ASYMPTOTIC BEHAVIOR IN A CHEMOTAXIS-GROWTH SYSTEM WITH NONLINEAR PRODUCTION OF SIGNALS

Yuanyuan Liu
College of Information Science & Technology
Dong Hua University
Shanghai 200051, China

Youshan Tao
Department of Applied Mathematics
Dong Hua University
Shanghai 200051, China

(Communicated by Michael Winkler)

Abstract. We consider the chemotaxis-growth system

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u (1 - u), \quad x \in \Omega, \quad t > 0, \\
    v_t &= \Delta v - v + h(u), \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

under no-flux boundary conditions, in a convex bounded domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \chi > 0 \) and \( \mu > 0 \) are given parameters, and \( h(s) \) is a prescribed function on \([0, \infty)\).

It is shown that under the assumption that

\[
4|h'| < \sqrt{2\mu - 7\chi^2},
\]

for any given nonnegative \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,\infty}(\Omega) \) the system possesses a global classical solution which is bounded in \( \Omega \times (0, \infty) \). Moreover, whenever

\[
\chi |h'| < \sqrt{8\mu},
\]

any bounded classical solution constructed above stabilizes to the constant stationary solution \((1, h(1))\) as the time goes to infinity.

1. Introduction. We consider the initial-boundary value problem

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u (1 - u), \quad x \in \Omega, \quad t > 0, \\
    v_t &= \Delta v - v + h(u), \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(1.1)

in a convex bounded domain \( \Omega \subset \mathbb{R}^3 \) with smooth boundary, where \( \chi > 0 \) and \( \mu > 0 \) are given parameters, and \( h(s) \) is a prescribed function on \([0, \infty)\). In (1.1), \( u \) denotes the density of a cell population, \( v \) stands for the concentration of a chemical substance, and \( h \) represents the production of the chemical substance by the cells. The model (1.1) was proposed in [12] for modeling the first steps of tumor-related

2010 Mathematics Subject Classification. Primary: 35B40, 35K57, 35Q92, 92C17.

Key words and phrases. Large time behavior, chemotaxis-growth system, nonlinear cue production.
angiogenesis, and it was also established in [8] and [6] for describing the pattern formation of bacteria.

Depending on various biological processes, the function $h$ takes the following three typical forms:

- A linear function $h(u) = u$, see [4].
- A saturating function $h(u) = \frac{u}{1 + u}$, see [12] and [8].
- A general nonlinear function $h(u)$ fulfilling $h'(u) > 0$, see [6].

When $h(u) = u$, it is known that any blow-up phenomenon can be completely suppressed for arbitrarily small $\mu > 0$ in (1.1) in the two-dimensional case ([13]) and that the blow-up can be prevented for appropriately large $\mu > 0$ in (1.1) in the higher-dimensional situations ([19]). Moreover, the global well-posedness for (1.1) with $h(u) = u(1 + u)^{\beta - 1}$ and a slightly general source term given by $\mu u(1 - u^{\alpha - 1})$ satisfying $\alpha > 1$ and $0 < \beta \leq 2$ was also discussed in [9, 10, 11].

Very recently, when the second parabolic equation in (1.1) is replaced by the elliptic counterpart $0 = \Delta v - v + h(u)$, Chaplain and Tello [1] studied the asymptotic behavior of solutions to the corresponding simplified parabolic-elliptic chemotaxis system under the assumption that

$$2\chi|h'| < \mu.$$ 

The approach used in [1] is the sup- and sub-solutions method, which seems not employable for the fully parabolic chemotaxis system (1.1). Therefore, the main purpose of this paper is to extend the result in [1] to a new one for the parabolic-parabolic system (1.1) based on a Lyapunov functional technique in [16].

Throughout this paper we assume that the signal production function $h$ is $C^1$-regular fulfilling

$$h(s) \geq 0, \quad h(0) = 0, \quad \text{and} \quad 0 \leq h'(s) \leq L \quad (1.2)$$

where $L > 0$ is a constant.

Our first result on the global existence and boundedness reads as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with smooth boundary, and suppose that the parameters $\chi$ and $\mu$ are positive, and assume that $h$ is a prescribed function from $C^1([0, \infty))$ satisfying (1.2). Then whenever

$$\mu > \frac{7}{2} \chi^2 \quad \text{and} \quad L \leq \sqrt{\frac{2\mu - 7\chi^2}{4}}, \quad (1.3)$$

for any given nonnegative $u_0 \in C^0(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ the problem [1.1] possesses a global classical solution $(u, v)$ which is bounded in $\Omega \times (0, \infty)$ in the sense that there exists $C > 0$ fulfilling

$$\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C \quad \text{for all} \ t > 0. \quad (1.4)$$

Here we note that the convexity assumption on the domain $\Omega$ in Theorem 1.1 can actually be removed (cf. [7] and [3], for instance); however, to shorten our presentation, we refrain from addressing this issue here.

If $\mu > 0$ is suitably large, then the global classical solution constructed above stabilizes to the homogeneous steady state $(1, h(1))$. More precisely, we have the following:

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with smooth boundary, and assume that $h$ satisfies (1.2). Then whenever

$$\mu > \frac{\chi^2 L^2}{8}, \quad (1.5)$$
for any bounded classical solution \((u, v)\) of (1.1) with \(u_0 \neq 0\) constructed in Theorem 1.1 has the property that
\[
\|u(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty \tag{1.6}
\]
and
\[
\|v(\cdot, t) - h(1)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{1.7}
\]

We should mention that one can further claim the exponential convergence rate of the solution in Theorem 1.2 by establishing some higher regularity estimates and making full use of the dissipation relation (4.6) below, as done in [16]. However, we refrain from repeating the details here.

Since a strong chemotaxis or high production of chemotactic signal may give rise to the formation of singularity of solutions to (1.1), the condition (1.3) or (1.5) biologically implies that a suitably considerable logistic damping can balance the above-said effects caused by the chemotactic term or the signal production term.

Finally, we remark that all generic constants \(C\) and \(c_i\) \((i = 1, 2, \cdots)\) throughout this paper may depend on \(|\Omega|, \|u_0\|_{C^0(\bar{\Omega})}, \|v_0\|_{W^{1,\infty}(\Omega)}, \chi, \mu\) and \(L\), but they are independent of \(t\) or \(T_{max}\) given in Lemma 2.1 below.

2. Preliminaries. The following local existence and extensibility result can be found in the literature ([19]).

Lemma 2.1. Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain with smooth boundary, let \(\chi\) and \(\mu\) be positive, suppose that \(h \in C^1([0, \infty))\) and assume that \(u_0\) and \(v_0\) are nonnegative functions from \(C^0(\bar{\Omega})\) and \(W^{1,\infty}(\Omega)\) respectively. Then there exist \(T_{max} \in (0, \infty]\) and a classical solution \((u, v)\) of (1.1) in \(\Omega \times (0, T_{max})\) such that
\[
(u, v) \in \left( C^0(\bar{\Omega} \times [0, T_{max})) \cap C^2(\bar{\Omega} \times (0, T_{max})) \right)^2,
\]
and that \(u\) and \(v\) are nonnegative in \(\Omega \times (0, T_{max})\), and such that
\[
either T_{max} = \infty, or \\
\lim_{t \uparrow T_{max}} \sup_{t' \leq T_{max}} \|u(\cdot, t')\|_{L^\infty(\Omega)} = \infty. \tag{2.1}
\]

Some basic properties of any such solution are readily checked.

Lemma 2.2. The solution of (1.1) satisfies
\[
\frac{d}{dt} \int_{\Omega} u = \mu \int_{\Omega} u - \mu \int_{\Omega} u^2 \quad \text{for all} \quad t \in (0, T_{max}), \tag{2.2}
\]
and in particular we have
\[
\int_{\Omega} u(\cdot, t) \leq m := \max \left\{ \int_{\Omega} u_0, |\Omega| \right\} \quad \text{for all} \quad t \in (0, T_{max}) \tag{2.3}
\]
and
\[
\int_{t}^{t + \tau} \int_{\Omega} u^2 \leq K := \left( 1 + \frac{1}{\mu} \right) m \quad \text{for all} \quad t \in (0, T_{max} - \tau) \tag{2.4}
\]
where
\[
\tau := \min \left\{ 1, \frac{1}{2} T_{max} \right\}. \tag{2.5}
\]
Proof. From and integration of the first equation in (4.1) we immediately obtain (2.2). Since $u$ is nonnegative, and since $\int_{\Omega} u^2 \geq \frac{1}{|\Omega|} (\int_{\Omega} u)^2$ for all $t \in (0, T_{\max})$ by the Cauchy-Schwarz inequality, this implies that
\[
\frac{d}{dt} \int_{\Omega} u \leq \mu \int_{\Omega} u - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u \right)^2 \quad \text{for all } t \in (0, T_{\max}).
\]
On an ODE comparison, this yields (2.3), whereas (2.4) results from a time integration of (2.2).

Later on, we shall need the following auxiliary lemma (cf. [15, Lemma 3.4]).

Lemma 2.3. Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous and such that
\[
y'(t) + ay(t) \leq f(t) \quad \text{for a.e. } t \in (0, T)
\]
with some nonnegative function $f \in L^1_{\text{loc}}([0, T))$ satisfying
\[
\int_{t + \tau}^{t} f(s) ds \leq b \quad \text{for all } t \in [0, T - \tau).
\]
Then
\[
y(t) \leq \max \left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\} \quad \text{for all } t \in (0, T).
\]

With the help of Lemma 2.3 we can derive an elementary estimate on $\nabla v$.

Lemma 2.4. There exists $C > 0$ such that
\[
\int_{\Omega} |\nabla v|^2 \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{2.6}
\]

Proof. Testing the second equation in (4.1) by $-\Delta v$, integrating by parts and using Young’s inequality we see that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 = -\int_{\Omega} h(u) \Delta v \\
\leq \int_{\Omega} |\Delta v|^2 + \frac{1}{4} \int_{\Omega} h(u) \\
\leq \int_{\Omega} |\Delta v|^2 + \frac{L^2}{4} \int_{\Omega} u^2
\]
for all $t \in (0, T_{\max})$, due to the fact that by (1.2) and the mean value theorem,
\[
0 \leq h(s) = h(s) - h(0) = h'(\eta)s \leq Ls \tag{2.7}
\]
where $\eta \in [0, s]$. This implies that for functions defined by $y(t) := \int_{\Omega} |\nabla v(\cdot, t)|^2, t \in (0, T_{\max})$, and $f(t) := \frac{L^2}{4} \int_{\Omega} u^2(\cdot, t), t \in (0, T_{\max})$, we have
\[
y'(t) + 2y(t) \leq f(t) \quad \text{for all } t \in (0, T_{\max}).
\]

Since Lemma 2.2 provides $c_1 := \frac{KL^2}{2}$ such that $\int_{t + \tau}^{t + \tau} f(s) ds \leq c_1$ for all $t \in (0, T_{\max} - \tau)$ with $\tau = \min(1, \frac{1}{2}T_{\max})$, this along with Lemma 2.3 provides (2.6) with $C := \max\{\int_{\Omega} |\nabla v_0|^2 + c_1, \frac{c_1}{2\tau} + 2c_1\}$.

\[\square\]
3. Global existence. Proof of Theorem 1.1 In order to prove Theorem 1.1 we first estimate $\int_{\Omega} u^2$ and $\int_{\Omega} |\nabla v|^4$. We begin with the following.

Lemma 3.1. The solution of (1.1) satisfies
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq \chi^2 \int_{\Omega} u^2 |\nabla v|^2 + 2\mu \int_{\Omega} u^2 - 2\mu \int_{\Omega} u^3
\]
for all $t \in (0, T_{\text{max}})$.

Proof. Testing the first equation in (1.1) against $u$ we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 = -\int_{\Omega} |\nabla u|^2 + \chi \int_{\Omega} u\nabla u \cdot \nabla v + \mu \int_{\Omega} u^2 - \mu \int_{\Omega} u^3
\]
for all $t \in (0, T_{\text{max}})$, which yields (3.1) because
\[
\chi \int_{\Omega} u\nabla u \cdot \nabla v \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\chi^2}{2} \int_{\Omega} u^2 |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$.

Lemma 3.2. There holds
\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla \nabla v|^2 \leq 7L^2 \int_{\Omega} u^2 |\nabla v|^2 - 4 \int_{\Omega} |\nabla v|^4
\]
for all $t \in (0, T_{\text{max}})$.

Proof. Using the second equation in (1.1), invoking the identity $\nabla v \cdot \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2$ and integrating by parts for several times we obtain
\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 = \int_{\Omega} |\nabla v|^2 \nabla \cdot (\Delta v - v + h(u))
\]
\[
= \frac{1}{2} \int_{\Omega} |\nabla v|^2 \Delta |\nabla v|^2 - \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} |\nabla v|^4
\]
\[
- \int_{\Omega} h(u) \nabla \cdot (|\nabla v|^2 \nabla v)
\]
\[
= -\frac{1}{2} \int_{\Omega} |\nabla v|^2 |\nabla v|^2 + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial v}
\]
\[
- \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} |\nabla v|^4
\]
\[
- \int_{\Omega} h(u) |\nabla v|^2 \Delta v - \int_{\Omega} h(u) \nabla v \cdot |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$. Here the convexity of $\Omega$ along with $\frac{\partial w}{\partial v} = 0$ on $\partial \Omega$ guarantees (3.5) that
\[
\frac{\partial |\nabla v|^2}{\partial v} \leq 0, \quad x \in \partial \Omega, \quad t \in (0, T_{\text{max}}),
\]
and since $|\Delta v| \leq \sqrt{3} |D^2 v|$, by (2.7) and Young’s inequality we can estimate
\[
- \int_{\Omega} h(u) |\nabla v|^2 \Delta v \leq \sqrt{3} L \int_{\Omega} u |\nabla v|^2 |D^2 v|
\]
\[
\leq \int_{\Omega} |\nabla v|^2 |D^2 v|^2 + \frac{3L^2}{4} \int_{\Omega} u^2 |\nabla v|^2
\]
for all \( t \in (0, T_{\text{max}}) \) and, similarly,
\[
- \int_{\Omega} h(u) \nabla v \cdot \nabla |\nabla v|^2 \leq \frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2 + L^2 \int_{\Omega} u^2 |\nabla v|^2
\]
for all \( t \in (0, T_{\text{max}}) \), from (3.3) we find that
\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 \leq -\frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{7L^2}{4} \int_{\Omega} u^2 |\nabla v|^2 - \int_{\Omega} |\nabla v|^4
\]
for all \( t \in (0, T_{\text{max}}) \), which yields (3.2).

To absorb the first integral on the right sides of (3.1) and (3.2), we shall make use of the logistic dampening effects in the first equation in (1.1).

**Lemma 3.3.** We have
\[
\frac{d}{dt} \int_{\Omega} u |\nabla v|^2 + (\mu - \frac{1}{2} \chi^2 - L^2) \int_{\Omega} u^2 |\nabla v|^2 \leq 3 \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla |\nabla v|^2|^2 + (\mu - 2) \int_{\Omega} u |\nabla v|^2
\]
for all \( t \in (0, T_{\text{max}}) \).

**Proof.** By a straightforward computation using the first two equations in (1.1), integrating by parts and employing the identity \( \nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2 \) we obtain
\[
\frac{d}{dt} \int_{\Omega} u |\nabla v|^2 = \int_{\Omega} |\nabla v|^2 \left\{ \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u - \mu u^2 \right\}
+ 2 \int_{\Omega} u \nabla v \cdot \nabla \left\{ \Delta v - v + h(u) \right\}
= \int_{\Omega} |\nabla v|^2 \Delta u - \chi \int_{\Omega} |\nabla v|^2 \nabla \cdot (u \nabla v)
+ \mu \int_{\Omega} u |\nabla v|^2 - \mu \int_{\Omega} u^2 |\nabla v|^2
+ \int_{\Omega} u \Delta |\nabla v|^2 - 2 \int_{\Omega} u |D^2 v|^2
- 2 \int_{\Omega} u |\nabla v|^2 + 2 \int_{\Omega} uh'(u) \nabla u \cdot \nabla v
= -2 \int_{\Omega} \nabla u \cdot \nabla |\nabla v|^2 + \chi \int_{\Omega} u \nabla v \cdot \nabla |\nabla v|^2
+ (\mu - 2) \int_{\Omega} u |\nabla v|^2 - \mu \int_{\Omega} u^2 |\nabla v|^2
+ \int_{\partial \Omega} u \frac{\partial |\nabla v|^2}{\partial \nu} - 2 \int_{\Omega} u |D^2 v|^2 + 2 \int_{\Omega} uh'(u) \nabla u \cdot \nabla v
\]
for all \( t \in (0, T_{\text{max}}) \). Here by (3.4) and nonnegativity of \( u \),
\[
\int_{\partial \Omega} u \frac{\partial |\nabla v|^2}{\partial \nu} - 2 \int_{\Omega} u |D^2 v|^2 \leq 0
\]
and by Young’s inequality
\[
-2 \int_{\Omega} \nabla u \cdot \nabla |\nabla v|^2 \leq 2 \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2
\]
and
\[ \chi \int_{\Omega} u \nabla v \cdot \nabla |v|^2 \leq \frac{\chi^2}{2} \int_{\Omega} u^2 |\nabla v|^2 + \frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2 | \]  
(3.9)
as well as
\[ 2 \int_{\Omega} u h' u \nabla v \cdot \nabla v \leq 2L \int_{\Omega} u |\nabla u \cdot \nabla v| \leq \int_{\Omega} |\nabla u|^2 + L^2 \int_{\Omega} u^2 |\nabla v|^2 \]  
(3.10)
for all \( t \in (0, T_{\text{max}}) \). In light of (3.7)-(3.10), the identity (3.6) immediately implies (3.5).

A simple linear combination of (3.1), (3.2) and (3.5) leads to the following.

**Corollary 3.4.** There holds
\[ \frac{d}{dt} \left\{ 3 \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} u |\nabla v|^2 \right\} + (\mu - \frac{7}{2} \chi^2 - 8L^2) \int_{\Omega} u^2 |\nabla v|^2 + 6\mu \int_{\Omega} u^3 \leq 6\mu \int_{\Omega} u^2 + (\mu - 2) \int_{\Omega} u |\nabla v|^2 - 4 \int_{\Omega} |\nabla v|^4 \]  
(3.11)
for all \( t \in (0, T_{\text{max}}) \).

Under a suitable parameter assumption, from this we can derive a bound for \( \int_{\Omega} u^2 \) and \( \int_{\Omega} |\nabla v|^4 \).

**Lemma 3.5.** Suppose that
\[ \mu \geq \frac{7}{2} \chi^2 + 8L^2. \]  
(3.12)
Then there exists \( C > 0 \) such that
\[ \int_{\Omega} u^2(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\text{max}}) \]  
(3.13)
and
\[ \int_{\Omega} |\nabla v(\cdot, t)|^4 \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \]  
(3.14)

**Proof.** Since \( \mu \geq \frac{7}{2} \chi^2 + 8L^2 \), Corollary 3.4 implies that
\[ y(t) := 3 \int_{\Omega} u^2(\cdot, t) + \int_{\Omega} |\nabla v(\cdot, t)|^4 + \int_{\Omega} u(\cdot, t) |\nabla v(\cdot, t)|^2, \quad t \in (0, T_{\text{max}}), \]  
satisfies
\[ y'(t) + y(t) + 6\mu \int_{\Omega} u^3 \leq (6\mu + 3) \int_{\Omega} u^2 - 3 \int_{\Omega} |\nabla v|^4 + (\mu - 1) \int_{\Omega} u |\nabla v|^2. \]  
(3.15)
In order to deal with the rightmost three terms therein, we first use Young’s inequality to find \( c_1 := c_1(\mu) > 0 \) such that
\[ (6\mu + 3) \int_{\Omega} u^2 - 3 \int_{\Omega} |\nabla v|^4 + (\mu - 1) \int_{\Omega} u |\nabla v|^2 \leq \left[ 6\mu + 3 + \frac{(\mu - 1)^2}{12} \right] \int_{\Omega} u^2 \leq 6\mu \int_{\Omega} u^3 + c_1 \]  
(3.16)
for all \( t \in (0, T_{\text{max}}) \). In view of (3.16), (3.15) entails that
\[
y'(t) + y(t) \leq c_1 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (3.17)
Upon an ODE comparison this yields
\[
y(t) \equiv 3 \int_\Omega u^2 + \int_\Omega |\nabla v|^4 + \int_\Omega u|\nabla v|^2 \leq c_2 \quad \text{for all } t \in (0, T_{\text{max}})
\]
with \( c_2 := \max\{y(0), c_1\} \), which proves both (3.13) and (3.14).

**Proof of Theorem 1.1.** The first equation in (1.1) can be rewritten in the form
\[
 u_t = \Delta u - \chi \nabla \cdot (uG) + \mu u - \mu u^2 \quad \text{in } \Omega \times (0, T_{\text{max}})
\] (3.18)
where \( G := \nabla v \) fulfills
\[
\|G(\cdot, t)\|_{L^4(\Omega)} \leq c_1 \quad \text{for all } t \in (0, T_{\text{max}})
\] (3.19)
thanks to Lemma 3.5. Relying on this, noting that \( \mu s - \mu s^2 \leq \frac{\mu}{4} \) for all \( s \in \mathbb{R} \), invoking the known smoothing properties of the Neumann heat semigroup (cf. e.g. [20, Lemma 1.3 (iv)]) and using the maximum principle we can obtain
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_2 \quad \text{for all } t \in (0, T_{\text{max}})
\] (3.20)
(cf. [17, Lemma 4.2] for details). Finally, in view of the extensibility criterion in Lemma 2.1, the estimates (3.20) asserts \( T_{\text{max}} = \infty \), and this completes the proof of Theorem 1.1.

4. Asymptotic behavior. **Proof of Theorem 1.2.** We start from establishing higher regularity of the solution.

**Lemma 4.1.** Let the conditions of Theorem 1.1 hold, then we have
\[
\|u(\cdot, t)\|_{C^{\theta, \frac{2}{3}}(\bar{\Omega} \times [t, t+1])} + \|v(\cdot, t)\|_{C^{\theta, \frac{2}{3}}(\bar{\Omega} \times [t, t+1])} \leq C \quad \text{for all } t > 1
\] (4.1)
with some \( \theta \in (0, 1) \) and \( C > 0 \) being independent of \( t \).

**Proof.** By (1.4) and the regularity assumption on \( h \), one can rely on standard regularity arguments applied to the second equation in (1.1) to obtain
\[
\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq c_1 \quad \text{for all } t > 0
\] (4.2)
(cf. e.g. [2] for details). We rewrite the first equation in (1.1) according to
\[
u_t = \nabla \cdot A(x, t, \nabla u) + B(x, t), \quad x \in \Omega, t > 0,
\]
with
\[
A(x, t, \xi) := \xi - \chi u(x, t) \nabla v(x, t), \quad (x, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^3
\]
and
\[
B(x, t) := \mu u(x, t)(1 - u(x, t)), \quad (x, t) \in \Omega \times (0, \infty).
\]
Here by Young’s inequality and the estimates (1.4) and (4.2) we see that
\[
A(x, t, \xi) \cdot \xi \geq \frac{1}{2} \xi^2 - \chi u(x, t) \nabla v(x, t) \cdot \xi \\
\geq \frac{1}{2} \xi^2 - \frac{1}{2} \|u(\cdot, t)\|_{L^\infty(\Omega)}^2 \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)}^2 \\
\geq \frac{1}{2} \xi^2 - c_2 \quad \text{for all } (x, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^3
\]
with some $c_2 > 0$, and moreover by (1.4) and (4.2) we find $c_3 > 0$ and $c_4 > 0$ such that

$$|A(x, t, \xi)| \leq |\xi| + c_3 \quad \text{for all } (x, t, \xi) \in \Omega \times (0, \infty) \times \mathbb{R}^3$$

and that

$$|B(x, t)| \leq c_4 \quad \text{for all } (x, t) \in \Omega \times (0, \infty).$$

Then, the claim for $u$ readily results from [14, Theorem 1.3], whereas the assertion for $v$ can easily be obtained from (4.2) and the Sobolev embedding.

We then focus on constructing a Lyapunov functional which implies some weak convergence information for the solution. As a preparation, we begin with the following differential inequalities.

**Lemma 4.2.** Assume that $u_0 \not\equiv 0$, then we have

$$\frac{d}{dt} \int_{\Omega} \ln u \geq -\frac{\chi}{4} \int_{\Omega} |\nabla v|^2 + \mu|\Omega| - \mu \int_{\Omega} u \quad \text{for all } t > 0. \quad (4.3)$$

**Proof.** Since $u$ is positive in $\bar{\Omega} \times (0, \infty)$ according to the strong maximum principle and the assumption that $u_0 \not\equiv 0$, we may test the first equation in (1.1) against $\frac{1}{u}$ and use integration by parts to obtain

$$\frac{d}{dt} \int_{\Omega} \ln u = \int_{\Omega} \frac{|\nabla u|^2}{u^2} - \chi \int_{\Omega} \frac{1}{u} \nabla u \cdot \nabla v + \mu |\Omega| - \mu \int_{\Omega} u \quad \text{for all } t > 0.$$ Here by Young’s inequality we can estimate

$$\left| - \chi \int_{\Omega} \frac{1}{u} \nabla u \cdot \nabla v \right| \leq \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2 \quad \text{for all } t > 0,$$

which implies (4.3).

**Lemma 4.3.** There holds

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( v - h(1) \right)^2 \leq - \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\Omega} \left( v - h(1) \right)^2 + \frac{L^2}{2} \int_{\Omega} (u - 1)^2 \quad (4.4)$$

for all $t > 0$.

**Proof.** Testing the second equation in (1.1) against $(v - h(1))$ we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( v - h(1) \right)^2 = - \int_{\Omega} |\nabla v|^2 - \int_{\Omega} \left( v - h(1) \right)^2 + \int_{\Omega} \left( v - h(1) \right) \left( h(u) - h(1) \right)$$

for all $t > 0$. Here by the mean value theorem, the assumption (1.2) and Cauchy-Schwarz inequality we can estimate

$$\left| \int_{\Omega} \left( v - h(1) \right) \left( h(u) - h(1) \right) \right| \leq \frac{1}{2} \int_{\Omega} \left( v - h(1) \right)^2 + \frac{1}{2} \int_{\Omega} \left( h(u) - h(1) \right)^2 \leq \frac{1}{2} \int_{\Omega} \left( v - h(1) \right)^2 + \frac{L^2}{2} \int_{\Omega} (u - 1)^2$$

for all $t > 0$, this yields (4.4).

We now can construct a Lyapunov functional for our purpose.
Lemma 4.4. We let
\[
F(t) := \int_{\Omega} \left( u(\cdot, t) - 1 - \ln u(\cdot, t) \right) + \frac{\chi^2}{8} \int_{\Omega} \left( v(\cdot, t) - h(1) \right)^2, \quad t > 0. \tag{4.5}
\]
Then we have
\[
F'(t) \leq -\left( \mu - \frac{\chi^2 L^2}{8} \right) \int_{\Omega} \left( u(\cdot, t) - 1 \right)^2 - \frac{\chi^2}{8} \int_{\Omega} \left( v(\cdot, t) - h(1) \right)^2 \tag{4.6}
\]
for all \( t > 0 \).

Proof. In view of the identity (2.2) and the differential inequalities established in Lemma 4.2 and Lemma 4.3 we have
\[
F'(t) \leq \mu \int_{\Omega} u - \mu \int_{\Omega} u^2 - \left\{ - \frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2 + \mu |\Omega| - \mu \int_{\Omega} u \right\} \\
+ \frac{\chi^2}{4} \left\{ - \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\Omega} \left( v - h(1) \right)^2 + \frac{L^2}{2} \int_{\Omega} (u - 1)^2 \right\}
\]
\[
= -\left( \mu - \frac{\chi^2 L^2}{8} \right) \int_{\Omega} (u - 1)^2 - \frac{\chi^2}{8} \int_{\Omega} \left( v - h(1) \right)^2 \quad \text{for all } t > 0.
\]
This yields (4.6).

Finally, we are in a position to prove the convergence assertion.

Proof of Theorem 1.2. From Lemma 4.4 we infer that
\[
\left( \mu - \frac{\chi^2 L^2}{8} \right) \int_0^t \int_{\Omega} (u - 1)^2 + \frac{\chi^2}{8} \int_0^t \int_{\Omega} \left( v - h(1) \right)^2 \leq F(0) \quad \text{for all } t > 0
\]
due to the fact that \( s - 1 - \ln s \geq 0 \) for any \( s > 0 \) and that hence
\[
\int_0^\infty \int_{\Omega} (u - 1)^2 < \infty \quad \text{and} \quad \int_0^t \int_{\Omega} \left( v - h(1) \right)^2 < \infty \tag{4.7}
\]
thanks to \( \mu - \frac{\chi^2 L^2}{8} > 0 \) guaranteed by the assumption (1.5). Using the first weak convergence information provided in (4.7), along with the Hölder continuity established in Lemma 4.1 which implies the spatio-temporal equicontinuity property of \( u \), yields (1.6) (see [18] for details). Similarly, the convergence statement for \( v \) can be verified.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No. 11571070). The authors also thank the referees for their valuable comments.

REFERENCES
[1] M. A. J. Chaplain and J. I. Tello, On the stability of homogeneous steady states of a chemotaxis system with logistic growth term, Appl. Math. Lett., 57 (2016), 1–6.
[2] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), 52–107.
[3] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, J. Differential Equations, 256 (2014), 2993–3010.
[4] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.
[5] P. L. Lions, *Résolution de problèmes elliptiques quasilinéaires*, Arch. Ration. Mech. Anal., 74 (1980), 335–353.

[6] M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth, *Physica A*, 230 (1996), 499–543.

[7] N. Mizoguchi and P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaré, Analyse Non Linéaire, 31 (2014), 851–875.

[8] M. R. Myerscough, P. K. Maini and J. Painter, Pattern formation in a generalized chemotactic model, *Bull. Math. Biol.*, 60 (1998), 1–26.

[9] E. Nakaguchi and K. Osaki, Global solutions and exponential attractors of a parabolic-parabolic system for chemotaxis with subquadratic degradation, Discrete Contin. Dyn. Syst. B, 18 (2013), 2627–2646.

[10] E. Nakaguchi and K. Osaki, $L^p$-estimates of solutions to n-dimensional parabolic-parabolic system for chemotaxis with subquadratic degradation, Funkcialaj Ekvacioj, 59 (2016), 51–66.

[11] E. Nakaguchi and K. Osaki, Global existence of solutions to n-dimensional parabolic-parabolic system for chemotaxis with subquadratic degradation, Preprint.

[12] M. E. Orme and M. A. J. Chaplain, A mathematical model of the first steps of tumour-related angiogenesis: Capillary sprout formation and secondary branching, IMA J. Math. Appl. Med. Biol., 13 (1996), 73–98.

[13] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Analysis, 51 (2002), 119–144.

[14] M. M. Porzio and V. Vespri, Holder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, J. Differential Equations, 103 (1993), 146–178.

[15] C. Stinner, C. Surulescu and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, SIAM J. Math. Anal., 46 (2014), 1969–2007.

[16] Y. Tao and M. Winkler, Large time behavior in a multidimensional chemotaxis-haptotaxis model with slow signal diffusion, SIAM J. Math. Anal., 47 (2015), 4229–4250.

[17] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, Z. Angew. Math. Phys. 66 (2015), 2555–2573.

[18] Y. Tao and M. Winkler, Boundedness and competitive exclusion in a population model with cross-diffusion for one species, Preprint.

[19] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, Commun. Partial Differential Equations, 35 (2010), 1516–1537.

[20] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.

Received March 2016; revised April 2016.

E-mail address: 774146061@qq.com
E-mail address: taoys@dhu.edu.cn