Singular fibers in barking families of degenerations of elliptic curves

Takayuki Okuda

Abstract.

Takamura [Ta3] established a theory of splitting families of degenerations of complex curves of genus \( g \geq 1 \). He introduced a powerful method for constructing a splitting family, called a barking family, in which the resulting family of complex curves has a singular fiber over the origin (the main fiber) together with other singular fibers (subordinate fibers). He made a list of barking families for genera up to 5 and determined the main fibers appearing in them. This paper determines most of the subordinate fibers of the barking families in Takamura’s list for the case \( g = 1 \). (There remain four undetermined cases.) Also, we show that some splittings never occur in a splitting family.

§1. Introduction

Let \( \pi : M \to \Delta \) be a proper surjective holomorphic map from a smooth complex surface \( M \) to an open disk \( \Delta := \{ s \in \mathbb{C} : |s| < \delta \} \) in \( \mathbb{C} \) with radius \( \delta > 0 \). We call \( \pi : M \to \Delta \) a family of complex curves of genus \( g \geq 1 \) over \( \Delta \) if \( \pi \) has at most finitely many singular fibers and the other fibers are smooth complex curves of genus \( g \). In particular, \( \pi : M \to \Delta \) is called a degeneration of complex curves of genus \( g \) if the fiber \( X_0 := \pi^{-1}(0) \) over the origin is singular and the other fibers \( X_s := \pi^{-1}(s) (s \neq 0) \) are all smooth.

In this paper, we consider the following problem: How does a singular fiber split in a deformation? Let us recall the concept of a splitting family of degenerations. Let \( \mathcal{M} \) be a smooth complex 3-manifold and set \( \Delta^\uparrow := \{ t \in \mathbb{C} : |t| < \varepsilon \} \), an open disk with sufficiently small radius \( \varepsilon > 0 \).
Consider a proper flat surjective holomorphic map $\Psi : \mathcal{M} \to \Delta \times \Delta^!$. For $t \in \Delta^!$, set $\Delta_t := \Delta \times \{t\}$, $M_t := \Psi^{-1}(\Delta_t)$ and $\pi_t := \Psi\big|_{M_t} : M_t \to \Delta_t$. Suppose that $\pi_0 : M_0 \to \Delta_0$ coincides with a given degeneration $\pi : M \to \Delta$. Then we call $\Psi : \mathcal{M} \to \Delta \times \Delta^!$ a deformation family of the degeneration $\pi : M \to \Delta$ and each $\pi_t : M_t \to \Delta_t$ ($t \in \Delta^! \setminus \{0\}$) a deformation of the degeneration $\pi : M \to \Delta$. In particular, $\Psi : \mathcal{M} \to \Delta \times \Delta^!$ is called a splitting family if every deformation $\pi_t : M_t \to \Delta_t$ of the degeneration $\pi : M \to \Delta$ is a family of complex curves with at least two singular fibers. Set $X_{s,t} := \Psi^{-1}(s,t) (= \pi_t^{-1}(s))$. Clearly $X_{0,0}$ is the original singular fiber $X_0$ of the degeneration $\pi : M \to \Delta$. For a fixed $t \in \Delta^! \setminus \{0\}$, let $s_1, s_2, \ldots, s_N$ ($N \geq 2$) be the singular values of $\pi_t$, that is, $X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t}$ are the singular fibers of $\pi_t : M_t \to \Delta_t$. (Note: The singular values $s_1, s_2, \ldots, s_N$ depend on $t$, but the number of them and the types of the singular fibers do not.) In this case, we say that the singular fiber $X_0$ splits into the singular fibers $X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t}$.

To classify atomic degenerations — degenerations admitting no splitting family — Takamura [Ta3] introduced a powerful method for constructing splitting families. Splitting families obtained by this construction are called barking families. In a barking family, the original singular fiber $X_0$ of the degeneration $\pi : M \to \Delta$ is deformed to a simpler singular fiber of its deformation $\pi_t : M_t \to \Delta_t$ in such a way that a part of $X_0$ looks "barked" off from $X_0$. See Fig. 2 in Section 2. The resulting singular fiber appears over the origin of $\Delta_t$ under Takamura's construction, so we denote it by $X_{0,t}$. In such a situation, we write\textsuperscript{1}

\[
X_0 \overset{\text{bark}}{\longrightarrow} X_{0,t},
\]

and call $X_{0,t}$ the main fiber.

In [Ta3], for genera up to 5, Takamura made a list of barking families which enabled him to show that a degeneration is absolutely atomic — that is, any degeneration topologically equivalent to it is atomic — if and only if its singular fiber is either a Lefschetz fiber or a multiple of a smooth complex curve. For instance, he listed thirty five barking families for degenerations of complex curves of genus $g = 1$, that is, for degenerations of elliptic curves, and determined the type of the main fiber of each of them as follows, where we use Kodaira's notation\textsuperscript{2} for

\textsuperscript{1}In the same situation, Takamura [Ta3] wrote $X_0 \to X_{0,t}$. In this paper, we use "\longrightarrow" only for splittings and distinguish "\overset{\text{bark}}{\longrightarrow}" from it.

\textsuperscript{2}See Table 1 in Section 2.
singular fibers (see also the list in Section 12):

(1.1) **Takamura’s list**

\[
\begin{align*}
[II.1] & \quad II \xrightarrow{\text{bark}} I_1 \\
[II.2] & \quad II \xrightarrow{\text{bark}} I_1 \\
[II*.1] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[II*.2] & \quad II^* \xrightarrow{\text{bark}} IV^* \\
[II*.3] & \quad II^* \xrightarrow{\text{bark}} I_2^* \\
[II*.4] & \quad II^* \xrightarrow{\text{bark}} I_5 \\
[II*.5] & \quad II^* \xrightarrow{\text{bark}} I_3^* \\
[II*.6] & \quad II^* \xrightarrow{\text{bark}} I_3^* \\
[II*.7] & \quad II^* \xrightarrow{\text{bark}} I_8 \\
[II*.8] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[II*.9] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[III.1] & \quad III \xrightarrow{\text{bark}} I_2 \\
[III.2] & \quad III \xrightarrow{\text{bark}} I_1 \\
[III.3] & \quad III \xrightarrow{\text{bark}} I_2 \\
[III*.1] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[III*.2] & \quad III^* \xrightarrow{\text{bark}} I_1^* \\
[III*.3] & \quad III^* \xrightarrow{\text{bark}} I_2^* \\
[III*.4] & \quad III^* \xrightarrow{\text{bark}} I_0^* \\
[III*.5] & \quad III^* \xrightarrow{\text{bark}} I_6 \\
[III*.6] & \quad III^* \xrightarrow{\text{bark}} I_2^* \\
[III*.7] & \quad III^* \xrightarrow{\text{bark}} I_7 \\
[III*.8] & \quad III^* \xrightarrow{\text{bark}} I_6 \\
[III*.9] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[IV.1] & \quad IV \xrightarrow{\text{bark}} I_3 \\
[IV.2] & \quad IV \xrightarrow{\text{bark}} I_2 \\
[IV.3] & \quad IV \xrightarrow{\text{bark}} I_2 \\
[IV.4] & \quad IV \xrightarrow{\text{bark}} II \\
[IV*.1] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[IV*.2] & \quad IV^* \xrightarrow{\text{bark}} I_0^* \\
[IV*.3] & \quad IV^* \xrightarrow{\text{bark}} I_6 \\
[IV*.4] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[I_0*.1] & \quad I_0^* \xrightarrow{\text{bark}} I_4 \\
[I_0*.2] & \quad I_0^* \xrightarrow{\text{bark}} I_3 \\
[I_n*.1] & \quad I_n^* \xrightarrow{\text{bark}} I_{n-1}^* \\
[I_n*.2] & \quad I_n^* \xrightarrow{\text{bark}} I_{n+4} \\
[III^*] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
\end{align*}
\]

In a barking family, there appear not only the main fiber but also other singular fibers, which are called _subordinate fibers_. In what follows, when the original singular fiber $X_0$ splits into the main fiber $X_{0,t}$ and subordinate fibers $X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t}$ ($s_i \neq 0$), we write

\[
X_0 \rightarrow X_{0,t} + X_{s_1,t} + X_{s_2,t} + \cdots + X_{s_N,t}
\]

—we always put the main fiber $X_{0,t}$ on the initial term to distinguish it from the subordinate fibers. The main fiber of a barking family is explicitly described. On the other hand, it is not clear what subordinate
fibers will appear. The aim of this paper is to determine the subordinate fibers of Takamura’s barking families for degenerations of elliptic curves.

Our results are summarized in two theorems. Firstly, the following theorem determines the subordinate fibers of most of the barking families in the above list (note: four cases remain undetermined, see Remark 1.1 below):

**Main Theorem A** (Theorem 10.10). Each barking family in Takamura’s list (1.1) except \([III*\, .8], [IV.3], [IV.4], [I*\, .2]\) splits the singular fiber as follows:

\[
\begin{align*}
[II.1] & \ II \rightarrow I_1 + I_1 & [III*\, .2] & \ III^* \rightarrow I_1 + I_2 \\
[II.2] & \ II \rightarrow I_1 + I_1 & [III*\, .3] & \ III^* \rightarrow I_2^* + I_1 \\
[II*\, .1] & \ II^* \rightarrow III^* + I_1 & [III*\, .4] & \ III^* \rightarrow I_0^* + I_1 + I_1 + I_1 \\
[II*\, .2] & \ II^* \rightarrow IV^* + II & [III*\, .5] & \ III^* \rightarrow I_6 + I_1 + I_1 + I_1 \\
[II*\, .3] & \ II^* \rightarrow I_2^* + I_1 + I_1 & [III*\, .6] & \ III^* \rightarrow I_2^* + I_1 \\
[II*\, .4] & \ II^* \rightarrow I_5^* & [III*\, .7] & \ III^* \rightarrow I_7 + I_1 + I_1 \\
& \rightarrow I_1 + I_1 + I_1 + I_1 + I_1 & [III*\, .9] & \ III^* \rightarrow IV^* + I_1 \\
[II*\, .5] & \ II^* \rightarrow I_3^* + I_1 & [IV.1] & \ IV \rightarrow I_3 + I_1 \\
[II*\, .6] & \ II^* \rightarrow I_3^* + I_1 & [IV.2] & \ IV \rightarrow I_2 + I_1 + I_1 \\
[II*\, .7] & \ II^* \rightarrow I_8 + I_1 + I_1 & [IV*\, .1] & \ IV^* \rightarrow I_1 + I_1 \\
[II*\, .8] & \ II^* \rightarrow III^* + I_1 & [IV*\, .2] & \ IV^* \rightarrow I_0^* + I_1 + I_1 \\
[II*\, .9] & \ II^* \rightarrow III^* + I_1 & [IV*\, .3] & \ IV^* \rightarrow I_6 + I_1 + I_1 \\
[III.1] & \ III \rightarrow I_2 + I_1 & [IV*\, .4] & \ IV^* \rightarrow I_7 + I_1 \\
[III.2] & \ III \rightarrow I_1 + I_2 & [I*\, .1] & \ I_0^* \rightarrow I_4 + I_1 + I_1 \\
[III.3] & \ III \rightarrow I_2 + I_1 & [I^*\, .1] & \ I_n^* \rightarrow I_{n-1}^* + I_1 \\
[III*\, .1] & \ III^* \rightarrow IV^* + I_1 & [I^*\, .2] & \ I_n^* \rightarrow I_{n+4} + I_1 + I_1.
\end{align*}
\]

**Remark 1.1.** We have not been able to determine the subordinate fibers of the four exceptional barking families \([III*\, .8], [IV.3], [IV.4], [I*\, .2]\) (see also Remark 6.6):

\[
\begin{align*}
[III*\, .8] & \ III^* \rightarrow I_6 + II + I_1, I_6 + I_2 + I_1, \text{ or } I_6 + I_1 + I_1 + I_1 \\
[IV.3] & \ IV \rightarrow I_2 + II, \text{ or } I_2 + I_1 + I_1 \\
[IV.4] & \ IV \rightarrow II + II, II + I_2, \text{ or } II + I_1 + I_1 \\
[I*\, .2] & \ I_0^* \rightarrow I_3 + II + I_1, \text{ or } I_3 + I_1 + I_1 + I_1.
\end{align*}
\]
In contrast, there are splittings that never occur in a splitting family. If in a splitting family for a degeneration of elliptic curves the singular fiber $X_0$ splits into $N$ singular fibers $X_1, X_2, \ldots, X_N$, then we have $e(X_0) = e(X_1) + e(X_2) + \cdots + e(X_N)$, where $e(X_i)$ denotes the topological Euler characteristic of the underlying reduced curve of $X_i$ (Lemma 3.1 (b)). However the converse does not hold. Even if the singular fibers satisfy this equation, the splitting $X_0 \to X_1 + X_2 + \cdots + X_N$ does not always occur. In fact:

**Main Theorem B** (Theorem 5.8). *None of the following splittings occurs:*

- $IV \to I_2 + I_2$,
- $II^* \to I_8 + II$,  $I_7 + III$,  $I_6 + IV$,
- $I_4 + I_0^*$,  $I_3 + I_1^*$,
- $I_u + I_v (u + v = 10)$,
- $III^* \to I_7 + II$,  $I_6 + III$,  $I_5 + IV$,
- $I_3 + I_0^*$,  $I_u + I_v (u + v = 9)$,
- $IV^* \to I_6 + II$,  $I_5 + III$,  $I_4 + IV$,
- $I_2 + I_0^*$,  $I_u + I_v (u + v = 8)$,
- $I_n^* (n \geq 0) \to I_{n+4} + II$,  $I_{n+3} + III$,  $I_{n+2} + IV$,
- $I_u + I_v (u + v = n + 6$ and $(n, u, v) \neq (2, 4, 4))$.
- $I_0^* \to I_3 + I_2 + I_1$.

**Organization of this paper.**

This paper is organized as follows. In Section 2, we first review Takamura’s theory of barking families, mainly for degenerations with stellar (star-shaped) singular fibers. In fact, most of the degenerations of elliptic curves may be assumed to have stellar singular fibers.

To determine the subordinate fibers of the barking families in Takamura’s list (1.1), we investigate the singular fibers in three steps: (1) In Section 3, we first consider the Euler characteristics of the singular fibers and give a list of the sets of subordinate fibers that can appear in each of the barking families. (2) In Section 4, we recall the concept of monodromies around singular fibers, and in Section 5, by comparing the traces of monodromies, we prove Main Theorem B — we give a list of splittings that never occur. In Section 6, based on the result of Section 5, we determine the subordinate fibers of five of Takamura’s barking families. (3) Sections 7, 8, 9 are devoted to study of the singularities of subordinate fibers. We investigate the singularities near proportional
subbranches in Section 7 and those near the core in Section 8. In Section 9, we show useful lemmas which give us the number of the subordinate fibers and that of their singularities. In Section 10, we determine the subordinate fibers of the remaining barking families, and complete the proof of Main Theorem A.

In Section 11, we give monodromy decompositions corresponding to the splittings induced from Takamura’s barking families.

In Section 12, we provide Takamura’s list of barking families for genus 1 with figures of the singular fibers, which will help the reader comprehend the barking deformations.

§2. Takamura’s theory

Let us review Takamura’s theory of barking families. For details see [Ta3].

First we recall the concept of linear degenerations. We begin with preparation. Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus $g \geq 1$ and express its singular fiber as $X_0 = \sum_i m_i \Theta_i$, where $\Theta_i$ is an irreducible component of $X_0$ with multiplicity $m_i$. In what follows, we assume that the underlying reduced curve $X_0^{\text{red}} := \sum_i \Theta_i$ of $X_0$ has at most simple normal crossings, that is, (i) any singularity of $X_0^{\text{red}}$ is a node and (ii) any irreducible component $\Theta_i$ is not self-intersecting (so $\Theta_i$ is smooth).

For an irreducible component $\Theta_i$ of $X_0$, we denote by $N_i$ the normal bundle of $\Theta_i$ in $M$. Let $\{ p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(h)} \}$ be the set of the intersection points on $\Theta_i$ with other irreducible components of $X_0$ and $m_i^{(j)} (j = 1, 2, \ldots, h)$ be the multiplicity of the irreducible component intersecting $\Theta_i$ at $p_i^{(j)}$. Then there exists a holomorphic section $\sigma_i$ of the line bundle $N_i^{\otimes (-m_i)}$ on $\Theta_i$ such that

$$\text{div}(\sigma_i) = \sum_{j=1}^{h} m_i^{(j)} p_i^{(j)},$$

where $\text{div}(\sigma_i)$ denotes the divisor defined by $\sigma_i$. Here $\sigma_i$ has a zero of order $m_i^{(j)}$ at $p_i^{(j)}$. Note that $\sigma_i$ is uniquely determined up to multiplication by a constant. We call $\sigma_i$ the standard section of $N_i^{\otimes (-m_i)}$ on $\Theta_i$.

Take an open covering $\Theta_i = \bigcup_{\alpha} U_{\alpha}$ such that $U_{\alpha} \times \mathbb{C}$ is a local trivialization of the normal bundle $N_i$ on $\Theta_i$. We denote by $(z_{\alpha}, \zeta_{\alpha})$ coordinates of $U_{\alpha} \times \mathbb{C}$. Now define holomorphic functions $\pi_{i,\alpha} : U_{\alpha} \times \mathbb{C} \to$
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\[ \pi_{i,\alpha}(z_\alpha, \zeta_\alpha) := \sigma_{i,\alpha}(z_\alpha)z_\alpha^{m_i}, \]
where \( \sigma_{i,\alpha} \) is the local expression of \( \sigma_i \) on \( U_\alpha \). Then the set \( \{ \pi_{i,\alpha} \}_\alpha \) of holomorphic functions defines a global holomorphic function \( \pi_i : N_i \to \mathbb{C} \).

**Definition 2.1.** A degeneration \( \pi : M \to \Delta \) is said to be *linear* if for any irreducible component \( \Theta_i \) of its singular fiber \( X_0 \),

(i): a tubular neighborhood \( N(\Theta_i) \) of \( \Theta_i \) in \( M \) is biholomorphic to a tubular neighborhood of a zero-section of the normal bundle \( N_i \), and

(ii): under the identification by the biholomorphic map of (i), the following conditions are satisfied:

- The restriction \( \pi|_{N(\Theta_i)} \) coincides with the holomorphic function \( \pi_i \) defined above.
- If \( \Theta_i \) intersects \( \Theta_j \) at a point \( p \), then there exist local trivializations \( U_\alpha \times \mathbb{C} \) of \( N_i \) and \( U_\beta \times \mathbb{C} \) of \( N_j \) around \( p \) such that neighborhoods of \( p \) in \( N(\Theta_i) \) and \( N(\Theta_j) \) are identified by plumbing \( (z_\alpha, \zeta_\alpha) = (\zeta_\beta, z_\beta) \) and \( \pi \) is locally expressed as

\[ \pi|_{N(\Theta_i)}(z_\alpha, \zeta_\alpha) = z_\alpha^{m_j} \zeta_\alpha^{m_i}, \quad \pi|_{N(\Theta_j)}(z_\beta, \zeta_\beta) = z_\beta^{m_i} \zeta_\beta^{m_j}, \]
where \( (z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C} \) and \( (z_\beta, \zeta_\beta) \in U_\beta \times \mathbb{C} \).

**Remark 2.2.** Any degeneration of complex curves (even if the underlying reduced curve of its singular fiber does not have at most simple normal crossings), after successive blowing up and down, becomes a degeneration topologically equivalent to some linear degeneration.

If \( \pi : M \to \Delta \) is linear, then we may express \( M \) locally as a hypersurface in some space as follows: We first identify \( M \) with the graph of \( \pi \) in \( M \times \Delta \)

\[ \text{Graph}(\pi) = \{(x, s) \in M \times \Delta : \pi(x) - s = 0\} \]
via the natural projection \( \text{Graph}(\pi) \ni (x, s) \mapsto x \in M \). Recall that for any irreducible component \( \Theta_i \) of the singular fiber \( X_0 \), the map \( \pi \) is expressed around \( \Theta_i \) as

\[ \pi(z_i, \zeta_i) = \sigma_i(z_i)\zeta_i^{m_i}, \]
where \( \sigma_i \) is the standard section of \( N_i^{\otimes (-m_i)} \) on \( \Theta_i \). Then we obtain the local expression of \( M \) around \( \Theta_i \):

\[ \sigma_i(z_i)\zeta_i^{m_i} - s = 0 \quad \text{in} \ N_i \times \Delta. \]
Fig. 1. A singular fiber of type $II^*$ of a degeneration of elliptic curves is stellar. Each circle denotes a complex projective line, the number stands for its multiplicity, and each intersection point is a node.

Note that these hypersurfaces are glued around the intersection points by plumbings $(z_j, \zeta_j, s) = (\zeta_i, z_i, s)$ where $(z_i, \zeta_i, s) \in N_i \times \Delta$ and $(z_j, \zeta_j, s) \in N_j \times \Delta$.

For a linear degeneration $\pi : M \to \Delta$, its singular fiber $X_0$ consists of three kinds of parts: cores, branches and trunks. An irreducible component $\Theta_i$ of $X_0$ is called a core if $\Theta_i$ intersects other irreducible components at at least three points or the genus of $\Theta_i$ is positive. A branch is a chain $\sum m_i \Theta_i$ of complex projective lines attached with a core on one hand, while a trunk is a chain $\sum m_i \Theta_i$ of complex projective lines connecting other irreducible components on both hands. We say that $X_0$ is a stellar singular fiber if $X_0$ consists of one core and branches emanating from the core. See Fig. 1. Otherwise $X_0$ is said to be constellar. If $X_0$ is normally minimal, that is, (i) any singularity of $X_0^{\text{red}}$ is a node and (ii) any irreducible component that is a $(-1)$-curve (an exceptional curve of the first kind) intersects other irreducible component at at least three points, then all the branches and trunks of $X_0$ contain no $(-1)$-curves.

A degeneration whose singular fiber is a (fringed) branch can be constructed explicitly and associated to a sequence of nonnegative integers (the multiplicity sequence):
Lemma 2.3. Let $m_0, m_1, \ldots, m_{\lambda+1} \ (\lambda \geq 1)$ be nonnegative integers\(^3\) satisfying the following conditions:

$$
\begin{cases}
  m_0 > m_1 > \cdots > m_\lambda > m_{\lambda+1} = 0 \\
  r_i := \frac{m_i-1+m_{i+1}}{m_i} \ (i = 1, 2, \ldots, \lambda) \text{ is an integer greater than 1.}
\end{cases}
$$

Then there exists a degeneration $\pi : M \to \Delta$ with the singular fiber

$$X_0 = m_0 \Delta_0 + m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_\lambda \Theta_\lambda,$$

where $\Delta_0 = \mathbb{C}$, and $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ are complex projective lines, and each pair of $\Theta_i$ and $\Theta_{i+1} \ (i = 1, 2, \ldots, \lambda - 1)$ and $\Delta_0$ and $\Theta_1$ intersect transversely at one point.

Proof. We take $\lambda$ copies $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ of the complex projective line. Let $\Theta_i = U_i \cup V_i$ be an open covering by two complex lines $U_i, V_i (= \mathbb{C})$ with coordinates $w_i \in U_i \setminus \{0\}$ and $z_i \in V_i \setminus \{0\}$ satisfying $z_i = 1/w_i$. Then we obtain a line bundle $N_i$ on $\Theta_i$ of degree $-r_i$ from $U_i \times \mathbb{C}$ and $V_i \times \mathbb{C}$ by identifying $(z_i, \zeta_i) \in (V_i \setminus \{0\}) \times \mathbb{C}$ with $(w_i, \eta_i) \in (U_i \setminus \{0\}) \times \mathbb{C}$ via

$$g_i : z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i} \eta_i.$$

Now consider the hypersurface $W_i$ in $N_i \times \Delta$ defined by

$$
\begin{align*}
H_i & : w_i^{m_i-1} \eta_i^{m_i} \cdot s = 0, \quad \text{in } U_i \times \mathbb{C} \times \Delta, \\
H'_i & : z_i^{m_i+1} \zeta_i^{m_i} \cdot s = 0, \quad \text{in } V_i \times \mathbb{C} \times \Delta.
\end{align*}
$$

Under plumbings $(w_{i+1}, \eta_{i+1}, s) = (\zeta_i, z_i, s)$ of $N_i \times \Delta$ and $N_{i+1} \times \Delta \ (i = 1, 2, \ldots, \lambda - 1)$, the hypersurfaces $W_1, W_2, \ldots, W_\lambda$ are glued, so that they together define a smooth complex surface $M$. Letting $\pi : M \to \Delta$ be the natural projection, the central fiber is

$$\pi^{-1}(0) = m_0 \Delta_0 + m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_\lambda \Theta_\lambda,$$

where $\Delta_0 := \{0\} \times \mathbb{C} \subset U_1 \times \mathbb{C}$. Thus the holomorphic map $\pi : M \to \Delta$ is the desired degeneration. Q.E.D.

Remark 2.4. Precisely speaking, the holomorphic function $\pi : M \to \Delta$ obtained in Lemma 2.3 does not satisfy the condition to be a degeneration. Indeed $\pi$ is not proper. Note that we consider the restriction of a degeneration to a tubular neighborhood of a branch.

\(^3\)In this paper, by convention, we append $m_{\lambda+1} = 0$ to the sequence $m_0, m_1, \ldots, m_\lambda$ of positive integers, so that $r_\lambda := (m_{\lambda-1} + m_{\lambda+1})/m_\lambda$ equals $m_{\lambda-1}/m_\lambda$. See [Ta3] Section 5.1.
Next we define a special subdivisor of a stellar singular fiber. Let \( \pi : M \to \Delta \) be a linear degeneration of complex curves with the stellar singular fiber \( X_0 = m_0\Theta_0 + \sum_{j=1}^{h} \mathcal{B}r^{(j)} \), where \( \Theta_0 \) is the core and \( \mathcal{B}r^{(j)} \) \((j = 1, 2, \ldots, h)\) is a branch. Write \( \mathcal{B}r^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda(j)}^{(j)}\Theta_{\lambda(j)}^{(j)} \) and let \( \mathcal{B}r^{(j)} = m_0\Delta_0^{(j)} + m_1^{(j)}\Theta_1^{(j)} + \cdots + m_{\lambda(j)}^{(j)}\Theta_{\lambda(j)}^{(j)} \) be a fringed branch. Consider a connected subdivisor \( Y = n_0\Theta_0 + \sum_{j=1}^{h} \mathcal{B}r^{(j)} \) of \( X_0 \), where \( \mathcal{B}r^{(j)} := n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{\lambda(j)}^{(j)}\Theta_{\lambda(j)}^{(j)} \) \((j = 1, 2, \ldots, h)\). Here \( Y \) satisfies \( 0 \leq \nu^{(j)} \leq \lambda^{(j)} \) and \( 0 < n_i^{(j)} \leq m_i^{(j)} \) for each \( i \) and \( j \). Set \( \mathcal{B}r^{(j)} := n_0\Delta_0^{(j)} + n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{\lambda(j)}^{(j)}\Theta_{\lambda(j)}^{(j)} \). For the time being, we consider \( \overline{\mathcal{B}r}^{(j)} \) and \( \overline{\mathcal{B}r}^{(j)} \), omitting the superscript \( (j) \) to simplify notation. We call \( \mathcal{B}r \) a subbranch of \( \overline{\mathcal{B}r} \) if one of the following conditions is satisfied:

- \( \nu = 0, 1 \), or
- \( \nu \geq 2 \) and \( n_{i+1} = r_in_i - n_{i-1} \) \((i = 1, 2, \ldots, \nu - 1)\),

where \( r_i := (m_{i-1} + m_{i+1})/m_i \) (see Lemma 2.3). Set \( n_{\nu+1} := r_{\nu}n_{\nu} - n_{\nu-1} \). If \( \nu = 0 \), then we set \( n_{\nu+1} = n_1 := 0 \). Define the three types of subbranches for a positive integer \( l \) as follows:

**Type A**\(_l\): A subbranch \( \mathcal{B}r \) of \( \overline{\mathcal{B}r} \) is of type \( A \_l \) if \( l_n_i \leq m_i \) for each \( i \) and \( n_{\nu+1} \leq 0 \).

**Type B**\(_l\): A subbranch \( \mathcal{B}r \) of \( \overline{\mathcal{B}r} \) is of type \( B \_l \) if \( l_n_i \leq m_i \) for each \( i \), \( n_1 = 1 \) and \( m_{\nu} = l \).

**Type C**\(_l\): A subbranch \( \mathcal{B}r \) of \( \overline{\mathcal{B}r} \) is of type \( C \_l \) if \( l_n_i \leq m_i \) for each \( i \), \( n_\nu = n_{\nu+1} \) and \( m_{\nu} - m_{\nu+1} \) divides \( l \).

Now we return to a connected subdivisor \( Y \) of the stellar singular fiber \( X_0 \).

**Definition 2.5.** Let \( Y = n_0\Theta_0 + \sum_{j=1}^{h} \mathcal{B}r^{(j)} \) be a connected subdivisor of \( X_0 \) such that \( n_0 < m_0 \) and each \( \mathcal{B}r^{(j)} \) is a subbranch of \( \overline{\mathcal{B}r}^{(j)} \). \( Y \) is called a crust of \( X_0 \) if there exists a meromorphic section \( \tau \) of the line bundle \( N_0^{\otimes n_0} \) on \( \Theta_0 \) such that for some nonnegative divisor \( D = \sum_{i=1}^{k} a_iq_i \) on \( \Theta_0 \),

\[
\text{div}(\tau) = -\sum_{j=1}^{h} n_1^{(j)}p^{(j)} + D,
\]

where \( N_0 \) denotes the normal bundle of \( \Theta_0 \) in \( M \), \( \{p^{(j)}\} \) is the set of the attachment points on \( \Theta_0 \) with the branches \( \mathcal{B}r^{(j)} \). Moreover, for a positive integer \( l \), if each \( \overline{\mathcal{B}r}^{(j)} \) is a subbranch of \( \overline{\mathcal{B}r}^{(j)} \) of either type \( A \_l \),
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Fig. 2. In the barking family \([II^* \cdot 1]\), the singular fiber of type \(II^*\) is deformed to the main fiber of type \(III^*\). It seems that the simple crust \(Y\) is “barked” (peeled) off from the original singular fiber.

type \(B_l\) or type \(C_l\), then we call \(Y\) a \textit{simple crust} of \(X_0\) with \textit{barking multiplicity} \(l\).

We call the meromorphic section \(\tau\) a \textit{core section}. Note that \(\tau\) is not uniquely determined by \(Y\). Setting \(r_0 := \sum_{j=1}^{h} m_1^{(j)}/m_0\) and \(r'_0 := \sum_{j=1}^{h} n_1^{(j)}/n_0\), the following holds:

**Lemma 2.6.** Suppose that \(\Theta_0\) is a complex projective line. Then a connected subdivisor \(Y\) is a crust of \(X_0\) (equivalently, \(Y\) has a core section \(\tau\)) if and only if \(r_0 \leq r'_0\). Moreover \(\tau\) has no zero, that is, \(D = 0\) exactly when \(r_0 = r'_0\).

Takamura constructed a deformation family of \(\pi : M \rightarrow \Delta\) associated with a simple crust \(Y\). We call a deformation family obtained by his method a \textit{barking family}. In a barking family, the original singular fiber \(X_0\) is deformed to a simpler singular fiber in such a way that a part of \(X_0\) looks “barked” off from \(X_0\). The resulting singular fiber appears over the origin of \(\Delta_t\), so we denote it by \(X_{0,t}\) and call it the \textit{main fiber}. See Fig. 2.

In a barking family, there appear not only the main fiber but also other singular fibers over some points away from the origin of \(\Delta_t\), which are called \textit{subordinate fibers}. It is easy to see this. Under the deformation, the topological type of the singular fiber over the origin changes, so the local monodromy around it also changes (see Section 4 for details). On the other hand, the global monodromies before and after the deformation — that is, the two monodromies each of which is induced by a loop in \(\Delta\) (resp. \(\Delta_t\)) parallel to its boundary \(\partial \Delta\) (resp. \(\partial \Delta_t\)) —
coincide. We then deduce that there appear other singular fibers with nontrivial monodromies. Thus every barking family turns out to be a splitting family. Therefore:

**Theorem 2.7** (Takamura [Ta3]). Let $\pi : M \to \Delta$ be a linear degeneration with the stellar singular fiber $X_0$. If $X_0$ has a simple crust $Y$, then $\pi : M \to \Delta$ admits a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$.

**Remark 2.8.** In this paper, for a degeneration which is not necessarily relatively minimal, a splitting family of it is defined to satisfy that each deformation has at least two singular fibers (see Section 1). Thus singular fibers of a deformation in a splitting family possibly become smooth fibers after blowing down. Such singular fibers are said to be fake.

**Kodaira’s notation.**

Before proceeding, we supply Kodaira’s list of singular fibers of (relatively) minimal degenerations of elliptic curves [Ko]. See Table 1. For a singular fiber $X$, we denote by $e(X)$ the topological Euler characteristic of the underlying reduced curve $X^{\text{red}}$ of $X$. $A_X \in SL(2, \mathbb{Z})$ is the standard monodromy matrix of $X$ and its trace is denoted by $\text{Tr}(A_X)$.

Note that minimal singular fibers of type $I^*_n$, $II^*$, $III^*$ and $IV^*$ in this table are normally minimal and their underlying reduced curves have at most simple normal crossings. In contrast, minimal singular fibers of type $II$, $III$ and $IV$ have a singularity that is not a node. However, after successive blowing up, they become normally minimal degenerations such that $X^{\text{red}}$ has at most simple normal crossings. In this paper, such degenerations are also referred to be of type $II$, $III$ and $IV$.

§3. Constraints from Euler characteristics

In [Ta3], Takamura listed thirty five barking families for degenerations of complex curves of genus $g = 1$, that is, for degenerations of elliptic curves, and determined the type of the main fiber of each of them as follows (see also the list in Section 12):
| $mI_0$ ($m \geq 2$) | a multiple torus | $e(X)$ | $A_X$ | $\text{Tr}(A_X)$ |
|-------------------|-----------------|--------|-------|-----------------|
| | 0 | | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 2 |
| $mI_1$ | a (multiple) projective line with one node | 1 | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | 2 |
| $mI_n$ | | $n$ | $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ | 2 |
| $II$ | a projective line with one cusp | 2 | $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ | 1 |
| $III$ | two projective lines with second order contact | 3 | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | 0 |
| $IV$ | three projective lines intersecting transversally at one point | 4 | $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ | -1 |
| $I_0^*$ | | 6 | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | -2 |
| $I_n^*$ | | $6 + n$ | $\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$ | -2 |
| $II^*$ | | 10 | $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | 1 |
| $III^*$ | | 9 | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 0 |
| $IV^*$ | | 8 | $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ | -1 |

Table 1. Kodaira’s notation.
The aim of this paper is to determine the subordinate fibers of the above barking families. In this section, we give a list of the sets of subordinate fibers that can appear in each of the barking families, using results on Euler characteristics of singular fibers of degenerations.

For a singular fiber $X$, we denote by $e(X)$ the topological Euler characteristic of the underlying reduced curve of $X$.

**Lemma 3.1.** Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus $g \geq 1$ with the singular fiber $X_0$ and let $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ be a splitting family of $\pi : M \to \Delta$, say, $X_0$ splits into singular fibers $X_1, X_2, \ldots, X_N (N \geq 2)$ of a deformation $\pi_t : M_t \to \Delta_t$. 

\begin{align*}
[II.1] & \quad II \xrightarrow{\text{bark}} I_1 \\
[II.2] & \quad II \xrightarrow{\text{bark}} I_1 \\
[III^*.1] & \quad III^* \xrightarrow{\text{bark}} III^* \\
[III^*.2] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[III^*.3] & \quad III^* \xrightarrow{\text{bark}} I_2^* \\
[III^*.4] & \quad III^* \xrightarrow{\text{bark}} I_5 \\
[III^*.5] & \quad III^* \xrightarrow{\text{bark}} I_2^* \\
[III^*.6] & \quad III^* \xrightarrow{\text{bark}} I_3^* \\
[III^*.7] & \quad III^* \xrightarrow{\text{bark}} I_8 \\
[III^*.8] & \quad III^* \xrightarrow{\text{bark}} III^* \\
[III^*.9] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[IV.1] & \quad IV \xrightarrow{\text{bark}} I_3 \\
[IV.2] & \quad IV \xrightarrow{\text{bark}} I_2 \\
[IV.3] & \quad IV \xrightarrow{\text{bark}} I_2 \\
[IV.4] & \quad IV \xrightarrow{\text{bark}} I_1 \\
[IV^*.1] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[IV^*.2] & \quad IV^* \xrightarrow{\text{bark}} I_0^* \\
[IV^*.3] & \quad IV^* \xrightarrow{\text{bark}} I_6 \\
[IV^*.4] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[I_{0.1}] & \quad I_0^* \xrightarrow{\text{bark}} I_4 \\
[I_{0.2}] & \quad I_0^* \xrightarrow{\text{bark}} I_3 \\
[I_{n.1}] & \quad I_n^* \xrightarrow{\text{bark}} I_{n-1}^* \\
[I_{n.2}] & \quad I_n^* \xrightarrow{\text{bark}} I_{n+4}.
\end{align*}
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(a): Then the following formula holds:

\[ e(X_0) - 2(1 - g) = \sum_{i=1}^{N} \{e(X_i) - 2(1 - g)\}. \]

(b): In particular, if \( g = 1 \), then the following holds:

\[(3.1) \quad e(X_0) = e(X_1) + e(X_2) + \cdots + e(X_N). \]

Proof. (a) The left hand side equals the Euler characteristic \( e(M) \) of \( M \), while the right hand side equals \( e(M_t) \) (see [BPV, p. 97]). Since \( M_t \) is diffeomorphic to \( M \), we have \( e(M) = e(M_t) \), which confirms the assertion.

(b) clearly follows from (a). Q.E.D.

Consider a barking family \( \Psi : \mathcal{M} \to \Delta \times \Delta^+ \) of the degeneration \( \pi : M \to \Delta \) of elliptic curves. Recall that for a singular fiber \( X_{s,t} := \Psi^{-1}(s,t) (t \neq 0) \), we call \( X_{s,t} \) the main fiber if \( s = 0 \), and a subordinate fiber if \( s \neq 0 \). Suppose that \( \Psi : \mathcal{M} \to \Delta \times \Delta^+ \) splits the original singular fiber \( X_0 \) into the main fiber \( X_{0,t} \) and subordinate fibers \( X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t} (N \geq 1) \). In these notations, we restate (3.1) in Lemma 3.1 as

\[(3.2) \quad e(X_0) = e(X_{0,t}) + \sum_{i=1}^{N} e(X_{s_i,t}). \]

This confirms (a) of the following:

**Lemma 3.2.** Let \( \pi : M \to \Delta \) be a degeneration of elliptic curves with the singular fiber \( X_0 \). Suppose that a barking family \( \Psi : \mathcal{M} \to \Delta \times \Delta^+ \) splits the original singular fiber \( X_0 \) into the main fiber \( X_{0,t} \) and subordinate fibers \( X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_N,t} (N \geq 1) \). Then:

(a): The sum of the Euler characteristics of the subordinate fibers is \( e(X_0) - e(X_{0,t}) \):

\[ \sum_{i=1}^{N} e(X_{s_i,t}) = e(X_0) - e(X_{0,t}). \]

(b): If \( e(X_0) - e(X_{0,t}) = 1 \) holds, then \( \Psi \) splits \( X_0 \) into the main fiber \( X_{0,t} \) and one subordinate fiber \( I_1 \):

\[ X_0 \to X_{0,t} + I_1. \]
Proof. It remains to show the second statement (b). From the assumption \( e(X_0) - e(X_{0,t}) = 1 \) together with (a), we have

\[
e(X_{s_1,t}) + e(X_{s_2,t}) + \cdots + e(X_{s_N,t}) = 1.
\]

Note that every subordinate fiber of any barking family is a reduced curve only with \( A \)-singularities (Lemma 7.1). In particular, each subordinate fiber \( X_{s_{i,t}} \) is not a multiple torus (whose Euler characteristic is 0), thus \( e(X_{s_{i,t}}) \geq 1 \). Hence we have \( N = 1 \) (that is, \( X_{s_{1,t}} \) is the unique subordinate fiber) and \( e(X_{s_{1,t}}) = 1 \). This equality holds exactly when \( X_{s_{1,t}} \) is \( mI_1 \) \((m \geq 1)\). By Lemma 7.1 again, \( X_{s_{1,t}} \) is a reduced curve, so \( m = 1 \). Accordingly \( X_{s_{1,t}} \) is \( I_1 \). Q.E.D.

Lemma 3.2 (b) immediately yields the following:

**Proposition 3.3** (Case: \( e(X_0) - e(X_{0,t}) = 1 \)). In each of the following barking families, the subordinate fiber is \( I_1 \).

\[
\begin{align*}
[II.1] & \quad II \xrightarrow{\text{bark}} I_1 \\
[II.2] & \quad II \xrightarrow{\text{bark}} I_1 \\
[II*.1] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[II*.5] & \quad II^* \xrightarrow{\text{bark}} I_3^* \\
[II*.6] & \quad II^* \xrightarrow{\text{bark}} I_3^* \\
[II*.8] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[II*.9] & \quad II^* \xrightarrow{\text{bark}} III^* \\
[III.1] & \quad III \xrightarrow{\text{bark}} I_2 \\
[III.3] & \quad III \xrightarrow{\text{bark}} I_2 \\
[III*.1] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[III*.3] & \quad III^* \xrightarrow{\text{bark}} I_2^* \\
[III*.6] & \quad III^* \xrightarrow{\text{bark}} I_3^* \\
[III*.9] & \quad III^* \xrightarrow{\text{bark}} IV^* \\
[IV.1] & \quad IV \xrightarrow{\text{bark}} I_3 \\
[IV*.1] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[IV*.4] & \quad IV^* \xrightarrow{\text{bark}} I_1^* \\
[I_.n.1] & \quad I_{n}^* \xrightarrow{\text{bark}} I_{n-1}^*.
\end{align*}
\]

If \( e(X_0) - e(X_{0,t}) \geq 2 \), then we need another criterion to determine the subordinate fibers. However by Lemma 3.2 (a) we can narrow down candidates.

**Lemma 3.4** (Case: \( e(X_0) - e(X_{0,t}) = 2 \)). In each of the following barking families, the set of subordinate fibers is one of \( \{II\} \), \( \{I_2\} \), and
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\{I_1, I_1\}.

\begin{align*}
[III^*.2] & \quad II^* \xrightarrow{\text{bark}} IV^* \\
[III^*.3] & \quad II^* \xrightarrow{\text{bark}} I_2^* \\
[III^*.7] & \quad II^* \xrightarrow{\text{bark}} I_8 \\
[III^*.2] & \quad III \xrightarrow{\text{bark}} I_1 \\
[III^*.2] & \quad III^* \xrightarrow{\text{bark}} I_1^* \\
[III^*.7] & \quad III^* \xrightarrow{\text{bark}} I_7 \\
[IV^*.2] & \quad IV^* \xrightarrow{\text{bark}} I_0^* \\
[IV^*.3] & \quad IV^* \xrightarrow{\text{bark}} IV \\
[IV^*.4] & \quad IV^* \xrightarrow{\text{bark}} I_0^* \\
[I^*.1] & \quad I_0^* \xrightarrow{\text{bark}} I_4 \\
[I^*.2] & \quad I_n^* \xrightarrow{\text{bark}} I_{n+4}. \\
[IV^*.2] & \quad IV \xrightarrow{\text{bark}} I_2 \\

\textbf{Lemma 3.5} \text{ (Case: } e(X_0) - e(X_{0,t}) = 3) \cdot \text{ In each of the following barking families, the set of subordinate fibers is one of } \{I_1, I_1\}, \{II, I_1\}, \{I_2, I_1\}, \text{ and } \{I_1, I_1, I_1\}.

\begin{align*}
[III^*.4] & \quad III^* \xrightarrow{\text{bark}} I_0^* \\
[III^*.5] & \quad III^* \xrightarrow{\text{bark}} I_6 \\
[III^*.8] & \quad III^* \xrightarrow{\text{bark}} I_6

\textbf{Lemma 3.6} \text{ (Case: } e(X_0) - e(X_{0,t}) = 5). \text{ The sum of the Euler characteristics of the subordinate fibers of the following barking family is } 5:

\begin{align*}
[II^*.4] & \quad II^* \xrightarrow{\text{bark}} I_5.

\section{Monodromies around singular fibers}

Next we consider the monodromies around singular fibers of splitting families \textit{(not necessarily} barking families).

Let } \pi : M \rightarrow \Delta \text{ be a (relatively) minimal degeneration of elliptic curves with the singular fiber } X_0. \text{ We take a base point } s_0 \text{ in } \Delta \setminus \{0\} \text{ and a loop (simple closed curve) } l_0 \text{ in } \Delta \setminus \{0\} \text{ passing through the base point } s_0 \text{ and circuiting around the origin with the counterclockwise orientation. Then } \pi^{-1}(l_0) \text{ is a real 3-manifold and the restriction } \pi : \pi^{-1}(l_0) \rightarrow l_0 \text{ is a } \Sigma \text{-bundle over } S^1, \text{ where } \Sigma \text{ is an elliptic curve. Here } \pi^{-1}(l_0) \text{ is obtained from } \Sigma \times [0, 1] \text{ by the identification of the boundaries } \Sigma \times \{0\} \text{ and } \Sigma \times \{1\} \text{ via an orientation-preserving homeomorphism } f \text{ of } \Sigma. \text{ The isotopy class } [f] \text{ of } f \text{ is called the } \textit{topological monodromy} \text{ around } X_0. \text{ Then } f \text{ induces an automorphism } \rho := f_* \text{ on } H_1(\Sigma, \mathbb{Z}), \text{ which is called}
the (homological) monodromy around $X_0$. Under an identification of $\Sigma$ and $\mathbb{R}^2/\mathbb{Z}^2$, fixing a basis of $H_1(\Sigma, \mathbb{Z})$, we obtain an isomorphism

$$\text{Aut}(H_1(\Sigma, \mathbb{Z})) \rightarrow SL(2, \mathbb{Z}).$$

In the subsequent discussion, we consider $\rho$ as an element of $SL(2, \mathbb{Z})$.

Next suppose that $\Psi : M \rightarrow \Delta \times \Delta^\dagger$ is a splitting family of the degeneration $\pi : M \rightarrow \Delta$, that is, the deformation $\pi_t : M_t \rightarrow \Delta_t$ of $\pi : M \rightarrow \Delta$ for a fixed $t \neq 0$ has singular fibers $X_1, X_2, \ldots, X_N$ ($N \geq 2$). Then we say that $X_0$ splits into $X_1, X_2, \ldots, X_N$ and express $X_0 \rightarrow X_1 + X_2 + \cdots + X_N$. Now we define the local monodromies around the singular fibers $X_k$ ($k = 1, 2, \ldots, N$) as follows: Set $s_k := \pi_t(X_k)$. We take a base point $s'_0$ in $\Delta_t \setminus \{s_1, s_2, \ldots, s_N\}$ (so the fiber $X_{s'_0} = \pi_t^{-1}(s'_0)$ is smooth). For each $k = 1, 2, \ldots, N$, we take a loop $l_k$ in $\Delta_t \setminus \{s_1, s_2, \ldots, s_N\}$ passing through the base point $s'_0$ and circuiting around $s_k$ with the counterclockwise orientation. Then the loop $l_k$ induces an orientation-preserving homeomorphism $f_k$ of $\Sigma$, which defines the local topological monodromy $[f_k]$ and the local (homological) monodromy $\rho_k \in SL(2, \mathbb{Z})$ around $X_k$.

The following is known (see [U]):

**Lemma 4.1.** The monodromy $\rho$ around $X_0$ (resp. the local monodromy $\rho_k$ around $X_k$ for each $k = 1, 2, \ldots, N$) is conjugate to the standard monodromy matrix\(^4\) corresponding to the singular fiber $X_0$ (resp. $X_k$).

Possibly after renumbering, we may assume that $l_1 \circ l_2 \circ \cdots \circ l_N$ is homotopic to a loop rounding all the singular values $s_1, s_2, \ldots, s_N$ with the counterclockwise orientation. Let $D \subset \Delta \times \Delta^\dagger$ be the set of singular values of $\Psi$. We now take a path $l$ in $(\Delta \times \Delta^\dagger) \setminus D$ connecting $s_0 \in \Delta_0$ and $s'_0 \in \Delta_t$. Note that for any point $(s, t) \in l$, the fiber $X_{s, t} = \Psi^{-1}(s, t)$ is smooth. Since the loop $l^{-1} \circ l_1 \circ l_2 \circ \cdots \circ l_N \circ l$ is homotopic to the loop $l_0$, the topological monodromy $[f]$ is conjugate to the composition of the local topological monodromies $[f_1] \circ [f_2] \circ \cdots \circ [f_N]$. Similarly:

**Lemma 4.2.** The monodromy $\rho$ is conjugate to the composition of the local monodromies $\rho_1, \rho_2, \ldots, \rho_N$.

We prepare notation. $SL(2, \mathbb{Z}) = \langle a, b \mid a^3 = b^2 = -E \rangle$ is generated by

$$a := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
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Setting

\[ s_0 := a^{-1}b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 := ba^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \]

then \( s_0 \) and \( s_2 \) are also generators of \( SL(2, \mathbb{Z}) \): indeed we have \( a = s_0 s_2 \) and \( b = s_0 s_2 s_0 = s_2 s_0 s_2 \). Since \( s_2 = (s_0 s_2) s_0 (s_0 s_2)^{-1} \), \( s_2 \) is conjugate to \( s_0 \).

Next we express the standard monodromy matrices of singular fibers as a product of \( s_0 \) and \( s_2 \) as follows (see \([U]\)):

\[
\begin{align*}
A_{I_1} &= (s_0)^n \quad (n \geq 1) \\
A_{II} &= s_0 s_2 \\
A_{III} &= s_0 s_2 s_0 = s_2 s_0 s_2 \\
A_{IV} &= s_0 s_2 s_0 s_2 \\
A_{II^*} &= (s_0 s_2)^5 \\
A_{III^*} &= (s_0 s_2)^4 s_0 \\
A_{IV^*} &= (s_0 s_2)^4 \\
A_{n^*} &= (s_0 s_2)^3 (s_0)^n \quad (n \geq 0).
\end{align*}
\]

The number of \( s_0, s_2 \) contained in each product coincides with the Euler characteristic of the corresponding singular fiber. Note that \( s_0 \) is the standard monodromy matrix \( A_{I_1} \) of the singular fiber \( I_1 \). It is known that for any degeneration of elliptic curves except with \( m \bar{I}_0 \) \((m \geq 2)\), the singular fiber splits into singular fibers of type \( I_1 \) (whose Euler characteristic \( e(I_1) \) is equal to 1) after successive deformations. See \([Ka]\), \([M]\).

**Example 4.3.** The barking family \([III.1]\) splits the singular fiber \( III \) into the main fiber \( I_2 \) and a subordinate fiber \( I_1 \):

\[ III \rightarrow I_2 + I_1. \]

Lemma 4.2 states that, if \( X_0 \) splits into \( X_1, X_2, \ldots, X_N \), then a monodromy matrix of \( X_0 \) is conjugate to the composition of monodromy matrices of \( X_1, X_2, \ldots, X_N \) (that is, conjugacies of the standard monodromy matrices corresponding to \( X_1, X_2, \ldots, X_N \) respectively). In this case, the standard monodromy matrix \( A_{III} \) of \( III \) is decomposed into conjugacies of the standard monodromy matrices corresponding to \( I_2 \)

---

5The notations \( s_0 \) and \( s_2 \) are used in \([FM]\) Section 2.4, where ‘\( s_1 \)’ is defined as \( s_1 := aba \).
and $I_1$:
\[
A_{III} = s_0s_2s_0 \\
= s_0^2(s_0^{-1}s_2s_0) \\
= s_0^2(s_2s_0s_2^{-1}) \\ (s_0s_2s_0 = s_2s_0s_2) \\
= A_{II} \cdot (s_2A_{II}^{-1}).
\]

In Section 11, we will give decompositions of the standard monodromy matrix corresponding to the splittings induced from Takamura's barking families.

§5. Constraints from monodromies

From Lemmas 4.1 and 4.2, it is a necessary condition for a singular fiber $X_0$ to split into singular fibers $X_1, X_2, \ldots, X_N$ ($N \geq 2$) that some monodromy matrix of $X_0$ is conjugate to the composition of monodromy matrices of $X_1, X_2, \ldots, X_N$, which means that monodromies give some constraints to splittings. In this section, we prove that none of the following splittings occurs (Theorem 5.8):

- $IV \rightarrow I_2 + I_2$,
- $II^* \rightarrow I_8 + II, \quad I_7 + III, \quad I_6 + IV, \quad I_4 + I_0, \quad I_3 + I_1^*$, \\
  $I_u + I_v (u + v = 10)$,
- $III^* \rightarrow I_7 + II, \quad I_6 + III, \quad I_5 + IV, \quad I_3 + I_0^*$, \\
  $I_u + I_v (u + v = 9)$,
- $IV^* \rightarrow I_6 + II, \quad I_5 + III, \quad I_4 + IV, \quad I_2 + I_0^*$, \\
  $I_u + I_v (u + v = 8)$,
- $I_n^* (n \geq 0) \rightarrow I_{n+4} + II, \quad I_{n+3} + III, \quad I_{n+2} + IV, \quad I_u + I_v (u + v = n + 6$ and $(n, u, v) \neq (2, 4, 4))$.
- $I_0^* \rightarrow I_3 + I_2 + I_1$.

We begin with preparation.

**Lemma 5.1.** If matrices $A_1, A_2 \in SL(2, \mathbb{Z})$ are conjugate, then $\text{Tr}(A_1) = \text{Tr}(A_2)$, where $\text{Tr}(A_i)$ denotes the trace of $A_i$.

**Proof.** By assumption, we may write $A_1 = PA_2P^{-1}$ for some $P \in SL(2, \mathbb{Z})$. Hence
\[
\text{Tr}(A_1) = \text{Tr}((PA_2P^{-1}) = \text{Tr}(P^{-1}(PA_2)) = \text{Tr}(A_2).
\]
Q.E.D.
The following is useful:

**Lemma 5.2.** Suppose that a singular fiber $X$ splits into two singular fibers $I_n (n \geq 1)$ and $Y$:

$$X \rightarrow I_n + Y.$$ 

Then

$$\text{Tr}(A_X) \equiv \text{Tr}(A_Y) \mod n,$$

where $A_X$ and $A_Y$ are the standard monodromy matrices of $X$ and $Y$.

**Proof.** If $X$ splits into $I_n$ and $Y$, then for some monodromy matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $Y$,

$$B := A_{I_n} C \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}$$

is a monodromy matrix of $X$. Then we have

$$\text{Tr}(B) = a + nc + d = \text{Tr}(C) + nc.$$

Thus

$$\text{Tr}(B) \equiv \text{Tr}(C) \mod n.$$

Where $A_X$ and $A_Y$ denote the standard monodromy matrices corresponding to $X$ and $Y$ respectively, $B$ is conjugate to $A_X$, while $C$ is conjugate to $A_Y$. By Lemma 5.1 we have $\text{Tr}(B) = \text{Tr}(A_X)$ and $\text{Tr}(C) = \text{Tr}(A_Y)$. Accordingly

$$\text{Tr}(A_X) \equiv \text{Tr}(A_Y) \mod n.$$

Q.E.D.

We now consider the singular fiber $IV$. Since the Euler characteristic of $IV$ is 4 and that of $I_2$ is 2,

$$e(IV) = e(I_2) + e(I_2)$$

holds. Note that, if a singular fiber $X_0$ splits into two singular fibers $X_1$ and $X_2$, then $e(X_0) = e(X_1) + e(X_2)$ (Lemma 3.1 (b)). So it is plausible that some deformation family splits the singular fiber $IV$ into two $I_2$. However this is not the case. If $IV$ splits into two $I_2$, by Lemma 5.2, we have

$$\text{Tr}(A_{IV}) \equiv \text{Tr}(A_{I_2}) \mod 2,$$

which contradicts that $\text{Tr}(A_{IV}) = -1$ and $\text{Tr}(A_{I_2}) = 2$. Thus the splitting

$$IV \rightarrow I_2 + I_2$$
does not occur. We have shown the first statement of the following lemma, and we can show the others by the same argument:

**Lemma 5.3.**

(a): The singular fiber $IV$ never splits as follows:

$$IV \rightarrow I_2 + I_2.$$  

(b): The singular fiber $II^*$ never splits as follows:

$$II^* \rightarrow I_8 + II, \ I_7 + III, \ I_6 + IV, \ I_4 + I_0^*, \ I_3 + I_1^*, \ I_u + I_v (u + v = 10).$$

(c): The singular fiber $III^*$ never splits as follows:

$$III^* \rightarrow I_7 + II, \ I_6 + III, \ I_5 + IV, \ I_3 + I_0^*, \ I_u + I_v (u + v = 9).$$

(d): The singular fiber $IV^*$ never splits as follows:

$$IV^* \rightarrow I_6 + II, \ I_5 + III, \ I_4 + IV, \ I_2 + I_0^*, \ I_u + I_v (u + v = 8).$$

(e): The singular fiber $I_n^* (n \geq 1)$ never splits as follows:

$$I_n^* \rightarrow I_{n+4} + II, \ I_{n+3} + III, \ I_{n+2} + IV, \ I_u + I_v (u + v = n + 6, \ (n, u, v) \neq (2, 4, 4)).$$

Next we consider splittings of $I_0^*$. The standard monodromy matrix of $I_0^*$ is $A_{I_0^*} = -E$, where $E$ is the identity matrix.

**Lemma 5.4.** Suppose that the singular fiber $I_0^*$ splits into two singular fibers $X$ and $Y$:

$$I_0^* \rightarrow X + Y.$$  

Then

$$\text{Tr}(A_X) + \text{Tr}(A_Y) = 0.$$  

**Proof.** If $I_0^*$ splits into $X$ and $Y$, then for monodromy matrices $B$ and $C$ of $X$ and $Y$, we have $A_{I_0^*} = BC$, where $A_{I_0^*}$ is the standard monodromy matrix of $I_0^*$. Since $A_{I_0^*} = -E$, we have $-E = BC$, that is, $B = -C^{-1}$. In particular,

$$\text{Tr}(B) = -\text{Tr}(C).$$
Singular fibers in barking families of degenerations

Since $B$ (resp. $C$) is conjugate to $A_X$ (resp. $A_Y$), by Lemma 5.1 we have $\text{Tr}(B) = \text{Tr}(A_X)$ and $\text{Tr}(C) = \text{Tr}(A_Y)$. Thus

$$\text{Tr}(A_X) + \text{Tr}(A_Y) = 0.$$  

Q.E.D.

**Lemma 5.5.** Suppose that the singular fiber $I_0^*$ splits into three singular fibers $I_n (n \geq 1)$, $X$ and $Y$:

$$I_0^* \rightarrow I_n + X + Y.$$  

Then $\text{Tr}(A_X) + \text{Tr}(A_Y) \equiv 0 \mod n$.

**Proof.** If $I_0^*$ splits into $I_3$, $X_1$ and $X_2$, then for monodromy matrices $B$ and $C$ of $X$ and $Y$, we have $A_{I_0^*} = A_{I_n} BC$. Since $A_{I_0^*} = -E$ and $A_{I_n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, writing $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$-C^{-1} = A_{I_n} B = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}.$$  

Then we have

$$- \text{Tr}(C) = a + nc + d = \text{Tr}(B) + nc.$$  

Thus

$$\text{Tr}(B) + \text{Tr}(C) \equiv 0 \mod n.$$  

Since $B$ (resp. $C$) is conjugate to $A_X$ (resp. $A_Y$), by Lemma 5.1 we have $\text{Tr}(B) = \text{Tr}(A_X)$ and $\text{Tr}(C) = \text{Tr}(A_Y)$. Accordingly

$$\text{Tr}(A_X) + \text{Tr}(A_Y) \equiv 0 \mod n.$$  

Q.E.D.

**Lemma 5.6.**

(a): The singular fiber $I_0^*$ never splits as follows:

$$I_0^* \rightarrow I_4 + II, \quad I_3 + III, \quad I_2 + IV,$$
$$I_5 + I_1, \quad I_4 + I_2, \quad I_3 + I_3.$$  

(b): The singular fiber $I_0^*$ never splits as follows:

$$I_0^* \rightarrow I_3 + I_2 + I_1.$$
Proof. (a) we only show that the splitting \( I_0^* \rightarrow I_4 + II \) does not occur, because we can give the proof for the other splittings by the same argument. If \( I_0^* \) splits into \( I_4 \) and \( II \), by Lemma 5.4, we have

\[ \text{Tr}(A_{I_4}) + \text{Tr}(A_{II}) = 0, \]

which contradicts that \( \text{Tr}(A_{I_4}) = 2 \) and \( \text{Tr}(A_{II}) = 1 \). Thus the splitting \( I_0^* \rightarrow I_4 + II \) does not occur.

(b) If \( I_0^* \) splits into \( I_3, I_2 \) and \( I_1 \), by Lemma 5.5, we have

\[ \text{Tr}(A_{I_2}) + \text{Tr}(A_{I_1}) \equiv 0 \mod 3. \]

which contradicts that \( \text{Tr}(A_{I_2}) = \text{Tr}(A_{I_1}) = 2 \). Thus the splitting \( I_0^* \rightarrow I_3 + I_2 + I_1 \) does not occur.

Q.E.D.

Remark 5.7. We can give an alternative proof of Lemma 5.6 (a) except for the splitting \( I_0^* \rightarrow I_4 + I_2 \) as follows; For instance, suppose that \( I_0^* \) splits into \( I_4 \) and \( II \). By Lemma 5.2, we then have

\[ \text{Tr}(A_{I_0^*}) \equiv \text{Tr}(A_{II}) \mod 4, \]

which contradicts that \( \text{Tr}(A_{I_0^*}) = -2 \) and \( \text{Tr}(A_{II}) = 1 \). Thus the splitting \( I_0^* \rightarrow I_4 + II \) does not occur.

We summarize Lemmas 5.3 and 5.6 as follows:
Theorem 5.8. None of the following splittings occurs:

\[
\begin{align*}
IV & \rightarrow I_2 + I_2, \\
II^* & \rightarrow I_8 + II, \quad I_7 + III, \quad I_6 + IV, \\
I_4 + I_0^*, \quad I_3 + I_1^*, \\
I_u + I_v (u + v = 10), \\
III^* & \rightarrow I_7 + II, \quad I_6 + III, \quad I_5 + IV, \quad I_3 + I_0^*, \\
I_u + I_v (u + v = 9), \\
IV^* & \rightarrow I_6 + II, \quad I_5 + III, \quad I_4 + IV, \quad I_2 + I_0^*, \\
I_u + I_v (u + v = 8), \\
I_n^* (n \geq 0) & \rightarrow I_{n+4} + II, \quad I_{n+3} + III, \quad I_{n+2} + IV, \\
I_u + I_v (u + v = n + 6 \text{ and } (n, u, v) \neq (2, 4, 4)), \\
I_0^* & \rightarrow I_3 + I_2 + I_1.
\end{align*}
\]

§6. Determination of subordinate fibers, 1

In this section, based on the result of the previous section, we determine the subordinate fibers of Takamura’s barking families \([II^*.7], [III^*.7], [IV^*.3], [I_0^*.1], [I_n^*.2]\).

Proposition 6.1. The barking family \([II^*.7]\) splits the singular fiber \(II^*\) as follows:

\[
II^* \rightarrow I_8 + I_1 + I_1,
\]

where \(I_8\) is the main fiber and the two \(I_1\) are subordinate fibers.

Proof. In the barking family \([II^*.7]\), \(II^*\) is deformed to \(I_8\):

\[
II^* \xrightarrow{\text{bark}} I_8.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{II\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Now Lemma 5.3 (b) eliminates the cases (i) and (ii). Thus the subordinate fibers are two \(I_1\). Q.E.D.

Proposition 6.2. The barking family \([III^*.7]\) splits the singular fiber \(III^*\) as follows:

\[
III^* \rightarrow I_7 + I_1 + I_1,
\]

where \(I_7\) is the main fiber and the two \(I_1\) are subordinate fibers.
Proof. In the barking family \([\text{III}^*.7]\), \(\text{III}^*\) is deformed to \(I_7\):

\[
\text{III}^* \xrightarrow{\text{bark}} I_7.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{I_1\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Now Lemma 5.3 (c) eliminates the cases (i) and (ii). Thus the subordinate fibers are two \(I_1\).

Q.E.D.

**Proposition 6.3.** The barking family \([\text{IV}^*.3]\) splits the singular fiber \(\text{IV}^*\) as follows:

\[
\text{IV}^* \rightarrow I_6 + I_1 + I_1,
\]

where \(I_6\) is the main fiber and the two \(I_1\) are subordinate fibers.

Proof. In the barking family \([\text{IV}^*.3]\), \(\text{IV}^*\) is deformed to \(I_6\):

\[
\text{IV}^* \xrightarrow{\text{bark}} I_6.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{I_1\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Now Lemma 5.3 (d) eliminates the cases (i) and (ii). Thus the subordinate fibers are two \(I_1\).

Q.E.D.

**Proposition 6.4.** The barking family \([I_0^*.1]\) splits the singular fiber \(I_0^*\) as follows:

\[
I_0^* \rightarrow I_4 + I_1 + I_1,
\]

where \(I_4\) is the main fiber and the two \(I_1\) are subordinate fibers.

Proof. In the barking family \([I_0^*.1]\), \(I_0^*\) is deformed to \(I_4\):

\[
I_0^* \xrightarrow{\text{bark}} I_4.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{I_1\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Now Lemma 5.6 (a) eliminates the cases (i) and (ii). Thus the subordinate fibers are two \(I_1\).

Q.E.D.

**Proposition 6.5.** The barking family \([I_n^*.2]\) splits the singular fiber \(I_n^*\) as follows:

\[
I_n^* \rightarrow I_{n+4} + I_1 + I_1,
\]

where \(I_{n+4}\) is the main fiber and the two \(I_1\) are subordinate fibers.

Proof. In the barking family \([I_n^*.2]\), \(I_n^*\) is deformed to \(I_{n+4}\):

\[
I_n^* \xrightarrow{\text{bark}} I_{n+4}.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{I_1\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Now Lemma 5.3 (e) eliminates the cases (i) and (ii). Thus the subordinate fibers are two \(I_1\).

Q.E.D.
Remark 6.6. For the barking families [IV.3], [III*.8], [I*.2], we cannot determine the subordinate fibers but we can narrow down candidates:

- The splitting of IV induced from the barking family [IV.3] is one of the following:
  \[ IV \rightarrow I_2 + I_1, \]
  \[ IV \rightarrow I_2 + I_1 + I_1. \]
  In fact, by Lemma 3.4, the set of subordinate fibers is one of (i) \{I_2\}, (ii) \{I_1, I_1\}, and (iii) \{I_1, I_1\}, and Lemma 5.3 (a) eliminates the case (ii).

- The splitting of III* induced from the barking family [III*.8] is one of the following:
  \[ III* \rightarrow I_6 + I_2 + I_1, \]
  \[ III* \rightarrow I_6 + I_1 + I_1 + I_1. \]
  In fact, by Lemma 3.5, the set of subordinate fibers is one of (i) \{III\}, (ii) \{I_3\}, (iii) \{I_1, I_1\}, (iv) \{I_2, I_1\}, and (v) \{I_1, I_1, I_1\}, and Lemma 5.3 (c) eliminates the cases (i) and (ii).

- The splitting of I*_0 induced from the barking family [I*.2] is one of the following:
  \[ I^*_0 \rightarrow I_3 + I_1 + I_1, \]
  \[ I^*_0 \rightarrow I_3 + I_1 + I_1 + I_1. \]
  In fact, by Lemma 3.5, the set of subordinate fibers is one of (i) \{III\}, (ii) \{I_3\}, (iii) \{I_1, I_1\}, (iv) \{I_2, I_1\}, and (v) \{I_1, I_1, I_1\}, and Lemma 5.6 eliminates the cases (i), (ii) and (iv).

§7. Singularities near proportional subbranches

Let \( \pi : M \rightarrow \Delta \) be a linear degeneration of complex curves with a stellar singular fiber \( X_0 = m_0 \Theta_0 + \sum_{j=1}^{h} Br^{(j)} \). If there exists a simple crust \( Y \) of \( X_0 \), then we can construct a splitting family of \( \pi : M \rightarrow \Delta \), which is called a barking family associated with \( Y \) (Theorem 2.7). Suppose that \( Y = n_0 \Theta_0 + \sum_{j=1}^{h} br^{(j)} \) is a simple crust of \( X_0 \) with barking multiplicity \( l \).

Recall that each subbranch of \( Y \) is of type \( A_l, B_l \) or \( C_l \). A subbranch \( Br^{(j)} \) is said to be proportional if \( m_0 n_1^{(j)} = n_0 m_1^{(j)} \) (equivalently...
\[ n_0/m_0 = n_1^{(j)}/m_1^{(j)} = \cdots = n_\nu^{(j)}/m_\nu^{(j)} \]. Note that every proportional subbranch of simple crusts is of type \( A_t \). Indeed, any proportional subbranch of type \( B_t \) is of type \( A_t \), and no proportional subbranch is of type \( C_t \). Moreover every proportional subbranch \( \overline{Br}^{(j)} \) has the same length as that of \( \overline{Br}^{(j)} \) (that is, \( \nu^{(j)} = \lambda^{(j)} \)) and satisfies \( n_{\lambda^{(j)}+1} = 0 \).

The following lemma is important ([Ta3] Proposition 16.2.6):

**Lemma 7.1.** Suppose that \( \Psi : M \to \Delta \times \Delta^\dagger \) is a barking family of the degeneration \( \pi : M \to \Delta \) associated with a simple crust \( Y \). Then any subordinate fiber of \( \Psi \) is a reduced curve only with \( A \)-singularities\(^6\). Moreover these singularities lie (i) near the core or (ii) near the edge\(^7\) of each proportional subbranch if it exists.

**Remark 7.2.** By Lemma 7.1, every subordinate fiber in barking families is a reduced curve only with isolated singularities. In particular, for degenerations of elliptic curves, none of \( ml_n \) (\( m \geq 2 \)), \( IV^*, III^*, II^* \), \( ml_n^I \) (\( m \geq 2 \)) appears as a subordinate fiber.

The rest of this section investigates the singularities of subordinate fibers near a proportional subbranch. Let \( \pi : M \to \Delta \) be a linear degeneration of complex curves with a stellar singular fiber \( X_0 \) and \( \Psi : M \to \Delta \times \Delta^\dagger \) be a barking family associated with a simple crust \( Y \) with barking multiplicity \( l \). Suppose that \( Y \) has a proportional subbranch \( \overline{Br} \) of a branch \( Br \) of \( X_0 \). First recall that near the branch \( Br \), \( M \) is given by the following data (see [Ta3] Chapter 7): for \( i = 1, 2, \ldots, \lambda \),

\[
\begin{align*}
H_i : w_i^{m_i-1-n_i-1} \eta_i^{m_i-n_i} \left( w_i^{n_i-1} \eta_i^{n_i} + t^d f_i \right) - s &= 0, \\
&\text{in } U_i \times \mathbb{C} \times \Delta \times \Delta^\dagger, \\
H_i' : z_i^{m_i+1-n_i+1} \xi_i^{m_i-n_i} \left( z_i^{n_i+1} \xi_i^{n_i} + t^d f_i \right) - s &= 0, \\
&\text{in } V_i \times \mathbb{C} \times \Delta \times \Delta^\dagger.
\end{align*}
\]

Note that, substituting \( t = 0 \) into these equations, we obtain

\[
\begin{align*}
H_i \big|_{t=0} : w_i^{m_i-1} \eta_i^{m_i} - s &= 0, \\
H_i' \big|_{t=0} : z_i^{m_i+1} \xi_i^{m_i} - s &= 0,
\end{align*}
\]

which are the local expressions of \( M \) near \( Br \). See the proof of Lemma 2.3. For a fixed \((s, t) \in \Delta \times \Delta^\dagger\), we consider the fiber \( X_{s, t} = \Psi^{-1}(s, t) \) of \( \Psi \). The following is required ([Ta3] Section 7.2):

---

\(^6\)An \( A \)-singularity is a singularity analytically equivalent to \( y^2 = x^{\mu+1} \) for some positive integer \( \mu \).

\(^7\)To be precise, near the ‘terminal’ irreducible component \( \Theta_{\lambda^{(j)}} \) of the branch \( Br^{(j)} \) corresponding to each proportional subbranch \( Br^{(j)} \).
Lemma 7.3. Let \( m, n, l \) be positive integers satisfying \( m - ln > 0 \) and \( m', n' \) be nonnegative integers satisfying \( m' - ln' \geq 0 \). Set \( h(z, \zeta) := f(z^p \zeta^p) \) for a non-vanishing holomorphic function \( f \) and positive integers \( p, p' (p < p') \). Then a complex curve \( C_{s,t} \) in \( \mathbb{C}^2 \) defined by

\[
C_{s,t} : z^{m'} - ln' \zeta^{m-ln} \left( z^{n'} \zeta^n + th \right)^l - s = 0
\]

is singular if and only if

(i): \( s = 0 \) or

(ii): \( m' = n' = 0 \) and \( \left( \frac{ln - m}{ln} \right)^{ln} s^n = \left( \frac{ln - m}{m} t c \right)^m \),

where \( c := h(0,0) \) and \( \bar{m} \) and \( \bar{n} \) are the relatively prime integers satisfying \( \bar{m}/\bar{n} = n/m \). In the case (ii), \((z, \zeta) \in C_{s,t} \) is a singularity exactly when

\[
z = 0 \quad \text{and} \quad \zeta^n = \frac{ln - m}{m} t c.
\]

Since \( \text{br} \) is proportional, we have \( m_{\lambda+1} = n_{\lambda+1} = 0 \), so

\[
\mathcal{H}'_{\lambda, s,t} : \zeta^{m_{\lambda}-ln_{\lambda}} \left( \zeta^{n_{\lambda}} + t^d \tilde{f}_{\lambda} \right)^l - s = 0.
\]

Lemma 7.3 ensures that for some \((s, t) \) \((s, t \neq 0) \), the curve \( \mathcal{H}'_{\lambda, s,t} \) has singularities. In what follows, we write \( m := m_{\lambda} \) and \( n := n_{\lambda} \), and denote by \( \bar{m} \) and \( \bar{n} \) the relatively prime integers satisfying \( \bar{n}/\bar{m} = n/m \).

For a fixed \( t \neq 0 \), the equation

\[
\left( \frac{ln - m}{ln} \right)^{ln} s^n = \left( \frac{ln - m}{m} t^d c \right)^m
\]

for \( s \) has \( \bar{n} \) solutions, say, \( s_1, s_2, \ldots, s_{\bar{n}} \). Since \((0, \zeta) \) satisfying \( \zeta^n = \frac{ln - m}{m} t^d c \) is a singularity of \( \mathcal{H}'_{\lambda, s_k, t} \) for some \( s_k \), each \( \mathcal{H}'_{\lambda, s_k, t} \) has \( n/\bar{n} = \gcd(m, n) \) singularities.

The above result is summarized as follows:

**Proposition 7.4.** Let \( \pi : M \to \Delta \) be a linear degeneration of complex curves with a stellar singular fiber \( X_0 \) and let \( \Psi : M \to \Delta \times \Delta^l \) be a barking family associated with a simple crust \( Y \) with barking multiplicity \( l \). Suppose that \( Y \) has a proportional subbranch \( \text{br}^{(j)} \) of a branch \( \text{Br}^{(j)} \) of \( X_0 \). Write \( \overline{\text{Br}}^{(j)} := m_0 \Delta_0 + m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_{\lambda} \Theta_{\lambda} \) and \( \overline{\text{br}}^{(j)} := n_0 \Delta_0 + n_1 \Theta_1 + n_2 \Theta_2 + \cdots + n_{\lambda} \Theta_{\lambda} \) and let \( \bar{m} \) and \( \bar{n} \) be the relatively prime positive integers satisfying \( \bar{n}/\bar{m} = n/\lambda \). Then in the deformation \( \pi_t : M_t \to \Delta_t \) for a fixed \( t \neq 0 \), there exist \( \bar{n} \) subordinate
fibers that have singularities near the edge of $\text{Br}^{(j)}$. Moreover, each of these subordinate fibers has $n/\bar{n} (= \gcd(m, n))$ singularities near the edge of $\text{Br}^{(j)}$.

§8. Singularities near the core

We next investigate the singularities of subordinate fibers near the core.

Let $\pi : M \to \Delta$ be a linear degeneration of complex curves with a stellar singular fiber $X_0 = m_0 \Theta_0 + \sum_{j=1}^{h} \text{Br}^{(j)}$ and let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family of the degeneration $\pi : M \to \Delta$ associated with a simple crust $Y = n_0 \Theta_0 + \sum_{j=1}^{h} \text{br}^{(j)}$. Write $\overline{\text{Br}}^{(j)} = m_0 \Delta^{(j)}_0 + m_1^{(j)} \Theta_1^{(j)} + \cdots + m^{(j)}_{\lambda(j)} \Theta^{(j)}_{\lambda(j)}$, $\text{br}^{(j)} = n_0 \Delta^{(j)}_0 + n_1^{(j)} \Theta_1^{(j)} + \cdots + n^{(j)}_{\nu(j)} \Theta^{(j)}_{\nu(j)}$, and let $p^{(j)}$ be the attachment point on $\Theta_0$ with $\text{Br}^{(j)}$. For brevity, we assume that the subbranches $\text{br}^{(1)}, \text{br}^{(2)}, \ldots, \text{br}^{(v)}$ are proportional and $\text{br}^{(v+1)}, \text{br}^{(v+2)}, \ldots, \text{br}^{(h)}$ are not.

Let $N_0$ be the normal bundle of $\Theta_0$ in $M$. Recall that the local expression of $\mathcal{M}$ near the core $\Theta_0$ is given by

$$
\sigma(z)\zeta^{m_0} - s + \sum_{k=1}^{l} C_k t^k d \sigma(z) \tau(z)^k \zeta^{m_0 - kn_0} = 0 \quad \text{in} \ N_0 \times \Delta \times \Delta^\dagger,
$$
equivalently

$$
\sigma(z)\zeta^{m_0 - kn_0} (\zeta^{n_0} + t^d \tau(z))^l - s = 0,
$$
where $\sigma$ is the standard section of $N_0^\otimes(-m_0)$ and $\tau$ is a core section of $N_0^\otimes n_0$ for $Y$ (see [Ta3] Chapter 16). Substituting $t = 0$ into this equation, we obtain

$$
\sigma(z)\zeta^{m_0} - s = 0 \quad \text{in} \ N_0 \times \Delta \times \{0\},
$$
which is the local expression of $M$ around $\Theta_0$. See the paragraph subsequent to Remark 2.2. Note that $\sigma$ has a zero of order $m_1^{(j)}$ at $p^{(j)}$, while $\tau$ has a pole of order $n_1^{(j)}$ at $p^{(j)}$. Suppose that $\tau$ has a zero of order $a_i$ at $q_i (i = 1, 2, \ldots, k)$ on $\Theta_0$.

Fixing $s, t \neq 0$, consider a fiber $X_{s,t} := \Psi^{-1}(s,t)$ of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. Set $F := \sigma(z)\zeta^{m_0 - kn_0} (\zeta^{n_0} + t^d \tau(z))^l$. Then $(z, \zeta) \in X_{s,t}$ is a singularity if and only if

$$
\frac{\partial}{\partial z} F(z, \zeta) = \frac{\partial}{\partial \zeta} F(z, \zeta) = 0,
$$
Singular fibers in barking families of degenerations

equivalently
\[
\begin{align*}
\left( \zeta_n^{m_0-n_0} \left( \zeta^{n_0} + t^d \tau(z) \right)^{l-1} \sigma(z) \zeta^{n_0} + t^d \left( \sigma(z) \tau(z) + l \sigma(z) \tau(z) \right) \right) &= 0, \\
\left( \zeta_n^{m_0-n_0-1} \left( \zeta^{n_0} + t^d \tau(z) \right)^{l-1} \sigma(z) \left( m_0 \zeta^{n_0} + (m_0 - l n_0) t^d \tau(z) \right) \right) &= 0,
\end{align*}
\]
where \( \sigma(z) := \frac{d}{dz} \sigma \) and \( \tau(z) = \frac{d}{dz} \tau \). Set \( K(z) := n_0 \sigma(z) \tau(z) + m_0 \sigma(z) \tau(z) \), which is called the plot function. Then the above equations hold precisely when
\[
\begin{cases}
K(z) = 0, \quad \sigma(z) \neq 0, \quad \tau(z) \neq 0, \\
\zeta_n^{n_0} = \frac{\ln_0 - m_0}{m_0} t^d \tau(z). 
\end{cases}
\]
In particular, whether \((z, \zeta) \in X_{s,t}\) is a singularity does not depend on \(s\). Noting that every point \((z, \zeta)\) in \(X_{s,t}\) satisfies
\[
\sigma(z) \zeta_n^{m_0-n_0} \left( \zeta^{n_0} + t^d \tau(z) \right)^l - s = 0,
\]
s is given by
\[
s = \sigma(z) \zeta_n^{m_0-n_0} \left( \zeta^{n_0} + t^d \tau(z) \right)^l \\
= \sigma(z) \zeta_n^{m_0-n_0} \left( \zeta^{n_0} + \left( \frac{m_0}{\ln_0 - m_0} \zeta^{n_0} \right) \right)^l \\
= \left( \frac{\ln_0}{\ln_0 - m_0} \right)^l \sigma(z) \zeta^{m_0}_n.
\]
Hence:

**Lemma 8.1.** Fix \( t \neq 0 \). A point \((z, \zeta) \in N_0\) is a singularity of some subordinate fiber \(X_{s,t}\) of the deformation \(\pi_t : M_t \to \Delta_t\) if and only if the following condition is satisfied:
\[
\begin{cases}
K(z) = 0, \quad \sigma(z) \neq 0, \quad \tau(z) \neq 0, \\
\zeta_n^{n_0} = \frac{\ln_0 - m_0}{m_0} t^d \tau(z). 
\end{cases}
\]
In this case, the following holds:
\[
s = \left( \frac{\ln_0}{\ln_0 - m_0} \right)^l \sigma(z) \zeta^{m_0}_n.
\]

\(\text{Note that } K(z) \text{ is } \text{not} \text{ a function on } \Theta_0 \text{ but a meromorphic section of a line bundle } N_0^{\otimes(n-m)} \otimes \Omega^1_{\Theta_0} \text{ on } \Theta_0, \text{ where } \Omega^1_{\Theta_0} \text{ is the cotangent bundle of } \Theta_0.\)
We call a zero \( \alpha \) of the plot function \( K(z) \) an **essential zero** if \( \sigma(\alpha) \neq 0 \) and \( \tau(\alpha) \neq 0 \). For an essential zero \( \alpha \) of \( K(z) \), Lemma 8.1 implies that \((\alpha, \beta) \in N_0\) is a singularity of a subordinate fiber \( X_{s,t} \) if and only if

\[
\begin{aligned}
\beta_{n_0} &= \frac{\ln{a} - m_0}{m_0} t^d \tau(\alpha), \\
\frac{s}{\ln{a} - m_0} &= \left( \frac{\ln{a} - m_0}{m_0} \right) t^d \sigma(\alpha) \beta_{m_0}.
\end{aligned}
\]

Eliminating \( \beta \), we have

\[
s_n = \left( \frac{\ln{a} - m_0}{m_0} \right)^{m_0} \frac{\ln{a} - m_0}{m_0} t^d \sigma(\alpha) \beta_{m_0},
\]

where \( m_0 \) and \( n_0 \) are the relatively prime integers satisfying \( \frac{n_0}{m_0} = \frac{n_0}{m_0} \). This equation for \( s \) has \( n_0 \) solutions, say, \( s_1, s_2, \ldots, s_{n_0} \). Observe that the equation

\[
\beta_{n_0} = \frac{\ln{a} - m_0}{m_0} t^d \tau(\alpha)
\]

for \( \beta \) has \( n_0 \) solutions, say \( \beta_1, \beta_2, \ldots, \beta_{n_0} \). Then \( n_0/\bar{n}_0 (= \gcd(m_0, n_0)) \) points among \((\alpha, \beta_1), (\alpha, \beta_2), \ldots, (\alpha, \beta_{n_0})\) lie on one of the subordinate fibers \( X_{s_1, t}, X_{s_2, t}, \ldots, X_{s_{n_0}, t} \).

**Lemma 8.2.** Let \( \alpha \) be an essential zero of \( K(z) \). Then:

(a): There exist \( n_0 \) subordinate fibers \( X_{s_1, t}, X_{s_2, t}, \ldots, X_{s_{n_0}, t} \) that have singularities with \( z \)-coordinate \( \alpha \). (In fact, \( s_1, s_2, \ldots, s_{n_0} \) are given as the solutions of the following equation for \( s \):

\[
s_n = \left( \frac{\ln{a} - m_0}{m_0} \right)^{m_0} \frac{\ln{a} - m_0}{m_0} t^d \sigma(\alpha) \beta_{m_0}.
\]

(b): Moreover the number of such singularities on each of these subordinate fibers is \( n_0/\bar{n}_0 \).

Next we write \( K(z) = \sigma \tau \omega \), where \( \omega(z) := \frac{d \log(\sigma_{n_0} \tau_{m_0})}{dz} \). Here \( \omega \) is a meromorphic section of the cotangent bundle \( \Omega^1_{\Theta_0} \) on \( \Theta_0 \). Recall the assumption that the subbranches \( \overline{br}^{(j)}_p (j = 1, 2, \ldots, v) \) are proportional (so \( m_0 n_1^{(j)} - m_1^{(j)} n_0 = 0 \)) and the others are not. Then \( \omega(z) \) is holomorphic at \( p^{(1)}, p^{(2)}, \ldots, p^{(v)} \), whereas \( \omega(z) \) has a pole of order 1 at \( p^{(v+1)}, p^{(v+2)}, \ldots, p^{(h)} \). On the other hand, \( \omega(z) \) has a pole of order 1 at \( q_1, q_2, \ldots, q_k \) (which are zeros of the core section \( \tau \)). Moreover

\[
\begin{aligned}
K(z) &= 0, \\
\sigma(z) &= 0, \quad \tau(z) &= 0
\end{aligned} \iff \begin{aligned}
\omega(z) &= 0, \\
z \notin \{ p^{(1)}, p^{(2)}, \ldots, p^{(v)} \}
\end{aligned}
\]
Lemma 8.3 ([Ta3] Lemma 21.3.5). Let $g_0$ denote the genus of the core $\Theta_0$. Then

$$\sum_{K(\alpha)=0, \sigma(\alpha) \neq 0, \tau(\alpha) \neq 0} \text{ord}_\alpha(K(z)) = (h-v) + k + (2g_0 - 2) - \sum_{j=1}^{v} \text{ord}_{p(j)}(\omega).$$

We set $\chi := (h-v) + k + (2g_0 - 2) - \sum_{j=1}^{v} \text{ord}_{p(j)}(\omega)$, which is called the core invariant.

Corollary 8.4. Let $\kappa$ denote the number of essential zeros of $K(z)$. Then we have

$$\kappa \leq \chi,$$

where the equality holds precisely when the order of any essential zero of $K(z)$ equals 1.

Proof. For any essential zero $\alpha$ of the plot function $K(z)$ we have

$$\text{ord}_\alpha(K(z)) \geq 1,$$

thus

$$\sum_{K(\alpha)=0, \sigma(\alpha) \neq 0, \tau(\alpha) \neq 0} \text{ord}_\alpha(K(z)) \geq \kappa.$$

From Lemma 8.3, the left hand side of this inequality is equal to the core invariant $\chi$, which confirms the assertion.

Q.E.D.

Let $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ be the essential zeros of $K(z)$, where $\kappa$ is the number of essential zeros of $K(z)$. By Lemma 8.2 (a), for each $\alpha_i$, there exist $\bar{n}_0$ subordinate fibers that have singularities with $z$-coordinate $\alpha_i$, and their singular values are given by

$$s^{\bar{n}_0} = \left(\frac{ln_0}{ln_0 - m_0}\right)^{\bar{n}_0} \left(\frac{ln_0 - m_0}{m_0}\right)^{\bar{m}_0} t^d \sigma(\alpha_i)^{\bar{n}_0} \tau(\alpha_i)^{\bar{m}_0}.$$

Thus, if $\alpha_i$ and $\alpha_j$ satisfy

$$\sigma(\alpha_i)^{\bar{n}_0} \tau(\alpha_i)^{\bar{m}_0} = \sigma(\alpha_j)^{\bar{n}_0} \tau(\alpha_j)^{\bar{m}_0},$$

then the singularities with $z$-coordinate $\alpha_i$ and $\alpha_j$ lie on the same subordinate fiber. We denote by $\bar{k}$ the number of the distinct values of the set $\{\sigma(\alpha_i)^{\bar{n}_0} \tau(\alpha_i)^{\bar{m}_0} : i = 1, 2, \ldots, \kappa\}$. Then for a fixed $t \neq 0$, the deformation $\pi_t : M_t \rightarrow \Delta_t$ has exactly $\bar{n}_0 \bar{k}$ subordinate fibers that have singularities near the core. This result together with Lemma 8.2 and Corollary 8.4 confirms the following:
Proposition 8.5. Let us consider the deformation $\pi_t : M_t \to \Delta_t$ of $\pi : M \to \Delta$ for a fixed $t \neq 0$. Then we have the following.

(a): \( \left( \text{The number of subordinate fibers in } M_t \text{ that have singularities near } \Theta_0 \right) \leq \bar{n}_0 \chi. \)
Here the equality holds precisely when the order of any essential zero equals 1 and $\bar{k} = k$.

(b): \( \left( \text{The number of singularities near } \Theta_0 \text{ on each subordinate fiber in } M_t \right) \leq \frac{n_0}{\bar{n}_0} \chi. \)
Here the equality holds precisely when the order of any essential zero equals 1 and $\bar{k} = 1$.

§9. Constraints from the numbers of singularities

In this section, we show two useful lemmas which give us the number of the subordinate fibers and that of their singularities. See Lemmas 9.2 and 9.4.

Let $\pi : M \to \Delta$ be a linear degeneration of complex curves with a stellar singular fiber $X_0 = m_0 \Theta_0 + \sum_{j=1}^h Br^{(j)}$. Suppose that $X_0$ has a simple crust $Y = n_0 \Theta_0 + \sum_{j=1}^h Br^{(j)}$ of with barking multiplicity $l$. For brevity, we assume that the subbranches $Br^{(1)}, Br^{(2)}, \ldots, Br^{(v)}$ are proportional and the others are not (so $v$ is the number of the proportional subbranches). Let $\Psi : M \to \Delta \times \Delta^\dag$ be a barking family of $\pi : M \to \Delta$ associated with $Y$. We define the core invariant of $Y$ as

$$\chi := (h - v) + k + (2g_0 - 2) - \sum_{j=1}^v \text{ord}_{p(j)}(\omega),$$

where $g_0$ is the genus of the core $\Theta_0$ and $\omega := \frac{d}{dz} \log(\sigma^{n_0} \tau^{m_0})$.

First we assume that $Y$ has no proportional subbranches. Since $v = 0$, we have $\chi = h + k + (2g_0 - 2)$. Then Lemma 7.1 ensures that the subordinate fibers have singularities only near the core.

Lemma 9.1. Suppose that $Y$ has no proportional subbranch. Set $c := \gcd(m_0, n_0)$ and $\bar{n}_0 := n_0 / c$. If $\chi = 1$, then for a fixed $t \neq 0$, we have the following.

(a): $\pi_t : M_t \to \Delta_t$ has exactly $\bar{n}_0$ subordinate fibers.

(b): Each subordinate fiber of $\pi_t : M_t \to \Delta_t$ has $c$ singularities.

(c): The number of singularities of all the subordinate fibers of $\pi_t : M_t \to \Delta_t$ is $n_0$.

Proof. First note that the plot function $K(z)$ has at least one essential zero. Otherwise, from Lemma 8.1, there would exist no singularities
around the core, which implies that \( \pi_t : M_t \to \Delta_t \) has no subordinate fibers. Accordingly

\[
1 \leq \text{(the number of essential zeros of } K(z)) \text{.}
\]

On the other hand, Corollary 8.4 states that

\[
\text{(the number of essential zeros of } K(z)) \leq \chi.
\]

From the assumption \( \chi = 1 \), we obtain

\[
\text{(the number of essential zeros of } K(z)) = 1.
\]

Namely \( K(z) \) has exactly one zero of order 1. By Proposition 8.5, we have

\[
\text{(the number of subordinate fibers of } \pi_t : M_t \to \Delta_t) = \tilde{n}_0,
\]

\[
\text{(the number of singularities on each subordinate fiber)} = c,
\]

confirming (a) and (b).

(c) clearly follows from (a) and (b). Q.E.D.

In particular:

**Lemma 9.2.** Suppose that (i) \( \Theta_0 \) is a complex projective line, (ii) \( X_0 \) has three branches, (iii) the core section \( \tau \) has no zero and (iv) \( Y \) has no proportional subbranches. Set \( c := \gcd(m_0, n_0) \) and \( \tilde{n}_0 := n_0/c \).

Then for a fixed \( t \neq 0 \), we have the following.

(a): \( \pi_t : M_t \to \Delta_t \) has exactly \( \tilde{n}_0 \) subordinate fibers.

(b): Each subordinate fiber of \( \pi_t : M_t \to \Delta_t \) has \( c \) singularities.

(c): The number of singularities of all the subordinate fibers of \( \pi_t : M_t \to \Delta_t \) is \( \tilde{n}_0 \).

**Proof.** By assumption, we have \( g_0 = 0, h = 3, k = 0 \), and so \( \chi = 1 \). Hence Lemma 9.1 confirms the assertion. Q.E.D.

**Remark 9.3.** By Lemma 2.6, we can restate the condition (iii) of Lemma 9.2 as “\( r_0 = r'_0 \),” where \( r_0 := \sum_{j=1}^{h} m_1^{(j)}/m_0 \) and \( r'_0 := \sum_{j=1}^{h} n_1^{(j)}/n_0 \).

Next we assume that \( Y \) has a proportional subbranch.

**Lemma 9.4.** Suppose that (i) \( \Theta_0 \) is a complex projective line, (ii) \( X_0 \) has three branches, (iii) the core section \( \tau \) has no zero and (iv) \( Y \) has a proportional subbranch \( \overline{br}^{(1)} = n_0 \Delta_0 + n_1 \Theta_1 + n_2 \Theta_2 + \cdots + n_\lambda \Theta_\lambda \) of \( \overline{Br}^{(1)} \). Then \( \overline{br}^{(1)} \) is the unique proportional subbranch of \( Y \) (that is, \( v = 1 \)). Moreover for a fixed \( t \neq 0 \), we have the following.
(a): \( \pi_t : M_t \to \Delta_t \) has exactly \( \tilde{n}_\lambda \) subordinate fibers.
(b): Each subordinate fiber of \( \pi_t : M_t \to \Delta_t \) has \( c \) singularities.
(c): The number of singularities of all the subordinate fibers of \( \pi_t : M_t \to \Delta_t \) is \( n_\lambda \).

Here \( c := \gcd(m_\lambda, n_\lambda) \) and \( \tilde{n}_\lambda := n_\lambda / c \).

**Proof.** By assumption, we have \( g_0 = 0 \), \( h = 3 \), \( k = 0 \). Thus

\[
\chi = 1 - v - \sum_{j=1}^{v} \text{ord}_{p(j)}(w),
\]

so

\[
\chi + \sum_{j=1}^{v} (\text{ord}_{p(j)}(w) + 1) = 1.
\]

Recall that \( \omega(z) \) is holomorphic at \( p(j) \) for \( j = 1, 2, \ldots, v \), that is, \( \text{ord}_{p(j)}(\omega) \geq 0 \). Noting that \( \chi \geq 0 \) and \( v \geq 1 \), we deduce that \( \chi = 0 \), \( v = 1 \) and \( \text{ord}_{p(1)}(\omega) = 0 \). Hence \( \overline{br}^{(1)} \) is the unique proportional sub-branch. Since \( \chi = 0 \), from Proposition 8.5, every subordinate fiber of \( \pi_t : M_t \to \Delta \) has no singularities near the core \( \Theta_0 \). Therefore Proposition 7.4 confirms (a), (b) and (c).

Q.E.D.

§10. Determination of the subordinate fibers, 2

We now determine the subordinate fibers of the remaining barking families.

We first consider barking families whose simple crust has no proportional subbranches. In the barking family \([III.2]\), \( III \) is deformed to \( I_1 \):

\[
III \xrightarrow{\text{bark}} I_1.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \( \{II\} \), (ii) \( \{I_2\} \), and (iii) \( \{I_1, I_1\} \). Note that the simple crust for this family has no proportional subbranches. See \([III.2]\) of the list in Section 12. Applying Lemma 9.2, since \( c = 2 \) and \( \tilde{n}_0 = 1 \), we deduce that there appears exactly one subordinate fiber and it has two singularities. This condition is satisfied only for the case (ii). Hence:

**Proposition 10.1.** The barking family \([III.2]\) splits the singular fiber \( III \) as follows:

\[
III \to I_1 + I_2,
\]

where \( I_1 \) is the main fiber and \( I_2 \) is a subordinate fiber.
Similarly:

**Proposition 10.2.** The barking family \([III^*.2]\) splits the singular fiber \(III^*\) as follows:

\[
III^* \rightarrow I_1^* + I_2,
\]

where \(I_1\) is the main fiber and \(I_2\) is a subordinate fiber.

In the barking family \([IV.2]\), \(IV\) is deformed to \(I_2\):

\[
IV \xrightarrow{\text{bark}} I_2.
\]

By Lemma 3.4, the set of subordinate fibers is one of (i) \{\(I\)\}, (ii) \{\(I_2\)\}, and (iii) \{\(I_1, I_1\)\}. Applying Lemma 9.2, since \(c = 1\) and \(\bar{n}_0 = 2\), we deduce that there appear two subordinate fibers and each of them has one singularity. This condition is satisfied only for the case (iii). Hence:

**Proposition 10.3.** The barking family \([IV.2]\) splits the singular fiber \(IV\) as follows:

\[
IV \rightarrow I_2 + I_1 + I_1,
\]

where \(I_2\) is the main fiber and the two \(I_1\) are subordinate fibers.

Similarly:

**Proposition 10.4.** The barking family \([IV^*.2]\) splits the singular fiber \(IV^*\) as follows:

\[
IV^* \rightarrow I_0^* + I_1 + I_1,
\]

where \(I_0^*\) is the main fiber and the two \(I_1\) are subordinate fibers.

In the barking family \([III^*.4]\), \(III^*\) is deformed to \(I_0^*\):

\[
III^* \xrightarrow{\text{bark}} I_0^*.
\]

By Lemma 3.5, the set of subordinate fibers is one of (i) \{\(III\)\}, (ii) \{\(I_3\)\}, (iii) \{\(II, I_1\)\}, (iv) \{\(I_2, I_1\)\}, and (v) \{\(I_1, I_1, I_1\)\}. Applying Lemma 9.2, since \(c = 1\) and \(\bar{n}_0 = 3\), we deduce that there appear three subordinate fibers and each of them has one singularity. This condition is satisfied only for the case (v). Hence:

**Proposition 10.5.** The barking family \([III^*.4]\) splits the singular fiber \(III^*\) as follows:

\[
III^* \rightarrow I_0^* + I_1 + I_1 + I_1,
\]

where \(I_0^*\) is the main fiber and the three \(I_1\) are subordinate fibers.
Similarly:

**Proposition 10.6.** The barking family \([III^*.5]\) splits the singular fiber \(III^*\) as follows:

\[ III^* \rightarrow I_6 + I_1 + I_1 + I_1, \]

where \(I_6\) is the main fiber and the three \(I_1\) are subordinate fibers.

In the barking family \([II^*.4]\), \(II^*\) is deformed to \(I_5\):

\[ II^* \xrightarrow{\text{bark}} I_5. \]

By Lemma 3.6, the sum of the Euler characteristics of the subordinate fibers is 5. Applying Lemma 9.2, since \(c = 1\) and \(\tilde{n}_0 = 5\), we deduce that there appear five subordinate fibers and each of them has one singularity. Hence:

**Proposition 10.7.** The barking family \([II^*.4]\) splits the singular fiber \(II^*\) as follows:

\[ II^* \rightarrow I_5 + I_1 + I_1 + I_1 + I_1, \]

where \(I_5\) is the main fiber and the five \(I_1\) are subordinate fibers.

In the following cases, the simple crust has a proportional subbranch.

In the barking family \([II^*.2]\), \(II^*\) is deformed to \(IV^*\):

\[ II^* \xrightarrow{\text{bark}} IV^*. \]

By Lemma 3.4, the set of subordinate fibers is one of (i) \(\{II\}\), (ii) \(\{I_2\}\), and (iii) \(\{I_1, I_1\}\). Note that the simple crust for this family has a proportional subbranch of length 2. See \([II^*.2]\) of the list in Section 12. Applying Lemma 9.4, since \(c = 1\) and \(\tilde{n}_2 = 1\), we deduce that there appears exactly one subordinate fiber and it has one singularity. This condition is satisfied only for the case (i). Hence:

**Proposition 10.8.** The barking family \([II^*.2]\) splits the singular fiber \(II^*\) as follows:

\[ II^* \rightarrow IV^* + II, \]

where \(IV^*\) is the main fiber and \(II\) is a subordinate fiber.

In the barking family \([II^*.3]\), \(II^*\) is deformed to \(I_2^*\):

\[ II^* \xrightarrow{\text{bark}} I_2^*. \]
By Lemma 3.4, the set of subordinate fibers is one of (i) \{II\}, (ii) \{I_2\}, and (iii) \{I_1, I_1\}. Note that the simple crust for this family has a proportional subbranch of length 1. See [II*.3] of the list in Section 12. Applying Lemma 9.4, since \(c = 1\) and \(\bar{n}_1 = 2\), we deduce that there appear two subordinate fibers and each of them has one singularity. This condition is satisfied only for the case (iii). Hence:

**Proposition 10.9.** The barking family [II*.3] splits the singular fiber \(II^*\) as follows:

\[ II^* \rightarrow I_2^* + I_1 + I_1, \]

where \(I_2^*\) is the main fiber and the two \(I_1\) are subordinate fibers.

We summarize Propositions 3.3, 6.1–6.5, 10.1–10.9 as follows:

**Theorem 10.10.** Each barking family in Takamura's list (1.1) except [III*.8], [IV.3], [IV.4], [I*_0.2] splits the singular fiber as follows:

\[
\begin{align*}
[II.1] & \ II \rightarrow I_1 + I_1 \\
[II.2] & \ II \rightarrow I_1 + I_1 \\
[II*.1] & \ II^* \rightarrow III^* + I_1 \\
[II*.2] & \ II^* \rightarrow IIV^* + II \\
[II*.3] & \ II^* \rightarrow I_2^* + I_1 + I_1 \\
[II*.4] & \ II^* \rightarrow I_5 \\
& \quad + I_1 + I_1 + I_1 + I_1 + I_1 \\
[II*.5] & \ II^* \rightarrow I_3^* + I_1 \\
[II*.6] & \ II^* \rightarrow I_3^* + I_1 \\
[II*.7] & \ II^* \rightarrow I_8 + I_1 + I_1 + I_1 \\
[II*.8] & \ II^* \rightarrow III^* + I_1 \\
[II*.9] & \ II^* \rightarrow III^* + I_1 \\
[III.1] & \ III \rightarrow I_2 + I_1 + I_1 \\
[III*.1] & \ III^* \rightarrow I_2^* + I_1 + I_1 \\
[I_0*.1] & \ I_0^* \rightarrow I_0 + I_1 + I_1 \\
[I_n*.1] & \ I_n^* \rightarrow I_n + I_1 + I_1 \\
[I_n*.2] & \ I_n^* \rightarrow I_{n+4} + I_1 + I_1.
\end{align*}
\]

§11. Supplement: Monodromy decompositions

In this section, we give decompositions of the standard monodromy matrices corresponding to the splittings of the singular fibers induced
by Takamura’s barking families. Recall that $SL(2, \mathbb{Z})$ is generated by

$$s_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$ 

Note that, since $s_0 s_2 s_0 = s_2 s_0 s_2$, we have

$$s_2 = (s_0 s_2) s_0 (s_0 s_2)^{-1}.$$

**Decomposition of $A_{II}$.** The standard monodromy matrix of $II$ is $A_{II} = s_0 s_2$. $A_{II}$ is decomposed into two conjugacies of $A_{I_1}$ as follows:

$$A_{II} = s_0 s_2 = A_{I_1} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1}.$$

In fact, the splitting $II \rightarrow I_1 + I_1$ occurs in the barking families $[II.1]$ and $[II.2]$.

**Decomposition of $A_{III}$.** The standard monodromy matrix of $III$ is $A_{III} = s_0 s_2 s_0$. $A_{III}$ is decomposed into $A_{I_2}$ and a conjugacy of $A_{I_1}$:

$$A_{III} = s_0 s_2 s_0 = s_0^2 (s_0^{-1} s_2 s_0) = A_{I_2} \cdot s_2 A_{I_1} s_2^{-1}.$$

In fact, the splitting $III \rightarrow I_2 + I_1$ occurs in the barking families $[III.1]$, $[III.2]$, $[III.3]$.

$A_{III}$ has other monodromy decompositions as follows (but we have not found barking families that admit the corresponding splittings):

$$A_{III} = (s_0 s_2) s_0 = A_{II} \cdot A_{I_1},$$

$$\quad (III \rightarrow II + I_1)$$

$$A_{III} = s_0 s_2 s_0 = A_{I_1} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1} \cdot A_{I_1},$$

$$\quad (III \rightarrow I_1 + I_1 + I_1)$$

**Decomposition of $A_{IV}$.** The standard monodromy matrix of $IV$ is $A_{IV} = s_0 s_2 s_0 s_2$. $A_{IV}$ is decomposed into $A_{I_3}$ and a conjugacy of $A_{I_1}$:

$$A_{IV} = s_0 s_2 s_0 s_2 = s_0^3 (s_0^{-1} s_2 s_0)$$

$$= A_{I_3} \cdot s_2 A_{I_1} s_2^{-1}.$$

In fact, the splitting $IV \rightarrow I_3 + I_1$ occurs in the barking family $[IV.1]$. $A_{IV}$ has another monodromy decomposition

$$A_{IV} = s_0 s_2 s_0 s_2 = s_0^2 s_2 s_0$$

$$= A_{I_2} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1} \cdot A_{I_1},$$
while the barking family [IV.2] induces the splitting \( IV \to I_2 + I_1 + I_1 \).

We have other monodromy decompositions of \( A_{IV} \) as follows (but we have not found splitting families that admit the corresponding splittings):

\[
A_{IV} = (s_0 s_2 s_0) s_2 = A_{III} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1},
\]
(\( IV \to III + I_1 \))

\[
A_{IV} = (s_0 s_2)^2 = A_{II} \cdot A_{II},
\]
(\( IV \to II + II \))

\[
A_{IV} = (s_0 s_2) s_0 s_2 = A_{II} \cdot A_{I} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1},
\]
(\( IV \to II + I_1 + I_1 \))

\[
A_{IV} = s_0^2 s_2 (s_0 s_2) s_2^{-1} = A_{I_2} \cdot s_2 A_{II} s_2^{-1},
\]
(\( IV \to I_2 + II \))

\[
A_{IV} = s_0 s_2 s_0 s_2
\]

\[
= A_{I} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1} \cdot A_{I} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1}.
\]
(\( IV \to I_1 + I_1 + I_1 + I_1 \))

**Decomposition of \( A_{II^*} \).** The standard monodromy matrix of \( II^* \) is \( A_{II^*} = (s_0 s_2)^5 \). \( A_{II^*} \) is decomposed into \( A_{III^*} \) and a conjugacy of \( A_{I} \):

\[
A_{II^*} = (s_0 s_2)^4 s_0 s_2 = A_{III^*} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1}.
\]

In fact, the splitting \( II^* \to III^* + I_1 \) occurs in the barking families \([II^*.1],[II^*.8],[II^*.9]\).

\( A_{II^*} \) is also decomposed into \( A_{I^*} \) and a conjugacy of \( A_{I_1} \):

\[
A_{II^*} = (s_0 s_2)^3 s_0 s_2 s_0 s_2 = (s_0 s_2)^3 s_0^3 (s_0^{-1} s_2 s_0) = A_{I_2^*} \cdot s_2 A_{I} s_2^{-1}.
\]

Note that the barking families \([II^*.5] \) and \([II^*.6] \) induce the splitting \( II^* \to I_3^* + I_1 \).

We have other monodromy decompositions of \( A_{II^*} \) which respectively correspond to the splittings induced by Takamura's barking families as follows:

\[
A_{II^*} = (s_0 s_2)^4 (s_0 s_2) = A_{IV^*} \cdot A_{II},
\]
(\( [II^*.2] II^* \to IV^* + II \))

\[
A_{II^*} = (s_0 s_2)^3 s_0^2 (s_0^{-1} s_2 s_0) s_2 = A_{I_2^*} \cdot s_2 A_{I} s_2^{-1} \cdot (s_0 s_2) A_{I} (s_0 s_2)^{-1},
\]
(\( [II^*.3] II^* \to I_2^* + I_1 + I_1 \))
\[ A_{III^*} = s_0^5 (s_0^{-1} s_2 s_0^2) s_0 s_2 s_2 s_0 \]
\[ = A_{I_5} \cdot s_2 A_{I_1} s_2^{-1} \cdot A_{I_1} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1} \cdot A_{I_1}, \]
\[ (III*.4) III^* \rightarrow I_5 + I_1 + I_1 + I_1 + I_1 \]
\[ A_{II^*} = s_0^8 (s_0^{-2} s_2 s_0^2) (s_0^{-1} s_2^2 s_0^2) (s_0^{-1} s_2^2 s_0) \]
\[ = A_{I_8} \cdot (s_0^{-1} s_2) A_{I_1} (s_0^{-1} s_2)^{-1} \cdot (s_0^{-1} s_2^2) A_{I_1} (s_0^{-1} s_2^2)^{-1}, \]
\[ (III*.7) II^* \rightarrow I_8 + I_1 + I_1 \]

**Decomposition of** \( A_{III^*}. \) **The standard monodromy matrix of** \( III^* \) **is** \( A_{III^*} = (s_0 s_2)^4 s_0.\) **\( A_{III^*} \) is decomposed into** \( A_{IV^*} \) **and** \( A_{I_1}:**

\[ A_{III^*} = (s_0 s_2)^4 s_0 = A_{IV^*} \cdot A_{I_1}. \]

In fact, the splitting \( III^* \rightarrow IV^* + I_1 \) occurs in the barking families [III*.1] and [III*.9].

\( A_{III^*} \) is also decomposed into \( A_{I_2} \) and a conjugacy of \( A_{I_1}:**

\[ A_{III^*} = (s_0 s_2)^3 s_0 s_2 s_0 = (s_0 s_2)^3 s_0^2 (s_0^{-1} s_2 s_0) \]
\[ = A_{I_2} \cdot s_2 A_{I_1} s_2^{-1}. \]

**Note** that the barking families [III*.3] and [III*.6] induce the splitting \( III^* \rightarrow I_2^* + I_1.\)

We have other monodromy decompositions of \( A_{III^*} \) which respectively correspond to the splittings induced by Takamura’s barking families as follows:

\[ A_{III^*} = s_2^{-1} (s_0 s_2)^3 s_0 s_2 s_0^2 = s_2^{-1} A_{I_1} s_2 \cdot A_{I_2}, \]
\[ (III*.2) III^* \rightarrow I_1^* + I_2 \]
\[ A_{III^*} = (s_0 s_2)^3 s_0 s_2 s_0 = A_{I_5} \cdot A_{I_1} \cdot (s_0 s_2) A_{I_1} (s_0 s_2)^{-1} \cdot A_{I_1}, \]
\[ (III*.4) III^* \rightarrow I_0^* + I_1 + I_1 + I_1 \]
\[ A_{III^*} = s_0^5 (s_0^{-3} s_2 s_0^3) (s_0^{-1} s_2 s_0) \]
\[ = A_{I_6} \cdot (s_0^{-2} s_2) A_{I_1} (s_0^{-2} s_2)^{-1} \cdot s_2 A_{I_1} s_2^{-1} \cdot s_2 A_{I_1} s_2^{-1}, \]
\[ (III*.5) III^* \rightarrow I_6 + I_1 + I_1 + I_1 \]
\[ A_{III^*} = s_0^7 (s_0^{-5} s_2 s_0^5) (s_0^{-2} s_2 s_0^2) \]
\[ = A_{I_7} \cdot (s_0^{-4} s_2) A_{I_1} (s_0^{-4} s_2)^{-1} \cdot (s_0^{-1} s_2) A_{I_1} (s_0^{-1} s_2)^{-1}. \]
\[ (III*.7) III^* \rightarrow I_7 + I_1 + I_1 \]
Decomposition of $A_{IV^*}$. The standard monodromy matrix of $IV^*$ is $A_{IV^*} = (s_0s_2)^4$. $A_{IV^*}$ is decomposed into $A_{I_1^+}$ and a conjugacy of $A_{I_1}$:

$$A_{IV^*} = (s_0s_2)^3 s_0s_2 = A_{I_1^+} \cdot (s_0s_2)A_{I_1} (s_0s_2)^{-1}.$$ 

In fact, the splitting $IV^* \to I_1^* + I_1$ occurs in the barking families $[IV^*\cdot 1]$ and $[IV^*\cdot 4]$.

We have other monodromy decompositions of $A_{IV^*}$ which respectively correspond to the splittings induced by Takamura's barking families as follows:

$$A_{IV^*} = (s_0s_2)^3 s_0s_2 = A_{I_1^+} \cdot (s_0s_2)A_{I_1} (s_0s_2)^{-1},$$

$([IV^*\cdot 2] IV^* \to I_0^* + I_1 + I_1)$

$$A_{IV^*} = s_0^6 (s_0^{-4}s_2s_0^4)(s_0^{-1}s_2s_0) = A_{I_0} \cdot (s_0^{-3}s_2)A_{I_1} (s_0^{-3}s_2)^{-1} \cdot s_2A_{I_1} s_2^{-1}.$$ 

$([IV^*\cdot 3] IV^* \to I_0 + I_1 + I_1)$

Decomposition of $A_{I_n^*} (n \geq 0)$. The standard monodromy matrix of $I_n^* (n \geq 0)$ is $A_{I_n^*} = (s_0s_2)^3 s_0^n$. $A_{I_n^*}$ is decomposed into $A_{I_{n+4}}$ and two conjugacies of $A_{I_1}$ as follows:

$$A_{I_{n+4}} = s_0s_2s_0s_2s_0s_2s_0^n = s_0^2 (s_0s_2s_0^{-2}) s_0^4s_0^n = (s_0s_2)A_{I_1} (s_0s_2)^{-1} \cdot (s_0^3s_2)A_{I_1} (s_0^3s_2)^{-1} \cdot A_{I_{n+4}}.$$

In fact, the splitting $IV^* \to I_1^* + I_1$ occurs in the barking families $[I_n^*\cdot 1]$ and $[I_n^*\cdot 2]$.

For $n \geq 1$, note that the barking family $[I_n^*\cdot 1]$ induces the splitting $I_n^* \to I_{n-1}^* + I_1$. Then $A_{I_n^*}$ is also decomposed into $A_{I_{n-1}}$ and $A_{I_1}$ as follows:

$$A_{I_n^*} = (s_0s_2)^3 s_0^{n-1}s_0 = A_{I_{n-1}} \cdot A_{I_1}.$$

§12. Appendix: Takamura's list for genus $g = 1$

In [Ta3], for genera up to 5, Takamura made a list of barking families — precisely speaking, a list of simple crusts (and weighted crustal sets) for constructing barking families — which enables him to show that a degeneration is absolutely atomic if and only if its singular fiber is either a Lefschetz fiber or a multiple of a smooth curve. Recall that in a barking family, for a fixed $t \neq 0$, the singular fiber $X_{0,t}$ over the origin is called the main fiber and other singular fibers $X_{s,t} (s \neq 0)$ are called subordinate fibers. As we saw in Section 2, the main fibers of barking
families are explicitly described. In this paper, when the original singular fiber $X_0$ is deformed to the main fiber $X_{0,t}$, we express $X_0 \xrightarrow{\text{bark}} X_{0,t}$.

For the convenience of the reader, we provide Takamura’s list of barking families for genus 1 with figures of the singular fibers:

$[II.1] \quad II \xrightarrow{\text{bark}} I_1$

$[II.2] \quad II \xrightarrow{\text{bark}} I_1$

$[II*.1] \quad II^* \xrightarrow{\text{bark}} III^*$
Singular fibers in barking families of degenerations

\[ [II^* \cdot 2] \quad II^* \xrightarrow{\text{bark}} IV^* \]

\[
\begin{array}{c}
X_0 \\
3 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \\
Y \\
1 \quad 2 \quad 3 \quad 3 \quad 3
\end{array}
\]

\[ [II^* \cdot 3] \quad II^* \xrightarrow{\text{bark}} I_2^* \]

\[
\begin{array}{c}
X_0 \\
3 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \\
Y \\
2 \quad 4 \quad 4 \quad 4
\end{array}
\]

\[ [II^* \cdot 4] \quad II^* \xrightarrow{\text{bark}} I_5^* \]

\[
\begin{array}{c}
X_0 \\
3 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \\
Y \\
1 \quad 3 \quad 2 \quad 5 \quad 5
\end{array}
\]
\[ [II^* .5] \ II^* \xrightarrow{\text{bark}} I_3^* \]

\[ X_0 \begin{array}{c} 2 \\ 4 \\ 3 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \xrightarrow{Y=2} \begin{array}{c} 2 \\ 4 \\ 3 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \]

\[ [II^* .6] \ II^* \xrightarrow{\text{bark}} I_3^* \]

\[ X_0 \begin{array}{c} 2 \\ 4 \\ 3 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \xrightarrow{Y=2} \begin{array}{c} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{array} \]

\[ [II^* .7] \ II^* \xrightarrow{\text{bark}} I_8 \]

\[ X_0 \begin{array}{c} 2 \\ 4 \\ 3 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \xrightarrow{Y=2} \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \]
Singular fibers in barking families of degenerations

\[ II^{*}.8 \] \( II^* \xrightarrow{\text{bark}} III^* \)

\( X_0 \)

\[ 2 \]

\[ 4 \]

\[ 3 \]
\[ 6 \]
\[ 5 \]
\[ 4 \]
\[ 3 \]
\[ 2 \]
\[ 1 \]

\( Y \)

\[ 1 \]
\[ 1 \]
\[ 1 \]
\[ 1 \]
\[ 1 \]

\( Q = 3 \)

\( X_{0,t} \)

\[ 2 \]

\[ 4 \]

\[ 3 \]
\[ 2 \]
\[ 1 \]
\[ 3 \]
\[ 2 \]
\[ 1 \]

\[ III^{*}.9 \] \( II^* \xrightarrow{\text{bark}} III^* \)

\( X_0 \)

\[ 2 \]

\[ 4 \]

\[ 3 \]
\[ 6 \]
\[ 5 \]
\[ 4 \]
\[ 3 \]
\[ 2 \]
\[ 1 \]

\( Y \)

\[ 1 \]
\[ 1 \]
\[ 1 \]
\[ 1 \]

\( Q = 4 \)

\[ III.1 \] \( III \xrightarrow{\text{bark}} I_2 \)

\( X_0 \)

\[ 1 \]

\[ 4 \]
\[ 2 \]

\( Y \)

\[ 1 \]
\[ 1 \]

\( X_{0,t} \)

\[ 1 \]
\[ 3 \]
\[ 2 \]

\[ III.2 \] \( III \xrightarrow{\text{bark}} I_1 \)

\( X_0 \)

\[ 1 \]

\[ 4 \]
\[ 2 \]

\( Y \)

\[ 1 \]
\[ 1 \]
\[ 2 \]

\( X_{0,t} \)

\[ 1 \]
\[ 2 \]
\[ 2 \]
[III.3] $III \xrightarrow{\text{bark}} I_2$

\[
\begin{array}{c}
X_0 \\
\begin{array}{c}
1 \\
4 \\
2 \\
1 \\
2 \\
1 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
X_{0,t} \\
\begin{array}{c}
1 \\
2 \\
1 \\
1 \\
2 \\
1 \\
\end{array}
\end{array}
\]

\[
Y \\
\begin{array}{c}
1 \\
2 \\
1 \\
\end{array}
\]

[III*.1] $III^* \xrightarrow{\text{bark}} IV^*$

\[
\begin{array}{c}
X_0 \\
\begin{array}{c}
2 \\
1 \\
3 \\
4 \\
3 \\
2 \\
1 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
X_{0,t} \\
\begin{array}{c}
2 \\
1 \\
2 \\
3 \\
2 \\
1 \\
1 \\
\end{array}
\end{array}
\]

\[
Y \\
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1 \\
\end{array}
\]

[III*.2] $III^* \xrightarrow{\text{bark}} I_1^*$

\[
\begin{array}{c}
X_0 \\
\begin{array}{c}
2 \\
1 \\
3 \\
4 \\
3 \\
2 \\
1 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
X_{0,t} \\
\begin{array}{c}
2 \\
1 \\
1 \\
2 \\
1 \\
1 \\
1 \\
\end{array}
\end{array}
\]

\[
Y \\
\begin{array}{c}
2 \\
2 \\
2 \\
2 \\
2 \\
\end{array}
\]

[III*.3] $III^* \xrightarrow{\text{bark}} I_2^*$

\[
\begin{array}{c}
X_0 \\
\begin{array}{c}
2 \\
1 \\
3 \\
4 \\
3 \\
2 \\
1 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
X_{0,t} \\
\begin{array}{c}
1 \\
2 \\
2 \\
2 \\
1 \\
1 \\
1 \\
\end{array}
\end{array}
\]

\[
Y \\
\begin{array}{c}
1 \\
1 \\
2 \\
2 \\
2 \\
\end{array}
\]
Singular fibers in barking families of degenerations

$[\text{III}^*.4] \text{ III}^* \xrightarrow{\text{bark}} I_0$

$X_0$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

$Y$

\[
\begin{array}{ccc}
3 & 3 & 3 \\
\end{array}
\]

$[\text{III}^*.5] \text{ III}^* \xrightarrow{\text{bark}} I_6$

$X_0$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

$Y$

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
\]

$[\text{III}^*.6] \text{ III}^* \xrightarrow{\text{bark}} I_2$

$X_0$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

$Y$

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

$q=2$

$[\text{III}^*.7] \text{ III}^* \xrightarrow{\text{bark}} I_7$

$X_0$

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

$Y_1$

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

$Y_2$

\[
\begin{array}{cccc}
1 & 2 & 2 & 2 \\
\end{array}
\]

$Y_3$

\[
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
\end{array}
\]

$Y_4$

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 \\
\end{array}
\]
[III*.8] $III^* \xrightarrow{\text{bark}} I_6$

$X_0$

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 3 & 2 \\
\end{array}
\xrightarrow{	ext{X}_0, t}
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

$Y_1$

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

$Y_2$

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

$Y_3$

\[
\begin{array}{cccc}
2 & 1 & 2 & 1 \\
\end{array}
\]

$Y_4$

\[
\begin{array}{cccc}
2 & 1 & 2 & 3 \\
\end{array}
\]

[III*.9] $III^* \xrightarrow{\text{bark}} IV^*$

$X_0$

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 3 & 2 \\
\end{array}
\xrightarrow{	ext{X}_0, t}
\begin{array}{cccccc}
1 & 2 & 3 & 1 & 3 & 2 \\
\end{array}
\]

$Y$

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\end{array}
\]

$\ell = 3$

[IV.1] $IV \xrightarrow{\text{bark}} I_3$

$X_0$

\[
\begin{array}{cccc}
1 & 1 & 3 & 1 \\
\end{array}
\xrightarrow{	ext{X}_0, t}
\begin{array}{cccc}
1 & 2 & 1 \\
\end{array}
\]

$Y$

\[
\begin{array}{cccc}
1 & 1 \\
\end{array}
\]

[IV.2] $IV \xrightarrow{\text{bark}} I_2$

$X_0$

\[
\begin{array}{cccc}
1 & 1 & 3 & 1 \\
\end{array}
\xrightarrow{	ext{X}_0, t}
\begin{array}{cccc}
1 & 1 \\
\end{array}
\]

$Y$

\[
\begin{array}{cccc}
1 & 2 \\
\end{array}
\]
Singular fibers in barking families of degenerations

[IV.3] \( IV \xrightarrow{\text{bark}} I_2 \)

\[
\begin{array}{c}
X_0 \quad 1 \\
1 \quad 3 \quad 1 \\
1 \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
X_{0,t} \quad 1 \\
1 \quad 2 \quad 1 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
Y \quad 1 \\
1 \\
1 \\
\end{array}
\]

[IV.4] \( IV \xrightarrow{\text{bark}} II \)

\[
\begin{array}{c}
X_0 \quad 1 \\
1 \quad 3 \quad 1 \\
1 \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
X_{0,t} \quad 1 \\
1 \quad 2 \quad 1 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
Y \quad 1 \\
1 \\
1 \\
\end{array}
\]

[IV*.1] \( IV^* \xrightarrow{\text{bark}} I_1^* \)

\[
\begin{array}{c}
X_0 \quad 1 \\
2 \\
1 \quad 2 \quad 3 \quad 2 \quad 1 \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
X_{0,t} \quad 1 \\
1 \quad 1 \quad 2 \quad 2 \quad 1 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
Y \quad 1 \\
1 \\
1 \\
\end{array}
\]

[IV*.2] \( IV^* \xrightarrow{\text{bark}} I_0^* \)

\[
\begin{array}{c}
X_0 \quad 1 \\
2 \\
1 \quad 2 \quad 3 \quad 2 \quad 1 \\
\end{array} \quad \rightarrow \quad \begin{array}{c}
X_{0,t} \quad 1 \\
1 \quad 1 \quad 2 \quad 1 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
Y \quad 2 \\
2 \\
\end{array}
\]
{$[IV^*.3] IV^* \xrightarrow{bark} I_6$}

\[
X_0 \quad 1 \quad 1 \\
2 \quad 1 \\
1 \quad 2 \quad 3 \quad 2 \quad 1 \\
\quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\]

{$[IV^*.4] IV^* \xrightarrow{bark} I_1^*$}

\[
X_0 \quad 1 \\
2 \\
1 \quad 2 \quad 2 \\
Y \\
1 \quad 1 \quad 1
\]

\[
X_0 \quad 1 \quad 1 \\
2 \\
1 \quad 2 \quad 3 \quad 2 \quad 1 \\
\quad 1 \quad 2 \quad 1 \quad 2 \quad 1
\]

\[ \alpha = 2 \]

{$[I_0^*.1] I_0^* \xrightarrow{bark} I_4$}

\[
X_0 \quad 1 \\
2 \quad 1 \\
1 \quad 2 \\
1 \quad 1 \quad 1 \quad 1 \\
\quad 1 \quad 1 \quad 1 \\
\quad 1
\]

{$[I_0^*.2] I_0^* \xrightarrow{bark} I_3$}

\[
X_0 \quad 1 \\
2 \quad 1 \\
1 \quad 2 \\
1 \quad 1 \quad 1 \quad 1 \\
\quad 1 \quad 1 \quad 1
\]

\[
Y \\
1 \quad 1 \quad 1
\]
Remark 12.1.  
(a): Takamura [Ta3] introduced not only a barking family associated with one simple crust (which we reviewed in Section 2) but also a barking family associated with several crusts. The latter is called a compound barking family. Note that the barking families [II* .6], [II* .7], [III* .7], [III* .8] are compound barking families.

(b): The singular fiber \( I_n^* \) \((n \geq 1)\) is constellar (constellation-shaped), that is, it is obtained by bonding stellar singular fibers. So \([I_n^*.1]\) and \([I_n^*.2]\) are barking families for constellar case rather than for stellar case. See [Ta3] for details.

(c): This list contains no barking families for a degeneration with the singular fiber \( mI_n \). In fact, for \( mI_n \) \((m \geq 2)\), we use another method to construct a splitting family, which splits \( mI_n \) into \( mI_{n-1} \) and \( I_1 \). See [Ta1] for details.

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Graduate School of Mathematics
Kyushu University
Motoooka 744, Nishi-ku
Fukuoka 819-0395
Japan
E-mail address: t-okuda@math.kyushu-u.ac.jp