ON THE GOODNESS OF “QUANTUM BLOBS” AS SYMPLECTIC INVARIANT QUANTUM CELLS

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Abstract

We introduce a notion of quantum cell which is invariant under linear canonical transformations. These cells, which we dub “quantum blobs”, have the essential and characteristic property that their intersection with any plane of conjugate coordinates passing through their center is an ellipse with area equal to one-half of the quantum of action. We show that this notion is particularly suitable for expressing various quantum mechanical properties, including a new geometric setting for the Weyl–Wigner–Moyal formalism.

1 Introduction

It is customary in quantum physics and thermodynamics to “coarse grain” phase space in cubic “quantum cells” with volume $h^n$ ($n$ is the number of degrees of freedom). This is consistent with the Weyl rule for the number $N(\lambda \leq E)$ of quantum states with energy $\lambda \leq E$, which says that

$$N(\lambda \leq E) \sim E \rightarrow \infty \frac{\text{Vol}(\Omega)}{h^n}$$

(1)

where $\Omega$ is the phase space volume within the corresponding energy shell (see for instance Ozorio de Almeida [19]). The integer $N(\lambda \leq E)$ thus appears as the number of quantum cells that can be “packed” inside $\Omega$. A
major drawback of these cubic cells is however that they have not enough symmetries; in particular they do not enjoy any interesting invariance properties under phase space transformations (canonical or not) and they are therefore rather a heuristic tool than a really useful concept. In this paper we propose a tractable substitute for these quantum cells, which has the overwhelming advantage of being invariant under symplectic (= linear canonical) transformations. We will call these substitutes quantum blobs. (We introduced a slightly different notion, under the same name, in [10] to express the quantization of integrable Hamiltonian systems in terms of properties from symplectic topology; also see [8, 9] for related constructions).

By definition, a quantum blob is the image of a phase space ball with radius $\sqrt{\hbar}$ by a (linear) symplectic transformation. (In particular, when $n = 1$, quantum blobs are just phase plane ellipses with area equal to $\frac{1}{2} \hbar$). This is a simple definition, but we will see that it is also useful because it will allow us to formulate various results of phase space quantization in a both precise and concise way, and to deduce new results. One of the main properties of quantum blobs is the following: quantum blobs are phase space ellipsoids, but of a very special kind:

The intersection of a quantum blob $Q$ by any plane passing through its center and parallel to a plane of conjugate variables $x_j, p_j$ is always an ellipse with area $\frac{1}{2} \hbar$.

Thus, quantum blobs can be interpreted as phase space sets of “minimum uncertainty”. This fundamental property (in fact a more general necessary and sufficient condition) will be proved in detail in Section 3 using properties of the symplectic group $Sp(n)$, but here is a simple dynamical proof. We may without restricting the generality of the argument assume that the quantum blob $Q$ is centered at the origin, so that $Q$ is the image of the phase space ball $B^{2n}(\sqrt{\hbar}) : |x|^2 + |p|^2 \leq \hbar$ by a linear symplectic transformation $S$. The intersection of $Q$ by any conjugate plane, say the $x_1, p_1$ plane is an ellipse $\varepsilon$, and that ellipse is the image by $S$ of a big circle $\gamma$ of the sphere $|x|^2 + |p|^2 = \hbar$. We can view $\gamma$ as a Hamiltonian orbit determined by, say, the Hamiltonian function $H = |z|^2$, and the action of that orbit (for one turn) is the area enclosed by $\gamma$, that is

$$\oint_{\gamma} px = \pi (\sqrt{\hbar})^2 = \frac{1}{2} \hbar.$$

Now, the action integral along closed curves is a symplectic invariant, so
that
\[ \oint_{\gamma} p \, dx = \oint_{\varepsilon} p \, dx = \frac{1}{2} \hbar; \]
since \( \varepsilon \) lies in the \( x_1, p_1 \) plane we have
\[ \oint_{\varepsilon} p \, dx = \oint_{\varepsilon} p_1 \, dx_1 = \frac{1}{2} \hbar, \]
showing that the area enclosed by \( \varepsilon \) is \( \frac{1}{2} \hbar \), as claimed.

We urge the reader to note that this property is neither trivial, nor obvious: if we deform a ball using a linear transformation this ball will get stretched out and the ellipses we get by intersecting that ellipsoid by arbitrary planes have no reason in general to have equal areas: it is not the same thing to cut an egg in two equal parts along its long axis or its short axis! What we actually have proven above is a linear version of Gromov’s [11] famous non-squeezing theorem: it is impossible to send a ball inside a cylinder with smaller radius based on a plane of conjugate variables.

Besides the property above quantum blobs have two other interesting features. The first is obvious: even though their volume \( \hbar^n/(n!2^n) \) is much smaller than that of a cubic cell, they can have arbitrarily large spatial extensions (this is, by the way, the reason for which we prefer to call our objects “blobs” instead of cells: a blob is something that can spread out); in that sense quantum blobs are good candidates to account for EPR quantum non-locality. The other property is that they are related to Gaussian states: let \( \text{Quant}_0(n) \) denote the set of all quantum blobs centered at the origin \( z_0 = 0 \) of phase space: \( \text{Quant}_0(n) \) is thus the orbit of the ball \( B^{2n}(0, \sqrt{\hbar}) \) under the action of the symplectic group \( \text{Sp}(n) \). The stabilizer of \( B^{2n}(0, \sqrt{\hbar}) \) under this action is the unitary group \( U(n) \) hence we have the identification

\[ \text{Quant}_0(n) \equiv \text{Sp}(n)/U(n). \]

It turn out that there exists a canonical identification between the coset space \( \text{Sp}(n)/U(n) \) and the space of all Wigner transforms of Gaussians (Littlejohn [18], p. 265). This suggests that there is a strong relation between quantum blobs and the Gaussian states of quantum mechanics. That this is indeed the case, and in a very explicit way, will be shown in Section 5: the ball \( B^{2n}(0, \sqrt{\hbar}) \) correspond to coherent states while the elements of \( \text{Quant}_0(n) \) corresponds to squeezed states. The reader should however not draw hastily the conclusion that the consideration of quantum blobs only leads to a geometric “Doppelgänger” of the WWM formalism: we will see
that they will allow us to prove properties of quantum states which are not immediately analytically obvious.

This paper is organized as follows:

- In Section 2 we review Williamson’s symplectic diagonalization theorem and complement it with a uniqueness result modulo symplectic rotations. We take the opportunity to study in some detail the notion of symplectic spectrum of a symmetric positive definite matrix and deduce from its properties a linear version of Gromov’s celebrated non-squeezing theorem;

- In Section 3 we state and prove the main properties of quantum blobs; in particular we show that the property of being a quantum blob is “hereditary” in the sense that an ellipsoid in a symplectic subspace of phase space is a quantum blob if and only if it is the intersection of a quantum blob in the total phase space containing the given subspace;

- In Section 4 we analyze the notion of quantum blob from the point of view of the statistical covariance matrix, so useful both in quantum mechanics and optics. We introduce for that purpose the notion of “quantum mechanically admissible” phase-space ellipsoid;

- In Section 5 we apply the notions introduced above to the Wigner transform of quantum states and prove that there exists a canonical bijection between minimum uncertainty states (also called “coherent states” in quantum optics) and the set of all quantum blobs; this justifies a posteriori their definition and interpretation as minimum uncertainty sets. We also show that if we smooth out a phase-space Gaussian using the coherent state associated with a quantum blob then we obtain the Wigner transform of a mixed state.

Notations. The standard symplectic form on phase space \( \mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_p \) is denoted by \( \omega \):

\[
\omega(z, z') = (z')^T J z
\]

where \( J \) is the block matrix

\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

The corresponding symplectic group is denoted by \( \text{Sp}(n) \):

\[
S \in \text{Sp}(n) \iff S^T JS = J \iff SJS^T = J.
\]
As usual the unitary group $U(n, \mathbb{C})$ is identified with the group $U(n) = \text{Sp}(n) \cap O(2n)$ of symplectic rotations via the monomorphism

$$A + iB \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$ 

When $M$ is a positive definite symmetric matrix we will write $M > 0$, and when it is semidefinite $M \geq 0$.

The closed ball $|z - z_0| \leq r$ will be denoted by $B^{2n}(z_0, r)$; when $z_0 = 0$ we will simply write $B^{2n}(r)$.

## 2 Williamson’s Theorem and Related Results

This Section is devoted to technical results related to Williamson’s [25] symplectic diagonalization theorem.

### 2.1 Williamson’s theorem

Let $M$ be a real $m \times m$ symmetric matrix: $M = M^T$. Elementary linear algebra tells us that all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $M$ are real, and that $M$ can be diagonalized using an orthogonal transformation. Williamson’s theorem provides us with the symplectic variant of this result. It says that every symmetric and positive definite matrix $M$ can be diagonalized using symplectic matrices, and this in a very particular way:

**Williamson’s theorem:** For every $2n \times 2n$ matrix $M > 0$ there exists $S \in \text{Sp}(n)$ such that $M = S^T DS$ with

$$D = \begin{bmatrix} \Lambda_\omega & 0 \\ 0 & \Lambda_\omega \end{bmatrix}, \quad \Lambda_\omega = \text{diag}[\lambda_{\omega,1}, \ldots, \lambda_{\omega,n}], \quad \lambda_{\omega,j} > 0$$

where the numbers $\pm i\lambda_{\omega,j}$ are the eigenvalues of $JM$.

This result was initially proved by Williamson [25], and has been periodically rediscovered by mathematicians; Folland [5] or Hofer–Zehnder [14] give “modern” proofs (Hofer and Zehnder claim that the result goes back to Weierstrass). We notice that Simon et al. [20] have proved a much more general result using elementary methods.

An important observation is the following: while the diagonal entries of $\Lambda_\omega$ are uniquely determined (up to a reordering) by $M$, there is no reason for the diagonalizing symplectic matrix $S$ to be unique. However:
Lemma 1 Assume that $S$ and $S'$ are two elements of $\text{Sp}(n)$ such that

$$M = (S')^T DS' = S^T DS.$$ 

There exists $U \in \mathbb{U}(n)$ such that $S = US'$.

Proof. Set $U = S(S')^{-1}$; clearly $U$ is symplectic: $U^T J U = UJU^T = J$; in addition $U^T D U = D$. We are going to show that these relations imply $UJ = JU$; this will prove our claim since then $U^T = U^{-1}$ so that $U$ is both orthogonal and symplectic. Set $R = D^{1/2} U D^{-1/2}$; we have

$$R^T R = D^{-1/2} (U^T D U) D^{-1/2}$$

$$= D^{-1/2} D D^{-1/2}$$

$$= I$$

hence $R \in O(2n)$. Since $J$ commutes with each power of $D$ we have (since $JU = (U^T)^{-1} J$)

$$JR = D^{1/2} J U D^{-1/2} = D^{1/2} (U^T)^{-1} J D^{-1/2}$$

$$= D^{1/2} (U^T)^{-1} D^{-1/2} J = (R^T)^{-1} J$$

hence $R \in \text{Sp}(n)$; $R$ is thus a symplectic rotation and therefore $JR = RJ$.

Now $U = D^{-1/2} RD^{1/2}$ and therefore

$$JU = J D^{-1/2} RD^{1/2} = D^{1/2} J RD^{1/2}$$

$$= D^{1/2} RJD^{1/2} = D^{1/2} RD^{1/2} J$$

$$= UJ$$

which was to be proven. 

When the matrix $M$ is itself symplectic, it can be diagonalized using a symplectic rotation:

Lemma 2 Let $S \in \text{Sp}(n)$ be symmetric and positive definite. There exists $U \in \mathbb{U}(n)$ such that

$$S = U^T \Delta U \quad , \quad \Delta = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}$$

(2)

where $\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_n]$ and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 1$ is a choice of $n$ first eigenvalues of $S$ (counted with their multiplicities).
Proof. Since $S$ is symmetric and positive definite its eigenvalues occur in pairs $(\lambda, 1/\lambda)$ with $\lambda > 0$ so that if $\lambda_1 \leq \cdots \leq \lambda_n$ are $n$ eigenvalues then $1/\lambda_1, \ldots, 1/\lambda_n$ are the other $n$ eigenvalues. Let now $U$ be an orthogonal matrix such that $S = U^T \Delta U$ where

$$D = \text{diag}[\lambda_1, \ldots, \lambda_n; 1/\lambda_1, \ldots, 1/\lambda_n];$$

let us prove that $U \in U(n)$. Let $u_1, \ldots, u_n$ be orthonormal eigenvectors of $U$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Since $SJ = JS^{-1}$ ($S$ is both symplectic and symmetric) we have, for $1 \leq k \leq n$,

$$SJ u_k = JS^{-1} u_k = \frac{1}{\lambda_j} Ju_k$$

hence $\pm Ju_1, \ldots, \pm Ju_n$ are the orthonormal eigenvectors of $U$ corresponding to the remaining $n$ eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Write now the $2n \times n$ matrix $[u_1, \ldots, u_n]$ as

$$[u_1, \ldots, u_n] = \begin{bmatrix} A \\ B \end{bmatrix}$$

where $A$ and $B$ are of order $n \times n$; we have

$$[-Ju_1, \ldots, -Ju_n] = -J \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -B \\ A \end{bmatrix}$$

hence $U$ is of the type

$$U = [u_1, \ldots, u_n; -Ju_1, \ldots, -Ju_n] = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$  

Since $U^T U = I$ the blocks $A$ and $B$ are such that

$$AB^T = B^T A, \quad AA^T + BB^T = I_n \quad (3)$$

hence $U$ is also symplectic, that is $U \in U(n)$. \hfill \blacksquare

Remark 3 Observe that the diagonalization formula (2) is not a Williamson diagonalization. However, writing

$$S = (\Delta^{1/2} U)^T (\Delta^{1/2} U) \quad (4)$$

where $\Delta^{1/2}$ is the positive square root of $\Delta$ we obtain the Williamson diagonalization of $S$. This is because the eigenvalues of $JS$ are those of $S^{1/2} JS^{1/2} = J$ ($S^{1/2}$ is symplectic and symmetric), and hence their moduli are all $+1$.  

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2.2 A property of the symplectic spectrum

We will always use the following convention: the eigenvalues $\lambda_{\omega,j}$ of $JM$ will be ranked in decreasing order:

$$\lambda_{\omega,1} \geq \lambda_{\omega,2} \geq \cdots \geq \lambda_{\omega,n} > 0$$  \hspace{1cm} (5)$$

and we will call the $n$-tuple

$$\text{Spec}_\omega(M) = (\lambda_{\omega,1}, \lambda_{\omega,2}, ..., \lambda_{\omega,n})$$

the \textit{symplectic spectrum} of $M$. The symplectic spectrum is a symplectic invariant in the following sense:

$$\text{Spec}_\omega(S^TMS) = \text{Spec}_\omega(M) \quad \text{for every} \ S \in \text{Sp}(n).$$  \hspace{1cm} (6)$$

This property is of course an immediate consequence of the definition of $\text{Spec}_\omega(M)$.

A very important result is the following; it allows to compare the symplectic spectra of two positive definite symmetric matrices.

\textbf{Theorem 4} Let $M$ and $M'$ be two symmetric positive definite matrices. If $M \leq M'$ then $\text{Spec}_\omega(M) \leq \text{Spec}_\omega(M')$.

\textbf{Proof.} We are following almost verbatim the clever argument of Giedke et al. [6] (for a proof using a variational argument see Hofer and Zehnder [14]). When two matrices $M$ and $M'$ have the same eigenvalues we will write $M \simeq M'$. Of course $MM' \simeq M'M$. We thus have to show that if the eigenvalues of $M$ are smaller or equal to those of $M'$, then the eigenvalues of $(JM)^2$ are inferior or equal to those of $(JM')^2$. The relation $M \leq M'$ is equivalent to $z^T M z \leq z^T M' z$ for every $z \in \mathbb{R}^{2n}$. Replacing $z$ par $(JM^{1/2})z$ we thus have

$$z^T (JM^{1/2})^T M' (JM^{1/2}) z \leq z^T (JM^{1/2})^T M (JM^{1/2}) z$$

that is, equivalently,

$$M^{1/2} J M M^{1/2} \leq M^{1/2} J M' M^{1/2}$$  \hspace{1cm} (7)$$

(note that both sides of (7) are symmetric and positive definite). Similarly, replacing $z$ by $(JM^{1/2})z$ in $z^T M' z \leq z^T M z$ we get

$$M'^{1/2} J M M'^{1/2} \leq M'^{1/2} J M' M'^{1/2}.$$  \hspace{1cm} (8)$$

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Noting that we have
\[ M^{1/2}J M' J M^{1/2} \simeq M J M' J \text{ and } M' J M J' M^{1/2} \simeq M' J M J \simeq M J M' J \]
we can rewrite (7) and (8) as
\[ M^{1/2}J M J M^{1/2} \leq M J M' J \text{ and } M J M' J \leq M' J M' J M^{1/2} \]
and hence
\[ M^{1/2}J M J M^{1/2} \leq M' J M' J M^{1/2}. \]
Since we have
\[ M^{1/2}J M J M^{1/2} \simeq (M J)^2, \quad M' J M' J M^{1/2} \simeq (M' J)^2 \]
we finally get \((M J)^2 \leq (M' J)^2\), which was to be proven.

This result has the following consequence from which one can easily derive Gromov’s non-squeezing theorem [11] in the linear case:

**Corollary 5** There exists \( S \in \text{Sp}(n) \) sending an ellipsoid \( \mathbb{B} : z^T M z \leq 1 \) into an ellipsoid \( \mathbb{B}' : z^T M' z \leq 1 \) if and only if \( \text{Spec}_\omega(M) \geq \text{Spec}_\omega(M') \).

**Proof.** The condition \( \text{Spec}_\omega(M) \geq \text{Spec}_\omega(M') \) is necessary: assume that there exists \( S \in \text{Sp}(n) \) such that \( S(\mathbb{B}) \subset \mathbb{B}' \); then
\[ (S^{-1} z)^T M S^{-1} z \geq z^T M' z \]
for all \( z \) and hence \((S^{-1})^T M S^{-1} \geq M'\). It follows from Theorem 4 that
\[ \text{Spec}_\omega((S^{-1})^T M S^{-1}) \geq \text{Spec}_\omega(M') \]
hence \( \text{Spec}_\omega(M) \geq \text{Spec}_\omega(M') \) taking into account the symplectic invariance of \( \text{Spec}_\omega(M) \). Let us prove the sufficiency. Williamson’s theorem allows us to choose \( S_1, S_2 \in \text{Sp}(n) \) such that
\[ S_1(\mathbb{B}) : \sum \lambda_{\omega,j}(x_j^2 + p_j^2) \leq 1 \]
\[ S_2(\mathbb{B}') : \sum \lambda_{\omega,j}'(x_j^2 + p_j^2) \leq 1. \]
The condition \( \text{Spec}_\omega(M) \geq \text{Spec}_\omega(M') \) implies that \( S_1(\mathbb{B}) \subset S_2(\mathbb{B}') \) and hence \( S(\mathbb{B}) \subset \mathbb{B}' \) with \( S = (S_2)^{-1} S_1 \).
3 Properties of Quantum Blobs

Let us begin by proving two preliminary results which show that quantum blobs can be obtained using only simple symplectic transformations, such as translations, symplectic rotations and rescalings.

3.1 Preliminaries

A quantum blob in \( \mathbb{R}_z^{2n} \) is the image of a ball \( B_2^{2n}(z_0, \sqrt{\hbar}) \) by some \( S \in \text{Sp}(n) \).

We can reformulate this definition in terms of affine symplectic transformations. Let \( T(z_0) \) be the translation operator \( z \mapsto z_0 \). In view of the obvious intertwining property

\[
S \circ T(z_0) = T(Sz_0) \circ S
\]

valid for all \( S \in \text{Sp}(n) \) and all \( z_0 \) we have

\[
S(B_2^{2n}(z_0, \sqrt{\hbar})) = T(Sz_0) \circ S(B_2^{2n}(\sqrt{\hbar})).
\]  (9)

A quantum blob is thus the image of the ball \( B_2^{2n}(\sqrt{\hbar}) \) centered at 0 by an element of the affine (or inhomogeneous) symplectic group \( \text{ISp}(n) \) (\( \text{ISp}(n) \) consists of all \( S \circ T \) (or \( T \circ S \)), \( S \in \text{Sp}(n) \), \( T \) a translation).

The following result complements this description:

**Proposition 6** Every quantum blob can be obtained from \( B_2^{2n}(\sqrt{\hbar}) \) by a symplectic rescaling \( (x_j, p_j) \mapsto (\lambda_j x_j, p_j/\lambda_j) \) \( (\lambda_j > 0, j = 1, 2, ..., n) \) followed by a symplectic rotation and a translation.

**Proof.** In view of (9) it is sufficient to assume that \( Q = S(B_2^{2n}(\sqrt{\hbar})) \); the set \( Q \) thus consists of all \( z \) such that \( z^T(SST)^{-1}z \leq \hbar \). In view of Lemma 2 there exists \( U \in U(n) \) such that \( SST = U^T \Delta U \) where \( \Delta \) is a diagonal matrix of the type

\[
\Delta = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}.
\]

Writing \( U^T \Delta U = (\Delta^{1/2} U^T \Delta^{1/2} U) \) (formula (4) in Remark 3) the inequality \( z^T(SST)^{-1}z \leq \hbar \) is equivalent to \( |\Delta^{1/2} U z|^2 \leq \hbar \) and hence \( Q = U^T \Delta^{1/2}(B_2^{2n}(\sqrt{\hbar})). \) Now \( U^T \in U(n) \) and \( \Delta \in \text{Sp}(n) \) is such that if \( z' = \Delta^{1/2}z \) then \( (x'_j, p'_j) = (\lambda_j x_j, p_j/\lambda_j) \) for numbers \( \lambda_j > 0, j = 1, 2, ..., n. \)

As observed in the Introduction the quantum blobs centered at the origin can be identified with elements of the coset space \( \text{Sp}(n)/U(n) \). Since in view of formula (9) the set \( \text{Quant}(n) \) of all quantum blobs can be obtained by
letting the inhomogeneous symplectic group $\text{ISp}(n)$ act on the ball $B^{2n}(\sqrt{\hbar})$ we get the identification

$$\text{Quant}(n) = \text{ISp}(n)/U(n).$$

The dimension of $\text{ISp}(n)$ and of $U(n)$ being respectively $n(2n + 1) + 2n$ and $n^2$ it follows that

$$\dim \text{Quant}(n) = n(n + 3).$$

For instance $\dim \text{Quant}(1) = 4$: given a disk in the phase plane we need two parameters to move it around in the phase plane using translations, one parameter to dilate it without changing its area and one parameter (angle!) to rotate it.

**Remark 7** *The group $\text{ISp}(n)$ being arcwise connected, so is $\text{Quant}(n)$.***

### 3.2 The characteristic property of a quantum blob

We will denote phase space ellipsoids by the “blackboard bold” letter $\mathbb{B}$ and quantum blobs by $Q$.

As already emphasized in the Introduction, a quantum blob has the property that if we cut it through its center by a plane which is parallel to a plane of conjugate coordinates $x_j, p_j$ we will always obtain an ellipse with area $\frac{1}{2} \hbar$. In fact, we have a more precise result, which will produce a necessary and sufficient condition for an ellipsoid to be a quantum blob. Before we state and prove this condition let us introduce some terminology:

- A *symplectic plane* is any two-dimensional linear subspace $P$ of $\mathbb{R}^{2n}_z$ on which the symplectic form is non-degenerate; in other words $P$ is a symplectic vector space when equipped with the restriction of $\omega$.

- An *affine symplectic plane* $P_{z_0}$ is the image of a symplectic plane $P$ by the phase-space translation $z \mapsto z + z_0$. (It is, in particular, a symplectic submanifold of $\mathbb{R}^{2n}_z$.)

**Theorem 8** *A phase space ellipsoid $\mathbb{B}$ with center $z_0$ is a quantum blob if and only if its intersection with any affine symplectic plane $P_{z_0}$ is an ellipse with area $\frac{1}{2} \hbar$.***

**Proof.** It is of course no restriction to choose $z_0 = 0$. (i) Assume that $Q$ is a quantum blob: $Q = S(B^{2n}(\sqrt{\hbar}))$ for some $S \in \text{Sp}(n)$. Let $P$ be a symplectic plane; the set

$$S^{-1}(Q \cap P) = B^{2n}(\sqrt{\hbar}) \cap S^{-1}(P)$$

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is the disk $B^2(\sqrt{\hbar})$ in $S^{-1}(\mathbb{P})$ and has thus area $\frac{1}{2}h$. Since the restriction $S_{|\mathbb{P}'}$ of $S$ to $\mathbb{P}' = S^{-1}(\mathbb{P})$ is symplectic, and hence area preserving, we have

$$\text{Area}(S(B^{2n}(\sqrt{\hbar})) \cap \mathbb{P}) = \text{Area}(S_{|\mathbb{P}'}(B^2(\sqrt{\hbar}))) = \frac{1}{2}h$$

which was to be proven [one can give an alternative proof of $(i)$ using Remark 3 following Lemma 2]. $(ii)$ Suppose conversely that $B$ is an ellipsoid centered at the origin and such that

$$\text{Area}(\mathbb{B} \cap \mathbb{P}) = \frac{1}{2}h$$

for every symplectic plane $\mathbb{P}$. Since $S(\mathbb{B} \cap \mathbb{P}) = S(\mathbb{B}) \cap S(\mathbb{P})$ and since $\text{Sp}(n)$ acts transitively on symplectic planes we have

$$\text{Area}(S(\mathbb{B}) \cap \mathbb{P}) = \frac{1}{2}h$$

for every $S \in \text{Sp}(n)$. Let $M > 0$ be such that the ellipsoid $\mathbb{B}$ is defined by the inequality $z^T M z \leq 1$; in view of Williamson’s theorem we can find $S \in \text{Sp}(n)$ such that

$$M = S^T D S \, , \, D = \begin{bmatrix} \Lambda_{\omega} & 0 \\ 0 & \Lambda_{\omega} \end{bmatrix}$$

where $\Lambda_{\omega} = \text{diag} g[\lambda_{\omega,1}, ..., \lambda_{\omega,n}]$. The ellipsoid $S^{-1}(\mathbb{B})$ is thus defined by $z^T D z \leq 1$, that is

$$\sum_{j=1}^{n} \lambda_{\omega,j} (x_j^2 + p_j^2) \leq 1.$$ 

It follows that the intersections of $S^{-1}(\mathbb{B})$ by the symplectic coordinate planes $\mathbb{P}_j = \{z : z_k = 0; k \neq j\}$ are the disks

$$\mathbb{D}_j : \lambda_{\omega,j} (x_j^2 + p_j^2) \leq 1$$

hence the equality

$$\text{Area} \, \mathbb{D}_j = \frac{\pi}{\lambda_{\omega,j}} = \frac{1}{2}h$$

implies that we must have $\lambda_{\omega,j} = 1/h$ for $j = 1, 2, ..., n$ and $S^{-1}(\mathbb{B})$ is thus the ball $B^{2n}(\sqrt{\hbar})$ so that $\mathbb{B} = S(B^{2n}(\sqrt{\hbar})$. 

Theorem 8 has the following straightforward consequence which is a linear version of Gromov’s famous non-squeezing theorem [11]:
Corollary 9 The orthogonal projection of a quantum blob $Q$ on any symplectic affine plane $\mathbb{P}_{z_1}$ has an area at least equal to $\frac{1}{2}h$.

Proof. Let $\mathbb{P}_{z_0}$ be the affine plane parallel to $\mathbb{P}_{z_1}$ and passing through the center of $Q$. That plane is also an affine symplectic plane hence $Q \cap \mathbb{P}_{z_0}$ has area $\frac{1}{2}h$. The orthogonal projection of $Q$ on $\mathbb{P}_{z_1}$ contains that of $Q \cap \mathbb{P}_{z_0}$ and is hence superior or equal to $\frac{1}{2}h$.

3.3 Extension to symplectic subspaces

The observant reader will have noticed that the main ingredient in the proof of the necessary condition in Theorem 8 was the following property:

*The restriction of a symplectic transformation to a symplectic plane $\mathbb{P}$ is area-preserving.*

This suggests that we might go one step further and study the intersection of quantum blobs by arbitrary symplectic affine subspaces passing through their center. This will allow us to associate to any quantum blob in $\mathbb{R}^{2n}$ a sort of “hierarchy” of quantum blobs with smaller dimensions by cutting the blob with symplectic (affine) spaces with smaller and smaller dimensions.

Before we proceed to do so, recall that:

- A **symplectic subspace** is a linear subspace $V$ of $\mathbb{R}^{2n}$ which is a symplectic vector space when equipped with the restriction of $\omega$. An **affine symplectic subspace** is obtained by translation in $\mathbb{R}^{2n}$ of a symplectic subspace.

- A **symplectic basis** of a symplectic space $V$ is a basis $(e_1, \ldots, e_m; f_1, \ldots, f_m)$ such that $\omega(f_j, e_k) = \delta_{jk}$ and $\omega(e_j, e_k) = \omega(f_j, f_k) = 0$ for $1 \leq j, k \leq m$.

- An **orthosymplectic basis** of a symplectic subspace is a basis which is both symplectic and orthogonal with respect to the scalar product determined by $\omega$.

A symplectic subspace always has even dimension $2m$ due to the non-degeneracy of the symplectic form. Given two symplectic subspaces $V$ and $V'$ of same dimension there exists a linear mapping $S_{V', V} : V \longrightarrow V'$ such that

$$\omega_{V'}(S_{V', V}v, S_{V', V}v') = \omega_V(v, v')$$

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for every \( v \in V \); \( \omega \) and \( \omega' \) are the restrictions of the symplectic form \( \omega \) to \( V \) and \( V' \), respectively (to construct such a \( S_{\omega, \omega'} \) it suffices to choose symplectic bases in \( V \) and \( V' \)). We will say that \( S_{\omega, \omega'} \) is as symplectic mapping. Note that it is volume preserving when \( V \) and \( V' \) are equipped with their canonical volume elements. When \( V = V' \) the symplectic mappings \( V \to V' \) form a group, the symplectic group \( \text{Sp}(V) \). Using orthosymplectic bases of \( V \) and \( V' \) one can as well construct mappings \( U_{V', V} : V \to V' \) that are both symplectic and orthogonal (orthogonality referring here to the fact that \( U_{V', V} \) takes one symplectic basis of \( V \) to an orthogonal basis of \( V' \). The following Lemma summarizes two results we shall need:

**Lemma 10** (i) The unitary group \( U(n) \) (and hence \( \text{Sp}(n) \)) acts transitively on the manifold of all symplectic subspaces of \( \mathbb{R}^{2n}_2 \) with same dimension; (ii) Every symplectic mapping \( S_{\omega, \omega'} : V \to V' \) is the restriction to \( V \) of an element of \( \text{Sp}(n) \). In particular, for every \( S \in \text{Sp}(V) \) there exists \( S \in \text{Sp}(n) \) such that \( S_{\omega} = S_{\omega'} \).

**Proof.** (i) Let \( V \) and \( V' \) be two symplectic subspaces of \( \mathbb{R}^{2n}_2 \) such that \( \dim V = \dim V' = 2k \) and let

\[
S_{2k} = (e_1, \ldots, e_m; f_1, \ldots, f_m), \quad S'_{2k} = (e'_1, \ldots, e'_m; f'_1, \ldots, f'_m)
\]

be orthosymplectic bases of \( V \) and \( V' \), respectively. Completing these bases into full symplectic bases \( S_{2n} \) and \( S'_{2n} \) of \( \mathbb{R}^{2n}_2 \), the linear mapping \( S : \mathbb{R}^{2n}_2 \to \mathbb{R}^{2n}_2 \), defined by \( S(S_{2n}) = S'_{2n} \), is symplectic and such that \( S(\mathbb{V}) = \mathbb{V}' \); the restriction of \( S \) to \( \mathbb{V} \) is orthogonal since \( S(S_{2n}) = S'_{2n} \). The bases \( S_{2n}, S'_{2n} \) are unitary. (ii) Let \( S_{\omega, \omega'} \) be a symplectic mapping of \( \mathbb{V} \to \mathbb{V}' \). Choose a symplectic basis \( S'_{2k} \) of \( \mathbb{V} \) as above; then \( S'_{2m} = (e'_1, \ldots, e'_m; f'_1, \ldots, f'_m) \) with \( e'_j = S_{\omega, \omega'}(e_j), f'_j = S_{\omega, \omega'}(f_j), 1 \leq j \leq m \), is also a symplectic basis of \( \mathbb{V} \). Let us now again complete \( S_{2m} \) and \( S'_{2m} \) into full symplectic bases \( S_{2n} \) and \( S'_{2n} \) of \( \mathbb{R}^{2n}_2 \) and define \( S \in \text{Sp}(n) \) by \( S(e_j) = e'_j, S(f_j) = f'_j, 1 \leq j \leq n \). The restriction of \( S \) to \( \mathbb{V} \) is just \( S_{\omega, \omega'} \).

**Proposition 11** (i) The intersection of a quantum blob \( Q = S(B^{2n}(z_0, \sqrt{\hbar})) \) with any affine symplectic subspace \( V_{z_0} = T(z_0)V \) of \( \mathbb{R}^{2n}_2 \) is a quantum blob in \( V_{z_0} \), that is the image of a ball of \( V_{z_0} \) with radius \( \sqrt{\hbar} \) by an element of the symplectic group of the symplectic space \( V \). (ii) Conversely, any quantum blob \( Q_{z_0} \) in \( V_{z_0} \) is the intersection of a quantum blob in \( \mathbb{R}^{2n}_2 \) with \( V_{z_0} \).

**Proof.** (i) It is a straightforward adaptation of the proof of the necessary condition in Theorem 8. It again suffices to assume \( z_0 = 0 \). Let \( V \) be a
symplectic subspace, \( \dim \mathbb{V} = 2m \); then

\[
S^{-1}(Q \cap \mathbb{V}) = B^{2n}(\sqrt{\hbar}) \cap S^{-1}(\mathbb{V})
\]

is the ball \( B^{2m}(\sqrt{\hbar}) \) in \( S^{-1}(\mathbb{V}) \). The restriction \( S|_{\mathbb{V}'} \) of \( S \) to \( \mathbb{V}' = S^{-1}(\mathbb{V}) \) being symplectic, the set

\[
S|_{\mathbb{V}'}(B^{2m}(\sqrt{\hbar})) = S(B^{2n}(\sqrt{\hbar})) \cap \mathbb{V}
\]

is a quantum blob in \( \mathbb{V} \). (\( ii \)) Let \( Q_0 = S_\mathbb{V}(B^{2n}(\sqrt{\hbar})) \) and choose \( S \in \text{Sp}(n) \) such that \( S|_{\mathbb{V}} = S_\mathbb{V} \) (Lemma 10, (\( ii \))). Let \( Q = S(B^{2n}(\sqrt{\hbar})) \); we have

\[
Q \cap S(\mathbb{V}) = S(B^{2n}(\sqrt{\hbar}) \cap \mathbb{V}) = Q_0.
\]

\[\blacksquare\]

**Remark 12** One might wonder whether it could be possible to find a ball \( B^{2n}(r) \) with radius \( r \) smaller than \( \sqrt{\hbar} \) and \( S \in \text{Sp}(n) \) such that the intersection of \( S(B^{2n}(r)) \) with a symplectic subspace plane is a quantum blob. Such a possibility is however ruled out by Theorem 8.

This discussion can be extended *mutatis mutandis* to the general case: every quantum blob of an affine symplectic subspace is the intersection of a quantum blob in the whole space with that subspace. This result can be interpreted in the following way: assume that a Hamiltonian system consists of two parts, Alice and Bob; each of these parts has its own phase space, which is equipped with the restriction of the overall symplectic form to each part. The discussion above says that a quantum blob in the whole phase space is perceived as an own private quantum blob by Alice as well as by Bob. It would certainly be interesting to pursue this analysis from the point of view of the so important entangled states of quantum mechanics; we hope to come back to this issue in a not too distant future.

Let us end this section with the following observation. Planck’s constant does of course not play any particular role in all what we have done and can therefore be replaced by any number \( r > 0 \). The images of arbitrary balls by symplectic transformations could perhaps be used as measures of the information contained in parts of Hamiltonian systems along the lines suggested in the thesis of Wolf [26] where a similar situation is discussed from the point of view of the BBGKY hierarchy.
4 Admissible Ellipsoids and Quantum Uncertainty

What about ellipsoids containing quantum blobs? It turns out that these also are of genuine interest (we will use them copiously in our study of uncertainty and of the WWM formalism).

We will call a phase space ellipsoid containing a quantum blob a *quantum mechanically admissible ellipsoid* or, for short: *admissible ellipsoid*.

4.1 Properties of admissible ellipsoids

The following property of admissible ellipsoids is very simple and “obvious”; the proof of its first part however requires Williamson’s theorem, and the proof of the second part the highly non-trivial Theorem 4!

**Proposition 13** (i) A phase space ellipsoid $\mathbb{B}$ is quantum mechanically admissible if and only if its intersection with every affine symplectic plane $\mathbb{P}$ passing through its center has area at least $\frac{1}{2}\hbar$. (ii) An admissible phase space ellipsoid $\mathbb{B} : z^T M z \leq 1$ can be sent in another phase space ellipsoid $\mathbb{B}' : z^T M' z \leq 1$ by an (affine) symplectic transformation if and only if $\text{Spec}_\omega(M') \leq \text{Spec}_\omega(M)$.

**Proof.** (i) That the condition is necessary is indeed obvious: if $\mathbb{B}$ contains a quantum blob $\mathbb{Q}$ then every plane through its center will also cut $\mathbb{Q}$, and if this plane is affine symplectic then the area of this section will be $\frac{1}{2}\hbar$. The hard part is to prove that it is also sufficient. Here is why: while it is true that if every section $\mathbb{B} \cap \mathbb{P}$ has area $\geq \frac{1}{2}\hbar$ then it will contain a smaller ellipse with area $\frac{1}{2}\hbar$, it is not clear that these ellipses are the “cuts” by symplectic planes of a same quantum blob contained in $\mathbb{B}$. That it is actually true is a consequence of Williamon’s theorem: choose $S \in \text{Sp}(n)$ such that

$$S^{-1}(\mathbb{B}) : \sum_{j=1}^{n} \frac{1}{R_j^2}(x_j^2 + p_j^2) \leq 1$$

(the $R_j$ are just the numbers $1/\sqrt{\lambda_{\omega,j}}$). If we now cut $S^{-1}(\mathbb{B})$ with the plane $\mathbb{P}_j$ of coordinates $x_j, p_j$ we get the disk

$$B^2(R_j) : x_j^2 + p_j^2 \leq R_j^2$$

The intersection $\mathbb{B} \cap S(\mathbb{P}_j)$ is an ellipse with same area $\pi R_j^2$ as that disk hence $\pi R_j^2 \geq \frac{1}{2}\hbar$ that is $R_j \geq \sqrt{\hbar}$. This implies that $S^{-1}(\mathbb{B})$ contains the ball $B^{2n}(\sqrt{\hbar})$ and hence $\mathbb{B} \supset S(B^{2n}(\sqrt{\hbar}))$ which was to be proven. (ii) Since
the symplectic spectrum is insensitive to phase space translations we may assume that \( \mathbb{B} \) and \( \mathbb{B}' \) have same center \( z_0 = 0 \). The result then immediately follows from the Corollary 5 of Theorem 4.

In [8, 9] we showed that the notion of symplectic capacity can be fruitfully used to describe in a concise way quantization of integrable systems. The notion also leads to an economical way of describing admissible ellipsoids.

Let us recall how the (Gromov) symplectic capacity \( c \) is defined: if \( \Omega \) is a subspace of \( \mathbb{R}^{2n}_z \) we denote by \( R \) the supremum of all \( r \geq 0 \) such that there exists a canonical transformation \( f \) of \( \mathbb{R}^{2n}_z \) sending a phase space ball with radius \( r \) inside \( \Omega \). We call that \( R \) (which can be \( +\infty \)) the symplectic radius of \( \Omega \) and \( \pi R^2 \) its Gromov width. The Gromov width of \( \Omega \), which we will denote by \( c(\Omega) \), is a particular and useful case of the more general notion of symplectic capacity. A symplectic capacity \( c \) on \( \mathbb{R}^{2n}_z \) assigns to every subset \( \Omega \subset \mathbb{R}^{2n}_z \) a non-negative number or \( +\infty \) and satisfies the following properties:

1. If \( f : \Omega \rightarrow \Omega' \) is canonical then \( c(f(\Omega)) \leq c(\Omega') \);
2. \( c(k\Omega) = k^2 c(\Omega) \) for every real number \( k \);
3. \( c(B^{2n}(r)) = \pi r^2 = c(Z_j(r)) \) where \( Z_j(r) : x_j^2 + p_j^2 \leq r^2 \).

Notice that (1) implies, in particular, that \( c(\Omega) \leq c(\Omega') \) if \( \Omega \subset \Omega' \) and that \( c(f(\Omega)) = c(\Omega) \) for every canonical transformation \( f \).

All these properties are trivial except the last: the equality \( c(Z_j(r)) = \pi r^2 \) is namely equivalent to Gromov’s theorem [11] which asserts that it is impossible to “squeeze”, using canonical transformations, a ball inside a cylinder \( Z_j(r) \) with smaller radius; all known proofs of Gromov’s theorem are notoriously difficult.

**Remark 14** It is easy to check (see for instance [14]) that the Gromov width is the smallest of all symplectic capacities: \( c \leq c \) for every other symplectic capacity \( c \).

Let \( \mathbb{B} : z^T M z \leq 1 \) be a phase space ellipsoid; in view of Williamson’s theorem there exists \( S \in \text{Sp}(n) \) such that

\[
S^{-1}(\mathbb{B}) : \sum_{j=1}^n \frac{1}{R_j^2}(x_j^2 + p_j^2) \leq 1;
\]

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the numbers $R_j^2$ are the inverses of the elements $\lambda_{\omega,j}$ of the symplectic spectrum of $M$, so that according to the convention (5) we have

$$0 < R_1 \leq R_2 \leq \cdots \leq R_n.$$  \hfill (10)

An essential property (see for instance [14]) is that

$$c(B) = c_{\text{aff}}(B) = \pi R_1^2$$  \hfill (11)

where $c_{\text{aff}}$ is the affine symplectic capacity: $c_{\text{aff}}(B)$ is the number $\pi R_1^2$ where $R$ is the supremum of the radii all balls $B^{2n}(r)$ that can be sent in $B$ using translations and elements of $\text{Sp}(n)$.

The following result characterizes admissible ellipsoids in terms of the symplectic capacity $c$:

**Proposition 15** A phase space ellipsoid $B$ is quantum mechanically admissible if and only if $c(B) \geq \frac{1}{2}\hbar$.

**Proof.** In view of (11) the condition $c(B) \geq \frac{1}{2}\hbar$ is equivalent to $R_1 \geq \sqrt{\hbar}$, that is to

$$\sqrt{\hbar} \leq R_1 \leq R_2 \leq \cdots \leq R_n.$$  

This is equivalent to saying that $S^{-1}(B)$ contains the ball $B^{2n}(\sqrt{\hbar})$ that is $S(B^{2n}(\sqrt{\hbar})) \subset B$. \hfill \( \blacksquare \)

If $B$ is a quantum blob then we have equality in the Proposition above: $c(B) = \frac{1}{2}\hbar$ because

$$c(B) = c(S(B^{2n}(z_0, \sqrt{\hbar})) = c(B^{2n}(z_0, \sqrt{\hbar})).$$

The converse of this property is however not true: property (3) of symplectic capacities says that any ellipsoid $B$ such that

$$B^{2n}(\sqrt{\hbar}) \subset B \subset Z_j(\sqrt{\hbar})$$

($Z_j(\sqrt{\hbar})$ the cylinder $x_j^2 + p_j^2 \leq \hbar$) will have capacity $\frac{1}{2}\hbar$: symplectic capacities are in a sense to imprecise to detect the whole symplectic spectrum of an ellipsoid. Notice, however, that if $n = 1$ then the condition $c(B) = \frac{1}{2}\hbar$ means that the area of $B$ is $\frac{1}{2}\hbar$ and hence a phase plane ellipse with symplectic capacity $\frac{1}{2}\hbar$ is just a quantum blob: it is only in more than one degree of freedom that the notion becomes interesting.

Here is another important observation: If an admissible ellipsoid $B$ has symplectic capacity $\frac{1}{2}\hbar$ then it can of course contain several distinct quantum
blobs. However, and this is an important property we will use in Section 5 when dealing with the Wigner transform, is that one can always associate a canonical concentric “quantum blob companion” to such an ellipsoid $B$. Since the symplectic capacity is invariant under translations it suffices to check this property when $B$ is centered at $z_0 = 0$. In view of Williamson’s theorem and the fact that $\sqrt{\hbar} = R_1$ there exists $S \in \text{Sp}(n)$ such that

$$S(B) : \frac{1}{\hbar}(x_1^2 + p_1^2) + \sum_{j=2}^{n} \frac{1}{R_j^2}(x_j^2 + p_j^2) \leq 1;$$

(12)

since $\sqrt{\hbar} = R_1 \leq R_2 \leq \cdots \leq R_n$ the ball $B^{2n}(\sqrt{\hbar})$ is contained in $S(B)$ and hence $S^{-1}(B^{2n}(\sqrt{\hbar})) \subset B$. The quantum blob $Q = S^{-1}(B^{2n}(\sqrt{\hbar}))$ is unambiguously defined, because if $S'$ is another element of $\text{Sp}(n)$ putting $B$ in the form (12) then $S = US'$ with $U \in \text{U}(n)$ in view of Lemma 1 and hence

$$S^{-1}(B^{2n}(\sqrt{\hbar})) = (S')^{-1}(B^{2n}(\sqrt{\hbar})).$$

4.2 Relation with the uncertainty principle

The usual quantum cells appearing in the literature are related to the Heisenberg uncertainty principle. So are our quantum blobs, but in a more precise way.

A generic ellipsoid $B$ in $\mathbb{R}^{2n}_z$ can always be viewed as the set of all $z$ such that $z^TMz \leq 1$ for some $M > 0$. From a quantum mechanical point of view it will be advantageous to replace $M$ by $F = \frac{1}{\hbar}M$ so that $B : z^TFz \leq \hbar$.

Let $F > 0$, be a phase space ellipsoid. We associate to $B$ the matrix

$$\Sigma = \frac{\hbar}{2}F^{-1}$$

which will be viewed as a statistical covariance (or “noise”) matrix. This interpretation is motivated by the properties of the Wigner transform that will be investigated in next Section.

**Proposition 16** The ellipsoid $B : z^TFz \leq \hbar$ is quantum mechanically admissible if and only if the three following equivalent conditions are satisfied:

(A) $\Sigma + \frac{i\hbar}{2}J$ is Hermitian positive semi-definite;

(B) $F^{-1} + iJ$ is Hermitian positive semi-definite.

(C) The eigenvalues of $J\Sigma$ are $\geq \frac{1}{2}\hbar$

(D) The eigenvalues of $JF$ are $\leq \frac{1}{2}$.

(13)
Proof. The conditions (A), (B) and (C), (D) are trivially equivalent in view of the definition of $F$. It thus suffices to prove that (B)$\iff$(D). Using Williamson’s theorem we can write

$$F = S^T DS, \quad S \in \text{Sp}(n), \quad D = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}$$

where $\Lambda = \text{diag}[\lambda_{\omega,1}, \ldots, \lambda_{\omega,n}]$. Since $S^T JS = J$ condition (A) is in turn equivalent to

$$D^{-1} + iJ \text{ is Hermitian positive semi-definite.} \quad (14)$$

The characteristic polynomial $P(\lambda)$ of $D^{-1} + iJ$ is the product

$$P(\xi) = P_1(\xi)P_2(\xi)\cdots P_n(\xi)$$

where $P_j$ is, for $j = 1, 2, \ldots, n$, the polynomial

$$P_j(\xi) = \xi^2 - 2\lambda_{\omega,j}^{-1}\xi + \lambda_{\omega,j}^{-2} - 1.$$ 

The roots of $P$ are the numbers $\xi_j = \pm 1 + (1/\lambda_{\omega,j})$ and we have $\xi_j \geq 0$ if and only if $\lambda_{\omega,j} \leq 1$. \[\blacksquare\]

The Proposition above says that:

An ellipsoid is admissible if and only if its associated covariance matrix satisfies the Heisenberg uncertainty principle \((13)\).

Conditions (13) are in fact symplectically invariant statements of Heisenberg’s uncertainty principle; see Arvind \textit{et al.} see \[1\] and Simon \textit{et al.} \[21\] (Trifonov \[22\] gives an interesting discussion, containing a comprehensive list of references, of various generalizations of the uncertainty principle). For instance when $n = 1$ the covariance matrix is

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_p \\ \rho \sigma_x \sigma_p & \sigma_p^2 \end{bmatrix} \quad \text{with} \quad -1 \leq \rho \leq 1 \quad (15)$$

and the eigenvalues of $J \Sigma$ are

$$\mu = \pm i\sigma_x \sigma_p \sqrt{1 - \rho^2}$$

hence $B$ is quantum-mechanically admissible if and only if we have

$$\sigma_x \sigma_p \sqrt{1 - \rho^2} \geq \frac{\hbar}{2} \quad (16)$$
Let us end this Section by briefly discussing the notion of squeezed state. Squeezed states are of a particular interest in quantum optics because they allow to beat the quantum limit in optical measurements (Walls and Milburn [24]). Assume that \( n = 1 \); a quantum state with covariance matrix

\[
\Sigma = \begin{bmatrix}
\sigma_x^2 & \rho \sigma_x \sigma_p \\
\rho \sigma_x \sigma_p & \sigma_p^2
\end{bmatrix}
\]

is sometimes said to be squeezed if either \( \sigma_x^2 \) or \( \sigma_p^2 \) is smaller than \( \frac{1}{2} \hbar \). However, as noticed and discussed in Arvind et al. [1], such a definition has no interesting or useful invariance property; in particular it is no more preserved by symplectic transformations than is the “popular” form \( \sigma_x \sigma_p \geq \frac{1}{2} \hbar \) of the uncertainty principle. This motivates the following definition (\textit{ibid.}), valid for any number of degrees of freedom:

A state is said to be squeezed if the smallest eigenvalue of the covariance matrix \( \Sigma \) is less than \( \frac{1}{2} \hbar \).

The condition above is clearly \( U(n) \)-invariant (\( U^T \Sigma U \) has the same eigenvalues as \( \Sigma \) for every \( U \in U(n) \)), but it is not \( \text{Sp}(n) \)-invariant: we can thus use non-unitary elements of the symplectic group to change the squeezed or non-squeezed status of a quantum state. Let us discuss this from the point of view of admissible ellipsoids. First, all quantum blobs which are not a ball correspond to a squeezed state. Let in fact \( Q = S(B^{2n}(\sqrt{\hbar})) \); in view of Lemma 2 for every \( S \in \text{Sp}(n) \) we can find \( U \in U(n) \) such that

\[
SS^T = U^T \Delta U
\]

where

\[
\Delta = \begin{bmatrix}
\Lambda & 0 \\
0 & \Lambda^{-1}
\end{bmatrix}, \quad \Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]
\]

the \( \lambda_j \), \( 1 \leq j \leq n \) being the first \( n \) eigenvalues of \( SS^T \); in particular \( 0 < \lambda_j \leq 1 \). The corresponding covariance matrix is

\[
\Sigma = \frac{\hbar}{2} SS^T = \frac{\hbar}{2} U^T \Delta U
\]

whose eigenvalues are the numbers

\[
\frac{\hbar}{2} \lambda_1 \leq \cdots \leq \frac{\hbar}{2} \lambda_n \leq \frac{\hbar}{2} \leq \frac{\hbar}{2 \lambda_n} \leq \cdots \leq \frac{\hbar}{2 \lambda_1}
\]

The only possibility for the state not to be squeezed is thus when all the \( \lambda_j \) are equal to one, and this corresponds to the trivial case \( Q = B^{2n}(\sqrt{\hbar}) \). More generally, Proposition 16 shows that:

An ellipsoid \( B : z^T F z \leq \hbar \) represents a squeezed quantum state if and only if the eigenvalues of \( JF \) are \( \leq 1 \) and the largest eigenvalue of \( F \) is \( > 1 \).
As announced in the Introduction, there is a one-to-one correspondence between quantum blobs and Gaussian pure states. Perhaps even more interestingly, quantum mechanically admissible ellipsoids correspond to Gaussian mixed states, and if such an admissible ellipsoid has symplectic capacity exactly one-half of the quantum of action we can associate to it a canonical “companion Gaussian pure state”.

Let us begin by recalling some basic results about the Wigner transform (for proofs and details see for instance Littlejohn [18]; interesting additional results can be found in Folland [5] (the latter however uses different normalizations); also see Grossmann et al. [12], and [2, 3, 19] for applications to integrable and non-integrable systems.

The Wigner transform $W\Psi$ of a function $\Psi \in L^2(\mathbb{R}^n_x)$ is defined by

$$W\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int \exp \left(-\frac{i}{\hbar} py\right) \Psi(x + \frac{1}{2}y)\Psi^*(x - \frac{1}{2}y) d^n y$$

(17)

and if $\Psi \in L^1(\mathbb{R}^n_x) \cap L^2(\mathbb{R}^n_x)$ then

$$\int W\Psi(z) d^n p = |\Psi(x)|^2 , \quad \int W\Psi(z) d^n x = |\hat{\Psi}(p)|^2$$

where $\hat{\Psi}$ is the quantum Fourier transform:

$$\hat{\Psi}(p) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} \int \exp \left(-\frac{i}{\hbar} px\right) \Psi(x) d^n x.$$  

(18)

The Wigner transform is a real function; however $W\Psi \geq 0$ if and only if $\Psi$ is a Gaussian (Hudson [16]); it is therefore not a true phase-space probability but only a “quasi probability distribution” (see the discussion in Simon et al. [21]). We will use the following properties of $W\Psi$:

- **Translations:**
  $$W\Psi(z - z_0) = W(\hat{T}(z_0)\Psi)(z)$$
  (19)
  where $\hat{T}(z_0)$ is the Heisenberg operator defined by
  $$\hat{T}(z_0) \Psi(x) = \exp \left[\frac{i}{\hbar}(p_0x - \frac{1}{2}p_0x_0)\right] \Psi(x - x_0).$$
  (20)

- **Metaplectic covariance:** for every $S \in \text{Sp}(n)$
  $$W\Psi(S^{-1}z) = W(\hat{S}\Psi)(z)$$
  (21)
  where $\hat{S}$ is any of the two operators $\pm \hat{S} \in Mp(n)$ ($Mp(n)$ is the metaplectic group) associated with $S$. 

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The Wigner transform is not invertible; however
\[ W\Psi = W\Psi' \iff \Psi = c\Psi' \quad , \quad |c| = 1 \quad (22) \]
so that the \( W\Psi \) determines \( \Psi \) only up to an overall phase.

Before we study the fundamental correspondence between quantum blobs and Gaussians, let us prove a technical result.

5.1 A reduction result

It turns out that we do not need all the elements of \( \text{Sp}(n) \) to generate the set \( \text{Quant}_{0}(n) \) of quantum blobs centered at \( z_0 = 0 \). This has already been proved above, where we showed that all quantum blobs can be obtained from the ball \( B^{2n}(\sqrt{\hbar}) \) by translations, rescalings, and rotations. We are going to prove here another “reduction result”, taking into account the fact that the ball \( B^{2n}(\sqrt{\hbar}) \) is insensitive to the action of \( U(n) \). (This is of course already reflected by the identification \( \text{Quant}_{0}(n) \equiv \text{Sp}(n)/\text{U}(n) \).)

Let \( S \) be a symplectic matrix in block-form
\[ S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (23) \]

**Lemma 17** Every symplectic matrix (23) can be written, in a unique way, as a product
\[ S = \begin{bmatrix} A_0 & 0 \\ C_0 & A_0^{-1} \end{bmatrix} \begin{bmatrix} X_0 & -Y_0 \\ Y_0 & X_0 \end{bmatrix}, \quad (24) \]
where \( A_0, C_0, X_0 \) and \( Y_0 \) are real \( n \times n \) matrices such that
\[ A_0 > 0, \quad A_0C_0 = C_0^TA, \quad \text{and} \quad X_0 + iY_0 \in U(n); \]
when \( S \) is written in block form (23) these matrices are given by the formulas
\[ A_0 = (AA^T + BB^T)^{1/2}, \quad (25) \]
\[ C_0 = (CA^T + DB^T)(AA^T + BB^T)^{-1/2}, \quad (26) \]
\[ X_0 + iY_0 = (AA^T + BB^T)^{-1/2}(A - iB). \quad (27) \]

**Proof.** The unitary group \( U(n) \) acts transitively on the Lagrangian Grassmannian (i.e. the set of all Lagrangian planes). This implies that there exists \( U \in U(n) \) such that \( S\ell_p = U\ell_p \) (recall that \( \ell_p = 0 \times \mathbb{R}^n_p \)) and hence
$S = RU$ for some $R \in \text{Sp}(n)$ such that $R\ell_p = \ell_p$. This implies that $R$ must be of the form
\[
R = \begin{bmatrix} A_0 & 0 \\ C_0 & A_0^{-1} \end{bmatrix}
\]
hence (24). Let us show that $AA^T + BB^T$ is invertible; formulae (25), (26), (27) then easily follow using the relations imposed on $A, B, C,$ and $D$ by the fact that $S$ is symplectic. Expanding the product (24) we get
\[
A = A_0X_0 \ , \ B = -A_0Y_0
\]
and hence
\[
AA^T + BB^T = A_0(X_0X_0^T + Y_0Y_0^T)A_0^T = A_0A_0^T
\]
that is, since $A_0$ is invertible,
\[
\det(AA^T + BB^T) \neq 0
\]
as claimed. ■

It follows that:

**Corollary 18** Every quantum blob $Q \in \text{Quant}_0(n)$ is the image of the ball $B^{2n}(\sqrt{\hbar})$ by a symplectic transformation of the type
\[
S = \begin{bmatrix} A_0 & 0 \\ C_0 & A_0^{-1} \end{bmatrix}
\]
with $A_0 > 0$ and $A_0C_0 = C_0^TA$.

**Proof.** It is an obvious consequence of Lemma 17 since $U(B^{2n}(\sqrt{\hbar})) = B^{2n}(\sqrt{\hbar})$ for every $U \in U(n)$. ■

### 5.2 The Fundamental Correspondence

Let us take a close scrutiny on the relationship between quantum blobs (and, more generally, admissible ellipsoids) and Wigner transforms.

We will denote by $\text{Gauss}_0(n)$ the set of all centered and normalized Gaussian functions given by
\[
\Psi(x) = \left(\frac{\det X}{(\pi \hbar)^n}\right)^{1/4} \exp \left[-\frac{1}{2\hbar}x^T(X + iY)x\right]
\]
(28)
where \( X \) and \( Y \) are real symmetric \( n \times n \) matrices, \( X > 0 \). The Wigner transform \( W\Psi \) of such a function is the phase space Gaussian
\[
W\Psi(z) = \left( \frac{1}{\pi \hbar} \right)^n \exp \left( -\frac{1}{\hbar} z^T G z \right)
\]
where \( G \) is the matrix
\[
G = \begin{bmatrix}
X + YY^{-1}Y & YX^{-1}Y \\
X^{-1}Y & X^{-1}
\end{bmatrix}
\]
(Littlejohn, [18]). A fundamental remark is that \( G \) is both symmetric positive definite and symplectic. The latter property is actually straightforward, checking that \( G^T J G = J \), and the positive definiteness is seen by noting that if \( z \neq 0 \) then, setting \( y = Yx + p \),
\[
z^T G z = x^T Xx + y^T X^{-1} y > 0.
\]
The associated covariance matrix is defined, as in Subsection 4.2, by
\[
\Sigma = \hbar^{-2} G^{-1}.
\]
We next notice that an immediate calculation shows that \( G \) can be factored as follows:
\[
G = (SS^T)^{-1} \quad \text{with} \quad S = \begin{bmatrix}
X^{-1/2} & 0 \\
-YX^{-1/2} & X^{1/2}
\end{bmatrix}
\]
(31)
hence the set \( \mathbb{Q} : z^T (SS^T)^{-1} z \leq \hbar \) is the quantum blob \( \mathbb{Q} = S(B^{2n}(\sqrt{\hbar})) \). Thus, to a Gaussian \( \Psi \) we have associated in a canonical way a quantum blob \( \mathbb{Q} = \mathbb{Q}(\Psi) \). The remarkable fact is that this correspondence is a bijection:

**Theorem 19** The mapping \( \mathbb{Q}(\cdot) : \text{Gauss}_0(n) \rightarrow \text{Quant}_0(n) \) which to every centered Gaussian \( \Psi \) associates the quantum blob \( \mathbb{Q}(\Psi) = S(B^{2n}(\sqrt{\hbar})) \) where \( S \) is defined by (31) is a bijection.

**Proof.** Assume that \( \mathbb{Q} \in \text{Quant}_0(n) \) is given by \( \mathbb{Q} = S(B^{2n}(\sqrt{\hbar})) \), that is \( z^T (SS^T)^{-1} z \leq \hbar \); in view of Lemma 17 in Subsection 5.1 there exist \( A_0 > 0 \) and \( C_0 \) such that \( A_0 C_0 = C_0^T A_0 \) and
\[
SS^T = \begin{bmatrix}
A_0 & 0 \\
C_0 & A_0^{-1}
\end{bmatrix} \begin{bmatrix}
A_0 & C_0^T \\
0 & A_0^{-1}
\end{bmatrix} = \begin{bmatrix}
A_0^2 & A_0 C_0^T \\
C_0 A_0 & C_0 C_0^T + A_0^{-2}
\end{bmatrix}
\]
and hence

$$(SS^T)^{-1} = \begin{bmatrix} C_0 C_0^T + A_0^{-2} & -C_0 A_0 \\ -A_0 C_0^T & A_0^2 \end{bmatrix}. $$

The equation

$$\begin{bmatrix} X + Y X^{-1} Y & Y X^{-1} \\ X^{-1} Y & X^{-1} \end{bmatrix} = \begin{bmatrix} C_0 C_0^T + A_0^{-2} & -C_0 A_0 \\ -A_0 C_0^T & A_0^2 \end{bmatrix}$$

has the unique solutions

$$X = A_0^{-2}, \quad Y = -A_0^{-1} C_0^T.$$

### Remark 20

This result shows that there is a bijective correspondence between the set $\Sigma(n)$ all complex symmetric $n \times n$ matrices with definite positive real part ("Siegel half-plane") and $Sp(n)/U(n)$. This has also been noticed in [5] in a different context.

The extension of Theorem 19 to the case of Gaussians with an arbitrary center is straightforward and left to the reader; on establishes the existence of a bijection

$$\mathbb{Q}(\cdot) : \text{Gauss}(n) \rightarrow \text{Quant}(n)$$

between the set of all Gaussians obtained from (28) by translation and the set of all quantum blobs.

Another remarkable fact is that we can associate in a canonical manner a “companion coherent state” to every quantum mechanically admissible ellipsoid with minimum capacity $\frac{1}{2} \hbar$. In this sense, those ellipsoids appear as “quasi quantum-blobs”.

### Corollary 21

Let $B$ be an admissible ellipsoid with $c(B) = \frac{1}{2} \hbar$. Then there exists a unique centered Gaussian (28), denoted by $\Psi_B$, such that

$$W\Psi_B(z) = \left(\frac{1}{\pi \hbar}\right)^n \exp\left(-\frac{1}{\hbar} z^T G z\right), \quad G = S^T S$$

(32)

where $S \in \text{Sp}(n)$ is any symplectic matrix such that $S(B^{2n}(\sqrt{\hbar})) \subset \mathbb{B}$.

### Proof.

We must check that the right-hand side of (32) is independent of the choice of the symplectic matrix $S$ putting $M$ in the Williamson diagonal form. Let $S$ and $S'$ be two such choices; in view of Lemma 1 (Section 2) there exists $U \in U(n)$ such that $S = US'$ and hence

$$z^T S^T S z = z^T S'^T U U^T S' S z = z^T S'^T S' z.$$
whence the result. ■

The generalization of the two results above to Gaussians and admissible sets centered at arbitrary points is straightforward using the property (19) of Wigner transforms. We leave it to the reader to restate Theorem 19 and its Corollary 21 in this more general setting.

5.3 Averaging on quantum blobs

Assume that \( \hat{\rho} \) is some arbitrary (mixed) state: \( Tr(\hat{\rho}^2) \leq 1 \). By the usual Weyl calculus [5, 18] one can associate to \( \hat{\rho} \) its Wigner transform \( W\hat{\rho} \). The state \( \hat{\rho} \) is said to be Gaussian if there exists \( F > 0 \) such that

\[
W\hat{\rho}(z) = \left( \frac{1}{\pi \hbar} \right)^n (\det F)^{1/2} \exp \left( -\frac{1}{\hbar} z^T F z \right); \tag{33}
\]

the eigenvalues of the covariance matrix \( \Sigma = \frac{\hbar}{2} F^{-1} \) are all \( \geq \frac{1}{2} \hbar \). This is precisely condition (13)(C) in Proposition 16, thus:

A phase space function (33) is the Wigner transform of a (mixed) quantum state if and only if the ellipsoid \( \mathbb{B} : z^T F z \leq \hbar \) is admissible.

Consider now a quite general Gaussian of the type

\[
W(z) = \left( \frac{1}{\pi \hbar} \right)^n (\det H)^{1/2} \exp \left( -\frac{1}{\hbar} z^T H z \right) \tag{34}
\]

with \( H > 0 \). We are going to see that if we “average” that function \( W \) over a quantum blob (in a sense that will be made precise below), then we will always obtain the Wigner transform of a (mixed) Gaussian state.

**Theorem 22** Let \( Q = S(B^{2n}(\sqrt{\hbar})) \) be a quantum blob and \( \Psi_Q \) the associated Gaussian (28). The convolution product \( W_Q = W \ast W\Psi_Q \) where \( W \) is any Gaussian (34) is the Wigner transform of some mixed state \( \hat{\rho} \). In fact,

\[
W\hat{\rho}(z) = \left( \frac{1}{\pi \hbar} \right)^n (\det F)^{1/2} \exp \left( -\frac{1}{\hbar} z^T F z \right) \tag{35}
\]

where the symplectic spectrum of \( F \) is the image of that of \( F \) by the mapping \( \lambda \mapsto \lambda/(1 + \lambda) \).

**Proof.** We have, by definition,

\[
W\Psi_Q(z) = \left( \frac{1}{\pi \hbar} \right)^n \exp \left( -\frac{1}{\hbar} z^T G z \right) , \quad G > 0 , \quad G \in \text{Sp}(n)
\]

where \( G = (SS^T)^{-1} \) hence

\[
W_Q(z) = \left( \frac{1}{\pi \hbar} \right)^{2n} (\det F)^{1/2} \int e^{-\frac{1}{\hbar}(z - z')^T H(z - z')} e^{-\frac{1}{\hbar}(z')^T G z'} d^2n z'.
\]
Setting \( z'' = S^{-1}z' \) we have

\[
W_Q(Sz) = \left(\frac{1}{\pi \hbar}\right)^{2n} (\det H)^{1/2} \int e^{-\frac{1}{\hbar}(z-z'')^T S^T H S (z-z'')} e^{-\frac{1}{\hbar}|z''|^2} d^{2n}z''.
\]

Replacing if necessary \( S \) by another symplectic matrix we may assume, in view of Williamson’s theorem, that

\[
S^T H S = D = \begin{bmatrix} \Lambda_\omega & 0 \\ 0 & \Lambda_\omega \end{bmatrix}
\]

and hence

\[
W_Q(Sz) = \left(\frac{1}{\pi \hbar}\right)^{2n} (\det D)^{1/2} \int e^{-\frac{1}{\hbar}(z-z'')^T D (z-z'')} e^{-\frac{1}{\hbar}|z''|^2} d^{2n}z''.
\]

Using the elementary formula

\[
\int_{-\infty}^{\infty} e^{-a(u-t)^2} e^{-bt^2} dt = \sqrt{\pi} \frac{a}{a+b} \exp \left( -\frac{ab}{a+b} u^2 \right)
\]

valid for all \( a, b > 0 \) together with the fact that the matrix \( D \) is diagonal we find

\[
W_Q(Sz) = \left(\frac{1}{\pi \hbar}\right)^n (\det D(I + D)^{-1})^{1/2} \exp \left[ -\frac{1}{\hbar}z^T D(I + D)^{-1} z \right]
\]

that is

\[
W_Q(z) = \exp \left[ -\frac{1}{\hbar}z^T (S^{-1})^T D(I + D)^{-1} S^{-1} z \right].
\]

Setting

\[
F = (S^{-1})^T D(I + D)^{-1} S^{-1}
\]

we obtain formula (35). There remains to show that the moduli of the eigenvalues of \( JF \) are not superior to one. Since \( S \in \text{Sp}(n) \) we have \( J(S^{-1})^T = SJ \) and hence

\[
JF = SJD(I + D)^{-1} S^{-1}
\]

has the same eigenvalues as \( JD(I + D)^{-1} \). Now,

\[
JD(I + D)^{-1} = \begin{bmatrix} 0 & \Lambda_\omega(I + \Lambda_\omega)^{-1} \\ -\Lambda_\omega(I + \Lambda_\omega)^{-1} & 0 \end{bmatrix}
\]

so that the eigenvalues \( \mu_{\omega,j} \) of \( JF \) are the numbers

\[
\mu_{\omega,j} = \pm i \frac{\lambda_{\omega,j}}{1 + \lambda_{\omega,j}}
\]

(36)

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which are such that $|\mu_{\omega,j}| \leq 1$ which was to be proven.

This result is reminiscent of (and related to) the well-known property of the Husimi transform (Husimi [17]). It is optimal in the following sense: let $\alpha$ and $\beta$ be two positive numbers and set

$$\Phi_{\alpha,\beta}(z) = \exp \left[ -\frac{1}{\hbar} \left( \frac{|x|^2}{\alpha^2} + \frac{|p|^2}{\beta^2} \right) \right].$$

As de Bruijn [4] has shown, the convolution product $W\Psi * \Phi_{\alpha,\beta}$ is always positive as long as $\alpha \beta \geq 1$, but it can take negative values when $\alpha \beta < 1$.

Now, the ellipsoid

$$\mathbb{B} : \frac{|x|^2}{\alpha^2} + \frac{|p|^2}{\beta^2} \leq \hbar$$

is the image of the ball $B^{2n}(\alpha \beta \sqrt{\hbar})$ by the symplectic transformation

$$(x, p) \mapsto (x \sqrt{\alpha/\beta}, p \sqrt{\beta/\alpha});$$

$\mathbb{B}$ is thus a quantum mechanically admissible ellipsoid if and only if $\alpha \beta \geq 1$.

6 Concluding Remarks and Perspectives

We have presented a general framework for the study of theoretical phase space properties associated to quantum mechanical systems. The fact that our treatment is essentially linear is not per se a real limitation because the uncertainty principle is only invariant under linear canonical transformations. On the other hand, it is also well-known that linear symplectic transformations correspond to those operations that preserve the Gaussian character of states and that can be implemented by means of optical elements such as beam splitters, phase shifts, and squeezers together with homodyne measurements (see for instance the discussion in Giedke et al. [6]). Still, it would be interesting, at least from a mathematical viewpoint, to have a closer look on non-linear blobs, images by arbitrary canonical transformations of the ball $B^{2n}(\sqrt{\hbar})$. One step in that direction is to note that while Theorem 8 probably does not remain true for non-linear blobs, its Corollary 9 does: Gromov’s non-squeezing theorem namely says that the area of the orthogonal projection of a ball $f(B^{2n}(r))$ (a canonical transformation) on any symplectic plane is always at least $\pi r^2$. It would be interesting to see whether one could deduce from this property non-linear analogues of Proposition 16, Theorem 19, and its Corollary 21.

It would certainly also be interesting, and perhaps useful, to recast our constructions in the context of Howe’s “oscillator semigroup” calculus [15].
(reviewed in the last Chapter in Folland [5]); Gaussians playing a fundamental role in both theories seems to indicate that there might be some intimate connection between quantum blobs and the Gaussian kernel operators introduced by Howe.

But perhaps the most fashionable – and interesting! – application of our methods would still be the study of entanglement of multi-partite systems, so essential in the understanding of various EPR-type phenomena like teleportation. What makes us believe that the notion of quantum blob could play a pivotal role in this question is that, as we already pointed out in the Introduction, quantum blobs have an arbitrarily large phase space extension, which makes them good tools to use in questions involving non-locality. We will return to these so fascinating and important questions in a near future.

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