TATE CONJECTURE AND MIXED PERVERSE SHEAVES

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Abstract. Using the theory of mixed perverse sheaves, we extend arguments on
the Hodge conjecture initiated by Lefschetz and Griffiths to the case of the Tate
conjecture, and show that the Tate conjecture for divisors is closely related to the
de Rham conjecture for nonproper varieties, finiteness of the Tate-Shafarevich
groups, and also to some conjectures in the analytic number theory.

Dedicated to John Tate

Introduction

Let $k$ be a finitely generated field over $\mathbb{Q}$, $\overline{k}$ an algebraic closure of $k$, and $G_k = \text{Gal}(\overline{k}/k)$. Let $X$ be a smooth projective variety of pure dimension $n$ over $k$, and $X_{\overline{k}} = X \otimes_k \overline{k}$. We will denote by $\text{TC}(X/k, p)$ the Tate conjecture ([35], [37]) which
states the surjectivity of the cycle map

\[ (0.1) \quad cl : \text{CH}^p(X) \otimes \mathbb{Q}_l \to H^{2p}(X_{\overline{k}}, \mathbb{Q}_l(p))^{G_k}. \]

Here $\text{CH}^p(X)$ is the Chow group of algebraic cycles of codimension $p$ on $X$, and the
right-hand side is the invariant part of the (Tate twisted) étale cohomology group
by the action of $G_k$. Let $D$ be a smooth divisor on $X$, and set

\[ H^j(D_{\overline{k}}, \mathbb{Q}_l)_{X} = \text{Coker}(H^j(X_{\overline{k}}, \mathbb{Q}_l) \to H^j(D_{\overline{k}}, \mathbb{Q}_l)), \]
\[ H^j(X_{\overline{k}}, \mathbb{Q}_l)^D = \text{Ker}(H^j(X_{\overline{k}}, \mathbb{Q}_l) \to H^j(D_{\overline{k}}, \mathbb{Q}_l)), \]

so that we have an exact sequence compatible with the Galois action

\[ (0.2) \quad 0 \to H^{2p-1}(D_{\overline{k}}, \mathbb{Q}_l)_{X} \to H^{2p}_c(D_{\overline{k}} \setminus D_{\overline{k}}, \mathbb{Q}_l) \to H^{2p}(X_{\overline{k}}, \mathbb{Q}_l)^D \to 0. \]

For $c \in (H^{2p}(X_{\overline{k}}, \mathbb{Q}_l(p))^D)^{G_k} (= \text{Hom}_{G_k}(\mathbb{Q}_l, H^{2p}(X_{\overline{k}}, \mathbb{Q}_l(p))^D))$, we denote by
\[ e(c) \in \text{Ext}^1(\mathbb{Q}_l, H^{2p-1}(D_{\overline{k}}, \mathbb{Q}_l(p))_{X}) \]
the extension class in the category of $\mathbb{Q}_l$-modules with action of $G_k$, which is
obtained by taking the pull-back of (0.2) by $c$ (see [26], [28] for the Hodge case). Let
\[ \text{CH}^p_{\text{hom}}(D) = \text{Ker}(cl : \text{CH}^p(D) \to H^{2p}(D_{\overline{k}}, \mathbb{Q}_l(p))). \]

By a similar argument, we get the Abel-Jacobi map
\[ \text{CH}^p_{\text{hom}}(D) \to \text{Ext}^1(\mathbb{Q}_l, H^{2p-1}(D_{\overline{k}}, \mathbb{Q}_l(p))), \]

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which has been obtained in [19], see also (2.1) below. In the Hodge setting, this construction is essentially due to Deligne, see [14] (and also [26], [28]). Taking the composition with the canonical projection we get

\[(0.3) \quad \text{CH}^p_{\text{hom}}(D) \otimes \mathbb{Q}_l \to \text{Ext}^1(\mathbb{Q}_l, H^{2p-1}(D, \mathbb{Q}_l(p)))_X,\]

where the extension group may be replaced with Galois cohomology. We have the following

0.4. Conjecture. For any \(c \in (H^{2p}(X_{\overline{K}}, \mathbb{Q}_l(p))^D)^{G_k}\), the above extension class \(e(c)\) belongs to the image of (0.3).

Note that (0.4) follows from TC\((X/k, p)\), because \(e(c)\) coincides with the image of the restriction of \(\zeta\) to \(D\) by (0.3) if \(c\) is the cycle class of a cycle \(\zeta\) on \(X\), see [28], 1.8. Conversely, we can reduce the Tate conjecture to (0.4) using a Lefschetz pencil (by induction on \(\dim X\)), where \(D\) is the generic fiber of the Lefschetz pencil and the base field \(k\) is replaced by the rational function field \(k(t)\) of one variable.

Indeed, \(X\) can be embedded in a projective space, and we have a Lefschetz pencil \(f : \tilde{X} \to S = \mathbb{P}^1\), where \(\pi : \tilde{X} \to X\) is the blow up along the intersection \(A\) of two general hyperplane sections defined over \(k\). Then \(\pi \times f : \tilde{X} \to X \times_k S\) is a closed embedding so that the closed fibers of \(\overline{f} = f \otimes_k \overline{k} : \tilde{X}_{\overline{k}} \to S_{\overline{k}}\) are identified with the hyperplane sections of \(X_{\overline{k}}\) containing \(A_{\overline{k}}\). For a closed point \(s\) of \(S\), we will denote by \(X_s\) the fiber of \(f\) over \(s\). The generic fiber of \(f\) will be denoted by \(Y\). It is smooth projective over \(\text{Spec} K\) with \(K : = k(S) = k(t)\). Let \(X_K = X \otimes_k K\) so that \(Y\) is a closed subvariety of \(X_K\). Let \(U\) be a nonempty open subvariety of \(S\) on which \(f\) is smooth, and \(|U|\) the set of closed points of \(U\). Then we can prove (see (2.5)):

0.5. Lemma. (i) In case \(n < 2p\), TC\((X/k, p)\) is true if TC\((X_s/k(s), p-1)\) is true for some \(s \in |U|\).

(ii) In case \(n > 2p\), TC\((X/k, p)\) is true if TC\((Y/K, p)\) and TC\((X_s/k(s), p-1)\) are true for some \(s \in |U|\).

So the Tate conjecture is reduced to the case \(n = 2p\) by induction. Assume the projective embedding of \(X\) is sufficiently ample so that we have by [20], 6.3, 6.4

\[(0.6) \quad H^{n-1}(Y_{\overline{k}}, \mathbb{Q}_l)_X \neq 0,\]

which is equivalent to the constantness of \(R^j f_* \mathbb{Q}_l\) on \(S_{\overline{k}}\) for \(j \neq n-1\), see (2.4) below. We have the following (see (2.6) below):

0.7. Theorem. For \(n = 2p\), TC\((X/k, p)\) is true, if the following two conditions are satisfied:

(i) TC\((X_s/k(s), p-1)\) is true for some \(s \in |U|\).

(ii) (0.4) is true for \(Y \subset X_K\) over \(K\).

Here we choose an embedding \(\overline{k} \to \overline{K}\) so that \(G_k\) is identified with a quotient of \(G_K := \text{Gal}(\overline{K}/K)\), because \(K \cap \overline{k} = k\). We may assume \(s \in U(k)\) replacing \(k\) with
a finite Galois extension (using the Galois action) if necessary. For \( s \in U(k) \) we have the canonical isomorphism
\[
(H^2p(X_K, \mathbb{Q}_l)(p))^G_L = (H^2p(X_{\bar{K}}, \mathbb{Q}_l(p))^Y)^G_L.
\]
In particular, the Tate conjecture TC\((X/k, p)\) implies condition (ii) by the remark after (0.4). For the proof of (0.7), we use the Leray spectral sequence
\[
E_2^{i,j} = H^i(S_F, R^jT_s\mathbb{Q}_l) \Rightarrow H^{i+j}(\bar{X}_F, \mathbb{Q}_l),
\]
which is compatible with the Galois action, and degenerates at \( E_2 \). The Hodge analogue of (0.7) is given in [26], [28], see also [22] and [41], etc.

In the case of divisors (i.e. \( p = 1 \)) with \( n = 2 \), S. Bloch and K. Kato informed us that the conjecture (0.4) in the case \( k \) is a number field is closely related with the finiteness of the \( l \)-primary torsion part of the Tate-Shafarevich group. Then J. Nekovar told us that it is also related with the de Rham conjecture for nonproper varieties [15] after I explained him the construction of \( e(c) \), see also [24]. Indeed, using the Kummer sequence, we get the isomorphism
\[
H^1(D_{\bar{F}}, \mathbb{Z}_l(1)) = T_lJ_D(\bar{k}),
\]
and similarly for \( X \), where \( J_D \) denotes the Picard variety of \( D = Y_s \), and \( T_l \) the Tate module. Let
\[
J_{D,X} = \text{Coker}(J_X \to J_D), \quad E_{D,X} = H^1(D_{\bar{F}}, \mathbb{Z}_l(1))_X,
\]
and \( V_{D,X} = E_{D,X} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \), so that
\[
E_{D,X} = T_lJ_{D,X}(\bar{k}).
\]
Then we have the short exact sequence
\[
0 \to J_{D,X}(k) \otimes \mathbb{Z}_l \to H^1(G_k, E_{D,X}) \to T_lH^1(G_k, J_{D,X}(\bar{k})) \to 0,
\]
using Galois cohomology, and the conjecture (0.4) is equivalent to the assertion that \( e(c) \) belongs to the image of \( J_{D,X}(k) \otimes \mathbb{Q}_l \) by the first morphism of (0.8).

As remarked by Bloch, Kato and Nekovar, the conjecture (0.4) is thus reduced to the conjecture on the finiteness of the \( l \)-primary torsion part of the Tate-Shafarevich group of \( J_{D,X} \) and the de Rham conjecture for nonproper varieties, using the theory of Bloch-Kato on \( H^1_\eta \) ([3], Remark before 3.8), see (3.5) below. (In the case \( k \) is a finite field, the relation between the Tate conjecture for \( X \) and the finiteness of the Tate-Shafarevich group of the Jacobian of the generic fiber of \( X/S \) was proved by Tate [36] where \( S \) is a curve over which \( X \) is dominant.)

The above two conjectures are, however, still insufficient to deduce the Tate conjecture in our case. Indeed, let
\[
J_{X_s,K} = \text{Coker}(J_X \otimes_k k(s) \to J_{X_s}), \quad J_{Y,K} = \text{Coker}(J_X \otimes_k K \to J_Y),
\]
and \( L_{\mathbb{Q}_l} \) the quotient of \( R^1f_*\mathbb{Q}_l(1) \vert_U \) by the geometrically constant part so that \( L_{\mathbb{Q}_l} \) is identified with \( (T_lJ_{X_s,K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \), see (3.1). We define \( \bar{e}(c) \in \text{Ext}^1(\mathbb{Q}_l, L_{\mathbb{Q}_l}) \) for \( c \in (H^2(X_{\bar{K}}, \mathbb{Q}_l(1))^{X_s})^{G_L} \) using the sheaf version of the short exact sequence (0.2). (Note that \( H^2(X_{\bar{K}}, \mathbb{Q}_l(1))^{X_s} \) is independent of \( s \in U \).) Then the restriction of \( \bar{e}(c) \) to \( s \in \vert U \vert \) coincides with \( e(c) \) in (0.4). Here the stalk of \( L_{\mathbb{Q}_l} \) at a geometric
point over $s$ is $V_{X_s,X}$ by the invariant cycle theorem. Applying the above remarks of Bloch, Kato and Nekovar to the smooth closed fibers $X_s$ of the Lefschetz pencil, and assuming the conjectures mentioned there, we would get $\zeta_s \in J_{X_s,X}(k(s)) \otimes \mathbb{Q}_l$ whose image by the Abel-Jacobi map coincides with the restriction of $\tilde{e}(c)$ to $s$ (in the case $k$ is a number field). But it is still unclear whether the $\zeta_s$ for $s \in |U|$ determine an element of $J_{Y,X,K}(K) \otimes \mathbb{Q}_l$. Using exact sequences similar to (0.8) for $s \in |U|$ together with the canonical morphism of short exact sequences, this problem is equivalent to

0.9. Conjecture. $T_l H^1(G, J_{Y,X,K}(K)) \to \prod_{s \in |U|} T_l H^1(G_s, J_{X_s,X}(k))$ is injective.

Here $G = \text{Gal}(\tilde{K}/K)$ with $\tilde{K}$ the maximal subfield of $K$ that is unramified over $U$, and $G_s = \text{Gal}(\tilde{K}/k(s))$. (Actually the Tate conjecture is equivalent to $T_l H^1(G, J_{Y,X,K}(K)) = 0$, see Remark (3.4)(ii).) Set $E = \Gamma(\text{Spec } \tilde{K}, \mathcal{L}), V = E \otimes \mathbb{Z}_l \otimes \mathbb{Q}_l$. As a much weaker (and easier) version of (0.9), we have at least the injectivity of

\[(0.10)\quad H^1(G, V) \to \prod_{s \in |U|} H^1(G_s, V).\]

This is indispensable for not losing information by taking the restrictions to the closed points of $U$ (see [26] for the Hodge case). This injectivity is informed from A. Tamagawa in a more general case, using Hilbert’s irreducibility theorem and the theory of Frattini subgroups (see also [30]). We are also informed that a similar argument was used in [38]. In our case, however, this injectivity follows almost immediately from arguments in [33], 9.1, see Remark (3.2)(iii) below. Furthermore we can prove (see (3.3)):

0.11. Proposition. There exists a thin subset $\Sigma$ of $U(k)$ in the sense of [33] such that $H^1(G, E) \to H^1(G_s, E)$ is injective for $s \in U(k) \setminus \Sigma$.

This is an analogue of Néron’s injectivity theorem (see [21], [33]). As a corollary of (0.11), we can solve (0.9) using exact sequences similar to (0.8), if

\[(0.12)\quad \text{rank } J_{Y,X,K}(K) = \text{rank } J_{X_s,X}(k) \text{ for some } s \in U(k) \setminus \Sigma.\]

Note that the last condition is not satisfied for certain elliptic surfaces over $\mathbb{P}^1$ (see [4] assuming Selmer’s conjecture [29]). However, this might occur only in the isotrivial case, assuming some conjectures in the analytic number theory, see Appendix of [5] for details.

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In Section 1 we explain some basic facts from the theory of $l$-adic mixed perverse sheaves. Using this we prove (0.5), (0.7) in Section 2. The divisor case is treated in Section 3.

In this paper, a variety means a separated scheme of finite type over a field.
1. Mixed Perverse Sheaves

Since the theory of mixed perverse sheaves is presented in [2] only for varieties over a finite field, we give a short account in the case of varieties $X$ defined on a finitely generated field $k$ over $\mathbb{Q}$ (see [8], [12], [19] for the case of $X = \text{Spec} \ k$). In this paper we restrict to the characteristic zero case. In the case $k$ is finitely generated over a finite field, [2] would be essentially sufficient taking a formula similar to (1.16.3) for definition.

1.1. Definition. Let $k$ be a field of characteristic zero, $\overline{k}$ an algebraic closure of $k$, and $G_k = \text{Gal}(\overline{k}/k)$. Let $X$ be a variety over $k$, and $l$ a prime number. Let $\operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l)$ be the category of perverse sheaves on $X_{\overline{k}}$ with $\mathbb{Q}_l$-coefficients, see [2]. 2.2.18. Let $\operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$ denote the category of objects of $\operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l)$ endowed with an action of $G_k$, i.e. an object $\mathcal{F}$ of $\operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l)$ consists of $(\mathcal{F}_{\overline{k}}, u)$, where $\mathcal{F}_{\overline{k}} \in \operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l)$ and $u$ is a collection of isomorphisms $u(\gamma) : \gamma^* \mathcal{F}_{\overline{k}} \to \mathcal{F}_{\overline{k}}$ for $\gamma \in G_k$,

satisfying the compatibility

(1.1.1) $u(\gamma \gamma') = u(\gamma) \circ \gamma^* u(\gamma').$

Here $\gamma$ denotes also the (contravariant) action of $\gamma \in G_k$ on $X_{\overline{k}} = X \otimes_k \overline{k}$ defined by the base change.

Similarly we denote by $\operatorname{Sh}_{\operatorname{sm}}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$ the category of (étale) smooth $\mathbb{Q}_l$-sheaves on $X_{\overline{k}}$ with an action of $G_k$ as above. Here a smooth sheaf means that it corresponds to an $l$-adic representation, i.e. a continuous morphism $\pi_1(X_{\overline{k}}) \to \text{Aut}(V)$ where $V$ is a finite dimensional $\mathbb{Q}_l$-vector spaces, see e.g. [23]. (The base point of the fundamental group is an algebraic closure of the function field unless otherwise stated.)

For $\mathcal{F} = (\mathcal{F}_{\overline{k}}, u) \in \operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$, $\mathcal{F}_{\overline{k}}$ will be called the underlying perverse sheaf on $X_{\overline{k}}$. We have the same for $\mathcal{L} = (\mathcal{L}_{\overline{k}}, u) \in \operatorname{Sh}_{\operatorname{sm}}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$.

1.2. Remarks. (i) In the above definition, there is no condition on the continuity of the action of $G_k$. Later we will consider full subcategories of $\operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$ where the continuity of the action follows from other conditions. (See [2], 5.1.2 for the finite field case.) This is considered in order to prove an analogue of [2], 5.3.8 (see (1.12)(ii)) below).

(ii) If $X$ is smooth and pure dimensional, let $\operatorname{Sh}_{\operatorname{sm}}(X, \mathbb{Q}_l)$ denote the category of $l$-adic smooth sheaves on $X$ (which correspond to $l$-adic representations of $\pi_1(X)$). Then we have fully faithful functors

(1.2.1) $\operatorname{Sh}_{\operatorname{sm}}(X, \mathbb{Q}_l) \to \operatorname{Sh}_{\operatorname{sm}}(X_{\overline{k}}, \mathbb{Q}_l; G_k) \to \operatorname{Perv}(X_{\overline{k}}, \mathbb{Q}_l; G_k),$

where the last functor associates $\mathcal{L} [\dim X]$ to $\mathcal{L} \in \operatorname{Sh}_{\operatorname{sm}}(X_{\overline{k}}, \mathbb{Q}_l; G_k)$. The full faithfulness of the first functor is proved by using

$$H^0(X, \mathcal{H}om(\mathcal{L}, \mathcal{L}')) = H^0(X_{\overline{k}}, \mathcal{H}om(\rho^* \mathcal{L}, \rho^* \mathcal{L}'))^{G_k},$$

where $\mathcal{L}, \mathcal{L}'$ are smooth sheaves and $\rho : X_{\overline{k}} \to X$ is the canonical morphism. (This will be used for example in (1.12)(ii).)
(iii) The forgetful functor
\[
Perv(X_{\mathbb{F}}, \mathbb{Q}_l; G_k) \rightarrow Perv(X_{\overline{\mathbb{F}}}, \mathbb{Q}_l)
\]
is exact and faithful. This induces the forgetful functor
\[
D^b Perv(X_{\mathbb{F}}, \mathbb{Q}_l; G_k) \rightarrow D^b Perv(X_{\overline{\mathbb{F}}}, \mathbb{Q}_l) \rightarrow D^b_c(X_{\overline{\mathbb{F}}}, \mathbb{Q}_l),
\]
where the last functor is given in [2], 3.1.9.

(iv) Let \(X_{\text{red}}\) be the reduced variety associated with \(X\). Then we have an equivalence of categories
\[
Perv(X_{\mathbb{F}}, \mathbb{Q}_l; G_k) = Perv((X_{\text{red}})_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k).
\]

1.3. Proposition. We have the canonical functors \(f_*, f_!, f^*, f^!, \psi_g, \varphi_g, \mathbb{D}, \boxtimes, \mathcal{H}om\) between the bounded derived categories \(D^b Perv(X_{\mathbb{F}}, \mathbb{Q}_l; G_k)\) for morphisms \(f\) of \(k\)-varieties and functions \(g\). Furthermore, these functors commute with the forgetful functor \((1.2.3)\).

Proof. The category of objects of \(D^b_c(X_{\overline{\mathbb{F}}}, \mathbb{Q}_l)\) with an action of \(G_k\) as in \((1.1.1)\) is stable by \(\boxtimes\), \(\mathcal{H}om\), and the pull-back by projections. So we get the assertion on the external product \(\boxtimes\) and the dual \(\mathbb{D}\).

For the direct images, we can apply the same argument as in [1] if we have the direct images of perverse sheaves with Galois action for the embedding of the complement of a Cartier divisor, see [27], 2.4. But the stability for this direct image easily follows from the above definition of perverse sheaves with Galois action.

The pull-backs can be defined as the adjoint functor of the direct images, and their existence is reduced to the case of a closed embedding where we can use Cech-type complexes associated with an affine open covering of the complement of the image, see [27], 3.3.

For the nearby and vanishing cycle functors, we can apply a generalization of Deligne’s construction [10] which uses finite determination sections, see [27], 5.2 and also Remark (1.4)(ii) below.

The last two functors are expressed using other functors.

1.4. Remarks. (i) Let \(g\) be a function on \(X\), and put \(Y = g^{-1}(0), X' = X \setminus Y\). Let \(Perv((X', Y, g)_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k)\) be the category whose objects are \((\mathcal{F}', \mathcal{F}'', u, v)\) where \(\mathcal{F}' \in Perv(X'_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k)\), \(\mathcal{F}'' \in Perv(Y_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k)\), and
\[
u : \psi_{g,1} \mathcal{F}' \rightarrow \mathcal{F}'', \quad \nu : \mathcal{F}'' \rightarrow \psi_{g,1} \mathcal{F}'(1),
\]
such that \(\nu \nu = N\). (Here \(\psi_{g,1}, \varphi_{g,1}\) denote the unipotent monodromy part of \(\psi_g, \varphi_g\).)

Then, by the Deligne-MacPherson-Verdier type extension theorem [39], we have an equivalence of categories
\[
Perv(X_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k) \sim \rightarrow Perv((X', Y, g)_{\overline{\mathbb{F}}}, \mathbb{Q}_l; G_k),
\]
induced by the functor
\[
\mathcal{F} \mapsto (\mathcal{F}|_{X'}, \varphi_{g,1} \mathcal{F}, \text{can}, \text{Var}).
\]
Indeed, an inverse functor is constructed by using the functor \(\xi_g\), see [25], 2.28.
Let \( \Perv(X, \mathbb{Q}; G_k)_{sm} \) (resp. \( \Perv((X', Y, g)_{\mathbb{Q}}; G_k)_{sm} \)) be the full subcategory of \( \Perv(X, \mathbb{Q}; G_k) \) (resp. \( \Perv((X', Y, g)_{\mathbb{Q}}; G_k) \)) defined by the condition: \( \mathcal{F}|_{X'} \) (resp. \( \mathcal{F}' \)) is smooth. Then (1.4.1) induces an equivalence of categories

\[
(1.4.2) \quad \Perv(X, \mathbb{Q}; G_k)_{sm} \xrightarrow{\sim} \Perv((X', Y, g)_{\mathbb{Q}}; G_k)_{sm}.
\]

So an object of \( \Perv(X, \mathbb{Q}; G_k) \) can be obtained by gluing smooth sheaves inductively.

(ii) We define the nearby and vanishing cycle functors \( \psi_g, \varphi_g \) so that they preserve perverse sheaves (i.e., they correspond to \( R\Psi[-1], R\Phi[-1] \) in [10]). With the notation of Remark (i) above, let \( j : X' \to X \) denote the inclusion, and \( g' : X' \to S' = \text{Spec} \ k[t, t^{-1}] \) the restriction of \( g \). Let \( E_i (i \geq 0) \) be a standard inductive system of indecomposable smooth sheaves on \( S' \) with a weight filtration \( W \) such that \( \text{Gr}^E_i E_i = \mathbb{Q}_l S'(-k) \) for \( j = 2k \) with \( 0 \leq k \leq i \) and 0 otherwise. These are constructed geometrically, see [27], 5.1. Then we have by definition ([27], 5.2)

\[
(1.4.3) \quad \psi_{g,1} \mathcal{F}' = \ker(j_!(j^* \mathcal{F} \otimes g^* E_i) \to j_*(j^* \mathcal{F} \otimes g^* E_i)) \quad \text{if} \quad i \gg 0.
\]

For \( \varphi_{g,1} \mathcal{F} \) we take the cohomology of the single complex associated to

\[
\begin{array}{ccc}
j_! j^* \mathcal{F} & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
j_!(j^* \mathcal{F} \otimes g^* E_i) & \longrightarrow & j_*(j^* \mathcal{F} \otimes g^* E_i).
\end{array}
\]

1.5. Definition. Assume \( k \) is finitely generated over \( \mathbb{Q} \). Let \( \Perv(X/k, \mathbb{Q}) \) denote the full subcategory of \( \Perv(X, \mathbb{Q}; G_k) \) consisting of objects \( \mathcal{F} \) satisfying the following condition (by increasing induction of \( \dim X \)):

For each point \( x \) of \( X \), there exists a Zariski-open neighborhood \( U \) of \( x \) in \( X \) together with a function \( g \) on \( U \) such that \( Y' := g^{-1}(0) \) has dimension \( < \dim X \), \( U' := U \setminus Y' \) is pure dimensional, \( \mathcal{F}|_{U'}[-\dim U'] \) is the image of \( \mathcal{L} \in \text{Sh}_{sm}(U', \mathbb{Q}) \), and \( \varphi_{g,1} \mathcal{F}|_U \in \Perv(Y'/k, \mathbb{Q}) \).

We define a full subcategory \( \Perv(X/k, \mathbb{Q})_{\text{gnr}} \) of \( \Perv(X/k, \mathbb{Q}) \) consisting of perverse sheaves generically unramified over \( k \) by the following conditions:

In the case \( X \) is pure dimensional and \( \mathcal{L} := \mathcal{F}[-\dim X] \) is a smooth sheaf, there exist a finitely generated \( \mathbb{Z}[[t]] \)-subalgebra \( R \) of \( k \) whose fractional field is \( k \) and an \( R \)-scheme \( X_R \) of finite type whose generic fiber over \( k \) is isomorphic to \( X \) and such that the \( l \)-adic representation corresponding to \( \mathcal{L} \) is unramified over \( X_R \), i.e. it factors through \( \pi_1(X_R) \).

In general, there exists for each \( x \in X \) a Zariski-open neighborhood \( U \) of \( x \) in \( X \) together with a function \( g \) on \( U \) such that \( Y' := g^{-1}(0) \) has dimension \( < \dim X \), \( U' := U \setminus Y' \) is pure dimensional, \( \mathcal{F}|_{U'}[-\dim U'] \) is a smooth sheaf and is generically unramified over \( k \), and \( \varphi_{g,1} \mathcal{F}|_U \in \Perv(Y'/k, \mathbb{Q})_{\text{gnr}} \) (by increasing induction on \( \dim X \)).

1.6. Remarks. (i) Let \( A \) a complete discrete valuation ring with the maximal ideal \( m \) such that \( A/m \) is a finite field of characteristic \( l \). Set \( A_i = A/m^{i+1} \), and let \( K \) be the fractional field of \( A \). Let \( (M_i)_{i \in \mathbb{N}} \) be a projective system of complexes
of $A_i$-modules such that $M_j^i = 0$ for $j \gg 0$ (independently of $i$), $H^j M_0$ are finite $A_0$-modules, and the transition morphism $M_{i+1} \to M_i$ induces an isomorphism in $D^{-}(A_i)$

\[(1.6.1) \quad M_{i+1} \otimes_{A_{i+1}}^L A_i \xrightarrow{\sim} M_i.\]

Then, as in [18], pp. 474–480, there exist a complex of finite $A$-modules $L$ which is bounded above and isomorphisms $L_i := L \otimes_A A_i \simeq M_i$ in $D^{-}(A_i)$ which are compatible with the isomorphisms (1.6.1) and the canonical isomorphisms $L_{i+1} \otimes_{A_{i+1}} A_i = L_i$. (Here we can take $L$ such that the differential of $L_0$ is zero.) Indeed, for a nonnegative integer $i$, set $R = A_i, R' = A_{i+1}$ in order to simplify the notation. Then it is enough to show the following:

Let $u : N \to L, v : N \to M$ be quasi-isomorphisms of complexes of flat $R$-modules which are bounded above, and $M'$ a complex of flat $R'$-modules which is bounded above. Assume the components of $L$ are finite free over $R$, and we have an isomorphism $M' \otimes_{R'} R = M$. Then there exist complexes of flat $R'$-modules $L', N'$ which are bounded above, together with morphisms $u : N' \to L', v : N' \to M'$ and isomorphisms of complexes $L' \otimes_{R'} R = L, N' \otimes_{R'} R = N$, such that $u \otimes_{R'} R$ and $v \otimes_{R'} R$ are identified with $u$ and $v$ up to homotopy.

This formulation may be slightly different from loc. cit. However the argument is essentially the same. Indeed, we may assume the components of $N, M'$ are projective over $R, R'$ by taking resolution, and furthermore $u, v$ are componentwise surjective by replacing $N$. Let $K = \text{Ker } u$. Then $K$ is acyclic and $N$ is identified with the mapping cone of $M[-1] \to K$. So the remaining argument is similar to loc. cit.

(ii) The theory of complexes of $l$-adic sheaves has become available in a general situation by T. Ekedahl, O. Gabber and U. Jannsen, see [13]. For our purpose, we can take the following formulation (which seems more down to the earth) by modifying some of the arguments in [7].

Let $X$ be a noetherian scheme on which $l$ is invertible, and $A, m, A_i, K$ be as in Remark (i) above. Let $M(X, A)$ denote the abelian category of projective systems $(F_i)_{i \in \mathbb{N}}$, where $F_i$ are étale sheaves of $A_i$-modules. Let $C(X, A)$ be the category of complexes of $M(X, A)$, and define $K(X, A), D(X, A)$ using homotopy and quasi-isomorphism as in [40]. Similarly we define $C^*(X, A), K^*(X, A), D^*(X, A)$ for $* = +, -, b$, so that $D^*(X, A)$ is naturally equivalent to a full subcategory of $D(X, A)$ (which is defined by a cohomological boundedness condition), using the truncations $\tau_{\leq n}, \tau_{\geq n}$.

We say that $(F_i) \in D^b(X, A)$ is strictly constructible, if $F_0$ has $A_0$-constructible cohomology and the transition morphism $F_{i+1} \to F_i$ induces an isomorphism as in (1.6.1) with $M$ replaced by $F$ in $D^b(X, A_i)$, see [7], 1.1.2. We will denote by $D^b_c(X, A)$ the full subcategory of $D^b(X, A)$ whose objects are strictly constructible. Then $D^b_c(X, A)$ has the truncation $\tau_{\leq n}$ in loc. cit. as follows:

In the usual definition of $\tau_{\leq n} F_i$, we replace $\text{Ker } d \subset F_i^n$ with the subsheaf $K^n_i$ of $\text{Ker } d$, containing $\text{Im } d$, such that

$$\text{Im}(K^n_i \to H^n F_i) = \text{Im}(H^n F_j \to H^n F_i) \quad \text{for } j \gg i.$$
Here the right-hand side is independent of \( j \gg i \) by the strict constructibility of \((\mathcal{F}_i)\). Since \((\tau'_\leq n^i \mathcal{F}_i) \in D(X, A)\) is well defined for \((\mathcal{F}_i) \in D_c^b(X, A)\), it is enough to verify (1.6.1) for \((\tau'_\leq n^i \mathcal{F}_i)\). By Remark (i) above, we may replace the stalk \((\mathcal{F}_{i,x})\) at each geometric point \(x\) with \((L_i)\), where \(L_i = L \otimes_A A_i\) with \(L\) a bounded complex of finite free \(A\)-modules. Then

\[
H^n L = \lim_{\leftarrow} H^n L_i
\]

by the Mittag-Leffler condition, and

\[
\text{Im}(\lim_{\leftarrow} H^n L_j \to H^n L_i) = \text{Im}(H^n L_j \to H^n L_i) \quad \text{for} \quad j \gg i,
\]

by the finiteness of \(H^n L_j\) for any \(j\). This means

\[
\tau'_\leq n L_i = (\tau_\leq n L) \otimes_A A_i.
\]

Thus \((\tau'_\leq n \mathcal{F}_i)\) is strictly constructible.

Let \(\mathcal{C}\) denote the heart of this \(t\)-structure. Then \(\mathcal{C}\) is naturally equivalent to \(\text{Sh}_c(X, A)\) the category of \(A\)-constructible sheaves on \(X\) by \([7]\), 1.1.2. Indeed, we have naturally \(\alpha : \mathcal{C} \to \text{Sh}_c(X, A)\) by \(\alpha(\mathcal{F}_i) = (\mathcal{H}_0^i \mathcal{F}_i)\). For \(\beta : \text{Sh}_c(X, A) \to \mathcal{C}\), let \((\mathcal{S}_j) \in \text{Sh}_c(X, A)\) with \(\mathcal{S}_j = \mathcal{S}_j \otimes_A A_i\), for \(j > i\), and take a quasi-isomorphism \((\mathcal{E}_i) \to (\mathcal{S}_i)\) in \(C^-(X, A)\) such that the stalks of the components of \(\mathcal{E}_i\) at geometric points are flat over \(A_i\). Let \(k\) be a positive integer such that the torsion of \(\lim_{\leftarrow} \mathcal{S}_{i,x}\) at geometric points \(x\) is annihilated by \(m^k\). Then we have as in loc. cit.

\[
\beta(\mathcal{S}_i) = (\tau_{2-1}(\mathcal{E}_{i+k} \otimes_{A_i} A_i)).
\]

Indeed, \(\alpha \beta \simeq \text{id}\) is clear, and \(\beta \alpha \simeq \text{id}\) is induced by \((\mathcal{F}_i) \to (\mathcal{H}_0^i \mathcal{F}_i)\) for \((\mathcal{F}_i) \in \mathcal{C}\), combined with the quasi-isomorphisms \(\mathcal{F}_{i+k} \otimes_{A_i} A_i \to \mathcal{F}_i\) where we may assume that the stalks of the components of \(\mathcal{F}_i\) are flat over \(A_i\).

(iii) Let \(D_c^b(X_R, \mathbb{Q}_l)\) be the derived category of bounded complexes of \(l\)-adic sheaves on \(X_R\) with constructible cohomology (see Remark (ii) above), where \(R\) and \(X_R\) are as in (1.5). We have a canonical functor

\[
\iota_{R,\mathcal{F}} : D_c^b(X_R, \mathbb{Q}_l) \to D_c^b(X_{\mathbb{Q}_l}; G_k),
\]

where the target is the category consisting of objects of \(D_c^b(X_{\mathbb{Q}_l}; \mathbb{Q}_l)\) with an action of \(G_k\) as in (1.1). (This category is considered only to define \(\text{Perv}(X/k, \mathbb{Q}_l)\)^{gen} in Remark (v) below, and it is not clear whether an object of \(D_c^b(X_{\mathbb{Q}_l}; G_k)\) is represented by a complex with an action of \(G_k\) in \(C(X_{\mathbb{Q}_l}; \mathbb{Q}_l)\).) The functor \(\iota_{R,\mathcal{F}}\) commutes with direct images and pull-backs by the generic base change theorem \([11]\), 2.9.

(iv) The category \(\text{Perv}(X, \mathbb{Q}_l)\) of \(l\)-adic perverse sheaves on \(X\) can be defined as a full subcategory of \(D_c^b(X, \mathbb{Q}_l)\) using \([2]\) since the latter category is defined as in Remark (ii) above. We have the gluing of perverse sheaves as in (1.4.1) by the same argument, and hence this category is equivalent to \(\text{Perv}(X/k, \mathbb{Q}_l)\) by increasing induction on \(\text{dim} \ X\). (Note that perverse sheaves can be defined locally, see \([2]\), 3.2.4). As a corollary, (1.3) holds also for \(D_c^b \text{Perv}(X/k, \mathbb{Q}_l)\). If the reader prefers, the category \(\text{Perv}(X/k, \mathbb{Q}_l)\) in (1.5) may be defined by the above
Perv($X, \mathbb{Q}_l$) (however, the inductive argument using (1.4.1) will be needed to define certain full subcategories of Perv($X/k, \mathbb{Q}_l$)).

By Remark (iii) above, a similar argument applies to Perv($X/k, \mathbb{Q}_l$)_{gen}, and this category is stable by the cohomological functors associated with the functors in (1.3), i.e., $H^if_*$, etc. Here the stability by dual and external product follows from the commutativity of these functors with $\varphi_{g,1}$. Then the assertion of (1.3) holds also for $D^b$Perv($X/k, \mathbb{Q}_l$)_{gen}, and the functors in (1.3) commute with the forgetful functors

$$D^b\text{Perv}(X/k, \mathbb{Q}_l)_{gen} \to D^b\text{Perv}(X/k, \mathbb{Q}_l) \to D^b\text{Perv}(X_\mathbb{F}_k, \mathbb{Q}_l; G_k).$$

We have the stability of Perv($X/k, \mathbb{Q}_l$)_{gen} and Perv($X/k, \mathbb{Q}_l$) by subquotients in Perv($X_\mathbb{F}_k, \mathbb{Q}_l; G_k$), because $\varphi_{g,1}$ is an exact functor. Indeed, in case of smooth sheaves, a subobject of the image of a smooth sheaf $L$ in $\text{Sh}_{\text{sm}}(X_\mathbb{F}_k, \mathbb{Q}_l; G_k)$ defines a subsheaf of $L$ by reducing to the finite coefficient case (because any $l$-adic representation of a profinite group $G$ on a $\mathbb{Q}_l$-vector space $V$ has a finitely generated $\mathbb{Z}_l$-submodule which generates $V$ over $\mathbb{Q}_l$ and is stable by the action of $G$) and using the action of $G_k$ on the variety representing the torsion sheaf over $X_\mathbb{F}_k$ (because the stability of a subvariety by the Galois action means that the subvariety is defined over $k$). This will be used for example in (1.12)(ii).

(v) Let Perv($X/k, \mathbb{Q}_l$)_{gen} denote the full subcategory of Perv($X_\mathbb{F}_k, \mathbb{Q}_l; G_k$) consisting of objects whose image in $D^b_c(X_\mathbb{F}_k, \mathbb{Q}_l; G_k)$ belongs to the essential image of the functor $\iota_{R,\mathbb{F}_k}$ in Remark (iii) above for some $R$ (which may depend on the objects). Then Perv($X/k, \mathbb{Q}_l$)_{gen} is stable by cohomological direct images and pull-backs by [11], 2.9. (Here we may assume that every stratum of the stratification of $X_R$ associated with a complex is dominant over Spec $R$ replacing $R$ if necessary.) Hence it is also stable by vanishing cycle functors. By induction on dim $X$ and using the gluing as in (1.4.1), we get an equivalence of categories

$$\text{Perv}(X/k, \mathbb{Q}_l)_{gen} = \text{Perv}(X/k, \mathbb{Q}_l)_{gen}.$$ 

1.7. Definition. Let $Z$ be an irreducible closed subvariety of $X$. We say that $F \in \text{Perv}(X/k, \mathbb{Q}_l)$ has strict support $Z$, if supp $F = Z$ or $\emptyset$, and if $F$ has no nontrivial sub or quotient objects with smaller support. We will denote by $\text{Perv}(X/k, \mathbb{Q}_l)_Z$ the full subcategory of $\text{Perv}(X/k, \mathbb{Q}_l)$ consisting of objects with strict support $Z$.

We say that $F \in \text{Perv}(X/k, \mathbb{Q}_l)$ admits a strict support decomposition, if we have a decomposition

$$(1.7.1) \quad F = \bigoplus_Z F_Z \quad \text{with} \quad F_Z \in \text{Perv}(X/k, \mathbb{Q}_l)_Z.$$ 

We will call $F_Z$ the direct factor of $F$ with strict support $Z$. We have the same for $\text{Perv}(X_\mathbb{F}_k, \mathbb{Q}_l)$.

1.8. Remarks. (i) The strict support decomposition (1.7.1) is unique, because

$$(1.8.1) \quad \text{Hom}(F_Z, F_{Z'}) = 0 \quad \text{if} \quad Z \neq Z'.$$ 

(ii) Assume that $F \in \text{Perv}(X/k, \mathbb{Q}_l)$ admits strict support decomposition and $F_\mathbb{F}_k$ is semisimple. Then any filtration of $F$ is compatible with the strict support
decomposition (1.7.1). Indeed, for an exact sequence
\[ 0 \to F' \to F \to F'' \to 0 \]
in \( \text{Perv}(X/k, \mathbb{Q}_l) \), \( F' \) and \( F'' \) admit strict support decomposition, and we have an exact sequence
\[ 0 \to F'_Z \to F_Z \to F''_Z \to 0. \]
This can be proved using (1.8.1) (or (1.8)(iv) below).

(iii) For a locally closed embedding \( j : X \to Y \), we have the intermediate direct image \( j_! \) (see [2]) defined by
\[ j_! F = \text{Im}(H^0 j_! F \to H^0 j_* F) \quad \text{for} \quad F \in \text{Perv}(X/k, \mathbb{Q}_l). \]
Let \( F \in \text{Perv}(X/k, \mathbb{Q}_l)_Z \), and \( Z' \) be a dense open subvariety of \( Z \) with the inclusion morphism \( j : Z' \to X \). Then we have a canonical isomorphism
\[ (1.8.2) \quad F = j_! j^* F, \]
using the morphisms \( H^0 j_! j^* F \to F \to H^0 j_* j^* F \). Furthermore, (1.8.2) holds with \( F \) replaced by \( F_\mathbb{T} \), and \( j \) by its base extension over \( k \). So, if \( F \in \text{Perv}(X/k, \mathbb{Q}_l) \) admits strict support decomposition, then \( F_\mathbb{T} \) does.

(iv) For \( F \in \text{Perv}(X/k, \mathbb{Q}_l) \), it admits strict support decomposition if and only if the composition
\[ (1.8.3) \quad i_* H^0 i^! F \to F \to i_* H^0 i^* F \]
is an isomorphism for any closed embedding \( i : Y \to X \), because \( i_* H^0 i^! F \) (resp. \( i_* H^0 i^* F \)) is identified with the largest sub (resp. quotient) object of \( F \) supported in the image of \( Y \). This is the same for \( \text{Perv}(X_\mathbb{T}, \mathbb{Q}_l) \). Then, combining this with Remark (iii) above, \( F \in \text{Perv}(X/k, \mathbb{Q}_l) \) admits strict support decomposition, if and only if \( F_\mathbb{T} \) does. (Here we can use also (1.8.1).)

1.9. Definition. We denote by \( \mathbb{Q}_l \) the constant object of \( \text{Perv}(\text{Spec } k/k, \mathbb{Q}_l) \) with trivial Galois action. For a variety \( X \) with a canonical morphism \( a_X : X \to \text{Spec } k \), we define
\[ \mathbb{Q}_{l,X} = (a_X)^* \mathbb{Q}_l \in D^b \text{Perv}(X/k, \mathbb{Q}_l). \]
\[ H^j(X/k, \mathbb{Q}_l) = H^j(a_X)_* \mathbb{Q}_{l,X} \in \text{Perv}(\text{Spec } k/k, \mathbb{Q}_l). \]
More generally, we set for \( F \in D^b \text{Perv}(X/k, \mathbb{Q}_l) \):
\[ (1.9.1) \quad H^j(X/k, F) = H^j(a_X)_* F, \quad H^*_c(X/k, F) = H^j(a_X)_! F \]
If \( X \) is smooth and pure dimensional, then \( \mathbb{Q}_{l,X} \in \text{Sh}_{\text{sm}}(X, \mathbb{Q}_l) \) and \( \mathbb{Q}_{l,X}[\text{dim } X] \in \text{Perv}(X/k, \mathbb{Q}_l) \).

Let \( X \) be a pure dimensional variety, and \( U \) a dense smooth open subvariety of \( X \) with the inclusion \( j : U \to X \). Then, using (1.2.1), we define for \( \mathcal{L} \in \text{Sh}_{\text{sm}}(U, \mathbb{Q}_l) \) the intersection complex with coefficients in \( \mathcal{L} \) by
\[ \text{IC}_X \mathcal{L} = j_{!*} \mathcal{L}[\text{dim } X] \in \text{Perv}(X/k, \mathbb{Q}_l). \]
If $\mathcal{L}$ is the constant sheaf $\mathbb{Q}_l,X$, we will denote
\[
\text{IC}_X \mathbb{Q}_l = j_* \mathbb{Q}_l,U[\dim X],
\]

1.10 Remark. For $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)_Z$, there exist a dense smooth open affine subvariety $Z'$ of $Z$ and $\mathcal{L} \in \text{Sh}_{\text{sm}}(Z', \mathbb{Q}_l)$ such that we have an isomorphism
\[
(1.10.1) \quad \mathcal{F} = \text{IC}_Z \mathcal{L}.
\]
This follows from Remark (1.8)(iii). Note that, if (1.10.1) holds for $Z'$ and $\mathcal{L}$, it holds also for any dense open subvariety of $Z'$ and the restriction of $\mathcal{L}$.

1.11. Definition. With the notation of (1.1), assume $k$ is finitely generated over $\mathbb{Q}$. In the case $X$ is smooth, we say that $\mathcal{L} \in \text{Sh}_{\text{sm}}(X, \mathbb{Q}_l)$ is generically unramified and pure of weight $n$, if there exist a finitely generated $\mathbb{Z}[1/l]$-subalgebra $R$ of $k$ whose fractional field is $k$, a smooth scheme $X_R$ of finite type over $\text{Spec} R$ whose generic fiber is isomorphic to $X$, and a smooth $\mathbb{Q}_l$-sheaf $\mathcal{L}_R$ on $X_R$ whose restriction to $X$ is isomorphic to $\mathcal{L}$ and whose stalk at each closed point is pure of weight $n$ in the sense of [7], see Remark (1.12)(i) below.

We say that $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)_Z$ is pure of weight $n$, if $\mathcal{L}$ in (1.10.1) is generically unramified and pure of weight $n - \dim Z$. We say that $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)$ is pure of weight $n$ if $\mathcal{F}$ admits the strict support decomposition (1.7.1) and each $\mathcal{F}_Z$ is pure of weight $n$. We say that $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)$ is mixed, if $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)^{\text{gnr}}$ and $\mathcal{F}$ has a finite increasing filtration $W$, which is called the weight filtration, such that each $\text{Gr}_n^W \mathcal{F}$ is pure of weight $n$.

We will denote by $\text{Perv}(X/k, \mathbb{Q}_l)^m$ the full subcategory of $\text{Perv}(X/k, \mathbb{Q}_l)^{\text{gnr}}$ consisting of mixed perverse sheaves.

1.12. Remarks. (i) The condition in the pure case means that the $l$-adic representation of $\pi_1(X)$ corresponding to $\mathcal{L}$ is unramified over $X_R$ (i.e., it factors through $\pi_1(X_R)$), and the eigenvalues of Frobenius at each closed point $x$ of $X_R$ are algebraic numbers whose any conjugates over $\mathbb{Q}$ have absolute value $q^{n/2}$, where $q = |\kappa(x)|$. In particular, pure perverse sheaves are stable by subquotients in $\text{Perv}(X/k, \mathbb{Q}_l)$ using Remark (1.8)(ii). Note that pure perverse sheaves are generically unramified over $k$, and hence are mixed, since $\text{Perv}(X/k, \mathbb{Q}_l)^{\text{gnr}}$ is stable by intermediate direct images, see Remark (1.6)(iv).

(ii) If $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)^{\text{gnr}}$ is pure, then $\mathcal{F}_{\overline{k}}$ is semisimple. The argument is essentially the same as in [2], 5.3.8. Indeed, for a short exact sequence of pure objects of $\text{Perv}(X/k, \mathbb{Q}_l)^{\text{gnr}}$
\[
(1.12.1) \quad 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,
\]
we have a commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^1(\mathcal{F}'', \mathcal{F}') & \longrightarrow & \text{Ext}^1(\mathcal{F}{''}_{\overline{k}}, \mathcal{F}{'}_{\overline{k}}) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathbb{Q}_l, H^1(X/k, \text{Hom}(\mathcal{F}'', \mathcal{F}'))) & \longrightarrow & \text{Hom}(\mathbb{Q}_l, H^1(X_{\overline{k}}, \text{Hom}(\mathcal{F}{''}_{\overline{k}}, \mathcal{F}{'}_{\overline{k}}))),
\end{array}
\]
using the adjunction for $X \to \text{Spec} \, k$. Here the right vertical morphism is an isomorphism. Since $H^1(X/k, \mathcal{H}om(F''', F'''))$ has weights $> 0$ by the generic base change theorem ([11], 2.9, 2.10), the horizontal morphisms are zero, and the exact sequence of the underlying perverse sheaves of (1.12.1) on $X_\eta$ splits. Then it is enough to apply this to the largest semisimple subobject of $\mathcal{F}_\eta$, which is stable by the Galois action, and determines a subobject of $\mathcal{F}$, see (1.2)(ii) and (1.6)(iv).

Note that in the above argument, we have used only the following condition:

(C) The mod $p$ reduction of $\mathcal{F}$ over any closed point of a sufficiently small open subvariety of $\text{Spec} \, R$ in (1.11) is pure.

Here the mod $p$ reduction exists by (1.6)(v). Condition (C) implies thus the strict support decomposition (1.7.1) for $\mathcal{F}_k$, and hence for $\mathcal{F}$, see (1.8)(iv). So we may take this as the definition of pure perverse sheaf.

(iii) $\text{Perv}(X/k, \mathbb{Q}_l)^m$ is stable by subquotients in $\text{Perv}(X/k, \mathbb{Q}_l)$, and the weight filtration on a mixed perverse sheaf $\mathcal{F}$ is unique. Indeed, any filtration on $\text{Gr}_n^W \mathcal{F}$ is compatible with the strict support decomposition (1.7.1) by Remark (ii) above and Remark (1.8)(ii), and induces a filtration on each $(\text{Gr}_n^W \mathcal{F})_Z$. This implies the uniqueness of $W$ and the stability of mixed perverse sheaves by subquotients, see Remark (1.6)(iv).

(iv) By the next proposition, the equivalence of categories (1.4.1) holds also for mixed perverse sheaves, because they are stable by the functor $\zeta_g$ which is used to construct the inverse functor of (1.4.1). As a corollary, we can also define the mixed perverse sheaves as in (1.5) by induction on $\dim X$ using the functor $\varphi_g$.

1.13. Proposition. The assertion of (1.3) holds with $\text{Perv}(X_\eta/k, \mathbb{Q}_l; G_k)$ replaced by $\text{Perv}(X/k, \mathbb{Q}_l)$, $\text{Perv}(X/k, \mathbb{Q}_l)^\text{gnr}$ and $\text{Perv}(X/k, \mathbb{Q}_l)^m$.

Proof. The assertion for $\text{Perv}(X/k, \mathbb{Q}_l)$, $\text{Perv}(X/k, \mathbb{Q}_l)^\text{gnr}$ is shown in (1.6)(iv), and it is enough to consider the case of $\text{Perv}(X/k, \mathbb{Q}_l)^m$. We have the stability by the dual and external product, because they are exact functors and the stability of pure objects by these functors follows easily from the definition. By the same argument as in the proof of (1.3), it is enough to show the stability by $j_*$ when $j$ is the open embedding of the complement of a divisor defined by a function $g$.

In this case, we have to show the existence of the relative monodromy filtration $W$ on $\psi_{g,1} F$ (see [7], 1.6.13), because $W$ defines the weight filtration on $j_* \mathcal{F}$ by a formula of Steenbrink-Zucker ([34], 4.11), see [25], 2.11. Note that $j_* \mathcal{F}$ corresponds to $(\mathcal{F}, \psi_{g,1} \mathcal{F}, N, id)$ by (1.4.1). The existence of $W$ is reduced to the finite field case by restricting to the fiber over a sufficiently general closed point of $\text{Spec} \, R$, and using a criterion for the existence of $W$ in [34], 2.20 (see also [25], 1.2). Then the assertion is reduced to Gabber’s result on the coincidence of the monodromy and weight filtrations (up to a shift) in the case $\mathcal{F}$ is pure. (The last result does not seem to have been published, but a (possibly) similar argument can be found in [27], 6.11.)

1.14. Remark. If $\mathcal{F} \in \text{Perv}(X/k, \mathbb{Q}_l)^m$ and $\mathcal{F}' \in \text{Perv}(Y/k, \mathbb{Q}_l)^m$ have weights $\geq n$ (resp. $\leq n$), then $H^i f_* \mathcal{F}, H^i f^! \mathcal{F}'$ (resp. $H^i f_* \mathcal{F}, H^i f^* \mathcal{F}'$) have weights $\geq n + i$.
(resp. \( \leq n + i \)). For the direct images, it is enough to show the assertion for \( H^i f_* \mathcal{F} \) (using \( f_* \mathcal{D} = \mathcal{D} f_* \)). If \( f \) is proper and \( \mathcal{F} \) is pure of weight \( n \), then \( H^i f_* \mathcal{F} \) is pure of weight \( i + n \) using the stability of pure complexes by the direct image under a proper morphism in [7]. So the assertion is reduced to the case \( f \) is an affine open embedding \( j \) as above, and moreover \( \mathcal{F} \) is pure. Then the assertion follows from the construction of \( W \) using the monodromy filtration, see [25], 2.11.

For pull-backs, we may assume \( X \) is a closed subvariety of \( Y \) defined by a function (by factorizing \( f \)) since the assertion is local and the case of a smooth morphism is easy. Then we may assume that \( \mathcal{F}' \) is pure of weight \( n \) with strict support and \( \text{supp} \mathcal{F}' \) is not contained in \( X \). In this case we have to show that the cokernel of \( \mathcal{F}' \to j_* j^* \mathcal{F}' \) has weights \( > n \), where \( j \) denotes the inclusion of \( Y \setminus X \). But this also follows from the construction of \( W \) using the monodromy filtration, see loc. cit.

1.15. Definition. Let \( S \) be an integral affine variety over \( k \), and \( K = k(S) \). For a variety \( X \) over \( K \), let \( X_S \) be a \( k \)-variety over \( S \) whose generic fiber is isomorphic to \( X \) (restricting \( S \) if necessary). Let

\[
(1.15.1) \quad \text{Perv}(X/K/k, \mathbb{Q}_l) = \lim_{\rightarrow} \text{Perv}(X_U/k, \mathbb{Q}_l)
\]

where the inductive limit is taken over nonempty open subvarieties \( U \) of \( S \), and \( X_U = X_S|_U \). Similarly, \( \text{Perv}(X/K/k, \mathbb{Q}_l)^{\text{gmr}} \) and \( \text{Perv}(X/K/k, \mathbb{Q}_l)^m \) are defined by replacing \( \text{Perv}(X_U/k, \mathbb{Q}_l) \) with \( \text{Perv}(X_U/k, \mathbb{Q}_l)^{\text{gmr}} \) and \( \text{Perv}(X_U/k, \mathbb{Q}_l)^m \) in (1.15.1).

1.16. Proposition. We have a fully faithful functor

\[
(1.16.1) \quad \text{Perv}(X/K/k, \mathbb{Q}_l) \to \text{Perv}(X/K, \mathbb{Q}_l)
\]

whose essential image is stable by subquotients in \( \text{Perv}(X/K, \mathbb{Q}_l) \). Furthermore, (1.16.1) induces equivalences of categories

\[
(1.16.2) \quad \text{Perv}(X/K/k, \mathbb{Q}_l)^{\text{gmr}} \to \text{Perv}(X/K, \mathbb{Q}_l)^{\text{gmr}},
\]

\[
(1.16.3) \quad \text{Perv}(X/K/k, \mathbb{Q}_l)^m \to \text{Perv}(X/K, \mathbb{Q}_l)^m.
\]

Proof. To define the functor (1.16.1), we choose an embedding \( \overline{k} \to \overline{K} \) which induces a morphism \( R' \otimes_k \overline{k} \to \overline{K} \), where \( S = \text{Spec} R' \). Then (1.16.1) is given by the base change by this morphism. Since the assertions are local, we may assume \( X \) affine. Let \( g \) be a function on \( X \), and put \( Y = g^{-1}(0), X' = X \setminus Y \). We may assume \( g \) (and \( Y, X' \)) defined over \( S \), shrinking \( S \) if necessary. Applying (1.4.1) to \( X, X', Y \) over \( K \) and also to \( X_U, X'_U, Y_U \) over \( k \), and using the generic base change theorem ([11], 2.9), the assertion is reduced to the case of smooth sheaves on smooth varieties by induction on \( \dim X \). So the assertion follows.

2. Tate Conjecture

In this section, \( k \) is a finitely generated field over \( \mathbb{Q} \), and we prove (0.5) and (0.7). Although the arguments are similar to those in [28] in some places, there are certain differences between the Hodge and \( l \)-adic settings, and we repeat some of the arguments here.
2.1. Cycle map. Let $X$ be a smooth variety over $k$. With the notation of (1.9), we have a cycle map

$$\text{CH}^p(X) \to \text{Ext}^{2p}(\mathbb{Q}_l, X, \mathbb{Q}_l(p)),$$

where Ext is taken in $D^b_{\text{Perv}}(X/k, \mathbb{Q}_l)$. For a closed irreducible reduced subvariety $Z$ of $X$, the image of the cycle $[Z]$ by (2.1.1) is given by the composition of the canonical morphism

$$\mathbb{Q}_l(X) \to \mathbb{Q}_l(Z) \to \text{IC}_{Z}\mathbb{Q}_l[-\dim Z]$$

with its dual using

$$\mathbb{D}(\text{IC}_{Z}\mathbb{Q}_l) = \text{IC}_{Z}\mathbb{Q}_l(\dim Z), \quad \mathbb{D}(\mathbb{Q}_l) = \mathbb{Q}_l(\dim X)[2\dim X].$$

We have the adjunction isomorphism

$$\text{Ext}^{2p}(\mathbb{Q}_l, X, \mathbb{Q}_l(p)) = \text{Ext}^{2p}(\mathbb{Q}_l, (a_X)_*\mathbb{Q}_l(p)).$$

If $X$ is smooth projective, we have a (noncanonical) decomposition by [6]:

$$\text{Ext}^{2p}(\mathbb{Q}_l, X, \mathbb{Q}_l(p)) = \otimes\mathbb{H}^j(X/k, \mathbb{Q}_l)[-j].$$

(This holds also in the case $X$ is smooth proper, see [27], 6.8.) So (2.1.1) induces naturally the cycle map

$$\text{cl} : \text{CH}^p(X) \to \text{Hom}(\mathbb{Q}_l, \mathbb{H}^{2p}(X/k, \mathbb{Q}_l)(p)),$$

which coincides with (0.1). Let

$$\text{CH}^p_{\text{hom}}(X) = \text{Ker}\text{cl}.$$

Then (2.1.1) induces naturally

$$\text{CH}^p_{\text{hom}}(X) \to \text{Ext}^1(\mathbb{Q}_l, \mathbb{H}^{2p-1}(X/k, \mathbb{Q}_l)(p)),$$

which is called the Abel-Jacobi map. Note that the cycle map (2.1.1) can also be defined by using the Gysin morphism

$$(a_Z)_*\mathbb{D}(\mathbb{Q}_l, Z) \to (a_X)_*\mathbb{D}(\mathbb{Q}_l),$$

whose mapping cone is isomorphic to $(a_U)_*\mathbb{Q}_l(X)[2\dim X]$, where $Z$ is the support of a cycle and $U = X \setminus Z$. So (2.1.3) can be obtained also by taking the pull-back of the exact sequence

$$0 \to \mathbb{H}^{2p-1}(X/k, \mathbb{Q}_l) \to \mathbb{H}^{2p-1}(U/k, \mathbb{Q}_l) \to \mathbb{H}^{2p}_Z(X/k, \mathbb{Q}_l),$$

see [19] (and also [14], [16], [26], [28] for the Hodge case.)

2.2. Remarks. (i) Let $\text{TC}(X/k, p)$ denote the Tate conjecture as in Introduction. Then $\text{TC}(X/k, p)$ depends only on $X$ and $p$, and is essentially independent of $k$. Indeed, for a finite extension $k'$ of $k$ with degree $d$, $\text{Spec} k' \otimes_k \overline{k}$ consists of $d$ points on which $\text{Gal}(\overline{k}/k)$ acts transitively, and its stabilizer can be identified with $\text{Gal}(\overline{k}/k')$.

(ii) Let $k'$ be a finite extension of $k$. Then $\text{TC}(X/k, p)$ follows from $\text{TC}(X \otimes_k k'/k', p)$. This can be proved using the action of $\text{Gal}(k''/k)$ on the cycles, where $k''$ is a Galois extension of $k$ containing $k'$. In particular, we may assume $X$ absolutely irreducible as in [35].
(iii) More generally, let $k \to K$ be an arbitrary extension, and $\overline{K}$ an algebraic closure of $K$ with an inclusion $\overline{k} \to \overline{K}$. We have a canonical isomorphism

\begin{equation}
H^{2p}(X_{\overline{K}}, \mathbb{Q}_l)(p) = H^{2p}(X_{\overline{K}}, \mathbb{Q}_l)(p)
\end{equation}

by the base change theorem. Let $c \in H^{2p}(X_{\overline{K}}, \mathbb{Q}_l)(p)$ which is invariant by the Galois action. Then, to show that $c$ is algebraic, it is enough to construct a cycle on $X_K$ whose cycle class coincides with $c$ by the isomorphism (2.1.4). Indeed, we may assume $K$ finitely generated over $k$. Let $R$ be a finitely generated algebra over $k$ such that $K$ is the fractional field of $R$ and the cycle is defined over $R$. Then the assertion is reduced to the above Remark (ii) by restricting to the fiber over a general closed point of Spec $R$.

(iv) The Tate conjecture over a $p$-adic field is not true. Indeed, there exist elliptic curves without complex multiplication, but having a formal complex multiplication, as remarked by J.-P. Serre ([31], Remark (i) in A.2.2). According to Y. Ihara, this can be verified by using a theory of Honda [17] on formal groups and a theory of Serre and Tate [32] on $p$-divisible groups.

**2.3.** Let $f : X \to S$ be a projective morphism of $k$-varieties. Let $\mathcal{F}$ be a pure object of $\text{Perv}(X/k, \mathbb{Q}_l)^m$. With the notation of (1.9.1), we have the Leray spectral sequence in $\text{Perv}(\text{Spec } k/k, \mathbb{Q}_l)$:

\begin{equation}
E_2^{p,q} = H^p(S/k, H^q(f_*\mathcal{F})) \Rightarrow H^{p+q}(X/k, \mathcal{F}),
\end{equation}

which degenerates at $E_2$. Indeed, the spectral sequence is induced by the filtration $\tau$ on $f_*\mathcal{F}$, and the $E_2$-degeneration follows from [6]. (If $f$ is assumed only proper, we can use [27], 6.8.) We denote by $L$ the associated filtration on $H^j(X/k, \mathcal{F})$ so that

\begin{equation}
\text{Gr}_L^iH^j(X/k, \mathcal{F}) = H^i(S/k, H^{j-i}f_*\mathcal{F}).
\end{equation}

If $X$ is smooth and purely $n$-dimensional, and $\mathcal{F}$ is the constant perverse sheaf $\mathbb{Q}_l, X[n]$, then we have by definition (see (1.9))

\begin{equation}
H^j(X/k, \mathbb{Q}_l, X[n]) = H^{j+n}(X/k, \mathbb{Q}_l),
\end{equation}

and we denote also by $L$ the induced filtration on $H^{j+n}(X/k, \mathbb{Q}_l)$. Note that $L$ induces also the filtration $L$ on

$$
\text{Hom}(\mathbb{Q}_l, H^{2p}(X/k, \mathbb{Q}_l)(p)),
$$

because $L$ on $H^j(X/k, \mathbb{Q}_l)$ splits in $\text{Perv}(\text{Spec } k/k, \mathbb{Q}_l)$ using the decomposition theorem for the direct image $f_*\mathcal{F}$ as above.

**2.4.** With the above notation, assume $f$ is a Lefschetz pencil $\tilde{X} \to S$ as in Introduction, and $\mathcal{F} = \mathbb{Q}_l, \tilde{X}[n]$ as in (2.3.3). Note that (0.6) is equivalent to

\begin{equation}
H^0f_*(\mathbb{Q}_l, \tilde{X}[n])_{\tilde{X}}[-1] \text{ is an } t\text{-adic sheaf on } S_\overline{k},
\end{equation}

i.e., $H^0f_*(\mathbb{Q}_l, \tilde{X}[n])$ has no direct factor whose support is zero-dimensional. Indeed, (0.6) is equivalent to the surjectivity (or nonvanishing) of

$$
can : \psi_{t,1}H^0f_*(\mathbb{Q}_l, \tilde{X}[n])_{\tilde{K}} \to \varphi_{t,1}H^0f_*(\mathbb{Q}_l, \tilde{X}[n])_{\tilde{K}} (= \mathbb{Q}_l(-p)),
$$

where $\psi_{t,1}$ and $\varphi_{t,1}$ are the $t$-adic and $\varphi$-adic analogues of $f_*$, respectively.
(where \(t\) is a local coordinate at a critical value of \(f\)), since \(H^{n-1}(Y, \mathbb{Q}_l)\) is generated by the vanishing cycles, which are locally identified with the image of the dual of the above morphism can. (Note that the vanishing cycles are conjugate to each other by the global monodromy group since the discriminant in the dual projective space parametrizing the singular hyperplane sections is the image of a projective bundle over \(X\), and hence is irreducible.) So the equivalence follows from (1.4.1).

By the Picard-Lefschetz formula [9], we have:

\[
H^j f_*(\mathbb{Q}_l, \tilde{X}[n])[-1] \text{ is a constant sheaf on } S_\mathbb{K} \text{ for } j \neq 0.
\]

So we get in the notation of (2.3)

\[
E_2^{p,q} = 0 \quad \text{for } p = 0, q \neq 0,
\]

because \(S = \mathbb{P}^1\). Let \(X_s\) be the fiber of a \(k\)-rational point \(s\) of \(U\) with the inclusion \(\tilde{t} : X_s \to \tilde{X}\), where \(U\) is as in Introduction. Then we have

\[
L^0 H^j(\tilde{X}/k, \mathbb{Q}_l) = \text{Ker}(\tilde{i}^* : H^j(\tilde{X}/k, \mathbb{Q}_l) \to H^j(X_s/k, \mathbb{Q}_l)),
\]

\[
L^1 H^j(\tilde{X}/k, \mathbb{Q}_l) = \text{Im}(\tilde{i}_*: H^{j-2}(X_s/k, \mathbb{Q}_l)(-1) \to H^j(\tilde{X}/k, \mathbb{Q}_l)),
\]

where \(\tilde{i}^*\) and \(\tilde{i}_*\) are the restriction and Gysin morphisms.

\section{Proof of (0.5).} By Remark (2.2)(ii), we may assume \(k(s) = k\) for the assertion (i) replacing \(X\) with \(X \otimes_k k(s)\) if necessary. Then the assertion (i) follows from the weak Lefschetz theorem. For (ii), we use the Leray spectral sequence as above. Take

\[
c \in \text{Hom}(\mathbb{Q}_l, H^{2p}(X/k, \mathbb{Q}_l)(p)),
\]

and let \(c' \in \text{Hom}(\mathbb{Q}_l, H^{2p}(\tilde{X}/k, \mathbb{Q}_l)(p))\) be its pull-back to \(\tilde{X}\). It is enough to show \(c'\) algebraic. Using the Leray spectral sequence and passing to the generic point of \(S\), \(c'\) induces

\[
c'' \in \text{Hom}(\mathbb{Q}_l, H^{2p}(Y/K, \mathbb{Q}_l)(p)).
\]

By TC\((Y/K, p)\), we get a cycle on \(Y\) whose cycle class is \(c''\). This induces a cycle on \(\tilde{X}\) whose cycle class coincides with \(c' \mod L^0\). Here \(c' \mod L^0\) belongs to

\[
\text{Hom}(\mathbb{Q}_l, H^{-1}(S/k, H^{2p-n+1}f_*(\mathbb{Q}_l, \tilde{X}[n])(p))) = \text{Hom}(\mathbb{Q}_l, S[1], H^{2p-n+1}f_*(\mathbb{Q}_l, \tilde{X}[n])(p)).
\]

So we may assume \(c' \in L^0\) by modifying \(c'\). Then \(c' \in L^1\) by (2.4.2), and the assertion follows from TC\((X_s/k(s), p - 1)\) using (2.4.4). Here we may assume \(k(s) = k\) (after reducing to the case \(c' \in L^1\)) by the same argument as above.

\section{Proof of (0.7).} Let \(c \in \text{Hom}(\mathbb{Q}_l, H^n(X/k, \mathbb{Q}_l)(p))\), and \(c'\) as above. It is enough to show \(c'\) algebraic. We first reduce to the case \(c' \in L^0\). For this, we may assume \(k(s) = k\) with \(s\) as in condition (i) (replacing \(X\) with \(X \otimes_k k(s)\), and using an argument similar to Remark (2.2)(ii)), because the Leray spectral sequence is
compatible with the base extension by \( k \to k' \). Let \( i : X_s \to X \) denote the inclusion morphism. Then \( c' \in L^0 \) is equivalent to
\[
c \in \text{Ker}(i^* : H^n(X/k, \mathbb{Q}_l)(p) \to H^n(X_s/k, \mathbb{Q}_l)(p)),
\]
by (2.4.3). Using condition (i) together with the hard Lefschetz theorem, we may assume \( c' \in L^0 \) by modifying \( c' \), because
\[
i^* i_* : H^{n-2}(X_s/k, \mathbb{Q}_l) \to H^n(X_s/k, \mathbb{Q}_l)(1)
\]
coincides with the cup product with the hyperplane section class. (The last fact can be verified by taking the composition with \( i^* : H^{n-2}(X/k, \mathbb{Q}_l) \to H^{n-2}(X_s/k, \mathbb{Q}_l) \) and \( i_* : H^n(X_s/k, \mathbb{Q}_l)(1) \to H^{n+2}(X/k, \mathbb{Q}_l)(2) \) which are isomorphisms.)

We now reduce the assertion to the case \( c' \in L^1 \). By the Poincaré duality, we have a canonical pairing on the \( l \)-adic sheaf \( H^0 f_*(\mathcal{Q}_{l, \overline{X}}[n])_\mathbb{F}_l[-1] \) (see (2.4.1)), which corresponds to a self duality isomorphism of \( H^0 f_*(\mathcal{Q}_{l, \overline{X}}[n]) \). Using this, we get a canonical decomposition in \( \text{Perv}(S/k, \mathcal{Q}) \):
\[
H^0 f_*(\mathcal{Q}_{l, \overline{X}}[n]) = \mathcal{F}_{\text{inv}} \oplus \mathcal{F}_{\text{van}}
\]
such that \( (\mathcal{F}_{\text{inv}})_\mathbb{F}_l[-1] \) is a constant sheaf and \( \Gamma(S, (\mathcal{F}_{\text{van}})_\mathbb{F}_l[-1]) = 0 \), i.e.,
\[
H^{-1}(S/k, \mathcal{F}_{\text{van}}) = 0.
\]
Then \( H^1(S/k, \mathcal{F}_{\text{van}}) = 0 \) by the duality. Since \( S = \mathbb{P}^1 \), we have
\[
H^0(S/k, \mathcal{F}_{\text{inv}}) = 0,
\]
(i.e., \( H^1(S, (\mathcal{F}_{\text{inv}})_\mathbb{F}_l[-1]) = 0 \)), and we get
\[
H^0(S/k, H^0 f_*(\mathcal{Q}_{l, \overline{X}}[n])) = H^0(S/k, \mathcal{F}_{\text{van}}).
\]

So \( c' \mod L^1 \) is uniquely lifted to
\[
c' \in \text{Hom}(\mathbb{Q}_l, (a_S)_*, \mathcal{F}_{\text{van}}),
\]
which corresponds to
\[
e \in \text{Ext}^1(\mathbb{Q}_l, S, \mathcal{F}_{\text{van}}),
\]
by the adjunction isomorphism for \( a_S : S \to \text{Spec} k \). Using the restriction morphism by \( Y \to X_K \), we define \( H^j(Y/K, \mathcal{Q}_l)_{X_K}, H^j(X_K/K, \mathcal{Q}_l)^Y \) as in Introduction so that we have an exact sequence
\[
0 \to H^{n-1}(Y/K, \mathcal{Q}_l)_{X_K} \to H^c((X_K \setminus Y)/K, \mathcal{Q}_l) \to H^n(X_K/K, \mathcal{Q}_l)^Y \to 0.
\]
Then \( H^{n-1}(Y/K, \mathcal{Q}_l)_{X_K} \) is isomorphic to the restriction of \( \mathcal{F}_{\text{van}}[-1] \) to the generic point \( \text{Spec} K \) of \( S \) (using (1.16)), and the restriction of \( e \) to \( \text{Spec} K \) coincides with the pull-back of the above short exact sequence by \( c \).

Indeed, the adjunction isomorphism for \( a_S : S \to \text{Spec} k \) is induced by the morphism
\[
(a_S)^*(a_S)_* = (p_2)_*(p_1)^* \to (p_2)_* \delta_* \delta^*(p_1)^* = id
\]
where \( p_\alpha : S \times_k S \to S \) are canonical projections, and \( \delta : S \to S \times_k S \) is a diagonal embedding. We get the coincidence using the exact sequence as above with \( X \) replaced by \( \overline{X} \) together with a canonical morphism of the exact sequences induced by the restriction morphism for \( \pi : \overline{X} \to X \).
So condition (ii) (together with (1.16)) implies that there exist a nonempty open subvariety $S'$ of $S$ and a cycle $\zeta$ on $f^{-1}(S')$ such that the cycle class of the restriction of $\zeta$ to the generic fiber is zero and the restriction of $\epsilon'$ mod $L^1$ to $f^{-1}(S')$ is induced by $\zeta$. Since $H^0 f_* (\mathbb{Q}_{l,\tilde{X}}[n])$ is an intersection complex $\mathcal{F}$ with strict support $S$, the natural morphism

$$H^0(S/k, H^0 f_* (\mathbb{Q}_{l,\tilde{X}}[n])) \to H^0(S'/k, H^0 f_* (\mathbb{Q}_{l,\tilde{X}}[n]))$$

is injective. (Indeed, $j_* j^* \mathcal{F}/\mathcal{F}$ is supported on $S \setminus S'$ where $j: S' \to S$ is the inclusion, and hence its cohomology vanishes except for the degree 0). So we may assume $S' = S$ by taking an extension of $\zeta$ to $\tilde{X}$, and the assertion is reduced to the case $\epsilon' \in L^1$ by modifying $\epsilon'$. Then we may assume again $k(s) = k$ by the same argument as in Remark (2.2)(ii), and the assertion follows from condition (i) and (2.4.4).

3. Divisor Case

In this section we treat the divisor case, and relate the Tate conjecture to the de Rham conjecture for nonproper varieties and finiteness of the Tate-Shafarevich groups.

3.1. Let $k$ be a finitely generated field over $\mathbb{Q}$, $S$ an integral curve over $k$, and $A$ an abelian scheme over a smooth dense open subvariety $U$ of $S$. Let $\mathcal{L} = T_1 A$ which is the projective system of étale sheaves defined by $\text{Ker}(l^m : A \to A)$. We assume

$$(3.1.1) \quad H^0(U_{k'}, \mathcal{L}) = 0. $$

Let $K = k(S)$, and $\bar{K}$ be the maximal subfield of $\bar{K}$ that is unramified over $U$. Let $G = \text{Gal}(\bar{K}/K)$ which is identified with $\pi_1(U, \text{Spec} \bar{K})$. Then $\mathcal{L}$ corresponds to the $\mathbb{Z}_l$-module

$$E := \Gamma(\text{Spec} \bar{K}, \mathcal{L}) = T_1 A(\bar{K})$$

with a continuous action of $G$, and we have natural isomorphisms

$$(3.1.2) \quad H^1(G, E/l^m) = H^1(U, \mathcal{L}/l^m), \quad H^1(G, E) = H^1(U, \mathcal{L}).$$

Here $\mathcal{L}/l^m$ denotes $\mathcal{L}/l^m\mathcal{L}$ for simplicity (same for $E$), and $H^1(G, E)$ is the projective limit of $H^1(G, E/l^m)$ (same for $H^1(U, \mathcal{L})$). Using the Kummer sequence, we get an exact sequence

$$(3.1.3) \quad 0 \to A(K) \otimes \mathbb{Z}_l \to H^1(G, E) \to T_1 H^1(G, A(\tilde{K})) \to 0.$$

Note that $A(K)$ is a finitely generated abelian group by (a generalization of) the Mordell-Weil theorem, and $H^1(G, E)$ is a finite $\mathbb{Z}_l$-module by (3.1.5) below.

Let $m$ be a positive integer. By the Hochschild-Serre spectral sequence, we have a long exact sequence

$$(3.1.4) \quad 0 \to H^1(G_k, H^0(U_{k'}, \mathcal{L}/l^m)) \to H^1(U, \mathcal{L}/l^m) \to H^0(G_k, H^1(U_{k'}, \mathcal{L}/l^m))$$

$$\to H^2(G_k, H^0(U_{k'}, \mathcal{L}/l^m)).$$
Passing to the limit, we get an exact sequence by the Mittag-Leffler condition, and
\[
H^1(U, \mathcal{L}) = H^1(U_K, \mathcal{L})^{G_k},
\]
because (3.1.1) implies that \(H^0(U_K, \mathcal{L}/l^{m+k}) \to H^0(U_K, \mathcal{L}/l^m)\) is zero for \(k \gg 0\).

3.2. Remarks. (i) There exists a positive integer \(a\) independent of \(m\) such that
\(H^0(U_K, \mathcal{L}/l^m)\) is annihilated by \(l^a\), because \(H^0(U_K, \mathcal{L} \otimes_{\mathbb{Z}_l} (\mathbb{Q}_l/\mathbb{Z}_l))\) is finite by (3.1.1) (using also the finiteness of \(H^1(U_K, \mathcal{L})\) over \(\mathbb{Z}_l\)).

(ii) Assume \(S = \mathbb{P}^1\). Let \(b\) be a positive integer such that there is no elements of \(A(K)\) with order \(l^b\). Then there exists a thin subset \(\Sigma\) of \(U(k)\) in the sense of [33] such that there is no element of \(A_s(k)\) with order \(l^b\) for \(s \in U(k) \setminus \Sigma\), where \(A_s\) is the fiber of \(A\) over \(s\). (It is enough to apply Hilbert’s irreducibility theorem ([21], [33]) to the irreducible components of the kernel of \(l^b : A \to A\).) In particular, the \(l\)-primary torsion part of \(A_s(k)\) is annihilated by \(l^{b-1}\). We have the same for the torsion part of \(H^1(G_s, E)\) using the exact sequence
\[
0 \to A_s(k) \otimes \mathbb{Z}_l \to H^1(G_s, E) \to T_l H^1(G_s, A(\overline{k})) \to 0,
\]
because the last term is torsion-free.

(iii) The injectivity of (0.10) in Introduction follows immediately from Hilbert’s irreducibility theorem (see [21], [33]), if \(V\) is replaced with \(E/l^m\), see also (3.3.2) below. Taking the projective limit for \(m\), we have the assertion for \(E\). (For this we do not have to assume that \(E\) is associated with an abelian variety.) Then the injectivity of (0.10) for \(V\) (with \(|U|\) replaced by \(|U(k) \setminus \Sigma|\)) follows from Remark (ii) above since we may assume that the thin subsets used in Hilbert’s irreducibility theory for the above argument contain the thin subset \(\Sigma\) in Remark (ii) above. Note that the assertion is true without assuming \(S = \mathbb{P}^1\) (using a morphism to \(\mathbb{P}^1\)). A. Tamagawa has informed us of the injectivity for any smooth sheaf \(\mathcal{L}\) (not necessarily associated with an abelian scheme) using the theory of Frattini subgroups, see also [30]. We are also informed that a similar argument was used in [38].

3.3. Proposition. With the notation and the assumptions of (3.1), assume further \(S = \mathbb{P}^1\). Then there exists a thin subset \(\Sigma\) of \(U(k)\) in the sense of [33] such that \(H^1(G, E) \to H^1(G_s, E)\) is injective for \(s \in U(k) \setminus \Sigma\), where \(G_s = \text{Gal}(\overline{k}(s)/k(s))\).

Proof. Let \(a, b\) be positive integers as in Remarks (i) and (ii) above, and take an integer \(m > a + b\). For a subfield \(K'\) of \(\overline{K}\) which is Galois over \(K\), let \(U' \to U\) be the corresponding finite étale morphism. For each \(s \in |U|\), we choose a lift \(s' \in |U'|\) in a compatible way with the change of \(K'\) so that we get an inclusion \(G_s \to G\) and a morphism of (3.1.3) to (3.2.1). We assume \(K'\) sufficiently large so that the pull-back of \(\mathcal{L}/l^m\) to \(U'\) is trivial, and the image of the composition of
\[
H^1(\text{Gal}(K'/K), E/l^m) \to H^1(G, E/l^m)
\]

\[
= H^1(U, \mathcal{L}/l^m) \to H^1(U_K, \mathcal{L}/l^m)^{G_k}
\]

coincides with the image of \(H^1(G, E/l^m) \to H^1(U_K, \mathcal{L}/l^m)^{G_k}\). Combined with (3.1.4) and Remark (i) above, this implies that for any \(e \in H^1(G, E/l^m)\), there
exists $e' \in H^1(\text{Gal}(K'/K), E/l^m)$ such that $e - e'$ is annihilated by $l^a$. (Since the first morphism of (3.3.1) is injective, the image of $e'$ is also denoted by $e'$.)

By Hilbert’s irreducibility theorem, there exists a thin subset $\Sigma$ of $U(k)$ such that for $s \in U(k) \setminus \Sigma$, a point $s'$ of $U'$ over $s$ is unique and hence $\text{Gal}(K'/K) = \text{Gal}(k(s')/k(s))$, see [21], [33]. So we get the isomorphism

$$H^1(\text{Gal}(K'/K), E/l^m) \cong H^1(\text{Gal}(k(s')/k(s)), E/l^m),$$

and hence any $e \in \text{Ker}(H^1(G, E/l^m) \to H^1(G_s, E/l^m))$ is annihilated by $l^a$ (taking $e'$ as above). We may assume moreover the torsion part of $H^1(G_s, E)$ is annihilated by $l^{b-1}$ for $s \in U(k) \setminus \Sigma$ by Remark (ii) above, replacing $\Sigma$ if necessary. Set

$$H = H^1(G, E), \quad H' = H^1(G_s, E).$$

Then $\text{Ker}(H/l^m \to H'/l^m)$ is also annihilated by $l^a$, because the natural morphism

$$H^1(G, E)/l^m \to H^1(G, E/l^m)$$

is injective. (This can be verified easily by taking the limit of the exact sequence associated with $0 \to E/l^i \to E/l^{i+m} \to E/l^m \to 0$.) Since $\text{Ker}(A(K) \to A_0(k))$ and $T_iH^1(G, A(\bar{K}))$ are torsion-free, $\text{Ker}(H \to H')$ is also torsion-free by (3.1.3) and (3.2.1). Finally, $H$ is finite over $\mathbb{Z}_l$ by (3.1.2) and (3.1.5). So the assertion follows from

**Sublemma.** A morphism of $\mathbb{Z}_l$-modules $H \to H'$ is injective, if $\text{Ker}(H \to H')$ is torsion-free, $H$ is finite over $\mathbb{Z}_l$, and $\text{Ker}(H/l^m \to H'/l^m)$ and the torsion part of $H'$ are annihilated by $l^a$ and $l^b$ respectively with $a + b < m$.

**Proof.** We may assume the surjectivity of the morphism by replacing $H'$ with the image so that we get an exact sequence. (Here $\text{Ker}(H/l^m \to H'/l^m)$ becomes smaller by replacing $H'$.) Then the assertion follows from the snake lemma applied to the endomorphism of the short exact sequence defined by the multiplication by $l^m$.

### 3.4. Remarks.

(i) Let $f : X \to S$ be a surjective morphism of smooth irreducible projective varieties over $k$. (We may assume $\dim X = 2$ since the Tate conjecture for divisors is reduced to the surface case by (0.5) replacing $k$ if necessary.) Let $U$ be a dense open subvariety of $S$ such that the restriction $f' : X' \to U$ of $f$ over $U$ is smooth. We have a closed embedding $X' \to X \times U$ by the graph of $f$. Let

$$A = \text{Coker}(J_{X \times U/U} \to J_{X'/U}),$$

$$\mathcal{L} = \text{Coker}(R^1(pr_2)_*\mathbb{Z}_l \times X \times U(1) \to R^1f'_*\mathbb{Z}_l \times X(1)),$$

where $J_{X \times U/U}, J_{X'/U}$ denote the identity component of the Picard scheme, and the cokernel $A$ can be defined by using the Néron model of the cokernel over the generic point of $U$. Then we have a canonical isomorphism $\mathcal{L} = T_1A$, and (3.1.1) is satisfied by the global invariant cycle theorem.

(ii) With the notation of Remark (i) above, the Tate conjecture for divisors on $X$ is equivalent to

$$T_1H^1(G, A(\bar{K})) = 0. \quad \text{(3.4.1)}$$
Indeed, let $Y$ be the generic fiber of $X \to S$, and $\text{CH}_1^{\text{hom}}(Y)$ as in Introduction. Since the Leray spectral sequence for $X'_k \to U'_k \to \text{Spec} k$ degenerates at $E_2$ (see [6]), the cycle map induces the surjective morphism

$$\text{CH}_1^{\text{hom}}(Y) \otimes \mathbb{Q}_l \to H^1(U'_k, \mathcal{L})^G_k \otimes \mathbb{Z}_l \mathbb{Q}_l,$$

if the Tate conjecture is true. It is known that this morphism coincides with the composition of the canonical morphism $\text{CH}_1^{\text{hom}}(Y) \to \mathcal{A}(K)$ with the first morphism of (3.1.3) with $\mathbb{Q}_l$-coefficients. So we get the assertion by (3.1.3), because $T_l H^1(G, \mathcal{A}(\widetilde{K}))$ is torsion-free.

3.5. Remark (S. Bloch, K. Kato, J. Nekovar). Let $k$ be a number field, and $A$ an abelian variety over $k$. Let $E = T_l A(\overline{k})$. For a place $v$ of $k$, let $k_v$ be the completion of $k$ at $v$, $\overline{k}_v$ its algebraic closure, and $E_v = T_l A(\overline{k}_v)$, where $E_v$ is isomorphic to $E$ by choosing $\overline{k} \to \overline{k}_v$. Then we have a canonical morphism of exact sequences

$$0 \longrightarrow A(k) \otimes \mathbb{Z}_l \longrightarrow H^1(G_k, E) \longrightarrow T_l H^1(G_k, A(\overline{k})) \longrightarrow 0$$

If $e \in H^1(G_k, E)$ is defined geometrically by taking a pull-back of an exact sequence like (0.2) in Introduction (where $A = J_{D,X}$), then we have the same for the restriction $e_v \in H^1(G_{k_v}, E_v)$ of $e$, and $e_v$ belongs to the image of $A(k_v) \otimes \mathbb{Z}_l$ for any $v$, using the theory of Bloch and Kato on $H^1_g$ (see [3], Remark before 3.8), provided that the de Rham conjecture [15] holds for the nonproper variety $(X \setminus D)_v$, see also [24]. If this is true, then $e$ belongs to the image of $A(k) \otimes \mathbb{Z}_l$ if we have furthermore the injectivity of

$$T_l H^1(G_k, A(\overline{k})) \to \prod_v T_l H^1(G_{k_v}, A(\overline{k}_v)),$$

(i.e., the $l$-primary torsion part of the Tate-Shafarevich group of $A$ is finite).

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