The decomposition of permutation module for infinite Chevalley groups

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Abstract  Let $G$ be a connected reductive group defined over $F_q$, the finite field with $q$ elements. Let $B$ be a Borel subgroup defined over $F_q$. In this paper, we completely determine the composition factors of the induced module $M(tr) = kG \otimes_B tr$ (where $tr$ is the trivial $B$-module) for any field $k$.

Keywords  reductive group, abstract representation, permutation module, flag variety

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1 Introduction

The representations of reductive algebraic groups is an interesting and fundamental topic. It has deep connections to other areas of mathematics, for example, algebraic geometry and number theory. The earlier attention to this topic concentrated on the rational representations of algebraic groups and the representations of finite groups of Lie type. The cohomology theory of flag varieties and Deligne-Lusztig varieties control the rational representations of algebraic groups and ordinary representations of finite groups of Lie type, respectively.

One important class of irreducible modules of a reductive group (Lie algebra, respectively) comes from certain induced modules from a one-dimensional character of a Borel subgroup (Borel subalgebra, respectively). For the rational representations of algebraic groups and the representations of Lie algebras in the BGG category $O$ (Bernstein-Gelfand-Gelfand category $O$), it is known that all the irreducible modules are the simple quotients of Weyl modules and Verma modules, respectively. Moreover, the decomposition of Weyl modules and Verma modules motivates the famous Lusztig’s conjecture (see [9,10]) and Kazhdan-Lusztig conjecture (see [8]), respectively. For the representations of finite groups of Lie type in defining characteristic, such induced modules have been deeply investigated. For example, Carter and Lusztig [2] classified simple modules via certain homomorphisms between such induced modules. Moreover, Jantzen [7] and Pillen [11] indicated that the decomposition of such induced modules is closely related to the decomposition of Weyl modules.

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Despite the fruitful results above, little is known about the abstract representations of algebraic groups. Assume that \( k \) is a field and let \( \theta \) be a one-dimensional \( kB \)-module. It was observed in [14] that the induced module \( M(\theta) = kG \otimes kB \theta \) will give some new infinite-dimensional abstract representations of \( G \). In particular, \( M(\theta) \) contains a submodule \( S_t \) which is called infinite-dimensional Steinberg module. The irreducibility of \( S_t \) was proved in [14] for the defining characteristic, and in [15] for cross characteristic. Thus \( S_t \) is irreducible for any field \( k \) which is surprising. For the nontrivial character \( \theta \), it was proved in [3] that \( M(\theta) \) is irreducible if \( \theta \) is strongly antidominant, and in [4] that a certain submodule of \( M(\theta) \) is irreducible when \( \theta \) is antidominant. Xi [14] constructed a filtration of \( M(\theta) = kG \otimes kB \theta \) whose subquotients are indexed by the subsets of simple reflections. The second author of this paper proved in [6] that some of these subquotients are irreducible when the groups are of type \( A \) or rank 2 when \( \text{char} k \neq \text{char} \mathbb{F}_q \). Later it was proved in [5] that all of these subquotients are irreducible and pairwise non-isomorphic if \( \text{char} k \neq \text{char} \mathbb{F}_q \). This paper shows that the same result holds if \( \text{char} k = \text{char} \mathbb{F}_q \).

Thus we completely determine the composition factors of \( S_t \) for any field \( k \) (see Theorem 4.1). The constructions of these subquotients are uniform for all the fields, but the proof of irreducibility depends on the characteristic of \( k \). It would be interesting to find a characteristic free proof.

The rest of this paper is organized as follows: In Section 2 we recall some notations and basic facts about the structure of reductive groups. Section 3 recalls some basic properties of the induced modules \( M(\theta) \). Section 4 gives the proof of the main theorem and in Section 5 we give another approach to prove our main theorem. Section 6 lists some open problems for further study.

## 2 Reductive groups with Frobenius maps

In this section, we recall the basic notations and facts about the structure of reductive groups. Let \( G \) be a connected reductive group defined over \( \mathbb{F}_q \) with the standard Frobenius map \( F \). Let \( B \) be an \( F \)-stable Borel subgroup, \( T \) be an \( F \)-stable maximal torus contained in \( B \), and \( U = R_u(B) \) be the (\( F \)-stable) unipotent radical of \( B \). We denote \( \Phi = \Phi(G; T) \) the corresponding root system, and \( \Phi^+ (\Phi^-) \), respectively, is the set of positive (negative, respectively) roots determined by \( B \). Let \( W = N_G(T)/T \) be the corresponding Weyl group. For each \( w \in W \), let \( \bar{w} \) be a representative in \( N_G(T) \). One denotes \( \Delta = \{ \alpha_i \mid i \in I \} \) the set of simple roots and \( S = \{ s_i \mid i \in I \} \) the corresponding simple reflections in \( W \).

For each \( \alpha \in \Phi \), there is a unique unipotent subgroup \( U_\alpha \) of \( G \) which is isomorphic to \( \bar{\mathbb{F}}_q \) and is stable under the conjugation by \( T \). For each \( \alpha \), we fix an isomorphism \( \varepsilon_\alpha : \bar{\mathbb{F}}_q \to U_\alpha \) so that \( t \varepsilon_\alpha(c)t^{-1} = \varepsilon_\alpha(\alpha(t)c) \). For any \( w \in W \), we set

\[
\Phi^-_w = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^- \}, \quad \Phi^+_w = \{ \alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+ \}.
\]

Now assume \( \Phi^-_w = \{ \beta_1, \beta_2, \ldots, \beta_k \} \) and \( \Phi^+_w = \{ \gamma_1, \gamma_2, \ldots, \gamma_l \} \) for a given \( w \in W \), and denote

\[
U_w = U_{\beta_1} U_{\beta_2} \cdots U_{\beta_k} \quad \text{and} \quad U'_w = U_{\gamma_1} U_{\gamma_2} \cdots U_{\gamma_l}.
\]

The following properties are well known (see [1]):

(a) For \( w \in W \) and \( \alpha \in \Phi \) we have \( \bar{w}U_\alpha \bar{w}^{-1} = U_{w(\alpha)} \).

(b) \( U_w \) and \( U'_w \) are subgroups and \( \bar{w}U'_w \bar{w}^{-1} \subseteq U_w \).

(c) The multiplication map \( U_w \times U'_w \to U \) is a bijection.

(d) Each \( u \in U_w \) is uniquely expressible in the form \( u = u_{\beta_1} u_{\beta_2} \cdots u_{\beta_k} \) with \( u_{\beta_i} \in U_{\beta_i} \).

(e) (Commutator relations) Given two positive roots \( \alpha \) and \( \beta \), there exist a total ordering on \( \Phi^+ \) and integers \( c_{m,n}^{\alpha,\beta} \) such that

\[
[\varepsilon_\alpha(a), \varepsilon_\beta(b)] := \varepsilon_\alpha(a)\varepsilon_\beta(b)\varepsilon_\alpha(a)^{-1}\varepsilon_\beta(b)^{-1} = \prod_{m,n>0} \varepsilon_{m\alpha+n\beta}(c_{m,n}^{\alpha,\beta} a^m b^n)
\]

for all \( a, b \in \bar{\mathbb{F}}_q \), where the product is over all integers \( m, n > 0 \) such that \( m\alpha + n\beta \in \Phi^+ \), taken according to the chosen ordering.

In the following sections, we will often use the properties of root subgroups. Except the properties above, we have the following technical but useful lemma.
Lemma 2.1. Let \( s = s_\alpha \) be a simple reflection and \( ws > w \). If \( U_w = (U_w)^s \), then \( ws = tw \) for some \( t \in S \).

Proof. Let \( \Phi_w^- = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \). Since \( U_{ws} = U_\alpha (U_w)^s = U_\alpha U_w \), we have
\[
\Phi_{ws}^- = \{ \alpha, \alpha_1, \alpha_2, \ldots, \alpha_m \}.
\]
Let \( \Phi_w^+ = \Phi^+ \Phi_w^- = \{ \beta_1 = \alpha, \beta_2, \ldots, \beta_l \} \). Denote \( \alpha'_i = w(\alpha_i) \in \Phi^- \) and \( \beta'_i = w(\beta_i) \in \Phi^+ \). Then we have
\[
\Phi^+ = \{ -\alpha'_1, -\alpha'_2, \ldots, -\alpha'_m, \beta'_1, \beta'_2, \ldots, \beta'_l \}.
\]
Since \( U_w = (U_w)^s \), there is a permutation \( \sigma \) of \( \{ 1, 2, \ldots, m \} \) such that \( s(\alpha_i) = \alpha_{\sigma(i)} \). Therefore we have
\[
ws^{-1}(\alpha'_i) = ws(-\alpha_i) = w(-\alpha_{\sigma(i)}) = -\alpha'_{\sigma(i)}.
\]
Similarly, there is a permutation \( \tau \) of \( \{ 2, 3, \ldots, l \} \) such that \( ws^{-1}(\beta'_j) = \beta'_{\tau(j)} \) for \( j = 2, 3, \ldots, l \).

The above discussion implies \( \ell(ws^{-1}) \leq 1 \). But \( ws^{-1} \neq 1 \) and hence \( ws^{-1} = t \in S \) which completes the proof. \( \square \)

For \( J \subset I \), let \( W_J \) be the standard parabolic subgroup of \( W \) and assume that \( w_J \) is the longest element in \( W_J \). For \( w \in W \), set \( \mathscr{A}(w) = \{ s \in S \mid ws < w \} \) and denote
\[
W^J = \{ x \in W \mid x \text{ has minimal length in } xW_J \},
\]
\[
Y^J = \{ w \in W^J \mid \mathscr{A}(ww_J) = J \}.
\]

Corollary 2.2. Let \( s \in S \) and \( w \in Y^J \). If \( sw \in Y^J \) and \( sw > w \), then \( U_{wsw^{-1}} = (U_{wsw^{-1}})^s \).

Proof. Suppose \( U_{wsw^{-1}} = (U_{wsw^{-1}})^s \). Then by Lemma 2.1 there exists a simple reflection \( r \in S \) such that \( wJw^{-1}s = rwJw^{-1} \) which is a contradiction to \( sw \in Y^J \). \( \square \)

3 The permutation module

In this section, we recall some basic facts in [14] and [5]. Assume that \( k \) is a field. Let \( \mathbb{M}(tr) = kG \otimes_k B \text{ tr} \) where \( tr \) is the trivial \( kB \)-module, and call it the permutation module. Let \( 1_{tr} \) be a nonzero element in \( tr \). For convenience, we abbreviate \( x \otimes 1_{tr} \in \mathbb{M}(tr) \) to \( x1_{tr} \). Since \( T \) acts trivially on \( 1_{tr} \), the notation \( w1_{tr} = \hat{w}1_{tr} \) is well defined for any \( w \in W \). By using the Bruhat decomposition of \( G \), it is easy to see
\[
\mathbb{M}(tr) = \sum_{w \in W} kU_{w^{-1}w}1_{tr}.
\]
Moreover, the set \( \{ uw1_{tr} \mid w \in W, u \in U_{w^{-1}} \} \) forms a basis of \( \mathbb{M}(tr) \).

Remark 3.1. Let \( G = G^F \) and \( B = B^F \). Naturally, we have a “finite version” of \( \mathbb{M}(tr) \), namely, \( kG1_{tr} \), which is isomorphic to the induced module \( \text{Ind}_{G}^{G_1}1_B \), where \( 1_B \) is the trivial \( kB \)-module. For \( \mathbb{F} = \mathbb{C} \), the decomposition of \( \text{Ind}_{G}^{G_1}1_B \) is closely related to the representation of \( \mathcal{H} = \text{End}_G(\text{Ind}_{G}^{G_1}1_B) \) which is known as the Hecke algebra. For \( \mathbb{F} = \mathbb{F}_q \), it is known that \( \text{Ind}_{G}^{G_1}1_B \) decomposes into a direct sum of indecomposable modules, each with simple socle, and there is a bijection between the direct summands and the subsets of \( I \) (see [16, Proposition 4.5]). However, we have
\[
\text{End}_k(\mathbb{M}(tr)) \simeq k
\]
for any field \( k \), since it is clear that \( f(1_{tr}) \in \mathbb{M}(tr)^U = k1_{tr} \). Therefore the induced \( kG \)-module \( \mathbb{M}(tr) \) is indecomposable for any field \( k \).

For any \( J \subset I \), let \( W_J \) be the subgroup of \( W \) generated by \( s_i \) with \( i \in J \). We set
\[
\eta_J = \sum_{w \in W_J} (-1)^{\ell(w)}w1_{tr},
\]
and let \( \mathbb{M}(tr)_J = kG\eta_J \). It was proved in [14] that \( \mathbb{M}(tr)_J = kUW\eta_J \). The following lemma is well known and is very useful in our arguments later. The proof can be found in [14, Proposition 2.3] (see also [5, Lemma 2.1]).
Lemma 3.2. Let $u \in U^*_s = U_{s_0} \setminus \{1\}$ and $w \in W^J$. Then

1. There exist unique $x, y \in U^*_s$ and $t \in T$ such that $s_iu{t}_i^{-1} = xs_iyt$. Note that if we denote $x = f_i(u)$, then $f_i$ is an automorphism on $U^*_s$.
2. If $ww_J < sww_J$, then $s_iu{t}_i = s_iu{t}_iJ$.
3. If $s_iw < w$, then $s_iu{t}_i = xw{t}_i$, where $x$ is defined in (1).
4. If $s_iw > w$ and $s_iu{t}_i < uw{t}_i$, then $s_iu{t}_i = (x - 1)w{t}_i$, where $x$ is defined in (1).

Since $M(\text{tr})_J \supseteq M(\text{tr})_K$ if $J \subseteq K$, following [14, Subsection 2.6], we define

$$E_J = M(\text{tr})_J/M(\text{tr})_J'$$

where $M(\text{tr})_J'$ is the sum of all $M(\text{tr})_K$ with $J \subseteq K$. The following lemma was proved in [14].

Lemma 3.3 (See [14, Proposition 2.7]). If $J$ and $K$ are different subsets of $I$, then $E_J$ and $E_K$ are not isomorphic as $\mathbb{k}G$-modules.

We denote by $C_J$ the image of $\eta_J$ in $E_J$. Combining [6, Lemma 2.6] and [6, Lemma 2.7] we have the following proposition.

Proposition 3.4. The set \{uuC_J \mid w \in Y^J, u \in U_{w_Jw^{-1}}\} forms a basis of $E_J$.

For any subset $J \subset I$, the $\mathbb{k}G$-modules $E_J$ is a subquotient of $M(\text{tr})$. We can also realize $E_J$ as a $\mathbb{k}G$-submodule of a parabolic induced module. For $K \subset I$, let $P_K$ be the standard parabolic subgroup of $G$ generated by $B$ and $s_i$ with $i \in K$, and $M_K = \mathbb{k}G \otimes_{\mathbb{k}P_K} tr_K$, where $tr_K$ is the trivial $\mathbb{k}P_K$-module. Then $M_K$ is the quotient $\mathbb{k}G$-module of $M(\text{tr})$. Let $1_K$ be a nonzero element in $tr_K$. For convenience, we abbreviate $x \otimes 1_K \in M_K$ to $x1_K$. For $J \subset I$, we denote $J' = I \setminus J$. Let $E_J'$ be the $\mathbb{k}G$-submodule of $M_J'$ generated by $D_J := \sum_{w \in W_J}(-1)^{\ell(w)}w1_{J'}$. Combining Proposition 3.4 and [5, Proposition 3.2], we get the following proposition.

Proposition 3.5. For any $J \subset I$, the set \{uuD_J \mid w \in Y^J, u \in U_{w_Jw^{-1}}\} forms a basis of $E_J'$. In particular, $E_J' \cong E_J$ as $\mathbb{k}G$-modules.

4 Composition factors of $M(\text{tr})$

In this section we prove that $E_J$ is irreducible for any subset $J \subset I$. So we completely determine the composition factors of $M(\text{tr})$ for any field $\mathbb{k}$. The main theorem is the following Theorem 4.1.

Theorem 4.1. Let $\mathbb{k}$ be any field. Then all the $\mathbb{k}G$-modules $E_J$ ($J \subset I$) are irreducible and pairwise non-isomorphic. In particular, $M(\text{tr})$ has exactly $2^r$ composition factors, where $r$ is the rank of $G$.

Remark 4.2. Theorem 4.1 reflects a new phenomenon for infinite reductive groups. In other words, it does not hold when $\mathbb{k}G$ is replaced by $\mathbb{k}G_q^r$. When $\mathbb{k} = \mathbb{C}$, it is known that there is a bijection between the composition factors of $\mathbb{k}G_q^r1_{tr}$ and the composition factors of the regular module $\mathbb{k}W$, which preserves multiplicities. But the number of composition factors of $\mathbb{k}W$ is not equal to $2^r$ in general. When $\mathbb{k} = \mathbb{F}_q$, let $G = SL_2(\mathbb{F}_q)$. Then Theorem 4.1 says that $M(\text{tr})$ has 4 composition factors. But it was shown in [2, p. 382] that $G_q1_{tr}$ has 6 composition factors, where $G_q = SL_2(\mathbb{F}_q)$.

Theorem 4.1 was proved in [5] in the case where $\text{char} \mathbb{k} \neq \text{char} \mathbb{F}_q$. In this section we will prove Theorem 4.1 in the case where $\text{char} \mathbb{k} = \text{char} \mathbb{F}_q$. From here to the end of this section, we always assume that $\text{char} \mathbb{k} = \text{char} \mathbb{F}_q$.

For any finite subset $H$ of $G$, \[ H := \sum_{h \in H} h \in \mathbb{k}G \] (this is a frequently used notation in the arguments below). It is clear that $H \cdot H = 0$ if $H$ is a subgroup and $\text{char} \mathbb{k}$ divides $|H|$. For each $F$-stable subgroup $H$ of $G$, denote $H_{\mathfrak{g}} := H^F$.

Although Theorem 4.1 works for any field $\mathbb{k}$, the arguments in this paper are significantly different from that in the case where $\text{char} \mathbb{k} \neq \text{char} \mathbb{F}_q$ in [5]. The following arguments, especially Propositions 4.3 and 4.4, rely heavily on the condition that $\text{char} \mathbb{k} = \text{char} \mathbb{F}_q$. While [5, Lemma 2.4], one of the key steps of the arguments in [5], relies heavily on the condition $\text{char} \mathbb{k} \neq \text{char} \mathbb{F}_q$. [5, Lemma 2.4] says that for any $\mathbb{T}$-fixed nonzero element $\eta$ in a $\mathbb{k}G$-module $M$, we have $\mathbb{k}G\eta = \mathbb{k}G_U\eta$ if $\text{char} \mathbb{k} \neq \text{char} \mathbb{F}_q$. However, this
does not hold when char $k = \text{char } F_q$. For example, let $k = \bar{F}_q$, $M = M(\text{tr})$ and $\eta = 1_{\text{tr}} \in M$. Then it is clear that $kG_{1_{\text{tr}}} = M(\text{tr})$, while $kGU_{1_{\text{tr}}} = 1_{\text{tr}}$ for any $u \in U_q$ and char $k = \text{char } F_q$. Therefore, we cannot apply [5, Lemma 2.4] to prove Theorem 4.1 when char $k = \text{char } F_q$. So in this paper we use new ideas and techniques to deal with the defining characteristic case. Firstly, we list two key technical results (Propositions 4.3 and 4.4 below) used in the proof of Theorem 4.1.

For each nonempty subset $Y$ of $Y^J$, set $\Phi_Y = \bigcup_{w \in Y} \Phi^-_{w J w^{-1}}$. We fix a linear order on $\Phi_Y$ such that $\Phi^-_{Y^J} = \{\beta_1, \ldots, \beta_m\}$ with $|\beta(\beta_1)| \geq \cdots \geq |\beta(\beta_m)|$, and assume that the linear order of each $\Phi_Y$ (in particular, each $\Phi^-_{w J w^{-1}}$) is inherited from $\Phi_Y$.

Let $a, b \in \mathbb{N}$ such that $a \mid b$. For each $w \in Y^J$, write $\Phi^-_{w J w^{-1}} = \{\gamma_1, \ldots, \gamma_t\}$ with respect to the above order (in particular $|\gamma(\gamma_1)| \geq \cdots \geq |\gamma(\gamma_t)|$). For such $a, b, w$ and $0 \leq d \leq t$, set

$$\Theta(w, d, b, a) := U_{\gamma_t, q^b} \cdots U_{\gamma_d, q^b} \cdot U_{\gamma_{d+1}, q^a} \cdots U_{\gamma_{t}, q^a}.$$  

With the above notations, we have the following proposition.

**Proposition 4.3.** Assume that char $k = \text{char } F_q$, and $M$ is a nonzero $kG$-module. Let $Y$ be a nonempty subset of $Y^J$ and write $\Phi_Y = \{\alpha_1, \ldots, \alpha_n\}$ with respect to the above order. If

$$\{a_1, \ldots, a_d\} \subset Y \quad (d = |\Phi^-_Y| - 1)$$

for $a, b \in \mathbb{N}$ such that $a \neq b$ and $a \mid b$ (all $a_w \in k$ here are nonzero), then $U_{w J w^{-1}, q^c} C_{J} \in M$ for some $w \in Y^J$ and $c \in \mathbb{N}$.

**Proposition 4.4.** Assume that char $k = \text{char } F_q$, and $M$ is a nonzero $kG$-module. If $U_{w J w^{-1}, q^c} C_{J} \in M$ for some $a \in \mathbb{N}$, where $sw \in Y^J$ and $sw > w$ (this implies $w \in Y^J$), then $U_{w J w^{-1}, q^c} C_{J} \in M$ for some $b \in \mathbb{N}$.

Once Propositions 4.3 and 4.4 are proved, we can prove Theorem 4.1 in the case where char $k = \text{char } F_q$ as follows.

**Proof of Theorem 4.1.** For a fixed $J \subset I$, assume that $M$ is a nonzero $kG$-submodule of $E_J$. Let $E_{J, q^a} := kG_q C_J$. Choose a nonzero element $x \in M$. Then $x \in E_{J, q^a}$ for some $a \in \mathbb{N}$ since $E_J = \bigcup_{a \in \mathbb{N}} E_{J, q^a}$.

It is clear that $(kG_q x)^{U_{q^a}} \subset (E_{J, q^a})^{U_{q^a}} \subset \bigoplus_{w \in Y^J} kU_{w J w^{-1}, q^c} C_J$ by Lemma 3.4. Moreover, $(kG_q x)^{U_{q^a}} \neq 0$ by [12, Proposition 26]. There exists a nonzero element

$$\xi = \sum_{w \in Y^J} c_w U_{w J w^{-1}, q^c} C_J \in (kG_q x)^{U_{q^a}} \subset M, \quad c_w \in k.$$  

(4.1)

Choose an integer $b \neq a$ and $a \mid b$. Then $\xi = \sum_{w \in Y^J} c_w \Theta(w, 0, b, a) w C_J$. We apply Proposition 4.3 to $Y = \{w \in Y^J \mid c_w \neq 0\}$, $d = 0$ and $\xi = \xi_d$. Then $U_{w J w^{-1}, q^c} w C_J \in M$ for some $w \in Y^J$ and $c \in \mathbb{N}$. Applying Proposition 4.4 repeatedly, we see that $U_{w_1 J_1, q^m} C_J \in M$ for some $m \in \mathbb{N}$.

By [13, Lemma 2.1], since char $k = \text{char } F_q$, we have

$$\sum_{w \in W_J} (-1)^{\ell(w)} w U_{w J w^{-1}, q^c} C_J = \sum_{w \in W_J} q^{m \ell(w)} C_J = C_J \in M,$$

which implies that $E_J$ is irreducible. The set $J$ in the above arguments can be any subset of $I$, so all $E_J$ ($J \subset I$) are irreducible.

Therefore, we devote to proving Propositions 4.3 and 4.4 in the sequel. In order to prove these two propositions, we need the following technical lemma.

**Lemma 4.5.** Fix $w \in Y^J$ and let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be two disjoint subsets of $\Phi^-_{w J w^{-1}}$. Assume that $\sum_i l_i \alpha_i \in A$ whenever $\sum_j l_i \alpha_i \in \Phi^+$ for some $l_i \in \mathbb{Z}_{\geq 0}$. Let $a, b \in \mathbb{N}$ with $a \mid b$, and denote

$$\delta := U_{\alpha_1, q^b} \cdots U_{\alpha_m, q^b} \cdot U_{\beta_1, q^a} \cdots U_{\beta_n, q^a} w C_J.$$
Then we have

(i) Assume that \( k\beta_1 + \sum_i l_i\alpha_i \in A \) whenever \( k\beta_1 + \sum_i l_i\alpha_i \in \Phi^+ \) for some \( k \in \mathbb{Z}_{>0} \) and \( l_i \in \mathbb{Z}_{>0} \). Then

\[
\delta = U_{\alpha_1,q^h} \cdots U_{\alpha_m,q^h} \cdot x(U_{\beta_1,q^h} \cdots U_{\beta_n,q^h} w)C_f
\]

for any \( x \in U_{\beta_1,q^h} \).

(ii) Let \( \gamma \in \Phi^{+,+}_{w_j,w_{j-1}} \). Assume that \( k\gamma + \sum_i l_i\alpha_i + \sum m_i\beta_i \in A \) whenever \( k\gamma + \sum_i l_i\alpha_i + \sum m_i\beta_i \in \Phi^{+,+}_{w_j,w_{j-1}} \) for some \( k \in \mathbb{N} \) and \( l_i,m_i \in \mathbb{Z}_{>0} \). Then \( y\delta = \delta \) for any \( y \in U_{\gamma,q^h} \).

Proof. (i) By the commutator formula and the assumption, it is easy to show that \( \Phi \) for any \( \sigma \).

Indeed, assume that \( \sigma \) proves the claim. Since \( U \) Let \( g \)

\( y \) satisfies the equation \( (4.3) \) that \( g_{l-1}^{-1} g_{l}^{-1} g_{l} \in U_{w_j,w_{j-1}} \cap U_{w_j,w_{j-1}} = \{1\} \), and hence \( q_1 = q' \) which proves the claim. Since \( zwD_j = wD_j \) for any \( z \in U_{w_j,w_{j-1}} \), we have \( y\delta = \delta \) for any \( y \in U_{\gamma,q^h} \) thanks to the equation \( (4.2) \) and the injectivity of \( \sigma \).

With this preparation in hand, we can give the following proof.

Proof of Proposition 4.3. We will prove the proposition by the induction on \( |Y| \). If \( |Y| = 1 \), then \( |\xi_d| = c\Theta(w,d,b,a)wC_f \in M \) for some \( c \in \mathbb{k}^* \) and \( w \in Y^J \). We consider the \( \mathbb{k}U_{q^h} \)-module \( N = \mathbb{k}U_{q^h}\Theta(w,d,b,a)wC_f \subset M \). Clearly, \( N^{U_{q^h}} \neq 0 \) by [12, Proposition 26]. Note that \( N^{U_{q^h}} \subset \mathbb{k}U_{q^h}wC_f \).

Assume that \( |Y| > 1 \). Let \( \{I_i\} \) be a set of left coset representatives of \( U_{\alpha_i,q^h} \) in \( U_{\alpha_i,q^h} \). Let \( b \) be the minimal such that \( \alpha_{d+l} \notin \Phi^{+,+}_{w_j,w_{j-1}} \) for some \( w \in Y \). Since \( \Phi^{+,+}_{w_j,w_{j-1}} \neq \Phi^{+,+}_{w_2,w_{2-1}} \) if \( w_1 \neq w_2 \), such an \( l \) always exists.

If \( w \in Y \) and \( \alpha_{d+l} \notin \Phi^{+,+}_{w_j,w_{j-1}} \), combining our assumption on the order in each \( \Phi^{+,+}_{w_j,w_{j-1}} \) and Lemma 4.5(iii)

\[
I_{d+i+1} \Theta(w,d+i,b,a)wC_f = \Theta(w,d+i+1,b,a)wC_f
\]

for all \( 0 \leq i < l-1 \), and Lemma 4.5(ii)

\[
I_{d+i} \Theta(w,d+l-1,b,a)wC_f = q^{b-a} \Theta(w,d+l-1,b,a)wC_f = 0
\]

since \( \text{char} \mathbb{k} = \text{char} \mathbb{F}_q \) and \( b \neq a \). Thus, combining (4.4) and (4.5) yields

\[
I_{d+l}I_{d+l} \cdots I_{d+l} \Theta(w,d,b,a)wC_f = 0.
\]

If \( w \in Y \) and \( \alpha_{d+l} \in \Phi^{+,+}_{w_j,w_{j-1}} \), we have

\[
I_{d+l}I_{d+l} \cdots I_{d+l} \Theta(w,d,b,a)wC_f = \Theta(w,d+l,b,a)wC_f
\]

by Lemma 4.5(i). Denote \( \xi_{d+l} := I_{d+l}I_{d+l} \cdots I_{d+l} \xi_d \in M \) and let \( Y' \) be the set of \( w \in Y^J \) such that the coefficient of \( \Theta(w,d+l,b,a)wC_f \) in \( \xi_{d+l} \) is nonzero. Combining (4.6), (4.7) and the minimality of \( l \), we see that \( \xi_{d+l} \neq 0 \) (equivalently, \( Y' \) is nonempty) and \( Y' \subseteq Y \) (in particular \( |Y'| < |Y| \)). Notice that \( \{\alpha_1, \ldots, \alpha_{d+l}\} \subset \bigcap_{w \in Y'} \Phi^{+,+}_{w_j,w_{j-1}} \). The lemma follows from applying the induction hypothesis to \( Y' \) and \( \xi_{d+l} \).
Proof of Proposition 4.4. We may assume that $a$ is big enough such that each $w \in W$ has a representative $\dot{w}$ in $G_{q^a}$. Fix a representative $\dot{s}$ of $s = s_\alpha$ in $G_{q^a}$. Since $U_{w, j w^{-1}} = U_\alpha(U_{j w^{-1}})^a$, we have

$$\dot{s} U_{w, j w^{-1}, q, s} w C_J = \dot{s} U_{\alpha, q, s} w C_J.$$

By Lemma 3.2(1), the above equation equals

$$(U_{\alpha, q, s})^s w C_J = U_{w, j w^{-1}, q, s} w C_J + U_{w, j w^{-1}, q, s} w C_J - (U_{w, j w^{-1}, q, s})^s w C_J.$$

By the assumption $U_{w, j w^{-1}, q, s} w C_J \in M$, we get

$$U_{w, j w^{-1}, q, s} w C_J - (U_{w, j w^{-1}, q, s})^s w C_J \in M.$$

Let $\Phi^-_{w, j w^{-1}} \cap \Phi^-_{w, j w^{-1}} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. By Corollary 2.2 we have $U_{w, j w^{-1}} \neq (U_{w, j w^{-1}})^a$, which implies $\Phi^-_{w, j w^{-1}} \cap \Phi^-_{w, j w^{-1}} = 0$. Let $\Phi^-_{w, j w^{-1}} \cap \Phi^-_{w, j w^{-1}} = \{\beta_1, \beta_2, \ldots, \beta_n\}$. Hence $U_{w, j w^{-1}}$ is the product of $U_{\alpha_i}$ and $U_{\beta_j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Write $\gamma_i = s(\beta_i)$. Then $(U_{w, j w^{-1}})^s$ is the product of $U_{\alpha_i}$ and $U_{\beta_j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Choose $\beta_H \in \{\beta_1, \beta_2, \ldots, \beta_n\}$ such that

$$\text{ht}(\beta_H) = \max\{\text{ht}(\beta_1), \text{ht}(\beta_2), \ldots, \text{ht}(\beta_n)\}.$$ 

Then the following property holds: (♣) $\beta_H + \gamma_i \neq \gamma_j$ for any $i, j$.

Indeed, we have

$$w, j w^{-1}(\beta_H) = w, j w^{-1}(\beta_H) - (\beta_H, \alpha^\vee) w, j w^{-1}(\alpha) \in \Phi^+.$$

Since $w, j w^{-1}(\beta_H) \in \Phi^-$ and $w, j w^{-1}(\alpha) \in \Phi^+$, this forces $(\beta_H, \alpha^\vee) < 0$. If $\beta_H + \gamma_i = \gamma_j$, then

$$\beta_j = s(\beta_H) + \beta_i = \beta_H + \beta_i - (\beta_H, \alpha^\vee) \alpha.$$

It follows that $\text{ht}(\beta_j) > \text{ht}(\beta_H)$ which contradicts the choice of $\beta_H$. This proves Property (♣).

We consider the following set:

$$V = \prod_{\text{ht}(\alpha_i) > \text{ht}(\beta_H)} U_{\alpha_i}.$$

It is clear that $V$ is a subgroup of $U_{w, j w^{-1}}$ and also a subgroup of $(U_{w, j w^{-1}})^a$. Let

$$V_1 = \prod_{1 \leq i \leq n} U_{\alpha_i} \prod_{1 \leq i \leq n} U_{\beta_i}, \quad V_2 = \prod_{1 \leq i \leq n} U_{\alpha_i} \prod_{1 \leq i \leq n} U_{\gamma_i}.$$

Then $U_{w, j w^{-1}} = V_1$ and $(U_{w, j w^{-1}})^a = V_2$. Let $b \in \mathbb{N}$ such that $b \neq a$ and $a \mid b$ and $I$ be a set of the left coset representatives of $V_{q^a}$ in $V_{q^b}$, and write

$$\xi := I \cdot (U_{w, j w^{-1}, q, s} w C_J) \quad \text{and} \quad \eta := I \cdot (U_{w, j w^{-1}, q, s})^s w C_J.$$

We set

$$V_{1, q^a} = \prod_{1 \leq i \leq n} U_{\alpha_i, q^a} \prod_{1 \leq i \leq n} U_{\beta_i, q^a}, \quad V_{2, q^a} = \prod_{1 \leq i \leq n} U_{\alpha_i, q^a} \prod_{1 \leq i \leq n} U_{\gamma_i, q^a}.$$

It is clear that

$$\xi = V_{q^b} V_{1, q^a} w C_J \quad \text{and} \quad \eta = V_{q^b} V_{2, q^a} w C_J.$$

Since $U_{w, j w^{-1}, q, s} w C_J - (U_{w, j w^{-1}, q, s})^s w C_J \in M$, we have $\xi - \eta \in M$.

Let $I_H$ be a set of the left coset representatives of $U_{\beta_H, q^a}$ in $U_{\beta_H, q^b}$. Using Property (♣) and Lemma 4.5(ii), we obtain $I_{Hq} = q^{-b-1} \eta = 0$ since char $k$ = char $F_q$. Therefore by Lemma 4.5(i), $I_{Hq} \xi \in M$ is nonzero. Let $N = k U_{q^b} I_{Hq} \xi \subset M$. Then $N U_{q^b} \neq 0$ by [12, Proposition 26]. Since $N U_{q^b} \subset (k U_{q^b} w C_J)^{q^b} = k U_{w, j w^{-1}, q^b} w C_J$, we have $U_{w, j w^{-1}, q^b} w C_J \in M$ which completes the proof. \qed
5 Another proof of Theorem 4.1

In this section, we assume that \( \text{char } k = \text{char } \mathbb{F}_q \). Let \( w_0 \) be the longest element in \( W \) and write \( w_0 = v_I w_J w_H \) with \( \ell(w_0) = \ell(v_I) + \ell(w_J) + \ell(w_H) \) (recall that \( J' = I \setminus J \)). In this section, we combine Proposition 4.4 and Proposition 5.1 below to give another proof of Theorem 4.1.

**Proposition 5.1.** Let \( M \) be a nonzero \( kG \)-submodule of \( E_J \). Then

\[
U_{w, q} w_{J, q}^{-1} v_J D_J \in M
\]

for some \( a \in \mathbb{N} \).

To prove this proposition, we make some preparation. Following [2, Proposition 3.16], for any \( a \in \mathbb{N} \) and \( w \in W \) there is a \( T_w \in \text{End}_{kG, q} (kG, q_1) \) such that \( T_w 1_{tr} = U_{w, q} w_{J, q}^{-1} 1_{tr} \). For any \( J \subset I \), denote

\[
f_J^q = \sum_{w \in w_0 W_J} T_w 1_{tr} = \sum_{w \in w_0 W_J} U_{w, q} w_{J, q}^{-1} 1_{tr}.
\]

Combining [2, Theorem 7.1], [2, Theorem 7.4] and [2, Corollary 7.5] yields the following lemma.

**Lemma 5.2.** The map \( J \mapsto kG_q f_J^q \) is a bijection between the subsets of \( I \) and the irreducible summands of \( \text{Soc} G_q kG_q 1_{tr} \). Moreover, the stabilizer of the space \( kG_q f_I^q \) in \( G_q \) is \( P_{J, q} \), and \( kG_q f_I^q \) is the unique one-dimensional \( U_q \)-invariant space in \( kG_q f_I^q \).

Keep the notation \( P_K, 1_K, M_K \) and \( J' \end{section} \) in the end of Section 3. For any \( a \in \mathbb{N} \), let

\[
f_{K, q} = \sum_{w \in W_K} U_{w_1 - a, q} w_{1_{tr}} \in M(1)\text{.}
\]

Since \( f_{K, q} \) is \( P_{K, q} \)-invariant and all \( u w f_{K, q} \) (\( w \in W_K, u \in U_{w_1 - a, q} \)) are linearly independent, the \( kG_q \)-module \( M_{K, q} = kG_q 1_K \subset \mathcal{M}_K \) is isomorphic to the \( kG_q \)-submodule of \( kG_q 1_{tr} \) generated by \( f_{K, q} \) (via \( 1_K \mapsto f_{K, q} \)). The \( kG_q \)-module \( E_{J, q}^J = kG_q D_J \) is isomorphic to the submodule of \( kG_q 1_{tr} \) generated by the element \( \sum_{w \in W_J} (-1)^{\ell(w)} w_{J, q} = (-1)^{\ell(J)} w_{J, q} \) (via \( D_J \mapsto \sum_{w \in W_J} (-1)^{\ell(w)} w_{J, q} \)). We denote \( \varphi \) for this isomorphism in the sequel.

Since the conjugation by \( w_0 \) permutes the simple reflections, this induces a permutation \( \sigma \) on \( I \). Notice that \( W_{\sigma J} = w_0 W_J w_0 \). By definition we have

\[
f_{J', q} = \sum_{w \in W_{J'} w_0} U_{w, q} w_{J, q}^{-1} 1_{tr} = \sum_{w \in W_{J'} w_0} U_{w_1 - a, q} w_{1_{tr}}\text{.}
\]

The above formula implies

\[
f_{J', q} = \sum_{w \in W_{J'} w_0} U_{w, q} w_{J, q}^{-1} v_J w_J w_1 = \sum_{w \in W_{J'} w_0} U_{w_1 - a, q} v_J w_J 1_{tr}\text{.}
\]

(5.1) By the definition of \( \varphi \), we have

\[
\varphi \left( \sum_{w \in W_{J'}} (-1)^{\ell(w)} U_{w_1 - a, q} v_J w_J \right) = U_{w_1 - a, q} v_J D_J\text{.}
\]

(5.2) Assume that \( w \notin w_J \). Then there exists a \( \gamma \in \Phi^+ \) such that \( w_J v_J^{-1} \gamma \in \Phi^- \) and \( w_1 v_J^{-1} \gamma \in \Phi^+ \) and hence

\[
U_{w, q} v_J f_{J', q} = q a v_J f_{J', q} = 0.
\]

(5.3) It follows that

\[
U_{w_1 - a, q} v_J f_{J', q} = 0\text{.}
\]

(5.4) if \( w \notin w_J \). Combining (5.1)–(5.4) yields

\[
\varphi (f_{J', q}^-) = (-1)^{\ell(w)} U_{w_1 - a, q} v_J D_J \in E_{J', q}^J.
\]
Lemma 5.3. The $\kG_{q^a}$-socle of $E'_{J,q^a}$ is simple and generated by $\varphi(f_{q^a}^{J'})$.

Proof. By the above discussion and Lemma 5.2, $\text{Soc}_{G_{q^a}}E'_{J,q^a} \supset \kG_{q^a}\varphi(f_{q^a}^{J'})$. It remains to show that $\varphi(f_{q^a}^{J'}) \notin E'_{J,q^a}$ for $K \neq J$ by Lemma 5.2. Suppose that $\varphi(f_{q^a}^{J'}) \in E'_{J,q^a}$. Then we have $D_K \in E'_{J,q^a}$, by the same arguments in the previous section and the above discussion. It follows that $E'_K \subset E'_J$ and taking the $T$-fixed points yields the inclusion $\phi: (E'_K)^T \rightarrow (E'_J)^T$. But $D_K \in (E'_K)^T$ is uniquely determined by the following two conditions: (i) $\delta_i D_K = -D_K$ if and only if $i \in K$, and (ii) $U_{\alpha_i} D_K = D_K$ if and only if $i \notin K$. Therefore, $K \neq J$ implies any nonzero element in $(E'_J)^T$ does not satisfy the above conditions for $D_K$, and such $\phi$ does not exist. This contradiction completes the proof. \hfill \Box

With the above preparation, we can give the following proof.

Proof of Proposition 5.1. Let $0 \neq x \in M$. Then $x \in M \cap E'_{J,q^a}$ for some $a \in \mathbb{N}$, and hence

$0 \neq \kG_{q^a} x \supset \text{Soc}_{G_{q^a}} E'_{J,q^a} = \kG_{q^a} \varphi(f_{q^a}^{J'})$

by Lemma 5.3. It follows that $\varphi(f_{q^a}^{J'}) = (-1)^{(w_{J'})U_{w_Jw_J^{-1}q^a}v_J} D_J \in M$ which completes the proof. \hfill \Box

Using Proposition 5.1 and the same discussion in Section 4, we can also prove that $E'_J$ is irreducible which implies the irreducibility of $E_J$ by Proposition 3.5.

6 Further developments

In this section we propose some questions on infinite-dimensional abstract representations of reductive groups with Frobenius maps. Any one-dimensional representation $\theta$ of $T$ is regarded as a representation of $B$ through the homomorphism $B \rightarrow T$. Let $M(\theta) = kG \otimes_B \theta$. If $k = \bar{F}_q$ and $\theta$ is a rational character of $T$, the first author of this paper gave in [3] a necessary and sufficient condition for irreducibility of $M(\theta)$, and found some $M(\theta)$ with infinitely many irreducible subquotients. The following questions naturally arise:

(1) Can one give a characteristic free proof of Theorem 4.1?

(2) What is the necessary and sufficient condition for $M(\theta)$ to have finitely many composition factors? If so, how does $M(\theta)$ decompose?

(3) Besides the irreducibility of $E_J$, Proposition 3.5 is more interesting in its own right. Now that $E_J$ can be realized as a submodule of a parabolic induced module, can one give a geometric construction of $E_J$ (probably using the geometry of partial flag varieties $G/P_K, K \subset I$)?

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References
1. Carter R W. Finite Groups of Lie Type: Conjugacy Classes and Complex Characters. New York: John Wiley & Sons, 1985
2. Carter R W, Lusztig G. Modular representations of finite groups of Lie type. Proc London Math Soc, 1976, 32: 347–384
3. Chen X Y. On the principal series representations of semisimple groups with Frobenius maps. arXiv:1702.05686v2, 2017
4. Chen X Y. Some non quasi-finite irreducible modules of semisimple groups with Frobenius maps. arXiv:1705.04845v1, 2017
5. Chen X Y, Dong J B. The permutation module on flag varieties in cross characteristic. Math Z, 2019, 293: 475–484
6. Dong J B. Irreducibility of certain subquotients of spherical principal series representations of reductive groups with Frobenius maps. arXiv:1702.01888v2, 2017
7. Jantzen J C. Filtrierungen der Darstellungen in der Hauptserie endlicher Chevalley-Gruppen. Proc Lond Math Soc (3), 1984, 49: 445–482
8 Kazhdan D, Lusztig G. Representations of Coxeter groups and Hecke algebras. Invent Math, 1979, 53: 165–184
9 Lusztig G. Hecke algebras and Jantzen’s generic decomposition patterns. Adv Math, 1980, 37: 121–164
10 Lusztig G. Some problems in the representation theory of finite Chevalley groups. In: Proceedings of the Symposium on Pure Mathematics, vol. 37. Providence: Amer Math Soc, 1980, 313–317
11 Pillen C. Loewy series for principal series representations of finite Chevalley groups. J Algebra, 1997, 189: 101–124
12 Serre J P. Linear Representations of Finite Groups. Graduate Texts in Mathematics, vol. 42. New York-Heidelberg: Springer-Verlag, 1977
13 Steinberg R. Prime power representations of finite linear groups II. Canada J Math, 1957, 9: 347–351
14 Xi N H. Some infinite dimensional representations of reductive groups with Frobenius maps. Sci China Math, 2014, 57: 1109–1120
15 Yang R T. Irreducibility of infinite dimensional Steinberg modules of reductive groups with Frobenius maps. J Algebra, 2019, 533: 17–24
16 Yutaka Y. A generalization of Pillen’s theorem for principal series modules II. J Algebra, 2015, 429: 177–191