Lattice Dynamics in the Half–Space, II.
Energy Transport Equation

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Abstract

We consider the lattice dynamics in the half-space. The initial data are random according to a probability measure which enforces slow spatial variation on the linear scale $\varepsilon^{-1}$. We establish two time regimes. For times of order $\varepsilon^{-\gamma}$, $0 < \gamma < 1$, locally the measure converges to a Gaussian measure which is time stationary with a covariance inherited from the initial measure (non-Gaussian, in general). For times of order $\varepsilon^{-1}$, this covariance changes in time and is governed by a semiclassical transport equation.

Key words and phrases: harmonic crystal in the half-space, random initial data, covariance matrices, weak convergence of measures, hydrodynamic limit, energy transport equation.

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1 Introduction

The paper concerns a mathematical problem of foundations of statistical physics and continues the work [9] devoted to the derivation of a limiting "hydrodynamic" (Euler type) equation from the Hamilton dynamics. We refer the reader to [2, 4, 14, 15] for a detailed discussion of the results and methods on this problem.

As the model we consider the harmonic crystals in the half-space $\mathbb{Z}^d_+ = \{ z \in \mathbb{Z}^d : z_1 > 0 \}$. In the harmonic approximation, the crystal is characterized by the displacements $u(z, t) \in \mathbb{R}^n$, $z \in \mathbb{Z}^d_+$, of the crystal atoms from their equilibrium positions. The field $u(z, t)$ is governed by a discrete wave equation.

The derivation of hydrodynamic equations is connected with the problem of convergence to an equilibrium measure. Hence, the first step in our investigation is the proof of such convergence. This step was done in [10]. We assume that a probability measure $\mu_0$ giving the distribution of initial data has some mixing properties. If $\mu_t$ denotes the time-evolved measure at time $t$, then the limit

$$\lim_{t \to \infty} \mu_t = \mu_\infty$$

is established, where $\mu_\infty$ is an equilibrium Gaussian measure. (The precise formulation of this assertion is given by Theorem 2.5). In [5, 6], we have analyzed the long-time convergence to an equilibrium distribution for systems described by partial differential equations in $\mathbb{R}^d$. In [7]–[9], we extended the results to harmonic crystals. In the above-mentioned papers the systems were considered in the entire space. In [10], the dynamics of the harmonic crystals is studied first in the half-space $\mathbb{Z}^d_+$.

To derive the hydrodynamic equation we apply the special so-called hydrodynamic limit procedure. Given a matrix function $\{R(r, \cdot), r \in \mathbb{R}^d\}$ (so-called "spectral density matrix function" in the terms of R.L. Dobrushin and others, [3]) we consider a family of measures $\{\mu_0^\varepsilon, \varepsilon > 0\}$ which satisfies the following conditions: (i) For any $r \in \mathbb{R}^d_+$, the covariance $Q_\varepsilon(z, z')$ of the measure $\mu_0^\varepsilon$ at points $z, z' \in \mathbb{Z}^d_+$ close to the $[r/\varepsilon]$ is approximately (as $\varepsilon \to 0$) described by $R(r, \cdot)$; (ii) the covariance $Q_\varepsilon(z, z')$ vanishes as $|z - z'| \to \infty$ uniformly in $\varepsilon$ (see conditions $V1$ and $V2$ in Section 2.2 below). Given nonzero $\tau \in \mathbb{R}$ and $r \in \mathbb{R}^d_+$, we study the distribution $\mu_{\tau/\varepsilon, r}^\varepsilon$ of the random solution $u(z, t)$ at time moments $\tau/\varepsilon$ and close to the spatial point $[r/\varepsilon]$. We establish the limit

$$\lim_{\varepsilon \to 0} \mu_{\tau/\varepsilon, r}^\varepsilon = \mu_{\tau, r}^G,$$

where $\mu_{\tau, r}^G$ is a Gaussian measure (see Theorem 2.14). In particular, we derive the explicit formulas for covariance matrix $q_{\tau, r}^G(z - z')$ of the limit measure $\mu_{\tau, r}^G$. These formulas allow us to conclude that the matrix function $q_{\tau, r}^G(\theta)$ evolves according to the following equation:

$$\partial_\tau f_{\tau, r}(\theta) = i C(\theta) \nabla \Omega(\theta) \cdot \nabla_r f_{\tau, r}(\theta), \quad r \in \mathbb{Z}^d_+, \quad \tau > 0,$$

where $C(\theta) = \begin{pmatrix} 0 & \Omega^{-1}(\theta) \\ -\Omega(\theta) & 0 \end{pmatrix}$, and, roughly, $\Omega(\theta)$ is the dispersion relation of the harmonic crystal. The boundary and initial conditions for (1.3) are written in terms of the function $R(r, \cdot)$. The equation of type (1.3) is called a hydrodynamic (or Euler) equation (see [3]). The result (1.3) is a continuation of the works [3] and [9]. In [3], the problem has
been studied for the infinite chain of harmonic oscillators on one-dimensional lattice $\mathbb{Z}^1$. In [9], the result has been extended to the many-dimensional case.

In phonon physics it is standard practice to use the Wigner function $W(t, r, \theta)$ as density of phonons with wave number $\theta$ at location $r$ and at specified time $t$. $W$ evolves according to the semiclassical energy transport equation

$$\partial_t W(t, r, \theta) = -\nabla \Omega(\theta) \cdot \nabla_r W(t, r, \theta)$$

(see Theorem 2.12 and Corollary 2.13). $W(t, r, \theta)$ at fixed $r, t$ are expressed by the covariance $\hat{q}^G_{t, \rho}(\theta)$ which is invariant under the lattice dynamics. Thus (1.3) or (1.4) can be understood as the equations governing the motion of the parameters which characterize the locally stationary measures.

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1.1 Model

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d_+$, $d \geq 1$,

$$\ddot{u}(z, t) = - \sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z})) u(z', t), \quad z \in \mathbb{Z}^d_+, \quad t \in \mathbb{R},$$

(1.5)

with zero boundary condition,

$$u(z, t)|_{z_1 = 0} = 0,$$

(1.6)

and with the initial data

$$u(z, 0) = u_0(z), \quad \dot{u}(z, 0) = u_1(z), \quad z \in \mathbb{Z}^d_+.$$  

(1.7)

Here $\mathbb{Z}^d_+ = \{z \in \mathbb{Z}^d : z_1 > 1\}$, $\tilde{z} = (-z_1, z_2, \ldots, z_d)$, $V(z)$ is the interaction (or force) matrix, $(V_{kl}(z))$, $k,l = 1, \ldots, n$, $u(z, t) = (u_1(z, t), \ldots, u_n(z, t))$, $u_0(z) = (u_0_1(z), \ldots, u_0_n(z)) \in \mathbb{R}^n$, and correspondingly for $u_1(z)$. To coordinate the boundary and initial conditions, we assume that $u_0(z) = u_1(z) = 0$ for $z_1 = 0$.

Write $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$ and $Y_0 = (Y^0_0, Y^1_0) \equiv (u_0(\cdot), u_1(\cdot))$. Then (1.5)–(1.7) becomes the evolution equation

$$\dot{Y}(t) = \mathcal{A}_+ Y(t), \quad t \in \mathbb{R}, \quad z \in \mathbb{Z}^d_+, \quad Y^0(t)|_{z_1 = 0} = 0, \quad Y(0) = Y_0.$$  

(1.8)

Here $\mathcal{A}_+ = \left( \begin{array}{cc} 0 & 1 \\ -\mathcal{V}_+ & 0 \end{array} \right)$ with $\mathcal{V}_+ u(z) := \sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z}'))u(z')$.

Let us assume that

$$V(z) = V(\tilde{z}), \quad \text{where} \quad \tilde{z} = (-z_1, \tilde{z}), \quad \tilde{z} = (z_2, \ldots, z_d) \in \mathbb{Z}^{d-1}.$$  

(1.9)

Then the solution to the problem (1.8) can be represented as the restriction of the solution to the Cauchy problem with odd initial data on the half-space. More precisely, consider the following Cauchy problem for the harmonic crystal in the entire space $\mathbb{Z}^d$:

$$\begin{cases}
\ddot{v}(z, t) = - \sum_{z' \in \mathbb{Z}^d} V(z - z') u(z', t), \quad z \in \mathbb{Z}^d, \quad t \in \mathbb{R}, \\
v(z, 0) = v_0(z), \quad \dot{v}(z, 0) = v_1(z), \quad z \in \mathbb{Z}^d.
\end{cases}$$  

(1.10)
Write \( X(t) = (X^0(t), X^1(t)) \equiv (v(\cdot, t), \dot{v}(\cdot, t)) \) and \( X_0 = (X_0^0, X_0^1) \equiv (v_0(\cdot), v_1(\cdot)) \). Then (1.10) becomes
\[
\dot{X}(t) = AX(t), \quad t \in \mathbb{R}, \quad X(0) = X_0.
\] (1.11)

Here \( \mathcal{A} = \begin{pmatrix} 0 & -\mathcal{V} \\ \mathcal{V} & 0 \end{pmatrix} \), where \( \mathcal{V} \) is a convolution operator with the matrix kernel \( V \).

Assume that the initial data \( X_0(z) \) form an odd function with respect to \( z_1 \in \mathbb{Z}^d \), i.e., let \( X_0(z) = -X_0(\hat{z}) \). Then the solution \( v(z, t) \) of (1.10) is also an odd function with respect to \( z_1 \in \mathbb{Z}^d \). Restrict the solution \( v(z, t) \) to the domain \( \mathbb{Z}^d_+ \) and set \( u(z, t) = v(z, t)|_{z_1=0} \). Then \( u(z, t) \) is the solution to the problem (1.5) with the initial data \( Y_0(z) = X_0(z)|_{z_1=0} \).

Assume that the initial data \( Y_0 \) for (1.8) belong to the phase space \( \mathcal{H}_{\alpha, +} \), \( \alpha \in \mathbb{R} \), defined below.

**Definition 1.1** \( \mathcal{H}_{\alpha, +} \) is the Hilbert space of \( \mathbb{R}^n \times \mathbb{R}^n \)-valued functions of \( z \in \mathbb{Z}^d_+ \) endowed with the norm
\[
\|Y\|_{\alpha, +}^2 = \sum_{z \in \mathbb{Z}^d_+} |Y(z)|^2 (1 + |z|^2)^\alpha < \infty.
\]

In addition, it is assumed that the initial data vanish \( Y_0 = 0 \) at \( z_1 = 0 \).

We impose the following conditions **E1–E6** on the matrix \( V \).

**E1.** There are positive constants \( C \) and \( \gamma \) such that \( \|V(z)\| \leq Ce^{-\gamma|z|} \) for \( z \in \mathbb{Z}^d \), where \( \|V(z)\| \) stands for the matrix norm.

Let \( \hat{V}(\theta) \) be the Fourier transform of \( V(z) \) with the convention
\[
\hat{V}(\theta) = \sum_{z \in \mathbb{Z}^d} V(z)e^{isz}, \quad \theta \in \mathbb{T}^d,
\]
where "\( \cdot \)" stands for the inner product in Euclidean space \( \mathbb{R}^d \) and \( \mathbb{T}^d \) for the \( d \)-torus \( \mathbb{R}^d/(2\pi\mathbb{Z})^d \).

**E2.** \( V \) is real and symmetric, i.e., \( V_{kl}(-z) = V_{lk}(z) \in \mathbb{R}, \ k, l = 1, \ldots, n, \ z \in \mathbb{Z}^d \).

The two conditions imply that \( \hat{V}(\theta) \) is a real-analytic Hermitian matrix-valued function of \( \theta \in \mathbb{T}^d \).

**E3.** The matrix \( \hat{V}(\theta) \) is non-negative definite for every \( \theta \in \mathbb{T}^d \).

Let us define the Hermitian non-negative definite matrix,
\[
\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0.
\] (1.12)

The matrix \( \Omega(\theta) \) has the eigenvalues \( 0 \leq \omega_1(\theta) < \omega_2(\theta) \cdots < \omega_n(\theta), \ s \leq n \), and the corresponding spectral projections \( \Pi_s(\theta) \) with multiplicity \( r_\sigma = \text{tr} \Pi_\sigma(\theta) \). The mapping \( \theta \mapsto \omega_\sigma(\theta) \) is the \( \sigma \)-th band function. There are special points in \( \mathbb{T}^d \) at which the bands cross, which means that \( s \) and \( r_\sigma \) jump to some other value. Away from such crossing points, \( s \) and \( r_\sigma \) are independent of \( \theta \). More precisely, the following lemma holds.
Lemma 1.2 (see [7, Lemma 2.2]). Let conditions E1 and E2 hold. Then there exists a closed subset $C_\ast \subset \T^d$ such that the following assertions hold:

(i) the Lebesgue measure of $C_\ast$ is zero;

(ii) for any point $\Theta \in \T^d \setminus C_\ast$, there exists a neighborhood $\O(\Theta)$ such that each band function $\omega_\sigma(\theta)$ can be chosen as a real-analytic function on $\O(\Theta)$;

(iii) the eigenvalue $\omega_\sigma(\theta)$ has constant multiplicity in $\T^d \setminus C_\ast$;

(iv) the following spectral decomposition holds:

$$\Omega(\theta) = \sum_{\sigma=1}^{s} \omega_\sigma(\theta) \Pi_\sigma(\theta), \quad \theta \in \T^d \setminus C_\ast,$$

where $\Pi_\sigma(\theta)$ is an orthogonal projection in $\R^n$, and $\Pi_\sigma$ is a real-analytic function on $\T^d \setminus C_\ast$.

For $\theta \in \T^d \setminus C_\ast$, denote by Hess $(\omega_\sigma)$ the matrix of second partial derivatives. The next condition on $V$ is as follows.

E4. Let $D_\sigma(\theta) = \det (\text{Hess}(\omega_\sigma(\theta)))$. Then $D_\sigma(\theta)$ does not vanish identically on $\T^d \setminus C_\ast$, $\sigma = 1, \ldots, s$.

Let us write

$$C_0 = \{ \theta \in \T^d : \det \hat{V}(\theta) = 0 \} \quad \text{and} \quad C_\sigma = \{ \theta \in \T^d \setminus C_\ast : D_\sigma(\theta) = 0 \}, \quad \sigma = 1, \ldots, s. \quad (1.14)$$

Then the Lebesgue measure of $C_\sigma$ vanishes, $\sigma = 0, 1, \ldots, s$ (see [7, Lemma 2.3]).

The last two conditions on $V$ look as follows.

E5. For each $\sigma \neq \sigma'$, the identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_{\pm}$ for $\theta \in \T^d \setminus C_\ast$, do not hold with $\text{const}_{\pm} \neq 0$.

This condition holds trivially for $n = 1$.

E6. $\|\hat{V}^{-1}(\theta)\| \in L^1(\T^d)$.

If $C_0 = \emptyset$, then $\|\hat{V}^{-1}(\theta)\|$ is bounded, and E6 holds trivially.

Remark 1.3 Conditions E1–E6 are satisfied, in particular, in the case of the nearest neighbor crystal in which the interaction matrix $V(z) = (V_{kl}(z))_{k,l=1}^{n}$ is of the form

$$V_{kl}(z) = 0 \quad \text{for} \quad k \neq l, \quad V_{kk}(z) = \begin{cases} -\gamma_k & \text{for} \quad |z| = 1, \\ 2\gamma_k + m_k^2 & \text{for} \quad z = 0, \quad k = 1, \ldots, n, \\ 0 & \text{for} \quad |z| \geq 2, \end{cases}$$

with $\gamma_k > 0$ and $m_k \geq 0$. In this case, equation (1.5) becomes

$$\ddot{u}_k(z,t) = (\gamma_k \Delta_L - m_k^2)u_k(z,t), \quad k = 1, \ldots, n.$$

Here $\Delta_L$ stands for the discrete Laplace operator on the lattice $\Z^d$,

$$\Delta_L u(z) := \sum_{e, |e|=1} (u(z + e) - u(z)).$$
Therefore, the eigenvalues of $\hat{V}(\theta)$ are

$$\tilde{\omega}_k(\theta) = \sqrt{2\gamma_1(1 - \cos \theta_1) + \ldots + 2\gamma_d(1 - \cos \theta_d) + m_k^2}, \quad k = 1, \ldots, n.$$  \hspace{0.5cm} (1.15)

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 1.2. Clearly, conditions E1-E5 hold with $C_\ast = \emptyset$. If $m_k > 0$ for any $k$, then the set $C_0$ is empty and condition E6 holds automatically. Otherwise, if $m_k = 0$ for some $k$, then $C_0 = \{0\}$. In this case, E6 is equivalent to the condition $\omega_k^{-2}(\theta) \in L^1(T^d)$, which holds if $d \geq 3$. Therefore, conditions E1-E6 hold if either (i) $d \geq 3$ or (ii) $d = 1, 2$ and $m_k > 0$ for any $k$.

**Lemma 1.4** (see [10, Corollary 2.4]) Let conditions E1 and E2 hold. Choose some $\alpha \in \mathbb{R}$. Then (i) for any $Y_0 \in \mathcal{H}_{\alpha,+}$, there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$ to the mixed problem (1.8);

(ii) the operator $U_+(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{H}_{\alpha,+}$.

The proof is based on the following formula for the solution $X(t)$ of (1.11):

$$X(t) = \sum_{z' \in \mathbb{Z}^d} G_t(z - z')X_0(z'),$$  \hspace{0.5cm} (1.16)

where the function $G_t(z)$ has the Fourier representation

$$G_t(z) := \hat{F}_{\theta \rightarrow z}^{-1}[\exp(\hat{A}(\theta)t)] = (2\pi)^{-d} \int_{T^d} e^{-iz\theta} \exp(\hat{A}(\theta)t) \, d\theta$$  \hspace{0.5cm} (1.17)

with

$$\hat{A}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{V}(\theta) & 0 \end{pmatrix}, \quad \theta \in T^d.$$  \hspace{0.5cm} (1.18)

Therefore, the solution $Y(t)$ of (1.8) admits the representation

$$Y(t) = \sum_{z' \in \mathbb{Z}^d} G_{t,+}(z, z')Y_0(z'), \quad z \in \mathbb{Z}^d_+,$$  \hspace{0.5cm} (1.19)

where $G_{t,+}(z, z') := G_t(z - z') - G_t(z - z').$  \hspace{0.5cm} (1.20)

## 2 Main results

### 2.1 Convergence to equilibrium

Denote by $\mu_0$ a Borel probability measure on $\mathcal{H}_{\alpha,+}$ giving the distribution of $Y_0$. The expectation with respect to $\mu_0$ is denoted by $E_0$.

Assume that the initial measure $\mu_0$ has the following properties S1-S4.

**S1.** $Y_0(z)$ has zero expectation value, $E_0(Y_0(z)) = 0$ for $z \in \mathbb{Z}^d_+$.

For $a, b, c \in \mathbb{C}^n$, denote by $a \otimes b$ the linear operator $(a \otimes b)c = a \sum_{j=1}^n b_j c_j$. Denote by $\mathcal{H}_\alpha$ the Hilbert space of $\mathbb{R}^n \times \mathbb{R}^n$-valued functions of $z \in \mathbb{Z}^d$ endowed with the norm

$$\|X\|_\alpha^2 = \sum_{z \in \mathbb{Z}^d} |X(z)|^2(1 + |z|^2)^\alpha < \infty.$$
**Definition 2.1** A measure $\nu$ is said to be translation invariant if $\nu(T_h B) = \nu(B)$ for $B \in \mathcal{B}(\mathcal{H}_a)$ and $h \in \mathbb{Z}^d$, where $T_h X(z) = X(z - h)$ for $z \in \mathbb{Z}^d$.

S2. The correlation matrices of the measure $\mu_0$ have the form
\[
Q_{ij}^0(z, z') = E_0 \left( Y_0^i(z) \otimes Y_0^j(z') \right) = q_{ij}^0(z_1, z_1', z - z'), \quad z, z' \in \mathbb{Z}_+^d, \quad i, j = 0, 1, \tag{2.1}
\]
where (i) $q_{ij}^0(z_1, z_1', z) = 0$ for $z_1 = 0$ or $z_1' = 0$,
(ii) $\lim_{y \to +\infty} q_{ij}^0(z_1 + y, y, z) = q_{ij}^0(z), \ z = (z_1, z) \in \mathbb{Z}^d$. Here $q_{ij}^0(z)$ are correlation functions of some translation invariant measure $\nu_0$ with zero mean value on $\mathcal{H}_a$.

S3. The measure $\mu_0$ has finite variance and finite mean energy density,
\[
e_0(z) = E_0(|Y_0^0(z)|^2 + |Y_0^1(z)|^2) = \text{tr} \left[ Q_{00}^0(z, z) + Q_{11}^0(z, z) \right] \leq e_0 < \infty, \quad z \in \mathbb{Z}_+^d. \tag{2.2}
\]

Finally, it is assumed that the measure $\mu_0$ satisfies a mixing condition. To formulate this condition, denote by $\sigma(\mathcal{A}), \ \mathcal{A} \subset \mathbb{Z}_+^d$, the $\sigma$-algebra on $\mathcal{H}_{a,+}$ generated by $Y_0(z)$ with $z \in \mathcal{A}$. Define the Ibragimov mixing coefficient of the probability measure $\mu_0$ on $\mathcal{H}_{a,+}$ by the rule (cf. [11, Definition 17.2.2])
\[
\varphi(r) = \sup_{\mathcal{A}, B \subset \mathbb{Z}_+^d} \sup_{\text{dist}(\mathcal{A}, B) \geq r} \sup_{A \in \sigma(\mathcal{A}), B \in \sigma(\mathcal{B}), \mu_0(B) > 0} \frac{|\mu_0(A \cap B) - \mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.3}
\]

**Definition 2.2** A measure $\mu_0$ is said to satisfy the strong uniform Ibragimov mixing condition if $\varphi(r) \to 0$ as $r \to \infty$.

S4. The measure $\mu_0$ satisfies the strong uniform Ibragimov mixing condition with
\[
\int_0^\infty r^{d-1} \varphi^{1/2}(r) \, dr < \infty. \tag{2.4}
\]

**Remark 2.3** The uniform Rosenblatt mixing condition [13] is also sufficient, together with a higher power $> 2$ in the bound (2.2). Namely, there is a $\delta > 0$ such that
\[
E_0 \left( |Y_0^0(z)|^{2+\delta} + |Y_0^1(z)|^{2+\delta} \right) \leq C < \infty, \quad z \in \mathbb{Z}_+^d.
\]
Condition (2.4) needs a modification, namely, $\int_0^{+\infty} r^{d-1} \alpha^p(r) \, dr < \infty$ with $p = \min(\delta/(2 + \delta), 1/2)$. Here $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.3) but without $\mu_0(B)$ in the denominator. Under these modifications, the statements of Theorem 2.5 and their proofs remain essentially unchanged.

The uniform Rosenblatt mixing condition can also be weakened, see Remarks 3.4 in [10].

**Definition 2.4** We define $\mu_t$ as the Borel probability measure on $\mathcal{H}_{a,+}$ which gives the distribution of the random solution $Y(t)$,
\[
\mu_t(B) = \mu_0(U_+(-t)B), \quad \text{where } B \in \mathcal{B}(\mathcal{H}_{a,+}) \text{ and } t \in \mathbb{R}.
\]
In [10], we prove the weak convergence of the measures \( \mu_t \) on the space \( \mathcal{H}_{\alpha,+} \) with \( \alpha < -d/2 \) to a limit measure \( \mu_\infty \),

\[
\mu_t \xrightarrow{\mathcal{H}_{\alpha,+}} \mu_\infty \quad \text{as} \quad t \to \infty,
\]

where \( \mu_\infty \) is an equilibrium Gaussian measure on \( \mathcal{H}_{\alpha,+} \). This means the convergence

\[
\lim_{t \to \infty} \int f(Y) \, \mu_t(dY) = \int f(Y) \, \mu_\infty(dY)
\]

for any bounded continuous functional \( f \) on \( \mathcal{H}_{\alpha,+} \).

**Theorem 2.5** (see [10]). Let \( d, n \geq 1, \alpha < -d/2 \). Assume that conditions (1.9), E1–E6, and S1–S4 hold. Then

(i) the convergence in (2.5) holds.

(ii) The limit measure \( \mu_\infty \) is a Gaussian measure on \( \mathcal{H}_{\alpha,+} \).

(iii) The correlation matrices of the measures \( \mu_t \) converge to a limit for \( i, j = 0, 1 \),

\[
Q_{i}^{ij}(z, z') = \int (Y^i(z) \otimes Y^j(z')) \, \mu_t(dY) \to Q_{i}^{ij}(z, z'), \quad t \to \infty, \quad z, z' \in \mathbb{Z}^d.
\]

The correlation matrix \( Q_\infty(z, z') = (Q_{i}^{ij}(z, z'))_{i,j=0}^{1} \) of the limit measure \( \mu_\infty \) has the form

\[
Q_\infty(z, z') = q_\infty(z - z') - q_\infty(z - z'') + q_\infty(z' - z) + q_\infty(z' - z''), \quad z, z' \in \mathbb{Z}^d.
\]

Here \( q_\infty(z) = q_+^\infty(z) + q^-_\infty(z) \), where in the Fourier transform, we have

\[
\hat{q}_+^\infty(\theta) = \frac{1}{4} \sum_{\sigma=1}^{8} \Pi_\sigma(\theta) (\hat{a}_0(\theta) + C(\theta)\hat{a}_0(\theta)C(\theta)^*) \Pi_\sigma(\theta),
\]

\[
\hat{q}^-_\infty(\theta) = \frac{i}{4} \sum_{\sigma=1}^{8} \text{sign} (\partial_{\theta} \omega_\sigma(\theta)) \Pi_\sigma(\theta) (C(\theta)\hat{a}_0(\theta) - \hat{a}_0(\theta)C(\theta)^*) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C},
\]

\( \Pi_\sigma(\theta) \) is the spectral projection in Lemma 1.2 (iv), and

\[
C(\theta) = \begin{pmatrix}
0 & \Omega(\theta)^{-1} \\
-\Omega(\theta) & 0
\end{pmatrix}, \quad C(\theta)^* = \begin{pmatrix}
0 & -\Omega(\theta)^{-1} \\
\Omega(\theta)^{-1} & 0
\end{pmatrix}.
\]

(iv) The measure \( \mu_\infty \) is time stationary, i.e., \( [U_+(t)]^* \mu_\infty = \mu_\infty, \quad t \in \mathbb{R} \).

**Remark 2.6** (i) From formulas (2.8)–(2.10) it follows that \( \hat{q}_\infty(\theta) \) satisfies the "equilibrium condition", i.e., one has the form

\[
\hat{q}_\infty(\theta) = \begin{pmatrix}
h(\theta) & g(\theta) \\
-g(\theta) & \Omega^2(\theta)h(\theta)
\end{pmatrix}.
\]

Moreover, \( (\hat{q}_\infty^{ij}(\theta))^* = \hat{q}_\infty^{ji}(\theta) \), \( i, j = 0, 1 \), and \( \hat{q}_\infty^{00}(\theta) \geq 0, \hat{q}_\infty^{11}(\theta) \geq 0 \).

(ii) Introduce the complex-valued field

\[
a(x) = \frac{1}{\sqrt{2}} \left( \mathcal{V}_+^{1/4} u_0(x) + i \mathcal{V}_+^{-1/4} u_1(x) \right) \in \mathbb{C}^n, \quad x \in \mathbb{Z}^d,
\]

\[\text{(2.12)}\]
Let \( a(x)^* \) stand for complex conjugate field. Obviously \( E_t(a(x)) = 0 \). The covariance \( Q_\infty(x, y) \) has two parts. By Theorem 2.5, the \( aa^- \), equivalently the \( a^*a^- \), covariance satisfies

\[
\lim_{t \to \infty} E_t(a(x) \otimes a(y)) = 0.
\]

For the \( a^*a^- \) covariance, Theorem 2.5 (iii) implies

\[
\lim_{t \to \infty} E_t(a(x) \otimes a(y)) = W(x - y) - W(x - \tilde{y}) - W(\tilde{x} - y) + W(\tilde{x} - \tilde{y}), \quad x, y \in \mathbb{Z}_+^d,
\]

where in Fourier transform

\[
\hat{W}(\theta) = \Omega(\theta) \hat{q}_0^0(\theta) + i \hat{q}_1^0(\theta) = \frac{1}{2} \sum_{\sigma=1}^s \left( 1 - \text{sign}(\nabla_{\theta, \omega_0}(\theta)) \right) \Pi_{\sigma}(\theta) \hat{W}_0(\theta) \Pi_{\sigma}(\theta),
\]

with \( \hat{W}_0(\theta) = \frac{1}{2} [\omega_\sigma(\theta) \hat{q}_0^0(\theta) + \omega^{-1}_\sigma(\theta) \hat{q}_1^0(\theta) + i \hat{q}_0^1(\theta) - i \hat{q}_1^0(\theta)] \).

### 2.2 Initial measure with slow variation

Let \( \{\mu_0, \varepsilon > 0\} \) be a family of initial measures. Roughly, in a linear region of size \( \varepsilon^{-1} \), \( \varepsilon \ll 1 \), \( \mu_0^\varepsilon \) looks like the initial measure from Section 2.1. To formulate the main conditions \( \text{V1} - \text{V2} \) on the initial covariance, let us introduce the complex \( 2n \times 2n \) matrix-valued function \( R(r, x, y) = (R_{ij}(r, x, y))_{1 \leq i, j \leq 0}, r \in \mathbb{R}^d, x, y \in \mathbb{Z}_+^d \), with the following properties.

**I0.** \( R(r, x, y) = 0 \) for \( x_1 = 0 \) or \( y_1 = 0 \). The \( n \times n \) matrix-valued functions \( R_{ij}(r, x, y) \) have the form

\[
R_{ij}(r, x, y) = R_{ij}^0(r, x_1, y_1, \tilde{x} - \tilde{y}), \quad \text{where} \; x = (x_1, \tilde{x}), \; y = (y_1, \tilde{y}), \; i, j = 0, 1.
\]

Moreover,

\[
\lim_{y_1 \to +\infty} R_{ij}(r, y_1 + z_1, y_1, \tilde{z}) = R_{ij}^0(r, z), \quad z = (z_1, \tilde{z}) \in \mathbb{Z}_+^d, \quad i, j = 0, 1. \tag{2.13}
\]

**I1.** For every fixed \( r \in \mathbb{R}^d \) and \( i, j = 0, 1 \), the bound holds,

\[
|R_{ij}(r, x, y)| \leq C(1 + |x - y|)^{-\gamma}, \quad x, y \in \mathbb{Z}_+^d,
\]

where \( C \) is some positive constant, \( \gamma > d \). In particular, for every \( r \in \mathbb{R}^d \),

\[
|R_{ij}^0(r, z)| \leq C(1 + |z|)^{-\gamma}, \quad z \in \mathbb{Z}^d. \tag{2.15}
\]

**I2.** For every fixed \( r \in \mathbb{R}^d \), the matrix-valued function \( R \) satisfies

\[
R_{ii}(r, \cdot, \cdot) \geq 0, \quad R_{ij}(r, x, y) = (R_{ij}^0(r, y, x))^T, \quad x, y \in \mathbb{Z}_+^d.
\]
In particular, for every fixed \( r \in \mathbb{R}^d \), \( \hat{R}_0(r, \theta) \) satisfies
\[
\hat{R}_0^{00}(r, \theta) \geq 0, \quad \hat{R}_0^{01}(r, \theta) \geq 0, \quad \hat{R}_0^{11}(r, \theta) = \hat{R}_0^{10}(r, \theta)^*, \quad \theta \in \mathbb{T}^d. \tag{2.16}
\]

\textbf{I3.} For every fixed \( r \in \mathbb{R}^d \) and \( \theta \in \mathbb{T}^d \), the matrix \( \hat{R}_0(r, \theta) \) is nonnegative definite.

\textbf{I4.} For every \( \theta \in \mathbb{T}^d \), \( \hat{R}_0^{ij}(\cdot, \theta) \), \( i, j = 0, 1 \), are \( C_d \) functions and the function
\[
r \rightarrow \sup_{\theta \in \mathbb{T}^d} \max_{i,j=0,1, a=(\alpha_1,\ldots,\alpha_d):|\alpha_j|\leq 1} \left| \frac{\partial^{\alpha}}{\partial \theta_1^{\alpha_1} \cdots \partial \theta_d^{\alpha_d}} \hat{R}_0^{ij}(r, \theta) \right|
\]
is bounded uniformly on bounded sets.

\textbf{Remark 2.7} For simplicity of proof, we could assume that \( R(r, x, y) \) has the simpler form, namely,
\[
R(r, x, y) = \zeta(x_1)\zeta(y_1)R_0(r, x - y), \tag{2.18}
\]
where \( \zeta(x), \ x \in \mathbb{Z}^d \), is a nonnegative bounded function such that \( \zeta(x) = 0 \) for \( x \leq 0 \) and \( \zeta(x) = 1 \) for \( x > a \) with some \( a \geq 1 \), and \( R_0(r, x) \) satisfies conditions \((2.15)-(2.17)\). Then \( R(r, x, y) \) satisfies \textbf{I1}–\textbf{I4}. However, the limit covariance \( Q_{\infty}(x, y) \) in \textbf{II} has not the form \textbf{II} in general. Formula \textbf{II} implies that \( Q_{\infty}(x, y) \) satisfies the bound similar to \textbf{II},
\[
Q_{\infty}(x, y) = Q_{\infty}(x_1, y_1, \bar{x} - \bar{y}), \quad \lim_{y_1 \to +\infty} Q_{\infty}(y_1 + z_1, y_1, \bar{z}) = q_{\infty}(z) + q_{\infty}(\bar{z}), \quad z \in \mathbb{Z}^d.
\]
Therefore, we will prove the main results under condition \textbf{II} which is weaker than \textbf{III}.

Let \( E_0^{\varepsilon} \) stand for expectation with respect to the measure \( \mu_0^{\varepsilon} \). Assume that \( E_0^{\varepsilon}(Y_j(x)) = 0 \) and define the covariance
\[
Q_{\varepsilon}^{ij}(x, x') = E_0^{\varepsilon}(Y_i(x) \otimes Y_j(x')), \quad x, x' \in \mathbb{Z}^d, \quad i, j = 0, 1.
\]

\textbf{Definition 2.8} We call a family of measures \( \{\mu_0^{\varepsilon}, \varepsilon > 0\} \) a family of slow variation for \( R \) if \( \{Q_{\varepsilon}^{ij}(x, x'), \varepsilon > 0\} \) satisfies conditions \textbf{V1}–\textbf{V3} listed below.

\textbf{V1.} For any \( \varepsilon > 0 \), there exists an even integer \( N_\varepsilon \) such that
(i) for all \( M \in \mathbb{R}^d \) and \( x, x' \in I_M \cap \mathbb{Z}^d_+ \), then
\[
\left| Q_{\varepsilon}^{ij}(x, x') - R^{ij}(\varepsilon M, x, x') \right| \leq C \min[(1 + |x - x'|)^{-\gamma}, \varepsilon N_\varepsilon], \tag{2.19}
\]
where \( C, \gamma \) are the constants from \textbf{II}, and \( I_M \) is the cube centered at the point \( M \) with edge length \( N_\varepsilon \),
\[
I_M = \{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d: |x_j - M_j| \leq N_\varepsilon/2, \ M = (M_1, \ldots, M_d)\}. \tag{2.20}
\]
(ii) \( N_\varepsilon \sim \varepsilon^{-\beta} \) as \( \varepsilon \to 0 \), with some \( \beta \in (1/2, 1) \).

\textbf{V2.} For any \( \varepsilon > 0 \) and all \( x, x' \in \mathbb{Z}^d_+ \), \( i, j = 0, 1 \),
\[
|Q_{\varepsilon}^{ij}(x, x')| \leq C(1 + |x - x'|)^{-\gamma} \quad \text{with constants } C, \gamma \text{ as in \textbf{II}}.
\]

To prove the weak convergence of the measures (Theorem 2.14 below) we need the stronger condition \textbf{V3}:

\textbf{V3.} The measures \( \mu_0^{\varepsilon} \) satisfy the Ibragimov mixing condition \textbf{S4} (see section 2.1) with the mixing coefficients \( \varphi_\varepsilon \). Moreover, it is assumed that
\[
\sup_{\varepsilon > 0} \left| \varphi_\varepsilon^{1/2}(x) \right| \leq C(1 + |x|)^{-\gamma}, \quad \text{with a } \gamma > d.
\]

Note that condition \textbf{V3} implies \textbf{V2}.
Definition 2.9 (i) $\mu^\varepsilon_t$ is a Borel probability measure on $\mathcal{H}_{\alpha,+}$ which gives the distribution of $Y(t)$,
\[ \mu^\varepsilon_t(B) = \mu^\varepsilon_0(U_+(-t)B), \text{ where } B \in \mathcal{B}(\mathcal{H}_{\alpha,+}) \text{ and } t \in \mathbb{R}. \]
(ii) The correlation functions of the measure $\mu^\varepsilon_t$ are defined by
\[ Q^{ij}_\varepsilon(x,y) = \int \left(Y^i(x) \otimes Y^j(y)\right) \mu^\varepsilon_t(dY) = E^\varepsilon_0(Y^i(x,t) \otimes Y^j(y,t)), \quad i,j = 0,1, \quad x,y \in \mathbb{Z}^d_+. \]
Here $Y^i(x,t)$ are the components of the random solution $Y(t) = (Y^0(\cdot,t),Y^1(\cdot,t))$ to the problem (1.8).

2.3 Covariance in the kinetic scaling limit

Let us use a time span of order $\tau/\varepsilon^\alpha$, $\tau \neq 0$, with $0 < \alpha \leq 1$, and study the asymptotics of the covariance $Q_{\varepsilon,\tau/\varepsilon^\alpha}(x,y)$ as $\varepsilon \to +0$. For $0 < \alpha < 1$, the result is given by Theorem 2.15.

To formulate the result for $\alpha = 1$ let us introduce the matrix $q_{r,r}(z)$, $z \in \mathbb{Z}^d_+$, $r \in \mathbb{R}^d$, $\tau \neq 0$, by the Fourier transform, $q_{r,r}(z) = F_{\theta}^{-1}[\hat{q}_{r,r}(\theta)]$, where
\[ \hat{q}_{r,r}(\theta) = \sum_{\sigma = 1}^s \Pi_{\sigma}(\theta) \left[ M_+^\sigma(\tau;\cdot,\theta) + iM_-^\sigma(\tau;\cdot,\theta) \right] \Pi_{\sigma}(\theta), \quad \theta \in \mathbb{T}^d \setminus C_s. \quad (2.21) \]
Here $\Pi_{\sigma}(\theta)$ is the spectral projection introduced in Lemma 1.2 (iv),
\begin{align*}
M_+^\sigma(\tau;\cdot,\theta) &= \frac{1}{2}(R_+^\sigma(\tau;\cdot,\theta) + C(\theta)T_0^\sigma(\tau;\cdot,\theta)C^*(\theta)), \\
M_-^\sigma(\tau;\cdot,\theta) &= \frac{1}{2}(C(\theta)T_0^\sigma(\tau;\cdot,\theta) - R_-^\sigma(\tau;\cdot,\theta)C^*(\theta)),
\end{align*}
with $C(\theta)$ defined in (2.10), and
\[ R_+^\sigma(\tau;\cdot,\theta) = \frac{1}{2}\left( R_0(r + \nabla \omega_{\sigma}(\theta)\tau,\theta) \chi^+_{r,r_1}(\theta) \pm R_0(r - \nabla \omega_{\sigma}(\theta)\tau,\theta) \chi^-_{r,r_1}(\theta) \right), \quad (2.23) \]
where
\[ \chi^\pm_{r,r_1}(\theta) = \frac{1}{2}(1 + \text{sign}(r_1 + \tau \nabla \omega_{\sigma}(\theta))). \quad (2.24) \]

Theorem 2.10 Let conditions V1–V2 and E1–E6 hold. Then for any $\tau \neq 0$, $r \in \mathbb{R}^d$ with $r_1 \geq 0$, the correlation functions converge to a limit,
\[ \lim_{\varepsilon \to 0} Q_{\varepsilon,\tau/\varepsilon^\alpha}([r/\varepsilon] + z,[r/\varepsilon] + z') = Q^G_{r,r}(z,z'), \quad (2.25) \]
where
\[ Q^G_{r,r}(z,z') = \begin{cases} 
q^R_{r,r}(z-z') = q_{r,r}(z-z') + q_{r,\tau}(\bar{z} - \bar{z}'), & \text{if } r_1 > 0, \ z,z' \in \mathbb{Z}_+^d, \\
q^L_{r,r}(z-z') - q_{r,r}(z-\bar{z}') + q_{r,\tau}(\bar{z} - z'), & \text{if } r_1 = 0, \ z,z' \in \mathbb{Z}_+^d,
\end{cases} \]
with $q_{r,r}(z)$ defined by (2.21)–(2.23).

This theorem is proved in Section 3.
Corollary 2.11 Let $r_1 > 0$ and $\tau \neq 0$. (i) From formulas (2.21)–(2.23) it follows that the $\sigma$-band of $q_{r,\tau}(\theta)$ satisfies the following "hydrodynamic" equation:

$$\partial_{\tau}f(\tau, r; \theta) = iC_\sigma(\theta)\nabla\omega_\sigma(\theta) \cdot \nabla_{\tau}f(\tau, r; \theta), \quad r_1 > 0, \quad \tau > 0,$$

where $C_\sigma = \begin{pmatrix} 0 & \omega^{-1}_\sigma \\
-\omega_\sigma & 0 \end{pmatrix}$, $\sigma = 1, \ldots, s$, the boundary and initial conditions are given by the $\hat{R}_0$.

(ii) From formulas (2.21)–(2.23) it follows that $q_{r,\tau}(\theta)$ satisfies the equilibrium condition (2.11), i.e., $q_{r,\tau}^{\sigma,11}(\theta)^* = \Omega^2(\theta)q_{r,\tau}^{G,01}(\theta)$, $q_{r,\tau}^{G,01}(\theta) = -q_{r,\tau}^{G,10}(\theta)$. Moreover, $q_{r,\tau}^{\sigma,ii}(\theta) \geq 0$, $q_{r,\tau}^{G,01}(\theta)^* = q_{r,\tau}^{G,10}(\theta)$.

Let us introduce the scaled $n \times n$ Wigner matrix through

$$W^\varepsilon(\tau; r, \theta) = \sum_{y \in \mathbb{Z}^d} e^{i\theta \cdot y} E^\varepsilon_{\tau/\varepsilon}(a^*([r/\varepsilon + y/2]) \otimes a([r/\varepsilon - y/2])), \quad r \in \mathbb{R}_+^d,$$

where $a(x)$ is given in (2.12). By conditions V1 and V2, the following limit exists

$$\lim_{\varepsilon \to 0} W^\varepsilon(0; r, \theta) = \frac{1}{2}(\Omega^{1/2}\hat{R}_0^{00}(r, \theta)\Omega^{1/2} + \Omega^{-1/2}\hat{R}_0^{11}(r, \theta)\Omega^{-1/2})$$

$$+ i\Omega^{1/2}\hat{R}_0^{10}(r, \theta)\Omega^{-1/2} - i\Omega^{-1/2}\hat{R}_0^{01}(r, \theta)\Omega^{1/2})$$

$$= W(0; r, \theta), \quad r_1 > 0.$$

We also define the limit Wigner matrix as follows.

$$W^p(\tau; r, \theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta)\left\{W(0; r - \tau\nabla\omega_\sigma(\theta), \theta)\chi^-_{\tau, r_1}(\theta)
+W(0; -r_1 + \tau\nabla\omega_\sigma(\theta), \tilde{r} - \tau\nabla\omega_\sigma(\theta), \tilde{\theta})\chi^+_{\tau, -r_1}(\theta)\right\} \Pi_\sigma(\theta)$$

$$= \begin{cases} \sum_{\sigma=1}^s \Pi_\sigma(\theta)W(0; r - \tau\nabla\omega_\sigma(\theta), \theta)\Pi_\sigma(\theta), & \text{if } r_1 > \tau\nabla\omega_\sigma(\theta), \\
\sum_{\sigma=1}^s \Pi_\sigma(\theta)W(0; -r_1 + \tau\nabla\omega_\sigma(\theta), \tilde{r} - \tau\nabla\omega_\sigma(\theta), \tilde{\theta})\Pi_\sigma(\theta), & \text{if } r_1 < \tau\nabla\omega_\sigma(\theta), \end{cases}$$

where $\tilde{\theta} = (-\theta_1, \tilde{\theta}), \tilde{\theta} = (\theta_2, \ldots, \theta_d), \tilde{r} = (r_2, \ldots, r_d)$.

Theorem 2.12 Let conditions V1–V2 and E1–E6 hold. Then for any $r \in \mathbb{R}_+^d$ and $\tau > 0$, the following limit exists in the sense of distributions,

$$\lim_{\varepsilon \to 0} W^\varepsilon(\tau; r, \theta) = W^p(\tau; r, \theta).$$

In addition, for the remaining part of the covariance,

$$\lim_{\varepsilon \to 0} \sum_{y \in \mathbb{Z}^d} e^{i\theta \cdot y} E^\varepsilon_{\tau/\varepsilon}(a([r/\varepsilon + y/2]) \otimes a([r/\varepsilon - y/2])) = 0.$$
Corollary 2.13 Denote by $W^p_\sigma(\tau; r, \theta)$, $\sigma = 1, \ldots, s$, the $\sigma$-th band of the Wigner function $W^p(\tau; r, \theta)$. Then $W^p_\sigma$ is a solution of the "energy transport" equation
\[
\partial_\tau W^p_\sigma(\tau; r, \theta) + \nabla \omega_\sigma(\theta) \cdot \nabla_r W^p_\sigma(\tau; r, \theta) = 0, \quad \tau > 0, \; r \in \mathbb{R}^d,
\]
where the boundary and initial conditions are given by the initial Wigner matrix projected onto the $\sigma$-th band,
\[
W^p_\sigma(\tau; r, \theta)|_{\tau=0} = \Pi_\sigma(\theta) W(0; r, \theta) \Pi_\sigma(\theta), \; r \in \mathbb{R}^d,
\]
\[
W^p_\sigma(\tau; \bar{r}, \bar{\theta})|_{r_1=0} = b(\tau; \bar{r}, \bar{\theta}), \; \bar{r} \in \mathbb{R}^{d-1}, \; \tau > 0.
\]
Here
\[
b(\tau; \bar{r}, \bar{\theta}) := \begin{cases} 
\Pi_\sigma(\theta) W(0; -\tau \nabla_1 \omega_\sigma(\theta), \bar{r} - \tau \tilde{\nabla} \omega_\sigma(\theta)) \Pi_\sigma(\theta), & \text{if } \nabla_1 \omega_\sigma(\theta) < 0, \\
\Pi_\sigma(\theta) W(0; \tau \nabla_1 \omega_\sigma(\theta), \bar{r} - \tau \tilde{\nabla} \omega_\sigma(\theta)) \Pi_\sigma(\theta), & \text{if } \nabla_1 \omega_\sigma(\theta) > 0.
\end{cases}
\]

2.4 Weak convergence of measures family

Let us consider the random field $Y$ at the kinetic time $\tau/\varepsilon$, $\tau \neq 0$, and close to the spatial point $[r/\varepsilon] \in \mathbb{Z}_+^d$. Denote by $T_h$, $h \in \mathbb{Z}_+^d$, the group of space translations. The measure at $r/\varepsilon$ is then defined through
\[
\mu^{\varepsilon/\varepsilon}(B) = \mu^{\varepsilon}(T_{[r/\varepsilon]} B), \quad \text{where } B \in \mathcal{B}(\mathcal{H}_{\alpha, +}) \text{ and } \mu^{\varepsilon/\varepsilon} \text{ is defined in Definition 2.9}.
\]

Theorem 2.14 Let conditions V1–V3 and E1–E6 hold. Then for $\tau \neq 0$, $r \in \mathbb{R}^d$ with $r_1 \geq 0$, in the sense of weak convergence on $\mathcal{H}_{\alpha, +}$,
\[
\lim_{\varepsilon \to 0} \mu^{\varepsilon/\varepsilon} = \mu^{G}_{\tau, r}.
\]

The measure $\mu^{G}_{\tau, r}$ is a Gaussian measure on $\mathcal{H}_{\alpha, +}$, which is invariant under the time translation $U_+(t)$. $\mu^{G}_{\tau, r}$ has mean zero and covariance
\[
Q^{G,ij}_{\tau, r}(z, z') = \int (Y^i(z) \otimes Y^j(z')) \mu^G_{\tau, r}(dY),
\]
defined by Theorem 2.10. If $r_1 > 0$, the covariance $Q^{G,ij}_{\tau, r}(z, z') = q^{G,ij}_{\tau, r}(z - z')$ is determined through $W^p(\tau; r, \theta)$ as
\[
\Omega(\theta) \tilde{q}^{G,00}_{\tau, r}(\theta) = \Omega(\theta)^{-1/2} \tilde{q}^{G,11}_{\tau, r}(\theta) = \frac{1}{2} (W^p(\tau; r, \theta) + W^p(\tau; r, -\theta)^T) \tag{2.30}
\]
and
\[
\tilde{q}^{G,01}_{\tau, r}(\theta) = -i \frac{1}{2} (W^p(\tau; r, \theta) - W^p(\tau; r, -\theta)^T). \tag{2.31}
\]

Theorem 2.14 is proved in Section 5.

From Theorem 2.14 we conclude that close to $r/\varepsilon$ in space and close to $\tau/\varepsilon$ in time the random field $Y^j(x, t)$ is a stationary Gaussian field. Its distribution at fixed local time $\tau$ is given by $\mu^{G}_{\tau, r}$, while in time it evolves deterministically according $U_+(t)$. In this sense locally in space and time the random field is stationary with statistics determined through the Wigner matrix at $(r, \tau)$ and the microscopic dynamics, compare with (2.30), (2.31).

Let us use a time span of order $\tau/\varepsilon^\alpha$ with an $\alpha \in (0, 1)$. In this case, change condition V1 (ii) as follows: $N_\varepsilon \sim \varepsilon^{-\beta}$ as $\varepsilon \to 0$ with some $\beta \in (\alpha/2, \alpha)$. Then the following result holds.
Theorem 2.15 Let $\alpha \in (0,1)$, conditions V1-V2 and E1-E6 hold. Then for $\tau \neq 0$, (i) the correlation functions of measures $\mu_{r/\varepsilon^\alpha_r}$ converge to a limit,

$$\lim_{\varepsilon \to 0} Q_{\varepsilon \cdot \tau \cdot \frac{r}{\varepsilon^\alpha}}([r/\varepsilon] + z, [r/\varepsilon] + z') = Q_r(z, z'), \quad z, z' \in \mathbb{Z}^d,$$

where $Q_r(z, z')$ does not depend on $\tau$ and has the form

$$Q_r(z, z') = \begin{cases} q_r(z - z'), & \text{if } r_1 > 0; \\
q_r(z - z') - q_r(z - z') - q_r(z - z') + q_r(z - z'), & \text{if } r_1 = 0;
\end{cases}$$

where in Fourier space, $\hat{q}_r(\theta) = \sum_{\sigma=1}^{\infty} \Pi_{\sigma}(\theta) M_r(\theta) \Pi_{\sigma}(\theta)$, with

$$M_r(\theta) = \begin{cases} \frac{1}{2} [\hat{R}_0(r, \theta) + C(\theta) \hat{R}_0(r, \theta) \hat{C}^*(\theta)], & \text{if } r_1 > 0; \\
\frac{1}{4} [\hat{R}_0(r, \theta) + C(\theta) \hat{R}_0(r, \theta) \hat{C}^*(\theta)] \\
+ i(\hat{C}(\theta) \hat{R}_0(r, \theta) - \hat{R}_0(r, \theta) \hat{C}^*(\theta)) \text{sign} (\nabla_1 \omega_0(\theta)), & \text{if } r_1 = 0.
\end{cases}$$

(ii) The measures $\mu_{r/\varepsilon^\alpha_r}$ converge weakly on the space $\mathcal{H}_{\alpha_r^+}$ to a limit measure $\mu_r$ as $\varepsilon \to 0$. Moreover, $\mu_r$ is a Gaussian measure on $\mathcal{H}_{\alpha_r^+}$, which is invariant under the time translation $U_+(t)$, has mean zero and covariance $Q_r(z, z')$ defined above.

We omit the proof of Theorem 2.15 since it can be proved by using the technique of Theorems 2.10 and 2.14.

3 Convergence of correlation functions

In this section we prove Theorem 2.10. Before we outline the strategy of the proof. At first, we use the cutting strategy from [7]-[9] combined with some techniques from [3], where Theorem 2.10 has proved for the case when $d = n = 1$ and in the entire space (see [3, Theorem 3.1]). Note that in [3] it is assumed the stronger conditions on matrix $V$ than E3, E4, namely, $\omega(\theta) > 0$, and the set

$$\{ \theta \in [-\pi, \pi] : \omega''(\theta) = \omega'''(\theta) = 0 \}$$

is empty. Under these conditions, in [3] the uniform asymptotics of the Green function is proved,

$$\sup_{x \in \mathbb{Z}^d} |\mathcal{G}_t(x)| \leq C(1 + |t|)^{-1/3}. \quad (3.1)$$

This bound plays an important role in the proof of [3]. However, if $n > 1$, then $\omega_\alpha$ may be non-smooth because of band crossing, and if $d > 1$, the set where the Hessian vanishes does not consist of isolated points. Therefore a strong estimate as (3.1) is unlikely to be valid, in general. To cope with such a situation, we split $\mathcal{G}_t(x)$ into two summands: $\mathcal{G}_t(x) = \mathcal{G}_t^f(x) + \mathcal{G}_t^g(x)$, where $\mathcal{G}_t^f(x)$ has a support in the neighborhood of a “critical set” $\mathcal{C} \subset \mathbb{T}^d$, and $\mathcal{G}_t^g(x)$ vanishes in the neighborhood of $\mathcal{C}$. The set $\mathcal{C}$ includes all points $\theta \in \mathbb{T}^d$ either with a degenerate Hessian of $\omega_\alpha(\theta)$, or with non-smooth $\omega_\alpha(\theta)$ (see formula (3.4)). We show
that the contribution of \( G_t^\ell(x) \) is negligible uniformly in \( t \) (see (3.9)). Hence, it allows us to represent correlations functions \( Q_{\varepsilon,\tau/\varepsilon} \) in the form: \( Q_{\varepsilon,\tau/\varepsilon} = Q^{g}_{\varepsilon,\tau/\varepsilon} + Q^{r}_{\varepsilon,\tau/\varepsilon} \), such that

\[
Q^{g}_{\varepsilon,\tau/\varepsilon}(x,y) = \sum_{x',y'\in\mathbb{Z}^d} G^{g}_{\varepsilon,\tau}(x,x')Q_{\varepsilon,\tau/\varepsilon}(x',y')G^g_{\varepsilon,\tau}(y,y').
\]

For the remainder \( Q^{r}_{\varepsilon,\tau/\varepsilon} = Q_{\varepsilon,\tau/\varepsilon} - Q^{g}_{\varepsilon,\tau/\varepsilon} \) we prove that \( Q^{r}_{\varepsilon,\tau/\varepsilon}(x,y) = o(1) \) uniformly in \( \tau \neq 0, \varepsilon > 0 \) and \( x,y \in \mathbb{Z}^d \). The last fact follows from two key observations: (i) \( \text{mes} C = 0 \) (Lemma 1.2) and (ii) the correlation quadratic form is continuous in \( \varepsilon,\tau/\varepsilon \), see Corollary 3.3. Up to this point we apply the “cutting strategy” from [7]–[9]. Finally, we prove that \( Q^{g}_{\varepsilon,\tau/\varepsilon} \) converges to a limit as \( \varepsilon \to 0 \), using the techniques of [3] and [8]. In addition, the asymptotics of \( G_t^g(x) \), (see Lemma 3.4) of the form \( G_t^g(x) \sim (1 + |t|)^{-d/2} \) plays the important role, since it replaces the asymptotics (3.1) and also simplifies some steps of the proof of [3]. However, in our case the structure of \( R(r,x,y) \) is more complex than in [3] or [9], in which \( R(r,x,y) = R_0(r,x-y) \). We apply the approach of [8, 10], where convergence to equilibrium was proved for non translation-invariant initial measures, and combine with the technique of [9].

### 3.1 Bounds for initial covariance

**Definition 3.1** By \( \ell^p \equiv \ell^p(\mathbb{Z}^d) \otimes \mathbb{R}^n \) (by \( \ell^p_+ \equiv \ell^p(\mathbb{Z}^d_+) \otimes \mathbb{R}^n \)), where \( p \geq 1 \) and \( n \geq 1 \), denote the space of sequences \( f(z) = (f_1(z), \ldots, f_n(z)) \) endowed with norm \( \|f\|_{\ell^p} = \left(\sum_{z\in\mathbb{Z}^d} |f(z)|^p\right)^{1/p} \), respectively, \( \|f\|_{\ell^p_+} := \left(\sum_{z\in\mathbb{Z}^d_+} |f(z)|^p\right)^{1/p} \).

The following lemma follows from condition \( \text{V2} \).

**Lemma 3.2** Let condition \( \text{V2} \) hold. Then, for \( i,j = 0,1 \), the following bounds hold:

\[
\sum_{z'\in\mathbb{Z}^d_+} |Q^{ij}_{\varepsilon}(z,z')| \leq C < \infty \quad \text{for all } z \in \mathbb{Z}^d_+,
\]

\[
\sum_{z\in\mathbb{Z}^d_+} |Q^{ij}_{\varepsilon}(z,z')| \leq C < \infty \quad \text{for all } z' \in \mathbb{Z}^d_+.
\]

Here the constant \( C \) does not depend on \( z,z' \in \mathbb{Z}^d_+ \) and \( \varepsilon > 0 \).

**Corollary 3.3** By the Shur lemma, it follows from Lemma 3.2 that

\[
|\langle Q_{\varepsilon}(z,z'), \Phi(z) \otimes \Psi(z') \rangle_{\ell^2_+}| \leq C\|\Phi\|_{\ell^2_+} \|\Psi\|_{\ell^2_+}, \quad \text{for any } \Phi,\Psi \in \ell^2_+,
\]

where the constant \( C \) does not depend on \( \varepsilon > 0 \).

### 3.2 Stationary phase method

By (1.17) and (1.18) we see that \( \hat{G}_t(\theta) \) is of the form

\[
\hat{G}_t(\theta) = \begin{pmatrix}
\cos \Omega t & \sin \Omega t \\
-\sin \Omega t & \cos \Omega t
\end{pmatrix},
\]
where \( \Omega = \Omega(\theta) \) is the Hermitian matrix defined by (1.12). Hence, we can rewrite \( \mathcal{G}_t(x) \) in the form
\[
\mathcal{G}_t(x) = \sum_{\pm, \sigma = 1}^{s} \int_{T^d} e^{-ix \cdot \theta} e^{ \pm i \omega_\sigma(\theta) t} a_{\sigma}^\pm(\theta) \, d\theta, 
\]
by (1.13). We are going to apply the stationary phase arguments to the integral (3.3) which require a smoothness in \( \theta \). Then we have to choose certain smooth branches of the functions \( a_{\sigma}^\pm(\theta) \) and \( \omega_\sigma(\theta) \) and cut off all singularities. First, introduce the critical set as
\[
\mathcal{C} = \mathcal{C}_0 \cup \bigcup_{\sigma = 1}^{s} \left( \mathcal{C}_\sigma \cup \bigcup_{i = 1}^{d} \{ \theta \in T^d \setminus \mathcal{C}_\sigma : \frac{\partial^2 \omega_\sigma(\theta)}{\partial \theta_i^2} = 0 \} \right) \bigcup \{ \theta \in T^d \setminus \mathcal{C} : \nabla \omega_\sigma(\theta) = 0 \},
\]
with \( \mathcal{C}_\sigma \) as in Lemma 1.2 and sets \( \mathcal{C}_0 \) and \( \mathcal{C}_\sigma \) defined by (1.14). Obviously, \( \text{mes} \mathcal{C} = 0 \) (see [8, lemma 7.3]). Secondly, fix an \( \delta > 0 \) and choose a finite partition of unity,
\[
f(\theta) + g(\theta) = 1, \quad g(\theta) = \sum_{k=1}^{K} g_k(\theta), \quad \theta \in T^d,
\]
where \( f, g_k \) are nonnegative functions in \( C_0^\infty(T^d) \), and
\[
\text{supp } f \subset \{ \theta \in T^d : \text{dist}(\theta, \mathcal{C}) < \delta \}, \quad \text{supp } g_k \subset \{ \theta \in T^d : \text{dist}(\theta, \mathcal{C}) \geq \delta/2 \}.
\]
Then we represent \( \mathcal{G}_t(x) \) in the form \( \mathcal{G}_t(x) = \mathcal{G}^f_t(x) + \mathcal{G}^g_t(x) \), where
\[
\mathcal{G}^f_t(x) = (2\pi)^{-d} \int_{T^d} e^{-ix \cdot \theta} f(\theta) \hat{\mathcal{G}}_t(\theta) \, d\theta, \quad \mathcal{G}^g_t(x) = (2\pi)^{-d} \int_{T^d} e^{-ix \cdot \theta} g(\theta) \hat{\mathcal{G}}_t(\theta) \, d\theta = \sum_{\pm, \sigma = 1}^{s} \sum_{k=1}^{K} \int_{T^d} g_k(\theta) e^{-ix \cdot \theta \pm i \omega_\sigma(\theta) t} a_{\sigma}^\pm(\theta) \, d\theta.
\]
By Lemma 1.2 and the compactness arguments, we can choose the supports of \( g_k \) so small that the eigenvalues \( \omega_\sigma(\theta) \) and the amplitudes \( a_{\sigma}^\pm(\theta) \) are real-analytic functions inside the \( \text{supp } g_k \) for every \( k \). (We do not label the functions by the index \( k \) to simplify the notation.) For the function \( \mathcal{G}^f_t(x) \), the Parseval identity, (3.2), and condition \textbf{E6} imply
\[
\|\mathcal{G}^f_t(\cdot)\|_{l^2}^2 = C \int_{T^d} |\hat{\mathcal{G}}_t(\theta)|^2 |f(\theta)|^2 \, d\theta \leq C \int_{\text{dist}(\theta, \mathcal{C}) < \delta} |\hat{\mathcal{G}}_t(\theta)|^2 \, d\theta \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0,
\]
uniformly in \( t \in \mathbb{R} \). For the function \( \mathcal{G}^g_t(x) \) the following lemma holds.

**Lemma 3.4** (see [9, Lemma 4.5]) Let conditions \textbf{E1}–\textbf{E4} and \textbf{E6} hold. Then the following bounds hold.
\[
(i) \quad \sup_{x \in \mathbb{Z}^d} |\mathcal{G}^g_t(x)| \leq C t^{-d/2}.
\]
\[
(ii) \quad \text{For any} \quad p > 0, \quad \text{there exist} \quad C_p, \gamma_g > 0 \quad \text{such that} \quad |\mathcal{G}^g_t(x)| \leq C_p(|t| + |x| + 1)^{-p} \quad \text{for} \quad |x| \geq \gamma_g t.
\]
3.3 Proof of Theorem 2.10

The representation (1.19) yields

\[ Q_{\varepsilon,t}(z, z') = E_0^\varepsilon(Y(z, t) \otimes Y(z', t)) = \sum_{x,y \in \mathbb{Z}_+^d} G_{t,\delta}(z, x)Q_\varepsilon(x, y)G_{t,\delta}(z', y)^T, \quad z, z' \in \mathbb{Z}_+^d, \quad (3.10) \]

for any \( t \in \mathbb{R}_+^1 \). It follows from condition (1.9) and from formulas (1.17) and (1.18) that \( G_t(z) = G_t(\delta) \) with \( \delta = (-z_1, z_2, \ldots, z_d) \). In this case, by (2.20), the covariance \( Q_{\varepsilon,t}(z, z') \) can be decomposed into the sum of fourth terms,

\[ Q_{\varepsilon,t}(z, z') = S_{\varepsilon,t}(z, z') - S_{\varepsilon,t}(\delta, z') - S_{\varepsilon,t}(z, \delta') + S_{\varepsilon,t}(\delta, \delta'), \quad z, z' \in \mathbb{Z}_+^d, \]

where

\[ S_{\varepsilon,t}(z, z') := \sum_{x,y \in \mathbb{Z}_+^d} G_t(z - x)Q_\varepsilon(x, y)G_t(z' - y)^T. \quad (3.11) \]

Proposition 3.5 Let \( r \in \mathbb{R}^d \) and \( z, z' \in \mathbb{Z}^d \). Then

\[ S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') \to q_{r,\varepsilon}(z - z'), \quad \varepsilon \to +0, \quad (3.12) \]

where \( q_{r,\varepsilon}(z) \) is defined in (2.21)-(2.23).

This proposition implies Theorem 2.10. Indeed, let \( r_1 = 0 \). Then \( \bar{r}/\varepsilon = r/\varepsilon = (0, \bar{r}/\varepsilon) \) and for \( z, z' \in \mathbb{Z}_+^d \),

\[ Q_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') = S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') - S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + \bar{z}, [r/\varepsilon] + z') - S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + \bar{z}') + S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + \bar{z}, [r/\varepsilon] + \bar{z}'). \]

Therefore, convergence (3.12) implies (2.25).

Let \( r_1 > 0 \). In this case, the matrix-valued functions \( S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + \bar{z}, [r/\varepsilon] + z') \) and \( S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + \bar{z}') \) vanish as \( \varepsilon \to +0 \), and

\[ S_{\varepsilon,\tau/\varepsilon}([\bar{r}/\varepsilon] + \bar{z}, [\bar{r}/\varepsilon] + z') \to q_{r,\varepsilon}(\bar{z} - z'), \quad \varepsilon \to +0. \]

It can be proved by similar way as Proposition 3.5.

**Proof of Proposition 3.5.** \( \text{Step (i)} \) Let us denote

\[ Q_\varepsilon(x, y) = \begin{cases} Q_\varepsilon(x, y) & \text{for } x, y \in \mathbb{Z}_+^d, \\ 0 & \text{otherwise} \end{cases} \]

Corollary 3.3 and (3.9) imply that

\[ S_{\varepsilon,t}(z, z') = \sum_{x,y \in \mathbb{Z}_+^d} G_t^\varepsilon(z - x)Q_\varepsilon(x, y)G_t^\varepsilon(z' - y)^T + o(1), \]

where \( o(1) \to 0 \) as \( \delta \to 0 \) uniformly in \( t \in \mathbb{R} \) and \( z, z' \in \mathbb{Z}_+^d \). In particular, setting \( t = \tau/\varepsilon \), \( z = [r/\varepsilon] + l \) and \( z' = [r/\varepsilon] + p \) we get

\[ S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{x,y \in \mathbb{Z}_+^d} G_t^\varepsilon([r/\varepsilon] + l - x)Q_\varepsilon(x, y)G_t^\varepsilon(p + [r/\varepsilon] - y)^T + o(1) \]

\[ = \sum_{x,y \in \mathbb{Z}_+^d} G_t^\varepsilon(l + x)Q_\varepsilon([r/\varepsilon] - x, [r/\varepsilon] - y)G_t^\varepsilon(p + y)^T + o(1). \]
Let \( c = \gamma + \max(|l|, |p|) \). Then Lemma 3.4 (ii) and condition V2 imply that

\[
S_{\varepsilon, \tau / \varepsilon}([r / \varepsilon] + l, [r / \varepsilon] + p) = \sum_{x, y \in [-c\tau / \varepsilon, c\tau / \varepsilon]^d} G_{r / \varepsilon}^d (l + x) \bar{Q}_\varepsilon([r / \varepsilon] - x, [r / \varepsilon] - y) G_{r / \varepsilon}^d (p + y)^T + r_1(\varepsilon, \tau) + o(1), \quad \varepsilon \to 0,
\]

where \( \lim_{\varepsilon \to 0} \varepsilon^{-p} r_1(\varepsilon, \tau) = 0 \) for any \( p > 0 \) and \( \tau \in \mathbb{R}^1 \).

**Step (ii)** We divide the cube \([-c\tau / \varepsilon, c\tau / \varepsilon]^d\) onto the cubes \( I_{\varepsilon N_\varepsilon} \) (see (2.20)),

\[
[-c\tau / \varepsilon, c\tau / \varepsilon]^d \subset \bigcup_{n \in J} I_{n \varepsilon N_\varepsilon},
\]

where \( J = \{ n = (n_1, \ldots, n_d) \in \mathbb{Z}^d, |n_j| \leq [c\tau / (\varepsilon N_\varepsilon)] + 1 \} \). Therefore,

\[
S_{\varepsilon, \tau / \varepsilon}([r / \varepsilon] + l, [r / \varepsilon] + p) = \sum_{m, n \in J} \sum_{x \in n \varepsilon N_\varepsilon, y \in I_{n \varepsilon N_\varepsilon}} G_{r / \varepsilon}^d (l + x) \bar{Q}_\varepsilon([r / \varepsilon] - x, [r / \varepsilon] - y) G_{r / \varepsilon}^d (p + y)^T + o(1).
\]

Now we prove that the contribution of the sums over pairs \( m, n \in J \) with \( m \neq n \) vanishes as \( \varepsilon \to 0 \). Let us denote

\[
r_2(\varepsilon, \tau) = \sum_{m, n \in J, m \neq n} \sum_{x \in n \varepsilon N_\varepsilon, y \in I_{n \varepsilon N_\varepsilon}} G_{r / \varepsilon}^d (l + x) \bar{Q}_\varepsilon([r / \varepsilon] - x, [r / \varepsilon] - y) G_{r / \varepsilon}^d (p + y)^T \quad (3.13)
\]

and prove that

\[
r_2(\varepsilon, \tau) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (3.14)
\]

for any \( \tau \in \mathbb{R}^1 \). We divide the sum in the RHS of (3.13) onto two sums \( S_1 \) and \( S_2 \), where the first sum \( S_1 \) is taken over all \( x \in I_{m N_\varepsilon} \), \( y \in I_{n N_\varepsilon} \) and \( m, n \in J \) such that \( \exists j \in \{ 1, \ldots, d \} : |m_j - n_j| \geq 2 \); the sum \( S_2 \) is taken over all \( x \in I_{n N_\varepsilon}, y \in I_{n N_\varepsilon} \) and \( m, n \in J \) such that \( m \neq n \) and \( \forall j = 1, \ldots, d : |m_j - n_j| \leq 1 \). The number of points \( m \in J \) is order of \( (\tau / (\varepsilon N_\varepsilon))^d \), the number of points \( x \in I_{m N_\varepsilon} \) is \( \sim N_\varepsilon^d \). Therefore, by Lemma 3.4 (i) and condition V2, the sum \( S_1 \) is estimated by

\[
C(1 + \tau / \varepsilon)^{-d}(\tau / \varepsilon)^d \sum_{s \in \mathbb{Z}^d, |s| \geq N_\varepsilon} (1 + |s|)^{-\gamma},
\]

which vanishes as \( \varepsilon \to 0 \), since \( N_\varepsilon \to +\infty \) as \( \varepsilon \to 0 \) and \( \gamma > d \). To estimate the second sum \( S_2 \) (the contribution of nearest neighbors \( I_{m N_\varepsilon} \) and \( I_{n N_\varepsilon} \)) we choose a number \( p > d + 1 \) and divide the sum onto two sums: \( S_2 = S_{21} + S_{22} \), where the sum \( S_{21} \) is taken over all \( m \in J \) and \( x \in I_{m N_\varepsilon} \), \( n \in \{ n \in J : n \neq m, \forall j : |m_j - n_j| \leq 1 \} \) and \( y \in I_{n N_\varepsilon} \) such that \( |x - y| \geq N_\varepsilon^{1/p} \) and the second sum \( S_{22} \) is taken, respectively, over \( y \) such that \( |x - y| \leq N_\varepsilon^{1/p} \). The contribution of “non-boundary zones” \( S_{21} \) is

\[
C(1 + \tau / \varepsilon)^{-d}(\tau / \varepsilon)^d \sum_{s \in \mathbb{Z}^d, |s| \geq N_\varepsilon^{1/p}} (1 + |s|)^{-\gamma}
\]

which vanishes as \( \varepsilon \to 0 \). The contribution of “boundary zones” \( S_{22} \) is order of

\[
C(1 + \tau / \varepsilon)^{-d}(\tau / \varepsilon)^d N_\varepsilon^{1/p + d - 1} N_\varepsilon^{d/p} \sim C N_\varepsilon^{(d+1)/p-1}.
\]

(3.15)
Indeed, the number of points $m \in J$ is order of $(\tau/(\varepsilon N_\varepsilon))^d$, the number of points $\{ n : |m_j-n_j| \leq 1, \forall j, m \neq n \}$ is finite. For fixed $m, n$ the number of points $x \in I_{mN_\varepsilon}$ such that $|x-y| \leq N_\varepsilon^{1/p}$ is order of $N_\varepsilon^{d-1}N_\varepsilon^{1/p}$. For fixed $x$ the number of points $y$ such that $|x-y| \leq N_\varepsilon^{1/p}$ is $\sim N_\varepsilon^{d/p}$. The number $p$ is chosen such that $(d+1)/p - 1 < 0$. Hence, (3.15) vanishes as $\varepsilon \to 0$ by condition V1 (ii). The decay (3.14) is proved.

Therefore,

$$S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{m \in J} \sum_{x,y \in I_{mN_\varepsilon}} \mathcal{G}_{\tau/\varepsilon}(l+x)\bar{Q}_\varepsilon([r/\varepsilon] - x, [r/\varepsilon] - y)\mathcal{G}_{\tau/\varepsilon}(p+y)^T + o(1).$$

Step (iii) Now we can apply condition V1 (i) at the points $[r/\varepsilon] - x, [r/\varepsilon] - y$ of the same cube $I_{[r/\varepsilon]-mN_\varepsilon}$ and obtain

$$|\bar{Q}_\varepsilon([r/\varepsilon] - x, [r/\varepsilon] - y) - \bar{R}(\varepsilon [r/\varepsilon] - \varepsilon mN_\varepsilon, [r/\varepsilon] - x, [r/\varepsilon] - y)| \leq C \min\{(1 + |x-y|)^{-\gamma}, \varepsilon N_\varepsilon\},$$

where, by definition, the function $\bar{R}$ is equal to

$$\bar{R}(r,x,y) = \begin{cases} R(r,x,y) & \text{if } x,y \in \mathbb{Z}_+^d, \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{m \in J} \sum_{x,y \in I_{mN_\varepsilon}} \mathcal{G}_{\tau/\varepsilon}^d(l+x)\bar{R}(\ldots)\mathcal{G}_{\tau/\varepsilon}^d(p+y)^T + o(1),$$

where $\bar{R}(\ldots) \equiv \bar{R}(\varepsilon [r/\varepsilon] - \varepsilon mN_\varepsilon, [r/\varepsilon] - x, [r/\varepsilon] - y)$. Indeed, for fixed $x \in I_{mN_\varepsilon}$,

$$\sum_{y \in I_{mN_\varepsilon}} \min\{(1 + |x-y|)^{-\gamma}, \varepsilon N_\varepsilon\} = \sum_{y : |x-y| \geq \varepsilon N_\varepsilon} \varepsilon N_\varepsilon + \sum_{y : 1 + |x-y| \geq \varepsilon N_\varepsilon} (1 + |x-y|)^{-\gamma}$$

$$= \sum_{s : |x| \leq (\varepsilon N_\varepsilon)^{-1/\gamma}} \varepsilon N_\varepsilon + \sum_{s : |x| \geq (\varepsilon N_\varepsilon)^{-1/\gamma}} (1 + |s|)^{-\gamma} \sim (\varepsilon N_\varepsilon)^{1-d/\gamma}.$$

By Lemma 3.4 (i) and condition V1 (ii), we obtain

$$\sum_{m \in J} \sum_{x,y \in I_{mN_\varepsilon}} |\mathcal{G}_{\tau/\varepsilon}^d(l+x)| \min\{(1 + |x-y|)^{-\gamma}, \varepsilon N_\varepsilon\}|(\mathcal{G}_{\tau/\varepsilon}^d(p+y)^T)|$$

$$\leq C(1 + \tau/\varepsilon)^{-d}(\tau/(\varepsilon N_\varepsilon))^d N_\varepsilon^d (\varepsilon N_\varepsilon)^{1-d/\gamma} \sim \varepsilon^{(1-\beta)(1-d/\gamma)} \to 0, \ \varepsilon \to 0,$$

since $\beta < 1$ and $\gamma > d$.

By the similar arguments as in steps (i) and (ii) of the proof, the sums in the RHS of (3.16) can be taken over $\{ m \in J, x \in I_{mN_\varepsilon}, y \in \mathbb{Z}_+^d \}$, i.e.,

$$S_{\varepsilon,\tau/\varepsilon}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{m \in J} \sum_{x \in I_{mN_\varepsilon}} \sum_{y \in \mathbb{Z}_+^d} \mathcal{G}_{\tau/\varepsilon}^d(l+x)\bar{R}(\ldots)\mathcal{G}_{\tau/\varepsilon}^d(p+y)^T + o(1).$$

Step (iv) Let us split the function $\bar{R}$ into the following three matrix functions:

$$R^+(r,x,y) := \frac{1}{2} R_0(r,x-y),$$

$$R^-(r,x,y) := \frac{1}{2} R_0(r,x-y) \text{sign}(y_1),$$

$$R^0(r,x,y) := \bar{R}(r,x,y) - R^+(r,x,y) - R^-(r,x,y).$$
Next, introduce the matrices

\[ S^a_{\epsilon, \tau/\epsilon} \equiv S^a_{\epsilon, \tau/\epsilon}([r/\epsilon] + l, [r/\epsilon] + p) = \sum_{m \in J} \sum_{x \in \mathbb{I}_{mN_{\epsilon}}} \sum_{y \in \mathbb{Z}^d} \mathcal{G}^g_{\tau/\epsilon}(l + x) R^a(\ldots) \mathcal{G}^g_{\tau/\epsilon}(p + y)^T, \quad (3.20) \]

for each \( a = \{+, -, 0\} \) and split \( S_{\epsilon, \tau/\epsilon} \) into three terms, \( S_{\epsilon, \tau/\epsilon} = S^+_{\epsilon, \tau/\epsilon} + S^-_{\epsilon, \tau/\epsilon} + S^0_{\epsilon, \tau/\epsilon} \). The convergence (3.12) results now from the following three lemmas.

**Lemma 3.6** \( \lim_{\epsilon \to 0} S^+_{\epsilon, \tau/\epsilon}([r/\epsilon] + l, [r/\epsilon] + p) = \frac{1}{2} s^+_{\tau,r}(l - p), \) \( l, p \in \mathbb{Z}^d \), where \( s^+_{\tau,r} \) is defined as in (2.21)–(2.22) but with \( \frac{1}{2}(\hat{R}_0(r + \nabla \omega_\sigma(\theta)\tau, \theta) \pm \hat{R}_0(r - \nabla \omega_\sigma(\theta)\tau, \theta)) \) instead of \( R^\pm_\tau \) (cf. (2.23)).

**Proof.** By (3.17) and (3.20), the function \( S^+_{\epsilon, \tau/\epsilon} \) can be represented as

\[ S^+_{\epsilon, \tau/\epsilon} = \frac{1}{2} \sum_{m \in J} \sum_{x \in \mathbb{I}_{mN_{\epsilon}}} \mathcal{G}^g_{\tau/\epsilon}(l + x) \sum_{y \in \mathbb{Z}^d} R_0(\epsilon[r/\epsilon] - \epsilon m N_{\epsilon}, y - x) \mathcal{G}^g_{\tau/\epsilon}(p + y)^T. \]

Using Fourier transform and the Parseval equality we can rewrite \( S^+_{\epsilon, \tau/\epsilon} \) as

\[ S^+_{\epsilon, \tau/\epsilon} = (2\pi)^{-2d} \frac{1}{2} \sum_{m \in J} \sum_{x \in \mathbb{I}_{mN_{\epsilon}}} \int e^{-i(l \cdot \theta - p \cdot \theta')} e^{i x \cdot (\theta - \theta')} \hat{\mathcal{G}}^g_{\tau/\epsilon}(\theta) \hat{R}_0(\epsilon[r/\epsilon] - \epsilon m N_{\epsilon}, \theta') \times \hat{\mathcal{G}}^g_{\tau/\epsilon}(\theta')^* d\theta' d\theta. \quad (3.21) \]

Therefore, the proof of Lemma 3.6 reduces to the finding the limit value of (3.21), that is done in Theorem 4.1 from [9] (the detailed proof see in Appendix A).

**Lemma 3.7** \( \lim_{\epsilon \to 0} S^-_{\epsilon, \tau/\epsilon}([r/\epsilon] + l, [r/\epsilon] + p) = \frac{1}{2} s^-_{\tau,r}(l - p), \) \( l, p \in \mathbb{Z}^d \), where \( s^-_{\tau,r} \) is defined as in (2.21)–(2.22) but with

\[ \frac{1}{2}(\hat{R}_0(r + \nabla \omega_\sigma(\theta)\tau, \theta) \text{sign}(r_1 + \tau \nabla_1 \omega_\sigma(\theta)) \pm \hat{R}_0(r - \nabla \omega_\sigma(\theta)\tau, \theta) \text{sign}(r_1 - \tau \nabla_1 \omega_\sigma(\theta))) \]

instead of \( R^\pm_\tau \).

**Lemma 3.8** \( \lim_{\epsilon \to 0} S^0_{\epsilon, \tau/\epsilon}([r/\epsilon] + l, [r/\epsilon] + p) = 0, \) \( l, p \in \mathbb{Z}^d \).

The proofs of Lemmas 3.7 and 3.8 see in Appendices B and C, resp.

### 4 Convergence of Wigner matrices

Here we prove Theorem 2.12. Theorem 2.10 implies that for any \( r \in \mathbb{R}^d_+, \tau \neq 0 \) and \( y \in (2\mathbb{Z})^d \), the following convergence holds,

\[ \lim_{\epsilon \to 0} E^\epsilon_{\tau/\epsilon}(a([r/\epsilon] + y/2)^* \otimes a([r/\epsilon] - y/2)) = W^p(\tau; r, y), \quad (4.1) \]
where in the Fourier space one has
\[
\hat{\mathcal{W}}^p(\tau; r, \theta) = \frac{1}{2} \left( \Omega^{1/2} \hat{g}_{r_r}^{G,00}(\theta) \Omega^{1/2} + \Omega^{-1/2} \hat{g}_{r_r}^{G,11}(\theta) \Omega^{-1/2} \right.
\]
\[
+ i \Omega^{1/2} \hat{g}_{r_r}^{G,01}(\theta) \Omega^{-1/2} - i \Omega^{-1/2} \hat{g}_{r_r}^{G,10}(\theta) \Omega^{1/2} \right)
\]
\[
= W^p(\tau; r, \theta),
\]
by formulas (2.26) and (2.21)–(2.23). Therefore, convergence (2.27) follows from (4.1), (4.2) and Lemma 4.1.

**Lemma 4.1** Let conditions \( \text{V2} \) and \( \text{E1–E3}, \text{E6} \) hold and let \( \alpha < -d/2 \). Then the following bound holds:
\[
\sup_{t \in \mathbb{R}} \sup_{z, z' \in \mathbb{Z}^d_+} \|Q_{\varepsilon,t}(z, z')\| \leq C < \infty.
\]

**Proof** The representation (1.19) gives
\[
Q_{\varepsilon,t}^{ij}(z, z') = \mathbb{E}_0^{*} \left( Y^i(z, t) \otimes Y^j(z', t) \right) = \sum_{y, y' \in \mathbb{Z}^d_+} \sum_{k, l = 0, 1} G_{t+}^{0k}(z, y) Q_k^{j}(y, y') G_{t+}^{lj}(z', y)
\]
\[
= \langle Q_y(y, y'), \Phi^{i}_z(y, t) \otimes \Phi^{j}_{z'}(y', t) \rangle_{+},
\]
where \( \Phi^{i}_z(y, t) \) is given by
\[
\Phi^{i}_z(y, t) = \left( G_{t+}^{00}(z, y), G_{t+}^{11}(z, y) \right)
\]
\[
= \left( G_{t}^{00}(z - y) - G_{t}^{00}(z - y), G_{t}^{11}(z - y) - G_{t}^{11}(z - y) \right), \quad i = 0, 1.
\]
The Parseval identity, formula (3.2), and condition \( \text{E6} \) imply that
\[
\|\Phi^{i}_z(\cdot, t)\|_{l_2}^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |\hat{\Phi}^{i}_z(\theta, t)|^2 d\theta \leq C \int_{\mathbb{T}^d} \left( |\hat{G}_{t}^{00}(\theta)|^2 + |\hat{G}_{t}^{11}(\theta)|^2 \right) d\theta \leq C_0 < \infty.
\]

Corollary 3.3 gives now
\[
|Q_{\varepsilon,t}^{ij}(z, z')| = |\langle Q_y(y, y'), \Phi^{i}_z(y, t) \otimes \Phi^{j}_{z'}(y', t) \rangle_{+}| \leq C \|\Phi^{i}_z(\cdot, t)\|_{l_2} \|\Phi^{j}_{z'}(\cdot, t)\|_{l_2} \leq C_1 < \infty,
\]
where the constant \( C_1 \) does not depend on \( z, z' \in \mathbb{Z}^d_+, t \in \mathbb{R}, \) and \( \varepsilon > 0. \)

**5 Proof of Theorem 2.14**

Theorem 2.14 follows from Propositions 5.1 and 5.2. Proposition 5.1 ensures the existence of the limit measures of the family \( \{\mu^\varepsilon_{r/\varepsilon, r}, \varepsilon > 0\} \), while Proposition 5.2 provides the uniqueness.

**Proposition 5.1** Let conditions \( \text{V2} \) and \( \text{E1–E3}, \text{E6} \) hold. Then for any \( r \in \mathbb{R}^d \) with \( r_1 \geq 0, \tau \neq 0, \) the family of measures \( \{\mu^\varepsilon_{r/\varepsilon, r}, \varepsilon > 0\} \) is weakly compact on \( \mathcal{H}_{\alpha+} \) for any \( \alpha < -d/2, \) and the following bounds hold:
\[
\sup_{\varepsilon \geq 0} \int \|Y_0\|_{\alpha}^2 \mu^\varepsilon_{r/\varepsilon, r}(dY_0) < \infty.
\]
We derive (5.5) by using the explicit representation (1.19) of the solution \( \Psi_0 \) below. In this case, (5.5) follows from the central limit theorem under a Lindeberg-type theorem \([16, \text{Lemma II.3.1}]\) by using a method applied in \([16, \text{Theorem XII.5.2}]\) because the embedding \( H_{\alpha,+} \subset H_{\beta,+} \) is compact for \( \alpha > \beta \).

Set \( S_+ = [S(Z^d_+) \times \mathbb{R}^n]^2 \), where \( S(Z^d_+) \) stands for the space of rapidly decreasing real sequences on \( Z^d_+ \). Write \( \langle Y, \Psi \rangle_+ = \langle Y^0, \Psi^0 \rangle_+ + \langle Y^1, \Psi^1 \rangle_+ \) for \( Y = (Y^0, Y^1) \in H_{\alpha,+} \) and \( \Psi = (\Psi^0, \Psi^1) \in S_+ \), where \( \langle Y^i, \Psi^j \rangle_+ = \sum_{z \in Z^d_+} Y^i(z) \cdot \Psi^j(z) \), \( i, 0, 1 \).

**Proposition 5.2** Let conditions V1–V4 and E1–E6 hold. Then for any \( r \in \mathbb{R}^d \) with \( r_1 \geq 0 \), \( \tau \neq 0 \) and \( \Psi \in S_+ \), the characteristic functionals converge to a Gaussian one,

\[
\hat{\mu}^\varepsilon_{\tau, r}(\Psi) := \int e^{i\langle Y, \Psi \rangle_+} \mu^\varepsilon_{\tau, r}(dY) \to \exp \left\{ -\frac{1}{2} Q_{\tau, r}(\Psi, \Psi) \right\} =: \hat{\mu}_G^\varepsilon(\Psi) \text{ as } \varepsilon \to 0,
\]

where \( Q_{\tau, r} \) is the quadratic form with the matrix kernel \( (Q_{\tau, r}^G(x, y))_{i,j=0,1} \),

\[
Q_{\tau, r}(\Psi, \Psi) = \sum_{i,j=0,1} \sum_{z, z' \in Z^d_+} (Q_{\tau, r}^{G, ij}(z, z'), \Psi^i(z) \otimes \Psi^j(z')), \quad \Psi \in S_+.
\]

To prove Theorem 2.14 it remains to check Proposition 5.2. Let us rewrite (5.3) as

\[
\hat{\mu}^\varepsilon_{\tau, r}(\Psi) = E_0(\exp\{i\langle T_{-\tau/\varepsilon} U_+ (\tau/\varepsilon) Y_0, \Psi \rangle_+\}) \to \hat{\mu}_G^\varepsilon(\Psi), \quad \varepsilon \to 0.
\]

We derive (5.5) by using the explicit representation (1.19) of the solution \( Y(t) \), the Bernstein ‘room–corridor’ technique and the approach of \([9, 10]\). The approach gives a representation of \( \langle T_{-\tau/\varepsilon} U_+ (\tau/\varepsilon) Y_0, \Psi \rangle_+ \) as a sum of weakly dependent random variables (see formula (5.12) below). In this case, (5.5) follows from the central limit theorem under a Lindeberg-type condition.

### 5.1 Duality arguments

In this section, we evaluate the inner product \( \langle T_{-\tau/\varepsilon} U_+ (\tau/\varepsilon) Y_0, \Psi \rangle_+ \). Introduce the function \( \Psi_*(z) \) as \( \Psi_*(z) = \Psi(z) \) for \( z_1 \geq 0 \), and \( \Psi_*(z) = 0 \) otherwise. Therefore,

\[
\langle T_{-\tau/\varepsilon} U_+ (\tau/\varepsilon) Y_0, \Psi \rangle_+ = \langle Y_0(z'), \Phi_r(z', \tau/\varepsilon) \rangle_+,
\]

where, by definition, the function \( \Phi_r(z', \tau/\varepsilon) \) is equal to

\[
\Phi_r(z', \tau/\varepsilon) = \sum_{z \in Z^d_+} G^T_{r/\varepsilon,+}(z, z') T_{r/\varepsilon} \Psi_*(z)
\]

\[
= (2\pi)^{-d} \int_{\mathbb{T}^d} (e^{-iz' \cdot \theta} - e^{-iz \cdot \theta}) \hat{G}^T_t(\theta) e^{i[r/\varepsilon] \cdot \theta} \hat{\Psi}_*(\theta) d\theta.
\]
Let us denote
\[ S^0 = \{ \Psi \in S = [S(Z^d) \otimes R^n]^2 : \Psi(\theta) = 0 \text{ in a neighborhood of } C \}, \]  
where \( C \) is defined in (3.4). Since \( \text{mes } C = 0 \) it suffices to prove (5.5) for \( \Psi_* \in S^0 \) only. For the function \( \Phi_r(z, \tau/\varepsilon) \), the following lemma holds.

**Lemma 5.3** (cf. Lemma 6.3 from [10], Lemma 5.2 from [9]). Let conditions \( \text{E1–E4} \) and \( \text{E6} \) hold. Then, for any chosen \( \Psi_* \in S^0 \), the following bounds hold.

(i) \( \sup_{z \in Z^d} |\Phi_r(z, \tau/\varepsilon)| \leq C \varepsilon^{d/2} \).

(ii) For any \( p > 0 \), there exist \( C_p > 0 \) and \( \gamma = \gamma(\tau, r) > 0 \) such that

\[ |\Phi_r(z, \tau/\varepsilon)| \leq C_p (1 + |z| + \tau/\varepsilon)^{-p}, \quad |z| \geq \gamma \tau/\varepsilon. \]  

This lemma follows from (5.7), (5.8), (3.2), and the standard stationary phase method.

### 5.2 Bernstein’s room-corridor’ partition

Write \( t = \tau/\varepsilon \). Let us introduce a ‘room–corridor’ partition of the half-ball \( \{ z \in Z^d_+ : |z| \leq \gamma t \} \) with \( \gamma \) in (5.9). For \( t > 0 \), we choose \( \Delta_t \) and \( \rho_t \in N \). Choose a \( \delta \), \( 0 < \delta < 1 \), and

\[ \rho_t \sim t^{1-\delta}, \quad \Delta_t \sim \frac{t}{\log t}, \quad t \to \infty. \]  

Write \( h_t = \Delta_t + \rho_t \) and \( a^j = j h_t, b^j = a^j + \Delta_t, j = 0, 1, 2, \ldots, n_t = [(\gamma t)/h_t] \). We refer to the slabs \( R^j_t = \{ z \in Z^d_+ : |z| \leq n_t h_t, a^j \leq z < b^j \} \) as the ‘rooms’, to \( C^j_t = \{ z \in Z^d_+ : |z| \leq n_t h_t, b^j \leq z < a^{j+1} \} \) as the ‘corridors’, and to \( L_t = \{ z \in Z^d_+ : |z| > n_t h_t \} \) as the ‘tail’. Here \( z = (z_1, \ldots, z_d) \), \( \Delta_t \) stands for the width of the room, and \( \rho_t \) for that of the corridor. Denote by \( \chi^j_t \) the indicator of the room \( R^j_t \), by \( \xi^j_t \) that of the corridor \( C^j_t \), and by \( \eta_t \) that of the tail \( L_t \). In this case,

\[ \sum_j [\chi^j_t(z) + \xi^j_t(z)] + \eta_t(z) = 1, \quad z \in Z^d_+, \]

where the symbol \( \sum_j \) stands for the sum \( \sum_{j=0}^{n_t-1} \). Hence, we obtain the following Bernstein-type representation:

\[ \langle Y_0, \Phi_r(\cdot, t) \rangle_+ = \sum_j [\langle Y_0, \chi^j_t \Phi_r(\cdot, t) \rangle_+ + \langle Y_0, \xi^j_t \Phi_r(\cdot, t) \rangle_+] + \langle Y_0, \eta_t \Phi_r(\cdot, t) \rangle_+. \]  

Introduce the random variables \( r^j_t, c^j_t, l_t \) by

\[ r^j_t = \langle Y_0, \chi^j_t \Phi_r(\cdot, t) \rangle_+, \quad c^j_t = \langle Y_0, \xi^j_t \Phi_r(\cdot, t) \rangle_+, \quad l_t = \langle Y_0, \eta_t \Phi_r(\cdot, t) \rangle_. \]

Therefore, it follows from (5.6) and (5.11) that

\[ \langle T_{[r/\varepsilon]} U_+(t) Y_0, \Psi \rangle_+ = \langle Y_0, \Phi_r(\cdot, t) \rangle_+ = \sum_j (r^j_t + c^j_t) + l_t. \]
Lemma 5.4 Let conditions V1 and V2 hold and \( \Psi_* \in \mathcal{S}^0 \). The following bounds hold for \( t > 1 \):
\[
E_0^\varepsilon |r_i|^2 \leq C(\Psi) \Delta_i/t, \quad \forall j, \quad \tag{5.13}
\]
\[
E_0^\varepsilon |c_i|^2 \leq C(\Psi) \rho_i/t, \quad \forall j, \quad \tag{5.14}
\]
\[
E_0^\varepsilon |l_i|^2 \leq C_p(\Psi) t^{-p}, \quad \forall p > 0. \tag{5.15}
\]

The proof is based on Lemmas 3.2 and 5.3 (see also [9, Lemma 7.1]).

Further, to prove (5.5), we use a version of the central limit theorem developed by Ibragimov and Linnik. If \( Q_{\tau,r}(\Psi, \Psi) = 0 \), then the convergence (5.3) is obvious. Indeed, then,
\[
\left| E_0^\varepsilon \left( \exp \{ i Y_0, \Phi_r(\cdot, \tau/\varepsilon) \} \right) - \mu_{\tau,r}^G(\Psi) \right| = E_0^\varepsilon \left( \exp \{ i Y_0, \Phi_r(\cdot, \tau/\varepsilon) \} \right) - 1
\]
\[
\leq E_0^\varepsilon \left( \left\{ Y_0, \Phi_r(\cdot, \tau/\varepsilon) \right\} \right) \leq \left( E_0^\varepsilon \left( \left\{ Y_0, \Phi_r(\cdot, \tau/\varepsilon) \right\} \right)^2 \right)^{1/2}
\]
\[
= \left( Q_\varepsilon(x, y), \Phi_r(x, \tau/\varepsilon) \otimes \Phi_r(y, \tau/\varepsilon) \right) \right)^{1/2} = (Q_{\varepsilon,\tau/\varepsilon,r}(\Psi, \Psi))^{1/2}, \tag{5.16}
\]
where \( Q_{\varepsilon,\tau/\varepsilon,r}(\Psi, \Psi) \to Q_{\tau,r}(\Psi, \Psi) = 0, \varepsilon \to 0 \). Therefore, (5.3) follows from Theorem 2.10.

Thus, we may assume that, for a given \( \Psi_* \in \mathcal{S}^0 \),
\[
Q_{\tau,r}(\Psi, \Psi) \neq 0. \tag{5.17}
\]

Lemma 5.5 The following limit holds,
\[
n_t \left[ \left( \frac{\rho_t}{t} \right)^{1/2} + (1 + \rho_t)^{-\gamma} \right] + n_t^2 \frac{\rho_t}{t} \to 0, \quad t \to \infty. \tag{5.18}
\]

Indeed, (5.10) implies that \( h_t = \rho_t + \Delta_t \sim \frac{t}{\log t}, t \to \infty \). Therefore, \( n_t \sim \frac{t}{h_t} \sim \log t \). Then (5.18) follows by (5.10). \( \blacksquare \)

For simplicity, we put \( t = \tau/\varepsilon \). By the triangle inequality,
\[
\left| E_0^\varepsilon \left( \exp \{ i Y_0, \Phi_r(\cdot, t) \} \right) - \mu_{\tau,r}^G(\Psi) \right| \leq \left| E_0^\varepsilon \left( \exp \{ i Y_0, \Phi_r(\cdot, t) \} \right) - E_0^\varepsilon \left( \exp \{ i \sum_j r^j \} \right) \right|
\]
\[
+ \left| \exp \{ - \frac{1}{2} \sum_j E_0^\varepsilon(|r^j|^2) \} - \exp \{ - \frac{1}{2} Q_{\tau,r}(\Psi, \Psi) \} \right|
\]
\[
+ \left| E_0^\varepsilon \left( \exp \{ i \sum_j r^j \} \right) - \exp \{ - \frac{1}{2} \sum_j E_0^\varepsilon(|r^j|^2) \} \right|
\]
\[
= I_1 + I_2 + I_3. \tag{5.19}
\]

We are going to show that all summands \( I_1, I_2, I_3 \) tend to zero as \( t \to \infty \).

Step (i) Eqn (5.12) implies
\[
I_1 = \left| E_0^\varepsilon \left( \exp \{ i \sum_j r^j \} \left( \exp \{ i \sum_j c^j + il^j \} - 1 \right) \right) \right|
\[
\leq \sum_j E_0^\varepsilon(|c^j|^2) + E_0^\varepsilon(|l^j|) \leq \sum_j \left( E_0^\varepsilon(|c^j|^2) \right)^{1/2} + \left( E_0^\varepsilon(|l^j|)^2 \right)^{1/2}. \tag{5.20}
\]

From (5.20), (5.14), (5.15) and (5.18) we obtain that
\[
I_1 \leq C n_t (\rho_t/t)^{1/2} + C p t^{-p} \to 0, \quad t \to \infty.
\]
Step (ii) By the triangle inequality,

\[
I_2 \leq \frac{1}{2} \left| \sum_j E_0^\varepsilon(|r^\varepsilon_j|^2) - Q_{\tau,r}(\Psi, \Psi) \right| \leq \frac{1}{2} \left| \mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) - Q_{\tau,r}(\Psi, \Psi) \right| \\
+ \frac{1}{2} \left| E_0^\varepsilon((\sum_j r^\varepsilon_j)^2) - \sum_j E_0^\varepsilon(|r^\varepsilon_j|^2) \right| + \frac{1}{2} \left| E_0^\varepsilon((\sum_j r^\varepsilon_j)^2) - Q_{\varepsilon,t,r}(\Psi, \Psi) \right| \\
= I_{21} + I_{22} + I_{23},
\]

(5.21)

where \( Q_{\varepsilon,t,r} \) is the quadratic form with the matrix kernel \( Q_{\varepsilon,t,r}^\varepsilon(x, y) \). Theorem 2.12 implies that \( I_{21}|_{t=\tau/\varepsilon} \to 0 \) as \( \varepsilon \to 0 \). As for \( I_{22} \), we first obtain that

\[
I_{22} \leq \sum_{j<l} \left| E_0^\varepsilon(r^\varepsilon_j r^\varepsilon_l) \right|.
\]

(5.22)

The distance between the different rooms \( R^\varepsilon_j \) is greater or equal to \( \rho_t \). Then, by Lemma 5.3 (i) and condition \( \text{V2} \),

\[
I_{22} \leq \sum_{j<l} \left| \langle \mathcal{Q}_\varepsilon(x, y), \chi^\varepsilon_j \Phi_r(x, t) \otimes \chi^\varepsilon_l \Phi_r(y, t) \rangle \right| \\
\leq Ct^{-d} \sum_{j<l} \sum_x \chi^\varepsilon_j(x) \sum_y \chi^\varepsilon_l(y) (1 + |x - y|)^{-\gamma} \\
\sim t^{-d} n^2 \Delta t \int_{\rho_t}^{\infty} (1 + s)^{-\gamma} s^{d-1} ds \sim n_t (1 + \rho_t)^{-\gamma+d},
\]

(5.23)

which vanishes as \( t \to \infty \) because of (5.18) and \( \gamma > d \). Finally, it remains to check that \( I_{23} \to 0 \), \( t \to \infty \). We have

\[
\mathcal{Q}_{\varepsilon,t,r}(\Psi, \Psi) = E_0^\varepsilon(|Y_0, \Phi_r(\cdot, t)|^2) = E_0^\varepsilon\left( \sum_j (r^\varepsilon_j + c^\varepsilon_j + l^\varepsilon_j)^2 \right),
\]

according to (5.12). Therefore, by the Cauchy-Schwarz inequality,

\[
I_{23} \leq \left| E_0^\varepsilon((\sum_j r^\varepsilon_j)^2) - E_0^\varepsilon((\sum_j r^\varepsilon_j + \sum_j c^\varepsilon_j + l^\varepsilon_j)^2) \right| \\
\leq Cn_t \sum_j E_0^\varepsilon(|c^\varepsilon_j|^2) + C_1 \left( E_0^\varepsilon((\sum_j r^\varepsilon_j)^2) \right)^{1/2} \left( n_t \sum_j E_0^\varepsilon(|c^\varepsilon_j|^2) + E_0^\varepsilon(|l^\varepsilon_j|^2) \right)^{1/2} \\
+ C E_0^\varepsilon(|l^\varepsilon_j|^2).
\]

(5.24)

Then (5.13), (5.22) and (5.23) imply

\[
E_0^\varepsilon((\sum_j r^\varepsilon_j)^2) \leq \sum_j E_0^\varepsilon(|r^\varepsilon_j|^2) + 2\sum_{j<l} \left| E_0^\varepsilon(r^\varepsilon_j r^\varepsilon_l) \right| \\
\leq Cn_t \Delta t / t + C_1 n_t (1 + \rho_t)^{-\gamma+d} \leq C_2 < \infty.
\]

Now (5.14), (5.15), (5.24), and (5.18) yield

\[
I_{23} \leq C_1 n^2 \rho_t / t + C_2 n_t (\rho_t / t)^{1/2} + C_3 t^{-p} \to 0, \quad t \to \infty.
\]

So, the terms \( I_{21}, I_{22}, I_{23} \) in (5.21) tend to zero. Then (5.21) implies that for \( t = \tau / \varepsilon \)

\[
I_2 \leq \frac{1}{2} \left| \sum_j E_0^\varepsilon(|r^\varepsilon_j|^2) - Q_{\tau,r}(\Psi, \Psi) \right| \to 0, \quad \varepsilon \to 0.
\]

(5.25)
Step (iii) It remains to verify that for \( t = \tau/\varepsilon \)

\[
I_3 = \left| E_0^{\varepsilon} \left( \exp \{ i \sum_j r_j^i \} \right) - \exp \left\{ - \frac{1}{2} \sum_j E_0^{\varepsilon}(|r_j^i|^2) \right\} \right| \to 0, \quad \varepsilon \to 0.
\]

Condition \textbf{V3} yields

\[
\left| E_0^{\varepsilon} \left( \exp \{ i \sum_j r_j^i \} \right) - \prod_{-n_t}^{n_t-1} E_0^{\varepsilon} \left( \exp \{ ir_t^i \} \right) \right|
\leq \left| E_0^{\varepsilon} \left( \exp \{ ir_t^{-n_t} \} \exp \{ i \sum_{-n_t+1}^{n_t-1} r_j^i \} \right) - E_0^{\varepsilon} \left( \exp \{ ir_t^{-n_t} \} \right) E_0^{\varepsilon} \left( \exp \{ i \sum_{-n_t+1}^{n_t-1} r_j^i \} \right) \right|
\]

\[
+ \left| E_0^{\varepsilon} \left( \exp \{ ir_t^{-n_t} \} \right) E_0^{\varepsilon} \left( \exp \{ i \sum_{-n_t+1}^{n_t-1} r_j^i \} \right) - \prod_{-n_t}^{n_t-1} E_0^{\varepsilon} \left( \exp \{ ir_t^i \} \right) \right|
\]

\[
\leq C(1 + \rho_t)^{-2\gamma} + \left| E_0^{\varepsilon} \left( \exp \{ i \sum_{-n_t+1}^{n_t-1} r_j^i \} \right) - \prod_{-n_t}^{n_t-1} E_0^{\varepsilon} \left( \exp \{ ir_t^i \} \right) \right|.
\]

We then apply condition \textbf{V3} recursively and obtain, according to Lemma 5.5,

\[
\left| E_0^{\varepsilon} \left( \exp \{ i \sum_j r_j^i \} \right) - \prod_{-n_t}^{n_t-1} E_0^{\varepsilon} \left( \exp \{ ir_t^i \} \right) \right|_{t=\tau/\varepsilon} \leq C n_{\tau/\varepsilon} (1 + \rho_{\tau/\varepsilon})^{-2\gamma} \to 0, \quad \varepsilon \to 0.
\]

Hence, it remains to show that for \( t = \tau/\varepsilon \)

\[
\left| \prod_{0}^{n_t-1} E_0^{\varepsilon} \exp \{ \text{ir}_t^j \} - \exp \left\{ - \frac{1}{2} \sum_j E_0^{\varepsilon}(|r_j^i|^2) \right\} \right| \to 0, \quad \varepsilon \to 0.
\]

According to the standard statement of the central limit theorem (see, e.g. [12, Theorem 4.7]), it suffices to verify the Lindeberg condition:

\[
\forall \delta > 0, \quad \frac{1}{\sigma_t} \sum_j E_0^{\varepsilon, \delta \sqrt{\sigma_t}} |r_j^i|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Here \( \sigma_t \equiv \sum_j E_0^{\varepsilon}|r_j^i|^2 \), and \( E_0^{\varepsilon, a} f \equiv E_0^{\varepsilon}(X_a f) \), where \( X_a \) is the indicator of the event \( |f| > a^2 \). Note that (2.6) and (5.17) imply that \( \sigma_{\tau/\varepsilon} \to (\Psi, \Psi) \neq 0 \) as \( \varepsilon \to 0 \). Hence, it remains to verify the limit relation

\[
\sum_j E_0^{\varepsilon, a} |r_{\tau/\varepsilon}^j|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for any} \quad a > 0.
\]

This condition can be proved by using the technique of [9].

6 Appendix A: Proof of Lemma 3.6

Step (i). Let us study the sum in (3.21) over \( I_{mN_\varepsilon} \equiv \{ x \in \mathbb{Z}^d : (m_j - 1/2)N_\varepsilon \leq x_j < (m_j + 1/2)N_\varepsilon, \ j = 1, \ldots d \} \):

\[
\sum_{x \in I_{mN_\varepsilon}} e^{ix \cdot (\theta' - \theta)} = \prod_{j=1}^{d} \frac{F(\theta'_j - \theta_j, N_\varepsilon, m_j)}{e^{(\theta'_j - \theta_j)} - 1}.
\]
where \( F(\theta_j, N_\varepsilon, m_j) := \exp\{i\theta_j N_\varepsilon (m_j + 1/2)\} - \exp\{i\theta_j N_\varepsilon (m_j - 1/2)\} \). Define the function \( \alpha(z) : \alpha(z) = \frac{iz}{e^{iz} - 1} \) if \( z \in (-\pi, \pi) \setminus 0 \) and \( \alpha(0) = 1 \). Changing variables in (3.21): \((\theta, \theta') \rightarrow (\theta - z, \theta)\), we obtain

\[
S_{\varepsilon, \tau/\varepsilon}^+ = (2\pi)^{-d} \frac{1}{2} \sum_{m \in I_{[-\pi, \pi]}^d} \int e^{-i(\ell \cdot \theta + i\varepsilon \cdot z)} \prod_{j=1}^d \frac{\alpha(z_j) F(z_j, N_\varepsilon, m_j)}{iz_j} \times \hat{G}^g_{\tau/\varepsilon}(\theta - z) \hat{R}_0(\kappa_{r, \varepsilon, m}, \theta) \hat{G}^g_{\tau/\varepsilon}(\theta)^* d\theta dz, \tag{6.1}
\]

where \( \kappa_{r, \varepsilon, m} := \varepsilon[r/\varepsilon] - \varepsilon m N_\varepsilon \). Let \( C(\theta) \) be defined by (2.10) and \( I \) be the identity matrix. Then

\[
\hat{G}^g_{\tau/\varepsilon}(\theta) = g(\theta) \left( \cos \Omega(\theta) t I + \sin \Omega(\theta) t C(\theta) \right),
\]

by (3.2) and (3.8). Let us define

\[
\hat{G}^g_{\tau/\varepsilon}(\theta) = g(\theta) \left( \cos \omega_\sigma(\theta) t I + \sin \omega_\sigma(\theta) t C_\sigma(\theta) \right) \quad \text{with} \quad C_\sigma(\theta) = \begin{pmatrix} 0 & \omega^{-1}_\sigma(\theta) \\ -\omega_\sigma(\theta) & 0 \end{pmatrix}.
\]

Hence, applying the projections \( \Pi_\sigma(\theta) \) from Lemma 1.2, we rewrite the product of matrices in the integrand from (6.1) as (for \( t = \tau/\varepsilon \))

\[
\hat{G}^g_{\tau/\varepsilon}(\theta - z) \hat{R}_0(\kappa_{r, \varepsilon, m}, \theta) \hat{G}^g_{\tau/\varepsilon}(\theta)^* = \sum_{\sigma, \sigma' = 1}^s \Pi_\sigma(\theta - z) \hat{G}^g_{\tau/\varepsilon}(\theta - z) \hat{R}_0(\kappa_{r, \varepsilon, m}, \theta) \hat{G}^g_{\tau/\varepsilon}(\theta)^* \Pi_{\sigma'}(\theta)
\]

\[
= \sum_{\sigma, \sigma' = 1}^s \Pi_\sigma(\theta - z) g(\theta - z) \left( \sum_\pm e^{\pm i\omega_\sigma(\theta - z)t} \frac{I \mp iC_\sigma(\theta - z)}{2} \right) \times \hat{R}_0(\kappa_{r, \varepsilon, m}, \theta) g(\theta) \left( \sum_\pm e^{\pm i\omega_{\sigma'}(\theta)t} \frac{I \mp iC_{\sigma'}(\theta)}{2} \right) \Pi_{\sigma'}(\theta). \tag{6.2}
\]

Step \((ii)\). Let us consider the one of the terms in (6.1) (denote it by \( I^\pm_\varepsilon \)). The proof for the remaining terms is similar.

\[
I^\pm_\varepsilon = (2\pi)^{-d} \frac{1}{2} \sum_{m \in I_{[-\pi, \pi]}^d} \int e^{-i(\ell \cdot \theta + i\varepsilon \cdot z)} g(\theta) \left( \int_{-\pi}^{\pi} e^{izd_\delta} \frac{\alpha(z_d) F(z_d, N_\varepsilon, m_d)}{iz_d} \right) \times \left( \int_{\delta}^{\pi} e^{iz_2} \frac{\alpha(z_2) F(z_2, N_\varepsilon, m_2)}{iz_2} \right) \times \left( \int_{-\pi}^{\pi} e^{iz_1} \frac{\alpha(z_1) F(z_1, N_\varepsilon, m_1)}{iz_1} \right) e^{\pm i\omega_\sigma(\theta - z) \tau/\varepsilon} \times g(\theta - z) \Pi_\sigma(\theta - z) \hat{R}_0(\kappa_{r, \varepsilon, m}, \theta) \Pi_{\sigma'}(\theta) d\varepsilon_1 \ldots d\varepsilon_d \right) d\theta. \tag{6.3}
\]

Let us write

\[
\nu_1 \equiv \nu_1(\theta_1, \theta_2 - z_2, \ldots) = \pm [\nabla_1 \omega_\sigma(\theta_1, \theta_2 - z_2, \ldots) \tau/(\varepsilon N_\varepsilon)], \quad \nu_2 \equiv \nu_2(\theta_1, \theta_2, \theta_3 \ldots) = \pm [\nabla_2 \omega_\sigma(\theta_1, \theta_2, \theta_3 - z_3, \ldots) \tau/(\varepsilon N_\varepsilon)], \ldots, \quad \nu_d \equiv \nu_d(\theta) = \pm [\nabla_d \omega_\sigma(\theta) \tau/(\varepsilon N_\varepsilon)],
\]

The first step in evaluating the limit value of \( I^\pm_\varepsilon \) is the following assertion.

**Proposition 6.1** Let condition I4 hold. Then

\[
I^\pm_\varepsilon = (2\pi)^{-d} \frac{1}{8} \int_{-\pi}^{\pi} e^{-i(\ell \cdot \theta + i\varepsilon \cdot z)} g(\theta) \left( \sum_{|m_d - \nu_d| \leq 2\pi} \int_{\pi}^{\pi} e^{izd_\delta} \frac{\alpha(z_d) F(z_d, N_\varepsilon, m_d)}{iz_d} \right) \ldots
\]
where \( o_{\tau}(1) \to 0 \) as \( \varepsilon \to 0 \) for any \( \tau \neq 0 \).

**Proof.** We generalize the strategy of the proof of Proposition 3.6 from [3], where this assertion is proved for \( d = 1 \). To prove the asymptotics (6.4), we will show that the series in (6.3) over \( \max_j |m_j - \nu_j| \geq 3 \) vanishes as \( \varepsilon \to 0 \).

Write \( J_0 = \{ n \in \mathbb{Z}^1 : |n| \leq \lfloor \varepsilon N \nu \rfloor + 1 \} \) and \( h_j = m_j - \nu_j \). Note that in integrand from (6.3) the elements of matrix product have of the form

\[
\left( \Pi_\sigma(\theta - z) \tilde{R}_0(\kappa_{r,\varepsilon,m},\theta) \Pi_{\sigma'}(\theta) \right)_{\alpha\gamma} = \sum_{\alpha,\beta,\gamma,\delta}^{d} \Pi_{\alpha\beta}(\theta) \tilde{R}_0(\kappa_{r,\varepsilon,m},\theta) \Pi_{\gamma\delta}(\theta), \quad \alpha, \gamma = 1, \ldots, d.
\]

For simplicity of exposition, we omit the sum over \( \alpha, \beta, \gamma, \delta \) and assume that \( d = 2 \). Let us denote \( \phi_1(\theta) = g(\theta) \Pi_{\alpha\beta}(\theta), \phi_2(\theta) = g(\theta) \Pi_{\gamma\delta}(\theta), \) and \( \mathcal{R}(\nu_1, \nu_2, \theta) = \tilde{R}_0(\kappa_{r,\varepsilon,m},\theta) \). Hence, instead of \( I_\varepsilon' \) we evaluate the following integral:

\[
I_\varepsilon' = C \sum_{h_2 \in h_0 - \nu_2} \left( \sum_{h_1 \in h_0 - \nu_1} \int_{\pi}^{\pi} e^{-i\theta - (l - p)\varepsilon} \alpha_2(\theta) \phi_2(\theta) \left( \int_{-\pi}^{\pi} e^{iz_2(l_2 + N_\varepsilon \nu_2)} \frac{\alpha(z_2) F(z_2, N_\varepsilon, h_2)}{iz_2} \right) d\theta \right) \sum_{h_2 \in h_0 - \nu_2} \left( \sum_{h_1 \in h_0 - \nu_1} \int_{\pi}^{\pi} e^{-iz_2(l_2 + N_\varepsilon \nu_2)} \phi_1(\theta) \mathcal{R}(h_1, h_2, \theta) d\theta \right) d\theta.
\]

Here we use the fact that \( F(z_2, N_\varepsilon, m_j) = e^{iz_2 N_\varepsilon \nu_2} F(z_2, N_\varepsilon, h_2) \). Decompose the series over \( h_1 \) and \( h_2 \) in \( I_\varepsilon' \) into the sums: over \( h_j \leq -3 \), over \( |h_j| \leq 2 \) and \( h_j \geq 3 \), \( j = 1, 2 \). Therefore,

\[
I_\varepsilon' = \sum_{i,j=1}^{3} I_\varepsilon^{(i,j)}, \quad (6.5)
\]

where

\[
I_\varepsilon^{(1,1)} = \sum_{h_1 \in J_0 - \nu_1} \sum_{h_2 \in J_0 - \nu_2} \ldots, \quad I_\varepsilon^{(1,2)} = \sum_{h_1 \in J_0 - \nu_1} \sum_{h_2 \in J_0 - \nu_2} \ldots, \quad I_\varepsilon^{(1,3)} = \sum_{h_1 \in J_0 - \nu_1} \sum_{h_2 \in J_0 - \nu_2} \ldots,
\]

and so on. We want to prove that the series in \( I_\varepsilon' \) over \( \max_j |h_j| \geq 3 \) vanishes as \( \varepsilon \to 0 \), i.e., \( I_\varepsilon^{(i,j)} \) vanish as \( \varepsilon \to 0 \) if \( i = 1, 3 \) or \( j = 1, 3 \). We prove this fact only for \( I_\varepsilon^{(1,1)} \). For remaining integrals the proof is similar. Let us write

\[
h_j^{\min} = \min \{ h_j \in \mathbb{Z}^1 : h_j \leq -3 \text{ and } h_j \in J_0 - \nu_j \}.
\]

Hence,

\[
I_\varepsilon^{(1,1)} = C \int_{-\pi}^{\pi} e^{-i\theta - (l - p)\varepsilon} \phi_2(\theta) \left( \sum_{h_2 \in h_0 - \nu_2} \sum_{h_2 \leq -3 - \pi} \int_{-\pi}^{\pi} e^{iz_2(l_2 + N_\varepsilon \nu_2)} \frac{\alpha(z_2) F(z_2, N_\varepsilon, h_2)}{iz_2} I_1 d\theta \right) d\theta,
\]

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where

\[ I_1 \equiv I_1(\theta, z_2, h_2) = \sum_{h_{\min}^1 \leq h_1 \leq -3} \int_\pi e^{iz_1(l_1 + N_\varepsilon \nu_1)} \frac{\alpha(z_1) F(z_1, N_\varepsilon, h_1)}{iz_1} e^{i\omega_\varepsilon(\theta - z)\tau/\varepsilon} \phi_1(\theta - z) \, dz_1 \mathcal{R}(h_1, h_2, \theta). \] (6.6)

To rewrite the sums over \( h_1 \) and \( h_2 \) we use the following “discrete integration-by-parts formula” (see [3, p.594])

\[ \sum_{h_{\min}^1 \leq h_1 \leq -3} \left[ e^{i(h+1/2)N_{\varepsilon z}} - e^{i(h-1/2)N_{\varepsilon z}} \right] f(h) = \left[ e^{-i\omega/2N_{\varepsilon z}} - e^{i(h_{\min}^1 - 1/2)N_{\varepsilon z}} \right] f(-3) + \sum_{h_{\min}^1 \leq h_1 \leq -4} \left[ e^{i(h+1/2)N_{\varepsilon z}} - e^{i(h_{\min}^1 - 1/2)N_{\varepsilon z}} \right] \left( f(h) - f(h + 1) \right). \] (6.7)

Let us apply (6.7) to the sum over \( h_2 \):

\[ \sum_{h_{\min}^2 \leq h_2 \leq -3} F(z_2, N_\varepsilon, h_2) I_1(\theta, z_2, h_2) = \left[ e^{-i\omega/2N_{\varepsilon z}} - e^{i(h_{\min}^1 - 1/2)N_{\varepsilon z}} \right] I_1(\theta, z_2, -3) + \sum_{h_{\min}^2 \leq h_2 \leq -4} \left[ e^{i(h+1/2)N_{\varepsilon z}} - e^{i(h_{\min}^1 - 1/2)N_{\varepsilon z}} \right] \left( I_1(\theta, z_2, h_2) - I_1(\theta, z_2, h_2 + 1) \right). \] (6.8)

Hence,

\[ |I_1^{(1, 1)}| \leq C_1 \sup_{\theta \in [-\pi, \pi]^2} \sup_{z_2 \in [-\pi, \pi]} \left( |I_1(\theta, z_2, -3)| + \sum_{h_{\min}^2 \leq h_2 \leq -4} |I_1(\theta, z_2, h_2) - I_1(\theta, z_2, h_2 + 1)| \right). \] (6.9)

Applying the formula (6.7) to the sum over \( h_1 \), we rewrite \( I_1 \) in the form

\[ I_1 = C(\theta, z_2, -5/2, h_{\min}^1 - 1/2) \mathcal{R}(-3, h_2, \theta) + \sum_{h_{\min}^1 \leq h_1 \leq -4} C(\theta, z_2, h_1 + 1/2, h_{\min}^1 - 1/2) \left( \mathcal{R}(h_1, h_2, \theta) - \mathcal{R}(h_1 + 1, h_2, \theta) \right), \] (6.10)

where, by definition, the function \( C(\theta, z_2, m, m') \) is equal to

\[ C(\theta, z_2, m, m') = \int_{-\pi}^\pi e^{iz_1(l_1 + \nu_1 N_\varepsilon) + i\omega_\varepsilon (\theta - z)\tau/\varepsilon} \frac{\alpha(z_1)}{iz_1} \left( e^{imN_{\varepsilon z_1}} - e^{im'N_{\varepsilon z_1}} \right) \phi_1(\theta - z) \, dz_1. \] (6.11)

Substituting (6.10) in (6.9) we obtain

\[ |I_1^{(1, 1)}| \leq C_1 \sup_{\theta \in [-\pi, \pi]^2} \sup_{z_2 \in [-\pi, \pi]} \left[ \sup_{h_{\min}^1 \leq h_1 \leq -3} |C(\theta, z_2, h_1 + 1/2, h_{\min}^1 - 1/2)| \right] \times \left( \sum_{h_{\min}^1 \leq h_1 \leq -4} |R(h_1, -3, \theta) - R(h_1 + 1, -3, \theta)| \right) + \sum_{h_{\min}^2 \leq h_2 \leq -4} |R(-3, h_2, \theta) - R(-3, h_2 + 1, \theta)| + \sum_{h_{\min}^1 \leq h_1 \leq -4} \sum_{h_{\min}^2 \leq h_2 \leq -4} \sum_{h_{\min}^1 \leq h_1 \leq -4} |R(h_1, h_2, \theta) - R(h_1 + 1, h_2, \theta)| - R(h_1, h_2 + 1, \theta) + R(h_1 + 1, h_2 + 1, \theta) \right). \]
By condition I4, all sums in (6.12) are bounded uniformly on \( \theta \), since they do not exceed the variations of \( \mathbf{R}_0(\cdot, \theta) \) on the set \([r_1 - 4 - c\tau, r_1 + 4 + c\tau] \times [r_2 - 4 - c\tau, r_2 + 4 + c\tau] \). Hence,

\[
|I^{(1,1)}_{\varepsilon}| \le C_2 \sup_{\theta \in [-\pi, \pi]^2} \sup_{z_2 \in [-\pi, \pi]} \sup_{h^1_{\text{min}} \le h_1 \le -3} |C(\theta, z_2, h_1 + 1/2, h^1_{\text{min}} - 1/2)|. \tag{6.12}
\]

**Lemma 6.2** (see Lemma 3.7 from [3]) Let condition I1 (ii) hold. Then

\[
e^{\pm i\omega (\theta_1, \theta_2 - z_2)\tau/\varepsilon} C(\theta, z_2, m, m') \to \pi(\text{sign } m - \text{sign } m')\phi(\theta_1, \theta_2 - z_2), \quad \varepsilon \to 0, \tag{6.13}
\]

uniformly in \( \theta \in [-\pi, \pi]^2, z_2 \in [-\pi, \pi] \) and \( |m|, |m'| > 2 \).

Since \( h_1 + 1/2 \le \pm 2, h^1_{\text{min}} - 1/2 \le -2 \), the integral \( I^{(1,1)}_{\varepsilon} \) vanishes as \( \varepsilon \to 0 \) by Lemma 6.2 and (6.12). Similarly, the remaining integrals in (6.5) with \( i = 1, 3 \) or \( j = 1, 3 \), vanish as \( \varepsilon \to 0 \), i.e., the series in \( I_{\varepsilon} \) over \( \max_j |h_j| \ge 3 \) vanish as \( \varepsilon \to 0 \). Proposition 6.1 is proved.

**Step (iii)** The next step in the proof is to prove the following asymptotics for the RHS of (6.4) as \( \varepsilon \to 0 \):

\[
I_{\varepsilon}^\pm = \frac{(2\pi)^{-2d}}{8} \int_{[-\pi, \pi]^d} e^{-i\theta \cdot (l_{\varphi} + i\omega(\theta)\tau/\varepsilon)} g(\theta) \Pi_{\sigma}(\theta) \hat{\mathbf{R}}_0(r + \nabla \omega(\theta)\tau, \theta) \Pi'_{\sigma}(\theta)
\]

\[
\left( \int_{-\pi}^{\pi} e^{iz_d(l_{d} + i\nu_d N_{d})} \alpha(z_d) \frac{e^{i5/2N_{d} z_d} - e^{-i5/2N_{d} z_d}}{iz_d} \left( \ldots \left( \int_{-\pi}^{\pi} e^{iz_2(l_2 + i\nu_2 N_2)} \alpha(z_2) \frac{e^{i5/2N_{z_2}} - e^{-i5/2N_{z_2}}}{iz_2} \right) dz_2 \right) \ldots dz_d \right) d\theta + o(1), \tag{6.14}
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \) for any \( \tau \neq 0 \). Formula (6.14) was proved in [3, Lemma 3.8] for the case when \( d = 1 \). This formula is based on the formula \( \sum_{|m_j - \nu_j| \le 2} F(z_j, N_{z_j}, m_j) = e^{i\nu_j N_{z_j}}(e^{i5/2N_{z_j}} - e^{-i5/2N_{z_j}}) \) and the following inequality

\[
|\hat{\mathbf{R}}_0(\varepsilon[\tau/\varepsilon] - \nu \varepsilon N_{\varepsilon} - h \varepsilon N_{\varepsilon}, \theta) - \hat{\mathbf{R}}_0(r + \nabla \omega(\theta)\tau, \theta)| \le C(\varepsilon + N_{\varepsilon}).
\]

This inequality follows because \( |h| \le 2 \) and \( \hat{\mathbf{R}}_0(r, \theta) \) satisfies condition I4. The proof of [3, Lemma 3.8] admits extension to the case when \( d > 1 \).

**Step (iv)** Let us apply Lemma 6.2 to the inner integrals in the RHS of (6.14) over \( z_1, \ldots, z_d \) and obtain

\[
I_{\varepsilon}^\pm = \frac{(2\pi)^{-d}}{8} \int_{T^d} e^{-i\theta \cdot (l_{\varphi} + i\omega(\theta)\tau/\varepsilon)} g(\theta) \Pi_{\sigma}(\theta) \hat{\mathbf{R}}_0(r + \nabla \omega(\theta)\tau, \theta) \Pi'_{\sigma}(\theta) d\theta + o(1), \tag{6.15}
\]

as \( \varepsilon \to 0 \). Note that \( \hat{\mathbf{R}}_{ij}^\pm(r, \cdot) \in C(T^d), \ i, j = 0, 1, \) by condition (2.15). Moreover, the identities \( \omega(\theta) + \omega'(\theta) \equiv \text{const}_+ \) or \( \omega(\theta) - \omega'(\theta) \equiv \text{const}_- \) with the \( \text{const}_\pm \neq 0 \) are impossible by condition E5. Furthermore, the oscillatory integrals with \( \omega(\theta) \pm \omega'(\theta) \neq \text{const}_\pm \) vanish as \( \varepsilon \to 0 \) by the Lebesgue–Riemann theorem. Hence, only the integrals with
Let us write $\kappa$ where

This completes the proof of Lemma 3.6.

7 Appendix B: Proof of Lemma 3.7

We first apply (3.18) and obtain

Let us write $\kappa_{r,e,m} = \varepsilon [r/e] - \varepsilon m N_e$. The Parseval equality yields

Note that

where $\theta' = (\theta'_1, \theta'_2)$, and $\text{PV}$ stands for the Cauchy principal part. Hence,

We change variables: $\theta'_1 \rightarrow \varphi = \theta'_1 - z$, and then denote $z = \theta'_1$. Therefore,

where

$$I_\varepsilon(\theta') := \text{PV} \int_{\mathbb{T}^1} e^{i[r_1/e + p_1]/\varphi} \hat{G}_{r/e}(\theta' + \varphi, \theta')^* g(\theta'_1 + \varphi, \theta') d\varphi, \quad \theta' = (\theta'_1, \theta'_2).$$
By formula (3.2), the matrix \( \hat{G}_t(\theta) \) has the form

\[
\hat{G}_t(\theta) = \cos \Omega(\theta)t I + \sin \Omega(\theta)t C(\theta),
\]

where \( I \) is the identity matrix and \( C(\theta) \) is introduced in (2.10). By Lemma 1.2 (iv),

\[
\hat{G}_t(\theta) = \sum_{\sigma=1}^{s} (\cos \omega_\sigma(\theta)t + \sin \omega_\sigma(\theta)t C_\sigma^*(\theta)) \Pi_\sigma(\theta)
\]

\[
= \sum_{\sigma=1}^{s} e^{\pm i\omega_\sigma(\theta)\tau/\epsilon} \frac{I \mp iC_\sigma^*(\theta)}{2} \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_*,
\]

(7.3)

since \( \cos \omega_\sigma(\theta)t = (e^{i\omega_\sigma t} + e^{-i\omega_\sigma t})/2 \) and \( \sin \omega_\sigma(\theta)t = (e^{i\omega_\sigma t} - e^{-i\omega_\sigma t})/(2i) \). Applying the partition of unity (3.5), (3.6), and formula (7.3) we rewrite \( I_\epsilon(\theta) \) in the form

\[
I_\epsilon(\theta) = \sum_{k, \pm \sigma=1} \text{PV} \int_{\mathbb{T}^d} \frac{g_k(\theta + \varphi, \theta) e^{i(\theta + \varphi)\tau/\epsilon} I \mp iC_\sigma^*(\theta + \varphi, \theta)}{\Pi_\sigma(\theta + \varphi, \theta)} d\varphi.
\]

Lemma 7.1 (i) \( \sup_{\theta \in \mathbb{T}^d, r_1 \in \mathbb{R}} \sup_{\epsilon > 0} |I_\epsilon(\theta)| < \infty \).

(ii) Let \( \nabla_1 \omega_\sigma(\theta) \neq \pm r_1 \), for \( \theta \in \text{supp} g_k \) and for fixed \( r_1 \in \mathbb{R} \). Then

\[
I_\epsilon(\theta) - 2\pi i \sum_{k, \pm \sigma=1} g_k(\theta + \varphi, \theta) e^{\pm i\omega_\sigma(\theta)\tau/\epsilon} f_{r_1}^\pm(\theta) \Pi_\sigma(\theta) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow +0,
\]

(7.4)

for fixed \( \tau > 0 \) and \( r_1 \in \mathbb{R} \). Here \( f_{r_1}^\pm(\theta) \) is a matrix-valued function of the form

\[
f_{r_1}^\pm(\theta) = \text{sign}(r_1 \pm \nabla_1 \omega_\sigma(\theta)\tau)(I \mp iC_\sigma^*(\theta))/2.
\]

Lemma 7.1 can be proved by using the technique of [8, Lemma 8.3] or of [1, Proposition A.4 (i), (ii)]. The proof is based on the following well-known assertion

\[
\lim_{\lambda \to +\infty} \text{PV} \int_{-\pi}^{\pi} \frac{e^{i\lambda f(z)} \chi(z)}{z} dz - \pi i e^{i\lambda f(0)} \chi(0) \text{sign } f'(0) = 0,
\]

where \( \chi \in C^1 \), \( f \in C^2 \) and \( f'(0) \neq 0 \). Lemma 7.1 gives

\[
S_{\epsilon, \tau/\epsilon}^- = (2\pi)^{-2d} \frac{1}{2} \sum_{\sigma=1}^{s} \sum_{m \in J} \sum_{x \in \mathbb{Z}^d} \int_{\mathbb{T}^d} g_k(\theta) e^{-i(\theta - p \cdot \theta')} e^{ix \cdot (\theta' - \theta)} \hat{G}_{r/k}(\theta)
\]

\[
\times \mathcal{R}_0(\kappa, \epsilon, m, \theta') e^{\pm i\omega_\sigma(\theta')\tau/\epsilon} f_{r_1}^\pm(\theta') \Pi_\sigma(\theta') d\theta' d\theta.
\]

(7.5)

Comparing (3.21) and (7.5) we see that the problem of evaluating the limit value of (7.5) is solved by the similar way as in Lemma 3.6.
8 Appendix C: Proof of Lemma 3.8

By (3.20) we write
\[ S_{\varepsilon, \tau/\varepsilon}^0 = \sum_{m \in J} \sum_{x \in I_m N_{\varepsilon}} \sum_{y \in \mathbb{Z}^d} G_{\tau/\varepsilon}^\prime (l + x) R^0 (\kappa_{r, \varepsilon, m}, [r/\varepsilon] - x, [r/\varepsilon] - y) G_{\tau/\varepsilon}^\prime (p + y)^T, \]
where \( \kappa_{r, \varepsilon, m} = \varepsilon [r/\varepsilon] - \varepsilon m N_{\varepsilon} \). Change variables \( y \to z = y - x \) and denote the sum over \( m \) and \( x \) by \( \Phi_\varepsilon (z) \),
\[ \Phi_\varepsilon (z) = \sum_{m \in J} \sum_{x \in I_m N_{\varepsilon}} G_{\tau/\varepsilon}^\prime (l + x) R^0 (\kappa_{r, \varepsilon, m}, [r/\varepsilon] - x, [r/\varepsilon] - x - z) G_{\tau/\varepsilon}^\prime (p + x + z)^T. \]

Therefore,
\[ S_{\varepsilon, \tau/\varepsilon}^0 = \sum_{z \in \mathbb{Z}^d} \Phi_\varepsilon (z). \]

The estimate (2.14) and definition (3.19) imply the same estimate for \( R^0 \):
\[ |R^0 (r, x, y)| \leq C (1 + |x - y|)^{\gamma}. \]

Next, the Cauchy–Schwartz inequality yields
\[ \sum_{m \in J} \sum_{x \in I_m N_{\varepsilon}} \sum_{y \in \mathbb{Z}^d} |G_{\tau/\varepsilon}^\prime (l + x)||G_{\tau/\varepsilon}^\prime (p + x + z)| \leq \sum_{x \in \mathbb{Z}^d} |G_{\tau/\varepsilon}^\prime (l + x)||G_{\tau/\varepsilon}^\prime (p + x + z)| \leq \|G_{\tau/\varepsilon}^\prime\|^2_{L^2 (\mathbb{Z}^d)}. \]

Hence, condition E6 and estimate (8.3) imply that \( |\Phi_\varepsilon (z)| \leq C (1 + |z|)^{\gamma} \). Since \( \gamma > d \),
\[ \sum_{z \in \mathbb{Z}^d} |\Phi_\varepsilon (z)| \leq C < \infty, \]
and the series in (8.2) converges uniformly in \( \varepsilon \) (and also in \( \tau, r, l, p \)). Therefore, it suffices to prove that
\[ \lim_{\varepsilon \to 0} \Phi_\varepsilon (z) = 0 \quad \text{for each} \quad z \in \mathbb{Z}^d. \]

Let us consider the series in (8.1). At first, note that by definitions I0 and (3.19), the function \( R^0 (r, x, y) \) depends on \( \bar{x} - \bar{y} \), i.e., has the form \( R^0 (r, x, y) = R^0 (r, x_1, y_1, \bar{x} - \bar{y}) \), and \( R^0 (r, x_1, y_1, \bar{z}) = 0 \) for \( y_1 < 0 \). Hence,
\[ R^0 (r, [r_1/\varepsilon] - x_1, [r_1/\varepsilon] - x_1 - z_1, \bar{z}) = 0 \quad \text{for} \quad x_1 \geq [r_1/\varepsilon] - z_1. \]

Further, from condition (2.13) it follows that \( \forall \delta > 0 \ \exists K_\delta > 0 \) such that for any \( y_1 > K_\delta \)
\[ |R^0 (r, y_1, y_1 - z_1, \bar{z})| < \delta. \] Hence, \( \forall \delta > 0 \ \exists M_\delta = \max (K_\delta, z_1) > 0 \) such that
\[ \left| \sum_{m \in J} \sum_{x \in I_m N_{\varepsilon}} G_{\tau/\varepsilon}^\prime (l + x) R^0 (\kappa_{r, \varepsilon, m}, [r_1/\varepsilon] - x_1, [r_1/\varepsilon] - x_1 - z_1, \bar{z}) G_{\tau/\varepsilon}^\prime (p + x + z)^T \right| \leq \delta \sum_{m \in J} \sum_{x \in I_m N_{\varepsilon}} \left| G_{\tau/\varepsilon}^\prime (l + x) G_{\tau/\varepsilon}^\prime (p + x + z)^T \right| \leq C \delta, \]
by estimate (8.4). It remains to prove that for fixed $M \delta > 0$,
\[
\sum_{m \in J} \sum_{x \in A_m} G_{\tau/\varepsilon}(l+x) R^0(\kappa_{r,\varepsilon,m}, [r_1/\varepsilon]-x_1, [r_1/\varepsilon]-x_1-z_1, \bar{z}) G^0_{\tau/\varepsilon}(p+x+z)^T \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (8.7)
\]
where $A_m = \{ x = (x_1, \bar{x}) : x_1 \in ([r_1/\varepsilon] - M \delta, [r_1/\varepsilon] - z_1) \cap I_{m \varepsilon N_e} ; \bar{x} \in I_{m \varepsilon N_e} \}$. For enough small an $\varepsilon > 0$, there is a $m^\varepsilon \in Z$ such that $([r_1/\varepsilon] - M \delta, [r_1/\varepsilon] - z_1) \subset I_{m^\varepsilon N_e}$. Hence, $|\varepsilon[r_1/\varepsilon] - \varepsilon m^\varepsilon N_e| \leq C(\varepsilon N_e + \varepsilon)$. Therefore, by condition I4, we have
\[
|R^0(\varepsilon[r_1/\varepsilon] - \varepsilon m^\varepsilon N_e, \ldots) - R^0(0, \ldots)| \leq C \varepsilon^{1-\beta} \quad \text{with some } \beta \in (0, 1).
\]
Hence, by the estimate (8.4), we can replace $R^0(\varepsilon[r_1/\varepsilon] - \varepsilon m^\varepsilon N_e, \ldots)$ into $R^0(0, \ldots)$ in the series (8.7).

Further, let us change $x_1 \rightarrow [r_1/\varepsilon] - x_1$ in (8.7). Therefore, $x_1$ runs the finite number of points, $x_1 \in (z_1, M \delta)$. To derive (8.6) it suffices to prove that for every fixed $z \in Z^d$, $x_1 \in Z$, $r \in R^d$, and $\tau \neq 0$,
\[
\sum_{m \in J} R^0(0, \bar{r} - \varepsilon m N_e, x_1, x_1-z_1, \bar{z}) \beta_{\gamma} \sum_{z \in I_{m N_e}} G^0_{\tau/\varepsilon}(l+x_r)_{\alpha\beta} G^0_{\tau/\varepsilon}(p+x_r+z)^T_{\gamma\delta} \rightarrow 0 \quad (8.8)
\]
as $\varepsilon \rightarrow 0$, where $\alpha, \beta, \gamma, \delta = 1, \ldots, d$, $x_r := ([r_1/\varepsilon] - x_1, \bar{x})$. For simplicity of exposition, we omit indices $i, j, k, l$ in (8.8) and assume that $d = 1$. In this case, to prove (8.8) it suffices to show that
\[
G^0_{\tau/\varepsilon}(l_1 + [r_1/\varepsilon] - x_1) G^0_{\tau/\varepsilon}(p + [r_1/\varepsilon] - x_1 + z)^T \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\]
Indeed, applying the Fourier transform, we have
\[
G^0_{\tau/\varepsilon}(l_1 + [r_1/\varepsilon] - x_1) = (2\pi)^{-1} \int_{T^1} e^{-i(l_1 + [r_1/\varepsilon] - x_1)\theta_1} \hat{G}_{\tau/\varepsilon}(\theta_1) g(\theta_1) d\theta_1. \quad (8.9)
\]
Using the decomposition (3.8) we rewrite (8.9) as
\[
G^0_{\tau/\varepsilon}(l_1 + [r_1/\varepsilon] - x_1) = (2\pi)^{-1} \sum_{k, \pm, \sigma} \int_{T^1} g_k(\theta_1) a_{\pm}(\theta_1) g(\theta_1) e^{-i(l_1 + [r_1/\varepsilon] - x_1)\theta_1 \pm \omega_{\sigma}(\theta_1) \tau/\varepsilon} d\theta_1. \quad (8.10)
\]
The eigenvalues $\omega_{\sigma}(\theta_1)$ and the matrices $a_{\pm}(\theta_1)$ are real-analytic functions inside the supp $g_k$ for every $k$. Moreover, conditions E4 and E6 imply that for fixed $r_1 \in R^1$ and $\tau \neq 0$, mes $\{ \theta_1 \in T^1 : \nabla_1 \omega_{\sigma}(\theta_1) = \pm r_1/\tau \} = 0$. Hence, the integrals in (8.10) vanish as $\varepsilon \rightarrow 0$ by the Lebesgue–Riemann theorem. The proof of convergence (8.8) in the case when $d > 1$ is similar and based on condition I4.

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