Locally Connected HL Compacta

Kenneth Kunen†‡

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Abstract

It is consistent with MA + ¬CH that there is a locally connected hereditarily Lindelöf compact space which is not metrizable.

1 Introduction

All spaces discussed in this paper are assumed to be Hausdorff. A question attributed in 1982 by Nyikos [8] to M. E. Rudin asks whether MA + ¬CH implies that every locally connected hereditarily Lindelöf (HL) compact space is metrizable (equivalently, second countable); see Gruenhage [5] for further discussion. Filippov [4] had constructed such a space in 1969 under CH, and his space is also hereditarily separable (HS). Since Filippov used a Luzin set in his construction, and MA + ¬CH implies that there are no Luzin sets, it might have been hoped that MA + ¬CH refutes the existence of such a space, but that turns out to be false; we shall show in Section 3:

Theorem 1.1 It is consistent with MA + 2^{ℵ₀} = ℵ₂ that there is a non-metrizable locally connected compactum which is both HS and HL.

Our proof shows in ZFC that the Filippov construction succeeds provided that there is a weakly Luzin set; details are in Section 2. Weakly Luzin sets are related to entangled sets, and our proof of Theorem 1.1 shows that weakly Luzin sets are consistent with MA + 2^{ℵ₀} = ℵ₂. We can show that PFA refutes spaces which are “like” the Filippov space (see Section 4), but we do not know whether PFA refutes all non-metrizable locally connected HL compacta.

The Filippov space may be viewed as a connected version of the double arrow space $D$, which was described in 1929 by Alexandroff and Urysohn [2]. This is a ZFC

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†University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu

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example of a non-metrizable compactum which is both HS and HL, but it is totally disconnected. The cone over $D$ yields a connected example, but this is not locally connected.

$D$ is constructed from $[0,1]$ by replacing the points of $(0,1)$ by neighboring pairs of points. To construct the Filippov space, start with $[0,1]^2$, choose a set $E \subseteq (0,1)^2$, and replace the points of $E$ by circles, obtaining a space $\Phi_E$. This $\Phi_E$ is compact and locally connected. $\Phi_E$ is metrizable iff $E$ is countable. Furthermore, if $E$ is a Luzin set, then, as Filippov showed, $\Phi_E$ is HL, and a similar proof shows that $\Phi_E$ is HS as well.

Actually, by Juhász [7] and Szentmiklóssy [9], HS and HL are equivalent for compacta under MA$(\aleph_1)$, but that result is not needed here. We shall show in ZFC (Theorem 2.5) that $\Phi_E$ is HS iff $\Phi_E$ is HL iff $E$ is weakly Luzin.

## 2 Weakly Luzin Sets

We begin by describing Filippov’s example [4]. We start with $[0,1]^n$ (where $1 \leq n < \omega$), rather than $[0,1]^2$, to show that the construction does not depend on accidental features of two-dimensional geometry. As usual, $S^{n-1} \subseteq \mathbb{R}^n$ denotes the unit sphere, and $\|x\|$ denotes the length of $x \in \mathbb{R}^n$, using the standard Pythagorean metric. Given $E \subseteq (0,1)^n$, we shall obtain the space $\Phi_E$ by replacing all points in $E$ by $((n-1))$-spheres and leaving the points in $[0,1]^n \setminus E$ alone.

**Definition 2.1** $\rho : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the perpendicular retraction: $\rho(x) = x/\|x\|$.

So, $\rho(y - x)$ may be viewed as the direction from $x$ to $y$.

**Definition 2.2** Fix $E \subseteq (0,1)^n$ and let $E' = [0,1]^n \setminus E$. The Filippov space $\Phi_E$, as a set, is $(E \times S^{n-1}) \cup E'$. Define $\pi = \pi_E : \Phi_E \to [0,1]^n$ so that $\pi(x,w) = x$ for $(x,w) \in E \times S^{n-1}$, and $\pi(x) = x$ for $x \in E'$. For $\varepsilon > 0$, define, for $x \in E'$:

$$B(x,\varepsilon) = \{p \in \Phi_E : \|\pi(p) - x\| < \varepsilon\},$$

and define, for $x \in E$ and $W$ an open subset of $S^{n-1}$:

$$B(x,W,\varepsilon) = \{x\} \times W \cup \{p \in \Phi_E : 0 < \|\pi(p) - x\| < \varepsilon \land \rho(\pi(p) - x) \in W\}.$$

Give $\Phi_E$ the topology which has all the sets $B(x,\varepsilon)$ and $B(x,W,\varepsilon)$ as a base.

**Lemma 2.3** For each $E \subseteq (0,1)^n$: $\Phi_E$ is compact and first countable. $\pi_E$ is a continuous irreducible map from $\Phi_E$ onto $[0,1]^n$. $\Phi_E$ is metrizable iff $E$ is countable. If $n \geq 2$, then $\Phi_E$ is connected and locally connected, and $\pi_E$ is monotone.

The proof of this last sentence uses the connectedness of $S^{n-1}$. When $n = 1$, $S^0 = \{\pm 1\}$, and $\Phi_E$ is just the double arrow space obtained by doubling the points of $E$, so $\Phi_E$ is always HS and HL. When $n > 1$, the argument of Filippov shows that $\Phi_E$ is HL if $E$ is a Luzin set, but actually something weaker than Luzin suffices:
Every subset of a skinny set is skinny, and $T$ is skinny iff $\overline{T}$ is skinny. Each skinny set is nowhere dense, so every Luzin set is weakly Luzin. When $n = 1$, $T$ is skinny iff $|T| \leq 1$, every uncountable set is weakly Luzin, and the proof of the following theorem reduces to the usual proof that the double arrow space is HS and HL.

When $n > 1$: Under CH, it is easy to construct a weakly Luzin set which is not Luzin. PFA implies that there are no weakly Luzin sets. We shall show in Section 3 that a weakly Luzin set is consistent with MA + $\mathfrak{c} = \aleph_2$. Clearly, if there is a weakly Luzin set in $\mathbb{R}^n$, then there is one in $(0, 1)^n$.

**Theorem 2.5** For $n \geq 1$ and uncountable $E \subseteq (0, 1)^n$, the following are equivalent:

1. $E$ is weakly Luzin.
2. $\Phi_E$ is HS.
3. $\Phi_E$ is HL.
4. $\Phi_E$ has no uncountable discrete subsets.

**Proof.** For $(4) \rightarrow (1)$: If $E$ is not weakly Luzin, fix an uncountable skinny $T \subseteq E$. Let $W = S^{n-1}\setminus \overline{\rho(T^*)}$, and fix $w \in W$. Then $\{(x, w) : x \in T\} \subseteq \Phi_E$ is discrete.

Since $(2) \rightarrow (4)$ and $(3) \rightarrow (4)$ are obvious, it is sufficient to prove $(1) \rightarrow (2)$ and $(1) \rightarrow (3)$. So, assume $(1)$, and let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be an $\omega_1$–sequence of distinct points from $\Phi_E$; we show that it can be neither left separated nor right separated. To do this, fix an open neighborhood $N_\alpha$ of $p_\alpha$ for each $\alpha$; we find $\alpha < \beta < \gamma$ such that $p_\beta \in N_\alpha$ and $p_\beta \in N_\gamma$. This is trivial if $\aleph_1$ of the $\pi(p_\alpha)$ lie in $E'$, or if $\aleph_1$ of the $\pi(p_\alpha)$ are the same point of $E$. So, thinning the sequence (discarding some points), and shrinking the neighborhoods (replacing them by smaller ones), we may assume that each $p_\alpha = (x_\alpha, w_\alpha) \in E \times S^{n-1}$ and that $N_\alpha = B(x_\alpha, W, \varepsilon)$, where the $x_\alpha$ are distinct points in $E$, $W$ is open in $S^{n-1}$, and each $w_\alpha \in W$. Let $T = \{x_\alpha : \alpha < \omega_1\}$. Thinning further, we may assume that $\text{diam}(T) < \varepsilon$, so that $p_\beta \in N_\alpha$ iff $\rho(x_\beta - x_\alpha) \in W$. Thinning again, we may assume that every point of $T$ is a condensation point of $T$. Since $E$ is weakly Luzin, $T$ cannot be skinny, so $\rho(T^*)$ is dense in $S^{n-1}$, so fix $\xi \neq \eta$ such that $\rho(x_\eta - x_\xi) \in W$. There are then open $U \ni x_\xi$ and $V \ni x_\eta$ such that $\rho(z - y) \in W$ for all $y \in U$ and $z \in V$. Since $|U \cap T| = |V \cap T| = \aleph_1$, we may fix $\alpha < \beta < \gamma$ with $x_\alpha, x_\gamma \in U$ and $x_\beta \in V$; then $\rho(x_\beta - x_\alpha) \in W$ and $\rho(x_\beta - x_\gamma) \in W$, so $p_\beta \in N_\alpha$ and $p_\beta \in N_\gamma$. \hfill $\blacksquare$

Entangled subsets of $\mathbb{R}$ were discussed by Avraham and Shelah [3] (see also [1]). The weakly Luzin sets and the entangled sets have a common generalization:
Definition 2.6 For $1 \leq n < \omega$ and $1 \leq k < \omega$:

1. If $E \subseteq \mathbb{R}^n$, then $\bar{E} \subseteq (\mathbb{R}^n)^k$ is derived from $E$ iff $\bar{E} \subseteq E^k$ and whenever $\bar{x} = (x_0, \ldots, x_{k-1}) \in \bar{E}$ and $\bar{y} = (y_0, \ldots, y_{k-1}) \in \bar{E}$: $x_i \neq y_j$ unless $i = j$ and $\bar{x} = \bar{y}$.

2. $E$ is $(n,k)$-entangled iff $E \subseteq \mathbb{R}^n$ is uncountable and whenever $\bar{E} \subseteq (\mathbb{R}^n)^k$ is uncountable and derived from $E$, and, for $i < k$, $W_i$ is open in $S_n^{n-1}$ with $W_i \neq \emptyset$: there exist $\bar{x}, \bar{y} \in \bar{E}$ with $\bar{x} \neq \bar{y}$ and $\rho(x_i - y_i) \in W_i$ for all $i$.

Then “weakly Luzin” is equivalent to “$(n,1)$-entangled”, and “$k$-entangled” is equivalent to “$(1,k)$-entangled”. $E \subseteq \mathbb{R}$ is $(1,1)$-entangled iff $E$ is uncountable. If $E$ is $(n,k)$-entangled and $\bar{E}$ and the $W_i$ are as in (2), then there are actually uncountable disjoint $X, Y \subseteq \bar{E}$ such that $\forall i \rho(x_i - y_i) \in W_i$ whenever $\bar{x} \in X$ and $\bar{y} \in Y$. In (2), when $k = 1$, WLOG we may assume that $W_0 = -W_0$.

3 Preserving Failures of SOCA

The Semi Open Coloring Axiom (SOCA) is a well-known consequence of the PFA; see Abraham, Rubin, and Shelah [1]. We shall show that certain classes of failures of SOCA can be preserved in ccc extensions satisfying $\text{MA} + 2^{\aleph_0} = \aleph_2$. This is patterned after the proof (see [1, 3]) that an entangled set is consistent with $\text{MA} + 2^{\aleph_0} = \aleph_2$.

Definition 3.1 For any set $E$: Let $E^\dagger = (E \times E) \setminus \{(x, x) : x \in E\}$. Fix $W \subseteq E^\dagger$ with $W = W^{-1}$. Then $T \subseteq E$ is $W$–free iff $T^\dagger \cap W = \emptyset$ and $T$ is $W$–connected iff $T^\dagger \subseteq W$.

Definition 3.2 $(E,W)$ is good iff $E$ is an uncountable separable metric space, $W = W^{-1}$ is an open subset of $E^3$, and no uncountable subset of $E$ is $W$–free.

Then, the SOCA is the assertion that whenever $(E,W)$ is good, there is an uncountable $W$–connected set. An uncountable $E \subseteq \mathbb{R}^n$ is weakly Luzin iff $(E,W)$ is good for all $W$ of the form $\{(x, y) \in E^\dagger : \rho(x - y) \in A\}$, where $A \subseteq S_n^{n-1}$ is open and $A = -A \neq \emptyset$. We shall prove:

Theorem 3.3 Assume that in the ground model $V$, $\text{CH} + 2^{\aleph_1} = \aleph_2$ holds and $E$ is a separable metric space. Then there is a ccc extension $V[G]$ satisfying $\text{MA} + 2^{\aleph_0} = \aleph_2$ such that for all $W \in V$, if $(E,W)$ is good in $V$ then $(E,W)$ is good in $V[G]$.

A good $(E,W)$ does not by itself contradict SOCA, since there may be an uncountable subset of $E$ which is $W$–connected. But, if $(E,U)$ and $(E,W)$ are both good and $U \cap W = \emptyset$, then SOCA is contradicted, since any $W$–connected set is $U$–free. Such $E,U,W$ are provided by a weakly Luzin $E \subseteq \mathbb{R}^n$ (for $n \geq 2$). The following combinatorial lemma will be used in the proof of Theorem 3.3.
Lemma 3.4 Assume the following:

1. CH holds.
2. \( m \in \omega \); and \((E, W_i)\) is good for each \( i \leq m \).
3. \( \theta \) is a suitably large regular cardinal and \((M_\xi : \xi < \omega_1)\) is a continuous chain of countable elementary submodels of \( H(\theta) \), with \( E \in M_0 \) and each \( M_\xi \in M_{\xi+1} \).
4. For \( x \in \bigcup_{\xi} M_\xi \setminus M_0 \): \( \text{ht}(x) \) is the \( \xi \) such that \( x \in M_{\xi+1} \setminus M_\xi \).
5. \( x_\alpha^i \in E \setminus M_0 \) for \( \alpha < \omega_1 \) and \( i \leq m \).
6. \( \text{ht}(x_\alpha^i) \neq \text{ht}(x_\beta^j) \) unless \( \alpha = \beta \) and \( i = j \).

Then there are \( \alpha \neq \beta \) such that \((x_\alpha^i, x_\beta^j) \in W_i \) for all \( i \).

Proof. Induct on \( m \). When \( m = 0 \), this is immediate from the fact that \((E, W_0)\) is good. Now, assume the lemma for \( m-1 \), and we prove it for \( m \). Let \( \bar{x}_\alpha = \langle x_\alpha^0, \ldots, x_\alpha^m \rangle \in E^{m+1} \). Let \( \xi(\alpha, i) = \text{ht}(x_\alpha^i) \). Thinning the \( \omega_1 \)-sequence and rearranging each \( \bar{x}_\alpha \) if necessary, we may assume that \( \xi(\alpha, 0) < \xi(\alpha, 1) < \cdots < \xi(\alpha, m) \) and that \( \alpha < \beta \rightarrow \xi(\alpha, m) < \xi(\beta, 0) \). Let \( F = \text{cl}\{\bar{x}_\alpha : \alpha < \omega_1\} \subseteq E^{m+1} \), and fix \( \mu < \omega_1 \) such that \( F \in M_\mu \); there is such a \( \mu \) by CH.

For \( \alpha \geq \mu \): Let \( K_\alpha = \{ z \in E : \langle x_\alpha^0, \ldots, x_\alpha^m, z \rangle \in F \} \). \( K_\alpha \) is uncountable because \( K_\alpha \in M_{\xi(\alpha, m)} \) but \( K_\alpha \) contains the element \( x_\alpha^m \notin M_{\xi(\alpha, m)} \). Since \((E, W_m)\) is good, choose \( u_\alpha, v_\alpha \in K_\alpha \) with \( (u_\alpha, v_\alpha) \in W_m \), and then choose disjoint basic open sets \( U_m, V_m \subseteq E \) with \( u_\alpha \in U_m, v_\alpha \in V_m \), and \( (x, y) \in W_m \) for all \( x \in U_m \) and \( y \in V_m \).

Of course, \( U_m, V_m \) depend on \( \alpha \), but there are only \( \aleph_0 \) possible choices, so fix an uncountable set \( I \subseteq \{ \alpha : \mu \leq \alpha < \omega_1 \} \) such that the \( U_m, V_m \) are the same for \( \alpha \in I \). By the lemma for \( m-1 \), fix \( \gamma, \delta \in I \) such that \( \gamma \neq \delta \) and \((x_\gamma^i, x_\delta^j) \in W_i \) for all \( i < m \). Now choose disjoint open neighborhoods \( U_i \) of \( x_\gamma^i \) and \( V_i \) of \( x_\delta^i \) for \( i < m \) so that \( (x, y) \in W_i \) whenever \( x \in U_i \) and \( y \in V_i \). Note that the two open sets \( \prod_{i \leq m} U_i \) and \( \prod_{i \leq m} V_i \) both meet \( F \), since \( u_\gamma \in K_\gamma \) and \( v_\delta \in K_\delta \), so \( \langle x_\gamma^0, \ldots, x_\gamma^{m-1}, u_\gamma \rangle \in F \cap \prod_{i \leq m} U_i \) and \( \langle x_\delta^0, \ldots, x_\delta^{m-1}, v_\delta \rangle \in F \cap \prod_{i \leq m} V_i \). We may then choose \( \alpha, \beta \) such that \( \bar{x}_\alpha \in \prod_{i \leq m} U_i \) and \( \bar{x}_\beta \in \prod_{i \leq m} V_i \). But then \((x_\alpha^i, x_\beta^j) \in W_i \) for all \( i \).

Lemma 3.5 In the ground model \( V \): Assume CH, let \((E, W)\) be good, and let \( Q \) be any forcing poset such that \( q \Vdash \text{"}(E, W) \text{ is not good" for some } q \in Q \).

Then, in \( V \): there is a ccc poset \( P \) of size \( \kappa_1 \) such that \( Q \times P \) is not ccc and such that for all \( U \in V \): If \((E, U)\) is good then \( 1 \Vdash \text{"}(E, U) \text{ is good"} \).

Proof. Extending \( q \), we may assume that for some \( Q \)-name \( \dot{Z} \): \( q \Vdash \text{"}\dot{Z} \subseteq E \text{ is uncountable and } W\text{-free"} \). Fix \( \theta \) and the \( M_\xi \) so that (3)(4) of Lemma 3.4 hold.
Now, inductively choose \( q_\alpha \leq q \) and \( x^0_\alpha, x^1_\alpha \in E \backslash M_0 \) for \( \alpha < \omega_1 \) so that \( q_\alpha \models x^0_\alpha, x^1_\alpha \in \hat{Z} \) and such that \( \text{ht}(x^0_\alpha) < \text{ht}(x^1_\alpha) < \text{ht}(x^0_\beta) \) whenever \( \alpha < \beta < \omega_1 \). Let 
\[
\mathbb{P} = \{ p \in [\omega_1]^{<\omega} : \forall (\alpha, \beta) \in [p]^2 \, [(x^0_{\alpha}, x^0_{\beta}) \in W \text{ or } (x^1_{\alpha}, x^1_{\beta}) \in W] \}.
\]
\( \mathbb{P} \) is ordered by reverse inclusion, with \( 1 = \emptyset \). Each \( \{ \alpha \} \in \mathbb{P} \), and the pairs \( (q_\alpha, \{ \alpha \}) \in Q \times \mathbb{P} \) are incompatible, so \( Q \times \mathbb{P} \) is not ccc.

Now, suppose that we have some good \((E, U)\) and \( p \models \equiv \) \("(E, U) \text{ is not good}\"; we shall derive a contradiction. Extending \( p \), we may assume that for some \( \mathbb{P}-\text{name} \hat{T} : p \models \equiv \) \("\hat{T} \subseteq E \text{ is uncountable and } U-\text{free}\". Then, inductively choose \( p_\mu \leq p \) and \( t_\mu \in E \backslash M_0 \) for \( \mu < \omega_1 \) so that \( p_\mu \models t_\mu \in \hat{T} \) and such that \( \text{ht}(t_\mu) < \text{ht}(t_\nu) \) whenever \( \mu < \nu < \omega_1 \). Our contradiction will use the observation:

\[
\mu \neq \nu \rightarrow (t_\mu, t_\nu) \notin U \text{ or } p_\mu \perp p_\nu. \tag{\ast}
\]

Thinning the sequence and extending \( p \) if necessary, we may assume that the \( p_\mu \) form a \( \Delta \) system with root \( p \); so \( p_\mu = p \cup \{ \alpha(0, \mu), \ldots, \alpha(c, \mu) \} \), with \( \alpha(0, \mu) < \ldots < \alpha(c, \mu) \). We also assume that \( \max(p) < \alpha(0, 0) \) and \( \mu < \nu \rightarrow \alpha(c, \mu) < \alpha(0, \nu) \). Since \( p_\mu \in \mathbb{P} \),

\[
i \neq j \rightarrow (x^0_{\alpha(i, \mu)}, x^0_{\alpha(i, \mu)}) \in W \text{ or } (x^1_{\alpha(i, \mu)}, x^1_{\alpha(i, \mu)}) \in W
\]

for each \( \mu \). Let \( \bar{x}_\mu = (x^0_{\alpha(0, \mu)}, x^1_{\alpha(0, \mu)} \ldots x^0_{\alpha(c, \mu)}, x^1_{\alpha(c, \mu)}) \in E^{2(c+1)} \). Since \( W \) is open, we may thin again and assume that all \( \bar{x}_\mu \) are sufficiently close to some condensation point of \( \{ \bar{x}_\mu : \mu < \omega_1 \} \) so that for all \( \mu, \nu \):

\[
i \neq j \rightarrow (x^0_{\alpha(i, \mu)}, x^0_{\alpha(i, \mu)}) \in W \text{ or } (x^1_{\alpha(i, \mu)}, x^1_{\alpha(i, \mu)}) \in W.
\]

Thus, if \( p_\mu \perp p_\nu \) then the incompatibility must come from the same index \( i \), so that \( \ast \) becomes

\[
\mu \neq \nu \rightarrow (t_\mu, t_\nu) \notin U \text{ or } \exists i \leq c \, [(x^0_{\alpha(i, \mu)}, x^0_{\alpha(i, \mu)}) \notin W \text{ and } (x^1_{\alpha(i, \mu)}, x^1_{\alpha(i, \mu)}) \notin W].
\]

This comes close to contradicting Lemma 3.4. With an eye to satisfying hypothesis (6), we thin the sequence again and assume that \( \text{ht}(t_\mu) \neq \text{ht}(x^\ell_{\alpha(i, \nu)}) \) whenever \( \mu \neq \nu \).

It is still possible to have \( \text{ht}(t_\mu) = \text{ht}(x^\ell_{\alpha(i, \mu)}) \), but for each \( \mu \), \( \text{ht}(t_\mu) = \text{ht}(x^\ell_{\alpha(i, \mu)}) \) can hold for at most one pair \((\ell, i)\). Thinning once more, we can assume WLOG that this \( \ell \) is always 1, so that \( \text{ht}(t_\mu) \neq \text{ht}(x^0_{\alpha(i, \nu)}) \) for all \( \mu < \omega_1 \) and all \( i \leq c \). But now the \((c+2)\)-tuples \((t_\mu, x^0_{\alpha(0, \mu)}, \ldots x^0_{\alpha(c, \mu)}) \) (for \( \mu < \omega_1 \)) contradict Lemma 3.4 where \( W_0 = U \) and the other \( W_i = W \).

We also need to show that \( \mathbb{P} \) is ccc. If this fails, then choose the \( p_\mu \) to enumerate an antichain. Derive a contradiction as before, but replace \( \ast \) by the stronger fact \( \mu \neq \nu \rightarrow p_\mu \perp p_\nu \), and delete all mention of \( \hat{T} \) and the \( t_\mu \). ☐

We remark that a simplification of the above proof yields the standard proof that an instance of SOCA can be forced by a ccc poset. Forget about \( Q \) and just assume...
that \((E,W)\) is good. Choose the \(x_\alpha \in E \setminus M_0\) for \(\alpha < \omega_1\) so that \(\text{ht}(x_\alpha) < \text{ht}(x_\beta)\) whenever \(\alpha < \beta < \omega_1\). \(P\) is now \(\{p \in [\omega_1]^{<\omega} : \forall \{\alpha, \beta\} \in [p]^2 ([x_\alpha, x_\beta] \in W)\}\). Then some \(p \in P\) forces an uncountable \(W\)-connected set.

**Proof of Theorem 3.3.** In the ground model \(V\), we build a normal chain of ccc posets, \(\langle F_\alpha : \alpha \leq \omega_2 \rangle\), where \(\alpha < \beta \rightarrow F_\alpha \subseteq c F_\beta\) and we take unions at limits. So, our model will be the generic extension \(V[G]\) given by \(F_{\omega_2}\). \(|F_\alpha| \leq \aleph_1\) for all \(\alpha < \omega_2\), while \(|F_{\omega_2}| = \aleph_2\). Given \(F_\alpha\), we choose \(P_\alpha\), which is an \(F_\alpha\)-name forced by \(1\) to be a ccc poset of size \(\aleph_1\); then \(F_{\alpha+1} = F_\alpha \ast P_\alpha\).

The standard bookkeeping which is used to guarantee that \(V[G] \models \text{MA} + 2^{\aleph_0} = \aleph_2\) is modified slightly here, since we need to assume inductively that \(1 \models F_\alpha \models \text{"}(E,W)\) is good" for all \(W\) such that \((E,W)\) is good in \(V\). This is easily seen (similarly to Theorem 49 of [9]) to be preserved at limit \(\alpha\). For the successor stage, assume that we have \(F_\alpha\) and the standard bookkeeping says that we should use \(Q_\alpha\), which is an \(F_\alpha\)-name which is forced by \(1\) to be a ccc poset of size \(\aleph_1\). Roughly, we ensure that either MA holds for \(Q_\alpha\) or \(Q_\alpha\) ceases to be ccc. More formally, choose \(P_\alpha\) as follows:

Consider this from the point of view of the \(F_\alpha\)-extension \(V[G \cap F_\alpha]\). In this model, \(CH\) holds, and we have a ccc poset \(Q_\alpha\), and we must define another ccc poset \(P_\alpha\). We know (using our inductive assumption) that for all \(W \in V\), if \((E,W)\) good in \(V\) then it is still good. If for all such \(W\), \(1 \models Q_\alpha \models \text{"}(E,W)\) is good", then let \(P_\alpha = Q_\alpha\). If not, then fix \(W \in V\) with \((E,W)\) good in \(V\) such that \(q \models Q_\alpha \models \text{"}(E,W)\) is not good" for some \(q \in Q_\alpha\). Still working in \(V[G \cap F_\alpha]\), we apply Lemma 3.3 and let \(P\) be a ccc poset of size \(\aleph_1\) such that \(Q_\alpha \times P\) is not ccc and such that for all \(U \in V[G \cap F_\alpha]\) (and hence for all \(U \in V\)): If \((E,U)\) is good then \(1 \models P \models \text{"}(E,U)\) is good". Since \(Q_\alpha \times P\) is not ccc, we may fix \(p_0 \in P\) such that \(p_0 \models P \models \text{"}Q_\alpha\) is not ccc". We cannot claim that \(1 \models P \models \text{"}Q_\alpha\) is not ccc", so let \(P_\alpha = P_{p_0} = \{p \in P : p \leq p_0\}\). Then \(1_{P_\alpha} = p \models P_\alpha \models \text{"}Q_\alpha\) is not ccc" and all good \((E,U)\) from \(V\) remain good in the \(P_\alpha\) extension.

Now, in \(V\), let \(P_\alpha\) be the name for this \(P_\alpha\) as chosen above.

**Proof of Theorem 1.1.** In the ground model \(V\), assume that \(2^{\aleph_0} = \aleph_1\) and \(2^{\aleph_1} = \aleph_2\). By \(CH\), we may fix a (weakly) Luzin set \(E \subseteq \mathbb{R}^n\) (where \(n \geq 2\)). Now, apply Theorem 3.3.

### 4 Use of SOCA

It is easily seen directly that a weakly Luzin set contradicts SOCA, so that the Filippov space cannot exist under SOCA. Somewhat more generally,

**Theorem 4.1** Assume SOCA. Let \(X\) be compact, with a continuous map \(\pi : X \rightarrow Y\), where \(Y\) is compact metric. Assume further that there is an uncountable \(E \subseteq Y\) such that for \(y \in E\), there are three points \(x^i_y \in \pi^{-1}\{y\}\) for \(i = 0, 1, 2\) and disjoint open neighborhoods \(U^i_y\) of \(x^i_y\) such that \(\pi(U^i_y) \cap \pi(U^j_y) = \{y\}\) whenever \(i \neq j\).

Then \(X\) has an uncountable discrete subset.
Note that the double arrow space satisfies these hypotheses with “three” weakened to “two”, while the Filippov space satisfies these hypotheses with “three” strengthened to “omega”.

**Proof.** Let \( F^i_y = \text{cl}(\pi(U^i_y)) \), which is a closed set in \( Y \) containing \( y \). Shrinking the \( U^i_y \), we may assume that the three sets \( F^i_y \{y\} \) are pairwise disjoint.

We use CSM, which is a consequence of SOCA; see [10]. Call \( T \subseteq E \) \( i \)-connected iff for all \( \{y, z\} \in [T]^2 \), either \( y \in F^i_z \) or \( z \in F^i_y \). Call \( T \) \( i \)-free iff for all \( \{y, z\} \in [T]^2 \), both \( y \notin F^i_z \) and \( z \notin F^i_y \). Applying CSM three times, we get an uncountable \( T \subseteq E \) such that for each \( i \), either \( T \) is \( i \)-connected or \( T \) is \( i \) free. By the disjointness of the \( F^i_y \{y\} \), \( T \) can be \( i \)-connected for at most two values of \( i \). Fixing \( i \) such that \( T \) is \( i \)-free, we see that \( \{x^i_y : y \in T\} \) is discrete.

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