Evolution of Real-world Hypergraphs: Patterns and Models without Oracles

Yunbum Kook  
Dept. of Mathematical Sciences, KAIST  
yb.kook@kaist.ac.kr

Jinhoon Ko  
Graduate School of AI, KAIST  
jihoonko@kaist.ac.kr

Kijung Shin  
Graduate School of AI & School of EE, KAIST  
kijungs@kaist.ac.kr

Abstract—What kind of macroscopic structural and dynamical patterns can we observe in real-world hypergraphs? What can be underlying local dynamics on individuals, which ultimately lead to the observed patterns, beyond apparently random evolution?

Graphs, which provide effective ways to represent pairwise interactions among entities, fail to represent group interactions (e.g., collaboration of three or more researchers, etc.). Regarded as a generalization of graphs, hypergraphs allowing for various sizes of edges prove fruitful in addressing this limitation. The increased complexity, however, makes it challenging to understand hypergraphs as thoroughly as graphs.

In this work, we closely examine seven structural and dynamical properties of real hypergraphs from six domains. To this end, we define new measures, extend notions of common graph properties to hypergraphs, and assess the significance of observed patterns by comparison with a null model and statistical tests.

We also propose HYPERFF, a stochastic model for generating realistic hypergraphs. Its merits are three-fold: (a) Realistic: it successfully reproduces all seven patterns, in addition to five patterns established in previous studies, (b) Self-contained: unlike previously proposed models, it does not rely on oracles (i.e., unexplainable external information) at all, and it is parameterized by just two scalars, and (c) Emergent: it relies on simple and interpretable mechanisms on individual entities, which do not trivially enforce but surprisingly lead to macroscopic properties.

I. INTRODUCTION

Which structural patterns do real-world hypergraphs have and how do they evolve over time? Are there simple mechanisms on individual nodes that these patterns emerge from?

Datasets based on relationships between two objects have naturally arisen from a wide range of domains in the real world: friendships between two users in online social networks, hyperlinks from a web page to another, and citations from a publication to another, to name a few. Graph representations give rise to an easy and extensive analysis of this type of relational datasets, and they have been used to understand such datasets in many respects.

A thorough understanding of those datasets via graph analysis has resulted in insights behind them and facilitated the development of effective algorithms building on the insights. Some of the well-known static properties are a small-world phenomenon, also known as six degrees of separation [1], and power law distributions of spectra [2] and degrees [3] of graphs in various domains. Temporal properties prevalent in real graphs include triadic closure [4], temporal locality [5], densification, and shrinking diameter over time [6]. These informative properties actually serve as useful tools for designing and analyzing graph algorithms [7]–[10].

Previous studies on fundamental mechanisms leading to these patterns have also proceeded with the development of generative models for realistic graphs. Various models [6], [11]–[13] have been proposed to reproduce the examined patterns and lent themselves to some applications, such as simulations [14], [15], benchmarking [16], and sampling [17].

Not all relational networks are restricted to have only pairwise relationships, and polyadic relationships are also ubiquitous [18]–[22]; for instance, publishing a paper may involve three or more authors, and an online group chat may involve tens of participants.

Hypergraphs [23]–[25] are a natural extension of the conventional notion of graphs by allowing for various sizes of edges. Formally, a hypergraph consists of a set of nodes and a set of hyperedges, where each hyperedge is a non-empty subset containing any number of nodes. They can represent high-order interactions (i.e., interactions among any number of objects), not just between two as in the graphs.

Taking into account polyadic interactions proves fruitful and indeed inevitable for challenging tasks. According to [26], the harder tasks are, the more useful leveraging information on high-order interactions is, and a simple reduction from high-order to pairwise interactions may lead to significant performance degradation. Hence, hypergraphs, which naturally represent such complex interplays, have drawn considerable attention in many domains, including computer vision [27]–[29], recommendation [30]–[32], and graph learning [33]–[39].

Despite its importance, the understanding of real hypergraphs is still not as concrete as that of real graphs. The daunting complexity due to the variability in hyperedge sizes prevents straightforward extensions of concepts and tools used for graphs to hypergraphs. Nevertheless, there have been fruitful and insightful attempts. In [26], [40], hypergraphs...
are decomposed into conventional graphs, and well-known properties of graphs on the decomposed graphs are examined. In [41], the triadic closure theory is extended to hypergraphs, and in [22], repetitive patterns of hyperedges are investigated in terms of subset correlation and recency bias.

Driven by the importance of hypergraphs and of extensive studies, we scrutinize additional four structural and three dynamical patterns inherent in real-world hypergraphs at the macroscopic level. To this end, we come up with measures to capture new aspects, revisit well-known properties of ordinary graphs, and study their hypergraph analogs. We thoroughly validate the significance of each observed pattern by relying on qualitative and quantitative analyses.

The established structural and dynamical patterns constitute a meaningful step toward advancing the partial understanding of real-world hypergraphs. For structural patterns, we suggest that distributions of (S1) degrees, (S2) hyperedge sizes, (S3) intersection sizes of hyperedges, and (S4) singular values of incidence matrices fall under the class of heavy-tailed distributions, of which the first three and the last are close to a truncated power law and log-normal distribution, respectively, among probable candidates. For dynamical patterns, we observe that, over time, (T1) the overlapping of hyperedges becomes less frequent, (T2) the number of hyperedges grows faster than that of nodes (i.e., densification), and (T3) the distances between nodes decrease (i.e., shrinking diameter).

Using these patterns as criteria, we propose a stochastic model HYPERFF (Hypergraph Forest Fire) for realistic hypergraph generation. In HYPERFF, where a hypergraph grows with new nodes, each new node goes through two stages. In the first stage, a node is randomly chosen and a ‘forest fire’ starts there. The forest fire is spread through existing hyperedges stochastically, and the new node forms a hyperedge with each of the burned nodes. In the second stage, each new hyperedge expands similarly through a forest fire.

The benefits of HYPERFF are three-fold. First, HYPERFF successfully reproduces all seven observed patterns, whereas previously proposed models focus only on a narrow scope of patterns. We also apply the decomposition technique [40] to generated hypergraphs and confirm the five known properties of decomposed graphs. Second, while the previous models rely on unexplainable external information (e.g., the number of new hyperedges along with each new node [40] and the size of each new hyperedge [22]), HYPERFF requires no such oracles, being parameterized simply by two scalars: burning and expanding probabilities. Lastly, HYPERFF leads to a deeper understanding of complex systems, giving a simple underlying mechanism that imposes the non-trivial macroscopic patterns.

As in the exploration of real hypergraphs, we validate HYPERFF through quantitative and qualitative analyses. We also explore its parameter space and suggest values of parameters for generating realistic hypergraphs.

Our contributions are summarized as follows:

- **Establishment of structural and dynamical patterns in real-world hypergraphs.**
  1) **Structural:** Heavy-tailed distributions of degrees, hyperedge sizes, intersection sizes, and singular values of incidence matrices.
  2) **Dynamical:** Diminishing overlaps of hyperedges, densification, and shrinking diameter.

- **Stochastic model HYPERFF for hypergraph generation.**
  1) **Realistic:** It exhibits all seven observed patterns and the five structural patterns reported in the previous study.
  2) **Self-contained:** It does not rely on oracles or external information, and it is parameterized by just two scalars.
  3) **Emergent:** Its simple and interpretable mechanisms on individual nodes non-trivially produce the examined patterns at the macroscopic level.

**Reproducibility:** We make the source code and datasets used in this work publicly available at [https://git.io/Jt8Ar](https://git.io/Jt8Ar).

In Section [II] we survey a line of research on properties and generative models of real-world (hyper)graphs. In Section [III] we introduce some notations and concepts. In Section [IV] we examine the structural and dynamical patterns in real-world hypergraphs. In Section [V] we propose our model HYPERFF and validate it. We conclude our work in Section [VI].
II. RELATED WORK

Below, we review previous work on patterns in real-world hypergraphs and generation of realistic (hyper)graphs.

A. Patterns in Real-world Hypergraphs.

There have been considerable efforts to find out fundamental properties in the structures and dynamics of real-world hypergraphs. Structural patterns are investigated in static hypergraphs or a few snapshots of evolving hypergraphs. Dynamical patterns are investigated in time-evolving hypergraphs.

Instead of dealing directly with the complexity of hypergraphs, [26], [40], [41] reduce hypergraphs to a set of ordinary graphs, which contains partial or all information on the hypergraphs, and then identify the properties of the graphs, using well-defined graph measures. The most basic way is to convert a hypergraph into the graph with the same set of nodes and the set of edges induced by replacing each hyperedge with a clique [41], referred to as the 2-projected graph. On 2-projected graphs, [41] investigates simplicial closure, the extension of triadic closure to hypergraphs, and [42] defines sub-hypergraph centrality and extends the notion of clustering coefficients to hypergraphs. This simple abstraction is extended to n-projected graphs [26] and n-level decomposed graphs [40], which aim to capture all or information between the subsets of n nodes in distinct ways. Especially for the n-level decomposed graphs, we elaborate on their definition in Section V-B. Based on these alternatives, [40] shows that the n-level decomposed graphs obtained from real hypergraphs retain five structural patterns: giant connected components, heavy-tailed degree distributions, small diameters, high clustering coefficients, and skewed singular-value distributions. Besides these approaches, [43] focuses on local structural patterns, specifically the patterns of interactions among three hyperedges; and [22] focuses on dynamical patterns regarding the subset correlation, recency bias, and repeat patterns of hyperedges. More patterns concerning diffusion, synchronization, and games are surveyed in [44]. In this work, we study seven additional generic patterns in real hypergraphs at the macroscopic level.

B. Generation of Realistic (Hyper)graphs.

A number of generative models have been proposed for realistic graph generation. For example, preferential attachment [45], [46] and copying models [11] provide simple underlying principles leading to the power law in degree distributions. The forest fire model [6] reproduces two dynamical patterns in graphs: densification (i.e., increase in the number of edges per node) and shrinking diameter (i.e., decrease in the distances between nodes) over time. The key idea behind the model is to connect a new node with a random node and the nodes burned by a ‘forest fire’, which starts at the random node and is spread stochastically along the edges. The Kronecker model [13] is a tractable model that enjoys a well-developed theory for the Kronecker product and thus provides theoretical guarantees as well as reproduce several established patterns. Darwini [16] aims to generate realistic large-scale graphs whose distributions of degrees and clustering coefficients fit explicitly given distributions. GraphRNN [47] takes a machine learning approach to train an autoregressive model generating graphs structurally similar to a given set of graphs.

Compared to realistic graph generation, relatively little attention has been paid to realistic hypergraph generation. The stochastic model proposed by [22] takes into consideration recurrent patterns of hyperedges; however, it requires an oracle which provides the sizes of the next hyperedges, and the number of new nodes in them. The generative model suggested in [40] extends the preferential attachment model to hypergraphs. In the model called HYPERPA, for each new node, the group of nodes forming a hyperedge with the new node is chosen randomly, proportionally to the number of existing hyperedges that contain the group. While HYPERPA generates a hypergraph whose n-decomposed graphs exhibit the five empirical structural patterns, it requires in advance the distribution of hyperedge sizes and the number of hyperedges related to each new node. However, our model described in Section V is able to reproduce the seven patterns examined in this paper without any oracles, and its decomposed graphs also maintain the five structural patterns.

III. PRELIMINARIES

In this section, we introduce some basic notations and important concepts used throughout the paper.

**Hypergraph.** A hypergraph $G = (V, E)$ consists of a set $V$ of nodes and a set $E \subseteq 2^V$ of hyperedges. Each hyperedge $e$ is a subset of $V$, and we define the size of a hyperedge $e$ as the number $|e|$ of nodes in the hyperedge. The degree of a node $v$, denoted by $\text{deg}(v)$, is defined as the number of hyperedges containing $v$. The neighborhood of a node $v$, denoted by $\mathcal{N}(v)$, is defined as the set of neighbors, each of which is connected to $v$ in one or more hyperedges.

**Incidence Matrix.** The incidence matrix $I \in \{0, 1\}^{V \times |E|}$ of a graph $G = (V, E)$ indicates the membership of the nodes $V$ in the hyperedges $E$. Each $(i, j)$-th entry $I_{ij}$ of $I$ is $1$ if and only if the $j$-th hyperedge in $E$ contains the $i$-th node in $V$.

**Effective Diameter.** A path in a hypergraph is a sequence of hyperedges in which any two immediate hyperedges have a non-empty intersection and the length of the path is the length of the sequence. The distance between two nodes is defined as the length of a shortest path in which the first and the last hyperedges include one of the two nodes and the other, respectively. The diameter of the hypergraph is a maximum distance between any pairs of nodes. Since any disconnected nodes in a hypergraph makes the diameter infinite, we consider the effective diameter [6], which is defined as the smallest $d$ such that the paths of length at most $d$ connect $90\%$ of all reachable pairs of nodes.\(^1\)

**Heavy-tailed Distribution.** Tails of heavy-tailed distributions decay slower than exponential distributions (i.e., they are not

\(^1\)Linear interpolation is used to find such $d$.}
Fig. 3: **Structural properties shown in six real datasets: Heavy-tailed distributions of four quantities.** The first three properties mostly show well-suited straight lines on the log-log scale, implying the potential for a stronger claim: they follow a power law distribution. This tendency is more apparent in larger hypergraphs. See Section IV-A for details.

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Intersection sizes|                    |                 |                 |            |         |                  |
| Intersection size  | Count              | Count           | Count           | Count      | Count   | Count            |
| Intersection size  | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Intersection size  | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Intersection size  | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Singular Values    |                    |                 |                 |            |         |                  |
| Rank              | Count              | Count           | Count           | Count      | Count   | Count            |
| Rank              | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Rank              | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Rank              | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Edge sizes        |                    |                 |                 |            |         |                  |
| Edge size         | Count              | Count           | Count           | Count      | Count   | Count            |
| Edge size         | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Edge size         | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Edge size         | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Intersection sizes|                    |                 |                 |            |         |                  |
| Intersection size  | Count              | Count           | Count           | Count      | Count   | Count            |
| Intersection size  | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Intersection size  | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Intersection size  | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Singular Values    |                    |                 |                 |            |         |                  |
| Rank              | Count              | Count           | Count           | Count      | Count   | Count            |
| Rank              | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Rank              | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Rank              | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Intersection sizes|                    |                 |                 |            |         |                  |
| Intersection size  | Count              | Count           | Count           | Count      | Count   | Count            |
| Intersection size  | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Intersection size  | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Intersection size  | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

| Data              | contact (smallest) | email           | tags            | substances | threads | coauth (largest) |
|-------------------|--------------------|-----------------|-----------------|------------|---------|------------------|
| Singular Values    |                    |                 |                 |            |         |                  |
| Rank              | Count              | Count           | Count           | Count      | Count   | Count            |
| Rank              | $10^0$             | $10^1$          | $10^2$          | $10^3$     | $10^4$  | $10^5$           |
| Rank              | $10^6$             | $10^7$          | $10^8$          | $10^9$     | $10^{10}$| $10^{11}$        |
| Rank              | $10^{12}$          | $10^{13}$       | $10^{14}$       | $10^{15}$  | $10^{16}$| $10^{17}$        |

TABLE I: The real-world hypergraphs used in our study.

| Dataset          | # of Nodes | # of Hyperedges | Summary          |
|------------------|------------|-----------------|------------------|
| contact          | 327        | 172,035         | Social Interaction|
| email            | 1,005      | 235,263         | Email            |
| tags             | 3,029      | 271,233         | Q&A              |
| substances       | 5,556      | 112,919         | Drug             |
| threads          | 176,445    | 719,792         | Q&A              |
| coauth           | 1,924,991  | 3,700,067       | Coauthorship     |

exponentially bounded). Power law distributions are typical examples of the heavy-tailed distributions. For instance, a discrete power law distribution $P$ of a random variable $X$ satisfies, possibly in a limited range, the relationship $P(X) \propto 1/|X|^\alpha$ for some constant $\alpha > 0$. Note that this relationship appears as a straight line on the log-log plot of the probability distribution over the range of the random variable.

**Goodness of Fit.** It is difficult to argue that an empirical dataset genuinely follows a target probability distribution, since other statistical models unexamined yet may describe the dataset with a better fit. Thus, it is more sound to rely on comparative tests which compare the goodness of fit of candidate distributions [48], [49]. We especially utilize the log likelihood-ratio test [50], [51] to this end. When a dataset $D$ with two candidate distributions $A$ and $B$ is given, the test computes $\log \left( \frac{L_A(D)}{L_B(D)} \right)$, where $L_A(D)$ stands for the likelihood of the dataset $D$ with respect to the candidate distribution $A$. Positive ratios imply the distribution $A$ is a better description available for the dataset between the two distributions, and negative ratios imply the opposite.

**Hypergraph Sequence.** Allowing for multiple hyperedges created at the same time, we use $e_i^t$ to denote the set of hyperedges created at time $t$ and $e_i$ to denote a hyperedge in $e_i^t$. Given a sequence $(e_i^t)_{t=1}^T$ of a set of time-stamped hyperedges for some $T > 0$, we define a sequence $(G_t = (V_t, E_t))_{t=1}^T$ of hypergraphs evolving under the hyperedge sequence, where the nodes $V_t$ of $G_t$ is $\bigcup_{i=1}^e e_i^t$ and the hyperedges $E_t$ of $G_t$ is $\bigcup_{i=1}^e e_i^t$. While the final snapshot $G_T$ of the hypergraph sequence is of our interest in exploring structural patterns, subsequences of the hypergraph sequence are examined for the understanding of dynamical patterns.

**Null Model.** We make use of a null model as a counterpart of an evolving hypergraph $(G_t)_{t=1}^T$ so as to emphasize the significance of examined patterns in real-world hypergraphs in Section IV. Formally, the null model is constructed from a sequence of a set of hyperedges $\{(e_i')^T\}_{t=1}^T$ such that there is a one-to-one correspondence between $e_i$ and $(e_i')^T$. Specifically, $e_i$ in $(e_i')^T$ has the same size with its corresponding hyperedge $e_i$ in $e_i^T$, while $e_i'$ consists of randomly selected $|e_i'|$ nodes from $V_t$. Note that this null model has the same distribution of hyperedge sizes with its corresponding dataset.

**IV. Structural and Dynamical Patterns**

In this section, we examine characteristics of real-world hypergraphs prevalent across 6 distinct domains and shed light on common four structural and three dynamical patterns at the macroscopic level. We summarize some basic statistics on the datasets [9], [41], [52] in Table I and briefly describe the characteristics of each dataset below.
We report the log-likelihood ratio normalized by its standard deviation and make the largest one among the three candidates boldfaced. We also provide the p-value of the boldfaced one if it is larger than 0.05, where the p-value is the significance value for the acceptance of the heavy-tailed distribution involved in the ratio. As suggested in Figure 3, the first three patterns generally have the best fit to (truncated) power law distributions, while singular values are best described by log-normal distributions. Remark that HYPERFF also shows the same trend.

### Table II: Log-likelihood ratio of two competing distributions: three heavy-tailed dists., versus exponential dist.

| Heavy-tailed Dist. | Degrees | Hyperedge Sizes* | Intersection Sizes | Singular Values |
|--------------------|---------|------------------|--------------------|-----------------|
|                    | pw      | tpw              | logn              | pw              | tpw | logn | pw | tpw | logn |
| contact            | -0.612  | 0.495†           | -0.011            | -                | -   | -    | -  | -   | -    |
| email              | 0.013   | 2.01             | 1.72              | 28.6             | 34.0 | 32.8 | 454 | 463 | 461  |
| substances         | 8.60    | 9.51             | 9.45              | -713             | -111 | 103  | -  | -   | -    |
| threads            | 3.69    | 3.90             | 3.83              | 29.5             | 31.8 | 30.2 | 39.1| 39.9| 40.0 |
| coauth             | 38.0    | 38.5             | 38.4              | 0.786            | 1.04†| 1.02 | -  | -   | -    |
| HYPERFF (Proposed) | 187     | 206              | 204               | 4.14             | 4.14 | 4.15 | 2.28| 2.36| 2.36 |

|                    |         |                  |                   |                  |         |      |     |     |     |
|                    |         |                  |                   |                  |         |      |     |     |     |

† Contact, tags, threads, and HYPERFF have a small range of hyperedge sizes and intersection sizes, which make some ratio not available.  
‡ The p-value is 0.62.  
§ The p-value is 0.30.  
¶ The p-value is 0.17.

- **contact-high-school (contact):** each node is a student, and each hyperedge is a set of individuals interacting each other as a group during an unit interval.
- **email-Eu (email):** each node is an email address at an European research institution, and each hyperedge consists of the sender and all recipients of an e-mail.
- **tags-ask-ubuntu (tags):** askubuntu.com is a question-and-answer website, where one can ask a question with up to 5 tags attached. Each node and hyperedge correspond to a tag and the set of tags attached to a question, respectively.
- **NDC-substances (substances):** each node is a substance, and each hyperedge indicates the set of substances which a drug is made of.
- **threads-math-sx (threads):** math.stackexchange.com is a question-and-answer website. Each node is a user on the website, and each hyperedge corresponds to the set of users participating in a thread that lasts for at most 24 hours.
- **coauth-DBLP (coauth):** each node is an author, and each hyperedge corresponds to the set of authors in a publication recorded on DBLP.

We first observe the heavy-tailed distributions of degrees, hyperedge sizes, intersection sizes, and singular values of incidence matrices as the structural patterns. Then, we elaborate on diminishing overlaps of hyperedges, densification, and decreasing diameter as the dynamical patterns.

### A. Structural Patterns

In this section, we study four tendencies in the final snapshots of the 6 real hypergraph sequences and present all the results in Figure 3. We demonstrate that (S1) degrees, (S2) hyperedge sizes, (S3) intersection sizes, and (S4) singular values of incidence matrices obey heavy-tailed distributions. Moreover, out of three probable candidates - power law, truncated power law, and log-normal distributions - quantitative validation based on the log-likelihood test supports that the first three are described best by the truncated power law distribution, and the last is closest to the log-normal distribution. These patterns are peculiar in the sense that the
In the following two subsections, we consider two patterns in the hypergraphs.

- **S3. Heavy-tailed intersection size distribution.** The intersection sizes of two hyperedges (i.e., the number of nodes commonly contained in two hyperedges), which offer insights into pairwise intersections of hyperedges, also follow heavy-tailed distributions. As shown in Table II, the heavy-tailed distributions are probab|les for this pattern, and the reported likelihood ratios in Table II together with the straight lines in Figure 3 back up the goodness of fit to the (truncated) power law distribution.

In contrast with the null model, whose intersection-size distribution is shown in Figure 4(b) although its overall distribution seems similar, the range of intersection sizes and the frequency of each intersection size are significantly different from those of the corresponding hypergraph.

- **S4. Skewed singular values.** This pattern means that the singular values of the incidence matrices of real hypergraphs are generally heavy-tailed. As certified in Table II, the singular-value distribution is best described by the log-normal distribution. In contrast, the tails of the singular-value distributions decay faster than those of the three distributions in (S1-S3).

### B. Dynamical Patterns

Now we move on to the investigation of three dynamical patterns in the six real hypergraphs sequences. We devise a quantity to measure how much the hyperedges are overlapped in overall and show that (T1) overall overlaps of the hyperedges decrease over time. Just as phenomena of densification and shrinking diameter are well known to take place in real graphs, the two dynamical patterns are still prevalent in real hypergraphs: (T2) average degrees increase over time and (T3) effective diameters decrease over time.

#### T1. Diminishing overlaps.

We first define the density of intersections as follows to capture the overall overlaps of hyperedges: for a hypergraph $G_t = (V_t, E_t)$ at time $t$,

$$\text{DoI}(G_t) := \frac{|\{e_i, e_j\} \cap e_i \neq \emptyset \text{ for } e_i, e_j \in E_t|}{|\{e_i, e_j\}||e_i, e_j \in E_t|}$$  \hspace{1em} (1)

In words, it is simply the ratio of the number of intersecting pairs of hyperedges to the number of all possible pairs of hyperedges, which amounts to $\binom{|E_t|}{2}$. Note that this quantity can range from 0 to 1.

Using $y(t)$ and $x(t)$ to indicate the numerator and the denominator, respectively, in Equation 1 we take a closer look at the log-log plot of $y(t)$ over $x(t)$. As shown in the first row of Figure 5 the slopes $s$ of fitted lines on the plots imply the formula $\text{DoI}(G_t) = y(t)/x(t) = O(x(t)^{-\langle 1-s \rangle})$. As the slopes $s$ are usually smaller than 1, the density of interactions decreases over time. Decreasing $\text{DoI}(G_t)$ indicates that the ratio of overlapping pairs of hyperedges decreases and that the intersections of hyperedges become less frequent in overall.

#### T2. Densification.

In the following two subsections, we confirm that dynamical phenomena in real graphs - densification and shrinking diameter established in a seminal work [6] - still take place in real hypergraphs.

We proceed with the same manner as in Section IV-B: For a hypergraph $G_t = (V_t, E_t)$ at time $t$, we plot $|E_t|$ over $|V_t|$ on the log-log scale in the second row of Figure 5 and compute

### Table

| Data | contact | email | tags | substances | threads | coauth |
|------|---------|-------|------|------------|---------|--------|
| # of intersecting pairs | # of intersecting pairs | # of intersecting pairs | # of intersecting pairs | # of intersecting pairs | # of intersecting pairs | # of intersecting pairs |
| # of all pairs | # of all pairs | # of all pairs | # of all pairs | # of all pairs | # of all pairs | # of all pairs |
| # of nodes | # of nodes | # of nodes | # of nodes | # of nodes | # of nodes | # of nodes |

**Fig. 5:** *Dynamical patterns shown in six real datasets: diminishing overlaps, densification, and shrinking diameter.*

The slopes smaller than 1 in the first row indicate decreasing ratios of intersecting pairs. The slopes larger than 1 for the edge density imply increasing average degrees of the hypergraphs. The effective diameters eventually decrease. For the threads dataset, the effective diameter starts to shrink almost in the end. See Section IV-B for details.
the slopes $s$ of the fitted lines, which lead to the formula $|E_t| \propto |V_t|^s$. The slopes larger than 1 in general indicate that the average degrees of hypergraphs, formulated as $2|E_t|/|V_t|$, increase over time, and thus the densification is also valid for real hypergraphs.

**T3. Shrinking diameter.** In the third row of Figure 5, we show how the effective diameters of hypergraphs change over time. We observe that the effective diameters of all real hypergraphs eventually decrease over time. In the threads dataset, while the decrease is marginal, the effective diameter starts to decrease almost in the end.

It is worthy to note that although shrinking diameter and densification seem intuitively compatible with one another, one can construct counterexamples that have only one of the two dynamical patterns, which are illustrated in the second and third row in Figure 5.

V. PROPOSED MODEL: HYPERGRAPH FOREST FIRE

In parallel with the establishment of underlying patterns in real hypergraphs, we propose a growth model HYPERFF (Hypergraph Forest Fire) for realistic hypergraph generation. In a nutshell, for each new node, HYPERFF simulates a 'forest fire', which is spread stochastically through existing hyperedges, to decide the nodes each of which forms a size-2 hyperedge with the new node. Then, HYPERFF simulates a forest fire again for each created hyperedge to expand it.

HYPERFF reproduces all seven examined patterns and additionally reproduces all five structural patterns found in previous work [40] without relying on oracles, i.e., unexplainable outside information. Notably in HYPERFF, the mechanisms on individual nodes are simple and intuitive, while they do not directly impose but eventually lead to the examined patterns.

In Section V-A we describe HYPERFF in detail with a pseudocode and rationales behind it. In Section V-B we confirm that it successfully reproduces all the seven patterns as well as five additional patterns. Lastly in Section V-C we explore the parameter space of HYPERFF.

A. Description of model

We provide the pseudocode of HYPERFF in Algorithm 1. Note that, except the target number of nodes (i.e., $T$), it needs only two parameters: burning probability $p$ and expanding probability $q$. HYPERFF starts with a hypergraph with one node and no hyperedge; and it initializes tie strengths, which measure the closeness of two nodes, to 0. Then, it repeats the following steps for each new node $u$.

1. The new node $u$ chooses a random ambassador $w$ from the hypergraph so far and burns the ambassador.
2. Burn $n$ neighbors of the ambassador $w$ in descending order of tie strength, where $n$ is sampled from the geometric distribution with mean $p/(1-p)$. In case of burning a part of neighbors with the same tie strength, proceed with random selection.
3. Recursively apply (2) to each burned neighbor by viewing a burned neighbor as a new ambassador of the new node $u$. Nodes cannot be burned twice until the recursion ends.

(4) For each burned node $v$, form a hyperedge $\{u, v\}$ and increase tie strength of $\{u, v\}$ by 1.

(5) For each hyperedge created in (4), reset the burning history and start the burning process at the burned node $v$ in which we use the geometric distribution with mean $q/(1-q)$, expand the hyperedge until the process ends.

Each step has a straightforward rationale, and it may be well understood through an example of a coauthorship network. Suppose a student joins a research community, advised by a supervisor ( ambassador). When introducing colleagues to the student for coworking, the supervisor is more likely to introduce intimate peers (considering tie strength). Those who cowork with the student recursively introduce their close peers for follow-up research (recursive burning). When a group of researchers looks for a new researcher to work with (hyperedge expansion), a referrer in the group is likely to introduce those who the referrer has worked with many times before than a totally new researcher (considering tie strength).

Comparison with [6]. Our model HYPERFF has several differences from the forest fire model [6], which HYPERFF is inspired by. Most importantly, the forest fire model generates conventional graphs instead of hypergraphs. Moreover, it relies

**Algorithm 1: HYPERFF: the proposed model for realistic hypergraph generation.**

**Input:** burning probability: $p$, expanding probability: $q$, timespan (i.e., the target number of nodes): $T$

**Output:** evolving hypergraph: $\{G_t\}_{t=1}^T$

**Algorithm Generator** $(p, q, T)$$
\begin{align*}
& G_0 \seteq \text{hypergraph with 1 node and 0 hyperedges} \\
& \text{Initialize tie measuring how close two nodes are} \\
& \text{foreach time } t \in [1, \ldots, T] \text{ do} \\
& \quad V_t \seteq V_{t-1} \cup \{\text{new node } u\} \& e^*_{t} \seteq \emptyset \\
& \quad w \seteq \text{random ambassador from } V_{t-1} \\
& \quad Burned \seteq \text{Burning } (w, p) \\
& \quad \text{foreach } v \in Burned \text{ do} \\
& \quad \quad \text{Increase tie} \{u, v\} \text{ by } 1 \\
& \quad \quad Burned \seteq \text{Burning } (v, q) \\
& \quad \quad e^*_{t} \seteq \text{hyperedge } Burned \cup \{u\} \\
& \quad E_t \seteq E_{t-1} \cup e^*_{t} \\
& \text{return } \{G_t\}_{t=1}^T \\
\end{align*}$

**Subroutine Burning** $(source, \text{prob})$$
\begin{align*}
& Burned = \emptyset \& \text{Queue} \seteq \emptyset \text{ empty queue} \\
& \text{Queue} \seteq source \\
& \text{while } \text{Queue} \neq \emptyset \text{ do} \\
& \quad s \seteq \text{node popped from } \text{Queue} \\
& \quad Burned \seteq s \\
& \quad n \sim \text{geometric dist. with mean } \frac{prob}{1-prob} \\
& \quad \text{Candidates} \seteq N(s) \\setminus Burned \setminus \text{Queue} \\
& \quad \text{Queue} \seteq n \text{ neighbors in } \text{Candidates} \text{ in decreasing order of } \text{tie} \{s, \text{neighbor}\} \\
& \text{return Burned} \\
\end{align*}
Fig. 6: Comparison between a real hypergraph and those generated by HYPERFF with distinct parameters. Each row for HYPERFF (p, q) with the burning probability p and expanding probability q shows holistic behaviors of the model. With suitable probabilities (e.g., p = 0.51 and q = 0.2), HYPERFF successfully reproduces all examined patterns. The overall comparison indicates larger p and q lead to faster densification and shrinking diameter, respectively. See Section V for details.

| Structural Patterns | Dynamical Patterns |
|--------------------|--------------------|
| Degrees | Hyperedge sizes | Intersection Sizes | Singular Values | Intersecting Pairs | Edge Density | Diameter |
| ![Graph](image1) | ![Graph](image2) | ![Graph](image3) | ![Graph](image4) | ![Graph](image5) | ![Graph](image6) | ![Graph](image7) |

on the ‘orientation’ of burning, requiring two parameters: forward and backward burning probabilities. Without relying on the orientation, it fails to achieve both shrinking diameters and power law distributions of degrees. On the other hand, HYPERFF achieves both by additionally taking tie strength (instead of the orientation) into account, without requiring an additional parameter. As a result, HYPERFF successfully copes with the complexity due to hyperedges of any size, while maintaining the number of parameters to two (i.e., one probability for neighbor selection and the other for expansion).

**Remarks.** HYPERFF suggests possible local dynamics on individual nodes of hypergraphs that give rise to the *macroscopic* patterns examined in this paper. Thus, predicting an order of hyperedges at the microscopic patterns examined in this paper. Thus, predicting an order of hyperedges at the microscopic level goes out of scope. Also, for simplicity, it does not have a particular parameter for repetition of hyperedges. Constructing a model tackling these sides is left for future work.

**B. Experimental results**

We examine the proposed model HYPERFF thoroughly, in terms of the established patterns, as in Section IV. We also examine its n-level decomposed graphs, as suggested in [40].

**Observed patterns.** We demonstrate that HYPERFF with suitable parameter values[4] reproduces four structural and three dynamical patterns investigated in Section IV.

3We investigate a synthetic hypergraph with 10,000 nodes and set the burning probability to 0.51 and the expanding probability to 0.2.

In Figure 6 we provide overall patterns of models with four distinct parameter settings. The model in the first row apparently exhibits all the patterns consistent with the representative results of one representative real dataset. Especially for the four structural patterns, including the distribution of degrees, hyperedge sizes, intersection sizes, and singular values, we confirm in Table II that all the distributions of the model have the same tendency with real hypergraphs. The four distributions are best suited to heavy-tailed distributions, and among probable heavy-tailed distributions, the best description for each pattern is also consistent with that of real hypergraphs; the first three and the last are close to the (truncated) power law distributions and log-normal distribution, respectively.

We also check that the model achieves the three dynamical patterns - diminishing overlap, densifying graph, and decreasing diameter - in Figure 6.

**Structural patterns in decomposed graphs.** The multi-level decomposition method [40] provides a way of reducing hypergraphs to m ordinary graphs for the largest size m of hyperedges. Each reduced graph is referred to as an n-level decomposed graph[4] which focuses on the interplays between
Fig. 7: Compatibility with previous work [40]. HYPERFF and a real dataset have similar heavy-tailed degree distributions at each decomposition level, and they even have the same tendency of deviating from the fitted lines at higher decomposition levels. Singular values of HYPERFF and the dataset basically lie on their fitted lines, regardless of decomposition levels. See Section VI-B for details.

| Level | Degrees | Singular Values |
|-------|---------|-----------------|
|       | Node    | Edge            | Triangle   |
| HYPERF (Proposed) | Count | Count | Count |
| substances (Real) | Count | Count | Count |

TABLE III: Size of the largest connected component at each decomposition level, and effective diameter and clustering coefficient at the node level. The red colors indicate the level at which decomposed graphs no longer retain a giant connected component. Note that HYPERFF also shows a similar trend with real datasets; its giant connected component starts to shatter after the edge level, its effective diameter is small, and it has similar clustering coefficient with the real datasets.

C. Exploring parameter space

In Figure 3 we present qualitative patterns in HYPERFF with parameters set from all the combinations of two burning probability and two expanding probability. We observe that the burning probability and the expanding probability are mainly concerned with how fast a hypergraph densifies and its effective diameter decreases, respectively. Note that with low burning probabilities, the effective diameter of a hypergraph is actually increasing in spite of the densification.

VI. Conclusions

Despite the omnipresence of hypergraphs, relatively little attention has been paid to structural and dynamical patterns of real-world hypergraphs. Toward more extensive and thorough understanding of the real-world hypergraphs, we closely examine four structural and three dynamical patterns prevalent in them. The former includes the heavy-tailed distributions of (S1) degrees, (S2) hyperedge sizes, (S3) intersection sizes, and (S4) singular values of incidence matrices. The latter includes (T1) diminishing overlaps, (T2) densification, and (T3) shrinking diameter. We validate the significance of these patterns by comparison with a null model and statistical tests.

We also propose a generative model HYPERFF, which captures all seven observed patterns and also shows results compatible with previous findings. Especially, it has only
two scalars (e.g., burning and expanding probabilities) as parameters, and it does not rely on any external information for imitating realistic patterns. Surprisingly, HYPERFF is made up of simple and intuitive mechanisms on individual nodes, which non-trivially lead to all the examined macroscopic patterns.

For reproducibility, we make the source code and datasets used in this work publicly available at https://git.io/JUfAr.

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