Abstract

Standard Bayesian analyses can be difficult to perform when the full likelihood, and consequently the full posterior distribution, is too complex and difficult to specify or if robustness with respect to data or to model misspecifications is required. In these situations, we suggest to resort to a posterior distribution for the parameter of interest based on proper scoring rules. Scoring rules are loss functions designed to measure the quality of a probability distribution for a random variable, given its observed value. Important examples are the Tsallis score and the Hyvärinen score, which allow us to deal with model misspecifications or with complex models. Also the full and the composite likelihoods are both special instances of scoring rules.

The aim of this paper is twofold. Firstly, we discuss the use of scoring rules in the Bayes formula in order to compute a posterior distribution, named SR-posterior distribution, and we derive its asymptotic normality. Secondly, we propose a procedure for building default priors for the unknown parameter of interest that can be used to update the information provided by the scoring rule in the SR-posterior distribution. In particular, a reference prior is obtained by maximizing the average $\alpha$–divergence from the SR-posterior distribution. For $0 \leq |\alpha| < 1$, the result is a Jeffreys-type prior that is proportional to the square root of the determinant of the Godambe information matrix associated to the scoring rule. Some examples are discussed.

Keywords: $\alpha$–divergences, Composite likelihood, Godambe information, $M$-estimating function, Reference prior, Robustness, Scoring rule.
1 Introduction

In the Bayesian setting, when the full likelihood is too complex and difficult to specify or when robustness with respect to data or to model misspecifications is required, several authors have proposed the use of surrogate likelihoods in the Bayes formula, in place of the full likelihood. Although this approach is non-orthodox, it is widely used in the recent statistical literature and theoretically motivated in several papers. See, among others, Pauli et al. (2011) and Ribatet et al. (2012) for the use of composite likelihoods, Lazar (2003), Lin (2006), Schennach (2005), Greco et al. (2008), Chang and Mukerjee (2008), Ventura et al. (2010), and Yang and He (2012) for the use of empirical and quasi-likelihoods; see also the review by Ventura and Racugno (2016), which addresses the use of pseudo-likelihoods in the Bayesian framework.

To deal with complex models or model misspecifications, useful surrogate likelihoods are also given by proper scoring rules. A scoring rule (see, for instance, the recent overviews by Machete, 2013, and Dawid and Musio, 2014, and references therein) is a special kind of loss function designed to measure the quality of a probability distribution for a random variable, given its observed value. It is proper if it encourages honesty in the probability evaluation. Proper scoring rules supply unbiased estimating equations for any statistical model, which can be chosen to increase robustness or for ease of computation. The Brier score (Brier, 1950), the logarithmic score (Good, 1952), the Tsallis score (Tsallis, 1988), and the Hyvärinen score (Hyvärinen, 2005) are widely known. In particular, when using the logarithmic score, the full likelihood and the composite likelihood (Varin et al., 2011) are obtained as special cases of proper scoring rules (see for instance Dawid and Musio, 2014).

Frequentist scoring rule inference has been widely discussed (see Dawid et al., 2016, and references therein), while Bayesian inference based on scoring rules has been considered only in Dawid and Musio (2015) for Bayesian model selection, in Ghosh and Basu (2016) for robust Bayes estimation using the density power divergence measure, and in Musio et al. (2017) for an illustration of Bayesian inference for directional data through the Hyvärinen score; see also Pauli et al. (2011) and Ribatet et al. (2012) for the use of composite likelihoods in the Bayes formula.

To perform Bayesian inference, a suitable prior distribution on the parameter of interest must be elicited. In this paper we focus on the class of default priors which are frequently used in Bayesian applications and which are still an active area of research (see, among others, Berger, 2006, Berger et al., 2009, 2012, Ghosh, 2011, Walker, 2016, Leisen et al., 2017). The two most common objective priors are the Jeffreys prior (Jeffreys, 1961), which uses the information about the parameter contained in the Fisher information, and the reference prior (Bernardo, 1979), which is based on the maximization of a distance in information between the prior and the posterior.

The goal of this paper is twofold. The first aim is to discuss the use of scoring rules in order to compute a posterior distribution, called here the SR-posterior distribution. In particular, we suggest a SR-posterior distribution obtained by extending to the general scoring rule setting the curvature adjustment of the composite likelihood proposed by Chandler and
Bate (2007) and Ribatet et al. (2012). We show that the SR-posterior distribution is, up to order $O_p(n^{-1/2})$, normally distributed, with the same asymptotic variance of the scoring rule estimator. A calibrated scoring rule is needed to reach the right asymptotic variance in the normal approximation, as well as a correct shape of the posterior distribution. Indeed, as discussed in Ghosh and Basu (2016), the variability of the non-calibrated scoring rule in a posterior distribution, may lead to a falsely precise inference.

The second aim is to propose the elicitation of a default prior for the unknown parameter of interest. In particular, we focus on reference priors as pioneered by Bernardo (1979); for a review see Bernardo (2005) and Ghosh (2011). Our purpose is to construct reference priors obtained by maximizing $\alpha-$divergences from the SR-posterior distribution. The $\alpha-$divergences are a well known class of discrepancy functions which include as a special case the Kullback-Leibler divergence. We show that, for $0 \leq |\alpha| < 1$, the maximizer is a Jeffreys-type prior that is proportional to the square root of the determinant of the Godambe information matrix.

The paper unfolds as follows. Section 2 reviews some background on scoring rules. Section 3 discusses the properties of the SR-posterior distribution. Section 4 presents the construction of reference priors based on $\alpha-$divergences using SR-posteriors. Examples are presented in Section 5 both in the context of complex models and in the context of robustness theory. Finally, suggestions and comments on further developments can be found in the conclusions.

## 2 Scoring rules

A scoring rule is a loss function which is used to measure the quality of a given probability distribution $Q$ for a random variable $X$, in view of the result $x$ of $X$; see Dawid (1986). The function $S(x; Q)$ takes values in $\mathbb{R}$ and its expected value under $P$ will be denoted by $S(P; Q)$.

The scoring rule $S$ is called proper relative to the class of distributions $\mathcal{P}$ if the following inequality is satisfied for all $P, Q \in \mathcal{P}$:

$$S(P; Q) \geq S(P; P).$$  \hspace{1cm} (1)

It is strictly proper relative to $\mathcal{P}$ if equation (1) is satisfied with equality if and only if $Q = P$. Note that in the following we identify a distribution $Q$ by its probability density $q$ with respect some measure $\mu$; so the two notations $S(x; q)$ and $S(x; Q)$ are indistinguishable.

An important example of proper scoring rules is the log-score, which is defined as $S(x; Q) = -\log q(x)$ (Good, 1952) and which corresponds to minus the log-likelihood function. The Tsallis score (Tsallis, 1988, Ghosh and Basu, 2013) is given by

$$S(x; Q) = (\gamma - 1) \int q(y)^\gamma d\mu(y) - \gamma q(x)^{\gamma - 1}, \ \gamma > 1.$$  \hspace{1cm} (2)

The Tsallis score gives in general robust procedures (Ghosh, 2011, Ghosh and Basu, 2013, Dawid et al., 2016), and the parameter $\gamma$ is a trade-off between efficiency and robustness.
The Hyvärinen score in its original formulation (Hyvärinen, 2005) for variables in \( \mathbb{R}^k \) is defined as
\[
S(x, Q) = \Delta \log q(x) + \frac{1}{2} \| \nabla \log q(x) \|^2,
\] (3)
where \( \| \cdot \| \) is the standard norm, \( \nabla \) denotes the gradient vector and \( \Delta \) the Laplacian operator. The Hyvärinen score has been extended to handle distributions defined on Riemannian manifolds (Dawid and Lauritzen, 2005, Mardia et al., 2016).

Proper scoring rules can also be extended to the case of a random vector in analogy with composite likelihoods (Varin et al., 2011). Let \( \{X_k\} \) be a set of marginal or conditional variables with associated proper scoring rule \( S_k \). A proper scoring rule for the random vector \( X \) is defined as
\[
S(x; Q) = \sum_k S_k(x_k; Q_k),
\] (4)
where \( X_k \sim Q_k \) when \( X \sim Q \), and \( x \) and \( x_k \) are the values assumed by \( X \) and \( X_k \), respectively. Scoring rules of the form (4) are called composite scoring rules; see Dawid and Musio (2014) and Dawid et al. (2016). Note that when each \( S_k \) is the logarithmic score, equation (4) is a negative composite log-likelihood; see Varin et al. (2011).

2.1 Inference based on scoring rules

Let \( x = (x_1, \ldots, x_n) \) be a random sample of size \( n \) from a distribution function \( P_\theta = P(x; \theta) \), with \( \theta \in \Theta \subseteq \mathbb{R}^k, k \geq 1 \), and let \( L(\theta; x) \) be the likelihood function based on model \( P_\theta \).

The validity of inference about \( \theta \) using scoring rules can be justified by invoking the general theory of unbiased \( M \)-estimating functions. Indeed, inference based on proper scoring rules is a special kind of \( M \)-estimation (see, e.g., Dawid et al., 2016, and references therein). The class of \( M \)-estimators is broad and includes a variety of well-known estimators. For example it includes the maximum likelihood estimator (MLE), the maximum composite likelihood estimator (see e.g. Varin et al., 2011), and robust estimators (see e.g. Hampel et al., 1986, Huber and Ronchetti, 2009) among others.

Given a proper scoring rule \( S(x; P_\theta) = S(x; \theta) \), let us denote by \( S(\theta) = \sum_{i=1}^{n} S(x_i; \theta) \) the total empirical score. Moreover, let \( s(x; \theta) \) be the gradient vector of \( S(x; \theta) \) with respect to \( \theta \), i.e. \( s(x; \theta) = \partial S(x; \theta) / \partial \theta \). Under broad regularity conditions (see Mameli and Ventura, 2015, and references therein), the scoring rule estimator \( \hat{\theta} \) is the solution of the unbiased estimating equation
\[
S_\theta(\theta) = \sum_{i=1}^{n} s(x_i; \theta) = 0
\]
(see Dawid and Lauritzen, 2005, Dawid, 2007, Dawid et al., 2016) and it is asymptotically normal, with mean \( \theta \) and covariance matrix \( V(\theta)/n \), with
\[
V(\theta) = K(\theta)^{-1} J(\theta)(K(\theta)^{-1})^T,
\] (5)
where \( K(\theta) = E_\theta(\partial s(X; \theta)/\partial \theta^T) \) and \( J(\theta) = E_\theta(s(X; \theta)s(X; \theta)^T) \) are the sensitivity and the variability matrices, respectively. The matrix \( G(\theta) = V(\theta)^{-1} \) is known as the Godambe information and its form is due to the failure of the information identity since, in general, \( K(\theta) \neq J(\theta) \).

From the general theory of \( M \)-estimators, the influence function (IF) of the estimator \( \tilde{\theta} \) is given by

\[
IF(x; \tilde{\theta}, P_\theta) = K(\theta)^{-1}s(x; \theta),
\]

and it measures the effect on the estimator \( \tilde{\theta} \) of an infinitesimal contamination at the point \( x \), standardised by the mass of the contamination. The estimator \( \tilde{\theta} \) is B-robust if and only if \( s(x; \theta) \) is bounded in \( x \) (see Hampel et al., 1986). Note that the IF of the MLE is proportional to the score function; therefore, in general, MLE has unbounded IF, i.e. it is not B-robust. Sufficient conditions for the robustness of the Tsallis score are discussed in Basu et al. (1998) and Dawid et al. (2016). Finally, note that the IF can also be used to evaluate the asymptotic covariance matrix of \( \tilde{\theta} \), since \( V(\theta) = E_\theta(IF(X; \tilde{\theta}, P_\theta)IF(X; \tilde{\theta}, P_\theta)^T) \).

Asymptotic inference on the parameter \( \theta \) can be based on the Wald-type statistic

\[
W_w(\theta) = n(\tilde{\theta} - \theta)^T V(\tilde{\theta})^{-1}(\tilde{\theta} - \theta),
\]

which has an asymptotic chi-square distribution with \( k \) degrees of freedom. The same asymptotic result also holds for the score-type statistic \( W_s(\theta) = S_\theta(\theta)^T(nJ(\tilde{\theta}))^{-1}S_\theta(\theta) \); see Dawid et al. (2016). In contrast, the asymptotic distribution of the scoring rule ratio statistic

\[
W_{rs}(\theta) = 2 \left\{ S(\theta) - S(\tilde{\theta}) \right\}
\]

is a linear combination of independent chi-square random variables with coefficients related to the eigenvalues of the matrix \( J(\theta)K(\theta)^{-1} \) (Dawid et al., 2016). More formally,

\[
W_{rs}(\theta) \overset{L}{\rightarrow} \sum_{j=1}^{k} \mu_j Z_j^2,
\]

with \( \mu_1, \ldots, \mu_k \) eigenvalues of \( J(\theta)K(\theta)^{-1} \) and \( Z_1, \ldots, Z_k \) independent standard normal variables. Adjustments of the scoring rule ratio statistic have received consideration in Dawid et al. (2016), extending results of Pace et al. (2011) for composite likelihoods. Higher-order asymptotic expansions for scoring rules, which encompass the classical results for likelihood quantities while allowing for the failing of the information identity, have been recently discussed by Mameli and Ventura (2015) and Mameli et al. (2017).

### 3 Posterior distributions based on scoring rules

In the Bayesian framework, the use of surrogate likelihoods in the Bayes formula has received great attention in the last decade (see the review by Ventura and Racugno, 2016, and references therein). In particular, the use of the composite likelihood, which is a special
instance of scoring rule, in the Bayes formula has been considered for instance in Smith and Stephenson (2009), Pauli et al. (2011) and Ribatet et al. (2012). Since the composite likelihood does not satisfy the information identity, it is necessary to calibrate it in order to reach the right asymptotic variance in the normal approximation, as well as a correct shape of the posterior distribution. The correct curvature of the posterior distribution based on composite likelihoods can also be reached by using the composite score function in the Approximate Bayesian Computation (ABC) procedure, as discussed in Ruli et al. (2016).

While the use of surrogate likelihoods for Bayesian inference has been widely discussed in the recent literature to deal with complex models or robustness, the use of scoring rules in the Bayesian framework has been considered only in Dawid and Musio (2015) for Bayesian model selection, in Ghosh and Basu (2016) for robust Bayes estimation using the density power divergence measure, and in Musio et al. (2017) for an illustration of Bayesian inference for directional data through the Hyvärinen score.

Paralleling the derivation of posterior distributions based on composite likelihoods, a SR-posterior distribution can be obtained by using a scoring rule instead of the full likelihood in Bayes formula. However, since the scoring rule does not satisfy the information identity, it must be suitably calibrated before deriving the SR-posterior distribution. In particular, here we suggest a SR-posterior distribution obtained by extending to the scoring rule setting the curvature adjustment of the composite likelihood proposed by Chandler and Bate (2007) and Ribatet et al. (2012).

Let $\pi(\theta)$ be a prior distribution for $\theta$. The proposed SR-posterior distribution is defined as

$$\pi_{SR}(\theta|x) \propto \pi(\theta) \exp \{-S(\theta^*)\},$$

with $\theta^* = \theta^*(\theta) = \tilde{\theta} + C(\theta - \tilde{\theta})$, where $C$ is a $k \times k$ fixed matrix such that $C^T K(\theta) C = G(\theta)$. A possible choice of the matrix $C$ is given by $C = M^{-1} M_A$, with $M_A^T M_A = G$ and $M^T M = K$; for details, see Ribatet et al. (2012) and references therein.

Under the same regularity conditions necessary for the asymptotic results for scoring rule inference, as $n \to \infty$, it can be shown that the scoring rule posterior distribution (8) is, up to order $O_p(n^{-1/2})$, normally distributed with mean $\tilde{\theta}$ and variance $H(\tilde{\theta})^{-1}/n$, i.e.

$$\pi_{SR}(\theta|x) \sim N_k\left(\tilde{\theta}, \frac{H(\tilde{\theta})^{-1}}{n}\right),$$

where $H(\theta) = C^T (\partial^2 S(\theta)/\partial \theta^2) C/n$. Note that $H(\tilde{\theta})$ converges almost surely to $G(\theta)$ as $n \to \infty$. This result is stated in the next Theorem 3.1 that gives the expansion of the scoring rule posterior distribution (8) up to third order.

In the sequel, a tilde over a quantity means evaluation of that quantity in $\tilde{\theta}$. We denote by $c_{ij}$ and $h^{ij}$ the elements of the matrices $C$ and $H^{-1}$, respectively. We use indices to denote derivatives of $S(x; \theta)$ and $\pi(\theta)$ with respect to the components of the parameter. Moreover, we use the Einstein summation convention so that when an index appears twice in an expression, summation on that index is intended.
Theorem 3.1. Let \( w = (w^1, \ldots, w^k)^T = n^{1/2}(\theta - \tilde{\theta}) \). The SR-posterior distribution for \( w \) can be written as

\[
\pi_{SR}(w|x) = \phi_k(w; \tilde{H}^{-1}) \left[ 1 + n^{-1/2}A_1(w) + n^{-1}A_2(w) \right] + O_p(n^{-3/2}),
\]

where \( \phi_k(w; \tilde{H}^{-1}) \) is the density of a \( k \)-variate normal distribution with zero mean vector and variance matrix \( H(\tilde{\theta})^{-1} \), and

\[
A_1(w) = \frac{\tilde{\pi}_iw^i}{\tilde{\pi}} - \frac{1}{6} \frac{\tilde{S}_{ijk}}{n} c_{ir}c_{js}c_{kt}w^r w^s w^t
\]

and

\[
A_2(w) = \frac{1}{2} \frac{\tilde{\pi}_{ij}(w^iw^j - \tilde{h}^{ij})}{\tilde{\pi}} - \frac{1}{6} \frac{\tilde{S}_{ijk}}{n} c_{ir}c_{js}c_{kt}c_{hu}(w^iw^j w^k w^l - 3 \tilde{h}^{ir} \tilde{h}^{st})
\]

\[
- \frac{1}{24} \frac{\tilde{S}_{ijk}}{n} c_{ir}c_{js}c_{kt}c_{hu}(w^iw^j w^k w^l - 3 \tilde{h}^{ir} \tilde{h}^{st})
\]

\[
+ \frac{1}{72} \frac{\tilde{S}_{ijk}}{n^2} \left[ \tilde{c}_{ia}c_{jb}c_{kc}c_{rd}c_{se}c_{tf}(w^aw^bw^cw^dw^ew^f - 9 \tilde{h}^{ab} \tilde{h}^{cd} \tilde{h}^{ef})
\right.
\]

\[
+ c_{ia}c_{rb}c_{jc}c_{sd}c_{ke}c_{tf}(w^aw^bw^cw^dw^ew^f - 6 \tilde{h}^{ab} \tilde{h}^{cd} \tilde{h}^{ef}) \right] .
\]

The proof is given in the Appendix. In practice, Theorem 3.1 shows that the SR-posterior distribution (8) has the correct curvature and that the variance in (9) is asymptotically equivalent to that of the minimum scoring rule estimator. When in particular \( S(x; \theta) \) is the log-score, it can be shown that (10) reduces to the expansion given in Datta and Mukerjee (2004) for the classic posterior distribution \( \pi(\theta|x) \propto \pi(\theta)L(\theta; x) \).

4 Reference priors obtained from \( \alpha \)-divergences

The information on \( \pi(\theta|x) \) induced by \( \pi(\theta) \) may be measured in terms of a divergence \( D(\cdot, \cdot) \) between the prior and the posterior distribution: the higher the divergence, the lower the influence of the prior on the posterior. Let \( D_\pi(x) = D(\pi(\theta), \pi(\theta|x)) \), and let \( p(x) \) and \( p(x|\theta) \) be the marginal and conditional distributions of \( X \) given \( \theta \), respectively. Minimizing the information in a prior is equivalent to maximize the expected divergence \( D_\pi \) from the corresponding posterior, i.e. the functional

\[
T(\pi) = \int_X D_\pi(x)p(x)dx,
\]

\[
= \int_{\theta} \int_X [D_\pi(x)p(x|\theta)dx] \pi(\theta)d\theta .
\]

(11)
Here, we focus on the well-known family of $\alpha$-divergences, defined as

$$D_\pi(x) = \frac{1}{\alpha(1-\alpha)} \int_\Theta \left\{ 1 - \left( \frac{\pi(\theta)}{\pi(\theta|x)} \right)^\alpha \right\} \pi(\theta|x) d\theta,$$

which for $\alpha \to 0$ reduces to the Kullback-Liebler divergence, for $\alpha = 1/2$ corresponds to twice the Hellinger distance and for $\alpha = -1$ is equivalent to the $\chi^2$-divergence.

In this section we extend the results of Ghosh et al. (2011) to the context of scoring rules with multidimensional parameters. In particular, in the following theorem we propose reference priors obtained by maximizing the expected $\alpha$-divergence from the SR-posterior distribution \[8\].

In the sequel, we denote by $g_{ij}$ and $g^{ij}$ the components of matrices $G(\theta)$ and $G(\theta)^{-1}$ respectively. Moreover, let $E[S_{ijk}/n|\theta] = B_{ijk}(\theta) + o(n^{-1/2})$ and let $a = (a_1, \ldots, a_k)^T$, with $a_i = B_{klh}c_{ki}c_{hs}c_{lt}g^{st}$.

**Theorem 4.1.** When $0 \leq |\alpha| < 1$, the prior which asymptotically maximizes the expected $\alpha$-divergence to the SR-posterior distribution \[8\] is

$$\pi_G(\theta) \propto |G(\theta)|^{1/2}.$$  \hspace{1cm} (12)

When $\alpha = -1$, the prior which asymptotically maximizes the expected $\chi^2$-divergence to the SR-posterior distribution \[8\] is such that

$$\frac{\partial \log \pi_\chi(\theta)}{\partial \theta} = \frac{1}{4} \left[ 2a + |G|^{-1} \frac{\partial |G|}{\partial \theta} + 2G \nabla \cdot G^{-1} \right],$$  \hspace{1cm} (13)

where $\nabla \cdot G^{-1} = (\partial g^{ij}/\partial \theta^j, \ldots, \partial g^{kj}/\partial \theta^j)^T$.

The proof is given in the Appendix.

Notice that for $|\alpha| > 1$ a maximizer for the expected $\alpha$-divergence does not exist.

For $\alpha = -1$ and in the case of a one dimensional parameter $\theta$, we can explicitly find the prior distribution satisfying \[13\] as

$$\pi_\chi(\theta) \propto \exp \left\{ \int_\Theta \frac{2BC^3 - G'}{4G} d\theta \right\}.$$  \hspace{1cm} (14)

It is important noticing that, in the case of a scalar parameter $\theta$, if we use the full likelihood in the Bayes formula, \[12\] and \[13\] reduce to the results obtained in Ghosh et al. (2011).

### 4.1 Properties of the prior

The proposed prior distributions \[12\] and \[13\] share some important properties with Jeffreys prior. The most relevant is invariance with respect to one-to-one changes in the parameterization. As shown in Liu et al. (2014), when the functional $T(\pi)$ is invariant, the
maximizer is also invariant. Thus, invariance of the proposed prior distributions follows from the well known invariant properties of α-divergences.

For $0 \leq |\alpha| < 1$, it can be easily seen that $G(\theta)$ is a second order tensor so that, if $\psi(\theta)$ is a new parametrisation, $G_{rs}(\psi) = G_{ij}(\theta(\psi))\theta_i^r(\psi)\theta_j^s(\psi)$, where $\theta_i^r = \partial \theta^i / \partial \psi^r$. Thus it follows that $\pi_G(\psi) = \pi_G(\theta(\psi)) |\partial \theta(\psi)/\partial \psi|$. Some other interesting properties of the prior distribution depend on the scoring rule under consideration.

### 4.1.1 The Tsallis scoring rule

Let $X$ be distributed as a location model with parameter $\mu \in \mathbb{R}$, so that $p(x; \mu) = p_0(x - \mu)$. The Tsallis scoring rule is defined as

$$S(x; \mu) = (\gamma - 1) \int p_0(x - \mu)\gamma^x dx - \gamma p_0(x - \mu)^{\gamma - 1}$$

$$= \text{const} - \gamma p_0(x - \mu)^{\gamma - 1}.$$

Its derivatives with respect to $\mu$ are all functions of $x - \mu$, so that the corresponding expected values are independent of $\mu$. So are the quantities $B(\mu)$ defined at Theorem 4.1 and the sensitivity and variability matrices $K(\mu)$ and $J(\mu)$ and, as a consequence, $G(\mu)$. Thus, $\pi_G(\mu) \propto 1$, like the Jeffreys prior with the log-score. Instead, the prior obtained from the Chi-squared divergence is of the form $\pi_\chi(\mu) \propto \exp\{k\mu\}$ for some constant $k$. However, if $p_0$ is symmetric, i.e. $p_0(x) = p_0(-x) \forall x$, then $B = 0$ and in this case we have $\pi_\chi(\mu) \propto 1$.

Also for scale models, $\pi_G(\sigma)$ with the Tsallis score coincides with the Jeffreys prior with the log-score. Indeed, let $X$ be distributed as a scale model with parameter $\sigma > 0$, so that $p(x; \sigma) = p_0(x/\sigma)/\sigma$. The Tsallis scoring rule is defined as

$$S(x; \sigma) = (\gamma - 1) \int \frac{1}{\sigma^\gamma} p_0 \left( \frac{x}{\sigma} \right)^\gamma dx - \gamma \frac{1}{\sigma^{\gamma - 1}} p_0 \left( \frac{x}{\sigma} \right)^{\gamma - 1}$$

$$= \frac{1}{\sigma^{\gamma - 1}} \left[ \text{const} - \gamma p_0 \left( \frac{x}{\sigma} \right)^{\gamma - 1} \right].$$

It is easy to show that the first three derivatives of $S(x; \sigma)$ can be written as

$$S_\sigma = \frac{1}{\sigma^\gamma} f_1 \left( \frac{x}{\sigma} \right), \quad S_{\sigma\sigma} = \frac{1}{\sigma^{\gamma + 1}} f_2 \left( \frac{x}{\sigma} \right), \quad S_{\sigma\sigma\sigma} = \frac{1}{\sigma^{\gamma + 2}} f_3 \left( \frac{x}{\sigma} \right),$$

where $f_1$, $f_2$ and $f_3$ are suitable functions. Notice that the expectation of any function of $X/\sigma$ with respect to $p(x; \sigma)$ does not depend on $\sigma$ itself. Thus, $B(\sigma) \propto 1/\sigma^{\gamma + 2}$, $J(\sigma) \propto 1/\sigma^{2\gamma}$ and $K(\sigma) \propto 1/\sigma^{\gamma + 1}$, so that $G(\sigma) \propto 1/\sigma^2$ and $\pi_G(\sigma) \propto 1/\sigma$. The Chi-squared divergence delivers $\pi_G(\sigma) \propto \sigma^{k_1} \exp\{k_2\sigma^{1-\gamma}\}$, for suitable constants $k_1$ and $k_2.$
4.1.2 Hyvärinen scoring rule

Let $X$ be a non-degenerate random variable belonging to the one-parameter natural exponential family

$$p(x; \theta) = \exp \{ \theta x - k(\theta) + a(x) \}, \; x \in \mathbb{R}. \quad (15)$$

The Hyvärinen total empirical score in this case reduces to

$$S(\theta) = - \left\{ 2 \sum_{i=1}^{n} a''(x_i) + \sum_{i=1}^{n} \left[ \theta + a'(x_i) \right]^2 \right\}. \quad (16)$$

The Hyvärinen score estimator for this family is

$$\hat{\theta} = - \frac{\sum_{i=1}^{n} a'(X_i)}{n},$$

which can be computed without knowledge of $k(\theta)$; see Barndorff-Nielsen (1976), Hyvärinen (2007) and Parry et al. (2012). The Hyvärinen score estimator is an unbiased estimator for the parameter $\theta$ and its variance coincides with the inverse of the Godambe information $G(\theta)^{-1/2} = \text{Var}(a'(X))$; see Mameli and Ventura (2015). Thus, $\pi_G(\theta) \propto \text{Var}(a'(X))^{-1/2}$, while the Jeffreys prior is $\pi_J(\theta) \propto k''(\theta)^{1/2}$ which requires the knowledge of $k(\theta)$. In models in which $k(\theta)$ is difficult to evaluate, the Jeffreys prior is not available.

5 Examples

Now we illustrate the behaviour of the SR-posterior distribution and of the corresponding reference prior by means of three examples. In the examples different non-informative priors are also considered. The R code for the examples of this section can be found in the Electronic Supplementary Material.

5.1 Hyvärinen scoring rule for the von Mises-Fisher distribution

Inference for models for directional data is difficult because typically the density function contains an intractable normalization constant. In this setting, Mardia et al. (2016) resorted to the Hyvärinen scoring rule for estimation. An important feature of this score is related to the homogeneity property for which the distribution needs to be known up to the normalization constant.

Let us consider the von Mises-Fisher density, which is a directional distribution defined on the unit sphere $\mathcal{S}_{q-1} \subset \mathbb{R}^q$ given by

$$p(z; \kappa) \propto \exp(-\kappa \mu_0^T z), \; z \in \mathcal{S}_{q-1},$$
with \( \kappa \in \mathbb{R}^+ \) scalar concentration parameter and \( \mu_0 = (\cos \theta_0, \sin \theta_0) \), with \( \theta_0 \in \mathbb{R} \) known. When \( q = 2 \) and the data are represented in polar coordinates \((z_{h1}, z_{h2}) = (\cos \theta_h, \sin \theta_h), h = 1, \ldots, n\), the Hyvärinen estimator for \( \kappa \) can be derived in closed form and it is given by

\[
\tilde{\kappa} = \frac{2\sqrt{R^2(1 + R^2)} + 2(C^2 - S^2)C_2 + 4CSS_2}{(1 - R^2)},
\]

where \( C = \frac{1}{n} \sum_{h=1}^{n} \cos(\theta_h), S = \frac{1}{n} \sum_{h=1}^{n} \sin(\theta_h), C_2 = \frac{1}{n} \sum_{h=1}^{n} \cos(2\theta_h), S_2 = \frac{1}{n} \sum_{h=1}^{n} \sin(2\theta_h) \) and \( R = \sqrt{C^2 + S^2} \).

For \( \theta_0 = 0 \), Mardia et al. (2016) show that the asymptotic variance of \( \tilde{\kappa} \) is

\[
\frac{V(\kappa)}{n} = \frac{\kappa [2\kappa - 3A_1(\kappa)]}{nA_1^2(\kappa)},
\]

with \( A_1(\kappa) = I_1(\kappa)/I_0(\kappa) \), where \( I_1 \) and \( I_0 \) are the modified Bessel functions of order 0 and 1, respectively. The proposed reference prior (12) is thus

\[
\pi_G(\kappa) \propto \sqrt{\frac{A_1^2(\kappa)}{\kappa [2\kappa - 3A_1(\kappa)]}}.
\]

Figure 1 shows the plot of the reference prior \( \pi_G(\kappa) \) and, for comparison, it also presents the classical non-informative prior \( \pi(\kappa) \propto 1/\kappa \) for the scale parameter. It can be seen that \( \pi_G(\kappa) \) puts finite mass at 0, giving rise to a proper posterior and leading to more appropriate inference when the true parameter values are close to the boundary of the parametric space.

In order to illustrate that the calibration of the Hyvärinen scoring rule is necessary to obtain a posterior distribution \( \hat{\kappa} \) with the right curvature, consider a sample of size \( n = 50 \).
from the von Mises-Fisher distribution with \((\kappa, \mu_0) = (3, 0)\). Hereafter, we take the location parameter \(\mu_0\) to be fixed and equal to zero, and consider the problem of estimating \(\kappa\). Figure 2 compares the full posterior (black line) based on the likelihood function of the von Mises-Fisher model, with the calibrated (red) and non calibrated (green) SR-posteriors based on the Hyvärinen scoring rule. Here, the non-informative prior \(\pi(\kappa) \propto 1/\kappa\) is used. The vertical line corresponds to the Hyvärinen scoring rule estimate of \(\kappa\), i.e. \(\tilde{\kappa} = 2.36\). It can be noted that the non calibrated SR-posterior is too narrow compared with the genuine posterior, while the calibrated SR-posterior shows a curvature similar to the genuine posterior.

\[
\text{Genuine, cal. and no cal (default prior)}
\]

![Diagram](image)

Figure 2: Comparison of calibrated SR-posterior vs non calibrated and proper posteriors.

To compare the behaviour of the SR-posterior distributions, based on the two priors \(\pi_G(k)\) and \(\pi(\kappa) \propto 1/\kappa\), two scenarios have been considered. In the first one, the true value of \(\kappa\) approaches the boundary and/or the sample size is small. In particular, we consider a sample of size \(n = 10\), with \(\kappa = 1.0\), and a sample of size \(n = 50\), with \(\kappa = 0.5\), and we compute the SR-posteriors with the two priors. The SR-posteriors are given in Figure 3. It can be noted that for small samples sizes or for the true value of \(\kappa\) near zero, the SR-posterior with \(\pi(\kappa) \propto 1/\kappa\) may not be proper or puts too much mass at zero, i.e. it has a vertical asymptote at zero. On the other hand, the SR-posterior with the reference prior \(\pi_G(\kappa)\) is proper and unimodal.

In the second scenario, the true value of \(\kappa\) is away from the boundary and the sample size is moderate or large. The SR-posteriors in these cases look very similar, as it can be noted from Figure 4. In particular, for high values of \(\kappa\) the two SR-posteriors are indistinguishable.
5.2 Pairwise likelihood for the multivariate equi-correlated normal model

Let $X$ be a $q$-dimensional random vector with mean $\mu$ and covariance matrix $\Sigma$, with $\Sigma_{rr} = \sigma^2$ and $\Sigma_{rs} = \rho \sigma^2$ for $r \neq s$, with $r, s = 1, \ldots, q$ and $\rho \in (-1/(q-1), 1)$. The pairwise log-likelihood for $\theta = (\mu, \sigma^2, \rho)$ is (see Pace et al., 2011)

$$S_p(\theta) = -\frac{nq(q-1)}{2} \log \sigma^2 - \frac{nq(q-1)}{4} \log (1 - \rho^2) - \frac{q-1+\rho}{2\sigma^2(1-\rho^2)} SSW +$$

$$- \frac{q(q-1)SSB + nq(q-1)(\bar{y} - \mu)^2}{2\sigma^2(1+\rho)},$$

where $SSW = \sum_{i=1}^n \sum_{r=1}^q (x_{ir} - \bar{x}_i)^2$, $SSB = \sum_{i=1}^n (\bar{x}_i - \bar{x})^2$, $\bar{x}_i = \sum_{r=1}^q x_{ir}/q$ and $\bar{x} = \sum_{i=1}^n \sum_{r=1}^q x_{ir}/(nq)$.

The reference prior (12) is proportional to the square root of the determinant of the Godambe information matrix whose components are given in Pace et al. (2011).

Consider, first, the simplest situation with $\mu$ and $\sigma^2$ known. The SR-posterior distribution (8) with both the uniform prior and with the proposed reference prior is computed in different scenarios for varying values of $\rho$ and $q$ (see Figure 4). We notice that the two priors give very similar results when the parameter value is away from zero.

Consider now the multi-parametric case. We compare the SR-posteriors with the reference prior and with a non-informative prior derived in two alternative parametrizations:

(a) $\theta = (\mu, \sigma, \rho)$, for which we assume the non-informative prior $\pi(\theta) \propto 1/\sigma$.

(b) $\xi = (\mu, \tau, \kappa)$, with $\tau = \log(\sigma)$ and $\kappa = \logit(\rho)$, for which we assume the flat prior $\pi(\xi) \propto 1$.

The calibrated SR-posterior distribution has been computed with the parametrization $\xi$, in order to avoid the constraints on $\theta \in \Theta$. As a first example, consider $n = 10$, $q = 10$ and ...
Figure 4: SR-posteriors with the $\pi_G(\rho)$ and with the uniform prior in $(0,1)$ for different values of $\rho$ and $q$ where $n = 10$, $\sigma^2 = 1$ and $\mu = 0$. 
a sample generated under the equi-correlated normal model with \( \theta = (0, 1, 0.5) \). The SR-posterior distributions with the reference prior and the two non-informative priors in cases (a) and (b) are shown in Figure 5. The SR-posterior distributions, described by means of histograms, show no appreciable differences.

Figure 5: SR-posterior with the reference prior \( \pi_G \) compared with the two alternative flat priors (a) and (b). The data are generated from the equi-correlated model with \( n = 10, \mu = 0, \sigma^2 = 1 \) and \( \rho = 0.5 \).

As a second example, consider the more problematic scenario with \( n = 10, q = 4 \) and data generated under the equi-correlated normal model with \( \theta = (0, 0.5, 0.1) \). The SR-posteriors with the reference prior and the two non-informative priors are shown in Figure 6.

It can be noted that a flat prior on \( \theta \) may not necessarily be flat also on \( \xi \), at least not for \( \sigma \) and \( \rho \). Indeed, the uniform prior on \( \rho \) implies a standard logistic prior on \( \kappa \), which is quite informative. This explains the "regularised" behaviour of the marginal posteriors of \( \rho \) and \( \sigma \) in case (a). On the other hand, the proposed reference prior behaves quite similarly.
Figure 6: SR-posterior with the reference prior $\pi_G$ (third row) compared with the two alternative flat priors (a) (first row) and (b) (second row). The data are generated from the equi-correlated model with $n=10$, $\mu = 0$, $\sigma^2 = 0.5$ and $\rho = 0.1$. 
to the uniform improper prior $\pi(\xi) \propto 1$.

### 5.3 Tsallis scoring rule for linear regression models

Consider the linear regression model

$$y = X\beta + \sigma \epsilon,$$

where $X$ is a fixed $n \times p$ matrix, $\beta \in \mathbb{R}^p$ ($p \geq 1$) is the vector of unknown regression coefficients, $\sigma > 0$ is a scale parameter, and $\epsilon$ is an $n$-dimensional vector of random errors from a standard normal distribution.

Let $\theta = (\beta, \sigma^2)$. The Tsallis total empirical score is (Ghosh and Basu 2013)

$$s_T(\theta) = \frac{\gamma}{(2\pi\sigma^2)^{\gamma/2-1}} \sum_{i=1}^{n} e^{-\frac{(y_i-x_i^T\beta)^2}{2\sigma^2}} - \frac{n(\gamma - 1)}{\sqrt{\gamma}(2\pi\sigma^2)^{(\gamma-1)/2}}.$$

The asymptotic variance of the Tsallis estimator of $\theta$ is given in Ghosh and Basu (2013). In particular, the asymptotic distribution of $(X^TX)^{1/2}(\tilde{\beta} - \beta)$ is a $N_p(0, v^\beta_{\gamma})$, with $v^\beta_{\gamma} = \sigma^2 \left(1 + \frac{(\gamma-1)^2}{2\gamma-1}\right)^{3/2}$, while $\sqrt{n}(\tilde{\sigma}^2 - \sigma^2)$ follows a normal distribution with mean 0 and variance $v^\epsilon_{\gamma}$, where

$$v^\epsilon_{\gamma} = \frac{4\sigma^4}{(2 + (\gamma - 1)^2)^2} \left(2(1 + 2(\gamma - 1)^2) \left(1 + \frac{(\gamma - 1)^2}{2\gamma - 1}\right)^{5/2} - (\gamma - 1)^2\gamma^2\right).$$

Since the asymptotic distributions of $\tilde{\beta}$ and $\tilde{\sigma}^2$ are independent, the proposed reference prior is proportional to the square root of the determinant of the Godambe information, i.e.

$$\pi_G(\theta) \propto \left(v^\beta_{\gamma} v^\epsilon_{\gamma}\right)^{-1/2}.$$  \hfill (18)

Moreover, a prior for $\beta$ is $\pi_G(\beta) \propto \left(v^\beta_{\gamma}\right)^{-1/2}$, while a prior for $\sigma^2$ is $\pi_G(\sigma^2) \propto \left(v^\epsilon_{\gamma}\right)^{-1/2}$.

In order to illustrate the behaviour of the Tsallis scoring rule in the context of the linear regression model, we consider the GFR dataset (Heritier et al., 2009), which contains measurements of the glomerular filtration rate (GFR) and serum creatinine (CR) on $n = 30$ subjects. The GFR is the volume of fluid filtered from the renal glomerular capillaries into the Bowmans capsule per unit of time (typically in millilitres per minute) and, clinically, it is often used to determine renal function. Its estimation, when not measured, is of clinical importance and several techniques are used for that purpose. One of them is based on CR, an endogenous molecule, synthesized in the body, which is freely filtered by the glomerulus (but also secreted by the renal tubules in very small amounts). Several models have been proposed in the literature to explain the logarithm of GFR as a function of CR. Here, following Heritier et al. (2009), we consider a model for GFR based on $CR^{-1}$ and AGE, i.e.

$$GFR = \beta_0 + \beta_1 \frac{1}{CR} + \beta_2 \text{AGE} + \epsilon.$$
Figure 7: Scatterplot diagram of GFR data.

Both covariates are scaled to have mean zero and unit variance, whereas the response is scaled to have unit variance. The data are illustrated in Figure 7; note that there are some observations which look like outliers. The need and consequence to resort to robust procedures to deal with these data have been highlighted in several papers; see, among others, Heritier et al. (2009) and Farcomeni and Ventura (2012).

Figure 8 gives the violin plots of the marginal SR-posterior distributions based on the Tsallis score, with $\gamma = 1.25$, and of the classical posterior distribution. For the parameter $\theta = (\beta, \log \sigma)$ we assume both the usual non-informative flat prior $\pi(\theta) \propto 1$ and the proposed reference prior (18) in the SR-posterior distribution, while the classical posterior distribution is based on the non-informative flat prior. From Figure 8 it can be noted that the proposed reference prior behaves similarly to the non-informative prior. Moreover, the classical marginal posterior distribution shows in general heavier tails and, in particular, the robust and the classical posterior distributions give different inferences on $\beta_2$ and $\sigma$.

Figure 9 gives useful monitoring plots in robustness studies. In particular, these plots illustrate the SR-posterior distributions based on the reference prior as a function of the robustness constant $\gamma$. This approach (see, e.g., Riani et al., 2014) provides tools for gaining knowledge and better understanding of the properties of robust procedures. The horizontal lines in Figure 9 correspond to the posterior mode of the genuine posterior based on the flat prior.

The value $\gamma = 1$ corresponds to complete overlap of the two methods. As $\gamma$ increases robustness is achieved; the value $\gamma = 1.25$ gives approximately 0.95% efficiency under the normal distribution.

As a final remark, note that the Tsallis scoring rule provides a cogent framework for dealing with robust procedures in the Bayesian framework.
Figure 8: SR-posteriors with $\gamma = 1.25$ (with the flat or the reference prior) vs the genuine posterior for the GFR dataset.
Figure 9: Marginal SR-posteriors for $\beta_0, \beta_1, \beta_2$ and $\sigma$ with the reference prior for varying values of $\gamma$. 
6 Conclusions

In this paper, we discuss the use of scoring rules in order to compute a posterior distribution, useful to deal with complex models or if robustness with respect to data or to model misspecification is required. Indeed, scoring rules provide a flexible and robust way of combining data-driven information with prior distributions, either subjective or non-informative.

One should devise a scoring rule that properly captures the structure of the data, otherwise the resulting posterior inferences are not reliable. In this respect, the result of Section 3 shows that a suitably calibrated SR-posterior distribution is, up to order $O_p(n^{-1/2})$, normally distributed, with the same asymptotic variance of the scoring rule estimator.

When dealing with default priors, in Section 4 reference priors for a vector parameter based on maximizing $\alpha-$divergences are discussed in the framework of scoring rules. We show that, for $0 \leq |\alpha| < 1$, the result is a Jeffreys-type prior that is proportional to the square root of the determinant of the Godambe information matrix.

Some extensions of the proposed results can also be considered. One possible direction of further research is to extend the proposed methodology considering the class of monotone and regular divergences which is a broad family of divergences asymptotically equivalent to $\alpha-$divergences; see Corcuera and Giummolè (1998).

When dealing with the multidimensional parameter case, the use of the square root of the determinant of the Godambe information matrix may be questionable. Following the advices for default priors, one possibility is to assume that the components of $\theta$ are a priori independent and to use the one-dimensional reference prior for each of the parameters. An alternative is to consider a sequential scheme as in Berger and Bernardo (1992). It should be noted that the ordering of the parameters is relevant. Unless the practitioner has a specific ordering in mind, different orderings may lead to different reference priors.

Appendix

Proof of Theorem 3.1

Proof. The posterior (8) can equivalently be written as

$$
\pi_{SR}(\theta|x) = \frac{\pi(\theta) \exp \{-S(\theta^*) + S(\tilde{\theta})\}}{\int_{\Theta} \pi(\theta) \exp \{-S(\theta^*) + S(\tilde{\theta})\} d\theta},
$$

with $\theta^* = \tilde{\theta} + C(\theta - \tilde{\theta})$ and $C$ fixed such that $C^T K(\theta) C = G(\theta)$.

Let $w = (w_1^1, \ldots, w_k^k)^T = n^{1/2}(\theta - \tilde{\theta})$. Then, $\theta = \tilde{\theta} + n^{-1/2} w$ and $\theta^* = \tilde{\theta} + n^{-1/2} C w$. A posterior for $w$ is

$$
\pi_{SR}(w|x) = \frac{b(w, x)}{\int b(w, x) dw},
$$

with

$$
b(w, x) = \pi(\tilde{\theta} + n^{-1/2} w) \exp \{-S(\tilde{\theta} + n^{-1/2} C w) + S(\tilde{\theta})\}.
$$
Now
\[ \pi(\tilde{\theta} + n^{-1/2} w) = \tilde{\pi} \left( 1 + n^{-1/2} R_1(w) + \frac{1}{2} n^{-1} R_2(w) \right) + O_p(n^{-3/2}), \]
with
\[ R_1(w) = \frac{\bar{\pi}}{\bar{n}} w^i \quad \text{and} \quad R_2(w) = \frac{\bar{\pi} i j}{\bar{n}} w^{ij}, \]
where \( w^{ij} = w^i w^j \ldots \) is a product of components of \( w \). Moreover,
\begin{align*}
-S(\tilde{\theta} + n^{-1/2} C w) + S(\tilde{\theta}) &= -n^{-1} \left( \frac{1}{2} (C w)^{ij} \tilde{S}_{ij} + \frac{1}{6} n^{-1/2} (C w)^{ijk} \tilde{S}_{ijk} + \frac{1}{24} n^{-1} (C w)^{ijh} \tilde{S}_{ijh} \right) \\
&= -n^{-1} w^T \left( C^T \frac{\partial^2 \tilde{S}}{\partial \theta \partial \theta^T} C \right) w - \frac{1}{6} n^{-1/2} R_3(w) \\
&\quad - \frac{1}{24} n^{-1} R_4(w) + O_p(n^{-3/2}) \\
&= -n^{-1} \left( \frac{1}{2} w^T \tilde{H} w - \frac{1}{6} n^{-1/2} R_3(w) - \frac{1}{24} n^{-1} R_4(w) + O_p(n^{-3/2}) \right), \quad (19)
\end{align*}
where
\[ R_3(w) = n^{-1} (C w)^{ijk} \tilde{S}_{ijk} \quad \text{and} \quad R_4(w) = n^{-1} (C w)^{ijh} \tilde{S}_{ijh}. \]

The numerator in Bayes’ formula can thus be written as
\begin{align*}
b(w, x) &= \tilde{\pi} \left( 1 + n^{-1/2} R_1(w) + \frac{1}{2} n^{-1} R_2(w) \right) \\
&\quad \exp \left\{ -\frac{1}{2} w^T \tilde{H} w \right\} \left( 1 - \frac{1}{6} n^{-1/2} R_3(w) - \frac{1}{24} n^{-1} R_4(w) + \frac{1}{72} n^{-1} R_3(w)^2 \right) + O_p(n^{-3/2}) \\
&= \tilde{\pi} \exp \left\{ -\frac{1}{2} w^T \tilde{H} w \right\} \left[ 1 + n^{-1/2} \left( R_1(w) - \frac{1}{6} R_3(w) \right) \\
&\quad + n^{-1} \left( \frac{1}{2} R_2(w) - \frac{1}{6} R_1(w) R_3(w) - \frac{1}{24} R_4(w) + \frac{1}{72} R_3(w)^2 \right) \right] + O_p(n^{-3/2}). \quad (20)
\end{align*}

The denominator can be approximated using the moments of the \( k \)-variate normal distribution \( N_k(0, \tilde{H}^{-1}) \). We have that
\begin{align*}
\int b(w, x) dw &= \tilde{\pi}(2\pi)^{k/2}|\tilde{H}|^{-1/2} \left[ 1 + n^{-1} \left( \frac{1}{2} E(R_2(W)) - \frac{1}{6} E(R_1(W) R_3(W)) \right) \\
&\quad - \frac{1}{24} E(R_4(W)) + \frac{1}{72} E(R_3(W)^2) \right] + o(n^{-1}), \quad (21)
\end{align*}
being \( E(R_1(W)) \) and \( E(R_3(W)) \) both related to odd moments and thus equal to zero. Now,
\begin{align*}
T_1 &= E(R_2(W)) = \frac{\bar{\pi} i j}{\bar{n}} \tilde{S}_{ij}, \\
T_2 &= E(R_1(W) R_3(W)) = \frac{3 \bar{\pi} i j k h}{\bar{n}} c_{ij} c_{ks} c_{ht} \tilde{S}_{ij} \tilde{S}_{kh}.
\end{align*}
$$T_3 = E(R_4(W)) = 3\frac{S_{ijkh}}{n}c_{ir}c_{js}c_{kt}c_{hu}\tilde{h}_{rs}^rh_{tu}$$

and

$$T_4 = E(R_3(W)^2) = \frac{S_{ijk}S_{rst}}{n^2}(9c_{ia}c_{jb}c_{ke}c_{rd}c_{se}c_{tf}\tilde{h}_{ab}^rh_{cd}^rh_{ef}^r + 6c_{ia}c_{rb}c_{jc}c_{sd}c_{ke}c_{tf}\tilde{h}_{ab}^rh_{cd}^rh_{ef}^r).$$

Putting together expressions (20) and (21), we finally obtain the result with

$$A_1(w) = R_1(w) - \frac{1}{6}R_3(w)$$

and

$$A_2(w) = \frac{1}{2}(R_2(w) - T_1) - \frac{1}{6}(R_1(w)R_3(w) - T_2) - \frac{1}{24}(R_4(w) - T_3) + \frac{1}{72}(R_3(w)^2 - T_4).$$

Proof of Theorem 4.1

Proof. The proof follows the same steps as in Liu et al. (2014), generalized for the use of a SR-posterior (8) instead of the classic posterior distribution.

First of all, notice that the functional (11) associated to an $\alpha-$divergence between the prior and the posterior can be written as

$$T(\pi) = \frac{1}{\alpha(1 - \alpha)} \left[ 1 - \int_{\Theta} \int_{X} \pi(\theta)^\alpha \pi_{SR}(\theta|x)^{1-\alpha} \frac{\pi(\theta)p(x|\theta)}{\pi_{SR}(\theta|x)} d\theta dx \right]$$

$$= \frac{1}{\alpha(1 - \alpha)} \left[ 1 - \int_{\Theta} \int_{X} \pi(\theta)^{\alpha+1} \pi_{SR}(\theta|x)^{-\alpha} p(x|\theta) d\theta dx \right]$$

$$= \frac{1}{\alpha(1 - \alpha)} \left[ 1 - \int_{\Theta} \pi(\theta)^{\alpha+1} \left( \int_{X} \pi_{SR}(\theta|x)^{-\alpha} p(x|\theta) dx \right) d\theta \right]$$

$$= \frac{1}{\alpha(1 - \alpha)} \left[ 1 - \int_{\Theta} \pi(\theta)^{\alpha+1} E[\pi_{SR}(\theta|X)^{-\alpha}|\theta] \right],$$

where $E(\cdot|\theta)$ denotes expectation with respect to the conditional distribution of $X$ given $\theta$.

Now, we use the shrinkage argument for evaluating the expected value $E[\pi_{SR}(\theta|X)^{-\alpha}|\theta]$. See Datta and Mukerjee (2004) for details on the shrinkage method. Here we use a modified shrinkage argument taking into account for the context.

Let us first consider the case of $0 < |\alpha| < 1$. The first step of the shrinkage method involves fixing a suitable prior distribution $\tilde{\pi}(\theta)$ and calculating the expected value of $\pi_{SR}(\theta|X)^{-\alpha}$ with respect to the corresponding posterior $\tilde{\pi}_{SR}(\theta|X)$, i.e. $E^*[\pi_{SR}(\theta|X)^{-\alpha}|X]$. Here, we use
the scoring rule posterior as $\tilde{\pi}_{SR}(\theta|X)$. Noting that $\tilde{H}$ tends to $G$ as $n \to \infty$, we have that

$$E^\pi \left[ \pi_{SR}(\theta|X)^{-\alpha}|X \right] = E^\pi \left[ (2\pi)^{k\alpha/2} n^{-\alpha/2} |\tilde{H}|^{-\alpha/2} \exp \left\{ \frac{n\alpha}{2} (\theta - \tilde{\theta})^T \tilde{H} (\theta - \tilde{\theta}) \right\} |X \right] + O(n^{-1})$$

$$= (2\pi)^{k\alpha/2} n^{-\alpha/2} |\tilde{H}|^{-\alpha/2} E^\pi \left[ \exp \left\{ \frac{n\alpha}{2} (\theta - \tilde{\theta})^T \tilde{H} (\theta - \tilde{\theta}) \right\} |X \right] + O(n^{-1})$$

$$= (2\pi)^{k\alpha/2} n^{-\alpha/2} |\tilde{H}|^{-\alpha/2} \int_\Theta \exp \left\{ \frac{n\alpha}{2} (\theta - \tilde{\theta})^T \tilde{H} (\theta - \tilde{\theta}) \right\} \phi_k(\theta - \tilde{\theta}, \tilde{H}) d\theta + O(n^{-1})$$

$$= (2\pi)^{k\alpha/2} n^{-\alpha/2} |G(\theta)|^{-\alpha/2} (1 - \alpha)^{-k/2} + O(n^{-1}).$$

By integrating again with respect to the distribution of $X$ given $\theta$ and with respect to the prior $\tilde{\pi}(\theta)$ and by letting the prior go to $\theta$, we finally obtain

$$E \left[ \pi(\theta|X)^{-\alpha}|\theta \right] = (2\pi)^{k\alpha/2} n^{-\alpha/2} |G(\theta)|^{-\alpha/2} (1 - \alpha)^{-k/2} + O(n^{-1}). \quad (23)$$

Substituting (23) in (22) we can see that the selection of a prior $\pi(\theta)$ corresponds to the minimization with respect to $\pi(\theta)$ of the functional

$$\frac{1}{\alpha(1 - \alpha)} \int_\Theta \pi^{\alpha+1}(\theta)|G(\theta)|^{-\alpha/2} d\theta. \quad (24)$$

When $0 < |\alpha| < 1$, it is easy to show that the prior $\pi_G(\theta)$ which asymptotically minimizes (24) is proportional to $|G(\theta)|^{1/2}$, i.e. the square root of the determinant of the Godambe information matrix.

A similar proof also holds for $\alpha \to 0$, which corresponds to the Kullback-Leibler divergence. The result is again the Jeffreys-type prior $\pi_G(\theta) \propto |G(\theta)|^{-1/2}$.

The case $\alpha = -1$ corresponds to the Chi-squared divergence. For this case the proof requires higher order terms in the expansion of the scoring rule posterior distribution. Thus, using (10) up to order $O_p(n^{-1/2})$ and the shrinkage argument, we can first evaluate

$$E^\pi \left[ \pi_{SR}(\theta|X)|X \right] = n^{1/2} \int \pi_{SR}(w|X) \tilde{\pi}_{SR}(w|X) dw$$

$$= (2\pi)^{-k/2} \left( \frac{|\tilde{H}|}{2} \right)^{1/2} \left\{ 1 + \frac{1}{4n} \left[ -\left( \frac{\tilde{\pi}_{ij}}{\pi} + \frac{\tilde{\pi}_{ij}}{\pi} \right) \tilde{h}^{ij} \right. \right.$$  

$$\left. + \left( \frac{\tilde{\pi}_i}{\pi} + \frac{\tilde{\pi}_i}{\pi} \right) \tilde{S}_{khl} \frac{c_{kij}c_{ljs}c_{tst}}{n} \tilde{h}^{st} \tilde{h}^{ij} + 2 \frac{\tilde{\pi}_{ij}}{\pi} \tilde{\pi}_{ij} \tilde{h}^{ij} + A \right\} + O(n^{-1}),$$

where $A(\theta)$ is a function that does not involve the prior nor its derivatives.

Now, let $E[S_{ijk}/n|\theta] = B_{ijk}(\theta) + o(n^{-1/2})$ and recall that $E[\tilde{H}|\theta] = G(\theta) + o(n^{-1/2})$. By integrating again the above expression with respect to the distribution of $X$ given $\theta$ we get

$$E^\pi \left[ \pi_{SR}(\theta|X)|X \right| \theta] = (2\pi)^{-k/2} \left( \frac{|G(\theta)|}{2} \right)^{1/2} \left\{ 1 + \frac{1}{4n} \left[ -\left( \frac{\tilde{\pi}_{ij}}{\pi} + \frac{\tilde{\pi}_{ij}}{\pi} \right) g^{ij} \right. \right.$$  

$$+ \left( \frac{\tilde{\pi}_i}{\pi} + \frac{\tilde{\pi}_i}{\pi} \right) B_{khl} c_{kij} c_{ljs} c_{tst} g^{st} g^{ij} + 2 \frac{\tilde{\pi}_{ij}}{\pi} \tilde{\pi}_{ij} g^{ij} + F(\theta) \right\} + O(n^{-1}), \quad (24)$$
where \( g_{ij} \) and \( g^{ij} \) are the components of \( G \) and \( G^{-1} \) respectively and \( F(\theta) \) is a function that does not involve the prior nor its derivatives.

By integrating again with respect to the prior \( \bar{\pi}(\theta) \) and by letting the prior go to \( \theta \), we finally obtain the expected value needed in (22) for \( \alpha = -1 \):

\[
E[\pi_{SR}(\theta|X)|\theta] = (2\pi)^{-k/2} \left( \frac{n|G|}{2} \right)^{1/2} \left\{ 1 + \frac{1}{4n} \left[ -3g^{ij} \frac{\pi_{ij}}{\pi} \right. \right. \\
+ \left. \left. a_j g^{ij} - |G|^{-1} \frac{\partial |G|}{\partial \theta^j} g^{ij} - 2 \frac{\partial g^{ij}}{\partial \theta^j} \right] \frac{\pi_i}{\pi} + 2g^{ij} \frac{\pi_i}{\pi} \frac{\pi_j}{\pi} + O(n^{-1}) \right\}
\]

(25)

where \( a_j = B_{khl}c_{kj}c_{hs}c_{lt}g^{st} \) and \( M(\theta) \) is a function that does not involve the prior nor its derivatives.

Substituting (25) in (22) with \( \alpha = -1 \), we find that maximization of the average Chi-squared divergence is equivalent to maximization of

\[
\int_{\Theta} |G|^{1/2} \left[ -3g^{ij} \frac{\pi_{ij}}{\pi} + \left( a_j g^{ij} - |G|^{-1} \frac{\partial |G|}{\partial \theta^j} g^{ij} - 2 \frac{\partial g^{ij}}{\partial \theta^j} \right) \frac{\pi_i}{\pi} + 2g^{ij} \frac{\pi_i}{\pi} \frac{\pi_j}{\pi} \right] d\theta.
\]

Let \( y = y(\theta) = (y_1(\theta), \ldots, y_k(\theta))^T \), with \( y_i(\theta) = \pi_i(\theta)/\pi(\theta), \ i = 1, \ldots, k \), and let \( y' \) be the matrix of partial derivatives of \( y \) with respect to \( \theta \), i.e. \( y' = (y_{ij})_{ij} \), \( y_{ij} = \partial y_i/\partial \theta^j \), \( i, j = 1, \ldots k \). Then, \( \pi_{ij}/\pi = y_{ij} + y_i y_j \) and the quantity to be maximized can be rewritten as

\[
\int_{\Theta} |G|^{1/2} \left[ -3g^{ij} y_{ij} + \left( a_j g^{ij} - |G|^{-1} \frac{\partial |G|}{\partial \theta^j} g^{ij} - 2 \frac{\partial g^{ij}}{\partial \theta^j} \right) y_i - g^{ij} y_i y_j \right] d\theta.
\]

Let us denote the integrand function by \( U(\theta, y, y') \). The solution to the maximization problem is found by solving the system of Euler-Lagrange equations:

\[
\frac{\partial U}{\partial y_i} - \sum_{j=1}^k \frac{\partial}{\partial \theta^j} \left( \frac{\partial U}{\partial y_{ij}} \right) = 0, \quad i = 1, \ldots, k.
\]

(26)

After some calculations, we obtain the solution to the variational problem as

\[
y(\theta) = \frac{\partial \log \pi(\theta)}{\partial \theta} = \frac{1}{4} \left[ 2a + |G|^{-1} \frac{\partial |G|}{\partial \theta} + 2G \nabla \cdot G^{-1} \right],
\]

where \( a = (a_1, \ldots, a_k)^T \) and \( \nabla \cdot G^{-1} = (\partial g^{ij}/\partial \theta^j, \ldots, \partial g^{kj}/\partial \theta^j)^T \).

\[\Box\]

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