ON BOTT-CHERN FORMS WITH APPLICATIONS TO DIFFERENTIAL $K$-THEORY

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Abstract. We use Chern-Weil theory for Hermitian holomorphic vector bundles with canonical connections for explicit computation of the Chern forms for trivial bundles with special non-diagonal Hermitian metrics. We prove that every $\bar{\partial}\partial$-exact real form of the type $(k, k)$ on an $n$-dimensional complex manifold $X$ arises as a difference of the Chern character forms on trivial Hermitian vector bundles with canonical connections, and that modulo $\text{Im}\partial + \text{Im}\bar{\partial}$ every real form of type $(k, k)$, $k < n$, arises as a Bott-Chern form for two Hermitian metrics on some trivial vector bundle over $X$. The latter result is a complex manifold analogue of Proposition 2.6 in the paper [SS10] by J. Simons and D. Sullivan.

1. Introduction

In [SS10] J. Simons and D. Sullivan have constructed a simple geometric model for differential $K$-theory (see [HS05] and [BS10] for review). The model uses a codification of complex vector bundles with connection over a smooth manifold, by introducing the notion of a structured vector bundle — a pair $(V, \{\nabla\})$ consisting of a complex vector bundle $V$ over a smooth manifold $X$ and the equivalence class of a connection $\nabla$. Two connections $\nabla^0$ and $\nabla^1$ are said to be equivalent if the corresponding Chern-Simons differential form is exact. The main technical innovation in [SS10] was Proposition 2.6 which states that all odd forms on $X$, modulo some natural relations, arise as the Chern-Simons forms between the trivial connection and an arbitrary connection on trivial bundles over $X$. It allows one to prove that differential $K$-theory has a natural analogue of the celebrated Character Diagram for the ring of Cheeger-Simons differential characters (see [SS08a] and [CS85]).

For Hermitian holomorphic vector bundles — holomorphic vector bundles over the complex manifold $X$ with Hermitian metrics — analogues of the Chern-Simons forms are the Bott-Chern forms, which were introduced in [BC65] earlier than the Chern-Simons forms in [CS74]. The corresponding differential $K$-theory was defined by H. Gillet and C. Soulé in [GS86].

In this paper we use Chern-Weil theory for Hermitian holomorphic vector bundles with canonical connections for explicit computation of the Chern forms for trivial bundles with special non-diagonal Hermitian metrics. This can be considered as the first step for the problem of explicitly computing
Chern forms of Hermitian holomorphic vector bundles, which we plan to address in the forthcoming paper. Here our goal is to obtain the analogue of Proposition 2.6 in [SS10] for complex manifolds. Namely, we prove that all real forms of type \((k, k)\) on an \(n\)-dimensional complex manifold \(X, k < n\), modulo \(\text{Im} \partial + \text{Im} \bar{\partial}\), arise as Bott-Chern forms for Hermitian metrics on trivial vector bundles over \(X\). As in the smooth manifold case, we deduce this statement from the result about Chern character forms which says that every \(\partial\bar{\partial}\)-exact real form of type \((k, k)\) on a complex manifold \(X\) arises as a difference of the Chern character forms on trivial Hermitian vector bundles. Our proof is based on several explicit computations of Chern forms for trivial vector bundles which may have interesting applications on their own.

Here is the more detailed content of the paper. In Section 2 for the convenience of the reader, we give a brief review of [SS10]. Namely, we use the definition of the Chern-Simons forms inspired by the approach of H. Gillet and C. Soulé for the complex manifold case [GS86], and deduce a somewhat stronger analogue of Proposition 2.6 in [SS10] — Corollary 2.2 — from Proposition 2.1. The latter states that for every exact even form \(\omega\) on a smooth manifold \(X\) there is a trivial vector bundle \(V\) over \(X\) with a connection \(\nabla\) such that

\[
\text{ch}(V, \nabla) - \text{ch}(V, d) = \omega,
\]

where \(d\) stands for the trivial connection on \(V\). Another small technical innovation is a different proof of Theorem 1.15 in [SS10] — Corollary 2.3 in the present paper — which does not appeal to the existence of universal connections.

In Section 3 we prove the main result, Theorem 3.2, which states that for every \(\partial\bar{\partial}\)-exact real form \(\omega\) of type \((k, k)\) on a general complex manifold \(X\) there is a trivial vector bundle \(E\) over \(X\) with two Hermitian metrics \(h_1\) and \(h_2\) such that

\[
\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.
\]

The proof in the general case is based on Lemma 3.3 where we explicitly compute the Chern form of a trivial vector bundle over \(X\) of arbitrary rank with a special non-diagonal Hermitian metric, and on Lemma 3.4 where we express real forms of type \((k, k)\) as finite linear combinations of wedge products of real \((1, 1)\)-forms of special type. We believe that these lemmas may have interesting applications on their own. When \(X\) is compact or is a submanifold of \(\mathbb{C}^n\), we give another proof of Theorem 3.2 based on Lemma 3.5. We deduce the complex manifold analogue of Proposition 2.6 in [SS10] — Corollary 3.3 — by using the Gillet-Soulé definition of the Bott-Chern forms [GS86]. Finally, we discuss how using Corollary 3.3 one can get rid of the differential form in the complex manifold version of differential \(K\)-theory [GS86]. However, developing differential \(K\)-theory for the complex manifolds in the spirit of [SS10] is an open and difficult problem since, in general, ‘inverses’ for the holomorphic bundles do not exist.
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2. Complex vector bundles over a smooth manifold

2.1. Chern-Simons secondary forms. Let \(X\) be a smooth \(n\)-dimensional manifold, let \(\mathcal{A}(X) = \bigoplus_{k=1}^{n} \mathcal{A}^k(X, \mathbb{C}) = \mathcal{A}^{\text{even}}(X) \oplus \mathcal{A}^{\text{odd}}(X)\) be the graded commutative algebra of smooth complex differential forms on \(X\), and let \(V\) be a \(C^\infty\)-complex vector bundle over \(X\) with a connection \(\nabla\). Recall that the Chern character form \(\text{ch}(V, \nabla)\) for the pair \((V, \nabla)\) is defined by

\[
\text{ch}(V, \nabla) = \text{Tr} \exp \left( \frac{\sqrt{-1}}{2\pi} \nabla^2 \right) \in \mathcal{A}^{\text{even}}(X).
\]

Here \(\nabla^2\) is the curvature of the connection \(\nabla\) — an \(\text{End} V\)-valued 2-form on \(X\) — and \(\text{Tr}\) is the trace in the endomorphism bundle \(\text{End} V\). The Chern character form is closed, \(d\text{ch}(V, \nabla) = 0\), and its cohomology class in \(H^*(X, \mathbb{C})\) does not depend on the choice of \(\nabla\).

Let \(\nabla^0\) and \(\nabla^1\) be two connections on \(V\). In [CS74], S.S. Chern and J. Simons introduced secondary characteristic forms — the Chern-Simons forms. Namely, the Chern-Simons form \(\text{cs}(\nabla^1, \nabla^0) \in \mathcal{A}^{\text{odd}}(X)\) defined modulo \(\mathcal{A}^{\text{even}}(X)\), satisfies the equation

\[
d\text{cs}(\nabla^1, \nabla^0) = \text{ch}(V, \nabla^1) - \text{ch}(V, \nabla^0),
\]

and enjoys a functoriality property under the pullbacks with smooth maps.

Here we present a construction of the Chern-Simons form \(\text{cs}(\nabla^1, \nabla^0)\), which is similar to the construction of Bott-Chern forms for holomorphic vector bundles, given by H. Gillet and C. Soulé in [GS86]. Namely, for a given \(V\) put \(\tilde{V} = \pi^*(V)\), where \(\pi : X \times S^1 \mapsto X\) is a projection, and \(S^1 = \{e^{i\theta}, 0 \leq \theta < 2\pi\}\). Explicitly, \(\tilde{V}\) is a bundle over \(X \times S^1\) whose fibre at every point \((x, \theta) \in X \times S^1\) is \(V_x \otimes \mathbb{C} \simeq V_x\). For every \(\theta\) define the map \(i_\theta : X \mapsto X \times S^1\) by \(i_\theta(x) = (x, e^{i\theta})\), and let \(\tilde{\nabla}\) be a connection on \(\tilde{V}\) such that

\[
i^{0*}_\theta(\tilde{\nabla}) = \nabla^0, \quad i^{1*}_\theta(\tilde{\nabla}) = \nabla^1.
\]

Denote by \(g\) a function defined by

\[
g(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta < \pi, \\ 1 & \text{if } \pi \leq \theta < 2\pi \end{cases}
\]
and extended $2\pi$-periodically to $\mathbb{R}$. It defines a function $g : S^1 \mapsto \mathbb{R}$, which is discontinuous at 0 and $\pi$.

**Definition.** The Chern-Simons form is defined as

$$
\text{cs}(\nabla^1, \nabla^0) = \pi_* (g(\theta) \text{ch}(\tilde{\nabla}, \tilde{\nabla})) = \int_{S^1} g(\theta) \text{ch}(\tilde{\nabla}, \tilde{\nabla}) \in \mathcal{A}^{\text{odd}}(X)
$$

— direct image of $g(\theta) \text{ch}(\tilde{\nabla}, \tilde{\nabla})$ under the projection $\pi : X \times S^1 \mapsto X$ (integration over the fibres of $\pi$).

**Remark 2.1.** Connection $\tilde{\nabla}$ is trivial to construct. If in local coordinates $\nabla^i = d_x + A^i(x)$, where $d_x$ is deRham differential on $X$ and $i = 0, 1$, then

$$
\tilde{\nabla} = d_x + d_\theta + A(x, \theta),
$$

where $A(x, \theta)$ is $2\pi$-periodic and $A(x, 0) = A^0(x)$, $A(x, \pi) = A^1(x)$.

**Lemma 2.1.** The Chern-Simons form $\text{cs}(\nabla^1, \nabla^0)$ satisfies the equation (2.1), and modulo $d\mathcal{A}^{\text{even}}(X)$ it does not depend on the choice of connection $\tilde{\nabla}$.

**Proof.** Using

$$(d_x + d_\theta) \text{ch}(\tilde{\nabla}) = 0 \quad \text{and} \quad d_\theta g = (\delta_\pi - \delta_0) d\theta,$$

we obtain

$$
d\text{cs}(\nabla^1, \nabla^0) = \int_{S^1} \left( (d_x + d_\theta) - d_\theta \right) (\text{ch}(\tilde{\nabla})) g(\theta) = - \int_{S^1} d_\theta (\text{ch}(\tilde{\nabla})) g(\theta)
$$

$$
= \int_{S^1} \text{ch}(\tilde{\nabla}) d_\theta g = \text{ch}(\nabla^1) - \text{ch}(\nabla^0).
$$

It is also easy to see that modulo exact forms $\text{cs}(\nabla^1, \nabla^0)$ does not depend on the choice of $\tilde{\nabla}$. Namely, let $\nabla = d_x + d_\theta + A(x, \theta)$, $\nabla' = d_x + d_\theta + A'(x, \theta)$ be two such connections. Define a connection $\tilde{\nabla}$ in the bundle $\tilde{\nabla}$ over $X \times S^1 \times S^1$ by

$$
\tilde{\nabla} = d_x + d_{\theta_1} + d_{\theta_2} + \hat{A}(x, \theta_1, \theta_2),
$$

where

$$
\hat{A}(x, \theta_1, 0) = A(x, \theta_1), \quad \hat{A}(x, \theta_1, \pi) = A'(x, \theta_1) \quad \text{for all} \quad \theta_1 \in [0, 2\pi],
$$

and $\hat{A}(x, 0, \theta_2) = A_0(x)$, $\hat{A}(x, \pi, \theta_2) = A_1(x)$ for all $\theta_2 \in [0, 2\pi]$. Then

$$
\int_{S^1} (\text{ch}(\nabla) - \text{ch}(\nabla')) g(\theta_1) = \int_{S^1} (d_x + d_{\theta_1}) \text{cs}(\tilde{\nabla}, \tilde{\nabla'}) g(\theta_1)
$$

$$
= d_x \int_{S^1} \text{cs}(\tilde{\nabla}, \tilde{\nabla'}) g(\theta_1) = \text{cs}(\tilde{\nabla}, \tilde{\nabla'})|_{\theta_1=0}^{\theta_1=\pi}
$$

$$
= d_x \int_{S^1} \text{cs}(\tilde{\nabla}, \tilde{\nabla'}) g(\theta_1) = \int_{S^1} \text{ch}(\tilde{\nabla}) g(\theta_2)|_{\theta_1=0}^{\theta_1=\pi}
$$

$$
= d_x \int_{S^1} \text{cs}(\tilde{\nabla}, \tilde{\nabla'}) g(\theta_1).\]
Here we have used that at $\theta_1 = 0$ and $\theta_1 = \pi$ the restriction of the form $\text{ch}(\nabla) \in \mathcal{A}^{\text{even}}(X \times S^1 \times S^1)$ to $X \times S^1$ has no components along $S^1$ and its integral over $S^1$ is zero.

**Remark 2.2.** Formula (2.2) is similar to formula (1.5) in [SS10] but has different applications than definition (1.2) in [SS10].

**Definition.** Put

$$
\text{CS}(\nabla^1, \nabla^0) = \text{cs}(\nabla^1, \nabla^0) \mod d\mathcal{A}^{\text{even}}(X),
$$

which, according to Lemma 2.1, is a well-defined element in $\tilde{\mathcal{A}}^{\text{odd}}(X) = \mathcal{A}^{\text{odd}}(X)/d\mathcal{A}^{\text{even}}(X)$.

**Remark 2.3.** Formula (2.1) can be written as

$$
\text{cs}(\nabla^1, \nabla^0) = \int_{\pi}^{2\pi} \text{ch}(\nabla),
$$

and the choice of points $\pi$ and $2\pi$ on the unit circle is immaterial. Using the change of variables, for every $\alpha < \beta$ on $S^1$ we get

$$
(2.3) \quad \text{cs}(\nabla^1, \nabla^0) = \int_{\alpha}^{\beta} \text{ch}(\nabla),
$$

where now $i^*_\alpha(\nabla) = \nabla^0$, $i^*_\beta(\nabla) = \nabla^1$.

Using (2.3) and Lemma 2.1, we immediately get

**Corollary 2.1.**

$$
\text{CS}(\nabla^2, \nabla^0) = \text{CS}(\nabla^2, \nabla^1) + \text{CS}(\nabla^1, \nabla^0).
$$

2.2. **K-theory.** Let $K_0(X)$ be the Grothendieck group of $X$ — a quotient of the free abelian group generated by the isomorphism classes $[V]$ of complex vector bundles $V$ over $X$ by the relations $[V] + [W] = [V \oplus W]$.

In [SS10], the authors defined a version of differential $K$-theory using the notion of a structured bundle. Namely, connections $\nabla^0$ and $\nabla^1$ on a complex vector bundle $V$ over $X$ are called equivalent, if $\text{CS}(\nabla^1, \nabla^0) = 0$. It follows from Corollary 2.1 that it is an equivalence relation.

**Definition.** A pair $\mathcal{V} = (V, \{\nabla\})$, where $\{\nabla\}$ is an equivalence class of connections on $V$, is called a structured bundle.

Denote by $\text{Struct}(X)$ the set of all equivalence classes of structured bundles over $X$. It is shown in [SS10] that it is a commutative semi-ring with respect to the direct sum $\oplus$ and tensor product $\otimes$ operations, and we denote by $\hat{K}_0(X)$ the corresponding Grothendieck ring.

We have two natural ring homomorphisms: the “forgetful map”

$$
\delta : \hat{K}_0(X) \to K_0(X),
$$
given by $[\mathcal{V}] \mapsto [V]$ for $\mathcal{V} = (V, \{\nabla\})$, and

$$
\text{ch} : \hat{K}_0(X) \to \mathcal{A}^{\text{even}}(X),
$$
given by the Chern character map $|V| \mapsto \text{ch}(V, \nabla)$. For a trivial bundle $V$ with trivial connection $\nabla = d$, $\text{ch}(V, d) = \text{rk}(V)$ — the rank of $V$.

The mapping $\delta$ is surjective and for compact $X$ its kernel consists of all differences $|V| - |F|$ such that $V = (V, \{\nabla\})$, where $V$ is stably trivial: $V \oplus M = N$ for some trivial bundles $M$ and $N$, and $F = (F, \{\nabla^F\})$, where $F$ is trivial and $\text{rk}(F) = \text{rk}(V)$. Indeed, by definition,

$$\ker \delta = \{[U] - [W] | U \oplus M = W \oplus M \text{ for some trivial bundle } M\}.$$

Now let $W'$ be a vector bundle satisfying $W \oplus W' = K$, where $K$ is a trivial bundle\footnote{Such bundle $W'$ exists since $X$ is compact.}. Choose arbitrary connections $\nabla^W$ and $\nabla^K$ on the bundles $W'$ and $K$, and introduce the bundles $V = U \oplus W'$ and $F = K$ with the connections $\nabla = \nabla^U \oplus \nabla^{W'}$ and $\nabla^F = \nabla^W \oplus \nabla^{W'}$. The bundle $V$ is stably trivial by construction: $V \oplus M = N$, where $N = F \oplus M$, and

$$[U] - [W] = [U \oplus W'] - [W \oplus W'] = [V] - [F].$$

It is an outstanding problem to describe the image of the Chern character map. The following simple result is crucial for our approach to the differential $K$-theory of Simons-Sullivan [SS10].

**Proposition 2.1.** The image of the Chern character map

$$\text{ch} : K_0(X) \to \mathcal{A}^{\text{even}}(X)$$

contains all exact forms. Specifically, for every exact even form $\omega$ there is a trivial vector bundle $V = X \times \mathbb{C}^r$ with a connection $\nabla = d + A$ such that

$$\text{ch}(V, \nabla) - \text{ch}(V, d) = \omega.$$

The proof is based on the following useful result (see [dR84] and [Con85]); for the convenience of the reader, we prove it here as well.

**Lemma 2.2.** Every $\eta \in \mathcal{A}^k(X)$ can be represented as a finite sum of the basic forms $f_1 df_2 \wedge \cdots \wedge df_{k+1}$, where $f_1, \ldots, f_{k+1}$ are smooth functions on $X$.

If the form $\eta$ is real, one can choose the basic forms such that all functions $f_i$ are real-valued, and if $\eta$ is zero on an open $U \subset X$, there is a representation such that all functions $f_i$ vanish on $U$.

**Proof.** When $X$ is a compact, one can choose a finite coordinate open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ and a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinated to it. Then $\eta = \sum_{\alpha \in A} \rho_\alpha \eta|_{U_\alpha}$, where in local coordinates $x^1, \ldots, x^n$ on $U_\alpha$,

$$\eta|_{U_\alpha} = \sum_I f_{\alpha, I} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $I = \{i_1, \ldots, i_k\}$ and $1 \leq i_1 < \cdots < i_k \leq n$. Let $K_\alpha$ be a compact set such that $\text{supp } \rho_\alpha \subset K_\alpha \subset U_\alpha$ and let $b_\alpha$ be a “bump function” — a smooth function on $X$ which is 1 on $\text{supp } \rho_\alpha$ and zero outside $K_\alpha$. Then

$$\eta_\alpha = \sum_I \rho_\alpha f_{\alpha, I} d(x^{i_1}b_\alpha(x)) \wedge \cdots \wedge d(x^{i_k}b_\alpha(x)) \in \mathcal{A}^k(X)$$
is of required form and \( \eta = \sum_{\alpha \in A} \eta_\alpha \). The second statement of the lemma is obvious from this construction.

In the general case one can use an embedding \( f : X \to \mathbb{R}^N \) of the manifold \( X \) into Euclidean space (say the Whitney embedding into \( \mathbb{R}^{2n} \)). By considering the tubular neighbourhood of \( f(X) \) in \( \mathbb{R}^N \), it is easy to show that the pullback map \( f^* : A(\mathbb{R}^N) \to A(X) \) is onto, which proves the result. \( \square \)

**Proof of Proposition 2.1.** Induction by the degree in \( d \).

According to Lemma 2.2, it is sufficient to consider only basic forms in \( A^{\text{odd}}(X) \).

For a basic 1-form \( \alpha = f_1 df_2 \) we have \( \omega = d\alpha = df_1 \wedge df_2 \), so that

\[
\text{ch}(L, \nabla) - \text{ch}(L, d) = \text{ch}(L, \nabla) - 1 = \omega,
\]

where \( L \) is a trivial line bundle over \( X \) with \( \nabla = d - 2\pi \sqrt{-1}f_1 df_2 \).

Now suppose that all exact forms of degree \( \leq 2k \) are in the image of \( \text{ch} \). For a basic \((2k+1)\)-form \( \alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2} \) we have \( \omega = d\alpha = df_1 \wedge df_2 \wedge \cdots \wedge df_{2k+2} \), which can be also written as

\[
\omega = \frac{1}{(k+1)!}(df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2})^{k+1}.
\]

Let \( V \) be a trivial line bundle over \( X \) with

\[
\nabla = d - 2\pi \sqrt{-1}(f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}),
\]

so that

\[
\nabla^2 = -2\pi \sqrt{-1}(df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2}).
\]

Then \( \text{ch}(V, \nabla) - 1 - \omega \) is an exact form of degree \( \leq 2k \) and, by induction, is in the image of \( \text{ch} \). \( \square \)

**Remark 2.4.** It the form \( \omega \) is real then the connection \( \nabla \) in Proposition 2.1 is compatible with the metric on \( V \) given by the standard Hermitian metric on \( \mathbb{C}^r \).

**Remark 2.5.** It immediately follows from the second statement of Lemma 2.2 and the proof of Proposition 2.1 that if form \( \omega \) vanishes on open \( U \subset X \), then connection \( \nabla = d + A \) can be chosen such that \( A = 0 \) on \( U \).

**Corollary 2.2.** For every \( \alpha \in \tilde{A}^{\text{odd}}(X) \) there is a trivial vector bundle \( V \) with connection \( \nabla \) such that \( CS(\nabla, d) = \alpha \).

**Proof.** For the given \( \alpha \in A^{\text{odd}}(X) \) let \( \Theta \in A^{\text{odd}}(X \times S^1) \) be such that under the inclusion map \( i_\theta : X \to X \times S^1 \) one has \( i_\theta^*(\Theta) = \alpha \) and \( i_\theta^*(\Theta) = 0 \) for all \( \theta \) in some neighborhood of 0. Applying Proposition 2.1 to the manifold \( X \times S^1 \) and the exact even form \(-d\Theta\), we have

\[
\text{ch}(\tilde{V}, \tilde{\nabla}) - \text{rk}(\tilde{V}) = -(d_x + d_\theta)\Theta.
\]

Integrating over \( \theta \) from \( \pi \) to \( 2\pi \) we get

\[
\alpha = \text{cs}(\nabla^1, \nabla^0) + d_x \int_{\pi}^{2\pi} \Theta,
\]
for connections $\nabla^0 = i^*_0(\tilde{\nabla})$ and $\nabla^1 = i^*_0(\tilde{\nabla})$ on a trivial bundle $V$ — a pullback of the trivial bundle $\tilde{V}$ to $X$. Finally, it follows from Remark 2.5 that one can choose connection $\tilde{\nabla}$ on $\tilde{V}$ such that $i^*_0(\tilde{\nabla}) = d$. Thus putting $\nabla = \nabla^1$ we obtain $\text{CS}(\nabla, d) = \alpha \mod d\text{A}^{\text{even}}(X)$. □

**Remark 2.6.** Corollary 2.2 gives somewhat stronger form of Proposition 2.6 in [SS10], the so-called “Venice lemma” of J. Simons\(^2\). It has been used in [SS10] to prove that one can remove the differential form from the definition of the differential $K$-theory given by M. Hopkins and I. Singer [HS05].

**Remark 2.7.** Here is a direct proof of Corollary 2.2 which is close to the original argument in [SS10]. Let $\eta$ be a 1-form on $X$ and let $L$ be a trivial line bundle with the connection $\nabla = d - 2\pi \sqrt{-1} \eta$. It follows from the homotopy formula in [SS10] that

$$\text{CS}(\nabla, d) = \int_0^1 \exp\{dt \wedge \eta + t d\eta\} = \sum_{l \geq 1} \frac{1}{l!} \eta \wedge (d\eta)^{l-1}.$$

Thus for the basic 1-form $\alpha = f_1 df_2$ putting $\eta = \alpha$ we get $\alpha = \text{CS}(\nabla, d)$. Now suppose that the result is valid for all odd forms of degree $\leq 2k - 1$, and let $\alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2}$ be a basic form of degree $2k + 1$. Putting $\eta = f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}$ we obtain in $\tilde{\text{A}}^{\text{odd}}(X)$,

$$\text{CS}(\nabla, d) = \frac{1}{(k + 1)!} \eta \wedge (d\eta)^k - \xi = \alpha - \xi,$$

where $\xi$ is a sum of odd forms of degrees $\leq 2k - 1$. By the induction hypothesis, there is a trivial vector bundle $V$ with the connection $\tilde{\nabla}$ such that $\text{CS}(\tilde{\nabla}, d) = \xi$; so that $\alpha = \text{CS}(\nabla \oplus \tilde{\nabla}, d)$.

Recall that a flat connection $\nabla$ on a trivial vector bundle $F$ is a connection with trivial holonomy around any closed path in $X$. Equivalently, $\nabla = d^g = d + g^{-1} dg$, where $g : X \to \text{GL}(r, \mathbb{C})$, $r = \text{rk}(F)$, is a global parallel frame. The corresponding structured bundle $\mathcal{F} = (F, \{\nabla\})$ is called flat. Since any two flat connections on a trivial bundle $F$ are gauge equivalent, flat bundles of a fixed rank $r$ correspond to a single point in $\text{Struct}(X)$, which following [SS10], we denote by $[r]$. Also, denote by $\mathcal{T}(X)$ a subgroup in $\tilde{\text{A}}^{\text{odd}}(X)$ consisting of $\text{CS}(\nabla, \nabla')$ for all trivial bundles $F$ and flat connections $\nabla$, $\nabla'$ on $F$.

**Remark 2.8.** According to Lemma 2.3 in [SS10], the group $\mathcal{T}(X)$ has the following description. Let $\Theta$ be the bi-invariant closed odd form on the stable general linear group $\text{GL}(\infty)$ such that the free abelian group generated by all distinct products of its components represent the entire cohomology ring of $\text{GL}(\infty)$ over $\mathbb{Z}$. Then

$$\mathcal{T}(X) = \{g^*(\Theta) \mid \text{for all smooth } g : X \to \text{GL}(\infty)\}/d\text{A}^{\text{even}}(X).$$

\(^2\)D. Sullivan, private communication.
Now following [SS10], let

\[
\text{Struct}_{ST}(X) = \{ [\mathcal{V}] = [(V, \{\nabla\})] \in \text{Struct}(X) \mid V \text{ is stably trivial} \}
\]

be the stably trivial sub-semigroup of \text{Struct}(X), and for \( \mathcal{V} \in \text{Struct}_{ST}(X) \)

\[
\widehat{\text{CS}}(\mathcal{V}) = \text{CS}(\nabla^N, \nabla \oplus \nabla^F) \in \mathcal{A}^{\text{odd}}(X)/d\mathcal{A}^{\text{even}}(X),
\]

where \( V \oplus F = N \) with trivial bundles \( F \) and \( N \), and \( \nabla^F, \nabla^N \) are flat connections on these bundles. According to Proposition 2.4 in [SS10], for another choice of trivial bundles \( \bar{F} \) and \( \bar{N} \) with flat connections \( \nabla^{\bar{F}}, \nabla^{\bar{N}} \) we have

\[
\text{CS}(\nabla^N, \nabla \oplus \nabla^F) - \text{CS}(\nabla^{\bar{N}}, \nabla \oplus \nabla^{\bar{F}}) \in T(X),
\]

so that the mapping \( \widehat{\text{CS}} : \text{Struct}_{ST}(X) \to \bar{\mathcal{A}}^{\text{odd}}(X)/T(X) \) is a well-defined homomorphism of semigroups.

**Remark 2.9.** One can choose \( F = X \times \mathbb{C}^r \) with \( \nabla^F = d \) and \( \nabla^N = dg \), where \( g \) is the isomorphism between \( V \oplus F \) and \( N = X \times \mathbb{C}^k \), and put \( \widehat{\text{CS}} = \text{CS}(\nabla \oplus d, dg) \).

According to Corollary 2.2 the map \( \widehat{\text{CS}} \) is surjective, and according to Proposition 2.5 in [SS10], \( \ker \widehat{\text{CS}} = \text{Struct}_{SF}(X) \) — the subgroup of stably flat structured bundles. By definition, \( \mathcal{V} \in \text{Struct}_{ST}(X) \) is stably flat, if

\[
\mathcal{V} \oplus \mathcal{F} = \mathcal{N},
\]

where \( \mathcal{F} = (F, \{\nabla^F\}) \) and \( \mathcal{N} = (N, \{\nabla^N\}) \) are trivial bundles with equivalence classes of flat connections. Since map \( \widehat{\text{CS}} \) is onto and \( \bar{\mathcal{A}}^{\text{odd}}(X)/T(X) \) is a group, for every \( \mathcal{V} \in \text{Struct}_{ST}(X) \) there is \( \mathcal{W} \in \text{Struct}_{ST}(X) \) such that \( \mathcal{V} \oplus \mathcal{W} \in \text{Struct}_{SF}(X) \). This introduces a group structure on the coset space \( \text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \), and we arrive at the following statement.

**Proposition 2.2.** The map \( \widehat{\text{CS}} \) induces a group isomorphism

\[
\widehat{\text{CS}} : \text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \to \bar{\mathcal{A}}^{\text{odd}}(X)/T(X).
\]

From this result we immediately obtain Theorem 1.15 in [SS10].

**Corollary 2.3.** Every structured bundle over a compact manifold \( X \) has a structured inverse: for every \( \mathcal{V} = (V, \{\nabla\}) \in \text{Struct}(X) \) there exists \( \mathcal{W} = (W, \{\nabla^W\}) \in \text{Struct}(X) \) such that

\[
\mathcal{V} \oplus \mathcal{W} = \mathcal{N},
\]

where \( \mathcal{N} = (N, \{\nabla^N\}) \) is a trivial bundle with flat connection.

**Proof.** For \( \mathcal{V} = (V, \{\nabla\}) \in \text{Struct}(X) \) let \( U \) be such that \( V \oplus U = F \) — a trivial bundle over \( X \). Then \( \mathcal{F} = (F, \{\nabla \oplus \nabla^U\}) \in \text{Struct}_{ST}(X) \) for any choice of connection \( \nabla^U \) on \( U \). By Proposition 2.2, there exists \( \mathcal{H} = (H, \{\nabla^H\}) \in \text{Struct}_{ST}(X) \) such that \( \mathcal{F} \oplus \mathcal{H} \in \text{Struct}_{SF}(X) \), i.e., there are trivial bundles
$M$ and $N$ with flat connections $\nabla^M$ and $\nabla^M$ such that $\mathcal{F} \oplus \mathcal{H} \oplus M = \mathcal{N}$. Putting

$$W = (U \oplus H \oplus M, \{\nabla^U \oplus \nabla^H \oplus \nabla^M\}),$$

we obtain $V \oplus W = N$. □

From Corollary 2.3 we immediately obtain, as in [SS10], that

• The Grothendieck group $\hat{K}(X)$ consists of elements $[V] - [r].$
• The element $[V] - [r] = 0$ if and only if $V = (V(\nabla))$ is stably flat and $r = \text{rk}(V)$.
• The mapping $\Gamma : \text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \to \hat{K}(X)$, defined by

$$\Gamma([V]) = [V] - [\text{rk}(V)],$$

gives an isomorphism $\text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \simeq \ker \delta$.

Denoting by $i = \Gamma \circ \tilde{CS}^{-1} : \tilde{A}^{\text{odd}}(X)/T(X) \to \ker \delta$, we obtain the main part of the result in [SS10], the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{A}^{\text{odd}}(X)/T(X) & \overset{i}{\longrightarrow} & \hat{K}(X) & \overset{\delta}{\longrightarrow} & K(X) & \longrightarrow & 0 \\
& & \downarrow d & & \downarrow \text{ch} & & \downarrow \text{ch} \\
0 & \longrightarrow & d\tilde{A}^{\text{odd}}(X) & \overset{j}{\longrightarrow} & Z^\text{even}(X) & \longrightarrow & H^\text{even}_{\text{dR}}(X) & \longrightarrow & 0,
\end{array}
$$

where $Z^\text{even}(X)$ is a subspace of closed forms in $A^{\text{even}}(X)$, and the map $j$ is an inclusion.

3. Holomorphic vector bundles over a complex manifold

3.1. Bott-Chern secondary forms. Let $(E, h)$ be a holomorphic Hermitian vector bundle — a holomorphic vector bundle of rank $r$ over a complex manifold $X$, $\text{dim}_\mathbb{C} X = n$, with the Hermitian metric $h$. For a given open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$, the bundle $E$ can be defined in terms of the transition functions: holomorphic maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(r, \mathbb{C})$, satisfying the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma,$$

where $I$ is $r \times r$ identity matrix. In terms of the transition functions, a Hermitian metric $h$ on $E$ is the collection $h = \{h_\alpha\}_{\alpha \in A}$, where $h_\alpha$ are positive-definite Hermitian $r \times r$ matrix-valued functions on $U_\alpha$, satisfying

$$h_\beta = g^*_{\alpha\beta}h_\alpha g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta,$$

and $g^*$ stands for the Hermitian conjugation.

\[\text{For the convenience of the reader who is not a geometer, here we briefly recall the basic definitions (see, e.g., [Wel08]).}\]
Denote by $\nabla$ the canonical connection on the holomorphic Hermitian bundle $(E, h)$. In terms of the open cover $\{U_\alpha\}_{\alpha \in A}$ and the transition functions it is given by the collection $\nabla = \{\nabla_\alpha\}_{\alpha \in A}$,

$$\nabla_\alpha = d + A_\alpha = \partial + \bar{\partial} + A^{1,0}_\alpha + A^{0,1}_\alpha,$$

where $A^{0,1}_\alpha = 0$ and $A^{1,0}_\alpha = h^{-1}_\alpha \partial h_\alpha$ are $r \times r$ matrix-valued $(1, 0)$-forms on $U_\alpha$, satisfying

$$A^\beta_\beta = g^{-1}_\alpha \beta A^\alpha_\alpha g^\alpha_\beta = 0 \quad \text{on} \quad U_\alpha \cap U_\beta.$$

The curvature of the canonical connection $\nabla = d + A$ on holomorphic Hermitian vector bundle $(E, h)$ is a collection $\Theta = \{\Theta_\alpha\}_{\alpha \in A}$, where $\Theta_\alpha = \partial A^\alpha_\alpha$ are $r \times r$ matrix-valued $(1, 1)$-forms on $U_\alpha$, satisfying

$$\Theta^\beta_\beta = g^{-1}_\alpha \beta \Theta^\alpha_\alpha g^\alpha_\beta \quad \text{on} \quad U_\alpha \cap U_\beta. \quad (3.1)$$

Chern-Weil theory associates to any polynomial $\Phi$ on GL($r$, $\mathbb{C}$), invariant under conjugation, a collection $\{\Phi(\Theta_\alpha)\}_{\alpha \in A}$ which, according to $(3.1)$, defines a global differential form $\Phi(\Theta)$ on $X$. The total Chern form $c(E, h)$ and the Chern character form $\text{ch}(E, h)$ of a holomorphic Hermitian vector bundle $(E, h)$ are special cases of this construction and are defined, respectively, by

$$c(E, h) = \det \left( I + \frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_{k=0}^{r} c_k(E, h)$$

and

$$\text{ch}(E, h) = \text{Tr} \exp \left( \frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_{k=0}^{n} \text{ch}_k(E, h).$$

Since

$$\text{ch}(E, h) \in \mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X, \mathbb{R}), \quad \mathcal{A}(X, \mathbb{C}) = \bigoplus_{p=0}^{n} \mathcal{A}^{p,p}(X, \mathbb{C}),$$

the Chern character form is $\partial$ and $\bar{\partial}$ closed.

Let $h_1$ and $h_2$ be two Hermitian metrics on a holomorphic vector bundle $E$ over a complex manifold $X$. In the classic paper [BC65], Bott and Chern have shown that there exist secondary characteristic forms, the so-called Bott-Chern forms — even differential forms $\text{bc}(E; h_1, h_2) \in \check{\mathcal{A}}(X, \mathbb{C}) = \mathcal{A}(X, \mathbb{C})/(\text{Im} \partial + \text{Im} \bar{\partial})$, satisfying

$$\text{ch}(E, h_2) - \text{ch}(E, h_1) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \text{bc}(E; h_1, h_2) \quad (3.2)$$

and the natural functorial property

$$\text{bc}(f^*(E), f^*(h_1), f^*(h_2)) = f^*(\text{bc}(E; h_1, h_2)) \quad (3.3)$$
for holomorphic maps $f: Y \to X$ of complex manifolds. The proof in [BC65, Proposition 3.15] uses the analogue of the homotopy formula in Chern-Weil theory.

**Remark 3.1.** In the smooth manifold case, using linear homotopy of connections $\nabla_t$, it is possible to integrate over $t$ in the homotopy formula in a closed form and to obtain explicit formulas for the Chern-Simons forms (see, e.g., [SS10]). However in the complex manifold case for any homotopy $h_t$ of Hermitian metrics, due to the presence of inverses $h_t^{-1}$ in $\Theta_t$, it is not possible to integrate over $t$ in the homotopy formula in a closed form and to obtain explicit formulas for the Bott-Chern forms in terms of the Hermitian metrics $h_1$ and $h_2$ only.\footnote{In a separate publication we will show that one can get explicit formulas for Bott-Chern forms using some natural coordinates on the space of positive-definite Hermitian matrices.}

In [GS86], Gillet and Soulé gave another definition of the Bott-Chern secondary classes which is also well-suited for short exact sequences of holomorphic vector bundles over $X$, which are used for defining the $K$-theory of $X$. Namely, let $E$ be a holomorphic vector bundle over $X$ with Hermitian metrics $h_1$ and $h_2$, let $\mathcal{O}(1)$ be the standard holomorphic line bundle of degree 1 over the complex projective line $\mathbb{P}^1$, and let $\tilde{E} = E \otimes \mathcal{O}(1)$ be the corresponding vector bundle over $X \times \mathbb{P}^1$. If $i_p : X \to X \times \mathbb{C}P^1$ is the natural inclusion map $i_p(x) = (x, p)$ then $i_p^*(\tilde{E}) \simeq E$ for all $p \in \mathbb{P}^1$. Let $\tilde{h}$ be a Hermitian metric on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_1$ and $i_\infty^*(\tilde{h}) = h_2$ (such a metric is constructed using partition of unity).

**Definition.** The Bott-Chern secondary form is defined as

$$bc(E; h_1, h_2) = \int_{\mathbb{P}^1} \text{ch}(\tilde{E}, \tilde{h}) \log |z|^2$$

— direct image of $\log |z|^2 \text{ch}(\tilde{E}, \tilde{h})$ under the projection $\pi : X \times \mathbb{P}^1 \to X$ (integration over the fibres of $\pi$). The integral is convergent since $\log |z|^2 \omega(z)$, where $\omega$ is any smooth $(1, 1)$-form on $\mathbb{P}^1$, is integrable.

**Lemma 3.1 (H. Gillet and C. Soulé).** The Bott-Chern form $bc(E; h_1, h_2)$ satisfies equations (3.2) and (3.3), and modulo $\text{Im} \partial + \text{Im} \bar{\partial}$ does not depend on the choice of Hermitian metric $\tilde{h}$.

The proof of (3.2) uses Poincaré-Lelong formula:

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log |z|^2 = \delta_\infty - \delta_0$$

(see [GS86, BGS88]), and Lemma 2.1 uses a simplified version of this argument. As in the previous section, we put

$$BC(E; h_1, h_2) = bc(E; h_1, h_2) \mod(\text{Im} \partial + \text{Im} \bar{\partial}).$$
Remark 3.2. Note that formula (3.4) for the Bott-Chern forms uses the Green’s function $\log |z|^2$ of the Laplace operator on $\mathbb{P}^1$, whereas formula (2.2) for the Chern-Simons form uses the Green’s function $g(\theta)$ of the operator $\frac{d}{d\theta}$ on $S^1$.

Remark 3.3. In fact, Gillet and Soulé in [GS86] (and with J.-M. Bismut in [BGS88]) defined Bott-Chern forms for short exact sequences of holomorphic vector bundles over $X$. Namely, let $\mathcal{E}$

\[ 0 \longrightarrow F \longrightarrow E \longrightarrow H \longrightarrow 0 \]

be such an exact sequence, where holomorphic bundles $F,E$ and $H$ are equipped with Hermitian metrics $h_F,h_E$ and $h_H$. Put $F(1) = F \otimes \mathcal{O}(1)$ and consider the map $\text{id} \otimes \sigma : F \rightarrow F(1)$, where $\sigma$ is a holomorphic section of the bundle $\mathcal{O}(1)$ over $X$. Namely, let $	ilde{E} = (F(1) \oplus E)/F$ be the quotient bundle over $X \times \mathbb{P}^1$, where $F$ is mapped diagonally into $F(1) \oplus E$ by $(\text{id} \otimes \sigma) \otimes i$. Then under the embedding $i_p : X \rightarrow X \times \mathbb{P}^1$ we will have $i_0^*(\tilde{E}) = E$ and $i_\infty^*(\tilde{E}) = F \oplus H$ since $E/F \simeq H$. There exists a Hermitian metric $\tilde{h}$ on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_E$ and $i_\infty^*(\tilde{h}) = h_F \oplus h_H$, and the Bott-Chern secondary form for the exact sequence $\mathcal{E}$ and Hermitian metrics $h_F,h_E,h_H$ is defined by Gillet and Soulé [GS86] by the formula

\[ \text{bc}(\mathcal{E}; h_E, h_F, h_H) = \int_{\mathbb{P}^1} \text{ch}(\tilde{E}, \tilde{h}) \log |z|^2. \]

Similar to (3.2), the Bott-Chern forms satisfy the equation

\[ \text{ch}(F \oplus H, h_F \oplus h_H) - \text{ch}(E, h_E) = \frac{1}{2\pi} \partial \bar{\partial} \text{bc}(\mathcal{E}; h_E, h_F, h_H), \]

are functorial, and vanish when the exact sequence $\mathcal{E}$ holomorphically splits and $h_E = h_F \oplus h_H$ (see [GS86, BGS88]).

3.2. Chern forms of trivial bundles. We start with the following simple linear algebra result.

Lemma 3.2. Let $\alpha_i, \beta_i$, $i = 1, \ldots, k$, be odd elements in some graded commutative algebra $A$ over $\mathbb{C}$ (e.g., the algebra of complex differential forms on $X$), and let $A$ be a $k \times k$ matrix with even elements $A_{ij} = \alpha_i \beta_j$, and put $a = \text{Tr} A = \sum_{i=1}^k \alpha_i \beta_i$. Then for every $\lambda \in \mathbb{C}$,

\[ (I - \lambda A)^{-1} = I + \frac{\lambda}{1 + \lambda a} A = I + (\lambda - \lambda^2 a + \cdots + (-1)^k \lambda^{k+1} a^k) A, \]

5Here and in what follows we use the same notation for bundles over $X$ and their pullbacks under the projection $\pi : X \times \mathbb{P}^1 \rightarrow X$.

6It is for this construction that it is necessary to twist the bundle $F$ by $\mathcal{O}(1)$. 
where $I$ is $k \times k$ identity matrix, and also
\[
\det(I - \lambda A) = \frac{1}{1 + \lambda a} = 1 - \lambda a + \cdots + (-1)^k \lambda^k a^k.
\]

**Proof.** Consider the following identity
\[
\frac{d}{d\lambda} \log \det(I - \lambda A) = -\text{Tr} A(I - \lambda A)^{-1},
\]
whose validity for matrices over $\mathbb{C}$ and small $\lambda$ follows from Jordan canonical form, and for matrices with even nilpotent entries — from the definition of the determinant.\(^7\) Since $A^2 = -aA$, we obtain
\[
(I - \lambda A)^{-1} = I + \frac{\lambda}{1 + \lambda a} A,
\]
so that
\[
\frac{d}{d\lambda} \log \det(I - \lambda A) = -\frac{a}{1 + \lambda a} = -\frac{d}{d\lambda} \log(1 + \lambda a).
\]
Integrating and using $\det I = 1$, we get the result. \(\Box\)

The next result is an explicit computation of the total Chern form of a trivial vector bundle with a special non-diagonal Hermitian metric.

**Lemma 3.3.** Let $E_r = X \times \mathbb{C}^r$ be a trivial rank $r$ vector bundle over $X$ with a Hermitian metric $h$ given by

\[
h = h(\sigma, f_1, \ldots, f_{r-1}) = g^* g, \quad \text{where} \quad g = \begin{pmatrix}
1 & 0 & \cdots & 0 & f_1 \\
0 & 1 & \cdots & 0 & f_2 \\
0 & 0 & 1 & \cdots & 0 & f_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & f_{r-1} \\
0 & 0 & 0 & \cdots & 0 & e^{\sigma/2}
\end{pmatrix},
\]

and $f_1, \ldots, f_{r-1} \in C^\infty(X, \mathbb{C})$, $\sigma \in C^\infty(X, \mathbb{R})$. Then
\[
c(E_r, h) = c(E_1, e^\sigma) + \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log \left(1 - \frac{\sqrt{-1}}{2\pi} U\right),
\]
where $U = e^{-\sigma} \sum_{i=1}^{r-1} \partial f_i \wedge \bar{\partial} f_i$, $E_1 = \det E_r$ is a trivial line bundle over $X$, and for a nilpotent element $a$ of order $r$, $\log(1 - a) = -(a + \frac{a^2}{2} + \cdots + \frac{a^{r-1}}{r-1})$.

Equivalently,
\[
c_1(E_r, h) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \sigma, \quad c_k(E_r, h) = -\frac{1}{k-1} \left(\frac{\sqrt{-1}}{2\pi}\right)^k \bar{\partial} \partial U^{k-1}, \quad k = 2, \ldots, r.
\]

\(^7\)In the latter case log and inverse are given, correspondingly by the finite sums — truncated power series.
Proof. Let $\Theta = \bar{\partial}(h^{-1}\partial h)$ be the curvature form associated with the Hermitian metric $h$. We need to prove that for every $\lambda \in \mathbb{C}$,

$$
\det(I + \lambda \Theta) = 1 + \lambda \bar{\partial}\partial \sigma + \lambda \bar{\partial}\partial \log(1 - \lambda U)
$$

$$
= 1 + \lambda \bar{\partial}\partial \sigma - \lambda^2 \frac{\bar{\partial}\partial U}{1 - \lambda U} - \lambda^3 \frac{\bar{\partial}U \wedge \partial U}{(1 - \lambda U)^2},
$$

where

$$
\frac{1}{1 - \lambda U} = \sum_{k=0}^{r-1} \lambda^k U^k \quad \text{and} \quad \frac{1}{(1 - \lambda U)^2} = \sum_{k=0}^{r-1} (k + 1) \lambda^k U^k.
$$

It is convenient to represent the matrix $I + \lambda \Theta$ in the following block form

$$
I + \lambda \Theta = \begin{pmatrix}
I + \lambda \Theta_{11} & \lambda \Theta_{12} \\
\lambda \Theta_{21} & 1 + \lambda \Theta_{22}
\end{pmatrix},
$$

where $(r - 1) \times (r - 1)$ matrix $\Theta_{11}$, $(r - 1)$-vectors $\Theta_{12}, \Theta_{21}$, and the scalar $\Theta_{22}$ are given by

$$
\Theta_{11} = \{-\bar{\partial}(\bar{f}_i e^{-\sigma} \partial f_j)\}_{i,j=1}^{r-1}, \quad \Theta_{12} = \{\bar{\partial}\partial \bar{f}_i - \bar{\partial}(\bar{f}_i F) - \bar{\partial}(\bar{f}_i \partial \sigma)\}_{i=1}^{r-1},
$$

$$
\Theta_{21} = \{\bar{\partial}(e^{-\sigma} \partial f_i)\}_{i=1}^{r-1}, \quad \Theta_{22} = \bar{\partial}\partial \sigma + \bar{\partial}F,
$$

and $F = e^{-\sigma} \sum_{i=1}^{r-1} \bar{f}_i \partial f_i$. The row operations $R_i \mapsto R_i + \bar{f}_i R_r$ transform the matrix $I + \lambda \Theta$ to the form

$$
\begin{pmatrix}
I - \lambda A & b \\
c & d
\end{pmatrix},
$$

where

$$
A = \{e^{-\sigma} \bar{\partial} \bar{f}_i \wedge \partial f_j\}_{i,j=1}^{r-1}, \quad b = \{\bar{f}_i + \lambda(\bar{\partial}\partial \bar{f}_i - \bar{\partial}\bar{f}_i \wedge \bar{f}_i \wedge \partial \sigma)\}_{i=1}^{r-1},
$$

and we put $c = \lambda \Theta_{21}$, $d = 1 + \lambda \Theta_{22}$.

Now it follows from the representation

$$
\begin{pmatrix}
I - \lambda A & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
I & b \\
c(I - \lambda A)^{-1} & d
\end{pmatrix} \begin{pmatrix}
I - \lambda A & 0 \\
0 & 1
\end{pmatrix}
$$

that

$$
\det(I + \lambda \Theta) = \det(I - \lambda A) \left(d - c(I - \lambda A)^{-1} b\right),
$$

which we compute explicitly using Lemma 3.2. Namely,

$$
\det(I + \lambda \Theta) = \frac{1}{1 - \lambda U} \left(1 + \lambda \bar{\partial}\partial \sigma + \bar{\partial}F\right)
$$

$$
- \sum_{i,j=1}^{r-1} \lambda \bar{\partial}(e^{-\sigma} \partial f_i) \wedge \left(\delta_{ij} + \frac{\lambda e^{-\sigma} \bar{\partial} \bar{f}_i \wedge \partial f_j}{1 - \lambda U}\right) \wedge (\bar{f}_j + \lambda(\bar{\partial}\partial \bar{f}_j - \bar{\partial}\bar{f}_j \wedge (F + \partial \sigma))).
$$

Using equations

$$
\bar{\partial}F = -U - \bar{\partial}\sigma \wedge F + e^{-\sigma} \sum_{i=1}^{r-1} \bar{f}_i \partial f_i,
$$
Proof. Replacing the functions $f_i$ in the definition of the Hermitian metric $h(\sigma; f_1, \ldots, f_{r-1})$ by $t_i f_i$ with real $t_i$, we can consider Chern forms $c_k(E_r, h)$ as polynomials in $t_1, \ldots, t_{r-1}$ with coefficients in the commutative ring $\mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X, \mathbb{R})$. In follows from the explicit formulas in Lemma
that these are polynomials in variables $\alpha_i = t_i^2$, which can be considered as nilpotent elements of order 2 since the forms $\delta f_i \wedge \delta \bar{f}_i$ have the same property, and that the Chern forms $c_k(E_i, h(\sigma; t_i f_1, \ldots, t_{i-1} f_{i-1})$ can be obtained from the Chern form $c_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{r-1} f_{r-1})$ by setting $t_j = 0$ for $j \in J$ — a complementary subset to $I = \{i_1, \ldots, i_{r-1}\}$ in the set $\{1, \ldots, r-1\}$.

The Chern character forms are related to the Chern forms by Newton’s identities

$$(-1)^k k! \text{ch}_k = -k c_k + c_{k-1} \text{ch}_1 - 2! c_{k-2} \text{ch}_2 + \cdots + (-1)^k (k-1)! c_1 \text{ch}_{k-1},$$

so that the same relation holds between Chern character forms for the vector bundles $E_l$ and $E_r$. Now represent the Chern character form as

$$\text{ch}_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{r-1} f_{r-1})) = \sum_{l=1}^{k-1} \sum_{|I|=l} \alpha_{i_1} \cdots \alpha_{i_{r-1}} \omega_I,$$

where summation goes over all subsets of $\{1, \ldots, r-1\}$ of cardinality $l$, and substitute it to the left hand side of the identity we want to prove, with $f_i$ replaced by $t_i f_i$. We obtain a sum of monomials, and by the relation between the Chern character forms for the bundles $E_l$ and $E_r$, for $k < r$ every term is a monomial $\alpha_{i_1} \cdots \alpha_{i_{r-1}} \omega_I$, taken with coefficient

$$1 - \sum_{j=1}^{r-k} (-1)^j \binom{r-k}{j} = 0,$$

which follows from the exclusion-inclusion principle. For the case $k = r$ the term $-c_r(E_r, h)/(k-1)!$ is the only term to survive, and for the case $k = r+1$ such term is given by $-c_1(E_r, h) c_r(E_r, h)/(r+1)!$. \hfill \Box

**Remark 3.4.** For the rank 2 trivial vector bundle $E_2$ with the Hermitian metric

$$h = h(\sigma, f) = \begin{pmatrix} 1 & 0 \\ f & e^{\sigma/2} + e^{-\sigma/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f & e^{\sigma/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\sigma/2} \end{pmatrix},$$

the main identity in the Corollary 3.1 takes the form

$$\text{ch}_2(E_2, h(\sigma, f)) - \text{ch}_2(E_1, e^\sigma) = -\frac{1}{(2\pi)^2} \bar{\partial} \partial (e^{-\sigma} \partial f \wedge \bar{\partial} \bar{f}),$$

and can be verified by a straightforward computation\(^8\). For the bundles of rank 3 and 4 corresponding identities were first verified using special Mathematica package for computing Chern character forms, written by Michael Movshev. Based on these results, Michael Movshev [Mov10] was able, with the help of Mathematica, to obtain Lemma 3.3. Here we give its complete algebraic proof and show that it implies the identities, originally conjectured by the second author.

\(^8\)This computation of the second author was the starting point of the paper.
Remark 3.5. Setting $\bar{\theta} = h^{-1}\partial h$, it is easy to obtain
\[
\text{Tr}(\theta \wedge \bar{\theta}) = e^{-\sigma} \partial f \wedge \bar{\partial} \bar{f} + e^{-\sigma} \bar{\partial} \bar{f} \wedge \partial f.
\]
To get rid of the second term and to write down the simplest nontrivial Bott-Chern form $bc_1(h, I)$, where $I$ is a trivial Hermitian metric on $E_2$, we need to add the “Wess-Zumino term” (rather its $(1, 1)$-component) to the “kinetic term” $\text{Tr}(\theta \wedge \bar{\theta})$. Such formula was first obtained by A. Alekseev and S. Shatashvili in [AS89], where for the case of Minkowski signature the decomposition
\[
\begin{pmatrix}
1 & f \\
|f|^2 & e^{\sigma}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & e^{\sigma}
\end{pmatrix}
\begin{pmatrix}
1 & \bar{f} \\
0 & 1
\end{pmatrix}
\]
is replaced by the Gauss decomposition for $SL(2, \mathbb{C})$.

Remark 3.6. The Bott-Chern forms (or rather their exponents) also appear quite naturally in supersymmetric quantum field theories as ratios of non-chiral partition functions for the higher dimensional analogues of the $bc$-systems [LMNS97].

3.3. The main result. The first result is an analogue of Lemma 2.2 for general complex manifolds.

Lemma 3.4. Let $X$ be a complex manifold. Every $\omega \in A^{k,k}(X, \mathbb{C}) \cap A^{2k}(X, \mathbb{R})$ can be written as a finite linear combination over $\mathbb{R}$ of wedge products of real $(1, 1)$-forms of the type $\sqrt{-1} e^{\sigma} \partial f \wedge \bar{\partial} \bar{f}$, where $\sigma \in C^\infty(X, \mathbb{R})$ and $f \in C^\infty(X, \mathbb{C})$. Moreover, if $\omega$ is zero on open $U \subset X$, than one can choose these forms such that all functions $\sigma$ and $f$ vanish on $U$.

Proof. Let $\omega$ be a real form of type $(k, k)$. According to Lemma 2.2, it is a finite sum of the terms
\[
\partial h \wedge \bar{\partial} g + \bar{\partial} h \wedge \partial g = \sqrt{-1} (\partial f \wedge \bar{\partial} \bar{f} - \partial f \wedge \bar{\partial} \bar{f} - \partial g \wedge \bar{\partial} \bar{g}),
\]
where $f = h + \sqrt{-1} g$. \qed

For compact complex manifolds (or rather for manifolds admitting a finite coordinate open cover) and for submanifolds of $\mathbb{C}^n$, there is a different version of Lemma 3.4.

Lemma 3.5. Let $X$ be a compact complex manifold or a submanifold of $\mathbb{C}^n$. Every $\omega \in A^{k,k}(X, \mathbb{C}) \cap A^{2k}(X, \mathbb{R})$ can be written as a finite linear combination of wedge products of real $(1, 1)$-forms of the type $\sqrt{-1} h \partial \bar{\partial} \rho$ where $h$ and $\rho$ are smooth real functions on $X$. 

Proof. We will follow the proof of Lemma 2.2. Namely, let \( \{U_\alpha\}_{\alpha \in A} \) be a finite coordinate open cover of \( X \) and \( \{\rho_\alpha\}_{\alpha \in A} \) be a partition of unity subordinate to it, so that \( \omega = \sum_{\alpha \in A} \rho_\alpha \omega \vert U_\alpha \). Denoting by

\[
z^1 = x^1 + \sqrt{-1} y^1, \ldots, z^n = x^n + \sqrt{-1} y^n
\]

each local complex coordinates in \( U_\alpha \), we can write

\[
\omega \vert U_\alpha = \sum_{I,J} f_{\alpha,IJ} dx^{i_1} \cdots dx^{i_I} \wedge dy^{j_1} \cdots dy^{j_J},
\]

where \( I = \{i_1, \ldots, i_I\}, J = \{j_1, \ldots, j_J\}, f_{\alpha,IJ} \in C^\infty(U_\alpha, \mathbb{R}) \) and \( 1 \leq i_1 < \cdots < i_I \leq n, 1 \leq j_1 < \cdots < j_J \leq n, I + J = 2k \). Since the form \( \omega \) was supposed to be of \((k, k)\) type, so are the forms \( \omega \vert U_\alpha \). On the other hand, the \((k, k)\)-component of these forms can be obtained by rewriting them in complex coordinates using

\[
dx^i = \frac{1}{2}(dz^i + d\bar{z}^i), \quad dy^j = \frac{1}{2\sqrt{-1}}(dz^i - d\bar{z}^i), \quad i = 1, \ldots, n,
\]

and collecting terms of the type \((k, k)\). If one of such terms has a factor \( dz^i \wedge d\bar{z}^j, i, k \in I \), then it necessarily has a factor

\[
(dz^i \wedge d\bar{z}^j + d\bar{z}^i \wedge dz^j),
\]

if it comes from \( dx^i \wedge dx^j \). Similarly, one has factors \( dz^j \wedge d\bar{z}^m + d\bar{z}^i \wedge dz^m \), \( j, m \in J \), coming from \( dy^j \wedge dy^m \), and \( \sqrt{-1}(dz^i \wedge d\bar{z}^j - d\bar{z}^i \wedge dz^j) \), coming from \( dx^i \wedge dy^j, i \in I \) and \( j \in J \).

In the first two cases corresponding factors can be written as

\[
2\sqrt{-1}\partial \bar{\partial} (\text{Im}(z^i \bar{z}^j)) \quad \text{and} \quad 2\sqrt{-1}\partial \bar{\partial} (\text{Im}(z^j \bar{z}^m)),
\]

whereas in the third case it takes the form \( 2\sqrt{-1}\partial \bar{\partial} (\text{Re}(z^i \bar{z}^j)) \). As in the proof of Lemma 2.2, let \( K_\alpha \) be a compact set such that \( \text{supp} \rho_\alpha \subseteq K_\alpha \subset U_\alpha \) and let \( b_\alpha \) be the corresponding bump function. Then we see that all the terms will take the form \( 2\sqrt{-1}\partial \bar{\partial} \rho \), where \( \rho(z) = \text{Im}(b_\alpha(z)z^i \bar{z}^j) \) or \( \rho(z) = \text{Re}(b_\alpha(z)z^i \bar{z}^j) \). This proves the first part of the statement. The second statement of the lemma is obvious from the construction. For submanifolds of \( C^n \), the local part of the argument above carries over globally using a tubular neighbourhood argument just as in Lemma 2.2. \( \square \)

Remark 3.7. Let \( \omega \) be a real differential form of pure type on a complex manifold \( X \), \( \omega \in \mathcal{A}^{k,k}(X, \mathbb{C}) \cap \mathcal{A}^{2k}(X, \mathbb{R}) \). We call the form \( \omega \) elementary, if it is a wedge product of \((1, 1)\)-forms \( \sqrt{-1} h \partial \bar{\partial} \rho \) with real-valued \( h \) and \( \rho \), and we call the form \( \omega \) composite, if it is a wedge product of \((1, 1)\)-forms \( \sqrt{-1} e^\theta \partial f \wedge \bar{\partial} \bar{f} \). According to Lemmas 3.4 and 3.5, on a compact complex manifold \( X \) every composite form is a finite linear combination of elementary forms and, conversely, every elementary form is a finite linear combination of composite forms. This is reminiscent of the “nuclear democracy” in the bootstrap model of the S-matrix theory in particle physics.

We have the following complex manifold analogue of Proposition 2.1.
Theorem 3.2. For every $\bar{\partial}\partial$-exact form $\omega \in A(X, \mathbb{C}) \cap A^{\text{even}}(X, \mathbb{R})$ on a complex manifold $X$ there is a trivial vector bundle $E$ over $X$ with two Hermitian metrics $h_1$ and $h_2$ such that

$$\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.$$ 

Proof. It is convenient to introduce a virtual Hermitian bundle $\mathcal{E} = E - E$ with corresponding Hermitian metrics $h_1$ and $h_2$, and to rewrite the above equation as $\text{ch} \mathcal{E} = \omega$. Thus defined Chern character form for virtual Hermitian bundles is obviously additive: if $\mathcal{W}_1 = W_1 - W_1$ with Hermitian metrics $h_1$ and $h_12$, and $\mathcal{W}_2 = W_2 - W_2$ with Hermitian metrics $h_2$ and $h_22$, then

$$\text{ch} \mathcal{W}_1 + \text{ch} \mathcal{W}_2 = \text{ch} \mathcal{W},$$

where $\mathcal{W} = W - W$ and $W = W_1 \oplus W_2$ with corresponding Hermitian metrics $h_1 = h_11 \oplus h_21$ and $h_2 = h_12 \oplus h_22$. Slightly abusing notations, we will write $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. The Chern character form for virtual Hermitian bundles is also multiplicative:

$$\text{ch} \mathcal{W}_1 \text{ch} \mathcal{W}_2 = \text{ch} \mathcal{W},$$

where $\mathcal{W} = W - W$ and $W = (W_1 \otimes W_2) \oplus (W_1 \otimes W_2)$ with corresponding Hermitian metrics

$$h_1 = (h_11 \otimes h_21) \oplus (h_12 \otimes h_22) \quad \text{and} \quad h_2 = (h_11 \otimes h_22) \oplus (h_12 \otimes h_21).$$

Slightly abusing notations, we will write $\mathcal{W} = \mathcal{W}_1 \otimes \mathcal{W}_2$.

Let $\omega$ be a real form of degree $(k, k)$, $k > 1$ which is a $\bar{\partial}\partial$ of a composite form:

$$(3.5) \quad \omega = \frac{1}{k - 1} \left( \frac{\sqrt{-1}}{2\pi} \right)^k \bar{\partial}\partial \left( e^{(k-1)\sigma} \partial f_1 \wedge \bar{\partial} \bar{f}_1 \wedge \ldots \wedge \partial f_{k-1} \wedge \bar{\partial} \bar{f}_{k-1} \right).$$

It follows from Corollary 3.1 that

$$(3.6) \quad \omega = \text{ch}_k \mathcal{F}_k \quad \text{and} \quad \text{ch}_i \mathcal{F}_k = 0, \quad i = 1, \ldots, k - 1,$$

where $\mathcal{F}_k = F_k - F_k$ and

$$F_k = \bigoplus_{l=1}^{k} \frac{1}{2} (1 + (-1)^l) \binom{k-1}{l-1} E_l$$

$$= \bigoplus_{l=1}^{k} \frac{1}{2} (1 - (-1)^l) \binom{k-1}{l-1} E_l$$

with Hermitian metrics

$$h_{1k} = \bigoplus_{l=1}^{k} \frac{1}{2} (1 + (-1)^l) \bigoplus_{1 \leq i_1 < \ldots < i_{l-1} \leq k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}})$$

$$h_{2k} = \bigoplus_{l=1}^{k} \frac{1}{2} (1 - (-1)^l) \bigoplus_{1 \leq i_1 < \ldots < i_{l-1} \leq k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}})$$

for $k > 2$, whereas $F_2 = E_2 = E_1 \oplus E_1$ and $h_{12} = h(\sigma, f), h_{22} = e^\sigma \oplus 1$. 
In particular, if \( \omega \) is a composite form of the top degree \((n,n)\), then
\[
\omega = \text{ch}\, \mathcal{F}_n.
\]

Now, we may use induction argument to finish the proof. Namely, suppose that the statement holds for all forms of degrees \((l,l)\), \(k < l \leq n\), and let \( \omega \) be a composite \((k,k)\)-form given by (3.5). According to (3.6), \( \omega - \text{ch}\, \mathcal{F}_k \) is a sum of forms of degrees \((l,l)\) with \(l > k\), so that by the induction hypothesis there exists a virtual Hermitian bundle \( \mathcal{F} \) such that \( \omega - \text{ch}\, \mathcal{F}_k = \text{ch}\, \mathcal{F} \). Thus
\[
\omega = \text{ch}\, \mathcal{E}, \quad \text{where} \quad \mathcal{E} = \mathcal{F}_k \oplus \mathcal{F}.
\]

Remark 3.8. When \( X \) is compact or is a submanifold of \( \mathbb{C}^n \) we can give another proof using Lemma 3.5. Firstly, the statement holds for \((1,1)\)-forms. Namely, since every real \( \bar{\partial}\partial\)-exact \((1,1)\) form is given by \( \omega = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial\sigma \), where \( \sigma \in \mathcal{C}^\infty(X,\mathbb{R}) \), consider the trivial holomorphic line bundle \( E_1 \) with the Hermitian metric \( h = e^{\sigma} \), so that
\[
\text{ch}(E_1, h) = \exp \omega = 1 + \omega + \frac{1}{2!} \omega^2 + \cdots + \frac{1}{n!} \omega^n.
\]

To get rid of all terms in this expression except \( \omega \), consider Hermitian metrics \( e^{\alpha_i \sigma}, i = 1, \ldots, n+1 \), and choose pair-wise distinct \( \alpha_i \) such that the following system of equations
\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+1} \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{n+1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1^n & \alpha_2^n & \alpha_3^n & \cdots & \alpha_{n+1}^n
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_{n+1}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

has an integer solution \( c_1, \ldots, c_{n+1} \). Namely, for any choice of \( n+1 \) different rational numbers \( \alpha_i \) the numbers \( c_i \) are also rational. If their least common denominator is \( N > 1 \), then for the numbers \( \beta_i = \alpha_i/N \) the corresponding solution is integral. Now putting
\[
(E, h_1) = \bigoplus_{c_i > 0} c_i(E_1, e^{h_i \sigma}) \quad \text{and} \quad (E, h_2) = \bigoplus_{c_i < 0} (-c_i)(E_1, e^{h_i \sigma}),
\]
where \( n(L, h) \) stands for the direct sum of \( n \) copies of a line bundle \( L \) with the Hermitian metric \( h \), we get
\[
\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.
\]

Now let \( \omega \) be a real form of degree \((k,k)\) which is a \( \bar{\partial}\partial \) of an elementary form:
\[
\omega = \left( \frac{\sqrt{-1}}{2\pi} \right)^k \bar{\partial}\partial (\rho_1 \bar{\partial}\partial \rho_2 \wedge \cdots \wedge \bar{\partial}\partial \rho_k) = \left( \frac{\sqrt{-1}}{2\pi} \right)^k \bar{\partial}\partial \rho_1 \wedge \bar{\partial}\partial \rho_2 \wedge \cdots \wedge \bar{\partial}\partial \rho_k. 
\]

Then
\[
\omega = \text{ch}\, \mathcal{E}, \quad \text{where} \quad \mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_k.
\]
Remark 3.9. It immediately follows from the second statement of Lemma 3.4 and the proof of Theorem 3.2, that if form $\omega$ vanishes on open $U \subset X$, then Hermitian metrics $h_1$ and $h_2$ can be chosen such that $h_1 = h_2 = I$ — identity matrix — on $U$.

**Corollary 3.3.** For every $\omega \in \mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X, \mathbb{R})$ of degree not greater than $2n - 2$, there is a trivial vector bundle $E$ over $X$ with two Hermitian metrics $h_1$ and $h_2$ such that in $\tilde{\mathcal{A}}(X, \mathbb{C})$

$$BC(E; h_1, h_2) = \omega.$$ 

**Proof.** It is analogous to the proof of Corollary 2.2. Namely, let $\Omega \in \mathcal{A}(X \times \mathbb{P}^1, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X \times \mathbb{P}^1, \mathbb{R})$ be such that under the inclusion map $i_p : X \to X \times \mathbb{P}^1$ one has $i^*_p(\Omega) = -\omega$ and $i^*_p(\Omega) = 0$ in some neighborhood of $0$ in $\mathbb{P}^1$. It follows from Theorem 3.2 that there is a trivial vector bundle $\tilde{E}$ over $X \times \mathbb{P}^1$ with two Hermitian metrics $\tilde{h}_1$ and $\tilde{h}_2$ such that

$$\sqrt{-1} \partial \bar{\partial} \Omega = \text{ch}(\tilde{E}, \tilde{h}_1) - \text{ch}(\tilde{E}, \tilde{h}_2),$$

where the metrics $\tilde{h}_1$ and $\tilde{h}_2$ can be chosen such that $i^*_p(\tilde{h}_1) = i^*_p(\tilde{h}_2) = I$.

Denoting by $E$ a trivial vector bundle over $X$ — a pullback of $\tilde{E}$ — and putting $h_1 = i^*_p(\tilde{h}_1), h_2 = i^*_p(\tilde{h}_2)$, we obtain, modulo $\text{Im} \partial + \text{Im} \bar{\partial}$,

$$\text{bc}(E; I, h_1) - \text{bc}(E; I, h_2) = \int_{\mathbb{P}^1} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Omega \log |z|^2$$

$$= \int_{\mathbb{P}^1} \frac{\sqrt{-1}}{2\pi} \partial_2 \bar{\partial}_2 \Omega \log |z|^2$$

$$= \int_{\mathbb{P}^1} \frac{\sqrt{-1}}{2\pi} \Omega \partial_2 \bar{\partial}_2 \log |z|^2$$

$$= i^*_\infty(\Omega) - i^*_0(\Omega)$$

$$= -\omega.$$

Therefore in $\tilde{\mathcal{A}}(X, \mathbb{C})$,

$$\omega = -\text{BC}(E; I, h_1) + \text{BC}(E; I, h_2) = \text{BC}(E; h_1, h_2). \quad \square$$

3.4. **Application to differential $K$-theory.** Recall that according to the definition of differential $K$-theory in [GS86], the $K$-group $\tilde{K}_0(X)$ for complex manifold $X$ is defined as the free abelian group generated by the triples $(E, h, \eta)$, where $E$ is a holomorphic vector bundle over $X$ with Hermitian metric $h$ and $\eta \in \tilde{\mathcal{A}}(X, \mathbb{C})$ with the following relations. For every exact sequence $\varepsilon$

$$0 \longrightarrow F \overset{i}{\longrightarrow} E \longrightarrow H \longrightarrow 0$$

of holomorphic vector bundles over $X$, endowed with arbitrary Hermitian metrics $h_F, h_E$ and $h_H$, impose

$$\text{bc}(\varepsilon, h_E, h_F, h_H),$$

$$\text{bc}((F, h_F, \eta') + (H, h_H, \eta'')) = (E, h_E, \eta' + \eta'') - \text{BC}(\varepsilon, h_E, h_F, h_H)).$$
where \( BC(\mathcal{E}, h_E, h_F, h_H) = bc(\mathcal{E}, h_E, h_F, h_H) \mod (\text{Im} \partial + \text{Im} \bar{\partial}) \) (see Remark 3.3). It follows from (3.7) that in \( \hat{K}_0(X) \)

\[
(E, h_1, \eta) = (E, h_2, \eta_2 - BC(E, h_1, h_2)).
\]

(3.8)

Now following [SS10], we define two Hermitian metrics \( h_1 \) and \( h_2 \) on the holomorphic vector bundle \( E \) to be equivalent, if \( BC(E, h_1, h_2) = 0 \), and define a \textit{structured} holomorphic Hermitian vector bundle \( \mathcal{E} \) as a pair \((E, \{h\})\), where \( \{h\} \) is the equivalence class of a Hermitian metric \( h \). Our goal is to impose a relations on a free abelian group generated by \( \mathcal{E} \) such the resulting group \( H\hat{K}_0(X) \) is isomorphic to the “reduced” differential \( K \)-theory group \( \hat{K}^{\text{rd}}_0(X) \), a subgroup of \( \hat{K}_0(X) \) with forms \( \eta \) of degrees not greater than \( 2n - 2 \).

First we observe that it follows from (3.8) that the mapping

\[
\mathcal{E} = (E, \{h\}) \mapsto \varepsilon(\mathcal{E}) = (E, h, 0) \in \hat{K}^{\text{rd}}_0(X)
\]

(3.9)

is well-defined. Next we show that when extended to to the free abelian group generated by the structured holomorphic Hermitian bundles, this mapping is onto. Indeed, for every \( \eta \in \hat{A}(X, \mathbb{C}) \cap A^{\text{even}}(X, \mathbb{R}) \) of degree not greater than \( 2n - 2 \), let \( F \) be the trivial vector bundle over \( X \) with two Hermitian metrics \( h_1 \) and \( h_2 \) such that, according to Corollary 3.3,

\[
(F, h_1, \eta) = (F, h_2, 0).
\]

in \( \hat{K}_0(X) \). Since

\[
(E \oplus F, h \oplus h_1, \eta) = (E, h, 0) + (F, h_1, \eta)
\]

\[
= (E, h, \eta) + (F, h_1, 0),
\]

we obtain

\[
(E, h, \eta) = (E, h, 0) + (F, h_2, 0) - (F, h_1, 0).
\]

Finally, we define the group \( H\hat{K}_0(X) \) as the quotient of the free abelian group generated by \( \mathcal{E} \) modulo the relations — pullbacks of the defining relations for \( \hat{K}^{\text{rd}}_0(X) \) by the mapping \( \varepsilon \). Explicitly, for every exact sequence \( \mathcal{E} \) of holomorphic vector bundles over \( X \) with Hermitian metrics \( h_F, h_E \) and \( h_H \) satisfying \( BC(\mathcal{E}; h_E, h_F, h_H) = 0 \), we impose

\[
(F, \{h_F\}) + (H, \{h_H\}) = (E, \{h_E\}).
\]

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ON BOTT-CHERN FORMS WITH APPLICATIONS TO DIFFERENTIAL $K$-THEORY

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Abstract. We use Chern-Weil theory for Hermitian holomorphic vector bundles with canonical connections for explicit computation of the Chern forms for trivial bundles with special non-diagonal Hermitian metrics. We prove that every $\bar{\partial}\partial$-exact real form of type $(k,k)$ on an $n$-dimensional complex manifold $X$ arises as a difference of the Chern character forms on trivial Hermitian vector bundles with canonical connections, and that modulo $\text{Im}\,\partial + \text{Im}\,\bar{\partial}$ every real form of type $(k,k)$, $k < n$, arises as a Bott-Chern form for two Hermitian metrics on some trivial vector bundle over $X$. The latter result is a complex manifold analogue of Proposition 2.6 in the paper [SS08b] by J. Simons and D. Sullivan.

1. Introduction

In [SS08b] J. Simons and D. Sullivan have constructed a simple geometric model for differential $K$-theory (see [HS05] and [BS10] for review). The model uses a codification of complex vector bundles with connection over a smooth manifold, by introducing the notion of a structured vector bundle — a pair $(V, \{\nabla\})$ consisting of a complex vector bundle $V$ over a smooth (that is, $C^\infty$) manifold $X$ and the equivalence class of a connection $\nabla$. Two connections $\nabla^0$ and $\nabla^1$ are said to be equivalent if the corresponding Chern-Simons differential form is exact. The main technical innovation in [SS08b] was Proposition 2.6 which states that all odd forms on $X$, modulo some natural relations, arise as the Chern-Simons forms between the trivial connection and arbitrary connections on trivial bundles over $X$. It allows one to prove that differential $K$-theory has a natural analogue of the celebrated Character Diagram for the ring of Cheeger-Simons differential characters (see [SS08a] and [CS85]).

For Hermitian holomorphic vector bundles — holomorphic vector bundles over the complex manifold $X$ with Hermitian metrics — analogues of the Chern-Simons forms are the Bott-Chern forms, which were introduced in [BC65] earlier than the Chern-Simons forms in [CS74]. The corresponding differential $K$-theory was defined by H. Gillet and C. Soulé in [GS86].

In this paper we use Chern-Weil theory for Hermitian holomorphic vector bundles with canonical connections for explicit computation of the Chern forms for trivial bundles with special non-diagonal Hermitian metrics. This
can be considered as the first step for explicitly computing Chern forms of the Hermitian holomorphic vector bundles using the transition functions, which we plan to address in the forthcoming paper. Here our goal is to obtain the analogue of Proposition 2.6 in [SS08b] for complex manifolds. Namely, we prove that all real forms of type $(k, k)$ on an $n$-dimensional complex manifold $X$, $k < n$, modulo $\text{Im} \bar{\partial} + \text{Im} \partial$, arise as Bott-Chern forms for Hermitian metrics on trivial vector bundles over $X$. As in the smooth manifold case, we deduce this statement from the result about Chern character forms which says that every $\bar{\partial}\partial$-exact real form of type $(k, k)$ on a complex manifold $X$ arises as a difference of the Chern character forms on trivial Hermitian vector bundles. Unlike the smooth manifold case, where the latter result follows by considering direct sums of line bundles with connections, in the complex manifold case one needs to consider vector bundles with non-diagonal Hermitian metrics. Our proof is based on several explicit computations of Chern forms for trivial vector bundles which may have interesting applications on their own.

Here is the more detailed content of the paper. In Section 2 for the convenience of the reader, we give a brief review of [SS08b]. Namely, we use the definition of the Chern-Simons forms inspired by the approach of H. Gillet and C. Soulé for the complex manifold case [GS86], and deduce a somewhat stronger analogue of Proposition 2.6 in [SS08b] — Corollary 2.2 — from Proposition 2.1. The latter states that for every exact even form $\omega$ on a smooth manifold $X$ there is a trivial vector bundle $V$ over $X$ with a connection $\nabla$ such that

$$\text{ch}(V, \nabla) - \text{ch}(V, d) = \omega,$$

where $d$ stands for the trivial connection on $V$. Another small technical innovation is a different proof of Theorem 1.15 in [SS08b] — Corollary 2.3 in the present paper — which does not appeal to the existence of universal connections.

In Section 3 we prove the main result, Theorem 3.2, which states that for every $\bar{\partial}\partial$-exact real form $\omega$ of type $(k, k)$ on a complex manifold $X$ there is a trivial vector bundle $E$ over $X$ with two Hermitian metrics $h_1$ and $h_2$ such that

$$\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.$$ 

The proof is based on Lemma 3.3 where we explicitly compute the Chern form of a trivial vector bundle over $X$ of arbitrary rank with special non-diagonal Hermitian metric, and on Lemma 3.4 where we express real forms of type $(k, k)$ as finite linear combinations of wedge products of real $(1, 1)$-forms of special type. We believe that these lemmas may have interesting applications on their own. We deduce the complex manifold analogue of Proposition 2.6 in [SS08b] — Corollary 3.3 — by using the Gillet-Soulé definition of the Bott-Chern forms [GS86]. Finally, we discuss how using Corollary 3.3 one can get rid of the differential form in the complex manifold
version of differential $K$-theory [GS86]. However, developing differential $K$-theory for the complex manifolds in the spirit of [SS08b] is an open and difficult problem since, in general, ‘inverses’ for the holomorphic bundles do not exist.

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2. Complex vector bundles over a smooth manifold

2.1. Chern-Simons secondary forms. Let $X$ be a smooth $n$-dimensional manifold, let $A(X) = \bigoplus_{k=1}^{n} A^k(X, \mathbb{C}) = A^{\text{even}}(X) \oplus A^{\text{odd}}(X)$ be the graded commutative algebra of smooth complex differential forms on $X$, and let $V$ be a $C^\infty$-complex vector bundle over $X$ with a connection $\nabla$. Recall that the Chern character form $\text{ch}(V, \nabla)$ for the pair $(V, \nabla)$ is defined by

$$\text{ch}(V, \nabla) = \text{Tr} \exp \left( \frac{\sqrt{-1}}{2\pi} \nabla^2 \right) \in A^{\text{even}}(X).$$

Here $\nabla^2$ is the curvature of the connection $\nabla$ — an $\text{End} V$-valued 2-form on $X$ — and $\text{Tr}$ is the trace in the endomorphism bundle $\text{End} V$. The Chern character form is closed, $d\text{ch}(V, \nabla) = 0$, and its cohomology class in $H^*(X, \mathbb{C})$ does not depend on the choice of $\nabla$.

Let $\nabla^0$ and $\nabla^1$ be two connections on $V$. In [CS74], S.S. Chern and J. Simons introduced secondary characteristic forms — the Chern-Simons forms. Namely, the Chern-Simons form $\text{cs}(\nabla^1, \nabla^0) \in A^{\text{odd}}(X)$ is defined modulo $A^{\text{even}}(X)$, satisfies the equation

$$d\text{cs}(\nabla^1, \nabla^0) = \text{ch}(V, \nabla^1) - \text{ch}(V, \nabla^0),$$

and enjoys a functoriality property under the pullbacks with smooth maps.

Here we present a construction of the Chern-Simons form $\text{cs}(\nabla^1, \nabla^0)$, which is similar to the construction of Bott-Chern forms for holomorphic vector bundles, given by H. Gillet and C. Soulé in [GS86]. Namely, for a given $V$ put $\tilde{V} = \pi^*(V)$, where $\pi : X \times S^1 \mapsto X$ is a projection, and $S^1 = \{e^{i\theta}, 0 \leq \theta < 2\pi\}$. Explicitly, $\tilde{V}$ is a bundle over $X \times S^1$ whose fibre at every point $(x, \theta) \in X \times S^1$ is $V_x \otimes \mathbb{C} \simeq V_x$. For every $\theta$ define the map $i_{\theta} : X \mapsto X \times S^1$ by $i_{\theta}(x) = (x, e^{i\theta})$, and let $\tilde{\nabla}$ be a connection on $\tilde{V}$ such that

$$i_{\theta}^*(\tilde{\nabla}) = \nabla^0, \quad i_{\theta}^*(\tilde{\nabla}) = \nabla^1.$$
Denote by $g$ a function defined by
\[
g(\theta) = \begin{cases} 
0 & \text{if } 0 \leq \theta < \pi, \\
1 & \text{if } \pi \leq \theta < 2\pi 
\end{cases}
\]
and extended $2\pi$-periodically to $\mathbb{R}$. It defines a function $g : S^1 \mapsto \mathbb{R}$, which is discontinuous at $0$ and $\pi$.

**Definition.** The Chern-Simons form is defined as
\[
(2.2) \quad cs(\nabla^1, \nabla^0) = \pi_* (g(\theta) \text{ch}(\tilde{V}, \tilde{\nabla})) = \int_{S^1} g(\theta) \text{ch}(\tilde{V}, \tilde{\nabla}) \in \mathcal{A}^{\text{odd}}(X)
\]
— direct image of $g(\theta) \text{ch}(\tilde{V}, \tilde{\nabla})$ under the projection $\pi : X \times S^1 \mapsto X$ (integration over the fibres of $\pi$).

**Remark 2.1.** Connection $\tilde{\nabla}$ is trivial to construct. If in local coordinates $\nabla^i = d_x + A^i(x)$, where $d_x$ is deRham differential on $X$ and $i = 0, 1$, then $\tilde{\nabla} = d_x + d_\theta + A(x, \theta)$,
where $A(x, \theta)$ is $2\pi$-periodic and $A(x, 0) = A^0(x)$, $A(x, \pi) = A^1(x)$.

**Lemma 2.1.** The Chern-Simons form $cs(\nabla^1, \nabla^0)$ satisfies the equation (2.1), and modulo $d\mathcal{A}^{\text{even}}(X)$ it does not depend on the choice of connection $\tilde{\nabla}$.

**Proof.** Using
\[
(d_x + d_\theta) \text{ch}(\tilde{\nabla}) = 0 \quad \text{and} \quad d_\theta g = (\delta_\pi - \delta_0) d\theta,
\]
we obtain
\[
d cs(\nabla^1, \nabla^0) = \int_{S^1} ((d_x + d_\theta) - d_\theta) (\text{ch}(\tilde{\nabla}))(\theta) = -\int_{S^1} d_\theta (\text{ch}(\tilde{\nabla}))(\theta) \]
\[
= \int_{S^1} \text{ch}(\tilde{\nabla}) d_\theta g = \text{ch}(\nabla^1) - \text{ch}(\nabla^0).
\]

It is also easy to see that modulo exact forms $cs(\nabla^1, \nabla^0)$ does not depend on the choice of $\tilde{\nabla}$. Namely, let $\tilde{\nabla} = d_x + d_\theta + A(x, \theta)$, $\tilde{\nabla}' = d_x + d_\theta + A'(x, \theta)$ be two such connections. Define a connection $\hat{\nabla}$ in the bundle $\hat{V}$ over $X \times S^1 \times S^1$ by
\[
\hat{\nabla} = d_x + d_\theta_1 + d_\theta_2 + \hat{A}(x, \theta_1, \theta_2),
\]
where
\[
\hat{A}(x, \theta_1, 0) = A(x, \theta_1), \quad \hat{A}(x, \theta_1, \pi) = A'(x, \theta_1) \quad \text{for all } \theta_1 \in [0, 2\pi],
\]
and \( \hat{A}(x, 0, \theta_2) = A_0(x), \hat{A}(x, \pi, \theta_2) = A_1(x) \) for all \( \theta_2 \in [0, 2\pi] \). Then
\[
\int_{S^1} (\text{ch}(\nabla) - \text{ch}(\nabla'))g(\theta_1) = \int_{S^1} (d_x + d_{\theta_1}) \text{cs}(\nabla, \nabla')g(\theta_1)
\]
\[
= d_x \int_{S^1} \text{cs}(\nabla, \nabla')g(\theta_1) - \text{cs}(\nabla, \nabla')|_{\theta_1=0}^{\theta_1=\pi}
\]
\[
= d_x \int_{S^1} \text{cs}(\nabla, \nabla')g(\theta_1) - \int_{S^1} \text{ch}(\nabla)g(\theta_2)|_{\theta_1=0}^{\theta_1=\pi}
\]
\[
= d_x \int_{S^1} \text{cs}(\nabla, \nabla')g(\theta_1).
\]
Here we have used that at \( \theta_1 = 0 \) and \( \theta_1 = \pi \) the restriction of the form \( \text{ch}(\hat{\nabla}) \in \mathcal{A}_{\text{even}}(X \times S^1 \times S^1) \) to \( X \times S^1 \) has no components along \( S^1 \) and its integral over \( S^1 \) is zero. \( \square \)

**Remark 2.2.** Formula (2.2) is similar to formula (1.5) in [SS08b] but has different applications than definition (1.2) in [SS08b].

**Definition.** Put
\[
\text{CS}(\nabla^1, \nabla^0) = \text{cs}(\nabla^1, \nabla^0) \mod d\mathcal{A}_{\text{even}}(X),
\]
which, according to Lemma 2.1, is a well-defined element in \( \tilde{\mathcal{A}}_{\text{odd}}(X) = \mathcal{A}_{\text{odd}}(X)/d\mathcal{A}_{\text{even}}(X) \).

**Remark 2.3.** Formula (2.1) can be written as
\[
\text{cs}(\nabla^1, \nabla^0) = \int_{\pi}^{2\pi} \text{ch}(\nabla),
\]
and the choice of points \( \pi \) and \( 2\pi \) on the unit circle is immaterial. Using the change of variables, for every \( \alpha < \beta \) on \( S^1 \) we get
\[
(2.3) \quad \text{cs}(\nabla^1, \nabla^0) = \int_{\alpha}^{\beta} \text{ch}(\nabla),
\]
where now \( i^\alpha_\beta(\nabla) = \nabla^0, i^\alpha_\beta(\nabla) = \nabla^1 \).

Using (2.3) and Lemma 2.1, we immediately get

**Corollary 2.1.**
\[
\text{CS}(\nabla^2, \nabla^0) = \text{CS}(\nabla^2, \nabla^1) + \text{CS}(\nabla^1, \nabla^0).
\]

**2.2. K-theory.** Let \( K_0(X) \) be the Grothendieck group of \( X \) — a quotient of the free abelian group generated by the isomorphism classes \([V]\) of complex vector bundles \( V \) over \( X \) by the relations \([V] + [W] = [V \oplus W]\).

In [SS08b], the authors defined a version of differential \( K \)-theory using the notion of a structured bundle. Namely, connections \( \nabla^0 \) and \( \nabla^1 \) on a complex vector bundle \( V \) over \( X \) are called equivalent, if \( \text{CS}(\nabla^1, \nabla^0) = 0 \). It follows from Corollary 2.1 that it is an equivalence relation.

**Definition.** A pair \( \mathcal{V} = (V, \{\nabla\}) \), where \( \{\nabla\} \) is an equivalence class of connections on \( V \), is called a structured bundle.
Denote by $\text{Struct}(X)$ the set of all equivalence classes of structured bundles over $X$. It is shown in [SS08b] that it is a commutative semi-ring with respect to the direct sum $\oplus$ and tensor product $\otimes$ operations, and we denote by $\hat{K}_0(X)$ the corresponding Grothendieck ring.

We have two natural ring homomorphisms: the “forgetful map”

$$\delta : \hat{K}_0(X) \to K_0(X),$$

given by $[\mathcal{V}] \mapsto [V]$ for $\mathcal{V} = (V, \{\nabla\})$, and

$$\text{ch} : \hat{K}_0(X) \to A^{\text{even}}(X),$$

given by the Chern character map $[\mathcal{V}] \mapsto \text{ch}(V, \nabla)$. For a trivial bundle $V$ with trivial connection $\nabla = d$, $\text{ch}(V, d) = \text{rk}(V)$ — the rank of $V$.

The mapping $\delta$ is surjective and for compact $X$ its kernel consists of all differences $[\mathcal{V}] - [\mathcal{F}]$ such that $\mathcal{V} = (V, \{\nabla\})$, where $V$ is stably trivial: $V \oplus M = N$ for some trivial bundles $M$ and $N$, and $\mathcal{F} = (F, \{\nabla_F\})$, where $F$ is trivial and $\text{rk}(F) = \text{rk}(V)$. Indeed, by definition,

$$\ker \delta = \{[U] - [W] | U \oplus M = W \oplus M \text{ for some trivial bundle } M\}.$$ 

Now let $W'$ be a vector bundle satisfying $W \oplus W' = K$, where $K$ is a trivial bundle\(^1\). Choose arbitrary connections $\nabla^{W'}$ and $\nabla^K$ on the bundles $W'$ and $K$, and introduce the bundles $V = U \oplus W'$ and $F = K$ with the connections $\nabla = \nabla^U \oplus \nabla^{W'}$ and $\nabla^F = \nabla^W \oplus \nabla^{W'}$. The bundle $V$ is stably trivial by construction: $V \oplus M = N$, where $N = F \oplus M$, and

$$[U] - [W] = [U \oplus W'] - [W \oplus W'] = [\mathcal{V}] - [\mathcal{F}].$$

It is an outstanding problem to describe the image of the Chern character map. The following simple result is crucial for our approach to the differential $K$-theory of Simons-Sullivan [SS08b].

**Proposition 2.1.** The image of the Chern character map

$$\text{ch} : \hat{K}_0(X) \to A^{\text{even}}(X)$$

contains all exact forms. Specifically, for every exact even form $\omega$ there is a trivial vector bundle $V = X \times \mathbb{C}^r$ with a connection $\nabla = d + A$ such that

$$\text{ch}(V, \nabla) - \text{ch}(V, d) = \omega.$$ 

The proof is based on the following useful result (see [dR84] and [Con85]); for the convenience of the reader, we prove it here as well.

**Lemma 2.2.** Every $\eta \in A^k(X)$ can be represented as a finite sum of the basic forms $f_1 df_2 \wedge \cdots \wedge df_{k+1}$, where $f_1, \ldots, f_{k+1}$ are smooth functions on $X$. If the form $\eta$ is real, one can choose the basic forms such that all functions $f_i$ are real-valued, and if $\eta$ is zero on an open $U \subseteq X$, there is a representation such that all functions $f_i$ vanish on $U$.

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\(^1\)Such bundle $W'$ exists since $X$ is compact.
Proof. When $X$ is a compact, one can choose a finite coordinate open cover \( \{ U_a \}_{a \in A} \) of $X$ and a partition of unity \( \{ \rho_a \}_{a \in A} \) subordinated to it. Then \( \eta = \sum_{a \in A} \rho_a \eta|_{U_a} \), where in local coordinates \( x^1, \ldots, x^n \) on $U_a$,

\[
\eta|_{U_a} = \sum_I f_{a,I} dx^{i_1} \wedge \cdots \wedge dx^{i_k},
\]

where \( I = \{ i_1, \ldots, i_k \} \) and \( 1 \leq i_1 < \cdots < i_k \leq n \). Let \( K_\alpha \) be a compact set such that \( \text{supp} \rho_\alpha \subseteq K_\alpha \subseteq U_\alpha \) and let \( b_\alpha \) be a “bump function” — a smooth function on $X$ which is 1 on \( \text{supp} \rho_\alpha \) and zero outside \( K_\alpha \). Then

\[
\eta_\alpha = \sum_I \rho_a f_{a,I} d(x^{i_1} b_\alpha(x)) \wedge \cdots \wedge d(x^{i_k} b_\alpha(x)) \in \mathcal{A}^k(X)
\]

is of required form and \( \eta = \sum_{a \in A} \eta_\alpha \). The second statement of the lemma is obvious from this construction.

In the general case one can use an embedding $f : X \to \mathbb{R}^N$ of the manifold $X$ into Euclidean space (say the Whitney embedding into $\mathbb{R}^{2n}$). By considering the tubular neighbourhood of $f(X)$ in $\mathbb{R}^N$, it is easy to show that the pullback map $f^* : \mathcal{A}(\mathbb{R}^N) \to \mathcal{A}(X)$ is onto, which proves the result. \qed

Proof of Proposition 2.1. Induction by the degree in $d\mathcal{A}^{\text{odd}}(X) \subset \mathcal{A}^{\text{even}}(X)$. According to Lemma 2.2, it is sufficient to consider only basic forms in $\mathcal{A}^{\text{odd}}(X)$.

For a basic 1-form $\alpha = f_1 df_2$ we have $\omega = d\alpha = df_1 \wedge df_2$, so that

\[
\text{ch}(L, \nabla) - \text{ch}(L, d) = \text{ch}(L, \nabla) - 1 = \omega,
\]

where $L$ is a trivial line bundle over $X$ with $\nabla = d - 2\pi \sqrt{-1} f_1 df_2$.

Now suppose that all exact forms of degree $\leq 2k$ are in the image of $\text{ch}$. For a basic $(2k + 1)$-form $\alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2}$ we have $\omega = d\alpha = df_1 \wedge df_2 \wedge \cdots \wedge df_{2k+2}$, which can be also written as

\[
\omega = \frac{1}{(k + 1)!} (df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2})^{k+1}.
\]

Let $V$ be a trivial line bundle over $X$ with

\[
\nabla = d - 2\pi \sqrt{-1} (f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}),
\]

so that

\[
\nabla^2 = -2\pi \sqrt{-1} (df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2}).
\]

Then $\text{ch}(V, \nabla) - 1 - \omega$ is an exact form of degree $\leq 2k$ and, by induction, is in the image of $\text{ch}$. \qed

Remark 2.4. It the form $\omega$ is real then the connection $\nabla$ in Proposition 2.1 is compatible with the metric on $V$ given by the standard Hermitian metric on $\mathbb{C}^n$.

Remark 2.5. It immediately follows from the second statement of Lemma 2.2 and the proof of Proposition 2.1 that if form $\omega$ vanishes on open $U \subset X$, then connection $\nabla = d + A$ can be chosen such that $A = 0$ on $U$. 
Corollary 2.2. For every $\alpha \in \tilde{A}^{\text{odd}}(X)$ there is a trivial vector bundle $V$ with connection $\nabla$ such that $\text{CS}(\nabla, d) = \alpha$.

Proof. For the given $\alpha \in \tilde{A}^{\text{odd}}(X)$ let $\Theta \in \tilde{A}^{\text{odd}}(X \times S^1)$ be such that under the inclusion map $i_\theta : X \to X \times S^1$ one has $i_\theta^*(\Theta) = \alpha$ and $i_\theta^*(\Theta) = 0$ for all $\theta$ in some neighborhood of 0. Applying Proposition 2.1 to the manifold $X \times S^1$ and the exact even form $-d\Theta$, we have

$$\text{ch}(\tilde{V}, \tilde{\nabla}) - \text{rk}(\tilde{V}) = -(d_x + d_\theta)\Theta.$$ 

Integrating over $\theta$ from $\pi$ to $2\pi$ we get

$$\alpha = \text{cs}(\nabla^1, \nabla^0) + d_x \int_0^{2\pi} \Theta,$$

for connections $\nabla^0 = i_0^*(\tilde{\nabla})$ and $\nabla^1 = i_\pi^*(\tilde{\nabla})$ on a trivial bundle $V$ — a pullback of the trivial bundle $\tilde{V}$ to $X$. Finally, it follows from Remark 2.5 that one can choose connection $\tilde{\nabla}$ on $\tilde{V}$ such that $i_0^*(\tilde{\nabla}) = d$. Thus putting $\nabla = \nabla^1$ we obtain $\text{CS}(\nabla, d) = \alpha \mod d\tilde{A}^{\text{even}}(X)$.

Remark 2.6. Corollary 2.2 gives somewhat stronger form of Proposition 2.6 in [SS08b], the so-called “Venice lemma” of J. Simons. It has been used in [SS08b] to prove that one can remove the differential form from the definition of the differential $K$-theory given by M. Hopkins and I. Singer [HS05].

Remark 2.7. Here is a direct proof of Corollary 2.2 which is close to the original argument in [SS08b]. Let $\eta$ be a 1-form on $X$ and let $L$ be a trivial line bundle with the connection $\nabla = d - 2\pi \sqrt{-1} \eta$. It follows from the homotopy formula in [SS08b] that

$$\text{CS}(\nabla, d) = \int_0^1 \exp\{dt \wedge \eta + td\eta\} = \sum_{l \geq 1} \frac{1}{l!} \eta \wedge (d\eta)^l.$$

Thus for the basic 1-form $\alpha = f_1 df_2$ putting $\eta = \alpha$ we get $\alpha = \text{CS}(\nabla, d)$. Now suppose that the result is valid for all odd forms of degree $\leq 2k - 1$, and let $\alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2}$ be a basic form of degree $2k + 1$. Putting $\eta = f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}$ we obtain in $\tilde{A}^{\text{odd}}(X)$,

$$\text{CS}(\nabla, d) = \frac{1}{(k+1)!} \eta \wedge (d\eta)^k - \xi = \alpha - \xi,$$

where $\xi$ is a sum of odd forms of degrees $\leq 2k - 1$. By the induction hypothesis, there is a trivial vector bundle $V$ with the connection $\tilde{\nabla}$ such that $\text{CS}(\tilde{\nabla}, \tilde{d}) = \xi$, so that $\alpha = \text{CS}(\nabla \oplus \tilde{\nabla}, \tilde{d})$.

Recall that a flat connection $\nabla$ on a trivial vector bundle $F$ is a connection with trivial holonomy around any closed path in $X$. Equivalently, $\nabla = d\gamma = d + g^{-1}dg$, where $g : X \to GL(r, \mathbb{C})$, $r = \text{rk}(F)$, is a global parallel frame. The corresponding structured bundle $\mathcal{F} = (F, \{\nabla\})$ is called flat. Since any two flat connections on a trivial bundle $F$ are gauge equivalent, flat bundles

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of a fixed rank $r$ correspond to a single point in $\text{Struct}(X)$, which following \cite{SS08b}, we denote by $[r]$. Also, denote by $\mathcal{T}(X)$ a subgroup in $\tilde{A}^{\text{odd}}(X)$ consisting of $\text{CS}(\nabla, \nabla')$ for all trivial bundles $F$ and flat connections $\nabla$, $\nabla'$ on $F$.

Remark 2.8. According to Lemma 2.3 in \cite{SS08b}, the group $\mathcal{T}(X)$ has the following description. Let $\Theta$ be the bi-invariant closed odd form on the stable general linear group $\text{GL}(\infty)$ such that the free abelian group generated by all distinct products of its components represent the entire cohomology ring of $\text{GL}(\infty)$ over $\mathbb{Z}$. Then

$$\mathcal{T}(X) = \{ g^*(\Theta) \mid \text{for all smooth } g : X \to \text{GL}(\infty) \}/dA^{\text{even}}(X).$$

Now following \cite{SS08b}, let

$$\text{Struct}_{ST}(X) = \{ [V] = [(V, \{\nabla\})] \in \text{Struct}(X) \mid V \text{ is stably trivial} \}$$

be the stably trivial sub-semigroup of $\text{Struct}(X)$, and for $V \in \text{Struct}_{ST}(X)$ define

$$\tilde{\text{CS}}(V) = \text{CS}(\nabla^N, \nabla \oplus \nabla^F) \in A^{\text{odd}}(X)/dA^{\text{even}}(X),$$

where $V \oplus F = N$ with trivial bundles $F$ and $N$, and $\nabla^F$, $\nabla^N$ are flat connections on these bundles. According to Proposition 2.4 in \cite{SS08b}, for another choice of trivial bundles $\tilde{F}$ and $\tilde{N}$ with flat connections $\nabla^{\tilde{F}}$, $\nabla^{\tilde{N}}$ we have

$$\text{CS}(\nabla^N, \nabla \oplus \nabla^F) - \text{CS}(\nabla^{\tilde{N}}, \nabla \oplus \nabla^{\tilde{F}}) \in \mathcal{T}(X),$$

so that the mapping $\tilde{\text{CS}} : \text{Struct}_{ST}(X) \to \tilde{A}^{\text{odd}}(X)/\mathcal{T}(X)$ is a well-defined homomorphism of semigroups.

Remark 2.9. One can choose $F = X \times \mathbb{C}^r$ with $\nabla^F = d$ and $\nabla^N = dg$, where $g$ is the isomorphism between $V \oplus F$ and $N = X \times \mathbb{C}^k$, and put $\tilde{\text{CS}} = \text{CS}(\nabla \oplus d, dg)$.

According to Corollary 2.2 the map $\tilde{\text{CS}}$ is surjective, and according to Proposition 2.5 in \cite{SS08b}, $\ker \tilde{\text{CS}} = \text{Struct}_{SF}(X)$ — the subgroup of stably flat structured bundles. By definition, $V \in \text{Struct}_{ST}(X)$ is stably flat, if

$$V \oplus \mathcal{F} = \mathcal{N},$$

where $\mathcal{F} = (F, \{\nabla^F\})$ and $\mathcal{N} = (N, \{\nabla^N\})$ are trivial bundles with equivalence classes of flat connections. Since map $\tilde{\text{CS}}$ is onto and $\tilde{A}^{\text{odd}}(X)/\mathcal{T}(X)$ is a group, for every $V \in \text{Struct}_{ST}(X)$ there is $\mathcal{W} \in \text{Struct}_{SF}(X)$ such that $V \oplus \mathcal{W} \in \text{Struct}_{SF}(X)$. This introduces a group structure on the coset space $\text{Struct}_{ST}(X)/\text{Struct}_{SF}(X)$, and we arrive at the following statement.

Proposition 2.2. The map $\tilde{\text{CS}}$ induces a group isomorphism

$$\tilde{\text{CS}} : \text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \to \tilde{A}^{\text{odd}}(X)/\mathcal{T}(X).$$

From this result we immediately obtain Theorem 1.15 in \cite{SS08b}.
Corollary 2.3. Every structured bundle over a compact manifold $X$ has a structured inverse: for every $V = (V, \{\nabla\}) \in \text{Struct}(X)$ there exists $W = (W, \{\nabla^W\}) \in \text{Struct}(X)$ such that

$$V \oplus W = \mathcal{N},$$

where $\mathcal{N} = (N, \{\nabla^N\})$ is a trivial bundle with flat connection.

Proof. For $V = (V, \{\nabla\}) \in \text{Struct}(X)$ let $U$ be such that $V \oplus U = F$ — a trivial bundle over $X$. Then $F = (F, \{\nabla \oplus \nabla^U\}) \in \text{Struct}_{ST}(X)$ for any choice of connection $\nabla^U$ on $U$. By Proposition 2.2, there exists $H = (H, \{\nabla^H\}) \in \text{Struct}_{SF}(X)$, i.e., there are trivial bundles $M$ and $N$ with flat connections $\nabla^M$ and $\nabla^M$ such that $F \oplus H \oplus M = \mathcal{N}$. Putting

$$W = (U \oplus H \oplus M, \{\nabla \oplus \nabla^H \oplus \nabla^M\}),$$

we obtain $V \oplus W = \mathcal{N}$. $\square$

From Corollary 2.3 we immediately obtain, as in [SS08b], that

- The Grothendieck group $\hat{K}(X)$ consists of elements $[V] - [r]$.
- The element $[V] - [r] = 0$ if and only if $V = (V, \{\nabla\})$ is stably flat and $r = \text{rk}(V)$.
- The mapping $\Gamma : \text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \to \hat{K}(X)$, defined by

$$\Gamma([V]) = [V] - [\text{rk}(V)],$$

gives an isomorphism $\text{Struct}_{ST}(X)/\text{Struct}_{SF}(X) \simeq \ker \delta$.

Denoting by $i = \Gamma \circ \hat{CS}^{-1} : \hat{A}^{\text{odd}}(X)/T(X) \to \ker \delta$, we obtain the main part of the result in [SS08b], the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \hat{A}^{\text{odd}}(X)/T(X) & \overset{i}{\longrightarrow} & \hat{K}(X) & \overset{\delta}{\longrightarrow} & K(X) & \longrightarrow & 0 \\
0 & \longrightarrow & d\hat{A}^{\text{odd}}(X) & \overset{j}{\longrightarrow} & Z^{\text{even}}(X) & \longrightarrow & H^r_{\text{dR}}(X) & \longrightarrow & 0,
\end{array}
$$

where $Z^{\text{even}}(X)$ is a subspace of closed forms in $\hat{A}^{\text{even}}(X)$, and the map $j$ is an inclusion.

3. Holomorphic vector bundles over a complex manifold

3.1. Bott-Chern secondary forms. Let $(E, h)$ be a holomorphic Hermitian vector bundle — a holomorphic vector bundle of rank $r$ over a complex manifold $X$, $\dim_{\mathbb{C}} X = n$, with the Hermitian metric $h$. For a given open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$, the bundle $E$ can be defined in terms of the transition functions: holomorphic maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(r, \mathbb{C})$, satisfying the

\[\text{(For the convenience of the reader who is not a geometer, here we briefly recall the basic definitions (see, e.g., [Wel08]).)}\]
cycocycle condition
\[ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad \text{on} \quad U_{\alpha} \cap U_{\beta} \cap U_{\gamma}, \]
where \( I \) is \( r \times r \) identity matrix. In terms of the transition functions, a Hermitian metric \( h \) on \( E \) is the collection \( h = \{ h_\alpha \}_{\alpha \in A} \), where \( h_\alpha \) are positive-definite Hermitian \( r \times r \) matrix-valued functions on \( U_\alpha \), satisfying
\[ h_\beta = g^{*}_{\alpha\beta}h_\alpha g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta, \]
and \( g^* \) stands for the Hermitian conjugation.

Denote by \( \nabla \) the canonical connection on the holomorphic Hermitian bundle \( (E, h) \). In terms of the open cover \( \{ U_\alpha \}_{\alpha \in A} \) and the transition functions it is given by the collection \( \nabla = \{ \nabla_\alpha \}_{\alpha \in A} \),
\[ \nabla_\alpha = d + A_\alpha = \partial + \bar{\partial} + A^{1,0}_\alpha + A^{0,1}_\alpha, \]
where \( A^{0,1}_\alpha = 0 \) and \( A^{1,0}_\alpha = h^{-1}_\alpha \partial h_\alpha \) are \( r \times r \) matrix-valued \((1,0)\)-forms on \( U_\alpha \), satisfying
\[ A_\beta = g^{-1}_{\alpha\beta}A_\alpha g_{\alpha\beta} + g^{-1}_{\alpha\beta}\partial g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \]

The curvature of the canonical connection \( \nabla = d + A \) on holomorphic Hermitian vector bundle \( (E, h) \) is a collection \( \Theta = \{ \Theta_\alpha \}_{\alpha \in A} \), where \( \Theta_\alpha = \partial A_\alpha \) are \( r \times r \) matrix-valued \((1,1)\)-forms on \( U_\alpha \), satisfying
\[ \Theta_\beta = g^{-1}_{\alpha\beta}\Theta_\alpha g_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \]

The Chern-Weil theory associates to any polynomial \( \Phi \) on \( \text{GL}(r, \mathbb{C}) \), invariant under conjugation, a collection \( \{ \Phi(\Theta_\alpha) \}_{\alpha \in A} \) which, according to (3.1), defines a global differential form \( \Phi(\Theta) \) on \( X \). The total Chern form \( c(E, h) \) and the Chern character form \( \text{ch}(E, h) \) of a holomorphic Hermitian vector bundle \( (E, h) \) are special cases of this construction and are defined, respectively, by
\[ c(E, h) = \det \left( I + \frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_{k=0}^{r} c_k(E, h) \]
and
\[ \text{ch}(E, h) = \text{Tr} \exp \left( \frac{\sqrt{-1}}{2\pi} \Theta \right) = \sum_{k=0}^{n} \text{ch}_k(E, h). \]

Since
\[ \text{ch}(E, h) \in \mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X, \mathbb{R}), \quad \mathcal{A}(X, \mathbb{C}) = \bigoplus_{p=0}^{n} \mathcal{A}^{p,p}(X, \mathbb{C}), \]
the Chern character form is \( \partial \) and \( \bar{\partial} \) closed.

Let \( h_1 \) and \( h_2 \) be two Hermitian metrics on a holomorphic vector bundle \( E \) over a complex manifold \( X \). In the classic paper [BC65], Bott and Chern have shown that there exist secondary characteristic forms, so-called Bott-Chern
forms — even differential forms $bc(E; h_1, h_2) \in \tilde{A}(X, \mathbb{C}) = A(X, \mathbb{C})/(\text{Im} \partial + \text{Im} \bar{\partial})$, satisfying

$$
(3.2) \quad \text{ch}(E, h_2) - \text{ch}(E, h_1) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log |z|^2 ch(\tilde{E}, \tilde{h})
$$

and the natural functorial property

$$
(3.3) \quad bc(f^*(E), f^*(h_1), f^*(h_2)) = f^*(bc(E; h_1, h_2))
$$

for holomorphic maps $f : Y \to X$ of complex manifolds. The proof in [BC65, Proposition 3.15] uses the analogue of the homotopy formula in Chern-Weil theory.

Remark 3.1. In the smooth manifold case, using linear homotopy of connections $\nabla_t$, it possible to integrate over $t$ in the homotopy formula in a closed form and to obtain explicit formulas for the Chern-Simons forms (see, e.g., [SS08b]). However in the complex manifold case for any homotopy $h_t$ of Hermitian metrics, due to the presence of inverses $h_t^{-1}$ in $\Theta_t$, it is not possible to integrate over $t$ in the homotopy formula in a closed form and to obtain explicit formulas for the Bott-Chern forms in terms of the Hermitian metrics $h_1$ and $h_2$ only.

In [GS86], Gillet and Soulé gave another definition of the Bott-Chern secondary classes which is also well-suited for short exact sequences of holomorphic vector bundles over $X$, which are used for defining the $K$-theory of $X$. Namely, let $E$ be a holomorphic vector bundle over $X$ with Hermitian metrics $h_1$ and $h_2$, let $\mathcal{O}(1)$ be the standard holomorphic line bundle of degree 1 over the complex projective line $\mathbb{P}^1$, and let $\tilde{E} = E \otimes \mathcal{O}(1)$ be the corresponding vector bundle over $X \times \mathbb{P}^1$. If $i_p : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the natural inclusion map $i_p(x) = (x, p)$ then $i_p^*(\tilde{E}) \simeq E$ for all $p \in \mathbb{P}^1$. Let $\tilde{h}$ be a Hermitian metric on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_1$ and $i_{\infty}^*(\tilde{h}) = h_2$ (such a metric is constructed using partition of unity).

**Definition.** The Bott-Chern secondary form is defined as

$$
(3.4) \quad bc(E; h_1, h_2) = \int_{\mathbb{P}^1} \text{ch}(\tilde{E}, \tilde{h}) \log |z|^2
$$

— direct image of $\log |z|^2 \text{ch}(\tilde{E}, \tilde{h})$ under the projection $\pi : X \times \mathbb{P}^1 \to X$ (integration over the fibres of $\pi$). The integral is convergent since $\log |z|^2 \omega(z)$, where $\omega$ is any smooth $(1,1)$-form on $\mathbb{P}^1$, is integrable.

**Lemma 3.1** (H. Gillet and C. Soulé). *The Bott-Chern form $bc(E; h_1, h_2)$ satisfies equations (3.2) and (3.3), and modulo $\text{Im} \partial + \text{Im} \bar{\partial}$ does not depend on the choice of Hermitian metric $\tilde{h}$.***

---

4In a separate publication we will show that one can get explicit formulas for Bott-Chern forms using some natural coordinates on the space of positive-definite Hermitian matrices.
The proof of (3.2) uses Poincaré-Lelong formula:

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log |z|^2 = \delta_\infty - \delta_0$$

(see [GS86, BGS88]), and Lemma 2.1 uses a simplified version of this argument. As in the previous section, we put

$$\text{BC}(E; h_1, h_2) = \text{bc}(E; h_1, h_2) \mod(\text{Im} \partial + \text{Im} \bar{\partial}).$$

Remark 3.2. Note that formula (3.4) the for Bott-Chern forms uses the Green’s function $\log |z|^2$ of the Laplace operator on $\mathbb{P}^1$, whereas formula (2.2) for the Chern-Simons form uses the Green’s function $g(\theta)$ of the operator $d/d\theta$ on $S^1$.

Remark 3.3. In fact, Gillet and Soulé in [GS86] (and with J.-M. Bismut in [BGS88]) defined Bott-Chern forms for short exact sequences of holomorphic vector bundles over $X$. Namely, let $\mathcal{E}$

$$0 \longrightarrow F \overset{i}{\longrightarrow} E \longrightarrow H \longrightarrow 0$$

be such an exact sequence, where holomorphic bundles $F, E$ and $H$ are equipped with Hermitian metrics $h_F, h_E$ and $h_H$. Put $F(1) = F \otimes \mathcal{O}(1)$ and consider the map $\text{id} \otimes \sigma : F \to F(1)$, where $\sigma$ is a holomorphic section of the bundle $\mathcal{O}(1)$ over $\mathbb{P}^1$ with a single zero at $\infty$. Let

$$\tilde{E} = (F(1) \oplus E)/F$$

be the quotient bundle over $X \times \mathbb{P}^1$, where $F$ is mapped diagonally into $F(1) \oplus E$ by $(\text{id} \otimes \sigma) \oplus i$. Then under the embedding $i_\pi : X \to X \times \mathbb{P}^1$ we will have $i_0^*(\tilde{E}) = E$ and $i_\infty^*(\tilde{E}) = F \oplus H$ since $E/F \simeq H$. There exists a Hermitian metric $\tilde{h}$ on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_E$ and $i_\infty^*(\tilde{h}) = h_F \oplus h_H$, and the Bott-Chern secondary form for the exact sequence $\mathcal{E}$ and Hermitian metrics $h_F, h_E, h_H$ is defined by Gillet and Soulé [GS86] by the formula

$$\text{bc}(\mathcal{E}; h_E, h_F, h_H) = \int_{\mathbb{P}^1} \text{ch}(\tilde{E}, \tilde{h}) \log |z|^2.$$

Similar to (3.2), the Bott-Chern forms satisfy the equation

$$\text{ch}(F \oplus H, h_F \oplus h_H) - \text{ch}(E, h_E) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \text{bc}(\mathcal{E}; h_E, h_F, h_H),$$

are functorial, and vanish when the exact sequence $\mathcal{E}$ holomorphically splits and $h_E = h_F \oplus h_H$ (see [GS86, BGS88]).

---

5 Here and in what follows we use the same notation for bundles over $X$ and their pullbacks under the projection $\pi : X \times \mathbb{P}^1 \to X$.

6 It is for this construction that it is necessary to twist the bundle $F$ by $\mathcal{O}(1)$. 

---

ON BOTT-CHERN FORMS
3.2. Chern forms of trivial bundles. We start with the following simple linear algebra result.

**Lemma 3.2.** Let \( \alpha_i, \beta_i, i = 1, \ldots, k \), be odd elements in some graded commutative algebra \( A \) over \( \mathbb{C} \) (e.g., the algebra of complex differential forms on \( X \)), and let \( A \) be a \( k \times k \) matrix with even elements \( A_{ij} = \alpha_i \beta_j \), and put \( a = \text{Tr} A = \sum_{i=1}^{k} \alpha_i \beta_i \). Then for every \( \lambda \in \mathbb{C} \),

\[
(I - \lambda A)^{-1} = I + \frac{\lambda}{1 + \lambda a} A = I + (\lambda - \lambda^2 a + \cdots + (-1)^k \lambda^{k+1} a^k) A,
\]

where \( I \) is \( k \times k \) identity matrix, and also

\[
\det(I - \lambda A) = \frac{1}{1 + \lambda a} = 1 - \lambda a + \cdots + (-1)^k \lambda^k a^k.
\]

**Proof.** Consider the following identity

\[
\frac{d}{d\lambda} \log \det(I - \lambda A) = -\text{Tr} A(I - \lambda A)^{-1},
\]

whose validity for matrices over \( \mathbb{C} \) and small \( \lambda \) follows from Jordan canonical form, and for matrices with even nilpotent entries — from the definition of the determinant. Since \( A^2 = -aA \), we obtain

\[
(I - \lambda A)^{-1} = I + \frac{\lambda}{1 + \lambda a} A,
\]

so that

\[
\frac{d}{d\lambda} \log \det(I - \lambda A) = -\frac{a}{1 + \lambda a} = -\frac{d}{d\lambda} \log(1 + \lambda a).
\]

Integrating and using \( \det I = 1 \), we get the result. \( \square \)

The next result is an explicit computation of the total Chern form of a trivial vector bundle with a special non-diagonal Hermitian metric.

**Lemma 3.3.** Let \( E_r = X \times \mathbb{C}^r \) be a trivial rank \( r \) vector bundle over \( X \) with a Hermitian metric \( h \) given by

\[
h = h(\sigma, f_1, \ldots, f_{r-1}) = g^* g, \quad \text{where} \quad g = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & f_1 \\
0 & 1 & 0 & \cdots & 0 & f_2 \\
0 & 0 & 1 & \cdots & 0 & f_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & f_{r-1} \\
0 & 0 & 0 & \cdots & 0 & e^{\sigma/2}
\end{pmatrix},
\]

and \( f_1, \ldots, f_{r-1} \in C^\infty(X, \mathbb{C}), \sigma \in C^\infty(X, \mathbb{R}) \). Then

\[
c(E_r, h) = c(E_1, e^\sigma) + \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \left( 1 - \frac{\sqrt{-1}}{2\pi} U \right),
\]

In the latter case log and inverse are given, correspondingly by the finite sums — truncated power series.
where \( U = e^{-\sigma} \sum_{i=1}^{r-1} \partial f_i \wedge \bar{\partial} f_i, \) \( E_1 = \det E_r \) is a trivial line bundle over \( X, \) and for a nilpotent element \( a \) of order \( r, \) \( \log(1 - a) = -(a + \frac{a^2}{2} + \cdots + \frac{a^{r-1}}{r-1}). \)

Equivalently,

\[
c_1(E_r, h) = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \sigma, \quad c_k(E_r, h) = -\frac{1}{k-1} \left( \frac{\sqrt{-1}}{2\pi} \right)^k \partial \bar{\partial} U^{k-1}, \quad k = 2, \ldots, r.
\]

**Proof.** Let \( \Theta = \bar{\partial}(h^{-1}\partial h) \) be the curvature form associated with the Hermitian metric \( h. \) We need to prove that for every \( \lambda \in \mathbb{C}, \)

\[
\det(I + \lambda \Theta) = 1 + \lambda \bar{\partial}\partial \sigma + \lambda \bar{\partial}\partial \log(1 - \lambda U)
\]

\[
= 1 + \lambda \bar{\partial}\partial \sigma - \lambda^2 \frac{\partial \bar{\partial} U}{1 - \lambda U} - \lambda^3 \frac{\partial \bar{\partial} U \wedge \partial U}{(1 - \lambda U)^2},
\]

where

\[
\frac{1}{1 - \lambda U} = \sum_{k=0}^{r-1} \lambda^k U^k \quad \text{and} \quad \frac{1}{(1 - \lambda U)^2} = \sum_{k=0}^{r-1} (k + 1) \lambda^k U^k.
\]

It is convenient to represent the matrix \( I + \lambda \Theta \) in the following block form

\[
I + \lambda \Theta = \begin{pmatrix}
I + \lambda \Theta_{11} & \lambda \Theta_{12} \\
\lambda \Theta_{21} & 1 + \lambda \Theta_{22}
\end{pmatrix},
\]

where \((r - 1) \times (r - 1)\) matrix \( \Theta_{11}, (r - 1)\)-vectors \( \Theta_{12}, \Theta_{21} \), and the scalar \( \Theta_{22} \) are given by

\[
\Theta_{11} = \left\{ -\bar{\partial}(\bar{f}_i e^{-\sigma} \partial f_j) \right\}_{i,j=1}^{r-1}, \quad \Theta_{12} = \left\{ \bar{\partial}\partial \bar{f}_i - \bar{\partial}(\bar{f}_i F) - \bar{\partial}(\bar{f}_i \partial \sigma) \right\}_{i=1}^{r-1},
\]

\[
\Theta_{21} = \left\{ \bar{\partial}(e^{-\sigma} \partial f_1) \right\}_{i=1}^{r-1}, \quad \Theta_{22} = \bar{\partial}\partial \sigma + \bar{\partial} F,
\]

and \( F = e^{-\sigma} \sum_{i=1}^{r-1} \bar{f}_i \partial f_i. \) The row operations \( R_i \mapsto R_i + \bar{f}_i R_r \) transform the matrix \( I + \lambda \Theta \) to the form

\[
\begin{pmatrix}
I - \lambda A & b \\
c & d
\end{pmatrix},
\]

where

\[
A = \{ e^{-\sigma} \bar{\partial} \bar{f}_i \wedge \partial f_j \}_{i,j=1}^{r-1}, \quad b = \{ \bar{f}_i + \lambda (\bar{\partial}\partial \bar{f}_i - \bar{\partial}\bar{f}_i \wedge F - \bar{\partial} \bar{f}_i \wedge \partial \sigma) \}_{i=1}^{r-1},
\]

and we put \( c = \lambda \Theta_{21}, d = 1 + \lambda \Theta_{22}. \)

Now it follows from the representation

\[
\begin{pmatrix}
I - \lambda A & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
I & b \\
c(I - \lambda A)^{-1} & d
\end{pmatrix} \begin{pmatrix}
I - \lambda A & 0 \\
0 & 1
\end{pmatrix}
\]

that

\[
\det(I + \lambda \Theta) = \det(I + \lambda A) \left( d - c(I - \lambda A)^{-1} b \right),
\]
which we compute explicitly using Lemma 3.2. Namely,

\[
\det(I + \lambda \Theta) = \frac{1}{1 - \lambda U} \left( 1 + \lambda \tilde{\partial} \partial \sigma + \tilde{\partial} F \right)
- \sum_{i,j=1}^{r-1} \lambda \tilde{\partial}(e^{-\sigma} \partial f_i) \wedge \left( \delta_{ij} + \frac{\lambda e^{-\sigma} \tilde{\partial} f_i \wedge \partial f_j}{1 - \lambda U} \right) \wedge (\tilde{f}_j + \lambda(\tilde{\partial} \partial f_j - \tilde{\partial} \bar{f}_j \wedge (F + \partial \sigma)) \right) .
\]

Using equations

\[
\tilde{\partial} F = -U - \tilde{\partial} \sigma \wedge F + e^{-\sigma} \sum_{i=1}^{r-1} \bar{f}_i \partial \partial f_i ,
\]
and

\[
\partial U = -\partial \sigma \wedge U + \Psi_+ , \quad \tilde{\partial} U = -\tilde{\partial} \sigma \wedge U + \Psi_- ,
\]

\[
\tilde{\partial} \partial U = -\tilde{\partial} \partial \sigma \wedge U + \tilde{\partial} \sigma \wedge U + \partial \sigma \wedge \Psi_- - \partial \sigma \wedge \Psi_+ + \Phi ,
\]
where

\[
\Psi_+ = e^{-\sigma} \sum_{i=1}^{r-1} \partial f_i \wedge \tilde{\partial} \partial f_i , \quad \Psi_- = e^{-\sigma} \sum_{i=1}^{r-1} \partial \partial f_i \wedge \tilde{\partial} \partial f_i , \quad \Phi = e^{-\sigma} \sum_{i=1}^{r-1} \partial f_i \wedge \tilde{\partial} \partial f_i ,
\]
and simplifying, we obtain

\[
\det(I + \lambda \Theta) = 1 + \frac{\lambda}{1 - \lambda U} \left( \tilde{\partial} \partial \sigma - \lambda \Phi + \lambda \tilde{\partial} \sigma \wedge \Psi_+ - \lambda \tilde{\partial} \sigma \wedge U \wedge F + \lambda \Psi_- \wedge F - \lambda \tilde{\partial} \sigma \wedge \Psi_+ \wedge \partial \sigma - \lambda \tilde{\partial} \sigma \wedge U \wedge \Psi_- \wedge F + \lambda \Psi_- \wedge U \wedge \partial \sigma - \lambda \tilde{\partial} \sigma \wedge U \wedge \Psi_+ \wedge \partial \sigma - \lambda \tilde{\partial} \sigma \wedge U \wedge \Psi_- \wedge U \wedge F \right) - \frac{\lambda}{1 - \lambda U} \left( -\tilde{\partial} \sigma \wedge U \wedge F + \Psi_- \wedge F \right)
\]

\[
= 1 + \frac{\lambda}{1 - \lambda U} \left( \tilde{\partial} \partial \sigma + \lambda(\Phi + \tilde{\partial} \sigma \wedge \Psi_+ - \partial \sigma \wedge \partial \sigma \wedge U + \Psi_- \wedge \partial \sigma) \right) - \frac{\lambda^3 \tilde{\partial} U \wedge \tilde{\partial} U}{(1 - \lambda U)^2} = 1 + \lambda \tilde{\partial} \partial \sigma - \lambda^2 \frac{\tilde{\partial} \partial U}{1 - \lambda U} - \lambda^3 \frac{\tilde{\partial} U \wedge \tilde{\partial} U}{(1 - \lambda U)^2} . \tag*{\square}
\]

Corollary 3.1. The following identities hold

\[
\sum_{l=1}^{r} (-1)^l \sum_{1 \leq t_1 < \cdots < t_{l-1} \leq r-1} \text{ch}_k(E_l, h(\sigma, f_{t_1}, \ldots, f_{t_{l-1}})) = \frac{\delta_{kr}}{r-1} \left( \frac{\sqrt{-1}}{2\pi} \right)^r \tilde{\partial} \left( \frac{U^{r-1}}{(r-1)!} \right)
= \frac{\delta_{kr}}{r-1} \left( \frac{\sqrt{-1}}{2\pi} \right)^r \tilde{\partial} \left( e^{(r-1)\sigma} \partial f_1 \wedge \tilde{\partial} f_1 \wedge \cdots \wedge \partial f_{r-1} \wedge \tilde{\partial} f_{r-1} \right) ,
\]
for \( k = 1, \ldots, r \), and

\[
\sum_{l=1}^{r} (-1)^l \sum_{1 \leq i_1 < \ldots < i_{l-1} \leq r-1} \text{ch}_{r+1}(E_l, h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}})) = -\frac{1}{(r+1)!} c_1(E_r, h)c_r(E_r, h) = -\frac{1}{(r-1)(r+1)!} \left( \frac{\sqrt{-1}}{2\pi} \right)^{r+1} \partial \bar{\partial} \sigma \wedge \partial \bar{\partial} U^{r-1},
\]

where it is understood that \( h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}) = e^\sigma \) for \( l = 1 \).

**Proof.** Replacing the functions \( f_i \) in the definition of the Hermitian metric \( h(\sigma; f_1, \ldots, f_{r-1}) \) by \( t_i f_i \) with real \( t_i \), we can consider Chern forms \( c_k(E_r, h) \) as polynomials in \( t_1, \ldots, t_{r-1} \) with coefficients in the commutative ring \( A(X, \mathbb{C}) \cap A^{\text{even}}(X, \mathbb{R}) \). In follows from the explicit formulas in Lemma 3.3, that these are polynomials in variables \( \alpha_i = t_i^2 \), which can be considered as nilpotent elements of order 2 since the forms \( \partial f_i \wedge \bar{\partial} f_i \) have the same property, and that the Chern forms \( c_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{l-1} f_{l-1})) \) can be obtained from the Chern form \( c_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{r-1} f_{r-1})) \) by setting \( t_j = 0 \) for \( j \in J \) — a complementary subset to \( I = \{i_1, \ldots, i_{l-1}\} \) in the set \( \{1, \ldots, r-1\} \).

The Chern character forms are related to the Chern forms by Newton’s identities

\[
(-1)^k k! \text{ch}_k = -kc_k + c_{k-1}\text{ch}_1 - 2!c_{k-2}\text{ch}_2 + \cdots + (-1)^k(k-1)!c_1\text{ch}_{k-1},
\]

so that the same relation holds between Chern character forms for the vector bundles \( E_l \) and \( E_r \). Now represent the Chern character form as

\[
\text{ch}_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{r-1} f_{r-1})) = \sum_{l=1}^{k-1} \sum_{|I|=l} \alpha_{i_1} \ldots \alpha_{i_{l-1}} \omega_I,
\]

where summation goes over all subsets of \( \{1, \ldots, r-1\} \) of cardinality \( l \), and substitute it to the left hand side of the identity we want to prove, with \( f_i \) replaced by \( t_i f_i \). We obtain a sum of monomials, and by the relation between the Chern character forms for the bundles \( E_l \) and \( E_r \), for \( k < r \) every term is a monomial \( \alpha_{i_1} \ldots \alpha_{i_{l-1}} \omega_I \), taken with coefficient

\[
1 - \sum_{j=1}^{r-k} (-1)^j \binom{r-k}{j} = 0,
\]

which follows from the exclusion-inclusion principle. For the case \( k = r \) the term \(-c_r(E_r, h)/(k-1)!\) is the only term to survive, and for the case \( k = r+1 \) such term is given by \(-c_1(E_r, h)c_r(E_r, h)/(r+1)!\). \( \square \)

**Remark 3.4.** For the rank 2 trivial vector bundle \( E_2 \) with the Hermitian metric

\[
h = h(\sigma, f) = \begin{pmatrix} 1 & f \hat{f} \\ f^* & |f|^2 + e^{\sigma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f & e^{\sigma/2} \end{pmatrix} \begin{pmatrix} 1 & \hat{f} \\ 0 & e^{\sigma/2} \end{pmatrix}.
\]
the main identity in the Corollary 3.1 takes the form
\[ \text{ch}_2(E_2, h(\sigma, f)) - \text{ch}_2(E_1, e^\sigma) = \frac{-1}{(2\pi)^2} \overline{\partial}\partial(e^{-\sigma} \partial f \wedge \overline{\partial} f), \]
and can be verified by a straightforward computation\(^8\). For the bundles of rank 3 and 4 corresponding identities were first verified using special Mathematica package for computing Chern character forms, written by Michael Movshev. Based on these results, Michael Movshev [Mov10] was able, with the help of Mathematica, to obtain Lemma 3.3. Here we give its complete algebraic proof and show that it implies the identities, originally conjectured by the second author.

Remark 3.5. Setting \( \tilde{\theta} = h^{-1} \overline{\partial} h \), it is easy to obtain
\[ \text{Tr}(\theta \wedge \overline{\theta}) = e^{-\sigma} \partial f \wedge \overline{\partial} f + e^{-\sigma} \overline{\partial} f \wedge \partial f. \]
To get rid of the second term and to write down the simplest nontrivial Bott-Chern form \( \text{bc}_1(h, I) \), where \( I \) is a trivial Hermitian metric on \( E_2 \), we need to add the “Wess-Zumino term” (rather its \((1,1)\)-component) to the “kinetic term” \( \text{Tr}(\theta \wedge \overline{\theta}) \). Such formula was first obtained by A. Alekseev and S. Shatashvili in [AS89], where for the case of Minkowski signature the decomposition
\[ \begin{pmatrix} 1 & \bar{f} \\ f & |f|^2 + e^\sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & e^\sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
is replaced by the Gauss decomposition for \( \text{SL}(2, \mathbb{C}) \).

Remark 3.6. The Bott-Chern forms (or rather their exponents) also appear quite naturally in supersymmetric quantum field theories as ratios of non-chiral partition functions for the higher dimensional analogues of the bc-systems [LMNS97].

3.3. The main result. The first result is an analogue of Lemma 2.2 for complex manifolds.

**Lemma 3.4.** Let \( X \) be a compact complex manifold. Every \( \omega \in \mathcal{A}^{k,k}(X, \mathbb{C}) \cap \mathcal{A}^{2k}(X, \mathbb{R}) \) can be written as a finite linear combination of wedge products of real \((1,1)\)-forms of the type \( \sqrt{-1} \overline{\partial} \overline{\partial} \rho \) and \( \sqrt{-1} e^\sigma \partial f \wedge \overline{\partial} f \), where \( \rho, \sigma \in C^\infty(X, \mathbb{R}) \) and \( f \in C^\infty(X, \mathbb{C}) \). Moreover, if \( \omega \) is zero on open \( U \subset X \), than one can choose these forms such that all functions \( \rho, \sigma \) and \( f \) vanish on \( U \).

**Proof.** We will follow the proof of Lemma 2.2. Namely, let \( \{U_\alpha\}_{\alpha \in A} \) be a finite coordinate open cover of \( X \) and \( \{\rho_\alpha\}_{\alpha \in A} \) be a partition of unity subordinate to it, so that \( \omega = \sum_{\alpha \in A} \rho_\alpha \omega|_{U_\alpha} \). Denoting by
\[ z^1 = x^1 + \sqrt{-1} y^1, \ldots, z^n = x^n + \sqrt{-1} y^n \]

\(^8\)This computation of the second author was the starting point of the paper.
local complex coordinates in $U_\alpha$, we can write
\[
\omega|_{U_\alpha} = \sum_{I,J} f_{\alpha,IJ} dx^i \wedge \cdots \wedge dx^i \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_m},
\]
where $I = \{i_1, \ldots, i_l\}$, $J = \{j_1, \ldots, j_m\}$, $f_{\alpha,IJ} \in C^\infty(U_\alpha, \mathbb{R})$ and $1 \leq i_1 < \cdots < i_l \leq n$, $1 \leq j_1 < \cdots < j_m \leq n$, $l + m = 2k$. Since the form $\omega$ was supposed to be of $(k,k)$ type, so are the forms $\omega|_{U_\alpha}$. On the other hand, the $(k,k)$-component of these forms can be obtained by rewriting them in complex coordinates using
\[
dx^i = \frac{1}{2}(dz^i + d\bar{z}^i), \quad dy^i = \frac{1}{2\sqrt{-1}}(dz^i - d\bar{z}^i), \quad i = 1, \ldots, n,
\]
and collecting terms of the type $(k,k)$. If one of such terms has a factor $dz^i \wedge d\bar{z}^i$, $i, k \in I$, then it necessarily has a factor
\[
(dz^i \wedge d\bar{z}^i + d\bar{z}^i \wedge dz^i),
\]
if it comes from $dx^i \wedge dx^j$. Similarly, one has factors $(dz^j \wedge d\bar{z}^m + d\bar{z}^j \wedge dz^m)$, $j, m \in J$, coming from $dy^j \wedge dy^m$, and $\sqrt{-1}(dz^i \wedge d\bar{z}^j - d\bar{z}^i \wedge dz^j)$, coming from $dx^i \wedge dy^j$, $i \in I$ and $j \in J$.

In the first two cases corresponding factors can be written as
\[
2\sqrt{-1}\bar{\partial}\partial(\text{Im}(z^j \bar{z}^i)) \quad \text{and} \quad 2\sqrt{-1}\bar{\partial}\partial(\text{Im}(z^j \bar{z}^m)),
\]
whereas in the third case it takes the form $2\sqrt{-1}\bar{\partial}\partial(\text{Re}(z^j \bar{z}^i))$ for $i \neq j$, and $2\sqrt{-1}\bar{\partial}\partial(|z|^2) = 2\sqrt{-1}dz^i \wedge d\bar{z}^i$ for $i = j$. As in the proof of Lemma 2.2, let $K_\alpha$ be a compact set such that $\text{supp } \rho_\alpha \subset K_\alpha \subset U_\alpha$ and let $b_\alpha$ be the corresponding bump function. Then we see that the first three terms will take the form $2\sqrt{-1}\bar{\partial}\partial f$, where $\rho(z) = \text{Im}(b_\alpha(z)z^i \bar{z}^j)$ or $\rho(z) = \text{Re}(b_\alpha(z)z^i \bar{z}^j)$ for all $i \neq j$, and the last term will take the form $\sqrt{-1}\bar{\partial}\partial f$ with $f(z) = b_\alpha(z)z^i$. Noting that every $g \in C^\infty(X, \mathbb{R})$ can be written as $g = e^{\sigma_1} - e^{\sigma_2}$ with $\sigma_1, \sigma_2 \in C^\infty(X, \mathbb{R})$ completes the proof of the first statement. The second statement of the lemma is obvious from the construction. \hfill \Box

Remark 3.7. One can also derive Lemma 3.4 from Lemma 2.2 by using the decomposition $d = \partial + \bar{\partial}$ and noticing that $d^c = \sqrt{-1}(\bar{\partial} - \partial)$ is obtained from the complex structure operator $J : TX \to TX$ by $(d^c f)(v) = df(Jv)$, and by using similar formulas for higher degree forms.

Definition. A form $\omega \in \mathcal{A}^{k,k}(X, \mathbb{C}) \cap \mathcal{A}^{2k}(X, \mathbb{R})$ is called a form of pure type if it is a wedge product of the forms $\sqrt{-1}e^{\sigma}\partial f \wedge \bar{\partial}f$; if one or more factors contain $\sqrt{-1}\bar{\partial}\partial f$, it is called a composite form.

We have the following complex manifold analogue of Proposition 2.1.

Theorem 3.2. For every $\bar{\partial}\partial$-exact form $\omega \in \mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\text{even}}(X, \mathbb{R})$ there is a trivial vector bundle $E$ over $X$ with two Hermitian metrics $h_1$ and $h_2$ such that
\[
\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.
\]
Proof. Firstly, the statement holds for \((1, 1)\)-forms. Namely, since every real \(\partial\bar{\partial}\)-exact \((1, 1)\) form is given by \(\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\sigma\), where \(\sigma \in C^\infty(X, \mathbb{R})\), consider trivial holomorphic line bundle \(E\) with the Hermitian metric \(h = e^\sigma\), so that

\[
\text{ch}(E, h) = \exp \omega = 1 + \omega + \frac{1}{2} \omega^2 + \cdots + \frac{1}{n!} \omega^n.
\]

To get rid of all terms in this expression except \(\omega\), consider Hermitian metrics \(e^{\alpha_i\sigma}, i = 1, \ldots, n+1\), and choose pair-wise distinct \(\alpha_i\) such that the following system of equations

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+1} \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{n+1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1^n & \alpha_2^n & \alpha_3^n & \cdots & \alpha_{n+1}^n
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_{n+1}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\]

has an integer solution \(c_1, \ldots, c_{n+1}\). Namely, for any choice of \(n+1\) different rational numbers \(\alpha_i\) the numbers \(c_i\) are also rational. If their least common denominator is \(N > 1\), then for the numbers \(\beta_i = \alpha_i/N\) the corresponding solution is integral. Now putting

\[
(E, h_1) = \bigoplus_{c_i > 0} c_i(E_1, e^{\beta_i\sigma}) \quad \text{and} \quad (E, h_2) = \bigoplus_{c_i < 0} (-c_i)(E_1, e^{\beta_i\sigma}),
\]

where \(n(L, h)\) stands for the direct sum of \(n\) copies of a line bundle \(L\) with the Hermitian metric \(h\), we get

\[
\text{ch}(E, h_1) - \text{ch}(E, h_2) = \omega.
\]

It is convenient to introduce a virtual Hermitian bundle \(E = E - E\) with corresponding Hermitian metrics \(h_1\) and \(h_2\), and to rewrite the above equation as \(\text{ch} E = \omega\). We observe that the thus defined Chern character form for virtual Hermitian bundles is multiplicative: if \(\mathcal{W}_1 = W_1 - W_1\) with Hermitian metrics \(h_{11}\) and \(h_{12}\), and \(\mathcal{W}_2 = W_2 - W_2\) with Hermitian metrics \(h_{21}\) and \(h_{22}\), then

\[
\text{ch} \mathcal{W}_1 \text{ch} \mathcal{W}_2 = \text{ch} \mathcal{W},
\]

where \(\mathcal{W} = W - W\) and \(W = W_1 \otimes W_2 \oplus (W_1 \otimes W_2)\) with corresponding Hermitian metrics

\[
h_1 = (h_{11} \otimes h_{21}) \oplus (h_{12} \otimes h_{22}) \quad \text{and} \quad h_2 = (h_{11} \otimes h_{22}) \oplus (h_{12} \otimes h_{21}).
\]

Slightly abusing notations, we will write \(\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2\).

Secondly, it is sufficient to prove the statement for the forms of pure type only. Indeed, suppose that it holds for pure forms, and let \(\omega\) be a composite form of the type

\[
\omega = \left(\frac{\sqrt{-1}}{2\pi}\right)^l \partial\bar{\partial}\rho_1 \wedge \cdots \wedge \partial\bar{\partial}\rho_l \wedge \eta,
\]
where \( \eta \) is a pure form. Since
\[
\frac{-1}{2\pi} \partial \bar{\partial} \rho_i = \text{ch} \, \mathcal{W}_i, \quad j = 1, \ldots, l, \quad \text{and} \quad \eta = \text{ch} \, \mathcal{F},
\]
we obtain \( \omega = \text{ch} \, \mathcal{E} \), where \( \mathcal{E} = \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_l \otimes \mathcal{F} \).

Finally, let \( \omega \) be a pure form of degree \((k, k)\), \( k > 1 \), given by
\[
\omega = \frac{1}{k-1} \left( \frac{-1}{2\pi} \right)^{\frac{k}{2}} \partial \bar{\partial} \left( e^{(k-1)\sigma} \partial f_1 \wedge \cdots \wedge \partial f_{k-1} \wedge \bar{\partial} f_{k-1} \right).
\]
It follows from Corollary 3.1 that
\[
(3.5) \quad \omega = \text{ch}_k \mathcal{F}_k \quad \text{and} \quad \text{ch}_i \mathcal{F}_k = 0, \quad i = 1, \ldots, k-1,
\]
where \( \mathcal{F}_k = F_k - F_k \) and
\[
F_k = \bigoplus_{l=1}^{k} \left( \frac{1}{2} \right) \left( 1 + (-1)^l \right) \left( \frac{k-1}{l-1} \right) E_l
\]
with Hermitian metrics
\[
h_{1k} = \bigoplus_{l=1}^{k} \left( \frac{1}{2} \right) \left( 1 + (-1)^l \right) \bigoplus_{1 \leq i_1 < \cdots < i_{l-1} \leq k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}})
\]
\[
h_{2k} = \bigoplus_{l=1}^{k} \left( \frac{1}{2} \right) \left( 1 - (-1)^l \right) \bigoplus_{1 \leq i_1 < \cdots < i_{l-1} \leq k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}})
\]
for \( k > 2 \), whereas \( F_2 = E_2 = E_1 \oplus E_1 \) and \( h_{12} = h(\sigma, f), h_{22} = e^\sigma \oplus 1 \).

Now we can finish the proof by using “Gaussian elimination” starting with \( -\text{ch}_{k+1} \mathcal{F}_k \). Namely, according to the second identity in Corollary 3.1,
\[
-\text{ch}_{k+1} \mathcal{F}_k = \frac{-1}{2\pi} \bar{\partial} \rho \wedge \omega
\]
with real \( \rho \), so that introducing \( \mathcal{H}_{k+1} = \mathcal{F}_k \ominus (\mathcal{W} \otimes \mathcal{F}_k) \) we get
\[
\omega = \text{ch}_k \mathcal{H}_{k+1} \quad \text{and} \quad \text{ch}_i \mathcal{H}_{k+1} = 0, \quad i = 1, \ldots, k-1, k+1.
\]

Next, consider \(-\text{ch}_{k+2} \mathcal{H}_{k+1}\) and represent its pure component by \( \text{ch}_{k+2} \mathcal{F}_{k+2}\), and its composite components by the Chern character forms of the tensor products of virtual bundles \( \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_l \otimes \mathcal{F}_{k+2} \). Taking the direct sum of these virtual Hermitian bundles with \( \mathcal{H}_{k+1} \), we get a virtual Hermitian bundle \( \mathcal{H}_{k+2} \) satisfying
\[
\omega = \text{ch}_k \mathcal{H}_{k+2} \quad \text{and} \quad \text{ch}_i \mathcal{H}_{k+2} = 0, \quad i = 1, \ldots, k-1, k+1, k+2.
\]
Repeating this process, we obtain the result. \( \square \)
Remark 3.8. It immediately follows from the second statement of Lemma 3.4 and the proof of Theorem 3.2, that if form \( \omega \) vanishes on open \( U \subset X \), then Hermitian metrics \( h_1 \) and \( h_2 \) can be chosen such that \( h_1 = h_2 = I \) — identity matrix — on \( U \).

Corollary 3.3. For every \( \omega \in \mathcal{A}(X,\mathbb{C}) \cap \mathcal{A}^{\text{even}}(X,\mathbb{R}) \) of degree not greater than \( 2n-2 \), there is a trivial vector bundle \( E \) over \( X \) with two Hermitian metrics \( h_1 \) and \( h_2 \) such that in \( \tilde{\mathcal{A}}(X,\mathbb{C}) \)

\[
\text{BC}(E; h_1, h_2) = \omega.
\]

Proof. It is analogous to the proof of Corollary 2.2. Namely, let \( \Omega \in \mathcal{A}(X \times \mathbb{P}^1,\mathbb{C}) \cap \mathcal{A}^{\text{even}}(X \times \mathbb{P}^1,\mathbb{R}) \) be such that under the inclusion map \( i_p : X \to X \times \mathbb{P}^1 \) one has \( i_p^*(\Omega) = -\omega \) and \( i_0^*(\Omega) = 0 \) in some neighborhood of 0 in \( \mathbb{P}^1 \). It follows from Theorem 3.2 that there is a trivial vector bundle \( \tilde{E} \) over \( X \times \mathbb{P}^1 \) with two Hermitian metrics \( \tilde{h}_1 \) and \( \tilde{h}_2 \) such that

\[
\sqrt{-1} \frac{\partial}{\partial z} \Omega = \text{ch} (\tilde{E}, \tilde{h}_1) - \text{ch} (\tilde{E}, \tilde{h}_2),
\]

where the metrics \( \tilde{h}_1 \) and \( \tilde{h}_2 \) can be chosen such that \( i_p^*(\tilde{h}_1) = i_p^*(\tilde{h}_2) = I \). Denoting by \( E \) a trivial vector bundle over \( X \) — a pullback of \( \tilde{E} \) — and putting \( h_1 = i_\infty^*(\tilde{h}_1), h_2 = i_\infty^*(\tilde{h}_2) \), we obtain, modulo \( \text{Im} \partial + \text{Im} \bar{\partial} \),

\[
\text{bc}(E; I, h_1) - \text{bc}(E; I, h_2) = \int_{\mathbb{P}^1} \sqrt{-1} \frac{\partial}{\partial z} \Omega \log |z|^2
\]

\[
= \int_{\mathbb{P}^1} \sqrt{-1} \frac{\partial}{\partial z} \bar{z} \log |z|^2
\]

\[
= i_\infty^*(\Omega) - i_0^*(\Omega)
\]

\[
= -\omega.
\]

Therefore in \( \tilde{\mathcal{A}}(X,\mathbb{C}) \),

\[
\omega = -\text{BC}(E; I, h_1) + \text{BC}(E; I, h_2) = \text{BC}(E; h_1, h_2). \quad \square
\]

3.4. Application to differential \( K \)-theory. Recall that according to the definition of differential \( K \)-theory in [GS86], the \( K \)-group \( \tilde{K}_0(X) \) for complex manifold \( X \) is defined as the free abelian group generated by the triples \( (E, h, \eta) \), where \( E \) is a holomorphic vector bundle over \( X \) with Hermitian metric \( h \) and \( \eta \in \tilde{\mathcal{A}}(X,\mathbb{C}) \) with the following relations. For every exact sequence \( \mathcal{E} \)

\[
0 \longrightarrow F \stackrel{i}{\longrightarrow} E \longrightarrow H \longrightarrow 0
\]

of holomorphic vector bundles over \( X \), endowed with arbitrary Hermitian metrics \( h_F, h_E \) and \( h_H \), impose

\[
(F, h_F, \eta') + (H, h_H, \eta'') = (E, h_E, \eta' + \eta'' - \text{BC}(\mathcal{E}, h_E, h_F, h_H)),
\]
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where $BC(\mathcal{E}, h_E, h_F, h_H) = bc(\mathcal{E}, h_E, h_F, h_H) \mod (\text{Im} \partial + \text{Im} \overline{\partial})$ (see Remark 3.3). It follows from (3.6) that in $\hat{K}_0(X)$

\begin{equation}
(E, h_1, \eta_1) = (E, h_2, \eta_2 - BC(E, h_1, h_2)).
\end{equation}

Now following [SS08b], we define two Hermitian metrics $h_1$ and $h_2$ on the holomorphic vector bundle $E$ to be equivalent, if $BC(E, h_1, h_2) = 0$, and define a \textit{structured} holomorphic Hermitian vector bundle $\mathcal{E}$ as a pair $(E, \{h\})$, where $\{h\}$ is the equivalence class of a Hermitian metric $h$. Our goal is to impose a relations on a free abelian group generated by $\mathcal{E}$ such the resulting group $H\hat{K}_0(X)$ is isomorphic to the “reduced” differential $K$-theory group $\hat{K}_0^{rd}(X)$, a subgroup of $\hat{K}_0(X)$ with forms $\eta$ of degrees not greater than $2n - 2$.

First we observe that it follows from (3.7) that the mapping

\begin{equation}
\mathcal{E} = (E, \{h\}) \mapsto \varepsilon(\mathcal{E}) = (E, h, 0) \in \hat{K}_0^{rd}(X)
\end{equation}

is well-defined. Next we show that when extended to to the free abelian group generated by the structured holomorphic Hermitian bundles, this mapping is onto. Indeed, for every $\eta \in \tilde{A}(X, \mathbb{C}) \cap A^{\text{even}}(X, \mathbb{R})$ of degree not greater than $2n - 2$, let $F$ be the trivial vector bundle over $X$ with two Hermitian metrics $h_1$ and $h_2$ such that, according to Corollary 3.3,

\[(F, h_1, \eta) = (F, h_2, 0).\]

Finally, we define the group $H\hat{K}_0(X)$ as the quotient of the free abelian group generated by $\mathcal{E}$ modulo the relations — pullbacks of the defining relations for $\hat{K}_0^{rd}(X)$ by the mapping $\varepsilon$. Explicitly, for every exact sequence $\mathcal{E}$ of holomorphic vector bundles over $X$ with Hermitian metrics $h_F, h_E$ and $h_H$ satisfying $BC(\mathcal{E}; h_E, h_F, h_H) = 0$, we impose

\[(F, \{h_F\}) + (H, \{h_H\}) = (E, \{h_E\}).\]

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