Quantum Statistics and Spacetime Surgery

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We apply the geometric-topology surgery theory on spacetime manifolds to study the constraints of quantum statistics data in 2+1 and 3+1 spacetime dimensions. First, we introduce the fusion data for worldline and worldsheet operators capable creating anyon excitations of particles and strings, well-defined in gapped states of matter with intrinsic topological orders. Second, we introduce the braiding statistics data of particles and strings, such as the geometric Berry matrices for particle-string Aharonov-Bohm and multi-loop adiabatic braiding process, encoded by submanifold linkings, in the closed spacetime 3-manifolds and 4-manifolds. Third, we derive “quantum surgery” constraints analogous to Verlinde formula associating fusion and braiding statistics data via spacetime surgery, essential for defining the theory of topological orders, and potentially correlated to bootstrap boundary physics such as gapless modes, conformal field theories or quantum anomalies.

Decades ago, the fractional quantum Hall effect was discovered [1]. The intrinsic relation between the topological quantum field theories (TQFT) and the topology of manifolds was found [2, 3] years after. The two breakthroughs partially motivated the study of topological order [4] as a new state of matter in quantum many-body systems and in condensed matter systems [5]. Topological orders are defined as the gapped states of matter with physical properties depending on global topology (such as the ground state degeneracy (GSD)), robust against any local perturbation and any symmetry-breaking perturbation. Accordingly, topological orders cannot be characterized by the old paradigm of symmetry-breaking phases of matter via the Ginzburg-Landau theory [6, 7]. The systematic studies of 2+1 dimensional (2+1D) topological orders enhance our understanding of the real-world plethora phases including quantum Hall states and spin liquids [8]. In this work, we explore the constraints between the 2+1D and 3+1D topological orders and the geometric-topology properties of 3- and 4-manifolds. We only focus on 2+1D / 3+1D topological orders with GSD insensitive to the system size and with a finite number of types of topological excitations creatable from 1D line and 2D surface operators.

We apply the tools of quantum mechanics in physics and surgery theory in mathematics [9, 10]. Our main results are: (1) We provide the fusion data for worldline and worldsheet operators creating excitations of particles (i.e. anyons [11]) and strings (i.e. anyonic strings) in topological orders. (2) We provide the braiding statistics data of particles and strings encoded by submanifold linkings in the 3- and 4-dimensional closed spacetime manifolds. (3) By “cutting and gluing” quantum amplitudes, we derive constraints between the fusion and braiding statistics data analogous to Verlinde formula [12, 13] for 2+1 and 3+1D topological orders.

Quantum Statistics: Fusion and Braiding Data

- Imagine a renormalization-group-fixed-point topologically ordered quantum system on a spacetime manifold \(\mathcal{M}\). The manifold can be viewed as a long-wavelength continuous limit of certain lattice regularization of the system. We aim to compute the quantum amplitude from “gluing” one kete-state \(|R\rangle\) with another bra-state \( \langle L|\), such as \(\langle L|R\rangle\). A quantum amplitude also defines a path integral or a partition function \(Z\) with the linking of worldlines/worldsheets on a d-manifold \(\mathcal{M}^d\), read as

\[
\langle L|R\rangle = Z(\mathcal{M}^d; \text{Link}[\text{worldline, worldsheet, . . .}]).
\]

For example, the \(|R\rangle\) state can represent a ground state of 2-torus \(T^2\), if we put the system on a solid torus \(D^2_t \times S^1_y\) [14] (see Fig.1(a) as the product space of 2-dimensional disk \(D^2\) and 1-dimensional circle \(S^1\)). Note that its boundary is \(\partial(D^2 \times S^1) = T^2\), and we can view the time \(t\) along the radial direction. We label the trivial vacuum sector without any operator insertions as \(|0_{D^2_t \times S^1_y}\rangle\), which is trivial respect to the measurement of any contractible line operator along \(S^1_x\). A worldline operator creates a pair of anyon and anti-anyon at its end points, if it forms a closed loop then it can be viewed as creating then annihilating a pair of

![FIG. 1.](image-url)
anyons in a closed trajectory [15]. Inserting a line operator $W^S_\sigma$ in the interior of $D^2 \times S^1_y$ gives a new state $W^S_\sigma|0_{D^2_z \times S^1_y}\rangle \equiv |\sigma_{D^2_z \times S^1_y}\rangle$. Here $\sigma$ denotes the anyon type [16] along the oriented line, see Fig.1. Insert all possible line operators of all $\sigma$ can completely span the ground state sectors for 2+1D topological order. The gluing of $|0_{D^2_z \times S^1_y}\rangle|0_{D^2_z \times S^1_y}\rangle$ computes the path integral $Z(S^2 \times S^1)$. If we view the $S^1$ as a compact time, this counts the ground state degeneracy (GSD) on a 2D spatial sphere $S^2$ without quasiparticle insertions, thus it is a 1-dimensional Hilbert space with $Z(S^2 \times S^1) = 1$. Similar relations hold for other dimensions, e.g. 3+1D topological orders on a $S^3$ without quasi-excitation yields $\langle 0_{D^3_z \times S^1_y}|0_{D^3_z \times S^1_y}\rangle = Z(S^3 \times S^1) = 1$.

2+1D Data – In 2+1D, we consider the worldline operators creating particles. We define the fusion data via fusing worldline operators:

$$W^S_\sigma W^S_\alpha = F^\sigma_\sigma \sigma_\alpha W^S_\alpha, \quad \text{and} \quad G^\sigma_\sigma = \langle \alpha | \sigma_{D^2_z \times S^1_y}\rangle. \quad (2)$$

Here $G^\sigma_\sigma$ is read from the projection to a complete basis $\langle \alpha |$. Indeed the $W^S_\sigma$ generates all the canonical bases from $|0_{D^2_z \times S^1_y}\rangle$. Thus the canonical projection can be

$$\langle \alpha | = \langle 0_{D^2_z \times S^1_y}|W^S_\sigma\rangle = \langle 0_{D^2_z \times S^1_y}|W^S_\alpha\rangle = \langle \sigma_{D^2_z \times S^1_y}| = \sigma_{D^2_z \times S^1_y},$$

then we have $G^\sigma_\sigma = \langle 0_{D^2_z \times S^1_y}|W^S_\sigma\rangle W^S_\alpha = \langle \sigma_{D^2_z \times S^1_y}| D^2_z \times S^1; \alpha, \sigma \rangle = \delta_{\sigma \sigma}$, where a pair of particle-antiparticle $\sigma$ and $\bar{\sigma}$ can fuse to the vacuum. We derive

$$F^\sigma_\sigma = \langle 0_{D^2_z \times S^1_y}|W^S_\sigma W^S_\alpha W^S_\sigma|0_{D^2_z \times S^1_y}\rangle = Z(S^2 \times S^1; \alpha, \sigma_1, \sigma_2) \equiv N^\alpha_\sigma_{\sigma_1, \sigma_2},$$

where this path integral counts the dimension of the Hilbert space (namely the GSD or the number of channels $\sigma_1$ and $\sigma_2$ can fuse to $\alpha$) on the spatial $S^2$. This shows the fusion data $F^\sigma_\sigma$ is equivalent to the fusion rule $N^\alpha_\sigma_{\sigma_1, \sigma_2}$, symmetric under exchanged $\sigma_1$ and $\sigma_2$.

More generally we can glue the $T^2_{xy}$-boundary of $D^2_z \times S^1_y$ via its mapping class group (MCG), namely MCG($T^2$) = SL(2, Z) generated by

$$\hat{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

The $\hat{S}$ identifies $(x, y) \rightarrow (-y, x)$ while $\hat{T}$ identifies $(x, y) \rightarrow (x + y, y)$ of $T^2_{xy}$. Based on Eq.(1), we write down the quantum amplitudes of the two SL(2, Z) generators $\hat{S}$ and $\hat{T}$ projecting to degenerate ground states. We denote gluing two open-manifolds $M_1$ and $M_2$ along their boundaries $\mathcal{B}$ under the MCG-transformation $\mathcal{U}$ to a new manifold as $M_1 \cup_{\mathcal{B}} M_2$ [17]. Then it is amusing to visualize the gluing $D^2 \times S^1 \cup_{T^2_{xy}} D^2 \times S^1 = S^3$ shows that the $S_\sigma_{\sigma_2}$ represents the Hopf link of two $S^1$ worldlines $\sigma_1$ and $\sigma_2$ (e.g. Fig.1(b)) in $S^3$ with the given orientation (in the canonical basis $S_{\sigma_1, \sigma_2} = \langle \sigma_1 | S_{\sigma_2} \rangle$):

$$S_{\sigma_1, \sigma_2} = \langle \sigma_1_{D^2_z \times S^1_y}| S_{\sigma_2_{D^2_z \times S^1_y}} \rangle = Z(\mathcal{B}^{S^3}) \quad (5)$$

Use the gluing $D^2 \times S^1 \sqcup_{T^2_{xy}} D^2 \times S^1 = S^2 \times S^1$, we can derive a well known result written in the canonical bases, $T_{\sigma_1, \sigma_2} = \langle \sigma_1_{D^2_z \times S^1_y}| T_{\sigma_2_{D^2_z \times S^1_y}} \rangle = \delta_{\sigma_1 \sigma_2} e^{i\theta_{\sigma_2}}. \quad (6)$

Its spacetime configuration is that two unlinked closed worldlines $\sigma_1$ and $\sigma_2$, with the worldline $\sigma_2$ twisted by $2\pi$. The amplitude of a twisted worldline is given by the amplitude of untwisted worldline multiplied by $e^{i\theta_{\sigma_2}}$, where $\theta_{\sigma_2}/2\pi$ is the spin of the $\sigma_2$ excitation. It means that $S_{\sigma_1, \sigma_2}$ measures the mutual braiding statistics of $\sigma_1$- and $\sigma_2$- while $T_{\sigma_1, \sigma_2}$ measures the spin and self-statistics of $\sigma_2$.

We can introduce additional data, the Borromean rings (BR) linking between three $S^1$ circles in $S^3$, written as $Z(\mathcal{B}^{S^3}; \text{BR} \{\sigma_1, \sigma_2, \sigma_3\})$.

Although we do not know a bra-ket expression for this amplitude, we can reduce this configuration to an easier one $Z[T^3_{xy}; \sigma_{1x}, \sigma_{2y}, \sigma_{3z}]$, a path integral of 3-torus with three orthogonal line operators each inserting along a non-contractible $S^1$ direction. The later is a simpler expression because we can uniquely define the three line insertions exactly along the homology group generators of $T^3$, namely $H_1(T^3; Z) = Z^3$. The two path integrals are related by three consecutive modular $S$ surgeries done along the $T^2$-boundary of $D^2 \times S^1$ tubular neighborhood around three $S^1$ rings [18]. Namely, $Z[T^3_{xy}; \sigma_{1x}, \sigma_{2y}, \sigma_{3z}] = \sum_{\sigma_1, \sigma_2, \sigma_3} S_{\sigma_1, \sigma_2, \sigma_3} S_{\sigma_2, \sigma_3, \sigma_1} S_{\sigma_3, \sigma_1, \sigma_2} Z[S^3; \text{BR} \{\sigma_1, \sigma_2, \sigma_3\}]$.

3+1D Data – In 3+1D, there are intrinsic meanings of braidsings of string-like excitations. We need to consider both the worldline and the worldsheet operators which create particles and strings. In addition to the $S^3$-worldline operator $W^S_\sigma$, we introduce $S^2$- and $T^2$-worldsheet operators as $V^S_\mu$ and $V^{T^2}_\mu$, which create closed-string(s) (or loops) at their spatial cross sections. We consider the vacuum sector ground state on open 4-manifolds: $|0_{D^3_z \times S^1_y}\rangle$, $|0_{D^3_z \times S^1_y}\rangle$, $|0_{D^3_z \times S^1_y}\rangle$, and $|0_{S^3 \times D^2_z \times T^2}\rangle$, while their boundaries are $\partial(D^3 \times S^1) = \partial(D^2 \times S^2) = S^2 \times S^1$ and $\partial(D^2 \times T^2) = S^1 \times D^2 \times S^1 = T^3$. Here $M_1 \times M_2$ means the complement space of $M_2$ out of $M_1$. Similar to 2+1D, we define the fusion data $\mathcal{F}^M$ by fusing operators:

$$W^S_\sigma W^S_\alpha = (F^S_\sigma)_\sigma \sigma_\alpha W^S_\alpha, \quad (7)$$

$$V^{S^2}_\mu V^{S^2}_\nu = (F^{S^2}_\mu)_\mu \nu \nu V^{S^2}_\nu, \quad (8)$$

$$V^{T^2}_\mu V^{T^2}_\nu = (F^{T^2}_\mu)_\mu \nu \nu V^{T^2}_\nu. \quad (9)$$
Notice that we introduce additional upper indices in the fusion algebra $\mathcal{F}^M$ to specify the topology of $M$ for the fused operators [19]. We require normalizing worldline/sheet operators for a proper basis, so that the $\mathcal{F}^M$ is also properly normalized in order for $Z(Y^{d-1} \times S^1, \ldots)$ as the GSD on a spatial closed manifold $Y^{d-1}$ always be a positive integer. In principle, we can derive the fusion rule of excitations in any closed spacetime 4-manifold. For instance, the fusion rule for fusing three particles on a spatial $S^3$ is $Z(S^3 \times S^1; \bar{\sigma}, \sigma_1, \sigma_2) = \langle 0_{D^2 \times S^1} | W^S_{\bar{\sigma}} W^S_{\sigma_1} W^S_{\sigma_2} | 0_{D^2 \times S^1} \rangle = ( F^{S^1})^{\bar{\sigma}, \sigma_1, \sigma_2}$. Many more examples of fusion rules can be derived from computing $Z(M^4; \sigma, \mu, \ldots)$ [20] by using $\mathcal{F}^M$ and Eq.(1), here the worldline and worldsheet are submanifolds parallel not linked with each other.

If the worldline and worldsheet are linked as Eq.(1), then the path integral encodes the braiding data. Below we discuss the important braiding processes in 3+1D. First, the Aharonov-Bohm particle-loop braiding can be represented as a $S^1$-worldline of particle and a $S^2$-worldsheet of loop linked in $S^4$ spacetime,

$$L^{(S^2, S^1)}_{\mu \sigma} \equiv \langle 0_{D^2 \times S^2} | V^{S^2}_\mu W^{S^1}_\sigma | 0_{D^2 \times S^1} \rangle = Z \left( \frac{S^2}{S^1} \right),$$

where we design the worldsheets $V^{T^2}_{\mu \nu}$ along the generator of homology group $H_2(D^2_{\mu} \times T^2_{\nu}, \mathbb{Z}) = \mathbb{Z}$ while $V^{T^2}_{\mu \nu}$ along the two generators of $H_2(S^1 \times D^2_{\mu} \times T^2_{\nu}, \mathbb{Z}) = \mathbb{Z}^2$ respectively. We require normalizing worldsheet $\langle S^1 \rangle_{\sigma_1, \sigma_2}$.

Third, we can represent a three-loop braiding process [25] as three $T^2$-worldsheets $\text{triple-linking}$ [28] in the spacetime $S^4$ (as the first figure in Eq.(12)). We find that

$$L_{\mu_1, \mu_2, \mu_3}^{\text{Tri}} \equiv \langle 0_{S^4 \times D^2_{\mu_1} \times T^2_{\nu_1}} | V^{T^2}_{\mu_1} V^{T^2}_{\mu_2} V^{T^2}_{\mu_3} | 0_{D^2_{\mu_1} \times T^2_{\nu_1}} \rangle = Z \left( \frac{S^4}{3} \right),$$

where $\langle S^4 \rangle_{\sigma_1, \sigma_2}$

If we design the worldline and worldsheet along the generators of $H_1(S^4 \setminus D^2 \times T^2, \mathbb{Z}) = H_2(D^2 \times T^2, \mathbb{Z}) = \mathbb{Z}$ respectively. Compare Eqs.(10) and (11), the loop excitation of $S^2$-worldsheet is shrinkable [21], while the loop of $T^2$-worldsheet needs not to be shrinkable.

Third, we can represent a three-loop braiding process [22–27] as three $T^2$-worldsheets $\text{triple-linking}$ [28] in the spacetime $S^4$ (as the first figure in Eq.(12)). We find that

$$L_{\mu_1, \mu_2, \mu_3}^{\text{Tri}} \equiv \langle 0_{S^4 \times D^2_{\mu_1} \times T^2_{\nu_1}} | V^{T^2}_{\mu_1} V^{T^2}_{\mu_2} V^{T^2}_{\mu_3} | 0_{D^2_{\mu_1} \times T^2_{\nu_1}} \rangle = Z \left( \frac{S^4}{3} \right),$$

where $\langle S^4 \rangle_{\sigma_1, \sigma_2}$

Fourth, the four-loop braiding process, where three loops dancing in the Borromean ring trajectory while linked by a fourth loop [30], can characterize certain 3+1D non-Abelian topological orders [25]. We find it is also the spin surgery construction of Borromean rings of three loops linked by a fourth torus in the spacetime picture, and its path integral $Z[S^4; \text{Link}[\text{Spin}[\text{BR}][\mu_4, \mu_3, \mu_2]], \mu_1]$ can be transformed:

$$\begin{align*}
\text{surgery} & \quad Z[T^4 \# S^2 \times S^2; \mu_4', \mu_3', \mu_2', \mu_1'] = Z[T^4 \# S^2 \times S^2],
\end{align*}$$

where the surgery contains four consecutive modular $S$-transformations done along the $T^3$-boundary of $D^2 \times T^2$ tubular neighborhood around four $T^2$-worldsheets [31]. The final spacetime manifold is $T^3 \# S^2 \times S^2$, where $\#$ stands for the connected sum.

We can glue the $T^3$-boundary of 4-submanifolds (e.g. $D^2 \times T^2$ and $S^1 \times D^2 \times T^2$) via $\text{MCG}(T^3) = \text{SL}(3, \mathbb{Z})$ generated by [32]

$$\hat{S}^{xy} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{T}^{xy} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

In this work, we define their representations as [32]

$$S^{xy}_{\mu_1, \mu_1} = \langle 0_{D^2_{\mu_1} \times T^2_{\nu_1}} | V^{T^2}_{\mu_1} \hat{S}^{xy} V^{T^2}_{\mu_1} | 0_{D^2_{\mu_1} \times T^2_{\nu_1}} \rangle, \quad T^{xy}_{\mu_1, \mu_1} = \langle 0_{D^2_{\mu_1} \times T^2_{\nu_1}} | V^{T^2}_{\mu_1} \hat{T}^{xy} V^{T^2}_{\mu_1} | 0_{D^2_{\mu_1} \times T^2_{\nu_1}} \rangle,$$

while $S^{xy}$ is a spin-Hopf link in $S^3 \times S^1$, and $T^{xy}$ is related to the topological spin and self-statistics of closed strings [25].

Quantum surgery and general Verlinde formulas

Now we like to derive a powerful identity for fixed-point path integrals of topological orders. If the path integral formed by disconnected manifolds $M$ and $N$, denoted as $M \sqcup N$, we have $Z(M \sqcup N) = Z(M)Z(N)$. Assume that
(1) we divide both $M$ and $N$ into two pieces such that $M = M_U \cup B M_D$, $N = N_U \cup B N_D$, and their cut topology (dashed $B$) is equivalent $B = \partial M_D = \partial M_U = \partial N_D = \partial N_U$, and (2) the Hilbert space on the spatial slice is 1-dimensional (namely the GSD=1)\(33\), then we obtain

$$Z \left( \begin{array}{c} M_U \noindent B \noindent M_D \noindent N_U \noindent B \noindent N_D \end{array} \right) = Z \left( \begin{array}{c} N_U \noindent B \noindent M_D \noindent N_U \noindent B \noindent N_D \end{array} \right)$$

$$\Rightarrow Z(M_U \cup B M_D)Z(N_U \cup B N_D) = Z(N_U \cup B M_D)Z(M_U \cup B N_D).$$

In 2+1D, we can derive the renowned Verlinde formula \([3, 12, 13]\) by one version of Eq.(17):

$$Z \left( \begin{array}{c} S \noindent S \noindent S \noindent S \noindent S \noindent S \end{array} \right) = Z \left( \begin{array}{c} S \noindent S \noindent S \noindent S \noindent S \noindent S \end{array} \right)$$

$$\Rightarrow L_{\mu_1 \sigma_4}^{S^2,S^1} \sum_{\sigma_4} L_{\mu_1 \sigma_4}^{S^2,S^1} (F^{S^2})_{\mu_1 \sigma_4 \sigma_2 \sigma_3} = L_{\mu_1 \sigma_3}^{S^2,S^1} L_{\mu_1 \sigma_3}^{S^2,S^1}. \quad (19)$$

$$Z \left( \begin{array}{c} 1 \noindent S \noindent 1 \noindent S \noindent 1 \noindent S \end{array} \right) = Z \left( \begin{array}{c} 1 \noindent S \noindent 1 \noindent S \noindent 1 \noindent S \end{array} \right)$$

$$\Rightarrow L_{\mu_4 \sigma_1}^{S^2,S^1} \sum_{\mu_4} L_{\mu_4 \sigma_1}^{S^2,S^1} (F^{S^2})_{\mu_4 \mu_3 \sigma_2 \sigma_1} = L_{\mu_3 \sigma_1}^{S^2,S^1} L_{\mu_3 \sigma_1}^{S^2,S^1}. \quad (20)$$

Here the gray areas mean $S^2$-spheres. All the data are well-defined in Eqs.(7),(8),(10). Notice that Eqs.(19) and (20) are symmetric by exchanging worldsheets/worldlines indices: $\mu \leftrightarrow \sigma$, except that the fusion data is different: $F^{S^1}$ fuses worldlines, while $F^{S^2}$ fuses worldsheets.

We also derive a quantum surgery constraint formula \([35]\) for the three-loop braiding in terms of $S^4$-spacetime path integral Eq.(12) via the $S^{xyz}$-surgery and its matrix representation:

$$Z \left( \begin{array}{c} T^2 \noindent T^2 \noindent T^2 \noindent T^2 \noindent T^2 \noindent T^2 \end{array} \right) = Z \left( \begin{array}{c} T^2 \noindent T^2 \noindent T^2 \noindent T^2 \noindent T^2 \noindent T^2 \end{array} \right)$$

$$\Rightarrow L_{\mu_1 \sigma_1}^{\text{Tri}} \sum_{\mu_1 \sigma_1} (F^{T^2})_{\mu_1 \sigma_1 \sigma_2 \sigma_3} S_{\mu_2 \sigma_2 \sigma_3}^{xyz} = L_{\mu_1 \sigma_1}^{\text{Tri}} L_{\mu_1 \sigma_1}^{\text{Tri}}.$$  \quad (21)

where each spacetime manifold is $S^3$, with the line operator insertions such as an unlink and Hopf links. Each $S^3$ is cut into two $D^3$ pieces, so $D^3 \cup \Sigma D^3 = S^3$, while the boundary dashed cut is $B = S^2$. The GSD for this spatial section $S^2$ with a pair of particle-antiparticle must be 1, so our surgery satisfies the assumptions for Eq.(17).

In 3+1D, the particle-string braiding in terms of $S^4$-spacetime path integral Eq.(10) has constraint formulas:

$$\Rightarrow S_{\mu_1 \sigma_4}^{xyz} \sum_{\sigma_4} S_{\mu_1 \sigma_4}^{xyz} N_{\sigma_2 \sigma_3}^{\mu_2} = S_{\mu_1 \sigma_2} S_{\sigma_1 \sigma_3}, \quad (18)$$

where $\mu_1$-worldsheet in gray represents a $T^2$ torus, while $\mu_2-\mu_3$-worldsheets and $\mu_4-\mu_5$-worldsheets are both a pair of two $T^2$ tori obtained by spinning the Hopf link. All our data are well-defined in Eqs.(9),(12),(15) introduced earlier. For example, the $L_{\mu_1 \sigma_1}^{\text{Tri}}$ is defined in Eq.(12) with 0 as a vacuum without insertion, so $L_{\mu_0 \mu_1}^{\text{Tri}}$ is a path integral of a $T^2$ worldsheet $\mu_1$ in $S^4$. The index $\xi$ is obtained from fusing $\mu_2-\mu_3$-worldsheets, and $\chi$ is obtained from fusing $\mu_4-\mu_5$-worldsheets. Only $\mu_1, \xi_2, \chi_4$ are the fixed indices, other indices are summed over.

For all path integrals of $S^4$ in Eqs.(19), (20) and (21),
each $S^1$ is cut into two $D^4$ pieces, so $D^4 \cup_S D^4 = S^4$. We choose all the dashed cuts for 3+1D path integral representing $B = S^3$, while we can view the $S^3$ as a spatial slice, with the following excitation configurations: A loop in Eq.(19), a pair of particle-antiparticle in Eq.(20), and a pair of loop-antiloop in Eq.(21). Here we require a stronger criterion that all loop excitations are gapped without zero modes, then the GSD is 1 for all above spatial section $S^3$. Thus all our surgeries satisfy the assumptions for Eq.(17).

The above Verlinde-like formulas constrain the fusion data (e.g. $N$, $F^{S^1}$, $F^{S^2}$, $F^{T^2}$, etc.) and braiding data (e.g. $S$, $T$, $L^{(S^2, S^1)}$, $L^{T^2}$, $S^{T^2}$, etc.). Moreover, we can derive constraints between the fusion data itself. Since a $T^2$-worldsheet contains two non-contractible $S^1$-worldlines along its two homology group generators in $H_1(T^2, \mathbb{Z}) = \mathbb{Z}^2$, the $T^2$-worldsheet operator $V_{T^2}$ contains the data of $S^1$-worldline operator $W_{S^1}$. More explicitly, we can compute the state $W_{S^1}W_{S^2}V_{T^2}|0_{D^2}\times T^2\rangle$ by fusing two $W_{S^1}$ operators and one $V_{T^2}$ operator in different orders, then we obtain a consistency formula [35]:

$$\sum_{\sigma_3}(F^{S_1})^{\sigma_3}_{\sigma_1,\sigma_2}(F^{T^2})^{\mu_3}_{\mu_2,\mu_1} = \sum_{\mu_1}(F^{T^2})^{\mu_1}_{\sigma_1,\sigma_2}(F^{T^2})^{\mu_3}_{\mu_2,\mu_1}. \quad (22)$$

We organize our quantum statistics data of fusion and braiding, and some explicit examples of topological orders and their topological invariances in terms of our data in the Supplemental Material.

CONCLUSION

It is known that the quantum statistics of particles in 2+1D begets anyons, beyond the familiar statistics of bosons and fermions, while Verlinde formula [12] plays a key role to dictate the consistent anyon statistics. In this work, we derive a set of quantum surgery formulas analogous to Verlinde’s constraining the fusion and braiding quantum statistics of anyon excitations of particle and string in 3+1D.

A further advancement of our work, comparing to the pioneer work Ref.[3] on 2+1D Chern-Simons gauge theory, is that we apply the surgery idea to generic 2+1D and 3+1D topological orders without assuming quantum field theory (QFT) or gauge theory description. Although many lattice-regularized topological orders happen to have TQFT descriptions at low energy, we may not know which topological order derives which TQFT easily. Instead we simply use quantum amplitudes written in the bra and ket (over-)complete bases, obtained from inserting worldline/sheet operators along the cycles of non-trivial homology group generators of a spacetime submanifold, to cut and glue to the desired path integrals. Consequently our approach, without the necessity of any QFT description, can be powerful to describe more generic quantum systems. While our result is originally based on studying specific examples of TQFT in Dijkgraaf-Witten gauge theory [35, 36], we formulate the data without using QFT. We have incorporated the necessary generic quantum statistic data and new constraints to characterize some 3+1D topological orders (including Dijkgraaf-Witten’s), we will leave the issue of their sufficiency and completeness for future work. Formally, our approach can be applied to any spacetime dimensions.

It will be interesting to study the analogous Verlinde formula constraints for 2+1D boundary states, such as highly-entangled gapless modes, conformal field theories (CFT) and anomalies, for example through the bulk-boundary correspondence [3, 37–39]. The set of consistent quantum surgery formulas we derive may lead to an alternative effective way to bootstrap [40, 41] 3+1D topological states of matter and 2+1D CFT.

Note added: The formalism and some results discussed in this work have been partially reported in the first author’s Ph.D. thesis [42]. Readers may refer to Ref.[42] for other discussions.

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Supplemental Material

A. Summary of quantum statistics data of fusion and braiding

| Quantum statistics data of fusion and braiding |
|-----------------------------------------------|
| Data for 2+1D topological orders:             |
| • Fusion data: \( N_{\sigma_1 \sigma_2} = F_{\sigma_1 \sigma_2} \) (fusion tensor), |
| • Braiding data: \( S^{xy}, T^{xy} \) (modular SL(2,Z) matrices from MCG(T^2)), |
| \( Z[T^3_{xy}; \sigma_1^x, \sigma_2^y, \sigma_3^y] \) or \( Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]] \), etc. |
| Data for 3+1D topological orders:             |
| • Fusion data: \( (F^S)^{\mu_1 \mu_2}, (F^T)^{\mu_1 \mu_2} \) (fusion tensor), |
| • Braiding data: \( S^{xy}, T^{xy} \) (modular SL(3,Z) matrices from MCG(T^3), |
| including \( S^{xy} \)), \( L^{yt}_{\alpha_0,0,\mu} \) (from \( L^{yt}_{\alpha_0,0,\mu} \)), \( L^{yt}_{\mu_4,0,\mu_1} \), \( L^{yt}(S^2,S^1) \), |
| \( Z[T^4 \times S^2 \times S^2; \mu_4, \mu_5, \mu_2, \mu_1] \) (from \( Z[S^3; \text{Link}[\text{Spin}[\text{BR}[\mu_4, \mu_5, \mu_2]], \mu_1]] \)), etc. |

TABLE I. Some data for 2+1D and 3+1D topological orders. We organize the quantum statistics data of fusion and braiding introduced in the main text in Table I. We pose using the set of data in Table I to label topological orders. We also remark that Table I may not contain all sufficient data to characterize and classify all topological orders. What can be the missing data in Table I? Clearly, there is the chiral central charge \( c_- = c_L - c_R \) for 2+1D topological orders. The \( c_- \) is essential for describing 2+1D topological orders with 1+1D boundary gapless chiral edge modes. The gapless chiral edge modes cannot be fully gapped out by adding scattering terms between different modes, because they are protected by the net chirality. So our 2+1D data only describes 2+1D non-chiral topological orders. Similarly, our 2+1D/3+1D data may not be able to fully classify 2+1D/3+1D topological orders whose boundary modes are protected to be gapless. We may need additional data to encode boundary degrees of freedom for their boundary modes.

In some case, some of our data may overlap with the information given by other data. For example, the 2+1D topological order data \( (S^{xy}, T^{xy}, N^{\alpha_0,0}_L) \) may contain the information of \( Z[T^3_{xy}; \sigma_1^x, \sigma_2^y, \sigma_3^y] \) (or \( Z[S^3; \text{BR}[\sigma_1, \sigma_2, \sigma_3]] \)), etc. For example, we know that the former set of data may fully classify 2+1D topological orders.

Although it is possible that there are extra required data beyond what we list in Table I, we find that Table I is sufficient enough for a large class of topological orders, at least for those described by Dijkgraaf-Witten twisted gauge theory [36] and those gauge theories with finite Abelian gauge groups. In the next Appendix, we will give some explicit examples of 2+1D and 3+1D topological orders described by Dijkgraaf-Witten theory, which can be completely characterized and classified by the data given in Table I.

B. Examples of topological orders and their topological invariances in terms of our data

In Table II, we give some explicit examples of 2+1D and 3+1D topological orders from Dijkgraaf-Witten twisted gauge theory. We like to emphasize that our quantum-surgery Verlinde-like formulas apply to generic 2+1D and 3+1D topological orders beyond the gauge theory or field theory description. So our formulas apply to quantum phases of matter or theories beyond the Dijkgraaf-Witten twisted gauge theory description. We list down these examples only because these are famous examples with a more familiar gauge theory understanding. In terms of topological order language, Dijkgraaf-Witten theory describes the low energy physics of certain bosonic topological orders which can be regularized on a lattice Hamiltonian [23, 25, 44] with local bosonic degrees of freedom (without fermions).

We also clarify that what we mean by the correspondence between the items in the same row in Table II:

• (i) Quantum statistic braiding data,
• (ii) Group cohomology cocycles
• (iii) Topological quantum field theory (TQFT).

What we mean is that we can distinguish the topological orders of given cocycles of (ii) with the low energy TQFT of (iii) by measuring their quantum statistic Berry phase under the prescribed braiding process in the path integral of (i). The path integral of (i) is defined through the action $S$ of (iii) via

$$ Z = \int [DB_I][DA_I] \exp[iS]. $$

For example, the mutual braiding (Hopf linking) measures the $S$ matrix distinguishing different types of $(\frac{1}{2\pi}B^I \wedge dA^I + \frac{1}{2\pi}A^I \wedge dA^I)$ with different $p_{IJ}$ couplings; while the Borromean ring braiding can distinguish different types of $(\frac{1}{2\pi}B^I \wedge dA^I + ic_{123}A^I \wedge A^2 \wedge A^3)$ with different $c_{123}$ couplings. However, the table does not mean that we cannot use braiding data in one row to measures the TQFT in another row. For example, $S$ matrix can also distinguish the $\int \frac{1}{2\pi}B^I \wedge dA^I + \frac{1}{2\pi}A^I \wedge dA^I$ of (iii) by measuring their quantum statistic Berry phase under the prescribed braiding process in the path integral of (i).

C. Derivations of some quantum surgery formulas

In this Appendix, we derive some Verlinde-like quantum surgery formulas, which are constraints of fusion and braiding data of topological orders. We will work out the derivations of Eqs.(18), (19), (20) and then later we will derive Eq.(21) step by step. We will also derive the fusion constraint Eq.(22) more explicitly.

First we derive a generic formula for our use of surgery. We consider a closed manifold $M$ glued by two pieces $M_U$ and $M_D$ so that $M = M_U \cup_B M_D$ where $B = \partial M_U = \partial M_D$. We consider there are insertions of operators in $M_U$ and $M_D$. We denote the generic insertions in $M_U$ as $\alpha_{M_U}$ and the generic insertions in $M_D$ as $\beta_{M_D}$. Here both $\alpha_{M_U}$ and $\beta_{M_D}$ may contain both worldline and worldsheet operators. We write the path integral as $Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U}\beta_{M_D}\rangle_M$ to specify the glued manifold is $M_U \cup_B M_D = M$. Now we like to do surgery by cutting out the submanifold $M_D$ out of $M$ and glue it back to $M_U$ via its mapping class group generator $K \in \text{MCG}(B) = \text{MCG}(\partial M_U) = \text{MCG}(\partial M_D)$. We now give some additional assumptions.

Assumption 1: The operator insertions in $M$ are well-separated into $M_U$ and $M_D$, so that no operator insertions cross the boundary $B$. Namely, at the boundary cut $B$ there are no defects of point or string excitations from the cross-section of $\alpha_{M_U}, \beta_{M_D}$ or any other operators.

Assumption 2: We can generate the complete bases of degenerate ground states fully spanning the dimension of Hilbert space for the spatial section of $B$, by inserting distinct operators (worldline/worldsheet, etc.) into $M_D$. Namely, we insert a set of operators $\Phi$ in the interior of $[0_M]$ to obtain a new state $\Phi|0_M\rangle \equiv |\Phi_{M_D}\rangle$, such that these states $\{|\Phi_{0_M}\rangle\}$ are orthonormal canonical bases, and the dimension of the vector space $\text{dim}(\{|\Phi_{0_M}\rangle\})$ equals to the ground state degeneracy (GSD) of the topological order on the spatial section $B$.

If both assumptions hold, then we find a relation:

$$ Z(M; \alpha_{M_U}, \beta_{M_D}) = \langle \alpha_{M_U}\beta_{M_D}\rangle_M = \sum_{\Phi} \langle \alpha_{M_U}\hat{K}\Phi|0_{M_D}\rangle\langle0_{M_D}|(\hat{K}\Phi)^\dagger|\beta_{M_D}\rangle $$

$$ = \sum_{\Phi} \langle \alpha_{M_U}\hat{K}\Phi|0_{M_D}\rangle\langle0_{M_D}|\Phi^\dagger\hat{K}^{-1}|\beta_{M_D}\rangle = \sum_{\Phi} \langle \alpha_{M_U}\hat{K}\Phi|0_{M_D}\rangle\sum_{M_D\cup_B K} \langle \Phi_{M_D}\hat{K}^{-1}|\beta_{M_D}\rangle $$

$$ = \sum_{\Phi} Z(M_U \cup_B K; M_D; \alpha_{M_U}, \Phi_{M_D}) \langle \Phi_{M_D}\hat{K}^{-1}|\beta_{M_D}\rangle = \sum_{\Phi} \hat{K}^{-1}_{\Phi, \beta} Z(M_U \cup_B K; M_D; \alpha_{M_U}, \Phi_{M_D}). $$

We note that in the second equality we write the identity matrix as $\mathbb{1} = \sum_{\Phi} \langle \hat{K}\Phi|0_{M_D}\rangle\langle0_{M_D}|(\hat{K}\Phi)^\dagger$. In the third and fourth equalities that we have $\hat{K}^{-1}$ in the inner product $\langle \Phi_{M_D}\hat{K}^{-1}|\beta_{M_D}\rangle$, because $\hat{K}$ as a
(i). Path-integral linking invariants; Quantum statistic braiding data
distinguished by the braiding in (i) (ii). Group-cohomology cocycles
(iii). TQFT actions $S$ characterized
by the spacetime-braiding in (i)

| 2+1D |
|-------|
| $Z_{\mathcal{S}^3, \text{Hopf}([\sigma_1, \sigma_2])} = \mathcal{S}_{\sigma_1, \sigma_2}$ |
| $\exp \left( \frac{2\pi i p_{12}}{N_{1234}} a_1 b_2 c_3 \right)$ |
| $\int \sum_{I,J,K}^{N_{1234}} \frac{N_{IJK}}{2} B^I \land dA^J \land \cdots$ |
| $\mathcal{N}_B \rightarrow \mathcal{N}_B + d\eta^I.$ |

| 3+1D |
|-------|
| $Z_{\mathcal{S}^3, \text{BR}([\sigma_1, \sigma_2, \sigma_3])}$ |
| $\exp \left( \frac{2\pi i p_{1234}}{N_{1234}} a_1 b_2 c_3 d_4 \right)$ |
| $\int \sum_{I,J,K,L}^{N_{1234}} \frac{N_{IJKL}}{2} B^I \land dA^J \land \cdots$ |
| $\mathcal{N}_B \rightarrow \mathcal{N}_B + d\eta^I + d\eta^J + d\eta^K.$ |

| TABLE II. Examples of topological orders and their topological invariants in terms of our data in the spacetime dimension $d+1$. Here some explicit examples are given as Dijkgraaf-Witten twisted gauge theory [36] with finite gauge group, such as $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times \mathbb{Z}_{N_4} \times \ldots$, although our quantum statistics data can be applied to more generic quantum systems without gauge or field theory description. The first column shows the path integral form which encodes the braiding process of particles and strings in the spacetime. In terms of spacetime picture, the path integral has nontrivial linkings of worldlines and worldsheets. The geometric Berry phases produced from this adiabatic braiding process of particles and strings yield the measurable quantum statistics data. This data also serves as topological invariants for topological orders. The second column shows the group-cohomology cocycle data $\omega$ as a certain partition-function solution of Dijkgraaf-Witten theory, where $\omega$ belongs to the group-cohomology group, $\omega \in \mathcal{H}^{d+1}[G, \mathbb{R}/\mathbb{Z}] = \mathcal{H}^{d+1}[G, \mathbb{U}(1)]$. The third column shows the proposed continuous low-energy field theory action form for these theories and their gauge transformations. In 2+1D, A and B are 1-forms, while $g$ and $\eta$ are 0-forms. In 3+1D, A is a 2-form, $g$ and $\eta$ are 1-forms, while $g$ is a 0-form. Here $I, J, K \in \{1, 2, 3, \ldots\}$ belongs to the gauge subgroup indices, $N_{12\ldots u} \equiv \gcd(N_1, N_2, \ldots, N_u)$ is defined as the greatest common divisor (gcd) of $N_1, N_2, \ldots, N_u$. Here $p_{IJ} \in \mathbb{Z}_{N_{12}}, p_{123} \in \mathbb{Z}_{N_{123}}, p_{12JK} \in \mathbb{Z}_{N_{12JK}}, p_{1234} \in \mathbb{Z}_{N_{1234}}$ are integer coefficients. The $c_{1234}, c_{123}, c_{124}, c_{134}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \left( \frac{1}{B_{12}} \frac{N_{12} N_{21}}{N_{12}}, c_{13} = \left( \frac{1}{B_{13}} \frac{N_{13} N_{31}}{N_{13}}, c_{1234} = \left( \frac{1}{B_{1234}} \frac{N_{1234}}{N_{1234}} \right) \right)$. Be aware that we define both $p_{IJ}$... and $c_{IJ}$... as constants with fixed-indices $I, J, \ldots$; they are additionally defined $c_{IJ}... \equiv c_{i_1 j_1}... c_{i_d j_d}...$ with the $c_{i_j}... = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where $I, J, \ldots$ are free indices needed to be Einstein-summed over, but $c_{i_j}$ is fixed. The lower and upper indices need be summed-over, for example $\int \frac{N_{IJ}}{2} B^I \land dA^J$ means that $\sum_{I,J,K,L}^{N_{1234}} \frac{N_{IJKL}}{2} B^I \land dA^J$ where the value of $s$ depends on the total number $s$ of gauge subgroups $G = \prod_i \mathbb{Z}_{N_i}$. The quantization labelings are described and derived in [25, 43].

MCG generator acts on the spatial manifold $B$ directly. The evolution process from the first $\hat{K}^{-1}$
on the right and the second $\hat{K}$ on the left can be viewed as the adiabatic evolution of quantum states in the case of fixed-point topological orders. In the fifth equality we rewrite $(\alpha_{M_U}\{\hat{K}_{M_D}\}_{M_U\cup B;K} M_D = Z(M_U \cup B;K M_D;\alpha_{M_U}, \Phi_{M_D})$ where $\alpha_{M_U}$ and $\Phi_{M_D}$ may or may not be linked in the new manifold $M_U \cup B;K M_D$. In the sixth equality, we assume that both $|\beta_{M_D}\rangle$ and $|\Phi_{M_D}\rangle$ are vectors in a canonical basis, then we can define

$$\langle\Phi_{M_D}| \hat{K}^{-1}|\beta_{M_D}\rangle_{M_D \cup B;K\setminus \{1\} M_D} = K_{\Phi, \beta}^{-1} \quad (24)$$

To summarize, so far we derive,

$$Z(M;\alpha_{M_U}, \beta_{M_D}) = \sum_{\Phi} K_{\Phi, \beta}^{-1} Z(M_U \cup B;K M_D;\alpha_{M_U}, \Phi_{M_D}). \quad (25)$$

We can also derive another formula by applying the inverse transformation,

$$Z(M_U \cup B;K M_D;\alpha_{M_U}, \Phi'_{M_D}) = \sum_{\Phi'} K_{\Phi, \Phi'} Z(M;\alpha_{M_U}, \beta_{M_D}). \quad (26)$$

if it satisfies $KK^{-1} = 1$. Again we stress that $K_{\Phi, \Phi'}$ is a quantum amplitude computed in the specific spacetime manifold $M_D \cup B;K\setminus \{1\} M_D$.

We now go back to derive Eqs.(18),(19) and (20). For Eq.(18), the only path integral we need to compute more explicitly is this:

$$Z\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}\right) = \langle 0_{D_2t} \times S_2 \left| (W_{\sigma_1})^{S_1} \hat{W}_{\sigma_2}^{S_1} W_{\sigma_3}^{S_1} | 0_{D_2t} \times S_2 \rangle
\end{array}\right)
\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}\right) = \langle 0_{D_2t} \times S_2 \left| (W_{\sigma_1})^{S_1} \hat{W}_{\sigma_2}^{S_1} W_{\sigma_3}^{S_1} | 0_{D_2t} \times S_2 \rangle
\end{array}\right)
\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}\right) = \sum_{\alpha_{S_1}} G_{\alpha_{S_1}}^{\sigma_1} S_{\sigma_1 \sigma_2} F_{\sigma_2 \sigma_3} = \sum_{\alpha_{S_1}} S_{\sigma_1 \sigma_2} N_{\sigma_2 \sigma_3}, \quad (27)$$

where the last equality we use the canonical basis. Together with the previous data, we can easily derive Eq.(18).

Since it is convenient to express in terms of canonical bases, below for all the derivations, we will implicitly project every quantum amplitude into canonical bases when we write down its matrix element.

For Eq.(19), the only path integral we need to compute more explicitly is this:

$$Z\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}\right) = \langle 0_{D_2 t} \times S_2 \left| (V_{\mu_1}^{S_1})^{S_1} W_{\sigma_2}^{S_1} W_{\sigma_3}^{S_1} | 0_{D_2t} \times S_2 \rangle
\end{array}\right) = \langle 0_{D_2 t} \times S_2 \left| (V_{\mu_1}^{S_1})^{S_1} W_{\sigma_2}^{S_1} W_{\sigma_3}^{S_1} | 0_{D_2t} \times S_2 \rangle
\end{array}\right) = \sum_{\sigma_4} L^{S_1, S_1}_{\mu_1, \sigma_4} (F^{S_1}_{\sigma_2 \sigma_3} \sigma_4), \quad (28)$$

again we use the canonical basis. Together with the previous data, we can easily derive Eq.(19). Similarly, we can derive Eq.(20) using the almost equivalent computation.

Now let us derivate Eq.(21). In the first path integral, we create a pair of loop $\mu_1$ and anti-loop $\bar{\mu}_1$ excitations and then annihilate them, in terms of the spacetime picture,

$$Z\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}\right) = Z\left(\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}\right)
\end{array}\right) = L_{0,0,\mu_1}^{\text{Tri}}, \quad (29)$$

based on the data defined earlier.

In the third path integral $L_{\mu_3, \mu_2, \mu_4}^{\text{Tri}}$ of Eq.(21), there are two descriptions to interpret it in terms of the braiding process in spacetime. Here is the first description, we create a pair of loop $\mu_1$ and anti-loop $\bar{\mu}_1$ excitations and then there a pair of $\mu_2 - \bar{\mu}_2$ and another pair of $\mu_3 - \bar{\mu}_3$ are created while both pairs are thread by $\mu_1$. Then the $\mu_1 - \mu_2 - \mu_3$ will do the three-loop braiding process, which gives the most important Berry phase or Berry matrix information into the path integral. After then the pair of $\mu_2 - \bar{\mu}_2$ is annihilated and also the pair of $\mu_3 - \bar{\mu}_3$ is annihilated, while all the four loops are threaded by $\mu_1$ during the process. Finally we annihilate the pair of $\mu_1$ and $\bar{\mu}_1$ in the end [23]. The second description is that...
we take a Hopf link of $\mu_2$-$\mu_3$ linking spinning around the loop of $\mu_1$ [26, 27]. We denote the Hopf link of $\mu_2$-$\mu_3$ as $\text{Hopf(}\mu_3, \mu_2\text{)}$, denote its spinning as $\text{Spun(Hopf(}\mu_3, \mu_2\text{))}$, and denote its linking with the third $T^2$-worldsheet of $\mu_1$ as $\text{Link(}\text{Spun(Hopf(}\mu_3, \mu_2\text{)), } \mu_1\text{)}$. Thus we can define $\Gamma_{\mu_3, \mu_2, \mu_1}^{T^2} \equiv Z[S^4; \text{Link(}\text{Spun(Hopf(}\mu_3, \mu_2\text{)), } \mu_1\text{)}]$. From the second description, we immediately see that $\Gamma_{\mu_3, \mu_2, \mu_1}^{T^2}$ as $Z[S^4; \text{Link(}\text{Spun(Hopf(}\mu_3, \mu_2\text{)), } \mu_1\text{)}]$ are symmetric under exchanging $\mu_2 \leftrightarrow \mu_3$, up to an overall conjugation due to the orientation of quasi-exclusions.

We can view the spacetime $S^4$ as $\mathbb{R}^4 + \{\infty\}$, the Cartesian coordinate $\mathbb{R}^4$ plus a point at the infinity $\{\infty\}$. Similar to the embedding of Ref. [26], we embed the $T^2$-worldsheets $\mu_1, \mu_2, \mu_3$ into the $(X_1, X_2, X_3, X_4) \in \mathbb{R}^4$ as follows:

\[
\begin{align*}
X_1(u, \vec{x}) &= [r_1(u) + (r_2(u) + r_3(u) \cos x) \cos y) \cos z, \\
X_2(u, \vec{x}) &= [r_1(u) + (r_2(u) + r_3(u) \cos x) \cos y) \sin z, \\
X_3(u, \vec{x}) &= [r_2(u) + r_3(u) \cos x) \sin y, \\
X_4(u, \vec{x}) &= r_3(u) \sin x,
\end{align*}
\]

here $\vec{x} \equiv (x, y, z)$. We choose the $T^2$-worldsheets as follows.

The $T^2$-worldsheet $\mu_1$ is parametrized by some fixed $u_1$ and free coordinates of $(x, y)$ while $z = 0$ is fixed. The $T^2$-worldsheet $\mu_2$ is parametrized by some fixed $u_2$ and free coordinates of $(x, y)$ while $z = 0$ is fixed. The $T^2$-worldsheet $\mu_3$ is parametrized by some fixed $u_3$ and free coordinates of $(y, z)$ while $x = 0$ is fixed. We can set the parameters $u_1 > u_2 > u_3$. Meanwhile, a $T^3$-surface can be defined as $M^3(u, \vec{x}) \equiv (X_1(u, \vec{x}), X_2(u, \vec{x}), X_3(u, \vec{x}), X_4(u, \vec{x}))$ with a fixed $u$ and free parameters $\vec{x}$. The $T^3$-surface $M^3(u, \vec{x}) \equiv (X_1(u, \vec{x}), X_2(u, \vec{x}), X_3(u, \vec{x}), X_4(u, \vec{x}))$ encloses a 4-dimensional volume. We define the enclosed 4-dimensional volume as the $M^3(u, \vec{x}) \times I^1(s)$ where $I^1(s)$ is the 1-dimensional radius interval along $r_3$, such that $I^1(s) = \{s|s = [0, r_3(u)]\}$, namely $0 \leq s \leq r_3(u)$. Here we can define $r_3(0) = 0$. The topology of the enclosed 4-dimensional volume of $M^3(u, \vec{x}) \times I^1(s)$ is of course the $T^3 \times I^1 = T^2 \times (S^1 \times I^1) = T^2 \times D^2$.

For a $M^3(u_{\text{large}}, \vec{x}) \times I^1(s)$ prescribed by a fixed larger $u_{\text{large}}$ and free parameters $\vec{x}$, the $M^3(u_{\text{large}}, \vec{x}) \times I^1(s)$ must enclose the 4-volume spanned by the past history of $M^3(u_{\text{small}}, \vec{x}) \times I^1(s)$, for any $u_{\text{large}} > u_{\text{small}}$. Here we set $u_1 > u_2 > u_3$. And we also set $r_1(u) > r_2(u) > r_3(u)$ for any given $u$.

One can check that the three $T^2$-worldsheet $\mu_1, \mu_2$ and $\mu_3$ indeed have the nontrivial triple-linking number [28]. We can design the triple-linking number to be: $\text{Tlk}(\mu_2, \mu_1, \mu_3) = \text{Tlk}(\mu_3, \mu_1, \mu_2) = 0$, $\text{Tlk}(\mu_1, \mu_2, \mu_3) = +1$, $\text{Tlk}(\mu_3, \mu_2, \mu_1) = -1$, $\text{Tlk}(\mu_2, \mu_3, \mu_1) = +1$, $\text{Tlk}(\mu_1, \mu_3, \mu_2) = -1$.

Below we will frequently use the surgery trick by cutting out a tubular neighborhood $D^2 \times T^2$ of the $T^2$-worldsheet and re-glue this $D^2 \times T^2$ back to its complement $S^4 \setminus D^2 \times T^2$ via the modular $S^{xy}z$-transformation. The $S^{xy}z$-transformation sends

\[
\begin{pmatrix}
x_{\text{out}} \\
y_{\text{out}} \\
z_{\text{out}}
\end{pmatrix} =
\begin{pmatrix}
x_{\text{in}} \\
y_{\text{in}} \\
z_{\text{in}}
\end{pmatrix} \rightarrow
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

Thus, the $S^{xy}z$-identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (z_{\text{in}}, x_{\text{in}}, y_{\text{in}})$. The $(S^{xy}z)^{-1}$-identification is $(x_{\text{out}}, y_{\text{out}}, z_{\text{out}}) \leftrightarrow (y_{\text{in}}, z_{\text{in}}, x_{\text{in}})$. The surgery on the initial $S^4$ outcomes a new manifold,

\[
(D^2 \times T^2) \cup_{T^3,S^{xy}z} (S^4 \setminus D^2 \times T^2) = S^3 \times S^1 \# S^2 \times S^2.
\]

In terms of the spacetime path integral picture, use Eqs. (25) and (26), we derive:

\[
\begin{align*}
Z\left(\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}\right) = & \text{L}_{\mu_3; \mu_2, \mu_1}^{T^2} = Z[S^4; \text{Link(}\text{Spun(Hopf(}\mu_3, \mu_2\text{)), } \mu_1\text{)}] \\
= & \sum_{\mu_3} S^{xyz}_{\mu_3; \mu_3} Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \mu_2 \parallel \mu_3) \\
= & \sum_{\mu_3; \mu_2} S^{xyz}_{\mu_3; \mu_3} (F^{T^2})_{\mu_2}^{\Gamma_2} Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \mu_2) \\
= & \sum_{\mu_3} S^{xyz}_{\mu_3; \mu_3} (F^{T^2})_{\mu_2}^{\Gamma_2} (S^{xyz})_{\Gamma_2}^{-1} \Gamma_2 \mu_2 \Gamma_2 Z(S^4; \mu_1, \Gamma_2) \\
= & \sum_{\mu_3} S^{xyz}_{\mu_3; \mu_3} (F^{T^2})_{\mu_2}^{\Gamma_2} (S^{xyz})_{\Gamma_2}^{-1} \Gamma_2 \mu_2 \Gamma_2 Z(S^3 \times S^1 \# S^2 \times S^2; \mu_1, \mu_2) \\
= & \sum_{\mu_3} S^{xyz}_{\mu_3; \mu_3} (F^{T^2})_{\mu_2}^{\Gamma_2} (S^{xyz})_{\Gamma_2}^{-1} \Gamma_2 \mu_2 \Gamma_2 Z(S^3 \times S^1 \# S^2 \times S^2; \eta_2)
\end{align*}
\]
As usual, the repeated indices are summed over. With the trick of \( S^{\bar{y} z} \)-transformation in mind, here is the step-by-step sequence of surgeries we perform.

Step 1: We cut out the tubular neighborhood \( D^2 \times T^2 \) of the \( T^2 \)-worldsheet of \( \mu_3 \) and re-glue this \( D^2 \times T^2 \) back to its complement \( S^3 \setminus D^2 \times T^2 \) via the modular \( (S^{\bar{y} z})^{-1} \)-transformation. The \( D^2 \times T^2 \) neighborhood of \( \mu_3 \)-worldsheet can be viewed as the 4-volume \( \mathcal{M}^3(u_3, \bar{u}) \times \hat{I}^4(s) \), which encloses neither \( \mu_1 \)-worldsheet nor \( \mu_2 \)-worldsheet. The \( (S^{\bar{y} z})^{-1} \)-transformation sends \((y_{in}, \bar{z}_{in})\) of \( \mu_3 \) to \((x_{out}, y_{out})\) of \( \mu_2 \). The gluing however introduces the summing-over new coordinate \( \mu'_3 \), based on Eq.(25). Thus Step 1 obtains Eq.(33).

In Step 1, as Eq.(33) and thereafter, we write down \( S^{\bar{y} z}_{\mu_3, \mu_3} \) based on Eq.(24), we stress that the \( S^{\bar{y} z}_{\mu_3, \mu_3} \) is projected to the \( |0_{D^2 \times T^2}\rangle\)-states with operator-insertions for both bra and ket states.

\[
S^{\bar{y} z}_{\mu_3, \mu_3} \equiv \langle \mu'_3 | D^2 \times T^2 | \bar{S}^{\bar{y} z} | \mu_3 D^2 \times T^2 \rangle D^2 \times T^2 \cup_{T^3, \bar{y} \neq y} D^2 \times T^2 \\
= \langle 0_{D^2 \times T^2} | V^{T^2}_{\mu_3} \bar{S}^{\bar{y} z} V^{T^2}_{\mu_3} | 0_{D^2 \times T^2} \rangle \langle \mu_3 | S^3 \times S^1 \rangle 
\]

(39)

Here we use the surgery fact

\[
D^2 \times T^2 \cup_{T^3, \bar{y} \neq y} D^2 \times T^2 = S^3 \times S^1. 
\]

(40)

So our \( S^{\bar{y} z}_{\mu_3, \mu_3} \) is defined as a quantum amplitude in \( S^3 \times S^1 \). Two \( T^2 \)-worldsheets \( \mu'_3 \) and \( \mu_3 \) now become a pair of Hopf link resides in \( S^3 \) part of \( S^3 \times S^1 \), while share the same \( S^1 \) circle in the \( S^3 \) part of \( S^3 \times S^1 \). We can view the shared \( S^1 \) circle as the spinning circle of the spin surgery construction on the Hopf link in \( D^3 \), the spin-topology would be \( D^3 \times S^1 \), then we glue this \( D^3 \times S^1 \) contains \( \text{Spin}[\text{Hopf}][\mu'_3, \mu_3] \) to another \( D^3 \times S^1 \), so we have \( D^3 \times S^1 \cup_{S^2 \times S^1} D^3 \times S^1 = S^3 \times S^1 \) as an overall new spacetime topology. Hence we also denote

\[
S^{\bar{y} z}_{\mu'_3, \mu_3} \equiv Z[S^3 \times S^1, \text{Spin}[\text{Hopf}][\mu'_3, \mu_3]]. 
\]

(41)

Step 2: The earlier surgery now makes the inner \( \mu'_3 \)-worldsheet \( \parallel \mu'_3 \) to the outer \( \mu_2 \)-worldsheet, since they share the same coordinates \((x_{out}, y_{out}) = (y_{in}, \bar{z}_{in})\). We denote their parallel topology as \( \mu_2 \parallel \mu'_3 \). So we can fuse the \( \mu_2 \)-worldsheet and \( \mu'_3 \)-worldsheet via the fusion algebra, namely

\[
V^{T^2}_{\mu'_3} V^{T^2}_{\mu_2} = (F^{T^2})_{\mu_2 \mu_3} V^{T^2}_{\mu_2} V^{T^2}_{\mu' \mu_3}. 
\]

Thus Step 2 obtains Eq.(34).

Step 3: We cut out the tubular neighborhood \( D^2 \times T^2 \) of the \( T^2 \)-worldsheet of \( \Gamma_2 \) and re-glue this \( D^2 \times T^2 \) back to its complement \( S^3 \setminus D^2 \times T^2 \) via the modular \( S^{\bar{y} z} \)-transformation. The \( D^2 \times T^2 \) neighborhood of \( \Gamma_2 \)-worldsheet can be viewed as the 4-volume \( \mathcal{M}^3(u_2, \bar{u}) \times \hat{I}^4(s) \) in the new manifold \( S^3 \times S^1 \# S^2 \times S^2 \), which encloses no worldsheet inside. After the surgery, the \( S^{\bar{y} z} \)-transformation sends the redefined \((x_{in}, y_{in})\) of \( \Gamma_2 \) back to \((y_{out}, z_{out})\) of \( \Gamma'_2 \). The gluing however introduces the summing-over new coordinate \( \Gamma'_2 \), based on Eq.(25). We also transform \( S^3 \times S^1 \# S^2 \times S^2 \) back to \( S^4 \) again. Thus Step 3 obtains Eq.(35).

Step 4: We cut out the tubular neighborhood \( D^2 \times T^2 \) of the \( T^2 \)-worldsheet of \( \Gamma'_2 \) and re-glue this \( D^2 \times T^2 \) back to its complement \( S^3 \setminus D^2 \times T^2 \) via the modular \( S^{\bar{y} z} \)-transformation. The \( D^2 \times T^2 \) neighborhood of \( \Gamma'_2 \)-worldsheet viewed as the 4-volume in the manifold \( S^4 \) encloses no worldsheet inside. After the surgery, the \( S^{\bar{y} z} \)-transformation sends the \((x_{in}, y_{in})\) of \( \Gamma'_2 \) to \((z_{out}, x_{out})\) of \( \mu_1 \). The gluing however introduces the summing-over new coordinate \( \Gamma''_2 \), based on Eq.(25). We also transform \( S^4 \) to \( S^3 \times S^1 \# S^2 \times S^2 \) again. Thus Step 4 obtains Eq.(36).

Step 5: The earlier surgery now makes the inner \( \Gamma''_2 \)-worldsheet \( \parallel \) to the outer \( \mu_1 \)-worldsheet, since they share the same coordinates \((z_{out}, x_{out}) = (x_{in}, y_{in})\). We denote their parallel topology as \( \mu_1 \parallel \Gamma''_2 \). We now fuse the \( \mu_1 \)-worldsheet and \( \Gamma''_2 \)-worldsheet via the fusion algebra, namely

\[
V^{T^2}_{\mu_1} V^{T^2}_{\mu_2} = (F^{T^2})_{\mu_1 \mu_2} V^{T^2}_{\mu_2} V^{T^2}_{\mu_1} \Rightarrow \text{Thus Step 5 obtains Eq.(37).}
\]

Step 6: We should do the inverse transformation to get back to the \( S^4 \) manifold. Thus we cut out the tubular neighborhood \( D^2 \times T^2 \) of the \( T^2 \)-worldsheet of \( \eta_2 \) and re-glue this \( D^2 \times T^2 \) back to its complement via the modular \( (S^{\bar{y} z})^{-1} \)-transformation. We relate the original path integral to the final one \( Z(S^4, \eta'_2) = L^{T^1}_{\eta_1 \eta_2^{\prime}} \). Thus Step 6 obtains Eq.(38).
Similarly, in the fourth path integral of Eq.(21), we derive

\[ Z \left( \begin{array}{c} 1 \\ 2 \\ \sigma_1 \\ \sigma_2 \end{array} \right) = \mathcal{L}_{\text{Tri}}^{\mu_5, \mu_4, \mu_1} = Z[S^4; \text{Link[Spun[Hopf[\mu_5, \mu_4]], \mu_1]]} \]

\[ = \sum_{\mu_5', \mu_4', \mu_1} S^{xyz}_{\mu_5, \mu_4, \mu_1} (F_T^{\sigma_1})_{\mu_5} (S^{xyz})_{\mu_1, \mu_4}^{-1} (F_T^{\sigma_2})_{\mu_5'} (S^{xyz})_{\mu_1}^{-1} L_{\text{Tri}}^{\mu_0, \mu_1'} \]

In the second path integral of Eq.(21), we have the Hopf link of Hopf[\mu_3, \mu_2] and the Hopf link of Hopf[\mu_5, \mu_4]. In the spacetime picture, all \mu_2, \mu_3, \mu_4, \mu_5 are \mathcal{T}^2-worldsheets under the spun construction. We can locate the the spun object named Spun[Hopf[\mu_3, \mu_2], Hopf[\mu_5, \mu_4]] inside a \(D^3 \times S^1\), while this \(D^3 \times S^1\) is glued with a \(S^2 \times D^2\) to a \(S^4\). Here the \(S^2 \times D^2\) contains a \(\mathcal{T}^2\)-worldsheet \(\mu_1\). We can view the \(\mathcal{T}^2\)-worldsheet \(\mu_1\) contains a \(S^2\)-sphere of the \(S^2 \times D^2\) but attached an extra handle. We derive:

\[ Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ \sigma_1 \\ \sigma_2 \end{array} \right) = Z[S^4; \text{Link[Spun[Hopf[\mu_3, \mu_2], Hopf[\mu_5, \mu_4]], \mu_1]}] \]

\[ = \sum_{\mu_5', \mu_4', \mu_1} S^{xyz}_{\mu_3, \mu_2, \mu_1} (F_T^{\sigma_1})_{\mu_5} (S^{xyz})_{\mu_1, \mu_4}^{-1} (F_T^{\sigma_2})_{\mu_5'} (S^{xyz})_{\mu_1}^{-1} L_{\text{Tri}}^{\mu_0, \mu_1'} \]

Here we do the Step 1, Step 2 and Step 3 surgeries on Spun[Hopf[\mu_3, \mu_2]] first, then do the same 3-step surgeries on Spun[Hopf[\mu_5, \mu_4]] later, then we obtain Eq.(43). While in Eq.(43), the new \(\mathcal{T}^2\)-worldsheets \(\Gamma_2\) and \(\Gamma_4\) have no triple-linking with the worldsheet \(\mu_1\). Here \(\Gamma_2\) and \(\Gamma_4\) are arranged in the \(D^3 \times S^1\) part of the \(S^4\) manifold, while \(\mu_1\) is in the \(S^2 \times D^2\) part of the \(S^4\) manifold. Indeed, \(\Gamma_2\) and \(\Gamma_4\) can be fused together in parallel to a new \(\mathcal{T}^2\)-worldsheet \(\Gamma\) via the fusion algebra \((F_T^{\sigma_1})_{\mu_5} (S^{xyz})_{\mu_1}^{-1}\), so we obtain Eq.(44). Then we apply the Step 4, Step 5 and Step 6 surgeries on the \(\mathcal{T}^2\)-worldsheets \(\Gamma\) and \(\mu_1\) of \(Z[S^4; \text{Spun[\Gamma]}, \mu_1] = Z[S^4; \Gamma, \mu_1]\) in Eq.(44), we obtain the final form Eq.(45).

Use Eqs.(29),(38),(42) and (45), and plug them into the path integral surgery relations, we derive a new quantum surgery formula (namely Eq.(19) in the main text):

\[ Z \left( \begin{array}{c} 1 \\ 2 \\ \sigma_1 \\ \sigma_2 \end{array} \right) Z \left( \begin{array}{c} 1 \\ 2 \\ \sigma_1 \\ \sigma_2 \end{array} \right) = Z \left( \begin{array}{c} 1 \\ 2 \\ \sigma_1 \\ \sigma_2 \end{array} \right) Z \left( \begin{array}{c} 1 \\ 2 \\ \sigma_1 \\ \sigma_2 \end{array} \right) \]

\[ \Rightarrow \Gamma_{0, \mu_1}^{\text{Tri}} \cdot \sum_{\Gamma_2, \Gamma_4, \Gamma_1'} (F_T^{\sigma_1})_{\mu_5} (S^{xyz})_{\mu_1}^{-1} (F_T^{\sigma_2})_{\mu_5'} (S^{xyz})_{\mu_1}^{-1} L_{\text{Tri}}^{\mu_0, \mu_1'} \]

\[ = \sum_{\Gamma_2, \Gamma_4, \Gamma_1'} (S^{xyz})_{\mu_1}^{-1} (F_T^{\sigma_1})_{\mu_5} (S^{xyz})_{\mu_1}^{-1} L_{\text{Tri}}^{\mu_0, \mu_1'} \]

Here only \(\mu_1, \Gamma_2, \Gamma_4\) are the fixed indices, other indices are summed over.

Lastly we provide more explicit calculations of Eq.(22), recall that the constraint between the fusion data itself. First, we

\[ W_{\sigma_1} S^1_{\sigma_2} W_{\sigma_2} = (F^{S^1}_{\sigma_1, \sigma_2} W_{\sigma_2})_{\sigma_1} \]
\[ V_{\mu_1}^T V_{\mu_2}^T = (F^T)_{\mu_1 \mu_2}^3 V_{\mu_3}^T, \quad (48) \]
\[ W_{\sigma_1}^T V_{\mu_2}^T = (F^T)_{\sigma_1 \mu_2}^3 V_{\mu_3}^T. \quad (49) \]

Of course, the fusion algebra is symmetric respect to exchanging the lower indices, \((F^T)^{\mu_1 \mu_2}_{\mu_3} = (F^T)^{\mu_2 \mu_1}_{\mu_3}\). We can regard the fusion algebra \((F^T)_{\sigma_1 \sigma_2}^3 \) and \((F^T)_{\mu_1 \mu_2}^3 \) with worldlines as a part of a larger algebra of the fusion algebra of worldsheets \((F^T)_{\sigma_1 \sigma_2}^3 \) and \((F^T)_{\mu_1 \mu_2}^3 \). We compute the state \(W_{\sigma_1}^S V_{\sigma_2}^S V_{\mu_2}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}}\) by fusing two \(V_{\sigma_1}^S\) operators and one \(V_{\mu_2}^T\) operator in different orders.

On one hand, we can fuse two worldlines first, then fuse with the worldsheets,
\[ W_{\sigma_1}^S W_{\sigma_2}^S V_{\mu_2}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}}\]
\[ = \sum_{\sigma_3} (F^S)_{\sigma_1 \sigma_3}^3 W_{\sigma_3}^S V_{\mu_2}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}} \] (50)

On the other hand, we can fuse a worldline with the worldsheets first, then fuse with another worldline,
\[ W_{\sigma_1}^S W_{\sigma_2}^S V_{\mu_2}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}}\]
\[ = \sum_{\mu_1} W_{\sigma_1}^S (F^T)_{\mu_1 \mu_2}^3 V_{\mu_3}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}}\]
\[ = \sum_{\mu_1 \mu_3} (F^T)_{\mu_1 \mu_3}^3 W_{\sigma_1}^S V_{\mu_3}^T |0_{D^2_{\sigma_2} \times T^2_{\sigma_1}}\] (51)

Therefore, by comparing Eqs.(50) and (51), we derive a consistency condition for fusion algebra:
\[ \sum_{\sigma_3} (F^S)_{\sigma_1 \sigma_3}^3 (F^T)_{\mu_2 \mu_3}^3 = \sum_{\mu_1} (F^T)_{\mu_1 \mu_3}^3 (F^T)_{\sigma_1 \mu_1}^3. \] (52)

[16] If there is a gauge theory description, then the quasi-excitation type of particle \(\sigma\) and string \(\mu\) would be labeled by the representation for gauge charge and by the conjugacy class of the gauge group for gauge flux.

[17] We may simplify the gluing notation \(M_1 \cup_B M_2\) to \(M_1 \cup_B M_2\) if the mapping class group’s generator \(U\) is trivial or does not affect the glued manifold. We may simplify the gluing notation further to \(M_1 \cup M_2\) if the boundary \(B = \partial M_1 = \partial M_2\) is obvious or stated in the text earlier.

[18] The three-step surgeries describe earlier sequentially send the initial 3-sphere configuration with Möbius rings insertion \(S^1 \times S^1\) first surgery \(S^2 \times S^1\) second surgery \(S^2 \times S^1 \times S^1\) third surgery \(T^2\) to a 3-torus configuration. Here we use the notation \(M_1 \# M_2\) means the connected sum of manifolds \(M_1\) and \(M_2\).

[19] To be even more specific, we can also specify the whole open-manifold topology \(M \times V\) corresponding to the ground state sector \(|0_{M \times V}\rangle\), namely we can rewrite the data in new notations to introduce the refined data: \((F^{S^1}) \rightarrow (F^{D^3 \times S^1})\), \((F^{T^2}) \rightarrow (F^{D^2 \times T^2})\) and \((F^{S^2}) \rightarrow (F^{D^2 \times S^2})\). However, because \((F^M)\) are the fusion data in the local neighborhood around the worldline/worldsheet operators with topology \(M\), physically it does not encode the information from the remained product space \(M \times V\). Namely, at least for the most common and known theory that we can regularize on the lattice, we understand that \((F^M) = (F^M \times V_1) = (F^M \times V_2) = \ldots\) for any topology \(V_1, V_2, \ldots\). So here we make a physical assumption that \((F^{S^1}) = (F^{D^3 \times S^1})\), \((F^{T^2}) = (F^{D^2 \times T^2})\) and \((F^{S^2}) = (F^{D^2 \times S^2})\).

[20] Throughout our work, we consistently use \(\sigma\) to represent the quasi-particle label (such as charge or flux) for worldlines \(V_{\sigma}^S\), and we use \(\mu\) to represent the quasi-string label for worldsheets, e.g. \(V_{\mu}^S\) and \(V_{\mu}^T\).

[21] If there is a gauge theory description, then the loop is a pure flux excitation with-
out any net charge.

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[29] By the spin surgery, hereafter we mean that taking an object (such as a Hopf link in this case) inside a $d$-disk $D^d$ and spinning it around a fixed axis, so to make it a $D^{d-1} \times S^1$. Then we will glue this $D^{d-1} \times S^1$ with another manifold $M^{d-1}$ which shares a common boundary as $\partial(M^{d-1}) = \partial(D^{d-1} \times S^1)$. The final manifold is constructed as $D^{d-1} \cup M^{d-1}$. In this case, we have $D^3 \times S^1 \cup S^2 \times D^2 = S^4$, while we insert a $T^2$ inside the $S^2 \times D^2$ part, so that the $T^2$ is linked with the spin Hopf link.

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[31] The four-step surgeries here sequentially send the initial surfaces in 4-space $\langle 3 \rangle \rightarrow \langle 2 \rangle \rightarrow \langle 1 \rangle \rightarrow \langle 0 \rangle$, while we insert a $T^2$ along the generator of homology group $H_1(D^2 \times T^2, Z) = \mathbb{Z}^2$, while $V^y_{T^2}$ is along the unique one generator of homology group $H_2(D^2 \times T^2, Z) = \mathbb{Z}^2$. So our projection here is different from the one-charge (representation) and two-charge (conjugacy class) basis $\langle 2, \mu_2, \mu_3 \rangle$ used in Ref. [23–25]. The $\langle 2, \mu_2, \mu_3 \rangle$-basis can be obtained through $W_{S^1} \mu_2 V_{xy} T_{yz} \mu_3 \big| D^2_{x,y} \times T_{z} \big)$, where $W_{S^1}$ is along the generator of homology group $H_1(S^1 \times D^2 \times T^2, Z) = \mathbb{Z}$, while $V^y_{T^2}$ and $V^x_{T^2}$ are along the two generators of homology group $H_2(S^1 \times D^2 \times T^2, Z) = \mathbb{Z}^2$ via the Alexander duality. See also Supplemental Material.

[32] Presumably there may be defect-like excitation of particles and strings on the spatial slice cross-section $B$. If the dimension of Hilbert space on the spatial slice $B$ is $n$, namely the ground state degeneracy (GSD) is $1$, then we can derive the gluing identity $\langle M_{D} | M_{D} \rangle = \langle N_{0} | N_{0} \rangle \Rightarrow \langle M_{D} | M_{D} \rangle = \langle N_{0} | M_{D} \rangle$ because the vector space is $1$-dimensional and all vectors are parallel in the inner product.

[33] In the canonical basis when $S$ is invertible, we can massage our formula to $N_{S^1}^{-1}_{\mu_2, \mu_3} = \sum_{\phi_1, \phi_2} S_{\mu_2, \mu_3}(\phi_1) S_{\phi_2, \mu_3}(S^{-1}_S \phi_2) \mu_3$.

[34] See further discussions in Supplemental Material.

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