Haldane exclusion statistics and the charge fractionalisation in chiral bags

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Abstract

It is proposed that the phenomenon of charge fractionalisation of the spatially confined particle in a topological chiral bag may be interpreted as a manifestation of the exclusion statistics proposed by Haldane. The fractional exclusion statistics parameter is just the fractional baryon charge $Q$ of the particle in this case. We also state the necessary conditions for Haldane fractional to occur for systems in higher dimensions.

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Haldane \[1\] has recently proposed a new exclusion statistics for quasi particles that has attracted a lot of attention \[2\, 3\]. His definition holds for systems with finite dimensional Hilbert spaces and is motivated by physical examples, such as quasi-particles in the fractional quantum Hall systems and spinons in antiferromagnetic spin chains. Haldane first defines \(d_N\), the dimension of the single particle space when there are \(N\) particles present, as follows: Consider the \(N\) particle wave function \(\psi(x_1, x_2, ..., x_N)\). Fix any \(N - 1\) of the variables at any \(N - 1\) points and look upon the wavefunctions as functions of the remaining single particle variable. The dimension of the space spanned by these functions, \(d_N\), then defines the single particle Hilbert space of dimension \(d_N\) in the presence of \(N - 1\) other particles. To clarify the definition, consider particles on a lattice of dimension \(d_1\). For fermions, if there are \(N\) particles and we fix the position of \(N - 1\) particles on the lattice, then the remaining single particle can only occupy any of the \(d_1 - (N - 1)\) remaining positions due to Pauli blocking. Thus the fermionic single particle dimension as defined by Haldane is \(d^F_N = d_1 - (N - 1)\). Bosons on the other hand can occupy any of the \(d_1\) sites irrespective of how many other particles are there. Therefore \(d^B_N = d_1\). Haldane then considers systems with a generalized Pauli blocking whose single particle space is given by,

\[d_N = d_1 - g(N - 1), \quad (1)\]

where \(d_N\), as stated above, is the number of states accessible to a particle in the presence of \(N - 1\) other particles. The exclusion statistics parameter \(g\) is then given by

\[g = -\frac{\Delta d}{\Delta N} \quad (2)\]

where \(\Delta d\) is the change in the dimension of the single particle space and \(\Delta N\) is the change in the number of particles when the size of the system and the boundary conditions are unchanged. Thus \(g\) is a measure of the (partial) Pauli blocking in the system, \(g = 0(1)\) corresponds to bosons (fermions). Recently this new definition of statistics has been generalised to the case of systems with infinite dimensional Hilbert spaces \[4\]. It was found that in such cases the exclusion statistics of a system of interacting fermions (or bosons) was
non-trivial if the interaction was such that the addition of a particle to the system caused a scale invariant shift to the energy all the of other particles in the system. Interacting systems in one-dimension, like the quasiparticles of Calogero-Sutherland Model, are generic examples of this phenomenon. Anyons (as defined by the exchange phases) in two dimensions is another example [4]. In this letter, by considering a class of relativistic models we are able to relate the Haldane statistical interpolating factor \( g \) to the fractional charge. We do this by first demonstrating the relation between fractional baryon charge and the Haldane statistical parameter in a 1+1 dimensional chiral bag model due to Zahed [6] and then generalize the result to the chiral bag model in 3+1 dimensions. This connection between the statistical parameter and the fractional charge indicates that \( g \) may have topological significance. A hint of this already emerges from the elegant proof [7] that the second virial coefficient of a nonrelativistic anyon gas in two space dimensions is determined by the axial anomaly of a 1+1- dimensional fermionic system gauged by a vortex vector field. This is so since \( g \) itself has been shown [4] to be determined by the high temperature limit of the second virial coefficient in cases where a virial expansion exists. In the end we also remark about an analogous phenomenon in the Kondo system.

To make this connection between the Haldane parameter \( g \) and fractional charge we first give a regulated definition of \( d_N \) applicable to systems of interacting fermions where the many particle spectrum admits an interpretation in terms of effective single particle levels (which may be \( N \) dependent). Suppose these single particle energies are denoted by \( \epsilon^N_n \), then the regulated definition of \( d_N(\beta) \) is given by,

\[
d_N(\beta) = \sum_{n(\text{unocc})} e^{-\beta \epsilon^N_n},
\]

(3)

where \( \beta \) is the inverse temperature and \( d_N = \lim_{\beta \to 0} d_N(\beta) \). The sum is taken over all the unoccupied levels. The occupancies could correspond to any state in the \( N \)-particle sector. For convenience, we will always work with the ground state. We can then compute the quantity,

\[
\Delta d = \lim_{\beta \to 0} (d_{N+1}(\beta) - d_N(\beta)).
\]

(4)
If this limit exists and is $N$ independent, then the system can be interpreted in terms of Eq.(1) and we have $g = -\Delta d$.

To be more specific we first apply this definition to the Calogero-Sutherland Model (CSM) which can be looked upon either as a system of interacting fermions or ideal exclusion anyons in one dimension. It is a system of non-relativistic fermions in one dimension confined in a harmonic oscillator potential with frequency $\omega$ and interacting via an inverse-square potential $\sum_{i<j} \lambda(\lambda+1)/(r_{ij})^2$, where $i,j$ runs over all particle positions and $\lambda$ is the interaction parameter. Note that the range of $\lambda$ is limited to $\lambda \geq -1/2$ in the fermionic basis. Otherwise the system does not lead to bound states. The states can be labelled by sets of occupied harmonic oscillator levels. The $N$ particle energy spectrum is given by,

$$E[\{n_i\}] = E_0[\{n_i\}] + \lambda\omega N(N-1)/2,$$  \hspace{1cm} (5)

where $E_0$ is the energy of the $N$ non-interacting fermionic system. This can be interpreted in terms of effective single particle levels $\epsilon_n^N = \omega(n + \frac{1}{2}) + \lambda\omega(N-1)$. (Note that the interaction energy is divided by a factor 2 in the total energy to avoid overcounting.) The $d_N(\beta)$ can now be calculated using eq.(3) and we find that,

$$d_N(\beta) = \frac{e^{-\beta\omega(\frac{1}{2}+\lambda+1)(N-1))}}{1-e^{-\beta\omega}}.$$

Using eq.(4) it follows, in the high temperature limit ($\beta \to 0$), that $\Delta d = -(1 + \lambda)$ and hence we obtain $g = 1 + \lambda$ which is known to be the correct result and demonstrates that our approach is consistent.

As advertised, we now relate the Haldane statistical parameter $g$ to the vacuum charge (under certain conditions) in a relativistic system. The model with which we first illustrate this connection is the (1+1)-dimensional chiral bag model due to Zahed. This simple model is chosen because it is completely solvable, and brings about the main features of the (3+1)-dimensional chiral bag in this context. For our purposes, it is sufficient to consider a flavour-singlet Dirac particle with unit baryon number or charge, $N=1$. The model is characterised by dividing the spatial dimension into three sections, the exterior solitonic
coat, the interior bag with the confined fermion and the boundary between the two where the fermion interacts with the soliton. In the notation of ref. [6], the Lagrangian density is given by,

\[ \mathcal{L} = \left[ i \frac{\gamma_\mu \partial^\mu q - B}{\theta_v} - \frac{1}{2} \bar{q} e^{-i \theta_v} \Delta_s + \frac{1}{2} (\partial_\mu \theta)(\partial^\mu \theta) - k^2 (1 + \cos(\theta)) \right] (1 - \theta_v), \]  

\hspace{1cm} (6)

where \( \theta_v \) is a theta function at the boundary defined at a radius R which ensures that the quarks remain inside the bag and \( \Delta_s \) is the surface delta-function.

The one baryon number solution then consists of the \( \theta \) field being the 1-kink solitonic solution of the mesonic action and with one extra fermion in the bag. As emphasised by Zahed, the chiral boundary condition gives a global rotation to the solutions of the Dirac equation and causes all the energy levels to shift by \( \theta(R)/2R \). They are given by,

\[ \epsilon_n^1 = \left( \frac{2n + 1}{4} \right) \frac{\pi}{R} \]  

\[ + \frac{\theta(R)}{2R} = \frac{\pi}{2R} (n + 1) + \frac{\theta(R)}{\pi}. \]  

\hspace{1cm} (7)

For the purposes of regularization we also define the so called “empty bag” as follows: Consider the situation in which particles are confined in a region of radius R. The energy levels of the particles are obtained by solving the Dirac equation and are given by,

\[ \epsilon_n^0 = \left( \frac{2n + 1}{4} \right) \frac{\pi}{R}. \]  

\hspace{1cm} (8)

Now consider a situation where all the negative energy states are fully occupied with no valence particle present. We refer to this system as the “empty bag”. The states inside the empty bag therefore correspond to the solutions of the Zahed model (eq.(6)) in the zero baryon number sector. The solution of the sine-Gordon equation in this sector is \( \theta(R) = \) constant which may be set equal to zero. Hence there is no chiral coupling. Thus for a fixed R, we have an \( N \)-particle\( (N \to \infty) \) system which is nothing but the filled negative energy sea (the zero baryon number sector). As soon as the valence particle is introduced, the solitonic field deforms due to the chiral coupling at the boundary R and we regard this as the \( (N + 1) \)-particle system (one baryon number sector). We are now interested in computing \( \Delta d \) a la eq.(4).
The total number of states of the empty bag, for a fixed boundary at R, is obtained as the sum over occupied and unoccupied levels and is given by,
\[ d_+ (\beta) + d_- (\beta) = \sum_{n=-\infty}^{n=\infty} e^{-\beta \frac{\pi}{2} |n+\frac{1}{2}|} = \frac{4R}{\pi \beta} + O(\beta), \tag{9} \]
where - and + subscripts denote the occupied and unoccupied states. We now consider placing a valence particle keeping R the same as before. The particles are now free in the finite confined region but interact with the meson fields outside at the boundary. The total number of states is given by,
\[ d^v_+ (\beta) + d^v_- (\beta) = \sum_{n=-\infty}^{n=\infty} e^{-\beta \pi R |n+\frac{1}{2} + \frac{\theta(R)}{\pi}|} = \frac{4R}{\pi \beta} + O(\beta), \tag{10} \]
where the dependence on the chiral angle occurs at \( O(\beta) \). Therefore in the high temperature limit,
\[ \lim_{\beta \to 0} d_+ (\beta) + d_- (\beta) = \lim_{\beta \to 0} d^v_+ (\beta) + d^v_- (\beta) \tag{11} \]
Hence the change in the number of available single particle states with the addition of a valence particle to the system is given by,
\[ \Delta d = \lim_{\beta \to 0} (d^v_+ (\beta) - d_+ (\beta)) = - \lim_{\beta \to 0} (d^v_- (\beta) - d_- (\beta)) = \frac{1}{2} \lim_{\beta \to 0} (d^v_+ (\beta) - d^v_- (\beta)) = - \frac{\theta(R)}{\pi}. \tag{12} \]
Note that the last part of the above equation is just the spectral asymmetry in the presence of the chiral coupling and \( \theta(R)/\pi \) is precisely the result one obtains for the baryon charge residing in the bag (see below). Therefore by definition (see eq.(1) and eq.(4))
\[ g = \frac{\theta(R)}{\pi}. \tag{13} \]
Essentially what is happening in this model is that the addition of one fermion in the bag causes the \( \theta \)- field to deform to a soliton thus causing all the single particle energies to shift.

The above results for the Haldane statistical parameter may also be interpreted in terms of the vacuum charge. In a relativistic problem, the vacuum charge (or equivalently the topological charge) of the vacuum is given by the spectral asymmetry [1].
\[ N_v = -\frac{1}{2} \lim_{\beta \to 0} \left[ \sum_{\epsilon_n \geq 0} e^{-\beta |\epsilon_n|} - \sum_{\epsilon_n < 0} e^{-\beta |\epsilon_n|} \right]. \] (14)

For \( \theta < \pi/2 \), the vacuum charge is given by \( N_v = \theta/\pi \) which is therefore the same as the Haldane \( g \) parameter. However, for \( \theta > \pi/2 \), the vacuum charge is given by \( N_v = \theta/\pi - 1 \).

The discontinuity at \( \pi/2 \) is caused by the null mode at \( \theta = \pi/2 \). However, in the computation of the net baryon number \( Q \), there is no discontinuity since the number of particles in the vacuum is decreased by 1 for \( \theta > \pi/2 \) as the valence particle has emerged out of the vacuum. The excess charge is then given by

\[
Q = N_v = \frac{\theta}{\pi}, \quad \theta < \pi/2 \\
Q = N_v + 1 = \frac{\theta}{\pi}, \quad \theta \geq \pi/2.
\]

Therefore the Haldane statistical parameter \( g \) is the same as the fractional baryon charge residing in the bag. Note that the net baryon charge is still unity since the sum of the baryon charge carried by the chiral field outside and that inside the bag is equal to 1.

We now consider the physically relevant 3+1 dimensional SU(2) chiral bag model \([12]\). This model is exactly analogous to the Zahed model described before. As in that case the space is divided into three regions. Inside a sphere of radius \( R \), we have non-interacting quarks obeying the massless Dirac equation. Outside the bag are mesons described by the SU(2) Skyrme model. The boundary conditions satisfied by the Dirac fermions are

\[
-i \bar{\gamma} \cdot \hat{n}(\vec{x})q(\vec{x}) = (U(\vec{x})P_L + U^\dagger(\vec{x})P_R)q(\vec{x}),
\] (15)

where \( q \) are the quark fields, \( \vec{x} \) is any point on the boundary such that \( |\vec{x}| = R \), \( U(\vec{x}) \) is the SU(2) valued field of the Skyrme model and \( P_L, P_R \) are left and right projection operators. The one baryon state is described by putting one extra quark (per colour) in the bag and a winding number 1 skyrmion outside the bag. The skyrmion is given by the hedgehog solution \( U(\vec{x}) = e^{i\theta(\vec{r}) \vec{z} \cdot \vec{x}/2} \) with \( \theta(\infty) = 0 \), \( \theta(0) = -\pi \). Goldstone and Jaffe \([13]\) have calculated the vacuum charge, \( N_v \) of the single particle Dirac hamiltonian with the boundary condition in eq.(15). They find
\[ N_v(\theta) = \frac{1}{\pi}(\theta - \sin \theta \cos \theta), \quad -\pi/2 < \theta < \pi/2 \]  
(16)

\[ N_v(\theta + \pi) = N_v(\theta). \]  
(17)

Hence, just as in the Zahed model, the excess charge is given by,

\[ Q = 1 + N_v(\theta), \quad -\pi/2 \leq \theta \leq 0 \]  
(18)

\[ Q = N_v(\theta), \quad -\pi \leq \theta \leq -\pi/2. \]  
(19)

Using the eq.(16), we find,

\[ Q = \frac{1}{\pi}(\pi + \theta - \sin \theta \cos \theta). \]  
(20)

Since \( g = \Delta Q \) as demonstrated with the Zahed model explicitly, we obtain the Haldane statistics parameter in the 3-dimensional chiral bag case; indeed this is the first time an example of nontrivial statistics in dimensions higher than 2 has been discussed. In the end we also point out another example of Haldane fractional statistics in higher dimensions, namely the Kondo system. We note that in the 3+1 dimensional example we did not make use of the effective single particle spectrum but obtained \( g \) by directly relating it to the excess charge. While it is difficult to obtain analytically the spectrum of states in 3+1 dimensional case, numerical calculations [14] indeed indicate a flow similar to that in the Zahed model further buttressing the connection between fractional charge and the statistics parameter.

To summarize, we have shown that in chiral bag models, the dimension of the single particle Hilbert space of quarks changes by a fraction, equal to the fractional charge residing in the bag, when one quark is added to the system. This indicates that they satisfy the generalized Pauli principle as defined by Haldane with the exclusion statistics parameter \( g \). However there is one important distinction to be made before accepting this interpretation: Haldane had defined the generalised Pauli principle for systems of fixed size and boundary conditions. In our example, the boundary conditions on the quarks apparently change and indeed this is the cause of charge fractionalization. We should, however, remember, that
the boundary conditions are not being changed by hand but by the dynamics of the model. The physical condition at the boundary, that is the conservation of the axial current, is not changed when a particle is added. Therefore the phenomenon of charge fractionalization in chiral bags, which we have shown is the same as the phenomenon of fractional Pauli blocking, is not put in by hand but is a consequence of the dynamics of the system. While the results appear very general, it should be remembered that the statistics of many body systems that we are investigating is really vacuum plus one valence particle. We are not able, even in the simple 1+1 model, to deal with more than one kink solution.

Our analysis clearly shows that in chiral bag models, the Haldane statistical parameter is a topological quantity. The fractional charge $Q$ of the quark plus the bag and the fractional charge of the soliton are not independent of each other since the sum has to be equal to the total integral baryon number which is a global quantum number of the system and hence conserved. The solitonic fractional charge is determined from the topological current and hence $Q$ is a topological quantity. Clearly then the Haldane statistics parameter $g$ has topological significance. In a more general context, the same result may be gleaned from the work of Comtet and Ouvry [7] who relate the second virial coefficient of an anyon gas to the chiral anomaly in a 1+1 dimensional chiral invariant theory. Such anomalies usually arise from topological interactions in the Lagrangian. The second virial coefficient of the anyon gas is also related to the Haldane statistics parameter $g$ albeit in a nontrivial way [4]. In the 3+1 dimensional chiral bag model, discussed above, the following features are responsible for the fractional Haldane statistics: (1) the particles have internal degrees of freedom (isospin) which couple to the condensate field that can carry a topological charge and (2) the system is defined in a region with a boundary and the particles are coupled to the condensates only at the boundary.

Infact, it is possible that any system in which the above two features occur the system obeys Haldane fractional statistics. An interesting analogous system is the Kondo system. This system consists of $k$ species of band electrons in three dimensions coupled to a spin $S$ impurity at the origin. In a recent work [15] it was shown that for the low energy physics
of this system, the interaction can be replaced by a dynamical boundary condition. Thus
the problem has features (1) and (2) if we assume that the electron system is defined in
all space except the origin. The Kondo problem maps on to a one dimensional effective
system since only the s-wave orbitals see the impurity. The solution for the low energy
physics is a conformal field theory. It has been shown that the scaling dimensions of vertex
operators in conformal field theories are related to the Haldane statistics parameter of the
quasiparticles created by them \[4,16\]. In the overscreened case, \(k > 2S\), the solution to
the Kondo problem has operators with scaling dimensions leading to quasiparticles with
fractional exclusion statistics. For example, for \(k = 2, S = 1/2\) there are spinon excitations
with \(g = 1/2\). It thus appears that (1) and (2) are general features of higher dimensional
systems with fractional statistics.

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