Zeroth WKB Approximation in Quantum Mechanics

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Abstract

Solution of the Schrödinger’s equation in the zero order WKB approximation is analyzed. We observe and investigate several remarkable features of the WKB method. Solution in the whole region is built with the help of simple connection formulas we derive from basic requirements of continuity and finiteness for the wave function in quantum mechanics. We show that, for conservative quantum systems, not only total energy, but also momentum is the constant of motion. We derive the quantization conditions for two and more turning point problems. Exact energy eigenvalues for solvable and some “insoluble” potentials are obtained. The eigenfunctions have the form of a standing wave, \( A_n \cos(k_n x + \delta_n) \), and are the asymptote of the exact solution.

1 Introduction

The wave equation in quantum mechanics can be derived with the help of the Bohr’s correspondence principle. This fundamental principle has been used at the stage of creation of quantum theory. It is used to establish correspondence between classical functions and operators of quantum mechanics, and to derive the apparent form of the operators [1, 2]. Moreover, the correspondence principle points out the way to a simplest asymptotic solution of the Schrödinger’s equation.

The correspondence principle states that the laws of quantum physics must be so chosen that in the classical limit, where many quanta are involved, the quantum laws lead to the classical equations as an average. In this way, in Ref. [3] this principle has been used to derive the nonrelativistic semiclassical wave equation appropriate in the quasiclassical region and, in Ref. [4], the corresponding relativistic wave equation. It was shown that application of the standard WKB method to the semiclassical wave equation allows to solve the so called “insoluble” problems [1, 2, 3] and results in the existence quantum fluctuations of the angular momentum, which contribute to the energy of the ground state [4].

There is a mathematical realization of the correspondence principle known as the quasiclassical approximation which is widely used mainly as the Wentzel-Kramers-Brillouin (WKB) method [5, 6, 7, 8] applicable in the case when the de Broglie wavelength, \( \lambda = h/p \) \((h = 2\pi \hbar)\), is changing slowly. This method is usually used as a tool to obtain the approximate solution of the one-dimensional Schrödinger’s equation in the quasiclassical region at large values of quantum numbers.

In this work, we analyze the classical limit of the Schrödinger’s wave mechanics with the help of the correspondence principle. We investigate solution of the wave equation...
in the zero-order WKB approximation (WKB₀) and observe several remarkable features and important advantages of the zeroth approximation with reference to the commonly used first-order WKB approximation (WKB₁). These advantages are: 1) the WKB₀ wave function (ψ₀(x)), has no divergencies at the turning points; 2) the WKB₀ approximation allows a considerably simpler (than the standard approach based on linear approximation) derivation of the connection formulas and quantization condition; 3) the WKB₀ method allows to solve the wave equation for many-turning-point problems; 4) for stationary states, not only energy, but also momentum is the constant of motion; and, finally, 5) the most important advantage is that the WKB₀ approach reproduces the exact energy spectra for all known solvable potentials and “insoluble” potentials as well.

Solution of the wave equation in the whole region is built with the help of connection formulas we derive in this work, which give the main term of the asymptotic series. To demonstrate efficiency of the WKB₀ approach, we calculate exact energy eigenvalues for solvable and some “insoluble” spherically symmetric potentials. For discrete spectrum, the WKB₀ eigenfunctions are written in terms of elementary functions in the form of a standing wave, \( A_n \cos(k_n x + \delta_n) \), and give the asymptote of the exact solution.

2 The zero order WKB solution

It is well known that the exact eigenvalues can be defined with the help of the asymptotic solution, i.e. the exact solution and its asymptote correspond to the same exact eigenvalues of the problem under consideration. The asymptotic solution in quantum mechanics can be obtained by the WKB method. This means that the quasiclassical method can be considered as a tool to reproduce the exact energy spectrum. This is true for some potentials [12, 13, 14, 15] and has proven for many others with the help of specially developed techniques, improvements, or modifications of the quasiclassical method on the real axis and in the complex plane [8, 12, 16, 17, 18, 19].

The main problem concerning the applicability of the WKB approximation is the problem of exactness of the method. Proofs of varying degrees of rigor have been advanced that demonstrate the exactness of the standard WKB quantization condition [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The question of exactness of the WKB approximation is usually reduced to the estimation of the high-order correction terms. The earliest development of the WKB method for obtaining the high-order corrections has been considered in Ref. [12]. Then, in [13], the radial generalization of Dunham’s one-dimensional WKB quantization condition was derived with the help of the Langer transformation [18]. It was shown that the second- and third-order integrals identically vanish for the hydrogen atom and the three-dimensional harmonic oscillator [13]. As we show below, the first order term also does not affect the exactness of the quasiclassical quantization condition.

Consider the static Schrödinger’s equation with the arbitrary potential \( V(x) \) in one dimension,

\[
\left[ \frac{1}{2m} \left(-i\hbar \frac{d}{dx}\right)^2 + V(x) \right] \psi(x) = E \psi(x). \tag{1}
\]
Let us write the equation (1) in the form,

\[
\left(-i\hbar \frac{d}{dx}\right)^2 \psi(x) = \left[P^2 - U(x)\right] \psi(x),
\]

where \( P^2 = 2mE \) and \( U(x) = 2mV(x) \). Equations (1) and (2) are equivalent, but the physical sense of their solutions is different. The Schrödinger’s equation (1) is the energy eigenvalue problem, and equation (2) is the eigenvalue problem for the squared momentum \( P^2 \). Equation (2) has two solutions for each eigenstate (as it should be for the second-order differential equation) with the corresponding eigenvalues \( P_{n}^{1,2} = \pm \sqrt{2mE_n} \), i.e. eigenvalues for these two problems are connected via the equality \( P_{n}^2 = 2mE_n \) for a free particle.

In the conventional WKB method, solution of Eq. (2) is searched for in the form

\[
\psi(t, x) = A \exp \left[\frac{i}{\hbar} S(t, x)\right],
\]

where \( A \) is the arbitrary constant and \( S(t, x) \) is the complex function. Substituting (3) into Eq. (2) one finds that \( S(t, x) \) satisfies the equation

\[
\left(\frac{dS}{dx}\right)^2 - \frac{i}{\hbar} \frac{d}{dx} \left(\frac{dS}{dx}\right) = P^2 - U(x).
\]

The Hamilton-Jacobi equation in quantum mechanics (4) is equivalent to the wave equation (2). In the limit \( \hbar \to 0 \) Eq. (4) reduces to the classical Hamilton-Jacobi equation.

In the WKB approximation one expands \( S(t, x) \) in power series of \( \hbar \),

\[
S = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots
\]

and usually keeps two first terms of the expansion, i.e. \( S_0(t, x) \) and \( S_1(t, x) \) (the WKB\(_1\) approximation). This gives, after simple calculations, \( (dS_0/dx)^2 = P^2 - U(x) \), and \( S_1(t, x) = i \ln \sqrt{|p(x)|} \), where \( p(x) = dS_0/dx \). For conservative systems, the classical action \( S_0(t, x) \) can be written as \( S_0(t, x) = -Et + W(x) \), where \( W(x) \) is the Hamilton’s function (reduced classical action). Hence, one gets

\[
\left(\frac{dW}{dx}\right)^2 = P^2 - U(x).
\]

Here \( dW/dx \equiv p(x) = \pm \sqrt{P^2 - U(x)} \) is the generalized momentum and the WKB\(_1\) \( \psi \) is

\[
\psi^{(1)}_{WKB}(x) = \frac{1}{\sqrt{|p(x)|}} \left[C_1 e^{i\phi(x)} + C_2 e^{-i\phi(x)}\right],
\]

where \( \phi(x) \) is dimensionless phase variable,

\[
\phi(x) = \frac{1}{\hbar} W(x) \equiv \frac{1}{\hbar} \int^x \sqrt{P^2 - U(x)} dx.
\]

There are several disadvantages of the WKB\(_1\) approximation. The most essential of them is that the WKB\(_1\) \( \psi \) (6) diverges at the classical turning points given by \( p(x) = 0 \), where the condition for its applicability breaks down. Although this divergence is understandable in the classical limit, since a classical particle has zero speed at the turning
points, it is certainly not present in a quantum mechanical treatment. Since \( \psi^{(1)}_{WKB}(x) \) is singular, tricky matching the \( \psi^{(1)}_{WKB}(x) \) and its first derivative at the classical turning points have been developed \[18\] that yields the famous connection formulas and well known WKB quantization condition. The exact mathematical conditions under which the connection formulas can be proved rigorously to apply are fairly complex. Other difficulties arising in the WKB\(_1\) approximation have been considered elsewhere \[3\].

The connection formulas and quantization condition can be obtained much simpler in the WKB\(_0\) approximation \[20, 21\], i.e. in the classical limit \( \hbar \to 0 \). In this case \( S(t, x) \approx S_0(t, x) \) and the general solution of the wave equation \( \Phi \) for the potential \( V(x) \) can be written in terms of the dimensionless phase variable \( \phi(x) \),

\[
\psi^{(0)}_{WKB}(x) \equiv \psi_0[\phi(x)] = C_1 e^{i\phi(x)} + C_2 e^{-i\phi(x)},
\]

where \( C_1 \) and \( C_2 \) are taken constant in the classical limit.

The WKB\(_0\) solution \( \Phi \) has several important advantages with reference to the commonly used WKB\(_1\) \( \psi \) \[4\]. The most important is that the solution \( \Phi \) has no divergence at turning points. In the phase space, the function \( \Phi \) has the form of superposition of two plane waves; it has also such a form in the configuration space (see below). Note, that the WKB\(_0\) approximation corresponds to the main term of the asymptotic series in the theory of the second-order differential equations.

### 3 Connection formulas. Quantization

Basic requirements for the \( \psi \) in quantum mechanics are continuity and finiteness in the whole region. The famous connection formulas \[18\] have been derived to connect the WKB solution in the classical region with the one in non-classical region. These formulas are based on a linear approximation to the potential in the regions around the classical turning points where the WKB\(_1\) \( \psi \) diverges; the exact solution involves Bessel’s functions of order \( 1/3 \). Even though the connection formulas themselves turn out to be simple enough, their derivation is quite tedious.

One of the most important problems to which the connection formulas apply is that of penetration of a potential barrier. Interaction with the potential barrier can be of two types - reflection and transmission. There are several problems in which the conventional WKB\(_1\) formulas break down \[1\]. One kind of problems occurs near the top of a barrier, where the slope of the potential is small. As a result, the straight-line approximation to the potential breaks down, and the connection formulas must be altered. The second type of problems arises when the potential changes too rapidly, for example, in the case of a square well potential. The connection formulas also break down for some other type of potentials \[1\].

Consider the WKB\(_0\) \( \psi \) \[8\] for the two-turning-poing (2TP) problem. To build the physical solution in the whole region we need to connect the oscillating solution in classically allowed region with the exponentially decaying solutions in classically inaccessible regions. For the 2TP problem, the whole interval \( (-\infty, \infty) \) is divided by the turning points \( x_1 \) and \( x_2 \) into three regions, \( (-\infty, x_1] \) (region I), \( [x_1, x_2] \) (region II), and \( [x_2, \infty) \)
(region III). In the classically allowed region II, right from the turning point \(x_1\), the real part of solution \(\psi\) can be written in the form

\[
\psi_{II}^0(\phi) = C_1 \cos(\phi - \phi_1 + \delta_1),
\]

and left from the turning point \(x_2\),

\[
\psi_{II}^0(\phi) = C_2 \cos(\phi - \phi_2 + \delta_2),
\]

where \(\phi_1 = \phi(x_1), \phi_2 = \phi(x_2)\), and \(\delta_{1,2}\) are the phase shifts at the turning points \(x_1\) and \(x_2\), respectively. In the classically inaccessible regions I and III, to guarantee finiteness of the \(\psi\) at \(x \to \pm\infty\), we choose the exponentially decaying solutions, i.e.,

\[
\psi_{I}^0(\phi) = Ae^{\phi - \phi_1},
\]

left from the turning point \(x_1\), and

\[
\psi_{III}^0(\phi) = Be^{-\phi + \phi_2},
\]

right from the turning point \(x_2\).

To construct solution in the whole region we need to connect functions \(\psi_{II}^0(\phi)\), \(\psi_{II}^0(\phi)\), and \(\psi_{II}^0(\phi)\) at the turning points \(x_1\) and \(x_2\). Matching the functions \(\psi_{II}^0(\phi)\) and \(\psi_{II}^0(\phi)\) and their first derivatives at the turning point \(x = x_1\) gives

\[
\begin{align*}
C_1 \cos \delta_1 &= A, \\
C_1 \sin \delta_1 &= -A,
\end{align*}
\]

and for \(\psi_{II}^0(\phi)\) and \(\psi_{II}^0(\phi)\) at the turning point \(x = x_2\) we have

\[
\begin{align*}
C_2 \cos \delta_2 &= B, \\
C_2 \sin \delta_2 &= B,
\end{align*}
\]

which yields

\[
\begin{align*}
\tan \delta_1 &= -1, \\
\tan \delta_2 &= 1,
\end{align*}
\]

and

\[
\begin{align*}
C_1 &= \sqrt{2}A, \\
C_2 &= \sqrt{2}B.
\end{align*}
\]

The connection formulas \((13)\) and \((16)\) supply the continuous transition of the oscillating solutions \((3)\) and \((10)\) into exponential solutions \((11)\) and \((12)\) at the turning points \(x_1\) and \(x_2\).

The function \(\psi_{II}^0(\phi)\) [given by Eqs. \((3)\) and \((10)\)] should match itself at each point of the interval \([x_1, x_2]\). Putting \(x = x_2\) we have, from Eqs. \((3)\) and \((10)\),

\[
C_1 \cos(\phi_2 - \phi_1 + \delta_1) = C_2 \cos \delta_2.
\]

This equation is valid if
φ₂ − φ₁ + δ₁ = δ₂ + πn,  n = 0, 1, 2, . . . \hspace{1cm} (18)

and \( C₂ = (−1)^n C₁ \). Equation (18) is the condition of the existence of continuous finite solution in the whole region. This condition being, at the same time, the quantization condition. Taking into account the notation (7), we have, from Eq. (18),

\[
\int_{x₁}^{x₂} \sqrt{p² - U(x)} dx = π\hbar \left( n + \frac{1}{2} \right). \hspace{1cm} (19)
\]

Condition (19) solves the 2TP eigenvalue problem given by Eq. (2). We see that the WKB₀ approximation results in the quantization of the reduced classical action, \( W(x) \), and coincides with the well known WKB quantization condition. It is important to underline that the first order term does not affect the exactness of the quantization condition. In our case, the condition (19) is obtained by product from general requirements of continuity and finiteness to the wave function in the whole region, i.e. from the requirements of a smooth transition of the oscillating solution given by Eqs. (4) and (10) to the exponentially decaying solutions (11) and (12) in the classically inaccessible regions. Quantization condition (19) reproduces the exact eigenvalues for all known 2TP problems (see Refs. [3, 4, 19]).

The derivation given above is evidently much simpler than the usual textbook approach for deriving connection formulas and quantization condition. In the general case of three or more turning points, phase shifts \( δ₁ \) and \( δ₂ \) will be different [22].

Combining the above results, we can write the finite continuous WKB₀ solution in the whole region (\( δ₂ = −δ₁ = π/4 \)),

\[
ψ₀[φ(x)] = C \begin{cases} 
\frac{1}{\sqrt{2}} e^{φ(x)−φ₁}, & x < x₁, \\
\cos[φ(x) − φ₁ − π₄], & x₁ ≤ x ≤ x₂, \\
\frac{−1}{\sqrt{2}} e^{−φ(x)+φ₂}, & x > x₂.
\end{cases} \hspace{1cm} (20)
\]

Oscillating part of solution (20) corresponds to the main term of the asymptotic series in theory of the second-order differential equations. In quantum mechanics, the oscillating part of Eq. (20) gives the asymptote of the exact solution of the Schrödinger’s equation (see below).

4 Momentum conservation for discrete spectrum

It is well known that, for discrete spectrum, the asymptote of Eq. (2) has the form

\[ ψₙ(x) = Aₙ \cos(kₙx + δₙ), \]

where \( Aₙ, kₙ = Pₙ/\hbar \), and \( δₙ \) are constant values; here \( kₙ \) is the wave number and \( Pₙ \) is the momentum eigenvalue. The asymptote has the form of a standing wave and can be written as a superposition of two plane waves, \( \exp(ikₙx) \) and \( \exp(−ikₙx) \), each of which describes free motion. Therefore, for the discrete spectrum, the asymptote of Eq. (2) describes free motion of a particle-wave in the enclosure and the eigenmomentum \( Pₙ = \hbar kₙ \) is the constant of motion.

The quasiclassical approximation is the asymptotic method, which takes advantage of the fact that the wave length is changing slowly, by assuming that the wave function is not changed...
much from the form it would take if the potential $V$ was constant. The WKB$_0$ solution (8) is justified in the classical limit $\hbar \to 0$, which corresponds to the short-wave asymptote of the Schrödinger’s equation (3). Our basic requirements to the solution (8) are: 1) $C_1$ and $C_2$ are constant values and 2) the generalized momentum $p(x) = dW/dx$ is the adiabatically slowly changing function, i.e.,

$$\frac{dW}{dx} \simeq \text{const.}$$  \hspace{1cm} (21)

These are the same assumptions which have been originally used by Schrödinger to derive the wave equation from the optical-mechanical analogy (see, for instance, Ref. [1]).

In this analogy, the most important aspect is the relation to the limiting case of wave optics, i.e. geometrical optics. The limiting case of geometrical optics is analogous to the classical limit of quantum mechanics: the amplitude of the $(wf)$ is a constant and momentum is the adiabatically slowly changing function. Note, the constraint (21) supplies the Hermiticity of the operator $\hat{p}^2 = [-i\hbar(d/dx)]^2$ in Eq. (2) [3].

Consider the WKB$_0$ eigenfunction (20) in the classically allowed region given by $p(x) > 0$. Show that the oscillating part of the $wf$ (20) is in agreement with the asymptote of the corresponding exact solution of Eq. (2) and can be written in the form of a standing wave. In equations (1) and (2), the total energy $E$ and the momentum $P$ are connected by means of the equality $P^2 = 2mE$ for a free particle. Integrating (21), we obtain, for the action $W(x)$,

$$W(x) \simeq Px + \text{const},$$ \hspace{1cm} (22)

where $P = \sqrt{2mE}$ is the total momentum. Solution of Eq. (1) [or (3)] results in the energy quantization, i.e. $E = E_n$. Therefore, $W(x) \simeq P_n x + \text{const}$, where $P_n = \hbar k_n = \sqrt{2mE_n}$. Underline, this form of the action is valid only for allowed motions in quantum mechanics, i.e. for the energies $E = E_n$, which can be found with the help of the quantization condition (19).

Thus, the oscillating part of the $wf$ (20) takes the form

$$\psi^I_0(x) = C_n \cos \left( \frac{1}{\hbar} P_n x + \frac{\pi}{2} n \right).$$ \hspace{1cm} (23)

The normalization coefficient,

$$C_n = \sqrt{\frac{2P_n}{\pi \hbar (n + \frac{1}{2}) + \hbar}},$$ \hspace{1cm} (24)

is calculated from the normalization condition

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 \, dx = 1.$$ \hspace{1cm} (25)

In Eq. (23), we have took into account the fact that, in the stationary states, the phase-space integral (4) at the TP $x_1$ and $x_2$ is $\phi_1 = -\pi(n + 1/2)/2$ and $\phi_2 = \pi(n + 1/2)/2$, respectively, so that $\phi_2 - \phi_1 = \pi(n + 1/2)$ [3, 19]. The phase integrals $\phi_1$ and $\phi_2$ depend on quantum number and do not depend on the form of the potential. This form of $\phi_1$ and
φ₂ guartanies that the eigenfunctions are necessarily either symmetrical \((n = 0, 2, 4, \ldots)\) or antisymmetrical \((n = 1, 3, 5, \ldots)\).

Solution \((23)\) has the form of a standing wave, which can be written as the superposition of two plane waves. The fact that, for stationary states, the oscillating part of solution \((23)\) can be written in such form means that (for conservative quantum-mechanical systems) the particle momentum is the constant of motion and the corresponding coordinate is cyclical. Therefore, solution \((23)\) describes free motion of a particle-wave in the enclosure, where the enclosure is the interaction potential. This means that interaction of the particle-wave with the potential reduces to reflection of the wave by the walls of the potential.

As other integrals of motion in quantum mechanics, the total momentum \(P\) can take only discrete values, \(P = P_n\), corresponding to the discrete values of the classical action \(W(x)\), which can be found from the quantization condition \((19)\). Note that, for a particle moving in the Coulomb potential, the energy eigenvalues can be written in the form of the total energy for a free particle, i.e. \(E_n = P_n^2/2m\), where \(P_n = iαm(νh)^{-1}\) is the momentum eigenvalue and \(n\) is the principal quantum number. It is easy to show that the energy eigenvalues for other solvable potentials also can be written in such form.

There are the following physical grounds to the above discussions. According to classical theory, accelerated electrons radiate energy at a rate equal to \(\frac{2}{3}e^2c^{-3}x^2\). But electrons in atomic orbits are always accelerated (??) and should, therefore, lose energy continuously until they fall into the nucleus. Actually, it is known that this never happens. To resolve this problem, Bohr postulated that 1) the electrons are in stationary states, i.e. do not radiate, despite their acceleration and 2) electrons can take discontinuous transitions from one allowed orbit to another. However, the postulate is not an answer to the question: why the accelerated electrons do not radiate?

Acceleration means change in momentum. In quantum mechanics, the change in momentum (which is associated with the de Broglie wave \(λ = h/p, h = 2π\hbar\)) appears as radiation of frequency \(ν = (E - E')/h\) that means change of the stationary state. As we have shown above, in the stationary states, particles are not accelerated, because the momentum is the constant of motion. This means that, in the stationary states, the electrons move as free particles, particles-waves in enclosures.

The well known quasiclassical condition,

\[
\frac{h}{p^2} \left| \frac{dp}{dx} \right| \ll 1,
\]

defines the region of applicability of the WKB₁ approximation. Conventional treatment of this condition implies that the momentum \(p(x)\) is large enough, i.e. quantum number \(n\) take large values. At the same time, the WKB method yields the exact eigenvalues for all values of \(n\), including \(n = 0\) (see Ref. [3, 4, 19]). This means that the condition \((24)\) can be treated differently: it supplies the hermiticity of the squared momentum operator, \(\hat{p}^2\), that implies \(dp/dx \simeq 0\), and does not imply that the momentum \(p(x)\) takes large values \([3]\).
5 The multi-turning-point problems

The WKB method is usually used to solve one-dimensional two turning point problems. Within the framework of the WKB method the solvable potentials mean those potentials for which the eigenvalue problem has two turning points. However the WKB₀ solution is general for all types of problems. The zeroth WKB approximation allows a considerably simpler derivation of the connection formulas and the corresponding quantization condition not only for two but, also, for many-turning-point problems (µTP, µ > 2). This is the class of the so-called “insoluble” problems, which cannot be solved by standard methods.

The quasiclassical formulas can be written both on the real axis and in the complex plane. Most general form of the WKB solution and quantization condition can be written in the complex plane. In the complex plane, the 2TP problem has one cut between turning points \( x_1 \) and \( x_2 \), and the phase-space integral (13) can be written as the contour integral about the cut. The µTP problems contain (in general case) bound state regions and the potential barriers, i.e. several cuts. The corresponding contour \( C \) should enclose all cuts. The WKB₀ \( \psi \) in the whole region can be built similarly to the 2TP one considered above.

Let the problem has \( \nu \) finite cuts, i.e. \( \mu = 2\nu \) turning points, and the potential is infinite between cuts. Then the integral about the contour \( C \) can be written as a sum of contour integrals about each of the cut,

\[
\oint_C \sqrt{P^2 - U(x)} \, dx = \sum_{i=1}^\nu \oint_{C_i} \sqrt{P^2 - U(x)} \, dx,
\]

where each term of the sum is the 2TP problem,

\[
\oint_{C_i} \sqrt{P^2 - U(x)} \, dx = 2\pi \hbar \left( n_i + \frac{1}{2} \right).
\]

Therefore, the µTP quantization condition is

\[
\oint_C \sqrt{P^2 - U(z)} \, dz = 2\pi \hbar \sum_{i=1}^\nu \left( n_i + \frac{1}{2} \right).
\]

The condition (29) is in agreement with the Maslov’s theory [23]. This means that the right-hand side of the equation (27) can be written as

\[
2\pi \hbar \sum_{i=1}^\nu \left( n_i + \frac{1}{2} \right) = 2\pi \hbar \left( N + \frac{\mu}{4} \right),
\]

where \( N = \sum_{i=1}^\nu n_i \) is the total number of zeroes of the \( \psi \) on the \( \nu \) cuts. In this interpretation, the number of turning points \( \mu \) is the Maslov’s index, i.e. number of reflections of the \( \psi \) on the walls of the potential. Thus the condition (29) takes the form

\[
\oint_C \sqrt{P^2 - U(z)} \, dz = 2\pi \hbar \left( N + \frac{\mu}{4} \right).
\]

\(^1\)In case where the potential is finite in the whole region, the quantization condition will be more complicate [22].
6 Multi-dimensional problems

In most practical applications we deal with the multi-dimensional problems. For known solvable spherically symmetric potentials, the conventional WKB method does not reproduce the exact energy levels unless one supplements it with Langer-like correction terms.

To overcome this problem in particular case of the Coulomb potential, a special technique has been developed. In order for the WKB approximation to give the exact eigenvalues, the quantity \( l(l+1) \) in the radial equation must be replaced by \( (l + 1/2)^2 \) \[18\]. The reason for this modification (for the special case of the Coulomb potential) was pointed out by Langer (1937) \[18\] from the Langer transformation \( r = e^x, U(r) = e^{x/2}X(x) \).

However, for other spherically symmetric potentials, in order to obtain the appropriate Langer-like correction terms, another special transformation of the \( \psi \) and its arguments is required.

There are several other related problems in the semiclassical consideration of the radial Schrödinger equation. (i) The WKB solution of the radial equation is irregular at \( r \to 0 \), i.e. \( R_{WKB}(r) \propto r^{\lambda}/\sqrt{r} \), whereas the exact solution in this limit is \( R(r) \propto r^l \). (ii) The radial equation has no the centrifugal term when \( l = 0 \), i.e. the radial problem has only one turning point and one cannot use the WKB quantization condition derived for two-turning-point problems. However, solving the equation for \( l > 0 \) by known exact methods one obtains energy eigenvalues for all \( l \). (iii) The WKB solution of the angular equation has analogous to the radial one, incorrect behavior at \( \theta \to 0 \): \( \Theta_{WKB}(\theta) \propto \theta^\mu \), \( \mu^2 = m^2 - \hbar^2/4 \), while the exact regular solution in this limit is \( \Theta_{reg}(\theta) \propto \theta^{|m|} \). Angular eigenfunction \( Y_{00}(\theta, \phi) = \text{const} \), i.e. no nontrivial solution exists.

Practical use shows the standard leading-order WKB approximation always reproduces the exact spectrum for the solvable spherically symmetric potentials \( V(r) \) if the centrifugal term in the radial Schrödinger’s equation has the form \( (l + 1/2)^2 \hbar^2/r^2 \). As was shown in Ref. \[19\] the centrifugal term of such a form can be obtained from the WKB solution of the angular Schrödinger’s equation if separation of the original three-dimensional equation has performed with the help of the correspondence principle.

In Refs. \[3, 19\] we have fulfilled the quasiclassical analysis of the three-dimensional Schrödinger’s equation and suggested the quasiclassical approach for multi-dimensional problems. In this approach the original wave equation is reduced to the form of the classical Hamilton-Jacobi equation without first derivatives. It was shown that, in the quasiclassical region the wave equation can be written in the form \[3, 19\],

\[
(-i\hbar)^2 \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \tilde{\psi}(\vec{r}) = 2m [E - V(r)] \tilde{\psi}(\vec{r}).
\]

Separation of this equation is performed with the help of the correspondence principle between classical and quantum-mechanical quantities. As a result of the separation, we have obtained a system of reduced second-order differential equations,

\[
\left( -i\hbar \frac{d}{dr} \right)^2 \tilde{R}(r) = \left[ 2m (E - V(r)) - \frac{\tilde{M}^2}{r^2} \right] \tilde{R}(r),
\]

\[33\]
where \( \tilde{M}^2, M_z^2 \) are the constants of separation and, at the same time, constants of motion.

Each of Eqs. (33)-(35) has the general WKB\(_0\) solution of the form (8). Application of the quantization condition (19) to the angular equation (34) results in the squared angular momentum eigenvalues \[ \tilde{M}^2 = \left( l + \frac{1}{2} \right)^2 \hbar^2. \] (36)

The same quantization condition appropriate to the radial equation (33) yields the exact energy spectra for all known solvable potentials and “insoluble” potentials, as well.

7 Some practical applications

To illustrate efficiency of the WKB\(_0\) method consider several classic problems. The general solution (8) is the same for all problems in quantum mechanics. In case of multi-dimensional problems, we use the corresponding multi-dimensional classical action. The oscillating part of the WKB\(_0\) \( \psi \) gives the asymptote of the exact solution of the Schrödinger’s equation. Most complicate computational part in this approach is calculation of the phase-space integral. In case of multi-dimensional problems, separation should be performed with the help of the correspondence principle between classical and quantum-mechanical quantities \[ \text{(7, 19)}. \]

7.1 The linear harmonic oscillator \( V(x) = \frac{1}{2}m\omega^2x^2 \)

Consider first the linear harmonic oscillator. Many problems in physics can be reduced to a harmonic oscillator with appropriate conditions. The general differential equation for oscillator potential can be solved using a technique that is frequently exploited in solving quantum mechanics problems. The eigenfunctions that are the solutions of the Schrödinger equation are the Hermite polynomials. In our approach, we directly obtain the asymptote of the exact solution and the corresponding well known exact energy eigenvalues.

The quantization condition (19) appropriate to the Schrödinger equation with the linear oscillator potential is

\[ I = \int_{r_1}^{r_2} \sqrt{2mE - (m\omega x)^2}dx = \pi\hbar \left( n + \frac{1}{2} \right). \] (37)

The phase-space integral (37) is calculated in closed form, \( I = E/\omega \), that results in the exact energy eigenvalues, \( E = \omega\hbar(n+1/2) \). The WKB\(_0\) \( \psi \) of the oscillator (the asymptote of the problem) is given by Eq. (23) and has the form of the standing wave.
\[ \psi_0(x) = C_n \cos \left( \frac{1}{\hbar} P_n x + \frac{\pi}{2} n \right). \] (38)

The normalization constant is calculated according to Eq. (24) with \( P_n = \sqrt{2mE_n} \).

### 7.2 The Coulomb problem \( V(r) = -\alpha/r \)

The Coulomb potential is a classic example of the exactly solvable problems in quantum mechanics. For this 2TP problem, the standard WKB method does not reproduce the exact energy levels unless one supplements it with Langer-like correction terms. In our approach, separation of the original 3D Schrödinger’s equation with the help of the correspondence principle [19] results in the centrifugal term \( \vec{M}^2/r^2 \) with \( \vec{M}^2 \) given by Eq. (36), which does not require any Langer type corrections. The quantization condition (19), for the Coulomb problem, is

\[ I = \oint_C \left[ 2mE - \frac{2m\alpha}{r} - \frac{\vec{M}^2}{r^2} \right] dr = 2\pi\hbar \left( n_r + \frac{1}{2} \right), \] (39)

where the integral is taken about a contour \( C \) inclosing the turning points \( r_1 \) and \( r_2 \). To calculate the integral (39) we use the method of stereographic projection. To do this, we should exclude the singularities outside the contour \( C \), i.e. at \( r = 0 \) and \( \infty \). Excluding these infinities we have, for the integral (39), \( I = I_0 + I_\infty \), where \( I_0 = -2\pi M \) and \( I_\infty = 2\pi i\alpha m/\sqrt{2mE} \). The sequential simple calculations result in the exact energy spectrum

\[ E_n = -\frac{\alpha^2 m}{2[(n_r + \frac{1}{2})\hbar + M]^2}. \] (40)

Note, for the energy of zeroth oscillations we have, from Eq. (11), \( E_0 = -\alpha^2 m/[2(\hbar/2 + M_0)^2] \), that apparently shows the contribution of the quantum fluctuations of the angular momentum, \( M_0 = \hbar/2 \), into the energy of the ground state \( E_0 \) [4]. The radial WKB eigenfunctions, \( \tilde{R}^{(0)}(r) \), inside the classical region \( [r_1, r_2] \) are written according to Eq. (23) and give the asymptote of the problem,

\[ \tilde{R}^{(0)}(r) = C_n \cos \left( \frac{1}{\hbar} P_n r + \frac{\pi}{2} n_r \right), \] (41)

The normalization constant, \( C_n \), is calculated with the help of Eq. (24), where \( P_n = \sqrt{2mE_n} \) and \( E_n \) is given by Eq. (10).

### 7.3 The isotropic oscillator \( V(r) = \frac{1}{2}m\omega^2 r^2 \)

The three-dimensional harmonic oscillator is another classic example of the exactly solvable problems in quantum mechanics. As in case of the Coulomb potential, the WKB method does not reproduce the exact energy levels unless one supplements it with Langer-like correction terms. This 4TP problem is usually solved with the help of the replacement \( z = r^2 \), which reduces the problem to the 2TP one.
This problem can be solved as the 4TP problem in the complex plane. Because of importance of the oscillator potential in many applications and with the purpose of further development of the WKB method, in Ref. [19] the problem has been solved by two methods, on the real axis as 2TP problem and then in the complex plane as 4TP problem.

Consider the WKB solution of the problem. In the complex plane, the problem has two cuts, between turning points \( r_1, r_2 \) and \( r_3, r_4 \). To apply residue theory for the phase space integral we need to take into account all zeroes of the \( \psi \) in the complex plane, i.e. the contour \( C \) has to include both cuts. The quantization condition (31) has the form

\[
I = \oint_C \sqrt{2mE - \left(m\omega r\right)^2 - \frac{\vec{M}^2}{r^2}} dr = 2\pi\hbar \left( N + \frac{\mu}{4} \right),
\]

where \( \mu = 4 \) and \( C \) is the contour about the cuts at \( r < 0 \) and \( r > 0 \), respectively. The number \( N = n_{r<0} + n_{r>0} \), where \( n_{r<0} \) and \( n_{r>0} \) are the numbers of zeroes of the \( \psi \) at \( r < 0 \) and \( r > 0 \), respectively. For the 3D harmonic oscillator, because of symmetry of the potential, we have \( n_{r<0} = n_{r>0} = n_r \), i.e. the total number of zeroes is \( N = 2n_r \).

Integral (42) is reduced to the above case of the Coulomb potential with the help of the replacement \( z = r^2 \). Integration result is \( I = \pi(E/\omega - M)/2 \) and we obtain, for the energy eigenvalues,

\[
E_n = \omega \left[ 2\hbar \left( n_r + \frac{1}{2} \right) + M \right].
\]

So far, as \( M = (l + 1/2)\hbar \), we get the exact energy spectrum for the isotropic oscillator. Energy of the ground state is \( E_0 = \omega(\hbar + M_0) \), where \( M_0 \) is the contribution of quantum fluctuations of the angular momentum [7]. The asymptotic eigenfunctions have the form (38), where the eigenmomentum \( P_n \) is calculated with the use of the energy eigenvalues (43).

### 7.4 The Hulthén potential \( V(r) = -V_0 e^{-r/r_0} / (1 - e^{-r/r_0}) \)

The Hulthén potential is of a special interest in atomic and molecular physics. The potential is known as an “insoluble” by the standard WKB method potentials, unless one supplements it with Langer-like corrections. The radial problem for this potential is usually considered at \( l = 0 \). However, in the approach under consideration, the quasiclassical method results in the nonzero centrifugal term at \( l = 0 \) and allows to obtain the analytic result for all \( l \).

The quantization condition (31) for the Hulthén potential is

\[
I = \oint_C \sqrt{2m \left( E + V_0 \frac{e^{-r/r_0}}{1 - e^{-r/r_0}} \right) - \frac{\vec{M}^2}{r^2}} dr = 2\pi\hbar \left( n_r + \frac{1}{2} \right).
\]

In the region \( r > 0 \), this problem has two turning points \( r_1 \) and \( r_2 \). The phase-space integral (44) is calculated analogously to the above case. Introducing the new variable \( \rho = r/r_0 \), we calculate the contour integral in the complex plane, where the contour \( C \) encloses the classical turning points \( \rho_1 \) and \( \rho_2 \). Using the method of stereographic
projection, we should exclude the infinities at $r = 0$ and $\infty$ outside the contour $C$. Excluding these infinities we have, for the integral (44),

$$I = I_0 + I_\infty,$$

where $I_0 = -2\pi M$ and $I_\infty$ is calculated with the help of the replacement $z = e^{\rho} - 1$ [3].

$$I = \oint \sqrt{2mr_0^2 \left( E + V_0 \frac{e^{-\rho}}{1 - e^{-\rho}} \right) - \frac{\vec{M}^2}{\rho^2}} d\rho = -2\pi M + 2\pi r_0 \sqrt{-2m \left[ -\sqrt{-E} + \sqrt{-E + V_0} \right]}.$$  \hspace{1cm} (45)

Substituting the integration result into Eq. (44), we immediately get the exact energy spectrum

$$E_n = -\frac{1}{8mr_0^2} \left( \frac{2mV_0r_0^2}{N} - N \right)^2.$$ \hspace{1cm} (46)

where $N = (n_r + 1/2)\hbar + M$ is the principal quantum number. Setting in (16) $M = 0$, we arrive at the energy eigenvalues obtained from known exact solution of the Schrödinger’s equation at $l = 0$. However, in our case $M_{\text{min}} \equiv M_0 = \hbar/2$ at $l = 0$ and the principal quantum number is $N = (n_r + 1/2)\hbar + M_0$. As in the previous examples, this apparently shows the contribution of the quantum fluctuations of the angular momentum into the energy of the ground state, $E_0$.

### 7.5 The Morse potential $V(r) = V_0 [e^{-2\alpha(r/r_0-1)} - 2e^{-\alpha(r/r_0-1)}]$

The Morse potential is usually considered as one-dimensional problem at $l = 0$. In this case the problem has two turning points (note that the left turning point, $r_1$, is negative) and can be solved exactly. In the general case, for $l > 0$, we have an “insoluble” 4TP problem.

For this potential, let us consider, first, the radial Schrödinger equation, which does not contain the centrifugal term at $l = 0$,

$$\left(-i\hbar \frac{d}{dr}\right)^2 R(r) = 2m \left[ E - V_0 e^{-2\alpha(r-r_0)/r_0} + 2V_0 e^{-\alpha(r-r_0)/r_0} \right] R(r).$$ \hspace{1cm} (47)

The quantization condition (19) appropriate to this equation is

$$\oint_C \sqrt{2m \left[ E - V_0 e^{-2\alpha(r-r_0)/r_0} + 2V_0 e^{-\alpha(r-r_0)/r_0} \right]} dr = 2\pi \hbar \left( n_r + \frac{1}{2} \right).$$ \hspace{1cm} (48)

Introducing a variable $x = e^{-\alpha(r-r_0)/r_0}$, we reduce the phase-space integral to the well known one. Sequential simple calculations result in the exact energy eigenvalues

$$E_n = -V_0 \left[ 1 - \frac{\alpha \hbar (n_r + \frac{1}{2})}{r_0 \sqrt{2mV_0}} \right]^2.$$ \hspace{1cm} (49)

Now, let us deal with Eq. (33) for this potential, which contains the non-vanishing centrifugal term, $\hbar^2/4r^2$, at $l = 0$. In this case we have an “insoluble” 4TP problem. In
the complex plane, the problem has two cuts \((\nu = 2)\), at \(r < 0\) and \(r > 0\). Therefore, we apply the 4TP quantization condition (31),

\[
\oint \sqrt{2m r_0^2 [E - V_0 e^{-2\alpha (\rho - 1)} + 2V_0 e^{-\alpha (\rho - 1)}]} - \frac{\chi^2}{\rho^2} d\rho = 4\pi \hbar \left( n_r + \frac{1}{2} \right) = I, \tag{50}
\]

where we have introduced the new variable, \(\rho = r/r_0\). The contour \(C\) encloses the two cuts, but does not enclose the point \(r = 0\). To calculate the integral (50) we use the method of stereographic projection. For this, we should exclude the singularities outside the contour \(C\), i.e. at \(r = 0\) and \(\infty\). Excluding these infinities we have, for the integral (50) \(3\),

\[
I = -2\pi M - \frac{2\pi r_0}{\alpha} \left( \sqrt{-2mE} - \sqrt{2mV_0} \right), \tag{51}
\]

and for the energy eigenvalues this gives

\[
E_n = -V_0 \left[ 1 - \alpha \left( \frac{2\hbar (n_r + \frac{1}{2}) + M}{r_0 \sqrt{2mV_0}} \right)^2 \right]. \tag{52}
\]

Setting in (52) \(l = 0\), we obtain,

\[
E_n = -V_0 \left[ 1 - \alpha \left( \frac{2\hbar (n_r + \frac{1}{2}) + M_0}{r_0 \sqrt{2mV_0}} \right)^2 \right]. \tag{53}
\]

Equation (53) for \(E_n\) is different from the expression (49) obtained from solution of radial Schrödinger equation for the Morse potential at \(l = 0\). This difference is caused by the nonzero centrifugal term \(\hbar^2/4r^2\) in the radial equation (33) at \(l = 0\). Thus we obtain two results for the Morse potential: the known exact eigenvalues (49) obtained from solution of radial equation at \(l = 0\) and another result (52) obtained from solution of Eq. (33) for all \(l\).

### 7.6 The relativistic Cornell problem, \(V(r) = -\tilde{\alpha}/r + \kappa r\)

In this paragraph, we use the WKB\(_0\) approach to reproduce the exact energy spectrum for the famous funnel type potential (Cornell potential) \[24\] known as one which is “insoluble” exactly in terms of special functions. This potential is one of a special interest in high energy hadron physics, quarkonium physics, and quark potential models. Its parameters are directly related to basic physical quantities of hadrons: the universal Regge slope \(\alpha' \simeq 0.9 \text{(GeV/c)}^{-2}\) of light flavours and one-gluon-exchange coupling strength \(\alpha_s\) for heavy quarkonia. A closer inspections reveals \[25\] that all phenomenologically acceptable ”QCD-inspired” potentials are only variations around the funnel potential.

In relativistic theory, the Cornell potential represents the so-called “insoluble” 4TP problem. It is insoluble also from the viewpoint of the conventional 2TP WKB approximation. Show that the WKB\(_0\) method and the quantization condition (31) derived above give the asymptote of the exact solution and yield the exact energy spectrum for the
potential. To solve the problem, we use the relativistic semiclassical wave equation obtained in Ref. [4]. For a two-particle system of equal masses, \( m_1 = m_2 = m \), interacting by means of the Cornell potential, the relativistic radial semiclassical wave equation is \((\hbar = c = 1)\)

\[
\left( -i \frac{d}{dr} \right)^2 \tilde{R}(r) = \left[ \frac{E^2}{4} - \left( m - \frac{\tilde{\alpha}}{r} + kr \right)^2 - \frac{(l + \frac{1}{2})^2}{r^2} \right] \tilde{R}(r), \tag{54}
\]

where \( \tilde{\alpha} = 4\alpha_s/3 \). The quantization condition \((31)\) appropriate to Eq. \((54)\) is

\[
I = \oint_C \sqrt{\frac{E^2}{4} - \left( m - \frac{\tilde{\alpha}}{r} + kr \right)^2 - \frac{(l + \frac{1}{2})^2}{r^2}} dr = 4\pi \left( n_r + \frac{1}{2} \right). \tag{55}
\]

As in above examples, to calculate the phase-space integral \((55)\) in the complex plane, we use the method of stereographic projection. Chose a contour \( C \) enclosing the cuts (and, therefore, zeros of the \( \psi_f \)) at \( r > 0 \) and \( r < 0 \) between the turning points \( r_1, r_2 \) and \( r_3, r_4 \), respectively. Outside the contour \( C \), the problem has two singularities, i.e. at \( r = 0 \) and \( \infty \). Using the method of stereographic projection, we have, for the integral \((55)\),

\[
I = I_0 + I_\infty,
\]

where

\[
I_0 = -2\pi \sqrt{(l + 1/2)^2 + \tilde{\alpha}^2}, \tag{56}
\]

and the integral \( I_\infty \) is calculated in closed form with the help of the replacement of variable, \( z = 1/r \), that gives

\[
I_\infty = 2\pi \left( \frac{E^2}{8\kappa} + \tilde{\alpha} \right). \tag{57}
\]

Therefore, for \( E_{n_r}^2 \), we obtain

\[
E_{n_r}^2 = 8\kappa \left[ 2 \left( n_r + \frac{1}{2} \right) + \sqrt{(l + 1/2)^2 + \tilde{\alpha}^2 - \tilde{\alpha}} \right], \tag{58}
\]

which is the exact result for the Cornell potential.

It is an experimental fact that the dependence \( E_{n_r}^2(l) \) is linear for light mesons (linear Regge trajectories [26]). However, at present, the best way to reproduce the experimental masses of particles is to rescale the entire spectrum assuming that the masses \( M_n \) of the mesons are expressed by the relation \( M_n^2 = E_{n_r}^2 - C^2 \), where \( E_{n_r}^2 = 8\kappa(2n_r + l + 3/2) \) is the exact energy eigenvalues for the linear potential and \( C \) is a constant energy (additional free parameter) [27]. The equation for \( M_n^2 \) is used to shift the spectra and appears as a means to simulate the effects of unknown structure approximately.

At the same time, the required shift of the spectra naturally follows from Eq. \((58)\). The Coulomb-like term, \(-\tilde{\alpha}/r\) of the Cornell potential takes into account the week coupling effect that allows to describe the Regge trajectories observed at experiment. For light mesons one may expect that the \( q\bar{q} \) bound states will feel only the linear part of the potential, which gives the main contribution to the binding energy. We thus assume that the Coulomb term can be considered as a small perturbation. Then we obtain, from Eq. \((58)\),
\[ E_n^2 = 8\kappa \left( 2n_r + l - \frac{4}{3}\alpha_s + \frac{3}{2} \right). \]  

Equation (59) does not require any additional free parameter and reproduces the linear Regge trajectories as it observed at experiment. We obtain the necessary shift with the help of the term \(-32\kappa\alpha_s/3\) which is the result of interference of the Coulomb and linear terms of the interquark potential. Note that we obtain the correct sign (minus) for this term only in the case of the scalar-like potential. Equation (58) describes the light meson trajectories with the accuracy \(\simeq 1\%\) [26, 28] (see Table 1).

| State | \(E_n^{\text{theor}}\) | \(E_n^{\text{exp}}\) |
|-------|----------------|----------------|
| \(1^3S_1\) | 0.763 | 0.768* |
| \(1^3P_2\) | 1.319 | 1.318* |
| \(1^3D_3\) | 1.703 | 1.691* |
| \(1^3F_4\) | 2.014 | 2.037 |
| \(1^3G_5\) | 2.284 | – |
| \(2^3S_1\) | 1.703 | 1.700* |
| \(2^3P_2\) | 2.014 | – |
| \(2^3D_3\) | 2.284 | – |
| \(2^3F_4\) | 2.525 | – |

The symbol * denotes the masses of the fitted states; parameters of the fit are: \(\alpha_s = 0.75 \pm 0.03\) and \(\kappa = (0.14 \pm 0.01) \text{ GeV}^2\).

### 8 Conclusion

We have considered solution of the Schrödinger’s equation in the zero order WKB approximation. One might treat the WKB\(_0\) solution as a very crude quasiclassical approximation. However, the WKB\(_0\) solution represents the main term of the asymptotic series in theory of the second-order differential equations. A proper treatment of the WKB\(_0\) approximation opens the way to a simple asymptotic solution of the wave equation.

It is a well known fact that the exact eigenvalues can be defined with the help of the asymptotic solution. The WKB\(_0\) approach gives a simplest way to find the asymptote of the exact solution directly, without solution of the Schrödinger’s equation, and results in quantization of the classical action.

We have observed several important advantages of the zeroth approximation with reference to the commonly used first-order WKB approximation. Because the general WKB\(_0\) solution (8) has no divergencies at the classical turning points, this allows a considerably simpler derivation of connection formulas and quantization condition. Using the general requirements of continuity and finiteness for WKB\(_0\) function \(\psi_0(x)\) and its derivative \(\psi'_0(x)\), we have derived a simple connection formulas, which allowed us to build the WKB\(_0\) \(wf\) in the whole region and the corresponding quantization condition for the classical action.
If WKB$_1$ approximation is applicable at a distance from the turning points satisfying the condition $|x - x_0| \gg \lambda/4\pi$, the WKB$_0$ solution considered in this work is applicable in the whole region. To demonstrate advantages of the WKB$_0$ approximation we have derived not only well known 2TP but, also, multi-turning-point quantization condition. This approach has been also used to derive the quantization conditions for periodic potentials and can be easily generalized for nonseparable problems.

The most important advantage of the WKB$_0$ approach is that the WKB$_0$ quantization condition successfully reproduces the exact energy spectra for all known solvable potentials and “insoluble” potentials, too. To demonstrate efficiency of the WKB$_0$ method, we have solved several classic problems. The method successfully reproduces the exact energy spectrum not only for solvable one-dimensional and spherically symmetric potentials but, also, for “insoluble” potentials with more than two turning points.

For discrete spectrum, the WKB$_0$ eigenfunctions have written in terms of elementary functions in the form of a standing wave, $A_n \cos(k_n x + \delta_n)$. The fact the eigenfunctions can be written in such form means the solution describes free motion of a particle-wave in the enclosure. This, in turn, means that the eigenmomentum is the constant of motion and the corresponding coordinate is cyclical.

The zeroth approximation reproduces all basic results of quantum mechanics: eigenvalues, asymptotes, probabilities, etc. Obviously, this is more than just an approximation. This is, at least, an alternative quantization method. What is more, we do not even need to solve the Schrödinger’s equation to find the asymptote. After we have fixed the classical action, using the existence/quantization formulas, we can write down the quantization condition for an arbitrary potential.

This approach can be easily generalized for the non-separable problems. In this case the classical action can not be written as sum of actions for each variable (degree of freedom). As a result, we cannot have a unique quantum number for each degree of freedom. We will have one quantum number for the non-separable variables, and unique quantum number for each separable variable. Asymptote will have the same form, but the existence formulas (quantization condition) will be multi-dimensional integral over the non-separable variables.

In the WKB$_0$ method, we use the same technique for all types of problems. The same simple rules formulated for 2TP problems work for $\mu$TP problems, as well. In this sense, the WKB$_0$ approach can be considered as a general method for finding the asymptote of the Schrödinger’s equation.

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