Perfect Matchings in Claw-free Cubic Graphs

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Abstract

Lovász and Plummer conjectured that there exists a fixed positive constant \(c\) such that every cubic \(n\)-vertex graph with no cutedge has at least \(2^{cn}\) perfect matchings. Their conjecture has been verified for bipartite graphs by Voorhoeve and planar graphs by Chudnovsky and Seymour. We prove that every claw-free cubic \(n\)-vertex graph with no cutedge has more than \(2^{n/12}\) perfect matchings, thus verifying the conjecture for claw-free graphs.

1 Introduction

A graph is claw-free if it has no induced subgraph isomorphic to \(K_{1,3}\). A graph is cubic if every vertex has exactly three incident edges. A well-known classical theorem of Petersen [9] states that every cubic graph with no cutedge has a perfect matching. Sumner [10] and Las Vergnas [6] independently showed that every connected claw-free graph with even number of vertices has a perfect matching. Both theorems imply that every claw-free cubic graph with no cutedge has at least one perfect matching.

In 1970s, Lovász and Plummer conjectured that every cubic graph with no cutedge has exponentially many perfect matchings; see [7, Conjecture 8.1.8]. The best lower bound has been obtained by Esperet, Kardoš, and Král’ [5]. They showed that the number of perfect matchings in a sufficiently large cubic graph with no cutedge always exceeds any fixed linear function in the number of vertices.

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So far the conjecture is known to be true for bipartite graphs and planar graphs. For bipartite graphs, Voorhoeve [11] proved that every bipartite cubic $n$-vertex graph has at least $6(4/3)^{n/2-3}$ perfect matchings. Recently, Chudnovsky and Seymour [2] proved that every planar cubic $n$-vertex graph with no cutedge has at least $2^{n/65578752}$ perfect matchings.

We prove that every claw-free cubic $n$-vertex graph with no cutedge has more than $2^{n/12}$ perfect matchings. The graph should not have any cutedge; in Figure 1 we provide an example of a claw-free cubic graph with only 9 perfect matchings.

Our approach is to use the structure of 2-edge-connected claw-free cubic graphs. The cycle space $C(H)$ of $H$ is a collection of the edge-disjoint union of cycles of $H$. It is well known that $C(H)$ forms a vector space over $GF(2)$ and

$$\dim C(H) = |E(H)| - |V(H)| + 1$$

if $H$ is connected, see Diestel [3]. Roughly speaking, almost all 2-edge-connected claw-free cubic graph $G$ can be built from a 2-edge-connected cubic multigraph $H$ by certain operations so that every member of $C(H)$ can be extended to 2-factors of $G$. We will have two cases to consider; either $H$ is big or small. If $H$ is big, then $C(H)$ is big enough to prove that $G$ has many 2-factors. If $H$ is small, then we find a 2-factor of $H$ using many of the specified edges of $H$ so that when transforming this 2-factor of $H$ to that of $G$, each of those edges of $H$ has many ways to make 2-factors of $G$.

2 Structure of 2-edge-connected claw-free cubic graphs

Graphs in this paper have no parallel edges and no loops, and multigraphs can have parallel edges and loops. We assume that a loop is counted twice when measuring a degree of a vertex in a multigraph. Every 2-edge-connected cubic multigraph can not have loops because if it has a loop, then it must have a cutedge.

We describe the structure of claw-free cubic graphs given by Palmer et al. [8]. A triangle of a graph is a set of three pairwise adjacent vertices. Replacing a vertex
with a triangle in cubic graph is to replace $v$ with three vertices $v_1, v_2, v_3$ forming a triangle so that if $e_1, e_2, e_3$ are three edges incident with $v$, then $e_1, e_2, e_3$ will be incident with $v_1, v_2, v_3$ respectively.

Every vertex in a claw-free cubic graph is in 1, 2, or 3 triangles. If a vertex is in 3 triangles, then the component containing the vertex is isomorphic to $K_4$. If a vertex is in exactly 2 triangles, then it is in an induced subgraph isomorphic to $K_4 \setminus e$ for some edge $e$ of $K_4$. Such an induced subgraph is called a diamond. It is clear that no two distinct diamonds intersect.

A string of diamonds is a maximal sequence $D_1, D_2, \ldots, D_k$ of diamonds in which, for each $i \in \{1, 2, \ldots, k - 1\}$, $D_i$ has a vertex adjacent to a vertex in $D_{i+1}$. A string of diamonds has exactly two vertices of degree 2, which are called the head and the tail of the string. Replacing an edge $e = uv$ with a string of diamonds with the head $x$ and the tail $y$ is to remove $e$ and add edges $ux$ and $vy$.

A connected claw-free cubic graph in which every vertex is in a diamond is called a ring of diamonds. We require that a ring of diamonds contains at least 2 diamonds. It is now straightforward to describe the structure of 2-edge-connected claw-free cubic graphs as follows.

**Proposition 1.** A graph $G$ is 2-edge-connected claw-free cubic if and only if either

(i) $G$ is isomorphic to $K_4$,

(ii) $G$ is a ring of diamonds, or

(iii) $G$ can be built from a 2-edge-connected cubic multigraph $H$ by replacing some edges of $H$ with strings of diamonds and replacing each vertex of $H$ with a triangle.

**Proof.** Let us first prove the “if” direction. It is easy to see that $G$ is 2-edge-connected cubic and has no loops or parallel edges. If $G$ is built as in (iii), then clearly $G$ has neither loops nor parallel edges, and every vertex of $G$ is in a triangle and therefore $G$ is claw-free. Note that since $H$ is 2-edge-connected, $H$ can not have loops.

To prove the “only if” direction, let us assume that $G$ is a 2-edge-connected claw-free cubic graph. We may assume that $G$ is not isomorphic to $K_4$ or a ring of diamonds. We claim that $G$ can be built from a 2-edge-connected cubic multigraph as in (iii). Suppose that $G$ is a counter example with the minimum number of vertices.

If $G$ has no diamonds, then every vertex of $G$ is in exactly one triangle and therefore $V(G)$ can be partitioned into disjoint triangles. By contracting each triangle, we obtain a 2-edge-connected cubic multigraph $H$.

So $G$ must have a string of diamonds. Let $D$ be the set of vertices in the string of diamonds. Since $G$ is cubic, $G$ has two vertices not in $D$, say $u$ and $v$, adjacent
to $D$. If $u = v$, then because the degree of $u$ is 3, $u$ must have another incident edge $e$ but $e$ will be a cutedge of $G$. Thus $u \neq v$.

If $u$ and $v$ are adjacent in $G$, then $u$ and $v$ must have a common neighbor $x$, because otherwise $G$ will have an induced subgraph isomorphic to $K_{1,3}$. However one of the edges incident with $x$ will be a cutedge of $G$, a contradiction.

Thus $u$ and $v$ are nonadjacent in $G$. Let $G' = (G \setminus D) + uv$, that is obtained from $G$ by deleting $D$ and adding an edge $uv$. Then $G'$ has no parallel edges or loops and moreover $G'$ is 2-edge-connected claw-free cubic. Since $G$ has a vertex not in a diamond, so does $G'$ and therefore $G'$ can be built from a 2-edge-connected cubic multigraph $H$ by replacing some edges with strings of diamonds and replacing each vertex of $H$ with a triangle. Since $D$ is chosen maximally, $u$ and $v$ are not in diamonds and therefore $H$ has the edge $uv$. So we can obtain $G$ from $H$ by doing all replacements to obtain $G'$ and then replacing the edge $uv$ with a string of diamonds. This completes the proof.

We remark that Proposition 1 can be seen as a corollary of the structure theorem of quasi-line graphs by Chudnovsky and Seymour [1]. A graph is a quasi-line graph if the neighborhood of each vertex is expressible as the union of two cliques. It is obvious that every claw-free cubic graph is a quasi-line graph. Chudnovsky and Seymour [1] proved that every connected quasi-line graph is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips. For 2-edge-connected claw-free cubic graphs, a fuzzy circular interval graph corresponds to a ring of diamonds and a composition of fuzzy linear interval strips corresponds to the construction (iii) of Proposition 1.

3 Main theorem

Theorem 2. Every claw-free cubic $n$-vertex graph with no cutedge has more than $2^{n/12}$ perfect matchings.

Proof. Let $G$ be a claw-free cubic $n$-vertex graph with no cutedge. We may assume that $G$ is connected. If $G$ is isomorphic to $K_4$, then the claim is clearly true. If $G$ is a ring of diamonds, then $G$ has $2^{n/4} + 1$ perfect matchings. Thus we may assume that $G$ is obtained from a 2-edge-connected cubic multigraph $H$ by replacing some edges with strings of diamonds and replacing each vertex of $H$ with a triangle.

Let $k = |V(H)|$. In other words, $3k$ is the number of vertices not in a diamond of $G$.

Suppose that $k \geq n/6$. Since $H$ has $3k/2$ edges, the cycle space of $H$ has dimension $3k/2 - k + 1 = k/2 + 1$ and therefore $|C(H)| = 2^{k/2+1}$. To obtain a 2-factor from $C \in C(H)$, we transform $C$ into a member $C' \in C(G)$ so that it meets all 3 vertices of $G$ corresponding to $v$ for each vertex $v$ of $H$ incident with
As well as it meets all the vertices in each diamond that corresponds to an edge in $C$. Then for each vertex $w$ of $G$ unused yet in $C'$, we add a cycle of length 3 or 4 depending on whether the vertex is in a diamond; see Figure 2. Then this is a 2-factor of $G$ because it meets every vertex of $G$. Since the complement of the edge-set of a 2-factor is a perfect matching, we conclude that $G$ has at least $2^{k/2+1} \geq 2^{n/12+1}$ perfect matchings.

Now let us assume that $k < n/6$. We know that $G$ has $(n - 3k)/4$ diamonds. The length of an edge $e$ of $H$ is the number of diamonds in the string of diamonds replaced with $e$. (If the edge $e$ is not replaced with a string of diamonds, then the length of $e$ is 0.)

Edmonds’ characterization of the perfect matching polytope [4] implies that there exist a positive integer $t$ depending on $H$ and a list of $3t$ perfect matchings $M_1, M_2, \ldots, M_{3t}$ in $H$ such that every edge of $H$ is in exactly $t$ of the perfect matchings. (In other words, $H$ is fractionally 3-edge-colorable.) By taking complements, we have a list of $3t$ 2-factors of $H$ such that each edge of $H$ is in exactly $2t$ of the 2-factors in the list. Since $G$ has $(n - 3k)/4$ diamonds, the sum of the length of all edges of $H$ is $(n - 3k)/4$. Therefore there exists a 2-factor $C$ of $H$ whose length is at least $n - 3k/2 = (n - 3k)/6$.

We claim that $G$ has at least $2^{(n-3k)/6}$ 2-factors corresponding to $C$. For each diamond in the string replacing an edge $e$ of $C$, there are two ways to route cycles of $C$ through the diamond, see Figure 2. Since $C$ passes through at least $(n - 3k)/6$ diamonds, $G$ has at least $2^{(n-3k)/6}$ 2-factors. Since $k < n/6$, $G$ has more than $2^{n/12}$ 2-factors. Thus $G$ has more than $2^{n/12}$ perfect matchings.

We remark that every 3-edge-connected claw-free cubic $n$-vertex graph $G$ has exactly $2^{n/6+1}$ perfect matchings, unless $G$ is isomorphic to $K_4$. That is because $G$ has no diamonds and so, from the idea of the above proof, there is a one-to-one correspondence between the set of all 2-factors of $G$ and the cycle space of a multigraph $H$ obtained by contracting each triangle of $G$. 

Figure 2: Transforming a member of $C(H)$ into a 2-factor of $G$ (Solid edges represent edges in a member of $C(H)$ or a 2-factor of $G$.)
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