An optical fusion gate for W-states

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Abstract. We introduce a simple optical gate to fuse arbitrary-size polarization entangled W-states to prepare larger W-states. The gate requires a polarizing beam splitter (PBS), a half-wave plate (HWP) and two photon detectors. We study, numerically and analytically, the necessary resource consumption for preparing larger W-states by fusing smaller ones with the proposed fusion gate. We show analytically that resource requirement scales at most sub-exponentially with the increasing size of the state to be prepared. We numerically determine the resource cost for fusion without recycling where W-states of arbitrary size can be optimally prepared. Moreover, we introduce another strategy that is based on recycling and outperforms the optimal strategy for the non-recycling case.

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1. Introduction

Quantum entanglement is at the heart of many quantum information processing (QIP) tasks such as quantum teleportation [1], quantum key distribution (QKD) [2] and quantum computation [3]. Among the multipartite entangled states, W, GHZ and cluster states form inequivalent classes in that they cannot be transformed into each other by local operations and classical communication (LOCC) [4]. Recent studies have shown that for a specific type of QIP tasks, specially designed multi-partite entangled states are required: cluster states have been proposed as a universal substrate for measurement-based quantum computation [5], GHZ class has not only been shown to be useful for quantum teleportation [6], quantum secret sharing [7, 8] and QKD [9] but also shown to be the only one for reaching consensus in distributed networks when no classical post-processing is allowed [10]. On the other hand, W-class is proposed as a resource for QKD [11] and for the optimal universal quantum cloning machine [12], as well as shown to be the only pure state to exactly solve the problem of leader election in anonymous quantum networks [10]. It is thus important to efficiently prepare states of different classes not only for practical applications but also for the fundamental study of quantum information.

It is known that starting with Bell pairs, probabilistic quantum parity checking gates, the so-called fusion gates, can be efficiently used to grow large-scale cluster states [13]. A GHZ state can be prepared and expanded by a photon with each successful application of the same fusion gate [13, 14]. So far a similar study on n-partite W-states, defined as $|W_n\rangle = |n-1, 1\rangle / \sqrt{n}$, where $|n-k, k\rangle$ is the sum of all states with $n-k$ zeros (H-polarized photon) and $k$ ones (V-polarized photons), has not been carried out. Given that W-states are optimal in the amount of pairwise entanglement when $n-2$ parities are discarded [4, 15] and have persistency of $n-1$, which is much larger than those of GHZ and cluster states [16], it becomes a necessity to probe the bounds on the efficiency of preparing large-scale W-states and the resource requirements. It is also worth noting that $|W_n\rangle$ constitute an entangled web and fully interconnected quantum network. Therefore, such a study may shed light on the scalability of entangled webs of W-states [17].

There have been many proposals on the preparation and manipulation of W-states and their experimental implementations in photons [18–26], trapped ions [27–29] and NMR systems [30]. Recently, we demonstrated experimentally that a three-partite W-state can be prepared from
two Bell pairs using LOCC [26]. Moreover, we proposed two schemes for the preparation and expansion of N-partite W-states [31, 32].

In [31], we proposed an optical gate formed by a pair of 50:50 beamsplitters, which accepts one photon from \(|W_n⟩\) to expand it into \(|W_{n+2}⟩\) with a success probability of \((n+2)/16n\) using an ancillary state of two H-polarized photons \(|2_H⟩\). The gate can be cascaded, in which each successful gate operation increases the size of W-state by two photons: cascading this gate \(k\) times prepares the state \(|W_{n+2k}⟩\) with a success probability of \(2^{-4k}(1+2k/n)\). We experimentally demonstrated this optical gate and successfully generated three-photon and four-photon polarization-based W-states [33]. In [32, 34], on the other hand, it was shown that \(|W_n⟩\) can be expanded to \(|W_{n+1}⟩\) with a success probability of \((n+1)/5n\) using a polarization-dependent beamsplitter and an ancillary state of H-polarized single photon \(|1_H⟩\). The cascade operation of this gate expands a W-state by a photon at each successful step: cascaded application of this gate \(k\) times prepares the state \(|W_{n+k}⟩\) with a success probability of \(5^{-k}(1+k/n)\). Since \(e^x \geq 1+x\) leads to \(a^{-k}(1+bk) \leq e^{-Mk}\) with \(M = \ln a - b\), we see that for both of these gates, the success probability decreases exponentially as \(k\) increases. Recycling in the case of failure is not possible; thus the resource required to prepare a large W-state scales exponentially.

In this paper, we present a theoretical proposal for preparing large-scale W-state networks using a fusion mechanism. We have previously introduced the basic principles of this W-state fusion mechanism and the gate for its realization in [37–39]. Here, we compare resource requirements for preparing arbitrarily large W-states using this fusion mechanism under various scenarios. We show that the resource for the proposed fusion mechanism scales sub-exponentially. Although we phrase our proposal for qubits encoded in horizontal (H) and vertical (V) photon polarization, the fusion mechanism is of wider applicability. The primary resource we will make use of is three-photon polarization entangled W-states \(|W_3⟩\). These can be prepared using the gate given in [31], with a success probability of \(3/16\) starting with a single photon \(|1_V⟩\) and two photons in the Fock state \(|2_H⟩\), and the gate given in [32], with a success probability of \(3/10\) with a single photon and a Bell pair. Alternatively, two polarization entangled Bell states can be used to create this initial W-state as shown in [26].

2. Fusion gate for W-states

Let us consider the following scenario: two spatially separated administrators, Alice and Bob, decide to merge their small-scale entangled webs \(|W_n⟩_A\) and \(|W_m⟩_B\) into a larger entangled web \(|W_{f,r}⟩_{A∪B}\) with the help of a trusted third party Claire. In order to do this, each transmits one qubit of their web to Claire who acts locally on the received two qubits with the fusion gate and informs them when the task is successful. The question is whether such a local manipulation is possible or not, and if possible, then what does the scheme look like. The polarization entangled W-states of Alice and Bob are

\[
|W_n⟩_A = \frac{1}{\sqrt{n}}((|n-1⟩_H)_a|1_V⟩_1 + \sqrt{n-1}|W_{n-1}⟩_a|1_H⟩_1),
\]

\[
|W_m⟩_B = \frac{1}{\sqrt{m}}((|m-1⟩_H)_b|1_V⟩_2 + \sqrt{m-1}|W_{m-1}⟩_b|1_H⟩_2),
\]

where the photons in mode 1 (2) are sent to Claire by Alice (Bob) and those in mode a (b) are kept at Alice’s (Bob’s) side. We see that the photon pairs Claire receives are \(|1_V⟩_1|1_V⟩_2\),
Figure 1. Non-deterministic fusion gate for W-states: two spatial modes (1 and 2) with photons coming from separate W-states are mixed on a polarizing beam splitter (PBS), which reflects vertically (V) polarized photons and transmits horizontally (H) polarized photons. The half-wave plate (HWP $\pi/2$) at mode 2 transforms $|H\rangle \rightarrow |V\rangle$ and $|V\rangle \rightarrow |H\rangle$. The HWPs $\pi/4$ perform the transformation $|H\rangle \rightarrow (|H\rangle + |V\rangle)/\sqrt{2}$ and $|V\rangle \rightarrow (|H\rangle - |V\rangle)/\sqrt{2}$. The output modes 3 and 4 of the first PBS are measured by detectors $D_1$ and $D_2$, which are formed by an HWP, a PBS and a pair of photon counters.

$|1_H\rangle_1|1_V\rangle_2$, $|1_V\rangle_1|1_H\rangle_2$ and $|1_H\rangle_1|1_H\rangle_2$ with the respective probabilities $P_{VV} = 1/nm$, $P_{HV} = (n - 1)/nm$, $P_{VH} = (m - 1)/nm$ and $P_{HH} = (n - 1)(m - 1)/nm$.

We start by describing the fusion mechanism, which is a parity check operation (see figure 1). The photons in two spatial modes are mixed on a polarizing beam splitter (PBS) after exchanging the polarization of the photon in one of the modes by $\pi/2$. After the PBS, photons in the output modes are measured in $\{|D\rangle, \overline{|D\rangle}\}$ basis where $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $\overline{|D\rangle} = (|H\rangle - |V\rangle)/\sqrt{2}$. The combined action of the HWP and PBS on the input photons is as follows: $|1_H\rangle_1|1_H\rangle_2 \rightarrow |0\rangle_3|1_H\rangle_1|1_V\rangle_2$, $|1_V\rangle_1|1_V\rangle_2 \rightarrow |1_H\rangle_1|1_V\rangle_2|0\rangle_3$, $|1_H\rangle_1|1_V\rangle_2 \rightarrow |1_H\rangle_1|1_V\rangle_2|1_H\rangle_3|1_H\rangle_4$ and $|1_V\rangle_1|1_H\rangle_2 \rightarrow |1_V\rangle_1|1_V\rangle_2|1_V\rangle_1|1_V\rangle_4$ where the subscript numbers denote the spatial modes of the fusion gate. It is clear that a coincidence detection between detectors $D_1$ and $D_2$ takes place when the photons in modes 1 and 2 have orthogonal polarizations, and no coincidence is observed when the photons in modes 1 and 2 have the same polarizations. Moreover, the detectors cannot discriminate between the two cases that lead to coincidence detection.

The case when $D_1$ detects photons but not $D_2$ implies that both of the initial W-states have lost their V-polarized photons. Therefore, the remaining photons will all be H-polarized. Thus, we end up with a product state (networks are destroyed). Such events, that we call failure, take place with a probability of $P_f(W_n, W_m) = P_{VV} = 1/nm$. On the other hand, the case when $D_2$ detects photons but $D_1$ does not implies that each of the initial W-states has lost one H-polarized photon. Thus, we will have two separate W-states with a smaller number of qubits, $|W_{n-1}\rangle$ and $|W_{m-1}\rangle$, with probability $P_r(W_n, W_m) = P_{HH} = (n - 1)(m - 1)/nm$. These shortened W-states can be recycled using the same fusion mechanism later.

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Now let us look closely at the cases that lead to a coincidence detection. Using the expressions in equations (1) and (2), we find that when both D1 and D2 detect photons in the same state $|D\rangle$ (or $|\bar{D}\rangle$), the state of the remaining photons becomes

$$\frac{1}{2\sqrt{nm}}(\sqrt{n-1}|W_{n-1}\rangle_a|(m-1)_H\rangle_b + \sqrt{m-1}|(n-1)_H\rangle_a|W_{m-1}\rangle_b) = \frac{1}{2\sqrt{nm}}|W_{n+m-2}\rangle,$$

where we have used $\sqrt{k}|W_k\rangle = \sqrt{i}|W_i\rangle|(k-i)_H\rangle + \sqrt{j-i}|j_H\rangle|W_{j-i}\rangle$. When one of the detectors detects a photon in $|D\rangle$ and the other in $|\bar{D}\rangle$, the state of the remaining photons will be the same as equation (3) but with a minus sign that can be corrected by applying a $\pi$-phase shift in one of the modes. Thus a coincidence detection signals the successful fusion operation and the preparation of a W-state with $n+m-2$ photons with the success probability $P_s(W_n, W_m) = (n+m-2)/nm$. Note that an attempt to fuse $|W_2\rangle$ with $|W_n\rangle$ will not expand the W-state; successful events will prepare only the state $|W_n\rangle$. Expansion requires that both $n$ and $m$ are greater than or equal to 3.

We give an example of fusion operation in figure 2, which shows the success, failure or recycling processes. Recycling is performed until either a Bell pair $|W_2\rangle = (|HV\rangle + |VH\rangle)/\sqrt{2}$
or a product state is obtained. In principle, the resultant Bell states can be further recycled to prepare a $|W_3\rangle$ using the gate introduced in [26]; however, in this study we do not consider such recycling.

As the above discussions show, the fusion gate for W-states differs from that for cluster states in two ways: (i) fusion gates for cluster states operate with a constant success probability of $P_s$ regardless of the size of the cluster states attempted to be fused. However, the success probability of fusion gate for W-state depends on the size of the W-states: as $n$ and $m$ increase, $P_f$ decreases but this does not necessarily lead to an increase of the same order in $P_s$; instead the probability $P_r$ of recyclable events increases. (ii) Failure events in cluster state fusion lead to two cluster states shortened by one qubit, similarly to the recyclable events in the fusion of W-states. On the other hand, a failure in fusion for W-states leads to complete destruction of both W-states. This makes the analysis of fusion and expansion of W-states much more difficult.

3. Cost of preparing arbitrary-size W-states

In this section, we compare the performances of various strategies using the proposed fusion gate for the preparation of arbitrary-size W-states. We will answer the question ‘How does the required resource to prepare a W-state of $N$-photons $|W_n\rangle$ scale?’ in various scenarios with and without the recycling process.

In this section, we switch to a new index to represent the size of W-states, which differs from the original by 2:

$$|w_n\rangle \equiv |W_{n+2}\rangle.$$  \hspace{1cm} (4)

The benefit of the lower-case notation is to make the result of successful fusion more intuitive: successful fusion of $w_m$ and $w_n$ simply produces $w_{m+n}$. The recyclable outcome leaves the states in the same form as in the original notation, namely $w_{m-1}$ and $w_{n-1}$. The probabilities associated with the three outcomes are now written as

$$P_s(w_n, w_m) = \frac{(n + m + 2)/[(n + 2)(m + 2)]},$$  \hspace{1cm} (5)

$$P_f(w_n, w_m) = \frac{(n + 1)(m + 1)/[(n + 2)(m + 2)]},$$  \hspace{1cm} (6)

$$P_f(w_n, w_m) = 1/[(n + 2)(m + 2)].$$  \hspace{1cm} (7)

We use the notation $R[w_m]$ for the resource cost (i.e. the number of $|w_1\rangle$ states required) of producing state $w_m$, which is an $(m + 2)$ qubit W-state $|W_{m+2}\rangle$. In our analysis, we consider $|W_3\rangle \equiv |w_1\rangle$ as the basic resource provided with unit cost, i.e. $R[w_1] = R[W_3] = 1$. For $m \geq 2$, the value of $R[w_m]$ will vary depending on the strategies.

When we do not use recycling and try to produce $w_{m+n}$ from $w_n$ and $w_m$, the costs are simply related as

$$R[w_{n+m}] = \frac{1}{P_s(w_n, w_m)}(R[w_n] + R[w_m])$$

$$= \frac{(m + 2)(n + 2)}{n + m + 2} (R[w_n] + R[w_m]),$$  \hspace{1cm} (8)

which is frequently used in the later analysis. The cases with recycling are more complicated and will be treated separately below.

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In the following, we treat this problem for linear (see figure 3) and exponential (see figure 4) growth strategies with and without recycling and derive analytical bounds for resource complexity. We provide the optimal strategy for fusion without recycling and also introduce a strategy based on fusing W-states of similar sizes which turns out to provide the best resource scaling among all the strategies considered here.

3.1. Linear growth strategies

Here, we analyze the resource requirements for linear growth strategies with and without recycling. The strategy is based on repeated fusion of a fixed-size W-state to an already existing W-state (see figure 3). Let us assume that we want to expand $w_m$ by fusing it with $w_n$ repeatedly. If fusing is successful, bring another $w_n$ and fuse it with the state prepared in the previous level. In this way, after the $k$th level of successively successful gate operations, the state $w_{m+kn}$ is prepared with the probability $(m + kn + 2)/(m + 2)(n + 2)^k$. At each level of successful operation, the size of the state increases by $n$, i.e. $\{m, n\} \rightarrow \{m + n, n\} \rightarrow \{m + 2n, n\} \rightarrow \{m + (k - 1)n, n\} \rightarrow \{m + kn\}$.

3.1.1. Linear growth without recycling. The resource required for expanding the state $w_m$ by $n$ is given in equation (8) as $R[w_{m+n}] = P_s^{-1}(w_m, w_n)(R[w_m] + R[w_n])$, which can be re-written

Figure 3. Linear growth of W-states using the proposed fusion gate. Consecutive successful events are shown and recycling is not performed. Dashed boxes denote the fusion operation. Note that actual sizes of the W-states are found by adding 2 to the sizes given in the figure.
Figure 4. Exponential growth of W-states using the proposed fusion gate. W-states of the same size are fused. Consecutive successful events are shown and recycling is not performed. Dashed boxes denote the fusion operation. Note that actual sizes of the W-states are found by adding 2 to the sizes given in the figure.

as

\[(m + n + 2)R[w_{m+n}] = (m + 2)(n + 2)(R[w_m] + R[w_n]).\]  \(9\)

Similarly, the cost \(R[w_{m+(k+1)n}]\) of preparing the state by fusing \(w_n\) with the state \(w_{m+kn}\) prepared in a previous fusion operation becomes

\[(m + (k + 1)n + 2)R[w_{m+(k+1)n}] = (m + kn + 2)(n + 2)(R[w_{m+kn}] + R[w_n]).\]  \(10\)

Defining \(r_k = (m + kn + 2)R[w_{m+kn}]\) and \(\xi = (n + 2)R[w_n]\), we can re-write equation (10) as

\[r_{k+1} = (n + 2)r_k + \xi(m + kn + 2),\]  \(11\)

which is a first-order inhomogeneous recurrence equation. Solving equation (11) yields

\[r_k = (r_0 - \beta)(n + 2)^k + \alpha k + \beta,\]  \(12\)

where \(\alpha = -\xi n / (n + 1), \beta = (\alpha - \xi (m + 2)) / (n + 1)\) and \(r_0 = (m + 2)R[m]\). Subsequently, we can find the cost \(R[w_{m+kn}]\) of preparing the state \(w_{m+kn}\) as \(R[w_{m+kn}] = r_k / (m + kn + 2)\). Here we consider the case \(m = n = 1\) where \(|w_1⟩ = |W_3⟩\) is repeatedly fused. Then we have \(R[w_{k+1}] = r_k / (k + 3)\) where \(r_k\) is found by setting \(n = 1\) and \(m = 1\) in equation (12). Consequently, we can find the cost \(R[w_N]\) of preparing the state \(w_N\) by setting \(k \to N - 1\) as

\[R[w_N] = \frac{1}{N + 2} \left( \frac{11}{4}3^N - \frac{3}{2}N - \frac{15}{4} \right),\]  \(13\)
Figure 5. Comparison of the expected amount of resources $R[N]$ to prepare a $W_N$ for strategies introduced in the text: linear growth by one, i.e. $\{n, 1\} \rightarrow \{n+1\}$, with and without recycling, exponential growth, i.e. $\{n, n\} \rightarrow \{2n\}$, without recycling, fusing W-states of similar sizes and optimal strategy without recycling. Each point corresponds to the average of 1000 trials. Note that sizes given here are the actual sizes of the states.

which is depicted in figure 5 (red + sign). Thus we conclude that the cost of preparing the state $w_N$ by the linear growth strategy is of the order $O(3^N)$; that is, it increases exponentially with $N$.

3.1.2. Linear growth with recycling. As shown in section 2, recyclable failure of the fusion gate, which takes place with probability $P_r(w_m, w_m) = (n+1)(m+1)/(n+2)(m+2)$, leads to a reduction in the size of the initial W-states by one qubit, e.g. $\{m, n\} \rightarrow \{m-1, n-1\}$. Therefore, the remaining W-states can be recycled. Here we include this recycling in the linear-growth strategy and see how the cost $R[w_m]$ changes.

The averaged cost $R[w_{m+1}]$ can be written as $R[w_{m+1}] = R[w_m] + \Delta_{m+1}$, where $\Delta_{m+1}$ is the averaged cost of creating $w_{m+1}$ when a shorter W-state $w_m$ is given. Let us calculate $\Delta_{m+1}$ as follows. Suppose that state $w_m$ is given, and we apply a fusion gate to it with $w_1$, by paying a unit cost. At probability $p_m \equiv P_s(w_m, w_1)$, the gate succeeds, and no additional cost is required. At probability $q_m \equiv P_r(w_m, w_1)$, we are left with the state $w_{m-1}$, and it takes an additional cost of $\Delta_m + \Delta_{m+1}$ to obtain $w_{m+1}$. Finally, at probability $1 - p_m - q_m$, the gate fails completely and we need full cost $R[w_{m+1}]$ to produce $w_{m+1}$. These observations lead to

$$\Delta_{m+1} = 1 + q_m(\Delta_m + \Delta_{m+1}) + (1 - p_m - q_m)R[w_{m+1}]$$

and thus we have

$$p_m R[w_{m+1}] = R[w_m] + 1 - q_m R[w_{m-1}]$$

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This recursive formula can be numerically solved with \( R[w_1] = 1 \) and \( R[w_2] = 9/2 \), which is depicted in figure 5 (pink dotted box). When \( m \) is large, \( p_m \sim 1/3 \) and \( q_m \sim 2/3 \) lead to \( R[w_{m+1}] - R[w_m] \sim 2(R[w_m] - R[w_{m-1}]) \). We thus conclude that although recycling reduces the required resources from \( O(3^m) \) to \( O(2^m) \), it does not change the resource scaling law: regardless of whether recycling is performed or not, the required amount of resource to prepare a desired state using linear-growth strategies scales exponentially.

### 3.2. Optimal strategy without recycling

We have seen that strategies to grow W-states by a constant amount at each step are not as efficient, even if recycling is introduced. We should thus turn to other strategies for efficiency. In this subsection, we numerically determine the optimal cost over all the strategies without recycling.

Let us start from simple examples. In order to produce \( w_2 \), the only way is to fuse two \( w_1 \) states, since \( w_2 \) cannot be produced when larger W-states are fused under the assumption of no recycling. The optimal cost \( R[w_2]_{\text{opt}} \) is thus given by \( P_1(w_1, w_1)^{-1}(1 + 1) = 9/2 \). Similarly, there is only one way to produce \( w_3 \), leading to \( R[w_3]_{\text{opt}} = P_1(w_2, w_1)^{-1}(1 + 9/2) = 66/5 \). In the case of \( w_4 \), on the other hand, there are two possible ways, \( \{w_1, w_3\} \) and \( \{w_2, w_2\} \), with the respective successful fusion probabilities of \( 2/5 \) and \( 3/8 \). Since we know the optimal costs for preparing \( w_2 \) and \( w_3 \), we calculate the cost of preparing \( w_4 \) from \( \{w_1, w_3\} \) as \((5/2)(1 + 66/5) = 71/2\), whereas that from \( \{w_2, w_2\} \) as \((8/3)(9/2 + 9/2) = 24\). The latter strategy is better, and hence \( R[w_4]_{\text{opt}} = 24 \). In this way, the optimal cost of any state can be numerically calculated using the recursive formula

\[
R[w_m]_{\text{opt}} = \min_{k=1,...,m-1} \frac{R[w_k]_{\text{opt}} + R[w_{m-k}]_{\text{opt}}}{P_1(w_k, w_{m-k})}.
\]

The calculated values of \( \{R[w_N]_{\text{opt}}\} \) are presented in figure 5 (green dotted circle), which suggests a sub-exponential resource scaling.

### 3.3. Exponential growth strategy without recycling

Although the discussion in the previous subsection enables us to numerically calculate the optimal cost in the case of no recycling, it does not tell us what the optimal strategy looks like or how the cost scales in the limit of a large target size. Here we consider a specific strategy without recycling, based on fusing two states of the same size to double it. We show that this strategy works under the optimal cost, and derive an analytical expression for the cost in order to see the scaling over the target size.

Here we only consider the production of a state \( w_N \) whose size is written as \( N = 2^k \). In what we call an exponential-growth strategy (see figure 4), the state \( w_N \) is produced by the fusion of two W-states \( w_{N/2} \) with equal size. The state \( w_{N/2} \) is in turn generated from the fusion of the state \( w_{N/4} \). When no recycling is performed, equation (8) leads to a simple relation among the costs in this strategy:

\[
R[w_{2i+1}] = \frac{(2^i + 2)^2}{2^i + 1} R[w_{2i}].
\]

If we define \( a_i \equiv (1 + 2^{i-1})R[w_{2i}] \), we have \( a_{i+1} = 2^{i+1}(1 + 2^{1-i})a_i \) and thus

\[
R[w_N] = R[w_{2^k}] = \frac{\gamma_k}{1 + 2^{2^{-k/2}}} 2^{k(k+1)/2},
\]

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Initially, let all the sets $S_m$ depending on the values of $l$. Depending on the values of $l$, the resultant states $S_m$ belong to the same set $S_m$. In the description below, $\mu$ and $\xi$ represent the number of recycled states and the number of consumed states, respectively. Apart from the case of no recycling, both are based on the same working principle, i.e. connecting a pair of ingredients having the same ‘size’ (distance or number).

3.4. Fusing states of similar size with recycling

We have seen that fusing two W-states of the same size is advantageous in the case of no recycling. Here we propose a strategy with recycling, which tries to fuse W-states of similar size. The performance of the strategy is then evaluated through Monte Carlo simulations.

Let us classify generated W-states into sets $S_m$ according to their sizes, such that the state $w_m$ belongs to the set $S_l$ when $m \in (2^{l-1}, 2^l]$. The idea is to perform fusion operation between two states belonging to the same set. It is easy to see that the fusion of two states $w_m$ and $w_n$ belonging to the same set $S_l$ will produce the state $w_{m+n}$ in $S_{l+1}$ upon successful operation. If the operation is complete failure, the states $w_n$ and $w_m$ are discarded. In the case of recyclable outcome, the resultant states $w_{m-1}$ and $w_{n-1}$ belong to either $S_l$ or $S_{l-1}$.

More precisely, our strategy to prepare a state belonging to the set $S_{k+1}$ is described in the following algorithm, which dictates the order in which the recycled W-states should be used. In the description below, $\mu$ represents the number (0, 1 or 2) of states already generated in the set $S_l$, $R$ is the cost (the number of consumed states $w_1$), and $\xi$ is a pointer to the ‘current’ working set.

1. Initially, let all the sets $S_0, \ldots, S_{k+1}$ be empty, $\xi = 0$, $R = 0$ and $\mu_0, \ldots, \mu_{k+1} = 0$.
2. Depending on the values of $\xi$ and $\mu_\xi$, perform one of the following procedures.
   (1) $\mu_\xi \leq 1$ and $\xi = 0$: add $w_1$ to $S_0$, $\mu_0 \rightarrow \mu_0 + 1$, $R \rightarrow R + 1$, and repeat step 2.
   (2) $\mu_\xi \leq 1$ and $\xi \geq 1$: $\xi \rightarrow \xi - 1$ (decrement $\xi$) and repeat step 2.
   (3) $\mu_\xi = 2$: proceed to step 3.
3. $S_\xi$ should have two states, which we denote $w_n$ and $w_m$, and we apply a fusion gate on them. Empty $S_\xi$ and set $\mu_\xi \rightarrow 0$. Perform one of the following procedures depending on the result of the gate operation.
   (Complete failure): go back to step 2.
   (Recyclable): add $w_{n-1}$ and $w_{m-1}$ to appropriate sets ($S_\xi$ or $S_{\xi-1}$), and update $\mu_\xi$ and $\mu_{\xi-1}$ accordingly. Go back to step 2.
   (Success): if $\xi = k$, the goal has been achieved with cost $R$ and the procedure ends here. Otherwise, add $w_{n+m}$ to $S_{\xi+1}$, $\mu_{\xi+1} \rightarrow \mu_{\xi+1} + 1$, $\xi \rightarrow \xi + 1$, and go back to step 2.

\[ \gamma_k \equiv \frac{3}{2} \prod_{l=0}^{k-1} (1 + 2^{1-l}), \]

which is plotted in figure 5. We see that the cost coincides with the optimal cost for no recycling derived in the previous section, indicating that the strategy of fusing two states of the same size is very cost-effective. Since the coefficient $\gamma_k$ is finite, namely $\gamma_k \leq \lim_{k \rightarrow \infty} \gamma_k = 21.458$, \ldots, the scaling of the cost in the limit of large $k$ is $O(2^{k(k+1)/2})$, or equivalently,

\[ R[w_N] = O(\sqrt{N^{\log_2 N/2}}) \]

in the limit of large $n$, which is sub-exponential in $n$.

We remark in passing that this strategy is, in a way, analogous to quantum repeater protocols [35], especially to those in which the efficiency in entanglement connection decreases slowly as the distance gets larger [36]. There the goal is to create an entangled pair at a larger distance, while here we aim at creating entanglement among a larger number of qubits. Apart from this difference, both are based on the same working principle, i.e. connecting a pair of ingredients having the same ‘size’ (distance or number).
The above strategy creates a W-state of (actual) size \( N = 2^k + 3 \) or larger. We have performed Monte Carlo simulations for \( k = 0, \ldots, 6 \), and figure 5 (black square) shows the cost averaged over 1000 runs for each value of \( k \). It is clearly seen that this strategy, based on fusing the states of similar sizes with recycling, outperforms all the other strategies considered in previous sections.

For this strategy, we also performed simulations for a situation where a limited number of \( w_3 \) (i.e. \( w_1 \)) states are initially given to see the success probability of producing one final state of \( w_n \). For each of the initial resource budget, we performed fusing operations until either the desired W-state is obtained or all the resources are consumed. We repeated the process 10 000 times and recorded the successful runs to estimate the probability of success. We found that to prepare the final state \( N = 10 \) with a success probability of \( p_s \geq 0.25 \), at least 50 \( W_3 \) states should be available, and \( p_s \geq 0.50 \) is achievable with an initial budget of 100 or more \( W_3 \) states. For \( N = 20 \), we find that \( p_s \geq 0.20 \) requires an initial budget of 400 or more \( W_3 \) states, and the success probability becomes \( p_s \geq 0.57 \) when the budget is doubled. For \( N = 40 \), the required number of \( W_3 \) initial resources becomes 1600 or more for \( p_s \geq 0.21 \) and more than 3200 for \( p_s \geq 0.57 \).

4. Conclusion

In this paper, we introduced an optical fusion gate to fuse W-states for preparing a W-state of larger size and discussed the scaling laws for the resources required for preparing a W-state using the proposed fusion gate.

We introduced four different strategies with different resource requirements depending on whether recycling is allowed or not. We first demonstrated both analytically and numerically that resource requirement for linear growth strategies, which consider the repeated fusion of a fixed-size W-state with the already existing W-state. These strategies scale exponentially regardless of whether recycling is performed or not, although recycling allows a reduction in the required resources. Then we calculated the optimal cost for fusion without recycling. Next, we considered exponential growth strategies in which the fusion gate is always applied to two states of the same size. We derived analytical expressions showing that the required resources scale sub-exponentially. Interestingly, the non-recycling exponential growth strategy appears to have the same resource scaling as the optimal strategy with non-recycling, implying that the former is the optimal solution for fusion without recycling. Finally, through numerical simulations we demonstrated a strategy, in which states with the closest sizes are fused, that provides the best resource scaling for the proposed fusion gate among the strategies investigated in this study.

The proposed fusion gate and the discussed fusion strategies outperform the previously proposed W-state preparation and expansion gates in terms of the required resources when the size of the state to be prepared is large. Our study does not exclude the possibility of the presence of a better strategy or a strategy with a polynomial scaling for the proposed fusion gate. We hope that this study will initiate further work and continuing discussions on the optimal methods of fusing/preparing/expanding W-states and understanding the structure of larger multipartite entangled states.

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Note Added. The fusion gate for W-state studied in this paper was first proposed by us in [37] and further detailed in [38, 39]. During the final stage of preparation of the manuscript, we became aware of [40].

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