Correspondence between momentum dependent relaxation time and field redefinition of relativistic hydrodynamic theory

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In this article a correspondence has been established between the out of equilibrium system dissipation and the thermodynamic field redefinition of the macroscopic variables through the momentum dependent relaxation time approximation (MDRTA) solution of relativistic transport equation. Here, it has been shown that the out of equilibrium thermodynamic fields are not uniquely defined and are subjected to include dissipative effects from the medium. A second order relativistic hydrodynamic theory has been developed including such dissipative effects. The necessary conditions for developing a hydrodynamic theory has been fulfilled, (i) the thermodynamic identities incorporating such redefined fields have been shown to conserve the energy-momentum tensor perfectly under MDRTA, (ii) the non-negativity of entropy production remains unaffected by the inclusion of such dissipative contributions in hydro fields as long as the independent transport coefficients remain positive.

I. INTRODUCTION

Relativistic dissipative hydrodynamic theory has been proved to be reasonably successful in describing the out of equilibrium dynamics of a system in the long wavelength limit. The evolution of the relevant macroscopic quantities such as temperature and charge chemical potential is given by a set of coupled differential equations where the out of equilibrium effects are included by certain dissipative fluxes. These fluxes are quantitatively manifested by quantities called transport coefficients such as viscosity and conductivities. The macroscopic thermodynamic quantities such as energy density and particle number density in these equations are set to their equilibrium values even in the dissipative medium by imposing certain matching or fitting conditions. Recently, in a number of studies the inclusion of dissipative effects into these quantities have been explored where the out of equilibrium field contributions are either estimated phenomenologically or microscopically such as from the renormalization-group technique. In this article those dissipative corrections have been estimated from gradient expansion technique of solving the relativistic transport equation using momentum dependent relaxation time approximation (MDRTA).

In recent literature, a number of studies have been carried out related to the application of momentum dependent relaxation time in solving the relativistic transport equation and extracting the macroscopic thermodynamic quantities thereafter. Relaxation time approximation, originally proposed by Anderson and Witting in 1974 for relativistic transport equation, has been used widely since then with a constant relaxation time almost everywhere. But in doing so two major issues arise that need to be addressed. First, as very nicely demonstrated in [4] that the microscopic interaction theory relevant for the medium can be related to the momentum dependence of relaxation time, so taking a constant one undermines the momentum transfer of underlying interactions. Secondly, it gives rise to identical relaxation times for microscopic particle distributions and macroscopic fields like viscous flow, while the latter is expected to have a slower relaxation rate. Hence, in order to extract relativistic hydrodynamic theory, the application of momentum dependent relaxation time in transport equation is very indicative.

The current work demonstrates that the fundamental thermodynamic quantities such as energy density, pressure, particle number density and hydrodynamic four velocity are redefined at each order of gradient expansion in an out of equilibrium situation including dissipative corrections from the medium. These corrections are estimated with momentum dependent medium interaction using MDRTA for the collision integral of the relativistic transport equation. The transport coefficients for these corrections are estimated for the first and second order theory which turn out to be sensitively dependent upon the momentum dependence of medium interaction. Finally, the hydrodynamic evolution equations are obtained up to second order of gradient expansion. The separation of relaxation time scales for microscopic particle distribution and macroscopic viscous fields have also been depicted which again turns out to be crucially dependent upon the momentum dependence of medium interaction.

The manuscript is organised as follows. Section II provides a basic framework for hydrodynamic field redefinition obtained from relativistic transport equation with a general collision integral. In section III the coefficients for first order field corrections are estimated using MDRTA for the collision term in the transport equation. Section IV provides the second order corrections along with the coefficients and hydro evolution equations. In section V the work has been summarized with necessary discussions and remarks.
II. FIELD REDEFINITION IN RELATIVISTIC HYDRODYNAMIC THEORY

The formalism begins with the microscopic relativistic transport equation for the single particle distribution function \( f(x,p) \) with particle four-momenta \( p^\mu \) and space-time variable \( x^\mu \),

\[
p^\mu \partial_{x^\mu} f(x,p) = C[f] = -\mathcal{L}[\phi] .
\]  

(1)

\( C[f] \) is the collision term which has been linearized to \( \mathcal{L}[\phi] \) over the out of equilibrium distribution deviation \( \phi \),

\( (f = f^{(0)} + f^{(0)}(1 \pm f^{(0)}) \phi) \) with \( f^{(0)} \) as the equilibrium distribution; as follows,

\[
\mathcal{L}[\phi] = \int d\Gamma_{p} \left\{ \phi + \phi^\prime - \phi^\prime \right\} W(p'p_1 | pp_1) .
\]  

(2)

Here, \( d\Gamma_{p} = \frac{d^4 p}{(2\pi)^3} \) is the phase space factor and \( W \) is the interaction rate carrying microscopic cross section. Before applying the MDRTA formalism, let us proceed with the general form of \( \mathcal{L}[\phi] \) given in Eq. (2) for which I briefly review few of its properties. Energy-momentum and particle number conservation gives \( \mathcal{L}[p^\mu] = 0 \) and \( \mathcal{L}[1] = 0 \) respectively. Self adjoint properties \( \int d\Gamma_p \psi \mathcal{L}[\phi] = \int d\Gamma_p \phi \mathcal{L}[\psi] \) with \( \psi = \psi(x, p^\mu) \) gives rise to summational invariant property \( \int d\Gamma_p \mathcal{L}[\phi] = 0 \) and \( \int d\Gamma_p \phi \mathcal{L}[\phi] = 0 \). Finally, applying non-negative entropy production rate it can be proved that \( \int d\Gamma_p \phi \mathcal{L}[\phi] \geq 0 \) where the equality holds for \( \phi = \{1, p^\mu\} \). If one proceeds to solve Eq. (1) with order by order gradient expansion method, for each order \( r = \phi = \sum_r \phi^{(r)} \), the left hand side of (1) turns out to be a linear combination of thermodynamic forces with different tensorial ranks as follows,

\[
\sum_r Q_i^{(r)} X_{i}^{(r)} + \sum_m R_m^{(r)} Y_{m}^{(r)} + \sum_n S_n^{(r)} Z_{n\mu\nu} = -\mathcal{L}[\phi^{(r)}] .
\]  

(3)

Here, \( X_{i}^{(r)} \), \( Y_{m}^{(r)} \), and \( Z_{n\mu\nu} \) respectively are the scalar, vector and rank-2 tensorial thermodynamic forces of gradient expansion order \( r \). The indices \( l, m \) and \( n \) denote number of independent thermodynamic forces of each kind respectively for each order. For eg., for scalar forces with \( r = 1 \), \( l \) has only one value corresponding to \( \partial \cdot u \). For \( r = 2 \), \( l \) runs over \( D(\partial \cdot u), (\partial \cdot u)^2, \sigma_{\mu\nu} \sigma^\mu\nu, \ldots \) and so on. From here on the macroscopic thermodynamic quantities are needed to be addressed. \( T, u^\mu \) and \( \tilde{\mu} / T \) denote the temperature, hydrodynamic four-velocity and scaled chemical potential of the system respectively. \( D = u^\mu \partial_{x^\mu} \) and \( \nabla^\mu = \Delta^\mu_{\nu} \partial_{x^\nu} \) are temporal and spatial counterparts of the total space-time derivative \( \partial_t = u^\mu \partial_{x^\mu} + \nabla^\mu = \Delta^\mu_{\nu} \partial_{x^\nu} \). \( g^{\mu\nu} - \frac{1}{3} \delta_{\mu\nu} \Delta_{\alpha\beta} \Delta_{\alpha\beta} \) respectively with \( \Delta_{\mu\nu} = \frac{1}{2} \left\{ \Delta_{\mu\alpha} \Delta_{\nu\beta} + \Delta_{\mu\beta} \Delta_{\nu\alpha} - \frac{1}{2} \Delta_{\mu\nu} \Delta_{\alpha\beta} \right\} \). \( Q_i^{(r)}, R_m^{(r)} \) and \( S_n^{(r)} \) are the contracted parts which carry the particle signature being function of \( p^\mu \) and its scaled mass \( z = m/T \). The general solution for \( \phi^{(r)} \) is a linear combination of the thermodynamic forces as the following,

\[
\phi^{(r)} = \sum_l A_l^{(r)} X_{l}^{(r)} + \sum_m B_m^{(r)} Y_{m}^{(r)} + \sum_n C_n^{(r)} Z_{n\mu\nu} = -\mathcal{L}[\phi^{(r)}] .
\]  

(4)

where the unknown coefficients \( A_l^{(r)}, B_m^{(r)} \) and \( C_n^{(r)} \) are needed to be estimated from the transport equation itself. Since the thermodynamic forces are independent it is straightforward to derive that,

\[
Q_i^{(r)} = -\mathcal{L}[A_l^{(r)}], R_l^{(r)} = -\mathcal{L}[B_l^{(r)}], S_l^{(r)} = -\mathcal{L}[C_l^{(r)}].
\]  

(5)

In order to extract the coefficients, here they are expanded in a simple polynomial basis following as,

\[
A_l^{(r)} = \sum_{s=0}^{p} (A_l^{(r)})^s (z, x) \tau_s ,
\]  

(6)

\[
B_l^{(r)} = \sum_{s=0}^{p} (B_l^{(r)})^s (z, x) \tau_s ,
\]  

(7)

\[
C_l^{(r)} = \sum_{s=0}^{p} (C_l^{(r)})^s (z, x) \tau_s .
\]  

(8)

where the series is expanded up to any desired degree of accuracy. Here \( \tilde{\mu}^\mu = p^\mu / T \) is the scaled particle four-momenta and \( \tau_s = p^\mu u_\mu / T \) is scaled particle energy at local rest frame. Following the previous mentioned properties of \( \mathcal{L}[\phi] \), it can be observed that \( (A_l^{(r)})^0, (A_l^{(r)})^1 \) and \( (B_l^{(r)})^0 \) can not be determined by the transport equation and hence called the homogeneous solutions. Beyond that, \( A_l^{(r)}, B_l^{(r)} \) and \( C_l^{(r)} \) can be estimated from the microscopic transport equation and can be called interaction solutions. Keeping up first nonvanishing contribution in thermodynamic fluxes, the interaction solutions can be extracted as the following,

\[
\langle A_l^{(r)} \rangle^2 = -\left\{ \int d\Gamma_p \tau_{s}^2 Q_i^{(r)} \right\} / \langle \tau_{s}^2 \rangle ,
\]  

(9)

\[
\langle B_l^{(r)} \rangle^1 = -\left\{ \int d\Gamma_p \tau_{p} \tilde{\mu}^\mu P^{\mu}_{l}^{(r)} \right\} / \langle \tilde{\mu}^\mu P^{\mu}_{l}^{(r)} \rangle ,
\]  

(10)

\[
\langle C_l^{(r)} \rangle^0 = -\left\{ \int d\Gamma_p \tilde{\mu}^\mu S_l^{(r)} \right\} / \langle \tilde{\mu}^\mu S_l^{(r)} \rangle .
\]  

(11)

The bracket quantity is defined as, \( \langle \phi, \phi \rangle = \int d\Gamma_p \phi \mathcal{L}[\phi] \) which are always non-negative.

I next proceed to estimate the out of equilibrium corrections for particle number density, energy density and thermodynamic pressure with the help of Eq. (4). These quantities are conventionally defined respectively as, \( \rho = \rho_{\mu} N_{\mu} \), \( \epsilon = \epsilon_{\mu} u_{\mu} T_{\mu\nu} \) and \( P = -\frac{1}{2} \Delta_{\mu\nu} T_{\mu\nu} \) with the help of energy-momentum tensor \( T_{\mu\nu} = \int d\Gamma_p p^\mu p^\nu f \) and particle 4-flow \( N^\mu = \int d\Gamma_p p^\mu f \). Following this prescription the respective corrections are \( \delta \rho = \sum \delta \rho^{(r)}, \delta \epsilon = \sum \delta \epsilon^{(r)}, \delta P = \sum \delta P^{(r)}, \delta N^\mu = \sum \delta N^\mu \) respectively.
\[\delta \epsilon = \sum_{l} \delta \epsilon^{(r)} \text{ and } \delta P = \sum_{l} \delta P^{(r)} \text{ where the corrections for each order are,}\]

\[
\delta \rho^{(r)} = \sum_{l} (c_{l})^{(r)} X_{l}^{(r)}, \quad \delta \epsilon^{(r)} = \sum_{l} (c_{l})^{(r)} X_{l}^{(r)},
\]

\[
\delta P^{(r)} = \sum_{l} (c_{l})^{(r)} X_{l}^{(r)}. \quad (12)
\]

The correction coefficients are given by,

\[
(c_{l})^{(r)} T = T\{(A_{l}^{(r)})^{0}a_{1} + (A_{l}^{(r)})^{1}a_{2} + (A_{l}^{(r)})^{2}a_{3}\}, \quad (13)
\]

\[
(c_{l})^{(r)} T = T\{(A_{l}^{(r)})^{0}a_{2} + (A_{l}^{(r)})^{1}a_{3} + (A_{l}^{(r)})^{2}a_{4}\}, \quad (14)
\]

\[
(c_{l})^{(r)} = \frac{T^{2}}{3}\{(A_{l}^{(r)})^{0}(a_{2} - z^{2}a_{0}) + (A_{l}^{(r)})^{1}(a_{3} - z^{2}a_{1})
+(A_{l}^{(r)})^{2}(a_{4} - z^{2}a_{2})\}. \quad (15)
\]

The corresponding vector corrections for mean-particle velocity \(\rho_{0}\delta u_{\nu}^{(r)} = V^{\mu} = \sum_{r} V^{(r)\mu}\) and energy flow or momentum density \((c_{0} + P_{0})\delta u_{E}^{(r)} = W^{\alpha} = \sum_{r} W^{(r)\alpha}\) belonging to \(r^{th}\) order are respectively given by,

\[
W^{(r)\alpha} = \Delta_{\mu}^{\nu} u_{\nu}^{(r)} A^{(r)\mu \nu} = \sum_{l} (c_{l})^{(r)} Y_{l}^{(r)\alpha}, \quad (16)
\]

\[
V^{(r)\alpha} = \Delta_{\alpha}^{\beta} \delta N^{(r)\beta} = \sum_{l} (c_{l})^{(r)} Y_{l}^{(r)\alpha}, \quad (17)
\]

with,

\[
(c_{l})^{(r)} T = T^{2}\{(B_{l}^{(r)})^{0}(b_{1}) + (B_{l}^{(r)})^{1}(b_{2}\} , \quad (18)
\]

\[
(c_{l})^{(r)} = T\{(B_{l}^{(r)})^{0}(b_{0}) + (B_{l}^{(r)})^{1}(b_{1}\} . \quad (19)
\]

where \(\delta N^{(r)\mu}\) and \(\delta T^{(r)\mu \nu}\) are the \(r^{th}\) order out of equilibrium correction to particle four-flow and energy-momentum tensor respectively. The moment integrals are given by,

\[
a_{n} = \int dF_{p} \tilde{a}_{p}^{n}, \quad (20)
\]

\[
\Delta_{\mu \nu} b_{n} = \int dF_{p} \tilde{b}_{(p)} \tilde{a}_{(p)}^{(\nu)} \tilde{a}_{p}^{(\mu)}, \quad (21)
\]

\[
\Delta_{\alpha \beta \mu \nu} c_{n} = \int dF_{p} \tilde{b}_{(p)} \tilde{a}_{(p)} \tilde{a}_{(p)}^{(\alpha \beta \mu \nu)}, \quad (22)
\]

with \(dF_{p} = dT_{p} f^{(0)}(1 \pm f^{(0)})\).

Including these corrections the most general expressions for particle four-flow and energy-momentum tensor are given by,

\[
N^{\mu} = (\rho_{0} + \delta \rho) u^{\mu} + V^{\nu}, \quad (23)
\]

\[
T^{\mu \nu} = (c_{0} + \delta \epsilon) u^{\mu} u^{\nu} - (P_{0} + \delta P) \Delta^{\mu \nu}
+(W^{\mu} u^{\nu} + W^{\nu} u^{\mu}) + \pi^{\mu \nu}, \quad (24)
\]

with 0 subscript defining the respective equilibrium scalar quantities, \(u^{\mu}\) being the equilibrium velocity and \(\pi^{\mu \nu} = \Delta^{\mu \nu \alpha \beta} \delta T_{\alpha \beta} = \sum_{l} (c_{l})^{(r)} Y_{l}^{(r)\mu \nu}\) as the shear stress tensor with \(\sum_{l} (c_{l})^{(r)} Y_{l}^{(r)\mu \nu} = T^{(C^{(r)})^{\mu \nu}} c_{0}\), Eq. (23) and (24) along with \([12, 19]\) set the out of equilibrium thermodynamic field definition of a system that includes dissipation.

Several issues need to be addressed here. First, in Eq. (13), only the interaction solutions \((A^{(r)})^{2}, (B^{(r)})^{1}\) (as well as \((C^{(r)}))^{0}\) can be obtained from the transport equation (11), the homogeneous solutions are fully arbitrary and can not be extracted from a microscopic theory. Secondly, in (23) and (24), the number of transport coefficients have been significantly increased as in the usual cases only the pressure correction and any one of the vector fluxes do exist [11]. To show that these two issues are connected the two following identities are obtained,

\[
(c_{0})^{(r)} I - \left(\frac{\partial P_{0}}{\partial \rho_{0}}\right)_{c_{0}} (c_{l})^{(r)} I - \left(\frac{\partial P_{0}}{\partial \rho_{0}}\right)_{c_{0}} (c_{l})^{(r)} I = -\left(-c_{r}\right)^{(r)} I, \quad (25)
\]

\[
(c_{0})^{(r)} I - \hat{h} T (c_{l})^{(r)} I = -\frac{T}{\hat{h}} (c_{l})^{(r)} I, \quad (26)
\]

with \(\hat{h} = (c_{0} + P_{0})/\rho_{0} T, \hat{\mu}\) belonging to the equilibrium value of \(\epsilon\) and \(\rho\). Here,

\[
(c_{0})^{(r)} I = T^{2} \int dF_{p} \hat{Q}_{A_{l}}^{(r)} , \quad (27)
\]

\[
(c_{l})^{(r)} I = -\frac{1}{3} c_{0} + P_{0}/n_{0} \int dF_{p} \hat{p}^{(r)} (\hat{\tau}_{p} - \hat{h}) B_{l}^{(r)} , \quad (28)
\]

are the \(r^{th}\) order coefficients of bulk viscous flow II and diffusion flow \(Q_{\mu}^{(r)}\) respectively for \(l^{th}\) kind of term with \(Q = \hat{Q} + \hat{q}^{2}\). This leads to the fact that the out of equilibrium corrections to the thermodynamic quantities add up to produce dissipative fluxes,

\[
\delta P - \frac{\partial P_{0}}{\partial \rho_{0}} (c_{0}) \delta \epsilon - \left(\frac{\partial P_{0}}{\partial \rho_{0}}\right)_{c_{0}} \delta \rho = \Pi, \quad W^{\mu} - \hat{h} T V^{\mu} = q^{\mu}, \quad (29)
\]

with \(\Pi = -\sum (c_{l})^{(r)} Y_{l}^{(r)\mu\nu}\) and \(q^{\mu} = -\sum_{l} \sum_{r} \sum_{l} (c_{l})^{(r)} Y_{l}^{(r)\mu\nu}\). Since, for \(r = 1\) it will be seen later that, \(Q f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T} L[\hat{A}_{I}^{(r)}] + \frac{1}{T} L[\hat{A}_{I}^{(r)}] \hat{h} f^{(0)}(1 \pm f^{(0)}) = \frac{1}{T} L[\hat{B}_{I}^{(r)}]^{2}\), applying collision operator properties it is found that \((c_{l})^{(r)} I\) and \((c_{l})^{(r)} I\) do not depend on the homogeneous solutions \((A^{(r)})^{0}, (A^{(r)})^{1}, (B^{(r)})^{0}\) but uniquely specified by the interaction solution \((A^{(r)})^{2}\) and \((B^{(r)})^{1}\). It can be trivially shown that, all the shear coefficients can be specified by \((C^{(r)})^{0}\). The correction in thermodynamic quantities \(\delta \epsilon, \delta \rho, \delta P, W^{\mu}\) and \(V^{\mu}\) due to the arbitrary homogeneous part of \(\phi\) is attributed solely to the hydrodynamic frame choice recently extensively studied in [14, 19]. From Eq. (25) and (26) it can be seen that the homogeneous part of the individual scalar and vector correction, i.e the frame information in thermodynamic quantities exactly cancels to retain only the interaction part in the transport coefficients of the dissipative fluxes at any order. This agrees with [17] that not all transport coefficients but their certain combinations remain invariant under field redefinition due to hydrodynamic frame choice. This part as already mentioned can not be extracted from the microscopic dynamics of the system and remains arbitrary to certain
choice. The interaction part of the dissipative correction will be next estimated using the MDRTA technique from the relativistic transport equation. Frame choice and matching conditions will be addressed in the later part of the work again.

### III. FIRST ORDER FIELD CORRECTIONS WITH MDRTA

The relaxation time approximation is a simple method to linearize the collision term with the help of relaxation time $\tau_R$ of single particle distribution function $f(x, p)$ as follows,

$$\hat{p}^{\nu} \partial_\mu f = -\frac{\tau_p f^{(0)}}{\tau_R} (1 \pm f^{(0)}) \phi, \quad \tau_R(x, p) = \tau^0_R(x) \rho_n, \tag{30}$$

where the momentum dependence of $\tau_R$ is expressed as a power law of the scaled particle energy in a comoving frame with $\tau^0_R$ as the momentum independent part and $n$ as a number specifying the power of the scaled energy. In order to solve Eq. (30), here the well known iterative technique of gradient expansion, the Chapman-Enskog (CE) method has been adopted [10]. Following that, the first order correction to the particle distribution function is obtained as follows,

$$\frac{\phi^{(1)}}{\tau^0_R} = \rho_n \left[ \hat{Q} \cdot \dot{u} + \left\{ \frac{\tau_p}{\tau_R} - 1 \right\} \hat{p}^{(\mu)} \nabla_\mu \hat{\mu} + \hat{\rho}^{(\mu)} \hat{\rho}^{(\nu)} \sigma_\mu \nu \right]. \tag{31}$$

In deriving Eq. (31), the equilibrium thermodynamic identities have been used such as $\dot{P}_0 \partial_0 \cdot u = 0, D_0 \partial_0 \cdot u = 0$ and $(\epsilon_0 + P_0) D_{0n} = \nabla \mu P_0$ without the inclusion of any dissipative effects.

Before proceeding further, the conservation of particle four-flow and energy-momentum tensor for $r = 1$ needs to be checked. It can be proved,

$$\partial_\mu N^\mu = \int dT_p \hat{p}^{(\mu)} \partial_\mu f = -\frac{T}{\tau^2_R} \int dF_p \tau^1_p \phi^{(1)} = 0, \tag{32}$$

$$\partial_\mu T^{\mu \nu} = \int dT_p \hat{p}^{(\mu)} p^{\mu} \partial_\mu f = -\frac{T^2}{\tau^2_R} \int dF_p \left\{ \dot{u}^{(\mu)} x^{(\nu - n)} + \hat{p}^{(\mu)} x^{(\nu - n)} \right\} \phi^{(1)} = 0, \tag{33}$$

for all values of $n$. After achieving the conservation properties, the corresponding first order correction in thermodynamic quantities are given by,

$$\delta \epsilon^{(1)} = c_1^\lambda (\partial \cdot u), \quad \delta \rho^{(1)} = c_1^1 (\partial \cdot u), \quad \delta \rho^{(1)} = c_0^1 (\partial \cdot u), \quad W^{(1)\alpha} = -c_2 \hat{h} (\nabla^{\alpha} T / T - D u^{\alpha}), \quad V^{(1)\alpha} = c_2 (\nabla^{\alpha} \hat{\mu}), \tag{34}$$

with associated correction coefficients,

$$\frac{c_1^\lambda}{T^2 \tau^0_R} = \frac{z^2}{3} a_{n+1} + \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n - \frac{1}{3} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+3}, \tag{35}$$

$$\frac{c_1^1}{T^2 \tau^0_R} = \frac{z^2}{3} a_n + \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n - \frac{1}{3} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+2}, \tag{36}$$

$$\frac{c_0^1}{T^2 \tau^0_R} = \frac{z^2}{3} a_{n+1} + \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n - \frac{1}{3} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+3} + \frac{1}{T} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+2}, \tag{37}$$

$$\frac{c_2}{T^2 \tau^0_R} = \frac{z^2}{9} a_{n+1} + \frac{1}{3} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+3} + \frac{1}{3T} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+2} \left( \frac{z^4}{9} \right) a_{n+1} - \frac{z^2}{9} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+1}, \tag{38}$$

$$\frac{c_3}{T^2 \tau^0_R} = \frac{1}{h} b_{n+1} - b_n. \tag{39}$$

It is to be noted here that for momentum independent case $n = 0, c_1^\lambda = c_1^1 = c_2^1 = 0$. $c_2^1$ vanishes for $n = 1$ with $c_3^1 = 3c_1^1$. The $\tau^0_R$ in the denominator of the correction coefficients in Eq. (38) can be replaced by expressing it in terms of the independent transport coefficients associated with the dissipative fluxes of corresponding tensorial rank. Putting $\phi^{(1)}$ in the expression of first order dissipative fluxes namely bulk viscous flow, diffusion flow and shear viscous flow respectively,

$$\Pi^{(1)} = -T^2 \int dF_p \hat{Q} \phi^{(1)} = -T^2 \int dF_p \hat{Q} \phi^{(1)} = -\frac{\Lambda T}{h} \nabla^{\alpha} \hat{\mu}, \tag{40}$$

$$q^{(1)\alpha} = T^2 \int dF_p \hat{p}^{(\mu)} (\tau_p - \hat{h}) \phi^{(1)} = -\frac{\Lambda T}{h} \nabla^{\alpha} \hat{\mu}, \tag{41}$$

$$\pi^{(1)\mu \nu} = T^2 \int dF_p \hat{p}^{(\alpha)} \hat{p}^{(\nu)} \phi^{(1)} = 2\eta \sigma^{\mu \nu}, \tag{42}$$

the corresponding first order transport coefficients bulk viscosity ($\zeta$), thermal conductivity ($\lambda$) and shear viscosity ($\eta$) in MDRTA are given by,

$$\frac{\zeta}{T^2 \tau^0_R} = \frac{z^4}{9} a_{n+1} + \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+3} + \frac{2z^2}{3T} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+2} + \frac{1}{T^2} \left\{ \frac{\partial P_0}{\partial \epsilon_0} \right\}_n a_{n+1}, \tag{43}$$

$$\frac{\eta}{T^2 \tau^0_R} = -\left\{ b_{n+1} - 2h b_n + \hat{h}^2 b_{n-1} \right\}, \tag{44}$$

$$\frac{\lambda}{T^2 \tau^0_R} = \frac{1}{2} c_{n-1}. \tag{45}$$

It has been checked that $\zeta / \tau^0_R, \lambda / \tau^0_R, \eta / \tau^0_R > 0$ for all values of $n$ for various combinations of $z, T$ and $\hat{\mu}$. They
have been plotted in Fig. 1 as a function of $z$ for several $n$ values.

![Graph showing scaled bulk viscosity and thermal conductivity as a function of $z$ with different $n$.](image)

**FIG. 1:** Scaled bulk viscosity and thermal conductivity as a function of $z$ with different $n$.

Shear viscosity has not been plotted since tensor contribution is not entering in the dissipative correction of any thermodynamic quantity. As predicted earlier the following two relations hold for any $n$ value,

\[
\begin{align*}
\zeta_n &= \zeta_1 - \zeta_1 \left( \frac{\partial P_0}{\partial \epsilon_0} \right)_{\rho_0} - \zeta_1 \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\epsilon_0} = -\zeta, \\
\zeta_2 &= \left( \frac{\epsilon_0 + P_0}{\rho_0} \right)_{\epsilon_0} = -\frac{\lambda T}{\hbar},
\end{align*}
\]

such that,

\[
\begin{align*}
\delta \rho_n &= \left( \frac{\partial P_0}{\partial \epsilon_0} \right)_{\rho_0} \delta \epsilon_n - \left( \frac{\partial P_0}{\partial \rho_0} \right)_{\epsilon_0} \delta \rho_n = \Pi_n, \\
W_n &= -\frac{\lambda T}{\hbar} \nabla T \cdot \nabla \rho_n = q_n.
\end{align*}
\]

Putting (44) and (46) into (25) and (26), the particle four-flow and energy-momentum tensor is respectively obtained including the out of equilibrium dissipative effects in all the thermodynamic quantities up to first order of gradient expansion. Eqs. (46) and (47) exhibit that the number of independent transport coefficients are still the same as for the usual case ($\zeta$, $\lambda$, and $\eta$). So it is observed that MDRTA introduces non-equilibrium dissipative contributions in all the thermodynamic quantities essential to define $N^\mu$ and $T^{\mu\nu}$ through the exponent $n$ without changing the number of independent transport coefficients. For $n = 0$ situation, i.e., without taking any momentum dependence in collision integral, one returns to the usual scenario where energy correction, particle number correction and energy flux vanishes leaving the entire scalar dissipation to pressure correction and vector dissipation to particle flux. So it can be said that in a general situation the dissipative corrections in thermodynamic field variables is determined by how the medium interaction distributes the respective dissipative fluxes among the scalar and vector fields. The coefficients of first order dissipative correction in (45) scaled by independent transport coefficients have been plotted for $m = 0.3$ GeV and $T = 0.3$ GeV as a function of $n$ in Fig. (2). Fig. (2) shows that the individual field corrections take how much fractional part of the dissipative flux, is decided by the value of $n$.

![Graph showing correction coefficients as a function of $n$.](image)

**FIG. 2:** Correction coefficients as a function of $n$.

The entropy production $\partial_\mu S^\mu = -\int d\Gamma_p (ln f)p^\mu \partial_\mu f$ under first order MDRTA turns out to be,

\[
T \partial_\mu S^\mu = \zeta (\partial \cdot u)^2 + 2\eta \sigma^{\mu\nu} \sigma_{\mu\nu} + \frac{\lambda T}{\hbar^2} (\nabla \mu)^2,
\]

which is non-negative as long as $\zeta$, $\lambda$, $\eta \geq 0$. So the individual values of dissipative correction coefficients do not affect the positive entropy production rate. In this context it needs to be mentioned that for some $n$ values the coefficients in Eq. (45) can take negative values as can be seen from Fig. (2), but as argued in [1], they are acceptable as long as $\zeta$, $\lambda$, $\eta$ and hence the entropy production is non-negative.

**IV. SECOND ORDER FIELD CORRECTIONS WITH MDRTA**

Before estimating the second order out of equilibrium correction to particle distribution function, one needs to estimate the equation of particle number density, energy density and equation of motion including up to second order of gradient corrections. Applying $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$, the following thermodynamic identities are
obtained,

\[
D(\rho_0 + \delta \rho^{(1)}) + (\rho_0 + \delta \rho^{(1)})(\partial \cdot u) - V^{(1)\mu} D u_\mu + \nabla_\mu V^{(1)\mu} = 0,
\]

(51)

\[
D(\epsilon_0 + \delta \epsilon^{(1)}) + (\epsilon_0 + P_0 + \delta P^{(1)})(\partial \cdot u) - \pi^{(1)\mu\nu} \sigma_{\mu\nu} - 2 W^{(1)\mu} D u_\mu + \nabla_\mu W^{(1)\mu} = 0,
\]

(52)

\[
(\epsilon_0 + P_0 + \delta \epsilon^{(1)} + \delta P^{(1)}) D u^\alpha - \nabla^\alpha (P_0 + \delta P^{(1)})
+ \Delta^\nu \nabla_\nu \pi^{(1)\mu\nu} - \pi^{(1)\alpha\mu} D u_\mu
+ W^{(1)\alpha} (\partial \cdot u) + \Delta^\nu D W^{(1)\nu} + W^{(1)\mu} \partial_\mu u^\alpha = 0.
\]

(53)

Using these identities and applying the second order iteration of CE method in Eq. (50), the second order correction in particle distribution function is obtained as follows,

\[
\phi^{(2)}(\tau_R) =
- \frac{c_s^2}{\epsilon_0 + P_0} \tau_p^{n+1} \left\{ \pi^{(1)\mu\nu} \sigma_{\mu\nu} - \Pi^{(1)} (\partial \cdot u) \left( 1 - c_A^1 (c_s^2 + 1) + c_A^2 T \partial \tau c_A^1 \right) + c_A^1 D \Pi^{(1)} \right\}
- \frac{1}{\epsilon_0 + P_0} \tau_p^n \nabla_\mu \pi^{(1)\mu\nu} - \left( 1 - c_A^2 c_A^1 \right) \nabla_\mu \Pi^{(1)} \right\}
- \frac{\tau_R^0}{2 \eta} \tau_p^{n+1} \left\{ \pi^{(1)\mu\nu} \sigma_{\mu\nu} - \Pi^{(1)} (\partial \cdot u) \left[ \frac{c_s^2}{3} \frac{1}{3} \tau_p^{2n} + \frac{n-1}{3} c_s^2 (n+1) + c_s^2 T \partial \tau (\eta_R \eta) \right] \tau_p^{2n-1} + \frac{1}{3} \tau_p^{2n-2} - \frac{1}{3} \tau_p^{2n-3} \right\} \Pi^{(1)}
+ \frac{\tau_R^0}{2 \eta} \frac{\pi_R^0}{\zeta} \pi^{(1)\mu\nu} (\partial \cdot u) \left[ \left( c_s^2 - \frac{1}{3} \right) \tau_p^{2n} + \frac{n-1}{3} c_s^2 (n+1) + c_s^2 T \partial \tau (\eta_R \eta) \right] \tau_p^{2n-1} + \frac{1}{3} \tau_p^{2n-2} - \frac{1}{3} \tau_p^{2n-3} \right\} \Pi^{(1)}
+ \frac{\tau_R^0}{2 \eta} \frac{\pi_R^0}{\zeta} \pi^{(1)\mu\nu} (\partial \cdot u) \left[ \frac{2}{3} \tau_p^{2n+2} + \frac{2}{3} (n+1) \tau_p^{2n+2} + \frac{2}{3} (n-1) \tau_p^{2n+2} \right] \Pi^{(1)}
+ \tau_p^{2n+1} \left\{ - c_s^2 \left( c_s^2 - \frac{1}{3} \right) \frac{T \partial \tau (\eta_R \eta)}{\tau_R^0} + (n+1) (c_s^2 - \frac{1}{3}) \tau_p^{2n-1} - \frac{2}{3} \tau_p^{2n-2} + \frac{2}{3} (n+1) \tau_p^{2n-2} \right\}
+ \tau_p^{2n+1} \left\{ - c_s^2 \left( c_s^2 - \frac{1}{3} \right) \frac{T \partial \tau (\eta_R \eta)}{\tau_R^0} + (n+1) (c_s^2 - \frac{1}{3}) \tau_p^{2n-1} - \frac{2}{3} \tau_p^{2n-2} + \frac{2}{3} (n+1) \tau_p^{2n-2} \right\}.
\]

(54)

The diffusive fluxes have been ignored in the above expression due to calculational complexity. \( c_A^2 = (\partial P_0)/(\partial \sigma)_{\dot{u}} \) is defined as the squared velocity of sound.

Putting \( \phi^{(2)} \) in Eq. (53) one again obtains it to be zero, giving rise to \( \partial_\alpha T^{\mu\nu} = 0 \). This demonstrates that dissipative corrections under MDRTA conserves the energy-momentum perfectly if the thermodynamic variables are redefined properly in the previous order and that modification is incorporated in the thermodynamic identities accordingly.

Utilising Eq. (54), it is now customary to obtain the second order hydrodynamic equations for bulk and shear viscous flow by putting \( \phi^{(2)} \) in \( \Pi^{(2)} \) and \( \pi^{(2)\mu\nu} \) respectively. The bulk viscous and shear viscous pressure equations for a second order hydrodynamic theory with MDRTA is respectively given below along with their transport coefficients.

\[
\Pi = -\zeta (\partial \cdot u - \tau_\Pi D \Pi + c_A^\Pi \pi^{(1)\mu\nu} \sigma_{\mu\nu} + c_A^\Pi \Pi (\partial \cdot u))
\]

(55)

\[
\pi^{\mu\nu} = 2 \eta \sigma^{\mu\nu} - \tau_\pi D \pi^{(\mu\nu)} + c_\pi^{\omega \rho} (\omega^{\rho}\rho) + c_\pi^{\sigma \rho} (\pi^{\rho\nu})
+ c_\pi^{\rho \pi^{(\mu\nu)}} (\partial \cdot u) + c_A^\pi \Pi \sigma^{\mu\nu},
\]

(56)
The $\tau_R^n$ in the denominator can be related to $\zeta$ and $\eta$ respectively from (63) and (65). Form Eq. (67) and (58) it can be observed that $\tau_\pi = \tau_\Pi = \tau_R^n$ holds only for $n = 0$. For all other $n$, the three times are evidently separate. In Fig. (3) it has been shown that with increasing $n$ both $\tau_\Pi$ and $\tau_\pi$ become larger with respect to $\tau_R^n$ which is expected for the macroscopic time scale. This separation of time scales with MDRTA itself provides a strong motivation for the study. Analogous to the first order, the second order correction in thermodynamic quantities can be estimated as well. The second order energy and pressure corrections are respectively given by,

$$\delta e_2 = c_\Delta^2 D\Pi(1) + \Pi(1)\eta(1)^{\mu\nu}\sigma_{\mu\nu} + \lambda_2\Pi(1)(\partial \cdot u) ,$$

$$\delta P_2 = c_\Delta^2 D\Pi(1) + \Pi(1)\eta(1)^{\mu\nu}\sigma_{\mu\nu} + \lambda_2\Pi(1)(\partial \cdot u) .$$

For any $n$ value it can be shown that

$$c_\Delta^2 - c_\pi^2 = -\Pi_1 , \quad c_\pi^2 - c_\Delta^2 = c_\Pi_1 , \quad \lambda_2 - c_\pi^2 \lambda_2 = c_\Pi_1^2 ,$$

keeping the number of independent scalar transport coefficients same for all $n$ values. This gives $\delta P_2 - c_\Delta^2 \delta e_2 = \Pi_2$. For $n = 0$, $c_\Delta^2, t_\Delta^2, \lambda_2^2$ vanish as before and $\delta P^{(2)}$ becomes just $\Pi_2$. The corresponding vector correction $W_\mu$ is,

$$W_\mu^{(2)} = r_2^2 \Delta_n^{\nu\alpha} \nabla_\nu n^{(1)}_{\mu\nu} + r_2^2 \nabla^{\alpha\nu} \Pi^{(1)} ,$$

with,

$$r_\Pi_2 = -\left[T^2 \frac{b_{n+1}}{c_\Pi + c_{n-1}}\right] ,$$

$$r_\Delta_2 = T^2 (1 - c_\Pi c_\Delta^2) \frac{b_{n+1}}{c_\Pi + c_{n-1}} ,$$

$$r_\pi_2 = \frac{2}{3} \frac{z^2}{c_\pi^2 - \frac{1}{3}} a_{n+1} + (c_s - \frac{1}{3})^2 a_{n+3} ,$$

both of which are zero for $n = 0$. Note that $\omega^{\mu\nu} \delta u_{\mu} = 0$, which is essential for maintaining velocity normalization. Putting (64), (65) and (67) in Eq. (24), the second order $T^{\mu\nu}$ is obtained including dissipative corrections using MDRTA.
FIG. 3: $\tau_\pi/\tau_R^0$ and $\tau_H/\tau_R^0$ as a function of $z$ with different $n$. 

V. SUMMARY AND DISCUSSIONS

In this work momentum dependent relaxation time approximation has been used to redefine the thermodynamic fields in order to include the out of equilibrium dissipative effects up to second order in gradient correction. The key finding is that these corrections are not independent but constrained to give the dissipative flux of same tensorial rank where the associated coefficients are sensitive to the interaction. The derived equations can be applied for hydrodynamic simulations since they can be uniquely solved and their phenomenological consequences can be significant as observed in [1].

Here comes the question regarding frame choice and matching conditions. Frame choice is a vector condition that defines the out of equilibrium velocity flow, i.e., putting constraints on $W^\mu$ or $V^\mu$. The flow can be defined either by setting $W^\mu = 0$ such that $T^{\mu\nu}u_\nu = (\epsilon + \delta\epsilon)u^\mu$ (Landau frame) or by $V^\mu = 0$ such that $N^\mu = (\rho + \delta\rho)u^\mu$ (Eckart frame). Conventionally, in both the frames $\delta\epsilon$ and $\delta\rho$ are both set to zero in order to define the out of equilibrium temperature and chemical potential. But in a number of recent studies [1, 20, 21] it has been shown that in presence of dissipation $\epsilon$ and $\rho$ can have extended matching conditions including dissipative effects both for Landau and Eckart frame retaining positive entropy production rate and causality and stability of the theory. The frame condition is not hampered by their presence since the flow direction $u^\mu$ is not directly influenced by them. However, the constitutive equations for the dissipative currents [32, 33] are frame independent as always [22] since the associated coefficients do not include the homogeneous part of the solution.

It is interesting to note here that though the conventional frame definition [14, 17] does not include microscopic dynamics and entirely decided by macroscopic constraints (otherwise Eq. (26) and (20) will not be invariant under frame redefinition since $(c_\gamma)^n_1$ and $(c_\lambda)^n_1$ are sensitive to interaction), [2, 3] present a different perspective regarding hydrodynamic frame choice. It suggests that instead of being arbitrary, the macroscopic frame vector can be related to the underlying microscopic theory via particle momenta and applies renormalization group method to establish such a relation. Interestingly, this microscopic frame definition exactly agrees with the results obtained here. $n = 0$ gives $u_\mu u_\nu \delta T^{\mu\nu} = 0, u_\mu \delta N^\mu = 0$ and $W^\mu = 0$ which indicates the Landau frame. $n = 1$ gives $u_\mu \delta N^\mu = 0, V^\mu = 0$ and $\delta T^\mu_\nu = 0$ which are the constraints for Eckart frame proposed by Stewart [23] and obtained by their analysis. In [24] a stable first order theory has been established in Eckart frame with this constraints where not only $\delta P$, but also $\delta\epsilon$ includes contribution from the bulk flow. However, defining hydrodynamic frames in terms of underlying microscopic kinetic theories is a debatable issue since in [25] it has been argued that the Landau-Lifshitz frame is the unique relativistic hydrodynamic frame, since from a macroscopic perspective the frame vector should be independent of particle momenta.

Here, few aspects need to be clarified. First, the field redefinition derived here is purely dissipative correction taken care by the medium interaction and should not be confused with that due to thermodynamic frame choice mentioned in [13, 18]. The corrections are expressed in terms of the dissipative forces which is why only the spatial gradients over fields are appearing in the corrections and not the time derivatives as for the other case. Second, the conservation of energy momentum and particle number shown here is purely macroscopic. Eq. (32) and (33) do not hold for any arbitrary $\phi$. If only the field corrections are properly implemented in thermodynamic identities, the $\phi$ obtained from transport equation for next order satisfies those equations such that it can be said that the dissipative field correction and conservation laws are compatible with each other. The only case where conservation holds at microscopic level (form of $\phi$ does not matter) known to author is the new collision term proposed in [3] where the mentioned integrals are identically zero without the need of extracting $\phi$ from order by order gradient expansion.

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