KLEIN’S GROUP DEFINES AN EXCEPTIONAL SINGULARITY OF DIMENSION 3

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INTRODUCTION

The aim of this paper is to construct examples of canonical exceptional singularities. Canonical (as well as terminal, log terminal and log canonical) singularities appear naturally in the minimal model theory and were studied by Reid, Mori, Kollár, Shokurov and others. Recently Shokurov [Sh1] introduced the notion of exceptional singularity, see Definition 1.2. This notion is closely connected with the inductive approach to the classification of singularities, flips, divisorial contractions, etc. The key ingredient of the inductive approach in its modern setting is the search of complements, that is, of good divisors in the multiple anticanonical systems (see Definition 1.4). In fact the main result of [Sh1] is that two-dimensional complements can be divided into two parts: regular and exceptional. Regular ones occur in 1-, 2-, 3-, 4- or 6-uple anticanonical systems and have a rather simple structure. Exceptional ones can only occur in the neighborhood of an exceptional singularity; they are more complicated to study but they belong, up to birational isomorphisms, to a finite number of families and, at least in principle, can be classified [Sh1]. By using standard arguments with Kawamata-Viehweg vanishing and the inversion of adjunction (see [Ut, 19.6]) these results can be applied to study three-dimensional log canonical singularities and, even more generally, extremal contractions [Sh1, §7].

In dimension 3, Shokurov [Sh1, §7] gave an example of a log Del Pezzo surface with no numerical obstructions to the existence of its blow down to an exceptional canonical singularity, but at the moment it is not clear whether such a singularity really exists. In the present paper, we construct first examples of 3-dimensional canonical exceptional singularities.

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Our examples belong to the class of quotient singularities: they are quotients of $\mathbb{C}^3$ by the action of Klein’s simple group $J$ of order 168, and of its central extension $J'$ by the 3-rd roots of unity, of order 504. Remark, that these quotient singularities have been already investigated in relation to another problem in [Mar] and [Ro]. It was proved, that they and, more generally, any quotient of $\mathbb{C}^3$ by a finite subgroup of the special linear group, are crepant, that is, admit resolutions with trivial canonical class. Thus, they are in a sense opposite to terminal, or totally discrepant singularities inside the class of canonical ones. By Example 1.3, the terminal singularities are always nonexceptional, so, we have a reason to start the search of exceptional ones among crepant quotients.

We use the Miller–Blichfeldt–Dickson classification [MBD] of finite subgroups of $SL_3(\mathbb{C})$, and show, that for some classes of this classification the quotient is not exceptional. These include the reducible, imprimitive groups, the icosahedral group $H$ and its central extension $H'$ of order 180. All these groups have a semiinvariant of degree $\leq 3$. It is plausible, that the quotients by the subgroups without semiinvariants of degree $\leq 3$ are exceptional. But the proof of the exceptionality is not so easy as that of the nonexceptionality: in the first case, one has to verify the minimal discrepancies for all possible boundaries, whereas, in the second case, it suffices to find one nonexceptional boundary. We prove the exceptionality of our examples by the rule of contraries, in using the existence of a nonexceptional 1-, 2-, 3-, 4- or 6-complement of the canonical divisor (according to [Sh1]) for a nonexceptional singularity. This implies that it suffices to verify only the boundaries given by the semiinvariants of degree $\leq 18$. The rest of the proof relies heavily upon the classical results of Klein on the group $J$: enumeration of the orbits, list of invariants, the configuration of the non-free locus of its action on the projective plane.

We will describe now briefly the contents of the article by sections. Section 1 is preliminary, it contains definitions and some facts for later use. In Section 2 we describe our approach to the proof of the exceptionality of quotient singularities. In particular we prove (Corollary 2.3) that a quotient singularity $\mathbb{C}^3/G$ can be exceptional only if $G$ is primitive. In Section 3 the main result (Theorem 3.1) is proved.

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1. Definitions and preliminary results

We follow essentially the terminology and notation of [Ut], [Sh] and [Sh1] (see also [Ko] for a nice introduction to the subject).

**Definition 1.1.** Let \((X \ni P)\) be a normal singularity (not necessarily isolated) and let \(D = \sum d_i D_i\) be a divisor on \(X\) with real coefficients. 

- \(D\) is called a boundary if \(0 \leq d_i \leq 1\) for all \(i\).
- \(D\) is called a subboundary, if it is majorated by a boundary.

A proper birational morphism \(f : Y \rightarrow X\) is called a log resolution of \((X, D)\) at \(P\), if \(Y\) is nonsingular near \(f^{-1}(P)\) and \(\text{Supp}(D) \cup E\) is a normal crossing divisor on \(Y\) near \(f^{-1}(P)\), where \(D\) is used to denote both the subboundary on \(X\) and its proper transform on \(Y\), and \(E = \bigcup E_i\) is the exceptional divisor of \(f\). The pair \((X, D)\) or, by abuse of language, the divisor \(K_X + D\) is called terminal, canonical, Kawamata log terminal (klt), purely log terminal (plt), and, respectively, log canonical (lc) near \(P\), if the following conditions are verified:

1. \(K_X + D\) is \(\mathbb{R}\)-Cartier.
2. Let us write for any proper birational morphism \(f : Y \rightarrow X\)
   \[
   K_Y \equiv f^*(K_X + D) + \sum a(E, X, D)E,
   \]
   where \(E\) runs over prime divisors on \(Y\), \(a(E, X, D) \in \mathbb{R}\), and \(a(D_i, X, D) = -d_i\) for each component \(D_i\) of \(D\). Then, for some log resolution of \((X, D)\) at \(P\) and for all prime divisors \(E\) on \(Y\) near \(P\), we have:
   - \(a(E, X, D) > 0\) (for terminal), \(a(E, X, D) \geq 0\) (for canonical), \(a(E, X, D) > -1\) and no \(d_i = 1\) (for klt), \(a(E, X, D) > -1\) (for plt, without any restriction on the subboundary \(D\)), and, respectively, \(a(E, X, D) \geq -1\) (for lc).

The coefficients \(a(E, X, D)\) are called discrepancies of \(f\), or of \((X, D)\); they depend only on the discrete valuations of the function field of \(X\) associated to the prime divisors \(E\), and not on the choice of \(f\).

We will identify prime divisors with corresponding discrete valuations, when speaking about ‘divisors \(E\) over \(X\)’ without indicating, on which birational model \(E\) is realized. The conditions given by inequalities in part (ii) of the above definition do not depend on the choice of a log resolution. The lc (as well as terminal, canonical, klt, plt) condition is obviously monotonic on \(D\): if \((X, D)\) is lc at \(P\), then \((X, D')\) is also lc at \(P\) for any \(D' \leq D\) such that \(K_X + D'\) is \(\mathbb{R}\)-Cartier. Thus, for any boundary \(D\), which is a \(\mathbb{R}\)-Cartier divisor on a \(\mathbb{Q}\)-Gorenstein variety we can define the log canonical threshold of \((X, D)\) at \(P\):

\[
c_P(X, D) = \max\{\alpha \in \mathbb{R} \mid (X, \alpha D) \text{ is log canonical}\}
\]
Definition 1.2 ([Sh1, 1.5]). Let \((X \ni P)\) be a normal singularity and let \(D = \sum d_i D_i\) be a boundary on \(X\) such that \(K_X + D\) is log canonical. The pair \((X, D)\) is said to be **exceptional** if there exists at most one exceptional divisor \(E\) over \(X\) with discrepancy \(a(E, X, D) = -1\). The singularity \((X, P)\) is said to be **exceptional** if \((X, D)\) is exceptional for any \(D\) whenever \(K_X + D\) is log canonical.

Remark 1.3. By the connectedness result [Sh, 5.7], [Ut, 17.4], the set of divisors with discrepancies \(\leq -1\) in \(Y\) for any proper birational morphism \(f: Y \rightarrow X\) is connected. Therefore, if a log canonical pair \((X, D)\) is nonexceptional, then there exist infinitely many divisors with discrepancy \(-1\), which can be constructed by blowing up the components of intersections of pairs of such divisors.

Definition 1.4 ([Sh, 5.1]). Let \((X, P)\) be a normal singularity, \(D = S + B\) a subboundary, such that \(B, S\) have no common components, \(S\) is a reduced divisor, and \(B = \sum b_i B_i\) with all \(b_i < 1\), that is, \([B] = 0\), where \([\cdot]\) denotes the integer part. Then one says that \(K_X + D\) is n-complemented, if there exists a \(\mathbb{Q}\)-divisor \(D^+\), such that the following conditions are verified:

(i) \(nD^+\) has integer coefficients ;
(ii) \(nD^+ \sim -nK_X\);
(iii) \(nD^+ - nS - [(n + 1)B] \geq 0\);
(iv) \(K_X + D^+\) islc.

The divisor \(K_X + D^+\) is called a **n-complement** of \(K_X + D\). Remark, that if \(D\) is a boundary, then so is \(D^+\).

Example 1.5 ([Sh, 5.2.3, 5.6], [Sh1, 1.5]). Let \((X, P)\) be a two-dimensional quotient singularity. Then the following conditions are equivalent:

(i) \((X \ni P)\) is exceptional,
(ii) \((X \ni P)\) is of type \(E_6, E_7\) or \(E_8\) (in the generalized sense of [1], [Br]); this means that the dual (weighted) graph of the minimal resolution of such a singularity has a single 3-valent vertex \(-b\), \(b \geq 2\) with three chains issued from it, exactly one of them being of type \(-3\) (see [Br], [1], or [Ut, Ch. 3] for a more precise description),
(iii) there are no 1- or 2-complements \(K_X + D\) such that \(P \in \text{Supp}(D)\) (but there is a 3-, 4- or 6-complement),
(iv) \((X \ni P)\) is analytically isomorphic to a quotient \(\mathbb{C}^2/G\), where \(G\) is a finite subgroup of \(GL_3(\mathbb{C})\) without reflections of dihedral, tetrahedral or icosahedral type \([\text{Br}]_4\). This means that the image of \(G\)
in $PGL_2(\mathbb{C}) \simeq SO_3(\mathbb{C})$ is the dihedral, tetrahedral or icosahedral group in the usual sense.

The following theorem is a consequence of the proof of Shokurov’s theorem [Sh1, 7.1].

**Theorem 1.6.** Let $(X \ni P)$ be a nonexceptional three-dimensional log canonical singularity. Then $K_X$ is either 1-, 2-, 3-, 4- or 6-complemented. Moreover, there exists such a nonexceptional complement $K_X + D$.

**Lemma 1.7.** Assume that there exists a reduced divisor $S = \sum S_i$ passing through $P$ such that $K_X + S$ is log canonical. Then $(X \ni P)$ is nonexceptional.

**Proof.** Take a general hyperplane section through $P$. Then, for some $0 \leq \alpha \leq 1$, the log divisor $K_X + S + \alpha H$ is log canonical, but not purely log terminal. By Remark 1.3, the set of divisors with discrepancy $a(\cdot, X, S + \alpha H) = -1$ on a resolution $Y \to X$ is connected, and by construction, we have at least two of them, one coming from $S$ and the other one, say $E$, exceptional over $X$. So we can get infinitely many divisors with discrepancy $-1$ in blowing up the curves of intersections.

**Example 1.8.** By definition and Lemma 1.7 any three-dimensional cDV-singularity is nonexceptional.

**Example 1.9.** Let $(X \ni P)$ be a three-dimensional terminal singularity and $S \in |-K_X|$ a general element (in the Gorenstein case, we should additionally suppose that $S \ni P$). By [R, 6.4] and the inversion of adjunction [Sh, 3.3, 3.12, 5.13], [Ut, 17.6] $K_X + S$ is purely log terminal (and even canonical). Therefore, all terminal singularities are nonexceptional. Of course, these arguments use the classification of terminal singularities. Shokurov (cf. [R, 6.5]) posed the problem to prove this fact directly.

## 2. Quotient singularities

Now let $(X \ni P)$ be a three-dimensional quotient singularity, i. e. $(X \ni P) = (\mathbb{C}^3 \ni 0)/G$, where $G \subset GL_3(\mathbb{C})$ is a finite subgroup. We may assume that $G$ contains no quasi-reflections, for if $G$ contains quasi-reflections, then there exists another subgroup $G'$ of $GL_3(\mathbb{C})$, which does not contain a quasi-reflection and such that $\mathbb{C}[y_1, y_2, y_3]^G \simeq \mathbb{C}[y_1, y_2, y_3]^{G'}$ as $\mathbb{C}$-algebras.
Let \( \pi : V \to X \) be the quotient morphism, where \( V = \mathbb{C}^3 \). Let \( D \) be a boundary on \( X \) and let \( D' := \pi^* D \). By [Sh2, 2.2], [Ut, 20.3], \( K_X + D \) is log canonical (resp., plt, klt) iff so is \( K_V + D' \).

**Lemma 2.1.** In the above notation, \((X, D)\) is exceptional iff \((V, D')\) is.

**Proof.** Assume that \((V, D')\) is exceptional. Then by [Sh2, 3.1], [Ut, 17.10] there exists a blow-up \( g : W \to V \) such that \( K_W + S + B' = g^*(K_V + D') \) is purely log terminal, where \( S \) is the (irreducible) exceptional divisor of \( g \) and \( B' \) is the proper transform of \( D' \). This blow-up is unique up to isomorphism, because \( g \) is projective and \( \rho(W/V) = 1 \) [Ut, 6.2]. So we can define an action of \( G \) on \( W \) making \( g \) equivariant. Let \( \varphi : W \to Y \) be the quotient morphism and put \( E := \varphi(S) \) and \( B := \varphi(B') \). We have the following commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\varphi} & Y \\
g \downarrow & & f \downarrow \\
V & \xrightarrow{\pi} & X \\
\end{array}
\]

(1)

By the ramification formula

\[
K_W + S + B' = \varphi^*(K_Y + E + B),
\]

whence

\[
K_Y + E + B = f^*(K_X + D).
\]

By [Sh2, 2.2], [Ut, 20.3] \( K_Y + E + B \) is purely log terminal. Therefore, \( K_X + D \) is exceptional.

Conversely, assume that \( K_X + D \) is exceptional. Let \( W \) be the normalization of \( V \times_X Y \). We again have the diagram (1) and relations (2), (3). Similarly, by [Sh2, 2.2], [Ut, 20.3], \( K_W + S + B' \) is purely log terminal, hence \( K_V + D' \) is exceptional. \( \square \)

**Lemma 2.2.** In the notation of Lemma 2.1, assume that \( G \) has a semiinvariant of degree \( \leq 3 \). Then \((X \ni P)\) is nonexceptional.

**Proof.** Let \( \psi \) be such a semiinvariant of minimal degree \( d \leq 3 \), let \( D' \) be its zero locus, and let \( D := \pi(D')_{\text{red}} \). Then \( D' = \pi^*(D) \). By Lemma 1.7 it is sufficient to show that \( K_V + D' \) is log canonical. Note that \( \psi \) is homogeneous, so \( D' \) is a cone over a plane curve \( C \) of degree \( d \leq 3 \). If \( C \) is nonsingular, then \( D' \) has a unique singular point which can be resolved by only one blow-up with \( a(v, V, D') = 2 - \text{mult}_0 \psi \geq -1 \), so \( K_V + D' \) is log canonical in this case. If \( C \) has a unique singular point, then \( G \) has an eigenvector, say \( v \in \mathbb{C}^3 \). But then \( G \) has an invariant
plane, orthogonal to \( v \) and we can take \( d = 1 \) and \( C \) is nonsingular. The same arguments work if \( C \) has exactly two singular points. In the remaining case \( C \) is the union of three lines in general position. Then \((V, D')\) is already log nonsingular (that is, may be taken as its own log resolution), and so \( K_V + D' \) is log canonical.

The subgroup \( G \in GL_3(\mathbb{C}) \) is said to be reducible, if it has a proper invariant subspace in \( \mathbb{C}^3 \). \( G \) is called imprimitive, if there exists a triple of lines \( L_1, L_2, L_3 \) in \( \mathbb{C}^3 \), permuted by \( G \) via a representation \( G \to S_3 \), where \( S_n \) denotes the symmetric group of permutations on \( n \) elements.

**Corollary 2.3.** If \((X \ni P)\) is exceptional, then \( G \) is irreducible and primitive.

*Proof.* If \( G \) is reducible, it has a semiinvariant of degree 1. If it is imprimitive, it has a semiinvariant of degree 3, defining the union of three planes spanned by pairs of lines from \( L_1, L_2, L_3 \). The result follows by Lemma 2.2.

The finite subgroups of \( G \subset GL_3(\mathbb{C}) \) were classified by Miller–Blichfeldt–Dickson [MBD] modulo extension by certain scalar matrices (compare with [P]). There are 9 types of such groups, denoted by A, B, \ldots, J in [MBD]. As soon as we are looking for those which yield exceptional quotient singularities, we have to test only primitive irreducible ones. They belong to the 6 types E, F, G, H, I, J. The orders of the associated collineation groups \( PG = G/(G \cap \mathbb{C}^*) \subset PGL_3(\mathbb{C}) \) are 36, 72, 216, 60, 360, 168; the first three are solvable, and the last three are simple. The collineation groups from G to J have their names: the Hessian group, the icosahedral one, the alternating group of degree 6, and, finally, Klein’s simple group.

**Proposition 2.4.** The quotients of \( \mathbb{C}^3 \) by the subgroups of \( SL_3(\mathbb{C}) \) of type H are nonexceptional.

*Proof.* There are two such groups (see, e. g., [P] or [Ro]): the icosahedral group \( G \) of order 60, and its central extension \( G' \) of order 180. The invariants of \( G \) are well-known; see, for example, [MBI, Sect. 116]. There is an invariant of degree 2, which is a semiinvariant of \( G' \). (This also follows from the fact that \( G \) is a subgroup of \( SL_3(\mathbb{R}) \) and therefore has an invariant quadratic form [Sp, 4.2.15]). The result follows by Lemma 2.2.

2.5. Now we will explain the logic of our approach to the proof of the exceptionality of a quotient singularity, which will be applied in the next section to Klein’s group. Assume that \((X \ni P)\) is nonexceptional. By Theorem 1.6 there exists a nonexceptional log canonical \( K_X + D \)
such that $n(K_X + D) \sim 0$ for $n \in \{1, 2, 3, 4, 6\}$. Further, we will use notations of Lemma 2.1. The integer divisor $F := nD'$ locally near 0 can be defined by a seminvariant function, say $\psi$.

Moreover $nD \sim -nK_X$ iff the form $\psi(dx_1 \wedge dx_2 \wedge dx_3)^n$ is invariant, i.e.

$$g(\psi) = \det(g)^n \psi \quad \text{for all} \quad g \in G. \quad (4)$$

Denote $d := \text{mult}_0(\psi)$. Let $\sigma : W \to V = \mathbb{C}^3$ be the blow-up of the origin and let $S \simeq \mathbb{P}^2$ be the exceptional divisor. Then $K_W = \sigma^*K_V + 2S$ and $\sigma^*F = R + dS$, where $R$ is the proper transform of $F$. Therefore the discrepancy of $S$ is

$$a(S, V, D') = 2 - \frac{1}{n} \text{mult}_0(\psi) \geq -1.$$  

So we have $d = \text{mult}_0(\psi) \leq 3n \leq 18$.

Further

$$K_W + S + \frac{3}{d}R = \sigma^*(K_V + \frac{3}{d}F). \quad (5)$$

By [Ko, Lemma 3.10] $K_V + \frac{3}{d}F$ is log canonical iff so is $K_W + S + \frac{3}{d}R$. In this case the pair $(V, \alpha F)$ is exceptional for all $0 \leq \alpha \leq \frac{3}{d}$ iff $K_W + S + \frac{3}{d}R$ is purely log terminal; by the above, we need this assertion only for $\alpha = \frac{1}{n}$ with $n \in \{1, 2, 3, 4, 6\}$, but we will have a stronger property with $\alpha$ not necessarily of this form. The plt condition for $K_W + S + \frac{3}{d}R$ is equivalent to that $K_S + \frac{3}{d}C$ is Kawamata log terminal, where $C = R \cap S$ [Sh, 5.13], [Ut, 17.6]. It is clear that $C$ is given by the equation $\psi_{\min} = 0$, where $\psi_{\min}$ is the homogeneous component of $\psi$ of minimal degree $d$. Therefore we have

**Proposition 2.6.** In the above notations, if $K_S + \frac{3}{d}C$ is Kawamata log terminal, then $(V, \alpha F)$ is exceptional for any $0 \leq \alpha \leq \frac{3}{d}$.

**Remark 2.7.** If $K_S + \frac{3}{d}C$ is log canonical, but not Kawamata log terminal, then $(V, F)$ is nonexceptional. If $K_S + \frac{3}{d}C$ is not log canonical, then we can conclude nothing (possibly in this situation we have to consider some weighted blow-up).

**Lemma 2.8.** Notations as above. Assume that $K_S + \frac{3}{d}C$ is not Kawamata log terminal and $C$ is a singular irreducible curve. Then there exists an orbit of $G$ consisting of at most 10 singular points of $C$.

**Proof.** Let $P \in C$ be a singular point of maximal multiplicity $m$ and let $r$ be the number of points in the orbit $G \cdot P$. Denote by $c = c(S, C)$ the (global) log canonical threshold of $(S, C)$. By our assumption $c \leq 3/d$. 


On the other hand, \( c \geq 1/m \) [Ko, Lemma 8.10]. This gives us \( d \leq 3m \).

Let \( g \) be the genus of the normalization of \( C \). Then

\[
0 \leq g \leq \frac{(d-1)(d-2)}{2} - \frac{m(m-1)}{2}.
\]

Taking into account that \( d \leq 3m \), we obtain

\[-2 \leq 2g - 2 \leq (9-r)m(m-1) .\]

This implies the assertion.

3. Klein’s group

The aim of this section is to prove the exceptionality of the quotients of \( \mathbb{C}^3 \) by the subgroups of \( SL_3(\mathbb{C}) \) of type J in the classification of [MBD]. There are two such groups, see [P] or [RG]: Klein’s simple group \( J_{168} \) of order 168 and its central extension \( J_{504}' \) of order 504.

**Theorem 3.1.** Let \( G \subset SL_3(\mathbb{C}) \) be \( J_{168} \) or \( J_{504}' \). Then the singularity of the quotient \( \mathbb{C}^3/G \) at the origin is exceptional.

We will briefly describe the irreducible representation of \( J_{168} \) in \( \mathbb{C}^3 \) following [W] and [Kl]. Another description of \( J_{168} \) see in [Sp, Sect. 4.6]. Let \( y_1, y_2, y_3 \) be coordinates in \( \mathbb{C}^3 \). The group \( J_{168} \) is generated by 3 elements \( \tau, \chi, \omega \) of orders 7, 3, 2 respectively, and the representation is defined by

\[
\tau = \begin{pmatrix} \epsilon & 0 \\ \epsilon^2 & \epsilon^4 \\ 0 & \epsilon^4 \end{pmatrix}, \quad \epsilon = \exp \left( \frac{2\pi i}{7} \right),
\]

\[
\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix},
\]

\[
\alpha = -\frac{2\sin \frac{8\pi}{\sqrt{7}}}{\sqrt{7}}, \quad \beta = -\frac{2\sin \frac{4\pi}{\sqrt{7}}}{\sqrt{7}}, \quad \gamma = -\frac{2\sin \frac{2\pi}{\sqrt{7}}}{\sqrt{7}}.
\]

The second group \( J_{504}' \) is generated by \( J_{168} \) and the scalar matrix with \( \exp \frac{2\pi i}{3} \) on the diagonal.

Now, we will describe the semiinvariants of these groups. First of all, since \( J_{168} \) is simple, all its semiinvariants are indeed invariants. They are also semiinvariants of \( J_{504}' \). According to [Kl] (see also [W]), the algebra of invariants \( A = \mathbb{C}[y_1, y_2, y_3]_{J_{168}} \) is generated by four homogeneous polynomials \( f, \Delta, C, K \) of degrees 4, 6, 14, 21 respectively, with one basic relation between them:
\[ f = y_1^3y_3 + y_2^3y_1 + y_3^3y_2, \]
\[ \Delta = \frac{1}{54} \text{Hess} (f), \]
\[ C = \frac{1}{9} \begin{vmatrix} f_{y_1y_1}'' & f_{y_1y_2}'' & f_{y_1y_3}'' & \Delta_{y_1}' & \Delta_{y_2}' & \Delta_{y_3}' \\ f_{y_2y_1}'' & f_{y_2y_2}'' & f_{y_2y_3}'' & \Delta_{y_1}' & \Delta_{y_2}' & \Delta_{y_3}' \\ f_{y_3y_1}'' & f_{y_3y_2}'' & f_{y_3y_3}'' & \Delta_{y_1}' & \Delta_{y_2}' & \Delta_{y_3}' \end{vmatrix}, \]
\[ \mathcal{K} = \frac{1}{14} \begin{vmatrix} f_{y_1}' & \Delta_{y_1}' & C_{y_1}' \\ f_{y_2}' & \Delta_{y_2}' & C_{y_2}' \\ f_{y_3}' & \Delta_{y_3}' & C_{y_3}' \end{vmatrix}. \]

\[ \mathcal{K}^2 = C^3 + 1728\Delta^7 + 1008C\Delta^4f - 88C^2\Delta f^2 - 60032\Delta^5f^3 + 1088C\Delta^2f^4 + 22016\Delta^3f^6 - 256Cf^7 - 2048\Delta f^9. \tag{6} \]

In the proof of the Theorem, we will follow the logic scheme outlined in Subsect. 2.5. In the notations of 2.3, let \( F \) be given by the equation \( \psi = 0 \), where \( \psi = \sum a_{ijkl} f^i \Delta^j C^k \mathcal{K}^l \) (\( i \geq 0, j \geq 0, k \geq 0, l = 0 \) or 1), such that the lowest degree of non-zero terms \( d = \text{mult}_0\psi = \deg \psi_{\text{min}} \leq 18 \). For \( G = J_{168} \), up to a constant factor of proportionality, \( \psi_{\text{min}} \) can be only one of the following functions:
\[ f, \Delta, f^2, f\Delta, \lambda f^3 + \mu \Delta^2, \lambda f^2 \Delta + \mu \mathcal{K}, \lambda f^4 + \mu \Delta^2 f, \lambda \Delta^3 + \mu f^3 \Delta + \nu f \mathcal{K}, \]
where \( \lambda, \mu, \nu \) are arbitrary complex constants. The list of initial forms for \( G = J'_{504} \) is even shorter: according to [3], the character of \( \psi \) is equal to \( g \mapsto (\det g)^n \), hence it is trivial, because \( G \subset SL_2(\mathbb{C}) \). So, only polynomials of degree divisible by 3 should be kept in the case of \( J'_{504} \). The rest of the proof depends only on the pair \( (S, C) = (\mathbb{P}^2, (\psi_{\text{min}})) \), and is done simultaneously for the two groups.

**Case 1.** \( \psi_{\text{min}} = f^k \) or \( \Delta^k \); \( \deg \psi_{\text{min}} = 4k \) or, resp. \( 6k \). The reduced curve \( C_{\text{red}} \) is nonsingular, so the pair \( (S, C_{\text{red}}) \) is log nonsingular. Hence \( (S, \alpha C_{\text{red}}) \) is klt for any \( \alpha < 1 \). Hence \( (S, \frac{2\alpha}{d} C) = (S, \frac{2\alpha}{d} C_{\text{red}}) \) is klt. By Proposition 2.6, the pair \( (V, \alpha F) \) is exceptional for any \( 0 \leq \alpha \leq \frac{3}{d} \), and we are done.

**Case 2.** \( \psi_{\text{min}} = f^i \Delta^j \), \( 10 \leq d = 4i + 6j \leq 18 \). Here \( C_{\text{red}} = \{f \Delta = 0\} \) is singular only at the points of intersection of Klein’s quartic \( C_1 = \{f = 0\} \) and its Hessian curve \( C_2 = \{\Delta = 0\} \), that is at the inflection points of Klein’s curve. It is known that they form one orbit of length 24 under the action of \( J_{168} \) with representative \( (1 : 0 : 0) \). Hence they are
ordinary double points of $C_{\text{red}}$, and thus, $(S, C_{\text{red}})$ is log nonsingular. Hence $(S, \alpha C_1 + \beta C_2)$ is klt for any $\alpha < 1, \beta < 1$. Hence $(S, \frac{3}{d}(iC_1 + jC_2))$ is klt, and we are done.

Cases 1 and 2 cover all the invariant curves of degree $< 12$, as well as multiples of $C_1$ and $C_2$.

**Case 3.** $C$ is a reduced irreducible curve of degree $d \geq 12$. If it is nonsingular, we are done, because $(S, C)$ is log nonsingular, and $\frac{3}{d} < 1$. Assume that $(S, \frac{3}{d}C)$ is not klt. Then $C$ cannot be singular by Lemma 2.8, because $J_{168}$ does not have orbits in $\mathbb{P}^2$ of length $\leq 10$. For the orbits of Klein’s group, see, e. g. [W, Sect. 120]; the possible lengths are 21, 24, 28, 42, 56, 84, 168. Hence $C$ is nonsingular, and this is a contradiction. Hence $(S, \frac{3}{d}C)$ is always klt in this case, and we are done.

**Case 4.** $C$ is reducible or non-reduced of degree $12 \leq d \leq 18$. The irreducible components of $C$ are permuted by the action of $J_{168}$. The length of the orbit (if not 1, for an invariant component), is $\geq 7$, for $J_{168}$ has no non-trivial homomorphisms to symmetric groups $S_p$ with $p < 7$. Moreover, there are no straight lines as irreducible components, because the action on the dual projective plane of lines in $\mathbb{P}^2$ is the same as on $\mathbb{P}^2$ itself, and hence the minimal orbit length is 21 (there is indeed an invariant of degree 21 which factors into the product of linear forms, namely, $K$). So, discarding the multiples or combinations of $C_1$ and $C_2$ covered by Cases 1 and 2, we have three subcases: A. $d \nmid \psi_{\text{min}}$ and $\Delta \nmid \psi_{\text{min}}$. Then $d = 14$, 16, or 18, and $C$ is the union of $d/2$ conics. B. $f \mid \psi_{\text{min}}$. Then $d = 16$ or 18, and $\psi_{\text{min}} = f(\lambda f^3 + \mu \Delta^2)$ with $\lambda \mu \neq 0$ or, resp., $\psi_{\text{min}} = f(\lambda f^2 \Delta + \mu C)$ with $\mu \neq 0$. C. $\Delta \mid \psi_{\text{min}}$. Then $\psi_{\text{min}} = \Delta(\lambda f^3 + \mu \Delta^2)$ with $\lambda \mu \neq 0$.

**Subcase A.** $d = 18$ can be eliminated, because $9 \nmid 168$. If $d = 16$, then the stabilizer of any conic component $\Gamma$ of $C$ is of order 21. Hence, the orbit of the generic point of $\Gamma$ is of length 21, which is impossible, because there is only one orbit of length 21 in $\mathbb{P}^2$: ‘die achtzahlige Pole’ in the classical terminology. A similar argument shows that $d = 14$ is also impossible.

**Subcase B.** If $d = 16$, then $C = C_1 \cup C'$, where $C_1$ is Klein’s curve, and $C' = \{\lambda f^3 + \mu \Delta^2 = 0\}$ is an irreducible curve of degree 12. The 24 points $Q_i$ of $C_1 \cap C'$ are ordinary cusps of $C'$, and the local indices $(C_1 \cdot C')_{Q_i} = 2$. By Bezout Theorem, these are the only intersection points. Assume that there is a singular point $R \in C'$, different from $Q_i$. Let $\delta$ be the length of the orbit of $R$, and $m'$ its multiplicity. Then $p_0(C') = \ldots$
55 \geq 24 + \delta \frac{m'(m' - 1)}{2}. Taking into the account the possible lengths of orbits, and the uniqueness of orbits of any length < 84, we obtain two possible values $\delta = 21$ or $28$, and $m' = 2$. (Remark, that $\delta = 28$ really occurs for the dual of Klein’s curve, and the corresponding singular points are ordinary double). So, the maximal multiplicity of singular points of $C$ is $m = 3$, attained at the points $Q_i$. By [Ko, Lemma 8.10], we have the following estimate for the log canonical threshold: $c_Q(S, C) \geq \frac{1}{\text{mult}_{Q} C} \geq \frac{1}{3}$. As $\frac{3}{d} < \frac{1}{3}$ in our case, we are done.

If $d = 18$, then $C = C_1 \cup C''$, where $C'' = \{\lambda f^2 \Delta + \mu C = 0\}$ is irreducible of degree 14. Similarly to the above, we see that the only possibility for the intersection locus $C_1 \cap C''$ is the orbit of the point $\{y_1 = \exp 2\pi i, y_2 = \exp 4\pi i, y_3\}$ of length 56 (the other candidate, the orbit of length 28 taken with multiplicity 2, is eliminated because it is not contained in $C_1$!). So, the intersections are transversal. Assuming that there is an extra singular point of $C$, we find at once that the length of its orbit should be 21, and in this case the genus of $C''$ is 1. Klein’s group cannot act non-trivially on an elliptic curve. So, $C$ has only 56 ordinary double points as singularities, and hence $(S, \alpha C)$ is klt for all $\alpha < 1$.

**Subcase C.** $C = C_2 \cup C'$, where $C_2 = \{\Delta = 0\}$ is the Hessian curve, and $C' = \{\lambda f^3 + \mu \Delta^2 = 0\}$ is the irreducible curve of degree 12 from Subcase B. The 24 points $Q_i$ are the points of triple intersection of $C_2, C'$, so by Bezout, there are no other points of intersection. By the argument of Subcase B, $C'$ cannot acquire singularities, worse than double points. So, the points $Q_i$ are the singular points of $C$ of maximal multiplicity $m = 3$. The same argument as in Subcase B ends the proof.

The cases 1-3 cover all possible invariant curves of degree $\leq 18$, and in all these cases the pair $(V, \alpha F)$ is exceptional as soon as it is log canonical. By [2.3], this ends the proof of the Theorem.

**Remark 3.2.** The exceptional divisor $E$ of $f : Y \to X = \mathbb{C}^3/G$ of Sect. 2 (see [1]) for Klein’s group $G$ is the weighted projective plane $\mathbb{P}(4, 6, 14) = \mathbb{P}^2/G$. The different $\text{Diff}_E(0)$ (see [Sh1, Ch. 1] for the definition) in this case is $(1/2)\Gamma$, where $\Gamma$ is an irreducible curve, the image of 21 lines of fixed points of the elements of order 2 in $G$. The curve $\Gamma$ is given by the equation

$$0 = \mathcal{C}^3 + 1728\Delta^7 + 1008\mathcal{C}\Delta^4 f - 88\mathcal{C}^2\Delta f^2 - 60032\Delta^5 f^3 + 1088\mathcal{C}\Delta^2 f^4 + 22016\Delta^3 f^6 - 256\mathcal{C} f^7 - 2048\Delta f^9$$

of weighted degree 42 (see [3]). It is clear that $(E, \text{Diff}_E(0))$ is a log Del Pezzo surface. The weighted projective plane $E = \mathbb{P}(4, 6, 14)$ has
three singular points of types $A_1$, $A_2$ and $\mathbb{C}^2/\mathbb{Z}_7(2,3)$. The curve $\Gamma$ passes through the first of them and has two singular points: a simple cusp and a tacnode point (lying in the non singular part of $E$). It is easy to compute that $K_E + (1/2)\Gamma$ is $1/7$-log terminal, so Shokurov’s invariant $\delta$ (see \cite{Sh1}) is $0$ in our case. A very interesting question is to compute the minimal complement of $K_E + \text{Diff}_E(0)$.

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