Proof search in multi-succedent sequent calculi for intuitionistic logic

Toshiyasu Arai *
Graduate School of Science, Chiba University
1-33, Yayoi-cho, Inage-ku, Chiba, 263-8522, JAPAN
tosarai@faculty.chiba-u.jp

Abstract

Following G. Mints [4,5], we present a complete proof search in multi-succedent sequent calculi for intuitionistic propositional and predicate logic in the spirit of Schütte’s schema [6].

Schütte’s schema in [6] is described as follows.

Given a logic calculus, e.g., a sequent calculus and a semantics for the logic, search recursively a cut-free derivation of a given sequent in a bottom-up manner. This results in a (computable but possibly infinite) deduction of the given sequent. If the deduction is a (finite) derivation, then it tells us that the sequent is cut-free derivable in the sequent calculus. Otherwise it yields a counter model of the sequent with respect to the semantics (Schütte’s dichotomy). Thus the schema shows simultaneously the completeness of the (cut-free fragment of) the sequent calculus with respect to the semantics and the Hauptsatz for the calculus. The schema has been successfully applied to (first-order and higher-order) classical logic calculi.

In this note let us give complete proof search procedures in the schema for multi-succedent sequent calculi for intuitionistic propositional logic LJpm and predicate logic LJm, and the Kripke semantics [2].

In section 1 we consider a proof search in intuitionistic propositional logic LJpm. As a corollary we see in subsection 1.4 that the intuitionistic propositional logic is in PSPACE, a half of R. Statman’s result [7].

In section 2 we introduce proof search procedures for predicate logic LJm and LJm + (cut) with the cut rule. Given a sequent, the procedure yields a tree of deductions. In subsection 2.4 we consider a transformation of trees of deductions similar to the transfer rule in [2] to avoid an inconsistency in the definition of Kripke models. From the transformed tree of deductions, we obtain a Schütte’s dichotomy, cf. Theorem 2.7. If we need only to show the completeness of LJm with the cut rule (cut), then it turns out in subsection 2.5 that we don’t need to transform trees of deductions. This is done by using

*I’d like to thank Grisha Mints for his interests and helpful suggestions.
characteristic formulas as in [2]. In subsection 2.6 we sharpen the decidability result due to Mints [3] by showing that the positive fragment of intuitionistic predicate logic is in PSPACE.

1 Propositional case

The language of the propositional logic consists of propositional variables or atoms denoted \( p, q, r, \ldots \), propositional connectives \( \bot, \lor, \land, \rightarrow \). \( \text{Atm} \) denotes the set of atoms. Formulas are denoted by Greek letters \( \alpha, \beta, \gamma, \ldots \). \( \neg \alpha \) is defined as \( (\alpha \rightarrow \bot) \).

The set of all formulas is denoted \( \text{Fml} \). Finite sets of formulas are cedents denoted \( \Gamma, \Delta, \ldots \). Sequents are ordered pairs of cedents denoted \( \Gamma \Rightarrow \Delta \), where \( \Gamma \) is the antecedent and \( \Delta \) the succedent of the sequent.

\( \text{Ip} \) denotes the intuitionistic propositional logic.

1.1 Sequent calculus \( \text{LJpm} \) for \( \text{Ip} \)

Axioms.

\((T) \) \( \Gamma \Rightarrow \Delta \) if \( \Gamma \cap \Delta \cap \text{Atm} \neq \emptyset \)

\((\bot) \) \( \Gamma \Rightarrow \Delta \) if \( \bot \in \Gamma \)

Inference rules.

\[ \frac{\alpha_0, \Gamma \Rightarrow \Delta \quad \alpha_1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (\lor \Rightarrow) \]

\[ \frac{\Delta \Rightarrow \Gamma, \alpha_0, \alpha_1}{\Delta \Rightarrow \Gamma} \quad (\Rightarrow \lor) \text{ with } (\alpha_0 \lor \alpha_1) \in \Gamma \]

\[ \frac{\alpha_0, \alpha_1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (\land \Rightarrow) \]

\[ \frac{\Delta \Rightarrow \Gamma, \alpha_0 \quad \Delta \Rightarrow \Gamma, \alpha_1}{\Delta \Rightarrow \Gamma} \quad (\Rightarrow \land) \text{ with } (\alpha_0 \land \alpha_1) \in \Gamma \]

\[ \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (\Rightarrow \rightarrow) \]

\[ \frac{\alpha, \Delta \Rightarrow \beta \quad \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma} \quad (\Rightarrow \supset) \text{ with } (\alpha \supset \beta) \in \Gamma \]

Each sequent above the line is an upper sequent, and the sequent below the line is the lower sequent of an inference rule. For example in a \( (\lor \Rightarrow) \), \( \alpha_i, \Gamma \Rightarrow \Delta \) is an upper sequent for \( i = 0, 1 \), and \( \Gamma \Rightarrow \Delta \) is the lower sequent. Observe that the antecedent of the lower sequent is a subset of the antecedent of each upper sequent in any inference rules, and the succedent of the lower sequent is a subset of the succedent of each upper sequent in an inference rule other than \( (\Rightarrow \supset) \).

Derivations in \( \text{LJpm} \) are defined as usual. These are labelled trees whose leaves are labelled axioms and which are locally correct with respect to inference rules. While deductions are labelled trees which are locally correct with respect to inference rules. In deductions labels of leaves may be any sequents.

A Kripke frame is a quasi order \( \langle W, \preceq \rangle \). This means that \( W \neq \emptyset \) and \( \preceq \) is a reflexive and transitive relation on \( W \). A Kripke model is a triple \( \langle W, \preceq, V \rangle \), where \( \langle W, \preceq \rangle \) is a Kripke frame, and \( V : W \rightarrow \mathcal{P}(\text{Atm}) \) such that \( V(\sigma) \subseteq V(\tau) \) if \( \sigma \preceq \tau \).

For formulas \( \alpha \) and \( a \in W \), \( a \models \alpha \) is defined recursively.
1. \(\sigma \models p\) if and only if \(p \in V(\sigma)\).
2. \(\sigma \models \alpha \lor \beta\) if \(\sigma \models \alpha\) or \(\sigma \models \beta\).
3. \(\sigma \models \alpha \land \beta\) if \(\sigma \models \alpha\) and \(\sigma \models \beta\).
4. \(\sigma \not\models \bot\).
5. \(\sigma \models (\alpha \supset \beta)\) if for any \(\tau \succeq \sigma\), if \(\tau \models \alpha\), then \(\tau \models \beta\).

**Proposition 1.1** (Monotonicity)
\[
\sigma \preceq \tau \land \sigma \models \alpha \Rightarrow \tau \models \alpha
\]

A sequent \(\Gamma \Rightarrow \Delta\) is **intuitionistically valid** if \(a \models \bigwedge \Gamma\), then \(a \models \bigvee \Delta\) for any Kripke model \(\langle W, \preceq, V \rangle\) and any \(a \in W\).

**Proposition 1.2** Any derivable sequent in \(LJ_{pm}\) is intuitionistically valid.

### 1.2 Proof search in \(LJ_{pm}\)

Given a sequent \(S = (\Gamma \Rightarrow \Delta)\), let \(Tr_S\) denote the deduction of \(S\) constructed bottom-up manners. We analyze each formula in a sequent. A formula is marked with a circle to indicate that the formula has not yet been analyzed. \(\alpha^o\) indicates that the formula \(\alpha\) has not been analyzed. \(\Gamma^x\) denotes the set of formulas obtained from formulas in \(\Gamma\) by erasing the circle. A formula of the form \(\alpha \supset \beta\) is an **implicational formula**, and it is also called **non-invertible**.

A sequent \(\Gamma \Rightarrow \Delta\) is **saturated** if the following conditions are met:

1. if \((\alpha \lor \beta) \in \Gamma\), then \(\{\alpha, \beta\} \cap \Gamma^x \neq \emptyset\),
2. if \((\alpha \lor \beta) \in \Delta\), then \(\{\alpha, \beta\} \subset \Delta^x\),
3. if \((\alpha \land \beta) \in \Gamma\), then \(\{\alpha, \beta\} \subset \Gamma^x\),
4. if \((\alpha \land \beta) \in \Delta\), then \(\{\alpha, \beta\} \cap \Delta^x \neq \emptyset\), and
5. if \((\alpha \supset \beta) \in \Gamma\), then \(\alpha \in \Delta^x\) or \(\beta \in \Gamma^x\).

Note that these conditions are only for unmarked formulas, and there is no condition on succedent implicational formula in the definition of the saturation. A saturated sequent \(\Gamma \Rightarrow \Delta\) is **fully analyzed** if it is not an axiom, and any marked formula \(\alpha^o\) in \(\Gamma \cup \Delta\) is either an atom \(p^o\) or \(\bot^o\) (\(\bot^o \notin \Gamma\)).

A saturated sequent \(\Gamma \Rightarrow \Delta, \Delta_\sim\) is **non-invertible** if \(\Gamma \Rightarrow \Delta\) is fully analyzed and the set \(\Delta_\sim\) of succedent implicational formulas is non-empty.

Put the given sequent \(S\) at the root of the tree \(Tr_S\).

If an antecedent \(\Delta\) contains a conjunction \((\alpha_0 \land \alpha_1)^o\) with a circle, then erase the circle and add marked conjuncts: for \(\Delta_1 = \Delta \setminus \{(\alpha_0 \land \alpha_1)^o\}\),
\[
\frac{\alpha_0^o, \alpha_1^o, \Delta_1, \alpha_0 \land \alpha_1 \Rightarrow \Gamma}{(\alpha_0 \land \alpha_1)^o, \Delta_1 \Rightarrow \Gamma}
\]
If the succedent $\Gamma$ contains a conjunction $(\alpha_0 \land \alpha_1)^\circ$, then erase the circle and add marked conjuncts: for $\Gamma_1 = \Gamma \setminus \{(\alpha_0 \land \alpha_1)^\circ\}$,

\[
\Delta \Rightarrow \alpha_0 \land \alpha_1, \Gamma_1, \alpha^\circ_0 \quad \Delta \Rightarrow \alpha_0 \land \alpha_1, \Gamma_1, \alpha^\circ_1
\]

\[
\Delta \Rightarrow \Gamma_1, (\alpha_0 \land \alpha_1)^\circ
\]

Inversions are similar for $\lor$ and implicational formulas in antecedents.

For example with $(\alpha \supset \beta)^\circ \not\in \Gamma$

\[
\alpha \supset \beta, \Gamma \Rightarrow \Delta, \alpha^\circ \supset \beta, \alpha \supset \beta, \Gamma \Rightarrow \Delta
\]

\[
(\alpha \supset \beta)^\circ, \Gamma \Rightarrow \Delta
\]

As far as one of the inversions can be performed, continue it.

In each case the number of marked connectives $\lor$, $\land$, $\supset$ decreases when we go up in the tree $T_{RS}$. Hence the process terminates, and $T_{RS}$ is a finite binary tree. Each leaf in $T_{RS}$ is a saturated sequent, which is an axiom, or a fully analyzed sequent or a non-invertible one. By convention when $S$ is saturated, $T_{RS}$ denotes the deduction with a ‘repetition’ rule:

\[
S \quad (Rep)
\]

A non-invertible sequent $\Gamma, \Gamma_\supset \Rightarrow \Delta, \Delta^\circ_\supset$ is extended by an ‘inference rule’ branching (br), where $\Delta^\circ_\supset \not\in \{\Gamma \supset\}$ is the set of succedent (marked) implicational formulas [the set of antecedent unmarked implicational formulas], resp., and $\Gamma^\circ_\supset$ is obtained from $\Gamma_\supset$ by marking each formula to analyze these again, $\Gamma^\circ_\supset = \{\alpha^\circ : \alpha \in \Gamma_\supset\}$.

\[
\{\gamma^\circ, \Gamma, \Gamma^\circ_\supset \Rightarrow \delta^\circ : (\gamma \supset \delta)^\circ \in \Delta^\circ_\supset\}
\]

\[
\Gamma, \Gamma_\supset \Rightarrow \Delta, \Delta^\circ_\supset
\]

(br)

This inference rule is a disjunctive one since if one of upper sequents is derivable, then so is the lower sequent using the inference rule $(\Rightarrow \supset)$.

### 1.3 Construction of the tree of deductions

**Definition 1.3** Given a sequent $S_0 = (\Gamma_0 \Rightarrow \Delta_0)$, let us define a tree $T_{R}(S_0) \subset \ll^{<\omega} \omega$, and a labeling function $(S(\sigma), g(\sigma), d(\sigma))$ for $\sigma \in T_{R}(S_0)$, where $S(\sigma)$ is a sequent, $g(\sigma) \in \{\lor, \land, 0, 1\}$ is a gate and $d(\sigma)$ is a deduction possibly with the branching rule such that

1. for each leaf $\sigma$ in $T_{R}(S_0)$, $d(\sigma)$ consists solely of the saturated sequent $S(\sigma)$, and either
   
   (a) $S(\sigma)$ is an axiom and $g(\sigma) = 1$ indicating that $S(\sigma)$ is derivable, or
   (b) $S(\sigma)$ is fully analyzed and $g(\sigma) = 0$ indicating that $S(\sigma)$ is underviable, and

2. for each internal node $\sigma$ in $T_{R}(S_0)$, $g(\sigma) \in \{\lor, \land\}$ and
(a) if \( g(\sigma) = \land \), then \( S(\sigma) \) is either non-saturated or non-invertible, and \( d(\sigma) = Tr_{S(\sigma)} \).

(b) if \( g(\sigma) = \lor \), then \( S(\sigma) \) is a non-invertible sequent, and \( d(\sigma) \) is a deduction with a single inference rule (br) with its lower sequent \( S(\sigma) \).

Thus \( TR(S_0) \) is a \( (\land, \lor) \)-tree, and it is constructed inductively according to \( \land \)-stage or to \( \lor \)-stage below.

**initial.** First the empty sequence \( \emptyset \in TR(S_0) \) and \( S(\emptyset) = S_0 \) where each formula in \( S_0 \) is marked. If \( S_0 \) is an axiom, then \( g(\emptyset) = 1 \). If \( S_0 \) is fully analyzed, then \( g(\emptyset) = 0 \). If \( g(\emptyset) \in \{0, 1\} \), then \( d(\emptyset) \) is the deduction consisting solely of \( S_0 \). Otherwise \( g(\emptyset) = \land \) and the tree is extended according to \( \land \)-stage.

**\( \land \)-stage.**

Suppose \( \sigma \in TR(S_0) \) and \( g(\sigma) = \land \). Let \( S(\sigma) = S \).

Let \( \{S_i\}_{i<\ell} \) \((\ell > 0)\) be an enumeration of all leaves in \( d(\sigma) = TR_S \). For each \( i < \ell \), let \( \sigma \ast (i) \in TR(S_0) \) with \( S(\sigma \ast (i)) = S_i \). If \( S_i \) is an axiom, then \( g(\sigma \ast (i)) = 1 \). If \( S_i \) is fully analyzed, then \( g(\sigma \ast (i)) = 0 \). Otherwise \( S_i \) is non-invertible, and let \( g(\sigma \ast (i)) = \lor \) and the tree is extended according to \( \lor \)-stage.

**\( \lor \)-stage.**

Suppose \( \sigma \in TR(S_0) \) and \( g(\sigma) = \lor \). Let \( S(\sigma) = S \). \( S \) is a non-invertible sequent \( \Gamma, \Gamma' \Rightarrow \Delta, \{(\beta_j \supsetin \gamma_j)^o\}_{j<J} \) \((J > 0)\) where \( \Gamma' \) is the set of antecedent unmarked implicational formulas, and \( \Gamma, \Gamma' \Rightarrow \Delta \) is fully analyzed. Then \( \sigma \ast (j) \in TR(S_0) \) for each \( j < J \). Also let \( S(\sigma \ast (j)) = (\beta_j^o, \Gamma, \Gamma' \Rightarrow \gamma_j^o) \), where each unmarked and non-invertible formula \( \alpha \in \Gamma' \) is marked in \( \Gamma' \) to be analyzed again. \( d(\sigma) \) denotes the following deduction:

\[
\frac{\{\beta_j^o, \Gamma, \Gamma' \Rightarrow \gamma_j^o\}_{j<J}}{\Gamma, \Gamma' \Rightarrow \Delta, \{(\beta_j \supsetin \gamma_j)^o\}_{j<J}} \quad \text{(br)}
\]

Let \( g(\sigma \ast (j)) = 1 \) if \( S(\sigma \ast (j)) \) is an axiom, and \( g(\sigma \ast (j)) = 0 \) if it is fully analyzed. Otherwise let \( g(\sigma \ast (j)) = \land \) and the tree is extended according to \( \land \)-stage.

**Proposition 1.4** *The whole process generating the tree \( TR(S_0) \) terminates.*

**Proof.** We see the termination as follows. Let \( De(S_0) \) denote the deduction with the branching rule (br), obtained from \( TR(S_0) \) by fulfilling intermediate deductions \( TR_{S(\sigma)} \) for \( \sigma \in TR(S_0) \) with \( g(\sigma) = \land \). For each sequent \( S \) in the deduction \( De(S_0) \), let \( a_S \) be the number of positive occurrences of marked connectives \( \supsetin \) in \( S \), and \( b_S \) the total number of occurrences of marked connectives \( \lor, \land, \supsetin \) in \( S \). Moreover let \( N \) denote the the total number of occurrences of marked connectives \( \lor, \land, \supsetin \) in the given sequent \( S_0 \). Then the number \( dp(S) = (N + 1)a_S + b_S \) decreases when we go up inference rules in \( De(S_0) \).
Specifically $dp(S') < dp(S)$ for an upper sequent $S'$ of an inference rule with its lower sequent $S$.

Let us compute the value of the $(\land, \lor)$-tree $TR(S_0)$ with gates $g(\sigma)$. Let $v(\sigma)$ denote the value of $\sigma \in TR(S_0)$. If the value $v(\emptyset)$ is 1, then $S_0$ is derivable where a derivation of $S_0$ is obtained by putting the deduction $TR_{S_0}$ of $S(\sigma)$ from $\{S_i\}_{i < t}$

$$\ldots S_i \ldots$$

$$\downarrow \quad \downarrow \quad \check{\downarrow}$$

$$(S(\sigma), \land)$$

to the $\land$-node $\sigma$, and choosing one of upper sequents $S(\sigma * (i, j))$ such that $v(\sigma * (i, j)) = 1$ for each lower sequent $S(\sigma * (i))$ of (br), i.e., $g(\sigma * (i)) = \lor$.

In what follows consider the case when the value $v(\emptyset)$ of $TR(S_0)$ is 0. In a bottom-up manner let us shrink the tree $TR(S_0)$ to a tree $T \subset TR(S_0)$ as follows. Simultaneously a set $V_T(\sigma)$ of atoms is assigned. For each node $\sigma \in T$ $g(\sigma) \neq \lor$. First $\emptyset \in T$.

Suppose $v(\sigma) = 0$ for a node $\sigma \in T$ with $g(\sigma) = \land$. Pick a son $\sigma * (i)$ such that $v(\sigma * (i)) = 0$, and identify the node $\sigma * (i)$ with $\sigma$. This means that we have chosen an undervisible sequent $S(\sigma * (i))$, which is a non-axiom leaf in the deduction $TR_{S_0}$. Let $V_T(\sigma) = \Gamma(\sigma)^{\land} \land Atm$ where $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ denotes the leaf sequent $S(\sigma * (i))$ chosen from $TR_{S_0}$. If $g(\sigma * (i)) = \lor$, then $\sigma$ will be a leaf in $T$. Otherwise $g(\sigma * (i)) = \lor$, and keep its sons in a shrunken tree, i.e., $\sigma * (i, j) \in T$.

Thus we have defined a Kripke model $\langle T, c_e, V_T \rangle$ where $s \subset e$ $t$ iff $s$ is an initial segment of $t$.

For each $\sigma \in T$, $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ denotes the leaf sequent $S(\sigma * (i))$ chosen from $TR_{S_0}$.

From the construction we see readily the followings for any $\sigma, \tau \in T$.

1. $\sigma \subset_e \tau \Rightarrow \Gamma(\sigma)^{\lor} \subset \Gamma(\tau)^{\lor}$.
2. $\Gamma(\sigma) \Rightarrow \Delta(\sigma)$ is saturated.
3. if $\alpha \cup \beta^\circ \in \Delta(\sigma)$, then there exists an extension $\tau \in T$ of $\sigma$ such that $\alpha^\circ \in \Gamma(\tau)$ and $\beta^\circ \in \Delta(\tau)$.
4. $\Gamma(\sigma)^{\lor} \cap \Delta(\sigma)^{\lor}$ has no common atom, and $\perp \not\in \Gamma(\sigma)^{\lor}$.

**Proposition 1.5** If $\alpha \in \Gamma(\sigma)^{\lor} [\alpha \in \Delta(\sigma)^{\lor}]$, then $\sigma \models \alpha [\sigma \models \alpha]$, resp. in the Kripke model $\langle T, c_e, V_T \rangle$. Hence $\sigma \models \bigwedge \Gamma(\sigma)$ and $\sigma \not\models \bigvee \Delta(\sigma)$.

**Proof.** By simultaneous induction on $\alpha$ using the above facts.

**Theorem 1.6** If $v(\emptyset) = 0$, then each Kripke model $\langle T, c_e, V_T \rangle$ falsifies the given sequent $S_0$, no matter which non-axiom leaves are chosen from $TR_{S_0}$.

On the contrary, if $v(\emptyset) = 1$, then we can extract a (cut-free) derivation of $S_0$ by choosing a derivable sequent from each (br).

Hence $LJ_{pm}$ is intuitionistically complete in the sense that any intuitionistically valid sequent is derivable in $LJ_{pm}$, and $LJ_{pm}$ admits the Hauptsatz.
Corollary 1.7 Let \( (W, \leq, V) \) be a Kripke model falsifying the sequent \( S_0 = (\Gamma_0 \Rightarrow \Delta_0), a_0 \models \Gamma \) and \( a_0 \not\models \Delta \) for some \( a_0 \in W \).

Then we can choose a Kripke model \( (T, \subset, V_T) \) for which there exists an \( h : T \rightarrow W \) such that \( \sigma \subset \tau \Rightarrow a_0 \leq h(\sigma) \leq h(\tau) \) and \( V_T(\sigma) \subset V(h(\sigma)) \) for any \( \sigma, \tau \in T \).

Proof. Choose \( T \), i.e., pick leaves of deductions bottom-up manner according to the given Kripke model \( (W, \leq, V) \). For example if \( a \not\models (\alpha_0 \land \alpha_1) \), then pick an \( i \) such that \( a \not\models \alpha_i \), and go to the \( i \)-th branch. Suppose \( a \models (\alpha \supset \beta) \). If \( a \not\models \alpha \), then go to the left. Otherwise we have \( a \models \beta \), and go to the right.

When we reach a non-invertible leaf \( \sigma * (i) \) of \( TR_{S(\sigma)} \) and \( (\alpha \supset \beta)^{\circ} \) is one of the succedent formulas in \( S(\sigma * (i)) \), supposing \( a \not\models (\alpha \supset \beta) \), and let \( b \geq a \) be such that \( b \models \alpha \) and \( b \not\models \beta \). Put \( h(\sigma * (i, j)) = b \) where in \( \sigma * (i, j) \) \( \alpha \supset \beta \) is analyzed.

When we reach a fully analyzed leaf \( \Gamma(\sigma) \Rightarrow \Delta(\sigma) \) of \( TR_{S(\sigma)} \) and \( a = h(\sigma) \), then \( a \models p \) with \( p^\circ \in \Gamma(\sigma) \). This means that any \( p \in V_T(\sigma) \) is in \( V(a) = V(h(\sigma)) \), and hence \( V_T(\sigma) \subset V(h(\sigma)) \).

\[ \square \]

1.4 The intuitionistic propositional logic \( \text{ip} \) is in \( \text{PSPACE} \)

In this subsection we describe a \( \text{PSPACE} \)-algorithm deciding the deducibility of the given sequent \( S_0 \) in \( \text{LJm} \). This is a half of the R. Statman’s result (or observation) in [7].

Corollary 1.8 (Statman [7])
The intuitionistic propositional logic \( \text{ip} \) is in \( \text{PSPACE} \).

Recall that \( De(S_0) \) denotes the deduction obtained from \( TR(S_0) \) by fulfilling intermediate deductions \( TR_{S(\sigma)} \) for \( \sigma \in TR(S_0) \) with \( g(\sigma) = \land \). Let \( \#S \) be the size of the sequents \( S \), which is the total number of occurrences of symbols in \( S \).

Proposition 1.9 Let \( \vec{S} = S_0, S_1, \ldots, S_{n-1} \) be a branch in \( De(S_0) \), where \( S_{i+1} \) is an upper sequent of an inference rule with its lower sequent \( S_i \). Then \( \# \vec{S} := \sum_{i<n} \#S_i \) is bounded by a (quartic) polynomial of the size \( \#S_0 \) of the given sequent \( S_0 \).

Proof. From the proof of Proposition 1.4 we see that the length \( n \) of branches \( \vec{S} \) is bounded by a quadratic polynomial of \( \#S_0 \). On the other side the maximal size of sizes \( \#S_i \) of sequents \( S_i \) is bounded by a quadratic polynomial, too, since each \( S_i \) is essentially a sequence of subformulas of formulas in \( S_0 \).

Let us traverse sequents in the tree \( De(S_0) \) starting from the root \( S_0 \) as follows. Let \( S \) be the current sequent.

Case 1 If \( S \) is a lower sequent of an inference rule in \( De(S_0) \), then visit the leftmost upper sequent next.
Case 2 Otherwise $S$ is a leaf in $De(S_0)$, and in $TR(S_0)$. Let $\sigma \in TR(S_0)$ be the node such that $S(\sigma) = S$. $S(\sigma)$ is a leaf in a deduction $Tr_{S(\tau)}$ for a \( \tau \in TR(S_0) \) with $g(\tau) = \land$.

Case 2.1 First consider the case when $g(\sigma) = 1$.

Case 2.1.1 Let $T$ be the uppermost sequent below $S$ in $Tr_{S(\tau)}$ such that there is an upper sequent $S'$ of the inference rule with its lower sequent $T$, which we have not yet visited. Next let us visit the leftmost such sequent $S'$ if such a sequent exists.

\[
\begin{array}{c}
S \\
\vdots \\
S' \\
T
\end{array}
\]

Case 2.1.2 Suppose that there is no such sequent. This means that $S(\tau)$ is derivable. If $\tau = \emptyset$, then we are done. Otherwise $S(\tau)$ is an upper sequent of a $(br)$, and we see that the lower sequent $S(\rho)$ of the $(br)$ is derivable. Let us change the gate $g(\rho) = \lor$ to $g(\rho) = 1$, and continue the search for the next visiting sequent in the deduction $Tr_{S(\kappa)}$, where $S(\rho)$ is a leaf in the deduction $Tr_{S(\kappa)}$.

Case 2.2 Second consider the case when $g(\sigma) = 0$. This means that $S(\tau)$ is underivable. If $\tau = \emptyset$, then we are done. Otherwise $S(\tau)$ is an upper sequent of a $(br)$.

Case 2.2.1 If $S(\tau)$ is not the rightmost upper sequent, then visit the next right one.

Case 2.2.2 Otherwise the lower sequent $S(\rho)$ of the $(br)$ is underivable. Let us change the gate $g(\rho) = \lor$ to $g(\rho) = 0$, and continue the search for the next visiting sequent in the deduction $Tr_{S(\kappa)}$, where $S(\rho)$ is a leaf in the deduction $Tr_{S(\kappa)}$.

In the PSPACE-algorithm, we record sequences $\vec{S} = S_0, S_1, \ldots, S_{n-1}$ of sequents on a tape, where $\vec{S}$ is an initial segment of a branch in $De(S_0)$. The next sequence $\vec{S}'$ is recursively computed as follows. If the tail $S_{n-1}$ is a lower sequent of an inference rule in $De(S_0)$, then $\vec{S}' = \vec{S} * (S_n)$, i.e., extend the sequence $\vec{S}$ by adding the leftmost upper sequent $S_n$ as a tail, cf. Case 1 in the traversal.

Suppose $S_{n-1} = S(\sigma)$ is a leaf in a deduction $Tr_{S(\tau)}$.

First consider the case when $g(\sigma) = 1$. If there is an $S_i$ ($i < n - 1$) such that $S_{i+1}$ is not the rightmost upper sequent, then break the sequence $\vec{S}$ at $S_i$ and put the next right upper sequent $S'$, $\vec{S}' = S_0, \ldots, S_i, S'$ for the maximal such $i$, cf. Case 2.1.1.
2.1 Sequent calculus

the intuitionistic predicate logic.

In this section

for

variables

connectives

S

of the input

with their

eigenvariables

A language

2 Predicate case

f, . . .

g, . . .

function symbols

and free variables

a, . . .

and bound

variables

predicate symbols

and function symbols

is a Kripke frame. This has to enjoy the following for \( \sigma \preceq \tau \).

A Kripke model for a language \( \mathcal{L} \) is a quadruple \( \langle W, \preceq, D, I \rangle \), where \( \langle W, \preceq \rangle \)
is a Kripke frame. This has to enjoy the following for \( \sigma \preceq \tau \).
For closed formulas $\alpha \in \mathcal{L}(X)$ and $\sigma \in W$, $\sigma \models \alpha$ is defined recursively.

1. $\sigma \models R(c_1, \ldots, c_n)$ iff $(c_1, \ldots, c_n) \in R^\sigma$.
2. $\sigma \models \alpha \lor \beta$ iff $\sigma \models \alpha$ or $\sigma \models \beta$.
3. $\sigma \models \alpha \land \beta$ iff $\sigma \models \alpha$ and $\sigma \models \beta$.
4. $\sigma \not\models \bot$.
5. $\sigma \models (\alpha \supset \beta)$ iff for any $\tau \succeq \sigma$, if $\tau \models \alpha$, then $\tau \models \beta$.
6. $\sigma \models \exists x \alpha(x)$ iff there exists a $c \in D(\sigma)$ such that $\sigma \models \alpha(c)$.
7. $\sigma \models \forall x \alpha(x)$ iff for any $\tau \succeq \sigma$ and any $c \in D(\tau)$, $\tau \models \alpha(c)$.

2.2 Proof search in $\mathcal{LJm}$

For sequents $S$, the search tree $Tr_S$ is in general infinite due to the presence of universal formulas $\forall x \alpha(x)$ in antecedents and existential formulas $\exists y \beta(y)$ in succedents, cf subsection 2.6. A formula is non-invertible if it is either an implicational formula or a universal formula. It is desirable for us that each stage in constructing the tree of deductions is executed in a finite number of steps. In order to do so, each stage tests only a finite number of terms for universal formulas in antecedents and for existential formulas in succedents. Let $\{t_i\}_i$ be an enumeration of all terms. $Tm(A)$ denotes the set of all terms over a set $A \subset FV$ of free variables, and

$$Tm(A) \upharpoonright n := \{t_i \in Tm(A) : i < n\}.$$ 

Let $n < \omega$ and $A$ a set of free variables. A sequent $\Gamma \Rightarrow \Delta$ is $(n, A)$-saturated iff it is saturated with respect to propositional connectives $\lor, \land, \supset$ as in subsection 1.2 and it enjoys the following:

1. if $(\exists y \beta(y)) \in \Gamma$, then $(\beta(a)) \in \Gamma^x$ for a free variable $a \in FV$.
2. if $(\exists y \beta(y)) \in \Delta$, then $(\beta(t)) \in \Delta^x$ for any $t \in Tm(A) \upharpoonright n$.
3. if $(\forall x \alpha(x)) \in \Gamma$, then $(\alpha(t)) \in \Gamma^x$ for any $t \in Tm(A) \upharpoonright n$.

An $(n, A)$-saturated sequent $\Gamma \Rightarrow \Delta$ is $(n, A)$-analyzed if it is not an axiom, any marked formula $\alpha^\circ$ in $\Gamma \cup \Delta$ is one of atomic formulas $R(t_1, \ldots, t_n)^\circ$, $\bot^\circ$ ($\bot^\circ \notin \Gamma$), or non-invertible formulas in $\Delta$. A $(0, \emptyset)$-saturated sequent $\Gamma \Rightarrow \Delta$ is fully analyzed if it is not an axiom, any marked formula $\alpha^\circ$ in $\Gamma \cup \Delta$ is either an atomic formula $R(t_1, \ldots, t_n)^\circ$ or $\bot^\circ$ ($\bot^\circ \notin \Gamma$), and there is no existential succedent formula [no universal antecedent formula], resp.

A deduction $Tr_{S}^{(n, A)}$ is constructed as for propositional case by leaving any non-invertible succedent formulas and applying $(\exists \Rightarrow), (\Rightarrow \exists), (\forall \Rightarrow)$ up to $n$th terms in $Tm(A)$. Put the given sequent $S$ at the root of the tree $Tr_{S}^{(n, A)}$. 

10
where the eigenvariable \( a \) does not occur in the lower sequent nor in the finite set \( A \) of free variables. Moreover \( a \) is chosen so that the condition 1 below is met.

\[
\frac{\Delta \Rightarrow \exists x \alpha(x), \Gamma, \{\alpha(t)\}_{t \in Tm(A)\cap n}}{\Delta \Rightarrow \Gamma, \exists x \alpha(x)^o} \quad (\Rightarrow \exists)
\]

\[
\frac{\{\alpha(t)^o\}_{t \in Tm(A)\cap n}, \Gamma, \forall x \alpha(x) \Rightarrow \Delta}{\forall x \alpha(x)^o, \Gamma \Rightarrow \Delta} \quad (\forall \Rightarrow)
\]

Each leaf in \( T_{TS}^{(n,A)} \) is one of axioms, fully analyzed sequent or \((n,A)\)-analyzed if \( S \) is \((n,A)\)-saturated and any non-invertible formula in its succedent is marked.

### 2.3 Extensions for non-invertible succedent formulas and postponed instantiations

In a \( \lor \)-stage of our proof search for the predicate logic we examine all possibilities with succedent non-invertible formulas by introducing a branching rule (br) as for the propositional case. Consider a sequent \( \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3, \Delta, \Delta_5, \Delta_\forall \) where \( \Gamma \cup \Delta \) consists solely of marked atomic formula \( R(t_1, \ldots, t_n)^o \) or \( \bot^o \) with \( \bot^o \in \Gamma, \Gamma_\lor \) is a set of unmarked implicational formulas, \( \Gamma_\forall \) is a set of unmarked universal formulas, \( \Delta_3 \) a set of unmarked existential formulas, \( \Delta_5^o \) a set of marked implicational formulas, and \( \Delta_\forall^o \) a set of marked universal formulas.

Assume that \( \Gamma_\forall \cup \Delta_3 \cup \Delta_5^o \cup \Delta_\forall^o \neq \emptyset \). (Otherwise it is either an axiom or fully analyzed.) Then the sequent follows from several sequents. Let us depict the several possibilities as an ‘inference rule’ as follows.

\[
\Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3, \Delta, \Delta_5, \Delta_\forall \quad \frac{\{\gamma^o, \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3^o, \Delta, \Delta_5, \Delta_\forall \}}{\{\gamma, \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3^o, \Delta, \Delta_5^o, \Delta_\forall \}} \quad (br)
\]

where

1. for sets of unmarked formulas \( \Gamma, \Gamma^o = \{\gamma^o : \gamma \in \Gamma\}. \) The sequent \( \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3^o, \Delta, \Delta_5, \Delta_\forall \) is absent when \( \Gamma_\forall \cup \Delta_3 = \emptyset \).
2. \( a \) is an eigenvariable distinct each other for universal formulas \( (\forall y \gamma(y)) \in \Delta_\forall \) and such that the condition 1 below is met.

Again if one of upper sequents of (br) is derivable, then so is the lower sequent possibly using a non-invertible inference rule \((\Rightarrow \lor)\) or \((\forall \Rightarrow)\). Each upper sequent \( \gamma^o, \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3^o \) with \( (\gamma \lor \delta)^o \in \Delta_5^o \) and each \( \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \gamma(a)^o \) with \( (\forall y \gamma(y)) \in \Delta_\forall^o \) is said to be a \textit{non-invertible upper sequent} of the inference rule (br). While the leftmost upper sequent \( \Gamma, \Gamma_\lor, \Gamma_\forall \Rightarrow \Delta_3^o, \Delta, \Delta_5^o, \Delta_\forall \) is the \textit{continued sequent}. 11
Each eigenvariable is distinct each other, and occurs only above or in the right part of the inference where the variable is introduced. (1)

This means that if \( \sigma \) is the node of the lower sequent of the inference rule where an eigenvariable \( a \) is introduced, and \( a \) occurs in the sequent at a node \( \tau \), then either \( \sigma \subset_e \tau \) (i.e., \( \tau \) is above \( \sigma \)) or \( \rho \ast (i) \subset_e \sigma \) and \( \rho \ast (j) \subset_e \tau \) for some \( \rho \) and \( i < j \) (i.e., \( \tau \) is right to \( \sigma \)).

As in the propositional case, Definition 2.1 let us construct a tree \( TR(S_0) \) for a given sequent \( S_0 \). The tree \( TR(S_0) \) is constructed in \( \omega \)-steps. \( TR(S_0) \) denotes the piece of \( TR(S_0) \) in the \( n \)-th step such that for each \( \sigma \in TR(S_0)_n \), the length \( lh(\sigma) \leq n \). The labeling function \( (S(\sigma), d(\sigma)) \) for \( \sigma \in \bigcup_n TR(S_0)_n \) is independent of the step \( n \), but the gate label \( g(\sigma, n) \in \{ \lor, \land, 0, 1, \times \} \) depend on \( n \) with \( \sigma \in TR(S_0)_n \).

\( TR(S_0) \) defined to be a subset of the union \( \bigcup_{n \in \omega} TR(S_0)_n \).

Let

\[
\sigma \subset^0 \tau \iff \sigma \subset_e \tau & \land \forall \rho[\sigma \subset_e \rho \subset^0 \tau & \land g(\rho, n(\rho)) = \lor \Rightarrow \tau(lh(\rho)) = 0]
\]

where \( \tau(i) \) denotes the \( i \)-th component of sequences \( \tau \) for \( i < lh(\tau) \). \( \sigma \subset^0 \tau \) means that \( \tau \) continues to substitute terms for \( \forall y \) in antecedents and \( \exists x \) in succedent for any \( \lor \)-stage after \( \sigma \). Let \( S(\tau) \) be the sequent at the node \( \tau \) in \( TR(S_0) \), and \( FV(\tau) \) the set of free variables occurring in the sequent \( S(\tau) \).

Furthermore a finite set \( FV_{\subset_e}(\sigma) \) of free variables is assigned to sequences \( \sigma \in TR(S_0) \) so that

\[
FV_{\subset_e}(\sigma) = \bigcup\{FV(\tau) : \exists \rho[\sigma \subset_e \sigma & \land \rho \subset^0 \tau \in TR(S_0)_{lh(\sigma)}]\}
\]

Let us denote \( Tr^{lh(\sigma), FV_{\subset_e}(\sigma)}_{S(\sigma)} \) by \( Tr_{\sigma} \).

\( De(S_0) \) denotes the whole tree of deductions obtained from \( TR(S_0) \) by fulfilling intermediate deductions, and is constructed recursively. Each \( \land \)-stage analyzes the current leaves parallel as in the propositional case. After the \( \land \)-stage, we extend the tree by non-invertible (br) inference rules in \( \lor \)-stage. In each moment \( De(S_0) \) is constructed so that the condition (1) on eigenvariables is met.

**Definition 2.1** Given a sequent \( S_0 = (\Gamma_0 \Rightarrow \Delta_0) \), let us define trees \( TR(S_0)_n \subset \omega \), and a labeling function \( (S(\sigma), g(\sigma, n), d(\sigma)) \) for \( \sigma \in TR(S_0)_n \), where \( S(\sigma) \) is a sequent, \( g(\sigma) \in \{ \lor, \land, 0, 1, \times \} \) is a gate and \( d(\sigma) \) is a deduction possibly with the branching rule.

First the empty sequence \( \emptyset \in TR(S_0)_0 = \{ \emptyset \} \) and \( S(\emptyset) = S_0 \) where each formula in \( S_0 \) is marked. Let \( FV(\emptyset) \) be the set of free variables occurring in
If the set is non-empty. Otherwise $FV(\emptyset) = \{a_0\}$. If $S_0$ is an axiom, then $g(\emptyset, 0) = 1$. If $S_0$ is fully analyzed, then $g(\emptyset, 0) = 0$. If $g(\emptyset, 0) \in \{0, 1\}$, then $d(\emptyset)$ is the deduction consisting solely of $S_0$. Otherwise $g(\emptyset, 0) = \land$ and the tree is extended according to $\land$-stage.

Suppose that $TR(S_0)_n$ has been constructed, and there exists a leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma, n) \in \{\land, \lor\}$. (Otherwise we are done, and $TR(S_0)_{n+1}$ is not defined.) If $n$ is even [odd], the tree is extended according to $\land$-stage [$\lor$-stage], resp.

$\land$-stage.

Consider each leaf $\sigma \in TR(S_0)_n$ with $g(\sigma, n) = \land$. Extend the tree $De(S_0)$ by putting the deduction $d(\sigma) = Tr_\land(TR(S_0))$ for each such $\sigma$. Let $\{S_i\}_{i < I}$ be an enumeration of all leaves in $d(\sigma)$. For each $i < I$, let $\sigma(i) (i) \in TR(S_0)_{n+1}$ with $S(\sigma(i)) = S_i$. If $S_i$ is an axiom, then $g(\sigma(i), n + 1) = 1$. If $S_i$ is fully analyzed, then $g(\sigma(i)) = 0$. Otherwise $S_i$ is not fully analyzed, but $(ib(x), FV_{\land}(\sigma; TR(S_0)))$-analyzed. This means that either its antecedent contains a universal formula, or its succedent contains an existential formula or a non-invertible formula. Let $g(\sigma(i), n + 1) = \lor$, and the tree is extended according to $\lor$-stage.

$TR(S_0)_{n+1}$ is defined to be the union of $TR(S_0)_n$ and nodes $\sigma(i)$ for each leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma, n) = \land$ and $i < I$.

$\lor$-stage.

Consider each leaf $\sigma \in TR(S_0)_n$ with $g(\sigma, n) = \lor$. Extend the tree $De(S_0)$ by the inference rule (br) parallel for each such $\sigma$.

Let $S(\sigma) = S$. $S$ is a non-invertible sequent $\Gamma, \Gamma_\lor \Rightarrow \Delta_3, \Delta, \Delta_5^\lor, \Delta_\lor^\lor$ where $\Gamma \cup \Delta$ consists solely of unmarked formulas and marked atomic formula $R(t_1, \ldots, t_n)^\lor$ or $\bot^\lor$ with $\bot^\lor \notin \Gamma$, and $\Gamma_\lor \cup \Delta_3 \cup \Delta_5^\lor \cup \Delta_\lor^\lor \neq \emptyset$.

Let $S(\sigma \ast (0))$ be the sequent $\Gamma, \Gamma_\lor \Rightarrow \Delta_3, \Delta, \Delta_5^\lor, \Delta_\lor^\lor$. Let $\Delta_\lor^\lor = \{\beta_j^\lor\}_{0 < j \leq J_\lor}$, and $\Delta_5^\lor = \{\beta_j^5\}_{0 < j \leq J_\lor}$. Then $\sigma \ast (j) \in TR(S_0)_{n+1}$ for each $j$ with $0 < j \leq J_\lor$. For $j > 0$ the sequent $S(\sigma \ast (j))$ is defined by analyzing the $j$th non-invertible formula $\beta_j$. Namely if $\beta_j \equiv (\gamma \supset \delta)$, then $S(\sigma \ast (j)) = (\gamma^\lor, \Gamma_\lor \Rightarrow \delta^\lor)$. If $\beta_j \equiv (\forall y \gamma(y))$, then $S(\sigma \ast (j)) = (\Gamma, \Gamma_\lor \Rightarrow (\forall a)\gamma^\lor)$ where the eigenvariables $a$ are fresh, i.e., do not occur in any $S(\tau)$ for $\tau \in TR(S_0)_n$, and distinct each other for $(\forall y \gamma(y)) \in \Delta_\lor$ and leaves $\sigma$. $d(\sigma)$ denotes the deduction consisting of a (br) with its lower sequent $S = S(\sigma)$.

Let $g(\sigma \ast (j), n + 1) = 1$ if $S(\sigma \ast (j))$ is an axiom, and $g(\sigma \ast (j), n + 1) = 0$ if it is fully analyzed. Otherwise let $g(\sigma \ast (j), n + 1) = \land$ and the tree is extended according to $\land$-stage.

$TR(S_0)_{n+1}$ is defined to be the union of $TR(S_0)_n$ and nodes $\sigma \ast (j)$ for each leaf $\sigma \in TR(S_0)_n$ such that $g(\sigma, n) = \lor$ and $j \leq J_\lor$.

In each stage, the gate $g(\tau, n + 1)$ is computed for $\tau \in TR(S_0)_{n+1}$ as follows. The computation is done in two rounds recursively. $g_0(\tau, n + 1)$ is the gate in the first round, and $g(\tau, n + 1)$ in the second round.
First consider the case when \( g(\tau, n) = \lor \) and \( \tau \in TR(S_0)_n \). If there exists an \( i \) such that \( g_0(\tau * (i), n + 1) = 1 \), then \( g_0(\tau, n + 1) = 1 \). If \( g_0(\tau * (i), n + 1) = 0 \) for any \( i \), then \( g_0(\tau, n + 1) = 0 \). Second consider the case when \( g(\tau, n) = \land \) and \( \tau \in TR(S_0)_n \). If there exists an \( i \) such that \( g_0(\tau * (i), n + 1) = 0 \), then \( g_0(\tau, n + 1) = 0 \). If \( g_0(\tau * (i), n + 1) = 1 \) for any \( i \), then \( g_0(\tau, n + 1) = 1 \). In all other cases \( g_0(\tau, n + 1) = g(\tau, n) \) for and \( \tau \in TR(S_0)_{n+1} \).

Next the second round. First consider the case when \( g(\tau, n) = \lor \) and \( g(\tau, n + 1) = 1 \). Let \( i \) be the least number such that \( g_0(\tau * (i), n + 1) = 1 \), then \( g(\tau * (i), n + 1) = \land \) and \( g(\tau * (j), n + 1) = \land \) for any \( j \neq i \). Second consider the case when there exists a \( \rho \subset_e \tau \) such that \( g(\rho, n + 1) = \land \). Then \( g(\tau, n + 1) = \land \). In all other cases \( g(\tau, n + 1) = g(\tau, n) \).

1. For \( \sigma \in TR(S_0) \)
   \[
   g(\sigma) := \lim_{n \to \infty} g(\sigma, n).
   \]

2. \[
   TR(S_0) = \{ \sigma \in \bigcup_{n \in \omega} TR(S_0)_n : g(\sigma) \neq \land \}.
   \]

It is easy to see that if \( g(\emptyset) = 1 \), then \( \mathbb{L} \text{Im} \vdash S_0 \).

Let \( T \subset TR(S_0) \) be a subtree of \( TR(S_0) \). Let us construct a Kripke model \( \langle T, \subset_e, D_T, I_T \rangle \) as follows.

**Definition 2.2** 1. For \( S(\sigma) = (\Gamma(\sigma) \Rightarrow \Delta(\sigma)) \), let
   \[
   \Gamma^\infty(\tau; T) = \bigcup \{ \Gamma(\rho)^\infty : \tau \subset_e \rho \in T, g(\rho) = \lor \}
   \]
   \[
   \Gamma_{\subset_e}^\infty(\sigma; T) = \bigcup \{ \Gamma^\infty(\tau) : T \ni \tau \subset_e \sigma, g(\tau) = \lor \}
   \]
   \[
   \Delta^\infty(\tau; T) = \bigcup \{ \Delta(\rho)^\infty : \tau \subset_e \rho \in T, g(\rho) = \lor \}
   \]

Note that \( \Gamma(\sigma; T)^\infty \subset \Gamma^\infty(\sigma; T) \subset \Gamma_{\subset_e}^\infty(\sigma; T) \) and \( \Delta(\sigma; T)^\infty \subset \Delta^\infty(\sigma; T) \).

2. Let
   \[
   D_T(\sigma) = Tm(FV_{\subset_e}^\infty(\sigma; T))
   \]

with
   \[
   FV^\infty(\tau; T) = \bigcup \{ FV(\rho) : \tau \subset_e \rho \in T, g(\rho) = \lor \}
   \]
   \[
   FV_{\subset_e}^\infty(\sigma; T) = \bigcup \{ FV^\infty(\tau; T) : T \ni \tau \subset_e \sigma, g(\tau) = \lor \}
   \]

3. For \( \sigma \in T \) and predicate symbol \( R \), let
   \[
   R^\sigma = \{ (t_1, \ldots, t_n) : t_1, \ldots, t_n \in D_T(\sigma) \land R(t_1, \ldots, t_n) \in \Gamma_{\subset_e}^\infty(\sigma; T) \}
   \]

and \( f^{\sigma}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \) for \( t_1, \ldots, t_n \in D_T(\sigma) \).
Proposition 2.3 \((T, \subset^e, D_T, I_T)\) is a Kripke model.

Proof. Let \(\sigma \subset^e \tau\) for \(\sigma, \tau \in T\). Then \(\text{FV}_{\subset^e}^\infty(\sigma; T) \subset \text{FV}_{\subset^e}^\infty(\tau; T)\), and hence \(D_T(\sigma) \subset D_T(\tau)\). Moreover we have \(\Gamma^\infty_{\subset^e}(\sigma; T) \subset \Gamma^\infty_{\subset^e}(\tau; T)\), and \(R^\infty \subset R^\tau\). \(\square\)

A pair \(\Gamma \Rightarrow \Delta\) of (possibly infinite) sets \(\Gamma, \Delta\) of formulas is \(A\)-saturated for a set \(A\) of free variables iff it is \((n, A)\)-saturated for any \(n\). This means that

1. if \((\exists y \beta(y)) \in \Delta\), then \((\beta(t)) \in \Delta^x\) for any \(t \in Tm(A)\).
2. if \((\forall x \alpha(x)) \in \Gamma\), then \((\alpha(t)) \in \Gamma^x\) for any \(t \in Tm(A)\).

\(A\)-saturated pair \(\Gamma \Rightarrow \Delta\) is \(A\)-analyzed if \(\bot \notin \Gamma^x\), \(\Gamma\) and \(\Delta\) has no common atomic formula.

Proposition 2.4 Suppose that \(T\) enjoys the following conditions.

1. \(\Gamma^\infty_{\subset^e}(\sigma; T) \Rightarrow \Delta^\infty(\sigma; T)\) is \(\text{FV}_{\subset^e}^\infty(\sigma; T)\)-analyzed.
2. (a) if \((\alpha \supset \beta) \in \Delta^\infty(\sigma; T)\), then there exists an extension \(\tau \in T\) of \(\sigma\) such that \(\alpha \in \Gamma^\infty_{\subset^e}(\tau; T)\) and \(\beta \in \Delta^\infty(\tau; T)\).
   (b) if \((\forall x \alpha(x)) \in \Delta^\infty(\sigma; T)\), then there exist an extension \(\tau \in T\) of \(\sigma\) and an \(a \in \text{FV}_{\subset^e}^\infty(\tau; T)\) such that \((\alpha(a)) \in \Delta^\infty(\tau; T)\).

Moreover assume that \(\Gamma^\infty_{\subset^e}(\sigma; T) \cap \Delta^\infty(\sigma; T)\) has no common atomic formula for any \(\sigma \in T\).

Let \(\sigma \in T\) and \(\alpha\) be a formula all of whose free variables are in the set \(D_T(\sigma)\). In the Kripke model \((T, \subset^e, D_T, I_T)\), if \(\alpha \in \Gamma^\infty_{\subset^e}(\sigma; T)\), then \(\sigma \models \alpha\), and if \(\alpha \in \Delta^\infty(\sigma; T)\), then \(\sigma \not\models \alpha\).

Hence \(\sigma \models \bigwedge \Gamma(\sigma)\) and \(\sigma \not\models \bigvee \Delta(\sigma)\).

Proof. This is shown by simultaneous induction on \(\alpha\).

Consider the case when \(\alpha\) is an atomic formula \(R(t_1, \ldots, t_n)\). By the assumption we have \(\alpha \notin \Gamma^\infty_{\subset^e}(\sigma; T) \cap \Delta^\infty(\sigma; T)\). Hence if \(\alpha \in \Delta^\infty(\sigma; T)\), then \(\alpha \notin \Gamma^\infty_{\subset^e}(\sigma; T)\), i.e., \(\sigma \not\models \alpha\). On the other side if \(\alpha \in \Gamma^\infty_{\subset^e}(\sigma; T)\), then \(t_1, \ldots, t_n \in D_T(\sigma)\) by the assumption, then \(\sigma \models \alpha\).

Next consider the case when \(\alpha \equiv (\forall x \beta(x))\). Suppose \(\alpha \in \Gamma^\infty_{\subset^e}(\sigma; T)\). For any extension \(\tau\) of \(\sigma\) in \(T\), i.e., \(\sigma \subset^e \tau \in T\), \(\alpha \in \Gamma^\infty_{\subset^e}(\sigma; T) \cap \Gamma^\infty_{\subset^e}(\tau; T)\). \(\beta(t) \in \Gamma^\infty_{\subset^e}(\tau; T)\) for any \(t \in D_T(\tau)\) by the supposition. By IH \(\tau \models \beta(t)\). Next suppose \(\alpha \in \Delta^\infty(\sigma; T)\). Then by the supposition, for an extension \(\tau \in T\) of \(\sigma\) and a free variable \(a \in \text{FV}_{\subset^e}^\infty(\tau; T)\), we have \(\beta(a) \in \Delta^\infty(\tau; T)\). Hence \(a \in D_T(\rho)\) as long as \(a\) occurs in \(\beta(a)\), and \(\tau \not\models \beta(a)\) by IH. \(\square\)

The first condition (analyzed) and second one (existences of extensions) in Proposition 2.4 are easily enjoyed when \(T\) has sufficiently many nodes, i.e., when nodes are prolonged in \(T\) unlimitedly. The third condition (no common atomic formula) is hard to satisfy. Obviously the tree \(TR(S_0)\) of the deductions enjoys the first and second conditions. But it may be the case that for a \(\sigma \in TR(S_0)\) with \(g(\sigma) = \forall, \Gamma^\infty_{\subset^e}(\sigma; TR(S_0)) \cap \Delta^\infty(\sigma; TR(S_0))\) has a common atomic formula.
Proposition 2.5 Suppose that each $\land$-gate has a unique $\lor$-gate son in $T$. Then $\Gamma^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ has no common atomic formula for any $\sigma \in T$.

Proof. Suppose that $\Gamma^\infty(\sigma; T) \cap \Delta^\infty(\sigma; T)$ has a common atomic formula $\alpha$. Let $\sigma \subseteq^0 \rho_a, \rho_s \in T$ be such that $g(\rho_a) = g(\rho_s) = \forall$ and $\alpha \in \Gamma^\times(\rho_a; T) \cap \Delta^\times(\rho_s; T)$. We see that $\rho_a$ and $\rho_s$ are compatible in the order $\subseteq^0$, since each $\land$-gate has a unique $\lor$-gate son in $T$, and each $\lor$-gate has a unique son, i.e., the leftmost continued one in $\subseteq^0$. Then $\alpha \in \Gamma^\times(\rho) \cap \Delta^\times(\rho)$ for a common extension $\rho \in \{\rho_a, \rho_s\}$, and $S(\rho)$ is an axiom. This means a contradiction $1 = g(\rho) = \forall$. $\Box$

2.4 Transfer

We consider a transformation of deductions similar to the transfer rule in Kripke

The tree $TR(S_0)$ of deductions is transformed to another tree $TTR(S_0)$ in which there is no $\sigma$ such that $\Gamma^\infty_c(\sigma; TR(S_0)) \cap \Delta^\infty(\sigma; TTR(S_0))$ has a common atomic formula.

Let us introduce the following inference rules.

\[
\begin{align*}
\Gamma \Rightarrow \Delta & \quad \Pi \Rightarrow \Delta, \Lambda \quad \text{(weak)} \\
\Gamma, \Gamma_y \Rightarrow \Delta_3, \Delta & \quad \Gamma, \Gamma_y \Rightarrow \Delta_3, \Delta_3, \Delta \quad \text{($\circ$)}
\end{align*}
\]

where $\Gamma_y$ is the set of unmarked universal formulas in the antecedent of lower sequent, $\Delta_3$ a set of unmarked existential formulas in the succedent.

Let $T$ be a (finite or infinite) labelled tree of deductions in which the inference rules $(Rep), (br), (weak), (\circ)$ may occur besides inference rules in $\mathbf{LJm}$, and $g(\sigma)$ is a gate in $\{\lor, \land, 0, 1, \times\}$, $S(\sigma)$ is a sequent and $d(\sigma)$ is a (finite) deduction of $S(\sigma)$ for $\sigma \in T$. Suppose that the tree $T$ of deductions enjoys the condition $[\Pi]$ on the eigenvariables.

For $\rho, \sigma_0, \sigma_1 \in T$ let $g(\rho) = g(\sigma_0) = g(\sigma_1) = \forall$. We say that a pair $(\sigma_0, \sigma_1)$ is a transferable pair if the following conditions are met:

1. The antecedent of $S(\sigma_0)$ and the succedent of $S(\sigma_1)$ have a common marked atomic formula $\alpha^\circ \equiv (R(t_1, \ldots, t_n))^\circ$. This means that $S(\sigma_0)$ is a sequent $\alpha^\circ, \Pi_0 \Rightarrow \Lambda_0$, and $S(\sigma_1)$ is $\Pi_1 \Rightarrow \Lambda_1, \alpha^\circ$ for some cedents $\Pi_1, \Lambda_1$.

2. There is no non-invertible upper sequent from $S(\sigma_0)$ to the sequent $S(\rho \ast (0\ast_i))$ in the deduction, i.e., $\rho \subseteq^0 \sigma_0 = \rho \ast \kappa_0$ for some $\kappa_0 = (0) \ast \kappa'_0$.

And $\rho \subseteq_c \sigma_1 = \rho \ast \kappa_1$ with $\rho \nsubseteq^0 \sigma_1$ for some $\kappa_1 = (i) \ast \kappa'_1$ and $i \neq 0$.

\[
\begin{align*}
\vdots \quad \vdots \\
\rho \ast \kappa_0 : \alpha^\circ, \Pi_0 \Rightarrow \Lambda_0 & \quad \rho \ast \kappa_1 : \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ \\
\vdash d_0 & \vdash d_1 \\
\Gamma, \Gamma_2, \Gamma_3^\circ \Rightarrow \Delta_3, \Delta_2, \Delta_1^\circ & \quad \ldots \\
\Gamma, \Gamma_2, \Gamma_3^\circ \Rightarrow \delta_1 & \quad (br)
\end{align*}
\]
where \( d_0 \) denotes a deduction of the leftmost continued sequent up to \( \rho \ast \kappa_0 \), and \( d_1 \) a deduction of a non-invertible upper sequent \( \Gamma, \Gamma^0 \vdash \delta_1 \) up to \( \rho \ast \kappa_1 \).

Then let us combine these two deductions as follows.

\[
\begin{align*}
\rho \ast \kappa_0 \ast \kappa_1 & : \alpha^\circ, \alpha^\circ, \Pi_0 \Rightarrow \Lambda_0' \quad \rho \ast \kappa_0 \ast \kappa_1 : \alpha^\circ, \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ \\
\vdots & \vdots \\
\alpha^\circ, \Gamma, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha} \quad \cdots \\
\vdots & \vdots \\
\vdots & \vdots \\
\rho \ast \kappa_0 & : \alpha^\circ, \Gamma, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}
\end{align*}
\]

\[\frac{\alpha^\circ, \Pi_0, \Gamma, \Gamma^0, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}}{\vdots \vdots \vdots \vdots \vdots} \quad (br)\]

\[\frac{\alpha^\circ, \Gamma, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}}{\Gamma, \Gamma^0, \Gamma^0, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}} \quad (weak)\]

\[\frac{\Gamma, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}}{\Gamma, \Gamma^0, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}, \Delta^0_{\alpha}} \quad (o)\]

where \( \rho \ast \kappa_0 \ast \kappa_1 : \alpha^\circ, \Pi_1 \Rightarrow \Lambda_1, \alpha^\circ \) is an axiom, \( \alpha^\circ \ast d_1 \) is a deduction obtained from \( d_1 \) by appending \( \alpha^\circ \) to antecedents, \( (weak) \) is a weakening. \( \Gamma_1 \ast d_0 \ast \Delta_1 \) is a deduction obtained from \( d_0 \) by appending \( \Gamma_1 = \Gamma \cup \Gamma^0 \) to antecedents and \( \Delta_1 = \Delta_{\alpha} \cup \Delta \ast \Delta^0_{\alpha} \cup \Delta^0_{\alpha} \) to succedents.

\( \alpha^\circ \ast d_0' \) is a deduction obtained from \( d_0 \) by appending \( \alpha^\circ \) to the antecedents and renaming eigenvariables from ones in \( d_0 \) for the condition (1) on eigenvariables. \( \alpha^\circ \ast (\Pi_0') [\Lambda_0] \) is obtained from \( \alpha^\circ \ast (\Pi_0) [\Lambda_0] \) by renaming free variables which are introduced as eigenvariables of \( (\exists \Rightarrow) \) in \( d_0 \), resp.

The above deduction is said to be transferred by the pair \((\sigma_0, \sigma_1)\) (and the common atomic formula \( \alpha^\circ \)).

In the transferred deduction, each sequent up to \( \rho \ast \kappa_0 : \alpha^\circ, \Gamma, \Gamma^0, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta, \Delta^0_{\alpha}, \Delta^0_{\alpha} \) receives the same label \( \mu \) and the same gate \( g(\mu) \) as in the original deduction, e.g., \( g(\rho) = \forall \) and \( g(\rho \ast (\emptyset)) = \land \). On the other side in the above part of \( \rho \ast \kappa_0 : \alpha^\circ, \Gamma, \Gamma^0, \Gamma^0 \Rightarrow \Delta^0_{\alpha}, \Delta, \Delta^0_{\alpha}, \Delta^0_{\alpha} \), insert \( \kappa \) to each label where \( \sigma_0 = \rho \ast \kappa_0 \). Then \( g(\rho \ast \kappa_0 \ast \kappa_1) = 1 \). The gate \( g(\rho \ast \kappa_0 \ast \kappa_0) \) is unchanged, and \( g(\rho \ast \kappa_0) = \forall \).

Let us check the conditions on multi-succedents and eigenvariables. Since there is no non-invertible upper sequent from \( \rho \) to \( \rho \ast \kappa_0 \), we can append formulas in \( \Delta_1 = \Delta_{\alpha} \cup \Delta \ast \Delta^0_{\alpha} \cup \Delta^0_{\alpha} \) to the succedents. Next consider a free variable \( a \) occurring in \( \alpha \). Assume that \( a \) is introduced as an eigenvariable of a \( (\exists \Rightarrow) \) or a \( (\forall \Rightarrow) \) (which subsumes \( (\Rightarrow \forall) \)) between \( \rho \) and \( \rho \ast \kappa_1 \) in \( d_1 \). But the variable \( a \) occurs in \( \rho \ast \kappa_0 \), and this is not the case by (1) since \( \rho \ast \kappa_0 \) is not above the inference rule nor to the right of it. Furthermore free variables occurring in \( \Gamma_1 \cup \Delta_1 \) is distinct from eigenvariables of \( (\exists \Rightarrow) \) occurring in \( d_0 \) since \( \rho \) is below \( d_0 \), (1).

All of transferrable pairs have to be removed from the constructed deduction in proof search to avoid an inconsistency in the definition of Kripke models, cf. Proposition 2.4 and Lemma 2.6.

For this, let us introduce a new stage transfer stage in the construction of the tree of deductions. Recall that \( TR(S_0) \) is defined in stages \( TR(S_0)_n \). Let \( TTR(S_0)_n \) denote the \( n \)th piece of a tree \( TTR(S_0) \) of deductions, which we are now going to define.
First $TTR(S_0) = TR(S_0) = \{ \emptyset \}$. Suppose that $g(\emptyset, 0) = \land$, i.e., the given sequent $S_0$ is neither an axiom nor fully analyzed. Then enter the $\land$-stage.

Suppose that $TTR(S_0)_n$ has been constructed, and there exists a leaf $\sigma$ in $TTR(S_0)_n$ such that $g(\sigma, n) \in \{ \land, \lor \}$. When $n \equiv 0 \pmod{3}$, we are in a $\land$-stage, and if we are in a $\lor$-stage, then $n \equiv 2 \pmod{3}$. When $n \equiv 1 \pmod{3}$, perform the following transfer stage. In other words, after a $\land$-stage, enter a transfer stage, and after that the tree is extended according to $\lor$-stage.

**transfer stage.** For the transferable pairs $(\sigma_0, \sigma_1), (\tau_0, \tau_1)$ in $TTR(S_0)_n$, let

$$(\sigma_0, \sigma_1) \prec (\tau_0, \tau_1) :\iff \sigma_0 \subseteq^c \tau_0 \lor \sigma_1 \subseteq^c \tau_0.$$ 

The relation $\prec$ is wellfounded, i.e., there is no cycle on the set of transferable pairs in $TTR(S_0)_n$.

Pick a $\prec$-minimal pair $(\sigma_0, \sigma_1)$. We have $g(\sigma_0, n) = g(\sigma_1, n) = \lor$. Let $\rho$ denote the infimum of $\sigma_0, \sigma_1$, and $\sigma_i = \rho \ast \kappa_i$ for $i = 0, 1$. Then transfer $TTR(S_0)_n$ by the pair $(\sigma_0, \sigma_1)$. In the transferred deduction the gate of the node $\rho \ast \kappa_0 \ast \kappa_1$ becomes 1 since $S(\rho \ast \kappa_0 \ast \kappa_1)$ is an axiom. Below the axiom $\rho \ast \kappa_0 \ast \kappa_1$, modify the values of gates as in Definition 2.1. Some gates might receive the value 1 by this modifications.

Iterate the transformations to yield a tree of deductions $TTR(S_0)_{n+1}$, in which there is no transferable pair. Then extend the tree according to $\lor$-stage.

This ends the construction of $TTR(S_0)$.

**Lemma 2.6** For $\sigma \in TTR(S_0)$, $\Gamma_\infty^\sigma(\sigma; TTR(S_0)) \cap \Delta^\infty(\sigma; TTR(S_0))$ has no common atomic formula.

**Proof.**

Suppose $\alpha$ is a common atomic formula in $\Gamma_\infty^\sigma(\sigma; TTR(S_0))$ and $\Delta^\infty(\sigma; TTR(S_0))$. Let $\sigma_1 \in TTR(S_0)$ be such that $g(\sigma_1) = \lor, \sigma \subseteq^c \sigma_1$ and $\alpha \in \Delta(\sigma_1)^\times$. Also let $\rho, \sigma_0 \in TTR(S_0)$ be such that $g(\rho) = g(\sigma_0) = \lor, \rho \subseteq \sigma, \rho \subseteq^c \sigma_0$ and $\alpha \in \Gamma(\sigma_0)^\times$. We see $\rho \subseteq^c \sigma \in$ from Proposition 2.5 $\alpha \in \Gamma^\infty(\sigma; TTR(S_0)) \cap \Delta^\infty(\sigma; TTR(S_0))$. Then this means that $(\sigma_0, \sigma_1)$ is a transferable pair. Such a pair has been removed from $TTR(S_0)_n$ for an $n$ in transfer stage, and from $TTR(S_0)$. Hence this is not the case. 

**Theorem 2.7** (Schütte’s dichotomy)

In $TTR(S_0)$, $g(\emptyset) = 1$ iff $LJm \vdash S_0$.

**Proof.** If $g(\emptyset) = 1$, then it is plain to see that $LJm \vdash S_0$.

In what follows assume $g(\emptyset) \neq 1$. Extract a subtree $T \subset TTR(S_0)$ as follows. First $\emptyset \in T$. The nodes $\sigma \in T$ with $g(\sigma) = 0$ are leaves in $T$. Suppose $\sigma \in T$ has been chosen so that $g(\sigma) = \land$. Then in the deduction $d(\sigma)$, there is a leaf $\sigma \ast (j)$ such that $g(\sigma \ast (j)) \neq 1$. If $g(\sigma \ast (j)) = 0$, then we would have $g(\sigma) = 0$. Hence $g(\sigma \ast (j)) = \lor$, and $g(\sigma \ast (j, i)) \neq 1$ for any son $\sigma \ast (j, i)$. Let $\sigma \ast (i), \sigma \ast (j, i) \in T$ for any $i$.

Then $T$ enjoys the three conditions in Proposition 2.4. The third condition follows from Lemma 2.6. Hence for $S_0 = S(\emptyset) = (\Gamma(\emptyset) \Rightarrow \Delta(\emptyset))$, $\emptyset \models \land \Gamma(\emptyset)$.
and \( \emptyset \not\models \Delta(\emptyset) \) in the Kripke model \( \langle T, C_e, D_T, I_T \rangle \) defined from the tree \( T \). Therefore \( \text{LJm} \not\models S_0 \). \( \square \)

**Corollary 2.8** Cut inference

\[
\Gamma \Rightarrow \alpha, \Delta \quad \Gamma, \alpha \Rightarrow \Delta
\]

\[(\text{cut})\]

is permissible in \( \text{LJm} \).

### 2.5 Completeness

In order to have the Schütte’s dichotomy in Theorem 2.4, we need to remove all of transferable pairs, i.e., to transform the tree \( TR(S_0) \) of deductions to \( TTR(S_0) \). However when we need only to show the completeness of \( \text{LJm} \) with the cut rule \((\text{cut})\), \( \text{LJm} + (\text{cut}) \), one can extract a consistent tree \( T \) from \( TR(S_0) \) by which \( S_0 \) is refuted provided that \( S_0 \) is not derivable in \( \text{LJm} + (\text{cut}) \).

In this subsection we consider the tree \( TR(S_0) \).

For formulas \( \alpha, \alpha^\forall \) denotes ambiguously formulas obtained from \( \alpha \) by binding some (possibly none) free variables by universal quantifiers.

For formulas \( \alpha, \beta, \oplus(\alpha, \beta) \) denotes the formula \( \forall \vec{x}(\gamma(\vec{x}) \supset (\delta(\vec{x}) \lor \beta)) \) if \( \alpha \) is a universally bound implicational formula \( \forall \vec{x}(\gamma(\vec{x}) \supset \delta(\vec{x})) \) for a list (possibly empty) list \( \vec{x} \) of bound variables. \( \vec{x} \) does not occur in \( \beta \). Otherwise \( \oplus(\alpha, \beta) \equiv (\alpha \lor \beta) \). For sequences \( \vec{\alpha} = \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) of formulas \( \alpha_i \), let \( \oplus(\vec{\alpha}, \beta) \equiv \oplus(\alpha_0, \oplus(\alpha_1, \ldots, \oplus(\alpha_{n-1}, \beta)) \) ), and \( \oplus^\forall(\vec{\alpha}, \beta) \) denotes formulas \( \oplus^\forall(\alpha_0, \oplus^\forall(\alpha_1, \ldots, \oplus^\forall(\alpha_{n-1}, \beta)) \) ).

For the sequent \( S(\sigma) = (\Gamma(\sigma) \Rightarrow \Delta(\sigma)) \) let

\[\chi(\sigma) \equiv (\bigwedge \Gamma(\sigma) \supset \bigvee \Delta(\sigma))\]

Let \( T \subset TR(S_0) \) be a finite subtree such that for each \( \sigma \in T \), if \( g(\sigma) = \land \), then \( \sigma \) has a single son \( \sigma * (i) \) in \( T \), and if \( g(\sigma) = \lor \), then \( \sigma \) has all of sons \( \sigma * (j) \) in \( T \), i.e., \( \forall j [\sigma * (j) \in T \iff \sigma * (j) \in TR(S_0)] \). Moreover \( g(\sigma) = \lor \) for any leaves \( \sigma \) in \( T \). Such a tree \( T \) is called a selected tree. Following [2] let us introduce characteristic formulas \( \chi(\sigma; T) \) for \( \sigma \in T \) recursively as follows. Recall that \( FV(\sigma) \) denotes the set of free variables occurring in the sequent \( S(\sigma) \) for \( \sigma \in TR(S_0) \).

For leaves \( \sigma \) in \( T \),

\[\chi(\sigma; T) \equiv \chi(\sigma)\]

Let \( \sigma \in T \) be an internal node with \( g(\sigma) = \land \), and \( \sigma * (i) \) be the unique son of \( \sigma \) in \( T \). Also let \( \chi(\sigma * (i); T) \equiv \alpha(\vec{a}) \) where \( \vec{a} \) is the set of eigenvariables of inference rules \( (\exists \Rightarrow) \) occurring between the leaf \( S(\sigma * (i)) \) and the root \( \sigma \) of the deduction \( TR_{\sigma} \). Then for \( \chi^\forall(\sigma * (i); T) \equiv \forall \vec{x} \alpha(\vec{x}) \)

\[\chi(\sigma; T) \equiv \chi^\forall(\sigma * (i); T)\]

Let \( \sigma \in T \) be an internal node with \( g(\sigma) = \lor \), and \( \sigma * (j) (j \leq J_\sigma) \) be all of sons of \( \sigma \) in \( T \), where \( \Delta^\forall_\sigma = \{ \beta_j \}_{J_\sigma < j \leq J_\sigma} \) for the set \( \Delta^\forall_\sigma \) of marked universal formulas.
in the succedent of $S(\sigma)$. For each $j$ with $J_2 < j \leq J_3$ let $\chi(\sigma * (j); T) \equiv \alpha(a)$ for the eigenvariable $a$ introduced at the $j$th upper sequent $S(\sigma * (j))$. Then for $\chi^*(\sigma; T) \equiv \forall x \alpha(x)$, let

$$\chi(\sigma; T) \equiv \oplus(\chi(\sigma * (0); T), \bigvee_{0 < j \leq J_2} \chi(\sigma * (j); T)) \vee \bigvee_{J_2 < j \leq J_3} \chi^*(\sigma * (j); T)).$$

Finally let $\chi(T) \equiv \chi(\sigma; T)$ for the root $\sigma$ in $T$.

**Proposition 2.9**

1. Let $T \subset TR(S_0)$ be a selected tree, and $\sigma$ be a leaf in $T$. Then there exist formulas $\vec{\alpha}$ such that $L\Delta m + (cut) \vdash \chi(T) \leftrightarrow \oplus(\vec{\alpha}, \chi(\sigma))$. 
2. Let $T \subset TR(S_0)$ be a selected tree, and $\sigma$ be a leaf in $T$. Then there exist formulas $\vec{\alpha}$ such that $L\Delta m + (cut) \vdash \chi(T) \leftrightarrow \bigoplus(\vec{\alpha}, \chi(\sigma), \beta)$. 
3. Let $g(\sigma) = \land$ and $L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma), \beta))$ for some formulas $\vec{\alpha}, \beta$. Then there exists a leaf $\sigma * (i)$ in the deduction $TR_\sigma$ such that $L\Delta m + (cut) \vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma * (i)), \beta))$. 
4. Let $g(\sigma) = \lor$ and $L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma), \beta))$ for some formulas $\vec{\alpha}, \beta$. Then for each $j$ there exists a leaf $\sigma * (j, i_j)$ in the deduction $TR_{\sigma * (j)}$ such that $L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma * (0), i_0)), \bigvee_{0 < j \leq J_2} \chi(\sigma * (j, i_j)) \vee \bigvee_{J_2 < j \leq J_3} \chi^*(\sigma * (j, i_j)), \beta))$. 
5. Let $T \subset TR(S_0)$ be a selected tree. Assume $L\Delta m + (cut) \not\vdash \chi(T)$. Then for each leaf $\sigma \in T$ and each $j$ there exists a leaf $\sigma * (j, i_j)$ in the deduction $TR_{\sigma * (j)}$ such that $L\Delta m + (cut) \not\vdash \chi(T')$ for the tree $T'$ obtained from $T$ by extending each leaf $\sigma$ to $\sigma * (j), \sigma * (j, i_j)$.

**Proof.** \[2.9\] is seen by induction on the size of the tree $T$ using Proposition \[2.1\]. \[2.9\] is seen by inspection to inference rules in $L\Delta m$ except non-invertible ones $(\Rightarrow \lor)$ and $(\Rightarrow \forall)$. \[2.9\] Assume $g(\sigma) = \land$ and $L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma), \beta))$. Then

$$L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma * (0), i_0)), \bigvee_{j > 0} \chi^*(\sigma * (j))), \beta))$$

by the definition of the rule (br). In other words,

$$L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi(\sigma * (0), i_0)), \bigvee_{j > 0} \chi^*(\sigma * (j)) \lor \beta)).$$

By Proposition \[2.9\] pick an $i_0$ such that

$$L\Delta m + (cut) \not\vdash \bigoplus(\vec{\alpha}, \oplus(\chi^*(\sigma * (0), i_0)), \bigvee_{j > 0} \chi^*(\sigma * (j)) \lor \beta)).$$
Hence

$$\text{LJm} + \text{(cut)} \not\vdash (\alpha, \oplus(\chi^y(\sigma * (0, i_0)), \bigvee_{j>1} \chi^y(\sigma * (j))) \lor \beta, \chi(\sigma * (1)))$$.

Then again by Proposition 2.9.3 pick an \(i_1\) such that

$$\text{LJm} + \text{(cut)} \not\vdash (\alpha, \oplus(\chi^y(\sigma * (0, i_0)), \bigvee_{j>1} \chi^y(\sigma * (j))) \lor \beta, \chi(\sigma * (1, i_1)))$$.

In this way we can pick numbers \(i_j\) so that

$$\text{LJm} + \text{(cut)} \not\vdash (\alpha, \oplus(\chi^y(\sigma * (0, i_0)), \bigvee_{j>0} \chi^y(\sigma * (j, i_j))) \lor \beta, \chi(\sigma * (1, i_j)))$$.

Let \(\sigma\) be a leaf in \(T\). By Proposition 2.9.2 we have \(\text{LJm} + \text{(cut)} \vdash \chi(T) \leftrightarrow \oplus^y(\alpha, \chi(\sigma)) \leftrightarrow \oplus^y(\alpha, \chi(\sigma), \bot)\) for some formulas \(\alpha\). On the other side the formula \(\chi(T')\) is obtained from \(\chi(T)\) by replacing \(\chi(\sigma)\) by \(\oplus(\chi^y(\sigma * (0, i_0)); \{0\})\) for each leaf \(\sigma\) in \(T\). Thus the proposition is seen from Proposition 2.9.4. \(\qed\)

Supposing the given sequent \(S_0\) is underivable in \(\text{LJm} + \text{(cut)}\), let us pick a tree \(T \subset TR(S_0)\) for which the following hold. Let \(T_n = \{\sigma \in T : \text{lh}(\sigma) \leq 2n + 1\}\).

1. for any \(\sigma \in T\), \(g(\sigma) \in \{0, \land, \lor\}\),
2. \(\emptyset \in T\), and there exists a unique son \((i)\) of \(\emptyset\) in \(T\) such that \(\text{LJm} + \text{(cut)} \vdash S((i))\). Namely \(T_0 = \{\emptyset, (i)\}\).

Let

$$\chi(T_0) := \chi((i)) \equiv (\bigwedge \Gamma((i)) \lor \bigvee \Delta((i)))$$

Then \(\text{LJm} + \text{(cut)} \vdash \chi(T_0)\).

3. for any \(\sigma \in T\) with \(g(\sigma) = \lor\), every son \(\sigma * (j) \in TR(S_0)\) is in \(T\), and there exists a unique son \(\sigma * (j, i_j)\) for each \(j\). Namely \(T_{n+1} = T_n \cup \{\sigma * (j, i_j) : \sigma \in T_n, \text{lh}(\sigma) = 2n + 1\}\).

Let \(\text{lh}(\sigma) = 2n + 1\) and assume \(\text{LJm} + \text{(cut)} \vdash \chi(T_n)\). The sons \((j, i_j)\) are chosen so that \(\text{LJm} + \text{(cut)} \not\vdash \chi(T_{n+1})\). Such an extension is possible by Proposition 2.9.5.

**Lemma 2.10** For \(\sigma \in T, \Gamma^\infty_\sigma(\sigma; T) \to \Delta^\infty_\sigma(\sigma; T)\) is \(FV^\infty_\sigma(\sigma; T)\)-analyzed, and \(\Gamma^\infty_\sigma(\sigma; T) \cap \Delta^\infty_\sigma(\sigma; T)\) has no common atomic formula

**Proof.** It is easy to see that for any \(\tau \in T\), \(\bot \notin \Gamma^\infty_\sigma(\sigma; T)\), and hence \(\bot \notin \Gamma^\infty_\sigma(\sigma; T)\).

Since each term over the set \(FV^\infty_\sigma(\sigma; T)\) is eventually tested for universal antecedent formula and existential succedent formula in the extensions \(\tau\) of \(\sigma\) with \(\sigma \sqsubseteq \tau, \Gamma^\infty_\sigma(\sigma; T) \to \Delta^\infty_\sigma(\sigma; T)\) is \(FV^\infty_\sigma(\sigma)\)-saturated.
Suppose \( \alpha \) is a common atomic formula in \( \Gamma^\infty_{\subset_e}(\sigma;T) \) and \( \Delta^\infty(\sigma;T) \). Let \( \tau \in T \) be such that \( g(\tau) = \vee, \sigma \subset^0 \tau \) and \( \alpha \in \Delta(\tau)^\times \). Also let \( \rho, \lambda \in T \) be such that \( g(\rho) = g(\lambda) = \vee, \rho \subset_e \sigma, \rho \subset^0 \lambda \) and \( \alpha \in \Gamma(\lambda)^\times \). We see \( \rho \subset_e \sigma \) from Proposition \( 2.6 \) \( \alpha \notin \Gamma^\infty(\sigma;T) \cap \Delta^\infty(\sigma;T) \).

\[
S(\lambda) : \alpha, \Pi_0 \Rightarrow \Lambda_0 \quad S(\tau) : \Pi_1 \Rightarrow \Lambda_1, \alpha \\
\begin{array}{c}
\vdots \\
\vdots \\
\Gamma_0 \Rightarrow \Delta_0 \\
\vdots \\
\Gamma_1 \Rightarrow \Delta_1 \\
\vdots \\
S(\rho) : \Gamma \Rightarrow \Delta, (br)
\end{array}
\]

We can assume that \( lh(\lambda) = lh(\tau) \). Otherwise extend the shorter node along the leftmost branch \( \subset^0 \). Let \( 2n + 1 := lh(\lambda) = lh(\tau) \). Let \( a \) be an eigenvariable of an inference rule occurring between \( \tau \) and \( \rho \). If the variable \( a \) occurs in the formula \( \alpha \), then so does in the sequent \( S(\lambda) : \alpha, \Pi_0 \Rightarrow \Lambda_0 \), and this contradicts \( \Box \). Hence the variable \( a \) does not occur in the formula \( \alpha \).

There are formulas \( \beta \) and \( \varphi(\alpha) \) such that \( \text{LJm} + (\text{cut}) \vdash \chi(\rho;T_n) \leftrightarrow ((\alpha \land \beta) \supset \varphi(\alpha))^\gamma \), where \( \alpha \) occurs in \( \varphi(\alpha) \) possibly in the scopes of \( \forall \) and \( \exists \) and in the succedents of \( \supset \), but no universal quantifier binds free variables in \( \alpha \). Namely \( \varphi(\alpha) \) is in the class \( \Phi(\alpha) \) such that \( \alpha \in \Phi(\alpha), \varphi(\alpha) \in \Phi(\alpha) \Rightarrow \{ \beta \lor \varphi(\alpha), (\beta \supset \varphi(\alpha))^\gamma \} \subset \Phi(\alpha) \) for any \( \beta \). No universal quantifier in \( (\beta \supset \varphi(\alpha))^\gamma \) binds free variables in \( \alpha \) since a universal quantifier in \( (\beta \supset \varphi(\alpha))^\gamma \) binds an eigenvariable of an inference rule occurring between \( \tau \) and \( \rho \).

We see inductively that \( \text{LJm} + (\text{cut}) \vdash \alpha \supset \varphi(\alpha) \) for any \( \varphi(\alpha) \in \Phi(\alpha) \). Hence \( \text{LJm} + (\text{cut}) \vdash \chi(\rho;T_n) \), and \( \text{LJm} + (\text{cut}) \vdash \chi(\tau;T_n) \) by Proposition \( 2.9 \). \( \Box \) \( \text{LJm} + (\text{cut}) \vdash \alpha \lor \beta \supset \oplus(\alpha, \beta) \). This is a contradiction.

**Theorem 2.11** Each Kripke model \( \langle T, \subset_e, D_T, I_T \rangle \) falsified the given sequent \( S_0 \), no matter which tree \( T \) is chosen. Hence \( \text{LJm} + (\text{cut}) \) is intuitionistically complete in the sense that any intuitionistically valid sequent is derivable in \( \text{LJm} + (\text{cut}) \).

**Proof.** \( T \) enjoys the three conditions in Proposition \( 2.4 \). The third condition follows from Lemma \( 2.10 \). Hence for \( S_0 = S(\emptyset) = (\Gamma(\emptyset) \Rightarrow \Delta(\emptyset)), \emptyset = \bigwedge \Gamma(\emptyset) \) and \( \emptyset \neq \bigvee \Delta(\emptyset) \) in the Kripke model \( \langle T, \subset_e, D_T, I_T \rangle \) defined from the tree \( T \). Therefore \( \text{LJm} + (\text{cut}) \not\vdash S_0 \). \( \Box \)

**2.6 The positive fragment \( \text{Lq}^+ \) is in PSPACE**

A formula is said to be **positive** (with respect to quantifiers) iff any universal quantifier [existential quantifier] occurs only positively [negatively] in it, resp. Here positive/negative occurrence of quantifiers is meant in the usual classical sense. A formula \( \alpha \) is **negative** if its negation \( \alpha \supset \bot \) is positive. A sequent is said to be **positive** iff any succedent formula [antecedent formula] in it is positive [negative], resp.

Then G. Mints \( 13 \) showed that it is decidable whether or not a given positive formula is intuitionistically derivable. In other words the positive fragment \( \text{Lq}^+ \) of intuitionistic predicate logic is decidable, cf. \( \Box \) for an alternative proof.
In this final subsection we sharpen the decidability result by showing that \( \mathsf{Iq}^+ \) is in PSPACE.

In a proof-search of positive sequents, only positive sequents are produced. Hence the branching rule \((\text{br})\) is of the form:

\[
\begin{array}{c}
\{\gamma^0, \Gamma_0, \Gamma_S \vdash \delta^0 : (\gamma \supset \delta)^0 \in \Delta^0\}_i \\
\{\Gamma, \Gamma_S \vdash \gamma(a)^0 : (\forall y \gamma(y)) \in \Delta^0\}_i
\end{array}
\]

\(\Gamma, \Gamma_S \vdash \Delta, \Delta^0, \Delta^0\)

where and elsewhere we don’t need the condition \((\text{I})\) on eigenvariables. Specifically it suffices for each eigenvariable of \((\exists \Rightarrow)\) and \((\text{br})\) (subsuming \((\Rightarrow \forall)\)) not to occur in the lower sequent.

Let \(S_0\) be a given positive sequent. The tree of deductions \(\mathsf{TR}(S_0)\) is constructed exactly as for propositional case, subsection \([1,3]\) Let us spell out modifications needed. A sequent \(\Gamma \Rightarrow \Delta\) is saturated if it is saturated in the sense of propositional formula, and if \((\exists y \beta(y)) \in \Gamma\), then \((\beta(a)) \in \Gamma^x\) for a free variable \(a \in \mathsf{FV}\). Then the deduction \(\mathsf{TR}_S\) of a sequent \(S\) is constructed so that \(\mathsf{TR}_S\) is a finite binary tree whose leaves are saturated sequents. From the deductions \(\mathsf{TR}_S, \mathsf{TR}(S_0)\) is defined to be a \((\land, \lor)\)-tree.

As in Proposition \([1,3]\) we see that the whole process generating the tree \(\mathsf{TR}(S_0)\) terminates, where the number \(dp(S) = (N + 1)\alpha_S + \beta_S\) is defined by letting \(\alpha_S\) be the sum of the number of positive occurrences of marked connectives \(\lor\) and the number of (positive) occurrences of marked universal quantifiers \(\forall\) in \(S\), and \(\beta_S\) be the total number of occurrences of marked connectives \(\land, \lor, \supset, \exists\) in \(S\). Moreover let \(N\) denote the the total number of occurrences of marked connectives \(\land, \lor, \supset, \exists\) in the given sequent \(S_0\).

Let \(\mathsf{De}(S_0)\) denote the finite deduction with the branching rule \((\text{br})\), obtained from the finite tree \(\mathsf{TR}(S_0)\) by fulfilling intermediate deductions \(\mathsf{TR}_{S(\sigma)}\) for \(\sigma \in \mathsf{TR}(S_0)\) with \(q(\sigma) = \land\). Note that the number of eigenvariables introduced on a branch in \(\mathsf{De}(S_0)\) is bounded by the length of the branch, which is bounded by a quadratic polynomial of \(#S_0\).

Suppose that the value \(v(\emptyset) = 0\). As in the propositional case let us shrink \(\mathsf{TR}(S_0)\) to a tree \(T\) with sets \(V_T(\sigma)\) of atoms (atomic formulas), where \(V_T(\sigma) = \Gamma(\sigma) \land \mathsf{Atm}\). Thus a Kripke model \((T, \subset_e, D_T, I_T)\) is defined by letting \(D_T(\sigma) = \mathsf{Tr}(\mathcal{FV}(\tau) : \tau \subset_e \sigma, g(\tau) = \lor)\) and for predicate symbol \(R\), \(R^x = \{(t_1, \ldots, t_n) : t_1, \ldots, t_n \in D_T(\sigma) \& R(t_1, \ldots, t_n) \in V_T(\sigma)\}\). Function symbols are interpreted literally.

Then as in Theorem \([1,6]\) we see that if \(v(\emptyset) = 0\), then each Kripke model \((T, \subset_e, D_T, I_T)\) falsifies the given positive sequent \(S_0\), and if \(v(\emptyset) = 1\), then we can extract a (cut-free) derivation of \(S_0\).

**Corollary 2.12** The positive fragment \(\mathsf{Iq}^+\) of intuitionistic predicate logic is in PSPACE.

Our proof of Corollary \([2,12]\) is similar to one for Corollary \([1,8]\) The size \(#S\) of the sequents \(S\) is defined to be the total number of occurrences of symbols in \(S\), where the \(n\)th free variable \(a_n\) is identified with the sequence \(ab_{|a_n|−1} \cdots b_0\).
for the binary representation $b_{n|−1} \cdots b_0$ of the number $n$. Hence the size of $a_n$ is equal to the binary length $|n|$ of $n$.

**Proposition 2.13** Let $\vec{S} = S_0, S_1, \ldots, S_{n−1}$ be a branch in $De(S_0)$, where $S_{i+1}$ is an upper sequent of an inference rule with its lower sequent $S_i$. Then $\#\vec{S} := \sum_{i<n} \#S_i$ is bounded by a (quintic) polynomial of the size $\#S_0$ of the given sequent $S_0$.

**Proof.** As in Proposition 1.9 we see that the length $n$ of branches $\vec{S}$ is bounded by a quadratic polynomial of $\#S_0$. On the other side the maximal size of sizes $\#S_i$ of sequents $S_i$ is seen to be bounded by a cubic polynomial as follows. First let us regard each free variable $a_n$ as a single symbol. Then the total number of occurrences of symbols in $S_i$ is bounded by a quadratic polynomial of $\#S_0$. Second the number of eigenvariables $a_n$ introduced on the branch is bounded by a quadratic polynomial of $\#S_0$. Hence the size of the variable $a_n$ is bounded linearly. $\square$

From Proposition 2.13 we can conclude Corollary 2.12 as in subsection 1.4.

**References**

[1] G. Dowek and J. Ying, Eigenvariables, bracketing and the decidability of positive minimal predicate logic, Theor. Comp. Sci., 360(2006), 193-208.

[2] S. Kripke, Semantic analysis of intuitionistic logic, in: J. Crossley and M. A. E. Dummett, eds., Formal Systems and Recursive Functions, North-Holland, Amsterdam (1965), pp. 92-130.

[3] G. Mints, Solvability of the problem of deducibility in LJ for a class of formulas not containing negative occurrences of quantifiers, Proc. Steklov Inst. Math. 98(1968), 135-145.

[4] G. Mints, A short introduction to intuitionistic logic, Kluwer Academic/Plenum Publishers, New York, 2000.

[5] G. Mints, Structure of tableau derivations and cut-elimination for intuitionistic logic, draft.

[6] K. Schütte, Syntactical and semantical properties of simple type theory, Jour. Symb. Logic 25(1960), 305-326.

[7] R. Statman, Intuitionistic propositional logic is polynomial-space complete, Theret. Comp. Sci., 9(1979), 67-72.