ALEXANDER POLYNOMIALS AND SIGNATURES OF SOME HIGH-DIMENSIONAL KNOTS

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Abstract. We give necessary and sufficient conditions for an integer to be the signature of a $4q - 1$-knot in $S^{4q+1}$ with a given square-free Alexander polynomial.

0. Introduction

What are the possibilities for the signatures of knots with a given Alexander polynomial? This question is answered in [B 21] for "classical" knots, i.e. knots $K^1 \subset S^3$, with some restrictions on the Alexander polynomial, and the same results hold for knots $K^m \subset S^{m+2}$ if $m \equiv 1 \pmod{4}$. In the present paper, we consider high-dimensional knots $K^m \subset S^{m+2}$ with $m \equiv -1 \pmod{4}$. In the introduction, we describe the results for $m > 3$; the case $m = 3$ is somewhat different (see Section 9).

Let $m \geq 7$ be an integer with $m \equiv -1 \pmod{4}$. An $m$-knot $K^m \subset S^{m+2}$ is by definition a smooth, oriented submanifold of $S^{m+2}$, homeomorphic to $S^m$; in the following, a knot will mean an $m$-knot as above. We refer to the book of Michel and Weber [MW 17] for a survey of high-dimensional knot theory.

Let $K^m$ be a knot. The Alexander polynomial $\Delta = \Delta_K \in \mathbb{Z}[X]$ is a polynomial of even degree; set $2n = \deg(\Delta)$. It satisfies the following three properties (see for instance [Le 69], Proposition 1):

1. $\Delta(X) = X^{2n}\Delta(X^{-1})$,
2. $\Delta(1) = (-1)^n$,
3. $\Delta(-1)$ is a square.

Conversely, if $\Delta \in \mathbb{Z}[X]$ is a degree $2n$ polynomial satisfying conditions (1)-(3), then there exists a knot with Alexander polynomial $\Delta$ (cf. Levine [Le 69], Proposition 2 and Lemma 3; note that Lemma 3 is based on a result of Kervaire, [K 65], Théorème II.3).

Let $F^{m+1}$ be a Seifert hypersurface of $K^m$ (see for instance [MW 17], Definition 6.16), and let $L = H_n(F^{m+1}, \mathbb{Z})/\text{tors}$, where Tors is the $\mathbb{Z}$-torsion subgroup of $H_n(F^{m+1}, \mathbb{Z})$. Let $S: L \times L \rightarrow \mathbb{Z}$ be the intersection form; since $m \equiv -1 \pmod{4}$, the form $S$ is symmetric. The signature of $K^m$ is by definition the signature of the symmetric form $S$; it is an invariant of the knot. The form $S$ is even and unimodular, therefore its signature is $\equiv 0 \pmod{8}$.

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Levine’s construction (see [Le 69], Proposition 2 and Lemma 3) shows the existence of a knot with Alexander polynomial $\Delta$ and signature $0$. It is natural to ask: what other signatures occur?

Let us denote by $\rho(\Delta)$ the number of roots $z$ of $\Delta$ such that $|z| = 1$. If a knot has Alexander polynomial $\Delta$ and signature $s$, then $|s| \leq \rho(\Delta)$. This shows that the conditions $s \equiv 0 \pmod{8}$ and $|s| \leq \rho(\Delta)$ are necessary for the existence of a knot with Alexander polynomial $\Delta$ and signature $s$; however, these conditions are not sufficient, as shown by the following example, taken from [GM 02], Proposition 5.2:

**Example 1.** Let $\Delta(X) = (X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1)(X^4 - X^2 + 1)$; we have $\rho(\Delta) = 8$, hence $s = -8, 0$ and $8$ satisfy the above necessary conditions. However, there does not exist any knot with Alexander polynomial $\Delta$ and signature $-8$ or $8$.

Let $\Delta \in \mathbb{Z}[X]$ be a polynomial satisfying conditions (1)-(3), and suppose that $\Delta$ is square-free. We associate to $\Delta$ a finite abelian group $G_\Delta$ that controls the signatures of the knots with Alexander polynomial $\Delta$ (see §4 - §7). In particular, we have (cf. Corollary 7.4):

**Theorem 1.** Assume that $G_\Delta = 0$, and let $s$ be an integer with $s \equiv 0 \pmod{8}$ and $|s| \leq \rho(\Delta)$. Then there exists a knot with Alexander polynomial $\Delta$ and signature $s$.

The vanishing of the group $G_\Delta$ has other geometric consequences: we show the existence of indecomposable knots with Alexander polynomial $\Delta$ (see §8).

1. **Seifert forms and Seifert pairs**

Seifert forms are well-known objects of knot theory; the aim of this section is to recall this notion, and to show that it is equivalent to the one of *Seifert pairs*; this notion was introduced, under a different name, by Kervaire in [K 71] in the context of knot cobordism; see also Stoltzfus ([St 77]) and [B 82], §5.

**Definition 1.1.** A *Seifert form* is by definition a pair $(L, A)$, where $L$ is a free $\mathbb{Z}$-module of finite rank and $A : L \times L \to \mathbb{Z}$ is a $\mathbb{Z}$-bilinear form such that the symmetric form $L \times L \to \mathbb{Z}$ sending $(x, y)$ to $A(x, y) + A(y, x)$ is unimodular (i.e. has determinant $\pm 1$); the *signature* of $(L, A)$ is by definition the signature of this symmetric form.

The *Alexander polynomial* of $(L, A)$, denoted by $\Delta_A$, is by definition the determinant of the form $L \times L \to \mathbb{Z}[X]$ given by

$$(x, y) \mapsto A(x, y)X + A(y, x).$$

**Definition 1.2.** A *Seifert pair* is by definition a triple $(L, S, a)$, where $L$ is a free $\mathbb{Z}$-module of finite rank, $S : L \times L \to \mathbb{Z}$ is an even (i.e. $S(x, x)$ is an even integer for all $x \in L$), unimodular, symmetric $\mathbb{Z}$-bilinear form, and $a : L \to L$ is an injective $\mathbb{Z}$-linear map such that

$$S(ax, y) = S(x, (1 - a)y)$$

for all $x, y \in L$. 
Let \((L, S, a)\) be a Seifert pair. Since \(S\) is even and unimodular, the rank of \(L\) is an even integer; let \(n \in \mathbb{Z}\) be such that \(\text{rank}(L) = 2n\). Let \(A : L \times L \to \mathbb{Z}\) be defined by

\[ A(x, y) = S(ax, y); \]

note that \((L, A)\) is a Seifert form, and we have

**Proposition 1.3.** Sending \((L, S, a)\) to \((L, A)\) as above induces a bijection between isomorphism classes of Seifert pairs and of Seifert forms. Let \(P_a\) be the characteristic polynomial of \(a\). We have

\[ P_a(X) = (-1)^n X^{2n} \Delta_A(1 - X^{-1}). \]

Note that \(\Delta_A(X) = X^{2n} \Delta_A(X^{-1})\), and that \(P_a(X) = P_a(1 - X)\).

**Definition 1.4.** A lattice is a pair \((L, S)\), where \(L\) is a free \(\mathbb{Z}\)-module of finite rank, and \(S : L \times L \to \mathbb{Z}\) is a symmetric bilinear form with \(\det(S) \neq 0\). We say that \((L, S)\) is unimodular if \(\det(S) = \pm 1\), and even if \(S(x, x)\) is an even integer for all \(x \in L\).

Note that a Seifert pair consists of an even, unimodular lattice \((L, S)\) and an injective endomorphism \(a : L \to L\) such that \(S(ax, y) = S(x, (1 - a)y)\) for all \(x, y \in L\).

## 2. Involutions of \(K[X]\), symmetric polynomials and bilinear forms compatible with a module

Let \(K\) be a field, let \(R\) be a commutative \(K\)-algebra, and let \(\sigma : R \to R\) be an involution; we say that \(\lambda \in R\) is \(\sigma\)-symmetric (or symmetric, if the choice of \(\sigma\) is clear from the context) if \(\sigma(\lambda) = \lambda\).

**Example 2.1.** (1) Let \(\sigma : K[X, X^{-1}] \to K[X, X^{-1}]\) be the involution sending \(X\) to \(X^{-1}\). If \(\Delta\) is the Alexander polynomial of a Seifert form of rank \(2n\), then \(X^{-n} \Delta(X)\) is symmetric.

(2) Let \(\sigma : K[X] \to K[X]\) be the involution sending \(X\) to \(1 - X\); the symmetric polynomials are the \(f \in K[X]\) such that \(f(1 - X) = f(X)\). The characteristic polynomial of a Seifert pair is symmetric.

Let \(M\) be an \(R\)-module that is a finite dimensional \(K\)-vector space. Recall from [B21], §1, that a non-degenerate symmetric bilinear form \(b : M \times M \to K\) is called an \((R, \sigma)\)-bilinear form if

\[ b(\lambda x, y) = b(x, \sigma(\lambda)y) \]

for all \(x, y \in V\) and for all \(\lambda \in R\).

**Example 2.2.** Let \((L, S, a)\) be a Seifert pair and set \(V = L \otimes_{\mathbb{Z}} \mathbb{Q}\); we denote by \(S : V \times V \to \mathbb{Q}\) and \(a : V \to V\) the symmetric bilinear form and the \(\mathbb{Q}\)-linear map induced by \(S\) and \(a\). Let \(\sigma : \mathbb{Q}[X] \to \mathbb{Q}[X]\) be the involution sending \(X\) to \(1 - X\). We endow \(V\) with a structure of \(\mathbb{Q}[X]\)-module by setting \(X.x = a(x)\) for all \(x \in V\); note that \(S : V \times V \to \mathbb{Q}\) is a \((\mathbb{Q}[X], \sigma)\) bilinear form.
Let $V$ be a finite dimensional $K$-vector space, and let $q : V \times V \to K$ be a non-degenerate symmetric bilinear form. Following [B 21], §1, we say that $M$ and $(V,q)$ are compatible if there exists a $K$-linear isomorphism $\phi : M \to V$ such that the bilinear form $b_\phi : M \times M \to K$, defined by $b_\phi(x,y) = q(\phi(x), \phi(y))$, is an $R$-bilinear form.

**Example 2.3.** Let $\sigma : K[X] \to K[X]$ be the involution sending $X$ to $1 - X$, and let $P \in K[X]$ be a monic, $\sigma$-symmetric polynomial. Assume $P$ is a product of distinct monic, symmetric, irreducible factors; let us denote by $I$ the set of these polynomials. Set $\sigma^2(P) = P$.

The set of these polynomials. Set

$$\text{Example 2.3.} \quad \text{Let } \sigma : K[X] \to K[X] \text{ be the involution sending } X \text{ to } 1 - X, \quad \text{and let } P \in K[X] \text{ be a monic, } \sigma\text{-symmetric polynomial. Assume } P \text{ is a product of distinct monic, symmetric, irreducible factors; let us denote by } I \text{ the set of these polynomials. Set } \sigma^2(P) = P. \quad \text{Set}$$

$\sigma^2(P)$

3. **Milnor signatures**

We recall the notion of Milnor signatures, introduced by Milnor in [M 68], in the context of Seifert pairs. Let $(L,S,a)$ be a Seifert pair, and let $P \in \mathbb{Z}[X]$ be the characteristic polynomial of $a$. Assume that the polynomial $P$ is square-free, i.e. has no repeated factors; we also suppose that if $f \in \mathbb{Z}[X]$ is a monic, irreducible factor of $P$, then $f(X) = f(1 - X)$.

Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. Let $f \in \mathbb{R}[X]$ be a monic, irreducible factor of degree 2 of $P \in \mathbb{R}[X]$; note that this implies that $f(X) = f(1 - X)$.

**Definition 3.1.** The signature of $(L,S,a)$ at $f$ is by definition the signature of the restriction of $S$ to $\text{Ker}(f(a))$.

**Notation 3.2.** Let $\text{Irr}_R(P)$ be the set of monic, irreducible factors $f \in \mathbb{R}[X]$ of degree 2 of $P$. Let $s \in \mathbb{Z}$. We denote by $\text{Mil}(P)$ the set of maps

$$\text{Irr}_R(P) \to \{-2, 2\},$$

and by $\text{Mil}_s(P)$ the set of $\tau \in \text{Mil}(P)$ such that

$$\sum_{f \in \text{Irr}_R(P)} \tau(f) = s.$$

Let $n \geq 1$ be an integer, and let $\Delta \in \mathbb{Z}[X]$ be a polynomial of degree $2n$ such that $\Delta(X) = X^{2n}\Delta(X^{-1})$, $\Delta(1) = (-1)^n$ and that $\Delta(-1)$ is a square of an integer. Suppose that $P(X) = (-1)^n X^{2n}\Delta(1 - X^{-1})$. We define $\text{Mil}_s(\Delta)$ as in [B 21], §26; note that there are obvious bijections between $\text{Irr}_R(P)$ and $\text{Irr}_R(\Delta)$, $\text{Mil}_s(P)$ and $\text{Mil}_s(\Delta)$, and that we recover the usual notion of Milnor signature.

If $P$ and $\Delta$ are as above, set $\rho(P) = \rho(\Delta)$; alternatively, $\rho(P)$ can be defined as the number of roots $z$ of $P$ with $z + \overline{z} = 1$, where $\overline{z}$ denotes the complex conjugate of $z$. Note that $\rho(\Delta) = |\text{Irr}_R(\Delta)|$ and $\rho(P) = |\text{Irr}_R(P)|$. 


4. The obstruction group

Let \( P \in \mathbb{Z}[X] \) be a monic polynomial such that \( P(1 - X) = P(X) \). Assume that \( P \) is a product of distinct irreducible monic polynomials \( f \in \mathbb{Z}[X] \) such that \( f(1 - X) = f(X) \). We associate to \( P \) an elementary abelian 2-group \( G_P \) that will be useful in the following sections; this construction is similar to the one of [B 21], §21.

Let \( I \) be the set of irreducible factors of \( P \). If \( f, g \in I \), let \( \Pi_{f,g} \) be the set of prime numbers \( p \) such that \( f \mod p \) and \( g \mod p \) have a common factor \( h \in \mathbb{F}_p[X] \) such that \( h(1 - X) = h(X) \). Let \( C(I) \) be the set of maps \( I \to \mathbb{Z}/2\mathbb{Z} \), and let \( C_0(I) \) be the set of \( c \in C_0(I) \) such that \( c(f) = c(g) \) if \( f, g \notin \emptyset \). Note that \( C_0(I) \) is a group with respect to the addition of maps, and let \( G_P \) be the quotient of the group \( C_0(I) \) by the subgroup of the constant maps.

**Example 4.1.** Let \( f_1(X) = X^4 - 2X^3 + 5X^2 - 4X + 1 \) and \( f_2(X) = X^4 - 2X^3 + 11X^2 - 10X + 3 \);
set \( P = f_1f_2 \). We have \( \Pi_{f_1,f_2} = \{2\} \), hence \( G_P = 0 \).

If \( P(X) = (-1)^nX^{2n}\Delta(1 - X^{-1}) \) for some polynomial \( \Delta \in \mathbb{Z}[X] \), set \( G_\Delta = G_P \). If moreover \( \Delta(0) = \pm 1 \), then the group \( G_\Delta \) is equal to the obstruction group \( III_{\Delta(0)\Delta} \) of [B 21], §21 and §25. In particular, 25.8 - 25.11, 31.4 and 31.5 of [B 21] provide examples of obstruction groups in our context as well. This is also the case for the following example, given in the introduction:

**Example 4.2.** Let \( g_1(X) = X^6 - 3X^5 - X^4 + 5X^3 - X^2 - 3X + 1 \) and \( g_2(X) = X^4 - X^2 + 1 \); set \( \Delta = g_1g_2 \), as in Example 1. Set \( f_1(X) = -X^6g_1(1 - X^{-1}) \), and \( f_2(X) = X^4g_2(1 - X^{-1}) \), and let \( P = f_1f_2 \). The polynomials \( f_1 \) and \( f_2 \) are relatively prime over \( \mathbb{Z} \), hence \( \Pi_{f_1,f_2} = \emptyset \); therefore \( G_P \simeq \mathbb{Z}/2\mathbb{Z} \).

5. Seifert pairs with a given characteristic polynomial and signature

Let \( n \geq 1 \) be an integer, and let \( \Delta \in \mathbb{Z}[X] \) be a polynomial of degree \( 2n \) such that \( \Delta(X) = X^{2n}\Delta(X^{-1}) \), \( \Delta(1) = (-1)^n \) and that \( \Delta(-1) \) is a square of an integer. Set \( P(X) = (-1)^nX^{2n}\Delta(1 - X^{-1}) \). Assume that \( P \) is a product of distinct irreducible monic polynomials \( f \in \mathbb{Z}[X] \) such that \( f(1 - X) = f(X) \), and let \( I \) be the set of irreducible, monic factors of \( P \).

Let \( G_P \) be the group introduced in §11 and set \( G_\Delta = G_P \).

Let \( s \) be an integer such that \( s \equiv 0 \pmod{8} \), and that \( |s| \leq \rho(P) \). Let \( \tau \in \text{Mil}_s(P) \). The aim of this section is to give a necessary and sufficient condition for the existence of a Seifert pair with characteristic polynomial \( P \) and Milnor signature \( \tau \).

Let \( V \) be a \( \mathbb{Q} \)-vector space of dimension \( 2n \), and let \( S: V \times V \to \mathbb{Q} \) be a non-degenerate quadratic form of signature \( s \) containing an even, unimodular lattice; such a form exists and is unique up to isomorphism (see for instance [B 21], Lemma 25.5).
Let $M = \bigoplus_{f \in I} \mathbb{Q}[X]/(f)$, considered as a $\mathbb{Q}[X]$-module. Let $\sigma : \mathbb{Q}[X] \to \mathbb{Q}[X]$ be the $\mathbb{Q}$-linear involution such that $\sigma(X) = 1 - X$. The Milnor signature $\tau \in \text{Mil}_s(P)$ determines an $(\mathbb{R}[X], \sigma)$-quadratic form (cf. [B 21], Example 24.1). The local conditions of [B 21], §24 are satisfied. Indeed, the $\mathbb{R}[X]$-module $M \otimes \mathbb{R}$ is compatible with $(V, S)$ by [B 15], Proposition 8.1. Using a result of Levine (see [Le 69], Proposition 2) and the bijection between Seifert forms and Seifert pairs (see §1), we see that there exists a Seifert pair of characteristic polynomial $P$. This implies that for all prime numbers $p$, the $\mathbb{Q}_p[X]$-module $M \otimes \mathbb{Q}_p$ and the quadratic form $(V, S) \otimes \mathbb{Q}_p$ are compatible.

As in [B 21], §24, we define a homomorphism $\epsilon_{\tau} : G_P \to \mathbb{Z}/2\mathbb{Z}$.

**Theorem 5.1.** There exists a Seifert pair with characteristic polynomial $P$ and Milnor signature $\tau$ if and only if $\epsilon_{\tau} = 0$.

**Proof.** By [B 21], Theorem 24.2, the global conditions are satisfied if and only if $\epsilon_{\tau} = 0$. Using [B 21], Proposition 6.2 this is equivalent with the existence of a Seifert pair having characteristic polynomial $P$ and Milnor signature $\tau$.

**Corollary 5.2.** Assume that $G_P = 0$. Then for all $\tau \in \text{Mil}_s(P)$ there exists a Seifert pair with characteristic polynomial $P$ and Milnor signature $\tau$.

6. **Seifert forms with a given Alexander polynomial and signature**

We keep the notation of the previous section. Using Proposition 1.3, Theorem 5.1 and Corollary 5.2 can be reformulated as follows:

**Theorem 6.1.** There exists a Seifert form with Alexander polynomial $\Delta$ and Milnor signature $\tau$ if and only if $\epsilon_{\tau} = 0$.

**Corollary 6.2.** Assume that $G_{\Delta} = 0$. Then for all $\tau \in \text{Mil}_s(\Delta)$ there exists a Seifert form with Alexander polynomial $\Delta$ and Milnor signature $\tau$.

7. **Knots with a given Alexander polynomial and signature**

We keep the notation of the previous two sections. Let $m \geq 7$ be an integer with $m \equiv -1 \pmod{4}$. We refer to [MW 17], 6.5 for the definition of the Seifert form associated to an $m$-knot. The results of this section rely on a result of Kervaire:

**Theorem 7.1.** Let $(L, A)$ be a Seifert form. Then there exists an $m$-knot with associated Seifert form isomorphic to $(L, A)$.

**Proof.** This is proved by Kervaire in [K 65], Theorem II.3, and formulated more explicitly by Levine in [Le 69], Lemma 3 and [Le 70], Theorem 2. A different proof is given by Michel and Weber in [MW 17], Theorem 7.3 (see also the remark at the end of [MW 17], §7.1).

Combining Theorem 6.1 and Corollary 6.2 with Theorem 7.1, we have the following applications:

**Theorem 7.2.** There exists an $m$-knot with Alexander polynomial $\Delta$ and Milnor signature $\tau$ if and only if $\epsilon_{\tau} = 0$. 
Corollary 7.3. Assume that $G_\Delta = 0$. Then for all $\tau \in \text{Mil}_s(\Delta)$ there exists an $m$-knot with Alexander polynomial $\Delta$ and Milnor signature $\tau$.

Recall that $s$ is an integer such that $s \equiv 0 \pmod{8}$, and that $|s| \leq \rho(\Delta)$.

Corollary 7.4. Assume that $G_\Delta = 0$. Then there exists an $m$-knot with Alexander polynomial $\Delta$ and signature $s$.

8. Indecomposable knots with decomposable Alexander polynomial

As an application of Corollary 7.4, we give some examples of indecomposable knots with decomposable Alexander polynomials. Let $m \geq 7$ be an integer with $m \equiv -1 \pmod{4}$.

Example 8.1. Let $\Delta = \Delta_1 \Delta_2$, where $\Delta_1(X) = X^4 - X^2 + 1$ and $\Delta_2(X) = 3X^4 - 2X^3 - X^2 - 2X + 3$; we have $\rho(\Delta) = 8$. The corresponding polynomial $P(X) = (-1)^4 X^8 \Delta(1 - X^{-1})$ is the one of example 4.1; it is equal to $f_1 f_2$, where $f_1(X) = X^4 - 2X^3 + 5X^2 - 4X + 1$, and $f_1(X) = X^4 - 2X^3 + 11X^2 - 10X + 3$.

We have $\Pi_{f_1, f_2} = \{2\}$, hence $G_\Delta = G_P = 0$. Corollary 9.3 implies that there exists an $m$-knot with Alexander polynomial $\Delta$ and signature 8; but such a knot is indecomposable, since an $m$-knot with Alexander polynomial $\Delta_i$ has signature 0 for $i = 1, 2$.

Example 8.2. Let $a \geq 0$ be an integer, and set $\Delta_a(X) = X^6 - aX^5 - X^4 + (2a - 1)X^3 - X^2 - aX + 1$.

The polynomial $\Delta_a$ is irreducible, and $\rho(\Delta_a) = 4$ (see [GM 02], §7.3, Example 1 on page 284). This implies that all $m$-knots with Alexander polynomial $\Delta_a$ have signature 0.

Let $b \geq 0$ be an integer with $b \neq a$. We have $\rho(\Delta_a \Delta_b) = 8$, and if moreover $G_{\Delta_a \Delta_b} = 0$, then there exist $m$-knots with Alexander polynomial $\Delta_a \Delta_b$ and signature 8; these knots are indecomposable. We can take for instance $a = 0$ and $b = 2$; then $\Pi_{\Delta_a, \Delta_b} = \{2\}$, hence $G_{\Delta_a \Delta_b} = 0$.

9. 3-knots in the 5-sphere

The signature of a 3-dimensional knot $K^3 \subset S^5$ is divisible by 16 (see for instance [KW 78], §3, page 95). The aim of this section is to show that with this additional restriction, the results of §7 extend to 3-knots.

Let $n \geq 1$ be an integer, and let $\Delta \in \mathbb{Z}[X]$ be a polynomial of degree $2n$ such that $\Delta(X) = X^{2n} \Delta(X^{-1})$, $\Delta(1) = (-1)^n$ and that $\Delta(-1)$ is a square of an integer. Set $P(X) = (-1)^n X^{2n} \Delta(1 - X^{-1})$. Assume that $P$ is a product of
distinct irreducible monic polynomials $f \in \mathbb{Z}[X]$ such that $f(1 - X) = f(X)$.

Let $G_\Delta = G_P$ be the group introduced in \S 4.

Let $s$ be an integer such that $s \equiv 0 \pmod{16}$, and that $|s| \leq \rho(P)$. Let $\tau \in \text{Mil}_s(P)$.

**Theorem 9.1.** There exists a 3-knot with Alexander polynomial $\Delta$ and Milnor signature $\tau$ if and only if $\epsilon_\tau = 0$.

**Proof.** This follows from Theorem 6.1 and from a result of Levine (see [Le 70], Theorem 2): if $A$ is a Seifert form of signature divisible by 16, then there exists a 3-knot in the 5-sphere with Seifert form $S$-equivalent to $A$. Since $S$-equivalent Seifert forms have the same Alexander polynomial and Milnor signature, this completes the proof of the theorem.

**Corollary 9.2.** Assume that $G_\Delta = 0$. Then for all $\tau \in \text{Mil}_s(\Delta)$ there exists a 3-knot with Alexander polynomial $\Delta$ and Milnor signature $\tau$.

**Corollary 9.3.** Assume that $G_\Delta = 0$. Then there exists a 3-knot with Alexander polynomial $\Delta$ and signature $s$.

10. **Unimodular Seifert forms**

We conclude by some remarks on a special case, which was already treated in detail in [B 21]. Let $A : L \times L \to \mathbb{Z}$ be a unimodular Seifert form, i.e. $\det(A) = \pm 1$, and let $S : L \times L \to \mathbb{Z}$, defined by $S(x, y) = A(x, y) + A(y, x)$, be the associated even, unimodular lattice.

Let $t : L \to L$ be defined by $A(tx, y) = -A(y, x)$ for all $x, y \in L$; note that $t$ is an isometry of $A$, and hence of $S$, and that the characteristic polynomial of $t$ is $\det(A)\Delta_A$.

**Proposition 10.1.** Sending $(L, A)$ to $(L, S, t)$ induces a bijection between isomorphism classes of unimodular Seifert forms and isomorphism classes of even, unimodular lattices with an isometry.

Hence the existence of a unimodular Seifert form with a given Alexander polynomial and Milnor signature is equivalent to the existence of an even, unimodular lattice having an isometry of a given characteristic polynomial and Milnor signature. This question is treated in [B 21], §25, 27 and 31; we recover the results of §6 in this special case.

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