Harmonic quasi-isometries of pinched Hadamard surfaces are injective

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Abstract

We prove that a harmonic quasi-isometric map between pinched Hadamard surfaces is a quasi-conformal diffeomorphism.

1 Introduction

1.1 Main result

The main result of this paper is the following.

Theorem 1.1 Let \( h : S_1 \to S_2 \) be a harmonic quasi-isometric map between pinched Hadamard surfaces. Then, \( h \) is a quasi-conformal diffeomorphism.

A pinched Hadamard manifold is a complete simply-connected Riemannian manifold whose curvature satisfies \(-b^2 \leq K \leq -a^2\) for some positive constants \(0 < a \leq b\). For instance, the hyperbolic disk \( \mathbb{D} \) is a pinched Hadamard surface with constant curvature \(-1\).

A map \( f : M_1 \to M_2 \) between two metric spaces is quasi-isometric if there exists a constant \( c \geq 1 \) such that, for every \( x, x' \in M_1 \),

\[
c^{-1} d(x, x') - c \leq d(f(x), f(x')) \leq c d(x, x') + c. \tag{1.1}
\]

A smooth map \( h : M_1 \to M_2 \) between Riemannian manifolds is harmonic if it is a critical point for the Dirichlet energy integral \( E(h) = \int |Dh|^2 dv_{M_1} \) with respect to variations with compact support.

A diffeomorphism \( h : M_1 \to M_2 \) between \( n\)-dimensional Riemannian manifolds is quasi-conformal, if there exists a constant \( C > 0 \) such that \( ||Dh||^n \leq C |\text{Jac}(h)| \) where \( \text{Jac}(h) := \det(Dh) \) is the Jacobian of \( h \).

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1.2 A few comments

The special case of Theorem 1.1 where both $S_1$ and $S_2$ are the hyperbolic disk $\mathbb{D}$, is due to Li-Tam [19] and Markovic [20].

The main issue in Theorem 1.1 is the injectivity of $h$. The quasi-conformality of $h$ is but our way to prove injectivity.

In Theorem 1.1 we only deal with surfaces. Indeed the analog in higher dimension is not true. A counterexample due to Farrell, Ontaneda and Raghunathan is given in [9].

Given two pinched Hadamard surfaces $S_1$ and $S_2$, there exist many harmonic quasi-isometric maps from $S_1$ to $S_2$ (see [4] or Theorem 2.2 below). Theorem 1.1 asserts that all these maps are injective.

Theorem 1.1 extends the Schoen-Yau injectivity theorem in [22] which says that a harmonic map between two compact Riemannian surfaces with negative curvature, when homotopic to a diffeomorphism, is also a diffeomorphism. This injectivity theorem is used in the parametrization due to J. Sampson and M. Wolf of the Teichmüller space by the Hopf quadratic differentials, see [21] and [19].

From a historical point of view, the first injectivity theorem for harmonic maps is due to Rado-Kneser-Choquet, almost 100 years ago. It states that, in the Euclidean plane, the harmonic extension of an homeomorphism of the unit circle is a diffeomorphism of the unit disk, see [13, Lemma 5.1.10]. The analog statement in dimension $d \geq 3$ is not true. A counterexample is given by R. Laugesen in [17]. Later on, injective harmonic maps between surfaces were studied by H. Lewy in [18] who proved that their Jacobian does not vanish, by R. Heinz in [12] and by J. Jost and H. Karcher in [16, Chapter 7] who found a lower bound for their Jacobian. There is also an extension of the Schoen-Yau injectivity theorem by J. Jost and R. Schoen that allows some positive curvature in [16, Chapter 11].

1.3 Structure of the paper

In Chapter 2 we recall classical facts concerning Hadamard surfaces, quasi-isometric maps and harmonic maps between surfaces. We will see that we can assume that the source $S_1$ is the hyperbolic disk $\mathbb{D}$. Recall that the special case of Theorem 1.1 where the target $S_2$ is the hyperbolic disk $\mathbb{D}$ is due to Li–Tam and Markovic.

In Chapter 3 we give an overview of the proof of Theorem 1.1. This proof uses a deformation $(g_t)$ of the metric on $S_2$, starting with the hyperbolic metric, and a deformation $(h_t)$ of the harmonic map $h$. The key point will be to obtain a uniform upper bound for the norm of the differential of $h_t$ and a uniform lower bound for the Jacobian of $h_t$. 
In Chapter 4 we gather compactness results for Hadamard surfaces and harmonic maps.
In Chapter 5 we obtain a uniform lower bound for the Jacobian of harmonic quasi-conformal diffeomorphisms.
In Chapter 6 we prove that the family \((h_t)\) varies continuously with \(t\) and we complete the proof of Theorem 1.1.
In Chapter 7 we include a short new proof of the special case of Theorem 1.1 where \(S_1 = S_2 = \mathbb{D}\).

This paper is as self-contained as possible, the main tools being the Bland-Kalka uniformization theorem in [5], the Bochner equations for harmonic maps between surfaces in [15], the existence and uniqueness of quasi-isometric harmonic maps in [4], and the PDE elliptic regularity in [11].

2 Background

We recall well-known properties of pinched Hadamard surfaces, quasi-isometric maps and harmonic maps between surfaces.

2.1 Pinched Hadamard surfaces

The first example of a pinched Hadamard surface is the hyperbolic disk \(\mathbb{D} = (D, g_{hyp})\), where \(D = \{ |z| < 1 \} \subset \mathbb{C}\) is the unit disk equipped with the hyperbolic metric \(g_{hyp} = \rho^2(\overline{z})|dz|^2\) with conformal factor \(\rho^2 = 4(1-|z|^2)^{-2}\). It is a Hadamard manifold with constant curvature \(-1\).

Any pinched Hadamard surface is conformal to the disk, namely reads as \((D, \sigma^2(\overline{z})|dz|^2)\). Moreover the conformal factors \(\rho^2\) and \(\sigma^2\) are in a bounded ratio: if the curvature \(K\) of this surface satisfies \(-b^2 \leq K \leq -a^2 < 0\), then \(a^2\sigma^2 \leq \rho^2 \leq b^2\sigma^2\). See Proposition 3.1.

Also observe that, for maps defined on a Riemannian surface \(S_1\), the Dirichlet energy functional is invariant under a conformal change of metric on \(S_1\). Hence, the harmonicity of such a map depends only on the conformal class of the source surface.

We infer from this discussion that, to prove Theorem 1.1 we can assume that \(S_1\) is the hyperbolic disk \(\mathbb{D}\).

2.2 Quasi-isometric maps

Let \(S = (D, \sigma^2(\overline{z})|dz|^2)\) be a pinched Hadamard surface. It is a proper Gromov hyperbolic space (a general reference for Gromov hyperbolic spaces is [10]). The boundary at infinity \(\partial_{\infty} S\) of \(S\) is defined as the set of equivalence classes of geodesic rays, where two geodesic rays are identified whenever they remain within bounded distance from each other. The union \(\overline{S} = S \cup \partial_{\infty} S\) provides a compactification of \(S\) (see [1]).
The boundary at infinity $\partial_\infty \mathbb{D}$ naturally identifies with the boundary $SS^1 = \{ z \in \mathbb{C}, |z| = 1 \}$ of $D$. Since the identity map $\text{Id} : D \to D$ is a quasi-isometry between the hyperbolic disk $\mathbb{D} = (D, \rho^2(z)|dz|^2)$ and the surface $S = (D, \sigma^2(z)|dz|^2)$, the boundary at infinity $\partial_\infty S$ also identifies canonically with $\partial_\infty \mathbb{D} = SS^1$.

A quasi-isometric map $f : \mathbb{D} \to S$ admits a boundary value at infinity $\partial_\infty f : \partial_\infty \mathbb{D} \to \partial_\infty S$, that we read as $\partial_\infty f : SS^1 \to SS^1$ through the above identifications. Two quasi-isometric maps share the same boundary value at infinity if and only if they remain within bounded distance from each other. The maps $\varphi : SS^1 \to SS^1$ that appear as boundary values at infinity of quasi-isometric maps $f : \mathbb{D} \to S$ are exactly the quasi-symmetric homeomorphisms. For convenience, we identify $SS^1$ with $\mathbb{R}/2\pi \mathbb{Z}$.

**Definition 2.1** Let $k \geq 1$. An homeomorphism $\varphi : SS^1 \to SS^1$ is a $k$-quasi-symmetric map if

$$\frac{1}{k} \leq \frac{\varphi(\theta + \alpha) - \varphi(\theta)}{\varphi(\theta) - \varphi(\theta - \alpha)} \leq k$$

holds for every $\theta, \alpha$ with $0 < \alpha \leq \pi$.

Note that any quasi-isometric map $f : \mathbb{D} \to S$ is actually a quasi-isometry. Namely, there exists $C > 0$ such that $d(y, f(D)) \leq C$ holds for all $y$ in $S$. Indeed, the inverse $f^{-1}$ of its boundary map is also a quasi-symmetric homeomorphism, hence $f^{-1}$ is the boundary map of a quasi-isometric map $f' : S \to \mathbb{D}$, and the map $f \circ f' : S \to S$ is within bounded distance from the identity map.

In a previous paper, we studied harmonic quasi-isometric maps between pinched Hadamard manifolds. Our result, when specialized to surfaces, asserts that any quasi-isometric map $f : \mathbb{D} \to S$ has the same boundary value at infinity as a unique harmonic quasi-isometric map. In other words, the following holds.

**Theorem 2.2** [4] Let $S = (D, \sigma^2(z)|dz|^2)$ be a pinched Hadamard surface and $\varphi : SS^1 \to SS^1$ be a quasi-symmetric map. Then, there exists a unique harmonic quasi-isometric map $h : \mathbb{D} \to S$ such that $\partial_\infty h = \varphi$.

### 2.3 Harmonic maps between surfaces

We introduce some notation that will be used throughout the paper, and recall some classical results concerning harmonic maps between surfaces. A general reference for this section is Jost [15].

Let $h : \mathbb{D} \to S$ be a smooth map from the hyperbolic disk $\mathbb{D} = (D, \rho^2(z)|dz|^2)$ to a pinched Hadamard surface $S = (D, \sigma^2(z)|dz|^2)$ with pinching condition
\(-b^2 \leq K \leq -a^2 \leq 0\). Recall that the curvature \(K\) of \(S\) is given by

\[K = -\sigma^{-2} \Delta_e \log \sigma\]

where \(\Delta_e = 4\partial_e \partial_{\overline{e}}\) is the Euclidean Laplacian. For such a map \(h\), we introduce as usual the functions \(h_z, \overline{h}_z : \mathbb{D} \to \mathbb{C}\) defined by

\[h_z = \frac{1}{2}(h_x - ih_y), \quad \overline{h}_z = \frac{1}{2}(h_x + ih_y)\]

where the conformal parameter reads as \(z = x + iy\), and the subscript \(x\) or \(y\) indicates a directional derivative. The map \(h\) is holomorphic (or anti-holomorphic) if \(\overline{h}_z = 0\) (or \(h_z = 0\)). It is worth noting that \(\overline{h}_z = \overline{h}_z\).

**Proposition 2.3** [15, Section 3.6] *The map \(h : \mathbb{D} \to S\) is harmonic if and only if it satisfies*

\[h_{zz} + 2 \left( \frac{\sigma_z}{\sigma} \circ h \right) h_z \overline{h}_z = 0.\]

If the map \(h\) is either holomorphic, or anti-holomorphic, then it is harmonic. Introduce the square norms of the complex derivatives of \(h\):

\[H = \|\partial h\|^2 := \frac{\sigma_z^2 \circ h}{\rho^2} |h_z|^2 \quad \text{and} \quad L = \|\overline{\partial} h\|^2 := \frac{\sigma_z^2 \circ h}{\rho^2} |h_{\overline{z}}|^2,\]

so that one has \(\|Dh\|^2 = H + L\). Observe that \(h\) is a local diffeomorphism if the Jacobian \(J = H - L\) does not vanish, and is moreover orientation preserving if \(J > 0\).

**Lemma 2.4** [15, Section 3.10] *Let \(h : \mathbb{D} \to S\) be a harmonic map. On the open subsets where they are non zero, the functions \(H\) and \(L\) satisfy the Bochner equations*

\[
\begin{align*}
(1/2) \Delta \log H &= (-K \circ h) J - 1, \quad (2.2) \\
(1/2) \Delta \log L &= (K \circ h) J - 1. \quad (2.3)
\end{align*}
\]

Here \(\Delta = 4 \rho^{-2} \partial_e \partial_{\overline{e}}\) is the Laplace operator relative to the hyperbolic metric.

On the open set \(\Omega := \{h_z \neq 0\}\), we introduce the conformal distortion \(\mu : \Omega \to \mathbb{C}\) by letting \(h_{\overline{z}} = \mu h_z\), so that one has the useful equalities

\[|\mu|^2 = L/H, \quad 1 - |\mu|^2 = J/H \quad \text{and} \quad \frac{1 - |\mu|^2}{1 + |\mu|^2} = \frac{J}{\|Dh\|^2}. \quad (2.4)\]

### 3 A family of metrics and harmonic maps

In this section we explain the continuity method that will be used to prove Theorem [11].
Let $S = (D, \sigma^2(z)|dz|^2)$ be a pinched Hadamard surface, with curvature bounds $-b^2 \leq K \leq -a^2 < 0$. Choose an increasing quasi-symmetric homeomorphism $\varphi : SS^1 \to SS^1$, and let $h : D \to S$ be the unique harmonic quasi-isometric map with boundary value at infinity $\partial_\infty h = \varphi$. We want to prove that $h$ is a quasi-conformal diffeomorphism.

In case the surface $S$ is the hyperbolic disk, that is for a harmonic quasi-isometric map $h : D \to D$, the result is due to Li-Tam and Markovic (see Chapter 7 for a proof). To prove it for a harmonic map $h : D \to S$ with values in a general pinched Hadamard surface $S$, we use the method of continuity, involving a family of pinched Hadamard surfaces $S_t = (D, e^{2u_t}g_{\text{hyp}})$, for $0 \leq t \leq 1$, starting with $S_0 = D$ and such that $S_1 = S$.

### 3.1 Construction of the metrics $g_t$

We construct the metric $g_t$ by prescribing its curvature. More specifically, we introduce for $0 \leq t \leq 1$ the unique complete conformal metric $g_t = e^{2u_t}g_{\text{hyp}}$ on the unit disk $D$ with curvature $K_t := -(1-t)+tK$. Each function $K_t$ being pinched between two negative constants, the existence and uniqueness of such a metric is granted by the following.

**Proposition 3.1** [5] Let $k$ be a smooth function on the unit disk $D$ such that $-\beta^2 \leq k \leq -\alpha^2$ for some constants $0 < \alpha \leq \beta$. Then, there exists a unique complete conformal metric $g = e^{2u}g_{\text{hyp}}$ on $D$ with curvature $k$. Moreover, the conformal factor $e^{2u}$ is controlled, with $\beta^{-2} \leq e^{2u} \leq \alpha^{-2}$.

We do not reproduce here the proof that is given in [5] and that relies on the sub-supersolution method for the curvature equation

$$\Delta u = (-k)e^{2u} - 1, \quad (3.1)$$

where, as above, $\Delta$ is the Laplace operator for the hyperbolic metric $g_{\text{hyp}}$.

The proof also uses the generalized maximum principle of Yau in [25]. We will need later a light form of this principle that reads as follows.

**Lemma 3.2** Let $v : S \to \mathbb{R}$ be a smooth function defined on a pinched Hadamard surface $S$. Assume that $v$ is bounded above.

Then, there exists a sequence $(x_n)$ in $S$ such that

$$v(x_n) \to \sup_S v, \quad |\nabla v|(x_n) \to 0 \quad \text{and} \quad \limsup \Delta v(x_n) \leq 0. \quad (3.2)$$

**Proof** We can assume that $\sup_S v = 1$. We fix a point $x_0 \in S$ where this supremum is not achieved and we introduce the function $v_n$ on $S$ given by $v_n(x) = v(x)e^{-d(x,x_0)/n}$. This function is smooth, except maybe at $x_0$, and it achieves its supremum at a point $x_n \neq x_0$ for $n$ large. This sequence $(x_n)$ satisfies (3.2) since $v_n(x_n) \to 1$, $\nabla v_n(x_n) = 0$ and $\Delta v_n(x_n) \leq 0$. $\square$
3.2 Construction of the harmonic maps $h_t$

We construct the harmonic map $h_t$ by prescribing its boundary map.

By construction, one has $\mathbb{D} = (D, g_0)$ and $S = (D, g_1)$. For $0 \leq t \leq 1$, we let $h_t : \mathbb{D} \to S_t$ be the unique harmonic quasi-isometric map whose boundary value at infinity is $\varphi : SS^1 \to SS^1$. Recall that the existence and uniqueness of those $h_t$ are granted by Theorem 2.2.

Here are some basic information concerning these harmonic maps $h_t$. For $0 \leq s, t \leq 1$, let $d(h_s, h_t) := \sup_{z \in D} d(h_s(z), h_t(z))$ denote the uniform distance between these two maps, where the distance is taken with respect to the hyperbolic metric $g_{hyp}$ on the target.

**Lemma 3.3** There exists $c^* > 0$ such that, for all $t \in [0, 1]$, the map $h_t$ is $c^*$-quasi-isometric, one has $d(h_t, h_0) \leq c^*$, and the map $h_t$ is $c^*$-Lipschitz.

Remark that, since the functions $u_t$ are uniformly bounded (Proposition 3.1), it was not really necessary to specify with respect to which one of the metrics $g_t$ the above distances were being estimated.

**Proof** As explained in Section 2.2, there exists a $c$-quasi-isometric map $f : \mathbb{D} \to \mathbb{D}$ whose boundary value at infinity is our quasi-symmetric map $\partial \infty f = \varphi$. By taking a larger constant $c$, we may assume that each map $f : \mathbb{D} \to S_t$ (that is, the same map $f$ now seen with values in one of the Riemannian surfaces $S_t$, with $t \in [0, 1]$) is $c$-quasi-isometric.

Thus the main result of [4] asserts that there exists a constant $C > 0$ such that $d(f, h_t) \leq C$. This constant $C$ depends only on $c$ and on the pinching constants $a$ and $b$, hence it does not depend on $t \in [0, 1]$. Thus the first two claims hold if $c_4 \geq 2c + 2C$.

The map $f$ being $c$-quasi-isometric, each harmonic map $h_t : D \to S_t$ sends any ball $B(z, 1) \subset D$ with radius 1 inside the ball $B(h_t(z), R) \subset S_t$ with radius $R = 2c + 2C$. Now the uniform Lipschitz continuity of the maps $h_t$ follows from the Cheng lemma, that we recall below. □

**Lemma 3.4** [8] Let $S$ be a Hadamard surface with $-b^2 \leq K \leq 0$. There exists a constant $\kappa$, that depends only on $b$, such that if a harmonic map $h : D \to S$ satisfies $h(B(z, 1)) \subset B(h(z), R)$ for some radius $R$, then

$$\|Dh(z)\| \leq \kappa R.$$  

3.3 An injectivity criterion

The following lemma tells us that a uniform lower bound for the Jacobian $J_t = \text{Jac}(h_t)$ is enough to ensure that $h_t$ is a quasi-conformal diffeomorphism.

**Lemma 3.5** If $\inf_{z \in D} J_t(z) > 0$ then $h_t$ is a quasi-conformal diffeomorphism.
Proof. By assumption the Jacobian $J_t$ does not vanish, hence the map $h_t : \mathbb{D} \to S_t$ is a local diffeomorphism. By construction, the map $h_t$ is quasi-isometric, hence it is a proper map. It thus follows that $h_t$ is a covering map. Hence, since $S$ is simply connected, the map $h_t$ is a diffeomorphism. Since, by Lemma 3.3, $h_t$ is Lipschitz, the lower bound for its Jacobian $J_t$ ensures that $h_t$ is quasi-conformal. \qed

3.4 Strategy of proof of Theorem 1.1

We will need the following two propositions.

Proposition 3.6 There exists $j_\ast > 0$ such that, for all $t \in \left[0, 1\right]$ for which $h_t$ is a quasi-conformal diffeomorphism, one has $J_t \geq j_\ast$.

Proposition 3.6 is a straightforward consequence of Proposition 5.2 that will be proven in Chapter 5. Indeed Lemma 3.3 ensures that the maps $h_t$ are $c_\ast$-Lipschitz.

Let $C_b(\mathbb{D}, \mathbb{R})$ be the space of bounded continuous functions $\psi$ endowed with the sup norm: $\|\psi\|_\infty = \sup_{z \in \mathbb{D}} |\psi(z)|$.

Proposition 3.7 The map $t \in \left[0, 1\right] \mapsto J_t \in C_b(\mathbb{D}, \mathbb{R})$ is continuous.

Proposition 3.7 will be proven in Chapter 6 as part of Proposition 6.2.

Proof of Theorem 1.1 using Propositions 3.6 and 3.7 Let $A$ be the set of parameters $t \in \left[0, 1\right]$ such that the harmonic map $h_t : \mathbb{D} \to S_t$ is a quasi-conformal diffeomorphism. We want to prove that $1 \in A$. We already know that $0 \in A$ (this is Theorem 7.1 due to Li-Tam and Markovic). It is enough to check that $A$ is open and closed. Let $j$ be the function on $\left[0, 1\right]$ given by $j(t) := \inf_{z \in \mathbb{D}} J_t(z) \in \mathbb{R}$.

By Proposition 3.7 the function $j$ is continuous. By Lemma 3.5 and Proposition 3.6 one has both $A = j^{-1}(\left[0, \infty\right[)$ and $A = j^{-1}(\left[j_\ast, \infty\right[)$. Hence $A$ is both open and closed. \qed

4 Sequences of metrics and harmonic maps

In order to obtain the uniform lower bounds in Chapter 5 or the continuity properties in Chapter 6 we will have to consider sequences of conformal metrics on the unit disk $D$, and sequences of harmonic maps. In this chapter, we state compactness results for such sequences.
These compactness results also hold in higher dimension (see [21], or [4]). Since we will only deal here with conformal metrics on the disk \(D\), the complex parameter \(z \in D\) naturally provides a global harmonic chart for these metrics so that the statements and the proofs are more elementary.

### 4.1 Sequence of Hadamard surfaces

Let us begin with sequences of conformal Riemannian structures on the unit disk \(D\).

Convergence in the following lemma is a special case of the Gromov-Hausdorff convergence for isometry classes of pointed proper metric spaces using the base point \(0 \in D\). See [4, §5.3] or [7] for a short introduction to this notion.

**Lemma 4.1** Let \(g_n = e^{2u_n} g_{\text{hyp}}\) be a sequence of complete conformal metrics on the unit disk \(D\) with curvature \(\ -b^2 \leq K_n \leq -a^2 < 0\). Then there is a subsequence of \((u_n)\) that converges to a \(C^1\) function \(u_\infty\) in the \(C^1\) topology.

The limit metric \(g_\infty = e^{-2u_\infty} g_{\text{hyp}}\) is a \(C^1\) complete conformal metric on \(D\), and \(S_\infty := (D, g_\infty)\) is a CAT-space with curvature between \(-b^2\) and \(-a^2\).

**Proof** Proposition 3.1 ensures that the logarithms \(u_n : D \to \mathbb{R}\) of the conformal factors are uniformly bounded. The curvature equation

\[
\Delta u_n = (-K_n) e^{2u_n} - 1
\]

for \(g_n\) ensures that the Laplacians \(\Delta u_n\) are also uniformly bounded.

Pick 0 \(\leq \alpha < 1\). We may now apply to the sequence \((u_n)\) the following first Schauder estimates (see [11, Theorem 3.9] or [21, Theorem 70]). These estimates state that there exists a constant \(c\) such that, for any smooth function \(v : D \to \mathbb{R}\) on the hyperbolic disk, the inequality

\[
\|v\|_{C^{1,\alpha}(B_1)} \leq c_\alpha (\|\Delta v\|_{C^0(B_2)} + \|v\|_{C^0(B_2)})
\]

holds for any pair of concentric hyperbolic balls \(B_1 \subset B_2 \subset D\) with respective radii 1 and 2. This provides a uniform local bound for the norms \(\|u_n\|_{C^{1,\alpha}}\).

Going if necessary to a subsequence, we may thus assume that the sequence \((u_n)\) converges in the \(C^1\) topology. Let \(u_\infty = \lim u_n\) and \(g_\infty = e^{-2u_\infty} g_{\text{hyp}}\) and introduce \(S_\infty := (D, g_\infty)\). As a limit of such, the length space \(S_\infty\) is a CAT-space with curvature between \(-b^2\) and \(-a^2\) (see [6, Corollary II.3.10] and [7, Theorem 10.7.1]).

**Remark** Under the hypothesis of Lemma 4.1, after extraction, the sequence of bounded functions \(K_n : D \to \mathbb{R}\) converges weakly to a bounded measurable function \(K_\infty : D \to \mathbb{R}\) with \(-b^2 \leq K_\infty \leq -a^2\), and the \(C^1\) function \(u_\infty\) is a weak solution of

\[
\Delta u_\infty = (-K_\infty) e^{2u_\infty} - 1.
\]
4.2 Sequence of harmonic maps

Now turn to sequences of maps between such Riemannian surfaces.

**Lemma 4.2** Let \( S_n = (D, g_n) \) be a sequence converging to \( S_\infty = (D, g_\infty) \) as in Lemma 4.1. Let \( c > 0 \), and let \( h_n : \mathbb{D} \to S_n \) be \( c \)-Lipschitz maps satisfying

\[
d_n(h_n(0), 0) \leq c.
\]

Then there is a subsequence of \( (h_n) \) that converges locally uniformly to a \( c \)-Lipschitz map \( h_\infty : \mathbb{D} \to S_\infty \).

a) If all the maps \( h_n \) are \( C \)-quasi-isometric, then \( h_\infty \) is \( C \)-quasi-isometric.

b) If all the maps \( h_n \) are harmonic, then \( h_\infty \) is \( C^2 \) and is harmonic too.

**Proof** Observe that, on any fixed compact set, the maps \( h_n : \mathbb{D} \to S_n \) are \( c_n \)-Lipschitz for some constants \( c_n \) converging to \( c \). Indeed these are the initial maps \( h_n \), albeit with the limit metric on the target. Since we assumed that \( d_n(h_n(0), 0) \leq c \), these maps \( h_n \) are locally uniformly bounded (this means locally in \( z \) and uniformly in \( n \)). It thus follows from the Ascoli lemma that we may assume the sequence \( (h_n) \) to converge uniformly on compact sets to a \( c \)-Lipschitz map \( h_\infty : \mathbb{D} \to S_\infty \).

a) If all \( h_n : \mathbb{D} \to S_n \) are \( C \)-quasi-isometric, then, on any fixed compact set, the maps \( h_n : \mathbb{D} \to S_\infty \) are \( C_n \)-quasi-isometric for some constant \( C_n \) converging to \( C \), and so \( h_\infty \) is \( C \)-quasi-isometric.

b) Now assume that each map \( h_n : \mathbb{D} \to S_n \) is harmonic, namely that each function \( h_n : \mathbb{D} \to D \subset \mathbb{C} \) satisfies the equation

\[
(h_n)_{zz} + 2((u_n)_z \circ h_n)(h_n)_z(h_n)_{\bar{z}} = 0. \tag{4.2}
\]

We want to prove that \( h_\infty \) is harmonic, namely that it is \( C^2 \) and satisfies

\[
(h_\infty)_{zz} + 2((u_\infty)_z \circ h_\infty)(h_\infty)_z(h_\infty)_{\bar{z}} = 0. \tag{4.3}
\]

The maps \( h_n : \mathbb{D} \to S_n \) are \( c \)-Lipschitz, so that all the derivatives \( (h_n)_z \) and \( (h_n)_{\bar{z}} \) are locally uniformly bounded. We have seen in the proof of Lemma 4.1 that the gradients \( \nabla u_n \) are locally uniformly bounded, hence \( (u_n)_z \circ h_n \) are locally uniformly bounded. Then (4.2) ensures that the functions \( \Delta h_n \) are also locally uniformly bounded. We apply the first Schauder estimates (4.1) to the functions \( v = h_n \). This implies that, for \( 0 < \alpha < 1 \), the functions \( h_n \) are uniformly bounded in the \( C^{1,\alpha}_{\text{loc}} \) topology.

Plugging this information in (4.2), and remembering from the proof of Lemma 4.1 that the gradients \( \nabla u_n \) are also uniformly bounded in the \( C^0_{\text{loc}} \) topology, we see that the functions \( \Delta h_n \) are uniformly bounded in the \( C^0_{\text{loc}} \) topology. We will now apply the second Schauder estimates to the functions \( v = h_n \) (see [21, Theorem 70]). With the same notation as (4.1), these estimates state

\[
\|v\|_{C^{2,\alpha}(B_1)} \leq c_\alpha (\|\Delta v\|_{C^\alpha(B_2)} + \|v\|_{C^0(B_2)}). \tag{4.4}
\]

Hence the functions \( h_n \) are uniformly bounded in the \( C^{2,\alpha}_{\text{loc}} \) topology.
Therefore \((h_n)\) admits a subsequence which converges in the \(C^2_{\text{loc}}\) topology. This proves that \(h_\infty\) is \(C^2\) and going to the limit in (1.2) ensures that the limit map \(h_\infty\) is harmonic, as claimed. \(\square\)

5 A lower bound for the Jacobian

In this section we provide a lower bound for the Jacobian \(J_t\) of \(h_t\) when \(h_t\) is a quasi-conformal diffeomorphism (Proposition 3.6).

The notation are those of Section 2.3: \(S\) is a pinched Hadamard surface and \(h : \mathbb{D} \to S\) is an harmonic map. We assume moreover that \(h\) is an orientation preserving diffeomorphism. The Jacobian of \(h\), which is \(J = H - L\) with \(H := \|\partial h\|^2\) and \(L := \|\overline{\partial} h\|^2\), is positive. The function \(w := \frac{1}{2} \log H\) satisfies Equation (2.2), that we may also write as

\[
\Delta w = (-K \circ h)(1 - |\mu|^2) e^{2w} - 1,
\]

where \(\mu := h_\overline{z}/h_z\) is the conformal distortion. By (2.4) the diffeomorphism \(h\) is quasi-conformal if and only if there exists a \(\delta < 1\) such that \(|\mu| \leq \delta\).

5.1 Controlling the norm of the differential

The next lemma tells us that the norm of the differential \(\|Dh\|\) of a harmonic quasi-conformal diffeomorphism is uniformly bounded below (see also [23]).

**Lemma 5.1** Let \(h : \mathbb{D} \to S\) be a quasi-conformal harmonic diffeomorphism, where \(S\) is a pinched Hadamard surface with curvature \(-b^2 \leq K \leq -a^2 < 0\). Then one has \(e^{2w} \geq b^{-2}\).

**Proof** Introduce the conformal metric \(\tilde{g} = e^{2w} g_{\text{hyp}}\) on \(D\). We first prove that \(\tilde{g}\) is complete with pinched negative curvature. Proposition 3.1 will then provide the lower bound on \(w\).

Let \(S = (D, \sigma^2(z)|dz|^2)\). The map \(h : \mathbb{D} \to S\) being a diffeomorphism and \(S\) being complete, the pull back metric \(G = h^*(\sigma^2(z)|dz|^2)\) is complete. This pull-back metric reads as \(G = (\sigma^2 \circ h)|h_z|^2|dz|^2\) and \(|\mu| \leq 1\), one easily checks that \(G \leq 4\tilde{g}\). This ensures that the metric \(\tilde{g}\) is complete.

Comparison of Equation (5.1) satisfied by \(w\) and the curvature equation (3.1) yields that the metric \(\tilde{g}\) has curvature \(\tilde{K} = (K \circ h)(1 - |\mu|^2)\). It follows that \(-b^2 \leq \tilde{K} \leq -a^2(1 - \delta^2) < 0\), where \(\delta := \|\mu\|_\infty < 1\). Proposition 3.1 thus ensures that \(w\) satisfies \(b^{-2} \leq e^{2w} \leq a^{-2}(1 - \delta^2)^{-1}\). \(\square\)
5.2 Controlling the Jacobian

The following proposition tells us that the Jacobian of a harmonic quasi-conformal diffeomorphism is controlled by its Lipschitz constant.

**Proposition 5.2** Let $0 < a \leq b$. Then, for every $c > 0$, there exists $j_+ = j_+(a,b,c) > 0$ such that, if $S$ is a pinched Hadamard surface with curvature $-b^2 \leq K \leq -a^2$, the Jacobian $J$ of any $c$-Lipschitz quasi-conformal harmonic diffeomorphism $h : \mathbb{D} \to S$ satisfies $J \geq j_+$.

**Proof** Assume by contradiction that there exist a sequence of pinched Hadamard surfaces $S_n = (D,e^{2u_n}\text{hyp})$ with curvatures $-b^2 \leq K_n \leq -a^2$, a sequence $h_n : \mathbb{D} \to S_n$ of $c$-Lipschitz harmonic quasi-conformal diffeomorphisms and a sequence $(x_n)$ of points of $D$ such that the Jacobian $J_n$ of $h_n$ satisfy $J_n(x_n) \to 0$.

Choosing sequences $(\gamma_n)$ and $(\gamma'_n)$ of isometries of the hyperbolic disk such that $\gamma_n(x_n) = 0$ and $\gamma'_n(h_n(x_n)) = 0$, and replacing $u_n$ by $u_n \circ \gamma'_n^{-1}$ and $h_n$ by $\gamma'_n h_n \gamma_n^{-1}$, we can assume that $x_n = 0$ and $h_n(x_n) = 0$.

By Lemmas 4.1 and 4.2, going to a subsequence, one may assume that:

- the sequence $(u_n)$ converges to a $C^1$ function $u_\infty$ in the $C^1_{\text{loc}}$ topology.
- the sequence $(h_n)$ converges to a $C^2$ map $h_\infty$ in the $C^2_{\text{loc}}$ topology.

Recall from (2.4) that $J_n = (1 - |\mu_n|^2)e^{2u_n}$ where $\mu_n = (h_n)_{z}/(h_n)_{\bar{z}}$ is the conformal distortion and where $e^{2u_n} = ||\partial h_n||^2$. Lemma 5.1 ensures that

$$e^{2u_\infty} = \lim_{n \to \infty} e^{2u_n} \geq b^{-2}.$$  \hspace{1cm} (5.2)

Thus $(h_\infty)_{z}$ does not vanish. Hence the functions $\mu_n$ also converge to a $C^1$ functions $\mu_\infty$ in the $C^1_{\text{loc}}$ topology, and one has $||\mu_\infty|| = 1$ and $|\mu_\infty(0)| = 1$.

**First step** We claim that $|\mu_\infty| \equiv 1$.

Indeed, we introduce the non negative $C^1$ functions $\ell_n := -\log |\mu_n|^2$ defined on $\Omega_n := \{\mu_n \neq 0\}$ and their limit $\ell_\infty := -\log |\mu_\infty|^2$, which is defined on $\Omega_\infty := \{\mu_\infty \neq 0\}$. By assumption, the function $\ell_\infty$ is a non-negative function that achieves its minimum $\ell_\infty(0) = 0$ at the origin. We will prove that the set $\{\ell_\infty = 0\}$ is open in $\Omega_\infty$, so that $\ell_\infty \equiv 0$ as claimed.

The function $\ell_n$ satisfies the equation on $\Omega_n$, difference of (2.2) and (2.3):

$$\Delta \ell_n = 4(-K_n \circ h_n)(1 - e^{-\ell_n})e^{2u_n}.$$  \hspace{1cm} (5.3)

Since $|K_n| \leq b^2$, $1 - e^{-\ell_n} \leq \ell_n$ and $e^{2u_n} \leq c^2$, we infer that

$$\Delta \ell_n \leq 4b^2c^2 \ell_n.$$  \hspace{1cm} (5.3)

Hence $\ell_\infty$ is a $C^1$ function on $\Omega_\infty$ that satisfies in the weak sense

$$\Delta \ell_\infty \leq 4b^2c^2 \ell_\infty.$$  \hspace{1cm} (5.3)
In particular, one has bounds $|\Delta_{\ell} \leq C K \ell \leq \infty$ on compact sets $K$ of $\Omega_{\infty}$ and, by Lemma 5.3 below, the set $\{ \ell = 0 \}$ is open. This proves $|\mu_{\infty}| \equiv 1$.

**Second step** We reach a contradiction.

We recall that the functions $w_n$ satisfy (5.1), namely

$$\Delta w_n = (-K_n \circ h_n) (1 - |\mu_n|^2) e^{2w_n} - 1.$$ 

Since the functions $(-K_n \circ h_n)$ and $e^{2w_n}$ are uniformly bounded and since $\lim_{n \to \infty} |\mu_n| = 1$, the limit function $w_{\infty} = \lim w_n$ satisfies $\Delta w_{\infty} = -1$ in the weak sense. In particular $w_{\infty}$ is smooth. Note also that (5.2) yields the lower bound $w_{\infty} \geq \log b - 2$.

In conclusion, $w_{\infty}$ is a smooth function on $D$ which is bounded below and satisfies $\Delta w_{\infty} = -1$. By the generalized maximum principle of Lemma 3.2, such a function $w_{\infty}$ does not exist. Contradiction. □

In the previous proof we have used the following lemma as in [12].

**Lemma 5.3** Let $C > 0$ and $\ell$ be a non-negative continuous function on an open set $U \subset \mathbb{R}^2$ such that $\Delta \ell \leq C \ell$ weakly. Then the set $\{ \ell = 0 \}$ is open.

**Proof** We can assume that $\ell(0) = 0$. By a standard convolution argument, in a small ball $B(0, R) \subset \Omega$, we can write $\ell$ as a uniform limit of non-negative $C^2$-functions $\ell_n$ that also satisfy

$$\Delta_{\ell_n} \leq C \ell_n.$$ 

We introduce the mean values of $\ell_n$ and $\ell$ on circles of radius $r \leq R$,

$$M_n(r) := \frac{1}{2\pi} \int_0^{2\pi} \ell_n(r e^{i\theta}) \, d\theta \quad \text{and} \quad M(r) := \frac{1}{2\pi} \int_0^{2\pi} \ell(r e^{i\theta}) \, d\theta.$$

The Green representation formula (see Hörmander [13, p.119]) gives

$$\ell_n(0) = M_n(r) - \frac{1}{2\pi} \int_{B(0,r)} \Delta \ell_n(y) \log \frac{r}{|y|} \, dy.$$ 

Since $\ell_n$ converges uniformly to $\ell$ and $\ell(0) = 0$ we infer, using (5.4), that

$$M(r) \leq \frac{C}{2\pi} \int_{B(0,r)} \ell(y) \log \frac{r}{|y|} \, dy,$$

so that, for every $r \leq R$,

$$M(r) \leq \frac{CR^2}{4} \sup_{[0,R]} M(t).$$

Choosing $R^2 < 4/C$, we obtain that $\ell \equiv 0$ on the ball $B(0, R)$. □
6 Continuity of the Jacobian

In this section we prove that the metrics $g_t$, the harmonic maps $h_t$ and their Jacobians $J_t$ depend continuously on $t$, thus proving Proposition 3.7.

6.1 A continuous family of metric

In Chapter 3, we introduced pinched Hadamard surfaces $S_t = (D, e^{2u_t} g_{\text{hyp}})$ with curvature $K_t = (t - 1) + tK$, where $-b^2 \leq K \leq -a^2 < 0$ ($t \in [0, 1]$).

In particular, $S_0 = D$. We have seen that all the metrics $g_t$ are uniformly bi-Lipschitz to each other. This means that the functions $u_t : D \to \mathbb{R}$ are uniformly bounded.

Lemma 6.1 tells us that they are uniformly bounded in norm $C^1$ and that the map $t \in [0, 1] \to u_t \in C^1$ is continuous. Here the gradients $\nabla$, as well as their norms, are taken with respect to the hyperbolic metric $g_{\text{hyp}}$.

Lemma 6.1 There exists a constant $c$ such that, for every $0 \leq t \leq 1$

\[
\|u_t\|_{\infty} + \|\nabla u_t\|_{\infty} \leq c \quad (6.1)
\]

\[
\|u_t - u_s\|_{\infty} + \|\nabla(u_t - u_s)\|_{\infty} \leq c |t - s|. \quad (6.2)
\]

Proof We argue as in the proof of Lemma 4.1. Let us first prove (6.1).

Each conformal factor $e^{2u_t}$ is solution of the curvature equation (3.1), here

\[
\Delta u_t = (-K_t)e^{2u_t} - 1. \quad (6.3)
\]

Since the metrics $g_t$ are complete, and the $K_t$ satisfy a uniform pinching condition $-B^2 \leq K_t \leq -A^2 < 0$ for all $0 \leq t \leq 1$, Proposition 3.1 ensures that the functions $u_t$ are uniformly bounded. Plugging into (6.3), we infer that the Laplacians $\Delta u_t$ are also uniformly bounded. Hence the Schauder estimates (4.1) with $\alpha = 0$ and $v = u_t$ yield the uniform bound (6.1).

We now prove (6.2). Using the curvature equations (6.3) satisfied by $u_s$ and $u_t$ ($0 \leq s < t \leq 1$), we obtain

\[
\Delta(u_t - u_s) = (K_s - K_t)e^{2u_t} + K_s(e^{2u_s} - e^{2u_t})
\]

that we rewrite as:

\[
\Delta(u_t - u_s) = (s - t)(1 + K)e^{2u_t} + (-K_s)(e^{2u_t} - e^{2u_s}). \quad (6.4)
\]

Since the functions $u_t$ are uniformly bounded, there exists a constant $m_0 > 0$, such that one has $|u_t - u_s| \leq m_0 |e^{2u_t} - e^{2u_s}|$ for all $s, t$ in $[0, 1]$.

The generalized maximum principle applied to $u_t - u_s$ combined with (6.4) ensures the existence of a constant $c$ such that $\|u_t - u_s\|_{\infty} \leq c |t - s|$ for every $s, t$ in $[0, 1]$. 

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Plugging this information into (6.4) yields a similar bound for $\Delta(u_t - u_s)$, and (6.2) follows from the Schauder estimates (4.1) with $v = u_t - u_s$. \hfill \Box)

**Remark** Since the curvature function $K$ is smooth, one could improve Lemma 6.1 and prove that all $u_t$ are smooth and that, for all $p \geq 2$ the maps $t \in [0,1] \to u_t \in C^p_{\text{loc}}$ is continuous. But the $p^{\text{th}}$ derivatives of $u_t$ might not be bounded.

### 6.2 A continuous family of harmonic maps

Recall that we have natural identifications $\partial_\infty S_t \simeq S^1$. We fix an increasing quasi-symmetric homeomorphism $\varphi : S^1 \to S^1$. In Chapter 3 we introduced the unique harmonic quasi-isometric map $h_t : \mathbb{D} \to S_t$ with boundary value at infinity $\partial_\infty h_t = \varphi$.

Here are the continuity properties of this family of maps $h_t$ that we used in the proof of Theorem 1.1.

**Proposition 6.2** (a) The map $t \in [0,1] \to h_t \in C(\mathbb{D}, \mathbb{D})$ is continuous.
(b) The map $t \in [0,1] \to J_t \in C(\mathbb{D}, \mathbb{R})$ is continuous.

This means that $\lim_{s \to t} d(h_s, h_t) = 0$ and $\lim_{s \to t} \|J_s - J_t\|_{\infty} = 0$, for all $t \in [0,1]$.

**Proof** Assume this is not the case. Then there exist a sequence $(t_n)$ in $[0,1]$ and a sequence $(x_n)$ of points in $\mathbb{D}$ such that

$$\lim_{n \to \infty} d(h_{t_n}(x_n), h_t(x_n)) > 0 \quad \text{or} \quad \lim_{n \to \infty} |J_{t_n}(x_n) - J_t(x_n)| > 0. \quad (6.5)$$

We want to get a contradiction by applying Lemmas 4.1 and 4.2 to re-centered surfaces and re-centered harmonic maps. We thus choose sequences $(\gamma_n)$ and $(\gamma'_n)$ of isometries of the hyperbolic disk $\mathbb{D}$ such that $\gamma_n(x_n) = 0$ and $\gamma'_n(h_t(x_n)) = 0$. Let $S_n = (D, g_n)$ and $S'_n = (D, g'_n)$ be the conformal surfaces where $g_n = e^{2u_n}_{\text{hyp}}$ and $g'_n = e^{2u'_n}_{\text{hyp}}$ with $u_n := u_t \circ \gamma_n$ and $u'_n := u_{t_n} \circ \gamma_n$. By Lemma 4.1 we may assume, after extraction, that the sequence $(u_n)$ converges to a $C^1$ function $u_\infty$ in the $C^1_{\text{loc}}$ topology, and that the limit $C^1$ metric space $S_\infty := (D, e^{2u_\infty})$ is a CAT space with pinched curvature $-b^2 \leq K_\infty \leq -a^2 < 0$.

By Lemma 6.1 one has

$$\lim_{n \to \infty} \|u'_n - u_n\|_\infty + \|\nabla u'_n - \nabla u_n\|_\infty = 0.$$

Hence the sequence $(u'_n)$ also converges in the $C^1_{\text{loc}}$ topology to the function $u_\infty$. We now introduce the sequence of maps

$$h_n := \gamma'_n \circ h_t \circ \gamma_n^{-1} : \mathbb{D} \to S_n, \quad (6.6)$$
$$h'_n := \gamma'_n \circ h_{t_n} \circ \gamma_n^{-1} \quad : \mathbb{D} \to S'_n. \quad (6.7)$$

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These maps $h_n$ and $h'_n$ are harmonic and (6.5) can be rewritten as
\[
\lim_{n \to \infty} d(h_n(0), h'_n(0)) > 0 \quad \text{or} \quad \lim_{n \to \infty} |J_n(0) - J'_n(0)| > 0,
\] (6.8)
where $J_n$ is the Jacobian of $h_n$ and $J'_n$ the Jacobian of $h'_n$. By Lemma 3.3 all these maps $h_n$ and $h'_n$ are uniformly Lipschitz and uniformly quasi-isometric. Hence Lemma 4.2 ensures that, after extraction, the sequences $(h_n)$ and $(h'_n)$ converge respectively, in the $C^2$ loc topology, to harmonic quasi-isometric maps $h_\infty, h'_\infty : \mathbb{D} \to S_\infty$.

Since Lemma 3.3 also asserts that $d(h_n, h'_n) \leq 2c_*$ for all $n$, the limit harmonic quasi-isometric maps $h_\infty, h'_\infty : \mathbb{D} \to S_\infty$ are within bounded distance from each other. Then the uniqueness theorem for quasi-isometric harmonic maps in [4, §5] ensures that $h_\infty = h'_\infty$. This contradicts (6.8). □

This also ends the proof of both Proposition 3.7 and Theorem 1.1.

7 The injectivity theorem in constant curvature

This chapter is an appendix in which we prove the injectivity theorem 7.1 that we used as a starting point in the proof of our main theorem 1.1.

7.1 The Li-Tam-Markovic injectivity theorem

**Theorem 7.1** Let $\mathbb{D}$ be the hyperbolic disk. Any harmonic quasi-isometric map $h : \mathbb{D} \to \mathbb{D}$ is a quasi-conformal harmonic diffeomorphism.

This theorem is an output of Markovic solution of the Schoen conjecture in [20]. It relies on a previous injectivity result of Li-Tam in [19] when the boundary map of $h$ is smooth, which is Proposition 7.4 below. The proof of Li-Tam itself relies on the Schoen-Yau injectivity theorem in [22].

We would like to give in this appendix a short new proof of Theorem 7.1 that does not rely on this Schoen-Yau theorem and that uses instead a continuity method combined with a simple topological fact (Lemma 7.8).

**Proof** The proof will last till the end of this appendix. We know (see Section 2.2) that the boundary value $\varphi = \partial_\infty h : SS^1 \to SS^1$ is a $k$-quasi-symmetric homeomorphism of $SS^1 = \partial_\infty \mathbb{D}$, where $k$ depends only on the constant $c$ of quasi-isometry of $h$. For $k \geq 1$, we introduce the set

$$
\mathcal{M}_k = \{ k\text{-quasi-symmetric homeomorphism } \varphi : SS^1 \to SS^1 \}
$$
equipped with the uniform distance $d(\varphi_1, \varphi_2) = \sup_{\xi \in SS^1} |\varphi_1(\xi) - \varphi_2(\xi)|$.

We also know that, for all $\varphi$ in $\mathcal{M}_k$, there exists a unique harmonic quasi-isometric map $h_\varphi : \mathbb{D} \to \mathbb{D}$ whose boundary map is $\varphi$. We want to prove that all these maps $h_\varphi$ are quasiconformal diffeomorphisms. This will follow from the next Lemma 7.2 Proposition 7.3 and Proposition 7.4 □

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Lemma 7.2 The $k$-quasi-symmetric $C^1$ diffeomorphisms are dense in $\mathcal{M}_k$.

Proof Choose a smooth approximation of unity $(\alpha_n)$ on $SS^1$. For $\varphi$ in $\mathcal{M}_k$, each function $\alpha_n \ast \varphi$ is a $k$-quasi-symmetric $C^1$ diffeomorphism while the sequence $(\alpha_n \ast \varphi)$ converges uniformly to $\varphi$. □

Proposition 7.3 Let $\mathcal{F}_k$ be the set of those $\varphi \in \mathcal{M}_k$ such that $h_\varphi$ is a quasi-conformal diffeomorphism. Then $\mathcal{F}_k$ is a closed subset of $\mathcal{M}_k$.

The proof of Proposition 7.3 will be given in Section 7.3. It relies on continuity properties of the boundary map $h \mapsto \partial_\infty h$ proven in Section 7.2.

Proposition 7.4 When $\varphi$ is a $C^1$ diffeomorphism of $SS^1$, its quasi-isometric harmonic extension $h_\varphi : \mathbb{D} \to \mathbb{D}$ is a quasi-conformal diffeomorphism.

The proof of Proposition 7.4 will be given in Section 7.5. It uses a deformation $\varphi_t$ of $\varphi$ starting with the identity. Let $G$ be the group of isometries of $\mathbb{D}$ acting on $SS^1$. The proof relies on the fact that the only homeomorphisms which are limits of elements of $G\varphi_tG$ belong to $G$. This is Lemma 7.8 which will be proven in Section 7.4.

7.2 Continuity of the boundary map

Let $c > 1$. Endow the space $Q_c$ of $c$-quasi-isometric maps $f : \mathbb{D} \to \mathbb{D}$ with the topology of uniform convergence on compact sets, and the space $\mathcal{C}$ of continuous maps $\varphi : SS^1 \to SS^1$ with the topology of uniform convergence.

Lemma 7.5 The map $f \in Q_c \to \partial_\infty f \in \mathcal{C}$ is continuous.

Proof We use the quasi-invariance of the Gromov product under quasi-isometric maps. We fix a point 0 in $\mathbb{D}$. For $n \in \mathbb{N} \cup \{\infty\}$, let $f_n \in Q_c$ be $c$-quasi-isometric maps, with boundary values at infinity $\varphi_n$. Assume that the sequence $(f_n)$ converges uniformly to $f_\infty$ on compact sets. In particular, the quantity $R := \sup_n d(f_n(0), 0)$ is finite. We want to prove that the sequence $(\varphi_n)$ converges uniformly to the boundary map $\varphi_\infty$ of $f_\infty$.

For $\xi \in SS^1$, denote by $t \in [0, \infty[ \to x_t^\xi \in \mathbb{D}$ the geodesic ray with origin 0 and endpoint $\xi$. By [10] Proposition 5.15, there exists a constant $\lambda > 1$ such that the following lower bound for the Gromov product seen from 0 holds when $s \geq t > 0$ and $n \in \mathbb{N} \cup \{\infty\}$. Letting $s \to \infty$, we obtain

$$ (f_n(x_t^\xi), f_n(x_s^\xi))_0 \geq (x_t^\xi, x_s^\xi)_0/\lambda - \lambda = t/\lambda - \lambda $$
for \( n \in \mathbb{N} \cup \{\infty\} \). Since \( \mathbb{D} \) is \( \delta \)-hyperbolic for a constant \( \delta > 0 \), each Gromov product \( (\varphi_n(\xi), \varphi_\infty(\xi))_0 \) is bounded below by

\[
\min\{ (\varphi_n(\xi), f_n(x_\xi^t))_0, (f_n(x_\xi^t), f_\infty(x_\xi^t))_0, (f_\infty(x_\xi^t), \varphi_\infty(\xi))_0 \} - 2\delta
\]

for every \( \xi \in SS^1 \) and \( n \in \mathbb{N} \) (see [10, Chap. 2]). The sequence \( (f_n) \) converging uniformly to \( f_\infty \) on compact sets there exists, for all \( t > 0 \), an integer \( n_t \geq 1 \) such that one has, for \( n \geq n_t \) and \( \xi \in SS^1 \),

\[
d(f_n(x_\xi^t), f_\infty(x_\xi^t)) \leq 1,
\]

and hence

\[
(\varphi_n(\xi), \varphi_\infty(\xi))_0 \geq \min\{t/\lambda - \lambda; t/c - c - R - 1/2\} - 2\delta.
\]

This proves that the sequence \( (\varphi_n) \) converges uniformly to \( \varphi_\infty \). \( \square \)

### 7.3 A continuous inverse to the boundary map

The following lemma is a variation of Lemma [3.3] Fix \( k \geq 1 \).

**Lemma 7.6** There exist a compact subset \( L_k \subset \mathbb{D} \) and a constants \( c_k \) such that the harmonic quasi-isometric extension \( h_\varphi \) of any \( \varphi \in \mathcal{M}_k \) is \( c_k \)-quasi-isometric, the point \( h_\varphi(0) \) is in \( L_k \), and the map \( h_\varphi \) is \( c_k \)-Lipschitz.

**Proof** We introduce the Douady-Earle extension \( f_\varphi : \mathbb{D} \to \mathbb{D} \) of \( \varphi \) and we recall some of their properties that can be found in J. Hubbard’s book [14, §5.1]. By definition, the image \( f_\varphi(z) \) of \( z \in \mathbb{D} \) is the barycenter of the measure \( \varphi_*(m_z) \) where \( m_z \) is the visual measure on \( SS^1 \) seen from \( z \). This map \( f_\varphi \) is smooth, and is \( C_k \)-quasi-isometric for some constant that depends only on \( k \) (it is even \( \delta_k \)-quasi-conformal or some constant that depends only on \( k \)). The map \( \varphi \to f_\varphi \) is continuous hence, since \( \mathcal{M}_k \) is compact, the points \( f_\varphi(0) \) belong to a fixed compact set of \( \mathbb{D} \).

By the main result of [20] or [3], the distance \( d(h_\varphi, f_\varphi) \) is bounded by a constant \( M_k \) that depends only on \( C_k \). The first two claims follow. The Lipschitz continuity of \( h_\varphi \) then follows from the Cheng lemma [5.4] \( \square \)

**Corollary 7.7** The map \( \varphi \in \mathcal{M}_k \to h_\varphi \in C^2(\mathbb{D}, \mathbb{D}) \) is continuous in the \( C^2_{\text{loc}} \) topology.

**Proof** Let \( (\varphi_n) \) be a sequence in \( \mathcal{M}_k \) converging to \( \varphi \). By Lemma 7.6 the harmonic maps \( h_n := h_{\varphi_n} \) are uniformly bounded and uniformly Lipschitz. By Lemma [7.2] after extraction, the sequence \( (h_n) \) converges in the \( C^2_{\text{loc}} \) topology to a harmonic quasi-isometric map \( h_\infty : \mathbb{D} \to \mathbb{D} \). To reach the conclusion, we need to prove that such a limit \( h_\infty \) is always equal to \( h_\varphi \).

Since the maps \( h_n \) are uniformly quasi-isometric, the continuity lemma yields that the limit \( \varphi \) of the boundary maps \( \varphi_n \) of \( h_n \) must be the boundary map of \( h_\infty \). This proves that \( h_\infty = h_\varphi \). \( \square \)
Proof of Proposition 7.3 Let \((\varphi_n)\) be a sequence in \(M_k\) converging to \(\varphi\) such that all the harmonic quasi-isometric extensions \(h_{\varphi_n}\) are quasiconformal diffeomorphisms. We want to prove that the harmonic map \(h_\varphi\) is also a quasiconformal diffeomorphism.

Corollary 7.7 ensures that the sequence \((h_{\varphi_n})\) converges to \(h_\varphi\) in the \(C^2_{loc}\) topology. Lemma 7.6 ensures that these maps \(h_{\varphi_n}\) are uniformly Lipschitz. Hence, by Proposition 5.2, there exists a uniform lower bound \(j_\ast > 0\) for the Jacobians of all these harmonic quasi-isometric diffeomorphisms \(h_{\varphi_n}\).

Therefore \(h_\varphi\) is also a Lipschitz harmonic map whose Jacobian is bounded below by \(j_\ast\). Hence, by the injectivity criterion in Lemma 3.5, the harmonic map \(h_\varphi\) is also a quasiconformal diffeomorphism. □

7.4 Orbit closure in the group of homeomorphisms of \(SS^1\)

Recall that \(D\) is the hyperbolic disk and \(SS^1\) is its boundary at infinity. Let \(G\) be the group of isometries of \(D\) acting on \(SS^1\). It is isomorphic to \(PGL(2, \mathbb{R})\).

In order to prove Proposition 7.4 in the next section we will need the following lemma.

Lemma 7.8 Let \(\varphi_n\) be a sequence of \(C^1\) diffeomorphisms of \(SS^1\) converging in the \(C^1\) topology to a \(C^1\) diffeomorphism \(\varphi_\infty\) of \(SS^1\). Let \(\gamma_n\) and \(\gamma'_n\) be two unbounded sequences in \(G\) such that the sequence \(\psi_n := \gamma'_n \circ \varphi_n \circ \gamma_n^{-1}\) converges to an homeomorphism \(\psi_\infty\) of \(SS^1\). Then this limit \(\psi_\infty\) belongs to \(G\).

Proof We recall the Cartan decomposition \(G = KA^+K\) of \(G\) where \(K\) is the group \(PO(2, \mathbb{R})\) and \(A^+ = \{\text{diag}(s, s^{-1}) \mid s \geq 1\}\). Since \(K\) is compact, we can assume that both \(\gamma_n\) and \(\gamma'_n\) are in \(A^+\). We write

\[
\gamma_n = \text{diag}(s_n^{1/2}, s_n^{-1/2}) \quad \text{and} \quad \gamma'_n = \text{diag}(s'_n^{1/2}, s'_n^{-1/2})
\]

with both \(s_n\) and \(s'_n\) converging to \(\infty\). Here it will be convenient to use the identification \(SS^1 \simeq \mathbb{R} \cup \{\infty\}\) given by the upper half-plane model of \(D\), so that, for \(x \in \mathbb{R}\), one has \(\gamma_n(x) = s_n x\) and \(\gamma'_n(x) = s'_n x\).

We notice that \(\varphi_\infty(0) = 0\). Indeed if this were not the case, we would have \(\psi_\infty(x) = \infty\) for all \(x \in \mathbb{R}\), contradicting the injectivity of \(\psi_\infty\).

Similarly we have \(\psi_\infty(\infty) = \infty\). Indeed if this were not the case, we would have \(\varphi_\infty(x) = 0\) for all \(x \in \mathbb{R}\), contradicting the injectivity of \(\varphi_\infty\).

Since the sequence \(\varphi_n\) converges in the \(C^1\) topology to \(\varphi_\infty\), we can write for all \(n \geq 1\) and all \(x \in \mathbb{R}\) with \(|x| \leq 1\)

\[
\varphi_n(x) = \alpha_n + (\beta_n + r_n(x)) x \quad \text{with} \quad \lim_{x \to 0} \sup_{n \in \mathbb{N}} |r_n(x)| = 0. \quad (7.1)
\]
Since \( \varphi_\infty(0) = 0 \) and \( \beta_\infty := \varphi'_\infty(0) \) is non zero, one has
\[
\lim_{n \to \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \beta_\infty > 0.
\] (7.2)

Therefore we can write for all \( n \geq 1 \) and all \( x \in \mathbb{R} \) with \( |x| \leq s_n \)
\[
\psi_n(x) = s'_n \alpha_n + (\beta_n + r_n(\frac{x}{s_n})) s'_n x \quad \text{with} \quad \lim_{n \to \infty} |r_n(\frac{x}{s_n})| = 0.
\] (7.3)

Since the sequences \( \psi_n(0) \) and \( \psi_n(1) \) converge, the following limits exist
\[
\alpha'_\infty := \lim_{n \to \infty} s'_n \alpha_n \in \mathbb{R} \quad \text{and} \quad \beta'_\infty := \lim_{n \to \infty} \beta_n \frac{s'_n}{s_n} > 0,
\] (7.4)

Hence one has \( \psi_\infty(x) = \alpha'_\infty + \beta'_\infty x \) for all \( x \in \mathbb{R} \), and \( \psi_\infty \) belongs to \( G \). \( \Box \)

**Remark** - As can be seen in the proof, the assumption on \( \psi_n \) can be weakened: it is sufficient to assume that there are three points \( \xi_0, \xi_1, \xi_\infty \) in \( SS^1 \) whose images \( \psi_n(\xi_0), \psi_n(\xi_1), \psi_n(\xi_\infty) \) converge to three distinct points. This ensures that the sequence \( \psi_n \) converges uniformly to an element \( \psi_\infty \) of \( G \).

- However, it is important to assume that the limit \( \varphi_\infty \) is of class \( C^1 \) and that the convergence \( \varphi_n \to \varphi_\infty \) is in the \( C^1 \) topology.

Here is a direct corollary of Lemma 7.8 in the spirit of [2].

**Corollary 7.9** For all \( C^1 \) diffeomorphism \( \varphi \) of \( SS^1 \), one has the equality
\[
G\varphi G \cap \text{Homeo}(SS^1) = G\varphi G \cup G.
\]

**7.5 When the boundary map is a \( C^1 \) diffeomorphism**

We now conclude the proof of Theorem 7.1 by giving the last argument:

**Proof of Proposition 7.4** Let \( \varphi \) be a \( C^1 \) diffeomorphism of \( SS^1 \). We want to prove that the harmonic quasi-isometric extension \( h_\varphi \) of \( \varphi \) is a quasi-conformal diffeomorphism. For convenience we identify here \( SS^1 \) with \( \mathbb{R}/2\pi \mathbb{Z} \). For \( t \in [0,1] \), we introduce the \( C^1 \) diffeomorphism \( \varphi_t \) given by
\[
\varphi_t(\xi) = \xi + (\varphi(\xi) - \xi) t \quad \text{for all} \xi \in SS^1.
\]

This is well defined since the map \( \xi \to \varphi(\xi) - \xi \) lifts as a map from \( SS^1 \) to \( \mathbb{R} \).

We argue as in Section 3.4. For \( t \in [0,1] \) we introduce the harmonic quasi-isometric extension \( h_t = h_{\varphi_t} : \mathbb{D} \to \mathbb{D} \) of \( \varphi_t \). Let \( A \) be the set of parameters \( t \in [0,1] \) for which \( h_t \) is a quasi-conformal diffeomorphism. By the injectivity criterion of Lemma 3.3 one has
\[
A = \{ t \in [0,1] | \inf_{z \in \mathbb{D}} J_t(z) > 0 \}
\]

where \( J_t \) is the Jacobian of \( h_t \). We want to prove that \( 1 \in A \). We already know that \( 0 \in A \) because \( h_0 \) is the identity. Since the maps \( \varphi_t \) are uniformly
quasi-symmetric, Proposition 7.3 tells us that $A$ is closed. Therefore it is enough to check that $A$ is open.

Assume by contradiction that there exists a sequence $t_n \not\in A$ converging to $t_\infty \in A$. By assumption there exists a sequence $(z_n)$ in $\mathbb{D}$ such that $\liminf_{n \to \infty} J_{t_n}(z_n) \leq 0$. After extraction we are in one of the two cases:

**First case** The sequence $(z_n)$ converges to a point $z_\infty \in \mathbb{D}$. Since the maps $\varphi_t$ are uniformly quasi-symmetric, Corollary 7.7 ensures that the map $t \in [0, 1] \to h_t \in C^2(\mathbb{D}, \mathbb{D})$ is continuous in the $C^2_{\text{loc}}$ topology. Therefore, one has $J_{t_\infty}(z_\infty) = \lim_{n \to \infty} J_{t_n}(z_n) \leq 0$, and $t_\infty$ is not in $A$. Contradiction.

**Second case** The sequence $(z_n)$ goes to infinity. To simplify we set $\varphi_n = \varphi_{t_n}$ and $h_n = h_{t_n}$ for all $n \in \mathbb{N} \cup \{\infty\}$. By Lemma 7.6 the sequence $h_n(z_n)$ goes to infinity. We choose sequences $(\gamma_n)$ and $(\gamma'_n)$ in $G$ with $\gamma_n(z_n) = 0$ and $\gamma'_n(h_n(z_n)) = 0$. We introduce the harmonic maps

$$h'_n := \gamma'_n \circ h_n \circ \gamma^{-1}_n : \mathbb{D} \to \mathbb{D}$$

and their boundary values $\psi_n := \gamma'_n \circ \varphi_n \circ \gamma^{-1}_n$. By construction, one has

$$h'_n(0) = 0 \quad \text{and} \quad \liminf_{n \to \infty} J'_{t_n}(0) \leq 0,$$  \hspace{1cm} (7.5)

where $J'_{t_n}$ is the Jacobian of $h'_n$. Moreover by Lemma 7.6 these maps $h'_n$ are uniformly Lipschitz. Therefore, after extraction, they converge in the $C^2_{\text{loc}}$ topology to a harmonic quasi-isometric map $h'_\infty$. By the continuity lemma 7.5 the sequence of boundary maps $\psi_n$ converge to the boundary map $\psi_\infty$ of $h'_\infty$. Now, by Lemma 7.8 this limit $\psi_\infty$ belongs to $G$. Therefore the harmonic map $h'_\infty$ is an isometry and its Jacobian is $J'_\infty \equiv 1$. This contradicts (7.5). \hfill $\square$

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