Perverse schobers and 3d mirror symmetry

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Abstract

The proposed physical duality known as 3d mirror symmetry relates the geometries of dual pairs of holomorphic symplectic stacks. It has served in recent years as a guiding principle for developments in representation theory. However, due to the lack of definitions, thus far only small pieces of the subject have been mathematically accessible. In this paper, we formulate abelian 3d mirror symmetry as an equivalence between a pair of 2-categories constructed from the algebraic and symplectic geometry, respectively, of Gale dual toric cotangent stacks.

In the simplest case, our theorem provides a spectral description of the 2-category of spherical functors – i.e., perverse schobers on the affine line with singularities at the origin. We expect that our results can be extended from toric cotangent stacks to hypertoric varieties, which would provide a categorification of previous results on Koszul duality for hypertoric categories. Our methods also suggest approaches to 2-categorical 3d mirror symmetry for more general classes of spaces of interest in geometric representation theory.

Along the way, we establish two results that may be of independent interest: (1) a version of the theory of Smith ideals in the setting of stable ∞-categories; and (2) an ambidexterity result for co/limits of presentable enriched ∞-categories over ∞-groupoids.

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Introduction

Mirror symmetry, which began life as a duality of 2-dimensional quantum field theories, entered mathematics in [CDGP91] as a predicted equality between Gromov–Witten invariants and period integrals associated to a dual pair of Calabi–Yau manifolds. In [Kon95], this equality of numbers was upgraded to a proposed equivalence of categories. These categories model the boundary conditions in a pair of topological field theories – the A-model and B-model – determined by the data of a symplectic or algebraic manifold. Kontsevich’s proposal prefigured the result [BD95, Cos07, Lur09b] that an \( n \)-dimensional topological field theory is determined by its \( (n-1) \)-category of boundary conditions.

The program of 3d mirror symmetry [IS96] (also known as symplectic duality [BLPW16]), which entails a duality between 3-dimensional quantum field theories, has so far been understood mathematically as expressing a relation between invariants associated to dual pairs of symplectic resolutions.\(^1\) As in (2d) mirror symmetry, these invariants probe topological field theories – the 3d A- and B-models.\(^2\) Therefore the deepest formulation of 3d mirror symmetry is as a conjectural equivalence of 2-categories.

Unfortunately, the mathematical existence of these 2-categories is also conjectural. The B-type 2-category, first studied in [KRS09, KR10], is expected to admit a formulation in the emerging mathematical language of coherent sheaves of categories [Gai15, Ari], but the A-type 2-category, an expected categorification of the Fukaya category, is much more mysterious. The inspirational ICM address [Tel14] first formulated the A-type 2-category associated to pure gauge theory and described the equivalence of 2-categories predicted by 3d mirror symmetry.

\(^1\)The literature on this subject is vast; some representative research directions are described in [BFN18b, BDGH16, Oko18, Kam], and deeper references to the field may be found there.

\(^2\)The 3d B-model is also known as Rozansky–Witten theory [RW97]. The 3d A-model studied in this paper is essentially 3d generalized Seiberg-Witten theory. In general, this latter theory is different than that considered in [KV, KSV].
In this paper, we take the first steps beyond pure gauge theory, by proposing 2-categories associated to abelian gauge theories with matter. In mathematical terms, we associate 2-categories to toric cotangent stacks – quotients of the form $T^*(\mathbb{C}^n/G)$ for $G \subset (\mathbb{C}^\times)^n$ a torus – and prove equivalences between the 2-categories associated to dual pairs of stacks. These equivalences, which can be understood in purely mathematical terms, reveal heretofore hidden structure inside the world of stable higher category theory.

In §0.1, we provide a purely mathematical account of our results, focusing on the basic duality between $T^*\mathbb{C}$ and $T^*(\mathbb{C}/\mathbb{C}^\times)$. In this case, we understand 3d mirror symmetry as an equivalence between a 2-category of perverse schobers on $\mathbb{C}$ and a 2-category of coherent sheaves of categories on $\mathbb{C}/\mathbb{C}^\times$. In §0.2, which may be skipped by a reader interested only in the mathematical results of this paper, we situate our results in the context of abelian gauge theories with matter and give a preview of our future applications to hypertoric categories.

0.1 | Mathematical overview

The notion of spherical functor, introduced in [Ann, AL17], has proven remarkably useful as an organizing principle in contexts throughout algebraic and symplectic geometry.

**Definition 0.1.** An adjunction $S : \mathcal{C}_\Phi \rightleftarrows \mathcal{C}_\Psi : S^R$ between stable categories is a spherical adjunction if it satisfies the requirement that the endofunctors $T_\Phi := \text{fib}(\text{id}_{\mathcal{C}_\Phi} \xrightarrow{\eta} S^R S)$ and $T_\Psi := \text{cofib}(S S^R \xleftarrow{\varepsilon} \text{id}_{\mathcal{C}_\Psi})$ are invertible. The left adjoint $S$ is the underlying spherical functor of the spherical adjunction.

Examples include the pushforward of coherent sheaves on a divisor in algebraic geometry, or the “Orlov/cup” functor in symplectic geometry; examples coming from representation theory are also discussed in [KS, §4] and [KS22, §8]. It was recognized in [KS] that spherical functors categorify the quiver presentation of perverse sheaves on $\mathbb{C}$ with stratification $S_{\text{toric}}$ given by $\mathbb{C} = 0 \sqcup \mathbb{C}^\times$.

**Theorem 0.2 ([Ver85]).** The category $\text{Perv}(\mathbb{C}, S_{\text{toric}})$ is equivalent to the category of diagrams of vector spaces $u : V_\Phi \simeq V_\Psi : v$ satisfying the requirement that the endomorphisms $T_\Phi := 1_\Phi - vu$ and $T_\Psi := 1_\Psi - uv$ are invertible.

For this reason, spherical functors are also referred to as perverse schobers on $\mathbb{C}$ with stratification $S_{\text{toric}}$ [BKS18].

The collection of all spherical functors (not fixing the source or target category) may be usefully organized into a 2-category $\text{Sph}$, a full sub-2-category of the 2-category $\text{Fun(Adj, St)}$ of stable adjunctions. As we shall see, this 2-category is remarkably rich: the simple step of demanding

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3Throughout this paper, we take the “implicit $\infty$” convention (see §0.3).

4We prefer the term “spherical adjunction” over the more common term “spherical functor”, since it is more symmetric and in particular adheres more closely to the intuition coming from symplectic topology. On the other hand, the terminology “spherical functor” will also be useful for us.
that the twists be invertible imposes a great deal of structure, including many symmetries not present for arbitrary stable adjunctions.

The mathematical content of our paper is a spectral description of the 2-category of spherical functors. To give a concrete explanation of what we mean by a “spectral” description, we consider the main precursor to our work, the equivalences of 2-categories discussed in [Tel14].

**Definition 0.3.** Let $X$ be a stack which is 1-affine in the sense of [Gai15]. We may associate to $X$ the 2-categories $\operatorname{QCoh}^2(X) := \operatorname{Mod}_{\operatorname{Perf}(X)}(\operatorname{St})$ of quasicoherent sheaves of categories and $\operatorname{Loc}^2(X) := \operatorname{Fun}(X_B, \operatorname{St})$ of local systems of categories, where we write $X_B$ for the homotopy type ("Betti stack") underlying $X$.

**Example 0.4.** Fix a torus $T := (\mathbb{C}^\times)^n$, and let $T^\vee$ be its Langlands dual. The (categorified) Fourier transform [Tel14, Remark 4.6] is the equivalence of 2-categories

$$\operatorname{QCoh}^2(BT) \simeq \operatorname{Loc}^2(T^\vee),$$

which follows from the observation that $\operatorname{QCoh}^2(BT)$ and $\operatorname{Loc}^2(T^\vee)$ are both equivalent to the 2-category of categories equipped with $n$ commuting automorphisms.

There is also a dual Fourier transform [Tel14, Proposition 4.1]

$$\operatorname{Loc}^2(BT) \simeq \operatorname{QCoh}^2(T^\vee),$$

where now the classifying stack appears on the automorphic side instead of the spectral side. To see this, note that $\operatorname{Loc}^2(BT)$ is equivalent to the 2-category of categories $\mathcal{C}$ equipped with a topological action of the torus $T$. Such an action is equivalent to the data of an $E_2$ map $\mathcal{C}[T^\vee] \cong C_*(\Omega T) \to HH^*(\mathcal{C})$, as explained in [Tel14, Theorem 2.5].

Example 0.4 expressed a duality between the spaces $BT$ and $T^\vee$. Similarly, our results will relate the space $\mathbb{C}$, equipped with stratification $0 \sqcup \mathbb{C}^\times$, to a dual geometry given by the space $\mathbb{C}/\mathbb{C}^\times$ equipped with stratification $0/\mathbb{C}^\times \sqcup \mathbb{C}^\times/\mathbb{C}^\times$. (In §0.2, we will give context for the geometric duality relating both these stratified spaces and the spaces described in Example 0.4.)

Consider the functor

$$\operatorname{Coh}(\mathbb{C}/\mathbb{C}^\times) \xrightarrow{i^*} \operatorname{Coh}(B\mathbb{C}^\times)$$

of pullback along the inclusion $i : 0/\mathbb{C}^\times \hookrightarrow \mathbb{C}/\mathbb{C}^\times$. This functor is spherical, and we prove that it is the free spherical functor on a categorified vanishing cycle, in the following sense.

**Theorem A** (Theorem 2.6). The functor $\Phi : \operatorname{Sph} \to \operatorname{St}$ taking a spherical adjunction $\mathcal{C}_\Phi \rightleftharpoons \mathcal{C}_\Psi$ to the category $\mathcal{C}_\Phi$ is corepresented by the spherical functor $\operatorname{Coh}(\mathbb{C}/\mathbb{C}^\times) \xrightarrow{i^*} \operatorname{Coh}(B\mathbb{C}^\times)$.

As a spherical functor, $i^*$ admits both adjoints $i^! \dashv i^* \dashv i_*$, which are themselves spherical functors. Moreover, these adjoints $i^!$ and $i_*$ are in fact equivalent, and (considered as objects
of \(\text{Sph}\) corepresent the functor \(\Psi\) taking a spherical adjunction \(\mathcal{C}_\Phi \leftrightarrow \mathcal{C}_\Psi\) to the category \(\mathcal{C}_\Psi\). Since the objects corepresenting the functors \(\Phi\) and \(\Psi\) jointly generate \(\text{Sph}\), by calculating their endomorphisms we obtain a presentation of the full 2-category \(\text{Sph}\).

**Theorem B** (Corollary 2.17). The 2-category of spherical functors admits a canonical equivalence \(\text{Sph} \simeq \text{Mod}_A(\text{St})\) with the 2-category of modules for the convolution monoidal category

\[
\mathcal{A} := \text{Coh} \left( (\mathcal{C}/\mathcal{C}^\times \sqcup 0/\mathcal{C}^\times) \times_{\mathcal{C}/\mathcal{C}^\times} (\mathcal{C}/\mathcal{C}^\times \sqcup 0/\mathcal{C}^\times) \right).
\]

As in Example 0.4, there is a dual form of Theorem B in which \(\mathcal{C}/\mathcal{C}^\times = 0/\mathcal{C}^\times \sqcup \mathcal{C}^\times/\mathcal{C}^\times\) appears on the automorphic side and \(\mathcal{C} = 0 \sqcup \mathcal{C}^\times\) appears on the spectral side.

**Theorem C** (Corollary 3.17). The 2-category \(\text{Sph}\) admits a topological \(S^1\)-action, and the invariant 2-category admits a canonical equivalence \(\text{Sph}^{S^1} \simeq \text{Mod}_{\mathcal{A}'}(\text{St})\) with the 2-category of modules for the convolution monoidal category

\[
\mathcal{A}' := \text{Coh} ((\mathcal{C} \sqcup 0) \times_\mathcal{C} (\mathcal{C} \sqcup 0)).
\]

We deduce Theorem C from Theorem B by applying the following (twice categorified) Fourier transform.

**Theorem D** (Theorem 3.10). For any torus \(T\), there is an equivalence of 3-categories

\[
\text{Loc}^{(3)}(BT) \simeq \text{QCo}h^{(3)}(BT^T).
\]

This equivalence is suitably functorial; in particular, it intertwines the functor of \(T\)-invariants with the functor that forgets the \(T^T\)-action.

Towards establishing the above theorems, we prove two auxiliary results that may be of independent interest, which we now describe.

Our first auxiliary result is an \(\infty\)-categorical version of the theory of Smith ideals. In ordinary algebra, given a ring \(R\), there is a bijective correspondence between ideals of \(R\) and surjective ring homomorphisms out of \(R\). Underlying this is the fact that a surjective homomorphism of abelian groups is likewise determined by its kernel. By contrast, in higher algebra, every homomorphism of derived abelian groups is determined by its fiber. Thus, one may expect that every homomorphism of derived rings should be equivalent data to its fiber equipped with suitable additional structure; and this is what we establish.

**Theorem E** (Proposition 1.5). Fix a stably monoidal \(\infty\)-category \(\mathcal{C}\). For each algebra object \(A \in \text{Alg}(\mathcal{C})\), the fiber \(\text{fib}(1_\mathcal{C} \xrightarrow{\eta} A) \in \mathcal{C}\) admits a canonical enhancement to an \textbf{ideal}, i.e. a monoidal functor \(\mathbb{Z}_{\leq 0} \to \mathcal{C}\). Moreover, this construction defines an equivalence

\[
\text{Alg}(\mathcal{C}) \xrightarrow{\sim} \text{Idl}(\mathcal{C}) := \text{Fun}^{\text{mon}}(\mathbb{Z}_{\leq 0}, \mathcal{C}).
\]
Our second auxiliary result is an ambidexterity theorem for limits and colimits of presentable enriched \(\infty\)-categories.

**Theorem F** (Theorem 3.2). Fix a presentably symmetric monoidal \(\infty\)-category \(W \in \text{CAlg}(\text{Pr}^L)\). For any \(\infty\)-groupoid \(A\) and any functor \(A \xrightarrow{F} \text{Pr}^L_W\) to the \(\infty\)-category of presentably \(W\)-enriched \(\infty\)-categories, there is a canonical equivalence

\[
\text{colim}_A(F) \simeq \lim_A(F).
\]

### 0.2 Abelian gauge theories and hypertoric category \(\emptyset\)

It is expected that a symplectic stack \(X\) admitting a \(\mathbb{C}^*\) action for which the symplectic form has weight 2 determines a pair of 3-dimensional topological field theories, the 3d A-model and B-model, respectively, relating to the symplectic and algebraic geometry of \(X\). This situation encompasses all cotangent stacks \(X = T^*(M/G)\), for \(M\) a smooth variety with action of a reductive group \(G\), and all examples we consider will be of this form. Moreover, to a choice of Lagrangian \(L \subset X\), each of these theories should associate a 2-category of boundary conditions supported on \(L\).

The **3d mirror symmetry conjecture** suggests that a 3d A-model associated to some geometry \(X\) may also be realized as a 3d B-model, and vice versa. More precisely, in good cases, given a stack \(X\), we can find a “3d dual” space \(X^\vee\) such that the A-model with target \(X\) is equivalent to the B-model with target \(X^\vee\).\(^5\) (When \(X = T^*(N/G)\) for a linear \(G\)-representation \(N\), the ring of functions on the dual space is produced by the Coulomb branch construction \([\text{Nak}16, \text{BFN}18b]\).) In this case, we expect moreover that given a Lagrangian \(L \subset X\), there ought to exist a Lagrangian \(L^\vee \subset X^\vee\) such that there is an equivalence between the respective 2-categories of boundary conditions supported on \(L\) and \(L^\vee\).

**Example 0.5.** Let \(X = T^*(pt/T)\) for some \(n\)-torus \(T \simeq (\mathbb{C}^*)^n\). The dual space in this case is the variety \(X^\vee = T^*(T^\vee)\) given by the cotangent bundle to the dual torus. Each of these spaces has a natural Lagrangian given by the zero-section, \(L = pt/T \subset X\) and \(L^\vee = T^\vee \subset X^\vee\).

To the geometric data \((L \subset X)\) and \((L^\vee \subset X^\vee)\), the 3d A-model assigns the 2-categories \(\text{Loc}^{(2)}(pt/T)\) and \(\text{Loc}^{(2)}(T^\vee)\), respectively, whereas the 3d B-model assigns \(\text{QCoh}^{(2)}(pt/T)\) and \(\text{QCoh}^{(2)}(T^\vee)\). The equivalences of 2-categories predicted by 3d mirror symmetry are described in Example 0.4.

The results of this paper take place in the situation where the space \(X\) is of the form \(T^*(N/G)\), where \(N\) is a vector space with linear action of the torus \(G\). Such a stack may be specified by the

\(^5\)Note that not every 3d A-model and B-model arises in this way; thus, even a naïve version of the 3d mirror symmetry conjecture will allow that the theory dual to a 3d A-model or B-model associated to \(X\) may not arise from the data of a symplectic stack \(X^\vee\). This is analogous to the phenomenon, familiar from 2d mirror symmetry, that a non-Calabi–Yau Kähler manifold will in general be mirror to a Landau–Ginzburg model (rather than another Kähler manifold).
data of an exact sequence of tori

\[
1 \longrightarrow G \xrightarrow{i} (\mathbb{C}^\times)^n \xrightarrow{p} F \longrightarrow 1,
\]

where we take \( N = \mathbb{C}^n \) equipped with with the action of \( G \) induced from the inclusion \( i \).

**Definition 0.6.** The toric cotangent stack determined by the exact sequence (1) is

\[ \mathfrak{X}(N, G) := \mathcal{T}^*(N/G). \]

If \( \mathfrak{X} = \mathfrak{X}(N, G) \) is specified as above, then the 3d dual \( \mathfrak{X}^\vee \) is known [dBHO09, BLPW10, BFN18b] to be a space of the same form; namely, it is the toric cotangent stack associated to the dual exact sequence of tori

\[
1 \longrightarrow F^\vee \xrightarrow{p^\vee} (\mathbb{C}^\times)^n \xrightarrow{i^\vee} G^\vee \longrightarrow 1.
\]

In other words, we identify the 3d dual as \( \mathfrak{X}(N, G)^\vee = \mathfrak{X}(N^\vee, F^\vee) \).

Now, a toric cotangent stack \( \mathfrak{X}(N, G) \) admits a distinguished Lagrangian subspace.

**Definition 0.7.** We write \( \mathbb{L}_{\text{toric}} \subset \mathfrak{X}(N, G) \) for the Lagrangian

\[ \mathbb{L}_{\text{toric}} = \bigcup_{\alpha} T^*_{\mathring{O}_\alpha} (N/G) \]

given by the union of conormals to closures \( \mathring{O}_\alpha \) of \((\mathbb{C}^\times)^n\)-orbits \( O_\alpha \) in \( N/G \). When we discuss the dual cotangent stack \( \mathfrak{X}^\vee = \mathfrak{X}(N^\vee, F^\vee) \), we will write \( \mathbb{L}^\vee_{\text{toric}} \) for the analogously defined Lagrangian in \( \mathfrak{X}^\vee \).

**Example 0.8.** If \( n = 1 \) and \( G = \{1\} \), then the space \( \mathfrak{X}(N, G) = \mathcal{T}^*\mathbb{C} \) with the Lagrangian \( \mathbb{L}_{\text{toric}} = \mathbb{C} \cup T^*_0\mathbb{C} \) whose irreducible components are the zero-section and the conormal to \( 0 \in \mathbb{C} \). The dual space \( \mathfrak{X}^\vee \) is the stack \( \mathfrak{X}(N^\vee, F^\vee) = \mathcal{T}^*(\mathbb{C}/\mathbb{C}^\times) \), equipped with Lagrangian \( \mathbb{L}^\vee_{\text{toric}} = \mathbb{C}/\mathbb{C}^\times \cup T^*_0(\mathbb{C}/\mathbb{C}^\times) \).

In general, it is not understood how to associate a 2-category of boundary conditions to the data of a symplectic stack \( \mathfrak{X} \) and a Lagrangian \( \mathbb{L} \subset \mathfrak{X} \): the assignment described in Example 0.5 is one of the only cases where it has been possible to formulate 3d mirror symmetry as an equivalence of 2-categories. The main difficulty is in understanding the symplectic (A-type) 2-category; the B-type 2-category has been explored in several works [KRS09, KR10, Ari].

In this paper, we propose the following definitions for the 2-categories of boundary conditions associated to the data of a toric cotangent stack \( \mathfrak{X}(N, G) \) equipped with the Lagrangian \( \mathbb{L}_{\text{toric}} \subset \mathfrak{X}(N, G) \).
**Proposal 0.9.** We model boundary conditions for the 3d A-model with target $T^*(N/G)$ supported on $\mathbb{L}_{toric}$ by the 2-category

$$Perv^{(2)}(N/G, S_{toric}) := Perv^{(2)}(N, S_{toric})^G$$

of $G$-invariant perverse schobers on $N$ equipped with stratification $S_{toric}$ by $(\mathbb{C}^\times)^n$-orbits (incarnated as $Sph^G_n$ in §3.3). We model boundary conditions for the 3d B-model with target $T^*(N/G)$ supported on $\mathbb{L}_{toric}$ by the 2-category

$$IndCoh^{(2)}(N/G, S_{toric})$$

of $G$-invariant Ind-coherent sheaves of categories with singular support in the union of conormals to strata in $S_{toric}$. This latter 2-category may be defined [Ari] as the 2-category $\text{Mod}_A(\text{St})$ of modules for the monoidal category of coherent sheaves

$$\mathcal{A} := \text{Coh} \left( \bigsqcup_{\alpha} \overline{O}_\alpha \times_{N/G} \left( \bigsqcup_{\alpha} \overline{O}_\alpha \right) \right),$$

where we write $\bigsqcup_{\alpha} \overline{O}_\alpha$ for the disjoint union of closures of $(\mathbb{C}^\times)^n$-orbits in $N/G$.

**Remark 0.10.** Physically, the Hom categories in 3d A- and B-models should be Fukaya–Seidel and matrix factorization categories, respectively, on path spaces, as can be deduced by compactifying the 3d theory on an interval.\(^6\) The resulting 2d theories are twists of the gauged Landau–Ginzburg model described in [BDGH16, Appendix A], whose superpotential is a complexification of the equivariant symplectic action functional on path spaces [CGMiRS02, Fra04]. On the B-side, the resulting matrix factorization categories were computed in [KRS09, KR10] and shown to match a $(\mathbb{Z}/2$-graded) version of Hom categories in $IndCoh^{(2)}$. On the A-side, the “algebra of the infrared” [GMW15, KKS16, KSS] presents the resulting Fukaya–Seidel categories as categorifications of Floer groups.\(^7\) This justifies our proposal to formalize the 2-category of boundary conditions in the 3d A-model as a categorification of the Fukaya category. (For the relation between perverse sheaves and the Fukaya category, see [GPSa, Jin15].)

**Remark 0.11.** In Proposal 0.9, we have expressed the 2-categories which 3d mirror symmetry associates to a cotangent bundle $T^*(N/G)$ in terms of the geometry of the base $N/G$. We expect that there will eventually be more “microlocal” definitions of these 2-categories which do not privilege a particular polarization.

The prediction that (topologically twisted) dual abelian 3d theories have equivalent boundary conditions may now be formulated precisely as an equivalence between the 2-categories described above. This is what we prove:

\(^6\)By instead compactifying on an $S^1$, one obtains the categories of line operators studied in [DGGH20, HR21, BN].

\(^7\)More precisely, these are categorifications of the moment Floer groups of [Fra04].
Theorem G (Theorem 3.22). There is an equivalence

$$\text{Perv}^{(2)}(N/G, \text{Storic}) \simeq \text{IndCoh}^{(2)}(N^\vee/F^\vee, \text{Storic})$$

between the A-model 2-category associated to $\mathbb{L}_{\text{toric}} \subset \mathfrak{X}(N, G)$ and the B-model 2-category associated to $\mathbb{L}_{\text{toric}}^\vee \subset \mathfrak{X}(N^\vee, F^\vee)$.

Remark 0.12. As we have mentioned previously, the proof of Theorem G exploits extra structure in these 3d theories: their residual symmetries enhance them to boundary conditions for 4-dimensional abelian gauge theories [GW09]. The equivalence described in Theorem D is a form of the Betti Geometric Langlands equivalence, relating the A- and B-type twists of these 4-dimensional theories. This idea was exploited in [BFN18a, BFGT21, HR21] to compute rings of local operators and categories of line operators and forms the underpinning for the relative Langlands program of Ben-Zvi, Sakellaridis, and Venkatesh.

From the perspective of supersymmetric gauge theory or representation theory, the support Lagrangian $\mathbb{L}_{\text{toric}} \subset \mathfrak{X} = \mathfrak{X}(N, G)$ may seem unnatural; usually, as in the theory of categories $\mathcal{O}$ for symplectic resolutions [BLPW16], one studies a Lagrangian defined as the attracting set for some Hamiltonian $\mathbb{C}^\times$-action on $\mathfrak{X}$. Note that a toric cotangent stack $\mathfrak{X}(N, G)$ defined via the exact sequence (1) admits a Hamiltonian action by the residual torus $F$; therefore, a Hamiltonian $\mathbb{C}^\times$-action may be determined from a cocharacter $m \in X_*(F)$ of the torus $F$, specifying a 1-parameter subgroup acting on $\mathfrak{X}(N, G)$.

Definition 0.13. Given a cocharacter $m \in X_*(F)$, the \textbf{relative skeleton} $\mathbb{L}^m \subset \mathfrak{X}(N, G)$ is obtained by Hamiltonian reduction from the Lagrangian $\mathbb{L}^m \subset T^*N$ which is the union of attracting sets for all cocharacters of $(\mathbb{C}^\times)^n$ which map to $m$ under the projection $p_*$ [Web17, Definition 2.8]. Following [GPSb], we refer to the process of obtaining $\mathbb{L}^m$ by removing components of $\mathbb{L}_{\text{toric}}$ as \textit{stop removal}.

Our choice of Lagrangian $\mathbb{L}_{\text{toric}}$ is therefore not so unnatural: it is a sort of “master Lagrangian” containing the relative skeleta $\mathbb{L}^m$ for all possible choices of parameter $m$.

At this point it is reasonable to ask how the passage from $\mathbb{L}_{\text{toric}}$ to the sub-Lagrangian $\mathbb{L}^m$ is reflected on the 3d mirror dual. We first recall the parameter $t$ which is dual to $m$.

Definition 0.14. Given a character $t \in X^\ast(G)$ for the torus $G$, we may treat $t$ as a GIT stability parameter to define the \textbf{hypertoric orbifold} $\mathfrak{X}(N, G)_t := T^*N/\!/tG$ [Got92, BD00, HS02], which embeds as an open substack in $\mathfrak{X}(N, G)$. We write $\mathbb{L}_t := \mathbb{L}_{\text{toric}} \cap \mathfrak{X}(N, G)_t$ for the restriction of the toric Lagrangian $\mathbb{L}_{\text{toric}}$ to this subset. We refer to the passage from $\mathfrak{X}(N, G)$ to $\mathfrak{X}(N, G)_t$ as \textit{microrestriction}.

Definition 0.15. Given $m \in X_*(F)$ and $t \in X^\ast(G)$, we may simultaneously impose the GIT stability condition $t$ and pass to the relative skeleton $\mathbb{L}^m_t := \mathbb{L}^m \cap \mathfrak{X}(G, N)_t$. This is the \textbf{category of Lagrangian} associated to the exact sequence (1) and the parameters $t$ and $m$. 

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Observe that when we dualize exact sequence (1) to (2), the roles of the parameters \( t \) and \( m \) are exchanged: \( m \) becomes a character of the subtorus \( F^\vee \), while \( t \) becomes a cocharacter of the quotient torus \( G^\vee \). The prediction of 3d mirror symmetry is that \( \mathcal{X}(N, G) \), equipped with Lagrangian \( L_m^t \) is dual to \( \mathcal{X}(N^\vee, F^\vee)_m \) equipped with Lagrangian \( (L^\vee)_m^t \). In slogan form: “Stop removal is dual to microrestriction.”

In the sequel to this paper, we will explain the effects of stop removal and microrestriction on both the 3d A-model and B-model 2-categories of boundary conditions. We will then be able to make the above slogan into a rigorous mathematical statement, through which we will establish 3d mirror symmetry as an equivalence between “2-categories \( \mathcal{O} \)” associated to \( L_m^t \) and \( (L^\vee)_m^t \).

### 0.3 Notation and conventions

Our categorical conventions, which are largely standard, are articulated in §A. Here we highlight a few of those and lay out the other conventions that we will use.

We reemphasize here that we take the “implicit \( \infty \)” convention; we use the term “discrete” to refer to a non-homotopical object. We also often suppress the adjective “stable,” to simplify our language.

We work \( \mathbb{k} \)-linearly for \( \mathbb{k} \) a commutative ring or commutative ring spectrum, which will largely be suppressed from our notation. We write \( \hat{\mathcal{V}} := \text{Mod}_{\mathbb{k}} \) for its category of modules, and \( \mathcal{V} := \text{Perf}_{\mathbb{k}} \subseteq \text{Mod}_{\mathbb{k}} \) for the subcategory of its perfect modules. Hence, for \( \mathbb{k} \) an ordinary ring we write \( \hat{\mathcal{V}}^{\text{discrete}} \) (resp. \( \mathcal{V}^{\text{discrete}} \)) for its discrete category of (resp. finitely-generated projective) discrete modules.

For \( 0 \leq n \leq 2 \), we write \( \text{St}_n \) for the \((n + 1)\)-category of (\( \mathbb{k} \)-linear) small stable \( n \)-categories (with \( \text{St}_0 := \hat{\mathcal{V}} \), and often simply writing \( \text{St} := \text{St}_1 \)), and we write

\[
\text{Cat}_n \xrightarrow{\gamma^{(\infty,n)}} \text{St}_n
\]

for the left adjoint to the forgetful functor. We will occasionally wish to refer to the 2-category of presentable stable (\( \mathbb{k} \)-linear) categories, which we denote by \( \text{Pr}^{\text{st}} := \text{Pr}^{\text{st}}_{\mathbb{k}} \simeq \text{Mod}_{\hat{\mathcal{V}}}^{\text{discrete}}(\text{Pr}^{\mathbb{k}}) \).

We write \( \mathbb{Z} \) for the poset of integers with its usual ordering, which is symmetric monoidal by addition. We write \( \mathbb{Z}_{\leq 0} \subseteq \mathbb{Z} \) for the full symmetric monoidal subcategory on the nonpositive elements, and we write \( \mathbb{Z}_{\delta} \subseteq \mathbb{Z} \) for its (symmetric monoidal) maximal subgroupoid.

We write

\[
f\mathcal{V} := \text{Fun}(\mathbb{Z}^{\text{op}}, \hat{\mathcal{V}})^\omega \simeq \Sigma_+^{(\infty, 1)}(\mathbb{Z}) \quad \text{and} \quad g\mathcal{V} := \text{Fun}((\mathbb{Z}_{\delta})^{\text{op}}, \hat{\mathcal{V}})^\omega \simeq \Sigma_+^{(\infty, 1)}(\mathbb{Z}_{\delta}).
\]

Note the implicit finiteness conditions:

- A filtered \( \mathbb{k} \)-module lies in \( f\mathcal{V} \) if and only if its values are perfect, it is eventually constant to the left (i.e. at large negative integers), and it is eventually constant at \( 0_V \) to the right (i.e. at large positive integers).


A graded $k$-module lies in $gV$ if and only if its values are perfect and it is eventually constant at 0 both to the left and to the right.

Nevertheless, we simply refer to these as the categories of filtered and graded modules, respectively.

Given a category $\mathcal{A}$ and an object $a \in \mathcal{A}$, we write

$$h_a := \Sigma^{(+,0)}_{\mathcal{A}} \text{hom}_{\mathcal{A}}(-, a) \in \Sigma^{(+,1)}_{\mathcal{A}} \quad \text{and} \quad h^a := \Sigma^{(+,0)}_{\mathcal{A}} \text{hom}_{\mathcal{A}}(a, -) \in \Sigma^{(+,1)}_{\mathcal{A}^{op}}$$

for its stabilized Yoneda functors. In particular, for any $n \in \mathbb{Z}$ we have the filtered $k$-module

$$fV \ni h_n := \Sigma^{(+,0)}_{\mathbb{Z}^d} \text{hom}_{\mathbb{Z}}(-, n) = \left( i \mapsto \begin{cases} \mathbb{k}, & i \leq n \\ 0, & i > n \end{cases} \right)$$

(with nontrivial structure maps $\mathbb{k} \xrightarrow{\text{id}_k} \mathbb{k}$). On the other hand, in the case that $\mathcal{A} = \mathbb{Z}^d$, we introduce the particular notation $\delta_n := h_n$ for the graded $k$-module

$$gV \ni \delta_n := \Sigma^{(+,0)}_{\mathbb{Z}^d} \text{hom}_{\mathbb{Z}^d}(-, n) = \left( i \mapsto \begin{cases} \mathbb{k}, & i = n \\ 0, & i \neq n \end{cases} \right).$$

We write $\langle 1 \rangle$ for the shift automorphism of both $fV$ and $gV$, given in both cases by the formula $(V(1))^n := V^{n-1}$. This convention is motivated by the formulas for the application of shift automorphism to standard objects:

$$h_n(1) \cong h_n \otimes h_1 \cong h_{n+1} \in fV \quad \text{and} \quad \delta_n(1) \cong \delta_n \otimes \delta_1 \cong \delta_{n+1} \in gV.$$
1 Algebras and ideals

In this section, we study algebra objects in a stably monoidal category $C := (C, \otimes) \in \text{Alg}(\text{St})$. Our main result here (Proposition 1.5) proves that an algebra object $A \in \text{Alg}(C)$ is equivalent data to its corresponding ideal: viz., the object $I := \text{fib}(\mathbb{1} \xrightarrow{\eta} A) \in C$, equipped with suitable multiplicative structure.

On the one hand, this is basically the theory of Smith ideals [Hov], ported over from model categories to $\infty$-categories. On the other hand, this can also be viewed as an enhancement of [MNN17, Proposition 2.14], which says that the Lurie–Dold–Kan equivalence

$$\text{Fun}(\Delta, C) \xrightarrow{\sim} \text{Fun}(\mathbb{Z}_{\leq 0}, C)$$

carries the cobar complex $\text{coBar}(A) \in \text{Fun}(\Delta, C)$ to the object $\text{cofib}(T_\bullet(A) \to \text{const}(\mathbb{1})) \in \text{Fun}(\mathbb{Z}_{\leq 0}, C)$, where we write $T_\bullet(A)$ for the Adams tower of $A$. Namely, we upgrade $\text{coBar}(A)$ to the augmented cobar complex $\text{coBar}_+(A) \in \text{Fun}(\Delta_+, C)$, which we moreover equip with the data of monoidality. This becomes equivalent data to the object $A \in \text{Alg}(C)$, because $\Delta_+$ is the free monoidal ($\infty$-)category containing an algebra object.

1.1 Monoidal structures on arrow categories

Because $C$ is a stable category, taking cofibers and fibers are inverse operations: they assemble into an equivalence

$$\text{Ar}(C) \xrightarrow{\sim \text{cofib}} \text{Ar}(C). \quad (3)$$

Using the monoidal structure of $C$, we will upgrade the equivalence (3) to a monoidal equivalence (Lemma 1.3) with respect to monoidal structures that we introduce presently.

Notation 1.1. Note that the category $[1] \in \text{Cat}$ admits two distinct monoidal structures: minimum and maximum. Day convolution endows the arrow category $\text{Ar}(C)$ with two corresponding monoidal structures, which we denote by

$$(\text{Ar}(C), \square) := \text{Fun}([1], \min), (C, \boxtimes) \quad \text{and} \quad (\text{Ar}(C), \boxtimes) := \text{Fun}([1], \max), (C, \boxtimes).$$

Observation 1.2. Unwinding the definitions, we see that these monoidal structures on $\text{Ar}(C)$ are

---

8Note however that it is not a priori clear what exactly an $\infty$-categorical ideal should be – a drawback of working in the model-categorical setting.

9Note that [MNN17, Construction 2.2] defines the Adams tower asymmetrically (denoted as $T_\bullet(A, \mathbb{1})$ there), privileging left over right. By contrast, ideals are required essentially by definition to have these maps be equivalent (see [Hov, Proposition 1.7]). Such symmetry is implicitly present in our work as well.

10The minimum is also the categorical product (as well as the usual product of integers), while the maximum is also the categorical coproduct.
respectively the pushout product and pointwise monoidal structures: we have natural equivalences

\[(A \to B) \boxdot (A' \to B') \simeq \left( A \boxtimes B' \coprod_{A \boxtimes A'} B \boxtimes A' \right) \to B \boxtimes B' \]

and

\[(A \to B) \boxast (A' \to B') \simeq (A \boxast A' \to B \boxast B') .\] 

We use these facts without further comment.

**Lemma 1.3.** The equivalence (3) canonically upgrades to a monoidal equivalence

\[
\begin{array}{c}
\xymatrix{ 
(\text{Ar}(\mathcal{C}), \boxdot) 
& \simeq 
\xymatrix{ \text{cofib} 
\ar[r]^-{\sim} 
& \text{fib} 

(\text{Ar}(\mathcal{C}), \boxast) 
}
} 
\end{array}
\]

**Proof.** Consider the product monoidal structure \(([1] \times [1], \min \times \max) := ([1], \min) \times ([1], \max)\). Day convolution yields a monoidal structure on the category \(\text{Ar}^2(\mathcal{C}) := \text{Fun}([1] \times [1], \mathcal{C})\), which is given by the formula

\[
\begin{pmatrix}
C 
& \to D \\
\uparrow 
& \uparrow \\
A 
& \to B
\end{pmatrix} , \quad \begin{pmatrix}
C' 
& \to D' \\
\uparrow 
& \uparrow \\
A' 
& \to B'
\end{pmatrix} \mapsto \begin{pmatrix}
C \boxtimes D' \coprod_{C \boxtimes C'} D \boxtimes C' 
& \to D \boxtimes D' \\
\uparrow 
& \uparrow \\
A \boxtimes B' \coprod_{A \boxtimes A'} B \boxtimes A' 
& \to B \boxtimes B'
\end{pmatrix} \]

its unit object is

\[
\mathbb{1}_{\text{Ar}^2(\mathcal{C})} \simeq \begin{pmatrix}
0 
& \to 1 \\
\id 
& \uparrow \\
0 
& \to 1
\end{pmatrix} .
\]

Let us write \(\text{Exact}(\mathcal{C}) \subseteq \text{Ar}^2(\mathcal{C})\) for the subcategory on the co/fiber sequences in \(\mathcal{C}\) – i.e., the functors that select exact squares and that moreover carry the object \((0, 1) \in [1]^{\times 2}\) to the zero object \(0 \in \mathcal{C}\). We claim that this is a monoidal subcategory. First of all, it clearly contains the unit object. To show that it is stable under the monoidal structure, choose any pair of objects

\[11\text{So in other words, we use notation } \boxast \text{ both for the monoidal structure of } \mathcal{C} \text{ and for the pointwise monoidal structure of } \text{Ar}(\mathcal{C}), \text{ while we use the symbol } \boxdot \text{ for the pushout product monoidal structure of } \text{Ar}(\mathcal{C}).
\]

\[12\text{This argument is essentially contained in the proof of [Hov, Theorem 1.4].}\]
\((A \xrightarrow{\varphi} B), (A' \xrightarrow{\varphi'} B') \in \text{Ar}(\mathcal{C})\) and consider the diagram

\[
\begin{array}{c}
0 \\ id \downarrow \\
A \boxtimes B' \xrightarrow{\varphi \boxtimes \text{id}} B \boxtimes B' \\
\downarrow \id \varphi' \\
A \boxtimes A' \xrightarrow{\varphi \boxtimes \text{id}} B \boxtimes A' \\
\downarrow \id \\
0 \\ id \\
B \boxtimes A' \xrightarrow{\text{id}} B \boxtimes A' \\
\end{array}
\]

in \(\mathcal{C}\) (a span of spans). Because left Kan extensions compose, we can compute its colimit either by taking the pushout of the vertical pushouts or by taking the pushout of the horizontal pushouts. Hence, we compute on the one hand that

\[
\text{colim } ((5)) \simeq \text{colim } \left( 0 \leftarrow A \boxtimes B' \coprod_{A \boxtimes A'} B \boxtimes A' \xrightarrow{\varphi \boxtimes \varphi'} B \boxtimes B' \right) =: \text{cofib}(\varphi \boxtimes \varphi'),
\]

and on the other hand that

\[
\text{colim } ((5)) \simeq \text{colim } \left( \begin{array}{c}
\text{cofib}(\varphi) \boxtimes B' \\
\downarrow \id \varphi' \\
\text{cofib}(\varphi) \boxtimes A' \\
\downarrow 0
\end{array} \right) \simeq \text{cofib}(\varphi) \boxtimes \text{cofib}(\varphi').
\]

It follows that the formula (4) indeed preserves the subcategory \textbf{Exact}(\mathcal{C}) \subseteq \text{Ar}^2(\mathcal{C}) (setting \(C = C' = 0, D = \text{cofib}(A \xrightarrow{\varphi} B), \) and \(D' = \text{cofib}(A' \xrightarrow{\varphi'} B')\)).

To conclude the proof, observe the monoidal functors

\[
([1], \text{min}) \xrightarrow{\text{id, const}_0} ([1], \text{min}) \times ([1], \text{max}) \xleftarrow{\text{const}_1, \text{id}} ([1], \text{max}),
\]

which by restriction induce monoidal functors

\[
(\text{Ar}(\mathcal{C}), \boxtimes) \leftarrow \text{Ar}^2(\mathcal{C}) \rightarrow (\text{Ar}(\mathcal{C}), \boxtimes).
\]

Because \(\mathcal{C}\) is stable, these restrict to monoidal equivalences

\[
(\text{Ar}(\mathcal{C}), \boxtimes) \sim \text{Exact}(\mathcal{C}) \sim (\text{Ar}(\mathcal{C}), \boxtimes),
\]

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which by construction compose to the inverse equivalences (3) on underlying categories.

1.2 Algebras and ideals

Definition 1.4. An ideal in (the unit object of) \( C \) is a monoidal functor \( \mathbb{Z}_{\leq 0} \xrightarrow{I} C \).\(^{13}\) We simply write \( I := I_{-1} \in C \) and refer to this as the underlying object of the ideal; using this notation, we have canonical equivalences \( I_n \simeq I^{\mathbb{Z}(-n)} \) for all \( n \in \mathbb{Z}_{\leq 0} \). We write \( \text{Idl}(C) := \text{Fun}^{\text{mon}}(\mathbb{Z}_{\leq 0}, C) \) for the category of ideals in \( C \).

Proposition 1.5. There are canonical equivalences \( \text{Idl}(C) \xrightarrow{\sim} \text{Fun}^{\text{ex},\text{mon}}(\Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}), C) \) and \( \text{Alg}(C) \xrightarrow{\sim} \text{Fun}^{\text{ex},\text{mon}}(\Sigma^{(\infty,1)}(\mathbb{Delta}^+), C) \),

Proof. Observing the equivalences

\[
\begin{array}{c}
\text{Idl}(C) \\
\text{Alg}(C)
\end{array} \xrightarrow{\sim} \text{Fun}^{\text{ex},\text{mon}}(\Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}), C) \quad \text{and} \quad \text{Fun}^{\text{ex},\text{mon}}(\Sigma^{(\infty,1)}(\mathbb{Delta}^+), C),
\]

we see that it is (necessary and) sufficient to construct an equivalence \( \Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}) \simeq \Sigma^{(\infty,1)}(\mathbb{Delta}^+) \) in \( \text{Alg}(\text{St}) \) that behaves as described. For this, we first construct a monoidal functor

\[
\Delta^+ \xrightarrow{\varphi} \Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}),
\]

or equivalently an object of \( \text{Alg}(\Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}), \otimes) \), where we write \( \otimes \) for the monoidal structure of \( \Sigma^{(\infty,1)}(\mathbb{Z}_{\leq 0}) \) (inherited from the monoidal structure \( + \) of \( \mathbb{Z}_{\leq 0} \) by Day convolution). In turn, for this we begin with the evident laxly monoidal strictly unital functor

\[
([1], \text{min}) \xrightarrow{\varphi'} (\mathbb{Z}_{\leq 0}, +)
\]

between monoidal (discrete) categories, whose underlying functor classifies the morphism \(-1 \rightarrow 0\); its nontrivial laxness data consists of the morphism

\[
\varphi'(0) + \varphi'(0) = -2 \longrightarrow -1 = \varphi'(\text{min}(0,0)) .
\]

We then define \( \varphi \) to be the image of \( \varphi' \) under the composite left vertical functor in the commutative

\(^{13}\)More generally (but actually not), an ideal in an algebra \( A \in \text{Alg}(C) \) is a monoidal functor \( \mathbb{Z}_{\leq 0} \rightarrow \text{BiMod}_{(A,A)}(C) \).
which we explain as follows.

- For any stably monoidal category \((\mathcal{D}, \boxtimes)\), Lemma 1.3 evidently upgrades to a commutative diagram

\[
\begin{array}{ccc}
\text{Alg}(\mathcal{D}, \boxtimes) & \text{cofib} \sim \text{fib} & \text{Alg}(\mathcal{D}, \boxtimes) \\
\text{Ar}(\mathcal{D}) & \sim & \text{Ar}(\mathcal{D}) \\
\mathcal{D} & \text{fib} & \mathcal{D}
\end{array}
\]

among stably monoidal categories. Applying the functor \(\text{Alg}(\text{St}) \xrightarrow{\text{Alg}(\_)} \text{Cat}\) and taking fibers over the initial object \(\mathbb{1} \in \text{Alg}(\mathcal{D}, \boxtimes)\), we obtain an equivalence

\[
\text{Alg}(\mathcal{D}, \boxtimes) \mid \mathbb{1} \sim \text{Alg}(\mathcal{D}, \boxtimes) .
\]

This explains the third, fourth, and fifth rows.

- The uppermost vertical functors are postcomposition with the monoidal functor \((\mathbb{Z}_{\leq 0}, +) \rightarrow (\Sigma^\infty_+ (\mathbb{Z}_{\leq 0}), \otimes)\).

- The right vertical equivalence between the second and third rows follows from the universal property of Day convolution. Tracing through the definitions, one obtains its factorization as the left vertical equivalence: strictly unital functors correspond to algebra objects in \((\text{Ar}(\Sigma^\infty_+ (\mathbb{Z}_{\leq 0})), \boxtimes)\) whose target satisfies the condition of being the initial object \(\mathbb{1} \in \text{Alg}(\Sigma^\infty_+ (\mathbb{Z}_{\leq 0}), \otimes)\).
Having constructed the monoidal functor \( \varphi \), we obtain our desired exact monoidal extension

\[
\begin{array}{ccc}
\Delta_+ & \xrightarrow{\varphi} & \Sigma_+^{(\infty,1)}(\mathbb{Z}_{\leq 0}) \\
\downarrow & \multicolumn{2}{c}{\varphi} \\
\Sigma_+^{(\infty,1)}(\Delta_+) & \multicolumn{2}{c}{\ddots}
\end{array}
\]

by the universal property of the left adjoint \( \Sigma_+^{(\infty,1)} \); it remains only to show that this functor is an equivalence in \( \text{Alg} \( \text{St} \) \). Since the forgetful functor \( \text{Alg} \( \text{St} \) \xrightarrow{\text{fgt}} \text{St} \) is conservative, it suffices to show that \( \tilde{\varphi} \) is an equivalence on underlying stable categories. By Lemma 1.6, there is a distinguished equivalence

\[
\tilde{\text{LDK}}_+ \in \text{Fun}^{\text{ex}}(\Sigma_+^{(\infty,1)}(\Delta_+), \Sigma_+^{(\infty,1)}(\mathbb{Z}_{\leq 0}))
\]

(the universal coaugmented Lurie–Dold–Kan equivalence), and it suffices to show that there exists an equivalence

\[
\tilde{\varphi} \simeq \tilde{\text{LDK}}_+.
\]

Applying Lemma 1.6 again, we find that it suffices to establish the assignment

\[
\begin{array}{ccc}
\text{Fun}^{\text{ex}}(\Sigma_+^{(\infty,1)}(\Delta_+), \Sigma_+^{(\infty,1)}(\mathbb{Z}_{\leq 0})) & \xrightarrow{\sim} & \text{Fun}(\Delta_+, \Sigma_+^{(\infty,1)}(\mathbb{Z}_{\leq 0})) \\
\tilde{\varphi} & \xrightarrow{\psi} & \varphi \\
\tilde{\text{LDK}}_+ & \xrightarrow{\sim} & h_\bullet.
\end{array}
\]

Now, by definition \( \varphi \) is the coaugmented cobar complex of an algebra object structure on \( \text{cofib}(h_{-1} \to h_0) \). Using the formula for \( \text{LDK}_+ \) given in Lemma 1.6, the claimed assignment \( \varphi \mapsto h_\bullet \) follows exactly as in the proof of [MNN17, Proposition 2.14].

\[\square\]

**Lemma 1.6.** For any stable category \( \mathcal{D} \), there is a natural equivalence

\[
\text{Fun}(\Delta_+, \mathcal{D}) \xrightarrow{\text{LDK}_+} \text{Fun}(\mathbb{Z}_{\leq 0}, \mathcal{D})
\]

which carries a coaugmented cosimplicial object \( (X \to Y^\bullet) \in \text{Fun}(\Delta_+, \mathcal{D}) \) to the tower

\[
\cdots \to \text{fib}(X \to \text{Tot}_1(Y^\bullet)) \to \text{fib}(X \to \text{Tot}_0(Y^\bullet)) \to X.
\]

\[14\text{Note that } \mathbb{Z}_{\leq 0} \text{ is the directed colimit of its subcategories } \mathbb{Z}_{(-n)/0} \text{ for } n \geq 0, \text{ each of which is constructed from the former by adjoining a morphism. Hence, in order to construct an equivalence in } \text{Fun}(\mathbb{Z}_{\leq 0}, \Sigma_+^{(\infty,1)}(\mathbb{Z}_{\leq 0})), \text{ it suffices to construct it on objects and on minimal morphisms: there are no higher coherences to specify.}\]
Proof. We write

\[
\begin{array}{ccc}
\text{Fun}(\Delta, \mathcal{D}) & \xrightarrow{\text{LDK}} & \text{Fun}(\mathbb{Z}_{\leq 0}, \mathcal{D}) \\
\cup & & \cup \\
Y^\bullet & \rightarrow & ((-n) \mapsto \text{Tot}_n(Y^\bullet))
\end{array}
\]

for the Lurie–Dold–Kan equivalence [Lur, Theorem 1.2.4.1]. Observe that both categories \(\Delta\) and \(\mathbb{Z}_{\leq 0}\) are weakly contractible, so that it is merely a condition for functors out of them to be constant; observe too that the equivalence LDK restricts to an equivalence between full subcategories of constant functors. Hence, we obtain the lower composite equivalence

\[
\begin{array}{ccc}
\text{Fun}([1], \text{Fun}(\Delta, \mathcal{D})) & \xrightarrow{\text{Fun}(1, \text{LDK})} & \text{Fun}([1], \text{Fun}(\mathbb{Z}_{\leq 0}, \mathcal{D})) \\
\uparrow & & \uparrow \\
\text{Fun}(\Delta^\circ, \mathcal{D}) & \rightarrow & \text{Fun}((\mathbb{Z}_{\leq 0})^<, \mathcal{D}) \\
\text{Fun}((\mathbb{Z}_{\leq 0})^\circ, \mathcal{D}) & \rightarrow & \text{Fun}((\mathbb{Z}_{\leq 0})^\circ, \mathcal{D})
\end{array}
\]

among full subcategories on those natural transformations whose source, source, and target functors are respectively constant.\(^{15}\) Applying the evident equivalences \(\Delta^+ \simeq \Delta^d\) and \((\mathbb{Z}_{\leq 0})^\circ \simeq \mathbb{Z}_{\leq 0}\), this is precisely our desired equivalence LDK\(_+\).

\[\square\]

Remark 1.7. Let us write Env([1], min) for the monoidal envelope of ([1], min), and Env\(_{\text{unital}}([1], \text{min})\) for its unitalization; the latter is the initial monoidal category equipped with a laxly monoidal strictly unital functor from ([1], min). The proof of Proposition 1.5 implicitly constructs a monoidal functor Env\(_{\text{unital}}([1], \text{min}) \rightarrow (\mathbb{Z}_{\leq 0}, +)\), and shows that it becomes an equivalence upon applying \(\Sigma^\infty_{(\infty,1)}\). We believe that it is an equivalence unstably, but checking this appears to involve a nontrivial amount of combinatorics, so our proof proceeds by other means.

2 | Spherical adjunctions

We now use the work done in §1 to prove our main results, culminating with our spectral description of the 2-category spherical adjunctions (Theorem 2.14).

2.1 | Definition and properties

Definition 2.1. Given an adjunction

\[
\begin{array}{ccc}
\mathcal{C}_\psi & \xrightarrow{S} & \mathcal{C}_\psi \\
\downarrow & & \downarrow \\
\mathcal{C}_\psi & \xrightarrow{S^R} & \mathcal{C}_\psi
\end{array}
\]

\[\text{Here, } (-)^< \text{ (resp. } (-)^\circ \text{) denotes the left (resp. right) cone construction, given by adjoining an initial (resp. terminal) object. The weak contractibility e.g. of } \Delta \text{ yields a localization } [1] \times \Delta \rightarrow \Delta^\circ \text{ at the morphisms whose first component is } \text{id}_0.\]

18
in \text{St}, consider the endomorphisms

\[ T_\Phi := T_{S,\Phi} := \text{fib} \left( \text{id}_{C\Phi} \xrightarrow{\eta} S^R S \right) \in \text{end}_{\text{St}}(C\Phi) \quad \text{and} \quad T_\Psi := T_{S,\Psi} := \text{cofib} \left( S S^R \xrightarrow{\varepsilon} \text{id}_{C\Psi} \right) \in \text{end}_{\text{St}}(C\Psi). \]

We say that the adjunction \( S \dashv S^R \) is a \textit{spherical adjunction} if the endomorphisms \( T_\Phi \) and \( T_\Psi \) are automorphisms, in which case we refer to them as \textit{(spherical) twists}.\(^{16}\) We write

\[ \text{Sph} \subset \text{Fun} (\text{Adj, St}) \]

for the full sub-2-category on the spherical adjunctions. To simplify our notation, we will often denote a spherical adjunction simply by its left adjoint, which we call its underlying \textit{spherical functor}. To simplify our notation, we often denote a spherical adjunction by its underlying spherical functor (as e.g. in the notation \( T_{S,\Phi} \) and \( T_{S,\Psi} \) above). Moreover, we write \( \Phi, \Psi \in \text{Fun} (\text{Sph, St}) \) for the functors carrying a spherical adjunction to its source and target (i.e., to the source and target of its underlying spherical functor). So for instance, given a spherical adjunction (6), we may write \( C\Phi = \Phi(S) \).

**Observation 2.2.** We will require the following basic facts about spherical adjunctions; see [DKSS] for more details.

1. The functors \( S \) and \( S^R \) intertwine the twists, up to a shift:

\[ ST_\Phi \simeq \Sigma^{-2} T_\Psi S \quad \text{and} \quad S^R T_\Psi \simeq \Sigma^2 T_\Phi S^R. \]

2. Given a spherical adjunction \( S \dashv S^R \), the functor \( S \) admits a left adjoint \( S^L \), given by the formulas

\[ S^L \simeq \Sigma^{-1} T^{-1}_\Phi S^R \simeq \Sigma S^R T^{-1}_\Psi. \]

Moreover, the adjunction \( S^L \dashv S \) is also spherical. Reversely, we can write \( S^R \) in terms of \( S^L \) by the formulas

\[ S^R \simeq \Sigma T_\Phi S^L \simeq \Sigma^{-1} S^L T_\Psi. \]

It follows that \( S^R \) is also a spherical functor.

**Definition 2.3.** We define the \textit{Fourier transform} to be the autoequivalence

\[ \begin{array}{ccc}
\text{Sph} & \xrightarrow{\text{F}} & \text{Sph} \\
\psi & \downarrow & \psi \\
(S \dashv S^R) & \xleftarrow{\psi} & (S^L \dashv S)
\end{array} \]

\(^{16}\)It is reasonable to write \( T_{\Phi, R} := T_\Phi \) and \( T_{\Psi, R} := T_\Psi \), and thereafter denote the inverses by \( T_{\Phi, L} := (T_{\Phi, R})^{-1} \) and \( T_{\Psi, L} := (T_{\Psi, R})^{-1} \) since they are similarly extracted from the adjunction \( S^L \dashv S \) of Observation 2.2(2).
resulting from Observation 2.2(2). In other words, the Fourier transform of a spherical adjunction (6) is the spherical adjunction
\[ \mathcal{C}_\Phi \xrightarrow{S^L} \mathcal{C}_\Phi. \]
(In particular, we have equivalences \( \Phi \circ \mathfrak{F} \simeq \Psi \) and \( \Psi \circ \mathfrak{F} \simeq \Phi \) in \( \text{Fun(Sph, St)} \).) When we refer to a spherical adjunction by its underlying spherical functor \( \mathcal{C}_\Phi \xrightarrow{S} \mathcal{C}_\Psi \), we may treat \( \mathfrak{F} \) and \( \mathfrak{S}(S) \) as synonyms for the same spherical adjunction.

Remark 2.4. Given a spherical adjunction \((\mathcal{C}_\Phi \xrightarrow{S} \mathcal{C}_\Psi) \in \text{Sph}\), the twists of its Fourier transform are
\[ T_{\mathfrak{F}(S), \Phi} \simeq T_{S, \Phi}^{-1} \quad \text{and} \quad T_{\mathfrak{F}(S), \Psi} \simeq T_{S, \Psi}^{-1}. \]

2.2 The universal spherical adjunction

Recall from §0.3 the categories \( fV \) and \( gV \) of filtered and graded vector spaces. The latter category is equivalent to \( \text{Coh}(BC^\times) \), and the former also has a geometric interpretation via the Rees construction, as we now describe. Consider the free commutative \( k \)-algebra as an object \( k[x] \in \text{CAlg}(gV) \) by declaring that \( x \) has weight \(-1\). Then there is a symmetric monoidal equivalence (described for instance in [Lur15, Proposition 3.1.6])
\[ fV \cong \Perf_{k[x]}(gV) \]
\[ h_n = h_0(n) \underset{\psi}{\longrightarrow} k[x]\langle n \rangle \]
presenting \( fV \) as the category \( \text{Coh}(C/C^\times) \).

Observation 2.5. The adjunction
\[ fV = \Perf_{k[x]}(gV) \xrightarrow{gr} \Perf_k(gV) = gV \]
corresponding to the augmentation \( k[x] \to k \) is spherical: its twist automorphisms are
\[ T_\Phi = (-1) = (-) \otimes h_{-1} \in \text{aut}_S(fV) \quad \text{and} \quad T_\Psi = \Sigma^2(-1) = \Sigma^2(-) \otimes \delta_{-1} \in \text{aut}_S(gV). \]

Theorem 2.6. The object \( \text{gr} \in \text{Sph} \) corepresents the functor \( \text{Sph} \xrightarrow{\Phi} \text{St}: \) evaluation at the object \( h_0 \in fV \) defines an equivalence
\[ \text{hom}_{\text{Sph}}(fV, gV), (\mathcal{C}_\Phi \xrightarrow{S} \mathcal{C}_\Psi)) \xrightarrow{\sim} \mathcal{C}_\Phi. \]
The inverse sends an object \( c \in \mathcal{C}_\Phi \) to the morphism \( F^c : \text{gr} \to S \) with components

\[
\begin{align*}
  f\mathcal{V} & \xrightarrow{F^c_\Phi} \mathcal{C}_\Phi \\
  g\mathcal{V} & \xrightarrow{F^c_\Psi} \mathcal{C}_\Psi
\end{align*}
\]

and

\[
\begin{align*}
  h_n & \xrightarrow{(T_\Phi)^{-n}c} \\
  \delta_n & \xrightarrow{\Sigma^{2n}(T_\Psi)^{-n}Sc} .
\end{align*}
\]

**Proof.** Consider the adjunction

\[
\text{Fun}(\text{Mnd}, \text{St}) \xrightarrow{\text{Kl}} \text{Fun}(\text{Adj}, \text{St}) \tag{7}
\]

given by restriction and left Kan extension along the inclusion \( \text{Mnd} \hookrightarrow \text{Adj} \): its left adjoint carries a stable category \( \mathcal{C} \) equipped with a monad \( T \in \text{Alg}(\text{end}_{\text{St}}(\mathcal{C})) \) to the adjunction \( \mathcal{C} \rightleftarrows \text{Kl}_T(\mathcal{C}) \) between \( \mathcal{C} \) and the Kleisli category of \( T \). We note for future use that the adjunction \( \text{gr} \dashv \text{triv} \) is in the image of \( \text{Kl} \), because the left adjoint \( f \mathcal{V} \xrightarrow{\text{gr}} g\mathcal{V} \) is surjective; in symbols,

\[
\text{Kl}(f\mathcal{V}, g\mathcal{V}) \simeq (\text{gr} \dashv \text{triv}) . \tag{8}
\]

We note too that \( \text{Mnd} := \mathfrak{B}\Delta_+ \); combining this with Proposition 1.5 we obtain an equivalence

\[
\Sigma^+(\infty, 2)\text{Mnd} \simeq \Sigma^+(\infty, 2)\mathfrak{B}\mathbb{Z}_{\leq 0}
\]

in \( \text{St}_2 \), which yields the composite equivalence

\[
\text{Fun}(\text{Mnd}, \text{St}) \simeq \text{Fun}^{2\text{ex}}(\Sigma^+(\infty, 2)\text{Mnd}, \text{St}) \simeq \text{Fun}^{2\text{ex}}(\Sigma^+(\infty, 2)\mathfrak{B}\mathbb{Z}_{\leq 0}, \text{St}) \simeq \text{Fun}(\mathfrak{B}\mathbb{Z}_{\leq 0}, \text{St}) . \tag{9}
\]

Now, the inclusion \( \mathbb{Z}_{\leq 0} \hookrightarrow \mathbb{Z} \) is evidently a monoidal localization at the object \((-1) \in \mathbb{Z}_{\leq 0}\), so that by restriction we obtain a full subcategory

\[
\text{Fun}(\mathfrak{B}\mathbb{Z}, \text{St}) \subseteq \text{Fun}(\mathfrak{B}\mathbb{Z}_{\leq 0}, \text{St}) .
\]

Moreover, by definition (namely the assumption that the twist \( T_\Phi \) is invertible) we have a factorization

\[
\begin{array}{ccc}
\text{Fun}(\mathfrak{B}\mathbb{Z}_{\leq 0}, \text{St}) & \xleftarrow{\sim} & \text{Fun}(\text{Mnd}, \text{St}) \xleftarrow{\text{fgt}^{-1}} \text{Fun}(\text{Adj}, \text{St}) \\
\uparrow & & \uparrow \\
\text{Fun}(\mathfrak{B}\mathbb{Z}, \text{St}) & \xleftarrow{\sim} & \text{Sph} .
\end{array} \tag{10}
\]

We now conclude by fixing an object \((\mathcal{C}_\Phi \xrightarrow{S} \mathcal{C}_\Psi) \in \text{Sph} \) and observing the natural composite equivalence

\[
\text{hom}_{\text{Sph}}(\text{gr}, S) \simeq \text{hom}_{\text{Fun}(\text{Adj}, \text{St})}(\text{gr} \dashv \text{triv}, S \dashv S^R) . \tag{11}
\]
\begin{align*}
\simeq & \hom_{\text{Fun}(\text{Adj}, \text{St})}(\text{Kl}(fV, \text{triv} \circ \text{gr}), S \rightrightarrows S^R) \\
\simeq & \hom_{\text{Fun}(\text{Mnd}, \text{St})}((fV, \text{triv} \circ \text{gr}), (C_\Phi, S^R S)) \\
\simeq & \hom_{\text{Fun}(\text{BZ}_{\leq 0}, \text{St})}((fV, \langle -1 \rangle), (C_\Phi, T_\Phi)) \\
\simeq & \hom_{\text{St}}(V, C_\Phi) \\
\simeq & C_\Phi ,
\end{align*}

in which

- equivalence (11) follows from the definition of \text{Sph} \subset \text{Fun}(\text{Adj}, \text{St}) as a full subcategory,
- equivalence (12) follows from equivalence (8),
- equivalence (13) follows from adjunction (7),
- equivalence (14) follows from equivalence (9),
- equivalence (15) follows from the factorization (10),
- equivalence (16) follows from the evident adjunction \text{St} \rightleftarrows \text{Fun}(\text{BZ}, \text{St}) and the fact that its left adjoint acts as \text{V} \mapsto (fV, \langle -1 \rangle), and
- equivalence (17) follows from the fact that \text{V} \in \text{St} is free on an object.

Tracing through these equivalences, we see that the composite is indeed implemented by evaluation at \( h^0 \in fV \), as desired. \qed

\textbf{Remark 2.7.} The adjoint triple

\begin{equation}
\begin{array}{c}
\text{Fun}(\text{Mnd, St}) \leftarrow \text{ev}_v \rightarrow \text{Fun}(\text{Adj, St}) \\
\text{KI} \quad \downarrow \quad \text{EM} \\
\text{Mod}_{fV}(\text{St}) \leftarrow \Phi \rightarrow \text{Sph ,}
\end{array}
\end{equation}

restricts to an adjoint triple

\begin{equation}
\begin{array}{c}
\text{Mod}_{fV}(\text{St}) \leftarrow \Phi \rightarrow \text{Sph ,}
\end{array}
\end{equation}

using the fact (established during the proof of Theorem 2.6) that the functor \( \Phi : \text{Sph} \rightarrow \text{St} \) factors through \text{Mod}_{fV}(\text{St}). Furthermore, the forgetful functor \( U : fV \cong \text{Perf}_{k[x]}(gV) \rightarrow gV \) induces an
adjoint triple

\[
\begin{array}{c}
\Mod_{\mathcal{V}(\text{St})} \\
\downarrow U^* \\
\Mod_{\mathcal{F}\mathcal{V}(\text{St})}
\end{array}
\]

The composition of the two adjoint triples (18) and (19) categorifies the well-known adjoint triple

\[
\begin{array}{c}
\text{Loc}(\mathbb{C}^\times) \simeq \text{Perv}(\mathbb{C}^\times) \\
\downarrow j^! \\
\text{Perv}(\mathbb{C}, \text{S_toric})
\end{array}
\]

coming from the open affine embedding \( j : \mathbb{C}^\times \hookrightarrow \mathbb{C} \).

**Observation 2.8.** Let \( \sigma \in \text{aut}(\mathcal{V}) \) denote the symmetric monoidal autoequivalence of \( \mathcal{V} \) defined as the composite

\[
g\mathcal{V} \xrightarrow{\text{rev}} g\mathcal{V} \xrightarrow{\text{shear}} g\mathcal{V} \\
\delta_n \xrightarrow{} \delta_{-n} \xrightarrow{} \Sigma^{-2n}\delta_{-n}
\]

of grading-reversal and the “shearing” autoequivalence [Gai15, §13.4] which shifts the homological degree according to the grading. The automorphism \( \hat{\sigma} \) is symmetric monoidal and intertwines \( \langle n \rangle \) with \( \Sigma^{-2n}\langle -n \rangle \). Let \( \kappa[\beta] = \sigma(\kappa[x]) \). Because \( \sigma \) sends the augmentation of \( \kappa[x] \) to an augmentation of \( \kappa[\beta] \), it induces an equivalence of spherical adjunctions \( \text{gr} \simeq \sigma(\text{gr}) \), depicted vertically in the following diagram:

\[
\begin{array}{c}
f\mathcal{V} = \text{Perf}_{\kappa[x]}(g\mathcal{V}) \\
\downarrow \sigma
\end{array}
\]

The twist automorphisms of the spherical functor \( \sigma(\text{gr}) \) are

\[
T_\Phi = \Sigma^{-2}\langle 1 \rangle \in \text{aut}_{\text{St}}(\sigma(f\mathcal{V})) \quad \text{and} \quad T_\Psi = \langle 1 \rangle \in \text{aut}_{\text{St}}(g\mathcal{V})
\]

See [Lur15, §3.3] for a more detailed study of the category \( \sigma(f\mathcal{V}) \), which is denoted by \( \Theta \) there.

**Corollary 2.9.** The object \( \mathfrak{F}(\sigma(\text{gr})) \in \text{Sph} \) corepresents the functor \( \text{Sph} \xrightarrow{\Psi} \text{St} \): evaluation at the object \( \kappa[\beta] \in \sigma(f\mathcal{V}) \) defines an equivalence

\[
\text{hom}_{\text{Sph}}(\mathfrak{F}(\sigma(f\mathcal{V})) \xrightarrow{\sigma(\text{gr})} g\mathcal{V}), \mathcal{C}_\mathcal{F} \xrightarrow{S} \mathcal{C}_\mathcal{V} \xrightarrow{\sim} \mathcal{C}_\Psi.
\]
The inverse sends an object \( c \in \mathcal{C}_\Psi \) to the morphism \( G^c : \text{gr} \to S \) with components

\[
\begin{array}{ccc}
\sigma(fV) & \xrightarrow{G^c} & \mathcal{C}_\Psi \\
gV & \xrightarrow{G^c} & \mathcal{C}_\Phi
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\kappa[\beta](n) & \xrightarrow{\Sigma^{2n}(T_\Psi)^{-n}c} & \delta_n \xrightarrow{(T_\Phi)^{-n}S^Rc} \\
& & \\
& & 
\end{array}
\]

**Proof.** This follows from Theorem 2.6 via the equivalences

\[
\text{hom}_{\text{sph}}(\mathfrak{F}(\sigma(\text{gr})), S) \simeq \text{hom}_{\text{sph}}(\sigma(\text{gr}), \mathfrak{F}^{-1}(S)) \simeq \text{hom}_{\text{sph}}(\text{gr}, \mathfrak{F}^{-1}(S)) \simeq \Phi(\mathfrak{F}^{-1}(S)) \simeq \mathcal{C}_\Psi.
\]

**Observation 2.10.** We also could have used \( \mathfrak{F}(\text{gr}) \) or \( \mathfrak{F}^{-1}(\text{gr}) \) in Corollary 2.9 to corepresent the functor \( \Psi \). Our choice of \( \mathfrak{F}(\sigma(\text{gr})) \) is meant to ensure that the twist automorphism \( T_\Phi \) of the left-hand category \( gV \) is \((-1)\), which will be essential in the proof of Proposition 2.16.

**Remark 2.11.** Tautologically, there exists a natural adjunction

\[
\Phi \xleftarrow{S} \xrightarrow{S^R} \Psi
\]

(20)

internal to the 2-category of functors \( \text{Fun}(\text{Sph}, \text{St}) \). Using Theorem 2.6 and Corollary 2.9, we see that the adjunction (20) is corepresented by an adjunction

\[
\text{gr} \xleftarrow{S_{\text{univ}}} \xrightarrow{S^R_{\text{univ}}} \mathfrak{F}(\sigma(\text{gr}))
\]

(21)

internal to \( \text{Sph} \) which may be called the universal spherical adjunction as it corepresents the identity functor \( \text{Sph} \xrightarrow{id} \text{Sph} \). In the notation of Theorem 2.6 and Corollary 2.9, the maps underlying this adjunction are \( S^R_{\text{univ}} = F^\delta_0 \) and \( S_{\text{univ}} = G^\delta_0 \).

**Observation 2.12.** There is a natural automorphism \( T : \text{id}_{\text{Sph}} \xrightarrow{\sim} \text{id}_{\text{Sph}} \) which acts on a spherical adjunction \( \mathcal{C}_\Phi \xrightarrow{S} \mathcal{C}_\Psi \) as

\[
\begin{array}{ccc}
\mathcal{C}_\Phi & \xrightarrow{S} & \mathcal{C}_\Psi \\
\text{T}_\Psi = T_\Psi & & \text{T}_\Psi = \Sigma^{-2}T_\Psi \\
\mathcal{C}_\Phi & \xleftarrow{S^R} & \mathcal{C}_\Psi
\end{array}
\]

(22)

(That this is a map of spherical adjunctions follows from Observation 2.2 (1).) We can think of this map as the universal twist automorphism. In the spirit of Remark 2.11, we may also understand \( T \) through its corepresenting map, namely the automorphism \( T_{\text{univ}} \) of the spherical functor \( \text{gr} \oplus \mathfrak{F}(\sigma(\text{gr})) \) (commuting with the universal spherical adjunction described in Remark 2.11).
which we may write componentwise (in the notation of Theorem 2.6 and Corollary 2.9) as

\[ T_{\text{univ}} = \begin{pmatrix} F_{h-1} & 0 \\ 0 & G_{k[\delta](-1)} \end{pmatrix}. \]

2.3 A spectral description of spherical adjunctions

**Notation 2.13.** We denote by \( A := \text{end}_{\text{Sph}}(\text{gr} \oplus \mathfrak{F}(\sigma(\text{gr}))) \in \text{Alg}(\text{St}) \) the endomorphism algebra of the direct sum.

**Theorem 2.14.** The functor

\[
\text{Sph} \xrightarrow{\text{hom}_{\text{Sph}}(\text{gr} \oplus \mathfrak{F}(\sigma(\text{gr})), -)} \text{Mod}_A(\text{St}) \quad (23)
\]

is an equivalence.

**Proof.** We first prove that the functor (23) is an equivalence on \( \iota_1 \), which amounts to proving that the functor

\[
\iota_1 \text{Sph} \xrightarrow{\text{hom}_{\text{Sph}}(\text{gr} \oplus \mathfrak{F}(\sigma(\text{gr})), -)} \iota_1 \text{St} \quad (24)
\]

is the right adjoint of a monadic adjunction. Observe that by Theorem 2.6 and Corollary 2.9 we can identify the functor (24) as the composite

\[
\iota_1 \text{Sph} \xrightarrow{(ev_{\ell}, ev_r)} \iota_1 \text{St} \times \iota_1 \text{St} \xrightarrow{\oplus} \iota_1 \text{St}. \quad (25)
\]

We apply the Lurie–Barr–Beck theorem \([Lur, \text{Theorem 4.7.3.5}]\) to the composite (25). This functor is clearly conservative; to conclude, it suffices to show that it admits both a left adjoint and a right adjoint. First of all, the functor \( \oplus \) is biadjoint to the diagonal functor. Then, observe that the functor \( (ev_{\ell}, ev_r) \) is restriction along a morphism

\[(\Sigma_{+}^{(\infty,1)} \text{Adj})' \leftarrow \Sigma_{+}^{(\infty,1)} S^0\]

in \( \iota_2 \text{St}_2 \) whose target is the walking spherical adjunction – the localization of the walking stably-enriched adjunction \( \Sigma_{+}^{(\infty,1)} \text{Adj} \) that freely adjoins inverses to the twists. Hence, left and right Kan extension define left and right adjoints to the functor \( (ev_{\ell}, ev_r) \).

We have shown that the functor (23) between 2-categories is an equivalence on \( \iota_1 \). Clearly its source and target both admit cotensors over small categories, computed pointwise. Hence, it suffices to observe that the functor (23) commutes with cotensors.

**Remark 2.15.** It follows immediately from Theorem 2.6 and Corollary 2.9 that we have an
identification

\[ \mathcal{A} \simeq \begin{pmatrix} fV & gV \\ gV & \sigma(fV) \end{pmatrix} \]

in \( \text{Alg}(\text{St}) \), where the right side denotes the “matrix algebra” whose underlying category is the direct sum of its entries, with monoidal structure given by matrix multiplication (using the functors \( \text{gr} \dashv \text{triv} \) and \( \sigma(\text{gr}) \dashv \sigma(\text{triv}) \)). In what follows, we will give a geometric presentation of this matrix algebra which will clarify the meaning of the automorphism \( \sigma \).

**Proposition 2.16.** The functor \( \Phi \) gives rise to an equivalence of monoidal categories

\[ \mathcal{A} \simeq \text{end}_{\text{Mod}_{fV}(\text{St})}(fV \oplus gV), \tag{26} \]

where, \( fV \) acts on itself as the rank-1 free module and on \( gV \) through the functor \( fV \text{gr} \to gV \).

**Proof.** In the proof of Theorem 2.6 we showed that the functor \( \Phi \in \text{Fun}(\text{Sph}, \text{St}) \) can be lifted to an object of \( \text{Fun}(\text{Sph}, \text{Mod}_{fV}(\text{St})) \). In other words, the left-hand category in a spherical adjunction carries an \( fV \)-module structure that is respected by maps of spherical adjunctions.

Applying this enhanced evaluation functor to the endomorphisms of the spherical adjunction \( \text{gr} \oplus \mathfrak{F}(\sigma(\text{gr})) \), we obtain the desired monoidal functor

\[ \mathcal{A} = \text{end}_{\text{Sph}}(\text{gr} \oplus \mathfrak{F}(\sigma(\text{gr}))) \to \text{end}_{\text{Mod}_{fV}(\text{St})}(\Phi(\text{gr}) \oplus \Phi(\mathfrak{F}(\sigma(\text{gr})))) = \text{end}_{\text{Mod}_{fV}(\text{St})}(fV \oplus gV). \]

The statement about \( fV \)-module structures follows from the description of the twists in Observation 2.5 and Observation 2.10.

As in Remark 2.15, we can think of this as a functor of \( 2 \times 2 \) matrix algebras

\[ \begin{pmatrix} fV & gV \\ gV & \sigma(fV) \end{pmatrix} \to \begin{pmatrix} \text{end}_{\text{Mod}_{fV}(\text{St})}(fV) & \text{hom}_{\text{Mod}_{fV}(\text{St})}(gV, fV) \\ \text{hom}_{\text{Mod}_{fV}(\text{St})}(fV, gV) & \text{end}_{\text{Mod}_{fV}(\text{St})}(gV) \end{pmatrix} \]

in \( \text{St} \). To show that this functor is an equivalence it is sufficient to observe that it induces an equivalence on each of the four components. The only interesting case is the bottom-right entry: this equivalence is the Koszul duality result [Lur15, Proposition 3.4.8].

**Corollary 2.17.** There is an equivalence of monoidal categories

\[ \mathcal{A} \simeq \text{Coh} \left( \left( \mathcal{C}/\mathcal{C}^\times \cup 0/\mathcal{C}^\times \right) \times_{\mathcal{C}/\mathcal{C}^\times} \left( \mathcal{C}/\mathcal{C}^\times \cup 0/\mathcal{C}^\times \right) \right) \]

\[ \simeq \text{Coh} \left( \begin{array}{cccc} \mathcal{C}/\mathcal{C}^\times & 0/\mathcal{C}^\times \\ 0/\mathcal{C}^\times & \mathcal{C}/\mathcal{C}^\times \end{array} \right) \]

\[ \simeq \text{Coh} \left( \begin{array}{cccc} \mathcal{C}/\mathcal{C}^\times & 0/\mathcal{C}^\times \\ 0/\mathcal{C}^\times & \mathcal{C}[\mathcal{C}^\times] \end{array} \right), \]

26
where the monoidal structure on the right-hand side is given by convolution.

Proof. We apply [BZN17, Proposition 1.2.1], which gives, for $X \to Y$ a proper map of perfect stacks with $X$ smooth, an equivalence

$$\text{Coh}(X \times_Y X) \simeq \text{end}_{\text{Mod}_{\text{Perf}}(Y)}(\text{St})(\text{Coh}(X))$$

by treating the left-hand side as a category of integral kernels. If we assume that $Y$ is also smooth, we may replace $\text{Perf}(Y)$ by the equivalent category $\text{Coh}(Y)$.

Now we specialize to the case

$$X := \left( \mathbb{C}/\mathbb{C}^\times \sqcup 0/\mathbb{C}^\times \right) \to \mathbb{C}/\mathbb{C}^\times =: Y.$$ 

The standard identifications $\text{Coh}(B\mathbb{C}^\times) \simeq g\mathcal{V}$ and $\text{Coh}(\mathbb{C}/\mathbb{C}^\times) \simeq f\mathcal{V}$ give an identification $\text{Coh}(X) \simeq f\mathcal{V} \oplus g\mathcal{V}$. Combining this with (27), we have an equivalence

$$\text{Coh}(X \times_Y X) \simeq \text{end}_{\text{Mod}_{f\mathcal{V}}}(f\mathcal{V} \oplus g\mathcal{V}).$$

We conclude that the algebra described in (28) is equivalent to the algebra $\mathcal{A}$ as defined in Notation 2.13.

Observation 2.18. Under the equivalence of Corollary 2.17, the universal twist automorphism $T_{\text{univ}} \in \text{end}_{\text{Sph}}(\text{gr} \oplus \mathbb{F}(\text{gr})) = \mathcal{A}$ computed in Observation 2.12 is sent to the object

$$\begin{pmatrix} 0 & 0 \\ 0 & \delta_0/\mathbb{C}^\times \langle -1 \rangle \end{pmatrix} \in \text{Coh} \begin{pmatrix} \mathbb{C}/\mathbb{C}^\times & 0/\mathbb{C}^\times \\ 0/\mathbb{C}^\times & \mathbb{C}[-1]/\mathbb{C}^\times \end{pmatrix}.$$ 

3 Torus actions and toric stacks

The 2-categories we have studied so far (and some generalizations which we are to discuss soon) admit actions by tori, and we will study invariants for these actions. For instance, understanding the $S^1$-action on the 2-category $\text{Sph}$ and the procedure of taking invariants will allow us to give a notion of “$S^1$-equivariant spherical functors,” which are associated to the symplectic geometry of $T^*(\mathbb{C}/\mathbb{C}^\times)$ rather than that of $T^*\mathbb{C}$. Indeed, as we shall see in Corollary 3.17, the mirror spectral description of this 2-category $\text{Sph}^{S^1}$ will be stated in terms of $T^*\mathbb{C}$ rather than $T^*(\mathbb{C}/\mathbb{C}^\times)$.

We begin in §3.1, by establishing a technical result identifying limits and colimits of groupoid-shaped diagrams of presentable enriched categories. We apply this in §3.2 to deduce an equivalence between invariants and coinvariants for $G$-actions on 2-categories. Finally, in §3.3 we apply the preceding results to describe the 2-category of $S^1$-invariant spherical functors and higher-dimensional generalizations.
3.1 | Presentable ambidexterity

**Notation 3.1.** Given a 2-category $\mathcal{X}$, we write $\mathcal{X}^L \subseteq \mathcal{X}$ for the 1-full sub-2-category on the left adjoint 1-morphisms.\(^{17}\) We write

$$\mathcal{X} \xleftarrow{\text{fgt}} \mathcal{X}^L \xrightarrow{\text{r.adjt}} \mathcal{X}^{1\&2op}$$

for the evident forgetful functors (which are both inclusions of 1-full sub-2-categories).

**Theorem 3.2.** Fix any presentably symmetric monoidal category $\mathcal{W} \in \mathcal{CAlg}(\mathcal{Pr}_L)$, any 1-category $\mathcal{A}$, and any diagram $\mathcal{A} \xrightarrow{F} \mathcal{Pr}^L_{\mathcal{W}}$ of presentable $\mathcal{W}$-enriched categories. If the category $\mathcal{A}$ is a groupoid, then there is a canonical equivalence

$$\text{colim}_{\mathcal{A}}^{\mathcal{Pr}^L_{\mathcal{W}}}(F) \simeq \text{lim}_{\mathcal{A}}^{\mathcal{Pr}^L_{\mathcal{W}}}(F).$$

More precisely, there is a canonical dashed functor making the diagram

![Diagram](https://via.placeholder.com/150)

commutative.

**Proof.** We first address the unenriched case: i.e., we set $\mathcal{W} := \mathcal{S} = \mathcal{S}_{\mathcal{Pr}_L}$. For this, we recall the following constructions and functorialities of limits and colimits in $\mathcal{Pr}^L$ (indexed over 1-categories), which follow from the evident equivalence $\mathcal{Pr}^L \simeq (\mathcal{Pr}^R)^{1\&2op}$ along with the fact that both forgetful functors $\mathcal{Pr}^L \xrightarrow{\text{fgt}} \hat{\mathcal{Cat}}$ and $\mathcal{Pr}^R \xrightarrow{\text{fgt}} \hat{\mathcal{Cat}}$ commutes with limits [Lur09a, Proposition 5.5.3.13 and Theorem 5.5.3.18].

- The limit of a functor $\mathcal{A} \xrightarrow{F} \mathcal{Pr}^L$ can be computed (in $\hat{\mathcal{Cat}}$) as the category of cocartesian sections of the presentable fibration corresponding to $F$ (i.e., the cocartesian unstraightening of the composite $\mathcal{A} \xrightarrow{F} \mathcal{Pr}^L \xrightarrow{\text{fgt}} \hat{\mathcal{Cat}}$). Moreover, given a morphism

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\
\downarrow \phi & & \downarrow \psi \\
\mathcal{Pr}^L & & \\
\end{array}$$

\(^{17}\)This is consistent with the notation $\mathcal{Pr}^L$ if one takes $\mathcal{Pr} \subseteq \hat{\mathcal{Cat}}$ to denote the full sub-2-category on the presentable categories (or alternatively the 1-full sub-2-category on the accessible functors between presentable categories).
in $\text{Cat}_{/\mathcal{P}r^L}$, the canonical morphism

$$\lim_{\mathcal{A}}^{\mathcal{P}r^L}(F) \xleftarrow{\text{can}} \lim_{\mathcal{B}}^{\mathcal{P}r^L}(G)$$

in $\mathcal{P}r^L$ can be computed (in $\hat{\text{Cat}}$) as the restriction functor between categories of cocartesian sections.

- The colimit of a functor $\mathcal{A} \xrightarrow{F} \mathcal{P}r^L$ can be computed as the limit of the corresponding functor $\mathcal{A}^{\text{op}} \xrightarrow{F^{\text{op}}} (\mathcal{P}r^L)^{\text{op}} \simeq \mathcal{P}r^R$, which can be computed (in $\hat{\text{Cat}}$) as the category of cartesian sections of the presentable fibration corresponding to $F$ (which is also the cartesian unstraightening of the composite $\mathcal{A}^{\text{op}} \xrightarrow{F^{\text{op}}} \mathcal{P}r^R \xrightarrow{\text{fgt}} \hat{\text{Cat}}$). Moreover, given a morphism $(29)$ in $\text{Cat}_{/\mathcal{P}r^L}$, the canonical morphism

$$\text{colim}_{\mathcal{A}}^{\mathcal{P}r^L}(F) \longrightarrow \text{colim}_{\mathcal{B}}^{\mathcal{P}r^L}(G)$$

in $\mathcal{P}r^L$ is the left adjoint of the restriction functor between categories of cartesian sections.

The claim (in the case that $W := S$) now follows immediately from the observation that given a presentable fibration over a groupoid, all sections are both cocartesian and cartesian.

We now turn to the general case. For this, we use the equivalence $\mathcal{P}r_{W}^L \simeq \text{Mod}_{W}(\mathcal{P}r^L)$. Observe that the forgetful functor $\text{Mod}_{W}(\mathcal{P}r^L) \xrightarrow{\text{fgt}} \mathcal{P}r^L$ admits both a left and a right adjoint (namely $W \otimes (-)$ and $\text{Fun}^L(W, -)$, respectively). In particular, because it has a left adjoint, limits in $\text{Mod}_{W}(\mathcal{P}r^L)$ can be computed in $\mathcal{P}r^L$ — i.e., as cocartesian sections of the corresponding presentable fibration, with residual $W$-action computed fiberwise. In order to deduce the corresponding result for colimits in $\text{Mod}_{W}(\mathcal{P}r^L)$, introduce the commutative diagram

$$\begin{array}{ccc}
\text{Mod}_{W}(\mathcal{P}r^L) & \xleftarrow{\sim} & \text{coMod}_{W}((\mathcal{P}r^R)^{\text{op}}^{1\&2}) \\
\downarrow^{\text{fgt}} & & \downarrow^{\text{fgt}} \\
\mathcal{P}r^L & \xleftarrow{\sim} & (\mathcal{P}r^R)^{\text{op}}^{1\&2}
\end{array} \tag{30}$$

(considering $W$ as an object of $\text{CAlg}(\mathcal{P}r^L) \simeq \text{CAlg}((\mathcal{P}r^R)^{\text{op}}^{1\&2}) \simeq \text{coCAlg}(\mathcal{P}r^R)^{1\&2}$). The fact that the left vertical functor in (30) admits both adjoints implies that the right vertical functor does as well. Hence, limits in $\text{coMod}_{W}(\mathcal{P}r^R)$ can be computed in $\mathcal{P}r^R$ — i.e., as cartesian sections of the corresponding presentable fibration, with residual $W$-coaction computed fiberwise. Combining these observations with the unenriched case, the claim now follows.

\[\square\]

**Corollary 3.3.** The functor

$$S^{\text{op}} \xrightarrow{\text{Fun}(-, \mathcal{P}r_{W}^L)} \hat{\text{Cat}}$$

takes values in ambidextrous adjunctions: for any morphism $X \xrightarrow{f} Y$ in $S$, there is a canonical
equivalence \( f_! \simeq f_* \) between the left and right adjoints to the functor \( f^* = \text{Fun}(f, \Pr_{LW}^L) \).

**Proof.** The functor \( \text{Fun}(X, \Pr_Y^L) \xrightarrow{L^*} \text{Fun}(Y, \Pr_Y^L) \) admits both adjoints, given by left and right Kan extension (note that \( \Pr_{LW}^L \) admits all limits and colimits). Because \( X \xrightarrow{f} Y \) is both a cocartesian fibration and a cartesian fibration, these Kan extensions are respectively computed by fiberwise colimit and fiberwise limit. As the fibers are groupoids, the claim follows from Theorem 3.2. \( \square \)

### 3.2 Gauging and ungauging

We begin by recalling the notions of torus-equivariant A- and B-type 2-categories. Throughout, we fix a homomorphism \( G \xrightarrow{f} H \) of tori, and write \( G^\vee \xleftarrow{f^\vee} H^\vee \) for its dual.

**Definition 3.4.** A local system of 2-categories over \( BG \) is a presentable stable 2-category equipped with a topological \( G \)-action. These assemble into the 3-category

\[
\text{Loc}^{(3)}(BG) := \text{Fun}(BG_\bullet, \Pr_{2}^{Lst}) ,
\]

where we write \( BG_\bullet \) for the Betti stack of \( BG \), thought of as an \( \infty \)-groupoid. There is a natural functor

\[
(Bf)^* : \text{Loc}^{(3)}(BG) \to \text{Loc}^{(3)}(BH)
\]

given by pullback along the map \( BG \xrightarrow{Bf} BH \) of groupoids. We write \( (Bf)_! \dashv (Bf)^* \dashv (Bf)_* \) for its left and right adjoints (which exist since \( \Pr_{2}^{Lst} \) admits all limits and colimits).

**Remark 3.5.** An object of \( \text{Loc}^{(3)}(BG) \) is equivalently specified as the data of a presentable stable 2-category \( \mathcal{C} \in \Pr_{2}^{Lst} \) equipped with an \( E_2 \) homomorphism \( \pi_1 G \to \text{hom}(\text{id}_\mathcal{C}, \text{id}_\mathcal{C}) \).

**Definition 3.6.** A quasicoherent sheaf of 2-categories over \( BG^\vee \) is a presentable stable 2-category equipped with an action of \( \mathcal{B}(\text{Perf}(BG^\vee)) \). These assemble into the 3-category

\[
\text{QCoh}^{(3)}(BG^\vee) := \text{Fun}_{3ext}(\mathcal{B}^2(\text{Perf}(BG^\vee), \otimes), \Pr_{2}^{Lst}) .
\]

The 3-category \( \text{QCoh}^{(3)}(BG^\vee) \) is most naturally covariant in the variable \( G^\vee \): a homomorphism of tori \( H^\vee \xrightarrow{f^\vee} G^\vee \) determines a morphism \( \text{Perf}(BH^\vee) \xleftrightarrow{(Bf^\vee)^*} \text{Perf}(BG^\vee) \) in \( \text{CAlg}(\text{St}) \), and thereafter we obtain a functor

\[
(Bf^\vee)_* : \text{QCoh}^{(3)}(BH^\vee) \to \text{QCoh}^{(3)}(BG^\vee)
\]

by precomposition. As in the A-side case, this functor admits both adjoints, which we denote by \( (Bf^\vee)^* \dashv (Bf^\vee)_* \dashv (Bf^\vee)_! \).

**Remark 3.7.** There is a more general definition of the 3-category \( \text{QCoh}^{(3)}(X) \) for a stack \( X \) (which can be found in [Ste21]). The 3-category \( \text{QCoh}^{(3)}(BG^\vee) \) of Definition 3.6 agrees with...
that more general definition (up to size considerations) for 2-affine stacks. (See [Ste21, Theorem 14.3.9] for 2-affineness of \(BG^\vee\).)

**Example 3.8.** Let \(A\) be a stably monoidal category. The 2-category \(\text{Mod}_A(\text{St})\) can be made an object of \(\text{QCoh}^{(3)}(BG)\) by equipping \(A\) with a central homomorphism \(\text{Perf}(BG) \to A\), which is equivalent to promoting \(A\) to an algebra object in \(\text{QCoh}^{(2)}(BG)\). One source of such central homomorphisms may be found in the following lemma.

**Lemma 3.9.** Let \(p : X \to Y\) be a proper surjective map of smooth perfect stacks, and \(A\) the convolution category

\[ A := \text{Coh}(X \times_Y X) \simeq \text{end}_{\text{Mod}_{\text{Coh}(Y)}(\text{St})}(\text{Coh}(X)) , \]

where the equivalence follows from [BZN17, Proposition 1.2.1] as in the proof of Corollary 2.17. Then the functor

\[ \Delta_* p^* : \text{Coh}(Y) \to \text{Coh}(X \times_Y X) , \]

where \(\Delta : X \to X \times_Y X\) is the diagonal map, admits a canonical lift to an \(\mathbb{E}_2\) homomorphism into the center \(Z(A)\) of the algebra \(A\).

**Proof.** The center \(Z(A)\) is computed in [BZN17, Theorem 1.2.10]: there is an equivalence of \(\mathbb{E}_2\)-categories

\[ Z(A) \simeq \text{Coh}_{prop/Y}(LY) , \]

where \(LY := Y \times_Y Y\) is the algebraic loop space of \(Y\) and \(\text{Coh}_{prop/Y}(LY)\) denotes the category of coherent sheaves on \(LY\) whose pushforward to \(Y\) is coherent.

Pushforward along the inclusion of constant loops \(i : Y \to LY\) defines an \(\mathbb{E}_2\) map

\[ i_* : \text{Coh}(Y) \to \text{Coh}_{prop/Y}(LY) , \]

and the resulting functor \(\text{Coh}(Y) \to Z(A)\) has an underlying central functor \(\text{Coh}(Y) \to A\) given by (31).

The main result of this subsection is an identification of the 3-categories associated to topological (respectively, algebraic) actions of \(G\) (respectively \(G\)). This multiply-categorified Fourier transform (a categorification of the Fourier transform described in Example 0.4) can be understood as “fully extended Betti Langlands” (in the sense of an equivalence of 3-categories) for the group \(G\):

**Theorem 3.10.** There is an equivalence of 3-categories

\[ \text{Loc}^{(3)}(BG) \simeq \text{QCoh}^{(3)}(BG^\vee) . \]
Moreover, this equivalence is functorial in the sense that given a homomorphism \( f : G \to H \) of tori (with dual map \( f^\vee : G^\vee \leftarrow H^\vee \)), the diagram

\[
\begin{array}{ccc}
\text{Loc}^{(3)}(BH) & \xleftarrow{\sim} & \text{QCoh}^{(3)}(BH^\vee) \\
(Bf)^* & & (Bf^\vee)^*
\end{array}
\]

(33)

commutes.

Proof. Observe the equivalence

\[\Sigma_{+}^{(\infty,1)}(\pi_1 G) \simeq \text{Perf}(BG^\vee)\]

in \( \text{CAlg}(\text{St}) \); indeed, the right side is semisimple, with simple objects indexed by the characters of \( G^\vee \), or equivalently the cocharacters of \( G \), or equivalently \( \pi_1 G \). This yields the last equivalence in the composite equivalence

\[\Sigma_{+}^{(\infty,3)}BG \simeq \Sigma_{+}^{(\infty,3)}B^2(\pi_1 G) \simeq \mathfrak{B}^2\Sigma_{+}^{(\infty,1)}(\pi_1 G) \simeq \mathfrak{B}^2\text{Perf}(BG^\vee),\]

and thereafter applying \( \text{Fun}^{3\text{ex}}(-, \Pr_{2}^{L,\text{st}}) \) yields the equivalence (32). From here, the square (33) canonically commutes because the square

\[
\begin{array}{ccc}
\Sigma_{+}^{(\infty,1)}(\pi_1 H) & \xleftarrow{\sim} & \text{Perf}(BH^\vee) \\
\uparrow & & \uparrow
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma_{+}^{(\infty,1)}(\pi_1 G) & \xleftarrow{\sim} & \text{Perf}(BG^\vee)
\end{array}
\]

canonically commutes. \(\square\)

The equivalence (32) therefore also intertwines the right adjoints of the vertical maps in (33). However, we will find that we would like to establish that the right adjoint of the left vertical map \((Bf)^*\) is intertwined with the left adjoint of the right vertical map \((Bf^\vee)^*\). In other words, we will show that the left and right adjoints of these maps agree. This is a consequence of the presentable ambidexterity theorem proven above.

Lemma 3.11. The right and left adjoints of the functor

\[(Bf^\vee)_* : \text{QCoh}^{(3)}(BH^\vee) \to \text{QCoh}^{(3)}(BG^\vee)\]

are equivalent.

Proof. This follows from Theorem 3.10 and Corollary 3.3. (Note that \( \Pr_{2}^{L,\text{st}} := \Pr_{S^1}^{L,\text{st}} \)) \(\square\)
The utility of Lemma 3.11 is that the left adjoint \((f^!\vee)^*\) of \((f^\vee)_*\) is a categorified de-equivariantization.

**Observation 3.12.** The following diagram commutes:

\[
\begin{array}{ccc}
\text{Alg}(\text{Qcoh}^{(2)}(BH^\vee)) & \xrightarrow{(Bf^\vee)^*} & \text{Alg}(\text{Qcoh}^{(2)}(BH^\vee)) \\
\text{Mod}_(-)(\text{St}) & \downarrow & \text{Mod}_(-)(\text{St}) \\
\text{Qcoh}^{(3)}(BH^\vee) & \xrightarrow{(Bf^\vee)^*} & \text{Qcoh}^{(3)}(BG^\vee).
\end{array}
\]

**Lemma 3.13.** Consider an iterated pullback square

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{p}} & \bar{Y} & \xrightarrow{\bar{q}} & BG^\vee \\
\downarrow b & & \downarrow a & & \downarrow Bf^\vee \\
X & \xrightarrow{p} & Y & \xrightarrow{q} & BH^\vee,
\end{array}
\]

where \(p\) is a proper map between smooth algebraic stacks. There is an isomorphism of stably monoidal categories

\[
\text{Coh}(\bar{X} \times_{\bar{Y}} \bar{X}) \simeq (Bf^\vee)^* (\text{Coh}(X \times_Y X)).
\]

**Proof.** Given a map of algebraic stacks, the 2-categories \(\text{Qcoh}^{(2)}\) admit *-pushforward and *-pullback functors satisfying base change. (See [dF19, Chapter 5] or [Ste21, Chapter 14] for more details.) Using this, we obtain the sequence of equivalences

\[
(Bf^\vee)^* (\text{Coh}(X \times_Y X)) \simeq (Bf^\vee)^* \left( q_* \text{hom}_{\text{Qcoh}^{(2)}(Y)}(a_* \text{Perf}(X), b_* \text{Perf}(X)) \right) \quad (34)
\]

\[
\simeq q_* a^* \text{hom}_{\text{Qcoh}^{(2)}(Y)}(p_* \text{Perf}(X), p_* \text{Perf}(X)) \quad (35)
\]

\[
\simeq q_* \text{hom}_{\text{Qcoh}^{(2)}(Y)}(a_* p_* \text{Perf}(X), a_* p_* \text{Perf}(X)) \quad (36)
\]

\[
\simeq q_* \text{hom}_{\text{Qcoh}^{(2)}(Y)}(\bar{p}_* b^* \text{Perf}(X), \bar{p}_* b^* \text{Perf}(X)) \quad (37)
\]

in which equivalences (34) and (37) follow from [BZN17, Proposition 1.2.1] while equivalences (35) and (36) follow from base change. 

\[\square\]

### 3.3 Toric stacks

We now apply the preceding results to a monodromy action on \(\text{Sph}\), and then to a generalization \(\text{Sph}_n\) of \(\text{Sph}\) which has \(n\) commuting monodromy actions.

**Notation 3.14.** Up until Notation 3.20, we write \(G := \mathbb{C}^\times\).
Proposition 3.15. The 2-category \( \text{Sph} \) can be promoted to an object of \( \text{QCoh}^{(3)}(BG^\vee) \) so that the structure of \( \text{Sph} \) as an object of \( \text{Loc}^{(3)}(BG) \) (under the equivalence of Theorem 3.10) has underlying automorphism of \( \text{id}_{\text{Sph}} \) given by the universal twist automorphism

\[
\begin{array}{ccc}
\mathcal{C}_\Phi & \xleftarrow{S} & \mathcal{C}_\Psi \\
\downarrow^{T_\Phi} & & \downarrow^{\Sigma^{-2}T_\Psi} \\
\mathcal{C}_\Psi & \xrightarrow{S^R} & \mathcal{C}_\Phi
\end{array}
\] (38)

described in Observation 2.12.

Proof. Let \( X = (\mathbb{C} \sqcup 0)/G^\vee, Y = \mathbb{C}/G^\vee \). From Theorem 2.14, Proposition 2.16, and Corollary 2.17, we may present the 2-category \( \text{Sph} \) as a module 2-category \( \text{Sph} \simeq \text{Mod}_A(\text{St}) \), where

\[
\mathcal{A} := \text{Coh}(X \times_Y X) .
\] (39)

Lemma 3.9 provides a homomorphism

\[
3 : \text{Perf}(BG^\vee) \to Z(\mathcal{A}) ,
\] (40)

thus promoting \( \text{Mod}_A(\text{St}) \) to an object of \( \text{QCoh}^{(3)}(BG^\vee) \simeq \text{Loc}^{(3)}(BG) \). As described in Remark 3.5, this is equivalent to the data of an \( \mathbb{E}_2 \) map \( \pi_1G \to \text{end}(\text{id}_{\text{Sph}}) \), and we would like to check that this map sends a generator of \( \pi_1G = \mathbb{Z} \) to (38).

Due to our choice of sign in Definition 1.4, the generator of interest to us is the one which is naturally written as \( -1 \in \mathbb{Z} \). In the spectral picture, the image of that generator is the automorphism given by multiplication with the central object

\[
\gamma := 3(\mathcal{O}_{BG^\vee}(-1)) ,
\]

which we now compute.

Recall that \( 3 \) can be computed as the functor \( \Delta_*p^* \), where \( p \) is the map \( X \to Y \) and \( \Delta \) is the diagonal \( X \to X \times_Y X \). Since \( \Delta \) factors through the inclusion

\[
\left( \mathbb{C}/G^\vee \times_{\mathbb{C}/G^\vee} \mathbb{C}/G^\vee \right) \sqcup \left( 0/G^\vee \times_{\mathbb{C}/G^\vee} 0/G^\vee \right) \to X \times_Y X ,
\]

the object \( \gamma \) is a diagonal object in the matrix category \( \mathcal{A} \): we may write \( \gamma = \gamma_\Phi + \gamma_\Psi \) for its upper-left and lower-right components, respectively.

The object \( \gamma_\Phi \) is the image of \( \mathcal{O}_{BG^\vee}(-1) \) under the functor

\[
\text{Coh}(BG^\vee) \to \text{Coh}(\mathbb{C}/G^\vee \times_{\mathbb{C}/G^\vee} \mathbb{C}/G^\vee) \simeq \text{Coh}(\mathbb{C}/G^\vee)
\]
of pullback along the map $p : C/G^\vee \to BG^\vee$. Therefore $\gamma_\Phi = p^*O_{BG^\vee}(-1) = O_{C/G^\vee}(-1)$; in other words, considered as an object of $fV$, we have $\gamma_\Phi = h_{-1} \in fV$.

Similarly, $\gamma_\Psi$ is given by the image of $O _{BG^\vee}(-1)$ under the functor

$$\text{Coh}(BG^\vee) \longrightarrow \text{Coh}(0/G^\vee \times _C / G^\vee 0/G^\vee) \simeq \text{Coh}(C[-1]/G^\vee).$$

This functor sends $O _{BG^\vee}(-1)$ to $\delta _{0/C^\times}(-1)$, the ($(-1)$-twisted) skyscraper sheaf at the origin of $C[-1]/G^\vee$.

We conclude that the object $\gamma$ is presented by the matrix

$$\gamma = \begin{pmatrix} O_{C/C^\times}(-1) & 0 \\ 0 & \delta _{0/C^\times}(-1) \end{pmatrix} \in \text{Coh} \begin{pmatrix} C/C^\times & 0/C^\times \\ 0/C^\times & C[-1]/C^\times \end{pmatrix};$$

as described in Observation 2.18, this is the matrix describing the universal twist automorphism (38).

**Remark 3.16.** As we have checked in Observation 2.12, the map (38) does define an invertible object of $\text{end}(\text{id}_{\text{Sph}})$ and therefore an $E_1$ map $\pi _1(G) \simeq Z \to \text{end}(\text{id}_{\text{Sph}})$. However, to promote $\text{Sph}$ to an object of $\text{Loc}^{(3)}(BG)$ would require upgrading this $E_1$ map to an $E_2$ map, which is difficult to accomplish manually; this is why we proved Proposition 3.15 via applying our main theorem and passing to the spectral description of $\text{Sph}$. However, a sufficiently functorial theory of perverse schobers would have given us this $E_2$ map for free, coming from the geometric $S^1$ action on $(C, 0)$. We can thus interpret Proposition 3.15 as evidence for a rich yet-to-be-developed structure on perverse schobers. (This $S^1$ action may also be manifest in the description of spherical functors as paracyclic Segal categories conjectured in [DKSS].)

As an immediate corollary of Proposition 3.15, we see that passing to $G$-invariants on the A-side undoes the $G^\vee$-quotient on the B-side:

**Corollary 3.17.** The 2-category $\text{Sph}^G$ of $G$-invariant spherical functors is equivalent to the 2-category of modules over the monoidal convolution category $\text{Coh}((C \sqcup 0) \times _C (C \sqcup 0))$.

**Proof.** This follows by combining Proposition 3.15 with Lemmas 3.11 and 3.13.

**Remark 3.18.** Intuitively, an object of $\text{Sph}^G$ may be understood as a spherical adjunction equipped with a trivialization of the monodromy automorphism (38). The canonical maps $T_\Phi \to \text{id}_{C_\Phi}$ and $\text{id}_{C_\Psi} \to T_\Psi$ then become the natural transformations

$$\text{id}_{C_\Phi} \xrightarrow{\beta} \text{id}_{C_\Phi} \quad \text{and} \quad \Sigma^{-2}\text{id}_{C_\Phi} \xrightarrow{\beta} \text{id}_{C_\Phi};$$

underlying the $\text{Coh}(C) \simeq \text{Perf}_{k[x]}(V)$-module structure and $\text{Coh}(C[-1]) \simeq \text{Perf}_{k[\beta]}(V)$-module structure on $C_\Phi$ and $C_\Psi$, respectively.
Corollary 3.17 becomes more geometrically interesting when we take invariants for the actions of higher-dimensional tori. To do this, we will need to replace \( \text{Sph} \) by a more interesting 2-category \( \text{Sph}_n \). (In terms of the geometry described in §0.2: just as \( \text{Sph} \) was associated to the geometry of the space \( \mathbb{C} \) with its toric stratification, the 2-category \( \text{Sph}_n \) will be associated to \( \mathbb{C}^n \) with its toric stratification.)

**Definition 3.19.** We write \( \text{Sph}_n \) for the 2-category \((\text{Sph})^\otimes n\) obtained as the \( n \)-fold tensor product (taken in \( \text{Pr}^{L, st}_2 \)) of \( \text{Sph} \) with itself. Concretely, this is the 2-category of “spherical \( n \)-cubes”: functors \( \text{Adj}^\times n \rightarrow \text{St} \) such that all 1-morphisms in \( \text{Adj}^\times n \) are carried to spherical functors. This 2-category has \( n \) commuting automorphisms of the identity, coming from the monodromies (as described in Proposition 3.15) associated to the \( n \) spherical adjunctions.

**Notation 3.20.** In what follows, we write \( D := (\mathbb{C}^\times)^n \) for the \( n \)-torus, with dual \( D^\vee \sim (\mathbb{C}^\times)^n \).

We will also fix a subtorus \( G \subset D \) with quotient \( F \), so that we have an exact sequence of tori

\[
1 \longrightarrow G \overset{i}{\longrightarrow} D \overset{p}{\longrightarrow} F \longrightarrow 1
\]

with dual exact sequence of tori

\[
1 \longrightarrow F^\vee \overset{p^\vee}{\longrightarrow} D^\vee \overset{i^\vee}{\longrightarrow} G^\vee \longrightarrow 1.
\]

**Proposition 3.21.** Let \( X^n := (\mathbb{C} \sqcup 0)^n / D^\vee \) and \( Y^n := \mathbb{C}^n / D^\vee \). Then \( \text{Sph}_n \) is equivalent to the 2-category of modules over the monoidal category \( \text{Coh}(X^n \times Y^n X^n) \).

Moreover, these 2-categories may be lifted to objects of the 3-category \( \text{Loc}^{(3)}(BD) \simeq \text{QCoh}^{(3)}(BD^\vee) \), where the first structure is as described in Definition 3.19 and the second comes from the \( D^\vee \)-linear structure of the category \( \text{Coh}(Y^n \times X^n Y^n) \).

**Proof.** Observe the identification \( X^n \times Y^n X^n \simeq (X^1 \times Y^1 X^1)^\times n \). Using Proposition 3.15, we obtain the composite equivalence

\[
\text{Coh}(X^n \times Y^n X^n) \simeq \text{Coh}((X^1 \times Y^1 X^1)^\times n) \simeq \text{Coh}(X^1 \times Y^1 X^1)^\otimes n \simeq \text{Sph}^\otimes n =: \text{Sph}_n.
\]

It follows that that the left-hand category’s structure as an object in \( \text{QCoh}^{(3)}(BD^\vee) \simeq \text{Loc}^{(3)}(BD) \) has underlying monodromy map \( \pi_1 D \rightarrow \text{end}(\text{id}_{\text{Sph}_n}) \) selecting the monodromy automorphisms of Proposition 3.15 in each of the \( n \) tensor factors. \( \square \)

As in Corollary 3.17, we may therefore deduce a description of the invariant 2-category \( \text{Sph}^D_n \) by \( D^\vee \)-deequivariantization in the spectral description of \( \text{Sph}_n \). More generally, we can describe the invariant 2-category \( \text{Sph}^G_n \) by \( G^\vee \)-dequivariantization.

**Theorem 3.22.** Let \( X^n_G := (\mathbb{C} \sqcup 0)^n / F^\vee \) and \( Y^n_G = \mathbb{C}^n / F^\vee \). Then \( \text{Sph}^G_n \) is equivalent to the 2-category of modules for the monoidal category \( \text{Coh}(X^n_G \times Y^n_G X^n_G) \).

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Proof. Let \( \pi : G \to \{1\} \) be the projection, and consider the iterated pullback squares

\[
\begin{array}{c}
X^n_G \longrightarrow Y^n_G \longrightarrow BF^\vee \longrightarrow B\{1\} \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow B_{p}^\vee \quad \quad \downarrow B_{\pi}^\vee \\
X^n \longrightarrow Y^n \longrightarrow BD^\vee \longrightarrow BG^\vee .
\end{array}
\]

We now deduce the equivalences

\[
(B\pi)_* (Bi)^* \text{Sph}_n \simeq (B\pi^\vee)_! (Bi^\vee)_* \text{Coh}(X^n \times Y^n, X^n) \quad (41)
\]

\[
\simeq (B\pi^\vee)_! (Bi^\vee)_* \text{Coh}(X^n \times Y^n, X^n) \quad (42)
\]

\[
\simeq \text{Coh}(X^n_G \times Y^n_G, X^n_G) ,
\]

where (41) follows from Propositions 3.21 and 3.15, (42) follows from Lemma 3.11, and (43) follows from Lemma 3.13.

An interpretation of Theorem 3.22 as 3d mirror symmetry between Gale dual toric cotangent stacks is given as Theorem G in §0.2.

A  |  Categorical conventions

A.1  |  Generalities

Throughout this paper, we adhere to the “implicit \( \infty \) convention: all definitions and constructions should be interpreted homotopy-coherently, so that for example by “\( n \)-category” we mean “\((\infty, n)\)-category”. We nevertheless occasionally use “\( \infty \)” for emphasis; we use the word “discrete” to emphasize that we are referring to a non-homotopical object.

We take Lurie’s books [Lur09a, Lur] as background references for category theory, although we also give specific citations where relevant. We take [GH15] for our foundations of enriched categories.

Given an \( \mathbb{E}_{k+1} \)-monoidal category \( \mathcal{W} \) for some \( 0 \leq k \leq \infty \), we write \( \text{Cat}(\mathcal{W}) \) for the category of \( \mathcal{W} \)-enriched categories, which is \( \mathbb{E}_k \)-monoidal. In all cases of interest to us, if \( \mathcal{W} \) is in fact a monoidal \( n \)-category then \( \text{Cat}(\mathcal{W}) \) is an \((n+1)\)-category. We write \( \text{Alg}(\mathcal{W}) \xrightarrow{\mathbb{B}} \text{Cat}(\mathcal{W}) \) for the functor taking an algebra object in \( \mathcal{W} \) to the corresponding one-object \( \mathcal{W} \)-enriched category. We refer the reader to [MGS, §§A and B] for a discussion of enriched and higher categories.

We write \( \text{Cat}_n := \text{Cat}(\text{Cat}_{n-1}) \) for the \((n+1)\)-category of small \( n \)-categories. As a special case, we write \( \mathcal{S} := \text{Cat}_0 \) for the 1-category of spaces. We write \( \iota_k \) to refer to the maximal sub-\( k \)-category; for any \( n \geq k \) this assembles as a functor \( \iota_{k+1}\text{Cat}_n \xrightarrow{\mathbb{B}} \text{Cat}_k \).
We write \( \text{Adj} \) for the 2-category that corepresents adjunctions [RV16]. We write \( \ell, r \in \text{Adj} \) for its two objects, and we respectively write \( \ell \xrightarrow{L} r \) and \( \ell \xleftarrow{R} r \) for the universal left and right adjoint 1-morphisms. We write \( \text{Mnd} \subset \text{Adj} \) for the full subcategory on the object \( \ell \in \text{Adj} \); this is the 2-category that corepresents monads.

Recall that among monoidal categories one can contemplate both strictly and laxly monoidal functors. We note that the right adjoint of a strictly monoidal functor is canonically laxly monoidal. We note too that a laxly monoidal functor can satisfy the condition of being strictly unital. Identical remarks apply in the symmetric monoidal case. Moreover, given an operad \( \mathcal{O} \), an adjunction in which the left adjoint is symmetric monoidal lifts to an adjunction on \( \mathcal{O} \)-algebras.

We use the device of Grothendieck universes, employing the terms “small”, “large”, and “huge” accordingly. These terms should always be interpreted inclusively, in the sense that e.g. the term “large space” is shorthand for “a space that is either small or large”. In all cases of interest to us, every large (resp. huge) category is locally small (resp. locally large); this applies both to enriched and unenriched categories. We use hats to denote passage to a larger context, e.g. we write \( \hat{\mathcal{S}} \) to denote the huge category of large spaces. We mostly suppress discussion of size, except where we find it clarifying.

### A.2 Stable \( n \)-categories

We work \( k \)-linearly, for \( k \) a fixed commutative ring or commutative ring spectrum. We will mostly suppress \( k \) in both our notation and our terminology, as indicated in what follows.

We write \( \text{St}_\mathcal{S} \subset \text{Cat} \) for the 1-full sub-2-category of small stable categories. We consider this as equipped with the symmetric monoidal structure that corepresents multiexact functors.

We write \( \mathcal{V} := \text{Perf}_k \in \text{CAlg}(\text{St}_\mathcal{S}) \) for the stably symmetric monoidal category of perfect \( k \)-modules. For the present discussion it will also be useful to write \( \hat{\mathcal{V}} := \text{Mod}_k \simeq \text{Ind}(\mathcal{V}) \) for the presentably symmetric monoidal stable category of \( k \)-modules.

We write \( \text{St} := \text{St}_k \simeq \text{Mod}_\mathcal{V}(\text{St}_\mathcal{S}) \) for the 2-category of small stable \( k \)-linear categories. We simply write \( \otimes := \otimes_\mathcal{V} \) for its symmetric monoidal structure, and \( \text{Fun}^{\text{ex}} \) for its adjoint self-enrichment.

We denote by

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\Sigma_+^{(\infty,0)}} & \hat{\mathcal{V}} \\
\Omega^{(\infty,0)} & \bot & \\
\text{Cat} & \xleftarrow{\Sigma_+^{(\infty,1)}} & \text{St}
\end{array}
\]

the indicated free/forget adjoints.\(^\text{18}\) Both left adjoints are symmetric monoidal. The second adjunction is obtained by applying \( \text{Cat}(\mathcal{V}) \) to the first and then composing with the adjunction

\[
\begin{array}{ccc}
\text{Cat}(\hat{\mathcal{V}}) & \xrightarrow{\text{Env}} & \text{St}
\end{array}
\]

whose (symmetric monoidal) left adjoint is given by the stable envelope and whose right adjoint

\(^{18}\)The left adjoint \( \Sigma_+^{(\infty,0)} \) may be referred to as the “\( k \)-linear chains” functor.
is fully faithful.

Given a category \( \mathcal{A} \in \text{Cat} \), we note that \( \Sigma_{(\infty,1)}^+ \mathcal{A} \in \text{St} \) can be characterized as the \( k \)-linear presheaves on \( \mathcal{A} \): the smallest stable subcategory containing the image of the stabilized Yoneda functor

\[
\mathcal{A} \hookrightarrow \text{Fun}(\mathcal{A}^{\text{op}}, S) \xrightarrow{\Sigma_{(\infty,0)}^+} \text{Fun}(\mathcal{A}^{\text{op}}, \hat{V}).
\]

In these terms, the symmetric monoidality of \( \Sigma_{(\infty,1)}^+ \) is incarnated via Day convolution.

For any \( n \geq 2 \), we inductively define a \textit{stable} \( n \)-category to be a category enriched in the symmetric monoidal \( n \)-category \( \text{St}_{n-1} \) of stable \( (n-1) \)-categories; these assemble into a (self-enriched) \((n+1)\)-category \( \text{St}_n := \text{Cat}(\text{St}_{n-1}) \).\(^{19}\) We use this definition for \( n = 2, 3 \). We inductively obtain an adjunction

\[
\text{Cat}_n \xleftarrow{\Sigma_{(\infty,n)}^+} \text{St}_n \xrightarrow{\text{fgt}}
\]

by applying \( \text{Cat}(\cdot) \), whose left adjoint is symmetric monoidal. We generally omit the right adjoint \( \text{fgt} \) from our notation.\(^{20}\)

We note here that stable 2-categories are “pre-semiadditive”: inside of any stable 2-category, any initial object is also terminal and reversely, and thereafter any finite coproduct is also a finite product and reversely.

### A.3 Presentability

We write \( \text{Pr}^L \subset \hat{\text{Cat}} \) for the 1-full sub-2-category whose objects are presentable categories and whose morphisms are left adjoint functors. We consider this as equipped with the symmetric monoidal structure that corepresents multicocontinuous functors. We write \( \text{Fun}^L \) for its adjoint self-enrichment.

We employ the theory of presentable enriched categories of [MGS, §A.3]. Our usage thereof may be summarized as follows.

Given a presentably symmetric monoidal category \( \mathcal{W} \in \text{CAlg}(\text{Pr}^L) \), we write \( \iota_1 \text{Pr}^L_W \subseteq \hat{\text{Cat}}(\mathcal{W}) \) for the huge category of \textit{presentable} \( \mathcal{W} \)-enriched categories. Rather than give a definition here, we simply note the canonical equivalence

\[
\iota_1 \text{Pr}^L_W \xrightarrow{\sim} \text{Mod}_W(\iota_1 \text{Pr}^L) ;
\]

this carries a presentable \( \mathcal{W} \)-enriched category \( \mathcal{C} \in \iota_1 \text{Pr}^L_W \) to its underlying (unenriched) presentable category \( U(\mathcal{C}) \in \iota_1 \text{Pr}^L \) equipped with the \( \mathcal{W} \)-action given by tensoring, which is charac-

\(^{19}\)In this case that \( n = 2 \), this does quite not match the conventions of [MGS], which additionally require the existence of finite sums.

\(^{20}\)This forgetful functor is conservative, and is moreover faithful in the case that \( k = \mathbb{S} \) is the sphere spectrum (or more generally whenever \( k \) is idempotent).
terized by the universal property that
\[ \text{hom}_{U(\mathcal{E})}(W \cdot C, D) \simeq \text{hom}_{\mathcal{W}}(W, \text{hom}_{\mathcal{C}}(C, D)) \]
for all \( W \in \mathcal{W} \) and all \( C, D \in \mathcal{C} \). We write \( \text{Fun}^L_W \) for the self-enrichment of \( t_1 \text{Pr}^L_W \simeq \text{Mod}_W(t_1 \text{Pr}^L) \), which is adjoint to its symmetric monoidal structure \( \otimes_W \) (computed in \( \text{Mod}_W(t_1 \text{Pr}^L) \)).

As a special case, we write \( t_1 \text{Pr}^L_n := t_1 \text{Pr}^L_{\text{tg}_n} \), and refer to its objects as \textit{presentable} \( n \)-\textit{categories}. Similarly, we write \( t_1 \text{Pr}^L_{\text{tg}} := t_1 \text{Pr}^L_{\text{tg}_{\text{st}}} \) and \( t_1 \text{Pr}^L_{\text{tg}_{\text{st}}} := t_1 \text{Pr}^L_{\text{tg}_{\text{st}}} \), and respectively refer to objects of these as \textit{presentable stable} 1- or 2-\textit{categories}.

Given a morphism \( W \to W' \) in \( \text{CAlg}(\text{Pr}^L) \), we obtain adjoint functors
\[
\begin{array}{ccc}
\text{Fun}^L_W(W', -) & \xrightarrow{\text{fgt}} & \text{Mod}_W(t_1 \text{Pr}^L) \\
\downarrow & & \downarrow \\
\text{Mod}_W(t_1 \text{Pr}^L) & \xleftarrow{\text{fgt}} & \text{Fun}^L_W(W', -)
\end{array}
\]
In particular, it follows that the forgetful functor \( t_1 \text{Pr}^L_W \xrightarrow{\text{fgt}} t_1 \text{Pr}^L_W \) commutes with both limits and colimits.

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