DYNAMICS OF STRONGLY INTERACTING KINK-ANTIKINK PAIRS FOR SCALAR FIELDS ON A LINE

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Abstract. This paper concerns classical nonlinear scalar field models on the real line. If the potential is a symmetric double-well, such a model admits static solutions called kinks and antikinks, which are perhaps the simplest examples of topological solitons. We study pure multi-kinks, which are solutions that converge in one infinite time direction to a superposition of a finite number of kinks and antikinks, without radiation. Our main result is a complete classification of all kink-antikink pairs in the strongly interacting regime, which means the speeds of the kinks tend asymptotically to zero. We show that up to translation there is only one such solution, and we give a precise description of the dynamics of the kink separation. We also establish the existence of strongly interacting $K$-multi-kinks, for any natural number $K$.

1. Introduction

1.1. Setting of the problem. We study scalar field equations on the real line. Let $U : \mathbb{R} \to [0, +\infty)$ be a function of class $C^\infty$ and consider the Lagrangian action,

$$\mathcal{L}(\phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 - U(\phi) \right) \, dx \, dt,$$

for real valued functions $\phi = \phi(t, x)$. The Euler-Lagrange equation associated to $\mathcal{L}$ is the nonlinear wave equation,

$$\partial_t^2 \phi(t, x) - \partial_x^2 \phi(t, x) + U'(\phi(t, x)) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \ \phi(t, x) \in \mathbb{R},$$

(1.1)

We will study (1.1) for potentials $U$ that are even functions taking the global minimal value $U_{\text{min}} = 0$, and such that there are distinct real numbers $\phi_- < \phi_+$ so that

$$U(\phi_-) = U(\phi_+) = U_{\text{min}} = 0,$n$$

$$U(\phi) > 0 \text{ for } \phi \in (\phi_-, \phi_+),$$

$$U''(\phi_-) = U''(\phi_+) > 0.$$

(1.2)

Two classically studied examples of (1.1) with potentials as in (1.2) are the sine-Gordon equation,

$$\partial_t^2 \phi(t, x) - \partial_x^2 \phi(t, x) + \sin \phi = 0,$$

(1.3)

where we have taken $U(\phi) = 1 - \cos \phi$ above, and the $\phi^4$ model,

$$\partial_t^2 \phi(t, x) - \partial_x^2 \phi(t, x) - \phi + \phi^3 = 0,$$

(1.4)

where $U(\phi) = \frac{1}{4}(1 - \phi^2)^2$.

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The potential energy $E_p$, the kinetic energy $E_k$, and the total energy $E$ associated with the equation (1.1) are given by

$$E_p(\phi) = \int_{-\infty}^{+\infty} \left( \frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right) \, dx,$$

$$E_k(\phi) = \int_{-\infty}^{+\infty} \frac{1}{2} (\partial_t \phi)^2 \, dx,$$

$$E(\phi) = \int_{-\infty}^{+\infty} \left( \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right) \, dx.$$

We say that a solution to (1.1) is in the energy space if $E(\phi)$ is finite. For such a solution the energy is conserved, i.e., $E(\phi(t, \cdot)) = \text{constant}$. By a solution $\phi(t, x)$ of (1.1), we always mean a strong solution in the energy space. By standard arguments, the Cauchy problem for (1.1) is locally well-posed for initial data $(\phi_0, \phi_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and globally well-posed under additional assumptions on $U$, for instance if $U$ is globally Lipschitz or if $\lim_{\phi \to \pm \infty} U(\phi) = \infty$.

Stationary solutions of (1.1) are the critical points of the potential energy. The trivial ones include the vacuum fields $\phi(x) = \phi_{\pm}$, which are global minima of $E_p$. Importantly, there are also non-constant static solutions $\phi(t, x)$ called \textit{kinks} connecting the two vacua, that is for instance

$$\lim_{x \to -\infty} \phi(t, x) = \phi_- \quad \text{and} \quad \lim_{x \to \infty} \phi(t, x) = \phi_+ \quad \forall t \in \mathbb{R}. \quad (1.5)$$

All of these solutions are given by

$$\phi(t, x) = H(x - a), \quad (1.6)$$

where $H(x)$ is an increasing, smooth, odd function that minimizes the potential energy restricted to those functions $\phi(x)$ satisfying (1.5), and $a \in \mathbb{R}$ is a parameter. For the sine-Gordon equation (1.3) the kink is given by $H(x) = 4 \arctan(e^x)$ and for the $\phi^4$ model we have $H(x) = \tanh(x/\sqrt{2})$. We will study the function $H$ for general $U$ as in (1.2) in detail in Section 2.1.

In this paper we agree that solutions of the form (1.6) that are increasing will be called \textit{kinks} and those that are decreasing (i.e., that connect from $\phi_+$ at $-\infty$ to $\phi_-$ at $+\infty$) will be called \textit{antikinks}. The latter are all given by $\phi(t, x) = H(-x + a)$. ODE analysis shows that besides the vacuum fields, the kinks, and the antikinks, no other finite potential energy stationary solutions such that $\phi_- \leq \phi \leq \phi_+$ exist. We note that equation (1.1) is invariant by Lorentz transformations and applying a Lorentz boost we obtain moving kinks and antikinks:

$$\phi(t, x) = H(\gamma(x - vt - a)), \quad \phi(t, x) = H(\gamma(-x + vt + a))$$

where $v \in (-1, 1)$ and $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

Kinks and antikinks are the simplest examples of topological solitons (they are one-dimensional) and this perhaps explains why the wave equation (1.1) is widely studied both as a model problem in physics and due to its own merit as an interesting and challenging mathematical problem. For example, the question of nonlinear stability of the kink for the $\phi^4$-model (1.4) is classical, but still open for general smooth perturbations; see the recent work of the second author with Martel and Muñoz [17] where stability of the $\phi^4$ kink was proved under odd perturbations. For some other, special potentials this problem was studied in [16], [15]. On the mathematical physics side, we refer the reader to [18], [19] and the references therein for specific examples and their motivations.

1.2. \textbf{Main results}. In this paper we consider the question of multi-kink solutions to (1.1) in what we call the \textit{strongly interacting} regime. Multi-kinks are informally defined as solutions that converge to a superposition of a finite number of kinks and antikinks, without radiation, as $t \to \infty$. We will define “strongly interacting” precisely below, but informally this means the special class of multi-kinks for which the speeds of the kinks tend to zero as $t \to \infty$. An interesting aspect of this regime is that the dynamics are driven solely by nonlinear interactions between the kinks and antikinks.
We remark that if $\phi$ is any strongly interacting kink-antikink pair, then there exist solutions.

Moreover, $\phi$ in (1.2), we define, Definition 1.1. such solutions.

be, up to a very small correction, equal to twice the potential energy of the kink. We think of this interacting regime and to hint at their existence, consider the function

$$H(x) = H_{\text{static}}(x) - H_{\text{kinetic}}(x).$$

This is in contrast to a multi-kink configuration consisting of boosted kinks and antikinks (i.e., the kinks have a nontrivial asymptotic velocities), where the nonlinear interactions between the kinks are negligible as compared to the internal dynamics of each kink determined by the Lorentz boost.

1.2.1. Strongly interacting kink-antikink pairs. To motivate the study of solutions in the strongly interacting regime and to hint at their existence, consider the function

$$w(x; a) = \phi_+ - H(x + a) + H(x - a) = H(-(x + a)) + H(x - a) - \phi_-, \quad \text{where in the last equality we use the fact that, by the symmetry of } U, H(x + a) = (\phi_+ + \phi_-) - H(-x - a).$$

Note that $w$ satisfies, with some $c > 0$

$$-\partial_{xx} w + U'(w) = O(e^{-c|a|}), \quad E_p(w(; a)) = O(e^{-c|a|}).$$

In other words $w$, which is a superposition of a sufficiently separated kink and antikink, "nearly" solves (1.1) when $|a| \gg 1$. Taking $a = a(t), |a(t)| \gg 1$, to be a time dependent modulation function, if there were a solution $\phi(t, x) \approx w(x, a(t))$ with $|a'(t)| \to 0$ as $t \to \infty$, then its total energy would be, up to a very small correction, equal to twice the potential energy of the kink. We think of this as a threshold situation for the formation of kink-antikink pairs. Our goal is to find and classify all such solutions.

First, we give a precise definition of this threshold scenario.

Definition 1.1. We say that a solution $\phi(t, x)$ of (1.1) is a strongly interacting kink-antikink pair if there exist real-valued functions $x_1(t)$ and $x_2(t)$ such that

$$\lim_{t \to \infty} (\|\partial_t \phi(t)\|_{L^2} + \|\phi(t) - (\phi_+ - H(\cdot - x_1(t)) + H(\cdot - x_2(t)))\|_{H^1}) = 0, \quad (1.7)$$

$$\lim_{t \to \infty} (x_2(t) - x_1(t)) = \infty. \quad (1.8)$$

We remark that if $\phi(t, x)$ is a strongly interacting kink-antikink pair, then

$$E(\phi) = 2E_p(H).$$

Before stating the main theorems we introduce the following explicit constants. Given $U$ as in (1.2), we define,

$$\kappa := \exp \left( \int_0^{\phi_+} \left( \frac{\sqrt{U''(\phi_+)} - 1}{\sqrt{2U(y)}} \phi_+ - y \right) dy \right), \quad (1.9)$$

and

$$A := \sqrt{2\phi_+ \sqrt{U''(\phi_+)}} \|\partial_x H\|_{L^2}^{-1}. \quad (1.10)$$

where $H$ is the static kink solution to (1.1) ($H$ is defined precisely later in (2.5)). With this notation in hand we have:

Theorem 1 (Existence and uniqueness of the strongly interacting kink-antikink pair). There exist a $C^1$ function $x(t)$, a small constant $\epsilon > 0$, a large constant $T_0 > 0$, and a solution $\phi^{(2)}(t, x)$ of (1.1) such that for all $t > T_0$,

$$\|x(t) - (U''(\phi_+))^{-\frac{1}{2}} \log \left( \frac{A t}{\sqrt{2\phi_+}} \right) \|_{L^2} \lesssim t^{-2+\epsilon}, \quad \|x(t) - (U''(\phi_+))^{-\frac{1}{2}} \log \left( \frac{A t}{\sqrt{2\phi_+}} \right) \|_{L^2} \lesssim t^{-2+\epsilon} \quad (1.11)$$

and

$$\| \phi^{(2)}(t) - (\phi_+ - H(\cdot + x(t)) + H(\cdot - x(t)))\|_{H^1} + \|\partial_t \phi^{(2)}(t) + x'(t)(\partial_x H(\cdot + x(t)) + \partial_x H(\cdot - x(t)))\|_{L^2} \lesssim t^{-2+\epsilon}. \quad (1.12)$$

Moreover, $\phi^{(2)}$ is the unique strongly interacting kink-antikink pair up to translation, i.e., if $\phi(t, x)$ is any strongly interacting kink-antikink pair, then there exist $t_0, x_0 \in \mathbb{R}$ so that

$$\phi(t, x) = \phi^{(2)}(t - t_0, x - x_0).$$
Remark 1.2. We expect that the subset of the energy space given by $\mathcal{M} = \{ \phi_{2}(t - t_{0}, x - x_{0}) \mid (t_{0}, x_{0}) \in \mathbb{R} \times \mathbb{R} \}$ is in fact a smooth two dimensional manifold, but we chose not to pursue this issue here.

Remark 1.3. One can observe in (1.11) that the main order term of $x'(t)$, namely $(U''(\phi_{+}))^{-\frac{1}{2}} t^{-1}$, is the time derivative of $(U''(\phi_{+}))^{-\frac{1}{2}} \log(At)$, which is the main order term of $x(t)$. Similarly, in the estimate (1.12) the term $(\partial_{x} H(t + x(t))) + (\partial_{x} H(t - x(t)))$ in the second line is the time derivative of the term $-\left( \phi_{+} - \phi_{-} \right) \cdot \{ H(t + x(t)) + H(t - x(t)) \}$ from the first line. Thus $\phi_{2}(t, x)$ is in fact a strongly interacting kink-antikink pair in the sense of Definition 1.1. Such solutions are discussed in the mathematical physics literature. For instance, [19, Chapter 5.2] contains formal and numerical predictions about the evolution of an initial configuration composed of a stationary kink and anti-kink placed at a large distance. As we make the initial separation tend to infinity, the corresponding solutions converge to strongly interacting kink-antikink pairs.

Note that the uniqueness statement in Theorem 1 is novel even for the completely integrable sine-Gordon equation; see the further discussion of this case below.

Remark 1.4. The sine-Gordon equation (1.3) is a very special case of (1.1) as it is a canonical example of a completely integrable equation and one can write down explicit solutions. An example of a strongly interacting kink-antikink pair is furnished by

$$\phi_{SG,(2)}(t, x) = -4 \arctan \left( \frac{t}{\cosh x} \right),$$

and the family $\mathcal{M}$ of such pairs is given by time and space translations of $\phi_{SG,(2)}$. Note that here we have taken $\phi_{+} = 0$ and $\phi_{-} = -2\pi$. Although in this case $\mathcal{M}$ is explicit, the uniqueness part of our theorem is novel and does not seem to follow directly from the fact that the sine-Gordon equation is completely integrable.

Note that for $t \gg 1$ we have

$$\phi_{SG,(2)}(t, x) \approx -4 \arctan \left( e^{(x+\log 2)t} \right) + 4 \arctan \left( e^{(x-\log 2)t} \right)$$

As expected $\phi_{SG,(2)}$ is for large positive times approximated by the superposition of the sine-Gordon antikink $H(x) = 4 \arctan(e^{-x})$ and the kink $H(x) = 4 \arctanh(e^{-x})$ shifted respectively to $x_{1}(t) = -\log 2t$ and $x_{2}(t) = \log 2t$.

1.2.2. $K$-kink clusters. We also give a construction of $K$-kink clusters, for arbitrary $K \in \mathbb{N}$ in the strongly interacting regime. We begin with a definition.

Definition 1.5. Let $K \geq 1$. We say that a solution $\phi(t, x)$ of (1.1) is a $kink$ $K$-$cluster$ if there exist real-valued functions $x_{j}(t)$ for $j \in \{ 1, 2, \ldots, K \}$ such that

$$\lim_{t \to \infty} \| \partial_{t} \phi(t) \|_{L^{2}} + \| \phi(t) - \left( \phi_{+} + \frac{1}{2} \sum_{j} K^{j} H(\cdot - x_{j}(t)) \right) \|_{H^{1}} = 0,$$

$$\lim_{t \to \infty} (x_{j+1}(t) - x_{j}(t)) = \infty, \quad \text{for } j \in \{ 1, \ldots, K - 1 \}.$$  

For example, the only kink 1-clusters are the antikinks $\phi(t, x) = \phi_{+} + \phi_{-} - H(x - x_{1})$ for some $x_{1} \in \mathbb{R}$. In Theorem 1, we classify all the kink 2-clusters – there is only one, given by $\phi_{2}$, up to translation. For $K \geq 2$ we have the following result.

Theorem 2. Fix $K \geq 1$ and set $c_{j} := \sum_{i=j}^{K} \log i$ for $1 \leq j \leq (K + 1)/2$, and $c_{j} := -c_{K+1-j}$ for $(K + 1)/2 < j \leq K$. There exist $C^{1}$ functions $x_{1}(t), \ldots, x_{K}(t)$, and a small number $\epsilon > 0$ satisfying

$$|x_{j}(t) - (U''(\phi_{+}))^{-\frac{3}{2}} (2j - K - 1) \log(At) + c_{j}| \lesssim t^{-2+\epsilon}, \quad |x'(t) - \left( U''(\phi_{+}) \right)^{-\frac{3}{2}} \frac{2j - K - 1}{t} | \lesssim t^{-3+\epsilon}.$$
and a solution \( \phi_{(K)}(t, x) \) of (1.1) such that

\[
\left\| \phi_{(K)}(t) - \left( \phi_+ + \frac{1 - (-1)^K}{2} \phi_- + \sum_{j=1}^{K} (-1)^j H(\cdot - x_j(t)) \right) \right\|_{H^1} + \left\| \partial_t \phi_{(K)}(t) + \sum_{j=1}^{K} x_j'(t) (-1)^j \partial_x H(\cdot - x_j(t)) \right\|_{L^2} \lesssim t^{-2+\epsilon}.
\]

**Remark 1.6.** We expect that for any \( K \geq 2 \) a result similar to Theorem 1 holds, that is the constructed \( K \)-cluster is unique up to translations in space and time. However, the proof seems quite technical and to keep the paper at reasonable length we do not pursue this issue here.

### 1.3. Further discussion of Theorem 1 and Theorem 2.

#### 1.3.1. Strong vs. weak soliton interactions.

Multi-kinks in the strongly interacting regime considered in Theorems 1 and 2 are threshold solutions in the sense that they have the minimal energy \( E = KE_p(H) \) needed to contain \( K \) distinct kink structures. Alternatively, one could consider solutions that are approximately the superposition of Lorentz boosted kinks and antikinks with nontrivial velocities, which we dub the weakly interacting regime. Any \( K \)-kink solution of the latter type would have nontrivial asymptotic kinetic energy, and thus total energy strictly above \( KE_p(H) \). The weakly interacting regime should be accessible given the existing literature (or via the techniques introduced in this paper), in particular given the landmark works of Merle [26], Martel [20], and Martel, Merle [21], who proved the existence of \( N \)-soliton solutions to g-KDV and NLS with *distinct, nontrivial velocities*; see also Martel, Merle, Tsai [25] and Côte, Martel, Merle [3]. Note that in [20], Martel also established uniqueness of the weakly interacting \( N \)-soliton for each given set of distinct velocities. In the context of nonlinear waves, see the work of Côte, Muñoz [4], who constructed \( N \)-solitons solutions with distinct velocities for nonlinear Klein-Gordon equations. We emphasize a key distinction: in the strongly interacting regime considered here, the dynamics are driven solely by nonlinear interactions between the kinks, whereas in the weakly interacting regime the soliton interactions are negligible to main order.

#### 1.3.2. Kink-antikink collisions.

The solution \( \phi_{(2)} \) in Theorem 1 contains an antikink moving to the left and a kink moving to the right in forward time. Since (1.1) is time-reversible, one may ask what happens when time is run backwards and the kink and antikink structures move towards each other and eventually collide (i.e., the distance \( x_2(t) - x_1(t) \) becomes \( \simeq 1 \)). The folklore conjecture is that whereas soliton collisions are known to be *elastic* for the integrable sine-Gordon equation, collisions should be *inelastic* for equations that are not completely integrable, i.e., for the \( \phi^4 \)-model (1.4) and for the general equation (1.1). Here *inelastic* means that the collision results in some quantum of energy radiating away freely as \( t \to -\infty \).

The threshold solution \( \phi_{(2)} \) is an interesting solution for which to consider the collision problem. Indeed, if any part of the solution breaks off as free radiation after the collision, the fact that it is the minimal energy topologically trivial kink-antikink structure suggests that the entire solution should disperse as \( t \to -\infty \). Such a phenomenon was established by the first and third authors for the minimal energy 2-bubble configuration for the \( k \)-equivariant \( \mathbb{R}^2 \to S^2 \) wave maps problem in [13]. A key ingredient in [13] is a so-called threshold theorem (proved earlier in [2]), which says that any topologically trivial \( k \)-equivariant wave map with energy less than twice the energy of the \( k \)-equivariant harmonic map \( Q \) must disperse freely in both time directions. Once it was shown in [13] (via a no-return analysis) that a solution could not form a minimal 2-bubble in both time directions, the threshold theorem could be applied to conclude that any two bubble in one direction had to completely disperse in the opposite direction, i.e., the collision had to be
completely inelastic. However, an analogous threshold theorem for (1.4) does not seem within reach. Even the small energy problem is extremely challenging given the slow dispersive decay of the 1d Klein-Gordon waves (which appear after linearization about the vacua φ±); see Delort [6] and Hayashi-Naumkin [7, 8] on the modified scattering procedure for NLKG solutions with cubic and quadratic nonlinearities and small, decaying initial data.

Another key difficulty in studying kink collisions is that any analysis must “see” the difference between the elastic collisions for (1.3) and the conjectured inelastic collisions for (1.4). For the sine-Gordon equation (1.3) we can use the explicit formula to find the asymptotic behavior of φSG,(2) when t → −∞. We see that

$$\phi_{SG,(2)}(t, x) \approx 4 \arctan \left( e^{x+\log 2|t|} \right) - 4 \arctan \left( e^{x-\log 2|t|} \right).$$

Hence, backward in time φSG,(2) is approximated by the superposition of the kink 4 arctan(e−x) and the antikink 4 arctan(e−x) shifted respectively to x1(t) = −log 2|t| and x2(t) = log 2|t|. At t ≈ 0 the antikink and the kink collide due to the attracting force between them. Viewing time running forward, before the collision φSG,(2)(t, x) connects the vacuum state 0 with itself through the vacuum state 2π and after the collision the 0 state is connected with itself through the vacuum state −2π. This means that the energy accumulated near the well φ = 2π of the sine-Gordon potential U(φ) = (1 − cos φ) during the collision is transferred as t → ∞ to the well φ = −2π. Up to the symmetry φSG,(2)(t, x) = −φSG,(2)(−t, x) the solution before and after the collision is identical and we see the perfectly elastic collision of the kink and the antikink.

Outside of the integrable setting, the inelastic soliton collision problem is a topic of interest to the physics community. Numerical evidence supports the conjecture that the elastic kink-antikink collision scenario is particular to the sine-Gordon equation. The general heuristic is that a strongly interacting kink-antikink pair arises as a result of a focusing of radiation near the left potential well φ− of U(φ). If the energy accumulated passes the threshold of 2E_p(H) then part of the energy is shifted to the right potential well φ+. This process can be seen as emission of the kink-antikink pair from the radiation. Reversing time it can be seen as a collision scenario in which some of the potential energy of the kink-antikink pair is converted to kinetic energy and is eventually radiated. We stress that Theorem 1 is only valid for $t > T_0$ and no conclusions can be drawn when the fronts of the kink and the antikink are close to one another, in this context see [29].

Global in time, rigorous descriptions of strongly interacting kink-antikink pairs is a challenging open problem. In general, there is very little known about the collision problem. The first mathematical examples of inelastic collisions were given for 2-soliton solutions to the gKDV equation in remarkable works by Martel and Merle [23, 22]. Recently in [24], Martel and Merle gave examples of inelastic collisions for the 5d energy critical nonlinear wave equation showing that some dispersion is produced after collision. The only other result is [13], which was already described above.

1.3.3. Comparison with the Allen-Cahn equation. This work was in part inspired by the construction of the multiple end solutions for the Allen-Cahn equation

$$\Delta u = u^3 - u, \quad \text{in } \mathbb{R}^2,$$

by the second author with del Pino, Pacard, and Wei in [5]. The analog of a strongly interacting kink-antikink pair in this case has approximate form

$$u(x, y) = 1 - H(x - f_1(y)) + H(x - f_2(y)) + \cdots \quad (1.13)$$

where the functions $f_1 \ll f_2$ solve the classical Toda system

$$c_0 f''_1 = -e \sqrt{2} (f_1 - f_2), \quad c_0 f''_2 = e \sqrt{2} (f_1 - f_2),$$
and $c_0 = \frac{\sqrt{2}}{\alpha}$. This problem has, for any $\alpha > 0$, a globally defined solution

$$f_1(y) = -\frac{1}{2^{3/2}} \left[ \log \left( \frac{1}{2} \cosh^2 \left( c_0^{-1/2} \alpha y \right) \right) - \log \alpha^2 \right] = -f_2(y).$$

The distance $z = f_2 - f_1$ between the kink and the antikink is at least $\sqrt{2} \log \frac{1}{\alpha} \gg 1$ for $\alpha \ll 1$ and therefore no collisions occur for this stationary problem (although the Toda system itself has colliding solutions). To explain this we note that $z(y)$ satisfies

$$c_0 z''(y) = 2e^{-\sqrt{2}z(y)},$$

which expresses the fact that the attracting force between the kink and the antikink is balanced by the curvature effect creating a kind of min-max situation on the level of the associated (formal) energy functional. At the same time for the wave equation the distance function between the kink and the antikink $z(t) = 2x(t)$ (in the notation of Theorem 1 with the normalizations $\phi_{\pm} = \pm 1$ and $U''(\phi_{\pm}) = 1$) satisfies formally

$$c_0 z''(t) = -2e^{-z(t)},$$

(see Lemma 3.5 and Lemma 2.8 for the precise formulation). The interaction force in this case remains attractive but its effect is no longer balanced by the kinetic energy in the (formal) action functional and as a consequence $z(t)$ vanishes for some $t < T_0$. In summary: by choosing a suitable solution to the Toda system it is possible to define globally in $\mathbb{R}^2$ an approximation of the solution to the Allen-Cahn equation in the form (1.13), while for the wave equation the analogous approximation is valid only for $t > T_0$ such that $z(t) \gg 1$.

1.4. A summary of the proof. In this section we give a brief outline of the paper, focusing on the proof of Theorem 1.

Section 2 gives a detailed study of the kink solution $H(x)$ and the coercivity properties of the operator obtained by linearization. We establish several technical lemmas, including a computation of the formal attraction force between a well separated kink-antikink pair. This section is technical in nature and can be skimmed on a first reading.

The argument used to prove Theorem 1 is then divided in two parts. First, in Section 3 we give a preliminary dynamical classification of all finite energy strongly interacting kink-antikink pairs. Then, in Section 4 we prove the existence of a kink-antikink pair while also establishing its uniqueness in a certain $t$-weighted function space. The dynamical classification result of Section 3 is then used to show that every strongly interacting kink-antikink pair lies in the function space in which uniqueness was established, thus giving uniqueness in the energy space and finishing the proof of Theorem 1. The structure of this argument, which establishes uniqueness of the multi-kink in addition to its existence, is novel and should be of independent interest. We give a rough sketch of how this works below.

**Part 1:** To establish the preliminary classification we use a scheme similar to the one introduced by the first and third authors to classify all two bubble wave maps in [13], and by the first author to classify strongly interacting two-solitons for gKdV in [10]. We assume that $\phi(t, x)$ is a strongly interacting kink-antikink pair, and without loss of generality that $\phi_{\pm} = \pm 1$. This means that for large enough times, $\phi$ admits a decomposition of the form

$$\phi(t, x) = 1 - H(x - x_1(t)) + H(x - x_2(t)) + g(t, x)$$

satisfying conditions (1.7) and (1.8), or equivalently

$$\lim_{t \to \infty} \|g(t)\|_{H^1 \times L^2(\mathbb{R})} = 0, \quad \lim_{t \to \infty} (x_2(t) - x_1(t)) = \infty. \quad (1.14)$$

The goal is to turn the qualitative assumptions above into quantitative information on the dynamics and decay of $(g(t), \partial_t g(t), x_1(t), x_1(t))$. 7
By standard modulation theoretic arguments, we fix the unique choice of \( x_1(t) \) and \( x_2(t) \) for which \( g(t) \) satisfies the orthogonality conditions
\[
\langle \partial_x H(\cdot - x_1(t)), g(t) \rangle = 0, \quad \langle \partial_x H(\cdot - x_2(t)), g(t) \rangle = 0. \tag{1.15}
\]

Differentiation of the orthogonality conditions, use of the equation satisfied by \( g(t,x) \), and an argument based on the Taylor expansion of the energy are enough to give preliminary estimates on the size of \( |x_j'(t)|, |x_j''(t)| \) and \( \|g(t), \partial_t g(t)\|_{H^1 \times L^2} \) in terms of the distance \( x_2(t) - x_1(t) \) between the kinks. However, as one might expect, these standard arguments are not sufficient to understand the dynamics in a useful way. At this point, we perform an ad hoc change of unknowns, replacing \( x_j'(t) \) with corrected variables \( p_j(t) \). The point is that while the \( p_j(t) \) are small perturbations of \( x_j'(t) \), the correction, which is built using a localized momentum functional, cancels terms of indeterminate sign in the equations for \( x_j''(t) \). We reveal that the dynamics of \( p_j(t) \), and hence of \( x_j'(t) \), are determined, up to negligible error, by the nonlinear interaction force \( F(x_2(t) - x_1(t)) \) between the two kinks; see Lemma 3.5. A study of the ODE satisfied by the \( p_j(t) \) yields bounds on the distance between the kinks, \( \simeq 2 \log t \), as well as decay rates for \( x_j'(t), x_j''(t) \). We remark that the technique of modifying a modulation parameter with a localized functional based on an underlying symmetry was used in a similar context by the first author in [9].

At the conclusion of Section 3, one could rather easily construct a strongly interacting kink-antikink pair. For example, see the construction performed in the recent work of the first and third authors with Rodriguez on singular wave maps in [14, Section 5], which used an analogous preliminary classification of the dynamics to pass to a weak limit of a sequence of well chosen approximations to the desired solution; see also previous work of Rodriguez [28]. However, such constructions fail to establish uniqueness, which is a main goal of this work. To this end, we introduce a new version of Liapunov-Schmidt reduction in the setting of dispersive equations, inspired in part by work of the second author on the 2d elliptic Allen-Cahn problem in [5]. That we can use this philosophy not just to construct but to prove unconditional uniqueness is novel, and relies crucially on the preliminary classification in Section 3.

**Part 2:** By Liapunov-Schmidt reduction, we simply mean that the process of finding the desired solution will be carried out in two steps described below. The implementation of these steps is of course quite different from the elliptic case, as we are here dealing with a nonlinear wave equation.

We assume *a priori* that
\[
\phi(t) = 1 - H(\cdot - x_1(t)) + H(\cdot - x_2(t)) + g(t)
\]
and that (1.14) and (1.15) hold. We project the equation (1.1) onto the space spanned by \( \partial_x H(\cdot - x_j(t)), j = 1, 2 \) and onto its orthogonal complement. This way we are lead to solving the projected equation
\[
\partial_t^2 \phi - \partial_x^2 \phi + U'(\phi) = \lambda_1(t)\partial_x H(\cdot - x_1(t)) + \lambda_2(t)\partial_x H(\cdot - x_2(t)) \tag{1.16}
\]
and what is referred to as the bifurcation equation
\[
\lambda_1(t) = 0, \quad \lambda_2(t) = 0, \tag{1.17}
\]
see for example [1, Section 2.4]. Any \((g(t,x), x_1(t), x_2(t))\) that solves both equations is the desired kink-antikink pair.

**Step 1:** The first step is to solve (1.16) by finding unique \((g(t,x), \lambda_j(t))\), for *given fixed* \( x_j(t) \)'s, within function spaces motivated by the classification result. The core ingredients in this step are energy-type estimates for the linearized equation followed by a contraction mapping argument. Of course the linearized potential is time dependent (the kinks are moving), so a naive definition of the energy functional is not sufficient. We design a modified energy, namely a mixed energylocalized momentum functional, where a local momentum term is added to remove terms of critical size but indeterminate sign after differentiation. The addition of the localized momentum correction term

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*Note: The text contains mathematical expressions that are not entirely rendered clearly in this representation.*
We define in the usual way the Banach spaces $N$ of smooth compactly supported functions $\|z\|_{N} = \sup_{t \in \mathbb{R}} \|z(t)\|_{\mathcal{C}^{0}}$. If the space $E$ is clear from the context, we write $N_{\gamma}$ instead of $N_{\gamma}(E)$ and $S_{\gamma}$ instead of $S_{\gamma}(E)$. We define the usual way the Banach spaces $N_{\gamma}$ and $S_{\gamma}$ as the completion of the space of smooth compactly supported functions $[T_{0}, \infty) \rightarrow E$ for the corresponding norm. Note that if $g$ before imposing orthogonality conditions. Here we pursue an alternative method to obtain the improvement, which consists of a further modification of the energy functional designed to exploit additional decay of the time derivative of the forcing; see Lemma 4.4. Crucial to the entire argument of course, is the design of the function spaces in which the contraction mapping arguments are performed.

By combining Parts 1 and 2 outlined above, the proof of Theorem 1 is completed at the end of Section 4.

The proof of Theorem 2 takes place in Section 5 and mimics the second part of the proof of Theorem 1, namely the Lyapunov-Schmidt part of the argument. We do not obtain uniqueness in the energy space for $K$-kink clusters because we do not perform the preliminary classification analysis from Section 3 in this setting. The immediate obstruction to such analysis is that we do not sufficiently understand the stability of solutions to the formal system satisfied by the modulation parameters.

1.5. Notation. Even if $v(x)$ is a function of one variable $x$, we often write $\partial_{x}v(x)$ instead of $v'(x)$ to denote the derivative. The prime notation is only used for the time derivative of a function of one variable $t$ and for the derivative of the potential $U$.

We now define some function spaces frequently used in the paper. Let $\gamma, \beta, \alpha \in \mathbb{R}$, $T_{0} > 0$ and $z: [T_{0}, \infty) \rightarrow \mathbb{R}$ a continuous function. We set
\[
\|z\|_{N_{\gamma}} := \sup_{t \geq T_{0}} \|t^{\gamma}z(t)\|_{1},
\]
\[
\|z\|_{W_{\alpha,\beta}} := \sup_{\tau \geq t \geq T_{0}} t^{\beta-\alpha} \left| \int_{\tau}^{t} s^{\alpha}z(s) \, ds \right|. \tag{1.18}
\]

If $z$ is twice continuously differentiable, we set
\[
\|z\|_{S_{\gamma}} := \|z\|_{N_{\gamma}} + \|z'\|_{N_{\gamma+1}} + \|z''\|_{N_{\gamma+1}}.
\]

Note that we are using the same time weight for $z'$ and $z''$.

If $z$ is a continuous function from $[T_{0}, \infty)$ to some Banach space $E$, we denote
\[
\|z\|_{N_{\gamma}(E)} := \|t \mapsto \|z(t)\|_{E}\|_{N_{\gamma}}.
\]

If $z$ is twice continuously differentiable function from $[T_{0}, \infty)$ to $E$, we denote
\[
\|z\|_{S_{\gamma}(E)} := \|z\|_{N_{\gamma}(E)} + \|z'\|_{N_{\gamma+1}(E)} + \|z''\|_{N_{\gamma+1}(E)}.
\]

If the space $E$ is clear from the context, we write $N_{\gamma}$ instead of $N_{\gamma}(E)$ and $S_{\gamma}$ instead of $S_{\gamma}(E)$. We define in the usual way the Banach spaces $N_{\gamma}(E)$ and $S_{\gamma}(E)$ as the completion of the space of smooth compactly supported functions $[T_{0}, \infty) \rightarrow E$ for the corresponding norm.
$z \in N_\gamma(E)$, then $z$ is a continuous function from $[T_0, \infty)$ to $E$, and if $z \in S_\gamma(E)$, then $z$ is a twice continuously differentiable function from $[T_0, \infty)$ to $E$.

**Remark 1.7.** We should keep in mind that all these norms depend on $T_0$. Often we can make some constants small by taking $T_0$ large enough. For example, if $\gamma_1 < \gamma_2$ and $c_0 > 0$ is a small constant, then $\| \cdot \|_{N_{\gamma_1}} \leq c_0 \| \cdot \|_{N_{\gamma_2}}$ if $T_0$ is large enough (depending only on $\gamma_1$, $\gamma_2$ and $c_0$). We will use this fact frequently.

We conclude this subsection with some additional notational conventions.

- If $\| \cdot \|_A$ and $\| \cdot \|_B$ are two norms, we denote $\| \cdot \|_{A \cap B} := \max(\| \cdot \|_A, \| \cdot \|_B)$.
- We denote $\mathcal{E} := H^1(\mathbb{R}) \times L^2(\mathbb{R})$ (the energy space).
- For $u, v : \mathbb{R} \to \mathbb{R}$ we write $\langle u, v \rangle := \int_\mathbb{R} u v \, dx$, whenever this expression makes sense.
- We denote $D$ and $D^2$ the first and second Fréchet derivatives of a functional.
- We denote $x_+$ the positive part of $x$, in other words $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ otherwise.
- We take $\chi : \mathbb{R} \to [0, 1]$ to be a decreasing $C^\infty$ function such that $\chi(x) = 1$ for $x \leq \frac{1}{2}$ and $\chi(x) = 0$ for $x \geq \frac{2}{3}$.

## 2. Potential energy and interaction of a kink-antikink pair

In this section, we analyse configurations close to a superposition of a well-separated kink and antikink at a fixed time. We prove coercivity of the potential energy and prove bounds on various interaction terms, which will be used in later sections.

We note that by changing $U(\phi) \mapsto U(\phi/\phi_+)$ without loss of generality we can assume that $\phi_+ = -\phi_- = 1$. Furthermore we may suppose that $U''(-1) = U''(1) = 1$. Indeed, $\phi(t,x)$ solves (1.1) if and only if

$$\tilde{\phi}(t,x) := \phi(t/\sqrt{U''(1)}, x/\sqrt{U''(1)})$$

solves the same equation, but with the potential $U(\phi)$ replaced by $U(\phi)/U''(1)$. For the kink $H$ of the original problem this amounts to

$$\tilde{H}(x) = \phi_+^{-1} H \left( \frac{x}{\sqrt{U''(\phi_+)}} \right).$$

Thus in the rest of this paper we always assume that $\phi_+ = 1$, $\phi_- = -1$ and $U''(-1) = U''(1) = 1$.

### 2.1. Stationary solutions

A stationary field $\phi(t,x) = \psi(x)$ is a solution of (1.1) if and only if

$$\partial_x^2 \psi(x) = U'(\psi(x)), \quad \text{for all } x \in \mathbb{R}.$$  \hfill (2.1)

We seek solutions of (2.1) having finite potential energy $E_p(\psi)$. Since $U(\psi) \geq 0$ for $\psi \in \mathbb{R}$, the condition $E_p(\psi) < \infty$ implies

$$\int_{-\infty}^{\infty} \frac{1}{2} (\partial_x \psi(x))^2 \, dx < \infty, \quad \hfill (2.2)$$

$$\int_{-\infty}^{\infty} U(\psi(x)) \, dx < \infty. \quad \hfill (2.3)$$

From (2.2) we have $\psi \in C(\mathbb{R})$, so (2.1) and $U \in C^\infty(\mathbb{R})$ yield $\psi \in C^\infty(\mathbb{R})$. Multiplying (2.1) by $\partial_x \psi$ we get

$$\partial_x \left( \frac{1}{2} (\partial_x \psi)^2 - U(\psi) \right) = \partial_x (\partial_x^2 \psi - U'(\psi)) = 0,$$

so $\frac{1}{2} (\partial_x \psi(x))^2 - U(\psi(x)) = k$ is a constant. But then (2.2) and (2.3) imply $k = 0$. We obtain the first order Bogomol’ny equations:

$$\partial_x \psi(x) = \sqrt{2U(\psi(x))} \quad \text{or} \quad \partial_x \psi(x) = -\sqrt{2U(\psi(x))}.$$  \hfill (2.4)
We consider the first case, since the second is obtained by changing $x$ to $-x$. If $\psi$ connects the two vacua $-1$ and $1$, then there exists $a \in \mathbb{R}$ such that $\psi(a) = 0$. The solution of (2.4) with this initial condition is $\psi(x) = H(x-a)$, where $H(x)$ is defined by

$$H(x) := G^{-1}(x), \quad \text{with} \ G(\psi) := \int_0^\psi \frac{dy}{\sqrt{2U(y)}}. \quad (2.5)$$

**Proposition 2.1.** The function $H(x)$ defined by (2.5) is of class $C^\infty(\mathbb{R})$ and there exist constants $\kappa > 0$ and $C > 0$ such that for all $x \in \mathbb{R}$

$$|H(x) + 1 - \kappa e^x| + |\partial_x H(x) - \kappa e^x| + |\partial_x^2 H(x) - \kappa e^x| \leq Ce^{2x}, \quad (2.6)$$

$$|H(x) - 1 + \kappa e^{-x}| + |\partial_x H(x) - \kappa e^{-x}| + |\partial_x^2 H(x) + \kappa e^{-x}| \leq Ce^{-2x}. \quad (2.7)$$

**Proof.** We only prove (2.7), which provides the asymptotic behavior of $H(x)$ for $x \to \infty$. The arguments for (2.6) are very similar.

Using the third order Taylor expansion of $U(y)$ around $y = 1$ one obtains $\frac{1}{\sqrt{2U(y)}} - \frac{1}{1-y} = O(1)$ as $y \to 1$, thus

$$G(\psi) = -\log(1-\psi) + \int_0^1 \left( \frac{1}{\sqrt{2U(y)}} - \frac{1}{1-y} \right) dy + O(1-\psi) = -\log \left( \frac{1-\psi}{\kappa} \right) + O(1-\psi),$$

where $\kappa$ is defined in (1.9). Let $1 - H(x) = \kappa e^{-z}$. We obtain

$$z - Ce^{-z} \leq x \leq z + Ce^{-z}.$$ 

This implies in particular $|z - x| \lesssim 1$, and once we know this we get

$$|z - x| \lesssim e^{-z} \lesssim e^{-x},$$

which implies

$$|1 - H(x) - \kappa e^{-x}| = \kappa e^{-x}|e^{x-z} - 1| \lesssim e^{-x}e^{-x} = e^{-2x}.$$ 

The bound for $\partial_x H(x)$ is obtained from (2.4) and the fact that $\sqrt{2U(\psi)} = (1-\psi) + O((1-\psi)^2)$. The bound for $\partial_x^2 H(x)$ is obtained from (2.1) and the fact that $U''(\psi) = (1-\psi) + O((1-\psi)^2)$. \[\square\]

We now compute two constants which will appear later in the proof. We claim that

$$\|\partial_x H\|_{L^2}^2 = \int_{-1}^1 \sqrt{2U(\phi)} \, d\phi, \quad \int_\mathbb{R} \partial_x H(x) \left( U''(H(x)) - U''(1) \right) e^x \, dx = -2\kappa. \quad (2.8)$$

The first formula follows from (2.4) and a change of variable $\phi = H(x)$. The second formula follows from

$$\int_{-\infty}^R (\partial_x H) U''(H(x)) e^x \, dx = \int_{-\infty}^R \partial_x^3 H(x) e^x \, dx$$

$$= e^R (\partial_x^2 H(R) - \partial_x H(R)) + \int_{-\infty}^R \partial_x H(x) e^x \, dx,$$

thus

$$\int_\mathbb{R} \partial_x H(U''(H(x)) - U''(1)) e^x \, dx = \lim_{R \to \infty} e^R (\partial_x^2 H(R) - \partial_x H(R)).$$
2.2. Coercivity. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a state such that \( E_p(\phi) < \infty \), \( \lim_{x \to -\infty} \phi(x) = -1 \) and \( \lim_{x \to +\infty} \phi(x) = 1 \). We have the classical Bogomolny coercivity:

\[
E_p(\phi) = \int_\mathbb{R} \left( \frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right) dx = \int_\mathbb{R} \sqrt{2U(\phi)} \partial_x \phi \, dx + \frac{1}{2} \int_\mathbb{R} (\partial_x \phi - \sqrt{2U(\phi)})^2 dx
\]

In particular,

\[
E_p(H) = \int_{-1}^1 \sqrt{2U(y)} \, dy, \quad E_p(\phi) \geq E_p(H).
\]

We define

\[
L := D^2E_p(H) = -\partial_x^2 + U''(H) = -\partial_x^2 + 1 + (U''(H) - 1).
\]

Observe that \( U''(H) - 1 \) is an exponentially decaying \( C^\infty \) function. Differentiating \( \partial_x^2 H(x - a) = U'(H(x - a)) \) with respect to \( a \) we obtain

\[
(-\partial_x^2 + U''(H(\cdot - a))) \partial_x H(\cdot - a) = 0, \quad (2.9)
\]

in particular for \( a = 0 \) we have \( L(\partial_x H) = 0 \). Differentiating (2.9) with respect to \( a \) at \( a = 0 \) we obtain

\[
L(\partial_x^2 H) = -U'''(H)(\partial_x H)^2. \quad (2.10)
\]

**Proposition 2.2.** The operator \( L \) is self-adjoint with domain \( H^2(\mathbb{R}) \), \( \text{spec}(L) \subset \{0\} \cup \{\lambda, +\infty\} \) for some \( \lambda > 0 \) and \( \ker L = \text{span}(\partial_x H) \).

**Proof.** This is a standard consequence of the Sturm-Liouville theory and the fact that \( \partial_x H(x) > 0 \) for all \( x \in \mathbb{R} \). \( \square \)

**Lemma 2.3.** There exists \( c > 0 \) such that for all \( v \in H^1(\mathbb{R}) \) the following inequality holds:

\[
\langle v, L v \rangle \geq c \|v\|^2_{H^1} - \lambda \|\partial_x H\|_{L^2}^{-2} \langle \partial_x H, v \rangle^2. \quad (2.11)
\]

**Proof.** By the definition of \( L \) we have

\[
\langle v, L v \rangle = \|v\|^2_{H^1} - \int_\mathbb{R} (1 - U''(H))v^2 \, dx,
\]

and by Proposition 2.2 we have

\[
\langle v, L v \rangle \geq \lambda \|v\|^2_{L^2} - \lambda \|\partial_x H\|_{L^2}^{-2} \langle \partial_x H, v \rangle^2.
\]

This implies

\[
\langle v, L v \rangle - c \|v\|^2_{H^1} \geq (1 - c)\lambda \|v\|^2_{L^2} - c \int_\mathbb{R} (1 - U''(H))v^2 \, dx - (1 - c)\lambda \|\partial_x H\|_{L^2}^{-2} \langle \partial_x H, v \rangle^2.
\]

Since \( 1 - U''(H) \) is a bounded function, (2.11) follows by taking \( c \) small. \( \square \)

Given \( X = (x_1, x_2) \) with \( x_1 < x_2 \), we denote

\[
H_j(x) := H(x - x_j), \quad L_X := D^2E_p(1 - H_1 + H_2) = -\partial_x^2 + U''(1 - H_1 + H_2).
\]

**Lemma 2.4.** There exist \( \lambda_0, z_0 > 0 \) such that for all \( X = (x_1, x_2) \) with \( x_2 - x_1 \geq z_0 \) and \( v \in H^1(\mathbb{R}) \)

\[
\langle v, L_X v \rangle \geq \lambda_0 \|v\|^2_{H^1} - \lambda \|\partial_x H\|_{L^2}^{-2} \left( \langle \partial_x H_1, v \rangle^2 + \langle \partial_x H_2, v \rangle^2 \right).
\]
Proof. We set
\[ \chi_1(x) := \chi \left( \frac{x - x_1}{x_2 - x_1} \right), \]
\[ \chi_2(x) := 1 - \chi_1(x), \]
\[ v_1 := \chi_1 v, \]
\[ v_2 := \chi_2 v. \]
We have
\[ \langle v, L_X v \rangle = \langle v_1, L_X v_1 \rangle + \langle v_2, L_X v_2 \rangle + 2 \langle v_1, L_X v_2 \rangle, \]
so it suffices to prove that
\[ \langle v_j, L_X v_j \rangle \geq c \| v_j \|^2_{H^1} - \lambda \| \partial_x H \|_{L^2}^{-2} (\partial_x H_j, v_j)^2 - o(1) \| v \|^2_{H^1}, \]
(2.12)
\[ \langle v_j, L_X v_2 \rangle \geq o(1) \| v \|^2_{H^1}, \]
(2.13)
\[ | (\partial_x H_j, v_j)^2 - (\partial_x H_j, v_j)^2 | \leq o(1) \| v \|^2_{H^1}, \]
(2.14)
where \( c > 0 \) is the constant in (2.11) and \( o(1) \to 0 \) as \( z_0 \to \infty \).

We prove (2.12) for \( j = 1 \) (the proof for \( j = 2 \) is similar). Without loss of generality we can assume \( x_1 = 0 \), so that \( x_2 \geq z_0 \). We then have
\[ L_X = L + V, \quad V := U''(1 - H + H_2) - U''(H). \]
We thus obtain
\[ \langle v_1, L_X v_1 \rangle = \langle v_1, L v_1 \rangle + \langle v_1, V v_1 \rangle \geq c \| v_1 \|^2_{H^1} - \lambda \| \partial_x H \|_{L^2}^{-2} (\partial_x H_1, v_1)^2 + \int \chi_1^2 V v^2 \, dx. \]
Note that \( \| \chi_1^2 V \|_{L^\infty} \ll 1 \). Indeed, if \( x \geq \frac{2}{3} x_2 \) then \( \chi_1(x) = 0 \). If \( x \leq \frac{2}{3} x_2 \), then \( |1 + H_2| \ll 1 \) which implies \( |V(x)| \ll 1 \). This proves (2.12).

Next, we show (2.13). Using the fact that \( \| \partial_x \chi_j \|_{L^\infty} \ll 1 \) we obtain
\[ \langle v_1, L_X v_2 \rangle = \int \partial_x (\chi_1 v) \partial_x (\chi_2 v) \, dx + V \chi_1 \chi_2 v^2 \]
\[ = \int \chi_1 \chi_2 (\partial_x v)^2 \, dx + o(1) \| v \|^2_{H^1} \geq o(1) \| v \|^2_{H^1}, \]
as the first term in the second line is positive.

Finally, the bound (2.14) follows from
\[ | (\partial_x H_1, v_1)^2 - (\partial_x H_1, v_1)^2 | \lesssim (\| v_1 \|_{L^2} + \| v \|_{L^2}) | (\partial_x H_1, v_1 - v) | \]
\[ \lesssim \| v \|_{L^2}^2 \| \chi_2 \partial_x H_1 \|^2_{L^2} \lesssim o(1) \| v \|_{L^2}^2, \]
and similarly for \( j = 2 \).

\[ \square \]

Remark 2.5. In a similar way one could treat a multi-kink case, namely the operator \( L_X := -\partial_x^2 + U''(1_K - H_1 + H_2 - \ldots + (-1)^K H_K) \), with \( 1_K = \frac{1}{2}(1 + (-1)^K), \) \( x_{j+1} - x_j \geq z_0 \gg 1 \) for \( j = 1, \ldots, K - 1 \).

2.3. Interaction of the kinks. Note that \( |1 - H_1| \lesssim e^{-(x-x_1)^+} \) and \( |1 + H_2| \lesssim e^{-(x_2-x)^+} \). The following lemma is often useful while estimating interactions.

Lemma 2.6. For any \( x_1 < x_2 \) and \( \alpha, \beta > 0 \) with \( \alpha \neq \beta \) the following bound holds:
\[ \int e^{-\alpha(x-x_1)+} e^{-\beta(x_2-x)+} \, dx \lesssim_{\alpha, \beta} e^{-\min(\alpha, \beta)(x_2-x_1)}, \quad \forall x_1, x_2 \in \mathbb{R}. \]
For any $\alpha > 0$, the following bound holds:

$$\int_{\mathbb{R}} e^{-\alpha(x-x_1)+} e^{-\alpha(x_2-x)+} \, dx \lesssim_\alpha (x_2 - x_1) e^{-\alpha(x_2-x_1)}, \quad \forall x_1, x_2 \in \mathbb{R}.$$  

Proof. Straightforward computation. \hfill \square

To measure the interaction between the kinks located at $x_1, x_2 \in \mathbb{R}$, we introduce a function $\Phi \in C^\infty(\mathbb{R}^3)$ by

$$\Phi(x_1, x_2, x) := -U'(1 - H_1(x) + H_2(x)) - U'(H_1(x)) + U'(H_2(x)). \quad (2.15)$$

Observe that

$$DE_j(1 - H_1 + H_2) = -\partial_2^2(1 - H_1 + H_2) + U'(1 - H_1 + H_2) = -\Phi(x_1, x_2, \cdot).$$

Lemma 2.7. There exists $C > 0$ (depending only on $U$) such that for all $x_1, x_2, x \in \mathbb{R}$ with $x_2 - x_1 \geq 1$ the following inequalities are true for all $j, k \in \{1, 2\}$:

$$|\Phi(x_1, x_2, x)| + |\partial_x \Phi(x_1, x_2, x)| \leq Ce^{-(x-x_1)+} e^{-(x_2-x)+}, \quad (2.16)$$

$$|\partial^2_x \Phi(x_1, x_2, x)| \leq Ce^{-(x-x_1)+} e^{-(x_2-x)+}, \quad (2.17)$$

$$|\partial_x H_1(x) \partial_x H_2(x)| \leq Ce^{-(x-x_1)+} e^{-(x_2-x)+}. \quad (2.18)$$

Proof. For $w_1, w_2 \in \mathbb{R}$ set

$$f(w_1, w_2) := -U'(1 - w_1 + w_2) - U'(w_1) + U'(w_2).$$

Then $\Phi(x_1, x_2, x) = f(H_1(x), H_2(x))$ so, by the Chain Rule and the fact that $\partial_{x_j} H_j = -\partial_x H_j$,

$$\partial_{x_j} \Phi(x_1, x_2, x) = -\partial_x H_j(x) \partial_{w_j} f(H_1(x), H_2(x)), \quad (2.19)$$

$$\partial^2_{x_j} \Phi(x_1, x_2, x) = \partial^2_x H_j(x) \partial_{w_j} f(H_1(x), H_2(x)) + (\partial_x H_j(x))^2 \partial^2_{w_j} f(H_1(x), H_2(x)),$$

$$\partial_{x_1} \partial_{x_2} \Phi(x_1, x_2, x) = \partial_x H_1(x) \partial_x H_2(x) \partial_{w_1} \partial_{w_2} f(H_1(x), H_2(x)).$$

Since $|1 - H_1| + |\partial_x H_1| + |\partial^2_x H_1| \lesssim e^{-(x-x_1)+}$ and $|1 + H_2| + |\partial_x H_2| + |\partial^2_x H_2| \lesssim e^{-(x_2-x)+}$, in order to prove (2.16) it suffices to check that for $-1 \leq w_1, w_2 \leq 1$

$$|f(w_1, w_2)| \lesssim |(1 - w_1)(1 + w_2)|, \quad (2.20)$$

$$|\partial_{w_1} f(w_1, w_2)| \lesssim |1 + w_2|, \quad (2.21)$$

$$|\partial_{w_2} f(w_1, w_2)| \lesssim |1 - w_1|, \quad (2.22)$$

$$|\partial^2_{w_1} f(w_1, w_2)| \lesssim |1 + w_2|, \quad (2.23)$$

$$|\partial^2_{w_2} f(w_1, w_2)| \lesssim |1 - w_1|, \quad (2.24)$$

$$|\partial_{w_1} \partial_{w_2} f(w_1, w_2)\| \lesssim 1. \quad (2.25)$$

We have $\partial_{w_1} f(w_1, w_2) = U''(1 - w_1 + w_2) - U''(w_1) = U''(1 - w_1 + w_2) - U''(w_1)$. Since $U''$ is locally Lipschitz, (2.21) follows. The bound (2.22) is similar, and (2.23) and (2.24) are proved similarly, using that fact that $U''$ is locally Lipschitz. We have $\partial_{w_1} \partial_{w_2} f(w_1, w_2) = U''(1 - w_1 + w_2)$, so (2.25) follows. Finally, in order to prove (2.20) we notice that

$$f(w_1, w_2) = f(1, w_2) - \int_{w_1}^1 \partial_{w_1} f(w, w_2) \, dw = - \int_{w_1}^1 \partial_{w_1} f(w, w_2) \, dw$$

and we conclude using (2.21).

Bound (2.17) follows easily from $U''$ being locally Lipschitz, and (2.18) is clear. \hfill \square
We will often denote \( z = x_2 - x_1 \) the distance between the kinks. We introduce the following function, which is the (renormalised) formally computed attraction force between a kink and an antikink at distance \( z \):

\[
F(z) := \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H, \Phi(0, z, \cdot) \rangle.
\]  

(2.26)

For future reference we note that

\[
F(x_2 - x_1) = \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H_1(\cdot), \Phi(x_1, x_2, \cdot) \rangle = \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H_2(\cdot), \Phi(x_1, x_2, \cdot) \rangle.
\]

Lemma 2.8. There exists \( C > 0 \) such that for all \( z \geq 1 \) the function \( F(z) \) satisfies

\[
|F(z) - 2\| \partial_x H \|_{L^2}^{-2} \kappa^2 e^{-z}| \leq Cze^{-2z},
\]  

(2.27)

\[
|F'(z) + 2\| \partial_x H \|_{L^2}^{-2} \kappa^2 e^{-z}| \leq Cze^{-2z},
\]  

(2.28)

where \( \kappa \) is defined by (1.9).

Proof. Lemma 2.7 implies \( \lim_{z \to \infty} F(z) = 0 \), hence (2.27) follows by integrating (2.28).

We now prove (2.28). Denote \( \tilde{H}(x) := H(x-z) \). Using the notations from Lemma 2.7 and (2.19), we have

\[
F'(z) = \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H, \partial_x \Phi(0, z, \cdot) \rangle = -\| \partial_x H \|_{L^2}^{-2} \int_\mathbb{R} \partial_x H(x) \partial_x \tilde{H}(x) \partial \omega f(H(x), \tilde{H}(x)) dx.
\]

In the computation which follows the symbol “\( \lesssim \)” means “up to terms of order ze\(^{-2z}\).”

The fundamental theorem of calculus together with (2.24) yields

\[
\left| -\partial \omega f(H, \tilde{H}) - (U''(H) - U''(1)) \right| = \left| \partial \omega f(H, \tilde{H}) - \partial \omega f(H, -1) \right| \lesssim |(1 + \tilde{H})(1 - H)|.
\]

Using \( |\partial_x H| + |1 - H| \lesssim e^{-x+} \) and \( |\partial_x \tilde{H}| + |1 + \tilde{H}| \lesssim e^{-(z-x)_+} \), we obtain

\[
\left| F'(z) + 2\| \partial_x H \|_{L^2}^{-2} \int_\mathbb{R} (\partial_x H)(\partial_x \tilde{H})(U''(H) - U''(1)) dx \right| \lesssim e^{-2x+} e^{-2(z-x)_+}.
\]

Lemma 2.6 yields

\[
F'(z) \lesssim \| \partial_x H \|_{L^2}^{-2} \int_\mathbb{R} (\partial_x H)(\partial_x \tilde{H})(U''(H) - U''(1)) dx.
\]

The function \( U'' \) is locally Lipschitz, thus \( |U''(H) - U''(1)| \lesssim |1 - H| \). We also have, by Proposition 2.1,

\[
|\partial_x \tilde{H}(x) - \kappa e^{x-z}| \lesssim \min(e^{2(x-z)}, e^{x-z}).
\]

Since

\[
\int_\mathbb{R} |(\partial_x H)(1 - H)| \min(e^{2(x-z)}, e^{x-z}) dx \lesssim \int_{-\infty}^z e^{-2x+} e^{-2(z-x)_+} dx + \int_z^{\infty} e^{-2x} e^{x-z} dx \lesssim ze^{-2z},
\]

we conclude that

\[
F'(z) \lesssim \kappa \| \partial_x H \|_{L^2}^{-2} e^{-z} \int_\mathbb{R} (\partial_x H)(U''(H) - U''(1)) e^x dx \lesssim -2\kappa^2 \| \partial_x H \|_{L^2}^{-2} e^{-z},
\]

where in the last step we use (2.8).

\[ \square \]

Remark 2.9. Let \( x_2 - x_1 \gg 1 \), \( H_1 := H(x-x_1) \) and \( H_2 := H(x-x_2) \). Then, by translation invariance and symmetry, it follows from the last lemma that

\[
\langle \partial_x H_1, \Phi(x_1, x_2, \cdot) \rangle = \langle \partial_x H_2, \Phi(x_1, x_2, \cdot) \rangle = 2\kappa^2 e^{-(x_2-x_1)} + O((x_2-x_1)e^{-2(x_2-x_1)}).
\]

In the next lemma, we compute the potential energy of a kink-antikink configuration \( \phi(x) = 1 - H(x-x_1) + H(x-x_2) \).
Lemma 2.10. For any $\epsilon > 0$ there exists $C > 0$ such that if $x_2 - x_1 \gg 1$ and $H_j(x) := H(x - x_j)$, then
\[
|E_p(1 - H_1 + H_2) - (2E_p(H) - 2\kappa^2 e^{-(x_2-x_1)})| \leq C(x_2 - x_1)e^{-2(x_2-x_1)},
\] (2.29)
where $\kappa$ is defined by (1.9).

Proof. Without loss of generality assume $x_1 = 0$ and $x_2 = z \gg 1$. Observe that
\[
\frac{d}{dz} \int_R \partial_x H \partial_z H \cdot (-z) \, dx = - \int_R \partial_x H \partial_z^2 H \cdot (-z) \, dx,
\]
\[
\frac{d}{dz} \int_R U(1 - H + H(z)) \, dx = - \int_R U(1 - H + H(z)) \partial_z H \cdot (-z) \, dx,
\]
\[
\frac{d}{dz} E_p(1 - H + H(z)) = - \int_R \left( U'(1 - H + H(z)) - U'(H(z)) \right) \partial_z H \, dx.
\]
By symmetry, we obtain
\[
\frac{d}{dz} E_p(1 - H + H(z)) = \int_R \left( U'(1 - H + H(z)) - U'(H(z)) \right) \partial_z H \, dx
\]
\[
= \int_R \left( U'(1 - H + H(z)) + U'(H) - U'(H(z)) \right) \partial_z H \, dx
\]
\[
= -\| \partial_z H \|_{L^2}^2 F(z),
\]
where $F(z)$ is defined by (2.26). It remains to check that
\[
\lim_{z \to \infty} E_p(1 - H + H(z)) = 2E_p(H),
\] (2.30)
and (2.29) will follow by integrating (2.27) in $z$.

Let $-1 \leq w_1, w_2 \leq 1$. Integrating the inequality $|U'(1 - w + w_2) + U'(w)| \lesssim |1 + w_2|$ for $w_1 \leq w \leq 1$, we get $|U(1 - w_1 + w_2) - U(w_1) - U(w_2)| \lesssim |(1 - w_1)(1 + w_2)|$, thus
\[
|U(1 - H(x) + H(x - z)) - U(H(x)) - U(H(x - z))| \lesssim e^{-x} + e^{-(z-x)}.
\] (2.31)

We also have
\[
\left| \left( \partial_x (1 - H(x) + H(x - z)) \right)^2 - \left( \partial_x H(x) \right)^2 - \left( \partial_x H(x - z) \right)^2 \right|
\lesssim \| \partial_x H(x) \partial_x H(z - x) \| \lesssim e^{-x} + e^{-(z-x)},
\] (2.32)
and (2.30) follows by integrating (2.31) and (2.32) in $x$. \qed

2.4. Taylor expansions. To finish this section, we prove a few estimates based on the Taylor expansion of the nonlinearity.

Lemma 2.11. There exists $C > 0$ such that for all $-1 \leq w, w^2 \leq 1$ and $|g| + |g^2| \ll 1$ the following bounds hold:
\[
|U(w + g) - U(w) - U'(w)g| \leq Cg^2,
\] (2.33)
\[
|U(w + g) - U(w) - U'(w)g - U''(w)g^2/2| \leq Cg^3,
\] (2.34)
\[
|U'(w + g) - U'(w) - U''(w)g| \leq Cg^2,
\] (2.35)
\[
|U''(w + g) - U''(w) - U'''(w)g| \leq Cg^2,
\] (2.36)
\[
\left| \left( U'(w + g^2) - U'(w^2) - U''(w^2)g^2 \right) - \left( U'(w + g) - U'(w) - U''(w)g \right) \right|
\leq C(|g^2| + |g|)(|g^2 - g| + |w^2 - w||g^2| + |g|).
\] (2.37)
Proof. Bounds (2.33), (2.34), (2.35) and (2.36) are clear, so we are left with (2.37). The Taylor formula yields
\[
U'(w + g) - U'(w) - U''(w)g = \int_0^g sU'''(w + g - s) \, ds,
\]
\[
U'(w^2 + g^2) - U'(w^2) - U''(w^2)g^2 = \int_0^{g^2} sU'''(w^2 + g^2 - s) \, ds.
\]
We observe that, since \( U''' \) is locally Lipschitz,
\[
\left| \int_0^{g^2} sU'''(w^2 + g^2 - s) \, ds - \int_0^{g^2} sU'''(w + g - s) \, ds \right| \lesssim \left| \int_0^{g^2} s|w^2 + g^2 - w - g| \, ds \right| \lesssim (g^2)^2(|w^2 - w| + |g^2 - g|).
\]
We also have
\[
\left| \int_0^{g^2} sU'''(w + g - s) \, ds - \int_0^{g^2} sU'''(w + g - s) \, ds \right| = \left| \int_g^{g^2} sU'''(w + g - s) \, ds \right| \lesssim \int_g^{g^2} |s| \, ds.
\]
If \( g \) and \( g^2 \) have the same sign, then the last integral equals \( \frac{1}{2}((g^2)^2 - g^2) = \frac{1}{2}(|g|)g^2 - g \). If \( g \) and \( g^2 \) have opposite signs, then we obtain \( \frac{1}{2}((g^2)^2 + g^2) \leq \frac{1}{2}(|g|)g^2 - g \).
This proves (2.37).

3. MAIN ORDER ASYMPTOTICS OF ANY KINK-ANTI KINK PAIR

We consider any solution \( \phi \) of (1.1) of the form
\[
\phi(t, x) = 1 - H(x - x_1(t)) + H(x - x_2(t)) + g(t, x),
\]
satisfying conditions (1.7) and (1.8), equivalently
\[
\lim_{t \to \infty} \|(g(t), \partial_x \phi(t))\|_\varepsilon = 0, \quad \lim_{t \to \infty} (x_2(t) - x_1(t)) = \infty.
\]

Lemma 3.1. Under the conditions (3.2) and (3.3), there exists a pair of twice continuously differentiable functions \((\bar{x}_1, \bar{x}_2) : [T_0, \infty) \to \mathbb{R}\) such that the following holds. Define \( \bar{g} : [T_0, \infty) \to H^1(\mathbb{R}) \) by the relation \( \phi(t, x) = 1 - H(x - \bar{x}_1(t)) + H(x - \bar{x}_2(t)) + \bar{g}(t, x) \). Then (3.2) and (3.3) hold with \((x_1, x_2, g)\) replaced by \((\bar{x}_1, \bar{x}_2, \bar{g})\) and, moreover, \( \bar{g}(t) \) satisfies the orthogonality conditions
\[
\langle \partial_x H(\cdot - \bar{x}_1(t)), \bar{g}(t) \rangle = 0, \quad \langle \partial_x H(\cdot - \bar{x}_2(t)), \bar{g}(t) \rangle = 0.
\]

Proof. The proof follows a well-known scheme based on a quantitative version of the Implicit Function Theorem, see for instance [11, Lemma 3.3].

Step 1. (Choice of parameters for a fixed time.) Fix fix \( t \) and write \((x_1, x_2, g, \phi)\) instead of \((x_1(t), x_2(t), g(t), \phi(t))\). We prove that there exists \( C_0, \eta_0 > 0 \) having the following property. If \( \phi(x) = 1 - H(x - x_1) + H(x - x_2) + g(x) \) with \( x_2 - x_1 \geq \eta^{-1}, \|g\|_{H^1} \leq \eta \) and \( \eta \leq \eta_0 \), then there exists a unique pair \((\bar{x}_1, \bar{x}_2)\) such that \( \bar{g}(x) := \phi(x) - (1 - H(x - \bar{x}_1) + H(x - \bar{x}_2)) \) satisfies \( \bar{x}_2 - \bar{x}_1 \geq (C\eta)^{-1}, \|\bar{g}\|_{H^1} \leq C\eta \) and the orthogonality conditions
\[
\langle \partial_x H(\cdot - \bar{x}_1), \bar{g} \rangle = 0, \quad \langle \partial_x H(\cdot - \bar{x}_2), \bar{g} \rangle = 0.
\]

We define \( \Gamma : \mathbb{R}^2 \times H^1(\mathbb{R}) \to \mathbb{R}^2 \) by
\[
\Gamma(x_1, x_2, \phi) := (\langle \partial_x H(\cdot - x_1), \phi - (1 - H(\cdot - x_1) + H(\cdot - x_2)) \rangle, \langle \partial_x H(\cdot - x_2), \phi - (1 - H(\cdot - x_1) + H(\cdot - x_2)) \rangle).
\]
Lemma 3.3. It is easy to check that $D_{x_1,x_2} \Gamma$ is a uniformly non-degenerate matrix, which implies the claim.

**Step 2.** (Time derivability of the modulation parameters.) Thus, if $T_0$ is large enough, by Step 1, there exist $\bar{x}_1(T_0)$ and $\bar{x}_2(T_0)$ such that (3.4) holds for $t = T_0$. We now define $(\bar{x}_1, \bar{x}_2)$ as the solution of the system of differential equations (with initial conditions at $t = T_0$)

$$\bar{x}'_1(t)(-\|\partial_x H\|_{L^2}^2 - \langle \partial^2_x \bar{H}_1(t), \bar{g}(t) \rangle) + \bar{x}'_2(t)(\|\partial_x H\|_{L^2}^2 - \langle \partial^2_x \bar{H}_2(t), \bar{g}(t) \rangle) + \bar{x}'_1(t)(\partial_x \bar{H}_1(t), \partial_x \bar{H}_2(t)) + \bar{x}'_2(t)(\partial_x \bar{H}_1(t), \partial_x \bar{H}_2(t)) = 0,$$

where we abbreviated $\bar{H}_j(t,x) := H(x - \bar{x}_j(t))$ and $\bar{g}(t) := \phi(t) - (1 - \bar{H}_1(t) + \bar{H}_2(t))$. The computation at the beginning of Lemma 3.3 below shows that (3.4) then holds for all $t \geq T_0$. By a straightforward bootstrap argument and using the uniqueness part of Step 1, $(\bar{x}_1, \bar{x}_2)$ satisfy $\bar{x}_2(t) - \bar{x}_1(t) \to \infty$ and $\|\bar{g}(t)\| \to 0$. Also, we deduce from the differential equations that $\bar{x}_1$ and $\bar{x}_2$ are twice continuously differentiable.

In the sequel, we write $(x_1, x_2, g)$ instead of $(\bar{x}_1, \bar{x}_2, \bar{g})$. In other words, we have (3.1), (3.2), (3.3) and, additionally, $x_1, x_2$ are twice continuously differentiable and satisfy

$$\langle \partial_x H(\cdot - x_1(t)), g(t) \rangle = 0, \quad \langle \partial_x H(\cdot - x_2(t)), g(t) \rangle = 0. \tag{3.5}$$

**Remark 3.2.** Passing from $\phi$ to the triple $(x_1, x_2, g)$ defines a diffeomorphism of a neighborhood of the kink-antikink pairs and a codimension two submanifold of $\mathbb{R}^2 \times H^1(\mathbb{R})$ determined by the conditions (3.5). Note that $(x_1, x_2, g)$ is not a system of coordinates; informally speaking, $g(t,x)$ and $(x_1(t), x_2(t))$ are “not independent”. This causes some trouble when one wants to compare two solutions corresponding to two different pairs of trajectories $X = (x_1, x_2)$ and $X^\sharp = (x_1^\sharp, x_2^\sharp)$, see Lemma 4.4 below.

Writing $H_j(t,x) := H(x - x_j(t))$, we have

$$\partial_t \phi(t) = x'_1(t)\partial_x H_1(t) - x'_2(t)\partial_x H_2(t) + \partial_t g(t), \tag{3.6}$$

thus (1.1) rewrites as

$$\partial^2_t g + x''_1\partial_x H_1 - (x'_1)^2 \partial^2_x H_1 - x''_2\partial_x H_2 + (x'_2)^2 \partial^2_x H_2,$$

$$+ \partial^2_x H_1 - \partial^2_x H_2 - \partial^2_x g + U'(1 - H_1 + H_2 + g) = 0,$$

or, using $\partial^2_x H_j = U'(H_j)$,

$$\partial^2_t g + x''_1\partial_x H_1 - (x'_1)^2 \partial^2_x H_1 - x''_2\partial_x H_2 + (x'_2)^2 \partial^2_x H_2,$$

$$- \partial^2_x g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) = 0. \tag{3.7}$$

**Lemma 3.3.** If $\phi$ is a solution of (1.1) such that (3.2) and (3.3) hold, then there exist $C_0$ and $T_0$ such that for all $t \geq T_0$ the following bounds hold:

$$\|g(t)\|_{H^1}^2 + \|\partial_x \phi(t)\|_{L^2}^2 \leq C_0 e^{-(x_2(t) - x_1(t))}, \tag{3.8}$$

$$|x'_1(t)| + |x'_2(t)| \leq C_0 e^{-\frac{1}{2}(x_2(t) - x_1(t))}, \tag{3.9}$$

$$|x''_1(t)| + |x''_2(t)| \leq C_0 e^{-(x_2(t) - x_1(t))}. \tag{3.10}$$
Proof. Differentiating in $t$ the first relation in (3.5), we obtain

$$0 = \frac{d}{dt} \langle \partial_x H_1(t), g(t) \rangle = -x'_1(t) \langle \partial^2_x H_1(t), g(t) \rangle + \langle \partial_x H_1(t), \partial_t g(t) \rangle$$

$$= -x'_1(t) \langle \partial^2_x H_1(t), g(t) \rangle + \langle \partial_x H_1(t), -x'_1(t) \partial_x H_1(t) + x'_2(t) \partial_x H_2(t) + \partial_t \phi(t) \rangle$$

$$= x'_1(t) (-\| \partial_x H \|_{L^2}^2 - \langle \partial^2_x H_1(t), g(t) \rangle) + x'_2(t) \langle \partial_x H_1(t), \partial_x H_2(t) \rangle + \langle \partial_x H_1, \partial_t \phi(t) \rangle. \quad (3.11)$$

Similarly, differentiating in $t$ the second relation in (3.5) yields

$$0 = x'_2(t) (\| \partial_x H \|_{L^2}^2 - \langle \partial^2_x H_2(t), g(t) \rangle) - x'_1(t) \langle \partial_x H_1(t), \partial_x H_2(t) \rangle + \langle \partial_x H_2(t), \partial_t \phi(t) \rangle. \quad (3.12)$$

This can be viewed as a linear system for $x'_1(t)$ and $x'_2(t)$. Note that $\lim_{t \to \infty} |\langle \partial_x H_1(t), \partial_x H_2(t) \rangle| = 0$ by (2.18), so the matrix of the system is diagonally dominant. In particular, we obtain

$$|x'_1(t)| + |x'_2(t)| \lesssim \| \partial_t \phi \|_{L^2}. \quad (3.13)$$

Observe that (3.6) and (3.13) yield

$$\| \partial_t g(t) \|_{L^2} \lesssim \| \partial_t \phi(t) \|_{L^2}. \quad (3.14)$$

In order to prove (3.8), we observe that (3.2) and (3.3) imply

$$E(\phi(t), \partial_t \phi(t)) = 2E_p(H). \quad (3.15)$$

Indeed, applying Cauchy-Schwarz we have

$$\left| \int \langle \partial_x (1 - H_1(t) + H_2(t) + g(t)) \rangle dx - \int \langle \partial_x (1 - H_1(t) + H_2(t)) \rangle dx \right|$$

$$\lesssim \| \partial_x g(t) \|_{L^2} \| \partial_x (1 - H_1(t) + H_2(t)) \|_{L^2} + \| \partial_x g(t) \|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (3.16)$$

Plugging $w = 1 - H_1 + H_2$ in (2.33) and integrating in $x$ we obtain

$$\left| \int U(1 - H_1(t) + H_2(t) + g(t)) dx - \int U(1 - H_1(t) + H_2(t)) dx \right|$$

$$\lesssim \| g \|_{L^2} \| U'(1 - H_1(t) + H_2(t)) \|_{L^2} + \| g(t) \|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty,$n

where in the last step we use boundedness of $\| U'(1 - H_1(t) + H_2(t)) \|_{L^2}$, easy to justify by (2.16). Since $\| \partial_t \phi(t) \|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$, from the last two estimates and (2.30) we deduce

$$E(\phi(t), \partial_t \phi(t)) = \lim_{t \to \infty} E(1 - H_1(t) + H_2(t) + g(t), \partial_t \phi(t)) = \lim_{t \to \infty} E_p(1 - H_1(t) + H_2(t)) = 2E_p(H).$$

Plugging $w = 1 - H_1(t) + H_2(t)$ in (2.34) we obtain

$$E_p(\phi(t)) = E_p(1 - H_1(t) + H_2(t)) + \langle \partial E_p(1 - H_1(t) + H_2(t)), g \rangle$$

$$+ \frac{1}{2} \langle g(t), D^2 E_p(1 - H_1(t) + H_2(t)) \rangle + O(\| g \|^3_{L^2}).$$

Applying Lemma 2.4 we get

$$\| g \|^2_{L^1} + \| \partial_t \phi \|^2_{L^2} \lesssim E(\phi(t), \partial_t \phi(t)) - E_p(1 - H_1(t) + H_2(t))$$

$$+ \| \partial E_p(1 - H_1(t) + H_2(t)) \|_{L^2} \| g \|_{L^2}. \quad (3.16)$$

By Lemma 2.10 and (3.15), the right hand side is bounded up to a constant by

$$e^{-(x_2 - x_1)} + \| \partial E_p(1 - H_1(t) + H_2(t)) \|_{L^2} \| g \|_{L^2}.$$n

From (2.16) and Lemma 2.6 we have $\| \partial E_p(1 - H_1(t) + H_2(t)) \|_{L^2} \lesssim \sqrt{x_2 - x_1} e^{-(x_2 - x_1)}$, hence (3.8) follows.

Bound (3.9) follows from (3.13) and (3.8).
In order to prove (3.10), we differentiate (3.11) and (3.12) in time. For example, from (3.9) we obtain,

\[
0 = x''_1(t)(-\|\partial_x H\|_{L^2}^2 - \langle \partial_x^2 H_1(t), g(t) \rangle) + (x'_1(t))^2 \langle \partial_x^2 H_1(t), g(t) \rangle - x'_1(t) \langle \partial_x^2 H_1(t), \partial_t g(t) \rangle \\
+ x''_2(t) \langle \partial_x H_1(t), \partial_x H_2(t) \rangle - x'_1(t)x'_2(t) \langle \partial_x^2 H_1(t), \partial_x H_2(t) \rangle - (x'_2(t))^2 \langle \partial_x H_1(t), \partial_x^2 H_2(t) \rangle \\
- x'_1(t) \langle \partial_x H_1(t), \partial_t \phi(t) \rangle + \langle \partial_x H_1(t), \partial_t^2 \phi(t) \rangle
\]

Rearranging, and using (3.6) we obtain,

\[
x''_1(t)(-\|\partial_x H\|_{L^2}^2 - \langle \partial_x^2 H_1(t), g(t) \rangle) + x''_2(t) \langle \partial_x H_1(t), \partial_x H_2(t) \rangle \\
= (x'_1(t))^2 \langle \partial_x^2 H_1(t), g(t) \rangle - x'_1(t) \langle \partial_x^2 H_1(t), \partial_t \phi(t) \rangle + (x'_2(t))^2 \langle \partial_x^2 H_1(t), \partial_x H_1(t) \rangle \\
- 2x'_1(t)x'_2(t) \langle \partial_x^2 H_1(t), \partial_x H_2(t) \rangle - (x'_2(t))^2 \langle \partial_x H_1(t), \partial_x^2 H_2(t) \rangle - x'_1(t) \langle \partial_x H_1(t), \partial_t \phi(t) \rangle \\
+ \langle \partial_x H_1(t), \partial_t^2 \phi(t) \rangle
\]

After similarly differentiating (3.12) it is clear that we obtain again a diagonally dominant linear system for \(x''_1(t)\) and \(x''_2(t)\). Almost all of the terms on the right-hand side are easily seen to be \(\lesssim e^{-(x_2(t) - x_1(t))}\), because they are at least quadratic with respect to \((g, \partial_t \phi, x'_1, x'_2)\). The only potentially problematic term is

\[
\langle \partial_x H_1(t), \partial_t^2 \phi(t) \rangle = \langle \partial_x H_1(t), \partial_t^2 \phi(t) - U'(\phi(t)) \rangle.
\]

(3.17)

However, this term is also \(\lesssim e^{-(x_2(t) - x_1(t))}\), due to the fact that \(\partial_x H \in \ker(-\partial_x^2 + U''(H))\). Indeed, we have

\[
\partial_t^2 \phi(t) - U'(\phi(t)) = \partial_t^2 g - U''(H_1)g + \Phi(x_1, x_2, \cdot) - \left(U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(H_1)g\right),
\]

and we observe that

\[
\langle \partial_x H_1(t), \partial_t^2 g - U''(H_1)g \rangle = 0,
\]

\[
\int_{\mathbb{R}} |\partial_x H_1(x)\Phi(x_1, x_2, x)| \, dx \lesssim \int_{\mathbb{R}} e^{-|x-x_1|} e^{-(x_2-x_1)} + e^{-(x_2-x)} \, dx \lesssim e^{-(x_2-x_1)},
\]

so we are left with the last term. From (2.35) we have

\[
\|U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(1 - H_1 + H_2)g\|_{L^2} \lesssim \|g\|_{L^2} \lesssim e^{-(x_2-x_1)}.
\]

From (2.17) it follows that

\[
\int_{\mathbb{R}} |\partial_x H_1| |U''(1 - H_1 + H_2) - U''(H_1)| |g| \, dx \lesssim \|g\|_{L^2} \|\partial_x H_1\| \|U''(1 - H_1 + H_2) - U''(H_1)\|_{L^2} \lesssim \|g\|_{L^2} \sqrt{x_2 - x_1} e^{-(x_2-x_1)} \ll e^{-(x_2-x_1)}.
\]

Combining the two estimates yields the conclusion. \(\square\)

The rest of this section closely follows the corresponding arguments in [10].

Lemma 3.4. For any \(M > 0\) there exists a constant \(C > 0\) such that for any functions \(\tilde{x}, w, g\) such that \(\|w\|_{H^1} \leq M, \|\tilde{x}\|_{W^{1,\infty}} < \infty\) and \(\|g\|_{H^1} \leq 1\) the following inequality is true:

\[
\left| \int_{\mathbb{R}} \tilde{x} \partial_x g(U'(w + g) - U'(w)) \, dx + \int_{\mathbb{R}} \tilde{x} \partial_x w(U'(w + g) - U'(w) - U''(w)g) \, dx \right| \leq C \|\partial_x \tilde{x}\|_{L^\infty} \|g\|_{H^1}^2.
\]

(3.18)
Proof. By the standard approximation procedure, we can assume that \( w, g, \bar{\chi} \in C^0(\mathbb{R}) \).

Consider the first line in (3.18). Rearranging the terms, we obtain
\[
\int_{\mathbb{R}} \left( \partial_x g(U'(w + g) - U'(w)) + \partial_x w(U'(w + g) - U'(w)) - U''(w)g \right) \, dx
= \int_{\mathbb{R}} \left( \partial_x(w + g)U'(w + g) - \partial_x w U'(w) - (\partial_x g U'(w) + g \partial_x w U''(w)) \right) \, dx
= \int_{\mathbb{R}} \partial_x(U(w + g) - U(w) - U'(w)g) \, dx,
\]
and we can integrate by parts. \( \square \)

Recall that \( \chi \in C^\infty \) is a decreasing function such that \( \chi(x) = 1 \) for \( x \leq \frac{1}{3} \) and \( \chi(x) = 0 \) for \( x \geq \frac{2}{3} \). We define
\[
\chi_1(t, x) := \chi \left( \frac{x - x_1(t)}{x_2(t) - x_1(t)} \right), \quad \chi_2 := 1 - \chi_1.
\]
Now the analysis is based on an ad-hoc change of unknowns in the modulation equations in order to remove some terms of low order. We consider the following real-valued functions:
\[
p_1(t) := \| \partial_x H \|_{L^2}^2 \langle \partial_x (H_1(t) - \chi_1(t)g(t)), \partial_t \phi \rangle,
p_2(t) := \| \partial_x H \|_{L^2}^2 \langle -\partial_x (H_2(t) + \chi_2(t)g(t)), \partial_t \phi \rangle.
\]

Lemma 3.5. If \( \phi \) is a strongly interacting kink-antikink pair, then there exist \( C, T_0 > 0 \) such that for all \( t \geq T_0 \)
\[
|\dot{x}_j(t) - p_j(t)| \leq C e^{-(x_2(t) - x_1(t))}, \quad j = 1, 2,
\]
\[
|\dot{p}_j(t) + (-1)^j F(x_2(t) - x_1(t))| \leq C(x_2(t) - x_1(t))^{-1} e^{-(x_2(t) - x_1(t))},
\]
where \( F \) is defined by (2.26).

Proof. We prove the inequalities for \( j = 1 \), the arguments for \( j = 2 \) being analogous. Using the estimates obtained in Lemma 3.3, (3.11) yields
\[
|\dot{x}_1(t) - \| \partial_x H \|_{L^2}^2 \langle \partial_x H_1(t), \partial_t \phi(t) \rangle| \lesssim e^{-(x_2(t) - x_1(t))}.
\]
We also have
\[
|\langle \partial_x(\chi_1(t)g(t)), \partial_t \phi(t) \rangle| \lesssim (\| g(t) \|_{L^2} + \| \partial_x g(t) \|_{L^2}) \| \partial_t \phi(t) \|_{L^2} \lesssim e^{-(x_2(t) - x_1(t))},
\]
so (3.19) is proved.

We have
\[
\| \partial_x H \|_{L^2}^2 p_1'(t) = \langle \partial_t (\partial_x H_1(t)), \partial_t \phi(t) \rangle - \langle \partial_x (g(t) \partial_t \chi_1(t)), \partial_t \phi(t) \rangle - \langle \partial_x (\chi_1(t) \partial_t g(t)), \partial_t \phi(t) \rangle
+ \langle \partial_x H_1(t), \partial_t^2 \phi(t) \rangle - \langle \partial_x \chi_1(t) g(t), \partial_t^2 \phi(t) \rangle - \langle \chi_1(t) \partial_x g(t), \partial_t^2 \phi(t) \rangle
= I + II + III + IV + V + VI,
\]
and we will estimate each term one by one. Until the end of this proof, we will say that some quantity is negligible if it is \( \lesssim (x_2 - x_1)^{-1} e^{-(x_2 - x_1)} \), and we use the symbol \( \simeq \) for equalities up to negligible quantities.

By the Chain Rule we have
\[
\partial_t \chi_1(t, x) = \frac{-x'_1(t)(x_2(t) - x_1(t)) - (x - x_1(t))(x'_2(t) - x'_1(t))}{(x_2(t) - x_1(t))^2} \partial_x \chi \left( \frac{x - x_1(t)}{x_2(t) - x_1(t)} \right),
\]
which yields, using (3.9), \( \| \partial_t \chi_1(t) \|_{L^\infty} \lesssim (x_2(t) - x_1(t))^{-1} e^{-\frac{1}{2} x_2(t) - x_1(t)} \). Thus II \( \simeq 0 \). Using (3.14) and
\[
\| \partial_x \chi_1(t) \|_{L^\infty} \lesssim (x_2(t) - x_1(t))^{-1}
\]
we obtain
\[
III \simeq - \langle \chi_1(t) \partial_x \partial_t g(t), \partial_t \phi(t) \rangle
\]
\[
= - \langle \chi_1(t) \partial_x \partial_t \phi(t), \partial_t \phi(t) \rangle + x'_1(t) \langle \chi_1(t) \partial_x^2 H_1(t), \partial_t \phi(t) \rangle - x'_2(t) \langle \chi_1(t) \partial_x^2 H_2(t), \partial_t \phi(t) \rangle
\]
Integrating by parts and using again (3.21), we see that the first term of the second line is negligible. The last term is negligible as well, because \( \partial_x^2 H_2(t) \) is (exponentially) small on the support of \( \chi_1(t) \). For a similar reason, we can remove \( \chi_1(t) \) from the second term, and obtain
\[
III \simeq x'_1(t) \langle \partial_x^2 H_1(t), \partial_t \phi(t) \rangle = -I,
\]
in other words we have \( I + II + III \simeq 0 \).

In order to estimate the remaining three terms, we write
\[
\partial^2_t \phi = \partial^2_x \phi(t) - U'(\phi(t)) = \Phi(x_1, x_2, \cdot) + \partial^2_x g - (U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2))
\]
In particular, examining the contribution of each term on the right above, and using (2.16), (3.8), (3.21) and the fact that \( U' \) is locally Lipschitz we see that the term \( V \) is negligible. Consider the term \( VI \). Integrating by parts, we see that \( \langle \chi_1 \partial_x g, \partial^2_x g \rangle \) is negligible. By (2.16) and Cauchy-Schwarz, \( \langle \chi_1 \partial_x g, \Phi(x_1, x_2, \cdot) \rangle \) is negligible as well. Hence, by Lemma 3.4 with \( \bar{\chi} = \chi_1 \) and \( w = 1 - H_1 + H_2 \), and using again that \( \partial_x^2 H_1(t) \) is exponentially small outside the support of \( \chi_1(t) \) we have
\[
VI \simeq \int_{\mathbb{R}} \chi_1(\partial_x^2 H_1(t) - \partial_x H_2(t))(U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(1 - H_1 + H_2)g) \, dx
\]
\[
\simeq \int_{\mathbb{R}} \partial_x H_1(U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(1 - H_1 + H_2)g) \, dx
\]
\[
\simeq \int_{\mathbb{R}} \partial_x H_1(U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(H_1)g) \, dx.
\]
We already encountered the term IV, see (3.17), where we obtained
\[
IV = \langle \partial_x^2 H_1, \Phi(x_1, x_2, \cdot) \rangle - \langle \partial_x H_1, (U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(H_1)g) \rangle.
\]
The last term cancels with the term VI and, recalling the definition of \( F \), we get (3.20).

**Proposition 3.6.** Let \( A \) be the constant defined by (1.10). If \( \phi \) is a strongly interacting kink-antikink pair, then there exist \( C, T_0 > 0 \) (depending on \( \phi \)) such that for all \( t \geq T_0 \)
\[
2t^{-1} - C(t \log t)^{-1} \leq x_2(t) - x_1(t) \leq 2t^{-1} + C(t \log t)^{-1},
\]
\[
2 \log(A t) - C(\log t)^{-1} \leq x_2(t) - x_1(t) \leq 2 \log(A t) + C(\log t)^{-1},
\]
\[
\| g(t) \|_{H^1} + \| \partial_t g(t) \|_{L^2} \leq C t^{-1}(\log t)^{-1/2}.
\]

**Remark 3.7.** The estimates given in Theorem 1 are stronger. However, proving the preliminary bounds above is crucial for our proof of Theorem 1 given in the next section. The fact that the distance between the kinks is estimated with precision \( (\log t)^{-1} \) is not crucial. In order for the arguments in the next section to work, this could be any function converging to 0 as \( t \to \infty \).

**Proof.** Set \( z(t) := x_2(t) - x_1(t) \) and \( p(t) := p_2(t) - p_1(t) \). Lemma 3.5 together with Lemma 2.8 yield,
\[
|z'(t) - p(t)| \lesssim e^{-z(t)}, \quad |p'(t) + 2A^2 e^{-z(t)}| \lesssim z(t)^{-1} e^{-z(t)}.
\]
By assumption (1.8) \( \lim_{t \to \infty} z(t) = \infty \). We claim that for \( t \) large enough \( z(t) \) is a strictly increasing function. Let \( t_1 \geq T_0 \), where \( T_0 \) is large (chosen later in the proof). We need to show that for all \( t > t_1 \) we have \( z(t) > z(t_1) \). Suppose this is not the case, and let
\[
t_2 := \sup \{ t : z(t) = \inf_{\tau \geq t_1} z(\tau) \}.
\]
Then \( t_2 > t_1 \) is finite, \( z(t_2) = \inf_{t_1 \leq \tau \leq t_2} z(\tau) \) and \( z'(t_2) = 0 \).

Let \( z_0 := z(t_2) \), \( t_3 := \inf \{ t \geq t_2 : z(t) = z_0 + 1 \} \). Since \( \lim_{t \to \infty} z(t) = \infty \), \( t_3 \) is finite. We will show that the inequalities (3.25) imply
\[
z(t_3) \leq z_0 + \frac{1}{2}, \tag{3.26}
\]
which is a contradiction. Note that \( z_0 \leq z(t) \leq z_0 + 1 \) for \( t \in [t_2, t_3] \), in particular \( e^{-z(t)} \approx e^{-z_0} \).

From (3.25) we have
\[
p'(t) \leq -A^2 e^{-z(t)} \leq -A^2 e^{-z_0} - 1 = -\frac{A^2}{e} e^{-z_0}, \quad \text{for all } t \in [t_2, t_3].
\]

Since \( p(t_2) = p(t_2) - z'(t_2) \leq C e^{-z(t_2)} = C e^{-z_0} \), we get
\[
p(t) \leq C e^{-z_0} - \frac{A^2}{e} (t - t_2) e^{-z_0}, \quad \text{for all } t \in [t_2, t_3].
\]

Using (3.25) again we obtain
\[
z'(t) \leq C e^{-z_0} - \frac{A^2}{e} (t - t_2) e^{-z_0}, \quad \text{for all } t \in [t_2, t_3].
\]

We now integrate for \( t \) between \( t_2 \) and \( t_3 \):
\[
\begin{align*}
z(t_3) - z(t_2) &\leq \int_{t_2}^{t_3} \left( -\frac{A^2}{e} (t - t_2) e^{-z_0} + C e^{-z_0} \right) dt \\
&= -\frac{1}{2} A^2 e^{-z_0} (t_3 - t_2)^2 + C e^{-z_0} (t_3 - t_2) \\
&\leq e^{-z_0} \sup_{s > 0} \left( -\frac{1}{2} A^2 e^{-z_0} s^2 + Cs \right) \\
&\leq \frac{e C^2}{2A^2} e^{-z_0},
\end{align*}
\]

so that (3.26) follows if \( T_0 \) (hence also \( z_0 \)) is large enough.

Set \( r(t) := p(t) - 2Ae^{-\frac{1}{2}z(t)} \). Using (3.25) we have
\[
r'(t) = p'(t) + Az'(t)e^{-\frac{1}{2}z(t)} = -2A^2 e^{-z(t)} + Ap(t)e^{-\frac{1}{2}z(t)} + O(z(t)^{-1}e^{-z(t)}) \\
= +Ae^{-\frac{1}{2}z(t)} \left( r(t) + O(z(t)^{-1}e^{-\frac{1}{2}z(t)}) \right). \tag{3.27}
\]

This implies that there exists \( C > 0 \) such that
\[
|r(t)| \leq C z(t)^{-1} e^{-\frac{1}{2}z(t)}, \tag{3.28}
\]
for \( t \) large enough. Indeed, suppose there exists \( t_1 \) arbitrarily large such that \( r(t_1) > C z(t_1)^{-1} e^{-\frac{1}{2}z(t_1)} \) (the case \( r(t_1) < -C z(t_1)^{-1} e^{-\frac{1}{2}z(t_1)} \) is similar). Let \( t_2 := \sup \{ t : r(t) = C z(t_1)^{-1} e^{-\frac{1}{2}z(t_1)} \} \).

Since \( \lim_{t \to \infty} r(t) = 0 \), we have \( t_2 \in (t_1, \infty) \) and \( r'(t_2) \leq 0 \). Since \( z(t) \) is non-decreasing, we have \( r(t_2) = C z(t_2)^{-1} e^{-\frac{1}{2}z(t_2)} \geq C z(t_2)^{-1} e^{-\frac{1}{2}z(t_2)} \). Thus, if we choose \( C \) large enough, (3.27) yields \( r'(t_2) > 0 \), a contradiction.
We deduce from (3.25), the definition of $r(t)$, and (3.28) that for some $t_0 > 0$ and all $t \geq t_0$ we have
\[
|z'(t) - 2A e^{-\frac{\lambda}{2}(t)}| \leq 2Cz(t)^{-1} e^{-\frac{\lambda}{2}(t)} \iff |(e^{\frac{\lambda}{2}(t)})' - A| \leq Cz(t)^{-1}, \tag{3.29}
\]
which implies, after integrating,
\[
(A - o(1))t \leq e^{\frac{\lambda}{2}(t)} \leq (A + o(1))t \iff 2 \log t + 2 \log(A - o(1)) \leq z(t) \leq 2 \log t + 2 \log(A + o(1)),
\tag{3.30}
\]
for $t$ large enough. Once we know that $z(t) \simeq \log t$, (3.23) follows by integrating (3.29) and taking the logarithm, similarly as in (3.30) but with $(\log t)^{-1}$ instead of $o(1)$. The bound (3.22) follows by inserting (3.23) into (3.29).

We are left with (3.24). We claim that
\[
\|\partial_t \phi(t)\|_{L^2} = ((x_1'(t))^2 + (x_2'(t))^2)\|\partial_x H\|^2_{L^2} + \|\partial_t g(t)\|_{L^2}^2 + O(t^{-3}).
\tag{3.31}
\]
Indeed, differentiating (3.5) we obtain $|\partial_x H_1(t), \partial_t g(t)| \lesssim t^{-2}$, so (3.31) follows by squaring (3.6) and using (2.18). Now, we observe that
\[
(x_1'(t))^2 + (x_2'(t))^2 \geq \frac{1}{2}(x_2'(t) - x_1'(t))^2 \geq 2t^{-2} - Ct^{-2}(\log t)^{-1},
\tag{3.32}
\]
where the last inequality follows from (3.22).

On the other hand, from (3.23) we deduce
\[
2\kappa^2 e^{-x_2(t)-x_1(t)} = 2\kappa^2 A^{-2}t^{-2} + O(t^{-2}(\log t)^{-1}) = t^{-2}\|\partial_x H\|^2_{L^2} + O(t^{-2}(\log t)^{-1}).
\]
By (3.16) and (2.29), for some $c > 0$ and $t$ large enough we have
\[
c\|g(t)\|^2_{H^1} + \frac{1}{2}\|\partial_t \phi(t)\|_{L^2}^2 \leq t^{-2}\|\partial_x H\|^2_{L^2} + O(t^{-2}(\log t)^{-1}),
\]
so (3.24) follows from (3.31) and (3.32). \qed

**Remark 3.8.** As a by-product of the proof of (3.24), we can deduce that $-t^{-1} - Ct^{-1}(\log t)^{-1/2} \leq x_1(t) \leq -t^{-1} + Ct^{-1}(\log t)^{-1/2}$ and $t^{-1} - Ct^{-1}(\log t)^{-1/2} \leq x_2(t) \leq t^{-1} + Ct^{-1}(\log t)^{-1/2}$. However, at this stage it is not clear whether $x_1(t) + \log t$ and $x_2(t) - \log t$ converge as $t \to \infty$.

4. The existence and uniqueness of the strongly interacting kink-antikink pair

4.1. **An implementation of the Lyapunov-Schmidt reduction.** Our strategy can be summarized as follows. Our aim is to find the strongly interacting kink-antikink pair $\phi(t, x)$ as solutions of (1.1). We assume a priori that $\phi = 1 - H_1 + H_2 + g$ and that (3.2)–(3.5) hold. As the first step we project the equation (1.1) onto the space spanned by $\partial_x H_j$, $j = 1, 2$ and onto its orthogonal complement. This way we are lead to solving the projected equation
\[
\partial_t \phi = \partial_{xx} \phi + U'(\phi) + \lambda_1(t) \partial_x H_1 + \lambda_2(t) \partial_x H_2
\tag{4.1}
\]
and the bifurcation equation
\[
\lambda_1(t) = 0, \quad \lambda_2(t) = 0.
\tag{4.2}
\]
Recalling that $H_j = H(x - x_j(t))$ we see that (4.1) amounts to finding the functions $g(t, x)$ and $\lambda_j(t)$ with $x_j(t)$ given. The second step is to solve the bifurcation equation. It turns out that (4.2) is a non local and nonlinear system of second order ODE for $x_1(t), x_2(t)$.

Writing more explicitly the projected equation (4.1) we get
\[
\partial_t^2 g + x_1'' \partial_x H_1 - (x_1')^2 \partial_x^2 H_1 - x_2'' \partial_x H_2 + (x_2')^2 \partial_x^2 H_2 - \partial_x^2 g + U'(1 - H_1 + H_2 + g) + U'(H_1) - U'(H_2) = \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2,
\tag{4.3}
\]
24
where \( \lambda_j = \lambda_j(t) \) are the Lagrange multipliers. Note that the right hand side depends on the chosen orthogonality conditions. In this section we will first study solutions \((g, \lambda_1, \lambda_2)\) of (4.3) for a given pair of trajectories \((x_1, x_2)\) satisfying
\[
2 \log t - C_0 \leq x_2(t) - x_1(t) \leq 2 \log t + C_0, \tag{4.4}
\]
\[
|x'_1(t)| + |x'_2(t)| \leq C_0 t^{-1}, \tag{4.5}
\]
\[
|x''_1(t)| + |x''_2(t)| \leq C_0 t^{-2} \tag{4.6}
\]
for some \( T_0 > 0 \) and all \( t \geq T_0 \).

The norms \( N_\gamma, S_\gamma, W_{\alpha, \beta} \) defined by (1.18) will play an important role in our proof as the fixed point scheme will be set up in these norms. The following simple lemma sheds some light on this.

**Lemma 4.1.** (i) For all \( \gamma > 0 \) and \( \alpha \in (-\infty, \gamma) \), the space \( N_\gamma \cap W_{\alpha, \gamma} := \{ g \in N_\gamma : ||g||_{W_{\alpha, \gamma}} < \infty \} \) with the norm \( ||\cdot||_{N_\gamma \cap W_{\alpha, \gamma}} := \max(||\cdot||_{N_\gamma}, \cdot||_{W_{\alpha, \gamma}}) \) is a Banach space, in which \( N_{\gamma+1} \) is continuously embedded.

(ii) For all \( \gamma > 0 \) and \( \alpha \in (-\infty, \gamma) \) there exists \( C = C(\gamma, \alpha) \) such that for all \( z \in C \)
\[
||z'||_{W_{\alpha, \gamma}} \leq C ||z||_{N_\gamma}.
\]

(iii) If \( \mu \geq 0 \) and \( \gamma > \frac{1}{2}(\sqrt{1 + 4\mu} - 1) \), then for any \( g \in N_{\gamma+1} \cap W_{\mu-\gamma+1} \cap W_{\mu+\gamma+1} \), where
\[
\mu^\pm := \frac{1}{2}(1 \pm \sqrt{1 + 4\mu}),
\]
the equation
\[
z'' = \mu t^{-2}z + g \tag{4.7}
\]
has a unique solution \( z \in S_\gamma \). The mapping \( g \mapsto z \) is a bounded linear operator \( N_{\gamma+1} \cap W_{\mu-\gamma+1} \cap W_{\mu+\gamma+1} \rightarrow S_\gamma \).

**Proof.** Point (i) is left to the reader. Point (ii) follows by integrating by parts in time. We now prove (iii). The definition of \( W_{\mu^\pm, \gamma+1} \) and the fact that \( \gamma + 1 > \mu^\pm \) imply
\[
\lim_{t \rightarrow \infty} \sup_{t \geq T_0} \left| \int_t^\infty s^{\mu^\pm} g(s) \, ds \right| = 0,
\]
thus for all \( t \geq T_0 \) the integral \( \int_t^\infty s^{\mu^\pm} g(s) \, ds \) exists as an improper Riemann integral. Moreover, we have
\[
\sup_{t \geq T_0} \int_t^\infty s^{\mu^\pm} g(s) \, ds \leq ||g||_{W_{\mu^\pm, \gamma+1}}. \tag{4.8}
\]

We observe that (4.7) is a standard Euler differential equation. A particular solution is given by
\[
z(t) := \frac{1}{\sqrt{1 + 4\mu}} \left( t^{\mu^-} \int_t^\infty s^{\mu^+} g(s) \, ds - t^{\mu^+} \int_t^\infty s^{\mu^-} g(s) \, ds \right).
\]

From (4.8) we easily deduce that
\[
\sup_{t \geq T_0} t^\gamma |z(t)| \leq \frac{1}{\sqrt{1 + 4\mu}} \left( ||g||_{W_{\mu^+, \gamma+1}} + ||g||_{W_{\mu^-, \gamma+1}} \right) \lesssim ||g||_{W_{\mu^+, \gamma+1} \cap W_{\mu^-, \gamma+1}}.
\]

We have
\[
z'(t) := \frac{1}{\sqrt{1 + 4\mu}} \left( (\mu^- t^{\mu^-} - 1) \int_t^\infty s^{\mu^+} g(s) \, ds - (\mu^+ t^{\mu^+} - 1) \int_t^\infty s^{\mu^-} g(s) \, ds \right),
\]
thus, analogously,
\[
\sup_{t \geq T_0} t^{\gamma+1} |z'(t)| \lesssim ||g||_{W_{\mu^+, \gamma+1} \cap W_{\mu^-, \gamma+1}}.
\]

The fact that \( ||z''||_{N_{\gamma+1}} \lesssim ||g||_{N_{\gamma+1} \cap W_{\mu^+, \gamma+1} \cap W_{\mu^-, \gamma+1}} \) follows from the equation (4.7). This finishes the proof that
\[
||z||_{S_\gamma} \lesssim ||g||_{N_{\gamma+1} \cap W_{\mu^+, \gamma+1} \cap W_{\mu^-, \gamma+1}}.
\]
Regarding uniqueness, the general solution of (4.7) is
\[ z_h(t) = z(t) + c^+ t^\mu^+ + c^- t^\mu^- . \]
Since \( \gamma > -\mu^- > -\mu^+ \), it is clear that \( c^+ t^\mu^+ + c^- t^\mu^- \notin N_\gamma \) unless \( c^+ = c^- = 0 \).

For \( \gamma > 2 \) we set
\[ W_\gamma := \bigcap_{\alpha \in \{-1,0,1,2\}} W_{\alpha,\gamma} . \]
As we will see, the four indices \( \alpha \) correspond to the characteristic exponents of certain differential equations of the form (4.7) appearing in the proof when we solve the bifurcation equation.

### 4.2. The linear equation associated to (4.3)

In this subsection we treat the linear equation associated to (4.3) for given trajectories \((x_1, x_2)\) satisfying (4.4), (4.5) and (4.6). We also compare solutions associated to two different sets of trajectories \((x_1, x_2)\) and \((x'_1, x'_2)\) in preparation for the contraction mapping argument performed in Section 4.3.

In the next lemma, we solve the linear problem corresponding to (4.3).

**Lemma 4.2.** For any \( \gamma > 1 \) and \( \beta \in (2, \gamma + 1) \) there exists \( C = C(\beta, \gamma) > 0 \) and \( T_0 = T_0(\beta, \gamma) \) such that for all \((x_1, x_2)\) satisfying (4.4), (4.5) and (4.6), and all \( f \in N_{\gamma+1}(L^2) \), the system
\[
\begin{align*}
\partial_{x_2}^2 h - \partial_{x_1}^2 h + U''(1 - H_1 + H_2)h &= f + \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2, \quad (4.9) \\
\langle \partial_x H_1, h \rangle &= \langle \partial_x H_2, h \rangle, \quad (4.10)
\end{align*}
\]
has a unique solution \((h, \lambda_1, \lambda_2)\) satisfying
\[
\| (h, \partial_t h) \|_{N_{\gamma}(\mathcal{E})} + \sum_{j=1}^2 \| \lambda_j + \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H_j, f \rangle \|_{W_{\beta \cap N_{\gamma+1}}} \leq C \| f \|_{N_{\gamma+1}(L^2)}.
\]
(4.11)

If \( \gamma = 1 \), the same result holds without the inclusion of the \( W_\beta \) norm on the left-hand side of (4.11).

**Proof.** We prove the theorem in the case of \( f \in C^\infty_0((T_0, \infty); \mathcal{S}) \) and then extend by density. Let us first assume the existence of a classical solution \((h, \partial_t h)(t)\) to (4.9), (4.10) satisfying
\[
\| (h, \partial_t h) \|_{N_{\gamma}(\mathcal{E})} < \infty
\]
(4.12)
Under these assumptions, we establish the bounds (4.11). We conclude by then proving the existence and uniqueness statement along with (4.12).

**Step 1.** (Computation of the Lagrange multipliers.) Differentiating in time the orthogonality conditions, we obtain
\[
0 = \frac{d}{dt} \langle \partial_x H_j, h \rangle = \langle \partial_x H_j, \partial_t h \rangle - x'_j \langle \partial_x^2 H_j, h \rangle.
\]
(4.13)
Differentiating again we get
\[
\langle \partial_x H_j, \partial_t^2 h \rangle = x'_j \langle \partial_x^2 H_j, \partial_t h \rangle + x''_j \langle \partial_x^2 H_j, h \rangle - (x'_j)^2 \langle \partial_x^2 H_j, h \rangle.
\]
Multiplying (4.9) by \( \partial_x H_1 \) and integrating in \( x \) we get
\[
\begin{align*}
x'_j \langle \partial_x^2 H_1, \partial_t h \rangle + x''_j \langle \partial_x^2 H_1, h \rangle - (x'_j)^2 \langle \partial_x^2 H_1, h \rangle &= \langle U''(1 - H_1 + H_2) - U''(H_1), \partial_x H_1, h \rangle \\
&= \langle \partial_x H_1, f \rangle + \lambda_1 \| \partial_x H \|_{L^2}^2 + \lambda_2 \langle \partial_x H_1, \partial_x H_2 \rangle.
\end{align*}
\]
(4.14)
Multiplying (4.9) by \( \partial_x H_2 \) and integrating in \( x \) we get
\[
\begin{align*}
x'_j \langle \partial_x^2 H_2, \partial_t h \rangle + x''_j \langle \partial_x^2 H_2, h \rangle - (x'_j)^2 \langle \partial_x^2 H_2, h \rangle &= \langle U''(1 - H_1 + H_2) - U''(H_2), \partial_x H_2, h \rangle \\
&= \langle \partial_x H_2, f \rangle + \lambda_2 \| \partial_x H \|_{L^2}^2 + \lambda_1 \langle \partial_x H_1, \partial_x H_2 \rangle.
\end{align*}
\]
(4.15)
These two equalities form a linear system for $\lambda_1$ and $\lambda_2$. By (2.18), its matrix is strictly diagonally dominant. By (4.4) and (2.17), we know that
\[
|U''(1 - H_1 + H_2) - U''(H_1))\partial_x H_1, h| \ll t^{-1}\|h\|_{L^2},
\]
hence we obtain
\[
|\lambda_1(t)| + |\lambda_2(t)| \lesssim (\|f(t)\|_{L^2} + t^{-1}(\|h(t), \partial_t h(t)\|_E)).
\] (4.16)

**Step 2.** (Energy estimate.) We now prove that
\[
\|(h, \partial_t h)\|_{N_{\gamma}(E)} \leq C\|f\|_{N_{\gamma+1}(L^2)}.
\]

Like in the previous section, we set
\[
\chi_1(t,x) := \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right), \quad \chi_2 := 1 - \chi_1.
\]

We introduce a modified energy functional
\[
I_t := \int_\mathbb{R} \left\{ \frac{1}{2}(\partial_t h(t))^2 + \frac{1}{2}(\partial_x h(t))^2 + \frac{1}{2}U''(1 - H_1(t) + H_2(t))h(t)^2 - \sum_{j=1}^2 x'_j(t)\chi_j(t)\partial_t h(t)\partial_x h(t) \right\} dx.
\]

The last term is sometimes called a correction term, because its size is negligible as compared to the other terms. However, we will see that, once we take the time derivative, this term is not negligible anymore. Denote $J_j(t) := \int_\mathbb{R} \chi_j(t)\partial_t h(t)\partial_x h(t) dx$.

By coercivity, see Lemma 2.4, and (4.10), we have
\[
I(t) \gtrsim \|(h(t), \partial_t h(t))\|_E^2.
\] (4.17)

After standard cancellations we obtain
\[
I'(t) = \langle \partial_t h(t), f(t), \lambda_1(t)\partial_x H_1(t) + \lambda_2(t)\partial_x H_2(t) \rangle
+ \frac{1}{2} \int_\mathbb{R} U''(1 - H_1(t) + H_2(t))(x'_1(t)\partial_x H_1(t) - x'_2(t)\partial_x H_2(t))h(t)^2 dx
- \sum_{j=1}^2 x'_j(t)J_1(t) - x'_1(t)J_1(t) - x'_2(t)J_2(t) - x'_2(t)J_2(t) / (4.18)

Observe that (4.13) yields $\langle \partial_x H_j, \partial_t h \rangle = x'_j(\partial^2_x H_j, h)$, so (4.5) and (4.16) yield
\[
|\langle \partial_t h, \lambda_1\partial_x H_1 + \lambda_2\partial_x H_2 \rangle| \lesssim t^{-1}(\|f\|_{L^2} + t^{-1}(\|h(t, \partial_t h)\|_E))\|(h, \partial_t h)\|_E
\]
\[
\lesssim t^{-1}\|f\|_{L^2}\|(h, \partial_t h)\|_E + t^{-2}\|(h, \partial_t h)\|_E^2.
\] (4.19)

The first and third term of the last line of (4.18) are negligible. The second and fourth term are not, and we will see that they cancel (up to negligible terms) the second line above.

We compute $J_1 '(t)$. The symbol “$\simeq$” means “up to terms $\leq c\|(h, \partial_t h)\|_E^2$ for an arbitrarily small constant $c > 0$”. We have
\[
J_1 '(t) = -x'_1 \int_\mathbb{R} \partial_x \chi_1 \partial_t h \partial_x h dx + \int_\mathbb{R} \chi_1 \partial_t^2 h \partial_x h dx + \int_\mathbb{R} \chi_1 \partial_t h \partial_t \partial_x h dx.
\]

Since $|x'_1| \lesssim t^{-1}$ and $|\partial_x \chi_1| \ll 1$, the first term is negligible. The third term is also negligible, since $\int_\mathbb{R} \chi_1 \partial_x (\partial_t h)^2 dx = -\int_\mathbb{R} (\partial_x \chi_1)(\partial_t h)^2 dx$. We compute the second term using (4.9):
\[
\int_\mathbb{R} \chi_1 \partial_t^2 h \partial_x h dx = \int_\mathbb{R} \chi_1(\partial_t^2 h - U''(1 - H_1 + H_2))h + f + \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2)\partial_x h dx.
\]
We have \( \int R \chi_1 \partial_x^2 h \partial_x h \, dx = -\frac{1}{2} \int R \partial_x \chi_1 (\partial_x h)^2 \, dx \), which is negligible. Using (4.16),
\[
\left| \int R \chi_1 (f + \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2) \partial_x h \, dx \right| \leq C \| f \|_{L^2} \| h \|_{H^1} + c \| h \|_{H^1}^2.
\]
Finally,
\[
- \int R \chi_1 U''(1 - H_1 + H_2) h \partial_x h \, dx = \frac{1}{2} \int R \chi_1 (-\partial_x H_1 + \partial_x H_2) U''(1 - H_1 + H_2) h^2 \, dx + \frac{1}{2} \int R \partial_x \chi_1 U''(1 - H_1 + H_2) h^2 \, dx.
\]
The second term is negligible. Since \( \partial_x H_2 \) is small on the support of \( \chi_1 \), we conclude that
\[
\left| J'_1 + \frac{1}{2} \int R \chi_1 \partial_x H_1 U''(1 - H_1 + H_2) h^2 \, dx \right| \leq C \| f \|_{L^2} \| h \|_{H^1} + c \| h \|_{H^1}^2.
\]
In a similar way,
\[
\left| J'_2 - \frac{1}{2} \int R \chi_2 \partial_x H_2 U''(1 - H_1 + H_2) h^2 \, dx \right| \leq C \| f \|_{L^2} \| h \|_{H^1} + c \| h \|_{H^1}^2.
\]
Combining these estimates with (4.18), we obtain
\[
| I' - \langle \partial_t h, f + \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2 \rangle | \leq Ct^{-1} \| f \|_{L^2} \| (h, \partial_t h) \|_\varepsilon + c t^{-1} \| (h, \partial_t h) \|_\varepsilon^2. \quad (4.20)
\]
In particular, using (4.19),
\[
| I' | \leq C \| f \|_{L^2} \| (h, \partial_t h) \|_\varepsilon + c t^{-1} \| (h, \partial_t h) \|_\varepsilon^2. \quad (4.21)
\]
Integrating in time, using (4.12) and (4.17) we obtain
\[
\| (h(t), \partial_t h(t)) \|_\varepsilon^2 \leq C \int_t^\infty \| (h, \partial_t h) \|_\varepsilon \| f \|_{L^2} \, ds + c \int_t^\infty s^{-1} \| (h, \partial_t h) \|_\varepsilon^2 \, ds,
\]
where \( c \) can be made as small as we wish choosing \( T_0 \) large. Invoking the definitions of the norm \( N_\gamma \), we get
\[
\| (h(t), \partial_t h(t)) \|_\varepsilon^2 \leq C \| (h, \partial_t h) \|_{N_\gamma} \| f \|_{N_{\gamma+1}} \int_t^\infty s^{-\gamma} s^{-1-\gamma} \, ds + c \| (h, \partial_t h) \|_\varepsilon^2 \int_t^\infty s^{-1} s^{-2\gamma} \, ds,
\]
thus \( \| h \|_{N_\gamma} \lesssim \| f \|_{N_{\gamma+1}} \), which is the required bound for the first term in (4.11).

**Step 3.** (Refined estimate of Lagrange multipliers.) Regarding the second term in (4.11), it is clear from (4.16) and the bound on \( \|(h, \partial_t h)\|_{N_\gamma} \) which we just proved that
\[
\| \lambda_j \|_{N_{\gamma+1}} \lesssim \| f \|_{N_{\gamma+1}} \Rightarrow \| \lambda_j + \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H_j, f \rangle \|_{N_{\gamma+1}} \lesssim \| f \|_{N_{\gamma+1}}.
\]
In order to obtain the bound on the \( W_{\alpha,\beta} \) norm for \( \alpha \in \{-1, 0, 1, 2\} \), we multiply (4.14) by \( t^\alpha \) and integrate. Since \( \| \langle \partial_x H_1, \partial_x H_2 \rangle \|_2 \lesssim t^{-1} \), we get
\[
\sup_{t \geq T_0} t^{\alpha+2} \chi_2(t) | \langle \partial_x H_1, \partial_x H_2 \rangle | \leq C \| f \|_{N_{\gamma+1}} \Rightarrow \| \lambda_2(t) \| \langle \partial_x H_1, \partial_x H_2 \rangle \|_{W_{\alpha,\gamma+1}} \leq C \| f \|_{N_{\gamma+1}}
\]
Similarly, we have
\[
\sup_{t \geq T_0} t^{\beta+1} \| U''(1 - H_1 + H_2) - U''(H_1) \partial_x H_1, h \| \leq C \| f \|_{N_{\gamma+1}},
\]
which allows to bound the last term of the first line of (4.14). The second and the third term are estimated in an analogous way (even a bound in \( W_{\alpha,\gamma+1} \) would be possible for these terms). The
where we have introduced above the temporary notation $\lambda$ to (4.9), which can then be extended to a global solution via Gronwall's inequality and the linear dependence of the right-hand side of (4.23) on the linear expression in $\lambda$. Note that since $\lambda^T$ and (4.15) from Step 1. We then solve by the standard iteration argument in $E$ with $\| (h_0, h_1) \|^2_2 = A$ and

$$\langle h_0, \partial_x H_1 \rangle = \langle h_0, \partial_x H_2 \rangle = 0$$

and view $(h_0, h_1)$ as initial data for (4.9) at time $T_0 > 0$. To find a local-in-time solution we set up an iteration argument in the closed ball of radius $2R$ in the space $E$, i.e., denote by

$$X_{R,[T_0,T_1]} := \{ (h, \partial_t h) \in C^0([T_0,T_1];E) \mid \sup_{t \in [T_0,T_1]} \| (h, \partial_t h)(t) \|^2_2 \leq 2R \}$$

for some $T_1 > T_0$ to be determined. Let $(h^{(0)}, \partial_t h^{(0)})(t)$ be the unique solution to the linear equation,

$$\partial^2_t h^{(0)} - \partial^2_x h^{(0)} + h^{(0)} = f$$

$$(h^{(0)}(T_0), \partial_t h^{(0)}(T_0)) = (h_0, h_1)$$

and iteratively define $(h_k, \partial_t h_k)$ as the unique solution to the inhomogeneous Klein-Gordon equation

$$\partial^2_t h^{(k)} - \partial^2_x h^{(k)} + h^{(k)} = (1 - U''(1 - H_1 + H_2))h^{(k-1)} + f + \lambda^{(k-1)}_1 \partial_x H_1 + \lambda^{(k-1)}_2 \partial_x H_2,$$

$$(h^{(k)}(T_0), \partial_t h^{(k)}(T_0)) = (h_0, h_1)$$

where we have introduced above the temporary notation $\lambda^{(k-1)}_j = \lambda_j(t, h^{(k-1)}, \partial_t h^{(k-1)}, f)$ to mean the linear expression in $(h^{(k-1)}, \partial_t h^{(k-1)}, f)$ obtained by solving the linear system determined by (4.14) and (4.15) from Step 1. We then solve by the standard iteration argument in $X_{R,[T_0,T_1]}$ based on energy estimates and taking $T_1 - T_0$ small. Passing to the limit in $k$ yields a local-in-time solution to (4.9), which can then be extended to a global solution $(h, \partial_t h) \in C^0([T_0, \infty), E)$ in the usual way via Gronwall's inequality and the linear dependence of the right-hand side of (4.23) on $(h, \partial_t h)$. Note that since $\lambda_j = \lambda_j(h, \partial_t h, f)$ is given by solving (4.14) and (4.15) in Step 1, the orthogonality conditions (4.10) hold.

It remains to show that there is a unique solution satisfying (4.12). At this stage we use crucially the functional $I(t)$ and the bound (4.21). We proceed as follows. Fix any smooth $(h_0, h_1) \in E$ with $h_0$ satisfying (4.22) and any sequence $t_n \to \infty$. Denote by

$$(h_{n,0}, h_{n,1}) := (0, 0)$$

Denote by $(h_n(t), \partial_t h_n(t))$ the unique solution to (4.9) on the time interval $[T_0, \infty)$ with $(h, \partial_t h)(t_n) = (0, 0)$. By (4.17) we can find a constant $C_0 > 0$ such that

$$\frac{1}{C_0} \| (h_n, \partial_t h_n)(t) \|^2_2 \leq I(t) \leq C_0 \| (h_n, \partial_t h_n)(t) \|^2_2$$

Denote by $C, c > 0$ the constants (4.21), and we allow ourselves to fix $c > 0$ as small as we like later in the proof by taking $T_0 > 0$ large enough. By continuity of the flow, and the definition of
the data \((h_n, h_n)(t_n) = (0, 0)\) we can find a sequence \(T_{n, 1} < t_n\) such that

\[
\sup_{t \in [T_{n,1}, t_n]} t^{2\gamma}\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 \leq C_0 \frac{C}{c} \|f\|_{N, \gamma+1(L^2)}^2
\]

(4.25)

where the \(N_{\gamma+1}(L^2)\) is measured on the time interval \([T_0, \infty)\) with \(T_0 < t_n\) to be determined below. We prove that by choosing \(T_0 > 0\) large enough we can in fact take \(T_{n, 1} = T_0\) uniformly in \(n\). Indeed, let \(t \in [T_{n,1}, t_n]\). Then using (4.17) and (4.21) we have

\[
\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 \leq C_0 I(t) \leq C_0 \int_t^{t_n} \|f(s)\|_{L^2} \|(h_n, \partial_t h_n)(s)\|_{\mathcal{E}} ds
\]

\[
\leq C_0 C \int_t^{t_n} \|f(s)\|_{L^2} \|(h_n, \partial_t h_n)(s)\|_{\mathcal{E}} ds + C_0 c \int_t^{t_n} s^{-1} \|(h_n, \partial_t h_n)(s)\|_{\mathcal{E}}^2 ds
\]

since \(I(t_n) = 0\). We obtain,

\[
\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 \leq C_0 \frac{C}{c} \|f\|_{N, \gamma+1(L^2)}^2 \int_t^{t_n} s^{-1-2\gamma} ds + 2C_0 c \int_t^{t_n} s^{-1} \|(h_n, \partial_t h_n)(s)\|_{\mathcal{E}}^2 ds
\]

Using (4.25) and (4.24) it follows that for any \(t \geq T_{n,1}\) we have

\[
\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 \leq C_0 \frac{C}{c} t^{-2\gamma} \|f\|_{N, \gamma+1(L^2)}^2 + 2C_0 c (C_0 \frac{C}{c})^{-\gamma} \|f\|_{N, \gamma+1(L^2)}^2
\]

Multiplying through by \(t^{2\gamma}\) and fixing \(c > 0\) small enough in (4.21) we obtain

\[
\sup_{t \in [T_{n,1}, t_n]} t^{2\gamma}\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 < 2C_0 \frac{C}{c} \|f\|_{N, \gamma+1(L^2)}^2
\]

which is an improvement to (4.25). Hence we obtain the uniform bounds,

\[
\sup_{t \in [T_0, t_n]} t^{2\gamma}\|(h_n, \partial_t h_n)(t)\|_{\mathcal{E}}^2 \leq C \|f\|_{N, \gamma+1(L^2)}^2
\]

Passing to a weak limit in \(N_{\gamma}(\mathcal{E})\) (after extending each \((h_n, \partial_t h_n)\) to be \((0, 0)\) on \((t_n, \infty)\)), we obtain a solution \((h, \partial_t h)(t)\) to (4.9) on \([T_0, \infty)\) satisfying (4.10) and such that

\[
\|(h, \partial_t h)\|_{N_{\gamma}(\mathcal{E})} \leq C \|f\|_{N, \gamma+1(L^2)}.
\]

(4.26)

That this solution is unique is an immediate consequence of (4.26) and the linearity of (4.9). \(\square\)

It turns out that if the time derivative of the forcing term decays, then we can substantially improve the bounds provided by the last lemma. This was pointed out to us by Y. Martel.

**Lemma 4.3.** For any \(\gamma > 1\) and \(\beta \in (2, \gamma + 1)\) there exists \(C = C(\beta, \gamma) > 0\) and \(T_0 = T_0(\beta, \gamma)\) such that for all \((x_1, x_2)\) satisfying (4.4), (4.5) and (4.6), and for all \(f \in N_{\gamma}(L^2)\) such that \(\partial_t f \in N_{\gamma+1}(L^2)\), the system (4.9)–(4.10) has a unique solution \((h, \lambda_1, \lambda_2)\) and

\[
\|(h, \partial_t h)\|_{N_{\gamma}(\mathcal{E})} + \sum_{j=1}^{2} \|\lambda_j + \|\partial_x H\|_{L^2}^2 \|\partial_x H, f\|_{W_{\beta} \land N_{\gamma+1}} \leq C(\|f\|_{N_{\gamma}(L^2)} + \|\partial_t f\|_{N_{\gamma+1}(L^2)})
\]

If \(\gamma = 1\), the same result holds without the inclusion of the \(W_{\beta}\) norm on the left-hand side of (4.11).
Proof. **Step 1.** (First estimate of Lagrange multipliers.) As in the proof of the previous lemma, we arrive at (4.16).

**Step 2.** (Energy estimate.) We prove the bound on \((h, \partial_h)\). We consider an energy functional slightly different than in the proof of the previous lemma:

\[
\tilde{I}(t) := I(t) - \langle h(t), f(t) \rangle.
\]

By coercivity \(\| (h, \partial_h) \|_E^2 \lesssim \tilde{I} + \| h \|_{L^2} \| f \|_{L^2}\). Observe that

\[
\left| \frac{d}{dt} \langle h, f \rangle - \langle \partial_h f, f \rangle \right| \leq \| h \|_{L^2} \| \partial_h f \|_{L^2}.
\]

From (4.20) we have

\[
\left| \tilde{I} - \langle \partial_h \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2 \rangle \right| \leq \| h \|_{L^2} \| \partial_h f \|_{L^2} + C t^{-1} \| f \|_{L^2} \| (h, \partial_h) \|_{E} + c t^{-1} \| (h, \partial_h) \|_E^2,
\]

so (4.19) yields

\[
\left| \tilde{I} \right| \leq \| h \|_{L^2} \| \partial_h f \|_{L^2} + C t^{-1} \| f \|_{L^2} \| (h, \partial_h) \|_{E} + c t^{-1} \| (h, \partial_h) \|_E^2,
\]

and we conclude as in Lemma 4.2.

**Step 3.** (Refined estimate of Lagrange multipliers.) This can be done similarly as in the proof of Lemma 4.2. □

In the next lemma, we compare solutions \((h, \lambda_1, \lambda_2)\) of (4.9), (4.10) as in Lemma 4.2 associated to different trajectories \((x_1, x_2)\) and different forcing \(f\). We first introduce some notation. Given trajectories \((x_1(t), x_2(t))\) satisfying (4.4), (4.5), (4.6), we define \(y(t) = (y_1(t), y_2(t))\) by

\[
y_1(t) := x_1(t) + \log(At), \quad y_2(t) := x_2(t) - \log(At)
\]

where we remind the reader that \(A = \sqrt{2} \| \partial_x H \|_{L^2}^2 \), see (1.10) and (1.9).

**Lemma 4.4.** For any \(\nu, \gamma > 1\) and \(\beta \in (2, \nu + \gamma)\) there exist \(C = C(\gamma, \nu, \beta)\) and \(T_0 = T_0(\gamma, \nu, \beta)\) such that the following holds. Let \((x_1, x_2)\) and \((x_1', x_2')\) be two pairs of trajectories satisfying (4.4), (4.5), (4.6) and \(\| y^2 - y \|_{S_\nu} \leq 1\), for \(y, y^2\) as in (4.27). Let \((h, \lambda_1, \lambda_2)\) be the solution of (4.9) and \((h^2, \lambda_1^2, \lambda_2^2)\) the solution of (4.9)–(4.10) with \((x_1', x_2')\) instead of \((x_1, x_2)\) and \(f^2\) instead of \(f\). Then

\[
\| (\lambda_1^2 + \| \partial_x H \|_{L^2}^2 \langle \partial_x H, j \rangle, f^2) \|_{N_{\gamma + \nu} \cap W_\beta} + \| h^2 - h \|_{N_{\gamma + \nu - 1}} \leq C(\| y^2 - y \|_{S_\nu} \| f \|_{N_{\gamma + 1}(L^2)} + \| f^2 \|_{N_{\gamma + 1}(L^2)}) (4.28)
\]

Proof. Let \((h^2)_{\perp} = h^2 + a_1 \partial_x H_1 + a_2 \partial_x H_2\) (4.29) be the projection of \(h^2\) on the subspace orthogonal to \(\partial_x H_1\) and \(\partial_x H_2\). The idea is to apply Lemma 4.2 in order to obtain an estimate of \((h^2)_{\perp} - h\).

**Step 1.** We prove that for \(j \in \{1, 2\}\)

\[
\| a_j \|_{N_{\gamma + \nu}} + \| a_j' \|_{N_{\gamma + \nu}} + \| a_j'' \|_{N_{\gamma + \nu}} \lesssim \| (h^2, \partial_h h^2) \|_{N_{\gamma}}(c) \| y^2 - y \|_{S_\nu} \lesssim \| f^2 \|_{N_{\gamma + 1}} \| y^2 - y \|_{S_\nu}
\]

(4.30)

In order to do this, we multiply (4.29) by \(\partial_x H_j\) and integrate:

\[
a_1 |\partial_x H_j|_{L^2}^2 + a_2 |\partial_x H_1, \partial_x H_2\rangle + |\partial_x H_1, h^2\rangle = 0,
\]

\[
a_1 \langle \partial_x H_1, \partial_x H_2 \rangle + a_2 |\partial_x H_2|_{L^2}^2 + |\partial_x H_2, h^2\rangle = 0,
\]

thus

\[
a_1 = \left( |\partial_x H|_{L^2}^4 - |\partial_x H_1, \partial_x H_2\rangle^2 \right)^{-1} \left( |\partial_x H_1, \partial_x H_2\rangle |\partial_x H_2, h^2\rangle - |\partial_x H_2|_{L^2}^2 |\partial_x H_1, h^2\rangle \right),
\]

\[
a_2 = \left( |\partial_x H|_{L^2}^4 - |\partial_x H_1, \partial_x H_2\rangle^2 \right)^{-1} \left( |\partial_x H_1, \partial_x H_2\rangle |\partial_x H_1, h^2\rangle - |\partial_x H_1|_{L^2}^2 |\partial_x H_2, h^2\rangle \right).
\]

(4.31)
We observe that
\[ |\langle \partial_x H_j, h^\sharp \rangle| = |\langle \partial_x H_j - \partial_x H_j^\sharp, h^\sharp \rangle| \leq \|\partial_x H_j - \partial_x H_j^\sharp\|_{L^2} \|h^\sharp\|_{L^2} \lesssim |y^\sharp - y|\|h^\sharp\|_{L^2}, \]
which yields the required bound on \( |a'_j| \). The bound on \( |a''_j| \) follows by differentiating in time (4.31). Indeed, from (4.31) we see
\[
a'_j = -\frac{1}{\|\partial_x H\|_{L^2}^2} \left\langle \frac{d}{dt}(\partial_x H_j - \partial_x H_j^\sharp), h^\sharp \right\rangle - \frac{1}{\|\partial_x H\|_{L^2}^2} \left\langle \partial_x H_j - \partial_x H_j^\sharp, \partial_t h^\sharp \right\rangle + \text{lower order terms}
\]
where by lower order terms we mean terms with similar structure to the first two terms on the right, but that come with an extra power of \( t \) decay. For the second term, we argue as above,
\[
\left| \left\langle \partial_x H_j - \partial_x H_j^\sharp, \partial_t h^\sharp \right\rangle \right| \lesssim |y^\sharp - y|\|\partial_t h^\sharp\|_{L^2}
\]
For the first term write
\[
\frac{d}{dt}(\partial_x H_j - \partial_x H_j^\sharp) = (x_j' - (x_j')')\partial_x^2 H_j + (x_j')'(\partial_x^2 H_j - \partial_x^2 H_j^\sharp)
\]
which yields the bound
\[
\left\| \frac{d}{dt}(\partial_x H_j - \partial_x H_j^\sharp) \right\|_{L^2} = \left\| (x_j' - (x_j')')\partial_x^2 H_j + (x_j')'(\partial_x^2 H_j - \partial_x^2 H_j^\sharp) \right\|_{L^2} \lesssim |y_j' - (y_j')'| + t^{-1} |y_j - y_j^\sharp|.
\]
The bound on \( |a''_j| \) follows by differentiating again in time. Indeed,
\[
a''_j = -\frac{1}{\|\partial_x H\|_{L^2}^2} \left\langle \frac{d^2}{dt^2}(\partial_x H_j - \partial_x H_j^\sharp), h^\sharp \right\rangle - \frac{1}{\|\partial_x H\|_{L^2}^2} \left\langle \partial_x H_j - \partial_x H_j^\sharp, \partial_t^2 h^\sharp \right\rangle + \text{lower order terms}
\]
For the first term on the right we argue similarly as for the first term in the expression for \( a'_j \). For the second term on the right above we invoke the equation (4.9) satisfied by \( h^\sharp \) and make use of the bounds (4.16) for \( \lambda^\sharp_1, \lambda^\sharp_2 \). We omit the (by now) routine details.

**Step 2.** We write the equation satisfied by \( (h^\sharp)^\perp - h \):
\[
\partial_x^2((h^\sharp)^\perp - h) - \partial_x^2((h^\sharp)^\perp - h) + U''(1 - H_1 + H_2)((h^\sharp)^\perp - h) = (f^\sharp)^\perp + (\lambda^\sharp)^\perp_1 \partial_x H_1 + (\lambda^\sharp)^\perp_2 \partial_x H_2,
\]
where
\[
(f^\sharp)^\perp := (\partial_x^2 h^\perp - \partial_x^2 h^\sharp + U''(1 - H_1^\perp + H_2^\perp)h^\sharp - \lambda^\sharp_1 \partial_x H_1 - \lambda^\sharp_2 \partial_x H_2)
\]
\[
- (\partial_x^2 h - \partial_x^2 h + U''(1 - H_1 + H_2)h - \lambda_1 \partial_x H_1 - \lambda_2 \partial_x H_2)
\]
\[
+ (\partial_x^2 (h^\perp)^\perp - \partial_x^2 h^\perp - a''_j \partial_x H_1 - a''_j \partial_x H_2) - (\partial_x^2 (h^\sharp)^\perp - \partial_x^2 h^\sharp) + U''(1 - H_1^\perp + H_2^\perp)((h^\perp)^\perp - h^\sharp) + U''(1 - H_1 + H_2) - U''(1 - H_1^\perp + H_2^\perp))(h^\perp)^\perp,
\]
and
\[
(\lambda^\sharp)^\perp_1 := \lambda^\sharp_1 - \lambda_j + a''_j.
\]

According to Lemma 4.2, we should
(i) estimate \((f^\sharp)^\perp\) in \( N_{\gamma + \nu}(L^2) \),
(ii) estimate \( \langle \partial_x H_j, (f^\sharp)^\perp \rangle \) in \( W_\beta \),
(iii) estimate \( a''_j \) in \( W_\beta \).
We see that (iii) follows from (4.30) since, upon integrating by parts, \[ \|a_j^\prime\|_{W^\beta} \lesssim \|a_j\|_{N_\beta} \lesssim \|a_j\|_{N_{\gamma+\nu}}. \]

We expand each line of (4.32). Most terms can be estimated in \( N_{\gamma+\nu+1}(L^2) \), which takes care of (i) and (ii) above, so we will call such terms “negligible”. The first line equals
\[ f^2 + \lambda_1^x (\partial_x H_1^x - \partial_x H_1) + \lambda_2^x (\partial_x H_2^x - \partial_x H_2). \]

The second and third term are negligible since
\[ \|\lambda_j^x (\partial_x H_j^x - \partial_x H_j)\|_{N_{\gamma+\nu+1}} \lesssim \|\lambda_j\|_{N_{\gamma+\nu}} \|y_j\|_{S_\nu} \lesssim \|y_j\|_{S_\nu} \|f^2\|_{N_{\gamma+1}}. \]

The second line of (4.32) equals \(-f\). Now we expand the third line. We have
\[ \partial_t(h^2)^\perp = \partial_t h^2 + \sum_{j=1}^2 (a_j^\prime \partial_x H_j - a_j x_j \partial_x^2 H_j), \]
\[ \partial_t^2(h^2)^\perp = \partial_t^2 h^2 + \sum_{j=1}^2 (a_j^\prime \partial_x H_j - 2a_j x_j \partial_x^2 H_j - a_j x_j \partial_x^2 H_j + a_j(x_j^2)^2 \partial_x^2 H_j). \]

Thus, it follows from (4.30) that the term \( \partial_t^2(h^2)^\perp - \partial_t^2 h^2 - a_j^\prime \partial_x H_j - a_j x_j \partial_x H_2 \) is negligible. The remaining part of the third line equals
\[ \sum_{j=1}^2 a_j (-\partial_x^2 + U''(1 - H_1^2 + H_2^2)) \partial_x H_j = \sum_{j=1}^2 a_j (U''(1 - H_1^2 + H_2^2) - U''(H_j)) \partial_x H_j \]
\[ = \sum_{j=1}^2 a_j (U''(1 - H_1^2 + H_2^2) - U''(1 - H_1 + H_2)) \partial_x H_j \]
\[ + \sum_{j=1}^2 a_j (U''(1 - H_1 + H_2) - U''(H_j)) \partial_x H_j. \]

The last line is negligible by (2.17) and (4.30). The other line can be estimated in \( N_{\gamma+2\nu} \), hence is negligible as well.

Finally, consider the fourth line of (4.32). We have
\[ \|(U''(1 - H_1 + H_2) - U''(1 - H_1^2 + H_2^2)) \partial_x H_j\|_{N_{\gamma+\nu}(L^2)} \lesssim \|y_j\|_{S_\nu} \|h^2\|_{N_\gamma} + \|a_1\|_{N_\gamma} + \|a_2\|_{N_\gamma} \lesssim \|y_j\|_{S_\nu} \|f^2\|_{N_{\gamma+1}}. \]

However, it turns out that this term is not negligible, and we have to estimate carefully its projection on \( \partial_x H_j \) for this term’s contribution to (ii). In what follows, “\( \lesssim \)” means “up to terms bounded in \( N_{\gamma+\nu+1}(L^2) \) by the right hand side of (4.28)”.

By (2.36) we have
\[ \|U''(1 - H_1 + H_2) - U''(1 - H_1^2 + H_2^2) - ((H_1^2 - H_1) - (H_2^2 - H_2)) U''(1 - H_1 + H_2)\|_{N_{2\nu}(L^\infty)} \lesssim \|y_j\|_{S_\nu} \|f^2\|_{N_{\gamma+1}}. \]

We also have, using Taylor expansions,
\[ \|H_j^2 - H_j + (y_j - y_j) \partial_x H_j\|_{N_{2\nu}(L^\infty)} \lesssim \|y_j\|_{S_\nu} \|y_j\|_{S_\nu} \lesssim \|y_j\|_{S_\nu}. \]
From these two inequalities and \( \| (h^\sharp)^\perp - h^\sharp \|_{N^{\gamma + \nu}} \lesssim \| y^\sharp - y \|_{S_\nu} \| f^\sharp \|_{N^{\gamma + 1}(L^2)} \) we deduce that, up to negligible terms, the fourth line of (4.32) equals
\[
h^\sharp U''(1 - H_1 + H_2)(- (y^2 - y_1) + (y^2 - y_2) \partial_x H_2) \simeq h^\sharp(U''(H_1) + (y^2 - y_2) \partial_x H_2),
\]
where the last approximate equality follows from the fact that \( U'' \) is locally Lipschitz, thus for instance
\[
\|(U''(1 - H_1 + H_2) + U''(H_1)) \partial_x H_1| \lesssim \| 1 + H_2 \| \partial_x H_1| \ll t^{-1}.
\]
(4.33)
Up to negligible terms, the projection on \( \partial_x H_1 \) is
\[
\int_{\mathbb{R}} h^\sharp((y^2 - y_1)U''(H_1)(\partial_x H_1)^2 dx = (y^2 - y_1)(\partial_x^2 H_1, (\partial_x^2 - U''(H_1))h^\sharp),
\]
where for the last equality we use (2.10). By an estimate analogous to (4.33) but with \( \partial_x^2 H_1 \) instead of \( \partial_x H_1 \) and \( U'' \) instead of \( U'' \), the right-hand side is approximately equal to
\[
(y^2 - y_1)((\partial_x^2 - U''(1 - H_1 + H_2))h^\sharp, \partial_x^2 H_1) = \]
\[
(y^2 - y_1)(\partial_x^2 h^\sharp - f^\sharp - \lambda_1^2 \partial_x H_1^1 - \lambda_2^2 \partial_x H_2^2).
\]
Integrating by parts allows us to bound \( (y^2 - y_1)(\partial_x^2 H_1, \partial_x H_1) \) in \( W_\beta \). The term \( (y^2 - y_1)(\partial_x^2 H_1, \lambda_1^2 \partial_x H_1^2) \) is negligible due to \( (\partial_x^2 H_1, \partial_x H_1) = 0 \). The term \( (y^2 - y_1)(\partial_x^2 H_1, \lambda_2^2 \partial_x H_2^2) \) is negligible as well. Finally, we observe that \( (y^2 - y_1)(\partial_x^2 H_1, f^\sharp) \simeq - (\partial_x H_1, f^\sharp) \). In conclusion,
\[
(\partial_x H_1, (f^\sharp)^\perp) \simeq (\partial_x H_1, f^\sharp - f) + (\partial_x H_1^1 - \partial_x H_1, \partial_x H_1, f^\sharp) = (\partial_x H_1^1, f^\sharp) - (\partial_x H_1, f),
\]
and similarly for the projection on \( \partial_x H_2 \). \( \square \)

**Remark 4.5.** Naively, one could expect that (4.28) should hold with \( \| y^\sharp - y \|_{S_{\nu - 1}} \) instead of \( \| y^\sharp - y \|_{S_\nu} \), but we do not know how to prove this stronger bound.

**Lemma 4.6.** For any \( \gamma, \nu > 1 \) and \( \beta \in (2, \gamma + \nu) \) there exist \( C = C(\gamma, \nu, \beta) \) and \( T_0 = T_0(\gamma, \nu, \beta) \) such that the following holds. Let \( (x_1, x_2) \) and \( (x'_1, x'_2) \) be two pairs of trajectories satisfying (4.4), (4.5) and (4.6). Let \( (h, \lambda_1, \lambda_2) \) be the solution of (4.9) and \( (h^\sharp, \lambda_1^\sharp, \lambda_2^\sharp) \) the solution of (4.9)–(4.10) with \( (x^1_1, x^1_2) \) instead of \( (x_1, x_2) \) and \( f^\sharp \) instead of \( f \). Then
\[
\|
\left(\| \lambda^\sharp + \| \partial_x H \|_{L^2}(\partial_x H_1^2, f^\sharp) \right) - \left(\| \lambda^\sharp + \| \partial_x H \|_{L^2}(\partial_x H_1^2, f) \right) \right\|_{N^{\gamma + \nu}} W_\beta + \| h^\sharp - h \|_{N^{\gamma + \nu - 1}(E)} \leq C \|
\left(\| y^\sharp - y \|_{S_\nu} \| f \|_{N^{\gamma}(L^2)} + \| \partial_x f \|_{N^{\gamma + 1}(L^2)} + \| f^\sharp \|_{N^{\gamma}} \| f \|_{N^{\gamma + 1}(L^2)} + \| f^\sharp + \| f \|_{N^{\gamma + 1}(L^2)} \right).
\]

**Proof.** The proof is similar to the previous one, but using Lemma 4.3 instead of Lemma 4.2. \( \square \)

### 4.3. Solving (4.3) for given trajectories \((x_1, x_2)\)

Let \( \gamma \geq 1 \). Given a pair of trajectories \((x_1, x_2)\) and \( g \in N_{\gamma}(E) \), we define
\[
(\Lambda_1(x_1, x_2, g), \Lambda_2(x_1, x_2, g), \Psi(x_1, x_2, g)) = (\lambda_1, \lambda_2, h)
\]
as the solution of the equation
\[
\partial_t^2 h - \partial_x^2 h + U''(1 - H_1 + H_2)h = (\lambda_1 - x'^1_1) \partial_x H_1 + (x'^2_1)^2 \partial_x^2 H_1 + (\lambda_2 + x'^2_2) \partial_x H_2 - (x'^2_2)^2 \partial_x^2 H_2
\]
\[
- U'(1 - H_1 + H_2) - U'(H_1) + U'(H_2)
\]
\[
- U'(1 - H_1 + H_2 + g) + U'(1 - H_1 + H_2) + U''(1 - H_1 + H_2)g
\]
satisfying the orthogonality conditions \( \langle \partial_x H_1, h \rangle = \langle \partial_x H_2, h \rangle = 0 \). Note that we do not require the argument \( g \) to satisfy any orthogonality conditions.
Proposition 4.7. The mapping \((\Lambda_1, \Lambda_2, \Psi)\) has the following properties.

(i) For any \(\gamma \in (1, 2)\) and \(\beta \in (2, \gamma + 1)\) there exist \(C_1 = C_1(\beta, \gamma) > 0\) and \(T_0 = T_0(\beta, \gamma)\) such that for all \((x_1, x_2)\) satisfying (4.4), (4.5) and (4.6)

\[
\|\Lambda_1(x_1, x_2, 0) - x_1'' + F(x_2 - x_1)\|_{W_\beta \cap N_{\gamma + 1}} + \|\Lambda_2(x_1, x_2, 0) + x_2'' + F(x_2 - x_1)\|_{W_\beta \cap N_{\gamma + 1}} + \|\Psi(x_1, x_2, 0)\|_{N_\gamma} \leq C_1, \tag{4.35}
\]

where \(F\) is the normalized attraction force defined in (2.26). For \(\gamma = 1\) the same conclusion holds without the inclusion of the \(W_\beta\) bound.

(ii) For any \(\gamma_1, \gamma_2 \geq 1\) there exist \(C = C(\gamma_1, \gamma_2) > 0\) and \(T_0 = T_0(\gamma_1, \gamma_2)\) such that for all \((x_1, x_2)\) satisfying (4.4), (4.5) and (4.6) and all \(g, g^2 \in N_{\gamma_1}(L^2) \cap N_{\gamma_2}(L^2)\)

\[
\|\Lambda_1(x_1, x_2, g^2) - \Lambda_1(x_1, x_2, g)\|_{N_{\gamma_1 + \gamma_2}} + \|\Lambda_2(x_1, x_2, g^2) - \Lambda_2(x_1, x_2, g)\|_{N_{\gamma_1 + \gamma_2}} + \|\Psi(x_1, x_2, g^2) - \Psi(x_1, x_2, g)\|_{N_{\gamma_1 + \gamma_2}} \leq C\left(\|g^2\|_{N_{\gamma_1}(L^2)} + \|g\|_{N_{\gamma_1}(L^2)}\right) \|g^2 - g\|_{N_{\gamma_2}(L^2)}, \tag{4.36}
\]

(iii) For any \(\gamma \in (1, 2), \nu > 1\) and \(\beta \in (2, \min(2\nu + 2, \nu + 2\gamma - 1))\) there exist \(C = C(\gamma, \nu, \beta) > 0\) and \(T_0 = T_0(\gamma, \nu, \beta)\) such that for all \((x_1, x_2)\) and \((x_1^2, x_2^2)\) satisfying (4.4), (4.5) and (4.6), and all \(g\) such that \(\|g\|\) and \(\|g^2\|\) are bounded by \(C\left(\|g^2\|_{N_{\gamma_1}(L^2)} + \|g\|_{N_{\gamma_1}(L^2)}\right) \|g^2 - g\|_{N_{\gamma_2}(L^2)}\)

\[
\left(\|\Lambda_1(x_1^2, x_2^2, g^2) - \Lambda_1(x_1, x_2, g^2)\|_{N_{\gamma_1 + \gamma_2}} + \|\Lambda_2(x_1^2, x_2^2, g^2) - \Lambda_2(x_1, x_2, g^2)\|_{N_{\gamma_1 + \gamma_2}} + \|\Psi(x_1^2, x_2^2, g^2) - \Psi(x_1, x_2, g^2)\|_{N_{\gamma_1 + \gamma_2}} \leq C\left(\|x_1^2 - x_1\|_{N_{\gamma_1}} + \|x_2^2 - x_2\|_{N_{\gamma_2}}\right) \|g^2 - g\|_{N_{\gamma_2}(L^2)} \right), \tag{4.37}
\]

Proof. If \(g = 0\), then the last line in (4.34) vanishes. Note also that the second line of the right hand side of (4.34) equals \(\Phi(x_1, x_2, \cdot)\), which is defined in (2.15). Note that in all estimates in Section 4.2 we are free to replace \(\lambda_j \partial_x H_j\) on the right-hand side of (4.9) with \(\lambda_j + (-1)^j x_j'' \partial_x H_j\). Thus, in the context of Lemma 4.3, we can take forcing term \(f\) in (4.34) with \(g = 0\) to be

\[
f = (x_1')^2 \partial_x^2 H_1 - (x_2')^2 \partial_x^2 H_2 + \Phi(x_1, x_2, \cdot).
\]

Hence, after noting that \(\langle \partial_x^2 H_j, \partial_x H_j \rangle = 0\), we see that up to negligible terms we have

\[
\lambda_j + (-1)^j x_j'' + \|\partial_x H\|_{L^2}^2 \partial_x H_j, f = \lambda_j + (-1)^j x_j'' + F(x_2 - x_1)
\]

with \(F\) defined in (2.26). In order to apply Lemma 4.3, we need to bound the terms in \(f\) in \(N_\gamma\) and their time derivatives in \(N_{\gamma + 1}\) for all \(\gamma < 2\). It is clear that \(\|\partial_t ((x_j')^2 \partial_x^2 H_j)\|_{L^2} \lesssim t^{-3}\). The second line of the right hand side of (4.34) equals \(\Phi(x_1, x_2, \cdot)\). The Chain Rule and Lemma 2.7 yield

\[
\|\partial_t \Phi(x_1(t), x_2(t), \cdot)\|_{L^2} \lesssim \sum_{j=1}^{\gamma} \|x_j'(t)\| \|\partial_x \Phi(x_1(t), x_2(t), \cdot)\|_{L^2} \lesssim t^{-\gamma - 1}, \quad \forall \gamma < 2,
\]

thus we have proved (4.35).

In order to prove (4.36), we observe that \(\lambda_j := \Lambda_j(x_1, x_2, g^2) - \Lambda_j(x_1, x_2, g)\) and \(h := \Psi(x_1, x_2, g^2) - \Psi(x_1, x_2, g)\) solve the equation

\[
\partial_t^2 h - \partial_x^2 h + U''(1 - H_1 + H_2)h = \lambda_1 \partial_x H_1 + \lambda_2 \partial_x H_2
\]

\[
- U'(1 - H_1 + H_2 + g^2) + U'(1 - H_1 + H_2) + U''(1 - H_1 + H_2)g^2
\]

\[
+ U'(1 - H_1 + H_2 + g) - U'(1 - H_1 + H_2) - U''(1 - H_1 + H_2)g
\]

and \(\langle \partial_x H_j, h \rangle = 0\). The second and third line constitute the forcing term. By (2.37), its \(L^2\) norm is bounded up to a constant by \((\|g^2\|_{H^1} + \|g\|_{H^1})\|g^2 - g\|_{H^1}\). Applying Lemma 4.2, we get (4.36).
We are left with (4.37). Let
\[ f := (x'_1)^2 \partial^2_x H_1 - (x'_2)^2 \partial^2_x H_2 - U'(1 - H_1 + H_2) + U'(H_1) - U'(H_2), \]
\[ f := -U'(1 - H_1 + H_2 + g) + U'(1 - H_1 + H_2) + U''(1 - H_1 + H_2)g, \]
\[ f^\sharp := ((x'_1)^2)\partial^2_x H_1 - ((x'_2)^2)\partial^2_x H_2 - U'(1 - H_1^2 + H_2^2) + U'(H_1^2) - U'(H_2^2), \]
\[ f^\sharp := -U'(1 - H_1^2 + H_2^2 + g) + U'(1 - H_1^2 + H_2^2) + U''(1 - H_1^2 + H_2^2)g. \]

Let \((h, \lambda_1, \lambda_2)\) solve (4.9)–(4.10), \((\tilde{h}, \tilde{\lambda}_1, \tilde{\lambda}_2)\) solve (4.9)–(4.10) with \(\tilde{f}\) instead of \(f\), \((h^\sharp, \lambda_1^\sharp, \lambda_2^\sharp)\) solve (4.9)–(4.10) with \(f^\sharp\) instead of \(f\) and \(H_j^\sharp\) instead of \(H_j\), and \((\tilde{h}^\sharp, \tilde{\lambda}_1^\sharp, \tilde{\lambda}_2^\sharp)\) solve (4.9)–(4.10) with \(\tilde{f}^\sharp\) instead of \(f\) and \(H_j^\sharp\) instead of \(H_j\). Then
\[ \Lambda_j(x_1, x_2, g) = (-1)^{j+1} x''_j + \lambda_j + \tilde{\lambda}_j, \]
\[ \Lambda_j(x'_1, x'_2, g) = (-1)^{j+1} (x''_j) + \lambda_j + \tilde{\lambda}_j^\sharp, \]
\[ \Psi(x_1, x_2, g) = h + \tilde{h}, \]
\[ \Psi(x'_1, x'_2, g) = h^\sharp + \tilde{h}^\sharp. \]

In order to estimate \(h^\sharp - h\) and \(\lambda_j^\sharp - \lambda_j\), take \(\gamma \in (\max(1, \beta - \nu), 2)\) in Lemma 4.6. We have already seen while proving (4.35) that
\[ \|f\|_{N_{\nu+1}(L^2)} + \|\partial_t f\|_{N_{\nu+1}(L^2)} + \|f^\sharp\|_{N_{\nu+1}(L^2)} + \|\partial_t f^\sharp\|_{N_{\nu+1}(L^2)} \lesssim 1, \]
so we only need to show that \(\|f^\sharp - f\|_{N_{\nu+1}(L^2)} \lesssim \|x^\sharp - x\|_{S_{\nu}}\). Estimating \(((x''_j)^2 \partial^2_{x} H_j^\sharp - (x''_j^\sharp) \partial^2_{x} H_j\) in \(N_{\nu+2}\) is straightforward, and the remaining part is estimated using (2.16). The projections of \(f\) and \(f^\sharp\) on \(\partial_x H_j\) and \(\partial_x H_j^\sharp\) yield the terms \(F(x_2 - x_1)\) and \(F(x_2^\sharp - x_1^\sharp)\) in (4.37).

Concerning the estimates of \(\tilde{h}^\sharp - \tilde{h}\) and \(\tilde{\lambda}_j^\sharp - \tilde{\lambda}_j\), we have, with the same choice of \(\gamma\),
\[ \|f^\sharp\|_{N_{\nu+1}(L^2)} + \|f^\sharp\|_{N_{\nu+1}(L^2)} \lesssim 1, \]
\[ \|f^\sharp - f\|_{N_{\nu+1}(L^2)} \lesssim \|x^\sharp - x\|_{S_{\nu}}, \]
and we obtain the conclusion using Lemma 4.4 and noting again that \(N_{\gamma+\nu+1} \subset W_{\beta}. \]

**Remark 4.8.** Since the distance between the kink and the antikink is \(2 \log t + O(1)\), the forcing term coming from the interaction of the kink and the anti-kink is of size \(t^{-2}\), hence too large to be handled directly by Lemma 4.2; see (2.16) and Lemma 2.6 for this computation. The same remark applies to the term \(x_k'(t)^2 \partial^2_{x} H_k\). However, taking the time derivative of these forcing terms, we gain one power of \(t\), and we can use Lemma 4.3, which is what happens in the proof above.

**Proposition 4.9.** For any \(C_0 > 0\) there exist \(T_0 > 0\) and \(\delta > 0\) such that the following is true. For any \(x_1, x_2 : [T_0, \infty) \to \mathbb{R}\) satisfying (4.4), (4.5) and (4.6), the equation (4.3) has a unique solution \((\lambda_1, \lambda_2, g) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2), g(x_1, x_2))\) such that \(\|(g, \partial_t g)\|_{N_1(\varepsilon)} \leq \delta\). For all \(\gamma \in [1, 2]\) there exist \(C = C(\gamma)\) and \(T_0 = T_0(\gamma)\) such that this solution satisfies
\[ \sum_{j=1}^{2} \|\lambda_j + (-1)^j x''_j + F(x_2 - x_1)\|_{N_{\gamma+1}(W_{\gamma+1})} + \|(g, \partial_t g)\|_{N_{\gamma}(\varepsilon)} \leq 1. \] (4.38)

Moreover, for all \(\nu > 1\) and \(\beta \in (2, \nu + 2)\) there exist \(C = C(\nu, \beta) > 0\) and \(T_0 = T_0(\nu, \beta)\) such that
\[ \|\lambda_j(x'_1, x'_2) - \lambda_j(x_1, x_2) + (-1)^j ((x''_j)^2 - x''_j) + (F(x'_2 - x'_1) - F(x_2 - x_1))\|_{N_{\gamma}(\varepsilon)} \leq C\|(x'_1, x'_2) - (x_1, x_2)\|_{S_\nu}. \] (4.39)
where \((x_1, x_2)\) and \((x_1^*, x_2^*)\) are any two pairs of admissible trajectories satisfying \(x - x^* \in S_{\nu}\).

\textbf{Proof.} We notice that for given \((x_1, x_2)\), \(g\) solves (4.3) if and only if \(g\) is a fixed point of the mapping \(\Psi(x_1, x_2, \cdot)\). By (4.36), this mapping is a contraction in a sufficiently small ball of \(N_1(\mathcal{E})\). Also, (4.35) implies that this ball is invariant, hence by the Contraction Principle there exists a unique fixed point \(g = g(x_1, x_2)\). By a similar argument, there exists a unique fixed point in the unit ball of \(N_\gamma(\mathcal{E})\) for any \(\gamma \in (1, 2)\). Since \(N_\gamma(\mathcal{E}) \subset N_1(\mathcal{E})\), we deduce that the unique fixed point in the small ball of \(N_1(\mathcal{E})\) in fact belongs to \(N_\gamma(\mathcal{E})\) for all \(\gamma \in (1, 2)\).

Let \((x_1, x_2)\) and \((x_1^*, x_2^*)\) be two pairs of trajectories such that \(x^* - x \in S_{\nu}\) for some \(\nu > 1\), \(g := g(x_1, x_2)\) and \(g^* := g(x_1^*, x_2^*)\). Let \(\beta \in (2, \nu + 2)\). We have

\[
\begin{align*}
\|g^* - g\|_{N_{\beta-1}(\mathcal{E})} &= \|\Psi(x_1^*, x_2^*, g^*) - \Psi(x_1, x_2, g)\|_{N_{\beta-1}(\mathcal{E})} \\
&\leq \|\Psi(x_1^*, x_2^*, g^*) - \Psi(x_1^*, x_2^*, g)\|_{N_{\beta-1}(\mathcal{E})} + \|\Psi(x_1^*, x_2^*, g) - \Psi(x_1, x_2, g)\|_{N_{\beta-1}(\mathcal{E})}.
\end{align*}
\]

Bound (4.36) implies that the first term is \(\ll \|g^* - g\|_{N_{\beta-1}(\mathcal{E})}\). Taking \(\gamma = \frac{2}{\beta}\) in Proposition 4.7 (iii), which is allowed by (4.38), we obtain that the second term is \(\lesssim \|x^* - x\|_{S_{\nu}}\). This proves the second bound in (4.39).

Take \(\gamma_1 \in (\max(1, \beta - \nu), 2)\) and \(\gamma_2 := \beta + 1 - \gamma_1 < \nu + 1\). As we have just proved, \(\|g^* - g\|_{N_{\gamma_2}(\mathcal{E})} \lesssim \|x^* - x\|_{S_{\nu}}\), thus (4.36) yields

\[
\|\Lambda_j(x_1^*, x_2^*, g^*) - \Lambda_j(x_1^*, x_2^*, g)\|_{N_{\beta} \cap \mathcal{W}_\beta} \lesssim \|\Lambda_j(x_1^*, x_2^*, g^*) - \Lambda_j(x_1^*, x_2^*, g)\|_{N_{\beta+1}} \lesssim \|x^* - x\|_{S_{\nu}}.
\]

Now the bound on the first line in (4.39) follows from (4.37).

\[\square\]

\textbf{4.4. Completion of the proof of Theorem 1.} First, we establish two preparatory lemmas.

\textbf{Lemma 4.10.} Let \(1 < \gamma < 2\). Suppose that \(v \in N_{\gamma+1} \cap \mathcal{W}_{\gamma+1}\) and let \(z(t)\) be a solution of

\[
z''(t) = -2F(z(t)) + v(t)
\]

such that \(|z(t) - 2 \log t| \lesssim 1\) and \(0 \leq z'(t) \lesssim t^{-1}\). Then there exists \(t_0 \in \mathbb{R}\) such that

\[
|z(t) - 2 \log(A(t - t_0))| \lesssim t^{-\gamma}, \quad \left|z'(t) - \frac{2}{t - t_0}\right| \lesssim t^{-\gamma - 1}, \quad \left|z''(t) + \frac{2}{(t - t_0)^2}\right| \lesssim t^{-\gamma - 1}.
\]

\textbf{Proof.} Since \(z(t)e^{-2z(t)} \in N_{\gamma+2} \subset \mathcal{W}_{\gamma+1}\) for all \(\gamma < 2\), Lemma 2.8 allows to replace \(F(z)\) by \(A^2e^{-z}\).

\textbf{Step 1.} We prove that

\[
z(t) = 2 \log(At) + O(t^{1-\gamma}).
\]

Multiplying the equation by \(z'(t)\) and integrating we obtain

\[
\frac{1}{2}(z'(t))^2 = 2A^2e^{-z(t)} - \int_t^\infty z'(\tau)v(\tau) \, d\tau = 2A^2e^{-z(t)} + O(t^{-\gamma - 1}),
\]

thus

\[
(z'(t) - 2Ae^{-\frac{1}{2}z(t)})(z'(t) + 2Ae^{-\frac{1}{2}z(t)}) \lesssim t^{-\gamma - 1} \Rightarrow z'(t) = 2Ae^{-\frac{1}{2}z(t)} + O(t^{-\gamma}),
\]

which in turn implies

\[
(e^{\frac{1}{2}z(t)})' = A + O(t^{1-\gamma}) \Rightarrow e^{\frac{1}{2}z(t)} = t(A + O(t^{1-\gamma})).
\]

Taking the logarithm, we obtain (4.41).

\textbf{Step 2.} We improve. We set \(z(t) = 2 \log(At) + u(t)\). In this step we prove \(|u(t)| \lesssim t^{-1+\epsilon}\), for a fixed small number \(\epsilon\). From the equation we obtain

\[
u''(t) = 2t^{-2}u(t) + v(t) + O(t^{-2}|u(t)|^2).
\]

Set

\[
\tilde{u}(s) := u(e^s).
\]
The Chain Rule yields
\[ \tilde{u}''(s) = 2\tilde{u}(s) + \tilde{u}'(s) + e^{2s}v(e^s) + O(\tilde{u}(s)^2). \]

This system diagonalises as follows:
\[
\begin{align*}
(\tilde{u}(s) + \tilde{u}'(s))' &= 2(\tilde{u}(s) + \tilde{u}'(s)) + e^{2s}v(e^s) + O(\tilde{u}(s)^2), \\
(2\tilde{u}(s) - \tilde{u}'(s))' &= -(2\tilde{u}(s) - \tilde{u}'(s)) - e^{2s}v(e^s) + O(\tilde{u}(s)^2).
\end{align*}
\tag{4.42}
\tag{4.43}
\]

Suppose \(|\tilde{u}(s)| \lesssim e^{-\beta s}\) for some \(\beta > 0\). We begin with \(\beta = \gamma - 1\) and in a finite number of steps we will bootstrap this to \(\beta = 1 - \epsilon\). The assumption \(v \in W_{2,\gamma+1}\) implies in particular that
\[
\lim_{s \to \infty} \int_{e^s}^{\infty} e^{3\sigma}v(e^\sigma)\,d\sigma = \lim_{T \to \infty} \int_{T}^{T} \tau^2v(\tau)\,d\tau
\]
exists and is finite. Thus (4.43) yields
\[
|e^s(2\tilde{u}(s) - \tilde{u}'(s))| \lesssim e^{(1-2\beta)s} \Rightarrow |2\tilde{u}(s) - \tilde{u}'(s)| \lesssim e^{-2\beta s}.
\]

From the assumption \(v \in W_{-1,\gamma+1}\) we have
\[
\left| \int_{e^s}^{\infty} v(e^\sigma)\,d\sigma \right| = \left| \int_{e^s}^{\infty} \tau^{-1}v(\tau)\,d\tau \right| \lesssim e^{-(\gamma+2)s},
\]
so integrating (4.42) we obtain
\[
|\tilde{u}(s) + \tilde{u}'(s)| \lesssim e^{-2\beta s} \Rightarrow |\tilde{u}(s)| \lesssim e^{-2\beta s}.
\]

Thus we can double the value of \(\beta\), which concludes Step 2.

**Step 3.** We improve again, using the information obtained in Step 2. We obtain from (4.42) and (4.43)
\[
\begin{align*}
(\tilde{u}(s) + \tilde{u}'(s))' &= 2(\tilde{u}(s) + \tilde{u}'(s)) + e^{2s}v(e^s) + O(e^{(-2+\epsilon)s}), \\
(2\tilde{u}(s) - \tilde{u}'(s))' &= -(2\tilde{u}(s) - \tilde{u}'(s)) - e^{2s}v(e^s) + O(e^{(-2+\epsilon)s}).
\end{align*}
\tag{4.44}
\tag{4.45}
\]

Equation (4.45) is equivalent to
\[
(\tilde{u}(s) + \tilde{u}'(s))' = -e^{3s}v(e^s) + O(e^{(-1+\epsilon)s}).
\]

From the assumption \(v \in W_{2,\gamma+1}\) we deduce that the limit \(b = \lim_{s \to \infty} (e^s(2\tilde{u}(s) - \tilde{u}'(s)))\) exists and
\[
2\tilde{u}(s) - \tilde{u}'(s) = be^{-s} + O(e^{-\gamma s}).
\]

Equation (4.44) is equivalent to
\[
(\tilde{u}(s) + \tilde{u}'(s))' = v(e^s) + O(e^{-(4+\epsilon)s}) \Rightarrow \tilde{u}(s) + \tilde{u}'(s) = O(e^{-\gamma s}).
\]

We conclude that
\[
\tilde{u}(s) = \frac{b}{3}e^{-s} + O(e^{-\gamma s}), \quad \tilde{u}'(s) = -\frac{b}{3}e^{-s} + O(e^{-\gamma s}),
\]

which after a straightforward transformation yield the bounds for \(z\) and \(z'\) in (4.40) with \(t_0 := \frac{b}{6}\). The bound on \(z''\) follows from the bound on \(z\), (2.27) and the differential equation. \(\square\)

**Lemma 4.11.** Let \(\gamma > 1\). For any \(f_1, f_2 \in N_{\gamma+1} \cap W_{\gamma+1}\) there exists a unique solution \((y_1, y_2) \in S_\gamma\) of the system
\[
\begin{align*}
y''_1 &= t^{-2}(-y_2 - y_1) + f_1, \\
y''_2 &= t^{-2}(y_2 - y_1) + f_2,
\end{align*}
\]
and this defines a bounded operator \(N_{\gamma+1} \cap W_{\gamma+1} \ni (f_1, f_2) \mapsto (y_1, y_2) \in S_\gamma\).
Proof. By setting $z_1 := y_2 + y_1$ and $z_2 := y_2 - y_1$, we transform the system to two decoupled second order equations:

\[
\begin{align*}
  z_1'' &= g_1, \\
  z_2'' &= 2t^{-2}z_2 + g_2,
\end{align*}
\]

Since $\gamma > 1 = \frac{1}{2}(\sqrt{1 + 4 \times 2} - 1)$, Lemma 4.1 (iii) applies. \qed

Proof of Theorem 1. Observe that $(x_1, x_2, g)$ solves (3.7) if and only if

\[
g = g(x_1, x_2), \quad \lambda_1(x_1, x_2) = \lambda_2(x_1, x_2) = 0. \tag{4.46}
\]

Step 1. Fix $\gamma \in (1, 2)$. We will prove that there exists a unique solution $(x_1, x_2, g)$ of (4.46) such that $\| (x_1(t), x_2(t)) - (-\log(At), \log(At)) \|_{S_\gamma} \leq 1$ and $\| g, \partial_t g \|_{N_\gamma(E)} \leq 1$. In particular, we obtain the same solution for all the values $\gamma \in (1, 2)$.

We define $(y_1, y_2)$ by $(x_1(t), x_2(t)) = (-\log(At) + y_1(t), \log(At) + y_2(t))$ and we set up a fixed point problem for $(y_1, y_2) \in S_\gamma$. Given $(y_1, y_2) \in S_\gamma$, we define $(\tilde{y}_1, \tilde{y}_2) = \Theta(y_1, y_2)$ as the solution of the following system of differential equations:

\[
\begin{align*}
  \tilde{y}_1'' &= -t^{-2}(\tilde{y}_2 - \tilde{y}_1) - \lambda_1(-\log(At) + y_1, \log(At) + y_2) + y_1'' + t^{-2}(y_2 - y_1), \\
  \tilde{y}_2'' &= t^{-2}(\tilde{y}_2 - \tilde{y}_1) + \lambda_2(-\log(At) + y_1, \log(At) + y_2) + y_2'' - t^{-2}(y_2 - y_1).
\end{align*}
\]

We see that $\lambda_j(-\log(At) + y_1, \log(At) + y_2) = 0$ for $j \in \{1, 2\}$ is equivalent to $(y_1, y_2)$ being a fixed point of $\Theta$. In this proof, we denote

\[
f_j(y_1, y_2, t) := (-1)^j \lambda_j(-\log(At) + y_1(t), \log(At) + y_2(t)) + y_j''(t) - (-1)^j t^{-2}(y_2(t) - y_1(t)).
\]

We first check that $\Theta(0, 0) \in S_\gamma$. By Lemma 4.11, it suffices to verify that

\[
\lambda_j(-\log(At), \log(At)) \in N_{\gamma + 1} \cap W_{\gamma + 1}. \tag{4.47}
\]

By Lemma 2.8 and recalling the definition of $A$ in Proposition 3.6, we have $\log(A)'' + F(2 \log(At)) \in N_{\gamma + 2} \subset N_{\gamma + 1} \cap W_{\gamma + 1}$, so (4.47) follows from Proposition 4.9.

We now prove that $\Theta$ is a contraction in $S_\gamma$. Again by Lemma 4.11, it suffices to verify that for any $c > 0$

\[
\| f_j(y_1^*, y_2^*, \cdot) - f_j(y_1, y_2, \cdot) \|_{N_{\gamma + 1} \cap W_{\gamma + 1}} \leq c \| (y_1^*, y_2^*) - (y_1, y_2) \|_{S_{\gamma}}, \tag{4.48}
\]

provided we take $T_3$ sufficiently large. Let $z := x_2 - x_1 = 2 \log(At) + y_2 - y_1$ and $z^* := x_2^* - x_1^* = 2 \log(At) + y_2^* - y_1^*$. For $z \gg 1$ and $|z^* - z| \ll 1$ we have, by (2.28),

\[
| (F(z^*) - F(z)) - 2A^2 e^{-z^*} - e^{-z} | \lesssim \left| \int_z^{z^*} we^{-2w} \, dw \right| \ll t^{-3} |z^* - z| \tag{4.49}
\]

and

\[
e^{-z^*} - e^{-z} = \frac{-(y_2^* - y_1^*) - (y_2 - y_1)}{1 + O(w)} d w = -(y_2^* - y_1^*) - (y_2 - y_1) + o(t^{-3} |y^* - y|),
\]

where the last inequality follows from $|y_2^* - y_1^*| + |y_2 - y_1| \lesssim t^{-\gamma} \ll t^{-1}$. Plugging this into (4.49) we obtain

\[
\| (F(x_2^* - x_1^*) - F(x_2 - x_1)) + t^{-2}((y_2^* - y_1^*) - (y_2 - y_1)) \|_{N_{\gamma + 2}} \ll \| y^* - y \|_{N_{\gamma}},
\]

Comparing this bound with (4.39), we get (4.48).

Invoking the Contraction Principle we obtain the unique solution $(x_1, x_2, g)$. Set $\tilde{x}_1(t) := -x_2(t)$, $\tilde{x}_2(t) := -x_1(t)$ and $\tilde{g}(t, x) := g(t, -x)$. Observe that, by the symmetry of the problem, $(\tilde{x}_1, \tilde{x}_2, \tilde{g})$ also satisfies the requirements stated at the beginning of Step 1. hence, by uniqueness, $g(t, -x) =$
Thus, using (3.22) and (3.23), the assumptions of Lemma 4.10 are satisfied, hence there exists $T_0 > 0$ such that

$$
\| (g, \partial_t g) \|_{N^1(E)} \leq \delta.
$$

Moreover, (3.22), (3.9) and (3.10) imply (4.4), (4.5) and (4.6). Thus, Proposition 4.9 yields

$$
(g, \partial_t g) \in N_2(E), \quad \text{for all } \gamma \in [1, 2),
$$

and, since $\lambda_j(x_1, x_2) = 0$,

$$
(-1)^j x_j''(t) + F(x_2(t) - x_1(t)) \in N_{\gamma + 1} \cap W_{\gamma + 1}, \quad \text{for all } \gamma \in [1, 2).
$$

Let $m(t) := x_2(t) + x_1(t)$, $z(t) := x_2(t) - x_1(t)$. The function $z(t)$ satisfies

$$
z''(t) + 2F(z(t)) \in W_{1+\gamma} \quad \text{for all } \gamma \in [1, 2).
$$

Thus, using (3.22) and (3.23), the assumptions of Lemma 4.10 are satisfied, hence there exists $t_0$ such that for all $\gamma \in [1, 2)$, (4.40) holds.

The function $m(t)$ satisfies

$$
m''(t) \in W_{1+\gamma} \quad \text{for all } \gamma \in [1, 2).
$$

Integrating in time and using $f \in W_{0,\gamma + 1}$, $m'(t) \to 0$ we get $|m'(t)| \lesssim t^{-\gamma - 1}$, in particular $x_0 := \frac{1}{\gamma} \lim_{t \to \infty} m(t)$ is well-defined, and $|m(t) - 2x_0| \lesssim t^{-\gamma}$.

We obtain

$$
(x_1, x_2) - (x_0 - \log(A(t - t_0)), x_0 + \log(A(t - t_0))) = \left(\frac{m - z}{2}, \frac{m + z}{2}\right) - (x_0 - \log(A(t - t_0)), x_0 + \log(A(t - t_0))) \in S_{\gamma},
$$

for all $\gamma \in (1, 2)$. We deduce that, after translating in time by $t_0$ and in space by $x_0$, the triple $(x_1, x_2, g)$ satisfies the requirements of Step 1, and the conclusion follows.

\[ \square \]

Remark 4.12. Our existence proof is constructive, as we can in principle obtain better and better approximate solutions by the usual iteration scheme. It can be seen from the proof that we obtain functions approximating the fixed point at arbitrary polynomial order in time. Indeed, our proof in fact yields

$$
\| \Theta(x^2) - \Theta(x) \|_{S_\beta} \lesssim \| x^2 - x \|_{S_{\gamma}}, \quad \text{for any } \beta < \gamma + 1.
$$

One could expect this to hold for any $\beta < \gamma + 2$, since in the formal expansion of the trajectory we expect only even powers of $t$, with logarithmic corrections. We believe that (4.50) indeed holds for all $\beta < \gamma + 2$, and the main reason we do not prove this is the loss of one power of $t$ in Lemmas 4.4 and 4.6, see Remark 4.5.

5. Construction of kink clusters

In this section, we explain how to adapt our arguments to construct configurations of more than two strongly interacting kinks and antikinks. That is, we prove Theorem 2.

Given a $K$-tuple of trajectories

$$
x_j(t) = (2j - K - 1) \log(A(t)) + c_j + y_j(t), \quad \text{with } y_j \in S_{\nu} \text{ for some } \nu > 0,
$$

we denote as in previous sections $H_j(t, x) := H(x - x_j(t))$. Following the case of the kink-anti-kink pair we assume \textit{a priori}

$$
\begin{align*}
|x'_1(t)| + \cdots + |x'_K(t)| & \leq C_0 t^{-1}, \\
|x''_1(t)| + \cdots + |x''_K(t)| & \leq C_0 t^{-2}
\end{align*}
$$

(5.2)
for some $T_0 > 0$ and all $t \geq T_0$. Assumptions (5.1)–(5.2) play the same role as (4.4)–(4.6).

Note that we immediately impose on the trajectories a constraint much stronger than in the case of kink-antikink pairs. This is allowed, because here we are only interested in a construction, and we can freely choose the spaces where this construction is to be carried out. We also denote the case of kink-antikink pairs. This is allowed, because here we are only interested in a construction.

For any $(x_1, \ldots, x_K)$ such that for all $j$, we have $x_j(t) \leq \beta > 1/2(1 + \sqrt{17}) \geq 1/2(1 + \sqrt{1 + 4\mu_K})$ we set

$$\mathbf{W}_\beta^K := \bigcap_{j=1}^K W_{\mu_j, \mu_j, \mu_j, \mu_j}^x \cap W_{\mu_j, \mu_j}^x, \quad \mu_j^\pm := \frac{1}{2}(1 \pm \sqrt{1 + 4\mu_j}).$$

We begin by stating the analog of Lemma 4.2.

**Lemma 5.1.** For any $\gamma \geq 1$ and $\beta \in (2, \gamma + 1)$ there exists $C = C(\beta, \gamma) > 0$ and $T_0 = T_0(\beta, \gamma)$ such that for all $(x_1, \ldots, x_K)$ as above and all $f \in \mathcal{N}_{\gamma+1}(L^2)$, the system

$$\partial_t^2 h - \partial_x^2 h + U''(H^K)h = f, \quad \langle \partial_x H_j, h \rangle = 0, \quad \text{for all } j \in \{1, \ldots, K\}$$

has a unique solution $(h, \lambda_1, \lambda_2)$ and

$$\| (h, \partial_t h) \|_{\mathcal{N}_{\gamma}(\mathcal{E})} + \sum_{j=1}^K \| \lambda_j + \| \partial_x H \|_{L^2}^{-2} \langle \partial_x H_j, f \rangle \|_{W_{\beta, \gamma+1}^{K \cap N_{\gamma+1}}(L^2)} \leq C\| f \|_{\mathcal{N}_{\gamma+1}(L^2)}.$$

**Proof.** The proof follows the lines of the proof of Lemma 4.2. The cut-off functions are defined by $\chi_j(t, x) := \chi((x - x_j(t))/(x_2(t) - x_1(t)))$, for $j = 2, \ldots, K - 1$

$$\chi_j(t, x) := \begin{cases} 1 - \chi_{j-1}(t, x) & \text{for } x \leq x_j(t), \\ \chi \left( \frac{x - x_j(t)}{x_{j+1}(t) - x_j(t)} \right) & \text{for } x \geq x_j(t), \end{cases}$$

and $\chi_K(t, x) := 1 - \chi_{K-1}(t, x)$. The appropriate energy functional is

$$I(t) := \int_{\mathbb{R}} \left( \frac{1}{2} (\partial_t h(t))^2 + \frac{1}{2} (\partial_x h(t))^2 + \frac{1}{2} U''(H^K(t))h(t)^2 - \sum_{j=1}^K x_j'(t) \chi_j(t) \partial_t h(t) \partial_x h(t) \right) dx.$$
Lemma 5.2. Let \( \gamma > \sqrt{17} - 1 \). For any \( f = (f_1, \ldots, f_K) \in N_{\gamma+1} \cap W_{\gamma+1}^K \) there exists a unique solution \( y = (y_1, \ldots, y_K) \in S_\gamma \) of the system

\[
y'' = t^{-2} \Delta y + f,
\]

and this defines a bounded operator \( N_{\gamma+1} \cap W_{\gamma+1}^K \geq f \mapsto y \in S_\gamma \).

Proof. Let \( b^{(l)} = (b_1^{(l)}, \ldots, b_{K}^{(l)}) \) for \( l = 1, \ldots, K \) be an orthonormal basis such that \( \Delta b^{(l)} = \mu b^{(l)} \).

Setting \( z_j(t) := \sum_{l=1}^{K} b_j^{(l)} y_l(t) \) and \( g_j(t) := \sum_{l=1}^{K} b_j^{(l)} f_l(t) \) transforms the system to

\[
z_j''(t) = \mu_j t^{-2} z_j(t) + g_j(t), \quad j \in \{1, \ldots, K\},
\]

and it suffices to apply Lemma 4.1 (iii). \( \square \)

Remark 5.3. The special case \( K = 2 \) was considered in Lemma 4.11 for a wider range of \( \gamma \). In order for our proof to work, it is important that \( \frac{1}{2}\sqrt{17} - 1 < 2 \) (otherwise we would be forced to refine the ansatz and work with larger time weights).

Let \( \gamma \geq 1 \). Given a \( K \)-tuple of trajectories \((x_1, \ldots, x_K)\) satisfying (5.1) and \( g \in N_\gamma(\mathcal{E}) \), we define

\[
(\Lambda_1(x_1, \ldots, x_K, g), \ldots, \Lambda_K(x_1, \ldots, x_K, g), \Psi(x_1, \ldots, x_K, g)) = (\lambda_1, \ldots, \lambda_K, \gamma)
\]

as the solution of the equation

\[
\begin{align*}
\partial_t^2 h - \partial_x^2 h + U''(H^{(K)}) h &= \sum_{j=1}^{K} ((\lambda_j + (-1)^j x_j''(t)) \partial_x H_j + (x_j')^2 \partial_x^2 H_j) \\
&- U'(H^{(K)}) - \sum_{j=1}^{K} (-1)^j U'(H_j) - U'(H^{(K)}) + g + U'(H^{(K)}) + U''(H^{(K)}) g
\end{align*}
\]

satisfying the orthogonality conditions \( \langle \partial_x H_j, h \rangle = 0 \).

Proposition 5.4. The mapping \((\Lambda_1, \ldots, \Lambda_K, \Psi)\) has the following properties.

(i) For any \( \gamma \in (1, 2) \) and \( \beta \in (2, \gamma + 1) \) there exist \( C = C(\beta, \gamma) > 0 \) and \( T_0 = T_0(\beta, \gamma) \) such that for all \((x_1, \ldots, x_K)\) satisfying (5.1)

\[
\sum_{j=1}^{K} \|\Lambda_j(x_1, \ldots, x_K, 0) + (-1)^j (x_j'' + F(x_j - x_{j-1}) - F(x_{j+1} - x_j))\|_{W_\beta^{K} \cap N_{\gamma+1}} + \\
+ \|\Psi(x_1, \ldots, x_K, 0)\|_{N_\gamma} \leq C_1,
\]

with the convention \( F(x_1 - x_0) = F(x_{K+1} - x_K) = F(\infty) = 0 \). For \( \gamma = 1 \) the same conclusion holds without the inclusion of the \( W_\beta^K \) bound.

(ii) For any \( \gamma_1, \gamma_2 \geq 1 \) there exist \( C = C(\gamma_1, \gamma_2) > 0 \) and \( T_0 = T_0(\gamma_1, \gamma_2) \) such that for all \((x_1, \ldots, x_K)\) satisfying (5.1) and all \( g \in N_{\gamma_1}(L^2) \cap N_{\gamma_2}(L^2) \),

\[
\sum_{j=1}^{K} \|\Lambda_j(x_1, \ldots, x_K, g^\sharp) - \Lambda_j(x_1, \ldots, x_K, g)\|_{N_{\gamma_1+\gamma_2}} + \\
+ \|\Psi(x_1, \ldots, x_K, g^\sharp) - \Psi(x_1, \ldots, x_K, g)\|_{N_{\gamma_1+\gamma_2-1}(\mathcal{E})} \\
\leq C(\|g^\sharp\|_{N_{\gamma_1}(L^2)} + \|g\|_{N_{\gamma_2}(L^2)}) \|g^\sharp - g\|_{N_{\gamma_1}(L^2)}.
\]
Lemma 2.8 yields

For all $x_1,\ldots,x_K$, and $(x_1^\gamma,\ldots,x_K^\gamma)$ satisfying (5.1), and all $g$ such that $\| (g, \partial_\gamma g) \|_{N_\gamma(\varepsilon)} \leq 1$

\[
\sum_{j=1}^K \left\| (A_j(x_1^\gamma,\ldots,x_K^\gamma), g - A_j(x_1,\ldots,x_K, g)) + \right. \\
\left. + \left( (-1)^j \left( \left( (x_j^\gamma)^{2\nu} - x_j^\gamma \right) + (F(x_j^\gamma) - x_j^{2\nu-1} - F(x_j - x_j^\gamma)) - (F(x_j^\gamma) - x_j^{2\nu-1} - F(x_j - x_j^\gamma)) \right) \right) \right\|_{N_\beta \cap W_\beta^K} \\
+ \left\| \Psi(x_1^\gamma,\ldots,x_K^\gamma) - \Psi(x_1,\ldots,x_K, g) \right\|_{N_{\beta-1}(\varepsilon)} \leq C \left\| (x_1^\gamma,\ldots,x_K^\gamma) - (x_1,\ldots,x_K) \right\|_{S_\nu}.
\]

We skip the proof, which is similar to the proof of Proposition 4.7.

Proposition 5.5. For any $C_0 > 0$ there exist $T_0 > 0$ and $\delta > 0$ such that the following is true. For any $x_1,\ldots,x_K: [0,T_0] \to \mathbb{R}$ satisfying (5.1)-(5.2), the equation (5.3) has a unique solution $(\lambda_1,\ldots,\lambda_K, g) = (\lambda_1(x_1,\ldots,x_K),\ldots,\lambda_K(x_1,\ldots,x_K), g(x_1,\ldots,x_K))$ such that $\| (g, \partial_\gamma g) \|_{N_\gamma(\varepsilon)} \leq \delta$.

For all $\gamma \in (1,2)$ there exist $C = C(\gamma)$ and $T_0 = T_0(\gamma)$ such that this solution satisfies

\[
\sum_{j=1}^K \left\| \lambda_j + (-1)^j (x_j^{2\nu} + F(x_j - x_j^\gamma) - F(x_j^\gamma) - F(x_j - x_j^\gamma)) \right\|_{W_{\nu+1}(\gamma) \cap N_{\gamma+1}} + \left\| (g, \partial_\gamma g) \right\|_{N_\gamma(\varepsilon)} \leq 1.
\]

Moreover, for all $\nu > 1$ and $\beta \in (2,\nu+2)$ there exist $C = C(\nu, \beta) > 0$ and $T_0 = T_0(\nu, \beta)$ such that

\[
\sum_{j=1}^K \left\| \left( \lambda_j (x_1^\gamma,\ldots,x_K^\gamma) - \lambda_j (x_1,\ldots,x_K) \right) + \right. \\
\left. + \left( (-1)^j \left( \left( (x_j^\gamma)^{2\nu} - x_j^\gamma \right) + (F(x_j^\gamma) - x_j^{2\nu-1} - F(x_j - x_j^\gamma)) - (F(x_j^\gamma) - x_j^{2\nu-1} - F(x_j - x_j^\gamma)) \right) \right) \right\|_{W_\beta \cap N_\beta} \\
+ \left\| g(x_1^\gamma,\ldots,x_K^\gamma) - g(x_1,\ldots,x_K) \right\|_{N_{\beta-1}(\varepsilon)} \leq C \left\| (x_1^\gamma,\ldots,x_K^\gamma) - (x_1,\ldots,x_K) \right\|_{S_\nu}.
\]

The proof follows the lines of the proof of Proposition 4.9, invoking Proposition 5.4 instead of Proposition 4.7.

Proof of Theorem 2. Fix $\gamma \in \left( \frac{1}{2}(\sqrt{17} - 1), 2 \right)$. We will prove that there exists a unique solution $(x_1,\ldots,x_K, g)$ of (4.46) such that $\left| x_j(t) - \left( (2j-K-1) \log(At) + c_j \right) \right| \leq 1$ and $\| (g, \partial_\gamma g) \|_{N_\gamma(\varepsilon)} \leq 1$.

We set $x_j^{\text{app}}(t) := (2j-K-1) \log(At) + c_j$ and define $y = (y_1,\ldots,y_K)$ by $x_j(t) = x_j^{\text{app}}(t) + y_j(t)$. We use the vectorial notation $\lambda := (-\lambda_1,\lambda_2,\ldots,-\lambda_K)$. Given $y \in S_\gamma$, we define $\vec{y} = \Theta(y)$ as the solution of the following system of differential equations:

\[
\vec{y}'(t) = t^{-2} \Delta \vec{y}'(t) + \lambda(x_1(t),\ldots,x_K(t)) + y''(t) - t^{-2} \Delta (K)(y)(t).
\]

We see that $\lambda(x_1(t),\ldots,x_K(t)) = 0$ for $j \in \{1,\ldots,K\}$ is equivalent to $y$ being a fixed point of $\Theta$. In this proof, we denote

\[
f(y,t) := \lambda(x_1(t),\ldots,x_K(t)) + y''(t) - t^{-2} \Delta (K)(y)(t).
\]

We first check that $\Theta(0,0) \in S_\gamma$. By Lemma 5.2, it suffices to verify that

\[
\lambda_j(x_1^{\text{app}},\ldots,x_K^{\text{app}}) \in N_{\gamma+1} \cap W_{\gamma+1}, \quad \text{for } k \in \{1,\ldots,K\}.
\]

Lemma 2.8 yields

\[
F(x_j^{\text{app}}(t) - x_j^{\text{app}}(t)) - A^2 e^{-\left( x_j^{\text{app}}(t) - x_j^{\text{app}}(t) \right)} \in N_{\gamma+2}.
\]

One checks easily that $c_{j+1} - c_j = - \log j - \log(K-j)$ for $j \in \{1,\ldots,K-1\}$, thus

\[
A^2 e^{-\left( x_j^{\text{app}}(t) - x_j^{\text{app}}(t) \right)} = \frac{j(K-j)}{t^2}, \quad j \in \{0,1,\ldots,K,K+1\},
\]

with the usual convention $x_0^{\text{app}}(t) = -\infty$ and $x_K^{\text{app}}(t) = \infty$. Since $j(K-j) - (j-1)(K-j+1) = -(2j-K-1)$, we obtain

\[
\left( x_j^{\text{app}}(t) \right)' + A^2 e^{-\left( x_j^{\text{app}}(t) - x_j^{\text{app}}(t) \right)} - A^2 e^{\left( x_j^{\text{app}}(t) - x_j^{\text{app}}(t) \right)} = 0 \quad \text{for } j \in \{1,\ldots,K\}.
\]
hence
\[ (x^{\text{app}}_j(t))^n + F(x^{\text{app}}_j(t) - x^{\text{app}}_{j-1}(t)) - F(x^{\text{app}}_{j+1}(t) - x^{\text{app}}_j(t)) \in N_{\gamma+2}, \]
and (5.4) follows from Proposition 5.5.

The proof that \( \Theta \) is a contraction in \( S_\gamma \) for \( \gamma > \frac{1}{2}(\sqrt{17} - 1) \) is almost the same as in the case \( K = 2 \). \( \square \)

References

[1] S.-N. Chow and J. K. Hale. Methods of Bifurcation Theory, volume 251 of Grundlehren der mathematischen Wissenschaften. Springer, 1982.
[2] R. Côte, C. Kenig, A. Lawrie, and W. Schlag. Characterization of large energy solutions of the equivariant wave map problem. I. Amer. J. Math., 137(1):139–207, 2015.
[3] R. Côte, Y. Martel, and F. Merle. Construction of multi-soliton solutions for the \( L^2 \)-supercritical gKdV and NLS equations. Rev. Mat. Iberoam., 27(1):273–302, 2011.
[4] R. Côte and C. Muñoz. Multi-solitons for nonlinear Klein-Gordon equations. Forum Math. Sigma, 2:e15, 2014.
[5] M. del Pino, M. Kowalczyk, F. Pacard, and J. Wei. Multiple-end solutions to the Allen-Cahn equation in \( \mathbb{R}^2 \). J. Funct. Anal., 258(2):458–503, 2010.
[6] J.-M. Delort. Existence globale et comportement asymptotique pour l’équation de Klein-Gordon quasi linéaire à données petites en dimension 1. Ann. Sci. École Norm. Sup. (4), 34(1):1–61, 2001.
[7] N. Hayashi and P. I. Naumkin. The initial value problem for the cubic nonlinear Klein-Gordon equation. Z. Angew. Math. Phys., 59(6):1002–1028, 2008.
[8] N. Hayashi and P. I. Naumkin. Quadratic nonlinear Klein-Gordon equation in one dimension. J. Math. Phys., 53(10):103711, 2012.
[9] J. Jendrej. Construction of two-bubble solutions for the energy-critical NLS. Anal. PDE, 10(8):1923–1959, 2017.
[10] J. Jendrej. Dynamics of strongly interacting unstable two-solitons for generalized Korteweg-de Vries equations. arXiv E-Prints, 2018.
[11] J. Jendrej. Nonexistence of two-bubbles with opposite signs for the radial energy-critical wave equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 18(2):735–778, 2018.
[12] J. Jendrej. Construction of two-bubble solutions for energy-critical wave equations. Amer. J. Math., 141(1):55–118, 2019.
[13] J. Jendrej and A. Lawrie. Two-bubble dynamics for threshold solutions to the wave maps equation. Invent. Math., 213(3):1249–1325, 2018.
[14] J. Jendrej, A. Lawrie, and C. Rodriguez. Bubbling dynamics for wave maps with prescribed radiation. arXiv E-Prints, 2019.
[15] E. A. Kopylova and A. I. Komech. On asymptotic stability of kink for relativistic Ginzburg-Landau equations. Arch. Ration. Mech. Anal., 202(1):213–245, 2011.
[16] E. A. Kopylova and A. I. Komech. On asymptotic stability of moving kink for relativistic Ginzburg-Landau equation. Comm. Math. Phys., 302(1):225–252, 2011.
[17] M. Kowalczyk, Y. Martel, and C. Muñoz. Kink dynamics in the \( \phi^4 \) model: asymptotic stability for odd perturbations in the energy space. J. Amer. Math. Soc., 30(3):769–798, 2017.
[18] M. A. Lohe. Soliton structures in \( P(\phi)_2 \). Phys. Rev. D, 20:3120–3130, Dec 1979.
[19] N. Manton and P. Sutcliffe. Topological solitons. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2004.
[20] Y. Martel. Asymptotic \( N \)-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. Amer. J. Math., 127(5):1103–1140, 2005.
[21] Y. Martel and F. Merle. Multi solitary waves for nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 23(6):849–864, 2006.
[22] Y. Martel and F. Merle. Description of two soliton collision for the quartic gKdV equation. Ann. of Math. (2), 174(2):757–857, 2011.
[23] Y. Martel and F. Merle. Inelastic interaction of nearly equal solitons for the quartic gKdV equation. Invent. Math., 183(3):563–648, 2011.
[24] Y. Martel and F. Merle. Inelasticity of soliton collisions for the 5d energy critical wave equation. arXiv E-Prints, 2017.
[25] Y. Martel, F. Merle, and T.-P. Tsai. Stability in \( H^1 \) of the sum of \( K \) solitary waves for some nonlinear Schrödinger equations. Duke Math. J., 133(3):405–466, 2006.
[26] F. Merle. Construction of solutions with exactly $k$ blow-up points for the Schrödinger equation with critical nonlinearity. *Commun. Math. Phys.*, 129(2):223–240, 1990.

[27] P. Raphaël and J. Szeftel. Existence and uniqueness of minimal mass blow up solutions to an inhomogeneous $L^2$-critical NLS. *J. Amer. Math. Soc.*, 24(2):471–546, 2011.

[28] C. Rodriguez. Threshold dynamics for corotational wave maps. *arXiv E-Prints*, 2018.

[29] I. Takyi and H. Weigel. Collective coordinates in one-dimensional soliton models revisited. *Phys. Rev. D*, 94:085008, Oct 2016.

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