ON THE VANISHING CRITERION FOR THE COHOMOLOGY GROUPS OF THE AUTOMORPHISM GROUP OF A FINITE ABELIAN $p$-GROUP

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ABSTRACT. For a partition $\lambda = (\lambda_1^p_1 > \lambda_2^p_2 > \lambda_3^p_3 > \ldots > \lambda_k^p_k)$ and its associated finite abelian $p$-group $A_\lambda = \bigoplus_{i=1}^k (\mathbb{Z}/p^{\lambda_i} \mathbb{Z})^{\rho_i}$, where $p$ is a prime, we consider two actions of its automorphism group $G_\lambda = \text{Aut}(A_\lambda)$ on $A_\lambda$. The first action is the natural action $g \cdot a = ga$ for all $g \in G_\lambda$ and $a \in A_\lambda$ where the action map is denoted by $\Lambda_1 = \text{Id}_{G_\lambda} : G_\lambda \rightarrow G_\lambda$ and the second action is the trivial action $g \cdot a = a$ for all $g \in G_\lambda$ and $a \in A_\lambda$ where the action map is denoted by $\Lambda_2 : G_\lambda \rightarrow \{e\} \subset G_\lambda$ the trivial map. For the natural action $\Lambda_1$, we show that the first and second cohomology groups $H^i_{\Lambda_1}(G_\lambda, A_\lambda)$, $i = 1, 2$ vanish for any partition $\lambda$ for an odd prime $p$. For the trivial action $\Lambda_2$ we show that, for an odd prime $p$, the first cohomology group $H^1_{\Lambda_2}(G_\lambda, A_\lambda)$ and for an odd prime $p \neq 3$, the second cohomology group $H^2_{\Lambda_2}(G_\lambda, A_\lambda)$ vanish if and only if the difference between two successive parts of the partition $\lambda$ is at most one. This is done by proving that the mod $p$ cohomologies $H^i(G_\lambda, \mathbb{Z}/p\mathbb{Z}), i = 1, 2$ vanish if and only if the difference between two successive parts of the partition $\lambda$ is at most one for an odd prime $p \neq 3$. The vanishing of the second cohomology in this context is proved using the extended Hochschild-Serre exact sequence for central extensions.

1. Introduction

The cohomology of finite groups is a very vast subject which has its interactions with other areas such as group theory, representation theory, homological algebra, number theory, K-theory, classifying spaces, group actions, characteristic classes, homotopy theory. It is because of its interactions with the other areas, the subject is of immense interest. The calculation of cohomology of finite groups is quite challenging as it involves a number of complicated ingredients. An article [1] by A. Adem mentions in Section 4.3, a calculational method for the computation of cohomology of finite groups. The calculation method mentioned here is also elaborated in the survey article [2] by A. Adem. The details of this method are mentioned in Section 2.4.

Among the finite groups, the class of symmetric groups and the class of general linear groups are particularly important and interesting as far as the cohomology computation is concerned. D. Quillen [12] has computed the mod $l$ cohomology
ring of $GL_n(Z/pZ)$ for a prime $l \neq p$ and gives partial results when $l = p$. Later B. M. Mann [10] has computed the mod $p$ cohomology ring of symmetric groups for an odd prime $p$. However, nothing much is known about the mod $p$ cohomology of the automorphism group $G_{\Lambda}$ of a finite abelian $p$ group $A_{\Lambda}$ except when the abelian $p$-group $A_{\Lambda}$ is elementary. Even when $\Lambda = (1^n)$ only a vanishing range for the cohomologies is known (see D. Quillen [1], E. M. Friedlander and B. J. Parshall [6]). So obtaining a vanishing criterion for $H^1_{\text{Trivial Action}}(G_{\Lambda}, Z/pZ)$ or $H^2_{\text{Trivial Action}}(G_{\Lambda}, Z/pZ)$ itself is a new result in this direction.

1.1. Motivation. There is a general principle that when cohomology of groups is used to classify obstructions to constructions then $H^2$ classifies the isomorphism classes of structures up to suitable equivalence and $H^1$ acts simply transitively on the set of equivalence classes of automorphisms of a given structure. When we have vanishing theorems for $H^1$ which occurs in most important situations then structures being studied do not have “nontrivial” automorphisms. When we have vanishing theorems for $H^2$ then there are no “nontrivial” obstructions to constructions. In this article, as mentioned in the abstract, we study $H^i_{\Lambda}(G_{\Lambda}, A_{\Lambda})$ for $i,j = 1,2$ and give vanishing results in the case of $\Lambda_1$ for odd primes and give vanishing criteria in the case of $\Lambda_2$ for odd primes $p \neq 3$. Here $\Lambda_1$ is the natural action of $G_{\Lambda}$ on $A_{\Lambda}$ and $\Lambda_2$ is the trivial action of $G_{\Lambda}$ on $A_{\Lambda}$. Henceforth we will specify the action more explicitly instead of using the notation $\Lambda_1, \Lambda_2$ to avoid any confusion.

Since this article concerns finite abelian $p$-groups and their automorphism groups which are characterized by partitions, the study of $H^1$ and $H^2$ is naturally of combinatorial interest. The vanishing criteria is expressed in terms of the combinatorics of the partitions.

In this article, for the computation of $H^i_{\text{Trivial Action}}(G_{\Lambda}, A_{\Lambda}), i = 1,2$, naturally the mod $p$ cohomologies $H^i_{\text{Trivial Action}}(G_{\Lambda}, Z/pZ), i = 1,2$ play an important role. Since automorphism group of a finite abelian $p$-group is a generalization of a finite general linear group over $Z/pZ$, the mod $p$ cohomologies $H^i_{\text{Trivial Action}}(GL_n(Z/pZ), Z/pZ), i = 1,2$ of the finite general linear groups $GL_n(Z/pZ), n \geq 1$ is of special interest.

D. Quillen in his article [12] gives a vanishing range for the mod $p$ cohomology of the finite general linear group and the range was further improved in the article [6] by E. M. Friedlander and B. J. Parshall. There is a landmark result due to D. Quillen [13] which tells us that the $p$-elementary abelian subgroups can be used to understand most of the mod $p$ cohomology of a group. This is also mentioned in the survey article [2] by A. Adem.
1.2. Ideas of Main Results and Brief Summary of the Article. The sections in this article are organized as follows. Section 2 contains the preliminaries required to understand most of the results of the article. Sections 3, 4 give vanishing results for $H^i_{\text{Natural Action}}(G_{\lambda}, A_{\lambda})$, $i = 1, 2$ in Theorems 3.1, 4.1 for odd primes respectively. Sections 5, 6 describe vanishing criteria for $H^i_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda})$, $i = 1, 2$ in terms of partitions in Theorems 5.9, 6.40 respectively. The vanishing and nonvanishing results in the case $H^i_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda})$, $i = 1, 2$ can be observed from the mod $p$ cohomologies $H^i_{\text{Trivial Action}}(G_{\lambda}/Z/pZ)$, $i = 1, 2$.

The proofs of Theorems 3.1, 4.1 are not long, though not obvious and follow with some tricky arguments. The nonvanishing cases of $H^i_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda})$ follow somewhat easily though they are not straightforward. The vanishing case of $H^1_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda})$ given in Theorem 5.5 and the vanishing case of $H^2_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda})$ given in Theorem 6.39 are involved. Much of the paper is devoted to proving these two theorems. So we will mention the ideas involved in proving them.

The proof of Theorem 5.5 requires a computation of the commutator subgroup of $G_{\lambda}$ in the case where the parts of the partition $\lambda$ differ by at most one. The proof of Theorem 6.39 is a long one which occupies most of Section 6. We mention briefly the method of the proof.

To give a vanishing range for $i \in \mathbb{N} \cup \{0\}$ of the mod $p$ cohomologies $H^i_{\text{Trivial Action}}(GL_n(Z/pZ), Z/pZ)$ of a general linear group $GL_n(Z/pZ)$, D. Quillen [12] considers the group of unipotent upper triangular matrices $U_n(Z/pZ) \subset GL_n(Z/pZ)$ and shows that the cohomology groups $H^i_{\text{Trivial Action}}(U_n(Z/pZ), Z/pZ)$ which are semisimple representations of the diagonal subgroup $T_n \subset GL_n(Z/pZ)$ has no nontrivial invariants, that is, $H^i_{\text{Trivial Action}}(U_n(Z/pZ), Z/pZ)^{T_n} = 0$ for $i$ in a certain vanishing range. For this he uses the Poincaré series of $H^*_n(U_n(Z/pZ), Z/pZ)$ as a representation of $T_n$ and estimates the series using the Hochschild-Serre spectral sequences for central extensions. The central extensions are obtained from a chief series of the $p$-group $U_n(Z/pZ)$ which is nilpotent. He shows that the trivial character does not occur in the character of the representation $H^i_{\text{Trivial Action}}(U_n(Z/pZ), Z/pZ)$ for $i$ in a certain vanishing range.

For proving $H^2_{\text{Trivial Action}}(G_{\lambda}, Z/pZ) = 0$ when the partition $\lambda$ has the property that any two consecutive parts of the partition differ by at most one, first we consider a suitable $p$-Sylow subgroup $P_{\lambda} \subset G_{\lambda}$ and observe that $H^2_{\text{Trivial Action}}(P_{\lambda}, Z/pZ)$ is a semisimple representation for the restricted diagonal subgroup $\mathcal{D}_{\lambda} \subset G_{\lambda}$ consisting of diagonal matrices whose orders divide $p - 1$. Then we prove that there are no nontrivial $\mathcal{D}_{\lambda}$ invariants in $H^2_{\text{Trivial Action}}(P_{\lambda}, Z/pZ)$, that is,
$H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^D_\Lambda = 0$. Here instead of using the method of Poincaré series and estimating the series with Hochschild-Serre spectral sequence as given in D. Quillen [12], we use the extended Hochschild-Serre exact sequence for central extensions (a result due to N. Iwahori and H. Matsumoto [7], Proposition 1.1, Page 132) repeatedly. The central extensions here are obtained from a chief series $N^s_\Lambda \triangleright P_\Lambda$ for $s \in S$ a totally ordered set. Then we compute the $D_\Lambda$ invariants $H^2_{\text{Trivial Action}}(P_\Lambda N^s_\Lambda, \mathbb{Z}/p\mathbb{Z})^D_\Lambda$ using the extended Hochschild-Serre exact sequence and show that there are no nontrivial invariants $H^2_{\text{Trivial Action}}(P_\Lambda, \mathbb{Z}/p\mathbb{Z})^D_\Lambda$. The chief series computation and calculation of the $D_\Lambda$ invariants are combinatorially technical as they involve the partition $\Lambda$.

2. Preliminaries

In this section we mention the required preliminaries needed in this article.

2.1. Commutator Subgroup of $GL_n(\mathbb{Z}/p^k\mathbb{Z})$.

We begin with a proposition.

**Proposition 2.1.** Let $p$ be a prime and $n, k$ be two positive integers. The group $SL_n(\mathbb{Z}/p^k\mathbb{Z})$ is generated by elementary matrices.

**Proof.** Clearly all elementary matrices have determinant 1. Let $A \in SL_n(\mathbb{Z}/p^k\mathbb{Z})$. There exists an entry in the first column of $A$ which is a unit in the ring $\mathbb{Z}/p^k\mathbb{Z}$. By using an elementary matrix bring this entry to $11^{th}$ position. Now again using elementary matrices clear all the entries in the first row and first column and make them zero except the $11^{th}$ entry which is a unit. Now continue this process to the remaining submatrix and reduce $A$ to a diagonal matrix $\text{Diag}(a_1, a_2, \cdots, a_n)$ of determinant one. Let $E_{ij}(\alpha) = I + e_{ij}(\alpha)$ for $1 \leq i \neq j \leq n$ where $e_{ij}(\alpha)$ is the $n \times n$ matrix with all zero entries except the $ij^{th}$ entry which is $\alpha$. Now we observe the following. If

$$B = E_{n, n-1}(1-a_n^{-1})E_{n-1, n-1}(1)a_n E_{n-1, n}(a_n^{-1})$$

then $B \text{Diag}(a_1, a_2, \cdots, a_n) = \text{Diag}(a_1, a_2, \cdots, a_{n-1}a_n, 1)$. Hence diagonal matrices of determinant one are product of elementary matrices. This completes the proof. $\blacksquare$

**Theorem 2.2.** Let $p$ be a prime and $n, k$ be two positive integers. The commutator subgroup of $GL_n(\mathbb{Z}/p^k\mathbb{Z})$ is $SL_n(\mathbb{Z}/p^k\mathbb{Z})$ unless $p = 2, n = 2$ in which case the commutator subgroup is strictly contained in $SL_2(\mathbb{Z}/2^k\mathbb{Z})$.

**Proof.** Let $E_{ij}(\alpha) = I + e_{ij}(\alpha)$ for $1 \leq i \neq j \leq n$ where $e_{ij}(\alpha)$ is the $n \times n$ matrix with all zero entries except the $ij^{th}$ entry which is $\alpha$. Then we have for $n \geq 3$ and...
r \neq i \neq j \neq r we have \[E_{ir}(\alpha), E_{rj}(\beta)] = E_{ij}(\alpha \beta).\] Hence for n \geq 3, the commutator subgroup contains SL_n(\mathbb{Z}/p^k\mathbb{Z}) using Proposition 2.1. Conversely since the group \[G \frac{GL_n(\mathbb{Z}/p^k\mathbb{Z})}{SL_n(\mathbb{Z}/p^k\mathbb{Z})} \cong (\mathbb{Z}/p^k\mathbb{Z})^* is abelian via the determinant homomorphism, we have the commutator subgroup is contained in SL_n(\mathbb{Z}/p^k\mathbb{Z}). Hence for n \geq 3, we have SL_n(\mathbb{Z}/p^k\mathbb{Z}) = [GL_n(\mathbb{Z}/p^k\mathbb{Z}), GL_n(\mathbb{Z}/p^k\mathbb{Z})].

Now assume n = 2. If p > 2 then there exists \( \beta \in \mathbb{Z}/p^k\mathbb{Z} \) such that both \( \beta, \beta - 1 \) are invertible. Here we have

\[
E_{12}(\alpha) = \text{Diag}(\beta, 1)E_{12}((\beta - 1)^{-1}a) \text{Diag}(\beta^{-1}, 1)E_{12}(-(\beta - 1)^{-1}a)
\]

So \( E_{12}(\alpha) \) and similarly \( E_{21}(\alpha) \) are commutators. Hence again we have \( SL_2(\mathbb{Z}/p^k\mathbb{Z}) = [GL_2(\mathbb{Z}/p^k\mathbb{Z}), GL_2(\mathbb{Z}/p^k\mathbb{Z})] \).

Now assume n = 2, p = 2. Suppose \( E_{12}(1) \) is in the commutator subgroup of \( GL_2(\mathbb{Z}/2^k\mathbb{Z}) \) then reducing modulo 2 we get that \( E_{12}(1) \) is in the commutator subgroup of \( GL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \). But \( E_{12}(1) \) has order 2 in \( GL_2(\mathbb{Z}/2\mathbb{Z}) \). The commutator subgroup \( A_3 \) of \( S_3 \) has no element of order 2 which is a contradiction. Hence \( [GL_2(\mathbb{Z}/2^k\mathbb{Z}), GL_2(\mathbb{Z}/2^k\mathbb{Z})] \neq SL_2(\mathbb{Z}/2^k\mathbb{Z}) \). We infact have that

\[
[GL_2(\mathbb{Z}/2^k\mathbb{Z}), GL_2(\mathbb{Z}/2^k\mathbb{Z})] \subsetneq SL_2(\mathbb{Z}/2^k\mathbb{Z})
\]

since \( GL_2(\mathbb{Z}/2^k\mathbb{Z}) \cong (\mathbb{Z}/2^k\mathbb{Z})^* \) is abelian via the determinant homomorphism. ■

2.2. A Vanishing Criterion for the second Cohomology.

We state the theorem.

**Theorem 2.3.** Let \( G \) be a finite group and \( p \) be a prime. Then the following are equivalent.

1. \( H^2_{\text{Trivial Action}}(G, A) = 0 \) for all finite abelian \( p \)-groups \( A \).
2. \( H^2_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z}) = 0 \).

**Proof.** Clearly (1) \( \Rightarrow \) (2). Now we prove (2) \( \Rightarrow \) (1). Suppose \( H^2_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z}) = 0 \). We prove \( H^2_{\text{Trivial Action}}(G, A) = 0 \) by induction on \( n \) where the cardinality of \( A \) is \( p^n \). For \( n = 1 \) the assertion holds. Now assume the assertion holds for \( n = m \). Let \( A \mid p^{m+1} \). Let \( B \) be an abelian subgroup of index \( p \) in \( A \). Then any 2-cocycle \( c : G \times G \rightarrow A \) gives rise to a 2-cocycle \( \overline{c} : G \times G \rightarrow \overline{A} \cong \mathbb{Z}/p\mathbb{Z} \). Hence \( \overline{c} \) is a coboundary. Let \( v : G \rightarrow A \) be such that \( \overline{c}(g_1, g_2) = \overline{v}(g_1) + \overline{v}(g_2) - \overline{v}(g_1g_2) \) where \( \overline{v} : G \rightarrow \overline{A} \) obtained from \( v \). Let \( \partial v : G \times G \rightarrow A \) be defined by \( \partial v(g_1, g_2) = v(g_1) + v(g_2) - v(g_1g_2) \), a 2-coboundary. Then we have \( c - \partial v : G \times G \rightarrow B \subset A \) is 2-cocycle cohomologous to \( c \). But by induction \( c - \partial v \) is 2-coboundary. Hence \( c \) is a 2-coboundary. Therefore \( H^2_{\text{Trivial Action}}(G, A) = 0 \). This proves (2) \( \Rightarrow \) (1). Hence the proposition follows. ■
2.3. On the Stable Cohomology Classes.
We state the theorem.

**Theorem 2.4.** Let \( r \) be a positive integer and \( p \) be a prime. Let \( c : \mathbb{Z}/p'\mathbb{Z} \times \mathbb{Z}/p'\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) be a 2-cocycle, that is, \( c \in Z^2_{\text{Trivial Action}}(\mathbb{Z}/p'\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \). Let \( \sigma \in (\mathbb{Z}/p'\mathbb{Z})^* \) such that \( \sigma^{p-1} = 1 \). Define another 2-cocycle \( d : \mathbb{Z}/p'\mathbb{Z} \times \mathbb{Z}/p'\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) such that \( d(x, y) = c(\sigma x, \sigma y) \). If \( c \) is not cohomologous to zero, that is, \( c \) is not a 2-coboundary then \( d \) is cohomologous to \( c \) if and only if \( \sigma = 1 \).

**Proof.** First note that \( H^2_{\text{Trivial Action}}(\mathbb{Z}/p'\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \). This follows from the periodic resolution

\[
\cdots \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/p'\mathbb{Z}] \xrightarrow{e-1} \mathbb{Z}[\mathbb{Z}/p'\mathbb{Z}] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/p'\mathbb{Z}] \xrightarrow{e-1} \mathbb{Z}[\mathbb{Z}/p'\mathbb{Z}] \xrightarrow{N} Z \rightarrow 0,
\]

where \( N = 1 + e + e^2 + \cdots + e^{p-1} \) for \( e \) a generator of \( \mathbb{Z}/p'\mathbb{Z} \), that is \( \mathbb{Z}/p'\mathbb{Z} = \langle e \rangle \). The group \( \mathbb{Z}/p^{r+1}\mathbb{Z} \) occurs as an extension in \( p-1 \) different ways. Let \( k_1, k_2 \in (\mathbb{Z}/p\mathbb{Z})^* \). Consider the following diagram where \( i^k(1) = p^r k_1 \) and \( i^{k_2}(1) = p^r k_2 \). The map \( \pi \) is reduction modulo \( p^r \).

\[
\begin{array}{c}
0 \xrightarrow{} \mathbb{Z}/p\mathbb{Z} \xrightarrow{i^k_1} \mathbb{Z}/p'^{r+1}\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\pi} 0 \\
| \downarrow \text{id}_{\mathbb{Z}/p\mathbb{Z}} | \quad \phi \quad \downarrow \text{id}_{\mathbb{Z}/p\mathbb{Z}} | \\
0 \xrightarrow{} \mathbb{Z}/p\mathbb{Z} \xrightarrow{i^k_2} \mathbb{Z}/p'^{r+1}\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\pi} 0
\end{array}
\]

If \( \phi \) is an automorphism then there exists \( s \in (\mathbb{Z}/p'^{r+1}\mathbb{Z})^* \) such that \( \phi(x) = sx \) for all \( x \in \mathbb{Z}/p'^{r+1}\mathbb{Z} \). Since \( \pi \circ \phi = \pi \) we have \( s \equiv 1 \mod p^r \). On the other hand we have \( \phi \circ i^k_1 = i^k_2 \). Therefore \( p^r s k_1 \equiv p^r k_2 \mod p^{r+1} \Rightarrow s k_1 \equiv k_2 \mod p \Rightarrow k_1 \equiv k_2 \mod p \). So we get \( p-1 \) inequivalent extensions for \( k \in (\mathbb{Z}/p\mathbb{Z})^* \).

Consider the standard 2-cocycle \( f : \mathbb{Z}/p'\mathbb{Z} \times \mathbb{Z}/p'\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) of the standard extension

\[
0 \rightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p'^{r+1}\mathbb{Z} \xrightarrow{\pi \mod p^r} \mathbb{Z}/p\mathbb{Z} \rightarrow 0
\]

where \( i(1) = p^r, i(a) = p^r a, \pi(\sum_{i=0}^{r} a_i p^i) = \sum_{i=0}^{r-1} a_i p^i. \) Here \( f(a, b) = [\frac{a + b}{p^r}] \) for \( a, b \in \{0, 1, \cdots, p^r - 1\} \) and \( 0 \leq a + b \leq 2p^r - 2 \). Now \( H^2_{\text{Trivial Action}}(\mathbb{Z}/p'\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \) and the 2-cocycles \( f, 2f, \cdots (p-1)f \) are mutually non-cohomologous representing all the nonzero cohomology classes of \( H^2_{\text{Trivial Action}}(\mathbb{Z}/p'\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \). For \( 1 \leq k \leq p-1 \), let \( A_{kf} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p'\mathbb{Z} \) as a set. Define a group structure on \( A_{kf} \) as follows. For \( (a, b), (a', b') \in A_{kf} \)

\[
(a, b) + (a', b') = (a + a' + kf(b, b'), b + b').
\]
Then the following diagram commutes for $k = 1$.

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i} & \mathbb{Z}/p^{r+1}\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \phi \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i_1} & A_f & \xrightarrow{\pi_1} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where $i_1(a) = (a,0), \phi(\sum_{i=0}^r a_i p^i) = (a_r, \sum_{i=0}^{r-1} a_i p^i), 0 \leq a_0, a_1, \cdots, a_r \leq p-1, \pi_1(a,b) = b$. Let $\kappa \in (\mathbb{Z}/p^r\mathbb{Z})^\ast$ be such that $\kappa^{p-1} \equiv 1 \pmod{p'}$. Let $k \in (\mathbb{Z}/p\mathbb{Z})^\ast$ be such that $\kappa \equiv k \pmod{p}$. Let $B_{k,f} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z}$ as a set. Define a group structure on $B_{k,f}$ as follows. For $(a,b), (a',b') \in B_{k,f}$

$$(a,b) + (a',b') = (a + a' + f(kb, kb'), b + b').$$

Let $\tilde{\kappa} \in (\mathbb{Z}/p^{r+1}\mathbb{Z})^\ast$ be such that $\tilde{\kappa} \equiv \kappa \pmod{p'}$.

Consider the following diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i_k} & A_{kf} & \xrightarrow{\pi_k} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \phi \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{j_k} & A_f & \xrightarrow{\pi_1} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \phi^{-1} \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{l_k} & \mathbb{Z}/p^{r+1}\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \phi_k \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i} & \mathbb{Z}/p^{r+1}\mathbb{Z} & \xrightarrow{\beta_k} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \phi \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i_1} & A_f & \xrightarrow{\gamma_k} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0 \\
& & 1d_{\mathbb{Z}/p\mathbb{Z}} & & \mu \cong & & 1d_{\mathbb{Z}/p^r\mathbb{Z}} & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{m_k} & B_{k,f} & \xrightarrow{\delta_k} & \mathbb{Z}/p^r\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where

- $i_k(a) = (a,0), j_k(a) = (k^{-1}a,0), l_k(a) = p^rk^{-1}a, m_k(a) = (a,0),$
- $\pi_k(a,b) = b, \beta_k(x) = (\kappa)^{-1}x \pmod{p'}, \gamma_k(a,b) = \kappa^{-1}b, \delta_k(a,b) = b,$
- $\psi(a,b) = (k^{-1}a,b), \phi_k(x) = \bar{x}, \mu(a,b) = (a, \kappa^{-1}b).$

The above diagram commutes and all vertical maps are group isomorphisms and all horizontal maps are group homomorphisms. The above diagram gives an equivalence of extensions $A_{kf}$ and $B_{k,f}$. So we have the cocycles $kf$ and $f_k$ are
cohomologous where \( f(κa, kb) = f(κ, kb) \). Therefore the set \( \{f_κ, f_κ, \cdots, f_κ_p-1\} \)
represent distinct nontrivial cohomology classes in \( H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \)
where \( κ_i \equiv i \mod p \). So we have for any \( 1 ≤ s, t ≤ p−1, σ, τ ∈ (\mathbb{Z}/p\mathbb{Z})^* \)
such that \( σ ≡ s \mod p \) and \( τ ≡ t \mod p \), the cocycles
\[
 sf_σ, tf_σ, stf, f_στ
\]
represent the same cohomology class. Finally note that we have
\[
\mathbb{Z}/(p−1)\mathbb{Z} \cong \{κ ∈ (\mathbb{Z}/p\mathbb{Z})^* \mid κ^{p−1} ≡ 1 \mod p\} \overset{\text{mod } p}{→} (\mathbb{Z}/p\mathbb{Z})^*
\]
is an isomorphism.
We have that \( c \) is a 2-coboundary if and only if \( d \) is a 2-coboundary. Assume that
\( c \) is not a 2-coboundary. So there exists \( 1 ≤ t ≤ p−1 \) such that \( c = tf \). So we
have \( d = tf_o = stf \). Hence \( c \) and \( d \) represent the same class if and only if \( σ = 1 \).
This proves the theorem.

2.4. Calculational Methods and Computations.
Calculating the cohomology of finite groups can be quite challenging, as it involves
a number of complicated ingredients. For a prime \( p \), we outline one technique of computing the \( \mod p \) cohomology which is used in this article.
Here in this section we assume that \( G \) is a finite group which acts trivially on
\( \mathbb{Z}/p\mathbb{Z} \).

Let \( \mathcal{F} \) denote a family of subgroups of \( G \) which satisfies the following two properties.

(1) If \( H ∈ \mathcal{F}, K ⊂ H \), then \( K ∈ \mathcal{F} \).
(2) If \( g ∈ G, H ∈ \mathcal{F} \) then \( gHg^{-1} ∈ \mathcal{F} \).

Then we can define for \( i = 1, 2 \)
\[
\lim_{H ∈ \mathcal{F}} H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}) = \{(a_H) \mid a_K = \text{res}^H_K a_H, \text{ if } K ⊂ H, \}
\]
\[
a_K = c_g(a_H) \text{ if } K = gHg^{-1} \subseteq \prod_{H ∈ \mathcal{F}} H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}),
\]
where \( c_g : H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}) → H^i_{\text{Trivial Action}}(gHg^{-1}, \mathbb{Z}/p\mathbb{Z}) \)
is the conjugation map which sends the cohomology class of \( c ∈ Z^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}) \)
to the cohomology class of \( d ∈ Z^i_{\text{Trivial Action}}(gHg^{-1}, \mathbb{Z}/p\mathbb{Z}) \) where \( d(g_1, \cdots, g_{i−1}) = c(g^{-1}g_1g, \cdots, g_{i−1}g, g) \).
Here \( g_i ∈ gHg^{-1}, g ∈ G \).

Remark 2.5. Let \( G \) be a finite group, \( N \) be a normal subgroup of \( G \), then \( G \) acts on
\( H^i_{\text{Trivial Action}}(N, \mathbb{Z}/p\mathbb{Z}) \) for \( i = 1, 2 \) via the action map \( g → c_g \). It is standard
result in group cohomology that \( c_n \) acts like identity for all \( n ∈ N \). Hence
\( H^i_{\text{Trivial Action}}(N, \mathbb{Z}/p\mathbb{Z}) \) becomes a \( \mathbb{Z}_N \)-module for \( i = 1, 2 \).
Theorem 2.6. Let $S_p(G)$ denote the family of all $p$-subgroups of $G$. Then for $i = 1, 2$ the restrictions induce an isomorphism

$$H^i_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z}) \cong \lim_{P \in S_p(G)} H^i_{\text{Trivial Action}}(P, \mathbb{Z}/p\mathbb{Z}).$$

Proof. For a proof of this theorem, refer to G. Karpilovsky [8], Chapter 9 on Group Cohomology or refer to H. Cartan, S. Eilenberg [5].

In order to compute $H^2_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z})$ we need to find out a collection of subgroups $H_i, 1 \leq i \leq l$ such that the map

$$H^2_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \bigoplus_{i=1}^l H^2_{\text{Trivial Action}}(H_i, \mathbb{Z}/p\mathbb{Z})$$

is injective in which case this collection is said to detect the cohomology. The philosophy used here is:

Reduce to the Sylow $p$-subgroup via the Cartan–Eilenberg result, and then combine information about the cohomology of $p$-groups with stability conditions.

For a group $G$ and a subgroup $H \subseteq G$, for $i = 1, 2$ we say an element $x \in H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z})$ is $G$-stable or simply stable if

$$\text{Res}^H_{gHg^{-1}\cap H}(x) = \text{Res}^{gHg^{-1}}_{gHg^{-1}\cap H}(c_g(x)) \text{ for all } g \in G.$$  

Remark 2.7. For a group $G$ and subgroup $H \subseteq G$, even though for $i = 1, 2$ the space $H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z})$ need not be a $G$-module, we can define a $G$-stable submodule consisting of $G$-stable elements of $H^i_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}).$

Theorem 2.8. Let $G$ be a finite group acting trivially on $\mathbb{Z}/p\mathbb{Z}$ for a prime $p$. Let $P \subseteq G$ be a $p$-Sylow subgroup of $G$. Then for $i = 1, 2$, the cohomology group $H^i_{\text{Trivial Action}}(G, \mathbb{Z}/p\mathbb{Z})$ restricts isomorphically onto the $G$-stable submodule of $H^i_{\text{Trivial Action}}(P, \mathbb{Z}/p\mathbb{Z}).$

Proof. For a proof of the theorem refer to G. Karpilovsky [8], Chapter 9, Section 2, Proposition 2.5(iv) on pages 399 – 400 and G. Karpilovsky [8], Chapter 9, Section 5, Theorem 5.3 on page 411.

2.5. The extended Hochschild-Serre Exact Sequence for Central Extensions.

First we mention a remark similar to Remark 2.5 but for general actions.

Remark 2.9. Let $G$ be a group and $N$ be a normal subgroup of $G$. Let $A$ be an abelian group on which $G$ acts via the map $g \rightarrow \cdot : a \rightarrow g\cdot a$. Then $G$ acts on $H^i(N, A), i = 1, 2$ via the action map $g \rightarrow c_g : H^i(N, A) \longrightarrow H^i(N, A)$ where $c_g$ sends the cohomology class of $c \in Z^i(N, A)$ to the cohomology class of $d \in Z^i(N, A)$ where $d(g_1, \cdots, g_i) = c(g^{-1}g_1g, \cdots, g^{-1}g_ig)$. So $H^i(N, A)$ is a $G$-module. In fact $N$ acts trivially on $H^i(N, A)$ which turns $H^i(N, A)$ into a $\mathbb{C}[G/N]$-module for $i = 1, 2$. 
The Hochschild-Serre exact sequence is given in the following theorem.

**Theorem 2.10.** Let $G$ be a group and $N$ be a normal subgroup of $G$. Let $A$ be an abelian group on which $G$ acts. Then the sequence

$$1 \longrightarrow H^1\left(\frac{G}{N}, A^N\right) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(N, A)^G \xrightarrow{\text{Tra}} H^2\left(\frac{G}{N}, A^N\right) \longrightarrow H^2(G, A)$$

is exact where $\text{Inf}$ is the inflation map, $\text{Res}$ is the restriction map and $\text{Tra}$ is the transgression map.

**Proof.** For a proof of the theorem refer to G. Karpilovsky [9], Chapter 1, Section 1, Theorem 1.12 on page 16 or refer to D. Benson [4], page 110 on the inflation restriction sequence which is obtained as a consequence of the spectral sequence of a group extension.

Let $G_1, G_2$ be two groups and $A$ be an abelian group. We define an abelian group $P(G_1, G_2, A)$ as follows:

$$P(G_1, G_2, A) = \{f : G_1 \times G_2 \longrightarrow A \mid f(xy, z) = f(x, z) + f(y, z), \quad f(x, zw) = f(x, z) + f(x, w) \text{ for all } x, y, z, w \in G_2\}. \quad (2.1)$$

The set $P(G_1, G_2, A)$ is a group under addition induced from that of $A$.

**Proposition 2.11.** Let $G_1, G_2$ be two subgroups of $G$ such that $xy = yx$ for all $x \in G_1, y \in G_2$. Let $A$ be an abelian group on which $G$ acts trivially. For any given $c \in Z^2_{\text{Trivial Action}}(G, A)$, let $\beta : G_1 \times G_2 \longrightarrow A$ be defined by $\beta(x, y) = c(x, y) - c(y, x)$ for $x \in G_1, y \in G_2$. Then $\beta \in P(G_1, G_2, A)$ and $\beta$ depends only on the cohomology class $[c] \in H^2_{\text{Trivial Action}}(G, A)$ and the map $\theta : [c] \longrightarrow \beta_\text{c} = \beta$ from $H^2_{\text{Trivial Action}}(G, A)$ to $P(G_1, G_2, A)$ is a well defined group homomorphism.

**Proof.** For a proof of the proposition refer to G. Karpilovsky [9], Chapter 1, Section 2, Lemma 2.2 on page 19.

**Theorem 2.12.** Let $G_1, G_2$ be two groups and $A$ be an abelian group on which $G_1, G_2, G_1 \times G_2$ act trivially. Then

$$H^2_{\text{Trivial Action}}(G_1 \times G_2, A) \cong H^2_{\text{Trivial Action}}(G_1, A) \times H^2_{\text{Trivial Action}}(G_2, A) \times P(G_1, G_2, A).$$

The isomorphism being

$$[c] \longrightarrow (\text{Res}_1([c]), \text{Res}_2([c]), \beta_{[c]})$$

where $\text{Res}_i : H^2_{\text{Trivial Action}}(G_1 \times G_2, A) \longrightarrow H^2_{\text{Trivial Action}}(G_i, A)$ are the restriction maps for $i = 1, 2$ and $\beta_{[c]}(x, y) = c((x, e_{G_2}), (e_{G_1}, y)) - c((e_{G_1}, y), (x, e_{G_2}))$ for all $x \in G_1, y \in G_2$ where $c \in Z^2_{\text{Trivial Action}}(G_1 \times G_2, A)$ is any cocycle representing the cohomology class $[c] \in H^2_{\text{Trivial Action}}(G_1 \times G_2, A)$. (Here $e_{G_i}$ is the identity element of the group $G_i, i=1,2$.)
Proof. For a proof of the theorem refer to G. Karpilovsky [9], Chapter 1, Section 2, Theorem 2.3 on page 20.

As a consequence of the above theorem we have the following corollary.

**Corollary 2.13.** Let $G_i, 1 \leq i \leq n$ be groups. Let $A$ be an abelian group on which $G_i$ act trivially for $1 \leq i \leq n$. Then

$$H^2_{\text{Trivial Action}}(\prod_{i=1}^{n} G_i, A) \cong \prod_{i=1}^{n} H^2_{\text{Trivial Action}}(G_i, A) \times \prod_{1 \leq i < j \leq n} P(G_i, G_j, A)$$

with the isomorphism being

$$[c] \rightarrow ((\text{Res}_i([c]))_{1 \leq i \leq n}, (\beta_{ij})_{1 \leq i < j \leq n})$$

where $\text{Res}_i : H^2_{\text{Trivial Action}}(\prod_{i=1}^{n} G_i, A) \rightarrow H^2_{\text{Trivial Action}}(G_i, A)$ is the restriction map for $1 \leq i \leq n$ and $\beta_{ij} : H^2_{\text{Trivial Action}}(\prod_{i=1}^{n} G_i, A) \rightarrow P(G_i, G_j, A)$ is a map defined as

$$\beta_{ij}([c])(x, y) = c((e_{G_1}, \ldots, e_{G_{i-1}}, x, e_{G_{i+1}}, \ldots, e_{G_n}),(e_{G_1}, \ldots, e_{G_{i-1}}, y, e_{G_{j+1}}, \ldots, e_{G_n})) - c((e_{G_1}, \ldots, e_{G_{j-1}}, y, e_{G_{j+1}}, \ldots, e_{G_n}),(e_{G_1}, \ldots, e_{G_{i-1}}, x, e_{G_{j+1}}, \ldots, e_{G_n}))$$

for $x \in G_i$, $y \in G_j$ for $1 \leq i < j \leq n$ where $c \in Z^2_{\text{Trivial Action}}(\prod_{i=1}^{n} G_i, A)$ is any cocycle representing the cohomology class $[c]$. (Here $e_{G_i}$ is the identity element of the group $G_i, 1 \leq i \leq n$).

In the following theorem we describe the extended Hochschild-Serre exact sequence for central extensions.

**Theorem 2.14.** Let $G$ be a finite group and $H \subseteq G$ be a central subgroup, that is, $H \subseteq Z(G)$. Let $A$ be a finite abelian group on which $G$ acts trivially. Then the following sequence

$$0 \rightarrow \text{Hom}(\frac{G}{H}, A) \xrightarrow{\text{Inf}} \text{Hom}(G, A) \xrightarrow{\text{Res}} \text{Hom}(H, A) \xrightarrow{\text{Tra}} H^2_{\text{Trivial Action}}(\frac{G}{H}, A)$$

$$\xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(G, A) \xrightarrow{\tau = \text{Res} \times \theta} H^2_{\text{Trivial Action}}(H, A) \times P(G, H, A)$$

is exact where $\text{Inf}$ is the inflation map, $\text{Res}$ is the restriction map, $\text{Tra}$ is the transgression map and $\theta : H^2_{\text{Trivial Action}}(G, A) \rightarrow P(G, H, A)$ is the map defined as

$$\theta([c])(g, h) = c(g, h) - c(h, g)$$

for $g \in G, h \in H$ with $c \in Z^2_{\text{Trivial Action}}(G, A)$ being any cocycle which represents the cohomology class $[c]$.

Proof. For a proof of the theorem, refer to G. Karpilovsky [9], Chapter 1, Section 1, Corollary 1.13 on page 17 and G. Karpilovsky [9], Chapter 1, Section 2, Theorem 2.8 on page 23.
2.6. An Algebraic Identity. In this section we prove an algebraic identity which is useful later. Let $X$ be a variable (indeterminate) such that $X^2 = 0$. Let $A, B$ be two $n \times n$ matrices with entries $a_{ij}, b_{ij}$ for $n \geq i > j \geq 1$ and with entries $Xa_{ij}, Xb_{ij}$ for $1 \leq i < j \leq n$ respectively and with diagonal entries being 0 in both of them, where $a_{ij}, b_{ij}, 1 \leq i \neq j \leq n$ are variables. All the variables $a_{ij}, b_{ij}, 1 \leq i \neq j \leq n, X$ commute with each other. Then $A$ and $B$ are nilpotent matrices because $X^2 = 0 \Rightarrow A^{2n} = B^{2n} = 0$. Moreover we have $\text{Trace}(A^i) = \text{Trace}(B^i) = 0$ for $i = 1, i \geq n + 1$. Let $g^a = A + \text{Diag}(1 + Xa_{11}, 1 + Xa_{22}, \ldots, 1 + Xa_{nn}), g^b = B + \text{Diag}(1 + Xb_{11}, 1 + Xb_{22}, \ldots, 1 + Xb_{nn})$ where the elements $a_{ii}, b_{ii}$ appearing in the diagonals of $g^a, g^b$ are also commuting variables which also commute with all the remaining variables. Then we have $g^c = g^a g^b = (L_A + XU_A + XD_A + I_n)(L_B + XU_B + XD_B + I_n)$ where $L_A, L_B$ are the strictly lower triangular part of $A, B$, and $XU_A, XU_B$ are the strictly upper triangular part of $A, B$, and $I_n + XD_A, I_n + XD_B$ are the diagonal parts of $g^a, g^b$ respectively. So $X^2 = 0 \Rightarrow g^c = (L_A + L_B + L_A L_B) + X(L_A U_B + U_A L_B + U_A + U_B) + X(L_AD_B + D_AL_B) + X(D_A + D_B) + I_n$. Now define a matrix $C = [c_{ij}]_{1 \leq i, j \leq n}$ where $c_{ii} = 0$ for $1 \leq i \leq n$, $c_{ij} = (L_A + L_B + L_A L_B)_{ij}$ for $n \geq i > j \geq 1$ the strictly lower triangular part of $g^c$ where we have ignored the coefficients of $X$ in these entries, $c_{ij} = X(L_A U_B + U_A L_B + U_A + U_B)_{ij}$ for $1 \leq i < j \leq n$ the strictly upper triangular part of $g^c$. Note that $C$ is also nilpotent, $C^{2n} = 0, \text{Trace}(C^i) = 0$ for $i = 1, i \geq n + 1$.

**Proposition 2.15.** With notations as above we have for $n \geq 2$,

1. $1 + \sum_{i=2}^{n} (-1)^{i-1} \text{Trace}(N_i) = \text{Det}(I_n + N)$ for $N = A, B, C$.
2. $\text{Det}(I_n + A) + \text{Det}(I_n + B) = 1 + \text{Det}(I_n + A) \text{Det}(I_n + B)$.
3. $\text{Det}(I_n + A) \text{Det}(I_n + B) = \text{Det}(I_n + C) + \text{Trace}(AB)$.

$\text{Trace}(AB) + \sum_{i=2}^{n} (-1)^{i-1} \text{Trace}(\frac{C^i}{i}) - \sum_{i=2}^{n} (-1)^{i-1} \text{Trace}(\frac{A^i}{i}) - \sum_{i=2}^{n} (-1)^{i-1} \text{Trace}(\frac{B^i}{i}) = 0$.

**Proof.** We prove (1) for $N = A$. Note $X^2 = 0$. Hence we have for $i \geq 2$,

$\text{Trace}(A^i) = iX \sum_{n \geq j_1 > j_2 > \ldots > j_i \geq 1} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_i-1,j_i} a_{j_i j_1}$.

Note that the product $Xa_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_i-1,j_i} a_{j_i j_1}$ appears $i$ times in the trace of $A^i$.

Now for $2 \leq i \leq n$ let $S_i = \{\sigma \in S_n \mid \sigma \text{ is an } i\text{-cycle of the form } (j_1 j_2 \cdots j_i) \text{ where } n \geq j_1 > j_2 > \cdots > j_i \geq 1\}$. Since $X^2 = 0$ we have

$\text{Det}(I_n + A) = 1 + X \sum_{i=2}^{n} \left( \sum_{\sigma = (j_1 j_2 \cdots j_i) \in S_i} \text{sign}(\sigma) a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_i-1,j_i} a_{j_i j_1} \right)$. 

Now it follows that $1 + \sum_{i=2}^{n} (-1)^{i-1} \operatorname{Trace}(A_i^T) = \operatorname{Det}(I_n + A)$. Similarly (1) follows for $N = B, C$.

We prove (2). This is clear because we have $(1 + Xu)(1 + Xv) = 1 + Xu + Xv$ since $X^2 = 0$. So (2) follows using (1).

We prove (3). Notice that for $g^a = A + I_n + X \operatorname{Diag}(a_{11}, a_{22}, \ldots, a_{nn})$ we have, since $X^2 = 0$

$$\det(g^a) = \det(I_n + A) + X \left( \sum_{i=1}^{n} a_{ii} \right).$$

(2.2)

Here $X \left( \sum_{i=1}^{n} a_{ii} \right) = \operatorname{Trace}(X \operatorname{Diag}(a_{11}, a_{22}, \ldots, a_{nn}))$. Also we have

$$\det(g^a + XL_M) = \det(g^a)$$

(2.3)

for any strictly lower triangular matrix $L_M$ as this follows since $X^2 = 0$. Now

$$\det(I_n + A) \det(I_n + B) = \det(I_n + A + B + AB).$$

The matrices $C$ and $I_n + A + B + AB$ differ in diagonal entries by a multiple of $X$ contributed by the term $AB$ and in the strictly lower triangular entries a multiple of $X$ again contributed by the term $AB$. Hence we get using Equations 2.2, 2.3 that

$$\det(I_n + A) \det(I_n + B) = \det(I_n + A + B + AB) = \det(I_n + C) + \operatorname{Trace}(AB).$$

This proves (3).

Now (4) is an immediate consequence of (1), (2), (3). Hence the proposition is proved. \[\blacksquare\]

**Theorem 2.16.** Let the matrices $A, B, C$ be as defined before with variables $a_{ij}, b_{ij}, 1 \leq i \neq j \leq n$.

$$\sum_{1 \leq i < j \leq n} a_{ij}b_{ji} + a_{ji}b_{ij} + \sum_{i=2}^{n} (-1)^{i-1} \left( \sum_{n \geq i_1 > i_2 > \ldots > i_j \geq 1} \left( c_{i_1i_2}c_{i_2i_3} \cdots c_{i_{j-1}i_j} \right) - a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{j-1}i_j}a_{i_ji_1} - b_{i_1i_2}b_{i_2i_3} \cdots b_{i_{j-1}i_j}b_{i_ji_1} \right) = 0.$$ 

Proof. The theorem follows from Proposition 2.15(4). The identity is a very general identity which holds in the ring $\mathbb{Z}[a_{ij}, b_{ij} : 1 \leq i \neq j \leq n]$. \[\blacksquare\]

**Remark 2.17.** We apply Theorem 2.16 in Theorem 6.37 later.
3. Computation of $H^1_{\text{Natural Action}}(G_\lambda, A_\lambda)$

**Theorem 3.1.** Let $A$ be an abelian group and $G = \text{Aut}(A)$. Let $q \in \mathbb{Z}$ be such that $q\text{Id}_A : A \to A$ is an automorphism. Then $(q - 1)$ annihilates $H^1_{\text{Natural Action}}(G, A)$. As a consequence, for a partition $A = (A_1 > A_2 > A_3 > \ldots > A_k)$ and $A_\lambda = \bigoplus_{i=1}^k (\mathbb{Z}/p^i\mathbb{Z})^q$, its associated finite abelian p-group where $p$ is a prime with $G_\lambda = \text{Aut}(A_\lambda)$ its automorphism group, we obtain the following.

(a) For $p \neq 2, H^1_{\text{Natural Action}}(G_\lambda, A_\lambda) = 0$.
(b) For $p = 2, H^1_{\text{Natural Action}}(G_\lambda, A_\lambda)$ is a finite elementary abelian 2-group.

**Proof.** Let $f \in Z^1_{\text{Natural Action}}(G, A)$. Then we have the following.

(1) $f(\text{Id}_A) = 0$.
(2) $f(g^{-1}) = -g^{-1}f(g)$.
(3) $(q - 1)f \in B^1_{\text{Natural Action}}(G, A)$.

We prove (1). $f(gh) = gf(h) + f(g)$ for all $g, h \in G$. Hence $f(\text{Id}_A) = f(\text{Id}_A \text{Id}_A) = \text{Id}_Af(\text{Id}_A) + f(\text{Id}_A) = 2f(\text{Id}_A) \Rightarrow f(\text{Id}_A) = 0$. This proves (1).

We prove (2). $0 = f(g^{-1}g) = gf(g^{-1}) + f(g) \Rightarrow f(g^{-1}) = -g^{-1}f(g)$ for all $g \in G$. This proves (2).

We prove (3). $f(q\text{Id}_A) = f(qgg^{-1}) = qgf(g^{-1}) + gf(q\text{Id}_A) + f(g) \Rightarrow (q - 1)f(g) = gf(q\text{Id}_A) - f(q\text{Id}_A)$. This implies that $(q - 1)f \in B^1_{\text{Natural Action}}(G, A)$. This proves (3).

Now let $q$ be a prime such that $q\text{Id}_{A_\lambda} : A_\lambda \to A_\lambda$ is an isomorphism. Here it is enough to choose $q = 2$ if $p$ is odd and $q = 3$ if $p = 2$. Then (a), (b) follow. This proves the theorem.

**Remark 3.2.** For more details about the group $H^1_{\text{Natural Action}}(G_\lambda, A_\lambda)$ when $p = 2$ and a description of a basis elements of the vector space $H^1_{\text{Natural Action}}(G_\lambda, A_\lambda)$, refer to W. H. Mills [11].

4. Computation of $H^2_{\text{Natural Action}}(G_\lambda, A_\lambda)$

**Theorem 4.1.** Let $A$ be an abelian group and $G = \text{Aut}(A)$. Let $q \in \mathbb{Z}$ be such that $q\text{Id}_A : A \to A$ is an automorphism. Then $(q - 1)^2$ annihilates $H^2_{\text{Natural Action}}(G, A)$. As a consequence, for a partition $A = (A_1 > A_2 > A_3 > \ldots > A_k)$ and $A_\lambda = \bigoplus_{i=1}^k (\mathbb{Z}/p^i\mathbb{Z})^q$, its associated finite abelian p-group where $p$ is a prime with $G_\lambda = \text{Aut}(A_\lambda)$ its automorphism group, we obtain the following.

(a) For $p \neq 2, H^2_{\text{Natural Action}}(G_\lambda, A_\lambda) = 0$.
(b) For $p = 2, H^2_{\text{Natural Action}}(G_\lambda, A_\lambda)$ is a direct sum of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$ and finitely many copies of $\mathbb{Z}/4\mathbb{Z}$. 

Proof. We have $H^2_{\text{Natural Action}}(G, A) = \frac{Z^2_{\text{Natural Action}}(G, A)}{B^2_{\text{Natural Action}}(G, A)}$ where $Z^2_{\text{Natural Action}}(G, A) = \{ u : G \times G \rightarrow A \mid xu(y, z) + u(x, yz) = u(x, y) + u(xy, z) \}$, $B^2_{\text{Natural Action}}(G, A) = \{ u : G \times G \rightarrow A \mid u(x, y) = xv(y) + v(x) - v(xy) \text{ for some } v : G \rightarrow A \}$.

Clearly if $u(x, y) = xv(y) + v(x) - v(xy)$ for some $v : G \rightarrow A$ then we have $xu(y, z) + u(x, yz) = u(x, y) + u(xy, z)$. Conversely let

$$(4.1) \quad xu(y, z) + u(x, yz) = u(x, y) + u(xy, z)$$

for some $u : G \times G \rightarrow A$. Then define $v : G \rightarrow A$ as

$$v(x) = u(qId_A, x) - u(x, qId_A).$$

We have by substituting $x = qId_A$ in Equation 4.1

$$(4.2) \quad qu(y, z) + u(qId_A, yz) = u(qId_A, y) + u(qy, z).$$

By substituting $z = qId_A$ we have

$$qu(y, qId_A) + u(qId_A, qy) = u(qId_A, y) + u(qy, qId_A)$$

which implies

$$(4.3) \quad (q - 1)u(y, qId_A) = v(y) - v(qy)$$

and

$$(4.4) \quad (q - 1)u(qId_A, y) = qv(y) - v(qy).$$

Replacing the variable $y$ by the variable $x$ and the variable $z$ by the variable $y$ in Equation 4.2 we get

$$(4.5) \quad qu(x, y) + u(qId_A, xy) = u(qId_A, x) + u(qx, y).$$

Using Equations 4.3, 4.4 we get

$$(4.6) \quad q(q - 1)u(x, y) + qv(xy) - v(qxy) = qv(x) - v(qx) + (q - 1)u(qx, y).$$

Substituting $y = qId_A$ in Equation 4.1 we get

$$xu(qId_A, z) + u(x, qz) = u(x, qId_A) + u(qx, z).$$

Now replace variable $z$ in the previous equation by the variable $y$ to get

$$xu(qId_A, y) + u(x, qy) = u(x, qId_A) + u(qx, y).$$

Using Equations 4.3, 4.4 we get

$$(4.7) \quad qxv(y) - xv(qy) + (q - 1)u(x, qy) = v(x) - v(qx) + (q - 1)u(qx, y).$$

By substituting $z = qId_A$ in Equation 4.1 we get

$$xu(y, qId_A) + u(x, qy) = u(x, y) + u(xy, qId_A).$$
Using Equations 4.3, 4.4 we get
\[(q - 1)^2 u(x, y) = (q - 1)v(x) - (q - 1)v(xy).
\]
Now eliminating \((q - 1)u(qx, y)\) and \((q - 1)u(x, qy)\) from Equations 4.6, 4.7, 4.8 and solving for \(u(x, y)\) we get
\[
(q - 1)^2 u(x, y) = (q - 1)xv(y) + (q - 1)v(x) - (q - 1)v(xy).
\]
This proves that \((q - 1)^2 u : G \times G \rightarrow A\) is a 2-coboundary, that is, \((q - 1)^2 u \in B_{\text{Natural Action}}^2(G, A)\). Therefore we have that \(H_{\text{Natural Action}}^2(G, A)\) is annihilated by \((q - 1)^2\).

Now let \(q\) be a prime such that \(q\text{Id}_{A_\perp} : A_\perp \rightarrow A_\perp\) is an isomorphism. Here it is enough to choose \(q = 2\) if \(p\) is odd and \(q = 3\) if \(p = 2\). Then (a), (b) follow. This proves the theorem.

**Remark 4.2.** In case \(p = 2\) in the previous theorem, the number of copies of \(\mathbb{Z}/2\mathbb{Z}\) and the number of copies of \(\mathbb{Z}/4\mathbb{Z}\) that occurs in \(H_{\text{Natural Action}}^2(G, A)\) is still not known. We provide two examples below illustrating the remark.

**Example 4.3.** Let \(A = \mathbb{Z}/4\mathbb{Z}\) and \(G = \mathbb{Z}/2\mathbb{Z}\). Then there are at least two possibilities for the extension \(E\) of \(G\) by \(A\) which gives rise to the natural action of \(G = \text{Aut}(A)\) on \(A\). They are \(E = D_8, Q_8\). Hence \(H_{\text{Natural Action}}^2(G, A) \neq 0\).

**Example 4.4.** Let \(A = (\mathbb{Z}/2\mathbb{Z})^2\) and \(G = \text{GL}_2(\mathbb{Z}/2\mathbb{Z})\). Then any extension \(E\) of \(G\) by \(A\) which gives rise to the natural action of \(G = \text{Aut}(A)\) on \(A\) is a nonabelian group of order 24 with trivial center. Hence \(E \cong S_4\) and there is only one normal subgroup \(V_4\) of \(S_4\) which is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2\). Any automorphism of \(V_4\) extends to an automorphism of \(E = S_4\). Also note that all automorphisms of \(S_4, S_3\) are inner automorphisms and we have \(S_3 \cong \text{Inn}(S_3) = \text{Aut}(S_3), S_4 \cong \text{Inn}(S_4) = \text{Aut}(S_4)\). Any exact sequence with \(E = S_4\) looks like
\[
0 \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow S_4 \rightarrow \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \rightarrow 1
\]
where \(\psi\) is an automorphism \((\mathbb{Z}/2\mathbb{Z})^2\) and \(\phi\) is an automorphism of \(\text{GL}_2(\mathbb{Z}/2\mathbb{Z}) = S_3\) and \(i((\mathbb{Z}/2\mathbb{Z})^2) = V_4\). But the following two extensions are equivalent.
\[
0 \rightarrow (\mathbb{Z}/2\mathbb{Z})^2 \overset{Id_A}{\longrightarrow} S_4 \overset{\phi \circ \pi}{\longrightarrow} \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \overset{1}{\longrightarrow} 1
\]
where \(\tilde{\psi}\) is an extension of \(\psi\). Let \(g \in S_4\) and \(\psi_g\) be inner automorphism of \(S_4\) induced by \(g\). If \(\phi_1, \phi_2\) are two automorphisms of \(\text{GL}_2(\mathbb{Z}/2\mathbb{Z})\) then the following
two extensions

\[ 0 \to (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{i} S_4 \xrightarrow{\phi_1 \circ \pi} GL_2(\mathbb{Z}/2\mathbb{Z}) \to 1 \]
\[ 0 \to (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{i} S_4 \xrightarrow{\phi_2 \circ \pi} GL_2(\mathbb{Z}/2\mathbb{Z}) \to 1 \]

are equivalent if and only if \( g \in V_4 = \{ \text{Identity}, (12)(34), (13)(24), (14)(23) \} \cap S_4 \) and \( \phi_1 = \phi_2 \). Among all exact sequences of the form

\[ 0 \to (\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{i} S_4 \xrightarrow{\phi \circ \pi} GL_2(\mathbb{Z}/2\mathbb{Z}) \to 1 \]

the group \( GL_2(\mathbb{Z}/2\mathbb{Z}) \) induces a natural action on \((\mathbb{Z}/2\mathbb{Z})^2\) if and only if \( \phi \) is the identity automorphism. Hence there is essentially only one way in which \( S_4 \) appears as an extension for the natural action of \( GL_2(\mathbb{Z}/2\mathbb{Z}) \) on \((\mathbb{Z}/2\mathbb{Z})^2\) which is the semidirect product \((\mathbb{Z}/2\mathbb{Z})^2 \rtimes GL_2(\mathbb{Z}/2\mathbb{Z})\). So we have \( H_2^{\text{Natural Action}}(G, A) = 0 \).

5. Computation of \( H_1^{\text{Trivial Action}}(G_{\lambda}, A_{\lambda}) \)

**Proposition 5.1.** Let \( \lambda = (\lambda^1_1 > \lambda^2_2 > \lambda^3_3 > \ldots > \lambda^k_k) \) be a partition. Let \( A_{\lambda} \) be the finite abelian \( p \)-group associated to \( \lambda \), where \( p \) is a prime. Let \( G_{\lambda} = \text{Aut}(A_{\lambda}) \) be its automorphism group. Then the following are equivalent.

1. \( H_1^{\text{Trivial Action}}(G_{\lambda}, A_{\lambda}) \neq 0 \).
2. \( \text{Hom}(G_{\lambda}, A_{\lambda}) \neq 0 \).
3. \( \text{Hom}(G_{\lambda}, \mathbb{Z}/p\mathbb{Z}) \neq 0 \), that is, there exists a normal subgroup \( N \trianglelefteq G_{\lambda} \) of index \( p \), that is, \( [G_{\lambda} : N] = p \).
4. The order of the abelian group \( \frac{G_{\lambda}}{[G_{\lambda}, G_{\lambda}]} \) is divisible by \( p \).

**Proof.** The proof is immediate. \( \blacksquare \)

**Proposition 5.2.**

1. We have \( p \) divides \( \frac{\text{GL}_n(\mathbb{Z}/p^n\mathbb{Z})}{[\text{GL}_n(\mathbb{Z}/p^n\mathbb{Z}), \text{GL}_n(\mathbb{Z}/p^n\mathbb{Z})]} \) if \( k \geq 2 \) for any prime \( p \) for all \( n \geq 1 \).
2. For \( k = 1 \) and \( p \) odd we have \( p \) does not divide \( \frac{\text{GL}_n(\mathbb{Z}/p^n\mathbb{Z})}{[\text{GL}_n(\mathbb{Z}/p^n\mathbb{Z}), \text{GL}_n(\mathbb{Z}/p^n\mathbb{Z})]} \) for all \( n \geq 1 \).
3. For \( k = 1, p = 2 \), we have \( 2 \) does not divide \( \frac{\text{GL}_n(\mathbb{Z}/2^n\mathbb{Z})}{[\text{GL}_n(\mathbb{Z}/2^n\mathbb{Z}), \text{GL}_n(\mathbb{Z}/2^n\mathbb{Z})]} \) if and only if \( n \geq 3 \) or \( n = 1 \).
Proof. We prove (1). If $k \geq 2$ and either $p \neq 2$ or $n \neq 2$ then the commutator subgroup is $SL_n(Z/p^kZ)$ using Theorem 2.2. Hence we have

$$|GL_n(Z/p^kZ)\left/SL_n(Z/p^kZ)\right| = p^{k-1}(p-1) \Rightarrow p$$

| divides | $GL_n(Z/p^kZ)$ \left/ [GL_n(Z/p^kZ),GL_n(Z/p^kZ)] \right. |

If $k \geq 2, n = 2$ and $p = 2$ then $[GL_2(Z/2^kZ),GL_2(Z/2^kZ)] \subseteq SL_2(Z/2^kZ)$ using Theorem 2.2. Hence we have

$$2^{k-1} | \left|GL_n(Z/p^kZ)\left/SL_n(Z/p^kZ)\right| \right| \text{ divides } GL_n(Z/p^kZ) \left/ [GL_n(Z/p^kZ),GL_n(Z/p^kZ)] \right. .$$

Since $k - 1 \geq 1$ we have $2$ divides $\left|GL_n(Z/p^kZ)\left/ [GL_n(Z/p^kZ),GL_n(Z/p^kZ)] \right. \right|$. This proves (1).

We prove (2). For $k = 1, p \neq 2$ we have that the commutator subgroup of $GL_n(Z/pZ)$ is $SL_n(Z/pZ)$ using Theorem 2.2. Hence $\left|GL_n(Z/pZ)\left/ [GL_n(Z/pZ),GL_n(Z/pZ)] \right. \right| = p - 1$. This proves (2).

Now we prove (3). For $k = 1, p = 2$ and $n = 1$, $GL_1(Z/2Z)$ is trivial. For $k = 1, p = 2, n \geq 3$, the commutator subgroup of $GL_n(Z/2Z)$ is $SL_n(Z/2Z) = GL_n(Z/2Z)$ using Theorem 2.2. Hence $\left|GL_n(Z/2Z)\left/ [GL_n(Z/2Z),GL_n(Z/2Z)] \right. \right| = 2^{k-1}$ is trivial. For $k = 1, n = 2, p = 2$, we have $\frac{GL_2(Z/2Z)}{[GL_2(Z/2Z),GL_2(Z/2Z)]} \cong Z/2Z$. This proves (3). □

**Theorem 5.3.** Let $p$ be a prime and $n,k$ be two positive integers. Then

$${H}^1_{\text{trivial Action}}(GL_n(Z/p^kZ),(Z/p^kZ)^n) \neq 0$$

if and only if $k \geq 2$ or $k = 1, p = 2, n = 2$.

Proof. This follows from Propositions 5.1, 5.2. □

Now we prove another required proposition.

**Proposition 5.4.** Let $p$ be an odd prime, $k,n$ be positive integers. Let $M_n(Z/p^kZ)$ be the space of $n \times n$ matrices over the ring $Z/p^kZ$. Let $K_n(Z/p^kZ)$ be the kernel of the composed homomorphism $GL_n(Z/p^kZ) \xrightarrow{\text{mod} \ p} GL_n(Z/pZ) \xrightarrow{\text{Det}} (Z/pZ)^*$. If $g \in K_n(Z/p^kZ)$ then there exists a matrix $h \in M_n(Z/p^kZ)$ such that

$$g = g' + ph.$$
So by induction there exists a $g'' \in SL_n(\mathbb{Z}/p^{k-1}\mathbb{Z})$ and $h'' \in M_n(\mathbb{Z}/p^{k-1}\mathbb{Z})$ such that $g \mod p^{k-1} = g'' + ph''$. Now there exists $g' \in SL_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $\phi(g') = g''$. Hence we have $g \mod p = g' \mod p \Rightarrow g = g' + ph$ for some $h \in M_n(\mathbb{Z}/p^k\mathbb{Z})$.

5.0.1. The Vanishing Case of $H^1_{\text{Trivial Action}}(G_\Delta, A_\Delta)$.

**Theorem 5.5.** Let $\lambda = (\lambda_1^p, \lambda_2^p > \lambda_3^p > \ldots > \lambda_k^p)$ be a partition such that $\lambda_i = k - i + 1$. Let $A_\Delta$ be the finite abelian group associated to $\Delta$, where $p$ is an odd prime. Let $G_\Delta = Aut(A_\Delta)$ be its automorphism group. Then $H^1_{\text{Trivial Action}}(G_\Delta, A_\Delta) = \text{Hom}(G_\Delta, A_\Delta) = 0$.

**Proof.** It suffices to show that $p$ does not divide the order of the group $G_\Delta$ using Proposition 5.1. We show that the commutator subgroup $G'_\Delta = [G_\Delta, G_\Delta]$ of $G_\Delta$ is the kernel of the surjective homomorphism

$$\Phi : G_\Delta \rightarrow (\mathbb{Z}/p^k\mathbb{Z})^* \times (\mathbb{Z}/p^k\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p^k\mathbb{Z})^*,$$

given as follows: If $g = [g_{im}]_{1 \leq m, n \leq k}$, $g_{mn} : (\mathbb{Z}/p^m\mathbb{Z})^\rho_n \rightarrow (\mathbb{Z}/p^n\mathbb{Z})^\rho_n$, then

$$\Phi(g) = (\text{Det}(g_{11}) \mod p, \text{Det}(g_{22}) \mod p, \ldots, \text{Det}(g_{kk}) \mod p).$$

It is clear that $\ker(\Phi)$ contains the commutator subgroup $G'_\Delta$. Now we show that $\ker(\Phi) \subseteq G'_\Delta$.

Let $I_\Delta$ be the identity element of $G_\Delta$. First it is clear for $1 \leq i < j \leq k$ the block elementary matrix $E_{ij}(p^{\lambda_i - \lambda_j}A) = I_\Delta + e_{ij}(p^{\lambda_i - \lambda_j}A) \in G'_\Delta$ where $e_{ij}(p^{\lambda_i - \lambda_j}A)$ is a matrix with all zero blocks except for the $ij^{\text{th}}$ block entry which is $p^{\lambda_i - \lambda_j}A$ where $p^{\lambda_i - \lambda_j}A : (\mathbb{Z}/p^i\mathbb{Z})^{\rho_i} \rightarrow (\mathbb{Z}/p^j\mathbb{Z})^{\rho_j}$. This is because if $I_l$ denotes the identity matrix in $GL_{p^l}(\mathbb{Z}/p^l\mathbb{Z})$ for $1 \leq l \leq k$ then

$$E_{ij}(p^{\lambda_i - \lambda_j}A) = \text{Diag}(I_1, \ldots, I_{i-1}, I_j, I_{i+1}, \ldots I_k)E_{ij}((\beta - 1)^{-1}p^{\lambda_i - \lambda_j}A)$$

$$\text{Diag}(I_1, \ldots, I_{i-1}, (\beta - 1)^{-1}I_j, I_{i+1}, \ldots I_k)E_{ij}((\beta - 1)^{-1}p^{\lambda_i - \lambda_j}A)$$

$$= [\text{Diag}(I_1, \ldots, I_{i-1}, I_j, I_{i+1}, \ldots I_k), E_{ij}((\beta - 1)^{-1}p^{\lambda_i - \lambda_j}A)]$$

where $\beta \in (\mathbb{Z}/p^k\mathbb{Z})^*$ such that $\beta - 1 \in (\mathbb{Z}/p^k\mathbb{Z})^*$.

It is also clear that for $1 \leq j < i \leq k$ the block elementary matrix $E_{ij}(A) = I_\Delta + e_{ij}(A) \in G'_\Delta$ where $e_{ij}(A)$ is a matrix with all zero blocks except for the $ij^{\text{th}}$ block entry which is $A$ where $A : (\mathbb{Z}/p^i\mathbb{Z})^{\rho_i} \rightarrow (\mathbb{Z}/p^j\mathbb{Z})^{\rho_j}$. This is because

$$E_{ij}(A) = \text{Diag}(I_1, \ldots, I_{j-1}, (\beta - 1)^{-1}I_j, I_{j+1}, \ldots I_k)E_{ij}((\beta - 1)^{-1}A)$$

$$\text{Diag}(I_1, \ldots, I_{j-1}, I_j, I_{j+1}, \ldots I_k)E_{ij}((\beta - 1)^{-1}A)$$

$$= [\text{Diag}(I_1, \ldots, I_{j-1}, I_j, I_{j+1}, \ldots I_k), E_{ij}((\beta - 1)^{-1}A)]$$

where $\beta \in (\mathbb{Z}/p^k\mathbb{Z})^*$ such that $\beta - 1 \in (\mathbb{Z}/p^k\mathbb{Z})^*$.
Now by induction on $k$ we observe that the matrix $g = \text{Diag}(g_{11}, g_{22}, \ldots, g_{kk}) \in G_\lambda'$, where $g_{11} \in SL_{p_1}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$ and $g_{ii} \in K_{P_i}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$ for $2 \leq i \leq k$.

Now consider $g = \text{Diag}(g_{11}, I_2, I_3, \ldots, I_k) \in G_\lambda'$ where $g_{11} \in K_{P_1}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$. We show that $g \in G_\lambda'$. Using Proposition 5.4 there exists $g_{11}' \in SL_{p_1}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$ and $h \in M_{p_1}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$ such that $g_{11} = g_{11}' + ph$. Now the matrix

$$\text{Diag}(g_{11}', I_2, I_3, \ldots, I_k) \in G_\lambda'.$$

Choose

$$pB : (\mathbb{Z}/p^{\lambda_2}\mathbb{Z})^{p_2} \longrightarrow (\mathbb{Z}/p^{\lambda_1}\mathbb{Z})^{p_1}, C : (\mathbb{Z}/p^{\lambda_1}\mathbb{Z})^{p_1} \longrightarrow (\mathbb{Z}/p^{\lambda_2}\mathbb{Z})^{p_2}$$

such that

$$pB g_{11}' = p e_{ij} : (\mathbb{Z}/p^{\lambda_1}\mathbb{Z})^{p_1} \longrightarrow (\mathbb{Z}/p^{\lambda_1}\mathbb{Z})^{p_1}, 1 \leq i, j \leq p_1.$$

Such a choice of matrices $B, C$ clearly exist. Note here $\lambda_1 - \lambda_2 = 1$.

Let $h_{ij}$ be the $ij$th element of $h \in M_{p_1}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$ for $1 \leq i, j \leq p_1$. We have

$$E_{21}(-C)E_{12}(ph_{ij}B) \text{Diag}(g_{11}', I_2, I_3, \ldots, I_k) = \begin{pmatrix}
g_{11}' & ph_{ij}B \\
-Cg_{11}' & I_2 - ph_{ij}CB \\
0 & I_{\mu}
ge_{11}' + ph_{ij}BCg_{11}' & 0 \\
0 & I_2 - ph_{ij}CB \\
0 & I_{\mu}
\end{pmatrix}E_{21}(Cg_{11}')E_{12}(-ph_{ij}(g_{11}')^{-1}(I_1 + ph_{ij}BC)^{-1}B)$$

where $I_\mu$ is the identity element of $G_\mu$ with $\mu = (\lambda_3^{p_3} > \cdots > \lambda_k^{p_k})$. We have

$$E_{21}(-C)E_{12}(ph_{ij}B)\text{Diag}(g_{11}', I_2, I_3, \cdots, I_k) = \begin{pmatrix}
g_{11}' + ph_{ij}BCg_{11}' & 0 \\
0 & I_2 - ph_{ij}CB \\
0 & I_{\mu}
\end{pmatrix}\text{Diag}(I_1, (I_2 - ph_{ij}CB)^{-1}, I_3, I_4, \cdots, I_k)$$

Now the matrix $I_2 - ph_{ij}CB \in K_{P_1}(\mathbb{Z}/p^{\lambda_2}\mathbb{Z})$. Hence we get

$$\begin{pmatrix}
g_{11}' + ph_{ij}BCg_{11}' & 0 \\
0 & I_2 - ph_{ij}CB \\
0 & I_{\mu}
\end{pmatrix}\text{Diag}(I_1, (I_2 - ph_{ij}CB)^{-1}, I_3, I_4, \cdots, I_k) = \text{Diag}(g_{11}', I_2, I_3, \ldots, I_k) \in G_\lambda'.$$

Now applying this procedure repeatedly for all $1 \leq i, j \leq p_1$ we get that

$$\text{Diag}(g_{11}' + p \sum_{i,j} h_{ij}e_{ij}, I_2, \ldots, I_k) = \text{Diag}(g_{11}' + ph, I_2, \ldots, I_k) = \text{Diag}(g_{11}, I_2, \ldots, I_k)$$

belongs to $G_\lambda'$. Hence we get that for $g_{ii} \in K_{P_i}(\mathbb{Z}/p^{\lambda_1}\mathbb{Z})$, $1 \leq i \leq k$ the matrix

$$\text{Diag}(g_{11}, g_{22}, \ldots, g_{kk}) = \text{Diag}(g_{11}, I_2, \ldots, I_k) \text{Diag}(I_1, g_{22}, g_{33}, \ldots, g_{kk}) \in G_\lambda'.$$
Now if \( g \in \text{Ker}(\Phi) \) with diagonal entries \( g_{ii} \in K_{p^j}(\mathbb{Z}/p^i\mathbb{Z}) \) then by using elementary block matrices which are in \( G'_\Lambda \) we can reduce \( g \) to the diagonal block matrix \( \text{Diag}(g_{11}, g_{22}, \ldots, g_{kk}) \) which is also in \( G'_\Lambda \). Hence we have \( g \in G'_\Lambda \). Therefore we have proved

\[
\text{Ker} \Phi = G'_\Lambda.
\]

Hence the theorem follows. \( \blacksquare \)

Remark 5.6. The above theorem does not hold if \( p = 2 \) as the following example illustrates. Also refer to Theorem 5.3 for another such example where \( k = 1, n = 2, p = 2 \).

Example 5.7. If \( k = 2, p = 2 \) and \( \Lambda = (2 > 1) \) then \( \text{Hom}(G_\Lambda, A_\Lambda) \neq 0 \). This is because \( G_\Lambda \) is a group of order 8. Hence there exists a normal subgroup of order 4 with quotient isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

5.0.2. The Nonvanishing Case of \( H^1_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \).

Theorem 5.8. Let \( \Lambda = (\lambda^1_1 > \lambda^2_2 > \lambda^3_3 > \ldots > \lambda^m_m) \) be a partition. Let \( A_\Lambda \) be the finite abelian \( p \)-group associated to \( \Lambda \) where \( p \) is any prime. Let \( G_\Lambda = \text{Aut}(A_\Lambda) \) be its automorphism group. If \( \lambda_i - \lambda_{i+1} \geq 2 \) for some \( 1 \leq i \leq k-1 \) or if \( \lambda_k \geq 2 \) then \( H^1_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \neq 0 \).

Proof. Define \( \lambda_{k+1} = 0 \). If \( g \in G_\Lambda \) then \( g = [g_{mn}]_{1 \leq m,n \leq k} \) where \( g_{mn} : (\mathbb{Z}/p^i\mathbb{Z})^{\rho_i} \rightarrow (\mathbb{Z}/p^j\mathbb{Z})^{\rho_j} \). Consider the homomorphism

\[
\phi : G_\Lambda \rightarrow GL_\rho(\mathbb{Z}/p^2\mathbb{Z})
\]

where \( \rho = \sum_{j=1}^i \rho_j \) given by

\[
g = [g_{mn}]_{1 \leq m,n \leq k} \rightarrow \overline{g} = [g_{mn}]_{1 \leq m,n \leq i} \mod p^2.
\]

Since \( \lambda_m - \lambda_n \geq 2 \) if \( n \geq i + 1 \) and \( m \leq i \) the map \( \phi \) is a homomorphism. Now consider the homomorphism \( \psi : GL_\rho(\mathbb{Z}/p^2\mathbb{Z}) \rightarrow (\mathbb{Z}/p^2\mathbb{Z})^* = \mathbb{Z}/p(p-1)\mathbb{Z} \) given by the determinant. The composed map \( \psi \circ \phi \) maps \( G_\Lambda \) onto a cyclic group of order \( p(p-1) \) which can further be mapped onto \( \mathbb{Z}/p\mathbb{Z} \). Hence using Proposition 5.1 we conclude that \( H^1_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \neq 0 \). \( \blacksquare \)

5.0.3. The Vanishing/Nonvanishing Criterion for \( H^1_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \) for Odd Primes. As a consequence of Theorems 5.5, 5.8 we have proved the following theorem for odd primes.

Theorem 5.9. Let \( \Lambda = (\lambda^1_1 > \lambda^2_2 > \lambda^3_3 > \ldots > \lambda^m_m) \) be a partition. Let \( A_\Lambda \) be the finite abelian \( p \)-group associated to \( \Lambda \) where \( p \) is an odd prime. Let \( G_\Lambda = \text{Aut}(A_\Lambda) \) be its automorphism group. Then \( H^1_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) = 0 \) if and only if the difference between two successive parts of \( \Lambda \) is at most 1.
6. **Computation of $H^2_{\text{Trivial Action}}(G_\Lambda, A_\Lambda)$**

In this section we compute the second cohomology group for the trivial action of $G_\Lambda$ on $A_\Lambda$.

6.1. **The Nonvanishing Case of $H^2_{\text{Trivial Action}}(G_\Lambda, A_\Lambda)$**.

**Theorem 6.1.** Let $\lambda = (\lambda_1^p > \lambda_2^p > \lambda_3^p > \ldots > \lambda_k^p)$ be a partition. Let $A_\Lambda$ be the finite abelian $p$-group associated to $\lambda$, where $p$ is a prime. Let $G_\Lambda = \text{Aut}(A_\Lambda)$ be its automorphism group. If $\lambda_i - \lambda_{i+1} \geq 2$ for some $1 \leq i \leq k-1$ or if $\lambda_k \geq 2$ then $H^2_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \neq 0$.

**Proof.** Define $\lambda_{k+1} = 0$. If $g \in G_\Lambda$ then $g = [g_{mn}]_{1 \leq m,n \leq k}$ where $g_{mn} : (\mathbb{Z}/p^{\lambda_m} \mathbb{Z})^p \to (\mathbb{Z}/p^{\lambda_n} \mathbb{Z})^p$. Consider the homomorphism

$$\phi : G_\Lambda \to GL_p(\mathbb{Z}/p^2 \mathbb{Z})$$

where $\rho = \sum_{j=1}^{i} \rho_j$ given by

$$g = [g_{mn}]_{1 \leq m,n \leq k} \mapsto \bar{g} = [g_{mn}]_{1 \leq m,n \leq i} \mod p^2.$$  

Since $\lambda_m - \lambda_n \geq 2$ if $n \geq i + 1$ and $m \leq i$ the map $\phi$ is a homomorphism. Now consider the homomorphism $\psi : GL_p(\mathbb{Z}/p^2 \mathbb{Z}) \to (\mathbb{Z}/p^2 \mathbb{Z})^* \to (\mathbb{Z}/p^2 \mathbb{Z})^*$ given by the determinant. The composed map $\psi \circ \phi$ maps $G_\Lambda$ onto a cyclic group of order $p(p-1)$. Consider a 2-cocycle $c : (\mathbb{Z}/p^2 \mathbb{Z})^* \times (\mathbb{Z}/p^2 \mathbb{Z})^* \to \mathbb{Z}/p \mathbb{Z}$ for the nontrivial central extension

$$0 \to \mathbb{Z}/p \mathbb{Z} \to \mathbb{Z}/p^2(p-1) \mathbb{Z} \to (\mathbb{Z}/p^2 \mathbb{Z})^* \to 0.$$  

Let $\sigma : \mathbb{Z}/p \mathbb{Z} \to \mathbb{Z}/p^{\lambda_k} \mathbb{Z}$ be defined as the inclusion homomorphism: $\sigma(1) = p^{\lambda_k-1}$ and $\sigma(a) = p^{\lambda_k-1}a$. Define a group structure on the set $E = \mathbb{Z}/p^{\lambda_k} \mathbb{Z} \times G_\Lambda$ as follows. Let $(x,g), (x',g') \in E$. Then define the multiplication on $E$ as:

$$(x,g).(x',g') = (x + x' + \sigma(c(\psi \circ \phi(g), \psi \circ \phi(g'))), gg').$$

**Claim 6.2.** Consider the following central extension,

$$0 \to \mathbb{Z}/p^\lambda \mathbb{Z} \to E \to G_\Lambda \to 0.$$  

Then $E$ is a nontrivial central extension of $G_\Lambda$ by $\mathbb{Z}/p^\lambda \mathbb{Z}$.

**Proof of Claim.** Consider the set $E_1 = \mathbb{Z}/p \mathbb{Z} \times G_\Lambda$. Define a group structure on the set $E_1 = \mathbb{Z}/p \mathbb{Z} \times G_\Lambda$ as follows. Let $(x,g), (x',g') \in E_1$. Then define the multiplication on $E_1$ as:

$$(x,g).(x',g') = (x + x' + c(\psi \circ \phi(g), \psi \circ \phi(g'))), gg').$$
Then \( E_1 \) is a nontrivial central extension of \( G_\Lambda \) by \( \mathbb{Z}/p\mathbb{Z} \) if and only if \( E \) is a nontrivial central extension of \( G_\Lambda \) by \( \mathbb{Z}/p^k\mathbb{Z} \). Now consider the diagonal subgroup \( D = (\mathbb{Z}/p^k\mathbb{Z})^* \oplus \bigoplus_{l=1}^k ((\mathbb{Z}/p^k\mathbb{Z})^*)^{p^l} \subset G_\Lambda \) where \( i \) is as mentioned in the theorem. Suppose the extension is given as:

\[
0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow E_1 \longrightarrow G_\Lambda \longrightarrow 0.
\]

Then we have a central extension.

\[
0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \pi_1^*(D) \longrightarrow D \longrightarrow 0.
\]

Note that \( \pi_1^*(D) \) is an abelian group and as a set \( \pi_1^*(D) = \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p^k\mathbb{Z})^* \). We have a group homomorphism \( \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p^k\mathbb{Z})^* \xrightarrow{\text{mod } p^2} \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p^2\mathbb{Z})^* \) where the group structure in both of them are given by the respective 2-cocycles.

So we have a commutative diagram

\[
\begin{array}{c}
0 \\ \downarrow \text{Id}_{\mathbb{Z}/p\mathbb{Z}}
\end{array}
\begin{array}{c}
\mathbb{Z}/p\mathbb{Z} \\ \downarrow \text{mod } p^2
\end{array}
\begin{array}{c}
0 \\ \pi_1^*(D) \\ \downarrow \text{mod } p^2
\end{array}
\begin{array}{c}
\mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p^2\mathbb{Z})^* \\ \downarrow \pi_1
\end{array}
\begin{array}{c}
0
\end{array}
\]

Hence now we observe that by a straight-forward calculation we have the set \( \mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p^2\mathbb{Z})^* \) as a group is isomorphic to \( \mathbb{Z}/p^2(p-1)\mathbb{Z} \) which is a nontrivial central extension. Hence \( \pi_1^*(D) \) and \( E_1 \) cannot be trivial central extensions implying that \( E \) is not a trivial extension. This proves the claim. \( \blacksquare \)

Continuing with the proof of the theorem, we have that the extension \( E \) gives rise to a nontrivial central extension

\[
0 \longrightarrow A_\Lambda \longrightarrow A_\mu \oplus E \longrightarrow G_\Lambda \longrightarrow 0
\]

where \( \mu = (\lambda_1^{p_1} > \lambda_2^{p_2} > \cdots > \lambda_k^{p_k-1}) \). Hence \( H^2_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \neq 0 \). \( \blacksquare \)

6.2. The Vanishing Case of \( H^2_{\text{Trivial Action}}(G_\Lambda, A_\Lambda) \).
Throughout this section we assume that \( \Lambda = (\lambda_1^{p_1} > \lambda_2^{p_2} > \cdots > \lambda_k^{p_k}) \) is a partition such that \( \lambda_i = k - i + 1 \). We also assume that \( p \) is an odd prime and \( p \neq 3 \). However we will mention where we require that the prime \( p \) must be odd and where we also require in addition that the odd prime \( p \) is not equal to 3.

6.2.1. Chief Series of a \( p \)-Sylow Subgroup of \( G_\Lambda \). Let \( P_\Lambda \subset G_\Lambda \) be the subgroup consisting of those automorphisms of \( A_\Lambda \) which are unipotent lower triangular
matrices modulo $p$ in $GL_p(\mathbb{Z}/p\mathbb{Z})$ where $\rho = \sum_{i=1}^{k} \rho_i$. Then $\mathcal{P}_{\Delta}$ is a $p$-Sylow subgroup of $G_{\Delta}$. We describe a chief series for $\mathcal{P}_{\Delta}$.

First we describe a typical element $g^a \in \mathcal{P}_{\Delta}$ where $g$ and $a$ are just two symbols. Let $g^a = [g_{mn}^a]_{1 \leq m,n \leq k}$ where $g_{mn}^a : (\mathbb{Z}/p^{\lambda_m} \mathbb{Z})^{\rho_m} \rightarrow (\mathbb{Z}/p^{\lambda_n} \mathbb{Z})^{\rho_n}$, that is, $g_{mn}^a = [(g_{mn}^a)^i]_{1 \leq i \leq \rho_m, 1 \leq j \leq \rho_n}$ where we have

\[
(g_{mn}^a)^i = \begin{cases} 
1 + a_{1,i,m,m}p + a_{2,i,m,m}p^2 + \cdots + a_{\lambda_m-1,i,m,m}p^{\lambda_m-1} & \text{if } m = n, i = j \\
0_{i,m,m} + a_{1,i,m,m}p + a_{2,i,m,m}p^2 + \cdots + a_{\lambda_m-1,i,m,m}p^{\lambda_m-1} & \text{if } m = n, i > j \\
a_{1,i,m,m}p + a_{2,i,m,m}p^2 + \cdots + a_{\lambda_m-1,i,m,m}p^{\lambda_m-1} & \text{if } m = n, i < j \\
a_{0,i,m,n} + a_{1,i,m,n}p + a_{2,i,m,n}p^2 + \cdots + a_{\lambda_m-1,i,m,n}p^{\lambda_m-1} & \text{if } m > n \\
a_{\lambda_m-\lambda_n,i,m,n}p^{\lambda_m-\lambda_n} + \cdots + a_{\lambda_m-1,i,m,n}p^{\lambda_m-1} & \text{if } m < n
\end{cases}
\]

with $a_{l,i,m,n} \in \{0, 1, 2, \ldots, p-1\}$ for $0 \leq l \leq \lambda_m - 1, 1 \leq i \leq \rho_m, 1 \leq j \leq \rho_n, 1 \leq m, n \leq k$. The value $a_{0,i,m,m}$ is defined to be 1 for $1 \leq i \leq \rho_m, 1 \leq m \leq k$ and the value $a_{l,i,j,m,n}$ is defined to be zero for $l = 0, 1 \leq m = n \leq k, 1 \leq i < j \leq \rho_m$ and for $0 \leq l \leq \lambda_m - \lambda_n - 1, 1 \leq i \leq \rho_m, 1 \leq j \leq \rho_n, 1 \leq m < n \leq k$. The matrix $g^a$ is of size $\rho \times \rho$. The position of $(g_{mn}^a)^i$ in the matrix $g^a$ is

\[
\text{Pos}(g_{mn}^a)^i = \left( \sum_{f=1}^{m-1} \rho_f + i, \sum_{h=1}^{n-1} \rho_h + j \right).
\]

We observe that $a_{l,i,j,m,n}$ is the coefficient of $p^l$ in the $p$-adic expansion of the element $(g_{mn}^a)^i$ in the matrix $g^a$ which occurs in the position $\text{Pos}(g_{mn}^a)^i$. We also write $\text{Pos}(a_{l,i,j,m,n}) = (\sum_{f=1}^{m-1} \rho_f + i, \sum_{h=1}^{n-1} \rho_h + j)$ and sometimes when the element $g^a$ is understood by default then we write

\[
\text{Pos}(l, i, j, m, n) = (\sum_{f=1}^{m-1} \rho_f + i, \sum_{h=1}^{n-1} \rho_h + j).
\]

Define a function $\chi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ as $\chi(x,y) = x - y$. Let $S' = \{(l, i, j, m, n) \mid 0 \leq l \leq \lambda_m - 1, 1 \leq i \leq \rho_m, 1 \leq j \leq \rho_n, 1 \leq m, n \leq k\}$ and $S'' = \{(0, i, j, m, n) \mid 1 \leq i \leq j \leq \rho_m, 1 \leq m \leq k\} \cup \{(l, i, j, m, n) \mid 0 \leq l \leq \lambda_m - \lambda_n - 1, 1 \leq i \leq \rho_m, 1 \leq j \leq \rho_n, 1 \leq m < n \leq k\}$. We define a total order $\leq_{TO}$ on the set
\[ S = (S' \setminus S'') \cup \{ \varnothing \} \] as follows. For \((l, i, j, m, n), (l', i', j', m', n') \in S \setminus \{ \varnothing \}, we say that
\[(l, i, j, m, n) \leq_{TO} (l', i', j', m', n')\]

- if \(l > l'\) or
- if \(l = l'\) and \(\chi(\text{Pos}(l, i, j, m, n)) > \chi(\text{Pos}(l', i', j', m', n'))\) or
- if \(l = l'\) and \(\chi(\text{Pos}(l, i, j, m, n)) = \chi(\text{Pos}(l', i', j', m', n'))\) and
\[
\sum_{f=1}^{m-1} \rho_f + i \geq \sum_{f=1}^{m'-1} \rho_f + i'.
\]

We also define that \(\varnothing \in S\) is the least element, that is, \(\varnothing \leq_{TO} (l, i, j, m, n)\) for all \((l, i, j, m, n) \in S \setminus \{ \varnothing \}. It is clear that \(\leq_{TO}\) is a total order on the set \(S\).

Now we define a chain of subgroups of \(\mathcal{P}_\Lambda^s\) indexed by the set \(S\). Define for \(s \in S\),
\[
\mathcal{P}_\Lambda^s = \{ g^a \in \mathcal{P}_\Lambda \mid a_{s'} = 0 \text{ for all } s' \in S \text{ and } s <_{TO} s' \}.
\]

Note that if \(s = \varnothing \in S\) then \(\mathcal{P}_\Lambda^\varnothing = \{1\}\), the trivial subgroup. From the definition of \(\mathcal{P}_\Lambda^s\) it is clear that \(\mathcal{P}_\Lambda^s \subseteq \mathcal{P}_\Lambda^s' \iff s <_{TO} s'\).

**Theorem 6.3.** The subset \(\mathcal{P}_\Lambda^s\) is indeed a subgroup for any \(s \in S\).

**Proof.** If \(s = \varnothing\) then the proof is trivial. So assume that \(s = (l, i, j, m, n) \in S \setminus \{ \varnothing \}\) and \(\text{Pos}(s) = (x, y)\). Suppose \(\chi(\text{Pos}(s)) = x - y < 0\), that is, \((x, y)\) corresponds to a strictly upper triangular entry. Let \(g^a, g^b \in \mathcal{P}_\Lambda^s\). Then certainly \(g^c = g^a g^b \in \mathcal{P}_\Lambda^s\). Moreover \(g^a \equiv g^b \equiv 1 \mod p^l\). Hence \(g^c \equiv 1 \mod p^l\).

Now consider Figure 1. Let region \(A = \{(u, v) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq u < v \leq y, u - v \leq x - y < 0 \text{ and } (u, v) \neq (x, y)\}\). Let region \(B = \{(u, v) \in \mathbb{N} \times \mathbb{N} \mid x \leq u < v \leq \rho = p_1 + \cdots + p_k, u - v < x - y\}\). The dotted boundary line of region \(B\) is not included in region \(B\). Let region \(C = \{(u, v) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq u < x, y < v \leq \rho = p_1 + \cdots + p_k\}\). To prove that \(\mathcal{P}_\Lambda^s\) is closed under group multiplication, it is enough to prove that, the entries in regions \(A, B, C\) are of the form \(p^{l+1}(*)\).

For \(u < v\), the \((u, v)^{th}\) entry of \(g^c = g^a g^b\) is of the form
\[
(g^c)_{(u,v)} = \sum_{z=1}^u (g^a)_{(u,z)} (g^b)_{(z,v)} + \sum_{z=u+1}^v (g^a)_{(u,z)} (g^b)_{(z,v)} + \sum_{z=v+1}^\rho (g^a)_{(u,z)} (g^b)_{(z,v)}.
\]

If \((u, v) \in \text{region } A \text{ or region } C\) then in the first sum of the RHS of 6.1, for \(1 \leq z \leq u, (z, v) \in \text{region } A \text{ or region } C\) and hence \((g^b)_{(z,v)}\) is of the form \(p^{l+1}(*)\).

If \((u, v) \in \text{region } B\) then in the first sum of the RHS of 6.1, for \(1 \leq z \leq u, (z, v) \in \text{region } B \text{ or region } C\). So \((g^b)_{(z,v)}\) is of the form \(p^{l+1}(*)\).

In the second sum of the RHS of 6.1, for \(u + 1 \leq z < v\) we have \((g^a)_{(u,z)}\) is of the form \(p(*)\) since it is lower triangular modulo \(p\) and \((g^b)_{(z,v)}\) is of the form \(p^l(*)\).
In the second sum of the RHS of 6.1, for \( z = v \), \((g^a)_{(u,v)}\) is of the form \( p^{l+1}(*)\) since \((u,v)\) belongs to region A or region B or region C.

In the third sum of the RHS of 6.1, we have \((g^a)_{(u,z)}\) is of the form \( p(*)\) since it is lower triangular modulo \( p \) and \((g^b)_{(z,v)}\) is of the form \( p^l(*)\). So we conclude that \((g^c)_{(u,v)}\) is of the form \( p^{l+1}(*)\).

The proof is similar if \((x,y)\) corresponds to a strictly lower triangular entry or a diagonal entry. Hence \( P^s \lambda \) is closed under group multiplication.

Now we prove that \( P^s \lambda \) is closed under inverses. This is immediate because any nonempty finite subset of a group which is also closed under group multiplication is closed under inverses and contains the identity element. Hence \( P^s \lambda \) is a subgroup.

Remark 6.4. We also observe in the above proof that if \( g^a, g^b \in P^s \lambda, s \in S \setminus \{\emptyset\}, s = (l, i, j, m, n), Pos(s) = (x, y) \) and \( g^c = g^a \cdot g^b \) then the entry

\[(g^c)_{(x,y)} \equiv (g^a)_{(x,y)} + (g^b)_{(x,y)} \mod p^{l+1}, \text{ that is, } c_{l,i,j,m,n} \equiv a_{l,i,j,m,n} + b_{l,i,j,m,n} \mod p \text{ if } x \neq y.\]
(ii) \((g^c)_{(x,y)} - 1 \equiv (g^a)_{(x,y)} - 1 + (g^b)_{(x,y)} - 1 \mod p^{l+1}\), that is, \((g^c)_{(x,y)} \equiv 1 + p^l a_{i,j,m,n} + p^l b_{i,j,m,n} \mod p^{l+1}, (g^a)_{(x,y)} \equiv 1 + p^l a_{i,j,m,n} \mod p^{l+1}, (g^b)_{(x,y)} \equiv 1 + p^l b_{i,j,m,n} \mod p^{l+1}\) if \(x = y\).

Remark 6.5. In the proof of Theorem 6.3, \(p\) can be any prime.

**Theorem 6.6.** Let \(t_1, t_2 \in S\) and \(t_1 < t_2\). Let \(S_{t_i} = \{ t \in S \mid t \leq t_i, t \neq \emptyset \}, i = 1, 2\).

Then

1. The set \(C_{t_2}^t = \{ g^c \mid g^c \in \mathcal{P}^t_{\Delta}, c_t = 0 \text{ for all } t \in S_{t_1} \}\) forms a set of distinct left and right coset representatives for the subgroup \(\mathcal{P}^t_{\Delta} \subset \mathcal{P}^t_{\Delta}\).
2. For \(g^c \in C_{t_2}^t\), \(g^d \in \mathcal{P}^t_{\Delta}\) if and only if \(c_t = d_t\) for all \(t_1 < t < t_2\).
3. \((g^c, g^d) \in \mathcal{P}^t_{\Delta} \times \mathcal{P}^t_{\Delta}\) if and only if \(c_t = d_t\) for all \(t_1 < t < t_2\).
4. The index \([\mathcal{P}^t_{\Delta} : \mathcal{P}^t_{\Delta}^s]\) = \(p^{|S_{t_2}| - |S_{t_1}|}\).

**Proof.** For \(s = \emptyset\) the proof is trivial. Assume that \(s \neq \emptyset\). Let \(s_i\) be the predecessor of \(s\) in the set \(S\) where \(\text{Pos}(s) = (x, y)\). Then using Remark 6.4 (i),(ii), we have that the set of elementary or diagonal matrices \(\{ E_{(x,y)}(p^l a_{i,j,m,n}) \mid 0 \leq a_{i,j,m,n} \leq p - 1 \}\) form a set of distinct left and right coset representatives for the subgroup \(\mathcal{P}^{s_1}_{\Delta} \subset \mathcal{P}^s_{\Delta}\). Here \(E_{(x,y)}(p^l a_{i,j,m,n}) = I + e_{(x,y)}(p^l a_{i,j,m,n})\) where \(e_{(x,y)}(p^l a_{i,j,m,n})\) is the matrix of zeroes except for the \((x, y)^{th}\) entry which is \(p^l a_{(i,j,m,n)}\). Note that if \((l, i, j, m, n)\) is in diagonal position, then \(x = y\) and \(l > 0\) and \(E_{(x,y)}(p^l a_{i,j,m,n})\) is a diagonal matrix. As a consequence we have \([\mathcal{P}^s_{\Delta} : \mathcal{P}^{s_1}_{\Delta}] = p\).

We prove the theorem by induction on the cardinality \(|S_{t_2} \setminus S_{t_1}|\). If \(|S_{t_2} \setminus S_{t_1}| = 1\) then it follows from Remark 6.4 (i),(ii). Suppose \(|S_{t_2} \setminus S_{t_1}| > 1\). Let \(t_1 < t_2\) where \(t_3\) is the successor of \(t_2\) and \(g^c, g^d \in C_{t_2}^t\). Suppose the two left cosets \(g^c \mathcal{P}^{t_1}_{\Delta} \mathcal{P}^t_{\Delta}\) and \(g^d \mathcal{P}^{t_1}_{\Delta} \mathcal{P}^t_{\Delta}\) are equal then \(g^c \mathcal{P}^{t_1}_{\Delta} = g^d \mathcal{P}^{t_1}_{\Delta}\). Hence we have \(c_t = d_t\) for all \(t_1 < t < t_2\) by induction on \(|S_{t_2} \setminus S_{t_3}| = |S_{t_2} \setminus S_{t_1}| - 1\). A similar conclusion follows when we consider right cosets. We now prove \(c_{t_3} = d_{t_3}\). Let \(t_4\) be the predecessor of \(t_2 = (l_2, i_2, j_2, m_2, n_2)\). Let \(\text{Pos}(t_2) = (u_2, v_2)\). Then we have

- \(E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2})g^c, E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2})g^d \in \mathcal{P}^{t_4}_{\Delta}\),
- \(g^c E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2}), g^d E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2}) \in \mathcal{P}^{t_4}_{\Delta}\),
- \(E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2})g^c \mathcal{P}^{t_1}_{\Delta} = E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2})g^d \mathcal{P}^{t_1}_{\Delta}\),
- \(g^c E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2}) \mathcal{P}^{t_1}_{\Delta} = g^d E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2})\),
- \(c_{l_2,i_2,j_2,m_2,n_2} = d_{l_2,i_2,j_2,m_2,n_2}\).

Hence by induction on \(|S_{t_4} \setminus S_{t_1}| = |S_{t_2} \setminus S_{t_1}| - 1\), we have that, if \(g^a = E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2})g^c\) and \(g^b = E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2})g^d\) then \(a_t = b_t\) for all \(t_1 < t \leq t_4\). Similarly if \(g^A = g^c E_{(u_2,v_2)}(-p^{l_2} c_{l_2,i_2,j_2,m_2,n_2})\) and \(g^B = g^d E_{(u_2,v_2)}(-p^{l_2} d_{l_2,i_2,j_2,m_2,n_2})\) then \(A_t = B_t\) for all \(t_1 < t \leq t_4\). We observe that
$g^a$ and $g^c$ differ only in the $(u_2)^{th}$-row. Also $g^d, g^c$ differ only in the $(v_2)^{th}$-column. A similar conclusion follows for the pairs $g^b, g^d$ and $g^b, g^d$. So if $\text{Pos}(t_3) \neq (u_2, v_2) = \text{Pos}(t_2)$ then $c_t = d_t$.

Now we assume that $\text{Pos}(t_3) = \text{Pos}(t_2) = (u_2, v_2)$. Let $t_3 = (l_3, i_3, j_3, m_3, n_3)$. We need to prove that the coefficient of $p^{3}$ in the $p$-adic expansions of $(g^c)_{(u_2,v_2)}, (g^d)_{(u_2,v_2)}$ are equal. We have

$$\begin{align*}
(g^d)_{(u_2,v_2)} &= (g^c)_{(u_2,v_2)} - p^{l_2}c_{l_2,i_2,j_2,m_2,n_2}(g^c)_{(v_2,v_2)} \\
\Rightarrow (g^c)_{(u_2,v_2)} &= (g^d)_{(u_2,v_2)} + p^{l_2}c_{l_2,i_2,j_2,m_2,n_2}(g^c)_{(v_2,v_2)}
\end{align*}$$

and

$$\begin{align*}
(g^b)_{(u_2,v_2)} &= (g^d)_{(u_2,v_2)} - p^{l_2}d_{l_2,i_2,j_2,m_2,n_2}(g^d)_{(v_2,v_2)} \\
\Rightarrow (g^d)_{(u_2,v_2)} &= (g^b)_{(u_2,v_2)} + p^{l_2}d_{l_2,i_2,j_2,m_2,n_2}(g^d)_{(v_2,v_2)}.
\end{align*}$$

There are two cases: $l_2 > 0$ and $l_2 = 0$. Suppose $l_2 > 0$. In Equations 6.2, 6.3, the coefficients of $p^l$ in $(g^a)_{(u_2,v_2)}$ and $(g^b)_{(u_2,v_2)}$ are equal for all $\bar{l} \leq l_3$ since $a_t = b_t$ for all $t_1 < t$. Also the coefficients of $p^l$ in $(g^c)_{(v_2,v_2)}$, $(g^d)_{(v_2,v_2)}$ are equal for $\bar{l} < l_3$ since $c_t = d_t$ for all $t_3 < t$. Hence in this case $l_2 > 0$ we have $c_{t_3} = d_{t_3}$.

Suppose $l_2 = 0$ then $u_2 > v_2$ because the coefficient of $p^0$ which can be nonzero and not equal to one can occur only in the strictly lower triangular entries. Hence the coefficients of $p^l$ in $(g^c)_{(v_2,v_2)}$, $(g^d)_{(v_2,v_2)}$ are equal for $\bar{l} \leq l_3$ and the coefficients of $p^l$ in $(g^a)_{(u_2,v_2)}$ and $(g^b)_{(u_2,v_2)}$ are equal for all $\bar{l} \leq l_3$. Again here we have $c_{t_3} = d_{t_3}$. The converse statements in (2),(3) also follow in a similar way. This proves the theorem.

Remark 6.7. In the proof of Theorem 6.6, $p$ can be any prime.

**Theorem 6.8.** We have

1. $P_{\lambda}^s$ is a normal subgroup of $P_{\lambda}$ for $s \in S$.
2. If $s'$ is the successor element of $s$ in $S$ then

$$\frac{P_{\lambda}^{s'}}{P_{\lambda}^s} \cong \mathbb{Z}/p\mathbb{Z}.$$

3. If $s'$ is the successor element of $s$ in $S$ then there is a central extension

$$0 \longrightarrow \frac{P_{\lambda}^{s'}}{P_{\lambda}^s} \longrightarrow \frac{P_{\lambda}}{P_{\lambda}^s} \longrightarrow \frac{P_{\lambda}}{P_{\lambda}^{s'}} \longrightarrow 0.$$

**Proof.** We prove Theorem 6.8 by induction on the elements of the totally ordered set $S \setminus \{\varnothing\}$. Let $(l, i, j, m, n) = s \in S \setminus \{\varnothing\}$ and $s_1 \in S$ be the predecessor of $s$. Let $\text{Pos}(s) = (x, y)$. We can assume that $P_{\lambda}^{s_1}$ is a normal subgroup of $P_{\lambda}$ by induction. Now let $g^b \in P_{\lambda}$. We have $g^b E_{(x,y)}(p^l a_{l,i,j,m,n}) = g^b + g^b E_{(x,y)}(p^l a_{l,i,j,m,n})$
and $E_{(x,y)}(p^la_{i,j,m,n})g^b = g^b + c_{(x,y)}(p^la_{i,j,m,n})g^b$. The elements of the matrices $g^bE_{(x,y)}(p^la_{i,j,m,n})$ and $E_{(x,y)}(p^la_{i,j,m,n})g^b$ are the same except for the $x^{th}$-row and $y^{th}$-column. In the $x^{th}$-row we have for any $1 \leq z \leq \rho = \rho_1 + \cdots + \rho_k$,

$$(E_{(x,y)}(p^la_{i,j,m,n})g^b)(x,z) = g^b_{(x,z)} + p^la_{i,j,m,n}g^b_{(y,z)}.$$ 

If $z > y$ then $p^{l+1}$ divides $p^la_{i,j,m,n}g^b_{(y,z)}$. If $z = y$ then $(E_{(x,y)}(p^la_{i,j,m,n})g^b)(x,y) = g^b_{(x,y)} + p^la_{i,j,m,n} + p^{l+1}$. If $z < y$ then $x - z > x - y = \chi(\text{Pos}(s))$. In the $y^{th}$-column we have for any $1 \leq w \leq \rho = \rho_1 + \cdots + \rho_k$,

$$(g^bE_{(x,y)}(p^la_{i,j,m,n}))(w,y) = g^b_{(w,y)} + p^la_{i,j,m,n}s_{(w,x)}.$$ 

Here if $w < x$ then $p^{l+1}$ divides $p^la_{i,j,m,n}s_{(w,x)}$. If $w = x$ then $(g^bE_{(x,y)}(p^la_{i,j,m,n}))(x,y) = g^b_{(x,y)} + p^la_{i,j,m,n} + p^{l+1}$. If $w > x$ then $w - y > x - y = \chi(\text{Pos}(s))$.

Hence we conclude that in the matrices $g^c = E_{(x,y)}(p^la_{i,j,m,n})g^b$ and $D^c = g^bE_{(x,y)}(p^la_{i,j,m,n})$, we have $C_t = D_t$ for all $s_1 < t$. So using Theorem 6.6 we conclude that $s_cP_{\lambda}^s = s_dP_{\lambda}^s$ and $P_{\lambda}^s \equiv s_c \equiv D_{\lambda}^s$. Hence $\frac{P_{\lambda}^s}{P_{\lambda}^s}$ is a central normal subgroup of $P_{\lambda}^s$. Hence Theorem 6.8(1), (2), (3) follow. This completes the proof of Theorem 6.8.





\[ \text{Remark 6.9. The chain of subgroups } P_{\lambda}^s, s \in S \text{ is a chief series for } P_{\lambda}. \]

\[ \text{Remark 6.10. In the proof of Theorem 6.8, } p \text{ can be any prime.} \]

6.2.2. Commutator Subgroup of $P_{\lambda} \subset G_{\lambda}$.

**Theorem 6.11.** Let $P'_{\lambda} = [P_{\lambda}, P_{\lambda}]$ be the commutator subgroup of $P_{\lambda} \subset G_{\lambda}$. If $k > 1$, define

$$(6.5) \quad T = \{(0, i, j, m, n) \in S \mid \chi(\text{Pos}(0, i, j, m, n)) = 1\}$$

$$\bigcup \{(1, 1, \rho_{m+1}, m, m+1) \in S, 1 \leq m \leq k - 1\}$$

and if $k = 1$, define

$$T = \{(0, i, j, 1, 1) \in S \mid \chi(\text{Pos}(0, i, j, 1, 1)) = 1\}.$$ 

For $g^a, g^b, g^c \in P_{\lambda},$ suppose $g^c = g^a \cdot g^b$. Then we have

1. $$(l, i, j, m, n) \in T \Rightarrow c_{(l,i,j,m,n)} = a_{(l,i,j,m,n)} + b_{(l,i,j,m,n)} \mod p,$$

2. $g^a \in P_{\lambda} \iff \text{for all } (l, i, j, m, n) \in T, a_{(l,i,j,m,n)} = 0.$
\( \frac{P_1}{\bar{P}_\lambda} \cong (\mathbb{Z}/p\mathbb{Z})^{\left|T\right|}. \)

**Proof.** We prove (1). Let \( s \in T \). Clearly if \( s = (0, i, j, m, n) \) such that \( \chi(\text{Pos}(s)) = 1 \) then \( s \) appears in the leading subdiagonal or first subdiagonal (just below the diagonal). \( a_s, b_s, c_s \) being the first coefficients in the \( p \)-adic expansion of the entry in position \( \text{Pos}(s) \), it is clear that \( c_s \equiv a_s + b_s \mod p \). Now assume that \( s = (1, 1, \rho_{m+1}, m, m+1) \) for some \( 1 \leq m \leq k - 1 \). Let \( \text{Pos}(s) = (x, y) \) where \( x = \sum_{f=1}^{m-1} \rho_f + 1, y = \sum_{h=1}^{m} \rho_h + \rho_{m+1} \). Then \( \chi(\text{Pos}(s)) = 1 - \rho_m - \rho_{m+1} < 0 \). Hence \( s \) appears in a strictly upper triangular entry, that is, \((x, y)\) corresponds to a strictly upper triangular entry. Moreover \((x, y)\) is the position corresponding to the top right corner of the \((m, m+1)^{th}\) block matrix. We have

\[
\mathcal{S}^c_{(x,y)} = \sum_{z=1}^{g} \mathcal{S}^a_{(x,z)} \mathcal{S}^b_{(z,y)}. 
\]

So for \( 1 \leq z < x \), we have \( p^2 \mid \mathcal{S}^b_{(z,y)} \) since \( 2 \leq \lambda_t - \lambda_{m+1} = m + 1 - t \) for all \( t < m \). For \( x < z < y \) we have \( p \mid \mathcal{S}^a_{(x,z)} \), \( p \mid \mathcal{S}^b_{(z,y)} \) since \( g^a, g^b \) are lower triangular modulo \( p \). If \( z > y \) then we have \( p^2 \mid \mathcal{S}^a_{(x,z)} \) since \( 2 \leq \lambda_m - \lambda_l = t - m \) for all \( t > m + 1 \). For \( z = x \) we have \( \mathcal{S}^a_{(x,x)} \mathcal{S}^b_{(y,y)} = pb_s + p^2(*) \) and for \( z = y \) we have \( \mathcal{S}^a_{(x,y)} \mathcal{S}^b_{(y,y)} = pa_s + p^2(*) \). Hence we conclude that \( c_s \equiv a_s + b_s \mod p \). This proves (1).

We prove (2). For \( g^a \in \mathcal{P}_\lambda \) if \( g^b = (g^a)^{-1} \) then using (1) we obtain that \( b_s \equiv -a_s \mod p \) for all \( s \in T \). Hence again using (1) for \( g^a, g^b \in \mathcal{P}_\lambda \) if \( g^c = [g^a, g^b] = g^a g^b (g^a)^{-1} (g^b)^{-1} \) then \( c_s = 0 \) for all \( s \in T \). Hence for \( g^a \in \mathcal{P}_\lambda \) if \([\mathcal{P}_\lambda, \mathcal{P}_\lambda], a_s = 0 \) for all \( s \in T \), that is, if we define the subgroup \( C_\lambda = \{ g^a \in \mathcal{P}_\lambda \mid a_s = 0 \} \) for all \( s \in T \), then \( \mathcal{P}_\lambda \subseteq C_\lambda \).

Now we prove the converse in (2), that is, \( C_\lambda \subseteq \mathcal{P}_\lambda \). Let \( s \in S \setminus T, s = (l, i, j, m, n) \) and let \( \text{Pos}(s) = (x, y) \). Consider the matrix \( E_{(x,y)}(p^l a_s) = I + e_{(x,y)}(p^l a_s) \). We show that it is a commutator or product of commutators.

First assume that \( x < y \), that is, \((x, y)\) corresponds to a strictly upper triangular entry. Then we have \( m \leq n \).

If \( n = m \) then \( i < j \) and \( \rho_m \geq 2 \) and position \((x, y)\) is in the diagonal block \((m, m)\). If \( m = k \) then the entry in the \((x, y)^{th}\) position is zero. So \( m < k \Rightarrow y < \rho = \rho_1 + \cdots + \rho_k \). We have \( E_{(x,y)}(p^l a_s) = [E_{(x,y+1)}(p^l a_s), E_{(y+1,y)}(1)] \). Position \((x, y + 1)\) occurs in either block \((m, m)\) or block \((m, m+1)\). So \( p^l \) is allowed in position \((x, y + 1)\) if \( p^l \) is allowed in position \((x, y)\).
If \( n > m + 1 \) then \( \lambda_m - \lambda_n = n - m \geq 2 \) and \( x + 1 < y \). So \( l \geq 2 \). We have \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,x+1)}(p \lambda_s), E_{(x+1,y)}(p^{l-1})] \). Note that position \( (x+1,y) \) occurs in either block \((m,n)\) or \((m+1,n)\). So \( p^{l-1} \) is allowed in position \( (x+1,y) \).

Suppose \( n = m + 1 \). If \( i > 1 \) then position \( (x-1,y) \) is also in the block \((m,m+1)\).

So \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,x-1)}(1), E_{(x,y-1)}(p^l \lambda_s)] \). If \( j < \rho_m + 1 \) then position \( (x,y+1) \) is also in the block \((m,m+1)\). So \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,y+1)}(p^l \lambda_s), E_{(y+1,y)}(1)] \). So assume \( i = 1, j = \rho_m + 1 \). If \( l \geq 2 \) then positive integer \( 3 \) is a part of \( \Lambda \) and \( k \geq 3 \). So if \( m + 1 < k \) then \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,y+1)}(p^l \lambda_s), E_{(y+1,y)}(1)] \). Position \( (x,y+1) \) occurs in block \((m,m+1)\) and \( p^l \) is allowed in position \( (x,y+1) \) since \( l \geq 2 \). If \( m + 1 = k \geq 3 \) then \( m \geq 2 \). So \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,x-1)}(1), E_{(x-1,y)}(p^l \lambda_s)] \).

Position \( (x-1,y) \) occurs in block \((m-1,m+1)\) = \((k-2,k)\) and \( p^l \) is allowed in position \( (x-1,y) \) since \( l \geq 2 \).

Finally in the scenario \( x < y \), we are left with the case \( n = m + 1, i = 1, j = \rho_m + 1, l = 1 \), that is, \( s = (1,1,\rho_m + 1, m, m + 1) \). But then we have that \( s \in T \).

Now consider the scenario \( x > y \), that is, \( (x,y) \) corresponds to a strictly lower triangular entry. Here we have \( m \geq n \).

If \( m = n \), that is, position \( (x,y) \) is in the diagonal block \((m,m)\) then \( i > j \) and \( \rho_m \geq 2 \). If \( \rho_m = 2 \) then \( i = 2, j = 1, x - y = 1 \). If \( l = 0 \) then \( s = (0,2,1, m, m) \in T \). So assume \( l > 0 \). In this case \( k > 1 \) and \( m < k \). So position \( (x+1,y) \) occurs in block \((m+1,m)\). So \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,x+1)}(p \lambda_s), E_{(x+1,y)}(p^{l-1})] \). Note that \( p^{l-1} \) is allowed in position \( (x+1,y) \) if \( p^l \) is allowed in position \( (x,y) \). If \( \rho_m > 2 \), then there exists \( z \neq x, z \neq y \) such that both positions \( (x,z), (z,y) \) are in block \((m,m)\).

If \( x > z > y \) then \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,z)}(1), E_{(z,y)}(p^l \lambda_s)] \). Now we assume that \( x - y = 1 \) and \( l > 0 \). If \( x > y > z \) then \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,z)}(1), E_{(z,y)}(p^l \lambda_s)] \). If \( z > x > y \) then \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,z)}(p^l \lambda_s), E_{(z,y)}(1)] \). If \( x - y = 1 \) and \( l = 0 \) then \( s = (0,i,j = i - 1, m, m) \in T \).

If \( m > n + 1 \) then position \( (x,y+1) \) is either in block \((m,n)\) or \((m,n+1)\). Also \( x > y + 1 > y \). We have \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,y+1)}(p^l \lambda_s), E_{(y+1,y)}(1)] \). Note that \( p^l \) is allowed in position \( (x,y+1) \).

If \( m = n + 1, x - y > 1 \) then position \( (x,y+1) \) is either in block \((m,n)\) or \((m,n+1)\).

Here again we have \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,y+1)}(p^l \lambda_s), E_{(y+1,y)}(1)] \). If \( x - y = 1 \) and \( l = 0 \) then \( (0,i,j,m,n) \in T \). If \( x - y = 1, l > 0 \) then \( k > 1, m < k \). So position \( (x+1,y) \) is in either in block \((m,n)\) or \((m+1,n)\). We have \( E_{(x,y)}(p^l \lambda_s) = [E_{(x,x+1)}(p), E_{(x+1,y)}(p^{l-1} \lambda_s)] \). Note that \( p^{l-1} \) is allowed in position \( (x+1,y) \).

Now we consider the case \( x = y \), that is, \( m = n, i = j \) and \( s = (l,i,i,m,m) \) where \( l > 0 \). Consider first \( E_{(x,x)}(p^l \lambda_s) \) for \( l \geq 2 \). In this case position \( (x+2,x+2) \) appears in the matrix since \( x + 2 \leq p = p_1 + \cdots + p_k \). So position \( (x+2,x) \) is
in one of the blocks \((m, m), (m + 1, m), (m + 2, m)\). Similarly \((x, x + 2)\) is in one of the blocks \((m, m), (m, m + 1), (m + 1, m)\). Also \((x + 2, x + 2)\) is in one of the blocks \((m, m), (m + 1, m + 1), (m + 2, m + 2)\). For \(2 \times 2\) matrices we have, for any symbol \(a\) and \(a^{-1}\) denoting the inverse of \(a\),

\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & a
\end{pmatrix} \begin{pmatrix}
1 & 1 - a \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 - a^{-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix}.
\]

Assume that \(E_{(z,z)}(p^l(*)), l \geq 2\) is a product of commutators for all \(x < z \leq \rho = \rho_1 + \cdots + \rho_k\). We prove that \(E_{(x,x)}(p^l(*))\) is a product of commutators for all \(l \geq 2\). We have using the above symbolic identity

\[
E_{(x,x)}(p^l a_s) = E_{(x+2,x+2)}(p^l a_s)E_{(x,x+2)}(-p^l a_s)E_{(x+2,x)}(-1)E_{(x,x+2)}(1 - (1 + p^l a_s)^{-1})E_{(x+2,x)}(p^l a_s).
\]

Note that here \(p^l\) for \(l \geq 2\) is allowed in position \((x, x + 2)\). Here if \(l\) is very large then \(p^l\) in that position becomes zero. We have \(E_{(x,x+2)}(-p^l a_s)\) is a commutator, \(E_{(x+2,x)}(-1)\), \(E_{(x+2,x)}(p^l a_s)\) are commutators. Also note that \(1 - (1 + p^l a_s)^{-1} = 1 - p^l a_s + p^{l+1}(*)\) with \(l \geq 2\). Hence

\[
E_{(x,x+2)}(1 - (1 + p^l a_s)^{-1}) = E_{(x,x+2)}(-p^l a_s)E_{(x,x+2)}(p^{l+1}(*))
\]

is a product of commutators which we have already shown. So \(E_{(x,x)}(p^l a_s)\) is a product of commutators for \(l \geq 2\). So from this we can show that \(E_{(x,x)}(p^l(*))\) is a product of commutators for all \(l \geq 2\). This is because if \(0 \leq a < p - 1, b\) is any non-negative integer then we have

\[
E_{(x,x)}(p^l a + p^{l+1} b) = E_{(x,x)}(p^l a)E_{(x,x)}(-1 + \frac{(1 + p^l a + p^{l+1} b)}{(1 + p^l a)})
\]

\[
= E_{(x,x)}(p^l a)E_{(x,x)}(p^{l+1}(*)).
\]

For any diagonal matrix \(D \in \mathcal{P}_\Delta\) if the diagonal entries are of the form \(1 + p^2(*)\) then \(D\) is a product of commutators. Now consider \(E_{(x,x)}(pa_s)\) where \(s = (1, i, i, m, m)\) with \(\text{Pos}(s) = (x, x)\). We can assume that \(x < \rho = \rho_1 + \cdots + \rho_k\). Position \((x + 1, x)\) occurs either in block \((m, m)\) or \((m + 1, m)\). Position \((x, x + 1)\) occurs either in block \((m, m)\) or \((m, m + 1)\). Position \((x + 1, x + 1)\) occurs either in block \((m, m)\) or \((m + 1, m + 1)\). Assume that \(E_{(z,z)}(p(*)\) is a product of commutators for all \(x < z \leq \rho = \rho_1 + \cdots + \rho_k\). We prove that \(E_{(x,x)}(p(*))\) is a product of commutators. First \(E_{(x,x)}(pa_s)\) can be expressed as the following long
product of matrices.

\[ E_{(x,x)}(p a_s) = E_{(x,x+1)}(- p a_s(1 + p a_s + p^2 a_s^2)^{-1}) E_{(x,x+1)}(1) \]

This can be checked by first evaluating the commutator \([E_{(x,x+1)}(p a_s), E_{(x,x+1)}(1)]\) and then reducing it by elementary matrices to the matrix \(E_{(x,x)}(p a_s)\). Note that here \(E_{(x,x)}(-1 + (1 + p a_s)(1 + p a_s + p^2 a_s^2)^{-1}) = E_{(x,x)}(p^2(*)\) which we have proved is a product of commutators. Moreover \(E_{(x+1,x+1)}(p a_s + p^2 a_s^2) = E_{(x+1,x+1)}(p(*))\) is a product of commutators by assumption. Also \(E_{(x+1,x)}(-p a_s(1 + p a_s + p^2 a_s^2)^{-1}) = E_{(x+1,x)}(p(*))\) is a product of commutators, \(E_{(x+1,x+1)}(p^2 a_s^2 (1 + p a_s + p^2 a_s^2)^{-1}) = E_{(x+1,x+1)}(p^2(*))\) is a product of commutators. So \(E_{(x,x)}(p a_s)\) is a product of commutators and hence \(E_{(x,x)}(p(*))\) is a product of commutators. So all diagonal matrices in \(\mathcal{P}_\Lambda\) are product of commutators.

Now every matrix in \(C_\Lambda\) is a product of elementary matrices \(E_{(x,y)}(p^l a_s)\) for \(s = (i,i,j,m,n) \in S \setminus T\). This can be observed as follows. We consider a matrix \(g^a \in C_\Lambda\) and reduce the diagonal blocks of \(g^a\) to identity matrices using elementary matrices \(E_{(x,y)}(p^l a_s)\) with \(s = (i,i,j,m,n) \in S \setminus T\) and \(\text{Pos}(s)\) occurs in the diagonal blocks only. Then we can reduce \(g^a\) further to identity matrix in \(C_\Lambda\) using \(E_{(x,y)}(p^l a_s)\) for \(s = (i,i,j,m,n) \in S \setminus T\) with \(\text{Pos}(s)\) occurring in the non-diagonal blocks. Hence we have proved that \(C_\Lambda \subseteq \mathcal{P}_\Lambda\). This proves the converse in (2). (3) is an immediate consequence of (2). This completes the proof of Theorem 6.11. \(\blacksquare\)

**Remark 6.12.** In the proof of Theorem 6.11, \(p\) can be any prime.

### 6.2.3. Modified Total Order on the Set \(S\)

Consider the two totally ordered subsets \((T, \leq_{\mathrm{TO}})\) and \((S \setminus T, \leq_{\mathrm{TO}})\) of the set \((S, \leq_{\mathrm{TO}})\) where \(T\) is as defined in Equation 6.5. The modified total order \(\leq_{\mathrm{MTO}}\) on the set \(S\) is defined as follows. Let \(s, s' \in S\). We say \(s \leq_{\mathrm{MTO}} s'\) if

- \(s, s' \in S \setminus T\) and \(s \leq_{\mathrm{TO}} s'\) or
- \(s, s' \in T\) and \(s \leq_{\mathrm{TO}} s'\) or
- \(s \in S \setminus T, s' \in T\).

**Remark 6.13.** The modified total order \(\leq_{\mathrm{MTO}}\) is introduced on the set \(S\), so that the elements of \(T \subseteq S\) become the larger elements of \(S\). This is going to be useful later because for \(t \in T, g^a \in \mathcal{P}_\Lambda', a_t = 0\).
Now we define another chain of subgroups of $N_\lambda$ indexed by the set $S$. Define for $s \in S$

$$N_s^s = \{g^a \in P_\lambda \mid a_s = 0 \text{ for all } s' \in S \text{ and } s <_{MTO} s'\}.$$ 

**Theorem 6.14.** We have

1. $N_s^s$ is a normal subgroup of $P_\lambda$ for $s \in S$.
2. $N_s^s \subseteq N_s^s' \iff s <_{MTO} s'$.
3. If $s'$ is the successor element of $s$ in $(S, \leq_{MTO})$ then
   $$N_s^s' / N_s^s \cong \mathbb{Z}/p\mathbb{Z}.$$
4. If $s'$ is the successor element of $s$ in $(S, \leq_{MTO})$ then there is a central extension

$$0 \rightarrow N_s^s' / N_s^s \rightarrow P_\lambda / N_s^s \rightarrow P_\lambda / N_s^s \rightarrow 0.$$ 

**Proof.** We prove (1). Let $T_s = \{t \in T \mid t \leq_{TO} s\}$. Let $S_s = \{t \in S \mid t \leq_{TO} s\}$. Then we have $T_s \subset S_s$.

$$\{t \in S \mid t \leq_{MTO} s\} = S_s \setminus T_s.$$

Now we conclude that for $s \in S \setminus T, N_s^s = P_s^s \cap P_s^s$ using Theorem 6.11(2). Hence it is a normal subgroup being the intersection of two normal subgroups. If $s_0$ is the maximal element of $S \setminus T$ then $N_s^{s_0} = P_s^s$. So $P_s^s / N_s^{s_0}$ is abelian. Hence if $t' \in T$ and $t$ is its predecessor in $(S, \leq_{MTO})$ then $N_t^{t'}$ is a subgroup of the abelian group $P_s^s / N_s^{s_0}$ and hence $N_t^{t'}$ is normal in $P_s^{s_0} / N_s^{s_0}$. So we conclude that $N_t^{t'}$ is normal in $P_s^s / N_s^{s_0}$ for all $t' \in T$. This proves (1).

(2) is clear. For $t \leq_{MTO} t'$, $t'$ the successor of $t$, the group $N_t^{t'}$ has index $p$ in $N_t^{t'}$. So (3) follows.

We prove (4). If $s' \in T$ then the proof is clear. If $s, s' \in S \setminus T$ and $s'$ is the successor of $s$ in $(S, \leq_{MTO})$ then for all $\bar{s} \in S$, such that $s <_{TO} \bar{s} <_{TO} s'$ we have $\bar{s} \in T$. Let $g \in P_s^s, h \in N_s^{s'} \subseteq P_s^s$. Let $t \leq_{TO} s'$ be the predecessor of $s'$ in $(S, \leq_{TO})$. Then using Theorem 6.8(3), we have $P_s^s / P_s^s$ is a central subgroup of $P_s^s / P_s^s$. So the cosets $gP_s^s, hP_s^s$ commute, that is, $ghg^{-1}h^{-1} \in P_s^s$ which implies that if $g^a = ghg^{-1}h^{-1}$ then $a_s = 0$. We also have $ghg^{-1}h^{-1} \in P_s^s$. So $a_\bar{s} = 0$ for all $s <_{TO} \bar{s} <_{TO} s'$ using Theorem 6.11(2). This implies that $ghg^{-1}h^{-1} \in P_s^s$. Being a commutator
\(ghg^{-1}h^{-1} \in \mathcal{P}_\Delta \cap \mathcal{P}_\lambda\), that is, \(ghg^{-1}h^{-1} \in \mathcal{N}_\Delta^s\) for all \(g \in \mathcal{P}_\Delta, h \in \mathcal{N}_\Delta^s\). Hence the cosets \(g\mathcal{N}_\Delta^s, h\mathcal{N}_\Delta^s\) commute. Consequentially we have \(\mathcal{N}_\Delta^s \subset \mathcal{G}_\Delta\) is a central subgroup of \(\mathcal{P}_\Delta\). This proves (4).

Hence the theorem follows. \(\blacksquare\)

**Remark 6.15.** In the proof of Theorem 6.14, \(p\) can be any prime.

### 6.2.4. The Action of the Restricted Diagonal Subgroup on the Cohomology.

Let \(\mathcal{D}_\Delta \subset \mathcal{G}_\Delta\) be the subgroup of diagonal matrices whose orders divide \(p - 1\). Then we have

- \(\mathcal{D}_\Delta \cap \mathcal{P}_\Delta = \{1\}\), the trivial subgroup,
- \(\mathcal{D}_\Delta\) normalizes \(\mathcal{P}_\Delta\), that is, \(\mathcal{D}_\Delta \subset \mathcal{N}_\mathcal{G}_\Delta(\mathcal{P}_\Delta)\).

As mentioned in Remark 2.5 the conjugation action of \(\mathcal{D}_\Delta\) on \(\mathcal{P}_\Delta\) gives rise to an action of \(\mathcal{D}_\Delta\) on the spaces \(H^2_{\text{Trivial Action}}(\mathcal{N}_\Delta^s, \mathbb{Z}/p\mathbb{Z})\) for \(s \leq_{\text{MTO}} s'\) and \(H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})\) for \(s \in S\). Since these are vector spaces over \(\mathbb{Z}/p\mathbb{Z}\), they are semisimple representations of \(\mathcal{D}_\Delta\) since \(p\) does not divide the cardinality of \(\mathcal{D}_\Delta\). Using Theorem 2.8, in order to prove \(H^2_{\text{Trivial Action}}(\mathcal{G}_\Delta, \mathbb{Z}/p\mathbb{Z})\) is zero it is enough to prove that the trivial subrepresentation

\[H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta} \subseteq H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})\]

is zero.

### 6.2.5. The Repeated Use of the Extended Hochschild-Serre Exact Sequence.

To compute the space \(H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta}\) we use the extended Hochschild-Serre exact sequences for the central extensions given in 6.6. Actually we use the following part of the exact sequence, for any \(s, s' \in S \setminus T\) such that \(s'\) is the successor element of \(s\) in \((S, \leq_{\text{MTO}})\).

\[
H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) \overset{\text{Inf}}{\longrightarrow} H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})
\]

\[
\overset{\tau = \text{Res} \times \theta}{\longrightarrow} H^2_{\text{Trivial Action}}(\mathcal{N}_\Delta^s, \mathbb{Z}/p\mathbb{Z}) \times P(\mathcal{P}_\Delta, \mathcal{N}_\Delta^s, \mathbb{Z}/p\mathbb{Z}).
\]

**Theorem 6.16.** Let \(p\) be an odd prime and let \(s, s' \in S\) be such that \(s'\) is the successor element of \(s\) in \((S, \leq_{\text{MTO}})\). Consider the restriction map \(H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) \overset{\text{Res}}{\longrightarrow} H^2_{\text{Trivial Action}}(\mathcal{N}_\Delta^s, \mathbb{Z}/p\mathbb{Z})\). Then we have

\[H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta} \subseteq \text{Ker}(\text{Res}).\]
Proof. If \( \text{Pos}(s') = (x, y) \) and \( 1 \leq x \neq y \leq \rho = \rho_1 + \cdots + \rho_k \) then we have \( H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z})D_\Lambda = 0 \) using Theorem 2.4 since \( p \) is odd. Hence

\[
\text{Res}(H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z})D_\Lambda) \subseteq H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z})D_\Lambda = 0.
\]

If \( \text{Pos}(s') = (x', y') \) for some \( 1 \leq x' \leq \rho = \rho_1 + \cdots + \rho_k \) where \( s' = (l', i', i', m', m') \) then we have \( l' > 0 \Rightarrow m' < k, 1 \leq x \leq \rho_1 + \cdots + \rho_{k-1} \). So position \( (x', x'+1) \) is in block \( (m', m') \) or \( (m', m' + 1) \), position \( (x' + 1, x') \) is in block \( (m', m') \) or \( (m' + 1, m') \), position \( (x' + 1, x' + 1) \) is in block \( (m', m') \) or \( (m' + 1, m' + 1) \). Let \( t \in S \) be such that \( \text{Pos}(t) = (x', x' + 1) \) and the first coordinate of \( t \) is \( l \) which is the first coordinate of \( s' \). Then \( s' <_{\text{MTO}} t, s' <_{\text{TO}} t \). Let the symbol \( A \) denote the entry in the position \( (x' + 1, x') \) of any typical matrix in \( P_\Lambda \). Consider the subgroup

\[
C^s_\Lambda = \{g^a + e_{(x',x'+1)}(p''a_t) + e_{(x'+1,x')}(A) \mid 0 \leq a_t \leq p - 1, g^a \in N^{s'}_\Lambda \} \subset P_\Lambda.
\]

This is indeed a subgroup, because if \( g_1 = g^a + e_{(x',x'+1)}(p''a_t) + e_{(x',x'+1)}(A), g_2 = g^b + e_{(x',x'+1)}(p''b_t) + e_{(x',x'+1)}(B) \) be two elements in \( C^s_\Lambda \). Then we have \( g_1g_2 \in C^s_\Lambda \). We can see this as follows.

\[
g_1g_2 = g^a + e_{(x',x'+1)}(p''a_t) + g^b + e_{(x',x'+1)}(p''b_t) + g^a e_{(x',x'+1)}(B)
\]

\[
= g^a + e_{(x',x'+1)}(p''a_t) + g^b + e_{(x',x'+1)}(p''b_t) + g^a e_{(x',x'+1)}(A) + e_{(x',x'+1)}(B)
\]

where \( g^c \in N^{s'}_\Lambda, c_{s'} \equiv a_{s'} + b_{s'} + a_tB \mod p, c_t \equiv a_t + b_t \mod p, C \equiv A + B \mod p'' \). Since the nonempty finite subset \( C^s_\Lambda \) is closed under group multiplication, it is also a subgroup. Now \( s \in S \setminus T \) is the predecessor of \( s' \). We have \( N^{s'}_\Lambda \subseteq C^s_\Lambda \). The group multiplication in \( C^s_\Lambda \) is given in terms of 2 \times 2 matrices as follows.

\[
\begin{pmatrix}
1 + p''a_{s'} & \mod p'' + 1 \\
A \mod p'' & 1 \mod p''
\end{pmatrix}
\begin{pmatrix}
1 + p''b_{s'} & \mod p'' + 1 \\
B \mod p'' & 1 \mod p''
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 + p''(a_{s'} + b_{s'} + a_tB) & \mod p'' + 1 \\
A + B \mod p'' & 1 \mod p''
\end{pmatrix}.
\]

We consider the normal abelian subgroup \( A^{s'}_\Lambda \subseteq C^s_\Lambda \) of upper triangular 2 \times 2 matrices in \( C^s_\Lambda \), that is,
\( A_{\Delta}^s = \left\{ \begin{pmatrix} 1 + p^\ell a_s & \mod p^{\ell+1} & p^\ell a_t & \mod p^{\ell+1} \\ 0 & \mod p^\ell & 1 & \mod p^\ell \end{pmatrix} \mid 0 \leq a_s, a_t \leq p - 1 \right\} \)

\( \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}. \)

Now consider the map \( \text{Res} \) as the following composition of maps \( \text{Res}_2, \text{Res}_1 \), that is, \( \text{Res} = \text{Res}_2 \circ \text{Res}_1 \) where,

\[
H^2_{\text{Trivial Action}}\left( \frac{\mathcal{P}_\Delta}{\mathcal{N}_\Delta}, \mathbb{Z}/p\mathbb{Z} \right) \xrightarrow{\text{Res}_1} H^2_{\text{Trivial Action}}(A_{\Delta}^s, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res}_2} H^2_{\text{Trivial Action}}\left( \frac{\mathcal{N}_\Delta}{\mathcal{P}_\Delta}, \mathbb{Z}/p\mathbb{Z} \right).
\]

To prove the theorem it is enough to show that

\( \text{Res}_1(H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{D\Delta}) = 0. \)

Let \( [c] = \text{Res}_1([d]) \) where \([d] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{D\Delta} \) then cohomology class \([d]\) and hence \([c]\) does not change under conjugation action induced by the elementary invertible matrix

\[
\overline{E}_{(x' + 1, x')} (C) = \begin{pmatrix} 1 & \mod p^{\ell+1} & 0 & \mod p^{\ell+1} \\ C & \mod p^\ell & 1 & \mod p^\ell \end{pmatrix} \in \frac{C^s}{\mathcal{N}_\Delta}
\]

using Remark 2.5. We observe that the conjugation on \( A_{\Delta}^s \) is given as follows.

\[
\overline{E}_{(x' + 1, x')} (C) \begin{pmatrix} 1 + p^\ell a_s & \mod p^{\ell+1} & p^\ell a_t & \mod p^{\ell+1} \\ 0 & \mod p^\ell & 1 & \mod p^\ell \end{pmatrix} \overline{E}_{(x' + 1, x')} (-C) = \begin{pmatrix} 1 + p^\ell (a_s - a_t C) & \mod p^{\ell+1} & p^\ell a_t & \mod p^{\ell+1} \\ 0 & \mod p^\ell & 1 & \mod p^\ell \end{pmatrix}.
\]

We also observe that cohomology class \([c] \in H^2_{\text{Trivial Action}}(A_{\Delta}^s, \mathbb{Z}/p\mathbb{Z})^{D\Delta}. \) We simplify the notation by denoting elements in \( A_{\Delta}^s = \{(a_s, \tilde{a}_t) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\}. \)

Using Theorem 2.12, we have \( c \) is cohomologous to \( c_1 + c_2 + \tilde{c} \) where \( c_1 \) and \( c_2 \) are cocycles obtained by restriction of the cocycle \( c \) to each of the components of \( A_{\Delta}^s \) and \( \tilde{c} : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) is a bilinear map which in particular is given as: For all \( a_s, \tilde{a}_t, b_s, \tilde{b}_t \in \mathbb{Z}/p\mathbb{Z}, \tilde{c}(a_s, \tilde{b}_t) = \beta a_s \tilde{b}_t, \)

\( c((a_s, \tilde{a}_t), (b_s, \tilde{b}_t)) \approx c_1(a_s, b_s) + c_2(\tilde{a}_t, \tilde{b}_t) + \beta a_s \tilde{b}_t \)

for some \( \beta \in \mathbb{Z}/p\mathbb{Z}. \) Now the action of \( D\Delta \) acts nontrivially on \( c_2 \) since \( t \) is in a nondiagonal position of any matrix in \( \mathcal{P}_\Delta \), where as \( D\Delta \) acts trivially on \( c_1 \) since \( s' \) is in a diagonal position. Using Theorem 2.4, since \( p \) is odd, we have \( c_2 \) is
We use conjugation by $\beta$ and $\gamma$. Then the subgroup of $D$-invariant cohomology classes of $H^2_c$. This proves the theorem.

**Remark 6.17.** In the proof of Theorem 6.16 we only require that $p$ is an odd prime and $p$ can be equal to 3.

**Theorem 6.18.** Let $p$ be an odd prime. Consider the following $p$-group of order $p^2$.

$$G_1 = \{ \begin{pmatrix} b \\ c \end{pmatrix} \mid b, c \in \mathbb{Z}/p\mathbb{Z} \}$$

where the group multiplication is given by:

$$\begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} b' \\ c' \end{pmatrix} = \begin{pmatrix} b + b' \\ c + c' \end{pmatrix}.$$ 

Consider the action of the group $D = (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^*$ on $G_1$ as follows. For $t_i \in (\mathbb{Z}/p\mathbb{Z})^*, i = 1, 2$,

$$(t_1, t_2) \cdot \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{t_1}{t_2} b \\ \frac{t_2}{t_1} c \end{pmatrix}.$$ 

Then the subgroup of $D$-invariant cohomology classes of $H^2_{\text{Trivial Action}}(G_1, \mathbb{Z}/p\mathbb{Z})$ is $H^2_{\text{Trivial Action}}(G_1, \mathbb{Z}/p\mathbb{Z})^D = \{ \beta[z] \mid \beta \in \mathbb{Z}/p\mathbb{Z} \} \cong \mathbb{Z}/p\mathbb{Z}$ where $z \in Z^2_{\text{Trivial Action}}(G_1, \mathbb{Z}/p\mathbb{Z})$ is defined as:

$$z \left( \begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b' \\ c' \end{pmatrix} \right) = bc'.$$

**Proof.** The theorem follows from an application of Theorem 2.12 and Theorem 2.4 and the fact that $p$ is odd.

**Remark 6.19.** In the proof of Theorem 6.18 we only require that $p$ is an odd prime and $p$ can be equal to 3.
Theorem 6.20. Let $\Lambda = (2 > 1)$ and $p$ be an odd prime. Then

$$H^2_{\text{Trivial Action}}(\mathcal{G}_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) = 0.$$ 

Proof. We show that $H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_{\Lambda}} = 0$. The group multiplication in $\mathcal{P}_{\Lambda}$ is given as follows.

$$\begin{pmatrix} 1 + ap & bp \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 + a'p & b'p \\ c' & 1 \end{pmatrix} = \begin{pmatrix} 1 + (a + a' + bc')p & (b + b')p \\ c + c' & 1 \end{pmatrix}.$$ 

Let $[y] \in H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_{\Lambda}}$ and consider $\text{Res} : H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\mathcal{Z}(\mathcal{P}_{\Lambda}), \mathbb{Z}/p\mathbb{Z})$. We observe using Theorem 6.16 where we choose $s = \emptyset, s' = (1, 1, 1, 1)$, we get $[y] \in \text{Ker}(\text{Res})$. Consider the map $\theta : H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \rightarrow \mathcal{P}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z})$. The class $[y] \in \text{Ker}(\theta)$. This is because $[y] \in H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_{\Lambda}}$ and the positions of $a, b$ are not mutually symmetric about the diagonal and the positions of $a, c$ are also not mutually symmetric about the diagonal.

Using the exactness of the sequence

$$H^2_{\text{Trivial Action}}(\frac{\mathcal{P}_{\Lambda}}{Z(\mathcal{P}_{\Lambda})}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res} \times \theta} H^2_{\text{Trivial Action}}(\mathcal{Z}(\mathcal{P}_{\Lambda}), \mathbb{Z}/p\mathbb{Z}) \times \mathcal{P}(\mathcal{P}_{\Lambda}, \mathbb{Z}/p\mathbb{Z}),$$

we get that there exists $[x] \in H^2_{\text{Trivial Action}}(\frac{\mathcal{P}_{\Lambda}}{Z(\mathcal{P}_{\Lambda})}, \mathbb{Z}/p\mathbb{Z})$ such that $\text{Inf}([x]) = [y]$.

So we have by averaging $[x]$ with respect to the action of the group $\mathcal{D}_{\Lambda}$

$$\text{Inf} \left( \frac{1}{|\mathcal{D}_{\Lambda}|} \sum_{t \in \mathcal{D}_{\Lambda}} t \cdot [x] \right) = [y]$$

and the cohomology class

$$\frac{1}{|\mathcal{D}_{\Lambda}|} \sum_{t \in \mathcal{D}_{\Lambda}} t \cdot [x] \in H^2_{\text{Trivial Action}}(\frac{\mathcal{P}_{\Lambda}}{Z(\mathcal{P}_{\Lambda})}, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_{\Lambda}}.$$ 

Now the group $\frac{\mathcal{P}_{\Lambda}}{Z(\mathcal{P}_{\Lambda})} \cong G_1$ (defined in Theorem 6.18) and the action of $\mathcal{D}_{\Lambda}$ on $\frac{\mathcal{P}_{\Lambda}}{Z(\mathcal{P}_{\Lambda})}$ and the action of $D$ in Theorem 6.18 on $G_1$ are compatible. Hence using Theorem 6.18 we get that $[y] = \beta [z]$ for some $\beta \in \mathbb{Z}/p\mathbb{Z}$ where

$$z \left( \begin{pmatrix} 1 + ap & bp \\ c & 1 \end{pmatrix}, \begin{pmatrix} 1 + a'p & b'p \\ c' & 1 \end{pmatrix} \right) = bc'.$$
But we observe that $z$ is a 2-coboundary because if $v : \mathcal{P}_\Delta \to \mathbb{Z}/p\mathbb{Z}$ is a 1-cochain defined as: $v(\left( \begin{array}{cc} 1 + ap & bp \\ c & 1 \end{array} \right)) = a$ then we have $v(gg') - v(g) - v(g') = bc'$ where $g = \left( \begin{array}{cc} 1 + ap & bp \\ c & 1 \end{array} \right)$, $g' = \left( \begin{array}{cc} 1 + a'p & b'p \\ c' & 1 \end{array} \right)$.

Hence we get $[y] = 0$ and so $H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{D_\Delta} = 0$. This proves the theorem.

\underline{Remark 6.21.} In the proof of Theorem 6.20 we only require that $p$ is an odd prime and $p$ can be equal to 3.

\textbf{Theorem 6.22.} Let $p$ be an odd prime. Consider the following $p$-group of order $p^4$.

$$G_2 = \left\{ \left( \begin{array}{ccc} b \\ c \\ e \\ f \end{array} \right) \mid b, c, e, f \in \mathbb{Z}/p\mathbb{Z} \right\}$$

where the group multiplication is given by:

$$\left( \begin{array}{ccc} c \\ e \\ f \end{array} \right) \left( \begin{array}{ccc} b' \\ c' \\ f' \end{array} \right) = \left( \begin{array}{ccc} c + c' & b + b' \\ e + e' + f c' & f + f' \end{array} \right).$$

Consider the action of the group $D = (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^*$ on $G_2$ as follows. For $t_i \in (\mathbb{Z}/p\mathbb{Z})^*$, $i = 1, 2, 3$,

$$(t_1, t_2, t_3) \cdot \left( \begin{array}{ccc} c \\ e \\ f \end{array} \right) = \left( \begin{array}{ccc} \frac{t_1 c}{t_2} \\ \frac{t_2 e}{t_3} \\ \frac{t_3 f}{t_1} \end{array} \right).$$

Then the subgroup of $D$-invariant cohomology classes of $H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z})$ is trivial, that is,

$$H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z})^{D} = 0.$$ 

\textbf{Proof.} Let $[y] \in H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z})^{D}$. Consider the central subgroup $Z = \left\{ \left( \begin{array}{cc} 0 \\ e \\ 0 \end{array} \right) \mid e \in \mathbb{Z}/p\mathbb{Z} \right\}$ and the map $\theta : H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z}) \to P(G_2, \mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ as defined in Proposition 2.11. We have

$$\theta([y]) \left( \begin{array}{ccc} b \\ c \\ e \end{array} \right), \left( \begin{array}{ccc} 0 \\ 0 \\ 0 \end{array} \right) = \alpha be' + \beta ce' + \gamma fe'$$
for some $\alpha, \beta, \gamma \in \mathbb{Z}/p\mathbb{Z}$. By the invariance of $[y]$ under the action of the group $D$ we get that $\beta = \gamma = 0$. Hence

\[(6.7) \quad \theta([y]) \left( \begin{pmatrix} c & b \\ e & f \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e' & 0 \end{pmatrix} \right) = \alpha be'.\]

Consider the normal abelian subgroup $H = \{ \begin{pmatrix} b \\ e \end{pmatrix} \mid b, ce \in \mathbb{Z}/p\mathbb{Z} \}$ and the map $\text{Res} : H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z})$. We have by using Corollary 2.13

\[
\text{Res}([y]) \left( \begin{pmatrix} c & b \\ e & 0 \end{pmatrix}, \begin{pmatrix} c' & b' \\ e' & 0 \end{pmatrix} \right) \approx \text{cohomologous} \ s_1(b, b') + s_2(c, c') + s_3(e, e') + \alpha' be' + \beta' ce'
\]

for some $\alpha', \beta' \in \mathbb{Z}/p\mathbb{Z}$, where $s_i, i = 1, 2, 3$ are the restrictions of $y$ to the respective components. Now using the invariance of $[y]$ under the action of $D$ and using Theorem 2.4, since $p$ is odd, we conclude that $s_i$ are cohomologous to zero for $i = 1, 2, 3$ and $\beta' = 0$. So we have

\[
\text{Res}([y]) \left( \begin{pmatrix} c & b \\ e & 0 \end{pmatrix}, \begin{pmatrix} c' & b' \\ e' & 0 \end{pmatrix} \right) \approx \text{cohomologous} \ \alpha' be'.
\]

Now consider the map $\theta_0 : H^2_{\text{Trivial Action}}(H, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(H, K, \mathbb{Z}/p\mathbb{Z})$ as defined in Proposition 2.11, where $K = \{ \begin{pmatrix} c & 0 \\ e & 0 \end{pmatrix} \mid c, e \in \mathbb{Z}/p\mathbb{Z} \}$. We have

\[
\theta_0(\text{Res}([y])) \left( \begin{pmatrix} c & b \\ e & 0 \end{pmatrix}, \begin{pmatrix} c' & 0 \\ e' & 0 \end{pmatrix} \right) = \alpha' be'.
\]

Substituting $c' = 0$ in the above equation and substuting $f = 0$ in Equation 6.7 we get that $\alpha' = \alpha$. 
Now $\text{Res}([y])$ is also invariant under the conjugation action of the element 
\[
\begin{pmatrix}
0 & 0 \\
0 & \bar{f}
\end{pmatrix}
\]. We observe that
\[
\begin{pmatrix}
0 & 0 \\
0 & \bar{f}
\end{pmatrix}
\begin{pmatrix}
c & b \\
e & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -\bar{f}
\end{pmatrix}
= \begin{pmatrix}
c & b \\
e + \bar{f}c & 0
\end{pmatrix}
\begin{pmatrix}
c' & 0 \\
e' & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & -\bar{f}
\end{pmatrix}
= \begin{pmatrix}
c' & 0 \\
e' + \bar{f}c' & 0
\end{pmatrix}
\]
So we get that
\[
\alpha b e' = \theta_0(\text{Res}([y])) \begin{pmatrix}
c & b \\
e + \bar{f}c & 0
\end{pmatrix} \begin{pmatrix}
c' & 0 \\
e' + \bar{f}c' & 0
\end{pmatrix} = \alpha b(e' + \bar{f}c').
\]
This implies that $\alpha = 0$. Hence we get $\theta([y]) = 0$ in Equation 6.7. Now $[y] \in \text{Ker}(\theta)$ and $[y] \in \text{Ker}(\text{Res} : H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z}) \to H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}))$ since $H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})^D = 0$ and here $\text{Res}([y]) \in H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})^D = 0$.

Using the exactness of the sequence
\[
H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res} \times \theta} H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \times P(G_2, \mathbb{Z}, \mathbb{Z}/p\mathbb{Z}),
\]
we get that there exists $[x] \in H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z})$ such that $\text{Inf}([x]) = [y]$. So we have by averaging $[x]$ with respect to the action of the group $D$
\[
\text{Inf} \left( \frac{1}{|D|} \sum_{t \in D} t \cdot [x] \right) = [y]
\]
and the cohomology class
\[
\frac{1}{|D|} \sum_{t \in D} t \cdot [x] \in H^2_{\text{Trivial Action}}(\mathbb{Z}/p\mathbb{Z})^D.
\]
But in $\hat{\frac{G_2}{\mathbb{Z}}}$ the positions of $b, c, f$ are nondiagonal and $D$ acts nontrivially. Also no two of $b, c, f$ are in mutually symmetric positions about the diagonal. Hence we conclude that
\[
H^2_{\text{Trivial Action}}(\hat{\frac{G_2}{\mathbb{Z}}}, \mathbb{Z}/p\mathbb{Z})^D = 0.
\]
This implies that $[y]$ is the inflation of the zero cohomology class which implies $H^2_{\text{Trivial Action}}(G_2, \mathbb{Z}/p\mathbb{Z})^D = 0$. This proves the theorem. ■
Remark 6.23. In the proof of Theorem 6.22 we only require that $p$ is an odd prime and $p$ can be equal to 3.

**Theorem 6.24.** Let $p$ be a prime. Consider the following $p$-group of order $p^5$.

$$G_3 = \left\{ \begin{pmatrix} a & b \\ c & e \\ d & f \end{pmatrix} \mid a, b, c, e, f \in \mathbb{Z}/p\mathbb{Z} \right\}$$

where the group multiplication is given by:

$$\begin{pmatrix} a & b \\ c & e \\ d & f \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & e' \\ d' & f' \end{pmatrix} = \begin{pmatrix} a + a' + bf' & b + b' \\ c + c' & e + e' + fc' \\ d + d' + ef' & f + f' \end{pmatrix}.$$  

Consider the action of the group $D = (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^*$ on $G_3$ as follows. For $t_i \in (\mathbb{Z}/p\mathbb{Z})^*$, $i = 1, 2, 3$,

$$(t_1, t_2, t_3) \cdot \begin{pmatrix} a & b \\ c & e \\ d & f \end{pmatrix} = \begin{pmatrix} a t_1 & b t_1 \\ c t_2 & e t_2 \\ d t_3 & f t_3 \end{pmatrix}.$$  

Then the subgroup of $D$-invariant cohomology classes of $H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})$ is

$$H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z}) = \{ \beta[z] \mid \beta \in \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \}$$

where $z \in Z^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})$ is defined as:

$$z\begin{pmatrix} a & b \\ c & e \\ d & f \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & e' \\ d' & f' \end{pmatrix} = ac' + be'.$$

**Proof.** Clearly $z$ is a 2-cocycle on $G_3$, that is, the cocycle condition is satisfied. This is because we have

$$(ac' + be') + (a + a' + bf')c'' + (b + b')e'' = a'c'' + b'e'' + a(c' + c'') + b(e' + e'' + f'c'')$$

for $a, b, c, e, f, a', b', c', e', f', a'', b'', c'', e'', f'' \in \mathbb{Z}/p\mathbb{Z}$. We also observe that $[z]$ is invariant under the action of the group $D$. So $[z] \in H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})^D$.

Let $[y] \in H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})^D$. Consider the central subgroup $Z = \{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}/p\mathbb{Z} \}$ and the map $\theta : H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z}) \to P(G_3, \mathbb{Z}/\mathbb{Z}/p\mathbb{Z})$. We observe that

$$\theta([y])\begin{pmatrix} a & b \\ c & e \\ d & f \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} = \beta ca'.$$
for some $\beta \in \mathbb{Z}/p\mathbb{Z}$. We have

$$\theta([z]) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \right) = -ca'.$$

So $[y + \beta z] \in \text{Ker}(\theta)$. Also $[y + \beta z] \in \text{Ker}(\text{Res}: H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z}) \to H^2_{\text{Trivial Action}}(Z, \mathbb{Z}/p\mathbb{Z}))$, since $H^2_{\text{Trivial Action}}(Z, \mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}$. Using the exact sequence

$$H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res} \times \theta} H^2_{\text{Trivial Action}}(Z, \mathbb{Z}/p\mathbb{Z}) \times P(G_3, \mathbb{Z}/p\mathbb{Z}),$$

we get that $[y] + [\beta z] = \text{Inf}([x])$ for some $[x] \in H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})$. Now $[y] + [\beta z] \in H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})^D$. So we have by averaging $[x]$ with respect to the action of the group $D$

$$\text{Inf}\left(\frac{1}{|D|} \sum_{t \in D} t \cdot [x]\right) = [y] + [\beta z]$$

and the cohomology class

$$\frac{1}{|D|} \sum_{t \in D} t \cdot [x] \in H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})^D.$$

We observe now that $\frac{G_3}{Z} \cong G_2$ and the actions of $D$ on $\frac{G_3}{Z}$ and $G_2$ are compatible. Using Theorem 6.24 we get that $H^2_{\text{Trivial Action}}(\frac{G_3}{Z}, \mathbb{Z}/p\mathbb{Z})^D = 0$. Hence we have $[y] = -\beta[z]$. Now the class $[z]$ is nonzero because $\theta([z]) \neq 0$. Hence $H^2_{\text{Trivial Action}}(G_3, \mathbb{Z}/p\mathbb{Z})^D \cong \mathbb{Z}/p\mathbb{Z}$. This proves the theorem. $\square$

Remark 6.25. In the proof of Theorem 6.24 we only require that $p$ is an odd prime and $p$ can be equal to 3.

Theorem 6.26. Let $\Lambda = (2^1 > 1^2)$ and $p$ be an odd prime. Then

$$H^2_{\text{Trivial Action}}(G_\Lambda, \mathbb{Z}/p\mathbb{Z}) = 0.$$
Proof. We show that \( H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta} = 0 \). The group multiplication in \( \mathcal{P}_\Delta \) is given as follows.

\[
\begin{pmatrix}
1 + pa & pb & pc \\
d & 1 & 0 \\
e & f & 1
\end{pmatrix}
\begin{pmatrix}
1 + pa' & pb' & pc' \\
d' & 1 & 0 \\
e' & f' & 1
\end{pmatrix}
= \begin{pmatrix}
1 + p(a + a' + bd' + ce') & (b + b' + cf')p & p(c + c') \\
d + d' & 1 & 0 \\
e + e' + fd' & f + f' & 1
\end{pmatrix}.
\]

We observe that \( \mathbb{Z}(\mathcal{P}_\Delta) = \left\{ \begin{pmatrix}
1 + pa & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \mid a \in \mathbb{Z}/p\mathbb{Z} \right\} \).

Let \([y] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta}\) and consider \( \text{Res} : H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\mathbb{Z}(\mathcal{P}_\Delta), \mathbb{Z}/p\mathbb{Z}) \). We observe using Theorem 6.16 where we choose \( s = \emptyset, s' = (1, 1, 1, 1) \), we get \([y] \in \ker(\text{Res})\). Consider the map \( \theta : H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(\mathcal{P}_\Delta, \mathbb{Z}(\mathcal{P}_\Delta), \mathbb{Z}/p\mathbb{Z}) \). The class \([y] \in \ker(\theta)\). This is because \([y] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta}\) and the positions of \( d, f, c \) are not symmetric about the diagonal with the position of \( a \).

Using the exactness of the sequence

\[
H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})
\]

\[
\xrightarrow{\text{Res} \times \theta} H^2_{\text{Trivial Action}}(\mathbb{Z}(\mathcal{P}_\Delta), \mathbb{Z}/p\mathbb{Z}) \times P(\mathcal{P}_\Delta, \mathbb{Z}(\mathcal{P}_\Delta), \mathbb{Z}/p\mathbb{Z}),
\]

we get that there exists \([x] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})\) such that \( \text{Inf}([x]) = [y] \).

So we have by averaging \([x]\) with respect to the action of the group \( \mathcal{D}_\Delta \)

\[
\text{Inf} \left( \frac{1}{|\mathcal{D}_\Delta|} \sum_{t \in \mathcal{D}_\Delta} t \cdot [x] \right) = [y]
\]

and the cohomology class

\[
\frac{1}{|\mathcal{D}_\Delta|} \sum_{t \in \mathcal{D}_\Delta} t \cdot [x] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Delta}.
\]

Now the group \( \frac{\mathcal{P}_\Delta}{\mathbb{Z}(\mathcal{P}_\Delta)} \cong G_3 \) (defined in Theorem 6.18) and the action of \( \mathcal{D}_\Delta \) on \( \frac{\mathcal{P}_\Delta}{\mathbb{Z}(\mathcal{P}_\Delta)} \) and the action of \( D \) in Theorem 6.24 on \( G_3 \) are compatible. Hence using
Theorem 6.24 we get that \([y] = \beta[z]\) for some \(\beta \in \mathbb{Z}/p\mathbb{Z}\) where

\[
z\left(\begin{pmatrix} 1 + pa & pb & pc \\ d & 1 & 0 \\ e & f & 1 \end{pmatrix}, \begin{pmatrix} 1 + pa' & pb' & pc' \\ d' & 1 & 0 \\ e' & f' & 1 \end{pmatrix}\right) = bd' + ce'.
\]

But we observe that \(z\) is a 2-coboundary because if \(v : \mathcal{P}_\Delta \rightarrow \mathbb{Z}/p\mathbb{Z}\) is a 1-cochain defined as:

\[
v\left(\begin{pmatrix} 1 + pa & pb & pc \\ d & 1 & 0 \\ e & f & 1 \end{pmatrix}\right) = a
\]

then we have \(v(gg') - v(g) - v(g') = bd' + ce'\) where \(g = \begin{pmatrix} 1 + pa & pb & pc \\ d & 1 & 0 \\ e & f & 1 \end{pmatrix}\) and \(g' = \begin{pmatrix} 1 + pa' & pb' & pc' \\ d' & 1 & 0 \\ e' & f' & 1 \end{pmatrix}\).

Hence we get \([y] = 0\) and so \(H^2_{\text{Trivial Action}}(\mathcal{P}_\Delta, \mathbb{Z}/p\mathbb{Z}) = 0\). This proves the theorem. 

**Remark 6.27.** In the proof of Theorem 6.26 we only require that \(p\) is an odd prime and \(p\) can be equal to 3.

**Theorem 6.28.** Let \(p\) be an odd prime. Consider the following \(p\)-group of order \(p^6\).

\[
G_4 = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| a, b, c, d, e, f \in \mathbb{Z}/p\mathbb{Z}\right\}
\]

where the group multiplication is given by:

\[
\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \\ e' & f' \end{pmatrix} = \begin{pmatrix} a + a' + bf' & b + b' \\ c + c' + d' + cb' & d + d' + cb' \\ e + e' + fc' & f + f' \end{pmatrix}.
\]

Consider the action of the group \(D = (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^* \times (\mathbb{Z}/p\mathbb{Z})^*\) on \(G_4\) as follows. For \(t_i \in (\mathbb{Z}/p\mathbb{Z})^*, i = 1, 2, 3,\)

\[
(t_1, t_2, t_3) \cdot \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} \frac{t_1}{t_2} a & \frac{t_1}{t_3} b \\ \frac{t_2}{t_1} c & \frac{t_2}{t_3} d \\ \frac{t_3}{t_1} e & \frac{t_3}{t_2} f \end{pmatrix}.
\]

Then the subgroup of \(D\)-invariant cohomology classes of \(H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})\) is

\[
H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})^D = \{ \alpha[y] + \beta[z] \mid \alpha, \beta \in \mathbb{Z}/p\mathbb{Z}\} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}
\]
where $y, z \in Z^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})$ are defined as:

\[
y \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \\ e' & f' \end{pmatrix} \right) = c' a'' + d' f'',
\]

\[
z \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \\ e' & f' \end{pmatrix} \right) = a' + b'.
\]

**Proof.** Clearly $y, z \in Z^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})$ are indeed cocycles and are invariant with respect to the action of $D$. We just show below that $y$ is a cocycle. We have

\[
ca' + df' + (c + c')a'' + (d + d' + cb')f' = c' a'' + d' f'' + c(a' + a'' + b' f'') + d(f' + f'')
\]

for $a, b, c, d, e, f, a', b', c', d', e', f', a'', b'', c'', d'', e'', f'' \in \mathbb{Z}/p\mathbb{Z}$.

Let $[x] \in H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})^D$. Let $\theta : H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})^D \rightarrow P(G_4, \mathbb{Z}(G_4), \mathbb{Z}/p\mathbb{Z})$ as defined in Proposition 2.11. Then we have

\[
\theta([x]) \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & d' \\ e' & 0 \end{pmatrix} \right) = \alpha f d' + \beta c a' + \gamma b e' + \delta f a' + \mu f e' + \epsilon c d' + \tau c e' + \omega b a' + \nu b d'.
\]

By invariance of $[x]$ under the action of $D$ we conclude that $\delta = \mu = \epsilon = \tau = \omega = \nu = 0$. So

\[
\theta([x]) \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & d' \\ e' & 0 \end{pmatrix} \right) = \alpha f d' + \beta c a' + \gamma b e'.
\]

We also have

\[
\theta([y]) \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & d' \\ e' & 0 \end{pmatrix} \right) = ca' - fd'.
\]

So we get that

\[
\theta([x + \alpha y]) \left( \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & d' \end{pmatrix} \right) = 0.
\]
We also have \([x + \alpha y] \in \ker (H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z}) \to H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}))\) where \(Z = \{(0, 0, d) \mid d \in \mathbb{Z}/p\mathbb{Z}\}\). This is because the position of \(d\) in \(Z\) is nondiagonal and \(D\) acts nontrivially in this position. Therefore \(H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})^D = \{0\}\) and \(\text{Res}([x + \alpha y]) \in H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})^D = \{0\}\).

Now using the exactness of the following sequence

\[
H^2_{\text{Trivial Action}}(\mathbb{Z}/\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res}_G} H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \times P(G_4, \mathbb{Z}/p\mathbb{Z})
\]

we conclude that the cohomology class \([x + \alpha y]\) is in the image of the inflation map, that is, there exists, \([\tilde{x}] \in H^2_{\text{Trivial Action}}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})\) such that \(\text{Inf}([\tilde{x}]) = [x + \alpha y]\).

Now by averaging \([\tilde{x}]\) under the action of \(D\) we get that

\[
\text{Inf}\left(\frac{1}{|D|} \sum_{t \in D} t \cdot [\tilde{x}]\right) = [x + \alpha y]
\]

and \(\frac{1}{|D|} \sum_{t \in D} t \cdot [\tilde{x}] \in H^2_{\text{Trivial Action}}(\mathbb{Z}/\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})^D\). We observe that \(\mathbb{Z}/\mathbb{Z} \cong G_3\) where \(G_3\) is as defined in Theorem 6.24 and the actions of \(D\) on \(\mathbb{Z}/\mathbb{Z}\) and \(G_3\) are compatible. Using Theorem 6.24, we get that there exists \(\kappa \in \mathbb{Z}/p\mathbb{Z}\) such that \(\frac{1}{|D|} \sum_{t \in D} t \cdot [\tilde{x}] = \kappa [z]\) where \([z]\) is as defined in Theorem 6.24. Upon inflation we get that \([x + \alpha y] = \kappa [z]\) where \([z]\) is as defined in Theorem 6.28. In other words we have \([x] = -\alpha [y] + \kappa [z]\).

Now the classes \([y] \neq 0 \neq [z]\) because \(\theta([y]) \neq 0 \neq \theta([z])\). We also have \([y] \neq \sigma [z]\) for any \(\sigma \in \mathbb{Z}/p\mathbb{Z}\) because \(\theta([y - \sigma z]) \neq 0\) for any \(\sigma \in \mathbb{Z}/p\mathbb{Z}\). Hence we conclude that \(H^2_{\text{Trivial Action}}(G_4, \mathbb{Z}/p\mathbb{Z})^D = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\). This proves the theorem. \(\blacksquare\)

**Remark 6.29.** In the proof of Theorem 6.28 we only require that \(p\) is an odd prime and \(p\) can be equal to 3.

**Theorem 6.30.** Let \(\lambda = (2^2 > 1^1)\) and \(p\) be an odd prime and \(p \neq 3\). Then

\[
H^2_{\text{Trivial Action}}(G_{\lambda}, \mathbb{Z}/p\mathbb{Z}) = 0.
\]
Proof. We show that \( H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} = 0 \). The group multiplication in \( \mathcal{P}_\Lambda \) is given as follows.

\[
\begin{pmatrix}
1 + pa & pb & pc \\
\frac{1}{h} + pe & 1 + pf & pg \\
\frac{1}{i} + 1 + p & pg' \\
\frac{1}{i'} + 1 + p & pg \\
\end{pmatrix} =
\begin{pmatrix}
1 + p(a + a' + bd' + cch' + d + d' + p(e + e' + dd' + fd' + gh') + 1 + p(f + f' + db' + ghi') & p(c + c') \\
\frac{1}{h + h' + id'} + i' + 1 + p & p(g + g' + dce') \\
\end{pmatrix}.
\]

We observe that \( Z(\mathcal{P}_\Lambda) = \{ \begin{pmatrix} 1 & 0 & 0 \\
pe & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} \} \). Let \( [x] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} \) and consider \( \text{Res} : H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(Z(\mathcal{P}_\Lambda), \mathbb{Z}/p\mathbb{Z}) \). We observe using Theorem 6.16 where we choose \( s = \emptyset, s' = (1, 1, 1, 1) \), we get \( [x] \in \ker(\text{Res}) \). Consider the map \( \theta : H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(\mathcal{P}_\Lambda, Z(\mathcal{P}_\Lambda), \mathbb{Z}/p\mathbb{Z}) \). The class \( [x] \in \ker(\theta) \). This is because \( [x] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} \) and the positions of \( c, i \) are not symmetric about the diagonal with the position of \( a \) and the position of \( d \) are same, but \( p \neq 3 \). Note if \( p \neq 3 \) is an odd prime then there exists \( t \in \mathbb{Z}/p\mathbb{Z} \) such that \( 0 \neq t^2 \neq 1 \).

Using the exactness of the sequence

\[
H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Res} \times \theta} H^2_{\text{Trivial Action}}(Z(\mathcal{P}_\Lambda), \mathbb{Z}/p\mathbb{Z}) \times P(\mathcal{P}_\Lambda, Z(\mathcal{P}_\Lambda), \mathbb{Z}/p\mathbb{Z}),
\]

we get that there exists \( [\tilde{x}] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} \) such that \( \text{Inf}(\tilde{x}) = x \).

So we have by averaging \( \tilde{x} \) with respect to the action of the group \( \mathcal{D}_\Lambda \)

\[
\text{Inf}
\left(\frac{1}{|\mathcal{D}_\Lambda|} \sum_{t \in \mathcal{D}_\Lambda} t \cdot [\tilde{x}] \right) = [x]
\]

and the cohomology class

\[
\frac{1}{|\mathcal{D}_\Lambda|} \sum_{t \in \mathcal{D}_\Lambda} t \cdot [\tilde{x}] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda} = H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda}.
\]

Hence we have

\[
\text{Inf}(H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda}) = H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathbb{Z}/p\mathbb{Z})^{\mathcal{D}_\Lambda}.
\]
By repeating the steps similarly once again we get that for $s' = (1, 2, 2, 1, 1)$, the successor of $s$ in $(S, \leq_{\text{MTO}})$,

$$\text{Inf}(H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda^\Delta, \mathbb{Z}/p\mathbb{Z})^{D\lambda}) = H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D\lambda}.$$ 

By repeating the steps similarly once again we get that for $s'' = (1, 1, 1, 1, 1)$, the successor of $s'$ in $(S, \leq_{\text{MTO}})$,

$$\text{Inf}(H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda^\Delta, \mathbb{Z}/p\mathbb{Z})^{D\lambda}) = H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D\lambda}.$$ 

Now we observe that

$$\mathcal{P}_\lambda^\Delta \frac{N^\Delta}{\mathcal{N}^\Delta} = \left\{ \begin{pmatrix} 1 & \mod p & pb & pc \\ d & \mod p & 1 & \mod p & pg \\ h & i & 1 \end{pmatrix} \mid b, d, c, g, h, i \in \mathbb{Z}/p\mathbb{Z} \right\} \cong G_4$$

where $G_4$ is as defined in Theorem 6.28. So using Theorem 6.28 we conclude that

$$[x] = \alpha[y] + \beta[z] \quad \text{for some } \alpha, \beta \in \mathbb{Z}/p\mathbb{Z}$$

where

$$y \left( \begin{pmatrix} 1 + pa & pb & pc \\ d + pe & 1 + pf & pg \\ h & i & 1 \end{pmatrix}, \begin{pmatrix} 1 + pa' & pb' & pc' \\ d' + pe' & 1 + pf' & pg' \\ h' & i' & 1 \end{pmatrix} \right) = db' + gi'$$

and

$$z \left( \begin{pmatrix} 1 + pa & pb & pc \\ d + pe & 1 + pf & pg \\ h & i & 1 \end{pmatrix}, \begin{pmatrix} 1 + pa' & pb' & pc' \\ d' + pe' & 1 + pf' & pg' \\ h' & i' & 1 \end{pmatrix} \right) = bd' + ch'.$$

But both $y$ and $z$ are 2-coboundaries on $\mathcal{P}_\lambda$ by observing the elements in positions $(1, 1)$ and $(2, 2)$ when we multiply two general elements of $\mathcal{P}_\lambda$.

So $H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D\lambda} = 0$. This completes the proof of the theorem.

Remark 6.31. In the proof of Theorem 6.30 we require that $p$ is an odd prime and $p$ is not equal to 3.

**Theorem 6.32.** Let $p$ be an odd prime and let $s, s' \in S \setminus T$ be such that $s'$ is the successor element of $s$ in $(S, \leq_{\text{MTO}})$. Consider the map $\theta : H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda^\Delta, \mathbb{Z}/p\mathbb{Z}) \rightarrow \mathcal{P}(\mathcal{P}_\lambda^\Delta, \mathcal{N}_\lambda^\Delta, \mathbb{Z}/p\mathbb{Z})$ which is defined in Theorem 2.14. Then we have

$$H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda^\Delta, \mathbb{Z}/p\mathbb{Z})^{D\lambda} \subseteq \text{Ker}(\theta),$$

in the following cases.
Theorem 6.11(3). Hence we have

\[ P(\mathcal{P}_\Lambda, \mathcal{N}_s^{\prime} / \mathcal{P}_\Lambda, \mathcal{Z} / p\mathcal{Z}) \cong \left\{ \tilde{\beta} : \mathcal{P}_\Lambda / [\mathcal{P}_\Lambda, \mathcal{P}_\Lambda] \times \mathcal{N}_s^{\prime} / \mathcal{N}_s \to \mathcal{Z} / p\mathcal{Z} | \tilde{\beta} \text{ is bilinear} \right\}. \]

If \([d] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\Lambda, \mathcal{Z} / p\mathcal{Z})\), then using Theorem 2.14, for \(g^a \in \mathcal{P}_\Lambda, g^b \in \mathcal{N}_s^{\prime}\)

\[ \theta([d])(g^a\mathcal{N}_s^{\prime}g^b\mathcal{N}_s^{\prime}) = d(g^a\mathcal{N}_s^{\prime}g^b\mathcal{N}_s^{\prime}) - d(g^b\mathcal{N}_s^{\prime}g^a\mathcal{N}_s^{\prime}) = \tilde{\beta}(a_i)_{i \in T}, b_{s'} = \sum_{t \in T} \beta_t a_t b_{s'} \text{ for some } \beta_t \in \mathcal{Z} / p\mathcal{Z}, t \in T. \]

Moreover \(\tilde{\beta}\) depends only on the cohomology class of \([d]\). Now we show that \(\tilde{\beta} = 0\), that is, \(\beta_t = 0\) for all \(t \in T\) in the cases (1), (2), (3), (4), (5) mentioned. We use the action of elements of \(\mathcal{D}_\Lambda\) on \([d]\). Since \([d]\) is invariant under this action, we have the following.

- If \(s'\) is in the diagonal position then \(\beta_t = 0\) for all \(t \in T\), that is, \(\tilde{\beta} = 0\).
- If \(s'\) is not in the diagonal position and \(\text{Pos}(s') = (x', y')\) then \(\beta_t = 0\) for all \(t \in T\) such that
  
  (i) \(\text{Pos}(t) \notin \{(x', y'), (y', x')\}\),
  
  (ii) \(\text{Pos}(t) = \text{Pos}(s') = (x', y')\) since \(p \neq 3\). Note if \(p \neq 3\) then there exists \(\gamma \in (\mathcal{Z} / p\mathcal{Z})^*\) such that \(\gamma^2 \neq 1\).

This establishes the theorem in cases (1), (2), (3).

Now consider the case (4): \(\text{Pos}(s') = (x', y'), x' - y' > 1\) and there exists \(t_0 \in T\) such that \(\text{Pos}(t_0) = (y', x')\). It remains to show that \(\beta_{t_0} = 0\) because in this case we have

\[ \theta([d])(g^a\mathcal{N}_s^{\prime}g^b\mathcal{N}_s^{\prime}) = \beta_{t_0} a_{t_0} b_{s'} \text{ for } g^a \in \mathcal{P}_\Lambda, g^b \in \mathcal{N}_s^{\prime}. \]

If \(x' - y' > 1\) then \(s'\) appears in a strictly lower triangular entry below the first subdiagonal. Here \(t_0\) appears in a strictly upper triangular entry above the first superdiagonal. So the first coordinate of \(t_0\) is 1. Let the first coordinate of \(s'\) be \(l' \geq 0\). Consider the positions in the set \(\{y', x' - 1, x'\} \times \{y', x' - 1, x'\}\).
Let $A_1, pA_2$ be typical entries of a matrix in $P_\Delta$ in positions $(x', 1, y')$ and $(y', x')$ respectively. Let $r \in S$ be such that the first coordinate of $r$ is $l'$ and $\text{Pos}(r) = (x', x' - 1)$. Consider the subgroup of $F_{\Delta}' = \{ g^{a} + e_{(x', y')}(pA_2) + e_{(x', x' - 1)}(p^{r}a_{r}) + e_{(x' - 1, y')}(A_1) \mid 0 \leq a_{r} \leq p - 1, g^{a} \in \mathcal{N}_{\Delta}' \}$. We can immediately see that $F_{\Delta}'$ is a subgroup and $\mathcal{N}_{\Delta}' \leq F_{\Delta}'$ is a normal subgroup. The group multiplication in $F_{\Delta}'$ is given in terms of $3 \times 3$ matrices as follows. For $g^{a}\mathcal{N}_{\Delta}' = g^{b}\mathcal{N}_{\Delta}' \in F_{\Delta}'$, the images of $g^{a}, g^{b} \in F_{\Delta}'$, the product is given as:

\[
\begin{pmatrix}
1 \mod p^{r+1} & 0 \mod p^{r+1} & pA_2 \mod p^{r+1} \\
A_1 \mod p^{r+1} & 1 \mod p^{r+1} & 0 \mod p^{r+1} \\
p^{r}a_{r} \mod p^{r+1} & p^{r}a_{r} \mod p^{r+1} & 1 \mod p^{r+1}
\end{pmatrix}
\begin{pmatrix}
1 \mod p^{r+1} & 0 \mod p^{r+1} & pB_1 \mod p^{r+1} \\
B_1 \mod p^{r+1} & 1 \mod p^{r+1} & 0 \mod p^{r+1} \\
p^{r}b_{r} \mod p^{r+1} & p^{r}b_{r} \mod p^{r+1} & 1 \mod p^{r+1}
\end{pmatrix}
\]

Here we have

\[
\theta([d])(g^{a}\mathcal{N}_{\Delta}', g^{b}\mathcal{N}_{\Delta}') = \beta_{10}\overline{A}_2b_{s'} \text{ for } g^{a} \in F_{\Delta}', g^{b} \in \mathcal{N}_{\Delta}'
\]

where $\overline{A}_2$ is the residue of $A_2$ modulo $p$. Now consider the normal abelian subgroup $\mathcal{H}_{\Delta}'$ of $F_{\Delta}'$ given by $3 \times 3$ matrices of the form

\[
\begin{pmatrix}
1 \mod p^{r+1} & 0 \mod p^{r+1} & pA_2 \mod p^{r+1} \\
A_1 \mod p^{r+1} & 1 \mod p^{r+1} & 0 \mod p^{r+1} \\
p^{r}a_{s'} \mod p^{r+1} & 1 \mod p^{r+1}
\end{pmatrix}
\]

where we have substituted $a_{r} = 0$. Consider the map $\text{Res} : H^{2}_{\text{Trivial Action}}(\mathcal{N}_{\Delta} / \mathbb{Z}, \mathbb{Z} / p\mathbb{Z}) \rightarrow H^{2}_{\text{Trivial Action}}(\mathcal{H}_{\Delta}', \mathbb{Z} / p\mathbb{Z})$. We have using Corollary 2.13, for $g^{a}\mathcal{N}_{\Delta}', g^{b}\mathcal{N}_{\Delta}' \in \mathcal{H}_{\Delta}'$

\[
\text{Res}(d)(g^{a}\mathcal{N}_{\Delta}', g^{b}\mathcal{N}_{\Delta}') \approx \text{cohomologous to } s_{1}(A_1 \mod p^{r+1}, B_1 \mod p^{r+1}) + s_{2}(a_{s'}, b_{s'}) + s_{3}(pA_2 \mod p^{r+1}, pB_2 \mod p^{r+1}) + \alpha\overline{A}_1\overline{B}_2 + \gamma\overline{A}_1b_{s'} + \beta\overline{A}_2b_{s'}
\]

for some $\alpha, \beta, \gamma \in \mathbb{Z} / p\mathbb{Z}$ where $s_{i}, i = 1, 2, 3$ are the restrictions of the cocycle $d$ in the respective components. Since $[d]$ is invariant under the action of $D_{\Delta}$ we have $s_{i}, i = 1, 2, 3$ are cohomologous to zero and $\alpha = \gamma = 0$. Hence we have

\[
(6.8) \quad \text{Res}(d)(g^{a}\mathcal{N}_{\Delta}', g^{b}\mathcal{N}_{\Delta}') \approx \text{cohomologous to } \beta\overline{A}_2b_{s'} \text{ for } g^{a}\mathcal{N}_{\Delta}', g^{b}\mathcal{N}_{\Delta}' \in \mathcal{H}_{\Delta}'.
\]

Let $\mathcal{K}_{s'}$ be the abelian subgroup of $\mathcal{H}_{\Delta}'$ consisting of matrices of the form

\[
\begin{pmatrix}
1 \mod p^{r+1} & 0 \mod p^{r+1} & 0 \mod p^{r+1} \\
A_1 \mod p^{r+1} & 1 \mod p^{r+1} & 0 \mod p^{r+1} \\
p^{r}a_{s'} \mod p^{r+1} & 0 \mod p^{r+1} & 1 \mod p^{r+1}
\end{pmatrix}
\]
obtained by substituting \(a_r = 0\) and \(A_2 = 0\) in matrices of \(\frac{F^t}{A^t}\). Consider the map \(\theta_0 : H^2_{\text{Trivial Action}}(H^d_{\chi}, \mathbb{Z}/p\mathbb{Z}) \to P(H^d_{\chi}, K_{s'}, \mathbb{Z}/p\mathbb{Z})\) as defined in Proposition 2.11. We have for \(g^aN_{s'}^x \in H^d_{\chi}, g^bN_{s'}^y \in K_{s'}\)

\[
\beta A_2 b_{s'} = \theta_0(\text{Res}([d]))(g^aN_{s'}^x, g^bN_{s'}^y)\text{ from Equation 6.8.}
\]

So, for \(g^aN_{s'}^x \in H^d_{\chi}, g^bN_{s'}^y \in K_{s'}\),

\[
\beta A_2 b_{s'} = \theta_0(\text{Res}([d]))(g^aN_{s'}^x, g^bN_{s'}^y) = d(g^aN_{s'}^x, g^bN_{s'}^y) - d(g^bN_{s'}^y, g^aN_{s'}^x)
\]
\[
= \theta([d])(g^aN_{s'}^x, g^bN_{s'}^y) = \beta_0A_2 b_{s'}.
\]

Hence \(\beta = \beta_0\). Therefore for \(g^aN_{s'}^x \in H^d_{\chi}, g^bN_{s'}^y \in K_{s'}\)

\[
\theta_0(\text{Res}([d]))(g^aN_{s'}^x, g^bN_{s'}^y) = \beta_0A_2 b_{s'}.
\]

Now \(\text{Res}([d])\) is invariant under the conjugation of the matrix \(g^c = E_{(x', y') \rightarrow (p'c_r)}\) where we have as a 3x3 matrix:

\[
g^cN_{s'}^x = \begin{pmatrix}
1 & \text{mod } p'^{r+1} & 0 & \text{mod } p'^{r+1} & 0 & \text{mod } p'^{r+1} \\
0 & \text{mod } p'^{r+1} & 1 & \text{mod } p'^{r+1} & 0 & \text{mod } p'^{r+1} \\
0 & \text{mod } p'^{r+1} & p'^{r}c_r & \text{mod } p'^{r+1} & 1 & \text{mod } p'^{r+1}
\end{pmatrix}.
\]

This conjugation detects the position \((x', y')\) and so we get for \(g^aN_{s'}^x \in H^d_{\chi}, g^bN_{s'}^y \in K_{s'}\)

\[
\beta_0A_2 b_{s'} = \theta_0(\text{Res}([d]))(g^aN_{s'}^x, g^bN_{s'}^y)
\]
\[
= \theta_0(\text{Res}([d]))(g^c, (g^c)^{-1}N_{s'}^x, g^c g^c (g^c)^{-1}N_{s'}^y)
\]
\[
= \beta_0A_2 (b_{s'} + c_r B_1).
\]

Choosing \(c_r \neq 0, B_1 \neq 0\) we get \(\beta_0 = 0\). Hence \(\theta([d]) = 0\) which implies \([d] \in \ker(\theta)\). This proves the theorem in case (4).

Now consider case (5) : \(\text{Pos}(s') = (x', y'), x' - y' = 1\) and there exists \(t_0 \in T\) such that \(\text{Pos}(t_0) = (y', x')\). Here the first coordinate of \(t_0\) is 1. In this scenario for \(g^a \in P_{\Delta}, g^b \in N_{s'}^y\), by invariance of \([d]\) under the action of \(D_{\Delta}\) and \(p \neq 3\), we conclude that

\[
\theta([d])(g^aN_{s'}^x, g^bN_{s'}^y) = d(g^aN_{s'}^x, g^bN_{s'}^y) - d(g^bN_{s'}^y, g^aN_{s'}^x) = \beta_0 a_{t_0} b_{s'}
\]

for some \(\beta_0 \in \mathbb{Z}/p\mathbb{Z}\). We show that \(\beta_0 = 0\).

Here if the first coordinate of \(s'\) is \(l'\) then \(s' \notin T, x' - y' = 1 \Rightarrow l' > 1\). So \(x' + 1 \leq \rho = \rho_1 + \cdots + \rho_k\). Let \(A_1, pA_2\) be two typical elements which occur in positions \((x'+1, y')\) and \((y', x')\) in the matrix respectively. Note that \(t_0 = (1, 1, m', m' + 1)\) for some \(m'\) where \(\rho_{m'} = \rho_{m'+1} = 1\) since \(t_0 \in T\) occurs in a first superdiagonal entry. Consider the abelian subgroup \(L_{\Delta}' = \{g^a + \)

We have by using Corollary 2.13 for \( g^a \in L_{\Delta}^{s'} \) the product is given as:

\[
\begin{pmatrix}
1 \mod p^\prime + 1 & pA_2 \mod p^\prime + 1 & 0 \mod p^\prime + 1 \\
p^\prime - A_1 \mod p^\prime & 1 \mod p^\prime + 1 & 0 \mod p^\prime + 1 \\
p^\prime - B_1 \mod p^\prime & 0 \mod p^\prime & 1 \mod p^\prime + 1
\end{pmatrix}
\begin{pmatrix}
1 \mod p^\prime + 1 & pB_2 \mod p^\prime + 1 & 0 \mod p^\prime + 1 \\
p^\prime - B_1 \mod p^\prime & 1 \mod p^\prime + 1 & 0 \mod p^\prime + 1 \\
0 \mod p^\prime & 0 \mod p^\prime & 1 \mod p^\prime + 1
\end{pmatrix}
\]

Consider the map \( \text{Res} : H^2_{\text{Trivial Action}}(\frac{\mathbb{P}}{N_{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\frac{\mathbb{P}}{N_{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \).

We have by using Corollary 2.13 for \( g^a \in L_{\Delta}^{s'}, g^b \in L_{\Delta}^{s'} \)

\[
\text{Res}(d)(g^aN_{\Delta}^{s'}, g^bN_{\Delta}^{s'}) \approx_{\text{cohomologous}} s_1(a_1, b_1) + s_2(pA_2 \mod p^\prime + 1, pB_2 \mod p^\prime + 1) + \alpha_1 A_1 B_1 + \gamma_1 A_1 b_1 + \beta_1 A_2 b_1
\]

for some \( \alpha_1, \beta_1, \gamma_1 \in \mathbb{Z}/p\mathbb{Z} \) where \( s_i, i = 1, 2, 3 \) are the restrictions of the cocycle \( d \) to the respective components, \( A_1, A_2, B_2 \) are the residues of \( A_1, A_2, B_2 \) modulo \( p \) respectively. By invariance of \([d]\) under the action of the group \( D_{\Delta} \) and using Theorem 2.4, since \( p \) is odd, we get that \( s_i, i = 1, 2, 3 \) are cohomologous to zero and also \( \alpha_1 = 0 = \gamma_1 \).

Let \( \theta_1 : H^2_{\text{Trivial Action}}(\frac{\mathbb{P}}{N_{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(\frac{\mathbb{P}}{N_{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \) where \( \theta_1 \) is given in Proposition 2.11. Now we have for \( g^a \in L_{\Delta}^{s'}, g^b \in N_{\Delta}^{s'} \)

\[
\beta_1 A_2 b_1 = \theta_1(\text{Res}(d))(g^aN_{\Delta}^{s'}, g^bN_{\Delta}^{s'}) = \theta([d])(g^aN_{\Delta}^{s'}, g^bN_{\Delta}^{s'}) = \beta_0 A_2 b_1 \Rightarrow \beta_1 = \beta_0.
\]

Hence, for \( g^a \in L_{\Delta}^{s'}, g^b \in L_{\Delta}^{s'} \)

\[
\text{Res}(d)(g^aN_{\Delta}^{s'}, g^bN_{\Delta}^{s'}) \approx_{\text{cohomologous}} \beta_0 A_2 b_1.
\]

Now \( \text{Res}([d]) \) is invariant under conjugation by the following \( 3 \times 3 \) matrix which is the image of \( E(x', x'' + 1)(pC) \in \mathcal{P}_\Delta \).

\[
\bar{E}(x', x'' + 1)(pC)
\begin{pmatrix}
1 \mod p^\prime + 1 & 0 \mod p^\prime + 1 & 0 \mod p^\prime + 1 \\
0 \mod p^\prime + 1 & 1 \mod p^\prime + 1 & pC \mod p^\prime + 1 \\
0 \mod p^\prime & 0 \mod p^\prime & 1 \mod p^\prime + 1
\end{pmatrix}
\]
We get for \( g^a \in L^2_{\Lambda}, g^b \in L^2_{\Lambda} \)

\[
\beta_{t_0} b_s = \theta_1(\text{Res}(|d|))(g^a N^s_{\Lambda} \cdot g^b N^s_{\Lambda} ) =
\theta_1(\text{Res}([d]))(E_{(x',x'+1)}(pC)g^a E_{(x',x'+1)}(pC)g^b E_{(x',x'+1)}(pC)N^s_{\Lambda} E_{(x',x'+1)}(-pC)N^s_{\Lambda})
= \beta_{t_0}(\overline{A_2} + C A_1)b_s \Rightarrow \beta_{t_0} = 0.
\]

Thus \( \theta([d]) = 0 \Rightarrow [d] \in \text{Ker}(\theta) \). This completes case (5) of the theorem. Hence the theorem follows. ■

**Definition 6.33 (Standard 2-Coboundaries on \( P_{\Lambda} \)).** Let \( s \in S \setminus \{ \emptyset \} \). We define the standard 1-cochain \( w_s : P_{\Lambda} \longrightarrow \mathbb{Z}/p\mathbb{Z} \) as:

\[
\text{For } g^a \in P_{\Lambda}, w_s(g^a) = a_s \mod p.
\]

Now we define the standard 2-coboundary \( v_s : P_{\Lambda} \times P_{\Lambda} \longrightarrow \mathbb{Z}/p\mathbb{Z} \) as:

\[
\text{For } g^a, g^b, g^c = g^a g^b \in P_{\Lambda}, v_s(g^a, g^b) = w_s(g^c) - w_s(g^a) - w_s(g^b) = c_s - a_s - b_s \mod p.
\]

Hence by definition \( v_s \in B^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \).

**Theorem 6.34.** Let \( p \neq 3 \) be an odd prime and let \( s, s' \in S \setminus T \) be such that \( s' \) is the successor element of \( s \) in \( S \setminus T \). Let \( \text{Pos}(s') = (x', x' + 1) \). Consider the map \( \theta : H^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow P(\mathbb{Z}/p\mathbb{Z}) \) which is defined in Theorem 2.14. Also consider the map \( \text{Inf} : H^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \).

Then given any cohomology class \([x] \in H^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z})D_\Lambda \) there exists a cohomology class \([v] \in H^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z})D_\Lambda \) such that \( \text{Inf}([v]) = 0 \) and \([x] - [v] \in \text{Ker}(\theta) \).

**Proof.** Let \( s' = (l', l', j', m', n') \) and let \( r \in S \) be such that the first coordinate of \( r \) is \( l' \) and \( \text{Pos}(r) = (x', x') \). Note that \( r \leq_{\text{MTO}} s >_{\text{MTO}} s' \) and \( l' > 0 \). So the first coordinate of \( s \) is also \( l' \). Define a cocycle \( v \in Z^2_{\text{VANISHING CRITERION}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z})D_\Lambda \) as follows. For \( g^a, g^b \in P_{\Lambda} \) with \( g^c = g^a g^b \in P_{\Lambda} \),

\[
\theta(g^a N^s_{\Lambda} \cdot g^b N^s_{\Lambda}) = c_r - a_r - b_r \mod p.
\]

We will show the following.

1. \( v \) is well defined.
2. \( v \) is a cocycle and \( v \in Z^2_{\text{VANISHING CRITERION}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z})D_\Lambda \).
3. \( \text{Inf}([v]) = [v_r] = 0 \) where \( v_r \) is the standard 2-coboundary in \( B^2_{\text{Trivial Action}}(P_{\Lambda}, \mathbb{Z}/p\mathbb{Z}) \).
4. There exists \( \beta \in \mathbb{Z}/p\mathbb{Z} \) such that \([x] - \beta[v] \in \text{Ker}(\theta) \).
We prove (1). Since $s \in S \setminus T$ we have $N^s \triangleleft \mathcal{P}^s_\lambda \cap \mathcal{P}'_\lambda$. This we have observed in the proof of Theorem 6.14. Using Theorem 6.6(3) and Theorem 6.11(2) we get that $g^a N^s = g^b N^s$ if and only if $a_t = b_t$ for all $s \not<_{TO} t, t \in S$ and also for all $t \in T$, that is, for all $t \in S$ such that $s \not<_{MTO} t$.

If $g^c = g^a g^b$ then we have

$$
(g^c)(x',x') = \sum_{z' < x'} (g^a)(x',z')(g^b)(z',x') + (g^a)(x',x')(g^b)(x',x') + \sum_{z' > x'} (g^a)(x',z')(g^b)(z',x').
$$

(6.9)

Here we observe that if $z' < x'$ then $p \mid (g^b)(z',x')$ and if $z' > x'$ then $p \mid (g^a)(x',z').$

So we observe that $c_t - a_t - b_t \mod p$ is depends only on

- $a_t, b_t$ for all $t = (l, i, j, m, n)$ where $l < l'$,
- those $b_t$ for all $t = (l, i, j, m, n)$ where $l = l'$ and which occur in positions $(z', x')$ with $z' < x'$,
- those $a_t$ for all $t = (l, i, j, m, n)$ where $l = l'$ and which occur in positions $(x', z')$ with $z' > x'$.

Hence in all cases $s \not<_{TO} t$. Therefore $v$ is well defined.

We prove (2). It is clear that the following cocycle identity is satisfied.

$$
v(g^a N^s, g^b N^s) + v(g^b N^s, g^d N^s) = v(g^b N^s, g^d N^s) + v(g^a N^s, g^b g^d N^s).
$$

This is because we have for $g^c = g^a g^b, g^c = g^b g^d, g^f = g^d g^b g^d$

$$
(c_r - a_r - b_r) + (f_r - c_r - d_r) \equiv (e_r - b_r - d_r) + (f_r - a_r - e_r) \mod p.
$$

Moreover $r$ appears in a diagonal position. Hence $v$ is invariant under the diagonal action. Therefore $v \in \mathcal{Z}_2^{Trivial Action}(\mathcal{P}^s_\lambda, \mathbb{Z}/p\mathbb{Z})D_\lambda$.

We prove (3). It is clear that $\text{Inf}([v]) = [\nu_r]$ where $\nu_r$ is the standard coboundary as defined in Definition 6.33. Hence $\text{Inf}([v]) = 0$.

We prove (4). Let $[x] \in H_2^{Trivial Action}(\mathcal{P}^s_\lambda, \mathbb{Z}/p\mathbb{Z})D_\lambda$. Let $t_0 \in T$ be such that the first coordinate of $t_0$ is 0 and $\text{Pos}(t_0) = (x' + 1, x')$. Let $\tilde{t} \in S$ be such that the first coordinate of $\tilde{t}$ is 1 and $\text{Pos}(\tilde{t}) = \text{Pos}(s')$. Then there are two cases.

(i) $\tilde{t} \in T$. (Here in this case, since $s' \in S \setminus T$, we have $s' \not< \tilde{t}$ though $\text{Pos}(s') = \text{Pos}(\tilde{t})$.)

(ii) $\tilde{t} \notin T$.

Then for $g^a \in \mathcal{P}^s_\lambda, g^b \in N^s_\lambda$, in both cases (i), (ii), by invariance of $[x]$ under the action of $D_\lambda$, we conclude that

$$
\theta([x])(g^a N^s g^b N^s) = x(g^a N^s g^b N^s) - x(g^b N^s g^a N^s) = \beta_{t_0 a_0 b_0} s'
$$

for some $\beta_{t_0} \in \mathbb{Z}/p\mathbb{Z}$. Note that in case (i) we need $p \neq 3$ and in case (ii) we just need $p$ to be an odd prime.
If \( g^c = s^a s^b \) with \( g^a \in \mathcal{P}_\Delta, g^b \in \mathcal{N}_\Delta^s \) then
\[
(g^b)(x', x'+1) = p' b_{s'} + p''+1(*)
\]
\[
(g^b)(z', x') = p' b_{s'} + p''+1(*) \text{ for } z' < x',
\]
\[
(g^b)(z', x') = p' b_{s'} + p''+1(*) \text{ for } z' > x',
\]
\[
(g^b)(x', x') = 1 + p' b_{s'} + p''+1(*)\]

Let \( (g^a)(x', x') = 1 + p a_{t_i} + p^2 a_{t_2} + \cdots + p'' a_{t_{i'}} + p''+1(*) \) where \( t_i \in S \) with first coordinate of \( t_i \) is \( i \) and \( Pos(t_i) = (x', x') \). Then we have \( t_{i'} = r \) and we get from Equation 6.9 that,
\[
(g^c)(x', x') \equiv (1 + p a_{t_i} + p^2 a_{t_2} + \cdots + p'' a_{t_{i'}})(1 + p' b_{r}) \mod p''+1.
\]
Therefore we have \( c_r \equiv a_{t'} + b_r \equiv a_r + b_r \mod p \Rightarrow v(g^a N_{\Delta}^s g^b N_{\Delta}^s) = 0 \). If \( g^d = s^b s^a \) with \( g^a \in \mathcal{P}_\Delta, g^b \in \mathcal{N}_\Delta^s \) then
\[
(g^d)(x', x') = \sum_{z' < x'} (g^b)(x', z') (g^a)(z', x') + (g^b)(x', x')(g^a)(x', x') + (g^b)(x', x'+1)(g^a)(x'+1, x')
\]
\[
+ \sum_{z' > x'+1} (g^b)(x', z') (g^a)(z', x')
\]
Here we have
\[
(g^b)(x', z') = p' b_{s'} \text{ for } z' < x',
\]
\[
(g^b)(x', z') = p''+1(*) \text{ for } z' > x' + 1.
\]
So from Equation 6.10 we get that,
\[
(g^d)(x', x') \equiv (1 + p' b_{r})(1 + p a_{t_i} + p^2 a_{t_2} + \cdots + p'' a_{t_{i'}}) + p'' b_{s'}(a_{t_0} + p(*) \mod p''+1
\]
So \( d_r \equiv b_r + a_{t'} + b_{s'} a_{t_0} \equiv b_r + a_r + b_{s'} a_{t_0} \mod p \Rightarrow v(g^b N_{\Delta}^s g^a N_{\Delta}^s) = a_{t_0} b_{s'} \). So
\[
\theta([v])(g^a N_{\Delta}^s g^b N_{\Delta}^s) = v(g^a N_{\Delta}^s g^b N_{\Delta}^s) - v(g^b N_{\Delta}^s g^a N_{\Delta}^s) = -a_{t_0} b_{s'}
\]
Hence we obtain \( \theta([x] + \beta_{t_0} [v]) = 0 \Rightarrow [x] + \beta_{t_0} [v] \in \ker(\theta) \).
This completes the proof of the theorem.

**Theorem 6.35.** Let \( p \) be a prime. Let \( s, s' \in S \setminus T \) be such that \( s' \) is the successor element of \( s \) in \( S \setminus T \). Consider the inflation map \( H^2_{Trivial \ Action}(p_{N_{\Delta}} Z/pZ) \overset{Inf}{\rightarrow} H^2_{Trivial \ Action}(p_{N_{\Delta}} Z/pZ) \). If \( [c] \in H^2_{Trivial \ Action}(p_{N_{\Delta}} Z/pZ)D_{\Delta} \) and \( [c] \in \Im(Inf) \) then there exists \( [d] \in H^2_{Trivial \ Action}(p_{N_{\Delta}} Z/pZ)D_{\Delta} \) such that \( Inf([d]) = [c] \).
Proof. Let $\text{Inf}([x]) = [c]$. Then define

$$[d] = \frac{1}{|D_\lambda|} \sum_{t \in D_\lambda} t \cdot [x].$$

We observe that $\text{Inf}([d]) = [c]$ and $[d] \in H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D_\lambda}$. Note here $p$ does not divide the cardinality of $D_\lambda$. \hfill \blacksquare

Remark 6.36. In the proof of Theorem 6.35, $p$ can be any prime.

**Theorem 6.37.** Let $s$ be the maximal element of $S \setminus T$. Consider the inflation map $H^2_{\text{Trivial Action}}(\mathcal{P}_s, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{Inf}} H^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})$. Then we have

$$\text{Inf}(H^2_{\text{Trivial Action}}(\mathcal{P}_s, \mathbb{Z}/p\mathbb{Z})) = 0.$$

**Proof.** From Theorem 6.11(3) we have $\mathcal{P}_s \cong (\mathbb{Z}/p\mathbb{Z})^{|T|}$. Consider the map $\theta : H^2_{\text{Trivial Action}}(\mathcal{P}_s, \mathbb{Z}/p\mathbb{Z}) \to \mathcal{P}_\lambda(\mathbb{Z}/p\mathbb{Z})$. If $[x] \in H^2_{\text{Trivial Action}}(\mathcal{P}_s, \mathbb{Z}/p\mathbb{Z})$, then we have, for $g^a, g^b \in \mathcal{P}_\lambda$,

$$x(g^a \mathcal{N}_\lambda^s g^b \mathcal{N}_\lambda^s) \approx_{\text{cohomologous}} \sum_{t \in T_1} \beta_t a_{\phi(t)} b_t$$

and

$$\theta([x])(g^a \mathcal{N}_\lambda^s g^b \mathcal{N}_\lambda^s) = \sum_{t \in T_1} \beta_t a_{\phi(t)} b_t$$

for some $\beta_t \in \mathbb{Z}/p\mathbb{Z}$ where $T_1 = \{t \in T \mid$ there exists $1 \leq m < k, t = (1,1,1,m,m+1,1) \}$ and $\phi : T_1 \to T$ such that $\phi(t) = (0,1,1,m+1,m)$ the entry in $T$ which is symmetric w.r.t $t$ about the diagonal.

Now for $t \in T_1$ define a 2-cocycle $u_t : \mathcal{P}_\lambda \times \mathcal{P}_\lambda \to \mathbb{Z}/p\mathbb{Z}$ given by $u_t(g^a, g^b) = a_{\phi(t)} b_t$. Indeed $u_t \in Z^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D_\lambda}$ because the cocycle identity is satisfied, that is, for $g^a, g^b, g^c \in \mathcal{P}_\lambda$

$$a_{\phi(t)} b_t + (a_{\phi(t)} + b_{\phi(t)}) c_t = b_{\phi(t)} c_t + a_{\phi(t)} (b_t + c_t)$$

and $u_t$ is invariant under the action of the group $D_\lambda$. Now we know that

$$\text{Inf}([x])(g^a, g^b) \approx_{\text{cohomologous}} \sum_{t \in T_1} \beta_t a_{\phi(t)} b_t.$$

So to prove the theorem it is enough to show that $u_{t_0}$ is a 2-coboundary for any $t_0 \in T_1$, that is, $u_{t_0} \in B^2_{\text{Trivial Action}}(\mathcal{P}_\lambda, \mathbb{Z}/p\mathbb{Z})^{D_\lambda}$ for $t_0 \in T_1$. Let $\text{Pos}(t_0) = (x, x+1), \text{Pos}(\phi(t_0)) = (x+1, x)$ for $1 \leq x < \rho = \rho_1 + \cdots + \rho_k$. 


Let
\[ S_0^x = \{ s = (l, i, j, m, n) \in S \mid l = 0, \chi(\text{Pos}(s)) > 0, \text{if } \text{Pos}(s) = (x', y') \text{ then } x' \leq x \}, \]
\[ \tilde{S}_1^x = \{ s = (l, i, j, m, n) \in S \mid l = 1, \chi(\text{Pos}(s)) < 0, \text{if } \text{Pos}(s) = (x', y') \text{ then } y' \leq x \}, \]
\[ S_2^x = \{ s = (l, i, j, m, n) \in S \mid l = 1, \chi(\text{Pos}(s)) = 0, \text{if } \text{Pos}(s) = (x', x') \text{ then } x' \leq x \}. \]

Let \( \phi : \tilde{S}_1^x \rightarrow S_0^x \) be defined as \( \phi(1, i, j, m, n) = (0, j, i, n, m) \). Note that the map \( \phi \) defined on \( T_1 \) agrees with the map \( \phi \) defined on \( \tilde{S}_1^x \) on the set \( \tilde{S}_1^x \cap T_1 \). Now it may happen that for \( (0, j, i, n, m) \in S_0^x \), the element \( (1, i, j, m, n) \) need not be in \( \tilde{S}_1^x \). So we enlarge the set \( \tilde{S}_1^x \) to \( S_1^x \) so that we include all such elements. Now the extended map \( \phi : S_1^x \rightarrow S_0^x, \phi(1, i, j, m, n) = (0, j, i, n, m) \) is a bijection. Define \( a_t = 0 = b_t \) for all \( t \in S_1^x \setminus \tilde{S}_1^x \).

We observe the following aspect of matrix multiplication.
\[
\sum_{s \in S_2^x} v_s(g^a, g^b) \equiv \sum_{s \in \tilde{S}_1^x} (a_s b_{\phi(s)} + a_{\phi(s)} b_s) + a_t b_{\phi(t_0)} \mod p
\]
(6.11)

where \( v_s \) is the standard coboundary on \( P_\lambda \) as in Definition 6.33. We also observe that for \( g^c = g^a g^b, s \in S_1^x, \text{Pos}(s) = (x', y') \)
\[
c_s \equiv a_s + b_s + \sum_{t_1 \in S_1^x, t_2 \in S_0^x, \text{Pos}(t_1) = (z', y') \text{ or } \text{Pos}(t_2) = (x', z')} b_{t_1} a_{t_2} \mod p.
\]

Also for \( g^c = g^a g^b, s \in S_0^x, \text{Pos}(s) = (x', y') \)
\[
c_s \equiv a_s + b_s + \sum_{t_1, t_2 \in S_0^x, \text{Pos}(t_1) = (x', z') \text{ or } \text{Pos}(t_2) = (z', y')} a_{t_1} b_{t_2} \mod p.
\]

Note that if \( s \in S_1^x \setminus \tilde{S}_1^x \) then \( c_s = 0 \) since \( g^c = g^a g^b \in P_\lambda \). The entries \( a_s, b_s, c_s \) with \( s \in S_1^x \cup S_0^x \) gives an instance for the application of Theorem 2.16. Hence we conclude that
\[
\sum_{s \in S_1^x} (a_s b_{\phi(s)} + a_{\phi(s)} b_s)
\]
is a coboundary since each of the following summands in the identity given in Theorem 2.16
\[
v_{t_1, t_2, \ldots, t_i}(g^a, g^b) = c_{t_1} c_{t_2} \cdots c_{t_i} - a_{t_1} a_{t_2} \cdots a_{t_i} - b_{t_1} b_{t_2} \cdots b_{t_i}
\]
is a 2-coboundary on \( P_\lambda \) where the positions of \( t_j, 1 \leq j \leq i \) are either strictly upper triangular or strictly lower triangular. Moreover we have \( \text{Pos}(t_1) = (x'_1, x'_2), \text{Pos}(t_2) = (x'_2, x'_3), \ldots, \text{Pos}(t_i) = (x'_i, x'_i) \) for some \( 1 \leq x'_1, x'_2, \ldots, x'_i \leq r \). Hence the above coboundary is a \( \mathcal{D}_\lambda \) invariant coboundary.
Using Equation 6.11 we get that $a_{t_0}b_{\phi(t_0)}$ is a coboundary. But now we have

$$a_{t_0}b_{\phi(t_0)} + a_{\phi(t_0)}b_{t_0} = (a_{t_0} + b_{t_0})(a_{\phi(t_0)} + b_{\phi(t_0)}) - a_{t_0}a_{\phi(t_0)} - b_{t_0}b_{\phi(t_0)}.$$

So $u_{t_0}(g^a, g^b) = a_{\phi(t_0)}b_{t_0}$ is a 2-coboundary which is what we wanted to prove. This proves the theorem.

\[
\text{Remark 6.38. In the proof of Theorem 6.37, } p \text{ can be any prime.}
\]

6.2.6. The Cohomology Vanishing Theorem.

**Theorem 6.39.** Let $\Delta = (\lambda_1^p \mid \lambda_2^p \mid \lambda_3^p \mid \ldots \mid \lambda_k^p)$ be a partition such that $\lambda_i = k - i + 1$. Let $A_{\Delta}$ be the finite abelian $p$-group associated to $\Delta$, where $p \neq 3$ is an odd prime. Let $G_\Delta = \text{Aut}(A_{\Delta})$ be its automorphism group. Then $H^2_{\text{Trivial Action}}(G_\Delta, \mathbb{Z}/p\mathbb{Z}) = 0$. As a consequence $H^2_{\text{Trivial Action}}(G_\Delta, A_{\Delta}) = 0$.

**Proof.** To prove $H^2_{\text{Trivial Action}}(G_\Delta, \mathbb{Z}/p\mathbb{Z}) = 0$ it is enough to prove that $H^2_{\text{Trivial Action}}(P_\Delta, \mathbb{Z}/p\mathbb{Z}) = 0$. For this we use the normal series $N_1^{\Delta}, s \in S$ and use repeatedly the extended Hochschild-Serre exact sequence for the central extensions given in exact sequence 6.6 for $s, s' \in S \setminus T$ with $s'$ the successor element of $s$ in $S \setminus T$.

If $[x] \in H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z})$ then $[x] \in \text{Ker}(\text{Res} : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\frac{N_1^{\Delta}}{N_2^{\Delta}}, \mathbb{Z}/p\mathbb{Z}))$ and one of the following (1), (2) occurs, using Theorems 6.16, 6.32, 6.34.

1. $[x] \in \text{Ker}(\theta : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}))$.

2. There exists a $D_{\Delta}$ invariant cohomology class $[v] \in H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z})$ such that $[x] - [v] \in \text{Ker}(\theta : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow P(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}))$ and $\text{Inf}([v]) = 0$ for the map $\text{Inf} : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_2^{\Delta}}, \mathbb{Z}/p\mathbb{Z})$.

By exactness of the extended Hochschild-Serre exact sequence and using Theorem 6.35 we can get a $D_{\Delta}$ invariant cohomology class $[y] \in H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z})$ such that either $\text{Inf}([y]) = [x]$ in case (1) or $\text{Inf}([y]) = [x] - [v]$ in case (2) where $\text{Inf} : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_1^{\Delta}}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(\frac{P_\Delta}{N_2^{\Delta}}, \mathbb{Z}/p\mathbb{Z})$.

We start with $[x] \in H^2_{\text{Trivial Action}}(P_\Delta, \mathbb{Z}/p\mathbb{Z})$ and by continuing this process repeatedly we get that there exists $[z] \in H^2_{\text{Trivial Action}}(\frac{P_\Delta}{P_\Delta'P_\Delta''}, \mathbb{Z}/p\mathbb{Z})$ such that we have $\text{Inf}([z]) = [x]$ where $\text{Inf} : H^2_{\text{Trivial Action}}(\frac{P_\Delta}{P_\Delta'P_\Delta''}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^2_{\text{Trivial Action}}(P_\Delta, \mathbb{Z}/p\mathbb{Z})$. Now using Theorem 6.37 we conclude that $[x] = \text{Inf}([z]) = 0$. 


Hence $H^2_{\text{Trivial Action}}(\mathcal{P}_{\lambda'}, \mathbb{Z}/p\mathbb{Z})_{\Delta} = 0 \Rightarrow H^2_{\text{Trivial Action}}(G_{\lambda'}, \mathbb{Z}/p\mathbb{Z})_{\Delta} = 0$. Now using Theorem 2.3 we conclude that $H^2_{\text{Trivial Action}}(G_{\lambda'}, A_{\lambda}) = 0$. This proves the theorem.

6.2.7. The Vanishing/Nonvanishing Criterion for $H^2_{\text{Trivial Action}}(G_{\lambda'}, A_{\lambda})$ for Odd Primes $p \neq 3$. As a consequence of Theorems 6.1, 6.39 we have proved the following theorem for odd primes $p \neq 3$.

Theorem 6.40. Let $\lambda = (\lambda_1^{P_1} > \lambda_2^{P_2} > \lambda_3^{P_3} > \ldots > \lambda_k^{P_k})$ be a partition. Let $A_{\lambda}$ be the finite abelian $p$-group associated to $\lambda$, where $p \neq 3$ is an odd prime. Let $G_{\lambda} = \text{Aut}(A_{\lambda})$ be its automorphism group. Then $H^2_{\text{Trivial Action}}(G_{\lambda}, A_{\lambda}) = 0$ if and only if the difference between two successive parts of $\lambda$ is at most 1.

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