A Note on the Second Spectral Gap Incompleteness Theorem

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Abstract

Pick a formal system. Any formal system. Whatever your favourite formal system is, as long as it’s capable of reasoning about elementary arithmetic. The First Spectral Gap Incompleteness Theorem of [CPGW15] proved that there exist Hamiltonians whose spectral gap is independent of that system: your formal system is incapable of proving that the Hamiltonian is gapped, and equally incapable of proving that it’s gapless.

In this note, I prove the Second Spectral Gap Incompleteness Theorem: I show how to explicitly construct, within the formal system, a concrete example of a Hamiltonian whose spectral gap is independent of that system. Just to be sure, I prove this result three times. Once with Gödel’s help. Once with Zermelo and Fraenkel’s help. And finally, doing away with these high-powered friends, I give a simple, direct argument which reveals the inherent self-referential structure at the heart of these results, by asking the Hamiltonian about its own spectral gap.

Gödel’s famous 1931 results [Göd31], nowadays known as his First and Second incompleteness theorems, taught us that we cannot expect to prove everything we might like to in mathematics. Gödel’s first incompleteness theorem proves that there exist statements in mathematics that can neither be proven nor disproven. The second incompleteness theorem extends this to statements we might actually care about. More specifically, the First Incompleteness Theorem states (technically this is the slightly stronger Rosser’s Theorem [Ros36]):

**Theorem 1 (Gödel’s 1st Incompleteness Theorem – informal)**

Let \( F \) be a consistent formal system capable of reasoning about elementary arithmetic. Then there exists a statement \( S \in F \) such that neither \( S \) nor \( \neg S \) are provable in \( F \).

This shows that there always exist statements that are beyond the reach of proof. But the first incompleteness theorem doesn’t pinpoint any specific such statement.

The Second Incompleteness Theorem states:

\[\text{Peano arithmetic + first-order logic? NBG set theory? New Foundations? Surely not boring old ZFC! And don’t think you can wiggle out of it by telling me you’re a constructivist: by the end of this abstract, you won’t have escaped either.}\]
Theorem 2 (Gödel's 2nd Incompleteness Theorem – informal)

Let \( F \) be a consistent formal system capable of reasoning about elementary arithmetic. Then \( F \) cannot prove its own completeness: the statement \( G = \langle \text{\textit{F is complete}} \rangle \), formalised within \( F \), cannot be proven in \( F \).

This proves that a mathematical statement of real significance is beyond the reach of proof. Namely, the statement of the system’s own completeness. (We use \( \langle \ldots \rangle \) to indicate the formalisation of a colloquial mathematical statement within a formal system, and leave writing these out in full as fun exercises for the reader.)

In [CPGW15], we showed that the same thing affects\(^1\) certain important questions in physics. Specifically, in [CPGW15] we proved that the spectral gap problem is undecidable in 2D or higher. Indeed, in a nod to Rice’s theorem [Ric53], we proved undecidability of any property that distinguishes gapped Hamiltonians with product ground states, from gapless Hamiltonians with algebraically-decaying connected correlation functions. In [Bau+18] we strengthened these result to include 1D systems.

More precisely, we considered translationally invariant, nearest-neighbour spin lattice Hamiltonians \( H^\Lambda(L) = \sum_{(i,j) \in \mathcal{E}} h^{(i,j)} + \sum_{k \in \Lambda(L)} h_1^{(k)} \) on a lattice \( \Lambda(L) \) of size \( L \), with “gapped” and “gapless” defined as follows:

Definition 3 (Gapped) We say that a family of Hamiltonians \( H^\Lambda(L) \), as described above, is gapped if there is a constant \( \gamma > 0 \) and a system size \( L_0 \) such that for all \( L > L_0 \), \( \lambda_0(H^\Lambda(L)) \) is non-degenerate and \( \Delta(H^\Lambda(L)) \geq \gamma \). In this case, we say that the spectral gap is at least \( \gamma \).

Definition 4 (Gapless) We say that a family of Hamiltonians \( H^\Lambda(L) \), as described above, is gapless if there is a constant \( c > 0 \) such that for all \( \epsilon > 0 \) there is an \( L_0 \in \mathbb{N} \) so that for all \( L > L_0 \) any point in \( [\lambda_0(H^\Lambda(L)), \lambda_0(H^\Lambda(L)) + c] \) is within distance \( \epsilon \) from spec \( H^\Lambda(L) \).

Under these mathematically precise definitions of “gapped” and “gapless”, we proved the following result for 2D (and higher) systems [CPGW15]:

Theorem 5 (2D Spectral Gap Undecidability) For any given Turing Machine \( M \) and any \( n \in \mathbb{N} \), we can explicitly construct Hermitian matrices \( h_1(n) \in \mathcal{B}(\mathbb{C}^d) \) and \( h_{\text{row}}(n), h_{\text{col}}(n) \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) \) with algebraic matrix elements, where \( d \) is a fixed numerical constant and \( \|h_1(n)\|, \|h_{\text{row}}(n)\|, \|h_{\text{col}}(n)\| \leq 1 \), such that:

(i). If \( M \) halts on input \( n \), then the associated family of nearest-neighbour, translationally invariant, 2D spin lattice Hamiltonians

\[
H^\Lambda(L)(n) = \sum_{i \in \text{lattice}} h_1^{(i)}(n) + \sum_{(i,j) \in \text{Rows}} h_{\text{row}}^{(i,j)}(n) + \sum_{(i,j) \in \text{Cols}} h_{\text{col}}^{(i,j)}(n)
\]

is gapped in the strong sense of Definition 3, with gap \( \gamma \geq 1 \).

\(^1\)Infects?
(ii). If $M$ does not halt on input $n$, then $H^{(L)}(n)$ is gapless in the strong sense of Definition 4.

We subsequently strengthened this to the following result for 1D chains [Bau+18]:

**Theorem 6 (1D Spectral Gap Undecidability)** For any given Turing Machine $M$ and any $n \in \mathbb{N}$, we can explicitly construct Hamiltonians $h_1(n) \in \mathcal{B}(\mathbb{C}^d)$ and $h(n) \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ with algebraic matrix elements, where $d$ is a fixed numerical constant and $\|h_1(n)\|, \|h(n)\| \leq 1$, such that:

(i). If $M$ halts on input $n$, then the associated family of nearest-neighbour, translationally invariant, 1D spin chain Hamiltonians

$$H^{(L)}(n) = \sum_{1 \leq i \leq L} h_1^{(i)}(n) + \sum_{0 \leq i \leq L-1} h^{(i,i+1)}(n)$$

is gapless in the strong sense of Definition 3.

(ii). If $M$ does not halt on input $n$, then $H^L(n)$ is gapped in the strong sense of Definition 4, with gap $\gamma \geq 1$.

Letting $M$ be a universal Turing Machine in the above results, this immediately implies (algorithmic) undecidability of the spectral gap problem, via the classic result by Turing [Tur36] that the Halting problem is undecidable. As explained in [CPGW15], by standard arguments this result also implies axiomatic independence à la Gödel:

**Corollary 7 (1st spectral gap incompleteness theorem)** Let $\mathcal{F}$ be a consistent formal system capable of reasoning about elementary arithmetic. Let $d \in \mathbb{N}$ be a sufficiently large constant. There exists a translationally invariant nearest-neighbour Hamiltonian on a 2D lattice, or on a 1D chain, with local dimension $d$ and algebraic matrix entries, for which neither the presence nor the absence of a spectral gap is provable or disprovable in $\mathcal{F}$.

This is a spectral-gap analogue of Gödel’s First Incompleteness Theorem: it proves existence of a spin lattice Hamiltonian with undecidable spectral gap. But, like the First Incompleteness Theorem, it isn’t able to pinpoint any specific Hamiltonian whose spectral gap is undecidable. We noted in [CPGW15] that:

“[T]here are particular Hamiltonians within these families for which one can neither prove nor disprove the presence of a gap, or of any other undecidable property. Unfortunately, our methods cannot pinpoint these cases.”

The purpose of this note is to point out that our methods can in fact pinpoint concrete cases. We’ll prove a spectral-gap analogue of Gödel’s Second Incompleteness Theorem: a concrete construction of a particular many-body quantum Hamiltonian whose spectral gap is undecidable.
I will assume¹ the reader understands (at least at an informal level) what a Turing Machine is, what the Halting Problem is and why it’s undecidable, what a formal system is, what it means to “formalise” mathematical statements within the system, and the distinction between proof and truth.² If not, one or both of [Hof+79] or [Obe19] are required background reading. The former inspired everyone whose read it. The latter was the inspiration for this note.

1 Second Spectral Gap Incompleteness Theorem
We deliberately formulate the Second Spectral Gap Incompleteness Theorem in a way that mirrors its hitherto more famous forebear:

**Theorem 8 (2nd Spectral Gap Incompleteness Theorem – informal)**
Let \( F \) be a consistent formal system capable of reasoning about elementary arithmetic. We can explicitly describe within \( F \) a specific, translationally invariant, nearest-neighbour Hamiltonian on a 2D spin-lattice (or 1D spin chain), for which neither the presence nor the absence of a spectral gap is provable or disprovable in \( F \).

More precisely:

**Theorem 9 (2nd Spectral Gap Incompleteness Theorem)**
For any consistent, recursive formal system \( F \) capable of reasoning about elementary arithmetic, we can explicitly construct within \( F \) a fixed set of Hermitian matrices \( h_1 \in \mathcal{B}(\mathbb{C}^d) \) and \( h_{\text{row}}, h_{\text{col}} \in \mathcal{B}(\mathbb{C}^d) \) with algebraic matrix elements, where \( d \) is a fixed numerical constant and \( \|h_1\|, \|h_{\text{row}}\|, \|h_{\text{col}}\|, \|h\| \leq 1 \), such that:

(i). The associated family of nearest-neighbour, translationally invariant, 2D spin lattice (1) or 1D spin chain Hamiltonians (2) \( H^{\Lambda(L)} \) is either gapped in the strong sense of Definition 3 with gap \( \gamma \geq 1 \), or gapless in the strong sense of Definition 4.

(ii). The statements \( G = \text{"}H^{\Lambda(L)} \text{ is gapped"} \) and \( C = \text{"}H^{\Lambda(L)} \text{ is gapless"} \), formalised within \( F \), are independent of \( F \).

In fact, we’ll see three ways to prove this. The first proof is a straightforward consequence of the original undecidability of the spectral gap results of Theorems 5 and 6. The second proof uses recent results by [YA16] to make the first proof more concrete and compelling. The main point of this note is to observe that there is an elegant, self-contained argument that gives a Second Spectral Gap Undecidability Theorem directly.

2 First proof³
The first proof makes use of none other than Gödel’s Second Incompleteness Theorem itself, to prove a spectral gap analogue.

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¹I.e. have already assumed.
²Nothing too deep, then.
³Tastefully numbered “2”.
Proof (of Theorem 9) Let $\mathcal{F}$ be any consistent, recursive formal system that is sufficiently powerful to reason about elementary arithmetic. Since $\mathcal{F}$ is recursive, we can construct a Turing Machine $M$ that enumerates over all the provable statements in $\mathcal{F}$, and halts if it ever finds a contradiction (i.e. if it ever finds both a proof of $T$ and a proof of $\neg T$). In pseudocode:

\begin{verbatim}
function M
    S ← ∅
    for all $x ∈$ all strings in lexicographic order do
        if $\exists T ∈ S$ such that $\neg T ∈ S$ then
            halt
        else if $x$ is a proof of $T$ then
            S ← S ∪ {T}
\end{verbatim}

$M$ will eventually halt if and only if $\mathcal{F}$ is inconsistent (it can prove a contradiction), and will run forever if and only if $\mathcal{F}$ is consistent. So deciding whether $\mathcal{F}$ halts or not is equivalent to deciding consistency of $\mathcal{F}$. However, by Theorem 2, consistency of $\mathcal{F}$ is independent of $\mathcal{F}$, so the Halting problem for $M$ must also be independent of $\mathcal{F}$. Thus $M$ is a particular Turing Machine whose Halting Problem is independent of $\mathcal{F}$.

This argument is standard and well-known (see e.g. [Obe19]). But the short step from this standard argument to the spectral gap problem is already provided by Theorems 5 and 6! Applying Theorem 5 or Theorem 6 to $M$, we obtain a description of a specific spin Hamiltonian $H$ which is gapped or gapless, respectively, depending on whether $M$ halts or not. (Or more precisely, a description of its local interactions.) Since Halting of $M$ is independent of $\mathcal{F}$, whether $H$ is gapped or gapless is also independent of $\mathcal{F}$. □

3 Second proof

In principle, if you pick your favourite formal system $\mathcal{F}$ sufficiently powerful to reason about elementary arithmetic – say ZFC – the above argument will give you back a concrete description of an $H$ whose spectral gap is independent of that formal system. The procedure for explicitly constructing $H$ from $M$ is essentially the content of [CPGW15] (plus [Bau+18] for the 1D case). But one still needs to construct $M$ from $\mathcal{F}$. It’s clear enough that one could in principle construct $M$ from any given $\mathcal{F}$. But we haven’t done so. You might quibble that this falls somewhat short of our goals. Strictly speaking, it doesn’t completely specify a concrete procedure for constructing $H$; it shows how one could fill in the gaps and come up with a complete procedure, were one so inclined.

Luckily, other people were so inclined. E.g. [YA16] recently gave an explicit construction of a concrete, 7910-state Turing Machine that implements $M$ for ZFC. Plugging their Turing Machine into [CPGW15; Bau+18] then gives a complete procedure for constructing a specific $H$ whose spectral gap is independent of ZFC.
4 Third proof

The above proof(s) rely on Gödel’s Second Incompleteness Theorem to prove that Halting of a specific $M$ is independent of $\mathcal{F}$. It’s a perfectly servicable proof. But as proofs go, it’s slightly unsatisfying. The self-referential structure that lies at the heart of all incompleteness proofs [Hof+79] isn’t evident. It’s there, but only indirectly, through bootstrapping off Gödel’s original Incompleteness Theorems (whose proofs of course do have this self-referential structure).

Moreover, Gödel’s Second Incompleteness Theorem follows from the First Incompleteness Theorem. But we already proved a First Spectral Gap Incompleteness Theorem in Corollary 7. Shouldn’t there be some simpler, more elegant, direct argument? One that lifts the First Spectral Gap Incompleteness Theorem to the Second Spectral Gap Incompleteness Theorem, instead of resorting to Gödel’s results?

Indeed there is! A delightful recent essay by Oberhoff [Obe19] explains how Gödel’s First and Second Incompleteness Theorems can be proven directly from the connection between algorithms and formal systems that we just saw. In a similar spirit, both the First and Second Spectral Gap Incompleteness Theorems can be proven directly from the connection between spectral gaps and formal systems established by [Spe; CPGW15].

The proof has to go via Turing Machines as an intermediate step, only because the results of [CPGW15; Bau+18] are phrased in terms of Turing Machines and not formal systems. If you have the patience and energy (I don’t!), you can redo the construction of [CPGW15, Section 4] to directly encode inference within a given formal system into the Hamiltonian. Instead of encoding inference into a Turing Machine, which is then encoded into a Hamiltonian. By [Tur36]’s results, we know these are all the same thing anyway.

I’ll give the argument for the 1D case, as this is the stronger result. The argument for the 2D case is almost identical, just flipping the construction in the obvious place since the relationship between gapped/gapless and halting/non-halting in the original 2D result is inverted compared to the 1D result.

For convenience, let’s give names to the statements that $H$ is gapped/gapless, formalised within $\mathcal{F}$. As before, we’ll denote formalisation within $\mathcal{F}$ by «...»). Let $G(H) = \text{«}H\text{ is gapped}\text{»}$ ($G$ for “gapped”) and $C(H) = \text{«}H\text{ is gapless}\text{»}$ ($C$ for “continuous spectrum”). Note that $G(H) \not= C(H)$, because our strong definitions of gapped and gapless (Definitions 3 and 4) are not negations of each other; they deliberately exclude ambiguous intermediate cases, such as Hamiltonians with degenerate ground states. So we’ll want to separately prove independence of both $G(H)$ and $C(H)$ from $\mathcal{F}$. It is true that $C(H) \Rightarrow \neg G(H)$ and $G(H) \Rightarrow \neg C(H)$, so a consistent theory cannot prove both $G(H)$ and $C(H)$. But, for any given $H$, that would still leave the possibility that both are false for the $H$ we construct, and that falsehood of both $C(H)$ and $G(H)$ is provable in $\mathcal{F}$. We want to rule out this possibility, too.

**Proof (of Theorem 9)** Let $\mathcal{F}$ be any consistent, recursive formal system that is sufficiently powerful to reason about elementary arithmetic. Let $M$ be a Turing Machine that takes as input a description of a translationally-invariant
spin-lattice Hamiltonian $H$, and enumerates over all possible proofs of $\mathcal{F}$ until it finds a proof of $G(H)$. In pseudocode:

\begin{verbatim}
function $M(H)$
    for $x \in$ all strings in lexicographic order do
        if $x$ proves $G(H)$ then halt
\end{verbatim}

Now, let $H_M$ be the Hamiltonian obtained by applying Theorem 6 to $M$ and setting $n$ in Theorem 6 to the description of $H_M$. I claim that $G(H_M)$ is independent of $\mathcal{F}$.\footnote{See where the self-referentiality is coming in?}

Assume for contradiction that $G(H_M) = \text{"$H_M$ is gapped"}$ is provable in $\mathcal{F}$. Then $M$ will eventually find a proof of this, and halt. By Theorem 6, if $M$ halts then $H_M$ is gapless. So far, we’ve just shown that $\mathcal{F}$ necessarily tells lies about some spectral gaps. Just because it lies, doesn’t mean that it’s inconsistent!

However, following [Obe19], once you strip away the window-dressing the entire proof of Theorem 6 in [CPGW15; Ban+18] involves nothing more than elementary arithmetic and first-order logic,\footnote{The only reason we followed the standard mathematical convention of not reducing the whole proof down to Peano’s axioms and first order logic was to keep the already-large page count of [CPGW15] down a bit. Quite a bit. In the British English sense of “quite”.
} hence can be formalised within $\mathcal{F}$. Since Theorem 6 proves that $H_M$ is gapless iff $M$ runs forever, this, together with the sequence of computational steps that led to $M$ halting, constitutes a proof within $\mathcal{F}$ of $C(H_M)$. But we already saw that $C(H_M) \Rightarrow \neg G(H_M)$, so this contradicts consistency of $\mathcal{F}$.

Now assume for contradiction that $\neg G(H_M) = \text{"$H_M$ is not gapped"}$ is provable in $\mathcal{F}$. Since we’re assuming $\mathcal{F}$ is consistent, this implies $M$ will never find a proof of $G(H_M)$, so will never halt, Theorem 6 tells us that if $M$ doesn’t halt, then $H_M$ is gapped, this gives us a proof formalisable within $\mathcal{F}$ of $G(H_M)$, and we again have a contradiction.

Exactly the same argument goes through, mutatis mutandis, for $C(H_M)$. Thus both $G(H_M)$ and $C_H(M)$ are independent of $\mathcal{F}$, as claimed. \hfill \Box

If you haven’t seen this type of thing before, you may be worried that there’s an issue here. The description of $H_M$ depends on $n$, which in turn depends on the description of $H_M$. Doesn’t this mean $H_M$ not well-defined? On the other hand, if you know too much computability theory, you may think we need to appeal here to Kleene’s second recursion theorem [Kle+52].

But this is no more of an issue than constructing a Turing Machine $B$ that, when fed the description of a Turing Machine $A$ as input, searches for proofs that the Turing Machine $A$ halts on input $A$ (as e.g. in [Obe19, Lemma 1]). This is obviously fine, as $B$ is given the string $A$ as input, and is free to interpret it both as a description of a Turing Machine $A$, and as a string $A$ to be input to that Turing Machine. If this still worries you for some reason, we could always have $B$ duplicate its input string $A$ as a first step, and use one copy as the description of a Turing Machine, the other as the input to that machine.

The Hamiltonian case is very similar. $M$ should take as input a string describing $H(n)$ with $n$ a free parameter, or “placeholder”, and have $M$ interpret
this input string both as a description of $H(n)$ and as the value of $n$. By now you’re probably not worried. But just in case you’re of a particularly nervous disposition, we could always have $M$ duplicate its input string as a first step, and use one copy as the description of $H(n)$, the other as the value of $n$.

This direct third proof is barely longer than the first one (upon which the second one builds), even though we’ve proven it directly as a stand-alone result, rather than reaching for the sledgehammer of Gödel’s Second Incompleteness Theorem.¹ It also makes the self-referential structure of the proof evident: essentially, the proof constructs a Hamiltonians which can answer questions about spectral gaps through its spectral properties, then asks this Hamiltonian about its own spectral gap.

References

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¹Indeed, we’ve independently proven a theorem that’s better² than Gödel’s original: we’ve proven that there are specific statements of interest to physicists – about spectral gaps, no less! – that are independent of whatever formal system floats your boat.

²Unless you consider consistency of ZFC and statements about the logical structure of all mathematics to be more interesting than spectral gaps. But that highly unlikely possibility scarcely seems worth addressing.
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