $\Sigma$ would be singular at $y$, which, as we know, is impossible (see [3, Theorems. 2.1]).

If $Z_s$ is one single line, then $Z_s$ contains $x$ by Lemma 5.8. Suppose finally $Z_s$ is a plane curve which is not a line and let $\alpha$ be the plane where $Z_s$ sits. Thus $Sec(Z_s) = \alpha$. Suppose $x \notin \alpha$. Then any tangent line $L$ at $x$ intersects $\alpha$ at a point $y \neq x$ and all the secants to $Z_s$ through $y$ meet $L$. Thus $L$ would be focal, hence $F(X)$ would contain $Tan(X)$, a contradiction.

Next assume that $Z_s$ is a plane curve. Let $\alpha := \alpha_x := < Z_s >$ be a plane. We denote by $\delta \geq 2$ the degree of the plane curve $Z_s$. We can consider the subvariety $F \in G(2, 7)$ described by the planes $\alpha_x$ when $x$ varies on $X$.

To better understand this variety we need the following Proposition, whose proof is demanded.

Proposition 5.10. Let $X \subset \mathbb{P}^r$, $r \geq 4$, be a smooth, irreducible, non degenerate, projective 3-fold which is neither a scroll over a curve nor a quadric in $\mathbb{P}^4$. Then there is no irreducible subvariety $F \subset G(2, r)$ of dimension 2, such that the general point $p \in F$ corresponds to a plane $\Pi \subset \mathbb{P}^r$ which is tangent to $X$ along an irreducible curve $\Gamma$ which is 2-filling for $X$.

We are in the condition to prove the following lemmas.

Lemma 5.11. In the above setting, one has $\dim(F) = 3$.

Proof. Suppose the assertion is not true. Given a general plane $\alpha \in F$, there would be an irreducible positive dimensional subvariety $\Delta$ of $X$ such that $\alpha$ would correspond to the general point $x \in \Delta$. By Lemma 5.8, $\Delta$ has to lie on $\alpha$. By Proposition 5.3 and since we are assuming that $X$ is not a scroll, we have that $\Delta$ is a curve. In conclusion the planes $\alpha$ would vary in a 2-dimensional family $F$.
and each of them would be tangent to $X$ along a curve $\Delta$, contrary to Proposition 5.10. □

Lemma 5.12. In the above setting, one has $\delta \leq 3$.

Proof. Assume by contradiction that $\delta \geq 4$. By Lemma 5.11, and by applying the results of [Mez, §3] to a general projection of $X$ to $\mathbb{P}^5$, we see that $X$ sits on a 4-dimensional scroll $W$ over a curve.

Let $\Pi$ be the general $\mathbb{P}^3$ of the ruling of $W$ and let $D$ be the surface cut out by $\Pi$ on $X$. Notice that $D$ is smooth in codimension 1, otherwise $\Pi$ would be tangent to $X$ along a curve, against Zak’s theorem on tangencies (see [Za1]). In particular $D$ is irreducible. Then by Proposition 2.6 we have that $X$ should be an Edge variety, which has no plane curves of degree $\delta \geq 3$, a contradiction. □

Lemma 5.13. In the above setting, if $\delta = 3$ then $Z_s$ cannot be an irreducible cubic.

Proof. For a general point point $y \in Z_s$ we can consider the plane $\alpha_y$, which, by Lemma 5.11 is different from $\alpha = \alpha_x$.

Notice that there is a unique point $w_y \neq y$ belonging to $T_y Z_s \cap Z_s$. Since by Lemma 5.9 one has $T_y X \cap X \subset \alpha_y$ and since both $y, w_y$ are points of $T_y X \cap X$, we have that $T_y Z_s \subset \alpha_y$. Since one has also $T_y Z_s \subset \alpha$ and $\alpha_y \neq \alpha$, then $T_y Z_s = \alpha_y \cap \alpha$.

We also remark that $w_y \in Z_y$ and that $\alpha_y \neq \alpha_{w_y}$. Indeed, $w_y$ is also a general point of $Z_s$ and therefore, as we saw, $T_{w_y} Z_s = \alpha_{w_y} \cap \alpha$. Since $T_{w_y} Z_s \neq T_y Z_s$, the assertion follows.

Consider now the Zariski closure $F_x$ in $F$ of the planes $\alpha_y$ with $y \in Z_s$ a general point. There are two possibilities:

i) the general hyperplane through $T_x X$ contains some general plane in $F_x$.

ii) $T_x X$ and all the planes in $F_x$ sit in one and the same $\mathbb{P}^4$ which we will denote by $F_x$.

Assume we are in case (i) and let us keep the notation introduced at the beginning of this section. The surface $\Sigma$ contains the curve $Z_y$ for $y \in Z_s$ a sufficiently general point. Thus the support $Z'$ of $Z_y$ is still an irreducible cubic. We let $Z'$ be the strict transform of $Z'$ on $S$. It is clear that $Z'$ is contracted by the morphism $\phi_L$, thus its arithmetic genus should be 0. On the other hand, the plane cubic $Z'$ does not contain $x$, hence $Z'$ is isomorphic to $Z'$ and therefore its arithmetic genus is 1, a contradiction.

Assume we are in case (ii). First notice that $F_x \cap X$ has dimension 2. Indeed it contains the irreducible surface $E_x$ described by the cubic curves $Z_y$ with $y$ a variable point in $Z_s$.

Consider the Zariski closure $V$ of the union of $F_y$ as $y \in Z_s$ varies. Remark that for $y$ general in $Z_s$ one has $F_y = \langle T_y X, \alpha_{w_y} \rangle$ hence $\langle \alpha_y, \alpha_{w_y} \rangle \subset F_y \cap F_x$, thus $\dim(F_x \cap F_y) \geq 3$. Therefore, if we consider the projection $\phi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^4$ from $T_x X$, then $\dim(\phi(V)) = 2$. On the other hand $\phi(X) = \mathbb{P}^3$, hence $V$ does not contain $X$, and this implies that the surface $E := E_y$ does not depend on $y \in Z_s$. This means that either $E$ has a 2-dimensional family of plane cubics or that the planes $\alpha_y$, with $y \in Z_s$, pairwise meet along a variable line. In either case $E$ has to be a cubic surface in a $\mathbb{P}^3$. By Proposition 2.6 we have a contradiction again. □

Now we go back to the characteristic system $C = Z + L$ of $\Lambda$ on $S$. By the previous lemmas, any irreducible component $Z'$ of $Z$ is rational and smooth. We have the following elementary fact.
Lemma 5.14. If \( Z' \) is a reduced, connected subcurve of \( Z \), then \( (\mathcal{L} + Z') \cdot Z' \leq -1 + p_a(Z') \) and \( Z'^2 \leq -1 + p_a(Z') \).

Proof. Consider the exact sequence

\[
0 \to \mathcal{O}_S(\mathcal{L}) \to \mathcal{O}_S(\mathcal{L} + Z') \to \mathcal{O}_{Z'}(\mathcal{L} + Z') \to 0
\]

Since \( h^0(S, \mathcal{O}_S(\mathcal{L})) = h^0(S, \mathcal{O}_S(\mathcal{L} + Z')) \) and \( h^1(S, \mathcal{O}_S(\mathcal{L})) = 0 \), the sequence yields \( h^0(\mathcal{L}', \mathcal{O}_{Z'}(\mathcal{L} + Z')) = 0 \). Hence \( \deg(\mathcal{L} + Z' \vert Z) \leq -1 + p_a(Z') \). The proof of the second assertion is similar. \( \square \)

We need two more results that will be proved later on.

Proposition 5.15. Let \( \tau : \Sigma \dashrightarrow \mathbb{P}^{r-4} \) be the projection of \( \Sigma \), the general hyperplane tangent section, from \( T_x X \). Then \( \tau \) contracts \( L \) to a point.

Finally a technical proposition on the normal sheaf of a reduced curve in \( X \), for more detail see the next section.

Proposition 5.16. Let \( C \subset X \) be any reduced curve with superficial singular points only, i.e. points of embedding dimension 1 or 2.

Let \( f: \tilde{C} \to C \) be the normalization morphism. We set \( p_n(C) = p_a(\tilde{C}) \). Then we have a natural exact sequence

\[
0 \to T \to N_f \to N'_f \to 0,
\]

where \( N'_f \) is locally free of rank \( n - 1 \). We denote by \( t_C \) the degree of \( T \), which can be called the number of cusps of \( C \). Then

\[
c_1(N_{C \vert X}) = c_1(N'_f) + t_C + 2(p_a(C) - p_n(C)).
\]

We can now prove the key Lemma in a strengthened form.

Lemma 5.17. If \( \dim(Z) = 1 \), then \( Z_s \) is a bunch of \( k \) lines through \( x \). For each irreducible component \( Z' \) of \( Z \) one has \( K_S \cdot Z' = -1 \), and \( K_S \cdot Z = -k \).

Proof. We separately discuss the various possibilities for \( Z_s \) given by the previous lemmas.

Case 5.18. \( Z_s \) is a bunch of lines through \( x \).

Let \( Z_i \) be the lines in \( Z_s \) and \( Z_i \) their strict transform on \( S \), with \( i = 1, \ldots, k \). We have \( Z = a_1 Z_1 + \ldots + a_k Z_k \), with \( a_i, i = 1, \ldots, k \), positive integers. By Proposition 5.15 we have \( L \cdot Z_i = 0, Z_i^2 \leq -1 \) for \( i = 1, \ldots, k \). Hence \( 1 = H_{|S} \cdot Z_i = \sum_{j=1}^k a_j Z_j \cdot Z_j + 2 = a_1 Z_1^2 + 2 \), thus \( a_i = 1 \) and \( Z_i^2 = -1 \), whence the assertion.

Case 5.19. \( Z_s \) is an irreducible conic.

If \( Z' \) is the strict transform of \( Z_s \) on \( S \), we have \( Z = a Z' \), with \( a \) a positive number. Arguing as before, we have \( 2 = Z' \cdot (L + a Z' + 2F) \leq 1 + (a - 1) Z'^2 \), which leads to a contradiction by Lemma 5.14.

Case 5.20. \( Z_s \) is a reducible conic.

Then \( Z_s \) consists of two lines \( Z_1, Z_2 \) which we may assume not both passing through \( x \). We denote by \( Z_1, Z_2 \) the strict transforms of \( Z_1, Z_2 \) on \( S \). Notice that \( Z_1 \cdot Z_2 = 1 \). Moreover \( Z = a_1 Z_1 + a_2 Z_2 \), with \( a_1, a_2 \) positive integers.

Set \( Z' = Z_1 + Z_2 \). By Lemma 5.14 we have \( (L + Z') \cdot Z' \leq -1 \). Hence \( 2 \leq (a_1 - 1) Z_1^2 + (a_2 - 1) Z_2^2 + (a_1 + a_2 - 1) \), a contradiction.
Case 5.21. $Z_s$ is a plane cubic which is the union of an irreducible conic and of a line.

Let $Z_1$ be the line and $Z_2$ the conic in $Z_s$ and let $Z_1$, $Z_2$ be the strict transforms on $S$ of $Z_1$ and $Z_2$ respectively. Notice that, by Lemma 5.8, $x$ belongs to $Z_1$. Furthermore $Z_1$ is at least 2-filling and, by Lemma 5.11, $Z_2$ is at least 3-filling for $X$.

Suppose both $Z_1$ and $Z_2$ contain $x$. Then $Z_s$ is a Cartier divisor on $\Sigma$, and one has $c_1(N_{Z_s}|X) = Z_s^2 + 3$. Consider the normalization map $f : \mathbb{P}^1 \amalg \mathbb{P}^1 \to Z_s$. By Lemma 6.2 we have $c_1(N_{Z_s}^f) \geq 1$, hence by Proposition 5.16 we have $c_1(N_{Z_s}|X) \geq 5$. Accordingly one has $Z_s^2 \geq 2$ and therefore $(Z_s')^2 \geq 0$, a contradiction.

Suppose $Z_1$ contains $x$ and $Z_2$ does not. Then $Z_2$ is a Cartier divisor on $\Sigma$ and, arguing as before we see that $Z_2^2 = -1$, thus $\mathcal{L} \cdot Z_2 = 0$ by Lemma 5.14. Furthermore $\mathcal{L} \cdot Z_1 = 0$ by Proposition 5.15. Since $Z_1 \cdot Z_2 = 2$ we have a contradiction to the last assertion of Proposition 5.4.

Case 5.22. $Z_s$ is a plane cubic which is the union of three lines, not all passing through $x$.

Let $Z_i$ be these lines, and let $Z_i$ be the strict transform of $Z_i$ on $S$, $i = 1, 2, 3$. By Lemma 5.8 $x$ has to lie on at least one of the lines $Z_i$, $i = 1, 2, 3$.

Suppose that $x \in Z_1$ but $x \notin C := Z_2 \cup Z_3$. Then $C$ is a Cartier divisor on $\Sigma$ and one has $c_1(N_{C}^{|X}) = C^2 + 2$. Consider the normalization map $f : \mathbb{P}^1 \amalg \mathbb{P}^1 \to C$. One has $c_1(N_{C}^f) \geq 0$, hence by Proposition 5.16 we have $c_1(N_{C}^{|X}) \geq 2$, thus $C^2 \geq 0$ which leads to a contradiction to Lemma 5.14.

Suppose that $x \notin Z_1$ but $x \in Z_2 \cap Z_3$. Again we set $C := Z_2 \cup Z_3$. As before, we see that $C^2 \geq 0$. Hence $(Z_2 + Z_3)^2 \geq -2$. Thus $Z_2^2 = Z_3^2 = -1$. Moreover, by a similar argument, we have that $Z_2^2 = -1$. Thus $\mathcal{L} \cdot Z_1 = 0$, $i = 1, 2, 3$.

Let $\mathcal{C} = \mathcal{L} + a_1Z_1 + a_2Z_2 + a_3Z_3$, with $a_i$ positive integers, $i = 1, 2, 3$. By intersecting $H_i^{|S}$ with $Z_i$, $i = 1, 2, 3$, we find the relations $a_1 = a_2 + a_3 - 1 = a_2 - 1 = a_3 - 1$, which lead to $a_2 = a_3 = 0$, $a_1 = -1$, a contradiction. \hfill \Box

We are finally in a position to prove Theorem 5.1.

Proof of Theorem 5.1. As already noticed, we may assume that $X$ is not a scroll over a curve. By Riemann-Roch theorem, one has:

\begin{equation}
3 - h^1(S, \mathcal{O}_S(\mathcal{C})) = \chi(\mathcal{O}_S(\mathcal{C})) = 1 + \frac{1}{2}(C^2 - K_S \cdot C).
\end{equation}

One has $C^2 = \deg(X) - 8$ and by Lemma 5.4 one has $K_S \cdot \mathcal{L} = -3$. Let $k$ be the number of lines through $x$. By Lemma 5.17 we have $K_S \cdot C = -3 - k$. Finally, by equation (5.1), we have $\deg(X) = 9 - k$. \hfill \Box

Remark 5.23. The proof of Theorem 5.1 shows that actually $d \leq 8$, unless for a general point $x \in X$, the scheme $Z_x$ cut out by $T_xX$ on $X$ is 0-dimensional supported at $x$ (see Proposition 5.14). Furthermore, if $X$ is not a scroll, i.e. $d \geq 6$, and if $d \leq 8$ then $Z$ consists of $9 - d$ lines through $x$. In particular in these cases $X$ is ruled by lines. We remark that this is in fact the situation in the known examples of Edge varieties, $d = 6, 7$ and of the scroll of degree 8 (see examples 2.4 and 2.7).
6. A few technical results

In this section we prove a few technical results we have been using in the previous section.

First we sketch the proof of Proposition 5.2 for lack of a modern reference.

**Proof of Proposition 5.2.** We prove only the non trivial implication. Assume that $n < r - 1$. By cutting $X$ with a general hyperplane we can reduce ourselves to the surface case $n = 2$. The general tangent hyperplane section $H$ of $X$ is reducible. If $H$ is non-reduced, then the dual variety of $X$ is degenerate and $X$ is a developable scroll (see [GH]). Otherwise write $H = A + B$. Since $H$ is a nodal curve, with a single node at the point $x$ of tangency, and since $H$ is connected, we see that $A$ and $B$ are smooth curves meeting at $x$. Namely $H$ is 1-connected but not 2-connected. This implies that either $A$ or $B$ is a line and $X$ is a scroll (see [vdV]). □

Notice that, in the case $X$ is smooth, as remarked in the proof of Proposition 5.6, the previous result follows from [K] and [Ei, Theorems 2.1 and 3.2]. As a consequence we have the following.

**Proposition 6.1.** Let $X \subset \mathbb{P}^{2n+1}$ be a OADP $n$-fold. Let $x$ be a general point of $X$. The intersection scheme of $X$ with $T_x X$ has an irreducible component of dimension $n - 1$ if and only if $X$ is one of the two scrolls $S(1^{n-1}, 3)$ and $S(1^{n-2}, 2^2)$.

**Proof.** This is a consequence of Lemma 5.2 and Proposition 2.3. □

Next we justify the basic proposition about linear systems of rational curves on surfaces.

**Proof of Proposition 5.4.** Assume $|\mathcal{M}|$ has some base point. There is no lack of generality in supposing that it has only one base point at $x \in S$ which is an ordinary multiple point of multiplicity $m$ for the general curve $C$ in $|\mathcal{M}|$. Indeed, we can always put ourselves in this situation after having performed a suitable sequence of blow-ups.

Let $f : S' \to S$ be the blow-up of $S$ at $x$ with exceptional divisor $E$, and consider the proper transform $C'$ of $C$ on $S'$, i.e. $C' = f^*(C) - mE$. Note that $C'$ is a smooth curve of genus $g$ on $S'$. If we set $f^*(\mathcal{M}) := \mathcal{M}'$, the curve $C'$ determines the line bundle $\mathcal{M}' \otimes \mathcal{O}_{S'}(-mE)$. Look at the exact sequence:

(6.1) \[ 0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'}(C') \to \mathcal{O}_{C'}(C') \to 0 \]

Since $h^1(S', \mathcal{O}_{S'}) = 0$, one has $h^0(C', \mathcal{O}_{C'}(C')) = h^0(S', \mathcal{O}_{S'}(C')) - 1 = h^0(S, \mathcal{M}) - 1$. Thus $h^0(C', \mathcal{O}_{C'}(C')) \geq g + 1$ and therefore $h^1(C', \mathcal{O}_{C'}(C')) = 0$. Hence $g + 1 \leq h^0(C', \mathcal{O}_{C'}(C')) = (C')^2 - g + 1$, i.e. $(C')^2 \geq 2g$ and then $K_{S'} \cdot C' < 0$, proving that all the plurigenera of $S'$ vanish. Since the same is true for $S$, we have that $S$ is rational. Now, from equation (6.1) we deduce $h^1(S', \mathcal{M}' \otimes \mathcal{O}_{S'}(-mE)) = h^1(S', \mathcal{O}_{S'}(C')) = 0$. Look at the exact sequence:

\[ 0 \to \mathcal{M}' \otimes \mathcal{O}_{S'}(-mE) \to \mathcal{M}' \to \mathcal{M}' \otimes \mathcal{O}_{mE} \simeq \mathcal{O}_{mE} \to 0 \]

Since $h^0(S', \mathcal{M}' \otimes \mathcal{O}_{S'}(-mE)) = h^0(S', \mathcal{M}')$, we see that $h^1(S', \mathcal{M}' \otimes \mathcal{O}_{S'}(-mE)) \geq h^0(S', \mathcal{M}' \otimes \mathcal{O}_{mE}) = m(m + 1)/2 > 0$, a contradiction. □
Here we introduce in a more detailed way the notion of number of cusps for a curve $C$ with only superficial singular points.

Let now $X$ be a smooth, projective variety of dimension $n$ and let $C \subset X$ be a reduced curve with superficial singular points only, i.e. points of embedding dimension 1 or 2.

First we remark that the normal sheaf $N_{C|X}$ of $C$ in $X$ is locally free of rank $n - 1$ (see [MP, Lemma 2.1]).

Let $f : \tilde{C} \to C$ be the normalization morphism. We set $p_a(C) = p_a(\tilde{C})$. We can also consider the normal sheaf to the map $f$, denoted by $N_f$, which is defined by the exact sequence:

$$0 \to T_{\tilde{C}} \xrightarrow{df} f^* (T_X) \to N_f \to 0$$

The sheaf $N_f$ is locally free, except at the points of $\tilde{C}$ where the differential of $f$ vanishes, where $N_f$ has torsion. If $T$ is the torsion subsheaf of $N_f$, we have the sequence

$$0 \to T \to N_f \to N'_f \to 0$$

where $N'_f$ is locally free of rank $n - 1$. We denote by $t_C$ the degree of $T$, which can be called the number of cusps of $C$.

Proof of Proposition 5.16. Notice that there is a smooth surface $S \subseteq X$ containing $C$. We denote by $\phi$ the map $\tilde{C} \to S$ induced by $f$. We have the sequences:

$$0 \to N_{C|S} \to N_{C|X} \to N_{S|X|C} \to 0$$

$$0 \to N'_\phi \to N'_f \to f^* N_{S|X|C} \to 0$$

which shows that we can reduce ourselves to the case $X$ itself is a surface. Then, by definition, one has:

$$c_1(N'_f) = -\deg(f^*(K_X)) + 2p_a(\tilde{C}) - 2 - t_C$$

and by adjunction:

$$c_1(N_{C|X}) = -K_X \cdot C + 2p_a(C) - 2$$

which proves the assertion. \Box

We are left to prove Proposition 5.10 and Proposition 5.15. These we believe to be of independent interest.

Let $C$ be an irreducible reduced rational curve, so that $\tilde{C} \simeq \mathbb{P}^1$. Then $N'_f \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})$, with $a_1 \leq \ldots \leq a_{n-1}$.

Under this hypothesis the above construction gives.

Lemma 6.2. If $C$ is a rational $k$-filling curve for $X$ then $a_1 \geq 0$ and $a_1 + \ldots + a_{n-1} \geq k - n + 1$. In particular $h^1(\tilde{C}, N'_f) = h^1(\tilde{C}, N'_\phi) = 0$.

Proof. A count of parameters shows that $k \geq n - 1$. Hence we have a dominant morphism $\mathbb{P}^1 \times B : X$ where $B$ is a suitable $(n - 1)$-dimensional variety. In this situation, given a general point $b \in B$, there is a focal square matrix $\Phi$ arising (see [FS]), whose rows are given by the sections of $N'_f$ corresponding to the $n - 1$ infinitesimal deformations of $f : \mathbb{P}^1 \times \{ b \} \to X$ determined by $n - 1$ independent
vectors of $T_b(B)$. By the filling hypothesis, $\Phi$ has maximal rank, which implies $a_1 \geq 0$. The final assertion translates the fact that $k \leq h^0(\tilde{C}, N_f')$ (see [AC]). □

Then a lemma about foci of certain families of planes in $\mathbb{P}^4$ (again, see [CS] for the basics in the theory of foci).

**Lemma 6.3.** Let $X \subset \mathbb{P}^4$ be a non degenerate, irreducible hypersurface. Suppose there is an irreducible subvariety $F \subset \mathbb{G}(2, 4) \times \mathbb{P}^4$ of dimension 2, such that the general point $\xi \in F$ corresponds to a pair $(\Pi, p)$ where $\Pi$ is a plane in $\mathbb{P}^4$, $p$ is a smooth point of $X$ and $\Pi$ is tangent to $X$ at $p$. Suppose the projection $G$ of $F$ to $\mathbb{G}(2, 4)$ has dimension 2. Then $p$ is a focus for the family $G$ on $\Pi$.

**Proof.** Let $x = (x_0, ..., x_4)$ be homogeneous coordinates on $\mathbb{P}^4$, let $a = (a_0, ..., a_4)$ be the dual coordinates and let $(u_1, u_2)$ be local parameters on $F$ around $\xi$. Thus we may assume that $(u_1, u_2)$ are local parameters on $G$ around $\Pi$, so that the equations of $\Pi$ are given by:

\[(6.2) \quad a_i \times x = 0, \quad i = 1, 2\]

where $a_i = a_i(u_1, u_2)$ are regular functions of $(u_1, u_2)$. We will denote by a subscript $j$ the differentiation with respect to $u_j$, $j = 1, 2$. Notice that the foci of $G$ on $\Pi$ are defined by the additional equation:

\[(6.3) \quad \det(a_{i,j} \cdot x)_{i,j=1,2} = 0\]

We can now assume that the homogeneous coordinates $p = [p_0, ..., p_4]$ of $p$ are also regular functions $p(u_1, u_2)$ of $(u_1, u_2)$. Hence, by differentiating the equations (6.2) with $x = p$, we get:

\[(6.4) \quad a_{i,j} \times p + a_i \times p_j = 0, \quad i, j = 1, 2\]

Let

\[(6.5) \quad f(x) = 0\]

be the equation of $X$. The equation of $T_pX$ is given by:

\[(6.6) \quad \sum_{h=0}^4 \frac{\partial f}{\partial x_h}(p)x_h = 0\]

By differentiating equation (6.5) with $x = p$, we also get:

\[(6.7) \quad \sum_{h=0}^4 \frac{\partial f}{\partial x_h}(p)p_{h,j} = 0, \quad j = 1, 2\]

Let $t = [t_1, t_2]$ be homogeneous coordinates on $\mathbb{P}^1$. By the hypothesis, $T_pX$ contains $\Pi$. Thus there is a regular function $t(u_1, u_2)$ of $(u_1, u_2)$ such that the equation (6.6) of $T_pX$ is also given by:

\[(t_1a_1 + t_2a_2) \times x = 0\]
By taking into account equation (6.3), we find:

\[(t_1 a_1 + t_2 a_2) \times p_j = 0, j = 1, 2\]

and using equation (6.4) we get:

\[(t_1 a_{1,j} + t_2 a_{2,j}) \times p = 0, j = 1, 2\]

The assertion follows by the equation 6.3 for the focal locus on \(\Pi\).

Now we can prove our first proposition.

**Proof of Proposition 5.14.** Suppose there is an irreducible subvariety \(F \subset \mathbb{G}(2, r)\) of dimension 2, such that the general point \(p \in F\) corresponds to a plane \(\Pi \subset \mathbb{P}^r\) which is tangent to \(X\) along a curve \(\Gamma\) which is 2-filling. Make a general projection in \(\mathbb{P}^4\). We thus get a hypersurface \(X'\) in \(\mathbb{P}^4\), the projection of \(X\), such that there is an irreducible subvariety \(G \subset \mathbb{G}(2, 4)\) of dimension 2, such that the general point \(p' \in G\) corresponds to a plane \(\Pi' \subset \mathbb{P}^4\), the projection of \(\Pi\), which is tangent to \(X'\) along a curve \(\Gamma'\), the projection of \(\Gamma\). By Lemma 6.3, every point of \(\Gamma'\) is a focus for the family \(G\) on \(\Pi'\). Hence \(\text{deg}(\Gamma) = \text{deg}(\Gamma') \leq 2\).

Assume that \(\text{deg}(\Gamma) = 1\). By lemma 6.2, we have \(N_{\Gamma'X} = \mathcal{O}_\Gamma(a) \oplus \mathcal{O}_\Gamma(b)\) with \(a, b \geq 0\) and \(\Gamma\) is unobstructed. By the results of \(\text{DC}\), we have an injection \(\mathcal{O}_\Gamma(1) \rightarrow N_{\Gamma'X}\), which implies \(a + b \geq 1\). Then \(X\) has a 3-dimensional family of lines at least, hence it is either \(\mathbb{P}^3\), or a quadric in \(\mathbb{P}^4\) or a scroll over a curve (see \(\text{Re}\)), a contradiction.

Assume now \(\text{deg}(\Gamma) = 2\). We may suppose \(\Gamma\) to be irreducible, hence \(\Gamma \simeq \mathbb{P}^1\). Set \(N_{\Gamma'X} = \mathcal{O}_{p_1}(a) \oplus \mathcal{O}_{p_1}(b)\). As above, we have \(a, b \geq 0\) and \(\Gamma\) is unobstructed. If \(a + b \geq 4\) then \(X\) has a 6-dimensional family of conics at least. This means that its general hyperplane section has a 3-dimensional family of conics (see \(\text{MP}\) Lemma 2.3)), hence \(X\) is a quadric in \(\mathbb{P}^4\), a contradiction. By the results of \(\text{DC}\) we have an injection \(\mathcal{O}_\Gamma(2) \rightarrow N_{\Gamma'X}\), which is incompatible with \(a + b \leq 3\).

**Remark 6.4.** If \(X\) is a smooth quadric in \(\mathbb{P}^4\), then for every line \(L\) on \(X\) there is a unique plane \(\Pi\) tangent to \(X\) along \(L\). Namely \(\Pi = \cap_{x \in L} T_x X\).

On the other hand one may prove that the hypothesis \(X\) not a scroll over a curve, in the previous proposition, is unnecessary. We will not dwell on this here.

Let \(X \subset \mathbb{P}^r, r \geq 7\), be a smooth, irreducible projective 3-fold which is not a scroll over a curve. Suppose that \(X\) is ruled by lines. Let \(x \in X\) be a general point and let \(L\) be a line through \(x\). Let \(\Sigma\) be a general tangent hyperplane section of \(X\) at \(x\). Notice that \(\Sigma\) contains \(L\). Let \(\tau : \Sigma \dashrightarrow \mathbb{P}^{r-4}\) be the projection of \(\Sigma\) from \(T_x X\). Note that \(\Sigma\) has a single node at \(x\) (see \(\text{Ro}\) Theorems 2.1]), thus it is normal and therefore \(\tau\) is well defined at the general point of \(L\).

We pay all our debts with the following enlarged version of Proposition 5.13.

**Proposition 6.5.** In the above setting, \(\tau\) contracts \(L\) to a point.

**Proof.** Let \(y \in L\) be the general point of \(L\). It is clear that \(\tau(y)\) is the intersection of the target \(\mathbb{P}^{r-5}\) with the \(\mathbb{P}^4\) spanned by \(T_y X\) and by \(T_y \Sigma\). If \(H\) is the hyperplane cutting \(\Sigma\) on \(X\), one has \(T_y X = T_y H \cap H\).

The variety \(S_{L,X} = \cup_{y \in L} T_y X\) is the so-called Segre cone of \(X\) along \(L\) (see \(\text{Ro}\) chapter 1, §3.1]). This is a quadric cone with vertex \(L\) lying in a \(\mathbb{P}^5\) containing \(L\) (see \(\text{Ro}\) chapter 1, §3.4]). A general tangent hyperplane \(H\) at \(x\) cuts \(S_{L,X}\).
along the union of $T_xX$ with another 3-space $\Pi$, intersecting $T_xX$ along a plane containing $L$. Thus $\Pi$ and $T_xX$ span a $\mathbb{P}^4$, whose projection from $T_xX$ is a fixed point. This proves that $\tau(y)$ stays fixed as $y$ varies on $L$. \hfill \Box

7. Classification of smooth 3-folds with one apparent double point

In this section we apply the previous results to classify smooth $OADP$ 3-folds. Our main result is the following theorem:

**Theorem 7.1.** Let $X \subset \mathbb{P}^7$ be an $OADP$ 3-fold. Let $d$ be the degree of $X$. Then:

i) either $d = 5$ and $X$ is a scroll in planes of type $S(1^2, 3)$ or $S(1, 2^2)$ as in Example 2.4,

ii) or $d = 6$, and $X$ is an Edge variety as in Example 2.4,

iii) or $d = 8$ and $X$ is a scroll in lines over $\mathbb{P}^2$ as described in Example 2.4.

Since we have shown before that $OADP$ 3-folds have degree $d$ less than or equal to 9, then these varieties are contained in the lists of [I01], [I03] and [I04]. Hence it could be possible to deduce Theorem 7.1 from Ionescu’s results. However we have already described, on the way, several geometrical properties of $OADP$ 3-folds. This simplifies a lot the classification. To finish the proof of the classification we only need a few basic properties of generalized adjunction maps and some of the ideas already described. Since we have shown before that $|K_X + 2H|$ is base point free and that it gives a morphism $\phi = (K_X + 2H): X \to \mathbb{P}^{g - 1}$, the so-called adjunction map of the polarized pair $(X, H)$.

The following proposition will conclude the classification.

**Proposition 7.2.** Let $X \subset \mathbb{P}^7$ be a smooth $OADP$ 3-fold of degree $d \geq 8$. Then $d = 8$ and $X$ is like in Example 2.4.

**Proof.** Assume $d = 8$ and therefore $g = 3$. Let $\phi: X \to \mathbb{P}^2$ be the adjunction map. Since $\phi(X)$ is non-degenerate, then either $\phi(X)$ is a curve or $\phi(X) = \mathbb{P}^2$. If $\phi(X)$ were a curve, then $X$ would be a hyperquadric fibration by [I01] Proposition 1.11] and by Proposition 2.4 this is impossible. Thus $\phi: X \to \mathbb{P}^2$ is surjective and endows $X$ with a structure of $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ (see again [I01]).

We now show that $X$ is like in Example 2.7. Let $H$ be a general hyperplane section of $X$. Then applying $\phi_*$ to the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H) \to \mathcal{O}_H(H) \to 0$$

we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \phi_*(\mathcal{O}_H(H)) \to R^1\phi_*(\mathcal{O}_X) = 0,$$

where $\mathcal{E}$ is locally free of rank 2 and $X = \mathbb{P}(\mathcal{E})$.

Since $C = H^2$ is a smooth curve of genus 3 and degree 8 on $H$, which is also smooth, we have that $C = \phi^*(\mathcal{O}_{\mathbb{P}^2}(4) - E_1 - \ldots - E_8)$, where $p_i = \phi(E_i)$ are 8 distinct points. Of course $E_i$ is a $(-1)$-curve and $C \cdot E_i = 1$, for all $i = 1, \ldots, 8$. 


It follows that \( \phi_*(\mathcal{O}_X(H)) = T_{p_1} \cdots T_{p_6}(4) \). Since \( H \) is smooth, the 8 points do not lie on a conic and no four point are collinear. Moreover, we claim that, by the generality of \( H \), we can suppose that no seven points \( p_i \) lie on a conic. Once this claim is proved we can conclude that \( X \) is like in Example \( 2.7 \).

To prove the claim we recall that \( Pic(X) = <H, \phi^*(\mathcal{O}_{P^2}(1))> \). If the claim were not true, for the general \( H \) we would have a line \( l_H \subset H \), the strict transform on \( H \) of the conic through the seven points \( p_i \). Notice that there would be a \( r \)-dimensional family of lines \( l_H \) on \( X \), with \( r \geq 2 \). Clearly \( H \cdot l_H = 1 \) and \( \phi^*(\mathcal{O}_{P^2}(1)) \cdot l_H = 2 \).

Remark that \( h^0(\mathcal{O}_X(H) \otimes \phi^*(\mathcal{O}_{P^2}(1)))) = h^0(\mathcal{E}(-1)) = 2 \), thus the system \( |H + \phi^*(\mathcal{O}_{P^2}(-1))| \) is a pencil. Take a divisor \( D \in |H + \phi^*(\mathcal{O}_{P^2}(-1))| \). From \( D \cdot l_H = -1 \) we conclude that the lines \( l_H \) cannot fill up \( X \), hence they fill up a plane \( \Pi \) which sits in the base locus \( F \) of the pencil \( |H + \phi^*(\mathcal{O}_{P^2}(-1))| \) and surjects onto \( P^2 \) via \( \phi \). Call \( |M| \) the movable part of the pencil. Set \( l_p = \phi^{-1}(p) \) for a \( p \in P^2 \). The relations \( 1 = D \cdot l_p = M \cdot l_p + F \cdot l_p \) and \( F \cdot l_p > 0 \) would imply \( M \cdot l_p = 0 \) so that \( M = \phi^*(\mathcal{O}_{P^2}(a)) \), \( a > 0 \). Then we would have \( 2 = h^0(X, \mathcal{O}_X(M)) = h^0(P^2, \mathcal{O}_{P^2}(a)) \), which is impossible.

Let us now suppose \( d \geq 9 \). Then by Theorem \( 5.4 \) \( d = 9 \) and therefore \( g = 4 \). Then \( 1 \leq \text{dim}(\phi(X)) \leq 3 \).

If \( \text{dim}(\phi(X)) = 1 \), then \( X \) would be a hyperquadric fibration and this is in contrast with Proposition \( 2.4 \).

If \( \text{dim}(\phi(X)) = 2 \), then the adjunction map would determine a structure of scroll in lines over a smooth surface, see loc. cit.. This is not possible by Theorem \( 5.1 \).

Let us now exclude the case \( \phi(X) = P^3 \). Since the sectional genus is \( g = 4 \) in this case, the only possibility left out by \( [o2 \ Theorem II] \) is that \( X \) could be the blow-up at finitely many points of a \( P^2 \)-bundle over \( P^1 \), which we denote by \( X' \). Moreover \( \phi \) would factor through \( X' \) and, by \([o2 \ Lemma 1.2]\) we would have a finite morphism \( \phi': X' \rightarrow P^3 \) embedding each fiber of the projection \( X' \rightarrow P^1 \) as a plane, which is clearly impossible.

We are finally ready for the proof of our classification theorem.

Proof of Theorem \( 7.4 \). We know by theorem \( 5.1 \) that \( 5 \leq d \leq 9 \). If \( d = 5 \), then \( X \) is either \( S(1, 2^2) \) or \( S(1^2, 3) \) by lemmas \( 2.2 \) and \( 2.3 \).

Then we can suppose \( 6 \leq d \leq 9 \), or equivalently \( 1 \leq g \leq 4 \) and \( h^0(K_X + 2H) = g \). Consider the adjunction morphism \( \phi = \phi_{K_X + 2H} : X \rightarrow P^{g-1} \).

If \( g = 1 \), then \( K_X = -2H \) and \( X \) is a Del Pezzo manifold of degree \( 6 \). Iskovskikh has shown that either \( X \) is the Segre variety \( Seg(1^3) \), i.e. the Edge 3-fold of degree \( 6 \) (see Example \( 2.4 \), or \( X = P(T_{P^2}) \), see \([s1]\) and \([s2]\). The last variety is the hyperplane section of \( Seg(2^2) \) and its secant variety is a hypersurface of degree 3, i.e. \( P(T_{P^2}) \) is defective and therefore is not an OADP-variety (see \([Za1], [CC2]\)).

If \( g = 2 \), then \( \phi : X \rightarrow P^1 \) gives to \( X \) the structure of hyperquadric fibration over \( P^1 \), see \([lo1 \ Proposition 1.11]\). By Proposition \( 2.6 \) \( X \) is an Edge variety.

If \( 3 \leq g \leq 4 \) we can apply Lemma \( 7.2 \) to conclude.

8. Mukai varieties with one apparent double point

In this section we apply Theorem \( 7.4 \) to the classification of smooth \( n \)-dimensional OADP-varieties \( X \subset P^{2n+1} \) having as a general curve section a canonical curve \( C \subset P^{n+2} \). Since \( X \) is regular and linearly normal, then \( C \) is linearly normal with
varieties with one apparent double point

By Theorem 7.1 we can suppose \( n \geq 4 \). Moreover, by the adjunction formula, \( -K_X = (n-2)H \), i.e. \( X \) is a Fano variety of coindex 3. These varieties have been recently called Mukai varieties. Mukai varieties have been classified in [Mu1] under the assumption of the existence of a smooth divisor in \( |H| \), a condition which is clearly satisfied in our case. In any event, this restriction has been removed in [Me].

We notice that the OADP-variety in the Example 2.8 and the one \( G_{\mathbb{R}}(2,5) \) from Example 2.9 are Mukai varieties.

The classification result of Mukai OADP-varieties is the following.

**Theorem 8.1.** Let \( X \subset \mathbb{P}^{2n+1} \) be a Mukai OADP-variety of dimension \( n \). Then either \( n = 4 \) and \( X \subset \mathbb{P}^9 \) is a linear section of the spinor variety \( S^{(4)} \subset \mathbb{P}^{15} \) as described in Example 2.8, or \( n = 6 \) and \( X = G_{\mathbb{R}}^{l}\text{lag}(2,5) \).

**Proof.** By theorems 4.10 and 7.1 we can suppose \( n \geq 4 \).

If \( b_2(X) = 1 \), then \( X \) is (a linear section of) a homogeneous manifold as described in theorems 1 and 2 of [Mu1] and necessarily \( g = 7, 8, 9 \). The first and third case correspond to the cases in the statement of the theorem. The Mukai manifold of dimension 8 and of sectional genus 8 is the Grassmannian \( G(1,5) \subset \mathbb{P}^{14} \), whose secant variety is a cubic hypersurface in \( \mathbb{P}^{14} \) (see [Za1]). Therefore its linear section with a general \( \mathbb{P}^9 \) is a defective variety \( X_5 \subset \mathbb{P}^{11} \).

We now show that the case \( b_2(X_n) \geq 2 \) is impossible. By [Mu1, Theorem 7] we know that either \( n = 4 \) and \( X \) is a Mukai 4-fold of product type, i.e. isomorphic to \( \mathbb{P}^1 \times Y \) with \( Y \) a Fano 3-fold of index 2 or 4, or \( X \) is (a linear section of) one of the nine varieties described in example 2 of [Mu1]. In all these cases one never has an \( n \) dimensional variety \( X \) in \( \mathbb{P}^{2n+1} \), except when either \( X \) is the Segre embedding of \( \mathbb{P}^1 \times F \), with \( F \) a smooth cubic hypersurface in \( \mathbb{P}^4 \), or \( X \) is the complete intersection of the cone over \( \text{Seg}(22) \) from a point with a quadric in \( \mathbb{P}^9 \). In both cases one has \( n = 4 \). However they do not lead to OADP-varieties. Indeed the former variety sits in \( \text{Seg}(1,4) \) as a divisor of type \((0,3)\) and, by Proposition 2.3, it is not an OADP-variety. In the latter case \( X \) is defective as well as \( \text{Seg}(22) \) (to see this, argue as in [CC2, Example 2.4]). \( \Box \)

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