The only true (2+1)-dimensional nonlocal KdV, fifth-order KdV, and Gardner equations derived from the ideal fluid model

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Abstract

We study the problem of gravity surface waves for an ideal fluid model in the (2+1)-dimensional case. We apply a systematic procedure to derive the Boussinesq equations for a given relation between the orders of four expansion parameters, the amplitude parameter $\alpha$, the long-wavelength parameter $\beta$, the transverse wavelength parameter $\gamma$, and the bottom variation parameter $\delta$. In three special cases, we derived the only true (2+1)-dimensional extensions to the Korteweg-de Vries equation, fifth-order KdV equation, and the Gardner equation. All these equations are non-local. When the bottom is flat, the (2+1)-dimensional KdV equation can be transformed into the Kadomtsev-Petviashvili-type equation.

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I. INTRODUCTION

The Korteweg-de Vries equation [1] is one of the most widely used nonlinear wave equations. It was derived by perturbation calculus from a model of an ideal fluid (non-viscous and incompressible) in which the motion is irrotational in (1+1)-dimension and the bottom of the fluid reservoir is flat. The simplest nonlinear wave equation in (2+1)-dimensions is the Kadomtsev-Petviashvili equation (KP for short) [2]. Both the KdV and KP equations are integral and satisfy many conservation laws. Their analytical solutions are also known.

Since nonlinear waves on the surface of seas and oceans are very important in practice, many papers on (2+1)-dimensional equations of KdV or KP type can be found in the literature, see e.g. [3–17]. The equations used in these works are also integral, which allows their authors to construct analytic solutions of various types (solitons, periodic solutions, lumps, breathers).

Extending the ideal fluid model to the (2+1)-dimensional case leads to much more complicated equations. The only such attempt known to us is the work [18]. Unfortunately, this particular paper is erroneous, as we have shown in the work [19]. To the best of our knowledge, there are no correct (2+1)-dimensional papers that consider the ideal fluid model in full detail. This is because the obtained equations are non-integrable.

In this paper, we generalise the perturbation approach described in [20] and [21] for the (1+1)-dimensional case. The authors of [20] have shown that for a flat bottom, one can derive the KdV, extended KdV, fifth-order KdV, and Gardner equations for different relations between small parameters determining nonlinearity ($\alpha$) and dispersion ($\beta$). In [21], we have generalised this approach for the case with an uneven bottom, assuming that the bottom changes are much smaller than the fluid’s depth. Here, following methods used in [21] and [22] we attempt to derive analogous wave equations for (2+1)-dimensional case.

In [22], we have discussed four cases of dependence between small parameters in detail. In all these cases, it was impossible to reduce the obtained Boussinesq equations to a single KdV-type equation for the wave profile. It was only possible to obtain a single wave-type nonlinear partial differential equation in which the argument is an auxiliary function determining the velocity potential. The solution of such an equation, if known, determines the (2+1)-dimensional function representing the time-dependent surface profile. In all these cases, however, the (2+1)-dimensional nonlinear wave equations are very complicated.

In this paper, we discuss three cases where the parameter $\gamma$ is in a higher order than the leading one, $\beta$ or $\alpha$. It turns out that for the flat bottom when $\alpha \approx \beta$ and $\gamma \approx \beta^2$ we can derive a true (2+1)-dimensional analogue of the KdV equation without any additional assumptions. In two other cases, $\alpha \approx \gamma \approx \beta^2$ and $\beta \approx \gamma \approx \alpha^2$ we are able to derive (2+1)-dimensional analogues to fifth-order KdV and Gardner equations, respectively. In these cases, the derivation requires imposing an additional condition on the wave profile function.

The paper is organised as follows. In section II, to keep the manuscript self-contained, we briefly repeat the description of the model from [22]. In section III we derive the only true (2+1)-dimensional KdV equation. Furthermore, if the bottom is equal, this equation trasformed to a
moving reference frame implies the Kadomtsev-Petviashvili (KP) equation. In section IV, for
the case \( \alpha \approx \gamma \approx \beta^2 \) and flat bottom, we derived the Boussinesq equations. With an additional
condition on the wave profile function, these Boussinesq equations can be made compatible and
reduced to a (2+1)-dimensional fifth-order KdV equation. In section V we considered the case
\( \beta \approx \gamma \approx \alpha^2 \) and flat bottom, we derived the Boussinesq equations. With an additional
condition on the wave profile function, these Boussinesq equations can be made compatible and
reduced to a (2+1)-dimensional fifth-order KdV equation. In section VI we extended the (2+1)-dimensional fifth-order KdV and Gardner
equations to the case with uneven bottom. Section VII contains the conclusions.

II. DESCRIPTION OF THE MODEL

Let us consider the inviscid and incompressible fluid model whose motion is irrotational in
a container with an impenetrable bottom. In dimensional variables, the set of hydrodynamical
equations consists of the Laplace equation for the velocity potential \( \phi(x, y, z, t) \) and boundary
conditions at the surface and the bottom.

1. \( \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \), in the volume,
2. \( \phi_z - (u_x \phi_x + u_y \phi_y + u_t) = 0 \), at the surface,
3. \( \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + gu = 0 \), at the surface,
4. \( \phi_z - h_x \phi_x - h_y \phi_y = 0 \), at the bottom.

Here \( u(x, y, t) \) denotes the surface profile function, \( g \) is the gravitational acceleration, and \( \rho \) is
fluid’s density. The bottom can be non-flat and is described by the function \( h(x, y) \). Indexes
denote partial derivatives, i.e. \( \phi_x \equiv \frac{\partial \phi}{\partial x} \), \( \phi_{yy} \equiv \frac{\partial^2 \phi}{\partial y^2} \), and so on.

The next step consists in introducing a standard scaling to dimensionless variables (in general, it could be different in \( x \)-, \( y \)-and \( z \)-direction)

\[
\tilde{x} = x/L, \quad \tilde{y} = y/l, \quad \tilde{z} = z/H, \quad \tilde{t} = t/(L/\sqrt{gH}),
\]
\[
\tilde{u} = u/A, \quad \tilde{\phi} = \phi/(L A \sqrt{gH}).
\]

Here, \( A \) is the amplitude of surface distortions from equilibrium shape (flat surface), \( H \) is average fluid depth, \( L \) is the average wavelength (in \( x \)-direction), and \( l \) is a wavelength in \( y \)-direction. In
general, \( l \) should be the same order as \( L \), but not necessarily equal. Then the set (1)-(4) takes in
scaled variables the following form (here and next, we omit the tilde signs)

1. \( \beta \phi_{xx} + \gamma \phi_{yy} + \phi_{zz} = 0 \),
2. \( u_t + \alpha(u_x \phi_x + \frac{\gamma}{\beta} u_y \phi_y) - \frac{1}{\beta} \phi_z = 0 \), for \( z = 1 + \alpha u \),
3. \( \phi_t + \frac{1}{2} \alpha \left( \phi_x^2 + \frac{\gamma}{\beta} \phi_y^2 + \frac{1}{\beta} \phi_z^2 \right) + u = 0 \), for \( z = 1 + \alpha u \),
4. \( \phi_z - \beta \delta h_x \phi_x - \gamma \delta h_y \phi_y = 0 \), for \( z = \delta h \).
Besides standard small parameters $\alpha = \frac{a}{H}$, $\beta = \left(\frac{H}{L}\right)^2$ and $\gamma = \left(\frac{H}{T}\right)^2$, which are sufficient for the flat bottom case, we introduced another one defined as $\delta = \frac{a_h}{H}$, where $a_h$ denotes the amplitude of bottom variations [21, 24]. In the perturbation approach, all these parameters, $\alpha, \beta, \gamma, \delta$, are assumed to be small but not necessarily of the same order. The standard perturbation approach to the system of Euler’s equations (6)-(9) consists of the following steps. First, the velocity potential is sought in the form of power series in the vertical coordinate

$$\phi(x, y, z, t) = \sum_{m=0}^{\infty} z^m \phi^{(m)}(x, y, t),$$

where $\phi^{(m)}(x, y, t)$ are yet unknown functions. The Laplace equation (6) determines $\phi$ in the form which involves only two unknown functions with the lowest $m$-indexes, $f(x, y, t) := \phi^{(0)}(x, y, t)$ and $F(x, y, t) := \phi^{(1)}(x, y, t)$ and their space derivatives. Hence,

$$\phi(x, y, z, t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} z^{2m} \left(\beta \partial_{xx} + \gamma \partial_{yy}\right)^m f(x, y, t)$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} z^{2m+1} \left(\beta \partial_{xx} + \gamma \partial_{yy}\right)^m F(x, y, t).$$

The explicit form of this velocity potential reads as

$$\phi = f - \frac{1}{2} z^2 (\beta f_{2x} + \gamma f_{2y}) + \frac{1}{24} z^4 (\beta^2 f_{4x} + 2\beta\gamma f_{2x2y} + \gamma^2 f_{4y}) + \cdots$$

$$+ zF - \frac{1}{6} z^3 (\beta F_{2x} + \gamma F_{2y}) + \frac{1}{120} z^5 (\beta^2 F_{4x} + 2\beta\gamma F_{2x2y} + \gamma^2 F_{4y}) + \cdots$$

Next, the boundary condition at the bottom (9) is utilized. For the flat bottom case, it implies $F = 0$, simplifying substantially next steps. In particular, $F = 0$ makes it possible to derive the Boussinesq equations up to arbitrary order. For an uneven bottom, the equation (9) determines a differential equation relating $F$ to $f$. This differential equation can be resolved to obtain $F(f, f_x, f_{xx}, h, h_x)$ but this solution can be obtained only up to some particular order in leading small parameter. Then, the velocity potential is substituted into kinematic and dynamic boundary conditions at the unknown surface (7)-(8). Retaining only terms up to a given order, one obtains the Boussinesq system of two equations for unknown functions $u, f$ valid only up to a given order in small parameters. The resulting equations, however, depend substantially on the ordering of small parameters.

In 2013, Burde and Sergiyeyev [20] demonstrated that for the case of (1+1)-dimensional and the flat bottom, the KdV, the extended KdV, fifth-order KdV, and Gardner equations can be derived from the same set of Euler’s equations (6)-(9). Different final equations result from the different ordering of small parameters and consistent perturbation approach up to first or second order in small parameters.

In 2020, we extended their results to cases with an uneven bottom in [21], but still in (1+1)-dimensional theory. We showed that the terms originating from the bottom have the same universal
form for all these four nonlinear equations. However, the validity of the obtained generalised wave equations is limited to the cases when the bottom functions are piecewise linear. On the other hand, the corresponding sets of the Boussinesq equations are valid for the arbitrary form of the bottom functions.

In the present paper there are four small parameters, \( \alpha, \beta, \gamma, \delta \). In order to make calculations easier, we will follow the idea from [20, 21], relating all small parameters to a single one, called leading parameter. This method allows easier control of the order of different terms, but the final forms of the resulting equations are presented in original parameters \( \alpha, \beta, \gamma, \delta \). In [27] we discussed several cases, which are listed in Table I. The table does not contain all possible second-order cases, but only those that lead to well-known KdV-type and Gardner equations when reduced to (1+1)-dimensions.

| Case | \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | (1+1), \( \delta = 0 \) |
|------|------------|------------|------------|------------|----------------------------|
| 1    | \( O(\beta) \) | lead | \( O(\beta) \) | \( O(\beta) \) | KdV |
| 2    | \( O(\beta) \) | lead | \( O(\beta) \) | \( O(\beta^2) \) | ext KdV |
| 3    | \( O(\beta^2) \) | lead | \( O(\beta^2) \) | \( O(\beta^2) \) | fifth-ord KdV |
| 4    | lead | \( O(\alpha^2) \) | \( O(\alpha^2) \) | \( O(\alpha^2) \) | Gardner |
| 5    | \( O(\beta) \) | lead | \( O(\beta^2) \) | \( O(\beta^2) \) | KdV |
| 6    | \( O(\beta^2) \) | lead | \( O(\beta^2) \) | 0 | fifth-ord KdV |
| 7    | lead | \( O(\alpha^2) \) | \( O(\alpha^2) \) | 0 | Gardner |

In Table I the abbreviations are used, lead means leading parameter, \( (1+1) \) means (1+1)-dimensions, ext KdV means the extended KdV.

Cases 1-4 were recently discussed by us in [22] where Boussinesq’s equations were derived for all of them. Below we study in detail cases 5-7. We will show that in these cases, it is possible to go further and derive (2+1)-dimensional nonlocal analogue to the KdV equation in first order perturbation approach. Next, we will show that in second order perturbation approach, it is impossible to derive (2+1)-dimensional nonlocal analogue to the extended KdV equation (KdV2). In the following, we will show that in the second order perturbation approach, we can derive (2+1)-dimensional nonlocal analogues to fifth-order KdV equation and Gardner equation, under additional condition imposed on the wave profile function. It is worth emphasizing that this is possible only when \( \gamma \) is of higher order than the leading parameter (\( \beta \) or \( \alpha \)) because only in such case zeroth order Boussinesq’s equations do not contain \( y \)-derivatives of the wave profile function (see Remark in section III).
III. THE ONLY KDV EXTENSION TO (2+1)-DIMENSIONS

CASE 5: \( \alpha = O(\beta), \quad \gamma = O(\beta^2), \quad \delta = O(\beta) \)

Derivations of the Kadomtsev-Petviashvili (KP) equation suggest that variations of the surface waves should be slower in the \( y \)-direction perpendicular to the wave propagation (\( x \)-direction). Therefore it is worth studying in detail the case when \( \gamma = O(\beta^2) \). Denote now

\[
\alpha = A \beta, \quad \gamma = G \beta^2, \quad \delta = D \beta, \tag{13}
\]

where, as earlier, \( A, G, D \) are arbitrary constants close to 1.

In this case, the boundary condition at the bottom (9) imposes the following relation

\[
F = \beta^2 D(hf_x)_x + \beta^3 \left[ DG(hf_y)_y + D^2 \frac{1}{2} (h^2 F_x)_x \right] + O(\beta^4). \tag{14}
\]

Therefore, we can eliminate \( F \) taking \( F = \beta^2 D(hf_x)_x \) which is valid only up to second order in small parameters. So, the explicit form of velocity potential, valid up to second order, is now

\[
\phi = f - \frac{1}{2} z^2 (\beta f_{xx} + \gamma f_{2y}) + z \beta^2 D(hf_x)_x + \frac{1}{24} z^4 \beta^2 f_{4x}. \tag{15}
\]

Now, if we substitute the velocity potential (15) into (7) and (8), the resulting equations are valid only up to first order. This fact seems to contradict the assumption that the parameter \( \gamma \) is of the second order. Therefore, it seems at first glance that when \( \alpha, \beta, \delta \) are of the first order, and \( \gamma \) is of the second order, consistent Boussinesq equations can not be derived. Fortunately, first order Boussinesq equations can be consistently derived as demonstrated below.

Inserting the velocity potential (15) into the kinematic boundary condition at the surface (7) and neglecting terms higher than the first order in small parameters yields

\[
u_t + f_{xx} + \alpha(u f_x)_x - \frac{1}{6} \beta f_{4x} + \frac{\gamma}{\beta} f_{yy} - \delta(h f_x)_x = 0. \tag{16}
\]

Analogous steps with the dynamic boundary condition at the surface (8) lead to the first order equation

\[
u + f_t + \frac{1}{2} \alpha f_x^2 - \frac{1}{2} \beta f_{xxt} = 0. \tag{17}
\]

Equations (16)-(17) constitute the first order Boussinesq’s equations for the case when \( \alpha \approx \beta, \quad \gamma \approx \beta^2 \) and \( \delta \approx \beta \), that is, for non-flat bottom. Despite the assumption that \( \gamma \) is of the second order, the term \( \frac{\gamma}{\beta^2} f_{yy} \) appears in the Boussinesq equation (16) as the first order one.

Let us try to apply a standard method for making the Boussinesq equations (16)-(17) compatible, which in (1+1)-dimensions leads to the Korteweg-de Vries equation.

By differentiating over \( x \) the equation (17) and denoting \( f_x = w, \quad f = \int w \, dx, \quad f_{yy} = \int w_{yy} \, dx \) we can write the equations (16)-(17) in the form

\[
u_t + w_x + \alpha(\nu w)_x - \frac{1}{6} \beta w_{4x} + \frac{\gamma}{\beta} \int w_{yy} \, dx - \delta(h w)_x = 0, \tag{18}
\]

\[
u_t + w_x + \alpha w w_x - \frac{1}{2} \beta w_{xxt} = 0. \tag{19}
\]
Equation (18) has a nonlocal form. When the problem is reduced to (1+1)-dimensions (u, w not dependent on y) equations (18)-(19) reduce to the classical Boussinesq equations leading to the KdV equation. Note, that additional term \( \frac{\gamma}{\beta} \int w_{yy} \, dx \) is the first order one because \( \frac{\gamma}{\beta} \approx \beta \). Then in zeroth-order the following holds

\[
  u_t + w_x = 0, \quad w_t + u_x \implies w = u, \quad w_t = -w_x, \quad u_t = -u_x.
\]

(20)

We seek such form of \( w \) function which allows us to make equations (16)-(17) compatible, that is reduce in first order to the same equation. We postulate \( w \) in the following form

\[
  w = u + w^{(1)} = u + \alpha Q^{(a)} + \beta Q^{(b)} + \frac{\gamma}{\beta} Q^{(g)} + \delta Q^{(d)},
\]

(21)

where \( \alpha Q^{(a)}, \beta Q^{(a)}, \frac{\gamma}{\beta} Q^{(g)}, \delta Q^{(d)} \) are first-order corrections. In principle, \( w \) in (21) can contain one more first order correction, namely \( \frac{\gamma}{\alpha} Q^{(aa)} \). The calculation shows, however, that \( Q^{(aa)} = 0 \).

Now, we insert \( w \) given by (21) into (18)-(19) and retain terms up to first order. The result from (18) is

\[
  u_x + u_t + \alpha \left( Q^{(a)}_x + 2uu_x \right) + \beta \left( Q^{(b)}_x - \frac{1}{6} u_{xxx} \right) + \frac{\gamma}{\beta} \left( Q^{(g)}_x + \int u_{yy} \, dx \right) + \delta \left( Q^{(d)}_x - (hu)_x \right) = 0.
\]

(22)

From (19) we obtain

\[
  u_x + u_t + \alpha \left( Q^{(a)}_t + uu_x \right) + \beta \left( Q^{(b)}_t - \frac{1}{2} u_{xxt} \right) + \frac{\gamma}{\beta} \left( Q^{(g)}_t \right) + \delta Q^{(d)}_t = 0.
\]

(23)

Subtracting (23) from (22) and replacing \( t \)-derivatives by \(( - )x\)-derivatives (thanks to (20), \( Q^{(a)}_t \rightarrow -Q^{(a)}_x, \ Q^{(g)}_t \rightarrow -Q^{(g)}_x, \ u_{xxt} \rightarrow -u_{xxx} \)) we receive

\[
  \alpha \left( 2Q^{(a)}_x + uu_x \right) + \beta \left( 2Q^{(b)}_x - \frac{2}{3} u_{xxx} \right) + \frac{\gamma}{\beta} \left( 2Q^{(g)}_x + \int u_{yy} \, dx \right) + \delta \left( Q^{(d)}_x - Q^{(d)}_t - (hu)_x \right) = 0.
\]

(24)

Due to arbitrariness of small parameters (24) is equivalent to four equations

\[
  Q^{(a)}_x = -\frac{1}{2} u u_x, \quad Q^{(b)}_x = \frac{1}{3} u_{xxx}, \quad Q^{(g)}_x = -\frac{1}{2} \int u_{yy} \, dx,
\]

(25)

and

\[
  Q^{(d)}_x - Q^{(d)}_t = (hu)_x.
\]

(26)

Integration of equations (25) over \( x \) yields

\[
  Q^{(a)} = -\frac{1}{4} u^2, \quad Q^{(b)} = \frac{1}{3} u_{xx} \quad Q^{(g)} = -\frac{1}{2} \int \left( \int u_{yy} \, dx \right) \, dx.
\]

(27)

These three first order corrections are common with the case of an even bottom. Correction function \( Q^{(d)} \) related to an uneven bottom allows for making the Boussinesq equations (18)-(19) compatible only when \( h_{xx} = 0 \), that is, when the dependence of bottom function \( h(x, y) \) on \( x \) is piecewise linear. In such a case one obtains

\[
  Q^{(d)} = \frac{1}{4} \left( 2hu + h_x \int u \, dx \right).
\]

(28)
The detailed discussion of this problem in (1+1)-dimensions, for all four cases presented in Table I is contained in the article [21].

Now, let us check that \(w\) given by (21),(27),(28), that is,
\[
w = u + \alpha \left( -\frac{1}{4} u^2 \right) + \beta \left( \frac{1}{3} u_x \right) + \frac{\gamma}{\beta} \left( -\frac{1}{2} \int (\int u_{yy} \, dx) \, dx \right) + \delta \left( \frac{1}{4} (2hu + h_x \int u \, dx) \right),
\]
does indeed reduce the Boussinseq equations to the same wave equation (when restricted up to first order terms). After substituting \(w\) into (18) and leaving the terms up to first order, we obtain
\[
\frac{\partial u}{\partial t} + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{6} \beta u_{xxx} + \frac{1}{2} \gamma \int u_{yy} \, dx - \frac{1}{4} \delta u_{xx} = 0.
\]
(30)
The same action with (19) gives (remember that \(h_{xx} = 0\))
\[
\frac{\partial u}{\partial t} + u_x + \alpha (uu_x - \frac{1}{2} uu_t) - \frac{1}{6} \beta u_{xxt} - \frac{1}{2} \beta \int (\int u_{yyt} \, dx) \, dx + \frac{1}{4} \delta (2hu + h_x \int u_t \, dx) = 0.
\]
(31)
Since in zeroth-order \(u_t = -u_x, u_{xxt} = -u_{xxx}, u_{yyt} = -u_{xyy}, \) and \(\int u_{yyt} \, dx = -u_{yy},\) then equation (31) does receive the same form as equation (30).

Equation (30) takes correctly into account an uneven bottom, but only under assumption that \(h_{xx} = 0.\) In the case when the bottom is even, \(\delta = 0,\) and the last term in (30) vanishes. In such case
\[
\frac{\partial u}{\partial t} + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{6} \beta u_{xxx} + \frac{1}{2} \gamma \int u_{yy} \, dx = 0.
\]
(32)
Equations (30) and (32) are indeed true (2+1)-dimensional nonlocal analogues to the Korteweg-de Vries equation. They are derived from Euler’s equations for ideal fluid model, when fluid’s motion is irrotational, within first order perturbative approach.

**Remark** In [22], we considered cases when parameter \(\gamma\) was the same order as \(\beta.\) In other words wavelength in \(x\) and \(y\) directions are not much different. Then the term \(\frac{1}{2} f_{yy}\) appears already in zeroth order in the Boussinesq equation originating from kinematic boundary condition at the surface (7). Its presence causes that in zeroth order there are no relations (20). The lack of such relations in zeroth order Boussinesq’s equations makes the derivation of the single wave equation of (2+1)-dimensional KdV-type impossible.

**A. Kadomtsev-Petviashvili - type equation**

For a flat bottom, (2+1)-dimensional KdV equation (32) implies the Kadomtsev-Petviashvili -type equation. Differentiating (32) over \(x\) yields
\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{6} \beta u_{xxx} + \frac{1}{2} \gamma \int u_{yy} \, dx \right) = -\lambda \frac{\partial^2 u}{\partial y^2},
\]
(33)
where \(\lambda = -\frac{1}{2} \gamma.\) Equation (33) can be named Kadomtsev-Petviashvili equation in a fixed reference frame. It is derived in first order perturbation approach from the ideal fluid model, when \(\alpha \approx \beta\) and \(\gamma \approx \beta^2.\)
The classical Kadomtsev-Petviashvili equation [2] has the following form

\[
\frac{\partial}{\partial \hat{x}} \left( \frac{\partial u}{\partial \hat{t}} + 6u \frac{\partial u}{\partial \hat{x}} + \frac{\partial^3 u}{\partial \hat{x}^3} \right) = -\lambda \frac{\partial^2 u}{\partial \hat{y}^2},
\]

(34)

where \( \lambda \) is a constant. Equation (34) with \( \lambda > 0 \) is called KP2 equation, whereas (34) with \( \lambda < 0 \) is called KP1. The expression in the bracket is the standard KdV equation in a moving reference frame.

Let us make a scaling transformation of (32) to a moving reference frame. Take

\[
\hat{x} = \sqrt{\frac{3}{2}} (x - t), \quad \hat{t} = \frac{1}{4} \sqrt{\frac{3}{2}} \alpha t, \quad \text{and} \quad \hat{y} = y.
\]

(35)

Then the (2+1)-dimensional KdV equation for flat bottom (32) takes the following form in scaled coordinates

\[
u_{\hat{t}} + 6 uu_{\hat{x}} + \frac{\beta}{\alpha} u_{\hat{x}\hat{x}} + \frac{1}{\sqrt{6}} \frac{\gamma}{\beta} \int u_{\hat{y}\hat{y}} d\hat{x} + \frac{4}{\sqrt{6}} \frac{\gamma}{\alpha \beta} \int u_{\hat{y}\hat{y}} d\hat{t} = 0.
\]

(36)

In case \( \alpha = \beta \), taking \( \hat{x} \)-derivative one receives

\[
\frac{\partial}{\partial \hat{x}} \left( \frac{\partial u}{\partial \hat{t}} + 6u \frac{\partial u}{\partial \hat{x}} + \frac{\partial^3 u}{\partial \hat{x}^3} \right) = -\lambda \frac{\partial^2 u}{\partial \hat{y}^2} - \frac{4}{\sqrt{6}} \frac{\gamma}{\beta^2} \frac{\partial}{\partial \hat{x}} \left( \int u_{\hat{y}\hat{y}} d\hat{t} \right),
\]

(37)

where \( \lambda = \frac{1}{\sqrt{6}} \frac{\lambda}{\beta} \geq 0 \). If the last term is neglected, the result is the classical Kadomtsev-Petviashvili equation (34).

Kadomtsev-Petviashvili equation possesses several kinds of analytic solutions. Let us begin with soliton solution. It is well known (see, e.g. [29]) that the function

\[
u(x, y, t) = A \text{Sech}^2 (kx + my - \omega t)
\]

(38)

fulfils equation KP (34) when \( A = 2k^2 \), \( \omega = 4k^3 + \lambda m^2/k \) and \( k, m \) are arbitrary constants.

B. Simple solutions to (2+1)-dimensional KdV equations

We can expect that solutions to KP equation can be simultaneously solutions to the (2+1)-dimensional KdV equation (32).

By direct calculation one can check that the function \( u \) given by (38) fulfills (2+1)-dimensional KdV equation (32) when

\[
A = \frac{4k^3 \beta}{3\alpha}, \quad \omega = k + \frac{2k^3 \beta}{3} + \frac{m^2 \gamma}{3k \beta}
\]

(39)

and, as previously, \( k, m \) are arbitrary constants.

In [30] Y. Stepanyants presented 2-soliton solution to KP2 equation. This solution should be the solution to (2+1)-dimensional KdV equation (32), as well.

(2+1)-dimensional KdV equation (32) is, however, more general than the KP equation. The latter is obtained (in a moving reference frame) by taking \( \hat{x} \)-derivative of the former. In such step an additional (nonlocal in time) term appears.
IV. CASE: \( \alpha = O(\beta^2), \quad \gamma = O(\beta^2), \quad \delta = 0. \) (2+1)-DIMENSIONAL FIFTH-ORDER KDV EQUATION

Denote now
\[
\alpha = A \beta^2, \quad \gamma = G \beta^2, \tag{40}
\]
where, as earlier, \( A, G \) are arbitrary constants close to 1.

In (1+1)-dimensional case perturbation approach allowed to derive the fifth-order KdV equation [20].

We consider the flat bottom case, \( \delta = 0 \), aiming to derive second order wave equation. Therefore it is enough to use the velocity potential in the form
\[
\phi = f - \frac{1}{2} z^2 (\beta f_{2x} + \gamma f_{2y}) + \frac{1}{24} z^4 (\beta^2 f_{4x} + 2 \beta \gamma f_{2x2y} + \gamma^2 f_{4y})
\]
\[
- \frac{1}{720} z^6 \left( \beta^3 f_{6x} + 3 \beta^2 \gamma f_{4x2y} + 3 \beta \gamma^2 f_{2x4y} + \gamma^3 f_{6y} \right) + \cdots
\]
(neglecting all terms of higher order than \( \beta^3 \)). Substituting velocity potential (41) into (7) and retaining only terms up to \( \beta^2 \) yields the first of Boussinesq’s equations for this case as
\[
u_t + f_{xx} - \frac{1}{6} \beta f_{4x} + \frac{\gamma}{\beta} f_{yy} + \alpha (u f_x)_x + \frac{1}{120} \beta f_{6x} - \frac{1}{3} \gamma f_{2x2y} = 0. \tag{42}
\]
Note that the last three terms in (42) are second order ones.

The second Boussinesq’s equation is obtained from (8). However, we will use this equation in the form which takes into account possible surface tension effects (see, [22, Eqs. (9) and (11)])
\[
\phi_t + \frac{1}{2} \alpha \left( \phi_x^2 + \frac{\gamma}{\beta} \phi_y^2 + \frac{1}{\beta} \phi_z^2 \right) + u - \tau (\beta u_{xx} + \gamma u_{yy}) = 0, \quad \text{for } z = 1 + \alpha u, \tag{43}
\]
where \( \tau = \frac{T}{\varrho g H^2} \) is the Bond number (\( T \) is surface tension coefficient, \( \varrho \) is fluid density, \( g \) is gravitational acceleration and \( H \) is fluid depth).

So, we substitute the velocity potential (41) into (43) and retain only terms up to \( \beta^2 \). After differentiation over \( x \) the resulting second Boussinesq’s equation receives the form
\[
u_x + f_{xt} + \beta \left( - \frac{1}{2} f_{3xt} - \tau u_{3x} \right) + \alpha f_x f_{xx} + \frac{1}{24} \beta^2 f_{5xt} - \frac{1}{2} \gamma f_{x2yt} = 0. \tag{44}
\]
Because Boussinesq’s equations (42) and (44) do not contain \( y \) dependence in zeroth order terms, we can follow the standard method used in section III. Denoting \( f_x = w, \quad f = \int w dx, \) and so on, we can write equations (42) and (44) as
\[
u_t + w_x - \frac{1}{6} \beta w_{3x} + \frac{\gamma}{\beta} \int w_{yy} dx + \alpha (uw)_x + \frac{1}{120} \beta^2 w_{5x} - \frac{1}{3} \gamma w_{x2y} = 0, \tag{45}
\]
\[
w_t + u_x - \beta \left( \frac{1}{2} w_{2xt} + \tau u_{3x} \right) + \alpha w w_x + \frac{1}{24} \beta^2 w_{4xt} - \frac{1}{2} \gamma w_{2yt} = 0. \tag{46}
\]
Note that the last three terms in both equations are second order ones.

Zeroth-order equations (45)-(46) are the same as those of (18)-(19). Therefore in zeroth-order relations (20) hold. In first order we have terms with $\beta$ and $\gamma$, so we seek for $w$ in the form

$$w = u + w^{(1)} = u + \beta Q^{(b)} + \frac{\gamma}{\beta} Q^{(gb)}.$$  \hfill (47)

Proceeding in the same way as in section III we obtain correction functions in first order as

$$Q^{(b)} = \left( \frac{2 - 3\tau}{6} \right) u_{xx}, \quad Q^{(gb)} = -\frac{1}{2} \int \left( \int u_{xx} \, dx \right) \, dx. \hfill (48)$$

So, with $w = u + \beta \left( \frac{2 - 3\tau}{6} \right) u_{xx} - \frac{\gamma}{2\beta} \int \left( \int u_{xx} \, dx \right) \, dx$ we obtain first order compatibility of Boussinesq’s equations (45)-(46) as

$$u_t + u_x + \beta \left( \frac{1 - 3\tau}{6} \right) u_{3x} + \frac{\gamma}{2\beta} \int u_{yy} \, dx = 0. \hfill (49)$$

Now, we consider second order corrections. We seek for $w$ in the form

$$w = u + w^{(1)} + w^{(2)} = u + \beta \left( \frac{2 - 3\tau}{6} \right) u_{xx} - \frac{1}{2} \frac{\gamma}{\beta} \int \left( \int u_{xx} \, dx \right) \, dx$$

$$+ \alpha Q^{(a)} + \beta^2 Q^{(b2)} + \gamma Q^{(g)}.$$ \hfill (50)

Substituting $w$ given by (50) into (45), and retaining terms up to second order we obtain

$$u_t + u_x + \beta \left( \frac{1 - 3\tau}{6} \right) u_{3x} + \frac{\gamma}{2\beta} \int u_{yy} \, dx + \alpha \left( Q_x^{(a)} + 2uu_x \right)$$

$$+ \beta^2 \left( Q_x^{(b2)} + u_{5x} \left( \frac{-17 + 30\tau}{360} \right) \right) + \gamma \left( Q_x^{(g)} + u_{x2y} \left( \frac{1 - 6\tau}{12} \right) \right)$$

$$- \frac{1}{2} \frac{\gamma^2}{\beta^2} \int \left( \int \left( \int u_{4y} \, dx \right) \, dx \right) \, dx = 0. \hfill (51)$$

The same steps with equation (46) give

$$u_t + u_x + \beta \left( \frac{1 - 3\tau}{6} \right) u_{3x} + \frac{\gamma}{2\beta} \int u_{yy} \, dx + \alpha \left( -Q_x^{(a)} + uu_x \right)$$

$$+ \beta^2 \left( -Q_x^{(b2)} + \frac{11 - 18\tau - 18\tau^2}{72} u_{5x} \right) + \gamma \left( -Q_x^{(g)} + \frac{5}{12} u_{x2y} \right)$$

$$+ \frac{1}{4} \frac{\gamma^2}{\beta^2} \int \left( \int \left( \int u_{4y} \, dx \right) \, dx \right) \, dx = 0. \hfill (52)$$

Subtraction of (52) from (51) yields

$$\alpha \left( 2Q_x^{(a)} + uu_x \right) + \beta^2 \left( 2Q_x^{(b2)} - \frac{3 - 5\tau}{15} u_{5x} \right) + \gamma \left( 2Q_x^{(g)} - \frac{2 + 3\tau}{6} u_{x2y} \right)$$

$$- \frac{3}{4} \frac{\gamma^2}{\beta^2} \int \left( \int \left( \int u_{4y} \, dx \right) \, dx \right) \, dx = 0. \hfill (53)$$
Under the condition \( u_{4y} = 0 \), equation (53), thanks to independence of small parameters, is equivalent to three equations for \( Q_x^{(i)} \) correction functions. Solving these equations and integrating the results one obtains

\[
Q^{(0)} = -\frac{1}{2} uu_x, \quad Q^{(02)} = \frac{12 - 20\tau - 15\tau^2}{120} u_{4x}, \quad Q^{(g)} = \frac{2 + 3\tau}{12} u_{yy}.
\] (54)

Then \( w \) which makes Boussinesq’s equations (45)-(46) compatible receives the form

\[
w = u + w^{(1)} + w^{(2)} = u + \beta \left( \frac{2 - 3\tau}{6} \right) u_{xx} - \frac{1}{2\beta} \int u_{xx} \, dx \, dx - \frac{1}{2\alpha} uu_x + \beta^2 \frac{12 - 20\tau - 15\tau^2}{120} u_{4x} + \gamma \frac{2 + 3\tau}{12} u_{yy}.
\] (55)

One has to emphasize that this compatibility is obtained only under the condition \( u_{4y} = 0 \).

Then substituting (55) into (45), and retaining only term up to second order, we obtain (again under condition \( u_{4y} = 0 \)) the nonlinear wave equation

\[
u_t + u_x + \beta \left( \frac{1 - 3\tau}{6} \right) u_{5x} + \gamma \frac{\beta}{2\beta} \int u_{yy} \, dx + \frac{3}{2} \alpha uu_x + \beta^2 \left( \frac{19 - 30\tau - 15\tau^2}{360} \right) u_{5x} - \gamma \left( \frac{1 - \tau}{4} \right) u_{x2y} = 0.
\] (56)

Indeed, proceeding in the same way with (46) we obtain the same wave equation.

The equation (56) is indeed the (2+1)-dimensional extension of the so-called fifth-order Korteweg-de Vries equation derived from the Euler equations describing the irrotational motion of an ideal fluid. The equation is true when \( \frac{\partial^4 u(x, y, t)}{\partial y^4} = 0 \).

V. CASE: \( \beta = O(\alpha^2), \quad \gamma = O(\alpha^2), \quad \delta = 0. \)

Denote now

\[
\beta = B \alpha^2, \quad \gamma = G \alpha^2,
\] (57)

where, as earlier, \( B, G \) are arbitrary constants close to 1. We consider flat bottom case, \( \delta = 0 \), aiming to derive second order wave equation. In (1+1)-dimensional case perturbation approach allowed to derive the Gardner equation [20].

Now, we have to express all perturbation equations with respect to parameter \( \alpha \). Then the velocity potential (12) can be rewritten as

\[
\phi = f - \frac{1}{2} z^2 \alpha^2 (B f_{2x} + G f_{2y}) + \frac{1}{24} z^4 \alpha^4 (B^2 f_{4x} + 2BG f_{2x2y} + G^2 f_{4y}) - \frac{1}{720} z^6 \alpha^6 (B^3 f_{6x} + 3B^2 G f_{4x2y} + 3BG^2 f_{2x4y} + G^3 f_{6y}) + \cdots
\] (58)

Substituting velocity potential (58) into (7) and retaining terms up to second order yields

\[
u_t + f_{xx} + \frac{\gamma}{\beta} f_{yy} + \alpha (uf_x)_x + \frac{\alpha \gamma}{\beta} (uf_y)_y - \frac{1}{6} \beta f_{4x} - \frac{1}{3} \beta f_{2x2y} - \frac{1}{6} \frac{\gamma^2}{\beta} f_{4y} = 0.
\] (59)
In the same way, from (8) we obtain

\[ u + f_t + \frac{1}{2} \alpha f_x^2 + \frac{1}{2} \alpha \gamma f_y^2 - \beta \left( \frac{1}{2} f_{xxx} + \tau u_{xx} \right) - \gamma \left( \frac{1}{2} f_{yyt} + \tau u_{yy} \right) = 0. \]  

(60)

Denoting \( f_x = w, f = \int w \, dx \), and so on, and differentiating (60) we can write equations (59)-(60) as

\[ u_t + w_x + \frac{\gamma}{\beta} \int w_{yy} \, dx + \alpha(uw)_x + \frac{\alpha \gamma}{\beta} (uw_y)_y - \frac{1}{6} \beta w_{3x} - \frac{1}{3} \beta w_{x2y} - \frac{1}{6} \frac{\gamma^2}{\beta} \int w_{4y} \, dx = 0, \]  

(61)

\[ w_t + u_x + \alpha uw_x + \frac{\alpha \gamma}{\beta} w_y \int w_y \, dx - \beta \left( \frac{1}{2} w_{xxt} + \tau u_{3x} \right) - \gamma \left( \frac{1}{2} w_{yyt} + \tau u_{x3y} \right) = 0. \]  

(62)

Note that when \( y \)-derivatives are set to zero Boussinesq’s equations (61)-(62) reduces to (1+1)-dimensions and are identical with [21, Eqs.(85)-(86)] with \( \delta = 0 \).

Since zeroth-order part of equations (59), (61) contain besides \( u_t + w_x \) the \( y \)-dependent term \( \frac{\gamma}{\beta} \int w_{yy} \, dx \), relations (20) do not hold. Therefore, derivation of single equation for surface wave in a way analogous to that used in sections III or IV is in this case impossible.

When surface tension effects can be neglected (case of shallow water waves, with the fluid depth of the order of meters), we can eliminate \( u \) and obtain a single differential equation for \( f \). Substituting \( u = -(f_t + \frac{1}{2} \alpha f_x^2 + \frac{1}{2} \alpha \gamma f_y^2 - \frac{1}{2} \beta f_{xx} - \frac{1}{2} \alpha \gamma f_{yy}) \) given by equation (60) into (59) and retaining terms up to second order one obtains

\[ f_{xx} + \frac{\gamma}{\beta} f_{xx} = f_{tt} - \alpha(2 f_x f_{xx} + f_{xxx}) - \gamma \left( \frac{1}{2} f_{yyt} + f_{yy} f_t \right) \]

\[ - \frac{3}{2} \alpha^2 f_x f_{xx} + \beta \left( \frac{1}{2} f_{xxx} - \frac{1}{6} f_{4x} \right) + \gamma \left( \frac{1}{2} f_{yyt} - \frac{1}{3} f_{xxyy} \right) \]

\[ - \alpha^2 \gamma \left( \frac{1}{2} f_x f_{yy} + 2 f_x f_y f_y + f_y f_{xx} \right) - \alpha^2 \gamma \left( \frac{3}{2} f_y f_{yy} + \frac{1}{6} f_{yy} \right) = 0. \]  

(63)

Note that terms in the second and third rows are second order ones.

If the solution of equation (63) is known, equation (60) determines the wave profile \( u(x, y, t) \). However, equation (63) is so complicated that it can be tough to find a solution, even limiting it to first order.

**A. Case:** \( \beta = O(\alpha^2), \ \gamma = O(\alpha^3), \ \delta = 0. \) (2+1)-dimensional Gardner equation

Since for \( \gamma = O(\alpha^2) \) \( y \)-dependent term \( \frac{\gamma}{\beta} \int w_{yy} \, dx \) is of zeroth order let us consider the case \( \gamma = O(\alpha^3) \). Denote now

\[ \beta = B \alpha^2, \ \gamma = G \alpha^3. \]  

(64)

The velocity potential (12) can be rewritten (up to fourth order) as

\[ \phi = f - \frac{1}{2} z^2 (\alpha^2 B f_{2x} + \alpha^3 G f_{2y}) + \frac{1}{24} z^4 \alpha^4 B^2 f_{4x} + \cdots \]  

(65)
Substituting velocity potential (58) into (7) and retaining terms up to second order yields

\[ u_t + f_{xx} + \frac{\gamma}{\beta} f_{yy} + \alpha(uf_x)_x + \frac{\alpha\gamma}{\beta} (uf_y)_y - \frac{1}{6} \beta f_{4x} = 0. \] (66)

Note, that term with \( \frac{\gamma}{\beta} \) is now first order one, and terms with \( \frac{\alpha\gamma}{\beta} \) and \( \beta \) are of second order.

In the same way, from (8) we obtain

\[ u + f_{t} + \frac{1}{2} \alpha f_{x}^{2} + \frac{1}{2} \frac{\alpha\gamma}{\beta} f_{y}^{2} - \beta \left( \frac{1}{2} f_{xxt} + \tau u_{xx} \right) = 0. \] (67)

Equations (66) and (67) are (2+1)-dimensional (second order) Boussinesq’s equations for the case \( \beta \approx \alpha^{2}, \gamma \approx \alpha^{3} \). In zeroth order they coincide with those which lead to KdV, so we can follow the same method that was used in sections III and IV.

Denoting \( f_{x} = w, f = \int w dx, \) and so on, and differentiating (67) we can write equations (66)-(67) as

\[ u_t + w_x + \frac{\gamma}{\beta} \int w_{yy} dx + \alpha(uw)_x + \frac{\alpha\gamma}{\beta} \left( u_y \int w_y dx + u \int w_{yy} dx \right) - \frac{1}{6} \beta w_{3x} = 0, \] (68)

\[ w_t + u_x + \alpha w w_x + \frac{\alpha\gamma}{\beta} w_y \int w_y dx - \beta \left( \frac{1}{2} w_{xxt} + \tau u_{xx} \right) = 0. \] (69)

Zeroth-order equations (68)-(69) are the same as those of (18)-(19). Therefore in zeroth-order relations (20) hold. In first order we have terms with \( \alpha \) and \( \frac{\gamma}{\beta} \), so we seek for \( w \) in the form

\[ w = u + w^{(1)} = u + \alpha Q^{(a)} + \frac{\gamma}{\beta} Q^{(gb)}. \] (70)

Proceeding in the same way as in section III we obtain correction functions in first order as

\[ Q^{(a)} = -\frac{1}{4} u^2, \quad Q^{(gb)} = -\frac{1}{2} \int \left( \int u_{xx} dx \right) dx. \] (71)

So, in first order perturbation approach we obtain the wave equation

\[ u_t + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{2} \frac{\gamma}{\beta} \int u_{yy} dx = 0. \] (72)

Seeking for second order wave equation we take

\[ w = u + w^{(1)} + w^{(2)} = u - \frac{1}{4} \alpha u^2 - \frac{\gamma}{\beta} \frac{1}{2} \int \left( \int u_{xx} dx \right) dx + \alpha^2 Q^{(a2)} + \frac{\alpha\gamma}{\beta} Q^{(agb)} + \beta Q^{(b)}. \] (73)

Then from (68) we obtain (retaining terms up to second order)

\[
\begin{align*}
\frac{\alpha\gamma}{\beta} \left( Q^{(agb)} - \frac{1}{2} \int u_{yy} dx - \frac{1}{2} \int uu_{yy} dx + \frac{1}{2} \int u_{y} dx + \frac{1}{2} \int u_{xy} dx \right) - \frac{1}{2} \frac{\gamma^2}{\beta^2} \left( \int (\int u_{yx} dx) dx \right) - \frac{1}{2} \frac{\gamma^2}{\beta^2} \int (\int (\int u_{xy} dx) dx) dx = 0, 
\end{align*}
\] (74)
whereas from (69) the result is
\[
\begin{align*}
  u_t + w_x + \frac{3}{2} \alpha uu_x + \frac{\gamma}{\beta} \int w_{yy} dx - \alpha^2 Q_x^{(a2)} + \beta \left( -Q_x^{(b)} + \frac{1 - 2\tau}{2} u_{3x} \right) \\
  + \frac{\alpha \gamma}{\beta} \left( -Q_x^{(agb)} + \frac{3}{4} \int u_y^2 dx + \frac{3}{4} \int uu_{yy} dx - \frac{1}{4} u \int u_{yy} dx + u_y \int u_y dx \\
  - \frac{1}{2} u_x \int (u_{yy} dx) dx \right) + \frac{1}{4} \frac{\gamma^2}{\beta^2} \int \left( \int \left( \int u_{4y} dx \right) dx \right) dx = 0.
\end{align*}
\]

Subtraction of (75) from (74) yields
\[
\begin{align*}
  \alpha^2 \left( 2Q_x^{(a2)} - \frac{3}{4} u^2 u_x \right) + \beta \left( 2Q_x^{(b)} - \frac{2 - 3\tau}{3} u_{3x} \right) \\
  + \frac{\alpha \gamma}{\beta} \left( 2Q_x^{(agb)} - \frac{5}{4} \int (u_y^2 + uu_{yy}) dx + \frac{3}{4} u \int u_{yy} dx \right) - \frac{3 \gamma^2}{4 \beta^2} \int \left( \int \left( \int u_{4y} dx \right) dx \right) dx = 0.
\end{align*}
\]

Only under the condition \( u_{4y} = 0 \) we can obtain the correction functions for second order \( w \)
\[
\begin{align*}
  Q_x^{(a2)} &= \frac{1}{8} u^3, \quad Q_x^{(b)} = \frac{2 - 3\tau}{6} u_{xx}, \\
  Q_x^{(agb)} &= \frac{5}{8} \int \left( \int (u_y^2 + uu_{yy}) \right) dx + \frac{3}{8} \int (u \int u_{yy} dx) dx.
\end{align*}
\]

With these correction functions Boussinesq’s equations (68)-(69) become compatible. The final wave equation receives the form (still under the condition \( u_{4y} = 0 \))
\[
\begin{align*}
  u_t + u_x + \frac{3}{2} \alpha uu_x + \frac{1}{2} \frac{\gamma}{\beta} \int w_{yy} dx - \frac{3}{8} \alpha^2 u^2 u_x + \beta \frac{1 - 3\tau}{6} u_{3x} \\
  + \frac{\alpha \gamma}{\beta} \left( \frac{1}{8} \int (u_y^2 + uu_{yy}) dx + \frac{1}{8} u \int u_{yy} dx + u_y \int u_y dx - \frac{1}{2} u_x \int (u_{yy} dx) dx \right) = 0.
\end{align*}
\]

Note that when \( u = u(x, t) \) does not depend on \( y \), equation (78) reduces to
\[
\begin{align*}
  u_t + u_x + \frac{3}{2} \alpha uu_x + \frac{3}{8} \alpha^2 u^2 u_x + \beta \frac{1 - 3\tau}{6} u_{3x} = 0,
\end{align*}
\]
which is well known Gardner equation.

Therefore, equation (79), the simplest extension of the Gardner equation to (2+1)-dimensions can be called \((2+1)\)-dimensional Gardner equation.

VI. EXTENSIONS TO CASES OF UNEVEN BOTTOM

In [21], we showed in (1+1)-dimensional theory that small variations of the bottom introduce additional terms in Boussinesq’s equations. These terms can be either first or second order depending on the relation of \( \delta \) parameter to the leading parameter \( \beta \) (or \( \alpha \) in the case leading to the Gardner equation). However, in each case, the Boussinesq equations could be reduced to single
wave equations for surface profile function $u(x, t)$, and the additional term in such equations have the universal form. It turned out that in each of the KdV, fifth-order KdV and Gardner equations extension to the case of uneven bottom adds the term $-\frac{1}{4}\delta(2hu_x + h_xu)$. However, compatibility is possible only if $h_{xx} = 0$ (for details, see [21]).

It turns out that in (2+1)-dimensional theory all these results hold. In section III we derived equation (30) which is (2+1)-dimensional KdV equation taking into account small bottom variations. Analogous procedures (not presented here) including $\delta \neq 0$ for the case discussed in section IV lead to the equation

$$u_t + u_x + \beta \left(\frac{1 - 3\tau}{6}\right) u_{3x} + \frac{\gamma}{2\beta} \int u_{yy} \, dx + \frac{3}{2} \alpha uu_x + \beta^2 \left(\frac{19 - 30\tau - 15\tau^2}{360}\right) u_{5x} - \gamma \left(\frac{1 - \tau}{4}\right) u_{x2y} - \frac{1}{4}\delta(2hu_x + h_xu) = 0,$$

which differs from (56) only by the last term. Note, that all terms in the second row are of second order. Equation (80) is (2+1)-dimensional 5th order KdV equation taking into account small bottom variations. For more details, see the Appendix A.

When the case $\delta \approx \alpha^2$ is considered together with $\beta \approx \alpha^2$, $\gamma \approx \alpha^3$ (like in subsection V A) the resulting equation is

$$u_t + u_x + 3\alpha uu_x + \frac{1}{2\beta} \int w_{yy} \, dx - \frac{3}{8}\alpha^2 u^2 u_x + \beta \left(\frac{1 - 3\tau}{6}\right) u_{3x} - \frac{1}{4}\delta(2hu_x + h_xu) + \frac{\alpha\gamma}{\beta} \left(\frac{1}{8} \int (u_y^2 + uu_{yy}) \, dx + \frac{1}{8} u \int u_{yy} \, dx + u_y \int u_y \, dx - \frac{1}{2}\int u_x \left(\int u_{yy} \, dx\right) \, dx\right) = 0.$$

Here, all terms in second and third rows are of second order. Equation (81) is (2+1)-dimensional Gardner equation taking into account small bottom variations. For more details, see the Appendix B.

It has to be remembered that single wave equations (80)-(81) can be obtained only under condition $u_{4y} = 0$.

**VII. CONCLUSIONS**

We have shown that for some particular relations between small parameters, it is possible to derive (2+1)-dimensional nonlinear wave equations of KdV-type from the ideal fluid model. The crucial condition allowing for such derivations is such that the parameter $\gamma$, responsible for wavelength in the direction perpendicular to the main wave propagation, is of order smaller than $\beta$ parameter related to the wavelength of the main wave. In such a case, zeroth-order Boussinesq’s equations do not contain $y$-derivatives and make it possible to use relations (20) that are true in zeroth order. Then, when $\alpha \approx \beta$, $\gamma \approx \beta^2$, limiting to first order perturbation approach, we obtain (2+1)-dimensional nonlocal extension to KdV equation. After transformation to moving reference
frame and differentiation over new \( x \)-variable this equation becomes the Kadomtsev-Petviashvili equation. These equations are obtained without any additional assumptions.

We derived (2+1)-dimensional nonlocal extensions to fifth-order KdV and Gardner equations in two other cases discussed. However, the derivation of these equations requires an additional condition. The fourth partial derivative, \( u_{yy} \), of the surface profile must be equal to zero. Otherwise, the corresponding Boussinesq equations cannot be reduced to a single wave equation.

When the condition \( u_{yy} = 0 \) is fulfilled, the (2+1)-dimensional nonlocal extension 5th order KdV equation is obtained when \( \alpha \approx \gamma \approx \beta^2 \) and (2+1)-dimensional nonlocal extension to Gardner equation is obtained when \( \beta \approx \alpha^2, \gamma \approx \alpha^3 \). In all cases discussed in the present paper, effects of the uneven bottom can be included if the bottom function fulfills the condition \( h_{xx} = 0 \). The related term, which takes into account small variations of the bottom, has a universal form.

**Appendix A: 5th-order KdV equation with an uneven bottom**

Denote now

\[
\alpha = A \beta^2, \quad \gamma = G \beta^2, \quad \delta = D \beta^2.
\] (A1)

In this case, the boundary condition at the bottom (9) imposes the following relation

\[
F = \beta^3 D (h_x f_x + h f_{xx}) + \beta^4 DG (h_y f_y + h f_{yy}) + \beta^5 D^2 \left( hh_x f_x - \frac{1}{2} h^2 F_{xx} \right) + O(\beta^6). \quad (A2)
\]

Therefore we can eliminate \( F \) using

\[
F = \beta^3 D (h_x f_x + h f_{xx}) + \beta^4 DG (h_y f_y + h f_{yy}),
\] (A3)

valid up to fourth order. Since we aim to derive wave equation valid up to second order in small parameters it is enough to keep terms in the velocity potential up to third order. This gives

\[
\phi = f - \frac{1}{2} z^2 (\beta f_{2x} + \beta^2 G f_{2y}) + \frac{1}{24} z^4 (\beta^2 f_{4x} + 2 \beta^3 G f_{2x2y}) - \frac{1}{720} z^6 \beta^3 f_{6x} + z \beta^3 D (h_x f_x + h f_{xx}) + O(\beta^4).
\] (A4)

Substituting the velocity potential (A4) into the kinematic boundary condition at the surface (7) and retaining terms up to second order yields equation (42) supplemented by term \(-\delta (h f_x)_x\). Like in other cases (see, e.g., equation (19) and all cases studied in [21]) no \( \delta \) dependent term appear in the Boussinesq equation originating from the dynamic boundary condition at the surface (8). Then the obtained bottom dependent term in the wave equation has the same form \(-\frac{1}{4} \delta (2 h u_x + h_x u)\).

**Appendix B: Gardner equation with an uneven bottom**

Now we have

\[
\beta = B \alpha^2, \quad \gamma = G \alpha^3, \quad \delta = D \alpha^2.
\] (B1)
The boundary condition at the bottom (9) implies

\[ F = \alpha^4 D (h_x f_x + h f_{xx}) + \alpha^5 D G (h_y f_y + h f_{yy}) + O(\alpha^6), \]  
(B2)

which allows us to express \( F \) by \( f \) and its derivatives. Then the velocity potential taken up to fourth order in \( \alpha \) is

\[ \phi = f - \frac{1}{2} z^2 \alpha^2 (B f_{2x} + \alpha G f_{2y}) + \frac{1}{24} z^4 \alpha^4 B^2 f_{4x} + z \alpha^4 B D (h f_x)_x + O(\beta^5). \]  
(B3)

In this case, to obtain Boussinesq’s equations valid up to second order, velocity potential valid to fourth order is necessary (due to term \( \frac{1}{\beta} \approx \frac{1}{\alpha} \) in (8)). Then with the velocity potential (B3), from (7) we obtain equation (66) supplemented by the term \( -\delta(h f_x)_x \). From (8) the result is equation (67) without any change. Next, with the same procedure we arrive to equation (78).

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