BOUNDARY ESTIMATES FOR THE RICCI FLOW.

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Abstract. In this paper we consider the Ricci flow on manifolds with boundary with appropriate control on its mean curvature and conformal class. We obtain higher order estimates for the curvature and second fundamental form near the boundary, similar to Shi’s local derivative estimates. As an application, we prove a version of Hamilton’s compactness theorem in which the limit has boundary. Finally, we show that in dimension three the second fundamental form of the boundary and its derivatives are a priori controlled in terms of the ambient curvature and some non-collapsing assumptions. In particular, the flow exists as long as the curvature remains bounded, in contrast to the general case where control on the second fundamental form is also required.

1. Introduction.

Let $g(t)$ be Ricci flow on a manifold $M$, namely a solution to
$$\frac{d}{dt} g(t) = -2 \text{Ric}(g(t)),$$
not necessarily complete. Shi in [18] obtained local a priori estimates for derivatives of the curvature along Ricci flow, depending only on a curvature bound. See also Theorem 13.1 in [11]. The following theorem states a global version of these estimates.

**Theorem 1.1.** Let $(M^n, g(t))$ be a Ricci flow on a closed manifold $M$, $t \in [0, \frac{1}{K}]$, and assume
$$|Rm(g(t))|_{g(t)} \leq K.$$ Then, for every positive integer $j$ there exists $C_j = C(n, j) > 0$ such that
$$|\nabla^j Rm(g(t))|_{g(t)} \leq \frac{C_j K}{t^{j/2}},$$ in $M$ for all $t \in (0, \frac{1}{K}]$.

Such estimates not only reveal the smoothing character of Ricci flow, but they are also an essential ingredient of a compactness theorem for sequences of Ricci flows, proven by Hamilton in [10]. This theorem allows the blow-up analysis of singularities, and is an important tool in the study of the global behaviour of the flow.

When $M$ is a manifold with boundary, although there have been several local existence results for the Ricci flow, very few is known regarding its global behaviour, even in dimension 3.

Regarding local existence, Shen in [17] and Pulemotov in [16] consider natural Neumann-type boundary conditions. On the other hand, in [9] the author considers a mixed Dirichlet-Neumann boundary value problem for the Ricci flow, motivated by work of Anderson on boundary value problems for Einstein metrics in [4], where the conformal class of the boundary and its mean curvature provide an elliptic boundary value problem for the Einstein equations.

On the global behaviour, Shen in [17] studies the case the initial metric has positive Ricci curvature and the boundary remains totally geodesic, while Cortissoz in [8] studies the same when the boundary
is convex and umbilic. Moreover, with Murcia in [7] they consider the 2-dimensional Ricci flow assuming positive Gauss curvature and convex boundary.

To understand interior singularities, where after rescaling the boundary is sent to infinity, a local version of Shi’s estimates and Hamilton’s compactness theorem would suffice. However, when singularities form close to the boundary one needs a compactness theorem which can handle the possibility that the limit is a manifold with boundary. Such compactness result would require some analogue of Shi’s estimates for the curvature to be valid on a neighbourhood of the boundary, together with higher order estimates of its second fundamental form.

In [8, 14, 7] the authors obtain estimates for derivatives of the curvature valid for the boundary value problems considered. However, it is not clear whether these techniques can be adapted to deal with boundary value problems like that in [9].

The main difficulty is essentially that the curvature doesn’t seem to satisfy boundary conditions which would allow an application of the maximum principle. Moreover, as mentioned above, the possible applications require control on higher derivatives of the second fundamental form as well.

In this paper, with a technique inspired by the work of Anderson in [3], we obtain both these higher order estimates at the same time for the boundary value problem introduced in [9]. Moreover, it seems that our approach could provide similar estimates for other boundary value problems for the Ricci flow as well.

To summarize the result in [9], let \((M, g_0)\) be a compact Riemannian manifold with boundary, \(\gamma(t)\) be any smooth one parameter family of metrics on \(\partial M\) and \(\eta(t)\) be any smooth function on \(\partial M \times [0, \infty)\), satisfying certain zeroth order compatibility conditions. Then, there exists a Ricci flow \(g(t)\) on \(M\), smooth for \(t > 0\), satisfying

\[
[g^T(t)] = [\gamma(t)], \quad \mathcal{H}(g(t)) = \eta(t).
\]

Here, \(g^T\) denotes the induced metric on \(\partial M\), \([\cdot]\) the conformal class, and \(\mathcal{H}(g)\) the mean curvature of \(\partial M\) with respect to \(g\). Moreover, \(g(t) \to g_0\) in the \(C^{1,\alpha}\) Cheeger-Gromov sense.

In [3] Anderson uses the ellipticity of the Ricci tensor in harmonic coordinates and elliptic regularity to obtain a \(C^{1,\alpha}\) compactness result for the class of manifolds with two sided Ricci curvature bounds and an injectivity radius bound. In particular, he introduces the notion of harmonic radius, the maximal radius of geodesic balls on which there exist harmonic coordinates such that the scale-invariant \(C^{1,\alpha}\) norms of the metric components are uniformly bounded. The compactness result is obtained by showing, via a blow up argument, that the Ricci curvature and injectivity radius bounds control the harmonic radius from below. This scheme was later used in [11] and [13] to obtain \(C^{1,\alpha}\) compactness results for manifolds with boundary.

The primary purpose of this paper, and the content of section 3, is to adapt this scheme to the parabolic setting, the Ricci flow, in order to estimate derivatives of the curvature near the boundary and the second fundamental form of the boundary. The motivation comes from the fact that the equivalent Ricci-DeTurck flow satisfies a parabolic boundary value problem for which one has the parabolic regularity estimates of Solonnikov from [19].

We briefly describe the main result. Let \(M\) be a manifold with compact boundary. In the following, \(i_{h,g(0)}\) denotes the size of a collar neighbourhood of the boundary diffeomorphic to \(\partial M \times [0, i_{h,g(0)}]\) via the normal exponential map of \(\partial M\) and \(i_{h,g^T(0)}\) the injectivity radius of the boundary with the induced metric, as described in Definitions 2.1 and 2.2. A Ricci flow with \(A\)-controlled boundary is, briefly, a Ricci flow with a choice \(\gamma(t)\) of representatives of \([g^T(t)]\), comparable to \(g^T(t)\). Moreover, in \(\gamma\)-harmonic coordinates, \(\gamma(t)\) and the mean curvature \(\mathcal{H}(g(t))\) are controlled in the Hölder sense.
of order $m + \epsilon$ and $m - 1 + \epsilon$ respectively, where $m$ is a large integer. See Definition 3.1 for more details.

**Theorem 1.2.** Let $(M, g(t), \gamma(t))$ be a complete Ricci flow with $\Lambda$-controlled boundary, with $t \in [0, T]$. Suppose

1. $|Rm(g(t))|_{g(t)} \leq K$ in $M$ and $|A(g(t))|_{g^\gamma(t)} \leq K$ on $\partial M$ for all $t$.
2. $i_{b,g(0)} \geq i_0$.
3. $inj_{g^\gamma(0)} \geq i_0$.

For any $j = 1, \ldots, m - 2$ and $\tau > 0$, there exists a constant $C = C(n, \tau, \Lambda, l, j, K, i_0) > 0$ such that for all $t \in [\tau, T]$

$$|\nabla^j Rm(g(t))|_{g(t)} \leq C \quad \text{in } M,$$

$$|\nabla^{j+1} A(g(t))|_{g^\gamma(t)} \leq C, \quad \text{on } \partial M.$$

The strategy is to define an analogous notion of “harmonic coordinates” suitable for our purposes, using a local harmonic map heat flow. Then, parabolic theory provides the higher order estimates near the boundary which allow us to obtain a compactness result under the assumption that the “parabolic radius”, the analogue of the harmonic radius, is bounded below. Finally, a blow up argument shows that the parabolic radius is controlled in terms of geometric data, thus obtaining the higher order estimates of Theorem 1.2. Moreover, in Theorem 3.2 we prove a local version of the estimate above.

In section 4 we apply these estimates to obtain a version of Hamilton’s compactness theorem for sequences of Ricci flows on manifolds with boundary, in Theorem 4.1. Then, in section 5 we show that in dimension three it is possible to obtain a priori control of the second fundamental form of the boundary and its derivatives along the Ricci flow, in terms of a curvature bound and some non-collapsing assumptions. This is Theorem 5.1. In particular, this allows us in Corollary 5.2 to improve the continuation principle proven in [9], in that only a curvature bound suffices for the continuation of the flow. Thus, we rule out the possibility that the flow will develop singularities where the second fundamental form blows up in finite time while the curvature remains bounded.

2. Preliminary definitions.

Let $M$ be a $(n + 1)$-dimensional manifold with non-empty boundary $\partial M$. Given $x \in \partial M$, $(M, x)$ will be called a pointed manifold with boundary. Let $h$ be a, possibly incomplete (in the sense of metric spaces), Riemannian metric on $M$. We will denote by $\overline{M}$ and $\partial\overline{M}$ the metric completions of $M$ and $\partial M$ respectively. $(M, h)$ is said to be complete, if $\overline{M} = M$.

**Definition 2.1.** Let $x \in \partial M$.

1. $i_{b,loc,h}(x)$ will denote the maximal number such that the normal exponential map over the geodesic ball $B^g_h(x, i_{b,loc,h}(x))$ on $\partial M$ induces a diffeomorphism between $B^g_h(x, i_{b,loc,h}(x)) \times [0, i_{b,loc,h}(x))$ and its image.
2. $inj_{\partial M,h^\tau}(x)$ will be the maximal radius such that the $h^\tau$-exponential map on $B(0, inj_{\partial M,h^\tau}(x)) \subset \mathbb{R}^n$ is a diffeomorphism onto its image. Similarly, we denote by $inj_{M,h}(x)$ the interior injectivity radius defined at an interior point $x$.

We will also need the concept of the boundary injectivity radius, defined below for complete manifolds with boundary (see also [1]).
Definition 2.2. Let \((M, h)\) be a complete Riemannian manifold with boundary \(\partial M\). The boundary injectivity radius \(i_b\) is the maximal number with the property that the normal exponential map of \(M\) is a diffeomorphism between \(\partial M \times [0, i_b)\) and \(\{x|\text{dist}_h(x, \partial M) < i_b\}\).

Now, let \(g(t)\) be a one parameter family of uniformly equivalent metrics on \(M\), \(t \in (a, b]\) and \(0 \in (a, b]\). Set \(\Sigma = \overline{\partial M} \setminus \partial M\) and define
\[
D_g(x, t) = \min \left\{ \text{dist}_g r(t)(x, \Sigma), (t-a)^{1/2} \right\}.
\]

We will also use the corresponding norms defined in smaller time intervals, namely in the domains \(B(0, r)\) and \(B(0, r)^+\). We are going to need a scale invariant version of the parabolic \(C^{l,l/2}\) norm on \(B(0, r) \times [0, r^2]\) and \(B(0, r)^+ \times [0, r^2]\). In the following definitions, we will denote both parabolic domains by \(G\), for simplicity. First, for any \(\mu \in (0, 1)\) we introduce the notation
\[
\langle v \rangle_{\mu, x} = \sup_{(x, t), (y, t) \in G} \frac{|v(x, t) - v(y, t)|}{|x-y|^\mu},
\]
\[
\langle v \rangle_{\mu, t} = \sup_{(x, t), (x', t') \in G} \frac{|v(x, t) - v(x', t')|}{|t-t'|^\mu}.
\]

Now, fix \(p > n + 3\), \(\epsilon = 1 - \frac{n+3}{2}\) and a large positive integer \(m\). For \(l = m + \epsilon\), the scale invariant \(C^{l,l/2}\) norm on \(G\) is defined by
\[
|v|^*_{l, r} = \sum_{k=0}^m \sum_{2q+|s|=k} r^k |\partial_t^q \partial_x^s v|_{C^0} + \sum_{2q+|s|=m} r^l \langle \partial_t^q \partial_x^s v \rangle_{\epsilon, x} + \sum_{0 < l - 2q - |s| < 2} r^l \langle \partial_t^q \partial_x^s v \rangle_{l-2q-|s|, t}.
\]

We will also use the corresponding norms defined in smaller time intervals, namely in the domains \(B(0, r) \times [(\eta r)^2, r^2]\) and \(B(0, r)^+ \times [(\eta r)^2, r^2]\) which will be denoted by \(|u|^*_{l, r, \eta}\) for \(0 < \eta < 1\).

Conventions. In the following, if the order of some convergence is higher than 3 it will be a function of \(m\) and will be referred as smooth throughout the paper. Also \(Q > 1\) will be a fixed constant and \(a < 0 \leq b\).

3. Boundary estimates.

In this section we prove high order estimates of the curvature close to the boundary, and the second fundamental form of the boundary. Such estimates will be essentially a consequence of parabolic regularity, provided that appropriate boundary data are controlled to higher order.

Here we will consider the boundary data in [9], namely the mean curvature and the conformal class of the boundary. In the following definition we make precise the control we assume on these data.

Definition 3.1. A Ricci flow on a manifold with boundary \((M, g(t))\) has \(\Lambda\)-controlled conformal class and mean curvature on the boundary, if there is a smooth one parameter family \(\gamma(t)\) of metrics on \(\partial M\), such that

1. \([g^T(t)] = [\gamma(t)]\) and \(\Lambda^{-2} \gamma(t) \leq g^T(t) \leq \Lambda^2 \gamma(t)\) for all \(t\).
2. at any \((\bar{x}, \bar{t}) \in \partial M \times (a, b]\) and any \(r \leq \rho_\Lambda(\bar{x}, \bar{t}) = \min\{1, \Lambda^{-1} D_\gamma(\bar{x}, \bar{t})\}\) there exist \(\gamma(t)\)-harmonic coordinates \(u : U \to B(0, r)\) around \(\bar{x}\) in which, after translating time so that \(\bar{t}\) corresponds to \(r^2\), one has the estimates
   a. \(Q^{-1} \delta \leq \gamma(s + \bar{t} - r^2) \leq Q \delta\), in \(B(0, r)\) and \(s \in [0, r^2]\).
\begin{proof}

(b) \(|\gamma_{ij}^s|_{r,s}^* \leq Q\),
(c) \(|H|_{t-1, r,s}^* \leq Q\),

where \(H(g)\) denotes the mean curvature of the boundary.

Such triplet \((M, g(t), \gamma(t))\) will be called a Ricci flow with \(\Lambda\)-controlled boundary.

3.1. Parabolic coordinates.

\begin{definition}
Let \((M, g(t), \gamma(t))\) be a Ricci flow with \(\Lambda\)-controlled boundary and \((\bar{x}, \bar{t}) \in \partial M \times (a, b)\).

Parabolic coordinates of radius \(r\) based at \((\bar{x}, \bar{t})\) of \((g(t), \gamma(t))\) will consist of a pair \((\Omega_s, \phi_s)\), where

1. For \(s \in [0, r^2]\), \(\phi_s : \Omega_s \to B(0, r)^+\) are coordinates on \(M\) around \(\bar{x}\) such that

\[
\frac{d}{ds} \phi_s = \Delta_{g(\bar{t} - r^2 + s)} \phi_s,
\]

\[
\phi_s |_{\Omega_s \cap \partial M} = \phi_0 |_{\Omega_0 \cap \partial M},
\]

2. \(\phi_s |_{\Omega_s \cap \partial M} : \Omega_s \cap \partial M \to B(0, r)\) form \(\gamma(\bar{t})\)-harmonic coordinates on \(\partial M\) around \(\bar{x}\).

To motivate this definition, notice that the push-forward flows \(\hat{g}(s) = (\phi_s)_* g(\bar{t} - r^2 + s)\) on \(B(0, r)^+\), evolve by the parabolic Ricci-DeTurck flow

\[
\frac{d}{ds} \hat{g} = -2 \text{Ric}(\hat{g}) + \mathcal{L}_{\mathcal{W}} \hat{g},
\]

where \(\mathcal{W}^i = \hat{g}^{pq} \hat{\Gamma}^i_{pq}\), see for example [3]. This may be considered as the parabolic analogue of the fact that the Ricci tensor becomes an elliptic operator in harmonic coordinates.

\begin{lemma} [existence of parabolic coordinates]
Given \((M, g(t), \gamma(t))\) and \((\bar{x}, \bar{t}) \in \partial M \times (a, b)\), there exists a small \(r > 0\) and parabolic coordinates \((\Omega_s, \phi_s)\) of radius \(r\) based at \((\bar{x}, \bar{t})\) such that \(\hat{g}(s) = (\phi_s)_* g(\bar{t} - r^2 + s)\) satisfies the bounds

\[
Q^{-1} \delta \leq \hat{g}(s) \leq Q \delta \quad \text{for} \ s \in [0, r^2],
\]

\[
|\hat{g}_{ij}|_{r,s}^* \leq Q,
\]

\[
\sup_{s \in (0,1)} \eta^8 |\hat{g}_{ij}|_{r,s}^* \leq Q,
\]

Proof. Let \(u : U \to B(0, 2r')^+\) be coordinates around \(\bar{x}\) which restrict to \(\gamma(\bar{t})\)-harmonic coordinates on \(\partial M\) and satisfy \(g(\bar{t})_{ij}|_{u=0} = \delta_{ij}\). Then, for small \(r'\) the condition \(Q^{-1} \delta \leq g(\bar{t}) \leq Q \delta\) holds in \(B(0, 2r')^+\). In the following, we use \(u\) to identify \(U\) with \(B(0, 2r')^+\).

Let \(B\) be a domain with smooth boundary such that \(B(0, \frac{r'}{2})^+ \subset B \subset B(0, \frac{5r'}{4})^+\). For appropriate \(r > 0\) which we will define later, consider solutions \(\varphi_s : (B, g(\bar{t} - r^2 + s)) \to (B, \delta)\) of the Dirichlet problem for the harmonic map heat flow

\[
\frac{d}{ds} \varphi_s = \Delta_{g(\bar{t} - r^2 + s)} \varphi_s,
\]

\[
\varphi_0 = \text{id}_B,
\]

\[
\varphi_s |_{\partial B} = \text{id}_{\partial B}.
\]

Since the metric on the target of \(\varphi_s\) is the Euclidean, the flow simplifies to the following initial-boundary value problem for a system of linear heat equations in \(B\).

\[
\frac{d}{ds} \varphi^i_s = g^k_l \partial^2_{kl} (\varphi^i_s) - g^k_l \Gamma^m_{s,kl} \partial_m (\varphi^i_s),
\]

\[
\varphi^i_0 = u^i,
\]

\[
\varphi^i_s |_{\partial B} = u^i |_{\partial B},
\]
Hence, by the embedding $W^2$ follows by parabolic estimates that there is a $C$ such that
\[ ||\varphi_s^i||_{W^2(B \times (0, r^2))} \leq C, \]
where
\[ ||u||_{W^2(B \times (0, r^2))} = \left( \int_{B \times (0, r^2)} (|u|^p + |Du|^p + |D^2u|^p + |\partial_t u|^p) dx dt \right)^{1/p}. \]
Hence, by the embedding $W^{2,1}_p \subset C^{1+\epsilon, \frac{1+\epsilon}{p}}$, for $p > n + 3$ and $\epsilon = 1 - \frac{n+3}{p}$ (see [15] for instance), there exists a $\tau > 0$ such that $\varphi_s$ is a diffeomorphisms for $s \in [0, \tau]$. In particular, $\tau$ does not depend on $r$ or $\bar{t}$ for any given $(g(t), \gamma(t))$. Moreover, a universal $C^{\epsilon / 2}$ bound of $\partial \varphi_s$ in $B \times [0, \tau]$ holds.

To obtain higher order control of $\varphi_s$ consider the map $v = (v^0, \ldots, v^n)$ where $v^i = u^i + s(\varphi_s^i - u^i)$. It satisfies the system
\[ \frac{d}{ds} v^i - \Delta_{g_s} v^i = (\varphi_s^i - u^i) - (1 - s) \Delta_{g_s} u^i \]
and the first order compatibility condition, namely
\[ \frac{d}{ds} v^i \bigg|_{\partial B \times 0} = \Delta g(v^i - u^i)|_{\partial B \times 0} = 0. \]

Thus, by parabolic regularity it belongs to $C^{3+\epsilon, \frac{3+\epsilon}{p}}(B \times [0, \tau])$. This shows that $\varphi_s$ is in $C^{3+\epsilon, \frac{3+\epsilon}{p}}(B \times [\tau', \tau])$ for any $0 < \tau' < \tau$. Similarly, one can show that $\varphi_s$ is in fact smooth for $s > \tau'$.

Now, let $r \leq \min(r', \tau^{1/2})$. Since $\partial_k \varphi_s^i = \frac{1}{s} \partial_k v^i + \frac{s-1}{s} \delta_k^i$, by the definition of the norm $|\cdot|_{2+\epsilon, r, \eta}^s$, there exists a $C = C(g(t), \gamma(t), r') > 0$ such that
\[ |\partial \varphi|_{2+\epsilon, r, \eta}^s \leq \frac{C}{\eta^3}, \]
and $\hat{g}(s) = (\varphi_s)_* g(\bar{t} - r^2 + s)$ satisfies
\[ \sup_{\eta \in (0, 1)} \eta^6 |\hat{g}|_{2+\epsilon, r, \eta}^s < C. \]

Now set, for $s \in [0, r^2]$,
\[ \Omega_s = (\varphi_s \circ u)^{-1}(B(0, r)^+) \]
\[ \phi_s = \varphi_s \circ u|_{\Omega_s}. \]
Thus we obtain parabolic coordinates $(\Omega_s, \phi_s)$ at $(\bar{x}, \bar{t})$. Moreover, by the uniform bounds above choosing $r$ small enough we obtain (3.3). \hfill \Box

**Remark 3.1.** We can define parabolic coordinates at interior points in a similar way. Then, instead of the bound (3.3) we would simply use the scale invariant $C^{2+\epsilon, \frac{2+\epsilon}{2}}$ norm. However, we will not need to do this, as Shi’s local estimates provide higher order estimates of the curvature in the interior.

**Remark 2.2.** The quantity in (3.3) in our case is only there to deal with the lack of higher order compatibility at the boundary, while solving the harmonic map heat flow.
In analogy to the elliptic setting where one has the concept of the harmonic radius, introduced in [3], we give the following definition.

**Definition 3.3 (Parabolic radius).** Consider a Ricci flow \((g(t), \gamma(t))\) on \(M\) with \(\Lambda\)-controlled boundary and let \((\vec{x}, \vec{t}) \in \partial M \times (a, b]\). We define the parabolic radius \(r_p^Q(\vec{x}, \vec{t})\) to be the maximal \(r > 0\) for which there exist parabolic coordinates \((\Omega, \phi)\) of radius \(r\) based at \((\vec{x}, \vec{t})\) such that the bounds (3.1)-(3.3) hold.

### 3.2. Curvature bounds

In the following lemma we show that on a Ricci flow with \(\Lambda\)-controlled boundary a lower bound of the parabolic radius yields bounds on derivatives of the ambient curvature \(Rm\) and the second fundamental form \(\delta\).

**Lemma 3.2.** There exists an \(\alpha = \alpha(Q) \in (0, 1)\) such that if \((M, g(t), \gamma(t))\) is a Ricci flow with \(\Lambda\)-controlled boundary and \((\vec{x}, \vec{t}) \in \partial M \times (a, b]\) then, for any \(\eta \in (0, 1)\) and \(0 \leq j \leq m - 2\), the estimates

\[
|\nabla^j Rm(g(t))|_{g(t)} \leq C(Q, \Lambda, l, j, \eta, \alpha)r_p^{-j+2}
\]

and

\[
|\nabla^{j+1} A(g(t))|_{g(t)} \leq C(Q, \Lambda, l, j, \eta, \alpha)r_p^{-j+3},
\]

hold for each \(t \in \left[\vec{t} - (\eta r_p)^2, \vec{t}\right]\) in \(B_{g(t)}(\vec{x}, \alpha r_p)\), where \(B_{g(t)}(\vec{x}, r)\) denotes a \(g(t)\)-distance ball in \(M\) centered at \(\vec{x}\) and \(r_p = r_p(Q, \vec{x}, \vec{t})\), the parabolic radius at \((\vec{x}, \vec{t})\).

**Proof.** We will assume that \(\vec{t} = 0\) and suppress the reference to \((\vec{x}, 0)\) and \(Q\). Moreover, by rescaling we may also assume that \(r_p = 1\).

The proof is essentially an application of parabolic regularity theory. Since \(r_p = 1\), there are parabolic coordinates \((\Omega, \phi)\) based at \((\vec{x}, 0)\) which define a flow \(\hat{g}(s) = (\phi_*)g(s - 1)\) on \(B(0, 1)^+\). Moreover, this flow satisfies the Ricci-DeTurck equation

\[
\partial_s \hat{g} = -2 \text{Ric}(\hat{g}) + \mathcal{L}_{W(\hat{g})} \hat{g},
\]

and the conditions

\[
W(\hat{g}(s)) = 0,
\]

\[
\mathcal{H}(\hat{g}(s)) = \eta(u^1, \ldots, u^n, s),
\]

\[
\hat{g}(s)^T - \frac{\text{tr}_s \hat{g}(s)^T}{n} \gamma(s) = 0,
\]

on \(B(0, 1)^+ \cap \{x^0 = 0\}\), \(s \in (0, 1]\) and \(W(\hat{g})^i = \hat{g}^{pq} \delta^i_{pq}\).

By the assumption on the parabolic radius \(\hat{g}\) satisfies the estimates (3.1)-(3.3), for \(r = 1\). Moreover, in the coordinates \(\phi_\|_{\partial M}, \gamma\) and \(\mathcal{H}\) satisfy bounds

\[
\Lambda^{-2}Q^{-1}\delta \leq \gamma \leq \Lambda^2 Q\delta,
\]

\[
|\gamma_{ij}|_{l, 1}^r \leq C(Q, \Lambda),
\]

\[
|\mathcal{H}|_{L, 1, 1}^r \leq C(Q, \Lambda).
\]

Thus, by parabolic regularity, for any \(\eta \in (0, 1)\) we obtain uniform \(C^{l,1/2}\) estimates of \(\hat{g}(s)\) on \(B(0, \alpha^*)^+ \times [1 - \eta^2, 1]\) for some \(\alpha^*(Q) \in (0, 1)\). Namely, we use cutoff functions to extend the Ricci DeTurck flow on \(B(0, 1)^+\) to a boundary value problem on the full \(\mathbb{R}^{n+1}_+\), as is done in [9]. This brings us exactly to the setting of [19]. Note that the argument in [9] requires \(\hat{g}(0)_{ij} = \delta_{ij}\) at \(u = 0\). However, we can achieve this by applying a linear transformation, which will be controlled in terms of \(Q\), by (3.1).
The higher order estimates of the curvature and the second fundamental form of $\tilde{g}(s)$ on $B(0, \alpha')^+$ follow immediately. Namely, we obtain

$$|\tilde{\nabla}^j \text{Rm}(\tilde{g}(s))|_{\tilde{g}(s)} \leq C(Q, \Lambda, l, j, \eta)$$

and

$$|\tilde{\nabla}^{j+1} A(\tilde{g}(s))|_{\tilde{g}(s)} \leq C(Q, \Lambda, l, j, \eta),$$

for $0 \leq j \leq m - 2$ and $s \in [1 - \eta^2, 1]$.

Now, $\ref{3.1}$ implies that $B_{\tilde{g}(s)}(0, \alpha'Q^{-1/2})^+ \subset B(0, \alpha')^+$. On the other hand, $B_{\tilde{g}(s)}(0, \alpha'Q^{-1/2})^+$ is isometric to the $g(s - 1)$-distance ball $B_{\tilde{g}(s-1)}(\tilde{x}, \alpha'Q^{-1/2})$ via $\phi_s$, which means that for any $t \in [-\eta^2, 0]$, $x \in B_{\tilde{g}(t)}(\tilde{x}, \alpha'Q^{-1/2})$ and $x' \in B_{\tilde{g}(t)}(\tilde{x}, \alpha'Q^{-1/2}) \cap \partial M$ one has the estimates

$$|\tilde{\nabla}^j \text{Rm}(g(t))(x, t)|_{g(t)} \leq C(Q, \Lambda, l, j, \eta),$$

$$|\tilde{\nabla}^{j+1} A(g(t))(x', t)|_{g(t)} \leq C(Q, \Lambda, l, j, \eta).$$

Setting $\alpha = \alpha'Q^{-1/2}$ we obtain the statement of the lemma. \hfill $\Box$

3.3. **Compactness.** Since we deal with incomplete manifolds, we define below a weak notion of Cheeger-Gromov convergence suitable for this setting. Moreover, we prove a compactness theorem which will allow us to extract good limits of sequences of incomplete Ricci flows with boundary. Such incomplete limits are not going to be unique. However, the compactness result will provide a natural criterion that guarantees the completeness of the limit.

**Definition 3.4 (Cheeger-Gromov convergence-weak form).** Let $(M_k, g_k(t), x_k)$, $(M_\infty, g_\infty(t), x_\infty)$ be, possibly incomplete, Ricci flows on pointed manifolds with boundary $M_k, M_\infty$, $t \in (a, b]$. Let $\gamma_k(t), \gamma_\infty(t)$ be one parameter families of metrics on $\partial M_k, \partial M_\infty$ such that $[g_k^T(t)] = [\gamma_k(t)]$ and $[g_\infty^T(t)] = [\gamma_\infty(t)]$.

We will say that $(M_k, g_k(t), \gamma_k(t), x_k)$ converge in the pointed $C^{m+1}$ Cheeger-Gromov sense to $(M_\infty, g_\infty(t), \gamma_\infty(t), x_\infty)$ if there exists an exhaustion $\{K_k\}$ of $M_\infty$ with compact sets and $C^{m, \alpha}$ diffeomorphisms $F_k : K_k \to F_k(K_k)$ such that

1. $F_k(x_\infty) = x_k$,
2. $F_k|_{K_k \cap \partial M_\infty} : K_k \cap \partial M_\infty \to F_k(K_k \cap \partial M_\infty) \subset \partial M_k$ is a diffeomorphism,
3. $F_k^* g_k(t)$ is smooth and locally in $M_\infty \times (a, b]$ in $C^m$.
4. $F_k^* \gamma_k(t)$ is smooth and locally in $\partial M_\infty \times (a, b]$ in $C^m$.

The $C^m$ topology here corresponds to $q$ space and $\tau$ time continuous derivatives, for $2 \tau + q \leq m$.

**Notation.** We will adopt the notation $(M_k, g_k(t), \gamma_k(t), x_k) \to_{\text{w}} (M_\infty, g_\infty(t), \gamma_\infty(t), x_\infty)$ for such convergence.

**Theorem 3.1 (compactness theorem).** Let $(M_k, g_k(t), \gamma_k(t), x_k)$ be a sequence of pointed Ricci flows with $\Lambda$-controlled boundary, $t \in (a, b]$. Suppose that for all $k$

1. $|\text{Rm}(g_k(t))| \leq K$ in $M_k \times (a, b]$,
2. $i_{b, \text{loc}, g_\infty(0)}(x_k) \geq 4i_0$,
3. $i_{g_k^T(0)}(x_k) \geq i_0$,
4. $r_{p, g_k(x, t)} \geq \zeta \left( \frac{D_{g_k(x, t)}}{\text{dist}_{g_k(0)}(x, x_k)} \right)$ for all $(x, t) \in \partial M_k \times (a, b]$,

for some $K, i_0 > 0$ and a positive nondecreasing continuous function $\zeta$ with $\zeta(0) = 0$.

Then there is a Ricci flow on a pointed manifold with boundary $(M_\infty, g_\infty(t), x_\infty)$ and a one parameter family of metrics $\gamma_\infty(t)$ on $\partial M_\infty$ such that
(1) Up to subsequence, \((M_k, g_k(t), \gamma_k(t), x_k) \to (M_\infty, g_\infty(t), \gamma_\infty(t), x_\infty)\) in the \(C^{m-3}\) topology.

(2) If \(i_{b,loc,g_k(0)}(x_k) \to 0\) then \((M_\infty, g_\infty(t))\) is complete and noncompact.

Before we are able to prove the compactness result, we will first need the following lemma. Suppose \((N, h)\) is a Riemannian manifold with boundary, possibly incomplete. Lemma 3.3 essentially reduces the control of the interior injectivity radius of \(N\) to that of its boundary, at least finite distance away from it (see also [13]).

For \(0 < r' < r < i_{b,loc,h}(x)\), denote by \(A(x, r', r)\) the image of \(B_h^0(x, r) \times \{s \nu, s \in [r', r]\}\) under the normal exponential map of the boundary, where \(\nu\) is the inward pointing unit normal.

**Lemma 3.3.** For any \(y \in A(x, r', r)\) there exist positive constants \(\kappa\) and \(\rho_0\), depending on \(i_{b,loc,h}(x)\), \(r', r\), a lower bound on \(\text{vol}(B_h^0(x, i_{b,loc,h}(x)))\), bounds of \(|\text{Rm}(h)|\) in \(A(x, 0, i_{b,loc,h}(x))\) and \(|\text{A}(h)|\) on \(B_h^0(x, i_{b,loc,h}(x))\), such that
\[
\frac{\text{vol}(B_h(y, \rho))}{\rho^{n+1}} \geq \kappa,
\]
for all \(\rho \leq \rho_0\).

**Proof.** It suffices to prove the statement for \(\tilde{y} = \exp_x\left(\frac{i_{b,loc,h}(x)}{2} \nu\right)\) since the full statement follows by volume comparison.

In appropriate coordinates on \(A(x, 0, r)\) we have \(h = dr^2 + h_r\), where \(h_r\) are metrics on \(B_h^0(x, i_{b,loc,h}(x))\), \(r\) here denoting distance from \(\partial N\). Setting \(\tilde{h} = dr^2 + h_0\), the bounds for the curvature and the second fundamental form provide a uniform \(C > 1\) such that \(C^{-1} \tilde{h} \leq h \leq C \tilde{h}\). Hence, there is a \(\rho_0 = \rho_0(C)\) such that, for all \(0 < \rho \leq \rho_0\), \(B_{\tilde{y}h}(\rho) \subset A(x, 0, r)\) and
\[
\frac{\text{vol}_h B_{\tilde{y}h}(\tilde{y}, \rho)}{\rho^{n+1}} \geq C^{-(n+1)} \frac{\text{vol}_h(B_{\tilde{y}h}(\tilde{y}, \rho))}{\rho^{n+1}} \geq C^{-(n+1)} \frac{\text{vol}_h(B_{\tilde{y}h}(\tilde{y}, C^{-1}(n+1)))}{\rho^{n+1}} \geq \kappa(C, v_0),
\]
where \(\frac{\text{vol}_h(B_{\tilde{y}h}(x, \rho))}{\rho^n} \geq v_0\) for all \(0 < \rho < i_{b,loc,h}(x)\). \(\square\)

**Proof of Theorem 2.7** The manifolds involved are incomplete and at this stage we only need the existence of a limit with the properties of the theorem. Thus we will make the assumption that each \(M_k\) is just the image \(C_k(x_k, 4i_0)\) of \(B_{g_k}^0(x_k, 4i_0) \times \{s \nu, s \in [0, 4i_0]\}\) under the normal exponential map with respect to \(g_k(0)\). Namely, we denote \(C(x, r') = A(x, 0, r)\).

Then, the assumption on the parabolic radius implies a uniform bound on the second fundamental form \(A(g_k(0))\) on \(B_{g_k(0)}^0(x_k, 2i_0)\). This, together with assumptions (1) and (3), controls \(g_k(0)\) in \(C_k(x_k, 3i_0)\) in the \(C^0\) sense (as in the proof of Lemma 3.3) thus the sets \(C_k(x_k, 2i_0)\) remain a uniform distance away from \(C_k(x_k, 3i_0)\).\(\cap C_k(x_k, 3i_0)\).

Now, by Shi’s estimates and Lemma 3.2 we obtain uniform control on derivatives of \(\text{Rm}(g_k(0))\) and \(A(g_k(0))\) in \(C_k(x_k, 2i_0)\). Also, the curvature bound and Lemma 3.3 controls the injectivity radius of interior points in \(C_k(x_k, 2i_0)\) in terms of their distance from \(B_{g_k}^0(x_k, 2i_0)\). The results in \(3\) and elliptic regularity then show that, for every \(\alpha \in (0,1)\), the \(C^{m-1,\alpha}\) harmonic radius of the interior points of \(C_k(x_k, i_0)\) is also controlled, in terms of their \(g_k(0)\)-distance from \(\partial M_k\).

Moreover, the work in \([1]\) implies that the \(C^{1,\alpha}\) boundary harmonic radius of points on the boundary is also uniformly bounded below.

Regarding the boundary, our assumptions provide control of the curvature up to order \(m-2\), via the Gauss equation. By \([3]\), it follows that in harmonic coordinates of uniform size the metrics \(g_k^T(0)\) are controlled in \(C^{m-1,\alpha}\).
By elliptic regularity (with the conformal class and mean curvature as boundary conditions) we obtain $C^{m-1,\alpha}$ control of $g_k$ in the interior, in boundary harmonic coordinates and a smaller domain.

The discussion above shows that $C_k(x_k, i_0)$ can be covered with harmonic coordinates of controlled size on which $g_k(0)$ is $C^{m-1,\alpha}$-controlled. Hence, standard arguments imply that $(C_k(x_k, i_0), g_k(0), x_k)$ converge, up to subsequence, to an incomplete limit $(M_\infty, x_\infty, g_\infty(0))$ in the $C^{m-1,\alpha}$ Cheeger-Gromov sense of Definition 3.1. For instance, see [10] and section 2.6 of [2] for sequences of incomplete manifolds.

Then, using the curvature bounds of Lemma 3.2 and Arzelà-Ascoli as in [10], we obtain that $F_k^* g_k(t) \to g_\infty(t)$ in the parabolic $C^{m-3}$ topology, i.e. one time derivative counts for two spatial.

If $i_{b,\text{loc},g_k(0)}(x_k) \to \infty$, we can consider a sequence $r_i \to \infty$, repeat the steps above to show that $C_k(x_k, r_i)$ converge as $k \to \infty$ and then take a diagonal subsequence to obtain a noncompact complete limit.

To show requirement (4) of Definition 3.1, first note that in the proof of Lemma 3.2 we obtained higher order estimates for the transition functions from $g_\infty(0)$ in parabolic coordinates at any $(\bar{x}, \bar{t}) \in \partial M \times (a,b)$. These coordinates restrict to $\gamma(\bar{t})$-harmonic coordinates around $\bar{x}$. Moreover, as the flow of the DeTurck vector field fixes the boundary, we also obtain higher order $C^{l,1/2}$ estimates of $g^T(t)$ in $\gamma(\bar{t})$-harmonic coordinates and $t \in (\bar{t} - r_\bar{t}^2, \bar{t})$.

Note that these estimates hold regardless the time $\bar{t}$ we choose for the $\gamma(\bar{t})$-harmonic coordinates, since $\gamma$ is controlled in $C^{l,1/2}$. Hence we have uniform $C^{m,\epsilon}$ control of $g^T(0)$ in $\gamma(0)$-harmonic coordinates.

By Schauder estimates, the transition functions from $g^T(0)$-harmonic coordinates to $\gamma(0)$-harmonic coordinates are controlled. Therefore, $\gamma(0)$ is controlled in $g^T(0)$-harmonic coordinates as well, in $C^{m,\epsilon}$. We also obtain $C^{l,1/2}$ control in the same coordinates.

Applying this for the sequence $(g_k(t), x_k(t))$ we obtain a limit $\gamma_\infty(t)$ by Arzelà-Ascoli, and (1) holds.

3.4. Lower semicontinuity of the parabolic radius.

**Lemma 3.4.** The parabolic radius is lower semicontinuous with respect the Cheeger-Gromov topology in the sense that if $(M_k, g_k(t), x_k(t)) \to (M_\infty, g_\infty(t), x_\infty(t))$ and $1 < Q' < Q$ then

$$\liminf_k r^Q_{p,g_k}(x_k, \bar{t}) \geq r^Q_{p,g_\infty}(x_\infty, \bar{t}) \quad \text{for all } \bar{t} \in (a,b).$$

**Proof.** For simplicity, let $\bar{t} = 0$ and denote $r_k = r^Q_{p,g_k}(x_k, 0)$, $r_\infty = r^Q_{p,g_\infty}(x_\infty, 0)$. Also, assume that $g_k(t)$ and $x_k(t)$ are defined on compact subsets $C \subset M_\infty$ and $C \cap \partial M_\infty$, that $x_k = x_\infty := \bar{x}$, and $g_k(t), x_k(t)$ converge to $g_\infty(t), x_\infty(t)$ smoothly and uniformly in compact subsets of $M_\infty$.

Let $r_\sigma = r_\infty - \sigma$, for $\sigma > 0$ small. The goal is to construct parabolic coordinates $(\Omega^k_s, \phi^k_s)$ of radius $r_\sigma$ at $(\bar{x}, 0)$ for the Ricci flow $(g_k(t), x_k(t))$ such the bounds (3.1)-(3.2) hold.

Consider $(g_\infty(t), x_\infty(t))$-parabolic coordinates $(\Omega_s, \phi_s)$ of radius $r_\infty$, $\phi_s : \Omega_s \to B(0, r_\infty)^+$, based at $(\bar{x}, 0)$ in which the bounds (3.1)-(3.2) hold for $Q' < Q$.

Let $u = (u^0, \ldots, u^n)$ denote the coordinates in $\mathbb{R}^{n+1}$, where superscripts $1 \leq i \leq n$ correspond to the directions along the boundary of the upper-half space $\mathbb{R}^{n+1}_+$. Now, let $u^1_k, \ldots, u^n_k$ be solutions to the following Dirichlet problems in $B(0, r_\sigma)$

$$\Delta_{x_k(0)} u^i_k = 0,$$

$$u^i_k |_{\partial B(0, r_\sigma)} = u^i.$$
By the convergence of $\gamma_k(0)$ to $\gamma_\infty(0)$ it follows that in the $u$-coordinates $|(\Delta_{\gamma_k} - \Delta_\infty)(f)|_{C^{3,\epsilon}} \to 0$, for any fixed function $f$ on $B(0, r_\sigma)$. Moreover, the functions $u_k^i - u^i$ satisfy

$$\Delta_{\gamma_k}(u_k^i - u^i) = (\Delta_{\gamma_k}(0) - \Delta_{\gamma_\infty}(0))u^i,$$

$$u_k^i - u^i|_{\partial B(0,r_\sigma)} = 0.$$

Since $\gamma_k(0)$ are uniformly bounded in $C^{2,\epsilon}$, by Schauder estimates we obtain that $|u_k^i - u^i|_{C^{3,\epsilon}} \to 0$. In particular, for large $k$, $u_k^i$ form harmonic coordinates for $\gamma_k$ and converge to $u^i$.

Now, let $\tau = r_\sigma^2 - r_\sigma^2$. For any $k$ and $s \in [0, r_\sigma^2]$ set $\Omega^k_s = \phi^{-1}_s(B(0,r_\sigma)^+) \subset \Omega_{s+\tau}$. We wish to construct $\phi^k_s : \Omega^k_s \to B(0, r_\sigma)^+$ satisfying

$$\frac{d}{ds} \phi^k_s = \Delta g_k(s-r_\sigma^2) \phi^k_s,$$

$$\phi^k_0 = \phi_{r_\sigma},$$

$$\phi^k_s|_{\partial \Omega^k_s} = \phi_{s+\tau}|_{\partial \Omega^k_s}.$$

In order to solve (3.4) we will use $\phi_{s+\tau}$ to pull-back the problem in a domain $B$ with smooth boundary, such that $B(0, r_\sigma)^+ \subset B \subset B(0, r_\infty)^+$, so we set $h_{k,\sigma}(s) = (\phi_{s+\tau})^*(g_k(s-r_\sigma^2))$. Note that $\phi_{s+\tau}$ is smooth even when $s = 0$. Now, to solve the problem (3.4)-(3.6) it is enough to solve the following.

$$\frac{d}{ds} \chi_k = \Delta h_{k,\sigma}(s) \chi_k + D\chi_k(Y(s)),$$

$$\chi_k|_{s=0} = \text{id}_{B_\sigma},$$

$$\chi_k|_{\partial B(0, r_\sigma)^+} = \text{id}_{\partial B_\sigma},$$

where $Y(t) = D\phi_{s+\tau} \frac{d}{ds} \phi^{-1}_{s+\tau}$ and $\chi_k(s) = \phi^k_s \circ \phi^{-1}_{s+\tau}|_{B_\sigma}$.

Now, let $h_s = (\phi_{s+\tau})^*(g_{\infty}(s-r_\sigma^2))$. The difference $\chi_k - u$ satisfies the linear problem

$$\frac{d}{ds}(\chi_k^i - u^i) - \Delta h_{k,\sigma}(\chi_k^i - u^i) - (D(\chi_k-u)(Y))^i = (\Delta h_{k,\sigma} - \Delta h_s) u^i,$$

$$\chi_k - u|_{s=0} = 0,$$

$$\chi_k - u|_{\partial B_\sigma} = 0.$$

Since $h_{k,\sigma} \to h_s$ smoothly enough, it follows from parabolic estimates that

$$|\chi_k - u|_{C^{3,\epsilon}} \to 0,$$

$$|\chi_k - u|_{C^{3,\epsilon}} \to 0,$$

uniformly for any $\eta \in (0, 1)$. In particular, $\phi^k_s$ are diffeomorphisms for large $k$.

Now, $(u_0^0, u_k^i, \ldots, u_k^i) \circ \phi^k_s$ define parabolic coordinates of radius $r_\sigma$. Moreover, (3.7) and (3.8) imply the bounds

$$Q^{-1} \delta \leq \hat{g}^k_s(s) \leq Q \delta$$

for $s \in [0, r_\sigma^2]$, and

$$|\hat{g}_{ij}|_{C^{3,\epsilon}} \leq Q,$$

$$\sup_{s \leq (0, 1)} \eta^k |\hat{g}_{ij}|_{C^{3,\epsilon}} \leq Q,$$

for large $k$, since $Q' < Q$. Hence $\lim \inf r_k \geq r_\sigma = r_\infty - \sigma$ for every $\sigma > 0$. 

\qed
3.5. Controlling the parabolic radius. In the following we show that certain geometric bounds suffice to control the parabolic radius from below. The proof is via a blow up argument, essentially on the same line as in [3], [1] and [13].

Lemma 3.5. Let \((g(t), \gamma(t))\) be a Ricci flow on \(M\) with \(\Lambda\)-controlled boundary, and assume the bounds

\[
\begin{align*}
(1) \quad & |Rm(g(t))| \leq K. \\
(2) \quad & |A(g(t))| \leq K. \\
(3) \quad & \text{inj}_{g(t)}(x) \geq i_0 D_g(x, t) \quad \text{and} \quad \text{inj}_{b,loc,g(t)}(x) \geq i_0 D_g(x, t) \quad \text{for all} \ (x, t) \in \partial M \times (a, b]. \\
(4) \quad & \text{diam}(\partial M, g^T(t)) \leq D.
\end{align*}
\]

Then, if \(\Lambda > 0\) is as in Definition 3.1, there exists \(c = c(K, i_0, D, \Lambda, l) > 0\) such that

\[
\frac{r^Q_p(x, t)}{D'_g(x, t)} \geq c.
\]

Proof. It suffices to show that given any precompact domain \(V \subset \partial M\) and \([a', b] \subset (a, b]\) the estimate

\[
\frac{r^Q_p(x, t)}{D'_g(x, t)} \geq c(1, K_2, i_0, D, \Lambda, l) > 0
\]

holds in \(V \times (a', b]\), where \(D'_g(x, t) = \min\{\text{dist}_{g^T(t)}(x, \bar{V} \setminus V), (t-a')^{1/2}\}\), assuming that condition (3) holds in \(V\) with \(D'_g\) replaced by \(D'_q\).

Suppose there is a sequence of counterexamples, namely manifolds with boundary \(M_k\), Ricci flows \((g_k(t), \gamma_k(t))\) with \(\Lambda\)-controlled boundary satisfying bounds (1)-(3) and \(V_k \times [a'_k, b] \subset \partial M \times (a, b]\), with \((y_k, t_k) \in V_k \times [a'_k, b]\) such that

\[
\frac{r^Q_{p,g_k}(y_k, t_k)}{D'_{g_k}(y_k, t_k)} = \epsilon_k \to 0.
\]

Since \(r^Q_{p,g_k}\) is bounded away from zero in \(V_k \times [a'_k, b]\) for any \(k\), by Lemma 3.4 we may assume that

\[
\frac{r^Q_{p,g_k}(x, t)}{D'_{g_k}(x, t)} \geq \frac{r^Q_{p,g_k}(y_k, t_k)}{D'_{g_k}(y_k, t_k)} \quad \text{for all} \ (x, t) \in V_k \times (a'_k, b].
\]

Moreover, \(r_k := r^Q_{p}(y_k, t_k) \to 0\).

Consider the pointed sequence \((M_k, y_k)\) and the rescaled flows \((h_k(t), \gamma_k(t))\), where

\[
\begin{align*}
\frac{h_k(t)}{r_k} &= g_k(t + tr_k^2), \\
\frac{\gamma_k(t)}{r_k} &= \gamma_k(t + tr_k^2).
\end{align*}
\]

These rescaled flows also have \(\Lambda\)-controlled boundary, and in addition they satisfy

\[
\begin{align*}
i. \quad & |Rm(h_k(t))| \to 0. \\
i. \quad & \text{A}_k(h_k(t)) \to 0. \\
i. \quad & r^Q_{p,h_k}(y_k, 0) = 1, \text{ since the parabolic radius scales like distance.} \\
i. \quad & \text{inj}_{V_k,h_k}(y_k) \geq i_0 D'_{h_k}(y_k, 0) \to \infty. \\
i. \quad & \text{inj}_{b,loc,h_k}(y_k) \geq i_0 D'_{h_k}(y_k, 0) \to \infty. \\
i. \quad & t \in \left(\frac{k-a'_k}{r^2_k}, \frac{b-t_k}{r^2_k}\right) \text{ and } \frac{k-a'_k}{r^2_k} \to \infty.
\end{align*}
\]

Moreover, we have \(r^Q_{p,h_k}(x, t) \geq \frac{D'_{h_k}(x, t)}{D'_{g_k}(y_k, 0)} r^Q_{p,h_k}(y_k, 0)\) which gives

\[
\frac{r^Q_{p,h_k}(x, t)}{r^Q_{p,h_k}(y_k, 0)} \geq c(\text{dist}_{h_k(0)}(x, y_k)),
\]

uniformly for large \(k\) on \(V_k \times [-1, 0]\).
By Theorem 3.1 the rescaled flows in $V_b \times (-1, 0]$ have a Cheeger-Gromov limit, a pointed manifold with boundary $(M_\infty, y_\infty)$ with a complete Ricci flow $h_\infty(t)$. Moreover, $(M_\infty, h_\infty)$ is flat and $\partial M_\infty$ is totally geodesic (and flat, from the Gauss equation). Since $\text{inj}_{V_b, h_\infty^T} \to \infty$ we conclude that $\partial M_\infty$ is isometric to $(\mathbb{R}^n, \delta)$.

Now, by (i), (ii) and (v) above and comparison geometry we have that $M_\infty$ is isometric to $(\mathbb{R}^{n+1}, \delta)$. Namely, the second fundamental form of the level sets of the distance functions from the boundary will converge to zero, uniformly in fixed distance from the boundary. The claim follows, as these level sets converge smoothly to the corresponding level sets in the limit.

Finally, $r_{p,\delta}^{Q''}(0, t) = \infty$, for any $1 < Q'' < Q'$, which contradicts Lemma 3.1 and the fact that $r_{p,\delta}^{Q'}(y_k, 0) = 1$. □

We finish this section by putting together Lemmata 3.2 and 3.5 to obtain local higher order estimates up to the boundary for the Ricci flow.

**Theorem 3.2.** Let $(M, g(t), \gamma(t))$ be a, possibly incomplete, Ricci flow with $\Lambda$-controlled boundary, with $t \in [0, T]$. Suppose

(1) $|\text{Rm}(g(t))|_{g(t)} \leq K$ in $M$ and $|\text{A}(g(t))|_{g^T(t)} \leq K$ on $\partial M$ for all $t$.

(2) $i_{b,\text{loc},g(t)}(q) \geq i_0$, for all $t \in [0, T]$.

(3) $\text{inj}(g^T(0)) \geq i_0$.

For any $j = 1, \ldots, m - 2$ and $\tau > 0$, there exists a constant $C = C(n, \tau, T, \Lambda, l, j, K, i_0) > 0$ such that for $t \in [\tau, T]$

(3.9) \[ |\nabla^j \text{Rm}(g(t))|_{g(t)} \leq C, \quad \text{in} \ C(q, i_0/2). \]

\[ |\nabla^{j+1} \text{A}(g(t))|_{g^T(t)} \leq C, \quad \text{on} \ C(q, i_0/2) \cap \partial M. \]

**Proof.** First we observe that for all $x \in B_{g(0)}^\delta(q, r_0), r_0 < i_0$,

(a) $i_{b,\text{loc},g(t)}(x) \geq i_0 - r_0$, for all $t \in [0, T]$.

This follows directly from the definition of $i_{b,\text{loc},g(t)}(q)$.

(b) $\text{inj}(g^T(t)) \geq c(i_0, r_0, K)$.

This follows from [4], since the Gauss equation gives a bound on the curvature of $(\partial M, g^T(t))$ and the fact that all the metrics $g^T(t)$ are comparable gives a uniform lower bound on the volume ratio of balls centered at $x$.

By Lemma 3.5 the parabolic radius is bounded below uniformly in intervals $[\tau, T]$ in the ball $B_{g(0)}^\delta(q, i_0/2)$. Therefore, there is a $\delta$-neighbourhood of $B_{g(0)}^\delta(q, i_0/2)$ in $M$ (with respect to the $g(0)$ metric) where estimates (3.9) hold for some $C$, by Lemma 3.2.

This proves that the required estimates hold on a neighbourhood of the boundary in $C(q, i_0/2)$. Shi’s interior estimates then handle the higher derivatives of the curvature in $C(q, i_0/2)$ away from the boundary. □

**Theorem 1.2** is the global version of the result above.

**Proof of Theorem 1.2.** The proof is essentially the same as that of Theorem 3.2. We only need to show that we can estimate on $i_{b,g(t)}$ in terms of $K$ and $i_0$.

Let $\nu(x, t)$ be the inward pointing unit normal to $\partial M$ and $\exp^t$ the exponential map with respect to the metric $g(t)$. We consider the geodesics $\gamma_{x,t}(s) = \exp^t_x(s\nu(x, t))$ and their $g(t)$-length $L_t(\gamma_{x,t})$.
Observe that there exists a $c(K) > 0$ such that the geodesics above don’t have focal points for $s < c(K)$. It follows that whenever $i_{b,g(t)} < c(K)$

$$i_{b,g(t)} = \frac{1}{2} \min \{ L_t(\gamma_{x,t}), x \in \partial M \}.$$  

In particular, there exists an $x_t \in \partial M$ such that $i_{b,g(t)} = \frac{1}{2}L_t(\gamma_{x_t,t})$. By the first variation formula of length $\gamma_{x_t,t}$ is $g(t)$-perpendicular to $\partial M$.

We define

$$\frac{d}{dt} i_{b,g(t)} = \liminf_{h \to 0^+} \frac{i_{b,g(t-h)} - i_{b,g(t)}}{h}.$$  

Since $\gamma_{x_t,t}$ are geodesics perpendicular to $\partial M$, we obtain that

$$\frac{d}{dt} \bigg|_{t=0} i_{b,g(t)} \geq \frac{d}{dt} \bigg|_{t=0} L_t(\gamma_{x_{t_0},t}) \geq C(n,K)L_{t_0}(\gamma_{x_{t_0},t_0}) = C(n,K)i_{b,g(t_0)},$$

where the last inequality follows from the curvature bound and the Ricci flow equation.

Therefore, using (3.10) we obtain control of the boundary injectivity radius for $t > 0$. □

4. A COMPACTNESS THEOREM.

In this section we prove a version of Hamilton’s compactness theorem for sequences of Ricci flows on manifolds with boundary. Below we define the notion of convergence we use, along the lines of [1].

**Definition 4.1** (Cheeger-Gromov convergence-strong form). Let $(M_k,g_k(t),x_k), (M_\infty,g_\infty(t),x_\infty)$ be complete Ricci flows on pointed manifolds with boundary $M_k, M_\infty$, $t \in (a,b]$. Let $\gamma_k(t), \gamma_\infty(t)$ be one parameter families of metrics on $\partial M_k, \partial M_\infty$ such that $[g_k^T(t)] = [\gamma_k(t)]$ and $[g_\infty^T(t)] = [\gamma_\infty(t)]$.

We will say that $(M_k, g_k(t), \gamma_k(t), x_k)$ converge in the strong pointed $C^m$ Cheeger-Gromov sense to $(M_\infty, g_\infty(t), \gamma_\infty(t), x_\infty)$ if there exists a sequence $R_k \to +\infty$ and an exhaustion $\{K_k\}$ of $M_\infty$ of compact sets such that $B_{g_\infty(0)}(x_\infty, R_k) \subset K_k$ and $C^{m+1}$ diffeomorphisms $F_k : K_k \to F_k(K_k)$ such that

1. $F_k(x_\infty) = x_k$.
2. $B_{g_k(0)}(x_k, R_k) \subset F_k(K_k)$.
3. $F_k|_{K_k \cap \partial M_\infty} : K_k \cap \partial M_\infty \to F_k(K_k \cap \partial M_\infty) \subset \partial M_k$ is a diffeomorphism.
4. $F_k^* g_k(t) \to g_\infty(t)$ smoothly and locally in $M_\infty \times (a,b]$ in $C^m$.
5. $F_k^* \gamma_k(t) \to \gamma_\infty(t)$ smoothly and locally in $\partial M_\infty \times (a,b]$ in $C^m$.

For such convergence we write

$$(M_k, g_k(t), \gamma_k(t), p_k) \to (M_\infty, g_\infty(t), \gamma_\infty(t), p_\infty).$$

**Remark 4.1.** Unlike Definition 3.1, Cheeger-Gromov limits in the strong sense defined above are complete and unique.

**Theorem 4.1.** Let $(M_k, p_k)$ be a pointed sequence of manifolds with boundary, and $(g_k(t), \gamma_k(t))$ be complete Ricci flows on $M_k$, $t \in (a,b]$ with $\Lambda$-controlled boundary. Assume

1. $|Rm(g_k)|_{g_k} \leq K$ in $M_k \times (a,b]$.
2. $|\mathcal{A}(g_k)|_{g_k^T} \leq K$ in $\partial M_k \times (a,b]$.
3. $i_{b,g_k(0)}(x) \geq i_0$, for all $x \in \partial M_k$.
4. $i_{b,g_k(0)} \geq i_0$.
for all $k$. Then there is a pointed manifold with boundary $(M_\infty,p_\infty)$, a Ricci flow $g_\infty(t)$ on $M_k$ and a family of metrics $\gamma_\infty(t)$ on $\partial M_\infty$ such that, up to subsequence,

$$(M_k,g_k(t),\gamma_k(t),p_k) \to (M_\infty,g_\infty(t),\gamma_\infty(t),p_\infty),$$

in the strong sense and in the $C^{m-3}$ topology.

**Proof.** We sketch the proof as it is essentially similar to the proof of Theorem 3.1 (following [10]). First, as in the proofs of Theorems 1.2 and 3.2, the assumptions of the theorem provide a uniform lower bound on $i^\cdot b,g_0$ and the injectivity radius of the boundary.

Then, Theorems 3.2, 3.5 and Shi’s local derivative of curvature estimates give

1. $|\nabla^j R^m(g_k(0))|_{g_k(0)} \leq C$,
2. $|\nabla^{j+1} A(g_k(0))|_{g_k^T(0)} \leq C$,

for some $C$ independent of $k$ (and appropriate order depending on $m$).

By the compactness result in [1], and elliptic regularity, one obtains a pointed smooth Cheeger-Gromov limit (in the strong sense) $(M_\infty,g_\infty(0))$. Here, the interior injectivity radius control follows from [5], the curvature bound and the volume bound of Lemma 3.3.

Now, as in [10], the curvature bound and Arzelà-Ascoli show that a subsequence converges to a limit flow $g_\infty(t)$ in the sense of Definition 4.1. The convergence of $\gamma_k(t)$ follows like in Theorem 3.1.

□

5. Estimates on 3-manifolds.

Now we assume that $M$ is a three dimensional manifold with compact boundary. The following theorem shows that along a three dimensional Ricci flow the second fundamental form of the boundary is essentially controlled by the ambient curvature. In the proof we use Liouville’s Theorem for bounded subharmonic functions in $\mathbb{R}^2$. It is unknown yet whether Theorem 5.1 holds in higher dimensions.

A Ricci flow $g(t)$ on $M$ with $t \in [0,T]$ will now be said to have $\Lambda$-controlled boundary, if in addition to the conditions of Definition 3.1 (putting $a = 0$ and $b = T$) the following holds. For all $\bar{x} \in \partial M$ and all $r \leq \Lambda^{-1}$ there exist $\gamma(0)$-harmonic coordinates around $\bar{x}$ in $\partial M$ in which

$$|\gamma_{\alpha\beta}|_{t,r}^* \leq Q,$$

$$|\mathcal{H}(g(t))|_{t-1,r}^* \leq Q.$$

**Theorem 5.1.** Let $(g(t),\gamma(t))$ be a three dimensional complete Ricci flow with compact $\Lambda$-controlled boundary, $t \in [0,T]$. Assume

$$|Rm(g(t))|_{g(t)} \leq K \quad \text{for all } t \in [0,T],$$

$$\text{inj}_{\partial M,g^T(0)} \geq i_0,$$

$$i_{b,g(0)} \geq i_0.$$

Then, for $0 \leq j \leq m-4$, there exist $C_j = C(K,T,\Lambda,i_0,j) > 0$ such that for all $t$ the second fundamental form $A$ of $\partial M$ and the boundary injectivity radius satisfy

$$|A(g(t))|_{g^T(t)} \leq C_0,$$

$$|\nabla^j A(g(t))|_{g^T(t)} \leq \frac{C_j}{t^{\frac{j}{2j+4}}},$$

$$i_{b,g(t)} \geq C_0^{-1}.$$
Proof. First, we show an estimate of \( i_{b,g}(t) \) for \( t > 0 \). From comparison geometry, if \( \lambda_{\text{max}}(t) \) is the largest eigenvalue of \( A(g(t)) \) and \( K(t) = \max\{\sqrt{K}, \lambda_{\text{max}}(t)\} \) then

\[
i_{b,g}(t) \geq \min\left\{ \frac{\pi}{2K(t)}, \frac{1}{2} \min\{L_t(\gamma_{x,t}), \ x \in \partial M\} \right\},
\]

where \( L_t(\gamma_{x,t}) \) is as in the proof of Theorem 1.2. Moreover, when \( i_{b,g}(t) < \frac{\pi}{2K(t)} \) we have

\[
d \frac{dt}{dt}i_{b,g}(t) \geq -Ci_{b,g}(t),
\]

for some positive constant \( C \) depending only on \( K \) and \( n \).

Let \( K_{\text{max}} = \max_{t \in [0,T]} K(t) \). It follows from (5.4) that there exists an \( 0 < \alpha = \alpha(K, i_0, T) < 1 \) with the property that

\[
i_{b,g}(t) \geq \alpha \frac{\pi}{2K_{\text{max}}},
\]

for all \( t \in [0, T] \).

Now, consider a sequence of counterexamples, namely \( \Lambda \)-controlled Ricci flows \((M_k, g_k(t), \gamma_k(t))\) and \((p_k, t_k) \in \partial M_k \times [0, T]\) such that

\[
|A(g_k(t_k))(p_k)| = \max_{\partial M_k \times [0, T]} |A(g_k(t))(p)| \to \infty.
\]

Set \( A_k = |A(g(t_k))(p_k)| \), and consider the pointed manifolds with boundary \((M_k, p_k)\) and the rescaled metrics

\[
h_k = A_k^2 g_k(t_k), \quad \bar{\gamma}_k = A_k^2 \gamma_k(t_k).
\]

Along this sequence, \( |Rm_k| \to 0, H_k \to 0, |A_k| \leq 1, \text{inj}_{h_k^T} \to \infty \) and the boundary injectivity radius is bounded below from the discussion above. Moreover, \( |A_k(p_k)| = 1 \), the interior injectivity radius is controlled from Lemma 3.3 and \( H_k \) is controlled in the Lipschitz sense, since \( g(t) \) is \( \Lambda \)-controlled. By the compactness result in [11] it follows that there is a subsequence converging in the \( C^{1,\epsilon}_r \) topology to a \( C^{1,\epsilon}_r \) Cheeger-Gromov limit \((M_\infty, h_\infty, p_\infty)\) satisfying \( |A_\infty(p_\infty)| = 1 \).

Moreover, \( h_k^T = e^{2U_k} \bar{\gamma}_k(t_k) \) for appropriate functions \( U_k \) on \( \partial M_k \) so \( U_k \) satisfy the elliptic equations

\[
-2\Delta h_k^T U_k = R_{h_k^T} - R_{\bar{\gamma}_k} e^{-2U_k}.
\]

By assumption, \( |U_k| \leq \ln \Lambda \) and by the convergence of \( h_k, h_k^T \) is controlled in \( C^{1,\epsilon}_r \) in \( h_k^T \)-harmonic coordinates. Elliptic regularity shows that \( U_k \) is controlled in the same coordinates in the \( C^{1,\epsilon} \) sense (but in a slightly smaller domain).

Therefore, after possibly passing to a subsequence, we may assume that \( F_k^* \bar{\gamma}_k \to \bar{\gamma}_\infty \) in \( C^{1,\epsilon}_r \), if \( F_k \) are the diffeomorphisms associated to the Cheeger-Gromov convergence (see Definition 4.1). It is clear that \( \bar{\gamma}_\infty \) is just the Euclidean metric \( \delta \) in \( \mathbb{R}^2 \), and that there exists a function \( U_\infty \) so that \( h_\infty^T = e^{2U_\infty} \delta \). Also, up to diffeomorphism, \( U_k \to U_\infty \) uniformly locally in \( C^{1,\epsilon}_r \).

From (5.6), it follows that for \( \psi \in C^\infty_c(\mathbb{R}^2) \),

\[
\int_{\mathbb{R}^2} \frac{1}{2} \left( e^{2U_k} R_{h_k^T} - R_{\bar{\gamma}_k} \right) \psi d\text{vol}_{\bar{\gamma}_k} = \int_{\mathbb{R}^2} \langle \nabla U_k, \nabla \psi \rangle d\text{vol}_{\bar{\gamma}_k} \to \int_{\mathbb{R}^2} \langle \nabla U_\infty, \nabla \psi \rangle d\text{vol}_{\delta}.
\]
From Gauss equation $R_{h^T} \leq \varepsilon_k$, for some positive $\varepsilon_k \to 0$. Thus, if $\psi \geq 0$ this gives
\[
\int_{\mathbb{R}^3} \frac{1}{2} \left( e^{2U_k} R_{h^T_k} - R_{\bar{\gamma}_k} \right) \psi d\text{vol}_{\bar{\gamma}_k} \leq \varepsilon'_k \to 0.
\]
Thus, it follows that $U_\infty$ is subharmonic in the weak sense. It is also bounded, since $|U_k| \leq \ln \Lambda$, hence by Liouville’s theorem $U_\infty$ is constant.

Since $|\text{Ric}_k| \to 0$ it follows that $\text{Ric}_\infty = 0$, in the weak sense in boundary harmonic coordinates (in the sense of [1]). Moreover, $h^T_\infty$ is isometric to $\delta$ and $H_\infty = 0$. Hence, by elliptic regularity we obtain that $h_\infty$ is smooth in harmonic coordinates, as in these coordinates $h^{0\beta}_\infty$ and $h^{0i}_\infty$ satisfy Dirichlet and Neumann elliptic boundary value problems, as shown in section 2 of [1].

Moreover, $H_\infty$ is bounded, since $|\text{Ric}_\infty| \to 0$ it follows that $\text{Ric}_\infty = 0$, in the weak sense in boundary harmonic coordinates (in the sense of [1]).

Thus, $U_\infty$ is subharmonic in the weak sense. It is also bounded, since $|U_k| \leq \ln \Lambda$, hence by Liouville’s theorem $U_\infty$ is constant.

Since $|\text{Ric}_k| \to 0$ it follows that $\text{Ric}_\infty = 0$, in the weak sense in boundary harmonic coordinates (in the sense of [1]). Moreover, $h^T_\infty$ is isometric to $\delta$ and $H_\infty = 0$. Hence, by elliptic regularity we obtain that $h_\infty$ is smooth in harmonic coordinates, as in these coordinates $h^{0\beta}_\infty$ and $h^{0i}_\infty$ satisfy Dirichlet and Neumann elliptic boundary value problems, as shown in section 2 of [1].

Now, the Gauss equation implies that $|A_\infty(p_\infty)| = 0$, which is a contradiction since, by the $C^{1,\varepsilon}$ convergence, $|A_\infty(p_\infty)| = 1$.

This shows that $|A(g(t))|$ is bounded above in terms of $K$, $i_0$ and $n$, proving (5.1). Then (5.3) follows from (5.5).

We sketch the proof of (5.2) briefly. Consider a sequence of counterexamples $(M_k, g_k(t), \gamma_k(t))$ such that there exist $(p_k, t_k) \in \partial M_k \times (0, T]$ so that
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |\nabla_j A(g_k(t))(p_k)|^2 \right) \to \infty.
\]
Under the assumptions of the theorem and estimate (5.1), setting $Q_k = |\nabla_j A(g_k(t))(p_k)|^2$ and rescaling
\[
\begin{align*}
h_k(t) &= Q_k g_k(t_k + tQ_k^{-1}), \\
\bar{\gamma}_k(t) &= Q_k \gamma_k(t_k + tQ_k^{-1}),
\end{align*}
\]
we may apply Theorem 4.1 to obtain
\[
(M_k, h_k(0), \bar{\gamma}_k(0), p_k) \to (M_\infty, h_\infty, \bar{\gamma}_\infty, p_\infty).
\]
The limit will have totally geodesic boundary, hence $|\nabla_j A_\infty| = 0$. This contradicts the fact that $|\nabla_j A_\infty(p_\infty)| = 1$, which holds if $m$ is large enough.

Remark 5.1. The proof of estimate (5.1) is using that $g(t)$ is a Ricci flow only to propagate forward in time the estimate on the boundary injectivity radius. Therefore Theorem 5.1 holds for any fixed Riemannian 3-manifold with compact boundary satisfying the same assumptions, a fact which may be of independent interest. Ricci flow was only used to obtain the higher order estimates (5.2).

Theorem 5.1 allows us to improve the continuation principle in [9] and Theorem 4.1 in dimension three.

**Corollary 5.1.** Let $g(t), t \in [0, T), T < \infty$, be a maximal Ricci flow on a compact 3-manifold with boundary $M$. Suppose that there exist smooth data $\gamma(t)$ and $\eta(t)$ defined for $t \in [0, T')$, $T' > T$, such that
\[
\begin{align*}
[g^T(t)] &= [\gamma(t)], \\
\mathcal{H}(g(t)) &= \eta.
\end{align*}
\]
Then,
\[ \sup_{M \times [0, T]} |Rm(g(t))|_{g(t)} = \infty. \]

**Proof.** By [9]
\[ \sup_{M \times [0, T]} |Rm(g(t))|_{g(t)} + \sup_{\partial M \times [0, T]} |A(g(t))|_{g(T)} = \infty. \]
However, Theorem 5.1 asserts that the second fundamental form remains bounded as \( t \to T \) which finishes the proof. \( \square \)

**Corollary 5.2.** If \( M \) is a compact 3-manifold, Theorem 4.1 holds without assumption (2) on the second fundamental form.

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