Lax and Phillips transformations for a sphere and two-point mean value formulas

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Abstract. We consider transformations, connecting the wave equation in Euclidean space and the wave equation over a sphere that has a metric of a hemisphere immersed in Euclidean space. Using these transformations, we prove a two-point mean value theorem for an elliptic equation on a sphere.

1. Introduction
This work is aimed at mean value formulas for solutions of an elliptic equation over a sphere. In various branches of pure and applied analysis, the terms ”mean value theorem” and ”mean value formula” may pertain to diverse phenomena. Nevertheless, various results for different types of equations have something in common, namely, the mean value of a sufficiently smooth function over a set, most often over a sphere.

The mean value theorems for elliptic equations are widely known. The following classical mean value theorem, referring to Gauss (see [1]), is fundamental in applications: a continuous function \( u(x) \) in a domain \( \Omega \) is harmonic if and only if for any \( x \in \Omega \) and any \( r > 0 \) such that the ball \( B(x, r) = \{ \xi \in \mathbb{R}^n : |\xi - x| \leq r \} \) belongs to \( \Omega \), its value at \( x \) is equal to its mean value over the ball (or its surface). Generalizations of this result have been established for solutions of second-order elliptic equations. V. A. Il’yn and E. I. Moiseev (see [2], [3]) obtained mean value formulas for more general elliptic operators. These formulas were utilized by these authors to study problems connected with spectral expansions with respect to eigenfunctions of elliptic operators. A mean value theorem for the Laplace equation in a circular sector is proved in [4]. Moreover, a mean value theorem of the said type can be established for Riemannian manifolds (see [5]). Among the mean value theorems for hyperbolic equations, we should mention, in the first place, the classical Asgeirsson’s principle for ultrahyperbolic equations (see [6]) and the Bitsadze-Nakhushev theorem about the mean value for the wave equation (see [7]). The Asgeirsson’s principle was also obtained for ultrahyperbolic singular equations (see [8], [9]).

The approach proposed below can be used to obtain some new mean value theorems.

2. Materials and methods
In this paper we use classical methods of mathematical analysis, differential geometry and theory of partial differential equations.
3. Mean value theorem for the wave equation in Euclidean space

In the space $R^{n+1}$, let us consider a pair of points $(\chi^{(j)}, \tau^{(j)}), j = 1, 2$, where $\chi^{(j)} = (\chi_1^{(j)}, \chi_2^{(j)}, \ldots, \chi_n^{(j)})$, satisfying the condition

$$|\chi^{(1)} - \chi^{(2)}| < |\tau^{(1)} - \tau^{(2)}|. $$

(1)

Let us construct the matrix $A$ according to the following rule. We fix a certain subscript $i \in \{1, \ldots, n\}$ and set $a_{ij} = (\chi_j^{(1)} - \chi_j^{(2)})/|\chi^{(1)} - \chi^{(2)}|$, $j = 1, \ldots, n$. We construct other entries of this matrix from the conditions $AA^T = I$, $\det A = 1$, where $A^T$ is the matrix transposed to the matrix $A$ and $I$ is the identity matrix. Let $A_i$ be the matrix obtained from $A$ when its $i$-th column replaces by zeros. Following [7], we introduce the averaging operator $S_\tau$ by the formula

$$S_\tau^n v = S_\tau^n v = \gamma(n) \int_{|\xi| = \tau} v(\eta, \sigma) \, d\omega, \quad |\chi^{(1)} - \chi^{(2)}| > 0,$$

where $\gamma(n) = \sqrt{\pi^{1-n}}$,

$$\eta = \frac{|\tau^{(1)} - \tau^{(2)}|}{|\chi^{(1)} - \chi^{(2)}|} \xi - \frac{|\chi^{(1)} + \chi^{(2)}|}{2} + A_i \xi,$$

$$\sigma = \frac{(\tau^{(1)} + \tau^{(2)})}{2} - \frac{|\chi^{(1)} - \chi^{(2)}|}{2} \xi,$$

and $d\omega$ is the area element of the surface of the sphere $|\xi| = t$ in $R^n$. When $\chi^{(1)} = \chi^{(2)}$, we define the operator $S_\tau$ by the formula

$$S_\tau^n v = \gamma(n) \int_{|\xi| = \tau} v(\chi^{(2)} + \xi, \frac{\tau^{(1)} + \tau^{(2)}}{2}) \, d\omega.$$

Furthermore, we introduce the operator $B_\tau$ by formulas

$$B_\tau^n v = B_\tau^n v = \tau \left( \frac{\partial}{2\tau \partial \tau} \right)^{n-1} \frac{1}{\tau} S_\tau v, \quad n = 1(\text{mod } 2),$$

(2)

$$B_\tau^n v = B_\tau^n v = \frac{1}{\sqrt{\tau}} \left( \frac{\partial}{2\tau \partial \tau} \right)^{\frac{n}{2}} \int_0^\tau \frac{S_\tau v \, d\theta}{\sqrt{\tau^2 - \theta^2}}, \quad n = 0(\text{mod } 2).$$

(3)

**Theorem 1.** (Mean value theorem for the wave equation, [7]). If a function $v(\chi_1, \ldots, \chi_n, \tau)$ is a regular solution of the wave equation

$$\partial^2 v/\partial \tau^2 = \sum_{k=1}^n \partial^2 v/\partial \chi_k^2,$$

(4)

then for any pair of points $(\chi^{(j)}, \tau^{(j)}), j = 1, 2$, satisfying condition (1), the following relation holds:

$$v(\chi^{(1)}, \tau^{(1)}) + v(\chi^{(2)}, \tau^{(2)}) = B_\delta v,$$

(5)

where $\delta = \sqrt{\Delta \tau^2 - |\Delta \chi|^2/2}$, $\Delta \tau = \tau^{(1)} - \tau^{(2)}$, $\Delta \chi = \chi^{(1)} - \chi^{(2)}$.

There is an inverse assertion to this theorem for sufficiently smooth functions. Let

$$l = \max\left\{ n - 1, \; (n + 3)/2 \right\}, \quad n = 1(\text{mod } 2),$$

(6)

$$l = \max\left\{ n, \; [1 + (n + 3)/2] \right\}, \quad n = 0(\text{mod } 2).$$

(7)

**Theorem 2.** (Inverse mean value theorem for the wave equation, [10], [11]). Let the function $v(\chi, \tau) \in C^l(R^{n+1})$ satisfy relation (5) for any pair of points $(\chi^{(j)}, \tau^{(j)}), j = 1, 2$, satisfying condition (1). Then $v(\chi, \tau)$ is a regular solution of equation (4).
4. The Hadamard descent method and the two-point mean value theorem for the Laplace equation in Euclidean space

A two-point mean value formula for the Laplace equation was obtained in [12] by means of the Hadamard descent method.

Let a function \( v(\chi) \) satisfy the Laplace equation
\[
\frac{\partial^2 v}{\partial \chi_1^2} + \ldots + \frac{\partial^2 v}{\partial \chi_n^2} = 0.
\]
in a neighborhood of the ball \( |\chi| < r \).

We introduce the fictitious variable \( \tau \), in accordance with the Hadamard descent method, setting
\[
v(\chi, \tau) = v(\chi, 0) = v(\chi), \quad |\chi| < \tau, \quad \tau \in \mathbb{R}.
\]

Without loss of generality, let us set \( \tau_1 = 0, \tau_2 = r > 0 \).

The function \( v(\chi, \tau) \) satisfies equation (4), therefore the mean value formula (5) holds for it. Taking into account our assumptions, the operator \( S_\tau \) acts as follows:
\[
S_\tau v = S^n_\tau v = \gamma(n) \int_{|\xi| = \tau} v(\eta) \, d\omega_\xi, \quad |\chi^{(1)} - \chi^{(2)}| > 0,
\]
\[
S_\tau = S^n_\tau v = \gamma(n) \int_{|\xi| = \tau} v\left(\chi^{(2)} + \xi\right) \, d\omega_\xi, \quad \chi^{(1)} = \chi^{(2)},
\]
and formula (5) is transformed to the form
\[
v(\chi^{(1)}) + v(\chi^{(2)}) = B\varrho v.
\]

Let us reveal the geometric meaning of formula (13). Without loss of generality, we set
\[
\chi^{(1)} = \alpha = (\alpha_1, 0, \ldots, 0), \quad \chi^{(2)} = (0, 0, \ldots, 0).
\]
In (11) the integration is performed over the sphere \( \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = \tau^2 \).

We replace the first variable by
\[
\nu = \alpha_1/2 + r\xi_1/\sqrt{r^2 - \alpha_1^2}.
\]
Expressing \( \xi_1 \) from formula (15) in terms of \( \nu \), we obtain
\[
\xi_1 = \sqrt{r^2 - \alpha_1^2}/(\nu - \alpha_1/2)/r.
\]
Substituting expression (16) in the equation \( |\xi| = r \) of the sphere, in new variables \( \nu, \xi_2, \ldots, \xi_n \), we obtain the equation of the ellipsoid \( \Phi \):
\[
4(\nu - \alpha_1/2)/r^2 + 4(\xi_2^2 + \cdots + \xi_n^2)/(r^2 - \alpha_1^2) = 1.
\]
The integration in (13) is performed over the ellipsoid \( \Phi \) given by equation (17). Therefore, we have proved the following assertion.

**Theorem 3.** (Two-point mean value theorem for a harmonic function). *Let a domain \( \Omega \subset \mathbb{R}^n \) contain two points \( \chi^{(1)} \) and \( \chi^{(2)} \), \( |\chi^{(1)} - \chi^{(2)}| < r \). Let the domain bounded by the ellipsoid \( \Phi \) defined (in suitable coordinate system) by equation (17), together with its closure, lie in the domain \( \Omega \). Then for any function \( v(\chi) \) harmonic in the domain \( \Omega \), the two-point mean value formula (13) holds.*

**Note.** The mean value formula (13) matches with the well-known Gauss mean value formula for a harmonic function and a sphere of radius \( \delta = r/2 \) with \( \chi^{(1)} = \chi^{(2)} = \chi^{(0)} \) in the case of \( n = 1(\text{mod} \ 2) \).
5. Lax and Phillips transformations, relationship of a wave equation on a sphere with a wave equation in Euclidean space

Let $\mathbb{R}^{n+1}$ be Euclidean space of points $(x_1, \ldots, x_{n-1}, x_n, z) = (x, z)$. Denote by $S_n$ the sphere in $\mathbb{R}^{n+1}$ defined by the equation

$$|x|^2 + z^2 = 1.$$  \hspace{1cm} (18)

Let $\Delta$ be the Laplace operator in $\mathbb{R}^{n+1}$:

$$\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2 + \partial^2/\partial z^2 = r^{-n} \partial/\partial r \ r^n \partial/\partial r + r^{-2} \Delta_\omega,$$  \hspace{1cm} (19)

where $r = \sqrt{|x|^2 + z^2}$ and $\Delta_\omega$ is the Laplace-Beltrami operator on the unit sphere $S_n$. Consider the wave equation on $S_n$

$$u_{tt} - \Delta_\omega u + ((n-1)/2)^2 u = 0.$$  \hspace{1cm} (20)

This equation was considered by P. Lax and R. Phillips in [13]. It was also studied by I. A. Kipriyanov and L. A. Ivanov (see [14], [15], [16]). These authors obtained Kirchhoff’s formulas for solutions of Cauchy problem for equation (20) and showed that the Huygens principle holds for odd $n$ for this equation. Note that the equation $u_{tt} - \Delta_\omega u = 0$ does not satisfy the Huygens principle, as shown by M. N. Olevsky in [17].

Further, we describe a local coordinate transformation that establishes a connection between (20) and the wave equation in Euclidean space. For these purposes, we write down the Laplace-Beltrami operator in the metric on the upper hemisphere of $S_n$, denoted by $S_n^+$ and determined by the equality

$$z = \sqrt{1 - |x|^2}. \hspace{1cm} (21)$$

The metric of the hemisphere $S_n^+$ can be expressed in coordinates as follows:

$$dl^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} g_{ik} dx_i dx_k,$$  \hspace{1cm} (22)

where

$$g_{ik} = \delta_{ik} + x_i x_k/z^2.$$

(23)

Let the matrix $||g^{ik}||$ be inverse to the matrix $||g_{ik}||$, then

$$g^{ik} = \delta^{ik} - x_i x_k.$$ \hspace{1cm} (24)

Further, we have (see [10]):

$$g = |det||g^{ik}|| = z^2, \quad \bar{g} = |det||g_{ik}|| = 1/z^2.$$  \hspace{1cm} (25)

The Laplace–Beltrami operator in the metric $||g_{ij}||$ is expressed by the well-known formula (see [5])

$$\Delta_\omega u = \frac{1}{\sqrt{\bar{g}}} \sum_{k=1}^{n} \partial_k \left( \sum_{i=1}^{n} g^{ik} \sqrt{\bar{g}} \partial_i u \right).$$ \hspace{1cm} (26)
From (24)–(26) we have

\[ \Delta_{\omega} u = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( \frac{1}{z} \frac{\partial u}{\partial x_k} \right) - \sum_{k,i=1}^{n} \frac{\partial}{\partial x_k} \left( \frac{1}{z} x_i x_k \frac{\partial u}{\partial x_i} \right). \]  

(27)

Let the function \( v(\chi, T) \) be a regular solution of the wave equation

\[ Lv = \frac{\partial^2 v}{\partial T^2} - \Delta_{\omega} v = 0 \]

(28)
in the cone \( |\chi| < |T| \) of Euclidean space \( \mathbb{R}^{n+1} \). The Lorentz metric

\[ ds^2 = dT^2 - |d\chi|^2 \]

(29)
corresponds to the wave operator defined in (28). Consider the following mappings:

\begin{align*}
T &= T_1(x, z, t) = \sin t/(z - \cos t), \\
\chi &= \chi_1(x, z, t) = x/(z - \cos t), \\
T &= T_2(x, z, t) = -\sin t/(z + \cos t), \\
\chi &= \chi_2(x, z, t) = x/(z + \cos t).
\end{align*}

(30)

(31)

In both cases the condition \( |\chi| < |T| \) gives

\[ |x| < |\sin t| \]

(32)
or

\[ |z| > |\cos t|. \]

(33)

Let us pass to the coordinates \( x, z, t \) by formulas (30) in the metric (29). We have

\[ d\chi_i = \sum_{j=1}^{n} \frac{\partial \chi_i}{\partial x_j} dx_j + \frac{\partial \chi_i}{\partial t} dt = \frac{dx_i}{z - \cos t} + \sum_{j=1}^{n} \frac{x_i x_j}{z(z - \cos t)} dx_j - \frac{x_i \sin t}{(z - \cos t)^2} dt. \]

Since

\[ dz = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} dx_j = -\frac{1}{z} \sum_{j=1}^{n} x_j dx_j, \]

(34)
it follows that

\[ d\chi_i = \frac{1}{z - \cos t} dx_i - \frac{x_i}{(z - \cos t)^2} dz - \frac{x_i \sin t}{(z - \cos t)^2} dt. \]

Squaring \( d\chi_i \) and summing over \( i \), we get

\[ |d\chi|^2 = \frac{|dx|^2}{(z - \cos t)^2} + \frac{|x|^2 dz^2}{(z - \cos t)^4} + \frac{|x|^2 \sin^2 t dz^2}{(z - \cos t)^4} + \frac{2 z dz^2}{(z - \cos t)^3} + \]

\[ + \frac{2 z \sin t dz dt}{(z - \cos t)^3} + \frac{x^2 \sin t}{(z - \cos t)^4} dz dt = \frac{(z - \cos t)^2 + \sin^2 t}{(z - \cos t)^4} dz^2 + \]

\[ + \frac{|dx|^2}{(z - \cos t)^2} + \frac{|x|^2 \sin^2 t}{(z - \cos t)^4} dt^2 + \frac{2 \sin t (1 - z \cos t)}{(z - \cos t)^4} dz dt. \]

For the differential \( dT \) we have:

\[ dT = \sum_{j=1}^{n} \frac{\partial T}{\partial x_j} dx_j + \frac{\partial T}{\partial t} dt = -\frac{\sin t}{(z - \cos t)^2} dz + \frac{z \cos t - 1}{(z - \cos t)^2} dt, \]

therefore,
Substituting (38)–(41) to (35), we get

\[ dt^2 = \frac{\sin^2 t}{(z - \cos t)^2}dz^2 + \frac{(z \cos t - 1)^2}{(z - \cos t)^4}dt^2 + 2\frac{\sin t(1 - z \cos t)}{(z - \cos t)^4}dzdt. \]

Thus, the metric (29) is converted by the formula

\[ ds^2 = \frac{1}{(z - \cos t)^2}(dt^2 - |dx|^2 - dz^2) = \frac{1}{(z - \cos t)^2} \left( dt^2 - |dx|^2 - \frac{1}{z^2} \sum_{i,j=1}^{n} x_ix_jdx_idx_j \right). \]

The Laplace – Beltrami operator, corresponding to the above metric, has the following form:

\[ L = z(z - \cos t)^{n+1} \left[ \frac{1}{z} \frac{\partial}{\partial t} \left( \frac{1}{z} \frac{\partial}{\partial t} \right) - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( \frac{1}{z(z - \cos t)^n} \frac{\partial}{\partial x_k} \right) \right] + \sum_{i,k=1}^{n} \frac{\partial}{\partial x_k} \left( \frac{x_i x_k}{z(z - \cos t)^n} \frac{\partial}{\partial x_i} \right]. \]  

We define a function \( f = f(z, t) \) by the formula

\[ f = (z - \cos t)^{(n-1)/2} \]  

and make the substitution

\[ v = fu. \]  

Consider expression (35) with the substitution (37). Mixed derivatives are transformed as follows:

\[ \sum_{k,i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{x_k x_i}{z} \frac{\partial (fu)}{\partial x_k} \right) = \sum_{k,i=1}^{n} \frac{\partial}{\partial x_i} \left[ \frac{x_k x_i}{z} \frac{1}{f^2} \frac{\partial f}{\partial x_k} u + \frac{x_k x_i}{z} \frac{\partial u}{f} \right] = \frac{1}{f} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{x_k x_i}{z} \frac{\partial u}{\partial x_k} \right) - \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{x_k x_i}{z} \frac{\partial}{\partial x_k} \left( \frac{1}{f} \right) \right) u. \]  

For repeated derivatives with respect to \( x_k \) we have

\[ \frac{\partial}{\partial x_k} \left( \frac{1}{f^2} \frac{\partial v}{\partial x_k} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{z f^2} \frac{\partial (fu)}{\partial x_k} \right) = \frac{\partial}{\partial x_k} \left( \frac{1}{z f^2} \frac{\partial f}{\partial x_k} u + \frac{1}{z f} \frac{\partial u}{\partial x_k} \right) = \frac{1}{f} \frac{\partial}{\partial x_k} \left( \frac{1}{z} \frac{\partial u}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \left( \frac{1}{z} \frac{\partial}{\partial x_k} \left( \frac{1}{f} \right) \right) u. \]  

For repeated derivative with respect to \( t \), in similar way, we obtain

\[ \frac{\partial}{\partial t} \left( \frac{1}{f^2} \frac{\partial v}{\partial t} \right) = \frac{1}{f} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial t^2} \left( \frac{1}{f} \right) u. \]  

Substituting (38)–(41) to (35), we get

\[ L = \frac{1}{f} \left( \frac{\partial^2 u}{\partial t^2} - \Delta \omega u + Qu \right) (z - \cos t)^{n+1} = \left( \frac{\partial^2 u}{\partial t^2} - \Delta \omega u + Qu \right) (z - \cos t)^{n+1}, \]  

where
\[ Q = f \left[ z \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( \frac{1}{z} \frac{\partial}{\partial x_k} \frac{1}{f} - \sum_{i=1}^{n} \frac{x_i x_k}{z} \frac{\partial}{\partial x_i} \frac{1}{f} \right) - \frac{\partial^2}{\partial t^2} \frac{1}{f} \right]. \]

Now we calculate the factor \( Q \). We have
\[ \frac{\partial}{\partial x_i} \frac{1}{f} = \frac{n-1}{2} x_i (z - \cos t)^{\frac{n-1}{2}}, \quad i = 1, \ldots, n. \tag{43} \]

Multiplying equation (43) by \( x_i x_k / z \) and summing over \( i \), we obtain
\[ \sum_{i=1}^{n} \frac{x_i x_k}{z} \frac{\partial}{\partial x_i} \frac{1}{f} = \frac{n-1}{2} \frac{x_k}{z^2} |x|^2 (z - \cos t)^{\frac{n-1}{2}}. \tag{44} \]

We divide equality (43) by \( z \), replace \( i \) by \( k \), and subtract (44) from the obtained equality. As a result, we get
\[ \frac{1}{z} \frac{\partial}{\partial x_k} \frac{1}{f} - \sum_{i=1}^{n} \frac{x_i x_k}{z} \frac{\partial}{\partial x_i} \frac{1}{f} = \frac{n-1}{2} x_k (z - \cos t)^{\frac{n-1}{2}}. \tag{45} \]

We differentiate equality (45) with respect to \( x_k \) and multiply by \( zf \). We have
\[ zf \frac{\partial}{\partial x_k} \left( \frac{1}{z} \frac{\partial}{\partial x_k} \frac{1}{f} - \sum_{i=1}^{n} \frac{x_i x_k}{z} \frac{\partial}{\partial x_i} \frac{1}{f} \right) = \frac{n-1}{2} (z - \cos t)^{-2} \left( \frac{n+1}{2} x_k^2 + z^2 - z \cos t \right). \tag{46} \]

Furthermore,
\[ \frac{\partial^2}{\partial t^2} \frac{1}{f} = \frac{n-1}{2} (z - \cos t)^{-\frac{n-3}{2}} \left( \frac{n+1}{2} \cos^2 t + \frac{n-1}{2} - z \cos t \right), \]
therefore,
\[ f \frac{\partial}{\partial t^2} \frac{1}{f} = \frac{n-1}{2} (z - \cos t)^{-2} \left( \frac{n+1}{2} - \frac{n-1}{2} \cos^2 t - z \cos t \right). \]

We sum (46) over \( k \), replace \( |x|^2 \) by \( 1 - z^2 \), subtract (40) from the obtained equality. As a result, we get
\[ Q = \frac{n-1}{2} (z - \cos t)^{-2} \left( n-1 \frac{z^2}{2} - (n-1)z \cos t + n \frac{1}{2} \cos^2 t \right) = \left( \frac{n-1}{2} \right)^2. \]

Substituting \( Q \) in (42), we finally obtain
\[ L u = (z - \cos t)^{n+3} \left( \frac{\partial^2 u}{\partial t^2} - \Delta u + \left( \frac{n-1}{2} \right)^2 u \right). \]
Thus, we have proved that if the function $v(\chi, T)$ satisfies the Euclidean wave equation in the cone $|\chi| < |T|$, then, taking into account (30) and (37), the function $u(x, z, t)$ satisfies in coordinates $x, z, t$ the spherical wave equation (20) in the domain described by the inequality (33). The inverse statement is also true. Similarly, we can prove that the substitution

$$f = (z + \cos t)^{\frac{n-1}{2}}$$

instead of (36) also transforms the Euclidean wave equation to the wave equation on the sphere.

The transformations, defined by relations (30), (31), (36), (37), (47) are considered in [10]. They constitute an adaptation of transformations suggested by Lax and Phillips [13] for the metric on the hemisphere $S_n^+$. Therefore, it is pertinent to name them the Lax and Phillips transformations.

6. Lax and Phillips transformations and a mean value theorem for the wave equation on a sphere

Now let us prove an analogue of formula (5) for the spherical wave equation (20). Let the function $u(x, z, t)$ be a regular solution of equation (20) in the domain $|z| > |\cos t|$. The mapping (30) converts this domain into the cone

$$|\chi| < |T|,$$

and the subsequent replacement of the function $u(x, z, t)$ by formula (37) transfers equation (20) to the Euclidean wave equation (28).

Theorem 3 holds for the solution $v(\chi, T)$ of equation (28). Let $(x_1, z_1, t_1), (x_2, z_2, t_2)$ be a pair of points in $S_n \times (0, \pi)$, satisfying the condition

$$\left(\frac{\sin t_2}{z_2 \mp \cos t_2} - \frac{\sin t_1}{z_1 \mp \cos t_1}\right)^2 - \left|\frac{x_2}{z_2 \mp \cos t_2} - \frac{x_1}{z_1 \mp \cos t_1}\right|^2 > 0.$$  

Let $T_{kj} = T_k(x_j, z_j, t_j), X_{kj} = X_k(x_j, z_j, t_j), k = 1, 2, j = 1, 2$ in accordance with formulas (30).

Then condition (49) in the variables $X, T$ has the form

$$|\Delta T_k|^2 - |\Delta X_k|^2 > 0,$$

where $\Delta T = T_{k1} - T_{k2}, \Delta X = X_{k1} - X_{k2}$.

The condition (50) shows that the points $(X_{k1}, T_{k1}), (X_{k2}, T_{k2})$ satisfy the condition of Theorem 3. Therefore, we have the equality

$$v(X_{k1}, T_{k1}) + v(X_{k2}, T_{k2}) = Bv.$$  

We replace the function $u$ by formula (37) on the left hand side of (51), and in coordinates $(x, z, t)$ we get:

$$v(X_{k1}, T_{k1}) + v(X_{k2}, T_{k2}) = (z_1 \mp \cos t_1)u(x_1, z_1, t_1) + (z_2 \mp \cos t_2)u(x_2, z_2, t_2).$$

Now we replace the function $u$ on the right hand side of formula (51). For this purpose, we find an inverse mapping of (30), (31). Let

$$R = |\chi| = (\chi_1^2 + \ldots + \chi_n^2)^{\frac{1}{2}}.$$  

Then
\[
\frac{T^2 - R^2 - 1}{2T} = \frac{\sin^2 t - |x|^2 - (z \mp \cos t)^2}{2 \sin t(z)} = \frac{\sin^2 t - |x|^2 - z^2 \pm 2z \cos t - \cos^2 t}{2 \sin t(z \mp \cos t)} = \\
= \frac{\sin^2 t - 1 \pm 2z \cos t - \cos^2 t}{2 \sin t(z \mp \cos t)} = \cot t,
\]
consequently,
\[t = \arccot \frac{T^2 - R^2 - 1}{2T}.\]

Further,
\[T^2 + R^2 + 1 = \frac{(\sin^2 t + |x|^2 + z^2 \mp 2z \cos t + \cos^2 t)/ (z \mp \cos t)^2 = (2 - 2z \cos t)/(z \mp \cos t^2)},
\]
therefore,
\[\mu \overset{\text{def}}{=} (T^2 + R^2 + 1)^2 - 4T^2 R^2 = \frac{4(1 \mp 2z \cos t + z^2 \cos^2 t - |x|^2 \sin^2 t)}{(z \mp \cos t)^4} = \\
= \frac{4(1 \mp 2z \cos t + z^2 \cos^2 t - |x|^2(1 - \cos^2 t))}{(z - \cos t)^4} = \frac{4}{(z \mp \cos t)^2} > 0.\]

Moreover,
\[T^2 - R^2 + 1 = \frac{\sin^2 t - |x|^2 + z^2 \mp 2z \cos t + \cos^2 t}{(z \mp \cos t)^2} = \frac{2z}{z \mp \cos t}.
\]

From (53) and (54) we get
\[z = \frac{T^2 - R^2 + 1}{\sqrt{\mu}}.\]

In order to express \(x\) from (30) and (31), we note that, according to formula (52), we have
\[\cos t = \pm \frac{T^2 - R^2 - 1}{\sqrt{\mu}}.
\]
Substituting (55) and (56) into (30), we obtain
\[x = \pm \frac{2}{\sqrt{\mu}} \chi.
\]

Now we express the function \(f\), defined by formula (36), in the coordinates \(\chi, T\). Taking into account (53), we get
\[f = \left(\frac{\mu}{2}\right)^{\frac{1-n}{4}}.
\]

In order to use formula (51) in applications, it is necessary to express arguments of the function \(f\), standing in the right hand side of (51) under the integral sign, through \(\xi, X_{kj}, T_{kj}\ (k, j = 1, 2)\). For these purposes, in formula (58), where \(\mu = (T^2 + R^2 + 1)^2 - 4T^2 R^2 = (T^2 - R^2 + 1)^2 + 4R^2 = (T^2 - R^2 - 1)^2 + 4T^2\), in accordance with the definition of the operator \(B\), we set:
\[ T = \frac{T_{k1} + T_{k2}}{2} - \frac{|\Delta X_k| \xi_i}{\sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}}, \quad \chi = \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} + \frac{X_{k1} + X_{k2}}{2} + A_i \xi. \]

Then
\[ T^2 = \left( \frac{T_{k1} + T_{k2}}{2} \right)^2 + \frac{|\Delta X_k|^2 \xi_i^2}{|\Delta T_k|^2 - |\Delta X_k|^2} - \frac{(T_{k1} + T_{k2})|\Delta X_k| \xi_i}{\sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}}. \]
\[ R^2 = \frac{|\Delta T_k|^2 \xi_i^2 |\Delta X_k|^2}{|\Delta X_k|^2(|\Delta T_k|^2 - |\Delta X_k|^2)} + \frac{|X_{k1} + X_{k2}|^2}{4} + |A_i \xi|^2 + \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} (X_{k1} + X_{k2}) + \]
\[ + (X_{k1} + X_{k2}) A_i \xi + \frac{2|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} A_i \xi. \]

Note that on the sphere \(|\xi| = t\) we have
\[ |A_i \xi|^2 = \sum_{\zeta=1}^{n} \left( \sum_{j \neq i} a_{\zeta j} \xi_j \right)^2 = \sum_{\zeta=1}^{n} \sum_{j \neq i} a_{\zeta j} a_{\zeta l} \xi_l \xi_i = \sum_{\xi=1}^{n} \xi_j^2 = t^2 - \xi_i^2, \]
whence
\[ \frac{|\Delta T_k|^2 \xi_i^2}{|\Delta T_k|^2 - |\Delta X_k|^2} + |A_i \xi|^2 = t^2 + \frac{|\Delta X_k|^2 \xi_i^2}{|\Delta T_k|^2 - |\Delta X_k|^2}. \]

Considering (60), we obtain
\[ R^2 = \frac{|X_{k1} + X_{k2}|^2}{4} + t^2 + \frac{|\Delta X_k|^2 \xi_i^2}{\Delta T_k^2 - |\Delta X_k|^2} + (X_{k1} + X_{k2}) A_i \xi + \]
\[ + \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} (X_{k1} + X_{k2} + 2A_i \xi). \]

From (59) and (60) we get
\[ T^2 - R^2 + 1 = \left( \frac{T_{k1} + T_{k2}}{2} \right)^2 - \frac{(X_{k1} + X_{k2})^2}{4} + 1 - t^2 - \frac{(T_{k1} + T_{k2})|\Delta X_k| \xi_i}{\sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} - \]
\[ (X_{k1} + X_{k2}) A_i \xi - \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} (X_{k1} + X_{k2} + 2A_i \xi). \]

Substituting (61) and (62) into (58), we find
\[ f = 2^{n-1} \left[ \left( \frac{(T_{k1} + T_{k2})^2 - (X_{k1} + X_{k2})^2}{4} + 1 - t^2 - \frac{(T_{k1} + T_{k2})|\Delta X_k| \xi_i}{\sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} - \right. \right. \right. \]
\[ - (X_{k1} + X_{k2}) A_i \xi - \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} (X_{k1} + X_{k2} + 2A_i \xi) \right)^2 + \]
\[ + |X_{x1} + X_{x2}| + 4 \left( t^2 + \frac{|\Delta X_k|^2 \xi_i^2}{\Delta T_k^2 - |\Delta X_k|^2} + (X_{k1} + X_{k2}) A_i \xi \right) + \]
\[ + \frac{|\Delta T_k| \xi_i \Delta X_k}{|\Delta X_k| \sqrt{|\Delta T_k|^2 - |\Delta X_k|^2}} (X_{k1} + X_{k2} + 2A_i \xi) \right]^{1-n}, \]
where \((X_{kj}, T_{kj}) = (\chi_k(x_j, z_j, t_j), T_k(x_j, z_j, t_j)), k, j = 1, 2\). Whereas the mapping defined by formulas (52), (54), (57) converts the characteristic surface of the Euclidean wave equation to the characteristic surface of the spherical equation, we conclude that surfaces

\[|\sin t/(\zeta \mp \cos t) - \sin t_0/(\zeta_0 \mp \cos t_0)| = |x/(\zeta \mp \cos t) - x_0/(\zeta_0 \mp \cos t_0)|\]

are characteristic.

Let us show that two conditions (49) can be replaced by one. We have

\[
\left(\frac{\sin t_2}{z_2 - \cos t_2} - \frac{\sin t_1}{z_1 - \cos t_1}\right)^2 - \left(\frac{x_2}{z_2 - \cos t_2} - \frac{x_1}{z_1 - \cos t_1}\right)^2 =
\frac{(\sin^2 t_1 - |x_1|^2)(z_2 - \cos t_2)^2 + (\sin^2 t_2 - |x_2|^2) - 2(\sin t_1 \sin t_2 - x_1 x_2)}{(z_1 - \cos t_1)(z_2 - \cos t_2)}. \tag{64}
\]

Taking into account \(|x_j|^2 = 1 - z_j^2, \sin^2 t_j = 1 - \cos t_j\), we obtain

\[
\sin^2 t_j - |x_j|^2 = z_j^2 - \cos^2 t_j. \tag{65}
\]

We substitute (65) into (64) and reduce by \((z_1 - \cos t_1)(z_2 - \cos t_2)\). As a result we get

\[
\left(\frac{\sin t_2}{z_2 - \cos t_2} - \frac{\sin t_1}{z_1 - \cos t_1}\right)^2 - \left(\frac{x_2}{z_2 - \cos t_2} - \frac{x_1}{z_1 - \cos t_1}\right)^2 =
\frac{2(z_2 x_2 + x_1 x_2 - \cos(t_1 - t_2))}{(z - 1 - \cos t)(z_2 - \cos t)}. \tag{66}
\]

Using the same calculations, we obtain

\[
\left(\frac{\sin t_2}{z_2 + \cos t_2} - \frac{\sin t_1}{z_1 + \cos t_1}\right)^2 - \left(\frac{x_2}{z_2 + \cos t_2} - \frac{x_1}{z_1 + \cos t_1}\right)^2 =
\frac{2(z_2 x_2 + x_1 x_2 - \cos(t_1 + t_2))}{(z_2 + \cos t)(z_1 + \cos t)}. \tag{67}
\]

Since the expressions \(z_j + \cos t_j\) and \(z_j - \cos t_j\) have the same sign with \(|x_j|^2 < \sin^2 t_j\), we see that each of conditions (49) is equivalent to both conditions (66) and (67), which are equivalent to each other. Thus, two conditions (49) can be replaced by one, for example, by the condition

\[
\frac{z_1 z_2 + x_1 x_2 - \cos(t_1 - t_2)}{(z_2 - \cos t_2)(z_1 - \cos t_1)} > 0. \tag{68}
\]

The set of points, satisfying condition (68), isn’t empty. It can easily be checked by setting \(\pi/2 < t_2 < \pi, t_1 = t_2 - \pi/2\), and taking the points \((x_k, z_k), k = 1, 2\), with positive coordinates.

The above reasonings can be summarized as the following theorem.

**Theorem 4.** (Mean value theorem for the spherical wave equation, [10]). Let the function \(u(x, z, t)\) be a regular solution of the spherical wave equation (20) in the domain \(|z| > |\cos t|\). Then for any pair of points \((x_j, z_j, t_j), j = 1, 2\), belonging to this domain and satisfying condition (68), the following equalities hold:

\[
(z_1 + \cos t_1) \frac{u_{k+1}}{2} u(x_1, z_1, t_1) + (z_2 + \cos t_2) \frac{u_{k+1}}{2} u(x_2, z_2, t_2) = B(f u), \tag{69}
\]

where \(u = u(x(\chi, T), z(\chi, T), t(\chi, T))\), the operator \(B\) defined by formulas (2), (3), acts in the space \(R^{n+1}\), and the function \(f\) defined by formula (58), has a form:
\[ f = \left( \left( (T^2 + R^2 + 1)^2 - 4R^2T^2 \right)/2 \right)^{(1-n)/4}. \]

We now apply the Hadamard descent method. Let a function \( u = u(x) \) be a regular solution of the equation

\[ \Delta_\omega u + \left( \frac{n - 1}{2} \right)^2 u = 0. \quad (70) \]

In accordance with the Hadamard descent method, we introduce the fictitious variable \( t \in \mathbb{R} \).

Then, technically, the function \( u(x, t) \) satisfies the spherical wave equation. Therefore, it also satisfies the mean value formula (69). In this formula \( t_1 \) and \( t_2 \) should already be perceived as two parameter values. Thus, we proved the two-point mean value formula for equation (70) (see [18]): for any two points \( (x_1, z_1), (x_2, z_2) \in S_n \) and any two values of \( t_1, t_2 \in (0, 2\pi] \), satisfying condition (68), the mean value formula (69) holds for any regular solution \( u(x) \) of equation (70).

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