SIMPLE AND OPTIMAL METHODS FOR STOCHASTIC VARIATIONAL INEQUALITIES, I: OPERATOR EXTRAPOLATION

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Abstract. In this paper we first present a novel operator extrapolation (OE) method for solving deterministic variational inequality (VI) problems. Similar to the gradient (operator) projection method, OE updates one single search sequence by solving a single projection subproblem in each iteration. We show that OE can achieve the optimal rate of convergence for solving a variety of VI problems in a much simpler way than existing approaches. We then introduce the stochastic operator extrapolation (SOE) method and establish its optimal convergence behavior for solving different stochastic VI problems. In particular, SOE achieves the optimal complexity for solving a fundamental problem, i.e., stochastic smooth and strongly monotone VI, for the first time in the literature. We also present a stochastic block operator extrapolations (SBOE) method to further reduce the iteration cost for the OE method applied to large-scale deterministic VIs with a certain block structure. Numerical experiments have been conducted to demonstrate the potential advantages of the proposed algorithms. In fact, all these algorithms are applied to solve generalized monotone variational inequality (GMVI) problems whose operator is not necessarily monotone. We will also discuss optimal OE-based policy evaluation methods for reinforcement learning in a companion paper.

Keywords: Variational inequality, operator extrapolation, acceleration, stochastic policy evaluation.

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1. Introduction. A unified framework to solve optimization, fixed-point equations, equilibrium and complementarity problems is provided by the concept of a variational inequality (VI) (see [10] for an extensive review and bibliography). In this paper, we consider a class of generalized monotone variational inequality (GMVI) problems:

Find \( x^* \in X : \langle F(x^*), x - x^* \rangle \geq 0, \; \forall x \in X, \) (1.1)

where \( X \subseteq \mathbb{R}^n \) is a nonempty closed convex set, and \( F : X \to \mathbb{R}^n \) is an \( L \)-Lipschitz continuous map, i.e. for some \( L > 0, \)

\[ \|F(x_1) - F(x_2)\| \leq L \|x_1 - x_2\|, \; \forall x_1, x_2 \in X. \] (1.2)

VIs satisfying (1.2) are often said to be smooth. In addition, we assume that \( F \) satisfies a generalized monotonicity condition

\[ \langle F(x), x - x^* \rangle \geq \mu \|x - x^*\|^2, \; \forall x \in X \] (1.3)

for some \( \mu \geq 0 \). Throughout this paper we assume the existence of the solution \( x^* \) to problem (1.1)-(1.3).

Clearly, condition (1.3) holds if \( F \) is monotone, i.e., there exists \( \mu \geq 0 \) such that

\[ \langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq \mu \|x_1 - x_2\|^2, \; \forall x_1, x_2 \in X. \] (1.4)

In particular, \( F \) is strongly monotone if \( \mu > 0 \) in (1.4). However, condition (1.3) does not necessarily imply the monotonicity of \( F \). For example, pseudo-monotone VIs satisfy (1.3), but not (1.4) [9, 16]. A notion related to strong monotonicity is the so-called weak sharpness condition, i.e., \( \exists \mu > 0 \) s.t.

\[ \langle F(x^*), x - x^* \rangle \geq \mu \|x - x^*\|^2, \; \forall x \in X. \] (1.5)

Note that if \( F \) is monotone and satisfies (1.5), then (1.3) must hold. Therefore, GMVI covers many VI problems that have been studied in the literature. As a special case of GMVI, we call problem (1.1)-(1.3) generalized strongly monotone VI (GSMVI) if \( \mu > 0 \) in (1.3). In this paper, we consider both deterministic

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GMVI with exact information about the operator $F$, and stochastic GMVI with only unbiased estimators of $F$ obtained through a stochastic oracle.

VI has attracted much interest recently in machine learning, statistics and artificial intelligence in addition to some more traditional applications, e.g., in transportation (see, e.g., [8] and Section 5.1). Specifically, GMVIs have been used to solve a class of minimax problems inspired by the Generative Adversary Networks (e.g., [24]). More recently, Juditsky and Nemirovski developed some interesting applications of stochastic strongly monotone VIs in signal estimation using generalized linear models (Chapter 5.2 of [12], and Section 5.2). Our study has also been motivated by the emergent application of stochastic strongly monotone VIs for policy evaluation in reinforcement learning (RL) [5]. For a given linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, the basic policy evaluation problem can be formulated as a fixed-point equation

$$\text{Find } x^* \in X : \quad x^* = T(x^*),$$

which is a special case of (1.1) with $X = \mathbb{R}^n$ and $F(x) = x - T(x)$. Observe that usually we do not have unbiased estimators of $F$ for the above VIs due to the existence of Markovian noise in RL. We will study this more challenging problem in a companion paper after developing some basic results in this work.

VI has been the focus of many algorithmic studies due to their relevance in practice. Classical algorithms for VI include, but not limited to, the gradient projection method (e.g., [4], [26]), Korpelevich’s extragradient method [14], and the proximal point algorithm (e.g., [25]). While earlier studies focused on the asymptotic convergence behavior of different algorithms, much recent research effort has been devoted to algorithms exhibiting strong performance guarantees in a finite number of iterations (a.k.a., iteration complexity). More specifically, Nemirovski in [20] presented a mirror-prox method by properly modifying the extragradient algorithm [15] and show that it can achieve an $\mathcal{O}(1/\epsilon)$ complexity bound for solving smooth monotone VI problems. Here, $\epsilon > 0$ denotes the target accuracy in terms of weak gap, i.e., $\text{gap}(\bar{x}) := \max_{x \in X} \langle F(x), x - \bar{x} \rangle$. This bound significantly improves the $\mathcal{O}(1/\epsilon^2)$ bound for solving VIs with bounded operators (i.e., nonsmooth VIs) (e.g., [3]). Nemirovski’s mirror-prox method has inspired many studies for solving VI problems under either deterministic (see, e.g., [2, 22, 23, 19, 9, 18]) or stochastic (see, e.g., [21, 13, 6, 27, 11, 7]) settings.

In spite of these efforts, there remain a few significant issues on the development of efficient solution methods for VIs. Firstly, there does not exist an optimal method for solving stochastic smooth and strongly monotone VIs, even though many VIs arising from different applications are given in this form [5, 12]. In the deterministic setting, Nesterov and Scimali [23] show that the dual extrapolation method [22] can achieve an optimal $\mathcal{O}(L/\mu \log(1/\epsilon))$ complexity bound in terms of the distance to the optimal solution, which significantly improves the $\mathcal{O}((L/\mu)^2 \log(1/\epsilon))$ complexity bound of the gradient (operator) projection method. However, only an $\mathcal{O}(1/\epsilon^2)$ sampling complexity bound has been reported for solving stochastic smooth and strongly monotone VIs [7]. The latter bound is actually worse than the $\mathcal{O}(1/\epsilon)$ complexity bound for solving stochastic nonsmooth strongly monotone VIs, which can be obtained by applying the stochastic approximation method to VI [21, 16]. However, it is well-known that these nonsmooth methods are not optimal for smooth VIs in terms of the dependence on the condition number and the initial distance to the optimal solution. They can not take advantage of a variety of variance reduction techniques available for smooth problems either.

Secondly, most existing optimal VI algorithms, including the mirror-prox method [20] and the dual extrapolation method [22], are somewhat complicated in the sense that they involve the updating of two or three intertwined sequences and the solution of two projection subproblems in each iteration. More recently, Malisiky [18] develops a novel variant of the mirror-prox method for smooth monotone VIs, called projected reflected gradient method, which only requires one projection step in each iteration. However, this method in [18] still has the following possible limitations: a) it requires $F$ to be defined over the $\mathbb{R}^n$ rather than the feasible region $X$; b) it still needs to maintain the updating of two sequences; and c) it does not easily extend for solving strongly monotone VIs. To the best of our knowledge, there does not exist an optimal method for VIs that requires the computation of only one sequence, similarly to the simple gradient projection method.

In this paper, we aim to address some of the aforementioned issues in the design and analysis of efficient VI solution methods. Our main contributions can be briefly summarized as follows. Firstly, we present a new and simple first-order algorithm, called operator extrapolation (OE) method, which only requires the updating of one sequence $\{x_t\}$ for solving VI problems. Each OE iteration involves the evaluation of the operator $F$ once at $x_t$, and the updating from $x_t$ to $x_{t+1}$ through only one projection subproblem:

$$x_{k+1} = \arg\min_{x \in X} \gamma_t \langle F(x_t) + \lambda_t (F(x_t) - F(x_{t-1})), x \rangle + \frac{1}{2} \|x - x_t\|^2.$$
Note that the above Euclidean projection can be generalized to the non-Euclidean setting (see Section 1.1). We show that OE exhibits an optimal monotone (i.e., $\text{res}(\bar{x}) := \min_{y \in -N_X(\bar{x})} \|y - F(\bar{x})\|_*$) complexity in terms of weak gap for solving smooth monotone VIs as a special case. Hence, it achieves the optimal complexity for the latter class of VIs in a much simpler way than [22], and the one for the more general GSMVIs for the first time in the literature. Moreover, it exhibits an optimal complexity for the latter class of VIs in this section.

Secondly, we present a stochastic OE (SOE) method obtained by replacing the operator $F(x_i)$ with its unbiased estimator in OE for solving stochastic VI. We show that SOE employed with different stepsize policies can achieve either nearly optimal or optimal complexity for stochastic GSMVI. More specifically, with only one sample of the random variables in each iteration, it achieves the $O((L/\mu) \log(1/\epsilon)) + \sigma^2 \log(1/\epsilon)/\mu^2)$ complexity bound. To the best of our knowledge, all these complexity bounds are new for solving stochastic GSMVIs. In fact, they significantly improve the existing stochastic VI methods, including the dual extrapolation method and stochastic block coordinate descent methods have been intensively studied for solving optimization problems.

Finally, we conduct numerical experiments on the proposed algorithms, including OE, SOE and SBOE, applied to solve a class of traffic assignment problems and signal estimation problems, and demonstrate their advantages over some existing VI methods, including the dual extrapolation method and stochastic approximation for solving smooth and strongly monotone VIs.

This paper is organized as follows. We discuss the OE method and its convergence properties for deterministic VIs in Section 2. We then present the SOE method for solving stochastic VIs in Section 3. The SBOE method for deterministic GSMVIs is discussed in Section 4, and the results from our numerical experiments are reported in Section 5. We complete this paper with some brief concluding remarks in Section 6.

1.1. Notation and terminology. Let $\mathbb{R}$ denote the set of real numbers. All vectors are viewed as column vectors, and for a vector $x \in \mathbb{R}^d$, we use $x^\top$ to denote its transpose. The identity matrix in $\mathbb{R}^d$ is denoted by $I_d$. Given a norm $\| \cdot \|$ in $\mathbb{R}^d$ the associated dual norm $\| \cdot \|_*$ is defined as $\|x\|_* = \sup\{\langle x, z \rangle : \|x\| \leq 1\}$. For any $n \geq 1$, the set of integers $\{1, \ldots, n\}$ is denoted by $[n]$. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the nearest integer to $s$ from above. We use $\mathbb{E}_s[X]$ to denote the expectation of a random variable $X$ on $\{i_1, \ldots, i_s\}$. For a given strongly convex function $\omega$ with modulus $1$, we define the prox-function (or Bregman’s distance) associated with $\omega$ as $V(x,y) := \omega(y) - \omega(x) - \langle \omega'(x), y - x \rangle$, $\forall x, y \in X$, where $\omega'(x) \in \partial \omega(x)$ is an arbitrary subgradient of $\omega$ at $x$. Note that by the strong convexity of $\omega$, we have

$$V(x,y) \geq \frac{1}{2} \|x - y\|^2. \quad (1.6)$$

With the definition of the Bregman’s distance, we can replace the generalized strong monotonicity assumption in (1.3) by

$$\langle F(x), x - x^* \rangle \geq 2\mu V(x,x^*), \quad \forall x \in X. \quad (1.7)$$

2. Deterministic VIs. We focus on the operator extrapolation (OE) method for solving deterministic VIs in this section.
2.1. The operator extrapolation method. As shown in Algorithm 1, the basic algorithmic scheme for the proposed OE method is conceptually simple. It only involves a single sequence of iterates \( \{x_t\} \), along with two sequences of nonnegative parameters \( \{\gamma_t\} \) and \( \{\lambda_t\} \), and a prox-function \( V : X \times X \to \mathbb{R} \). The parameters \( \{\lambda_t\} \) define the way we take extrapolation on the operators, while the parameters \( \{\gamma_t\} \) can be viewed as stepsizes. If \( F(x_t, x) = \|x - x_t\|^2/2 \) and \( \lambda_t = 0 \), then this method reduces to the well-known operator projection method for VI, or gradient projection method if \( F \) is a gradient field. A distinctive feature of the OE method is that, when \( \lambda_t > 0 \), it performs an operator extrapolation step given by \( F(x_t) + \lambda_t(F(x_t) - F(x_{t-1})) \), before the projection on \( X \). While the extrapolation of gradients and its relation with Nesterov’s acceleration had been studied before (see Section 3 of [17]), the extrapolation of operators for VIs has not been studied before in the literature to the best of our knowledge.

Algorithm 1 The Operator Extrapolation (OE) Method

Let \( x_0 = x_1 \in X \), and the nonnegative parameters \( \{\gamma_t\} \) and \( \{\lambda_t\} \) be given. For \( t = 1, \ldots, k \) do

\[
x_{t+1} = \text{argmin}_{x \in X} \gamma_t(F(x_t) + \lambda_t(F(x_t) - F(x_{t-1})), x) + V(x_t, x).
\]

End for

We add some remarks about the differences between the OE method with a few other existing VI methods, especially those with accelerated rate of convergence. Firstly, while the classic extragradient method [14] and mirror-prox method [20] require at least two operator evaluations and two projections, OE only requires one operator evaluation \( F(x_t) \) and one projection (or prox-mapping) over the set \( X \). The more recent projected reflected gradient method for VI in [18] requires a simple recursion at each iteration given by

\[
x_{t+1} = \text{argmin}_{x \in X} \gamma_t(F(x_t + \beta_t(x_t - x_{t-1}))) + V(x_t, x).
\]

This scheme implicitly maintains two sequences, i.e., \( \{x_t\} \) and \( \{x_t + \beta_t(x_t - x_{t-1})\} \). Because the sequence \( \{x_t + \lambda_t(x_t - x_{t-1})\} \) may sit outside the feasible region \( X \), it requires \( F \) to be well-defined over the whole \( \mathbb{R}^n \). Moreover, all these methods in [14, 18, 20] output a solution (e.g., \( x_t \)) different from the point where the operator is evaluated (e.g., \( x_t + \beta_t(x_t - x_{t-1}) \)). As a consequence, it is difficult to directly utilize the strong monotonicity conditions in (1.3) or (1.7). Secondly, Nesterov and Scimali [23] show that the dual extrapolation method in [22] can be used to solve strongly monotone problems in an optimal way. However, similar to [20], each iteration of this method requires two operator evaluations and two projections. In addition, this method requires the strong monotonicity in (1.4) rather than the generalized strong monotonicity in (1.3) or (1.7).

In order to analyze the convergence behavior of the OE method, we first need to discuss a few different termination criteria for the VI problem in (1.1). If \( F \) satisfies the generalized strong monotonicity condition in (1.7) for some \( \mu > 0 \), then the distance to the optimal solution \( \tilde{x} \) is a natural termination criterion. In addition we will use two other termination criteria. The first termination criterion is called the weak gap, defined as

\[
gap(\bar{x}) := \max_{x \in X} \langle F(x), \bar{x} - x \rangle
\]

for a given \( \bar{x} \in X \). We employ this criterion in the case of standard monotone VIs in Subsection 2.4. Our analysis in that subsection applies solely to the case when \( X \) is a bounded set. The case of an unbounded feasible set is investigated in section 5 of [19]. In the context of generalized monotone VIs in Subsection 2.3 we consider also the residual of a point as a termination criterion and our analysis applies also to possibly unbounded feasible sets. To this end let us denote the normal cone of \( X \) at \( \bar{x} \) by

\[
N_X(\bar{x}) := \{y \in \mathbb{R}^n | \langle y, x - \bar{x} \rangle \leq 0, \forall x \in X \}.
\]

Noting that \( \bar{x} \in X \) is an optimal solution for problem (1.1) if and only if \( F(\bar{x}) \in -N_X(\bar{x}) \), we define the residual of \( \bar{x} \) as

\[
\text{res}(\bar{x}) := \min_{y \in -N_X(\bar{x})} \|y - F(\bar{x})\|_*.
\]
In particular, if \( X = \mathbb{R}^n \), then \( N_X(\overline{x}) = \{0\} \) and \( \text{res}(\overline{x}) = \|F(\overline{x})\|_s \), which is exactly the residual of solving the nonlinear equation \( F(\overline{x}) = 0 \).

As stated, using these termination criteria, we will establish the convergence of the OE method applied for solving different VI problems, including the generalized strongly monotone VI (GSMVI), the generalized monotone VI (GMVI), and the standard monotone VI (MVI) in Subsections 2.2, 2.3 and 2.4, respectively. We will first show in Proposition 2.2 some important convergence properties associated with the OE method that hold for all these cases. Before doing so, we briefly state a well-known technical result (see, e.g., Lemma 3.1 of [16]), which, often referred to as the “three-point lemma”, characterizes the optimality condition of problem (2.1).

**Lemma 2.1.** Let \( x_{t+1} \) be defined in (2.1). Then,

\[
\gamma_t \langle F(x_t) + \lambda_t (F(x_t) - F(x_{t-1})), x_{t+1} - x \rangle + V(x_t, x_{t+1}) \leq V(x_t, x) - V(x_{t+1}, x), \forall x \in X. \tag{2.5}
\]

Henceforth for a given sequence of iterates \( \{x_t\} \) and \( x \in X \) we will use the notation

\[
\Delta F_t := F(x_t) - F(x_{t-1}) \quad \text{and} \quad \Delta V_t(x) := V(x_t, x) - V(x_{t+1}, x). \tag{2.6}
\]

**Proposition 2.2.** Let \( \{x_t\} \) be generated by Algorithm 1 and \( \{\theta_t\} \) a sequence of nonnegative numbers. If the parameters \( \{\gamma_t\} \) and \( \{\lambda_t\} \) in Algorithm 1 satisfy

\[
\theta_{t+1} \gamma_{t+1} \lambda_{t+1} = \gamma_t \theta_t, \tag{2.7}
\]

\[
\theta_{t-1} \geq 4L^2 \theta_t \gamma_t^2 \lambda_t^2 \tag{2.8}
\]

for all \( t = 1, \ldots, k \), then for any \( x \in X \),

\[
\sum_{t=1}^{k} \theta_t \left[ \gamma_t (F(x_{t+1}), x_{t+1} - x) + V(x_{t+1}, x) \right] - \sum_{t=1}^{k} \lambda_t \gamma_t \langle \Delta F_t, x_{t+1} - x \rangle + \sum_{t=1}^{k} \theta_t V(x_t, x_{t+1}). \tag{2.10}
\]

where

\[
Q_k := \sum_{t=1}^{k} \left[ \theta_t \gamma_t \lambda_t \langle \Delta F_t, x_{t+1} - x_t \rangle + \theta_t V(x_t, x_{t+1}) \right]. \tag{2.11}
\]

Using (1.6) and the Lipschitz condition (1.2) we can lower bound the term \( Q_k \) as follows:

\[
Q_k \geq \sum_{t=1}^{k} \left[ -\theta_t \gamma_t \lambda_t \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + \frac{\theta_t}{2} \|x_t - x_{t+1}\|^2 \right]
\]

\[
\geq \sum_{t=1}^{k} \left[ -\theta_t \gamma_t \lambda_t \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + \frac{\theta_t}{4} \|x_t - x_{t+1}\|^2 + \frac{\theta_t}{4} \|x_t - x_{t-1}\|^2 \right] + \frac{\theta_t}{4} \|x_k - x_{k+1}\|^2
\]

\[
\geq \frac{\theta_t}{4} \|x_k - x_{k+1}\|^2,
\]

where the last step follows by employing (2.8). Hence

\[
\sum_{t=1}^{k} \theta_t \Delta V_t(x) \geq \sum_{t=1}^{k} \left[ \theta_t \gamma_t \langle F(x_{t+1}), x_{t+1} - x \rangle \right] - \theta_k \gamma_k \langle \Delta F_{k+1}, x_{k+1} - x \rangle + \frac{\theta_t}{4} \|x_k - x_{k+1}\|^2.
\]

Using the fact that

\[
-\theta_k \gamma_k \langle \Delta F_{k+1}, x_{k+1} - x \rangle + \frac{\theta_t}{4} \|x_k - x_{k+1}\|^2 \geq -\theta_k \gamma_k \|x_k - x_{k+1}\| \|x - x_{k+1}\| + \frac{\theta_t}{4} \|x_k - x_{k+1}\|^2
\]

\[
\geq -\|x\| \|x_k - x_{k+1}\|^2
\]

in the above inequality, we obtain the desired result. \( \square \)
2.2. Convergence for GSMVIs. In this subsection, we consider the generalized strongly monotone VIs which satisfy (1.2) and (1.7) for some $\mu > 0$.

**Theorem 2.3.** Assume that (1.7) holds for some $\mu > 0$. Let $x^*$ be a solution of problem (1.1), and suppose that the parameters $\{\gamma_t\}, \{\theta_t\}, \{\lambda_t\}$ satisfy (2.7), (2.8) and additionally

$$
\theta_t \leq \theta_{t-1}(1 + 2\mu\gamma_{t-1}), \quad (2.12)
$$

$$
L^2\gamma_k^2 \leq \frac{1}{2}, \quad (2.13)
$$

Then for all $k \geq 1$,

$$
2\mu\theta_k\gamma_k V(x_{k+1}, x^*) \leq \theta_1 V(x_1, x^*). \quad (2.14)
$$

In particular by setting for $t = 1, \ldots, k$,

$$
\gamma_t = \frac{1}{2L}, \quad \lambda_t = \frac{\theta_{t-1}\gamma_{t-1}}{\theta_t\gamma_t} = \left(\frac{\mu}{L} + 1\right)^{-1}, \quad \text{and} \quad \theta_t = \left(\frac{\mu}{L} + 1\right)^t, \quad (2.15)
$$

we have

$$
V(x_{k+1}, x^*) \leq \frac{L}{\mu} \left(\frac{L}{\mu} + 1\right)^{k-1} V(x_1, x^*). \quad (2.16)
$$

**Proof.** By combining Proposition 2.2, (1.7) and (2.12), we obtain

$$
\sum_{t=1}^{k} \theta_t(2\mu\gamma_t + 1)V(x_{t+1}, x^*) - L^2\theta_k\gamma_k^2\|x_{k+1} - x^*\|^2 \leq \theta_1 V(x_1, x^*) + \sum_{t=2}^{k} \theta_t V(x_t, x^*)
$$

$$
\leq \theta_1 V(x_1, x^*) + \sum_{t=2}^{k} \theta_{t-1}(2\mu\gamma_{t-1} + 1)V(x_t, x^*),
$$

which together with (1.6) and (2.13) lead to the desired inequality in (2.14). The choice of algorithmic parameters in (2.15) is compatible with the conditions (2.7), (2.8), (2.12), and (2.13) respectively, thus by substituting into the above relation we obtain the linear rate of convergence in (2.16) for Algorithm 1. $\square$

In view of Theorem 2.3, the number of OE iterations required to find a solution $\bar{x} \in X$ s.t. $V(\bar{x}, x^*) \leq \epsilon$ for GSMVIs is bounded by $O(L/\mu \log(1/\epsilon))$. This bound appears to be optimal and significantly outperforms the $O((L/\mu)^2 \log(1/\epsilon))$ iteration complexity bound possessed by the projected operator (gradient) method in terms of their dependence on the condition number $L/\mu$ (see [23] for more discussions). To the best of our knowledge, this is the first time this optimal complexity has been obtained for GSMVIs, while [23] established a similar result for strongly monotone VIs by using a more involved algorithmic scheme.

2.3. Convergence for GMVI. In this subsection, we consider generalized monotone VIs which satisfy (1.2) and (1.7) with $\mu = 0$. Our goal is to show the OE method is robust in the sense that it converges even if the modulus $\mu$ is rather small. Throughout this subsection we assume that the distance generating function $\omega$ is differentiable and its gradient $\omega$ is Lipschitz continuous, i.e.,

$$
\|\nabla \omega(x_1) - \nabla \omega(x_2)\| \leq L_\omega \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X. \quad (2.17)
$$

We define the output solution $x_{R+1}$ of OE method as

$$
\|x_{R+1} - x_{R}\|^2 + \|x_{R} - x_{R-1}\|^2 = \min_{t=1,\ldots,k} \left(\|x_{t+1} - x_t\|^2 + \|x_t - x_{t-1}\|^2\right). \quad (2.18)
$$

Lemma 2.4 provides a technical result regarding the relation between the residual of $x_{R+1}$ and the summation of squared distances $\sum_{i=1}^{k} \|x_{t+1} - x_t\|^2$.

**Lemma 2.4.** Let $x_t, t = 1, \ldots, k + 1$, be generated by the OE method in Algorithm 1. Assume that $x_{R+1}$ is chosen according to (2.18). If

$$
\sum_{t=1}^{k} \|x_{t+1} - x_t\|^2 \leq \delta, \quad (2.19)
$$

then

$$
\text{res}(x_{R+1}) \leq 2(L + \frac{L_\omega}{\mu} + L\lambda_R)\frac{\sqrt{2\delta}}{\sqrt{k}},
$$
where \( \text{res}(\cdot) \) is defined in (2.4).

**Proof.** Observe that by the optimality condition of (2.1), we have

\[
\langle F(x_{R+1}) + \delta_R, x - x_{R+1} \rangle \geq 0 \quad \forall x \in X,
\]

(2.20)

with

\[
\delta_R := F(x_R) - F(x_{R+1}) + \lambda_R[F(x_R) - F(x_{R-1})] + \frac{1}{\gamma R}||\nabla \omega(x_{R+1}) - \nabla \omega(x_R)||.
\]

By (1.2) and (2.17), we have

\[
\|\delta_R\| \leq \|F(x_R) - F(x_{R+1})\| + \lambda_R\|F(x_R) - F(x_{R-1})\| + \frac{1}{\gamma_R}\|\nabla \omega(x_{R+1}) - \nabla \omega(x_R)\|
\]

\[
\leq (L + \frac{L}{\gamma_R})\|x_{R+1} - x_R\| + \lambda_R\|x_R - x_{R-1}\|.
\]

(2.21)

It follows from (2.19) that

\[
\sum_{t=1}^{k} \left( \|x_{t+1} - x_t\|^2 + \|x_t - x_{t-1}\|^2 \right) \leq 2\sum_{t=1}^{\infty} \|x_{t+1} - x_t\|^2 \leq 2\delta.
\]

(2.22)

The previous conclusion clearly implies that \( \|x_{R+1} - x_R\|^2 + \|x_R - x_{R-1}\|^2 \leq \frac{2\delta}{\gamma} \) and hence that

\[
\max\{\|x_{R+1} - x_R\|, \|x_R - x_{R-1}\|\} \leq \frac{\sqrt{2\delta}}{\sqrt{k}}.
\]

(2.23)

We then conclude from the definition of \( \text{res}(\cdot) \) in (2.4) and relations (2.20), (2.21), and (2.23) that

\[
\text{res}(x_{R+1}) = \|\delta_R\| \leq 2(L + \frac{L}{\gamma_R} + \lambda_R)\frac{\sqrt{2\delta}}{\sqrt{k}}.
\]

\( \square \)

We are now ready to show the convergence of the OE method for GMVMs.

**Theorem 2.5.** Let \( \{x_t\} \) be generated by Algorithm 1 and \( \{\theta_t\} \) be a sequence of nonnegative numbers. If the parameters \( \{\gamma_t\} \) and \( \{\lambda_t\} \) in Algorithm 1 satisfy (2.7) and

\[
\theta_{t-1} \geq 9L^2\theta_t^2\lambda_t^2
\]

(2.24)

for all \( t = 1, \ldots, k \), then

\[
\frac{1}{5}\sum_{t=1}^{k} \left[ \theta_t V(x_t, x_{t+1}) + \frac{\theta_t}{2} (1 - L^2\gamma_t^2)\|x_{k+1} - x^*\|^2 \right] \leq \theta_1 V(x_1, x^*).
\]

(2.25)

In particular, if

\[
\theta_t = 1, \lambda_t = 1 \quad \text{and} \quad \gamma_t = \frac{1}{3\sqrt{2}},
\]

(2.26)

and \( x_{R+1} \) is chosen according to (2.18), then

\[
\text{res}(x_{R+1}) \leq 4L(2 + 3L\omega)\frac{\sqrt{3V(x_1, x^*)}}{\sqrt{k}}.
\]

(2.27)

where \( \text{res}(\cdot) \) is defined in (2.4).

**Proof.** Observe that (2.10) still holds. However, we will bound \( Q_k \) in (2.11) differently from the proof of Proposition 2.2. By splitting \( V(x_t, x_{t+1}) \geq \|x_{t+1} - x_t\|^2/2 \) into three equal terms in \( Q_k \) we obtain

\[
Q_k \geq -\sum_{t=1}^{k} \left[ \theta_t \gamma_t \lambda_t L\|x_t - x_{t-1}\|\|x_{t+1} - x_t\| + \theta_t V(x_t, x_{t+1}) \right]
\]

\[
= \sum_{t=1}^{k} \left[ -\theta_t \gamma_t \lambda_t L\|x_t - x_{t-1}\|\|x_{t+1} - x_t\| + \frac{\theta_t}{6} \|x_t - x_{t+1}\|^2 + \frac{\theta_t}{6} \|x_t - x_{t-1}\|^2 \right] + \frac{\theta_k}{6} \|x_k - x_{k+1}\|^2
\]

\[
+ \sum_{t=1}^{k} \frac{\theta_t}{3} V(x_t, x_{t+1})
\]

\[
\geq \frac{\theta_k}{6} \|x_k - x_{k+1}\|^2 + \sum_{t=1}^{k} \frac{\theta_t}{3} V(x_t, x_{t+1}).
\]
Here, the first inequality follows from (1.2) and (1.6), and the second inequality follows from (2.24). Substituting the above bound into (2.10) with \( x = x^* \) and using the fact \( \langle F(x_{t+1}), x_{t+1} - x^* \rangle \geq 0 \) due to (1.7), we have:

\[
\sum_{t=1}^k \theta_t \Delta V_t(x^*) \geq \sum_{t=1}^k \left[ \theta_t \gamma_t \langle F(x_{t+1}), x_{t+1} - x^* \rangle \right] - \theta_k \gamma_k \langle \Delta F_{k+1}, x_{k+1} - x^* \rangle + \frac{\theta_k}{6} \| x_k - x_{k+1} \|^2 + \sum_{t=1}^k \frac{\theta_k}{3} V(x_t, x_{t+1})
\]

\[
\geq -\theta_k \gamma_k \langle \Delta F_{k+1}, x_{k+1} - x^* \rangle + \frac{\theta_k}{6} \| x_k - x_{k+1} \|^2 + \sum_{t=1}^k \frac{\theta_k}{3} V(x_t, x_{t+1}).
\]

Observe that:

\[
-\theta_k \gamma_k \langle \Delta F_{k+1}, x_{k+1} - x^* \rangle + \frac{\theta_k}{6} \| x_k - x_{k+1} \|^2 \geq -\theta_k \gamma_k L \| x_k - x_{k+1} \| \| x^* - x_k \| + \frac{\theta_k}{6} \| x_k - x_{k+1} \|^2 \\
\geq -\frac{3L}{2} \theta_k \gamma_k^2 \| x^* - x_{k+1} \|^2,
\]

we then conclude from the previous inequality that:

\[
\sum_{t=1}^k \left\{ \theta_t \left[ V(x_{t+1}, x^*) + \frac{1}{2} V(x_t, x_{t+1}) \right] \right\} - \frac{3}{2} \theta_k \gamma_k^2 \| x_{k+1} - x^* \|^2 \leq \sum_{t=1}^k \theta_t V(x_t, x^*),
\]

or equivalently:

\[
\frac{3}{2} \sum_{t=1}^k \left[ \theta_t V(x_t, x_{t+1}) \right] + \theta_k V(x_{k+1}, x^*) - \frac{3}{2} L \theta_k \gamma_k^2 \| x_{k+1} - x^* \|^2 \leq \theta_1 V(x_1, x^*).
\]

The result in (2.25) then follows from the above inequality and (1.6). Moreover, by choosing the parameter setting in (2.26), we have \( \sum_{t=1}^k \frac{1}{2} V(x_t, x_{t+1}) \leq V(x_1, x^*) \), which implies that \( \sum_{t=1}^k \| x_t - x_{t+1} \|^2 \leq 6V(x_1, x^*) \). The result in (2.27) immediately follows from the previous conclusion and Lemma 2.4.

In view of Theorem 2.5, the OE method can find a solution \( \bar{x} \in X \) s.t. \( \text{res}(\bar{x}) \leq \epsilon \) in \( O(1/\epsilon^2) \) iterations for GMVs.

**2.4. Convergence for MVIs.** In this subsection, we show that the OE method can achieve the optimal rate of convergence for solving standard monotone VIs, for which \( F \) satisfies

\[
\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in X.
\]

In this subsection it is assumed that the set \( X \) is bounded.

**Theorem 2.6.** Suppose that (2.28) holds. Let \( \{x_t\} \) be generated by Algorithm 1 and denote

\[
\bar{x}_{k+1} := \sum_{t=1}^k \theta_t \gamma_t x_{t+1} \sum_{t=1}^k \gamma_t \theta_t.
\]

If (2.7), (2.8) and (2.13) hold and \( \theta_t \leq \theta_{t-1} \), then

\[
\text{gap}(\bar{x}_{k+1}) \leq \frac{3}{2} \epsilon \sum_{t=1}^k \gamma_t \theta_t \text{max} V(x_1, x).
\]

In particular, If \( \gamma_t = \frac{1}{2t^2} \), \( \lambda_t = 1 \) and \( \theta_t = 1 \), then

\[
\text{gap}(\bar{x}_{k+1}) \leq \frac{3}{4} \epsilon \text{max} V(x_1, x)
\]

**Proof.** In view of Proposition 2.2,

\[
\sum_{t=1}^k \theta_t \left[ \gamma_t \langle F(x_{t+1}), x_{t+1} - x \rangle + V(x_{t+1}, x) \right] - L^2 \theta_k \gamma_k^2 \| x_{k+1} - x \|^2 \leq \sum_{t=1}^k \theta_t V(x_t, x).
\]

By using the monotonicity property in (2.28) and the definition of \( \bar{x}_{k+1} \) in (2.29),

\[
\sum_{t=1}^k \theta_t \left[ \gamma_t \langle F(x_{t+1}), x_{t+1} - x \rangle \right] \geq \sum_{t=1}^k \theta_t \left[ \gamma_t \langle F(x), x_{t+1} - x \rangle \right] = \left( \sum_{t=1}^k \theta_t \gamma_t \right) \langle F(x), \bar{x}_{k+1} - x \rangle.
\]

Combining the above two inequalities, we obtain:

\[
\left( \sum_{t=1}^k \theta_t \gamma_t \right) \langle F(x), \bar{x}_{k+1} - x \rangle + \sum_{t=1}^k \theta_t V(x_{t+1}, x) + \theta_k \left( \frac{1}{2} - L^2 \gamma_k^2 \right) \| x_{k+1} - x \|^2 \leq \theta_1 V(x, x) + \sum_{t=2}^k \theta_t V(x_t, x),
\]

which together with the assumption \( \theta_t \leq \theta_{t-1} \) and (2.13) imply that \( \sum_{t=1}^k \theta_t \gamma_t \langle F(x), \bar{x}_{k+1} - x \rangle \leq \theta_1 V(x, x) \). The result follows from the above inequality and the definition of \( \text{gap}(\bar{x}_{k+1}) \) in (2.2).

In view of Theorem 2.6, the OE method can achieve the same \( O(1/\epsilon) \) optimal complexity bound for MVI, similarly to the mirror-prox method in [20] and the projected reflected gradient method in [18]. However, the algorithmic scheme of the OE method appears to be much simpler than the mirror-prox method. In contrast to the projected reflected gradient method, the OE method does not require \( F \) to be well-defined over the whole \( \mathbb{R}^n \) and hence is applicable to a broader class of MVI problems.
3. Stochastic VIs. In comparison to deterministic VIs, the algorithmic studies for solving stochastic VIs, especially (generalized) strongly monotone VIs, are still quite limited. In this section we show that the stochastic OE method can significantly improve existing complexity for solving stochastic strongly monotone VIs, while achieving the optimal complexity for solving stochastic monotone VIs.

3.1. The stochastic OE method. In this section, we consider the stochastic VI for which the operator $F$ can be accessed only through a stochastic oracle. More specifically, given the current iterate $x_t$, the stochastic oracle can generate a random vector $\tilde{F}(x_t, \xi_t)$ such that
\[
\mathbb{E}[\tilde{F}(x_t, \xi_t)] = F(x_t) \quad \text{and} \quad \mathbb{E}[\|\tilde{F}(x_t, \xi_t) - F(x_t)\|^2] \leq \sigma^2,
\]
where $\xi_t \in \mathbb{R}^d$ denotes a random vector independent of $x_1, \ldots, x_t$.

We also consider the mini-batch setting that is widely used in practice. In this setting, at search point $x_t$ we call the stochastic oracle $m_t$ times to generate the i.i.d. samples $\{\xi_{t,i}\}_{i=1}^{m_t}$ and corresponding values of $\{\tilde{F}(x_t, \xi_{t,i})\}_{i=1}^{m_t}$, and compute an unbiased estimator of $F(x_t)$ according to
\[
\hat{F}(x_t) = \frac{1}{m_t} \sum_{i=1}^{m_t} \tilde{F}(x_t, \xi_{t,i}).
\]
Assuming that the norm $\|\cdot\|_*$ is Euclidean, we then conclude from the independence of the successive samples
\[
\mathbb{E}[\|\hat{F}(x_t) - F(x_t)\|^2] \leq \sigma^2_m := \frac{\sigma^2}{m_t}.
\]
Obviously, the above mini-batch setting reduces to the standard single-oracle setting if $m_t = 1$ for all $t \geq 1$. It should be noted, however, that the single-oracle setting does not require the Euclidean structure.

The stochastic operator extrapolation (SOE) method (see Algorithm 2) is obtained by replacing the exact operator $F$ with a stochastic estimator $\hat{F}$ in the OE method.

Algorithm 2 The Stochastic Operator Extrapolation (SOE) Method

Let $x_0 = x_1 \in X$, and the nonnegative parameters $\{\gamma_t\}$ and $\{\lambda_t\}$ be given.

for $t = 1, \ldots, k$
do
\[
x_{t+1} = \arg\min_{x \in X} \gamma_t \langle \hat{F}(x_t) + \lambda_t \hat{F}(x_t - \hat{F}(x_{t-1})), x \rangle + V(x_t, x).
\]
\endfor

Throughout this section for given sequences $\{x_t\}$ and $\{\xi_t\}$ we will use the notation
\[
\Delta \hat{F}_t := \hat{F}(x_t) - \hat{F}(x_{t-1}) \quad \text{and} \quad \delta_t := \hat{F}(x_t) - F(x_t),
\]
where $\delta_t$ denotes the error associated with the computation of the operator $F(x_t)$. We still employ the notation $\Delta F_t$ and $\Delta V_t(x)$ for $x \in X$, as introduced in (2.6). Similar to Lemma 2.5, we characterize the optimality condition of (3.3) as follows.

Lemma 3.1. Let $x_{t+1}$ be defined in (3.3), then $\forall x \in X$,
\[
\gamma_t \langle \hat{F}(x_t) + \lambda_t \hat{F}(x_t - \hat{F}(x_{t-1})), x_{t+1} - x \rangle + V(x_t, x_{t+1}) \leq V(x_t, x) - V(x_{t+1}, x).
\]

What follows is the counterpart of Proposition 2.2 for the stochastic case.

Proposition 3.2. Let $\{x_t\}$ be generated by the SOE method and $\{\theta_t\}$ be a sequence of nonnegative numbers. If the parameters in this method satisfy (2.7) and
\[
\theta_{t-1} \geq 16L^2 \gamma_t^{-2} \lambda_t^2 \theta_t
\]
for all $t = 1, \ldots, k$, then for any $x \in X$,
\[
\sum_{t=1}^{k} \left\{ \theta_t \|\gamma_t \langle \hat{F}(x_{t+1}), x_{t+1} - x \rangle + V(x_{t+1}, x) + \frac{1}{4} V(x_t, x_{t+1}) \| \right\} - 2L^2 \theta_k \gamma_k^2 \| x_{k+1} - x \|^2 \leq \sum_{t=1}^{k} \theta_t V(x_t, x) + 2 \sum_{t=1}^{k} \left( \theta_t \gamma_t^2 \lambda_t^2 \| \delta_t - \delta_{t-1} \|^2 \right),
\]

Proof. It follows from (3.5) after multiplying with $\theta_t$ that
\[
\theta_t \Delta V_t(x) \geq \theta_t \gamma_t \langle \hat{F}(x_{t+1}), x_{t+1} - x \rangle - \theta_t \gamma_t \langle \Delta \hat{F}_{t+1}, x_{t+1} - x \rangle + \theta_t \gamma_t \lambda_t \langle \Delta \hat{F}_t, x_t - x \rangle + \theta_t V(x_{t+1}, x_t).
\] (3.8)

Summing up from $t = 1$ to $k$, invoking (2.7) and $x_1 = x_0$, and assuming also $\delta_1 = \delta_0$ we obtain
\[
\sum_{t=1}^k \theta_t \Delta V_t(x) \geq \sum_{t=1}^k \left[ \theta_t \gamma_t \langle \hat{F}(x_{t+1}), x_{t+1} - x \rangle \right] - \theta_0 \gamma_k \langle \Delta \hat{F}_{k+1}, x_{k+1} - x \rangle + \hat{Q}_k, \tag{3.9}
\]
with
\[
\hat{Q}_k := \sum_{t=1}^k \left[ \theta_t \gamma_t \lambda_t \langle \Delta \hat{F}_t, x_{t+1} - x_t \rangle + \theta_t V(x_t, x_{t+1}) \right] = \sum_{t=1}^k \left[ \theta_t \gamma_t \lambda_t \langle \Delta F_t + \delta_t - \delta_{t-1}, x_{t+1} - x_t \rangle + \theta_t V(x_t, x_{t+1}) \right]. \tag{3.10}
\]

Using the Lipschitz condition (1.2) and $x_1 = x_0$, we can lower bound the term $\hat{Q}_k$ as follows
\[
\hat{Q}_k \geq \sum_{t=1}^k \left[ - \theta_t \gamma_t \lambda_t L \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + \theta_t V(x_t, x_{t+1}) + \theta_0 \gamma_k \lambda_k \langle \delta_t - \delta_{t-1}, x_{t+1} - x_t \rangle \right] \\
\geq \sum_{t=1}^k \left[ - \theta_t \gamma_t \lambda_t L \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + \frac{\theta_0}{8} \|x_t - x_{t+1}\|^2 + \frac{\theta_0}{8} \|x_t - x_{t-1}\|^2 \right] + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}) \\
+ \sum_{t=1}^k \left[ \theta_t \gamma_t \lambda_t \langle \delta_t - \delta_{t-1}, x_{t+1} - x_t \rangle + \frac{\theta_0}{8} \|x_t - x_{t+1}\|^2 \right] + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}) \\
\geq \frac{\theta_0}{8} \|x_k - x_{k+1}\|^2 + \sum_{t=1}^k \left[ \theta_t \gamma_t \lambda_t \langle \delta_t - \delta_{t-1}, x_{t+1} - x_t \rangle + \frac{\theta_0}{8} \|x_t - x_{t+1}\|^2 \right] + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}) \\
\geq \frac{\theta_0}{8} \|x_k - x_{k+1}\|^2 - 2 \sum_{t=1}^k \left( \theta_t \gamma_t \lambda_t^2 \|\delta_t - \delta_{t-1}\|^2 \right) + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}),
\]
where the second inequality follows from (1.6), the third inequality from (3.6) and the last one follows from Young’s inequality. Using the above bound of $\hat{Q}_k$ in (3.9), we obtain
\[
\sum_{t=1}^k \theta_t \Delta V_t(x) \geq \sum_{t=1}^k \left[ \theta_t \gamma_t \langle \hat{F}(x_{t+1}), x_{t+1} - x \rangle \right] - \theta_0 \gamma_k \langle \Delta \hat{F}_{k+1}, x_{k+1} - x \rangle + \frac{\theta_0}{8} \|x_k - x_{k+1}\|^2 \\
- 2 \sum_{t=1}^k \left( \theta_t \gamma_t \lambda_t^2 \|\delta_t - \delta_{t-1}\|^2 \right) + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}) \\
\geq \sum_{t=1}^k \left[ \theta_t \gamma_t \langle \hat{F}(x_{t+1}), x_{t+1} - x \rangle \right] - 2 L^2 \theta_k \gamma_k^2 \|x_k - x_{k+1}\|^2 - \theta_k \gamma_k \langle \delta_k - \delta_{k-1}, x_{k+1} - x \rangle - 2 \sum_{t=1}^k \left( \theta_t \gamma_t \lambda_t^2 \|\delta_t - \delta_{t-1}\|^2 \right) + \frac{1}{4} \sum_{t=1}^k \theta_t V(x_t, x_{t+1}), \tag{3.11}
\]
where the second inequality follows from
\[
- \gamma_k \langle \Delta \hat{F}_{k+1}, x_{k+1} - x \rangle + \frac{1}{8} \|x_k - x_{k+1}\|^2 = - \gamma_k \langle \Delta \hat{F}_{k+1}, x_{k+1} - x \rangle - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle + \frac{1}{8} \|x_k - x_{k+1}\|^2 \\
\geq - \gamma_k L \|x_k - x_{k+1}\| \|x - x_{k+1}\| + \frac{1}{8} \|x_k - x_{k+1}\|^2 - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle \\
\geq - 2 L^2 \gamma_k^2 \|x - x_{k+1}\|^2 - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle.
\]
The result is obtained by rearranging the terms in (3.12) □

3.2. Convergence for stochastic GSMVIs. We describe the main convergence properties of SOE for stochastic generalized strongly monotone VI, i.e., when (1.7) holds for some $\mu > 0$.

Theorem 3.3. Suppose that (1.7) holds for some $\mu \geq 0$. If the parameters in the SOE method satisfy (2.7), (3.6), and
\[
\theta_t \leq \theta_{t-1}(2\mu \gamma_{t-1} + 1), \quad t = 1, \ldots, k, \tag{3.13}
\]
\[
8 L^2 \gamma_k^2 \leq 1, \tag{3.14}
\]
then
\[
\theta_k (2 \mu \gamma_k + \frac{1}{2}) \mathbb{E}[V(x_{k+1}, x^*)] + \sum_{t=1}^{k-1} \theta_t \mathbb{E}[V(x_t, x_{t+1})] \leq \theta_1 V(x_1, x^*) + 4 \sum_{t=1}^k \left( \theta_t \gamma_t \lambda_t^2 (\sigma_t^2 + \sigma_t^2) \right) + 2 \theta \gamma_k^2 \sigma_k^2,
\]
Proof. Denote $\bar{\xi}_t = (\xi_{t,1}, \ldots, \xi_{t,m_t})$. Let us fix $x = x^*$ and take expectation on both sides of (3.7) w.r.t. $\bar{\xi}_1, \ldots, \bar{\xi}_k$. Then,

$$\sum_{t=1}^{k} \left\{ \theta_t \gamma_t \mathbb{E}[\langle \bar{F}(x_{t+1}), x_{t+1} - x^* \rangle] + \mathbb{E}[V(x_{t+1}, x^*)] + \frac{1}{4} \mathbb{E}[V(x_t, x_{t+1})] \right\} - 2L^2\theta_t\gamma_t^2\mathbb{E}[\|x_{k+1} - x^*\|^2]$$

$$- \theta_k \gamma_k \mathbb{E}[\langle \bar{\delta}_{k+1} - \delta_k, x_{k+1} - x^* \rangle] \leq \sum_{t=1}^{k} \theta_t \mathbb{E}[V(x_t, x^*)] + 2\sum_{t=1}^{k} (\theta_t \gamma_t^2 \lambda_t^2 (\sigma_{t-1}^2 + \sigma_t^2)). \quad (3.15)$$

First note that $x_t$ is a deterministic function of $\bar{\xi}_1, \ldots, \bar{\xi}_{t-1}$. By conditioning on $\bar{\xi}_1, \ldots, \bar{\xi}_{t-1}$ and using the law of iterated expectations it follows that $\mathbb{E}[\langle \delta_t, x_t - x^* \rangle] = 0$. This implies that

$$\mathbb{E}[\langle \bar{F}(x_{t+1}), x_{t+1} - x^* \rangle] = \mathbb{E}[\langle F(x_{t+1}), x_{t+1} - x^* \rangle],$$

$$\mathbb{E}[\langle \bar{\delta}_{k+1} - \delta_k, x_{k+1} - x^* \rangle] = \mathbb{E}[\langle -\delta_k, x_{k+1} - x^* \rangle] = \mathbb{E}[\langle -\delta_k, x_{k+1} - x^* \rangle].$$

In addition, we have

$$\mathbb{E}[\|\delta_t - \delta_{t-1}\|^2] \leq 2(\mathbb{E}[\|\delta_t\|^2] + \mathbb{E}[\|\delta_{t-1}\|^2]) \leq 2\sigma_t^2 + 2\sigma_t^2.$$

Using these observations in (3.15), we obtain

$$\sum_{t=1}^{k} \left\{ \theta_t \gamma_t \mathbb{E}[\langle F(x_{t+1}), x_{t+1} - x^* \rangle] + \mathbb{E}[V(x_{t+1}, x^*)] + \frac{1}{4} \mathbb{E}[V(x_t, x_{t+1})] \right\} - 2L^2\theta_t\gamma_t^2\mathbb{E}[\|x_{k+1} - x^*\|^2]$$

$$+ \theta_k \gamma_k \mathbb{E}[\langle \bar{\delta}_{k+1} - \delta_k, x_{k+1} - x^* \rangle] \leq \sum_{t=1}^{k} \theta_t \mathbb{E}[V(x_t, x^*)] + 4\sum_{t=1}^{k} (\theta_t \gamma_t^2 \lambda_t^2 (\sigma_{t-1}^2 + \sigma_t^2))$$

which together with the fact

$$\frac{1}{4} \mathbb{E}[V(x_t, x_{k+1})] + \gamma_k \mathbb{E}[\langle \delta_t, x_{k+1} - x_k \rangle] \geq \frac{1}{4} \mathbb{E}[\|x_t - x_{k+1}\|^2] + \gamma_k \mathbb{E}[\langle \delta_t, x_{k+1} - x_k \rangle]$$

$$\geq -2\gamma_k^2 \mathbb{E}[\|\delta_t\|^2] \geq -2\gamma_k^2 \sigma_k^2$$

imply after taking (1.7) into account

$$\sum_{t=1}^{k} \left\{ \theta_t (2\mu \gamma_t + 1) \mathbb{E}[V(x_{t+1}, x^*)] \right\} + \sum_{t=1}^{k-1} \left\{ \theta_t (2\mu \gamma_t + 1) \mathbb{E}[V(x_t, x_{t+1})] - 2L^2\theta_t\gamma_t^2\mathbb{E}[\|x_{k+1} - x^*\|^2] \right\}$$

$$\leq \sum_{t=1}^{k} \theta_t \mathbb{E}[V(x_t, x^*)] + 4\sum_{t=1}^{k} (\theta_t \gamma_t^2 \lambda_t^2 (\sigma_{t-1}^2 + \sigma_t^2)) + 2\theta_k \gamma_k^2 \sigma_k^2.$$

Invoking (3.13) and using (1.6), we have

$$\theta_k (2\mu \gamma_k + \frac{1}{2}) \mathbb{E}[V(x_{k+1}, x^*)] + \sum_{t=1}^{k-1} \left\{ \theta_t (2\mu \gamma_t + 1) \mathbb{E}[V(x_t, x_{t+1})] + (\frac{1}{4} - 2L^2\gamma_t^2) \theta_t \mathbb{E}[\|x_{k+1} - x^*\|^2] \right\}$$

$$\leq \theta_t \mathbb{E}[V(x_t, x^*)] + 4\sum_{t=1}^{k} (\theta_t \gamma_t^2 \lambda_t^2 (\sigma_{t-1}^2 + \sigma_t^2)) + 2\theta_k \gamma_k^2 \sigma_k^2,$$

which, in view of (3.14), clearly implies the result.

We now specify the selection of a few particular stepsize policies for solving stochastic GSMVI problems.

**Corollary 3.4.** Consider the single-oracle setting with $m_t = 1$. If

$$t_0 = \frac{4L}{\mu}, \quad \gamma_t = \frac{1}{\mu(t_0+t-1)}, \quad \theta_t = (t + t_0 + 1)(t + t_0), \quad \lambda_t = \frac{\mu(t_0+t-1)}{\theta_t},$$

then

$$\mathbb{E}[V(x_{k+1}, x^*)] \leq \frac{2t_0^2 + 1}{t_0(t_0 + 1)(t_0 + 2)} + \frac{8(4k+1)\sigma_t^2}{\mu^2(1+k)(1+k_0^2)}.$$

**Proof.** Note that (2.7) holds by the definition of $\lambda_t$. Observe that

$$8L^2\gamma_t^2 = \frac{8L^2}{\mu^2(t_0 + t - 1)^2} \leq \frac{8L^2}{\mu^2t_0^2} \leq 1,$$

$$16L^2\gamma_t^2 \lambda_t \theta_t = 16L^2\gamma_t^2 \theta_t \langle \delta_{t-1}, \gamma_t \rangle \leq 16L^2\gamma_t^2 \theta_t \leq \theta_{t-1},$$

and thus both (3.6) and (3.14) hold. In order to check (3.13), we observe that

$$\theta_{t-1}(2\mu \gamma_t + 1) = \theta_{t-1} \left( \frac{2t_0 + 1}{t_0(t_0+t-1)} + 1 \right) = \theta_{t-1} \frac{t_0(t_0+t-1)}{t_0(t_0+t-1)} = \theta_t.$$
The result then follows from Theorem 3.3 and the following simple calculations.

\[ \theta_k(2\mu\gamma_k + \frac{1}{2}) \geq \frac{2k}{2} = \frac{1}{2}(k + t_0 + 1)(k + t_0), \]

\[ \sum_{i=t}^{k} \theta_i \gamma_i^2 \lambda_i^2 = \sum_{i=t}^{k} \frac{\theta_i^2 \gamma_i^2 - \mu \gamma_i}{\theta_i} = \sum_{i=t}^{k} \frac{(t + t_0 - 1)^2}{2(t + t_0 - 2)^2}(t + t_0) \frac{1}{\mu^2} \leq \frac{2k}{\mu^2}, \]

\[ \theta_k \gamma_k^2 = \frac{(k + t_0 + 1)(k + t_0)}{\mu^2(k + t_0 - 1)^2} \leq \frac{2}{\mu^2}. \]

\[ \square \]

In view of Corollary 3.4, the number of iterations performed by the SOE method to find a solution \( \bar{x} \in X \) s.t. \( \mathbb{E}[V(\bar{x}, x^*)] \leq \epsilon \) is bounded by

\[ O\{\max(\frac{L\sqrt{V(x_1, x^*)}}{\mu \sqrt{\epsilon}}, \frac{\sigma^2}{\mu^2 \epsilon})\}. \]

In view of the above result one can expect the benefit of reducing the variance \( \sigma \) per iteration by increasing the size of \( m_t \) in terms of the resulting convergence rate.

Note that when \( \sigma = 0 \) (i.e., the deterministic case), the convergence rate achieved in Corollary 3.4 is not linear and hence not optimal. Assuming that the total number of iterations \( k \) is given in advance, we can select a novel stepsize policy that improves this convergence rate.

**Corollary 3.5.** Consider the single-oracle setting with \( m_t = 1 \). If \( k \) is fixed,

\[ \gamma_t = \gamma = \min\{\frac{1}{4L}, \frac{q \log k}{\mu k}\}, \quad \theta_t = (2\mu\gamma + 1)^t, \quad \lambda_t = \frac{1}{2\mu\gamma + 1}, \quad \text{with } q = 1 + \frac{\log(\mu^2 V(x_1, x^*))}{\log k} \]

then

\[ \mathbb{E}[V(x_{k+1}, x^*)] \leq 2(1 + \frac{\mu}{2L})^{-k} V(x_1, x^*) + \frac{2V(x_1, x^*)}{k^2} + \frac{4q^2(\log k)^2 \sigma^2}{\mu^2 k}. \]

**Proof.** Note that (2.7) holds by the definition of \( \lambda_t \). Observe that

\[ 8L^2 \gamma_t^2 \leq \frac{8L^2}{16L^2} < 1, \quad \text{and} \quad 16L^2 \gamma_t^2 \lambda_t \theta_t = 16L^2 \frac{\theta_t - \gamma_t - 1}{\theta_t} \leq 16L^2 \gamma_t^2 \theta_{t-1} \leq \theta_{t-1}, \]

and thus both (3.6) and (3.14) hold. Also, (3.13) holds, due to \( \theta_{t-1}(2\mu\gamma_{t-1} + 1) = \theta_t \). The result then follows from Theorem 3.3 and the following calculations.

\[ \frac{\theta_t - \gamma_t}{2\mu\gamma + 1} V(x_1, x^*) \leq 2(2\mu\gamma + 1)^{-k} V(x_1, x^*) \leq 2(1 + \frac{\mu}{2L})^{-k} V(x_1, x^*) + \frac{2V(x_1, x^*)}{k^2} \]

\[ = 2(1 + \frac{\mu}{2L})^{-k} V(x_1, x^*) + \frac{2\sigma^2}{\mu k}. \]

\[ \frac{\theta_t^2 \sigma^2}{2\mu\gamma + 1} \sum_{t=1}^{k} \theta_t \gamma_t^2 \lambda_t^2 = \frac{\theta_t^2 \sigma^2}{2\mu\gamma + 1} \sum_{t=1}^{k} \theta_t - \gamma_t - 1 \leq \frac{\sigma^2}{\mu} \leq \frac{2q^2(\log k)^2 \sigma^2}{\mu^2 k}. \]

\[ \square \]

In view of corollary 3.5, the number of iterations performed by the SOE method to find a solution \( \bar{x} \in X \) s.t. \( \mathbb{E}[V(\bar{x}, x^*)] \leq \epsilon \) is bounded by

\[ O\{\max(\frac{1}{\mu} \log \frac{V(x_1, x^*)}{\epsilon}, \frac{\sigma^2}{\mu^2 \epsilon} \log \frac{1}{\epsilon})\}. \]

This complexity bound is nearly optimal, up to a logarithmic factor, for solving GSMVI problems. In order to obtain the optimal convergence for GSMVIs, we need to develop a more advanced stepsize policy obtained by properly resetting the iteration index to zero for the stepsize policy in Corollary 3.4. More specifically, the OE iterations will be grouped into epochs indexed by \( s \), and each epoch contains \( k_s \) iterations. A local iteration index \( i \), which is set to 0 whenever a new epoch starts, will take place of \( t \) in the definitions of \( \gamma_t \) and \( \theta_t \) in Corollary 3.4.

**Corollary 3.6.** Consider the single-oracle setting with \( m_t = 1 \). Set \( t_0 = \frac{4L}{\mu} \), and let

\[ k_s = \left\lceil \max\{(2\sqrt{2} - 1)t_0 + 4, \frac{2^{s+6}\sigma^2}{\mu^2 V(x_1, x^*)}\} \right\rceil, \quad s \in \mathbb{Z}^+, \quad K_0 = 0, \quad \text{and} \quad K_s = \sum_{s'=1}^{s} k_{s'}. \]
For \( t = 1, 2, \ldots \), introduce the epoch index \( \tilde{s} \) and local iteration index \( \tilde{t} \) such that
\[
\tilde{s} = \arg\max_{s \in \mathbb{Z}^+} \mathbb{1}_{\{K_{s-1} < t \leq K_s\}} \quad \text{and} \quad \tilde{t} := t - K\tilde{s} - 1.
\]

For the stepsize policy
\[
\gamma_t = \frac{1}{\mu(t_0 + t - 1)}, \quad \theta_t = (\tilde{t} + t_0 + 1)(\tilde{t} + t_0), \quad \text{and} \quad \lambda_t = \begin{cases} 
0, & \tilde{t} = 1, \\
\frac{\theta_{\tilde{t}-1}\gamma_{\tilde{t}-1}}{\theta_{\tilde{t}}}, & \tilde{t} \geq 2,
\end{cases}
\]
it holds that \(\mathbb{E}[V(x_{K_{s+1}}, x^*)] \leq 2^{-s}V(x_1, x^*)\) for any \( s \geq 1 \).

**Proof.** First we note that in each epoch we use the stepsize policy of Corollary 3.4. This enables us to infer that, for \( s = 1, 2, \ldots \),
\[
\mathbb{E}[V(x_{K_{s+1}}, x^*)] \leq \frac{2(t_0 + 1)(t_0 + 2)\mathbb{E}[V(x_{K_{s-1}+1}, x^*)]}{(K_{s} + t_0 + 1)(K_{s} + t_0)} + \frac{8(4K_{s+1})\sigma^2}{\mu^2(t_{s} + t_0 + 1)(t_{s} + t_0)}.
\]
(3.16)

As such by taking the specification of \( k_s \) into account, we obtain
\[
\mathbb{E}[V(x_{K_{s+1}}, x^*)] \leq \frac{2(t_0 + 1)(t_0 + 2)\mathbb{E}[V(x_{K_{s-1}+1}, x^*)]}{(K_{s} + t_0 + 1)(K_{s} + t_0)} + \frac{8(4K_{s+1})\sigma^2}{\mu^2(K_{s} + t_0 + 1)(K_{s} + t_0)}
\leq \frac{2(t_0 + 1)(t_0 + 2)\mathbb{E}[V(x_{K_{s-1}+1}, x^*)]}{(2\sqrt{2}t_0 + 5)(2\sqrt{2}t_0 + 4)} + \frac{32\sigma^2}{\mu^2(K_{s} + t_0 + 1)}
\leq \frac{V(x_1, x^*)}{4} + \frac{V(x_1, x^*)}{4} = \frac{V(x_1, x^*)}{2}.
\]
The desired convergence result follows by recursively using (3.16). \( \Box \)

In view of corollary 3.6, the number of epochs performed by the SOE method to find a solution \( \bar{x} \in X \) s.t. \(\mathbb{E}[V(\bar{x}, x^*)] \leq \epsilon\) is bounded by \(\log_2(V(x_1, x^*)/\epsilon)\). Then together with the length of each epoch, the total number of iterations is bounded by
\[
\mathcal{O}(\max(\frac{1}{\mu} \log_2 \frac{V(x_1, x^*)}{\epsilon} \cdot \frac{\sigma^2}{\mu\gamma}))
\]
which is optimal for solving GSMVI problems. Note also that to simplify the implementation, we can replace the quantity \(\sigma^2/(\mu^2 V(x_1, x^*))\) in the definition of \( k_s \) by some estimation or simply by \(\mathcal{O}(1)\).

### 3.3. Convergence for stochastic GMVIs

In this subsection, we consider stochastic generalized monotone VIs which satisfy (1.2), (1.7) with \( \mu = 0 \) and (2.17). Our goal is to show the SOE method is robust in the sense that it converges when the modulus \( \mu \) is really small. We use constant size mini-batch method whose operator is defined in \( (3.1) \). Assuming the norm \( \| \cdot \|_1 \) is Euclidean, we then conclude from (3.2) that \(\mathbb{E}[\| \bar{F}(x_t) - F(x_t) \|_1^2] \leq \tilde{\sigma}^2 := \frac{\sigma^2}{m} \), where \( m \) is the constant mini-batch size. Throughout this subsection we assume that the distance generating function \( \omega \) is differentiable and satisfies (2.17).

We define the output solution of OE method as \( x_{R+1} \), where \( R \) is uniformly chosen from \( \{2, 3, \ldots, k\} \). Lemma 3.7 provides a technical result regarding the variation between the residual of \( x_{R+1} \) and the summation of squared distances \( \sum_{t=1}^{k} \mathbb{E}[\|x_{t+1} - x_t\|^2] \).

**Lemma 3.7.** Let \( x_t, t = 1, \ldots, k + 1, \) be generated by the SOE method in Algorithm 2 and fix the mini-batch size \( m \). If
\[
\sum_{t=1}^{k} \mathbb{E}[\|x_{t+1} - x_t\|^2] \leq \tilde{\delta}
\]
(3.17)
and \( R \) is uniformly chosen from \( \{2, 3, \ldots, k\} \), then
\[
\mathbb{E}[\text{res}(x_{R+1})^2] \leq \left[ 4(1 + \lambda_R)^2 + 4\lambda_R^2 \tilde{\sigma}^2 + 8 \left( L + \frac{Lm}{\gamma} \right)^2 + L^2 \lambda_R^2 \right] \frac{\tilde{\delta}}{k - 1},
\]
where \( \text{res}(\cdot) \) is defined (2.4).

**Proof.** Observe that by the optimality condition of (3.3), we have
\[
\langle F(x_{R+1}) + \tilde{\delta}_R, x - x_{R+1} \rangle \geq 0, \quad \forall x \in X,
\]
(3.18)
with
\[ \tilde{\delta}_R := \tilde{F}(x_R) - F(x_{R+1}) + \lambda_R [\tilde{F}(x_R) - \tilde{F}(x_{R-1})] + \frac{1}{\gamma_R} [\nabla \omega(x_{R+1}) - \nabla \omega(x_R)] \]
\[ = (1 + \lambda_R) [\tilde{F}(x_R) - F(x_R)] + \lambda_R [\tilde{F}(x_{R-1}) - F(x_{R-1})] + [F(x_R) - F(x_{R+1})] \]
\[ + \lambda_R [F(x_R) - F(x_{R-1})] + \frac{1}{\gamma_R} [\nabla \omega(x_{R+1}) - \nabla \omega(x_R)]. \]

By (1.2) and (2.17), we have
\[ \|\tilde{\delta}_R\|^2 \leq (1 + \lambda_R) \|\tilde{F}(x_R) - F(x_{R+1})\| + \lambda_R \|\tilde{F}(x_{R-1}) - F(x_{R-1})\| + (L + \frac{L}{\gamma_R}) \|x_{R+1} - x_R\| + L\lambda_R \|x_R - x_{R-1}\|. \]

By using Young’s inequality, we obtain
\[ \|\tilde{\delta}_R\|^2 \leq 4(1 + \lambda_R)^2 \|\tilde{F}(x_R) - F(x_{R+1})\|^2 + 4\lambda_R^2 \|\tilde{F}(x_{R-1}) - F(x_{R-1})\|^2 \]
\[ + 4(L + \frac{L}{\gamma_R})^2 \|x_{R+1} - x_R\|^2 + 4L^2 \lambda_R^2 \|x_R - x_{R-1}\|^2. \]

Taking expectation on both sides of the above inequality, then we have
\[ \mathbb{E}[\|\tilde{\delta}_R\|^2] \leq 4(1 + \lambda_R)^2 \mathbb{E}[\|\tilde{F}(x_R) - F(x_{R+1})\|^2] + 4\lambda_R^2 \mathbb{E}[\|\tilde{F}(x_{R-1}) - F(x_{R-1})\|^2] \]
\[ + 4(L + \frac{L}{\gamma_R})^2 \mathbb{E}[\|x_{R+1} - x_R\|^2] + 4L^2 \lambda_R^2 \mathbb{E}[\|x_R - x_{R-1}\|^2]. \]

Let \( \mathcal{F}_i = \sigma(x_1, \ldots, x_i) \) denote the sigma field generated by the first \( x_1, \ldots, x_i \) itterates. Since \( R \) is uniformly chosen from \{2, 3, \ldots, k\} independently of \( \mathcal{F}_k \), we have
\[ \mathbb{E}[\|x_{R+1} - x_R\|^2 + \|x_{R-1} - x_{R-2}\|^2] = \mathbb{E}[\mathbb{E}[\|x_{R+1} - x_R\|^2 + \|x_{R-1} - x_{R-2}\|^2 | \mathcal{F}_k]] \leq \frac{2^k}{k} \sum_{i=1}^{k} \mathbb{E}[\|x_{i+1} - x_i\|^2]. \]

As such
\[ \mathbb{E}[\|x_{R+1} - x_R\|^2 + \|x_{R-1} - x_{R-2}\|^2] \leq \frac{2^k}{k} \mathbb{E}[\|x_{i+1} - x_i\|^2]. \]

We then conclude from the definition of \( \text{res}(\cdot) \) in (2.4) and relations (3.18), (3.19), and (3.20) that
\[ \mathbb{E}[\text{res}(x_{R+1})^2] = \mathbb{E}[\|\tilde{\delta}_R\|^2] \leq 4(1 + \lambda_R)^2 + 4\lambda_R^2 \mathbb{E}[\|x_{R+1} - x_R\|^2] + 8(L + \frac{L}{\gamma_R})^2 + 4L^2 \lambda_R^2 \mathbb{E}[\|x_R - x_{R-1}\|^2]. \]

We can now show the convergence of the SOE method by establishing the convergence of \( \mathbb{E}[\text{res}(x_{R+1})]\).

**THEOREM 3.8.** Let \( \{x_i\} \) be generated by Algorithm 2 and \( \{\theta_i\} \) be a sequence of nonnegative numbers. If the parameters \( \{\theta_i\} \) and \( \{\lambda_i\} \) in Algorithm 2 satisfy (2.7), (3.6), (3.14) and \( \theta_t \geq \theta_{t-1} \), for all \( t = 1, \ldots, k \), and fix the mini-batch size \( m \), then
\[ \sum_{i=1}^{k} \theta_i \mathbb{E}[\|x_{i+1} - x_i\|^2] \leq \theta_1 V(x_1, x^*) + \sum_{i=1}^{k} 8\theta_i \gamma_i^2 \lambda_i^2 \sigma^2 + 4\theta_k \gamma_k^2 \sigma^2. \]

In particular, if
\[ m = k + 1, \theta_1 = 1, \lambda_1 = 1 \quad \text{and} \quad \gamma_1 = \frac{1}{4\pi}, \]
and \( R \) is uniformly chosen from \{2, 3, \ldots, k\}, then
\[ \mathbb{E}[\text{res}(x_{R+1})^2] \leq \frac{20\sigma^2}{k+1} + 32 \left\{ (L + 4LL\omega)^2 + L^2 \right\} \frac{2V(x_1, x^*) + \frac{L}{\gamma}}{k-1}, \]
where \( \text{res}(\cdot) \) is defined (2.4).

**Proof.** Observe that (3.11) still holds. Now we bound the terms \(-\theta_k \gamma_k \langle \Delta \tilde{F}_{k+1}, x_{k+1} - x \rangle + \frac{\theta_k}{\gamma} \|x_k - x_{k+1}\|^2\)
on the right hand side of (3.11) in a slightly different manner than before,
\[ -\gamma_k \langle \Delta \tilde{F}_{k+1}, x_{k+1} - x \rangle + \frac{\theta_k}{\gamma} \|x_k - x_{k+1}\|^2 = -\gamma_k \langle \Delta F_{k+1}, x_{k+1} - x \rangle - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle + \frac{1}{\gamma} \|x_k - x_{k+1}\|^2 \]
\[ \geq -\gamma_k L \|x_k - x_{k+1}\|^2 - \gamma_k \|x_k - x_{k+1}\|^2 + \frac{1}{16} \|x_k - x_{k+1}\|^2 - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle \]
\[ \geq -4L^2 \gamma_k \|x - x_{k+1}\|^2 + \frac{1}{16} \|x_k - x_{k+1}\|^2 - \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle. \]
Plugging the bound into (3.11) and rearranging the terms, we observe that
\[
\sum_{t=1}^{k} \{ \theta_t \left[ \gamma_t \langle \tilde{F}(x_{t+1}), x_{t+1} - x \rangle + V(x_{t+1}, x) + \frac{1}{4} V(x_t, x_{t+1}) \right] \} - 4L^2 \theta_k \gamma_k^2 \| x_{k+1} - x \|^2 + \frac{\theta_k}{16} \| x_{k+1} - x_k \|^2
\]
\[
- \theta_k \gamma_k \langle \delta_{k+1} - \delta_k, x_{k+1} - x \rangle \leq \sum_{t=1}^{k} \theta_t V(x_t, x) + 2\sum_{t=1}^{k} \left( \theta_t \gamma_t^2 \lambda_t^2 \| \delta_t - \delta_{t-1} \|_*^2 \right).
\]
Fixing \( x = x^* \), taking expectation on both sides of the inequality and using the fact \( \langle F(x_{t+1}), x_{t+1} - x^* \rangle \geq 0 \), we have
\[
\sum_{t=1}^{k} \{ \theta_t [E[V(x_{t+1}, x^*)] + \frac{1}{4} E[V(x_t, x_{t+1})]] \} - 4L^2 \theta_k \gamma_k^2 E[\| x_{k+1} - x^* \|^2]
\]
\[
\leq \sum_{t=1}^{k} \theta_t [E[V(x_t, x^*)] + \sum_{t=1}^{k} 8 \theta_t \gamma_t^2 \lambda_t^2 \delta_t^2 + \theta_k \gamma_k E[\langle \delta_{k+1} - \delta_k, x_{k+1} - x^* \rangle] - \frac{\theta_k}{16} E[\| x_{k+1} - x_k \|^2].
\]
By taking into account that
\[
\gamma_k E[\langle \delta_{k+1} - \delta_k, x_{k+1} - x^* \rangle] - \frac{1}{16} E[\| x_{k+1} - x_k \|^2] = \gamma_k E[\langle -\delta_k, x_{k+1} - x_k \rangle] - \frac{1}{16} E[\| x_{k+1} - x_k \|^2] \leq 4 \gamma_k \| \delta_k \|_*^2 \leq 4 \gamma_k \| \tilde{\sigma} \|^2,
\]
we obtain
\[
\sum_{t=1}^{k} \{ \theta_t [E[V(x_{t+1}, x^*)] + \frac{1}{4} E[V(x_t, x_{t+1})]] \} - 4L^2 \theta_k \gamma_k^2 E[\| x_{k+1} - x^* \|^2]
\]
\[
\leq \sum_{t=1}^{k} \theta_t [E[V(x_t, x^*)] + \sum_{t=1}^{k} 8 \theta_t \gamma_t^2 \lambda_t^2 \delta_t^2 + 4 \theta_k \gamma_k \| \tilde{\sigma} \|^2].
\]
Moreover, by using the condition (3.14), (1.6), and \( \theta_t \geq \theta_{t-1} \), we obtain
\[
\sum_{t=1}^{k} \frac{\theta_t}{4} E[\| x_{t+1} - x_t \|^2] \leq \theta_1 V(x_1, x^*) + \sum_{t=1}^{k} 8 \theta_t \gamma_t^2 \lambda_t^2 \delta_t^2 + 4 \theta_k \gamma_k \| \tilde{\sigma} \|^2.
\]
Finally, the choice of parameters in (3.22) implies that
\[
\sum_{t=1}^{k} E[\| x_{t+1} - x_t \|^2] \leq 8V(x_1, x^*) + \sum_{t=1}^{k} \frac{4}{\theta_t} \| \delta_t \|_*^2 + \frac{2}{\theta_k} \| \lambda \|^2 \leq 8V(x_1, x^*) + \frac{4}{\theta_k} \| \delta \|^2.
\]
The results in (3.23) readily follow from the previous conclusion and Lemma 3.7. \( \square \)

In view of Theorem 3.8, the SOE method can find a solution \( \bar{x} \in X \) such that \( E[\text{res}(\bar{x})^2] \leq \epsilon \) in \( O(1/\epsilon) \) iterations and \( O(1/\epsilon^2) \) overall sample complexity for solving generalized stochastic monotone VIs. Our result outperforms [11], improving the rate by a logarithmic factor.

### 3.4. Convergence for stochastic MVIs

In this subsection it is assumed that the set \( X \) is bounded. We start out with an auxiliary result that is helpful to the subsequent developments.

**Lemma 3.9.** For \( t = 1, 2, \ldots, k \), define \( \hat{x}_{t+1} \) as
\[
\hat{x}_{t+1} = \arg \min_{x \in X} \langle -\gamma_t \delta_t, x \rangle + V(\hat{x}_t, x),
\]
where \( \hat{x}_1 \in X \) is an arbitrary point and \( \{ \gamma_t \} \) a sequence of non-negative numbers. Then
\[
- \sum_{t=1}^{k} \langle \gamma_t \delta_t, \hat{x}_{t+1} - x \rangle \leq V(\hat{x}_1, x) + \sum_{t=1}^{k} \frac{\gamma_t^2}{2} \| \delta_t \|_*^2.
\]

**Proof.** The optimality condition is characterized as
\[
\langle -\gamma_t \delta_t, \hat{x}_{t+1} - x \rangle + V(\hat{x}_t, \hat{x}_{t+1}) + V(\hat{x}_{t+1}, x) \leq V(\hat{x}_t, x).
\]
As such
\[
\langle -\gamma_t \delta_t, \hat{x}_t - x \rangle - \gamma_t \| \delta_t \|_* \| \hat{x}_t - \hat{x}_{t+1} \| + \frac{1}{2} \| \hat{x}_t - \hat{x}_{t+1} \|^2 + V(\hat{x}_{t+1}, x) \leq V(\hat{x}_t, x)
\]
and therefore
\[
\langle -\gamma_t \delta_t, \hat{x}_t - x \rangle + V(\hat{x}_{t+1}, x) \leq V(\hat{x}_t, x) + \frac{\gamma_t^2}{2} \| \delta_t \|_*^2.
\]
Summing up from $t = 1$ to $k$ gives the desired result. □

**Proposition 3.10.** Assume $\max_{x_t, x_{t+1} \in X} V(x_t, x_{t+1}) \leq D_X$. Let $\{x_t\}$ be generated by the SOE method and set $\theta_t = 1$. If the parameters in this method satisfy (2.7), and

$$1 \geq 16L^2\gamma_t^2\lambda_t^2,$$

for all $t = 5, \ldots, k$, then

$$E[\text{gap}(\bar{x}_{k+1})] \leq \frac{1}{(\sum_{t=\lceil k/2 \rceil}^k \gamma_t)} \left(\frac{9}{2} \sum_{t=\lceil k/2 \rceil}^k \gamma_t^2 \sigma_t^2 + \frac{15}{4} D_X + \frac{7}{4} \gamma_t^2 \sigma_t^2 + \gamma_k \gamma_{k-1} \sigma_t^2\right).$$

(3.25)

In particular by setting $\gamma_t = \frac{1}{L\sqrt{\delta}}$, it follows that

$$E[\text{gap}(\bar{x}_{k+1})] \leq \frac{2L}{\sqrt{k+1}} \left(\frac{9}{2} \log 5 \sigma_t^2 + \frac{15}{4} D_X + \frac{7}{4} \sigma_t^2\right).$$

(3.26)

**Proof.** We start from (3.5) and sum from $t = \lceil k/2 \rceil$ to $k$, while invoking (2.7).

$$\sum_{t=\lceil k \rceil}^k \Delta V_t(x) \geq \sum_{t=\lceil k \rceil}^k \left[\gamma_t (\tilde{F}(x_{t+1}), x_{t+1} - x) - \gamma_k \lambda_k (\Delta \tilde{F}_k, x_k - x) + Q_k\right],$$

(3.27)

with

$$Q_k := \sum_{t=\lceil k \rceil}^k \left[\gamma_t \lambda_t (\Delta \tilde{F}_t, x_{t+1} - x_t) + V(x_t, x_{t+1})\right].$$

Using the Lipschitz condition (1.2), we can lower bound the term $Q_k$ as follows

$$\bar{Q}_k \geq \sum_{t=\lceil k \rceil}^k \left[-\gamma_t \lambda_t L \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + V(x_t, x_{t+1}) + \gamma_t \lambda_t (\delta_t - \delta_{t-1}, x_{t+1} - x_t)\right]$$

$$\geq \sum_{t=\lceil k \rceil}^k \left[-\gamma_t \lambda_t L \|x_t - x_{t-1}\| \|x_{t+1} - x_t\| + \frac{1}{2} \|x_t - x_{t+1}\|^2 + \frac{1}{2} \|x_t - x_{t-1}\|^2\right] + \gamma_k \lambda_k \gamma_{k-1} \Delta \tilde{F}_k, x_k - x)$$

$$+ \sum_{t=\lceil k \rceil}^k \left[\gamma_t \lambda_t (\delta_t - \delta_{t-1}, x_{t+1} - x_t) + \frac{1}{4} \|x_t - x_{t+1}\|^2\right] + \frac{4}{3} \sum_{t=\lceil k \rceil}^k V(x_t, x_{t+1})$$

$$\geq \frac{1}{2} \|x_k - x_{k-1}\|^2 - \frac{1}{2} \|x_k - x_{k-1}\|^2 - 2 \sum_{t=\lceil k \rceil}^k \left(\gamma_t^2 \lambda_t^2 \|\delta_t - \delta_{t-1}\|^2\right) + \frac{1}{2} \sum_{t=\lceil k \rceil}^k V(x_t, x_{t+1}),$$

where the individual steps are similar to the ones encountered in the proof of Proposition 3.2. Using the upper bound of $Q_k$ in (3.27), we obtain

$$\sum_{t=\lceil k \rceil}^k \Delta V_t(x) \geq \sum_{t=\lceil k \rceil}^k \left[\gamma_t (\tilde{F}(x_{t+1}), x_{t+1} - x) - \gamma_k \lambda_k (\Delta \tilde{F}_k, x_k - x) + \frac{1}{2} \|x_k - x_{k-1}\|^2\right]$$

$$+ \frac{1}{2} \|x_k - x_{k-1}\|^2 - \frac{1}{2} \|x_k - x_{k-1}\|^2 - 2 \sum_{t=\lceil k \rceil}^k \left(\gamma_t^2 \lambda_t^2 \|\delta_t - \delta_{t-1}\|^2\right) + \frac{1}{2} \sum_{t=\lceil k \rceil}^k V(x_t, x_{t+1}).$$

(3.28)

Note that

$$-\gamma_k (\Delta \tilde{F}_k, x_k - x) + \frac{1}{2} \|x_k - x_{k-1}\|^2 = -\gamma_k (\Delta F_k, x_k - x) - \gamma_k (\delta_k + 1 - \delta_k, x_k - x) + \frac{1}{2} \|x_k - x_{k-1}\|^2$$

$$\geq -\gamma_k L \|x_k - x_{k-1}\| \|x - x_{k-1}\| + \frac{1}{8} \|x_k - x_{k-1}\|^2$$

$$- \gamma_k (\delta_k + 1 - x) + \gamma_k (\delta_k, x_k - x) + \gamma_k (\delta_k, x_k + x_k - x) + \frac{1}{8} \|x_k - x_{k-1}\|^2$$

$$\geq -2L^2 \gamma_k^2 \|x_k - x_{k-1}\|^2 - \gamma_k (\delta_k + 1 - x) + \gamma_k (\delta_k, x_k - x) + \gamma_k (\delta_k, x_k + x_k - x) + \frac{1}{8} \|x_k - x_{k-1}\|^2$$

Similarly

$$\gamma_k \lambda_k (\Delta \tilde{F}_k, x_k - x) = \gamma_k \lambda_k (\Delta F_k, x_k - x) + \gamma_k \lambda_k (\delta_k - \delta_{k-1}, x_k - x)$$

$$\geq -\gamma_k \lambda_k \|x_k - x_{k-1}\| \|x_k - x_{k-1}\| + \gamma_k \lambda_k (\delta_k - \delta_{k-1}, x_k - x) + \frac{1}{2} \|x_k - x_{k-1}\|^2 - \frac{1}{2} \|x_k - x_{k-1}\|^2$$

$$\geq -2L^2 \gamma_k^2 \lambda_k^2 \|x_k - x_{k-1}\|^2 - \gamma_k \lambda_k (\delta_k - \delta_{k-1}, x_k - x) + 2L^2 \gamma_k^2 \lambda_k^2 \|x_k - x_{k-1}\|^2 - \gamma_k \lambda_k (\delta_k - \delta_{k-1}, x_k - x) - \frac{1}{4} \|x_k - x_{k-1}\|^2.$$
After rearranging terms
\[
2\sum_{t=k}^{k} \left( \gamma_i^2 \lambda_k^2 \| \delta_t - \delta_{t-1} \|^2 \right) + (V(x_k, x) + 2L^2 \gamma_i^2 \lambda_k^2 \| x_k - x \|^2) - (V(x_{k+1}, x) - 2L^2 \gamma_i^2 \| x - x_{k+1} \|^2) \geq \sum_{t=k}^{k} \left[ \gamma_i (F(x_{t+1}, x_{t+1} - x)) + \gamma_i \langle \delta_t, x_{t+1} - x \rangle + \gamma_i \langle \delta_k, x_k - x \rangle - \gamma_k \lambda_k \langle \delta_{k-1}, x_{k-1} - x \rangle \right] - 2\gamma_k^2 \lambda_k^2 \delta_{k-1}^2 - 2\gamma_k^2 \| \delta_k \|^2 - \frac{3}{8} \| x_k - x_{k-1} \|^2 + \frac{1}{4} \sum_{i=k}^{k-1} V(x_t, x_{t+1}).
\]

We now utilize Lemma 3.9. To this end we consider an auxiliary stochastic process \( \hat{x}_t \), initialized at \( \hat{x}_{k-1} = x_{k-1} \). We add and subtract the term \( \sum_{t=k}^{k} \gamma_i \langle \delta_t, \hat{x}_t - x \rangle \), while taking the particular selection of algorithmic parameters into account. Furthermore we set the stepsize \( \hat{\gamma}_t \) to
\[
\hat{\gamma}_t = \begin{cases} 
\gamma_k \lambda_k, & t = k - 1, \\
\gamma_t \lambda_t, & t = k, \\
\gamma_k \lambda_k + \gamma_k, & t = k.
\end{cases}
\]
to obtain
\[
2\sum_{t=k}^{k} \left( \gamma_i^2 \lambda_k^2 \| \delta_t - \delta_{t-1} \|^2 \right) + 2V(x_k, x) + V(x_{k-1}, x) + \sum_{t=k}^{k} \frac{\delta^2}{2} \| \delta_t \|^2 + \frac{3}{8} \| x_k - x_{k-1} \|^2 + 2\gamma_k^2 \lambda_k^2 \| \delta_{k-1} \|^2 + 2\gamma_k^2 \| \delta_k \|^2 \geq \sum_{t=k}^{k} \left[ \gamma_i (F(x_{t+1}, x_{t+1} - x)) + \gamma_i \langle \delta_t, x_t - x_t \rangle + \gamma_k \langle \delta_k, x_k - \hat{x}_k \rangle - \gamma_k \lambda_k \langle \delta_{k-1}, x_{k-1} - \hat{x}_{k-1} \rangle \right].
\]

Furthermore by using the monotonicity property in (2.28) and the definition of \( \bar{x}_{k+1} \) in (2.29),
\[
\sum_{t=k}^{k} \left[ \gamma_i (F(x_{t+1}, x_{t+1} - x)) \right] \geq \sum_{t=k}^{k} \left[ \gamma_i (F(x_t, x_{t+1} - x)) \right] = (\sum_{t=k}^{k} \gamma_i) (F(x_t, \bar{x}_{k+1} - x)).
\]

By employing the definition of the gap function and taking expectations, one obtains 3.25. Moreover by setting \( \gamma_i = \frac{1}{L \sqrt{t}} \) and noticing that for all \( k \geq 5 \), \( \sum_{t=k}^{k-1} \gamma_i^2 \leq \frac{\log(2)}{2} \), \( \sum_{t=k}^{b} \gamma_i \geq \frac{1}{L} \sqrt{t + 1} \) (3.26) follows.

In view of proposition 3.10, the SOE method finds a solution \( \bar{x} \in X \) s.t. \( E[\text{gap}(\bar{x})] \leq \epsilon \) in \( O(\frac{1}{\epsilon}) \) iterations.

4. Stochastic Block OE for Deterministic Problems. In this section, we assume \( X \subset \mathbb{R}^n \) is a nonempty closed convex set with a block structure, i.e., \( X = X_1 \times X_2 \times \cdots \times X_b \), where \( X_i \subset \mathbb{R}^{n_i} \), \( i \in [b] \), are closed convex sets with \( n_1 + n_2 + \cdots + n_b = n \). We introduce the matrices \( U_i \in \mathbb{R}^{n \times n_i} \), \( i \in [b] \), satisfying \( \langle U_i U_j \rangle = I_n \), so that the \( i \)-th block of \( x \in X \) is given by \( x^{(i)} = U_i^T x \). Similarly given the operator values at \( x \in X \) we denote by \( F_i(x) \equiv U_i^T F(x), i \in [b] \), the corresponding value associated to the \( i \)-th block. The map \( F_i : X \rightarrow \mathbb{R}^{n_i} \) is Lipschitz continuous, i.e., for some \( L_i > 0 \),
\[
\| F_i(x_1) - F_i(x_2) \|_{i, \ast} \leq L_i \| x_1 - x_2 \|_i, \quad \forall x_1, x_2 \in X.
\]

Let \( \bar{L} = \max_{i=1, \ldots, b} L_i \), then \( \forall i \in [b] \),
\[
\| F_i(x_1) - F_i(x_2) \|_{i, \ast} \leq \bar{L} \| x_1 - x_2 \|_i, \quad \forall x_1, x_2 \in X.
\]

Each Euclidean space \( \mathbb{R}^{n_i}, i \in [b] \), is equipped with inner product \( \langle \cdot, \cdot \rangle_i \) and norm \( \| \cdot \|_i \). (\( \| \cdot \|_i \)) is the conjugate norm). We define a norm on \( X \) as follows: For \( x = (x^{(1)}, \ldots, x^{(b)}) \in X, \| x \|^2 = \| x^{(1)} \|^2_1 + \cdots + \| x^{(b)} \|^2_b \), and similarly for the conjugate norm \( \| y \|_2^2 = \| y^{(1)} \|_1^2 + \cdots + \| y^{(b)} \|_b^2 \). For a given strongly convex function \( \omega_i \) on \( X_i \), we define the prox-function (Bregman distance) associated with \( \omega_i \) as
\[
V_i(x, y) := \omega_i(x) - \omega_i(y) - \langle \omega'_i(y), x - y \rangle, \quad \forall x, y \in X_i.
\]

where \( \omega'_i(y) \in \partial \omega_i(y) \) is an arbitrary subgradient of \( \omega_i \) at \( y \). For simplicity, let us use the notation \( V_i(x, y) \equiv V_i(x^{(i)}, y^{(i)}), \| x - y \|_i \equiv \| x^{(i)} - y^{(i)} \|_i, i = 1, \ldots, b \), for \( x, y \in X \). Given the individual prox-functions \( V_i(\cdot, \cdot) \) on each \( X_i, i = 1, \ldots, b \), we define \( V(x, y) := \sum_{i=1}^{b} V_i(x^{(i)}, y^{(i)}), \forall x, y \in X, \) and \( \omega(x) := \sum_{i=1}^{b} \omega_i(x^{(i)}), \forall x \in X \).

With the above definitions and notation, we can formally describe the stochastic block operator extrapolation algorithm.
Algorithm 3 Stochastic Block Operator Extrapolation (SBOE) for Variational Inequalities

Let \( x_0 = x_1 \in X \), the nonnegative parameters \( \{\gamma_t\}, \{\lambda_t\} \) and the positive probabilities \( p_i \in (0, 1), i = 1, \ldots, b \), s.t. \( \sum_{i=1}^{b} p_i = 1 \) be given.

for \( t = 1, \ldots, k \) do

1. Generate a random variable \( i_t \) according to \( P\{i_t = i\} = p_i, \quad i = 1, \ldots, b \).
2. Update \( x_t^{(i)} \), \( i = 1, \ldots, b \), by

\[
x_{t+1}^{(i)} = \begin{cases} \arg \min_{u \in X, i} \gamma_t \langle F_t(x_t) + \lambda_t [F_t(x_t) - F_t(x_{t-1})], u \rangle + V_t(x_t, u), & i = i_t; \\ x_t^{(i)}, & i \neq i_t. \end{cases} \tag{4.4} \]

end for

In each iteration, Algorithm 3 only updates the one block that is randomly chosen. For simplicity, the block to be updated is uniformly chosen in each iteration, i.e. \( p_i = 1/b, \forall i \in [b] \). In order to simplify the analysis, we introduce for \( t = 2, \ldots, k+1 \) the iterates \( \{\hat{x}_t\} \), where

\[
\hat{x}_{t+1} = \arg \min_{x \in X} \gamma_t \langle F(x_t) + \lambda_t [F(x_t) - F(x_{t-1})], x \rangle + V(x_t, x). \tag{4.5} \]

Proposition 4.1. Let \( \{x_t\} \) be generated by the SBOE method and \( \{\theta_t\} \) be a sequence of nonnegative numbers. If the parameters in this method satisfy

\[
p_i = \frac{1}{b}, \quad i = 1, \ldots, b, \tag{4.6} \]

\[
\theta_{t+1} \gamma_{t+1} \lambda_{t+1} = \theta_t \gamma_t b, \tag{4.7} \]

\[
\theta_{t-1} \geq 4L^2 \gamma_t^2 \lambda_t^2 \theta_t \tag{4.8} \]

for all \( t = 1, \ldots, k \), then for any \( x \in X \),

\[
\sum_{t=1}^{k-1} \left[ \theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1) \right] \langle F(x_{t+1}), x_{t+1} - x \rangle + \theta_k \gamma_k b \langle F(x_{k+1}), x_{k+1} - x \rangle + \sum_{t=1}^{k} \theta_t bV(x_{t+1}, x) - 2\theta_k \gamma_k^2 bL^2 \langle F(x_{k+1}), x \rangle + \sum_{t=1}^{k} \theta_t bV(x_t, x) + \theta_1 \gamma_1 (b - 1) \langle F(x_1), x_1 - x \rangle, \tag{4.9} \]

where

\[
\hat{D}_t := \gamma_t \langle F(x_t) + \lambda_t [F(x_t) - F(x_{t-1})], \hat{x}_{t+1} - [bx_{t+1} - (b - 1)x_t] \rangle \\
+ [V(x_t, \hat{x}_{t+1}) - bV(x_t, x_{t+1})] + [V(\hat{x}_{t+1}, x) - bV(x_{t+1}, x) + (b - 1)V(x_t, x)]. \tag{4.10} \]

Proof. We first note that, by the update of Algorithm 3, \( x_{t+1} \) and \( x_t \) only differ in the \( i_t \)th block. Using the definition of \( \hat{x}_{t+1} \) in (4.5) and the uniform sampling rule (4.6) we obtain the relations

\[
x_t^{(i_t)} = \hat{x}_{t+1}^{(i_t)}, \text{ and } E_t[x_{t+1}] = \sum_{i=1}^{b} p_i [U_i U_i^T (\hat{x}_{t+1} - x_t) + x_t] = \frac{1}{b} \hat{x}_{t+1} + \frac{b-1}{b} x_t. \tag{4.11} \]

By employing Lemma 2.1,

\[
\gamma_t \langle F(x_t) + \lambda_t [F(x_t) - F(\hat{x}_{t-1})], x_{t+1} - x \rangle + V(x_t, \hat{x}_{t+1}) \leq V(x_t, x) - V(\hat{x}_{t+1}, x). \tag{4.12} \]

Rearranging the terms, we have

\[
D_t + \hat{D}_t + bV(x_t, x_{t+1}) + bV(x_{t+1}, x) \leq bV(x_t, x), \tag{4.12} \]

where \( \hat{D}_t \) is defined in (4.10) and

\[
D_t := \gamma_t \langle F(x_t) + \lambda_t [F(x_t) - F(x_{t-1})], bx_{t+1} - (b - 1)x_t - x \rangle \\
= \gamma_t b \langle F(x_{t+1}), x_{t+1} - x \rangle - \gamma_t (b - 1) \langle F(x_t), x_t - x \rangle - \gamma_t b \langle F(x_{t+1}) - F(x_t), x_{t+1} - x \rangle \\
+ \gamma_t \lambda_t [F(x_t) - F(x_{t-1}), x_t - x] + \gamma_t \lambda_t [F(x_t) - F(x_{t-1}), x_{t+1} - x_t]. \tag{4.13} \]
Combining (4.12) and (4.13), multiplying with \( \theta_t \) on both sides and summing up from \( t = 1 \) to \( k \), we obtain
\[
\sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k} (\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1}(b-1))(F(x_{t+1}), x_{t+1} - x) - \theta_1 \gamma_1 (b-1) \langle F(x_1), x_1 - x \rangle \\
+ \theta_k \gamma_k b(F(x_{k+1}), x_{k+1} - x) + \sum_{t=1}^{k} (\theta_{t+1} \gamma_{t+1} \lambda_{t+1} - \theta_t \gamma_t b) \langle \Delta F_t, x_{t+1} - x \rangle \\
- \theta_k \gamma_k b(\Delta F_{k+1}, x_{k+1} - x) + D + \sum_{t=1}^{k} \theta_t \tilde{D}_t,
\]
where \( D := \sum_{t=1}^{k} \theta_t \gamma_t \lambda_t b \langle \Delta F_t, x_{t+1} - x_t \rangle + \sum_{t=1}^{k} \theta_t b V(x_t, x_{t+1}) \). Invoking condition (4.7), we have
\[
\sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k} (\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1}(b-1))(F(x_{t+1}), x_{t+1} - x) - \theta_1 \gamma_1 (b-1) \langle F(x_1), x_1 - x \rangle \\
+ \theta_k \gamma_k b(F(x_{k+1}), x_{k+1} - x) - \theta_k \gamma_k b(\Delta F_{k+1}, x_{k+1} - x) + D + \sum_{t=1}^{k} \theta_t \tilde{D}_t. \tag{4.14}
\]
The strong convexity of the individual prox-functions \( V_t \) together with the definition of \( V \) gives
\[
V(x, y) \geq \frac{1}{2} \| x - y \|^2, \quad \forall x, y \in X. \tag{4.15}
\]
Together with the Lipschitz condition (4.2) and \( x_1 = x_0 \), we can lower bound the term \( D \) as follows:
\[
D \geq \sum_{t=1}^{k} \left[ \theta_t \gamma_t b \lambda_t (\Delta F_t, x_{t+1} - x_t) + \frac{b_0}{4} \| x_t - x_{t+1} \|^2 + \frac{b_0}{4} \| x_t - x_{t-1} \|^2 \right] + \frac{b_0}{4} \theta_k \| x_k - x_{k+1} \|^2 \tag{4.16}
\]
Here, the first inequality follows from (4.15). The second inequality follows from the Lipschitz condition (4.2), the Cauchy-Schwarz inequality, and the relationship \( \| x_{t+1} - x_t \| = \| x_{t+1} - x_t \| \). The third inequality follows from (4.8). Using the above bound on \( D \) in (4.14), we obtain
\[
\sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k} \left[ \theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1}(b-1) \right](F(x_{t+1}), x_{t+1} - x) + \theta_k \gamma_k b(F(x_{k+1}), x_{k+1} - x) \\
- \theta_1 \gamma_1 (b-1) \langle F(x_1), x_1 - x \rangle - \theta_k \gamma_k b(\Delta F_{k+1}, x_{k+1} - x) + \frac{b_0}{4} \theta_k \| x_k - x_{k+1} \|^2 + \sum_{t=1}^{k} \theta_t \tilde{D}_t \tag{4.17}
\]
where the second inequality follows from
\[
- \gamma_k b(\Delta F_{k+1}, x_{k+1} - x) + \frac{b_0}{4} \| x_k - x_{k+1} \|^2 \geq -\gamma_k bL \| x_k - x_{k+1} \|^2 + \frac{b_0}{4} \| x_k - x_{k+1} \|^2 \tag{4.18}
\]
The result is obtained by rearranging the terms in (4.16). \( \square \)

### 4.1. Convergence for GSMVIs with block structure.

Given the new Bregman distance \( V(\cdot, \cdot) \) the generalized strong monotonicity condition in accordance to (1.7) still holds. We now describe the main convergence properties for generalized strongly monotone VIs with block structure.

**Theorem 4.2.** Suppose that (1.7) holds for some \( \mu > 0 \). If the parameters in the SBOE method satisfy (4.6), (4.7), (4.8),
\[
\theta_{t-1} \gamma_{t-1} b \geq \theta_t \gamma_t (b-1), \tag{4.17}
\]
\[
\theta_t (2\mu \gamma_t - 1 \gamma_t + 1) \leq \theta_{t-1} (2\mu \gamma_{t-1} + 1), \quad t = 1, \ldots, k, \tag{4.18}
\]
\[
4L^2 \gamma_k^2 \leq 1, \tag{4.19}
\]
then
\[
\theta_k (2\mu \gamma_k + \frac{1}{b}) V(x_{k+1}, x^*) \leq \theta_1 V(x_1, x^*) + \frac{b-1}{b} \theta_1 \gamma_1 \langle F(x_1), x_1 - x^* \rangle.
\]
Proof. Let us fix \( x = x^* \) and take expectation on both sides of (4.9) w.r.t. \( i_1, \ldots, i_k \). Note that \( x_t \) is a deterministic function of \( i_1, \ldots, i_{t-1} \). By conditioning on \( i_1, \ldots, i_{t-1} \) and using the law of iterated expectations and (4.11), it follows that \( \mathbb{E} [\hat{D}_t] = 0 \). As such

\[
\sum_{t=1}^{k-1} \left[ \theta_t \gamma_t \beta - \theta_{t+1} \gamma_{t+1} (b - 1) \right] \mathbb{E} [F(x_{t+1}, x_{t+1} - x^*)] + \theta_t \mu \gamma \frac{b}{1+\mu \gamma} \mathbb{E} [\langle F(x_{t+1}, x_{t+1} - x^*) \rangle] + \sum_{t=1}^{k} \theta_t b \mathbb{E} [V(x_t, x^*)] - 2 \theta_t \gamma_t^2 b^2 \mathbb{E} [V(x_{t+1}, x^*)] \leq \sum_{t=1}^{k} \theta_t b \mathbb{E} [V(x_t, x^*)] + \theta_1 \gamma_1 (b - 1) (F(x_1), x_1 - x^*). \]

Using the above inequality, (1.7), and (4.17), we obtain

\[
\sum_{t=1}^{k-1} 2 \left[ \theta_t \gamma_t \beta - \theta_{t+1} \gamma_{t+1} (b - 1) \right] \mathbb{E} [V(x_{t+1}, x^*)] + 2 \mu \theta_t \gamma_t \mathbb{E} [V(x_{t+1}, x^*)] + \sum_{t=1}^{k} \theta_t b \mathbb{E} [V(x_t, x^*)] - 2 \theta_t \gamma_t^2 b^2 \mathbb{E} [V(x_{t+1}, x^*)] \leq \sum_{t=1}^{k} \theta_t b \mathbb{E} [V(x_t, x^*)] + \theta_1 \gamma_1 (b - 1) (F(x_1), x_1 - x^*). \]

Invoking (4.18), we have

\[
\theta_1 (2 \mu \gamma + 1) \mathbb{E} [V(x_{k+1}, x^*)] - 2 \theta_1 \gamma_1^2 b^2 \mathbb{E} [V(x_{k+1}, x^*)] \leq \theta_1 V(x_1, x^*) + \frac{b-1}{b} \theta_1 \gamma_1 (F(x_1), x_1 - x^*),
\]

which, in view of (4.19), implies the result. \( \square \)

We now specify the selection of a particular stepsize policy for solving GSMVI problems with block structure. We show that by using a constant stepsize \( \gamma_t = \gamma_0 \), one can achieve a linear convergence rate.

**Corollary 4.3.** If

\[
\gamma_t = \gamma = \frac{1}{2Lb}, \quad \lambda_t = \frac{b+2(b-1) \mu \gamma}{1+2 \mu \gamma}, \quad \text{and} \quad \theta_t = \left( \frac{1+2 \mu \gamma}{1+2 \mu \gamma} \right)^k,
\]

then \( \mathbb{E} [V(x_{k+1}, x^*)] \leq 2 \left( \frac{1+2 \mu \gamma}{1+2 \mu \gamma} \right)^k \left[ V(x_1, x^*) + \frac{b-1}{b} \gamma (F(x_1), x_1 - x^*) \right] \).

**Proof.** Note that (4.7) holds by the definition of \( \lambda_t \). Observe that

\[
4L^2 \gamma_t^2 \leq \frac{\lambda_t^2}{L \beta^2} \leq 1, \quad 4L^2 \gamma_t^2 \lambda_t^2 \theta_t = 4L^2 b^2 \gamma_t^2 \theta_t^2 \left( \frac{b+1}{b} \right)^2 \theta_t = 4L^2 b^2 \frac{2 \theta_t^2}{b+1} \theta_t \leq 4L^2 \gamma_t^2 \theta_t - \theta_t \leq \theta_t - \theta_t,
\]

and thus both (4.8) and (4.19) hold. In order to check (4.18), we observe that \( \theta_t (2 \mu \gamma + 1) = \theta_t (2 \mu \gamma + 1) = \theta_t (2 \mu \gamma + 1) \). From the definition of \( \gamma_t \) and \( \theta_t \), (4.17) is satisfied. The result then follows from Theorem 4.2. \( \square \)

For the specific choice of algorithmic parameters \( \{ \gamma_t \}, \{ \lambda_t \}, \) and \( \{ \theta_t \} \) employed in Corollary 4.3, the total number of iterations required by the SBOE algorithm to find an \( \epsilon \)-solution can be bounded by \( O \left( b^2 \log \left( \frac{1}{\epsilon} \right) \right) \). This result shows that we can obtain linear rate of convergence for solving VI problems even though the algorithm is stochastic. Comparing with the deterministic OE algorithm, the iteration cost of SBOE is cheaper up to a factor of \( O(b) \). Note that the Lipschitz constant is \( \hat{L} = O \left( \frac{L}{\sqrt{b}} \right) \). In terms of the dependence on \( b \), the SBOE algorithm is \( O \left( \sqrt{b} \right) \) worse, while it is not evident to us that this dependence can be improved.

### 4.2 Convergence for MVIs with block structure

In this subsection we show that the SBOE method can achieve a \( \left( \frac{1}{t} \right) \) convergence rate for solving standard monotone VIs. In the following we assume the norm \( \| \cdot \| \) is Euclidean and \( \omega(x) = \frac{1}{2} \| x \|^2 \), \( \forall x \in X \). Furthermore we assume that the set \( X \) is bounded. The proof of this result relies on the construction of a novel auxiliary sequence \( \{ z_t \} \) that is related but different from those used in the VI literature [21, 1].

**Theorem 4.4.** Suppose that (2.28) holds. Let \( \{ x_t \} \) be generated by Algorithm 3 and denote

\[
\bar{x}_{k+1} := \frac{\sum_{t=1}^{k+1} [\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1)] x_{t+1} + \theta_t \gamma_t b x_{k+1}}{\sum_{t=1}^{k+1} [\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1)] + \theta_k \gamma_k b}.
\]

If (4.6), (4.7), (4.17), (4.19) hold, and \( \theta_{t-1} \geq 16L^2 \gamma_t^2 \lambda_t^2 \theta_t, \theta_t \leq \theta_{t-1} \), then

\[
\mathbb{E} \left[ \text{gap}(\bar{x}_{k+1}) \right] \leq \frac{\theta_1}{\sum_{t=1}^{k+1} [\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1)] + \theta_k \gamma_k b} \max_{x \in X} \left[ 5(b+1) V(x_1, x) + \gamma_1 (b - 1)(F(x_1), x_1 - x) \right].
\]

In particular, if \( \gamma_t = \frac{1}{4Lb}, \lambda_t = b \) and \( \theta_1 = 1 \), then

\[
\mathbb{E} \left[ \text{gap}(\bar{x}_{k+1}) \right] \leq \frac{4Lb}{k+1} \max_{x \in X} \left[ 5(b+1) V(x_1, x) + \frac{b-1}{4Lb} (F(x_1), x_1 - x) \right].
\]
Proof. Note that (4.14) still holds, and therefore
\[ \sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k-1} (\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1)) \langle F(x_{t+1}), x_{t+1} - x \rangle - \theta_1 \gamma_1 (b - 1) \langle F(x_1), x_1 - x \rangle \\
+ \theta_1 \gamma_1 b \langle F(x_{k+1}), x_{k+1} - x \rangle - \theta_k \gamma_k b \langle \Delta F_{k+1}, x_{k+1} - x \rangle + \sum_{t=1}^{k} \theta_t (A_t + B_t) \\
+ \sum_{t=1}^{k} \theta_t \lambda_t b \langle \Delta F_t, x_t - x_t \rangle + \sum_{t=1}^{k} \theta_t b V(x_t, x_{t+1}), \] (4.23)

where \( A_t = \gamma_t \langle F(x_t) + \lambda_t (F(x_t) - F(x_{t+1})), x_{t+1} - x \rangle \), \( \hat{x}_{t+1} = \lceil b x_{t+1} - (b - 1) x_t \rceil \), and \( B_t = V(x_{t+1}, x) - b V(x_{t+1}, x) + (b - 1) V(x_t, x) \).

By the definition of Bregman distance, we have \( B_t = B_t^{(1)} + B_t^{(2)} + \langle \delta_t, x \rangle \), where
\[ B_t^{(1)} = -\omega(\hat{x}_{t+1}) + b \omega(x_{t+1} - (b - 1) \omega(x_t), \]
\[ B_t^{(2)} = \langle \nabla \omega(\hat{x}_{t+1}), \hat{x}_{t+1} \rangle - b \langle \nabla \omega(x_{t+1}), x_{t+1} \rangle + (b - 1) \langle \nabla \omega(x_t), x_t \rangle, \]
\[ \delta_t = b \nabla \omega(x_{t+1}) - (b - 1) \nabla \omega(x_t) - \omega(\hat{x}_{t+1}). \]

We now introduce an auxiliary sequence \( \{z_t\} \), which satisfies \( z_1 = x_1 \) and for \( t = 1, 2, \ldots, k \),
\[ z_{t+1} = \arg\min_{x \in X} \langle \delta_t, x \rangle + 4(b + 1) V(z_t, x). \]

Utilizing Lemma 3.9 and \( \theta_t \leq \theta_{t-1} \), we obtain
\[ \sum_{t=1}^{k} \theta_t \delta_t (z_t - x) \leq 4 \theta_1 (b + 1) V(z_1, x) + \sum_{t=1}^{k} t \theta_t \delta_t^2 (z_t - x) \leq 4 \theta_1 (b + 1) V(z_1, x) + \sum_{t=1}^{k} \theta_t (\|x_0 - x_{t+1}\|^2 + \|x_t - x_{t+1}\|^2 \|x_t - x_{t+1}\|^2). \]

The second inequality follows from \( \omega(\cdot) = \frac{\|\cdot\|^2}{2} \) and Young’s inequality. Combining the relation above with (4.23) and invoking (4.15) gives us
\[ \sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k-1} (\theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1)) \langle F(x_{t+1}), x_{t+1} - x \rangle - \theta_1 \gamma_1 (b - 1) \langle F(x_1), x_1 - x \rangle \\
+ \theta_1 \gamma_1 b \langle F(x_{k+1}), x_{k+1} - x \rangle - \theta_k \gamma_k b \langle \Delta F_{k+1}, x_{k+1} - x \rangle + \sum_{t=1}^{k} \theta_t (A_t + B_t^{(1)} + B_t^{(2)}) \\
+ \sum_{t=1}^{k} \theta_t \langle \delta_t, z_t \rangle - 4 \theta_1 (b + 1) V(z_1) + \sum_{t=1}^{k} \theta_t \|x_0 - x_{t+1}\|^2 \|x_t - x_{t+1}\|^2 \|x_t - x_{t+1}\|^2 + C, \] (4.24)

where
\[ C = \sum_{t=1}^{k} \theta_t \gamma_t \lambda_t b \langle \Delta F_t, x_t - x_t \rangle + \sum_{t=1}^{k} \theta_t b \|x_t - x_{t+1}\|^2 \\
\geq \sum_{t=1}^{k} \left[ - \theta_t \gamma_t \lambda_t b L \|x_t - x_{t+1}\| \|x_{t+1} - x_t\| + \frac{b \theta_t}{b} \|x_t - x_{t+1}\|^2 + \frac{b \theta_t}{b} \|x_t - x_{t-1}\|^2 \right] + \frac{b \theta_t}{b} \|x_t - x_{t+1}\|^2. \]

Here the first inequality follows from (4.2), while the second inequality follows from the condition \( \theta_{t-1} \geq 16 L^2 \gamma^2 \lambda^2 \theta_t \). Using the above bound on \( C \) in (4.23) and with Young’s inequality, we obtain
\[ \sum_{t=1}^{k} \theta_t b \Delta V_t(x) \geq \sum_{t=1}^{k-1} \left[ \theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1) \right] \langle F(x_{t+1}), x_{t+1} - x \rangle + \theta_1 \gamma_1 b \langle F(x_{k+1}), x_{k+1} - x \rangle \\
- \theta_1 \gamma_1 (b - 1) \langle F(x_1), x_1 - x \rangle - 2 \theta_1 \gamma_1 \|x_{k+1} - x\|^2 + \frac{b \theta_1}{b} \|x_{k+1} - x_t\|^2 \\
+ \sum_{t=1}^{k} \theta_t (A_t + B_t^{(1)} + B_t^{(2)}) + \langle \delta_t, z_t \rangle + \frac{b \|x_t - x_{t+1}\|^2 \|x_t - x_{t+1}\|^2 \|x_t - x_{t-1}\|^2}{4(b + 1)} - 4 \theta_1 (b + 1) V(x_1, x). \]

After taking expectation on both sides of the inequality, it follows by (4.11) that \( E[A_t + B_t^{(1)} + B_t^{(2)} + \theta_t \langle \delta_t, z_t \rangle + \|x_t - x_{t+1}\|^2 \|x_t - x_{t+1}\|^2 \|x_t - x_{t-1}\|^2] = 0 \). Together with the condition \( \theta_t \leq \theta_{t-1} \) and (4.19) we obtain
\[ \sum_{t=1}^{k-1} \left[ \theta_t \gamma_t b - \theta_{t+1} \gamma_{t+1} (b - 1) \right] \mathbb{E}[\langle F(x_{t+1}), x_{t+1} - x \rangle] + \theta_1 \gamma_1 b \mathbb{E}[\langle F(x_{k+1}), x_{k+1} - x \rangle] \\
\leq \theta_1 b V(x_1, x) + 4 \theta_1 (b + 1) V(x_1, x) + \theta_1 \gamma_1 (b - 1) \langle F(x_1), x_1 - x \rangle. \]

Furthermore, similar as in the proof of Theorem 2.6, using the monotonicity property in (2.28) and the definition of \( \bar{x}_{k+1} \) we get (4.21). \( \square \)

In view of Theorem 4.4, the SOBIE method can achieve a \( O(b^2 \bar{L} / \epsilon) \) complexity rate for MVI problems. Similarly to the GSMVI case, the SOBIE algorithm is \( O(\sqrt{b}) \) worse than the OE algorithm in terms of the dependence on number of blocks.
5. Numerical experiments. In this section we report some preliminary numerical results for the operator extrapolation (OE) algorithm as well as its stochastic (SOE) and stochastic block (SBOE) variants.

5.1. Traffic assignment problem. We consider a classic problem in operations research, the general traffic assignment problem, where the travel cost on each link in the transportation network may depend on the flow on this link as well as other links in the network. Algorithmic design for the computation of traffic equilibrium patterns based on the theory of variational inequalities originated in the work of [8]. The traffic network is abstracted as a directed graph, consisting of a set of nodes $\mathcal{N}$, a set of directed arcs $\mathcal{A}$, $|\mathcal{A}| = n$, together with a set of ordered node pairs $\mathcal{W}$, $|\mathcal{W}| = N$, where an element $w = (o, d) \in \mathcal{W}$ is referred to as an origin-destination (OD) pair. The travel demand $d_w \geq 0$ associated to the OD pair $w \in \mathcal{W}$ is to be distributed among the paths of the network that connect $w$, the latter set is denoted by $P_w$. The set of feasible path flow vectors is $X \subset \mathbb{R}^N$, $X = \{x \in \mathbb{R}^n \mid \forall w \in \mathcal{W}, \forall p \in P_w, \sum_{p \in P_w} x_p = d_w, \ x_p \geq 0\}$.

Let $A$ denote the arc-chain incidence matrix of the network. A path flow pattern $x \in X$ induces an arc flow pattern $y \in \mathbb{R}^n$ and the set of feasible arc flows is $Y = \{y \in \mathbb{R}^n \mid y = Ax, x \in X\}$. We introduce the map $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $F(y)$ denotes the vector of costs associated to a particular arc flow pattern $y$. The traffic assignment problem admit a VI formulation: Given a transportation network and a demand vector $d \in [0, 1]^n$, we find $y^* \in Y : \langle F(y^*), y - y^* \rangle \geq 0, \ \forall y \in Y.

The equilibrium condition (5.1) is referred in the literature as the user-optimization principle. We consider an affine parameterization for the operator $F$, with $F(y) = Gy + b$, $G \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^n$. The matrix $G$ is not necessarily symmetric. We test our algorithms on randomly generated data sets, where $A = I$, $|\mathcal{W}| = 5$, $G + G^T \succ 0$, and $b = 5$, while the number of arcs is progressively increased. Given that the operator $F$ is affine, one can directly compute $L = \sigma_{\text{max}}(G)$, $\mu = \frac{1}{2}\lambda_{\text{min}}(G + G^T)$. The algorithm proposed in [23] abbreviated as (NS) is also considered for the purposes of illustration.

![Fig. 5.1](https://example.com/fig51.png)

**Fig. 5.1.** Traffic assignment problem for networks with 1000, 2500, 5000, 10000 arcs. Comparison of error trajectory $\|y_t - y^*\|^2 = \|x_t - x^*\|^2$ for OE, NS and SBOE algorithm, number of blocks is $b = 5$. 

In all four experiments the (OE) algorithm seems to exhibit faster convergence to the equilibrium point in terms of the iteration count. The (NS) algorithm requires the maintenance of two sequences of iterates which is reflected in approximately double the average computation time per iteration in comparison to the (OE) algorithm. As expected the (SBOE) algorithm is the least computationally intensive given that it has the advantage of solving a lower dimensional optimization problem at each iteration, while it can also benefit from a recursive update of the operator utilizing the underlying block structure, see also [16].

5.2. Signal estimation and generalized linear models. We consider a nonlinear signal estimation problem involving generalized linear models (GLMs) that is amenable to a variational inequality formulation (see Chapter 5.2 of [12]). An i.i.d. sequence of regressor-label observations

\[ \xi^K = \{\xi_k = (\eta_k, y_k), 1 \leq k \leq K\} \]

is generated according to a distribution \( P_{\xi^K} \), where \( x^* \in \mathcal{X} \subset \mathbb{R}^n \) is an unknown signal lying in a convex compact set. The regressor vector \( \eta \) has distribution \( Q \) independent of \( x^* \), while the conditional distribution of the label \( y \) given \( \eta \) is controlled by \( x^* \) and \( \mathbb{E}[y|\eta] = f(\eta^TAx^*) \), where \( f: \mathbb{R} \to \mathbb{R} \) is referred to in the statistics literature as the link function and \( A \in \mathbb{R}^{n \times n} \) is a full rank, known matrix, that satisfies the condition \( A + A^T > 0 \). The goal is to infer \( x^* \in \mathcal{X} \) from the i.i.d. realizations \( \xi^K \) generated by the unknown distribution \( P_{\xi^K} \). This inference problem admits the following VI formulation. Let the operator \( F: \mathcal{X} \to \mathcal{X} \) be defined as

\[ F(x) = \mathbb{E}[\eta f(\eta^T Ax)] - \mathbb{E}[\eta f(\eta^T Ax^*)]. \]

Then \( x^* \) is a zero of \( F \) and under mild regularity assumptions is given as the solution of the VI problem of finding \( x^* \in X \) s.t. \( \langle F(x^*), x - x^* \rangle \geq 0 \) for all \( x \in X \). An unbiased estimator of \( F(x) \) is given by \( \tilde{F}(x) = \eta f(\eta^T Ax) - \eta y \). We will apply the four versions of the (SOE) algorithm with the stepsize selection as in Corollary 3.4 (SOE-1), Corollary 3.5 (SOE-2), Corollary 3.6 (SOE-3) and Theorem 3.8 (SOE-4) and compare to the classic stochastic approximation (SA) algorithm. We consider the case where \( \mathcal{X} = \{x \in \mathbb{R}^n | \|x\| \leq R\} \), is a ball of radius \( R \), while the distribution \( Q \) of the regressor \( \eta \) is \( \mathcal{N}(0, I_n) \). There are two options for the link function:

|   |   |   |   |
|---|---|---|---|
| A | hinge function | \( y \in \mathbb{R} \) | \( f(s) = \max\{s, 0\} \) |
| B | ramp sigmoid   | \( y \in \mathbb{R} \) | \( f(s) = \min\{1, \max\{s, 0\}\} \) |

Table 5.2

link functions

In both cases the conditional distribution of the label \( y \) given the regressor \( \eta \) is Gaussian \( \mathcal{N}(f(\eta^T Ax^*), \sigma_y) \), while the operator \( F \) is Lipschitz continuous and strongly monotone. The following observation facilitates the analytic calculation of bounds to the Lipschitz constant as well as the modulus of strong monotonicity in each case. Fix \( x \in \mathbb{R}^n - \{0\} \), and set \( z = Ax \). Consider the operator \( G(x) = \mathbb{E}[\eta f(\eta^T Ax)] \), and note that \( F(x) = G(x) - G(x^*) \). Set \( u_1 = \frac{1}{\|x\|} \) and extend this vector to an orthonormal basis \( \{u_1, \ldots, u_n\} \) of \( \mathbb{R}^n \). The Gaussian vector \( \eta \) is written as \( \eta = \sum_{i=1}^n (u_i, \eta) u_i \). One observes that \( \langle \eta, z \rangle \) and \( \eta_u = \sum_{j=2}^n (u_j, \eta) u_i \) are independent, since for any \( j \in \{2, n\} \) the normal r.v.s \( \langle \eta, z \rangle \) and \( \langle \eta, u_j \rangle \) are uncorrelated. This realization allows one to write

\[ G(x) = \mathbb{E}[\eta f(\eta^T Ax)] = \mathbb{E}[\frac{\|z\|}{\|x\|} \eta f(\eta^T z)] = \frac{\|z\|}{\|x\|} \mathbb{E}[\zeta f(\zeta \|z\|)], \]

where \( \zeta = \frac{z}{\|z\|} \) is a standard normal.
5.2.1. Hinge function. In this case $G_A(x) = \frac{1}{2} A x$ and as such $L_A = \frac{1}{2} \sigma_{\text{max}}(A)$, $\mu_A = \frac{1}{2} \lambda_{\text{min}}(A + A^T)$. It is worth noting that in this example, while the link function is non-linear the corresponding operator is linear proportional to the one of the linear regression case, where the link function is $f(s) = s$. We test the algorithms on some randomly generated data sets with dimension $n = 100$ and radius $R = 100$. The vector $x^*$ is created by sampling each of its entries uniformly in $[0, 1]$ and subsequently normalizing such that $\|x^*\| = R$. The matrix $A$ is of the form $A = \text{diag}(d) + d_\times \times 10^{-2} \hat{A}$ where the entries of $\hat{A}$ are sampled uniformly in $[0, 1]$ and the vector $d \in \mathbb{R}^{100}$ has entries chosen equidistantly between $d_-$ and $d_+ = 1$. The parameter $d_-$ is progressively decreased in order to achieve lower $\mu$ and therefore higher condition numbers.

The theoretical analysis provides us with conservative stepsize policies which will ensure the convergence for all the algorithms above. In terms of the actual implementation we fine-tuned each stepsize policy, by changing the value of the Lipschitz constant $L$. For fairness purposes we maintain the same $L$ across all algorithms and stepsize policies. We fine-tuned the value of $L$ based on the first 50 iterations and set it to $L = 0.5$. Our choice of $L$ improves the convergence speed while ensuring that the error of the algorithm decreases steadily. The motivation behind our approach lies in the fact that the bound of (3.6) only requires a local Lipschitz constant for each time step $t$, which can be smaller than the global one.

As we can see in figure 5.2 above, the SOE method performs better than SA, with the contrast becoming more pronounced as $\mu$ decreases. In the next set of experiments (see figure 5.3), we wanted to manifest the significance of the index resetting scheme. For this purpose we lowered the standard deviation $\sigma_y$ associated to the observation label $y$ and increased the batch size to $m = 1000$. This step reduces the operator noise and therefore diminishes the size of each epoch, so that we are able to observe the effect of this scheme within the horizon under consideration.

![Fig. 5.2. Comparison between SA and SOE-1, SOE-2, SOE-3 and SOE-4 algorithms for the hinge function example. From left to right $d_-$ is set to $10^{-1}$, $10^{-2}$, $10^{-3}$ respectively. For each experiment the value of the operator at each iteration was averaged with $m = 100$ samples and we set $\sigma_y = 1$.](image)

![Fig. 5.3. Comparison between SA and SOE-1 and SOE-3 algorithms for the hinge function example. From left to right $d_-$ is set to $10^{-1}$, $10^{-2}$, $10^{-3}$, respectively. For each experiment we use a batch-size $m = 1000$ samples and we set $\sigma_y = 0.1$.](image)
5.2.2. Ramp sigmoid. In this case the computation of the operator $G_C(x)$ is more involved and we demonstrate it when $A = I_n$.

$$G_C(x) = \frac{x}{\|x\|} \mathbb{E}[\zeta \min \{1, \max \{\zeta \|x\|, 0\}\}] = \frac{1}{\sqrt{2\pi}} \frac{x}{\|x\|} \left( \int_0^{\|x\|} \zeta^2 \|x\| \exp(-\zeta^2) d\zeta + \int_1^{\infty} \zeta \exp(-\zeta^2/4) d\zeta \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{x}{\|x\|} \left( \sqrt{\frac{2}{\pi}} \|x\| \text{erf}\left(\frac{\sqrt{2}}{\|x\|}\right) \right) = \frac{1}{2} x \text{erf}\left(\frac{\sqrt{2}}{\sqrt{\|x\|}}\right).$$

The above calculation involves partial integration and employing the definition of the error function, namely $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$. The Jacobian $\nabla G_C(x)$ is given by $\nabla G_C(x) = \frac{1}{2} \text{erf}\left(\frac{1}{\sqrt{2\|x\|}}\right)I_n - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\|x\|^2}\right)xx^T$. As such

$$L_c = \max_{x \in X} \left\{ \frac{1}{2} \text{erf}\left(\frac{1}{\sqrt{\|x\|}}\right) \right\} = \frac{1}{2}, \quad \mu_C = \min_{x \in X} \left\{ \frac{1}{2} \text{erf}\left(\frac{1}{\sqrt{\|x\|}}\right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\|x\|^2}\right) \right\} = \frac{1}{2} \text{erf}\left(\frac{1}{\sqrt{2\|x^*\|}}\right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\|x^*\|^2}\right).$$

The modulus of strong monotonicity is a strictly decreasing function of the radius of the domain and the latter variable is going to be progressively increased in order to achieve more challenging cases from the condition number point of view. As before we choose $n = 100$ and the vector $x^*$ is created by sampling its entries i.i.d. in $[0, 1]$ and subsequently normalizing such that $\|x^*\| = R$.

This is a more challenging example, given the nonlinearity of the operator $F$. The results exhibit similar trend as in the hinge function case, in the sense of SOE outperforming SA as the condition number increases, exacerbated by the non-linearity of the operator.

6. Concluding remarks. This paper presents a new class of (stochastic) first-order methods obtained by incorporating operator extrapolation into the gradient (operator) projection methods. We show that the OE method can achieve the optimal convergence for solving deterministic VIs in a much simpler way than existing approaches. The stochastic counterpart of OE, i.e., SOE, achieves the optimal complexity for solving many stochastic VIs, including the stochastic smooth and strongly monotone VIs for the first time in the literature. As a smooth optimization method, SOE allows the application of mini-batch of samples for variance reduction and hence facilitates distributed stochastic optimization. Novel stochastic block operator extrapolation (SBOE) has been proposed, for the first time in the literature, for solving deterministic VIs with a certain block structure. Numerical experiments, conducted on a classic traffic assignment problem and a more recent generalized linear model for signal estimation, demonstrate the advantages of the proposed algorithms.

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