The Fokker–Planck Equation with Absorbing Boundary Conditions

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Abstract

We study the initial-boundary value problem for the Fokker–Planck equation in an interval with absorbing boundary conditions. We develop a theory of well-posedness of classical solutions for the problem. We also prove that the resulting solutions decay exponentially for long times. To prove these results we obtain several crucial estimates, which include hypoellipticity away from the singular set for the Fokker–Planck equation with absorbing boundary conditions, as well as the Hölder continuity of the solutions up to the singular set.

1. Introduction

We consider the initial boundary value problem for the following Fokker–Planck equation in an interval $[0, 1]$:  

\begin{align*}
  f_t + v f_x &= f_{vv}, \\
  f(x, v, 0) &= f_0(x, v), \\
  f(0, v, t) &= 0, \text{ for } v > 0, \ t > 0, \\
  f(1, v, t) &= 0, \text{ for } v < 0, \ t > 0,
\end{align*}

where $f(x, v, t) \geq 0$ is the distribution of particles at position $x$, velocity $v$, and time $t$ for $(x, v, t) \in [0, 1] \times \mathbb{R} \times \mathbb{R}_+$ and $f_0(x, v) \geq 0$ the initial charge distribution.

The kinetic boundary condition given in (1.3)–(1.4) is the so-called absorbing boundary condition or absorbing barrier (cf. [14,24]). If we interpret (1.1)–(1.4) as the equation for the density in the phase space of a system of particles, the meaning of the boundary conditions (1.3)–(1.4) is that the particles reaching the boundary of the domain containing them can escape but not re-enter it.

Equations with the form (1.1) and boundary conditions like (1.3) appear in the study of different problems of statistical physics. For instance, they arise in the...
study of Brownian particles moving in bounded domains (cf. [24]), or in the study of the statistics of polymer chains (cf. [6]).

The Fokker–Planck operator is a well-known hypoelliptic operator. Diffusion in $v$ together with the transport term $v \cdot \nabla x$ has a regularizing effect for solutions not only in $v$ but also in $t$ and $x$, which can be obtained by applying Hörmander’s commutator (cf. [19]) to the linear Fokker–Planck operator. For more details, see [2]. Note that these results were obtained in the whole space without boundaries.

On the other hand, the Fokker–Planck operator is also known as a hypocoercive operator, which concerns the rate of convergence to equilibria. Indeed, the trend to equilibria with a certain rate has been investigated in many papers (cf. [11,17,21,36]) in the Maxwellian regime and in the whole space or in the periodic box. For more details, we refer to [36].

The hypoelliptic and hypocoercive properties have also been explored for other kinetic equations. Among others, we briefly review theories of existence, regularity and asymptotic behaviors for the Vlasov–Poisson–Fokker–Planck system in the whole space, which is one of the important models in mathematical physics and has been widely studied. The global existence of classical solutions was studied in [3,30,35]. Asymptotic behaviors and time decay of the solutions in the vacuum regime were considered in [8,10,28]. We mention the works in [9,34], where the global weak solutions were constructed, and the work in [4], where the smoothing effect was observed.

Compared to the theory in the case of the whole space, little progress has been made towards the boundary-value problems for these equations. In [5,7], global weak solutions and asymptotic behaviors for the Vlasov–Poisson–Fokker–Planck equations were studied in bounded domains with absorbing and reflective type boundary conditions. In [27], a global stability of DiPerna–Lions renormalized solutions to some kinetic equations including the Vlasov–Fokker–Planck equation which was studied under the Maxwell boundary conditions.

However, to our knowledge, the hypoellipticity property of the Fokker–Planck equation has not been studied in bounded domains other than the periodic boundary condition, and no convergence rate for solutions of the Fokker–Planck equation has been investigated for an interval in the vacuum regime.

In this paper we develop a theory for classical solutions of (1.1)–(1.4). We will also prove that the solutions of this problem vanish exponentially fast as $t \to \infty$.

From the technical point of view, the main obstruction to developing a theory for classical solutions of (1.1)–(1.4) is the presence of the so-called singular set. This set can be defined for some kinetic equations (cf. [14,15,18]). In the case of (1.1)–(1.4), the singular set reduces to the points $(x, v) \in \{(0, 0), (1, 0)\}$. The fact that the solutions of kinetic equations cannot have arbitrary regularity near the singular set was first noticed by Guo in the Vlasov–Poisson system (cf. [14]). In this paper we will prove that the solutions of (1.1)–(1.4) are not $C^\infty$ in general near the singular set.

Notice that the equation (1.1) contains the second derivative that yields the regularizing effects only in the variable $v$. On the other hand, the presence of the transport term $vf_x$ has the following consequence that the solutions of (1.1) become $C^\infty$ for any $t > 0$ in the set $([0, 1] \times \mathbb{R}) \setminus \{(0, 0), (1, 0)\}$. This property is known as
hypoellipticity. However, such regularizing effects do not take place at the singular set. Indeed, it turns out that there exist some explicit solutions of (1.1) with boundary conditions (1.3), and it indicates that the maximum regularity that we can expect is $C_{x,v}^{1/6,1/2}$.

In order to prove the results of this paper we will make extensive use of maximum principles and comparison arguments combined with suitable sub and super-solutions. We will first construct a theory of weak solutions of (1.1)–(1.4) by maximum principles and comparison arguments combined with suitable sub and super-solutions. We will then study the regularity of the solutions of (1.1)–(1.4) near the singular set. We also define the incoming, outgoing, and grazing boundary of super-solutions. We will first construct a theory of weak solutions of (1.1)–(1.4) by maximum principles and comparison arguments combined with suitable sub and super-solutions. We will then study the regularity of the solutions of (1.1)–(1.4) near the singular set. This will be made using suitable sub and super-solutions and comparison arguments. We will prove in this way that the solutions of (1.1)–(1.4) belong to $C_{x,v}^{1/6-\varepsilon,1/2-\varepsilon}$ for any $t > 0$, with $\varepsilon > 0$ arbitrarily small.

We will also prove that the solutions of (1.1)–(1.4) decay exponentially fast as $t \to \infty$. The main idea used in the proof of this result is that, due to the hypoellipticity property, the particle fluxes along the boundaries $\{(0,v) : v < 0\} \cup \{(1,v) : v > 0\}$ are comparable to the total number of particles at a given time (cf. Section 4). This implies the exponential decay for the total number of particles of the system. However, in order to make this argument precise a careful treatment is needed in order to control the amount of mass near the singular set, because the hypoellipticity property is not valid there. To control the mass in such regions we will use again suitable sub and super-solutions.

1.1. Main Results

We first introduce notations for the domain and boundaries. Define

$$U_T = \Omega \times (0, T) := \{(x, v, t) \in (0, 1) \times (-\infty, \infty) \times (0, T)\},$$

where $\Omega = (0, 1) \times (-\infty, \infty)$.

We also define the incoming, outgoing, and grazing kinetic boundary of $U_T$ as

$$\Gamma_T^- := \Omega \times \{t = 0\} \cup \{x = 0\} \times (0, \infty) \times (0, T) \cup \{x = 1\} \times (-\infty, 0) \times (0, T),$$

$$\Gamma_T^+ := \Omega \times \{t = 0\} \cup \{x = 0\} \times (-\infty, 0) \times (0, T) \cup \{x = 1\} \times (0, \infty) \times (0, T),$$

$$\Gamma_T^0 := \Omega \times \{t = 0\} \cup \{x = 0\} \times \{v = 0\} \times (0, T) \cup \{x = 1\} \times \{v = 0\} \times (0, T).$$

In addition, we define the incoming, outgoing, and grazing boundary of $U_T$ as

$$\gamma_T^- := \{x = 0\} \times (0, \infty) \times (0, T) \cup \{x = 1\} \times (-\infty, 0) \times (0, T),$$

$$\gamma_T^+ := \{x = 0\} \times (-\infty, 0) \times (0, T) \cup \{x = 1\} \times (0, \infty) \times (0, T),$$

$$\gamma_T^0 := \{x = 0\} \times \{v = 0\} \times (0, T) \cup \{x = 1\} \times \{v = 0\} \times (0, T).$$
We give a definition of a weak solution of (1.1)–(1.4) in the following.

**Definition 1.** We say that \( f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) is a weak solution of (1.1)–(1.4) if the function

\[
t \to \int f(x, v, t) \psi(x, v, t) \, dx \, dv
\]

is continuous on \([0, T]\) for any test function \( \psi(x, v, t) \in C^{1,2,1}_{x,v,t}(U_T) \) such that \( \text{supp}(\psi(\cdot, \cdot, t)) \subset [0, 1] \times [-R, R] \) for some \( R > 0 \) and if it satisfies for every \( t \in [0, T] \) and any test function \( \psi(x, s) \in C^{1,2,1}_{x,v,s}(U_t) \) such that \( \text{supp}(\psi(\cdot, \cdot, s)) \subset [0, 1] \times [-R, R] \) for some \( R > 0 \) and \( \psi|_{\gamma_+} = 0 \),

\[
\int_{\Omega} f(x, v, t) \psi(x, v, t) \, dx \, dv - \int_{\Omega} f(x, v, 0) \psi(x, v, 0) \, dx \, dv
\]

\[
= \int_{U_t} f(x, v, s) [\psi_t (x, v, s) + v\psi_x (x, v, s) + \psi_{vv} (x, v, s)] \, dx \, dv \, ds.
\]

We are now ready to state our main results. The first result concerns the existence of a unique weak solution of (1.1)–(1.4).

**Theorem 1.1.** Let \( T > 0 \) and \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \) given. Then there exists a unique weak solution \( f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) with \( f \geq 0 \) of the Fokker–Planck equation with the absorbing boundary condition (1.1)–(1.4). Moreover, the weak solution \( f(t) \) satisfies the following bounds:

\[
\| f(t) \|_{L^\infty(\Omega)} \leq \| f_0 \|_{L^\infty(\Omega)} \quad \text{and} \quad \| f(t) \|_{L^1(\Omega)} \leq \| f_0 \|_{L^1(\Omega)}
\]

for each \( t \in [0, T] \).

The next results concern the regularity of weak solutions.

**Theorem 1.2.** Let \( f(x, v, t) \) be the weak solution of (1.1)–(1.4) with \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \). Then the following holds:

(i) For each \( t > 0 \), \( f \in H^{k,m}_{loc}(\widehat{\Omega} \setminus \{(0, 0), (1, 0)\}) \), where \( H^{k,m} = H^{k,m}_{x,v} \) and for any \( k, m \in \mathbb{N} \).

(ii) For all \( t > 0 \), \( f(x, v, t) \) is continuous in \( \widehat{\Omega} \) such that \( f(0, 0, t) = f(1, 0, t) = 0 \) for all \( t > 0 \) and \( \lim_{(x,v) \to (0,0),(1,0)} f(x, v, t) = 0 \) for all \( t > 0 \). In fact, \( f \) is Hölder continuous up to the singular set: \( f \in C^{a,3a}_{x,v} (\widehat{\Omega}) \) for any number \( 0 < a < 1/6 \).

The last main theorem shows the exponential trend of the solution to 0 in \( L^1 \) and \( L^\infty \) sense:

**Theorem 1.3.** Let \( f_0(x, v) \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \) and let \( f(x, v, t) \) be a solution to (1.1)–(1.4). Then the following holds.

(i) \( f \) decays exponentially in time in \( L^1(\Omega) \). In particular, there exists \( \kappa > 0 \) such that

\[
\| f(t) \|_{L^1(\Omega)} \leq \| f_0 \|_{L^1(\Omega)} \exp(-\kappa t).
\]
(ii) $f$ decays exponentially in time in $L^\infty(\Omega)$. In particular, there exist $\kappa > 0$ and $C > 0$ such that

$$\|f(t)\|_{L^\infty(\Omega)} \leq C \exp(-\kappa t),$$

where $C$ depends on $\|f_0\|_{L^1(\Omega)}$ and $\|f_0\|_{L^\infty(\Omega)}$.

Some of the first related results for the problem were obtained in [26], where the probability distribution for the velocity with which an accelerated Brownian particle—a Brownian particle, that is, a particle whose paths are the ones associated to the Orstein–Uhlenbeck process—exits from the domain $\{x > 0\}$ is computed. A consequence of that formula is the following: the probability that a particle leaves the origin with some positive velocity $v_0$ at time $t = 0$ but has not left the domain $\{x > 0\}$ at time $t$, decreases as $t^{-1/4}$. This power law in the problem which can be considered as a discrete analogue of the accelerated random walk problem and its derivation can be found in [31].

The Laplace transform of propagators of the Orstein–Uhlenbeck process in a half-line with absorbing boundary conditions at $x = 0$ was computed in [24] by using the Wiener–Hopf methods. This yields equations similar to (1.1) but with an additional friction term. In the limit case in which the friction coefficient tends to zero, the formulas in [24] reduce to the ones in [26]. The exit time of the Orstein–Uhlenbeck process in a suitable asymptotic limit was considered in [20].

The power law decay of the solutions of the equation (1.1) obtained by means of the explicit formulas mentioned above is in contrast with the exponential decay which we will derive in this paper for the case of solutions in the interval $0 < x < 1$. We remark that the mean exit time for an accelerated Brownian particle in an interval $0 < x < 1$ was obtained in [25].

The approach of the references above that concerned the asymptotics of the solutions of (1.1) in the half-line or an interval is based on explicit or semi-explicit representation formulas for the derived quantities. The approach of this paper relies more on PDE arguments, like mass balance equations and maximum principle arguments applied to arbitrary initial distributions, and hopefully can be applied to more general cases.

The paper proceeds as follows. In Section 2 we develop a theory of the existence and the uniqueness of weak solutions for (1.1)–(1.4). Section 3 contains a regularity theory which allows us to prove that the solutions are $C^\infty$ outside the singular set and have suitable Hölder estimates near the singular set. Section 4 proves that every solution of (1.1)–(1.4) decreases exponentially in $L^1$ and $L^\infty$ sense as $t \to \infty$.

2. Weak Solutions in an Interval

The goal of this section is to construct a weak solution to (1.1)–(1.4) for a given bounded and integrable initial data $f_0 \in L^1 \cap L^\infty(\Omega)$ with $f_0 \geq 0$.

2.1. Approximation

The first step for constructing a weak solution is to regularize the equation (1.1), in particular the transport term $v \partial_x f$ near the grazing boundary set $(x, v) \in$
{(0, 0), (1, 0)}. This will be achieved by approximating it with cut-off functions. Define \( \beta_\varepsilon(v) \in C^\infty(-\infty, \infty) \) and \( \eta_\varepsilon(x) \in C^\infty(0, 1) \) as follows:

\[
\beta_\varepsilon(v) = \begin{cases} 
0, & |v| < \varepsilon^2 \\
\in [-v, v], \varepsilon^2 \leq |v| \leq 2\varepsilon^2 & v, \\
|v| > 2\varepsilon^2 
\end{cases}
\]

\[
\eta_\varepsilon(x) = \begin{cases} 
0, & 0 \leq x < \varepsilon, 1 - \varepsilon \leq x \leq 1 \\
\in [0, 1], \varepsilon \leq x \leq 2\varepsilon, 1 - 2\varepsilon \leq x \leq 1 - \varepsilon & 1, \\
2\varepsilon < x < 1 - 2\varepsilon. 
\end{cases}
\]

We will also approximate the diffusion term \( f_{vv} \) by choosing a cut-off function \( \xi(\zeta) \in C^\infty_c(-\infty, \infty) \) such that

\[
\int_{-\infty}^{\infty} \xi(\zeta) \, d\zeta = 1, \quad \int_{-\infty}^{\infty} \zeta \xi(\zeta) \, d\zeta = 0, \quad \int_{-\infty}^{\infty} \zeta^2 \xi(\zeta) \, d\zeta = 1.
\]

Then \( f_{vv} \) can be approximated as

\[
Q^\varepsilon [f](x, v, t) := \frac{2}{\varepsilon^2} \int_{-\infty}^{\infty} [f(x, v + \varepsilon\zeta, t) - f(x, v, t)] \xi(\zeta) \, d\zeta.
\]

From Taylor’s Theorem, we see that \( Q^\varepsilon [f] \to f_{vv} \) as \( \varepsilon \to 0 \), at least formally.

We consider the approximate equation

\[
f^\varepsilon_t + [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(x)] f^\varepsilon_x = Q^\varepsilon [f^\varepsilon] \tag{2.1}
\]

with the same initial and boundary conditions:

\[
f^\varepsilon|_{t=0} = f_0, \quad f^\varepsilon|_{\gamma_T} = 0. \tag{2.2}
\]

The approximate equation (2.1) is essentially a transport equation combined with the jump process \( Q^\varepsilon \), where the transport term is truncated in the small neighborhood of the grazing set whose area is of \( O(\varepsilon^3) \). We first study the regularized version (2.1) of (1.1) by using the method of characteristics and prove its well-posedness by exploiting a weak maximum principle.

The corresponding equation of characteristics to (2.1) reads as follows:

\[
\frac{dX(s; x, v, t)}{ds} = \beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s)), \quad V(s; x, v, t) = v, \quad s < t, \tag{2.3}
\]

\[
X(t; x, v, t) = x.
\]

For simplicity, we will use \( X(s) \) and \( V(s) \) instead of \( X(s; x, v, t) \) and \( V(s; x, v, t) \) respectively. Due to the cut-off functions and the absorbing boundary condition, it is not trivial to write the backward characteristics explicitly. To get around it, for a given \( (x, v, t) \), we define \( 0 < t_0 \leq t \) if there exists \( t_0 = t_0(x, v, t) > 0 \) satisfying

\[
x = \int_{t_0}^{t} [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s))] \, ds, \text{ or }
\]

\[
x = 1 + \int_{t_0}^{t} [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s))] \, ds,
\]
otherwise, set $t_0 = 0$.

We first calculate the Jacobian $J(s; t)$, for $t_0 < s < t$, of the transformation

$$(x, v) \rightarrow (X(s), v).$$

Note that $J(s; t)$ measures the change rate of the unit volume in the phase space along the characteristics as follows. Let

$$J(s; t) := \det \begin{pmatrix} \frac{\partial X(s)}{\partial x} & \frac{\partial X(s)}{\partial v} \\ 0 & 1 \end{pmatrix} = \frac{\partial X(s)}{\partial x},$$

where $X(s)$ is the characteristics defined in (2.3). Then the following lemma holds.

**Lemma 1.** For $0 \leq t_0 < s < t \leq T$ with $T > 0$ given, we have the following estimates

$$1 - O(\varepsilon) \leq |J(s; t)| = \left| \frac{\partial X(s)}{\partial x} \right| \leq 1 + O(\varepsilon), \quad (2.4)$$

$$1 - O(\varepsilon) \leq |J(t; s)| = \left| J(s; t)^{-1} \right| \leq 1 + O(\varepsilon), \quad (2.5)$$

where $O(\varepsilon) = \varepsilon CT e^{\varepsilon CT}$.

**Proof.** We integrate (2.3) over time from $t$ to $s$ and differentiate the resulting equation with respect to $x$ to get

$$\frac{\partial X(s)}{\partial x} = 1 + (v - \beta_\varepsilon(v)) \int_t^s \eta'_\varepsilon(X(\tau)) \frac{\partial X(\tau)}{\partial x} \, d\tau.$$

Then using the definitions of the cut-off functions $(v - \beta_\varepsilon(v)) = O(\varepsilon^2)$, $\eta'_\varepsilon(X(\tau)) = O(1/\varepsilon)$ yields

$$1 - C\varepsilon \int_s^t \left| \frac{\partial X(\tau)}{\partial x} \right| \, d\tau \leq \left| \frac{\partial X(s)}{\partial x} \right| \leq 1 + C\varepsilon \int_s^t \left| \frac{\partial X(\tau)}{\partial x} \right| \, d\tau \quad (2.6)$$

for some constant $C > 0$. We now apply a standard Gronwall’s inequality to get

$$e^{-\varepsilon C|t-s|} \leq \left| \frac{\partial X(s)}{\partial x} \right| \leq e^{\varepsilon C|t-s|}.$$

Since $|\frac{\partial X(s)}{\partial x}| \leq e^{\varepsilon CT}$ for all $s < t \leq T$, using this in (2.6) leads to (2.4)–(2.5).

We now introduce the standard notion of a mild solution to (2.1)–(2.2).

**Definition 2.** We say that $F \in C([0, T]; L^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega))$ is a mild solution of (2.1)–(2.2) if it satisfies for every $t \in [0, T]$,

$$F(x, v, t) = \bar{f}_0(X(t_0), v) + \int_{t_0}^t Q^\varepsilon[F](X(s), v) \, ds =: T[F](x, v, t), \quad (2.7)$$

where $\bar{f}_0(X(t_0), v) = f_0(X(0), v)$ if $t_0 = 0$, otherwise $\bar{f}_0(X(t_0), v) = 0$. 

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We show in the following lemma the existence and the uniqueness of a mild solution (2.7) of (2.1)--(2.2).

**Lemma 2.** For any given \( \varepsilon > 0 \) and any \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \), there exist a time \( T = T(\varepsilon) > 0 \) independent of \( f_0 \) and a unique mild solution of (2.1)--(2.2) in \([0, T]\).

**Proof.** We will show the existence by a fixed point argument. Let

\[
\mathcal{U} = \left\{ F \in C \left( [0, T]; L^1(\Omega) \right) \cap L^\infty \left( [0, T]; L^\infty(\Omega) \right) : \sup_{0 \leq t \leq T} \| F(\cdot, \cdot, t) \|_{L^\infty(\Omega)} \leq 2 \right\}
\]

\[
\subset C \left( [0, T]; L^1(\Omega) \right) \cap L^\infty \left( [0, T]; L^\infty(\Omega) \right).
\]

We aim to show that \( T \) maps \( \mathcal{U} \) into \( \mathcal{U} \) and is a contraction if \( T = T(\varepsilon) \) is sufficiently small. We first estimate \( T[F] \) for \( F \in L^\infty(\Omega) \). It is easy to see, for \( F \in \mathcal{U} \),

\[
\left| Q^\varepsilon [F] (X(s), v) \right| \leq \frac{4}{\varepsilon^2} \| F(\cdot, \cdot, s) \|_{L^\infty(\Omega)} \int_{-\infty}^{\infty} \xi(\zeta) \, d\zeta \leq \frac{8}{\varepsilon^2} \| f_0 \|_{L^\infty(\Omega)}.
\]

Thus we get, for all \( t \in [0, T] \),

\[
\| T[F](\cdot, \cdot, t) \|_{L^\infty(\Omega)} \leq \| \tilde{f}_0 \|_{L^\infty(\Omega)} + \frac{8T}{\varepsilon^2} \| f_0 \|_{L^\infty(\Omega)} \leq 2 \| f_0 \|_{L^\infty(\Omega)},
\]

if \( \frac{8T}{\varepsilon^2} < 1 \). This implies \( T[F] \in L^\infty([0, T]; L^\infty(\Omega)) \). We now estimate \( T[F] \) for \( F \in L^1(\Omega) \). If \( F \in \mathcal{U} \), then

\[
\| F(\cdot, \cdot, t) \|_{L^1(\Omega)} \leq \int_{\Omega} \tilde{f}_0 (X(t_0), v) \, dx \, dv + \int_{t_0}^{t} \int_{\Omega} \left| Q^\varepsilon [F] \right| (X(s), v) \, ds \, dx \, dv = I + II.
\]

We first estimate \( I \) as follows. Using (2.5) in Lemma 1 yields

\[
I = \int_{\Omega} \tilde{f}_0 (X(t_0), v) \, dx \, dv = \int_{\Omega} f_0 (X(0), v) \, dx \, dv
\]

\[
= \int_{\Omega} f_0 (X(0), v) |J(t; 0)| \, dx \, dv
\]

\[
\leq (1 + O(\varepsilon)) \int_{\Omega} f_0 (y, v) \, dy \, dv
\]

\[
\leq \frac{3}{2} \| f_0 \|_{L^1(\Omega)},
\]

provided \( T \) is chosen in such a way that \( O(\varepsilon) \leq 1/2 \). Here we denote by \( \Omega_1 \), through the back-time characteristics,

\[
\Omega_1 = \{(x, v) \in \Omega \mid (x, v, t) \text{ connects with } (X(0), v, 0) \} \subset \Omega.
\]
For $II$, if $F \in \mathcal{U}$, then

$$II \leq \frac{2}{\varepsilon^2} \int_0^T \int_0^t \int_{-\infty}^{\infty} \xi(\zeta) \int_0^{\infty} |F|(X(s), v + \varepsilon \zeta, s) \, ds \, dv \, d\zeta \, ds$$

$$\leq \frac{2}{\varepsilon^2} \int_0^t \int_0^{\infty} \xi(\zeta) \, d\zeta \left[ \int_0^{\infty} |F|(X(s), v + \varepsilon \zeta, s) \, ds \right] + |F|(X(s), v, s) |J(t; s)| \, dX(s) \, dv \, ds$$

$$\leq \frac{4}{\varepsilon^2} \int_0^t \int_0^{\infty} |F|(X(s), v, s) |J(t; s)| \, dX(s) \, dv \, ds$$

$$\leq \frac{4}{\varepsilon^2} (1 + O(\varepsilon)) \int_0^t \int_0^{\infty} |F|(X(s), v, s) \, dX(s) \, dv \, ds$$

$$\leq \frac{4}{\varepsilon^2} \frac{1 + O(\varepsilon)}{(1 + O(\varepsilon))} \sup_{0 \leq s \leq T} \|F(\cdot, \cdot, t)\|_{L^1(\Omega)}$$

$$\leq \frac{8}{\varepsilon^2} \frac{(1 + O(\varepsilon))T}{(1 + O(\varepsilon))} \|f_0\|_{L^1(\Omega)}$$

$$\leq \frac{1}{2} \|f_0\|_{L^1(\Omega)}$$

if $T$ is further made in such a way that $\frac{8(1+O(\varepsilon))T}{\varepsilon^2} < 1/2$. Then by the estimates of $I$ and $II$, we obtain

$$\sup_{0 \leq t \leq T} \|F(\cdot, \cdot, t)\|_{L^1(\Omega)} \leq 2 \|f_0\|_{L^1(\Omega)}.$$  

For the continuity in time of $T[F]$ in $L^1(\Omega)$, we use the absolute continuity in $L^1(\Omega)$ of $L^1$ functions, the continuity in $t$ of $t_0$, $X(t_0)$ as functions of $t$, and the monotone convergence theorem. In particular, we treat the continuity of $T[F]$ in $L^1(\Omega)$ as $t$ goes to 0 in a more careful way, where the monotone convergence theorem applies. This can be seen in the following integral

$$\int_0^1 \int_0^{\infty} f_0(x, v) \, dv \, dx \to 0 \text{ as } t \to 0.$$  

Thus $T[F] \in C([0, T]; L^1(\Omega))$ and this implies that $T$ maps $\mathcal{U}$ into $\mathcal{U}$. Similar arguments yield

$$\|T[F_1](\cdot, \cdot, t) - T[F_2](\cdot, \cdot, t)\|_{L^1(\Omega) \cap L^\infty(\Omega)}$$

$$\leq \frac{8}{\varepsilon^2} \frac{1 + O(\varepsilon)}{(1 + O(\varepsilon))} \sup_{0 \leq t \leq T} \|F_1(\cdot, \cdot, t) - F_2(\cdot, \cdot, t)\|_{L^1(\Omega) \cap L^\infty(\Omega)}$$

so that $T$ is a contraction if $\frac{8(1+O(\varepsilon))T}{\varepsilon^2} < 1$. Therefore if $T = T(\varepsilon) < \frac{\varepsilon^2}{8(1+O(\varepsilon))}$, then there exists a unique mild solution by a fixed point theorem. □

We obtain in the following lemma the existence of solutions of (2.1)–(2.2) for an arbitrary time $T$, which does not depend on $\varepsilon$. 

Corollary 1. For any given $\varepsilon > 0$, $T > 0$ independent of $\varepsilon$, and $f_0 \in L^1 \cap L^\infty(\Omega)$ with $f_0 \geq 0$, there exists a unique mild solution of (2.1)–(2.2) in $[0, T]$.

Proof. We know the existence time $T = T(\varepsilon)$ and then we can apply the same argument as in the proof of Lemma 2 to extend the existence time to the given time $T$. □

2.2. Well-Posedness of the Approximate Equations

We will show in this subsection the existence and the uniqueness of weak solutions of the approximate equation (2.1) with the initial and boundary conditions (2.2). For that purpose, several maximum principles will be used in this subsection. Maximum principle properties have been extensively studied in the analysis of elliptic and parabolic problems (see [12,13,16,23,29]). We adapt them to the corresponding definitions and results of this paper.

Definition 3. We say that $F \in C([0, T]; L^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega))$ is a weak solution of (2.1) with (2.2) if for every $t \in [0, T]$ and any test function $\psi(x, v, s) \in C^1(U_t)$ such that $\text{supp}(\psi(\cdot, \cdot, s)) \subset [0, 1] \times [-R, R]$ for some $R > 0$ and $\psi|_{\gamma^+} = 0$, it satisfies

\[ -\int_{U_t} F(x, v, s) [\psi_t(x, v, s) + \partial_x (\beta_\varepsilon(v)) \psi(x, v, s)] \, dx \, dv \, ds 
+ \int_{\Omega} F(x, v, t) \psi(x, v, t) \, dx \, dv 
- \int_{\Omega} f_0(x, v) \psi(x, v, 0) \, dx \, dv 
= \frac{2}{\varepsilon^2} \int_{U_t} F(x, v, s) \int_{-\infty}^{\infty} [\psi(x, v - \varepsilon \xi, s) - \psi(x, v, s)] \xi(\xi) \, d\xi \, dx \, dv \, ds. \]

Then we have the lemma which states the existence of a weak solution to (2.1).

Lemma 3. Let $T > 0$ and let $f_0 \in L^1 \cap L^\infty(\Omega)$ with $f_0 \geq 0$. Then there exists a weak solution $f^\varepsilon \in C([0, T]; L^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega))$ to (2.1)–(2.2).

Proof. It is easy to see that a mild solution is a weak solution by multiplying to (2.7) a test function $\psi(x, v, s) \in C^1(U_t)$ with a compact support and $\psi|_{\gamma^+} = 0$ and by integrating the resulting equation over $x$, $v$, and $t$. We omit the details. □

We will use smooth solutions of the adjoint problem to (2.1) as test functions in the weak formulation although they may not have compact supports, since the smooth solutions can be approximated by test functions with compact supports in Definition 3. Thus the solutions indeed satisfy the formula in Definition 3. We first show the existence of such smooth solutions.

Define
\[ \tilde{L}\psi := \psi_t + \partial_x \left( \left[ \beta_\varepsilon (v) + (v - \beta_\varepsilon (v)) \eta_\varepsilon (x) \right] \psi \right) - \tilde{Q}_\varepsilon [\psi] (x, v, t) = 0, \quad (2.9) \]

\[ \psi|_{t=T} = \psi_T, \quad \psi|_{\gamma^-} = 0, \]

where \( \tilde{Q}[\psi](x, v, t) = \frac{2}{\varepsilon^2} \int_{-\infty}^{\infty} [\psi(x, v, t) - \psi(x, v - \varepsilon \zeta, t)] \xi(\zeta) d\zeta \) and \( \psi_T(x, v) \in C^\infty(\Omega) \) given.

We take the data \( \psi_T(x, v) \) so as to satisfy the following compatibility condition:

\[ \psi_T(x, v) = 0, \quad \text{for all } (x, v) \in N, \quad (2.10) \]

where \( N = \{(x^2 + \beta^2_\varepsilon(v) < \delta) \cup \{(x - 1)^2 + \beta^2_\varepsilon(v) < \delta\}\} \cap \Omega \) for some \( \delta > 0 \) small.

**Lemma 4.** Let \( \psi_T(x, v) \in C^\infty(\Omega) \) be a smooth data at \( t = T \) and satisfy \( (2.10) \) with \( \psi_T \geq 0 \). Then there exists a smooth solution \( \psi(x, v, t) \in C^\infty(UT) \) satisfying \( (2.9) \).

**Proof.** We solve the adjoint problem to \( (2.1) \) with the data \( \psi_T(x, v) \) at time \( t = T \) and we find a smooth solution \( \psi(x, v, t) \in C^\infty(UT) \) satisfying

\[ \psi_t + \partial_x \left( \left[ \beta_\varepsilon (v) + (v - \beta_\varepsilon (v)) \eta_\varepsilon (x) \right] \psi \right) - \tilde{Q}[\psi] (x, v, t) = 0, \quad (2.11) \]

\[ \psi(x, v, T) = \psi_T(x, v), \quad \psi|_{\gamma^+} = 0. \]

First we can show that there exists a mild solution

\[ \psi(x, v, t) \in C([0, T] ; L^1(\Omega)) \cap L^\infty([0, T] ; L^\infty(\Omega)) \]

of \( (2.11) \) by applying a method similar to that in Lemma 2 and Corollary 1. Then we can show that the mild solution of \( (2.11) \) is indeed smooth, that is, \( \psi \in C^\infty(UT) \).

This can be proved by observing that \( \psi_T(x, v) \in C^\infty(\Omega) \) satisfies the compatibility condition \( (2.10) \). Then we can apply a fixed point argument to the integral representation for the derivatives of \( \psi \) by differentiating the integral representation for \( \psi \) itself. \( \square \)

The next few lemmas concern the non-negativity of \( \psi \) and \( L^1 \) estimate.

**Lemma 5.** Let \( \psi(x, v, t) \in C^\infty(UT) \) be a solution of the adjoint equation \( (2.9) \) backwards in time. Then it satisfies \( \psi(x, v, t) \geq 0 \) for all \( (x, v, t) \in UT \).

**Proof.** Suppose that \( \psi \) is as in the statement of the lemma. We define \( \psi_k = \psi + [k - k(t - T)] e^{-L(t - T)} \) with \( k > 0 \) small and \( L \) depending on \( \varepsilon \) to be made precise later. Using \( (2.9) \) we obtain

\[ \tilde{L}\psi_k = -ke^{-L(t-T)} - L [k - k(t - T)] e^{-L(t-T)} + \psi_t \]

\[ + [\beta_\varepsilon (v) + (v - \beta_\varepsilon (v)) \eta_\varepsilon (x)] \partial_x \psi + (v - \beta_\varepsilon (v)) \eta'_\varepsilon (x) \psi \]

\[ + (k - k(t - T)) (v - \beta_\varepsilon (v)) \eta'_\varepsilon (x) e^{-L(t-T)} - \tilde{Q}_\varepsilon [\psi] \]

\[ = [k + (-L + (v - \beta_\varepsilon (v)) \eta'_\varepsilon (x)) (k - k(t - T))] e^{-L(t-T)}. \]
By using the fact that \((v - \beta_\varepsilon(v))\) and \(\eta_\varepsilon(x)\) are compactly supported as well as the fact that \((k - k(t - T)) > 0\) for \(t \leq T\), it then follows that, choosing \(L > 0\) sufficiently large, we obtain, for \(t \leq T\),

\[
\tilde{\mathcal{L}}\psi^k \leq -ke^{-L(t-T)} < 0.
\]  

(2.12)

We define domains \(U_{T_1,T} = \{(x,v,t) : x \in (0,1), v \in \mathbb{R}, 0 \leq T_1 \leq t \leq T\}\). Denote as \(T_\ast\) the infimum of the values of \(T_1 \geq 0\) such that \(\psi^k > 0\) in \(U_{T_1,T}\). Notice that, since \(\psi^k \geq k > 0\) at \(t = T\), we have \(T_\ast < T\) and by definition \(T_\ast \geq 0\). By continuity of \(\psi^k\) we have \(\psi^k \geq 0\) in \(\overline{U_{T_\ast,T}}\). We now apply maximum principle arguments in this set. We may assume that the minimum of \(\psi^k\) in \(U_{T_\ast,T}\) is 0, since otherwise \(\psi^k > 0\) in \(U_T\) and thus we are done. Suppose that the minimum 0 of \(\psi^k\) at \(\overline{U_{T_\ast,T}}\) is attained at one interior point of this set. Then \(\psi^k(x,v,t) = \psi^k(x,v,t) = 0\) while \(\tilde{\mathcal{L}}\psi^k(x,v,t) \geq 0\) so that \(\psi^k = 0\) at the point. Thus it cannot occur due to (2.12). On the other hand, we will prove now that minimum 0 cannot be obtained near the singular set. Suppose that \(\psi^k\) has its minimum in the set \(|v| \leq \varepsilon, 0 \leq x \leq \varepsilon, 1 - \varepsilon \leq x \leq 1\). Then the definition of \(\beta_\varepsilon(v)\) and \(\eta_\varepsilon(x)\) implies that \([\beta_\varepsilon(v) + (v - \beta_\varepsilon(v))\eta_\varepsilon(x)]\psi^k = 0\) in that set. We also have \(\psi^k \geq 0\) and \(\tilde{\mathcal{L}}\psi^k \leq 0\) at that minimum point so that \(\tilde{\mathcal{L}}\psi^k \geq 0\) at that point. This is again a contradiction. Therefore, the minimum of \(\psi^k\) in \(U_{T_\ast,T}\) cannot be achieved in that set. Now suppose that this minimum 0 is attained at \((x,v,T_\ast)\) with \((x,v) \in \Omega\). Then \(\psi^k(x,v,T_\ast) \geq 0\), \(\psi^k(x,v,T_\ast) = 0\), \(\tilde{\mathcal{L}}\psi^k(x,v,T_\ast) \leq 0\) so that \(\tilde{\mathcal{L}}\psi^k(x,v,T_\ast) \geq 0\). Again it cannot happen. Suppose that \(\psi^k\) has its minimum 0 at \((0,v,t)\) with \(v > 0\) and \(t > 0\) or at \((1,v,t)\) with \(v < 0\) and \(t > 0\). Then \(\psi^k = 0\), \([\beta_\varepsilon(v) + (v - \beta_\varepsilon(v))\eta_\varepsilon(x)]\psi^k \geq 0\), \(\tilde{\mathcal{L}}\psi^k \leq 0\) so that \(\tilde{\mathcal{L}}\psi^k \geq 0\). This leads to a contradiction. Therefore \(\psi^k\) has its minimum \(k = T\) or \(x = 0, v < 0\) or \(x = 1, v > 0\).

We then have obtained that \(\psi^k \geq k > 0\) in \(U_{T_\ast,T}\). If \(T_\ast > 0\) it would be possible to prove that \(\psi^k > 0\) in some set \(U_{T_\ast-\delta,T}\) for some \(\delta > 0\) and this would contradict the definition of \(T_\ast\). Therefore \(T_\ast = 0\). We then have \(\psi^k > 0\) in \(U_T\) for any \(k > 0\). Taking the limit \(k \to 0\) we obtain \(\psi \geq 0\) and complete the proof.  

We now have the following result.

**Lemma 6.** Let \(\psi(x,v,t) \in C^\infty(U_T)\) be a solution of the adjoint equation (2.9) and let \(F \in L^1(\Omega) \cap L^\infty(\Omega)\) be a weak solution of (2.1) with (2.2). Then we have for any \(t \in [0,T]\),

\[
\int_\Omega F(x,v,t) \psi (x,v,t) \, dx \, dv - \int_\Omega F(x,v,0) \psi(x,v,0) \, dx \, dv = 0.
\]

**Proof.** It follows from Definition 3 and from (2.9).  

Solutions to the adjoint problem satisfy the following property: the \(L^1\)-norm of a solution of (2.9) does not increase backward in time.

**Lemma 7.** Let \(\psi(x,v,t) \in C^\infty(U_T)\) be a solution of the adjoint equation (2.9) backwards in time. Then it satisfies

\[
\int_\Omega \psi(x,v,0) \, dx \, dv \leq \int_\Omega \psi(x,v,T) \, dx \, dv.
\]
Proof. We integrate (2.9) in \(x\) and \(v\) to get
\[
\frac{d}{dt} \int_{\Omega} \psi (x, v, t) \, dx \, dv = \int_{0}^{\infty} \beta_\varepsilon (v) \psi (0, v, t) \, dv - \int_{-\infty}^{0} \beta_\varepsilon (v) \psi (1, v, t) \, dv
\]
Now using Lemma 5, we can deduce the lemma. \(\Box\)

We now go back to our original approximated Fokker–Planck problem and establish the following maximum and minimum principles for weak solutions of (2.1). For that purpose, we first recall a basic lemma in measure theory.

Lemma 8. Let \(A\) be a set with a positive measure and let \(\delta > 0\) small be given. Then there exists a ball \(B\) such that
\[
\frac{\text{meas}(B \cap A)}{\text{meas}(B)} > 1 - \delta.
\]

Proof. Let \(A\) be a set with a positive measure. Let us denote as \(B_r(x_0)\) the ball \(|x - x_0| < r\). Then we get
\[
\frac{\text{meas}(B_r(x_0) \cap A)}{\text{meas}(B_r(x_0))} \to 1, \quad \text{almost everywhere } x_0 \in A \quad \text{as } r \to 0,
\]
where \(\chi_\Omega\) is the characteristic function on the set \(\Omega\) (cf. [32]), whence the result follows, choosing a suitable \(x_0 \in A\) and \(r > 0\) small. \(\Box\)

We now present the maximum principle for weak solutions of (2.1):

Lemma 9. If \(f_0 \in L^\infty (\Omega)\), then a weak solution \(f^\varepsilon\) to (2.1)–(2.2) satisfies
\[
f^\varepsilon (x, v, t) \leq \| f_0 \|_{L^\infty(\Omega)}
\]
up to a measure zero set.

Proof. Let \(M = \| f_0 \|_{L^\infty(\Omega)}\), then we want to prove that \(f^\varepsilon (x, v, t) \leq M\) for all \((x, v, t) \in (0, 1) \times (-\infty, \infty) \times (0, \infty)\) almost everywhere. We prove this by contradiction. Suppose the weak solution \(f^\varepsilon (\cdot, \cdot, T) > M\) at time \(t = T\) on a set with a positive measure. Then there is \(\kappa > 0\) small such that \(f^\varepsilon > M + \kappa\) on a set with a positive set, say \(A\). Since \(A\) has a positive measure, we apply Lemma 8 to ensure that for any given \(\delta > 0\) small there exists a ball \(B \subset \Omega\) such that
\[
\text{meas}(B \cap A) > \text{meas}(B) (1 - \delta) . \qquad (2.13)
\]
Then we choose a smooth function \(\psi_T (x, v) \in C^\infty (\Omega)\) such that \(\psi_T \geq 0\), \(\text{supp}(\psi_T)\) is contained in \(\bar{B}, \psi_T\) is uniformly bounded, and
\[
\left| \int_{\Omega} \frac{\chi_B}{\text{meas}(B)} \, dx \, dv - \int_{\Omega} \psi_T (x, v) \, dx \, dv \right| < \delta . \quad (2.14)
\]
Then by Lemma 4, Lemma 5, Lemma 7, there exists a smooth test function \( \psi(x, v, t) \in C^\infty(U_T) \) such that \( \psi \geq 0 \) and \( \int_\Omega \psi(x, v, 0) \, dx \, dv \leq \int_\Omega \psi(x, v, T) \, dx \, dv \) and

\[
\begin{align*}
\psi_t + \partial_x ([\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(x)] \psi) - \bar{Q}_\varepsilon[\psi](x, v, t) &= 0, \\
\psi|_{v^+} &= \psi_T(x, v), \quad \psi|_{v^-} = 0.
\end{align*}
\]

Then we estimate \( \int_\Omega f_\varepsilon(x, v, 0) \psi(x, v, 0) \, dx \, dv \) and \( \int_\Omega f_\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv \) respectively.

\[
\begin{align*}
\int_\Omega f_\varepsilon(x, v, 0) \psi(x, v, 0) \, dx \, dv &\leq M \int_\Omega \psi(x, v, 0) \, dx \, dv \\
&\leq M \int_\Omega \psi(x, v, T) \, dx \, dv \\
&= M \int_\Omega \psi_T(x, v) \, dx \, dv \\
&\leq M (1 + \delta) \leq M + C_1 \delta,
\end{align*}
\]

(2.15)

where we used Lemma 7 and (2.14). To estimate \( \int_\Omega f_\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv \)

\[
\begin{align*}
\int_\Omega f_\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv &= \int_B f_\varepsilon(x, v, T) \psi_T(x, v) \, dx \, dv \\
&= \int_{B \cap A} f_\varepsilon(x, v, T) \psi_T(x, v) \, dx \, dv \\
&\quad + \int_{B \setminus A} f_\varepsilon(x, v, T) \psi_T(x, v) \, dx \, dv =: I + II.
\end{align*}
\]

From the construction of \( \psi_T \) and the definition of the set \( A \) with a positive measure, we deduce

\[
I \geq (M + \kappa) \int_{B \cap A} \psi_T(x, v) \, dx \, dv
\]

\[
\geq (M + \kappa) \left[ \int_{B \cap A} \frac{\chi_B}{\text{meas}(B)} \, dx \, dv - \int_{B \cap A} \left\{ \frac{\chi_B}{\text{meas}(B)} - \psi_T(x, v) \right\} \, dx \, dv \right]
\]

\[
\geq (M + \kappa) \left[ \frac{\text{meas}(B \cap A)}{\text{meas}(B)} - \delta \right] > (M + \kappa) (1 - 2\delta) = M + \kappa - 2(M + \kappa) \delta.
\]

For \( II \), we use the fact that \( f_\varepsilon \) and \( \psi_T \) are bounded and (8) to get

\[
|II| \leq \|f_\varepsilon\|_{L^\infty} \|\psi_T\|_{L^\infty} \text{meas}(B \setminus A) < \|f_\varepsilon\|_{L^\infty} \|\psi_T\|_{L^\infty} \text{meas}(B) \delta.
\]

Combining the estimates for \( I \) and \( II \), we obtain

\[
\int_\Omega f_\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv \geq M + \kappa - C_2 \delta,
\]

(2.16)

where \( C_2 \) is independent of \( \delta \) and depends only on \( M + \kappa, \|f_\varepsilon\|_{L^\infty}, \|\psi_T\|_{L^\infty}, \) and \( \text{meas}(B) \).
Now suppose \( f^\varepsilon \) is a weak solution in Definition 3. Then we choose our test function \( \psi(x, v, t) \) as in the above to apply Lemma 6 and get
\[
\int_\Omega f^\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv = \int_\Omega f^\varepsilon(x, v, 0) \psi(x, v, 0) \, dx \, dv.
\]
Then using the estimates (2.15), (2.16), we deduce
\[
M + \kappa - C_2 \delta \leq \int_\Omega f^\varepsilon(x, v, T) \psi(x, v, T) \, dx \, dv = \int_\Omega f^\varepsilon(x, v, 0) \psi(x, v, 0) \, dx \, dv \leq M + C_1 \delta.
\]
Thus if \( \delta \) is chosen sufficiently small in such a way that \((C_1 + C_2)\delta < \kappa/2\), we can get a contradiction. Therefore, \( \|f^\varepsilon\|_{L^\infty(UT)} \leq M \), that is, \( f^\varepsilon \) satisfies the maximum principle. We complete the proof. \( \square \)

We also derive the non-negativity of solutions, which is the minimum principle for weak solutions.

**Lemma 10.** If \( f_0(x, v) \geq 0 \) and \( f^\varepsilon(x, v, t) \) is a weak solution of (2.1)–(2.2), then
\[
f^\varepsilon(x, v, t) \geq 0
\]
up to a measure zero set.

**Proof.** It can be proved similarly to the proof of Lemma 9 and we skip its proof. \( \square \)

The above two lemmas together provide the uniqueness result for solutions of the approximate Fokker–Planck equation (2.1) with (2.2).

**Corollary 2.** Let \( f_1^\varepsilon, f_2^\varepsilon \) be two weak solutions of (2.1) with the same initial and boundary conditions (2.2). Then \( f_1^\varepsilon = f_2^\varepsilon \) in \( L^\infty(UT) \).

**Proof.** It follows from Lemma 9 and Lemma 10. \( \square \)

Next, we show that the total mass is non-increasing up to a correction \( O(\varepsilon^2) \). The subtlety here is that we are not able to use the integration by parts since regularity has yet to be shown.

**Lemma 11.** Let \( f_0 \in L^1 \cap L^\infty(\Omega) \) and \( f_0 \geq 0 \) given. Then the total mass of a mild solution \( f^\varepsilon \) for \( 0 \leq t \leq T \) satisfies the following inequality.
\[
\int_\Omega f^\varepsilon(x, v, t) \, dx \, dv \leq \int_\Omega f_0(x, v) \, dx \, dv + r^\varepsilon(t) + q^\varepsilon(t),
\]
where \( 0 \leq r^\varepsilon(t) \leq 8\varepsilon^4 C T e^{CT} \|f_0\|_{L^\infty(\Omega)} \), \(|q^\varepsilon(t)| \leq 64\varepsilon^2 T^2 (1 + \varepsilon T) e^{CT} \|f_0\|_{\infty} \), and \( C \) is independent of \( T \) and \( \varepsilon \).
Proof. We will estimate the $L^1$ norm from the integral form (2.7). By integrating (2.7) over $\Omega$, we obtain
\[
\int_{\Omega} f^\varepsilon(x, v, t) \, dx \, dv = \int_{\Omega} \tilde{f}_0(X(t_0), v) \, dx \, dv
+ \int_{\Omega} \int_{t_0}^t Q^\varepsilon[f](X(s), v) \, ds \, dx \, dv =: I + II. \tag{2.18}
\]

We start with the estimation of $I$:
\[
I = \int_{\Omega} \tilde{f}_0(X(t_0), v) \, dx \, dv = \int_{\Omega_1} f_0(X(0), v) \, dx \, dv, \tag{2.19}
\]
where $\Omega_1$ is defined in (2.8), that is, we only treat the particles $(x, v, t)$ which connect to $t_0 = 0$. Our goal is to show that
\[
I \leq \int_{\Omega} f_0(x, v) \, dx \, dv + r^\varepsilon(t). \tag{2.20}
\]
By the change of variables,
\[
I = \int_{\Omega_1} f_0(X(0), v) \, dX(0) \, dv = \int_{\tilde{\Omega}_1} f_0(X(0), v) \, dX(0) \, dv,
\]
where
\[
J(t; 0) = 1 + \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s)) \, ds \right] = O(\varepsilon) \text{ as in Lemma 1 and hence } J(t; 0) > 0 \text{ for sufficiently small } \varepsilon > 0. \]
Thus we see that
\[
I = \int_{\tilde{\Omega}_1} f_0(X(0), v) \, dX(0) \, dv
= \int_{\tilde{\Omega}_1} f_0(X(0), v) \, dX(0) \, dv
+ \int_{\tilde{\Omega}_1} \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s)) \, ds \right] f_0(X(0), v) \, dX(0) \, dv
=: (i) + (ii). \tag{2.21}
\]
For (i), we write it as
\[
(i) = \int_{\Omega} f_0(X(0), v) \, dX(0) \, dv - \int_{\Omega \setminus \tilde{\Omega}_1} f_0(X(0), v) \, dX(0) \, dv \leq \int_{\Omega} f_0(x, v) \, dx \, dv. \tag{2.22}
\]
For the second term (ii), we write it as
\[
(ii) = \int_{\tilde{\Omega}_1} \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(s)) \, ds \right] f_0(X(0), v) \, dX(0) \, dv
= \int_{\tilde{\Omega}_1 \cap \{v > 0\}} + \int_{\tilde{\Omega}_1 \cap \{v < 0\}} =: (ii)^+ + (ii)^-. \]
Let us first estimate $(ii)^+$. Notice that the first factor $\frac{\partial}{\partial X(0)}[\int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds] = 0$ for $X(s) \leq \varepsilon$ or $2\varepsilon \leq X(s) \leq 1 - 2\varepsilon$ or $X(s) \geq 1 - \varepsilon$ or $v > 2\varepsilon^2$ due to the cutoffs. Thus we only need to integrate it over the set $A_0^+ = \{(X(0), v)\mid \varepsilon < X(s) < 2\varepsilon$ and $0 < v < 2\varepsilon^2$ for some $0 < s < t \}$ and $A_1^+ = \{(X(0), v)\mid 1 - 2\varepsilon < X(s) < 1 - \varepsilon$ and $0 < v < 2\varepsilon^2$ for some $0 < s < t \}$. On one hand, we see that $\frac{\partial}{\partial X(0)}[\int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds] \leq 0$ over $A_1^+$ since the cutoff function $\eta_\varepsilon$ decreases for $1 - 2\varepsilon < X(s) < 1 - \varepsilon$. Hence, we deduce that

\[
(ii)^+ \leq \int_{A_0^+ \cap \Omega_1} \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds \right] f_0(X(0), v) \, dX(0) \, dv =: r_+^\varepsilon(t),
\]

(2.23)

where $r_+^\varepsilon(t) \geq 0$. On the other hand, since $\beta_\varepsilon(v) + (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) > 0$, we see that if $X(0) \geq 2\varepsilon$, then $X(s) > 2\varepsilon$ for all $s > 0$. Therefore,

\[
A_0^+ \subset \{(X(0), v)\mid 0 < X(0) < 2\varepsilon, \ 0 < v < 2\varepsilon^2 \} =: B^+,
\]

where $|B^+| = 4\varepsilon^3$. Thus

\[
0 \leq r_+^\varepsilon(t) \leq \|f_0\|_{L^\infty(\Omega)} \int_{B^+} \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds \right] dX(0) \, dv \\
\leq \|f_0\|_{L^\infty(\Omega)} \int_{B^+} \varepsilon Cte^{\varepsilon Ct} \, dX(0) \, dv \text{ by Lemma 1} \\
\leq \varepsilon Cte^{\varepsilon Ct} \|f_0\|_{L^\infty(\Omega)} |B^+| \\
\leq 4\varepsilon^4 Cte^{\varepsilon Ct} \|f_0\|_{L^\infty(\Omega)}.
\]

(2.24)

For $(ii)^-$, as in the case of $v > 0$, we first note that the first factor vanishes $X(s) \leq \varepsilon$ or $2\varepsilon \leq X(s) \leq 1 - 2\varepsilon$ or $X(s) \geq 1 - \varepsilon$ or $v < -2\varepsilon^2$. Thus we only need to integrate it over the set $A_0^- = \{(X(0), v)\mid \varepsilon < X(s) < 2\varepsilon$ and $-2\varepsilon^2 < v < 0$ for some $0 < s < t \}$ and $A_1^- = \{(X(0), v)\mid 1 - 2\varepsilon < X(s) < 1 - \varepsilon$ and $-2\varepsilon^2 < v < 0$ for some $0 < s < t \}$. This time, we see that $\frac{\partial}{\partial X(0)}[\int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds] \leq 0$ over $A_0^-$ and hence we deduce that

\[
(ii)^- \leq \int_{A_1^- \cap \Omega_1} \frac{\partial}{\partial X(0)} \left[ \int_0^t (v - \beta_\varepsilon(v))\eta_\varepsilon(X(s)) \, ds \right] f_0(X(0), v) \, dX(0) \, dv =: r_-^\varepsilon(t).
\]

(2.25)

Moreover, we see that

\[
A_1^- \subset \{(X(0), v)\mid 1 - 2\varepsilon < X(0) < 1, \ -2\varepsilon^2 < v < 0 \} =: B^-,
\]

where $|B^-| = 4\varepsilon^3$ and hence by the same argument as in (2.24), we obtain

\[
0 \leq r_-^\varepsilon(t) \leq 4\varepsilon^4 Cte^{\varepsilon Ct} \|f_0\|_{L^\infty(\Omega)}.
\]

(2.26)
Combining (2.19), (2.21)–(2.26), we obtain

\[ I \leq \int_{\Omega} f_0(X(0), v) \, dX(0) \, dv + r^\varepsilon(t), \]

where \( r^\varepsilon(t) := r^\varepsilon_+(t) + r^\varepsilon_-(t) \) and complete the proof of (2.20).

We now turn to the second term \( II \) in (2.18):

\[ II = \int_{\Omega} \int_{t_0}^{t} Q^\varepsilon [f](X(s), v) \, ds \, dx \, dv \]

\[ = 2 \varepsilon^2 \left\{ \int_{\Omega} \int_{t_0}^{t} \int_{-\infty}^{\infty} \left\{ f^\varepsilon(X(s), v+\varepsilon \xi, s) - f^\varepsilon(X(s), v, s) \right\} \xi(\xi) \, d\xi \, ds \, dv \right\} \]

\[ = 2 \varepsilon^2 (iii) - (iv). \]

We will estimate (iv) first. Recall that for \( t_0 < s < t \),

\[ x = X(s) + \int_s^t [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(\tau))] \, d\tau. \]

Consider the following change of variables: \((x, v, s) \rightarrow (y = X(s), v, s)\). Then by the definition of \( t_0 \), we see that the domain of integration changes from \((x, v, s) \in (0, 1) \times (-\infty, \infty) \times (t_0, t)\) to \((y = X(s), v, s) \in (0, 1) \times (-\infty, \infty) \times (0, t)\). Hence, by Fubini’s Theorem,

\[ (iv) = \int_0^t \int_{\Omega} f^\varepsilon(y, v, s) \left( 1 + \frac{\partial}{\partial y} \left[ \int_s^t [\beta_\varepsilon(v) + (v - \beta_\varepsilon(v)) \eta_\varepsilon(X(\tau))] \, d\tau \right] \right) \, dy \, dv \, ds \]

\[ = \int_0^t \int_{\Omega} f^\varepsilon(y, v, s) \, dy \, dv \, dx \]

\[ + \int_0^t \int_{\Omega} f^\varepsilon(y, v, s) \left[ \int_s^t (v - \beta_\varepsilon(v)) \eta'_\varepsilon(X(\tau)) \frac{\partial X(\tau)}{\partial y} \, d\tau \right] \, dy \, dv \, ds \]

\[ = \int_0^t \int_{\Omega} f^\varepsilon(y, v, s) \, dy \, dv \, ds + (iv)_2. \]

Now for the second term \((iv)_2\), since \( |(v - \beta_\varepsilon(v)) \eta'_\varepsilon(X(\tau))| \leq C \varepsilon \) and \( (v - \beta_\varepsilon(v)) \eta'_\varepsilon(X(\tau)) = 0 \) if \( |v| > 2 \varepsilon^2 \) or \( X(\tau) \leq \varepsilon \) or \( 2\varepsilon \leq X(\tau) \leq 1 - 2\varepsilon \) or \( X(\tau) \geq 1 - \varepsilon \), by using Lemma 1 and Lemma 9

\[ |(iv)_2| = \left| \int_0^t \int_A \int_s^t (v - \beta_\varepsilon(v)) \eta'_\varepsilon(X(\tau)) \frac{\partial X(\tau)}{\partial y} f^\varepsilon(y, v, s) \, d\tau \, dy \, dv \, ds \right| \]

\[ \leq C \varepsilon t \varepsilon^2 e^{C t} \| f_0 \|_{\infty} \left| \int_A \, dy \, dv \right|, \]
where \( A = \{(y, v) : |v| \leq 2\varepsilon^2 \text{ and } \varepsilon < X(\tau) < 2\varepsilon \text{ or } 1 - 2\varepsilon < X(\tau) < 1 - \varepsilon \text{ for some } s < \tau < t\} \). Note that \( A \subset \{(y, v) : |v| \leq 2\varepsilon^2 \text{ and } y \leq 2\varepsilon(1 + \varepsilon t) \text{ or } 1 - y \leq 2\varepsilon(1 + \varepsilon t)\} \) and hence
\[
|\text{(iv)}_2| \leq 16C \varepsilon^4 t^2 (1 + \varepsilon t) e^{Ct} \|f_0\|_{\infty},
\]
where \( C \) is a constant independent of \( t \) and \( \varepsilon \).

For (iii), we consider the following change of variables: \( y = X(s) \), \( w = v + \varepsilon \xi \) and \( s = s \). Then similarly, by Fubini’s Theorem we obtain
\[
\text{(iii)} = \int_{-\infty}^{\infty} \int_0^t \int_{\Omega} f \varepsilon (y, w, s) \left( 1 + \frac{\partial}{\partial y} \int_s^t (w - \varepsilon \xi) \eta \varepsilon (X_{w - \varepsilon \xi}(\tau)) d\tau \right) dy \, dw \, ds \xi(\xi) \, d\xi
\]
\[
= \int_0^t \int_{\Omega} f \varepsilon (y, w, s) dy \, dw \, ds
\]
\[
+ \int_{-\infty}^{\infty} \int_0^t \int_{\Omega} f \varepsilon (y, w, s) \left[ \int_s^t (w - \varepsilon \xi) \eta_\varepsilon (X_{w - \varepsilon \xi}(\tau)) \frac{\partial X_{w - \varepsilon \xi}(\tau)}{\partial y} d\tau \right] dy \, dw \, ds \xi(\xi) \, d\xi
\]
\[
= \int_0^t \int_{\Omega} f \varepsilon (y, w, s) dy \, dw \, ds + (\text{iii})_2.
\]
As in the previous case, it is easy to see that the second term \( |(\text{iii})_2| \) is bounded by \( 16C \varepsilon^4 t^2 (1 + \varepsilon t) e^{Ct} \|f_0\|_{\infty} \). Therefore, we deduce that
\[
II = \int_{\Omega} \int_0^t Q \varepsilon [f] (X(s), v) \, dx \, dv = \frac{2}{\varepsilon^2} \{(\text{iii})_2 - (\text{iv})_2\} =: q \varepsilon (t),
\]
where \( |q \varepsilon (t)| \leq 64C \varepsilon^2 t^2 (1 + \varepsilon t) e^{Ct} \|f_0\|_{\infty} \) for a constant \( C \) independent of \( t \) and \( \varepsilon \). This completes the proof of the lemma. \( \square \)

2.3. Well-Posedness of Weak Solutions for the Fokker–Planck Equation

We now obtain a weak limit of the approximating sequence \( \{f \varepsilon \} \) as a candidate for a weak solution.

**Proposition 1.** Let \( T > 0 \) and let \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \). Then \( f \varepsilon \) converges weakly to \( f \) in \( L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) with \( f \geq 0 \). Moreover, the following holds:
\[
f(x, v, t) \leq \|f_0\|_{L^\infty(\Omega)} \text{ and } \int_{\Omega} f(x, v, t) \, dx \, dv \leq \int_{\Omega} f_0(x, v) \, dx \, dv.
\]

**Proof.** It follows from Lemma 9, Lemma 10, Lemma 11, and by taking the limit in the weak topology as \( \varepsilon \to 0 \). \( \square \)
We now prove Theorem 1.1, which is the existence of a weak solution of (1.1) in the following.

**Proof of Theorem 1.1 (Existence).** We show that \( f \) in the Proposition 1 above is indeed a weak solution of (1.1). We first show the weak continuity of \( f(t) \). Let a test function \( \psi(x, v, t) \) compactly supported be given and \( t_1, t_2 \in [0, T] \) and let \( \varepsilon > 0 \) be given. Note that for the solution \( f^\varepsilon \) of the regularized problem (2.1)–(2.2),

\[
\int_\Omega f^\varepsilon(t_1) \psi(t_1) \, dx \, dv - \int_\Omega f^\varepsilon(t_2) \psi(t_2) \, dx \, dv = \int_\Omega f^\varepsilon(t_1)[\psi(t_1) - \psi(t_2)] \, dx \, dv + \int_\Omega [f^\varepsilon(t_1) - f^\varepsilon(t_2)] \psi(t_2) \, dx \, dv.
\]

Since \( f^\varepsilon \) is in \( C([0, T]; L^1(\Omega)) \cap L^\infty([0, T]; L^\infty(\Omega)) \), both terms on the right-hand side can be made small uniformly in \( \varepsilon \) if \( |t_1 - t_2| \) is sufficiently small. Since \( f^\varepsilon \) converges weakly to \( f \), by taking \( \varepsilon \to 0 \), we deduce the weak continuity of \( f(t) \) and in particular, \( \int_\Omega f(t) \psi(t) \, dx \, dv \) is well-defined for each \( t \in [0, T] \).

We can now derive the maximum and minimum principles for the Fokker–Planck operator.

Define

\[
\mathcal{M} f := f_t + v f_x - f_{vv}.
\]

We can deduce the maximum and minimum principle for weak solutions of (1.1).

**Lemma 12.** The operator \( \mathcal{M} \) defined in (2.27) has a maximum principle for weak solutions: Let \( f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) be a weak solution of (1.1)–(1.4) as in Definition 1 and let \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \). Then we have

\[
f(x, v, t) \leq \|f_0\|_{L^\infty(\Omega)}
\]

up to a measure zero set.

**Proof.** It is analogous to Lemma 9. \( \Box \)

**Lemma 13.** The operator \( \mathcal{M} \) defined in (2.27) has a minimum principle for weak solutions: Let \( f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) be a weak solution of (1.1)–(1.4) as in Definition 1 and let \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \), then we have

\[
f(x, v, t) \geq 0
\]

up to a measure zero set.
Proof. It is analogous to Lemma 10. □

We then show the uniqueness of weak solutions of (1.1)–(1.4) as follows.

Proof of Theorem 1.2 (Uniqueness). Let $f_1, f_2$ be two weak solutions of (1.1) with the same initial and boundary conditions (1.2)–(1.4). Then $f_1 = f_2$ in $L^\infty(U_T)$. The proof is analogous to Corollary 2. □

Before we conclude this section, we present the maximum and minimum principles for classical solutions of (1.1). Some of the results will be used in Section 4 after we establish the regularity of weak solutions.

We begin with the maximum principle in bounded domains. Let $U_T^1 = (0, 1) \times (-L, L) \times (0, T)$ with $L > 0$, $T > 0$.

Lemma 14. The operator $\mathcal{M}$ defined in (2.27) has a maximum principle: Let $f$ be in $C^{1,2,1}_{x,v,t}(U_T) \cap C(\bar{U}_T)$ and satisfy $\mathcal{M}f \leq 0$, then $f$ attains its maximum either at $t = 0$ or at $x = 0$, $v > 0$ or at $x = 1$, $v < 0$ or at $v = \pm L$.

Proof. We extend $f$ to the domain outside of $(-L, L)$ with respect to $v$ by defining it to be zero. First we assume that $\mathcal{M}f < 0$. We prove this case by case. First we suppose the solution $f$ attains its maximum at an interior point $(x, v, t) \in U_T^1$. Then $f_t(x, v, t) = f_x(x, v, t) = 0$ while $f_{vv} \leq 0$ so that $\mathcal{M}f(x, v, t) \geq 0$. Thus it cannot occur. Now suppose its maximum is attained at $(x, v, T)$ with $(x, v) \in \mathcal{O}$. Then $f_t(x, v, T) = 0$, $f_t(x, v, T) \geq 0$ and $f_{vv} \leq 0$ so that $\mathcal{M}f(x, v, t) \geq 0$. Again, it cannot happen. Lastly we suppose that $f$ has its maximum at $(0, v, t)$ with $v < 0$ and $0 < t$ or at $(1, v, t)$ with $v > 0$ and $0 < t$. Then for $x = 0$, $v < 0$ or $x = 1$, $v > 0$, we have $f_t(x, v, t) \geq 0$, $v f_x(0, v, t) \geq 0$, and $f_{vv} \leq 0$ so that $\mathcal{M}f(x, v, t) \geq 0$. Therefore $f$ has a maximum at the kinetic boundary. Next we will show the lemma in the case of $\mathcal{M}f \leq 0$. In this case, we use $g = f - k t$, $k > 0$ to derive the maximum principle by letting $k \to 0$. We skip the details. This completes the proof. □

In order to prove the maximum principle for unbounded domains with respect to $v$, we will find a barrier function near $v = \infty$.

Lemma 15. There exists a super-solution $\phi (v, t) \in C^{2,1}_{v,t}((-\infty, \infty) \times (0, T))$ with $\phi \geq 0$ of $\mathcal{M}$ satisfying $\mathcal{M} \phi \geq 0$ and $\phi \to \infty$ as $|v| \to \infty$ uniformly in $t \in [0, T]$. This will be called a barrier function at infinity.

Proof. We find $\phi$ of the form $\phi (v, t) = a_0(t) + a_1(t) v^2$ and plug it into $\mathcal{M} \phi \geq 0$. Then we have

$$\frac{d}{dt} a_0(t) + \frac{d}{dt} a_1(t) v^2 \geq 2a_1(t).$$

Indeed, there exist many super-solutions which satisfy the equation above. For instance, $a_0(t) = e^{2t}$, $a_1(t) = e^{2t}$ or $a_0(t) = 2t + 1$, $a_1(t) = 1$ will work. □

We can obtain the maximum principle for unbounded domains in $v$. 

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Lemma 16. The operator $\mathcal{M}$ defined in (2.27) has a maximum principle: Let $f$ be in $C^{1,2,1}_{x,v,t}(U_T) \cap C(\bar{U}_T)$ and satisfy $\mathcal{M}f \leq 0$, then $f$ attains its maximum at its kinetic boundary $\Gamma_T^-$, that is, either at $t = 0$ or at $x = 0$, $v > 0$ or at $x = 1$, $v < 0$.

Proof. Fix $w \in \mathbb{R}$, $\lambda > 0$, and let $\phi$ be a barrier function at infinity as in Lemma 15. We then define

$$g(x,v,t) = f(x,v,t) - \lambda \phi(v-w,t).$$

Then we have $\mathcal{M}g \leq 0$. Thus Lemma 14 applies to $g$ in $U_T^1 = (0,1) \times (w-r, w+r) \times (0, T)$ for any $r > 0$. For any $0 < x < 1$, $-\infty < v < \infty$, $g(x,v,0) = f(x,v,0) - \lambda \phi(v-w,0) \leq f(x,v,0)$. For $0 < v < \infty$, $0 < t < T$, $g(0,v,t) = f(0,v,t) - \lambda \phi(v-w,t) \leq f(0,v,t)$, and for $-\infty < v < 0$, $0 < t < T$, $g(1,v,t) = f(1,v,t) - \lambda \phi(v-w,t) \leq f(1,v,t)$. Now for $v = w \pm r$, $g(x,v,t) = f(x,v,t) - \lambda \phi(\pm r,t) \leq \sup_{x,v} f(x,v,0)$ if $r > 0$ is sufficiently large. Thus $g(x,w,t) \leq \sup_{\Gamma_T^-} f(x,v,t)$ for all $(x,w,t) \in \bar{U}_T$. Letting $\lambda \to 0$, we get $f(x,w,t) \leq \sup_{\Gamma_T^-} f(x,v,t)$ for all $(x,w,t) \in \bar{U}_T$. This completes the proof. \sq

We can also derive a minimum principle for the Fokker–Planck operation.

Lemma 17. The operator $\mathcal{M}$ defined in (2.27) has a minimum principle: Let $f$ be in $C^{1,2,1}_{x,v,t}(U_T) \cap C(\bar{U}_T)$ and satisfy $\mathcal{M}f \geq 0$. Then $f$ has a minimum at its kinetic boundary $\Gamma_T^-$, that is, either at $t = 0$ or at $x = 0$, $v > 0$ or at $x = 1$, $v < 0$.

Proof. It is analogous to Lemma 16. \sq

3. Regularity

In this section, we will establish the regularity (hypoellipticity) of the weak solutions obtained in the previous section by studying the adjoint problem. As a preparation, we first recall the fundamental solution to the forward Fokker–Planck equation in the whole space.

3.1. Preliminaries

3.1.1. Fundamental Solution in the Absence of Boundary The fundamental solution $G$ for the Fokker–Planck equation (1.1) in the whole space $(x,v,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ is given by (for instance, see [22])

$$G(x,v,t; \xi,v,\tau) = G(x-\xi,v,v,t-\tau) = \frac{3^{\frac{1}{2}}}{2\pi(t-\tau)^2} \exp \left( -\frac{3|x-\xi-(t-\tau)(v+v)/2|^2}{(t-\tau)^3} - \frac{|v-v|^2}{4(t-\tau)} \right). \quad (3.1)$$

Any solution of the linear problem (1.1) with initial data $f_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$ has the integral expression

$$f(x,v,t) = \int_{\mathbb{R}^2} G(x,v,t; \xi,v,0)f_0(\xi,v) \, d\xi \, dv.$$
For the purpose of our study on the boundary hypoelliptic regularity, we further investigate in the following lemma the behavior of the fundamental solution $G$ near $x = 0$ in the integral form. Since the behavior near $x = 1$ can be studied in a similar manner, we will skip it.

**Lemma 18.** The fundamental solution $G$ given in (3.1) satisfies the following right limit at $x = 0$ in the integral form. Let $t > 0$ and $v > 0$ be given fixed positive time and velocity and let $\lambda$ be a given integrable and continuous function. Then we have

$$
\lim_{x \to 0^+} \int_0^t ds \int_\mathbb{R} d\omega \lambda(\omega, s) G(x, v, w, t - s)
= \frac{\lambda(v, t)}{v} + \int_0^t ds \int_\mathbb{R} d\omega \lambda(\omega, s) G(0, v, w, t - s). \quad (3.2)
$$

**Proof.** Let $\varepsilon > 0$ be a given arbitrary small number. Let $x > 0$ be sufficiently small, say $x = o(\varepsilon)$ so that $\lim_{\varepsilon \to 0^+} x/\varepsilon = 0$. We now divide the integral on the left-hand side of (3.2) into two parts. One is when $t - \varepsilon < s < t$ and the other is its complement: $0 < s < t - \varepsilon$. Then

$$
\int_0^t ds \int_\mathbb{R} d\omega \lambda(\omega, s) G(x, v, w, t - s)
= \int_{t - \varepsilon}^t ds \int_\mathbb{R} d\omega \lambda(\omega, s) G(x, v, w, t - s)
+ \int_0^{t - \varepsilon} ds \int_\mathbb{R} d\omega \lambda(\omega, s) G(x, v, w, t - s)
=: (i) + (ii). \quad (3.3)
$$

We first compute the first part $(i)$. Since $G$ is integrable, we can write $(i)$ as

$$(i) = \lambda(v, t) \int_{t - \varepsilon}^t ds \int_\mathbb{R} d\omega G(x, v, w, t - s)
+ \int_{t - \varepsilon}^t ds \int_\mathbb{R} d\omega (\lambda(\omega, s) - \lambda(v, t)) G(x, v, w, t - s) =: \lambda(v, t) (i)_1 + (i)_2. \quad (3.4)
$$

The second integral $(i)_2$ in (3.4) converges to 0 as $\varepsilon \to 0$. This can be established by splitting the integral into two parts: $|w - v| < \delta$ and $|w - v| > \delta$. When $w$ is close enough to $v$: $|w - v| < \delta$, one can use the continuity of $\lambda$ and when $|w - v| > \delta$ and $0 < t - s < \varepsilon$, it can be shown that the remaining integral can be made as small as possible. We omit the details.

In what follows, we will show that the first integral in (3.4), $(i)_1 \to 1/v$ as $\varepsilon \to 0$. To do so, we use the explicit expression for $G$: first, substituting $t - s$ with a new variable (denoted again as $s$) in $(i)_1$, we see that

$$(i)_1 = \frac{3\frac{1}{2}}{2\pi} \int_0^\varepsilon \int_\mathbb{R} \frac{1}{s^2} \exp\left(-\frac{3|x - s(v + w)|^2}{s^3} - \frac{|v - w|^2}{4s}\right) d\omega \, ds.$$
We first note that the \( w \)-integral can be explicitly computed: for any fixed \( x, s, v > 0 \),
\[
\int_{\mathbb{R}} \exp \left( -\frac{3|x-s(v+w)/2|^2}{s^3} - \frac{|v-w|^2}{4s} \right) \, dw \\
= \int_{\mathbb{R}} \exp \left( -\frac{|w-v-\frac{3(x-s)}{s}v|^2}{s} - \frac{3|\frac{x}{s}-v|^2}{4s} \right) \, dw \\
= \exp \left( -\frac{3|x-sv|^2}{4s^3} \right) \int_{\mathbb{R}} \exp \left( -\frac{|\tilde{w}|^2}{s} \right) \, d\tilde{w} = \pi^{\frac{1}{2}} s^\frac{1}{2} \exp \left( -\frac{3|x-sv|^2}{4s^3} \right)
\]
and thus (i) can be rewritten as
\[
(i)_1 = \frac{3^{\frac{1}{4}}}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{1}{s^\frac{1}{2}} \exp \left( -\frac{3|x-sv|^2}{4s^3} \right) \, ds \\
= \frac{3^{\frac{1}{4}}}{2\pi^{\frac{1}{2}}} \left( \int_0^\infty \frac{(1-\alpha)^{\frac{1}{4}}}{s^\frac{1}{2}} \, ds + \int_\infty^\infty \frac{(1+\alpha)^{\frac{1}{4}}}{s^\frac{1}{2}} \, ds \right)
\]
(3.5)
for a sufficiently small positive number \( \alpha \) to be determined. We will estimate each term respectively. For (i)_{11}, notice that \( 0 < s < (1-\alpha)^{\frac{1}{\alpha}}v \) implies \( x-vs > \frac{\alpha}{1-\alpha}vs > 0 \), which in turn implies \( -|x-sv|^2 < -(\frac{\alpha}{1-\alpha})^2v^2s^2 \). Hence
\[
(i)_{11} = \int_0^{(1-\alpha)^{\frac{1}{\alpha}}v} \frac{1}{s^\frac{1}{2}} \exp \left( -\frac{3|x-sv|^2}{4s^3} \right) \, ds \\
< \int_0^{(1-\alpha)^{\frac{1}{\alpha}}v} \frac{1}{s^\frac{1}{2}} \exp \left( -\frac{3\alpha^2v^2}{4(1-\alpha)^2} \frac{1}{s} \right) \, ds.
\]
Now to see the dependence on \( \alpha \) of the integral, we make the substitution, \( t = \frac{v}{\alpha^2v^2} \):
\[
(i)_{11} < \int_0^{(1-\alpha)^{\frac{1}{\alpha}}v} \frac{1}{\alpha^3v^3t^\frac{1}{2}} \exp \left( -\frac{3\alpha^2v^2}{4(1-\alpha)^2} \frac{1}{t} \right) \alpha^2v^2 \, dt \\
= \frac{1}{\alpha v} \int_0^{(1-\alpha)^{\frac{1}{\alpha}}v} \frac{1}{t^\frac{1}{2}} \exp \left( -\frac{3\alpha^2v^2}{4(1-\alpha)^2} \frac{1}{t} \right) \, dt.
\]
Since \( v \) is bounded away from zero, the integrand is uniformly bounded, and therefore, we deduce that
\[
(i)_{11} \leq \frac{C_1x}{\alpha^2v^4} \quad \text{for some uniform constant } C_1. \quad (3.6)
\]
For (i)_{13}, \( (1+\alpha)^{\frac{1}{\alpha}}v < s \) implies \( vs-x > \frac{\alpha}{1+\alpha}vs > 0 \) and thus \( -|x-vs|^2 < -\frac{\alpha^2}{(1+\alpha)^2}v^2s^2 \). Hence we get
\[
(i)_{13} = \int_{(1+\alpha)^{\frac{1}{\alpha}}v}^\infty \frac{1}{s^\frac{1}{2}} \exp \left( -\frac{3|x-sv|^2}{4s^3} \right) \, ds \\
< \int_{(1+\alpha)^{\frac{1}{\alpha}}v}^\infty \frac{1}{s^\frac{1}{2}} \exp \left( -\frac{3\alpha^2v^2}{4(1+\alpha)^2} \frac{1}{s} \right) \, ds.
\]
As before, letting $t = \frac{s}{\alpha^2 v^2}$, we obtain

\[
(i)_{13} < \int_{(1+\alpha)^{1/2} v}^{1/\alpha^2 v^2} \frac{1}{\alpha^3 v^3 t^2} \exp \left( -\frac{3}{4(1+\alpha)^2} \frac{1}{t} \right) \alpha^2 v^2 \, dt
\]

\[
< \frac{1}{\alpha v} \int_0^{1/\alpha^2 v^2} \frac{1}{\alpha^3 v^3 t^2} \exp \left( -\frac{3}{4(1+\alpha)^2 t} \right) \, dt
\]

and hence we deduce that

\[
(i)_{13} \leq \frac{C_2 \varepsilon}{\alpha^2 v^3} \quad \text{for some uniform constant } C_2. \tag{3.7}
\]

The integral $(i)_{12}$ is when $s$ is very close to $x/v$. Notice that $(1-\alpha)\frac{x}{v} < s < (1+\alpha)\frac{x}{v}$ as well as $|s - \frac{x}{v}| < \frac{\alpha x}{v}$. Therefore, we can bound $(i)_{12}$ as

\[
I^- \leq (i)_{12} = \int_{|s - \frac{x}{v}| < \frac{\alpha x}{v}} \frac{1}{s^2} \exp \left( -\frac{\sqrt{3}v}{2s^2} \left( s - \frac{x}{v} \right)^2 \right) \, ds \leq I^+, \tag{3.8}
\]

where

\[
I^\pm = \int_{|s - \frac{x}{v}| < \frac{\alpha x}{v}} \frac{1}{((1 \mp \alpha)^2 v^2)^{3/2}} \exp \left( -\frac{\sqrt{3}v}{2((1 \pm \alpha)^2 v^2)^{3/2}} \left( s - \frac{x}{v} \right)^2 \right) \, ds.
\]

Letting $z = \frac{\sqrt{3}v}{2((1\pm\alpha)^2 v^2)^{3/2}} (s - \frac{x}{v})$, we rewrite $I^\pm$ as follows:

\[
I^\pm = \frac{2}{\sqrt{3}v} \left( \frac{1 \pm \alpha}{1 \mp \alpha} \right)^{3/2} \int_{|z| < \frac{\sqrt{3}v}{2((1\pm\alpha)^2 v^2)^{3/2}}} e^{-z^2} \, dz. \tag{3.9}
\]

From (3.6)–(3.9), it is clear that if we choose $\alpha = \varepsilon^{1/4}$, the following holds

\[
\lim_{\varepsilon \to 0} (i)_{11} = \lim_{\varepsilon \to 0} (i)_{13} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} I^\pm = \lim_{\varepsilon \to 0} (i)_{12} = \frac{2\sqrt{\pi}}{\sqrt{3}v}
\]

and therefore, from (3.5) we conclude that

\[(i)_1 \to \frac{1}{v} \quad \text{as } \varepsilon \to 0.\]

It now remains to compute the limit of (ii) in (3.3). It is clear that for $t - s > \varepsilon > x = o(\varepsilon)$, there exists a uniform constant $C_3 > 0$ so that

\[
\frac{1}{(t-s)^2} \exp \left( -\frac{3|v - w|^2}{4(t-s)} \right) \leq \frac{1}{(t-s)^2} \exp \left( -C_3 \frac{|w|^2}{(t-s)} \right)
\]
and therefore, by the dominated convergence theorem, we can pass to the limit:

$$\lim_{\epsilon \to 0} (i) = \int_0^t ds \int_{\mathbb{R}} dw \lambda(w, s) G(0, v, w, t - s).$$

This completes the proof of Lemma. \quad \square

As Lemma 18 indicates, the fundamental solution for the Fokker–Planck equation is different from the heat kernel: due to the hyperbolic (transport) nature of the Fokker–Planck equation (1.1), \( G \) displays more singular behavior in \( x \) than the non-degenerate variable \( v \). This lemma will be used crucially for the boundary hypoellipticity result.

3.1.2. Adjoint problem We recall from Definition 1 that the weak solutions \( f \in L^\infty([0, T]; L^1 \cap L^\infty(\Omega)) \) to (1.1)–(1.4) with initial data \( f_0 \in L^1 \cap L^\infty(\Omega) \) satisfy

$$\int_{U_t} M^*(\psi) f + \int_\Omega \psi(t) f(t) = \int_\Omega \psi(0) f_0,$$

where

$$M^*(\psi) = -\psi_t - v \psi_x - \psi_{vv}$$

for every \( t \in [0, T] \) and any test function \( \psi(x, v, s) \in C^1(U_t) \) satisfying \( \text{supp}(\psi(\cdot, \cdot, s)) \subset [0, 1] \times [-R, R] \) for some \( R > 0 \) and \( \psi|_{r_t^+} = 0 \).

The adjoint problem is to solve the following adjoint equation (the backward Fokker–Planck equation)

$$M^*(\psi) = 0 \quad \text{for } t < T$$

for given data at \( t = T \) so that \( \psi \in L^1 \cap L^\infty(\Omega) \).

Notice that \( g(x, v, t) := f(x, -v, T - t) \), where \( f \) is the solution to the forward Fokker–Planck equation, solves the backward Fokker–Planck (3.12) for \( t < T \). Thus the transformation \( t \to t_0 - t \) and \( v \to -v \) in \( G \) yields the fundamental solution to the backward Fokker–Planck equation.

3.2. Hypoellipticity Away from the Singular Set

3.2.1. Interior Hypoellipticity The goal of this subsection is to prove the following interior hypoellipticity:

**Proposition 2.** Let \( f \) be the weak solution with given initial data \( f_0 \in L^1 \cap L^\infty(\Omega) \). Then for each \( t > 0 \), \( f \in H^{k,m}_{loc}(\Omega) \), where \( H^{k,m} = H^{k,m}_{x,v} \).

**Proof.** Let \((x_0, v_0, t_0)\), where \( x_0 > 0 \) and \( t_0 > 0 \), be given. Suppose the data \( \varphi \) at \( t = t_0 \) is supported in the interior: \( \text{supp} \varphi \subset B_\rho(x_0, v_0) \subset \Omega \) and \( \varphi \in C(B_\rho) \). We consider the following backward Fokker–Planck equation in the whole space:

$$M^*(\phi) = 0 \quad \text{for } t < t_0 \text{ where } \phi(x, v, t_0) = \varphi(x, v).$$

(3.13)

Notice that we can solve this equation in the whole space via the fundamental solution and moreover, the solution \( \phi \) of (3.13) is smooth due to the hypoellipticity
of the Fokker–Planck operator: \( \phi \in C^\infty(\mathbb{R}^2 \times (-\infty, t_0)) \). Moreover, we have \( \|\phi(0)\|_{L^\infty} \leq C\|\phi\|_{L^2} \), where \( C \) depends on \( t_0 \). Here we can extend a function in \( L^2(B_\rho) \) to a function in \( L^2(\mathbb{R}^N) \) just extending it by zero and this does not modify the \( L^2 \) norm. For a detailed discussion on hypoellipticity, see [19,21,36]. Choose \( 0 < \rho < \rho_1 < \rho_2 \) so that \( B_{\rho_2}(x_0, v_0) \) is contained in the interior and consider a smooth cutoff function \( \zeta \in C^\infty(\mathbb{R}^2) \) such that

\[
\zeta = \begin{cases} 
1 & \text{on } B_{\rho_1}, \\
0 & \text{on } \mathbb{R}^2 \setminus B_{\rho_2}.
\end{cases}
\]

Letting \( \psi = \phi \zeta \), we see that \( \psi \) satisfies the following

\[
-\mathcal{M}^*(\psi) = \psi_t + v\psi_x + \psi_{vv} = v\zeta_x \phi + 2\zeta_v \phi_v + \zeta_{vv} \phi.
\]

Since \( \zeta_x, \zeta_v, \zeta_{vv} \) are smooth, supported in \( \overline{B_{\rho_2}} \setminus B_{\rho_1} \), and \( \phi \) is also smooth in there, we deduce that \( R \) is smooth and \( \text{supp } R \subset \overline{B_{\rho_2}} \setminus B_{\rho_1} \). Note that \( \psi(0, v, t) = 0 \) for \( v < 0 \). We will use this \( \psi = \phi \zeta \) as a test function in (3.10) to get

\[
\int_{B_\rho} \varphi f(t_0) = \int_0^{t_0} \int_{\mathbb{R}^2 \setminus B_{\rho_1}} dx \, dv \, R f + \int_{\Omega} dx \, dv \, \psi(0) f_0.
\]

Notice that the right-hand side is bounded by \( \| f_0 \|_{L^1} \). Thus we deduce that

\[
\int_{B_\rho} \varphi f(t_0) \leq C \| \varphi \|_{L^2(B_\rho)},
\]

where \( C \) depends on \( t_0 \) and \( \| f_0 \|_{L^1} \). Since \( \varphi \in C(B_\rho) \) can be taken arbitrarily, by density argument, this can be extended for all functions in \( L^2(B_\rho) \). Thus by duality, \( f \in L^2(B_\rho) \). Now for regularity of higher order, we define a new function \( \phi \) solving (3.13) with initial data \( \phi(t_0) = \frac{\partial^{k'+m'}\varphi}{\partial x^{k'} \partial v^{m'}} \) (for \( k' \leq k, m' \leq m \)) and \( \varphi \in C^\infty \). Then, standard hypoellipticity results, which can be obtained by using the explicit kernel for the solutions of (3.13) and integration by parts, yield \( \|\phi(0)\|_{L^\infty} \leq C\|\varphi\|_{L^2} \). This is a regularizing effect from \( H_{x,v}^{-k',-m'} \) to \( L^\infty \). We then define \( \psi = \zeta \phi \). Arguing as in the case without derivatives we obtain \( \int_{B_\rho} \frac{\partial^{k'+m'}\varphi}{\partial x^{k'} \partial v^{m'}} f(t_0) \leq C \| \varphi \|_{L^2(B_\rho)} \) for \( k' \leq k, m' \leq m \) and this implies the desired estimate in \( H_{x,v}^{k,m}(B_\rho) \) by the duality characterization of \( H_{x,v}^{k,m}(B_\rho) \), we conclude that \( f \in H_{x,v}^{k,m}(B_\rho) \). This completes the proof. □

3.2.2. Boundary hypoellipticity The goal of this subsection is to prove the boundary hypoellipticity away from the singular set \( \{(0, 0), (1, 0)\} \). Before we prove it, we derive a lemma which will be used to obtain the boundary hypoellipticity.
Let \( v_0 < 0 \). Choose \( \delta > 0 \) such that \( 2\delta < |v_0| \). We consider the following backward Fokker–Planck problem:

\[
\begin{aligned}
\mathcal{M}^*(\phi) &= 0 \text{ for } t < t_0 \\
\phi(x, v, t_0) &= \varphi(x, v) \text{ where } \text{supp} \varphi \subset B_\delta(0, v_0) \text{ and } \varphi \in C(B_\delta) \\
\phi(0, v, t) &= 0 \text{ for } |v - v_0| < 2\delta.
\end{aligned}
\tag{3.16}
\]

We show the existence of a solution \( \phi \) to (3.16).

**Lemma 19.** There exists \( \lambda(w, t) \in L^1(\mathbb{R} \times [0, t_0]) \) such that its support in \( w \) lies in \( |w - v_0| \leq 2\delta \) and it is smooth in \( |w - v_0| < 2\delta \) and that \( \phi(x, v, t) \) defined by the following expression

\[
\phi(x, v, t) = \int_{\mathbb{R}^2} d\xi dw \varphi(\xi, w)G(x - \xi, -v, -w, t_0 - t) \\
+ \int_{t_0}^t ds \int_{v_0 - 2\delta}^{v_0 + 2\delta} dw \lambda(w, s)G(x, -v, -w, s - t)
\tag{3.17}
\]

solves the problem (3.16). Here \( G \) is the fundamental solution to the forward Fokker–Planck equation given in (3.1).

**Proof.** Denote the right-hand side of (3.17) as \( \Phi[\lambda](x, v, t) \). Then it is clear that \( \mathcal{M}(\Phi[\lambda]) = 0 \) in \( \Omega \) and \( \Phi[\lambda](t = t_0) = \varphi \). We want to show that there exists a \( \lambda = \lambda(w, t) \) such that \( \lambda = 0 \) for \( |v - v_0| > 2\delta \) (by defining \( \lambda = 0 \) for \( |v - v_0| > 2\delta \) for instance) and \( \Phi[\lambda] = 0 \) for \( x = 0, |v - v_0| < 2\delta \). Let us first see what equation \( \lambda \) would obey to satisfy the desired properties. \( \Phi[\lambda] = 0 \) for \( |v - v_0| < 2\delta \) at \( x = 0 \) is equivalent to

\[
0 = \bar{\Phi}(0, v, t) + \lim_{x \to 0^+} \int_{t_0}^t ds \int_{v_0 - 2\delta}^{v_0 + 2\delta} dw \lambda(w, s)G(x, -v, -w, s - t) \text{ for } |v - v_0| < 2\delta,
\tag{3.18}
\]

where \( \bar{\Phi} \) is the first term of \( \Phi[\lambda] \): the homogeneous solution in the whole space. Now by Lemma 18, (3.18) can be written as follows:

\[
0 = \bar{\Phi}(0, v, t) - \frac{\lambda(v, t)}{v} + \int_{t_0}^t ds \int_{v_0 - 2\delta}^{v_0 + 2\delta} dw \lambda(w, s)G(0, -v, -w, s - t).
\tag{3.19}
\]

Here instead of \( 1/v, -1/v \) comes out in front of \( \lambda \) because \( G \) is evaluated at \( -v \) and \( -w \).

Note that \( |G(0, v, w, s - t)| \) is bounded by \( e^{-\frac{A}{|v-w|}} \) for \( v, w \in (v_0 - 2\delta, v_0 + 2\delta) \). Thus \( \lambda \) satisfies the following integral equation: for \( |v - v_0| < 2\delta \) and \( t < t_0 \),

\[
\lambda(v, t) = q(v, t) + \int_{t_0}^t ds \int_{|w-v_0|<2\delta} dw \lambda(w, t)K(v, w, s - t),
\tag{3.20}
\]

where \( q \) is a given smooth function and the given smooth kernel \( K \) has the following bound:

\[
|K(v, w, s - t)| \leq Ce^{-\frac{A}{|v-w|}} \text{ for } |v - v_0| < 2\delta \text{ and } |w - v_0| < 2\delta
\]
for some positive $A > 0$. Hence by a fixed point argument, we can find a $\lambda$ satisfying (3.20). This completes the proof. \qed

**Proof of Theorem 1.3 (i).** Proposition 2 proves the hypoellipticity away from the boundary. It then suffices to establish the regularity near the boundary: $(0, v_0, t_0)$ where $v_0 \neq 0$ and $t_0 > 0$ since the other boundary $(1, v_0, t_0)$ where $v_0 \neq 0$ and $t_0 > 0$ can be treated similarly. We divide into two cases: when $v_0 < 0$ and $v_0 > 0$.

We first treat the case when $x_0 = 0$, $v_0 < 0$. To do so, we choose a smooth cutoff function $\zeta \in C^\infty(\mathbb{R}^2)$ such that

$$
\zeta = \begin{cases} 
1 & \text{on } B_\delta(0, v_0) \\
0 & \text{on } \mathbb{R}^2 \setminus B_\delta(0, v_0).
\end{cases}
$$

Following the same argument as for the interior regularity, we pick a test function $\psi$ as $\psi = \phi \zeta$ where $\phi$ is the solution to (3.16) in Lemma 19. First we see that $\psi$ satisfies the following

$$
-M^k(\psi) = \psi_t + v \psi_x + \psi_{vv} = v \xi_x \phi + 2 \xi_v \phi_v + \xi_{vv} \phi =: R.
$$

(3.21)

Since $\xi_x$, $\xi_v$, $\xi_{vv}$ are smooth and supported in $\overline{B}_\delta \setminus B_\delta$ and $\phi$ is also smooth in there, we deduce that $R$ is smooth and $\text{supp } R \subset \overline{B}_\delta \setminus B_\delta$. Moreover, since $\phi(0, v, t) = 0$ for $|v - v_0| < 2\delta$ and $\xi(0, v) = 0$ for $|v - v_0| \geq 2\delta > 0$, $\psi(0, v, t) = 0$ for all $v < 0$. Thus we can use this $\psi = \phi \zeta$ as a test function in (3.10) by restricting to

$$
\int_{B_\delta \cap \Omega} \varphi f(t_0) = \int_0^{t_0} \int_{\overline{B}_\delta \setminus B_\delta \cap \Omega} dx \, dv \, R f + \int_{\Omega} dx \, dv \, \psi(0) f_0.
$$

(3.22)

Notice that the right-hand side is bounded by $\|f_0\|_{L^1}$. Thus we deduce that

$$
\int_{B_\delta \cap \Omega} \varphi f(t_0) \leq C.
$$

Since $\varphi \in C(B_\delta)$ can be taken arbitrarily, by density argument, this can be extended for all functions in $L^2(B_\delta \cap \Omega)$. Thus by duality, $f \in L^2(B_\delta \cap \Omega)$. Now if we take a test function: $\psi = \phi \zeta$ with a new function $\phi$ solving (3.13) with initial data $\phi(t_0) = \frac{\partial^{k+m'} q}{\partial x^k \partial v^{m'}}$ (for $k' \leq k$, $m' \leq m$) and $\varphi \in C^\infty$ as in the interior regularity, by the duality characterization of $H_{x,v}^{k,m}(B_\delta \cap \Omega)$, we conclude that $f \in H_{x,v}^{k,m}(B_\delta \cap \Omega)$.

It now remains to treat the case when $x_0 = 0$ and $v_0 > 0$. This can be treated in the same way as in the interior case: since there is no restriction on the boundary values of the test functions $\psi$ for $v > 0$, by a suitable choice of a cutoff function, we can easily find an appropriate test function localized near $(0, v_0)$ for $v_0 > 0$. We omit the details. \qed

**Remark 1.** Notice that as in the proof of the hypoellipticity (see (3.15) and (3.22)) the supremum norm of $f$ away from the singular set is bounded by $\|f_0\|_{L^1}$.
3.2.3. Optimal estimates for the derivatives near the singular set  
We derive in this subsection the following estimates near the singular set for the derivatives of solutions to (1.1)–(1.4) by a scaling argument and the hypoellipticity.

Lemma 20. Let $f(x, v, t) \in C^\infty(U_T)$ be a solution of (1.1)–(1.4). Then it satisfies
\[
\left(|v|^3 + |x - x_0|\right) \|f_x\|_{L^\infty} + \left(|v|^3 + |x - x_0|\right)^{2/3} \|f_{vv}\|_{L^\infty} + \frac{1}{3} \left(|v|^3 + |x - x_0|\right)^{1/3} \|f_v\|_{L^\infty} \leq C,
\]
where $x_0 = 0$ or $1$ and $C$ depends only $\|f_0\|_{L^1(\Omega)}$ and $\|f_0\|_{L^\infty(\Omega)}$.

Proof. First we use the hypoellipticity to get, for $1/2 \leq (|V|^3 + |X|)^{1/3} \leq 1$ and for $1/2 \leq (|V|^3 + |X - 1|)^{1/3} \leq 1$,
\[
\|f_X\|_{L^\infty} + \|f_{VV}\|_{L^\infty} + \|f_V\|_{L^\infty} \leq C,
\]
where $C$ depends only on $\|f_0\|_{L^1(\Omega)}$ and $\|f_0\|_{L^\infty(\Omega)}$. We now scale $X, V, \tau$ as follows:
\[
v = RV, \ x - x_0 = R^3 (X - x_0), \ t = R^2 \tau.
\]
Then we have, for $R/2 \leq (|V|^3 + |X|)^{1/3} \leq R$ and for $R/2 \leq (|V|^3 + |X - 1|)^{1/3} \leq R$,
\[
R^3 \|f_X\|_{L^\infty} + R^2 \|f_{VV}\|_{L^\infty} + R \|f_V\|_{L^\infty} \leq C,
\]
where $C$ depends only on $\|f_0\|_{L^1(\Omega)}$ and $\|f_0\|_{L^\infty(\Omega)}$. This implies (3.23) and completes the proof. □

3.3. Power Law Estimates for the Solution

We now derive estimates for the solutions of the Fokker–Planck equation near the singular point $(x, v) \in \{(0, 0), (1, 0)\}$. The asymptotic behavior of these solutions has been found in some of the explicit solutions obtained in the physical literature (cf. [6,24]). We summarize the main properties of the relevant solutions and prove them in detail here.

We will use repeatedly in this subsection the usual asymptotic notation. More precisely, we will say that $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, for $x_0 \in [-\infty, \infty]$. On the other hand, we will use the symbol $\simeq$ in heuristic, nonrigorous arguments to indicate that two functions have a similar behavior in some region.

3.3.1. Construction of Super-solutions  
Our goal is to construct super-solutions which will allow us to control the singular set $\{(x, v) : (0, 0), (1, 0)\}$ based on the study of the self-similar behavior of solutions to the Fokker–Planck equation. We begin by recalling the steady Fokker–Planck equation:
\[
vF_x = F_{vv}.
\]
Lemma 21. There exist positive steady solutions $f_k^*$, with $k = 0, 1$, to (3.24), which blow up at the singular set, namely

1. $f_k^*(x, v) > 0$,
2. $\liminf_{r \to 0, (x,v) \in \partial B_r(k,0) \cap \Omega} f_k^*(x, v) = \infty$.

Proof. We seek a solution $F$ to (3.24) of the following form

$$F(x, v) = x^\alpha \Phi \left( \frac{-v^3}{9x} \right)$$

for $\alpha < 0$ to be determined. Plugging in the ansatz (3.25) into (3.24) and letting $z = -\frac{v^3}{9x}$, we deduce that $\Phi = \Phi(z)$ satisfies the following ODE

$$z \Phi'' + \left( \frac{2}{3} - z \right) \Phi' + \alpha \Phi = 0.$$  

(3.26)

It is well-known that the solutions to (3.26) are given by Kummer functions $M(-\alpha, \frac{2}{3}, z)$ and $U(-\alpha, \frac{2}{3}, z)$ (see [1]). We are interested in positive solutions for $z \in \mathbb{R}$ which grow at infinity. We recall the integral representation formula for $M$ (see 13.2.1 in [1]):

$$\frac{\Gamma\left(\frac{2}{3} + \alpha\right) \Gamma(-\alpha)}{\Gamma\left(\frac{2}{3}\right)} M\left(-\alpha, \frac{2}{3}, z\right) = \int_0^1 e^{zt} t^{-a-1}(1 - t)^{a-rac{1}{2}} \, dt,$$

(3.27)

which is valid as long as $\alpha < 0$ ($b = \frac{2}{3}$ is already positive). For sufficiently small negative $\alpha \sim 0$, we see that

$$\frac{\Gamma\left(\frac{2}{3} + \alpha\right) \Gamma(-\alpha)}{\Gamma\left(\frac{2}{3}\right)} > 0$$

and that for any real value $z \in \mathbb{R}$, the integral on the right-hand side of (3.27) is positive. Hence we deduce that for such negatively small $\alpha$ and for $z \in \mathbb{R}$, $M(-\alpha, \frac{2}{3}, z) > 0$, which implies that the corresponding $F$, denoted by $f_0^*$,

$$f_0^*(x, v) := x^\alpha M\left(-\alpha, \frac{2}{3}, -\frac{v^3}{9x}\right)$$

in (3.25) is also positive. It now remains to show that $f_0^*$ blows up at the origin, the item (2) in the above. This can verified by noting the asymptotic behavior of $M(a, b, z)$ (see 13.1.4 and 13.1.5 in [1]):

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{z^{a-b}} \quad \text{for } z \to \infty,$$

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(b - a)} (-z)^{-a} \quad \text{for } z \to -\infty.$$  

(3.28)
Therefore, applying (3.28) to our case: \( z = -\frac{v^3}{9x} \), \( a = -\alpha, \ b = \frac{2}{3} \), we obtain the following asymptotic behavior of \( f_0^* \):

\[
\begin{align*}
f_0^*(x, v) &\sim \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma(-\alpha)} x^{\alpha} e^{-\frac{3}{9x} \left(-\frac{v^3}{9x}\right)}^{-\alpha - \frac{2}{3}} \quad \text{for} \ v < 0, \\
f_0^*(x, v) &\sim \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3} + \alpha\right)} x^{\alpha} \left(\frac{v^3}{9x}\right)^{\alpha} \quad \text{for} \ v > 0,
\end{align*}
\]

as \( x \to 0 \). Thus \( f_0^*(x, v) \to \infty \) for \( v < 0 \) and \( x \to 0 \), and hence we conclude that

\[
\lim \inf_{r \to 0, (x, v) \in \partial B_r(0, 0) \cap \{0 \leq x \leq 1\}} f_0^*(x, v) = \infty.
\]

The next goal is to construct super-solutions that control the singular behavior near the singular set via self-similarity. The first step is to find the regular self-similar solution to (3.24). To this end, we define also \( \Lambda \) by means of

\[
\Lambda(\zeta) = \Phi\left(-\zeta^3\right)
\]

and we seek a solution to (3.24) of the form

\[
F(x, v) = x^\alpha \Lambda\left(\frac{v}{(9x)^{\frac{1}{3}}}\right)
\]

for \( \alpha > 0 \) this time. Then it is easy to check that \( \Lambda \) satisfies the following ODE

\[
\Lambda''(\zeta) + 3\zeta^2 \Lambda'(\zeta) - 9\alpha \zeta \Lambda(\zeta) = 0. \tag{3.29}
\]

We are interested in the construction of solutions of (3.29) which are polynomially bounded. We have the following result.

Claim. For any \( 0 < \alpha < \frac{1}{6} \), there exists a solution \( \Lambda(\zeta) \) of (3.29) with the form:

\[
\Lambda(\zeta) = U\left(-\alpha, \frac{2}{3}, -\zeta^3\right), \quad \zeta \in \mathbb{R}, \tag{3.30}
\]

where we denote as \( U(a, b, z) \) the Tricomi confluent hypergeometric function. The function \( \Lambda(\zeta) \) has the following properties.

1. \( \Lambda(\zeta) > 0 \) for any \( \zeta \in \mathbb{R} \).
2. there exists a positive constant \( K_+ > 0 \) such that

\[
\Lambda(\zeta) \sim \begin{cases}
K_+ |\zeta|^{3\alpha}, & \zeta \to \infty, \\
|\zeta|^{3\alpha}, & \zeta \to -\infty.
\end{cases} \tag{3.31}
\]

3. The function \( \Lambda(\zeta) \), up to a multiplicative constant, is the only solution of (3.29) which is polynomially bounded for large \( |\zeta| \).
**Proof of Claim 3.3.1.** Due to the relation between (3.29) and (3.26), we need to study the solutions of this last equation (3.29), which are algebraically bounded. The only solutions of the equation (3.26) which do not grow exponentially for large $z > 0$ are proportional to

$$\Phi(z) = U\left(-\alpha, \frac{2}{3}, z\right).$$  \hfill (3.32)

In order to study the properties of $\Phi(z)$ for negative values of $z$ we use that (cf. [1, 13.1.3]):

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left( \frac{M(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{M(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)} \right), \quad b \notin \mathbb{Z}. \hfill (3.33)$$

The function $M(a, b, z)$ is analytic for all $z \in \mathbb{C}$. Notice that, combining (3.33) and (3.30) we obtain the following representation formula for $\Lambda(\zeta)$:

$$\Lambda(\zeta) = \frac{\pi}{\sin(\frac{2}{3}\pi)} \left( \frac{M(-\alpha, \frac{2}{3}, -\zeta^3)}{\Gamma(\frac{1}{3} - \alpha)\Gamma(\frac{2}{3})} + \zeta \frac{M(\frac{1}{3} - \alpha, \frac{4}{3}, -\zeta^3)}{\Gamma(-\alpha)\Gamma(\frac{4}{3})} \right), \quad \zeta \in \mathbb{R}. \hfill (3.34)$$

Formula (3.34) provides a representation formula for $\Lambda(\zeta)$ in terms of the analytic functions $M(-\alpha, \frac{2}{3}, -\zeta^3)$, $M(\frac{1}{3} - \alpha, \frac{4}{3}, -\zeta^3)$. This formula shows that $\Lambda(\zeta)$ is analytic in $\zeta \in \mathbb{C}$.

We can compute the asymptotics of $\Lambda(\zeta)$ as $\zeta \to -\infty$ by using (3.30) and 13.5.2 in [1]. Then we deduce that

$$\Lambda(\zeta) \sim |\zeta|^{3\alpha} \quad \text{as} \quad \zeta \to -\infty. \hfill (3.35)$$

On the other hand, using 13.5.1 in [1] as well as (3.34) we obtain the asymptotics:

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(b - a)} \frac{e^{\pm i\pi a}}{(z)^{-a}} \quad \text{as} \quad z \to -\infty,$$

where the sign $+$ is used if $-\pi/2 < \arg(z) < 3\pi/2$ and the sign $-$ is used if $-3\pi/2 < \arg(z) < -\pi/2$. In the choice of the branch of the function $(\cdot)^\alpha$ we must choose the branch of the function $M$ which is analytic. We can obtain easily the asymptotics of the function $M$ which is analytic. This means that, choosing $z = re^{i\theta}$ with $\theta \in (-\pi/4, \pi/4)$ we obtain the asymptotics $M(a, b, r e^{\pm i\pi}) \sim \frac{\Gamma(b)}{\Gamma(b - a)} (r)^{-a}$, because the exponentials cancel out. Then, we must use the formulae:

$$M(a, b, -r) \sim \frac{\Gamma(b)}{\Gamma(b - a)} (r)^{-a}, \quad r \to \infty,$$

$$\zeta M\left(\frac{1}{3} - \alpha, \frac{4}{3}, -\zeta^3\right) \sim \frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma(1 + \alpha)} (\zeta)^{3\alpha}, \quad \zeta \to \infty.$$
Choose $\zeta > 0$. Then the phase factor cancels out. We then have, using (3.34),

$$\Lambda(\zeta) = \frac{\pi}{\sin(\frac{2}{3} \pi)} \left( \frac{M(-\alpha, \frac{2}{3}, -\zeta^3)}{\Gamma\left(\frac{1}{3} - \alpha\right) \Gamma\left(\frac{2}{3}\right)} + \zeta \frac{M\left(\frac{1}{3} - \alpha, \frac{4}{3}, -\zeta^3\right)}{\Gamma(-\alpha) \Gamma\left(\frac{4}{3}\right)} \right)$$

$$\sim \frac{\pi}{\sin(\frac{2}{3} \pi)} \left[ \frac{1}{\Gamma\left(\frac{1}{3} - \alpha\right)} \frac{1}{\Gamma\left(\frac{2}{3} + \alpha\right)} + \frac{1}{\Gamma(-\alpha) \Gamma(1 + \alpha)} \right] |\zeta|^{3\alpha}, \ zeta \to \infty.$$

Using the following formulae,

$$\Gamma\left(\frac{1}{3} - x\right) \Gamma\left(\frac{2}{3} + x\right) = \frac{\pi}{\sin\left(\pi\left(x + \frac{2}{3}\right)\right)} \quad \text{and} \quad \Gamma(-x) \Gamma(1 + x) = -\frac{\pi}{\sin\left(\pi x\right)},$$

we see that

$$\Lambda(\zeta) \sim \frac{1}{\sin(\frac{2}{3} \pi)} \left[ \sin\left(\pi\left(\alpha + \frac{2}{3}\right)\right) - \sin\left(\pi \alpha\right) \right] |\zeta|^{3\alpha}, \ zeta \to \infty.$$

Elementary trigonometric formulas show that

$$\Lambda(\zeta) \sim \frac{2 \sin\left(\frac{\pi}{3}\right)}{\sin(\frac{2}{3} \pi)} \cos\left(\pi\left(\alpha + \frac{1}{3}\right)\right) |\zeta|^{3\alpha}, \ zeta \to \infty.$$

Whence

$$\Lambda(\zeta) \sim K_+ |\zeta|^{3\alpha} \quad \text{as} \quad \zeta \to \infty$$

(3.36)

with $K_+ = 2 \cos(\pi (\alpha + \frac{1}{3}))$. Notice that $K_+ > 0$ if $0 < \alpha < \frac{1}{6}$.

It only remains to prove that $\Lambda(\zeta) > 0$ for any $\zeta \in \mathbb{R}$ and the considered range of values of $\alpha$. To this end, notice that if $\alpha \to 0$ we have $\Lambda(\zeta) \to 1 > 0$ uniformly in compact sets of $\zeta$. The functions $\Lambda(\zeta) \equiv \Lambda(\zeta, \alpha)$ considered as functions of $\alpha$, change in a continuous manner. On the other hand, the asymptotic behaviors (3.35), (3.36) imply that the functions $\Lambda(\zeta, \alpha)$ are positive for large values of $|\zeta|$. If $\Lambda(\cdot, \alpha)$ has a zero at some $\zeta = \zeta_0 \in \mathbb{R}$ and $0 < \alpha < \frac{1}{6}$, then there should exist, by continuity, $0 < \alpha_* < \frac{1}{6}$ and $\zeta_* \in \mathbb{R}$ such that $\Lambda(\zeta_*, \alpha_*) = \Lambda(\zeta_*, \alpha_*) = 0$. The uniqueness theorem for ODEs then implies that $\Lambda(\cdot, \alpha_*) = 0$, but this would contradict the asymptotics (3.35), (3.36), whence the result follows. \qed

We now let

$$F_0(x, v) := x^\alpha \Lambda\left(\frac{v}{(9x)^{\frac{2}{3}}}\right),$$

(3.37)

where $\Lambda$ is obtained in Claim 3.3.1. Then from (3.31) we deduce that $F_0$ is a positive steady solution to the Fokker–Planck equation and that when $x \to 0^+$, $F_0(x, v) \simeq x^\alpha + |v|^{3\alpha}$. By the same argument, one can find $F_1(x, v) > 0$, a steady solution to the Fokker–Planck equation such that when $x \to 1^-$, $F_1(x, v) \simeq (1 - x)^\alpha + |v|^{3\alpha}$.
These $F_0$ and $F_1$ will be used in the construction of a super-solution to (1.1), which now follows.

Let $\Psi = \Psi(y, \xi)$ be a self-similar type solution to (1.1) of the following form
\[
f(x, v, t) = \Psi \left( \frac{x}{t^{3/2}}, \frac{v}{t^{1/2}} \right).
\] (3.38)

If $\Psi$ is a solution to (1.1), it should satisfy the following PDE:
\[
-\frac{3}{2} y \Psi_y - \frac{1}{2} \xi \Psi_\xi + \xi \Psi_y = \Psi_\xi\xi.
\]

We will not attempt to solve this partial differential equation since we only need a super-solution of (1.1), but try to find a super-solution $Z$ to the self-similar equation above, namely satisfying
\[
Z_{\xi\xi} + \frac{1}{2} \xi Z_\xi + \left( \frac{3}{2} y - \xi \right) Z_y \leq 0.
\] (3.39)

We first show that one can construct $Z_0$ satisfying (3.39) in a small neighborhood of the singular set $(0, 0)$.

**Lemma 22.** There exists a sufficiently small $R_0(y, \xi)$ such that (i) $|R_0| \ll F_0$ for $|\xi|^3 + |y| \ll 1$ and that (ii) $Z_0(y, \xi) := F_0(y, \xi) + R_0(y, \xi)$, where $F_0$ is given by (3.37), satisfies (3.39) for $|\xi|^3 + |y| \ll 1$. Notice that $Z_0 > 0$.

**Proof.** Notice that $(F_0)_{\xi \xi} - \xi (F_0)_y = 0$, where $F_0(y, \xi) = y^\alpha Q(-\frac{\xi^3}{9y})$ with $Q$ obtained in Claim 3.3.1 (we use $Q$ instead of $\Lambda$ to distinguish the different argument) and hence
\[
(F_0)_{\xi \xi} + \frac{1}{2} \xi (F_0)_\xi + \left( \frac{3}{2} y - \xi \right) (F_0)_y = \frac{1}{2} \xi (F_0)_\xi + \frac{3}{2} y (F_0)_y.
\]

But then
\[
\frac{1}{2} \xi (F_0)_\xi + \frac{3}{2} y (F_0)_y = \frac{1}{2} \xi \left[ y^\alpha \left( -\frac{3\xi^2}{9y} \right) Q' \right] + \frac{3}{2} y \left[ y^\alpha \left( \frac{\xi^3}{9y^2} \right) Q' + \alpha y^{\alpha-1} Q \right] = \frac{3}{2} \alpha F_0, \text{ since the first two terms cancel each other out.}
\]

For $R_0$, we have the following inequality to be solved:
\[
(R_0)_{\xi \xi} + \frac{1}{2} \xi (R_0)_\xi + \left( \frac{3}{2} y - \xi \right) (R_0)_y \leq -\frac{3}{2} \alpha F_0.
\]

With the ansatz $R_0 = y^\beta \varphi(-\frac{\xi^3}{9y})$, the left-hand side reads
\[
(R_0)_{\xi \xi} + \frac{1}{2} \xi (R_0)_\xi + \left( \frac{3}{2} y - \xi \right) (R_0)_y
\]
\[
= -\xi y^{\beta-1} \left[ z \varphi_{zz} + \left( \frac{2}{3} - z \right) \varphi_z + \beta \varphi \right] + \frac{3}{2} \beta y^\beta \varphi
\]
and hence the above inequality for $R_0$ reduces to

$$-\xi y^{\beta-1} \left[ z\varphi_{zz} + \left( \frac{2}{3} - z \right) \varphi_z + \beta \varphi \right] + \frac{3}{2} \beta y^{\beta} \varphi \leq -\frac{3}{2} \alpha F_0.$$ 

Choose $\beta = \frac{2}{3} + \alpha$, then it suffices to find $\varphi(z)$ satisfying

$$z^{1/3} \left[ z\varphi_{zz} + \left( \frac{2}{3} - z \right) \varphi_z + \left( \frac{2}{3} + \alpha \right) \varphi \right] + \left( 1 + \frac{3}{2} \alpha \right) y^{2/3} \varphi \leq -\frac{3}{2} \alpha Q. \quad (3.40)$$

To do so, we will solve the following ODE

$$z\varphi_{zz} + \left( \frac{2}{3} - z \right) \varphi_z + \left( \frac{2}{3} + \alpha \right) \varphi = -\frac{\gamma}{z^{1/3}} Q \quad (3.41)$$

for some constant $\gamma > \frac{3}{2} \alpha$. Let $M(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, z)$ and $U(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, z)$ be two independent solutions to the homogeneous part. Then, by variation of constants, the solution of (3.41) is given by

$$\varphi(z) = -M \left( -\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, z \right) \int_{z}^{\infty} \frac{\gamma Q(\eta) U\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right)}{\eta^{4/3} W(\eta)} \, d\eta$$

$$-U \left( -\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, z \right) \int_{0}^{z} \frac{\gamma Q(\eta) M\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right)}{\eta^{4/3} W(\eta)} \, d\eta, \quad (3.42)$$

where

$$W(\eta) = \begin{vmatrix} M\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right) & U\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right) \\ M_{\eta}\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right) & U_{\eta}\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right) \end{vmatrix} = -\frac{\eta^{-\frac{2}{3}} e^{\eta}}{\Gamma\left(-\frac{2}{3} - \alpha \right)}.$$ 

Here we have used the fact that $W(\eta)$ satisfies the ODE:

$$\frac{dW(\eta)}{d\eta} + \left( \frac{2}{3z} - 1 \right) W(\eta) = 0$$

with the asymptotics:

$$W(\eta) \sim -\frac{\eta^{-\frac{2}{3}} e^{\eta}}{\Gamma\left(-\frac{2}{3} - \alpha \right)} \quad \text{as} \quad \eta \to 0,$$

which can be computed using the asymptotics of the functions $M\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right)$, $U\left(-\left( \frac{2}{3} + \alpha \right), \frac{2}{3}, \eta \right)$ as $\eta \to 0$ (cf. [1]). Then we have

$$|R_0| = \left| y^{\frac{2}{3} + \alpha} \varphi\left(-\frac{\xi^3}{9y} \right) \right| \ll F_0 = y^\alpha Q\left(-\frac{\xi^3}{9y} \right) \text{ for } |\xi|^3 + |y| \ll 1. \quad (3.43)$$

This can be checked by comparing the asymptotic behavior of $\varphi$ with that of $Q$. We first recall from (3.31)

$$Q(z) = O(|z|^\alpha), \quad |z| \to \infty.$$
Moreover, by (3.28) and 13.5.2 in [1], and from (3.42) we deduce that
\[ \varphi(z) = O(|z|^\frac{2}{3} + \alpha), \quad |z| \to \infty, \]
which implies (3.43). Finally, together with (3.43), we deduce that \( \varphi \) for \( \gamma > \frac{3}{2} \alpha \), which is a solution to (3.41), satisfies the inequality (3.40). This completes the proof of the lemma. \( \square \)

One can also construct a super-solution \( Z_1 \) of self-similar type near \((1, 0)\) in the same way. We omit the details.

### 3.4. Hölder Estimates for the Solution Near the Singular Set

In this section, we will prove that our solution \( f \) is continuous up to the singular set \( \{(0, 0), (1, 0)\} \), in fact Hölder continuous by means of maximum principles: we will apply comparison principles to the solution \( f \) with a suitable super-solution \( \bar{f} \) that controls the singular set. 

#### 3.4.1. The adjoint problem

We study in this subsection the adjoint problem \( M^* \varphi = 0 \) in \( UT \) backward in time with the corresponding absorbing boundary:

\[ \varphi_t + v \varphi_x + \varphi_{vv} = 0, \]
\[ \varphi(x, v, T) = \varphi_0(x, v) \geq 0, \]
\[ \varphi(0, v, t) = 0, \quad v < 0, \quad t > 0, \]
\[ \varphi(1, v, t) = 0, \quad v > 0, \quad t > 0. \]

Then we obtain the following results concerning the existence of solutions, the hypoellipticity away from the singular set \( \{(0, 0), (1, 0)\} \), and the non-negativity of solutions, similarly to the original Fokker–Planck problem. We will skip the proofs.

**Proposition 3.** Let data \( \varphi_0 \in L^\infty(\Omega) \cap L^1(\Omega) \) with \( \varphi_0 \geq 0 \) be given at \( t = T \). Then there exists a solution \( \varphi(x, v, t) \in C([0, T]; L^\infty(\Omega) \cap L^1(\Omega)) \) to (3.44)--(3.47). Moreover, for each \( t > 0, \varphi \in H^{k,m}_{loc}(\tilde{\Omega} \setminus \{(0, 0), (1, 0)\}) \), where \( H^{k,m} = H^{k,m}_{x,v} \).

**Proof.** It is analogous to Theorem 1.3 (i). \( \square \)

**Lemma 23.** Let \( \varphi(x, v, t) \in C^\infty(UT) \) be a solution of the adjoint equation (3.44)--(3.47) backward in time. Then it satisfies \( \varphi(x, v, t) \geq 0 \) for all \( (x, v, t) \in UT \).

**Proof.** It is analogous to Lemma 10. \( \square \)

We define a super-solution to the operator \( M \) in (2.27) as follows.

**Definition 4.** Let \( \psi \in C(\bar{UT} \setminus \{(0, 0), (1, 0) \times [0, T]\}) \). Then we say \( \psi \) is a super-solution of (3.44), that is, \( M\psi \geq 0 \) if for every \( t \in [0, T] \) and any test
function \( \varphi(x, v, s) \in C_{x,v,t}^{1,2,1}(U_t) \cap C(\overline{U_t}) \) with \( \varphi \geq 0 \) such that \( \text{supp}(\varphi(\cdot, \cdot, \cdot)) \subset [0, 1] \times [-R, R] \) for some \( R > 0 \) and \( \varphi |_{\gamma^+} = 0 \), it satisfies

\[
- \int_{U_t} \psi(x, v, s) \{ \varphi_t(x, v, s) + \partial_x (v \varphi(x, v, s)) + \varphi_{vv}(x, v, s) \} \, dx \, dv \, ds \\
+ \int_{\Omega} \psi(x, v, t) \varphi(x, v, t) \, dx \, dv - \int_{\Omega} \psi(x, v, 0) \varphi(x, v, 0) \, dx \, dv \\
+ \int_0^t \int_{-\infty}^0 v \psi(1, v, s) \varphi(1, v, s) \, dv \, ds - \int_0^t \int_0^\infty v \psi(0, v, s) \varphi(0, v, s) \, dv \, ds \\
\geq 0.
\]

**Remark 2.** In the Definition 4, we can extend the definition to a more general region. For instance, the interval could be smaller than \([0, 1]\), which will be used in Lemma 24.

**3.4.2. Maximum principle** Let

\[
\hat{f}_0(x, v, t) = \min \left\{ K Z_0 \left( \frac{x}{t^{3/2}}, \frac{v}{t^{1/2}} \right), 1 \right\},
\]

where \( K > 1 \) and \( Z_0 \) is obtained in Lemma 22. Then since \( Z_0 \) satisfies (3.39), \( \hat{f}_0 \) is a super-solution of the Fokker–Planck equation. We further define \( \tilde{f}_\epsilon \) by

\[
\tilde{f}_\epsilon = C \hat{f}_0 + \epsilon f_1^* + \epsilon f_2^*,
\]

where \( C = \| f_0 \|_\infty \), \( f_1^* \) is a singular solution near the singular set \((0, 0)\) and \( f_2^* \) is a singular solution near the singular set \((1, 0)\) constructed in Lemma 21.

The following maximum principle plays a key role.

**Lemma 24.** Let \( f(x, v, t) \) be a solution to (1.1)–(1.4). If \( f(x, v, 0) \leq C \hat{f}_0(x, v, 0) \), then \( f(x, v, t) \leq C \hat{f}_0(x, v, t) \) for all \( t > 0 \) and for all \( (x, v) \in \hat{Q} \setminus \{(0, 0), (1, 0)\} \). Here \( C = \| f_0 \|_\infty \) and \( \hat{f}_0 \) is given in (3.48).

**Proof.** We first introduce a cut-off function \( \zeta(x, v) \) near the singular set \( \{(0, 0), (1, 0)\} \) in the phase plane \( \hat{Q} = [0, 1] \times (-\infty, \infty) \) such that for any \( \rho > 0 \) small,

\[
\zeta_\rho(x, v) = \begin{cases} 
0, & (|v|^3 + |x|)^{1/3} < \rho, \\
\in [0, 1], & \rho \leq (|v|^3 + |x|)^{1/3} \leq 2\rho, \\
1, & \rho \leq (|v|^3 + |x - 1|)^{1/3} \leq 2\rho.
\end{cases}
\]

Let \( \psi(x, v, t) = \tilde{f}_\epsilon(x, v, t) - f(x, v, t) \) with \( \tilde{f}_\epsilon \) being the super-solution in Definition 4, given in (3.49). Let \( \varphi(x, v, t) \) be a solution of the following adjoint problem: for any given \( h > 0 \) small,

\[
\varphi_t + v \varphi_x + \varphi_{vv} = 0, \quad \text{for } h < x < 1 - h, \quad -\infty < v < \infty, \\
\varphi(x, v, T) = \varphi_0(x, v), \\
\varphi(h, v, t) = 0, \quad v < 0, \quad \varphi(1 - h, v, t) = 0, \quad v > 0,
\]

and let

\[
\varphi_t + \partial_x (v \varphi) + \varphi_{vv} = 0. \quad \text{for } 0 < x < 1 - h, \quad -\infty < v < \infty,
\]

\[
\varphi(x, v, T) = \varphi_0(x, v), \\
\varphi(h, v, t) = 0, \quad v < 0, \quad \varphi(1 - h, v, t) = 0, \quad v > 0.
\]
where \( \varphi_0 \in C(B_{\rho}(x_0, v_0)) \) is an arbitrary test function. Then we put \( \psi \) and \( \bar{\varphi} = \zeta_{\rho} \varphi \) with the cut-off function \( \zeta_{\rho} \) in (3.50) into the Definition 4 with a smaller domain \((h, 1-h) \times (-\infty, \infty) \times (0, t)\) to get

\[
I + II + III :=
\]

\[
- \int_0^T \int_0^1 \int_{-h}^{1-h} \int_0^\infty \psi(x, v, t) [\bar{\varphi}_t(x, v, t) + \partial_x (v \bar{\varphi}(x, v, t)) + \bar{\varphi}_{vv}(x, v, t)] \, dx \, dv \, dt
\]

\[
+ \int_{-\infty}^{\infty} \int_{h}^{1-h} \psi(x, v, t) \bar{\varphi}(x, v, t) \, dx \, dv - \int_{-\infty}^{\infty} \int_0^\infty \psi(x, v, 0) \bar{\varphi}(x, v, 0) \, dx \, dv
\]

\[
+ \int_0^T \int_{-\infty}^{\infty} v \psi(1-h, v, t) \bar{\varphi}(1-h, v, t) \, dv \, dt
\]

\[
- \int_0^t \int_{-\infty}^{\infty} v \psi(h, v, s) \bar{\varphi}(h, v, s) \, dv \, dt
\]

\[
\geq 0.
\]

Notice that the compact support restriction of test functions in the Definition 4 can be extended to \( \varphi \) without compact support by an approximation argument.

For \( I \), we use the estimates, similar to (3.23) for solutions of the Fokker–Planck problem, for the derivatives of solutions to the adjoint problem together with the estimates for the derivatives of the cut-off function \( \zeta_{\rho} \):

\[
\| \zeta_{\rho, x} \|_{L^\infty} \leq \frac{C}{\rho^3}, \quad \| \zeta_{\rho, v} \|_{L^\infty} \leq \frac{C}{\rho}, \quad \| \zeta_{\rho, vv} \|_{L^\infty} \leq \frac{C}{\rho^2}.
\]

Thus we get

\[
- \int_{UT} \psi(x, v, t) [\bar{\varphi}_t(x, v, t) + \partial_x (v \bar{\varphi}(x, v, t)) + \bar{\varphi}_{vv}(x, v, t)] \, dx \, dv \, dt
\]

\[
= - \int_{UT} \psi(x, v, t) \zeta(x, v)
\]

\[
[\varphi_t(x, v, t) + \partial_x (v \varphi(x, v, t)) + \varphi_{vv}(x, v, t)] \, dx \, dv \, dt
\]

\[
- \int_{UT} \psi(x, v, t) [v \varphi \zeta_{\rho, x} + 2 \zeta_{\rho, v} \varphi_v + \varphi \zeta_{\rho, vv}] \, dx \, dv \, dt
\]

\[
= - \int_{UT} \psi(x, v, t) [v \varphi \zeta_{\rho, x} + 2 \zeta_{\rho, v} \varphi_v + \varphi \zeta_{\rho, vv}] \, dx \, dv \, dt,
\]

which leads to

\[
|I| \leq C \rho^2,
\]

where \( C \) depends on \( T, \| \varphi_0 \|_{L^\infty(\Omega)} \), and \( h \).
Thus we have
\[
\int_{\Omega} \psi(x, v, T) \overline{\phi}(x, v, T) \, dx \, dv \\
\geq \int_{\Omega} \psi(x, v, T) \zeta(x, v) \phi_0(x, v) \, dx \, dv - I - III \\
\geq \int_{\Omega} \psi(x, v, 0) \zeta(x, v) \phi(x, v, 0) \, dx \, dv - III + O(\rho^2) \\
\geq -III + O(\rho^2).
\]

Letting \( \rho \to 0 \) and, we obtain
\[
\int_{\Omega} \psi(x, v, T) \overline{\phi}(x, v, T) \, dx \, dv \geq -III.
\]

Then we let \( h \to 0 \) to get
\[
III \leq 0
\]

since \( \psi(1, v, s) \geq 0 \) for \( v < 0, s > 0 \) and \( \psi(0, v, s) \geq 0 \) for \( v > 0, s > 0 \), and \( \varphi \geq 0 \). Noticing \( \varphi_0(x, v) \) is arbitrary, we can deduce that if \( \bar{f}_\varepsilon(x, v, 0) \geq C \hat{f}_0 \geq f(x, v, 0) \), then \( \bar{f}_\varepsilon(x, v, t) \geq f(x, v, t) \). Finally we let \( \varepsilon \to 0 \) to complete the proof. \( \square \)

We are now ready to prove Theorem 1.3 (ii).

Proof of Theorem 1.3 (ii). Recall (3.49). Since \( C = \| f_0 \|_\infty \), we see that \( f(x, v, 0) \leq \bar{f}_\varepsilon(x, v, 0) \) for all \( (x, v) \) and for any \( \varepsilon > 0 \). By the maximum principle as in the proof of Lemma 24, we deduce that
\[
f(x, v, t) \leq \bar{f}_\varepsilon(x, v, t) \text{ for } t > 0, \ (x, v) \in \hat{\Omega} \setminus [B_\delta(0, 0) \cup B_\delta(1, 0)],
\]
where \( B_\delta(x_0, v_0) := \{ (x, v) : |x - x_0| + |v - v_0|^3 < \delta^3 \} \) and \( \delta > 0 \) is arbitrary. Letting \( \varepsilon \to 0 \), we further deduce that
\[
f(x, v, t) \leq C \hat{f}_0(x, v, t) \text{ for all } (x, v) \in \hat{\Omega} \setminus \{(0, 0), (1, 0)\},
\]
which immediately implies that for \( |x| + |v|^3 \ll 1 \)
\[
f(x, v, t) \leq C (|x|^\alpha + |v|^{3\alpha}). \tag{3.51}
\]

We repeat the argument near \( (1, 0) \) by using \( \hat{f}_1 \), which is defined by using the super-solution \( Z_1 \) of self-similar type near \( (1, 0) \) to establish \( f(x, v, t) \leq C (|x - 1|^\alpha + |v|^{3\alpha}) \).

It now remains to show the Hölder continuity. We already know that \( f \) is smooth away from the singular set from the hypoellipticity result. Choose \( 1 \gg \delta_0 > 0 \) such that on
\[
B_{\delta_0} := \Omega \cap \{B_{\delta_0}(0, 0) \cup B_{\delta_0}(1, 0)\},
\]
$f$ satisfies (3.51). We will focus on the part near $(0, 0)$. Then it suffices to show that for each $t > 0$,
$$
|f(x_1, v_1, t) - f(x_2, v_2, t)| \leq C(|x_1 - x_2|^\alpha + |v_1 - v_2|^{3\alpha})
$$
for any $(x_i, v_i) \in B_{\delta_0}$, $i = 1, 2$.

If $(x_i, v_i)$ for either $i = 1$ or $i = 2$ is a singular point $(0, 0)$, we are done because of (3.51). Suppose not. We may assume that $(x_1, v_1) \in \partial B_{\delta_1} \cap B_{\delta_0}$ and $(x_2, v_2) \in \partial B_{\delta_2} \cap B_{\delta_0}$ for $0 < \delta_2 \leq \delta_1 \leq \delta_0$ and let $\rho^3 := |x_1 - x_2| + |v_1 - v_2|^3 > 0$. If
$$
\rho^3 \geq \frac{1}{100}(\delta_1^3 + \delta_2^3),
$$
then by the triangle inequality,
$$
|f(x_1, v_1, t) - f(x_2, v_2, t)| \leq |f(x_1, v_1, t) - f(0, 0, t)| + |f(0, 0, t) - f(x_2, v_2, t)| \leq C(\delta_1^{3\alpha} + \delta_2^{3\alpha}) \leq C\rho^{3\alpha}
$$
$$
\leq C(|x_1 - x_2|^\alpha + |v_1 - v_2|^{3\alpha}).
$$

If $\rho^3 < \frac{1}{100}(\delta_1^3 + \delta_2^3)$, then $\rho \ll \delta_1 \sim \delta_2$, for instance we have $\rho < \frac{1}{10}\delta_2$ and $\delta_2 \leq \delta_1 \leq 2\delta_2$. And hence, the distance between $B_{\rho^3}(x_2, v_2) \cap B_{\delta_0}$ and the singular point $(0, 0)$ is strictly positive and it contains $(x_1, v_1)$. Therefore, we can use the hypoellipticity result on $B_{\rho^3}(x_2, v_2) \cap B_{\delta_0}$ to conclude that it is in fact smooth. To make it precise, we introduce a rescaling as follows:
$$
f(x, v, t) = \delta^{3\alpha} \tilde{f}(\tilde{x}, \tilde{v}, \tilde{t}), \quad x = \rho^3 \tilde{x}, \quad v = \rho \tilde{v}, \quad t = t_0 + \rho^2 \tilde{t}, \quad t_0 \geq 0,
$$
where $\delta^3 := \max(\delta_1^3, \delta_2^3)$. Then $\tilde{f}(\tilde{x}, \tilde{v}, \tilde{t})$ solves the Fokker–Planck equation (1.1) and due to (3.51), it is bounded $0 \leq \tilde{f} \leq C$ in the set $|\tilde{x}| + |\tilde{v}|^3 \leq \delta^3 / \rho^3$. By applying Lemma 20, it then follows that
$$
(\delta^3 / \rho^3) |\partial_{\tilde{x}} \tilde{f}| + (\delta^3 / \rho^3)^{1/3} |\partial_{\tilde{v}} \tilde{f}| \leq C.
$$
Returning to the original variables $(x, v, t)$, we obtain the following estimates.
$$
(\delta^3 / \rho^3) |\partial_x f| + (\delta^3 / \rho^3)^{1/3} |\partial_v f| \leq C\delta^{3\alpha},
$$
for all $(x, v) \in \Omega$ with $|x| + |v|^3 \leq \delta^3$. We now estimate the difference $f(x_1, v_1, t) - f(x_2, v_2, t)$ as follows.
$$
|f(x_1, v_1, t) - f(x_2, v_2, t)| \leq |\partial_x f| (\tilde{x}, \tilde{v}, \tilde{t}) |x_1 - x_2| + |\partial_v f| (\tilde{x}, \tilde{v}, \tilde{t}) |v_1 - v_2|,
$$
for some $(\tilde{x}, \tilde{v})$ with $\min(\delta_1^3, \delta_2^3) \leq |\tilde{x}| + |\tilde{v}|^3 \leq \max(\delta_1^3, \delta_2^3) = \delta^3$. Thus we have
$$
|f(x_1, v_1, t) - f(x_2, v_2, t)| \leq C\delta^{3\alpha} \left[ \frac{|x_1 - x_2|}{\min(\delta_1^3, \delta_2^3)} + \frac{|v_1 - v_2|}{\min(\delta_1^3, \delta_2^3)} \right]
$$
$$
\leq C\delta^{3\alpha} \left[ \frac{\rho^3}{\min(\delta_1^3, \delta_2^3)} + \frac{\rho}{\min(\delta_1, \delta_2)} \right]
$$
$$
\leq C\delta^{3\alpha} \left[ \frac{\rho^{3\alpha} \delta^{3\alpha-3\alpha}}{\min(\delta_1^3, \delta_2^3)} + \frac{\rho^{3\alpha} \delta^{1-3\alpha}}{\min(\delta_1, \delta_2)} \right]
$$
$$
\leq C\rho^{3\alpha} \left[ \frac{\delta^3}{\min(\delta_1^3, \delta_2^3)} + \frac{\delta}{\min(\delta_1, \delta_2)} \right].
$$
Notice that the third inequality above is due to our assumption that $\rho^3 < \frac{1}{100}(\delta_1^3 + \delta_2^3)$ and $\alpha < 1/6$. Since $\delta_1 \sim \delta_2 \sim \delta$, we can conclude that

$$|f(x_1, v_1, t) - f(x_2, v_2, t)| \leq C\rho^{3\alpha} \leq C \left(|x_1 - x_2|^\alpha + |v_1 - v_2|^{3\alpha}\right).$$

This completes the proof of the Hölder continuity. $\square$

4. Exponential Convergence Rate on a Bounded Interval

In this section, we will show that the solutions for the Fokker–Planck equation (1.1) with the initial and absorbing boundary conditions (1.2)–(1.4) decay exponentially as time goes to infinity. We begin with the following technical lemma, which shows the exponential convergence under a certain set of assumptions to be verified later.

Lemma 25. Let $\{z_n\}, \{M_n\}$ be given such that $z_n \geq 0$ and $M_n \geq 0$ and that $M_n \leq M_{n-1}$ for all $n \in \mathbb{N}$. Suppose that there exist $0 < \theta, \beta < 1, A \geq C > 0, T > 0$ such that the following holds

1. If $z_n \geq AM_{n-1}$, $z_{n+1} \leq \theta z_n$ for $\theta < 1$,
2. If $z_n < AM_{n-1}$, $M_{n+1} \leq \beta M_{n-1}$ for $\beta < 1$,
3. $z_{n+1} \leq \theta \max\{z_n, CM_{n-1}\}$.

Then there exists $0 < \mu < 1$ satisfying $z_n + M_n \leq c\mu^n$.

Proof. Define a sequence

$$\omega_k := \max\left\{ \frac{z_{2k(T+1)+1}}{A}, M_{2k(T+1)} \right\}$$

for $k \geq 0$. We will show that there exists $0 < \gamma < 1$ independent of $k$ such that for each $k \geq 0$,

$$\omega_{k+1} \leq \gamma \omega_k. \quad (4.1)$$

We remark that (4.1) implies the exponential decay with the rate $\mu = O(\gamma^{1/2}(T+1))$.

What follows is the proof of (4.1). We first start with $M_{2k(T+1)+2(T+1)}$ part in $\omega_{k+1}$. Suppose there exists an $1 \leq n_0 \leq T + 1$ such that $z_{2k(T+1)+n_0} < AM_{2k(T+1)+n_0-1}$. Then, by the assumption (2), $M_{2k(T+1)+n_0+T} \leq \beta M_{2k(T+1)+n_0-1}$. Since $M_n$ is non-increasing in $n$, we deduce that

$$M_{2k(T+1)+2(T+1)} \leq M_{2k(T+1)+n_0+T} \leq \beta M_{2k(T+1)+n_0-1} \leq \beta M_{2k(T+1)}.$$ 

Suppose that $z_{2k(T+1)+n} \geq AM_{2k(T+1)+n-1}$ for all $1 \leq n \leq T + 1$. Then by the assumption (1), we deduce that $z_{2k(T+1)+n} + T \leq \theta^T z_{2k(T+1)+1}$. This immediately yields that

$$M_{2k(T+1)+2(T+1)} \leq M_{2k(T+1)+T} \leq \frac{z_{2k(T+1)+T+1}}{A} \leq \theta^T \frac{z_{2k(T+1)+1}}{A}$$

and thus we deduce that

$$M_{2k(T+1)+2(T+1)} \leq \max\left\{ \beta M_{2k(T+1)}, \theta^T \frac{z_{2k(T+1)+1}}{A} \right\}. \quad (4.2)$$
We now turn to $z_{2k(T+1)+2(T+1)+1}$ part in $\omega_{k+1}$. Using the assumption (3), we observe that

$$
\frac{z_{2k(T+1)+2(T+1)+1}}{A} \leq \max \left\{ \frac{C\theta}{A} M_{2k(T+1)+2(T+1)-1}, \frac{\theta z_{2k(T+1)+2(T+1)+1}}{A} \right\} 
$$

$$
\leq \max \left\{ \frac{C\theta}{A} M_{2k(T+1)+2(T+1)-1}, \frac{\theta z_{2k(T+1)+2(T+1)+1}}{A} \right\}.
$$

Since $M_n$ is non-increasing in $n$ and $C \leq A$, we deduce that

$$
\frac{z_{2k(T+1)+2(T+1)+1}}{A} \leq \max \left\{ \theta M_{2k(T+1)+2(T+1)+1}, \frac{\theta z_{2k(T+1)+2(T+1)+1}}{A} \right\}.
$$

Combining (4.2) and (4.3), and letting $\beta = \max \{b, \theta \} < 1$, we obtain (4.1).

We denote a neighborhood of the singular set by $S$:

$$
S := \{(x, v) \in \Omega : |x| + |v|^3 \leq \rho^3 \text{ or } |x - 1| + |v|^3 \leq \rho^3\}
$$

for $\rho > 0$ a fixed small (not necessarily too small) number. Let $Q$ be the complement of $S$:

$$
Q := \Omega \setminus S
$$

and we further introduce the extended $Q$ by

$$
Q_E := \Omega \setminus \frac{1}{2} S
$$

so that $Q \subset Q_E \subset \Omega$.

We use $\zeta_s(t)$ to denote the supremum of $f$ on $S$:

$$
\zeta_s(t) := \|f(\cdot, t)\|_{L^\infty(S)}
$$

and as before we will use $\|f(\cdot, t)\|_\infty$ to denote the supremum of $f$ on the entire phase space $\Omega$. We use $M(t)$ to denote the total mass at time $t$:

$$
M(t) := \int_{\Omega} f(x, v, t) \, dx \, dv.
$$

We know that $M(t)$ is non-increasing in $t$. Our goal is to prove that $M(t)$ decays exponentially to zero by showing that part of the mass escapes to the boundary and that the solution decays also on the singular set.

The following lemma concerns the behavior of a solution on $S$.

**Lemma 26.** There exist $\rho > 0$ and $0 < \theta < 1$ such that

$$
\zeta_s(t) \leq \theta \|f(\cdot, \tilde{t})\|_\infty \text{ for } t \geq \tilde{t} + 1,
$$

where $\theta < 1$. 

Proof. Recall that \( f(x, v, t) \leq C \hat{f}_0(x, v, t) \), where \( \hat{f}_0 \) is given in (3.48). Then by the self-similar structure of the super-solution \( \hat{f}_0 \), we see that there exists a small \( \rho > 0 \) so that for \( t \geq \bar{t} + 1 \),
\[
\sup_S f(x, v, t) \leq \theta \| f(\cdot, \bar{t}) \|_{\infty}.
\]
Since a similar argument holds near \((1, 0)\), this completes the proof. \(\square\)

Next we obtain the following estimate on \( \sup_Q f \) from the hypoellipticity of \( f \).

**Lemma 27.** There exists \( C_s > 0 \) such that
\[
\sup_{(x, v) \in Q} f(x, v, t) \leq C_s \int_{Q_E} f(x, v, \bar{t}) \, dx \, dv \text{ for } t \geq \bar{t} + 1,
\]
where \( C_s \) depends only on the size of the singular set \( S \).

**Proof.** It follows from Theorem 1.2 and the bound by \( L^1 \) norm was given in the proof of Theorem 1.2, as mentioned in Remark 1. \(\square\)

As a direct consequence of the two lemmas above, we derive the following property of \( \zeta_s(t) \).

**Lemma 28 (Verification of assumptions (1) and (3)).** For any \( t \geq 1 \),
\[
\zeta_s(t + 1) \leq \theta \max\{\zeta_s(t), C_s M(t - 1)\},
\]
where \( \theta > 0 \) is given in (4.4) and \( C_s \) is given in (4.5). In particular, if \( \zeta_s(t) > C_s M(t - 1) \), \( \zeta_s(t + 1) \leq \theta \zeta_s(t) \).

**Proof.** It follows from Lemma 26 and Lemma 27. \(\square\)

Lemma 28 asserts that if the amplitude of a solution on the singular set is much greater than the total mass at an earlier time, the amplitude at a later time should decrease.

In the next lemma, we show that mass does not move far away over time.

**Lemma 29.** (Tightness lemma) Let \( f \) be a strong solution of (1.1)-(1.4) with the initial data \( f_0 \in L^1 \cap L^\infty(\Omega) \) with \( f_0 \geq 0 \). For a given \( t > 0 \) and \( \delta > 0 \), there exists a constant \( B > 0 \) depending on \( t, \delta, \) and \( \int_{\Omega} f(x, v, t) \, dx \, dv \) such that
\[
\int_{|v| \leq B} f(x, v, t) \, dx \, dv \geq (1 - \delta) \int_{\Omega} f(x, v, t) \, dx \, dv.
\]

**Proof.** We may assume that \( \int_{\Omega} f_0 \, dx \, dv = 1 \). Let mass at time \( t \) be \( M: \int_{\Omega} f(x, v, t) \, dx \, dv = M \). We first split the initial data into two parts: \( f_0 = f_{0,1} + f_{0,2} \) with \( f_{0,1} \geq 0, f_{0,2} \geq 0 \), where \( \text{supp}(f_{0,1}) \subset [0, 1] \times [-B, B] \) and \( \int_{\Omega} f_{0,1} \, dx \, dv \geq 1 - \frac{\delta M}{2} \). Then there exist strong solutions \( f_1 \) and \( f_2 \) corresponding to the initial
data $f_{0,1}$ and $f_{0,2}$ respectively and $f = f_1 + f_2$. Let $\tilde{f}(x,v,t) = e^{\theta t} e^{-A\sqrt{v^2+1}}$, where $\theta = \theta(A)$. Then by a direct calculation, it is easy to see that

$$|\tilde{f}_{vv}(x,v,t)| \leq C(A) e^{\theta t} e^{-A\sqrt{v^2+1}}.$$

Thus it is now easy to see that $\tilde{f}(x,v,t)$ is a super-solution of (1.1) provided we choose $\theta(A) > 0$ sufficiently large. Since $f_{0,1}$ has a compact support, there exists $K > 0$ such that $f_{0,1}(x,v) \leq K \tilde{f}(x,v,0)$. Then using the comparison property for strong solutions in Lemma 16 and applying it to $g = f_1 - K\tilde{f}$, we get

$$f_1(x,v,t) \leq K \tilde{f}(x,v,t).$$

This implies that

$$\int_{|v| \geq B} f_1(x,v,t) \, dx \, dv \leq \int_{|v| \geq B} K \tilde{f}(x,v,t) \, dx \, dv$$

$$= Ke^{\theta t} \int_{|v| \geq B} e^{-A\sqrt{v^2+1}} \, dx \, dv \leq \frac{\delta M}{2}$$

if we choose $B > 0$ sufficiently large. Indeed, we can choose $B = C(1+\ln \frac{1}{\delta M} + t)$, where $C$ depends on $A$ and $K$. Thus we derive

$$\int_{|v| \geq B} f_1(x,v,t) \, dx \, dv \leq \frac{\delta M}{2}.$$

For $f_2$, we have

$$\int_{\Omega} f_2(x,v,t) \, dx \, dv \leq \int_{\Omega} f_{0,2}(x,v) \, dx \, dv \leq \frac{\delta M}{2}.$$

Therefore, we obtain

$$\int_{|v| \geq B} f(x,v,t) \, dx \, dv = \int_{|v| \geq B} f_1(x,v,t) \, dx \, dv$$

$$+ \int_{|v| \geq B} f_2(x,v,t) \, dx \, dv \leq \delta M.$$

This completes the proof. $\square$

In particular, we have the following.

**Corollary 3.** For any $t > 0$, there exists a $\tilde{B} > 0$ such that

$$\int_{|v| \leq \tilde{B}} f(x,v,t) \, dx \, dv \geq \frac{1}{2} \int_{\Omega} f(x,v,t) \, dx \, dv.$$

**Proof.** It follows immediately from Lemma 29. $\square$

Next we show that if $f$ is comparable to the mass in a small ball away from the singular set at the present time, then the amount of mass comparable to the mass at the present time escapes to the boundary at some later times.
Lemma 30 (Escape of mass to the boundary). Let \( f_0(x, v) \geq \varepsilon M \chi_{B_\rho(x_0, v_0)}(x, v) \) be given where \( B_\rho(x_0, v_0) \) is an interior ball with center \((x_0, v_0)\) and radius \( \rho > 0 \) and \( M = \int f_0(x, v) \, dx \, dv \). Let \( f(x, v, t) \) be a solution to the Fokker–Planck equation (1.1) in the unit interval \([0, 1]\) with absorbing boundary conditions. Then there exists \( \alpha = \alpha(\rho, \varepsilon) < 1 \), independent of \( x_0 \) and \( v_0 \) such that

\[
\int f(x, v, 1) \, dx \, dv \leq \alpha \int f_0(x, v) \, dx \, dv.
\]

Proof. Without loss of generality, we may assume \( v_0 > 0 \) since the other case can be treated similarly. First we show that \( f_0(x, v) \geq \varepsilon M \chi_{B_\rho(x_1, v_1)}(x, v) \) for some \((x_1, v_1) \in \Omega\) with \( v_1 \geq 1 \). We may assume that \( 0 < v_0 \leq 1 \) since we are done otherwise. We can also assume \( \rho \leq 1 \). To prove the statement above, let \( H(v, t) := \int_0^1 f(x, v, t) \, dx \). Then \( H(v, t) \) satisfies the following equation:

\[
H_t - H_{vv} = -vf(1, v, t) \chi_{[v>0]} + vf(0, v, t) \chi_{[v<0]} =: -g(v, t) \leq 0,
\]

where \( \chi \) is the characteristic function.

If \( g(v, t) > \gamma M \) for some \((v, t) \in (-\infty, \infty) \times [0, 1/2]\), where \( \gamma > 0 \) small and depending only on \( \varepsilon \) to be determined, then there exists a ball in which \( g(v, t) > \frac{\gamma M}{2} \) and this implies that \( \int_{1/2}^{\infty} \int_{-\infty}^{\infty} g(v, t) \, dv \, dt \geq \gamma_1 M \), where \( \gamma_1 > 0 \) depends on \( \gamma \). We then have \( M(1) = \int f(x, v, 1) \, dx \, dv \leq M(1/2) \leq (1 - \gamma_1) M \) and we are done.

If we now assume that \( g(v, t) \, dv \, dt \leq \gamma M \) for all \((v, t) \in (-\infty, \infty) \times [0, 1/2]\). Then we have the following integral representation for \( H(v, t) \):

\[
H(v, t) = \int_{-\infty}^{\infty} K(v - w, t) H(w, 0) \, dw
- \int_0^t \int_{-\infty}^{\infty} K(v - w, t - s) g(w, s) \, dw \, ds,
\]

where \( K(v, t) \) is the one dimensional heat kernel. We use the assumptions on \( f_0 \) and on \( g(v, t) \) to get, for \( 1 \leq v \leq 2 \),

\[
H(v, 1/2) \geq \int_{B_\rho(x_0, v_0)} K(v - w, 1/2) f(x, w, 0) \, dw \, dx - \frac{\gamma M}{2}
\geq \varepsilon M K(3, 1/2) - \frac{\gamma M}{2} = \left[ \varepsilon K(3, 1/2) - \frac{\gamma}{2} \right] M
\geq \frac{K(3, 1/2)}{2} \varepsilon M =: \varepsilon_0 M,
\]

if we choose \( \gamma = \varepsilon K(3, 1/2) \). Since \( \int_1^2 \int_0^1 f(x, v, 1/2) \, dx \, dv \geq \varepsilon_0 M \) and \( f \) is continuous on \([0, 1] \times [1, 2]\) from the hypoellipticity, there exists \( \rho_1 > 0 \) such that \( f(x, v, 1/2) \geq \frac{\rho_1}{M} \chi_{B_{\rho_1}(x_1, v_1)}(x, v) \) for all \( v_1 \geq 1 \) and \((x_1, v_1) \in [0, 1] \times [1, 2]\).

Now with abuse of notation, we use \( \rho, x_0, v_0 \) and \( t = 0 \) with \( f_0 \) being continuous instead of \( \rho_1, x_1, v_1 \) and \( t = 1/2 \). We look for a sub-solution \( F(x, v, t) \in C^{1,2,1}_{x,v,t}(\Omega_T) \) to the Fokker–Planck equation (1.1) of the form

\[
F(x, v, t) = e^{-\lambda t} h(x - vt, v),
\]
where \( \lambda > 0 \) will be chosen later and \( h \in C^{1,2}_{x,v}(\Omega) \). We plug in (4.6) into \( M f \leq 0 \) to get

\[
h_{vv} + t^2h_{xx} - 2th_{xt} \geq -\lambda h. \quad (4.7)
\]

Since \( \varepsilon M \chi_{B_{\rho}(x_0,v_0)}(x,v) \leq f_0(x,v) \) for all \((x,v) \in \Omega\), then there exists \( \delta > 0 \) such that \( f_0 > \varepsilon M \) for \((x,v) \in B_{\rho+\delta}(x_0,v_0)\) since \( f_0 \) is continuous. We then can find \( \varepsilon M \chi_{B_{\rho}(x_0,v_0)}(x,v) \leq h(x,v) \leq f_0(x,v) \) for all \((x,v) \in \Omega\). For instance, \( h = \varepsilon M \cos(\lambda^{1/4}(v - v_0)) \cos(\lambda^{1/4}(x - x_0)) + \varepsilon M \) on \( B_{\rho}(x_0,v_0) \), where \( \lambda = (\frac{2\pi}{\rho})^4 > 0 \) for \( \lambda \) sufficiently large so as to satisfy (4.7) and for \( 0 \leq t \leq 1 \) (This is possible since we can make \( \rho > 0 \) as small as possible). Then we take values of \( h \) in such a way that \( h_{vv} \geq 0 \). This can be done by first taking values of \( h \) as 0 for \( \mathbb{R}^2 \setminus B_{\rho+\delta}(x_0,v_0) \) and then by connecting the points on \( \partial B_{\rho}(x_0,v_0) \) and the points on \( \partial B_{\rho+\delta}(x_0,v_0) \) with the points \((v,h(v)) = (v_0 \pm \rho_0, \varepsilon M)\) in a convex way. Note that \( \lambda \) depends only on \( \rho \). Then by the maximum principle of \( M \) (Lemma 16), we have \( F(x,v,t) \leq f(x,v,t) \) for all \( t > 0 \). In particular, at \( x = 1 \), there exists \( t_0 > 0 \) such that \( x_0 + v_0t_0 = 1 \) (which implies \( t_0 \leq 1 \)) and

\[
f(1,v,t_0) \geq F(1,v,t_0) = e^{-\lambda t_0}h(1 - v_0, v) \geq \frac{e^{-\lambda t_0}M}{2} \geq \frac{e^{-\lambda \varepsilon}}{2} M =: \varepsilon_1 M,
\]

for \( |v - v_0| < \rho_1 \) with some \( \rho_1 = \frac{\rho}{\sqrt{1 + t_0}} \geq \frac{\rho}{\sqrt{2}} > 0, \varepsilon_1 = \frac{e^{-\lambda \varepsilon}}{2} > 0 \). Note that \( \varepsilon_1 \) does not depend on \((x_0,v_0)\) but on \( \rho \) and \( \varepsilon \). By the continuity of \( f \),

\[
f(1,v,t) \geq \frac{\varepsilon_1 M}{2} =: \varepsilon_2 M,
\]

for \( |v - v_0| < \frac{\rho_1}{2} \) and \( t \in [t_0 - \delta_0, t_0] \). Then integrating (1.1) in \( x, v, t \) yields

\[
\int f(x,v,1) \, dx \, dv \leq \int f_0(x,v) \, dx \, dv - \int_{t_0}^{t_0 + \rho/2\sqrt{2}} \int_{v_0 - \rho/2\sqrt{2}}^{v_0 + \rho/2\sqrt{2}} v f(1,v,t) \, dv \, dt \leq \left(1 - \frac{\rho \varepsilon_2}{\sqrt{2}}\right) M =: \alpha M,
\]

where \( \alpha < 1 \) depends only on \( \rho \) and \( \varepsilon \). We used \( v_0 \geq 1 \) and \( \rho \leq 1 \). This completes the proof. \( \square \)

We will now show that if \( \zeta(t) \) is bounded by a multiple of \( M(t - 1) \), then the total mass after a finite time should be decreasing by a uniform factor which is strictly less than one.

**Lemma 31.** (Verification of assumption (2)) *Given \( S, A \) (\( A \) arbitrarily large), there exist \( 0 < \beta = \beta(A, S) < 1 \) and \( T = T(A, S) \) such that if \( \zeta_s(n) < AM(n - 1) \) for some \( n \geq 1 \), then \( M(n + T) \leq \beta M(n - 1) \).*
Proof. We define $\varepsilon_0$ by means of $4C_s |S| \varepsilon_0 < 1$ and $4\varepsilon_0 < 1$. Let $T > 0$ be a given positive integer to be determined. If $M(n + T) \leq \frac{1}{2} M(n - 1)$, we are done. Suppose then that $M(n + T) > \frac{1}{2} M(n - 1)$. Since $M(n)$ is decreasing, we first see that

$$M(l) \geq M(n + T) > \frac{M(n - 1)}{2} \quad \text{for} \quad l = n, n + 1, \ldots, n + T. \quad (4.8)$$

We have two cases.

1. There exists an $l_0 \in \{n, n + 1, \ldots, n + T\}$ such that

$$\int_{Q_E} f \, dx \, dv \bigg|_{t = l_0} \geq \varepsilon_0 M(n - 1).$$

2. For all $l \in \{n, n + 1, \ldots, n + T\}$,

$$\int_{Q_E} f \, dx \, dv \bigg|_{t = l} < \varepsilon_0 M(n - 1).$$

In the case of (1), we apply Lemma 30 to prove that at least some part of the mass occupied in $Q_E$ at time $l_0$ should escape to the boundary at a later time $l_0 + 1$, which would in turn imply $M(l_0 + 1) \leq \beta_0 M(n - 1)$ for some $\beta_0 < 1$.

We now turn to the case (2). We will show that this case is impossible if $T$ is chosen appropriately. To show a contradiction, we exploit the property of $\zeta_s$. The first claim is the following

$$\zeta_s(n + 1) \leq A \theta M(n - 1). \quad (4.9)$$

To see it, notice that

$$\|f(\cdot, n)\|_\infty \leq \max\{\zeta_s(n), \|f(\cdot, n)\|_{L^\infty(Q)}\} \leq \max\{A M(n - 1), C_s M(n - 1)\} \text{ by the assumption on } \zeta_s(n) \text{ and (4.5)}$$

$$\leq A M(n - 1) \text{ by choosing } A > C_s.$$

Now by (4.4) we can easily deduce the above assertion (4.9).

For $l \geq n$, we observe that

$$M(l) = \int_S f \, dx \, dv + \int_Q f \, dx \, dv \leq \int_S f \, dx \, dv \bigg|_{t = l} + \int_{Q_E} f \, dx \, dv \bigg|_{t = l}.$$

Since $\int_{Q_E} f \, dx \, dv \bigg|_{t = l} < \varepsilon_0 M(n - 1)$ by the given assumption of the case (2) and since $M(n - 1) \leq 2M(l)$ by the assumption (4.8), we see that

$$M(l) \leq \int_S f \, dx \, dv \bigg|_{t = l} + 2 \varepsilon_0 M(l)$$

and hence for $2\varepsilon_0 < 1/2$, we deduce that for $l \geq n$,

$$M(l) \leq 2 \int_S f \, dx \, dv \bigg|_{t = l} \leq 2 |S| \zeta_s(l). \quad (4.10)$$
As a consequence of (4.10), we derive that for \( l \geq n + 1 \),
\[
\sup_{Q} f(x, v, l) \leq C_s \int_{Q_E} f \, dx \, dv \big|_{l=l-1} \quad \text{by (4.5)}
\]
\[
\leq C_s \varepsilon_0 M(n - 1) \quad \text{by the assumption in the case of (2)}
\]
\[
\leq 2C_s \varepsilon_0 M(l) \quad \text{by (4.8)}
\]
\[
\leq 4\varepsilon_0 C_s |S| \zeta_s(l) \quad \text{by (4.10)}.
\]
Then together with (4.4), we obtain for \( l \geq n + 1 \),
\[
\zeta_s(l + 1) \leq \theta \max \left\{ \zeta_s(l), \sup_{Q} f(x, v, l) \right\} \leq \theta \max\{\zeta_s(l), 4\varepsilon_0 C_s |S| \zeta_s(l)\}
\]
and since \( 4\varepsilon_0 C_s |S| < 1 \), we conclude that
\[
\zeta_s(l + 1) \leq \theta \zeta_s(l), \quad \text{for } l \geq n + 1. \tag{4.11}
\]
Hence, by iteration together with (4.9) we deduce that
\[
\zeta_s(n + T) \leq A \theta^T M(n - 1),
\]
which yields that from (4.10), for \( l = n + T \),
\[
M(n + T) \leq 2|S| A \theta^T M(n - 1).
\]
On the other hand, from (4.8), we have that
\[
\frac{M(n - 1)}{2} < M(n + T).
\]
But this is impossible for \( T \) sufficiently large since \( \theta < 1 \), which is the desired contradiction. This finishes the proof of the lemma. \( \square \)

We are now ready to prove Theorem 1.3 of the exponential decay in \( L^1 \) and \( L^\infty \) sense in time of solutions for (1.1)–(1.4).

**Proof of Theorem 1.4.** Part (i) is a consequence of Lemma 25, 28, and 31. Part (ii) follows immediately from Part (i) and the hypoellipticity. \( \square \)

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