A fully nonlinear version of the Yamabe problem and a Harnack type inequality

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We present some results in [9], a continuation of our earlier works [7] and [8]. One result is the existence and compactness of solutions to a fully nonlinear version of the Yamabe problem on locally conformally flat Riemannian manifolds, and the other is a Harnack type inequality for general conformally invariant fully nonlinear second order elliptic equations.

Let \((M, g)\) be an \(n\)-dimensional, compact, smooth Riemannian manifold without boundary, \(n \geq 3\), consider the Weyl-Schouten tensor

\[
A_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} g \right),
\]

where \(Ric_g\) and \(R_g\) denote respectively the Ricci tensor and the scalar curvature associated with \(g\). We use \(\lambda(A_g)\) to denote the eigenvalues of \(A_g\) with respect to \(g\).

Let \(\hat{g} = u^4 g\) be a conformal change of metrics, then (see, e.g., [17]),

\[
A_{\hat{g}} = -\frac{2}{n-2} u^{-1} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-2} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g + A_g. \quad (1)
\]

Let

\[
\Gamma \subset \mathbb{R}^n \text{ be an open convex cone with vertex at the origin ,}
\]

\[
\{ \lambda \in \mathbb{R}^n \mid \lambda_i > 0, 1 \leq i \leq n \} \subset \Gamma \subset \{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i > 0 \},
\]

\(\Gamma\) is symmetric in the \(\lambda_i\),

\[
f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ be concave and symmetric in the } \lambda_i,
\]

1
\[ f = 0 \text{ on } \partial \Gamma; \quad f_{\lambda_i} > 0 \text{ on } \Gamma \quad \forall \ 1 \leq i \leq n, \]  
\[ \lim_{s \to \infty} f(s\lambda) = \infty, \quad \forall \lambda \in \Gamma. \]  

**Theorem 1** ([9]) For \( n \geq 3 \), let \((f, \Gamma)\) satisfy (2), (3), (4), (5), (6) and (7), and let \((M, g)\) be an \( n \)-dimensional smooth compact locally conformally flat Riemannian manifold without boundary satisfying
\[ \lambda(A_g) \in \Gamma, \quad \text{on } M. \]  

Then there exists some smooth positive function \( u \) on \( M \) such that \( \hat{g} = u^{4/(n-2)} g \) satisfies
\[ f(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma, \quad \text{on } M. \]  

Moreover, if \((M, g)\) is not conformally diffeomorphic to the standard \( n \)-sphere, all solutions of the above satisfy, for any positive integer \( m \), that
\[ \|u\|_{C^m(M, g)} + \|u^{-1}\|_{C^m(M, g)} \leq C; \]  

where \( C \) is some constant depending only on \((M, g), (f, \Gamma)\) and \( m \).

**Remark 1** In the proof of Theorem 1, we see that the \( C^0 \) and \( C^1 \) apriori estimates above do not require the concavity of \( f \). More precisely, without the concavity assumption on \( f \) in the statement of Theorem 1, and when \((M, g)\) is not conformally diffeomorphic to the standard \( n \)-sphere, all solutions of (9) satisfy, for some constant \( C \) depending only on \((M, g), b, \delta_1 \) and \( \delta_2 \), that
\[ \|u\|_{C^1(M, g)} + \|u^{-1}\|_{C^1(M, g)} \leq C. \]  

For \( 1 \leq k \leq n \), let \( \sigma_k(\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), denote the \( k \)-th symmetric function, and let \( \Gamma_k \) denote the connected component of \( \{ \lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0 \} \) containing the positive cone \( \{ \lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0 \} \). Then (see [2]) \((f, \Gamma) = (\sigma_k^F, \Gamma_k)\) satisfies the hypothesis in Theorem 1.

For \((f, \Gamma) = (\sigma_1, \Gamma_1)\), hypothesis (8) is equivalent to \( R_g > 0 \) on \( M \), and therefore Theorem 1 in this case is the Yamabe problem for locally conformally flat manifolds with positive Yamabe invariants, and the result is due to Schoen ([12] and [13]). The Yamabe conjecture was proved through the work of Yamabe, Trudinger, Aubin and Schoen. For \((f, \Gamma) = (\sigma_k^F, \Gamma_k)\) with \( k = 2 \) and \( n = 4 \), the result was proved without the locally conformally flatness hypothesis of the manifold by Chang, Gursky and...
Yang \[3\]. For \((f, \Gamma) = (\sigma_k^\frac{1}{k}, \Gamma_k)\) with \(k = n \geq 3\), some existence result was established by Viaclovsky \[10\] for a class of manifolds which are not necessarily locally conformally flat. For \((f, \Gamma) = (\sigma_k^\frac{1}{k}, \Gamma_k)\), \(n \geq 3\), \(1 \leq k \leq n\), Theorem \[1\] was established in \[1\] and \[3\]; while the existence part in the case \(k \neq \frac{n}{2}\) was independently obtained by Guan and Wang in \[5\]. Subsequently, Guan, Viaclovsky and Wang \[4\] proved the algebraic fact that \(\lambda(A_g) \in \Gamma_k\) for \(k \geq \frac{n}{2}\) implies the positivity of the Ricci tensor, and therefore both the existence and compactness results in this case follow from known results. More recently, Gursky and Viaclovsky \[6\] have obtained existence results for \((f, \Gamma) = (\sigma_k^\frac{1}{k}, \Gamma_k)\), \(n = 3, 4\), on general Riemannian manifolds.

A Liouville type theorem for \((f, \Gamma) = (\sigma_k^\frac{1}{k}, \Gamma_k)\) was established in \[8\]. The crucial ingredient in our proof of the Liouville type theorem is a Harnack type inequality for \((f, \Gamma) = (\sigma_k^\frac{1}{k}, \Gamma_k)\) established in the same paper. In \[9\], we have established the Harnack type inequality for general conformally invariant fully nonlinear second order elliptic equations. In the following, \(S_n \times S_n\) denotes the set of \(n \times n\) real symmetric matrices, \(S^+_{n \times n} \subset S_{n \times n}\) denotes the set of positive definite matrices, \(O(n)\) denotes the set of \(n \times n\) real orthogonal matrices, and \(I\) denotes the \(n \times n\) identity matrix.

It was shown in \[8\] that 
\[
H(\cdot, u, \nabla u, \nabla^2 u) \text{ is conformally invariant on } \mathbb{R}^n
\]
(see \[8\] for the definition) if and only if
\[
H(\cdot, u, \nabla u, \nabla^2 u) \equiv F(A_u),
\]
and \(F\) is invariant under orthogonal conjugation, i.e.,
\[
F(O^{-1}MO) = F(M), \quad \forall M \in S^{n \times n}, \forall O \in O(n).
\]

Let \(U \subset S^{n \times n}\) be an open set satisfying
\[
O^{-1}UO = U, \quad \forall O \in O(n),
\]
\[
U \cap \{M + tN \mid 0 < t < \infty\} \text{ is convex } \forall M \in S^{n \times n}, N \in S^+_{n \times n}.
\]

Let \(F \in C^1(U)\) satisfy \[12\] and
\[
(F_{ij}(M)) > 0, \quad \forall M \in U,
\]
where \(F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M)\). We assume that for some \(\delta > 0\),
\[
F(M) \neq 1 \quad \forall M \in U \cap \{M \in S^{n \times n} \mid \|M\| := (\sum_{i,j} M_{ij}^2)^{\frac{1}{2}} < \delta\}.
\]
For $F_k(M) := \sigma_k^4(\lambda(M))$, and $U_k := \{M \in S^{n \times n} \mid \lambda(M) \in \Gamma_k\}$, it is well known that $(F, U) = (F_k, U_k)$ satisfies (13), (14), (12) and (16). In the following we use $B_R$ to denote a ball in $\mathbb{R}^n$ which is of radius $R$ and centered at the origin.

**Theorem 2** (9) For $n \geq 3$, let $U \subset S^{n \times n}$ satisfy (13) and (14), and let $F \in C^1(U)$ satisfy (12), (15) and (16). For $R > 0$, let $u \in C^2(B_{3R})$ be a positive solution of

$$F(A^u) = 1, \quad A^u \in U, \quad \text{in } B_{3R}. \tag{17}$$

Then

$$(\sup_{B_R} u)(\inf_{B_{2R}} u) \leq C(n) \delta_{2n}^2 R^{-n}, \tag{18}$$

where $C(n)$ is some constant depending only on $n$.

**Remark 2** In Theorem 2, there is no concavity assumption on $F$ and the constant $C(n)$ can be given explicitly.

**Remark 3** The Harnack type inequality (18) for $(F, U) = (F_1, U_1)$ was obtained by Schoen in [14] based on a Liouville type theorem of Caffarelli, Gidas and Spruck in [1]. Li and Zhang gave in [11] a different proof of Schoen’s Harnack type inequality without using the Liouville type theorem. For $(F, U) = (F_k, U_k)$, $1 \leq k \leq n$, the Harnack type inequality was established in our earlier work [8]. There are two new ingredients in our proof of Theorem 2. One is that we have developed, along the line of [8], new $C^0$ and $C^1$ estimates which allow us to extend the Harnack type inequality in [8] to this generality, and the other is that we have given a direct proof which makes it possible to give an explicit constant $C$ in (18). Arguments in [14], [11] and [8] were indirect and therefore no explicit value of $C$ was available, even in the case $(F, U) = (F_1, U_1)$.

We first present our proof of Theorem 2, more details can be found in [9]. As explained in [9], we may further assume without loss of generality that $f$ is homogeneous of degree 1. By (3) and (6), there exists a unique $b > 0$ such that $f(be) = 1$, where $e = (1, \cdots, 1)$. By (8), there exists some $\delta_1 > 0$ such that

$$f(\lambda) < 1, \quad \forall \lambda \in \Gamma, |\lambda| < \delta_1. \tag{19}$$

Fix some constant $\delta_2$ such that

$$0 < \delta_2 \leq \min_{x \in M} f(\lambda(A_g(x))). \tag{20}$$
Let \((\tilde{M}, \tilde{g})\) denote the universal cover of \((M, g)\), with \(i : \tilde{M} \to M\) a covering map and \(\tilde{g} = i^* g\). By a theorem of Schoen and Yau in [15], there exists an injective conformal immersion \(\Phi : (\tilde{M}, \tilde{g}) \to (S^n, g_0)\), where \(g_0\) denotes the standard metric on \(S^n\). Moreover, \(\Omega := \Phi(\tilde{M})\) is either \(S^n\) or an open and dense subset of \(S^n\). Fix a compact subset \(E\) of \(\tilde{M}\) such that \(i(E) = M\).

To prove Theorem 1, we will establish (11) first. In the following, let \(u \in C^\infty(M)\) be a positive solution of (11) with \(\tilde{g} = u^{\frac{4}{n-2}} g\).

**Step 1.** For some positive constant \(C\) depending only on \((M, g)\), \(b\), \(\delta_1\) and \(\delta_2\), we have
\[
\frac{1}{C} \leq u \leq C, \quad |\nabla_g u| \leq C \quad \text{on} \quad M. \tag{21}
\]

We will use notation \(F(A_g) := f(\lambda(A_g))\). We distinguish into two cases.

**Case 1.** \(\Omega = S^n\); **Case 2.** \(\Omega \neq S^n\).

In Case 1, \((\Phi^{-1})^* \tilde{g} = \eta^{\frac{4}{n-2}} g_0\) on \(S^n\), where \(\eta\) is a positive smooth function on \(S^n\). Let \(\tilde{u} = u \circ i\). Since \(F\left(A_{\tilde{g}}^{\frac{1}{u} - \frac{4}{n-2} g_0}\right) = 1\) on \(\tilde{M}\), we have \(F\left(A_{\eta^{\frac{4}{n-2}} g_0}\right) = 1\) on \(S^n\).

By corollary 1.1 in [8], \((\tilde{u} \circ \Phi^{-1})\eta = a |J\varphi|^\frac{4}{n-2}\) for some positive constant \(a\) and some conformal diffeomorphism \(\varphi : S^n \to S^n\). Since \(\varphi^* g_0 = |J\varphi|^\frac{2}{n-2} g_0\), we have, by the above equation, that \(f(a^{-\frac{4}{n-2}} (n-1)e) = f(a^{-\frac{4}{n-2}} \lambda(A_{g_0})) = 1\), i.e. \((n-1)a^{-\frac{4}{n-2}} = b\). With this explicit formula of \(\tilde{u}\), estimate (11) can be established without much difficulty.

In Case 2, \((\Phi^{-1})^* \tilde{g} = \eta^{\frac{4}{n-2}} g_0\) on \(\Omega\) where, by [15], \(\eta\) is a positive smooth function in \(\Omega\) satisfying \(\lim_{z \to \partial \Omega} \eta(z) = \infty\). Recall that \(\Omega\) is an open and dense subset of \(S^n\). Let \(u(x) = \max_M u\) for some \(x \in M\), and let \(i(\tilde{x}) = x\) for some \(\tilde{x} \in E\). By composing with a rotation of \(S^n\), we may assume without loss of generality that \(\Phi(\tilde{x}) = S^\prime\), the south pole of \(S^n\). Let \(P : S^n \to \mathbb{R}^n\) be the stereographic projection, and let \(v\) be the positive function on the open subset \(P(\Omega)\) of \(\mathbb{R}^n\) determined by \((P^{-1})^* \eta^{\frac{4}{n-2}} g_0 = v^{\frac{4}{n-2}} g_{\text{flat}}\), where \(g_{\text{flat}}\) denotes the Euclidean metric on \(\mathbb{R}^n\). Then for some \(\epsilon > 0\), depending only on \((M, g)\), we have \(B_{9\epsilon} := \{x \in \mathbb{R}^n \mid |x| < 9\epsilon\} \subset P(\Omega)\), and \(\text{dist}_{g_{\text{flat}}}(P(\Phi(E)), \partial P(\Omega)) > 9\epsilon\). Let \(\tilde{u} = (\tilde{u} \circ \Phi^{-1} \circ P^{-1}) v\) on \(P(\Omega)\), we have, by (11), \(f(\lambda(A^v)) = 1\) and \(\lambda(A^\tilde{u}) \in \Gamma\). By the property of \(\eta\), we know that
\[
\lim_{y \to \tilde{y}, y \in P(\Omega)} \tilde{u}(y) = \infty \quad \forall \tilde{y} \in \partial P(\Omega), \tag{22}
\]
and, if the north pole of \(S^n\) does not belong to \(\Omega\),
\[
\lim_{y \in P(\Omega), |y| \to \infty} (|y|^{n-2} \tilde{u}(y)) = \infty. \tag{23}
\]

By a moving sphere argument (i.e. moving plane method together with conformal invariance of the equation) as in [8], we have, for every \(x \in \mathbb{R}^n\) satisfying
where $u \in \mathcal{M}$, $g$ positive constant depending only on $(\bar{x}, \lambda)$, centered at the origin, $n \geq 3$. Assume that $u \in C^1(B_{8a})$ is a non-negative function satisfying $\text{dist}_{g_{flat}}(y, P(\Phi(E))) < 2\epsilon$, that

$$\hat{u}_{x, \lambda}(y) := \frac{\lambda^{n-2}}{|y - x|^{n-2}} \hat{u} \left( \frac{\lambda^2(y - x)}{|y - x|^2} \right) \leq \hat{u}(y),$$

$$\forall \ 0 < \lambda < 4\epsilon, |y - x| \geq \lambda, \ y \in P(\Omega). \tag{24}$$

The following calculus lemma is established in [9].

**Lemma 1** Let $a > 0$ be a constant and let $B_{8a} \subset \mathbb{R}^n$ be the ball of radius $8a$ and centered at the origin, $n \geq 3$. Assume that $u \in C^1(B_{8a})$ is a non-negative function satisfying $u_{x, \lambda}(y) \leq u(y)$, $\forall \ x \in B_{4a}$, $y \in B_{8a}$, $0 < \lambda < 2a$, $\lambda < |y - x|$, where $u_{x, \lambda}(y) := \left( \frac{\lambda}{|y|} \right)^{n-2} u \left( x + \frac{\lambda^2(y - x)}{|y - x|^2} \right)$. Then

$$|\nabla u(x)| \leq \frac{n-2}{2a} u(x), \ \forall |x| < a.$$  \tag{25}

By (24) and the above lemma, we have $|\nabla (\log \hat{u})(y)| \leq C(\epsilon)$, $\forall$ dist$_{g_{flat}}(y, P(\Phi(E))) < \epsilon$. Thus, for some positive constant $C$ depending only on $(M, g)$, $|\nabla g \log u| \leq C$ on $M$, and

$$\sup_{B_{\epsilon}} \hat{u} \leq C \inf_{B_{\epsilon}} \hat{u}. \tag{25}$$

Let $\beta > 0$ be the constant such that $\xi(y) := \beta(\epsilon^2 - |y|^2)$ has the property that $\hat{u} \geq \xi$ on $B_{\epsilon}$, and, for some $\bar{y} \in B_{\epsilon}$, $\hat{u}(\bar{y}) = \xi(\bar{y})$. It follows that $\nabla \hat{u}(\bar{y}) = \nabla \xi(\bar{y})$, $(D^2 \hat{u}(\bar{y})) \geq (D^2 \xi(\bar{y}))$, and $A^u(\bar{y}) \leq A^\xi(\bar{y})$. By (23) and the definition of $\xi$, we have $1 - (\frac{|\bar{y}|}{\epsilon})^2 \geq C^{-1}$, and $C^{-1} \sup_{B_{\epsilon}} \hat{u} \leq \beta \epsilon^2 \leq C \inf_{B_{\epsilon}} \hat{u}$, where $C$ is some positive constant depending only on $(M, g)$. Consequently, $A^u(\bar{y}) \leq A^\xi(\bar{y}) \leq C\beta^{-\frac{4}{n-2}} I$. This, together with the fact that $\lambda(A^u(\bar{y})) \in \Gamma \subset \Gamma_1$, implies that $|\lambda(A^u(\bar{y}))| \leq C\beta^{-\frac{4}{n-2}}$. Since $f(\lambda(A^u(\bar{y}))) = 1$, we have, by (19), that $\beta \leq C\delta_{\frac{4}{n}}$, where $C$ depends only on $(M, g)$. Again by (24), we have

$$\max_M u = \hat{u}(\bar{x}) \leq C\hat{u}(0) \leq C\hat{u}(\bar{y}) = C\xi(\bar{y}) \leq C\beta \leq C\delta_{\frac{4}{n}}.$$

Namely, we have proved, for some positive constant $C$ depending only on $(M, g)$, that $u \leq C\delta_{\frac{4}{n}}$ on $M$. Let $\bar{x} \in M$ be a maximum point of $u$, it was shown in [8] that
f(u(\bar{x})^{-\frac{1}{n-2}}\lambda(\bar{A}_g(\bar{x}))) \leq 1. This, together with (20), implies \( \max_M u = u(\bar{x}) \geq \delta_2^{\frac{n-2}{2}} \).

Using the upper bound of \(|\nabla_g \log u|\) on \( M \), we have, for some positive constant \( C \) depending only on \((M, g)\), that \( u \geq \frac{1}{\epsilon} \max_M u \geq \frac{1}{\epsilon}\delta_2^{\frac{n-2}{2}} \), on \( M \). Step 1 is established.

**Step 2.** For some positive constant \( C \) depending only on \((M, g)\), \( b, \delta_1 \) and \( \delta_2 \), we have \( |\nabla^2 u| \leq C \) on \( M \).

\( C^2 \) estimates for \((f, \Gamma) = (\sigma^\frac{1}{2}_k, \Gamma_k)\) were obtained by Viaclovsky [10]. The arguments can be adapted in our situation. Indeed, this is equivalent to setting \( \rho \equiv 1 \) in the definition of \( G(x) \) in the proof of theorem 1.6 in [3], so that \( G(x) \) is defined on \( M \), and Step 2 follows from the computation there (with \( h \equiv 1 \)) together with our \( C^0 \) and \( C^1 \) estimates of \( u \) and \( u^{-1} \) obtained in Step 1. Since \( f \) is concave in \( \Gamma \), and since we have established \( C^0, C^1 \) and \( C^2 \) estimates of \( u \) and \( u^{-1} \), higher derivative estimates of \( u \) and \( u^{-1} \) in (11) follow from the interior estimates of Evans and Krylov together with the Schauder estimates. Estimate (11) has been established.

For the existence part of Theorem 1, we only need to treat the case that \((M, g)\) is not conformally diffeomorphic to a standard sphere since it is obvious otherwise. The following homotopy was introduced in [3]: For \( 0 \leq t \leq 1 \), let \( f_t(\lambda) = f(t\lambda + (1 - t)\sigma_1(\lambda)e) \), be defined on \( \Gamma_t := \{ \lambda \in \mathbb{R}^n \mid t\lambda + (1 - t)\sigma_1(\lambda)e \in \Gamma \} \). We consider, for \( 0 \leq t \leq 1 \), and for \( \check{g} = u^\frac{4}{n-2} g \),

\[
\begin{align*}
  f_t(\lambda(A_{\check{g}})) &= 1, \\
  \lambda(A_{\check{g}}) &\in \Gamma_t, \\
  &\text{on } M.
\end{align*}
\]  

(26)

For \( 0 \leq t \leq 1 \), \((f_t, \Gamma_t)\) satisfies (2), (3), (4), (5), (6) and (7). Moreover estimate (11) holds for solutions of (20), uniform in \( 0 \leq t \leq 1 \). With this uniform estimates the degree argument in [3] yields a solution \( u \) of (3) in \( C^{4,\alpha} \). By standard elliptic theories, \( u \in C^\infty(M) \). Theorem 1 is established.

Next we present our proof of Theorem 2. By scaling, it is easy to see that we only need to prove the theorem for \( R = \delta = 1 \), which we assume below. Let \( u(\bar{x}) = \max_{B_1} u \). As in the proof of theorem 1.8 in [3], we can find \( \bar{x} \in B_\frac{1}{4}(\bar{x}) \) such that

\[
  u(\bar{x}) \geq 2^{\frac{2n}{n-2}} \sup_{B_\frac{1}{4}(\bar{x})} u, \quad \text{and} \quad \gamma := u(\bar{x})^{\frac{2}{n-2}} \sigma \geq \frac{1}{2} u(\bar{x})^{\frac{2}{n-2}},
\]

(27)

where \( \sigma = \frac{1}{2}(1 - |\bar{x} - \bar{x}|) \leq \frac{1}{2} \).

If \( \gamma \leq 2^{n+8} n^4 \), then \( (\sup_{B_1} u)(\inf_{B_2} u) \leq u(\bar{x})^2 \leq (2\gamma)^{\frac{n-2}{n+2}} \leq C(n) \), and we are done. So we always assume that \( \gamma > 2^{n+8} n^4 \). Let \( \Gamma := u(\bar{x})^{\frac{n-2}{2}} \geq 2\gamma \), and consider
\[ w(y) := \frac{1}{u(x)} u\left(\tilde{x} + \frac{y}{u(\tilde{x})} \right), \quad |y| < \Gamma. \] By supharmonicity of \( u \),

\[
\min_{\partial B_r} w = \inf_{B_r} w \geq \frac{1}{u(x)} \min_{\partial B_2} u, \quad 1 = w(0) \geq 2^{2-n} \sup_{B_r} w. \tag{28}
\]

By the conformal invariance of the equation satisfied by \( u \), \( F(A^n) = 1 \) on \( B_\Gamma \). Fix \( r = 2^{n+0} n^4 < \frac{1}{4} \gamma \). For \( |x| < r \), consider

\[
w_{x,\lambda}(y) := \left( \frac{\lambda}{|y - x|} \right)^{n-2} w(x + \frac{\lambda(y - x)}{|y - x|^2}), \quad y \in B_\Gamma.
\]

By the conformal invariance of the equation, we have \( F(A^{ww,\lambda}) = 1 \) on \( B_\Gamma \setminus B_\lambda(x) \). As in [8], there exists \( 0 < \lambda_x < r \) such that we have

\[
w_{x,\lambda}(y) \leq w(y), \quad \forall \ 0 < \lambda < \lambda_x, \ y \in B_\Gamma \setminus B_\lambda(x),
\]

and

\[
w_{x,\lambda}(y) < w(y), \quad \forall \ 0 < \lambda < \lambda_x, \ y \in \partial B_\Gamma.
\]

By the moving sphere argument in [8], we only need to consider the following two cases:

**Case 1.** For some \( |x| < r \) and some \( \lambda \in (0, r) \), \( w_{x,\lambda} \) touches \( w \) on \( \partial B_\Gamma \).

**Case 2.** For all \( |x| < r \) and all \( \lambda \in (0, r) \), we have

\[
w_{x,\lambda}(y) \leq w(y), \quad \forall \ |y - x| \geq \lambda, \ y \in B_\Gamma.
\]

In Case 1, let \( \lambda \in (0, r) \) be the smallest number for which \( w_{x,\lambda} \) touches \( w \) on \( \partial B_\Gamma \). By (28), we have, for some \( |y_0| = \Gamma, u(\tilde{x})^{-1} \min_{\partial B_2} u \leq \min_{\partial B_\Gamma} w = w_{x,\lambda}(y_0) \).

Using (28),

\[
w_{x,\lambda}(y_0) \leq \left( \frac{\lambda}{\Gamma - |x|} \right)^{n-2} \sup_{B_r} w \leq 2^{n-2} \left( \frac{\lambda}{\Gamma - |x|} \right)^{n-2} \leq 2^{n-2} \left( \frac{r}{\Gamma - r} \right)^{n-2}.
\]

Therefore

\[
\sigma^{\frac{n-2}{2}} u(\tilde{x}) \min_{\partial B_2} u \leq 2^{n-2} \sigma^{\frac{n-2}{2}} u(\tilde{x})^2 \left( \frac{r}{\Gamma - r} \right)^{n-2}.
\]

Since \( 4r < \gamma \leq \frac{\Gamma}{2} \) and \( \sigma \leq \frac{1}{2} \),

\[
\sigma^{\frac{n-2}{2}} u(\tilde{x}) \min_{\partial B_2} u \leq 2^{n-2} \sigma^{\frac{n-2}{2}} u(\tilde{x})^2 \left( \frac{r}{\Gamma} \right)^{n-2} = 2^{2/(n-2)} \sigma^{n-2} r^{n-2} \leq C(n). \tag{29}
\]
We deduce from (27) and (29) that \((\sup_{B_1} u)(\inf_{B_2} u) \leq 8^{n-2}r^{n-2} \leq C(n)\).

In Case 2, we have, by Lemma 1 and (28), that

\[|\nabla w(y)| \leq 2(n-2)r^{-1}w(y) \leq (n-2)2^{\frac{n}{2}}r^{-1}, \quad \forall |y| \leq r.\]

Let \(\epsilon\) be the number such that \(\xi(y) := \frac{1-\epsilon}{r}(r - |y|^2)\) satisfies \(w \geq \xi\) on \(B_{\sqrt{r}}\) and for some \(|\bar{y}| \leq \sqrt{r}, w(\bar{y}) = \xi(\bar{y})\). Since \(1 = w(0) \geq \xi(0) = 1 - \epsilon\) and \(\xi(\bar{y}) > 0\), we have \(0 \leq \epsilon < 1\).

By the estimates of \(|\nabla w|\) and the mean value theorem, \(|w(y) - 1| \leq (n-2)2^{\frac{n}{2}}r^{-\frac{1}{2}}\), for all \(|y| \leq \sqrt{r}\). So \(\frac{1}{2} \leq 1 - (n-2)2^{\frac{n}{2}}r^{-\frac{1}{2}} \leq w(\bar{y}) = \xi(\bar{y}) \leq 1 - \epsilon\), and therefore \(0 \leq \epsilon \leq (n-2)2^{\frac{n}{2}}r^{-\frac{1}{2}}\).

Clearly,

\[\nabla w(\bar{y}) = \nabla \xi(\bar{y}), \quad |\nabla \xi(\bar{y})| \leq \frac{2}{\sqrt{r}}, \quad D^2w(\bar{y}) \geq D^2\xi(\bar{y}) = -2(1-\epsilon)r^{-1}I.\]

It follows that

\[A^w(\bar{y}) \leq A^\xi(\bar{y}) \leq \frac{(10n+4)}{(n-2)^2}2^{\frac{2n}{n-2}}r^{-1}I.\]

Since \(F(A^w(\bar{y})) = 1\), we have, by (16) (recall that \(\delta = 1\)), \(\frac{(10n+4)}{(n-2)^2}2^{\frac{2n}{n-2}}r^{-1}I \geq 1\), violating the choice of \(r\). Thus we have shown that Case 2 can never occur. Theorem 2 is established.

The results in this note have been presented by the second author at his 45-minute invited talk at ICM 2002 in August 2002 in Beijing. The results have also been presented by the second author in a colloquium talk at Northwestern University on September 27, 2002, in the Geometric Analysis seminar at Princeton University on October 18, 2002, in a mini-course in late October 2002 at Università di Milano. On December 2 2002, the second author was informed by P. Guan that he, in collaboration with C.S. Lin and G. Wang, has obtained some related results.

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