Classification of pairs of rotations in
finite-dimensional Euclidean space

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Dedicated to Fred Van Oystaeyen, on the occasion of his sixtieth birthday

Abstract
A rotation in a Euclidean space $V$ is an orthogonal map $\delta \in O(V)$ which acts locally as a plane rotation with some fixed angle $a(\delta) \in [0, \pi]$. We give a classification of all finite-dimensional representations of the real algebra $\mathbb{R}\langle X, Y \rangle$ which are given by rotations, up to orthogonal isomorphism.

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1 Definitions and results

Let $V = (V, \langle \rangle)$ be a Euclidean space. A rotation in $V$ is an orthogonal endomorphism $\delta \in O(V)$ such that

1. $\delta^2(v) \in \text{span}\{v, \delta(v)\}$ and
2. $\langle v, \delta(v) \rangle = \langle w, \delta(w) \rangle$

for all unit vectors $v, w \in V$. By the angle of $\delta$ then is meant the number $a(\delta) = \arccos(\langle v, \delta(v) \rangle) \in [0, \pi]$. If $a(\delta) \in \{0, \pi\}$, then $\delta = \pm I_V$, where $I_V$ denotes the identity map on $V$. If $a(\delta) \in ]0, \pi[$, we call $\delta$ a proper rotation.

Given any orthogonal operator $\sigma$ on a finite-dimensional Euclidean space $V$, there exists an orthonormal basis $\mathfrak{e}$ of $V$ such that the matrix of $\sigma$ with respect to $\mathfrak{e}$ has the form

$$[\sigma]_{\mathfrak{e}} = R_{\alpha_1} \oplus \cdots \oplus R_{\alpha_k} \oplus I_l \oplus -I_m \quad \text{with} \quad \alpha_i \in ]0, \pi[ \quad \text{and} \quad k, l, m \in \mathbb{N}$$

(1)

1. The first axiom is redundant if $V$ is finite-dimensional. An orthogonal map satisfying only 1 is either a rotation or a reflection in a proper, non-trivial subspace of $V$. 

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where
\[ R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \in O_2(\mathbb{R}), \quad A \oplus B = \begin{pmatrix} A \\ B \end{pmatrix} \]
and \( I_n \) is the identity matrix of size \( n \times n \). The presentation in \( \{1\} \) is unique up to permutations of the summands \( R_\alpha \). This structure theorem for orthogonal operators \( [6] \) plays an important role in the present article.

In view of the above description, the map \( \sigma \) is a proper rotation if and only if there exists an orthonormal basis \( \{x\} \) of \( V \) such that \( [\sigma]_x = R_\alpha \oplus \cdots \oplus R_\alpha \) for some \( \alpha \in [0, \pi] \).

The structure theorem provides complete information about the behaviour of a single orthogonal operator on \( V \). Our main result, formulated in the three propositions below, gives a corresponding picture for pairs \((\delta, \epsilon)\) of rotations in a finite-dimensional Euclidean space.

**Proposition 1.1** Let \( \delta \) and \( \epsilon \) be rotations in a finite-dimensional Euclidean space \( V \). The space \( V \) decomposes into an orthogonal direct sum of subspaces, each of which is invariant under \( \delta \) and \( \epsilon \), and has dimension either 1, 2 or 4.

Every pair \((\sigma, \tau)\) of linear endomorphisms of a real vector space \( V \) gives rise to a representation in \( V \) of the free associative algebra \( \mathbb{R}\langle X, Y \rangle \) with generators \( X, Y \), via the algebra morphism \( \mathbb{R}\langle X, Y \rangle \rightarrow \text{End}(V) \) determined by \( X \mapsto \sigma \), \( Y \mapsto \tau \). We denote this representation again by \((\sigma, \tau)\). If \( V \) is Euclidean and \( \delta, \epsilon \) are rotations in \( V \), we call \((\delta, \epsilon)\) a *rotational representation*. We denote by \( \mathfrak{R} \) the category of all finite-dimensional rotational representations of \( \mathbb{R}\langle X, Y \rangle \). Given rotations \( \delta, \epsilon \in O(V) \) and \( \sigma, \tau \in O(W) \), a morphism \( (\delta, \epsilon) \rightarrow (\sigma, \tau) \) in \( \mathfrak{R} \) is a linear map \( \varphi : V \rightarrow W \) which decomposes as \( \varphi = \psi \oplus 0 : U \oplus \ker \varphi \rightarrow W \) where \( U = (\ker \varphi)^\perp \subset V \) is the orthogonal complement of \( \ker \varphi \), and the map \( \psi : U \rightarrow W \) is orthogonal. Thus \( \mathfrak{R} \) is a non-full subcategory of \( \text{rep}_f(\mathbb{R}\langle X, Y \rangle) \), the category of finite-dimensional representations of \( \mathbb{R}\langle X, Y \rangle \).

We remark that \( \mathfrak{R} \) is closed under direct summands, but not under direct sums: Any subrepresentation of a rotational representation is again rotational. However, the direct sum of two rotations with different angles is not a rotation, and therefore, the sum of two objects in \( \mathfrak{R} \) is in general not a rotational representation.

If \( \sigma \in O(V) \) and \( U \subset V \) is an invariant subspace for \( \sigma \), then the orthogonal complement \( U^\perp \subset V \) of \( U \) is also invariant under \( \sigma \). This means that if \( \delta, \epsilon \) are rotations in \( V \) and \( U \subset V \) is invariant under \( \delta \) and \( \epsilon \), then the representation \((\delta, \epsilon)\) decomposes as \((\delta, \epsilon) = (\delta|_U, \epsilon|_U) \oplus (\delta|_{U^\perp}, \epsilon|_{U^\perp})\). Thus a rotational representation of \( \mathbb{R}\langle X, Y \rangle \) is indecomposable if and only if it is irreducible, and every finite-dimensional rotational representation can be decomposed into a direct sum of irreducible representations. Moreover, the next proposition asserts that this decomposition is essentially unique.

**Proposition 1.2** The category \( \mathfrak{R} \) has the Krull-Schmidt property: If \((\delta, \epsilon) \in \mathfrak{R} \), and \((\delta', \epsilon') = (\sigma_1, \tau_1) \oplus \cdots \oplus (\sigma_k, \tau_k)\) and \((\delta, \epsilon) = (\sigma'_1, \tau'_1) \oplus \cdots \oplus (\sigma'_l, \tau'_l)\) are decompositions of \((\delta, \epsilon)\) into irreducible subrepresentations, then \( k = l \) and there
exists a permutation $f \in S_k$ such that $(\sigma_{f(i)}, \tau_{f(i)})$ is isomorphic to $(\sigma'_i, \tau'_i)$ for all $i \leq k$.

The above statement is not trivial. Since $\mathcal{R}$ is not a full subcategory of $\text{rep}_f(\mathbb{R}\langle X, Y \rangle)$, isomorphism classes in the former are a priori smaller than in the latter. Therefore the Krull-Schmidt theorem, although certainly true in $\text{rep}_f(\mathbb{R}\langle X, Y \rangle)$, does not automatically carry over to $\mathcal{R}$.

A consequence of Propositions 1.1 and 1.2 is the following: If $V$ is a finite-dimensional Euclidean space, then every pair $(\delta, \epsilon)$ of rotations in $V$ has a has a 2-dimensional invariant subspace $U \subset V$ if and only if $\dim V \equiv 2 \mod 4$. This result has been proven independently by Dieterich [5], using determinant calculus.

We complete the investigation of the category $\mathcal{R}$ by giving a classification of its irreducible objects. In view of Proposition 1.2, this amounts to classifying $\mathcal{R}$ itself. By a classification is meant a list of pairwise non-isomorphic objects, exhausting all isomorphism classes.

Let $\mathcal{IR}$ be the full subcategory of $\mathcal{R}$ consisting of all irreducible finite-dimensional rotational representations. We consider $\mathbb{R}^n$ as a Euclidean space, equipped with the standard scalar product. Linear maps $\mathbb{R}^n \to \mathbb{R}^m$ are identified with $m \times n$-matrices in the natural way. Given $\theta \in \mathbb{R}$, we write

$$T_\theta = \begin{pmatrix} 1 & R_\theta \\ \ast & 1 \end{pmatrix} \in O_4(\mathbb{R}).$$

**Proposition 1.3** The category $\mathcal{IR}$ is classified by the following list of objects:

- $(r, s)$ where $r, s \in \{-1, 1\}$,
- $(rI_2, R_\beta), (R_\alpha, sI_2), (R_\alpha, R_\beta)$ where $r, s \in \{-1, 1\}, \alpha, \beta \in [0, \pi[$,
- $(R_\alpha \oplus R_\beta, T_\theta (R_\beta \oplus R_\beta) T_{-\theta})$ where $\alpha, \beta, \theta \in [0, \pi[$.

Our results indicate the contrast between the categories $\text{rep}_f(\mathbb{R}\langle X, Y \rangle)$ and $\mathcal{R}$. The algebra $\mathbb{R}\langle X, Y \rangle$ is wild, and the number of parameters of non-isomorphic indecomposable representations increases heavily with the dimension. By Proposition 1.1, $\mathcal{R}$ has no indecomposable objects of dimension greater than 4. The category $\text{rep}_f(\mathbb{R}\langle X, Y \rangle)$ is immense, the objects in $\mathcal{R}$ are easily visualised with geometric intuition.

The author’s interest in rotational representations originates in the theory of real (not necessarily associative) division algebras. Proposition 1.3 plays an important role in the classification of the 8-dimensional absolute valued algebras which have either a non-zero central idempotent or a one-sided identity element [2]. This is an instance of what appears to be a general pattern: The connection between various classes of real division algebras and some “geometric” subcategory of the representation category of a real associative algebra determined by the class of division algebras. As examples can be mentioned division algebras...
of dimension 2, which are related to certain representations of $\mathbb{R}(X, Y)$, and 8-dimensional flexible quadratic division algebras, which are parametrised by a non-full subcategory of $\text{rep}_T(\mathbb{R}[X])$ (c.f. [1] and [2] respectively).

In Section 2 below, we prove Proposition 1.1. In addition a criterion is obtained, formulated in Proposition 2.2, for when a pair of rotations has a 2-dimensional invariant subspace. In Section 3 we show that Proposition 1.2 can be deduced from the Krull-Schmidt theorem for $\text{rep}_T(\mathbb{R}[X, Y])$. Finally, by classifying pairs of rotations in Euclidean spaces of dimension at most 4, we attain the proof of Proposition 1.3.

Henceforth, $V = (V, \langle \rangle)$ will always denote a finite-dimensional Euclidean space.

## 2 Decomposition of $V$ into invariant subspaces

Let $\delta$ and $\epsilon$ be rotations in $V$, with respective angles $\alpha$ and $\beta$. We may assume that both $\delta$ and $\epsilon$ are proper rotations, otherwise the decomposition of $V$ into 1- or 2-dimensional invariant subspaces follows directly from the structure theorem for orthogonal operators. We say that a subspace $U \subset V$ is $(\delta, \epsilon)$-invariant if it is invariant under both $\delta$ and $\epsilon$.

Consider the complexification $V^C = \mathbb{C} \otimes_\mathbb{R} V$ of $V$ equipped with the induced inner product: $\langle l \otimes u, \mu \otimes v \rangle = l\mu(u, v)$. We identify $V$ with the real subspace $\mathbb{R} \otimes V$ of $V^C$ in the canonical way. Elements in this subspace are called real vectors, whereas elements in $iV = \mathbb{R}i \otimes V$ are said to be purely imaginary. As a real vector space, $V^C = V \oplus iV$. The complex conjugate of a vector $v = v_1 + v_2 \in V \oplus iV$ is $\bar{v} = v_1 - v_2$, and the conjugation map $\kappa : V^C \rightarrow V^C$, $v \mapsto \bar{v}$ is an antilinear operator on $V^C$. By $T_C$ we denote the $\mathbb{C}$-linear endomorphism of $V^C$ induced by a real linear endomorphism $T$ of $V$. We will make use of the fact that the conjugation map $\kappa$ induces bijections $\ker(T_C - \bar{l}I_{V^C}) \rightarrow \ker(T_C - lI_{V^C})$ for all eigenvalues $l \in \mathbb{C}$ of $T_C$.

The maps $\delta_C$ and $\epsilon_C$ give rise to decompositions $V^C = A \oplus B$ and $V^C = C \oplus D$, where

$$
A = \ker(\delta_C - e^{i\alpha}I_{V^C}) \quad C = \ker(\epsilon_C - e^{i\beta}I_{V^C})
$$

$$
B = \ker(\delta_C - e^{-i\alpha}I_{V^C}) \quad D = \ker(\epsilon_C - e^{-i\beta}I_{V^C})
$$

and the summands in each decomposition are mutually orthogonal. As noted above, $\kappa(A) = B$ and $\kappa(C) = D$.

Suppose $A \cap C$ is non-trivial, and $v \in (A \cap C) \setminus \{0\}$. Then $\bar{v} \in B \cap D$, and $B \cap D \neq 0$. Hence both $v$ and $\bar{v}$ are common eigenvectors of $\delta_C$ and $\epsilon_C$, which in particular means that $\text{span}_C\{v, \bar{v}\}$ is invariant under $\delta_C$ and $\epsilon_C$. On the other hand, $v + \bar{v}, i(v - \bar{v})$ are real vectors, and $\text{span}_C\{v + \bar{v}, i(v - \bar{v})\} = \text{span}_C\{v, \bar{v}\}$. Hence $\text{span}_R\{v + \bar{v}, i(v - \bar{v})\} = V \cap \text{span}_C\{v, \bar{v}\}$. This is a 2-dimensional (real) subspace of $V$, invariant under $\delta_C|_V = \delta$ and $\epsilon_C|_V = \epsilon$.

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2A map $T : V \rightarrow W$ between complex vector spaces is called antilinear if it is additive and $T(lv) = \bar{l}Tv$ for all $l \in \mathbb{C}$, $v \in V$. A theory for antilinear maps is developed in [1] [2] [3] [10].
Certainly, the above argument goes through also when the roles of $C$ and $D$ are interchanged. Thus we have shown, that whenever $A \cap (C \cup D) \neq 0$, $V$ has a $2$-dimensional subspace which is invariant under $\delta$ and $\epsilon$.

The crucial property of the $2$-dimensional subspace span$_C\{v, \bar{v}\} \subset V^C$ above is, that its intersection with each one of the sets $A, B, C, D, V \subset V^C$ is non-zero. This is precisely what is needed for $V$ to have a $2$-dimensional $(\delta, \epsilon)$-invariant subspace. More generally, the following holds.

**Lemma 2.1** If $U \subset V^C$ is a $\kappa$-invariant subspace such that

$$U = (A \cap U) \oplus (B \cap U) = (C \cap U) \oplus (D \cap U) \quad (4)$$

then $U \cap V \subset V$ is invariant under $\delta$ and $\epsilon$, and $\dim_{\mathbb{R}}(U \cap V) = \dim_{\mathbb{C}}U$. Conversely, if $W \subset V$ is an invariant subspace for $\delta$ and $\epsilon$, then its complexification $U = W^C$ is a $\kappa$-invariant subspace satisfying (4).

**Proof:** Suppose $U \subset V^C$ is a $\kappa$-invariant subspace satisfying (4), and let $\mathbb{b}$ be a basis of $A \cap U$. Then $\mathbb{b}' = \{ \bar{u} | u \in \mathbb{b} \}$ is a basis of $B \cap U$ and $\mathbb{b} \cup \mathbb{b}'$ a basis of $U$. Taking $\mathbf{g} = \{ u + \bar{u}, i(u - \bar{u}) | u \in \mathbb{b} \}$, we get span$_\mathbb{C}\mathbf{g} = \text{span}_C(\mathbb{b} \cup \mathbb{b}') = U$. Hence $\mathbf{g}$ is a basis of $U$. Moreover, $\mathbf{g} \subset U \cap V$. A vector $v \in \text{span}_\mathbb{C}\mathbf{g} = U$ is real if and only if its coefficients in $\mathbf{g}$ are real numbers. Thus span$_\mathbb{R}\mathbf{g} = U \cap V$ and consequently dim$_\mathbb{R}(U \cap V) = \dim_{\mathbb{C}}U$.

The real space $V$ is invariant under $\delta|_V$ and $\epsilon|_V$, and from (4) it follows that so is $U$. Therefore, the intersection $U \cap V$ is invariant under $\delta = \delta|_V$ and $\epsilon = \epsilon|_V$.

The converse is immediate. \[\square\]

In the case when $A \cap (C \cup D) = 0$, $V^C$ may or may not have a $2$-dimensional $\kappa$-invariant subspace $U$ satisfying (4). Below we construct a map $T$, which will be our tool to find subspaces of this type, of dimension either 2 or 4.

Assume $A \cap C = A \cap D = 0$. Let $P_A : V^C \to A$ and $P_B : V^C \to B$ be the orthogonal projection maps onto $A$ and $B$ respectively. Since the subspaces $\ker P_A = B$ and $\ker P_B = A$ intersect $C$ trivially, the restricted maps $P_A|_C : C \to A$ and $P_B|_C : C \to B$ are bijective. We define $T : A \to A$ as the composition of the following chain of maps:

$$A \xrightarrow{P_A|^{-1}_C} C \xrightarrow{P_B|_C} B \xrightarrow{\kappa} A \quad (5)$$

i.e., $T = \kappa \circ P_B|_C \circ P_A|^{-1}_C$. Clearly, $T$ being the composition of two linear and one antilinear map, itself is antilinear. The factors of $T$ are bijections, hence $T$ is a bijection.

**Proposition 2.2** Let $V$ be a non-trivial Euclidean space of finite dimension, and $\delta, \epsilon$ rotations in $V$ with angles $\alpha, \beta \in [0, \pi]$ respectively. Let $A, B, C, D \subset V^C$ be the subspaces defined by (3). Now there exists a $2$-dimensional subspace of $V$ which is invariant under $\delta$ and $\epsilon$ if and only if one of the following conditions is satisfied:

1. Either of the intersections $A \cap C$ and $A \cap D$ is non-trivial.
2. Both $A \cap C$ and $A \cap D$ are zero, and the antilinear operator $T : A \to A$ defined by (6) has a 1-dimensional invariant subspace.

We remark that an antilinear operator $S$ has a 1-dimensional invariant subspace if and only if $S^2$ has a non-negative real eigenvalue. Moreover, this is always the case if the domain of $S$ is odd-dimensional. This is a consequence of the normal form for antilinear operators given in [7].

Proof: We have already shown, that the first alternative in the proposition implies existence of a 2-dimensional $(\delta, \epsilon)$-invariant subspace. Consider instead the case $A \cap (C \cup D) = 0$.

Assume $T$ has a 1-dimensional invariant subspace. This means that there exists a non-zero vector $v \in A$ such that $Tv = \mu v$ for some $\mu \in \mathbb{C}$. Setting $w = P_A^{-1}v$, we have $P_A w = v$ and $P_B w = P_{B|C}(P_A^{-1}v) = \kappa Tv = \mu \bar{v}$. Hence $w = P_A w + P_B w = v + \mu \bar{v}$. Since $w \in C$, this means that $C \cap \text{span}_C \{v, \bar{v}\} \neq 0$. Now $\bar{w} = \kappa(w) = \bar{v} + \mu v \in D \cap \text{span}_C \{v, \bar{v}\}$, whence $D \cap \text{span}_C \{v, \bar{v}\} \neq 0$.

So $U = \text{span}_C \{v, \bar{v}\}$ intersects each of $A, B, C, D$ non-trivially. Hence $U = (A \cap U) \oplus (B \cap U) = (C \cap U) \oplus (D \cap U)$. Obviously $\kappa(U) = U$, and Lemma 2.1 now asserts that $U \cap V$ is a 2-dimensional subspace of $V$, invariant under $\delta$ and $\epsilon$.

Assume instead there exists a real subspace $W \subset V$ of dimension 2, which is invariant under $\delta$ and $\epsilon$. By Lemma 2.1 $\kappa(W^c) = W^c$ and $W^c = (A \cap W^c) \oplus (B \cap W^c) = (C \cap W^c) \oplus (D \cap W^c)$. This implies that $W^c$ is invariant under $P_A$, $P_B$ and $P_A|_C^{-1}$ (the last of which is defined since we have assumed $A \cap (C \cup D) = 0$). Thus $A \cap W^c$ is invariant under $T$. As $\dim(A \cap W^c) = 1$, we are done. \qed

Along the same lines, we can now prove Proposition 1.1. Instead of the antilinear map $T$, which not necessarily has a 1-dimensional invariant subspace, we consider the linear map $T^2 : A \to A$.

Since every object in $\mathfrak{R}$ can be written as a sum of irreducibles, it suffices to show that for any two rotations $\delta, \epsilon$ in $V$, there exists a $(\delta, \epsilon)$-invariant subspace $W \subset V$ of dimension at most 4. We may assume that $A \cap (C \cup D) = 0$, since otherwise $V$ has a $(\delta, \epsilon)$-invariant subspace of dimension 2. Now $T : A \to A$ is defined, and $T^2$ is a linear endomorphism of the complex vector space $A$. Thus it has an eigenvalue $l \in \mathbb{C}$. Let $u \in A$ be a corresponding, non-zero eigenvector. Set $v = Tu$ and $w = P_A|_C^{-1}u$. We get $\kappa P_B w = \kappa(P_B|_C)(P_A|_C^{-1})u = Tu = v$ and $P_A v = u$, so $w = u + \bar{v}$. Similarly, on setting $z = P_A|_C^{-1}v$ we obtain

$$\kappa P_B z = \kappa(P_B|_C)(P_A|_C^{-1})u = \kappa(P_B|_C)(P_A|_C^{-1})Tu = T^2 u = l u.$$

Thus $P_B z = l \bar{u}$ and $z = P_A z + P_B z = v + \bar{v}$.

If $u$ and $v$ are linearly dependent, then $\text{span}_C \{u\}$ is an invariant subspace for $T$ of dimension 1 and hence, by Proposition 2.2, $V$ has a 2-dimensional $(\delta, \epsilon)$-invariant subspace.

If $u, v$ are linearly independent, then so are $w, z$. Set $U = \text{span}_C \{u, v, u + \bar{v}\}$. Since now $w, z \in C \cap U$, we have $w, z \in D \cap U$. Thus $\dim_C(C \cap U) = \dim_C(D \cap U) = 2$ and $U = (C \cap U) \oplus (D \cap U)$. Moreover $U = (A \cap U) \oplus (B \cap U)$, since
and a under conjugation with \( \phi \). This implies that a map (and thus an isomorphism in \( \text{rep} \)) is given, respectively, by proper rotations and rotations with angle \( \psi \). Indeed, this extends to be true for all \( \delta \in \mathbb{R} \) and \( \epsilon \in \mathbb{R} \). The following proposition shows that any pair of irreducible rotational representations which are isomorphic in \( \text{rep} \) are also isomorphic in \( \mathfrak{R} \). In view of the above, this establishes the proof of Proposition 3.1.

**Proposition 3.1** Let \( (\delta, \epsilon), (\sigma, \tau) \in \mathfrak{R} \). If \( \varphi : (\delta, \epsilon) \rightarrow (\sigma, \tau) \) is an isomorphism in \( \text{rep}_f(\mathbb{R}(\langle X, Y \rangle)) \), then there exists a number \( l \in \mathbb{R} \) such that \( l \varphi \) is an orthogonal map (and thus an isomorphism in \( \mathfrak{R} \)).

**Proof:** If \( \varphi : (\delta, \epsilon) \rightarrow (\sigma, \tau) \) is such an isomorphism, then \( \sigma = \varphi \delta \varphi^{-1} \) and \( \tau = \varphi \epsilon \varphi^{-1} \). The complexification \( \delta_C \) of \( \delta \) has eigenvalues \( e^{i\alpha(\delta)} \) and \( e^{-i\alpha(\delta)} \). This implies that \( a(\delta) \in [0, \pi] \) is determined by the spectrum of \( \delta_C \), which is invariant under conjugation with \( \varphi \). The same certainly holds for \( a(\epsilon) \). So \( a(\sigma) = a(\delta) \) and \( a(\tau) = a(\epsilon) \).

Now, as \( \rho_\delta = \frac{1}{\sin \alpha(\delta)}(\delta - \cos \alpha(\delta)) \), we have \( \rho_\sigma \varphi = \varphi \rho_\delta \). If \( v \) is any vector in the space \( V \) carrying \( (\delta, \epsilon) \), then \( \langle \varphi(v), \varphi(\rho_\delta(v)) \rangle = \langle \varphi(v), \rho_\sigma(\varphi(v)) \rangle = 0 \). This means that \( \varphi \) restricted to span\{\( v, \rho_\delta(v) \)\} preserves orthogonality, or equivalently, that there exists a number \( \mu \in \mathbb{R} \setminus \{0\} \) such that \( \|\varphi(u)\| = \mu \|u\| \) for all \( u \in \text{span}\{v, \rho_\delta(v)\} \).

A similar argument shows that \( \|\varphi(u)\| = \mu \|u\| \) whenever \( u \in \text{span}\{v, \rho_\sigma(v)\} \). Indeed, this extends to be true for all \( u \in \mathbb{R}(\delta, \epsilon)v \), the subrepresentation of \( (\delta, \epsilon) \) generated by \( v \). Since \( (\delta, \epsilon) \) is irreducible, \( \mathbb{R}(\delta, \epsilon)v = V \) if \( v \neq 0 \). Consequently, \( \|\varphi(u)\| = \mu \|u\| \) for all \( u \in V \), and thus the map \( \frac{1}{\mu} \varphi \) is orthogonal.

We proceed to prove Proposition 1.3.
Lemma 3.2  1. For any proper rotation \( \delta \) in \( V \), the map \( \rho_\delta \) is a rotation with angle \( \frac{\pi}{2} \).

2. The map \( \vartheta : \mathcal{R} \to \mathcal{R}_{\pi/2} \times ]0,\pi[ \), \( \delta \mapsto (\rho_\delta, a(\delta)) \) is a bijection.

3. Suppose \( \delta, \epsilon \in \mathcal{R} \). The representation \( (\delta, \epsilon) \) is irreducible if and only if \( (\rho_\delta, \rho_\epsilon) \) is irreducible.

4. If \( C \subset \mathcal{R}_{\pi/2}^2 \) classifies \( \mathfrak{M}_o \), then \( \{(\vartheta^{-1}(\sigma, \alpha), \vartheta^{-1}(\tau, \beta))\}_{(\sigma, \tau) \in C, \alpha, \beta \in ]0,\pi[} \) classifies \( \mathfrak{M}_p \).

Proof: Let \( \delta \in \mathcal{R}_o \). Clearly, \( \rho_\delta \) is linear. Since \( \delta(v) = \cos \alpha v + P_{o,\perp} \delta(v) \), we have \( \|P_{o,\perp} \delta(v)\| = \sin \alpha \|v\| \) and thus \( \|\rho_\delta(v)\| = \|v\| \). This means that \( \rho_\delta \) is orthogonal. Since \( \langle v, \rho_\delta(v) \rangle = 0 \) for all \( v \in V \), indeed \( \rho_\delta \in \mathcal{R}_{\pi/2} \).

Let \( \sigma \in \mathcal{R}_{\pi/2} \). There exists a basis \( \mathcal{B} \) of \( V \) such that \( [\sigma]_{\mathcal{B}} = R_\sigma \oplus \cdots \oplus R_\sigma \).

The preimage of \( \sigma \) under the map \( \mathcal{R} \to \mathcal{R}_{\pi/2} \), \( \delta \mapsto \rho_\delta \) is the set of \( \delta \in \mathcal{R} \) for which \( [\delta]_{\mathcal{B}} = R_{a(\delta)} \oplus \cdots \oplus R_{a(\delta)} \). Hence, for fixed \( \alpha \in ]0,\pi[ \) there exists precisely one \( \delta \in \mathcal{R}_o \) such that \( \rho_\delta = \sigma \). This proves the second statement in the lemma.

We have seen (Proposition 1.1) that every irreducible rotational representation of \( \mathbb{R}(X,Y) \) has dimension at most 4. On the other hand, every 2-dimensional representation given by proper rotations is necessarily irreducible. Thus, to prove 3 remains only the 4-dimensional case. Let \( \dim V = 4 \) and \( \delta, \epsilon \in \mathcal{R} \). The representation \( (\delta, \epsilon) \) is irreducible if and only if there exists a non-zero vector \( v \in V \) such that \( \delta(v) \) and \( \epsilon(v) \) are linearly independent. This is equivalent to that \( \rho_\delta(v) \) and \( \rho_\epsilon(v) \) are linearly independent, which happens if and only if the representation \( (\rho_\delta(v), \rho_\epsilon(v)) \) is irreducible.

Number 4 is an immediate consequence of 2 and 3.

The virtue of Lemma 3.2 is that it effectively reduces the classification problem for \( \mathfrak{M}_o \) to the corresponding problem for \( \mathfrak{M}_p \).

From the structure theorem for orthogonal endomorphisms immediately follows, that the finite-dimensional irreducible rotational representations which are not in \( \mathfrak{M}_p \) are classified by

\[
\{(r, s)\}_{r, s \in \{-1, 1\}} \cup \{(r, R_\alpha), (R_\alpha, r)\}_{r \in \{-1, 1\}, \alpha \in ]0,\pi[}.
\]

Let \( V \) be of dimension 2, and \( \sigma, \tau \in \mathcal{R}_{\pi/2} \). For any \( v \in V \setminus \{0\} \) we have \( \sigma(v), \tau(v) \in v^\perp \) and hence \( \sigma(v) = r \tau(v) \), where \( r = \pm 1 \). Since \( \sigma(v) = \tau(v) \) implies \( \sigma = \tau \), \( r \) does not depend on \( v \). Thus, every 2-dimensional object in \( \mathfrak{M}_o \) is isomorphic to either \( (R_{\bar{z}}, R_\bar{z}) \) or \( (R_{\bar{z}}, R_{\bar{z}}^-) \). By Lemma 3.2 this means that

\[
\{(R_\alpha, R_\beta), (R_\alpha, R_{\bar{\beta}})\}_{\alpha, \beta \in ]0,\pi[}
\]

classifies \( (\mathfrak{M}_p)_2 \).

It remains to consider the 4-dimensional case. Suppose \( \dim V = 4 \), \( \sigma, \tau \in \mathcal{R}_{\pi/2} \), and that the representation \( (\sigma, \tau) \) is irreducible. We shall construct a basis of \( V \), with respect to which the matrices of \( \sigma \) and \( \tau \) take certain, canonical
forms. Let $e_1 \in V$ be any unit vector, and $e_2 = \sigma(e_1)$. Since $(\sigma, \tau)$ is irreducible, $\tau(e_1) \notin \text{span}\{e_1, e_2\}$. Take $e_3 = \frac{1}{\sqrt{1 - \langle e_2, \tau(e_1) \rangle^2}} P_{e_2^\perp} \tau(e_1)$, which is a unit vector orthogonal to $e_1, e_2$. Finally, by setting $e_4 = \sigma(e_3)$ we get an orthonormal basis $\underline{e}$ of $V$, such that

$$[\sigma]_{\underline{e}} = R_{\underline{e}} \oplus R_{\underline{e}}.$$  

(6)

Now let $f_1 = e_1$, and $f_2 = \tau(f_1) = \tau(e_1)$. Since $f_2 \in \text{span}\{e_2, e_3\}$ and $\langle f_2, e_3 \rangle > 0$, we have $f_2 = \cos \theta e_2 + \sin \theta e_3$ for $\theta = \arccos(f_2, e_2) \in [0, \pi]$. Set $f_3 = -\sin \theta e_2 + \cos \theta e_3$ and $f_4 = \tau(f_3)$. As $f_4 \in \{f_1, f_2, f_3\}^\perp = \{e_1, e_2, e_3\}^\perp$, it follows that $f_4 = re_4$, where $r = \in \{-1, 1\}$. Denoting $f = (f_1, f_2, f_3, f_4)$, we get $[\tau]_{\underline{e}} = R_{\underline{e}} \oplus R_{\underline{e}}$. An easy calculation shows that

$$[\tau]_{\underline{e}} = \begin{pmatrix} 0 & -\cos \theta & -\sin \theta & 0 \\ \cos \theta & 0 & 0 & r \sin \theta \\ \sin \theta & 0 & 0 & -r \cos \theta \\ 0 & -r \sin \theta & r \cos \theta & 0 \end{pmatrix}$$  

(7)

and

$$[\sigma^{-1} \tau]_{\underline{e}} = \begin{pmatrix} \cos \theta & 0 & 0 & r \sin \theta \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -r \sin \theta & r \cos \theta & 0 \\ -\sin \theta & 0 & 0 & r \cos \theta \end{pmatrix}.$$  

If $r = -1$, then $u = \sin \theta e_1 + (\cos \theta - 1)e_2$ is an eigenvector of $\sigma^{-1} \tau$ with eigenvalue 1. This means that $\sigma(u) = \tau(u)$, and thus that $\text{span}\{u, \sigma(u)\}$ is invariant under both $\sigma$ and $\tau$, which contradicts the irreducibility of the representation $(\sigma, \tau)$. Hence $r = 1$.

For any unit vector $v = \sum_{i=1}^{4} v_i e_i \in V$, we have

$$\langle \sigma(v), \tau(v) \rangle = \langle v, \sigma^{-1} \tau(v) \rangle = \sum_{i,j=1}^{4} v_i v_j \langle e_i, \sigma^{-1} \tau(e_j) \rangle =$$

$$= \sum_{i=1}^{4} v_i^2 \langle e_i, \sigma^{-1} \tau(e_i) \rangle = \|v\|^2 \cos \theta = \cos \theta = \langle \sigma(e_1), \tau(e_1) \rangle.$$  

Hence, the number $\theta = \arccos(\langle \sigma(e_1), \tau(e_1) \rangle)$ is an invariant for the representation $(\sigma, \tau)$. On the other hand, it completely determines the representation. The identity (7), with $r$ now specified to 1, can be rewritten as

$$[\tau]_{\underline{e}} = T_\theta (R_{\underline{e}} \oplus R_{\underline{e}}) T_{-\theta}$$  

(8)

where $T_\theta$ is defined by (2). This, together with (6), implies that $\{(R_{\underline{e}} \oplus R_{\underline{e}}, T_\theta (R_{\underline{e}} \oplus R_{\underline{e}}) T_{-\theta})\}_{\theta \in [0, \pi]}$ classifies $(\mathfrak{M}_{\nu})_4$. Lemma 3.2 now gives the result for $(\mathfrak{M}_{\nu})_4$, completing the proof of Proposition 1.3.
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