Some simple biset functors

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Abstract: Let $p$ be a prime number, let $H$ be a finite $p$-group, and let $F$ be a field of characteristic 0, considered as a trivial $F\text{Out}(H)$-module. The main result of this paper gives the dimension of the evaluation $S_{H,F}(G)$ of the simple biset functor $S_{H,F}$ at an arbitrary finite group $G$. A closely related result is proved in the last section: for each prime number $p$, a Green biset functor $E_p$ is introduced, as a specific quotient of the Burnside functor, and it is shown that the evaluation $E_p(G)$ is a free abelian group of rank equal to the number of conjugacy classes of $p$-elementary subgroups of $G$.

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1. Introduction

Let $R$ be a commutative ring. The biset category $RC$ over $R$ has finite groups as objects, with morphisms $\text{Hom}_{RC}(G, H) = R \otimes_{\mathbb{Z}} B(H, G)$, where $B(H, G)$ is the Burnside group of $(H, G)$-bisets. The composition of morphisms is induced by the usual tensor product of bisets. A biset functor over $R$ is an $R$-linear functor from $RC$ to the category $R\text{-Mod}$ of $R$-modules. Biset functors over $R$ form an abelian category, where morphisms are natural transformations of functors. They have proved a useful tool in various aspects of the representation theory of finite groups (see [12], [7], [8], [9]), and they are still the object of active research ([15], [11], [10], [16], [13], [14], [5], [3], [2], ...).

The simple biset functors over $R$ are parametrized ([6], Proposition 2) by equivalence classes of pairs $(H, W)$, where $H$ is a finite group, and $W$ is a simple $R\text{Out}(H)$-module - the simple functor parametrized by $(H, W)$ being denoted $S_{H,W}$. However for a finite group $G$, the computation of the evaluation $S_{H,W}(G)$ is generally quite hard: in Theorem 4.3.20 of [9], this evaluation is shown to be equal to the image of a complicated linear map. Assuming that $R$ is a field - which is always possible when dealing with simple functors - the dimension of $S_{H,W}(G)$ is given by Theorem 7.1 of [11], as the rank of a yet complicated bilinear form with values in $R$.

Let $F$ be a field of characteristic 0, let $p$ be a prime number, and $H$ be a finite $p$-group. The present paper is mainly devoted to the computation of the dimension of the evaluation $S_{H,F}(G)$, where $G$ is an arbitrary finite group, and $F$ is the trivial $F\text{Out}(H)$-module. The result is as follows:
**Theorem:** Let $\mathbb{F}$ be a field of characteristic 0, let $p$ be a prime number, and $H$ be a finite $p$-group. Let moreover $G$ be a finite group.

1. If $H = 1$, the dimension of $S_{H,\mathbb{F}}(G)$ is equal to the number of conjugacy classes of cyclic subgroups of $G$.

2. If $H \cong C_p \times C_p$, the dimension of $S_{H,\mathbb{F}}(G)$ is equal to the number of conjugacy classes of non-cyclic $p$-elementary subgroups of $G$.

3. If $H$ is any other finite $p$-group, the dimension of $S_{H,\mathbb{F}}(G)$ is equal to the number of conjugacy classes of sections $(T, S)$ of $G$ such that $T/S \cong H$ and $T$ is $p$-elementary.

In the last section of this paper, for each prime number $p$, we introduce a Green biset functor $E_p$, closely related to the two first assertions of the above theorem. Green biset functors have been defined in [9], Section 8.5. They are ring objects in the category of biset functors. For a finite group $G$, we denote by $F_p(G)$ the set of elements of the Burnside group $B(G)$ which vanish when restricted to all $p$-elementary subgroups of $G$, and we show that this actually defines a biset subfunctor $F_p$ of $B$. The functor $E_p$ is defined as the quotient $B/F_p$, and it then inherits from $B$ a Green biset functor structure (over $\mathbb{Z}$). We show moreover that its evaluation $E_p(G)$ at a finite group $G$ is a free abelian group of rank equal to the number of conjugacy classes of $p$-elementary subgroups of $G$. We also show that the biset functor $\mathbb{F}E_p = \mathbb{F} \otimes \mathbb{Z} E_p$ fits in a non split short exact sequence

$$0 \to S_{(C_p)^2,\mathbb{F}} \to \mathbb{F}E_p \to S_{1,\mathbb{F}} \to 0$$

of biset functors over $\mathbb{F}$. In the case $\mathbb{F} = \mathbb{Q}$, the restriction of this sequence to $p$-groups is the short exact sequence of Theorem D of [12], involving the Dade functor $\mathbb{Q}D$, the Burnside functor $\mathbb{Q}B$, and the functor of rational representation $\mathbb{Q}RQ$.

2. Preliminary results

Recall (Sections 5.3 and 5.4 of [9]) that for a normal subgroup $N$ of a finite group $G$, the rational number $m_{G,N}$ is defined by

$$m_{G,N} = \frac{1}{|G|} \sum_{X \leq G, XN = G} |X| \mu(X, G),$$

where $\mu$ is the M"{o}bius function of the poset of subgroups of $G$. The group $G$ is called a $B$-group if $m_{G,N} = 0$ for any non-trivial normal subgroup $N$. 
of $G$. Any finite group $G$ has a largest quotient $B$-group $\beta(G)$, unique up to isomorphism. If $N \trianglelefteq G$, then $m_{G,N} = 0$ if and only if $\beta(G) \cong \beta(G/N)$.

2.1. Lemma: [M. Baumann [3] - See also [6], p 713] Let $L$ be a finite group, let $p$ be a prime, and let $E$ be an elementary abelian $p$-group on which $L$ acts irreducibly, faithfully, and such that $H^1(L, E) = \{0\}$. Then the group $G = E \rtimes L$ is a $B$-group.

Proof: First as $E$ is $L$-simple, it follows that $E$ is a minimal normal subgroup of $G$. Let $N$ be any normal subgroup of $G$. Then $N \cap E$ is equal to $E$ or $1$. So if $N \not\trianglelefteq E$, then $N \cap E = 1$, and $N$ centralizes $E$. But $L$ acts faithfully on $E$. Thus $N \leq E$, hence $N = 1$.

It follows that $E$ is the unique minimal normal subgroup of $G$. By Proposition 5.6.4 of [9], since $E$ is abelian,

$$m_{G,E} = 1 - \frac{|K_G(E)|}{|E|},$$

where $K_G(E)$ is the set of complements of $E$ in $G$. The group $E$ acts by conjugation on $K_G(E)$, and the normalizer in $E$ of $K \in K_G(N)$ is equal to the group $E^K$ of fixed points of $K$ on $E$. Since $E$ is $K$-simple, and $K$-faithful, this is equal to $1$. Thus $E$ acts freely on $K_G(E)$. Since $H^1(L, E) = \{0\}$, the set $K_G(E)$ is a single conjugacy class, i.e. a single $E$-orbit. Thus $|K_G(E)| = |E|$, and $m_{G,E} = 0$. It follows that $G$ is a $B$-group.

2.3. Recall that a finite group $G$ is called cyclic modulo a prime number $p$ if $G/O_p(G)$ is cyclic, and that $G$ is called $p$-elementary if $G \cong P \times C$, where $P$ is a $p$-group and $C$ is a cyclic group.

2.4. Lemma: Let $p$ be a prime number, and $G$ be a finite group.

1. [M. Baumann [3]] The group $\beta(G)$ is cyclic modulo $p$ if and only if $G$ is cyclic modulo $p$.

2. The group $\beta(G)$ is a $p$-group if and only if $G$ is $p$-elementary.

Proof: For Assertion 1, use the fact that by a theorem of Conlon, the subspace $NC_p(G)$ of $\mathbb{Q}B(G)$ generated by the idempotents $e^G_H$, where $H$ is not cyclic modulo $p$, is equal to the kernel of the morphism $\mathbb{Q}B(G) \to \mathbb{Q}pp_k(G)$. In particular, the correspondence $G \mapsto NC_p(G)$ is a biset subfunctor of $\mathbb{Q}B$. It follows that there exists a family $\mathcal{B}$ of $B$-groups such that for any group $G$, the space $NC_p(G)$ is the $\mathbb{Q}$-vector subspace of $\mathbb{Q}B(G)$ generated by the idempotents $e^G_H$, where $\beta(H) \in \mathcal{B}$. The family $\mathcal{B}$ consists of those
B-groups $H$ for which $e_H^p \in NC_p(H)$, i.e. the $B$-groups which are not cyclic modulo $p$. Now for any group $G$, and any subgroup $H$ of $G$, the idempotent $e_H^p$ is in $NC_p(G)$ if and only if $\beta(G) \in B$, on the one hand, but also if and only if $H$ is not cyclic modulo $p$. Hence $\beta(H)$ is not cyclic modulo $p$ if and only if $H$ is not cyclic modulo $p$. This proves Assertion 1.

For Assertion 2, clearly, if $G$-is $p$-elementary, one can assume $G \cong P \times C$, where $P$ is a $p$-group, and $C$ is a cyclic $p'$-group. By Proposition 5.6.6 of [9], this implies $\beta(G) \cong \beta(P) \times \beta(C) \cong \beta(P)$, since $\beta(C) = 1$. Hence $\beta(G)$ is a $p$-group.

Conversely, suppose that $\beta(G)$ is a $p$-group. In particular, it is cyclic modulo $p$, hence $G$ is cyclic modulo $p$, by Lemma 2.1. The Frattini subgroup $\Phi(P)$ of $P$ is a normal subgroup of $G$, and $G/\Phi(P) \cong \overline{P} \rtimes C$, where $\overline{P}$ is the elementary abelian group $P/\Phi(P)$. Suppose that the $\mathbb{F}_pC$-module $\overline{P}$ admits a simple quotient $E$ with non-trivial $C$-action (that is, not isomorphic to $\mathbb{F}_p$).

Then the action of $C$ on $E$ has a kernel $D < C$, and the group $E \rtimes (C/D)$ is a quotient of $\overline{P} \rtimes C$, hence a quotient of $G$. But $E \rtimes (C/D)$ is a $B$-group by Lemma 2.1. Indeed $E$ is $(C/D)$-simple and faithful by construction, and $H^1(C/D, E) = \{0\}$, since $C/D$ is a $p'$-group.

Now $E \rtimes (C/D)$ is a $B$-group, which is not a $p$-group, since $D \neq C$, and it is a quotient of $G$, hence of $\beta(G)$, which is a $p$-group. This is a contradiction.

Hence $C$ acts trivially on $\overline{P}$. But for any $p$-group $P$, the kernel of the morphism $\text{Aut}(P) \rightarrow \text{Aut}(P/\Phi(P))$ is a $p$-group. As $C$ is a $p'$-group, and acts trivially on $\overline{P}$, it acts trivially on $P$. Thus $G \cong P \times C$, as was to be shown.

\[ \Box \]

2.5. Lemma: Let $p$ be a prime number, and $P$ be a finite $p$-group.

1. Let $Q$ be a normal subgroup of $P$. Then $Q \cap \Phi(P) = 1$ if and only if $Q$ is elementary abelian and central in $P$, and admits a complement in $P$.

2. Let $Q$ and $R$ be normal subgroups of $P$, such that $|Q| = |R|$. Then $Q \cap \Phi(P) = 1 = R \cap \Phi(P)$ if and only if $Q$ and $R$ are elementary abelian and central in $P$, and admit a common complement in $P$.

In this case, set $H = P/R \cong P/Q$, and denote by $\gamma$ the rank of the group $H/\Phi(H)$. If $Q$ and $R$ have rank $m$, and if $Q \cap R \Phi(P)$ has rank $m - s$, the number of common complements of $Q$ and $R$ in $P$ is equal to

\[ (p^s - 1)(p^{s+1} - 1) \cdots (p - 1)p^{s(m-s)+\gamma-s}. \]

Proof: For Assertion 1, if $Q$ is elementary abelian and central in $P$, and
admits a complement $L$, then $P = Q \times L$. Thus $\Phi(P) = 1 \times \Phi(L)$, hence $Q \cap \Phi(P) = 1$. Conversely, if $Q \cap \Phi(P) = 1$, then $Q$ maps injectively into $P/\Phi(P)$, so $Q$ is elementary abelian. Let $L \geq \Phi(P)$ be a subgroup of $P$ such that $L/\Phi(P)$ is a complement of $Q\Phi(P)/\Phi(P)$ in the $\mathbb{F}_p$-vector space $P/\Phi(P)$. Then $Q\Phi(P)L = P$, thus $QL = P$, and $Q\Phi(P) \cap L = \Phi(P)$, i.e. $Q \cap L \leq Q \cap \Phi(P) = 1$. Since $L \geq \Phi(P)$, it follows that $L \leq P$, thus $[L, Q] \leq L \cap Q = 1$, and $Q$ is central in $P$.

For Assertion 2, let $Q$ and $R$ be normal subgroups of $P$ with $|Q| = |R|$. If $Q$ and $R$ are elementary abelian central subgroups of $P$ with a common complement in $P$, then $Q \cap \Phi(P) = R \cap \Phi(P) = 1$ by Assertion 1. Conversely, if $Q \cap \Phi(P) = 1$ and $R \cap \Phi(P) = 1$, then $Q$ and $R$ are elementary abelian and central in $P$ by Assertion 1. If $L$ is a complement of $Q$ in $P$, then $P = Q \times L$ for $Q$ is central in $P$, thus $L \leq P$, and $P/L \cong Q$ is elementary abelian. Thus $L \geq \Phi(P)$, and $L/\Phi(P)$ is a complement of $Q\Phi(P)/\Phi(P)$ in $P/\Phi(P)$. Conversely if $L/\Phi(P)$ is a complement of $Q\Phi(P)/\Phi(P)$ in $P/\Phi(P)$, then $L$ is a complement of $Q$ in $P$, by the argument used in the proof of Assertion 1.

So finding a common complement to $Q$ and $R$ in $P$ amounts to finding a common complement of $\tilde{Q} = Q\Phi(P)/\Phi(P)$ and $\tilde{R} = R\Phi(P)/\Phi(P)$ in $\tilde{P} = P/\Phi(P)$. Moreover $|\tilde{Q}| = |Q| = |R| = |\tilde{R}|$. The $\mathbb{F}_p$-vector space $\tilde{P}$ can be split as $\tilde{P} = I \oplus E \oplus F \oplus V$, where $I = \tilde{Q} \cap \tilde{R}$, where $E$ is a complement of $I$ in $\tilde{Q}$ and $F$ is a complement of $I$ in $\tilde{R}$, and $V$ is a complement of $\tilde{Q} + \tilde{R}$ in $\tilde{P}$. Then $L = F \oplus V$ is a complement of $\tilde{Q}$ in $\tilde{P}$, and all the other complements of $\tilde{Q}$ are of the form $\{(\varphi(x), x) \mid x \in L\}$, where $\varphi : L \to \tilde{Q}$ is a group homomorphism. In other words, any complement $L'$ of $\tilde{Q}$ is of the form

$$L' = \{(a(f) + b(v), c(f) + d(v), f, v) \mid f \in F, v \in V\},$$

where $a : F \to I$, $b : V \to I$, $c : F \to E$ and $d : V \to E$ are group homomorphisms. The group $L'$ is a complement of $\tilde{R}$ if and only if its intersection with $\tilde{R}$ is trivial, or equivalently if $c$ is injective, hence an isomorphism, since $|E| = |F|$.

It follows that the number of common complements of $\tilde{Q}$ and $\tilde{R}$ in $\tilde{P}$ is equal to the number of 4-tuples $(a, b, c, d)$, where $c$ is an isomorphism. Hence

$$|K_P(Q) \cap K_P(R)| = |\text{Aut}(E)||\text{Hom}(F, I)||\text{Hom}(V, I)||\text{Hom}(V, E)|$$

$$= |\text{Aut}(E)||\text{Hom}(F, I)||\text{Hom}(V, \tilde{Q})|.$$  

Moreover

$$I = (Q\Phi(P) \cap R\Phi(P))/\Phi(P) = (Q \cap R\Phi(P))\Phi(P)/\Phi(P) \cong Q \cap R\Phi(P)$$

has rank $m - s$, and $F \cong E \cong \tilde{Q}/I$ has rank $s$. Finally

$$V \cong \tilde{P}/(\tilde{Q}\tilde{R}) \cong (P/R\Phi(P))/(Q\Phi(P)/R\Phi(P)).$$
has rank $\gamma - s$, since $P/R\Phi(P) \cong H/\Phi(H)$, as $\Phi(P/R) = R\Phi(P)/R$, and since $QR\Phi(P)/R\Phi(P) \cong Q/(Q \cap R\Phi(P))$. This completes the proof.

2.6. Corollary: Let $P$ be a finite $p$-group, and $M$ be a normal subgroup of $P$. Then

$$P/(M \cap \Phi(P)) \cong E \times (P/M) ,$$

where $E = M/(M \cap \Phi(P))$ is elementary abelian.

Proof: The normal subgroup $M = M/(M \cap \Phi(P))$ of $P = P/(M \cap \Phi(P))$ intersect the Frattini subgroup $\Phi(P) = \Phi(P)/(M \cap \Phi(P))$ trivially, hence there exists a subgroup $L$ of $P$ such that $P = M \times L$. Moreover $L \cong P/M \cong P/M$.

3. Simple biset functors and bilinear forms

Let $F$ be any field. Recall (see [11]) that, given a finite group $H$, we defined, for any finite group $G$

$$\mathbb{F}\mathcal{B}(G, H) = \mathbb{F}B(G, H)/ \sum_{|K| < |H|} \mathbb{F}B(G, K) \circ \mathbb{F}B(K, H) ,$$

and that the correspondence $G \mapsto \mathbb{F}\mathcal{B}(G, H)$ is a quotient biset functor of the Yoneda functor $G \mapsto \mathbb{F}B(G, H)$ at the group $H$.

When $V$ is a $\mathbb{F}\text{Out}(H)$-module, we defined an $F$-valued bilinear form $\langle , \rangle_{V,G}$ on $\mathbb{F}\mathcal{B}(G, H)$ by

$$\forall \alpha, \beta \in \mathbb{F}\mathcal{B}(G, H), \langle \alpha, \beta \rangle_{V,G} = \chi_V(\pi_H(\hat{\alpha}^{\text{op}} \circ \hat{\beta})) ,$$

where $\hat{\alpha}, \hat{\beta}$ are elements of $\mathbb{F}B(G, H)$ lifting $\alpha, \beta \in \mathbb{F}\mathcal{B}(G, H)$, respectively, where $\pi_H : \mathbb{F}B(H, H) \to \mathbb{F}\mathcal{B}(H, H) \cong \mathbb{F}\text{Out}(H)$ is the projection map, and $\chi_V$ is the character of $V$, i.e. the trace function $\text{End}_F(V) \to F$. The main property of these constructions is that

$$\mathbb{F}\mathcal{B}(G, H)/\text{Rad}\langle , \rangle_{V,G} \cong S_{H,V}(G)^{\text{dim}_F V} .$$

Moreover, if $L$ is a finite group, then for any $\gamma \in B(L, G)$, any $\alpha \in \mathcal{B}(G, H)$, and any $\beta \in \mathcal{B}(L, H)$,

$$\langle \gamma(\alpha), \beta \rangle_{V,L} = \langle \alpha, \gamma^{\text{op}}(\beta) \rangle_{V,G} .$$
3.2. Suppose from now on that \( \mathbb{F} \) is a field of characteristic 0. Observe that \( \tilde{e}_K^G = (e_K^G)^{op} \) for any subgroup \( K \) of a finite group \( G \). By \([3.1]\), this implies that the decomposition

\[
\mathbb{F} \overline{B}(G, H) = \bigoplus_{K \in [s_G]} \tilde{e}_K^G \mathbb{F} \overline{B}(G, H),
\]

where \([s_G]\) is a set of representatives of conjugacy classes of subgroups of \( G \), is an orthogonal decomposition with respect to the form \( \langle \, , \rangle_{V,G} \). Moreover

\[
e_G^H S_{H,V}(G)^{dimr V} \cong \tilde{e}_G^G \mathbb{F} \overline{B}(G, H)/\text{Rad}\langle \, , \rangle_{V,G},
\]

and the isomorphism

\[
\mathbb{F} \overline{B}(G, H) \cong \bigoplus_{K \in [s_G]} \left( \tilde{e}_K^K \mathbb{F} \overline{B}(K, H) \right)^{N_G(K)}
\]

given by Proposition 6.5.5 of \([9]\) induces an isomorphism

\[
(3.3) \quad S_{H,V}(G)^{dimr V} \cong \bigoplus_{K \in [s_G]} \left( \tilde{e}_K^K \mathbb{F} \overline{B}(K, H)/\text{Rad}\langle \, , \rangle_{V,K} \right)^{N_G(K)}.
\]

Now \( \overline{B}(K, H) \) is generated by the images of the elements \( (K \times H)/L \), where \( L \) is a subgroup of \( K \times H \). If this image is non-zero, then \( L \) is of the form \( L = \{(x, s(x)) \mid x \in X\} \), where \( X \) is a subgroup of \( K \) and \( s : X \to H \) is a surjective group homomorphism.

The \((K, G)\)-biset \( U = (K \times H)/L \) factors as \( U = \text{Ind}_X^K \circ V \), for a suitable \((X, H)\)-biset \( V \) \([9] \), Lemma 2.3.26), and by \([9]\), Corollary 2.5.12

\[
\tilde{e}_K^K \circ \text{Ind}_X^K = \text{Ind}_X^K \circ \text{Res}_X^K \tilde{e}_K^K.
\]

Now \( \text{Res}_X^K \tilde{e}_K^K = 0 \) if \( X \) is a proper subgroup of \( K \). It follows that \( \tilde{e}_K^K \mathbb{F} \overline{B}(K, H) \) is generated by the images \( \pi_s \) of the elements

\[
u_s = \tilde{e}_K^K \times_K (K \times H)/\Delta_s^X(K),
\]

where \( \Delta_s^X(K) = \{(x, s(x)) \mid x \in K\} \), for a surjective group homomorphism \( s : K \to H \).

Let \( \varpi_H : \text{Aut}(H) \to \text{Out}(H) \) denote the projection map. Then:
3.4. Proposition: Let \( s, t : K \to H \) be two surjective group homomorphisms. Let \( M = \ker s \) and \( N = \ker t \). Then

\[
\langle \overline{u}_s, \overline{u}_t \rangle_{V,K} = m_{K,M \cap N} \sum_{Y \in \mathcal{K}(K,M,N)} \chi_V \left( \varpi_H([s,Y,t]) \right)
\]

where \( \mu_{\leq K} \) is the Möbius function of the poset of normal subgroups of \( K \), and

\[
\mathcal{K}(K,M,N) = \{ Y \leq K \mid YN = YM = K, \ Y \cap N = Y \cap M = M \cap N \}
\]

is the set of subgroups \( Y \) of \( K \), containing \( M \cap N \), such that \( Y/(M \cap N) \) is a common complement of \( M/(M \cap N) \) and \( N/(M \cap N) \) in \( K/(M \cap N) \). Moreover for \( Y \in \mathcal{K}(K,M,N) \), the symbol \([s,Y,t]\) denotes the automorphism of \( H \) defined by \([s,Y,t](t(y)) = s(y), \forall y \in Y\).

Proof: By definition, and since \( \tilde{e}_K^K \) is an idempotent

\[
\langle \overline{u}_s, \overline{u}_t \rangle_{V,K} = \chi_V \left( \pi_H(u_{s}^{op} \circ u_t) \right)
\]

\[
= \chi_V \left( \pi_H \left( (H \times K)/\Delta_s(K) \times_K \tilde{e}_K^K \times_K (K \times H)/\Delta_t^K(K) \right) \right),
\]

where \( \Delta_s(K) = \{(s(x),x) \mid x \in K \} \). Set

\[
a_{s,t} = (H \times K)/\Delta_s(K) \times_K \tilde{e}_K^K \times_K (K \times H)/\Delta_t^K(K).
\]

Then

\[
a_{s,t} = \frac{1}{|K|} \sum_{L \leq K} |L| \mu(L,K)(H \times H)/\Delta_{s,t}(L),
\]

where \( \Delta_{s,t}(L) = \{(s(l),t(l)) \mid l \in L \} \).

This subgroup of \( H \times H \) is equal to \( \Delta_{\theta}(H) \), for some automorphism \( \theta \) of \( H \), if and only if

(3.5) \[ L \cap M = L \cap N \quad \text{and} \quad LM = K = LN, \]

where \( M = \ker s \) and \( N = \ker t \). In this case the automorphism \( \theta \) is defined by \( \theta(t(l)) = s(l) \), for any \( l \in L \). The two conditions \( 3.5 \) and the automorphism \( \theta \) remain unchanged when \( L \) is replaced by \( L(M \cap N) \). Moreover
the conditions 3.5 are equivalent to saying that the group \( Y = L(M \cap N) \) is in \( \mathcal{K}(K, M, N) \), and in this case \( \theta = [s, Y, t] \). Conversely, fix some \( Y \in \mathcal{K}(K, M, N) \), and consider all the subgroups \( L \) of \( K \) such that \( L(M \cap N) = Y \).

Recall that
\[
\sum_{L \leq Y, L(M \cap N) = Y} |L| \mu(L, K) = m_{K, M \cap N} |Y| \mu(Y, K) .
\]

This gives:
\[
\pi_H(a_s,t) = m_{K, M \cap N} \sum_{Y \in \mathcal{K}(K, M, N)} \frac{|Y|}{|K|} \mu(Y, K) \varpi([s, Y, t]) .
\]

Now if \( Y/(M \cap N) \) is a complement of \( M/(M \cap N) \) in \( K/(M \cap N) \), and if \( K/M \cong H \), it follows that \( |Y| = |M \cap N||H| \). Moreover the poset \( |Y, K| \) is isomorphic to the poset \( |M \cap N, M| \). But since \( M \) and \( N \) are normal subgroups of \( K \), the commutator group \( [M, N] \) is contained in \( M \cap N \). It follows that \( |M \cap N, M| = |M \cap N, M|^Y = |M \cap N, M|^Y N = |M \cap N, M|^K \), and that \( \mu(Y, K) = \mu_{\leq K}(M \cap N, M) \). This completes the proof of the first equality of the proposition. The second one follows from the observation that the correspondences
\[
Y \mapsto [s, Y, t] \quad \text{and} \quad \theta \mapsto \{ k \in K \mid \theta(t(k)) = s(k) \}
\]
are mutual inverse bijections between \( \mathcal{K}(K, M, N) \) and the set of automorphisms \( \theta \) of \( H \) such that \( \Delta(\theta(H)) \leq (s \times t)(K) \) (see Section 8.3 of [6]).

\[
3.6. \text{Corollary:}
\]
1. If \( e^K_{S_{H,V}}(K) \neq \{0\} \), the group \( \beta(K) \) is isomorphic to \( \beta(L) \), where \( L \) is a subgroup of \( H \times H \) with the following properties:
   (a) \( p_1(L) = p_2(L) = H \).
   (b) \( k_1(L) \) and \( k_2(L) \) are direct products of minimal normal subgroups of \( H \).
   (c) There exist an automorphism \( \theta \) of \( H \) such that \( \theta(k_2(L)) = k_1(L) \).

2. In particular, if \( H \) is a \( p \)-group for some prime \( p \), then \( K \) is \( p \)-elementary. If \( K = P \times C \), where \( P \) is a \( p \)-group and \( C \) is a cyclic \( p' \)-group, then
\[
e^K_{S_{H,V}}(K) \cong e^p_P S_{H,V}(P) ,
\]
and this isomorphism is compatible with the action of \( \text{Aut}(K) \).
Proof: Indeed, if \( \check{K}_K S_{H, V}(K) \neq \{0\} \), then the bilinear form \( \langle , \rangle_{V,K} \) is not identically zero on \( \check{K}_K \overline{F}(K, H) \). It follows that there exist surjective group homomorphisms \( s, t : K \to H \) such that \( \langle \overline{s}, \overline{t} \rangle_{V,K} \neq 0 \).

Then \( m_{K,M \cap N} \neq 0 \), \( \mu_{\leq K}(M \cap N, M) \neq 0 \), and \( \overline{K}(K, M, N) \neq \emptyset \), where \( M = \ker s \) and \( N = \ker t \). Hence \( \beta(K) \cong \beta(K/(M \cap N)) \). Now \( K/(M \cap N) \) is isomorphic to \( L = (s \times t)(K) \), which is a subgroup of \( H \times H \) such that \( p_1(L) = p_2(L) = H \). Moreover \( k_1(L) = s(\ker t) \cong N/(M \cap N) \) and \( k_2(L) = t(\ker s) \cong M/(M \cap N) \). Then \( \mu_{\leq K}(M \cap N, M) = \mu_{\leq H}(1, t(\ker s)) \), and this is non zero if and only if the lattice \( [1, t(\ker s)]^H \) of normal subgroups of \( H \) contained in \( t(\ker s) \) is complemented, i.e. if \( t(\ker s) \) is a direct product of minimal normal subgroups of \( H \). Finally, let \( Y \in \overline{K}(K, M, N) \) and \( \theta = [s, Y, t] \). If \( u \in k_2(L) = t(\ker s) \), then there exist \( v \in \ker s \) and \( y \in Y \) such that \( u = t(v) = t(y) \). Then \( v^{-1}y \in \ker t \), and \( \theta(u) = s(y) = s(v^{-1}y) \in s(\ker t) = k_1(L) \). In other words \( \theta(k_2(L)) = k_1(L) \), which completes the proof of Assertion 1.

The first part of Assertion 2 follows from Assertion 2 of Lemma [2.4. Now if \( H \) is a \( p \)-group, if \( K = P \times C \), where \( P \) is a \( p \)-group and \( C \) is a \( p' \)-group, and if \( s : K \to H \) is a surjective group homomorphism, then \( C \leq \ker s \). In other words, there is a surjective homomorphism \( \overline{s} : P \to H \) such that \( s = \overline{s} \circ \pi \), where \( \pi : K \to P \) is the projection map. Moreover \( \ker s = \ker \overline{s} \times C \).

So with the notation of Proposition [3.4, \( M = \overline{M} \times C \), where \( \overline{M} = \ker \overline{s} \). Similarly \( N = \overline{N} \times C \), where \( \overline{N} = \ker \overline{t} \), and \( \overline{t} : P \to H \) is such that \( t = \overline{t} \circ \pi \). Clearly \( |M : M \cap N| = |\overline{M} : \overline{M} \cap \overline{N}| \). Moreover, one checks easily that

\[
m_{K,M \cap N} = m_{P \times C, (\overline{M} \cap \overline{N}) \cap C} = m_{P, \overline{M} \cap \overline{N}} m_{C, C} = m_{P, \overline{M} \cap \overline{N}} \frac{\phi(|C|)}{|C|},
\]

since \( P \) and \( C \) have coprime orders, and \( C \) is cyclic.

Also

\[
\mu_{\leq K}(M \cap N, M) = \mu_{\leq P}(\overline{M} \cap \overline{N}, \overline{M}) .
\]

Finally, the maps \( Q \mapsto Q \times 1 \) and \( Y \mapsto (Y \cap P) \) induce inverse bijections from \( \overline{K}(P, \overline{M}, \overline{N}) \) to \( \overline{K}(K, M, N) \), and for any \( Y \in \overline{K}(K, M, N) \),

\[
[s, Y, t] = [\overline{s}, Y \cap P, \overline{t}] .
\]

It follows that the matrix of the form \( \langle , \rangle_{V,K} \) on \( \check{K}_K \overline{F}(K, H) \) is equal to the matrix of the form \( \langle , \rangle_{V,P} \) on \( \check{K}_P \overline{F}(P, H) \), multiplied by the non-zero scalar \( \frac{\phi(|C|)}{|C|} \). Hence the two forms define isomorphic quadratic spaces. As all the above bijections are obviously compatible with the action of \( \text{Aut}(K) \).
and the canonical group homomorphism \( \text{Aut}(K) \to \text{Aut}(P) \), the induced isomorphism
\[
\tilde{e}_K^{S_{H,V}}(K) \cong \tilde{e}_P^{S_{H,V}}(P)
\]
is compatible with the action of \( \text{Aut}(K) \).

3.7. Notation: Let \( H \) and \( P \) be finite \( p \)-groups.

1. Let \( Q_{H}(P) \) denote the \( \mathbb{F} \)-vector space with basis the set
\[
\Sigma_{H}(P) = \{ s \mid s : P \to H \}
\]
of surjective group homomorphisms from \( P \) to \( H \), endowed with the \( \mathbb{F} \)-valued bilinear form \( \langle \cdot, \cdot \rangle_{V,P} \) defined as follows: For \( s, t \in \Sigma_{H}(P) \), set \( M = \text{Ker} s \) and \( N = \text{Ker} t \). If \( M \cap \Phi(P) \neq N \cap \Phi(P) \), set \( \langle s, t \rangle_{V,P} = 0 \). And if \( M \cap \Phi(P) = N \cap \Phi(P) \), then the groups \( M/(M \cap N) \) and \( N/(M \cap N) \) are central elementary abelian subgroups of the same rank of \( P/M \cap N \). In this case, set
\[
\langle s, t \rangle_{V,P} = m_{P,M \cap N}^{\mu(M \cap N,M)} [M : M \cap N] \chi_{V,H} \left( \sum_{Y \in \mathcal{K}(P,M,N)} \omega_H([s,Y,t]) \right).
\]

2. Let \( Q_{H}^{\sharp}(P) \) be the subspace of \( Q_{H}(P) \) with basis the subset
\[
\Sigma_{H}^{\sharp}(P) = \{ s \mid s : P \to H, \text{ Ker} s \cap \Phi(P) = 1 \}
\]
of \( \Sigma_{H}(P) \).

3. Set
\[
\mathcal{N}_{H}(P) = \{ N \mid N \leq P, N \cap \Phi(P) = 1 \}.
\]

4. Denote by \( \mathcal{E}_{H}(P) \) the set of normal subgroups \( R \) of \( P \), contained in \( \Phi(P) \), and such that \( P/R \cong E \times H \), for some elementary abelian \( p \)-group \( E \).

3.8. Proposition: Let \( H \) be a \( p \)-group, and \( K \) be a \( p \)-elementary group. Set \( P = O_{p}(K) \). Then:

1. There is an isomorphism of \( \mathbb{F}\text{Aut}(K) \)-modules
\[
\tilde{e}_K^{S_{H,V}}(K) \cong Q_{H}(P)/\text{Rad}\langle \cdot, \cdot \rangle_{V,P}.
\]
2. Let $\Gamma$ be a finite group acting on the group $K$. Then $\Gamma$ acts on the set $\mathcal{E}_H(P)$, and there is an isomorphism of $\mathbb{F}T$-modules

$$\hat{e}_K^{\mathcal{E}_H(V)}(K) \cong \bigoplus_{R \in [\Gamma \mathcal{E}_H(P)]} \text{Ind}_R^\Gamma (Q^\mathcal{E}_H(P/R)/\text{Rad}(\langle \cdot \rangle_{V,P/R})),$$

where $[\Gamma \mathcal{E}_H(P)]$ is a set of representatives of $\Gamma$-orbits on $\mathcal{E}_H(P)$, and $\Gamma_R$ denotes the stabilizer of $R$ in $\Gamma$.

**Proof:** The map $s \in \Sigma_H(P) \mapsto \bar{s} \in \hat{e}_P \mathcal{E}_H(P,H)$ induces a surjective linear map $Q_H(P) \to \hat{e}_P \mathcal{E}_H(P,H)$. Let $s, t \in \Sigma_H(P)$, and set $M = \text{Ker } s$ and $N = \text{Ker } t$. Then $|M| = |N|$. It follows from Lemma 2.5 that $\mathbf{K}(P, M, N) \neq \emptyset$ if and only if $M/(M \cap N)$ and $N/(M \cap N)$ are central elementary abelian subgroups of $P/(M \cap N)$, which intersect trivially the Frattini subgroup of $P/(M \cap N)$. But

$$\Phi(P/(M \cap N)) = \Phi(P)(M \cap N)/(M \cap N).$$

Hence $M/(M \cap N) \cap \Phi(P/(M \cap N)) = (M \cap \Phi(P))(M \cap N)/(M \cap N)$. This group is trivial if and only if $M \cap \Phi(P) \leq M \cap N$, i.e. if $M \cap \Phi(P) \leq N \cap \Phi(P)$. Hence $\mathbf{K}(P, M, N) \neq \emptyset$ if and only if $M \cap \Phi(P) = N \cap \Phi(P)$.

This holds in particular if $\langle \bar{s}, \bar{t} \rangle \neq 0$. In this case, by Proposition 3.4

$$\langle \bar{s}, \bar{t} \rangle_{V,P} = m_{P,M \cap N} \mu_{s,M,M \cap N} \chi_V \left( \sum_{Y \in \mathbf{K}(P,M,N)} \varphi_H([s,Y,t]) \right).$$

But $\mu_{s,P}(M \cap N, M) = \mu(M \cap N, M)$, since $P/(M \cap N)$ centralizes both $M/(M \cap N)$ and $N/(M \cap N)$. Hence

$$\langle \bar{s}, \bar{t} \rangle_{V,P} = \langle s, t \rangle_{V,P}$$

in this case, and Assertion 1 follows.

Since $\langle s, t \rangle_{V,P} = 0$ if $M \cap \Phi(P) \neq N \cap \Phi(P)$, the quadratic space $Q = \langle Q_H(P), \langle \cdot \rangle_{V,P} \rangle$ splits as the orthogonal sum of the subspaces $Q_R$ generated by the elements $s \in \Sigma_H(P)$ such that $\text{Ker } s \cap \Phi(P) = R$. These subspaces are permuted by the action of $\text{Aut}(K)$, and the space $Q_R$ is invariant by $\text{Aut}(K)_R$.

Let $\pi_R : P \to P/R$ be the canonical projection. The map

$$\theta_R : s \in \Sigma_H^2(P/R) \mapsto s \circ \pi_R$$

is a bijection from $\Sigma_H^2(P/R)$ to the set $\{s \in \Sigma_H(P) \mid \text{Ker } s \cap \Phi(P) = R\}$, and the map $Y \mapsto Y/R$ is a bijection from $\mathbf{K}(P, M, N)$ to $\mathbf{K}(P/R, M/R, N/R)$, such that

$$[s, Y/R, \bar{t}] = [\theta_R(s), Y, \theta_R(\bar{t})].$$

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for any $\bar{s}, \bar{t} \in \Sigma^2_H(P/R).
Moreover, if $M \cap \Phi(P) = N \cap \Phi(P) = R$, then

$$m_{P, M \cap N} = m_{P, R}m_{P/R, (M \cap N)/R} = m_{P/R, (M \cap N)/R},$$
as $R \leq \Phi(P)$. Also

$$M/(M \cap N) \cong (M/R)/(M \cap (N/R)),$$
It follows that

$$\forall \bar{s}, \bar{t} \in \Sigma^2_H(P/R), \quad \langle \theta_R(\bar{s}), \theta_R(\bar{t}) \rangle_{V, P} = \langle \bar{s}, \bar{t} \rangle_{V, P/R}.$$
Hence there is an isomorphism

$$Q_{R}/\text{Rad}\langle , \rangle_{V, P} \cong Q^2_H(P/R)/\text{Rad}\langle , \rangle_{V, P/R},$$
of $\mathbb{F}\Gamma_{R}$-modules.

To complete the proof of Assertion 2, it remains to observe that if $Q$ is a
$p$-group, the set $\Sigma^2_H(Q)$ is non-empty if and only if the group $Q$ is isomorphic
to $E \times H$, for some elementary abelian $p$-group $E$: Indeed, if $Q = E \times H$, where $E$ is elementary abelian, then $\Phi(Q) = 1 \times \Phi(H)$, and the projection
map $s : Q \to H = Q/E$ is an element of $\Sigma^2_H(Q)$. Conversely, if $s \in \Sigma^2_H(Q)$, then $E = \text{Ker} \ s$ is an elementary abelian central subgroup of
$Q$, which admits a complement $L$ in $Q$, by Lemma 2.5. Thus $Q = E \times L$, and $L \cong Q/E \cong H$. Hence $Q \cong E \times H$.

3.9. Theorem: Let $G$ be a finite group, let $H$ be a finite $p$-group, and let
$V$ be a simple $\mathbb{F}\text{Out}(H)$-module. Then

$$S_{H, V}(G)^{\dim V} \cong \bigoplus_{(K, R)} \left( Q^2_H(K_p/R)/\text{Rad}\langle , \rangle_{V, K_p/R} \right)^{N_G(K, R)}.$$
where $(K, R)$ runs through a set of $G$-conjugacy classes of pairs consisting
of a $p$-elementary subgroup $K$ of $G$, and a $p$-subgroup $R$ in $\mathcal{E}_H(K_p)$, where
$K_p = O_p(K)$, and $N_G(K, R) = N_G(K) \cap N_G(R)$.

Proof: This follows from Equation 3.3, Corollary 3.6 and Proposition 3.8.

4. Proof of the theorem

This section is devoted to the proof of the following theorem, announced in
the introduction:
4.1. Theorem: Let \( \mathbb{F} \) be a field of characteristic 0, let \( p \) be a prime number, and \( H \) be a finite \( p \)-group. Let moreover \( G \) be a finite group.

1. If \( H = 1 \), the dimension of \( S_{H,\mathbb{F}}(G) \) is equal to the number of conjugacy classes of cyclic subgroups of \( G \).

2. If \( H \cong C_p \times C_p \), the dimension of \( S_{H,\mathbb{F}}(G) \) is equal to the number of conjugacy classes of non-cyclic \( p \)-elementary subgroups of \( G \).

3. If \( H \) is any other finite \( p \)-group, the dimension of \( S_{H,\mathbb{F}}(G) \) is equal to the number of conjugacy classes of sections \((T,S)\) of \( G \) such that \( T/S \cong H \) and \( T \) is \( p \)-elementary.

Proof: Step 1: Let \( K \) be a subgroup of \( G \). Then \( e^K_{\mathbb{F}}S_{H,\mathbb{F}}(K) = \{0\} \), by Corollary 3.6, unless \( K \cong P \times C \), where \( P \) is a \( p \)-group and \( C \) is a cyclic \( p' \)-group, and in this case \( e^K_{\mathbb{F}}S_{H,\mathbb{F}}(K) \cong e^P_{\mathbb{F}}S_{H,\mathbb{F}}(P) \) as \( \mathbb{F} \text{Aut}(K) \)-modules.

By Proposition 3.8 there is an isomorphism of \( \mathbb{F} \text{Aut}(P) \)-modules

\[
e^P_{\mathbb{F}}S_{H,\mathbb{F}}(P) \cong \oplus \text{Ind}_{\text{Aut}(P)R}^{\text{Aut}(P)}(Q^t_H(P/R)/\text{Rad}, , )_{\mathbb{F}},
\]

where \( R \) runs through a set of representatives of \( \text{Aut}(P) \)-orbits of normal subgroups of \( P \) contained in \( \Phi(P) \), such that \( P/R \cong E \times H \), for some elementary abelian \( p \)-group \( E \). So the computation of \( e^P_{\mathbb{F}}S_{H,\mathbb{F}}(P) \) comes down to the computation of the \( \mathbb{F} \text{Aut}(Q) \)-module

\[
\mathbb{V}_H(Q) = Q^t_H(Q)/\text{Rad}(, )_{\mathbb{F}},
\]

for a \( p \)-group \( Q = P/R \) of the form \( E \times H \), where \( R \) is some normal subgroup of \( P \) contained in \( \Phi(P) \). Recall that \( Q^t_H(Q) \) is the \( \mathbb{F} \)-vector space with basis

\[
\Sigma_H^t(Q) = \{ s \mid s : Q \rightarrow H, \ \text{Ker } s \cap \Phi(Q) = 1 \},
\]

and that the bilinear form \( (, )_{\mathbb{F},Q} \) is defined for \( s, t \in \Sigma_H^t(Q) \) by

\[
(s, t)_{\mathbb{F},Q} = m_{Q,M\cap N} \frac{\mu(M\cap N, M)}{|M: M\cap N|} |K(Q, M, N)|,
\]

where \( M = \text{Ker } s \) and \( N = \text{Ker } t \).

This shows that \( (s, t)_{\mathbb{F},Q} \) depends only on \( M \) and \( N \). It follows that \( Q^t_H(Q)/\text{Rad}(, )_{\mathbb{F},Q} \) is also isomorphic to the quotient of the \( \mathbb{F} \)-vector space with basis the set \( \mathcal{N}_H(Q) = \{ N \subseteq Q \mid P/N \cong H, \ N \cap \Phi(Q) = 1 \} \) introduced in Notation 3.7, by the radical of the bilinear form \( (, )_{\mathbb{F},Q} \) defined by

\[
(M, N)_{\mathbb{F},Q} = m_{Q,M\cap N} \frac{\mu(M\cap N, M)}{|M: M\cap N|} |K(Q, M, N)|,
\]

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for $M, N \in \mathcal{N}_H(Q)$.

**Step 2:** Now Assertion 1 is well known (see e.g. Proposition 4.4.8 [9]), but it can also be recovered from the argument of Step 1: Indeed, if $H = 1$, there is a unique normal subgroup $N$ of $Q$ such that $Q/N \cong H$, namely $Q$ itself. If moreover $N \cap \Phi(Q) = 1$, then $\Phi(Q) = 1$, and $Q$ is elementary abelian. But as $Q = P/R$, for some $R \leq \Phi(P)$, it follows that $R = \Phi(P)$, and $Q = P/\Phi(P)$.

Moreover $\mathcal{N}_H(Q) = \{Q\}$, and

$$\langle Q, Q \rangle_{\mathbb{F}, Q} = m_{Q,Q},$$

which is equal to 0 if $Q$ is non-cyclic, and to $1 - 1/p$ otherwise. Hence $V_H(Q) = \{0\}$ if $Q$ is non-cyclic, and $V_H(Q)$ is one dimensional if $Q = P/\Phi(P)$ is cyclic, i.e. if $P$ is cyclic. But $P = O_p(K)$ for some $p$-elementary subgroup $K$ of $G$. Hence $P$ is cyclic if and only if $K$ itself is cyclic, and this leads to Assertion 1.

We can now assume that $H$ is a non-trivial $p$-group, of order $p^h$, and make a series of observations:

- Let $P$ be a $p$-group, and $Q$ be a normal subgroup of $P$. Example 5.2.3 of [9] shows that $m_{P,Q} = m_{P,\Phi(P)Q}$. By Proposition 5.3.1 of [9], it follows that

$$m_{P,Q} = m_{P,\Phi(P)m_{P/\Phi(P),Q\Phi(P)/\Phi(P)}} = m_{E,F},$$

where $E$ is the elementary abelian $p$-group $P/\Phi(P)$, and $F$ its subgroup $Q\Phi(P)/\Phi(P)$. If $E$ has rank $n \geq 2$ and $F$ has rank $k$, then

$$m_{E,F} = (1 - p^{n-2})(1 - p^{n-3})\cdots(1 - p^{n-k-1}),$$

(which is equal to 0 if $k \geq n - 1$, and non-zero otherwise). This follows from an easy induction argument on $k$, using Proposition 5.3.1 of [9], and starting with the case $k = 1$, which is a special case of Equation 2.2.

If $E$ has rank 1 and $F = 1$, then $m_{E,F} = 1$. In this case $m_{E,E} = 1 - 1/p$. This is the only case where $m_{E,F}$ is not an integer.

- Let $M, N \in \mathcal{N}_H(Q)$. Then in particular $M$ and $N$ have the same order. Recall that $m_{Q,M\cap N}$ is non-zero if and only if $\beta(Q) \cong \beta(Q/(M \cap N))$. So either $Q$ and $Q/(M \cap N)$ are both cyclic, or they are both non-cyclic. Equivalently, either $Q$ is cyclic, or $Q/(M \cap N)$ is non-cyclic. If $H$ is non-cyclic, then $Q/(M \cap N)$ is non-cyclic, as it maps surjectively on $Q/M \cong H$. 

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So if $m_{Q,M\cap N} = 0$, then $H$ is cyclic, $Q$ is non-cyclic, and $Q/(M \cap N)$ is cyclic. But then $M/(M \cap N) = N/(M \cap N)$, since the cyclic group $Q/(M \cap N)$ admits a unique subgroup of a given order. Thus $M = N$. Conversely, if $H$ is cyclic, if $Q$ is non-cyclic, and if $M = N$, then $m_{Q,M\cap N} = 0$ since $Q/(M \cap N) \cong H$ is cyclic.

- If $M, N \in \mathcal{N}_H(Q)$, then the subgroups $\overline{M} = M/(M \cap N)$ and $\overline{N} = N/(M \cap N)$ are central elementary abelian subgroups of the same order of $\overline{Q} = Q/(M \cap N)$. If $\overline{M}$ has rank $m$, then

$$\mu(M \cap N, M) = (-1)^m p\binom{m}{s}.$$ 

Now by Lemma 2.5 the product

$$(4.4) \quad \alpha_{M,N} = \mu(M \cap N, M)|K(Q, M, N)|$$

is equal to

$$\alpha_{M,N} = (-1)^m (p^s - 1)(p^{s-1} - 1) \cdots (p - 1)p^m \bigg( \frac{2}{s} \bigg) + (\gamma) + s(m-s) + m(\gamma-s)$$

$$= (-1)^m (1-p)(1-p^{s-1}) \cdots (1-p)p^m \bigg( \frac{2}{s} \bigg) + (\gamma) + s(m-s) + m(\gamma-s),$$

where $\gamma$ is the rank of $H/\Phi(H)$, and $s$ is the rank of $\overline{M}/(\overline{M} \cap \overline{N}\Phi(\overline{Q})) \cong M/(M \cap N\Phi(Q))$.

**Step 3:** Finally $\langle M, N \rangle_{F,Q}^2 = 0$ if and only if $m_{Q,M\cap N} = 0$, i.e. $H$ is cyclic, $M = N$, and $Q$ is not cyclic. In all other cases, the groups $\overline{M} = M/(M \cap N)$ and $\overline{N} = N/(M \cap N)$ are elementary abelian, and central in $\overline{Q} = Q/(M \cap N)$. Moreover $\overline{M} \cap \overline{N} = 1$. Let $m$ be the rank of $\overline{M}$, let $s$ denote the rank of $\overline{M}/(\overline{M} \cap \overline{N}\Phi(\overline{Q}))$, and let $\gamma$ denote the rank of $H/\Phi(H)$. Then

$$\langle M, N \rangle_{F,Q}^2 = m_{Q,M\cap N} \frac{\alpha_{M,N}}{|M:M\cap N|}$$

$$= m_{Q,M\cap N} \frac{\alpha_{M,N}}{|M|}$$

$$= (-1)^m m_{Q,M\cap N}(p^s - 1)(p^{s-1} - 1) \cdots (p - 1)p^m \bigg( \frac{2}{s} \bigg) + (\gamma) + s(m-s) + m(\gamma-s),$$

i.e. finally

$$(4.5) \quad \langle M, N \rangle_{F,Q}^2 = (-1)^m m_{Q,M\cap N}(p^s - 1)(p^{s-1} - 1) \cdots (p - 1)p^m \bigg( \frac{2}{s} \bigg) + (\gamma) + s(m-s) + m(\gamma-2).$$

Let $n$ denote the rank of $Q/\Phi(Q)$. By Equation 4.3

$$m_{Q,M\cap N} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p^{n-k-1})$$

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where \( k \) is the rank of \((M \cap N)\Phi(Q)/\Phi(Q) \cong M \cap N\). Since \( Q/M\Phi(Q) \cong H/\Phi(H) \) has rank \( \gamma \), it follows that \( M\Phi(Q)/\Phi(Q) \cong M \) has rank \( n - \gamma \). Since \( M/(M \cap N) \) has rank \( m \), it follows that \( k = n - m - \gamma \). Thus
\[
m_{Q,M \cap N} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p^{m+\gamma-1}) .
\]

It follows that
\[
(4.6) \quad (M, N)_{\mathbb{F}, Q}^\sharp = A_{M,N}(-1)^{m+s}p^{\frac{1}{2}(m-s)(m+s+1)+m(\gamma-2)} ,
\]
where
\[
A_{M,N} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p^{m+\gamma-1})(1 - p^{s})(1 - p^{s-1}) \cdots (1 - p)
\]
is an integer congruent to 1 modulo \( p \).

**Step 4:** Assume first that \( H \) is non-cyclic, i.e. that \( \gamma \geq 2 \). In this case \( (M, N)_{\mathbb{F}, Q}^\sharp \) is non-zero. If \( M = N \), then \( m = s = 0 \), and \( (M, N)_{\mathbb{F}, P}^\sharp = A_{M,M} \) is congruent to 1 modulo \( p \). And if \( M \neq N \), then \( m \geq 1 \). As \( \gamma \geq 2 \) and \( m \geq s \), the exponent
\[
\frac{1}{2}(m-s)(m+s+1) + m(\gamma-2)
\]
of \( p \) in the right hand side of (4.6) is non-negative. It is equal to 0 if and only if \( m = s \) and \( \gamma = 2 \). In this case \( \overline{M} \cap \overline{N}\Phi(\overline{Q}) = 1 \), so \( \overline{M} \) maps into \( \overline{Q}/\overline{N}\Phi(\overline{Q}) \cong H/\Phi(H) \), which has rank \( \gamma = 2 \). It follows that \( m \leq 2 \).

If \( m = 2 \), then \( \overline{M}\overline{N}\Phi(\overline{Q}) = \overline{Q} \), thus \( \overline{M}\overline{N} = \overline{Q} \), and \( H \cong \overline{Q}/\overline{N} \cong \overline{M} \) (since \( \overline{M} \cap \overline{N} = 1 \)), so \( H \) is elementary abelian of rank 2.

If \( m = 1 \), then \( \overline{M} \cong C_p \) maps into \( \overline{Q}/\overline{N}\Phi(\overline{Q}) \cong C_p \times C_p \), the group \( \overline{Q}/(\overline{M}\overline{N}\Phi(\overline{Q})) \) is cyclic, so \( \overline{Q}/\overline{M}\overline{N} \) is cyclic. But \( \overline{M}\overline{N} \) is a central subgroup of \( \overline{Q} \). It follows that \( \overline{Q} \) is abelian, so \( H \cong \overline{Q}/\overline{M} \) is abelian. Hence \( \overline{Q}/\overline{M} \) is non-cyclic, and it has a subgroup \( \overline{M}\overline{N}/\overline{M} \) of order \( p \) such that the corresponding quotient \( \overline{Q}/\overline{M}\overline{N} \) is cyclic. It follows that \( \overline{Q}/\overline{M} \cong H \cong C_p \times C_{p^{h-1}} \), for some \( h \geq 2 \).

**Step 5:** Assume that \( H \) is neither cyclic nor isomorphic to \( C_p \times C_{p^{h-1}} \), for some \( h \geq 2 \). Then the matrix of the bilinear form \( \langle \ , \ \rangle_{\mathbb{F}, Q}^\sharp \) is congruent to the identity matrix modulo \( p \). In particular, it is non-singular, and the \( \mathbb{F}\text{Aut}(Q) \)-module \( \mathcal{V}_H(Q) = Q_{\mathbb{F}, H}^\sharp(Q) \) is isomorphic to the permutation module on the set \( \mathcal{N}_H(Q) \).

It follows that the \( \text{Aut}(P) \)-module \( \mathcal{E}_P \mathcal{S}_{H,\mathbb{F}}(P) \) is isomorphic to the permutation module on the set of normal subgroups \( M \) of \( P \) such that \( P/M \cong H \).
Going back to Step 1 and to the $p$-elementary subgroup $K = P \times C$ of $G$, it follows that the space $\tilde{e}_K^*S_{H,F}(K)^{N_C(K)}$ has a basis in one to one correspondence with the $N_G(K)$-orbits of normal subgroups $M$ of $K$ such that $K/M \cong H$. Now the isomorphism (4.3) shows that $S_{H,F}(G)$ has a basis in one to one correspondence with the $G$-conjugacy classes of sections $(K,M)$ of $G$ such that $K$ is $p$-elementary and $K/M \cong H$. This proves the theorem, in the case where $H$ is neither cyclic nor isomorphic to $C_p \times C_{p^{h-1}}$, for some $h \geq 2$.

**Step 6:** Suppose now that $H$ is cyclic, of order $p^h > 1$. Assume first that $Q$ is cyclic. Then since $Q \cong E \times H$ for some elementary abelian $p$-group $E$, it follows that $E = 1$, i.e. $Q \cong H$. In this case $\mathcal{N}_H(Q) = \{1\}$, and $\langle 1,1 \rangle_{H,F} = 1$. Hence $\mathcal{V}_H(Q)$ is isomorphic to the trivial $\mathbb{F}\text{Aut}(Q)$-module in this case.

If $Q$ is non-cyclic, let $M,N \in \mathcal{N}_H(Q)$. Recall that

$$\langle M,N \rangle_{F,Q}^Z = m_{Q,M\cap N} \frac{\alpha_{M,N}}{|M:M\cap N|},$$

where $\alpha_{M,N}$ is defined in (4.4).

The groups $\overline{M} = M/(M \cap N)$ and $\overline{N} = N/(M \cap N)$ are non-trivial elementary abelian central subgroups of $\overline{Q} = Q/(M \cap N)$, and have a common complement in $\overline{Q}$. As $\overline{M}$ is isomorphic to the subgroup $MN/N$ of the cyclic group $Q/N \cong H$, it follows that $\overline{M} \cong C_p$. Moreover $\overline{M}$ has a complement in $\overline{Q}$, so $\overline{Q} \cong C_p \times C_{p^h}$. Hence if $Q/\Phi(Q)$ has rank $n$, then $\Phi(Q)(M\cap N)/\Phi(Q)$ has rank $n-2$ since

$$Q/(\Phi(Q)(M\cap N)) \cong \overline{Q}/\Phi(\overline{Q}) \cong C_p \times C_p .$$

By Equations 4.3 and 4.2 it follows that

$$m_{Q,M\cap N} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p) .$$

Moreover since $m = 1$ and $\gamma = 1$, Equation 4.5 gives

$$\langle M,N \rangle_{F,Q}^z = -m_{Q,M\cap N}(p^s-1)(p^{s-1}-1) \cdots (p-1)p^{\frac{1}{2}(1-s)(2+s)-1} .$$

Since $0 \leq s \leq m = 1$, there are two cases:

- If $s = 1$, then $\overline{M}$ maps into $\overline{Q}/\Phi(\overline{Q}) \cong H/\Phi(H) \cong C_p$, hence $MN = Q$ as above, and $Q/N \cong C_{p^h} \cong \overline{M} \cong C_p$, so $h = 1$. In this case

$$\langle M,N \rangle_{F,Q}^z = -(1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p)(p-1)/p = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p^2)(1 - p)^2/p .$$
If $s = 0$. Then $\overline{M} \leq \overline{N}\Phi(\overline{Q})$, so $\overline{M}\Phi(\overline{Q}) = \overline{N}\Phi(\overline{Q})$. If $h = 1$, then $\overline{Q}/\overline{M} \cong C_p$, so $\overline{M} \geq \Phi(\overline{Q})$, and it follows that $\overline{M} = \overline{N}$, a contradiction. Thus $h > 1$ in this case. Moreover

$$\langle M, N \rangle_{\mathbb{F}, Q} = - (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p).$$

So in any case, there is a non-zero rational number $\rho$, depending only on $Q$ (and $H$), such that $\langle M, N \rangle_{\mathbb{F}, Q}^2 = \rho$ when $\langle M, N \rangle_{\mathbb{F}, Q} \neq 0$. Moreover $\langle M, N \rangle_{\mathbb{F}, Q}^2 \neq 0$ if and only if $M \neq N$.

So the matrix of the form $\langle , \rangle_{\mathbb{F}, Q}^2$ is equal to $\rho J$, where $J$ is a matrix of size $|N_H(Q)|$, with zero diagonal, and non-diagonal coefficients equal to 1. Hence this matrix is non-singular if and only if $|N_H(Q)| > 1$.

But $Q = E \times L$, where $L \cong H$ and $E$ is a non-trivial elementary abelian $p$-group. The elements of $N_H(Q)$ are exactly the groups

$$E_{\varphi} = \{ (e, \varphi(e)) \mid e \in E \},$$

where $\varphi$ is a group homomorphism from $E$ to $L$. There are $|E|$ such homomorphisms, hence $|N_H(Q)| = |E| > 1$.

It follows that the matrix of the form $\langle , \rangle_{\mathbb{F}, Q}^2$ is non-singular, hence the form $\langle , \rangle_{\mathbb{F}, Q}^2$ is non-degenerate.

So either when $Q$ is cyclic, or when it is not, the form $\langle , \rangle_{\mathbb{F}, Q}^2$ is non-degenerate. By the same argument as at the end of Step 4, this proves that $S_{H, \mathbb{F}}(G)$ has a basis in one to one correspondence with the $G$-conjugacy classes of sections $(K, M)$ of $G$ for which $K$ is $p$-elementary and $K/M \cong H$. This proves the theorem in the case where $H$ is cyclic.

**Step 7:** Suppose now that $H \cong C_p \times C_p^{h-1}$, for some $h \geq 2$. Note that if $h = 2$, then $H$ is elementary abelian, so $Q = P/R \cong E \times H$ is elementary abelian. Since $R \leq \Phi(P)$, this forces $R = \Phi(P)$.

Now if $M, N \in N_H(Q)$, since $\gamma = 2$ in this case,

$$\langle M, N \rangle_{\mathbb{F}, Q}^2 = (-1)^m m_{Q,M \cap N}(p^s-1)(p^{s-1}-1) \cdots (p-1)p^{\frac{1}{2}(m-s)(m+s+1)},$$

and moreover

$$m_{Q,M \cap N} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p^{m+1}),$$

where $n$ is the rank of $Q/\Phi(Q)$, where $m$ is the rank of the elementary abelian groups $\overline{M} = M/(M \cap N)$ and $\overline{N} = N/(M \cap N)$ of $\overline{Q} = Q/(M \cap N)$, and $s$ is the rank of $M/(M \cap N\Phi(Q)) \cong \overline{M}/(\overline{M} \cap \overline{N}\Phi(Q))$. Since the exponent
\( \frac{1}{2}(m-s)(m+s+1) \) of \( p \) is non-negative, it follows that \( \langle M, N \rangle_{F, Q} \) is an integer. Moreover, if \( m > s \), this integer is a multiple of \( p \). On the other hand \( m = s \) if and only if \( M \cap N \cap \Phi(Q) = 1 \), or equivalently if \( M \cap N \cap \Phi(Q) = M \cap N \). Since \( M \cap \Phi(Q) = N \cap \Phi(Q) = 1 \), this is equivalent to \( MN \cap \Phi(Q) = 1 \). In this case

\[
\langle M, N \rangle_{F, Q} = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p)
\]

is congruent to 1 modulo \( p \). It follows that the matrix of the form \( \langle \cdot, \cdot \rangle_{F, Q} \) is congruent modulo \( p \) to the incidence matrix of the relation \( \sim \) on \( N \). There are now two cases:

- **Case 1:** Assume first that \( h \geq 3 \), i.e. that \( H \) is not elementary abelian of rank 2.

4.8. **Lemma:** Let \( H = C_p \times C_p^{h-1} \), for \( h \geq 3 \), and \( Q = E \times H \), where \( E \) is an elementary abelian \( p \)-group of rank \( e \). Let \( S \) denote the incidence matrix of the relation \( \sim \) on \( N_H(Q) \) defined by

\[
M \sim N \Leftrightarrow MN \cap \Phi(Q) = 1.
\]

Then:

1. if \( e = 0 \), the matrix \( S \) is the matrix (1).
2. if \( e \geq 1 \), the eigenvalues of \( S \) are \( p^{e+1} - p + 1, \ p^e - p + 1, \) and \( 1 - p, \) with respective multiplicities \( 1, \ p^{e+1} - p, \) and \( p^2 - p^{e+1} + p - 1. \)

In both cases \( M \) is invertible modulo \( p \).

**Proof:** If \( e = 0 \), then \( E = 1 \) and \( Q \cong H \), so \( N_H(Q) \) consists of the trivial subgroup \( E \) of \( Q \). Since \( E \sim E \), Assertion 1 follows.

If \( e \geq 1 \), then \( N_H(Q) \) consists of the subgroups

\[
E_\varphi = \{(x, \varphi(x)) \mid x \in E\},
\]

where \( \varphi : E \to H \) is a group homomorphism. Since \( E \) is elementary abelian, the image of \( \varphi \) is contained in the subgroup \( C_p \times C_p \) of \( H = C_p \times C_p^{h-1} \). So there are group homomorphisms \( a, b : E \to C_p \) such that \( \varphi = (a, b) \), i.e. \( \varphi(x) = (a(x), b(x)) \) for any \( x \in E \).

Let \( \varphi = (a, b) \) and \( \varphi' = (a', b') \) be two group homomorphisms from \( E \) to \( H \). Then, with an additive notation

\[
E_\varphi E_{\varphi'} = \{(x - x', a(x) - a(x'), b(x) - b(x')) \mid x, x' \in E\} \leq E \times C_p \times C_p.
\]
The element \((x - x', a(x) - a'(x'), b(x) - b'(x'))\) is in \(\Phi(Q) = 1 \times 1 \times C_p \times 2\) if and only if \(x = x'\) and \(a(x) = a'(x')\). Thus

\[
E_x \sim E_{x'} \iff \text{Ker}(a - a') \leq \text{Ker}(b - b') .
\]

Identifying \(E\) with the vector space \((\mathbb{F}_p)^c\), and \(C_p\) with \(\mathbb{F}_p\), the homomorphisms \(a, b, a', b'\) become elements of the dual vector space \(E^*\), and the condition \(\text{Ker}(a - a') \leq \text{Ker}(b - b')\) means that there is a scalar \(\lambda \in \mathbb{F}_p\) such that \(b - b' = \lambda(a - a')\). Hence the incidence matrix \(S\) is the matrix indexed by pairs \(((a, b), (a', b'))\) of pairs of elements of \(E^*\), defined by

\[
S(((a, b), (a', b'))) = \begin{cases} 1 & \text{if } \exists \lambda \in \mathbb{F}_p, \ b - b' = \lambda(a - a') \\ 0 & \text{otherwise} \end{cases} .
\]

Let \(T\) be the rectangular matrix indexed by the set of pairs \(((a, b), (c, \lambda))\), where \(a, b, c \in E^*\), and \(\lambda \in \mathbb{F}_p\), defined by

\[
T(((a, b), (c, \lambda))) = \begin{cases} 1 & \text{if } c = b - \lambda a \\ 0 & \text{otherwise} \end{cases} .
\]

Then for \(a, b, a', b' \in E^*\), consider the sum

\[
s = \sum_{\substack{c \in E^* \\
\lambda \in \mathbb{F}_p}} T(((a, b), (c, \lambda)))T(((a', b'), (c, \lambda))) .
\]

The non-zero terms in this summation correspond to pairs \((c, \lambda)\) such that \(c = b - \lambda a = b' - \lambda a'\). Hence \(s\) is equal to the number of \(\lambda \in \mathbb{F}_p\) such that \(b - \lambda a = b' - \lambda a'\). This is equal to \(1\) if \(a' \neq a\) and if \(b' - b\) is a scalar multiple of \(a' - a\), to \(p\) if \(a = a'\) and \(b = b'\), and to \(0\) if \(a = a'\) and \(b \neq b'\). In other words

\[
T \cdot \dagger T = S + (p - 1)\text{Id} .
\]

Since \(T \cdot \dagger T\) is symmetric, it is diagonalizable over \(\mathbb{C}\), with real eigenvalues. Let \(\mu\) be an eigenvalue of \(T \cdot \dagger T\), and \(u\) be a corresponding eigenvector. Then \(T \cdot \dagger Tu = \mu u\), thus \(\dagger T \cdot T \cdot \dagger Tu = \mu \dagger Tu\). So either \(\dagger Tu = 0\), and then \(\mu = 0\).

And if \(\mu \neq 0\), then \(\dagger Tu\) is an eigenvector of \(\dagger T \cdot T\) for the eigenvalue \(\mu\).

Moreover, the map \(u \mapsto \dagger Tu\) is an injection of the \(\mu\)-eigenspace of \(T \cdot \dagger T\) into the \(\mu\)-eigenspace of \(\dagger T \cdot T\). The same argument applied to \(\dagger T \cdot T\) instead of \(T \cdot \dagger T\) shows that these two matrices have the same non-zero eigenvalues, and the same multiplicities.

Now for \(c, c' \in E^*\) and \(\lambda, \lambda' \in \mathbb{F}_p\)

\[
\dagger T \cdot T(((c, \lambda), (c', \lambda'))) = \sum_{a, b \in E^*} T(((a, b), (c, \lambda)))T(((a, b), (c', \lambda'))) .
\]
The right hand side is the number of pairs \((a, b)\) of elements of \(E^*\) such that 
\[c = b - \lambda a\] and 
\[c' = b - \lambda' a,\] i.e. the number of elements \(a \in E^*\) such that 
\[c + \lambda a = c' + \lambda' a,\] or \(c - c' = (\lambda - \lambda') a.\) This is equal to 1 if \(\lambda \neq \lambda',\) to \(|E|\) if \(\lambda = \lambda'\) and \(c = c',\) and to 0 if \(\lambda = \lambda'\) and \(c \neq c'.\) Hence the matrix \(t' T \cdot T\) is a block matrix of the following form

\[
t' T \cdot T = \begin{pmatrix}
|E| Id & \Omega & \cdots & \Omega \\
\Omega & |E| Id & \cdots & \Omega \\
\vdots & \vdots & \ddots & \vdots \\
\Omega & \Omega & \cdots & |E| Id
\end{pmatrix},
\]

where all the \(p^2\)-blocks are square matrices of size \(|E|\), and \(\Omega\) is a matrix with all entries equal to 1. Let \(\mu\) be an eigenvalue of this matrix, and

\[
v = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_p
\end{pmatrix}
\]

be a corresponding eigenvector, where \(X_1, \ldots, X_p\) are column vectors of size \(|E|\). Equivalently, for each \(i \in \{1,\ldots, p\}\)

\[
|E| X_i + \sum_{j \neq i} \Omega X_j = \mu X_i.
\]

But \(\Omega X = s(X) \omega\) for any column vector \(X\) of size \(|E|\), where \(s(X)\) denotes the sum of the entries of \(X\), and \(\omega\) is a column vector of size \(|E|\) with all entries equal to 1. Setting \(\sigma = \sum_{j=1}^p s(X_j)\), this gives, since \(|E| = p^\epsilon\)

\[
p^\epsilon X_i + (\sigma - s(X_i)) \omega = \mu X_i.
\]

Hence if \(\mu \neq p^\epsilon\), the vector \(X_i\) is a multiple of \(\omega\), i.e. \(X_i = \alpha_i \omega\) for some scalar \(\alpha_i\). Then \(s(X_i) = \alpha_i p^\epsilon\), thus \(\sigma = \tau p^\epsilon\), where \(\tau = \sum_{j=1}^p \alpha_j\). Finally

\[
p^\epsilon \alpha_i + (\tau - \alpha_i) p^\epsilon = \tau p^\epsilon = \mu \alpha_i.
\]

Thus if \(\mu \neq 0\), all the \(\alpha_i\)'s are equal to \(\alpha\), say, and then \(\tau = p \alpha\), thus \(\mu = p^{\epsilon+1}\). Conversely, if \(X_i = \omega\) for all \(i\), then \(v\) is an eigenvector of \(t' T \cdot T\) with eigenvalue \(p^{\epsilon+1}\). So \(p^{\epsilon+1}\) is an eigenvalue of \(t' T \cdot T\), with multiplicity 1.

If \(\mu = 0\), then the vector \(v\) corresponding to \(X_i = \alpha_i \omega\) for \(i \in \{1, \ldots, p\}\) is in the kernel of \(t' T \cdot T\) if and only if \(\sum_{j=1}^p \alpha_i = 0\). Hence 0 is an eigenvalue of \(t' T \cdot T\) with multiplicity \(p - 1\).
Finally, if \( \mu = p^e \), then \( s(X_i) = \sigma \) for \( i \in \{1, \ldots, p\} \), hence \( \sigma = p\sigma = 0 \).

The vector \( v \) is in the \( p^e \)-eigenspace of \( T \cdot T \) if and only if \( s(X_i) = 0 \) for all \( i \).

Thus \( p^e \) is an eigenvalue of \( T \cdot T \), with multiplicity \( p(p^e - 1) \).

It follows that \( T \cdot T \) has eigenvalues \( p^{e+1} \), \( p^e \), and 0, with respective multiplicities \( 1 \), \( p^{e+1} - p \), and \( p^{2e} - p^{e+1} + p - 1 \). This completes the proof, since \( S = T \cdot T - (p - 1)\text{Id} \).

Lemma 4.8 shows that the form \( \langle \cdot, \cdot \rangle \) is non-degenerate whenever \( H \) is a quotient of \( Q \). By the argument of the end of Step 4, or the end of Step 6, this shows that \( S_{H,F}(G) \) has a basis in bijection with the \( G \)-conjugacy classes of sections \((T,S)\) of \( G \) such that \( T \) is \( p \)-elementary and \( T/S \approx H \).

**Case 2:** Suppose finally that \( H = C_p \times C_p \). As observed earlier, in this case, if \( R \) is a normal subgroup of \( P \) contained in \( \Phi(P) \) such that \( P/R \approx E \times H \) for some elementary abelian \( p \)-group \( E \), then in fact \( R = \Phi(P) \).

The group \( Q = P/R \) is elementary abelian, and decomposes as \( Q = E \times L \), where \( L \approx H \) is elementary abelian of rank 2. The set \( \mathcal{N}_H(Q) \) is the set of complements \( M \) of \( L \) in \( Q \), and \( MN \cap \Phi(Q) = 1 \) for any \( M, N \in \mathcal{N}_H(Q) \). Equation 4.7 shows that
\[
\langle M, N \rangle_{F,Q}^2 = (1 - p^{n-2})(1 - p^{n-3}) \cdots (1 - p),
\]
where \( n \) is the rank of \( P/\Phi(P) \). This is non-zero, and does not depend on \( M, N \in \mathcal{N}_H(Q) \). Hence the form \( \langle \cdot, \cdot \rangle_{F,Q}^2 \) has rank 1 in this case. Thus \( \tilde{e}_p^P S_{H,F}(P) \) is one dimensional if \( P \) is non-cyclic, and it is zero otherwise.

Saying that \( P \) is non-cyclic is equivalent to saying that the \( p \)-elementary group \( K = P \times C \) of Step 1 is non-cyclic. Hence \( S_{H,F}(G) \) has a basis in bijection with the conjugacy classes of non-cyclic \( p \)-elementary subgroups of \( G \). This completes the proof of Theorem 4.1. \( \square \)

4.9. **Remark:** As \( C_p \times C_p \) is a \( B \)-group, Case 2 above also follows from Proposition 11 of [3]: for a \( B \)-group \( H \) and a finite group \( G \), the dimension of \( S_{H,F}(G) \) is equal to the number of conjugacy classes of subgroups \( K \) of \( G \) such that \( \beta(K) \approx H \). Now by Lemma 2.3, if \( \beta(K) \approx C_p \times C_p \), then \( K \) is \( p \)-elementary, and non cyclic (for otherwise \( \beta(K) = 1 \)). Conversely, if \( K \) is \( p \)-elementary and non cyclic, then \( \beta(K) \) is a non trivial \( p \)-group, and also a \( B \)-group, hence \( \beta(K) \approx C_p \times C_p \).

**5. A Green biset functor for \( p \)-elementary groups**

The following theorem is closely related to Theorem 4.1. In particular, it yields an alternative proof of its Assertions 1 and 2. We refer to Section 8.5 of [9] for the basic definitions on Green biset functors.
5.1. **Theorem:** Let \( p \) be a prime number.

1. For a finite group \( G \), let \( \mathcal{E}_p(G) \) denote the set of \( p \)-elementary subgroups of \( G \). Set

\[
F_p(G) = \{ u \in B(G) \mid \forall H \in \mathcal{E}_p(G), \ Res^G_H u = 0 \} .
\]

Then the assignment \( G \mapsto F_p(G) \) is a biset subfunctor of the Burnside functor \( B \), and the quotient functor

\[
E_p = B/F_p
\]

is a Green biset functor (over \( \mathbb{Z} \)).

2. For a finite group \( G \), the evaluation \( E_p(G) \) is a free abelian group of rank equal to the number of conjugacy classes of \( p \)-elementary subgroups of \( G \).

3. Let \( \mathbb{F} \) be a field of characteristic 0. Then the biset functor \( \mathbb{F}E_p = \mathbb{F} \otimes_{\mathbb{Z}} E_p \) has a unique non zero proper subfunctor \( I \), isomorphic to \( S_{(C_p)^2, \mathbb{F}} \), and the quotient \( \mathbb{F}E_p/I \) is isomorphic to \( S_{1, \mathbb{F}} \cong \mathbb{F}R_Q \). In other words there is a non split short exact sequence

\[
(5.2) \quad 0 \to S_{(C_p)^2, \mathbb{F}} \to \mathbb{F}E_p \to S_{1, \mathbb{F}} \to 0
\]

of biset functors over \( \mathbb{F} \).

**Proof:** Let \( \mathbb{F}B = \mathbb{F} \otimes_{\mathbb{Z}} B \) be the Burnside functor over \( \mathbb{F} \). If we forget the \( \mathbb{F} \)-structure on \( \mathbb{F}B \), we get an inclusion \( B \to \mathbb{F}B \) of biset functors over \( \mathbb{Z} \). In particular, for each finite group \( G \), we get an inclusion

\[
f_p : F_p(G) \to \mathbb{F}B(G) .
\]

Now saying that \( u \in B(G) \) lies in \( F_p(G) \) amounts to saying that the restriction of \( f_p(u) \) to any \( p \)-elementary subgroup of \( G \) is equal to 0. Since any subgroup of a \( p \)-elementary group is again \( p \)-elementary, this amounts to saying that \( |f_p(u)^H| = 0 \) for any \( H \in \mathcal{E}_p(G) \). In other words \( f_p(u) \) is a linear combination of idempotents \( e_K^G \) of \( \mathbb{F}B(G) \), where \( K \) is a subgroup of \( G \) which is not \( p \)-elementary. By Lemma 2.4, we get that \( u \in F_p(G) \) if and only if \( f_p(u) \) is a linear combination of idempotents \( e_K^G \), for subgroups \( K \) such that \( \beta(K) \) is not a \( p \)-group, that is \( \beta(K) \) is non trivial and not isomorphic to \( (C_p)^2 \).

Let \( \mathcal{G}_p \) be the class of \( B \)-groups which are non trivial, and not isomorphic to \( (C_p)^2 \). Then \( \mathcal{G}_p \) is a closed class of \( B \)-groups ([9], Definition 5.4.13), that is,
if a $B$-group $L$ admits a quotient in $\mathcal{G}_p$, then actually $L \in \mathcal{G}_p$ (this is because the only quotient $B$-groups of $(C_p)^2$ are the trivial group and $(C_p)^2$, up to isomorphism). By Theorem 5.4.14 of [9], this closed class $\mathcal{G}_p$ is associated to a subfunctor $N_p$ of the Burnside functor $FB$, defined for a finite group $G$ by

$$N_p(G) = \sum_{K \leq G \beta(K) \in \mathcal{G}_p} FE_K^G.$$  

This shows that $f_p(F_p(G)) = f_p(B(G)) \cap N_p(G)$, and since $N_p$ is a biset subfunctor of $FB$, it follows that $F_p$ is a biset subfunctor of $B$. As biset subfunctors of $B$ are also ideals of the Green biset functor $B$ (see Lemma 2.5.8, Assertion 4 in [9]), we get that $F_p$ is an ideal of $B$. It follows that the quotient $E_p = B/F_p$ is a Green biset functor. This completes the proof of Assertion 1.

Moreover, the ghost map

$$\Phi : B(G) \to \prod_{K \leq G \text{ mod } G} \mathbb{Z}$$

sending $u \in B(G)$ to the sequence $|u^K|$, is injective by Burnside’s theorem. The above discussion shows that $\Phi$ induces an injective map

$$E_p(G) = B(G)/F_p(G) \to \prod_{K \in \mathcal{E}_p(G) \text{ mod } G} \mathbb{Z},$$

which becomes an isomorphism after tensoring with $\mathbb{F}$. Assertion 2 follows.

Finally, it follows from Theorem 5.4.14 of [9] that the lattice $[0, FE_p]$ of biset subfunctors of $FE_p$ is isomorphic to the set of closed classes of $B$-groups which contain $\mathcal{G}_p$. There are exactly three such classes: the class $\mathcal{G}_p$, the class of non-trivial $B$-groups, and the class of all $B$-groups. So $[0, FE_p]$ is a totally ordered set of cardinality 3. Hence $FE_p$ admits a unique non zero proper subfunctor $I$. The quotient $FE_p/I$ is the unique simple quotient of $FB$, hence it is isomorphic to $S_{1, \mathbb{F}} \cong \mathbb{F} R_Q$. Now $I$ is a simple biset functor, which is a subquotient of $FB$. By Proposition 5.5.1 of [9], it follows that $I \cong S_{H, \mathbb{F}}$ for some $B$-group $H$. Since the group $K = (C_p)^2$ has a unique non cyclic subgroup, it follows that $I(K)$ is one dimensional, and a trivial $\mathbb{F}Out(K)$-module. Moreover $K$ is a group of minimal order such that $I(K) \neq \{0\}$. Hence $H \cong K$, and $I \cong S_{(C_p)^2, \mathbb{F}}$. This completes the proof of Assertion 3, and the proof of Theorem 5.1.

5.3. Remark: One can show that the exact sequence (5.2) is essentially unique as a non split exact sequence in the category $\mathcal{F}$ of biset functors over $\mathbb{F}$: more precisely, one can show that $\text{Ext}^1_{\mathcal{F}}(S_{1, \mathbb{F}}, S_{(C_p)^2, \mathbb{F}}) \cong \mathbb{F}$. 25
5.4. Remark: If $G$ is a $p$-group (or even if $G$ is $p$-elementary), then $F_p(G) = \{0\}$, so $E_p(G) \cong B(G)$. So if we restrict the exact sequence (5.2) to finite $p$-groups, we get an exact sequence

$$0 \to S_{(C_p)^2,F} \to FB \to FRQ \to 0$$

of $p$-biset functors over $F$. This (restricted) exact sequence was introduced in [12], where it was shown that for a finite $p$-group $P$, the evaluation $S_{(C_p)^2,F}(P)$ is isomorphic to $FD(P)$, where $D(P)$ is the Dade group of endomutation modules. It was also shown that the dimension of $S_{(C_p)^2,F}(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of $P$, which is also the number of conjugacy classes of non-cyclic $p$-elementary subgroups of $P$. So this agrees with Assertion 2 of Theorem 4.1.

5.5. Remark: For a finite group $G$, let $M_p(G)$ be the $\mathbb{Z}$-submodule of $B(G)$ generated by the classes of the transitive $G$-sets $G/H$, where $H$ is a $p$-elementary subgroup of $G$. One can check easily that $M_p(G) \cap F_p(G) = \{0\}$, so comparing ranks, one might hope that $B(G) = M_p(G) \oplus F_p(G)$. This is false in general: for $p = 2$, when $G$ is the symmetric group $S_3$, there are three $p$-elementary subgroups in $G$, up to conjugation, namely the proper subgroups of $G$ (that is the trivial group, the alternating subgroup $A = A_3$, and the subgroup $C$ of order 2). Hence if $B(G) = M_p(G) \oplus F_p(G)$, then in particular $G/G \in M_p(G) \oplus F_p(G)$ so there exist integers $a, b, c$ such that the element $u = G/G - (b G/1 + a G/A + c G/C)$ is in $F_p(G)$. Taking fixed points by $A$ then gives $|u^A| = 0 = 1 - 2a$, a contradiction. One can show more precisely that $M_p(G) \oplus F_p(G)$ has index 2 in $B(G)$ in this case.

References

[1] L. Barker and I. A. Öğüt. Some deformations of the fibred biset category. *Turkish J. Math.*, 44(6):2062–2072, 2020.

[2] J. Barsotti. On the unit group of the Burnside ring as a biset functor for some solvable groups. *J. Algebra*, 508, 05 2018.

[3] M. Baumann. The composition factors of the functor of permutation modules. *J. Algebra*, 344:284–295, 2011.

[4] R. Boltje and O. Coşkun. Fibered biset functors. *Adv. Math.*, 339:540–598, 2018.

[5] R. Boltje, G. Raggi-Cárdenas, and L. Valero-Elizondo. The $-$ and $+$ constructions for biset functors. *J. Algebra*, 523:241–273, 2019.
[6] S. Bouc. Foncteurs d’ensembles munis d’une double action. *J. Algebra*, 183(0238):664–736, 1996.

[7] S. Bouc. The Dade group of a $p$-group. *Invent. Math.*, 164:189–231, 2006.

[8] S. Bouc. The functor of units of Burnside rings for $p$-groups. *Comment. Math. Helv.*, 82:583–615, 2007.

[9] S. Bouc. *Biset functors for finite groups*, volume 1990 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

[10] S. Bouc and N. Romero. The center of a Green biset functor. *Pacific Journal of Mathematics*, 303:459–490, 2019.

[11] S. Bouc, R. Stancu, and J. Thévenaz. Simple biset functors and double Burnside ring. *Journal of Pure and Applied Algebra*, 217:546–566, 2013.

[12] S. Bouc and J. Thévenaz. The group of endo-permutation modules. *Invent. Math.*, 139:275–349, 2000.

[13] O. Coşkun and E. Yakın. Obstructions for gluing biset functors. *J. Algebra*, 532:268–310, 2019.

[14] O. Coşkun and D. Yılmaz. Fibered $p$-biset functor structure of the fibered Burnside rings. *Algebr. Represent. Theory*, 22(1):21–41, 2019.

[15] B. García. Essential support of Green biset functors via morphisms. *Arch. Math. (Basel)*, 116(1):23–32, 2021.

[16] B. Rognerud. Around evaluations of biset functors. *Ann. Inst. Fourier (Grenoble)*, 69(2):805–843, 2019.

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