Hopf Bifurcation and Control of a Fractional-Order Delay Stage Structure Prey-Predator Model with Two Fear Effects and Prey Refuge

Yongzhong Lan, Jianping Shi* and Hui Fang

Department of System Science and Applied Mathematics, Kunming University of Science and Technology, Kunming 650500, China; math_lyz@stu.kust.edu.cn (Y.L.); fangh@kust.edu.cn (H.F.)
* Correspondence: sjp0207@kust.edu.cn

Abstract: A generalized delay stage structure prey-predator model with fear effect and prey refuge is considered in this paper via introducing fractional-order and fear effect induced by immature predators. Hopf bifurcation and control of this system are investigated though regarding the delay as the parameter. Firstly, by using the method of linearization and Laplace transform, the roots of the characteristic equation of the linearized system of the original system are discussed, and the sufficient conditions for the system exhibits an unstable state of symmetrical periodic oscillation (Hopf bifurcation) are explored. Secondly, a linear delay feedback controller is added to the system to increase the stability domain successfully. Thirdly, numerical simulations are performed to validate the theoretical analysis, and the various impacts on the dynamical behavior of the system occurring by fear effects, prey refuge, and each fractional-order are illustrated, respectively. Furthermore, the influence of feedback gain on the bifurcation critical point is analyzed. Finally, an analysis based on the results and in-depth research about this system under the biological background is stated in the conclusion.

Keywords: stage structure; fear effect; fractional-order delay system; Hopf bifurcation; feedback control

1. Introduction

The series of prey-predator models are paid widespread attention because they reflect the ecological phenomenon existing in the real world generally. The dynamic behavior of those models are investigated in-depth and a large number of valuable results have been obtained in the past few decades [1–4]. To keep the ecological balance, it is necessary to increase the survival rate of the prey in some ecosystems. The prey refuge is a suitable method to protect the prey population [5]. Prey-predator models with prey refuge are brought into focus and many worthy results are obtained [6–8].

Generally, the growth of many species are divided into two stages, immaturity and maturity, and the characteristics and behaviors of the different stage are quite distinguishing. For example, the predatory ability of mature predators is stronger than immature predators. In the past few years, researchers found that the prey-predator model with multi-stage structure is more reasonable than the one-stage model for describing the relationship between species [9–11].

Fear effect is another factor that impacts the dynamic behaviors of prey-predator models besides stage structure. For instance, the fear to predators could affect the birth rate of the prey, thereby affecting the population density of the prey [12]. Investigations have shown that the fear induced by the predator has an even greater effect on the prey than the direct killing [13–15]. Biologists discovered that many prey species have the inborn ability to identify predators in addition to acquired learning [16]. This means the fear effect may come from not only mature predators but also immature.
In many fields such as physics, mechanics, biology, chemistry, communication engineering, and control engineering, the evolution of a system not only depends on the current state but also be influenced by the past state. Therefore, it is of great significance to consider the delay in the system [17,18]. Because the population density of predator depends on the consumption of prey in the past partly, it is widely recognized for considering the delay in the prey-predator model. The effect of delay on the system has two sides: on the one hand, the enormous delay of the predator may lead to the extinction of the predator and the prey [19], on the other hand, the proper time delay can also increase the stability of the dynamical system [20,21].

Fractional calculus theory considers the mathematical properties and applications of differential and integral of arbitrary order. Fractional calculus operators have non-locality and are suitable for describing real-life materials and processes with memory and genetic properties [22–25], which are often ignored in classical integer-order models. Fractional calculus has a long history [26,27]. However, in the past three centuries, due to the computational complexity and the lack of practical background, the development of fractional calculus was very slow. Until the past few years, fractional calculus received extensive attention with the rapid development of computer technology and became an active research field. In biological models, the densities of species are in flux. It is related to both the current moment and some past state of the species. Those properties of the biological model coincide with the “memory” characteristic of fractional differentiation. Since each species has different degree of dependence on the past, incommensurate fractional models are more realistic.

Stability and bifurcation are important issues in the research of fractional-order differential equation models. Hopf bifurcation is widely focused on because it reflects the properties of periodic solutions near the steady-state of nonlinear systems [28–31]. However, the appearance of Hopf bifurcation is also a sign that the system appears periodic oscillation and enters an unstable state from a stable state. The Hopf bifurcation control of fractional systems has received more and more attention [32–34]. In Ding et al. [33], the dynamics of a fractional-order memristor-based chaotic system with delay were investigated. The authors confirmed that the delay feedback controller was valid in controlling chaos and Hopf bifurcation in the controlled system. Zheng et al. [34] proposed a linear delay feedback controller to put off the onset of Hopf bifurcation for a fractional-order paddy ecosystem. They observed that the delay could affect the dynamics of the system heavily, and the feedback gain and the fractional-order had significant impacts on the control effect.

For some integer-order delay prey-predator models, the Hopf bifurcation and control of their corresponding fractional-order models have never been studied in-depth, for example, the Crowley–Martin prey-predator model with fear effect and prey refuge [35]. Otherwise, many research considered that only mature predators could give rise to fear effect [35,36]. In fact, the fear effect may come from both mature and immature predators [16]. Inspired by these ideas, the incommensurate fractional-order and the fear factor induced by the immature predators are introduced to the Crowley–Martin prey-predator model [35] in this paper, and Hopf bifurcation and control of the generalized model are investigated by theoretical and numerical method.

The main contributions of this paper include: (1) By adding the fear factor induced by immature predators and introducing the fractional orders, an integer-order delay stage structure prey-predator model with fear effect and prey refuge is generalized. The existence conditions of the coexistence equilibrium point of the proposed system are deduced. (2) The conditions of emergence of Hopf bifurcation for the generalized system are determined. In other words, the critical value of delay that the system switches from asymptotical stability to symmetric periodic oscillation is deduced. (3) A linear delay feedback controller is added to put off the emergence of the Hopf bifurcation for the proposed system, and the stability domain of the system has increased. (4) From an ecological point of view, the effects of two fear factors, prey refuge, three fractional-orders, and the feedback gain to the bifurcation critical value of delay are analyzed in virtue of numeric simulations, respectively.
The organization of this paper is as follows. In Section 2, some definitions of fractional calculus and some basic knowledge are presented. In Section 3, the mathematical model is generalized, and the existence of the coexistence equilibrium point of the model is analyzed. In Section 4, Hopf bifurcation of the generalized system and the control of bifurcation of the controlled system are explored, respectively. In Section 5, numerical simulations are performed to further illustrate our theoretical results, and the influences of two fear effects, prey refuge, and fractional-order to the bifurcation of the system are given. Finally, a necessary conclusion explains the results and in-depth research about this system under the biological background.

2. Preliminary Knowledge

In this section, some basic definitions about fractional-order calculus and Hopf bifurcation used in the following sections are given.

Definition 1 (Riemann–Liouville Fractional Integral [37,38]). Fractional integral of order \( \alpha \) for the function \( f(t) : [a, \infty) \to \mathbb{R} \) can be expressed as follows:

\[
\alpha I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t > a,
\]

where \( a, a \in \mathbb{R}, \alpha > 0, \Gamma(\cdot) \) is Euler’s Gamma function.

Definition 2 (Caputo Fractional Derivative [37,38]). The Caputo fractional-order derivative is defined by

\[
^{\alpha}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha}} d\tau, \quad n-1 < \alpha \leq n,
\]

where \( f(t) \in C^n([a, \infty), \mathbb{R}) \). In particularly, if \( 0 < \alpha \leq 1, a = 0 \), Equation (2) can be written as:

\[
^{\alpha}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f^{(n)}(\tau) d\tau, \quad 0 < \alpha \leq 1, t > 0.
\]

Definition 3 (Laplace Transform of Fractional Derivation [39]). The Laplace transform of Caputo fractional derivation of order \( \alpha (n-1 < \alpha \leq n) \) for the function \( f(t) \in C^n([a, \infty), \mathbb{R}) \) is

\[
L\left\{^{\alpha}D_t^\alpha f(t) ; s\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(a),
\]

where \( F(s) \) is the Laplace transform of \( f(t) \), and \( f^{(k)}(a) (k = 0, 1, ..., n-1) \) are the initial conditions. Obviously, if \( f^{(k)}(a) = 0 \) for \( k = 0, 1, ..., n-1 \), Equation (4) can be written as

\[
L\left\{^{\alpha}D_t^\alpha f(t) ; s\right\} = s^\alpha F(s).
\]

Definition 4 ([40]). Consider the following \( n \)-dimensional fractional-order system with delay

\[
^{\alpha}D_t^\alpha u_i(t) = f_i(u_1(t), \cdots, u_n(t); \tau), i = 1, 2, \cdots, n,
\]

where \( 0 < \alpha \leq 1 \) and the delay \( \tau \geq 0 \). System (6) undergoes Hopf bifurcation at the equilibrium \( u^* = (u^*_1, u^*_2, \cdots, u^*_n) \) when \( \tau = \tau_0 \) if the following three conditions are satisfied:

C1: All the eigenvalues \( \lambda_j \) (\( j = 1, 2, ..., n \)) of the coefficient matrix \( A \) of the linearized system of Equation (6) with \( \tau = 0 \) satisfy \( |\arg(\lambda_j)| > \frac{\alpha \pi}{2} \).

C2: The characteristic equation of the linearized system of Equation (6) has a pure of imaginary roots \( s = \pm ia\omega \) when \( \tau = \tau_0 \).

C3: \( \text{Re}\left[ \frac{d\lambda_i(\gamma)}{d\tau}\right] \big|_{(\tau=\tau_0, \omega=a\omega_0)} > 0 \), where \( \text{Re}[\cdot] \) denotes the real part of the complex number.
Remark 1. (C3) in Definition 4 is the so-called transversality condition.

3. Model Description

Xiao et al. [41] studied the following Beddington–DeAngelis prey-predator model with stage structure and prey refuge

\[
\begin{align*}
\dot{x}(t) &= x(t) \left( r - cx(t) \right) - \frac{a(1-m)x(t)y_1(t)}{1+a(1-m)x(t)+by_2(t)}, \\
y_1(t) &= x(t)y_2(t) - \left( ny_1(t) - d_1y_1(t) \right), \\
y_2(t) &= ny_1(t) - d_2y_2(t),
\end{align*}
\]

where \( x(t), y_1(t), \) and \( y_2(t) \) represent the population densities of prey, immature predator, and mature predator at time \( t \), respectively. \( r \) is the birth rate of prey. \( d_1 \) and \( d_2 \) represent the natural mortality of immature predators and mature predators, respectively. \( c \) is the intraspecific competition rate of the prey. \( m \in [0, 1] \) is the prey refuge rate, and \( n \) represents the proportion of immature predators that grow into mature predators. \( a \) and \( b \) refer to the processing time of the mature predator and the strength of the interaction. \( \alpha \) and \( \beta \) refer to the capture rate of the prey and the conversion rate of nutrients into the production of predator species, respectively. \( \tau \) represents the delay due to the gestation of the mature predator. The authors investigated the local stability of the equilibrium point of the system and the influence of prey refuge on the densities of predator species and prey species.

In 2021, Wang and Hu [35] improved this model and discussed a Crowley–Martin prey-predator model with fear effect and prey refuge as follows

\[
\begin{align*}
\dot{x}(t) &= \frac{rx(t)}{1+k_2y_2(t)} - d_0x(t) - cx^2(t) - \frac{a(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)}, \\
y_1(t) &= \left( f_a(1-m)x(t) + by_2(t) + ab(1-m)x(t)y_2(t) \right) - ny_1(t) - d_1y_1(t), \\
y_2(t) &= ny_1(t) - d_2y_2(t),
\end{align*}
\]

where \( k \) is the prey’s fear factor induced by mature predators and \( d_0 \) represents the natural mortality of prey. The existence and stability of the equilibrium point of the system Equation (8) have been established in [35].

Considering that the evolution of prey-predators system related to both the current moment and some past state of the species, and each species has different degree of dependence on the past, incommensurate fractional-orders are added to the system (8). Otherwise, inspired by the reference [16], the prey’s fear effect is thought about not only mature predator but also immature predator. Thus, the system Equation (8) is generalized as the follows:

\[
\begin{align*}
D^{q_1}x(t) &= \frac{rx(t)}{1+k_1y_1(t)+k_2y_2(t)} - d_0x(t) - cx^2(t) - \frac{a(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)+ab(1-m)x(t)y_2(t)}, \\
D^{q_2}y_1(t) &= \left[ f_a(1-m)x(t) + by_2(t) + ab(1-m)x(t)y_2(t) \right] - ny_1(t) - d_1y_1(t), \\
D^{q_3}y_2(t) &= ny_1(t) - d_2y_2(t),
\end{align*}
\]

where \( q_i \in (0, 1] \ (i = 1, 2, 3) \) is fractional-order. \( k_1 \) and \( k_2 \) are the prey’s fear factors induced by immature predators and mature predators, respectively.

Obviously, the system has a zero equilibrium point \( E_0 = (0, 0, 0) \). When \( r > d_0 \), the system has a predator-extinction equilibrium point \( E_1 = (r-d_0, 0, 0) \). In fact, we are interested in the stability and stability switch at the coexistence equilibrium point of the system (9). Thus, it is necessary to find the conditions in that system (9) has a positive value equilibrium point.

**Lemma 1.** When the following four conditions are satisfied, the system (9) has a unique coexistence equilibrium point \( E^* = (x^*, y_1^*, y_2^*) \ (x^* > 0, y_1^* > 0, y_2^* > 0) \)

(H1) \( c \geq a(r - d_0)(1 - m) \);

(H2) \( r > d_0 \);
(H3) \( n\beta \alpha > d_2 a(n + d_1) \);

(H4) \( d_2 c(n + d_1) < (r - d_0)(n\beta \alpha - d_2 a(n + d_1))(1 - m) \).

**Proof.** In fact, system (9) exists an coexistence equilibrium point \( E^* = (x^*, y_1^*, y_2^*) \) means the following equation set has positive solution

\[
\begin{align*}
\begin{cases}
\frac{r}{1 + k_1 y_1 + k_2 y_2} - d_0 - c x - \frac{a(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2} = 0, \\
\frac{n + k_1 d_2 y_2 + nk_2 y_2}{\beta a(1-m)x} - d_2 - \frac{\alpha(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2} = 0, \\
y_1 - d_1 y_1 = 0.
\end{cases}
\end{align*}
\]

(10)

It is easy to obtain \( y_1 = \frac{d_1}{n} y_2 \), and substituting \( y_1 \) into the first and second equations of Equation (10), one has

\[
\begin{align*}
\begin{cases}
\frac{r n}{1 + k_1 d_2 y_2 + nk_2 y_2} - d_0 - c x - \frac{a(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2} = 0, \\
\frac{n + k_1 d_2 y_2 + nk_2 y_2}{\beta a(1-m)x} - d_2 - \frac{\alpha(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2} = 0.
\end{cases}
\end{align*}
\]

(11)

Let

\[
\begin{align*}
F(x, y_2) &:= \frac{r n}{1 + k_1 d_2 y_2 + nk_2 y_2} - d_0 - c x - \frac{a(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2}, \\
G(x, y_2) &:= \frac{n + k_1 d_2 y_2 + nk_2 y_2}{\beta a(1-m)x} - d_2 - \frac{\alpha(1-m)y_2}{1 + a(1-m)x + by_2 + ab(1-m)xy_2},
\end{align*}
\]

(12)

if curve \( F(x, y_2) = 0 \) intersects curve \( G(x, y_2) = 0 \) in the first quadrant, then system (10) has positive solution. According to the first equation of Equation (12), one has

\[
\frac{d y_2}{d x} = -\frac{F_x}{F_{y_2}} = \frac{a(1-m)^2 y_2 a(1+by_2)}{(1 + a(1-m)x + by_2 + ab(1-m)xy_2)^2} - \frac{c}{n + k_1 d_2 y_2 + nk_2 y_2}
\]

(13)

In the first quadrant, if \( \frac{d y_2}{d x} < 0 \) then

\[
\frac{a(1-m)^2 y_2 a(1+by_2)}{(1 + a(1-m)x + by_2 + ab(1-m)xy_2)^2} < c.
\]

(14)

From \( F(x, y_2) = 0 \), one can get

\[
\frac{a(1-m)^2 y_2 a(1+by_2)}{(1 + a(1-m)x + by_2 + ab(1-m)xy_2)^2} = \frac{(r n k_2 + k_1 d_2 y_2 + nk_2 y_2)}{(r n k_2 + k_1 d_2 y_2 + nk_2 y_2)^2} - \frac{d_0 - c x}{1 + a(1-m)x} a(1-m).
\]

(15)

Substituting Equation (15) into inequation Equation (14), one has

\[
\frac{l_1 x y_2 + l_2 x + l_3 y_2 + l_4}{(1 + a(1-m)x)(n + nk_2 y_2 + k_1 d_2 y_2)} > 0,
\]

(16)

where

\[
\begin{align*}
l_1 &= 2ac(1-m)(k_1 d_2 + nk_2), \\
l_2 &= 2acn(1-m), \\
l_3 &= (k_1 d_2 + nk_2)(c + a d_0 (1-m)), \\
l_4 &= n(c + a(d_0 - r)(1-m)).
\end{align*}
\]

For all \( x > 0, y_2 > 0 \), if \( l_4 = n(c + a(d_0 - r)(1-m)) > 0 \), that is

\[
c > a(r - d_0)(1-m),
\]

then the inequation Equation (16) is established, and it means \( \frac{d y_2}{d x} < 0 \).

If \( y_2 = 0 \), then \( x^{(1)} = \frac{-d_0}{c} \), and when \( x^{(1)} > 0 \), there is \( r > d_0 \).
When $x = 0$, there is
\begin{equation}
Ay_2^2 + By_2 + C = 0,
\end{equation}
where
\begin{align*}
A &= (d_2k_1 + k_2n)((1 - m)\alpha + bd_0), \\
B &= ((1 - m)\alpha + d_0(b + k_2) - br)n + d_0d_2k_1, \\
C &= n(d_0 - r).
\end{align*}
Obviously, if $r > d_0$,
Equation (17) has only one positive root $y_2^{(1)}$.
According to Definition 4, we analyze the conditions of emergence of Hopf bifurcation.
We are interested in the dynamical properties at the coexistence equilibrium point
Therefore, if $(H1)-(H4)$ are satisfied, $F(x, y_2)$ and $G(x, y_2)$ have a unique intersection $(x^*, y_2^*)$ in the first quadrant, and then $y_1^* > 0$ can be obtained by the third equation of Equation (10).

4. Hopf Bifurcation Analysis and Control of System (9)

We are interested in the dynamical properties at the coexistence equilibrium point $(x^*, y_1^*, y_2^*)$ of system (9). In this section, the Hopf bifurcation and control are analyzed in details.

4.1. Hopf Bifurcation Analysis of System (9)

Using the transformation $u(t) = x(t) - x^*, v(t) = y_1(t) - y_1^*, w(t) = y_2(t) - y_2^*$, and linearizing the converted system, we can have
\begin{equation}
\begin{cases}
D^3u(t) = a_{11}u(t) + a_{12}v(t) + a_{13}w(t), \\
D^3v(t) = a_{21}u(t) - \tau + a_{22}v(t) + a_{23}w(t - \tau), \\
D^3w(t) = a_{31}v(t) + a_{32}w(t),
\end{cases}
\end{equation}
where
\begin{align*}
a_{11} &= \frac{r}{1+k_1y_1^{*}+k_2y_2^{*}} - d_0 - 2c\tau - \frac{a(1-m)y_2^{*}}{(1-x^{*}(1-m)a)^{\tau}(by_2^{*}+1)^{\tau}}, \\
a_{12} &= -\frac{rx^{*}k_1}{(1+k_1y_1^{*}+k_2y_2^{*})^{1/2}}, \\
a_{13} &= \frac{\alpha(1-m)y_2^{*}}{(1-x^{*}(1-m)a)^{\tau}(by_2^{*}+1)^{\tau}}, \\
a_{21} &= \frac{\beta a(1-m)y_2^{*}}{(1-x^{*}(1-m)a)^{\tau}(by_2^{*}+1)^{\tau}}, \\
a_{22} &= -\frac{\beta a(1-m)y_2^{*}}{(1-x^{*}(1-m)a)^{\tau}(by_2^{*}+1)^{\tau}}, \\
a_{23} &= \frac{\beta a(1-m)y_2^{*}}{(1-x^{*}(1-m)a)^{\tau}(by_2^{*}+1)^{\tau}}, \\
a_{31} &= n, \\
a_{32} &= -d_2.
\end{align*}

According to Definition 4, we analyze the conditions of emergence of Hopf bifurcation for system (9) at the coexistence equilibrium point.

**Lemma 2.** When $\tau = 0$, all the eigenvalues $\lambda_j$ $(j = 1, 2, 3)$ of the coefficient matrix of the linearized system (19) have negative real parts, if the following assumptions (H5)-(H7) hold,
\begin{enumerate}
\item[(H5)] $a_{11} + a_{22} + a_{32} < 0,$
\item[(H6)] $-a_{11}a_{22}a_{32} + a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{12} > 0,$
\item[(H7)] $-a_{11} - a_{22} - a_{32} \alpha \beta > 0$.
\end{enumerate}
\textbf{Proof.} When τ = 0, the coefficient matrix \( A \) of the system (19) is

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    0 & a_{31} & a_{32}
\end{pmatrix},
\]

the corresponding characteristic equation of \( A \) is

\[
\lambda^3 + (-a_{11} - a_{22} - a_{32})\lambda^2 + (a_{11}a_{22} + a_{11}a_{23} - a_{12}a_{21} \\
+ a_{22}a_{32} - a_{23}a_{31})\lambda + (-a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{32}) = 0. \tag{20}
\]

If the assumptions \((H5)-(H7)\) are true, according to the Routh-Hurwitz criterion, all the characteristic roots of Equation (20) have negative real parts, that is, the characteristic root \( \lambda_j (j = 1, 2, 3) \) satisfies \(|\arg(\lambda_j)| > \frac{\pi}{2} \) \((q = \max(q_1, q_2, q_3))\).

Taking Laplace transform \([42]\) to system (19), one has

\[
\begin{aligned}
    s^2 F_1(s) &= a_{11} F_1(s) + a_{12} F_2(s) + a_{13} F_3(s), \\
    s^2 F_2(s) &= a_{21} e^{-s\tau} F_1(s) + a_{22} F_2(s) + a_{23} e^{-s\tau} F_3(s), \\
    s^2 F_3(s) &= a_{31} F_2(s) + a_{32} F_3(s). 
\end{aligned} \tag{21}
\]

The characteristic equation of Equation (21) is

\[
\begin{vmatrix}
    s^3 - a_{11} & -a_{12} & -a_{13} \\
    -a_{21} e^{-s\tau} & s^2 - a_{22} & -a_{23} e^{-s\tau} \\
    0 & -a_{31} & s^2 - a_{32}
\end{vmatrix} = 0. \tag{22}
\]

Equation (22) can be written as

\[
U_1(s) + U_2(s) e^{-s\tau} = 0, \tag{23}
\]

where

\[
U_1(s) = s^3 + q_1 s^2 + q_3 - a_{22} s^2 + q_2 - a_{22} a_{32} s - a_{11} s^2 + q_3 \\
+ a_{11} a_{23} s + a_{11} a_{23} s^2 - a_{11} a_{23} a_{23} - a_{12} a_{21} a_{32}, \\
U_2(s) = -a_{23} a_{31} s^2 + q_1 - a_{12} a_{21} s^2 + a_{11} a_{23} a_{31} - a_{13} a_{21} a_{31} + a_{12} a_{21} a_{32}.
\]

In order to find the critical value of delay that the stability of system (19) switches, one can assume

\((A1) \ |U_1(0)| < |U_2(0)|;\)

Assume that \( s = \imath \omega = \omega (\cos \frac{\pi}{2} + \imath \sin \frac{\pi}{2}) \) \((\omega > 0)\) is a root of Equation (23), substituting it into Equation (23) and separating the real and imaginary parts, we have

\[
\begin{aligned}
    a_1 \cos \omega \tau + a_2 \sin \omega \tau &= -a_3, \\
    a_2 \cos \omega \tau - a_1 \sin \omega \tau &= -a_4,
\end{aligned} \tag{24}
\]

where

\[
a_1 = \text{Re}(U_2(\imath \omega)), \quad a_2 = \text{Im}(U_2(\imath \omega)), \quad a_3 = \text{Re}(U_1(\imath \omega)), \quad a_4 = \text{Im}(U_1(\imath \omega)),
\]

the exact expressions of \( a_i \) \((i = 1, 2, 3, 4)\) are defined in Appendix A.

Solving Equation (24), it is easy to obtain the following result,

\[
\begin{aligned}
    \sin \omega \tau &= \frac{a_1 a_2 - a_2 a_3}{a_1^2 + a_2^2} = \frac{\text{Im}(U_1(\imath \omega) \cdot U_2(\imath \omega))}{|U_2(\imath \omega)|^2}, \\
    \cos \omega \tau &= -\frac{a_1 a_2 + a_2 a_3}{a_1^2 + a_2^2} = -\frac{\text{Re}(U_1(\imath \omega) \cdot U_2(\imath \omega))}{|U_2(\imath \omega)|^2},
\end{aligned} \tag{25}
\]
Without loss of generality, we assume that all positive roots are where 

\[ U \] 

If the hypothesis \((H8)\) the expressions of \(A\) respectively, one gets

\[ \frac{A_1 B_1 + A_2 B_2}{B_1^2 + B_2^2} > 0, \]

Therefore

\[ \lim_{\omega \to +\infty} \left( |U_2(i\omega)| - |U_1(i\omega)| \right) = -\infty. \]

According to assumption \((A1)\), the equation \(|U_1(i\omega)| = |U_2(i\omega)|\) has at least one positive root.

Combining with the formula \(\sin^2 \omega \tau + \cos^2 \omega \tau = 1\), the value of \(\omega\) can be solved. Without loss of generality, we assume that all positive roots are \(\omega_k\) \((k = 1, 2, ..., K)\). By substituting each \(\omega_k\) into Equation \((25)\) and the corresponding critical value of \(\tau_k\) can be obtained. In relation to the actual meaning of delay, we only pay attention to the value of \(\tau\) when Hopf bifurcation occurs firstly, so the bifurcation critical value of delay is

\[ \tau_0 = \min \{\tau_k\}, k = 0, 1, 2, ...K, \tag{26} \]

the critical value of frequency corresponding to \(\tau_0\) is denoted as \(\omega_0\).

According to Definition \(4\), we need to verify the transversality condition at the critical point \((\tau_0, \omega_0)\). Thus, it is necessary to give the following hypothesis

\[ (H8) \quad \frac{A_1 B_1 + A_2 B_2}{B_1^2 + B_2^2} > 0, \]

the expressions of \(A_i, B_i\) \((i = 1, 2)\) is in Appendix \(B\).

**Lemma 3.** If the hypothesis \((H8)\) holds, let \(s(\tau) = \gamma(\tau) + i\omega(\tau)\) be the root of Equation \((23)\) near \(\tau = \tau_j\) satisfying \(\gamma(\tau_j) = 0, \omega(\tau_j) = \omega_0\), then the following transversality conditions established

\[ \text{Re} \left[ \frac{ds(\tau)}{d\tau} \right]_{\tau=\tau_0,\omega=\omega_0} > 0. \tag{27} \]

**Proof.** According to the implicit function derivation rule, deriving \(\tau\) on both sides of Equation \((23)\) respectively, one gets

\[ U_1'(s) \frac{ds}{d\tau} + U_2'(s)e^{-st} \frac{ds}{d\tau} + U_2(s)e^{-st}(-\tau \frac{ds}{d\tau} - s) = 0, \]

where \(U_i'(s)\) is the derivative of \(U_i(s)\) \((i = 1, 2)\). Hence,

\[ \frac{ds}{d\tau} = \frac{A(s)}{B(s)}, \tag{28} \]

where

\[ A(s) = U_2(s)se^{-st}, \]

\[ B(s) = U_1'(s) + U_2'(s)e^{-st} - U_2(s)\tau e^{-st}. \]

It can be deduced from Equation \((28)\) that

\[ \text{Re} \left[ \frac{ds(\tau)}{d\tau} \right]_{\tau=\tau_0,\omega=\omega_0} = \frac{A_1 B_1 + A_2 B_2}{B_1^2 + B_2^2}, \tag{29} \]
where $A_1, A_2$ are the real and imaginary parts of $A(s)$, $B_1, B_2$ are the real and imaginary parts of $B(s)$. In terms of (H8), one has
\[
\operatorname{Re} \left[ \frac{d\omega_2(t)}{dt} \right] |_{t=t_0, \omega_2(0)=\omega_0} > 0.
\]

The proof of Lemma 3 is finished. \(\square\)

Based on Lemmas 2 and 3, we can get the following theorem:

**Theorem 1.** If (H1)–(H8) and (A1) hold, then the coexistence equilibrium point of system (9) is asymptotically stable when $\tau \in [0, \tau_0)$, and system (9) undergoes Hopf bifurcation at the coexistence equilibrium point when $\tau = \tau_0$. $\tau_0$ is the critical value of delay defined by Equation (26).

### 4.2. Hopf Bifurcation Control of System (9)

In this section, we focus on the control of Hopf bifurcation of system (9). From an ecological point of view, it is more effective to control the stability of the system by regulating the population density of mature predators than by regulating the population density of immature predators, as the mature predators play a dominant role in the ecosystem. A linear delay feedback controller $L[y_2(t) - y_2(t - \tau)]$ is added to the third equation of system (9) to control the emergence of Hopf bifurcation, i.e., the stability domain is regulated by controlling the population density of mature predators. The controlled system is

\[
\begin{align*}
D^\theta x(t) &= \frac{rx(t)}{1+u_1y_1(t)+u_2y_2(t)} - dx(t) - cx^2(t) - \frac{a(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+b_2y_2(t)+ab(1-m)x(t)y_2(t)}, \\
D^\theta y_1(t) &= \frac{b(1-m)x(t)}{1+a(1-m)x(t)+b_2y_2(t)+ab(1-m)x(t)y_2(t)}(t) - ny_1(t) - d_1y_1(t), \\
D^\theta y_2(t) &= ny_1(t) - d_2y_2(t) + L[y_2(t) - y_2(t - \tau)],
\end{align*}
\]

where $L \in \mathbb{R}$ is the feedback control gain.

Making a transformation $u(t) = x(t) - x^*$, $v(t) = y_1(t) - y_1^*$, $w(t) = y_2(t) - y_2^*$, and doing the linearization at zero equilibrium point to Equation (30), the linearization system of controlled system (30) can be achieved

\[
\begin{align*}
D^{\theta 1}_u(t) &= a_{11}u(t) + a_{12}v(t) + a_{13}w(t), \\
D^{\theta 2}_v(t) &= a_{21}u(t - \tau) + a_{22}v(t) + a_{23}w(t - \tau), \\
D^{\theta 3}_w(t) &= a_{31}v(t) + a_{32}w(t) + L[w(t) - w(t - \tau)],
\end{align*}
\]

where $a_{ij}$ ($i, j = 1, 2, 3$) is same as Equation (19).

Taking Laplace transform to system (31), one can get the characteristic equation as following

\[
\begin{vmatrix}
-s^{\theta 1} - a_{11} & -a_{12} & -a_{13} \\
-a_{21} e^{-s\tau} & s^{\theta 2} - a_{22} & -a_{23} e^{-s\tau} \\
0 & a_{31} & s^{\theta 3} - a_{32} - L + Le^{-s\tau}
\end{vmatrix} = 0.
\]

Obviously, Equation (32) is equivalent to

\[
V_1(s) + V_2(s) e^{-s\tau} + V_3 e^{-2s\tau} = 0,
\]

where

\[
\begin{align*}
V_1(s) &= s^{\theta 1} + s^{\theta 2} + s^{\theta 3} - a_{22}s^{\theta 1} + a_{32}s^{\theta 1} + a_{11}s^{\theta 2} + a_{13}s^{\theta 3} - Ls^{\theta 1} + a_{12}a_{21}s^{\theta 1} + a_{13}a_{23}s^{\theta 1} - a_{11}a_{22}a_{32} - L_{a_{11}a_{22}} \\
V_2(s) &= Ls^{\theta 1} + s^{\theta 2} - a_{22}s^{\theta 1} + a_{32}s^{\theta 1} + a_{11}s^{\theta 2} + a_{13}s^{\theta 3} - a_{11}a_{22}a_{32} - L_{a_{11}a_{22}} \\
V_3 &= -L_{a_{12}a_{21}}.
\end{align*}
\]
Multiplying $e^{s\tau}$ on both sides of Equation (33), one gets

$$V_1(s)e^{s\tau} + V_2(s) + V_3e^{-s\tau} = 0. \quad (34)$$

In order to find the critical value of delay that the stability of system (30) switches, one can assume

(A2) $|V_1(0)| - |V_2(0) + V_3| < 0$;

Let $s = i\omega = \omega(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})(\omega > 0)$ as a root of Equation (34), substituting it into Equation (34) and separating the real and imaginary parts, one has:

$$\left\{ \begin{array}{ll}
(\beta_1 + \beta_3) \cos \omega\tau - \beta_2 \sin \omega\tau = -\gamma_1, \\
\beta_2 \cos \omega\tau + (\beta_1 - \beta_3) \sin \omega\tau = -\gamma_2,
\end{array} \right. \quad (35)$$

where $\beta_1 = Re(V_1(i\omega))$, $\beta_2 = Im(V_1(i\omega))$, $\beta_3 = V_3$, $\gamma_1 = Re(V_2(i\omega))$, $\gamma_2 = Im(V_2(i\omega))$.

By Equation (34), it can get

$$|V_1(i\omega)| = |V_2(i\omega) + V_3e^{-i\omega\tau}|.$$

Set $G(\omega) = |V_1(i\omega)| - |V_2(i\omega) + V_3e^{-i\omega\tau}|$, then

$$G(\omega) = |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3} + (i\omega)^{q_1+q_2+q_3} - |V_2(i\omega) + V_3e^{-i\omega\tau}|$$

$$\geq |(i\omega)^{q_1+q_2+q_3} - |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3} - |V_2(i\omega) + V_3e^{-i\omega\tau}|$$

$$= \omega^{q_1+q_2+q_3} - |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3} - |V_2(i\omega) + V_3e^{-i\omega\tau}|.$$

Therefore,

$$\lim_{\omega \to +\infty} G(\omega) = \infty.$$

According to assumption (A2), the equation $G(\omega) = 0$ has at least one positive root. Same as Section 4.1, we can obtain the minimum bifurcation critical point $(\tau_0^*, \omega_0^*)$ of the controlled system (30).

It is necessary to get the transversality condition, thus we make the following assumption

(H9) $\frac{C_1D_2 + C_2D_1}{D_1^2 + D_2^2} > 0$,

the expressions of $C_i$, $D_i$ ($i = 1, 2$) are in Appendix D.

**Lemma 4.** If the hypothesis (H9) holds, let $s(\tau) = \delta(\tau) + i\omega(\tau)$ be the root of Equation (33) near $\tau = \tau_j$ satisfying $\delta(\tau_j) = 0, \omega(\tau_j) = \omega_0^*$, then the following transversality condition satisfied

$$Re \left[ \frac{ds(\tau)}{d\tau} \right]_{(\tau = \tau_j, \omega = \omega_0^*)} > 0.$$

**Proof.** Deriving on both sides of Equation (33) for the variable $\tau$, one gets

$$V_1'(s)\frac{ds}{d\tau} + V_2'(s)e^{-s\tau}\frac{ds}{d\tau} + V_2(s)e^{-s\tau}(-s - \frac{ds}{d\tau}) + V_3e^{-2s\tau}(-2\tau \frac{ds}{d\tau} - 2s) = 0,$$

where $V_i'(s)$ is the derivative of $V_i(s)$ ($i = 1, 2$). Hence,

$$\frac{ds}{d\tau} = \frac{C(s)}{D(s)}, \quad (36)$$

where
\[ C(s) = s[V_2(s)e^{-st} + 2V_3e^{-2st}], \]
\[ D(s) = V_1(s) + [V_2(s) - V_2(s)\tau]e^{-st} - 2V_3\tau e^{-2st}. \]

It can be deduced from Equation (36) that
\[ \text{Re}\left[ \frac{ds(\tau)}{d\tau} \right] |_{\tau = \tau_0, \omega = \omega_0} = \frac{C_1D_1 + C_2D_2}{D_1^2 + D_2^2}, \]
where \( C_1, C_2 \) are the real and imaginary parts of \( C(s) \), \( D_1, D_2 \) are the real and imaginary parts of \( D(s) \).

Obviously, if hypothesis (H9) is true, then the transversality condition is true. The proof of Lemma 4 is finished. \( \Box \)

Based on Lemmas 2 and 4, we can get the following theorem:

**Theorem 2.** If (H1)–(H7), (H9) and (A2) hold, the coexistence equilibrium point of controlled system (30) is asymptotically stable when \( \tau \in [0, \tau_0^*] \), controlled system (30) undergoes Hopf bifurcation at the coexistence equilibrium point when \( \tau = \tau_0^* \).

**Remark 2.** The influence of the linear delay feedback controller on the stability domain of system (9) is direct because the formula for calculating the delay critical value includes \( I \) (see Appendixes C and D).

### 5. Numerical Simulations

In this section, we use the Adama-Bashforth-Moulton predictive correction method [43] to validate the feasibility of theoretical analysis.

#### 5.1. Example 1

For better comparison, the parameters of system (9) refer to the literature [35]: \( r = 0.34, a = 1, b = 1, c = 0.3, \alpha = 1, \beta = 0.8, m = 0.05, n = 0.8, d_0 = 0.1, d_1 = 0.1, d_2 = 0.1, k_1 = 0.05, k_2 = 0.1 \), and fractional-orders are chosen as \( q_1 = 0.98, q_2 = 0.92, q_3 = 0.95 \), then system (9) is

\[
\begin{align*}
D^{0.98} x(t) &= \frac{0.3x(t)}{1 + 0.05y_1(t) + 0.1y_2(t) - 0.1x(t) - 0.3x^2(t)} - \frac{0.95x(t)y_2(t)}{1 + 0.95x(t) + y_2(t) + 0.95x(t)y_2(t)}, \\
D^{0.92} y_1(t) &= \frac{0.95x(t)y_2(t)}{1 + 0.95x(t) + y_2(t) + 0.95x(t)y_2(t)} - 0.9y_1(t), \\
D^{0.95} y_2(t) &= 0.8y_1(t) - 0.1y_2(t).
\end{align*}
\]

It can be verified that (H1)–(H8) and (A1) hold. It is easy to obtain that the coexistence equilibrium point is \((x^*, y_1^*, y_2^*) = (0.2130, 0.0246, 0.1968)\), the bifurcation critical point is \((\tau_0 = 35.533, \omega_0 = 0.038858)\), and transversality condition \( \text{Re}\left[ \frac{ds(\tau)}{d\tau} \right] |_{\tau = \tau_0, \omega = \omega_0} = 0.0000899288 > 0 \). By means of Theorem 1, the coexistence equilibrium point \((x^*, y_1^*, y_2^*)\) is asymptotically stable when \( \tau \in [0, \tau_0) \), and Hopf bifurcation occurs when \( \tau \geq \tau_0 \). These results are illustrated in Figures 1 and 2 by choosing \( \tau = 34.99 \) and \( \tau = 36.01 \), respectively. Moreover, we can see from Figure 2 that the system is in an unstable state of symmetrical periodic oscillation.

In what follows, the influences on bifurcation critical value of delay caused by the fear factors \( k_1, k_2 \), and the prey refuge rate \( m \) are discussed through numerical simulations, respectively. Furthermore, numerical simulations show that the fractional-order \( q_i \) (i = 1, 2, 3) has different effects on the stability region of system (38).

#### Case 1. The influences of fear factors on the stability region

In this paper, the fear factor is considered as two cases caused by mature predators and immature predators, respectively, i.e., \( k_1 \) and \( k_2 \). We are interested in which one makes an important role in the stability of system (38). When all parameters and fractional-orders remain unchanged except \( k_1 \), let \( k_1 \) increases continuously, we can get different bifurcation critical points \((\tau_0, \omega_0)\) presented in Table 1:
Next, in a similar way, remain all parameters and fractional-orders unchanged except $k_2$, and let $k_2$ increases continuously, we also get different bifurcation critical points $(\tau_0, \omega_0)$ presented in Table 2:

It can be viewed in Figure 3 that the occurrence of the Hopf bifurcation is put off slightly as $k_1$ increases. However, the relationship between the critical value of delay $\tau_0$ and fear factor $k_2$ shows a U-shaped curve. What calls for special attention is when $k_2$ increases from 0.1 to 0.6, $\tau_0$ descends by 9%. From the perspective of ecology, if the fear of predators is greater, the instability of the system will be more obvious, and the critical value of delay will decrease. In other words, the occurrence of Hopf bifurcation is advanced and the stability state is broken. However, when the intensity of the fear effect reaches a certain level, the limit effect will be produced, and the critical value of delay will decrease more and more weakly. On the other hand, Figure 3 shows that the fear effect on the stability region of the system mainly comes from mature predators.
Table 1. The relationship between $k_1$ and $\tau_0$.

| $k_1$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-------|------------|----------|-------------------------|
| 0.05  | 0.038858   | 35.533   | 0.0000899288            |
| 0.1   | 0.038850   | 35.608   | 0.0000900460            |
| 0.2   | 0.038833   | 35.737   | 0.0000902623            |
| 0.3   | 0.038815   | 35.906   | 0.0000904556            |
| 0.4   | 0.038796   | 36.054   | 0.0000906266            |
| 0.5   | 0.038776   | 36.202   | 0.0000907763            |
| 0.6   | 0.038755   | 36.348   | 0.0000909056            |
| 0.7   | 0.038732   | 36.494   | 0.0000910151            |
| 0.8   | 0.038709   | 36.640   | 0.0000911057            |
| 0.9   | 0.038685   | 36.784   | 0.0000911781            |
| 0.99  | 0.038663   | 36.914   | 0.0000912283            |

Table 2. The relationship between $k_2$ and $\tau_0$.

| $k_2$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-------|------------|----------|-------------------------|
| 0.1   | 0.038858   | 35.533   | 0.0000899288            |
| 0.2   | 0.039765   | 34.178   | 0.0000966910            |
| 0.3   | 0.040387   | 33.304   | 0.0001012262            |
| 0.4   | 0.040800   | 32.752   | 0.0001040106            |
| 0.5   | 0.041056   | 32.429   | 0.0001054188            |
| 0.6   | 0.041191   | 32.276   | 0.0001057446            |
| 0.7   | 0.041229   | 32.255   | 0.0001052195            |
| 0.8   | 0.041192   | 32.338   | 0.0001040259            |
| 0.9   | 0.041092   | 32.507   | 0.0001023092            |
| 0.99  | 0.040958   | 32.721   | 0.0001004132            |

Figure 3. (a) $k_2 = 0.1, \tau_0$ varies with the increase of $k_1$. (b) $k_1 = 0.05, \tau_0$ varies with the increase of $k_1$.

Case 2. The influence of prey refuge rate on the stability region

Prey refuge is an effective measure of ecosystem regulation. We are interested in how the prey refuge rate $m$ influences the stability of system (38). Same as Case 1, remain all the parameters and fractional-orders unchanged except $m$, let $m$ increases continuously, we can get different bifurcation critical points $(\tau_0, \omega_0)$ presented in Table 3.

It can be noticed easily from Table 3 and Figure 4 that when $m$ increases from 0.05 to 0.12, the critical value of delay $\tau_0$ increases from 35.533 to 262.802, i.e., the stability region of the system becomes 7.4 times the original. That is to say, to system (38), $m$ has extremely influence on stability at the coexistence.
Table 3. The relationship between $m$ and $\tau_0$.

| $m$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-----|------------|----------|--------------------------|
| 0.05 | 0.038858  | 35.533   | 0.0000899288              |
| 0.06 | 0.036354  | 40.349   | 0.0000686956              |
| 0.07 | 0.033570  | 46.673   | 0.0000499478              |
| 0.08 | 0.030428  | 55.381   | 0.0000338849              |
| 0.09 | 0.026812  | 68.210   | 0.0000207164              |
| 0.10 | 0.022526  | 89.220   | 0.0000106555              |
| 0.11 | 0.017200  | 130.850  | 0.0000038985              |
| 0.12 | 0.009920  | 262.802  | 0.000005387               |

equilibrium point, and it is a useful method to keep the ecosystem (38) stable development by changing the degree of prey refuge.

Figure 4. $\tau_0$ varies with the increase of $m$.

**Case 3. The influence of fractional-orders on the stability region**

Let $q_1 = 1, q_2 = 1, q_3 = 1$ and $k_1 = 0$, the system (38) becomes an integer-order system corresponding to system (8). The coexistence equilibrium point is $(0.2131, 0.0247, 0.1974)$, and the bifurcation critical point is $(\tau_0 = 22.77, \omega_0 = 0.0501)$, which is consistent with the results in Wang [35]. If $q_1 = 0.98, q_2 = 0.92, q_3 = 0.95$ and $k_1 = 0$, the bifurcation critical point is $(\tau_0 = 35.46, \omega_0 = 0.0389)$. These results validate that when other parameters of the model are consistent, the delay critical value of the emergence of Hopf bifurcation in the fractional-order system is obviously larger than that in the integer-order system, and the stability domain of the system expands from $[0, 22.77)$ to $[0, 35.46)$. Otherwise, Tables 4–6 further illustrate that fractional-order can effectively expand the stability domain of the system.

Next, we want to know which fractional-order has the obvious effect on the stability of the system (38). The main idea is to keep two fractional-orders unchanged and vary the third one. Tables 4–6 show the different bifurcation critical points $(\tau_0, \omega_0)$ along with $q_i$ ($i = 1, 2, 3$) varying, respectively. The varying curves are drawn in Figure 5 to compare the distinguishing influences to the stability region.
Table 4. The relationship between $q_1$ and $\tau_0$ ($q_2 = 0.92$, $q_3 = 0.95$).

| $q_1$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-------|-------------|-----------|--------------------------|
| 0.63  | 0.001594    | 1641.098  | 0.00000006631            |
| 0.67  | 0.002666    | 951.636   | 0.000001883              |
| 0.71  | 0.004259    | 573.477   | 0.000005115              |
| 0.75  | 0.006541    | 355.943   | 0.0000000877             |
| 0.79  | 0.009698    | 226.110   | 0.000000445              |
| 0.83  | 0.013900    | 146.548   | 0.0000068265             |
| 0.87  | 0.019235    | 96.960    | 0.0000145882             |
| 0.91  | 0.025648    | 65.706    | 0.0000296835             |
| 0.95  | 0.032949    | 45.759    | 0.0000571911             |
| 0.98  | 0.038858    | 35.533    | 0.0000899288             |
| 0.99  | 0.040891    | 32.769    | 0.0001037719             |
| 1     | 0.042951    | 30.268    | 0.0001192876             |

Table 5. The relationship between $q_2$ and $\tau_0$ ($q_1 = 0.98$, $q_3 = 0.95$).

| $q_2$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-------|-------------|-----------|--------------------------|
| 0.63  | 0.033009    | 46.423    | 0.0000585174             |
| 0.67  | 0.034213    | 43.729    | 0.0000646623             |
| 0.71  | 0.035273    | 41.558    | 0.0000702717             |
| 0.75  | 0.036199    | 39.808    | 0.0000752912             |
| 0.79  | 0.037000    | 38.400    | 0.0000796967             |
| 0.83  | 0.037686    | 37.275    | 0.0000834898             |
| 0.87  | 0.038266    | 36.382    | 0.0000866934             |
| 0.91  | 0.038751    | 35.681    | 0.0000893461             |
| 0.95  | 0.039149    | 35.141    | 0.0000914972             |
| 0.98  | 0.039398    | 34.824    | 0.0000928146             |
| 0.99  | 0.039472    | 34.733    | 0.0000932021             |
| 1     | 0.039542    | 34.649    | 0.0000935654             |

Table 6. The relationship between $q_3$ and $\tau_0$ ($q_1 = 0.98$, $q_2 = 0.92$).

| $q_3$ | $\omega_0$ | $\tau_0$ | Transversality Condition |
|-------|-------------|-----------|--------------------------|
| 0.63  | 0.010074    | 236.261   | 0.0000029780             |
| 0.67  | 0.013125    | 170.912   | 0.0000057800             |
| 0.71  | 0.016401    | 128.390   | 0.0000102249             |
| 0.75  | 0.019836    | 99.378    | 0.0000167178             |
| 0.79  | 0.023394    | 78.732    | 0.0000256033             |
| 0.83  | 0.027064    | 63.492    | 0.0000371544             |
| 0.87  | 0.030851    | 51.884    | 0.0000515967             |
| 0.91  | 0.034772    | 42.804    | 0.0000691252             |
| 0.95  | 0.038858    | 35.533    | 0.0000899288             |
| 0.98  | 0.042054    | 30.971    | 0.0001078058             |
| 0.99  | 0.043148    | 29.589    | 0.0001142184             |
| 1     | 0.044259    | 28.269    | 0.0001208652             |

It can be seen from Tables 4–6 and Figure 5 that $q_1$ has an important influence over $q_2$ and $q_3$ on the stability of system (38). That is to say, in an ecosystem such as model Equation (38), the prey is the main fact that affects the stability of the ecosystem. It can be expressed that the change of prey affects the population density not only of prey but also of predator, which intensifies the turbulence of the ecosystem. Moreover, immature predators have more influence than mature predators when fractional-order is less than 0.8 in this ecosystem. It can be seen that when the three fractional-orders tend to 1, respectively, the critical value of delay changes gradually converge. This further shows that it is more practical to use fractional-order to explain the evolution of the ecosystem.
Now, we add a linear delay feedback controller to the system to control the emergence of Hopf bifurcation. All the parameters of the controller system are the same as system (38), and the feedback gain is selected as $L = -0.01$, then the controlled system can be described as

$$
\begin{align*}
D^{0.98}x(t) &= \frac{0.3x(t)}{1+0.05y_1(t)+0.1y_2(t)} - 0.1x(t) - 0.3x^2(t) - \frac{0.95x(t)y_1(t)}{1+0.95x(t)+0.95y_1(t)+0.95y_2(t)} \\
D^{0.92}y_1(t) &= \frac{0.76x(t-\tau)y_1(t-\tau)}{1+0.95x(t-\tau)+0.95y_1(t-\tau)+0.95y_2(t-\tau)} - 0.9y_1(t) \\
D^{0.95}y_2(t) &= 0.8y_1(t) - 0.1y_2(t) + L(y_2(t) - y_2(t-\tau)).
\end{align*}
$$

(39)

The bifurcation critical point of controlled system (39) is $(\omega_0^* = 0.028071, \tau_0^* = 61.544)$. This means the emergence of Hopf bifurcation is put off obviously, and the stability region is enlarged successfully. The influence of feedback gain $L$ on the bifurcation critical point is illustrated by numeric simulations in Table 7 and Figure 6.

Table 7. The relationship between $L$ and $\tau_0^*$.

| $L$   | $\omega_0^*$ | $\tau_0^*$ | Transversality Condition |
|-------|--------------|------------|--------------------------|
| $-0.003$ | 0.036053 | 40.542 | 0.0000654495 |
| $-0.005$ | 0.034009 | 44.827 | 0.0000512848 |
| $-0.007$ | 0.031796 | 50.214 | 0.0000388544 |
| $-0.010$ | 0.028071 | 61.544 | 0.0000234772 |
| $-0.013$ | 0.023680 | 80.183 | 0.0000120318 |
| $-0.015$ | 0.020215 | 101.259 | 0.0000065762 |
| $-0.018$ | 0.013707 | 172.017 | 0.0000015764 |
| $-0.020$ | 0.007774 | 344.276 | 0.0000002167 |

It can be seen from Figure 6 that as the feedback gain decreases, the system converges to a steady state faster. In other words, the smaller the feedback gain, the better the control effect of the controller to the Hopf bifurcation.
Figure 6. Waveform plots of system (39) with $\tau = 50$, feedback gains are $L = -0.002$, $L = -0.007$ and $L = -0.012$, respectively. The control effect increases as the feedback gain decreases.

6. Conclusions

In this paper, the dynamic behaviors of a fractional-order delay stage structure prey-predator model with two fear effects and prey refuge are explored by the linearized method and Laplace transform of fractional-order delay differential equation. Firstly, the conditions for the existence of the coexistence equilibrium point of the system (9) is deduced through the implicit function derivation rule and the function monotonicity theory. Secondly, the stability of the coexistence equilibrium point of the system (9) is investigated with the delay as parameter, and sufficient conditions for the emergence of Hopf bifurcation of the system (9) are obtained. Thirdly, a linear delay feedback controller is added to the system (9) to control the emergence of Hopf bifurcation, and the result states that the system can be controlled successfully by selecting an appropriate feedback gain. Finally, two examples are introduced to validate the theoretical results with the help of numerical simulation.

Moreover, some numerical simulations are performed to explore the influence facts of stability of system (9). The results show that fear factors $k_1, k_2$, prey refuge rate $m$ and fractional-orders $q_i$ ($i = 1, 2, 3$) have distinguish effects to the bifurcation critical value of delay $\tau$, and then affect the stability region of the system. These results have some implications for the regulation and management of ecosystems described in system (9).

However, because of lacking the complete theory of fractional-order differential equation, all the theoretical analyses in this paper are performed on the linearization system of original system (9) and the rationality of theoretical analysis is verified by numerical simulation. We are trying to study the stability of fractional differential equations theoretically in the next work as it is a challenging problem.

Author Contributions: Conceptualization, Y.L.; formal analysis, J.S.; methodology, Y.L. and H.F.; software, Y.L. and H.F.; supervision, J.S.; validation, Y.L.; writing—original draft, Y.L.; writing—review and editing, Y.L., J.S. and H.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (No. 11561034 and No. 11761040).
Data Availability Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The authors are greatly thankful to the editor and the referees for their valuable suggestions, which help to improve this paper considerably.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Computation of the expressions $a_1, a_2, a_3$ and $a_4$ in Equation (24)

\[
\begin{align*}
\alpha_1 &= -\omega^{\theta_1} \cos\left(\frac{q_1 \pi}{2}\right) a_{23} a_{31} - \omega^{\theta_3} \cos\left(\frac{q_3 \pi}{2}\right) a_{12} a_{21} + a_{11} a_{23} a_{31} + a_{12} a_{21} a_{32} - a_{13} a_{21} a_{31}, \\
\alpha_2 &= -\omega^{\theta_1} \sin\left(\frac{q_1 \pi}{2}\right) a_{23} a_{31} - \omega^{\theta_3} \sin\left(\frac{q_3 \pi}{2}\right) a_{12} a_{21}, \\
\alpha_3 &= \omega^{\theta_1+\theta_2+\theta_3} \cos\left(\frac{\pi(q_1 + q_2 + q_3)}{2}\right) - \omega^{\theta_1+\theta_2} \cos\left(\frac{\pi(q_1 + q_2)}{2}\right) a_{32} + \omega^{\theta_1} \cos\left(\frac{q_1 \pi}{2}\right) a_{22} a_{32} \\
&\quad - \omega^{\theta_1+\theta_3} \cos\left(\frac{\pi(q_1 + q_3)}{2}\right) a_{22} + a_{11} (\omega^{\theta_2} \cos\left(\frac{q_2 \pi}{2}\right) a_{32} + \omega^{\theta_3} \cos\left(\frac{q_3 \pi}{2}\right) - a_{32}) \\
&\quad - \omega^{\theta_2+\theta_3} \cos\left(\frac{\pi(q_2 + q_3)}{2}\right) a_{11}, \\
\alpha_4 &= \omega^{\theta_1} \sin\left(\frac{q_1 \pi}{2}\right) a_{22} a_{32} + \omega^{\theta_3} \sin\left(\frac{q_3 \pi}{2}\right) a_{11} a_{22} + \omega^{\theta_2} \sin\left(\frac{q_2 \pi}{2}\right) a_{11} a_{32} \\
&\quad - \omega^{\theta_1+\theta_3} \sin\left(\frac{\pi(q_1 + q_3)}{2}\right) a_{22} - \omega^{\theta_1+\theta_2} \sin\left(\frac{\pi(q_1 + q_2)}{2}\right) a_{32} \\
&\quad - \omega^{\theta_2+\theta_3} \sin\left(\frac{\pi(q_2 + q_3)}{2}\right) a_{11} + \omega^{\theta_1+\theta_2+\theta_3} \sin\left(\frac{\pi(q_1 + q_2 + q_3)}{2}\right).
\end{align*}
\]

Appendix B

Computation of the expressions $A_1, B_1, A_2$ and $B_2$ in Equation (29)

\[
\begin{align*}
A_1 &= \omega_0\left[-\omega^{\theta_1} a_{23} a_{31} \sin\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) + ((a_{12} a_{32} - a_{13} a_{31}) a_{21} + a_{11} a_{23} a_{31}) \sin(\omega_0 \tau_0) \right. \\
&\quad - \omega^{\theta_3} a_{12} a_{21} \sin\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right), \\
B_1 &= \tau_0 a_{31} a_{23} \omega_0^{\theta_1} \cos\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) + \tau_0 a_{21} a_{12} \omega_0^{\theta_3} \cos\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right) \\
&\quad + q_1 a_{31} a_{23} \omega_0^{\theta_1-1} \sin\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) + q_3 a_{21} a_{12} \omega_0^{\theta_3-1} \sin\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right) \\
&\quad + \omega_0^{\theta_1-1+\theta_2+\theta_3} (q_1 + q_2 + q_3) \sin\left(\frac{\pi(q_1 + q_2 + q_3)}{2}\right) \\
&\quad - a_{32} \omega_0^{\theta_1-1+\theta_2} (q_1 + q_2) \sin\left(\frac{\pi(q_1 + q_2)}{2}\right) - \omega_0^{\theta_1-1+\theta_3} a_{22} (q_1 + q_3) \sin\left(\frac{\pi(q_1 + q_3)}{2}\right) \\
&\quad - \omega_0^{\theta_2-1+\theta_3} a_{11} (q_2 + q_3) \sin\left(\frac{\pi(q_2 + q_3)}{2}\right) + \omega_0^{\theta_1-1} \sin\left(\frac{\pi(q_1 \pi}{2}\right) q_1 a_{22} a_{32} \\
&\quad + \omega_0^{\theta_2-1} \sin\left(\frac{q_2 \pi}{2}\right) q_2 a_{11} a_{32} + \omega_0^{\theta_3-1} \sin\left(\frac{q_3 \pi}{2}\right) q_3 a_{11} a_{22} \\
&\quad - \tau_0 (a_{11} a_{23} a_{31} - a_{21} (a_{12} a_{32} + a_{13} a_{31})) \cos(\omega_0 \tau_0), \\
A_2 &= \omega_0\left[-\omega^{\theta_1} a_{23} a_{31} \cos\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) + ((a_{12} a_{32} - a_{13} a_{31}) a_{21} + a_{11} a_{23} a_{31}) \cos(\omega_0 \tau_0) \right. \\
&\quad - \omega^{\theta_3} a_{12} a_{21} \cos\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right), \\
B_2 &= q_1 a_{31} a_{23} \omega_0^{\theta_1-1} \cos\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) + q_3 a_{21} a_{12} \omega_0^{\theta_3-1} \cos\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right) \\
&\quad - \tau_0 a_{31} a_{23} \omega_0^{\theta_1} \sin\left(-\frac{q_1 \pi}{2} + \omega_0 \tau_0\right) - \tau_0 a_{21} a_{12} \omega_0^{\theta_3} \sin\left(-\frac{q_3 \pi}{2} + \omega_0 \tau_0\right).
\end{align*}
\]
\[
- \omega_0 q_1 - q_2 + q_3 (q_1 + q_2 + q_3) \cos \left( \frac{\pi (q_1 + q_2 + q_3)}{2} \right) \\
+ a_{32} \omega_0^2 q_1 - q_2 (q_1 + q_2) \cos \left( \frac{\pi (q_1 + q_2)}{2} \right) + \omega_0 q_1 - q_2 q_3 \cos \left( \frac{\pi (q_1 + q_3)}{2} \right) \\
+ \omega_0 q_2 - q_3 a_{11} (q_2 + q_3) \cos \left( \frac{\pi (q_2 + q_3)}{2} \right) - \omega_0 q_1 - q_2 \cos \left( \frac{q_1 \pi}{2} \right) q_1 a_{22} a_{32} \\
- \omega_0 q_2 - q_3 \cos (\frac{q_2 \pi}{2}) q_2 a_{11} a_{32} - \omega_0 q_1 - q_2 \cos (\frac{q_3 \pi}{2}) q_3 a_{11} a_{22} \\
+ a_{11} a_{23} a_{31} - a_{21} (-a_{12} a_{32} + a_{13} a_{31}) \sin (\omega_0 t_0).
\]

Appendix C

Computation of the expressions \(\beta_1, \beta_2, \beta_3, \gamma_1\) and \(\gamma_2\) in Equation (35)

\[
\beta_1 = \omega_0 q_1 + q_2 + q_3 \cos \left( \frac{\pi (q_1 + q_2 + q_3)}{2} \right) - \omega_0 q_1 + q_2 (L + a_{32}) \cos \left( \frac{\pi (q_1 + q_2)}{2} \right) \\
- \omega_0 q_1 + q_3 \cos \left( \frac{\pi (q_1 + q_3)}{2} \right) a_{22} - \omega_0 q_2 + q_3 \cos \left( \frac{\pi (q_2 + q_3)}{2} \right) a_{11} \\
+ (\omega_0 q_2 (L + a_{32}) \cos \left( \frac{q_2 \pi}{2} \right) - a_{22} (-\omega_0 q_3 \cos \left( \frac{q_3 \pi}{2} \right) + L + a_{32}) a_{11} \\
+ \omega_0 q_2 a_{22} (L + a_{32}) \cos \left( \frac{\pi q_1}{2} \right),
\]

\[
\beta_2 = -\omega_0 q_1 + q_2 (L + a_{32}) \sin \left( \frac{\pi (q_1 + q_2)}{2} \right) - \omega_0 q_1 + q_3 \sin \left( \frac{\pi (q_1 + q_3)}{2} \right) a_{22} \\
+ \omega_0 q_2 a_{22} (L + a_{32}) \sin \left( \frac{q_1 \pi}{2} \right) + \sin \left( \frac{q_2 \pi}{2} \right) (L + a_{32}) a_{11} \omega_0 q_3 + \sin \left( \frac{q_3 \pi}{2} \right) \omega_0 q_1 a_{12} a_{22} \\
+ \omega_0 q_1 + q_3 \sin \left( \frac{\pi (q_1 + q_2 + q_3)}{2} \right) - \omega_0 q_2 + q_3 \sin \left( \frac{\pi (q_2 + q_3)}{2} \right) a_{11},
\]

\[
\beta_3 = -L a_{12} a_{21},
\]

\[
\gamma_1 = -\omega_0 (L a_{22} + a_{23} a_{31}) \cos \left( \frac{q_1 \pi}{2} \right) - L a_{22} \cos \left( \frac{q_2 \pi}{2} \right) a_{11} - \omega_0 q_3 \cos \left( \frac{q_3 \pi}{2} \right) a_{12} a_{21} \\
+ (L a_{22} + a_{23} a_{31}) a_{11} + a_{21} (L + a_{32}) a_{12} + \omega_0 q_2 + q_3 \cos \left( \frac{\pi (q_1 + q_2)}{2} \right) L - a_{13} a_{21} a_{31},
\]

\[
\gamma_2 = L a_{22} + a_{23} a_{31} \sin \left( \frac{\pi (q_1 + q_2)}{2} \right) - \omega_0 (L a_{22} + a_{23} a_{31}) \sin \left( \frac{q_1 \pi}{2} \right) - L a_{22} \sin \left( \frac{q_2 \pi}{2} \right) a_{11} \\
- \omega_0 q_3 \sin \left( \frac{q_3 \pi}{2} \right) a_{12} a_{21}.
\]

Appendix D

Computation of the expressions \(C_1, D_1, C_2\) and \(D_2\) in Equation (37)

\[
C_1 = -\omega_0^2 (-L (\omega_0^0 q_1 + q_2) \sin \left( \frac{-q_1 - q_2}{2} \right) + \omega_0^2 t_0^0) + L (\omega_0^0) q_1 a_{11} \sin \left( \frac{-q_2 \pi}{2} \right) + \omega_0^2 t_0^0 \\
+ (\omega_0^0) q_1 \left( L a_{22} + a_{23} a_{31} \right) \sin \left( \frac{-q_1 \pi}{2} \right) + \omega_0^2 t_0^0 + (\omega_0^0) q_2 a_{12} \sin \left( \frac{-q_3 \pi}{2} \right) + \omega_0^2 t_0^0 \\
+ 2 \sin \left( \omega_0^0 t_0^0 \right) L a_{12} a_{21} - \sin \left( \omega_0^0 t_0^0 \right) (a_{11} a_{22} + a_{12} a_{21} L + (a_{12} a_{32} - a_{13} a_{31}) a_{21} \\
+ a_{11} a_{23} a_{31}),
\]

\[
D_1 = -L (\omega_0^0) q_1 - q_2 \sin \left( \frac{-q_1 - q_2}{2} \right) + \omega_0^2 t_0^0 + (\omega_0^0) q_3 - \sin \left( \frac{q_3 \pi}{2} \right) q_3 a_{11} a_{22} \\
- L (\omega_0^0) q_1 + q_2 \cos \left( \frac{-q_1 - q_2}{2} \right) + \omega_0^2 t_0^0 + a_{12} a_{21} (\omega_0^0) q_3 \sin \left( \frac{q_3 \pi}{2} \right) + \omega_0^2 t_0^0 \\
+ (L a_{22} + a_{23} a_{31}) t_0^0 (\omega_0^0) q_1 \sin \left( \frac{-q_1 \pi}{2} \right) + \omega_0^2 t_0^0 + (L + a_{32}) (\omega_0^0) q_2 - \sin \left( \frac{q_2 \pi}{2} \right) a_{11} q_2 \sin \left( \frac{q_2 \pi}{2} \right) + \omega_0^2 t_0^0 \\
+ (L a_{22} + a_{23} a_{31}) (\omega_0^0) q_1 \sin \left( \frac{-q_1 \pi}{2} \right) + \omega_0^2 t_0^0 + \omega_0^2 t_0^0 \right) L a_{11} \sin \left( \frac{-q_2 \pi}{2} \right) + \omega_0^2 t_0^0 \\
+ a_{12} a_{21} t_0^0 (\omega_0^0) q_3 \cos \left( \frac{-q_3 \pi}{2} \right) + \omega_0^2 t_0^0 + (\omega_0^0) q_2 - \sin \left( \frac{q_2 \pi}{2} \right) q_2 L a_{11} \sin \left( \frac{-q_2 \pi}{2} \right) + \omega_0^2 t_0^0.
\]
\[ C_2 = -L(\omega_0^* \sin(\frac{\omega_0^* q_2}{2}) + \omega_0^* \tau_0) + 2 \cos(2\omega_0^* \tau_0) LA_{12a21} \]

\[ D_2 = L(\omega_0^* q_2 \sin(\frac{\omega_0^* q_2}{2}) + \omega_0^* \tau_0) - a_{12a21} \tau_0 (\omega_0^* \sin(\frac{3\omega_0^* q_2}{2}) + \omega_0^* \tau_0) \]

\[ + \omega_0^* q_2 a_{11} \cos(\frac{\omega_0^* q_2}{2}) + (\omega_0^* \tau_0) \sin(\frac{3\omega_0^* q_2}{2}) + \omega_0^* \tau_0) \]

\[ -a_{12a21} (\omega_0^*)^3 \tau_0 \cos(\frac{3\omega_0^* q_2}{2}) + \omega_0^* \tau_0) - \tau_0 (\omega_0^*)^2 LA_{11} \sin(\frac{q_2}{2}) + \omega_0^* \tau_0) \]

\[ - (\omega_0^*)^2 \sin(\frac{q_2}{2}) + \omega_0^* \tau_0) + 2 \cos(2\omega_0^* \tau_0) LA_{12a21} \]

References

1. He, X.; Zhao, X.; Feng, T.; Qiu, Z. Dynamical behaviors of a prey-predator model with foraging arena scheme in polluted environments. Math. Slovaca 2021, 71, 235–250. [CrossRef]

2. Kumar, A.; Dubey, B. Stability and Bifurcation of a Prey-Predator System with Additional Food and Two Discrete Delays. Comput. Model. Eng. Sci. 2021, 126, 505–547. [CrossRef]

3. Panigoro, H.S.; Suryanto, A.; Kusumawinahyu, W.M.; Darti, I. Dynamics of an Eco-Epidemic Predator–Prey Model Involving Fractional Derivatives with Power-Law and Mittag–Leffler Kernel. Symmetry 2021, 13, 785. [CrossRef]

4. Satria, A.; Putri, A.R.; Syafwan, M. Stability Analysis and Numerical Simulation of a Diffusive Prey-Predator Holling Type II Model. J. Phys. Conf. Ser. 2021, 2040, 012013. [CrossRef]

5. Das, M.; Samanta, G.P. A delayed fractional order food chain model with fear effect and prey refuge. Math. Comput. Simul. 2020, 178, 218–245. [CrossRef]

6. Shireen, J.; Raid, K.N. Stability Analysis of Stage Structure Prey-Predator Model with a Partially Dependent Predator and Prey Refuge. Int. J. Environ. Manuf. 2022, 12, 1–11.

7. Al-Salti, N.; Al-Musalhi, F.; Gandhi, V.; Al-Moqbil, M.; Elmojtaba, I. Dynamical analysis of a prey-predator model incorporating a prey refuge with variable carrying capacity. Ecol. Complex. 2021, 45, 100888. [CrossRef]

8. Alabacy, Z.K.; Majeed, A.A. The Fear Effect on a Food Chain Prey-Predator Model Incorporating a Prey Refuge and Harvesting. J. Phys. Conf. Ser. 2021, 1804, 012077. [CrossRef]
9. Li, S. Hopf bifurcation, stability switches and chaos in a prey-predator system with three stage structure and two time delays. *Math. Biosci. Eng.* **2019**, *16*, 6934–6961. [CrossRef]
10. Panja, P.; Jana, S.; Mondal, S.K. Dynamics of a stage structure prey-predator model with ratio-dependent functional response and anti-predator behavior of adult prey. *Numer. Algebra Control. Optim.* **2021**, *11*, 391–405. [CrossRef]
11. Alkhasawneh, R.A. A New Stage Structure Predator-Prey Model with Diffusion. *Int. J. Appl. Comput. Math.* **2021**, *7*, 1–11. [CrossRef]
12. Sarkar, K.; Khajanchi, S. Impact of fear effect on the growth of prey in a predator-prey interaction model. *Ecol. Complex.* **2020**, *42*, 100826. [CrossRef]
13. Cresswell, W. Predation in bird populations. *J. Ornithol.* **2011**, *152*, 251–263. [CrossRef]
14. Yan, J.C.; Liu, X.N. A Predator-Prey System with Beddington–DeAngelis Functional Response and Fear Effect. *J. Southwest Univ.* **2018**, *40*, 109–114.
15. Zhang, H.; Cai, Y.; Fu, S.; Wang, W. Impact of the fear effect in a prey-predator model incorporating a prey refuge. *Appl. Math. Comput.* **2019**, *356*, 328–337. [CrossRef]
16. Du, Y.P.; Huang, Y.; Li, D.S.; Zhang, H. Predator Recognition in Prey Animals and the Application in Reintroduction Program. *Sichuan J. Zool.* **2012**, *31*, 332–335.
17. Kar, T.K.; Batabyal, A. Stability and bifurcation of a prey–predator model with time delay. *Comptes Rendus Biol.* **2009**, *332*, 642–651. [CrossRef]
18. Rao, F.; Castillo-Chavez, C.; Kang, Y. Dynamics of a stochastic delayed Harrison-type predation model: Effects of delay and stochastic components. *Math. Biosci. Eng.* **2018**, *15*, 1401–1423. [CrossRef]
19. Chang, L.; Sun, G.Q.; Wang, Z.; Jin, Z. Rich dynamics in a spatial predator–prey model with delay. *Math. Slovaca* **2015**, *256*, 540–550. [CrossRef]
20. Celik, C. The stability and Hopf bifurcation for a predator–prey system with time delay. *Chaos Solitons Fractals* **2008**, *37*, 87–99. [CrossRef]
21. Yang, Y. Hopf bifurcation in a two-competitor, one-prey system with time delay. *Appl. Math. Comput.* **2009**, *214*, 228–235. [CrossRef]
22. Bagley, R.L.; Torvik, P.J. A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity. *J. Rheol.* **1983**, *27*, 201–210. [CrossRef]
23. Bagley, R.L.; Torvik, P.J. Fractional calculus—A different approach to the analysis of viscoelastically damped structures. *Asian J. 1983*, *21*, 741–748. [CrossRef]
24. Sakthivel, R.; Ganesh, R.; Ren, Y.; Anthoni, S.M. Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 3498–3508. [CrossRef]
25. Li, C.; Ma, Y. Fractional dynamical system and its linearization theorem. *Nonlinear Dyn.* **2012**, *71*, 621–633. [CrossRef]
26. Miller, K.S.; Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations; Wiley: New York, NY, USA, 1993.
27. Brunner, H. Numerical Solution of Fractional Differential Equations. *Math. Sci. Eng.* **1999**, *198*, 223–242.
28. Zhao, H.; Zhang, X.; Huang, X. Hopf bifurcation and spatial patterns of a delayed biological economic system with diffusion. *Appl. Math. Comput.* **2015**, *266*, 462–480. [CrossRef]
29. Xiao, M.; Zheng, W.X.; Cao, J. Hopf bifurcation of an (n + 1)-neuron bidirectional associative memory neural network model with delays. *IEEE Trans. Neural Netw. Learn. Syst.* **2013**, *24*, 118–132. [CrossRef]
30. Shi, M.; Wang, Z.H. Stability and Hopf bifurcation control of a fractional-order small world network model. *Sci. Sin.* **2013**, *43*, 467–477.
31. Xiao, M.; Jiang, G.; Zhao, L. State feedback control at Hopf bifurcation in an exponential RED algorithm model. *Nonlinear Dyn.* **2014**, *76*, 1469–1484. [CrossRef]
32. Chen, S.; Zou, Y.; Zhang, X. An efficient method for Hopf bifurcation control in fractional-order neuron model. *IEEE Access* **2019**, *7*, 77490–77498. [CrossRef]
33. Ding, D.; Qian, X.; Hu, W.; Wang, N.; Liang, D. Chaos and Hopf bifurcation control in a fractional-order memristor-based chaotic system with time delay. *Eur. Phys. J. Plus* **2017**, *132*, 447. [CrossRef]
34. Zheng, K.; Zhou, X.; Wu, Z.; Wang, Z.; Zhou, T. Hopf bifurcation controlling for a fractional order delayed paddy ecosystem in the fallow season. *Adv. Differ. Ecu.* **2019**, *2019*, 307. [CrossRef]
35. Wang, C.X.; Hu, Z.X. Stability of a delayed prey-predatory model with fear effect and prey refuge. *J. Henan Norm. Univ.* **2021**, *49*, 10–17.
36. Mukherjee, D. Dynamical study of non-integer order predator–prey system with fear effect. *Int. J. Model. Simul.* **2022**, *42*, 441–449. [CrossRef]
37. Podlubny, I. *Fractional Differential Equations*; Academic Press: London, UK, 1999.
38. Caputo, M. Linear models of dissipation whose Q is almost frequency independent—II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
39. Phillips, M.J. Transform Methods with Applications to Engineering and Operations Research. *J. Oper. Res. Soc.* **1978**, *29*, 1038–1039. [CrossRef]
40. Liu, X.; Fang, H. Periodic pulse control of Hopf bifurcation in a fractional-order delay predator–prey model incorporating a prey refuge. *Adv. Differ. Equ.* **2019**, *2019*, 479. [CrossRef]
41. Xiao, Z.; Li, Z.; Zhu, Z.; Chen, F. Hopf bifurcation and stability in a Beddington–DeAngelis predator-prey model with stage structure for predator and time delay incorporating prey refuge. *Open Math.* **2019**, *17*, 141–159. [CrossRef]

42. Lin, S.D.; Lu, C.H. Laplace transform for solving some families of fractional differential equations and its applications. *Adv. Differ. Equ.* **2013**, *2013*, 137. [CrossRef]

43. Bhalekar, S.; Daftardar-Gejji, V. A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. *J. Fract. Calc. Appl.* **2011**, *1*, 1–9.