SOME BERWALD SPACES OF NON-POSITIVE FLAG CURVATURE

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Abstract. In this paper by using left invariant Riemannian metrics on some 3-dimensional Lie groups we construct some complete non-Riemannian Berwald spaces of non-positive flag curvature and several families of geodesically complete locally Minkowskian spaces of zero constant flag curvature.

1. Introduction

The study of Riemannian manifolds with some curvature properties is one of interesting problems in differential geometry. The family of Lie groups equipped with invariant Riemannian metrics is one of these important spaces. Riemannian Lie groups with some sectional curvature properties have been studied by many mathematicians and many interesting and important results have been found on these manifolds (for example see [12] and [13]).

In the recent years with extension of studies on Finsler manifolds and applications of Finsler spaces in physics, Finsler spaces with some curvature properties have been considered by many Finsler geometers ([1], [2], [3], [5] and [18]). Invariant Finsler metrics on Lie groups and homogeneous spaces are of the best spaces for finding spaces with some curvature properties. Some of these metrics and their flag curvatures have been studied in [6], [7], [9], [10], [15] and [16].

An important family of Finsler manifolds which can help us with studying Finsler geometry is the family of flat Finsler spaces (that is Finsler manifolds of zero constant flag curvature). In the present paper, after some preliminaries, by using left invariant Riemannian metrics and left invariant vector fields on some 3-dimensional Lie groups we construct some geodesically complete left invariant locally Minkowskian Randers metrics with zero constant flag curvature. Some Finsler spaces with non-positive flag curvature have been studied by Z. Shen in [18]. He showed that every Finsler metric with negative flag curvature and constant S-curvature must be Riemannian if the manifold is compact. Also S. Deng and Z. Hou have studied homogeneous Finsler spaces of non-positive curvature. They proved that a homogeneous Finsler space with non-positive flag curvature and strictly negative Ricci scalar is a simply connected manifold [8].

In the last section of the present paper we give some complete left invariant non-Riemannian Berwald spaces of non-positive flag curvature.

Key words and phrases. invariant metric, flag curvature, Berwald space, Randers space, 3-dimensional Lie group

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2. Preliminaries

Let $M$ be a smooth $n$-dimensional manifold and $TM$ be its tangent bundle. A Finsler metric on $M$ is a non-negative function $F : TM \rightarrow \mathbb{R}$ which has the following properties:

1. $F$ is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$,
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$,
3. the $n \times n$ Hessian matrix $[g_{ij}(x, y)] = \frac{1}{2} \frac{\partial^2 F^2}{\partial y_i \partial y_j}$ is positive definite at every point $(x, y) \in TM^0$.

An important family of Finsler metrics introduced by G. Randers ([14]) in 1941 is Randers metrics with following form:

$$F(x, y) = \sqrt{g_{ij}(x)y^iy^j} + b_i(x)y^i,$$

where $g = (g_{ij}(x))$ is a Riemannian metric and $b = (b_i(x))$ is a nowhere zero 1-form on $M$.

It has been shown that $F$ is a Finsler metric if and only if $\|b\| = b_i(x)b^i(x) < 1$, where $b^i(x) = g^{ij}(x)b_j(x)$ and $[g^{ij}(x)]$ is the inverse matrix of $[g_{ij}(x)]$.

For constructing Randers metrics we can also use vector fields as follows:

$$F(x, y) = \sqrt{g(x)(y, y)} + g(x)(X(x), y),$$

where $X$ is a vector field on $M$ such that $\|X\| = \sqrt{g(X, X)} < 1$.

A Riemannian Metric $g$ on the Lie group $G$ is called left invariant if

$$g(x)(y, z) = g(e)(T_xl^{-1}y, T_xl^{-1}z) \quad \forall x \in G, \forall y, z \in T_xG,$$

where $e$ is the unit element of $G$.

Similar to the Riemannian case, a Finsler metric is called left invariant if

$$F(x, y) = F(e, T_xl^{-1}y).$$

For constructing left invariant Randers metrics on Lie groups we can use left invariant Riemannian metrics and invariant vector fields. Suppose that $G$ is a Lie group, $g$ is a left invariant Riemannian metric and $X$ is a left invariant vector field such that $\sqrt{g(X, X)} < 1$, then we can define $F$ as the formula $2.2$. Clearly $F$ is left invariant. If $X$ be parallel with respect to the Levi-Civita connection induced by the Riemannian metric $g$ then $F$ is of Berwald type.

Flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is one of the fundamental quantities which associates with a Finsler space. Flag curvature is computed by the following formula:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y)g_Y(U, U) - g_Y^2(Y, U)},$$

where $g_Y(U, V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}(F^2(Y + sU + tV))|_{s=t=0}$, $P = \text{span}\{U, Y\}$, $R(U, Y)Y = \nabla_U\nabla_Y Y - \nabla_Y\nabla_U Y - \nabla_{[U, Y]} Y$ and $\nabla$ is the Chern connection induced by $F$ (see [1] and [17]).
3. Some geodesically complete locally Minkowskian spaces of zero constant flag curvature

We begin this section with a study about completeness of a special family of Randers metrics.

**Definition 3.1.** The Riemannian manifold \((M, g)\) is said to be homogeneous if the group of isometries of \(M\) acts transitively on \(M\) (see [4]).

**Theorem 3.2.** Let \((M, g)\) be a homogeneous Riemannian manifold. Suppose that \(F\) is a Randers metric of Berwald type defined by \(g\) and a 1-form \(b\). Then \((M, F)\) is geodesically complete.

*Proof.* Since \(F\) is of Berwald type therefore the Chern connection of \(F\) and the Levi-Civita connection of \(g\) coincide and hence their geodesics coincide. On the other hand \((M, g)\) is a homogeneous Riemannian manifold, hence \((M, g)\) is geodesically complete (see [4] page 185.). Therefore \((M, F)\) is geodesically complete. □

**Corollary 3.3.** In above theorem, suppose that \(M\) is connected then by using Hopf-Rinow theorem for Finsler manifolds, \((M, F)\) is complete.

**Corollary 3.4.** Let \(G\) be a Lie group and \(g\) be a left invariant Riemannian metric on \(G\). Also suppose that \(X\) is a parallel vector field with respect to the Levi-Civita connection of \(g\) such that \(\|X\| < 1\). Then the Randers metric defined by \(g\), \(X\) and the relation \(2.2\) is geodesically complete.

Finding Riemannian spaces of constant sectional curvature is an interesting problem in Riemannian geometry. A class of Riemannian spaces of constant sectional curvature is the class of flat manifolds (that is Riemannian spaces of zero constant sectional curvature). Lie groups equipped with invariant Riemannian metrics are suitable manifolds for finding spaces of constant sectional curvature. For example Abelian Lie groups with left invariant Riemannian metrics are flat. Therefore we can have the following theorem:

**Theorem 3.5.** Let \(G\) be an abelian Lie group equipped with a left invariant Riemannian metric \(g\) and let \(\mathfrak{g}\) be the Lie algebra of \(G\). Suppose that \(X \in \mathfrak{g}\) is a left invariant vector field with \(\sqrt{g(X,X)} < 1\). Then the Randers metric \(F\) defined by the formula \(2.2\) is a flat geodesically complete locally Minkowskian metric on \(G\).

*Proof.* Assume that \(U, V, W \in \mathfrak{g}\), now by using the formula

\[
2g(\nabla_U V, W) = g([U, V], W) - g([V, W], U) + g([W, U], V),
\]

(3.1)

and the fact that \(G\) is abelian we have \(\nabla_Y X = 0\) for any \(Y \in \mathfrak{g}\). Hence \(X\) is parallel with respect to \(\nabla\) and \(F\) is of Berwald type. Also the curvature tensor \(R\) of \(g\) coincides on the curvature tensor of \(F\) and therefore the flag curvature of \(F\) is zero. \(F\) is a flat Berwald metric therefore by proposition 10.5.1 (page 275) of [1], \(F\) is locally Minkowskian. □
A famous class of Lie groups is the class of unimodular Lie groups. A Lie group \( G \) is said to be unimodular if its left invariant Haar measure is also right invariant (see \([11]\) and \([12]\)).

**Proposition 3.6.** Let \( G \) be a 3-dimensional unimodular Lie group with a left invariant Riemannian metric. Then there exists an orthonormal basis \( \{x, y, z\} \) of the Lie algebra \( g \) such that

\[
\begin{align*}
[x, y] &= c_3 z , \\
[y, z] &= c_1 x , \\
[z, x] &= c_2 y ,
\end{align*}
\]

where \( c_i = \mathbb{R} \).

(see \([11]\).)

**Lemma 3.7.** Let \( G \) be a 3-dimensional unimodular Lie group with a left invariant Riemannian metric \( g \) and use the notations introduced above. \( G \) admits a parallel (with respect to the Levi-Civita connection of \( g \)) left invariant vector field \( U = u_1 x + u_2 y + u_3 z \) if and only if the following equations hold:

\[
\mu_1 u_1 = \mu_1 u_2 = \mu_2 u_1 = \mu_2 u_3 = \mu_3 u_1 = \mu_3 u_2 = 0,
\]

where \( \mu_i = \frac{1}{2} (c_1 + c_2 + c_3) - c_i, \ i = 1, 2, 3. \)

**Proof.** Suppose that \( \nabla \) is the Levi-Civita connection of \( g \). Let \( <, > \) be the inner product induced by \( g \) on \( g \). Then by using formula \([3.1]\) and some computations we have (also you can see \([11]\).):

\[
\begin{align*}
\nabla_x x &= 0 , \\
\nabla_x y &= \mu_1 z , \\
\nabla_x z &= -\mu_1 y , \\
\n\nabla_{y} x &= -\mu_2 z , \\
\n\nabla_{y} y &= 0 , \\
\n\nabla_{y} z &= \mu_2 x , \\
\n\nabla_{z} x &= \mu_3 y , \\
\n\nabla_{z} y &= -\mu_3 x , \\
\n\nabla_{z} z &= 0 .
\end{align*}
\]

Now it is suffix to let \( \nabla_x U = \nabla_{y} U = \nabla_{z} U = 0. \)

With K. Nomizu \([13]\) we consider the Lie groups \( \mathbb{H}, E(1, 1) \) and \( E(2) \) in the following theorem.

**Theorem 3.8.** Consider the following 3-dimensional Lie groups:

- \( \mathbb{H} \): the Heisenberg group,
- \( E(1, 1) \): group of rigid motions of Minkowski 2-space,
- \( E(2) \): group of rigid motions of Euclidean 2-space.

\( \mathbb{H} \) and \( E(1, 1) \) do not admit any non-Riemannian Berwald Randers metric of the form \([2.2]\), where \( g \) and \( X \) are left invariant. \( E(2) \) admits a non-empty family of flat geodesically complete locally Minkowskian Randers metrics of the form \([2.2]\) where \( g \) and \( X \) are left invariant.

**Proof.** Case 1 (\( \mathbb{H} \)). Let \( \mathfrak{h} \) be the Lie algebra of \( \mathbb{H} \) and \( g \) be any left invariant Riemannian metric on \( \mathbb{H} \). By using proposition \([3.6]\) and \([11]\) there exist an orthonormal basis \( \{x, y, z\} \) for \( \mathfrak{h} \) such that \( c_1 > 0, c_2 = c_3 = 0 \). Therefore we have \( \mu_1 = -\frac{1}{2} c_1 < 0, \mu_2 = \mu_3 = \frac{1}{2} c_1 > 0 \). Now let \( U = u_1 x + u_2 y + u_3 z \) be any left invariant vector field on \( \mathbb{H} \) such that is parallel with respect to the Levi-Civita connection \( \nabla \) of \( g \). By using lemma \([3.7]\) we have \( \mu_1 u_1 = \mu_1 u_2 = c_1 u_3 = 0 \)
and so $U = 0$. 

Case 2 ($E(1,1)$). Suppose that $\mathfrak{e}(1,1)$ is the Lie algebra of $E(1,1)$ and $g$ is a left invariant Riemannian metric on $E(1,1)$. By using proposition 3.6 and [11] there exist an orthonormal basis $\{x, y, z\}$ for $\mathfrak{e}(1,1)$ such that $c_1 > 0$, $c_2 < 0$ and $c_3 = 0$. Now we use lemma 3.7 and get $(c_2 - c_1)u_3 = (c_2 - c_1)u_2 = (c_1 - c_2)u_1 = 0$. Therefore $U = 0$.

Case 3 ($E(2)$). We consider the Lie group $E(2)$ as follows:

$$E(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, \theta \in \mathbb{R} \right\}. \quad (3.5)$$

The Lie algebra of $E(2)$ is of the form

$$\mathfrak{e}(2) = \text{span}\{x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \}, \quad (3.6)$$

where

$$[x, y] = 0 \quad , \quad [y, z] = x \quad , \quad [z, x] = y. \quad (3.7)$$

Now consider the following inner product and let $g$ be the left invariant Riemannian metric induced by this inner product on $E(2)$,

$$< x, x > = < y, y > = < z, z > = \lambda^2 \quad , \quad < x, y > = < y, z > = < z, x > = 0, \quad \lambda > 0. \quad (3.8)$$

By using formula 3.11 and some computations for the Levi-Civita connection $\nabla$ of $g$ we have:

$$\nabla_x x = 0 \quad , \quad \nabla_x y = 0 \quad , \quad \nabla_x z = 0, \quad (3.9)$$

$$\nabla_y x = 0 \quad , \quad \nabla_y y = 0 \quad , \quad \nabla_y z = 0, \quad \nabla_z x = y \quad , \quad \nabla_z y = -x \quad , \quad \nabla_z z = 0.$$

For curvature tensor of $\nabla$ we have $R = 0$ and so $(E(2), g)$ is a flat Riemannian manifold. Suppose that $U = u_1 x + u_2 y + u_3 z \in \mathfrak{e}(2)$ such that $U$ is parallel with respect to $\nabla$. A simple computation shows that $U = uz$. Assume that $\sqrt{< U, U >} < 1$, in other words let $0 < |u| < \frac{1}{\lambda}$. Hence, the left invariant Randers metric $F$ defined by $g$ and $U$ with formula 2.2 is of Berwald type. Also since $F$ is of Berwald type therefore the curvature tensor of $F$ and $g$ coincide and $F$ is of zero constant flag curvature. Hence $F$ is locally Minkowskian. $\square$

Another example of flat geodesically complete locally Minkowskian Randers spaces is described as follows.

Let $\mathfrak{g} = \text{span}\{x, y, z\}$ be a Lie algebra such that

$$[x, y] = \alpha y + \alpha z \quad , \quad [y, z] = 2\alpha x \quad , \quad [z, x] = \alpha y + \alpha z \quad , \quad \alpha \in \mathbb{R}. \quad (3.10)$$
Also consider the inner product described by 3.8 on $\mathfrak{g}$.
Suppose that $G$ is a Lie group with Lie algebra $\mathfrak{g}$, and $g$ is the left invariant Riemannian metric induced by the above inner product $\langle ., . \rangle$ on $G$.
A direct computation for the Levi-Civita connection $\nabla$ of $(G, g)$ shows that:

\[
\begin{align*}
\nabla_{x} x & = 0 , \quad \nabla_{xy} = 0 , \quad \nabla_{xz} = 0 , \\
\nabla_{y} x & = -\alpha y - \alpha z , \quad \nabla_{y} y = \alpha x , \quad \nabla_{y} z = \alpha x , \\
\nabla_{z} x & = \alpha y + \alpha z , \quad \nabla_{z} y = -\alpha x , \quad \nabla_{z} z = -\alpha x .
\end{align*}
\] (3.11)

The above equations for $\nabla$ show that $R = 0$, therefore $(G, g)$ is a flat Riemannian manifold.

Let $U = u_{1}x + u_{2}y + u_{3}z \in \mathfrak{g}$ be a left invariant vector field such that $\nabla U = 0$. By a short computation we have $U = uy - uz$. Now suppose that $\sqrt{2}|u|\lambda = \sqrt{<U, U>} < 1$ or equivalently let $0 < |u| < \frac{1}{\sqrt{2}\lambda}$. Therefore the invariant Randers metric $F$ defined by $g$ and $U$ is a flat geodesically complete locally Minkowskian metric on $G$. Also if we consider $G$ is connected, $(G, F)$ will be complete.

4. Examples of complete Berwald spaces of non-positive flag curvature

In this section we give three families of complete Berwald spaces of non-positive flag curvature also we give their explicit formulas for computing flag curvature.

**Example 1.** Suppose that $\mathfrak{g} = \text{span}\{x, y, z\}$ is a Lie algebra with the following structure:

\[
[x, y] = \alpha y + \alpha z , \quad [y, z] = 0 , \quad [z, x] = -\alpha y - \alpha z , \quad \alpha \in \mathbb{R}.
\] (4.1)

Also consider the inner product $\langle ., . \rangle$ described by 3.8 on $\mathfrak{g}$ with respect to the basis $\{x, y, z\}$. Suppose that $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$ and $g_{1}$ is the left invariant Riemannian metric induced by $\langle ., . \rangle$ on $G$.

By using formula 3.11 for the Levi-Civita connection of the left invariant Riemannian metric $g_{1}$ we have:

\[
\begin{align*}
\nabla_{x} x & = 0 , \quad \nabla_{xy} = 0 , \quad \nabla_{xz} = 0 , \\
\nabla_{y} x & = -\alpha y - \alpha z , \quad \nabla_{y} y = \alpha x , \quad \nabla_{y} z = \alpha x , \\
\nabla_{z} x & = -\alpha y - \alpha z , \quad \nabla_{z} y = \alpha x , \quad \nabla_{z} z = \alpha x .
\end{align*}
\] (4.2)

Also for curvature tensor of $(G, g_{1})$ we have:

\[
\begin{align*}
R(x, y)x & = R(x, z)x = 2\alpha^{2}(y + z), \\
R(x, y)y & = R(x, y)z = R(x, z)y = R(x, z)z = -2\alpha^{2}x \\
R(y, z)x & = R(y, z)y = R(y, z)z = 0 .
\end{align*}
\] (4.3)

Now let $Y = ax + by + cz$ and $V = \tilde{a}x + \tilde{b}y + \tilde{c}z$ be two vectors in $\mathfrak{g}$. A simple computation shows that

\[
R(V, Y)Y = 2\alpha^{2}(a(y + z) - x(b + c))(\tilde{a}(b + c) - a(\tilde{b} + \tilde{c})).
\] (4.4)
Suppose that \( \{Y, V\} \) is orthonormal with respect to \( <,> \), then a direct computation for sectional curvature of \((G, g_1)\) shows that:

\[
K(V, Y) = -2(\alpha \lambda (a(b + c) - a(b + c)))^2 \leq 0.
\]

Therefore the Riemannian manifold \((G, g_1)\) is of non-positive sectional curvature.

Now let \( U = u_1x + u_2y + u_3z \in \mathfrak{g} \) be a left invariant vector field and \( \nabla U = 0 \). So we have \( U = uy - uz \). Assume that \( \|U\| = \sqrt{\langle U, U \rangle} < 1 \) or equivalently let \( 0 < |u| < \frac{1}{\sqrt{2\lambda}} \).

Now consider the Randers metric \( F_1 \) defined by \( g_1 \) and \( U \). By using formula of \( g_Y \) (Also you can see [9]) we have the following equations:

\[
g_Y(R(V, Y)Y, V) = -2(\alpha \lambda (a(b + c) - a(b + c)))^2(1 + (b - c)u\lambda^2)
\]

\[
g_Y(Y, Y) = (1 + u\lambda^2(b - c))^2
\]

\[
g_Y(V, V) = 1 + u\lambda^2(y\lambda^2(\tilde{b} - \tilde{c})^2 + (b - c))
\]

\[
g_Y(Y, Y) = u\lambda^2(\tilde{b} - \tilde{c})(1 + u\lambda^2(b - c)).
\]

A simple direct computation shows that \( F_1 \) is of non-positive flag curvature and its flag curvature for the flag \( P = \text{span}\{Y, V\} \) is obtained by the following formula:

\[
K(P, Y) = \frac{-2(\alpha \lambda (a(b + c) - a(b + c)))^2}{(1 + u\lambda^2(b - c))^2} \leq 0.
\]

**Example 2.** The Lie algebra \( \mathfrak{g} \) in example 1 admits a basis \( \{x, y, z\} \) such that:

\[
[x, y] = 0, \ [y, z] = 0, \ [z, x] = \alpha x, \ \alpha \in \mathbb{R}.
\]

Now let \( <,> \) be the inner product introduced by [3.3] with respect to this new basis. Also suppose that \( g_2 \) is the left invariant Riemannian metric induced by \( <,> \) on \( G \). Similar to example 1 we can obtain the Levi-Civita connection of \((G, g_2)\) as follows:

\[
\nabla_x x = \alpha z, \ \nabla_x y = 0, \ \nabla_x z = -\alpha x,
\]

\[
\nabla_y x = 0, \ \nabla_y y = 0, \ \nabla_y z = 0,
\]

\[
\nabla_z x = 0, \ \nabla_z y = 0, \ \nabla_z z = 0.
\]

Also for curvature tensor we have:

\[
R(x, y)x = R(x, y)y = R(x, y)z = R(y, z)x = R(y, z)y = R(y, z)z = 0,
\]

\[
R(x, z)x = \alpha^2 z, \ R(x, z)z = -\alpha^2 x.
\]

Therefore for an orthonormal basis \( \{Y = ax + by + cz, V = \tilde{a}x + \tilde{b}y + \tilde{c}z\} \) we have:

\[
K(V, Y) = -(\alpha \lambda(\tilde{a}c - \tilde{c}a))^2 \leq 0,
\]

which shows \((G, g_2)\) is of non-positive sectional curvature.

It is easy to show that the only left invariant vector fields parallel with respect to \( \nabla \) are of the
form \( U = uy, u \in \mathbb{R} \). Assume that \( 0 < |u| < \frac{1}{\lambda} \), so we have \( \|U\| < 1 \).

Let \( F_2 \) be the Randers metric defined by \( g_2 \) and \( U \). Then for \( F_2 \) we have:

\[
    \begin{align*}
    g_Y(R(V,Y)Y,V) &= -(\alpha \lambda (\tilde{ac} - \tilde{ca}))^2(1 + bu\lambda^2) \\
    g_Y(Y,Y) &= (1 + ub\lambda^2)^2 \\
    g_Y(Y,V) &= 1 + (ub\lambda^2)^2 + ub\lambda^2 \\
    g_Y(Y,V) &= ub\lambda^2(1 + ub\lambda^2).
    \end{align*}
\]

Therefore \((G,F_2)\) is of non-positive flag curvature as follows:

\[
    K(P,Y) = \frac{-(\alpha \lambda (\tilde{ac} - \tilde{ca}))^2}{(1 + ub\lambda^2)^2} \leq 0,
\]

where \( P = \text{span}\{Y,V\} \).

**Example 3.** The Lie algebra \( g \) described in example 1 also has a basis of the form \( \{x, y, z\} \) such that:

\[
    [x,y] = 0 \quad , \quad [y,z] = \alpha y + \alpha z \quad , \quad [z,x] = 0 \quad , \quad \alpha \in \mathbb{R}.
\]

Now consider the inner product defined by 3.8 with respect to this basis. Also let \( g_3 \) be the left invariant Riemannian metric induced by this inner product. Similar to the above examples for \((G,g_3)\) we have:

\[
    \begin{align*}
    \nabla_x x &= 0 \quad , \quad \nabla_x y = 0 \quad , \quad \nabla_x z = 0, \\
    \nabla_y x &= 0 \quad , \quad \nabla_y y = -\alpha z \quad , \quad \nabla_y z = \alpha y, \\
    \nabla_z x &= 0 \quad , \quad \nabla_z y = -\alpha z \quad , \quad \nabla_z z = \alpha y, \\
    \end{align*}
\]

and,

\[
    \begin{align*}
    R(x,y)x &= R(x,y)y = R(x,y)z = R(x,z)y = R(x,z)x = R(y,z)x = R(x,z)z = 0, \\
    R(y,z)y &= 2\alpha^2 z \quad , \quad R(y,z)z = -2\alpha^2 y.
    \end{align*}
\]

Hence for an orthonormal basis \( \{Y = ax + by + cz, V = \tilde{a}x + \tilde{b}y + \tilde{c}z\} \) we have:

\[
    K(V,Y) = -2(\alpha \lambda (\tilde{bc} - \tilde{cb}))^2 \leq 0.
\]

Therefore \((G,g_3)\) is of non-positive sectional curvature.

The only left invariant vector fields parallel with respect to \( \nabla \) are of the form \( U = ux, u \in \mathbb{R} \). Let \( 0 < |u| < \frac{1}{\lambda} \), so we have \( \|U\| < 1 \).

Let \( F_3 \) be the Randers metric defined by \( g_3 \) and \( U \). Then for \( F_3 \) we have:

\[
    \begin{align*}
    g_Y(R(V,Y)Y,V) &= -2(\alpha \lambda (\tilde{bc} - \tilde{cb}))^2(1 + au\lambda^2) \\
    g_Y(Y,Y) &= (1 + u\alpha\lambda^2)^2 \\
    g_Y(Y,V) &= 1 + au\lambda^2 + a^2 u^2 \lambda^4 \\
    g_Y(Y,V) &= \tilde{a}u\lambda^2(1 + au\lambda^2).
    \end{align*}
\]
The above equations show that \((G, F_3)\) is of non-positive flag curvature with the following formula:

\[
K(P, Y) = \frac{-2(\alpha \lambda (\bar{b}c - \bar{c}b))^2}{(1 + au \lambda^2)^2} \leq 0,
\]

where \(P = \text{span}\{Y, V\}\).

All Finsler metrics introduced in examples 1, 2 and 3 are geodesically complete and the connected Lie group \(G\) with these metrics is complete.

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