A Comparison Study of the Classical and Modern Results of Semi-Local Convergence of Newton-Kantorovich Iterations

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Abstract: There are a plethora of semi-local convergence results for Newton’s method (NM). These results rely on the Newton–Kantorovich criterion. However, this condition may not be satisfied even in the case of scalar equations. For this reason, we first present a comparative study of established classical and modern results. Moreover, using recurrent functions and at least as small constants or majorant functions, a finer convergence analysis for NM can be provided. The new constants and functions are specializations of earlier ones; hence, no new conditions are required to show convergence of NM. The technique is useful on other iterative methods as well. Numerical examples complement the theoretical results.

Keywords: iterative methods; Banach space; semi-local convergence

MSC: 49M15; 65J15; 65G99

1. Introduction

The concern of this study is to solve the following nonlinear equation

\[ F(x) = 0, \]

using the celebrated Newton’s method (NM) in the following form

\[ x_{n+1} = x_n - F'(x_n)^{-1}F(x_n). \]

Here, \( F : \Omega \subset B_1 \rightarrow B_2 \) is differentiable according to Fréchet and operates between Banach spaces \( B_1 \) and \( B_2 \), whereas subset \( \Omega \neq \emptyset \).

Kantorovich provided the semi-local convergence analysis of NM utilizing Banach’s contraction mapping principle. In particular, he presented two different proofs using majorant functions or recurrent relations [1]. The Newton-Kantorovich Theorem is that no assumption as to the solution is made, while at the same time the existence of the solution \( x^\ast \) is established. Numerous researchers have used this theorem, both in applications and as a theoretical tool [2–10]. While the convergence criteria may not hold, NM may converge.

Assume that there exist constants \( \eta \geq 0 \), \( L_1 > 0 \), and a point \( z \in \Omega \) such that the Lipschitz condition

\[ \|F'(z)^{-1}(F'(v) - F'(u))\| \leq L_1\|v - u\| \]

and

\[ \|F'(z)^{-1}F(z)\| \leq \eta, \]

where ...
hold for each \( u, v \in \Omega \). The main convergence criterion is provided by

\[ 2L_1 \eta \leq 1. \]  

(5)

There are even simple scalar equations where criterion (5) is not satisfied. Indeed, let 
\( \Omega = U[z, 1 - q], \quad q \in (0, \frac{1}{2}), \ F(t) = t^3 - q, \) and set \( z = x_0 = 1 \). Then, conditions (3) and (4) are satisfied provided that \( L_1 = 2(2 - q) \) and \( \eta = \frac{1}{2-q} \). However, then condition (5) is not satisfied for any \( q \in (0, \frac{1}{2}) \), as \( 2L_1 \eta > 1 \). Hence, there is a need to replace (5) with a weaker criterion without adding conditions. The same extension is useful in the Hölder case or when \( L_1 \) is replaced by a majorant function. The last two cases are not considered in [11]. Moreover, the Lipschitz case is extended further than in [11] (see the end of Section 4). This is the motivation for presenting new results that both extend the convergence region and provide more precise error estimates and better knowledge as to the location of the solution. In this way, the use of NM can be extended. The novelty of the present article is that these benefits require no additional conditions. The same technique used here can be applied to extend other iterative methods along the same lines.

Majorizing scalar sequences for NM are provided in Section 2. The semi-local convergence of NM appears in Section 3. A discussion is provided in Section 4. Examples can be found in Section 5, and Conclusions in Section 6.

2. Majorizing Sequences

Real functions are utilized to develop sequences that are shown to be majorizing for NM in Section 3. Let \( S = [0, \infty) \). Assume:

There exist functions \( \bar{\psi}_R : S \rightarrow S, \ \psi_1 : S \rightarrow S \) non-decreasing and continuous such that function \( \psi_1(t) - 1 \) has a smallest zero \( b \in S - \{0\} \). Set \( S_0 = [0, b) \). Consider function \( \psi_\tau : S_0 \rightarrow S \) for \( \tau \in [0, 1] \) continuous and non-decreasing.

Let \( \eta \geq 0 \) be a given constant. Define scalar sequence \( \{t_n\} \) for \( t_0 = 0, t_1 = \eta \) by

\[
\begin{align*}
t_2 &= t_1 + \frac{\int_0^1 \bar{\psi}_\tau(\tau(t_1 - t_0))d\tau(t_1 - t_0)}{1 - \bar{\psi}_1(t_1)}, \\
t_{n+2} &= t_{n+1} + \frac{\int_0^1 \psi_\tau(\tau(t_{n+1} - t_n))d\tau(t_{n+1} - t_n)}{1 - \psi_1(t_{n+1})}, \quad n = 1, 2, \ldots.
\end{align*}
\]

(6)

Next, results appear on the convergence of sequence \( \{t_n\} \).

Lemma 1. Assume:

\[
\bar{\psi}_1(\eta) < 1, \quad (7)
\]

\[
t_n \leq b_0 \text{ for some } b_0 \in [0, b), \quad n = 2, 3, \ldots, \quad (8)
\]

and

\[
\psi_1(b_0) < 1. \quad (9)
\]

Then, sequence \( \{t_n\} \) is such that \( t_n \leq t_{n+1}, \quad n = 0, 1, 2, \ldots \text{ and } \lim_{n \to \infty} t_n = t^*, \) where \( t^* \) is the unique least upper bound of sequence \( \{t_n\} \) satisfying \( t^* \in [0, b_0]. \)

Proof. It follows from definition (6), condition (7), and the definition of functions \( \bar{\psi}_1 \) and \( \psi_\tau \) that \( t_1 \leq t_2 \). Similarly, the definitions of functions \( \psi_1 \) and \( \psi_\tau \) and condition (9) imply \( \psi_1(t_{n+1}) \leq \psi_1(b_0) < 1 \). Hence, \( t_{n+1} \leq t_{n+2}, \ n = 1, 2, \ldots \). Therefore, sequence \( \{t_n\} \) is non-decreasing and bounded from above by \( b_0 \), and thus converges to \( t^* \). \( \square \)

Remark 1. (i) If function \( \psi_1 \) is strictly increasing, then \( b_0 \) in conditions (8) and (9) can be defined by

\[
b := \psi_1^{-1}(1). \quad (10)
\]
(ii), although the conditions of Lemma 1 can be verified by constructing sequence \( \{t_n\} \) in advance. Next, convergence conditions which are stronger than (7)–(9) are provided, which are easier to verify. Let \( K_0, K, L_0, L_1 \) be non-negative constants and \( p \in [0, 1] \). Define functions
\[
\bar{\psi}_1(t) = K_0 t^p,
\bar{\psi}_\tau(t) = K(\tau t)^p,
\psi_1(t) = L_0 t^p,
\]
and
\[
\bar{\psi}_\tau(t) = L(\theta t)^p.
\]
Notice that for \( p = 1 \) the Lipschitz and for \( p \in (0, 1) \) the Hölder case is obtained. It follows from these definitions that sequence \( \{t_n\} \) is reduced for \( t_0 = a, t_1 = \eta \) to
\[
t_2 = t_1 + \frac{K(t_1 - t_0)^p}{(1 + p)(1 - K_0 t_1^p)},
\]
\[
t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^p}{(1 + p)(1 - L_0^p n_{n+1}^p)}, \quad n = 1, 2, \ldots .
\]
(11)

Define parameters \( \delta_0, \eta_0 \) and \( \delta_1 \) by
\[
\delta_0 = \frac{K(t_2 - t_1)^p}{(1 + p)(1 - K_0 t_2^p)}, \quad \eta_0 = t_2 - t_1,
\]
and
\[
\delta_1 = 1 - \frac{\eta_0}{(\frac{1}{t_1^p})^p - t_1}.
\]
Moreover, define functions on the interval \([0, 1)\) for \( k = 1, 2 \ldots \) by
\[
h_{k, p}(t) = \frac{L}{1 + p} t^p \eta_0 + t L_0 (t_1 + \frac{1 - t^{k+1}}{1 - t} \eta_0)^p - t,
\]
and
\[
g_{n+1, p}(t) = \frac{L}{1 + p} t^{(k+1)} \eta_0^p - \frac{L}{1 + p} t^p \eta_0^p + t L_0 [(t_1 + (1 + t + \ldots + t^{k+1}) \eta_0)^p
\]
\[-(t_1 + (1 + t + \ldots + t^k) \eta_0)]^p.
\] (13)

Next, the second convergence result for sequence \( \{t_n\} \) obtained by (11) is presented, using the preceding notation.

**Lemma 2.** Assume:

(H1) \( \exists \delta \in [\delta_0, 1), \quad p \in (0, 1) : \)
\[
g_{n+1, p}(\delta) \geq 0, \quad \forall n = 1, 2, 3, \ldots ,
\]
or

(H2) \( \exists \delta \in [\delta_0, 1), \quad p \in (0, 1) : \)
\[
g_{n+1, p}(\delta) \leq 0, \quad \forall n = 1, 2, 3, \ldots
\]
Then, the following assertions hold under condition (H1):
\[
h_{n, p}(\delta) \leq h_{n+1, p}(\delta),
\] (14)
and the following under condition (H2):

\[ h_{n,p}(\delta) \geq h_{n+1,p}(\delta). \] (15)

Moreover, if under condition (H1)

\[ 0 \leq \delta_0 \leq \delta \leq \delta_1, \] (16)

or under condition (H2)

\[ h_{1,p}(\delta) \leq 0 \] (17)

hold, then sequence \( \{t_n\} \) obtained by (11) is such that

\[ 0 \leq t_n \leq t_{n+1}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad \lim_{n \to \infty} t_n = t^*. \]

**Proof.** Induction is utilized to show

\[ 0 \leq \frac{L(t_{m+1} - t_m)p}{(1 + p)(1 - L_0 t_{m+1})} \leq \delta. \] (18)

Inequality (18) holds if \( m = 1 \) per the definition of \( \delta_0 \) and \( \delta \). Then, it follows per (11) that

\[ 0 \leq t_3 - t_2 = \delta_0 (t_2 - t_1) \implies t_3 = t_2 + \delta_0 (t_2 - t_1) = t_2 + (1 + \delta_0)(t_2 - t_1) - (t_2 - t_1) = t_1 + (1 + \delta_0)(t_2 - t_1) = t_1 + \frac{1 - \delta_0}{1 - \delta_0}(t_2 - t_1) \leq t_1 + \frac{t_2 - t_1}{1 - \delta_0} \leq t_1 + \frac{t_2 - t_1}{1 - \delta} = t^* . \] (20)

Assume estimates (19) and (20) hold for all values of \( m \leq n \). It follows by the induction hypotheses that

\[ 0 \leq t_{m+2} - t_{m+1} \leq \delta^m(t_2 - t_1), \] (21)

and

\[ t_{m+2} \leq t_{m+1} + \delta^m(t_2 - t_1) \leq t_m + \delta^{m-1}(t_2 - t_1) + \delta^m(t_2 - t_1) \leq \ldots \leq t_1 + (1 + \delta_0(1 + \delta + \ldots + \delta^{m-1}))(t_2 - t_1) \leq t_1 + (1 + \frac{1 - \delta^{m+1}}{1 - \delta})(t_2 - t_1) \leq t^* . \] (22)

Hence, per estimates (21) and (22), assertion (18) holds if

\[ \frac{L}{1 + p} \delta^m(t_2 - t_1)p^p + \delta L_0 t_1 \leq (1 + \delta + \ldots + \delta^m)\eta_0^p - \delta \leq 0, \] (23)

or

\[ h_{m,\delta}(t) \leq 0 \quad \text{at} \quad t = \delta. \] (24)
The recurrent functions are related, as
\[
    h_{m+1,p}(t) = h_{m+1,p}(t) - h_{m,p}(t) + h_{m,p}(t)
\]
\[
    = \frac{L}{1 + p} t^{(m+1)p} + L_0(t_1 + (1 + t + \ldots + t^{m+1})\eta_0)^p
\]
\[
    - t + h_{m,p}(t) - \frac{L}{1 + p} t^{mp}\eta_0^p
\]
\[
    + tL_0(t_1 + (1 + t + \ldots + t^{m})\eta_0)^p + t,
\]
\[
    = h_{m,p}(t) + g_{m+1,p}(t). \tag{25}
\]

**Case of Condition (H1):** Define function
\[
    h_{\infty,p}(t) = \lim_{m \to \infty} h_{m,p}(t). \tag{26}
\]
Per estimates (23) and (26), it follows that
\[
    h_{m+1,p}(\delta) \geq h_{m,p}(\delta), \tag{27}
\]
and
\[
    h_{\infty,p}(t) = t[L_0(t_1 + \frac{\eta_0}{1-t})^p - 1]. \tag{28}
\]
Thus, estimate (24) holds if
\[
    h_{\infty,p}(\delta) \leq 0, \tag{29}
\]
which is true per condition (16). The induction under (H1) is terminated.

**Case of Condition (H2):** It follows that estimate (24) holds, as
\[
    h_{m,p}(\delta) \leq g_{1,p}(\delta) \leq 0. \tag{30}
\]
That is, the induction is completed under condition (H2) as well. Hence, under either condition (H1) or (H2), sequence \( \{t_m\} \) is non-decreasing and bounded from above. Hence, it converges to \( t^* \).

**Remark 2.** Define function \( f_p : [0,1] \to S \) by
\[
    f_p(t) = L_0 t^{1+p} + \frac{L}{1+p} t^p - \frac{L}{1+p}. \tag{31}
\]
It follows that \( f_p(0) - \frac{L}{1+p} < 0 \) and \( f_p(1) = L_0 > 0. \) Hence, the intermediate value theorem ensures that the equation \( f_p(t) = 0 \) has zeros in \((0,1)\). Denote by \( \gamma_p \) the smallest such zero. Notice that
\[
    \gamma_1 = \frac{2L}{L + \sqrt{L^2 + 8L_0 L}}. \tag{32}
\]
The parameter \( \delta \) is preferred in closed form. This is possible in certain cases.

**Lipschitz Case:** Set \( p = 1. \) Then, the choice for \( \delta = \gamma_1. \)
In the case that \( p \in (0,1), \) the zero \( \gamma_p \) is not in closed form; instead
\[
    \gamma_p \leq \left( \frac{L}{L + (1+p)L_0} \right)^{\frac{1}{1+p}}. \tag{33}
\]
Indeed, because \( \gamma_p \) solves equation \( f_p(t) = 0, \) then
\[
    L = L_0(1+p)^{1+p} + L\gamma_p \geq (L_0(1+p) + L)\gamma_p^{1+p}. \tag{34}
\]
leading to condition (33). Under (H1), one obtains \( h_{k+1,1}(t) = h_{k,1}(t) + f_1(t) t^{n-1} \eta \), and condition (16) becomes
\[
\delta_0 \leq \gamma_1 \leq \delta_1.
\] (34)

By solving the left hand side inequality for \( \eta \),
\[
\eta \leq \frac{2 \gamma_1}{(1 + 2 \gamma_1)L_0}.
\] (35)

The right hand side inequality in condition (34) is solved as follows:

**Case:** \( 2(1 - \gamma_1)K_0 - K = 0 \). Set
\[
\eta_1 = \frac{1}{K_0 + L_0}.
\] (36)

Hence, (34) holds if
\[
\eta \leq \min \{ \frac{2 \gamma_1}{(1 + 2 \gamma_1)L_0}, \frac{1}{K_0 + L_0} \}.
\] (37)

To consider the rest of the cases, define function \( \varphi : S \rightarrow S \) by
\[
\varphi(t) = L_0(2 - \gamma_1)K_0 - K)t^2 + (2 - \gamma_1)(K_0 + L_0)t + 2(1 - \delta).
\] (38)

Denote by \( \Delta \) the discriminant of \( \varphi \). If \( 2(1 - \delta)K_0 - K > 0 \) and \( \Delta < 0 \), set
\[
\eta_2 = \frac{2 \gamma_1}{L_0(1 + 2 \gamma_1)}.
\] (39)

Then, condition (34) holds if
\[
\eta \leq \eta_2.
\] (40)

If \( 2(1 - \delta)K_0 - K > 0 \) and \( \Delta > 0 \), then \( \varphi \) has a positive zero denoted by \( \eta_3 \). Then, (34) holds if
\[
\eta \leq \min \{ \frac{2 \gamma_1}{L_0(1 + 2 \gamma_1)}, \eta_3 \}.
\] (41)

If \( 2(1 - \delta)K_0 - K < 0 \) and \( \Delta > 0 \), then denote by \( \eta_4 \) the positive zero of function \( \varphi \). Then, condition (34) holds if
\[
\eta \leq \min \{ \frac{2 \gamma_1}{L_0(1 + 2 \gamma_1)}, \eta_4 \}.
\] (42)

If \( 2(1 - \delta)K_0 - K < 0 \) and \( \Delta < 0 \), the right hand side of (34) is not solvable and does not hold.

**Hölder Case.** Let \( p \in (0, 1) \). Per the definition of functions \( h_{m,p}, \ell_{m+1,p} \), it follows that
\[
h_{m+1,p}(t) \leq h_{m,p}(t) + \frac{1}{1 + p} t^{(m+1)p} \eta_0^p
\]
\[+ \frac{L}{1 + p} t^{m} \eta_0^p + tL_0 t^{m+1} \eta_0^p, \]
\[\leq h_{m,p}(t) + f_p(t) t^{mp} \eta_0^p. \]

Thus, condition (H2) becomes
\[
h_{1,p}(\gamma_p) \leq 0 \text{ and } 0 \leq \delta_0 \leq \gamma_p,
\] (43)

where
\[
h_{1,p}(t) = \frac{L}{1 + p} t^p \eta_0^p + tL_0 (t_1 + (1 + t))^p \eta_0^p - t.
\] (44)

The convergence conditions in this Section are compared to existing ones in Section 4.
3. Convergence of NM

The notation $U(w, \rho), U[w, \rho]$ refers to the open and closed balls with radius $\rho > 0$ and center $w \in X$, respectively. The semi-local convergence of NM is presented under conditions (A). The functions $\psi$ are provided in Section 2. Assume:

(A1) There exist $x_0 \in \Omega, \eta \geq 0$ such that $F'(x_0)^{-1} \in L(B_2, B_1)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

(A2) $\|F'(x_0)^{-1}(F'(x_1) - F'(x_0))\| \leq \psi_1(\|x_1 - x_0\|)$ and

$$\|F'(x_0)^{-1}(F'(x_0 + \tau(x_1 - x_0)) - F'(x_0))\| \leq \psi_1(\|x_1 - x_0\|)$$

$\forall \tau \in [0, 1)$, where $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$.

(A3) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq \psi_1(\|x - x_0\|)$, $\forall x \in \Omega$. Set $\Omega_0 = U(x_0, b) \cap \Omega$.

(A4) $\|F'(x_0)^{-1}(F'(x+\tau(y-x)) - F'(x))\| \leq \psi_1(\|y - x\|) \forall x, y \in \Omega_0, \tau \in [0, 1)$.

(A5) Conditions of Lemma 1 hold and

(A6) $U[x_0, t^*] \subset \Omega$ (or $U[x_0, b_0] \subset \Omega$).

Next, the main semi-local convergence result for NM is developed.

Theorem 1. Assume conditions (A) hold. Then, the sequence $\{x_n\}$ generated by NM exists in $U[x_0, t^*]$, stays in $U[x_0, t^*]$ for all $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in U[x_0, t^*]$ of equation $F(x^*) = 0.$

Proof. Estimates

$$\|x_{i+1} - x_i\| \leq t_{i+1} - t_i$$

(43)

and

$$U[x_{i+1}, t^* - t_{i+1}] \subseteq U[x_i, t^* - t_i],$$

(44)

$\forall i = 0, 1, 2, \ldots$ shall be shown by induction. Let $u \in U[x_1, t^* - t_{i+1}] \subseteq U[x_i, t^* - t_i].$

It follows that

$$\|u - x_0\| \leq \|u - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 - t_0 = t^*.$$

Thus, $u \in U[x_0, t^* - t_0].$ Then, we obtain

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = t_1 - t_0$$

(45)

per condition (A1), implying that estimates (43) and (44) hold for $i = 0.$ Assume (43) and (44) hold for all $n \leq i.$ It follows that

$$\|x_{i+1} - x_0\| \leq \sum_{j=1}^{i+1} \|x_j - x_{j-1}\|,$$

$$\leq \sum_{j=1}^{i+1} (t_j - t_{j-1}) = t_{i+1} - t_0 = t_{i+1} \leq t^*,$$

and

$$\|x_{i+1} - x_0\| \leq t_{i+1} + \tau(t_{i+1} - t_i) \leq t^*, \forall \tau \in [0, 1].$$

Let $v \in U[x_0, t^*].$ Using (A2), we obtain

$$\|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq \psi_1(\|v - x_0\|),$$

$$\leq \psi_1(\eta) \leq \psi_1(t^*) < 1.$$
Thus, per condition (A4) and estimates (46) and (50) for \( v \), Consequently, estimate (43) holds \( \forall \).

Hence, \( F'(v)^{-1} \in L(B_2, B_1) \) and

\[
\|F'(v)^{-1}F'(x_0)\| \leq \frac{1}{1 - \psi_1(\|v - x_0\|)}.
\] (46)

are derived by Lemma on invertible linear operators due to Banach [2,9,12]. In particular, for \( v = x_1 \), iterate \( x_2 \) exists per the second substep of NM. Moreover, one can write

\[
F(x_1) = F(x_1) - F(x_0) - F'(x_0)(x_1 - x_0),
\]

\[
= \int_0^1 (F'(x_0 + \tau(x_1 - x_0))d\tau - F'(x_0))(x_1 - x_0).
\] (47)

Then, by (A2) and estimate (46), it follows that

\[
\|F'(x_0)^{-1}F(x_1)\| \leq \int_0^1 \psi_\tau(\|x_1 - x_0\|)d\tau \|x_1 - x_0\|,
\]

\[
\leq \int_0^1 \psi_\tau(t_1 - t_0)d\tau(t_1 - t_0).
\] (48)

It then follows from estimates (46) (for \( v = x_1 \)), (48) and NM for \( n = 2 \) that

\[
\|x_2 - x_1\| \leq \|F'(x_1)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_1)\|,
\]

\[
\leq \int_0^1 \psi_\tau(t_1 - t_0)d\tau(t_1 - t_0)
\]

\[
\frac{1}{1 - \psi_1(t_1)} = t_2 - t_1.
\] (49)

Similarly, we can write

\[
F(x_{i+1}) = F(x_{i+1}) - F(x_i) - F'(x_i)(x_{i+1} - x_i),
\]

\[
= \int_0^1 (F'(x_i + \tau(x_{i+1} - x_i))d\tau - F'(x_i))(x_{i+1} - x_i).
\] (50)

Thus, per condition (A4) and estimates (46) and (50) for \( v = x_{i+1} \),

\[
\|F'(x_0)^{-1}F(x_{i+1})\| \leq \int_0^1 \psi_\tau(\|x_{i+1} - x_i\|)d\tau \|x_{i+1} - x_i\|,
\]

\[
\leq \int_0^1 \psi_\tau(t_{i+1} - t_i)d\tau(t_{i+1} - t_i).
\] (51)

Consequently,

\[
\|x_{i+2} - x_{i+1}\| \leq \|F'(x_{i+1})^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_{i+1})\|,
\]

\[
\leq \int_0^1 \psi_\tau(t_{i+1} - t_i)d\tau(t_{i+1} - t_i)
\]

\[
\leq \frac{1}{1 - \psi_1(t_{i+1})} t_{i+2} - t_{m+1}.
\] (52)

Hence, estimate (43) holds \( \forall i \). Furthermore, if \( w \in U[x_{i+2}, t^* - t_{i+2}] \), one obtains

\[
\|w - x_{i+1}\| \leq \|w - x_{i+2}\| + \|x_{i+2} - x_{i+1}\|
\]

\[
\leq t^* - t_{i+2} + t_{i+2} - t_{i+1} = t^* - t_{i+1}.
\] (53)

thus, \( w \in U[x_{i+1}, t^* - t_{i+1}] \). Induction for estimates (43) and (44) is terminated. Sequence \( \{t_i\} \) is fundamental as convergent. Hence, per estimate (43) and (44), sequence \( \{x_n\} \) is fundamental in a Banach space \( B_1 \) as well, and thus converges to some \( x^* \in U[x_0, t^*] \). Per the continuity of \( F \), and letting \( i \to \infty \) in estimate (48), it follows that \( F(x^*) = 0 \). \( \square \)

Next, we present a uniqueness of the solution result.
Proposition 1. Assume:
(i) Point $x^*$ is a simple solution of equation $F(x) = 0$ in $U[x_0, r_0] \subset \Omega$ for some $r_0 > 0$, and condition (A3) holds.
(ii) There exists $r \geq r_0$ such that
\[
\int_0^1 \psi_1((1 - \tau)r_0 + \tau r)d\tau < 1. \tag{54}
\]
Let $\Omega_1 = U[x_0, r] \cap \Omega$. Then, the only solution of equation $F(x) = 0$ is $x^*$.

**Proof.** Assume there exists $x^* \in \Omega_1$ such that $F(x^*) = 0$. Set $J = \int_0^1 F'(x^* + \tau(x^* - x^*))d\tau$. Then, it follows from conditions (A3) and (54) that
\[
\|F'(x_0)^{-1}(J - F'(x_0))\| \leq \int_0^1 \psi_1((1 - \tau)\|x_0 - x^*\| + \tau\|x_0 - x_0\|)d\tau,
\]
\[
\leq \int_0^1 \psi_1((1 - \tau)r_0 + \tau r)d\tau < 1.
\]
Thus, $x^* = x^*$ is implied by the invertibility of $J$, and the approximation $J(x^* - x^*) = F(x^*) - F(x^*) = 0$. \qed

**Remark 3.** Proposition 1 uses only condition (A3). However, under conditions (A) set $r_0 = t_x^*$.  

4. Discussion and Conclusions

There are a plethora of results on NM. The sufficient convergence criteria in these studies are weakened by the new technique without adding conditions (see the numerical Section). Notice that with the developed technique there is an at least as tight subset $\Omega_0$ of $\Omega$ containing the iterates. Therefore, the Lipschitz–Hölder majorant functions are at least as tight as the ones provided in earlier studies, leading to the advantages stated in the introduction. Below, we provide several examples.

Assume \cite{2,9,13}
\[
\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|) \tag{55}
\]
for all $x, y \in \Omega$. Define function
\[
Q(t) = \int_0^t (r - t)\omega(t)dt - t + \eta \tag{56}
\]
and iteration $\{I_n\}$ per $I_0 = 0, I_1 = \eta$:
\[
I_{n+1} = I_n + \frac{Q(I_n)}{Q'(I_n)}. \tag{57}
\]

The sufficient convergence criterion is derived supposing that scalar function $Q$ has a smallest positive zero $r_Q$. However, under condition (A4), define instead for $I_0 = 0, I_1 = \eta$
\[
I_{n+1} = I_n + \frac{Q(I_n)}{Q'(I_n)}, \tag{58}
\]
\[
Q(t) = \int_0^t (r - t)\psi_1(t)dt - t + \eta. \tag{59}
\]
Notice that
\[
Q(t) \leq Q(t). \tag{60}
\]
In particular as $\psi_1(t) \leq \omega(t)$,
\[
Q(\bar{r}_\tau) \leq Q(\bar{r}_\tau) = 0. \tag{61}
\]
and \( Q(0) = 0 \), \( Q \) has an at least as small zero \( r_Q \). Hence, the new convergence criterion is at least as weak. This is the contribution in the case of majorant functions.

In particular, in the case where \( \omega \) and \( \psi_1 \) are constant functions, such as

\[
\omega(t) = L_1 t, \\
\psi_1(t) = Lt,
\]

then

\[
\eta \leq \frac{1}{2L_1}
\]

and

\[
\eta \leq \frac{1}{2L'},
\]

which is weaker, as \( L \leq L_1 \) and \( L_1 \) is the Lipschitz constant on \( \Omega \).

**(a) Lipschitz Case:** Convergence criteria already in the literature are listed below in the left column, and the new criteria in the right column.

\[
T_1 : \eta \leq \frac{1}{\mu_1}, [1], \quad \bar{T}_1 : \eta \leq \frac{1}{\bar{\mu}_1}, \tag{66}
\]

\[
T_2 : \eta \leq \frac{1}{\mu_2}, [12], \quad \bar{T}_2 : \eta \leq \frac{1}{\bar{\mu}_2}, \tag{67}
\]

\[
T_3 : \eta \leq \frac{1}{\mu_3}, [11,14], \quad \bar{T}_3 : \eta \leq \frac{1}{\bar{\mu}_3}, \tag{68}
\]

\[
T_4 : \eta \leq \frac{1}{\mu_4}, [12], \quad \bar{T}_4 : \eta \leq \frac{1}{\bar{\mu}_4}, \tag{69}
\]

where

\[
\mu_1 = 2L_1, \quad \bar{\mu}_1 = 2L, \\
\mu_2 = L_0 + L_1, \quad \bar{\mu}_2 = L_0 + L, \\
\mu_3 = \frac{1}{4}(L_1 + 4L_0 + \sqrt{L^2 + 8L_0L_1}), \quad \bar{\mu}_3 = \frac{1}{4}(L + 4L_0 + \sqrt{L^2 + 8L_0L}), \tag{70}
\]

and

\[
\mu_4 = \frac{1}{4}(4L_0 + \sqrt{L_0L_1 + 8L_0^2} + \sqrt{L_0L_1}), \quad \bar{\mu}_4 = \frac{1}{4}(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}). \tag{71}
\]

It follows from these definitions that

\[
L_0 \leq L_1 \text{ and } L \leq L_1. \tag{72}
\]

Consequently,

\[
\mu_4 \leq \mu_3 \leq \mu_2 \leq \mu_1, \quad \bar{\mu}_4 \leq \bar{\mu}_3 \leq \bar{\mu}_2 \leq \bar{\mu}_1, \tag{73}
\]

\[
(T_1) \Rightarrow (T_2) \Rightarrow (T_3) \Rightarrow (T_4), \text{ and } (\bar{T}_1) \Rightarrow (\bar{T}_2) \Rightarrow (\bar{T}_3) \Rightarrow (\bar{T}_4). \tag{74}
\]

although not necessarily vice versa unless \( L_0 = L = L_1 \). Parameter \( L_1 \) is the Lipschitz constant on \( \Omega \) used by Kantorovich and others \([1,7]\). Another comparison between convergence criteria can be made as follows. Notice that

\[
\frac{1}{\bar{\mu}_1} = \frac{1}{\bar{\mu}_2} \rightarrow \frac{1}{2}, \quad \eta \rightarrow \frac{1}{2} \text{ as } \frac{L_0}{L} \rightarrow 0.
\]

\[
\frac{1}{\mu_1} = \frac{1}{\mu_2} = \frac{1}{2} \left(1 + \frac{L_0}{L}\right) \rightarrow \frac{1}{2} \text{ as } \frac{L_0}{L} \rightarrow 0.
\]
Similarly,
\[
\frac{1}{\mu_3} \to \frac{1}{4}, \quad \frac{1}{\mu_3} \to \frac{1}{2} \quad \frac{1}{\mu_1} \to 0, \quad \frac{1}{\mu_4} \to 0, \text{ and } \frac{1}{\mu_4} \to 0 \text{ as } L \to 0.
\]

These limits show by how many times at most the applicability of NM can be extended. Moreover, under the new approach this applicability is extended even further. Indeed, consider again iterations \(r\) with weaker criteria and more precise majorizing sequences. Thus, the uniqueness of the solution ball is extended. The constant \(r\) becomes possible if \(\frac{L}{L_1} \to 0\).

Similarly,
\[
\frac{1}{\mu_3} \to 0 \text{ and } \frac{1}{\mu_4} \to 0 \text{ as } L \to 0 \text{ and } \frac{L}{L_1} \to 0.
\]

Clearly, the new convergence criterion can be arbitrarily many times weaker than the previous ones. This technique of replacing constants with smaller ones leads to both weaker criteria and more precise majorizing sequences. Indeed, consider again iterations \(\{s_n\}\) and \(\{t_n\}\) with the choice of functions \(\omega\) and \(\psi_1\) obtained by (62) and (63), respectively. It follows that
\[0 \leq t_n \leq \bar{t}_n,
\]

then
\[0 \leq t_{n+1} - t_n \leq \bar{t}_{n+1} - \bar{t}_n\]

and
\[s_1 = \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} \bar{t}_n = s_2,
\]

as
\[L \leq L_n,
\]

\[t_{n+1} - t_n = \frac{L(t_{n+1} - t_n)^2}{2(1 - Lt_n)},
\]

and
\[\bar{t}_{n+1} - \bar{t}_n = \frac{L_1(\bar{t}_{n+1} - \bar{t}_n)^2}{2(1 - L_1\bar{t}_n)}.
\]

where
\[r_1 = \frac{1 - \sqrt{1 - 2Lt}}{L} \text{ and } r_2 = \frac{1 - \sqrt{1 - 2L_1t}}{L_1}.
\]

The constant \(r_2\) was reported by Kantorovich. Hence, tighter error estimates on \(||x_{n+1} - x_n||\) and \(||x^* - x_n||\) become possible if \(L < L_1\).

Moreover, the information on the location of the solution is more accurate if \(s_1 < s_2\), which is possible for \(L < L_1\). Concerning the uniqueness of the solution balls, \(u(x_0, r_3) \subset u(x_0, r_4)\) where
\[r_3 = \frac{1 + \sqrt{1 - 2L\eta}}{L_1} \leq r_4 = \frac{1 + \sqrt{1 - 2L_1\eta}}{L_1}.
\]

Thus, the uniqueness of the solution ball is extended. The constant \(r_3\) is provided by Kantorovich. Furthermore, if we choose \(\psi_1(t) = L_0 t\) and assume \(\eta \leq \frac{1}{2L}\), set \(r_5 = \frac{1 - \sqrt{1 - 2L_0\eta}}{L_0}\). Then, \(r_4 < r_5\) if \(L_0 < L_1\), and the solution \(x^*\) is unique in the ball \(u(x_0, r_5)\).
according to Proposition 1. This is the contribution in the Lipschitz case.

(b) Hölder Cases:

\[(H_1) : \quad L_1 \eta^p \leq \left( \frac{p}{1+p} \right)^p, \]
\[(H_2) : \quad L_1 \eta^p \leq 2^{p-1} \left( \frac{p}{1+p} \right)^p, \quad [1, 2, 4, 6, 13, 15] \]

where the new contributions are, respectively,

\[(H_1)' : \quad L \eta^p \leq \left( \frac{p}{1+p} \right)^p, \]
\[(H_2)' : \quad L \eta^p \leq \left( \frac{p}{1+p} \right)^p. \]

Hence,

\[(H_1) \Rightarrow (H_1)', \]
\[(H_2) \Rightarrow (H_2)'. \]

Notice that the proofs of \((H_1)'\) and \((H_2)'\) are implied by \((H_1), (H_2)\), respectively, if \(\Omega_0\) is used instead of \(\Omega\). The conditions of Lemma 1 are weaker than all preceding, whereas those of Lemma 1 can be weaker as well, as \(L_0 \leq L_1\) and \(L \leq L_1\) (see the numerical Section).

(c) It turns out that the results can be weakened even more if Lipschitz or Hölder constants are replaced by constants at least as small in two different ways, namely:

(i) Noting that convergence conditions should not hold for all \(x, y \in \Omega_0\), only for \(x \in \Omega_0\) and \(y = x - F'(x)^{-1} F(x)\), the proofs of the results proceed in this weaker setting. However, the corresponding constant is at least as tight. Denote such a constant per \(L_2\); it follows that \(L_2 \leq L\).

(ii) Consider ball \(\Omega_2 = U(x_1, b_0 - \eta) \subset \Omega\) for \(b_0 > \eta\) and have \(\Omega_2\) replace \(\Omega_0\) in condition (A3). Then, as \(\Omega_2 \subset \Omega_0\), the constant is again at least as small. Denote such a constant per \(L_3\). It follows that \(L_3 \leq L\). The rest of the convergence results found in the literature can be immediately weakened under this technique as well.

Finally, it is worth noticing that the new constants \(L_0, L, L_2, L_3\) are special cases of \(L_1\). Hence, the benefits under the new technique are obtained without additional computational effort or conditions, which represents one contribution of this paper.

5. Numerical Experiments

In the first example, the new constants are smaller than those of earlier studies.

Example 1. Define scalar function

\[F(t) = \lambda_0 t + \lambda_1 + \lambda_2 \sin \lambda_3 t,\]

for \(t_0 = 0\) where \(\lambda_j\) and \(j = 0, 1, 2, 3\) are real parameters. It follows by this definition that for \(\lambda_3\) sufficiently large and for \(\lambda_2\) sufficiently small \(\frac{\lambda_2}{\lambda_1}\) can be arbitrarily small. In particular, \(\frac{\lambda_2}{\lambda_1} \to 0\).

In the last two examples, Kantorovich criterion (66) is not satisfied; however, the new criterion (66) holds.
Example 2. Let \( B_1 = B_2 = \mathbb{R}, \Omega = [q, 2-q] \) for \( q \in M = (0,0.5) \), and \( t_0 = 1 \). Define real function \( F \) on \( \Omega \) as
\[
F(t) = t^3 - q.
\]
Then, the parameters are \( \eta = \frac{1}{2}(1-q) \), \( K_0 = K = L_0 = 3 - q \), and \( L_1 = 2(2-q) \). Moreover, one obtains \( \Omega_0 = U(1,1) \cap U(1, \frac{1}{2})_0 = U(1, \frac{1}{2}) \), thus \( L = 2(1 + \frac{1}{2}) \). Denote by \( S_i, i = 1, 2, 3, 4 \) the set of values \( q \) for which conditions (\( T_i \)) are satisfied. Then, by solving inequalities (66)–(69) for \( q, S_1 = \emptyset, S_2 = [0.464816242, 0.5], S_3 = (0.45039002, 0.5), \) and \( S_4 = [0.4271907643, 0.5] \), respectively.

Next, the new conditions (\( T_i \)) are tested. Concerning the new criterion (66), the inequality
\[
\frac{4}{3} \left( 1 + \frac{1}{3-q} \right) (1-q) \leq 1
\]
must be solved, obtaining \( q \geq \frac{17 - \sqrt{177}}{8} \approx 0.461983163 \); thus,
\[
S_1 = [0.461983163, 0.5].
\]
Similarly, when solving for \( q \) in new criteria (67)–(69), the intervals \( S_2, S_3, S_4 \) are extended. This is due to the fact that \( L < \frac{L}{L_1} \) for all \( q \) in \( M \). Hence, the range of values \( q \) is extended in the new approach. Notice in particular that the Newton–Kantorovich criterion (66) is not satisfied for any \( q \in (0, 0.5) \).

Example 3. Let \( B_1 = B_2 = \mathbb{C}[0,1] \) be the domain of continuous real functions defined on the interval \( [0,1] \). The max-norm is used. Set \( \Omega = U[x_0,3] \) and define operator \( F \) on \( \Omega \) as
\[
F(v)(v_1) = v(v_1) - v(y(v_1) - \int_0^1 N(v_1,t)v(t)^3dt, v \in \mathbb{C}[0,1], v_1 \in [0,1])
\]
where \( y \) is given in \( \mathbb{C}[0,1] \) and \( N \) is a kernel obtained by Green’s function as
\[
N(v_1,t) = \begin{cases} (1-v_1)t, & t \leq v_1 \\ v_1(1-t), & v_1 \leq t. \end{cases}
\]
It follows per this definition that the derivative of \( F \)
\[
[F'(v)(z)](v_1) = z(v_1) - 3 \int_0^1 N(v_1,t)v(t)^2z(t)dt
\]
\( z \in \mathbb{C}[0,1], v_1 \in [0,1] \). Pick \( x_0(v_1) = \gamma(v_1) = 1 \). It then follows from (78)–(80) that \( F'(x_0)^{-1} \in L(B_2, B_1) \),
\[
\| I - F'(x_0) \| < 0.375, \quad \| F'(x_0)^{-1} \| \leq 1.6, \quad \eta = 0.2, \quad L_0 = 2.4, \quad L_1 = 3.6,
\]
and \( \Omega_0 = U(x_0,3) \cap U(x_0,0.4167) = U(x_0,0.4167) \), thus \( L = 1.5 \). Notice that \( L_0 < L_1 \) and \( L < L_1 \). Choose \( K_0 = K = L_0 \). The Kantorovich convergence criterion (66) is not satisfied, as \( 2L_1\eta = 1.44 > 1 \). Hence, convergence of converge NM is not assured by the Kantorovich criterion. However, new criterion (66) is satisfied, as \( 2L\eta = 0.6 < 1 \).

6. Conclusions
A comparison study between results on the semi-local convergence of NM is provided. The technique uses recurrent functions. In this way, the new sufficient convergence criteria are weaker in the Lipschitz, Hölder, and more general cases, as they rely on smaller constants. Other benefits include more precise error bounds and uniqueness of the solution results. The new constants are special cases of earlier ones. The technique is very general, rendering it useful to extend the usage of other iterative methods.
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