Stability Implies Computational Tractability: Locating a Tree in a Stable Network is Easy

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Abstract

In this work, we answer an open problem in the study of phylogenetic networks. Phylogenetic trees are rooted binary trees in which all edges are directed away from the root, whereas phylogenetic networks are rooted acyclic digraphs. For the purpose of evolutionary model validation, biologists often want to know whether or not a phylogenetic tree is contained in a phylogenetic network. The tree containment problem is NP-complete even for very restricted classes of networks such as tree-sibling phylogenetic networks. We prove that this problem is solvable in cubic time for stable phylogenetic networks. A linear time algorithm is also presented for the cluster containment problem.

1 Introduction

How life came to existence and evolved has been a key question in science in the past hundreds of years. Traditionally, a (phylogenetic) tree has been used to model the evolutionary history of species, in which each internal node represents a speciation event and the leaves represent the extant species under study. Such evolutionary trees are often reconstructed from the gene or protein sequences sampled from the extant species under study. Since more and more genomic studies demonstrate that genetic material was horizontally transferred from one species to another coexisting species via recombination, hybridization, and horizontal gene transfer [3, 16, 20], it has been commonly accepted that (phylogenetic) networks are more suitable for modelling genome evolution [5, 6, 10, 17, 18]. Mathematically, a network is a rooted acyclic digraph with labelled leaves. Algorithmic and combinatorial aspects of networks have been intensively studied in the past two decades [10, 12, 23].

A major issue in systematic biology is to check the “consistency” of two evolutionary models. A somewhat simpler (but nonetheless very important) version of this issue asks whether a given network is consistent with an existing tree model or not. This motivates researchers to study the problem of determining whether a tree is displayed by a network or not, called the tree containment problem (TCP). The cluster containment problem (CCP) is another related algorithmic problem that asks whether or not a subset of taxa is a cluster in a tree displayed by a network.

The TCP and CCP are NP-complete [13], even on a very restricted class of time-consistent tree-sibling regular networks [22]. It is an open problem whether or not the problem is solvable in...
polynomial time for stable networks \cite{7,12,21,22}. The stability property was originally introduced to capture an important feature of galled networks \cite{11} and real network models are often stable \cite{15}. Although great effort has been devoted to the study of the TCP, it has been shown to be polynomial-time solvable only for a couple of very restricted subclasses of stable networks, namely, tree-child networks \cite{22} and so-called genetically stable networks \cite{8}. Other studies related to the TCP include \cite{4} and \cite{14}.

In this paper, we shall give an affirmative answer to the open problem by presenting a cubic time algorithm for the TCP for binary stable networks. Additionally, we also present a time-optimal algorithm for the CCP for binary stable networks. These two algorithms are further modified into polynomial time algorithms for non-binary stable networks.

The rest of the paper is organized as follows. Section \ref{section2} introduces basic concepts and notation that are used throughout this paper. Section \ref{section3} lists our main results (Theorems \ref{thm3.1} and \ref{thm3.2}) and gives a brief summary of algorithmic methodologies that lead us to obtain the results. In Section \ref{section4} we present a decomposition theorem (Theorem \ref{thm4.1}) that reveals an important structural property of stable networks, based on which the two main theorems are respectively proved in Section \ref{section5} and \ref{section6}. Section \ref{section7} describes how to modify the two algorithms to solve the problems for non-binary stable networks. Finally, we conclude with two remarks in Section \ref{section8}.

\section{Basic Concepts and Notation}

\subsection{Phylogenetic Networks}

In phylogenetics, \textit{networks} are rooted acyclic digraphs in which a unique node (the \textit{root}) has a directed path to every other node in the network and the nodes of indegree one and outdegree zero (called the \textit{leaves}) are uniquely labeled. Leaves represent bio-molecular sequences, extant organisms or species under study. We also assume that each non-root node in a network has either an indegree of one or an outdegree of one. A node of indegree one and of outdegree zero or strictly greater than one is called a \textit{tree node} (and thus leaves \textit{are} tree nodes). A node is called a \textit{reticulation} node if its indegree is strictly greater than one and its outdegree is precisely one. Tree nodes and reticulation nodes represent speciation and horizontal transfer events, respectively.

The network model studied in this paper does allow, purely from a theoretical point of view, nodes that have both indegree and outdegree as one even though such nodes do \textit{not} represent any evolutionary events. Additionally, for convenience in describing the algorithms and proofs, we add an \textit{open} incoming edge to the root (see Figure \ref{fig1}) that makes the root the “highest” tree node in the network.

Let $N$ be a network. We use the following notation:

- $\rho(N)$ is the root of $N$.
- $\mathcal{L}(N)$ is the set of all leaves in $N$.
- $\mathcal{R}(N)$ is the set of all reticulation nodes in $N$.
- $\mathcal{T}(N)$ is the set of all tree nodes in $N$.
- $\mathcal{W}(N)$ is the set of all degree-two nodes in $N$.
- $\mathcal{V}(N) = \mathcal{R}(N) \cup \mathcal{T}(N) \cup \mathcal{W}(N)$, which is the set of all nodes in $N$.
- $\mathcal{E}(N)$ is the set of all edges in $N$.
- For two nodes $u, v$ in $N$:
  - $u$ is a \textit{parent} of $v$ or alternatively $v$ is a \textit{child} of $u$ if $(u, v)$ is a directed edge in $N$, and
Figure 1: The network in panel A displays the tree in panel D through the removal of four edges $e_1, e_2, (x, v), (x, u)$ and the node $x$. Here, reticulation nodes are represented by filled circles.

- $u$ is an ancestor of $v$ or alternatively $v$ is a descendant of $u$ if there is a directed path from $u$ to $v$.
- $p(u)$ is the set of the parents of $u \in R(N)$ or the unique parent of $u \in T(N) \setminus \{\rho(N)\}$.
- $c(u)$ is the set of the children for $u \in T(N)$ or the unique child for $u \in R(N)$.
- $D_N(u)$ is the subnetwork vertex-induced by $u \in V(N)$ and all descendants of $u$.
- For any $E \subseteq E(N)$, $N - E$ is the subnetwork of $N$ with the (same) node set $V(N)$ and the edge set $E(N) \setminus E$.
- For any subset $V$ of nodes of $N$, $N - V$ is the subnetwork of $N$ with the node set $V(N) \setminus V$ and the edge set $\{(x, y) \in E(N) | x \notin V, y \notin V\}$.

In the rest of the paper, we focus on binary networks and binary trees in which each non-leaf nodes has degree 3. Therefore, we assume all the networks and trees are binary unless explicitly mentioned otherwise.

2.2 The Stability Property

Let $N$ be a network and $u \in V(N)$. We say that $u$ is stable if there exist a leaf $\ell \in L(N)$ such that every path from the root $\rho(N)$ to $\ell$ must go through $u$ (also see [12, p. 165]). For instance, for the network in Figure 1A, the root and all reticulation nodes are stable, whereas all the non-root internal tree nodes are not stable.

Definition 2.1 (Stable Network) A network is stable (or reticulation visible) if every reticulation node is stable.

By definition, a network is stable if every reticulation node separates the network root from at least one leaf. Clearly, all trees are stable networks. But there are binary, non-planar stable networks. In fact, tree-child networks, galled trees, and galled networks are all stable [2] [23].
2.3 The TCP and CCP

We recall the following standard graph-theoretic terminologies for reader’s convenience (e. g., see [1], p. 55). Suppressing a node of indegree one and outdegree one means removing the node and merging the two edges incident to it into an edge with the same orientation between its two neighbors. A tree \( T' \) is called a subdivision of another tree \( T \) if \( T \) can be obtained by suppressing some degree-two nodes in \( T' \).

Consider a network \( N \). Since \( N \) is binary, each reticulation node has two incoming and one outgoing edges. Thus, removing one incoming edge for each reticulation node in \( N \) results in a directed tree (rooted out-arborescence). However, there may exist new (dummy) leaves in the obtained tree. For example, after removing \( e_1, e_2, (x, v) \), and \( (x, u) \) in the network given in Figure 1A, we obtain the tree shown in Figure 1B in which \( x \) is a new leaf besides the original leaves \( \ell_i \) (\( 1 \leq i \leq 4 \)). Therefore, if the obtained tree contains such dummy leaves we will have to remove them and some of their ancestors to obtain a subtree having the same set of leaves as \( N \).

Definition 2.2 (Network displaying a tree) Network \( N \) displays a directed tree \( T \) that satisfies \( \mathcal{L}(N) = \mathcal{L}(T) \), if there exist subsets \( E \subset \mathcal{E}(N) \) and \( V \subset \mathcal{V}(N) \) such that (i) \( E \) contains exactly one incoming edge for each \( u \in \mathcal{R}(N) \), and (ii) \( N - E - V \) is a subdivision of \( T \).

Because of the existence of dummy leaves, \( V \) is usually nonempty to guarantee that \( N - E - V \) has the same set of leaves as \( T \) (see Figure 1 for an example). Note that a network with \( k \) reticulation nodes can display as many as \( 2^k \) trees.

Definition 2.3 (TCP) It is to determine whether a given network displays a given tree or not.

In a tree \( T \), the set of all the labeled leaves in the subtree \( D_T(v) \) is called the cluster of a node \( v \). An internal node in a network may have different clusters in different trees displayed by the network. Given a subset of labeled leaves \( B \subseteq \mathcal{L}(N) \), \( B \) is contained in \( N \) if \( B \) is a cluster of a node in some tree displayed by the network. When \( B \) is contained in \( N \), we call \( B \) a soft cluster in \( N \).

Definition 2.4 (CCP) It is to determine whether a given subset \( B \) of \( \mathcal{L}(N) \) is a soft cluster in a network \( N \) or not.

3 Our Results

Theorem 3.1 (Main Result) Given a network \( N \) and a tree \( T \), the TCP for \( N \) and \( T \) can be solved in cubic time provided \( N \) is stable.

The CCP is another important problem that has a polynomial time algorithm for stable networks [12, pp. 168–171]. Using the powerful decomposition theorem presented in the next section, we are able to design an optimal algorithm for it.

Theorem 3.2 Given a network \( N \) and a subset \( B \) of labeled leaves in \( N \), the CCP for \( N \) and \( B \) can be solved in linear time provided \( N \) is stable.

Synopsis of Algorithmic Methodologies On the surface the reader may wonder why solving the TCP is hard since after all \( N \) is an acyclic digraph and \( T \) is a tree. First, it is closely related to the subgraph isomorphism problem (SIP). In general, it is very tricky to find out whether a special case of the SIP remains NP-complete or can be solved in polynomial time. For example, whether
or not a directed tree \( H \) is a subgraph of an acyclic digraph \( G \) is NP-complete, but can be solved in polynomial time provided that \( G \) is a forest \( [9] \). Second, the TCP remains NP-complete even if \( N \) is a binary network in which every reticulation node has a tree-node sibling and the subsets of descendant leaves of any two nodes are different \( [22] \).

The following intuition may explain some of the non-trivialness one may face in solving TCP. Note that each reticulation node \( u \) of \( N \) may be thought of encoding a Boolean variable \( x_u \): \( x_u \) being 0 or 1 depending on which incoming edge for \( u \) is removed. Thus, the selection of incoming edges for removal for a subset of reticulation nodes can be thought of assigning truth values to some Boolean variables to satisfy “arbitrary length” clauses. The TCP has been known to be solved in polynomial time only for a very restricted subclasses of stable networks \( [22] \) and the so-called nearly-stable networks \( [7] \). For tree-child or nearly-stable networks, one can determine which incoming edge for a reticulation node should be removed by comparing a small local structure around it and the structure of the given tree. However, any approach that works on reticulation nodes one by one is not enough for solving the TCP for a network shown in Figure 1. We need to deal with even the whole set of reticulation nodes simultaneously for some stable networks.

Our algorithms for the TCP and CCP rely primarily on a strong decomposition theorem (Theorem 4.1). Roughly speaking, the theorem states that, in a stable network, all the non-leaf tree nodes are partitioned into a collection of disjoint connected components each having a tree structure with at least two nodes and, most importantly, either containing a network leaf or being on the top of another component.

The topological property uncovered by this theorem allows us to solve the TCP and CCP by the divide-and-conquer approach: We work on the tree components one-by-one in a bottom-up fashion. In the TCP case, when working on a tree component, we simply call a dynamic programming algorithm to decipher all the reticulation nodes right below it. In the CCP case, a slightly structural complex (but faster) dynamic programming algorithm is called on a tree component.

### 4 A Decomposition Theorem

In this section, we shall present a decomposition theorem that plays a vital role in designing a fast algorithm for the TCP and CCP. We remind the reader that our given network is binary and there is no nodes of degree 2 in it. We first show that stable networks have two useful properties.

**Proposition 4.1** A stable network \( N \) has the following two properties:

(a) (Reticulation separability) The child and the parents of a reticulation node are tree nodes.

(b) (Stability inheritability) For any \( E \subseteq \mathcal{E}(N) \), then \( N - E \) is also stable if \( \mathcal{L}(N - E) = \mathcal{L}(N) \).

**Proof.** (a) The statement is equivalent to that there are no two reticulation nodes having the parent-son relation. Suppose on the contrary, there are \( u, v \in \mathcal{R}(N) \) such that \( (u, v) \in \mathcal{E}(N) \). Let \( w \) be the other parent of \( v \). By assumption on \( N \), there is a path \( P \) from \( \rho(N) \) to \( w \). If \( P \) goes through \( u \), \( P \) must also go through \( v \), as \( u \) is of outdegree 1, and then \( v \) is an ancestor of \( w \), contradicting to that \( N \) is acyclic. Hence, \( P \) does not pass \( u \).

For each leaf \( \ell \), if it is a descendant of \( u \), it it also a descendant of \( v \) and hence there exists a path \( P' \) from \( v \) to \( \ell \). \( P' \) does not pass through \( u \), as \( u \) is a parent of \( v \) and \( N \) is acyclic. Combining \( P \), the edge \( (w, v) \), and \( P' \), we obtain a path from \( \rho(N) \) to \( \ell \) that does not pass \( u \). This implies that for every leaf descendant \( \ell \) of \( u \), there exists a path from \( \rho(N) \) to \( \ell \) that does not pass \( u \). Thus, \( u \) is not stable, a contradiction.

(b) The proposition follows from the observation that removing an edge only eliminates some directed paths and does not add any new path from the root to a leaf. \( \square \)
Consider a stable network $N$. By Proposition 4.1, each reticulation node are incident to three tree nodes. Furthermore, each connected component $C$ of $N - R(N)$ is actually a subtree of $N$ in which edges are directed away from its root. Indeed, if $C$ contains two nodes $u$ and $v$ both of indegree 0, where indegree is defined over $N - R(N)$, the path between $u$ and $v$ (ignoring edge direction) must contain a node $x$ with indegree 2, contradicting that $x$ is a tree node in $N$. As such, we call $C$ a tree component of $N$. A tree component is said to be a leaf component if it consists of only one leaf in $N$. Additionally, a tree component with at least two nodes is called a big tree component, to distinguish it from a leaf component. The stable network in Figure 2A has four big tree components and five leaf components.

By definition, any two tree components $C'$ and $C''$ of $N$ are disjoint. We say $C'$ is below $C''$, denoted by $C' \prec_b C''$, if there is a directed path from the root of $C''$ to the root of $C'$ in $N$. Since $N$ is acyclic, $\prec_b$ is anti-symmetric and transitive (that is, a partial order) in the set of the tree components of $N$. For instance, $C_4 \prec_b C_2 \prec_b C_1$ in the stable network in Figure 2A.

**Theorem 4.1 (Decomposition Theorem)** Assume that there are $m$ tree components $C_1, C_2, \ldots, C_m$ in a stable network $N$. Then, the following statements are true:

(i) $T(N) = \biguplus_{k=1}^m V(C_k)$, where $V(C_k)$ is the set of its nodes for each $C_k$.

(ii) For each reticulation node $r$, its child $c(r)$ is the root of some $C_i$, and each of its parents is a node in some $C_j$.

(iii) For each $C_k$,

(a) $|V(C_k)| = 1$ if and only if it is a leaf component, and

(b) If $C_k$ is a big tree component, either there is a network leaf in $C_k$ or there exists at least one reticulation node $r$ such that $p(r) \subseteq V(C_k)$, that is, the incoming edges of $r$ are both incident to $C_k$.

(iv) There exists at least one big tree component $C$ below which there is no other big tree components.
Proof. (i) This is obvious from the definition of tree components.

(ii) Let \( r \in \mathcal{R}(N) \). By the Reticulation Separability property, \( c(r) \) is a tree node, and by (i), \( c(r) \) is in a unique tree component. Since \( c(r) \) has indegree 0 in \( N - \mathcal{R}(N) \), \( c(r) \) must be the root of the tree component.

Similarly, again by the Reticulation Separability property, each parent of \( r \) is a tree node. By (i), it is in a unique tree component \( C_j \).

(iii) (a) Suppose on the contrary that for some \( j \), \( C_j = \{u\} \), where \( u \in \mathcal{T}(N)\setminus \mathcal{L}(N) \). Since \( u \) is the only tree node in \( C_j \), \( p(u) \in \mathcal{R}(N) \) and \( c(u) \subseteq \mathcal{R}(N) \) and thus we can prove that \( p(u) \) is not stable using the same argument as in the proof of the part (a) of Proposition 4.1. This contradicts the fact that \( N \) is a stable network.

(b) Assume that \( C \) is a big tree component of \( N \), that is, \( |V(C)| \geq 2 \). Note that if a reticulation is stable on a leaf, its unique child is also stable on the leaf. Let \( p(C) \) be the root of \( C \). It is stable on some network leaf \( \ell \). If \( \ell \) is a leaf of \( C \), we are done.

If \( \ell \) is not a leaf in \( C \), we let \( C_\ell \) be the tree component containing \( \ell \) and define \( \mathcal{X} = \{C_i \mid C_i \preceq_b C \} \). Since \( (\mathcal{X}, \preceq_b) \) is a partially ordered set (poset), it contains a maximal tree component \( M \). Let \( p(M) \) be the root of \( M \). Since \( M \preceq_b C \), by definition, \( p(C) \) is an ancestor of \( p(M) \), implying that \( p(C) \) is also an ancestor of the reticulation parent \( r \) of \( p(M) \), whereas \( \ell \) is a descendant of \( r \). Note that \( r \) has two parents and one of them must be in \( C \), as \( M \) is in \( \mathcal{X} \). If the other parent of \( r \) is not in \( C \), it is either in a big tree component not in \( \mathcal{X} \), implying that \( p(C) \) is not stable on \( \ell \), or in a tree component in \( \mathcal{X} \), contradicting that \( M \) is maximal in \( \mathcal{X} \). Therefore, \( p(r) \subseteq C \).

(iv) Consider the poset defined by the partial order \( \preceq_b \) on the set of big tree components. A minimal element of this poset has the desired property. \( \square \)

Time complexity for finding tree decomposition Since \( N \) is a DAG and has at most \( 9|\mathcal{L}(N)| \) nodes \([2]\), we can determine the tree components using the breadth-first search technique in \( O(|\mathcal{L}(N)|) \) time.

Additionally, a topological ordering of its nodes can also be found in \( O(|\mathcal{L}(N)|) \) time. Using such a topological ordering, we can derive a topological ordering for the big tree components. With this ordering, we can identify a lowest tree component described in Theorem 4.1(iv) in constant time.

5 Proof of Theorem 3.1

Let \( N \) be a stable network and \( T \) be a tree. A reticulation in \( N \) is inner if its parents are both in the same tree component of \( N \); it is called a cross reticulation otherwise. By Theorem 4.1, there exists a “lowest” big tree component \( C \) below which there are only (if any) leaf components. We assume that \( C \) contains \( k \) network leaves, say \( \ell_1, \ell_2, \ldots, \ell_k \), and there are:

- \( m \) inner reticulations \( r_1, r_2, \ldots, r_m \), and
- \( n \) cross reticulations \( r'_1, r'_2, \ldots, r'_n \)

below \( C \). By Theorem 4.1(iii)(b), \( k + m \geq 1 \) and \( k + m + n \geq 2 \). We use IR(\( C \)) and CR(\( C \)) to denote the sets of inner and cross reticulations below \( C \), respectively.

Define:

\[
L_C = \{ \ell_1, \ell_2, \ldots, \ell_k, c(r_1), c(r_2), \ldots, c(r_m) \},
\]

\( L_C \) is non-empty and the root of \( C \) is stable on each network leaf in \( L_C \). Select an \( \ell \in L_C \). Since \( \ell \in \mathcal{L}(T) \), there is a unique path \( P_T \) from the root \( \rho(T) \) to \( \ell \) in \( T \). Let:

\[
P_T : v_1, v_2, \ldots, v_t, v_{t+1},
\]
where $v_1 = \rho(T)$ and $v_{t+1} = \ell$. Then, $T - P_T$ is a union of $t$ disjoint subtrees $T_1, T_2, \ldots, T_t$, where $T_i$ is the subtree rooted at the child of $v_i$ other than $v_{i+1}$ for each $i = 1, 2, \ldots, t$ (see Figure 2A).

For the sake of convenience, we consider the single leaf $\ell$ as a subtree, written $T_{t+1}$. We now define $s_C$ as:

$$s_C = \min\{s \mid \mathcal{L}(T_s) \cap L_C \neq \phi\} \quad (3)$$

Since $\ell \in \mathcal{L}(T_{t+1}) \cap L_C$, $s_C$ is well defined. Recall that $C$ is a tree. Let $p(C)$ be the root of $C$.

**Proposition 5.1** The index $s_C$ can be computed in time $O(|\mathcal{L}(N)|)$.

**Proof.** Since $T$ is a binary tree with the same set of labeled leaves as the network $N$. $T$ has $2|\mathcal{L}(N)| - 1$ nodes and $2|\mathcal{L}(N)| - 2$ edges. For each $u \in \mathcal{V}(T)$, we define a flag variable $y_u$ to indicate whether the subtree below $u$ contains a network leaf in $L_C$ or not. We first traverse $T$ in the post-order:

- For a leaf $u \in \mathcal{L}(T)$, $y_u = 1$ if $u \in L_C$ and 0 otherwise.
- For an internal node $u$ with children $v$ and $w$, $y_u = 1$ if either $y_v$ or $y_w$ is 1 and 0 otherwise.

Then, we compute $s_C$ as

$$s_C = \min\{i \mid y_{p(T_i)} = 1\}.$$  

Clearly, this algorithm correctly computes $s_C$ in time $O(|\mathcal{L}(N)|)$. $\square$

**Proposition 5.2** If $N$ displays $T$, then $\mathcal{D}_T(v_{s_C})$ is displayed in $\mathcal{D}_N(p(C))$. Equivalently, if $\mathcal{D}_T(v_{s_C})$ is not displayed in $\mathcal{D}_N(p(C))$, $T$ is not displayed in $N$.

**Proof.** When $s_C = t + 1$, then the claim is trivial, as $p(C)$ is stable on $\ell$ and thus every path from the network root to $\ell$ must go through $p(C)$.

When $s_C < t + 1$, there is a network leaf $\ell'$ in $\mathcal{L}(T_{s_C}) \cap L_C$ such that $\ell' \neq \ell$. If $N$ displays $T$, $T$ has a subdivision $T'$ in $N$. Recall that $p(C)$ is stable on both $\ell$ and $\ell'$. The paths from $\rho(T')$ to $\ell$ and to $\ell'$ in $T'$ must both pass through $p(C)$. Since $T'$ is a tree, the lowest common ancestor $a(\ell, \ell')$ of $\ell$ and $\ell'$ is a descendant of $p(C)$ in $T'$ and it is the node in $T'$ that corresponds with $v_{s_C}$. Therefore, the subnetwork of $T'$ below $a(\ell, \ell')$ is a subdivision of $\mathcal{D}_T(v_{s_C})$, that is, $\mathcal{D}_N(p(C))$ displays $\mathcal{D}_T(v_{s_C})$. $\square$

If $N$ displays $T$, then $C$ may display more than $\mathcal{D}_T(v_{s_C})$. In other words, it may display a subtree $\mathcal{D}_T(v_j)$ for some $j < s_C$. To handle such a scenario, we define

$$d_C = \min \{j \mid \mathcal{D}_T(v_j) \text{ is displayed in } \mathcal{D}_N(p(C))\} \quad (4)$$

**Proposition 5.3** If $N$ displays $T$, there must be a subdivision $T''$ of $T$ in $N$ such that the node (in $T''$) corresponding with $v_{d_C}$ is in $C$.

**Proof.** Assume that $N$ displays $T$ via a subdivision $T'$ of $T$. Let $u$ be the node in $T'$ that corresponds with $v_{d_C}$. Since $p(C)$ is stable on the leaf $\ell$, $p(C)$ is in the unique path $P$ from the root to $\ell$ in $T'$. If $u$ is $p(C)$ or below it, it is done.

Assume that $u$ is neither $p(C)$ nor below $p(C)$ in $T'$. Since $\ell$ is a network leaf below $v_{d_C}$ in $T$, $\ell$ is also below $u$ in $T'$. Hence, $P$ must pass $u$ as well. Since $u$ is not below $p(C)$, $p(C)$ is below $u$ in $P$.

On the other hand, by assumption, $\mathcal{D}_T(v_{d_C})$ is displayed in $\mathcal{D}_N(p(C))$. It has a subdivision $T^*$ in $\mathcal{D}_N(p(C))$. Let $v_{d_C}$ correspond with $u'$ in $T^*$. It is not hard to see that $u'$ is in the path from $p(C)$ to $\ell$ in $T^*$ (and hence in $C$).
Let \( P' \) be the subpath from \( u \) to \( p(C) \) of \( P \) and \( P'' \) be the path from \( p(C) \) to \( u' \) in \( C \). Since the subtree below \( u' \) in \( T^* \) and the subtree below \( u \) in \( T' \) has the same set of labeled leaves as the subtree below \( v_{d_C} \) in \( T \),

\[
T' = D_T(u) + P' + P'' + D_{T^*}(u')
\]

is also a subdivision of \( T \) in \( N \), in which \( v_{d_C} \) is mapped to \( u' \) in \( C \). Here, \( G + H \) is the graph with the same node set as \( G \) and the edge set being the union of \( E(G) \) and \( E(H) \) for graphs \( G \) and \( H \) such that \( V(H) \subseteq V(G) \).

To compute \( d_C \) defined in Eqn. (4), we create a tree \( T_C \) from \( C \) by attaching two identical copies of the network leaf below each \( r \in IR(C) \) to its parents in \( p(r) \) in \( C \) and one copy of the network leaf below \( r \in CR(C) \) to the parent in \( p(r) \cap V(C) \). That is, \( T_C \) has the node set:

\[
\mathcal{V}(T_C) = \mathcal{V}(C) \cap \{x_r,y_r \mid r \in IR(C)\} \cup \{z_r \mid r \in CR(C)\},
\]

and the edge set

\[
\mathcal{E}(T_C) = \mathcal{E}(C) \cup \{(u_r,x_r),(v_r,y_r) \mid p(r) = \{u_r,v_r\}, r \in IR(C)\} \\
\quad \cup \{(u_r,z_r) \mid \{u_r\} = p(r) \cap \mathcal{V}(C), r \in CR(C)\},
\]

where \( x_r, y_r, \) and \( z_r \) are new leaves with the same label as \( c(r) \) for each \( r \in IR(C) \) or in \( CR(C) \).

**Proposition 5.4** There is a dynamic programming algorithm that takes \( T_C \) and \( T \) as input and outputs \( d_C \) defined in Eqn. (4) in \( O(|\mathcal{V}(C)|^2|\mathcal{V}(T)|) \).

**Proof.** It is given in Appendix.

---

**The TCP Algorithm**

**Input:** A stable network \( N \) and a tree \( T \), which are binary.

1. Decompose \( N \) into tree components and order them by \( \prec_b \) defined in Sec. 4
2. \( N' \leftarrow N \) and \( T' \leftarrow T \)
3. **Repeat** unless \( (N' \) becomes a single node \( ) \{
4. 3.1. Select a lowest big tree component \( C \);
5. 3.2. Compute \( L_C \) in Eqn. (1) and select \( \ell \in L_C \);
6. 3.3. Compute the path \( P_T \) from the root to \( \ell \) in Eqn. (2);
7. 3.4. Determine the smallest index \( s_C \) defined by Eqn. (3);
8. 3.5. Determine the smallest index \( d_C \) defined by Eqn. (4);
9. 3.6. **If** \( (s_C > d_C) \), output “\( N \) does not display \( T' \)”; **else**
10. \( \quad \) For each \( r \in CR(C) \) \{
11. \( \quad \) if \( (c(r) \notin D_T(v_{d_C})) \), delete \( (z,r) \) for \( z \in p(r) \cap \mathcal{V}(C) \);
12. \( \quad \) if \( (c(r) \in D_T(v_{d_C})) \), delete \( (z,r) \) for \( z \in p(r) \cap \mathcal{V}(C) \);
13. \( \quad \) \}
14. \( \quad \) Replace \( C \) (and \( D_T(v_{d_C}) \)) by a leaf \( \ell_C \) in \( N' \) (and \( T' \));
15. \( \quad \) Remove \( C \) from the list of tree components;
16. \( \quad \) Update \( CR(C') \) for affected big tree components \( C' \);
17. \( \quad \) \}

We now examine the time complexity of the TCP Algorithm. Note that \( N \) has at most \( 10|\mathcal{L}(N)| \) nodes and the input tree contains \( 2|\mathcal{L}(N)| - 1 \) nodes. Step 1 can be done in \( O(|\mathcal{L}(N)|) \)
time if the breadth-first search is used.

Step 3 is a while-loop. During each execution of this step, the current network is obtained from the previous network by replacing the big tree component examined in the last execution with a new leaf node. Because of this, the modification done in the last two lines in Step 3.6 makes the tree decomposition of the current network available before the current execution. Hence, Step 3.1 takes a constant time. The time spent in Step 3.2 for each execution is linear in the sum of the numbers of the leaves in \( C \) and of the inner reticulations below \( C \). Hence, the total time spent in Step 3.2 is \( O(|L(N)|) \), as each reticulation is examined at most twice.

By Proposition 5.1, the total time spent in Step 3.4 is \( O(|\mathcal{L}(N)|^2) \).

By Proposition 5.4, the total time spent in Step 3.5 is \( \sum_i O(|\mathcal{V}(C_i)|^2|\mathcal{L}(T)|) \), which is \( O(|\mathcal{L}(N)|^3) \).

The time spent in Step 3.6 for each execution is \( O(|\mathcal{V}(C)|) \). Hence, the total time spent in Step 3.6 is \( O(|\mathcal{L}(N)|) \).

In summary, the TCP Algorithm takes time \( O(|\mathcal{L}(N)|^3) \). Hence, we have obtained a cubic time algorithm for the TCP.

6 Proof of Theorem 3.2

As another application of the Decomposition Theorem, we shall design a time-optimal algorithm for the CCP. Given a network \( N \) and a subset \( B \subseteq \mathcal{L}(N) \), the objective is to determine whether or not \( B \) is a cluster of some node in a tree displayed by \( N \).

Assume \( N \) has \( t \) big tree components \( C_1, C_2, \ldots, C_t \). Consider a lowest big tree component \( C \). We use the same notation as in the last section: \( L_C \) is defined in Eqn. (1); \( \text{IR}(C) \) and \( \text{CR}(C) \) denote the set of inner and cross reticulations below \( C \), respectively. We also set \( B = \mathcal{L}(N) \setminus B \).

When \( L_C \cap B \neq \emptyset \) and \( L_C \cap \bar{B} \neq \emptyset \), there are two network leaves \( \ell_1, \ell_2 \) in \( L_C \) such that \( \ell_1 \in B \), but \( \ell_2 \in B \). Recall that \( p(C) \) denotes the root of \( C \). If \( B \) is the cluster of a tree node \( z \) in a subtree \( T' \) of \( N \), \( z \) is in the path from the root \( p(T') \) to \( \ell_1 \). Assume that \( z \) is above \( p(C) \), no matter which incoming edge is removed for each \( r \in \text{IR}(C), \ell_2 \) is also in \( B \), as \( p(C) \) is stable on \( \ell_2 \), a contradiction. This implies that \( z \) is in \( C \). Therefore, we conclude that \( B \) must be the cluster of a node (in \( T' \) ) that is found in \( C \).

When \( B = L_C \), \( B \) is the cluster of \( p(C) \) after deleting the incoming edge whose tail is in \( C \) for each \( r \in \text{CR}(C) \). In this case, we conclude that \( B \) is a cluster contained in \( N \).

In general, when \( L_C \cap \bar{B} = \emptyset \) (that is, \( L_C \subseteq B \)), we define

\[
X = \{ r \in \text{CR}(C) \mid c(r) \notin B \}. \tag{7}
\]

Construct a tree \( T'_C \) from \( \mathcal{D}_N(p(C)) \) by deleting:

- a selected incoming edge for each \( r \in \text{IR}(C) \),
- the incoming edge whose tail is in \( C \) for each \( r \in X \), and
- the incoming edge whose tail is not in \( C \) for each \( r \in \text{CR}(C) \setminus X \).

We then define

\[
\hat{B} = L_C \cup \{ c(r) \mid r \in \text{CR}(C) \setminus X \}. \tag{8}
\]

It is not hard to see that \( \hat{B} \) is the cluster of the root of \( T'_C \) such that \( \hat{B} = B \cap \mathcal{L}(\mathcal{D}_N(p(C))) \subseteq B \). Hence, if \( B = \hat{B} \), then \( B \) is a cluster contained in \( \mathcal{D}_N(p(C)) \). If \( B \neq \hat{B} \), we reconstruct \( N' \) from \( N \) by:

- removing edges in \( \{(u, r) \mid r \in X, u \in C \} \),
- removing edges in \( \{(u, r) \mid r \in \text{CR}(C) \setminus X, u \notin C \} \), and
• replacing $D_N(p(C))$ by a new leaf $\ell_C$.

and set $B' = B \setminus \hat{B} \cup \{\ell_C\}$. We have the following fact, whose proof can be found in Appendix 2.

**Proposition 6.1** $B$ is contained in $N$ if and only if $B'$ is contained in $N'$ when $L_C \cap \hat{B} = \emptyset$.

**Proof.** Recall that $B' = (B \cup \{\ell_C\}) \setminus \hat{B}$. If $B'$ is contained in $N'$, then $B'$ is the cluster of a node $z$ in a tree displayed in $N'$. Note that during the construction of $N'$, $\ell_C$ actually replace a tree below $p(C))$ whose leaves are $\hat{B}$; so if we re-expand $\ell_C$ into this tree, the cluster of $z$ in $N$ becomes $(B \cup \{\ell_C\}) \setminus \hat{B} = B$, thus $B$ is contained in $N$.

Now assume $B$ is contained in $N$. $B$ is then the cluster of a node $z$ in a subtree $T'$. Let $E$ is the set of reticulation edges that are removed from $N$ to obtain $T'$. Since $B \neq \hat{B} = B \cap L(D_N(p(C)))$, there is a leaf $\ell'$ in $B$ that is not below $p(C)$. Note that $\ell'$ is below $z$ in $T'$, thus $z$ must be above $p(C)$ in $T'$. Note that any leaf $\ell''$ that is the child of a reticulation $r$ in $CR(C) \setminus X$ is a leaf in $B$, thus it must be a descendant of $z$ in $T'$. The node $r$ has two parents: one is in $C$, say $p_1$, while the other is not in $C$, say $p_2$. Either $(p_1, r)$ or $(p_2, r)$ is in $E$ exclusively. We can then construct a new subtree $T''$ of $N$, while keeping the cluster of $z$ unchanged.

Initially set $E' = E$, and for any $r \in CR(C) \setminus X$, if $(p_1, r) \in E$, let $E' := E' \cup \{(p_2, r)\} \setminus \{(p_1, r)\}$. Let $T'' = N - E' - V'$ ($V'$ is the set of vertices that are removed to eliminate dummy leaves from $N - E'$). It is easy to see that the cluster of $z$ in $T''$ is equal to $B$. It is also true that $\hat{B}$ is the cluster of $p(C)$ in $T''$. Thus, if we contract the subtree below $p(C)$ into a single leaf $\ell_C$, then the cluster of $z$ becomes $B \cup \{\ell_C\} \setminus \hat{B}$, which is $B'$, and thus $B'$ is contained in $N'$.

When $B \cap L_C = \emptyset$, we first check whether or not $B$ is contained in $D_N(p(C))$. If it is not, we use $X$ defined in Eqn. (7) to reconstruct $N'$ from $N$ by:

- removing edges in $\{(u, r) \mid r \in X, u \notin C\}$,
- removing edges in $\{(u, r) \mid r \in CR(C) \setminus X, u \in C\}$, and
- replacing $D_N(p(C))$ by a new leaf $\ell_C$.

Similar to the last case, we have the following fact.

**Proposition 6.2** $B$ is contained in $N$ if and only if $B$ is contained in $N'$ when $L_C \cap B = \emptyset$.

**Proof.** If $B$ is contained in $N'$, then similar as the previous proposition, it is easy to see that $B$ is also contained in $N$.

Conversely, assume that $B$ is contained in $N$. Let $B$ be the cluster of a node $z$ in a subtree $T'$ of $N$. Let $E$ be the set of reticulation edges removed from $N$ to obtain $E$. By assumption, $B$ is not contained in $D_N(p(C))$. Since $p(C)$ is stable on all leaves in $L_C$, $z$ cannot be an ancestor of $p(C)$.

Consider a reticulation $r \in X$. Since $c(r)$ is not a leaf in $B$, it must not be a descendant of $z$ in $T'$. Node $r$ has two parents: one is in $C$, say $p_1$, while the other is not in $C$, say $p_2$. Either $(p_1, r)$ or $(p_2, r)$ is in $E$ exclusively.

We define a new subtree $T''$ of $N$ in which the cluster of $z$ remains the same. Initially set $E' = E$, and for any $r \in X$, if $(p_1, r) \in E$, let $E' := (E' \cup \{(p_2, r)\}) \setminus \{(p_1, r)\}$. Define $T'' = N - E' - V'$, where $V'$ is the set of vertices removed to eliminate dummy leaves from $N - E'$. It is easy to see that the cluster of $z$ in $T''$ remains the same as the cluster of $z$ in $T$, which is equal to $B$. It is also true that the subtree below $p(C)$ in $T''$ is now the same as the one in $N'$ before being replaced by a single leaf $\ell_C$. So if we contract $D_{T''}(p(C))$ into a single leaf $\ell_C$, $T''$ is a subtree of $N'$, and hence $B$ is contained in $N'$. \qed
Taking all the above facts together, we are able to give a linear time algorithm for the CCP.

---

**The CCP Algorithm**

**Input:** A binary network $N$ and a subset $B \subseteq L(N)$;

1. Compute the big tree components: $C_1 \prec_b C_2 \prec_b \cdots \prec_b C_t$;

2. for $k = 1$ to $t$ do {
   2.1. Compute $L := L(C_k)$ defined in Eqn. (1);  
   2.2. $Y := \{B \text{ contained in } D_N(p(C_k))\}$;  
   2.3. if ($Y == 1$) output “Yes!” and exit;
   2.4. if ($Y == 0$) {
      $\bar{B} := L(N) \setminus B$;  
      if ($L \cap \bar{B} \neq \emptyset$ & $B \cap L \neq \emptyset$) output “No!” and exit;
      Compute $X$ defined in Eqn. (7);  
      if ($B \cap L == \emptyset$) {
         Remove edges in $\{(u, r) \mid r \in X, u \notin C_k\}$;  
         Remove edges in $\{(u, r) \mid r \in CR(C_k) \setminus X, u \in C_k\}$;
      }
      if ($L \cap \bar{B} == \emptyset$) {
         Remove edges in $\{(u, r) \mid r \in X, u \in C_k\}$;
         Remove edges in $\{(u, r) \mid r \in CR(C_k) \setminus X, u \notin C_k\}$;
         $B := (B \cup \{\ell_{C_k}\}) \setminus (L \cup \{c(r) \mid r \in CR(C_k) \setminus X\})$;
      }
      Replace $D_N(p(C_k))$ by a leaf $\ell_{C_k}$;  
      Remove $C_k$ from the list of big tree components;  
      Update CR($C'$) for affected big tree components $C'$;
   }
} /* for */

**Complexity analysis**

Step 1 takes $O(|V(N)|) = O(|L(N)|)$ time.

Step 2 executes $t$ times. Since the sum of the number of network leaves in $C_k$ and the numbers of inner and cross reticulations below $C_k$ is at most as many as three times the numbers of nodes in $C_k$. Step 2.1 takes $O(|V(C_k)|)$ for each execution. In Step 2.2, we can use a dynamic programming algorithm to compute $Y$ in time $O(|V(C_k)|)$. Step 2.3 takes constant time.

To implement Step 2.4 in linear time, we need to use an array $A$ to indicate whether a network leaf is in $B$ or in $\bar{B}$. $A$ can be established in $O(|L(N)|)$ time. After having $A$, each conditional clause in Step 2.4 can be determined in $|L|$ time, which is at most $O(|V(C_k)|)$. Since the numbers of inner and cross reticulation are at most as many as two times the number of nodes in $C_k$, each line in Step 2.4 takes at most $O(|V(C_k)|)$. Overall, each execution of step 2.4 takes $O(|V(C_k)|)$ time.

Taking all these together, we have that the total time taken by Step 2 is $\sum_{1 \leq k \leq t} O(|V(C_k)|) = O(|V(N)|) = O(|L(N)|)$. Thus, the algorithm takes linear time.

**7 Algorithms for Non-binary Stable networks**

We have presented fast TCP and CCP algorithms for binary stable networks. They can be easily modified into ones for binary trees and non-binary stable networks, in which each tree node has
indegree 1 and outdegree 2 or more, whereas each reticulation has outdegree 1 and indegree 2 or more. First, the decomposition theorem holds for such stable networks. Second, the concepts of inner and cross reticulation can be defined in the same manner. Third, the only modification we have to make is on $T_C$ used in Step 3.5. It is done as follows.

For each inner reticulation $r$ of indegree $d$ below the selected big tree component $C$, we will add $d$ leaves with the same label as the unique leaf below $r$ in $T_C$. Similarly, for a cross reticulation $r$ with $k$ parents in $C$, $T_C$ contains as many as $k$ leaves with the same label as the unique leaf below $r$.

It is not hard to see that the running times of the modified TCP and CCP algorithms are cubic and linear in the sum of all the node degrees in the input network, respectively. However, it is unclear whether this sum is bounded from above by a function linear in the number of leaves or not.

8 Conclusion

We have presented two efficient algorithms for the TCP and CCP for stable networks. Recall that a network is stable if every reticulation node separates the network root from some leaves. Our study raises some interesting problems for future research. For example, can the graph isomorphism problem be solved in polynomial time for two stable networks? Is whether or not two networks display the same set of binary trees determined in polynomial time? A solution to the latter is definitely valuable in phylogenetics.

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Appendix 1: Proof of Proposition 5.4

Recall that \( v_1 = \rho(T), v_2, \ldots, v_t \) is the sequence of nodes in the unique path from \( \rho(T) \) to leaf \( \ell \) in \( T \), \( \rho(C) \) is the root of the lowest big tree component \( C \) below which there is no big tree components, and our goal is to find the minimum index \( j \) such that \( D_N(p(C)) \) displays \( D_T(v_j) \). In order to do this, we convert \( C \) to a (possibly non-uniquely) leaf labeled tree \( T_C \) in the following manner:

- We replace an inner reticulation \( r \in \text{IR}(C) \) that is incident to a leaf \( \ell' \) by two leaves \( x_r \) and \( y_r \), each having the same label as \( \ell' \), attached separately to each parent of \( r \) in \( C \). Note that, for the purpose of realizing a display of \( D_T(v_j) \) in \( D_N(p_C) \), exactly one of these leaves will be kept and the other will be deleted. Such leaves will be referred to ambiguous leaves. We use \( A(C) \) to denote the set of ambiguous leaves in \( T_C \).

- For a cross reticulation \( r \in \text{CR}(C) \) attached to a leaf \( \ell' \), we replace \( r \) by a leaf \( z_r \) that has the same label as \( \ell' \). Similar to the case of ambiguous leaves, such a leaf may be kept or may be deleted. We call such a leaf an optional leaf. We use \( O(C) \) to denote the set of optional leaves in \( T_C \).

Since each node in \( C \) is of degree 3 in \( N \), the resulting tree \( T_C \) is a full binary tree has at most \( 2|V(C)| + 1 \) nodes and edges.

For our purpose, we shall present a dynamic programming to compute the following set \( S_u \) of nodes in \( T_C \):

\[
S_u = \{ x \in V(T) \mid D_{T_C}(u) \text{ displays } D_T(x) \text{ where } u \text{ is mapped to } v \}
\]

for each node \( u \) in \( T_c \). Here, that \( D_{T_C}(u) \text{ displays } D_T(x) \) means that there exists \( V_x \subset V(T_C) \) such that \( T_C - V_x \text{ is a subdivision of } D_T(v) \). Since \( T_C \) is a tree, \( V_x \) contains all the leaves in \( D_{T_C}(u) \) but not in \( D_T(x) \) and some ancestors of these leaves. We introduce a boolean variable \( f_{ux} \) to indicate whether or not \( D_{T_C}(u) \text{ displays } D_T(x) \) and a set variable \( M_{ux} \) to record these removed leaves if so. That is, \( M_{ux} = L_{T_C}(u) \setminus L_T(x) \) if \( f_{ux} = 1 \), where \( L_Y(z) \) denotes the set of leaves below \( z \) in the tree \( Y \).

When \( u \) is a leaf in \( T_C \), we consider whether \( x \) is a leaf or not to compute \( f_{ux} \).

If \( x \) is a leaf with the same label as \( u \), then \( D_{T_C}(u) \text{ displays } D_T(x) \) and thus we set \( f_{ux} = 1 \), \( M_{ux} = \emptyset \). If \( x \) is a leaf with a label different from that of \( u \), \( D_{T_C}(u) \) does not display \( D_T(x) \). Therefore, \( f_{ux} = 0 \) and \( M_{ux} \) is undefined.

If \( x \) is not a leaf, it is trivial that \( D_{T_C}(u) \) does not display \( D_T(x) \). Hence, we set \( f_{ux} = 0 \) and \( M_{ux} \) is undefined.

When \( u \) is an internal node with children \( v \) and \( w \) in \( T_C \), we consider similar cases. If \( x \) is a leaf, \( D_T(x) \) may be displayed in \( D_{T_C}(u) \) if it is displayed below a child of \( u \). If \( D_T(x) \) is displayed below \( v \), it is also displayed at \( u \) only if every ambiguous leaf in \( M_{uv} \) is not below \( w \) and all the leaves below \( w \) are each ambiguous or optional. If \( D_T(x) \) is displayed below neither \( v \) nor \( w \), it is not displayed at \( u \). The remaining cases can be found in Table I.

Our dynamic programming algorithm recursively computes \( f_{ux} \) for each \( u \) and \( x \) by traversing both \( T_C \) and \( T \) in the post-order. For each \( u \) and \( x \), we compute \( f_{ux} \) using the formulas listed in Table I. Note that \( M_{ux} \) is a subset of \( A(C) \cup O(C) \) and hence has at most \( 2|V(C)| \) elements. Therefore, each recursive step takes \( O(|V(T)|) \) time. Because of this, the total time taken by the algorithm is \( O(|V(T)| \cdot |V(C)|) \).

After we know the values of \( f_{ux} \) for every \( u \) in \( T_C \) and every \( x \) in \( T_C \), we can compute the minimum index \( d_C \) such that \( D_T(v_j) \) is displayed in \( D_N(p(C)) \) as \( d_C = \min_{1 \leq j \leq t+1} \min_{u \in V(T_C)} \{ j \mid f_{uj} = 1 \} \).
Table 1: The recursive formulas on $f_{ux}$ for different cases when $u$ is an internal node in $T_C$.

| $x \in \mathcal{L}(T)$? | $f_{vx}$ | $f_{wx}$ | $u$ |
|--------------------------|----------|----------|-----|
| Yes                      | 1 1      |                      |     |
|                          | 1 0      |                      |     |
|                          | 0 1      |                      |     |
|                          | 0 0      |                      |     |
| No.                      | 1 1      | same as the leaf case |     |
| $p(x) = \{y,z\}$        | 1 0      | same as the leaf case |     |
|                          | 0 1      | same as the leaf case |     |
|                          | 0 0      |                      |     |

| $x \in \mathcal{L}(T)$? | $f_{vx}$ | $f_{wx}$ | $u$ |
|--------------------------|----------|----------|-----|
| Yes                      | 1 1      | \{ \begin{align*}
    f_{ux} &= 1 & \text{if } M_{vx} \cap M_{wx} \cap A(T_C) = \emptyset, \\
    f_{ux} &= 0 & \text{otherwise.}
\end{align*} \} & M_{ux} = M_{vx} \cup M_{wx} \cup \{\ell_x\} |
|                          | 1 0      | \{ \begin{align*}
    f_{ux} &= 1 & \text{if } L(D_{TC}(w)) \subseteq A(T_C) \cup O(T_C), \\
    & \quad & \text{& } M_{vx} \cap L(D_{TC}(w)) \cap A(T_C) = \emptyset, \\
    f_{ux} &= 0 & \text{otherwise,}
\end{align*} \} & M_{ux} = M_{vx} \cup L(D_{TC}(w)) |
|                          | 0 1      | \{ \begin{align*}
    f_{ux} &= 1 & \text{if } L(D_{TC}(v)) \subseteq A(T_C) \cup O(T_C), \\
    & \quad & \text{& } M_{wx} \cap L(D_{TC}(v)) \cap A(T_C) = \emptyset, \\
    f_{ux} &= 0 & \text{otherwise,}
\end{align*} \} & M_{ux} = M_{wx} \cup L(D_{TC}(v)) |
|                          | 0 0      | $f_{ux} = 0$ |   |

| $x \in \mathcal{L}(T)$? | $f_{vx}$ | $f_{wx}$ | $u$ |
|--------------------------|----------|----------|-----|
| No.                      | 1 1      | same as the leaf case |
| $p(x) = \{y,z\}$        | 1 0      | same as the leaf case |
|                          | 0 1      | same as the leaf case |
|                          | 0 0      | \{ \begin{align*}
    f_{ux} &= 1 & \text{if } f_{vy} = 1 \& f_{wz} = 1 \\
    & \quad & \text{& } M_{vy} \cap M_{wz} \cap A(T_C) = \emptyset, \\
    f_{ux} &= 1 & \text{if } f_{vz} = 1 \& f_{wy} = 1 \\
    & \quad & \text{& } M_{vz} \cap M_{wy} \cap A(T_C) = \emptyset, \\
    f_{ux} &= 0 & \text{otherwise.}
\end{align*} \} & M_{ux} = \begin{cases}
    M_{vy} \cup M_{wz} & \text{if } f_{vy} = 1 \& f_{wz} = 1, \\
    M_{vz} \cup M_{wy} & \text{if } f_{vz} = 1 \& f_{wy} = 1.
\end{cases} |