ON THE $p$-ADIC PRO-ÉTALE COHOMOLOGY OF DRINFELD SYMMETRIC SPACES

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ABSTRACT. Via the relative fundamental exact sequence of $p$-adic Hodge theory, we determine the geometric $p$-adic pro-étale cohomology of the Drinfeld symmetric spaces defined over a $p$-adic field, thus giving an alternative proof of a theorem of Colmez-Dospinescu-Nizioł. Along the way, we describe, in terms of differential forms, the geometric pro-étale cohomology of the positive de Rham period sheaf on any connected, paracompact, smooth rigid-analytic variety over a $p$-adic field, and we do it with coefficients. A key new ingredient is the condensed mathematics recently developed by Clausen-Scholze.

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1. Introduction

Let $p$ be a fixed prime number. Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ denote its ring of integers. Let $\overline{K}$ be a fixed algebraic closure of $K$. Let us denote by $\hat{C} := \widehat{\overline{K}}$ the completion of $\overline{K}$, $\mathcal{O}_C$ its ring of integers, and $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$ the absolute Galois group of $K$.

1.1. History and motivation. In this article, using perfectoid methods, we study the geometric $p$-adic pro-étale cohomology of the Drinfeld upper half-spaces defined over $K$ (a non-archimedean variant of the Poincaré upper half-spaces),\(^1\) whose interest lies in the understanding of the $p$-adic local Langlands program.

\(^1\)See Definition 3.1.

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The $p$-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, envisioned by Breuil, and established in full generality by Colmez, [Col10], is given by a covariant exact functor $\Pi \mapsto V(\Pi)$ from the category of admissible unitary $\mathbb{Q}_p$-Banach representations of $GL_2(\mathbb{Q}_p)$, which are residually of finite length and admitting a central character, towards the category of finite-dimensional continuous $\mathbb{Q}$-representations of $GL_2(\mathbb{Q}_p)$ in the source, and the isomorphism classes of 2-dimensional absolutely irreducible $\mathbb{Q}_p$-Banach representations of $GL_2(\mathbb{Q}_p)$ in the source, and the isomorphism classes of 2-dimensional absolutely irreducible continuous $\mathbb{Q}_p$-representations of $\mathcal{H}_{\mathbb{Q}_p}$. As shown by Colmez-Dospinescu-Paskunas, [CDP14, Theorem 1.1], the functor $\Pi \mapsto V(\Pi)$ induces a bijection between the isomorphism classes of absolutely irreducible non-ordinary $\mathbb{Q}_p$-Banach representations of $GL_2(\mathbb{Q}_p)$ in the source, and the isomorphism classes of 2-dimensional absolutely irreducible continuous $\mathbb{Q}_p$-representations of $\mathcal{H}_{\mathbb{Q}_p}$. Moreover, by [CDP14, Theorem 1.3], such functor encodes the classical $\ell$-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, for a prime number $\ell \neq p$. The latter correspondence is known more generally for $GL_n(K)$, for any integer $n \geq 1$, thanks to the work of Harris-Taylor [HT01], Henniart [Hen00], and, more recently, Scholze [Sch13a]. The proof of Harris-Taylor, combined with [Fal02], [Far08], in particular shows that, for the supercuspidal representations, such correspondence can be realized in the geometric $\ell$-adic pro-étale cohomology of the Drinfeld upper half-space. In fact, Colmez-Dospinescu-Nizioł, [CDN20b], proved the following surprising result, of which we propose an alternative proof that is amenable to several generalizations.

For the Drinfeld tower in higher dimension, all that we can say so far is for the level 0 of the tower, i.e. for the Drinfeld upper half-space. In fact, Colmez-Dospinescu-Nizioł, [CDN20a], were able to show that the geometric $p$-adic étale cohomology of the Drinfeld tower over $\mathbb{Q}_p$ in dimension 1 realizes the $p$-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, for the 2-dimensional de Rham representations of $\mathcal{H}_{\mathbb{Q}_p}$ of Hodge-Tate weight 0 and 1, whose associated Weil-Deligne representation is irreducible. Moreover, their computation suggests that the geometric $p$-adic étale cohomology of the Drinfeld tower over $K$ in dimension 1 should encode a still hypothetical $p$-adic Langlands correspondence for $GL_2(K)$, for a general $K$.

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1.2. Main result. Given an integer $d \geq 1$, let $\mathbb{H}_K^d$ denote the Drinfeld upper half-space of dimension $d$ defined over $K$, and let $\mathbb{H}_C^d$ be its base change to $C$. Let $G := GL_{d+1}(K)$.

Theorem 1.1 ([CDN20b, Theorem 4.12], Theorem 8.1). For all $i \geq 0$, there is a strictly exact sequence of $G \times \mathcal{G}_K$-Fréchet spaces over $\mathbb{Q}_p$

$$0 \to \Omega^{d-1}(\mathbb{H}_C^d)/\ker d \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{Q}_p(i)) \to \text{Sp}_i(\mathbb{Q}_p)^* \to 0$$

where $\text{Sp}_i(\mathbb{Q}_p)^*$ denotes the weak topological dual of the locally constant special representations of $G$ with coefficients in $\mathbb{Q}_p$.

2Although there are by now at least two candidates for a $p$-adic local Langlands correspondence for $GL_n(K)$: Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin constructed a functor from representations of $\mathcal{H}_K$ to representations of $GL_n(K)$, [CEG+16], and Scholze produced a functor going in the opposite direction, [Sch18].

3See Definition 3.11.
Remark 1.2. In [CDN20b] this result is deduced from a general comparison theorem, between the geometric $p$-adic pro-étale cohomology and the Hyodo-Kato and de Rham cohomologies, for rigid-analytic Stein spaces over $K$ having a semistable weak formal model over $\mathcal{O}_K$ (e.g. $\mathbb{H}^d_K$); in turn, such comparison is obtained via the geometric syntomic cohomology of Fontaine-Messing. We recall that Colmez-Dospinescu-Nizioł also computed the rational and integral $p$-adic étale cohomology of $\mathbb{H}^d_{\ell}$ in [CDN20b] and [CDN21], respectively.

Moreover, recently, Orlik gave an alternative proof of Theorem 1.1, [Orl19], which is based on his strategy for describing global sections of certain $G$-equivariant vector bundles on $\mathbb{H}^d_K$, [Orl08].

Remark 1.3. The relation between Theorem 1.1 and a hypothetical $p$-adic local Langlands correspondence is still not so transparent, and it will not be explored in this paper.

Before summarizing the strategy we use to prove Theorem 1.1, we first recall its $\ell$-adic version due to Schneider-Stuhler, for a prime number $\ell \neq p$, and explain why the proof of the latter fails in the $p$-adic setting.

**Theorem 1.4** (Schneider-Stuhler, [SS91]). Let $\ell \neq p$ be a prime number. For all $i \geq 0$, there is an isomorphism of $G \times \mathcal{G}_K$-modules

$$H^i_{\text{pro\text{-}ét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(i)) \cong \text{Sp}_i(\mathbb{Q}_\ell)^*.$$ 

Schneider-Stuhler computed the cohomology groups of $\mathbb{H}^d_K$ for a general abelian sheaf cohomology theory, defined on the category of smooth rigid-analytic varieties over $K$, satisfying a number of axioms, the most restricting one being the homotopy invariance with respect to the 1-dimensional open unit disk $\mathbb{D}_K$. These axioms are satisfied by the geometric pro-étale cohomology with coefficients in $\mathbb{Q}_\ell$, but the homotopy invariance with respect to $\mathbb{D}_K$ fails with coefficients in $\mathbb{Q}_p$: in fact, $H^1_{\text{pro\text{-}ét}}(\mathbb{D}_C, \mathbb{Q}_p)$ is an infinite-dimensional $C$-vector space (see [CN20b, Theorem 3] and [LB18a, §3]). Therefore, a different strategy is needed.

### 1.3. Overview of the strategy

To prove Theorem 1.1 we consider the sheaf-theoretic version of the fundamental exact sequence of $p$-adic Hodge theory on the pro-étale site $\mathbb{H}^d_{\text{C,pro\text{-}ét}}$

$$0 \to \mathbb{Q}_p \to \mathbb{B}_e \to \mathbb{B}_{\text{dR}}/\mathbb{B}^+_{\text{dR}} \to 0. \quad (1.1)$$

Here, $\mathbb{B}^+_{\text{dR}}$ is the positive de Rham period sheaf, $\mathbb{B}_{\text{dR}}$ denotes the de Rham period sheaf, and $\mathbb{B}_e$ is defined as the Frobenius invariants $\mathbb{B}[1/t]^{\varphi=1}$ where $\mathbb{B}$ is the pro-étale sheaf-theoretic version of the ring $B$ of the analytic functions on the “punctured open unit disk” $\text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V(p[t])$, introduced by Fargues-Fontaine in their work on the curve, and $t$ is the Fontaine’s $2\pi i$ (see §4).

Therefore, in order to determine the $p$-adic pro-étale cohomology of $\mathbb{H}^d_C$, we reduce to study the pro-étale cohomology of $\mathbb{H}^d_C$ with coefficients in $\mathbb{B}^+_{\text{dR}}$, $\mathbb{B}_{\text{dR}}$ and $\mathbb{B}_e$.

Using Scholze’s Poincaré lemma for $\mathbb{B}^+_{\text{dR}}$ (Proposition 6.7), we determine, in terms of differential forms, the geometric pro-étale cohomology of the period sheaf $\mathbb{B}^+_{\text{dR}}$ on any connected, paracompact, smooth rigid-analytic variety over $K$ (such as $\mathbb{H}^d_K$), as explained in more details in §1.4.

On the other hand, for the geometric pro-étale cohomology of the period sheaves $\mathbb{B}_{\text{dR}}$ and $\mathbb{B}_e$, we show that they satisfy a slight variant of the above-mentioned axioms of Schneider-Stuhler, including

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4However, in loc. cit. the cohomology groups $H^i_{\text{pro\text{-}ét}}(\mathbb{H}^d_C, \mathbb{Q}_p(i))$ are only determined as $G \times \mathcal{G}_K$-modules, and not as topological $G \times \mathcal{G}_K$-vector spaces over $\mathbb{Q}_p$. 

the homotopy invariance with respect to the 1-dimensional open unit disk \( \hat{D}_K \) (see Proposition 6.15, and Proposition 7.20, respectively). The proof of the latter axiom for the geometric pro-étale cohomology of \( \mathbb{B}_e \) is essentially due to Le Bras (see Remark 1.5), and it is largely inspired by the strategy used by Bhatt-Morrow-Scholze to relate their \( A_{\inf} \)-cohomology theory with the \( q \)-de Rham cohomology, [BMS18].

This will be enough to show Theorem 1.1. We leave the complete study of the geometric pro-étale cohomology of \( \mathbb{B}_e \) for a future work, [Bos]. In fact, the strategy we have outlined here, and the results we show in this paper, are suitable to eventually extend [CDN20b, Theorem 1.8] to more general spaces and coefficients.

Remark 1.5. The fundamental exact sequence (1.1) has already been used by Le Bras to compute the geometric \( p \)-adic pro-étale cohomology of the rigid-analytic affine space, and the open polydisk of any dimension, [LB18a, §3].

1.4. **Relevance of the condensed and solid formalisms.** In the search for a geometric incarnation of the hypothetical \( p \)-adic Langlands correspondence for \( G = \text{GL}_{d+1}(K) \), in Theorem 1.1 it is crucial to describe the cohomology groups \( H^i_{\text{pro}\acute{e}t}(\mathbb{H}^d, \mathbb{Q}_p(i)) \) as topological \( G \times \mathcal{G}_K \)-vector spaces over \( \mathbb{Q}_p \), and not merely as \( G \times \mathcal{G}_K \)-modules. We note also that, from a purely geometric point of view, for many cohomology theories appearing in \( p \)-adic Hodge theory (such as the \( p \)-adic (pro-)étale, de Rham, Hyodo-Kato, etc.), the cohomology groups of non-proper rigid-analytic varieties (e.g. \( \mathbb{H}^d_K \)) are usually huge; therefore, it is important to exploit the topological structure that they may carry in order to study them. But, in doing so, one quickly runs into topological issues, mainly due to the fact that the category of topological abelian groups is not abelian.

In our case, one encounters an example of such a topological issue already at the start of the strategy we have outlined. In fact, we consider the long exact cohomology sequence of \( \mathbb{Q}_p \)-vector spaces associated to the fundamental exact sequence (1.1) on \( \mathbb{H}^d_{\text{pro}\acute{e}t} \)

\[
\cdots \to H^i_{\text{pro}\acute{e}t}(\mathbb{H}^d_{\text{et}}, \mathbb{Q}_p) \to H^i_{\text{pro}\acute{e}t}(\mathbb{H}^d_{\text{et}}, \mathbb{B}_e) \to H^i_{\text{pro}\acute{e}t}(\mathbb{H}^d_{\text{et}}, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) \to \cdots \tag{1.2}
\]

and we want to endow these cohomology groups with a natural topology in such a way that all the maps in (1.2) are continuous. One may try to work in the category of locally convex \( \mathbb{Q}_p \)-vector spaces, and put a topology à la Čech on such cohomology groups, using the fact that the global sections of a pro-étale period sheaf on an affinoid perfectoid space over \( \text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C}) \) carry a natural topology (cf. §4), but *a priori* it is not clear whether, in this way, the boundary maps of the long exact sequence (1.2) are continuous.

As we explain below, since the pro-étale cohomology groups appearing in (1.2) have a natural structure of condensed \( \mathbb{Q}_p \)-vector spaces, by their very definition, we found it convenient and fruitful to work in the condensed mathematics framework, recently introduced by Clausen-Scholze, [Sch19], which is precisely designed to overcome the kind of topological issues we have described.

We denote by \( \text{CondAb} \) the category of condensed abelian groups,\(^6\) which we identify with the category of pro-étale sheaves of abelian groups on the geometric point \( \text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C}) = \ast \) (see Remark 2.5). Recall that \( \text{CondAb} \) is a nice abelian category, containing most topological abelian groups of interest, [Sch19, Proposition 1.7, Theorem 2.2]. Then, we give the following definition.

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\(^5\)A classical solution to this particular issue is to work instead in an *abelian envelope* of the category of locally convex \( \mathbb{Q}_p \)-vector spaces. Cf. [CDN20b, §2.1.1].

\(^6\)See §1.6 for the set-theoretic conventions we adopt.
**Definition 1.6.** (Definition 2.3) Let \( f : X \to \text{Spa}(C, \mathcal{O}_C) \) be an analytic adic space, and let \( \mathcal{F} \) be a sheaf of abelian groups on \( X_{\text{pro\acute{e}t}} \). We define the complex of \( D(\text{CondAb}) \)

\[
R\Gamma_{\text{pro\acute{e}t}}^i(X, \mathcal{F}) := Rf_{\text{pro\acute{e}t}}^i\mathcal{F}
\]

with \( i \)th cohomology, for \( i \geq 0 \), the condensed pro-\'{e}tale cohomology group \( H^i_{\text{pro\acute{e}t}}(X, \mathcal{F}) = Rf_{\text{pro\acute{e}t}}^i\mathcal{F} \).

**Remark 1.7.** Note that the underlying abelian group \( H^i_{\text{pro\acute{e}t}}(X, \mathcal{F})(*) \) is the usual pro-\'{e}tale cohomology group \( H^i_{\text{pro\acute{e}t}}(X, \mathcal{F}) \).

Coming back to (1.2), if we denote by \( f : \mathbb{H}^d_C \to \text{Spa}(C, \mathcal{O}_C) \) the structure morphism, now we can simply apply the derived functor \( Rf_{\text{pro\acute{e}t}}^i \) to (1.1), and consider the the associated long exact sequence in cohomology, which will be an exact sequence of condensed \( \mathbb{Q}_p \)-vector spaces.

Let us now explain in more detail how we compare the geometric pro-\'{e}tale cohomology with coefficients in \( \mathbb{B}_{d\text{r}} \) and \( \mathbb{B}^+_{d\text{r}} \), with the de Rham cohomology, and the role that the solid formalism plays in this. In the following, we denote by \( T \to \mathcal{T} \) the functor from topological groups/rings/\( K \)-vector spaces/etc. to condensed groups/rings/\( K \)-vector spaces/etc.\(^7\) Moreover, we denote by \( \text{Vect}_{\text{K}}^{\text{cond}} \) the category of condensed \( K \)-vector spaces, and by \( \text{Vect}_{\text{K}}^{\text{solid}} \) the symmetric monoidal subcategory of solid \( K \)-vector spaces, endowed with the solid tensor product \( \otimes^\bullet_K \) (see Appendix A).

We will prove the following theorem, which extends results of Scholze [Sch13b, Theorem 7.11], and Le Bras [LB18a, Proposition 3.17]. We refer the reader to Theorem 6.5 for a version with coefficients of the statement below.

**Theorem 1.8** (Theorem 6.5). Let \( X \) be a smooth rigid-analytic variety\(^8\) defined over \( K \).

We define\(^9\) the de Rham cohomology of the base change of \( X \) to \( B_{\text{dR}}^\text{cond} \) as the complex of \( D(\text{Vect}_{\text{K}}^{\text{cond}}) \)

\[
R\Gamma_{\text{dR}}(X_{B_{\text{dR}}^\text{cond}}) := R\Gamma(X, \Omega^\bullet_X \otimes^K B_{\text{dR}}).
\]

(i) We have a \( \mathcal{G}_{\text{K}} \)-equivariant, compatible with filtrations, natural quasi-isomorphism in \( D(\text{Vect}_{\text{K}}^{\text{solid}}) \)

\[
R\Gamma_{\text{pro\acute{e}t}}(X_C, B_{\text{dR}}) \simeq R\Gamma_{\text{dR}}(X_{B_{\text{dR}}}).
\]

(ii) Assume that \( X \) is connected and paracompact. Then, for each \( r \in \mathbb{Z} \), we have a \( \mathcal{G}_{\text{K}} \)-equivariant quasi-isomorphism in \( D(\text{Vect}_{\text{K}}^{\text{solid}}) \)

\[
R\Gamma_{\text{pro\acute{e}t}}(X_C, \text{Fil}^r B_{\text{dR}}) \simeq \text{Fil}^r(R\Gamma_{\text{dR}}(X) \otimes^K B_{\text{dR}}).
\]

Here, \( R\Gamma_{\text{dR}}(X) \) denotes the de Rham cohomology complex in \( D(\text{Vect}_{\text{K}}^{\text{cond}}) \) (Definition 5.10).

**Remark 1.9.** The paracompact assumption (Definition 5.15) in Theorem 1.8 is not very restrictive, in fact most of the rigid-analytic varieties over \( K \) that arise in nature are paracompact. Examples include any separated rigid-analytic variety over \( K \) of dimension 1, the rigid analytification of a separated scheme of finite type over \( K \), a Stein space over \( K \) (e.g. \( \mathbb{H}^d_{\text{K}} \)), an admissible open of a quasi-compact and quasi-separated rigid-analytic variety over \( K \) (see Examples 5.18).

\(^7\)Again, see §1.6 for the set-theoretic conventions.

\(^8\)All rigid-analytic varieties will be assumed to be quasi-separated.

\(^9\)See Definition 5.5 for the relevant notation.
Let us briefly describe how we prove Theorem 1.8: we show part (i) using Scholze’s Poincaré lemma (Proposition 6.7); then, part (ii) follows from part (i) and the following base change result, recalling that the period ring \( B_{\text{dR}} \) is filtered by \( K \)-Fréchet algebras.

**Theorem 1.10 (Theorem 5.20).** Let \( X \) be a connected, paracompact, rigid-analytic variety defined over \( K \). Let \( \mathcal{F}^\bullet \) be a bounded below complex of sheaves of topological \( K \)-vector spaces whose terms are coherent \( \mathcal{O}_X \)-modules. Let \( A \) be a \( K \)-Fréchet algebra, regarded as a condensed \( K \)-algebra. Then,

we have a natural quasi-isomorphism in \( D(\text{Vect}_{\text{solid}}^K) \)

\[
R\Gamma(X, \mathcal{F}^\bullet) \otimes_K^\bullet A \xrightarrow{\sim} R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\bullet A).
\]

(1.3)

The proof of Theorem 1.10 is based on results of Clausen-Scholze on the category of nuclear \( K \)-vector spaces (see §A.3), which allow us to reduce the statement to the case \( X \) is affinoid.

**Remark 1.11.** Under the hypotheses of Theorem 1.10, by the flatness of \( K \)-Fréchet spaces with respect to the solid tensor product \( \otimes_K^\bullet \) (a result due to Clausen-Scholze), taking cohomology in (1.3) we have the following isomorphism in \( \text{Vect}_{\text{solid}}^K \) (Corollary 5.21)

\[
H^i(X, \mathcal{F}^\bullet) \otimes_K^\bullet A \cong H^i(X, \mathcal{F}^\bullet \otimes_K^\bullet A)
\]

for all \( i \in \mathbb{Z} \). We note that such a statement is characteristic of the condensed mathematics realm: in fact, its naive analogue in the category of locally convex \( K \)-vector spaces, with the completed projective tensor product replacing the solid tensor product, is trivially false, due to the fact that the cohomology group \( H^i(X, \mathcal{F}^\bullet) \) can be a non-Hausdorff locally convex \( K \)-vector space (see Remark 5.22). This is just an instance of the fact that the solid formalism works very well also for “non-Hausdorff objects” (more precisely, for non-quasi-separated condensed sets).

As a consequence of Theorem 1.8, in some special cases we can give a particularly nice description of the geometric pro-étale cohomology with coefficients in \( \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+ \), appearing in the fundamental exact sequence (1.1), including in cases where the de Rham cohomology groups are otherwise pathological as topological vector spaces. The following result explains in particular how, for a smooth rigid-analytic variety \( X \) over \( K \), the (non-)degeneration of the Hodge-de Rham spectral sequence is reflected in its geometric \( p \)-adic pro-étale cohomology.

**Corollary 1.12 (Corollary 6.14).** Let \( X \) be a smooth rigid-analytic variety over \( K \). Let \( i \geq 0 \).

(i) If \( X \) is proper, we have a \( \mathcal{G}^K \)-equivariant isomorphism in \( \text{Vect}_{\text{solid}}^K \)

\[
H^i_{\text{pro\-\text{ét}}}(X_C, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) \cong (H^i_{\text{dR}}(X_C) \otimes_{k} \mathbb{B}_{\text{dR}})/\text{Fil}^0.
\]

(ii) If \( X \) is an affinoid space, we have the following \( \mathcal{G}^K \)-equivariant exact sequence in \( \text{Vect}_{\text{solid}}^K \)

\[
0 \rightarrow H^i_{\text{dR}}(X) \otimes_{k} \mathbb{B}_{\text{dR}}/t^{-i} \mathbb{B}_{\text{dR}}^+ \rightarrow H^i_{\text{pro\-\text{ét}}}(X_C, \mathbb{B}_{\text{dR}}/\mathbb{B}_{\text{dR}}^+) \rightarrow \Omega^i(X)/\ker d \otimes_{k} C(-i - 1) \rightarrow 0.
\]

**Remark 1.13.** Let us shortly comment on the statement and the proof of Corollary 1.12.

(i) If \( X \) is proper, the proof uses crucially the degeneration of the Hodge-de Rham spectral sequence, [Sch13b, Corollary 1.8]. In this case, the de Rham cohomology groups are finite-dimensional \( K \)-vector spaces, therefore it not a surprise that \( H^i_{\text{dR}}(X) = H^i_{\text{dR}}(X) \) (Lemma 5.11), and Corollary 1.12(i) can be stated in classical topological terms.
(ii) If $X$ is an affinoid space, the proof relies on Tate’s acyclicity theorem (Lemma 5.6(i)), using which one can show that there is a $\mathcal{G}_K$-equivariant exact sequence in $\text{Vect}^{\text{solid}}_K$

$$0 \to H^i_{dR}(X) \otimes_K t^{-i+1} B^+_{dR} \to H^i_{\text{proét}}(X_C, \mathbb{B}^+_{dR}) \to \Omega^i(X)^{d=0} \otimes_K C(-i) \to 0.$$ (1.4)

Note that $H^i_{dR}(X) = \Omega^i(X)^{d=0}/d\Omega^{i-1}(X)$, but, in general, $H^i_{dR}(X) \neq \Omega^i(X)^{d=0}/d\Omega^{i-1}(X)$ (since the latter displayed quotient can be non-Hausdorff, see Remark 5.14).

(iii) If $X$ is a Stein space, using Kiehl’s acyclicity theorem (Lemma 5.9), one can prove that the exact sequence (1.4) also holds for such $X$. In this case, we have $H^i_{dR}(X) = \Omega^i(X)^{d=0}/d\Omega^{i-1}(X)$ (Lemma 5.13), and, a posteriori, the exact sequence (1.4) can be restated in the category of topological $K$-vector spaces with the completed projective tensor product replacing the solid tensor product (see Remark 6.13). On the contrary, in the affinoid case, it seems hard to get such a clean statement as the one of Corollary 1.12(ii) in the usual topological setting. This kind of issue is classically avoided replacing affinoid spaces with dagger affinoid spaces (cf. [CDN20b], [CN20a]); however, the point we want to make here is that, in our setting, there is no need for this.

Let us also mention that Heuer has recently observed in [Heu20, Remark 5.9] that, for $X$ a smooth Stein space over $K$, the exact sequence (1.4) can be used to describe the “exotic line bundles” appearing when passing from the analytic site to the pro-étale site of $X_C$.

As a final note, we want to add that, since the work of Clausen-Scholze [CS] is fairly recent, we had to put a particular accent on comparing topological spaces to condensed sets. However, it should be noted that the main results and proofs of this paper could be entirely formulated in the condensed language.

1.5. Leitfaden of the paper. We have organized the paper as follows. In §2, we discuss several ways of endowing an abelian sheaf cohomology group with a structure of condensed abelian group, and we explain the relations between them. In §3, we redefine in the condensed setting, and slightly modify (in order to include the geometric pro-étale cohomology of $\mathbb{B}_{dR}$ and $\mathbb{B}_e$), the cohomology theories for which Schneider-Stuhler computed the cohomology groups of the Drinfeld upper half-spaces in [SS91]; then, we outline the computation of loc. cit. highlighting the main points where, due to our modifications, an additional argument is needed.

We then proceed in §4 by recalling the definitions and the results we use on the pro-étale period sheaves. In §5, we prove Theorem 1.10; the latter will be then used in §6 to show Theorem 1.8. In §6, we also prove that the geometric pro-étale cohomology of $\mathbb{B}_{dR}$ satisfies the axioms of Schneider-Stuhler (3.4). In §7, we show that the geometric pro-étale cohomology of $\mathbb{B}_e$ satisfies the axioms of Schneider-Stuhler (3.4). Finally, in §8, we gather the results of the previous sections and prove Theorem 1.1.

We end with two appendices. In Appendix A, we recall some foundational results, due to Clausen-Scholze, on the condensed functional analysis over non-archimedean local fields. In appendix B, we recall the definition of condensed group cohomology, and, using the solid formalism, we relate it to Koszul complexes in some cases of particular interest to us.
1.6. Notation and conventions. \(^{10}\) Fix a prime number \(p\). We will denote by \((K, | \cdot |)\) a finite extension of \(\mathbb{Q}_p\), with ring of integers \(\mathcal{O}_K\), a fixed uniformizer \(\varpi \in \mathcal{O}_K\), and residue field \(k\).

We fix an algebraic closure \(\overline{K}\) of \(K\). We will denote \(C := \overline{K}\) the completion of \(\overline{K}\), \(\mathcal{O}_C\) its ring of integers, and \(\mathcal{G}_K := \text{Gal}(\overline{K}/K)\) the absolute Galois group of \(K\).

Throughout the paper, all Huber rings and pairs will be assumed to be complete.

To avoid set-theoretic issues, we fix an uncountable cardinal \(\kappa\) as in [Sch21, Lemma 4.1]: \(\kappa\) is a strong limit cardinal; the cofinality of \(\kappa\) is uncountable; for all cardinals \(\lambda < \kappa\), there is a strong limit cardinal \(\kappa_\lambda < \kappa\) such that the cofinality of \(\kappa_\lambda\) is greater than \(\lambda\).\(^{11}\)

We say that an analytic adic space \(X\) is \(\kappa\)-small if the cardinality of the underlying topological space \(|X|\) is less than \(\kappa\), and for all open affinoid subspaces \(\text{Spa}(R, R^+) \subset X\), the ring \(R\) has cardinality less than \(\kappa\). In this paper, all the analytic adic spaces will be assumed to be \(\kappa\)-small.

We say that a profinite set is \(\kappa\)-small if it has cardinality less than \(\kappa\). We denote by \(*_{\kappa-\text{profin}}\) the site of \(\kappa\)-small profinite sets, with coverings given by finite families of jointly surjective maps. Recall that the category of \(\kappa\)-condensed sets/groups/rings/etc. is defined as the category of sheaves on \(*_{\kappa-\text{profin}}\) with values in sets/groups/rings/etc., [Sch19, Definition 2.1], and it is equivalent to the category of contravariant functors from \(\kappa\)-small extremally disconnected sets to sets/groups/rings/etc. taking finite disjoint unions to finite products, [Sch19, Proposition 2.7].

Unless explicitly stated otherwise, all condensed sets will be \(\kappa\)-condensed sets (and often the prefix “\(\kappa\)” is tacit). We will denote by \(\text{CondSet}\) the category of \(\kappa\)-condensed sets, and by \(\text{CondAb}\) the category of \(\kappa\)-condensed abelian groups.

All condensed rings will be \(\kappa\)-condensed commutative unital rings. Given a \((\kappa\text{-})\)-condensed ring \(A\), we denote by \(\text{Mod}^\text{cond}_A\) the category of \(A\)-modules in \(\text{CondAb}\). We write \(\text{Hom}_A(\cdot, \cdot)\) for the internal Hom in the category \(\text{Mod}^\text{cond}_A\) (and in the case \(A = \mathbb{Z}\), we often omit the subscript \(\mathbb{Z}\)).

We denote by \(T \mapsto T\) the functor from the category of topological spaces/groups/rings/etc. to the category of \(\kappa\)-condensed sets/groups/rings/etc., where \(T\) is defined via sending a \(\kappa\)-small profinite set \(S\) to the set/group-ring/etc. of the continuous functions \(C^0(S, T)\).

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\(^{10}\)This notation, and these conventions, are adopted throughout the paper, except for Appendix A and Appendix B. In particular, in Appendix A, we allow \(K\) to be any non-archimedean local field. See §A.1.

\(^{11}\)There exists an arbitrary large such cardinal \(\kappa\).
2. Condensed cohomology groups

In this section, we endow the cohomology groups of a pro-étale abelian sheaf on an analytic adic space defined over $\mathbb{Z}_p$ with a structure of condensed abelian group. Then, we discuss several other situations in which a cohomology group has a natural condensed structure.

2.1. The pro-étale topology. Let $\kappa$ be a cut-off cardinal as in §1.6, and let $\text{Perf}_\kappa$ denote the category of $\kappa$-small perfectoid spaces of characteristic $p$. In the following, all diamonds are assumed to live as pro-étale sheaves on $\text{Perf}_\kappa$, [Sch21, Definition 11.1].

Given a diamond $Y$, we denote by $Y_{\acute{e}t}$ its étale site, and by $Y_{\text{qpro}\acute{e}t}$ its quasi-pro-étale site, [Sch21, Definition 14.1]. An important feature of the category underlying the site $Y_{\text{qpro}\acute{e}t}$ is that it is “tensored over $\kappa$-small profinite sets”, as we now recall.

Remark 2.1. Let $Y$ be a diamond. Given a $\kappa$-small profinite set $S$, we also denote by $S$ the functor on $\text{Perf}_\kappa$ sending a perfectoid space $Z$ to the continuous functions $\mathcal{C}^0(|Z|, S)$; this defines a pro-étale sheaf on $\text{Perf}_\kappa$. Then, we observe that $Y \times S \in Y_{\text{qpro}\acute{e}t}$. In fact, writing $S = \varprojlim_{i \in I} S_i$ as a cofiltered limit of finite sets $S_i$ along a $\kappa$-small index category $I$,\(^\text{12}\) and recalling [Sch21, Definition 10.1], we have

$$Y \times S = \varprojlim_{i \in I}(Y \times S_i) = \varprojlim_{i \in I} \prod_{s_i \in S_i} Y \in Y_{\text{qpro}\acute{e}t}.$$ 

In general, denoting $S_Y := Y \times S$, for any $Y' \in Y_{\text{qpro}\acute{e}t}$, we have $Y' \times_Y S_Y \in Y_{\text{qpro}\acute{e}t}$.

Next, we restrict our attention to the category of analytic adic spaces defined over $\mathbb{Z}_p$.

Recall that there is a natural functor $X \mapsto X^\Diamond$ from the category of analytic adic spaces defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ to the category of locally spatial diamonds, satisfying $|X| = |X^\Diamond|$ and $X_{\acute{e}t} \cong X^\Diamond_{\acute{e}t}$ (see [Sch21, Definition 15.5, Lemma 15.6]). We can then give the following definitions.

Definition 2.2. Let $X$ be an analytic adic space defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We denote by

$$X_{\text{pro}\acute{e}t} := X^{\Diamond}_{\text{qpro}\acute{e}t}$$

its pro-étale site.\(^\text{13}\)

Recalling §1.6, in what follows, we denote by $*_{\kappa-\text{pro}\acute{e}t}$ the site of $\kappa$-small profinite sets, with coverings given by finite families of jointly surjective maps. Moreover, we denote by $\text{CondSet}$ the category of $\kappa$-condensed sets, and by $\text{CondAb}$ the category of $\kappa$-condensed abelian groups.

Definition 2.3. Let $X$ be an analytic adic space defined over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$. We denote by

$$f_{\text{cond}} : X_{\text{pro}\acute{e}t} \to *_{\kappa-\text{pro}\acute{e}t}$$

the natural morphism of sites defined by sending $S \in *_{\kappa-\text{pro}\acute{e}t}$ to $X^\Diamond \times S \in X_{\text{pro}\acute{e}t}$.

Given $\mathcal{F}$ a sheaf of abelian groups on $X_{\text{pro}\acute{e}t}$, we define the complex of $D(\text{CondAb})$

$$R\Gamma_{\text{pro}\acute{e}t}(X, \mathcal{F}) := Rf_{\text{cond}}^* \mathcal{F},$$

with $i$th cohomology, for $i \geq 0$, the condensed pro-étale cohomology group $H^i_{\text{pro}\acute{e}t}(X, \mathcal{F}) = R^i f_{\text{cond}}^* \mathcal{F}$, an object of $\text{CondAb}$.

\(^{12}\)Such a presentation of $S$ exists by the proof of [Sta, Tag 08ZY].

\(^{13}\)Note that the definition of pro-étale site that we adopt is different from the one given in [Sch13b] (cf. [BS15, Remark 4.1.11]). However, the results on $X_{\text{pro}\acute{e}t}$ of loc. cit. that we will need still hold with this definition.
Remark 2.4. Note that the condensed pro-étale cohomology groups are an enrichment of the usual cohomology groups, in fact, for $i \geq 0$, we have $H^i_{\text{pro-ét}}(X, \mathcal{F})(*) = H^i_{\text{pro-ét}}(X, \mathcal{F})$ as abelian groups.

Remark 2.5. In the case $X$ is an analytic adic space over $\text{Spa}(C, \mathcal{O}_C)$, then Definition 2.3 agrees with Definition 1.6. In fact, the category of sheaves of sets on the site $\text{Spa}(C, \mathcal{O}_C)_{\text{pro-ét}}$ is equivalent to the category of $\kappa$-condensed sets. For this, we first note that the topos associated to the site $\text{Spa}(C, \mathcal{O}_C)_{\text{pro-ét}}$ is equivalent to the topos associated to the subsite $\text{Spa}(C, \mathcal{O}_C)_{\text{aff}}_{\text{pro-ét}}$, consisting of the affinoid pro-étale maps in $\text{Spa}(C, \mathcal{O}_C)_{\text{pro-ét}}$, [Sch21, Definition 7.8, (i)]. We recall that a map $U \to \text{Spa}(C^\flat, \mathcal{O}_{C^\flat})$, from a perfectoid space $U$ to the tilt of $\text{Spa}(C, \mathcal{O}_C)$, is an affinoid pro-étale map if and only if $U = \text{Spa}(C^\flat, \mathcal{O}_{C^\flat}) \times S$ for some $\kappa$-small profinite set $S$. Then, the functor sending $U \in \text{Spa}(C, \mathcal{O}_C)^{\text{aff}}_{\text{pro-ét}}$ to the underlying topological space $|U|$ induces an equivalence of categories between the category underlying $\text{Spa}(C, \mathcal{O}_C)^{\text{aff}}_{\text{pro-ét}}$ and the category of $\kappa$-small profinite sets. Now, the claim easily follows unraveling the definition of the covering families of $\text{Spa}(C, \mathcal{O}_C)^{\text{aff}}_{\text{pro-ét}}$.

2.2. Condensed structure à la Čech. In this paper, we will have to deal with sheaves of topological abelian groups/rings/etc. on analytic adic spaces (e.g. the structure sheaf). In some cases, it is reasonable to equip the cohomology groups of such sheaves with a condensed structure à la Čech, using the fact that their local sections have a natural topology, and then to ask how this compares with Definition 2.3. This is explained in the following remarks, that we will use to define a structure of condensed abelian group on the geometric cohomology of the pro-étale period sheaves, in §4 (checking its agreement with Definition 2.3), and on the cohomology of coherent sheaves on rigid-analytic varieties, in §5.

We start with some reminders on the category CondAb that we will use throughout the paper.

Remark 2.6. Recall that the category of $\kappa$-condensed abelian groups CondAb is an abelian category which satisfies the Grothendieck’s axioms (AB3), (AB3*), (AB4), (AB4*), (AB5) and (AB6), and it is generated by a set of compact projective objects, [Sch19, Theorem 2.2]. In particular, for any small site $\mathcal{C}$, the category of sheaves on $\mathcal{C}$ with values in CondAb has enough injectives.

Then, given $\mathcal{G}$ a complex in the bounded below derived category $D^-(\text{Shv}_{\text{CondAb}}(\mathcal{C}))$ of sheaves on $\mathcal{C}$ with values in CondAb, for any object $V \in \mathcal{C}$ we can define the sheaf cohomology complex

$$R\Gamma(V, \mathcal{G}) \in D(\text{CondAb})$$

by taking an injective resolution of $\mathcal{G}$. For $i \in \mathbb{Z}$, we denote by $H^i(V, \mathcal{G})$ its $i$th cohomology, which is an object of CondAb.

Remark 2.7. Let $\mathcal{C}$ be a Verdier site, i.e. a small site having a basis $\mathcal{B}$ such that, for every covering $U \to V$ of $\mathcal{B}$, the diagonal $U \to U \times_V U$ is also a covering of $\mathcal{B}$ (see [DHI04, Definition 9.1]). Let $\mathcal{F}$ be a sheaf of topological abelian groups on $\mathcal{B}$. Since the functor $T \mapsto T$ from topological abelian groups to CondAb preserves limits, the presheaf given by

$$\mathcal{B}_{\text{op}} \to \text{CondAb} : U \mapsto \mathcal{F}(U)$$

Recall also that, instead, the category of all condensed abelian groups ([Sch19, Definition 2.11]) has a class, and not a set, of compact projective generators.

More precisely, it has functorial injective embeddings, [Sta, Tag 0139, Tag 079H].

An alternative to “taking injective resolutions” would be to work with the $\infty$-category of sheaves on $\mathcal{C}$ with values in the derived $\infty$-category of condensed abelian groups $D(\text{CondAb})$. However, the theory of $\infty$-categories will not play a crucial role in this paper.

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16An alternative to “taking injective resolutions” would be to work with the $\infty$-category of sheaves on $\mathcal{C}$ with values in the derived $\infty$-category of condensed abelian groups $D(\text{CondAb})$. However, the theory of $\infty$-categories will not play a crucial role in this paper.
is a sheaf. We denote by $\mathcal{F}$ the associated sheaf on $\mathcal{C}$ with values in $\text{CondAb}$, and, for $V \in \mathcal{C}$, we consider the sheaf cohomology complex $R\Gamma(V, \mathcal{F}) \in D(\text{CondAb})$, defined following Remark 2.6. By the (generalized) Verdier’s hypercovering theorem, [DHI04, Theorem 8.6, Theorem 9.6], we have that $H^i(V, \mathcal{F})$ has the usual sheaf cohomology group as underlying abelian group, i.e.

$$H^i(V, \mathcal{F})(\ast) = H^i(V, \mathcal{F})$$

as abelian groups (here, $\mathcal{F}$ is regarded as a sheaf of abelian groups). Note that Verdier’s hypercovering theorem, combined with Remark A.1, also shows that, if the basis $\mathcal{B}$ has cardinality less than $\kappa$, and, for all $U \in \mathcal{B}$, the topological space $\mathcal{F}(U)$ is T1 with cardinality less than $\kappa$, then $H^i(V, \mathcal{F})$ does not depend on our choice of the cardinal $\kappa$. The previous discussion also holds replacing $\mathcal{F}$ with a bounded below complex of such sheaves on $\mathcal{C}$.

2.3. Comparison to the étale topology. In the rest of this section, we denote by $X$ an analytic adic space over $\text{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, and we identify it with the associated diamond $X^\diamond$.

We remark that Definition 2.3 takes advantage of Remark 2.1, which obviously fails for the étale site $X_{\text{ét}}$. In particular, by applying the pushforward functor along the natural morphism of sites $X_{\text{pro\-ét}} \to X_{\text{ét}}$, one may lose “topological information”. However, for the comparison theorems proven in this paper, we will need to pass from the pro-étale site to the étale site, still retaining the information captured by profinite sets. Thus, to remedy this issue, we give the following definition, which will play a crucial role later on (see Corollary 6.9).

Definition 2.8. Let $\tau \in \{\text{pro\-ét, ét}\}$. The site $X_{\tau,\text{cond}}$ has underlying category the pairs $(U, S)$ with $U \in X_{\tau}$ and $S \in \ast_{\kappa-\text{pro\-ét}}$, and coverings the families of morphisms $\{(f_i, s_j) : (U_i, S_j) \to (U, S)\}_{(i, j) \in I \times J}$ with $\{f_i : U_i \to U\}_{i \in I}$ a covering of $X_{\tau}$ and $\{s_j : S_j \to S\}_{j \in J}$ a covering of $\ast_{\kappa-\text{pro\-ét}}$.

Remark 2.9. Note that $X_{\tau,\text{cond}}$ is indeed a site. There is a natural projection of sites

$$\pi : X_{\tau,\text{cond}} \to X_{\tau}$$

given by sending $U \in X_{\tau}$ to the pair $(U, \ast) \in X_{\tau,\text{cond}}$. There is also a natural morphism of sites

$$\mu : X_{\text{pro\-ét}} \to X_{\tau,\text{cond}}$$

defined by sending a pair $(U, S) \in X_{\tau,\text{cond}}$ to $u(U) \times S \in X_{\text{pro\-ét}}$, where $u : X_{\tau} \to X_{\text{pro\-ét}}$ denotes the natural continuous functor.

Remark 2.10. We have an equivalence between the category of sheaves of sets on $X_{\tau,\text{cond}}$ and the category of sheaves of $\kappa$-condensed sets on $X_{\tau}$

$$\text{ShvSet}(X_{\tau,\text{cond}}) \sim \text{ShvCondSet}(X_{\tau}) : \mathcal{F} \mapsto \mathcal{F}^\vee$$

where the sheaf $\mathcal{F}^\vee$ is defined by assigning to $U \in X_{\tau}$ the $\kappa$-condensed set $\mathcal{F}^\vee(U) : S \mapsto \mathcal{F}(U, S)$. A quasi-inverse of the functor (2.1) is given by sending a sheaf of $\kappa$-condensed sets $\mathcal{G}$ on $X_{\tau}$ to the sheaf of sets on $X_{\tau,\text{cond}}$ defined by assigning to $(U, S) \in X_{\tau,\text{cond}}$ the set $\mathcal{G}(U)(S)$. Under this equivalence, an abelian group object/ring object/etc. in $\text{ShvSet}(X_{\tau,\text{cond}})$, i.e. a sheaf on $X_{\tau,\text{cond}}$ with values in abelian groups/rings/etc., corresponds to a sheaf on $X_{\tau}$ with values in $\kappa$-condensed abelian groups/rings/etc. Moreover, the functor (2.1) induces an equivalence of derived categories

$$D(\text{ShvAb}(X_{\tau,\text{cond}})) \sim D(\text{ShvCondAb}(X_{\tau})) : \mathcal{F} \mapsto \mathcal{F}^\vee.$$

Then, recalling Remark 2.6, we can give the following definition.
Definition 2.11. Given $F \in D^+(\text{Shv}_{\text{Ab}}(X_{\text{r,cond}}))$ we define the complex of $D(\text{CondAb})$
\[ R\Gamma_{\text{r,cond}}(X, F) := R\Gamma_{\tau}(X, F^\vee) \]
with $i$th cohomology, for $i \in \mathbb{Z}$, denoted by $H^i_{\text{r,cond}}(X, F)$, an object of $\text{CondAb}$.

In the case $\tau = \text{pro\text{é}t}$, the latter definition gives us only a slightly alternative point of view on Definition 2.3.

Lemma 2.12. Let $F$ be a sheaf of abelian groups on $X_{\text{pro\text{é}t}}$. Let $\mu : X_{\text{pro\text{é}t}} \to X_{\text{pro\text{é}t,cond}}$ be the natural morphism of sites. Then, for all $i \geq 0$, we have
\[ H^i_{\text{pro\text{é}t}}(X, F) = H^i_{\text{pro\text{é}t,cond}}(X, R\mu_*F) \]
as condensed abelian groups.\(^\text{17}\)

Proof. Let $f_{\text{cond}} : X_{\text{pro\text{é}t}} \to \kappa_{-\text{pro\text{é}t}}$ denote the morphism of sites of Definition 2.3. It suffices to observe that the pushforward functor $f_{\text{cond}}^*$ is given by the composition
\[ \text{Shv}_{\text{Ab}}(X_{\text{pro\text{é}t}}) \xrightarrow{\mu^*} \text{Shv}_{\text{Ab}}(X_{\text{pro\text{é}t,cond}}) \xrightarrow{\mathbf{v}} \text{Shv}_{\text{CondAb}}(X_{\text{pro\text{é}t}}) \xrightarrow{\Gamma(X_{\cdot})} \text{CondAb} \]
and then use the associated Grothendieck spectral sequence. In fact, for any $\kappa$-small extremally disconnected set $S$, we have
\[ (f_{\text{cond}}^*F)(S) = F(X \times S) = (\mu_*F)(X, S) = \Gamma \left( X, (\mu_*F)^\vee \right)(S) \]
and one readily checks that the restriction morphisms agree as well. \(\square\)

3. Schneider-Sthuler’s theorem

In [SS91] Schneider-Stuhler computed the cohomology groups of the Drinfeld upper half-spaces over $K$, for a general abelian sheaf cohomology theory that satisfies some natural axioms, the most important one requiring the homotopy invariance with respect to the 1-dimensional open unit disk. Examples of such cohomology theories include the de Rham cohomology and the étale cohomology with coefficients in a finite ring whose order is prime to $p$. In this section, we slightly change the axioms of Schneider-Stuhler, [SS91, §2], in order to include the (condensed) geometric pro-étale cohomology of the period sheaves $B_{dR}$ and $B_{e}$.\(^\text{18}\) Then, we highlight the main points of the proof of loc. cit. where, due to such changes, an additional argument is needed.

3.1. Drinfeld upper half-space. We start by recalling the following definition.

Definition 3.1. Let $d \geq 1$ be an integer. The Drinfeld upper half-space over $K$ of dimension $d$ is defined by
\[ \mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H \in \mathcal{H}} H \]
where $\mathcal{H} := \mathbb{P}((K^{d+1})^*)$ denotes the set of the $K$-rational hyperplanes in the rigid-analytic projective space $\mathbb{P}^d_K$.

\(^{17}\)Of course, we have $H^i_{\text{pro\text{é}t,cond}}(X, R\mu_*F)(\cdot) = H^i_{\text{pro\text{é}t}}(X, F)$ as abelian groups.

\(^{18}\)See §4 for the definition of such pro-étale period sheaves.
Remark 3.2. The Drinfeld upper half-space $\mathbb{H}_K^d$ is an admissible open of the rigid-analytic projective space $\mathbb{P}_K^d$, [SS91, §1, Proposition 1], in particular we can (and do) regard it as a rigid-analytic variety over $K$. Moreover, $\mathbb{H}_K^d$ has an increasing admissible covering $\{U_n\}_{n \in \mathbb{N}}$ defined as follows: for a fixed uniformizer $\varpi$ of $\mathcal{O}_K$ (see §1.6), and $n \in \mathbb{N}$, we set $\mathcal{H}_n := \mathbb{P}((\mathcal{O}_K^{d+1})^*/\varpi^n)$ and

$$U_n := \mathbb{P}_K^d \setminus \bigcup_{H \in \mathcal{H}_n} N_n(H)$$

(3.1)

where, for a hyperplane $H \in \mathcal{H}_n$ we denote by $N_n(H)$ the $|\varpi|^n$-neighbourhood of $H$.\(^\text{19}\)

We note that $\mathbb{H}_K^d$ admits a natural action of the group $\text{GL}_{d+1}(K)$ via homographies.

3.2. Schneider-Stuhler’s axioms. Next, we define a variant of the cohomology theories for which Schneider-Stuhler were able to compute the cohomology groups of the Drinfeld upper half-space.

Using that the category of $\kappa$-condensed abelian groups $\text{CondAb}$ is an abelian category satisfying the same Grothendieck’s axioms of the category of abelian groups, and it is generated by a set of compact projective objects, we will also be able to deal with cohomology theories defined by a complex of sheaves with values in $\text{CondAb}$ (see Remark 2.6).

Notation and conventions 3.3. Let $\text{RigSm}_K$ denote the category of quasi-separated smooth rigid-analytic varieties over $K$, endowed with a fixed Grothendieck topology $\tau$ that is finer than the analytic topology.\(^\text{20}\) We denote by $D^{\geq 0}(\text{RigSm}_K)$ the derived category of complexes of sheaves on $\text{RigSm}_K$, concentrated in non-negative degrees, with values in $\text{CondAb}$.

We fix a complex $\mathcal{F} \in D^{\geq 0}(\text{RigSm}_K)$, and we denote by

$$H^\bullet := H^\bullet(-, \mathcal{F}) : \text{RigSm}_K \rightarrow \text{CondAb}$$

the sheaf cohomology with coefficients in $\mathcal{F}$ defined on $\text{RigSm}_K$. Moreover, given $X \in \text{RigSm}_K$, and $U \subseteq X$ an open subvariety, we denote by $H^\bullet(X, U) := H^\bullet(X, U; \mathcal{F})$ the relative sheaf cohomology of the pair $(X, U)$ with coefficients in $\mathcal{F}$.\(^\text{21}\)

Definition 3.4. Let $A$ be a condensed ring. The sheaf cohomology theory $H^\bullet = H^\bullet(-, \mathcal{F})$ satisfies the axioms of Schneider-Stuhler (relative to $A$) if it takes values in the category of $A$-modules in $\text{CondAb}$

$$H^\bullet : \text{RigSm}_K \rightarrow \text{Mod}_A^{\text{cond}}$$

and the following conditions are satisfied.

(1) The homotopy invariance with respect to the 1-dimensional open unit disk $\tilde{D}_K$ over $K$.

i.e. for any affinoid space $X \in \text{RigSm}_K$, the natural projection $X \times \tilde{D}_K \rightarrow X$ induces an isomorphism in cohomology

$$H^\bullet(X) \cong H^\bullet(X \times \tilde{D}_K).$$

\(^\text{19}\)Namely, chosen a unimodular representative $\ell_H \in (\mathcal{O}_K^{d+1})^*$ among the linear forms defining $H$, the subset $N_n(H)$ of $\mathbb{P}_K^d$ is defined by identifying the conjugate points over $K$ of the set $\{z \in \mathbb{P}^d(C) : |\ell_H(z)| \leq |\varpi|^n\}$, where we use unimodular coordinates to represent a point of $\mathbb{P}^d(C)$. See also [SS91, §1, (C)].

\(^\text{20}\)For example, $\tau$ can be the analytic, étale, or pro-étale topology.

\(^\text{21}\)We recall that the relative sheaf cohomology of the pair $(X, U)$ with coefficients in $\mathcal{F}$ is the derived functor of the “sections of $\mathcal{F}$ on $X$ vanishing on $U$” functor. It can be thought as the “cohomology with support in $X \setminus U$.”
There is a morphism in $D^+(\mathrm{RigSm}_{K})$
\[ \cup : \mathcal{F} \otimes_{\mathbb{Z}} \mathcal{F} \to \mathcal{F} \]
called cup product, that is associative and commutative with unit $e : \mathbb{Z} \to \mathcal{F}$.

There exists a non-archimedean local field $F$ such that $A$ is a condensed $F$-algebra.\(^{22}\) Moreover, we have\(^{23}\)
\[ H^i(*_K) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \]

We have that $H^i(\mathbb{P}^d_K) = 0$ for $i$ odd, or $i > 2d$. Moreover, there exists a morphism in $D^{\geq 0}(\mathrm{RigSm}_{K})$, called cycle class map,\(^{24}\)
\[ c : \mathbb{G}_m[-1] \to \mathcal{F} \]
satisfying the following condition: if $\eta \in H^2(\mathbb{P}^d_K)$ denotes the image of the canonical line bundle $\mathcal{O}(1) \in H^1(\mathbb{P}^d_K, \mathbb{G}_m)$ under the map $H^1(\mathbb{P}^d_K, \mathbb{G}_m) \to H^2(\mathbb{P}^d_K)$ induced by $c$, then, for all $0 \leq i \leq d$, the map
\[ f^*(-) \cup \eta^i : A = H^0(*_K) \to H^2(\mathbb{P}^d_K) \]
is an isomorphism, where $f : \mathbb{P}^d_K \to *_K$ is the structure morphism, and $\eta^i := \eta \cup \cdots \cup \eta$ is the $i$ times repeated cup product of $\eta$.

In Schneider-Stuhler’s paper, \cite[§2]{SS91}, axiom (3) requires $A$ to be a (discrete) Artinian ring: this is the main condition we have changed. In loc. cit. this axiom guarantees the vanishing of $R^1 \operatorname{lim}$ in some cases of particular interest. In our case, such vanishing will follow from Lemma 3.5 below.

In the following, we let $F$ be a non-archimedean local field, and we denote by $
_{\text{cond}} := \text{Mod}_{\text{cond}}$
the category of condensed $F$-vector spaces. We say that $V \in \n_{\text{cond}}$ is finite-dimensional if there exists an integer $m \geq 0$, and an isomorphism $V \cong F^\oplus_m$ of condensed $F$-vector spaces.

**Lemma 3.5** (cf. Lemma A.18). Let $\{V_n\}$ be a countable inverse system of finite-dimensional condensed $F$-vector spaces. Let $A$ be a condensed $F$-algebra. Then, the inverse system $\{V_n \otimes_F A\}$ is Mittag-Leffler.\(^{25}\) In particular, $R^j \operatorname{lim}_{n}(V_n \otimes_F A) = 0$, for all $j > 0$.

**Proof.** By \cite[§13.1.2]{Gro61} any countable inverse system of finite-dimensional $F$-vector spaces is Mittag-Leffler; we deduce from Lemma A.15 that $\{V_n\}$ is Mittag-Leffler, and then that $\{V_n \otimes_F A\}$ is Mittag-Leffler too. Equivalently, for every extremally disconnected set $S$, the inverse system $\{(V_n \otimes_F A)(S)\}$ is Mittag-Leffler, hence, by \cite[Proposition 13.2.2]{Gro61} combined with \cite[Lemma 3.18]{Sch13b}, we have that $R^j \operatorname{lim}_{n}(V_n \otimes_F A) = 0$, for all $j > 0$. \(\square\)

\(^{22}\)The non-archimedean local field $F$ is regarded as a condensed ring.

\(^{23}\)Here, $*_K$ denotes the rigid-analytic point $\Sp(K)$.

\(^{24}\)Here, $\mathbb{G}_m$ denotes the multiplicative group sheaf, regarded as a sheaf of discrete condensed abelian groups.

\(^{25}\)Here, and in the following, we adopt Grothendieck’s definition of the Mittag-Leffler condition, \cite[§13.1]{Gro61}, which we now recall. Let $C$ be an abelian category satisfying (AB3*). We say that an inverse system $(A_i, f_{ij})$ of objects in $C$, indexed by a directed set $I$, is Mittag-Leffler if, for each $i \in I$, there exists $j \geq i$ such that for all $k \geq j$ we have $f_{ij}(A_j) = f_{ik}(A_k)$.
3.3. Tits buildings. Before stating Schneider-Stuhler’s result, [SS91, §3, Theorem 1], let us recall the following definition.

**Definition 3.6.** Let $d \geq 1$ be an integer. For $1 \leq i \leq d$, we denote by $\mathcal{T}_i$ the topological Tits $i$-building of $GL_{d+1}(K)$, i.e. the simplicial profinite set defined by

$$\mathcal{T}_j := \{\text{flags } W_0 \subseteq \cdots \subseteq W_j \text{ in } (K^{d+1})^* : 1 \leq \dim_K W_r \leq i \text{ for all } 0 \leq r \leq j\}$$

endowed with its natural profinite topology, with face/degeneracy maps given by omitting/doubling one vector subspace in a flag.\(^{26}\)

**Theorem 3.7.** Let $A$ be a condensed ring, and let $H^\bullet : \text{RigSm}_K \to \text{Mod}^\text{cond}_A$ be a cohomology theory that satisfies the axioms of Schneider-Stuhler (3.4). Let $d \geq 1$ be an integer, and $G = GL_{d+1}(K)$. Then, for all $i \geq 0$, there exists a natural isomorphism of $G$-modules in $\text{Mod}^\text{cond}_A$

$$H^i(\mathbb{H}_K) \cong \begin{cases} A & \text{if } i = 0 \\ \text{Hom}(\bar{H}^{i-1}(|\mathcal{T}_i|, \mathbb{Z}), A) & \text{if } 1 \leq i \leq d \\ 0 & \text{if } i > d \end{cases}$$

where $\bar{H}$ denotes the reduced cohomology, $|\mathcal{T}_i|$ denotes the geometric realization of the topological Tits $i$-building of $G$, and $\bar{H}^{i-1}(|\mathcal{T}_i|, \mathbb{Z})$ is regarded as a discrete condensed abelian group.

**Proof.** By axiom (4), considering the relative cohomology exact sequence

$$\cdots \to H^i(\mathbb{P}_K^d) \to H^i(\mathbb{H}_K) \to H^{i+1}(\mathbb{P}_K^d, \mathbb{H}_K) \to H^{i+1}(\mathbb{P}_K^d) \to \cdots$$

we can reduce to compute $H^i(\mathbb{P}_K^d, \mathbb{H}_K)$. Given $n \in \mathbb{N}$, we begin by studying $H^i(\mathbb{P}_K^d, U_n)$, with $U_n$ as in (3.1). Note that we can write

$$U_n = \bigcap_{H \in \mathcal{H}_n} U_n(H)$$

where $U_n(H) := \mathbb{P}_K^d \setminus N_n(H)$ is an open polydisk in the affine space $\mathbb{P}_K^d \setminus H$. Then, we can reduce to study the strongly convergent spectral sequence\(^ {27}\)

$$E_{-j,i}^{1,(n)} := \bigoplus_{(H_0, \ldots, H_j) \in \mathcal{H}_n} H^i(\mathbb{P}_K^d, U_n(H_0) \cup \cdots \cup U_n(H_j)) \Longrightarrow H^{i-j}(\mathbb{P}_K^d, U_n).$$

We first compute $H^i(\mathbb{P}_K^d, U_n(H_0) \cup \cdots \cup U_n(H_j))$. Given hyperplanes $H_0, \ldots, H_j \in \mathcal{H}_n$, we denote by $\text{rk}(H_0, \ldots, H_j)$ the rank of the following $\mathcal{O}_K$-module (i.e. its minimal number of generators over $\mathcal{O}_K$): $\sum_{r=0}^j (\mathcal{O}_K/\mathbb{Z}^n) \ell_{H_r} \subseteq (\mathcal{O}_K^{d+1})^*/\mathbb{Z}^n$, where $\ell_{H_r} \in (\mathcal{O}_K^{d+1})^*$ is a unimodular representative among the linear forms defining $H_r$.\(^{26}\)

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\(^{26}\)We warn the reader that a topological Tits building differs from a Bruhat-Tits building. See also the discussion before [SS91, §3, Lemma 3].

\(^{27}\)By the proof of [SS91, §2, Proposition 6, Lemma 7] and the discussion afterwards, for a sheaf cohomology theory $H^\bullet : \text{RigSm}_K \to \text{Mod}^\text{cond}_A$, given $X \in \text{RigSm}_K$, and $U_1, \ldots, U_m$ a finite number of open subvarieties of $X$, if we set $U := U_1 \cap \cdots \cap U_m$, then, there exists a strongly convergent spectral sequence

$$E_1^{i,s} = \bigoplus_{1 \leq i_0, \ldots, i_r \leq m} H^s(X, U_{i_0} \cup \cdots \cup U_{i_r}) \Longrightarrow H^{i+s}(X, U).$$
Then, one checks that, given $H_0, \ldots, H_j \in \mathcal{H}_n$, for $m = \text{rk}(H_0, \ldots, H_j) - 1$, there exists a locally trivial fibration $U_n(H_0) \cup \cdots \cup U_n(H_j) \to \mathbb{P}_K^m$, whose fibers are open polydisks of dimension $d - m$ (see [SS91, §1, Lemma 5, Proposition 6]). Using this observation, and considering the relative cohomology sequence associated to the pair $(\mathbb{P}_K^d, U_n(H_0) \cup \cdots \cup U_n(H_j))$, by axioms (1) and (4), we have that

$$H^i(\mathbb{P}_K^d, U_n(H_0) \cup \cdots \cup U_n(H_j)) = \begin{cases} A & \text{if } i \text{ even and } 2 \text{rk}(H_0, \ldots, H_j) \leq i \leq 2d \\ 0 & \text{otherwise} \end{cases}$$

(see [SS91, §3, Lemma 1]). Then, the first page of spectral sequence (3.4) can be rewritten as

$$E_{1,(n)}^{-j,i} = \bigoplus_{(H_0, \ldots, H_j) \in \mathcal{H}_n^{i+1} \atop \text{rk}(H_0, \ldots, H_j) \leq i/2} A$$

if $i \in [2, 2d]$ is even, and $E_{1,(n)}^{-j,i} = 0$ otherwise. Next, to have a cleaner picture of the differentials of (3.5), we introduce, for $1 \leq k \leq d$, the simplicial set $Y^{(n,k)}$ defined by

$$Y^{(n,k)}_j := \{(H_0, \ldots, H_j) \in \mathcal{H}_n^{i+1} : \text{rk}(H_0, \ldots, H_j) \leq k\}$$

with face/degeneracy maps given by omitting/doubling one hyperplane in a tuple. In addition, we denote by $\mathcal{K}(Y^{(n,k)}, A)$ the chain complex on $Y^{(n,k)}$ with coefficients in $A$. Then, the terms and the differentials of the first page of the spectral sequence (3.4) identify with

$$E_{1,(n)}^{-j,i} = \mathcal{K}(Y^{(n,i/2)}_j, A)$$

if $i \in [2, 2d]$ is even, and $E_{1,(n)}^{-j,i} = 0$ otherwise. Passing to the second page of the spectral sequence,

$$E_{2,(n)}^{-j,i} = H_j(Y^{(n,i/2)}_j, A)$$

if $i \in [2, 2d]$ is even, and $E_{2,(n)}^{-j,i} = 0$ otherwise. At this point, to compute $H^i(\mathbb{P}_K^d, \mathbb{P}_K^d)$, we want to take the inverse limit of (3.4) over $n \in \mathbb{N}$. For this, we will need the following observations.

**Remark 3.8.** Let $F$ be a non-archimedean local field as in axiom (3), i.e. such that $A$ is a condensed $F$-algebra. Clearly, the functor $- \otimes_F A$ is exact on finite-dimensional condensed $F$-vector spaces. One deduces the following points.

(i) Note that the complex $\mathcal{K}(Y^{(n,k)}, A)$ can be written as $\mathcal{K}(Y^{(n,k)}, A) = \mathcal{K}(Y^{(n,k)}_*, F) \otimes_A A$, where the terms of $\mathcal{K}(Y^{(n,k)}_*, F)$ are finite-dimensional condensed $F$-vector spaces. Then, we have that $H_j(Y^{(n,k)}, A) = H_j(Y^{(n,k)}_*, F) \otimes_A A$, where $H_j(Y^{(n,k)}_*, F)$ is finite-dimensional.

(ii) From the spectral sequence (3.4) it follows that $H^i(\mathbb{P}_K^d, U_n) = W_n \otimes_F A$, with $W_n$ a finite-dimensional condensed $F$-vector space.

Now, we note that we have a morphism of spectral sequences

$$E_{2,(n+1)}^{-j,i} \Longrightarrow H^{i-j}(\mathbb{P}_K^d, U_{n+1})$$

$$\downarrow$$

$$E_{2,(n)}^{-j,i} \Longrightarrow H^{i-j}(\mathbb{P}_K^d, U_n)$$
where the left arrow is induced by the map of simplicial sets $Y^{(n+1,k)}_* \to Y^{(n,k)}_*$, and the right arrow is induced by the inclusion $U_n \subset U_{n+1}$. Therefore, for varying $n \in \mathbb{N}$, we obtain an inverse system of spectral sequences; by Remark 3.8(i), and Lemma 3.5, we have that of spectral sequences; by Remark 3.8(ii), and Lemma 3.5, passing to the limit, we get the following spectral sequence

$$E_2^{-j,i} \implies \lim_n H^{i-j}(\mathbb{P}^d_K, U_n)$$

(3.6)

where $E_2^{-j,i} = \lim_n H_j(Y^{(n,i/2)}_*, A)$ if $i \in [2, 2d]$ is even, and $E_2^{-j,i} = 0$ otherwise. Moreover, by (the proof of) \cite[§2, Proposition 4]{SS91}, for all $i \geq 0$, we have the following natural exact sequence

$$0 \to R^1 \lim_n H^{i-1}(\mathbb{P}^d_K, U_n) \to H^i(\mathbb{P}^d_K, \mathbb{H}^d_K) \to \lim_n H^i(\mathbb{P}^d_K, U_n) \to 0.$$ 

Hence, by Remark 3.8(ii), and Lemma 3.5, we have that

$$\lim_n H^i(\mathbb{P}^d_K, U_n) = H^i(\mathbb{P}^d_K, \mathbb{H}^d_K).$$

In conclusion, the spectral sequence (3.6) can be rewritten as

$$E_2^{-j,i} = \begin{cases} \lim_n H_j(Y^{(n,i/2)}_*, A) & \text{if } i \in [2, 2d] \cap \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \implies H^{i-j}(\mathbb{P}^d_K, \mathbb{H}^d_K).$$

(3.7)

Next, we study the second page $E_2^{-j,i}$ of the obtained spectral sequence (3.7). For this, we introduce, for $1 \leq k \leq d$, the simplicial profinite set $Y_*^{(k)}$ defined by

$$Y_*^{(k)} := \left\{ (H_0, \ldots, H_j) \in \mathcal{H}^{j+1} : \dim_K \left( \sum_{r=0}^j K\ell_{H_r} \right) \leq k \right\}$$

endowed with the topology induced from the natural profinite one on $\mathcal{H}^{j+1}$, and with face/degeneracy maps given by omitting/doubling one hyperplane in a tuple.\textsuperscript{28}

The definition above is motivated by the following result.

**Lemma 3.9.**

(i) For each $i \geq 0$, and $1 \leq k \leq d$, there is a natural isomorphism of abelian groups\textsuperscript{29}

$$H^i(|Y_*^{(k)}|, \mathbb{Z}) \cong H^i(|T_*^{(k)}|, \mathbb{Z}).$$

(ii) We have $H^i(|T_*^{(k)}|, \mathbb{Z}) = 0$ for $i \neq 0, k-1$, and

$$H^0(|T_*^{(k)}|, \mathbb{Z}) = \begin{cases} \text{LC}(\mathbb{P}((K^{d+1})^*), \mathbb{Z}) & \text{if } k = 1 \\ \mathbb{Z} & \text{if } k > 1. \end{cases}$$

(iii) For each $j \geq 0$, we have a natural isomorphism in $\text{Mod}_{A}^{\text{cond}}$

$$\lim_n H_j(Y^{(n,k)}_*, A) \cong \text{Hom}(H^j(|T_*^{(k)}|, \mathbb{Z}), A)$$

where $H^j(|T_*^{(k)}|, \mathbb{Z})$ is regarded as discrete condensed abelian group.

\textsuperscript{28}Note that $Y_*^{(k)} = \lim_{n \to \infty} Y^{(n,k)}_*$, in fact, for $(H_0, \ldots, H_j) \in \mathcal{H}^{j+1}$, we have $\dim_K (\sum_{r=0}^j K\ell_{H_r}) \leq k$ if and only if $\text{rk}(\sum_{r=0}^j (O_K/\mathfrak{p}^n)\ell_{H_r}) \leq k$ for all $n \in \mathbb{N}$.

\textsuperscript{29}Here, $|\cdot|$ denotes the geometric realization.

\textsuperscript{30}Here, $\text{LC}(\cdot, \mathbb{Z})$ denotes the locally constant functions with values in $\mathbb{Z}$. 
Proof. Part (i) is [SS91, §3, Proposition 5], and part (ii) is [SS91, §3, Proposition 6, (iii)]. For part (iii), we adapt the argument of [SS91, §3, Proposition 6, (i)]. Given a field $F$ as in axiom (3), by the universal coefficient theorem we have, for each $n \in \mathbb{N}$, a natural exact sequence in $\text{Vect}_F^{\text{cond}}$

$$0 \to \text{Ext}^1(H^{j+1}(\mathfrak{y}^{(n,k)}, Z), F) \to H_j(\mathfrak{y}^{(n,k)}, F) \to \text{Hom}(H^j(\mathfrak{y}^{(n,k)}, Z), F) \to 0.$$  \hfill (3.8)

We recall that $H^j(\mathfrak{y}^{(n,k)}, Z)$ is a finite module in $\text{Mod}_Z^{\text{cond}}$, in particular $H^j(\mathfrak{y}^{(n,k)}, Z) \otimes F$ is a finite-dimensional condensed $F$-vector space, for all $j \geq 0$. We deduce that the left term of (3.8) vanishes, and thus we have an isomorphism

$$H_j(\mathfrak{y}^{(n,k)}, F) \cong \text{Hom}(H^j(\mathfrak{y}^{(n,k)}, Z), F).$$  \hfill (3.9)

Then, recalling Remark 3.8, applying the functor $- \otimes_F A$ to (3.9), and passing to the inverse limit over $n \in \mathbb{N}$, we have

$$\lim_{\longrightarrow} H_j(\mathfrak{y}^{(n,k)}, A) \cong \text{Hom}(\lim_{\longrightarrow} H^j(\mathfrak{y}^{(n,k)}, Z), A)$$  \hfill (3.10)

where we used that $\text{Hom}(H^j(\mathfrak{y}^{(n,k)}, Z), F) \otimes_F A = \text{Hom}(H^j(\mathfrak{y}^{(n,k)}, Z), A)$, which can be checked taking a finite presentation of the module $H^j(\mathfrak{y}^{(n,k)}, Z)$ in $\text{Mod}_Z^{\text{cond}}$. Now, for all $j \geq 0$, we have $\lim_{\longrightarrow} H^j(\mathfrak{y}^{(n,k)}, Z) = H^j(\mathfrak{y}^{(k)}, Z)$ (see the proof of [SS91, §3, Lemma 2]), therefore, the statement of part (iii) follows from (3.10) and part (i). \hfill \square

Now, we come back to the spectral sequence (3.7). In view of Lemma 3.9, it can we rewritten as

$$E_2^{-j,i} = \begin{cases} \text{Hom}(H^j(\mathfrak{T}(i/2)), Z), A) & \text{if } i \in [2, 2d] \cap 2\mathbb{Z}, j \in \{0, \ldots, i - 1\} \\ 0 & \text{otherwise} \end{cases} \implies H^{-j}(\mathbb{P}_K^d, \mathbb{H}_K^d).$$

We note that on the second page of the spectral sequence above there are no differentials between the non-zero terms $E_2^{-j,i}$ with $j = i/2 - 1$. In conclusion, this spectral sequence degenerates and computes $H^i(\mathbb{P}_K^d, \mathbb{H}_K^d)$. Then, Theorem 3.7 follows from the exact sequence (3.3) (see [SS91, §3, Lemma 7] for the details); the $G$-equivariance of the isomorphism in the statement is a consequence of the remarks after [SS91, §3, Theorem 1]. \hfill \square

3.4. Generalized Steinberg representations. As we will recall, it is possible to give a clearer representation-theoretic interpretation to the result of Theorem 3.7. In fact, the cohomology of the geometric realization of the topological Tits buildings of $\text{GL}_{d+1}(K)$ is related with a generalization of the Steinberg representations. See also [CDN20b, §5.2].

Notation 3.10. Let $d \geq 1$ be an integer, and $G = \text{GL}_{d+1}(K)$. Let $e_1^*, \ldots, e_{d+1}^*$ be the canonical basis of $(K^{d+1})^*$. Let $\Delta := \{1, \ldots, d\}$ and $I \subseteq \Delta$ a subset. We denote by $P_I$ the corresponding parabolic subgroup of $G$, i.e. the stabilizer in $G$ of the flag $\tau_I$ defined by $W_{i_1} \subset \cdots \subset W_{i_r} \subseteq W_{i_0}$, where $\Delta \setminus I = \{i_0 < \cdots < i_r\}$ and $W_{i_k} := \sum_{i=i_k+1}^{d+1} Ke_i^*$. Moreover, we denote $X_I := G/P_I$.

Definition 3.11. Let $M$ be an abelian group, and let $I \subseteq \Delta$. The locally constant special representations of $G$ (associated to $I$) with coefficients in $M$ are defined by

$$\text{Sp}_I(M) := \frac{\text{LC}(X_I, M)}{\sum_{J \in \Delta \setminus I} \text{LC}(X_{I \cup \{j\}}, M)}$$

where $\text{LC}(\cdot, M)$ denotes the locally constant functions with values in $M$. 


For $0 \leq i \leq d$, we denote 
\[ \mathrm{Sp}_i(M) := \mathrm{Sp}_{\{1, \ldots, d-i\}}(M) \]
and for $i > d$, we set $\mathrm{Sp}_i(M) = 0$.

**Remark 3.12.** We have that $\mathrm{Sp}_I(M)$ is a smooth $G$-module. Moreover, we observe that there is a natural isomorphism $\mathrm{Sp}_I(M) \cong \mathrm{Sp}_I(\mathbb{Z}) \otimes_{\mathbb{Z}} M$.

**Remark 3.13.** For $I = \emptyset$, we have that $\mathrm{Sp}_I(M)$ is the locally constant Steinberg representation of $G$ with coefficients in $M$.

**Proposition 3.14.** For all $1 \leq i \leq d$, we have a natural isomorphism 
\[ \widetilde{H}^{i-1}(|\mathcal{T}^{(i)}_{\infty}|, \mathbb{Z}) \cong \mathrm{Sp}_i(\mathbb{Z}) \]
where $\widetilde{H}$ denotes the reduced cohomology.

*Proof.* See [SS91, §4, Lemma 1] and [CDN20b, Proposition 5.6]. \qed

**Theorem 3.15.** Let $A$ be a condensed ring, and let $H^\bullet : \text{RigSm}_K \to \text{Mod}_A^{\text{cond}}$ be a cohomology theory that satisfies the axioms of Schneider-Stuhler (3.4). Then, for all $i \geq 0$, there exists a natural isomorphism of $G$-modules in $\text{Mod}_A^{\text{cond}}$
\[ H^i(\mathbb{H}_K^A) \cong \text{Hom}(\mathrm{Sp}_i(\mathbb{Z}), A) \]
where $\mathrm{Sp}_i(\mathbb{Z})$ is regarded as a discrete condensed abelian group.

*Proof.* This is a consequence of Theorem 3.7 combined with Proposition 3.14. \qed

**Remark 3.16.** In the case $A = F$ is a non-archimedean local field, we can describe in classical topological terms the internal dual $\text{Hom}(\mathrm{Sp}_I(\mathbb{Z}), F)$ that appears in Theorem 3.15 (here, we maintain Notation 3.10).

First, we observe that we can endow $\mathrm{Sp}_I(F)$ with a natural structure of topological $F$-vector space (see also [CDN20b, §5.2.2]). In fact, the quotient space $X_I = G/P_I$ is a profinite set; more precisely, we can write $X_I = \varprojlim_n X_{n,I}$ as a countable inverse limit of finite sets $X_{n,I}$, along surjective transition maps. Since each $\text{LC}(X_{n,I}, F)$ is a finite-dimensional $F$-vector space, it has a natural topology coming from the topology on $F$. We equip $\text{LC}(X_I, F) = \varprojlim_n \text{LC}(X_{n,I}, F)$ with the direct limit topology; and $\mathrm{Sp}_I(F)$ with the induced quotient topology. Note that we can write $\mathrm{Sp}_I(F) = \varinjlim_n V_n$ as a countable direct limit of finite-dimensional topological $F$-vector spaces $V_n$, along (closed) immersions; then, by Example A.17, we have $\mathrm{Sp}_I(F) = \varinjlim_n V_n$. We deduce that the internal dual $\text{Hom}(\mathrm{Sp}_I(\mathbb{Z}), F)$ identifies with 
\[ \text{Hom}_F(\mathrm{Sp}_I(F), F) = \varprojlim_n \text{Hom}_F(V_n, F) = \varprojlim_n \text{Hom}_{\text{cont}, F}(V_n, F) = \text{Hom}_{\text{cont}, F}(\mathrm{Sp}_I(F), F) \]
where the dual space $\mathrm{Sp}_I(F)^* := \text{Hom}_{\text{cont}, F}(\mathrm{Sp}_I(F), F)$ is endowed with the weak topology. We observe that $\mathrm{Sp}_I(F)^*$ is an $F$-Fréchet space, being a countable inverse limit of finite-dimensional topological $F$-vector spaces.

\[ ^{31} \text{Recall that, given } \{W_n\} \text{ a countable direct system of locally compact Hausdorff topological } K \text{-vector spaces, whose transitions maps are immersions, then, the } F \text{-vector space } \varinjlim_n W_n, \text{ endowed with the direct limit topology, is a topological } F \text{-vector space (see e.g. [HSTH01, Theorem 4.1]).} \]
4. Pro-étale period sheaves

In this section, we recall definitions and basic results on the pro-étale period sheaves. Moreover, we check that the condensed pro-étale cohomology groups (Definition 2.3) of the period sheaves are compatible with the “standard topology” on the relative period rings.

Following Scholze, [Sch13b, §6], we start by defining the de Rham pro-étale period sheaves, that are the sheaf-theoretic version of the classical de Rham period rings of Fontaine.

Definition 4.1. Let $X$ be an analytic adic space over $\text{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$.\(^{32}\) Let $\nu : X_{\text{pro-ét}} \to X_{\text{ét}}$ be the natural morphism of sites. The following are defined to be sheaves on $X_{\text{pro-ét}}$.

(i) The integral structure sheaf $\mathcal{O}_X^\chi = \nu^*\mathcal{O}_{X_{\text{ét}}}^\chi$, the structure sheaf $\mathcal{O}_X = \nu^*\mathcal{O}_{X_{\text{ét}}}$, the completed integral structure sheaf $\hat{\mathcal{O}}_X = \varprojlim_n \mathcal{O}_X^\chi/p^n$, and the completed structure sheaf $\hat{\mathcal{O}}_X = \hat{\mathcal{O}}_X^\chi[1/p]$.

Moreover, we define the tilted integral structure sheaf $\hat{\mathcal{O}}_X^{\tilde{+}} = \varprojlim_n \mathcal{O}_X^\chi/p$, where the inverse limit is taken along the Frobenius map $\varphi$.

(ii) The sheaves $\mathcal{A}_{\text{inf}} = W(\hat{\mathcal{O}}_X^{\tilde{+}})$ and $\mathcal{B}_{\text{inf}} = \mathcal{A}_{\text{inf}}[1/p]$. We have a morphism of pro-étale sheaves $\theta : \mathcal{A}_{\text{inf}} \to \hat{\mathcal{O}}_X^{\tilde{+}}$ that extends to $\theta : \mathcal{B}_{\text{inf}} \to \hat{\mathcal{O}}_X$.

(iii) We define the positive de Rham sheaf $\mathbb{B}_{dR}^+ = \varprojlim_{n \in \mathbb{N}} \mathcal{B}_{\text{inf}}/(\ker \theta)^n$, with filtration given by $\text{Fil}^r \mathbb{B}_{dR}^+ = (\ker \theta)^r \mathbb{B}_{dR}^+$.

(iv) Let $t$ be a generator of $\text{Fil}^1 \mathbb{B}_{dR}^+$.\(^{33}\) We define the de Rham sheaf $\mathbb{B}_{dR} = \mathbb{B}_{dR}^+[1/t]$, with filtration $\text{Fil}^r \mathbb{B}_{dR} = \sum_{j \in \mathbb{Z}} t^{-j} \text{Fil}^{r+j} \mathbb{B}_{dR}^+$.

Now, we recall the definition of the pro-étale sheaf-theoretic version of the ring $B$ introduced by Fargues and Fontaine in their work on the fundamental curve of $p$-adic Hodge theory, [FF18, §1.6]. See also [LB18a, §8].

Definition 4.2. Let $X$ be an analytic adic space over $\text{Spa}(C,\mathcal{O}_C)$. Let $I = [\rho_1, \rho_2]$ be a compact interval of $[0, 1]$ with $\rho_1, \rho_2 \in \mathbb{Q}$, and let $a, b \in \mathcal{O}_C$ such that $|a| = \rho_1$ and $|b| = \rho_2$. We define the following sheaves on $X_{\text{pro-ét}}$

$$A_{\text{inf}, I} = \mathcal{A}_{\text{inf}} \left( \frac{a}{p}, \frac{b}{p} \right), \quad A_I = \varprojlim_n A_{\text{inf}, I}/p^n, \quad \mathbb{B}_I = A_I[1/p].$$

Moreover, we define the sheaf on $X_{\text{pro-ét}}$

$$\mathbb{B} = \varinjlim_{I \subseteq [0, 1[} \mathbb{B}_I$$

where $I$ runs over all the compact intervals of $[0, 1]$ with endpoints in $\mathbb{Q}$.

Remark 4.3. In the notation of the definition above, the Frobenius $\varphi$ on $\mathcal{A}_{\text{inf}}$ induces an isomorphism

$$\varphi : \mathbb{B}_I \xrightarrow{\sim} \mathbb{B}_{\varphi(I)}$$

where $\varphi : [0, 1] \to [0, 1]$ denotes the bijection defined by $\varphi(\rho) = \rho^p$ (see [FF18, §1.6]). In particular, the Frobenius $\varphi$ extends to an automorphism of $\mathbb{B}$.

\(^{32}\)We remind the reader that all the analytic adic spaces are assumed to be $\kappa$-small (see §1.6).

\(^{33}\)Such a generator exists locally on $X_{\text{pro-ét}}$, it is not a zero-divisor and unique up to unit, by [Sch13b, Lemma 6.3].
Next, we recall that, on the affinoid perfectoid spaces over Spa($C, \mathcal{O}_C$), the period sheaves defined above are given by the expected relative period rings.

Let $(R, R^+)$ be an affinoid perfectoid algebra over $(C, \mathcal{O}_C)$, and let $(\bar{R}^+, \bar{R}^{+})$ be its tilt.\footnote{Here, $R^+$ is endowed with the $p$-adic topology, and $R^+ = \lim \frac{1}{p^n} R^+$ with the induced inverse limit topology.} We define $\mathcal{A}_{\text{inf}}(R, R^+) = W(R^+)/\mathcal{A}_{\text{inf}}(R, R^+)$ endowed with the inverse limit topology, $\mathcal{B}_{\text{inf}}(R, R^+) = \mathcal{A}_{\text{inf}}(R, R^+)[1/p]$ with the direct limit topology, and $\mathcal{B}_{\text{dR}}(R, R^+) = \lim \mathcal{B}_{\text{inf}}(R, R^+)/(\ker \theta)^n$ with the inverse limit topology. Given $\xi \in \mathcal{A}_{\text{inf}} = \mathcal{A}_{\text{inf}}(C, \mathcal{O}_C)$ a generator of the kernel of $\theta : \mathcal{A}_{\text{inf}} \to \mathcal{O}_C$,\footnote{We recall that such an element $\xi$ generates also the kernel of $\theta : \mathcal{A}_{\text{inf}}(R, R^+) \to R^+$ and it is not a zero-divisor in $\mathcal{A}_{\text{inf}}(R, R^+)$ (see [Sch13b, Lemma 6.3]).} we set $\mathcal{B}_{\text{dR}}(R, R^+) = \mathcal{B}_{\text{dR}}(R, R^+)[1/\xi]$ and endow it with the direct limit topology. Moreover, for $I \subset \{0, 1\}$ a compact interval with endpoints in $\mathbb{p}^Q$, in a similar fashion we define $\mathcal{A}_{\text{inf}, I}(R, R^+)$, $\mathcal{A}_{I}(R, R^+)$, $\mathcal{B}_{I}(R, R^+)$, $\mathcal{B}(R, R^+)$ and endowed them with the topologies induced by the one on $\mathcal{A}_{\text{inf}}(R, R^+)$.\footnote{We recall that such an element $\xi$ generates also the kernel of $\theta : \mathcal{A}_{\text{inf}}(R, R^+) \to R^+$ and it is not a zero-divisor in $\mathcal{A}_{\text{inf}}(R, R^+)$ (see [Sch13b, Lemma 6.3]).}

**Remark 4.4.** Recall that the topology defined above gives to $\mathcal{B}_{\text{dR}}(R, R^+)$ a structure of $\mathbb{Q}_p$-Fréchet algebra. Moreover, for $I \subset \{0, 1\}$ a compact interval with endpoints in $\mathbb{p}^Q$, we have that $\mathcal{B}_{I}(R, R^+)$ is a $\mathbb{Q}_p$-Banach algebra, and $\mathcal{B}(R, R^+)$ is a $\mathbb{Q}_p$-Fréchet algebra (see [FF18, §1.6]).

**Proposition 4.5.** Let $Z = \text{Spa}(R, R^+)$ be an affinoid perfectoid space over Spa($C, \mathcal{O}_C$).

(i) We have $\hat{\mathcal{O}}^+_Z(Z) = R^+$ and $\hat{\mathcal{O}}^{++}_Z(Z) = R^{++}$. Moreover, $H^i_{\text{proét}}(Z, \hat{\mathcal{O}}^+_Z)$ (resp. $H^i_{\text{proét}}(Z, \hat{\mathcal{O}}^{++}_Z)$) is almost zero, for all $i > 0$, with respect to the almost setting defined by the ring $\mathcal{O}_C$ (resp. $\mathcal{O}_C$) and its ideal of topologically nilpotent elements.

(ii) For $A \in \{\mathcal{A}_{\text{inf}}, \mathcal{A}_{\text{inf}, I}, \mathcal{A}_{I}\}$, we have $A(Z) = A(R, R^+)$, and $H^i_{\text{proét}}(Z, A)$ is almost zero, for all $i > 0$, with respect to the almost setting defined by the ring $\mathcal{A}_{\text{inf}}$ and its ideal $(|p|^1)^n_{n \geq 1}$.

(iii) For $B \in \{\mathcal{B}_{\text{dR}}^+, \mathcal{B}_{\text{dR}}, \mathcal{B}_I, \mathcal{B}_I[1/t], \mathcal{B}[1/t]\}$ we have $B(Z) = B(R, R^+)$, and $H^i_{\text{proét}}(Z, B)$ vanishes for all $i > 0$.

**Proof.** Part (i) is [Sch13b, Lemma 4.10, Lemma 5.10] (see also [MW20, Lemma 2.7]). The other statements are a corollary. The statement for $A = \mathcal{A}_{\text{inf}}$ and $B \in \{\mathcal{B}_{\text{dR}}^+, \mathcal{B}_{\text{dR}}\}$ is proven in [Sch13b, Proposition 6.5]. The statement for $A \in \{\mathcal{A}_{\text{inf}, I}, \mathcal{A}_{I}\}$ and $B = \mathcal{B}_I$ follows from (the proof of) [LB18a, Proposition 8.3]. In order to deduce the statement for $B = \mathcal{B}$, we reduce to the previous case, using [Sch13b, Lemma 3.18], together with Lemma 4.6 below. The statement for $B \in \{\mathcal{B}_I[1/t], \mathcal{B}[1/t]\}$ follows from the latter two cases using that $|Z|$ is quasi-compact and quasi-separated. \qed

We used the following lemma.

**Lemma 4.6.** Let $Z = \text{Spa}(R, R^+)$ be an affinoid perfectoid space over Spa($C, \mathcal{O}_C$). Then, for all $j > 0$, we have

$$R^j \lim_{\mathcal{B}_I(Z) = 0}$$

where $I$ runs over all the compact intervals of $[0, 1]$ with endpoints in $\mathbb{p}^Q$.

**Proof.** This is proven in [LB18b, Lemma 3.5]. \qed

Therefore, given $X$ an analytic adic space over Spa($C, \mathcal{O}_C$), and $\mathcal{F}$ any of the sheaves on $X_{\text{proét}}$ of Proposition 4.5, denoting by $\mathcal{B}$ the basis of $X_{\text{proét}}$ consisting of the affinoid perfectoid spaces $U \in X_{\text{proét}}$, one has that the values of $\mathcal{F}$ on $\mathcal{B}$ have a natural structure of topological abelian groups. Hence, following Remark 2.7, we associate to $\mathcal{F}$ a sheaf $\mathcal{F}$ on $X_{\text{proét}}$ with values in CondAb
such that, for $U \in \mathcal{B}$, we have $F(U) = F(U)$.\footnote{More precisely, Remark 2.7 applies to the big pro-étale site, [Sch21, Definition 8.1, (i)].} The following result shows that the condensed cohomology groups $H^i_{\text{proét}}(X, F)$ agree with the ones of Definition 2.3.

**Corollary 4.7** ([Sch13b, Corollary 6.6]). Let $X$ be an analytic adic space over $\text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C})$. Let $\mathcal{F}$ be any of the pro-étale sheaves on $X$ of Proposition 4.5. For any $U \in X_{\text{proét}}$ affinoid perfectoid, and $S$ profinite set, we have $\mathcal{F}(U \times S) = \mathcal{E}^0(S, \mathcal{F}(U))$. In particular, for all $i \geq 0$, we have

$$H^i_{\text{proét}}(X, \mathcal{F}) = H^i_{\text{proét}}(X, \mathcal{F})$$

as condensed abelian groups.\footnote{In the case $\mathcal{F} = \mathcal{B}$ of Proposition 4.5(iii), we have that 4.1 also holds as condensed $\mathbb{Q}_p$-vector spaces.}

**Example 4.8.** By Corollary 4.7, the pro-étale period sheaf $\mathcal{B}$ on $\text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C})_{\text{proét}}$ agrees with the condensed $\mathbb{Q}_p$-algebra $\mathcal{B}$, where $\mathcal{B} = \mathcal{B}(\mathbb{C}, \mathcal{O}_\mathbb{C})$ is Fargues-Fontaine’s period ring regarded as a $\mathbb{Q}_p$-Fréchet algebra.

Next, we state the pro-étale sheaf-theoretic version of the fundamental exact sequence of $p$-adic Hodge theory.

**Proposition 4.9.** Let $X$ be a locally noetherian adic space over $\text{Spa}(\mathbb{C}, \mathcal{O}_\mathbb{C})$. We have the following exact sequences of sheaves on $X_{\text{proét}}$

$$0 \to \mathbb{B}_e \to \mathbb{B}[1/t] \xrightarrow{\varphi - 1} \mathbb{B}[1/t] \to 0$$

and

$$0 \to \mathbb{Q}_p \to \mathbb{B}_e \to \mathbb{B}_{\text{dR}} / \mathbb{B}_+ \to 0$$

where $\mathbb{B}_e := \mathbb{B}[1/t]^{\varphi = 1}$.

**Proof.** The affinoid perfectoid spaces $\text{Spa}(A, A^\diamond) \in X_{\text{proét}}$, with $A$ a sympathetic $\mathbb{C}$-algebra, form a basis for $X_{\text{proét}}$: this is a consequence of [MW20, Lemma 2.6], and the proof of [Sch13b, Proposition 4.8] (which in turn uses a construction of Colmez [Col02, §4.4, §4.5]). Hence, it suffices to prove that, for any sympathetic $\mathbb{C}$-algebra $A$, we have exact sequences as in the statement over $\text{Spa}(A, A^\diamond)$. Then, the exact sequence (4.2) follows from [KL15, Proposition 6.2.2], and the exact sequence (4.3) follows from [Col02, Proposition 8.25, (SEF 3E)] combined with [KL15, Corollary 5.2.12].

5. Solid base change for coherent cohomology

In this section, for $X$ a rigid-analytic variety over $K$, we endow the cohomology groups of a coherent $\mathcal{O}_X$-module over $X$ with a structure of condensed $K$-vector space. Then, in the category of condensed $K$-vector spaces, we prove a base change result for the coherent cohomology of connected, paracompact, rigid-analytic varieties defined over $K$.

Following §1.6, all of our rigid-analytic varieties will be assumed to be $\kappa$-small. Moreover, we introduce the following convention and notation.

**Convention 5.1.** All rigid-analytic varieties will be assumed to be quasi-separated.

**Notation 5.2.** We denote by $\text{Vect}^\text{cond}_K$ the category of $\kappa$-condensed $K$-vector spaces, and by $\text{Vect}^\text{solid}_K \subset \text{Vect}^\text{cond}_K$ the symmetric monoidal subcategory of $\kappa$-solid $K$-vector spaces, endowed with the tensor product $\otimes^\kappa_K$ (and often the prefix “$\kappa$” is tacit). See §A.2.\footnote{In particular, see Remark A.1 and Remark A.5 on set-theoretic bounds.}
5.1. **Coherent cohomology in the condensed world.** As we will need to compute coherent cohomology both on the analytic site and the étale site, we start by recalling the following definition.

**Definition 5.3.** Let \( X \) be a rigid-analytic variety defined over \( K \), and let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-module over \( X \). We denote by \( \mathcal{F}_{\text{ét}} \) the associated \( \mathcal{O}_{X_{\text{ét}}} \)-module over \( X_{\text{ét}} \), given by sending an affinoid \( f : U \to X \) to the sections \( \Gamma(U, f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U) \).

By [dJvdP96, Proposition 3.2.2], \( \mathcal{F}_{\text{ét}} \) is indeed a sheaf on \( X_{\text{ét}} \).

**Remark 5.4.** In the situation of Definition 5.3, both \( \mathcal{F} \) and \( \mathcal{F}_{\text{ét}} \) are naturally sheaves of topological \( K \)-vector spaces. More precisely, the values of \( \mathcal{F} \) (resp. \( \mathcal{F}_{\text{ét}} \)) on an admissible open affinoid \( U \subset X \) (resp. on an affinoid étale \( U \to X \)) have a natural structure of \( K \)-Banach space: in fact, \( \mathcal{F}(U) \) is a finite module over the \( K \)-Banach algebra \( \mathcal{O}_U(U) \).

Then, following §2.2, we now study the coherent cohomology of rigid-analytic varieties from the point of view of condensed mathematics.

**Definition 5.5.** Let \( X \) be a rigid-analytic variety defined over \( K \). Let \( \mathcal{F}_{\text{an}} \) be a coherent \( \mathcal{O}_{X_{\text{an}}} \)-module over \( X_{\text{an}} \), and let \( \mathcal{F}_{\text{ét}} \) denote the associated \( \mathcal{O}_{X_{\text{ét}}} \)-module over \( X_{\text{ét}} \).\(^{39}\) Let \( \tau \in \{ \text{an, ét} \} \), and let \( \mathcal{B}_{\tau} \) denote the basis for the site \( X_{\tau} \) consisting of all \( U \subset X_{\tau} \) which are affinoid.

(i) We denote by \( \mathcal{F}_{\tau} \) the presheaf on \( X_{\tau} \), with values in \( \text{Vect}^{\text{cond}}_K \), extending the presheaf \( (\mathcal{B}_{\tau})^{\text{op}} \to \text{Vect}^{\text{cond}}_K : U \mapsto \mathcal{F}_{\tau}(U) \).

(ii) Given \( A \in \text{Vect}^{\text{solid}}_K \) flat,\(^{40}\) we denote by \( \mathcal{F}_{\tau} \otimes^\bullet_K A \) the presheaf on \( X_{\tau} \), with values in \( \text{Vect}^{\text{cond}}_K \), extending the following presheaf\(^{41}\)

\[
(\mathcal{B}_{\tau})^{\text{op}} \to \text{Vect}^{\text{cond}}_K : U \mapsto \mathcal{F}_{\tau}(U) \otimes^\bullet_K A.
\]

The next result shows that the presheaves defined above are indeed sheaves, and translates in our setting some classical results on coherent cohomology.

**Lemma 5.6.** Let \( X \) be a rigid-analytic variety defined over \( K \). Let \( \mathcal{F}_{\text{an}} \) be a coherent \( \mathcal{O}_{X_{\text{an}}} \)-module over \( X_{\text{an}} \), and let \( \mathcal{F}_{\text{ét}} \) denote the associated \( \mathcal{O}_{X_{\text{ét}}} \)-module over \( X_{\text{ét}} \).

(i) The presheaf \( \mathcal{F}_{\tau} \) is a sheaf on \( X_{\tau} \) with values in \( \text{Vect}^{\text{solid}}_K \). Moreover, if \( X \) is an affinoid space, then we have that \( H^i(X, \mathcal{F}_{\tau}) = 0 \) for all \( i > 0 \).

(ii) For \( A \in \text{Vect}^{\text{solid}}_K \) flat, the presheaf \( \mathcal{F}_{\tau} \otimes^\bullet_K A \) is a sheaf on \( X_{\tau} \) with values in \( \text{Vect}^{\text{solid}}_K \). Moreover, if \( X \) is an affinoid space, then we have that \( H^i(X, \mathcal{F}_{\tau} \otimes^\bullet_K A) = 0 \) for all \( i > 0 \).

**Proof.** First, we recall that, by Proposition A.12 (combined with Remark A.21), for any \( K \)-Banach space \( V \), we have \( V \in \text{Vect}^{\text{solid}}_K \). Then, from Remark 5.4 we deduce that \( \mathcal{F}_{\text{ét}} \) and \( \mathcal{F}_{\tau} \otimes^\bullet_K A \) take values in \( \text{Vect}^{\text{solid}}_K \), since the abelian subcategory \( \text{Vect}^{\text{solid}}_K \subset \text{Vect}^{\text{cond}}_K \) is stable under all limits and colimits, Proposition A.2(ii).

\(^{39}\)We denote by \( X_{\text{an}} \) the analytic site, i.e. the site of the admissible open subsets of \( X \).

\(^{40}\)See Definition A.6.

\(^{41}\)We warn the reader that it might not be true, in general, that for all \( V \in X_{\tau} \) we have \( (\mathcal{F}_{\tau} \otimes^\bullet_K A)(V) = \mathcal{F}_{\tau}(V) \otimes^\bullet_K A \), since the tensor product \( \otimes^\bullet_K \) does not commute with all limits.
For part (i), it suffices to show the exactness of the Čech complex

$$0 \to F_{\tau}(U) \to \prod_{i \in I} F_{\tau}(U_i) \to \prod_{i,j \in I} F_{\tau}(U_i \times_U U_j) \to \cdots$$  \hspace{1cm} (5.1)

for any $U \in B_{\tau}^{aff}$ and $U = \{U_i \to U\}_{i \in I}$ a finite covering of $U$ in $B_{\tau}^{aff}$. By Tate’s acyclicity theorem, and \cite[Proposition 3.2.5]{dJvdP96}, the Čech cohomology $H^i(U, F_{\tau})$ vanishes for all $i > 0$. Therefore, recalling Remark 5.4, the statement follows from Lemma A.15.

For part (ii), we observe that, since $A$ is a flat solid $K$-vector space, the complex (5.1) remains acyclic after applying the functor $- \otimes_K A$ (and then we use that $- \otimes_K A$ commutes with finite products).

\textit{Remark 5.7.} We keep the notation of Definition 5.5.

(i) By Remark 2.7, for all $i \geq 0$, we have $H^i_{\tau}(X, F_{\tau})(*) = H^i_{\tau}(X, F_{\tau})$ as $K$-vector spaces, and the condensed cohomology group $H^i_{\tau}(X, F_{\tau})$ does not depend on our choice of the cardinal $\kappa$ (here, we use Remark A.21, and the fact that $X$ is assumed to be $\kappa$-small).

(ii) We note that the sheaf cohomology complex $R\Gamma_{\tau}(X, F_{\tau}) \in D(Vect_{K}^{\text{cond}})$ lies in $D(Vect_{K}^{\text{solid}})$. Equivalently, by Proposition A.2(iv), $H^i_{\tau}(X, F_{\tau})$ lies in $\text{Vect}_{K}^{\text{solid}}$ for all $i$. This is true by Verdier’s hypercovering theorem, recalling that the subcategory $\text{Vect}_{K}^{\text{solid}} \subset \text{Vect}_{K}^{\text{cond}}$ is stable under all limits and colimits.

(iii) Let $A$ be a solid $K$-algebra. By Corollary A.8 (and Remark A.5), we can endow the condensed ring $A$ with an analytic ring structure $(A, M_A)$ such that $\text{Mod}_{(A, M_A)}$ is the category of $A$-modules in $\text{Vect}_{K}^{\text{solid}}$. Then, assuming that $A$ is flat as a solid $K$-vector space, the argument of point (ii) shows that, since the sheaf $F_{\tau} \otimes_K A$ (recall Lemma 5.6(ii)) has values in $\text{Mod}_{(A, M_A)}$, the complex $R\Gamma_{\tau}(X, F_{\tau} \otimes_K A) \in D(Vect_{K}^{\text{cond}})$ lies in $D(A, M_A)$.\footnote{Here, we denote $D(A, M_A) := D(\text{Mod}_{(A, M_A)})$.}

The present remark also holds replacing $F_{\tau}$ with a bounded below complex of such sheaves on $X_{\tau}$.

In the condensed setting, we also have a version of Kiehl’s acyclicity theorem for Stein spaces. First, let us recall the following definition.

\textbf{Definition 5.8.} A rigid-analytic variety $X$ over $K$ is called a \textit{Stein space} if it has an increasing admissible affinoid covering $\{U_j\}_{j \in \mathbb{N}}$ such that $U_j \subseteq U_{j+1}$, i.e. the inclusion $U_j \subset U_{j+1}$ factors over the adic compactification of $U_j$, for every $j \in \mathbb{N}$. We call $\{U_j\}_{j \in \mathbb{N}}$ a \textit{Stein covering} of $X$.

\textbf{Lemma 5.9.} Let $X$ be a Stein space over $K$, and let $F$ be a coherent $O_{X}$-module over $X$. Then, $H^i(X, F) = 0$ for all $i > 0$.

\textit{Proof.} By Lemma 5.6(i), choosing a countable admissible affinoid covering $\mathcal{U}$ of $X$, we have that $H^i(X, F) = H^i(\mathcal{U}, F)$. Then, the statement follows from Kiehl’s acyclicity theorem, \cite[Satz 2.4]{Kie67b},\footnote{Note that loc. cit. is stated for any quasi-Stein space (see \cite[Definition 2.3]{Kie67b}), and in particular it holds for Stein spaces (Definition 5.8).} and Lemma A.15 (recalling that a countable product of $K$-Fréchet spaces is a $K$-Fréchet space).

Next, we give some examples by defining a structure of condensed $K$-vector space on the de Rham cohomology groups.
Lemma 5.13. Let $K$ the differentials Frechét, and such structure does not depend on the choice of $K$.

Now, we want to compare the definition above with some other natural ways of putting a structure of topological $K$-vector space on the latter. By Lemma 5.6(i), we have $H^i_{\text{dR}}(X) = H^i(U, \Omega^*_X)$, where in the last step we used Lemma A.15. □

Remark 5.12. If $X$ is a smooth Stein space, then the global sections $\Omega^i(X)$, $i \geq 0$, have a natural structure of $K$-Fréchet spaces: in fact, given a Stein covering $\{U_j\}_{j \in \mathbb{N}}$ of $X$, each space $\Omega^i(U_j)$ is $K$-Banach, therefore the spaces $\Omega^i(X) = \lim_{\longrightarrow} \Omega^i(U_j)$ endowed with the inverse limit topology are $K$-Fréchet, and such structure does not depend on the choice of $\{U_j\}_{j \in \mathbb{N}}$. By [GK04, Corollary 3.2], all the differentials $d: \Omega^{i-1}(X) \to \Omega^i(X)$ have closed image, hence, endowing $\Omega^i(X)$ with the subspace topology, and $H^i_{\text{dR}}(X) = \Omega^i(X)^{d=0}/d\Omega^{i-1}(X)$ with the induced quotient topology, we obtain a $K$-Fréchet space structure on the latter.

Lemma 5.13. Let $X$ be a smooth Stein space over $K$. Then, for all $i \geq 0$, we have

$$H^i_{\text{dR}}(X) = \frac{\Omega^i(X)^{d=0}}{d\Omega^{i-1}(X)}.$$

Proof. First note that, by definition, for all $i \geq 0$, we have $\Omega^*_X(X) = \lim_{\longrightarrow} \Omega^i(U)$, with the structure of $K$-Fréchet space on $\Omega^i(X)$ as in Remark 5.12.

Then, by Lemma 5.9, we have that $H^i_{\text{dR}}(X) = \frac{\Omega^i(X)^{d=0}}{d\Omega^{i-1}(X)}$ and the statement follows from Lemma A.15. □

Remark 5.14. Let $X$ be a smooth affinoid over $K$. Then, by Lemma 5.6(i), for all $i \geq 0$, we have that $H^i_{\text{dR}}(X) = \frac{\Omega^i(X)^{d=0}}{d\Omega^{i-1}(X)}$. But, in general, $H^i_{\text{dR}}(X) \neq \frac{\Omega^i(X)^{d=0}}{d\Omega^{i-1}(X)}$: in fact, the subspace $d\Omega^{i-1}(X) \subseteq \Omega^i(X)^{d=0}$ can be non-closed, and then we can refer to Example A.16.

---

\[\text{Note that } d\Omega^{i-1}(X) = \frac{\Omega^{i-1}(X)}{\Omega^{i-1}(X)^{d=0}} = d\Omega^{i-1}(X), \text{ where in the first step we used Lemma A.15, and in the second one we used the open mapping theorem for } K\text{-Fréchet spaces.}\]

\[\text{For example, let } \mathbb{D} \text{ be the 1-dimensional closed unit disk over } \mathbb{Q}_p, \text{ then, one can check that the subspace } d\mathcal{O}(\mathbb{D}) \text{ of } \Omega^1(\mathbb{D}) \text{ is not closed.}\]
Indeed, the quotient topology on the de Rham cohomology groups is not the “correct” one in this case (cf. Remark 5.22).

5.2. Paracompact rigid-analytic varieties. Before stating and proving the main result of this section, namely Theorem 5.20, we first need to recall the notion of paracompact rigid-analytic variety, and give some examples.

Definition 5.15. A (quasi-separated)\(^{46}\) rigid-analytic variety \(X\) over \(K\) is \textit{paracompact} if it admits an admissible locally finite affinoid covering, i.e. there exists an admissible covering \(\{U_i\}_{i \in I}\) of \(X\) by affinoid subspaces such that for each index \(i \in I\) the intersection \(U_i \cap U_j\) is non-empty for at most finitely many indices \(j \in I\).

Remark 5.16.
(i) A paracompact rigid-analytic variety over \(K\) is taut, [Hub96, Definition 5.1.2, Lemma 5.1.3], and it is the admissible disjoint union of connected paracompact rigid-analytic varieties of countable type, i.e. having a countable admissible affinoid covering, [dJvdP96, Lemma 2.5.7].
(ii) Conversely, given \(X\) a taut rigid-analytic variety over \(K\) that is of countable type, then \(X\) is paracompact. In order to see this, recall that there is an equivalence between the category of Hausdorff strictly \(K\)-analytic Berkovich spaces and the category of taut rigid-analytic varieties over \(K\), [Hub96, Proposition 8.3.1], and, under such equivalence, Hausdorff, paracompact\(^{47}\) strictly \(K\)-analytic Berkovich spaces correspond to paracompact rigid-analytic varieties over \(K\), [Ber93, Theorem 1.6.1]. Now, if \(X^\text{Berk}\) corresponds to \(X\) under such equivalence, then \(X^\text{Berk}\) is a locally compact Hausdorff topological space that is a countable union of compact subspaces; we deduce that \(X^\text{Berk}\) is paracompact, and so \(X\) is a paracompact rigid-analytic variety.
(iii) Let us also recall that there are examples of (non-taut) separated rigid-analytic varieties over \(K\) of countable type that are not paracompact, [LvdP95, Remarks 4.4].

Remark 5.17. Any admissible open of a paracompact rigid-analytic variety \(X\) over \(K\) is paracompact. In fact, the Hausdorff strictly \(K\)-analytic Berkovich space \(X^\text{Berk}\) associated to \(X\) is metrizable, hence any of its subspaces is metrizable, in particular paracompact. In order to see that \(X^\text{Berk}\) is metrizable, we recall that, since \(K\) has a countable dense subfield (namely, the algebraic closure of the field of rational numbers \(\mathbb{Q}\) in \(K\)), \(X^\text{Berk}\) is locally metrizable (see [CL06, §2]), and a Hausdorff, paracompact, topological space that is locally metrizable is metrizable.\(^{48}\)

Examples 5.18. A lot of interesting rigid-analytic varieties over \(K\) are paracompact. Here is a list of examples: any separated rigid-analytic variety over \(K\) of dimension 1, [LvdP95]; the rigid analytification of a separated scheme of finite type over \(K\), [Bos14, §8.4, Proposition 7]; a Stein space over \(K\); an admissible open of a quasi-compact (and quasi-separated) rigid-analytic variety over \(K\) (by Remark 5.17).

Remark 5.19. Let \(X\) be a paracompact rigid-analytic variety over \(K\), and let \(\mathcal{F}\) a coherent \(\mathcal{O}_X\)-module \(\mathcal{F}\). If \(X\) has finite dimension \(n\), by [dJvdP96, Corollary 2.5.10] and Lemma 5.6(i), we have that \(H^i(X, \mathcal{F}) = 0\) for all \(i > n\).

\(^{46}\)Recall Convention 5.1.
\(^{47}\)Here, we mean \textit{paracompact} as a topological space.
\(^{48}\)We want to stress the importance of the assumptions on the base field \(K\) in this remark: it would not hold for a base field with uncountable residue field, e.g. \(K((t))\), (see [LvdP95, Proposition 4.3] for a counterexample).
5.3. Solid base change. We are ready to state and prove the main result of this section.

**Theorem 5.20.** Let $X$ be a connected, paracompact, rigid-analytic variety defined over $K$. Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves of topological $K$-vector spaces whose terms are coherent $\mathcal{O}_X$-modules. Let $A$ be a $K$-Fréchet algebra, regarded as a condensed $K$-algebra. Then, the natural map $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet \otimes^\mathbb{L}_K A)$ induces a quasi-isomorphism in $D(\text{Vect}^\text{solid}_K)$

$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A \xrightarrow{\sim} R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A).$$

**Proof.** First, we note that, by Proposition A.12, $A$ is a quasi-separated solid $K$-algebra, and in particular, by Corollary A.10, $A$ is flat as a solid $K$-vector space. Moreover, the natural map $R\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A)$ induces a morphism

$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A \to R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A).$$

in $D(\text{Vect}^\text{solid}_K)$. In fact, by Remark 5.7(iii), the complex $R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A)$ lies in $D(A, \mathcal{M}_A)$.

Now, we show that (5.2) is a quasi-isomorphism. We suppose first that $X$ is affinoid, and consider the hypercohomology spectral sequence

$$E_1^{ij} = H^j(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A) \Rightarrow H^{i+j}(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A).$$

By Lemma 5.6(ii), we have that $E_1^{ij} = 0$ for $j > 0$. Hence, the spectral sequence degenerates and it gives the quasi-isomorphism

$$\mathcal{F}^\bullet(X) \otimes_K^\mathbb{L} A \simeq R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A).$$

Since $R\Gamma(X, \mathcal{F}^\bullet) = \mathcal{F}^\bullet(X)$, recalling that $A$ is a flat solid $K$-vector space, we have that

$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A = \mathcal{F}^\bullet(X) \otimes_K^\mathbb{L} A = \mathcal{F}^\bullet(X) \otimes_K^\mathbb{L} A. \quad (5.3)$$

Thus, putting together (5.3) and (5.4), we have shown that (5.2) is a quasi-isomorphism for $X$ affinoid. For $X$ quasi-compact (and quasi-separated), we can reduce to the affinoid case by covering $X$ by a finite number of admissible affinoid subspaces. In general, for $X$ as in the statement, by Remark 5.16(i), we can choose a quasi-compact admissible covering $\{U_n\}_{n \in \mathbb{N}}$ of $X$ such that $U_n \subseteq U_{n+1}$. By Theorem A.20, each complex $R\Gamma(U_n, \mathcal{F}^\bullet)$ is representable by a complex of nuclear $K$-vector spaces; in fact, for $V \in \mathcal{B}_\text{affan}$, $R\Gamma(V, \mathcal{F}^\bullet)$ is representable by a complex of $K$-Banach spaces; for $U$ a quasi-compact admissible open of $X$, which is the colimit of a finite, full and complete subcategory $\{V_i\}_{i \in I}$ of $\mathcal{B}_\text{affan}$, we have that $R\Gamma(U, \mathcal{F}^\bullet) = R\lim_i R\Gamma(V_i, \mathcal{F}^\bullet)$, and the claim follows from the Bousfield-Kan formula for the derived limits.\footnote{For $U$ an admissible open of $X$ we denote $R\Gamma(U, \mathcal{F}^\bullet) := R\Gamma(U, \mathcal{F}^\bullet|_U)$.}

Then, by Corollary A.24, we have

$$R\Gamma(X, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A = R\lim_n (R\Gamma(U_n, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A) \simeq R\lim_n R\Gamma(U_n, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A) = R\Gamma(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A)$$

which is what we wanted. \qed

**Corollary 5.21.** Under the hypotheses of Theorem 5.20, we have the isomorphism in $\text{Vect}^\text{solid}_K$

$$H^i(X, \mathcal{F}^\bullet) \otimes_K^\mathbb{L} A \cong H^i(X, \mathcal{F}^\bullet \otimes_K^\mathbb{L} A)$$

for all $i \in \mathbb{Z}$.

\footnote{See [BK72, Chapter XI], or [Sch20b, Appendix to Lecture VIII] for a more recent reference.}
Proof. We want to show that, for all $i \in \mathbb{Z}$, we have $H^i(\Gamma(X, F^\bullet) \otimes^{\mathbb{L}}_K A) = H^i(X, F^\bullet) \otimes^K A$. Considering the spectral sequence

$$E_2^{ij} = H^j(H^i(X, F^\bullet) \otimes^{\mathbb{L}}_K A) \implies H^{i+j}(\Gamma(X, F^\bullet) \otimes^{\mathbb{L}}_K A),$$

it suffices to note that $H^i(X, F^\bullet) \otimes^{\mathbb{L}}_K A$ is concentrated in degree 0, recalling that, by Corollary A.10, $A$ is a flat solid $K$-vector space. \qed

Remark 5.22. Let $X$ be a smooth affinoid over $K$. We note that, taking for example $F^\bullet = \Omega^\bullet_x$ and $A = K$, Corollary 5.21 is trivially false, in general, if we work instead in the category of locally convex $K$-vector spaces, with the completed projective tensor product $\hat{\otimes}_K$,\footnote{See Footnote 80 for a reminder about the definition of $\hat{\otimes}_K$.} and put on $H^i_{\text{dR}}(X)$ its naturally locally convex quotient topology. In fact, $H^i_{\text{dR}}(X) \hat{\otimes}_K K$ is the Hausdorff completion of $H^i_{\text{dR}}(X)$, by definition of $\hat{\otimes}_K$, but $H^i_{\text{dR}}(X)$ can be non-Hausdorff (cf. Remark 5.14).

6. Pro-étale cohomology of $\mathbb{B}_{\text{dR}}$ and $\mathbb{B}_+^{\text{dR}}$

In this section, we study the geometric pro-étale cohomology of $\text{Fil}^r \mathbb{B}_{\text{dR}}$ on any connected, paracompact, smooth rigid-analytic variety defined over $K$. As a corollary, we also show that the geometric pro-étale cohomology of $\mathbb{B}_{\text{dR}}$ satisfies the axioms of Schneider-Stuhler (3.4).

Throughout this section, we maintain the notations and conventions of §4 and §5. In particular, all rigid-analytic varieties will be assumed to be quasi-separated (Convention 5.1).

Let $X$ be a smooth rigid-analytic variety over $K$. Given a filtered $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^*)$, [Sch13b, Definition 7.4], we denote by

$$d\text{R}^\mathcal{E}_X := (\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^n_X \xrightarrow{\nabla} \cdots)$$

its de Rham complex, which we equip with the filtration given by

$$\text{Fil}^r d\text{R}^\mathcal{E}_X := (\text{Fil}^r \mathcal{E} \xrightarrow{\nabla} \text{Fil}^{r-1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \text{Fil}^{r-m} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^m_X \xrightarrow{\nabla} \cdots)$$

for $r \in \mathbb{Z}$. Then, we give the following definitions.

**Definition 6.1.** We define the complex of $D(\text{Vect}^\text{cond}_K)$

$$\text{R}\Gamma_{\text{dR}}(X, \mathcal{E}) := \text{R}\Gamma(X, d\text{R}^\mathcal{E}_X)$$

whose $i$th cohomology, for $i \geq 0$, is the condensed de Rham cohomology group $H^i_{\text{dR}}(X, \mathcal{E})$ with coefficients in $\mathcal{E}$.

In the following, we put $B^+_{\text{dR}} = B^+_{\text{dR}}(\mathcal{C}, \mathcal{O}_C)$, we denote by $B_{\text{dR}} = B_{\text{dR}}(\mathcal{C}, \mathcal{O}_C)$ Fontaine’s field of $p$-adic periods, which we equip with the filtration $\text{Fil}^r B_{\text{dR}} = t^r B^+_{\text{dR}}$, for $r \in \mathbb{Z}$, induced from the filtration of Definition 4.1(iv).

**Remark 6.2.** Note that, by Example A.17, we have $B_{\text{dR}} = \lim_{j \in \mathbb{N}} t^{-j} B^+_{\text{dR}}$, and each $t^{-j} B^+_{\text{dR}}$ is a $K$-Fréchet algebra. In particular, $B_{\text{dR}}$ is a quasi-separated condensed $K$-vector space, being a filtered colimit along injections of quasi-separated condensed $K$-vector spaces, and a flat solid $K$-vector space (recall Proposition A.12 and Corollary A.10).
Definition 6.3. We define the complex of $D(Vect_R^{\text{cond}})$

$$R\Gamma_{dR}(X_{B_{dR}}, \mathcal{E}) := R\Gamma(X, dR_X^\mathbb{E} \otimes^B_{K} B_{dR})$$

and we endow it with the filtration induced from the tensor product filtration.

Remark 6.4. By Remark 5.7, the complexes $R\Gamma_{dR}(X, \mathcal{E})$ and $R\Gamma_{dR}(X_{B_{dR}}, \mathcal{E})$ lie in $D(Vect_K^{\text{solid}})$.

We will prove the following theorem, which generalizes results of Scholze [Sch13b, Theorem 7.11], and Le Bras [LB18a, Proposition 3.17].

Theorem 6.5. Let $X$ be a smooth rigid-analytic variety defined over $K$. Let $L$ be a de Rham lisse $\mathbb{Z}_p$-sheaf on $X_{\text{pro\-â©t}}$, with associated filtered $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^\bullet)$.

Denote $M_{dR}^\mathbb{E} := L \otimes_{\mathbb{Z}_p} B_{dR}$.

(i) We have a $\mathcal{G}_K$-equivariant, compatible with filtrations, natural quasi-isomorphism in $D(Vect_K^{\text{solid}})$

$$R\Gamma_{\text{pro\-â©t}}(X_C, M_{dR}^\mathbb{E}) \simeq R\Gamma_{dR}(X_{B_{dR}}, \mathcal{E}).$$

(ii) Assume that $X$ is connected and paracompact. Then, for each $r \in \mathbb{Z}$, we have a $\mathcal{G}_K$-equivariant quasi-isomorphism in $D(Vect_K^{\text{solid}})$

$$R\Gamma_{\text{pro\-â©t}}(X_C, \text{Fil}^r M_{dR}^\mathbb{E}) \simeq \text{Fil}^r (R\Gamma_{dR}(X, \mathcal{E}) \otimes^B_{K} B_{dR}).$$

One of the main ingredients that we will use to prove Theorem 6.5 is a version of the Poincaré lemma, due to Scholze, that we now recall.

Definition 6.6. Let $X$ be a smooth analytic adic space over $\text{Spa}(K, \mathcal{O}_K)$. Let $\nu : X_{\text{pro\-â©t}} \rightarrow X_{\text{ét}}$ denote the natural morphism of sites. We define the following sheaves on $X_{\text{pro\-â©t}}$.

(i) For $m \geq 1$, the sheaf of differentials $\Omega_X^m = \nu^! \Omega_X^m_{\text{ét}}$.

(ii) We define the sheaf $\mathcal{O}_{B_{dR}}^+$ as the sheafification of the presheaf that to $Z^{\diamond} \in X_{\text{pro\-â©t}}$, with $Z = \text{Spa}(R_\infty, R_\infty^+)$, associate $X$ a map, from an affinoid perfectoid space $Z$, that can be written as a cofiltered limit of étale maps $\text{Spa}(R_i, R_i^+) \rightarrow X$, with $i \in I$, along a $\kappa$-small index category $I$, associates the following direct limit

$$\lim_{i} \lim_{n} ((R_i^+ \hat{\otimes}_{W(k)} A^\inf(R_\infty, R_\infty^+))[1/p]) / (\ker \theta)^n.$$

Here, $\hat{\otimes}$ denotes the $p$-adic completion of the tensor product, and

$$\theta : (R_i^+ \hat{\otimes}_{W(k)} A^\inf(R_\infty, R_\infty^+))[1/p] \rightarrow R_\infty$$

is the tensor product of the maps $R_i^+ \rightarrow R_\infty^+$ and $\theta : A^\inf(R_\infty, R_\infty^+) \rightarrow R_\infty$. We define a filtration on $\mathcal{O}_{B_{dR}}^+$ by setting $\text{Fil}^r \mathcal{O}_{B_{dR}}^+ = (\ker \theta)^r \mathcal{O}_{B_{dR}}^+$.

(iii) Let $t$ be a generator of $\text{Fil}^1 B_{dR}^+$. We define the sheaf $\mathcal{O}_B_{dR} = \mathcal{O}_{B_{dR}}^+[1/t]$ with filtration

$$\text{Fil}^r \mathcal{O}_B_{dR} = \sum_{j \geq 0} t^{-j} \text{Fil}^{r+j} \mathcal{O}_{B_{dR}}^+.$$

\footnote{A lisse $\mathbb{Z}_p$-sheaf $L$ on $X_{\text{pro\-â©t}}$ is de Rham if the $B_{dR}^+$-local system $M_{dR}^B := L \otimes_{\mathbb{Z}_p} B_{dR}^+$ is associated to a filtered module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^\bullet)$. See [Sch13b, Theorem 7.6, Definition 8.3].}

\footnote{Recall that the set of all such $Z^{\diamond} \in X_{\text{pro\-â©t}}$ forms a basis of $X_{\text{pro\-â©t}}$ (see e.g. [MW20, Lemma 2.6]).}
Proposition 6.7 ([Sch13b, Corollary 6.13]). Let $X$ be a smooth analytic adic space over $\text{Spa}(K, \mathcal{O}_K)$ of dimension $n$. Then, we have an exact sequence of sheaves on $X_{\text{pro}\acute{e}t}$

$$0 \to \mathcal{B}^+_{dR} \to \mathcal{O} \mathcal{B}^+_{dR} \to \mathcal{O} \mathcal{B}^+_{dR} \otimes_{O_X} \Omega_X^1 \to \cdots \to \mathcal{O} \mathcal{B}^+_{dR} \otimes_{O_X} \Omega_X^n \to 0$$

where $\nabla : \mathcal{B}^+_{dR} \to \mathcal{O} \mathcal{B}^+_{dR} \otimes_{O_X} \Omega_X^1$ is the unique $\mathcal{B}^+_{dR}$-linear connection extending the differential $d : \mathcal{O}_X \to \Omega_X^1$. Moreover, for $r \in \mathbb{Z}$, we have compatible exact sequences of sheaves on $X_{\text{pro}\acute{e}t}$

$$0 \to \text{Fil}^r \mathcal{B}_{dR} \to \text{Fil}^r \mathcal{O} \mathcal{B}_{dR} \to \text{Fil}^{r-1} \mathcal{O} \mathcal{B}_{dR} \otimes_{O_X} \Omega_X^1 \to \cdots \to \text{Fil}^{-n} \mathcal{O} \mathcal{B}_{dR} \otimes_{O_X} \Omega_X^n \to 0.$$

We will also need the following lemma.

Lemma 6.8. Let $X = \text{Spa}(R, R^+)$ be an affinoid adic space of finite type over $\text{Spa}(K, \mathcal{O}_K)$ with an étale map

$$X \to \mathbb{T}_K^n := \text{Spa}(K(T_1^{\pm 1}, \ldots, T_n^{\pm 1}), \mathcal{O}_K (T_1^{\pm 1}, \ldots, T_n^{\pm 1}))$$

that can be written as a composite of of finite étale maps and rational embeddings. Let $S \in \ast_{\kappa-\text{pro}\acute{e}t}$. Then, for any $j \in \mathbb{Z}$, we have

$$H^i_{\text{pro}\acute{e}t}(X_C \times S, \text{gr}^j \mathcal{O} \mathcal{B}_{dR}) = \begin{cases} \mathcal{E}^0(S, R \hat{\otimes}_K C(j)) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

where $(j)$ denotes a Tate twist.$^{54}$

Proof. This is [Sch13b, Proposition 6.16] in the case $S = \ast$. For the general case, it suffices to slightly expand the argument of loc. cit. as follows. By twisting, we can suppose $j = 0$. Denote by

$$\mathbb{T}_C^n := \text{Spa}(C(T_1^{\pm 1/p^{\infty}}, \ldots, T_n^{\pm 1/p^{\infty}}), \mathcal{O}_C (T_1^{\pm 1/p^{\infty}}, \ldots, T_n^{\pm 1/p^{\infty}}))$$

the affinoid perfectoid $n$-torus over $C$, and let $\tilde{X}_C := X_C \times_{\mathbb{T}_C^n} \mathbb{T}_C^n = \text{Spa}(R_\infty, R^+_{\infty})$. We recall that $\mathbb{T}_C^n \to \mathbb{T}_C^n$ is a pro-(finite étale) $\mathbb{Z}_p/(1)^n$-cover. Then, since a version of Proposition 4.5 and Corollary 4.7 holds true for the pro-étale sheaf $\text{gr}^0 \mathcal{O} \mathcal{B}_{dR}$, considering the Cartan-Leray spectral sequence associated to the affinoid perfectoid $\mathbb{Z}_p/(1)^n$-cover $\tilde{X}_C \times S \to X_C \times S$, we have, for $i \geq 0$,

$$H^i_{\text{pro}\acute{e}t}(X_C \times S, \text{gr}^0 \mathcal{O} \mathcal{B}_{dR}) \cong H^i_{\text{cont}}(\mathbb{Z}_p, \text{gr}^0 \mathcal{O} \mathcal{B}_{dR}(\tilde{X}_C \times S)).$$

We note that $\tilde{X}_C \times S = \text{Spa}(\mathcal{E}^0(S, R_\infty), \mathcal{E}^0(S, R^+_{\infty}))$. Now, the proof of loc. cit. shows that $H^i_{\text{pro}\acute{e}t}(X_C \times S, \text{gr}^0 \mathcal{O} \mathcal{B}_{dR}) = 0$ if $i > 0$, and it is isomorphic to $\mathcal{E}^0(S, R) \hat{\otimes}_K C$ if $i = 0$. Then, we obtain the statement observing that $\mathcal{E}^0(S, R) \hat{\otimes}_K C = \mathcal{E}^0(S, R \hat{\otimes}_K C)$ (which follows from [PGS10, Corollary 10.5.4]).

In the following statement, we will keep using the notations introduced in §2 (in particular, see Remark 2.10). Moreover, given $X$ a rigid-analytic variety defined over $K$, $\mathcal{F}$ an $\mathcal{O}_X_{\text{ét}}$-module over $X_{\text{ét}}$ that is locally free of finite rank, and $A$ a $K$-algebra such that $A$ is a flat solid $K$-vector space,$^{56}$

$^{54}$See Footnote 80 for the definition of completed projective tensor product $\hat{\otimes}_K$.

$^{55}$Here, $\mathcal{E}^0(S, R)$ is endowed with the sup-norm.

$^{56}$Our main cases of interest will be $A \in \{ t^m B^+_dR/t^n B^+_dR : -\infty \leq m < n \leq \infty \}$. 

in addition to Definition 5.5, by abuse of notation we will also denote by $\mathcal{E} \otimes_k^\bullet A$ the sheaf with
values in $\text{Vect}_K^\text{solid} \subset \text{Vect}_K^\text{cond}$ regarded on $X_{C,\text{ét}}$ via the equivalence of topoi\footnote{As observed in [LB18a, §3.2], this equivalence follows from Elkik’s approximation theorem, [Elk73]. Compare with [ILZ17, Lemma 2.5].}
\begin{equation}
X_{C,\text{ét}} \cong \lim_{K'/K \text{ finite}} X_{K',\text{ét}}.
\end{equation}

**Corollary 6.9.** Let $X$ be a smooth rigid-analytic variety over $K$. Let $(\mathcal{E}, \nabla, \text{Fil}^*)$ be a filtered
$\mathcal{O}_X$-module with integrable connection, with associated $B_{\text{dR}}^+$-local system\footnote{See [Sch13b, Theorem 7.6].}
$M_{\text{dR}}^+ := \text{Fil}^0(\mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_{\mathcal{B}_{\text{dR}}}^\nabla)_{\nabla = 0}$
and let $M_{\text{dR}} := M_{\text{dR}}^+[1/t]$. Let us denote by $\lambda : X_{\text{proét}}/X_C \cong X_{C,\text{proét}} \to X_{C,\text{ét},\text{cond}}$ the natural
morphism of sites. Then, we have a natural quasi-isomorphism of complexes of sheaves on $X_{C,\text{ét}}$
with values in $\text{Vect}_K^\text{cond}$ which is compatible with filtrations
\begin{equation}
(R\lambda_* M_{\text{dR}})^\nabla \cong \text{dR}^\nabla_X \otimes_k B_{\text{dR}}
\end{equation}
where the right-hand side is endowed with the tensor product filtration.

**Proof.** We may assume that $X$ is connected of dimension $n$. Then, from Proposition 6.7, we have an exact sequence of sheaves on $X_{\text{proét}} \cong X_{\text{proét}}/X_C$
\begin{equation*}
0 \to M_{\text{dR}} \to \mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_{\mathcal{B}_{\text{dR}}} \to \mathcal{E} \otimes \Omega_X \otimes \mathcal{O}_\mathcal{B}_{\text{dR}} \to \cdots \mathcal{E} \otimes \Omega^n_X \otimes \mathcal{O}_\mathcal{B}_{\text{dR}} \to 0
\end{equation*}
which remains exact after taking $\text{Fil}^r$. In particular, this induces a quasi-isomorphism between $(R\lambda_* M_{\text{dR}})^\nabla$ and the complex\footnote{PN stands for “Poincaré.”}
\begin{equation}
P_{N}^\nabla_X := R\lambda_* (\mathcal{E} \otimes \mathcal{O}_{\mathcal{B}_{\text{dR}}} \to \mathcal{E} \otimes \Omega_X^1 \otimes \mathcal{O}_\mathcal{B}_{\text{dR}} \to \cdots \mathcal{E} \otimes \Omega^n_X \otimes \mathcal{O}_\mathcal{B}_{\text{dR}})^\nabla
\end{equation}
which is compatible with the natural filtrations. Now, we construct a natural morphism
\begin{equation}
\text{dR}^\nabla_X \otimes_k B_{\text{dR}} \to P_{N}^\nabla_X
\end{equation}
of filtered complexes of sheaves on $X_{C,\text{ét}}$, which we claim to be a quasi-isomorphism. We note that
it suffices to exhibit a natural morphism $\mathcal{O}_X \otimes_k^\bullet B_{\text{dR}} \to (\lambda_* \mathcal{O}_{\mathcal{B}_{\text{dR}}})^\nabla$ of sheaves on $X_{C,\text{ét}}$ (with values
in $\text{Vect}_K^\text{cond}$) that is compatible with filtrations. Let $S \in *_{\kappa-\text{proét}}$ be an extremally disconnected
set, let $K'/K$ be a finite extension with residue field $k'$, and let $V = X_{K'}$ be an étale morphism
with $V = \text{Spa}(R, R^+)$ affinoid. Moreover, let $V_C = \text{Spa}(R_{\infty}, R^+_{\infty}) \to V_C$ be a map from an affinoid
perfectoid space, that can be written as a cofiltered limit of étale maps $\text{Spa}(R_i, R^+_i) \to V_C$, $i \in I$,
along a $\kappa$-small index category $I$. We observe that
\begin{align*}
(\mathcal{O}_X \otimes_k^\bullet B_{\text{dR}})(V)(S) &= \lim_{j \in \mathbb{N}} (R \otimes_{K'} t^{-j} B_{\text{dR}}^+)(S) \\
&= \lim_{j \in \mathbb{N}} \mathcal{E}^0(S, R \otimes_{K'} t^{-j} B_{\text{dR}}^+) \\
&= \lim_{j \in \mathbb{N}} \mathcal{E}^0(S, R \otimes_{K'} t^{-j} B_{\text{dR}}^+) \\
&= \left( \lim_{n \in \mathbb{N}} \left( \mathcal{E}^0(S, R^+) \otimes_{W(k')} \mathcal{O}_{\mathcal{B}_{\text{dR}}}(1/p) \right) / \xi^n \right) [1/t]
\end{align*}
where in the first step we used Remark 6.2 together with the fact that the tensor product \( \otimes^K \) commutes with colimits, in the second one we used that filtered colimits of condensed \( K \)-vector spaces can be computed pointwise on extremally disconnected sets, and Proposition A.25, and, finally, in the third step we used [PGS10, Corollary 10.5.4]. Therefore, recalling Definition 6.6, we deduce that we have a natural map

\[
(\mathcal{O}_X \otimes^K B_{\text{dR}})(V)(S) \to \mathcal{O}_{B_{\text{dR}}}(\widetilde{V}_C \times S)
\]

since \( \widetilde{V}_C \times S = \text{Spa}(\mathcal{E}^0(S, R_{\infty}), \mathcal{E}^0(S, R_{\infty})) \). Then, one checks that the latter map descends to a natural map with target \( \mathcal{O}_{B_{\text{dR}}}(V_C \times S) \), and that it gives the desired morphism of sheaves.

Now, observing that the filtration on both \( dR_X^\varepsilon \otimes^K B_{\text{dR}} \) and \( PN_X^\varepsilon \) is separated and exhaustive,\(^{60} \) it suffices to show that the natural morphism \( (6.3) \) is a quasi-isomorphism on graded pieces, i.e. that, for all \( j \in \mathbb{Z} \), the induced morphism

\[
\text{gr}^j(dR_X^\varepsilon \otimes^K B_{\text{dR}}) \to \text{gr}^j PN_X^\varepsilon
\]

is a quasi-isomorphism. Since the morphism \( (6.3) \) is also compatible with respect to the naive filtration on both sides, it suffices to show that, for any \( \mathcal{O}_{X_{\text{ét}}} \)-module \( \mathcal{F} \) over \( X_{\text{ét}} \) that is locally free of finite rank, and for all \( j \in \mathbb{Z} \), the natural morphism

\[
\mathcal{F} \otimes^K \text{gr}^j B_{\text{dR}} \xrightarrow{\sim} R\lambda_* (\nu^* \mathcal{F} \otimes_{\mathcal{O}_X} \text{gr}^j \mathcal{O}_{B_{\text{dR}}})^\vee
\]

(6.4)

is a quasi-isomorphism of complexes of sheaves on \( X_{C,\text{ét}} \). Let us consider the commutative diagram of morphisms of sites

\[
\begin{array}{ccc}
X_{C,\text{proét}} & \xrightarrow{\lambda} & X_{C,\text{ét},\text{cond}} \xrightarrow{\pi} X_{C,\text{ét}} \\
\uparrow{\varepsilon'} & & \downarrow{\varepsilon} \\
X_{\text{proét}} & \xrightarrow{\nu} & X_{\text{ét}}
\end{array}
\]

where \( \pi \) is the natural projection and \( \varepsilon, \varepsilon' \) are the base change morphisms. Let \( \nu := \varepsilon \circ \pi \), then, by the projection formula, for all \( j \in \mathbb{Z} \), we have\(^{61} \)

\[
R\lambda_* (\nu^* \mathcal{F} \otimes_{\mathcal{O}_X} \text{gr}^j \mathcal{O}_{B_{\text{dR}}}) = \nu^* \mathcal{F} \otimes_{\nu^* \mathcal{O}_X} R\lambda_* (\text{gr}^j \mathcal{O}_{B_{\text{dR}}}).
\]

Now, note that we can check (6.4) locally on \( X_{C,\text{ét}} \). Since \( X \) is smooth, by [Sch13b, Lemma 5.2], we can assume that \( X \) is affinoid and there exists an étale map \( X \to T^n_K \) that can be written as a composite of finite étale maps and rational embeddings. By Lemma 6.8 and Proposition A.25, we have, for all \( j \in \mathbb{Z} \), \( R\lambda_* (\text{gr}^j \mathcal{O}_{B_{\text{dR}}})^\vee = \mathcal{O}_X \otimes^K C(j) \). By twisting we can assume \( j = 0 \), hence, it remains to show that we have

\[
(\nu^* \mathcal{F})^\vee \otimes_{(\nu^* \mathcal{O}_X)^\vee} (\mathcal{O}_X \otimes^K C) = \mathcal{F} \otimes^K C
\]

as sheaves on \( X_{C,\text{ét}} \).\(^{62} \) For this, we first observe that, by Proposition A.25, we have \( \mathcal{O}_X \otimes^K C = \mathcal{O}_{XC} \) and \( \mathcal{F} \otimes^K C = FC \). Then, let \( S \in *_{\text{proét}} \), be an extremally disconnected set, let \( K'/K \) be a finite

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\(^{60}\) The filtration on \( dR_X^\varepsilon \otimes^K B_{\text{dR}} \) is separated since the filtration on \( B_{\text{dR}} \) has such property. To deduce this we can use Corollary A.24, recalling that \( B_{\text{dR}} \) is filtered by \( K \)-Fréchet spaces, which are nuclear \( K \)-vector spaces by Corollary A.22.

\(^{61}\) Here, the sheaf \( \nu^* \mathcal{F} \otimes_{\mathcal{O}_X} \text{gr}^j \mathcal{O}_{B_{\text{dR}}} \) is regarded on \( X_{\text{proét}}/XC \cong X_{C,\text{proét}} \), hence it should be read as the sheaf \( \nu^* (\nu^* \mathcal{F} \otimes_{\nu^* \mathcal{O}_X} \text{gr}^j \mathcal{O}_{B_{\text{dR}}}) \).

\(^{62}\)  Note that, by Lemma A.15, for all \( j \in \mathbb{Z} \), we have \( C(j) = t^j B_{\text{dR}}/t^{j+1} B_{\text{dR}} = \text{gr}^j B_{\text{dR}} \).
extension, let \( V \to X_{K'} \) be an étale morphism with \( V = \text{Spa}(R, R^+) \) affinoid, and let \( M \) be the projective module of finite rank over \( R \) corresponding to \( \mathcal{F}_{K'}|_V \) (endowed with its natural \( K' \)-Banach space structure). Then,

\[
((\nu^*\mathcal{F})\otimes_{(\nu^*\mathcal{O}_X)\cdot\mathcal{O}_{X_C}})(VC) = \lim_{L/K' \text{ finite}} (M_L\otimes_{R_L}\mathcal{C}^0(S, R\hat{\otimes}_{K'} C)) = \mathcal{C}^0(S, M\hat{\otimes}_{K'} C) = (\mathcal{F}_C)(VC)(S)
\]

which is what we wanted. \( \square \)

Before proving Theorem 6.5, we need another preliminary result (cf. [LZ17, Lemma 3.1]).

**Lemma 6.10.** Let \( X \) be an affinoid space defined over \( K \), and let \( \mathcal{F} \) be an \( \mathcal{O}_{X_{\text{ét}}} \)-module over \( X_{\text{ét}} \) that is locally free of finite rank. Then, for \( A \in \{t^nB^+_{\text{dR}}/t^nB^+_{\text{dR}} : -\infty \leq m < n \leq \infty \} \), we have

\[
H^i_{\text{ét}}(X_C, \mathcal{F} \otimes_K A) = \begin{cases} 
\mathcal{F}(X) \otimes_K A & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

**Proof.** By twisting, it suffices to prove the statement for \( A \in \{B_{\text{dR}}, B^+_{\text{dR}}, B^+_{\text{dR}}/t^j : j \geq 1\} \).

Let us start from \( A = B^+_{\text{dR}}/t = C \). By Proposition A.25, we have \( \mathcal{F} \otimes_K C = \mathcal{F}_C \) as sheaves on \( X_{C,\text{ét}} \), and the result simply follows from the acyclicity of \( \mathcal{F}_C \) (cf. Lemma 5.6). In order to prove the statement for \( A = B^+_{\text{dR}}/t^j \), \( j \geq 1 \), we can proceed by induction on \( j \), using the following exact sequence of sheaves on \( X_{C,\text{ét}} \):

\[
0 \to \mathcal{F} \otimes_K C(j) \to \mathcal{F} \otimes_K (B^+_{\text{dR}}/t^{j+1}) \to \mathcal{F} \otimes_K (B^+_{\text{dR}}/t^j) \to 0.
\]

For \( A = B^+_{\text{dR}} \), observing that \( B^+_{\text{dR}} = \lim_{j \in \mathbb{N}} B^+_{\text{dR}}/t^j \), we deduce the statement from [Sch13b, Lemma 3.18].

Finally, for \( A = B_{\text{dR}} \), it suffices to recall that \( B_{\text{dR}} = \lim_{j \in \mathbb{N}} t^{-j}B^+_{\text{dR}} \) and use that, since \( |X_C| \) is quasi-compact and quasi-separated, the cohomology commutes with direct limit of sheaves on \( X_{C,\text{ét}} \) with values in CondAb. \( \square \)

**Proof of Theorem 6.5.** The natural morphism of sites \( \lambda : X_{C,\text{proét}} \to X_{C,\text{ét},\text{cond}} \) factors as

\[
X_{C,\text{proét}} \xrightarrow{\mu} X_{C,\text{proét},\text{cond}} \to X_{C,\text{ét},\text{cond}}.
\]

Then, by Lemma 2.12, we have

\[R\Gamma_{\text{proét}}(X_C, M_{\text{dR}}) \simeq R\Gamma_{\text{proét},\text{cond}}(X_C, R\mu_*M_{\text{dR}}) \simeq R\Gamma_{\text{ét},\text{cond}}(X_C, R\lambda_*M_{\text{dR}}) = R\Gamma_{\text{ét}}(X_C, (R\lambda_*M_{\text{dR}})^\vee).\]

Let \( \varepsilon : X_{C,\text{ét}} \to X_{\text{ét}} \) be the base change morphism. By Corollary 6.9, Lemma 6.10, and Lemma 5.6(ii), we have a natural quasi-isomorphism which is compatible with filtrations

\[R\Gamma_{\text{proét}}(X_C, M_{\text{dR}}) \simeq R\Gamma_{\text{ét}}(X, R\varepsilon_*(R\lambda_*M_{\text{dR}})^\vee) \simeq R\Gamma_{\text{ét}}(X, dR^\varepsilon_X \otimes_K B_{\text{dR}}) \simeq R\Gamma_{\text{an}}(X, dR^\varepsilon_X \otimes_K B_{\text{dR}}).
\]

This finishes the proof of part (i). For part (ii), we assume that \( X \) is connected and paracompact. It remains to show that, for each \( r \in \mathbb{Z} \), the natural morphism

\[\text{Fil}^r(R\Gamma_{\text{dR}}(X, E) \otimes_K L^r B_{\text{dR}}) \to R\Gamma(X, \text{Fil}^r(dR^\varepsilon_X \otimes_K B_{\text{dR}}))\]

\[^{63}\text{The exactness of this sequence can be checked using Corollary A.10.}\]

\[^{64}\text{We note that loc. cit. is stated for sheaves on a site with values in abelian groups, however it holds also for sheaves with values in CondAb.}\]
is a quasi-isomorphism. This follows from Theorem 5.20, observing that the complex $dR_X^\xi$ is bounded (since $X$ has finite dimension), and the period ring $B_{dR}$ is filtered by $K$-Fréchet algebras.

Let us make some comments on Theorem 6.5 and deduce some corollaries.

Remark 6.11. Let $X$ be a quasi-compact (and quasi-separated) smooth rigid-analytic variety defined over $K$. Retaining the notation of Theorem 6.5, we have the following $G_K$-equivariant quasi-isomorphism in $D(Vect^\text{solid}_K)$

$$RT_{\text{prof}}(X_C, \mathcal{M}_{dR}) = \colim_{j \in \mathbb{N}} RT_{\text{prof}}(X_C, \text{Fil}^{-j} \mathcal{M}_{dR})$$

$$\simeq \colim_{j \in \mathbb{N}} \text{Fil}^{-j}(RT_{dR}(X, \mathcal{E}) \otimes_{K} B_{dR})$$

$$= RT_{dR}(X, \mathcal{E}) \otimes_{K} B_{dR}$$

where in the first step we used that $|X_C|$ is quasi-compact and quasi-separated, in the second one Theorem 6.5(ii), and in the third step we used that $\otimes_K$ commutes with colimits, together with the fact that $\text{Vect}^\text{solid}_K$ satisfies Grothendieck’s axiom (AB5). In particular, in this case, for all $i \geq 0$, we have a $G_K$-equivariant isomorphism in $\text{Vect}^\text{solid}_K$

$$H^i_{\text{prof}}(X_C, \mathcal{M}_{dR}) \cong H^i_{dR}(X, \mathcal{E}) \otimes_{K} B_{dR}. \quad (6.5)$$

as it follows recalling that, by Remark 6.2, $B_{dR}$ is a flat solid $K$-vector space (see also the proof of Corollary 5.21).

Remark 6.12. Let $X$ be a smooth proper rigid-analytic variety over $K$. For any filtered $\mathcal{O}_X$-module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^*)$, by the finiteness of coherent cohomology of proper rigid-analytic varieties, [Kie67a], the same argument of Lemma 5.11 shows that $H^i_{dR}(X, \mathcal{E}) = H^i_{dR}(X, \mathcal{E})$. Then, in this case, the right-hand side of (6.5) is given by $H^i_{dR}(X, \mathcal{E}) \otimes_{K} B_{dR}$. Hence, the isomorphism (6.5) recovers [Sch13b, Theorem 7.11].

Remark 6.13. Let $X$ be a smooth affinoid or Stein space over $K$. By Theorem 6.5(ii) for $\mathcal{E} = \mathcal{O}_X$, recalling that $K$-Fréchet spaces are flat solid $K$-vector spaces (by Proposition A.12 and Corollary A.10), for $i \geq 0$, we have that

$$H^i_{\text{prof}}(X_C, \mathbb{B}^+_\text{dR}) = \colim_j \left(H^i(X, \Omega_X^{\geq j}) \otimes_{K} t^{-j} B^+_\text{dR} \right).$$

We note that $H^i(X, \Omega_X^{\geq j})$ vanishes for $i < j$, moreover, $H^i(X, \Omega_X^{\geq i}) = \Omega^i(X)^{d=0}$ and, by Lemma 5.6 and Lemma 5.9, respectively, we have $H^i(X, \Omega_X^{\geq j}) = H^i_{dR}(X)$ for $i > j$. Therefore, we have the following $G_K$-equivariant exact sequence in $\text{Vect}^\text{solid}_K$

$$0 \to H^i_{dR}(X) \otimes_{K} t^{-i} B^+_\text{dR} \to H^i_{\text{prof}}(X_C, \mathbb{B}^+_\text{dR}) \to \Omega^i(X)^{d=0} \otimes_{K} C(-i) \to 0. \quad (6.6)$$

In the case $X$ is a smooth Stein space, by Lemma 5.13, we have that $H^i_{dR}(X) = H^i_{dR}(X),^{65}$ where $H^i_{dR}(X)$ is endowed with its natural structure of $K$-Fréchet space as in Remark 5.12. Then, by Proposition A.25, the exact sequence (6.6) can be stated in classical topological terms, with the completed projective tensor product replacing the solid tensor product.

^{65}Let us remark that we don’t know whether, in the smooth Stein case and with non-trivial coefficients, the condensed de Rham cohomology group $H^i_{dR}(X, \mathcal{E})$ comes from a topological $K$-vector space.
Using Theorem 6.5, one can express the geometric pro-étale cohomology with coefficients in $\mathbb{B}_{dR}/\mathbb{B}_{dR}^+$, appearing in the fundamental exact sequence of $p$-adic Hodge theory (4.3), in terms of differential forms. In the following special cases, this takes a particularly nice form, which also illustrates how, for a smooth rigid-analytic variety $X$ over $K$, the (non-)degeneration of the Hodge-de Rham spectral sequence is reflected in its geometric $p$-adic pro-étale cohomology.

**Corollary 6.14.** Let $X$ be a smooth rigid-analytic variety over $K$. Let $i \geq 0$.

(i) If $X$ is proper, we have a $\mathcal{G}_K$-equivariant isomorphism in $\text{Vect}_K^{\text{solid}}$

\[ H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+) \cong (H^i_{\text{dR}}(X) \otimes_K \mathbb{B}_{dR})/\text{Fil}^0. \]

(ii) If $X$ is an affinoid space, we have the following $\mathcal{G}_K$-equivariant exact sequence in $\text{Vect}_K^{\text{solid}}$

\[ 0 \to H^i_{\text{dR}}(X) \otimes_K \mathbb{B}_{dR}/t^{-i}B^+_{dR} \to H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to \Omega^i(X)/\ker d \otimes_K C(-i-1) \to 0. \]

**Proof.** Let $f : X_C \to \text{Spa}(C, \mathcal{O}_C)$ denote the structure morphism, and let us consider the long exact sequence associated to the distinguished triangle $Rf_{\text{pro\acute{e}t}} \mathbb{B}_{dR}^+ \to Rf_{\text{pro\acute{e}t}} \mathbb{B}_{dR} \to Rf_{\text{pro\acute{e}t}} \mathbb{B}_{dR}/\mathbb{B}_{dR}^+$, i.e.

\[ \cdots \to H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}^+) \to H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}) \to H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+) \to \cdots \]

which gives the short exact sequence

\[ 0 \to \text{coker} \alpha_i \to H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+) \to \ker \alpha_{i+1} \to 0. \quad (6.7) \]

If $X$ is proper, recalling Remark 6.12, by the degeneration of the Hodge-de Rham spectral sequence at the first page, [Sch13b, Corollary 1.8], we have that $\ker \alpha_i = 0$, hence part (i).

If $X$ is an affinoid space, by Remark 6.11 and Remark 6.13, we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 \to H^i_{\text{dR}}(X) \otimes_K t^{-i+1}B^+_{dR} & \to & H^i_{\text{pro\acute{e}t}}(X_C, \mathbb{B}_{dR}^+) & \to & \Omega^i(X)_{d=0} \otimes_K C(-i) & \to & 0 \\
\| & & \downarrow \alpha_i & & & & \downarrow \beta_i \\
0 \to H^i_{\text{dR}}(X) \otimes_K t^{-i+1}B^+_{dR} & \to & H^i_{\text{dR}}(X) \otimes_K B_{dR} & \to & H^i_{\text{dR}}(X) \otimes_K B_{dR}/t^{-i+1}B^+_{dR} & \to & 0
\end{array}
\]

where the top right arrow is given by the quotient by $t^{-i+1}B^+_{dR}$, and the morphism $\beta_i$ is given by the natural projection $\Omega^i(X)_{d=0} \to H^i_{\text{dR}}(X)$ on the first factor and the inclusion on the second one. By the snake lemma, we have

\[
\ker \alpha_i = \ker \beta_i = d\Omega^{i-1}(X) \otimes_K C(-i),
\]

\[
\text{coker} \alpha_i = \text{coker} \beta_i = H^i_{\text{dR}}(X) \otimes_K B_{dR}/t^{-i}B^+_{dR}
\]

that, together with the short exact sequence (6.7), imply part (ii). \Box

From Theorem 6.5 we can also deduce the following result.

**Proposition 6.15.** The cohomology theories

\[ H^\bullet_{\text{dR}} : \text{RigSm}_{K, \text{an}} \to \text{Vect}^{\text{cond}}_K, \quad H^\bullet_{\text{pro\acute{e}t}}(-, \mathbb{B}_{dR}) : \text{RigSm}_{K, \text{pro\acute{e}t}} \to \text{Mod}^{\text{cond}}_{B_{dR}} \]

satisfy the axioms of Schneider-Stuhler (3.4).
Proof. The condensed de Rham cohomology $H^\bullet_{\text{dR}}$ is defined by the complex of sheaves $\Omega^\bullet$ on the site $\text{RigSm}_{K,\text{an}}$. By Corollary 4.7, $H^\bullet_{\text{proét}}(-C, \mathbb{B}_{\text{dR}})$ is defined by the complex of sheaves $R\varepsilon_*\varepsilon^*\mathbb{B}_{\text{dR}}$ on the site $\text{RigSm}_{K,\text{proét}}$, where we denote by $\varepsilon : \text{RigSm}_{C,\text{proét}} \to \text{RigSm}_{K,\text{proét}}$ the base change morphism.

We begin by proving axiom (1). Let $X$ be a smooth affinoid space defined over $K$. For the condensed de Rham cohomology, we need to show that the natural map

$$\Omega^\bullet(X) \to \Omega^\bullet(X \times \hat{D}_K)$$

is a quasi-isomorphism. From the knowledge of axiom (1) for the (classical) de Rham cohomology (see the discussion after [SS91, §1, Lemma 2]), we know that $\Omega^\bullet(X) \simeq \Omega^\bullet(X \times \hat{D}_K)$ is a quasi-isomorphism of complexes of $K$-Fréchet spaces; then, we can conclude using Lemma A.15. Now, axiom (1) for $H^\bullet_{\text{proét}}(-C, \mathbb{B}_{\text{dR}})$ follows observing that, writing the 1-dimensional open unit disk $\hat{D}$ over $\mathbb{Q}_p$ as a strictly increasing admissible union of closed disks $\{D_j\}_{j \in \mathbb{N}}$ of radius in $p^0$, we have

$$RT\Gamma_{\text{proét}}(X \times \hat{D}_C, \mathbb{B}_{\text{dR}}) = R\lim_{\longleftarrow j} RT\Gamma_{\text{proét}}(X \times D_{j,C}, \mathbb{B}_{\text{dR}})$$

$$\simeq R\lim_{\longleftarrow j} (RT\Gamma_{\text{dR}}(X \times D_{j,K}) \otimes_{K^1} \mathbb{B}_{\text{dR}})$$

$$\simeq R\lim_{\longleftarrow j} (RT\Gamma_{\text{dR}}(X \times \hat{D}_{j,K}) \otimes_{K^1} \mathbb{B}_{\text{dR}})$$

$$(6.9)$$

$$\simeq RT\Gamma_{\text{dR}}(X) \otimes_{K^1} \mathbb{B}_{\text{dR}}$$

$$(6.10)$$

$$\simeq RT\Gamma_{\text{proét}}(X, \mathbb{B}_{\text{dR}})$$

$$(6.11)$$

where in (6.8) and (6.11) we used Remark 6.11, in (6.9) we used that, for each $j \in \mathbb{N}$, the morphism $RT\Gamma_{\text{dR}}(X \times D_{j+1,K}) \to RT\Gamma_{\text{dR}}(X \times D_{j,K})$ factors through $RT\Gamma_{\text{dR}}(X \times \hat{D}_{j+1,K})$, and in (6.10) we used axiom (1) for the condensed de Rham cohomology.

Axioms (2) and (3) are satisfied. In order to verify axiom (4), we first need to construct the respective cycle class maps for the cohomology theories in the statement; we observe that it suffices to do this at the level of the complexes of abelian sheaves underlying the respective defining complexes of sheaves (having values in condensed abelian groups), i.e. it suffices to define $c^{\text{dR}} : \mathbb{G}_m[-1] \to \Omega^\bullet$ and $c^{B\text{dR}} : \mathbb{G}_m[-1] \to R\varepsilon_*\varepsilon^*\mathbb{B}_{\text{dR}}$. The cycle class map $c^{\text{dR}}$ is defined on $\text{RigSm}_{K,\text{an}}$ by

$$c^{\text{dR}} : \mathbb{G}_m[-1] \xrightarrow{d\log} \Omega^{\geq 1} \to \Omega^\bullet,$$  

$$c^{\text{dR}} : \mathbb{G}_m[-1] \xrightarrow{d\log} \Omega^{\geq 1} \to \Omega^\bullet,$$  

$$c^{B\text{dR}} : \mathbb{G}_m[-1] \to R\varepsilon_*\varepsilon^*\mathbb{G}_m[-1] \to R\varepsilon_*\varepsilon^*\mathbb{Z}_p(1) \to R\varepsilon_*\varepsilon^*\mathbb{B}_{\text{dR}}$$  

where the first arrow is the adjunction morphism, and the middle arrow comes from the boundary map of the Kummer exact sequence of sheaves on $\text{RigSm}_{K,\text{proét}}$

$$0 \to \mathbb{Z}_p(1) \xrightarrow{\prod_{x \neq p}} \lim_{\longleftarrow \mathbb{G}_m} \to \mathbb{G}_m \to 0.$$  

Then, axiom (4) for the cohomology theory $H^\bullet_{\text{dR}}$ follows from the knowledge of the same axiom for the (classical) de Rham cohomology, recalling that $H^\bullet_{\text{dR}}(\mathbb{P}^d_K) = H^\bullet_{\text{dR}}(\mathbb{P}^d_K)$ by Lemma 5.11. For the cohomology theory $H^\bullet_{\text{proét}}(-C, \mathbb{B}_{\text{dR}})$, we can reduce to the latter case, using the isomorphism (6.5) for the projective space $\mathbb{P}^d_K$, if we check the compatibility of loc. cit. with the cycle class maps $c^{\text{dR}}$.
and $\mathcal{O}_{X}^{\infty}$, up to a sign. For this, we show more generally that, for any $X \in \text{RigSm}_K$, the quasi-isomorphism (6.2) constructed in Corollary 6.9 (in the case of trivial coefficients) is compatible with the cycle class maps, up to a sign. Since it suffices to do it locally on $X$, we can assume that there exists an étale map $X \to T_n^K$ that can be written as a composite of finite étale maps and rational embeddings. Then, recalling that (6.2) is induced by the quasi-isomorphism $B^{\text{dR}} \sim \Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{dR}}$ of complex of sheaves on $X_{\text{pro-ét}}$, given by Proposition 6.7, we need to check that the following diagram of complexes of sheaves on $X_{\text{pro-ét}}$ is commutative, up to quasi-isomorphisms, and up to a sign,

$$
\begin{array}{cccc}
\mathcal{O}_X^{\infty}[-1] & \xrightarrow{d \log} & \Omega_X^{\geq 1} \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{dR}} & \xrightarrow{\nabla} & \Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{dR}} \\
\downarrow & & \downarrow & & \\
\left[ Z_p(1) \to \lim_{\xrightarrow{\longrightarrow} x_p} \mathcal{O}_X^{\infty} \right] & \xrightarrow{\nabla} & \left[ \mathcal{B}_{\text{dR}} \to \mathcal{O}_{\text{dR}} \right] & \xrightarrow{i} & \left[ \mathcal{B}_{\text{dR}} \to \mathcal{O}_{\text{dR}} \right] \\
\downarrow & & \downarrow & & \\
Z_p(1) & \xrightarrow{i} & \mathcal{B}_{\text{dR}} & \xrightarrow{i} & \mathcal{B}_{\text{dR}}
\end{array}
$$

where the middle horizontal map sends an “element” $x$ of $\lim_{\xrightarrow{\longrightarrow} x_p} \mathcal{O}_X^{\infty}$ to $\log(V/|U|) \in \mathcal{O}_{\text{dR}}$, with $V \in \mathcal{O}_X^{\infty}$ and $U \in \hat{\mathcal{O}}_X^{\infty}$ the respective “elements” defined by $x$. For this, it suffices to observe that the right square of the diagram above is anticommutative, and the top left square is commutative: in fact, using the Leibniz rule, and $d[U] = 0$, we have that $\nabla(\log(V/|U|)) = d \log(V)$.

7. Pro-étale cohomology of $\mathcal{B}_e$

In this section, we show that the pro-étale cohomology with coefficients in $\mathcal{B}_e = \mathcal{B}[1/t]^{\varphi=1}$, defined in §4, satisfies the axioms of Schneider-Stuhler (3.4). The crucial axiom to be proven is the homotopy invariance with respect to the 1-dimensional open unit disk.

7.1. Décalage functor and Koszul complexes. We begin by recalling the construction of the décalage functor, due to Deligne and Berthelot-Ogus, and defined more generally by Bhatt-Morrow-Scholze on any ringed site (or topos), [BMS18, §6]. For our purposes, only the case of the ringed site ($\mathfrak{s}_{\kappa-\text{pro-ét}}, A$), for $\kappa$ a cut-off cardinal as in §1.6, and $A$ a $\kappa$-condensed ring, will be relevant.

Let $A$ be a $(\kappa)$-condensed ring, and let $D(A) := D(\text{Mod}^{\text{fin}}_A)$ denote the derived category of $A$-modules in CondAb. Let $f$ be a non-zero-divisor in $A$, i.e. a generator of a principal invertible ideal sheaf of $A$; we denote by $(f) = fA$ the invertible ideal sheaf of $A$ it generates. We say that a complex $M^{\bullet}$ of $A$-modules in CondAb is $f$-torsion-free if the map $(f) \otimes_A M^i \to M^i$ is injective for all $i \in \mathbb{Z}$, and, in this case, we denote by $fM^i$ its image.

**Definition 7.1.** Let $M^{\bullet}$ be an $f$-torsion-free complex of $A$-modules in CondAb. We denote by $\eta_f(M^{\bullet})$ the subcomplex of $M^{\bullet}[1/f]$ defined by

$$
\eta_f(M^{\bullet})^i := \{ \alpha \in f^iM^i : da \in f^{i+1}M^{i+1} \}.
$$

---

66 A similar argument appears in the proof of [Sch13c, Lemma 3.24].
The endo-functor \( \eta_f(-) \), defined on the category of \( f \)-torsion-free complexes of \( A \)-modules in \( \text{CondAb} \), induces an endo-functor on \( D(A) \) that kills \( f \)-torsion in cohomology. More precisely, the following result holds true.

**Lemma 7.2** ([BMS18, Lemma 6.4]). Let \( M^\bullet \) be an \( f \)-torsion-free complex of \( A \)-modules in \( \text{CondAb} \). Then, for all \( i \in \mathbb{Z} \), there is a canonical isomorphism

\[
H^i(\eta_f(M^\bullet)) \cong (H^i(M^\bullet)/H^i(M^\bullet)[f]) \otimes_A (f^i)
\]

where \( H^i(M^\bullet)[f] \) denotes the \( f \)-torsion of \( H^i(M^\bullet) \), and \( (f^i) \subset A[1/f] \) denotes the invertible ideal sheaf generated by \( f^i \).

Every complex of \( A \)-modules in \( \text{CondAb} \) is quasi-isomorphic to an \( f \)-torsion-free complex of \( A \)-modules in \( \text{CondAb} \), [BMS18, Lemma 6.1]. Then, from Lemma 7.2, one can deduce the following key result.

**Corollary 7.3** ([BMS18, Corollary 6.5]). The functor \( \eta_f \) from \( f \)-torsion-free complexes of \( A \)-modules in \( \text{CondAb} \) to \( D(A) \) factors canonically over a functor

\[
L\eta_f : D(A) \rightarrow D(A)
\]

called the décalage functor, that satisfies \( H^i(L\eta_f(M)) \cong (H^i(M)/H^i(M)[f]) \otimes_A (f^i) \) functorially in \( M \).

**Remark 7.4.** For any \( M \in D(A) \), we have

\[
L\eta_f(M)[1/f] = M[1/f].
\]

**Remark 7.5.** The décalage functor \( L\eta_f \) is not exact (see [BMS18, Remark 6.6] for a counterexample).

Next, we check that the décalage functor preserves “solid complexes”, cf. [BMS18, Lemma 6.19]. In the following statement, given an analytic ring \((A,M)\),\(^{67}\) we denote \( D(A,M) := D(\text{Mod}_{(A,M)}) \) the derived category of \( \text{Mod}_{(A,M)} \).

**Lemma 7.6.** Let \((A,M)\) be an analytic ring. If \( M \in D(A,M) \), then \( L\eta_f M \in D(A,M) \).

**Proof.** By [Sch19, Proposition 7.5, (ii)], we know that \( L\eta_f M \in D(A,M) \) if and only if \( H^i(L\eta_f M) \in \text{Mod}_{(A,M)} \) for all \( i \). Then, the statement follows recalling that, by Corollary 7.3, we have a (non-canonical) isomorphism \( H^i(L\eta_f(M)) \cong H^i(M)/H^i(M)[f] \), and observing that \( H^i(M)[f] \) lies in \( \text{Mod}_{(A,M)} \) since, by definition, it is the kernel of a morphism in \( \text{Mod}_{(A,M)} \).

A favorable property of the décalage functor \( L\eta_f \) is that, in some suitable cases, it simplifies Koszul complexes, whose definition we now recall (translated in the condensed setting). See also [BMS18, §7].

**Definition 7.7.** Let \( M \) be a condensed abelian group and let \( f_i : M \rightarrow M, i = 1, \ldots, n \), be \( n \) endomorphisms of \( M \) that commute with each other. We define the Koszul complex

\[
\text{Kos}_M(f_1, \ldots, f_n) := M \otimes_{\mathbb{Z}[f_1, \ldots, f_n]} \bigotimes_{i=1}^n (\mathbb{Z}[f_1, \ldots, f_n] \xrightarrow{f_i} \mathbb{Z}[f_1, \ldots, f_n])
\]

as a complex of condensed abelian groups that sits in non-negative cohomological degrees.

---

\(^{67}\)See §A.1 for the notation, and Remark A.5 for the set-theoretic conventions.
Lemma 7.8. Let $A$ be a condensed ring and $f,g_1,\ldots,g_m \in A$ be non-zero-divisors, such that each $g_i$ either divides $f$ or is divisible by $f$. Let $M^\bullet$ be an $f$-torsion-free complex of $A$-modules in $\text{CondAb}$.

(i) If some $g_i$ divides $f$, then $\eta_f(M^\bullet \otimes_A \text{Kos}_A(g_1,\ldots,g_m))$ is acyclic.

(ii) If $f$ divides $g_i$ for all $i$, then

$$\eta_f(M^\bullet \otimes_A \text{Kos}_A(g_1,\ldots,g_m)) \cong \eta_f M^\bullet \otimes_A \text{Kos}_A(g_1/f,\ldots,g_m/f).$$

Proof. See the proof of [BMS18, Lemma 7.9].

We will also need to relate Koszul complexes with condensed group cohomology. For this, we refer the reader to Appendix B, in particular Proposition B.4.

7.2. Homotopy invariance with respect to the open disk. We are ready to prove that pro-étale cohomology with coefficients in $\mathbb{B}_e = \mathbb{B}[1/t]^{\varphi=1}$ satisfies the homotopy invariance with respect to the 1-dimensional open unit disk.

We refer the reader to §4 for the definitions of the pro-étale period sheaves used here. In addition, we introduce the following notation.

Notation 7.9. We fix a compatible system $(1,\varepsilon_p,\varepsilon_p^2,\ldots)$ of $p$-th power roots of unity in $\mathcal{O}_C$, which defines an element $\varepsilon \in \mathcal{O}_p^\times$. We denote by $[\varepsilon] \in A_{\text{inf}} = \mathcal{A}_{\text{inf}}(\mathcal{O}_C)$ its Teichmüller lift and $\mu = [\varepsilon] - 1 \in A_{\text{inf}}$. Furthermore, we let $\xi = \mu/\varphi^{-1}(\mu) \in A_{\text{inf}}$ and $t = \log[\varepsilon] \in B = \mathbb{B}(\mathcal{O}_C)$. Given a compact interval $I$ of $]0,1[$ with endpoints in $p\mathbb{Q}$, we let $A_I = \mathcal{A}_I(\mathcal{O}_C)$ and $B_I = \mathcal{B}_I(\mathcal{O}_C)$.

We will use several times the following fact.

Lemma 7.10. Let $I \subset ]1/p,1[$ a compact interval with endpoints in $p\mathbb{Q}$. Then, the element $t \in B_I$ can be written as $t = \mu \cdot u$ for some unit $u \in B_I$.

Proof. Since $I \subset ]1/p,1[$, we have $A_{\text{cris}} \subset B_I$ (see e.g. [CN17, §2.4.2]). Then, the statement follows from [BMS18, Lemma 12.2, (iii)].

To illustrate the ideas needed to prove the main result of this section, namely Corollary 7.19, we start by studying the geometric pro-étale cohomology of $\mathbb{B}[1/t]$ on the torus. We denote by

$$T^n_C := \text{Spa}(R, R^+) = \text{Spa}(C\langle T_1^{\pm 1},\ldots,T_n^{\pm 1} \rangle, \mathcal{O}_C\langle T_1^{\pm 1},\ldots,T_n^{\pm 1} \rangle)$$

the $n$-torus over $C$, and we write

$$\tilde{T}^n_C := \text{Spa}(R_{\infty}, R_{\infty}^+) = \text{Spa}(C\langle T_1^{\pm 1/p^{\infty}},\ldots,T_n^{\pm 1/p^{\infty}} \rangle, \mathcal{O}_C\langle T_1^{\pm 1/p^{\infty}},\ldots,T_n^{\pm 1/p^{\infty}} \rangle)$$

for the affinoid perfectoid $n$-torus over $C$.

Lemma 7.11. Let $I$ be a compact interval of $]0,1[$ with endpoints in $p\mathbb{Q}$.

(i) We have a canonical identification

$$\mathcal{A}_I(\tilde{T}^n_C) = A_I(V_1^{\pm 1/p^{\infty}},\ldots,V_n^{\pm 1/p^{\infty}}) := A_{\text{inf}}(V_1^{\pm 1/p^{\infty}},\ldots,V_n^{\pm 1/p^{\infty}}) \hat{\otimes}_{A_{\text{inf}}} A_I$$

where $V_i := [T_i]$, we denote by $\hat{\otimes}_{A_{\text{inf}}}$ the $p$-adically completed tensor product, and we write $A_{\text{inf}}(V_1^{\pm 1/p^{\infty}},\ldots,V_n^{\pm 1/p^{\infty}})$ for the $(p,\mu)$-adic completion of $A_{\text{inf}}[V_1^{\pm 1/p^{\infty}},\ldots,V_n^{\pm 1/p^{\infty}}]$. 

(ii) If the interval \( I \) is contained in \([1/p, 1]\), then we have

\[
A_I(V_1^{\pm 1/p^\infty}, \ldots, V_n^{\pm 1/p^\infty}) = \bigoplus_{(a_1, \ldots, a_n) \in \mathbb{Z}[1/p]^n} A_I \cdot \prod_{j=1}^n V_j^{a_j}
\]

where the completion over the direct sum is \((p, \mu)\)-adic.

**Proof.** By Proposition 4.5, we have \( A_I(\mathbb{T}_C^n) = A_I(R_\infty, R_\infty^+) \), hence, unraveling the definition of \( A_I(R_\infty, R_\infty^+) \), for part (i) it suffices to identify \( A_{\inf}(R_\infty, R_\infty^+) \) with \( A_{\inf}(V_1^{\pm 1/p^\infty}, \ldots, V_n^{\pm 1/p^\infty}) \). We recall from §4 that we have \( A_{\inf}(R_\infty, R_\infty^+) = W(R_\infty^+) \) and \( R_\infty^+ = \mathcal{O}_C^\flat ((T_1^\gamma)^{1/p^\infty}, \ldots, (T_n^\gamma)^{1/p^\infty}) \) where the completion is \( p^\flat \)-adic. We deduce that \( W(R_\infty^+) = W(\mathcal{O}_C^\flat)(V_1^{\pm 1/p^\infty}, \ldots, V_n^{\pm 1/p^\infty}) \) where the completion is \((p, [p^\flat])\)-adic,\(^{68}\) or equivalently \((p, \mu)\)-adic, as desired.

For part (ii) it suffices check that, for \( I \subset [1/p, 1] \), the \( p \)-adic topology is equivalent to the \((p, \mu)\)-adic topology on \( A_I \). For this, we recall that \( A_{\cris} \subset A_I \), by our assumption on \( I \), and hence by [BMS18, Proposition 12.2, (i)] we have \( \mu^{p-1} \subset pA_I \); in particular, \((p, \mu)^N A_I \subset pA_I \) for a large enough positive integer \( N \).

**Remark 7.12.** Recall that the affinoid perfectoid pro-(finite étale) cover of the \( n \)-torus \( \mathbb{T}_C^n \rightarrow \mathbb{T}_C^n \) has Galois group \( \mathbb{Z}_p(1)^n \). The choice of \( \epsilon \) (Notation 7.9) gives an isomorphism \( \mathbb{Z}_p(1)^n \cong \mathbb{Z}_p^n \). We denote \( \Gamma := \mathbb{Z}_p^n \) and by \( \gamma_1, \ldots, \gamma_n \) the canonical generators of \( \Gamma \). One can describe explicitly the Galois action of \( \Gamma \) on the coordinates \((R_\infty, R_\infty^+)\) of \( \mathbb{T}_C^n \), as follows. We can write

\[
R_\infty^+ = \mathcal{O}_C(T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}) = \bigoplus_{(a_1, \ldots, a_n) \in \mathbb{Z}[1/p]^n} \mathcal{O}_C \cdot \prod_{j=1}^n T_j^{a_j}
\]

where the completion over the direct sum is \( p \)-adic. Then, the action of \( \Gamma \) on \( R_\infty^+ \) preserves the decomposition above and \( \gamma_i \) acts on \( \prod_{j=1}^n T_j^{a_j} \) via multiplication by \( \epsilon_p^{a_i} \).

In the following, we maintain Notation 5.2.

**Proposition 7.13.** Let \( \mathbb{T}^n \) denote the \( n \)-dimensional torus defined over \( \mathbb{Q}_p \). Then, we have a Frobenius-equivariant quasi-isomorphism in \( D(\text{Vec}^{\text{solid}}_{\mathbb{Q}_p}) \)

\[
R_{\Gamma_{\text{proét}}}(\mathbb{T}_C^n, \mathcal{B}[1/t]) \simeq \Omega^*(\mathbb{T}^n) \otimes_{\mathbb{Q}_p} \mathcal{B}^\cdot[1/t].
\]

**Proof.** First, we prove that, for any compact interval \( I \) of \([0, 1]\) with endpoints in \( p^\mathbb{Q} \), we have a Frobenius-equivariant quasi-isomorphism in \( D(\text{Vec}^{\text{solid}}_{\mathbb{Q}_p}) \)

\[
L_{\eta_t}R_{\Gamma_{\text{proét}}}(\mathbb{T}_C^n, \mathcal{B}_I) \simeq \Omega^*(\mathbb{T}^n) \otimes_{\mathbb{Q}_p} \mathcal{B}_I.
\]

(7.1)

Fix a compact interval \( I \) of \([0, 1]\) with endpoints in \( p^\mathbb{Q} \). We compute \( R_{\Gamma_{\text{proét}}}(\mathbb{T}_C^n, \mathcal{B}_I) \) using the Cartan-Leray spectral sequence relative to the pro-étale cover \((\mathbb{T}_C^n)^\proet \rightarrow (\mathbb{T}_C^n)^\flat \), that is

\[
E_1^{i,j} = H^i_{\text{proét}}(\mathbb{T}_C^n, \mathcal{B}_I) \implies H^{i+j}_{\text{proét}}(\mathbb{T}_C^n, \mathcal{B}_I)
\]

\(^{68}\) One way to see this is to use that the Witt vectors \( W(\cdot) \) induce an equivalence of categories between the perfect \( \mathbb{F}_p \)-algebras and the category of \( p \)-adically complete, separated and \( p \)-torsion free \( \mathbb{Z}_p \)-algebras \( R \) such that \( R/p \) is perfect, with quasi-inverse the tilting functor (see e.g. [Bha18, Remark 2.5, Example 2.6]).
where, for $j \geq 1$, $\tilde{T}^n_{C,j}$ denotes the $j$-fold fibre product of $\tilde{T}^n_C$ over $T^n_C$. Note that $\tilde{T}^n_{C,j} \cong \tilde{T}^n_C \times \Gamma^{j-1}$, where $\Gamma = \mathbb{Z}_p$. By Proposition 4.5(iii), for any affinoid perfectoid $Z$ over $\text{Spa}(C, O_C)$, we have $H^0_{\text{proét}}(Z, B_I) = 0$ for all $i > 0$. Moreover, it follows from Corollary 4.7 that

$$H^0_{\text{proét}}(\tilde{T}^n_{C,j}, B_I) = \text{Hom}(Z[\Gamma^{j-1}], B_I(\tilde{T}^n_C)).$$

Then, the spectral sequence above degenerates, and, by Proposition B.2(i), we have a quasi-isomorphism

$$R\text{Hom}_{Z[\Gamma]}(Z, B_I(\tilde{T}^n_C)) \cong R\Gamma_{\text{proét}}(\tilde{T}^n_C, B_I).$$

Thus, we need to study the complex $L\eta R\text{Hom}_{Z[\Gamma]}(Z, B_I(\tilde{T}^n_C))$. By Lemma 7.11(i), we can write

$$A_I(\tilde{T}^n_C) = A_I(R) \oplus A_I(R_{\infty})^{n-\text{int}}, \quad B_I(\tilde{T}^n_C) = B_I(R) \oplus B_I(R_{\infty})^{n-\text{int}} \quad (7.2)$$

where $A_I(R) := A_I(V_1^{\pm 1}, \ldots, V_n^{\pm 1})$ denotes the “integral part”, $A_I(R_{\infty})^{n-\text{int}}$ denotes the “non-integral part” of $A_I(\tilde{T}^n_C)$, and similarly for $B_I(\tilde{T}^n_C)$ by inverting $p$. By Proposition B.4, we have

$$R\text{Hom}_{Z[\Gamma]}(Z, B_I(\tilde{T}^n_C)) \simeq \text{Kos}_{B_I(R)}(\gamma_1 - 1, \ldots, \gamma_n - 1) \oplus \text{Kos}_{B_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1)$$

Now, we first assume that $I \subset [1/p, 1]$.

- We begin by showing that we have $L\eta \text{Kos}_{B_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1) = 0$. For this, it suffices to prove that the multiplication by $t$ on the complex $\text{Kos}_{B_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1)$ is homotopic to 0.\(^{69}\) Let $\mu = [\varepsilon] - 1$ as in Notation 7.9, and recall that, by Lemma 7.10, we have that $t = \mu \cdot u$ for some unit $u \in B_I$. Then, it suffices to show that the multiplication by $\varphi^{-1}(\mu) = [\varepsilon]^{1/p} - 1$ on $\text{Kos}_{A_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1)$ is homotopic to 0. This can be done as in the proof of [BMS18, Lemma 9.6]: by Remark 7.12 and Lemma 7.11(ii), we have

$$\text{Kos}_{A_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1) = \bigoplus_{(a_1, \ldots, a_n)} \text{Kos}_{A_I(R)}(\gamma_1 [\varepsilon]^{a_1} - 1, \ldots, \gamma_n [\varepsilon]^{a_n} - 1)$$

where the completion is $(p, \mu)$-adic, and the direct sum runs over $a_1, \ldots, a_n \in \mathbb{Z}[1/p] \cap [0, 1]$ not all 0. Hence, we are reduced to show that for $a_i \in \mathbb{Z}[1/p] \cap (0, 1)$ the multiplication by $\varphi^{-1}(\mu)$ on the complex

$$A_I(R) \xrightarrow{[\varepsilon]^{a_i} - 1} A_I(R)$$

is homotopic to 0, i.e. we have to find $h$ that completes the following diagram

$$\begin{array}{ccc}
A_I(R) & \xrightarrow{[\varepsilon]^{a_i} - 1} & A_I(R) \\
\varphi^{-1}(\mu) \downarrow & & \downarrow \varphi^{-1}(\mu) \\
A_I(R) & \xrightarrow{h} & A_I(R)
\end{array}$$

\(^{69}\) In fact, this would imply that the cohomology groups of the complex of condensed $\mathbb{Q}_p$-vector spaces $\text{Kos}_{B_I(R_{\infty})^{n-\text{int}}}(\gamma_1 - 1, \ldots, \gamma_n - 1)$ are annihilated by $t$, and hence the claim by Corollary 7.3.
Let us write \( a_i = m/p^r \), with \( m \in \mathbb{Z} \setminus p\mathbb{Z} \). Up to changing the choice of \((1, \varepsilon_p, \varepsilon_p^2, \ldots)\) in Notation 7.9, we can suppose \( m = 1 \). Furthermore, since \( \gamma_i/e[1/p^r] - 1 \) divides \( \gamma_i^{p^r-1}[e]^{1/p} - 1 \), it suffices to show that the latter map is homotopic to 0. Then, one has to find \( h \) such that

\[
\gamma_i^{p^r-1}(h(x))[e]^{1/p} - h(x) = \varphi^{-1}(\mu)x.
\]

For this, we refer the reader to [BMS18, Lemma 9.6].

Next, we show that

\[
L\eta_! \text{Kos}_{B_I}(\gamma_1 - 1, \ldots, \gamma_n - 1) \simeq \bigotimes^\bullet_{\mathbb{Q}_p} B_I.
\]

Since \( \mu \) divides \( \gamma_i - 1 \), i.e. \( \gamma_i \) acts trivially on \( B_I(R)/\mu \), and by Lemma 7.10 \( t = \mu \cdot u \) for some unit \( u \in B_I \), using Lemma 7.8(ii) we have

\[
L\eta_! \text{Kos}_{B_I}(\gamma_1 - 1, \ldots, \gamma_n - 1) \simeq \text{Kos}_{B_I}(\frac{\gamma_1 - 1}{t}, \ldots, \frac{\gamma_n - 1}{t}).
\]

By the proof of [BMS18, Lemma 12.4], one has the following Taylor expansion in \( B_I(R) \)

\[
\gamma_i = \sum_{j \geq 0} \frac{t^j}{j!} \left( \frac{\partial}{\partial \log(V_i)} \right)^j,
\]

from which we can write

\[
\frac{\gamma_i - 1}{t} = \frac{\partial}{\partial \log(V_i)}(1 + H), \quad \text{with} \quad H := \sum_{j \geq 1} \frac{t^j}{(j+1)!} \left( \frac{\partial}{\partial \log(V_i)} \right)^j.
\]

Note that \( H \) is topologically nilpotent, using again that \( A_{\text{cris}} \subset B_I \) since \( I \subset 1/p, 1/\varepsilon \), and [BMS18, Lemma 12.2, (ii)]. In particular, the factor \( 1 + H \) is an automorphism of \( B_I(R) \); moreover, the latter automorphism is Frobenius-equivariant recalling that \( \varphi(t) = pt \) and

\[
\frac{\partial}{\partial \log(V_i)} \circ \varphi = p \left( \varphi \circ \frac{\partial}{\partial \log(V_i)} \right).
\]

We deduce that we have a Frobenius-equivariant quasi-isomorphism

\[
\text{Kos}_{B_I(R)} \left( \frac{\gamma_1 - 1}{t}, \ldots, \frac{\gamma_n - 1}{t} \right) \simeq \text{Kos}_{B_I(R)} \left( \frac{\partial}{\partial \log(V_1)}, \ldots, \frac{\partial}{\partial \log(V_n)} \right) \simeq \bigotimes^\bullet_{\mathbb{Q}_p} B_I.
\]

Then, putting the above points together, we obtain a quasi-isomorphism as in (7.1) for any given \( I \subset 1/p, 1/\varepsilon \). Now, since the Frobenius \( \varphi \) induces an isomorphism \( \varphi : B_I \simeq B_{\varphi(t)} \) (Remark 4.3), applying \( \varphi^N \) to the latter quasi-isomorphism, for \( N \) a sufficiently big positive integer, we deduce that (7.1) holds true for a general compact interval \( I \subset 0, 1 \) with endpoints in \( p\mathbb{Q} \), as desired.

It remains to show that the statement follows from (7.1). Passing to the inverse limit over all compact intervals \( I \subset 0, 1 \) with endpoints in \( p\mathbb{Q} \), we obtain the following quasi-isomorphism

\[
L\eta_! \text{R} \Gamma_{\text{proét}}(\mathbb{P}^n, B) \simeq R\lim_I L\eta_! \text{R} \Gamma_{\text{proét}}(\mathbb{P}^n, B_I) \simeq R\lim_I \left( \bigotimes^\bullet_{\mathbb{Q}_p} B_I \right) \simeq \bigotimes^\bullet_{\mathbb{Q}_p} B.
\]

Here, in the first step we used \( R\lim_I B_I = B \) (which follows from Lemma 4.6), together with a condensed version of [LB18b, Lemma 3.10].\(^{70}\) In the last step, we used Corollary A.24(ii), which

\(^{70}\) In more detail, by loc. cit. we can reduce to show that \( R\Gamma_{\text{proét}}(\mathbb{P}^n, B)/t \simeq R\lim_I (R\Gamma_{\text{proét}}(\mathbb{P}^n, B_I)/t) \), which follows expressing both sides in terms of Koszul complexes, and then applying Lemma 4.6.
applies using that the $\mathbb{Q}_p$-Banach space $B_\mathfrak{t}$ is a nuclear $\mathbb{Q}_p$-vector space by Theorem A.20 (and a flat solid $\mathbb{Q}_p$-vector space by Corollary A.10). Then, the statement follows inverting $t$ in (7.3), using that the tensor product $\otimes_{\mathbb{Q}_p}$ commutes with colimits, and observing that $(\mathcal{L}_\mathfrak{t} R\Gamma_{\text{proét}}(\mathbb{T}_C^n, \mathbb{B}))[1/t] = R\Gamma_{\text{proét}}(\mathbb{T}_C^n, \mathbb{B}[1/t])$ (recall Remark 7.4, and use that $|\mathbb{T}_C^n|$ is quasi-compact and quasi-separated).

To reach the stated goal of this section, we will need in particular to study the pro-étale cohomology with coefficients in $\mathbb{B}[1/t]$ of the 1-dimensional unit closed disk $\mathbb{D}_C$. For this, the general strategy we will use is similar to the one we have seen for the torus (Proposition 7.13), but there is one slight difference, which is explained in the following remark.

Remark 7.14. If we denote by $\tilde{\mathbb{D}}_C = \text{Spa}(\mathbb{C}(T^{1/p^\infty}), \mathcal{O}_C(T^{1/p^\infty}))$ the affinoid perfectoid unit closed disk over $C$, then the cover $\tilde{\mathbb{D}}_C \to \mathbb{D}_C$ is not pro-étale. But, it is quasi-pro-étale. Hence, recalling Definition 2.2, we can still study the pro-étale cohomology of $\mathbb{D}_C$ with coefficients in $\mathbb{B}[1/t]$ using the Cartan-Leray spectral sequence relative to such cover.

Thus, we need to better understand $\tilde{\mathbb{D}}_{C,j}$, the $j$-fold fibre product of $\tilde{\mathbb{D}}_C$ over $\mathbb{D}_C$. The following result is due to Le Bras.

Lemma 7.15. Let $\mathbb{D}_C = \text{Spa}(\mathbb{C}(T), \mathcal{O}_C(T))$ denote the 1-dimensional unit closed disk over $C$. Let $\tilde{\mathbb{D}}_C = \text{Spa}(\mathbb{C}(T^{1/p^\infty}), \mathcal{O}_C(T^{1/p^\infty}))$ be the affinoid perfectoid unit closed disk over $C$. For $j \geq 1$, let $\mathbb{D}_{C,j}$ denote the $j$-fold fibre product of $\tilde{\mathbb{D}}_C$ over $\mathbb{D}_C$. Then, we have $\mathbb{D}_{C,j} \cong \text{Spa}(R^+_j[1/p], R^+_j)$ where

$$R^+_j := \{ f \in \mathcal{O}(\mathbb{Z}_p^{j-1}, \mathcal{O}_C(T^{1/p^\infty})): f|_{T=0} \text{ is constant} \}.$$

Proof. This is a particular case of [LB18a, Lemme 3.30].

Remark 7.16. Let $I$ be a compact interval of $]0, 1[$ with endpoints in $p^\mathbb{Q}$. Similarly to Lemma 7.11(i), from Lemma 7.15 we deduce that we have $A_I(\tilde{\mathbb{D}}_{C,j}) = \{ f \in \mathcal{O}(\mathbb{Z}_p^{j-1}, A_I(V^{1/p^\infty})): f|_{V=0} \text{ is constant} \}$ where $V := [T^0]$, and $A_I(V^{1/p^\infty}) := A_{\text{inf}}(V^{1/p^\infty}) \otimes_{\text{Aut}} A_I$ (here, we denote by $\otimes_{\text{Aut}}$ the $p$-adically completed tensor product, and we write $A_{\text{inf}}(V^{1/p^\infty})$ for the $(p, \mu)$-adic completion of $A_{\text{inf}}[V^{\pm 1/p^\infty}]$).

Proposition 7.17. Let $\mathbb{D}$ denote the 1-dimensional unit closed disk defined over $\mathbb{Q}_p$. Let $X$ be a smooth affinoid space defined over $C$ with an étale map $X \to \mathbb{T}_C^n$ that factors as a composite of rational embeddings and finite étale maps. Then, for any compact interval $I$ of $]0, 1[$ with endpoints in $p^\mathbb{Q}$, we have a Frobenius-equivariant quasi-isomorphism in $D(\text{Vect}_{\mathbb{Q}_p})$

$$L\mathcal{L}_\mathfrak{t} R\Gamma_{\text{proét}}(X \times \mathbb{D}_C, \mathbb{B}_I) \simeq L\mathcal{L}_\mathfrak{t} R\Gamma_{\text{proét}}(X, \mathbb{B}_I) \otimes_{\mathbb{Q}_p} \Omega^*(\mathbb{D}).$$

Proof. Fix a compact interval $I$ of $]0, 1[$ with endpoints in $p^\mathbb{Q}$. First, we compute $R\Gamma_{\text{proét}}(X \times \mathbb{D}_C, \mathbb{B}_I)$ using the Cartan-Leray spectral sequence associated to the cover $\tilde{X} \times \tilde{\mathbb{D}}_C \to \tilde{X} \times \mathbb{D}_C$, where $\tilde{X} := X \times_{\mathbb{T}_C^n} \mathbb{T}_C^n$ (see Remark 7.14). Recall that, by Proposition 4.5(iii), for any affinoid perfectoid
Z over \text{Spa}(C, \mathcal{O}_C)$, we have $H^j_{\text{pro\acute{e}t}}(Z, \mathbb{B}_I) = 0$ for all $i > 0$. Moreover, for $j \geq 1$, denoting by $\tilde{X}_j$ the $j$-fold fibre product of $X$ over $X$, by Lemma 7.15 and Remark 7.16, we have that

$$\mathbb{B}_I(\tilde{X}_j \times \mathbb{D}_{C,j}) = \{ f \in \mathcal{C}^0(\Gamma^j \times \mathbb{Z}_p, \mathbb{B}_I(\tilde{X}_j)(V^{1/p^\infty}): f(\gamma, -)|_{V=0} \text{ is constant for all } \gamma \in \Gamma^j \}$$

where $\Gamma = \mathbb{Z}_p$, $V = [T^b]$, and $\mathbb{B}_I(\tilde{X}_j)(V^{1/p^\infty}) := \mathcal{A}_I(\tilde{X}_j)(V^{1/p^\infty})[1/p]$. Hence, $H^0_{\text{pro\acute{e}t}}(\tilde{X}_j \times \mathbb{D}_{C,j}, \mathbb{B}_I)$ fits into the exact sequence

$$0 \rightarrow \text{Hom}(\mathbb{Z}[\Gamma^j \times \mathbb{Z}_p], N_j) \rightarrow H^0_{\text{pro\acute{e}t}}(\tilde{X}_j \times \mathbb{D}_{C,j}, \mathbb{B}_I) \rightarrow \text{Hom}(\mathbb{Z}[\Gamma^j \times \mathbb{Z}_p], \mathbb{B}_I(\tilde{X}_j)) \rightarrow 0$$

where $N_j := V \cdot \mathbb{B}_I(\tilde{X}_j)(V^{1/p^\infty})$, and the last map is given by $f \mapsto f|_{V=0}$. Then, from Proposition B.2(i), we deduce that we have a distinguished triangle as follows

$$R\text{Hom}_{\mathbb{Z}[\Gamma \times \mathbb{Z}_p]}(\mathbb{Z}, N) \rightarrow R\Gamma_{\text{pro\acute{e}t}}(X \times \mathbb{D}_C, \mathbb{B}_I) \rightarrow R\text{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \mathbb{B}_I(\tilde{X})) \quad (7.4)$$

where $N := V \cdot \mathbb{B}_I(\tilde{X})(V^{1/p^\infty})$. Next, we want to study $L\eta_I R\Gamma_{\text{pro\acute{e}t}}(X \times \mathbb{D}_C, \mathbb{B}_I)$.

Let us begin by handling $L\eta_I R\text{Hom}_{\mathbb{Z}[\Gamma \times \mathbb{Z}_p]}(\mathbb{Z}, N)$. We can write $N = N^{\text{int}} \oplus N^{\text{n-int}}$, where $N^{\text{int}} := V \cdot \mathbb{B}_I(\tilde{X})(V)$ denotes the “integral part”, and $N^{\text{n-int}}$ the “non-integral part” of $N$. Denoting by $\gamma_1, \ldots, \gamma_{n+1}$ the canonical generators of $\Gamma \times \mathbb{Z}_p$, by Proposition B.4 we have that

$$R\text{Hom}_{\mathbb{Z}[\Gamma \times \mathbb{Z}_p]}(\mathbb{Z}, N) \simeq \text{Kos}_{\mathbb{N}^{\text{int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1) \oplus \text{Kos}_{\mathbb{N}^{\text{n-int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1)$$

In the following, we assume first that $I \subset [1/p, 1]$. Then, similarly to the proof of Proposition 7.13, one checks that the multiplication by $t$ on the complex $\text{Kos}_{\mathbb{N}^{\text{n-int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1)$ is homotopic to 0, and hence

$$L\eta_I \text{Kos}_{\mathbb{N}^{\text{n-int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1) = 0$$

Moreover, by Corollary A.10, we have

$$\text{Kos}_{\mathbb{N}^{\text{int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1) \simeq M^* \otimes_{\mathbb{Q}_p} \text{Kos}_{\mathbb{Q}_p}(V)(\gamma_{n+1} - 1)$$

where $M^* := \text{Kos}_{\mathbb{B}_I}(\tilde{X})(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1)$. Therefore, arguing again as in the proof of Proposition 7.13, by the proof of Lemma 7.8(ii), we have a Frobenius-equivariant quasi-isomorphism

$$L\eta_I \text{Kos}_{\mathbb{N}^{\text{int}}}(\gamma_1 - 1, \ldots, \gamma_{n+1} - 1) \simeq L\eta_I M^* \otimes_{\mathbb{Q}_p} \text{Kos}_{\mathbb{Q}_p}(V) \left( \frac{\partial}{\partial \log(V)} \right)$$

where, by Proposition B.4, $M^* \simeq R\text{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \mathbb{B}_I(\tilde{X})) \simeq R\Gamma_{\text{pro\acute{e}t}}(X, \mathbb{B}_I)$.

Putting everything together, and recalling the definition of the last arrow of (7.4), we obtain the following Frobenius-equivariant quasi-isomorphism

$$L\eta_I R\Gamma_{\text{pro\acute{e}t}}(X \times \mathbb{D}_C, \mathbb{B}_I) \simeq L\eta_I R\Gamma_{\text{pro\acute{e}t}}(X, \mathbb{B}_I) \otimes_{\mathbb{Q}_p} \text{Kos}_{\mathbb{Q}_p}(V) \left( \frac{\partial}{\partial \log(V)} \right) \simeq L\eta_I R\Gamma_{\text{pro\acute{e}t}}(X, \mathbb{B}_I) \otimes_{\mathbb{Q}_p} \Omega^*(\mathbb{D})$$

Arguing as in the proof of Proposition 7.13, such quasi-isomorphism extends to a general compact interval $I \subset [0, 1]$ with endpoints in $p^{\mathbb{Q}}$, as desired. \hfill \Box

\textbf{Remark 7.18.} We note that Proposition 7.17 holds replacing the 1-dimensional unit closed disk $\mathbb{D}$ with any 1-dimensional closed disk $\mathbb{D}(\rho)$ over $\mathbb{Q}_p$ of radius $\rho \in p^{\mathbb{Q}}$.  


Corollary 7.19. Let \( \hat{D} \) denote the 1-dimensional open unit disk defined over \( \mathbb{Q}_p \). Let \( X \) be a smooth affinoid space defined over \( C \). Then, the natural projection map \( X \times \hat{D}_C \rightarrow X \) induces a quasi-isomorphism in \( D(\text{Vect}_{\mathbb{Q}_p}^{\mathrm{solid}}) \)

\[
\Gamma_{\text{pro ét}}(X, \mathbb{B}[1/t]) \cong \Gamma_{\text{pro ét}}(X \times \hat{D}_C, \mathbb{B}[1/t]).
\]

**Proof.** Using [Sch13b, Lemma 5.12], we can reduce to the case in which there exists an étale map \( X \rightarrow \mathbb{T}_C^n \) that factors as a composite of rational embeddings and finite étale maps. We write \( \hat{D} \) as a strictly increasing admissible union of closed disks \( \{D_j\}_{j \in \mathbb{N}} \) of radius in \( p^\mathbb{Q} \). Then, we have

\[
\Gamma_{\text{pro ét}}(X \times \hat{D}_C, \mathbb{B}[1/t]) \cong \Gamma_{\text{pro ét}}(1/\mathbb{B}_j, \mathbb{B}[1/t])
\]

\[
\cong \Gamma_{\text{pro ét}}(1/\mathbb{B}_j, \mathbb{B}[1/t])
\]

\[
\cong \Gamma_{\text{pro ét}}(1/\mathbb{B}_j, \mathbb{B}[1/t])
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\[
\cong \Gamma_{\text{pro ét}}(1/\mathbb{B}_j, \mathbb{B}[1/t])
\]

\[
\cong \Gamma_{\text{pro ét}}(1/\mathbb{B}_j, \mathbb{B}[1/t])
\]

Here, for (7.5) we used that \( |X \times \mathbb{D}_j| \) is quasi-compact and quasi-separated; in (7.6) we used \( R\text{lim}_j \mathbb{B}_j = \mathbb{B} \) (Lemma 4.6), together with a condensed version of [LB18b, Lemma 3.10];\(^{72}\) the quasi-isomorphism (7.7) follows from Proposition 7.17 (and Remark 7.18); in (7.8) we used that, for each \( j \in \mathbb{N} \), the morphism \( R\Gamma_{\text{DR}}(\mathbb{D}_{j+1}) \rightarrow R\Gamma_{\text{DR}}(\mathbb{D}_j) \) factors through \( R\Gamma_{\text{DR}}(\mathbb{D}_j) \); the quasi-isomorphism (7.9) follows from axiom (1) of Schneider-Stuhler for the condensed de Rham cohomology (Proposition 6.15); finally, (7.10) follows using the same ingredients of (7.5) and (7.6). \(\square\)

We are ready to show the following result.

**Proposition 7.20.** The cohomology theory

\[
H^*_{\text{pro ét}}(-C, \mathbb{B}_e) : \text{RigSm}_{K,\text{pro ét}} \rightarrow \text{Mod}_{\mathbb{B}_e}^{\text{cond}}
\]

satisfies the axioms of Schneider-Stuhler (3.4).

**Proof.** By Corollary 4.7, \( H^*_{\text{pro ét}}(-C, \mathbb{B}_e) \) is defined by the complex of sheaves \( R\varepsilon_* \mathbb{e}^* \mathbb{B}_e \) on the site \( \text{RigSm}_{K,\text{pro ét}} \), where we denote by \( \varepsilon : \text{RigSm}_{C,\text{pro ét}} \rightarrow \text{RigSm}_{K,\text{pro ét}} \) the base change morphism.

Axiom (1) follows from Corollary 7.19 and the exact sequence (4.2). Axioms (2) and (3) are satisfied. For axiom (4), as in the proof of Proposition 6.15, we first construct the cycle class map \( c^B : \mathbb{G}_m[-1] \rightarrow R\varepsilon_* \mathbb{e}^* \mathbb{B}_e \), with target the complex of sheaves of abelian groups underlying \( R\varepsilon_* \mathbb{e}^* \mathbb{B}_e \); we define it on \( \text{RigSm}_{K,\text{pro ét}} \) as the composite

\[
c^B : \mathbb{G}_m[-1] \rightarrow R\varepsilon_* \mathbb{e}^* \mathbb{G}_m[-1] \rightarrow R\varepsilon_* \mathbb{e}^* \mathbb{Z}_p(1) \rightarrow R\varepsilon_* \mathbb{e}^* \mathbb{B}_e(1) \cong R\varepsilon_* \mathbb{e}^* \mathbb{B}_e
\]

\(^{72}\)Cf. Footnote 70 for more details.
where the first arrow is the adjunction morphism, the middle arrow comes from the boundary map of the Kummer exact sequence (6.14), and the trivialization $\mathcal{B}_e(1) \cong \mathcal{B}_e$ is given by the choice of a compatible system of $p$-th power roots of unity in $\mathcal{O}_C$ (Notation 7.9).

Now, we need to check that $H^i_{proét}(\mathbb{P}^d_C, \mathcal{B}_e) = 0$ for $i$ odd, or $i > 2d$, and that, for all $0 \leq i \leq d$, the map $B_e \to H^{2i}_{proét}(\mathbb{P}^d_C, \mathcal{B}_e)$, defined in (3.2) and induced by $c^{B_e}$, is an isomorphism. We note that, since $\mathbb{P}^d_K$ is a proper smooth rigid-analytic variety, by well-known results in (relative) $p$-adic Hodge theory, for $\mathcal{B} \in \{ \mathcal{B}_{dR}, \mathcal{B}_e \}$, for all $j \geq 0$ we have an isomorphism $^{73}$

$$H^j_{proét}(\mathbb{P}^d_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{B}(C, \mathcal{O}_C) \cong H^j_{proét}(\mathbb{P}^d_C, \mathcal{B}).$$

(7.11)

By the crystalline comparison theorem for proper smooth schemes over $\mathcal{O}_K$, [Fal89], combined with the GAGA for étale cohomology, [Hub96, Theorem 3.2.10], we have $H^i_{proét}(\mathbb{P}^d_C, \mathbb{Q}_p) = 0$ for $i$ odd, or $i > 2d$, and $H^{2i}_{proét}(\mathbb{P}^d_C, \mathbb{Q}_p) = \mathbb{Q}_p$ for all $0 \leq i \leq d$. Then, axiom (4) follows from the knowledge of the same axiom for $H^*_{proét}(-C, \mathcal{B}_{dR})$, which was shown in Proposition 6.15, and from the compatibility of the cycle class map $c^{B_{dR}}$ of loc. cit. with the cycle class map $c^{B_e}$, under the inclusion $B_e \hookrightarrow B_{dR}$.

$\square$

Remark 7.21. We note that, in the proof of Proposition 7.20, we identified $\mathcal{B}_e(1)$ with $\mathcal{B}_e$, in order to make $\mathcal{H}^*_{proét}(-C, \mathcal{B}_e)$ satisfy the axiom (4) of Schneider-Stuhler (3.4). Keeping track of the Galois action on the geometric pro-étale cohomology of $\mathbb{P}^d_K$ with coefficients in $\mathcal{B}_e$, by Theorem 3.15 and Proposition 7.20 above, for all $i \geq 0$, we have a $G \times \mathcal{G}_K$-equivariant isomorphism in $\text{Vect}^{\text{ord}}_{\mathbb{Q}_p}$

$$H^i_{proét}(\mathbb{H}^d_C, \mathcal{B}_e) \cong \text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathcal{B}_e)(-i)$$

where $\mathbb{H}^d_C$ denotes the base change to $C$ of the Drinfeld upper half-space $\mathbb{H}^d_K$.

8. An application

We are ready to reprove [CDN20b, Theorem 4.12].

Theorem 8.1. Given an integer $d \geq 1$, let $G = \text{GL}_{d+1}(K)$. Let $\mathbb{H}^d_K$ be the Drinfeld upper half-space of dimension $d$ defined over $K$, and let $\mathbb{H}^d_C$ be its base change to $C$. For all $i \geq 0$, we have the following commutative diagram of $G \times \mathcal{G}_K$-Fréchet spaces over $\mathbb{Q}_p$, with strictly exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & \Omega^{i-1}(\mathbb{H}^d_C)/\ker d & \rightarrow & H^i_{proét}(\mathbb{H}^d_C, \mathbb{Q}_p(i)) & \rightarrow & \text{Sp}_i(\mathbb{Q}_p)^* & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Omega^{i-1}(\mathbb{H}^d_C)/\ker d & \rightarrow & \Omega^i(\mathbb{H}^d_C)^{d=0} & \rightarrow & \text{Sp}_i(K)^{\ast, \otimes K C} & \rightarrow & 0
\end{array}
$$

where $(-)^*$ denotes the weak topological dual.$^{74}$

Proof. By Proposition 4.9, we have a commutative diagram of sheaves on $\mathbb{H}^d_{C,proét}$ with exact rows

\footnote{For $\mathcal{B} = \mathcal{B}_{dR}$, the isomorphism (7.11) follows from [Sch13b, Theorem 8.8, (i)]. For $\mathcal{B} = \mathcal{B}_e$ it follows combining the proof of loc. cit. with [LB18b, Proposition 3.4] and the exact sequence (4.2).}

\footnote{See §3.4.}
$0 \to \mathbb{Q}_p \to \mathbb{B}_e \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to 0$

$0 \to \mathbb{B}_{dR}^+ \to \mathbb{B}_{dR} \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to 0$

Let $f : \mathbb{H}_C^d \to \text{Spa}(C, \mathcal{O}_C)$ be the structure morphism. Then, applying the derived functor $Rf_{\text{proét}} \ast$ to the diagram above, and taking the long exact sequences in cohomology, we obtain the following $G \times \mathcal{G}_K$-equivariant commutative diagram in $\text{Vect}_{\mathbb{Q}_p}^{\text{cond}}$ with exact rows

$$\cdots \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{Q}_p) \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_e) \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+) \to \cdots$$

$$\cdots \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}^+) \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}) \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}/\mathbb{B}_{dR}^+) \to \cdots$$

from which we obtain the commutative diagram with exact rows

$$0 \to \text{coker } \alpha_{i-1} \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{Q}_p) \to \ker \alpha_i \to 0$$

$$0 \to \text{coker } \beta_{i-1} \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}^+) \to \ker \beta_i \to 0$$

(8.1)

First, we determine $\ker \beta_i$ and $\text{coker } \beta_{i-1}$. By Theorem 3.15 for the cohomology theories $H^*_{\text{dR}}$ and $H^*_{\text{proét}}(-C, \mathbb{B}_{dR})$, which applies thanks to Proposition 6.15, we have the following compatible $G \times \mathcal{G}_K$-equivariant isomorphisms in $\text{Vect}_{\mathbb{Q}_p}^{\text{cond}}$

$$H^i_{\text{dR}}(\mathbb{H}_K^d) \cong \text{Hom}(\text{Sp}_i(\mathbb{Z}), K), \quad H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}) \cong \text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathbb{B}_{dR}).$$

Then, since $\mathbb{H}_K^d$ is a Stein space, by Remark 6.13 we have the following $G \times \mathcal{G}_K$-equivariant commutative diagram in $\text{Vect}_{\mathbb{Q}_p}^{\text{cond}}$ with exact rows

$$0 \to H^i_{\text{dR}}(\mathbb{H}_K^d) \otimes_K t^{-i+1} \mathbb{B}_{dR}^+ \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}^+) \to \Omega^i(\mathbb{H}_K^d)^{\text{d}=0} \otimes_K C(-i) \to 0$$

$$0 \to \text{Hom}(\text{Sp}_i(\mathbb{Z}), t^{-i+1} \mathbb{B}_{dR}^+) \to H^i_{\text{proét}}(\mathbb{H}_C^d, \mathbb{B}_{dR}) \to \text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathbb{B}_{dR}) \to 0$$

where the lower row is exact, and the left vertical arrow is an isomorphism, thanks to Remark 8.2. Here, the morphism $\pi_i$ is defined as the composite

$$\pi_i : \Omega^i(\mathbb{H}_K^d)^{\text{d}=0} \otimes_K C(-i) \to H^i_{\text{dR}}(\mathbb{H}_K^d) \otimes_K C(-i) \cong \text{Hom}(\text{Sp}_i(\mathbb{Z}), C(-i)) \to \text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathbb{B}_{dR}) \to 0$$

We deduce that

$$\ker \beta_i = \text{im } \gamma_i = \text{Hom}(\text{Sp}_i(\mathbb{Z}), t^{-i} \mathbb{B}_{dR}^+) \quad \text{and} \quad \text{coker } \beta_{i-1} = \ker \gamma_i = \Omega^i(\mathbb{H}_K^d)^{\text{d}=0}/ \text{ker } C^{-i}.$$
Next, we determine $\ker \alpha_i$ and $\coker \alpha_{i-1}$. By Theorem 3.15 for the cohomology theory $H^\bullet_{proét}(\mathbb{C}, \mathbb{E}_c)$, which applies thanks to Proposition 7.20, we have the following $G \times \mathcal{G}_K$-equivariant commutative diagram in $\text{Vect}_{\mathbb{Q}_p}^{\text{cond}}$ with exact rows

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \text{Hom}(\text{Sp}_i(\mathbb{Z}), B_{e}(-i)) & \longrightarrow & H^i_{proét}(\mathbb{H}^d_{\mathbb{C}}, \mathbb{E}_e) & \longrightarrow & 0 & \longrightarrow 0 \\
& & \downarrow \delta_i & & \downarrow \alpha_i & & & \\
0 & \longrightarrow & \text{Hom}(\text{Sp}_i(\mathbb{Z}), B_{\text{dR}}/t^{-i}B^+_{\text{dR}}) & \longrightarrow & H^i_{proét}(\mathbb{H}^d_{\mathbb{C}}, \mathbb{E}_{\text{dR}}/\mathbb{B}^+_\text{dR}) & \longrightarrow & \Omega^i(\mathbb{H}^d_{\mathbb{K}})/\ker d \otimes_K C(-i-1) & \longrightarrow 0
\end{array}
$$

where we recalled Remark 7.21. Now, using the fundamental exact sequence (4.3) over $\text{Sp}_i(C, O_C)_{proét}$ together with Corollary 4.7, we have a short exact sequence

$$
0 \longrightarrow \text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathbb{Q}_p(-i)) \longrightarrow \text{Hom}(\text{Sp}_i(\mathbb{Z}), B_{e}(-i)) \longrightarrow \text{Hom}(\text{Sp}_i(\mathbb{Z}), B_{\text{dR}}/t^{-i}B^+_{\text{dR}}) \longrightarrow 0
$$

where we observed that $R\text{Hom}(\text{Sp}_i(\mathbb{Z}), \mathbb{Q}_p(-i))$ is concentrated in degree 0 (see again Remark 8.2). In other words, $\coker \delta_i = 0$, and, by Remark 3.16, $\ker \delta_i = \text{Sp}_i(\mathbb{Q}_p)^*(-i)$, where the weak topological dual $\text{Sp}_i(\mathbb{Q}_p)^*$ is a $\mathbb{Q}_p$-Fréchet space. Therefore, by the snake lemma applied to the diagram above, we have

$$
\ker \alpha_i = \ker \delta_i = \text{Sp}_i(\mathbb{Q}_p)^*(-i)
$$

$$
\coker \alpha_{i-1} = (\Omega^{i-1}(\mathbb{H}^d_{\mathbb{C}})/\ker d)(-i).
$$

Hence, putting everything together, from the diagram (8.1), twisting by $(i)$ we obtain the following $G \times \mathcal{G}_K$-equivariant commutative diagram in $\text{Vect}_{\mathbb{Q}_p}^{\text{cond}}$ with exact rows

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \Omega^{i-1}(\mathbb{H}^d_{\mathbb{C}})/\ker d & \longrightarrow & H^i_{proét}(\mathbb{H}^d_{\mathbb{C}}, \mathbb{Q}_p(i)) & \longrightarrow & \text{Sp}_i(\mathbb{Q}_p)^* & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{i-1}(\mathbb{H}^d_{\mathbb{C}})/\ker d & \longrightarrow & H^i_{proét}(\mathbb{H}^d_{\mathbb{C}}, \mathbb{B}^+_{\text{dR}}(i)) & \longrightarrow & \text{Sp}_i(K)^* \otimes_K B^+_{\text{dR}} & \longrightarrow 0 \\
& & & & \downarrow q & & & \\
0 & \longrightarrow & \Omega^{i-1}(\mathbb{H}^d_{\mathbb{C}})/\ker d & \longrightarrow & \Omega^i(\mathbb{H}^d_{\mathbb{C}})^{d=0} & \longrightarrow & \text{Sp}_i(K)^* \otimes_K C & \longrightarrow 0
\end{array}
$$

where $q$ is the quotient by $tB^+_{\text{dR}}$ map. By Proposition A.25, we have $\text{Sp}_i(K)^* \otimes_K C = \text{Sp}_i(K)^* \otimes_K C$. From the diagram above we deduce in particular that $H^i_{proét}(\mathbb{H}^d_{\mathbb{C}}, \mathbb{Q}_p(i))$ is a $\mathbb{Q}_p$-Fréchet space, since it is the pullback of the following diagram of $\mathbb{Q}_p$-Fréchet spaces

$$
\text{Sp}_i(\mathbb{Q}_p)^* \longrightarrow \text{Sp}_i(K)^* \otimes_K C \longleftarrow \Omega^i(\mathbb{H}^d_{\mathbb{C}})^{d=0}.
$$

Then, the statement follows recalling Remark A.14. \hfill \Box

We used the following observation.
Remark 8.2. Let $W$ be a $K$-Fréchet space in $\text{Vect}_K^{\text{cond}}$. Writing $\text{Sp}_i(K) = \lim_{\longrightarrow} V_n$ as a countable direct limit of finite-dimensional topological $K$-vector spaces $V_n$ along immersions, as in Remark 3.16, we deduce that
\[
\mathcal{R}\text{Hom}(\text{Sp}_i(\mathbb{Z}), W) = \mathcal{R}\lim_{\longrightarrow} \mathcal{R}\text{Hom}_K(V_n, W) = \mathcal{R}\lim_{\longrightarrow} (\text{Hom}_K(V_n, K) \otimes_K W) = \text{Hom}_K(\text{Sp}_i(K), K) \otimes_K W
\]
concentrated in degree 0, where, in the last step, we used Corollary A.24, and the fact that $\mathcal{R}^j \lim_{\longrightarrow} \text{Hom}_K(V_n, K) = 0$ for all $j > 0$ (see Lemma 3.5).
Appendix A. Non-archimedean condensed functional analysis

This appendix is devoted to the condensed functional analysis over a non-archimedean local field, developed by Clausen-Scholze. Our main references will be [Sch19], [Sch20b], and [CS].

A.1. Notation and conventions. In this appendix, contrary to §1.6, we work in the category of all condensed sets (and not only $\kappa$-condensed sets), [Sch19, Definition 2.11].

We denote by $T \mapsto T$ the natural functor from the category of topological spaces/groups/rings/etc. to the category of sheaves of sets/groups/rings/etc. on the site of profinite sets with coverings given by finite families of jointly surjective maps; here, $T$ is defined via sending a profinite set $S$ to the set/group/ring/etc. of the continuous functions $C^0(S,T)$. Recall that if $T$ is a T1 topology space, then $T$ is a condensed set, [Sch19, Proposition 2.15].

Remark A.1 (On set-theoretic bounds). Let $\kappa$ be an uncountable strong limit cardinal as in §1.6.

(i) Recall that the category of $\kappa$-condensed sets embeds fully faithfully into the category of all condensed sets, via left Kan extension along the full embedding of the category of $\kappa$-small extremally disconnected sets into the category of all extremally disconnected sets. By [Sch19, Proposition 2.9], and the choice of the cardinal $\kappa$, the inclusion of the category of $\kappa$-condensed sets into the category of condensed sets commutes with all colimits, and with limits along an index category with cardinality less than $\kappa$.

(ii) If we temporarily denote by $T \mapsto T_\kappa$ the functor from the category of topological spaces to the category of $\kappa$-condensed sets, then, the proof of [Sch19, Proposition 2.15] shows that, given a T1 topological space $T$ with cardinality less than $\kappa$, we have $T_\kappa = T$ (in particular, in this case, $T_\kappa$ does not depend on our choice of the cardinal $\kappa$).

We denote by CondAb the category of condensed abelian groups. Given a condensed associative unital ring $A$, we denote by $\text{Mod}_A^{\text{cond}}$ the category of $A$-modules in CondAb, and we write $\text{Hom}_A(\cdot, \cdot)$ for the internal Hom in the category $\text{Mod}_A^{\text{cond}}$.

We write ExtrDisc for the category of extremally disconnected sets. For an analytic ring $(A, \mathcal{M})$, [Sch19, Definition 7.4], with underlying condensed ring $A$, and measures given by the functor $\mathcal{M}[-] : \text{ExtrDisc} \to \text{Mod}_A^{\text{cond}}$ we denote by $\text{Mod}_{(A, \mathcal{M})}$ the full subcategory of $\text{Mod}_A^{\text{cond}}$ whose objects are the $M \in \text{Mod}_A^{\text{cond}}$ such that, for all extremally disconnected sets $S$, the natural map

$\text{Hom}_A(\mathcal{M}[S], M) \to \text{Hom}_A(A[S], M) = M(S)$

is an isomorphism.

Recall that the pre-analytic ring $(\mathbb{Z}, \mathcal{M}_{\mathbb{Z}})$ given by the condensed ring $\mathbb{Z}$, together with the functor sending an extremally disconnected set $S$ to the condensed abelian group $\mathcal{M}_{\mathbb{Z}}[S] := \mathbb{Z}[S]^{\bullet}$ is an analytic ring, [Sch19, Theorem 5.8]. Here, $\mathbb{Z}[S]^{\bullet} := \lim_{\leftarrow} \mathbb{Z}[S_i]$ for $S = \lim_{\rightarrow} S_i$, written as a cofiltered limit of finite sets $S_i$, denotes the solidification of the free condensed abelian group $\mathbb{Z}[S]$. Recall that the category of solid abelian groups $\text{Solid} := \text{Mod}_{(\mathbb{Z}, \mathcal{M}_{\mathbb{Z}})}$ has a symmetric monoidal tensor product $\otimes_{\mathbb{Z}}$ (see [Sch19, Theorem 6.2]).

We will denote by $(K, |\cdot|)$ a non-archimedean local field, i.e. a field that is complete with respect to a non-trivial discrete valuation $|\cdot|$, and having finite residue field. We let $\mathcal{O}_K$ denote the ring

$\text{76}A$ topological space $T$ is T1 if for all $x \in T$, the set $\{x\}$ is closed.
of integers of $K$, and we fix $\varpi$ a uniformizer of $\mathcal{O}_K$. By abuse of notation, $K$ (resp. $\mathcal{O}_K$) will also denote the condensed ring $K$ (resp. $\mathcal{O}_K$).

A.2. Solid $K$-vector spaces. We begin by recalling some basic results on the category of solid $K$-vector spaces, that we use throughout the paper.

In the following, we denote by $\text{Vect}^\text{cond} := \text{Mod}^\text{cond}_K$ the category of condensed $K$-vector spaces.

**Proposition A.2.** Let $(K, \mathcal{M}_K)$ be the pre-analytic ring given by the condensed ring $K$, together with the functor

$$\mathcal{M}_K[-] : \text{ExtrDisc} \to \text{Vect}^\text{cond}_K : S \mapsto \mathcal{M}_K[S] \otimes^\mathbb{L} K.$$  

(i) The pre-analytic ring $(K, \mathcal{M}_K)$ is analytic.

(ii) The category $\text{Mod}_{(K, \mathcal{M}_K)}$ is the category of solid $K$-vector spaces, i.e. $K$-modules in Solid, which will be denoted by $\text{Vect}^\text{solid}_K$. It is a full abelian subcategory of $\text{Vect}^\text{cond}_K$ stable under all limits, colimits, and extensions. The objects $\mathcal{M}_K[S]$, for varying extremally disconnected $S$, form a family of compact projective generators of $\text{Vect}^\text{solid}_K$.

(iii) The inclusion $\text{Vect}^\text{solid}_K \subset \text{Vect}^\text{cond}_K$ admits a left adjoint

$$\text{Vect}^\text{cond}_K \to \text{Vect}^\text{solid}_K : V \mapsto V^\bullet$$

(A.1)

that is the unique functor sending $K[S] \mapsto \mathcal{M}_K[S]$ and preserving all colimits. Moreover, there is a unique symmetric monoidal tensor product $\otimes^\mathbb{L}_K$ on $\text{Vect}^\text{solid}_K$ making the functor (A.1) symmetric monoidal.

(iv) The functor $D(\text{Vect}^\text{solid}_K) \to D(\text{Vect}^\text{cond}_K)$ is fully faithful, and its essential image is stable under all limits and colimits. It admits a left adjoint

$$D(\text{Vect}^\text{cond}_K) \to D(\text{Vect}^\text{solid}_K) : \mathcal{V} \mapsto \mathcal{V}^{\text{left adj}}$$

(A.2)

that is the left derived functor of (A.1). The left derived functor of $\otimes^\mathbb{L}_K$ is the unique symmetric monoidal tensor product $\otimes^\mathbb{L}_K$ on $D(\text{Vect}^\text{solid}_K)$ making the functor (A.2) symmetric monoidal. Moreover, an object $\mathcal{V} \in D(\text{Vect}^\text{cond}_K)$ lies in $D(\text{Vect}^\text{solid}_K)$ if and only if $H^i(\mathcal{V})$ lies in $\text{Vect}^\text{solid}_K$ for all $i$.

(v) The objects $(\prod I \mathcal{O}_K)[1/\varpi]$, for varying sets $I$, form a family of compact projective generators of $\text{Vect}^\text{solid}_K$. We have $(\prod I \mathcal{O}_K)[1/\varpi] \otimes^\mathbb{L}_K (\prod J \mathcal{O}_K)[1/\varpi] = (\prod I \times J \mathcal{O}_K)[1/\varpi]$ for any sets $I$ and $J$.

**Proof.** Since $K = \mathcal{O}_K[1/\varpi]$, and $\mathcal{O}_K$ is a profinite abelian group (recall that $K$ has finite residue field, by assumption), by Lemma A.3, for any set $I$, we have that $(\prod I \mathbb{Z}) \otimes^\mathbb{L}_K K = (\prod I \mathcal{O}_K)[1/\varpi]$ (concentrated in degree 0). Hence, recalling [Sch19, Corollary 5.5], part (i) follows by adjunction from the analyticity of $(\mathbb{Z}, \mathcal{M}_\mathbb{Z})$. Parts (ii)-(iv) are a consequence of part (i), and [Sch19, Proposition 7.5]. For part (v), the first statement follows by adjunction from the fact that the objects $\prod I \mathbb{Z}$, for varying sets $I$, form a family of compact projective generators of Solid, [Sch19, Theorem 5.8, (i)], and the final statement follows from [Sch19, Proposition 6.3].

We used the following general lemma.

**Lemma A.3 ([CS]).** Let $M$ be a profinite abelian group. For any set $I$, we have $(\prod I \mathbb{Z}) \otimes^\mathbb{L}_\mathbb{Z} M = \prod I M$, which is concentrated in degree 0.
Proof. Recall that any discrete abelian group has a 2-term free resolution; then, since $M$ is a compact abelian group, by Pontrjagin duality, it admits a resolution of the form

$$0 \to M \to \prod_{J_0} \mathbb{R}/\mathbb{Z} \to \prod_{J_1} \mathbb{R}/\mathbb{Z} \to 0 \quad (A.3)$$

where $\mathbb{R}/\mathbb{Z}$ is the circle group. Moreover, since (A.3) is a strictly exact sequence of locally compact abelian groups, it remains exact after applying the functor $T : \mathcal{T} \to \mathcal{T}$ (see [Sch19, page 26]). Hence, applying the left derived solidification functor to (A.3), we get the distinguished triangle

$$M \xrightarrow{L} \prod_{J_0} \mathbb{Z}^1 \to \prod_{J_1} \mathbb{Z}^1 (A.4)$$

where we used that $\mathbb{R}^\bullet = 0$ (see [Sch19, Corollary 6.1, (iii)]). Then, applying the functor $(\prod_I \mathbb{Z}) \otimes^\bullet \mathbb{Z}$ to (A.4), we obtain the statement from [Sch19, Proposition 6.3]. □

Remark A.4. The proof of Proposition A.2 also shows that the pre-analytic ring $(\mathcal{O}_K, \mathcal{M}_{\mathcal{O}_K})$ given by the condensed ring $\mathcal{O}_K$, together with the functor

$$\mathcal{M}_{\mathcal{O}_K}[-] : \text{ExtrDisc} \to \text{Mod}_{\text{cond}}^\mathcal{O}_K : S \mapsto \mathcal{M}_\mathbb{Z}[S] \otimes^\bullet \mathcal{O}_K$$

is analytic. Moreover, $\text{Mod}^{\text{solid}}_{\mathcal{O}_K} := \text{Mod}_{(\mathcal{O}_K, \mathcal{M}_{\mathcal{O}_K})}$ is the category of $\mathcal{O}_K$-modules in Solid, and it is generated by the family of compact projective objects $\prod_I \mathcal{O}_K$, for varying sets $I$. Note that the category $\text{Vect}^{\text{solid}}_K$ is the full subcategory of $\text{Mod}^{\text{solid}}_{\mathcal{O}_K}$ of the objects on which $\varpi$ acts invertibly.

In the main body of the paper we need to work in the category of $\kappa$-condensed sets (see §1.6), therefore, we make the following remark on set-theoretic bounds.

Remark A.5. Let $\kappa$ be an uncountable strong limit cardinal. It is possible to define the notion of (pre-)analytic ring for $\kappa$-condensed associative unital rings, replacing extremally disconnected sets with $\kappa$-small extremally disconnected sets in [Sch19, Definition 7.1, Definition 7.4]. Then, with this definition, the pre-analytic ring $(\mathbb{Z}, \mathcal{M}_\mathbb{Z})_\kappa$ given by the $\kappa$-condensed ring $\mathbb{Z}$, together with the functor sending a $\kappa$-small extremally disconnected set $S$ to the $\kappa$-condensed abelian group $\mathcal{M}_\mathbb{Z}[S] := \mathbb{Z}[S]^\bullet$ is an analytic ring. We observe that the category $\text{Solid}_\kappa := \text{Mod}_{(\mathbb{Z}, \mathcal{M}_\mathbb{Z})_\kappa}$ of $\kappa$-solid abelian groups embeds fully faithfully in Solid. Moreover, Proposition A.2 (with the obvious modifications) holds true for the $\kappa$-condensed ring $K$, except for the first assertion of part (v). The $K$-modules in Solid$_\kappa$ are called $\kappa$-solid $K$-vector spaces.

Now, we recall some important results about the following notion of flatness for the objects of the symmetric monoidal category $\text{Vect}^{\text{solid}}_K$, endowed with the tensor product $\otimes^\bullet_K$.

Definition A.6. We say that an object $V \in \text{Vect}^{\text{solid}}_K$ is flat if, for all $W \in \text{Vect}^{\text{solid}}_K$, we have that $V \otimes^\bullet_K W$ is concentrated in degree 0.

Proposition A.7 ([CS]). For any set $I$, the solid $K$-vector space $(\prod_I \mathcal{O}_K)[1/\varpi]$ is flat.

As a consequence of Proposition A.7, by the same argument used in the proof of Proposition A.2(i), we have the following result.

\footnote{In fact, the right arrow of (A.3) is an open map by the following well-known fact: suppose that $f : G \to H$ is a surjective morphism of Hausdorff topological groups and $G$ is compact, then $f$ is open.}
Corollary A.8. Let $A$ be a solid $K$-algebra. The pre-analytic ring $(A, \mathcal{M}_A)$ given by the condensed ring $A$, together with the functor
\[
\mathcal{M}_A[-] : \text{ExtrDisc} \to \text{Mod}^\text{cond}_A : S \mapsto \mathcal{M}_K[S] \otimes^K A
\]
is analytic. The category $\text{Mod}_{(A, \mathcal{M}_A)}$ is the category of $A$-modules in $\text{Vect}^\text{solid}_K$.

Clausen-Scholze proved the following nice characterization of the quasi-separated solid $K$-vector spaces, i.e. the objects of the category $\text{Vect}^\text{solid}_K$ whose underlying condensed set is quasi-separated.

Proposition A.9 ([CS]). A solid $K$-vector space is quasi-separated if and only if it is the filtered union of its subobjects isomorphic to $(\prod_I \mathcal{O}_K)[1/\varpi]$ for some set $I$.

Corollary A.10 ([CS]). Any quasi-separated solid $K$-vector space is flat.

Proof. Using that the abelian category $\text{Vect}^\text{solid}_K$ satisfies Grothendieck’s axiom (AB5), the statement follows from Proposition A.7 and Proposition A.9. □

Next, we study some familiar objects of the classical theory of locally convex $K$-vector spaces, from the point of view of condensed mathematics. Let us begin with the following crucial observation.

Lemma A.11. Let $V$ be a $K$-Banach space. Then, there exists a profinite set $S$, and an isomorphism of condensed $K$-vector spaces
\[
V \cong \text{Hom}(\mathbb{Z}[S], K).
\]

Proof. Since the valuation of the field $K$ is discrete, by [PGS10, Theorem 2.5.4], the $K$-Banach space $V$ admits a basis. Therefore, by [PG15, Proposition 4.6], there exists a profinite set $S$ such that $V$ is isomorphic to the $K$-Banach space $\mathcal{C}^0(S, K)$, endowed with the sup-norm. Then, we have $V \cong \mathcal{C}^0(S, K) \cong \text{Hom}(\mathbb{Z}[S], K)$, recalling that, for any profinite set $S'$,
\[
\mathcal{C}^0(S', \mathcal{C}^0(S, K)) \cong \mathcal{C}^0(S' \times S, K)
\]
(see [PGS10, Theorem 10.5.6]). □

Proposition A.12 ([CS]). Let $V$ be a complete locally convex $K$-vector space. Then, $V$ is a quasi-separated solid $K$-vector space.

Proof. Recall that any complete locally convex $K$-vector space is isomorphic to a cofiltered limit of $K$-Banach spaces, in the category of topological $K$-vector spaces (cf. [SW99, Chapter II, §5.4]). Then, since the subcategory of quasi-separated solid $K$-vector spaces in $\text{Vect}^\text{cond}_K$ is stable under limits, we can reduce to the case that $V$ is a $K$-Banach space. In the latter case, $V$ is quasi-separated by [Sch19, Theorem 2.16].

To show that $V$ is a solid $K$-vector space, by Lemma A.11, we can suppose $V = \text{Hom}(\mathbb{Z}[S], K)$, with $S$ a profinite set. Since a solid $K$-vector space is a $K$-module in Solid, we need to prove that $\text{Hom}(\mathbb{Z}[S], K)$ is a solid abelian group. This follows formally from the fact that $K$ is a solid abelian group: in fact, by adjunction, for all profinite sets $T$, there is a natural isomorphism
\[
\text{Hom}(\mathbb{Z}[T], \text{Hom}(\mathbb{Z}[S], K)) \cong \text{Hom}(\mathbb{Z}[T], K)(S) \cong \text{Hom}(\mathbb{Z}[T], \text{Hom}(\mathbb{Z}[S], K)).
\]

Now, we focus our attention on the category of $K$-Fréchet spaces, i.e. the category of complete metrizable locally convex $K$-vector spaces.
Remark A.13. A locally convex $K$-vector space is a $K$-Fréchet space if and only if it is isomorphic, in the category of topological $K$-vector spaces, to the limit of a countable inverse system of $K$-Banach spaces along transition maps having dense image (cf. [SW99, Chapter II, §5.4, Corollary 1]).

Remark A.14. Any $K$-Fréchet space is compactly generated, being metrizable, hence, by [Sch19, Proposition 1.7, Proposition 2.15], the category of $K$-Fréchet spaces embeds fully faithfully into $\text{Vect}_K^{\text{cond}}$, via the functor $V \mapsto V$.

The next result is often useful when passing from $K$-Fréchet spaces to the associated condensed $K$-vector spaces.

Lemma A.15. The functor $V \mapsto V$ sends acyclic complexes of $K$-Fréchet spaces to acyclic complexes of condensed $K$-vector spaces.

Proof. Note that the kernel of a map of $K$-Fréchet spaces is a closed subspace of the source, hence it is a $K$-Fréchet space. Then, since the functor $V \mapsto V$ is left exact, by the open mapping theorem for $K$-Fréchet spaces, [PGS10, Theorem 3.5.10], it suffices to prove the following general statement: given an open surjective map $f : V \to W$ of complete metrizable topological $K$-vector spaces, then the induced map $V \to W$ is surjective.

We need to show that, for $S$ an extremally disconnected set, we have $C^0(S,V) \to C^0(S,W)$, i.e. given $\psi \in C^0(S,W)$, there exists $\tilde{\psi} \in C^0(S,V)$ making the following diagram commute

\[
\begin{array}{ccc}
V & \to & W \\
\downarrow & & \downarrow \\
S & \xrightarrow{\psi} & W
\end{array}
\]

By [Tre67, Lemma 45.1], since $\psi(S)$ is compact in $W$, it is the image $f(H)$ of a compact subset $H$ of $V$. We conclude by recalling that the extremally disconnected sets are the projective objects of the category of compact Hausdorff topological spaces. □

In particular, by Lemma A.15, given $V$ a $K$-Fréchet space, and $W \subset V$ a closed subspace, we have that $V/W = V/W$. However, in general, given $V$ a topological $K$-vector space, and $W \subset V$ a subspace, then, taking the quotient $V/W$ in the category of topological $K$-vector spaces may cause a high loss of topological information; instead, taking the quotient $V/W$ in the category of condensed $K$-vector spaces keeps track of the topological information of both $V$ and $W$. This phenomenon is illustrated in the following example.

Example A.16. Let $V$ be a Hausdorff topological $K$-vector space, having a subspace $W \subset V$ such that the topological $K$-vector quotient space $V/W$ is non-Hausdorff. Recall that this is the case if and only if $W \subset V$ is non-closed, if and only if $V/W$ is not T1. Then, we have that $V/W \neq V/W$, since $V/W$ is not even a condensed set (see [Sch19, Warning 2.14]).

We note that this is not a pathology of the definition of the category of condensed sets, [Sch19, Definition 2.11]. In fact, let us denote by $T \mapsto T^\kappa$ the functor from the category of topological spaces to the category of $\kappa$-condensed sets; then, we claim that, given an uncountable strong limit cardinal $\kappa > |V|$, we have that $V^\kappa/W^\kappa \neq V/W^\kappa$ as $\kappa$-condensed sets. In order to show this, we

\[\text{More precisely, it is also } \kappa\text{-compactly generated, for any uncountable cardinal } \kappa\text{ (see [Sch19, Remark 1.6]).}\]
need to find a $\kappa$-small extremally disconnected set $S$ such that the map $\mathcal{E}^0(S,V) \to \mathcal{E}^0(S,V/W)$, induced by the quotient morphism $V \to V/W$, is not surjective. Let $\lambda$ be the cardinality of $V$, let $S_0$ be a discrete set with cardinality $\lambda$, and let $S := \beta S_0$ be the Stone-Čech compactification of $S_0$. Note that $|S| = 2^{2\lambda} < \kappa$. Since the (continuous) inclusion map $S_0 \to S$ has dense image, a continuous function from $S$ to the Hausdorff space $V$ is determined by its values on $S_0$, in particular the set $\mathcal{E}^0(S,V)$ has cardinality at most $\lambda^\lambda = 2^\lambda$. On the other hand, denoting by $\overline{W}$ the closure of $W$ in $V$, we observe that the subspace $\overline{W}/W \subseteq V/W$ has the indiscrete topology, therefore, every function $S \to \overline{W}/W$ is continuous; recalling that $|S| = 2^{2\lambda}$, we deduce that the set $\mathcal{E}^0(S,V/W)$ has cardinality at least $2^{2^{2\lambda}}$. Hence, there cannot be a surjective map from $\mathcal{E}^0(S,V)$ onto $\mathcal{E}^0(S,V/W)$.

Relatively, in general, the functor $V \mapsto V$ does not commute with filtered colimits; however, it does in the following special case.

**Example A.17.** Let $\{V_n\}$ be a countable direct system of Hausdorff topological $K$-vector spaces, whose transitions maps are closed immersions. Let us define the $K$-vector space $V := \lim_{\longrightarrow} V_n$, and suppose that, endowing it with the direct limit topology, $V$ is a topological $K$-vector space. Then, by [BS15, Lemma 4.3.7] we have that $V = \lim_{\longrightarrow} V_n$ as condensed $K$-vector spaces.

Let us also record the following condensed version of the classical topological Mittag-Leffler lemma for $K$-Banach spaces.

**Lemma A.18 ([CS]).** Let $\{V_n, f_{nm}\}$ be an inverse system of $K$-Banach spaces indexed by $\mathbb{N}$. Suppose that for each $n \in \mathbb{N}$, there exists $k \geq n$ such that, for every $m \geq k$, $f_{nm}(V_m)$ is dense in $f_{nk}(V_k)$. Then, for all $j > 0$, we have

$$R^j \lim_{\longrightarrow} V_n = 0.$$  

**Proof.** By [Sch13b, Lemma 3.18], it suffices to show that, for any $S$ extremally disconnected set, the inverse system of $K$-Banach spaces $\{\mathcal{E}^0(S,V_n)\}$ (endowed with the sup-norm), satisfies

$$R^1 \lim_{\longrightarrow} \mathcal{E}^0(S,V_n) = 0.$$  

Let $S$ be an extremally disconnected set, and let us denote by $\psi_{nm} : \mathcal{E}^0(S,V_m) \to \mathcal{E}^0(S,V_n)$ the transition map given by the composition with $f_{nm}$. Recall that, for any $n \in \mathbb{N}$, the locally constant functions $\text{LC}(S,V_n)$ are dense in the continuous functions $\mathcal{E}^0(S,V_n)$, and observe that, for any $m \geq n$, we have $\psi_{nm}(\text{LC}(S,V_m)) = \text{LC}(S,f_{nm}(V_m))$. We deduce that, for each $n \in \mathbb{N}$, there exists $k \geq n$ such that, for every $m \geq k$, $\psi_{nm}(\mathcal{E}^0(S,V_m))$ is dense in $\psi_{nk}(\mathcal{E}^0(S,V_k))$. Then, the statement follows from the topological Mittag-Leffler lemma [Gro61, Remarques 13.2.4].

**A.3. Nuclear $K$-vector spaces.** In this section, we recall a useful characterization of the category of nuclear $K$-vector spaces, and explain some of its consequences.

Clausen-Scholze introduced the following important class of solid $K$-vector spaces.

**Definition A.19.** An object $V \in \text{Vect}_{K}^\text{solid}$ is a nuclear $K$-vector space if, for all profinite sets $S$, the natural map of condensed $K$-vector spaces

$$\Hom(\mathbb{Z}[S], K) \otimes_K^\wedge V \to \Hom(\mathbb{Z}[S], V)$$  

is an isomorphism. We denote by $\text{Vect}_{K}^\text{nuc}$ the full subcategory of $\text{Vect}_{K}^\text{solid}$ whose objects are the nuclear $K$-vector spaces.
We warn the reader that the definition of nuclear $K$-vector space recalled above is quite different from the one adopted in the classical non-archimedean functional analysis' literature (see e.g. [Sch02, Chapter IV, §19]). In fact, using the latter definition, a $K$-Banach space is nuclear if and only if it is finite-dimensional over $K$, [Sch02, Chapter IV, §19, p. 120]. Instead, adopting Definition A.19, one has the following result.

**Theorem A.20 ([CS]).** The subcategory $\text{Vect}_{\text{mc}}^K \subset \text{Vect}_{\text{solid}}^K$ is an abelian category, stable under finite limits, countable products, and all colimits. Moreover, it is generated under colimits by the $K$-Banach spaces.\(^{(79)}\)

**Remark A.21.** Let $\kappa$ be an uncountable strong limit cardinal. For any $K$-Banach space $V$, we have that $V$ is a $\kappa$-condensed set ([CS]). In particular, by Theorem A.20, $\text{Vect}_{\text{mc}}^K$ is a full subcategory of the category of $\kappa$-condensed $K$-vector spaces.

**Corollary A.22.** Any $K$-Fréchet space is a nuclear $K$-vector space.

**Proof.** The statement follows immediately from Remark A.13 and Theorem A.20. \(\square\)

By Proposition A.2, we know that the tensor product $\otimes_K^*$ commutes with colimits in both variables. Next, we recall a particular case in which it commutes with limits.

**Proposition A.23 ([CS]).** Let $\{V_n\}$ be a countable family of nuclear $K$-vector spaces, and let $W$ be a $K$-Fréchet space. Then, the natural map of condensed $K$-vector spaces
$$\left(\prod_n V_n\right) \otimes_K^* W \to \prod_n (V_n \otimes_K^* W)$$
is an isomorphism.

Let us collect some corollaries of Proposition A.23.

**Corollary A.24.**

(i) Let $\{V_n\}$ be a countable inverse system of nuclear $K$-vector spaces, and let $W$ be a $K$-Fréchet space. Then, we have
$$\left(\lim_n V_n\right) \otimes_K^* W = \lim_n (V_n \otimes_K^* W).$$

(ii) Let $\{V_n\}$ be a countable inverse system of objects in $D(\text{Vect}_{\text{solid}}^K)$ such that each $V_n$ is representable by a complex of nuclear $K$-vector spaces. Let $W \in D(\text{Vect}_{\text{solid}}^K)$ be representable by a bounded above complex of $K$-Fréchet spaces. Then, we have
$$\left(R\lim_n V_n\right) \otimes^L_K W = R\lim_n (V_n \otimes^L_K W).$$

**Proof.** We first prove part (ii). Consider the distinguished triangle in $D(\text{Vect}_{\text{solid}}^K)$
$$R\lim_n V_n \to \prod_n V_n \to \prod_n V_n$$
(A.6)
where the last arrow is the difference of the identity and the transition morphisms. Applying the functor $- \otimes^L_K W$ to (A.6), we see that it suffices to prove that the natural map
$$\left(\prod_n V_n\right) \otimes^L_K W \to \prod_n (V_n \otimes^L_K W).$$
(A.7)

\(^{(79)}\)Here, and in the following, by abuse of terminology, we call a $K$-Banach/Fréchet space in $\text{Vect}_{\text{cond}}^K$ an object of the form $\underline{V}$ for a $K$-Banach/Fréchet space $V$. 

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is an isomorphism. Let $W^\bullet$ be a bounded above complex of $K$-Fréchet spaces, representing $W$. By Proposition A.12, a $K$-Fréchet space, regarded as a condensed set, is quasi-separated solid $K$-vector space, then, by Corollary A.10, $W^\bullet$ is a bounded above complex of flat solid $K$-vector spaces (Definition A.6); in particular, for any complex $K^\bullet \in D(Vect^\text{solid}_K)$, the derived tensor product $K^\bullet \otimes^\mathbb{L}_K W^\bullet$ is represented by the total complex $\text{Tot}(K^\bullet \otimes^\mathbb{L}_K W^\bullet)$. Moreover, since the category $\text{Vect}^\text{solid}_K$ satisfies Grothendieck’s axiom (AB4*), by [Sta, Tag 07KC], we know that the countable product $\prod_n V_n$ in $D(\text{Vect}^\text{solid}_K)$ is computed by taking the termwise products of any of the complexes representing the $V_n$. Therefore, the assertion that the natural map (A.7) is an isomorphism reduces to the statement of Proposition A.23.

Part (i) can be shown using a similar argument, or it can be deduced from part (ii), recalling that the $K$-Fréchet space $W$ is a flat solid $K$-vector space. \hfill $\square$

Also the following result is hidden in the statement of Proposition A.23: on $K$-Fréchet spaces the completed projective tensor product $\hat{\otimes}_K$ agrees with $\otimes^\mathbb{L}_K$.\hfill $80$

**Proposition A.25 ([CS]).** Let $V$ and $W$ be $K$-Fréchet spaces. Then, the natural map of condensed $K$-vector spaces

$$V \otimes^\mathbb{L}_K W \to V \hat{\otimes}_K W$$

is an isomorphism.\hfill $81$

*Proof.* By Remark A.13, we can suppose that $V$ (resp. $W$) is the limit of a countable inverse system of $K$-Banach spaces $\{V_n\}$ (resp. $\{W_m\}$) with transition maps having dense image. By Corollary A.24(i), combined with Corollary A.22, we have

$$V \otimes^\mathbb{L}_K W = \lim_{\longrightarrow n,m} V_n \otimes^\mathbb{L}_K W_m.$$ 

Moreover, we have $V \hat{\otimes}_K W = \lim_{\longrightarrow n,m} V_n \hat{\otimes}_K W_m$.\hfill $82$ Therefore, to show the statement, we can reduce to the case that $V$ and $W$ are $K$-Banach spaces. In the latter case, by (the proof of) Lemma A.11, there exists a profinite set $S$, such that $V \cong C^0(S, K)$ (endowed with the sup-norm), and we have $V \cong \text{Hom}(\mathbb{Z}[S], K)$. Then, the statement follows from the fact that $W$ is a nuclear $K$-vector space (Theorem A.20), and [PGS10, Corollary 10.5.4, Corollary 10.5.7]. \hfill $\square$

**Remark A.26.** Let $A$ and $B$ be two $K$-Banach algebras, and denote by $A \hat{\otimes}_K B$ their completed tensor product in the category of $K$-Banach algebras. Recall that the topological $K$-vector space underlying the $K$-Banach algebra $A \hat{\otimes}_K B$ is the completed projective tensor product of $A$ and $B$, regarded as $K$-Banach spaces (see e.g. [Bos14, Appendix B]). Therefore, by Proposition A.25, the natural map of condensed $K$-algebras

$$A \otimes^\mathbb{L}_K B \to A \hat{\otimes}_K B$$

is an isomorphism.

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$80$ Let $V$ and $W$ be two (possibly non-Hausdorff) locally convex $K$-vector spaces. Recall that the completed projective tensor product $V \hat{\otimes}_K W$ is the Hausdorff completion of the projective tensor product of $V$ and $W$, [PGS10, Definition 10.3.2].

$81$ Here, the map (A.25) is constructed using that, by Proposition A.12, $V \hat{\otimes}_K W$ is a solid $K$-vector space.

$82$ This follows from [SJF+72, Proposition 9, p. 192] and (the proof of) [Var07, Corollary 1.7, (b)].
Our goal in this appendix is to revisit the definition of condensed group cohomology, due to Bhatt-Scholze, and explain its relation to Koszul complexes (Definition 7.7), in some cases of particular interest to us.

We keep the notation and the conventions of Appendix A, §A.1. In particular, in this appendix, contrary to §1.6, we work in the category of all condensed sets (and not only $\kappa$-condensed sets).

Let $G$ be a condensed group, and let $M$ be a $G$-module in condensed abelian groups, i.e. a condensed abelian group $M$ endowed with a left $G$-action $G \times M \to M$ in the category of condensed sets. Denoting by $\mathbb{Z}[G]$ the condensed group ring of $G$ over $\mathbb{Z}$, we can regard $M$ as a $\mathbb{Z}[G]$-module in CondAb, and give the following definition (cf. [BS15, §4.3]).

**Definition B.1.** We define the $i$th condensed group cohomology of $G$ with coefficients in $M$ as the condensed abelian group $\text{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) := R^i \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ where $\mathbb{Z}$ is endowed with the trivial $G$-action.\(^{83}\)

As we now explain, in many cases, the condensed group cohomology recovers the continuous group cohomology (cf. [BS15, Lemma 4.3.9]). However, Definition B.1 has some favorable extra features: for example, given a short exact sequence of $G$-modules in condensed abelian groups, one always gets long exact sequences in cohomology. Note also that, contrary to the continuous group cohomology, the condensed group cohomology has a natural “topological structure” (more precisely, a condensed structure) by definition, and one can show it admits a Hochschild-Serre spectral sequence.

We learned the following proposition, which is certainly well-known to experts, from Anschütz and Le Bras.

**Proposition B.2.** Let $G$ be a profinite group, and let $M$ be a $T1$ topological $G$-module over $\mathbb{Z}$, such that $M$ is a solid abelian group.

(i) The complex $R\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ is quasi-isomorphic to the complex of solid abelian groups $M \to \text{Hom}(\mathbb{Z}[G], M) \to \text{Hom}(\mathbb{Z}[G \times G], M) \to \cdots$ sitting in non-negative cohomological degrees.\(^{84}\)

(ii) For all $i \geq 0$, we have a canonical isomorphism of abelian groups $\text{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) \cong H^i_{\text{cont}}(G, M)$.

As preparation, we need a formal lemma on Solid. Denote by $R\text{Hom}_{\text{Solid}}(-, -)$ the derived internal Hom in the symmetric closed monoidal category $D(\text{Solid})$, endowed with the derived solid tensor product $\otimes^L_{\mathbb{Z}}$: for all $P, M, N \in D(\text{Solid})$, we have the adjunction

$$\text{Hom}_{D(\text{Solid})}(P, R\text{Hom}_{\text{Solid}}(M, N)) \cong \text{Hom}_{D(\text{Solid})}(P \otimes^L_{\mathbb{Z}} M, N).$$

\(^{83}\)Contrary to the category of $\kappa$-condensed abelian groups, the category of all condensed abelian groups has no non-zero injective objects (see [Sch20a]). However, we can compute $\text{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ by taking a projective resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$.

\(^{84}\)The differentials of the complex are described in the proof.
Lemma B.3. For all \( M, N \in D(\text{Solid}) \), the natural map

\[
R\text{Hom}_{\text{Solid}}(M, N) \to R\text{Hom}_{\text{CondAb}}(M, N)
\]

is an isomorphism.

Proof. Recalling that the category Solid is generated under colimits by \( \mathbb{Z}[S]^\bullet \), for varying extremally disconnected sets \( S \), we can assume that \( M = \mathbb{Z}[T]^\bullet \), for some extremally disconnected set \( T \). In this case, we note that \( R\text{Hom}_{\text{Solid}}(\mathbb{Z}[T]^\bullet, N) \) is concentrated in degree 0, since \( \mathbb{Z}[T]^\bullet \) is a projective object of Solid. By [Sch19, Corollary 6.1, (iv)], the natural map

\[
R\text{Hom}_{\text{CondAb}}(\mathbb{Z}[T]^\bullet, N) \to R\text{Hom}_{\text{CondAb}}(\mathbb{Z}[T], N)
\]

is an isomorphism; we deduce that \( R\text{Hom}_{\text{CondAb}}(\mathbb{Z}[T]^\bullet, N) \) is also concentrated in degree 0 (since \( \mathbb{Z}[T] \) is a projective object of CondAb), and it is a solid abelian group (cf. the last part of the proof of Proposition A.12). Then, we can reduce to show that, for all extremally disconnected sets \( S \), the natural map

\[
\text{Hom}_{D(\text{Solid})}(\mathbb{Z}[S]^\bullet, R\text{Hom}_{\text{Solid}}(\mathbb{Z}[T]^\bullet, N)) \to \text{Hom}_{D(\text{Solid})}(\mathbb{Z}[S]^\bullet, R\text{Hom}_{\text{CondAb}}(\mathbb{Z}[T]^\bullet, N))
\]

(B.1)

is an isomorphism. By adjunction, the source of (B.1) is isomorphic to

\[
\text{Hom}_{D(\text{Solid})}(\mathbb{Z}[S \times T]^\bullet, N) \cong \text{Hom}_{D(\text{CondAb})}(\mathbb{Z}[S \times T]^\bullet, N) \cong \text{Hom}_{D(\text{CondAb})}(\mathbb{Z}[S \times T], N)
\]

recalling that the functor \( D(\text{Solid}) \to D(\text{CondAb}) \) is fully faithful, [Sch19, Theorem 5.8, (ii)]. Using the same ingredients, one readily checks that also the target of the natural map (B.1) is isomorphic to \( \text{Hom}_{D(\text{CondAb})}(\mathbb{Z}[S \times T], N) \), as desired. \( \square \)

Proof of Proposition B.2. We consider the bar resolution\(^{85}\) of \( \mathbb{Z} \) over \( \mathbb{Z}[G] \)

\[
\cdots \to \mathbb{Z}[G \times G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.
\]

Applying \( R\text{Hom}_{\mathbb{Z}[G]}(-, M) \), we obtain the spectral sequence

\[
E_1^{i,j} = \text{Ext}_\mathbb{Z}[G]^j(\mathbb{Z}[G]^i, M) \Rightarrow \text{Ext}_\mathbb{Z}[G]^{i+j}(Z, M).
\]

Note that \( E_1^{i,j} = \text{Ext}_\mathbb{Z}[G]^{j}(\mathbb{Z}[G]^{i-1}, M) \), for all \( j \geq 0 \) and \( i > 0 \). Since \( M \) is a solid abelian group, by [Sch19, Corollary 6.1, (iv)], and Lemma B.3, for all profinite sets \( S \), and for all \( j > 0 \), we have

\[
\text{Ext}_\mathbb{Z}[G]^{j}(\mathbb{Z}[S], M) \cong \text{Ext}_\mathbb{Z}[G]^{j}(\mathbb{Z}[S]^\bullet, M) \cong \text{Ext}_{\text{Solid}}^{j}(\mathbb{Z}[S]^\bullet, M) = 0
\]

where in the last step we used that \( \mathbb{Z}[S]^\bullet \) is a projective object of the category Solid (see [Sch19, Corollary 5.5] and [Sch19, Corollary 6.8, (i)]). In particular, we have that \( E_1^{i,j} = 0 \), for all \( j > 0 \) and \( i > 0 \), which gives part (i).

Part (ii) follows from part (i). In fact, by [Sch19, Proposition 1.7], for every integer \( n \geq 0 \), we have that \( \text{Hom}(\mathbb{Z}[G^n], M) = \mathcal{E}^0(G^n, M) \). We conclude observing that the differentials of the complexes computing respectively \( \text{Ext}_{\mathbb{Z}[G]}^{i}(Z, M) \) and \( H^i_{\text{cont}}(G, M) \) agree as well. \( \square \)

The following result generalizes [BMS18, Lemma 7.3].

\(^{85}\)Let \( \kappa \) be an uncountable strong limit cardinal such that the condensed set \( \mathcal{G} \) is the left Kan extension of its restriction to \( \kappa \)-small extremally disconnected sets. Then, the resolution (B.2) is the left Kan extension of the sheafification (on the site of \( \kappa \)-small extremally disconnected sets, with coverings given by finite families of jointly surjective maps) of the functor sending a \( \kappa \)-small extremally disconnected set \( S \) to the bar resolution of \( \mathbb{Z} \) over \( \mathbb{Z}[\mathcal{G}(S)] \), [Bro82, Chapter I, §5].
**Proposition B.4.** Given an integer \( n \geq 1 \), let \( \Gamma := \mathbb{Z}_{p}^{n} \), and let \( \gamma_{1}, \ldots, \gamma_{n} \) denote the canonical generators of \( \Gamma \). Let \( M \) a \( \Gamma \)-module in \( \text{Mod}_{\mathbb{Z}_{p}}^{\text{solid}} \). Then, we have a quasi-isomorphism

\[
R\text{Hom}_{\mathbb{Z}[[\Gamma]]}(\mathbb{Z}, M) \simeq \text{Kos}_{M}(\gamma_{1} - 1, \ldots, \gamma_{n} - 1).
\]

First, we prove a general lemma.

**Lemma B.5.** Let \( G \) be a profinite group, and let \( R \) be a profinite commutative unital ring.

We define the Iwasawa algebra of \( G \) over \( R \) as the condensed \( R \)-algebra

\[
R[G] := \lim_{U} R[G/U]
\]

where \( U \) runs over all the open normal subgroups of \( G \).

(i) We have \( R[G] = R[G]^{\bullet} \), i.e. \( R[G] \) is the solidification of the condensed \( R \)-algebra \( R[G] \).

(ii) The pre-analytic ring \( (R[G], \mathcal{M}_{R,G}) \) given by the condensed ring \( R[G] \) together with the functor

\[
\mathcal{M}_{R,G}[-] : \text{ExtrDisc} \rightarrow \text{Mod}_{R[G]}^{\text{cond}} : S \mapsto \mathcal{M}_{Z[S]} \otimes_{Z} R[G]
\]

is an analytic ring. The category \( \text{Mod}_{(R[G], \mathcal{M}_{R,G})} \) is the category of \( R[G] \)-modules in Solid, and it is generated by the compact projective objects \( \prod_{I} R[G] \), for varying sets \( I \).

(iii) Assume, in addition, that \( G \) is an abelian group. Then, the derived category \( D(R[G], \mathcal{M}_{R,G}) \) endowed with the derived tensor product \( \otimes^{L} \) is a symmetric closed monoidal category, with derived internal Hom denoted by \( R\text{Hom}_{\mathcal{M}_{R,G}}(-,-) \). For all \( M, N \in D(R[G], \mathcal{M}_{R,G}) \), the natural map

\[
R\text{Hom}_{\mathcal{M}_{R,G}}(M, N) \rightarrow R\text{Hom}_{R[G]}(M, N)
\]

(B.3)

is an isomorphism.

**Proof.** For part (i), we first note that \( Z[G]^{\bullet} = \lim_{U} Z[G/U] \), where \( U \) runs over all the open normal subgroups of \( G \). Then, the statement follows observing that, by [Sch19, Corollary 5.5] and Lemma A.3, for any profinite set \( S \), we have

\[
R[S]^{L} \otimes_{Z} R \simeq (\prod_{I} Z) \otimes_{Z} R = \prod_{I} R
\]

(B.4)

concentrated in degree 0, for some set \( I \) depending on \( S \).

For part (ii), we note that, by the proof of the previous point, \( R[G] \) is profinite: in fact, we have \( R[G] \cong \prod_{J} R \), for some set \( J \). Hence, applying again Lemma A.3, for any set \( I \), we have \( (\prod_{I} Z) \otimes_{Z} R[G] = \prod_{I} R[G] \) (concentrated in degree 0). Then, the first statement of part (ii) follows by adjunction from the analyticity of \((Z, \mathcal{M}_{Z})\), and the second assertion follows from [Sch19, Theorem 5.8, (i)].

For part (iii), we use the analyticity of \((R[G], \mathcal{M}_{R,G})\) from part (ii), together with [Sch19, Proposition 7.5]. The first assertion is clear, then, it remains to prove that the natural map (B.3) is an

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86 Recall that \( \text{Mod}_{\mathbb{Z}_{p}}^{\text{solid}} \) is the category of \( \mathbb{Z}_{p} \)-modules in Solid.
87 Here, and in the following, to keep notation light, we write \( R \) (resp. \( G \)) to denote \( R \) (resp. \( G \)).
88 Here, we denote \( D(R[G], \mathcal{M}_{R,G}) := D(\text{Mod}_{R[G]}^{\text{cond}}) \).
isomorphism. By adjunction from Lemma B.3, for all extremally disconnected sets $S$, and for all $N \in D(R[G],\mathcal{M}_{R,G})$, we have a natural isomorphism

$$\text{RHom}_{\mathcal{M}_{R,G}}(\mathcal{M}_{R,G}[S], N) \simeq \text{RHom}_{\mathcal{M}_{R,G}}(\mathcal{Z}[S]^\bullet \otimes_{\mathcal{Z}} R[G], N)$$

$$\simeq \text{RHom}_{\text{Solid}}(\mathcal{Z}[S]^\bullet, N) \simeq \text{RHom}_{\text{CondAb}}(\mathcal{Z}[S]^\bullet, N)$$

$$\simeq \text{RHom}_{\text{CondAb}}(\mathcal{Z}[S], N) \simeq \text{RHom}_{\mathcal{R}[G]}(\mathcal{Z}[S] \otimes_{\mathcal{Z}} R[G], N)$$

$$\simeq \text{RHom}_{\mathcal{R}[G]}(\mathcal{M}_{R,G}[S], N).$$

We conclude recalling that the category $\text{Mod}_{\mathcal{R}[G],\mathcal{M}_{R,G}}$ is generated under colimits by $\mathcal{M}_{R,G}[S]$, for varying extremally disconnected sets $S$.

\textbf{Proof of Proposition B.4.} Applying Lemma B.5 for $G = \Gamma$, and $R = \mathbb{Z}_p$, we have that

$$\text{RHom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}, M) \simeq \text{RHom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, M) \simeq \text{RHom}_{\mathbb{Z}_p[\Gamma]}(\mathbb{Z}_p, M).$$

Now, we note that the condensed $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[\Gamma]$ is given by applying the functor $T \mapsto T$ to the topological Iwasawa algebra of $\Gamma$ over $\mathbb{Z}_p$. Then, we have the following projective resolution of $\mathbb{Z}_p$ as a $\mathbb{Z}_p[\Gamma]$-module in Solid

$$\bigotimes_{i=1}^n (\mathbb{Z}_p[\Gamma] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Z}_p.$$

Taking $\text{RHom}_{\mathbb{Z}_p[\Gamma]}(-, M)$ gives the statement. \qed

\textbf{References}

[Ber93] Vladimir G. Berkovich, \textit{Étale cohomology for non-Archimedean analytic spaces}, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161 (1994).

[Bha18] Bhargav Bhatt, \textit{Specializing varieties and their cohomology from characteristic 0 to characteristic $p$}, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 43–88.

[BK72] A. K. Bousfield and D. M. Kan, \textit{Homotopy limits, completions and localizations}, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, Berlin-New York, 1972.

[BS15] Bhargav Bhatt and Peter Scholze, \textit{Integral p-adic Hodge theory}, Publ. Math. Inst. Hautes Études Sci. \textbf{128} (2018), 219–397.

[Bos] Guido Bosco, \textit{Rational p-adic Hodge theory for rigid-analytic varieties}, in preparation.

[Bos14] Siegfried Bosch, \textit{Lectures on formal and rigid geometry}, Lecture Notes in Mathematics, vol. 2105, Springer, Cham, 2014.

[Bro82] Kenneth S. Brown, \textit{Cohomology of groups}, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982.

[BS15] Bhargav Bhatt and Peter Scholze, \textit{The pro-étale topology for schemes}, Astérisque (2015), no. 369, 99–201.

[CDN20a] Pierre Colmez, Gabriel Dospinescu, and Wiesława Nizioł, \textit{Cohomologie p-adique de la tour de Drinfeld: le cas de la dimension 1}, J. Amer. Math. Soc. \textbf{33} (2020), no. 2, 311–362.

[CDN20b] \textit{Cohomologie of p-adic Stein spaces}, Invent. Math. \textbf{219} (2020), no. 3, 873–985.

[CDN21] \textit{Integral p-adic étale cohomology of Drinfeld symmetric spaces}, Duke Math. J. \textbf{170} (2021), no. 3, 575–613.

[CDP14] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paškūnas, \textit{The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$}, Camb. J. Math. \textbf{2} (2014), no. 1, 1–47.

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, \textit{Patchning and the p-adic local Langlands correspondence}, Camb. J. Math. \textbf{4} (2016), no. 2, 197–287.

[CL06] Antoine Chambert-Loir, \textit{Mesures et équidistribution sur les espaces de Berkovich}, J. Reine Angew. Math. \textbf{595} (2006), 215–235.
[LZ17] Ruochuan Liu and Xinwen Zhu, *Rigidity and a Riemann-Hilbert correspondence for $p$-adic local systems*, Invent. Math. **207** (2017), no. 1, 291–343.

[MW20] Lucas Mann and Annette Werner, *Local systems on diamonds and $p$-adic vector bundles*, https://arxiv.org/abs/2005.06655v1, 2020, Preprint.

[Orl08] Sascha Orlik, *Equivariant vector bundles on Drinfeld’s upper half space*, Invent. Math. **172** (2008), no. 3, 585–656.

[Orl19] ______, *The pro-etale cohomology of Drinfeld’s upper half space*, https://arxiv.org/abs/1908.10591v1, 2019, Preprint.

[PG15] C. Perez-Garcia, *Non-Archimedean countably injective Banach spaces*, J. Convex Anal. **22** (2015), no. 3, 733–746.

[PGS10] C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, Cambridge, 2010.

[Sch02] Peter Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

[Sch13a] Peter Scholze, *The local Langlands correspondence for $GL_n$ over $p$-adic fields*, Invent. Math. **192** (2013), no. 3, 663–715.

[Sch13b] ______, *$p$-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77.

[Sch13c] ______, *Perfectoid spaces: a survey*, Current developments in mathematics 2012, Int. Press, Somerville, MA, 2013, pp. 193–227.

[Sch16] ______, *$p$-adic Hodge theory for rigid-analytic varieties – Corrigendum*, Forum Math. Pi **4** (2016), e6, 4.

[Sch18] ______, *On the $p$-adic cohomology of the Lubin-Tate tower*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 4, 811–863, With an appendix by Michael Rapoport.

[Sch19] ______, *Lectures on Condensed Mathematics*, https://www.math.uni-bonn.de/people/scholze/Condensed.pdf, 2019.

[Sch20a] ______, *Are there (enough) injectives in condensed abelian groups? (answer)*, https://mathoverflow.net/q/356261, 2020.

[Sch20b] ______, *Lectures on Analytic Geometry*, https://www.math.uni-bonn.de/people/scholze/Analytic.pdf, 2020.

[Sch21] ______, *Étale cohomology of diamonds*, https://arxiv.org/abs/1709.07343, 2021, Preprint.

[SJF+72] G. Schiffmann, H. Jacquet, J.P. Ferrier, L. Gruson, and C. Houzel, *Séminaire Banach*, Lecture Notes in Mathematics, Vol. 277, Springer-Verlag, 1972.

[SS91] P. Schneider and U. Stuhler, *The cohomology of $p$-adic symmetric spaces*, Invent. Math. **105** (1991), no. 1, 47–122.

[Sta] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu.

[SW99] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, second ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999.

[Tre67] François Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967.

[Var07] Oğuz Varol, *On the derived tensor product functors for (DF)- and Fréchet spaces*, Studia Math. **180** (2007), no. 1, 41–71.

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