On using empirical null distribution in Benjamini-Hochberg procedure

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Abstract: When performing multiple testing, adjusting the distribution of the null hypotheses is ubiquitous in applications. However, the cost of such an operation remains largely unknown in terms of false discovery proportion (FDP) and true discovery proportion (TDP). We explore this issue in the most classical case where the null hypotheses are Gaussian with an unknown rescaling parameters (mean and variance) and where the Benjamini-Hochberg (BH) procedure is applied after a data-rescaling step. Our main result identifies the following sparsity boundary: an asymptotically optimal rescaling (in some specific sense) exists if and only if the sparsity $k$ (number of false nulls) is of order less than $n/\log(n)$, where $n$ is the total number of tests. Our proof relies on new non-asymptotic lower bounds on FDP/TDP, which are of independent interest and share similarities with those developed in the minimax robust statistical theory. Further sparsity boundaries are derived for general location models for which the shape of the null distribution is not necessarily Gaussian.

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1. Introduction

1.1. Background

In large-scale data analysis, the practitioner routinely faces the problem of simultaneously testing a large number $n$ of null hypotheses. In the last decades, a wide spectrum of multiple testing procedures have been developed, providing theoretically-founded control of the amount of false rejections, notably by controlling the false discovery rate (FDR), that is, the average proportion of errors among the rejections, as introduced in Benjamini and Hochberg (1995). Among these procedures, various types of power enhancements have been proposed by taking into account the underlying structure of the data. For instance, let us mention adaptation to the quantity of signal Benjamini et al. (2006); Blanchard and Roquain (2009); Sarkar (2008); Li and Barber (2019), to the signal strength Roquain and van de Wiel (2009); Cai and Sun (2009); Hu et al. (2010); Ignatiadis and Huber (2017); Durand (2019), to the spatial structure Perone Pacifico et al. (2004); Sun and Cai (2009); Ramdas et al. (2019); Durand et al. (2018), or to data dependence structure Leek and Storey (2008); Friguet et al. (2009); Fan et al. (2012); Guo et al. (2014); Delattre and Roquain (2015); Fan and Han (2017), among others. However, most of these theoretical studies assume that the null distribution is exactly known (either for finite $n$ or asymptotically). By contrast, in common practice, the null distribution is often mis-specified, as it is implicitly defined as the “predominant behavior” of the measurements, and is adjusted via some pre-processing steps. For instance, this occurs in genomics Consortium et al. (2007); Zablocki et al. (2014); Jiang and Yu (2016); Amar et al. (2017) neuro-imaging Lee et al. (2016) and astrophysics Szalay et al. (1999); Miller et al. (2001); Sulis et al. (2017).
The issue of finding an appropriate null distribution has been popularized by a famous series of papers by Efron, see Efron (2004, 2007b, 2008, 2009), where he has introduced the concept of "empirical null distribution". Via concrete examples in large-scale data sets, he showed that the theoretical null distribution is often wrong which can lead to questionable discoveries (see, e.g., Table 3 in Efron (2008)). By contrast, he argued that choosing a null distribution fitted from the data, typically $N(θ, σ^2)$ for some parameters $θ, σ^2$, is much more meaningful and leads to a better interpretation of the results.

We adopt in this paper a frequentist point of view on Efron’s empirical Bayes approach. Assume that there exists a true underlying null distribution $N(θ, σ^2)$, for some unknown parameters $θ$ and $σ^2$, and let us consider the classical Benjamini-Hochberg (BH) procedure Benjamini and Hochberg (1995) applied to $p$-values computed from Gaussian null distribution $N(u, s^2)$, for potentially mis-specified values $u, s^2$ of $θ, σ^2$. Figure 1 displays the behavior of that procedure for different values of $u, s^2$ (true, mis-specified or estimated). This simple example illustrates that using a wrong null distribution can lead to poor multiple testing performances, with either an uncontrolled increase of false discoveries (top-right panel), or an uncontrolled decrease of true discoveries (bottom-left panel). By contrast, fitting the null distribution (here with robust estimator of the parameters) can nearly mimic the ideal situation where the true null is known. In a nutshell, the aim of the paper is to address whether the procedure using the empirical null $N(\hat{θ}, \hat{σ}^2)$ can mimic the performances of the procedure using the true null $N(θ, σ^2)$.

Only few work have provided theoretical guarantees for using an empirical null distribution into a multiple testing procedure, even for the simple Gaussian case. The work Jin and Cai (2007); Cai and Jin (2010) proposed a method to estimate the null in a particular context, but without evaluating the cost of such an operation when plugged into a multiple testing procedure. Such an attempt has been made in Ghosh (2012), who showed that the FDR control is maintained under the assumption that incorporating the empirical null distribution is an operation that can make the BH procedure only more conservative. Nevertheless, this assumption is admittedly difficult to check. Other studies have been developed in the one-sided context, for which contaminations (that is non-null measurements) are assumed to come only from the right-side (say) of the global measurement distribution. In that case, the left-tail of the distribution can be used to learn the null. Such an idea has been exploited in Carpentier et al. (2018) to estimate the scaling parameters $θ$ and $σ^2$ within the null $N(θ, σ^2)$ from the left-quantiles of the observed data. Doing so, they show that the plugged BH procedure has performances close (asymptotically in $n$) to those of the BH procedure using the true unknown scaling. Meanwhile, relaxing the Gaussian-null assumption, an FDR controlling procedure has been introduced in Barber and Candès (2015); Arias-Castro and Chen (2017), by only assuming the symmetry of the null. In that case, the null is implicitly learned by estimating the number of false discoveries occurring at the right-side of the null from its left-side. However, the one-sided contamination model is not the most common practical situation where signal can arise at both sides of the null distribution. The case for which the alternative distributions are let arbitrary and potentially two-sided is more difficult than the one-sided case and will be considered throughout the paper.
1.2. Formalization of the problem

1.2.1. Framework for testing a mis-specified null

Let us observe independent real random variables $Y_i, 1 \leq i \leq n$. The distribution of the vector $Y = (Y_i)_{1 \leq i \leq n}$ in $\mathbb{R}^n$ is denoted by $P = \otimes_{i=1}^{n} P_i$.

Following the setting used by Efron (2004), we shall assume in this manuscript that the unknown null distribution is of the form $N(\theta, \sigma^2)$ for some unknown scaling $(\theta, \sigma) \in \mathbb{R} \times (0, \infty)$ (except in Section 5 where extensions are considered). To formalize the idea that the $i$-th null hypothesis is "$P_i$ is equal to the predominant element of $\{P_j, 1 \leq j \leq n\}$" and make the problem well-defined, we assume henceforth that $P = \otimes_{i=1}^{n} P_i$ belongs to the collection $\mathcal{P}$ of all distributions satisfying

$$\left\{ (\theta, \sigma) \in \mathbb{R} \times (0, \infty) : \sum_{i=1}^{n} \mathbb{1}\{P_i = N(\theta, \sigma^2)\} > n/2 \right\} \neq \emptyset. \quad (1)$$

In other words, we assume that there exists a scaling $(\theta, \sigma)$ such that at least half of the $P_i$'s...
are $\mathcal{N}(\theta, \sigma^2)$ so that the null distribution can be uniquely defined as $\mathcal{N}(\theta, \sigma^2)$. For $P \in \mathcal{P}$, we denote $(\theta(P), \sigma(P))$, the unique couple $(\theta, \sigma)$ such that $\sum_{i=1}^{n} 1\{P_i = \mathcal{N}(\theta, \sigma^2)\} > n/2$.

This allows us to formulate the multiple testing problem with unknown scaling of the (Gaussian) null distribution as testing

$$H_{0,i} : "P_i = \mathcal{N}(\theta(P), \sigma^2(P))" \text{ against } H_{1,i} : "P_i \neq \mathcal{N}(\theta(P), \sigma^2(P))",$$

for all $1 \leq i \leq n$. We underline that $H_{0,i}$ is not a point mass null hypotheses, that is, “$P_i = P^{0_{\theta}}$, for some known distribution $P^{0_{\theta}}$, nor a composite null of the type “$P_i$ is a Gaussian distribution”, but rather a point mass null hypothesis with unknown value. Note that the condition (1) is minimal for defining our testing problem; otherwise, quantities $\theta(P)$ and $\sigma(P)$ corresponding to most $P_i$’s are not necessarily uniquely defined.

Let us introduce some notation. We denote $\mathcal{H}_0(P) = \{1 \leq i \leq n : P \text{ satisfies } H_{0,i}\}$ the set of true null hypotheses, $n_0(P) = |\mathcal{H}_0(P)|$ its cardinal, $\mathcal{H}_1(P)$ the complement of $\mathcal{H}_0(P)$ in $\{1, \ldots, n\}$, and $n_1(P) = |\mathcal{H}_1(P)| = n - n_0(P)$. Sometimes, we will consider an asymptotic situation where $n$ tends to infinity. In that case, the quantities $\mathcal{P}$, $P$, $Y$ (and those related) are all depending on $n$, but we remove such dependence in the notation for the sake of clarity.

A multiple testing procedure is defined as a measurable function $R$ taking as input the data $Y$ and returning a subset $R(Y) \subset \{1, \ldots, n\}$ corresponding to the set of rejected null hypotheses among $(H_{0,i}, 1 \leq i \leq n)$. The amount of false positives of $R$ (type I errors) is classically measured by the false discovery proportion of $R$ (see Benjamini and Hochberg (1995)):

$$\text{FDP}(P, R(Y)) = \frac{|R(Y) \cap \mathcal{H}_0(P)|}{|R(Y)| \lor 1},$$

(2)

The expectation $\text{FDR}(P, R) = \mathbb{E}_{Y \sim P}[\text{FDP}(P, R(Y))]$ is the false discovery rate of the procedure $R$. The amount of true positives of $R$ is measured by

$$\text{TDP}(P, R(Y)) = \frac{|R(Y) \cap \mathcal{H}_1(P)|}{n_1(P) \lor 1},$$

(3)

and corresponds to the proportion of (correctly) rejected nulls among the set of false null hypotheses. It has been often used as a power metric for multiple testing procedures, see, e.g. Benjamini and Hochberg (1995); Roquain and van de Wiel (2009); Arias-Castro and Chen (2017); Rabinovich et al. (2017).

1.2.2. Plug-in BH procedures

Our work is largely devoted to the analysis of BH procedures with rescaled $p$-values. This corresponds to the natural approach advocated in Efron (2004) of first estimating the null distribution $(\theta(P), \sigma(P))$ and then plugging it into BH.

Since Benjamini-Hochberg (BH) procedure is defined through the $p$-value family, we first define, for $u \in \mathbb{R}$ and $s > 0$, the rescaled $p$-values

$$p_i(u, s) = 2\Phi \left( \frac{|Y_i - u|}{s} \right), \quad u \in \mathbb{R}, \quad s > 0, \quad 1 \leq i \leq n,$$

(4)

which corresponds to the situation where $\theta(P), \sigma(P)$ have been estimated by $u$, $s$, respectively. By convention, the value $s = +\infty$ is allowed here, which gives a rescaled $p$-value always equal to 1. The oracle $p$-values are then given by

$$p_i^* = p_i(\theta(P), \sigma(P)), \quad 1 \leq i \leq n.$$
Definition 1.1. Let $\alpha \in (0,1)$, $u \in \mathbb{R}$, $s > 0$ and $P \in \mathcal{P}$. The plug-in BH procedure (of level $\alpha$) with scaling $u$ and $s$ is given by

$$BH_\alpha(Y; u, s) = \{1 \leq i \leq n : p_i(u, s) \leq T_\alpha(Y; u, s)\};$$

$$= \{1 \leq i \leq n : p_i(u, s) \leq T_\alpha(Y; u, s) \vee (\alpha/n)\};$$

$$T_\alpha(Y; u, s) = \max \left\{t \in [0, 1] : \sum_{i=1}^n \mathbb{1}\{p_i(u, s) \leq t\} \geq nt/\alpha\right\}. \quad (7)$$

In particular, the ($P$-)oracle BH procedure (of level $\alpha$) is defined as the plug-in BH procedure (of level $\alpha$) with scaling $\theta(P)$ and $\sigma(P)$, that is, is defined by $BH_\alpha^*(Y) = BH_\alpha(Y; \theta(P), \sigma(P))$.

When not ambiguous, we will sometimes drop $Y$ in the notation $BH_\alpha(Y; u, s)$, $T_\alpha(Y; u, s)$, $BH_\alpha^*(Y)$ for short. The oracle procedure $BH_\alpha^*$ corresponds to the situation where the true scaling $(\theta(P), \sigma(P))$ is directly plugged into the BH procedure and is therefore the ideal benchmark procedure in our study. In our framework, the $p$-values $p_i^*$ are all independent, with the property $p_i^* \sim U(0, 1)$ whenever $i \in \mathcal{H}_0(P)$. Hence, it is well known Benjamini and Hochberg (1995); Benjamini and Yekutieli (2001) that its FDR satisfies the following:

$$\forall P \in \mathcal{P}, \quad \text{FDR}(P, BH_\alpha^*) = \alpha n_0(P)/n. \quad (8)$$

To mimic $BH_\alpha^*$, natural candidates are the plug-in BH procedures $BH_{\alpha_0}(\hat{\theta}, \hat{\sigma})$, for some suitable estimators $\hat{\theta}$, $\hat{\sigma}$ of $\theta(P)$, $\sigma(P)$ (by convention, the value $\hat{\sigma} = \infty$ is allowed here). In the sequel, $(\hat{\theta}, \hat{\sigma})$ is called a rescaling.

Next, to evaluate how a rescaling is mimicking $BH_\alpha^*$ on some sparsity range, let us define the following notation: for any procedure $R(Y) \subset \{1, \ldots, n\}$, any sparsity parameter $k \in [1, n/2]$ and any level $\alpha \in (0,1)$, we let

$$\mathbf{I}(R, k) = \sup_{P \in \mathcal{P}, \ n_1(P) \leq k} \{\text{FDR}(P, R)\};$$

$$\mathbf{II}(R, k, \alpha) = \sup_{P \in \mathcal{P}, \ n_1(P) \leq k} \{\mathbb{P}_Y \sim P (\text{TDP}(P, R) < \text{TDP}(P, BH_\alpha^*))\}. \quad (10)$$

Note that $\mathbf{I}(BH_\alpha^*, k) = \alpha$ for any $k$ by (8). In particular, the control $\mathbf{I}(BH_\alpha^*, n) \leq \alpha$ is uniform on $P \in \mathcal{P}$ meaning that any least favorable configuration does not deteriorate the FDR. This is often referred to as a strong control of the FDR in the general multiple testing theory, see, e.g., Dickhaus (2014). The criterion $\mathbf{II}(R, k, \alpha)$ is a type II risk defined relatively to $BH_\alpha^*$; it is small when the TDP of $R$ is at least as large as the one of $BH_\alpha^*$, with a large probability. In particular, the map $\alpha \mapsto \mathbf{II}(R, k, \alpha)$ is nondecreasing. Then, an asymptotically optimal rescaling is defined as a rescaling $(\hat{\theta}, \hat{\sigma})$ such that the plugged BH procedure $BH_{\alpha_0}(\hat{\theta}, \hat{\sigma})$ asymptotically maintains the strong FDR control while having a small relative type II risk.

Definition 1.2. Let $\hat{\theta}$ and $\hat{\sigma}$ be two (sequence of) estimators of $\theta(P)$ and $\sigma(P)$, respectively. For a given sparsity sequence $k_n \in [1, n/2]$, the rescaling $(\hat{\theta}, \hat{\sigma})$ is said to be asymptotically optimal whenever the two following properties hold: there exists a positive sequence $\eta_n \rightarrow 0$ such that

$$\limsup_{n} \sup_{\alpha \in (1/n, 1/2)} \{\mathbf{I}(BH_\alpha(\hat{\theta}, \hat{\sigma}), k_n) - \alpha\} \leq 0; \quad (11)$$

$$\lim_{n} \sup_{\alpha \in (1/n, 1/2)} \{\mathbf{II}(BH_\alpha(\hat{\theta}, \hat{\sigma}), k_n, \alpha(1-\eta_n))\} = 0. \quad (12)$$
Hence, a rescaling is optimal if the corresponding BH procedure mimics the oracle one, both in terms of FDR and TDP. Note that the power statement is made slightly weaker than one could expect at first sight, with a slight decrease of the level in $\text{BH}^*_{\alpha(1-\eta_n)}$. Since $\eta_n$ converges to 0, this modification is very light. Finally, we underline that, while the statements (11) and (12) are formulated in an asymptotic manner for compactness, all our results will be non-asymptotic.

1.3. Presentation of the results

1.3.1. Main result

We now state the main result of the paper.

**Theorem 1.3.** In the setting of Section 1.2 and according to Definition 1.2, the following holds:

(i) for a sparsity $k_n \gg n/\log(n)$, there exists no (sequence of) estimators $(\hat{\theta}, \hat{\sigma})$ such that the scaling $(\hat{\theta}, \hat{\sigma})$ is asymptotically optimal.

(ii) for a sparsity $k_n \ll n/\log(n)$, the scaling $(\tilde{\theta}, \tilde{\sigma})$ given by standard robust estimators (22) is asymptotically optimal.

The part (i) of Theorem 1.3 (lower bound) is proved in Section 2. In a nutshell, this result means that the null distribution cannot be incorporated into the BH procedure in a proper way when there are two many false hypotheses, that is, $k_n \gg n/\log(n)$. Markedly, we show in addition that this impossibility holds for *any* multiple testing procedure $R(Y) \subset \{1, \ldots, n\}$, not necessarily of the plug-in BH type, see Theorem 2.1 below. Obtaining negative results on FDR control has received recently some attention in multiple testing literature Arias-Castro and Chen (2017); Rabinovich et al. (2017); Castillo and Roquain (2018) in various contexts. Here, our lower bound relies on a Le Cam’s two-point reduction scheme. Namely, it is derived by identifying two mixture distributions on $\mathbb{R}^n$ that are indistinguishable while corresponding to distant null hypotheses (see Figure 2) and by studying the impact of such fuzzy configuration on the FDR and TDP metrics. While this argument is classical in the estimation or (single) testing literature (see, e.g., (Tsybakov, 2009) and (Donoho and Jin, 2006)), it is to our knowledge new in the multiple testing context.

The part (ii) of Theorem 1.3 (upper bound) is proved in Section 3. For this, we extend the ideas used in Carpentier et al. (2018) to accommodate the new two-sided geometry of the test statistics. In particular, correcting the $Y_i$’s by $\hat{\theta}$ changes the order of the $p$-values, which was not the case in the one-sided situation. Our proof relies on the symmetry of the Gaussian distribution and on special properties of the BH procedure rejection set when removing one element of the $p$-value family, see, e.g., Ferreira and Zwinderman (2006). Also note that the scaling $(\tilde{\theta}, \tilde{\sigma})$ does not use the knowledge of $k_n$, which means that these estimators are adaptive with respect to the sparsity $k_n$ on the range $k_n \ll n/\log(n)$.

1.3.2. Additional results

We provide three complementary results. First, in the testing literature, type I error rate controls are generally favored over type II error rate controls. In our framework, we can always design a plug-in BH procedure that controls the FDR by simply setting $\hat{\sigma} = \infty$, which
is equivalent to taking $R(Y) = \emptyset$ (no rejection). In view of this remark, we can re-interpret the statement of Theorem 1.3 as follows:

(i) in the dense regime ($k_n \gg n/\log(n)$), it is possible to achieve (11) but not with (12);
(ii) in the sparse regime ($k_n \ll n/\log(n)$), it is possible to achieve both (11) and (12).

A natural question is then: can we achieve the best of the two worlds? Is that possible to find a rescaling satisfying (11) in the dense regime and both (11) and (12) in the sparse regime? We establish in Section 2.2 that such a procedure does not exist, see Corollary 2.2. As a consequence, any procedure controlling the FDR in the dense regime is not optimal in the sparse regime. Conversely, any optimal procedure in the sparse regime is not able to control the FDR in the dense regime. This is the case in particular for the plug-in procedure BH$_n(\hat{\theta},\hat{\sigma})$ considered in Theorem 1.3 (ii). Precisely, combining Corollary 2.2 ($\alpha = c_3/2$) and Theorem 3.1 below establishes the following result.

**Corollary 1.4.** There exist numerical constants $\alpha_0 \in (0,1/2)$ and $c > 0$ such that for any sequence $u_n \to \infty$,

$$\lim \inf \{ I(BH_{\alpha_0}(\hat{\theta},\hat{\sigma}), u_n n/\log(n)) - \alpha_0 \} > c.$$  

Second, in Section 4, we show an analogue of Theorem 1.3 when $\sigma$ is supposed to be known. Hence, the only unknown null parameter is $\theta$ and the class of rescaling is restricted to those of the form $(\hat{\theta},\sigma)$, where $\hat{\theta}$ is an estimator of $\theta$. We establish that the sparsity boundary is slightly modified in this case: impossibility is shown for $k_n \gg n/\log^{1/2}(n)$, while $(\hat{\theta},\sigma(P))$ is asymptotically optimal for $k_n \ll n/\log^{1/2}(n)$ (Theorem 4.1). While the upper-bound part is similar to the upper-bound part of Theorem 1.3 above, the lower bound argument should be adapted to the case where only the location parameter is unknown. More precisely, we establish two types of lower-bounds. We first develop a lower bound valid for any multiple testing procedure (Theorem 4.2), which follows the same philosophy as the lower-bound developed in Theorem 1.3 (via Theorem 2.1). Next, we provide a refined lower bound specifically valid for plug-in BH type procedures. Contrary to the previous lower bounds, it does not state type I error/type II error trade-offs but it establishes that uniform control of the FDR is alone already out of reach. Namely, this result shows that, on the sparsity range $k_n \gg n/\log^{1/2}(n)$, any plug-in procedure exhibits a FDP close to 1/2 and makes around $n^{3/4}$ false discoveries, this on an event of probability close to 1/2 (see Theorem 4.4). Intuitively, this comes from the fact that $\hat{\sigma} = \sigma$ is fixed to the true value and thus cannot compensate the estimation error of $\hat{\theta}$, which irredeemably leads to many false discoveries in that regime.

Third, we extend our results to the case where the null is driven by a known symmetric density $g$ with an unknown location parameter, see Section 5. Therein, we derive lower bounds in two different regimes, when $k_n/n$ tends to zero (Theorem 5.1) and when $k_n/n$ is of order constant (Theorem 5.2). Also, we provide a general upper bound matching the lower bounds under assumptions on $g$ (Theorem 5.3). As expected, the sparsity boundary depends on $g$. For instance, for $\zeta$-Subbotin null $g(x) = L_{\zeta}^{-1} e^{-|x|^\zeta/\zeta}$, $\zeta > 1$, the boundary is proved to be $k_n \asymp n/((\log(n))^{1-1/\zeta}$ (Corollary 5.4), which recovers the Gaussian case for $\zeta = 2$. For the Laplace distribution $g(x) = e^{-|x|/2}$, optimal scaling is possible as long as $k_n \ll n$ (Corollary 5.5). Finally, we further explore the behavior of any procedure for the Laplace distribution on the boundary when $k_n$ is of the same order as $n$ (Proposition 5.6).
1.3.3. Take-home message

Coming back to the issue raised by Efron, our results shed lights on the theoretical behavior of BH procedure using a Gaussian empirical null hypothesis with robust estimators:

- On the feasibility side, it is shown that a small sparsity effect $k_n \ll n/\log(n)$ is enough to ensure that it correctly mimics the performances of the known-null case, both in terms of false discovery rate control and power.
- On the impossibility side, such procedure is shown to irremediably violate the FDR control above the sparsity boundary. In particular, this shows limitation of the empirical null approach in dense regimes, where the data contain more than a constant portion of signal (say, 10%).

1.4. Other related work

A tool closely related to the empirical null approach is the two group model, as introduced in Efron et al. (2001). It has been extensively used in the statistical literature under various forms or extensions, see Sun and Cai (2009); Cai and Sun (2009); Padilla and Bickel (2012); Nguyen and Matias (2014); Heller and Yekutieli (2014); Zablocki et al. (2017); Amar et al. (2017); Cai et al. (2019); Rebafka et al. (2019), among others. In the two-group model, the distribution is assumed to be identical across all the alternatives. The inference generally consists in computing the local FDR (probability of being null given the data) by estimating the null distribution, the alternative distribution and the proportion of nulls. The framework followed here is markedly different, because we adopt a more pessimistic robust and minimax angle: the alternative are let arbitrary and potentially adversarial. Hence, the theoretical guarantees that we demand here are stronger than one would ask in the two-group model. This originates from the Neyman-Pearson’s approach that puts much more emphasis on the type I error rate. In the multiple testing setting, this corresponds to a strong control of the FDR, that is, an FDR control valid uniformly over any parameter configuration, and is the original setting of the seminal paper Benjamini and Hochberg (1995).

Next, let us underline that studying pure frequentist properties of empirical Bayes procedure is a well identified statistical research field, that has received a considerable attention in the last decades for various inference processes (see, e.g., van der Pas et al. (2017); Castillo and Mismer (2018) and references therein). Some work are also flourishing for FDR control in that direction Salomond (2017); Castillo and Roquain (2018), but they only consider known null distributions.

As originally discussed in Efron (2007a), the parameters of the null can act as a random effect that induces dependencies among the observations. In this view, estimating and correcting the null can be used to remove dependencies in the observations, which is a task of independent interest. To tackle this issue, several methods have been proposed see, e.g., Leek and Storey (2008); Friguet et al. (2009); Fan et al. (2012); Fan and Han (2017) in a more challenging multivariate factor model. While the authors provide error bounds for the inferred factor models, none of this work establishes FDR controls on the corresponding corrected BH procedure.

Let us mention few additional related studies with mis-specified null: in Blanchard et al. (2010), the null is unknown and estimated from an independent sample, so the setting is completely different. In Jing et al. (2014), the authors study the effect of non-normality over the
BH procedure using *p*-values calibrated with the Gaussian distribution. This is substantially different from our problem, where the null is assumed Gaussian with an uncertainty in the parameters.

From a technical perspective, our proofs of the impossibility results borrow some ideas from the literature on robust estimation and classical Huber contamination model Huber (1964, 2011).

1.5. Notation and presentation of the paper

**Notation.** For two sequences $u_n$ and $v_n$, $u_n \gg v_n$ means $v_n = o(u_n)$. Given a real number $x$, $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively denote the lower and upper integer parts of $x$. Given $x, y$, $x \land y$ (resp. $x \lor y$) stands the minimum (resp. maximum) of $x$ and $y$. For $Y \sim P$, the corresponding probability is denoted $\mathbb{P}_{Y \sim P}$ or simply $\mathbb{P}$ when there is no confusion. The density of the standard normal distribution is denoted $\phi$ whereas $\Phi$ stands for its tail distribution function. Finally, given a vector $v \in \mathbb{R}^n$, we denote by $v(i)$ the $i$-th order statistic of $v$, that is, the $i$-th smallest entry of $v$.

**Organization of the paper.** Impossibility results are stated and proved in Section 2. A matching upper-bound is stated and proved in Section 3 by using classical robust estimators. Section 4 is devoted to study the situation where the variance of the null is known, while Section 5 provides extensions to a general location model. A discussion is given in Section 6. Additional proofs, lemmas and auxiliary results are deferred to appendices.

2. Lower bound

2.1. General result

To prove the part (i) of Theorem 1.3, we establish a more general impossibility result.

**Theorem 2.1.** There exist numerical positive constants $c_1$–$c_5$ such that the following holds for all $n \geq c_1$ and any $\alpha \in (0, 1)$. Consider any two positive numbers $k_1 \leq k_2$ satisfying

$$c_2 \frac{n \log \left( \frac{2}{\alpha} \right)}{\log(n)} \left[ 1 + \log \left( \frac{k_2}{k_1} \right) \right] \leq k_2 < n/2 .$$

(13)

For any multiple testing procedure $R$ such that

$$FDR(P, R) \leq c_3 , \text{ for any } P \in \mathcal{P} \text{ with } n_1(P) \leq k_2 ,$$

there exists some $P \in \mathcal{P}$ with $n_1(P) \leq k_1$ such that we have

$$\mathbb{P}_{Y \sim P}(|R(Y) \cap \mathcal{H}_1(P)| = 0) \geq 2/5 ;$$

$$\mathbb{P}_{Y \sim P} \left[ |BH_{\alpha/2} \cap \mathcal{H}_1(P)| \geq c_4 \alpha^{-1} \sqrt{\frac{n}{\log{n}}} \right] \geq 1 - e^{-c_5 \alpha^{-1} \sqrt{n/\log(n)}} \geq 4/5 .$$

(14)

In particular, we have that $I(R, k_2) \leq c_3$ implies $II(R, k_1, \alpha/2) \geq 1/5$. 

Theorem 2.1 states that, for any procedure $R$, either the FDR is not controlled at the nominal level $\alpha \leq c_3$ for all $P$ with $n_1(P) \leq k_2$ or that there exists a distribution $P$ with $n_1(P) \leq k_1$ such that $R$ does not make any correct rejection with positive probability while the oracle procedure $BH^*_{n/2}$ make at least (of the order of) $\sqrt{n/\log(n)}$ correct rejections with probability close to one.

Let us show that Theorem 2.1 implies part (i) of Theorem 1.3. Consider any sequence $k_n$ with $n/2 > k_n \gg n/(\log n)$, any sequence $\eta_n \to 0$, an arbitrary sequence of estimators $(\hat{\theta}, \hat{\sigma})$, and choose $\alpha = (c_3 \wedge 1)/2$. Clearly, for $n$ large enough, $k_1 = k_2 = k_n$ satisfies the requirements of Theorem 2.1 and taking $R = BH_n(\hat{\theta}, \hat{\sigma})$ gives that either $I(BH_n(\hat{\theta}, \hat{\sigma}), k_n) - \alpha > (c_3 \wedge 1)/2$ or $II(BH_n(\hat{\theta}, \hat{\sigma}), k_n, \alpha/2) \geq 1/5$. This entails that (11) and (12) cannot hold simultaneously.

2.2. Relation between FDR control in the dense regime and power optimality in the sparse regime

Theorem 1.3 establishes that it is impossible to perform as well as the oracle BH procedure when $k_n \gg n/\log(n)$. As simultaneously controlling the FDR and ensuring optimal power is out of reach, one may require that, at least, the FDR is controlled. Theorem 2.1, applied with $k_1 < k_2$, shows that controlling the FDR in the dense case has consequences on the relative type II risk in the sparse case. More precisely, for some $\epsilon > 0$, Condition (13) and $k_1 \leq k_2$ is satisfied for $k_2 = \log(1/\epsilon)n/\log(n)$ and $k_1 = e^{(c_2 \log(2/\alpha))^{-1} \log(1/\epsilon)e^{n/\log n}}$ (for $\epsilon$ in a specific range), which entails the following result.

**Corollary 2.2.** Consider the same numerical constants $c_1 \sim c_3$ as in Theorem 2.1 above. Take any $\alpha \in (0, c_3)$, any $n \geq c_1$ and fix any $\epsilon \in (n^{-1/2}; (\alpha/2)^{c_2})$. Then for any procedure $R$ with $I(R, k_2) \leq c_3$ for a sparsity $k_2 = \log(1/\epsilon)n/\log(n)$, we have $II(R, k_1, \alpha/2) \geq 1/5$ for a sparsity $k_1 = e^{(c_2 \log(2/\alpha))^{-1} \log(1/\epsilon)e^{n/\log n}}$. In particular, if $n^{-1/4} < (\alpha/2)^{c_2}$, we have for any procedure $R$, 

- if $I(R, n/4) \leq \alpha$, then $II(R, n^{1-\delta}e/4, \alpha/2) \geq 1/5$;
- if $II(R, n^{1-\delta}e/4, \alpha/2) < 1/5$ then $I(R, n/4) > c_3$,

where we let $\delta = 1/(4c_2 \log(2/\alpha)) > 0$.

In plain words, the above corollary entails that a procedure $R$ controlling the FDR up to a sparsity $\log(1/\epsilon)e^{n/\log n}$ (that is of order larger than or equal to the boundary $n/\log(n)$ of Theorem 1.3), suffers from a suboptimal power in a sparse setting where $n_1(P)$ is of order $e^{(c_2 \log(2/\alpha))^{-1} \log(1/\epsilon)e^{n/\log n}}$, for which asymptotic optimality is theoretically possible (as stated in Theorem 1.3). As $\epsilon$ decreases, $R$ is assumed to control the FDR in denser settings and becomes over-conservative in sparser settings. The case $\epsilon = n^{-1/4}$, requiring that the FDR is controlled at the nominal level up to a sparsity $n/4$ enforces that the power is suboptimal in some "easy" settings where $n_1(P)/n$ is polynomially small. In other words, if we require FDR control in the dense regime, we will pay a high power price in the "easy" regime where asymptotic optimality is achievable. Conversely, if a procedure do not pay that price in the sparse regime, it means that it violates the FDR control in the dense regime.

2.3. Proof of Theorem 2.1

The proof is based on the following general argument. We build two collections $\mathcal{P}_1$ and $\mathcal{P}_2$ of distributions. For any $P \in \mathcal{P}_1$, the null distribution is $\mathcal{N}(0, 1)$ and the distribution of the
alternative is fairly separated from the null (see the red and blue curves in the left panel of Figure 2). For any \( P \in \mathcal{P}_2 \), the null distribution is \( \mathcal{N}(0, \sigma^2) \) (with \( \sigma > 1 \)) and the alternative distribution is more concentrated around zero (right panel of Figure 2). We first establish that any multiple testing procedure \( R \) behaves similarly on \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \). Then, we prove that, under \( P \in \mathcal{P}_2 \), if \( |R(Y)| > 0 \), then its FDP is bounded away from zero. In contrast, under \( P \in \mathcal{P}_1 \), if \( |R(Y)| = 0 \), then its TDP is much smaller than that of oracle BH\(_*\). This will allow us to conclude that \( R \) either does not control the FDR under some \( P \in \mathcal{P}_2 \) or has a suboptimal TDP under some \( P \in \mathcal{P}_1 \).

\[
\begin{align*}
\text{Left: } & \quad h = (1 - \pi_1)\phi + \pi_1 f_1 \\
\text{Right: } & \quad h = (1 - \pi_2)\phi^V_{\sigma^2} + \pi_2 f_2
\end{align*}
\]

**Figure 2.** Left: the density \( h \) given by (15) (black), interpreted as a mixture between the null \( \mathcal{N}(0, 1) \) ((1−\( \pi_1 \))\( \phi \) in blue) and the alternative \( f_1 \) (\( \pi_1 f_1 \) in red). Right: the same \( h \) interpreted as a mixture between the null \( \mathcal{N}(0, \sigma^2) \) ((1−\( \pi_2 \))\( \phi^V_{\sigma^2} \) in blue) and the alternative \( f_2 \) (\( \pi_2 f_2 \) in red). \( \pi_1 = 1/8, \pi_2 = 1/4, \sigma_2 \approx 1.51 \). The distance between the vertical dashed gray lines and 0 is \( t_0 \approx 1.47 \) given by (17). See the text for the definitions.

**Step 1: Building a least favorable mixture distribution** Let us denote \( \phi^V_{\sigma}(x) = \phi(x/\sigma)/\sigma \) for all \( x \in \mathbb{R} \) and \( \sigma > 0 \). Fix some \( \pi_1 = k_1/(2n) \) and \( \pi_2 = k_2/(2n) \). For any \( \sigma_2 \geq 1 \), define \( \mu \), the real measure with density

\[
h = (1 - \pi_1)\phi + \pi_1 f_1 = (1 - \pi_2)\phi^V_{\sigma_2} + \pi_2 f_2 = \max((1 - \pi_1)\phi, (1 - \pi_2)\phi^V_{\sigma_2})
\]

where

\[
f_1 = \frac{1}{\pi_1} \left[ (1 - \pi_2)\phi^V_{\sigma_2} - (1 - \pi_1)\phi \right]_+; \quad f_2 = \frac{1}{\pi_2} \left[ (1 - \pi_1)\phi - (1 - \pi_2)\phi^V_{\sigma_2} \right]_+.
\]

Since \( \int (1 - \pi_1)\phi - (1 - \pi_2)\phi^V_{\sigma_2} = \pi_2 - \pi_1 \), we deduce that, if \( \sigma_2 \) is chosen in such a way that \( \int f_2(u)du = 1 \), we have \( \int f_1(u)du = 1 \). Let us prove that \( \int f_2(u)du = 1 \) for a suitable \( \sigma_2 > 1 \). For \( \sigma_2 = 1 \), we have \( \int f_2(u)du = 1 - \pi_1/\pi_2 \in [0, 1) \) (because 0 < \( \pi_1 \leq \pi_2 \)), whereas

\[
\int f_2(u)du \geq \frac{(1 - \pi_2)}{\pi_2} \int [\phi(u) - \phi^V_{\sigma_2}(u)]_+ du \geq 3 \int [\phi(u) - \phi^V_{\sigma_2}(u)]_+ du
\]

since \( \pi_2 \leq 1/4 \). The above expression is larger than 1 for \( \sigma_2 \) large enough (compared to some universal constant). Since \( \int f_2(u)du \) is a continuous function with respect to the variable \( \sigma_2 \), there exists at least one value of \( \sigma_2 > 1 \), depending only on \( \pi_1 \) and \( \pi_2 \), such that both \( f_1 \) and
Recall that $\mu$ denotes the probability measure on $\mathbb{R}$ with density $h$ given by (15). Let $Q = \mu^{\otimes n}$ be the corresponding product distribution on $\mathbb{R}^n$. Let $Z_{1,i}, 1 \leq i \leq n,$ be i.i.d. and all following a Bernoulli distribution with parameter $\pi_1$. Let $Q_{1,z}$ the distribution on $\mathbb{R}^n$ of density $\prod_{i=1}^n((1 - z_i)\phi + z_if_1)$ for $z \in \{0,1\}^n$, so that $Y \sim Q_{1,z_1}$ is distributed as $Q$ unconditionally on $Z_1$. If $\sum_{i=1}^n z_i < n/2$, we have $\theta(Q_{1,z}) = 0, \sigma_1(Q_{1,z}) = 1$ and $H_1(Q_{1,z}) = \{1 \leq i \leq n : z_i = 1\}$. If $\sum_{i=1}^n z_i \geq n/2, P_{1,z} \notin \mathcal{P}$. Nevertheless, we still let $\theta(Q_{1,z}) = 0, \sigma_1(Q_{1,z}) = 1$ and $H_1(Q_{1,z}) = \{1 \leq i \leq n : z_i = 1\}$ by convention. Define similarly $Z_2$ and $Q_{2,z}$, so that $Y \sim Q_{2,z_2}$ has the same distribution as $Y \sim Q_{1,z_1}$ (that is, $Q$) unconditionally on $Z_1$ and $Z_2$. In the sequel, we denote $n_1(Z_1) = \sum_{i=1}^n 1\{Z_{1,i} = 1\}$ and $n_1(Z_2) = \sum_{i=1}^n 1\{Z_{2,i} = 1\}$, so that $n_1(Q_{1,z_1}) = n_1(Z_1)$ and $n_1(Q_{2,z_2}) = n_1(Z_2)$.

As a consequence, $Q$ can be both interpreted as a mixture of $Q_{1,z}$ (with $\theta(Q_{1,z}) = 0$ and $\sigma(Q_{1,z}) = 1$) and as a mixture of $Q_{2,z}$ (with $\theta(Q_{2,z}) = 0$ and $\sigma(Q_{2,z}) = \sigma_2$).

Consider any multiple testing procedure $R$ and define the event $A = \{|R(Y)| > 0\}$. Since $1 = Q(A) + Q(A^c) = \mathbb{E}Z_1Q_{1,Z_1}(A) + \mathbb{E}Z_2Q_{2,Z_2}(A^c)$, this entails that either $\mathbb{E}Z_1Q_{1,Z_1}(A^c) \geq 1/2$ or $\mathbb{E}Z_2Q_{2,Z_2}(A) \geq 1/2$. We show in Step 2 that, if $\mathbb{E}Z_2Q_{2,Z_2}(A) \geq 1/2$, $R$ does not control the FDR under some $Q_{2,z}$ with $n_1(Q_{2,z}) \leq k_2$, whereas we establish in Step 3 that, if $\mathbb{E}Z_1Q_{1,Z_1}(A^c) \geq 1/2$, $R$ is over-conservative under some $Q_{1,z}$ with $n_1(Q_{1,z}) \leq k_1$.

**Step 2:** if $\mathbb{P}_{Y \sim Q}(|R(Y)| > 0) \geq \frac{1}{2}$ then FDR($P_2,R) \geq c_3$ for some $P_2$ with $n_1(P_2) \leq k_2$

We consider the mixture distribution where $Z_2$ is sampled according to a Bernoulli distribution with parameters $\sigma_2$ and $Y \sim Q_{2,z_2}$. We have by the Fubini theorem,

$$
\mathbb{E}_{Z_2}\left[\mathbb{E}_{Y \sim Q_{2,z_2}}[\text{FDP}(Q_{2,z_2}, R(Y))]\right]
= \mathbb{E}_{Z_2}\left[\mathbb{E}_{Y \sim Q_{2,z_2}}\left[\frac{\sum_{i \in R(Y)} 1\{Z_{2,i} = 0\}}{|R(Y)|} 1\{|R(Y)| > 0\}\right]\right]
= \mathbb{E}_{Y \sim Q}\left[\mathbb{E}_{Z_2}\left[\frac{\sum_{i \in R(Y)} 1\{Z_{2,i} = 0\}}{|R(Y)|} 1\{|R(Y)| > 0\} \bigg| Y\right]\right]
= \mathbb{E}_{Y \sim Q}\left[\frac{\sum_{i \in R(Y)} \mathbb{P}(Z_{2,i} = 0 | Y)}{|R(Y)|} 1\{|R(Y)| > 0\}\right].
$$

Next, we have

$$
\mathbb{P}(Z_{2,i} = 0 | Y) = \frac{(1 - \pi_2)\phi_{\sigma_2}(Y_i) h(Y_i)}{\max((1 - \pi_2)\phi(Y_i), (1 - \pi_2)\phi_{\sigma_2}(Y_i))}
\geq 1 \land \frac{(1 - \pi_2)\phi_{\sigma_2}(Y_i)}{(1 - \pi_2)\phi(Y_i)} \geq 1 \land \frac{(1 - \pi_2)}{(1 - \pi_2)\sigma_2} \exp\left[\frac{Y_i^2}{2} \left(1 - \frac{1}{\sigma_2^2}\right)\right].
$$

Since $\sigma_2 \geq 1$, $\pi_2 \leq 1/4$ and since we proved in step 1 that $\sigma_2 \leq c_1'$ (for some numerical constant $c_1'$), we obtain

$$
\mathbb{P}(Z_{2,i} = 0 | Y) \geq 1 \land \frac{(1 - \pi_2)}{(1 - \pi_2)\sigma_2} \geq 1 \land \frac{3}{4c_1'}.
$$
Combining this with (16), we obtain
\[
\mathbb{E}_{Z_2} \left[ \mathbb{E}_{Y \sim Q_{2,Z_2}} \left[ \text{FDP}(Q_{2,Z_2}, R(Y)) \right] \right] \geq \left( 1 \wedge \frac{3}{4c_1} \right) \mathbb{P}_{Y \sim Q}(R(Y) > 0) \geq \left( 1 \wedge \frac{3}{4c_1} \right)/2.
\]

Recall that \( n_1(Z_2) \) follows a Binomial distribution with parameter \( n \) and \( \pi_2 = k_2/(2n) \). By Chebyshev inequality, we have
\[
\mathbb{P}(|n_1(Z_2) - k_2/2| > k_2/4) \leq \frac{n\pi_2(1 - \pi_2)}{(k_2/4)^2} \leq \frac{8}{k_2}.
\]

This implies that there exists \( z_2 \) with \( n_1(z_2) \in [k_2/4; k_2] \) such that
\[
\text{FDR}(Q_{2,z_2}, R) = \mathbb{E}_{Y \sim Q_{2,z_2}} \left[ \text{FDP}(Q_{2,z_2}, R(Y)) \right] \geq \left( 1 \wedge \frac{3}{4c_1} \right)/2 - \frac{8}{k_2},
\]

which is bounded away from zero for \( k_2 \) large enough, this last condition being ensured by (13) and \( n \) large enough. To summarize, we have proved that, for \( P_2 = Q_{2,z_2} \), we have \( \text{FDR}(P_2, R(Y)) \geq c'_3 \) for some universal constant \( c'_3 \in (0, 1) \) and \( n_1(P_2) \leq k_2 \).

**Step 3:** If \( \mathbb{P}_{Y \sim Q}(|R(Y)| = 0) \geq 1/2 \), then \( R \) is over-conservative, for some \( P_1 \) with \( n_1(P_1) \leq k_1 \) - Applying Chebyshev inequality as in Step 2, we deduce that \( \mathbb{P}(n_1(Z_1) \in [k_1/4; k_1]) \geq 1 - 8/k_1 \). Since \( \mathbb{E}_{Z_1} \mathbb{P}_{Y \sim P_{1,z_1}}(|R(Y)| = 0) \geq 1/2 - 8/k_1 \), this implies that there exists \( z_1 \) with \( n_1(z_1) \leq k_1 \) such that \( \mathbb{P}_{Y \sim P_{1,z_1}}(|R(Y)| = 0) \geq 1/2 - 8/k_1 \). Since (13) can be satisfied only if \( \log(k_2/k_1) \leq (2c_2 \log 2)^{-1} \log(n) \), that is, \( k_1 \geq k_2 n^{(2c_2 \log 2)^{-1}} \), by choosing \( c_1 \) and \( c_2 \) large enough, we may assume that \( 1/2 - 8/k_1 \geq 2/5 \). In the sequel, we fix \( P_1 = P_{1,z_1} \).

For such \( P_1 \) with \( n_1(P_1) \leq k_1 \), we have therefore \( \mathbb{P}_{Y \sim P_1}(|R(Y)| = 0) \geq 2/5 \). In contrast, we claim that \( \text{BH}^* \) rejects many false null hypotheses with positive probability.

Before this, let us provide further properties of \( \sigma_2 \) and \( h \). The positive number \( u_0 \) satisfying \( (1 - \pi_1)\phi(u_0) = (1 - \pi_2)\phi_{\sigma_2}^V(u_0) \) is defined as
\[
u_0 = \frac{2\sigma_2^2}{\sigma_2^2 - 1} \log \left( \frac{\sigma_21 - \pi_1}{\sigma_21 - \pi_2} \right).
\]

We easily check that we have \( (1 - \pi_1)\phi(u) > (1 - \pi_2)\phi_{\sigma_2}^V(u) \) if and only if \( |u| < u_0 \), and \( (1 - \pi_1)\phi(u) < (1 - \pi_2)\phi_{\sigma_2}^V(u) \) if and only if \( |u| > u_0 \), so that \( f_1(u) > 0 \) if and only if \( |u| > 0 \) and \( f_2(u) > 0 \) if and only if \( |u| < u_0 \).

**Lemma 2.3.** There exists a numerical constant \( c'_0 \in (0, 1) \) such that \( \sigma_2 \) satisfies
\[
\sigma_2 - 1 \geq c'_0 \pi_2 [1 + \log(\pi_2/\pi_1)]^{-1}.
\]

Also, there exists another numerical constant \( c'_{0} \in (0, 1) \) such that, for \( n \) large enough, we have \( \mathbb{P}(u_0/\sigma_2) \geq 10 \sqrt{2 \log(2n)/n} \) provided that \( \log(\pi_2/\pi_1) \leq c'_{0} \log(n) \).

Let \( u_1 \) be the smallest number such that for all \( u \geq u_1 \), one has
\[
\pi_1 f_1(u) \geq 8\alpha^{-1} \phi(u).
\]

This implies that for all \( u \geq u_1 \), \( f_2(u) = 0 \). From the definition of \( f_1 \), we derive that \( (1 - \pi_2)\phi_{\sigma_2}^V(u_1) = \phi(u_1) \times 8\alpha^{-1} + (1 - \pi_1) \), which is again equivalent to
\[
u_1 = \frac{2\sigma_2^2}{\sigma_2^2 - 1} \log \left( \frac{\sigma_28\alpha^{-1} + (1 - \pi_1)}{1 - \pi_2} \right).
\]
Lemma 2.4. There exists a positive numerical constant $c'_3$ such that the following holds for all $\alpha \in (0, 1)$. If
\[
\frac{1 + \log(\pi_2/\pi_1)}{\pi_2} \log \left( \frac{2}{\alpha} \right) \leq c'_3 \log(n),
\]
then, we have $u_1 \leq \sqrt{\log(n)}$.

Now, recall that $\text{BH}_{\alpha/2}^*$ procedure does use some knowledge of the true underlying distribution $P_1$, namely, $\theta(P_1) = 0$ and $\sigma(P_1) = 1$. Hence, it can be written as $\text{BH}_{\alpha/2}^* = \{i \in \{1, \ldots, n\} : |Y_i| \geq \hat{u}\}$ for
\[
\hat{u} = \min \left\{ u \in \mathbb{R}_+ : \sum_{i=1}^n \mathbbm{1}\{|Y_i| \geq u\} \geq 4n\alpha^{-1}\overline{\Phi}(u) \right\}.
\]

Hence, we shall prove that $\sum_{i \in \mathcal{H}_1(P_1)} \mathbbm{1}\{|Y_i| \geq \hat{u}\} > 0$ is large with high probability. For this, let us consider $N = \sum_{i \in \mathcal{H}_1(P_1)} \mathbbm{1}\{|Y_i| \geq u_1\}$ with $u_1$ as in (19). By (18), we have $2\int_{u_1}^{\infty} f_1(u)du \geq 16\pi_1^{-1}u^{-1}\overline{\Phi}(u_1)$. Since $n_1(P_1) \geq k_1/4 = \pi_1 n/2$, $N$ stochastically dominates the binomial distribution with parameters $[\pi_1 n/2]$ and $16\pi_1^{-1}u^{-1}\overline{\Phi}(u_1)$. Applying the Bernstein inequality yields $\mathbb{P}_{Y \sim P_1}(N \leq q/2) \leq e^{-3q/28}$ for $q = \pi_1 n/2 \times 16\pi_1^{-1}u^{-1}\overline{\Phi}(u_1) = 8n\alpha^{-1}\overline{\Phi}(u_1)$. By (20), $N \geq q/2$ implies $\hat{u} \leq u_1$. This leads us to
\[
\mathbb{P}_{Y \sim P_1}(\hat{u} \leq u_1, N \geq 4n\alpha^{-1}\overline{\Phi}(u_1)) \geq 1 - e^{-3q/28} = 1 - e^{-(6/7)\alpha^{-1}\overline{\Phi}(u_1)}.
\]

In view of condition (13), we can apply Lemma 2.4 which gives $u_1 \leq \sqrt{\log(n)}$. Next, by Lemma D.2, for $n$ larger than a numerical constant, we have $n\overline{\Phi}(\sqrt{\log(n)}) \geq c'\sqrt{n/(\log n)}$, for some other numerical constant $c' > 0$. Hence, for $n$ larger than a numerical constant, with probability at least $1 - 1/n$, we have
\[
|\text{BH}_{\alpha/2}^* \cap \mathcal{H}_1(P_1)| = \sum_{i \in \mathcal{H}_1(P_1)} \mathbbm{1}\{|Y_i| \geq \hat{u}\} \geq N \geq 4n\alpha^{-1}\overline{\Phi}(u_1) \geq c'\alpha^{-1}\sqrt{n/(\log n)}.
\]

Conclusion Step 1 entails that either $\mathbb{E}_{Z_2} Q_{2,Z_2}(A) \geq 1/2$ or $\mathbb{E}_{Z_1} Q_{1,Z_1}(A^c) \geq 1/2$. In the former case, Step 2 implies that $\sup_{P \in \mathcal{P}, n_1(P) \leq k_1} \text{FDR}(P, R) \geq c_3$. In the latter case, we deduce from Step 3 that, for some $P \in \mathcal{P}$ with $n_1(P) \leq k_1$, we have $\mathbb{P}_{Y \sim P_1}(|R(Y)| = 0) \geq 2/5$, whereas
\[
\mathbb{P}_{Y \sim P_1} \left[ |\text{BH}_{\alpha/2}^* \cap \mathcal{H}_1(P_1)| \geq c'\alpha^{-1}\sqrt{n/(\log n)} \right] \geq 1 - e^{-c_5\alpha^{-1}\sqrt{n/(\log n)}}.
\]
This concludes the proof by choosing appropriately the numerical positive constants $c_1\cdots c_4$.

3. Upper bound

In this section, we prove (ii) of Theorem 1.3. Since our framework allows arbitrary alternative distributions, we consider simple robust estimators for $(\theta(P), \sigma(P))$ defined by
\[
\tilde{\theta} = Y_{(\lfloor n/2 \rfloor)}; \quad \overline{\sigma} = U_{(\lfloor n/2 \rfloor)}/\overline{\Phi}^{-1}(1/4),
\]
where $U_i = |Y_i - Y_{(\lfloor n/2 \rfloor)}|$ and $\overline{\Phi}^{-1}(1/4) \approx 0.674$. While the empirical median $\tilde{\theta}$ is a standard robust estimator for the mean, let us give some rationale behind the quantile estimator $\overline{\sigma}$:
under the null, the variables $|Y_i - \theta|/\sigma$ are i.i.d. and distributed as the absolute value of a standard Gaussian variable. Hence, taking the median of the $|Y_i - \theta|$ should be a robust estimator of $\sigma$ times the median of the absolute value of a standard Gaussian variable, that is, of $\sigma \Phi^{-1}(1/4)$. Rescaling suitably this quantity and replacing $\theta$ by $\tilde{\theta}$ leads to the definition of $\tilde{\sigma}$.

These two estimators are shown to be minimax in Chen et al. (2018) (as a matter of fact, in a slightly different mixture model). We will use here specific properties of these estimators, to be found in Section C.1 further on.

### 3.1. FDR and TDP bounds

**Theorem 3.1.** In the setting of Section 1.2, there exist universal constants $c_1$, $c_2 > 0$ such that the following holds for all $n \geq c_1$ and $\alpha \in (0, 0.5)$. Consider any number $k \leq 0.1n$ such that $\eta = c_2 \log(n/\alpha) \left( (k/n) \lor n^{-1/6} \right) \leq 0.05$. Then, we have

\[
\mathbf{I}(BH_\alpha(\tilde{\theta}, \tilde{\sigma}), k) \leq \alpha (1 + \eta) + e^{-\sqrt{n}}; \quad (23)
\]

\[
\mathbf{II}(BH_\alpha(\tilde{\theta}, \tilde{\sigma}), k, \alpha(1 - \eta)) \leq e^{-\sqrt{n}}. \quad (24)
\]

Let us check that Theorem 3.1 implies (ii) of Theorem 1.3. If $\log(n)k_n/n$ tends to zero and $\alpha \in (1/n, 1/2)$, we have $\eta \leq 2c_2 \log(n) \left( \frac{k_n}{n} \lor n^{-1/6} \right)$ which is smaller than 0.05 for $n$ large enough, and by (23) above and (8),

\[
\sup_{\alpha \in (1/n, 1/2)} \{ \mathbf{I}(BH_\alpha(\tilde{\theta}, \tilde{\sigma}), k) - \alpha \} \leq \eta + e^{-\sqrt{n}},
\]

which converges to 0 as $n$ grows to infinity. This gives (11) for $(\tilde{\theta}, \tilde{\sigma}) = (\tilde{\theta}, \tilde{\sigma})$. Similarly,

\[
\sup_{\alpha \in (1/n, 1/2)} \{ \mathbf{II}(BH_\alpha(\tilde{\theta}, \tilde{\sigma}), k, \alpha(1 - \eta)) \} \leq e^{-\sqrt{n}} \to 0,
\]

which gives (12) for $(\tilde{\theta}, \tilde{\sigma}) = (\tilde{\theta}, \tilde{\sigma})$ and $\eta_n = \eta$.

The remainder of the section is devoted to the proof of Theorem 3.1. The general argument can be summarized as follows. Observing that the estimators $\tilde{\theta}, \tilde{\sigma}$ converge at the rate $n_1(P) / n + n^{-1/2}$ (Lemma C.1), we mainly have to quantify the impact of these errors on the FDR/TDP metrics. To show (23), we establish that the FDR metric is at worst perturbed by the estimation rate multiplied by $\log(n/\alpha)$. Here, $\alpha/n$ corresponds to the smallest threshold value of the $p$-value by a BH procedure. This can be shown by studying how the $p$-value process is affected by misspecifying the scaling parameters (Lemma 3.3). A difficulty stems from the fact that the FDR metric is not monotonic in the rejection set, so that specific properties of BH procedure and of the estimators $\tilde{\theta}, \tilde{\sigma}$ should be used (Lemmas 3.2 and 3.4).

The second result (24) is proved similarly, the main difference being that we need a slight decrease in the level $\alpha$ (Lemma 3.3) of the oracle procedure $BH^*_\alpha$ to compare the BH thresholds $T_\alpha(\theta(P), \sigma(P))$ and $T_\alpha(\tilde{\theta}, \tilde{\sigma})$. This results in a level $\alpha(1 - \eta)$ instead of $\alpha$ in (24).
3.2. Proof of (23) in Theorem 3.1

Fix $P \in \mathcal{P}$ with $n_0(P)/n \geq 0.9$. First denote

$$
\delta = c \left( \frac{n_1(P) + 1}{n} + n^{-1/6} \right)
$$

$$
\Omega = \left\{ \frac{|\hat{\theta} - \theta|}{\sigma} \leq \delta, \frac{|\hat{\sigma} - \sigma|}{\sigma} \leq \delta, \hat{\sigma} \in [\sigma/2; 2\sigma], |\hat{\theta} - \theta| < 0.3 \hat{\sigma} \right\},
$$

with $c > 0$ being a universal constant chosen small enough so that $\mathbb{P}(\Omega^c) \leq 6e^{-n^{2/3}}$. This is possible according to Lemma C.1 (used with $x = n^{2/3}$).

For any $i \in \{1, \ldots, n\}$, we introduce $Y(i)$ as the vector of $\mathbb{R}^n$ such that $Y(j) = Y_j$ for $j \neq i$ and $Y(i) = \text{sign}(Y_i - \theta) \times \infty$. Hence $Y(i)$ is such that the $i$-th observation has been set $-\infty$ or $+\infty$ depending on the sign of $Y_i - \theta$. The estimators based on the modified sample $Y(i)$ are then defined by

$$
\tilde{\theta}(i) = Y(i)_{(1/2)}; \quad \tilde{\sigma}(i) = U(i)_{(1/2)}/\Phi^{-1}(1/4),
$$

for $U(j) = |Y(j) - \tilde{\theta}(i)|$, $1 \leq j \leq n$. As justified at the end of the proof, the purpose of these modified samples is to introduce some independence between the oracles orimsart-generic ver. 2014/07/30 file: RV2019_Arxiv.tex date: December 9, 2019

It turns out that, for small rescaled $p$-values $p_i(\tilde{\theta}, \tilde{\sigma})$, the estimators $\tilde{\theta}(i)$ and $\tilde{\sigma}(i)$ are not modified. Furthermore, the BH threshold does not change when replacing $Y$ by $Y(i)$. These two facts lead to the following lemma.

**Lemma 3.2.** Consider any $i \in \{1, \ldots, n\}$ and any $\alpha \in (0, 0.5)$. Provided that $|\hat{\theta} - \theta| < 3\hat{\sigma}$, we have

$$
1\{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y; \tilde{\theta}, \tilde{\sigma})\} = 1\{p_i(\tilde{\theta}(i), \tilde{\sigma}(i)) \leq T_\alpha(Y(i); \tilde{\theta}(i), \tilde{\sigma}(i))\}.
$$

Moreover, if $p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y; \tilde{\theta}, \tilde{\sigma})$, we have $\tilde{\theta}(i) = \tilde{\theta}$, $\tilde{\sigma}(i) = \tilde{\sigma}$ and $T_\alpha(Y(i); \tilde{\theta}(i), \tilde{\sigma}(i)) = T_\alpha(Y; \tilde{\theta}, \tilde{\sigma}) \geq \alpha/n$.

Combining this lemma with the definition of the FDP, we get

$$
\text{FDP}(P, BH_\alpha(Y; \tilde{\theta}, \tilde{\sigma})) = \frac{1}{n} \sum_{i \in \mathcal{H}_0(P)} \frac{1\{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y; \tilde{\theta}, \tilde{\sigma})\}}{T_\alpha(Y; \tilde{\theta}, \tilde{\sigma}) \lor (\alpha/n)}/\Omega_\Omega
$$

$$
= \frac{1}{n} \sum_{i \in \mathcal{H}_0(P)} \frac{1\{p_i(\tilde{\theta}(i), \tilde{\sigma}(i)) \leq T_\alpha(Y(i); \tilde{\theta}(i), \tilde{\sigma}(i))\}}{T_\alpha(Y(i); \tilde{\theta}(i), \tilde{\sigma}(i))}/\Omega_\Omega
$$

Now, note that, on $\Omega$, when $\tilde{\theta}(i) = \tilde{\theta}$, $\tilde{\sigma}(i) = \tilde{\sigma}$, we have $|\tilde{\theta}(i) - \theta| \leq \sigma \delta$, $|\tilde{\sigma}(i) - \sigma| \leq \sigma \delta$, $\tilde{\sigma}(i) \geq \sigma/2$. The following key lemma compares the hypotheses rejected by the oracle BH procedure and the rescaled procedure.

**Lemma 3.3.** For arbitrary estimators $\tilde{\theta}, \tilde{\sigma}$, any $\theta \in \mathbb{R}$, $\sigma > 0$, $\delta > 0$, $\alpha \in (0, 0.8)$, $t_0 \in (0, \alpha)$, define

$$
\eta = \delta c \left( (2 \log(1/t_0))^{1/2} + 2 \log(1/t_0) \right),
$$
with the constant $c > 0$ of Corollary C.6. Assume that $\hat{\sigma} \in (\sigma/2; 2\sigma)$, $|\hat{\theta} - \theta| \leq (\sigma \land \hat{\sigma})\delta$, $|\hat{\sigma} - \sigma| \leq (\sigma \land \hat{\sigma})\delta$, and $\eta \leq 0.05$. Then, for all $i \in \{1, \ldots, n\}$,

- if $T_\alpha(\hat{\theta}, \hat{\sigma}) \lor (\alpha/n) \geq t_0$, we have
  \[
  \mathbb{I}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma})\} \leq \mathbb{I}\{p_i(\theta, \sigma) \leq (1 + \eta)T_\alpha(\hat{\theta}, \hat{\sigma})\} \leq \mathbb{I}\{p_i(\theta, \sigma) \leq T_\alpha(1 + \eta)(\theta, \sigma)\};
  \] (26)

- if $T_{0.95}(\theta, \sigma) \lor (0.95\alpha/n) \geq t_0$, we have
  \[
  \mathbb{I}\{p_i(\theta, \sigma) \leq T_\alpha(1 - \eta)(\theta, \sigma)\} \leq \mathbb{I}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma})\}. \] (27)

Intuitively, (26) above implies that any rescaled procedure is more conservative than the oracle procedure $BH^{*}_{\alpha(1 + \eta)}$ with an enlarged parameter $\alpha(1 + \eta)$.

By definition of $\eta$ in the statement of Theorem 3.1 and taking $\hat{\theta} = \theta^{(i)}$, $\hat{\sigma} = \sigma^{(i)}$, $\theta = \theta(P)$, $\sigma = \sigma(P)$, $t_0 = \alpha/n$, we are in position to apply (26). For all $i \in \{1, \ldots, n\}$, we have
\[
\mathbb{I}\{p_i(\theta^{(i)}, \sigma^{(i)}) \leq T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})\} \leq \mathbb{I}\{p^*_i \leq (1 + \eta)T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})\},
\] where we recall that $p^*_i = p_i(\theta(P), \sigma(P))$ is the $i$-th oracle $p$-value. This gives
\[
\text{FDP}(P, BH_\alpha(Y; \tilde{\theta}, \tilde{\sigma})) \mathbb{I}_\Omega \leq \frac{\alpha}{n} \sum_{i \in H_0(P)} \mathbb{E}_P \left[ \mathbb{I}\{p^*_i \leq (1 + \eta)T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})\} \right].
\]

The following lemma stems from the symmetry of the normal distribution.

**Lemma 3.4.** For any $P \in \mathcal{P}$, any $i \in H_0(P)$, $|Y_i - \theta(P)|$ is independent of $Y^{(i)}$, and thus also of the estimators $(\bar{\theta}^{(i)}, \bar{\sigma}^{(i)})$.

By Lemma 3.4, for $i \in H_0(P)$, the oracle $p$-value $p^*_i = 2\Phi \left( \frac{|Y_i - \theta(P)|}{\sigma(P)} \right)$ is independent of $(Y^{(i)}, \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})$ and thus of $T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})$. As a result, we obtain by integration
\[
\mathbb{E}_P \left[ \text{FDP}(P, BH_\alpha(Y; \tilde{\theta}, \tilde{\sigma})) \mathbb{I}_\Omega \right] \leq \frac{\alpha}{n} \sum_{i \in H_0(P)} \mathbb{E}_P \left[ \mathbb{I}\{p^*_i \leq (1 + \eta)T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})\} \right] \leq \frac{\alpha}{n} \sum_{i \in H_0(P)} \mathbb{E}_P \left[ \mathbb{P}\{p^*_i \leq (1 + \eta)T_\alpha(Y^{(i)}; \bar{\theta}^{(i)}, \bar{\sigma}^{(i)})\} \mid Y^{(i)}, \bar{\theta}^{(i)}, \bar{\sigma}^{(i)} \right] \leq \frac{\alpha n_0(P)}{n} (1 + \eta),
\] where we used that $p^*_i \sim U(0, 1)$ for $i \in H_0(P)$. This entails (23) of Theorem 3.1.

### 3.3. Proof of (24) in Theorem 3.1

Take $P, \delta, \Omega$ as in the previous section. On the event $\Omega$, the conditions of Lemma 3.3 are satisfied with $\hat{\theta} = \theta, \hat{\sigma} = \sigma, \theta = \theta(P), \sigma = \sigma(P)$, and $t_0 = 0.95\alpha/n$. Hence, (27) ensures
that, for all $i \in \{1, \ldots, n\}$, $\mathbb{1}\{p_i^* \leq T_{\alpha(1-\eta)}(\theta(P), \sigma(P))\} \leq \mathbb{1}\{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_{\alpha}(\tilde{\theta}, \tilde{\sigma})\}$ and thus $\text{TDP}(P, \text{BH}^*_\alpha(1-\eta)) \leq \text{TDP}(P, \text{BH}_\alpha(\theta, \sigma))$. Hence, we have

$$
\mathbb{P}(\text{TDP}(P, \text{BH}^*_\alpha(1-\eta)) > \text{TDP}(P, \text{BH}_\alpha(\tilde{\theta}, \tilde{\sigma}))) \leq \mathbb{P}(\Omega^c) \leq 6e^{-n^{2/3}} \leq e^{-\sqrt{n}},
$$

for $n$ larger than a universal constant.

4. Known variance

This section is dedicated to the simpler case where $\sigma(P)$ is known to the statistician, so that only the mean $\theta(P)$ has to be estimated. In this setting, it turns out that the boundary for asymptotic optimality is $n/\sqrt{\log(n)}$ instead of $n/\log(n)$.

**Theorem 4.1.** In the setting of Section 1.2 and according to Definition 1.2, the following holds:

(i) for a sparsity $k_n$ with $k_n \log^{1/2}(n)/n \gg 1$, there exists no (sequence of) estimators $\tilde{\theta}$ such that the scaling $(\tilde{\theta}, \sigma(P))$ is asymptotically optimal.

(ii) for a sparsity $k_n$ with $k_n \log^{1/2}(n)/n = o(1)$, the scaling $(\tilde{\theta}, \sigma(P))$ given by (22) is asymptotically optimal.

The upper bound (ii) is proved similarly to the upper bound of Theorem 1.3, but with the weaker condition $k_n \log^{1/2}(n)/n = o(1)$. For this, one readily checks that Theorem 3.1 extends to the case where $\tilde{\sigma} = \sigma(P)$ up to replacing $\eta$ by $\eta = c_2 \log^{1/2}(n/\alpha) \left(\frac{n_1(P)+1}{n} + n^{-1/6}\right)$ (and possibly modifying the constants $c_1$ and $c_2$). The proofs are exactly the same, except that Lemma 3.3 has to be replaced by Lemma A.3. See Section A.3 for the details. In the remainder of this section, we focus on the impossibility results. We first establish in Theorem 4.2 the counterpart of Theorem 2.1. This lower bound is valid for arbitrary testing procedures. Next, we provide a sharper lower bound for plug-in procedures.

4.1. General lower bound

**Theorem 4.2.** There exist numerical positive constants $c_1$–$c_5$ such that the following holds for all $n \geq c_1$ and any $\alpha \in (0, 1)$. Consider two positive numbers $k_1 \leq k_2$ satisfying

$$
c_2 \frac{n \log^{2}(\alpha)}{\log(n)} \sqrt{1 + \log \left(\frac{k_2}{k_1}\right)} \leq k_2 < n/2,
$$

(28)

For any multiple testing procedure $R$ satisfying

$$
\text{FDR}(P, R) \leq c_3, \quad \text{for any } P \in \mathcal{P} \text{ with } n_1(P) \leq k_2,
$$

there exists some $P \in \mathcal{P}$ with $n_1(P) \leq k_1$ such that we have

$$
\mathbb{P}_{Y \sim P}(|R(Y) \cap \mathcal{H}_1(P)| = 0) \geq 2/5;
$$

$$
\mathbb{P}_{Y \sim P} \left[|\text{BH}^*_\alpha/2 \cap \mathcal{H}_1(P)| \geq c_4 \alpha^{-1} \sqrt{\frac{n}{\log(n)}} \right] \geq 1 - e^{-c_5 \alpha^{-1} \sqrt{n/\log(n)}} \geq 4/5.
$$

(29)

In particular, we have that $I(R, k_2) \leq c_3$ implies $I(R, k_1, \alpha/2) \geq 1/5$. 

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This result is qualitatively similar to Theorem 2.1, up to the change the boundary condition (13) into (28). Taking $k_1 = k_2 = k_n \gg n/\sqrt{\log(n)}$, we deduce point (i) of Theorem 4.1. Let us provide an heuristic to explain the value of the boundary. Roughly, the oracle BH procedure is equivalent to the plug-in BH procedure if the corrected observations $Y_i - \hat{\theta}$ can be compared to the Gaussian quantiles $\Phi^{-1}(\alpha k/(2n))$ in the same way as the $Y_i - \theta$ do. Hence, the plug-in operation will mimic the oracle if

$$|\hat{\theta} - \theta| \ll \min_k \left\{ \Phi^{-1}(\alpha k/(2n)) - \Phi^{-1}(\alpha(k - 1)/(2n)) \right\} \approx \frac{\alpha/n}{\phi(\Phi^{-1}(\alpha/n))},$$

which leads to $k/n \ll 1/\sqrt{\log n}$, by using the standard properties on the Gaussian tail distribution (Section D) and the estimation rate of $\hat{\theta}$ (Section C.1).

As in Section 2, we also deduce from Theorem 4.2 that no procedure $R$ can simultaneously control the FDR at the nominal level up to some $k_n \gg n/\sqrt{\log(n)}$ while being also asymptotically optimal for all sequences $k_n \ll n/\sqrt{\log(n)}$.

**Corollary 4.3.** Consider the same numerical constants $c_1$–$c_5$ as in Theorem 4.2 above. Take any $\alpha \in (0, c_3)$, any $n \geq c_1$ and fix any $\epsilon \in (n^{-1/4}/(\alpha/2)^2, e^{-(c_2 \log(2/\alpha))^2})$. Then for any procedure $R$ with $I(R, k_2) \leq c_3$ for a sparsity $k_2 = \sqrt{\log(1/\epsilon)} \frac{n}{\log n}$, we have $I(R, k_1, \alpha/2) \geq 1/5$ for a sparsity $k_1 = \epsilon e^{(c_2 \log(2/\alpha))^2} \sqrt{\log(1/\epsilon)} e^{-n/\log n}$. In particular, if $n^{-1/16} < (\alpha/2)^2 e^{-(c_2 \log(2/\alpha))^2}$, we have for any procedure $R$,

- if $I(R, n/4) \leq \alpha$, then $I(R, n^{1-\delta} e/4, \alpha/2) \geq 1/5$;
- if $I(R, n^{1-\delta} e/4, \alpha/2) < 1/5$ then $I(R, n/4) > c_3$,

where we let $\delta = 1/(16c_2^2 \log^2(2/\alpha)) > 0$.

### 4.2. Lower bound for plug-in procedures

In the previous section, we established an impossibility result for all multiple testing procedures $R$. In this section, we turn our attention to the special case of plug-in procedures $B_{1,2}(\hat{\theta}, \sigma(P))$ where $\hat{\theta}$ is any estimator of $\theta(P)$.

**Theorem 4.4.** There exist positive numerical constants $c_1$–$c_3$ such that the following holds for all $\alpha \in (0, 1)$, all $n \geq N(\alpha)$, any estimator $\hat{\theta}$, and all $k$ satisfying

$$c_1 \frac{n \log(2/\alpha)}{\sqrt{\log(n)}} \leq k < \frac{n}{2}.$$  \hspace{1cm} (30)

There exists $P \in \mathcal{P}$ with $n_1(P) \leq k$ and an event $\Omega$ of probability higher than $1/2 - c_2/n$ such that, on $\Omega$, the plug-in procedure $B_{1,2}(Y; \hat{\theta}, \sigma(P))$ satisfies

$$|B_{1,2}(Y; \hat{\theta}, \sigma(P)) \cap \mathcal{H}_0(P)| \geq 0.5 n^{3/4};
$$

$$\text{FDP}(P, B_{1,2}(Y; \hat{\theta}, \sigma(P))) \geq \frac{1}{2 + c_3 n^{-1/5}}. \hspace{1cm} (31)$$

This theorem enforces that no plug-in procedure $B_{1,2}(Y; \hat{\theta}, \sigma(P))$ is able to control the FDR at the nominal level in dense settings ($k_n \gg n/\sqrt{\log(n)}$). In fact, the FDP of plug-in
procedures $\text{BH}_\alpha(Y; \hat{\theta}, \sigma(P))$ is even shown to be at least of the order of $1/2$ with probability close to $1/2$. On the same event, the plug-in procedure $\text{BH}_\alpha(Y; \hat{\theta}, \sigma(P))$ makes many false rejections.

In contrast to the previous lower bounds, the proof of Theorem 4.4 relies on a tighter control of the shifted p-value process and quantifies its impact on the BH threshold.

5. Extension to general location model

In this section, we generalize our approach to the case where the null distribution is not necessarily Gaussian. For simplicity, we focus here on the location model. Let $G$ denote the collection of densities on $\mathbb{R}$ that are symmetric, continuous and non-increasing on $\mathbb{R}_+$. Given any $g \in G$, we extend the setting of Section 1.2, by now assuming that $P = \otimes_{i=1}^n P_i$ belongs to the collection $P_g$ of all distributions on $\mathbb{R}^n$ satisfying

$$\left\{ \theta \in \mathbb{R} : \sum_{i=1}^n \mathbb{1}\{P_i \text{ has density } g(\cdot - \theta)\} > n/2 \right\} \neq \emptyset. \quad (32)$$

In other words, we assume that there exists $\theta$ such that at least half of the $P_i$’s have for density $g(\cdot - \theta)$. Such $\theta$ is therefore uniquely defined from $P$, and we denote it again by $\theta(P)$.

The testing problem becomes

$$H_{0,i} : "P_i \sim g(\cdot - \theta(P))" \text{ against } H_{1,i} : "P_i \not\sim g(\cdot - \theta(P))", \text{ for all } 1 \leq i \leq n.$$

The rescaled p-values are now defined by

$$p_i(u) = 2\mathcal{G}(|Y_i - u|), \; u \in \mathbb{R}, \; 1 \leq i \leq n, \quad (33)$$

where $\mathcal{G}(y) = \int_y^{+\infty} g(x)dx$, $y \in \mathbb{R}$. The oracle p-values are given by $p_i^* = 2\mathcal{G}(|Y_i - \theta(P)|)$, $1 \leq i \leq n$. The BH procedure at level $\alpha$ using p-values $p_i(u)$, $1 \leq i \leq m$, is denoted $\text{BH}_\alpha(u)$, whereas the oracle version is still denoted $\text{BH}^*_\alpha$.

Let $\hat{\theta}$ be a sequence of estimators of $\theta(P)$, respectively. For a given sparsity sequence $k_n \in [1,n/2)$, the rescaling $\hat{\theta}$ is said to be asymptotically optimal if there exists a positive sequence $\eta_n \to 0$ such that

$$\limsup_{n} \sup_{\alpha \in (1/n,1/2)} \{I_g(\text{BH}_\alpha(\hat{\theta}),k_n) - \alpha\} \leq 0; \quad (34)$$

$$\lim_{n} \sup_{\alpha \in (1/n,1/2)} \{\Pi_g(\text{BH}_\alpha(\hat{\theta}),k_n,\alpha(1-\eta_n))\} = 0, \quad (35)$$

where $I_g(\cdot)$ and $\Pi_g(\cdot)$ are respectively defined as (9) and (10), except that $\mathcal{P}$ is replaced by $\mathcal{P}_g$ therein.

5.1. Lower bounds

We first state two conditions implying that (34) and (35) cannot hold together.
Theorem 5.1. Consider \( g \in \mathcal{G} \). There exist numerical positive constants \( c_1 \) and \( c_2 \) and a constant \( c_g \) (only depending on \( g \)) such that the following holds for all \( n > 2k \geq 1 \) and any \( \alpha \in (0, 1/2) \). Assume that
\[
\frac{k}{nc_g} \geq \min_{t \in [\frac{\alpha}{2n}; \frac{\alpha}{12}]} \left[ \overline{G}^{-1} \left( \frac{t}{2} \right) - \overline{G}^{-1} \left( \frac{12t}{\alpha} \right) \right],
\]
and consider
\[
t_0 = \max \left\{ t \in \left[ \frac{\alpha}{2n}; \frac{\alpha}{12} \right] : \text{s.t. } \overline{G}^{-1} \left( \frac{t}{2} \right) - \overline{G}^{-1} \left( \frac{12t}{\alpha} \right) \leq \frac{k}{nc_g} \right\}.
\]
For any multiple testing procedure \( R \) satisfying
\[
\text{FDR}(P, R) \leq \frac{1}{5}, \quad \text{for all } P \in \mathcal{P}_g \text{ with } n_1(P) \leq k,
\]
there exists some \( P \in \mathcal{P}_g \) with \( n_1(P) \leq k \) such that we have
\[
\mathbb{P}_{Y \sim P}(|R(Y) \cap \mathcal{H}_1(P)| = 0) \geq 2/5;
\]
\[
\mathbb{P}_{Y \sim P} \left[ |B_{1/2}^{\pi} \cap \mathcal{H}_1(P)| \geq \frac{2nt_0}{\alpha} \right] \geq 1 - e^{-c_2\alpha^{-1}nt_0}.
\]
In particular, \( I_g(R, k) \leq 1/5 \) implies \( I_g(R, k, \alpha/2) \geq 2/5 - e^{-c_2\alpha^{-1}nt_0} \).

A consequence of Theorem 5.1 is that, for some sparsity sequence \( k_n \), if for all \( n > 2k_n \geq c_1 \), Condition (36) holds with \( e^{-c_2nt_0} \leq 1/5 \), it is not possible to achieve any asymptotically optimal scaling in the sense defined above. Interestingly, Condition (36) depends on the variations of \( \overline{G}^{-1}(t) \) for small \( t > 0 \). Taking \( g = \phi \) and \( t = 1/\sqrt{n \log(n)} \) and using the relations stated in Lemma D.2, we recover Theorem 4.2 (case \( k_1 = k_2 \)) obtained in the Gaussian location model and the corresponding optimal condition \( k \geq n/\sqrt{\log(n)} \).

Now consider the Laplace function \( g(x) = e^{-|x|^2}/2 \), so that \( \overline{G}^{-1}(t) = \log(1/(2t)) \). Then Condition (36) cannot be guaranteed even when \( k/n \) is of the order of a constant. More generally, Theorem 5.1 is silent for any \( g \) such that \( \min_{t \in [\frac{\alpha}{2n}, \frac{\alpha}{12}]} |\overline{G}^{-1}(\frac{t}{2}) - \overline{G}^{-1}(\frac{12t}{\alpha})| \) is of the order of a constant.

The next result is dedicated to this case. Remember that the case \( k/n \geq 1/2 \) is not identifiable. The next result shows that there exists a threshold \( \pi_\alpha < 1/2 \), such that optimal scaling is impossible when \( k/n \) belongs to the region \((\pi_\alpha, 1/2)\). Markedly, \( \pi_\alpha \) does not depend on \( g \). For \( \alpha \in (0, 1) \), it is defined by
\[
\pi_\alpha = \sqrt{(1 - \alpha) - (1 - \alpha) \overline{\alpha}} \in (0, 1/2).
\]

Theorem 5.2. Consider any \( \alpha \in (0, 1) \) and \( \pi_\alpha \) given by (38). There exist a positive constant \( c_\alpha \) (only depending on \( \alpha \)) such that following holds for any \( \pi \in (\pi_\alpha, 1/2) \), any \( g \in \mathcal{G} \) and \( n \) larger than a constant depending on \( \alpha \) and \( \pi \). For any multiple testing procedure \( R \) satisfying
\[
\text{FDR}(P, R) \leq 1/4, \quad \text{for all } P \in \mathcal{P}_g \text{ and } n_1(P) \leq \pi n,
\]
there exists \( P \in \mathcal{P}_g \) with \( n_1(P) \leq \pi n \) such that we have
\[
\mathbb{P}_{Y \sim P}(|R(Y)| = 0) \geq 1/3;
\]
\[
\mathbb{P}_{Y \sim P} \left[ |B_{\pi}^{\pi} \cap \mathcal{H}_1(P)| \geq \frac{n_1\pi}{4} \right] \geq 1 - 10e^{-c_\alpha n(\pi - \pi_\alpha)^2} \geq 3/4.
\]
In particular, \( I_g(R, \pi n) \leq 1/4 \) implies \( I_g(R, \pi n, \alpha) \geq 1/12 \).
To illustrate the above result, take $\alpha \in (0, 1/4]$ and $\bar{\pi} \in (\pi_\alpha, 1/2)$. Applying the above result for $\alpha' < \alpha$ with $\pi_{\alpha'} < \bar{\pi}$, we obtain that for any procedure $R$ with $I_g(R, \bar{\pi} n) \leq \alpha$, we have $\Pi_g(R, \bar{\pi} n, \alpha') \geq 1/12$. In particular, this shows that there exists no optimal scaling in the regime $k_n = n\bar{\pi}$, for $\bar{\pi} \in (\pi_\alpha, 1/2)$. In addition, this holds uniformly over all $g$ in the class $\mathcal{G}$.

### 5.2. Upper bound

Since any $g \in \mathcal{G}$ is symmetric, $\theta(P)$ is the median of the null distribution. We consider, again, $\tilde{\theta} = Y_{\lceil n/2 \rceil}$ as the estimator of $\theta(P)$ and plug it into BH to build BH$_\alpha(\tilde{\theta})$. The following result holds for any $g \in \mathcal{G}$.

**Theorem 5.3.** There exist constants $c_1(g), c_2(g) > 0$ only depending on $g$ such that the following holds for all $n \geq c_1(g)$ and $\alpha \in (0, 0.5)$. Consider an integer $k \leq 0.1n$ such that

$$\eta = c_2(g) \left( (k/n) \lor n^{-1/6} \right) \max_{t \in [0.95\alpha/n, \alpha]} \left\{ \frac{1}{G^{-1}(t/2) - G^{-1}(t)} \frac{g(G^{-1}(t))}{g(G^{-1}(t/2))} \right\} \leq 0.05. \quad (40)$$

Then we have

$$I_g(\text{BH}_\alpha(\tilde{\theta}), k) \leq \alpha(1 + \eta) + e^{-\sqrt{n}}; \quad (41)$$

$$\Pi_g(\text{BH}_\alpha(\tilde{\theta}), k, \alpha(1 - \eta)) \leq e^{-\sqrt{n}}. \quad (42)$$

If we consider any asymptotic setting where $\eta$ in (40) converges to 0, then it follows from the above theorem that $\tilde{\theta}$ is an optimal scaling.

Comparing (36) of the lower bound in the previous section with (40), we observe that those are matching up to the term

$$\max_{t \in [0.95\alpha/n, \alpha]} \left\{ \frac{g(G^{-1}(t))}{g(G^{-1}(t/2))} \right\}.$$ 

The latter is of the order of a constant for the Subbotin-Laplace cases as illustrated below.

### 5.3. Application to Subbotin distributions

We now apply our general results to the class of Subbotin distributions.

**Corollary 5.4.** Consider the location Subbotin null model for which $g(x) = L^{-1}_{\zeta} e^{-|x|^{\zeta}/\zeta}$, for some fixed $\zeta > 1$ and the normalization constant $L_{\zeta} = 2\Gamma(1/\zeta)\zeta^{1/\zeta - 1}$. Then

(i) for a sparsity $k_n \gg n/(\log(n))^{1-1/\zeta}$, there exists no (sequence of) estimators $\tilde{\theta}$ such that the scaling $\tilde{\theta}$ is asymptotically optimal.

(ii) for a sparsity $k_n \ll n/(\log(n))^{1-1/\zeta}$, the scaling $\tilde{\theta} = Y_{\lceil n/2 \rceil}$ is asymptotically optimal.

**Corollary 5.5.** Let us consider the Laplace density $g(x) = 0.5 e^{-|x|}$. Then for a sparsity $k_n \ll n$, the scaling $\tilde{\theta} = Y_{\lceil n/2 \rceil}$ is asymptotically optimal.
5.4. An additional result for the Laplace location model

Our general theory implies that, in the Laplace location model, an optimal scaling is possible when \( k_n \ll n \) (Corollary 5.5) and is impossible if \( \liminf k_n/n > \pi_\alpha \) (Theorem 5.2). However, it is silent when \( k_n/n \) converges to a small constant \( \pi \in (0, 1) \). In this section, we establish that optimal scaling is impossible and that one needs to incur a small loss in that regime. Let for any \( \alpha \in (0, 1) \),

\[ \pi_\alpha^* = \frac{1 - \sqrt{\alpha}}{2 - \sqrt{\alpha}} \in (0, 1/2). \]  

Proposition 5.6 (Lower Bound for the Laplace distribution). There exists a positive and increasing function \( \zeta : (0, 1/2) \rightarrow \mathbb{R}_+ \) with \( \lim_{1/2} \zeta = +\infty \) such that the following holds for any \( \alpha \in (0, 1) \), any \( \pi < \pi_\alpha^* \) and for any \( n \) larger than a constant depending only on \( \alpha \) and \( \pi \).

For any procedure \( R \) satisfying

\[ \text{FDR}[P, R] \leq \alpha n_0(P)/n, \text{ for all } P \in \mathcal{F}_g \text{ with } n_1(P) \leq \pi n, \]

there exists a distribution \( P \in \mathcal{F}_g \) with \( n_1(P) \leq \pi n \) such that

\[ \mathbb{P}_{Y \sim P}[|BH^*_{\alpha}| > 0] - \mathbb{P}_{Y \sim P}[|R(Y)| > 0] \geq \alpha \zeta(\pi) - c_\pi n^{-1/3}, \]

where \( c_\pi \) only depends on \( \pi \).

Recall that, for any distribution \( P \), the FDR of \( BH^*_{\alpha} \) is \( \alpha n_0(P)/n \). Hence, the above proposition states that any procedure achieving the same FDR bound as the oracle procedure is strictly more conservative than the oracle, in the sense that \( \mathbb{P}_{Y \sim P}[|BH^*_{\alpha}| > 0, |R(Y)| = 0] \geq \alpha \zeta(\pi) + o(1) > 0 \). In addition, the amplitude of \( \alpha \zeta(\pi) \) is increasing with \( \pi \), which is expected.

The assumption \( \pi \geq \pi_\alpha^* \) is technical. In particular, we can easily prove that, for larger \( \pi \), the result remains true by replacing \( \zeta(\pi) \) by \( \zeta(\pi \land \pi_\alpha^*) \).

Remark 5.7. On the feasibility side, we can show that in the regime where \( n_1(P)/n \) converges to a small constant, the plug-in BH procedure at level \( \alpha \) is yet not asymptotically optimal, but is comparable to oracle BH procedures with modified nominal levels \( \alpha' \neq \alpha \). Recall that \( p_i(u) = 2\overline{G}(\overline{Y}_i - u) = e^{-|\overline{Y}_i - u|} \) and \( p_i^* = e^{-|\overline{Y}_i - \theta(P)|} \). As a consequence, given an estimator \( \hat{\theta} \), the ratio \( p_i(\hat{\theta})/p_i^* \) belongs to \( [e^{-|\theta - \theta(P)|}; e^{\theta - \theta(P)}] \). Assuming that \( \alpha e^{-|\theta - \theta(P)|} < 1 \), it follows from the definition of \( BH_{\alpha}(u) \) that

\[ BH^*_{\alpha e^{-|\theta - \theta(P)|}} \subset BH_{\alpha}(\hat{\theta}) \subset BH^*_{\alpha e^{|\theta - \theta(P)|}}. \]

As a consequence, as long as \( |\hat{\theta} - \theta(P)| \leq \log(1/\alpha) \), \( BH_{\alpha}(\hat{\theta}) \) is sandwiched between two oracle BH procedures with modified type I error. As an example, the median estimator \( \hat{\theta} = Y_{\lceil n/2 \rceil} \) satisfies \( |\hat{\theta} - \theta| \leq c_n(P)/n \) with high probability when \( n_1(P)/n \) is small enough (see the proof of Theorem 5.3). As a consequence, with high probability, we have

\[ BH^*_{\alpha e^{-cn_1(P)/n}} \subset BH_{\alpha}(\hat{\theta}) \subset BH^*_{\alpha e^{cn_1(P)/n}}. \]

6. Discussion

Elaborating upon Efron’s problem, we have presented a general theory to assess whether one can estimate the null and use it into a plug-in BH procedure, while keeping optimal properties.
in terms of FDP and TDP. As expected, the sparsity parameter \( k \) played a central role, and matching lower bounds and upper bounds were established. The obtained sparsity boundaries were shown to depend 1) on the fact that the null variance is known or not and 2) on the variations of the quantile function that defines the null distribution.

This work paves the way for several extensions. First, one direction is to investigate the sparsity boundary when the model is reduced, e.g., by considering more constrained alternatives. A first hint has been given for one-sided alternatives in Carpentier et al. (2018), where both a uniform FDR control and power results can be achieved in dense settings, e.g., \( k = n/2 \) (say), which is markedly different from what we obtained here. In future work, many more structured setting can be considered, e.g., decreasing alternative densities, temporal/spatial structure on the signal, and so on. Conversely, the problem could also be made more difficult by considering a larger model, for instance, dropping the assumption that \( g \) is known, but assuming instead that it belongs to some parametric or non-parametric class. These avenues are both exciting and challenging for future investigations.

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Appendix A: Proofs

**A.1. Proof of Theorem 4.2**

We follow the same general approach as for proving Theorem 2.1 (see Section 2.3).

**Step 1: Building a least favorable mixture distribution** Given \( \mu \in \mathbb{R} \), let \( \phi_{\mu}(x) = \phi(x - \mu) \) for all \( x \in \mathbb{R} \). Let us consider the real measure with density

\[
h = (1 - \pi_1)\phi + \pi_1 f_1 = (1 - \pi_2)\phi + \pi_2 f_2 = \max\{(1 - \pi_1)\phi, (1 - \pi_2)\phi_{\mu}\},
\]

for \( \pi_1 = k_1/(2n) \) and \( \pi_2 = k_2/(2n) \) (with \( \pi_1 \leq \pi_2 \) by (28)) and where

\[
f_1 = \frac{1}{\pi_1} [(1 - \pi_2)\phi_{\mu} - (1 - \pi_1)\phi]_+; \quad f_2 = \frac{1}{\pi_2} [(1 - \pi_1)\phi - (1 - \pi_2)\phi_{\mu}]_+.
\]

Now, we can choose \( \mu \in (0,2) \) (as a function of \( \pi_1 \) and \( \pi_2 \)) such that \( f_1, f_2 \) and \( h \) are probability densities. To see this, it is sufficient to choose \( \mu \) with \( \int f_2(u)du = 1 \). Such a \( \mu \) always exists because, as a function of \( \mu \geq 0, \int f_2(u)du \) is continuous with value \( (\pi_2 - \pi_1)/\pi_2 < 1 \) for \( \mu = 0 \) and value larger than \( \pi_2^{-1}(1 - \pi_2)\int [\phi(u) - \phi_{\mu}(u)]_+ du \geq 3 \int_{\infty}^{\infty} (\phi(u) - \phi_{\mu}(u))du = 3(\Phi(\mu/2) - \Phi(-\mu/2)) > 1 \) for \( \mu \geq 2 \). Hence, we fix in the sequel such a \( \mu \in (0,2) \).

Define \( \kappa_0 = \mu^{-1}\log[(1 - \pi_1)/(1 - \pi_2)] \geq 0 \) and \( u_0 = \kappa_0 + \mu/2 \). We deduce from straightforward computations that \( f_1(x) > 0 \) if and only if \( x > u_0 \) and \( f_2(x) > 0 \) if and only if \( x < u_0 \).

The following lemma states a lower bound for \( \mu \) (to be proved at the end of the section).

**Lemma A.1.** There exists a numeric constant \( \epsilon' > 0 \) such that

\[
\mu \geq \epsilon' \frac{\pi_2}{\sqrt{1 + \log \left(\frac{\pi_2}{\pi_1}\right)}}.
\]
Let $Q$ be the distribution on $\mathbb{R}^n$ associated to the product density $\prod_{i=1}^{n} h(x_i)$. Let $Q_{1,z}$ (resp. $Q_{2,z}$) be the distribution on $\mathbb{R}^n$ of density $\prod_{i=1}^{n} ((1-z_i)\phi + z_i f_1)$ (resp. $\prod_{i=1}^{n} ((1-z_i)\phi + z_i f_2)$), for $z \in \{0, 1\}^n$. Let $Z_{1,i}$ (resp. $Z_{2,i}$), $1 \leq i \leq n$, be i.i.d. variables with common distribution being a Bernoulli distribution with parameter $\pi_1$ (resp. $\pi_2$). Hence, $Y \sim Q_{1,z}$ (resp. $Y \sim Q_{2,z}$) is distributed as $Q$ unconditionally on $Z_1$ (resp. $Z_2$). Note that, for any $z$, $\theta(Q_{1,z}) = 0$ whereas $\theta(Q_{2,z}) = \mu$. Besides, we have $\mathcal{H}_1(Q_{1,z} = \{i : z_i = 1\})$. As in the proof of Theorem 2.1, for $z$ such that $n_1(z) \geq n/2$, we have $Q_{j,z} \not\in \mathcal{P}$, but we readily extend the definition of $\theta(Q_{1,z})$ and $\sigma(Q_{1,z})$ to that setting.

**Step 2:** if $\mathbb{P}_{Y \sim Q}(|R(Y)| > 0) \geq \frac{1}{5}$ then $\text{FDR}(P,R) \geq \frac{1}{5}$ for $n$ larger than a numeric constant, for some $P$ with $n_1(P) \leq k_2$ Recall that, for any $j \in \{1, 2\}$, we have

$$\text{FDP}(Q_{j,z}, R(Y)) = \frac{\sum_{i \in R(Y)} 1\{Z_{1,i} = 0\} |R(Y)|}{|R(Y)|} 1\{|R(Y)| > 0\}.$$ 

We derive from the Fubini theorem that

$$\begin{align*}
\mathbb{E}_{Z_1}[\text{FDR}(Q_{1,z}, R)] + \mathbb{E}_{Z_2}[\text{FDR}(Q_{2,z}, R)] &= \mathbb{E}_{Z_1}\left[\mathbb{E}_{Y \sim Q}(\text{FDP}(Q_{1,z}, R(Y)))\right] + \mathbb{E}_{Z_2}\left[\mathbb{E}_{Y \sim Q}(\text{FDP}(Q_{2,z}, R(Y)))\right] \\
&= \mathbb{E}_{Y \sim Q}\left[\sum_{i \in R(Y)} (1 - \pi_1 f_1(Y_i)/h(Y_i)) + (1 - \pi_2 f_2(Y_i)/h(Y_i))\right] 1\{|R(Y)| > 0\}.
\end{align*}$$

From Step 1, we deduce that $\mathbb{P}(Z_{1,i} = 0 | Y) = 1 - \pi_1 f_1(Y_i)/h(Y_i)$ and $f_1(y) = 0$ for $y \leq u_0$. Similarly, we have $\mathbb{P}(Z_{2,i} = 0 | Y) = 1$ for $Y_i \geq u_0$. This entails that, for all $Y$, we have $\mathbb{P}(Z_{1,i} = 0 | Y) + \mathbb{P}(Z_{2,i} = 0 | Y) \geq 1$ for all $i$. Hence, we obtain

$$\mathbb{E}_{Z_1}[\text{FDR}(Q_{1,z}, R)] + \mathbb{E}_{Z_2}[\text{FDR}(Q_{2,z}, R)] \geq \mathbb{P}_{Y \sim Q}(R(Y) > 0) \geq 1/2.$$ 

Hence, we may assume that $\mathbb{E}_{Z_{j_0}}[\text{FDR}(Q_{j_0,z_{j_0}}, R)] \geq 1/4$ for some $j_0 \in \{1, 2\}$. Then, we apply Chebychev’s inequality to obtain

$$\mathbb{E}_{Z_{j_0}}\left[\mathbb{E}_{Y \sim Q_{j_0,z_{j_0}}}(\text{FDP}(Q_{j_0,z_{j_0}}, R(Y))) 1\{n_1(Z_{j_0}) \in [k_{j_0}/4; k_{j_0}]\}\right] \geq 1/4 - 8/k_{j_0}.$$
and thus

\[ E_{Z_{j_0}} \left[ \mathbb{E}_{Y \sim Q_{j_0, z_{j_0}}} \left[ \text{FDP}(Q_{j_0, z_{j_0}}, R(Y)) \right] \mathbb{I}\{n_1(Z_{j_0}) \leq k_2\} \right] \geq 1/4 - 8/k_1. \]

As a result, for \( k_1 \) large enough (by Condition (28) for \( c_1, c_2 \) large enough), there exists \( z \in \{0, 1\}^n \) such that \( n_1(z) \leq k_2 \) and \( \text{FDR}(Q_{1,z}, R) > 1/5 \).

**Step 3:** If \( P_{Y \sim Q}(|R(Y)| = 0) \geq 1/2, \) then \( R \) is over-conservative, for some \( P \) with \( n_1(P) \leq k_1 \). Since \( E_{Z_i}[P_{Y \sim Q_{1,z_i}}(|R(Y)| = 0)] \geq 1/2, \) it follows again from Chebychev’s inequality, that for some \( z \in \{0, 1\}^n \) such that \( n_1(z) \in [k_1/4; k_1] \), we have \( P_{Y \sim Q_{1,z_i}}(|R(Y)| = 0) \geq 1/2 - 8/k_1 \geq 2/5 \) (\( k_1 \) being large enough). In the sequel, we fix \( P = Q_{1,z} \) for such a \( z \), so that \( \theta(P) = 0 \).

Let \( u_1 \) be the smallest number such that for all \( u \geq u_1 \), one has

\[ \pi_1 f_1(u) \geq 16 \alpha^{-1} \phi(u). \] (47)

From the definition of \( f_1 \), we derive that \( (1 - \pi_2)\phi_{\mu}(u_1) = \phi(u_1)[16 \alpha^{-1} + \pi_1] \), which is again equivalent to

\[ u_1 = \frac{\mu}{2} + \frac{1}{\mu} \log \left( \frac{1 - \pi_1 + 16 \alpha^{-1}}{1 - \pi_2} \right) \leq \frac{\mu}{2} + \frac{1}{\mu} \log \left( 2 + \frac{16}{\alpha} \right), \]

Since \( \mu \leq 2 \) and by Lemma A.1, we have

\[ u_1 \leq 1 + \frac{\sqrt{1 + \log \left( \frac{\pi_2}{\pi_1} \right)}}{c' \pi_2} \log \left( 2 + \frac{32}{\alpha} \right) = 1 + 2n \frac{\sqrt{1 + \log \left( \frac{\pi_2}{\pi_1} \right)}}{c' k_2} \log \left( 2 + \frac{32}{\alpha} \right). \] (48)

For a suitable constant \( c_2 \) in Condition (28) and for \( n \) large enough, we therefore have \( u_1 \leq \sqrt{\log(n)} \).

Then, it remains to prove that \( \text{BH}_{\alpha/2}^* \) rejects many false null hypotheses with probability close to one. Recall that \( \text{BH}_{\alpha/2}^* = \{ i \in \{1, \ldots, n\} : |Y_i| \geq \hat{u} \} \) for

\[ \hat{u} = \min \left\{ u \in \mathbb{R}_+ : \sum_{i=1}^n \mathbb{I}\{|Y_i| \geq u\} \geq 4n \alpha^{-1} \Phi(u) \right\}. \]

Define \( N = \sum_{i \in \mathcal{H}_1(P)} \mathbb{I}\{|Y_i| \geq u_1\} \). Arguing exactly as in Step 3 of the proof of Theorem 2.1 (see Section 2.3), we conclude as in (21) that

\[ \mathbb{P}_{Y \sim P}(\hat{u} \leq u_1, N \geq 4n \alpha^{-1} \Phi(u_1)) \geq 1 - e^{-3q/28} = 1 - e^{-c' n \alpha^{-1} \Phi(u_1)}, \]

where \( n \Phi(u_1) \geq c'' \sqrt{n/\log(n)} \). We have proved that

\[ \mathbb{P}_{Y \sim P}\left( |\text{BH}_{\alpha/2}^* \cap \mathcal{H}_1(P)| \geq c_4 \alpha^{-1} \sqrt{n/\log(n)} \right) \geq 1 - e^{-c' \alpha^{-1} c'' \sqrt{n/\log(n)}}, \]

whereas \( |R(Y)| = 0 \) with probability higher than 2/5.
Proof of Lemma A.1. Since $\int f_1(x)dx = 1$, we deduce from the definition of $\kappa_0$ that

\[
\pi_1 = \int [(1 - \pi_2)\phi(x) - (1 - \pi_1)\phi(x)]_+ dx
= [1 - \pi_2][\Phi[\kappa_0 - \mu/2] - [1 - \pi_1][\Phi[\kappa_0 + \mu/2]
= -[\pi_2 - \pi_1][\Phi[\kappa_0 + \mu/2] + [1 - \pi_2] \int_{\kappa_0 - \mu/2}^{\kappa_0 + \mu/2} \phi(x)dx .
\]

(49)

Recall that $\pi_2 > \pi_1$. By integration, we derive that

\[
\frac{\pi_1}{1 - \pi_2} \leq \int_{\kappa_0 - \mu/2}^{\kappa_0 + \mu/2} \phi(x)dx \leq \phi(\kappa_0) \int_{-\mu/2}^{\mu/2} e^{\kappa_0 x}dx
\leq \frac{\phi(\kappa_0)}{\kappa_0} \frac{\pi_2 - \pi_1}{(1 - \pi_1)(1 - \pi_2)} \leq \frac{\phi(\kappa_0)}{\kappa_0} \frac{\pi_2 - \pi_1}{1 - \pi_2}.
\]

Hence, we conclude that

\[
\frac{\phi(\kappa_0)}{\kappa_0} \geq \frac{\pi_1}{\pi_2 - \pi_1} .
\]

(50)

Case 1: $\frac{\pi_1}{\pi_2 - \pi_1} \geq \phi(0)e^{-1/2}$. Since $\phi(\kappa_0) \leq \phi(0)$, we deduce from the definition of $\kappa_0$

\[
\mu \geq \frac{\pi_1 \log \left(1 + \frac{\pi_2 - \pi_1}{1 - \pi_2}\right)}{\phi(0)(\pi_2 - \pi_1)} \geq \pi_1 \geq c' \frac{\pi_2}{\sqrt{1 + \log(\pi_2/\pi_1)}},
\]

(51)

for a suitable constant $c'$ since $\pi_2 \leq 1/2$, log$(1 + x) \geq x/2$ for $x \in [0, 1]$ and we assume that $\pi_2/\pi_1 \leq 1 + e^{1/2}/\phi(0)$. Note that, for $\pi_1 = \pi_2$, we also easily derive from (49) that (51) also holds.

Case 2: $\frac{\pi_1}{\pi_2 - \pi_1} < \phi(0)e^{-1/2}$. We deduce from (50) that either $\kappa_0 \leq 1$ or $\phi(\kappa_0) \geq \frac{\pi_1}{\pi_2 - \pi_1}$, which in turn implies that

$\kappa_0 \leq \sqrt{2 \log \left(\frac{\phi(0)(\pi_2 - \pi_1)}{\pi_1}\right)}$.

From the definition of $\kappa_0$, we derive that

$\mu \geq \frac{\log \left(1 + \frac{\pi_2 - \pi_1}{1 - \pi_2}\right)}{2 \log \left(\frac{\phi(0)(\pi_2 - \pi_1)}{\pi_1}\right)} \geq \frac{\pi_2 - \pi_1}{2 \sqrt{2[1 + \log(\pi_2/\pi_1)]}} \geq c' \frac{\pi_2}{2 \sqrt{2[1 + \log(\pi_2/\pi_1)]}}$,

for a suitable constant $c'$ since $\pi_2/\pi_1$ is bounded away from one.

\[\square\]

A.2. Proof of Theorem 4.4

Without loss of generality, we restrict ourselves to distributions $P$ such that $\sigma(P) = 1$. Let $\hat{\theta}$ be any estimator of $\theta(P)$ and assume that Condition (30) holds.
Step 1: building a least favorable mixture distribution  We use the same mixture distribution as in the proof of Theorem 4.2. Consider the density \( h(44) \) with \( \pi_1 = \pi_2 = \pi = k/2n \), and
\[
h = (1 - \pi) \phi + \pi f_1 = (1 - \pi) \phi + \pi f_2 = \max\{(1 - \pi) \phi, (1 - \pi) \phi\},
\]
where
\[
f_1 = \frac{1}{\pi} [(1 - \pi) \phi - (1 - \pi) \phi]_+; \quad f_2 = \frac{1}{\pi} [(1 - \pi) \phi - (1 - \pi) \phi]_+.
\]
Recall that \( \mu \in (0, 2) \) is chosen in such a way that \( f_1 \) and \( f_2 \) are densities. Also recall the probability measures \( Q, Q_{1,z} \), and \( Q_{2,z} \) introduced in the previous proof (see Section A.1). Also let \( Z_i, 1 \leq i \leq n \), be i.i.d. variables with common distribution being a Bernoulli distribution with parameter \( \pi \). For any event \( A \), we have \( \Pr_{Y \sim Q}[A] = \mathbb{E}_Z[\Pr_{Y \sim Q_{1,z}}(A)] = \mathbb{E}_Z[\Pr_{Y \sim Q_{2,z}}(A)] \).

Consider the events
\[
\Omega^-=\{\hat{\theta}(Y) \geq \mu/2\}; \quad \Omega^+=\{\hat{\theta}(Y) \leq \mu/2\}
\]
Either \( \mathbb{E}_Z[\Pr_{Y \sim Q_{1,z}}(\Omega^-)] \geq 1/2 \) or \( \mathbb{E}_Z[\Pr_{Y \sim Q_{2,z}}(\Omega^+)] \geq 1/2 \). We assume without loss generality that \( \mathbb{E}_Z[\Pr_{Y \sim Q_{1,z}}(\Omega^-)] \geq 1/2 \) the other case being handled similarly.

Since \( n_1(Z) \) follows a Binomial distribution with parameters \( n \) and \( \pi \), it follows from Bernstein’s inequality that
\[
|n_1(Z) - \pi n| \leq \sqrt{2n \pi \log(n)} + \frac{\log(n)}{3} \leq n/4,
\]
with probability higher than 1 - 2/n, for \( n \) large enough. Hence, there exists \( z \in \{0,1\}^n \) such that for \( P = Q_{1,z} \) we have
\[
n_1(P) \in \left[\pi n - \sqrt{2n \pi \log(n)} - \frac{\log(n)}{3}; n/2\right] \quad \text{and} \quad \Pr_{Y \sim P}[\Omega^-] \geq \frac{1}{2} - \frac{2}{n}. \tag{52}
\]

Note that \( \theta(P) = 0 \) whereas, on \( \Omega^- \), \( \hat{\theta} \) is larger or equal to \( \mu/2 \). In the remainder of the proof, we quantify how this estimation error shifts the distribution of the rescaled \( p \)-values.

Step 2: translated \( p \)-values  Since, under \( \Omega^- \), we have \( \theta(P) = 0 \) and \( \hat{\theta} \geq \mu/2 \), the rescaled \( p \)-values are shifted and do not follow an uniform distribution. Let us characterize this shift. We have for all \( t \in [0, 1] \), and all \( i \in \{1, \ldots, n\} \),
\[
\mathbb{I}\{p_i(Y; \hat{\theta}, 1) \leq t\} = \mathbb{I}\{|Y_i - \hat{\theta}| \geq \Phi^{-1}(t/2)\} \\
\geq \mathbb{I}\{Y_i - \hat{\theta} \leq -\Phi^{-1}(t/2)\} \\
\geq \mathbb{I}\{Y_i \leq \hat{\theta} - \Phi^{-1}(t/2)\} \\
\geq \mathbb{I}\{p_i^- \leq \Phi[\Phi^{-1}(t/2) - \hat{\theta}]\},
\]
where we have denoted \( p_i^- = \Phi(-Y_i) \). Let \( \Psi(t) = \Phi(\Phi^{-1}(t/2) - \hat{\theta}) \) and \( \Psi_1(t) = \Phi(\Phi^{-1}(t/2) - \mu/2) \), \( t \in [0, 1] \). On the event \( \Omega^- \), we have \( \Psi(t) \geq \Psi_1(t) \) for any \( t \in [0, 1] \). This entails that, for all \( i \in \{1, \ldots, n\} \) and \( t \in [0, 1] \),
\[
\mathbb{I}\{p_i(Y; \hat{\theta}, 1) \leq t\} \geq \mathbb{I}\{p_i^- \leq \Psi(t)\} \geq \mathbb{I}\{p_i^- \leq \Psi_1(t)\} \geq \mathbb{I}\{p_i^- \leq \Psi_1(t)\} \quad \text{for } Y \in \Omega^- \tag{53}
\]
Interestingly, for \( i \in \mathcal{H}_0 \), the \( p_i^- \)'s are all i.i.d. \( U(0,1) \). In contrast to \( \Psi \), the function \( \Psi_1 \) does not depend on the \( Y_i \)'s.
Step 3: with high probability, on $\Omega^-$, the threshold of $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ is large Since the rescaled $p$-values are shifted, one should expect that a large number of them are small enough so that $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ rejects many hypotheses. Let us denote by $\tilde{T}_\alpha = T_\alpha(Y; \hat{\theta}, \sigma(P)) \vee (\alpha/n)$ the $p$-value threshold of $\text{BH}_\alpha(Y; \hat{\theta}, 1)$. In view of (53), we consider the empirical distribution function of $p_i^-$, $i \in \mathcal{H}_0$, given by
\[
\hat{G}_0^-(t) = (n_0(P))^{-1} \sum_{i \in \mathcal{H}_0(P)} 1\{p_i^- \leq t\}, \quad t \in [0, 1].
\]
Relying on the DKW inequality (Lemma D.1), we derive that this process is uniformly bounded. Precisely, we have $\mathbb{P}(\Omega_0^-) \geq 1 - 1/n$, where
\[
\Omega_0^- = \left\{ \sup_{t \in [0,1]} |\hat{G}_0^-(t) - t| \leq \sqrt{\log(2n)/(2n_0(P))} \right\}.
\]
Now, a consequence of (53) is that
\[
\alpha \geq \tilde{T}_\alpha = \max\left\{ t \in [0, 1] : \sum_{i=1}^n 1\{p_i(Y; \hat{\theta}, \sigma(P)) \leq t\} \geq nt/\alpha \right\}
\geq \max\left\{ t \in [0, 1] : \sum_{i=1}^n 1\{p_i^- \leq \Psi_1(t)\} \geq nt/\alpha \right\}
\geq \max\left\{ t \in [0, 1] : (n_0(P)/n)\hat{G}_0^-(\Psi_1(t)) \geq t/\alpha \right\} \geq T_0^-,
\]
by letting $T_0^- = \max\{t \in [0, 1] : \hat{G}_0^-(\Psi_1(t)) \geq 2t/\alpha\}$. On $\Omega^-$, $\hat{G}_0^-(\Psi_1(t))$ is uniformly close to $\Psi_1(t)$, which will allow us to get a lower bound of $\Psi_1(T_0^-)$. This argument is formalized in Lemma A.2 below.

**Lemma A.2.** There exists an integer $N = N(\alpha)$ such that if $n_0(P) \geq N$ and
\[
\mu \geq \frac{4\log(32/\alpha)}{\sqrt{0.25 \log(n_0(P)) + \log(8/\alpha)}},
\]
we have on the event $\Omega_0^-$,
\[
\Psi_1(T_0^-) \geq n_0(P)^{-1/4}.
\]
By Lemma A.1, we have $\mu \geq \epsilon' \pi = \epsilon' k/(2n)$. Hence, Condition (55) is satisfied by Condition (30) together with $n_0(P) \geq 0.5n$. Combining (54) and (56) and since $\Psi_1$ is increasing, we finally have on the event $\Omega^- \cap \Omega_0^-$,
\[
\Psi(\tilde{T}_\alpha) \geq \Psi_1(\tilde{T}_\alpha) \geq \Psi_1(T_0^-) \geq n_0(P)^{-1/4}.
\]

Step 4: $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ makes many false rejections Since the threshold $\tilde{T}_\alpha$ is large enough, one can then prove that $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ makes many false rejections. By (53), we have, on the event $\Omega^- \cap \Omega_0^-$, that for all $t \in [0, 1],$
\[
\sum_{i \in \mathcal{H}_0(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} = \sum_{i \in \mathcal{H}_0(P)} 1\{Y_i \leq -\Phi^{-1}(t/2) + \hat{\theta}\} + 1\{Y_i \geq \Phi^{-1}(t/2) + \hat{\theta}\}
= \sum_{i \in \mathcal{H}_0(P)} 1\{p_i^- \leq \Psi(t)\} + 1\{p_i^+ \leq \Psi^+(t)\}.
\]
where $\Psi^+(t) = \Phi[\Phi^{-1}(t/2) + \hat{\theta}]$ and $p_i^+ = \Phi(Y_i)$. Define the process

$$\hat{G}_0^+(t) = (n_0(P))^{-1} \sum_{i \in \mathcal{H}_0(P)} 1\{p_i^+ \leq t\}, \ t \in [0, 1].$$

Relying again on the DKW inequality (Lemma D.1)), we derive that this process is uniformly bounded in the sense that $\mathbb{P}(\Omega_0^+) \geq 1 - 1/n$, where

$$\Omega_0^+ = \left\{ \sup_{t \in [0,1]} |\hat{G}_0^+(t) - \Psi(t)| \leq \sqrt{\log(2n)/(2n_0(P))} \right\}.$$

Hence, on $\Omega^- \cap \Omega_0^- \cap \Omega_0^+$, we have, uniformly over all $t \in [0, 1],$

$$\sum_{i \in \mathcal{H}_0(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} \geq n_0(P)[\Psi(t) + \Psi^+(t)] - \sqrt{2n \log(2n)}.$$

By (52), $n_0(P) \geq n(1 - \pi) - 2n \pi \log(n) - \log(n)/3 \geq n(1 - \pi) - 2\sqrt{2n \log(n)}$. Hence, we conclude that, uniformly over all $t \in [0, 1],$

$$\sum_{i \in \mathcal{H}_0(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} \geq n(1 - \pi)[\Psi(t) + \Psi^+(t)] - 5\sqrt{2n \log(n)} . \quad (58)$$

In the previous step (see (57)), we have proved that $\Psi(\hat{T}_0) \geq n_0(P)^{-1/4}$. This implies that

$$|BH_n(Y; \hat{\theta}, 1) \cap \mathcal{H}_0(P)| \geq \frac{3}{4} n^{3/4} - 5\sqrt{2n \log(n)} \geq \frac{1}{2} n^{3/4},$$

for $n$ large enough. This proves the first statement of the theorem.

**Step 5: with high probability, on $\Omega^-$, $BH_n(Y; \hat{\theta}, 1)$ cannot make too many true rejections** In this step, we bound the number of true rejections uniformly with respect to the threshold $t$ of the testing procedure. We have for all $t \in [0, 1]$, and all $i \in \{1, \ldots, n\},$

$$1\{p_i(Y; \hat{\theta}, 1) \leq t\} = 1\{Y_i \leq -\Phi^{-1}(t/2) + \hat{\theta}\} + 1\{Y_i \geq \Phi^{-1}(t/2) + \hat{\theta}\} = 1\{Y_i \leq -\Phi^{-1}(t/2) + \hat{\theta}\} + 1\{Y_i \geq \Phi^{-1}(t/2) + \hat{\theta}\}.$$

Now, recall that the variables $Y_i, i \in \mathcal{H}_1(P)$, are i.i.d. with common density $\frac{1-\pi}{\pi} (\phi - \phi_\mu)$. For $t \in [0, 1],$ define

$$\hat{G}_1^-(t) = (n_1(P))^{-1} \sum_{i \in \mathcal{H}_1(P)} 1\{\Phi(-Y_i) \leq t\}, \ G_1^-(t) = \mathbb{E}_P[\hat{G}_1^-(t)];$$

$$\Omega_1^- = \left\{ \sup_{t \in [0,1]} (\hat{G}_1^-(t) - G_1^-(t)) \leq \sqrt{\log(n)/(2n_1(P))} \right\};$$

$$\hat{G}_1^+(t) = (n_1(P))^{-1} \sum_{i \in \mathcal{H}_1(P)} 1\{\Phi(Y_i) \leq t\}, \ G_1^+(t) = \mathbb{E}_P[\hat{G}_1^+(t)];$$

$$\Omega_1^+ = \left\{ \sup_{t \in [0,1]} (\hat{G}_1^+(t) - G_1^+(t)) \leq \sqrt{\log(n)/(2n_1(P))} \right\}.$$
Applying twice the DKW inequality (Lemma D.1), we have $\mathbb{P}(\Omega_1^-) \geq 1 - 1/n$ and $\mathbb{P}(\Omega_1^+) \geq 1 - 1/n$, which gives that the event $\Omega_1 = \Omega_1^+ \cap \Omega_1^-$ is such that $\mathbb{P}(\Omega_1) \geq 1 - 2/n$. Furthermore, we have for all $t \in [0, 1]$, and $i \in \mathcal{H}_1(P)$,

$$G_1^-(t) = \mathbb{P}(\Phi(Y_i) \leq t) = \mathbb{P}(Y_i \leq \Phi^{-1}(t)) = \frac{1 - \pi}{\pi} \int_{-\infty}^{\Phi^{-1}(t)} (\phi(x - \mu) - \phi(x)) + dx$$

$$\leq \frac{1 - \pi}{\pi} \Phi[\Phi^{-1}(t) + \mu] ;$$

$$G_1^+(t) = \mathbb{P}(\Phi(Y_i) \leq t) = \mathbb{P}(Y_i \geq \Phi^{-1}(t)) = \frac{1 - \pi}{\pi} \int_{\Phi^{-1}(t)}^{\infty} (\phi(x - \mu) - \phi(x)) + dx$$

$$\leq \frac{1 - \pi}{\pi} \Phi[\Phi^{-1}(t) - \mu] .$$

As a result, on the event $\Omega_1$, we have for all $t \in [0, 1]$,

$$\sum_{i \in \mathcal{H}_1(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} = n_1(P) \frac{\hat{G}_1^- (\Psi(t)) + n_1(P) \hat{G}_1^+ (\Psi^+(t))}{\n_1(P) \left[ G_1^- (\Psi(t)) + G_1^+ (\Psi^+(t)) \right] + \sqrt{2n_1(P) \log(n)}}$$

$$\leq n_1(P) \frac{2 - \pi}{\pi} \left[ \Phi[\Phi^{-1}(t/2) - \hat{\theta} + \mu] + \Phi[\Phi^{-1}(t/2) + \hat{\theta} - \mu] \right] + \sqrt{2n_1(P) \log(n)} .$$

Now, since for any fixed $t \in [0, 1]$, the map

$$u \in [0, \infty] \mapsto \Phi\left[ \Phi^{-1}(t/2) + u \right] + \Phi\left[ \Phi^{-1}(t/2) - u \right]$$

is nondecreasing, we have on $\Omega^- \cap \Omega_0^- \cap \Omega_1$ that

$$\Phi\left[ \Phi^{-1}(t/2) + \hat{\theta} - \mu \right] + \Phi\left[ \Phi^{-1}(t/2) - \hat{\theta} + \mu \right]$$

$$\leq \Phi\left[ \Phi^{-1}(t/2) + \hat{\theta} \right] + \Phi\left[ \Phi^{-1}(t/2) - \hat{\theta} \right] = \Psi(t) + \Psi^+(t) ,$$

by using $|\hat{\theta} - \mu| \leq \hat{\theta}$ (since $\hat{\theta} \geq \mu/2$) and the definition of $\Psi(t)$. Finally, on $\Omega^- \cap \Omega_0^- \cap \Omega_1$, we obtain that, for all $t \in [0, 1]$,

$$\sum_{i \in \mathcal{H}_1(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} \leq \frac{n_1(P)(1 - \pi)}{\pi} \left[ \Psi(t) + \Psi^+(t) \right] + \sqrt{2 \log(n)} .$$

From (52) and since $\pi \geq \log^{-1/2}(n)$, we deduce that

$$\frac{n_1(P)(1 - \pi)}{\pi} \leq n(1 - \pi) + \sqrt{\frac{2n \log(n)}{\pi}} + \frac{\log(n)}{3\pi} \leq n(1 - \pi) + 2\sqrt{n \log(n)} .$$

Hence, we conclude that, for all $t \in [0, 1]$,

$$\sum_{i \in \mathcal{H}_1(P)} 1\{p_i(Y; \hat{\theta}, 1) \leq t\} \leq n(1 - \pi) \left[ \Psi(t) + \Psi^+(t) \right] + 5\sqrt{2n \log(n)} . \quad (59)$$
Step 6: On $\Omega^{-} \cap \Omega_0^{-} \cap \Omega_1$, the FDP of $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ is close to 1/2. We deduce from the previous steps a lower bound for the FDP. First, by definition of the FDP and of the threshold $\hat{T}_\alpha$ of $\text{BH}_\alpha(Y; \hat{\theta}, 1)$ (note that the procedure rejects at least one true null hypotheses by (57) and (58)), we deduce from (58) and (59) that, for $Y \in \Omega^{-} \cap \Omega_0^{-} \cap \Omega_1$,

$$
\text{FDP}(P, \text{BH}_\alpha(Y; \hat{\theta}, 1)) \geq \inf_{t \in [\hat{T}_\alpha, \alpha]} \frac{\sum_{i \in H_0(P)} \mathbb{1}\{p_i(Y; \hat{\theta}, 1) \leq t\}}{\sum_{i=1}^{n} \mathbb{1}\{p_i(Y; \hat{\theta}, 1) \leq t\}}
$$

$$
= \inf_{t \in [\hat{T}_\alpha, \alpha]} \left[ 1 + \frac{\sum_{i \in H_1(P)} \mathbb{1}\{p_i(Y, \hat{\theta}, 1) \leq t\}}{\sum_{i \in H_0(P)} \mathbb{1}\{p_i(Y, \hat{\theta}, 1) \leq t\}} \right]^{-1}
$$

$$
\geq \inf_{t \in [\hat{T}_\alpha, \alpha]} \left[ \frac{n(1-\pi)[\Psi(t) + \Psi^+(t)] + 5\sqrt{2n}\log(n)}{(n(1-\pi)[\Psi(t) + \Psi^+(t)] - 5\sqrt{2n}\log(n))_+} \right]^{-1}
$$

$$
\geq \inf_{t \in [\hat{T}_\alpha, \alpha]} \left[ 2 + \frac{10\sqrt{2n}\log(n)}{(n(1-\pi)[\Psi(t) + \Psi^+(t)] - 5\sqrt{2n}\log(n))_+} \right]^{-1}.
$$

We have proved in (56) that $\Psi(t) \geq \Psi(\hat{T}_\alpha) \geq n_0(P)^{-1/4} \geq (n/2)^{-1/4}$. Hence, for $n$ large enough, we obtain

$$
\text{FDP}(P, \text{BH}_\alpha(Y; \hat{\theta}, 1)) \geq \frac{1}{2 + c'n^{-1/5}}.
$$

Since the event $\Omega^{-} \cap \Omega_0^{-} \cap \Omega_1$ occurs with probability higher than $1/2 - c''/n$, we have proved the second statement of the theorem.

A.3. Proof of Theorem 4.1 (ii)

As explained above, Theorem 4.1 (ii) is shown by adapting Theorem 3.1 to the case $\hat{\sigma} = \sigma(P)$, the only difference in the proof being that we now quantify the impact of the mean $\theta(P)$ estimation error on the $p$-values and the corresponding threshold $\hat{T}_\alpha(\hat{\theta}, \sigma(P))$ of plug-in BH. In other words, Lemma 3.3 has to be replaced by Lemma A.3 below (to be proved in Appendix C).

Lemma A.3. For any estimator $\hat{\theta}$, let $\delta > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\alpha \in (0, 0.8)$, $t_0 \in (0, \alpha)$ and

$$
\eta = \delta c \sqrt{2\log(1/t_0)},
$$

with the constant $c > 0$ of Corollary C.6. Assume $|\hat{\theta} - \theta| \leq \sigma \delta$ and $\eta \leq 0.05$. Then, for all $i \in \{1, \ldots, n\}$,

- if $T_\alpha(\hat{\theta}, \hat{\sigma}) \lor (\alpha/n) \geq t_0$, we have

$$
\mathbb{1}\{p_i(\hat{\theta}, \sigma) \leq T_\alpha(\hat{\theta}, \sigma)\} \leq \mathbb{1}\{p_i(\theta, \sigma) \leq (1 + \eta)T_\alpha(\hat{\theta}, \sigma)\}
$$

$$
\leq \mathbb{1}\{p_i(\theta, \sigma) \leq T_{\alpha(1+\eta)}(\theta, \sigma)\};
$$

- if $T_{0.95\alpha}(\theta, \sigma) \lor (0.95\alpha/n) \geq t_0$, we have

$$
\mathbb{1}\{p_i(\theta, \sigma) \leq T_{\alpha(1-\eta)}(\theta, \sigma)\} \leq \mathbb{1}\{p_i(\hat{\theta}, \sigma) \leq T_{\alpha\hat{\theta}}(\hat{\theta}, \sigma)\}.
$$
A.4. Proofs for location model

Proof of Theorem 5.1. Consider any procedure \( R \). We extend the proof of Theorem 4.2 (see Section A.1) to general function \( g \) but in the specific case \( k_1 = k_2 = k \). Take \( \pi = k/(2n) \). Let \( \mu > 0 \) be such that

\[
\int_0^{\mu/2} g(x)dx = \frac{\pi}{2(1-\pi)} < 1/2 .
\]

Since \( g \) is non-increasing on \( \mathbb{R}_+ \), we have \( \mu = 2G^{-1}(\frac{1-2\pi}{2(1-\pi)}) \geq c'_g \pi \). Then, one can check that

\[
h = (1-\pi)g + \pi f_1 = (1-\pi)g_{\mu} + \pi f_2 = \max\{(1-\pi)g, (1-\pi)g_{\mu}\},
\]

is a density where

\[
f_1 = \frac{1}{\pi} [(1-\pi)g_{\mu} - (1-\pi)g]_+ ; \quad f_2 = \frac{1}{\pi} [(1-\pi)g - (1-\pi)g_{\mu}]_+ .
\]

Since \( g \) is non-increasing on \( \mathbb{R}_+ \), \( f_1(x) > 0 \) for \( x > \mu/2 \) and \( f_2(x) > 0 \) for \( x < \mu/2 \).

Defining the distributions \( Q, Q_{1, 2}, Q_{2, 2} \) and \( Z \) as in the proof of Theorem 4.2 (with \( k_1 = k_2, Z = Z_1 = Z_2 \)), we derive that either

\[
\mathbb{E}_Z[\mathbb{E}_{Y \sim Q_{1, 2}}[|R(Y)| > 0]] = \mathbb{E}_Z[\mathbb{E}_{Y \sim Q_{2, 2}}[|R(Y)| > 0]] \geq 1/2
\]

or

\[
\mathbb{E}_Z[\mathbb{E}_{Y \sim Q_{1, 2}}[|R(Y)| = 0]] = \mathbb{E}_Z[\mathbb{E}_{Y \sim Q_{2, 2}}[|R(Y)| = 0]] > 1/2.
\]

In the former case, we obtain by arguing exactly as in step 2 of the previous proof that there exists \( P \) with \( n_1(P) \leq k \) and \( \text{FDR}(P, R) \geq 2/5 \).

Let us now turn to the case \( \mathbb{E}_Z[\mathbb{E}_{Y \sim Q_{1, 2}}[|R(Y)| = 0]] \geq 1/2 \). Hence, there exists \( P \) with \( n_1(P) \in [k/4, k] \) and \( \theta(P) = 0 \) such that \( \mathbb{P}_{Y \sim P}[|R(Y)| = 0] \geq 2/5 \) (for \( k \) large enough). It remains to prove that the oracle Benjamini-Hochberg BH\(_{\alpha/2}^*\) rejects many null hypotheses with probability close to one. It suffices to prove that many oracle \( p \)-values \( p_i^* = 2G(|Y_i|) \) are small enough. Consider any \( i \in \mathcal{H}_1(P) \). The corresponding density of \( Y_i \) is given by \( f_1(x) = \frac{1-\pi}{\pi} [g(x - \mu) - g(x)]_+ \), which is positive for \( x \geq \mu/2 \). Consider some \( t \in [\alpha/(2n); \alpha/2] \) whose value will be fixed later. The event \( \{p_i^* \leq t\} \subset \{Y_i \geq G^{-1} \left( \frac{t}{2} \right) \} \) occurs with probability higher or equal to

\[
r_{\mu}(t) = \frac{1-\pi}{\pi} \left[ G \left[ G^{-1} \left( \frac{t}{2} \right) - \mu \right] - \frac{t}{2} \right] .
\]

Applying Bernstein inequality, we deduce that

\[
\mathbb{P}_{Y \sim P} \left[ \sum_{i \in \mathcal{H}_1(P)} \mathbbm{1}\{p_i^* \leq t\} \geq \frac{n_1(P)}{2} r_{\mu}(t) \right] \geq 1 - e^{-3n_1(P)r_{\mu}(t)/28} . \tag{62}
\]

Observe that

\[
\frac{n_1(P)r_{\mu}(t)}{2} \geq \frac{1-\pi}{4} n \left[ G \left[ G^{-1} \left( \frac{t}{2} \right) - \mu \right] - \frac{t}{2} \right] \geq \frac{3n}{16} \left[ G \left( \frac{t}{2} \right) - c'_g \pi \right] - \frac{t}{2} .
\]
By definition of BH procedure, on the event within (62), BH$_{\alpha/2}^*$ rejects each null hypothesis corresponding to a p-value $p_i^* \leq t$ as long as this last expression is higher than $2nt/\alpha$. Putting everything together we have proved that

$$
\Pr_{Y \sim P} \left[ |BH_{\alpha/2}^* \cap H_1(P)| \geq \frac{2nt}{\alpha} \right] \geq 1 - e^{-c'nt/\alpha}
$$

if some $t \in \left[ \frac{\alpha}{2n}, \alpha/2 \right]$ satisfies

$$
\mathcal{G} \left( \mathcal{G}^{-1} \left( \frac{t}{2} \right) - c'_g \pi \right) \geq \frac{12t}{\alpha}.
$$

(63)

Such a $t$ exists by Condition (36). Fixing $t = t_0$ leads to the desired conclusion.

Proof of Theorem 5.2. We consider the exact same density $h$ as above, but we now fix $\pi = \bar{\pi} - n^{-1/3}$. Arguing as in the proof of Theorem 4.2 (see (46) therein), we have

$$
\mathbb{E}_Z \text{FDR}[Q_{1,Z}, R] + \mathbb{E}_Z \text{FDR}[Q_{2,Z}, R] \geq \Pr_{Y \sim Q}[|R(Y)| > 0],
$$

where $Z \sim B(\pi)^{\otimes n}$ and with $Q_1, Q_2, z$ defined therein.

If $\Pr_{Y \sim Q}[|R(Y)| > 0] \geq 1/2$, we have either $\mathbb{E}_Z \text{FDR}[Q_{1,Z}, R] \geq 1/4$ or $\mathbb{E}_Z \text{FDR}[Q_{2,Z}, R] \geq 1/4$. Since $n_1(Z)$ follows a Binomial distribution with parameter $n$ and $\pi = \bar{\pi} - n^{-1/3}$, it follows from the Bernstein inequality, that

$$
\Pr_Z[n_1(Z) \geq \bar{\pi} n] \leq \exp(-c'_\alpha n^{1/3}),
$$

(64)

for some constant $c'_\alpha > 0$ that only depends on $\alpha$ (through $\pi_\alpha$). As a consequence, there exists $i \in \{0, 1\}$ and $z \in \{0, 1\}^n$ with $n_1(z) \leq \bar{\pi} n$ such that

$$
\text{FDR}[Q_{i,z}, R] \geq 1/4 - \exp(-c'_\alpha n^{1/3}).
$$

Now assume that

$$
\Pr_{Y \sim Q}[|R(Y)| = 0] = \mathbb{E}_Z [\Pr_{Y \sim Q_{1,Z}} = 0] \geq 1/2.
$$

(65)

We consider the behavior of $|BH_\alpha^*|$, the number of rejections of BH$_\alpha^*$, under $Q_{1,Z}$. Fix $t_0 = 2\mathcal{G}(\mu/2) = \frac{1 - 2\pi}{1 - \pi} \in (0, 1)$, so that $\mathcal{G}^{-1}(t_0/2) = \mu/2$. From Fubini’s Theorem and the definition of $|BH_\alpha^*|$, we derive that

$$
\mathbb{E}_Z \Pr_{Y \sim Q_{1,Z}} \left[ |BH_\alpha^*| \geq \frac{nt_0}{\alpha} \right] = \Pr_{Y \sim Q} \left[ |BH_\alpha^*| \geq \frac{n t_0}{\alpha} \right] \geq \Pr_{Y \sim Q} \left[ \sum_{i=1}^{n} \mathbf{1}\{p_i^* \leq t_0\} \geq \frac{nt_0}{\alpha} \right].
$$

Under $Q$, the random variables $\mathbf{1}\{p_i^* \leq t_0\}$, $1 \leq i \leq n$, are i.i.d. and follow a Bernoulli distribution with parameter $\pi$

$$
\Pr[p_i^* \leq t_0] = (1 - \pi) \left[ \mathcal{G}(\mu/2) + 1 - \mathcal{G}(\mu/2) \right] = 1 - \pi.
$$
Define the function \( \psi : x \mapsto (1 - x) - \frac{1 - 2x}{\alpha(1 - x)} \). For any \( \alpha \in (0, 1) \), \( \psi'(x) = -1 + 1/[\alpha(1 - x)^2] \). Hence, \( \psi \) is convex and strictly increasing in \([0, 1/2] \). Recall the definition of \( \pi_\alpha = \frac{\sqrt{1 - \alpha} - (1 - \alpha)}{\alpha} \in (0, 1/2) \). One then shows that \( \psi(\pi_\alpha) = 0 \) and \( \psi(x) \in (0, 1/2) \) for any \( x \in (\pi_\alpha, 1/2) \). Since \( \lim_{\pi} \pi = \bar{\pi} > \pi_\alpha \), it follows that, for \( n \) larger than a constant depending only on \( \bar{\pi} \) and \( \alpha \), we have \( \pi > \pi_\alpha \). Hence, we derive from Bernstein inequality that

\[
E_Z P_{Y \sim Q_{1, z}} \left[ |BH_\alpha^* \geq nt_0 / \alpha \right] \geq 1 - \exp \left[ -n \frac{\psi^2(\pi)}{2[1 - \pi] + \psi(\pi)/3} \right] \geq 1 - \exp \left[ -c_\alpha n(\pi - \pi_\alpha)^2 \right] \geq 1 - \exp \left[ -c_\alpha n(\pi - \pi_\alpha)^2 \right],
\]

since \( \psi(\pi) = \psi(\pi) = \psi(\pi_\alpha)(\pi - \pi_\alpha), \psi'(\pi_\alpha) > 0 \) only depends on \( \alpha \). Moreover, provided that \( |BH_\alpha^* \geq t_0 \pi_\alpha \), we have \( |BH_\alpha^* \cap H_1(Q_{1, z})| \geq \sum_{i=1}^n \mathbb{1}\{Z_i = 1\} \mathbb{1}\{p_i^* \leq t_0\} \). When \( Z_i = 1 \), it follows from the definition of \( t_0 \) that we have \( p_i^* \leq t_0 \) almost surely. As a consequence, for each \( i \), \( \mathbb{1}\{Z_i = 1\} \mathbb{1}\{p_i^* \leq t_0\} \) is a Bernoulli variable of parameter \( \pi \), we derive from the Bernstein inequality that

\[
E_Z P_{Y \sim Q_{1, z}} \left[ \sum_{i=1}^n \mathbb{1}\{Z_i = 1\} \mathbb{1}\{p_i^* \leq t_0\} \geq \frac{n\pi}{2} \right] \geq 1 - \exp \left[ -c_2 n\pi^2 \right],
\]

for \( n \) larger than a constant depending on \( \bar{\pi} \). Since \( \pi \geq \bar{\pi}/2 \) for \( n \) large enough, we deduce that by combining the two previous inequalities that

\[
E_Z P_{Y \sim Q_{1, z}} \left[ |BH_\alpha^* \cap H_1(P)| \geq \frac{n\bar{\pi}}{4} \right] \geq 1 - 2 \exp \left[ -c'\alpha n(\bar{\pi} - \pi_\alpha)^2 \right],
\]

and therefore that

\[
P_Z \left[ P_{Y \sim Q_{1, z}} \left[ |BH_\alpha^* \cap H_1(P)| \geq \frac{n\bar{\pi}}{4} \right] \geq 1 - 10 \exp \left[ -c'\alpha n(\bar{\pi} - \pi_\alpha)^2 \right] \right] \geq \frac{4}{5}.
\]

Also, it follows from (65) that

\[
P_Z \left[ P_{Y \sim Q_{1, z}} [|R(Y)| = 0] \geq 1/3 \right] \geq \frac{1}{4}.
\]

Combining the two previous inequalities with (64), we conclude that there exists \( P \in \mathcal{P}_g \) with \( n_1(P) \leq \bar{\pi} n \) such that \( P_{Y \sim P} [|R(Y)| = 0] \geq 1/3 \) and

\[
P_{Y \sim P} \left[ |BH_\alpha^* \cap H_1(P)| \geq \frac{n\bar{\pi}}{4} \right] \geq 1 - 10 \exp \left[ -c'\alpha n(\bar{\pi} - \pi_\alpha)^2 \right].
\]

The result follows.

**Proof of Proposition 5.6.** We follow the same approach as in the previous proof, but we can sharpen the bounds using the explicit form of \( g \). As in the previous proofs, we consider the same density \( h \), which now takes the form \( h(y) = (1 - \pi) \max\{e^{-|y|/2}, e^{-|y-\mu|/2}\} \) with \( \pi = \bar{\pi} - n^{-1/3} \). We also consider the same \( Q, Q_{1, z}, Q_{2, z}, z \in \{0, 1\}^n \), and \( Z \) with i.i.d. \( B(\pi) \) coordinates. Recall that \( g(x) = e^{-|x|/2}, \overline{G}(x) = e^{-x}/2 \) for \( x \geq 0 \) and \( \overline{G}^{-1}(t) = \log(1/(2t)) \) for
$t \leq 1/2$. As a consequence, $\mu = 2 \log(\frac{1-\pi}{1-2\pi})$. Note that for any $x > \mu$, we have $g(x-\mu)/g(x) = e^{-\mu}$. Now consider any multiple testing procedure $R$.

**Step 1:** Controlling $\mathbb{E}_Z\text{FDR}(Q_{1,Z}, R) + \mathbb{E}_Z\text{FDR}(Q_{2,Z}, R)$. For $Y_i \geq \mu$, then

$$
\mathbb{P}_{Z,Y \sim Q_{1,Z}}[Z_i = 0 \mid Y_i] = \frac{(1 - \pi)e^{-Y_i}/2}{h(Y_i)} = \frac{e^{-Y_i}}{\max\{e^{-Y_i}, e^{-Y_i+\mu}\}} = e^{-\mu} = \frac{(1 - 2\pi)^2}{(1-\pi)^2}.
$$

If $Y_i \in [\mu/2; \mu]$, then

$$
\mathbb{P}_{Z,Y \sim Q_{1,Z}}[Z_i = 0 \mid Y_i] = \frac{e^{-Y_i}}{\max\{e^{-Y_i}, e^{-\mu+Y_i}\}} = e^{\mu-2Y_i} \geq e^{-\mu} = \frac{(1 - 2\pi)^2}{(1-\pi)^2}.
$$

Finally, if $Y_i \leq \mu/2$, then $\mathbb{P}_{Z,Y \sim Q_{1,Z}}[Z_i = 0 \mid Y_i] = 1$. Arguing similarly for $Q_{2,Z}$, we derive that

$$
\mathbb{P}_{Z,Y \sim Q_{1,Z}}[Z_i = 0 \mid Y_i] + \mathbb{P}_{Z,Y \sim Q_{2,Z}}[Z_i = 0 \mid Y_i] \geq 1 + e^{-\mu}
$$

This allows us to derive that

$$
\mathbb{E}_Z[\text{FDR}(Q_{1,Z}, R)] + \mathbb{E}_Z[\text{FDR}(Q_{2,Z}, R)] \geq (1 + e^{-\mu})\mathbb{P}_{Y \sim Q}[|R(Y)| > 0].
$$

By symmetry, we may assume henceforth that

$$
\mathbb{E}_Z[\text{FDR}(Q_{1,Z}, R)] \geq (1 + e^{-\mu})\frac{\mathbb{P}_{Y \sim Q}[|R(Y)| > 0]}{2}. \tag{68}
$$

**Step 2:** Controlling the behavior of $|\text{BH}_\alpha^*|$. Under $Q_{1,Z}$, the oracle p-value $p_i^*$ is simply $2G(|Y_i|)$. Consider any $t \leq e^{-\mu}$. Under the mixture distribution $Q$, we have

$$
\mathbb{P}_{Y \sim Q}[p_i^* \leq t] = (1 - \pi)\left[\int_{-\infty}^{g^{-1}(t/2)} g(x)dx + \int_{g^{-1}(t/2)}^{\infty} g(x-\mu)dx\right]
$$

$$
= (1 - \pi) \left[\frac{t/2 + e\mu t/2}{2}\right] = t\eta(\pi), \tag{69}
$$

where

$$
\eta(\pi) = \frac{1 - \pi}{2}[1 + e\mu] = \frac{1 - \pi}{2}\left[1 + \frac{(1 - \pi)^2}{(1-2\pi)^2}\right]. \tag{70}
$$

The function $\eta$ is increasing and is larger than 1 for $\pi \in (0, 1/2)$ (to see this, we check that $\eta'(\pi) = (\pi/2)(10\pi^2 - 15\pi + 6)/(1-2\pi)^3$ ) . Besides, $\eta$ goes to $+\infty$ when $\pi$ converges to $1/2$. In addition, observe that $\pi_\alpha^*$ given by (43) satisfy $(1-2\pi_\alpha^*)^2 = \alpha$, so that $\alpha < e^{-\mu}$. This implies $\alpha\eta(\pi) < (1 - \pi) < 1$.

Now denote $T_\alpha^*(Y) = \max\{t \in [0, 1] : \sum_{i=1}^n 1\{p_i^* \leq t\} \geq nt/\alpha\}$ the threshold of $\text{BH}_\alpha^*$. We have

$$
\mathbb{P}_{Y \sim Q}[|\text{BH}_\alpha^*| > 0] = \frac{\alpha}{n}\sum_{i=1}^n \mathbb{E}_{Y \sim Q}\left[1\{p_i^* \leq T_\alpha^*(Y)\} \bigg/ T_\alpha^*(Y) \vee (\alpha/n)\right]
$$

$$
= \frac{\alpha}{n}\sum_{i=1}^n \mathbb{E}_{Y \sim Q}\left[\mathbb{P}(p_i^* \leq T_\alpha^*(Y(i)) \mid Y(i)) \bigg/ T_\alpha^*(Y(i))\right] = \alpha\eta(\pi)
$$
where we used Lemma C.3 (and the notation therein), the independence between $p_i^*$ and $Y^{(i)}$, combined with the fact that $T^*_a(Y^{(i)}) \leq \alpha < e^{-\mu}$ and (69).

Next, as assumed in the statement of the theorem, let us suppose that $R$ is such that

$$\sup_{P \in \mathcal{P}_0: n_1(P)/n \leq \pi} \{\text{FDR}(P, R) - \alpha n_0(P)/n \leq 0\}.$$

Then, it follows from (68) and Bernstein inequality that

$$\mathbb{E}_Z \mathbb{P}_{Y \sim P_1, Z} [|R(Y)| > 0] \leq \frac{2}{1 + e^{-\mu}} \mathbb{E}_Z [\text{FDR}(Q_1, Z, R)]$$

$$\leq \frac{2}{1 + e^{-\mu}} \left[ \mathbb{E} \left[ \left( \frac{\alpha n_0(Z)}{n} \right) 1_{\{\pi - 2n^{-1/3} \leq n_1(Z)/n \leq \pi\}} \right] + \mathbb{P}(|n_1(Z)/n - \pi| > n^{-1/3}) \right]$$

$$\leq \frac{2}{1 + e^{-\mu}} \alpha(1 - \pi) + 2n^{-1/3} + 4e^{-c_m n^{1/3}}$$

$$\leq \frac{2}{1 + e^{-\mu}} \alpha(1 - \pi) + c_\mu n^{-1/3}.$$

Combining the above bounds yields

$$\mathbb{E}_Z [\mathbb{P}_{Y \sim P_1, Z} [|\text{BH}^*_\alpha| > 0] - \mathbb{P}_{Y \sim P_1, Z} [|R(Y)| > 0]] \geq \alpha \eta(\pi) - \frac{2}{1 + e^{-\mu}} \alpha(1 - \pi) - c_\mu n^{-1/3}$$

$$= \alpha \zeta(\pi) - c_\mu n^{-1/3},$$

where $\zeta(u) = \frac{1-u}{2} [1 + \frac{(1-u)^2}{(1-2u)^2} - 4/\left(1 + \frac{(1-2u)^2}{(1-u)^2}\right)],$ for $u \in (0, 1/2)$. Since $1 + x > 4/(1 + 1/x)$ for any $x > 1$, it follows that $\zeta(u) > 0$ for any $u \in (0, 1/2)$. Besides, one can check that $\zeta$ is increasing on $(0, 1/2)$. Since the functions $\eta$ and $\mu$ are continuously differentiable in $\pi$, we conclude that

$$\mathbb{E}_Z [\mathbb{P}_{Y \sim P_1, Z} [|\text{BH}^*_\alpha| > 0] - \mathbb{P}_{Y \sim P_1, Z} [|R(Y)| > 0]] \geq \alpha \zeta(\pi) - c_\mu n^{-1/3},$$

Applying again Bernstein inequality to $n_1(Z)$, we conclude that there exists $P$ with $n_1(P)/n \leq \pi$ such that

$$\mathbb{P}_{Y \sim P} [|\text{BH}^*_\alpha| > 0] - \mathbb{P}_{Y \sim P} [|R(Y)| > 0] \geq \alpha \zeta(\pi) - c_\mu n^{-1/3}.$$

The result follows.

\[\square\]

**Proof of Theorem 5.3.** We adapt the proof of Theorem 3.1 (see Sections 3.2 and 3.3) to the location model. Following exactly the same proof, (41) and (42) hold provided that we modify the four following ingredients: Lemma C.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4. This is done below.

**Modification of Lemma C.1:** Arguing as for Lemma C.1, we can prove that the empirical median is close to $\theta(P)$. Precisely, there exist constants $c'_1, c'_2, c'_3 > 0$, only depending on $g$, such that for $n \geq 1$, all $P \in \mathcal{P}$ such that $n_0(P)/n \geq 0.9$, and all $x \in (0, c'_4 n)$,

$$\mathbb{P} \left( |\bar{\theta} - \theta(P)| \geq c'_2 \frac{n_1(P)}{n} + \frac{2}{n} + c'_3 \sqrt{\frac{x}{n}} \right) \leq 2e^{-x}. \quad (71)$$
The proof is mainly based on the fact $\mathcal{G}$ and $\mathcal{G}^{-1}$ are continuously differentiable and therefore locally lipschitz around 0 and 1/2. Hence, for $x = n^{2/3}$, we get for $n$ large enough, $\mathbb{P}(\Omega^n) \leq 2e^{-n^{2/3}}$ for

$$
\Omega = \left\{ |\hat{\theta} - \theta(P)| \leq \delta \right\}, \quad \delta = c_2 \left( \frac{n_1(P) + 2}{n} + n^{-1/6} \right),
$$

for some constant $c_2$ only depending on $g$.

**Modification of Lemma 3.2:** with the additional assumption $\eta \leq 1/2$, we easily check that the same result holds under the condition $|\hat{\theta} - \theta(P)| \leq \mathcal{G}^{-1}(1/4)/2$, which is ensured on $\Omega$ for $n$ larger than some constant (only depending on $g$).

**Modification of Lemma 3.3:** the following lemma is a modification of Lemma 3.3.

**Lemma A.4.** For an arbitrary estimator $\hat{\theta}$, let $\delta > 0$, $\theta \in \mathbb{R}$, $\alpha \in (0, 0.5)$, $t_0 \in (0, \alpha)$ and

$$
\eta = \delta \max_{t \in [t_0, \alpha]} \left\{ 1 - \frac{g(\mathcal{G}^{-1}(t))}{\mathcal{G}^{-1}(t/2) - \mathcal{G}^{-1}(t)} \right\}
$$

Assume that $|\hat{\theta} - \theta| \leq \delta$ and $\eta \leq 0.05$. Then, for all $i \in \{1, \ldots, n\}$,

- if $T_\alpha(\hat{\theta}) \vee (\alpha/n) \geq t_0$, we have
  $$
  \mathbb{I} \{ p_i(\hat{\theta}) \leq T_\alpha(\hat{\theta}) \} \leq \mathbb{I} \{ p_i(\theta) \leq (1 + \eta)T_\alpha(\hat{\theta}) \} \leq \mathbb{I} \{ p_i(\theta) \leq T_{\alpha(1+\eta)}(\theta) \}; \quad (72)
  $$
- if $T_{0.95\alpha}(\hat{\theta}) \vee (0.95\alpha/n) \geq t_0$, we have
  $$
  \mathbb{I} \{ p_i(\theta) \leq T_{\alpha(1-\eta)}(\theta) \} \leq \mathbb{I} \{ p_i(\theta) \leq T_\alpha(\hat{\theta}) \}. \quad (73)
  $$

To prove Lemma A.4, we follow the same strategy as in Section C.3. For all $u, u' \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, $\alpha \in (0, 0.5)$, $t_0 \in (0, \alpha)$, for all $t \in [t_0, \alpha]$, we have

$$
\mathbb{I} \{ p_i(u) \leq t \} = \mathbb{I} \{ 2\mathcal{G}(|Y_i - u'|) \leq t \}
\leq \mathbb{I} \{ 2\mathcal{G}(|Y_i - u| + |u - u'|) \leq t \}
= \mathbb{I} \{ 2\mathcal{G}(|Y_i - u|) \leq 2\mathcal{G}(\mathcal{G}^{-1}(t/2) - |u - u'|) \}
\leq \mathbb{I} \{ 2\mathcal{G}(|Y_i - u|) \leq t(1 + \eta') \},
$$

for

$$
\eta' = \max_{t \in [t_0, \alpha]} \left\{ \frac{\mathcal{G}(\mathcal{G}^{-1}(t/2) - \delta) - t/2}{t/2} \right\},
$$

provided that $|u - u'| \leq \delta$. Now, if $\eta \leq 0.05$, we can prove that

$$
\eta' \leq \eta. \quad (74)
$$

Indeed, $\eta \leq 1$ implies that the following inequality holds

$$
\delta \leq \min_{t \in [t_0, \alpha]} \left\{ \mathcal{G}^{-1}(t/2) - \mathcal{G}^{-1}(t) \right\}, \quad (75)
$$

Furthermore, since $\delta \leq \mathcal{G}^{-1}(\alpha/2)$, for all $t \in [t_0, \alpha]$,

$$
\mathcal{G}(\mathcal{G}^{-1}(t/2) - \delta) - t/2 \leq \delta \max_{u \in [\mathcal{G}^{-1}(t/2) - \delta, \mathcal{G}^{-1}(t/2)]} \{ g(u) \}
= \delta g(\mathcal{G}^{-1}(t/2) - \delta) \leq \delta g(\mathcal{G}^{-1}(t)),
$$

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by the monotonic property of $g$ and (75). Similarly, for all $t \in [t_0, \alpha]$, 
\[
G^{-1}(t/2) - G^{-1}(t) \leq \frac{t/2}{g(G^{-1}(t/2))}.
\]
Combining these inequalities leads to (74). In turn, (72) and (73) hold provided that $|\tilde{\theta} - \theta| \leq \delta$.

**Modification of Lemma 3.4:** The same results holds because $g$ is symmetric.

\[\Box\]

**Proof of Corollaries 5.4 and 5.5.** First, we assume that $\zeta > 1$. Define $g(x) = L^{-1}_\zeta e^{-|x|^{1/\zeta}}$, $\zeta > 1$, with the normalization constant $L_\zeta = 2\Gamma(1/\zeta)\zeta^{1/\zeta-1}$. In that case, we have (see Lemma S-5.1 in the supplement of Neuvial and Roquain (2012)),
\[
\forall q \in (0, G(1)), \quad G^{-1}(q) \leq (-\zeta \log q - \zeta \log L_\zeta)^{1/\zeta};
\]
\[
\forall y > 0, \quad \frac{g(y)}{y^{1-\zeta}} \geq \frac{g(y)}{y^{1-\zeta} y^{\zeta} + \zeta - 1}.
\]
The last inequality (used with $y = G^{-1}(q)$) implies that
\[
|G^{-1}(q)|^{\zeta} + \zeta \log(q) + \zeta \log L_\zeta + \zeta(\zeta - 1) \log(G^{-1}(q)) \in \left[\zeta \log \left(\frac{G^{-1}(q)^{\zeta}}{G^{-1}(q)^{\zeta} + \zeta - 1}\right); 0\right],\tag{77}
\]
and therefore that $[G^{-1}(t)] \sim [\zeta \log(1/t)]^{1/\zeta}$ for $t$ going to zero. As a consequence, for $t$ small enough, we have
\[
G^{-1}(t/2) - G^{-1}(t) = \zeta^{-1} [G^{-1}(t')]^{1-\zeta} \left[G^{-1}(t/2)^{\zeta} - [G^{-1}(t)]^{\zeta}\right],\tag{78}
\]
where $t' \in [t/2, t]$. In view of (77), this is of the order $\log^{1-\zeta}(1/t)$. Besides, (77) implies that $g(G^{-1}(t/2))/g(G^{-1}(t))$ is bounded away from 0. Since $[G^{-1}(t/2)] - [G^{-1}(t)]$ is bounded away from zero for large $t$, we deduce that $\eta$ in (40) is of the order
\[
\left(\frac{1}{n} \vee n^{-1/6}\right) \log^{1-\zeta}(n).
\]
Hence, $\eta$ goes to 0 when $k_n \ll n/\log^{1-\zeta}(n)$ and we deduce from Theorem 5.3 that the scaling $\tilde{\theta}$ is asymptotically optimal. Conversely, for $k_n \gg n/\log^{1-\zeta}(n)$, Condition (36) is satisfied and we deduce from Theorem 5.1 that optimal scaling is impossible.

Let us turn to the case $\zeta = 1$ (Laplace distribution). In that case, $g(x) = 0.5 e^{-|x|}$; $G(y) = 0.5 e^{-y}$ for $y \geq 0$; $G^{-1}(q) = -\log(2q)$ for $q \in [0, 1/2]$. Hence, $\eta$ in (40) satisfies
\[
\eta = c_2(g) \left(\frac{1}{n} \vee n^{-1/6}\right) \frac{2}{\log(2)}.
\]
Hence, we deduce from Theorem 5.3 that that the scaling $\tilde{\theta}$ is asymptotically optimal as long as $k_n \ll n$. 

\[\Box\]
Appendix B: Technical lemmas for Theorem 2.1

Proof of Lemma 2.3. We deduce from the definition of \( f_1 \) that

\[
\frac{\pi_1}{2} = \pi_1 \int f_1(u) du = \frac{1}{2} \left( (1 - \pi_2)\Phi(t_0/\sigma_2) - (1 - \pi_1)\Phi(t_0) \right)
= (1 - \pi_2) \left( \Phi(t_0/\sigma_2) - \Phi(t_0) \right) - (\pi_2 - \pi_1)\Phi(t_0)
\leq (1 - \pi_2)\phi(t_0/\sigma_2)\frac{\sigma_2 - 1}{\sigma_2} = (1 - \pi_1)\phi(t_0)\sigma_2(\sigma_1 - 1)
\]

where we used the definition of \( t_0 \) in the last line. Let \( \epsilon_0' \in (0, 1) \) be an absolute constant that will be fixed later. We prove the first result by contradiction. Assume that

\[ \sigma_2 - 1 \geq \epsilon_0'\pi_2[1 + \log(\pi_2/\pi_1)]^{-1} \leq 1/4 \] (79)

which, in view of the previous inequality, implies

\[ t_0\phi(t_0) \geq \frac{\pi_1[1 + \log(\pi_2/\pi_1)]}{2\epsilon_0'\pi_2(1 - \pi_1)}. \] (80)

**Case 1**: \( \pi_2 \leq 2\pi_1 \). Then, (80) implies that \( t_0\phi(t_0) \geq (4\epsilon_0')^{-1}(1 + \log(2)) \), because \( x \in [0, 1] \rightarrow x(1 + \log(1/x)) \) is nondecreasing. Since \( x\phi(x) \leq (2\pi)^{-1/2}e^{-1/2} \) for any \( x \in \mathbb{R} \), this last inequality cannot hold for \( \epsilon_0' \) sufficiently small and we have therefore

\[ \sigma_2 - 1 \geq \epsilon_0'\pi_2[1 + \log(\pi_2/\pi_1)]^{-1}. \]

**Case 2**: \( \pi_2 > 2\pi_1 \). We deduce from (79), the definition (17) of \( t_0 \), \( \log(1 + x) \geq x/(1 + x) \) for \( x \geq 0 \), and \( \sigma_2 \geq 1 \) that

\[
t_0 = \sqrt{\frac{2\log \left( \frac{1 - \pi_1}{1 - \pi_2} \right) \sigma_2^2}{\sigma_2^2 - 1} + \frac{2\log(\pi_2)\sigma_2^2}{\sigma_2^2 - 1}}
\geq \sqrt{\frac{2[1 + \log(\pi_2/\pi_1)]\log \left( \frac{1 - \pi_1}{1 - \pi_2} \right)\sigma_2^2}{c_0'\pi_2(\sigma_2 + 1)} + \frac{2\sigma_2}{\sigma_2 + 1}}
\geq \sqrt{\frac{2[1 + \log(\pi_2/\pi_1)](\pi_2 - \pi_1)}{c_0'\pi_2(1 - \pi_1)} + 1}
\geq \sqrt{\frac{1 + \log(\pi_2/\pi_1)}{c_0'} + 1}.
\]

Recall that \( x\phi(x) \) is decreasing for \( x \geq 1 \). Provided that we chose \( \epsilon_0' \leq 1/2 \), we have therefore \( \phi(t_0)\sigma_2(\sigma_1 - 1) \leq \frac{\pi_1}{c_0'}\sqrt{\frac{1 + \log(\pi_2/\pi_1)}{c_0'}} + 1 \) which contradicts (80) provided that \( \epsilon_0' \) is small enough (independently of \( \pi_1 \) and \( \pi_2 \)). As in Case 1, we conclude that

\[ \sigma_2 - 1 \geq \epsilon_0'\pi_2[1 + \log(\pi_2/\pi_1)]^{-1}. \]
Let us turn to the second part of the lemma. By concavity, we have $\log(1 + x)/x \in [1/(1 + x), 1]$ for any $x > 0$. From (17), $\pi_1 \leq 1/4$, and the last bound of $\sigma_2 - 1$, we deduce that

$$
\frac{t_0}{\sigma_2} \leq 1 + \sqrt{\frac{2(\pi_2 - \pi_1)}{(1 - \pi_1)(\sigma_2^2 - 1)}} \leq 1 + \sqrt{\frac{4(\pi_2 - \pi_1)}{3(\sigma_2 - 1)}}
$$

Provided that $\log(\pi_2/\pi_1) \leq c_0'' \log(n)$, we have $t_0/\sigma_2 \leq \sqrt{0.5 \log(n)}$ for $n$ large enough and $c_0''$ sufficiently small. Hence, we derive from Lemma D.2 that

$$
\Phi(t_0/\sigma_2) \geq \frac{\sqrt{0.5 \log(n)}}{1 + 0.5 \log(n)} \phi(\sqrt{0.5 \log(n)}) \geq (4\pi \log n)^{-1/2} n^{-1/4} \geq 10 \sqrt{2 \log(2n)}/n,
$$

for $n$ large enough.

Proof of Lemma 2.4. We check that $u_1 \leq \sqrt{\log(n)}$. From the definition (19) of $u_1$, $\pi_1 \leq \pi_2 \leq 1/4$, and Lemma 2.3, we deduce that

$$
u_1^2 \leq 2\frac{\sigma_2^2}{\sigma_2^2 - 1} \log(\sigma_2) + 2\frac{\sigma_2^2}{\sigma_2^2 - 1} \log \left(\frac{9}{\alpha(1 - \pi_2)}\right)
$$

$$
\leq 2\sigma_2 \left[1 + \frac{1 + \log(\pi_2/\pi_1)}{c_0\pi_2} \log \left(\frac{12}{\alpha}\right)\right],
$$

where we used that $\pi_2 \leq 1/4$. Besides, we have shown above (17) that $\sigma_2 \leq c_1$ for some universal constant $c_1$. All in all, we have proved that

$$
u_1^2 \leq c_1' + c_2' \frac{1 + \log(\pi_2/\pi_1)}{c_0\pi_2} \log \left(\frac{68}{3\alpha}\right),
$$

which, by assumption, is smaller than $\log n$. The result follows.

\appendix

Appendix C: Remaining proofs for Theorems 3.1 and 4.4

C.1. Estimation of $\theta(P)$, $\sigma(P)$

The following results are close to those of Chen et al. (2018) in dimension 1. The setting here is slightly different, because we are not considering a mixture model, so we provide a proof for completeness.

Lemma C.1. Consider the estimators defined by (22). Then there exists a constant $c > 0$ such that for $n \geq 16$, all $P \in \mathcal{P}$ such that $n_0(P)/n \geq 0.9$, and all $x \in (0, cn)$,

$$
P \left(\frac{\hat{\theta} - \theta(P)}{\sigma} \geq 2 \frac{n_1(P)}{n} + 2 \sqrt{\frac{x}{n}}\right) \leq 2e^{-x};
$$

$$
P \left(\frac{\hat{\sigma} - \sigma(P)}{\sigma} \geq 6 \frac{n_1(P)}{n} + 16 \sqrt{\frac{x}{n}}\right) \leq 4e^{-x}.
$$
Proof of Lemma C.1. Let $\xi = (Y - \theta)/\sigma$, so that $\xi_i$, $i \in \mathcal{H}_0$ are i.i.d. $\mathcal{N}(0, 1)$. Let $T = \sqrt{\frac{18}{n/2} + 2\frac{n_0 + 2}{n}}$. Since $n_0 \geq 0.9n$, to prove (81), it suffices to show that we prove $|\bar{\theta} - \theta| \geq \sigma T$ with probability smaller than $2e^{-x}$. We have

$$
P\left(\bar{\theta} - \theta \geq \sigma T\right) = P\left(\frac{Y_{(n/2)} - \theta}{\sigma} \geq T\right) = P\left(\xi_{(n/2)} \geq T\right) \leq P\left(\xi_{(n/2):\mathcal{H}_0} \geq T\right),$$

(83)

Note that $0.3 < 0.5 \leq \lceil \frac{n/2}{n_0} \rceil \leq \left(\frac{n}{2} + 1\right)/(0.9n) < 0.7$ by assumption. Hence, from Lemma D.3, we have

$$0 \leq -\overline{\Phi}^{-1}\left(\frac{n/2}{n_0}\right) \leq 3.6\left(\frac{n/2}{n_0} - 1/2\right) \leq 1.8\frac{n + 2 - n_0}{n_0} \leq 2\frac{n_0 + 2}{n}.$$ 

Using this and applying Lemma D.4, we get that for $x \leq cn$ (for some constant $c > 0$) the rhs in (83) is bounded by

$$P\left(\xi_{(n/2):\mathcal{H}_0} \geq T\right) \leq P\left[\xi_{(n/2):\mathcal{H}_0} + \overline{\Phi}^{-1}\left(\frac{n/2}{n_0}\right) \geq \sqrt{18x/n}\right] \leq P\left[\xi_{(n/2):\mathcal{H}_0} + \overline{\Phi}^{-1}\left(\frac{n/2}{n_0}\right) \geq 3\frac{\sqrt{2(n/2)x}}{n_0} \right] \leq e^{-x}.$$ (84)

This gives for all $x \in (0, cn)$, $P(\bar{\theta} - \theta \geq \sigma T) \leq e^{-x}$. Conversely, we have

$$P\left(\theta - \bar{\theta} \geq \sigma T\right) = P\left((-\xi)_{(n/2)} \geq T\right) \leq P\left((-\xi)_{(n/2):\mathcal{H}_0} > T\right) = P\left(\xi_{(n/2):\mathcal{H}_0} > T\right),$$

by symmetry of the Gaussian distribution. Bounding again $P(\xi_{(n/2):\mathcal{H}_0} > T)$, we obtain (81).

Let us now prove (82). Let $u_0 = \overline{\Phi}^{-1}(1/4) \in (0.6, 0.7)$ and $T' = (1 + n_1)/n + \sqrt{\frac{8(n+1)x}{n_0}}$. Since $n_0 \leq 0.9n$, we only have to prove that, with probability higher than $1 - 4e^{-x}$, we have $|\sigma - \bar{\sigma}| \geq \sigma (2T + 2T')$. By Definition (22) of $\bar{\sigma}$, we have

$$P\left(|\sigma - \bar{\sigma}| \geq \sigma (2T + 2T')\right) = P\left(\frac{|u_0 - U_{(n/2)}|/\sigma}{u_0} \geq 2T + 2T'\right).$$

Since $|\xi_i| - |\xi_{(n/2)}| \leq |\xi_i - \xi_{(n/2)}| \leq |\xi_i| + |\xi_{(n/2)}|$, we have $|U_{(n/2)}|/\sigma - |\xi|_{(n/2)} \in [-|\xi_{(n/2)}|, |\xi_{(n/2)}|]$. Thus, we have

$$P\left(|\sigma - \bar{\sigma}| \geq \sigma (2T + 2T')\right) \leq P\left(|\xi_{(n/2)}|/u_0 \geq 2T\right) + P\left(|u_0 - |\xi|_{(n/2)}|/u_0 \geq 2T'\right) \leq 2e^{-x} + P\left(|u_0 - |\xi|_{(n/2)}| \geq T'\right),$$

(85)

where we used (84) and $2u_0 \geq 1$. Since

$$|\xi|_{(n_0 - n + [n/2]:\mathcal{H}_0)} = -(-|\xi|)_{(n_0 + [n/2]:\mathcal{H}_0)} \leq |\xi|_{(n/2)} \leq |\xi|_{(n/2):\mathcal{H}_0}$$

we have

$$P\left(|u_0 - |\xi|_{(n/2)}| \geq T'\right) \leq P\left(|\xi|_{(n/2):\mathcal{H}_0} \geq u_0 + T'\right) + P\left(|\xi|_{n_0 - n + [n/2]:\mathcal{H}_0} \leq u_0 - T'\right).$$
We now apply Lemma D.5 to control the deviations of these order statistics. We easily check that $0.4n_0 \leq \lfloor n/2 \rfloor \leq \lceil n/2 \rceil \leq 0.6n_0$. Hence, for all $x \leq cn$ (for $c$ small enough), we have

$$
\Pr \left[ |\xi|_{(\lfloor n/2 \rfloor):\mathcal{H}_0} \geq \Phi^{-1} \left( \frac{n - \lfloor n/2 \rfloor}{2n_0} \right) + 4 \sqrt{\frac{n/2}{n_0}} \right] \leq e^{-x};
$$

$$
\Pr \left[ |\xi|_{(n_0 - n + \lfloor n/2 \rfloor):\mathcal{H}_0} \leq \Phi^{-1} \left( \frac{n - \lfloor n/2 \rfloor}{2n_0} \right) - 2 \sqrt{2(n_0 - n + \lfloor n/2 \rfloor)x} \right] \leq e^{-x}.
$$

It remains to compare these two rhs expression with $T'$. By Lemma D.3 and since $n_0 \geq 0.9n$, both $|\Phi^{-1} \left( \frac{n_0 - \lfloor n/2 \rfloor}{2n_0} \right) - \Phi^{-1}(1/4)|$ and $|\Phi^{-1} \left( \frac{n - \lfloor n/2 \rfloor}{2n_0} \right) - \Phi^{-1}(1/4)|$ are less or equal $(n_1 + 1)/n$. The deviation terms in the above deviation inequalities are also smaller than $\sqrt{6(n + 1)x}/n_0$. This concludes the proof.

\[ \square \]

### C.2. Proof of Lemma 3.2

We start with two lemmas. The first one ensures that $\tilde{\theta}$ and $\tilde{\sigma}$ are not perturbed when $p_i(\tilde{\theta}, \tilde{\sigma})$ is small. The second one compares the thresholds of plug-in BH procedures based on $Y$ and $Y^{(i)}$.

**Lemma C.2.** For any $i \in \{1, \ldots, n\}$, and $t \in (0, 0.5)$, if $p_i(\tilde{\theta}, \tilde{\sigma}) \leq t$ and $|\tilde{\theta} - \theta| < 0.3 \tilde{\sigma}$, then we have both $\tilde{\theta}^{(i)} = \tilde{\theta}$ and $\tilde{\sigma}^{(i)} = \tilde{\sigma}$.

**Lemma C.3.** For all $u \in \mathbb{R}$, $s > 0$, any $i \in \{1, \ldots, n\}$, we have for all $\alpha \in (0, 1)$,

$$
1 \{p_i(u, s) \leq T_\alpha(Y; u, s)\} = 1 \{T_\alpha(Y^{(i)}; u, s) = T_\alpha(Y; u, s)\} = 1 \{p_i(u, s) \leq T_\alpha(Y^{(i)}; u, s)\},
$$

where $T(\cdot)$ is defined by (7).

From Lemma C.3 with $u = \tilde{\theta}$ and $s = \tilde{\sigma}$ and Lemma C.2 with $t = T_\alpha(Y^{(i)}; \tilde{\theta}, \tilde{\sigma}) \leq \alpha < 0.5$, we deduce that

$$
1 \{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y; \tilde{\theta}, \tilde{\sigma})\} = 1 \{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y^{(i)}; \tilde{\theta}, \tilde{\sigma})\}
$$

$$
= 1 \{p_i(\tilde{\theta}, \tilde{\sigma}) \leq T_\alpha(Y^{(i)}; \tilde{\theta}, \tilde{\sigma}), \tilde{\theta} = \tilde{\theta}^{(i)}, \tilde{\sigma} = \tilde{\sigma}^{(i)}\}
$$

$$
= 1 \{p_i(\tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)}) \leq T_\alpha(Y^{(i)}; \tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)}), \tilde{\theta} = \tilde{\theta}^{(i)}, \tilde{\sigma} = \tilde{\sigma}^{(i)}\}
$$

$$
= 1 \{p_i(\tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)}) \leq T_\alpha(Y^{(i)}; \tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)})\}.
$$

The fourth equality uses again Lemma C.2 with $t = T_\alpha(Y^{(i)}; \tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)}) \leq \alpha < 0.5$. Finally, provided that the above event is true, we clearly have $T_\alpha(Y; u, s) = T_\alpha(Y^{(i)}; \tilde{\theta}^{(i)}, \tilde{\sigma}^{(i)})$ by Lemma C.3. We have proved Lemma 3.2

**Proof of Lemma C.2.** Assume that $p_i(\tilde{\theta}, \tilde{\sigma}) \leq t$. This gives $|Y_i - \tilde{\theta}| \geq \tilde{\sigma} \Phi^{-1}(t/2) > \tilde{\sigma} \Phi^{-1}(1/4)$ (because $t < 1/2$). If we further assume that $|\tilde{\theta} - \theta| \leq 0.3 \tilde{\sigma} < \tilde{\sigma} \Phi^{-1}(1/4)/2$, we have

- either $Y_i - \tilde{\theta} > \tilde{\sigma} \Phi^{-1}(1/4)$, which gives $Y_i - \theta > \tilde{\sigma} \Phi^{-1}(1/4)/2$ and thus $Y_i > Y_{(\lfloor n/2 \rfloor)} \lor \theta$.

In this case, $Y_i^{(i)} = \text{sign}(Y_i - \theta) \times \infty = \infty$ and $\tilde{\theta}^{(i)} = \tilde{\theta}$;
• or $Y_i - \tilde{\theta} < -\tilde{\sigma} \bar{\Phi}^{-1}(1/4)$, which gives $Y_i - \theta < -\tilde{\sigma} \bar{\Phi}^{-1}(1/4)/2$ and thus $Y_i < Y_{[n/2]} \wedge \theta$. In this case, $Y_i^{(i)} = \text{sign}(Y_i - \theta) \times \infty = -\infty$ and $\tilde{\theta}^{(i)} = \tilde{\theta}$.

Hence, in both cases, we have $\tilde{\theta}^{(i)} = \tilde{\theta}$. This implies that $\bar{\Phi}^{-1}(1/4) \tilde{\sigma}$ is the empirical median of the $|Y_j - \tilde{\theta}^{(i)}|$, $1 \leq j \leq n$ and that $|Y_i - \tilde{\theta}^{(i)}| > \bar{\Phi}^{-1}(1/4) \tilde{\sigma}$. Hence, $\bar{\Phi}^{-1}(1/4) \tilde{\sigma}$ is also the empirical median of the $|Y_j^{(i)} - \tilde{\theta}^{(i)}|$, $1 \leq j \leq n$ (whose element $j = i$ is infinite). Hence, $\tilde{\sigma} = \tilde{\sigma}^{(i)}$ and the result is proved.

\begin{proof}[Proof of Lemma C.3] Remember that

$$T_\alpha(Y; u, s) = \max \left\{ t \in [0, 1] : \sum_{j=1}^{n} 1\{p_j(u, s) \leq t\} \geq nt/\alpha \right\}$$

with $\sum_{j=1}^{n} 1\{p_j(u, s) \leq T_\alpha(Y; u, s)\} = nT_\alpha(Y; u, s)/\alpha$. Since $Y_i^{(i)} = \text{sign}(Y_i - \theta) \times \infty$, the corresponding $p$-value is equal to 0 and

$$T_\alpha(Y^{(i)}; u, s) = \max \left\{ t \in [0, 1] : 1 + \sum_{j \neq i} 1\{p_j(u, s) \leq t\} \geq nt/\alpha \right\}.$$ 

Hence, $T_\alpha(Y^{(i)}; u, s) \geq T_\alpha(Y; u, s)$ always holds.

Assume now that $p_i(u, s) \leq T_\alpha(Y^{(i)}; u, s)$. This gives $\sum_{j=1}^{n} 1\{p_j(u, s) \leq T_\alpha(Y^{(i)}; u, s)\} = 1 + \sum_{j \neq i} 1\{p_j(u, s) \leq T_\alpha(Y^{(i)}; u, s)\} \geq nT_\alpha(Y^{(i)}; u, s)/\alpha$ and thus the reverse inequality $T_\alpha(Y^{(i)}; u, s) \leq T_\alpha(Y; u, s)$ is also true, which gives $T_\alpha(Y^{(i)}; u, s) = T_\alpha(Y; u, s)$. This in turn implies $p_i(u, s) \leq T_\alpha(Y; u, s)$.

To conclude, it remains to check that $T_\alpha(Y^{(i)}; u, s) = T_\alpha(Y; u, s)$ implies $p_i(u, s) \leq T_\alpha(Y^{(i)}; u, s)$. If both thresholds are equal, we have

$$1 + \sum_{j \neq i} 1\{p_j(u, s) \leq T_\alpha(Y^{(i)}; u, s)\} = nT_\alpha(Y^{(i)}; u, s)/\alpha = \sum_{j=1}^{n} 1\{p_j(u, s) \leq T_\alpha(Y^{(i)}; u, s)\},$$

which implies $p_i(u, s) \leq T_\alpha(Y^{(i)}; u, s)$. The result follows.

\end{proof}

\section*{C.3. Proof of Lemma 3.3}

We start by gathering a few lemmas on the rescaled $p$-values process. Those are proved at the end of the section. The first lemma to quantifies how the rescaling affects the $p$-value process. For $x, y \geq 0$ and $t \in [0, 1)$, define

$$I_t(x, y) = 2\bar{\Phi}\left(\bar{\Phi}^{-1}(t/2) - x - y\bar{\Phi}^{-1}(t/2)\right).$$

The following lemma quantifies how the process $1\{p_i(u, s) \leq t\}$ fluctuates in $u$ and $s$, according to the functional $I_t(\cdot, \cdot)$.
Lemma C.4. For all \( u, u' \in \mathbb{R}, s, s' > 0, i \in \{1, \ldots, n\} \) and \( t \in [0, 1) \), we have
\[
\mathbb{1}\{p_i(u', s') \leq t\} \leq \mathbb{1}\{p_i(u, s) \leq I_t(|u' - u|s^{-1}, |s' - s|s^{-1})\}. \tag{87}
\]

Interestingly, \( t \mapsto I_t(x, y) \) is close to the identity function when \( x \) and \( y \) are small, as the following lemma shows.

Lemma C.5. There exists a universal constant \( c > 1 \) such that the following holds. For all \( \alpha \in (0, 0.8) \), for all \( x, y \geq 0 \) and \( t_0 \in (0, \alpha) \), we have
\[
\max_{0 \leq t \leq \alpha} \left\{ \frac{I_t(x, y) - t}{t} \right\} \leq c \left( x(2 \log(1/t_0))^{1/2} + 2y \log(1/t_0) \right). \tag{88}
\]

provided that this upper bound is smaller than 0.05.

Combining Lemma C.4 and Lemma C.5, we obtain the following corollary.

Corollary C.6. There exists a universal constant \( c > 1 \) such that the following holds. For all \( u, u' \in \mathbb{R}, s, s' > 0, i \in \{1, \ldots, n\}, \alpha \in (0, 0.8), t_0 \in (0, \alpha) \), let
\[
\eta = c \left( |u' - u|s^{-1}(2 \log(1/t_0))^{1/2} + |s' - s|s^{-1}2 \log(1/t_0) \right). \tag{89}
\]
Provided \( \eta \leq 0.05 \), we have for all \( t \in [t_0, \alpha] \),
\[
\mathbb{1}\{p_i(u', s') \leq t\} \leq \mathbb{1}\{p_i(u, s) \leq t(1 + \eta)\}. \tag{90}
\]

Let us first prove \((26)\). First, if \( T_\alpha(\hat{\theta}, \hat{\sigma}) < \alpha/n \), then \( \mathbb{1}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma})\} = 0 \) for all \( i \) and the result is trivial. Now assume that \( T_\alpha(\hat{\theta}, \hat{\sigma}) \geq t_0 \). By \((90)\) \((u' = \theta, u = \theta, s' = \hat{\sigma}, s = \sigma, t = T_\alpha(\hat{\theta}, \hat{\sigma}))\), we have for all \( i \),
\[
\mathbb{1}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma})\} \leq \mathbb{1}\{p_i(\theta, \sigma) \leq (1 + \eta)T_\alpha(\hat{\theta}, \hat{\sigma})\}, \tag{91}
\]
so we only have to prove \((1 + \eta)T_\alpha(\hat{\theta}, \hat{\sigma}) \leq T_{\alpha(1+\eta)}(\theta, \sigma)\). Since by definition
\[
T_{\alpha(1+\eta)}(\theta, \sigma) = \max \left\{ t \in [0, 1] : \sum_{i=1}^{n} \mathbb{1}\{p_i(\theta, \sigma) \leq t\} \geq nt/(\alpha(1+\eta)) \right\},
\]
we only have to prove \( \sum_{i=1}^{n} \mathbb{1}\{p_i(\theta, \sigma) \leq (1 + \eta)T_\alpha(\hat{\theta}, \hat{\sigma})\} \geq nT_\alpha(\hat{\theta}, \hat{\sigma})/\alpha \). For this, apply again \((91)\), to get
\[
\sum_{i=1}^{n} \mathbb{1}\{p_i(\theta, \sigma) \leq (1 + \eta)T_\alpha(\hat{\theta}, \hat{\sigma})\} \geq \sum_{i=1}^{n} \mathbb{1}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma})\}
= nT_\alpha(\hat{\theta}, \hat{\sigma})/\alpha,
\]
by using the definition of \( T_\alpha(\hat{\theta}, \hat{\sigma}) \). Hence, the first result is proved.

Exchanging \( \hat{\theta}, \hat{\sigma} \) by \( \theta, \sigma \) and replacing \( \alpha \) by \( \alpha(1 - \eta) \), \((26)\) implies that if \( T_{\alpha(1-\eta)}(\theta, \sigma) \lor (\alpha(1 - \eta)/n) \geq t_0 \), \( \mathbb{1}\{p_i(\theta, \sigma) \leq T_{\alpha(1-\eta)}(\theta, \sigma)\} \leq \mathbb{1}\{p_i(\hat{\theta}, \hat{\sigma}) \leq T_{\alpha(1-\eta)}(\hat{\theta}, \hat{\sigma})\} \). Since \( T_{\alpha(1-\eta)(1+\eta)}(\hat{\theta}, \hat{\sigma}) \leq T_\alpha(\hat{\theta}, \hat{\sigma}) \), this gives in turn \((27)\), which concludes the proof.
Proof of Lemma C.4. First, we can assume that $p_i(u', s') \leq t$ otherwise the inequality is trivial. By definition (4), this implies $|Y_i - u'|/s' \geq \Phi^{-1}(t/2)$. By triangular inequality, we have

$$\frac{|Y_i - u|}{s} \geq \frac{s'}{s} \Phi^{-1}(t/2) - \frac{1}{s}|u' - u|;$$

$$= \Phi^{-1}(t/2) - \frac{1}{s}|u' - u| - \frac{|s' - s|}{s} \Phi^{-1}(t/2),$$

which entails the upper bound in (87).

Proof of Lemma C.5. We have

$$I_t(x, y) = 2\Phi \left( \Phi^{-1}(t/2) - z(t) \right), \ z(t) = x + y\Phi^{-1}(t/2).$$

By Lemma D.2, we have for all $t \in [t_0, \alpha]$

$$z(t) \leq x + y(2 \log(1/t))^{1/2} \leq (0.05/c)(2 \log(1/t))^{-1/2} \leq 0.05/\Phi^{-1}(t/2),$$

since the rhs of (88) is smaller than or equal to 0.05. Now using that $\Phi(\sqrt{0.05}) \geq 0.4 \geq t/2$, we deduce $z(t) \leq \Phi^{-1}(t/2)$ for all $t \in [t_0, \alpha]$. Also deduce that for such a value of $t$,

$$\frac{\phi \left( \Phi^{-1}(t/2) - z(t) \right)}{\phi \left( \Phi^{-1}(t/2) \right)} = e^{-z^2(t)/2}e^{z(t)\Phi^{-1}(t/2)} \leq e^{z(t)\Phi^{-1}(t/2)} \leq e^{0.05} \leq 2.$$

Since $\Phi$ is decreasing and its derivative is $-\phi$, we have for all $t \in [t_0, \alpha]$

$$I_t(x, y) - t = 2\Phi \left( \Phi^{-1}(t/2) - z(t) \right) - 2\Phi \left( \Phi^{-1}(t/2) \right)$$

$$\leq z(t) \frac{2\Phi \left( \Phi^{-1}(t/2) - z(t) \right)}{\phi \left( \Phi^{-1}(t/2) \right)}$$

$$\leq z(t) \frac{\Phi \left( \Phi^{-1}(t/2) - z(t) \right)}{\phi \left( \Phi^{-1}(t/2) \right)}$$

$$\leq 4z(t)\phi \left( \Phi^{-1}(t/2) \right)$$

$$\leq 2tz(t) \left( 1 + \left( \Phi^{-1}(t/2) \right)^{-2} \right) \Phi^{-1}(t/2)$$

$$\leq tz(t_0)\Phi^{-1}(t_0/2) \left( 1 + \left( \Phi^{-1}(0.4) \right)^{-2} \right),$$

where we used inequality (92) of Lemma D.2 and $t \in [t_0, \alpha]$ in the last line. Finally, we invoke Lemma D.2 again to obtain that

$$z(t_0)\Phi^{-1}(t_0/2) \leq x(2 \log(1/t_0))^{1/2} + 2y \log(1/t_0),$$

which concludes the proof.
C.4. Proof of Lemma 3.4

By assumption, the \( Y_i \)'s are all mutually independent. In addition, when \( i \in \mathcal{H}_0, Y_i - \theta \) is a Gaussian distribution, which is symmetric. This implies that the variables of \( \{(\text{sign}(Y_i - \theta), i \in \mathcal{H}_0), (|Y_i - \theta|, i \in \mathcal{H}_0), (Y_i, i \in \mathcal{H}_1)\} \) are all mutually independent. In particular, for all fixed \( i \in \mathcal{H}_0 \), the variables of \( \{\text{sign}(Y_i - \theta), |Y_i - \theta|, (Y_j, j \neq i)\} \) are mutually independent. Since \( Y^{(i)} \) is a measurable function of \( \{\text{sign}(Y_i - \theta), (Y_j, j \neq i)\} \), it is in particular independent of \( |Y_i - \theta| \).

C.5. Proof of Lemma A.3

It is analogous to the proof of Lemma 3.3. The only change is that (90) is now used with \( s = \sigma \) (so \( s = s' \)) instead of \( s = \hat{\sigma} \).

C.6. Proof of Lemma A.2

To simplify the notation, we write \( \delta = \mu/2 \) and \( n_0 \) for \( n_0(P) \).

Lemma C.7. The function \( t \mapsto \Psi_1(t)/t \) is continuous and strictly decreasing on \((0, 1]\) and \( \Psi_1(t)/t \) goes to infinity when \( t \) converges to zero.

Since \( \Psi_1(1) = \Phi(-\delta) \leq 1 < 2/\alpha \), the equation \( \Psi_1(t) = 2t/\alpha \) has only one solution on \((0, 1]\), denoted \( t^*_\alpha \). Write \( t_0 = \frac{\alpha}{2} n_0^{-1/4} \), we claim that, for \( n_0 \geq N(\alpha) \), \( \Psi_1(t_0) \geq 4t_0/\alpha \). This claim is justified at the end of the proof. This implies \( \Psi_1(t_0)/t_0 \geq 4/\alpha = \Psi_1(t^*_\alpha/2)/t^*_\alpha/2 \).

Hence, we have \( t^*_\alpha/2 \geq t_0 \). On the event \( \Omega_0^- = \left\{ \sup_{t \in [0, 1]} |\hat{\Phi}_0^-(t) - t| \leq \sqrt{\log(2n)/(2n_0)} \right\} \), we have \( \hat{\Phi}_0^-(\Psi_1(t)) \geq \Psi_1(t) - \sqrt{\log(2n)/(2n_0)} \) for all \( t \in [0, 1] \), hence \( T_0^- = \max\{t \in [0, 1] : \hat{\Phi}_0^-(\Psi_1(t)) \geq 2t/\alpha\} \) is such that

\[
T_0^- \geq \max\left\{ t \in [0, 1] : \Psi_1(t) \geq 2t/\alpha + \sqrt{\log(2n)/(2n_0)} \right\}
\geq \max\left\{ t \in [0, 1] : \Psi_1(t) \geq 2t/\alpha + 2t_0/\alpha \right\}
\geq \max\left\{ t \in [0, 1] : \Psi_1(t) \geq 2t/\alpha + 2t^*_\alpha/\alpha \right\}
\geq t^*_\alpha/2 \geq \max\{t \in [0, 1] : \Psi_1(t) \geq 2t/\alpha \}.
\]

since \( \Psi_1(t^*_\alpha/2) = 4t^*_\alpha/\alpha \). Since \( \Psi_1 \) is non-decreasing, we conclude that

\[
\Psi_1(T_0^-) \geq \Psi_1(t_0) \geq \frac{4t_0}{\alpha} = n_0^{-1/4},
\]

which is the statement of the lemma.

It remains to prove the claim \( \Psi_1(t_0) \geq 4t_0/\alpha \) for \( n_0 \geq N(\alpha) \) and all \( \delta \geq \delta_0 = 2 \log(32/\alpha)/\sqrt{\log(2/t_0)} \) (Condition (55)). Define \( x_0 = \Phi^{-1}(t_0/2) \) so that, by (95) of Lemma D.2, \( \delta_0 x_0 \geq 2 \log(32/\alpha) \) for \( n_0 \geq N(\alpha) \). Since \( \Psi_1(t) \) is increasing with respect to \( \delta \), it suffices to prove the claim for
\( \delta = \delta_0 \). For \( n_0 \geq N(\alpha) \), we have \( \delta_0 \leq 1 \) and \( x_0 \geq 2 \). Applying twice Lemma D.2, we obtain

\[
\Psi_1(t_0) \geq \frac{(x_0 - \delta_0)}{1 + (x_0 - \delta_0)^2}\phi(x_0 - \delta_0) \geq \frac{(x_0 - \delta_0)e^{x_0\delta_0 - \delta_0^2/2}}{1 + (x_0 - \delta_0)^2}\phi(x_0) \\
\geq t_0 \frac{x_0(x_0 - \delta_0)}{2[1 + (x_0 - \delta_0)^2]}e^{x_0\delta_0 - \delta_0^2/2} \geq \frac{t_0x_0^2}{4(1 + x_0^2)}e^{x_0\delta_0/2} \\
\geq \frac{t_0}{8}e^{x_0\delta_0/2} \geq \frac{4t_0}{\alpha},
\]

where we used \( \delta_0x_0 \geq 2 \log(32/\alpha) \) in the last line.

**Proof of Lemma C.7.** For \( t \) going to 0, \( \Phi^{-1}(t/2) \) goes to infinity. Furthermore, Lemma D.2 ensures that \( \Phi(x) \sim \phi(x)/x \). Hence, for \( t \) converging to 0, we have

\[
\frac{\Psi_1(t)}{t} = \frac{\Psi_1(t)}{\Phi(\Phi^{-1}(t/2))} \sim \frac{\phi(\Phi^{-1}(t/2) - \delta)}{\phi(\Phi^{-1}(t/2))} = e^{\delta \Phi^{-1}(t/2) - \delta^2/2} \to \infty.
\]

To show that \( t \in (0,1] \mapsto \Psi_1(t)/t \) is decreasing, we prove that \( t \in (0,1] \mapsto \Psi_1(t) \) is strictly concave. This holds because

\[
\Psi_1'(t) = \frac{1}{2} \frac{\phi'\left(\Phi^{-1}(t/2) - \delta\right)}{\phi(\Phi^{-1}(t/2))} = e^{\delta \Phi^{-1}(t/2) - \delta^2/2}
\]

is decreasing in \( t \).

\[\square\]

**Appendix D: Auxiliary results**

**Lemma D.1** (DKW inequality Massart (1990)). Let \( X_1, \ldots, X_n \) be i.i.d. distributed with cumulative function \( F \). Denote \( F_n \) the empirical distribution function defined by \( F_n(x) = n^{-1} \sum_{i=1}^{n} I\{X_i \leq x\} \). Then, for any \( t \geq \log(2) \), we have

\[
\mathbb{P}
\left[
\sup_{x \in \mathbb{R}} (F_n(x) - F(x)) \geq \sqrt{\frac{t}{2n}}
\right] \leq e^{-t}.
\]

**Lemma D.2** (Carpentier et al. (2018)). We have

\[
\max\left(\frac{t\phi(t)}{1 + t^2}, \frac{1}{2} - \frac{t}{\sqrt{2\pi}}\right) \leq \Phi(t) \leq \phi(t) \min\left(\frac{2}{t}, \sqrt{\frac{\pi}{2}}\right), \quad \text{for all } t > 0.
\]

(92)

As a consequence, for any \( x < 0.5 \), we have

\[
\sqrt{2\pi(1/2 - x)} \leq \Phi^{-1}(x) \leq \sqrt{2 \log\left(\frac{1}{2x}\right)},
\]

(93)

\[
\log\left(\frac{[\Phi^{-1}(x)]^2}{[\Phi^{-1}(x)]^2 + 1}\right) \leq \frac{[\Phi^{-1}(x)]^2}{2} - \log\left(\frac{1}{x}\right) + \log\left(\frac{\sqrt{2\pi\Phi^{-1}(x)}}{2}\right) \leq 0,
\]

(94)

and if additionally \( x \leq 0.004 \), we have

\[
\Phi^{-1}(x) \geq \sqrt{\log\left(\frac{1}{x}\right)}.
\]

(95)
Lemma D.3. For \(0.2 \leq x \leq y \leq 0.8\), we have
\[
\Phi^{-1}(x) - \Phi^{-1}(y) \leq 3.6(y - x).
\] (96)

**Proof of Lemma D.3.** By the mean-value theorem, we have
\[
\frac{y - x}{\sup_{z \in [x,y]} \phi(\Phi^{-1}(z))} \leq \Phi^{-1}(x) - \Phi^{-1}(y) \leq \frac{y - x}{\inf_{z \in [x,y]} \phi(\Phi^{-1}(z))}.
\] (97)

The function \(t \mapsto \phi(\Phi^{-1}(t + 1/2))\) defined on \([-1/2,1/2]\) is symmetric and increasing on \([-1/2,0]\). Thus if \(0.2 \leq x \leq y \leq 0.8\), the above infimum equals \(\phi(\Phi^{-1}(0.2))\) which is larger than 1/3.6.

Lemma D.4 (Carpentier et al. (2018)). Let \(\xi = (\xi_1, \ldots, \xi_n)\) be a standard Gaussian vector of size \(n\). For any integer \(q \in (0.3n,0.7n)\) and for all \(0 < x \leq \frac{8}{225} q \vee \left(\frac{n^2}{18q} [\Phi^{-1}(q/n) - \Phi^{-1}(0.7)]^2\right)\), we have
\[
\mathbb{P}\left[\xi_{(1)} + \Phi^{-1}(q/n) \geq 3 \sqrt{\frac{2qx}{n}}\right] \leq e^{-x},
\] (98)
where we denote \(\xi_{(1)} \geq \cdots \geq \xi_{(n)}\) the values of \(\xi_1, \ldots, \xi_n\) ordered decreasingly.

Lemma D.5. Let \(\xi = (\xi_1, \ldots, \xi_n)\) be a standard Gaussian vector of size \(n\) and denote \(|\xi|_{(1)} \leq \cdots \leq |\xi|_{(n)}\) the values of \(|\xi_1|, \ldots, |\xi_n|\) ordered increasingly. For any integer \(q \in [0.2n,0.6n]\) and for all \(0 < x \leq 0.04q \vee \left(\frac{n^2}{14q} [\Phi^{-1}(0.2) - \Phi^{-1}((1 - q/n)/2)]^2\right)\), we have
\[
\mathbb{P}\left[|\xi|_{(q)} - \Phi^{-1}((1 - q/n)/2) \geq \frac{4 \sqrt{qx}}{n}\right] \leq e^{-x}.
\] (99)

For any integer \(q \in (0.4n, n)\) and for all \(0 < x \leq \frac{n^2}{2q} [\Phi^{-1}((1 - q/n)/2) - \Phi^{-1}(0.3)]^2\), we have
\[
\mathbb{P}\left[|\xi|_{(q)} - \Phi^{-1}((1 - q/n)/2) \leq -2 \sqrt{\frac{qx}{n}}\right] \leq e^{-x}.
\] (100)

**Proof of Lemma D.5.** Consider any \(t > 0\) and denote \(p = 1 - 2\Phi(\Phi^{-1}((1 - q/n)/2) + t)\) which belongs to \([q/n,1)\). Denote \(B(n,p)\) the binomial distribution with parameters \(n\) and \(p\). We have
\[
\mathbb{P}\left[|\xi|_{(q)} \geq \Phi^{-1}((1 - q/n)/2) + t\right] = \mathbb{P}\left[B(n,p) \leq q - 1\right] \leq \mathbb{P}\left[B(n,p) \leq q\right].
\] (101)

By the mean value theorem, we have
\[
p - q/n \geq t \inf_{x \in [0,t]} \phi[\Phi^{-1}((1 - q/n)/2) + x] = t\phi[\Phi^{-1}((1 - q/n)/2) + t],
\]
because \(\Phi^{-1}((1 - q/n)/2) \geq 0\). Assume that \(\Phi^{-1}((1 - q/n)/2) + t \leq \Phi^{-1}(0.2)\). Then, it follows from the previous inequality that \(p - q/n \geq 2t\phi[\Phi^{-1}(0.2)] \geq t/2\). Together with Bernstein’s inequality, we obtain
\[
\mathbb{P}\left[|\xi|_{(q)} \geq \Phi^{-1}((1 - q/n)/2) + t\right] \leq \mathbb{P}\left[B(n,q/n + t/2) \leq q\right] \leq \exp\left[\frac{-n^2t^2}{8[q/n + t/2](1 - q/n + nt/3)}\right]}
\]

Since $q \geq 0.2n$ and further assuming that $nt \leq 0.8q$, we conclude that
\[
\mathbb{P} \left[ |\xi_q| \geq \Phi^{-1}((1 - q/n)/2) + t \right] \leq e^{-n^2t^2/14},
\]
for any $0 < t \leq 0.8q/n \wedge [\Phi^{-1}(0.2) - \Phi^{-1}((1 - q/n)/2)]$. We have proved (99).

Next, we consider the left deviations. Assume that $q/n > 0.4$ (so that $(1 - q/n)/2 < 0.3$) and take $0 \leq t \leq \Phi^{-1}((1 - q/n)/2) - \Phi^{-1}(0.3)$. Write $p = 1 - 2\Phi^{-1}((1 - q/n)/2) - t$. We have $p \in [0.4, q/n]$ and
\[
\mathbb{P}[|\xi_q| \leq \Phi^{-1}((1 - q/n)/2) - t] = \mathbb{P}[B(n, p) \geq q].
\]

By the mean value theorem,
\[
q/n - p \geq 2t \inf_{x \in [0, t]} \phi \left( \Phi^{-1}((1 - q/n)/2) - x \right) \geq 2t\phi \left( \Phi^{-1}((1 - q/n)/2) \right) \geq 2t\phi(\Phi^{-1}(0.2)) \geq t/2,
\]
Then, Bernstein’s inequality yields
\[
\mathbb{P}[|\xi_q| \leq \Phi^{-1}((1 - q/n)/2) - t] \leq \exp \left[ -\frac{(q - np)^2}{2np(1 - p) + 2(q - np)/3} \right] \leq \exp \left[ -\frac{(q - np)^2}{2q} \right],
\]
because $2np(1 - p) + 2(q - np)/3 \leq (2 - 2/3)np + 2q/3 \leq 2q$ since $p \leq q/n$. which implies
\[
\mathbb{P} \left[ \xi_q \leq -\Phi^{-1}((1 - q/n)/2) - t \right] \leq \exp \left[ -\frac{n^2t^2}{8q} \right].
\]
We have shown (100).