TORIC IDEALS ASSOCIATED WITH GAP-FREE GRAPHS

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1. Introduction

Algebraic objects depending on combinatorial data have attracted a lot of interest among both algebraists and combinatorialists: some valuable sources to learn about this research area are the books by Stanley [Sta96], Villarreal [Vil01], Miller and Sturmfels [MS04], and Herzog and Hibi [HH10]. It is often a challenge to establish relationships between algebraic and combinatorial properties of these objects.

Let \( G \) be a simple graph and consider its vertices as variables of a polynomial ring over a field \( K \). We can associate with each edge \( e \) of \( G \) the squarefree monomial \( M_e \) of degree 2 obtained by multiplying the variables corresponding to the vertices of the edge. With this correspondence in mind, we can now introduce some algebraic objects associated with the graph \( G \):

- the **edge ideal** \( I(G) \) is the monomial ideal generated by \( \{ M_e \mid e \text{ is an edge of } G \} \);
- the **toric ideal** \( I_G \) is the kernel of the presentation of the \( K \)-algebra \( K[G] \) generated by \( \{ M_e \mid e \text{ is an edge of } G \} \).

An important result by Fröberg [Frö90] gives a combinatorial characterization of those graphs \( G \) whose edge ideal \( I(G) \) admits a linear resolution: they are exactly the ones whose complementary graph \( G^c \) is chordal. Another strong connection between the realms of commutative algebra and combinatorics is the one which links initial ideals of the toric ideal \( I_G \) to triangulations of the edge polytope of \( G \), see Sturmfels’s book [Stu96] and the recent article by Haase, Paffenholz, Piechnik and Santos [HPPS14]. Furthermore, Gröbner bases of \( I_G \) have been studied among others by Ohsugi and Hibi [OH99b] and Tatakis and Thoma [TT11]. A necessary condition for \( I_G \) to have a squarefree initial ideal is the normality of \( K[G] \), which was characterized combinatorially by Ohsugi and Hibi [OH98] and Simis, Vasconcelos and Villarreal [SVV98]. Normality, though, is not sufficient: Ohsugi and Hibi [OH99a] gave an example of a graph \( G \) such that \( K[G] \) is normal but all possible initial ideals of \( I_G \) are not squarefree.

The main goal of this paper is to prove that the toric ideal \( I_G \) has a squarefree lexicographic initial ideal, provided the graph \( G \) is gap-free (Theorem 3.7): moreover, the corresponding reduced Gröbner basis consists of circuits. In the particular case when \( I(G) \) has a linear resolution (Theorem 3.5) we are actually able to prove that the reduced Gröbner basis \( G \) we describe consists of circuits such that all monomials (both leading and trailing) in the support of \( G \) are squarefree.
In [HHZ04] Herzog, Hibi and Zheng proved that the following conditions are equivalent:

(a) $I(G)$ has a linear resolution;
(b) $I(G)$ has linear quotients;
(c) $I(G)^k$ has a linear resolution for all $k \geq 1$.

It is quite natural to ask (see for instance the article by Hoefel and Whieldon [HW11]) whether these conditions are in turn equivalent to the fact that

(d) $I(G)^k$ has linear quotients for all $k \geq 1$.

In Theorem 2.6 we prove that this is indeed the case, as can be deduced from results in [HH10]. Note that all the equivalences between conditions (a), (b), (c), (d) above hold more generally for monomial ideals generated in degree 2 which are not necessarily squarefree.

The computer algebra system CoCoA [CoC] gave us the chance of performing computations which helped us to produce conjectures about the behaviour of the objects studied.

2. Notation and known facts

First of all, let us fix some notation. $K$ will always be a field and $G$ a simple graph with vertices $V(G) = \{1, \ldots, n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. We can associate to each edge $e = \{i, j\}$ the degree 2 monomial (called edge monomial) $M_e := x_ix_j \in K[x_1, \ldots, x_n]$ and hence we can consider the edge ideal $I(G) := (M_{e_1}, \ldots, M_{e_m})$ and the subalgebra $K[G] := K[M_{e_1}, \ldots, M_{e_m}]$. In the following we will denote by $I_G$ the toric ideal associated with $G$, i.e. the kernel of the surjection

$$K[y_1, \ldots, y_m] \twoheadrightarrow K[G]$$

$$y_i \mapsto M_{e_i}$$

Since the algebraic objects we defined are not influenced by isolated vertices of $G$, we will always assume without loss of generality that $G$ does not have any isolated vertex. We will now introduce some terminology and state some well-known results about toric ideals of graphs: for reference, see for instance [HH10], Section 10.1.

A collection of (maybe repeated) consecutive edges

$$\Gamma = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{q-1}, v_q\}\}$$

(also denoted by $\{v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{q-1} \rightarrow v_q\}$) is called a walk of $G$. If $v_0 = v_q$, the walk is closed. If $q$ is even (respectively odd), the walk is an even (respectively odd) walk. A path is a walk having all distinct vertices; a cycle is the closed walk most similar to a path, i.e. such that vertices $v_0, \ldots, v_{q-1}$ are all distinct. A bow-tie is a graph consisting of two vertex-disjoint odd cycles joined by a single path. Given a walk $\Gamma$, we will denote by $|\Gamma|$ the subgraph of $G$ whose vertices and edges are exactly the ones appearing in $\Gamma$. If no confusion occurs, we will often write walks in more compact ways, such as by decomposing them into smaller walks. If $\Gamma$ is a walk, $-\Gamma$ denotes the walk obtained from $\Gamma$ by reversing the order of the edges.
Remark 2.1. Note that, given a primitive walk $\Gamma$, one can paint – using two colours – the edges of $|\Gamma|$ so that those appearing in an even position in $\Gamma$ are assigned the same colour and those appearing in an odd position are assigned the other one. If an edge were assigned both colours, then the walk $\Gamma$ would not be primitive: deleting inside both monomials one instance of the variable corresponding to that edge, one could construct a proper subwalk of $\Gamma$.

The support of a binomial $b = u - v \in I_G$ is the union of the supports of the monomials $u$ and $v$, that is to say the variables that appear in $u$ and $v$. A binomial $b \in I_G$ is called a circuit if it is irreducible and has minimal support, i.e., there does not exist $b' \in I_G$ such that $\text{supp}(b') \subseteq \text{supp}(b)$. The set of circuits of $I_G$ is denoted by $C_G$.

Let $I$ be an ideal of $S := K[x_1, \ldots, x_n]$. A Gröbner basis $G$ of $I$ is called reduced if every element of $G$ is monic, the leading terms of $G$ minimally generate $\text{in}_\tau(I)$ and no trailing term of $G$ lies in $\text{in}_\tau(I)$. Such a basis is unique and is denoted by $\text{RGB}_\tau(I)$. Generally speaking, changing the term order $\tau$ yields a different reduced Gröbner basis: we will denote by $\text{UGB}(I)$ the universal Gröbner basis of $I$, i.e., the union of all reduced Gröbner bases of $I$.

Proposition 2.2 ([Stu96], Proposition 4.11). One has that $C_G \subseteq \text{UGB}(I_G) \subseteq \text{Gr}_G$.

The second inclusion of Proposition 2.2 means that every reduced Gröbner basis $G$ of $I_G$ consists of binomials coming from primitive walks of $G$. Consider the set of monomials (both leading and trailing) in such a basis: if they are all squarefree, we will say that $G$ is doubly squarefree.

Complete characterizations of both $C_G$ (Villarreal [Vil95]) and $\text{UGB}(I_G)$ (Tatakis and Thoma [TT11]) are known. We recall the characterization of $C_G$ (using the phrasing in Ohsugi and Hibi’s article [OH13]) as a reference.

Proposition 2.3. A binomial $b \in I_G$ is a circuit of $G$ if and only if $b = b_\Gamma$, where $\Gamma$ is one of the following even closed walks:

1. an even cycle;
2. $\{C_1, C_2\}$ where $C_1$ and $C_2$ are odd cycles with exactly one common vertex;
3. $\{C_1, p, C_2, -p\}$ where $C_1$ and $C_2$ are vertex-disjoint odd cycles and $p$ is a path running from a vertex of $C_1$ to a vertex of $C_2$. 

If $\Gamma = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{q-1}, v_q\}\}$ is an even closed walk, one can associate with $\Gamma$ a binomial $b_\Gamma \in K[y_1, \ldots, y_m]$ in the following way:

$$b_\Gamma := \prod_{i=1}^{q} y_{\{v_{2i-2}, v_{2i-1}\}} - \prod_{i=1}^{q} y_{\{v_{2i-1}, v_{2i}\}},$$

where, if $e \in E(G)$, by $y_e$ we mean the variable which is mapped to $M_e$ by the standard surjection. A subwalk $\Gamma'$ of $\Gamma$ is an even closed walk such that all even edges of $\Gamma'$ are also even edges of $\Gamma$ and all odd edges of $\Gamma'$ are also odd edges of $\Gamma$. An even closed walk $\Gamma$ is called primitive if it does not have any proper subwalk. The set of binomials corresponding to primitive walks of a graph $G$ coincides with the so-called Graver basis of $I_G$ (see for instance [Stu96]) and is denoted by $\text{Gr}_G$. 

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Definition 2.4. Let $I \subseteq S := K[x_1, \ldots, x_n]$ be a graded ideal generated in degree $d$.

- If the minimal free resolution of $I$ as an $S$-module is linear until the $k$-th step, i.e. $\text{Tor}^S_i(I, K)_j = 0$ for all $i \in \{0, \ldots, k\}$, $j \neq i + d$, we say that $I$ is $k$-step linear.

- If $I$ is $k$-step linear for every $k \geq 1$, we say $I$ has a linear resolution.

- If $I$ is minimally generated by $f_1, \ldots, f_s$ and for every $1 < i \leq s$ one has that $(f_1, f_2, \ldots, f_{i-1}) : S (f_i)$ is generated by elements of degree 1, then $[f_1, \ldots, f_s]$ is called a linear quotient ordering and $I$ is said to have linear quotients.

- If $I = I(G)$ for some graph $G$ and $I$ has one of the properties above, we say that $G$ has that property.

Proposition 2.5 ([HH10], Proposition 8.2.1). Let $I \subseteq K[x_1, \ldots, x_n]$ be a graded ideal generated in degree $d$. Then

$I$ has linear quotients $\Rightarrow$ $I$ has a linear resolution.

We now recall an important result by Herzog, Hibi and Zheng ([HHZ04]) about the connection between linear quotients and linear resolution in the case when $I$ is a monomial ideal generated in degree 2. Condition (d) below did not appear in the original paper: its equivalence to other conditions, though, can be obtained quickly using results in [HH10].

Theorem 2.6. Let $I \subseteq K[x_1, \ldots, x_n]$ be a monomial ideal generated in degree 2. Then the following conditions are equivalent:

(a) $I$ has a linear resolution;

(b) $I$ has linear quotients;

(c) $I^k$ has a linear resolution for all $k \geq 1$;

(d) $I^k$ has linear quotients for all $k \geq 1$.

Proof. The implications (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b) are obvious, while (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) follow from Proposition 2.5. It is then enough to prove that (a) $\Rightarrow$ (d), but this follows at once from [HH10], Theorems 10.1.9 and 10.2.5 (since the lexicographic order $<_{\text{lex}}$ introduced in Theorem 10.2.5 is of the kind appearing in Theorem 10.1.9). \qed

Remark 2.7. Theorem 10.2.5 and the proof of Theorem 10.1.9 in [HH10] (or, as an alternative, just the proof of the implication (a) $\Rightarrow$ (b) in Theorem 10.2.6) tell us also that, if $I$ is a monomial ideal of degree 2 having a linear resolution and $\{m_1, m_2, \ldots, m_s\}$ is a minimal set of monomial generators for $I$, then there exists a permutation $\sigma$ of $\{1, \ldots, s\}$ such that $[m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(s)}]$ is a linear quotient ordering for $I$. As a consequence, if $I$ is the edge ideal of some graph $G$ having a linear resolution, there exists a way of ordering the edge monomials so that they form a linear quotient ordering.

We thank Aldo Conca for pointing out the following result:
Proposition 2.8. Let $f_1, \ldots, f_s$ be distinct homogeneous elements of degree $d$ in $S := K[x_1, \ldots, x_n]$ which are minimal generators for the ideal $(f_1, \ldots, f_s)$. The following conditions are equivalent:

(a) $[f_1, \ldots, f_s]$ is a linear quotient ordering;
(b) the ideal $(f_1, \ldots, f_i)$ is 1-step linear for all $i \leq s$.

Proof. Let us prove that (a) $\Rightarrow$ (b). Let $i \leq s$. If $[f_1, \ldots, f_s]$ is a linear quotient ordering, then $[f_1, \ldots, f_i]$ is too and hence, by Proposition 2.5, the ideal $(f_1, \ldots, f_i)$ has a linear resolution; in particular, it is 1-step linear.

To prove that (b) $\Rightarrow$ (a), let $i \in \{2, \ldots, s\}$. Consider the exact sequence

$$0 \rightarrow \text{Ker } \varepsilon \rightarrow S(-d)^i \xrightarrow{\varepsilon} (f_1, \ldots, f_i) \rightarrow 0,$$

where $\varepsilon$ is the map which sends $e_j$ to $f_j$ for all $j \in \{1, \ldots, i\}$. Then, by hypothesis, Ker $\varepsilon$ is generated in degree 1. Since $(f_1, \ldots, f_{i-1}) : S (f_i)$ is isomorphic to the $i$-th projection of Ker $\varepsilon$, we are done. \hfill $\square$

In what follows, we will denote by $G^c$ the complementary graph of $G$, i.e., the graph which has the same vertex set of $G$ and whose edges are exactly the non-edges of $G$.

The next result by Eisenbud, Green, Hulek and Popescu proves that, in our context, the algebraic concept of $k$-step linearity can be characterized in a purely combinatorial manner.

Proposition 2.9 ([EGHP05], Theorem 2.1). Let $G$ be a graph and let $k \geq 1$. The following conditions are equivalent:

- $G$ is $k$-step linear;
- $G^c$ does not contain any induced cycle of length $i$ for any $4 \leq i \leq k + 3$.

As a corollary, we recover the important result by Fröberg characterizing combinatorially graphs with a linear resolution.

Corollary 2.10 ([Frö90]). Let $G$ be a graph. Then $G$ has a linear resolution if and only if $G^c$ is chordal, i.e., $G^c$ does not contain any induced cycle of length greater than or equal to 4.

The following notation in [DHS13], we will call a graph $G$ gap-free if for any $\{v_1, v_2\}$, $\{w_1, w_2\}$ in $E(G)$ (where $v_1, v_2, w_1, w_2$ are all distinct) there exist $i, j \in \{1, 2\}$ such that $\{v_i, w_j\} \in E(G)$. In other words, in a gap-free graph any two edges with no vertices in common are linked by at least a bridge.

Remark 2.11. It is easy to see that $G$ is gap-free if and only if $G^c$ does not contain any induced cycle of length 4. It then follows from Proposition 2.9 that $G$ is gap-free if and only if $G$ is 1-step linear.

The following theorem holds more generally for affine semigroup algebras.

Theorem 2.12. Let $G$ be a graph.

1. (Hochster [Hoc72]) If $K[G]$ is normal, then it is Cohen-Macaulay.
2. (Sturmfels [Stu96], Proposition 13.15) If $I_G$ admits a squarefree initial ideal with respect to some term order $\tau$, then $K[G]$ is normal (and hence Cohen-Macaulay).

The problem of normality of graph algebras (and, as a consequence, of edge ideals, see [SVV98], Corollary 2.8) was addressed and completely solved by Ohsugi and Hibi [OH98] and Simis, Vasconcelos and Villarreal [SVV98]. One of the main results they found is the following:

**Theorem 2.13.** A connected graph $G$ is such that $K[G]$ is normal if and only if $G$ satisfies the odd cycle condition, i.e. for every couple of disjoint minimal odd cycles $\{C_1, C_2\}$ in $G$ there exists an edge linking $C_1$ and $C_2$.

Ohsugi and Hibi [OH99a] also found an example of a graph $G$ such that $K[G]$ is normal but $in_{\tau}(I_G)$ is not squarefree for every choice of $\tau$, hence the condition in Theorem 2.12.2 is sufficient but not necessary.

**Remark 2.14.** There is a strong connection between squarefree initial ideals of $I_G$ and unimodular regular triangulations of the edge polytope of $G$. To get more information about this topic, see [Stu96] and the recent work [HPPS14], in particular Section 2.4.

3. **Results**

We start by stating a result about the shape of primitive walks. This is a modification of [OH99b], Lemma 2.1: note that primitive walks were completely characterized by Reyes, Tatakis and Thoma in [RTT12], Theorem 3.1. In the rest of the paper we will often talk of primitive walks of type (i), (ii), (iii) referring to the classification below.

**Lemma 3.1.** Let $\Gamma$ be a primitive walk. Then $\Gamma$ is one of these:

(i) an even cycle;
(ii) $\{C_1, C_2\}$ where $C_1$ and $C_2$ are odd cycles with exactly one common vertex;
(iii) $\{C_1, p_1, C_2, p_2, \ldots, C_h, p_h\}$ where the $p_i$'s are paths of length greater than or equal to one and the $C_i$'s are odd cycles such that $C_i \pmod{h}$ and $C_{i+1} \pmod{h}$ are vertex-disjoint for every $i$.

**Proof.** It is straightforward to see that walks of types (i) and (ii) are primitive. Let $\Gamma$ be a primitive walk neither of type (i) nor (ii). Since $\Gamma$ is primitive, there exists a cycle $C_1$ inside $\Gamma$ (otherwise $\Gamma = \{p, -p\}$ where $p$ is a path and hence all edges of $p$ would appear both in odd and even position in $\Gamma$, thus violating the primitivity); moreover, since $\Gamma$ is not of type (i), $C_1$ has to be odd. Let $C_1 = \{v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{2k+1} \rightarrow v_1\}$; then

$$\Gamma = \{v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{2k+1} \rightarrow v_1 = u_0 \rightarrow u_1 \rightarrow u_2 \rightarrow u_3 \rightarrow \ldots\}.$$  

Let $s \geq 1$ be the least integer such that $u_s$ coincides with one of the vertices in $\{u_0 = v_1, v_2, \ldots, v_{2k+1}, u_1, \ldots, u_{s-1}\}$.  


Suppose \( u_s = v_i \) where \( i \neq 1 \). If \( s = 1 \) and \( i \in \{2, 2k + 1\} \), we get that the edge \( \{v_1, v_i\} \) is both an even and an odd edge of \( \Gamma \) (contradiction). In all other cases, paint the edges appearing in \( \Gamma \) red and black alternately and note that, since \( i \neq 1 \), there are both a red and a black edge of \( C_1 \) starting from \( v_i \). Then exactly one of \( \{v_1 = u_0 \to u_1 \to \cdots \to u_{s-1} \to u_s = v_i \to v_{i+1} \to \cdots \to v_{2k+1} \to v_1 = u_0\} \) and \( \{v_1 = u_0 \to u_1 \to \cdots \to u_{s-1} \to u_s = v_i \to v_{i-1} \to \cdots \to v_2 \to v_1 = u_0\} \) is an even closed subwalk, thus violating the primitivity of \( \Gamma \). This gives us a contradiction.

Suppose \( u_s = u_i \) where \( i \in \{0, \ldots, s - 2\} \) (since \( G \) has no loops, \( i \neq s - 1 \)). Note that one actually has that \( i < s - 2 \), since \( i = s - 2 \) would imply that the edge \( \{u_{s-1}, u_s\} \) is both an even and an odd edge of \( \Gamma \) (contradiction). Therefore there exists a cycle \( C_2 = \{u_s = u_i \to u_{i+1} \to \cdots \to u_{s-1} \to u_s = u_i\} \) disjoint from \( C_1 \) by construction. Since \( \Gamma \) is primitive, \( C_2 \) must be odd; moreover, since \( \Gamma \) is not of type (ii), one has that \( i \neq 0 \). This means that we have found a path \( p_1 = \{u_0 \to u_1 \to \cdots \to u_i\} \) linking the odd cycles \( C_1 \) and \( C_2 \). We can now repeat the whole procedure starting from the cycle \( C_2 \) to find a path \( p_2 \) and an odd cycle \( C_3 \) disjoint from \( C_2 \) and so on, hence proving the claim in a finite number of steps.

\[ \square \]

**Remark 3.2.** Note that, by Proposition 2.3, all binomials corresponding to primitive walks of type (i) and (ii) are circuits.

**Notation 3.3.** Let \( G \) be a graph with \( m \) edges and let \( \tau \) be a term ordering on \( K[y_1, \ldots, y_m] \). With a slight abuse of notation, we will often say that \( e \preceq \tau e' \) instead of \( y_e \preceq \tau e' \) (where \( e, e' \in E(G) \)). Moreover, if \( H \) is a subgraph of \( G \) and \( \tau \) is lexicographic, we will say that \( e \in E(H) \) is the leading edge of \( H \) with respect to \( \tau \) if \( y_{e'} \preceq \tau y_e \) for every \( e' \in E(H) \).

Next we introduce the main technical lemma of the paper. Note that, when dealing with the vertices \( v_1, \ldots, v_s \) of a cycle, for the sake of simplicity we will often write \( v_i \) instead of \( v_i \mod s \).

**Lemma 3.4.** Let \( \Gamma \) be a primitive closed walk of \( G \) of type (iii) and let \( \tau \) be a lexicographic term order on \( K[y_1, \ldots, y_m] \). Let \( e \) be the leading edge of \( \lceil |\Gamma| \rceil \) with respect to \( \tau \): by Lemma 3.1, \( e \) lies into a bow-tie \( \{C_1, p, C_2\} \). Let \( C_1 = \{v_1 \to v_2 \to \cdots \to v_{2k+1} \to v_1\} \), \( C_2 = \{v_1' \to v_2' \to \cdots \to v_{2l+1} ' \to v_1'\} \) and let \( v_1 \) and \( v_1' \) be the starting and ending vertices of the path \( p \). Suppose one of the following two conditions holds:

(a) \( e \in p \) and there exist \( i, j \) such that \( \bar{e} := \{v_i, v_j'\} \in E(G) \), \( \bar{e} \preceq \tau e \), \( \bar{e} \neq \{v_1, v_1'\} \);

(b) \( e = \{v_i, v_{i+1}\} \) and there exists \( j \) such that at least one between \( v_i, v_j' \) and \( \{v_i+1, v_j\} \) is an edge of \( G \) (call it \( \bar{e} \)) such that \( \bar{e} \preceq \tau e \) and \( \bar{e} \neq \{v_1, v_1'\} \).

Then \( b_1 \notin \text{RGB}_\tau(I_G) \).

**Proof.** First of all, by Remark 2.1 the primitivity of the walk \( \Gamma \) allows us to paint the edges of \( \lceil |\Gamma| \rceil \) red and black so that no two edges consecutive in \( \Gamma \) are painted the same colour. We can assume without loss of generality that the edge \( e \) is black.
Theorem 3.5. Let $G$ be a graph with linear resolution and let $[e_1, \ldots, e_m]$ be an ordering of the edges of $G$ such that $[M_{e_1}, \ldots, M_{e_m}]$ is a linear quotient ordering for $I(G)$ (such an ordering exists by Remark 2.7). Let $\tau$ be the lexicographic order on $K[y_1, \ldots, y_m]$ such that $y_1 <_\tau y_2 <_\tau \ldots <_\tau y_m$. Then the reduced Gröbner basis of $I_G$ with respect to $\tau$ is doubly squarefree.

(a) Paint $\hat{e}$ red. We can suppose without loss of generality that $i \neq 1$: hence, exactly one of $\{v_{i-1}, v_i\}$ and $\{v_i, v_{i+1}\}$ is black. This means that exactly one of the two paths going from $v_i$ to $v_1$ along $C_1$ has its first edge painted black: let $w$ be this path. We now need to define a path $w'$ going from $v_i'$ to $v_j'$.

- If $j \neq 1$, exactly one of $\{v_{j-1}', v_j'\}$ and $\{v_j', v_{j+1}'\}$ is black. Applying the same reasoning as before, let $w'$ be the path going from $v_i'$ to $v_j'$ along $C_2$ having its last edge painted black.
- If $j = 1$ and the last edge of $p$ is red, let $w' = \{v_i' \rightarrow v_2' \rightarrow \ldots \rightarrow v_{2k+1}' \rightarrow v_1'\}$ (in other words, the whole cycle $C_2$); if the last edge of $p$ is black, let $w'$ be the empty path in $v_1'$.

Let $\Gamma' = \{v_j' \xrightarrow{e} v_i, w, v_1 \xrightarrow{p} v_1' \xrightarrow{w'} v_j'\}$. By construction, $\Gamma'$ is an even closed walk, since its edges are alternately red and black and the first and the last one have different colours. Moreover, it is easy to check that $\Gamma'$ is primitive either of type (ii) (when $j = 1$ and the last edge of $p$ is red) or of type (i) (in all other cases); hence, $\Gamma' \in \text{Gr}_G$. Finally, since $\tau$ is a lexicographic term order, to get who the leading monomial of $b_{\Gamma'}$ is we just have to identify the leading edge of $\Gamma'$: since $\hat{e} \prec \tau e$ by hypothesis and the rest of the edges of $\Gamma'$ are edges of $\Gamma$, we get that the leading monomial of $b_{\Gamma'}$ is the one formed by black edges. Since the black edges of $\Gamma' \forall i$ lie in $\Gamma$, we have that $in_\tau(b_{\Gamma'})$ divides $in_\tau(b_{\Gamma})$. Since $b_{\Gamma} \neq b_{\Gamma'}$, we have that $b_{\Gamma'} \notin \text{RGB}_\tau(I_G)$.

(b) Paint $\hat{e}$ red and define $w'$ in the same way as in part (a). Let $w$ be defined the following way:

- if $\hat{e} = \{v_i, v_j\}$, let $w := \{v_{i+1} \rightarrow v_{i+2} \rightarrow \ldots \rightarrow v_{2k+1} \rightarrow v_1\}$ (if $i = 2k + 1$, $w$ is the empty path);
- if $\hat{e} = \{v_{i+1}, v_j\}$, let $w := \{v_i \rightarrow v_{i-1} \rightarrow \ldots \rightarrow v_2 \rightarrow v_1\}$ (if $i = 1$, $w$ is the empty path).

Let

$$
\Gamma' := \begin{cases} 
\{v_{i+1} \xrightarrow{w} v_1 \xrightarrow{p} v_1' \xrightarrow{w'} v_j' \xrightarrow{\hat{e}} v_i \xrightarrow{\hat{e}} v_{i+1}\} & \text{if } \hat{e} = \{v_i, v_j\} \\
\{v_i \xrightarrow{w} v_1 \xrightarrow{p} v_1' \xrightarrow{w'} v_j' \xrightarrow{\hat{e}} v_{i+1} \rightarrow v_i\} & \text{if } \hat{e} = \{v_{i+1}, v_j\}
\end{cases}
$$

Reasoning the same way as in part (a), we get that $\Gamma'$ is an even closed walk; moreover, it can be easily checked that $\Gamma'$ is primitive either of type (ii) (when $v_1'$ belongs to $\hat{e}$ and the last edge of $p$ is red or when $v_1$ belongs to $\hat{e}$, with no restrictions on the colour of the last edge of $p$) or type (i) (in all other cases), hence $b_{\Gamma'} \in \text{Gr}_G$. For the same reasons as in part (a), we get that $b_{\Gamma'} \notin \text{RGB}_\tau(I_G)$.

\[\square\]
Proof. By Proposition 2.8, the linear quotient property is equivalent to asking that each subgraph \( \{e_1, \ldots, e_l\} \) is 1-step linear, that is to say gap-free by Remark 2.11. Let \( \Gamma \) be a primitive walk such that at least one of the two monomials of \( b_\Gamma \) is not squarefree. This implies that \( \Gamma \) is primitive of type (iii). Hence, by Lemma 3.1, we know that the leading edge \( e \) of \( \Gamma \) lies into a bow-tie \( \{C_1, p, C_2\} \). Let \( G_{\leq e} \) be the subgraph of \( G \) obtained by considering all the edges \( e' \) such that \( e' \preceq e \). This means that \( G_{\leq e} = \{e_1, e_2, \ldots, e_s = e\} \); hence, \( G_{\leq e} \) is gap-free. Using the notation of Lemma 3.4, we have to consider two different cases.

- If \( e \in \rho \), consider the edges \( \{v_2, v_3\} \) and \( \{v'_2, v'_3\} \). Since \( G_{\leq e} \) is gap-free, there exists an edge \( \tilde{e} \in E(G) \) which links the edges we are considering and is such that \( \tilde{e} \prec e \). By applying Lemma 3.4.(a), we get that \( b_\Gamma \notin \text{RGB}_\tau(I_G) \).

- If \( e = \{v_i, v_{i+1}\} \), consider the edge \( \{v'_2, v'_3\} \). Reasoning as before, we discover the existence of an edge \( \tilde{e} \in E(G) \) linking these two edges and having the property that \( \tilde{e} \prec e \); hence, by applying Lemma 3.4.(b), we get that \( b_\Gamma \notin \text{RGB}_\tau(I_G) \).

This ends the proof.

Remark 3.6. We actually proved that \( \text{RGB}_\tau(I_G) \) does not contain any binomials corresponding to primitive walks of type (iii). This means in particular that \( \text{RGB}_\tau(I_G) \) consists entirely of circuits (and hence \( I_G \) is generated by circuits).

Theorem 3.7. Let \( G \) be a gap-free graph and order its edges the following way:
\[
\{v_{i_1}, v_{i_2}\} \preceq \{v_{j_1}, v_{j_2}\} \text{ if and only if } v_{i_1} v_{i_2} \preceq_{\sigma} v_{j_1} v_{j_2}, \text{ where } \sigma \text{ is an arbitrary graded reverse lexicographic order on } K[v_1, \ldots, v_n].
\] Rename the edges so that \( e_1 \succ e_2 \succ \ldots \succ e_m \). Let \( \tau \) be the lexicographic order on \( K[y_1, \ldots, y_m] \) such that \( y_1 \succ_{\tau} y_2 \succ_{\tau} \ldots \succ_{\tau} y_m \). Then \( \text{in}_{\tau}(I_G) \) is generated by squarefree elements.

Proof. By Lemma 3.1 there exists a bow-tie \( \{C_1, p, C_2\} \) containing \( e \). We will use the notation of Lemma 3.4 to denote the edges of this bow-tie.

- If \( e \in C_1 \), then no edges of \( C_2 \) have vertices in common with \( e \). In the following we will say that a vertex \( v \in V(\Gamma) \) satisfies condition \( (\prec) \) if
\[
v \prec_{\sigma} u_1, u_2 \prec_{\sigma} u_2.
\] Note that, by definition of \( \sigma \) and \( \tau \), if an edge \( \{w_1, w_2\} \in E(\Gamma) \) shares no vertices with \( e \), then at least one of \( w_1 \) and \( w_2 \) must satisfy condition \( (\prec) \). Since no edges of \( C_2 \) share vertices with \( e \), any pair of consecutive vertices in \( C_2 \) must include a vertex satisfying condition \( (\prec) \): since \( C_2 \) is odd, by pigeonhole principle we get that there exists an edge \( e' \) of \( C_2 \) whose vertices both satisfy condition \( (\prec) \).

Since \( G \) is gap-free, there exists \( \tilde{e} \in E(G) \) linking \( e \) and \( e' \): moreover, since both vertices of \( e' \) satisfy condition \( (\prec) \), one has that \( \tilde{e} \prec_{e'} e \). If \( \tilde{e} \neq \{v_1, v'_1\} \) then, by Lemma 3.4.(b), we get that \( b_\Gamma \notin \text{RGB}_\tau(I_G) \). If \( \tilde{e} = \{v_1, v'_1\} \), then \( v_1 \in e \) and we have to consider two different cases.
○ If $p$ is made of an even number of edges, then $\Gamma' := \{C_1, p, -\bar{e}\}$ is a primitive walk of type (ii) such that $in_\tau(b_{\Gamma'})$ divides $in_\tau(b_\Gamma)$. Hence $b_\Gamma \notin \text{RGB}_\tau(I_G)$.

○ If $p$ is made of an odd number of edges, then consider $\Gamma' := \{C_1, \bar{e}, C_2, -\bar{e}\}$. By Proposition 2.3, $b_{\Gamma'}$ is a circuit and hence $\Gamma'$ is a primitive walk. Since $in_\tau(b_{\Gamma'})$ is squarefree and divides $in_\tau(b_\Gamma)$, we get that $b_\Gamma \notin \text{RGB}_\tau(I_G)$.

• If $e \in p$, we have to discuss two different situations.

If $p$ is made of more than one edge, then at least one of the cycles $C_1$ and $C_2$ has no vertices in common with $e$ (let it be $C_1$ without loss of generality). Then, applying the same pigeonhole reasoning used in the previous case, we discover the existence of an edge $e'$ of $C_1$ whose vertices both satisfy condition $(<)$. Since $G$ is gap-free, there exists $\bar{e}$ linking $e'$ and $\{v_2', v_3'\}$. Since $\bar{e} \prec_\tau e$ by construction, applying Lemma 3.4.(a) we get that $b_\Gamma \notin \text{RGB}_\tau(I_G)$.

The last case standing is the one where $p = \{\{v_1, v_1'\}\} = \{e\}$. Let $\hat{C}_1 := \{v_2, v_3, \ldots, v_{2k+1}\}$, $\hat{C}_2 := \{v_2', v_3', \ldots, v_{2k+1}'\}$. If there exist two consecutive vertices belonging to either $\hat{C}_1$ or $\hat{C}_2$ and satisfying condition $(<)$, then we can apply Lemma 3.4.(a) to infer that $b_\Gamma \notin \text{RGB}_\tau(I_G)$. Suppose otherwise. Then condition $(<)$ is satisfied alternately: to be more precise, we have that the vertices of $\hat{C}_1$ (or $\hat{C}_2$) satisfying condition $(<)$ are either the ones with odd index or the ones with even index. We can suppose without loss of generality that $v_3, v_5, \ldots, v_{2k+1}, v_3', v_5', \ldots, v_{2k+1}'$ are the vertices in $\hat{C}_1 \cup \hat{C}_2$ satisfying condition $(<)$. Consider the edges $\{v_2, v_3\}$ and $\{v_2', v_3'\}$. Since $G$ is gap-free, these edges are surely linked by some edge $\bar{e}$: if one of $v_3$ and $v_3'$ belongs to $\bar{e}$ we have that $\bar{e} \prec_\tau e$ and hence, by Lemma 3.4.(a), we can conclude that $b_\Gamma \notin \text{RGB}_\tau(I_G)$. What happens if $\bar{e} = \{v_2, v_3\}$? If $\bar{e} \prec_\tau e$ we are done for the same reason as before. Suppose $\bar{e} \succ_\tau e$. Then, by definition of $\tau$, at least one of $v_1$ and $v_1'$ (call it $w$) must be such that $w \prec_\sigma v_2$ and $w \prec_\sigma v_2'$. Since $e$ is the leading edge of $|\Gamma|$, though, one has that $e \succ_\tau \{v_1, v_2\}$ and $e \succ_\tau \{v_1', v_2'\}$, hence $v_1 \succ_\sigma v_2$ and $v_1 \succ_\sigma v_2'$. This gives us a contradiction. \qed

**Remark 3.8.** The proof of Theorem 3.7 shows also that $\text{RGB}_\tau(I_G)$ consists of circuits and hence $I_G$ is generated by circuits. To see this, replace the hypothesis “$\Gamma$ primitive walk such that $in_\tau(b_\Gamma)$ is not squarefree” with “$\Gamma$ primitive walk of type (iii)” and note that the only primitive walks of type (iii) that may appear in $\text{RGB}_\tau(I_G)$ are bow-ties (more precisely, just those with a connecting path of length one). Since binomials associated with bow-ties are circuits by Proposition 2.3, we are done. In general, the construction appearing in Theorem 3.7 does not yield a doubly squarefree reduced Gröbner basis of $I_G$. 

REFERENCES

[CoC] CoCoATeam. CoCoA: a system for doing Computations in Commutative Algebra. URL: http://cocoa.dima.unige.it.

[DHS13] Hailong Dao, Craig Huneke, and Jay Schweig. “Bounds on the regularity and projective dimension of ideals associated to graphs”. In: Journal of Algebraic Combinatorics 38 (2013), pp. 37–55.

[EGHP05] David Eisenbud, Mark Green, Klaus Hulek, and Sorin Popescu. “Restricting linear syzygies: algebra and geometry”. In: Compos. Math. 141 (2005), pp. 1460–1478.

[Frö90] Ralf Fröberg. “On Stanley-Reisner rings”. In: Topics in Algebra, Banach Center Publ. 26 (2) (1990), pp. 55–70.

[HH10] Jürgen Herzog and Takayuki Hibi. Monomial Ideals. Graduate Texts in Mathematics 260. Springer-Verlag, 2010.

[HHZ04] Jürgen Herzog, Takayuki Hibi, and Xinxian Zheng. “Monomial ideals whose powers have a linear resolution”. In: Math. Scand. 95 (2004), pp. 23–32.

[Hoc72] Melvin Hochster. “Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes”. In: Annals of Mathematics 96 (1972), pp. 318–337.

[HPPS14] Christian Haase, Andreas Paffenholz, Lindsay C. Piechnik, and Francisco Santos. Existence of unimodular triangulations - positive results. Preprint. 2014. URL: http://arxiv.org/pdf/1405.1687v2.pdf.

[HW11] Andrew H. Hoefel and Gwyneth Whieldon. Linear quotients of the square of the edge ideal of the anticycle. Preprint. 2011. URL: http://arxiv.org/pdf/1106.2348v2.pdf.

[MS04] Ezra Miller and Bernd Sturmfels. Combinatorial Commutative Algebra. Graduate Texts in Mathematics 227. Springer-Verlag, 2004.

[OH13] Hidefumi Ohsugi and Takayuki Hibi. “Toric ideals and their circuits”. In: Journal of Commutative Algebra 5 (2) (2013), pp. 309–322.

[OH98] Hidefumi Ohsugi and Takayuki Hibi. “Normal polytopes arising from finite graphs”. In: Journal of Algebra 207 (1998), pp. 409–426.

[OH99a] Hidefumi Ohsugi and Takayuki Hibi. “A normal (0,1)-polytope none of whose regular triangulations is unimodular”. In: Discrete and Computational Geometry 21 (1999), pp. 201–204.

[OH99b] Hidefumi Ohsugi and Takayuki Hibi. “Toric ideals generated by quadratic binomials”. In: Journal of Algebra 218 (1999), pp. 509–527.

[RTT12] Enrique Reyes, Christos Tatakis, and Apostolos Thoma. “Minimal generators of toric ideals of graphs”. In: Advances in Applied Mathematics 48 (2012), pp. 64–78.
REFERENCES

[Sta96] Richard P. Stanley. *Combinatorics and Commutative Algebra*. Progress in Mathematics 41. Second edition. Birkhäuser, 1996.

[Stu96] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*. University Lecture Series 8. American Mathematical Society, 1996.

[SVV98] Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal. “The integral closure of subrings associated to graphs”. In: *Journal of Algebra* 199 (1998), pp. 281–289.

[TT11] Christos Tatakis and Apostolos Thoma. “On the universal Gröbner basis of toric ideals of graphs”. In: *Journal of Combinatorial Theory. Series A* 118 (2011), pp. 1540–1548.

[Vil01] Rafael H. Villarreal. *Monomial Algebras*. Monographs and Textbooks in Pure and Applied Mathematics 238. Marcel Dekker, Inc., 2001.

[Vil95] Rafael H. Villarreal. “Rees algebras of edge ideals”. In: *Communications in Algebra* 23 (1995), pp. 3513–3524.

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