Factorization Method for d-Dimensional Isotropic Harmonic Oscillator and the Generalized Laguerre Polynomials

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The factorization method of Infeld and Hull is applied to the radial Schrödinger equation for d-dimensional isotropic harmonic oscillator and various new ladder operators are defined. The radial energy eigenstates are expressed in terms of the generalized Laguerre polynomials and their properties are shown to follow from the expressions involving the ladder operators. In the same way as the harmonic oscillator we also obtain the bound energy eigenstates of the Morse oscillator.

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I. INTRODUCTION

The factorization method for solving the eigenvalue equations for the Schrödinger operators was first introduced by Schrödinger [1], which he also applied it to the hypergeometric differential equation [2]. Later, the method was generalized by Infeld and Hull [3], and an extensive list of references of the method can be found in [4]. It is an effective algebraic method which is related to the mathematical structure of supersymmetric quantum mechanics [5–7] and also to the concept of the shape invariance [8, 9].

The linear harmonic oscillator is the textbook-example for which the method is applied elegantly. In this case, all the analytical properties of the eigenfunctions, namely, the Hermite polynomials, can be derived from the factorization method itself. However, this is not the case for the multi-dimensional isotropic harmonic oscillator for which the special function related to the radial eigenfunctions is the generalized Laguerre polynomials. The generalized Laguerre polynomials occur in various other well-known quantum mechanical problems, such as the Morse oscillator and the Coulomb problem and thus their properties are essential for the study of these problems as well.

One of the algebraic methods to study the spectrum of the isotropic harmonic oscillator problem is to employ a realization of $so(2,1)$ Lie algebra as a spectrum generating algebra [10]. Another algebraic method is to transform the radial equation into a confluent hypergeometric equation and then factorize the resulting equation which requires the use of the properties of the Laguerre polynomials [3, 4]. Still another algebraic way is to start from the generalized Rodrigues formula for the Laguerre polynomials and define ladder operators from various recurrence relations [11, 12] which again requires the use of the properties of the Laguerre polynomials.

In the present paper, we will show that the properties of the generalized Laguerre polynomials can be obtained as a by product of the factorization of the radial equation for the isotropic harmonic oscillator in arbitrary dimensions. We also show that the same procedure can be applied to obtain bound energy eigenstates of the Morse oscillator as well.

The paper is organized as follows. In Section II we set our notation by defining the ladder operators that factorize the radial part of the Schrödinger equation for the isotropic harmonic oscillator, following the method of Infeld-Hull [3]. Using these ladder operators, we also define new type of $l$-changing ladder operators which leave value of radial quantum number $j$ fixed. We also show how to construct further new ladder operators that shift between any given pair of radial eigenstates labeled by angular and radial quantum numbers $(l, j)$.

In Section III we indicate how the properties of the generalized Laguerre polynomials can be obtained by using the various ladder operators and the radial eigenfunctions defined in Section II.

In Section IV we find the energy eigenstates of the Morse oscillator in terms of the generalized Laguerre polynomials by using the ladder operators obtained by factorizing the Morse hamiltonian. In Section V we present our conclusions.
II. ISOTROPIC HARMONIC OSCILLATOR IN $d$-DIMENSIONS

The details of the factorization method presented here can be found in [2], where the isotropic harmonic oscillator is classified as type-C factorization. We also refer to [3] for the factorization method applied to the radial Schrödinger equation for a non-relativistic particle with the central potential of the form $\frac{1}{2}m\omega^2 r^2 + \frac{k^2}{2mR^2}$ in $d$ dimensions from a different point of view.

We shall work with the variables expressed in natural units. The natural unit of energy for the harmonic oscillator is $\hbar\omega$. Dividing the Schrödinger equation by $\hbar\omega$ and also scaling length by $r_0 \equiv (\frac{\hbar}{m\omega})^{1/2}$; momentum by $\hbar/r_0$ one has the Schrödinger equation in terms of the natural units. In the natural unit system, the canonical commutation relations read $[x_m, p_n] = i\delta_{mn}$ where $m, n = 1, 2, \ldots d$.

Up to a sign, square of the momentum in coordinate representation is given by the generalized Laplacian operator. In Euclidian space $\mathbb{R}^d$, and in terms of spherical coordinates, it can be written as

$$\Delta^d = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta^{S^{d-1}}$$

where $\Delta^{S^{d-1}}$ is the Laplacian operator defined on the unit sphere $S^{d-1}$ with the metric induced from that of the ambient Euclidian space $\mathbb{R}^d$. The radial momentum, in accordance with the Weyl prescription of ordering of non-commuting operators, can be defined as

$$p_r = \frac{1}{2} \left( 1 \frac{\partial}{\partial r} - \frac{\partial}{\partial r} \frac{1}{r} \right) = -i \frac{1}{r^{(d-1)/2}} \frac{\partial}{\partial r} r^{(d-1)/2}$$

in the position representation. The radial momentum has the canonical commutator $[r, p_r] = i$ with the radial coordinate and one has $p_r = p_r^\dagger$ with respect to the inner product defined in terms of the the weight function $w(r) = r^{(d-1)}$ with $r \in [0, \infty)$. With this weight function, the inner product of two functions $|f\rangle$ and $|g\rangle$, in the Hilbert space of functions spanned by the square integrable energy eigenfunctions becomes

$$\langle f | g \rangle = \int_0^\infty dr r^{d-1} f^*(r)g(r).$$

The radial term of the Laplacian operator in (1) can be written in terms of the coordinate representation of the radial momentum as

$$-\frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} = p_r^2 + (d - 1)(d - 3) \frac{1}{4r^2}.$$ (4)

$(d-1)$ number of angular coordinates can be separated from the radial coordinate by introducing generalized spherical harmonics, $\mathcal{Y}_l(\hat{r})$, in $d$ dimensions [12]. Recalling that $r^l \mathcal{Y}_l(\hat{r})$ can be expressed in terms of homogeneous irreducible monomials of order $l$ in the Cartesian coordinates $x_m$. Hence, they satisfy $\Delta^d r^l \mathcal{Y}_l(\hat{r}) = 0$ and therefore using (1) one has

$$\Delta^{S^{d-1}} \mathcal{Y}_l(\hat{r}) = -l(l + d - 2) \mathcal{Y}_l(\hat{r}).$$ (5)

As a result, one obtains the set of effective Hamiltonians $h_l$, depending on the radial coordinate and also on the separation parameters $l$ and $d$. $h_l$ can be written as

$$h_l = \frac{1}{2} \left( p_r^2 + \left[ (l + \frac{1}{2}(d - 2))^2 - \frac{1}{4} \right] \frac{1}{r^2} + r^2 \right).$$ (6)

This form of $h_l$ implies that the eigenvalue equation $h_l |l, j\rangle = \varepsilon_j(l) |l, j\rangle$ admits factorization since it is of the form the sum of the squares of two operators up to a constant remainder. Note also that the radial Coulomb-type potential of the form $\frac{1}{r^2}$ also allows the corresponding effective radial Hamiltonian to be written as the of product of two hermitian conjugate operators up to a constant remainder.

The first label of radial eigenkets $|l, j\rangle$ refers to the angular momentum quantum number, whereas the second one $j$ refers to the radial quantum number. For convenience, the energy eigenvalues corresponding to the radial eigenfunctions $R_{lj}^0(r) =: (r|l,0\rangle$ are denoted by $\varepsilon_j(l)$, all of which depend on the parameters $l, j$ and $d$.

The ladder operators, which are the conjugates of each other with respect to the inner product [3] defined with respect to the weight function $w(r) = r^{(d-1)}$, are of the form

$$D_l = \frac{1}{\sqrt{2}} \left( +ip_r - (2l + d - 1) \frac{1}{2r} + r \right),$$ (7)

$$D_l^\dagger = \frac{1}{\sqrt{2}} \left( -ip_r - (2l + d - 1) \frac{1}{2r} + r \right).$$ (8)
By using the same symbols for the position representations for the ladder operators, the position representations of these operators can be written as

\[ D_l = \frac{1}{\sqrt{2}} r^l e^{-r^2/2} \frac{d}{dr} e^{r^2/2} r^{-l}, \]
\[ D_l^\dagger = -\frac{1}{\sqrt{2}} r^{-(l+1)} e^{-r^2/2} \frac{d}{dr} e^{r^2/2} r^{l+1}. \]

The particular forms of the position representations (9) and (10) for the first order ladder operators can be obtained from (7) and (8) respectively by introducing appropriate integrating factors for the multiplicative terms of the ladder operators and will be shown to be very useful for the discussions below.

For \( l \rightarrow 0 \) and \( d \rightarrow 1 \), the expressions in (9) and (10) become the position representations for the ladder operators of the one dimensional harmonic oscillator, namely \( a \) and \( a^\dagger \) respectively.

The choice of the ladder operators having the same algebraic structure is not unique. However, the eligible ladder operators can be obtained from the requirements: (i) The key functions which are annihilated by \( D_l \) should be normalizable, that is, \( \langle l, 0|l, 0 \rangle = 1 \), (ii) \( \varepsilon_j(l) < \varepsilon_{j+1}(l) \) for all \( l, j = 0, 1, 2, \ldots \) which ensures the normalization of the radial eigenkets \( |l, j \rangle \), that is, \( \langle l, j|l, j \rangle = 1 \) for \( j > 0 \). In terms of the ladder operators above, the hamiltonian can be written as

\[ h_l = D_l^\dagger D_l + \varepsilon(l) \]

where \( \varepsilon_0(l) = l + d/2 \) is the ground state energy eigenvalue corresponding to the key function \( R_0^{(l)} \) provided that \( D_l R_0^{(l)} = 0 \). The spectra of the sequence of the hamiltonians \( h_{l+j} \) with \( l, j = 0, 1, 2, \ldots \) follow from the algebra of the ladder operators (see Fig. 1). From the definitions of the ladder operators, one finds the recurrence relations

\[ D_{l+j+1}^\dagger D_{l+j+1} + 2 = D_{l+j} D_{l+j}^\dagger. \]

with \( l, j = 0, 1, 2, \ldots \). In terms of the sequence of the hamiltonians \( h_{l+j} \) which are naturally defined as

\[ h_{l+j} = D_{l+j}^\dagger D_{l+j} + \varepsilon_j(l). \]

The recurrence relation (12) for \( j = 0 \) can be written as

\[ h_l D_l^\dagger = D_l^\dagger h_{l+1}. \]

Iteration of this result in the radial quantum number \( j \) gives

\[ h_l D_l^\dagger D_{l+1}^\dagger \cdots D_{l+j-1}^\dagger = D_l^\dagger D_{l+1}^\dagger \cdots D_{l+j-1}^\dagger h_{l+j}. \]

The recurrence relation (13) implies that the ground state energy of \( h_{l+j} \) corresponds to the \( j^{th} \) excited level of \( h_l \) which is given by \( \varepsilon_j(l) = l + 2j + d/2 \). Thus, the radial energy eigenfunction corresponding to the \( j^{th} \) excited eigenstate of the hamiltonian \( h_l \) is given by

\[ |l, j \rangle \propto D_l^\dagger D_{l+1}^\dagger \cdots D_{l+j-1}^\dagger |l + j, 0 \rangle \]

up to an appropriate normalization constant to be found below. The radial key eigenfunctions are annihilated by the ladder operator

\[ D_{l+j} |l + j, 0 \rangle = 0. \]

This yields the normalized key radial eigenfunctions as

\[ \langle l + j, 0|l + j, 0 \rangle = R_0^{(l+j)}(r) = \left[ \frac{1}{2}\Gamma(l + j + d/2) \right]^{-1/2} r^{(l+j)} e^{-r^2/2} \]

by inspecting the position representation of the ladder operator in (9). It follows from both the definition of the key functions and of the ladder operators in harmony that

\[ D_{l+j} |l + j, 0 \rangle = r D_{l+j-1} r^{-1} |l + j, 0 \rangle = 0. \]
and as a result one finds $|l + j, 0\rangle \propto r^{\mp 1}|l + j \mp 1, 0\rangle$. Therefore, the operators $r^{\pm 1}$ can be identified as the ladder operators which shift between the successive key functions. Furthermore, the definition of the operator $D^\dagger_{(l+j)}$ in (10) allows the ladder operators $D^\dagger_{(l+j)}$ to be written in terms of $D^\dagger_0$ as
\[ D^\dagger_{(l+j)} = r^{-(l+j)}D^\dagger_0 r^{(l+j)}. \] (20)

The relation in (20) and its conjugate are very practical in the calculations below. Using (20), it is possible to rewrite all the excited radial eigenstates of the hamiltonians $h_l$ as
\[ |l, j\rangle \propto D^\dagger_l D^\dagger_{(l+1)} \cdots D^\dagger_{(l+j-1)} |l + j, 0\rangle, \] (21)
\[ R^{(l)}_j (r) \propto r^{-(l+d-2)} e^{r^2/2} \left( \frac{-1}{\sqrt{2r}} \frac{d}{dr} \right)^j r^{2(l+j-1)+d} e^{-r^2} \] (22)
up to a normalization constant. The normalization of the eigenfunctions $R^{(l)}_j (r)$ follows from the normalization of the eigenfunctions $R^{(l)}_0 (r)$. Using (12) and by induction one finds
\[ \langle l, j|l, j\rangle = \langle l + j, 0|D^\dagger_{(l+j-1)} \cdots D^\dagger_0 |l + j, 0\rangle \] (23)
\[ = \prod_{k=1}^{j} (\varepsilon_k(l) - \varepsilon_0(l)) |l + j, 0|l + j, 0\rangle \] (24)
\[ = 2^j j! |l + j, 0|l + j, 0\rangle. \] (25)

Therefore, the normalized radial eigenfunctions can be written in a convenient form as
\[ R^{(l)}_j (r) = \left[ 2^{(j-1)} j! \Gamma(l + j + d/2) \right]^{-1/2} e^{r^2/2} r^{-(l+d-2)} \left( -\sqrt{2} \frac{d}{dr} \right)^j r^{2(l+d/2-1+j)} e^{-r^2}. \] (26)

The expression (26) is a generalized Rodrigues-type formula for the radial eigenfunctions and it is easy to write it in terms of the generalized Laguerre polynomials in the variable $r^2$ as
\[ R^{(l)}_j (r) = (-1)^j \left( \frac{2 \Gamma(j + 1)}{\Gamma(l + j + d/2)} \right)^{1/2} e^{-r^2/2} r^d L^{(l+d/2-1)}_j (r^2), \] (27)
for $j \geq 1$ values of the radial quantum number. In the next section we will show that the functions $L^{(l+d/2-1)}_j (z)$ defined above do indeed satisfy the generalized Laguerre differential equation in the variable $z = r^2$ by using the recurrence relations provided by the ladder operators, cf. Eqs. (27)-(40) below. Note that the radial key eigenfunctions themselves are not generalized Laguerre polynomials and that the minimum value of $l + d/2 - 1$ is $-1/2$ which corresponds to values of $l = 0$ and $d = 1$. The Laguerre polynomials, namely, $L^{(d=0)}_j$-s, occur only for $d = 2$ and zero angular momentum radial energy eigenstates. The expressions (26) and (27) hint at the possibility that the ladder operators can be used to obtain the properties of the generalized Laguerre polynomials which are usually obtained by other means [11]. The functions $e^{-z^2/2} z^\mu L^{(l)}_j (z)$ are sometimes called the generalized Laguerre functions [12]. In Section IV, we will also show that for $z = 2e^{-x}$ these functions correspond to the bound eigenstates of the Morse oscillator. In the study of the coherent states defined in terms of the ladder operators, a hamiltonian whose eigenfunctions are generalized Laguerre functions were constructed in [17]. In a more general scheme, it is also possible to define ladder operators for the classical polynomials starting from the corresponding generalized Rodrigues formula and the recurrence relations [12,13].

It is possible to construct a new type of ladder operators, namely $l$-changing ladder operators, which shift between the radial eigenfunctions with a fixed value of the radial quantum number $j$. For this purpose, first recall that $|l + j \mp 1, 0\rangle \propto r^{\mp 1}|l + j, 0\rangle$ and therefore, using this property, the relation (10) can be written as
\[ |l + 1, j\rangle \propto D^\dagger_{(l+1)} D^\dagger_{(l+2)} \cdots D^\dagger_{(l+j)} |l + j + 1, 0\rangle \] (28)
\[ = r^{-1} D^\dagger_{(l+1)} D^\dagger_{(l+2)} \cdots D^\dagger_{(l+j-1)} r^2 |l + j, 0\rangle \]
up to a normalization constant. By commuting $r^2$ to the left of the product of the ladder operators, it is possible to rewrite the right hand side back in terms of $|l, j\rangle$ again. To this end, one needs the commutator identity
\[ [D^\dagger_{(l+1)} D^\dagger_{(l+2)} \cdots D^\dagger_{(l+j-2)} D^\dagger_{(l+j-1)}] r^2 = -\sqrt{2j} r D^\dagger_{(l+1)} D^\dagger_{(l+2)} \cdots D^\dagger_{(l+j-2)} D^\dagger_{(l+j-1)}. \] (29)
radial eigenkets can be written as operator which will be denoted by \( \mathcal{D}(l,j) \) and \( \mathcal{D}^\dagger(l,j) \), respectively, (ii) Nowhere in section II does the notation imply that \( \mathcal{D}(l,j) \) and \( \mathcal{D}^\dagger(l,j) \) depend on both of the quantum numbers \( l \) and \( j \) and also on the dimension \( d \) which is not indicated for convenience of the notation. Note here that, it is possible to define an operator composed of appropriate combinations of the \( l \)-changing ladder operators defined so far, namely, \( \mathcal{D}(l+1,j) \); \( \mathcal{D}^\dagger(l+1,j) \); \( \mathcal{D}(l,j) \) and \( \mathcal{D}^\dagger(l,j) \); \( \mathcal{D}(l-1,j) \), \( \mathcal{D}^\dagger(l-1,j) \), \( \mathcal{D}(l,j) \) and \( \mathcal{D}^\dagger(l,j) \), that shifts between any given pair of the eigenkets \( |l,j\rangle \) and \( |l',j'\rangle \) illustrated in Fig. 1. For instance, the combination \( \mathcal{D}^\dagger(l,j)\mathcal{D}(l,j) \) of the ladder operators acting on the eigenstate \( |l,j\rangle \) gives

\[
\mathcal{D}^\dagger(l,j)\mathcal{D}(l,j) |l,j\rangle \propto |l,j+1\rangle
\]

up to a normalization constant. Therefore, it is possible to identify \( \mathcal{D}^\dagger(l,j)\mathcal{D}(l,j) \) and its conjugate, namely \( \mathcal{D}(l,j)\mathcal{D}^\dagger(l,j) \), as \( j \)-changing ladder operators.

It is worth to emphasize at this point that (i) The ladder operators above can be used to construct new ladder operators that shift between any given states \( |l,j\rangle \) and \( |l',j'\rangle \) and \( l \)-changing and \( j \)-changing operators defined above are examples of these new of ladder operators which shift between \( |l+1,j\rangle \) and \( |l,j+1\rangle \) respectively, (ii) Nowhere in

FIG. 1: The lattice of the energy levels for the sequence of hamiltonians \( h(l,j) \). The column of the equally spaced eigenkets above the ground level \( |l+j,0\rangle \) for each \( j = 0, 1, 2, \ldots \) constitute the spectrum of the hamiltonian \( h(l,j) \). The energy difference between successive key eigenkets is one unit (of \( \hbar \omega \)) whereas the energy difference of the two successive eigenkets of the hamiltonian \( h(l,j) \) is two units. The actions of the ladder operators on the eigenkets are indicated by the arrows.
the above construction the properties of the generalized Laguerre polynomials have been used. Furthermore, various recurrence relations for the generalized Laguerre polynomials can be derived using any of the ladder operators defined above. From this point of view, the presentation above can be considered to unify the method of solving eigenvalue problem and obtaining the recurrence relations of the related special function as a by product. In contrast to the other approaches, the ladder operators are constructed by invoking the properties of related special functions \[11, 12\] and the matrix elements for any of the ladder operators defined above can be calculated using the mathematical tools constructed above without employing any of the properties of the Laguerre polynomials as additional mathematical property of the solutions. For example, integrals involving the generalized Laguerre polynomials can be calculated using the normalization of the radial eigenfunctions and the expressions of the radial eigenfunctions in terms of the ladder operators, in particular, the conventional normalization of the generalized Laguerre polynomials can be found from the normalization of the radial eigenfunctions.

As will be shown in the next section, the expressions involving the ladder operators and the energy eigenfunctions can be translated into the expressions involving the generalized Laguerre polynomials. Since it is possible to construct a number of different recurrence relations for the radial eigenfunctions by defining an appropriate composition of the ladder operators defined above, the factorization method also provides a new way of deriving new recurrence relations for the generalized Laguerre polynomials as well.

III. PROPERTIES OF THE GENERALIZED LAGUERRE POLYNOMIALS

For the definitions and the conventions of the Laguerre polynomials adopted here we refer to \[11, 16\]. First, we note that the standard normalization of the generalized Laguerre polynomials follows easily from the normalization of the radial eigenfunctions, all of which can be turned into Gamma function-type integrals. The radial eigenfunctions which are normalized to unity are given explicitly in (26) and they can be expressed in terms of the Laguerre polynomials

\[
\langle l, j | \phi_l \rangle = \frac{\Gamma(j + 1)}{\Gamma(l + d/2 + j)} \int_0^\infty wr^{l+d/2-1} L_j^{(l+d/2-1)}(w) dw
\]

where the variable \( r^2 = z \) has been introduced in the second line. From this expression it follows that

\[
\int_0^\infty e^{-z} z^\mu L_j^n(z) \, dz = \frac{\Gamma(\mu + j + 1)}{\Gamma(j + 1)}
\]

with \( \mu = l + d/2 - 1 \). This standard normalization can also be obtained by means of the Rodrigues formula for the Laguerre polynomials.

Second, we illustrate that any expression for the radial eigenfunctions involving the ladder operators can be translated into corresponding recurrence relation for the Laguerre polynomials and in particular we derive the Laguerre differential equation as follows. As the simplest example, consider the expression \[27\]. By using the relations among the radial eigenstates \( R_l^{(l)} \) and \( R_l^{(l+1)} \), namely,

\[
\langle l, j \rangle \propto D_l^j | l + 1, j - 1 \rangle = r^{-l} D_l^j | l + 1, j - 1 \rangle
\]

it is possible to derive corresponding recurrence relation for the generalized Laguerre polynomials. By the change of variable \( r^2 = z \) and using the definition \[27\], Eqn. \[36\] becomes

\[
L_j^{(l+d/2-1)}(z) = \frac{1}{j} z^\frac{1}{2} e^{-z} \frac{d}{dz} L_l^{(l+d/2)}(z).
\]

Similarly, by using the normalized ladder operator \( D_l \), the recurrence relation

\[
| l + 1, j - 1 \rangle = D_l | l, j \rangle,
\]

in terms of Laguerre polynomials, becomes

\[
L_j^{(l+d/2-1)}(z) = - \frac{d}{dz} L_j^{(l+d/2)}(z)
\]
where $z = r^2$. The recurrence relations (37) and (39) can be used to obtain the second order differential equation that $L_j^{(l+1+d/2)}(z)$ satisfy. By defining $\mu = l + d/2 - 1$ for convenience and using the relations (37) and (39) one finds
\[ z \frac{d^2}{dz^2} L_j^\mu(z) + (\mu + 1 - z) \frac{d}{dz} L_j^\mu(z) + j L_j^\mu(z) = 0 \]  
which justifies, a posteriori, that the $L_j^\mu$ introduced in (27) corresponds to the generalized Laguerre polynomials.

For $d = 1$ the quantum number corresponding to the angular momentum can take the value zero only while the radial quantum number becomes the principle quantum number for the 1-d harmonic oscillator and therefore for the values $d = 1, l = 0$ the radial eigenfunctions become the energy eigenfunctions of the 1-d harmonic oscillator. By using the expression (42) for the radial eigenfunctions $R_j^{(l)}$ and the key functions $R_0^{(l)}$ together with the definition of the Hermite polynomials, one has the relation
\[ H_{2j}(r) = (-1)^j 2^j j! L_j^{1/2}(r^2) \]  
where $H_j(r)$ is the Hermite polynomial of order $j$ and $L_j^\mu(z)$ is a polynomial of order $j$ in the variable $z$ respectively 10. Using this identification, it is easy to check the validity of the recursion relation (37). This can be done by putting $d = 1$ and $l = 0$ in (37). Thus, by the change of the variable $r^2 = z$, one finds
\[ L_j^{-1/2}(z) = -z^{-1/2} e^{-z} \frac{d}{dz} e^{-z} L_j^{1/2}(z) \]  
Using the relation (11), and the recursion relation between the Hermite polynomials, namely,
\[ H_{(2j+1)}(r) = -\frac{1}{\sqrt{2}} e^{r^2} \frac{d}{dr} e^{-r^2} H_{2j}(r) \]  
which already follows from the construction above, one finds that
\[ L_j^{1/2}(r^2) = \frac{(-1)^j}{2^j j! r} H_{(2j+1)}(r). \]  
Returning back to the recursion relation (37), it can be be put into a more useful form
\[ z^{-\mu} e^{-z} L_j^\mu(z) = \frac{1}{j} \frac{d}{dz} e^{-z} z^{\mu+1} L_{(j-1)}^{(\mu+1)}(z) \]  
with $\mu = l + d/2 - 1$ for convenience. This form of the recurrence relation shows that $\frac{d}{dz}$ is the ladder operator between the functions $e^{-z} z^\mu L_j^\mu(z)$ and can be iterated $k$ times to obtain
\[ L_j^\mu(z) = \frac{(j-k)!}{k!} z^{-\mu} e^z \frac{d^k}{dz^k} e^{-z} z^{\mu+k} L_{(j-k)}^{(\mu+k)}(z). \]  
In terms of the radial coordinate and in terms of the radial eigenfunctions, this recurrence relation corresponds to the coordinate representation of the expression
\[ |l, j\rangle \propto D_l^1 D_{(l+1)}^1 D_{(l+2)}^1 \cdots D_{(l+k-2)}^1 D_{(l+k-1)}^1 |l + k, j - k\rangle. \]  
For $k = j$, this yields the Rodrigues formula for the Laguerre polynomials rather then a recurrence relation among them. Similarly, the conjugate of the relation (47), namely,
\[ |l + k, j - k\rangle \propto D_{(l+k-1)} D_{(l+k-2)} \cdots D_{(l+2)} D_{(l+1)} D_l |l, j\rangle \]  
can be translated into the recurrence relation for the generalized Laguerre polynomials as
\[ L_{(j-k)}^{(\mu+k)}(z) = (-1)^k \frac{d^k}{dz^k} L_j^\mu(z). \]  
The expressions for the radial eigenfunctions involving $l$-changing operators $D_{(l,j)}^l$ and $D_{(l,l)}^l$ can also be translated into a recurrence relation for the generalized Laguerre polynomials. For example, using normalized ladder operators, the relation (39) can be used to derive recurrence relations for the generalized Laguerre polynomials as
\[ L_j^{(\mu-1)}(z) = z^{-\mu+1} \frac{d}{dz} e^{-z} L_j^\mu(z), \]  
where $\mu = l + d/2 - 1$ and $z = r^2$ as before. Evidently, it is possible to derive further recurrence relations for the Laguerre polynomials from the expressions involving the ladder operators acting on the radial eigenfunctions. As the above typical examples illustrate, the normalization and recurrence properties of the Laguerre polynomials are built into the factorization method by construction.
IV. MORSE OSCILLATOR

The above choice and the special way of writing the ladder operators $D_{(l+j)}$ and $D^\dagger_{(l+j)}$ for the harmonic oscillator make it possible to obtain the useful properties of the generalized Laguerre polynomials. We will illustrate that the above approach also provides a framework for expressing energy eigenstates of factorizable (or supersymmetric) hamiltonians in terms of related special function.

In this section we shall obtain the bound energy eigenstates of the Morse oscillator following the same approach as the harmonic oscillator. The Morse potential, which is important in the study of molecular vibrations, is given by

$$V(x) = V_0(e^{-2\beta x} - 2e^{-\beta x})$$

with the parameters $V_0 > 0$ and $\beta$ having dimension inverse length with $-\infty < x < \infty$ [3, 19, 20]. It is also possible to transform the Morse potential problem into the two dimensional harmonic oscillator by an appropriate change of variables [21]. The finite number energy eigenstates for the bound states of the Morse potential can be expressed in terms of the generalized Laguerre polynomials by constructing appropriate ladder operators as follows. The natural units of length, momentum and energy of the Morse hamiltonian can be taken as $1$ units of length, momentum and energy of the Morse hamiltonian can be taken as $1$ units of length, momentum and energy of the Morse hamiltonian can be taken as $1$

$$\lambda = \frac{2V_0m}{\hbar^2}$$

defining the parameter $\lambda$. Using the canonical commutation relation

$$[x, p] = i\hbar$$

the harmonic oscillator, by using the same symbols for the operators, the coordinate representation of the conjugate ladder operators can be written conveniently as

$$D_n = +ip - (e^{-x} - n), \quad D^\dagger_n = -ip - (e^{-x} - n).$$

As in the case of the harmonic oscillator, by using the recurrence relation (57) and the definition of the generalized Laguerre polynomials, one finds

$$h_{(n+1)}D^\dagger_{(n+1)} = D^\dagger_n D_n + 2n + 1, \quad n = 1, 2, 3\ldots$$

by using the canonical commutation relation $[x, p] = i$. It is convenient to work with the hamiltonian $h_n$ expressed in terms of the ladder operators $[55]$ and $[51]$, is given by

$$h_n = p^2 + [e^{-x} - (n + \frac{1}{2})]^2 = D^\dagger_n D_n + (n + \frac{1}{2}).$$

The single key eigenket, which corresponds to the ground state, will be denoted by $|n\rangle$. The key eigenket is annihilated by the lowering operator. Thus $D_n|n\rangle = 0$ yields the normalized eigenket

$$\psi_n(x) = \frac{2^n e^{-nx}e^{-x}}{[\Gamma(2n)]^{1/2}}$$

corresponding to the ground state energy $\varepsilon'_n = n + \frac{1}{4}$. Using the recurrence relation (57) and the definition of the Morse Hamiltonian $[55]$, one finds

$$h_{(n+j)}D^\dagger_{(n+j)}\cdots D^\dagger_{(n+1)} = D^\dagger_{(n+j)}\cdots D^\dagger_{(n+1)} \left\{ h_n + \sum_{k=1}^j [2(n + k) + 1] \right\}$$

(60)
which determines all the excited states and the corresponding eigenvalues. In turn, by using the key functions \( \beta \) and the expression \( \psi(n,j) \) and the position representation of the raising ladder operator \( \beta \), the excited states can be written as

\[
\psi(n+j)(x) \propto \langle x | D_{n+j} \cdots D_{n+1} | n \rangle = (-1)^j e^{(n+1)\frac{x-e^{-z}}{2}} e^{-(2n+1)z-2e^{-z}}
\]

(61)

with the corresponding energy \( \varepsilon'(n+j) = \varepsilon'_n + \sum_{k=1}^j [2(n + k) + 1] \). The expression (61) suggests that it is convenient to change to the variable \( z = 2e^{-x} \). Therefore, all the excited energy eigenstates can be written as

\[
\psi(n+j)(x) \propto z^{-(n+1)} e^{z/2} \left( \frac{d}{d(1/z)} \right)^j z^{(2n+1)} e^{-z}
\]

(62)

in the new variable \( z \). Note that with a suitable change of variables, the expression for excited states in terms of ladder operators resembles to the corresponding expression for the linear oscillator, where the excited states are obtained by repeated application of a single raising ladder operator on the ground state. Using the identity

\[
\frac{d^{j}}{dz^{j}} f(z) = (-1)^j \rho^{(j+1)} \frac{d^{j}}{d\rho^{j}} \rho^{(j-1)} f(\frac{1}{\rho})
\]

(63)

where \( z = 1/\rho \) and for any continuous function \( f(z) \), it is possible to rewrite the expression \( \psi(n,j) \) in terms of the generalized Laguerre polynomials as

\[
\psi(n,j)(x) = \left( \frac{\Gamma(j+1)}{\Gamma(2n+j+1)} \right)^{1/2} z^n e^{-z/2} L_j^{2n}(z)
\]

(64)

where \( z = 2e^{-x} \) and normalization constants of the eigenfunctions \( \psi(n,j)(x) \) can be found from \( \beta \). The eigenfunctions \( \beta \) can easily be expressed back in terms of the parameters \( \lambda \) and \( \beta \) of the Morse Hamiltonian. In contrast to the \( d \)-dimensional harmonic oscillator, for the Morse Hamiltonian there is only one quantum number yet the eigenfunctions are given in terms of the generalized Laguerre polynomials defined with two parameters.

V. CONCLUSION

The Schrödinger equations for isotropic harmonic oscillator and the Morse potential are studied using the factorization method of Infeld-Hull and for both of the potentials, the associated special function is identified as the generalized Laguerre polynomials and they are expressed in terms of appropriate independent variables which directly obtained from the factorization. In the factorization method of Infeld-Hull, the connection between the set of eigenfunctions and the special functions involved, is usually established, or rather verified, by comparing the results obtained from the algebraic methods with the results obtained by other methods [3]. In this respect, it is worth to emphasize that in the two cases presented above, the factorization is carried out without transforming the Schrödinger equation into a standard Hypergeometric-type differential equation by the change of the dependent and the independent variable at the outset and then employing the properties of the generalized Laguerre polynomials. In contrast, the generalized Laguerre polynomials are obtained directly using the expressions involving the ladder operators of the standard factorization method. Thus, we showed in the particular case of the harmonic oscillator that the feature of the Infeld-Hull factorization method which directly relates the energy eigenstates to the relevant special functions, that is, to the generalized Laguerre functions as in \( \beta \) and also showed that any expression involving the ladder operators defined for the radial eigenfunctions in Section II can be used to define a corresponding recurrence relation for the generalized Laguerre polynomials with the help of the relation \( \beta \).

Finally, we remind that the algebraic approach presented for the generalized Laguerre polynomials above is already known to apply to other special functions. A well-known example is the quantum mechanical angular momentum. Two different recurrence formulae for the associated Legendre polynomials \( P_l^m(\theta) \) can be obtained from the properties of the ladder operators \( L_\pm \) using the key functions \( P_l^m(\theta) e^{i\ell\theta} \) and the zero angular momentum states. By writing the coordinate representation of the ladder operators \( L_\pm \) in a form similar to those of the harmonic oscillator, it is possible to find the Rodriguez type formulae from both \( \langle L_\pm \rangle (l-m) | l, m \rangle \propto | l, m \rangle \) and \( (L_\pm)^{m} | l, 0 \rangle \propto | l, m \rangle \) for the \( P_l^m(\theta) \)’s. Another well-known example is the treatment of the quantum mechanical free particle in spherical coordinates. By omitting the \( r^2 \) term in the harmonic oscillator Hamiltonian in three dimensions, the factorization method can be used to obtain the spherical Bessel and Neumann functions as well as their recurrence formulae. The different feature
of the latter example is that the normalization of spherical Bessel and Neumann functions (which involve δ-function normalization) can be shown to follow from the properties of the key functions corresponding to the zero angular momentum state.

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