A Criterion for Univalent Meromorphic Functions

El Moctar Ould Beiba

D´epartement de Math´ematiques et Informatique
Facult´e des Sciences et Techniques
Universit´e de Nouakchott Al Aasriya
B.P 5026, Nouakchott, Mauritanie

Abstract. Let $D = \{z \in \mathbb{C}, |z| < 1\}$ and $A(p)$ be the set of meromorphic functions in $D$ possessing only simple pole at the point $p$ with $p \in (0, 1)$.

The aim of this paper is to give a criterion by mean of conditions on the parameters $\alpha, \beta \in \mathbb{C}$, $\lambda > 0$ and $g \in A(p)$ for functions in the class denoted $P_{\alpha, \beta}(p; \lambda)$ of functions $f \in A(p)$ satisfying a differential inequality of the form

$$\left| \alpha \left( \frac{z}{f(z)} \right)^{\prime\prime} + \beta \left( \frac{z}{g(z)} \right)^{\prime\prime} \right| \leq \lambda \mu, \quad z \in D$$

to be univalent in the disc $D$, where $\mu = \left( \frac{1-p}{1+p} \right)^2$.

1. Introduction

Let $M$ be the set of meromorphic functions in the region $\Delta = \{\zeta \in \mathbb{C}, |\zeta| > 1\} \cup \{\infty\}$ with the following Laurent development

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}, \quad \zeta \in \Delta. \quad (1.1)$$

Let $\Sigma$ be the subset of $M$ consisting of univalent functions. $A$ is the set of analytic functions $f$ in the unit disc $D$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. The subset of $A$ consisting of univalent functions is denoted by $S$. If $f \in A$, then the function $F$ defined by

$$F(\zeta) = \frac{1}{f(\frac{1}{\zeta})} \quad (1.2)$$

belongs to $M$ and $f$ is univalent in $D$ if and only if $F$ is univalent in $\Delta$. In [1], Aksentév proved that a function $F$ in $M$ is univalent if its derivative $F'$ satisfies the differential inequality:

$$|F'(\zeta) - 1| < 1, \quad \zeta \in \Delta. \quad (1.3)$$

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30D15

Keywords. Meromorphic Functions, Univalent Functions

Received: 01 August 2018; Accepted: 20 September 2018

Communicated by Dragan S. Djordjević

Email address: elbeiba@yahoo.fr (El Moctar Ould Beiba)
If $F$ and $f$ are as in (1.2) then the condition (1.3) is equivalent to
\[
\left| \left( \frac{z}{f(z)} \right) f'(z) - 1 \right| < 1, \quad z \in \mathbb{D}.
\] (1.4)

Hence, by virtue of the Aksent`ve criterion, a criterion for a function $f \in \mathcal{A}$ with $\frac{f'(z)}{z} \neq 0$ for $|z| < 1$ to be univalent is stated as follows:
\[
\left| U_f(z) \right| < 1, \quad z \in \mathbb{D},
\] (1.5)

where $U_f(z) := \left( \frac{z}{f(z)} \right)^2 f'(z) - 1$.

Ozaki and Nunokawa proved in [11], without using the theorem of Aksent`ev, that functions in $\mathcal{A}$ satisfying (1.4) are univalent.

For $\lambda \in (0, 1]$, let $\mathcal{U}(\lambda)$ be the subclass of $\mathcal{U} = \mathcal{U}(1)$ defined by
\[
\mathcal{U}(\lambda) = \{ f \in \mathcal{A}, \left| U_f(z) \right| < \lambda, \ z \in \mathbb{D} \}.
\] (1.6)

The classes $\mathcal{U}(\lambda)$ have been extensively studied by many authors and the results obtained cover a wide range of properties (starlikeness, convexity, coefficients properties, radius properties, etc.). For more details on this subject see [4] - [8] and references therein.

In their article [7], Obradovi´c and Ponnusamy considered the subclass $\mathcal{P}_{\alpha,\beta,\gamma}(\lambda)$ of functions $f$ in $\mathcal{A}$ such that $\frac{f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and satisfying the differential inequality
\[
\left| \frac{\alpha}{f(z)} + \frac{\beta}{g(z)} \right| \leq \lambda, \quad z \in \mathbb{D}
\] (1.7)

where $\alpha \neq 0, \beta$ are given complex numbers and $g$ is a given function in $\mathcal{A}$ with $\frac{g(z)}{z} \neq 0$ in $\mathbb{D}$. One of their main results was the following theorem:

**Theorem 1.1.** Let $g \in \mathcal{A}$ with $\frac{g'(z)}{z} \neq 0$ in $\mathbb{D}$ and $K = \sup_{z \in \mathbb{D}} \left| (\frac{z}{g(z)})^2 g'(z) - 1 \right|$. Then we have
\[
\mathcal{P}_{\alpha,\beta,\gamma}(2 \lambda |\alpha| - 2K |\beta|) \subset \mathcal{U}(\lambda).
\] (1.8)

In particular, we have
\[
\mathcal{P}_{\alpha,\beta,\gamma}(2 |\alpha| - 2K |\beta|) \subset \mathcal{U}(1).
\] (1.9)

Let $p \in (0, 1)$ and $\mathcal{A}(p)$ be the set of meromorphic functions in $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$ and possessing only simple pole at the point $p$. Each function $f$ in $\mathcal{A}(p)$ has a Laurent expansion of the form
\[
f(z) = \frac{m}{z-p} + \frac{m}{p^2 + 1} \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} \setminus \{p\}, \ m \neq 0,
\] (1.10)

where $m$ is the residue of $f$ at $p$ ($m \neq 0$). Our investigations will concern functions in $\mathcal{A}(p)$ satisfying the condition
\[
\left| 1 + \frac{p^2}{m} \right| < 1.
\] (1.11)

In a recent paper [2], Bhowmik and Parveen introduced, for $0 < \lambda \leq 1$, a meromorphic analogue of the class $\mathcal{U}(\lambda)$, namely the class $\mathcal{U}_\mu(\lambda)$ consisting of functions $f$ in $\mathcal{A}(p)$ satisfying
\[
\left| U_f(z) \right| \leq \lambda \mu, \quad z \in \mathbb{D},
\] (1.12)
where

\[ U_f(z) = \left( \frac{z}{f(z)} \right)^2 f'(z) - 1, \quad z \in D \quad \text{and} \quad \mu = \frac{(1 - p)^2}{1 + p} \quad (1.13) \]

They obtained some results for the class \( \mathcal{U}_p(\lambda) \), in particular they proved the following theorem:

**Theorem 1.2.** (Theorem 1, [2]) Let \( f \) be of the form (1.10). If

\[
\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| \leq \left( \frac{1 - p}{1 + p} \right)^2, \quad z \in D
\]

then \( f \) is univalent in \( D \).

Note that Ponnusamy and Wirths have proved by elegant method (Theorem 2, [12]), that functions in \( \mathcal{U}_p(\lambda) \) are univalent on the closure of the disc \( D \).

The main object of the present paper is to give, for the class \( \mathcal{A}(p) \), an analog result to the Theorem 1.1 obtained for the class \( \mathcal{A} \).

2. Main Results

We start by some “round trip” results between the classes \( \mathcal{A}(p) \) and \( \mathcal{A} \).

**Proposition 2.1.** Let \( f(z) = \frac{m}{z - p} + \frac{m + p}{p^2} z + \sum_{n=2}^{\infty} a_n z^n \) be a function in \( \mathcal{A}(p) \) such that \( \frac{f(z)}{z} \neq 0 \) in \( D \) and \(-c\) be an omitted value by \( f \). Let \( g \) be defined by

\[
g(z) = \frac{c f(z)}{c + f(z)} \quad (2.1)
\]

Then \( g \in \mathcal{A}(p) \) and we have

\[
g(p) = c, \quad g'(p) = -\frac{c^2}{m} = -\frac{g^2(p)}{m}, \quad (2.2)
\]

\[
U_f(p) = -1 - \frac{p^2}{m} \quad (2.3)
\]

and

\[
\lim_{z \to p} U_f(z) = U_f(p) = -1 - \frac{p^2}{m}. \quad (2.4)
\]

**Proof.** Since \( f \) is holomorphic in \( D \setminus \{ p \} \), \( g \) is also holomorphic in \( D \setminus \{ p \} \). It is easy to check that \( g(0) = g'(0) - 1 = 0 \).

For the value of \( g(p) \), we have

\[
g(p) = \lim_{z \to p} g(z) = \lim_{z \to p} \frac{c f(z)}{c + f(z)} = \lim_{z \to p} \frac{c (z - p) f(z)}{c(z - p) + (z - p) f(z)} = \frac{c m}{m} = c.
\]

To conclude that \( g \in \mathcal{A}(p) \), we have to prove that \( g'(p) \) exists.

We have, by (2.1, that

\[
\lim_{z \to p} \frac{g(z) - g(p)}{z - p} = \lim_{z \to p} \frac{g(z) - c}{z - p} = \lim_{z \to p} \frac{-c^2}{c(z - p) + (c - p) f(z)} = -\frac{c^2}{m}.
\]
Thus \( g'(p) \) exists and its value gives (2.2). Now, taking (2.2) in the expression of \( U_p \), we get
\[
U_p(p) = -\left(\frac{p^2 c}{m} - 1\right) = -1 - \frac{p^2}{m}.
\]

To prove (2.4), we have by a little calculation
\[
U_f(z) = U_1(z), \quad z \in \mathbb{D} \setminus \{p\}.
\]
(2.5)

Thus we have
\[
\lim_{z \to p} U_f(z) = U_1(p)
\]
which yields, by (2.3), the desired result. \( \square \)

**Remark 2.2.** We obtain from (2.4) that a necessary condition for \( f \) in \( \mathcal{A}(p) \) to be in \( \mathcal{U}_p(\lambda) \) is that
\[
|1 + \frac{p^2}{m}| \leq \lambda \mu,
\]
where \( m \) is the residue of \( f \) at \( p \).

**Proposition 2.3.** Let \( p \in (0, 1) \) and \( g \in \mathcal{A} \) such that \( g'(p) \neq 0 \) and \( g(z) - g(p) \) has no zero in \( \mathbb{D} \setminus \{p\} \). We suppose also that \( g \) satisfies the following condition
\[
|g'(p) - g'(p)p^2| < |g'(p)|.
\]
(2.6)

Then, the function \( f \) defined by
\[
f(z) = \frac{-g(p)g(z)}{g(z) - g(p)}
\]
belongs to \( \mathcal{A}(p) \) and satisfies (1.11). If in addition \( g \) is univalent, then \( f \) is also univalent.

**Proof.** It is obvious that \( f \) is holomorphic in \( \mathbb{D} \setminus \{p\} \) and that \( f(p) = \infty \). We get by a simple calculation
\[
\lim_{z \to p} (z - p)f(z) = -\frac{g'(p)}{g'(p)}.\]
From (2.6) we have \( g(p) \neq 0 \). Hence the limit above shows that \( f \) has a simple pole with residue \( m = -\frac{g'(p)}{g'(p)} \) at the point \( p \). By the condition (2.6) we have
\[
\left|1 - \frac{p^2 g'(p)}{g'(p)}\right| < 1
\]
and hence \( f \) satisfies the condition (1.11).

It is easy to verify that \( f \) is univalent if \( g \) is univalent. \( \square \)

**Remark 2.4.** The condition (2.6) is satisfied when \( g \in \mathcal{U}(1) \);

Let \( \mathcal{P}_{\alpha, \beta, \delta}(p; \lambda) \) be the set of functions \( f \) in \( \mathcal{A}(p) \) of the form (1.10) such that \( \frac{f(z)}{z} \neq 0 \) in \( \mathbb{D} \) and satisfying the condition
\[
\left|\alpha \left(\frac{z}{f(z)}\right)^\gamma + \beta \left(\frac{z}{h(z)}\right)^\gamma\right| \leq \lambda \mu, \quad z \in \mathbb{D}
\]
(2.7)

and
\[
|1 + \frac{p^2}{m}| \leq \lambda \mu,
\]
(2.8)
where \( \alpha \neq 0, \beta \) are given complex numbers and \( h \) is a given function in \( \mathcal{A}(p) \) with \( \frac{h(z)}{z} \neq 0 \) in \( \mathbb{D} \).

We observe that \( \mathcal{P}_{1,0,\lambda}(p; \lambda) \) doesn’t depend on the function \( h \) and thus will be simply noted \( \mathcal{P}(p; \lambda) \). The particular case where \( \lambda = 2 \) has been considered by Bhowmik and Parveen in [3].

We need the following Lemma:

**Lemma 2.5.** Let \( 0 < \lambda < \mu^{-1} \). If \( f \) belongs to \( \mathcal{U}_\rho(\lambda) \) then, \( f \) is univalent in \( \mathbb{D} \).

**Proof.** Let \( -c \) be an omitted value for \( f \) and let \( g(z) = \frac{cf}{z+c} \). As seen above we have

\[
\mathcal{U}_\rho(z) = \mathcal{U}_f(z)
\]

and hence \( g \in \mathcal{U}(\lambda \mu) \). Since \( \lambda \mu < 1 \), \( g \) belongs to \( \mathcal{U}(1) \) and thus it is univalent. This implies that \( f \) is univalent. \( \square \)

**Theorem 2.6.** Let \( h \in \mathcal{A}(p) \) be such that \( \frac{h(z)}{z} \neq 0 \) for \( z \in \mathbb{D} \) and

\[
K = \sup_{z \in \mathbb{D}} \left| \frac{z}{h(z)} \right|^2 h'(z) - 1 < +\infty.
\]

If \( f \in \mathcal{P}_{\alpha,\beta,\lambda}(p; 2 \lambda |\alpha| - 2 K |\beta|p) \), then \( f \in \mathcal{U}_\rho(\lambda) \). If in addition \( \lambda < \mu^{-1} \), the function \( f \) is univalent in the disc \( \mathbb{D} \). In particular, we have

\[
\mathcal{P}_{\alpha,\beta,\lambda}(p; 2 \mu |\alpha| - 2 K |\beta|p) \subset \mathcal{U}_\rho(1).
\]

**Proof.** Let \( f \in \mathcal{P}_{\alpha,\beta,\lambda}(p; 2 \lambda |\alpha| - 2 K |\beta|p) \). Let \( g \) and \( k \) be defined by

\[
g(z) = \frac{cf}{c+f} \quad \text{and} \quad k(z) = \frac{dh}{d+h},
\]

where \( -c \) and \( -d \) are omitted values respectively by \( f \) and \( h \). By Proposition 2.1, \( g \) and \( k \) belong to \( \mathcal{A} \). A little calculation shows that \( \frac{h(z)}{z} \neq 0 \) and \( \frac{k(z)}{z} \neq 0 \) in \( \mathbb{D} \) and

\[
\frac{z}{g(z)} = \frac{z}{f(z)} + \frac{z}{c} \quad \text{and} \quad \frac{z}{k(z)} = \frac{z}{h(z)} + \frac{z}{d'},
\]

which gives

\[
\left( \frac{z}{g(z)} \right)' = \left( \frac{z}{f(z)} \right)' \quad \text{and} \quad \left( \frac{z}{k(z)} \right)' = \left( \frac{z}{h(z)} \right)'.
\]

Since \( f \) belongs to \( \mathcal{P}_{\alpha,\beta,\lambda}(p; 2 \lambda |\alpha| - 2 K |\beta|p) \), we have by (2.11)

\[
g \in \mathcal{P}_{\alpha,\beta,\lambda}(2 \lambda \mu |\alpha| - 2 K |\beta|p).
\]

Applying (2.5) to \( h \) and \( k \), we obtain

\[
\sup_{z \in \mathbb{D}} \left| \frac{z}{k(z)} \right|^2 k'(z) - 1 = K.
\]

Moreover (2.12) and (2.13) give, by applying Theorem 1.1 to \( g \) and \( k \),

\[
g \in \mathcal{U}(\lambda \mu)
\]
which gives from (2.5) and (2.8) that \( f \in \mathcal{U}_p(\lambda) \).

If now \( 0 < \lambda < \mu^{-1} \), then \( f \) is univalent by Lemma 2.5.

The second assertion of the theorem follows by taking \( \lambda = 1 \) in the first one. \( \square \)

Let \( p \in (0, 1) \) and let \( h(z) = \frac{z}{(z-p)(z-\lambda)} \). A little calculation yields

\[
\sup \left| \left( \frac{z}{h(z)} \right)^2 h'(z) - 1 \right| = 1 \quad \text{and} \quad \left( \frac{z}{h(z)} \right)'' = 2, \quad z \in \mathbb{D}
\]

**Corollary 2.7.** Let \( 0 < p < 1 \) and \( f \in \mathcal{A}(p) \) with \( \frac{f(z)}{z} \neq 0 \) for \( z \in \mathbb{D} \). Let \( \alpha \neq 0 \) and \( \beta \) be two complex numbers. If \( f \) satisfies

\[
\left| \alpha \left( \frac{z}{f(z)} \right)'' + \beta \right| \leq 2 \left( \lambda \mu |\alpha| - \frac{|\beta|}{2} \right), \quad z \in \mathbb{D}
\]  

then \( f \in \mathcal{U}_p(\lambda) \). If in addition \( 0 < \lambda < \mu^{-1} \), then \( f \) is univalent in \( \mathbb{D} \).

**Proof.** Let \( h(z) = \frac{z}{(z-p)(z-\lambda)} \). We have, as shown above, that

\[
\sup \left| \left( \frac{z}{h(z)} \right)^2 h'(z) - 1 \right| = 1 \quad \text{and} \quad \left( \frac{z}{h(z)} \right)'' = 2.
\]

Now, if \( f \) satisfies (2.14) then \( f \in \mathcal{P}(a,\lambda,\beta,p;2(\lambda|\alpha| - \frac{|\beta|}{2})) \) and hence, by taking \( K = 1 \) in the first statement of Theorem 2.6, we get the desired conclusion. \( \square \)

If we take \( |\alpha| = 1 \) and \( \beta = 0 \) in Corollary 2.7, we obtain the following

**Corollary 2.8.** Let \( 0 < p < 1 \) and \( f \in \mathcal{A}(p) \) with \( \frac{f(z)}{z} \neq 0 \) for \( z \in \mathbb{D} \). If \( f \) satisfies

\[
\left| \left( \frac{z}{f(z)} \right)'' \right| \leq 2 \lambda \mu, \quad z \in \mathbb{D},
\]  

then \( f \in \mathcal{U}_p(\lambda) \), in other words, we have \( \mathcal{P}(p;2\lambda) \subset \mathcal{U}_p(\lambda) \). If in addition \( 0 < \lambda < \mu^{-1} \), then functions in \( \mathcal{P}(p;2\lambda) \) are univalent.

**Corollary 2.9.** If \( 0 < \lambda \leq 2 \), then \( \mathcal{P}(p;\lambda) \subset \mathcal{U}_p(1) \) and hence functions in \( \mathcal{P}(p;\lambda) \) are univalent.

**Proof.** Since \( \mu^{-1} > 1, 0 < \frac{1}{\lambda} < \mu^{-1} \). Hence, the desired conclusion follows by applying Corollary 2.8 to \( \frac{1}{\lambda} \). \( \square \)

**Remark 2.10.** If we take \( \lambda = 2 \) in Corollary 2.9, we obtain Theorem 2 in [3].

We need the two following lemmas:

**Lemma 2.11.** Let \( g \in \mathcal{P}_{a,\beta,\lambda}(\lambda) \). Then there exists a Schwarz function \( w \) in \( \mathbb{D} \) such that

\[
\frac{z}{g(z)} - 1 = -\frac{\beta}{\alpha} \left( \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right) - \frac{g''(0)}{2}z + \frac{\lambda z}{\alpha} \int_0^1 \frac{w(tz)}{t} (1-t) dt.
\]

**Proof.** The proof can be extracted of the proof of Theorem 1.3 ([7], p.186). \( \square \)

**Lemma 2.12.** Let \( h \in \mathcal{A}(p) \), \( -c \) be an omitted value for \( h \) and \( k = \frac{ch}{c+k} \). Then,

\[
1 - \frac{z}{k(z)} - \frac{k''(0)}{2}z = 1 - \frac{z}{h(z)} - \frac{h''(0)}{2}z, \quad z \in \mathbb{D}.
\]  

(2.16)
Proof. We have
\[
\frac{z}{k(z)} = \frac{z}{h(z)} + \frac{z}{c}
\] (2.17)
and
\[
k''(0) = h''(0) - \frac{2}{c}
\] (2.18)
Taking (2.17) and (2.18) in the left side of (2.17), we get the desired conclusion. □

The following theorem is an analogue result of Corollary 1.8 in [7].

Theorem 2.13. Let \( f \in P_{\alpha,\beta}(\lambda, \mu) \) and \( M = \sup_{z \in \mathbb{D}} |1 - \frac{z}{h(z)} - \frac{h''(0)}{2}z| \). Then
\[
\left| \frac{z}{f(z)} - 1 \right| \leq \left| \frac{\beta}{\alpha} M + \frac{|f''(0)|}{2} |z| + \frac{\lambda \mu}{2|\alpha|} |z|^2. \right|
\] (2.19)

Proof. Let \(-c\) and \(-d\) be omitted values by \( f \) and \( h \), respectively. Furthermore let \( g(z) = \frac{cf}{c+f} \) and \( k(z) = \frac{dh}{d+h} \), respectively. We have
\[
\frac{z}{f(z)} - 1 = \frac{z}{g(z)} - 1 - \frac{z}{c}
\] (2.20)
and
\[
g''(0) = \frac{f''(0)}{2} - \frac{1}{c}.
\] (2.21)
Since \( f \in P_{\alpha,\beta}(\lambda, \mu) \), we have \( g \in P_{\alpha,\beta}(\lambda, \mu) \). Applying Lemma 2.11, we obtain
\[
\frac{z}{g(z)} - 1 = -\frac{\beta}{\alpha} \left( \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right) - \frac{g''(0)}{2}z - \frac{z}{c} - \frac{\lambda \mu z}{\alpha} \int_0^1 \frac{w(tz)}{t} (1-t)dt,
\] (2.22)
where \( w \) is a Schwarz function in \( \mathbb{D} \). Taking (2.22) in (2.20), we obtain
\[
\frac{z}{f(z)} - 1 = -\frac{\beta}{\alpha} \left( \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right) - \frac{g''(0)}{2}z - \frac{z}{c} - \frac{\lambda \mu z}{\alpha} \int_0^1 \frac{w(tz)}{t} (1-t)dt.
\] (2.23)
Now, taking (2.21) in (2.23), we get
\[
\frac{z}{f(z)} - 1 = -\frac{\beta}{\alpha} \left( \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right) - \frac{f''(0)}{2}z - \frac{z}{c} - \frac{\lambda \mu z}{\alpha} \int_0^1 \frac{w(tz)}{t} (1-t)dt.
\] (2.24)
The last equality gives us, using the fact that \(|w(z)| \leq |z| \) in \( \mathbb{D} \),
\[
\left| \frac{z}{f(z)} - 1 \right| \leq \left| \frac{\beta}{\alpha} \sup_{z \in \mathbb{D}} \left| \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right| + \frac{|f''(0)|}{2} |z| + \frac{\lambda \mu}{2|\alpha|} |z|^2. \right|
\] (2.25)
We have, by Lemma 2.12,
\[
\sup_{z \in \mathbb{D}} \left| \frac{z}{k(z)} + \frac{k''(0)}{2}z - 1 \right| = \sup_{z \in \mathbb{D}} \left| \frac{z}{h(z)} + \frac{h''(0)}{2}z - 1 \right| = M
\] (2.26)
Taking (2.26) in (2.25), we get the desired result. □

As a consequence of Theorem 2.13, we have the following corollary:
Corollary 2.14. If $z$ is a given point in $\mathbb{D}$ then, we have

\begin{align*}
(1) \quad & \left| \frac{z}{f(z)} - 1 \right| \leq \left( \frac{1}{p} + \frac{\lambda \mu p^2}{2} \right) |z| + \frac{\lambda \mu}{2} |z|^2, \quad \forall f \in \mathcal{P}(p; \lambda); \\
(2) \quad & \left| \frac{z}{f(z)} - 1 \right| \leq \frac{1}{p} + \frac{\lambda \mu p^2}{2} + \frac{\lambda \mu}{2}, \quad \forall f \in \mathcal{P}(p; \lambda).
\end{align*}

Proof. Let $f \in \mathcal{P}(p; \lambda)$. Taking $\alpha = 1$, $\beta = 0$ and $h(z) = \frac{pz}{p^2 + (1 + p^2)z + p^*}$, the formula (2.24) gives

\begin{equation}
\frac{z}{f(z)} - 1 = -\frac{f''(0)}{2}z + \lambda \mu z \int_0^1 \frac{w(tz)}{t}(1-t)dt.
\end{equation}

Putting $z = p$ in the last equality, we obtain

\begin{equation}
\frac{f''(0)}{2} = \frac{1}{p}(1 + \lambda \mu p \int_0^1 \frac{w(tp)}{t}(1-t)dt).
\end{equation}

Since $w$ is a Schwarz function, the modulus of the integral in (2.28) is majored by $\frac{p^*}{2}$ and hence we have

\begin{equation}
\left| \frac{f''(0)}{2} \right| \leq \frac{1}{p} + \frac{\lambda \mu p^2}{2}.
\end{equation}

Now, taking (2.29) in (2.19), where $\alpha$, $\beta$ and $h$ as above, we obtain the estimation

\begin{equation}
\left| \frac{z}{f(z)} - 1 \right| \leq \left( \frac{1}{p} + \frac{\lambda \mu p^2}{2} \right) |z| + \frac{\lambda \mu}{2} |z|^2.
\end{equation}

This achieves the proof of (1). The estimation (2) is an immediate consequence of (1). \hfill \Box

3. Acknowledgement

The author thanks Karl-Joachim Wirths for his remarks and suggestions and careful reading of the first version of the manuscript.

References

[1] L.A. Aksentiev, Sufficient conditions for univalence of regular functions. Izv. Vys. Ucebn. Zaved. Matematika. 1958, 1958, 3 - 7.
[2] B. Bhowmik and F. Parveen, On a subclass of meromorphic univalent functions, Complex Var. Elliptic Equ., 62 (2017), 494-510.
[3] B. Bhowmik and F. Parveen, Sufficient conditions for univalence and study of a class of meromorphic univalent functions, Bull. Korean. Math. Soc. 55 (2018), No. 3, 999 - 1006.
[4] R. Fournier and S. Ponnusamy, A Class of Locally Univalent Functions defined by a differential Inequality, Complex Var. and Elliptic Equations, 52 (1) (2007), 1-8.
[5] M. Obradović and S. Ponnusamy, V. Singh and P. Vasundhara, Univalency, starlikeness and convexity applied to certain classes of rational functions. Analysis (Munich) 22 (2002), 225 - 242.
[6] M. Obradović and S. Ponnusamy, Univalence of quotient of analytic functions, Applied Mathematics and Computation 247 (2014) 689 - 694.
[7] M. Obradović and S. Ponnusamy, New criteria and distortion Theorems for univalent functions. Complex Var. Theory Appl. 44(3), 173191 (2001), 173 - 191.
[8] M. Obradović and S. Ponnusamy, Product of univalent functions, Math. Comput. Model. 57, 793799 (2013)
[9] M. Obradović and S. Ponnusamy, Univalence and starlikeness of certain transforms defined by convolution. J. Math. Anal. Appl. 336, 758767 (2007), 758 - 767.
[10] M. Obradović and S. Ponnusamy, Radius of univalence of certain combination of univalent and analytic functions. Bull. Malays. Math. Sci. Soc. 35(2), (2012), 325 - 334.
[11] S. Ozaki, and M. Nunokawa, The Schwarzian Derivative and Univalent Functions, Proc. Amer. Math. Soc. 33(2), Number 2, 1972, 392 - 394.
[12] S. Ponnusamy and K.-J. Wirths, Elementary considerations for classes of meromorphic univalent functions, Lobachevskii J. of Math. 39(5)(2018), 712-713.