Polynomials and number sets associated with the probability distribution of the hyperbolic cosine type for even values of the parameter

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Abstract. The polynomials used in the formation of the probability distribution density function of the hyperbolic cosine type are investigated. Earlier, on the basis of a hyperbolic cosine distribution, the author obtained numerical sets, among which not only new ones, but also, for example, the triangle of Stirling numbers, the triangle of the coefficients of Bessel polynomials, sequences of coefficients in the expansion of various functions, etc. In this paper, depending on the natural parameter \( m \) and the real distribution parameter \( \beta \), a new class of polynomials is obtained. For even and odd \( m \), the polynomials are constructed using similar, but different formulas. The article presents polynomials for even values \( m \). Structurally, polynomials consist of quadratic factors. The coefficients of the polynomials, ordered by \( m \), form numerical triangles depending on \( \beta \). Some relations are found between the coefficients. From the numerical triangles, a set of numerical sequences is obtained, which for integers \( \beta \) are integers. Also, polynomials with respect to \( x \) turn out to be polynomials with respect to \( \beta \). With this interpretation the variable \( x \) acts as a parameter. New numerical triangles and sequences for different \( x \) were found. The overwhelming majority of the obtained numerical sequences are new. The class of polynomials arising from problems of probability theory indicates the possibility of applying the results.

1. Introduction

The probability distribution of the hyperbolic cosine type \( Ch(m, \beta, \mu) \) of a random variable \( X \) appeared as a result of the characterization of the distribution by the condition of constancy of the regression of quadratic statistics on a linear form [1].

The three-parameter distribution \( Ch(m, \beta, \mu) \) generalizes the two-parameter Meixner distribution [2-3], while the parameter \( \mu \) is the mathematical expectation of a random variable \( X \). And the Meixner distribution is a generalization of the one-parameter distribution of the hyperbolic cosine (hyperbolic secant) [4-6].

The distribution \( Ch(m, \beta, \mu) \) of a random variable \( X \) has a characteristic function

\[
f(t) = \left( \frac{\text{ch} \frac{\beta}{m} t - i \frac{\mu}{\beta} \text{sh} \frac{\beta}{m} t}{\mu} \right)^{-m}, \quad \text{where} \quad \mu, \beta, m \in \mathbb{R}; \ m > 0, \beta \neq 0.
\]  

(1)

The probability distribution density \( p_m(x) \) for the characteristic function (1) at \( \beta > 0 \) is given by the relation, [7-8].
The initial moments \( M(X^n) \) are calculated using moment-generating polynomials \( P_n(m; b) \), where \( b = \frac{\beta}{\mu} \). Recursive differential relations for the connection of polynomials \( P_n(m; b) \) of the type of expressions presented in [9-10] are established, and algebraic recurrence relations for the coefficients \( \{U(n; k, j)\} \) of these polynomials are found, [11]:

\[
P_{n+1}(m; b) = mbP_n(m; b) + (1 + b^2) \frac{\partial P_n(m; b)}{\partial b}
\]

where

\[
P_n(m; b) = \sum_{k=1}^{n} \sum_{j=1}^{n} U(n; k, j)m^k b^j, \quad b = \frac{\mu}{\beta},\]

at \( n = 0, 1, 2, \ldots \).

Coefficients \( U(n; k, j) \) of polynomials are ordered by arguments \( n; k, j \) and systematized in the form of a numerical prism \( \{U(n; k, j)\} \). Fixing one of the arguments in \( U(n; k, j) \) leads to numerical triangles, for example, \( \{U(n; 1, j)\} \) a triangle of coefficients of some polynomials when expanding in degrees \( \tan z \) of derivatives [11]. Representation of one of the arguments \( n; k, j \) through the other again leads to numerical triangles, for example, \( \{U(n; k, n)\} \) the Stirling triangle for the number of cycles [11-13] of the Stirling number of the first kind \( \left[ \binom{R}{k} \right] \) and the prism section \( \{U(2n - j; n, j)\} \) is a numerical triangle of coefficients in Bessel polynomials [14-17]. When fixing two arguments in \( U(n; k, j) \), numerical sequences are obtained, both new and widely known, in particular, secant and tangential numbers, generalized Euler numbers, coefficients in the expansion of various functions [12, 18-22].

Sequences that are currently absent in the widely known and constantly updated electronic encyclopedia OEIS [23] are considered new.

In the presented article, the class of polynomials included in the structure of the distribution density of the hyperbolic cosine type \( Ch(m, \beta, \mu) \) where \( m \in \mathbb{N} \) is considered. Polynomials for even and odd \( m \) value have a similar structure, but are calculated using different formulas. In this paper, polynomials are investigated for even values of the parameter \( m \), namely: \( m = 2k \). For odd values of the parameter \( m \), the study is similar, and the corresponding article is in print.

Coefficients of polynomials, form numerical triangles and integer sequences. The found sets are simpler in origin than the above. However, the overwhelming majority of the obtained numerical sets, according to the OEIS encyclopedia [23], are new.
2. Main points

2.1. Representation of polynomials

The distribution density function $p_m(x)$ of the hyperbolic cosine $Ch(m, \beta, \mu)$ type of the form (2) for natural $m$ is expressed in elementary functions. For even values of the parameter $m$, the function $p_m(x)$ has the form [7-8]:

$$p_2(x) = \frac{x}{(2\beta^2 + \mu^2)^2};$$

$$p_m(x) = \frac{m^2 x}{2(\beta^2 + \mu^2)^2(m - 1)! \text{sh} \frac{m x}{2\beta}} \prod_{s=1}^{\frac{m}{2}-1} [(2s)^2 \beta^2 + m^2 x^2] \text{ for } m = 4, 6, 8, \ldots \tag{6}$$

where $\beta > 0$,

$$A = \begin{cases}
        e^{\text{arctg} \frac{2\beta \mu}{2\beta^2 - \mu^2}} & \text{for } \beta^2 - \mu^2 > 0, \\
        e^{\text{arctg} \frac{2\beta \mu}{2\beta^2 - \mu^2} + \pi \text{sign } \mu} & \text{for } \beta^2 - \mu^2 < 0, \\
        e^{\pi \text{sign } \mu} & \text{for } \beta^2 - \mu^2 = 0.
        \end{cases}$$

We select polynomial factors from the functions $p_m(x)$, $m = 2, 4, 6, \ldots$, in (5), (6) and write them out as separate polynomials $Q_{m\beta}(x)$:

$$Q_{2\beta}(x) = 1; \quad Q_{m\beta}(x) = \prod_{s=1}^{\frac{m}{2}-1} [(2s)^2 \beta^2 + m^2 x^2] \text{ for } m = 4, 6, \ldots \tag{7}$$

According to (7), the density function $p_m(x)$ has the form

$$p_m(x) = \frac{m^2 x}{2(\beta^2 + \mu^2)^2(m - 1)! \text{sh} \frac{m x}{2\beta}} Q_{m\beta}(x) \text{ for } m = 2, 4, 6, \ldots \tag{8}$$

Further, we put $\beta = 1$ and $Q_{m1}(x) \equiv Q_m(x)$. Let us consider the obtained polynomials in more detail $Q_m(x)$.

First-order $Q_m(x)$ polynomials with even indices are:

$$Q_2(x) = 1;$$

$$Q_4(x) = (4 + 16x^2);$$

$$Q_6(x) = (4 + 36x^2)(16 + 36x^2) = 64 + 720x^2 + 1296x^4;$$

$$Q_8(x) = (4 + 64x^2)(16 + 64x^2)(36 + 64x^2) = 2304 + 50176x^2 + 229376x^4 + 262144x^6;$$

$$Q_{10}(x) = (4 + 100x^2)(16 + 100x^2)(36 + 100x^2)(64 + 100x^2) = 147456 + 524800x^2 + 4368000x^4 + 12000000x^6 + 100000000x^8;$$

$$Q_{12}(x) = (4 + 144x^2)(16 + 144x^2)(36 + 144x^2)(64 + 144x^2)(100 + 144x^2) = 14745600 + 776945664x^2 + 10145710080x^4 + 48874586112x^6 + 94595973120x^8 + 61917364224x^{10};$$

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It is easy to see that the polynomials \( Q_m(x) \) are even functions and the order of the polynomial \( Q_m(x) \) is equal \( m - 2 \).

2.2. Polynomial coefficients. Recurrent formulas. Sequences

Let us introduce the notation for the coefficients of the polynomials: we write the polynomial \( Q_m(x) \) in degrees \( x \) in the form

\[
Q_m(x) = a_m^{(0)} + a_m^{(2)} x^2 + a_m^{(4)} x^4 + \cdots + a_m^{(m-2)} x^{m-2}, \quad m = 2, 4, 6, \ldots
\]

We arrange the coefficients of the polynomials \( Q_m(x) \) in the form of a numerical triangle (Table 1).

| Table 1. Number triangle of coefficients at degrees \( x \) of polynomials \( Q_m(x) \). |
|---|---|---|---|---|---|---|---|
| \( Q_2(x) \) | 1 | | | | | |
| \( Q_4(x) \) | 4 | 16 | | | | |
| \( Q_6(x) \) | 64 | 720 | 1296 | | | |
| \( Q_8(x) \) | 2304 | 50176 | 229376 | 262144 | | |
| \( Q_{10}(x) \) | 147456 | 5248000 | 4368000 | 12000000 | 10000000 | |
| \( Q_{12}(x) \) | 14745600 | 776945664 | 10145710080 | 339408848 | 115495973120 | 76317364224 |
| \( Q_{14}(x) \) | 212336400 | 155171487744 | 2913921826816 | 21431107489792 | 70908712562688 | 105288694411264 |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |

The coefficients of the polynomials \( Q_m(x), m = 2, 4, 6, \ldots \), for the same even degrees \( x \) are located in the columns: at the intersection \( Q_m(x) \) and \( x^k \) the coefficient is located \( a_m^{(k)} \). Coefficients \( \{a_m^{(k)}\} \) at changing \( m \) form integer sequences.

Theorem 1. The sequence of coefficients of polynomials \( Q_m(x) \) with even indices \( \{a_m^{(0)}\} m = 2, 4, 6, \ldots \) can be restored using the recursive formula

\[
a_m^{(0)} = a_{m+2}^{(0)} \cdot m^2, \quad (11)
\]

i.e.

\[
a_m^{(0)} = 1 \cdot 4 \cdot 16 \cdot 36 \cdot \ldots \cdot m^2 = (1 \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot m)^2. \quad (12)
\]

Proof of Theorem 1. The validity of the statement of the theorem follows from the structure of the polynomials \( Q_m(x) \), presented in (9). Each value \( a_m^{(0)} = Q_m(0) \) is the product of free terms of the factors of the polynomial, which is indicated in the statement of the theorem (see. (12)).

Theorem 2. The sequence of coefficients \( \{a_m^{(2k)}\} \) of polynomials \( Q_m(x), m = 2, 4, 6, \ldots \) can be restored by the recursive formula

\[
a_m^{(2k)} = \left( \frac{a_{m+2}^{(2k-2)} + a_m^{(2k)}}{m^{2k-2}} \right) \cdot (m + 2)^{2k}, \quad (13)
\]

Corollary 2.1. For a sequence of coefficients \( \{a_m^{(2)}\} \), the recurrence relation is true

\[
a_m^{(2)} = \left( a_m^{(0)} + a_m^{(2)} \right) \cdot (m + 2)^2, \quad m = 2, 4, 6, \ldots
\]
Corollary 2.2. For a sequence of coefficients \( \{a_m^{(4)}\} \), the recurrence relation is true

\[
d_{m+2}^{(4)} = \frac{(e_{m}^{(2)}+a_{m}^{(4)})}{m^2} \cdot (m + 2)^4.
\]

Further, from the numerical triangle (Table 1), we write down sequences of coefficients of polynomials at the same degrees \( x \) and identify them according to the OEIS encyclopedia.

\( \{a_m^{(0)}\} = 1, 4, 64, 2304, 147456, 14745600, 2123366400, 416179814400, 106542032486400, \ldots \)

The specified sequence is known as A002454 – central factorial numbers: \( a_n = 4^n(n!)^2, n = 0, 1, 2, \ldots \) [18], [19], [24], [25].

\( \{a_m^{(2)}\} = 16, 720, 50176, 5248000, 776945664, 155171487744, 40267482660864, 1318150664985536, \ldots \).

This sequence is absent in the OEIS, i.e. is new.

\( \{a_4^{(4)}\} = 1296, 229376, 43680000, 10145710080, 2913921826816, 1026204589096960, 437320203925192704, \ldots \)

The specified sequence is not present in the OEIS.

\( \{a_6^{(6)}\} = 262144, 1200000000, 48874586112, 21431107489792, 10632075577131008, 6050476630683942912, \ldots \)

The specified sequence is not present in the OEIS.

Other sequences of odds for even \( m \) and fixed \( k > 3 \) coefficients \( \{a_m^{(2k)}\} \) are also not currently available in the OEIS.

A sequence of all coefficients of polynomials \( Q_m(x) \), ordered at even \( m \):

\( (1, 4, 16, 64, 2304, 100176, 229376, 262144, 147456, 5248000, 43680000, \ldots ) \).

This sequence is also missing from the OEIS.

Let us introduce sequences of values of polynomials \( Q_m(x) \). \( m = 2, 4, 6, \ldots \) for some specific values of the variable \( x \).

The sequence \( \{Q_m(0)\} \) is the same as the above sequence \( \{a_0^{(0)}\} \).

\( \{Q_m(\pm 1)\} = 1, 20, 2080, 544000, 269075456, 216325324800, 257393643520000, 425327541616640000, \ldots \)

\( \{Q_m(\pm 2)\} = 1, 68, 23680, 20650240, 34000019456, 9091377484800, 3596053550288000, \ldots \)

\( \{Q_m(\pm 3)\} = 1, 148, 111520, 210136320, 747165459456, 4313261002752000, 36827754369748992000, \ldots \)

\( \{Q_m(\pm 4)\} = 1, 260, 343360, 1133267200, 7056386195456, 71327303855308800, \ldots \)

Further, for the combined set of polynomials with odd and even indices, we give sequences of values for some specific \( x \): \( \{Q_m(0)\}, \{Q_m(\pm 1)\}, \{Q_m(\pm 2)\}, \{Q_m(\pm 3)\}, \) \( m = 1, 2, 3, 4, \ldots \).

\( \{Q_m(0)\} = 1, 1, 1, 4, 9, 64, 225, 2304, 11025, 147456, 893025, 14745600, 108056025, \ldots \)

\( \{Q_m(\pm 1)\} = 1, 1, 10, 20, 884, 2080, 214600, 544000, 101696400, 269075456, 795163304, \ldots \)

\( \{Q_m(\pm 2)\} = 1, 1, 37, 68, 11009, 23680, 8925085, 20650240, 14088405825, 34000019456, \ldots \)
\[ \{Q_m(\pm 3)\} = 1, 1, 82, 148, 52884, 111520, 92687400, 210136320, 316031348880, \ldots \]

The found sequences, as well as other similar ones, are absent in the OEIS.

2.3. Study of polynomials
Polynomials \( Q_m(x), m = 2, 4, 6, \ldots \) are even functions.
For an arbitrary even index \( m, m = 2s \), according to (7), we obtain
\[
Q_m(x) = (4 + m^2x^2)(16 + m^2x^2)(36 + m^2x^2) \ldots ((m - 2)^2 + m^2x^2)
= 4^{s-1}(1 + s^2x^2)(4 + s^2x^2)(9 + s^2x^2) \ldots ((s - 1)^2 + s^2x^2).
\]
In particular, \( Q_4(x) = 4(1 + 4x^2), Q_6(x) = 4^2(1 + 9x^2)(4 + 9x^2), \)
\[
Q_8(x) = 4^3(1 + 16x^2)(4 + 16x^2)(9 + 16x^2).
\]
Polynomials \( Q_m(x) \) have degree \( (m - 2) \) and all \( (m - 2) \) roots are simple and purely imaginary. Moreover, due to the parity of the function, the roots are divided into pairs: if \( a \) is a root of a polynomial, then \( -a \) is also a root. Geometrically, the roots are located on the imaginary axis in pairs symmetrically to zero, and \( x = 0 \) are not a root of the polynomial.

For polynomials of the first indices, we list these roots:
\[
Q_4(x): \left\{ \pm \frac{1}{2}i \right\};
Q_6(x): \left\{ \pm \frac{1}{3}i; \pm \frac{2}{3}i \right\};
Q_8(x): \left\{ \pm \frac{1}{4}i; \pm \frac{2}{4}i; \pm \frac{3}{4}i \right\};
Q_{10}(x): \left\{ \pm \frac{1}{5}i; \pm \frac{2}{5}i; \pm \frac{3}{5}i; \pm \frac{4}{5}i \right\};
\]
In general, when \( m = 2s \) we get:
\[
Q_{2s}(x): \left\{ \pm \frac{1}{s}i; \pm \frac{2}{s}i; \pm \frac{3}{s}i; \ldots; \pm \frac{s-1}{s}i \right\}, \quad m = 4, 6, 8, \ldots \quad (14)
\]
Figure 1 shows a typical polynomial plot \( Q_m(x) \). Polynomial plots at other indices \( m \) retain their shape and differ only in size.

![Figure 1. Polynomial plot \( Q_{10}(x) \).](image-url)
2.4. Polynomials of the form $Q_{m\beta}(x)$ and $Q_{m\beta}(\beta x)$

In the relation (7), polynomials of the generalized form $Q_{m\beta}(x)$, are introduced where $\beta > 0$, as the components of the distribution density function $p_m(x)$ of the hyperbolic cosine type:

$$Q_{m\beta}(x) = 1; \quad Q_{m\beta}(x) = \prod_{s=1}^{m/2-1} ((2s)^2\beta^2 + m^2x^2) \text{ for } m = 4, 6, ...$$

In particular,

$$Q_{2\beta}(x) = 1; \quad Q_{4\beta}(x) = 4(\beta^2 + 4x^2);$$

$$Q_{6\beta}(x) = 4^2(\beta^2 + 9x^2)(4\beta^2 + 9x^2) = 64\beta^4 + 720\beta^2 x^2 + 1296x^4;$$

$$Q_{8\beta}(x) = 4^3(\beta^2 + 16x^2)(4\beta^2 + 16x^2)(9\beta^2 + 16x^2)$$
$$= 2304\beta^6 + 50176\beta^4x^2 + 229376\beta^2x^4 + 262144x^6;$$

$$Q_{10\beta}(x) = 4^4(\beta^2 + 25x^2)(4\beta^2 + 25x^2)(9\beta^2 + 25x^2)(16\beta^2 + 25x^2)$$
$$= 147456\beta^8 + 5248000\beta^6x^2 + 43680000\beta^4x^4 + 120000000\beta^2x^6 + 100000000x^8.$$

The roots of the polynomials $Q_{m\beta}(x)$ in comparison with $Q_m(x)$ at $\beta \neq 1$ will also change, in particular:

$$Q_{4\beta}(x): \{\pm \frac{\beta}{2}i\};$$
$$Q_{6\beta}(x): \{\pm \frac{\beta}{3}i; \pm \frac{2\beta}{3}i\};$$
$$Q_{8\beta}(x): \{\pm \frac{\beta}{4}i; \pm \frac{2\beta}{4}i; \pm \frac{3\beta}{4}i\};$$
$$Q_{10\beta}(x): \{\pm \frac{\beta}{5}i; \pm \frac{2\beta}{5}i; \pm \frac{3\beta}{5}i; \pm \frac{4\beta}{5}i\}.$$

In general, with $m = 2s$:

$$Q_{m\beta}(x): \{\pm \frac{\beta}{s}i; \pm \frac{2\beta}{s}i; \pm \frac{3\beta}{s}i; \ldots; \pm \frac{(s-1)\beta}{s}i\}, \text{ for } m = 4, 6, 8, ... (15)$$

The roots given in (15) differ from the corresponding roots in (14) by a factor $\beta$.

Denoting the coefficients of the polynomials $Q_{m\beta}(x)$, $m = 2, 4, 6, ..., at the same $k$-th degrees of the variable $x$ as $a_{m\beta}^{(k)}$, we obtain new numerical sequences.

For example,

$$a_{m\beta}^{(0)} = 1, 4\beta^2, 64\beta^4, 2304\beta^6, 147456\beta^8, 14745600\beta^{10}, 2123366400\beta^{12},$$
$$416179814400\beta^{14}, \ldots$$

The resulting sequence is a generalization of the sequence $\{a_{m}^{(0)}\}$, also known as central factorial numbers $a_n = 4^n(n!)^2$, [18], [19], [24], [25] (OEIS, A000254). In particular, at $\beta^2 = 2$ follows

$$a_{m2^0.5}^{(0)} = 1, 8, 256, 18432, 2359296, 471859200, 135895449600, 53271016243200, \ldots;$$

at $\beta^2 = 3$ follows

$$a_{m3^{0.5}}^{(0)} = 1, 12, 576, 62208, 11943936, 3583180800, 1547934105600, 910185254092800, \ldots.$$
\[ \{a_m^{(2)}\} = 16,720\beta^2, 50176\beta^4, 5248000\beta^6, 776945664\beta^8, 155171487744\beta^{10}, 40267482660864\beta^{12}, \ldots \]

In particular, at \( \beta^2 = 2 \) follows
\[ \{a_m^{(2)}\} = 16, 1440, 200704, 41984000, 12431130624, 4965487607808, 2577118890295300, \ldots \]

Note that, considering the polynomials \( Q_m\beta(x) \) with respect to \( \beta \), we get a numerical triangle of coefficients, different from the numerical triangle in Table 1. In particular, when \( x = \pm 1 \) the same numbers from table 1 are ordered differently (Table 2).

**Table 2.** Number triangle of coefficients at degrees \( \beta \) of polynomials \( Q_m\beta(1) \).

| \( \beta^0 \) | \( \beta^2 \) | \( \beta^4 \) | \( \beta^6 \) | \( \beta^8 \) | \( \beta^{10} \) | \( \beta^{12} \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( Q_{2p}(1) \) | 1           |             |             |             |             |             |
| \( Q_{4p}(1) \) | 16          | 4           |             |             |             |             |
| \( Q_{6p}(1) \) | 1296        | 720         | 64          |             |             |             |
| \( Q_{8p}(1) \) | 262144      | 229376      | 50176       | 2304        |             |             |
| \( Q_{10p}(1) \) | 100000000   | 120000000   | 43680000    | 5248000     | 147456      |             |
| \( Q_{12p}(1) \) | 61917364224 | 94595973120 | 48874586112 | 10145710080 | 776945664   | 14745600    |
| \( Q_{14p}(1) \) | 56693912375296 | 105288694411264 | 70908712562688 | 21431107489792 | 2913921826816 | 155171487744 | 2123366400 |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |

Let us give a sequence of coefficients at the same degrees \( \beta \).

\( \beta^0: 1, 16, 1296, 262144, 100000000, 61917364224, 56693912375296, \ldots \)

The specified sequence in the *OEIS* is present under the number A163395 with the formula for constructing elements \( (2k^{2k-2}, k = 1, 2, 3, \ldots \) \).

\( \beta^2: 4, 720, 229376, 120000000, 115495973120, 105288694411264, \ldots \)

This sequence, as well as others for degrees \( \beta \), is absent in the *OEIS*.

Assuming \( x \neq \pm 1 \), we arrive at a new set of numerical sequences. In particular, for \( x^2 = 2 \), i.e. \( x = \pm \sqrt{2} \), we get a new numerical triangle (Table 3).

**Table 3.** Number triangle of coefficients at degrees \( \beta \) of polynomials \( Q_m\beta(\sqrt{2}) \).

| \( \beta^0 \) | \( \beta^2 \) | \( \beta^4 \) | \( \beta^6 \) | \( \beta^8 \) | \( \beta^{10} \) | \( \beta^{12} \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( Q_{2p}(\sqrt{2}) \) | 1           |             |             |             |             |             |
| \( Q_{4p}(\sqrt{2}) \) | 32          | 4           |             |             |             |             |
| \( Q_{6p}(\sqrt{2}) \) | 5184        | 1440        | 64          |             |             |             |
| \( Q_{8p}(\sqrt{2}) \) | 2097152     | 917504      | 100352      | 2304        |             |             |
| \( Q_{10p}(\sqrt{2}) \) | 1600000000  | 960000000   | 174720000   | 1049600     | 147456      |             |
| \( Q_{12p}(\sqrt{2}) \) | 1981355655168 | 1513535569920 | 390996688896 | 40582840320 | 1553891328 | 14745600 |
| \( Q_{14p}(\sqrt{2}) \) | 3628410392018944 | 3369238221160448 | 1134539401003008 | 171448859918336 | 11655687307264 | 310342975488 | 2123366400 |
Let us give a sequence of coefficients at the same degrees $\beta$ in the found numerical triangle. When $\beta^0$: 1, 32, 5184, 2097152, 160000000, 1981355655168, 3628410392018944, …

When $\beta^2$: 4, 1440, 917504, 960000000, 151353569920, 3369238221160448, 10088063165309911040, …

The indicated sequences and other similar ones are absent in the OEIS and in other special cases with specific $\beta$.

The considered $Q_{m\beta}(x)$ polynomials for even $m$ as well as for odd $m$ are simplified for the argument $\beta x$, namely, the following property takes place:

$$Q_{m\beta}(\beta x) = \prod_{s=1}^{m} \beta^2[(2s)^2 + m^2x^2] = \beta^{m-2} Q_m(x) \quad \text{for } m = 4, 6, 8, ... \quad (16)$$

Obviously, the roots of the polynomials $Q_{m\beta}(\beta x)$ and $Q_m(x)$ coincide.

In accordance with (16), a factor $\beta^{m-2}$ is added to each of the elements of the sequence of coefficients $\{a_m^{(k)}\}$.

3. Conclusion

Let us summarize the results obtained. The class of polynomials $\{Q_{m\beta}(x)\}$, arose from a probabilistic problem [1, 7-8, 11]. The structure of the density function of a three-parameter distribution of the hyperbolic cosine type for an integer parameter $m$ contains polynomial factors. For $m \in \mathbb{N}$, these factors form a family of polynomials with a real parameter $\beta$. For even and odd $m$, the polynomials are constructed using different formulas.

The polynomials considered in the article for even $m$, as well as for odd $m$, have only purely imaginary roots. Polynomial graphs have the same configuration, which unites them, and differ only in numerical characteristics.

The previously indicated distribution is used by the author as a source of numerical triangles and integer sequences. The coefficients of the polynomials, ordered by $m$ and powers of the variable $x$, form numerical triangles with a parameter $\beta$. Fixing $\beta$, we come to specific number triangles and number sequences. The values of the polynomials, when fixed $x$, also form numerical sequences, and not necessarily integer ones. Some relations are found between the coefficients of the polynomials.

Considering polynomials as a function of the argument $\beta$, a different ordering of the coefficients, new number triangles and number sequences is obtained. This work supplements and develops the previous results of the author related to number sets and distributions of the hyperbolic cosine type [11, 14].

The results obtained can be used in mathematical fields such as number theory, algebra, and information technology.

References
[1] Tokmachev M S 1995 The constancy of the regression of quadratic statistics on linear statistics Vestnik NovSU 1 139–41
[2] Lai C D 1982 Meixner classes and Meixner hypergeometric distributions Aust. J. Stat. 24 221–33
[3] Grigoletto M and Provasi C 2009 Simulation and estimation of the Meixner distribution Communications in Statistics: Simulation and Computation 38 (1) 58–77
[4] Feller W 1984 Introduction to the theory of probability and its applications (Moscow: Mir) 2 p 738
[5] Johnson N, Kotz S and Balakrishnan N 1994 Continuous Univariate Distributions 1 (2nd Ed. New York: Wiley)
[6] Korn G and Korn T 1974 Handbook of Mathematics (for researchers and engineers) (Moscow: Nauka) p 832
[7] Tokmachev M S and Tokmachev A M 2001 Distribution of the hyperbolic cosine type *Vestnik NovSU* **17** 85–8

[8] Tokmachev M S 2005 Applied aspect of the generalized distribution of the hyperbolic cosine *Vestnik NovSU* **34** 96–9

[9] Hoffman Michael E 1995 Derivative polynomials for tangent and secant *Amer. Math. Monthly,* **102** 23–30

[10] Hoffman M E 1999 Derivative polynomials, Euler polynomials, and associated integer sequences, *Electronic Journal of Combinatorics* **6** #R21

[11] Tokmachev M S 2018 Probability distributions: from characterization to numerical sets (Saarbrücken: LAP LAMBERT) p 224

[12] Grekhem R, Knut D and Patashnik O 1998 Concrete Mathematics. Foundation of Informatics (Moscow: Mir) p 703

[13] Franssens G R 2006 On a number pyramid related to the binomial, Eleham, Eulerian, MacMahon and Stirling number triangles *Journal of Integer Sequences* **9** 06.4.1

[14] Tokmachev M S 2016 Sections of a numerical prism associated with Bessel polynomials *Bulletin of the South Ural State University series Mathematics. Mechanics. Physics* **8** (3) 64–71

[15] Krall H L and Fink O 1949 A new class of orthogonal polynomials: The bessel polynomials *Trans. Amer. Math. Soc.* **65** 100–15

[16] Carlitz L 1957 A note on the bessel polynomials *Duke Math. J.* **24** 151–62

[17] Kim T and Kim D 2016 Identities involving Bessel polynomials arising from linear differential equations *arXiv:1602.04106 [math.NT]*

[18] Sloane N J A and Plouffe Simon 1995 *The encyclopedia of integer sequences* (San Diego: Academic Press) p 587

[19] Sloane N J A 1973 *A Handbook of Integer Sequences* (NY: Academic Press) pp xiii+206

[20] Knuth D E and Buckholtz T J 1967 Computation of tangent, Euler and Bernoulli numbers, *Math. Comp.* **21** pp 663–88

[21] Arregui J L 2001 Tangent and Bernoulli numbers related to Motzkin and Catalan numbers by means of numerical triangles, *arXiv:math/0109108 [math.NT]*

[22] Brent R P and Harvey D 2011 Fast computation of Bernoulli, Tangent and Secant numbers *arXiv preprint arXiv: 1108.0286 [math.CO]*.

[23] The On-Line Encyclopedia of Integer Sequences™ (OEIS™) Available at: http://www.research.att.com/~njas/sequences/ (accessed on: 5.06.2021)

[24] Riordan J 1968 Combinatorial Identities Wiley p 217

[25] Bronstein-Semendjajew 1965 *Taschenbuch der Mathematik* 7th german ed. *ch. 4.4.7*