On A Theorem In Multi-Parameter Potential Theory

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Let $X$ be an $N$-parameter additive Lévy process in $\mathbb{R}^N$ with Lévy exponent $(\Psi_1, \cdots, \Psi_N)$ and let $\lambda_d$ denote Lebesgue measure in $\mathbb{R}^d$. We show that

$$E\{\lambda_d(X(\mathbb{R}^N_+))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty.$$  

This was previously proved by Khoshnevisan, Xiao and Zhong [1] under a sector condition.

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1. Introduction and Proof

Let $X_1, X_2, \cdots, X_N$ be $N$ independent Lévy processes in $\mathbb{R}^d$ with their respective Lévy exponents $\Psi_j$, $j = 1, 2, \cdots, N$. The random field

$$X_t = X_{t_1} + X_{t_2} + \cdots + X_{t_N}, \quad t = (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N_+$$

is called the additive Lévy process. Let $\lambda_d$ denote Lebesgue measure in $\mathbb{R}^d$.

**Theorem 1.1** Let $X$ be an additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \cdots, \Psi_N)$. Then

$$E\{\lambda_d(X(\mathbb{R}^N_+))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty. \quad (1.1)$$

Recently, Khoshnevisan, Xiao and Zhong [1] proved that if

$$\text{Re} \left( \prod_{j=1}^N \frac{1}{1 + \Psi_j(\xi)} \right) \geq \theta \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) \quad (1.2)$$

for some constant $\theta > 0$ then Theorem 1.1 holds. In fact the proof of Theorem 1.1 does not need any condition.

**Proof of Theorem 1.1:** Define

$$\mathcal{E}(\mu) = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi$$

where $\mu$ is a probability measure on a compact set $F \subset \mathbb{R}^d$ and $\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu(dx)$. Let $F = \{0\} \subset \mathbb{R}^d$ and $\delta_0$ be the point mass at $0 \in \mathbb{R}^d$. We first quote a key lemma of [1]:

**Lemma 5.5** Suppose $X$ is an additive Lévy process in $\mathbb{R}^d$ that satisfies Condition (1.3), and that $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$, where $\Psi = (\Psi_1, \cdots, \Psi_N)$ denotes the Lévy exponent of $X$. Then,
for all compact sets $F \subset \mathbb{R}^d$, and for all $r > 0$,

$$E\{\lambda_d(X([0, r]^N \oplus F)) \leq \theta^{-2}(4e^{2r})^N \cdot C_\Psi(F),$$

where $\theta > 0$ is the constant in Condition (1.3).

By reviewing the whole process of the proof of Theorem 1.1 of [1] given by Khoshnevisan, Xiao and Zhong, our Theorem 1.1 certainly follows if we instead prove the following statement:

Let $X$ be any additive Lévy process in $\mathbb{R}^d$. If $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$, then

$$E\{\lambda_d(X([0, r]^N)) \leq \frac{c_{N,d,r}}{E_\Psi(\delta_0)}$$

for some constant $c_{N,d,r} \in (0, \infty)$ depending on $N$, $d$, $r$ only.

Clearly, all we have to do is to complete Eq. (5.11) of [1] without bothering ourselves with Condition (1.3) of [1]. Since $\delta_0$ is the only probability measure on $F = \{0\}$, letting $\eta \to 0$, $k \to \infty$, and $\varepsilon \to 0$ and using the integrability condition $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$ yield

$$\mathcal{E}_\Psi(\delta_0) \geq c_1 \left| \int_{\mathbb{R}^d} \text{Re} \left( \prod_{i=1}^N \frac{1}{1 + \Psi_j(\xi)} \right) d\xi \right|^2 E\{\lambda_d(X([0, r]^N)) \} \tag{1.4}$$

where $c_1 \in (0, \infty)$ is a constant depending on $N$, $d$, $r$ only.

Consider the $2^{N-1}$ similar additive Lévy processes (including $X_t$ itself) $X_t^\pm = X_{t_1}^1 \pm X_{t_2}^2 \pm \cdots \pm X_{t_N}^N$. Here, $\pm$ is merely a symbol for each possible arrangement of the minus signs; e.g., $X^1 - X^2 + X^3$, $X^1 - X^2 - X^3$, $X^1 + X^2 + X^3$ and so on. Let $\Psi^\pm$ be the Lévy exponent for $X_t^\pm$. Since $-X^j$ has Lévy exponent $-\Psi_j$, $\mathcal{E}_{\Psi^\pm}(\mu) = \mathcal{E}_\Psi(\mu)$ for all $X_t^\pm$ and

$$\sum \text{Re} \left( \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N s_j - s_s \Psi_j(\xi)} ds \right) = 2^{N-1} \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) > 0$$

where the first summation $\sum$ is taken over the collection of all the $X_t^\pm$. On the other hand,

$$Q_\mu(\xi) = \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} e^{-\sum_{j=1}^N |t_j - s_j| \Psi_j(\text{sgn}(t_j - s_j) \xi)} \mu(ds)\mu(dt)$$

remains unchanged for all $X_t^\pm$ as long as $\mu$ is an $N$–fold product measure on $\mathbb{R}_+^N$. Proposition 10.3 of [1] and Theorem 2.1 of [1] together state that for any additive Lévy process $X$,

$$k_1 \left( \int_{\mathbb{R}^d} Q_\mu(\xi) d\xi \right)^{-1} \leq E\{\lambda_d(X([0, r]^N)) \} \leq k_2 \left( \int_{\mathbb{R}^d} Q_\mu(\xi) d\xi \right)^{-1},$$

where $\lambda^\prime$ is the restriction of the Lebesgue measure $\lambda_N$ in $\mathbb{R}^N$ to $[0, r]^N$ and $k_1$, $k_2 \in (0, \infty)$ are two constants depending only on $r$, $N$, $d$, $\pi$. Note that $\lambda^\prime$ is an $N$–fold product measure on $\mathbb{R}_+^N$. Thus, there exists a constant $c_2 \in (0, \infty)$ depending only on $N$ and $r$ such that

$$E\{\lambda_d(X([0, r]^N)) \} \leq c_2 E\{\lambda_d(X^\pm([0, r]^N)) \}$$
for all $X_t^\pm$. Since $|1 + z| = |1 + \bar{z}|$ where $z$ is a complex number, $\int_{\mathbb{R}^d} \prod_{j=1}^N |1 + \Psi_j(\xi)|^{-1} d\xi < +\infty$ as well. Therefore, by (1.4),

$$2^{N-1} \sqrt{c_2} \sqrt{\frac{\mathbb{E}_\Psi(\delta_0)}{\mathbb{E}\{\lambda_d(X([0,r]^N))\}}} \geq \sum \sqrt{\frac{\mathbb{E}_\Psi(\delta_0)}{\mathbb{E}\{\lambda_d(X^{\pm}([0,r]^N))\}}}$$

$$\geq \sqrt{c_1} \sum \int_{\mathbb{R}^d} \text{Re} \left( \int_{\mathbb{R}^d} e^{-\sum_{j=1}^N s_j - s \cdot \Psi_j(\xi)} d\xi \right) d\xi$$

$$\geq \sqrt{c_1} \sum \int_{\mathbb{R}^d} \text{Re} \left( \int_{\mathbb{R}^d} e^{-\sum_{j=1}^N s_j - s \cdot \Psi_j(\xi)} d\xi \right) d\xi$$

$$= 2^{N-1} \sqrt{c_1} \int_{\mathbb{R}^d} \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi$$

$$= 2^{N-1} \sqrt{c_1} (2\pi)^d \mathbb{E}_\Psi(\delta_0).$$

(1.3) follows, so does the theorem.

\[ \square \]

2. Applications

2.1 The Range of An Additive Lévy Process

As the first application, we use Theorem 1.1 to compute $\dim_H X(\mathbb{R}^N)$. Here, $\dim_H$ denotes the Hausdorff dimension. To begin, we introduce the standard $d$-parameter additive $\alpha$-stable Lévy process in $\mathbb{R}^d$ for $\alpha \in (0,1)$:

$$S^\alpha_t = S^1_{t_1} + S^2_{t_2} + \cdots + S^d_{t_d},$$

that is, the $S^j$ are independent standard $\alpha$-stable Lévy processes in $\mathbb{R}^d$ with the common Lévy exponent $|\xi|^\alpha$.

**Theorem 2.1** Let $X$ be any $N$-parameter additive Lévy process in $\mathbb{R}^d$ with Lévy exponent $(\Psi_1, \ldots, \Psi_N)$. Then

$$\dim_H X(\mathbb{R}^N) = \sup \left\{ \beta \in (0, d) : \int_{\mathbb{R}^d} |\xi|^\beta - d \prod_{j=1}^N \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty \right\} \text{ a.s.} \quad (2.1)$$

**Proof** Let $C_\beta$ denote the Riesz capacity. By Theorem 7.2 of [1], for all $\beta \in (0, d)$ and $S^{1-\beta/d}$ independent of $X$,

$$EC_\beta(X(\mathbb{R}^N)) > 0 \iff E\{\lambda_d(S^{1-\beta/d}(\mathbb{R}^d) + X(\mathbb{R}^N))\} > 0. \quad (2.2)$$
Note that $S^{1 - \beta/d} + X$ is a $(d + N, d)$–additive Lévy process. Thus, by Theorem 1.1 and the fact that $\beta < d$ and $\Re \left( \frac{1}{1 + \Psi_j(\xi)} \right) \in (0, 1]$, we have for all $\beta \in (0, d)$,

$$EC_\beta(X(\mathbb{R}_+^d)) > 0 \iff \int_{\mathbb{R}^d} |\xi|^{\beta - d} \prod_{j=1}^N \Re \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty. \quad (2.3)$$

Thanks to the Frostman theorem, it remains to show that $C_\beta(X(\mathbb{R}_+^N)) > 0$ is a trivial event. Let $\mathcal{E}_\beta$ denote the Riesz energy. By Plancherel’s theorem, given any $\beta \in (0, d)$, there is a constant $c_{d,\beta} \in (0, \infty)$ such that

$$\mathcal{E}_\beta(\nu) = c_{d,\beta} \int_{\mathbb{R}^d} |\hat{\nu}(\xi)|^2 |\xi|^{\beta - d} d\xi \quad (2.4)$$

holds for all probability measures $\nu$ in $\mathbb{R}^d$. Consider the 1-killing occupation measure

$$O(A) = \int_{\mathbb{R}_+^N} 1(X_t \in A)e^{-\sum_{j=1}^N t_j} dt, \quad A \subset \mathbb{R}^d.$$

Clearly, $O$ is a probability measure supported on $X(\mathbb{R}_+^N)$. It is easy to verify that

$$E|\hat{O}(\xi)|^2 = \prod_{j=1}^N \Re \left( \frac{1}{1 + \Psi_j(\xi)} \right).$$

It follows from (2.4) that

$$E\mathcal{E}_\beta(O) = c_{d,\beta} \int_{\mathbb{R}^d} |\xi|^{\beta - d} \prod_{j=1}^N \Re \left( \frac{1}{1 + \Psi_j(\xi)} \right) d\xi < \infty$$

when $EC_\beta(X(\mathbb{R}_+^N)) > 0$. Therefore, $\mathcal{E}_\beta(O) < \infty$ a.s. Hence, $C_\beta(X(\mathbb{R}_+^N)) > 0$ a.s. \hfill \Box

2.2 The Set of $k$-Multiple Points

First, we mention a $q$-potential density criterion: Let $X$ be an additive Lévy process and assume that $X$ has an a.e. positive $q$-potential density on $\mathbb{R}^d$ for some $q \geq 0$. Then for all Borel sets $F \subset \mathbb{R}^d$,

$$P \left\{ F \cap X((0, \infty)^N) \neq \emptyset \right\} > 0 \iff E \left\{ \lambda_d(F - X((0, \infty)^N)) \right\} > 0. \quad (2.5)$$

The argument is elementary but crucially hinges on the property: $X_{b+t} - X_b$, $t \in \mathbb{R}_+^N$ (independent of $X_b$) can be replaced by $X$ for all $b \in \mathbb{R}_+^N$; moreover, the second condition “a.e. positive on $\mathbb{R}^{dn}$” is absolutely necessary for the direction $\iff$ in (2.5); see for example Proposition 6.2 of [1].

Let $X^1, \ldots, X^k$ be $k$ independent Lévy process in $\mathbb{R}^d$. Define

$$Z_t = (X^2_{t_2} - X^1_{t_1}, \ldots, X^k_{t_k} - X^{k-1}_{t_{k-1}}), \quad t = (t_1, t_2, \ldots, t_k) \in \mathbb{R}_+^k.$$

$Z$ is a $k$-parameter additive Lévy process taking values in $\mathbb{R}^{d(k-1)}$. 

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Theorem 2.2  Let \((X^1; \Psi_1), \ldots, (X^k; \Psi_k)\) be \(k\) independent Lévy processes in \(\mathbb{R}^d\) for \(k \geq 2\). Assume that \(Z\) has an a.e. positive \(q\)-potential density for some \(q \geq 0\). [A special case is that if for each \(j = 1, \ldots, k\), \(X^j\) has a one-potential density \(u^j_1 > 0\), \(\lambda_d\)-a.e., then \(Z\) has an a.e. positive 1-potential density on \(\mathbb{R}^{d(k-1)}\).] Then

\[
P(\bigcap_{j=1}^{k} X^j((0, \infty)) \neq \emptyset) > 0 \iff \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^{k} \text{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1} < \infty \tag{2.6}
\]

with \(\xi_0 = \xi_k = 0\).

Proof  For any \(\mathbb{R}^d\)-valued random variable \(X\) and \(\xi_1, \xi_2 \in \mathbb{R}^d\), \(e^{i(\xi_1, \xi_2) \cdot (X, -X)} = e^{i(\xi_1 - \xi_2) \cdot X}\). In particular, the Lévy process \((X^j, -X^j)\) has Lévy exponent \(\Psi_j(\xi_1 - \xi_2)\). It follows that the corresponding integral in (1.1) for \(Z\) equals

\[
\int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^{k} \text{Re} \left( \frac{1}{1 + \Psi_j(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1}
\]

with \(\xi_0 = \xi_k = 0\). Clearly,

\[
P(\bigcap_{j=1}^{k} X^j((0, \infty)) \neq \emptyset) > 0 \iff P(0 \in Z((0, \infty)^k)) > 0.
\]

Since \(Z\) has an a.e. positive \(q\)-potential density, by (2.5)

\[
P(0 \in Z((0, \infty)^k)) > 0 \iff E\{\lambda_d^{d(k-1)}(Z((0, \infty)^k))\} > 0.
\]

(2.6) now follows from Theorem 1.1. \(\square\)

For each \(\beta \in (0, d)\) and \(S^{1-\beta/d}\) independent of \(X^1, \ldots, X^k\), define

\[
Z_{t}^{S,\beta} = (X^1_{t_0} - S^{1-\beta/d}_{t_0}, X^2_{t_2} - X^1_{t_1}, \ldots, X^k_{t_k} - X^{k-1}_{t_{k-1}}), \quad t = (t_0, t_1, t_2, \ldots, t_k) \in \mathbb{R}^{d+k}, \quad t_0 \in \mathbb{R}^d.
\]

\(Z_{t}^{S,\beta}\) is a \(k + d\) parameter additive Lévy process taking values in \(\mathbb{R}^{d+k}\).

Theorem 2.3  Let \((X^1; \Psi_1), \ldots, (X^k; \Psi_k)\) be \(k\) independent Lévy processes in \(\mathbb{R}^d\) for \(k \geq 2\). Assume that for each \(\beta \in (0, d)\), \(Z_{t}^{S,\beta}\) has an a.e. positive \(q\)-potential density on \(\mathbb{R}^{d+k}\) for some \(q \geq 0\). (\(q\) might depend on \(\beta\).) [A special case is that if for each \(j = 1, \ldots, k\), \(X^j\) has a one-potential density \(u^j_1 > 0\), \(\lambda_d\)-a.e., then \(Z_{t}^{S,\beta}\) has an a.e. positive 1-potential density on \(\mathbb{R}^{d+k}\) for all \(\beta \in (0, d)\).] If \(P(\bigcap_{j=1}^{k} X^j((0, \infty)) \neq \emptyset) > 0\), then almost surely \(\dim_H \bigcap_{j=1}^{k} X^j((0, \infty))\) is a constant on \(\bigcap_{j=1}^{k} X^j((0, \infty)) \neq \emptyset\) and

\[
\dim_H \bigcap_{j=1}^{k} X^j((0, \infty)) = \sup \left\{ \beta \in (0, d) : \int_{\mathbb{R}^{d+k}} | \sum_{j=1}^{k} \xi_j |^{\beta-d} \prod_{j=1}^{k} \text{Re} \left( \frac{1}{1 + \Psi_j(\xi_j)} \right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty \right\} \tag{2.7}
\]
Similarly, the corresponding integral in (1.1) for $Z$ equals

$$
\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^{\beta/d})^d} \prod_{j=1}^{k} \Re \left( \frac{1}{1 + \Psi_{j}(\xi_j)} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.
$$

Similarly, the corresponding integral in (1.1) for $Z^{S,\beta}$ equals

$$
\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi_0|^{\beta/d})^d} \prod_{j=1}^{k} \Re \left( \frac{1}{1 + \Psi_{j}(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.
$$

with $\xi_k = 0$. Since $Z^{S,\beta}$ has an a.e. positive $g$-potential density, by (2.5) and Theorem 1.1

$$
P \left[ \bigcap_{j=1}^{k} X^j((0,\infty)) \cap S^{1-\beta/d}((0,\infty)^d) \neq \emptyset \right] > 0 \iff P(0 \in Z^{S,\beta}((0,\infty)^{k+d})) > 0
$$

$$
\iff E \{ \lambda_{dk}(Z^{S,\beta}((0,\infty)^{k+d})) \} > 0 \iff
$$

$$
\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi_0|^{\beta/d})^d} \prod_{j=1}^{k} \Re \left( \frac{1}{1 + \Psi_{j}(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.
$$

Finally, use the cyclic transformation: $\xi_j - \xi_{j-1} = \xi_j^{\prime}, j = 1, \cdots, k-1, \xi_k = \xi_k^{\prime}$ to obtain

$$
\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi_0|^{\beta/d})^d} \prod_{j=1}^{k} \Re \left( \frac{1}{1 + \Psi_{j}(\xi_j - \xi_{j-1})} \right) d\xi_0 d\xi_1 \cdots d\xi_{k-1} < \infty.
$$

Let $X$ be a Lévy process in $\mathbb{R}^d$. Fix any path $X_t(\omega)$. A point $x^\omega \in \mathbb{R}^d$ is said to be a $k$-multiple point of $X(\omega)$ if there exist $k$ distinct times $t_1, t_2, \cdots, t_k$ such that $X_{t_1}(\omega) = X_{t_2}(\omega) = \cdots = X_{t_k}(\omega) = x^\omega$. Denote by $E_k^\omega$ the set of $k$-multiple points of $X(\omega)$. It is well known that $E_k$ can
be identified with $$\bigcap_{j=1}^{k} X^j((0, \infty))$$ where the $$X^j$$ are i.i.d. copies of $$X$$. Thus, Theorem 2.2 and Theorem 2.3 imply the next theorem.

**Theorem 2.4** Let $$(X, \Psi)$$ be any Lévy process in $$\mathbb{R}^d$$. Assume that $$X$$ has a one-potential density $$u^1 > 0$$, $$\lambda_d$$-a.e. Let $$E_k$$ be the $$k$$-multiple-point set of $$X$$. Then

$$P(E_k \neq \emptyset) > 0 \iff \int_{\mathbb{R}^{d(k-1)}} \prod_{j=1}^{k} \text{Re} \left( \frac{1}{1 + \Psi(\xi_j - \xi_{j-1})} \right) d\xi_1 \cdots d\xi_{k-1} < \infty \quad (2.9)$$

with $$\xi_0 = \xi_k = 0$$. If $$P(E_k \neq \emptyset) > 0$$, then almost surely $$\dim H E_k$$ is a constant on $$\{E_k \neq \emptyset\}$$ and

$$\dim H E_k = \sup \left\{ \beta \in (0, d) : \int_{\mathbb{R}^d} \left| \sum_{j=1}^{k} \xi_j \right|^{\beta - d} \prod_{j=1}^{k} \text{Re} \left( \frac{1}{1 + \Psi(\xi_j)} \right) d\xi_1 d\xi_2 \cdots d\xi_k < \infty \right\}. \quad (2.10)$$

### 2.3 Intersection of Two Independent Subordinators

Let $$X_t, t \geq 0$$ be a process with $$X_0 = 0$$, taking values in $$\mathbb{R}_+$$. First, we ask this question: What is a condition on $$X$$ such that for all sets $$F \subset (0, \infty),$$

$$P(F \cap X((0, \infty)) \neq \emptyset) > 0 \iff E\{\lambda_1(F - X((0, \infty)))\} > 0 \quad ?$$

For subordinators, still the existence and positivity of a $$q$$-potential density ($$q \geq 0$$) is the only known useful condition to this question.

Let $$\sigma$$ be a subordinator. Take an independent copy $$\sigma^-$$ of $$-\sigma$$. We then define a process $$\tilde{\sigma}$$ on $$\mathbb{R}$$ by $$\tilde{\sigma}_s = \sigma_s$$ for $$s \geq 0$$ and $$\tilde{\sigma}_s = \sigma_{-s}$$ for $$s < 0$$. Note that $$\tilde{\sigma}$$ is a process of the property: $$\tilde{\sigma}_{t+b} - \tilde{\sigma}_b, t \geq 0$$ (independent of $$\tilde{\sigma}_b$$) can be replaced by $$\sigma$$ for all $$b \in \mathbb{R}$$.

Let $$X_t, t \geq 0$$ be any process in $$\mathbb{R}^d$$. Then the $$q$$-potential density is nothing but the density of the expected $$q$$-occupation measure with respect to the Lebesgue measure. (When $$q = 0$$, assume that the expected 0-occupation measure is finite on the balls.) Since the reference measure is Lebesgue, one can easily deduce that if $$u$$ is a $$q$$-potential density of $$X$$, then $$u(-x)$$ is a $$q$$-potential density of $$-X$$. Consequently, if we define $$\tilde{X}_s = X_s$$ for $$s \geq 0$$ and $$\tilde{X}_s = X_{-s}$$ for $$s < 0$$ where $$X^-$$ is an independent copy of $$-X$$, then $$u(x) + u(-x)$$ is a $$q$$-potential density of $$\tilde{X}$$. Conversely, if $$\tilde{X}$$ has a $$q$$-potential density, then it has to be the form $$u(x) + u(-x)$$, where $$u$$ is a $$q$$-potential density of $$X$$. If $$\sigma$$ is a subordinator, after a little thought we can conclude that $$\tilde{\sigma}$$ has an a.e. positive $$q$$-potential density on $$\mathbb{R}$$ if and only if $$\sigma$$ has an a.e. positive $$q$$-potential density on $$\mathbb{R}_+$$.

**Lemma 2.5** If a subordinator $$\sigma$$ has an a.e. positive $$q$$-potential density for some $$q \geq 0$$ on $$\mathbb{R}_+$$, then for all Borel sets $$F \subset (0, \infty),$$

$$P(F \cap \sigma((0, \infty)) \neq \emptyset) > 0 \iff E\{\lambda_1(F - \sigma((0, \infty)))\} > 0. \quad (2.11)$$
Proof  Define $F^* = (-F) \cup F$. Since $F \subset (0, \infty)$, $(-F) \cap F = \emptyset$. Since $\sigma((0, \infty)) \cap \tilde{\sigma}((\infty, 0)) = \emptyset$ or at most $\{0\}$, by looking at the law of $\sigma^-$, it is clear that

$$P(F \cap \sigma((0, \infty)) \neq \emptyset) > 0 \iff P(F^* \cap \tilde{\sigma}(\mathbb{R}\{0\}) \neq \emptyset) > 0.$$  

Assume that $E\{\lambda_1(F - \sigma((0, \infty)))\} > 0$. Since $F \subset F^*$, $E\{\lambda_1(F^* - \sigma((0, \infty)))\} > 0$. From the above discussion, $\tilde{\sigma}$ has an a.e. positive $q$-potential density. Moreover, $\tilde{\sigma}$ is a process of the property: $\tilde{\sigma}_{t+b} - \tilde{\sigma}_b$, $t \geq 0$ (independent of $\tilde{\sigma}_b$) can be replaced by $\sigma$ for all $b \in \mathbb{R}$. It follows from the standard $q$-potential density argument that $P(F^* \cap \tilde{\sigma}(\mathbb{R}\{0\}) \neq \emptyset) > 0$. The direction $\implies$ in (2.11) is elementary since $\sigma$ has a $q$-potential density. \hfill \Box 

Theorem 2.6  Let $\sigma^1$ and $\sigma^2$ be two independent subordinators having the Lévy exponents $\Psi_1$ and $\Psi_2$, respectively. Assume that $\sigma^1$ has an a.e. positive $q$-potential density for some $q \geq 0$ on $\mathbb{R}_+$. Then

$$P[\sigma^1((0, \infty)) \cap \sigma^2((0, \infty)) \neq \emptyset] > 0 \iff \int_{-\infty}^{\infty} \text{Re} \left( \frac{1}{\Psi_1(x)} \right) \text{Re} \left( \frac{1}{1 + \Psi_2(x)} \right) dx < \infty. \quad (2.12)$$

Note that our result does not require any continuity condition on the $q$-potential density. 

Proof  By Lemma 2.5 and Theorem 1.1,

$$P[\sigma^1((0, \infty)) \cap \sigma^2((0, \infty)) \neq \emptyset] > 0 \iff \int_{-\infty}^{\infty} \text{Re} \left( \frac{1}{1 + \Psi_1(x)} \right) \text{Re} \left( \frac{1}{1 + \Psi_2(x)} \right) dx < \infty.$$  

Since $\sigma^1$ is transient, $\int_{|x| \leq 1} \text{Re} \left( \frac{1}{\Psi_1(x)} \right) dx < \infty$. The proof is therefore completed. \hfill \Box 

2.4 A Fourier Integral Problem

This part of content can be found in Section 6 of [1]. It is an independent Fourier integral problem. Neither computing the Hausdorff dimension nor proving the existence of 1-potential density needs the discussion below. [But this Fourier integral problem might be of novelty to those who want to replace the Lévy exponent by the 1-potential density.] Let $X$ be an additive Lévy process. Here is the question. Suppose that $K : \mathbb{R}^d \to [0, \infty]$ is a symmetric function with $K(x) < \infty$ for $x \neq 0$ that satisfies $K \in L^1$ and $\hat{K}(\xi) = k_1 \prod_{j=1}^{N} \text{Re} \left( \frac{1}{1 + \Psi_j(\xi)} \right)$. Under what conditions, can

$$\int \int K(x - y) \mu(dx) \mu(dy) = k_2 \int |\hat{\mu}(\xi)|^2 \prod_{j=1}^{N} \text{Re} \left( \frac{1}{1 + \Psi(\xi)} \right) d\xi \quad (2.13)$$

hold for all probability measures $\mu$ in $\mathbb{R}^d$? Here, $k_1$, $k_2 \in (0, \infty)$ are two constants. Consider the function $K$ in the following example. Define $\check{X}^1_{t_j} = -Y^j_{t_j}$ for $t_j < 0$ and $\check{X}^1_{t_j} = X^j_{t_j}$ for $t_j \geq 0$, where $Y^j$ is an independent copy of $X^j$ and the $Y^j$ are independent of each other and of $X$ as well. Then $\check{X}_t = \check{X}^1_{t_1} + \check{X}^2_{t_2} + \cdots + \check{X}^N_{t_N}$, $t \in \mathbb{R}^N$ is a random field on $\mathbb{R}^N$. Assume that $\check{X}$ has
a 1-potential density $K$. So, $K \in L^1$ and a direct check verifies that $K$ is symmetric. By the definition of $K$, $\tilde{K}(\xi) = \int_{\mathbb{R}^N} e^{-\sum_{j=1}^N |t_j|} e^{i \xi \cdot \tilde{X}_t} dt$. Evaluating this integral quadrant by quadrant and using the identity $\sum \prod_{j=1}^N \frac{1}{1+z_j} = 2^N \prod_{j=1}^N \text{Re} \left( \frac{1}{1+z_j} \right)$ for $\text{Re}(z_j) \geq 0$ (where $\sum$ is taken over the $2^N$ permutations of conjugate) yield $\tilde{K}(\xi) = k_1 \prod_{j=1}^N \text{Re} \left( \frac{1}{1+\Psi_j(\xi)} \right) > 0$.

If $\tilde{K} \in L^1$ (even though this case is less interesting), on one hand by Fubini,

$$\int |\hat{\mu}(\xi)|^2 \tilde{K}(\xi) d\xi = \int \int e^{-i \xi \cdot (x-y)} \tilde{K}(\xi) d\xi d\mu(dx) d\mu(dy)$$

and on the other hand by inversion (assuming the inversion holds everywhere by modification on a null set),

$$\int \int K(x-y) \mu(dx) \mu(dy) = (2\pi)^{-d} \int \int e^{-i \xi \cdot (x-y)} \tilde{K}(\xi) d\xi d\mu(dx) d\mu(dy).$$

Thus, (2.13) holds automatically in this case. If $K$ is continuous at $0$ and $K(0) < \infty$, then $\tilde{K} \in L^1$. This is a standard fact. Since $K \in L^1$ and $\tilde{K} > 0$, a bottom line condition needed to prove (2.13) is that $K$ is continuous at $0$ on $[0, \infty]$. This paper makes no attempt to solve the general case $K(0) = \infty$.

**Remark** Lemma 6.1 of [1] is not valid. (The authors of [1] looked like not having a clear idea how to prove a result of that sort.) The assumption that $\text{Re} \left( \prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)} \right) > 0$ cannot (by any means) justify either equation in (6.4) of [1]. Fortunately, Lemma 6.1 played no role in [1], because Theorem 7.2 of [1] is an immediate consequence of the well-known identity (2.4) of the present paper and Theorem 1.5 of [1]. Nevertheless [1] indeed showed that the 1-potential density of an isotropic stable additive process is comparable to the Riesz kernel at 0, and therefore the 1-potential density is continuous at 0 on $[0, \infty]$.

**REFERENCES**

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