On the one-dimensional representations of the
general linear supergroup

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Abstract. Because of its multiplicativity, the Berezinian is the character of
the one-dimensional representation of the general linear supergroup. We give
an explicit construction of this representation on a space of tensors. Similarly,
we construct the representation such that its character is the inverse of the
Berezinian.

0 Introduction

The following result belongs to the basic ones in the classic case. Let $V$
be an $n$-dimensional linear space over a field $K$ of zero characteristic. Then
the space $\text{Alt}(V \otimes \cdots \otimes V)$ of the skew-symmetric tensors of degree $n$ is the
one-dimensional module over the general linear group $GL V$. The character
of such representation of $GL V$ is the determinant. In the present paper we
construct a pair of analogous representations on spaces of tensors for the
supercase. The character of one of these representations is the Berezinian.
The other character is the inverse of the Berezinian.

Outline the way to our construction in general. Let $G$ be a Grassman
algebra with a countable set of generators, $V = V_0 \oplus V_1$ a free $\mathbb{Z}_2$-graded
$G$-module of dimension $m|n$, where $m = \dim_G V_0$, $n = \dim_G V_1$. Suppose
$W_{\lambda_h}$, $W_{\lambda_g}$ are the irreducible $GL V$-modules defined by the partitions
$\lambda_h = (n+1, \ldots, n+1)$, $\lambda_g = (n, \ldots, n)$ (respectively) and $b$ is a formal element
such that for an arbitrary $A \in GL V$ we have $A(b) = \text{Ber} A \cdot b$, where $A$ is a
matrix of $A$. We show that the $GL V$-modules $W_{\lambda_h}$ and $b \cdot W_{\lambda_g}$ are isomorphic
(see Theorem 2.1). To prove this theorem we use a natural correspondence
between the sets of the base vectors of $W_{\lambda_h}$ and $W_{\lambda_g}$. The tensor $\bar{b}$ generating
the one-dimensional $GL V$-module is constructed of the element $b$ and these base vectors of $W_{\lambda_h}$ and $W_{\lambda_g}$ (see Theorem 3.1). By analogy we construct the one-dimensional $GL V$-module such that its character is the inverse of the Berezinian.

This paper is organized as follows. We introduce the main concepts in §1. In §2 we prove that $W_{\lambda_h} \simeq b \cdot W_{\lambda_g}$ and obtain an important corollary of this result (see Theorem 2.2). Then in §3 the one-dimensional representations of $GL V$ are constructed.

In relation to the present paper, the result of H.M. Khudaverdian and Th.Th. Voronov must be mentioned. In [1], the Berezinian is expressed as the ratio of the two Hankel determinants such that these are the characters of the representations of $GL V$ defined by the partitions $\lambda_h$ and $\lambda_g$. On one hand, this is a simple corollary of Theorem 2.1, on the other directly implies this theorem (unfortunately, the last easy observation have come too late to be useful for the author).

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1 The basic concepts and auxiliary results

Let $K$ be a field of characteristic zero, $G = G_0 \oplus G_1$ the Grassman algebra over $K$ with a countable set of generators.

By definition, the notion of $G$-module includes the property to be free.

Let $V = V_0 \oplus V_1$ be a finite-dimensional $\mathbb{Z}_2$-graded $G$-bimodule and the structures of the left and right $G$-modules on $V$ are compatible (see, for example, [2], [3]). Denote $m = \dim_G V_0, n = \dim_G V_1$. The pair $m|n$ is the dimension of $V$.

We suppose that any operation of changing base of $V$ is even.

Let $\text{End} V$ be the algebra of all $G$-linear mappings from $V$ to $V$. The algebra $\text{End} V$ is isomorphic to the algebra $M = M_0 \oplus M_1$ of block matrices with Grassman elements. The even component $M_0$ of $M$ is called the full matrix superalgebra and is denoted by $M_{m,n}$. Actually the elements of $M_{m,n}$ are all matrices of the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{11}, A_{22}$ are square $G_0$-matrices of orders $m$ and $n$ respectively and $A_{12}, A_{21}$ are rectangular $G_1$-matrices of corresponding orders.

Let $GL V$ and $GL_{m,n}$ be the groups of all invertible elements of $\text{End} V$ and $M_{m,n}$ respectively. We have the isomorphism $GL V \simeq GL_{m,n}$. The group
$GL_{m,n}$ ($GLV$) is the general linear supergroup.

Recall that for any $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in GL_{m,n}$ the Berezinian of $A$ is given by the formula

$$\text{Ber} A = \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \det A_{22}^{-1} = \det A_{11} \det(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$$

(see [4]). The main feature of the Berezinian is its multiplicativity.

Denote $T_l(V) = V \otimes \cdots \otimes V$, where $l = 1, 2, \ldots$. Clearly, $T_l(V)$ is a $G$-module. By definition, for any $A \in GLV, v_i \in V, l = 1, 2, \ldots$ we have

$$A(v_1 \cdots v_l) = A(v_1) \cdots A(v_l). \quad (1.1)$$

Action (1.1) of $GLV$ on decomposable tensors is extended to the whole space $T_l(V)$ by linearity and we see that $T_l(V)$ is a $GLV$-module.

In what follows we consider the action induced by $GLV$ on some submodules of $T_l(V)$. Nevertheless again the corresponding operator is denoted by $\mathcal{A}$. This cannot confuse us forasmuch as from context is clear, what of submodules is considered.

Let $S_l$ be the symmetric group on the elements $1, 2, \ldots, l$.

By definition, for an arbitrary $\sigma \in S_l, v_i \in V, i = 1, 2, \ldots, l$ put

$$\sigma(v_1 \cdots v_l) = v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(l)}.$$

Then $T_l(V)$ is a left $S_l$-module.

To determine irreducible $GLV$-submodules of $T_l(V)$ the special element of the group algebra $K[S_l]$, the so-called Young symmetrizer, can be used. Recall this notion.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be positive integers such that $\lambda_1 \geq \ldots \geq \lambda_k$ and $l = \lambda_1 + \cdots + \lambda_k$. Then $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the partition of $l$. The corresponding Young diagram $\lambda$ has $\lambda_i$ boxes in its $i$-th row. The Young tableau $T_\lambda$ is obtained from $\lambda$ by filling its boxes with the numbers $1, 2, \ldots, l$ moving by the rows from left to right and from top to bottom. For example, if $\lambda = (4, 2, 1)$, then $T_\lambda$ has the form
Let $C(T_\lambda) \ (R(T_\lambda))$ be the subgroup of $S_i$ that preserves the numbers of $T_\lambda$ within their columns (rows, respectively). Then

\[
e_T = \sum_{\sigma \in C(T), \tau \in R(T)} \text{sgn}(\sigma) \tau \sigma
\]

is the Young symmetrizer and $W_\lambda = e_T \lambda T(V)$ is the irreducible $GLV$-module.

Denote

\[
\lambda_h = (n + 1, \ldots, n + 1)_m,
\]

\[
\lambda_g = (n, \ldots, n)_{m+1}.
\]

In other words, $\lambda_h$ is the $m \times (n + 1)$ rectangle and $\lambda_g$ is the $(m + 1) \times n$ rectangle. In the sequel, the $GLV$-modules $W_{\lambda_h}$, $W_{\lambda_g}$ play an important role.

**Remark.** One can say that $W_{\lambda_g}, W_{\lambda_h}$ are particular cases of the so-called external forms (see [2]).

Denote

\[
\lambda_{SR} = (n, \ldots, n)_m,
\]

\[
\lambda_{LR} = (n + 1, \ldots, n + 1)_{m+1},
\]

\[
\lambda_{AR} = (n + 1, \ldots, n + 1, n)_m.
\]

According to terminology of Issaia L. Kantor the diagrams $\lambda_{SR}$, $\lambda_{LR}$, $\lambda_{AR}$ are called the small rectangle, the large rectangle, the almost rectangle (respectively).
Evidently, to obtain $\lambda_h (\lambda_g)$ one must add the column of height $m$ (the row of length $n$) to the small rectangle. As mentioned above, in either case we obtain a kind of rectangle.

An arbitrary tensor of the form

$$e_T v_1 \cdots v_l,$$

(1.3)

where $v_1, v_2, \ldots, v_l \in V$, has the following properties:

i) if $i_1, i_2 \in \{1, 2, \ldots, l\}$ place in the same column of $T$ then tensor (1.3) is skew-symmetric by the elements $v_{i_1}, v_{i_2}$, that is,

$$e_T v_1 \cdots v_l = -e_T(i_1, i_2)v_1 \cdots v_l;$$

(1.4)

ii) let $i_1, \ldots, i_{k-1}$ be numbers that fill a column of length $(k - 1)$ of $T$, $i_k$ a number that belongs to a column of length not more than $(k - 1)$; then the Jacobi identity holds, that is,

$$e_T v_1 \cdots v_l = \sum_{j=1}^{k-1} e_T(i_j, i_k)v_1 \cdots v_l$$

(1.5)

(see [5]). Clearly, (1.4) and (1.5) are the applications of the corresponding properties of the symmetrizer $e_T$.

Suppose $f$ is a tensor of the form (1.3). We say that $(i, j)$ is an $f$-box if this box belongs to the tableau $T$ defining the tensor $f$. By analogy the concepts of an $f$-row and an $f$-column are introduced.

Let $\{e_1, \ldots, e_m\}, \{\varepsilon_1, \ldots, \varepsilon_n\}$ be some bases of the $G$-modules $V_0$ and $V_1$ respectively. Denote

$$\Omega_0 = \{e_1, \ldots, e_m\}, \quad \Omega_1 = \{\varepsilon_1, \ldots, \varepsilon_n\}, \quad \Omega = \Omega_0 \cup \Omega_1.$$

We say that an element $f \in T_l(V)$ is an $\Omega$-tensor if $f$ has the form (1.3), where $v_i \in \Omega$ for $i = 1, 2, \ldots, l$.

Recall that we use the unique way of filling of Young diagrams (see above). Thus there is a one-to-one correspondence between $\Omega$-tensors and fillings of the corresponding Young diagrams with the vectors $v_i \in \Omega$.

A filling of a rectangular Young diagram by vectors $v_i \in \Omega$ is canonical if every even basis vector $e_i \in \Omega_0$ cannot be placed anywhere except boxes of the $i$-th row and every odd basis vector $\varepsilon_j \in \Omega_1$ cannot be placed anywhere except boxes of the $j$-th column. In other words, a filling of a rectangular
diagram is canonical if a box $(i, j)$ that is within the small rectangle can be filled by $e_i$ or by $\varepsilon_j$ only.

For example, if the dimension of $V$ is $2|1$, $\Omega_0 = \{e_1, e_2\}$, $\Omega_1 = \{\varepsilon\}$, then the canonical fillings of the small rectangle are the following:

$\begin{array}{ccc}
  e_1 & e_1 & \varepsilon \\
  e_2 & \varepsilon & e_2 \\
\end{array}$

Suppose $\lambda$ is a rectangular Young diagram filled with elements of $\Omega$ in a canonical way. Then the contents of boxes that do not belong to the small rectangle is defined uniquely. More precisely, if these boxes belong to the $i$-th row, where $i \in \{1, 2, \ldots, m\}$ ($j$-th column, where $j \in \{1, 2, \ldots, n\}$), then they are filled with $e_i$ (with $\varepsilon_j$ respectively).

In particular, the canonical fillings of the diagrams $\lambda_g$, $\lambda_h$ have the following properties:

i) the $(n + 1)$-th column of the diagram $\lambda_h$ is filled with the vectors $e_1, \ldots, e_m$ from the top down;

ii) the $(m + 1)$-th row of the diagram $\lambda_g$ is filled with the vectors $\varepsilon_1, \ldots, \varepsilon_n$ from left to right.

For example, if the dimension of $V$ is $1|2$, $\Omega_0 = \{e\}$, $\Omega_1 = \{\varepsilon_1, \varepsilon_2\}$, then the canonical fillings of the diagram $\lambda_g$ are the following:

$\begin{array}{cccc}
  e & e & \varepsilon & \varepsilon \\
  \varepsilon_1 & \varepsilon_1 & \varepsilon_2 & \varepsilon_2 \\
\end{array}$

Evidently, the concept of canonical filling is defined for diagrams that do not contain the large rectangle. From the other hand, if a diagram contains the large rectangle, then tensor (1.3) is equal to zero (see, for example, [2]).

**Remark.** The concept of canonical filling is introduced in [2] and can be applied not only to rectangular diagrams.

We say that an $\Omega$-tensor $f$ is *canonical* if the filling of the corresponding diagram is canonical.

By $\Lambda_g$, $\Lambda_h$ denote the sets of all canonical tensors for the diagrams $\lambda_g$, $\lambda_h$. Note that there is one-to-one correspondence between the sets $\Lambda_g$, $\Lambda_h$: the corresponding elements have the same filling within the small rectangle. Elements of the sets $\Lambda_g$, $\Lambda_h$ are denoted by $g_i$ and $h_j$ respectively, where
\(i, j = 1, 2, \ldots, k_\Lambda, \ k_\Lambda = 2^{mn}\). Moreover we suppose that elements of \(\Lambda_g, \Lambda_h\) with the same numbers are corresponding, that is, for every \(i \in \{1, 2, \ldots, k_\Lambda\}\) tensors \(g_i, h_i\) have the same filling of the small rectangle.

**Theorem 1.1.** The sets \(\Lambda_g, \Lambda_h\) are the bases of the \(G\)-modules \(W_{\lambda_g}, \ W_{\lambda_h}\).

The proof of this theorem is found in [2].

2 The isomorphism of \(W_{\lambda_h}\) and \(b \cdot W_{\lambda_g}\) as the \(GL V\)-modules

Let \(b\) be a formal element such that for any \(A \in GL V\) we have

\[
A(b) = \text{Ber} A \cdot b,
\]

where \(A\) is the matrix of \(A\) in some base of \(V\). Recall that to change a base of \(V\) we use even operators only (see §1). Then taking into account the multiplicativity of \(\text{Ber}\), one can see that the concept of the element \(b\) is well defined.

Clearly, we assume the element \(b\) is homogeneous. At the same time now we do not define if this element even or odd.

Because of the multiplicativity of \(\text{Ber}\) formula (2.1) defines the one-dimensional representation of the group \(GL V\) on the \(G\)-module generated by the element \(b\).

Let \(f\) be an \(\Omega\)-tensor. By \(\kappa(f)\) denote the cardinality of the automorphism group of \(f\)-tableau by odd elements. Clearly, if an \(f\)-column does not contain any odd elements, then the corresponding factor in \(\kappa(f)\) is equal to 1.

By \(\rho(f)\) denote the number of \(\Omega_1\)-elements belonging to the small rectangle of \(f\). Let \(\alpha\) be the map from \(\Lambda_h\) to the rational numbers such that

\[
\alpha(h_i) = \frac{(-1)^{\rho(h_i)} \kappa(h_i)}{\kappa(g_i)}. \quad (2.2)
\]

By definition, put

\[
h'_i = \frac{h_i}{\alpha(h_i)}, \quad (2.3)
\]

where \(i = 1, 2, \ldots, k_\Lambda\).

The following result is the heart of the present paper.
Theorem 2.1. The mapping

\[ \varphi : h_i' \mapsto b \cdot g_i \]  

(2.4)
determines the isomorphism of the \( GL_V \)-modules \( W_{\lambda h} \) and \( b \cdot W_{\lambda g} \), that is, for any \( A \in GL_V \) the following equality holds

\[ A \varphi = \varphi A. \]  

(2.5)

In the sequel we assume that \( A \in GL_{m,n} \) is the matrix of \( A \in GL_V \) in the base \( e_1, \ldots, e_m, \varepsilon_1, \ldots, \varepsilon_n \).

The following proposition reduces the proof of Theorem 2.1 to some easier particular cases of the general situation.

Proposition 2.1. To prove Theorem 2.1 it suffices to check (2.5) for the cases when the matrix \( A \) of \( A \in GL_V \) belongs to one of the following classes of matrices:

\[ E_{m+n} + e_{ij} \eta_{ij}, \]  

(2.6)

where \( E_{m+n} \) is the unit matrix of order \( m+n \), \( e_{ij} \) is the matrix unit, \( \eta_{ij} \in G_1 \), \( m + 1 \leq i \leq m + n, 1 \leq j \leq m \);

\[ E_{m+n} + e_{ij} \xi_{ij}, \]  

(2.7)

where \( \xi_{ij} \in G_1, 1 \leq i \leq m, m + 1 \leq j \leq m + n \);

\[ \text{diag}(x_1, \ldots, x_m, y_1, \ldots, y_n), \]  

(2.8)

where \( x_i, y_j \in G_0 \);

\[ E_{m+n} + e_{ij} x_{ij}, \]  

(2.9)

where \( x_{ij} \in G_0, 1 \leq i, j \leq m, i \neq j \);

\[ E_{m+n} + e_{ij} y_{ij}, \]  

(2.10)

where \( y_{ij} \in G_0, m + 1 \leq i, j \leq m + n, i \neq j \).

The proof of this proposition is based on the following auxiliary result.

Lemma 2.1. Any matrix \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in GL_{m,n} \) can be represented in the form
\begin{equation}
A = \begin{pmatrix} E_m & 0 \\ B_{21} & E_n \end{pmatrix} \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} E_m & B_{12} \\ 0 & E_n \end{pmatrix}, \tag{2.11}
\end{equation}

where \( B_{11} = A_{11}, B_{12} = A_{11}^{-1}A_{12}, B_{21} = A_{21}A_{11}^{-1}, B_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}, \)
or in the form

\begin{equation}
A = \begin{pmatrix} E_m & C_{12} \\ 0 & E_n \end{pmatrix} \begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} E_m & 0 \\ C_{21} & E_n \end{pmatrix}, \tag{2.12}
\end{equation}

where \( C_{22} = A_{22}, C_{12} = A_{12}A_{22}^{-1}, C_{21} = A_{21}^{-1}A_{21}, C_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}. \)

**Proof.** Since \( A \in GL_{m,n} \), we see that \( A_{11}^{-1}, A_{22}^{-1} \) are exist (see [4]). Equalities (2.11), (2.12) can be checked directly.

This completes the proof.

**Proof of Proposition 2.1.** With the isomorphism \( GL_V \simeq GL_{m,n} \) a composition of linear operators corresponds to the composition of matrices (clearly, the inverse statement is correct). Hence it follows from Lemma 2.1 that it suffices to check (2.5) for \( A \in GL_V \) such that the matrix \( A \) of \( A \) has either the form

\begin{equation}
\begin{pmatrix} E_m & A_{12} \\ 0 & E_n \end{pmatrix}, \tag{2.13}
\end{equation}

or

\begin{equation}
\begin{pmatrix} E_m & 0 \\ A_{21} & E_n \end{pmatrix}, \tag{2.14}
\end{equation}

or

\begin{equation}
\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \tag{2.15}
\end{equation}

where (as above) \( A_{11}, A_{22} \) are square \( G_0 \)-matrices of orders \( m \) and \( n \) respectively, \( A_{12}, A_{21} \) are \( G_1 \)-matrices.

If \( i, j \) are integers such that \( 1 \leq i \leq m, m+1 \leq j \leq m+n \), then \( e_{ij}^2 = 0 \). Consequently an arbitrary matrix (2.13) can be represented as a product of matrices (2.7). Similarly any matrix (2.14) can be represented as a product of matrices (2.15).

By \( \mathcal{D} \) denote the group generated by all matrices of the form (2.8) - (2.10). Also by \( \overline{\mathcal{D}} \) denote the group of all matrices of the form (2.15). Just as for the regular subgroup in the classic case the set \( \mathcal{D} \) is open and dense in \( \overline{\mathcal{D}} \).

Thus we see that Proposition 2.1 is proved.
Applying Proposition 2.1 we use the coordinate form of equality (2.5). Let us obtain this form. With (2.1) and (2.4) we have

\[ \text{Ber} \ A \cdot b \cdot \mathcal{A}(g_i) = \mathcal{A}(b)\mathcal{A}(g_i) = \mathcal{A}(bg_i) = \mathcal{A}\varphi(h'_i). \]

Hence (2.5) is equivalent to

\[ \varphi(\mathcal{A}(h'_i)) = \text{Ber} \ A \cdot b \cdot \mathcal{A}(g_i). \quad (2.16) \]

The matrix elements of the action induced by \( \mathcal{A} \) on the spaces \( W_{\lambda h} \), \( W_{\lambda g} \) and written in the bases \( \{h'_i\} \) and \( \{g_j\} \) we denote by \( a'_{ij} \) and \( a''_{ij} \) (respectively), that is,

\[ \mathcal{A}(h'_j) = \sum_{i=1}^{k_{\Lambda}} h'_i a'_{ij}, \quad (2.17) \]
\[ \mathcal{A}(g_j) = \sum_{i=1}^{k_{\Lambda}} g_i a''_{ij}, \quad (2.18) \]

where \( 1 \leq j \leq k_{\Lambda} \).

Using (2.17), (2.18) rewrite condition (2.16) in the form

\[ \varphi \left( \sum_{i=1}^{k_{\Lambda}} h'_i a'_{ij} \right) = \text{Ber} \ A \cdot b \cdot \sum_{i=1}^{k_{\Lambda}} g_i a''_{ij}. \]

Hence by definition of \( \varphi \) we have

\[ b \cdot \sum_{i=1}^{k_{\Lambda}} g_i a'_{ij} = \text{Ber} \ A \cdot b \cdot \sum_{i=1}^{k_{\Lambda}} g_i a''_{ij}. \quad (2.19) \]

Since the elements \( g_i \) are \( G \)-linearly independent, we see that (2.19) is equivalent to

\[ a'_{ij} = \text{Ber} \ A \cdot a''_{ij}, \quad (2.20) \]

where \( 1 \leq i, j \leq k_{\Lambda} \). Equality (2.20) is the coordinate form of (2.5).

For an arbitrary \( \mathcal{A} \in GL V \) by \( A_h' \), \( A_g \) denote the matrices of the action induced by \( \mathcal{A} \) on the spaces \( W_{\lambda h} \), \( W_{\lambda g} \) and written in the bases \( \{h'_i\}, \{g_i\} \), that is, \( A_h' = (a'_{ij}), \ A_g = (a''_{ij}) \).
Lemma 2.2. If the matrix $A$ of $\mathcal{A} \in \text{GL } V$ has the form (2.6), then $A_{h'} = A_g$.

For any matrix $A$ of the form (2.6) we have $\text{Ber } A = 1$. Hence, according to Proposition 2.1, Lemma 2.2 is a part of the proof of Theorem 2.1 (see (2.20)).

Proof of Lemma 2.2. By assumption, there exist integers $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ such that $A(e_j) = e_j + \varepsilon_i \eta_{ij}$, where $\eta_{ij} \in G_1$, and $A(w) = w$ for $w \in \Omega \setminus \{e_j\}$.

Let $h_t$, $g_t$ be corresponding elements of the bases $\{h_i\}$, $\{g_i\}$, where $1 \leq t \leq k_A$, $Q_{g_t} = \{j_1, \ldots, j_l\}$ ($Q_{h_t}$) the set of the numbers of $g_t$-columns ($h_t$-columns, respectively) that contain $e_j$. In other words, $r \in Q_{g_t}$ ($Q_{h_t}$) iff the $g_t$-box ($h_t$-box, resp.) $(j, r)$ is filled with $e_j$. Evidently, $Q_{h_t} = Q_{g_t} \cup \{n + 1\}$.

Then we have
\[ A(h_t) = h_t + \sum_{r \in Q_{h_t}} \hat{h}_r \gamma_{j,r} \eta_{ij}, \tag{2.21} \]
where $\hat{h}_r$ is obtained from $h_t$ by the substitution of $\varepsilon_i$ for $e_j$ placed in the $h_t$-box $(j, r)$;
\[ \gamma_{j,r} = (-1)^{n_{j,r}}, \tag{2.22} \]
where $n_{j,r}$ is the number of odd base vectors in the $h_t$-tableau within the interval from the box $(j, r + 1)$ to the box $(m, n + 1)$ (recall that we move by the tableau from left to right and from top to bottom by rows). Clearly, the tensors $\hat{h}_r$ depend on $t$, $i$, $j$. Nonetheless, we omit these indices in the notation to simplify the last one. In general, the tensors $\hat{h}_r$ are not canonical.

Suppose $i \neq Q_{h_t}$, that is, the $h_t$-box $(j, i)$ is filled with $\varepsilon_i$.

Then any one of $\{\hat{h}_r | r \in Q_{h_t}\}$ is not canonical. To express $\hat{h}_r$ in terms of canonical tensors we use the Jacobi identity for the $i$-th column and for the box $(j, r)$ of the $\hat{h}_r$-tableau. Also we use (1.4) to transpose $e_p$ and $\varepsilon_r$ that are placed in the boxes $(j, r)$ and $(p, r)$ after applying (1.3).

Thus we obtain
\[ \hat{h}_r = -q \hat{h}_r + \sum_{p \in P_r} h_{r,p} \beta_{r,p}, \tag{2.23} \]
where $q$ is the number of $\varepsilon_i$ in the $i$-th $\hat{h}_r$-column (or, what is the same, in the $i$-th $\hat{h}_r$-column); $P_r \subseteq \{1, \ldots, m\}$ is the set of integers such that $z \in P_r$ iff $e_z$ belongs to the $i$-th $\hat{h}_r$-column and $e_z$ does not belong to the $r$-th $\hat{h}_r$-column. The tensor $h_{r,p}$ is obtained from $\hat{h}_r$ in the following two steps:

i) the elements $\varepsilon_i$ and $e_p$ disposed in the $\hat{h}_r$-boxes $(j, r)$ and $(p, i)$ (respectively) change places;

ii) the elements $e_p$ and $\varepsilon_r$ disposed in the boxes $(j, r)$ and $(p, r)$ (resp.) change places (see Fig. 1).

$$\beta_{r,p} = (-1)^{\hat{m}_{r,p} + 1},$$

where $\hat{m}_{r,p}$ is the number of pairs of odd elements that change their order under transpositions i) - ii); the additional term $(+1)$ in the exponent appears in the application of (1.4).

It is easy to see that the transformation of $\hat{h}_r$-tableau to $h_{r,p}$-tableau also can be realised in the following two steps: first the elements $\varepsilon_i$ and $\varepsilon_r$ disposed in the boxes $(j, r)$ and $(p, r)$ (respectively) change places; then the elements $\varepsilon_i$ and $e_p$ disposed in the boxes $(p, r)$ and $(p, i)$ (resp.) change places (see Fig. 2).

Then we see that

$$\beta_{r,p} = (-1)^{m_{r,p}},$$

where $m_{r,p}$ is the number of odd base vectors of the $p$-th row that are placed strictly between the $r$-th and the $i$-th columns.

Now it follows from (2.23) that

$$\hat{h}_r = \frac{1}{q + 1} \sum_{p \in P_r} h_{r,p} \beta_{r,p}. \quad (2.24)$$
Notice that the tensors $h_{r,p}$ in (2.24) are canonical. Note also that $P_{n+1} = \emptyset$. Hence using the Jacobi identity we get $\hat{h}_{n+1} = 0$.

Combining (2.24) with (2.21) we get

$$A(h_t) = h_t + \frac{1}{q+1} \sum_{r \in Q} \gamma_{j,r} \sum_{p \in P_r} h_{r,p} \beta_{r,p} \eta_{ij},$$

(2.25)

where $Q = Q_{h_t} \setminus \{n+1\}$.

By assumption, $i \notin Q$. We claim that if $(r_1, p_1) \neq (r_2, p_2)$, then $h_{r_1, p_1} \neq h_{r_2, p_2}$. Indeed, if $r_1 \neq r_2$ then the sets of columns containing $e_j$ are distinct for $h_{r_1, p_1}$ and $h_{r_2, p_2}$. Further, an element $e_p$, where $p \in P_r$, belongs to the $r$-th $h_{r,z}$-column iff $z = p$. Hence, $h_{r, p_1} \neq h_{r, p_2}$ for $p_1 \neq p_2$.

In terms of the “primed” tensors (see (2.3)) equality (2.25) takes the form

$$A(h'_t) = h'_t + \frac{1}{(q+1)\alpha_t} \sum_{r \in Q} \gamma_{j,r} \sum_{p \in P_r} h'_{r,p} \beta_{r,p} \alpha_{r,p} \eta_{ij},$$

(2.26)

where $\alpha_t = \alpha(h_t)$, $\alpha_{r,p} = \alpha(h_{r,p})$ (see (2.2)).

Similarly, still assuming that $i \notin Q$ we get

$$A(g_t) = g_t + \sum_{r \in Q} \hat{g}_r \tilde{\gamma}_{j,r} \eta_{ij},$$

(2.27)

where $\hat{g}_r$ is obtained from $g_t$ by replacing the vector $e_j$ in the box $(j, r)$ to $\varepsilon_i$;

$$\tilde{\gamma}_{j,r} = (-1)^{\bar{n}_{j,r}},$$

(2.28)

where $\bar{n}_{j,r}$ is the number of odd base vectors that are placed after the $g_t$-box $(j, r)$ when we move by the tableau from left to right and from the top down. Since the tableaux of $h_t$ and $g_t$ have the same filling within the small rectangle and the $(m+1)$-th $g_t$-row of length $n$ is filled with the odd base vectors $\varepsilon_1, \ldots, \varepsilon_n$, we have

$$\bar{n}_{j,r} = n_{j,r} + n.$$

(2.29)

With (2.28), (2.29), (2.22) equality (2.27) takes the form

$$A(g_t) = g_t + \sum_{r \in Q} \hat{g}_r \tilde{\gamma}_{j,r} (-1)^n \eta_{ij},$$

where $Q = Q_{g_t} = Q_{h_t} \setminus \{n+1\}$. 13
The tensors $\hat{g}_r$ are not canonical because the element $\varepsilon_i$ is placed not in its “native” $i$-th column but in the $r$-th column ($r \neq i$).

To express $\hat{g}_r$ in terms of canonical tensors we use again the Jacobi identity. By analogy with (2.23), writing this identity for the $i$-th column and the box $(j, r)$ we get

$$\hat{g}_r = -(q + 1)\hat{g}_r + \sum_{p \in P_r} g_{r,p} \bar{\beta}_{r,p},$$ \hspace{1cm} (2.30)

where $g_{r,p}$ is obtained from $\hat{g}_r$ in the same way as $h_{r,p}$ is obtained from $\hat{h}_r$ (see above);

$$\bar{\beta}_{r,p} = (-1)^{\bar{m}_{r,p}},$$

where $\bar{m}_{r,p}$ is defined by analogy with $m_{r,p}$ (see above) but now for $\hat{g}_r$. Since the tableaux of $\hat{h}_r$ and $\hat{g}_r$ have the same filling within the small rectangle, we have $\bar{m}_{r,p} = m_{r,p}$. Hence,

$$\bar{\beta}_{r,p} = \beta_{r,p}.$$

From (2.30) it follows that

$$\hat{g}_r = \frac{1}{q + 2} \sum_{p \in P_r} g_{r,p} \bar{\beta}_{r,p}. \hspace{1cm} (2.31)$$

With (2.31) equality (2.27) takes the form

$$\mathcal{A}(g_t) = g_t + \frac{(-1)^{n}}{q + 2} \sum_{r \in Q} \sum_{p \in P_r} g_{r,p} \bar{\beta}_{r,p} \eta_{ij}. \hspace{1cm} (2.32)$$

We see that the canonical tensors $h_{r,p}$ and $g_{r,p}$ are corresponding in that these have the same filling of the small rectangle.

When (2.26) is compared with (2.32), it is apparent that to conclude the proof it remains to check the equality

$$\frac{\alpha_{r,p}}{\alpha_t} = \frac{(-1)^n(q + 1)}{q + 2}. \hspace{1cm} (2.33)$$

With (2.3) the last equality is equivalent to

$$\frac{(-1)^{np(h_{r,p})} \kappa(h_{r,p}) \kappa(g_t)}{\kappa(g_{r,p})(-1)^{np(h_t)} \kappa(h_t)} = \frac{(-1)^n(q + 1)}{q + 2}. \hspace{1cm} (2.34)$$
First note that $\rho(h_{r,p}) - \rho(h_t) = 1$.

Further, for an arbitrary $z \in \{1, \ldots, n\}$, $z \neq i$ the number of odd base vectors placed in the $z$-th column is the same for $h_t$ and $h_{r,p}$ (also for $g_t$ and $g_{r,p}$). The numbers of odd base vectors in the $i$-th column of $h_t$, $h_{r,p}$, $g_t$, $g_{r,p}$ are equal to $q$, $q + 1$, $q + 1$, $q + 2$ respectively. Then we have

$$\frac{\kappa(h_{r,p})\kappa(g_t)}{\kappa(g_{r,p})\kappa(h_t)} = \frac{(q + 1)!(q + 1)!}{(q + 2)!q!} = \frac{q + 1}{q + 2}.$$  

Hence (2.34) is proved.

Thus Lemma 2.2 is proved for the case when $i \notin Q$.

Now suppose $i \in Q$, that is, the $h_t$-box $(j, i)$ is filled with $e_j$. Still we have (2.21). The tensors $\hat{h}_r$ are not canonical except when $r = i$.

The tensor $\hat{h}_{n+1}$ is not canonical because the element $\varepsilon_i$ is in the box $(j, n + 1)$, but the $(n + 1)$-th column must be filled with even base vectors only. To express $\hat{h}_{n+1}$ in terms of canonical tensors we use the Jacobi identity for the $i$-th column and the box $(j, n + 1)$. In such a manner we get

$$\hat{h}_{n+1} = -q\hat{h}_{n+1} + \bar{h}\beta_{n+1,j}, \quad (2.35)$$

where as above $q$ is the number of $\varepsilon_i$ placed in the $i$-th $h_t$-column; $\bar{h}$ is obtained from $\hat{h}_{n+1}$ by the transposition of $e_j$ and $\varepsilon_i$ placed in the $\hat{h}_{n+1}$-boxes $(j, i)$ and $(j, n + 1)$ (respectively).

Evidently, $\bar{h} = \hat{h}_i$. Also,

$$\beta_{n+1,j} = (-1)^{m_{n+1,j}},$$

where $m_{n+1,j}$ is the number of odd base vectors placed in the $j$-th $\hat{h}_{n+1}$-row strictly between $i$-th and $(n + 1)$-th columns.

Now it follows from (2.35) that

$$\hat{h}_{n+1} = \frac{\beta_{n+1,j}}{q + 1}\hat{h}_i. \quad (2.36)$$

Consider an arbitrary tensor $\hat{h}_r$, where $r \in Q \setminus \{i\}$. We claim that $j \in P_r$, where $P_r$ is the set defined above. Indeed, by assumption the $\hat{h}_r$-box $(j, i)$ is filled with $e_j$, also $\varepsilon_i$ is in the box $(j, r)$ and $e_j$ can be placed only in the $j$-th row in the tableau of $\hat{h}_r$.

By applying the Jacobi identity to the $i$-th column and the box $(j, r)$ of the tableau of $\hat{h}_r$, we get
\[ \hat{h}_r = -q\hat{h}_r + \sum_{p \in P_r\{j\}} h_{r,p}\beta_{r,p} + \hat{h}_i\beta_{r,j}, \]  

(2.37)

where \( h_{r,p}, \beta_{r,p} \) are defined above.

Recall that \( \hat{h}_r \) is canonical iff \( r = i \). By this reason (2.21) is conveniently rewritten in the form:

\[ A(h_t) = h_t + \sum_{r \in Q\{i\}} \hat{h}_r\gamma_{j,r}\eta_{ij} + \hat{h}_i\gamma_{j,i}\eta_{ij} + \hat{h}_{n+1}\gamma_{j,n+1}\eta_{ij}. \]  

(2.38)

With (2.36) and (2.37) equality (2.38) takes the form:

\[ A(h_t) = h_t + \sum_{r \in Q\{i\}} \hat{h}_r\gamma_{j,r}\eta_{ij} + \hat{h}_i\gamma_{j,i}\eta_{ij} + \sum_{r \in Q\{i\}} \frac{\gamma_{j,r}}{q+1} \hat{h}_i\beta_{r,j}\eta_{ij} + \hat{h}_i\gamma_{j,i}\eta_{ij} + \frac{\beta_{n+1,j}}{q+1} \hat{h}_i\gamma_{j,n+1}\eta_{ij}. \]  

(2.39)

Clearly (2.39) is equivalent to

\[ A(h_t) = h_t + \sum_{r \in Q\{i\}} \gamma_{j,r} \sum_{p \in P_r\{j\}} h_{r,p}\beta_{r,p}\eta_{ij} + \hat{h}_i\left( \frac{1}{q+1} \sum_{r \in Q\{i\}} \gamma_{j,r} \beta_{r,j} + \gamma_{j,i} + \frac{\beta_{n+1,j}}{q+1} \gamma_{j,n+1} \right) \eta_{ij}. \]  

(2.40)

Notice that \( h_{r,p} \neq \hat{h}_i \), where \( r \in Q\{i\}, p \in P_r\{j\} \).

We have

\[ \gamma_{j,r}\beta_{r,j} = \gamma_{j,i}, \quad \gamma_{j,n+1}\beta_{n+1,j} = \gamma_{j,i}. \]  

(2.41)

Then the coefficient at \( \hat{h}_i \) in (2.40) is equal to

\[ \frac{\gamma_{j,i}(l + q + 2)}{q + 1}, \]

where \( l = |Q| \). Therefore equality (2.40) takes the form
\[ \mathcal{A}(h_t) = h_t + \frac{1}{q+1} \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \sum_{p \in P \setminus \{j\}} h_{r,p} \beta_{r,p} \eta_{ij} + \frac{l + q + 2}{q+1} \hat{h}_i \eta_{ij}. \]

Going to the “primed” tensors (see (2.3)) we obtain

\[ \mathcal{A}(h'_t) = h'_t + \frac{1}{(q+1)\alpha_t} \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \sum_{p \in P \setminus \{j\}} h'_{r,p} \beta_{r,p} \alpha_{r,p} \eta_{ij} + \frac{l + q + 2}{(q+1)\alpha_t} \hat{h}'_i \hat{\gamma}_{j,i} \hat{\alpha}_t \eta_{ij}, \quad (2.42) \]

where \( \hat{\alpha}_t = \alpha(\hat{h}_i) \).

In its turn by analogy with (2.27) we have

\[ \mathcal{A}(g_t) = g_t + \sum_{r \in Q \setminus \{i\}} \hat{g}_r \gamma_{j,r} \eta_{ij} (-1)^n + \hat{g}_i \gamma_{j,i} \eta_{ij} (-1)^n, \quad (2.43) \]

where \( \hat{g}_r, \gamma_{j,r}, \gamma_{j,i} \) are defined as above for the case when \( i \not\in Q \). An important point is that the tensors \( \hat{g}_r \) are not canonical for \( r \in Q \setminus \{i\} \) and \( \hat{g}_i \) is canonical.

To express an arbitrary tensor \( \hat{g}_r \), where \( r \in Q \setminus \{i\} \), in terms of canonical tensors we use the Jacobi identity for the \( i \)-th column and the box \( (j, r) \). As it is mentioned above the condition \( i \in Q \) implies \( j \in P_r \). Then by analogy with (2.37), (2.30) we get

\[ \hat{g}_r = -(q+1)\hat{g}_r + \sum_{p \in P \setminus \{j\}} g_{r,p} \beta_{r,p} + \hat{g}_i \beta_{r,j}. \]

From the last equality it follows that

\[ \hat{g}_r = \frac{1}{q+2} \sum_{p \in P \setminus \{j\}} g_{r,p} \beta_{r,p} + \frac{1}{q+2} \hat{g}_i \beta_{r,j}. \]

With the last expression for \( \hat{g}_r \) equality (2.43) becomes:

\[ \mathcal{A}(g_t) = g_t + \frac{(-1)^n}{q+2} \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \sum_{p \in P \setminus \{j\}} g_{r,p} \beta_{r,p} \eta_{ij} + \hat{g}_i (-1)^n \left( \frac{1}{q+2} \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \beta_{r,j} + \gamma_{j,i} \right) \eta_{ij}. \quad (2.44) \]

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Using (2.41) we obtain

\[ \frac{1}{q+2} \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \beta_{r,j} + \gamma_{j,i} = \frac{\gamma_{j,i}(l + q + 2)}{q + 2}. \]  

(2.45)

With (2.45) equality (2.44) takes the form

\[ \mathcal{A}(g_t) = g_t + \left( -\frac{1}{q+2} \right) \sum_{r \in Q \setminus \{i\}} \gamma_{j,r} \sum_{p \in P \setminus \{j\}} g_{r,p} \hat{\beta}_{r,p} \eta_{ij} + \hat{g}_i (-1)^n \frac{l + q + 2}{q + 2} \eta_{ij}. \]  

(2.46)

Note that (2.33) is correct as above. Then comparison of (2.42) and (2.46) shows that to complete the proof of the lemma it suffices to check the equality

\[ \hat{\alpha}_i \frac{\hat{\alpha}_t}{\alpha_t} = (-1)^n (q + 1) \frac{q + 2}{q + 2}. \]

But the last one is correct for the same reason as (2.33) is.

Lemma 2.2 is completely proved.

Suppose \( A_{h'} \) and \( A_g \) are as above.

**Lemma 2.3.** If the matrix \( \mathcal{A} \in GL V \) has the form (2.7), then

\[ A_{h'} = A_g. \]

For any matrix \( A \) of the form (2.7) we have \( \text{Ber} A = 1 \). Hence, according to Proposition 2.1, Lemma 2.3 is the next part of the proof of Theorem 2.1 (see (2.20)).

**Proof of Lemma 2.3.** By assumption, there exist \( i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \) such that

\[ \mathcal{A}(\varepsilon_j) = \varepsilon_j + e_i \xi_{ij}, \]

where \( \xi_{ij} \in G_1 \), and for any \( w \in \Omega \setminus \{\varepsilon_j\} \) we have \( \mathcal{A}(w) = w \).

Let \( h_t \) be an element of \( \Lambda_h \). By \( R \) denote the subset of \( \{1, \ldots, m\} \) such that \( r \in R \) if and only if \( \varepsilon_j \) belongs to the box \( (r, j) \) of the \( h_t \)-tableau. Then we have
\[ A(h_t) = h_t + \sum_{r \in R} (-1)^{n_{r,j}} \tilde{h}_r \xi_{ij}, \]  
(2.47)

where \( \tilde{h}_r \) is obtained from \( h_t \) using the replacement of \( \varepsilon_j \) in the box \((r, j)\) by \( e_i \); as above, \( n_{r,j} \) is the number of odd base vectors that are placed from the box \((r, j + 1)\) to the box \((m, n + 1)\) when we move from left to right and from the top down by the rows of the \( h_t \)-tableau.

Suppose \( i \in R \), that is, the \( h_t \)-box \((i, j)\) is filled with \( \varepsilon_j \). Since \( h_t \) is canonical, we see that the \( j \)-th \( h_t \)-column does not contain \( e_i \).

Evidently, the tensor \( \tilde{h}_i \) is canonical. We claim that it is the only canonical tensor among \( \tilde{h}_r \), where \( r \in R \). Indeed, consider the \( j \)-th \( \tilde{h}_r \)-column, where \( r \neq i \). Then \( e_i \) is placed not in its “native” \( i \)-th row but in the \( r \)-th row.

To reduce these tensors to a canonical form we use their skew-symmetry by column elements (see (1.4)). To be precise for a given \( r \in R \) we transpose the vectors \( e_i \) and \( \varepsilon_j \) placed in the \( \tilde{h}_r \)-boxes \((i, j)\) and \((r, j)\) respectively.

As a result of this transposition the additional factor \( (-1)^{m_{r+1}} \) appears, where \( m_r \) is the number of odd base vectors placed strictly between the \( h_t \)-boxes \((r, j)\) and \((i, j)\) through the movement by the rows from left to right and from top to bottom.

Since

\[ (-1)^{m_{r+1}} = (-1)^{n_{i,j} - n_{r,j}} \]

we see that all terms of the sum in the right-hand side of (2.47) are equal to \( (-1)^{n_{i,j}} \tilde{h}_i \xi_{ij} \) and the number of these terms is \( n_R = |R| \). Then equality (2.47) takes the form

\[ A(h_t) = h_t + (-1)^{n_{i,j}} n_R \tilde{h}_i \xi_{ij}. \]

Going to the “primed” tensors we obtain

\[ A(h'_t) = h'_t + (-1)^{n_{i,j}} n_R \tilde{h}'_i \frac{\alpha(\tilde{h}_i)}{\alpha(h_t)} \xi_{ij}. \]  
(2.48)

If \( i \notin R \) then the \( h_t \)-box \((i, j)\) is filled with \( e_i \). As a result all tensors \( \tilde{h}_r \), where \( r \in R \), are equal to zero because the element \( e_i \) enters twice in theirs \( j \)-th column but the tensors are skew-symmetric by elements of an arbitrary column. Thus for the case when \( i \notin R \) we have

\[ A(h_t) = h_t. \]  
(2.49)
Now let $g_t \in \Lambda_g$ be the tensor corresponding to $h_t \in \Lambda_h$ considered above, that is, the tableaux of $h_t$ and $g_t$ have the same filling within the small rectangle. By $\bar{R}$ denote the set of integers such that $r \in \bar{R}$ if and only if the $g_t$-box $(r, j)$ is filled with $\varepsilon_j$. We claim that $\bar{R} = R \cup \{ n+1 \}$. In fact, since the tensor $g_t$ is canonical, we see that the $g_t$-box $(m+1, j)$ is filled with $\varepsilon_j$ and the tableaux of $h_t$ and $g_t$ are filled equally within the small rectangle.

Then we have
\begin{equation}
A(g_t) = g_t + \sum_{r \in \bar{R}} (-1)^{\bar{n}_{r,j}} \tilde{g}_r \xi_{ij},
\end{equation}
where $\tilde{g}_r$ is obtained from $g_t$ by the replacement of the element $\varepsilon_j$ disposed in the $g_t$-box $(r, j)$ to $e_i$; $\bar{n}_{r,j}$ is the number of odd base vectors placed in the $g_t$-tableau strictly after the box $(r, j)$ when we move by the rows from left to right and from top to bottom.

Evidently, we have $\bar{n}_{r,j} = n_{r,j} + n$ for $r \leq m$ and $\bar{n}_{m+1,j} = n - j$. Then separating the term for $r = m + 1$ in right-hand side of (2.50) we obtain
\begin{equation}
A(g_t) = g_t + \sum_{r \in R} (-1)^{n_{r,j}+n} \tilde{g}_r \xi_{ij} + (-1)^{n-j} \tilde{g}_{m+1} \xi_{ij}.
\end{equation}

Suppose that $i \in \bar{R}$, that is, the element $\varepsilon_j$ is placed in the $g_t$-box $(i, j)$ and the element $e_j$ does not enter in the $j$-th $g_t$-column.

The tensor $\tilde{g}_i$ is canonical and the rest tensors $\tilde{g}_r$, where $r \in \bar{R}\{i\}$, are not canonical because the element $e_i$ is not placed in its “native” $i$-th row for these tensors.

To reduce a tensor $\tilde{g}_r$, where $r \in \bar{R}\{i\}$, to a canonical form we use skew-symmetry of the tensor by elements of the column (see (1.4)).

Then for any $r \in \bar{R}\{i\}$ we obtain the same tensor $\tilde{g}_i$ with the additional factor $(-1)^{\bar{m}_{r}}$, where $\bar{m}_r$ is the number of odd base vectors placed strictly between the boxes $(r, j)$ and $(i, j)$ (as above, we move from one box to another by the rows from left to right and from top to bottom). Since
\begin{equation}
(-1)^{\bar{m}_{r+1}} = (-1)^{n_{r,j}-n_{i,j}}
\end{equation}
for $r \leq m$ and
\begin{equation}
(-1)^{\bar{m}_{m+1}} = (-1)^{n_{i,j}+j},
\end{equation}
we see that all terms (distinct from $g_t$) in the right-hand side of (2.51) are equal to $(-1)^{n_{i,j}+n} \tilde{g}_i \xi_{ij}$ and the number of these terms is equal to $|\bar{R}| = n_R + 1$.  

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Then equality (2.50) takes the form
\[ A(g_t) = g_t + (-1)^{n_{i,j} + n}(n_R + 1)\tilde{g}_i\xi_{ij}. \] (2.52)

Comparison of (2.48) and (2.52) shows that to complete the proof of the lemma it suffices to check the equality
\[ \frac{\alpha(\tilde{h}_i)}{\alpha(h_t)} = \frac{(-1)^n(n_R + 1)}{n_R}. \] (2.53)

By definition (see (2.2)) we have
\[ \frac{\alpha(\tilde{h}_i)}{\alpha(h_t)} = \frac{(-1)^{n\rho(\tilde{h}_i)}\kappa(\tilde{h}_i)\kappa(g_t)}{\kappa(\tilde{g}_i)(-1)^{n\rho(h_t)}\kappa(h_t)}. \]

When passing from \( h_t \) to \( \tilde{h}_i \) the number of odd base vectors within the small rectangle increases on 1, that is, \( \rho(h_t) - \rho(h_i) = 1 \).

The numbers of odd base vectors are the same for all respective columns of \( h_t \) and \( \tilde{h}_i \) (\( g_t \) and \( \tilde{g}_i \)) except the \( j \)-th one. Also, the numbers of odd base vectors in the \( j \)-th column of \( h_t, \tilde{h}_i, g_t, \tilde{g}_i \) are equal to \( n_R, n_R - 1, n_R + 1, n_R \) (respectively).

Hence we have
\[ \frac{\kappa(\tilde{h}_i)\kappa(g_t)}{\kappa(\tilde{g}_i)\kappa(h_t)} = \frac{(n_R - 1)!(n_R + 1)!}{(n_R!)^2} = \frac{n_R + 1}{n_R}. \]

and equality (2.53) is proved.

Suppose that \( i \in \tilde{R} \), that is, the box \((i, j)\) is filled with \( e_i \). By the same argument as above we get \( A(g_t) = g_t \). Also from (2.49) it follows that \( A(h_t') = h_t' \). Thus the case when \( i \in \tilde{R} \) is trivial.

This concludes the proof of Lemma 2.3.

A further step in the proof of Theorem 2.1 is the following lemma.

**Lemma 2.4.** *If the matrix \( A \) of \( A \in GL V \) has the form (2.10), then \( A h_t' = A g_t \).*

**Proof of Lemma 2.4.** By assumption, there exist unequal integers \( s, t \in \{1, \ldots, n\} \) such that
\[ A(\varepsilon_s) = \varepsilon_s + \varepsilon_{ty}, \]

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where \( y \in G \), and for any \( w \in \Omega \setminus \{\varepsilon_s\} \) we have \( A(w) = w \).

Let \( h \) be an element of \( \Lambda_h \). By \( F \) denote the set of integers such that an integer \( r \) belongs to \( F \) if and only if the \( h \)-box \((r, s)\) is filled with \( \varepsilon_s \). Suppose \( F \neq \emptyset \). Denote
\[
F_j = \{ \nu | \nu \subseteq F, |\nu| = j \},
\]
where \( j = 0, 1, \ldots, n_F, n_F = |F| \), that is, \( F_j \) is the set of all subsets of \( F \) with \( j \) elements. Then we have
\[
A(h) = h + \sum_{j=1}^{n_F} \sum_{\nu \in F_j} h_{j,\nu} y^j,
\]
where \( h_{j,\nu} \) is obtained from \( h \) using the replacement of \( \varepsilon_s \) placed in the \( h \)-boxes \((i, s)\), where \( i \in \nu \), by \( \varepsilon_t \).

For any \( j > 0 \) an arbitrary element \( h_{j,\nu} \) is not canonical, because \( j \) elements of the form \( \varepsilon_t \) are placed not in their “native” \( t \)-th column but in the \( s \)-th column. For an arbitrary \( h_{j,\nu} \) by \( P(h_{j,\nu}) \) denote the set of integers such that \( p \in P(h_{j,\nu}) \) if and only if \( e_p \) belongs to the \( t \)-th \( h_{j,\nu} \)-column and \( e_p \) does not belong to the \( s \)-th \( h_{j,\nu} \)-column. In other words, \( p \in P(h_{j,\nu}) \) iff the \( h_{j,\nu} \)-box \((p, t)\) is filled with \( e_p \) but the \( h_{j,\nu} \)-box \((p, s)\) is filled either with \( \varepsilon_s \) or with \( \varepsilon_t \). Notice that for all \( j > 0 \), \( \nu \in F_j \) the set \( P(h_{j,\nu}) \) is the same, that is, the composition of \( P(h_{j,\nu}) \) does not depend on \( j \) and \( \nu \). For this reason in what follows we write simply \( P \) instead of \( P(h_{j,\nu}) \).

Denote
\[
P_j = \{ \mu | \mu \subseteq P, |\mu| = j \},
\]
where \( j = 0, 1, \ldots, n_F \), that is, \( P_j \) is the set of all subsets of \( P \) with \( j \) elements.

We claim that
\[
h_{j,\nu} = \frac{1}{C_j} \sum_{\mu \in P_j} \delta_{\mu} \overline{h}_{j,\nu},
\]
where \( q \) is the number of odd base vectors (i.e., the number of \( \varepsilon_t \)) in the \( t \)-th column of \( h \) and \( \overline{h}_{j,\nu} \) is obtained from \( h_{j,\nu} \) as follows: for every \( z \in \mu \) the elements \( e_z \) and \( \varepsilon_t \) placed in the \( h_{j,\nu} \)-boxes \((z, t)\) and \((z, s)\) (respectively) change their places; \( \delta_{\mu} = (-1)^{m_{\mu}}, m_{\mu} = \sum_{z \in \mu} m_z, \) where \( m_z \) is the number of odd base vectors placed in the \( z \)-th \( \overline{h}_{j,\nu} \)-row strictly between the boxes \((z, t)\) and \((z, s)\). Note that we can assume that the \( h_{j,\nu} \)-box \((z, s)\) is filled with \( \varepsilon_t \). Indeed, otherwise the box \((z, s)\) is filled with \( \varepsilon_s \) and we use (1.4) to transpose \( \varepsilon_s \) and \( \varepsilon_t \) in the \( s \)-th column. Since the transposed elements are odd, we see that the additional sign does not appear.

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Now let us prove identity (2.55). To express $h_{j,\nu}$ in terms of canonical tensors we apply the Jacobi identity step by step. In any step an element $\varepsilon_t$ from the $s$-th column is moved to the $t$-th column. Also the application of the Jacobi identity in the next step brings the additional factor $1/(r + 1)$, where $r$ is the number of elements $\varepsilon_t$ in the $t$-th column in the previous step. Thus at last the factor

$$\frac{1}{(q + 1)(q + 2) \cdots (q + j)}$$

arises. Further, there is one-to-one correspondence between the set of canonical tensors obtained in the last step and the set of all subsets of $P$ with $j$ elements. Also, any canonical tensor enters $j!$ times in the final expression for $h_{j,\nu}$, because an element $e_z, z \in P$, can come to the $s$-th column in any step of the algorithm and the number of these steps is equal to $j$.

Notice that the right-hand side of (2.55) depends on $j$ and does not depend on $\nu$. In other words, all tensors $h_{j,\nu}$, where $\nu \in F_j$, have the same representation in terms of the canonical tensors. Then with (2.55) equality (2.54) takes the form

$$A(h) = h + \sum_{j=1}^{n_F} C_j^{n_F} \sum_{\mu \in P_j} \delta_{\mu} \bar{h}_{j,\mu} y^j$$

or in terms of the “primed” tensors we have

$$A(h') = h' + \frac{1}{\alpha(h)} \sum_{j=1}^{n_F} C_j^{n_F} \frac{\alpha(h_{j,\mu})}{C_{q+j}^{n_F}} \sum_{\mu \in P_j} \delta_{\mu} \bar{h}'_{j,\mu} y^j$$

where $h' = \frac{h}{\alpha(h)}$, $\bar{h}'_{j,\mu} = \frac{h_{j,\mu}}{\alpha(h_{j,\mu})}$.

Now let $g \in \Lambda_g$ be the tensor corresponding to $h$ and $A \in GL V$ as above. By $\bar{F}$ denote the set of integers such that $r \in \bar{F}$ if and only if the $g$-box $(r, s)$ is filled with $\varepsilon_s$. Any column of $g$ as compared with $h$ contains the additional box filled with a proper odd base element (to be precise the $g$-box $(m + 1, s)$ is filled with $\varepsilon_s$, where $s = 1, 2, \ldots, n$). Hence we have

$$\bar{F} = F \cup \{m + 1\}$$

and $n_F = |\bar{F}| = n_F + 1$. Define
\[ F_j = \{ \nu | \nu \subseteq F, |\nu| = j \}, \]
where \( j = 0, 1, \ldots, n_F \), that is, \( F_j \) is the set of all subsets of \( F \) with \( j \) elements. Then we have
\[
A(g) = g + \sum_{j=1}^{n_F+1} \sum_{\nu \in F_j} g_{j,\nu} y^j, \tag{2.57}
\]
where \( g_{j,\nu} \) are obtained from \( g \) just as \( h_{j,\nu} \) are obtained from \( h \) (see above).

Since the tableaux of \( h \) and \( g \) have the same filling within the small rectangle and the \((m + 1)\)-th \( g \)-row is filled with odd base vectors, we see that \( P(h_{j,\nu}) = P(g_{j,\nu}) \). Therefore we write \( P \) as above instead of \( P(g_{j,\nu}) \).

Let \( P_j \) be as above. We see that \( \delta_\mu \), where \( \mu \in P_j \), does not depend on either \( h_{j,\nu} \) or \( g_{j,\nu} \) is considered.

We have
\[
g_{j,\nu} = \frac{1}{C_{q+j+1}^{\nu \in P_j}} \sum_{\mu \in P_j} \delta_\mu \tilde{g}_{j,\mu}, \tag{2.58}
\]
where \( \tilde{g}_{j,\mu} \) are obtained from \( g_{j,\nu} \) just as \( \tilde{h}_{j,\mu} \) are obtained from \( h_{j,\nu} \). Indeed, by analogy with (2.55) equality (2.58) can be obtained and the only difference between these two cases is that \( g_{j,\nu} \) has not \( q \) but \((q + 1)\) elements \( \varepsilon_t \) in the \( t \)-th column. That is why the coefficient in (2.58) is equal not to \( 1/C_{q+j}^{j} \) (as in (2.56)) but to \( 1/C_{q+j+1}^{j} \).

With (2.58) equality (2.57) takes the form
\[
A(g) = g + \sum_{j=1}^{n_F} \frac{C_{q+j}^{\nu \in P_j}}{C_{q+j+1}^{\nu \in P_j}} \sum_{\mu \in P_j} \delta_\mu \tilde{g}_{j,\mu} y^j + g_{F} y^{n_F+1}, \tag{2.59}
\]
where \( g_{F} \) is obtained from \( g \) using the replacement of all \( \varepsilon_s \) in the \( s \)-th \( g \)-column by \( \varepsilon_t \). Let us show that
\[
g_{F} = 0. \tag{2.60}
\]

Indeed, applying the Jacobi identity we represent \( g_{F} \) as a linear combination of tensors such that theirs \( s \)-th column is filled with even base vectors only. But the height of the \( s \)-th column is greater by 1 than the number \( m \) of even base vectors. Since any tensor of the form (1.3) is skew-symmetric by elements of any column we get (2.60).
Comparison of \( (2.56) \) and \( (2.59) \) shows that to complete the proof it suffices to check the equality

\[
\frac{\alpha(\tilde{h}_{j,\mu})}{\alpha(h)} = \frac{C^j_{q+j} C^j_{n_F+1}}{C^j_{n_F} C^j_{q+j+1}}
\]

or, what is the same –

\[
\frac{\alpha(\tilde{h}_{j,\mu})}{\alpha(h)} = \frac{(n_F + 1)(q + 1)}{(n_F - j + 1)(q + j + 1)}.
\]

We have:

|                     | the numbers of odd base vectors |
|---------------------|--------------------------------|
|                     | \( s \)-th column | \( t \)-th column |
| \( h \)            | \( n_F \)          | \( q \)           |
| \( g \)            | \( n_F + 1 \)      | \( q + 1 \)       |
| \( h_{i,\nu} \)    | \( n_F - j \)      | \( q + j \)       |
| \( \tilde{g}_{j,\nu} \) | \( n_F - j + 1 \) | \( q + j + 1 \)   |

Also note that the number of odd base vectors within the small rectangle is the same for the tableaux of \( h \) and \( \tilde{h}_{j,\mu} \), that is, we have \( \rho(\tilde{h}_{j,\mu}) = \rho(h) \). Hence we obtain

\[
\frac{\alpha(\tilde{h}_{j,\mu})}{\alpha(h)} = \frac{\kappa(g)\kappa(\tilde{h}_{j,\mu})}{\kappa(h)\kappa(\tilde{g}_{j,\mu})} = \frac{(n_F + 1)!(q + 1)!(n_F - j)!(q + j)!(n_F - j + 1)!(q + j + 1)!}{n_F!q!(n_F - j + 1)!(q + j + 1)!}
\]

and we see that \( (2.61) \) is proved.

This concludes the proof of Lemma 2.4.

Actually the following lemma is the last step in the proof of Theorem 2.1.

**Lemma 2.5.** If the matrix \( A \) of \( A \in GL V \) has the form \( (2.9) \), then

\[
A_{h'} = A_g.
\]

**Proof.** By assumption, there exist integers \( s, t \in \{1, 2, \ldots, m\}, s \neq t \) and \( x \in G_0 \) such that

\[
A(e_s) = e_s + e_t x,
\]

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and for any \( w \in \Omega \setminus \{e_s\} \) we have \( A(w) = w. \)

Let \( h \) be an element of the set \( \Lambda_h \). By \( H \) denote the set of integers such that \( r \in H \) if and only if the \( h \)-boxes \( (s, r) \) and \( (t, r) \) are filled with \( e_s \) and \( e_t \) (respectively). Evidently we have \( (n + 1) \notin H \). By \( H_j \) denote the set of all subsets of \( H \) with \( j \) elements, where \( j = 1, 2, \ldots, k_H, k_H = |H| \).

We have

\[
A(h) = h + \sum_{j=1}^{k_H} \sum_{\mu \in H_j} h_\mu x^j,
\]

where \( h_\mu \) is obtained from \( h \) when all elements \( e_s \) placed in the \( h \)-boxes \( (s, r) \), where \( r \in \mu \), are replaced by \( e_t \).

An arbitrary tensor \( h_\mu \) is not canonical in so far as the \( h_\mu \)-boxes \( (s, r) \), where \( r \in \mu \), are filled with \( e_t \) but \( t \neq s \).

To reduce a tensor \( h_\mu \), where \( \mu \in H_j \), to a canonical form we use the skew-symmetry of this tensor by the elements of the columns. To be precise, for every \( r \in \mu \) we transpose the vectors \( e_t \) and \( e_r \) placed in the \( h_\mu \)-boxes \( (s, r) \) and \( (t, r) \) respectively. The sign that appears as a result of this permutation we denote by \( \sigma_\mu \).

The canonical tensor obtained from \( h_\mu \) is denoted by \( \bar{h}_\mu \). Then equality (2.62) takes the form

\[
A(h) = h + \sum_{j=1}^{k_H} \sum_{\mu \in H_j} \sigma_\mu \bar{h}_\mu x^j,
\]

and in terms of the “primed” tensors we have

\[
A(h') = h' + \frac{1}{\alpha(h)} \sum_{j=1}^{k_H} \sum_{\mu \in H_j} \sigma_\mu \bar{h}_\mu' \alpha(\bar{h}_\mu) x^j,
\]

where \( h' = h/\alpha(h) \), \( \bar{h}_\mu' = \bar{h}_\mu/\alpha(\bar{h}_\mu) \).

Note that for an arbitrary column the number of odd base vectors is the same for the tableaux of the tensors \( h, h_\mu \) and \( \bar{h}_\mu \). Hence we have \( \alpha(\bar{h}_\mu) = \alpha(h) \), where \( \mu \in H_j, j = 1, 2, \ldots, k_H \). Thus (2.63) becomes

\[
A(h') = h' + \sum_{j=1}^{k_H} \sum_{\mu \in H_j} \sigma_\mu \bar{h}_\mu' x^j.
\]
Now suppose $g$ is the tensor corresponding to $h$, that is, $g$ and $h$ have the same filling within the small rectangle. Evidently the set $H$ is the same for the tensors $h$ and $g$. Then we get

$$A(g) = g + \sum_{j=1}^{k_H} \sum_{\mu \in H_j} g_{\mu} x^j,$$

where $g_{\mu}$ is obtained from $g$ much as $h_{\mu}$ is obtained from $h$. The tensors $g_{\mu}$ are reduced to a canonical form by the same process as $h_{\mu}$ are. Then we obtain

$$A(g) = g + \sum_{j=1}^{k_H} \sum_{\mu \in H_j} \tilde{\sigma}_\mu \tilde{g}_{\mu} x^j,$$  \hspace{1cm} (2.65)

where $\tilde{g}_{\mu}$ are canonical tensors and we claim that $\tilde{\sigma}_\mu = \sigma_\mu$. Indeed, $g_{\mu}$-boxes touched by the permutation that reduces $g_{\mu}$ to a canonical form are placed strictly within the small rectangle.

Comparison of (2.64) and (2.65) shows that Lemma 2.5 is proved.

The case when the matrix $A$ of $A$ has the form (2.8) is trivial.

Thus we see that Theorem 2.1 is completely proved.

Let $b_*$ be a formal element such that for any $A \in GL V$ we have

$$A(b_*) = (\text{Ber } A)^{-1} \cdot b_*,$$

where $A$ is the matrix of $A$.

**Theorem 2.2.** The mapping

$$\varphi_* : \text{b}_* \cdot h'_i \mapsto g_i$$

determines the isomorphism of $GL V$-modules $\text{b}_* \cdot W_{\lambda_h}$ and $W_{\lambda_g}$.

**Proof.** Follows immediately from Theorem 2.1.

### 3 An explicit construction of the one-dimensional representations of the general linear supergroup

By definition, put $\vartheta_i = e_i$ and $\vartheta_j = \varepsilon_{j-m}$, where $i = 1, 2, \ldots, m$, $j = m+1, m+2, \ldots, m+n$. The last notation gives the through enumeration of the set $\Omega$. 

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By \( V^* \) denote the \( G \)-module dual to \( V \).

Let \( l \) be a positive integer.

Suppose \( u_i^*, w_j \) are arbitrary homogeneous elements of \( V^* \) and \( V \) (respectively), where \( i, j = 1, 2, \ldots, l \). By definition, for the decomposable tensors \( u_1^* \cdots u_l^* \in T_l(V^*), w_1 \cdots w_l \in T_l(V) \) put

\[
(u_1^* \cdots u_l^*, w_1 \cdots w_l) = (-1)^\chi(u_1^*, w_1) \cdots (u_l^*, w_l),
\]

where \( \chi \) is the number of pairs of odd elements that change their order when one pass from the sequence \( u_1^*, \ldots, u_l^*, w_1, \ldots, w_l \) to the sequence \( u_1^*, w_1, \ldots, u_l^*, w_l \). Equality (3.1) determines the inclusion

\[
T_l(V^*) \rightarrow (T_l(V))^*.
\]

By definition, for arbitrary \( u_1^*, \ldots, u_l^* \in V^*, \sigma \in S_l \) put

\[
(u_1^* \cdots u_l^*)\sigma = u_{\sigma(1)}^* \cdots u_{\sigma(l)}^*.
\]

Thus \( T_l(V^*) \) is the right \( S_l \)-module. With (1.2), (3.1), (3.2) we get

\[
((u_1^* \cdots u_l^*)\sigma, w_1 \cdots w_l) = (u_1^* \cdots u_l^*, \sigma(w_1 \cdots w_l)),
\]

where as above \( u_i^* \in V^*, w_j \in V \).

By \( \vartheta_i^* \) denote the elements of \( V^* \) dual to \( \vartheta_j \), that is,

\[
(\vartheta_i^*, \vartheta_j) = \delta_{ij},
\]

where \( i, j = 1, 2, \ldots, m + n \), \( \delta_{ij} \) is the Kronecker delta. Clearly, \( e_i^* = \vartheta_i^*, e_j^* = \vartheta_j^* \), \( e_j^* = \vartheta_{j+m}^* \), where \( i = 1, \ldots, m, j = 1, \ldots, n \). By analogy with the notation used above, \( \Omega^*_0 = \{e_1^*, \ldots, e_m^*\}, \Omega^*_1 = \{e_1^*, \ldots, e_m^*\}, \Omega^* = \Omega^*_0 \cup \Omega^*_1 \).

We say that \( f \in T_l(V^*) \) is an \( \Omega^* \)-tensor if there exist \( u_1^*, \ldots, u_l^* \in \Omega^* \) and a Young tableau \( T \) such that

\[
f = (u_1^* \cdots u_l^*)e_T.
\]

Let \( \Lambda_g \) be as above (see §1), \( g_i \) elements of \( \Lambda_g \) in some ordering of the last one, where \( i = 1, 2, \ldots, k_\Lambda \), \( k_\Lambda = 2^m \). For an arbitrary \( i \in \{1, \ldots, k_\Lambda \} \) by \( g_i^* \) denote the \( \Omega^* \)-tensor such that \( g_i^*- \)tableau is obtained from \( g_i \)-tableau by the formal change \( \vartheta_j \mapsto \vartheta_j^* \), where \( j = 1, 2, \ldots, (m + n) \).

By \( k_i \) denote the cardinality of the automorphism group of the \( g_i \)-tableau and by \( q_i \) denote the number of odd base elements in the \( g_i \)-tableau.
Lemma 3.1. For arbitrary $i, j \in \{1, 2, \ldots, k_\Lambda\}$ the following equality holds
\[
(g_i^*, g_j) = \delta_{ij}\zeta_i, \tag{3.4}
\]
where
\[
\zeta_i = k_i \mu_{\lambda_g} (-1)^{(q_i^2 - q_i)}/2, \\
\mu_{\lambda_g} = \frac{\prod_{t=1}^{m+1} (m + n + 1 - t)!}{\prod_{t=1}^{m+1} (m + 1 - t)!}. \tag{3.5}
\]

Proof. By definition, there exist $u_1^*, \ldots, u_l^* \in \Omega^*$, $w_1, \ldots, w_l \in \Omega$ such that
\[
g_i^* = (u_1^* \cdots u_l^*)e_T, \quad g_j = e_T(w_1 \cdots w_l),
\]
where $T$ is the Young tableau with the diagram $\lambda_g = (n, \ldots, n)_{m+1}, l = (m+1)n$.

With (3.3) we get
\[
(g_i^*, g_j) = ((u_1^* \cdots u_l^*)e_T, e_T(w_1 \cdots w_l)) = (u_1^* \cdots u_l^*, e_T^2(w_1 \cdots w_l)).
\]

For an arbitrary Young diagram $\lambda$ the following identity holds
\[
e_{T_\lambda}^2 = \mu_\lambda e_{T_\lambda},
\]
where $T_\lambda$ is a Young tableau with the diagram $\lambda$, $\mu_\lambda$ is a non-zero integer. In particular, $\mu_{\lambda_g}$ is given by (3.5) (see [6]). By assumption, $u_1^*, \ldots, u_l^*, w_1, \ldots, w_l$ fill the diagram $\lambda_g$ in a canonical way. Then to conclude the proof it suffices to use Lemma 1.1 from [2].

Thus Lemma 3.1 is proved.

An arbitrary linear transformation $A \in GLV$ determines, as it usually does, the corresponding linear transformation of $V^*$ by the equalities
\[
(A(\vartheta_i^*), A(\vartheta_j^*)) = (\vartheta_i^*, \vartheta_j^*) = \delta_{ij} \tag{3.6}
\]
(this transformation of $V^*$ we still denote by $A$).

Suppose $A$ is the matrix of $A$, that is, we have
\[
(A(\vartheta_1), \ldots, A(\vartheta_{m+n})) = (\vartheta_1, \ldots, \vartheta_{m+n}) \cdot A. \tag{3.7}
\]
Then with (3.6) we obtain
\[
\begin{pmatrix}
A(\vartheta_1^*) \\
A(\vartheta_2^*) \\
\vdots \\
A(\vartheta_{m+n}^*)
\end{pmatrix} = A^{-1} \cdot
\begin{pmatrix}
\vartheta_1^* \\
\vartheta_2^* \\
\vdots \\
\vartheta_{m+n}^*
\end{pmatrix}.
\tag{3.8}
\]

Denote
\[g_i'' = \frac{1}{(g_i^*, g_i)} g_i^*.
\]
From (3.4) it follows that
\[(g_i'', g_j) = \delta_{ij}.
\]
Recall that \(A\) is even. Then it follows from (3.1) and (3.6) that
\[(A(g_i''), A(g_j)) = (g_i'', g_j) = \delta_{ij}.
\]
In other words, the action of \(A\) on \(\langle g_i \rangle\) and \(\langle g_i'' \rangle\) is the particular case of changing base in a space and the dual one (see (3.7), (3.8)).

Consequently we have
\[
\sum_{i=1}^{k_\Lambda} A(g_i) A(g_i'') = (A(g_1), \ldots, A(g_{k_\Lambda})) \cdot
\begin{pmatrix}
A(g_1'') \\
\vdots \\
A(g_{k_\Lambda}'')
\end{pmatrix} =
\]
\[
= (g_1, \ldots, g_{k_\Lambda}) \tilde{A} A^{-1} \left( \begin{array}{c}
g_1'' \\
\vdots \\
g_{k_\Lambda}''
\end{array} \right) = (g_1, \ldots, g_{k_\Lambda}) \cdot
\begin{pmatrix}
g_1'' \\
\vdots \\
g_{k_\Lambda}''
\end{pmatrix} = \sum_{i=1}^{k_\Lambda} g_i g_i''.
\]

This completes the proof of the following result:
**Proposition 3.1.** The action of an arbitrary \(A \in \text{GLV}\) on the tensor
\[
\sum_{i=1}^{k_\Lambda} g_i g_i''
\]
is identical.

**Theorem 3.1.** The tensor
\[
\tilde{b} = \sum_{i=1}^{k_\Lambda} h_i g_i''
\]
is identical.
generates the one-dimensional $GL_V$-module such that 
\[ \mathcal{A}(\tilde{b}) = \text{Ber } A \cdot \tilde{b}, \]
where $A$ is a matrix of $A \in GL_V$.

**Proof.**Follows immediately from Theorem 2.1 and Proposition 3.1. Consider some examples.

**Example 3.1.** Suppose $n = 0$. In this case one can say that the small rectangle degenerates into the segment of height $m$. Then $\tilde{b} = \gamma h_1$, where $h_1$ is the tensor skew-symmetric in $e_1, \ldots, e_m$; $\gamma$ is an invertible element of $G_0$. Also, for $n = 0$ we have $\text{Ber } A = \det A$. Thus we arrive at the basic classic result.

**Example 3.2.** Let the dimension of $V$ be $1|1$; $V_0 = \langle \epsilon \rangle$, $V_1 = \langle \varepsilon \rangle$. By $\mathcal{T}_{h_i}$ ($\mathcal{T}_{g_i}$) denote the $h_i$-tableau ($g_i$-tableau, respectively). Then we have 
\[ \mathcal{T}_{h_1} = \begin{array}{c} e \\ e \end{array}, \quad \mathcal{T}_{h_2} = \begin{array}{c} \varepsilon \\ e \end{array}, \]
\[ h_1 = (e + (12))ee = 2ee, \]
\[ h_2 = (e + (12))\varepsilon e = \varepsilon e + \varepsilon e, \]
where $e$ is the unit of $S_2$. With (2.2) and (2.3) we get 
\[ h_1' = 2ee, \quad h_2' = -2(\varepsilon e + \varepsilon e). \quad (3.9) \]
Further, we have 
\[ \mathcal{T}_{g_1} = \begin{array}{c} e \\ \varepsilon \end{array}, \quad \mathcal{T}_{g_2} = \begin{array}{c} \varepsilon \\ \varepsilon \end{array}, \]
\[ q_1 = k_1 = 1, \quad q_2 = k_2 = 2, \quad \mu_{\lambda_g} = 2. \quad \text{Hence, } \zeta_1 = 2, \quad \zeta_2 = -4. \text{ Therefore,} \]
\[ g_1'' = \frac{1}{2}(e^*\varepsilon^* (e - (12))) = \frac{1}{2}(e^*\varepsilon^* - \varepsilon^* e^*), \quad (3.10) \]
\[ g_2'' = -\frac{1}{4}(\varepsilon^* e^* (e - (12))) = -\frac{1}{2}e^* e^*. \quad (3.11) \]
Now with (3.9), (3.10), (3.11) we obtain 
\[ \tilde{b} = \sum_{i=1}^{2} h_i' g_i'' = eee^* \varepsilon^* - eee^* e^* + e\varepsilon e^* e^* + e\varepsilon e^* \varepsilon^*. \]
Let $\Lambda_h = \{h_i\}$ be as above (see §1). For an arbitrary $i \in \{1, 2, \ldots, k_\Lambda\}$ by $h_i^*$ denote the $\Omega$-tensor such that the $h_i^*$-tableau is obtained from the $h_i$-tableau by the formal change $\vartheta_j \mapsto \vartheta_j^*$, where $j = 1, 2, \ldots, (m + n)$.

By $l_i$ denote the cardinality of the automorphism group of the $h_i$-tableau and by $p_i$ denote the number of odd elements of the $h_i$-tableau.

Then by analogy with Lemma 3.1 we obtain:

**Lemma 3.2.** For arbitrary $i, j \in \{1, 2, \ldots, k_\Lambda\}$ the following equality holds

$$(h_i^*, h_j) = \delta_{ij} \zeta_i',$$

where

$$\zeta_i' = l_i \mu_{\lambda_h}(-1)^{(\mu_1 - p_i)/2},$$

$$\mu_{\lambda_h} = \frac{\prod_{t=1}^m (m + n + 1 - t)!}{\prod_{t=1}^m (m - t)!}.$$ 

By definition, put

$$h_i^{**} = \frac{\alpha(h_i)}{(h_i^*, h_i)} h_i^*.$$ 

With (2.3) we have

$$(h_i^{**}, h_j') = \delta_{ij}.$$ 

By analogy with Proposition 3.1 and Theorem 3.1 we arrive at the following results:

**Proposition 3.2.** The action of an arbitrary $A \in GL V$ on the tensor

$$\sum_{i=1}^{k_\Lambda} h'_i h_i^{**}$$

is identical.

**Theorem 3.2.** The tensor

$$\tilde{b}_s = \sum_{i=1}^{k_\Lambda} g_i h_i^{**}$$

is identical.
generates the one-dimensional GLV-module such that
\[ A(\tilde{b}_s) = (\text{Ber} A)^{-1} \cdot \tilde{b}_s, \]
where \( A \) is a matrix of \( A \in \text{GL} V \).

**Example 3.3.** Let the dimension of \( V \) be \( 1|1; V_0 = \langle e \rangle, V_1 = \langle \varepsilon \rangle \). Then we have

\[ h_1^* = e^* e^* (e + (12)) = 2e^* e^*, \]
\[ h_2^* = \varepsilon^* \varepsilon^* (e + (12)) = \varepsilon^* e^* + e^* \varepsilon^* \]
(see the \( h_i \)-tableaux \( T_{h_i} \) in Example 3.2). Also, \( l_1 = 2, p_1 = 0, l_2 = p_2 = 1, \mu_{\lambda_h} = 2 \). Hence, \( \zeta_1' = 4, \zeta_2' = 2 \).

With \( \langle 2.3 \rangle \) we get \( \alpha(h_1) = 1, \alpha(h_2) = -\frac{1}{2} \). Thus we obtain
\[ h_1^{*'} = \frac{1}{2}e^* e^*, \quad h_2^{*'} = -\frac{1}{4}(\varepsilon^* e^* + e^* \varepsilon^*). \]

Further,
\[ g_1 = (e - (12))\varepsilon\varepsilon = e\varepsilon - \varepsilon e, \]
\[ g_2 = (e - (12))\varepsilon\varepsilon = 2\varepsilon\varepsilon \]
(see the \( g_i \)-tableaux \( T_{g_i} \) in Example 3.2). At last, we obtain
\[ \tilde{b}_s = \sum_{i=1}^{2} g_i h_i^{*'} = \frac{1}{2}(\varepsilon e^* e^* - e\varepsilon e^* - \varepsilon \varepsilon e^* - \varepsilon e e^*). \]

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