Spontaneous Symmetry Breaking and Reflectionless Scattering Data

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Abstract

We consider the question which potentials in the action of a (1+1)-dimensional scalar field theory allowing for spontaneous symmetry breaking have quantum fluctuations corresponding to reflectionless scattering data. The general problem of restoration from known scattering data is formulated and a number of explicit examples is given. Only certain sets of reflectionless scattering data correspond to symmetry breaking and all restored potentials are similar either to the Phi**4-model or to the sine-Gordon model.

1 Introduction

Quantum corrections to classical solutions like kinks [1, 2] and spontaneous symmetry breaking are fields of intensive study and have applications in many branches of theoretical physics ranging from the Standard model to solid state. Recent interest appeared from some subtleties connected with supersymmetry [3]. A number of models is usually considered in this connection. The most popular ones are the Φ^4-model and the sine-Gordon model. They result in a scattering
problem for the quantum fluctuations with reflectionless potentials. As a result calculations of quantum corrections to the mass become very explicite. In the present paper we investigate the question which models result in a reflectionless scattering potential. The surprising result is that all of them are very similar to the above mentioned ones.

The setup of the problem is as follows. We consider a scalar field \( \Phi(x, t) \) in (1+1) dimensions with action

\[
S[\Phi] = \frac{1}{2} \int dx dt \left( (\partial_\mu \Phi)^2 + U(\Phi)^2 \right). \tag{1}
\]

If the squared potential, \( U^2(\Phi)^2 \), has two (or more) minima of equal depth spontaneous symmetry breaking occurs and topological nontrivial kink solutions \( \Phi_k(x) \) exist. In order to calculate the quantum fluctuations \( \eta(x, t) \) in the background of the kink one has to solve the scattering problem for the potential \( V(x) \) which appears from the second derivative \( \delta^2 S[\Phi_k]/\delta \Phi^2_k(x) \) of the action, see below Eq. (7). In simple models like the mentioned above this potential \( V(x) \) is reflectionless.

In the present paper we try to describe all potentials \( U(\Phi) \) in (1) which correspond to a reflectionless scattering potential \( V(x) \) and calculate the corresponding classical energy \( E_{\text{class}} \) and the quantum energy \( E_0 \) which is the ground state energy of the field \( \eta \) in the background of \( \Phi_k(x) \).

In calculating these quantities it is usually assumed that the potential \( U(\Phi) \) is given. After that one solves the scattering problem related to \( V(x) \) and calculates the energies \( E_{\text{class}} \) and \( E_0 \). In the paper [4] the inverse approach had been proposed. One starts from the solution of the scattering problem given in terms of the so called scattering data \( \{r(k), \beta_i, \kappa_i\} \) known since [5] to be in a one-to-one correspondence with the potential \( V(x) \) (for a representation of these questions see [6] and references therein). Here \( r(k) \) is the reflection coefficient, \( \kappa_i \) are the bound state energies and \( \beta_i \) are numbers connected with the normalization of the bound state wave functions. As shown in [4] the ground state energy can be expressed in a simple way in terms of the scattering data even including the necessary ultraviolet renormalization, see Eq. (10) below. In order to find the classical energy one has to restore the potential \( V(x) \) from the scattering data. This is the so called inverse scattering problem which was solved in terms of certain integral equations (see, again, [4]). In this way, solving the inverse scattering problem the classical energy can be calculated from the scattering data. In [4] it was shown how this procedure works on the simplest example of reflectionless \( (r(k) = 0) \) scattering data containing only one bound state.

In the present paper we use this inverse approach to describe all potentials \( U(\Phi) \) corresponding to reflectionless scattering data and having topologically nontrivial solutions allowing in this way for spontaneous symmetry breaking. It turns out that not all scattering data correspond to such potentials \( U(\Phi) \) but only certain classes. So we can formulate the reconstruction problem: find the
mapping between scattering data and potentials $U(\Phi)$ allowing for spontaneous symmetry breaking.

A special consideration deserve the so called rational scattering data. Here the reflection coefficient $r(k)$ is a rational function of $k$ thus given by a finite number of parameters. For a rational $r(k)$ the inverse scattering problem is known to have an explicit, algebraic solution (in a similar way as in the reflectionless case) and the classical energy can be obtained then by integration. In addition, the rational scattering data form a dense subset in the set of all scattering data. In this way, the inverse approach may provide an approximation scheme for the general case.

The paper is organized as follows. In the next section we consider soliton potentials providing completely explicit formulas. In the third section we consider scattering data given by two bound states. In the fourth section we show how this can be generalized to the general reflectionless case. Conclusions are given in the last section. We use units with $\hbar = c = 1$.

## 2 Formulation of the reconstruction problem

We consider a scalar field $\Phi$ with action $S[\Phi]$, Eq. (1), in (1+1) dimensions. Static solutions $\Phi(x)$ are subject to the equation of motion $\Phi''(x) = U(\Phi)U'(\Phi)$ where the prime denotes differentiation with respect to the argument. We assume that $U^2(\Phi)$ has at least two minima of equal depth and we are free to denote two neighbored ones by $\pm \Phi_{\text{vac}}$. These fields, $\Phi(x) = \pm \Phi_{\text{vac}}$, are the vacuum solutions. In case $\Phi_{\text{vac}} \neq 0$ there exist topological nontrivial solutions $\Phi_k(x)$ called kink solutions which interpolate between the vacuum solutions by means of $\Phi_k(x \to \pm \infty) = \pm \Phi_{\text{vac}}$. These solutions obey the Bogomolny equations

$$\Phi'_k(x) = U(\Phi_k(x)) \quad (2)$$

and have the classical energy

$$E_{\text{class}} = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( (\Phi'_k(x))^2 + U^2(\Phi_k(x)) \right)$$

which by means of Eq. (3) can be written in the form

$$E_{\text{class}} = \int_{-\infty}^{\infty} dx \ U^2(\Phi_k(x)) \quad (4)$$

In order to have a finite energy of the kink we must assume that the potential $U(\Phi)$ is zero in its minima.

The quantization of the scalar field in the background of the kink solution by means of the shift

$$\Phi(x, t) = \Phi_k(x) + \eta(x, t)$$

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delivers in the Gaussian approximation the action
\[ S_{\text{fluct.}}[\eta] = \frac{1}{2} \int dx \, dt \, \eta(x,t) \left( \partial^2_t - \partial^2_x + \mu^2 + V(x) \right) \eta(x,t) \] (6)
for the fluctuations where the potential \( V(x) \) results from
\[ \frac{1}{2} \int \! \! \! \int dx \, dt \, \eta(x,t) \left( \partial^2_t - \partial^2_x + \mu^2 + V(x) \right) \eta(x,t) = (U'(\Phi))^2 + U(\Phi)U''(\Phi) \equiv \mu^2 + V(x). \] (7)

Here \( \mu \) is defined from demanding \( V(x \to \infty) = 0 \) and has the meaning of being the mass of the fluctuating field \( \eta(x,t) \).

The one loop quantum corrections to the energy are given by a functional determinant. For a static background they can be equivalently formulated in terms of the ground state energy \( E_0 \) of \( \eta(x,t) \) in the background of the kink,
\[ E_0 = \frac{1}{2} \sum_{(n)} \epsilon_{(n)}, \] (8)
where the \( \epsilon_{(n)} \) are the one particle energies of the fluctuations. They are eigenvalues of the corresponding Schrödinger equation
\[ (-\partial^2_x + \mu^2 + V(x)) \eta_{(n)}(x) = \epsilon_{(n)}^2 \eta_{(n)}(x). \] (9)

Here, the index \( (n) \) denotes the spectrum of the operator in the lhs of Eq. (9).

In fact, Eq. (8) defines \( E_0 \) only symbolically. One has to subtract the Minkowski space contribution and to perform the ultraviolet renormalization. These procedures are by now well known. We follow here the treatment in [4]. For a discussion of the relations to different renormalization schemes we refer to [7] where, for instance, the equivalence of the subtraction scheme based on the heat kernel expansion and the mass renormalization with the ‘no tadpole condition’ had been shown.

In terms of the scattering data the renormalized ground state energy can be written in the form [8]
\[ E_0 = \frac{-1}{4\pi^2} \int_0^\infty dq \, \frac{d\,q}{\sqrt{\mu^2 + q^2}} \log \frac{q + \sqrt{\mu^2 + q^2}}{\sqrt{\mu^2 + q^2} - q} \log \frac{1}{1 - r(q)^2} \]
\[ -\frac{1}{\pi} \sum_{i=1}^N \left( \kappa_i - \sqrt{\mu^2 - \kappa_i^2} \right) \arcsin \left( \frac{\kappa_i}{\mu} \right). \] (10)

Here, the \( \kappa_i \) are the binding energies of the bound states in the potential \( V(x) \),
\[ (-\partial^2_x + V(x)) \eta_i(x) = -\kappa_i^2 \eta_i(x), \] (11)
where the \( \eta_i(x) \) are the corresponding eigenfunctions. These are bound state wave functions and they are normalizable, \( \int_\infty^\infty dx \, \eta_i^2(x) < \infty \). The function \( r(k) \) is the
reflection coefficient and both, $\kappa_i$ and $r(k)$ belong to the scattering data. It should be underlined that in $E_0$, Eq. (10), the ultraviolet divergences are subtracted. This resulted in this quite simple form because the heat kernel coefficients could be expressed in terms of the scattering data. A nice consequence which can be read off from this formula is that the ground state energy is always negative.

As mentioned in the introduction, the problem of calculating quantum corrections can be inverted. One starts from the scattering data and by means of Eq. (10) the quantum corrections can be obtained by simple integration. The price one has to pay is a more complicated procedure to obtain the classical energy. One has to solve the inverse scattering problem, i.e., one has to reconstruct the potential $V(x)$ from the scattering data. This problem had been intensively studied in connection with the solution of nonlinear evolution equations in the 70ies. The last step in this procedure is then to restore the potential $U(\Phi)$ from $V(x)$ using Eq. (7) and finally to calculate the classical energy from Eq. (4).

In following this general procedure we make use of Eq. (7) and the Bogomolny equation (2). Differentiating Eq. (2) twice with respect to $x$ we obtain

$$\Phi'''(x) = \left((U'(x))^2 + U(x)U''(x)\right)\Phi'(x).$$

(12)

By means of Eq. (8) and with the notation $\eta(x) := \Phi'(x)$ we rewrite this equation in the form

$$\left(-\partial_x^2 + V(x)\right)\eta(x) = -\mu^2 \eta(x).$$

(13)

This equation shows that the derivative of the kink is a bound state solution of the scattering problem associated with the potential $V(x)$ and that the mass $\mu$ of the fluctuating field $\eta(x, t)$ in Eq. (3) is the corresponding binding energy, i.e., one of the $\kappa_i$’s in the scattering data. Note that $\eta(x)$ in Eq. (13) cannot be a scattering solution because in that case $\mu^2$ would be negative. The decrease of $\eta(x)$ for $x \to \pm \infty$ is by means of

$$\int_{-\infty}^{\infty} dx \eta(x) = \int_{-\infty}^{\infty} dx \frac{d}{dx} \Phi_k(x) = \Phi_k(\infty) - \Phi_k(-\infty) = 2\Phi_{\text{vac}}$$

(14)

c connected with a finite vacuum solution.

In this way, if we know $\eta(x)$, the field $\Phi(x)$ is given by

$$\Phi_k(x) = -\Phi_{\text{vac}} + \int_{-\infty}^{x} d\xi \eta(\xi)$$

(15)

and we restored $\Phi_k(x)$ from $\eta(x)$. The potential $U(\Phi)$ can be restored simply as

$$U(\Phi_k(x)) = \eta(x).$$

(16)

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1This is related to the fact that here the heat kernel coefficients are just the conservation laws of the Korteweg-de-Vries equation.
Note that the potential \( U(\Phi) \) can be restored only from the ground state wave function of the scattering potential \( V(x) \) because it is only this function which does not have zeros. In case \( \eta(x) \) vanishes for some finite \( x \), the function \( U(\Phi_k(x)) \) would do so in contradiction to our assumption that two neighbored zeros correspond to \( x \to \pm \infty \).

In this way, by means of equations (15) and (16) we obtained a parametric representation of the potential \( U(\Phi) \) in terms of the ground state wave function \( \eta(x) \). We note that this representation covers the region with \( \Phi \in [-\Phi_{\text{vac}}, \Phi_{\text{vac}}] \).

How to go beyond we consider in the following sections.

There is a freedom in the parametric representation, Eqs. (15), (16). The ground state wave function, \( \eta(x) \), which we obtain as a solution of the inverse scattering problem is determined up to a multiplicative factor, which has the meaning of the normalization of \( \eta(x) \) only. So we are free to multiply the function \( \eta(x) \) by a constant, \( \eta(x) \to \alpha \eta(x) \). After that we can assume \( \eta(x) \) to be normalized, \( \int_{-\infty}^{\infty} dx \eta(x) = 1 \). In doing so we express \( \alpha \) from Eq. (14) as

\[
\alpha = 2\Phi_{\text{vac}}.
\]

In this way the freedom in the normalization of \( \eta(x) \) is expressed in terms of \( \Phi_{\text{vac}} \). After this rescaling we rewrite Eqs. (15) and (16) in the final form

\[
\Phi_k(x) = -\Phi_{\text{vac}} + 2\Phi_{\text{vac}} \int_{-\infty}^{x} d\xi \eta(\xi) \quad (17)
\]

and

\[
U(\Phi_k(x)) = 2\Phi_{\text{vac}} \eta(x). \quad (18)
\]

Using the last line we obtain from Eq. (2) the classical energy which is the quantity we are interested in,

\[
E_{\text{class}} = 4\Phi_{\text{vac}}^2 \int_{-\infty}^{\infty} dx \eta^2(x) \quad (19)
\]

By the pair of equations, Eq. (14) and (19), we obtained the final expressions relating the complete energy

\[
E = E_{\text{class}} + E_0 \quad (20)
\]

to the scattering data.

However, it should be noticed that this is merely a formal solution. We restored \( U(\Phi) \) for a restricted range of \( \Phi \) only. We have to construct a continuation to all values of \( \Phi \) which must deliver a single valued function \( U(\Phi) \) having the necessary extrema in order to allow for spontaneous symmetry breaking. The investigation of this property is the main difficulty in the restoration problem.

We conclude this section with a discussion of the free parameters. First of all there are the scattering data which constitute a set of independent parameters.
Second, we have the vacuum solution, $\Phi_{\text{vac}}$, which is in fact the condensate of the field $\Phi$. As seen from the above formulas there is no more freedom in the restoration process. Together with the uniqueness of the restoration of $\eta(x)$ from the scattering data the above mentioned parameters are the only independent ones. As for the dimensions we note that $\Phi_{\text{vac}}$ is dimensionless (we work in (1+1) dimensions) and that the bound state levels $\kappa_i$ have the dimension of a mass. For reflectionless scattering data these are the only dimensional parameters and a rescaling $\kappa_i \to \lambda \kappa_i$ results in $E \to \lambda E$. In the remaining paper of the paper we put the mass scale equal to one.

3 Reconstruction from Soliton Potentials

In this section we consider the case of reflectionless scattering data ($r(k) = 0$) given by $N$ bound states with energy levels

$$\kappa_i = i \quad (i = 1, 2, \ldots, N). \quad (21)$$

Here the ground state is that with number $i = N$. The potential $V(x)$ belonging to these scattering date is well known,

$$V(x) = \frac{-N(N+1)}{\cosh^2 x}. \quad (22)$$

The solutions $\eta(x)$ of Eq. (11) are well known too. The ground state wave function reads

$$\eta(x) = \frac{1/\gamma_N}{\cosh^N x} \quad (23)$$

and the corresponding eigenvalue is $\kappa_N = N$. The normalization factor $\gamma_N$ is defined from $\int_{-\infty}^{\infty} dx \, \eta(x) = 1$ and will be calculated later in Eq. (29). We call these $V(x)$ soliton potentials because they are related to the soliton solutions of the Korteweg-de-Vries equation.

Now, in order to solve the restoration problem we first consider even $N$. Here it is useful to change the variable in Eq. (17) according to

$$x = \tanh t. \quad (24)$$

We introduce the notation $\Phi(t) = \Phi(x(t))$. After that the integral over $\xi$ in Eq. (17) can be calculated easily and we arrive at

$$\Phi(t) = -\Phi_{\text{vac}} + \frac{2\Phi_{\text{vac}}}{\gamma_N} \int_{-1}^{t} \frac{d\tau}{1 - \tau^2} (1 - \tau^2)^{N/2},$$

$$= -\Phi_{\text{vac}} + \frac{2\Phi_{\text{vac}}}{\gamma_N} \sum_{i=1}^{N/2-1} \left( \frac{N}{2} - 1 \right) \frac{(-1)^i}{2i + 1} \left( t^{2i+1} + 1 \right), \quad (25)$$

$$U(\Phi(t)) = \frac{2\Phi_{\text{vac}}}{\gamma_N} (1 - t^2)^{N/2}. \quad (26)$$
Now we observe that for \( t \in [-1, 1] \), or equivalently, for \( x \in (-\infty, \infty) \) we restored just the kink solution, \( \Phi_k(t) \) and the potential \( U(\Phi_k(t)) \) in a parametric representation. In this way we know \( U(\Phi) \) for \( \Phi \in [-\Phi_{\text{vac}}, \Phi_{\text{vac}}] \). However, the parametrization (24) together with the explicit formulas (25) and (26) allow us to go beyond the region \( t \in [-1, 1] \). Simply, we have to consider Eqs. (25) and (26) for \( |t| > 1 \). For that \( t \), the variable \( x \) becomes complex but \( \Phi(t) \) and \( U(\Phi(t)) \) remain real. We have to ensure that \( t \in (-\infty, \infty) \) covers the whole range \( \Phi \in (-\infty, \infty) \) and that the resulting \( U(\Phi) \) is a single valued function. For this end we consider the derivative

\[
\frac{d\Phi(t)}{dt} = \frac{2\Phi_{\text{vac}}}{\gamma_N} (1 - t^2)^{N-1}.
\]

It may change its sign in \( t = \pm 1 \). If it changes it sign the function \( \Phi(t) \) is not monotonous and, as a consequence, \( U(\Phi) \) is not single valued. If, in contrary, there is no change in the sign, \( \Phi(t) \) is monoton. Finally from the remark that \( \Phi(t) \) is a polynomial in \( t \) the coverage of the whole region for \( \Phi \) follows. This is the case for \( N = 2(2s + 1), (s = 1, 2, \ldots) \). From Eq. (26) it is seen that \( U(\Phi) \) is in that case a function with two minima like in the \( \Phi^4 \)-model. For large \( \Phi \), the asymptotic behavior is

\[
U(\Phi) \overset{\Phi \to \infty}{\sim} \Phi^\frac{N}{2}.
\]

Some examples for \( U(\Phi) \) are shown in Fig. 1.

![Figure 1: The squared potential \( U^2(\Phi) \) reconstructed from a soliton potential with even number of bound states, \( N = 2, 6, 10, 14 \), and \( \Phi_{\text{vac}} = 1 \).](image)

For \( N = 2 \) we reobtain the \( \Phi^4 \)-model. Here the explicit formulas read

\[
\Phi(t) = \Phi_{\text{vac}} t,
\]
\[ U(\Phi(t)) = \Phi_{\text{vac}} (1 - t^2), \]

which can be trivially resolved,

\[ U(\Phi) = \Phi_{\text{vac}} \left( 1 - \left( \frac{\Phi}{\Phi_{\text{vac}}} \right)^2 \right). \]

The next example is \( N = 6 \). Here the parametric representation reads

\[ \Phi(t) = \frac{1}{8} \Phi_{\text{vac}} t \left( 15 - 10t^2 + 3t^4 \right), \]
\[ U(\Phi(t)) = \frac{15}{8} \Phi_{\text{vac}} \left( 1 - t^2 \right)^3, \]

which for \( t \in (-\infty, \infty) \) defines the complete dependence \( U(\Phi) \). However, as can be seen, there is no explicit expression for \( U(\Phi) \). Only the inverse function can be given explicitly,

\[ \Phi(U) = \Phi(t) \bigg|_{t = \sqrt{1 - (8U/15\Phi_{\text{vac}})^3/4}}, \]

where the branches have to be chosen accordingly (the parametric representation is much simpler).

In this example we see explicitly how the continuation beyond the initial region works. The reason that it works at all is that we assumed the potential \( U(\Phi) \) to be a function of \( \Phi \) and not a more general object like, for instance, a functional.

Now we turn to odd \( N \). Here it is useful to change the variable \( x \) for \( \theta \) according to \( \frac{1}{\cosh x} = \cos \theta \). (27)

We obtain again an explicit parametric representation,

\[ \Phi(\theta) = \frac{\Phi_{\text{vac}}}{\gamma_N} \left( N - 1 \right) \left( \frac{N-1}{N-2} \right) \frac{\theta}{2^{N-1}} + \frac{2\Phi_{\text{vac}}}{\gamma_N} (N-3)/2 \sum_{k=0}^{\infty} \frac{1}{2^{2k-1}} \binom{N-1}{k} \frac{\sin(N-1-2k)\theta}{N-1-2k} \]
\[ U(\Phi(\theta)) = \frac{\Phi_{\text{vac}}}{\gamma_N} \cos^N \theta. \] (28)

The region \( x \in (-\infty, \infty) \) corresponds to \( \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) and (28) gives for that \( \theta \) the kink solution \( \Phi_k(\theta) = \Phi_k(x(\theta)) \). Again, we obtain from this explicit parametric representation all \( \Phi \) by going beyond this region to \( |\theta| > \frac{\pi}{2} \). From Eqs. (28) and (28) it is obvious that \( U(\Phi) \) defined in this way is a single valued function. It has neighbored zeros located in \( \Phi = \pm \Phi_{\text{vac}} \). It is a periodic function with period
So we see that for each odd $N$ the restoration delivers a periodic potential $U(\Phi)$. For $N = 1$ we note

$$\Phi(\theta) = \frac{2\Phi_{\text{vac}}}{\pi} \theta,$$

$$U(\Phi(\theta)) = \frac{2\Phi_{\text{vac}}}{\pi} \cos \theta,$$

which can be resolved to $U(\Phi) = \frac{2\Phi_{\text{vac}}}{\pi} \cos \left(\frac{\pi \Phi}{2\Phi_{\text{vac}}}\right)$ which is the sine-Gordon-model. For $N = 3$ we obtain

$$\Phi(\theta) = \frac{\Phi_{\text{vac}}}{\pi} (2\theta + \sin(2\theta)),$$

$$U(\Phi(\theta)) = \frac{4\Phi_{\text{vac}}}{\pi} \cos^3 \theta.$$

Again, there is an explicit expression for $\Phi(U)$ but no for $U(\Phi)$. Examples for some first odd $N$ are given in Fig. 2.

![Figure 2: The squared potential $U^2(\Phi)$ reconstructed from a soliton potential with odd number of bound states, $N = 1, 3, 5, 7$, and $\Phi_{\text{vac}} = 1$.](image)

It remains to calculate the corresponding energies. The normalization factor $\gamma_N$ in Eq. (23) can be calculated explicitly,

$$\gamma_N = \int_{-\infty}^{\infty} dx \frac{1}{\cosh^N x} = \frac{\sqrt{\pi} \Gamma \left(\frac{N}{2}\right)}{\Gamma \left(\frac{(N+1)}{2}\right)}.$$  

The asymptotics for large $N$ is $\gamma_n \sim \sqrt{\pi/(2N)}$. Further we note

$$\int_{-\infty}^{\infty} dx \, \eta^2(x) = \gamma_{2N}.$$
In this way we obtain
\[ E_{\text{class}} = 4 \Phi_{\text{vac}}^2 \frac{\gamma^{2N}}{(\gamma N)^2} \] (30)

and
\[ E_0 = -\frac{1}{\pi} \sum_{i=1}^{N} \left( i - \sqrt{N^2 - i^2} \arcsin \frac{i}{N} \right). \] (31)

As mentioned in [8], the renormalized vacuum energy is always negative in (1+1) dimensions which can be checked for Eq. (31) easily. The classical energy is of course positive so that these two contributions to the complete energy compete. For any finite \( N \) it depends on \( \Phi_{\text{vac}} \) which prevails. For large \( \Phi_{\text{vac}} \) which correspond to a weak coupling we have positive complete energy whereas for large \( N \) the quantum energy grows faster than the classical one. This is shown in Fig. 3.

![Figure 3: The complete energy for soliton potentials with \( N \) bound states, the value of the condensate is \( \Phi_{\text{vac}} = 1.5 \).](image)

4 Reconstruction from two bound states

In this section we consider reflectionless scattering data consisting of two bound states,
\[ \kappa_1 = N_1, \]
\[ \kappa_2 = N_2, \quad (\text{ground state}) \]
assuming \( N_2 > N_1 \). The ground state wave function reads

\[
\eta(x) = \frac{2 \cosh(N_1 x)}{(N_2 - N_1) \cosh((N_2 + N_1)x) + (N_2 + N_1) \cosh((N_2 - N_1)x)}
\]  

(32)

(up to the normalization factor). By means of Eqs. (17) and (18) we restore \( U(\Phi_k(x)) \) and \( \Phi_k(x) \). In this way we obtain information on \( U(\Phi) \) for \( \Phi \in [-\Phi_{\text{vac}}, \Phi_{\text{vac}}] \). To go beyond this region we used in the preceding section some specific parametrization. In fact we made an analytic continuation to complex \( x \). Indeed, for \(|t| > 1\) we note for the first parametrization, Eq. (24),

\[
x = \frac{1}{2} \ln \frac{1 + 1/t}{1 - 1/t} \pm \frac{i\pi}{2}
\]  

(33)

and for the second one, Eq. (27), for \( \theta \in \left[\pi, \frac{3\pi}{2}\right] \) (where \( \cos \theta < 0 \))

\[
x = \ln \left( -\frac{1}{\cos \theta} - \sqrt{\frac{1}{\cos^2 \theta} - 1} \right) \pm i\pi.
\]  

(34)

Here the signs of the imaginary parts depend on which side we bypass the corresponding branch point. Aimed by these examples we consider \( \eta(x + iy) \) (with real \( x \) and \( y \)). Now we have to ensure that both, \( U \) and \( \Phi \) are real. Because \( \Phi \) contains an additional integration as compared to \( U \) we need \( \eta(x + iy) \) to be real for all \( x \). Hence, only shifts in parallel to the real axis are allowed. From the structure of \( \eta \), Eq. (32), it is clear that this may happen only if \( N_1 \) and \( N_2 \) are integer numbers and if we take the shift in multiples of \( i\frac{\pi}{2} \). In general, rational number are possible too. But the denominators can be removed by a rescaling of \( x \), i.e. they can be absorbed into the mass scale. In this way we see that the two parametrization introduced in the preceding section provide just the required continuation.

As already mentioned we have to ensure that the parametrizations provide monotone functions \( \Phi(t) \) resp. \( \Phi(\theta) \) which cover the whole range \( \Phi \in (-\infty, \infty) \).
First we check the monotony. For that task we consider the derivative of \( \Phi \) with respect to the parameter. In the first parametrization we note \( dx/dt = 1/(1-t^2) \) and obtain

\[
\frac{d\Phi(t)}{dt} = \frac{\eta(x(t))}{1-t^2}
\]  

(35)

which must have a definite sign. A change in the sign may occur only in passing through \( t = 1 \), i.e., when going through \( x \to \infty \). Using

\[
U(x) \xrightarrow{x \to \infty} e^{-N_2 x}
\]  

(36)

and

\[
x(t) \xrightarrow{t \to 1} -\frac{1}{2} \ln(1-t)
\]
we obtain
\[
\frac{d\Phi(t)}{dt} \sim (1 - t)^{N_2 - 1}.
\] (37)
This derivative is nonnegative for \( t > 1 \) too only if \( N_2 = 2(2s + 1) \) \((s = 0, 1, 2, \ldots)\).

In the second parametrization we have to investigate the behavior in \( \theta = \frac{\pi}{2} \).
By means of \( dx/d\theta = 1/\cos \theta \) and
\[
x(\theta) \sim \theta - \ln \left( \frac{\pi}{2} - \theta \right)
\]
we obtain
\[
\frac{d\Phi(\theta)}{d\theta} = \frac{U(x(\theta))}{\cos \theta} \sim \left( \frac{\pi}{2} - \theta \right)^{N_2 - 1}
\] (38)
which is positive for \( \theta > \frac{\pi}{2} \) for odd \( N_2 \), \( N_2 = 2s + 1 \) \((s = 0, 1, 2, \ldots)\).

In this way we arrived with the result that for each second even \( N_2 \) by the first parametrization and for each odd \( N_2 \) by the second parametrization a monotone function \( \Phi(t) \) resp. \( \Phi(\theta) \) appears. It remains to check that the whole region \( \Phi \in (-\infty, \infty) \) is covered. For the second parametrization this is indeed the case simply by periodicity. However for the first one this turns out not to be the case for all even \( N_2 \). To check this we note that for \( t \to \infty \) the real part of \( x \) returns to zero as follows from Eq. (39). In \( \eta(x) \), Eq. (32), after \( x \to x + iy \), the cosh’s in the denominator turn into \( \pm \) sinh’s of the corresponding arguments. As a consequence, for \( x \to 0 \) there may be a cancellation of the contributions linear in \( x \). It is just this cancellation which lets \( U(x) \) grow up. It can be checked that this cancellation happens just for \( N_2 = 2(2s + 1) \), i.e., for that we selected from the sign of the derivative, and not for the other even \( N_2 \). There is no restriction on \( N_1 \). As a result we obtain that the potential \( U(\Phi) \) is again similar to that in the \( \Phi^4 \)-model, its asymptotic behavior is \( U(\Phi) \sim \Phi^2 \).

The classical energy can be calculated using Eq. (13). However there is no such simple explicit formula as in section 3. Results are shown in Fig. 4. As seen it depends on the value of the condensate which contribution prevails. For \( N_1 \) close to \( N_2 \), for any fixed value of the condensate, the energy becomes negative for sufficiently large \( N_2 \).

5 Reconstruction from a general reflectionless potential

In this section we consider a general reflectionless potential. Is is given by \( M \) bound states with energies \( \kappa_i = N_i \) \((i = 1, 2, \ldots, M)\). We assume \( N_1 < N_2 < \ldots < N_M \). The wave function of the ground state (its energy is \( N_M \)) can be obtained from the inverse scattering method or by Darboux-transformation. It is a quotient
\[
\eta(x) = \frac{P}{Q},
\]
Table 1: Allowed (1) and forbidden (0) combinations of the bound state levels for four bound states. This is independent on the ground state level, $N_4$.

| $N_2$ | $N_1$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 2  | 1  | 0 0 0 0 |
| 3  | 0 0 1 0 |
| 4  | 1 0 1 0 |
| (even $N_3$) | (odd $N_3$) |

where $P$ is a monomial in $\cosh((N_1 \pm N_2 \pm \ldots \pm N_{M-1})x)$ and $Q$ is a monomial in $\cosh((N_1 \pm N_2 \pm \ldots \pm N_M)x)$. $Q$ contains the ground state energy $\kappa_M = N_M$ and $P$ doesn’t. Following the discussion in the preceding section we conclude that all $N_i$ must be integer. For the behaviour at $x \to \infty$ from the largest in module eigenvalue

$$\eta(x) \sim e^{-N_Mx}$$

follows. Again, we conclude that for $N_M = 2(2s + 1)$ ($s = 0, 1, 2, \ldots$) using the first parametrization, Eq. (24), we obtain a monotone function $\Phi(t)$ and that for odd $N_M$ the second parametrization does the job. Whereas the second parametrization covers the whole region of $\Phi$ by periodicity, the first does this only for certain sets of numbers $N_1, N_2, \ldots, N_{M-1}$. Here it seems too hard or even impossible to give a general rule other than in special cases. So, for example, for three bound states ($M = 3$) and a ground state energy $N_3 = 2(2s + 1)$, the energy of the second level, $N_2$, must be an odd number and that of the first level, $N_1$, an even number. This is a conjecture from considering $N_3$ explicitely up to 20. For four bound states ($M = 4$) some allowed combinations are shown in Table 1.

The general behavior of $U(\Phi)$ is the same as seen before. For the ground state energy being an even number a potential like in the $\Phi^4$-model appears and for an odd number it is periodic. It seems that for reflectionless scattering data no other behavior of $U(\Phi)$ is possible.

6 Conclusions

We formulated the reconstruction problem on how to get the potential $U(\Phi)$ allowing for spontaneous symmetry breaking in the action, Eq. (1), for a scalar field in (1+1) dimensions from the scattering data related to the quantum fluctuation in the background of the corresponding kink solution. We considered reflectionless scattering data and solved the reconstruction problem explicitely for some classes, for soliton potentials and for two bound states. We gave a conjecture for the general reflectionless case. It states that $U(\Phi)$ reconstructed from reflectionless scattering data can be only like a $\Phi^4$-potential, i.e., with two minima, or
periodic like in the sine-Gordon model.

It would be interesting to give a proof of this conjecture. Furthermore, it would be interesting to consider scattering data including reflections, for example with a rational reflection coefficient and to see how other than the two mentioned types appear.

We wrote down the formulas for the classical and the quantum energies in terms of the scattering data resp. the ground state wave function. In the considered examples it is seen that in dependence on the free parameters, the complete energy may take both signs. In general, by an increase of the bound state energies the quantum energy (it is negative) grows faster than the classical one and the complete energy becomes increasingly negative.

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References

[1] R.F. Dashen, B. Hasslacher, and A. Neveu. Nonperturbative methods and extended-hadron models in field theory. I. Semiclassical functional methods. Physical Review D, 10:4114–29, 1974.

[2] R. Rajaraman. Solitons and Instantons. North-Holland Publishing Company, 1982. Amsterdam.

[3] A. Rebhan and P. van Nieuwenhuizen, No saturation of the quantum Bogomolnyi bound by two-dimensional N = 1 supersymmetric solitons, Nucl. Phys. B 508 (1997) 449; H. Nastase, M. A. Stephanov, P. van Nieuwenhuizen and A. Rebhan, Topological boundary conditions, the BPS bound, and elimination of ambiguities in the quantum mass of solitons, Nucl. Phys. B 542 (1999) 471.M. A. Shifman, A. I. Vainshtein and M. B. Voloshin, Anomaly and quantum corrections to solitons in two-dimensional theories with minimal supersymmetry, Phys. Rev. D 59 (1999) 045016.N. Graham and R. L. Jaffe, Energy, central charge, and the BPS bound for 1+1 dimensional supersymmetric solitons, Nucl. Phys. B 544 (1999) 432.

[4] M. Bordag. Vacuum energy in smooth background fields. J. Phys., A28:755–766, 1995.
[5] L. D. Faddeev. The inverse problem in the quantum theory of scattering. *J. Math. Phys.*, 4:72–104, 1963.

[6] K. Chadan and P.C. Sabatier. *Inverse Problems in Quantum Scattering Theory*. Springer-Verlag, 2nd. edition, 1989.

[7] Michael Bordag, Alfred Scharff Goldhaber, Peter van Nieuwenhuizen, and Dmitri Vassilevich. Heat kernels and zeta-function regularization for the mass of the susy kink. hep-th/0203066, 2002.

[8] M. Bordag. Vacuum energy density in arbitrary background fields. In A.M. Semikhatov I.M. Dremin, editor, *Proceedings of the Second Intern. Zakharov Conference, May 15th - 25th, 1996 in Moscow, Russia*, pages 378–381. World Scientific, 1997.
Figure 4: The complete energy for potentials restored from two bound states, the value of the condensate is (a), $\Phi_{\text{vac}} = 0.5$ and (b), $\Phi_{\text{vac}} = 0.45$. 