Abstract. Let $X$ be a closed, connected and oriented topological $n$-dimensional manifold. Suppose that:

1. $X$ supports an effective action of $(\mathbb{Z}/r)^n$ for arbitrarily large values of $r$,
2. the fundamental group $\pi_1(X)$ is virtually solvable, and
3. there exists a map $X \to T^n$ of nonzero degree.

We then prove that $X$ is homeomorphic to $T^n$. Without using the assumption on $\pi_1(X)$ we prove that the cohomology ring $H^*(X; \mathbb{Z})$ is isomorphic to $H^*(T^n; \mathbb{Z})$. We also prove that any closed connected and oriented $n$-dimensional manifold $X$ admitting a map to $T^n$ of nonzero degree has Jordan homeomorphism group, and we study for such manifolds the maximal number $k$ such that $X$ admits effective actions of $(\mathbb{Z}/r)^k$ for arbitrarily large $r$. Finally, we prove that if $X$ is a compact and connected Kaehler manifold of real dimension $n$ and $X$ supports effective holomorphic actions of $(\mathbb{Z}/r)^m$ for arbitrarily large $r$, then $m \leq n$, and if $m = n$ then $X$ is biholomorphic to a complex torus.

1. Introduction

1.1. Main theorems. Can one characterize a manifold by the collection of all finite groups that act effectively on it? Of course not: there are plenty of examples of asymmetric manifolds, i.e., manifolds which do not admit any effective action of a finite group [17]. It is actually expected that in some sense most manifolds are asymmetric [15]. But perhaps the question makes more sense if we consider manifolds supporting actions of arbitrarily large finite groups. Here ”large” has to be understood in an appropriate sense and not merely as having many elements, because assuming that a manifold supports an effective action of the circle (which implies it supports actions of arbitrarily large finite cyclic groups) is still very far from implying that the manifold can be characterized by the finite groups which act effectively on it.

The $n$-dimensional torus $T^n$ supports a continuous effective action of $(\mathbb{Z}/r)^n$ for arbitrarily large integers $r$, given by rotations of angle a multiple of $2\pi/r$ in each of the factors of $T^n = S^1 \times \cdots \times S^1$. Our main result is a sort of converse to this statement, that may be viewed as a partial positive answer to the previous question. Suppose that
a closed, connected and oriented manifold $X$ supports effective actions\footnote{Here and everywhere all actions of groups on a topological manifold are implicitly assumed to be continuous.} of $(\mathbb{Z}/r)^n$ for arbitrarily large integers $r$, meaning that there exists a sequence of integers $r_i \to \infty$ such that $X$ admits an effective action of $(\mathbb{Z}/r_i)^n$ for every $i$ (we emphasize that we are not assuming any compatibility between the actions of $(\mathbb{Z}/r_i)^n$ and $(\mathbb{Z}/r_j)^n$ for $i \neq j$). Under this assumption, we prove that either $X$ is homeomorphic to $T^n$, or $X$ is in some sense very different from $T^n$. We express this dichotomy in terms of a strengthening of the topological rigidity of tori for maps from $X$ to $T^n$.

Topological rigidity of tori is the statement that if $X$ is a closed connected manifold then any homotopy equivalence $X \to T^n$ is homotopic to a homeomorphism. If $n \leq 2$ this is a consequence of the classification of compact connected manifolds of dimensions at most 2. It was proved for $n \geq 5$ by Hsiang and Wall \cite{29}, for $n = 4$ by Freedman \cite{24, §11.5} (see also \cite{4}), and in dimension $n = 3$ it is a consequence of Thurston’s geometrisation conjecture proved by Perelman (see \cite{31, §4} for proofs of the geometrisation conjecture, and \cite{33, §5} for the proof that geometrisation implies topological rigidity of $T^3$). Topological rigidity of tori is a particular case of Borel’s conjecture (see e.g. \cite{37, §3} for a survey, \cite{22, 23} for the case of Riemannian manifolds with nonpositive curvature, and also the recent textbook \cite{15}).

In our main theorem, assuming a priori that $X$ supports effective actions of $(\mathbb{Z}/r)^n$ for arbitrarily large integers $r$, we replace homotopy equivalence by the weaker requirement that the map has nonzero degree and that $\pi_1(X)$ is virtually solvable. Without the assumption on $\pi_1(X)$ we prove that $X$ is cohomologically indistinguishable from $T^n$.

**Theorem 1.1.** Let $X$ be a closed, connected and oriented $n$-dimensional topological manifold supporting an effective action of $(\mathbb{Z}/r)^n$ for arbitrarily large integers $r$. Suppose that there exists a map $\phi : X \to T^n$ of nonzero degree. We have:

1. There is an isomorphism of rings $H^*(X; \mathbb{Z}) \simeq H^*(T^n; \mathbb{Z})$.
2. If $\pi_1(X)$ is virtually solvable then $X$ is homeomorphic to $T^n$.

The following result complements the previous theorem in the smooth category.

**Theorem 1.2.** Let $n \neq 4$ be a natural number. Let $X$ be a smooth manifold homeomorphic to $T^n$. We have:

1. $X$ supports effective smooth actions of $(\mathbb{Z}/r)^n$ for arbitrarily large values of $r$.
2. There exists a number $\delta(n)$ (depending on $n$ but not on $X$) such that if $X$ supports an effective smooth action of $(\mathbb{Z}/m\delta(n))^n$ for some nonzero integer $m$ then $X$ is diffeomorphic to $T^n$.

Statement (2) of the previous theorem is related to many other existing results in the literature showing that homeomorphic but not diffeomorphic manifolds need not support smooth effective actions of the same finite or compact groups (see \cite{30} and the references
Corollary 1.3. Let $n \neq 4$ be a natural number. Let $X$ be a closed, connected and oriented $n$-dimensional smooth manifold supporting an effective action of $(\mathbb{Z}/r)^n$ for every natural number $r$. If there exists a continuous map $\phi : X \to T^n$ of nonzero degree and $\pi_1(X)$ is virtually solvable then $X$ is diffeomorphic to $T^n$.

A closed, connected and oriented $n$-dimensional manifold $X$ admitting a continuous map $\phi : X \to T^n$ of degree $\pm 1$ is called hypertoral in [47, 48]. If the degree of $\phi$ is a nonzero integer, then we say that $X$ is rationally hypertoral. Equivalently, $X$ is hypertoral if it admits classes $\alpha_1, \ldots, \alpha_n \in H^1(X; \mathbb{Z})$ such that $\alpha_1 \cup \cdots \cup \alpha_n \neq 0$, because $T^n = (S^1)^n$ and $S^1$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 1)$. And similarly $X$ is hypertoral if it admits classes $\alpha_1, \ldots, \alpha_n \in H^1(X; \mathbb{Z})$ such that $\alpha_1 \cup \cdots \cup \alpha_n$ is a generator of $H^n(X; \mathbb{Z})$. Using this characterisation it follows from statement (1) in Theorem 1.1 that if a rationally hypertoral $n$-manifold supports effective actions of $(\mathbb{Z}/r)^n$ for arbitrarily large values of $r$ then the manifold must be hypertoral.

There are plenty of examples of hypertoral manifolds. For example, if $Y$ is any closed connected and oriented $n$-manifold, then the connected sum $Y \# T^n$ is a hypertoral manifold, as the map $Y \# T^n \to T^n$ that identifies all points coming from $Y$ has degree 1. There are also many rationally hypertoral manifolds which are not hypertoral.

Example 1.4. Let $\pi : Y \to T^2$ be a degree 2 covering ramified at 2 points. Then $Y$ is an orientable closed and connected surface of genus 2. Let $\sigma : Y \to Y$ be the nontrivial automorphism of $\pi$, and let $X$ be the mapping torus of $\sigma$. In Section 14 we prove that $X$ is rationally hypertoral but not hypertoral. Consequently, Theorem 1.1 implies the existence of some $r_0 \in \mathbb{N}$ such that if $X$ supports an effective action of $(\mathbb{Z}/r)^3$ for some integer $r$ then necessarily $|r| \leq r_0$.

Both Example 1.4 and the arguments in Section 14 can easily be generalized to yield infinitely many examples in any dimension bigger than 2 of rationally hypertoral manifolds which are not hypertoral.

1.2. Jordan property, bounds on stabilizers, and almost asymmetric manifolds. The tools used to prove Theorem 1.1 lead to other results of finite group actions on rationally hypertoral manifolds. Recall that a group $G$ is said to be Jordan if there exists a constant $C$ such that every finite subgroup $G \leq \mathfrak{S}$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$. The following result extends to the topological category the first part of [12, Theorem 1.4], and it also partially extends [54, Corollary 1.7].

Theorem 1.5. Let $X$ be a rationally hypertoral manifold. Then the homeomorphism group of $X$ is Jordan.

Arguing as in [13, Corollary 4], the previous theorem has the following implication, which was originally proved by Burghelea and Schultz in [12, Theorem A].
Corollary 1.6. Let $X$ be a rationally hypertoral manifold. If a compact connected Lie group $G$ acts effectively and continuously on $X$ then $G$ is abelian.

If $X$ is a set supporting an action of a group $G$ we denote $\text{Stab}(X, G) = \{G_x \mid x \in X\}$ the set of stabilizers of points in $X$. The following result gives a positive answer to [18, Question 1.8] for rationally hypertoral manifolds.

Theorem 1.7. Let $X$ be a rationally hypertoral manifold. There exists a constant $C$ such that every finite group $G$ acting on $X$ has a subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C$ and $|\text{Stab}(X, G)| \leq C$.

We say that a closed topological manifold $X$ is almost asymmetric if there exist a number $C$ such that any finite group $G$ acting effectively on $X$ satisfies $|G| \leq C$. The following extends to the continuous category the second part of [42, Theorem 1.4]:

Theorem 1.8. If a rationally hypertoral manifold $X$ satisfies $\chi(X) \neq 0$ then $X$ is almost asymmetric. More precisely, there exists a number $C$ depending only on $\sum b_j(X)$, where $b_j(X) = \dim \mathbb{Q}H_j(X; \mathbb{Q})$ is the $j$-th rational Betti number of $X$, such that any finite group acting effectively on $X$ has at most $C$ elements.

Using the previous theorem one can give a simple construction showing that the question at the beginning of the paper has a negative answer. Let $X_0 = T^4 \sharp T^4 \sharp T^4$. Then $X_0$ is hypertoral and $\chi(X_0) = -4$. Fix a nonzero class $c \in H_1(X_0; \mathbb{Z}) \simeq \mathbb{Z}^{12}$ which is not a non-trivial integral multiple. For any prime $p$ choose an embedded circle $\gamma \subset X_0$ representing $p$ times the class $c$, and denote by $X_p$ the result of performing surgery on $X_0$ along $\gamma$ (i.e., removing a tubular neighborhood of $\gamma$, which will be diffeomorphic to $S^1 \times D^3$, and gluing back $D^2 \times S^2$ using the identification of boundaries $\partial(S^1 \times D^3) = S^1 \times S^2 = \partial(D^2 \times S^2)$). We have $H_1(X_p; \mathbb{Z}) \simeq \mathbb{Z}/p \oplus \mathbb{Z}^{11}$ and $\chi(X_p) = -2$. Furthermore, by Mayer–Vietoris the sum of the rational Betti numbers of $X_p$ can be bounded by a constant independent of $p$. So the previous theorem combined with the pigeonhole principle gives an infinite sequence of pairwise non homeomorphic 4-dimensional manifolds each of which supports effective actions of the same collection of (isomorphism classes of) finite groups.

1.3. Discrete degree of symmetry. For any closed topological manifold $X$ we denote by $\mu(X)$ the set of all natural numbers $m$ such that $X$ supports effective actions of $(\mathbb{Z}/r)^m$ for arbitrarily large $r$. Note that $\mu(X)$ may be empty. Define the discrete degree of symmetry of $X$ to be

$$\text{disc-sym}(X) = \max(\{0\} \cup \mu(X)).$$

This is always finite and hence well defined thanks to a theorem of Mann and Su, see [38, Theorem 2.5]. This invariant is related to the previous notion in the following terms:

Lemma 1.9. For any closed manifold $X$ we have $\text{disc-sym}(X) = 0$ if and only if $X$ is almost asymmetric.
Proof. Clearly if $X$ is almost asymmetric then $\text{disc-sym}(X) = 0$. Conversely, suppose that $\text{disc-sym}(X) = 0$. Then there is a natural number $C$ such that any finite cyclic group acting effectively on $X$ has at most $C$ elements. Let $G$ be a finite group acting effectively on $X$. For any prime divisor $p$ of $|G|$ there is a cyclic subgroup of $G$ with $p$ elements, so all prime divisors $p$ of $|G|$ satisfy $p \leq C$. By \cite[Theorem 2.5]{38} there exists a natural number $d$ such that for any prime $p$ and any natural number $m$ such that $(\mathbb{Z}/p)^m$ acts effectively on $X$ we have $m \leq d$. Fix some prime $p$ dividing $|G|$ and let $H$ be a Sylow $p$-subgroup of $G$. Choose a maximal normal abelian subgroup $A \triangleleft H$. Then $A$ is isomorphic to the product of at most $d$ finite cyclic groups. Each of these cyclic groups has at most $C$ elements, so $|A| \leq C^d$. By \cite[§5.2.3]{38} the action of $H$ by conjugation on $A$ induces an injection $H/A \hookrightarrow \text{Aut} A$. Hence, $|H| \leq |A| \cdot |\text{Aut} A| \leq C^d(C^d)!$. Repeating the previous argument for all primes dividing $|G|$ we obtain the bound $|G| \leq [C^d(C^d)!]^C$. 

Recall that the rank of a finite group $G$ is the minimal size of a generating subset of $G$; we denote it by $\text{rk} G$. One can relate the discrete degree of symmetry to actions of arbitrary finite abelian groups as follows.

**Lemma 1.10.** Let $X$ be a closed manifold and let $k$ be a natural number. The following two conditions are equivalent.

1. $\text{disc-sym}(X) \leq k$;
2. there exists some constant $C$ such that any finite abelian group acting effectively on $X$ has a subgroup $A' \leq A$ satisfying $[A : A'] \leq C$ and $\text{rk} A' \leq k$.

Proof. If $\text{disc-sym}(X) \leq k$ then there exists a natural number $r_0$ such that if $(\mathbb{Z}/r)^{k+1}$ acts effectively on $X$ then $r \leq r_0$. Let $A$ be a finite abelian group acting effectively on $X$. For any prime $p$ let $A_p \leq A$ denote the subgroup of elements whose order is a power of $p$. Then $A \simeq \prod_p A_p$ and $\text{rk} A = \max_p \text{rk} A_p$. For any prime $p$ satisfying $p > r_0$ we have $\text{rk} A_p \leq k$, because if $\text{rk} A_p = s$ then the subgroup of $A_p$ consisting of elements whose order divides $p$ is isomorphic to $(\mathbb{Z}/p)^s$. Now suppose that $p$ is a prime satisfying $p \leq r_0$ and let $s = \text{rk} A_p$. We are going to define a subgroup $A'_p \leq A_p$. If $s \leq k$ then we set $A'_p := A_p$. Suppose now that $s > k$. By \cite[Theorem 2.5]{38} we have $s \leq C_0$ for some number $C_0$ depending only on $X$. Choose an isomorphism $\phi : A_p \simeq (\mathbb{Z}/p)^{s}$, where $e_1, \ldots, e_s$ are natural numbers written in nonincreasing order. By the definition of $r_0$ we have $p^{e_j} \leq r_0$ for every $j > k$. So if we set

$$A'_p = \phi^{-1}(\mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_k} \times \{0\} \times \cdots \times \{0\})$$

then $[A_p : A'_p] \leq r_0^{C_0}$ and $\text{rk} A'_p \leq k$. Let $A' = \prod_{p \leq r_0} A'_p \times \prod_{p > r_0} A_p$. Then $\text{rk} A' \leq k$ and $[A : A'] \leq C := (r_0^{C_0})^{r_0}$. This proves (1) $\Rightarrow$ (2). The converse implication follows from Lemma [6.4] below, the fact the rank does not increase when passing from a finite abelian group to a subgroup, and the formula $\text{rk}(\mathbb{Z}/a)^b = b$, which is valid for any natural numbers $a \geq 2$ and $b$. 


1.4. **Bounds on the discrete degree of symmetry.** In the proof of statement (2) in Theorem 1.11 we will reduce the general case to the case in which $\pi_1$ is solvable using the following result.

**Theorem 1.11.** Let $X$ be a closed connected manifold and let $X' \to X$ be a finite covering. We have $\text{disc-sym}(X') \geq \text{disc-sym}(X)$.

The inequality in Theorem 1.11 can be strict in some cases, as the following theorem proves. Here and in the rest of the paper we identify $T^n$ with $(\mathbb{R}/\mathbb{Z})^n$, so we use additive notation for the group structure on $T^n$.

**Theorem 1.12.** Fix natural numbers $k, n$ satisfying $1 \leq k \leq n - 1$. Consider the free involution $\sigma : T^n \to T^n$ defined by $\sigma(x_1, \ldots, x_n) = (x_1 + 1/2, \ldots, x_k + 1/2, -x_{k+1}, \ldots, -x_n)$. Let $X' = T^n$ and let $X = T^n/\sigma$. The natural projection $\rho : X' \to X$ is a covering map and we have $\text{disc-sym}(X') = n$ and $\text{disc-sym}(X) = k$.

For example, setting $n = 2$ and $k = 1$ the manifold $X$ is the Klein bottle and $X'$, the 2-torus, is the orientation 2-cover of $X$.

The following theorem gives bounds on $\text{disc-sym}(X)$ for rationally hypertoral $X$.

**Theorem 1.13.** Let $X$ be a rationally hypertoral manifold. We have:

1. $\text{disc-sym}(X) \leq \dim X$;
2. suppose that $X$ is homeomorphic to $Y \times (Z \sharp Z')$, where $Y, Z, Z'$ are closed connected topological manifolds satisfying $\dim Z = \dim Z' > 1$; if neither $Z$ nor $Z'$ are integral homology spheres, then $\text{disc-sym}(X) \leq \dim Y$.

For example, if $Z$ is a closed connected topological $l$-manifold and $H_*(Z; \mathbb{Z}) \not\cong H^*(S^1; \mathbb{Z})$ then $\text{disc-sym}(T^k \times (T^l \sharp Z)) \leq k$. Since $T^k \times (T^l \sharp Z)$ supports actions of $(\mathbb{Z}/r)^k$ for every $r$ (inherited from the actions on $T^k$) we conclude that $\text{disc-sym}(T^k \times (T^l \sharp Z)) = k$.

So among rationally hypertoral $n$-manifolds we can find examples realizing any possible value of $\text{disc-sym}$ between 0 and $n$.

We saw above how Theorem 1.11 implies that the manifold $X$ in Example 1.4 satisfies $\text{disc-sym}(X) \leq 2$. The previous results lead to a stronger inequality. The structure of mapping torus on $X$ defines a fibration $\rho : X \to S^1$. Let $q : S^1 \to S^1$ be a degree 2 covering and let $X' \to S^1$ be the pullback of $\rho$ through $q$. More precisely, $X' = \{(x, a) \in X \times S^1 \mid \rho(x) = q(a)\}$. The projection $X' \to X$ sending $(x, a)$ to $x$ is an unramified degree 2-covering, so by Theorem 1.11 we have $\text{disc-sym}(X) \leq \text{disc-sym}(X')$. Now, $X' \cong S^1 \times Y$, where $Y$ is a closed connected surface of genus 2. So we may write $X' \cong T^1 \times (T^2 \sharp T^2)$, and consequently Theorem 1.13 implies $\text{disc-sym}(X') \leq 1$. This inequality is sharp. Indeed, if $r$ is an odd integer and $q_r : S^1 \to S^1$ is a degree $r$ covering then the pullback $X_r$ of $X \to S^1$ via $q_r$ can be identified with the mapping torus of $\sigma^r = \sigma$, and hence is homeomorphic to $X$. But the projection $X_r \to X$ is a regular degree $r$ covering with group of deck transformations $\mathbb{Z}/r$. Consequently, $X_r$ supports a free action of $\mathbb{Z}/r$. Since $X_r \cong X$ we conclude that $X$ supports a free action of $\mathbb{Z}/r$ for every odd integer $r$. It follows that $\text{disc-sym}(X) = 1$. 
Denote by $\nu(X)$ the set of natural numbers $m$ such that $X$ supports effective actions of $T^m$, and define the **abelian degree of symmetry** of $X$ as

$$\text{ab-sym}(X) = \max(\{0\} \cup \nu(X)).$$

It is a standard fact that if $T^m$ acts effectively on $X$ then $m \leq \dim X$ (see e.g. the end of Subsection 13.2 for a proof), so $\text{ab-sym}(X)$ is well defined and satisfies $\text{ab-sym}(X) \leq \dim X$. In [48, Theorem 5.1] Schultz proves that if $Z$ is a closed connected $l$-dimensional topological manifold then $\text{ab-sym}(T^k \times (T^l\sharp Z)) \leq k$ unless $Z$ is a homotopy sphere. Clearly, $\text{disc-sym}(X) \geq \text{ab-sym}(X)$ for every $X$, so Theorem 1.13 is a strengthening of Schultz’s result, except that our hypothesis that neither $Z$ nor $Z'$ are integral homology spheres is stronger than the assumption that neither of them is a homotopy sphere.

A construction due to Cappell, Weinberger and Yan provides examples of manifolds $X$ with $\text{disc-sym}(X) > \text{ab-sym}(X)$:

**Theorem 1.14.** Let $X$ be any of the manifolds $T(h) \times H$ constructed in [14 §2]. We have $\text{disc-sym}(X) \geq 1$ and $\text{ab-sym}(X) = 0$.

This theorem is not new. The equality $\text{ab-sym}(X) = 0$ is the main result in [14], while the inequality $\text{disc-sym}(X) \geq 1$ stems from the existence of regular self coverings $X \to X$ of degree $d$ for every odd natural number $d$. The existence of such self coverings (for degree 3, and hence also every power of 3) is stated without proof in [51, Remark 1.3], and we prove it (for any odd $d$) in Section 15. There are obvious analogues of the invariants $\text{disc-sym}$ and $\text{ab-sym}$ for locally linear actions and for smooth actions on smooth manifolds, and in neither of these categories does one have the equality $\text{disc-sym} = \text{ab-sym}$ in general. For locally linear actions there are counterexamples in dimension 4, by the work of Edmonds [21] and Huck [31]. In the smooth category one may take $X = T^n\sharp \Sigma$, where $\Sigma$ is an exotic $n$-sphere. Then $X$ is homeomorphic to $T^n$, so $\text{disc-sym}(X) = n$ by Theorem 1.2, but $\text{ab-sym}(X) = 0$ by the main result in [1]. In contrast, for holomorphic actions on compact Kaehler manifolds one does have $\text{disc-sym} = \text{ab-sym}$ in general, as proved by Theorem 1.15 below.

Note that in contrast with the previous examples our main theorem proves that, in some very particular situations, if $\text{disc-sym}(X)$ is big then $\text{ab-sym}(X)$ is also big: namely if $X$ is an $n$-dimensional rationally hypertoral manifolds with virtually solvable $\pi_1$ then $\text{disc-sym}(X) = n$ implies $\text{ab-sym}(X) = n$.

The proof we give of Theorem 1.1 uses crucially both hypothesis that $X$ is rationally hypertoral and that $\pi_1(X)$ is virtually solvable (see Remark 10.5 for the latter). But it is tempting to believe that the result should be true in greater generality. If the theorem were true for every closed topological manifold $X$ (i.e., if one could prove that any closed topological $n$-manifold $X$ satisfying $\text{disc-sym}(X) \geq n$ is homeomorphic to $T^n$) then a remarkable consequence would be that for any closed topological $n$-manifold $X$ we have $\text{disc-sym}(X) \leq n$, a much stronger inequality than the elementary bound $\text{ab-sym}(X) \leq n$. 
1.5. **Holomorphic actions on Kaehler manifolds.** We don’t know whether the previous statement is true, but we can prove the following analogue for Kaehler manifolds:

**Theorem 1.15.** Let \( X \) be a compact connected Kaehler manifold of real dimension \( n \). Suppose that, for some natural number \( m \), \( X \) supports an effective holomorphic action of \((\mathbb{Z}/r)^m\) for arbitrarily large values of \( r \). Then \( X \) supports an effective holomorphic action of \( T^m \). Furthermore, \( m \leq n \), and if \( m = n \) then \( X \) is biholomorphic to a complex torus.

We will prove Theorem 1.15 using a result of Fujiki [25] on automorphism groups of compact Kaehler manifolds and the following result on Lie groups.

**Theorem 1.16.** Let \( G \) be a (finite dimensional) Lie group with finitely many connected components. For every natural number \( n \) the following properties are equivalent:

1. \( G \) has a Lie subgroup isomorphic to \( T^n \),
2. for arbitrary large integers \( r \) the group \( G \) has a subgroup isomorphic to \((\mathbb{Z}/r)^n\).

The previous theorem does not extend to the infinite dimensional setting. Indeed, if we take \( G = \text{Diff}(X) \) for some manifold \( X \) then the equivalence of statements (1) and (2) of the theorem applied to this \( G \) is the same thing as the equality of the smooth analogues of disc-sym(\( X \)) and ab-sym(\( X \)), which as we explained previously is false in general.

Theorem 1.16 seems to be new even for smooth projective varieties over the complex numbers. One can also ask the analogous question for birational transformation groups. Namely, if \( X \) is an \( n \)-dimensional variety defined over the complex numbers (or more generally a field of characteristic zero) and the birational transformation group \( \text{Bir}(X) \) contains subgroups isomorphic to \((\mathbb{Z}/r)^m\) for arbitrarily large values of \( r \), does it follow that \( m \leq 2n \)? And if \( m = 2n \), does it follow that \( X \) is birational to an abelian variety?

A partial result on the first question, due to Prokhorov and Shramov, appears in [44, Theorem 1.10]. An analogue of the second question for rationally connected varieties has been recently proved by Xu [53, Theorem 1.3]: namely, if \( X \) is a rationally connected \( n \)-dimensional variety and \( \text{Bir}(X) \) contains subgroups isomorphic to \((\mathbb{Z}/p)^n\) for sufficiently big primes \( p \) then \( X \) is rational.

Theorem 1.16 implies, when combined with the theorem of Myers–Steenrod, the following Riemannian analogue of Theorem 1.15: if \((M, g)\) is a compact \( n \)-dimensional Riemannian manifold whose isometry group contains subgroups isomorphic to \((\mathbb{Z}/r)^m\) for arbitrarily large values of \( r \) then \( m \leq n \), and if \( m = n \) then \((M, g)\) is isometric to a flat torus. One may wonder whether an analogous result holds for Lorentz manifolds (note that the isometry group of a Lorentz manifold, unlike the case of Riemannian manifolds, may have infinitely many connected components — see e.g. [19, §1]).

1.6. **Structure of the proofs and contents of the paper.** To prove of Theorem 1.15 we use two main tools. The first one is geometric. In the presence of an effective action of a finite group \( G \) on a closed manifold \( X \), it allows to substitute any map of nonzero
degree $\phi : X \to T^n$ by a homotopic map that is $G$-equivariant with respect to an "almost effective" action of $G$ on $T^n$. By "almost effective" we mean that the amount of elements of $G$ acting trivially on $T^n$ is bounded above by a constant depending only on $X$ and the degree of $\phi$. This fact is proved in Section 2. Immediate applications are Theorems 1.5 and 1.7 (proved in Section 3) and Theorem 1.8 (proved in Section 4).

The second tool we use is algebraic. Let $M$ be a finitely generated $A$-module, where $A = \mathbb{Z}[t^{\pm 1}_1, \ldots, t^{\pm 1}_n]$. Suppose that for every $1 \leq i \leq n$ there exists a nonzero integer $d_i$, a sequence of integers $(r_{i,j})_j$ satisfying $r_{i,j} \to \infty$ as $j \to \infty$, and automorphisms $w_{i,j} \in \text{Aut}_A M$ such that $w_{i,j}^{r_{i,j}}$ coincides with multiplication by $t^{d_i}_i$. Then $M$ is finitely generated as a $\mathbb{Z}$-module. This is proved in Section 5 (see Corollary 5.3). Section 6 contains some elementary lemmas on finite abelian groups, in Section 7 we prove Theorem 1.11, and in Section 8 we prove Theorem 1.12.

Theorem 1.1 is proved in Section 10. The idea is the following: suppose that a closed $n$-manifold $X$ supports effective actions of $(\mathbb{Z}/r)^n$ for arbitrary large values of $r$ and that $\phi : X \to T^n$ is a map of nonzero degree. Let $X_\phi \to X$ be the pullback of the principal $\mathbb{Z}^n$-bundle $\mathbb{R}^n \to \mathbb{Z}^n$ via $\phi$. Then $X_\phi$ supports an action of $\mathbb{Z}^n$, so its homology $H_*(X_\phi; \mathbb{Z})$ is a module over $\mathbb{Z}[\mathbb{Z}^n] \simeq A$. We prove in Section 9 that $H_*(X_\phi; \mathbb{Z})$ is finitely generated as an $A$-module. Then we use the geometric tool proved in Section 2 to deduce that $H_*(X_\phi; \mathbb{Z})$ satisfies the hypothesis allowing to apply the algebraic tool, so that $H_*(X_\phi; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. A simple argument using Serre’s spectral sequence applied to the Borel construction of the $\mathbb{Z}^n$-space $X_\phi$ implies that $H_*(X_\phi; \mathbb{Z})$ is zero in positive degrees. This leads almost immediately to statement (1) in Theorem 1.11. As for statement (2), if $\pi_1(X)$ is solvable then we conclude using Whitehead’s theorem that the arc connected components of $X_\phi$ are contractible, so $X$ is homotopy equivalent to a torus. At this point we invoke the topological rigidity of tori to deduce that $X \cong T^n$. Finally, the case of virtually solvable $\pi_1(X)$ is reduced to the solvable case using Theorem 1.11.

In Section 11 we prove Theorem 1.12. Theorem 1.13 is proved in Section 12 using the geometric tool introduced in Section 2, the finite generatedness proved in Section 9, and statement (T1) above. In Section 13 we prove Theorems 1.16 and 1.15. Finally, in Section 14 we prove that the manifold in Example 1.4 is rationally hypertoral but not hypertoral, and in Section 15 we prove Theorem 1.14.

1.7. **Notation.** For every finite set $S$ we denote by $|S|$ the cardinality of $S$.

1.8. **Acknowledgements.** I wish to thank Jordi Daura for pointing out to me reference [51], which led to Theorem 1.14. Thanks also to Costya Shramov for comments on a preliminary version of this paper and information on birational transformation groups.

2. **Equivariant maps to the torus: proof of Theorem 2.1**

In all this section $X$ denotes a closed, connected and oriented $n$-dimensional manifold. We identify $T^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$, and we use additive notation for the group structure on $T^n$. Suppose that $G$ is a group, $\eta : G \to T^n$ is a group homomorphism,
and $G$ acts on a space $X$. A map $\phi : X \to T^n$ will be called $\eta$-equivariant if it satisfies $\phi(g \cdot x) = \eta(g) + \phi(x)$ for every $x \in X$ and $g \in G$.

**Theorem 2.1.** Let $X$ be a closed, connected and oriented $n$-dimensional topological manifold. Let $\phi : X \to T^n$ be a continuous map of nonzero degree. Let $G$ be a finite group. Suppose that $X$ is endowed with an effective action of $G$ inducing the trivial action on $H^1(X; \mathbb{Z})$. Then there is a morphism of groups $\eta : G \to T^n$ with these properties:

1. the map $\phi$ is homotopic to an $\eta$-equivariant map $\psi : X \to T^n$,
2. $|\text{Ker } \eta| \leq |\text{deg } \phi|$.

Before proving the theorem we prove three auxiliary lemmas. The first lemma is a topological analogue of the construction at the beginning of [42, §2.1]. We identify $S^1$ with $\mathbb{R}/\mathbb{Z}$ and accordingly we use additive notation for the group structure on $S^1$.

**Lemma 2.2.** Let $\alpha : X \to S^1$ be a continuous map. Let $\theta$ be a generator of $H^1(S^1; \mathbb{Z})$. Suppose that a finite group $G$ acts continuously on $X$ preserving $\alpha \cdot \theta$. Let $r$ be the cardinal of $G$ and let $\mu_r \subset S^1$ denote the group of $r$-th roots of unity. There exists a morphism of groups $\xi : G \to \mu_r$ and a continuous map $\beta : X \to S^1$ homotopic to $\alpha$ such that $\beta(g \cdot x) = \xi(g) + \beta(x)$ for every $x \in X$ and $g \in G$.

**Proof.** Define $\zeta : X \to S^1$ by $\zeta(x) = \sum_{g \in G} \alpha(g \cdot x)$ for every $x \in X$. Then $\zeta$ is continuous and constant on $G$-orbits. Let $\rho_g : X \to X$ be the homeomorphism induced by the action of $g \in G$. By assumption $\rho_g^* \alpha \cdot \theta = \alpha \cdot \theta$ for every $g \in G$. We have $\zeta^* \theta = \sum_{g \in G} \rho_g^* \alpha \cdot \theta = r \alpha \cdot \theta$.

Let $\Gamma = \{(x, t) \in X \times S^1 | \zeta(x) = rt\}$. Let $\pi : \Gamma \to X$ be the restriction of the projection map $X \times S^1 \to X$. The action of $\mu_r$ on $\Gamma$ given by $(x, t) \cdot \theta = (x, t \theta)$ endows $\pi : \Gamma \to X$ with a structure of principal $\mu_r$-bundle. We claim that it is a trivial principal bundle. This is equivalent to the triviality of the monodromy of $\pi$, which we denote by $\nu : \pi_1(X, x_0) \to \mu_r$, where $x_0 \in X$ is an arbitrary base point. If there existed some $\lambda \in \pi_1(X, x_0)$ such that $\nu(\lambda) \neq 0$ then the pairing of $\zeta^* \theta$ with $[\lambda] \in H_1(X; \mathbb{Z})$ would not be divisible by $r$, which contradicts the fact that $\zeta^* \theta = r \alpha \cdot \theta$. Hence $\nu$ is trivial and consequently the bundle $\pi : \Gamma \to X$ is trivial, so we may choose a section $\sigma : X \to \Gamma$.

Define $\beta : X \to S^1$ by the condition that $\sigma(x) = (x, \beta(x))$. Then $\beta$ is continuous and we have $r \beta(x) = \zeta(x)$ for every $x \in X$. For any $g \in G$ define $\chi_g : X \to S^1$ by $\chi_g(x) = \beta(g \cdot x) - \beta(x)$. We have $r \chi_g(x) = r \beta(\rho_g \cdot x) - r \beta(x) = \zeta(\rho_g \cdot x) - \zeta(x) = 0$ because $\zeta$ is $G$-invariant. Hence $\chi_g$ takes values in $\mu_r$, and consequently, being continuous, it is a constant map. We may thus define a map $\xi : G \to \mu_r$ by the condition that $\xi(g) = \chi_g(x)$ for every $x \in X$. Let $g, g' \in G$ and let $x \in X$. We have

$$\chi_{gg'}(x) = \beta(gg' \cdot x) - \beta(x) = \beta(gg' \cdot x) - \beta(g' \cdot x) + \beta(g' \cdot x) - \beta(x) = \chi_g(g' \cdot x) + \chi_{g'}(x),$$

which proves that $\xi(gg') = \xi(g) + \xi(g')$, so $\xi$ is a morphism of groups. Now the formula $\beta(g \cdot x) = \xi(g) + \beta(x)$ follows immediately from the definition of $\xi$. To conclude the proof, note that $r \beta^* \theta = \zeta^* \theta = r \alpha \cdot \theta$, so $r (\beta^* \theta - \alpha \cdot \theta) = 0$. Since $H^1(X; \mathbb{Z})$ has no torsion
we conclude that $\beta^*\theta = \alpha^*\theta$. Hence $\beta$ and $\alpha$ are homotopic, because $S^1$ is a model for $K(\mathbb{Z}, 1)$.

Lemma 2.3. Let $G$ be a finite group acting effectively, continuously and preserving the orientation on $X$. Let $X^* \subseteq X$ be the set of points with trivial stabilizer. Then $X^*$ is connected.

Proof. For any $g \in G$ denote $X^g = \{x \in X \mid g \cdot x = x\}$. Let $g_1, \ldots, g_s$ be the (nontrivial) elements of $G$ of prime order. Since any nontrivial element of $G$ has some power belonging to the set $\{g_1, \ldots, g_s\}$, we have $X^* = X \setminus \bigcup_i X^{g_i}$. Define $X_1 = X$ and $X_i = X \setminus (X^{g_1} \cup \cdots \cup X^{g_{i-1}})$ for $2 \leq i \leq s + 1$. Then $X_i$ is an open subset of $X$ for each $i$. We prove that $X_i$ is connected for every $1 \leq i \leq s + 1$, using ascending induction on $i$. Since $X^* = X_{s+1}$, this will imply the lemma. Clearly $X_1$ is connected. Now suppose that $1 \leq i < s$ and that $X_i$ is connected. Let $p_i$ be the order of $g_i$. Since $G$ acts on $X$ preserving the orientation, by [9, Chap V, Theorem 2.3] and [9, Chap V, Theorem 2.5], $X^g$ is a $\mathbb{Z}/p_i$-cohomology manifold of dimension $d_i$, where $n - d_i$ is even. Arguing as in the proof of [9, Chap V, Theorem 2.6] we conclude that $d_i < n$, and consequently $d_i \leq n - 2$. Applying [9, Chap I, Corollary 4.7] we conclude that $X_{i+1} = X_i \setminus (X_i \cap X^{g_i})$ is connected, so the lemma is proved.

The following result generalizes [20, Lemma 2.5].

Lemma 2.4. Suppose that a finite group $G$ acts effectively, continuously, and preserving the orientation on $X$. Let $\pi : X \rightarrow X/G$ denote the quotient map. Let $r$ denote the cardinal of $G$. The image of the map $\pi^* : H^n(X/G; \mathbb{Z}) \rightarrow H^n(X; \mathbb{Z})$ is contained in $rH^n(X; \mathbb{Z})$.

Proof. Denote as in the previous lemma by $X^*$ the open subset of $X$ consisting of points with trivial stabilizer. Since $X/G$ is endowed with the quotient topology, $\pi(X^*) = X^*/G$ is an open subset of $X/G$. Let $F = X \setminus X^*$. Consider the following commutative diagram, where $H^*_c(\cdot; \mathbb{Z})$ denotes cohomology with compact support, the rows are portions of the long exact sequences for the inclusions $X^* \hookrightarrow X \hookrightarrow F$ and $X^*/G \hookrightarrow X/G \hookrightarrow F/G$ (see e.g. [9, Chap I, (2) in §1.1]), and the vertical arrows are pullback morphisms induced by proper maps:

$$
\begin{array}{cccccc}
H^n_c(X^*; \mathbb{Z}) & \xrightarrow{j} & H^n_c(X; \mathbb{Z}) & \xrightarrow{r} & H^n_c(F; \mathbb{Z}) & \rightarrow 0 \\
\pi^* & & \pi^* & & \pi^* & \\
H^n_c(X^*/G; \mathbb{Z}) & \xrightarrow{j_G} & H^n_c(X/G; \mathbb{Z}) & \xrightarrow{r_G} & H^n_c(F/G; \mathbb{Z}) & \rightarrow 0,
\end{array}
$$

The morphism $j$ is an isomorphism because, by Lemma 2.3, $X^*$ is connected (see [9, Chap I, Theorem 4.3]). By the exactness this implies that $H^n_c(F; \mathbb{Z}) = 0$. The long exact sequence for $X^* \hookrightarrow X \hookrightarrow F$ and the fact that $\dim X = n$ imply that $H^k_c(F; \mathbb{Z})$ for every $k \geq n$ (here we are using [9, Chap I, (3) in §1.2]). From [9, Chap III, Theorem 5.2] it follows that $H^k_c(F/G; \mathbb{Z})$ for every $k \geq n$. Hence $j_G$ is surjective. Consequently it suffices
to prove that the image of the morphism $\pi^* : H^n_c(X^*/G; \mathbb{Z}) \to H^n_c(X^*; \mathbb{Z})$ is contained in $rH^n_c(X^*; \mathbb{Z})$.

Denote $G^* = G \setminus \{1\}$. We are going to prove that there exists a connected open subset $U \subset X^*$ such that $gU \cap U = \emptyset$ for every $g \in G^*$. Fix some point $x \in X^*$. Since $X^*$ is Hausdorff, for every $g \in G^*$ there exists disjoint open subsets $A_g, B_g \subset X^*$ such that $x \in A_g$ and $gx \in B_g$. Let $C = \bigcap_{g \in G^*} A_g$. Then $gx \notin C$ for every $g \in G^*$, because $C \cap B_g = \emptyset$. This implies that $x$ belongs to the open set $D := C \cup \bigcup_{g \in G^*} gC$. Let $U \subset D$ be a connected open subset containing $x$. Then for every $g \in G^*$ we have $gU \subset gC$ because $U \subset C$, and consequently $gU \cap D = \emptyset$, which implies that $gU \cap U = \emptyset$.

Let $V = \pi(U)$, so that $\pi^{-1}(V) = GU = \bigcup_{g \in G} gU$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^n_c(GU; \mathbb{Z}) & \xrightarrow{j_{GU}} & H^n_c(X^*; \mathbb{Z}) \\
j_V \downarrow & & \downarrow \pi^* \\
H^n_c(V; \mathbb{Z}) & \xrightarrow{j_V} & H^n_c(X^*/G; \mathbb{Z}),
\end{array}
$$

where $j_{GU}$ and $j_V$ are the covariant morphisms induced by open embeddings and the vertical arrows are pullback morphisms induced by proper morphisms. By [9, Chap I, Theorem 4.3] $j_V$ is an isomorphism, so it suffices to prove that the image of $j_{GU} \circ \pi_V^*$ is contained in $rH^n_c(X^*; \mathbb{Z})$. Since $gU \cap U = \emptyset$ for every $g \in G^*$, the open subset $GU$ contains $r = |G|$ connected components, which are $\{gU \mid g \in G\}$. Denote by $i_g : gU \to GU$ the inclusion. The pullback morphisms $i_g^* : H^n_c(GU; \mathbb{Z}) \to H^n_c(gU; \mathbb{Z})$ combine to give an isomorphism $H^n_c(GU; \mathbb{Z}) \xrightarrow{\sim} \bigoplus_{g \in G} H^n_c(gU; \mathbb{Z})$. Let $j_{gU} : H^n_c(gU; \mathbb{Z}) \to H^n_c(X^*; \mathbb{Z})$ be the morphism induced by the open embedding $gU \hookrightarrow X^*$. We have $j_{gU} = \sum_{g \in G} j_{gU} \circ i_g^*$, so if we prove that $j_{gU} \circ i_h^* \circ \pi_V^* = j_{hU} \circ i_h^* \circ \pi_V^*$ for every $g, h \in G$ then we will be done. Take two elements $g, h \in G$ and let $\rho : X^* \to X^*$ be the map given by $\rho(x) = gh^{-1} \cdot x$. Then $\rho$ is a homeomorphism and it restricts to a homeomorphism $\rho : hU \to gU$. The induced morphism $\rho^* : H^n_c(X^*/Z) \to H^n_c(X^*/Z)$ is the identity because $G$ acts on $X$ (and hence on $X^*$) preserving the orientation. Now, the desired equality follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
H^n_c(hU; \mathbb{Z}) & \xrightarrow{j_{hU}} & H^n_c(X^*; \mathbb{Z}) \\
i_h^* \circ \pi_V^* \downarrow & & \downarrow \rho^* = \text{Id} \\
H^n_c(V; \mathbb{Z}) & \xrightarrow{i_h \circ \pi_V^*} & H^n_c(gU; \mathbb{Z}) & \xrightarrow{j_{gU}} & H^n_c(X^*; \mathbb{Z})
\end{array}
$$

The triangle commutes because $\pi_V \circ i_h = \pi_V \circ i_g \circ \rho$ and the square commutes because $\rho$ is a homeomorphism and the inclusion $hU \hookrightarrow X^*$ is equal to the composition of the inclusion $gU \hookrightarrow X^*$ and $\rho$. 

$\square$
We are now ready to prove Theorem 2.1. Let \( \phi_i : X \to S^1 \) be the composition of \( \phi \) with the projection to the \( i \)-th factor \( T^n = (S^1)^n \to S^1 \). Since \( G \) acts trivially on \( H^1(X; \mathbb{Z}) \), in particular it fixes \( \phi_i \theta \), where \( \theta \in H^1(S^1; \mathbb{Z}) \) is any generator. Applying Lemma 2.2 to \( \phi_i \) we obtain the existence of a morphism of groups \( \eta_i : G \to S^1 \) and a map \( \psi_i : X \to S^1 \) homotopic to \( \phi_i \) such that \( \psi_i(g \cdot x) = \eta_i(g) + \psi_i(x) \) for every \( x \in X \) and \( g \in G \). Define \( \psi = (\psi_1, \ldots, \psi_n) : X \to T^n \) and \( \eta = (\eta_1, \ldots, \eta_n) : G \to T^n \). Then \( \psi \) is homotopic to \( \phi \) and it is \( \eta \)-equivariant.

Let \( G_0 = \ker \eta \). The map \( \psi \) factors as a composition
\[
X \xrightarrow{\pi} X/G_0 \xrightarrow{\psi_0} T^n,
\]
where \( \pi \) is the natural quotient map. Hence \( \psi^* = \pi^* \circ \psi_0 \), so the image of \( \psi^* : H^n(T^n; \mathbb{Z}) \to H^n(X; \mathbb{Z}) \) is contained in the image of \( \pi^* : H^n(X/G_0; \mathbb{Z}) \to H^n(X; \mathbb{Z}) \).

By the definition of degree, the image of \( \psi^* : H^n(T^n; \mathbb{Z}) \to H^n(X; \mathbb{Z}) \) is equal to \( (\deg \psi) H^n(X; \mathbb{Z}) \), and by Lemma 2.4 the image of \( \pi^* \) is contained in \( [G_0] \cdot H^n(X; \mathbb{Z}) \). It then follows that \( |G_0| \leq |\deg \psi| \). But \( |\deg \psi| = |\deg \phi| \) because \( \psi \) and \( \phi \) are homotopic. So the proof of the theorem is now complete.

Remark 2.5. The results in this section have been independently proved by Csikós, Pyber and Szabó, lifting \( \phi \) to a map \( \zeta \) from the universal cover of \( X \) to that of \( T^n \), and defining \( \psi \) as the average the translates of \( \zeta \) by lifts of the action of elements of \( G \) to the universal covers of \( X \) and \( T^n \).

3. Jordan property and bounds on stabilizers for rationally hypertoral manifolds

The following is a classical result of Minkowski, originally published in [40]. It follows from the elementary observation that the morphism \( \rho_3 : \text{GL}(k, \mathbb{Z}) \to \text{GL}(k, \mathbb{Z}/3) \) defined by taking residues mod 3 componentwise is injective when restricted to finite groups (equivalently: if \( X \in \text{GL}(k, \mathbb{Z}) \) has finite order and \( \rho_3(X) = 1 \) then \( X = 1 \)). See [39] for a plethora of generalisations of this fact.

Lemma 3.1. For any \( n \) there exists a constant \( C_n \) with the property that any finite subgroup of \( \text{GL}(n, \mathbb{Z}) \) has at most \( C_n \) elements.

If \( X \) is a closed topological manifold then \( H^1(X; \mathbb{Z}) \) is a finitely generated torsion free abelian group, hence isomorphic to \( \mathbb{Z}^b \) for some \( b \in \mathbb{Z}_{\geq 0} \). So the previous lemma implies the following.

Lemma 3.2. Let \( X \) be a closed topological manifold. There exists a constant \( C_X \) such that for any finite group \( G \) acting continuously on \( X \) the kernel \( G' \) of the induced morphism \( G \to \text{Aut} H^1(X; \mathbb{Z}) \) satisfies \( |G : G'| \leq C_X \).

Let \( X \) be a closed, connected and oriented \( n \)-dimensional manifold and let \( \phi : X \to T^n \) be a continuous map of nonzero degree satisfying
\[
|\deg \phi| = \min\{|\deg \psi| : \psi : X \to T^n, \deg \psi \neq 0\}.
\]
Suppose that a finite group $G$ acts continuously on $X$. Let $G'$ be the kernel of the induced morphism $G \to \text{Aut} H^1(X; \mathbb{Z})$. By Lemma 3.2 we have $[G : G'] \leq C_X$ for some constant $C_X$ depending only on $X$. By Theorem 2.1 there is a morphism of groups $\eta : G' \to T^n$ satisfying $|\text{Ker } \eta| \leq |\text{deg } \phi|$ and an $\eta$-equivariant map $\psi : X \to T^n$ homotopic to $\phi$. The existence of the $\eta$-equivariant map $\psi$ implies that for every $x \in X$ we have $G_x \leq \text{Ker } \eta$ (if $g \in G_x$ then $\eta(g) = \psi(g \cdot x) - \psi(x) = \psi(x) - \psi(x) = 0$), so the previous bound implies that $|\text{Stab}(X, G)| \leq 2^{\text{deg } \phi}$. This proves Theorem 1.7.

To prove Theorem 1.5 note that since $\eta(G')$ is a subgroup of $T^n$, it is abelian and can be generated by $n$ or fewer elements. If $|\text{deg } \phi| = 1$ then $G' \cong \eta(G')$, so $G'$ is abelian. This implies that $\text{Homeo}(X)$ is Jordan in this case. For other values of $|\text{deg } \phi|$, we apply Lemma 2.2 to the exact sequence

$$1 \to \text{Ker } \eta \to G' \to \eta(G') \to 1$$

and since $|\text{Ker } \eta| \leq |\text{deg } \phi|$ we deduce the existence of an abelian subgroup $G'' \leq G'$ such that $[G' : G'']$ is bounded above by a constant depending only on $n$ and $\text{deg } \phi$. Since $n$ and $\text{deg } \phi$ only depend on $X$, it follows that $[G : G'']$ is bounded above by a constant depending only on $X$, so the proof that $\text{Homeo}(X)$ is Jordan is now complete.

If $|\text{deg } \phi| = 1$ Theorem 1.5 can also be proved using [26, Theorem 2.5].

4. RATIONALLY HYPERTORAL MANIFOLDS WITH $\chi \neq 0$ ARE ALMOST ASYMMETRIC

Here we prove Theorem 1.8. Let $X$ be a rationally hypertoral manifold satisfying $\chi(X) \neq 0$. By hypothesis there exists a morphism $\phi : X \to T^n$ satisfying $d = |\text{deg } \phi| \neq 0$. Let $b = \sum_j b_j(X)$ and let $\text{Tor } H^*(X; \mathbb{Z})$ denote the torsion of $H^*(X; \mathbb{Z})$. Then $H^*(X; \mathbb{Z})/\text{Tor } H^*(X; \mathbb{Z}) \cong \mathbb{Z}^b$. By Lemma 3.1 there exists a natural number $C$ such that any finite subgroup of $\text{GL}(b; \mathbb{Z})$ has more than $C$ elements. Let $G$ be a finite group acting effectively on $X$. Let $G_0$ be the kernel of the induced map

$$G \to \text{Aut}(H^*(X; \mathbb{Z})/\text{Tor } H^*(X; \mathbb{Z})) \cong \text{GL}(b, \mathbb{Z}).$$

Then $[G : G_0] \leq C$. Since $G_0$ acts trivially on $H^*(X; \mathbb{Z})/\text{Tor } H^*(X; \mathbb{Z})$, it also acts trivially on $H^*(X; \mathbb{Q})$. Since $H^1(X; \mathbb{Z})$ is torsion free, the action of $G_0$ on $H^1(X; \mathbb{Z})$ is also trivial. By Theorem 2.1 there exists a morphism of groups $\eta : G_0 \to T^n$ satisfying $|\text{Ker } \eta| \leq d$ and an $\eta$-equivariant map $\psi : X \to T^n$. Let $g \in G_0$. Since the action of $g$ on $H^*(X; \mathbb{Q})$ is trivial and $\chi(X) \neq 0$, there exists some $x \in X$ such that $g \cdot x = x$ (see e.g. [50, Chap III, Exercise (6.17) 3]; this applies to $X$ because topological manifolds are Euclidean Neighborhood Retracts, see e.g. [27, Corollary A.9]). It follows that every $g \in G_0$ belongs to $\text{Ker } \eta$, so $G_0 = \text{Ker } \eta$ and consequently $|G_0| \leq d$, so $|G| \leq C \cdot d$.

5. FINITELY GENERATED $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_1^{\pm 1}]$-MODULES

**Theorem 5.1.** Let $A$ be a Noetherian ring and let $M$ be a finitely generated $A[z]$-module. Suppose that there exists a sequence of integers $r_j \to \infty$ and $A[z]$-module morphisms
there exists a filtration by $A$ such that $w_j^i$ coincides with multiplication by $z$. Then $M$ is finitely generated as an $A$-module.

Proof. Let $S \subseteq M$ be a finite $A[z]$-generating set. Let $M_0 \subseteq M$ be the $A$-submodule generated by $S$. Define an increasing sequence of $A$-submodules of $M$

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d \subseteq \ldots$$

by the condition that $M_d = M_{d-1} + zM_{d-1}$ for every positive integer $d$. Define also $M_d = 0$ for negative integers $d$. For each $d$ the quotient $M_d/M_{d-1}$ is a finitely generated $A$-module, because $M_d$ is a finitely generated $A$-module.

For any $d \geq 1$, multiplication by $z$ gives a surjective morphism

$$\mu_d : M_{d-1}/M_{d-2} \to M_d/M_{d-1}.$$  

Consider the composition

$$\nu_d = \mu_d \circ \cdots \circ \mu_1 : M_0 \to M_d/M_{d-1},$$

and define $K_d = \text{Ker } \nu_d$. Each $K_d$ is an $A$-submodule of $M_0$, and there are inclusions

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots.$$  

Since $A$ is Noetherian and $M_0$ is a finitely generated $A$-module, there exists some $d_0$ such that $K_d = K_{d-1}$ for $d \geq d_0$. If for some $d$ the morphism $\mu_d$ fails to be an isomorphism, then $\text{Ker } \nu_d$ is strictly bigger than $\text{Ker } \nu_{d-1}$, because $\nu_d = \mu_d \circ \nu_{d-1}$ and $\nu_{d-1}$ is surjective. It follows that $\mu_d$ is an isomorphism for any $d \geq d_0$.

Let $N = M_{d_0}/M_{d_0-1}$. If $N = 0$ then $M = M_{d_0-1}$, so $M$ is finitely generated as an $A$-module and we are done.

Suppose from now on that $N \neq 0$. Since $A$ is Noetherian and $N$ is finitely generated, there exists a filtration by $A$-submodules

$$(1) \quad 0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$$

in such a way that $N_j/N_{j-1} \cong A/p_j$ for primes $p_1, \ldots, p_r \in \text{Spec } A$ (see [2, Chap 7, Exercise 18] or [39, Theorem 6.4]). Let $p$ be a minimal element of $\{p_1, \ldots, p_r\}$. Denote as usual by $A_p$ the localisation of $A$ at $p$ and by $k_p$ its residual field. For any $A$-module $R$ we denote $R_p = R \otimes_A A_p$. Since $A_p$ is a flat $A$-module, for any inclusion of $A$-modules $R' \subseteq R$ we have $R_p/R'_p \cong (R/R')_p$. Since $p$ is a minimal element of $\{p_1, \ldots, p_r\}$, for every $i$ we have

$$(A/p_i)_p \cong \begin{cases} k_p & \text{if } p_i = p, \\ 0 & \text{if } p_i \neq p. \end{cases}$$

Hence, $(A/p_i)_p$ is a simple $A_p$-module for every $i$. So if we tensor by $A_p$ the elements of the filtration (1) and we ignore the resulting inclusions that are actually equalities, we get a composition series for $N_p$ of length

$$\lambda := \#\{i \mid p_i = p\} \geq 1$$

(see e.g. the paragraph before Proposition 6.7 in [2]).
To conclude the proof of the theorem we are going to prove that there is no $A[z]$-module morphism $w : M \to M$ satisfying $w^r = z$ for any $r > \lambda$. Arguing by contradiction, let us assume that there exists an $A[z]$-module morphism $w : M \to M$ satisfying $w^r = z$ for some $r > \lambda$.

Let $M'_0 = M_0$ and define recursively $M'_\delta$ for positive integers $\delta$ as $M'_\delta = M_{\delta-1} + wM_{\delta-1}$. Define also $M'_\delta = 0$ for negative integers $\delta$. The action of $w$ defines a surjective $A$-module morphism $\mu'_\delta : M'_{\delta-1}/M'_{\delta-2} \to M'_\delta/M'_{\delta-1}$. Arguing as we did for $M_d$, we prove the existence of some $\delta_0$ such that $\mu'_\delta$ is an isomorphism for any $\delta \geq \delta_0$. Let $N' = M'_{\delta_0}/M'_{\delta_0-1}$.

Denote by $A[z]_{\leq d}$ (resp. $A[w]_{\leq d}$) the $A$-module of polynomials in $z$ (resp. $w$) of degree at most $d$ (resp. $d$). We have

$$M_d = A[z]_{\leq d}M_0, \quad M'_\delta = A[w]_{\leq d}M_0.$$  

Since $w^r = z$, we have $A[z]_{\leq d} \subseteq A[w]_{\leq rd}$ for every $d$. This implies that

$$M_d \subseteq M'_{rd}$$

for every nonnegative $d$. Suppose that $S = \{m_1, \ldots, m_s\}$. Since $m_1, \ldots, m_s$ generate $M$ as an $A[z]$-module, there exist polynomials $P_{ijk} \in A[z]$ for $i = 1, \ldots, k - 1$ and $j, k = 1, \ldots, s$ such that

$$w^i m_j = P_{ij1} m_1 + \cdots + P_{ijr} m_s.$$  

Let $e = \max_{i,j,k} \deg P_{ijk}$. We have $w^i m_j \in M_e$ for every $i = 1, \ldots, k - 1$ and $j = 1, \ldots, s$, and this implies that for any $d$ we have $A[w]_{\leq \delta}M_0 \subseteq A[z]_{\leq \delta + [d/r] + e}M_0$, or equivalently

$$M'_\delta \subseteq M_{[\delta/r] + e}.$$  

Following [2] Chap 6 (see the proof of [2] Proposition 6.7]) for any $A_p$-module $R$ we denote by $l(R)$ the length of $R$. This is defined to be $\infty$ if $R$ has no composition series of finite length and it is equal to $n$ if $R$ has a composition series of length $n$. This is well defined by [2] Proposition 6.7]. Furthermore, for any inclusion of $A_p$-modules $R \subseteq R'$ we have $l(R') = l(R) + l(R'/R)$ (see [2] Proposition 6.9]). For example, we have $l(N_p) = \lambda$, and if $d \geq d_0$ then $l((M_{d+k,p})/(M_d,p)) = k\lambda$ for every $k$.

To simplify our notation we will denote $M_{i,p} = (M_i)_p$ and $M'_{i,p} = (M'_i)_p$ for every $i$. Fix some value of $d$ satisfies both $d \geq d_0$ and $[d/r] \geq \delta_0$. Let $k$ be a big number, to be specified later. The inclusions $M_{d+k,p} \subseteq M'_{r(d+k),p} \subseteq M_{d+k+e,p}$ imply

$$0 \leq l(M_{d+k+e,p}/M'_{r(d+k),p}) = l(M_{d+k+e,p}/M_{d+k,p}) - l(M'_{r(d+k),p}/M_{d+k,p})$$

$$\leq l(M_{d+k+e,p}/M_{d+k,p}) = e\lambda.$$  

Using the additivity of $l$ and the filtration

$$M_{d,p} = M'_{rd,p} \subseteq M'_{rd+1,p} \subseteq \cdots \subseteq M'_{r(d+k),p} \subseteq M_{d+k+e,p}$$

we have

$$(k + e)\lambda = l(M_{d+k+e,p}/M_{d,p}) = l(M'_{r(d+k),p}/M'_{rd,p}) + l(M_{d+k+e,p}/M'_{r(d+k),p})$$

$$= rk l(N'_p) + l(M_{d+k+e,p}/M'_{r(d+k),p})$$
and using (3) we have
\[ \frac{\lambda}{r} = \frac{(k + e)\lambda - e\lambda}{rk} \leq l(N'_p) \leq \frac{(k + e)\lambda}{rk}. \]

The lower bound for \( l(N'_p) \) belongs to the interval (0, 1), and if \( k \) is big enough so that \( (k + e)/k < r/\lambda \) then the upper bound for \( l(N'_p) \) also belongs to (0, 1). This contradicts the fact that \( l(N'_p) \) is an integer, so the proof that \( r \) cannot be bigger than \( \lambda \) is now finished. \( \Box \)

**Corollary 5.2.** Let \( M \) be a finitely generated module over \( A := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \). Suppose that for every \( 1 \leq i \leq n \) there exists a sequence of integers \( (r_{i,j})_j \) satisfying \( r_{i,j} \to \infty \) as \( j \to \infty \), and \( A \)-module automorphism \( w_{i,j} : M \to M \) such that \( w_{i,j}^{r_{i,j}} \) coincides with multiplication by \( t_i \). Then \( M \) is finitely generated as a \( \mathbb{Z} \)-module.

**Proof.** Let \( B = \mathbb{Z}[z_1, \ldots, z_{2n}] \) and let \( \phi : B \to A \) be the morphism of rings defined by \( \phi(z_{2i-1}) = t_i \) and \( \phi(z_{2i}) = t_i^{-1} \). We can look at \( M \) as a finitely generated \( B \)-module via \( \phi \), and the automorphisms \( w_{i,j} \) in the statement are \( B \)-module automorphisms satisfying
\[ w_{i,j}^{r_{i,j}} = \text{multiplication by } z_{2i-1}, \quad (w_{i,j}^{-1})^{r_{i,j}} = \text{multiplication by } z_{2i}. \]

Let \( B_0 = \mathbb{Z} \) and \( B_j = \mathbb{Z}[z_1, \ldots, z_j] \) for \( 1 \leq j \leq 2n \). Then \( B_{2n} = B \), and we can prove that \( M \) is finitely generated as a \( B_j \)-module for any \( 0 \leq j \leq 2n \) using descending induction on \( j \), applying Theorem 5.1 in the induction step. It follows that \( M \) is finitely generated as a \( B_0 \)-module. \( \Box \)

**Corollary 5.3.** Let \( M \) be a finitely generated module over \( A := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \). Suppose that for every \( 1 \leq i \leq n \) there exists a nonzero integer \( d_i \), a sequence of integers \( (r_{i,j})_j \) satisfying \( r_{i,j} \to \infty \) as \( j \to \infty \), and \( A \)-module automorphisms \( w_{i,j} : M \to M \) such that \( w_{i,j}^{r_{i,j}} \) coincides with multiplication by \( t_i^{d_i} \). Then \( M \) is finitely generated as a \( \mathbb{Z} \)-module.

**Proof.** Suppose that \( M \) is generated as an \( A \)-module by \( s_1, \ldots, s_r \in M \). Let \( A' = \mathbb{Z}[t_1^{\pm d_1}, \ldots, t_n^{\pm d_n}] \). Then \( M \) is generated as an \( A' \)-module by the set
\[ \{ t_1^{a_1}t_2^{a_2} \ldots t_n^{a_n}s_i \mid 1 \leq i \leq r, 0 \leq a_i \leq |d_i| - 1 \text{ for each } i \}, \]
so \( M \) is finitely generated as an \( A' \)-module. Applying Corollary 5.2 to \( M \) viewed as an \( A' \)-module, we conclude that \( M \) is a finitely generated \( \mathbb{Z} \)-module. \( \Box \)

### 6. Some lemmas on finite abelian groups

For any natural numbers \( a, b \) we denote for convenience \( \Gamma_{a,b} = (\mathbb{Z}/a)^b \). The first three lemmas below refer to properties of the groups \( \Gamma_{a,b} \).

**Lemma 6.1.** Let \( a, b, C \) be natural numbers and suppose that \( \Gamma' \) is a subgroup of \( \Gamma_{a,b} \) satisfying \( [\Gamma_{a,b} : \Gamma'] \leq C \). There exists a subgroup \( \Gamma'' \leq \Gamma' \) which is isomorphic to \( \Gamma_{a',b} \) for some natural number \( a' \) satisfying \( C!a' \geq a \).
Proof. Let $d$ be the greatest common divisor of $a$ and $C!$ and let $a' = a/d$. Note that $d \leq C!$, so $C!a' \geq a$. For every $1 \leq j \leq a$ let $F_j = \{ (\gamma_1, \ldots, \gamma_b) \in \Gamma_{a,b} \mid \gamma_i = 0 \text{ for } i \neq j \}$. Then $F_j \simeq \mathbb{Z}/a$. The set of subgroups of $F_j$ can be put in bijection with the set of divisors of $a$ by assigning to each subgroup its index in $F_j$, and given two subgroups $F', F''$ of $F_j$ we have $F' \leq F''$ if and only if $[F_j : F'']$ divides $[F_j : F']$. Let $F''_j$ be subgroup of $F_j$ satisfying $[F_j : F''_j] = d$. Then $F''_j$ is a cyclic group of $a'$ elements. Let $F'_j = F_j \cap \Gamma'$. We have $[F_j : F'_j] \leq [\Gamma_{a,b} : \Gamma'] \leq C$, so $[F_j : F''_j]$ divides both $C!$ and $a$. Consequently $[F_j : F''_j]$ divides $d$ and hence $F''_j \leq F'_j$. Then $\Gamma'' := F'_1 \times \cdots \times F''_b$ is a subgroup of $\Gamma'$ isomorphic to $\Gamma_{a',b}$.

Lemma 6.2. Let $\Gamma$ be a finite abelian group, and suppose that there exists a surjection $f : \Gamma \to \Gamma_{a,b}$ for some natural numbers $a, b$. Then there exists a subgroup of $\Gamma$ which is isomorphic to $\Gamma_{a,b}$.

Proof. For any prime $p$ and any finite abelian group $A$ we denote by $A_p$ the Sylow $p$-subgroup of $A$, which coincides with the set of elements in $A$ whose order is a power of $p$. Any morphism of finite abelian groups $g : A \to B$ satisfies $g(A_p) \leq B_p$, and $g$ is surjective if and only if $g(A_p) = B_p$ for every prime $p$. The group $A$ contains a subgroup isomorphic to $B$ if and only there is a subgroup $A(p) \leq A$ isomorphic to $B_p$ for every prime $p$, because in this case $\prod_p A(p) \simeq B$. So it suffices to prove the lemma assuming that both $a$ and $|\Gamma|$ are powers of the same prime $p$. Let $p$ be a prime, let $\Gamma$ be a finite abelian $p$-group, and let $f : \Gamma \to \Gamma_{p^e,b}$ be a surjection. There is an isomorphism $\phi : \Gamma \to \mathbb{Z}/p^{e_1} \times \cdots \times \mathbb{Z}/p^{e_k}$ for some natural numbers $e_1 \geq e_2 \geq \cdots \geq e_k$. We have $f(p^{e-1}A) = p^{e-1}f(A) \simeq (\mathbb{Z}/p)^b$, so $e_b \geq e$. Then

$$
\phi^{-1}(p^{e_1-e}Z/p^{e_1} \times \cdots \times p^{e_b-e}Z/p^{e_b} \times \{0\} \cdots \times \{0\})
$$

is a subgroup of $\Gamma$ which is isomorphic to $\Gamma_{p^e,b}$.

Lemma 6.3. For any natural numbers $b$ and $C_1$ there exists a natural number $C_2$ with the following property. Suppose that $\Gamma$ is a finite group and that $N$ is a normal subgroup of $\Gamma$ satisfying $|N| \leq C_1$. Suppose that $\Gamma/N \simeq \Gamma_{a,b}$ for some natural number $a$. Then there is a subgroup $\Gamma' \leq \Gamma$ which is isomorphic to $\Gamma_{a',b}$ for some natural number $a'$ satisfying $C_2 a' \geq a$.

Proof. By assumption there is a surjective morphism $\pi : \Gamma \to Q := \Gamma_{a,b}$ whose kernel is $N$. Let $c : \Gamma \to \text{Aut } N$ be the morphism given by the conjugation action of $\Gamma$ on $N$. Let $\Gamma_0 = \text{Ker } c$. Then $[\Gamma_{a,b} : \pi(\Gamma_0)] \leq |\Gamma : \Gamma_0| \leq |\text{Aut } N| \leq C_1!$. By Lemma 6.1 there is a subgroup $Q_0 \leq \pi(\Gamma_0)$ which is isomorphic to $\Gamma_{a_0,b}$ for some natural number $a_0$ satisfying $(C_1!) a_0 \geq a$. The kernel of the restriction $\pi_0$ of $\pi$ to $\Gamma_0$ coincides with the center $Z$ of $N$. Let $\Gamma_1 := \pi_0^{-1}(Q_0)$. We have an exact sequence

$$
0 \to Z \to \Gamma_1 \xrightarrow{\pi_0} Q_0 \simeq \Gamma_{a_0,b} \to 0.
$$

Since this is a central extension, one can define a bilinear morphism $\beta : Q_0 \times Q_0 \to Z$ by setting, for any two elements $u, v \in Q_0$, $\beta(u, v) = [\bar{u}, \bar{v}]$, where $\bar{u}, \bar{v} \in \Gamma_1$ are arbitrary
lifts of $u, v$. Define a morphism of groups
\[
\phi_\beta : Q_0 \to \text{Hom}(Q_0, Z)
\]
setting $(\phi_\beta(u))(v) = \beta(u, v)$ for every $u, v \in Q_0$. Now, $Q_0$ can be generated by $b$ elements because $Q_0 \simeq \Gamma_{a_0, b}$, so we may bound
\[
|\text{Hom}(Q_0, Z)| \leq |Z|^b \leq |N|^b \leq C^b.
\]
Consequently, $Q_1 := \ker \phi_\beta$ satisfies $[Q_0 : Q_1] \leq C^b$. By construction, $\pi^{-1}_0(Q_1)$ is an abelian subgroup of $\Gamma_1$. Using again Lemma 6.1 we deduce the existence of a subgroup $Q_2 \leq Q_1$ which is isomorphic to $\Gamma_{a', b}$ for some natural number $a'$ satisfying $(C^b)!a' \geq a_0$. Then $\pi^{-1}_0(Q_2)$ is a finite abelian group surjecting onto $\Gamma_{a', b}$, and hence by Lemma 6.2 there is a subgroup $\Gamma' \leq \pi^{-1}_0(Q_2) \leq \Gamma$ which is isomorphic to $\Gamma_{a', b}$. Setting $C_2 = (C^b)!\Gamma'! \leq (C^b)!a_0 \geq a$.

This finishes the proof of the lemma. \qed

The next two lemmas refer to finite subgroups of tori. Recall that we use additive notation for the group structure on tori.

**Lemma 6.4.** Let $a, d$ be natural numbers and suppose that $\Gamma$ is a finite subgroup of $T^d$ with the property that $a\gamma = 0$ for every $\gamma \in \Gamma$. Then $|\Gamma| \leq a^d$. In particular, if $T^d$ contains a subgroup isomorphic to $\Gamma_{a, b}$ for some $a \geq 2$ then $b \leq d$.

**Proof.** Let $\pi : \mathbb{R}^d \to T^d = \mathbb{R}^d/\mathbb{Z}^d$ denote the projection. Then $\pi^{-1}(\Gamma)$ is a discrete subgroup of $\mathbb{R}^d$, so it can be generated by $d$ or fewer elements, say $g_1, \ldots, g_{d'}$ with $d' \leq d$, see e.g. [11, Chap I, Lemma (3.8)]. Let $\gamma_1 = \pi(g_1)$. Then $\gamma_1, \ldots, \gamma_{d'}$ is a generating set of $\Gamma$, so the morphism $\Gamma_{a, d'} \to \Gamma$ sending $(\alpha_1, \ldots, \alpha_{d'}) \in \Gamma_{a, d'}$ to $\sum \alpha_\gamma$ is surjective (here $\alpha_i \in \mathbb{Z}$ and $\alpha_i$ is the class of $\alpha_i$ in $\mathbb{Z}/a$). It follows that $|\Gamma| \leq |\Gamma_{a, d'}| = a^{d'} \leq a^d$. \qed

**Lemma 6.5.** Let $a, d$ be natural numbers and suppose that $\Gamma$ is a finite subgroup of $T^d$ satisfying $a\gamma = 0$ for every $\gamma \in \Gamma$. Let $1 \leq j \leq d$ be any integer and let
\[
S = \{ (\theta_1, \ldots, \theta_d) \in T^d \mid \theta_i = 0 \text{ for } i \neq j \}.
\]
Then $|\Gamma \cap S| \geq |\Gamma|/a^{d-1}$.

**Proof.** We can identify $\Gamma/(\Gamma \cap S)$ with a finite subgroup of $T^d/S \simeq T^{d-1}$, all of whose elements have order dividing $a$. Hence, by Lemma 6.4 we have $|\Gamma/(\Gamma \cap S)| \leq a^{d-1}$. The exact sequence $0 \to \Gamma \cap S \to \Gamma \to \Gamma/(\Gamma \cap S) \to 0$ then implies
\[
|\Gamma \cap S| = \frac{|\Gamma|}{|\Gamma/(\Gamma \cap S)|} \geq |\Gamma|/a^{d-1},
\]
as we wished to prove. \qed
7. Proof of Theorem [1.11]

Let $X$ be a closed connected manifold and let $n$ be a natural number. We say that two degree $n$ coverings $\rho_i : X_i \to X$ ($i = 1, 2$) are isomorphic if there is a homeomorphism $f : X_1 \to X_2$ such that $\rho_1 = \rho_2 \circ f$. Let $\text{Cov}_n(X)$ denote the set of isomorphism classes of degree $n$ coverings. We denote the isomorphism class of a degree $n$ covering $\rho : X' \to X$ as $[\rho : X' \to X]$. Fix a base point $x_0 \in X$ and write $\pi_1 = \pi_1(X, x_0)$. Let $S_n$ denote the symmetric group of permutations of $\{1, \ldots, n\}$. The group $\pi_1$ is finitely generated because $X$ is closed, so the set of group homomorphisms $\text{Hom}(\pi_1, S_n)$ is finite. The group $S_n$ acts on $\text{Hom}(\pi_1, S_n)$ by postcomposition, via the conjugation action of $S_n$ on itself. Denote by $\text{Hom}(\pi_1, S_n)/S_n$ the orbit set of this action. Since $\text{Hom}(\pi_1, S_n)$ is finite, so is $\text{Hom}(\pi_1, S_n)/S_n$.

Let $\rho : X' \to X$ be a degree $n$ covering. Taking a bijection between $\rho^{-1}(x_0)$ and $\{1, \ldots, n\}$, the monodromy of $\rho$ can be identified with an element in $\text{Hom}(\pi_1, S_n)$, whose class in $\text{Hom}(\pi_1, S_n)/S_n$ is independent of the chosen bijection. The resulting element in $\text{Hom}(\pi_1, S_n)/S_n$ only depends on the isomorphism class of $\rho : X' \to X$, so this construction gives a map $\mu : \text{Cov}_n(X) \to \text{Hom}(\pi_1, S_n)/S_n$. It is an elementary fact that $\mu$ is a bijection. Hence, the set $\text{Cov}_n(X)$ is finite.

Let $G$ be a group acting continuously on $X$. For any $g \in G$ denote by $h_g : X \to X$ the homeomorphism given by the action of $g$ on $X$. The group $G$ acts on $\text{Cov}_n(X)$ as follows: if $[\rho : X' \to X] \in \text{Cov}_n(X)$, then $g \cdot [\rho : X' \to X] = [h_g^* \rho : h_g^* X' \to X]$. Since $\text{Cov}_n(X)$ is finite, there exists a constant $C_n(X)$, which only depends on $X$, and a subgroup $G_0 \leq G$ whose action on $\text{Cov}_n(X)$ is trivial and which satisfies $[G : G_0] \leq C_n(X)$. Let $\rho : X' \to X$ be a finite covering of degree $n$. Define

$$G_0' = \{(g, f) \mid g \in G_0, f : X' \to X' \text{ homeomorphism lifting } h_g\}.$$ 

Then $G_0'$ is a group (with product defined componentwise) and it acts on $X'$ via the projection $(g, f) \mapsto f$. If the action of $G$ on $X$ is effective, then so is the action of $G_0$ on $X'$. The fact that the isomorphism class $[\rho : X' \to X]$ is preserved by $G_0$ implies that the projection $\pi : G_0' \to G_0$ sending $(g, f)$ to $g$ is surjective. Furthermore, the kernel of $\pi$ can be identified with a subgroup of the permutation group of $\rho^{-1}(x_0)$, and hence has at most $n!$ elements.

Theorem [1.11] follows immediately from the next statement.

**Lemma 7.1.** Given a finite covering $\rho : X' \to X$ and a natural number $b$ there is a constant $C$ such that, for every natural number $a$, if $\Gamma_{a,b}$ acts effectively on $X$ then there is a natural number $a'$ satisfying $Ca' \geq a$ and an effective action of $\Gamma_{a',b}$ on $X'$.

**Proof.** Let $n$ be the degree of $\rho$, and assume that $\Gamma := \Gamma_{a,b}$ acts effectively on $X$. By the previous considerations, there is a subgroup $\Gamma_0 \leq \Gamma$ satisfying $[\Gamma : \Gamma_0] \leq C_n(X)$ and an exact sequence

$$1 \to F \to \Gamma_0 \xrightarrow{\pi} \Gamma_0 \to 1$$
where $|F| \leq n!$ and $\Gamma_0$ acts effectively on $X'$. By Lemma 6.1 there is a subgroup $\Gamma_1 \leq \Gamma_0$ isomorphic to $\Gamma_{a_1,b}$ for some integer $a_1$ satisfying $C_n(X) a_1 \geq a$. Let $\Gamma_1 = \pi^{-1}(\Gamma_1)$. By Lemma 6.3 there is a subgroup $\Gamma'' \leq \Gamma'$ isomorphic to $\Gamma_{a',b}$ for some natural number $a'$ satisfying $C'a' \geq a_1$, where $C'$ only depends on $n$ (through the bound $|F| \leq n!$) and $b$. Setting $C = C_n(X)!C'$ we have $Ca' \geq a$. Since $\Gamma_0$ acts effectively on $X'$, so does $\Gamma''$, so the proof is complete. \qed

8. Proof of Theorem 1.12

First we introduce some notation. Let $\text{Aff}_{\mathbb{Z}^n} \mathbb{R}^n$ denote the group of affine transformations of $\mathbb{R}^n$ that send the lattice $\mathbb{Z}^n$ to some translate of itself. There an exact sequence

$$0 \to \mathbb{R}^n \to \text{Aff}_{\mathbb{Z}^n} \mathbb{R}^n \to \text{GL}(n, \mathbb{Z}) \to 1.$$  

The transformations in $\text{Aff}_{\mathbb{Z}^n} \mathbb{R}^n$ descend to give affine diffeomorphisms of $T^n$. We denote the resulting group of transformations of $T^n$ as $\text{Aff} T^n$. This coincides, as the notation suggests, with the group of affine transformations of $T^n$. There is an exact sequence

$$0 \to T^n \xrightarrow{\tau} \text{Aff} T^n \xrightarrow{\mu} \text{GL}(n, \mathbb{Z}) \to 1,$$

where $\tau$ sends $a \in T^n$ to the translation $b \mapsto a + b$. Using the identification $T^n = \mathbb{R}^n/\mathbb{Z}^n$ we may naturally identify $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$. The morphism $\sigma : \text{GL}(n, \mathbb{Z}) \to \text{Aff} T^n$ induced by the action of $\text{GL}(n, \mathbb{Z})$ on $\mathbb{R}^n$ satisfies $\mu \circ \sigma = \text{Id}_{\text{GL}(n, \mathbb{Z})}$. If $A \in \text{GL}(n, \mathbb{Z})$, the action of $\sigma(A)$ on $H_1(T^n; \mathbb{Z})$ coincides, via the isomorphism $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$, with $A$. The following lemma results from combining [36, Corollary to Lemma 1] and [36, Theorem 3].

**Lemma 8.1.** Suppose that a finite group $\Gamma$ acts effectively on $T^n$. Using the isomorphism $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$ the action of $\Gamma$ on $H_1(T^n; \mathbb{Z})$ gives a morphism $\rho : \Gamma \to \text{GL}(n, \mathbb{Z})$. Then there is an embedding of groups $\eta : \Gamma \hookrightarrow \text{Aff} T^n$ such that $\mu \circ \eta = \rho$.

Fix natural numbers $k, n$ satisfying $1 \leq k \leq n - 1$. Recall that $\sigma \in \text{Aff} T^n$ is the involution defined by $\sigma(x_1, \ldots, x_n) = (x_1 + 1/2, \ldots, x_k + 1/2, -x_{k+1}, \ldots, -x_n)$, and that $X' = T^n, X = T^n/\sigma$ and $\rho : X' \to X$ denotes the projection.

Since $\Gamma_{r,n}$ acts effectively on $X'$ for every $r$, we have disc-sym $X' \geq n$. The action of $\Gamma_{r,k}$ on $T^n$ given by

$$(a_1, \ldots, a_k) \cdot (x_1, \ldots, x_n) = (x_1 + a_1/r, \ldots, x_k + a_k/r, a_{k+1}, \ldots, a_n)$$

commutes with $\sigma$, and hence defines an action of $\Gamma_{r,k}$ on $X$. This action is effective if $r$ is odd, and hence we conclude that disc-sym $X \geq k$.

The inequality disc-sym $X' \leq n$ follows from (1) in Theorem 1.13 (see Section 12). Let us prove that disc-sym $X \leq k$. We will need the following result.

** Lemma 8.2.** Let $T^n_x = \{ x \in T^n \mid \tau(x) = \sigma(x) \}$. There is a short exact sequence of the form $0 \to T_k^n \xrightarrow{\iota} T^n_x \to \Gamma_{2,n-k} \to 0$. 
Proof. Let $\iota : T^k \to T^n$ be $\iota((x_1, \ldots, x_k)) = (x_1, \ldots, x_k, 0, \ldots, 0)$. Since $(x_1, \ldots, x_n) \in T^n$ belongs to $T^n$ if and only if $2x_i = 0$ for every $i \geq k + 1$, we have $T^n_\sigma / \iota(T^k) \simeq \Gamma_{2,n-k}$. □

Suppose that $\Gamma_{r,m}$ acts effectively on $X$. The arguments in the proof of Theorem 1.11 imply the existence of a constant $C'$ depending only on $k$ and $n$, a subgroup $\Gamma_0 \leq \Gamma_{r,m}$ satisfying $[\Gamma_{r,m} : \Gamma_0] \leq C'$, and a central extension of groups

$$1 \to Z \to \Gamma'_0 \to \Gamma_0 \to 1$$

such that $\Gamma'_0$ acts effectively on $X$, and $Z = \{\text{Id, } \sigma\}$. This implies that the order of every element of $\Gamma'_0$ is smaller than or equal to $2r$. By Lemma 8.1 there is an injective morphism $\eta : \Gamma'_0 \to \text{Aff } T^n$ satisfying $\rho = \mu \circ \eta$, where $\rho : \Gamma'_0 \to \text{GL}(n, \mathbb{Z})$ is the morphism induced by the action of $\Gamma'_0$ on $H_1(T^n; \mathbb{Z}) \simeq \mathbb{Z}^n$. The subgroup $\Gamma''_0 := \text{Ker } \rho$ satisfies $[\Gamma'_0 : \Gamma''_0] \leq C$, where $C$ is given by Lemma 6.1 applied to $\text{GL}(n, \mathbb{Z})$ (so $C$ only depends on $n$). Then $\Gamma''_0 \leq \tau(T^n)$, so by Lemma 8.2 there is a subgroup $\Gamma'''_0 \leq \Gamma''_0$ satisfying $[\Gamma''_0 : \Gamma'''_0] \leq 2^{n-k}$ and an embedding $\Gamma'''_0 \hookrightarrow T^k$. By Lemma 6.4 we have $[\Gamma''_0] \leq (2r)^k$ because $\Gamma''_0 \leq \Gamma'_0$, so

$$r^m = [\Gamma_{r,m}] \leq C' |\Gamma_0| = \frac{C'}{2} |\Gamma'_0| \leq \frac{C'}{2} C2^{n-k}(2r)^k = C' C2^{n-1}r^k.$$ 

Consequently, if $r > C'C2^{n-1}$ then $m \leq k$.

9. Finite generation of the cohomology of abelian covers

Denote by $\pi : \mathbb{R}^k \to T^k = \mathbb{R}^k / \mathbb{Z}^k$ the quotient map. Given a topological space $X$ and a continuous map $\phi : X \to T^k$ we denote by

$$X_\phi = \{(x, u) \in X \times \mathbb{R}^k \mid \phi(x) = \pi(u)\}$$

the pullback to $X$ of the covering $\mathbb{R}^k \to T^k$. The projection

$$\rho_\phi : X_\phi \to X, \quad \rho_\phi(x, u) = x$$

is an unramified covering map. We can also look at $\rho_\phi : X_\phi \to X$ as a principal $\mathbb{Z}^k$-bundle, where the action of $\mathbb{Z}^k$ on $X_\phi$ can be described as follows: if $\nu \in \mathbb{Z}^k$ and $(x, u) \in X_\phi$ then $\nu \cdot (x, u) = (x, u + \nu)$. Standard results on fiber bundles imply the following.

Lemma 9.1. Suppose that $X$ is paracompact. If two continuous maps $\phi, \psi : X \to T^k$ are homotopic then there is a $\mathbb{Z}^k$-equivariant homeomorphism $\zeta : X_\phi \to X_\psi$ such that $\rho_\phi = \rho_\psi \circ \zeta$.

The action of $\mathbb{Z}^k$ on $X_\phi$ induces an action of $\mathbb{Z}^k$ on $H_*(X_\phi; \mathbb{Z})$ by group automorphisms. Let $e_1, \ldots, e_k$ denote the canonical basis of $\mathbb{Z}^k$, and let $t_i : \mathbb{Z}^k \to \mathbb{Z}$ denote the characteristic function of $\{e_i\} \subset \mathbb{Z}^k$. Then $t_i$ belongs to the group ring $\mathbb{Z}[\mathbb{Z}^k]$, which we identify with the additive group of finitely supported functions $\mathbb{Z}^k \to \mathbb{Z}$ with ring structure given by convolution. Furthermore, $\mathbb{Z}[\mathbb{Z}^k]$ is isomorphic to $\mathbb{Z}[t_{1}^{\pm 1}, \ldots, t_{k}^{\pm 1}]$. The action of $\mathbb{Z}^k$ on $H_*(X_\phi; \mathbb{Z})$ defines a unique structure of module over the group ring $\mathbb{Z}[\mathbb{Z}^k] \simeq \mathbb{Z}[t_{1}^{\pm 1}, \ldots, t_{k}^{\pm 1}]$ on $H_*(X_\phi; \mathbb{Z})$. 
Lemma 9.2. Let $X$ be a closed topological manifold, and let $\phi : X \to T^k$ be a continuous map. Then $H_\ast(X_\phi; \mathbb{Z})$ is finite generated as a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$-module.

Proof. Since compact topological manifolds are Euclidean Neighborhood Retracts (see e.g. [27, Corollary A.9]) we can identify homeomorphically $X$ with a closed subset of some Euclidean space $\mathbb{R}^N$ in such a way that there exists an open subset $\emptyset \subset \mathbb{R}^N$ containing $X$ and a retraction $r : \emptyset \to X$. For any $x \in X$ let $B_x \subset \emptyset$ be an open ball centered at $x$. By compactness we may choose a finite set of points $x_1, \ldots, x_s \in X$ such that $X \subset Y := B_{x_1} \cup \cdots \cup B_{x_s}$. Let $B_i = B_{x_i}$ and let $\psi = \phi \circ r : Y \to T^k$.

Let $A := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$. For every $i$ the space $B_i$ is contractible, so by Lemma 9.1 the principal $\mathbb{Z}^k$-bundle $(B_i)_\psi \to B_i$ is trivial. Hence, for every subspace $S \subseteq B_i$ we have an isomorphism of $A$-modules $H_\ast(S_\psi; \mathbb{Z}) \cong H_\ast(S; \mathbb{Z}) \otimes \mathbb{Z} A$. So if $S \subseteq B_i$ has the property that $H_\ast(S; \mathbb{Z})$ is a finitely generated abelian group, then $H_\ast(S_\psi; \mathbb{Z})$ is a finitely generated $A$-module. It also follows that $H_\ast((B_i)_\psi; \mathbb{Z})$ is a free $A$-module of rank 1.

For any $1 \leq j \leq s$ we denote $B_{\leq j} = B_{j_1} \cup \cdots \cup B_{j_s}$. We claim that $H_\ast((B_{\leq j})_\psi; \mathbb{Z})$ is a finitely generated $A$-module for every $1 \leq j \leq s$. The case $j = 1$ has been already been proved. For general $j$ we use ascending induction on $j$. Suppose that $j > 1$ and that the claim is true for $j - 1$. The Mayer–Vietoris (MV) exact sequence:

$$\cdots \to H_k((B_{\leq j-1})_\psi \cap (B_j)_\psi; \mathbb{Z}) \to H_k((B_{\leq j-1})_\psi; \mathbb{Z}) \oplus H_k((B_j)_\psi; \mathbb{Z}) \to$$

$$\to H_k((B_{\leq j})_\psi; \mathbb{Z}) \to H_{k-1}((B_{\leq j-1})_\psi \cap (B_j)_\psi; \mathbb{Z}) \to \cdots$$

is actually an exact sequence of $A$-modules. This follows from the naturality of the MV exact sequence and the fact that the $A$-module structure on each term comes from an action of $\mathbb{Z}^k$ on each of the spaces commuting with the inclusions

$$(B_{\leq j-1})_\psi \hookrightarrow (B_{\leq j-1})_\psi \cap (B_j)_\psi \hookrightarrow (B_j)_\psi$$

and

$$(B_{\leq j-1})_\psi \hookrightarrow (B_{\leq j})_\psi \hookrightarrow (B_j)_\psi.$$
10. Proof of Theorem 1.1

Let $X$ be a rationally hypertorial $n$-dimensional manifold satisfying $\text{disc-sym}(X) \geq n$, so that $X$ supports effective actions of $\Gamma_{r,n} = (\mathbb{Z}/r)^n$ for arbitrarily large integers $r$. Fix a continuous map $\phi : X \to T^n$ of nonzero degree. Let $d$ denote the absolute value of the degree of $\phi$. Define the principal $\mathbb{Z}^n$-bundle $X_\phi \to X$ as in the previous section. The action of $\mathbb{Z}^n$ on $X_\phi$ induces an action of $\mathbb{Z}^n$ on $H_*(X_\phi; \mathbb{Z})$ by isomorphisms of $\mathbb{Z}$-modules. Hence $H_*(X_\phi; \mathbb{Z})$ is a module over the ring $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

The following lemma describes how certain homeomorphisms of $X$ lift to homeomorphisms of $X_\psi$ for suitable maps $\psi : X \to T^n$. Recall that for any $a \in T^n$ we denote by $\tau(a) : T^n \to T^n$ the translation $\tau(a)(t) = t + a$.

**Lemma 10.1.** Let $f : X \to X$ be a homeomorphism of order $r$, and suppose that $\tau(a) \circ \psi = \psi \circ f$ for some $a \in T^n$, which necessarily satisfies $ra = 0$. Let $v \in \mathbb{R}^n$ satisfy $\pi(v) = a$. Then there exist a lift $g : X_\psi \to X_\psi$ satisfying $g^r(x,u) = (x, u + rv)$ for every $(x, u) \in X_\psi$.

**Proof.** Recall that $X_\psi = \{(x, u) \in X \times \mathbb{R}^k \mid \psi(x) = \pi(u)\}$. Define $g : X_\psi \to X_\psi$ by $g(x, u) = (f(x), u + v)$. The equality $\tau(a) \circ \psi = \psi \circ f$ guarantees that this is indeed a well defined homeomorphism of $X_\psi$. The definition immediately implies that $g^r(x, u) = (x, u + rv)$ for every $(x, u) \in X_\psi$. \qed

**Lemma 10.2.** For every $1 \leq j \leq n$ there exists a nonzero integer $d_j$ and a sequence of natural numbers $o_{i,j}$ satisfying $o_{i,j} \to \infty$ as $i \to \infty$, together with isomorphisms of groups $w_{i,j} : H_*(X_\psi; \mathbb{Z}) \to H_*(X_\psi; \mathbb{Z})$, such that $w_{i,j}^{o_{i,j}}$ coincides with multiplication by $t_j^{d_j}$.

**Proof.** By Lemma 3.2 there exists a number $C$ such that for every action of a finite group $G$ on $X$ the kernel $G'$ of the natural map $G \to \text{Aut} H^1(X; \mathbb{Z})$ satisfies $[G : G'] \leq C$. Let $0 < r_1 < r_2 < \ldots$ be the infinite sequence of integers such that $X$ supports an effective action of $G_i := \Gamma_{r_i,n}$ for every $i$. For every $i$ let $G_i' = \text{Ker} (G_i \to \text{Aut} H^1(X; \mathbb{Z}))$. Then $[G_i : G_i'] \leq C$. By Lemma 6.1 there is a subgroup $G_i'' \leq G_i$ such that $G_i'' \simeq \Gamma_{s_i,n}$ for a natural number $s_i$ satisfying $C!s_i \geq r_i$. In particular, $s_i \to \infty$ as $i \to \infty$.

Applying Theorem 2.1 to the action of $G_i''$ on $X$ we obtain a morphism of groups

$$\eta_i : G_i'' \to T^n$$

and an $\eta_i$-equivariant map $\psi : X \to T^n$ which is homotopic to $\phi$. Also, $|\text{Ker} \eta_i| \leq d$, so $|\eta(G_i'')| \geq s_i^d/d$. For every $1 \leq j \leq n$ let $S_j = \{(\theta_1, \ldots, \theta_n) \in T^n \mid \theta_k = 0$ for $k \neq j\}$.

By Lemma 6.5 we have

$$\sigma_{i,j} := |\eta(G_i'') \cap S_j| \geq \frac{s_i^d}{d \cdot s_i^{n-1}} = \frac{s_i}{d}.$$

The element $e_{i,j} = \pi(e_{j}/\sigma_{i,j}) \in T^n$ is a generator of $\eta(G_i'') \cap S_j$. Let $f_{i,j} : X \to X$ be the homeomorphism given by the action of an element of $G_i''$ which is sent to $e_{i,j}$ by the
morphism $\eta$. Then we have $\psi_i \circ f_{i,j} = \tau(e_j/\sigma_{i,j}) \circ \psi_i$. Furthermore, the order of $f_{i,j}$ is a number of the form

$$o_{i,j} = d_{i,j} \sigma_{i,j},$$

where $d_{i,j}$ is a natural number satisfying $1 \leq d_{i,j} \leq d$, because $|\text{Ker} \eta_i| \leq d$. Passing to a subsequence and relabelling accordingly we may assume that all natural numbers $d_{1,j}, d_{2,j}, \ldots$ are equal to the same number $d_j$. By Lemma 10.1 there is a homeomorphism $g_{i,j} : X_{\psi_i} \to X_{\psi_i}$ such that $g_{i,j}^o : X_\psi \to X_\psi$ coincides with the action of $d_j e_j$ on $X_\psi$ given by the structure of principal $\mathbb{Z}$-bundle on $X_\psi$.

Since $\psi_i$ is homotopic to $\phi$, by Lemma 9.1 there is a $\mathbb{Z}$-equivariant homeomorphism $\zeta_i : X_\phi \to X_{\psi_i}$. Let $w_{i,j} : H_*(X_\phi; \mathbb{Z}) \to H_*(X_\phi; \mathbb{Z})$ be the isomorphism induced by the homeomorphism

$$\zeta_i^{-1} \circ g_{i,j} \circ \zeta_i : X_\phi \to X_\phi.$$

Then $w_{i,j}^o$ coincides with multiplication by $t_{d_j}^i$. Since $\sigma_{i,j} \geq s_i / d$ and $s_i \to \infty$ as $i \to \infty$, we conclude that $o_{i,j} \to \infty$ as $i \to \infty$. \hfill $\square$

Combining the previous lemma with Corollary 5.3 it follows that $H_*(X_\phi; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. (Note that we may apply Corollary 5.3 to the $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$-module $H_*(X_\phi; \mathbb{Z})$ thanks to Lemma 9.2.)

**Lemma 10.3.** We have $H_k(X_\phi; \mathbb{Z}) = 0$ for every $k > 0$.

**Proof.** Suppose that $H_k(X_\phi; \mathbb{Z}) \neq 0$ for some $k > 0$. By the universal coefficient theorem, there exists some prime $p$ such that $H_k(X_\phi; \mathbb{Z}/p) \neq 0$ for some $k > 0$. The action of $\mathbb{Z}^n$ on $X_\phi$ induces a morphism $\alpha_p : \mathbb{Z}^n \to \text{Aut}(H_*(X_\phi; \mathbb{Z}/p))$. Since $H_*(X_\phi; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module, $H_*(X_\phi; \mathbb{Z}/p)$ is a finite group (again by the universal coefficient theorem). Consequently, the group $\Lambda = \text{Ker} \alpha_p$ has finite index in $\mathbb{Z}^n$.

Consider the action of $\Lambda$ on $X_\phi \times \mathbb{R}^n$ given by $\lambda \cdot ((x, u), v) = (\lambda \cdot (x, u), v - \lambda) = ((x, u + \lambda), v - \lambda)$, and let $X_\phi \times_\Lambda \mathbb{R}^n$ denote the quotient of $X_\phi \times \mathbb{R}^n$ under this action. There are natural maps

$$X_\phi \times_\Lambda \mathbb{R}^n \xrightarrow{\Theta} X_\phi / \Lambda,$$

$$\Pi : \mathbb{R}^n / \Lambda$$

where $\Pi$ is induced by the projection $X_\phi \times \mathbb{R}^n \to \mathbb{R}^n$ and $\Theta$ is induced by the projection $X_\phi \times \mathbb{R}^n \to X_\phi$. The map $\Pi$ is a fibration with fiber $X_\phi$, and $\Theta$ is a fibration with fiber $\mathbb{R}^n$ and hence is a homotopy equivalence. (This is of course a general phenomenon: $X_\phi \times_\Lambda \mathbb{R}^n$ is the Borel construction for the action of $\Lambda$ on $X_\phi$, and the fact that $X_\phi \times_\Lambda \mathbb{R}^n$ is homotopy equivalent to the quotient $X_\phi / \Lambda$ is a consequence of the fact that $\Lambda$ acts freely on $X_\phi$.) Since $X_\phi / \Lambda$ is an $n$-dimensional manifold, we have

$$H_k(X_\phi \times_\Lambda \mathbb{R}^n; \mathbb{Z}/p) = H_k(X_\phi / \Lambda; \mathbb{Z}/p) = 0 \quad \text{for every } k > n.$$
The monodromy action of $\pi_1(\mathbb{R}^n/\Lambda) \simeq \Lambda$ on the homology over $\mathbb{Z}/p$ of any given fiber of the fibration $\Pi$ coincides with the action of $\Lambda$ on $H_*(X_\phi; \mathbb{Z}/p)$, and hence is trivial. Consequently, the homology Serre spectral sequence for the fibration $\Pi$ takes the form

$$H_a(\mathbb{R}^n/\Lambda; H_b(X_\phi; \mathbb{Z}/p)) \simeq H_a(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_b(X_\phi; \mathbb{Z}/p) \Rightarrow H_{a+b}(X_\phi \times \Lambda \mathbb{R}^n; \mathbb{Z}/p).$$

Let $l = \max\{k \mid H_k(X_\phi; \mathbb{Z}/p) \neq 0\}$. By our choice of $p$, we have $l > 0$. Then $H_n(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_1(X_\phi; \mathbb{Z}/p)$ is a nonzero entry in the second page of the spectral sequence, and for dimension reasons it is contained in the kernel of every differential and none of its elements is killed by any differential; consequently, $H_n(\mathbb{R}^n/\Lambda; \mathbb{Z}/p) \otimes_{\mathbb{Z}/p} H_1(X_\phi; \mathbb{Z}/p)$ can be identified with a subquotient of $H_{n+l}(X_\phi \times \Lambda \mathbb{R}^n; \mathbb{Z}/p)$. Hence \([1]\) implies that $l = 0$, which is a contradiction. \(\square\)

**Lemma 10.4.** $\pi_0(X_\phi)$ is finite, and the action of $\mathbb{Z}^n$ on $X_\phi$ induces a transitive action on $\pi_0(X_\phi)$.

**Proof.** Recall that $\rho_\phi : X_\phi \to X$ is the projection $\rho_\phi(x, p) = x$, and that $X_\phi$ has a structure of principal $\mathbb{Z}^n$-bundle over $X$ with projection $\rho_\phi$. Fix some base point $x_0 \in X$. Since $X$ is arc-connected, each arc-connected component of $X_\phi$ intersects $\rho^{-1}(x_0)$ nontrivially. This implies that the action of $\mathbb{Z}^n$ on $\pi_0(X_\phi)$ is transitive.

Again because $X$ is arc-connected, $\pi_0(X_\phi)$ can be identified with the set $\mathcal{O}$ of orbits of the monodromy action of $\pi_1(X, x_0)$ on $\rho_\phi^{-1}(x_0)$. Since the bundle $\rho_\phi : X_\phi \to X$ is the pullback of $\pi : \mathbb{R}^n \to T^n$ by $\phi$, the set $\mathcal{O}$ can be identified with the set $\mathcal{O}'$ of orbits of the action of $\phi_*\pi_1(X, x_0) \leq \pi_1(T^n, \phi(x_0))$ on $\pi^{-1}(\phi(x_0))$. Since $\pi^{-1}(\phi(x_0))$ is a torsor over $\pi_1(T^n, \phi(x_0))$ (via the monodromy action), the choice of any element $y \in \pi^{-1}(\phi(x_0))$ gives a bijection between $\mathcal{O}'$ and $\pi_1(T^n, \phi(x_0))/\phi_*\pi_1(X, x_0)$. Summarizing, we have proved that $\pi_0(X_\phi)$ can be put in bijection with $\pi_1(T^n, \phi(x_0))/\phi_*\pi_1(X, x_0)$.

We thus need to prove that $\pi_1(T^n, \phi(x_0))/\phi_*\pi_1(X, x_0)$ is finite. Since $\pi_1(T^n, \phi(x_0))$ is abelian, the previous quotient is isomorphic to $H_1(T^n, \mathbb{Z})/\phi_*H_1(X, \mathbb{Z})$. Now suppose that the index of $\phi_*H_1(X, \mathbb{Z})$ in $H_1(T^n, \mathbb{Z})$ is infinite. Then $H_1(T^n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = \phi_*H_1(X, \mathbb{Q})$ is a proper subspace of $H_1(T^n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(T^n, \mathbb{Q})$. Hence there exists some nonzero $\beta \in H^1(T^n; \mathbb{Q})$ inducing a morphism $H_1(T^n; \mathbb{Q}) \to \mathbb{Q}$ whose kernel contains $\phi_*H_1(X, \mathbb{Q})$. This implies that $\phi^*\beta = 0$.

Since $\beta \neq 0$ there exist classes $\beta_1 = \beta, \beta_2, \ldots, \beta_n \in H^1(T^n; \mathbb{Q})$ such that $\Omega = \beta_1 \cup \cdots \cup \beta_n$ is a generator of $H^n(T^n; \mathbb{Q})$. Then $\phi^*\beta = 0$ implies $\phi^*\Omega = 0$, so $\phi^* : H^n(T^n; \mathbb{Q}) \to H^n(X; \mathbb{Q})$ is the zero map. This contradicts the assumption that $\deg \phi \neq 0$.

Consequently, the index of $\phi_*H_1(X, \mathbb{Z})$ in $H_1(T^n; \mathbb{Z})$ is finite, so $\pi_0(X_\phi)$ is finite. \(\square\)

Fix an arc-connected component $X^0_\phi \subseteq X_\phi$. Since $H_k(X_\phi; \mathbb{Z}) = 0$ for every $k > 0$, $X^0_\phi$ has the same integral homology as the point:

$$H_0(X^0_\phi; \mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad H_k(X^0_\phi; \mathbb{Z}) = 0 \quad \text{for} \quad k > 0. \quad (5)$$

Let $V \leq \mathbb{Z}^n$ be the subgroup consisting of those elements of $\mathbb{Z}^n$ whose action of $X_\phi$ maps $X^0_\phi$ to itself. By the previous lemma, $V$ has finite index in $\mathbb{Z}^n$ and we have $X = X_\phi/\mathbb{Z}^n = \{x \in X : \phi(x) = x\}$.
Arguing as in the definition of $X$ to $X_H$, it follows that $X = X_0/V$ is homotopy equivalent to $X_0 \times V \mathbb{R}^n$, where the latter is defined exactly as $X_0 \times \mathbb{R}^n$ but replacing $X_0$ resp. $\Lambda$ with $X_0$ resp. $V$. The fibers of the projection $Q : X_0/\mathbb{R}^n \to \mathbb{R}^n/V$ can be identified with $X_0$, and hence are acyclic by [5]; it follows that $Q$ induces an isomorphism in integral homology. It follows that $H_*(X;\mathbb{Z}) = H_*(X_0/V;\mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n/V;\mathbb{Z}) \cong H_*(T^n;\mathbb{Z})$, which proves statement (1) of Theorem 1.1.

To prove statement (2) of Theorem 1.1 we momentarily assume that $\pi_1(X)$ is solvable. Since the projection $X_0 \to X$ is a covering space, it identifies $\pi_1(X_0)$ with a subgroup of $\pi_1(X)$, so $\pi_1(X_0)$ is solvable. The abelianization of $\pi_1(X_0)$ can be identified with $H_1(X_0)$, which we know is 0. A solvable group with trivial abelianization is the trivial group, so $X_0$ is simply connected. By the Hurewicz theorem, $X_0$ is contractible (see e.g. [27, Corollary 4.33]).

**Remark 10.5.** This is the only point where we use that $\pi_1(X)$ is (virtually) solvable. Note that an acyclic manifold can perfectly be noncontractible. Indeed, there are plenty of examples of finitely generated acyclic groups (see e.g. [5]) and by a result of Kervaire [32, Theorem 1] any such group is the fundamental group of an integral smooth homology sphere of dimension $n > 4$. Removing a point from such manifold we obtain an acyclic manifold with the same fundamental group. It is not clear, however, whether a non simply connected acyclic $n$-dimensional manifold can support a free action of $\mathbb{Z}^n$ with compact quotient (let alone that its quotient by $\mathbb{Z}^n$ supports free actions of $(\mathbb{Z}/r)^n$ for arbitrarily large $r$). So it could be the case that our assumption that $\pi_1(X)$ is virtually solvable is unnecessary.

It follows that $Q : X_0 \times V \mathbb{R}^n \to \mathbb{R}^n/V$ is a homotopy equivalence. Precomposing it with a homotopy inverse of the projection $X_0 \times V \mathbb{R}^n \to X_0/V = X$ we obtain a homotopy equivalence $X \to \mathbb{R}^n/V$. It then follows from the topological rigidity of tori (see the Introduction for references) that $X$ is homeomorphic to $T^n$.

To conclude, let us prove statement (2) of Theorem 1.1 in the general case. Suppose that $X$ is a rationally hypertoral manifold satisfying disc-sym$(X) \geq n$ and that $\pi_1(X)$ is virtually solvable. Then there is a finite covering $r : X' \to X$ such that $\pi_1(X')$ is solvable. Let $\phi : X \to T^n$ be a map of nonzero degree and let $\phi' = \phi \circ r$. Then $\deg \phi' = \deg \phi \cdot \deg r \neq 0$ and we have a Cartesian diagram

$$
\begin{array}{ccc}
X'_0 & \xrightarrow{r_{\phi}} & X_0 \\
\rho_{\phi'} \downarrow & & \downarrow \rho_{\phi} \\
X' & \xrightarrow{r} & X.
\end{array}
$$

In particular, $r_{\phi}$ is a finite (unramified) covering space. By Theorem 1.1 we have $\text{disc-sym}(X') \geq n$. Applying the previous discussion to $X'$ we conclude that the connected components of $X'_0$ are contractible. Let $X_0$ be any connected component of $X_0$. 


and denote by \((X_\phi')^0\) its preimage under \(r_\phi\). Let \(\pi = \pi_1(X_\phi^0)\), so that \(X_\phi^0 = (X_\phi')^0/\pi\), where \(\pi\) acts freely on \((X_\phi')^0\). Note that \(\pi\) is finite, because \(r_\phi\) is a finite covering. The freeness of the action implies that \(\pi\) acts freely on the set of connected components of \((X_\phi')^0\) because by Smith’s theory a homeomorphism of primer order of a contractible manifold has necessarily some fixed point (see e.g. [9, Chap III, Corollary 4.6]). This implies that \(X_\phi^0\) is also contractible, so the same argument as in the case of solvable fundamental group allows to finish the proof of statement (2) of Theorem 1.1 in the general case.

11. Proof of Theorem 1.2

In dimensions up to three any topological manifold has a unique smooth structure, so we will assume from now on in this section that \(n \geq 5\). According to [52, §15A], for any smooth \(n\)-manifold \(X\) and any simple homotopy equivalence \(h : X \to T^n\) one can define a "characteristic class"

\[
c(h : X \to T^n) \in A_n := H^3(T^n; \mathbb{Z}/2) \oplus \bigoplus_{i \leq n} H^i(T^n; \pi_i(PL/O))
\]

with the property that if \(h' : X' \to T^n\) is another simple homotopy equivalence and \(c(h : X \to T^n) = c(h' : X' \to T^n)\) then \(X\) and \(X'\) are diffeomorphic. The piece of the characteristic class in \(H^3(T^n; \mathbb{Z}/2)\) accounts for the PL structure of \(X\), whereas that in \(H^i(T^n; \pi_i(PL/O))\) accounts for the different choices of smooth structure compatible with the given PL structure. The identity map \(Id : T^n \to T^n\) has trivial characteristic class, so if \(c(h : X \to T^n) = 0\) then \(X\) is diffeomorphic to the standard torus. In addition, if \(\pi : T^n \to T^n\) is a covering and \(\pi^*h : \pi^*X \to T^n\) is the pullback of \(h\), then

\[
c(\pi^*h : \pi^*X \to T^n) = \pi^*c(h : X \to T^n)
\]

(note that \(\pi^*h : \pi^*X \to T^n\) is also a simple homotopy equivalence). The homotopy groups \(\pi_i(PL/O)\) are finite and hence so is the group \(A_n\).

Let \(X\) be a smooth \(n\)-manifold and suppose that \(h : X \to T^n\) is a homeomorphism. Then \(h\) is a simple homotopy equivalence by Chapman’s theorem (see e.g. the Appendix in [16]), so we have a well defined characteristic class \(c(h : X \to T^n) \in A_n\). Let \(k\) be any natural number and let \(r = k|A_n| + 1\). Multiplication by \(r\) is the identity map on \(A_n\), so if \(\pi_r : T^n \to T^n\) is the covering space defined by \(\pi_r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n)\) (where \(x_i \in \mathbb{R}/\mathbb{Z}\)) then \(\pi_r^*c(h : X \to T^n) = c(\pi_r^*h : \pi_r^*X \to T^n) = c(h : X \to T^n)\). Hence there exists a diffeomorphism \(\phi_r : X \to \pi_r^*X\). The manifold \(\pi_r^*X\) has a free and smooth action of \((\mathbb{Z}/r)^n\) given by deck transformations of the covering \(\pi_r^*X \to X\). This action can be transported via \(\phi_r\) to a free action of \((\mathbb{Z}/r)^n\) on \(X\). This proves statement (1) of Theorem 1.2.

Let us now prove (2). Let \(X\) be a smooth manifold homeomorphic to \(T^n\), and fix a homotopy equivalence \(h : X \to T^n\). By Lemma 3.2 and Theorem 2.1 there exists a constant \(C\) such that for any finite group \(\Gamma\) acting continuously on \(T^n\) there is a subgroup.
\[ \Gamma_0 \leq \Gamma \text{ satisfying } [\Gamma : \Gamma_0] \leq C, \text{ a map } \psi : X \to T^n \text{ homotopic to } h, \text{ and a monomorphism } \eta : \Gamma_0 \to T^n \text{ such that } \psi \text{ is } \eta\text{-equivariant. In particular, the action of } \Gamma_0 \text{ on } T^n \text{ is free.} \]

We next define \( \delta(n) \). By Lemma 6.1 there exists a constant \( C' \) such that, if \( \Gamma' \) is any subgroup of \( \Gamma_{r,n} \) satisfying \([\Gamma_{r,n} : \Gamma'] \leq C \), then \( \Gamma' \) contains a subgroup isomorphic to \( \Gamma_{s,n} \), where \( C's \geq r \). Suppose that \([A_n] = p_1^{e_1} \cdots p_k^{e_k} \), where \( p_1, \ldots, p_k \) are pairwise distinct prime numbers and the exponents \( e_i \) are natural numbers. For each \( i \) let \( f_i \) be the smallest natural number such that \( p_i^{f_i} \geq C' \). Define \( \delta(n) = p_1^{e_1+f_1} \cdots p_k^{e_k+f_k} \).

If \( r \) is divisible by \( \delta(n) \) and \( \Gamma' \) is a subgroup of \( \Gamma_{r,n} \) satisfying \([\Gamma_{r,n} : \Gamma'] \leq C \) then there exists a subgroup \( \Gamma'' \leq \Gamma' \) isomorphic to \( \Gamma_{s,n} \) for some \( s \) satisfying \( Cs' \geq r \). Since \( \Gamma_{r,n} \) has a subgroup isomorphic to \( \Gamma_{s,n} \), \( s^n = [\Gamma_{s,n}] \) divides \( r^n = |\Gamma_{r,n}| \), which implies that \( s \) divides \( r \). We next prove that \( s \) is divisible by \( p_i^{e_i} \) for every \( i \). Let \( p_i^{n_i} \) (resp. \( p_i^{h_i} \)) be the biggest power of \( p_i \) dividing \( s \) (resp. \( r \)). We have \( h_i \geq e_i + f_i \) because \( r \) is divisible by \( \delta(n) \), and \( h_i \geq g_i \) because \( s \) divides \( r \). The quotient \( r/s \) is divisible by \( p_i^{h_i-g_i} \). Since \( r/s \leq C' \), we have \( h_i - g_i \leq f_i \), so \( g_i \geq h_i - f_i \geq e_i \). Hence \( s \) is divisible by \( |A_n| \).

If the group \( \Gamma_{r,n} \) acts smoothly and effectively on \( X \) then there is a monomorphism \( \eta : \Gamma_{s,n} \to T^n \) and an \( \eta\text{-equivariant map } \psi : X \to T^n \text{ homotopic to } h. \) The quotient \( T^n \to T^n/\eta(\Gamma_{s,n}) \) is a covering map and \( T^n/\eta(\Gamma_{s,n}) \) is homeomorphic to \( T^n \). So the map \( \psi \) descends to a continuous map \( \zeta : X/\Gamma_{s,n} \to T^n/\eta(\Gamma_{s,n}) \cong T^n \). The projection map identifies \( \pi_1(X) \) (resp. \( \pi_1(T^n) \)) with a subgroup of \( \pi_1(X/\Gamma_{s,n}) \) (resp. \( \pi_1(T^n/\eta(\Gamma_{s,n})) \)), and via these identifications \( \zeta : \pi_1(X/\Gamma_{s,n}) \to \pi_1(T^n/\eta(\Gamma_{s,n})) \) can be seen as an extension of \( \psi_\ast : \pi_1(X) \to \pi_1(T^n) \). Since \( \psi_\ast \) is an isomorphism, \( [\pi_1(X/\Gamma_{s,n}) : \pi_1(X)] = [\pi_1(T^n/\eta(\Gamma_{s,n})) : \pi_1(T^n)] \) (both are equal to \( |\Gamma_{s,n}| \)), and \( \pi_1(X/\Gamma_{s,n}) \simeq \pi_1(T^n/\eta(\Gamma_{s,n})) \simeq \mathbb{Z}^n \); it follows that \( \zeta_\ast \) is an isomorphism. Both \( X/\Gamma_{s,n} \) and \( T^n/\eta(\Gamma_{s,n}) \) are aspherical spaces, so \( \zeta \) is a homotopy equivalence. By the topological rigidity of tori, \( \zeta \) is homotopic to a homeomorphism \( \xi : X/\Gamma_{s,n} \to T^n \). Since \( \xi \) is homotopic to \( \zeta \), it can be lifted to a homeomorphism \( \theta : X \to T^n \) that makes the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\theta} & T^n \\
\downarrow & & \downarrow q \\
X/\Gamma_{s,n} & \xrightarrow{\xi} & T^n.
\end{array}
\]

Since \( s \) is divisible by \( |A_n| \) and \( q \) can be written as \( q(x_1, \ldots, x_n) = (sx_1, \ldots, sx_n) \), it follows that \( c(\theta : X \to T^n) = 0 \), so \( X \) is diffeomorphic to \( T^n \).

**12. Proof of Theorem 1.13**

Statement (1) of Theorem 1.13 follows from combining Lemma 3.2, Theorem 2.1 and Lemma 6.4. For the proof of statement (2) we will need the following two lemmas.

**Lemma 12.1.** Let \( X, Y \) be two closed connected topological manifolds. Then \( X \times Y \) is rationally hypertoral if and only if both \( X \) and \( Y \) are rationally hypertoral.
Lemma 12.2. Let $X, Y$ be closed connected topological manifolds of the same dimension. Suppose that $X \# Y$ is rationally hypertoral. Then at least one of the manifolds $X, Y$ is also rationally hypertoral.

Proof. If $n = 1$ the statement is obvious, and if $n = 2$ it follows from the classification of closed connected surfaces. Let us suppose from now on that $n \geq 3$ and that $Z := X \# Y$ is rationally hypertoral. Let $X_0, Y_0$ be the complementaries of open balls in $X, Y$ respectively, so that $\partial X_0 \simeq S^{n-1} \simeq \partial Y_0$. We identify $Z$ with $X_0 \cup_{\partial X_0 \simeq \partial Y_0} Y_0$. Denote by $i_X : X_0 \hookrightarrow Z$ and $i_Y : Y_0 \hookrightarrow Z$ the natural inclusions.

Let $W := i_X(X_0) \cap i_Y(Y_0)$, choose a homeomorphism $\xi : W \to S^{n-1}$, and let $E$ be the result of attaching to $Z$ an $n$-disk $D^n$ along its boundary $\partial D^n = S^{n-1}$, using $\xi$. Let $\phi : Z \to T^n$ be a map of nonzero degree. Since $\pi_{n-1}(T^n) = 0$, the restriction of $\phi$ to $W$ is homotopically trivial, so $\phi$ extends to a continuous map $\psi : E \to T^n$. By contracting $D^n \subset E$ we obtain a map $c : E \to X \vee Y$ that is a homotopy equivalence. Composing $\psi$ with a homotopy inverse of $c$ we obtain a map $\zeta : X \vee Y \to T^n$. The map $\phi$ coincides up to homotopy with the composition

$$Z \xrightarrow{\zeta} X \vee Y \xrightarrow{\zeta} T^n.$$ 

Since the degree of $\phi$ is nonzero, it follows that the restriction of $\zeta$ to one of the two summands $X \subset X \vee Y$ or $Y \subset X \vee Y$ has to be nonzero, so the corresponding manifold is rationally hypertoral.

We now prove statement (2) in Theorem 12.13. Let $X = Y \times (Z \# Z')$ be a rationally hypertoral manifold, where dim $Y = k$ and dim $Z = \dim Z' = n - k$ for some integer $0 \leq k < n$. Since $X$ is orientable, the three manifolds $Y, Z, Z'$ are orientable. By Lemma 12.17 both $Y$ and $Z \# Z'$ are hypertoral, and by Lemma 12.2 at least one of the manifolds $Z, Z'$ (say, $Z$) is rationally hypertoral. Assume that $H^*(Z'; \mathbb{Z})$ is not isomorphic to $H^*(S^{n-k}; \mathbb{Z})$. Choose maps of nonzero degree $\phi_Y : Y \to T^k$, $\phi_Z : Z \to T^{n-k}$ and let $\phi = (\phi_Y, \phi_Z) : X = Y \times (Z \# Z') \to T^k \times T^{n-k} = T^n$, where $\phi_Z = \phi_Z \circ c_Z : Z \# Z' \to T^{n-k}$ and $c_Z : Z \# Z' \to Z$ is the map collapsing $Z'$. Then $d := \deg \phi$ is nonzero.
We prove that disc-sym\((X) \leq k\) by contradiction. Assume that there exists a sequence of natural numbers \(r_i \to \infty\) such that \(\Gamma_i := \Gamma_{r_i,k+1}\) acts effectively on \(X\). By Lemma 3.2, Theorem 2.1 and Lemma 6.1, there exists a constant \(C\) and, for every \(i\), a subgroup \(\Gamma_i'' \leq \Gamma_i\) isomorphic to \(\Gamma_{s_i,k+1}\), with \(s_i\) satisfying \(C s_i \geq r_i\), a morphism \(\eta_i : \Gamma_i' \to T^n\) satisfying \(|\text{Ker} \eta_i| \leq d\), and an \(\eta_i\)-equivariant map 
\[
\phi_i : X \to T^n
\]

homotopic to \(\phi\).

Let \(\lambda : T^n \to T^k\) be the projection that forgets the last \(n - k\) coordinates. By Lemma 6.4 we have \(|\lambda(\eta_i(\Gamma_i'))| \leq s_i^k\). Consequently

\[
\Gamma_{i,0} : = \eta_i(\Gamma_i') \cap \text{Ker} \lambda
\]
satisfies

\[
|\Gamma_{i,0}| = \frac{|\eta_i(\Gamma_i')|}{|\lambda(\eta_i(\Gamma_i'))|} \geq \frac{s_i^{k+1}/d}{|\lambda(\eta_i(\Gamma_i'))|} \geq s_i/d.
\]

Let \(\mu : T^n \to T^{n-k}\) be the projection that forgets the first \(k\) coordinates, and consider the sequence of projections

\[
T^n \xrightarrow{\mu} T^{n-k} \xrightarrow{\mu} T^{n-k-1} \to \ldots \xrightarrow{\mu} T^1 \xrightarrow{\mu} \{1\},
\]

where each \(\mu_j\) forgets the first coordinate. Denote the compositions \(\nu_j = \mu_j \circ \cdots \circ \mu_1\). Define \(G_{i,0} := \mu(\Gamma_{i,0})\) and, for every \(1 \leq j \leq n - k\), \(G_{i,j} = \nu_j(G_{i,0})\). Consider the set

\[
B = \{ j \in \{0, 1, \ldots, n - k - 1\} : |G_{i,j} \cap \text{Ker} \mu_{j+1}| \text{ is not bounded as } i \to \infty \}.
\]

For every \(j\) there is a short exact sequence

\[
0 \to G_{i,j} \cap \text{Ker} \mu_{j+1} \to G_{i,j} \to G_{i,j+1} \to 0,
\]

and since \(|G_{i,n-k}| = 1\), we have

\[
|G_{i,0}| = \prod_{j=0}^{n-k-1} |G_{i,j} \cap \text{Ker} \mu_{j+1}|.
\]

Since \(|G_{i,0}| \to \infty\) as \(i \to \infty\), it follows that the set \(B\) is nonempty. Let \(j_0 := \min B\). Replacing the sequence \((\Gamma_i')\) by a subsequence, we may assume that \(|G_{i,j_0} \cap \text{Ker} \mu_{j_0+1}| \to \infty\) as \(i \to \infty\). By the choice of \(j_0\), there is a constant \(C\) such that \(|G_{i,0} \cap \text{Ker} \nu_{j_0}| \leq C\) for every \(i\). Define

\[
\Gamma_i'' = \eta_i^{-1}(\Gamma_{i,0})
\]

and let

\[
\tilde{\eta}_i \circ \eta_i|_{\Gamma_i''} : \Gamma_i'' \to T^{n-k-j_0}.
\]

Then the fibers of \(\tilde{\eta}_i|_{\Gamma_i''}\) have at most \(\delta C\) elements.

Let \(S = \text{Ker} \nu_{j_0+1} \leq T^{n-k-j_0}\). This can be identified with the elements of \(T^{n-k-j_0}\) whose coordinates are all zero except possibly the first one. Let \(\Gamma_i''' = \tilde{\eta}_i^{-1}(S)\). By the
definition of \( j_0 \), we have \(|\eta_i(\Gamma''_i)| \to \infty \) as \( i \to \infty \). Let \( \xi : T^{n-k-j_0} \to S \) be the projection that forgets all coordinates except the first one, and let

\[
\psi_i = \xi \circ \nu_{j_0} \circ \mu \circ \phi_i : X \to S.
\]

Then \( X_{\psi_i} \to X \) is a \( \mathbb{Z} \)-principal bundle and \( H_*(X_{\psi_i}; \mathbb{Z}) \) is a finitely generated \( \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}] \) module by Theorem \ref{thm:5.1}. The map \( \psi_i \) is homotopic to the map

\[
\zeta = \xi \circ \nu_{j_0} \circ \mu \circ \phi : X \to S,
\]

so there exists a \( \mathbb{Z} \)-equivariant homeomorphism \( X_{\zeta} \cong X_{\psi_i} \). We are next going to prove that \( H_*(X_{\zeta}; \mathbb{Z}) \) is finitely generated as a \( \mathbb{Z} \)-module, using an argument similar to the proof of Lemma \ref{lem:10.2}.

Then for any \( \gamma \in \Gamma''_i \) and \( x \in X \) we have

\[
\psi_i(\gamma \cdot x) = \tilde{\eta}_i(\gamma) + \psi_i(x).
\]

Let \( o_i \) be the order of \( \tilde{\eta}_i(\Gamma''_i) \). We can identify \( S \) with \( \mathbb{R}/\mathbb{Z} \), and we denote by \( \pi : \mathbb{R} \to S \) the projection. There exists an element \( \gamma_i \in \Gamma''_i \) satisfying \( \tilde{\eta}_i(\gamma_i) = \pi(1/o_i) \). Then the order of \( \gamma_i \) is equal to \( C_i \), where \( C_i \) is a natural number satisfying \( 1 \leq C_i \leq \delta C \). Passing to a subsequence of \( (\Gamma_i) \) we may assume that all numbers \( C_i \) are equal to a natural number \( C' \). Then the homeomorphism \( f_i : X \to X \) given by the action of \( \gamma_i \) has order \( C'o_i \). By Lemma \ref{lem:10.1} the map \( f_i \) admits a lift \( g_i : X_{\psi_i} \to X_{\psi_i} \) such that the action of \( g_i^{C'o_i} \) on \( H_*(X_{\psi_i}; \mathbb{Z}) \) coincides with multiplication by \( t^{C'} \). Since \( o_i \to \infty \), Corollary \ref{cor:5.3} implies that \( H_*(X_{\zeta}; \mathbb{Z}) \) is a finitely generated \( \mathbb{Z} \)-module, as we wanted to see.

The map \( \zeta : X \to S \) factors through the composition

\[
X = Y \times (\mathbb{Z}^*_Z \mathbb{Z}') \to \mathbb{Z}^*_Z \mathbb{Z}' \xrightarrow{\zeta} \mathbb{Z},
\]

where the first map is the projection to the second factor. Let \( \kappa : X \to \mathbb{Z} \) be this composition. Define \( \psi_Z = \xi \circ \nu_{j_0} \circ \phi_Z \). Then \( \zeta = \psi_Z \circ \kappa \), so \( X_{\zeta} \) can be identified with \( Y \times (\mathbb{Z}^*_Z \mathbb{Z}') \), the product of \( Y \) with the connected sum of \( \mathbb{Z}_Z^* \mathbb{Z}' \) with countably many copies of \( \mathbb{Z}' \). Since \( H_*(\mathbb{Z}'^*; \mathbb{Z}) \neq H^*(\mathbb{Z}_Z^*; \mathbb{Z}) \), making connected sum with \( \mathbb{Z}' \) increases the size of the homology, so \( H_*(X_{\zeta}; \mathbb{Z}) \) is not finitely generated over \( \mathbb{Z} \), contradicting our previous conclusion. This finishes the proof that \( \text{disc-sym}(X) \leq k \).

13. Holomorphic finite group actions on Kaehler manifolds

13.1. Proof of Theorem \ref{thm:1.16}. The implication \((1) \Rightarrow (2) \) is immediate, as the \( r \)-torsion of \( T^n \) is isomorphic to \( (\mathbb{Z}/r)^n \). To prove the converse implication \((2) \Rightarrow (1) \) it is enough to consider the case in which \( G \) is compact. Indeed, if \( G \) is an arbitrary Lie group with finitely many connected components then the existence and uniqueness up to conjugation of maximal compact subgroups (see e.g. \cite[Theorem 14.13]{28}) implies the existence of a compact subgroup \( K \leq G \) with the property that any compact (in particular, any finite) subgroup of \( G \) is conjugate to a subgroup of \( K \). Hence, replacing \( G \) by \( K \) we assume from now on that \( G \) is a compact Lie group.
Let $\mathfrak{g}$ denote the Lie algebra of $G$. For every $\alpha \in \mathfrak{g}$ we denote by $e^\alpha$ the image of $\alpha$ under the exponential map $\mathfrak{g} \to G$. We claim that there exists a neighborhood of zero $U \subset \mathfrak{g}$ with the property that for any $\alpha, \beta \in U$ such that the elements $e^\alpha, e^\beta \in G$ commute we have $[\alpha, \beta] = 0$. To prove the claim, pick an $\text{Ad}$-invariant norm on $\mathfrak{g}$ and choose $\epsilon > 0$ small enough so that the exponential map sends the ball $U := B_\epsilon(0) \subset \mathfrak{g}$ diffeomorphically to a neighborhood of the identity in $G$. Choose elements $\alpha, \beta \in U$ such that $e^\alpha$ and $e^\beta$ commute. Then

$$e^\beta = e^\alpha e^\beta (e^\alpha)^{-1} = e^{\text{Ad}(e^\alpha)(\beta)},$$

so by the injectivity of $\exp|_U$ we have $\beta = \text{Ad}(e^\alpha)(\beta)$. Since $\text{Ad}(e^\alpha)$ is a linear automorphism of $\mathfrak{g}$, it follows that $t\beta = \text{Ad}(e^\alpha)(t\beta)$ for every $t \in \mathbb{R}$ and consequently $e^\alpha$ commutes with $e^{t\beta}$ for every $t \in [-1, 1]$. Repeating the same argument with $t\beta$ resp. $\alpha$ instead of $\alpha$ resp. $\beta$ we conclude that for every $s, t \in [-1, 1]$ we have $e^{s\alpha} e^{t\beta} = e^{t\beta} e^{s\alpha}$. This implies $[\alpha, \beta] = 0$.

Choose a neighborhood of the identity $V \subset G$ such that $V^{-1}V \subseteq e_U$. Since $G$ is compact, we can cover $G$ with finitely many translates of $V$, say $G = g_1V \cup \cdots \cup g_mV$ for some elements $g_1, \ldots, g_m \in G$. Suppose that $\Gamma$ is a finite group of $G$. Since $\Gamma = (g_1V \cap \Gamma) \cup \cdots \cup (g_mV \cap \Gamma)$, there exists some $1 \leq i \leq m$ such that $|g_iV \cap \Gamma| \geq |\Gamma|/m$. Take any element $\gamma_0 \in g_iV \cap \Gamma$ and let $S_U = \{\gamma_0^{-1}\gamma \mid \gamma \in g_iV \cap \Gamma\}$. Then $S_U \subseteq V^{-1}V \cap \Gamma \subseteq e_U \cap \Gamma$ and $|S_U| \geq |\Gamma|/m$, so the subgroup $\Gamma_U \leq \Gamma$ generated by $S_U$ satisfies $[\Gamma : \Gamma_U] \leq m$. Let $\mathfrak{s}_U = \{\alpha \in U \mid e^\alpha \in S_U\}$. By the construction of $U$, $\mathfrak{s}_U$ spans an abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Let $T$ be the closure of $e^\mathfrak{t}$. Then $T$ is a torus and $S_U \subseteq T$, so $\Gamma_U \subseteq T$. We have thus proved that for any finite subgroup $\Gamma$ of $G$ there is a torus $T \subseteq G$ such that $[\Gamma : T \cap \Gamma] \leq m$, where $m$ is independent of $\Gamma$.

Suppose that $n$ is a natural number such that $G$ contains subgroups isomorphic to $\Gamma_{r,n} = (\mathbb{Z}/r)^n$ for arbitrary large numbers $r$. By Lemma 6.1 there is a constant $C$ such that any subgroup $\Gamma' \leq \Gamma_{r,n}$ satisfying $[\Gamma_{r,n} : \Gamma'] \leq m$ contains a subgroup $\Gamma'' \leq \Gamma'$ such that $\Gamma'' \simeq \Gamma_{s,n}$ with $Cs \geq r$. By assumption, there exists some $r \geq 2C$ and some subgroup $\Gamma$ of $G$ isomorphic to $\Gamma_{r,n}$. By our previous result, there exists a torus $T \leq G$ such that $[\Gamma : T \cap \Gamma] \leq m$. Then there is a subgroup $\Gamma'' \leq \Gamma' := T \cap \Gamma$ such that $\Gamma'' \leq \Gamma_{s,n}$ for some $s$ satisfying $Cs \geq r \geq 2C$, so $s \geq 2$. Then Lemma 6.4 implies that $\dim T \geq n$, so the proof of Theorem 1.15 is complete.

13.2. Proof of Theorem 1.15. Let $X$ be a compact connected Kaehler manifold of real dimension $n$. Let $\text{Aut}_X$ denote the group of biholomorphisms of $X$. A theorem of Bochner and Montgomery [80, §9] states that $\text{Aut}_X$ has a natural structure of Lie group. Let $\omega$ denote the Kaehler form of $X$ and let $\overline{\mathcal{C}}$ denote the cohomology class in $H^2(X; \mathbb{R})$ represented by $\omega$ through the de Rham isomorphism. Let $\overline{\text{Aut}}_X$ denote the subgroup of $\text{Aut}_X$ consisting of those biholomorphisms fixing the class $\overline{\mathcal{C}}$. According to a theorem of Fujiki [25, Theorem 4.8], $\overline{\text{Aut}}_X$ has finitely many connected components.

By Lemma 3.1 there exists a constant $C$ such that any finite subgroup acting effectively and by group automorphisms on $H^2(X; \mathbb{Z})/\text{Tor} H^2(X; \mathbb{Z}) \simeq \mathbb{Z}^{2s}(X)$ has at most $C$ elements. This implies that if $\Gamma$ is a finite group acting on $X$ then there is a subgroup
Γ′ ≤ Γ satisfying [Γ : Γ′] ≤ C and whose action on \(H^2(X;\mathbb{Z})/\text{Tor} \ H^2(X;\mathbb{Z})\) is trivial. Equivalently, the action of Γ′ on \(H^2(X;\mathbb{R})\) is trivial. In particular, if Γ acts effectively on X by biholomorphic transformations, so that we can identify Γ with a subgroup of \(\text{Aut}_\mathbb{C} X\), then \([Γ : \text{Aut}_\mathbb{C} X \cap Γ] \leq C\).

Now assume that for some natural number m the group \(\text{Aut} X\) contains subgroups isomorphic to \(Γ_{r,m}\) for arbitrary large values of r. Arguing as in the preceding subsection (applying the existence of the constant C and using Lemma 6.1) we may conclude that \(\text{Aut}_\mathbb{C} X\) contains subgroups isomorphic to \(Γ_{s,m}\) for arbitrary large values of s. This implies, by Theorem 1.16, that \(\text{Aut}_\mathbb{C} X\) contains a torus T of satisfying \(\dim T \geq m\).

Now, if an m-dimensional torus T acts effectively on an n-dimensional connected topological manifold X then \(m \leq n\), and if \(m = n\) then X is homeomorphic to a torus. This is probably well known, but since we did not find a reference with a proof, we sketch it here. For any nontrivial closed subgroup \(K \leq T\) the fixed point set \(X^K\) is closed and has empty interior (see [10, Chap III, Theorem 9.5]). The set of closed subgroups of T is countable and topological manifolds are Baire spaces, so \(\bigcap_{t \neq s} X \setminus X^K \neq \emptyset\). Hence, there exists a point \(x \in X\) whose stabilizer is trivial, so the map \(f : T \ni t \mapsto t \cdot x \rightarrow X\) is continuous and injective. By the theorem of invariance of domain it follows that \(\dim T \leq \dim X\) and that, if \(\dim T = \dim X\), then f is an open map. In that case, \(f(T)\) is closed (because T is compact) and open in X and since X is connected it follows that \(f(T) = X\). Since f is open and injective, it gives a homeomorphism \(T \cong f(T)\), so we conclude that \(T \cong X\).

Finally, a Kaehler manifold homeomorphic to a torus is biholomorphic to a complex torus (see e.g. [3, Theorem B] for a very nice exposition of a more general result), so the proof of Theorem 1.15 is now finished.

14. The manifold in Example 1.14 is rationally hypertoral but not hypertoral

Recall that X is the mapping torus of \(σ : Y \rightarrow Y\), the involution of a degree 2 covering \(π : Y \rightarrow T^2\) ramified at two points. So \(X = ([0,1] \times Y)/\sim\), where \(\sim\) identifies \((0,y)\) with \((1,σ(y))\) for every \(y \in Y\). The projection \([0,1] \times Y \rightarrow [0,1]\) induces a fibration \(ρ : X \rightarrow S^1\). The map \(\text{Id} \times π : [0,1] \times Y \rightarrow [0,1] \times T^2\) descends to a map \(X \rightarrow T^3\), where we view \(T^3\) as the mapping torus of the identity on \(T^2\). So X is rationally hypertoral.

We prove that X is not hypertoral using the following fact, whose proof is left to the reader: if \(Z\) is a closed connected n-manifold such that \(H^1(Z;\mathbb{Z}) \simeq \mathbb{Z}^n\), then \(Z\) is hypertoral if and only if for any primitive class \(α \in H^1(Z;\mathbb{Z})\) there exist classes \(α_2, \ldots, α_n\) such that \(α \cup α_2 \cup \cdots \cup α_n\) is a generator of \(H^n(Z;\mathbb{Z})\). (The class \(α\) is primitive if it is not of the form \(λβ\) for \(β \in H^1(Z;\mathbb{Z})\) and \(λ \in \mathbb{Z} \setminus \{-1,0,1\}\).)

Let \(p \in \Sigma\) be one of the two fixed points of \(σ\). Let \(θ\) be a generator of \(H^1(S^1;\mathbb{Z})\), and let \(τ : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow X\) be the map sending \([t] \in \mathbb{R}/\mathbb{Z}\) with \(t \in [0,1]\) to the class of \((t,p)\) in \(([0,1] \times Y)/\sim\). Then \(ρ ◦ τ : S^1 \rightarrow S^1\) is a homeomorphism, so \(ρ*θ \in H^1(X;\mathbb{Z})\) is primitive. The Poincaré dual of \(ρ*θ\) is equal to the fundamental class of any fiber \(F\) of
$ho$. Let $j : F \to X$ be the inclusion. Given $\alpha_2, \alpha_3 \in H^1(X; \mathbb{Z})$, the product $\nu^* \theta \cup \alpha_2 \cup \alpha_3$ is a generator of $H^3(X; \mathbb{Z})$ if and only if $j^* \alpha_2 \cup j^* \alpha_3$ is a generator of $H^2(F; \mathbb{Z})$.

Now, the image of $j^* : H^1(X; \mathbb{Z}) \to H^1(F; \mathbb{Z}) \simeq H^1(Y; \mathbb{Z})$ is contained in the group of $\sigma$-invariant classes $H^1(Y; \mathbb{Z})^\sigma$. We claim that $H^1(Y; \mathbb{Z})^\sigma$ is contained in the image of $\pi^* : H^1(T^2; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$. This follows from applying Lemma 2.2 to the action of $G = \{\text{Id}, \sigma\}$ on $Y$, taking into account that the action of $\sigma$ on $Y$ has fixed points (and hence the morphism $\eta : G \to T^2$ given by the lemma is necessarily trivial). Hence, for any classes $\alpha_2, \alpha_3 \in H^1(X; \mathbb{Z})$ there exist classes $\beta_2, \beta_3 \in H^1(T^2; \mathbb{Z})$ such that $j^* \alpha_2 \cup j^* \alpha_3 = \pi^* \beta_2 \cup \pi^* \beta_3 = \pi^*(\beta_2 \cup \beta_3)$. But $\pi^*(\beta_2 \cup \beta_3)$ will never be a generator of $H^2(Y; \mathbb{Z})$, because $\pi : Y \to T^2$ has degree 2. This concludes the proof that $X$ is not hypertoral.

15. Regular self coverings of the manifolds in Theorem 1.14

Let $d$ be an odd natural number. The manifolds constructed in [14] are products $T(h) \times H$, where $T(h)$ is the mapping torus of a self homeomorphism $h$ of a closed topological manifold $V$ and $H$ is a closed hyperbolic manifold. So it suffices to prove that $T(h)$ supports a regular self covering of degree $d$. The structure of mapping torus on $T(h)$ gives a map $T(h) \to S^1$, and the regular self covering we claim to exist is the pullback, via this map, of the covering $S^1 \to S^1$ sending $\theta$ to $d \theta$. This pullback can be identified with the mapping torus $T(h^d)$, so all we need to prove is that $T(h)$ and $T(h^d)$ are homeomorphic.

The manifold $V$ is $W \cup_T (T^n \times [0, 1]) \cup_T W'$, where $W$ and $W'$ are $(n+1)$-dimensional manifolds with boundaries $T$ and $T'$ respectively, where $n \geq 5$. Here both $T$ and $T'$ denote the torus $T^n$ with involutions $h_T : T \to T$ and $h_{T'} : T' \to T'$. The involution $h_T$ is a linear involution, whereas $h_{T'}$ is exotic, i.e., not conjugate to a linear action. Both $h_T$ and $h_{T'}$ have nonempty fixed point set (see [7] §2.1 for a concrete description of the involutions $h_T$ and $h_{T'}$ used in [14]). The maps $h_T$ and $h_{T'}$ are homotopic. In the definition of $V$ we glue $T \subset W$ with $T^n \times \{0\}$ and $T' \subset W'$ with $T^n \times \{1\}$.

The involutions $h_T$ and $h_{T'}$ extend to involutions $h_W : W \to W$ and $h_{W'} : W' \to W'$ respectively, and there is a self homeomorphism $h_C$ of $C := T^n \times [0, 1]$ whose restriction to $C_0 := T^n \times \{0\}$ resp. $C_1 := T^n \times \{1\}$ coincides with $h_T$ resp. $h_{T'}$. We now briefly justify the existence of $h_C$. It suffices to prove the existence of a homeomorphism $\phi : C \to C$ such that $\phi|_{C_0} = \text{Id}_{T^n}$ and $\phi|_{C_1} = \psi := h_T^{-1} \circ h_{T'}$, for then $h_C := (h_T \times \text{Id}_{[0,1]}) \circ \phi$ has the desired property. Now, $\psi$ is homotopic to Id$_{T^n}$ so the mapping tori $T(\psi)$ and $T(\text{Id}_{T^n})$ are homotopy equivalent. Since $T(\text{Id}_{T^n}) = T^{n+1}$, the topological rigidity of tori implies that $T(\psi)$ and $T(\text{Id}_{T^n})$ are homeomorphic. The existence of $\phi$ now follows from combining: [35], Theorem 1, the observation that invertible cobordism are $h$-cobordisms, the s-cobordism theorem, and the vanishing of the Whitehead group of $\pi_1(T^n)$.

Unlike $h_T$ and $h_{T'}$, $h_C$ is not an involution. However, we have the following:

**Proposition 15.1.** If we chose $h_C$ suitably, then $h_C^2$ and Id$_C$ are homotopic rel. $\partial C$. 

Proof. We identify $T^n$ with $\mathbb{R}^n/\mathbb{Z}^n$, so the universal covering space $C^g$ of $C$ can be identified with $\mathbb{R}^n \times [0,1]$. Let $C^g_i = \mathbb{R}^n \times \{i\}$ for $i = 0, 1$. Let $f, g : C \to C$ be continuous maps such that $f|_{\partial C} = g|_{\partial C}$. Choose lifts $f^g, g^g : C^g \to C^g$. Then $g^g|_{C^g_i} = f^g|_{C^g_i}$ is equal to some constant $\lambda_i \in \mathbb{Z}^n$, because $g|_{C_i} - f|_{C_i}$ is identically zero. Let 

$$\lambda(g, f) := \lambda_1 - \lambda_0.$$ 

The vector $\lambda(g, f) \in \mathbb{Z}^n$ is independent of the chosen lifts of $f, g$.

**Lemma 15.2.** $f$ and $g$ are homotopic rel. $\partial C$ if and only if $\lambda(g, f) = 0$.

**Proof.** The "only if" part of the lemma is an easy exercise. For the "if" part, note that there is a linear map $\rho : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $f^g(p + \mu, s) = f^g(p, s) + (\rho(\mu), 0)$ and $g^g(p + \mu, s) = g^g(p, s) + (\rho(\mu), 0)$ for every $p \in \mathbb{R}^n, \mu \in \mathbb{Z}^n, s \in [0,1]$. Actually $\rho$ can be identified with the morphism $H_1(T^n) \to H_1(T^n)$ induced by $f$ or $g$. Then the map $C^g \times [0,1] \to C^g$ sending $((p, s), t)$ to $(1-t)f^g(p, s) + tg^g(p, s)$ is a homotopy between $f^g$ and $g^g$ that descends to a homotopy rel $\partial C$ between $f$ and $g$. \hfill $\Box$

Now suppose that $\zeta : C \to C$ is a homeomorphism satisfying $\zeta|_{C_0} = h_T$ and $\zeta|_{C_1} = h_{T'}$. Since both $h_T$ and $h_{T'}$ have fixed points, there exist $x \in C_0$ and $y \in C_1$ such that $\zeta(x) = x$ and $\zeta(y) = y$. Choose lifts $x^g, y^g \in C^g$ of $x, y$ respectively. There is a unique lift $\zeta^g : C^g \to C^g$ satisfying $\zeta^g(x^g) = x^g$. As before, there is a morphism of groups $\rho : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\zeta^g(p + \mu, s) = \zeta^g(p, s) + (\rho(\mu), 0)$ for all $p, \mu, s$. Let $\rho : C^g = \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ denote the projection and let $\lambda := o(\zeta^g(y^g) - y^g) \in \mathbb{Z}^n$, so that $\zeta^g(y^g) = y^g + (\lambda, 0)$. Then:

$$\lambda(\zeta^g, \text{Id}) = o(\zeta^g \zeta^g(y^g) - y^g) = o(\zeta^g \zeta^g(y^g) - \zeta^g(y^g) + \zeta^g(y^g) - y^g)$$

$$= o(\zeta^g(y^g + (\lambda, 0)) - \zeta^g(y^g) + (\lambda, 0)) = \rho(\lambda) + \lambda.$$

By the previous lemma, in order for $\zeta^g$ to be homotopic to $\text{Id}_C$ rel. $\partial C$ we need $\rho(\lambda) + \lambda$ to vanish. This need not be the case, but if we define $\xi^g : C^g \to C^g$ as $\xi^g(p, s) = \zeta^g(p) - s\lambda$ then we have $\xi^g(p + \mu, s) = \xi^g(p, s) + (\rho(\mu), 0)$ for all $p, \mu, s$, so $\xi^g$ descends to a homeomorphism $\xi : C \to C$ satisfying $\xi|_{\partial C} = \zeta|_{\partial C}$. Furthermore, $\xi^g(y^g) = y^g$, so $\lambda(\xi, \text{Id}) = 0$. Consequently, $h_C := \xi$ has the desired property. \hfill $\Box$

Assume from now on that $h_C$ has been chosen in such a way that $h_C^2$ is homotopic to $\text{Id}_C$ rel. $\partial C$, which implies that $h_C^d$ and $h_C$ are homotopic rel. $\partial C$. The involution $h : V \to V$ is defined by the condition that its restriction to the subspaces $W, T^n \times [0,1], W'$ is given by $h_W, h_C, h_{W'}$ respectively.

Since $h_W$ and $h_{W'}$ are involutions, the restrictions of the maps $h$ and $h^d$ to $W$ and $W'$ are equal. Hence, to prove that $T(h)$ and $T(h^d)$ are homeomorphic it suffices to prove the existence of a homeomorphism of mapping tori $\phi : T(h_C) \to T(h_C^d)$ whose restriction to $\partial T(h_C)$ is the natural homeomorphism $\partial T(h_C) \to \partial T(h_C^d)$ resulting from the equalities $h_T = h_T^d$ and $h_{T'} = h_{T'}^d$. We are going to prove the existence of $\phi$ using the topological rigidity of non-positively curved Riemannian manifolds proved in [22] [23].

By definition $T(h_C)$ is the quotient of $C \times [0,1]$ by the relation that identifies $(x, 1)$ with $(h_C(x), 0)$ for every $x \in C$. For $i = 0, 1$, let $T_i(h_C) \subset T(h_C)$ be the image of
prove the claim the existence of $\phi$ of non-positively curved $T$ and the existence of a homotopy $h$. Now, $T$ is a linear involution, $T$ supports a non-positively curved Riemannian metric. Now, $T$ is homotopy equivalent to $T$, because $h$ and $h$ are homotopic. Hence, by topological rigidity [22, Theorem 14.1], there is a homeomorphism $\psi : T(h_T) \to T(h_T)$. Choosing $\psi$ appropriately, we may and do assume that the compositions of maps $T(h_T) \xrightarrow{\psi^{-1}} T(h_T') = T_1(h_C) \to T(h_C)$ and $T(h_T) = T_0(h_C) \to T(h_C)$ are homotopic.

We claim the existence of a homeomorphism

$$\xi : T(h_C) \to T(h_T \times \text{Id}_{0,1}) = T(h_T) \times [0, 1]$$

whose restriction to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\text{Id}_{T(h_T)}$ resp. $\psi$. Once we prove the claim the existence of $\phi$ will follow immediately, since by [22, Theorem 14.1] topological rigidity applies to $T(h_T) \times [0, 1]$ (again because $T(h_T)$ supports a metric of non-positive curvature) and the existence of a homotopy $h^d_C \sim h_C$ rel. $\partial C$ gives a homotopy equivalence $T(h_C) \to T(h_C)$ whose restriction to $\partial T(h_C)$ is a homeomorphism.

To prove the existence of the homeomorphism $\xi : T(h_C) \to T(h_T) \times [0, 1]$ we rely once again on the topological rigidity of $T(h_T) \times [0, 1]$, so we only need to prove the existence of a continuous map $\chi : T(h_C) \to T(h_T) \times [0, 1]$ whose restriction to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\text{Id}_{T(h_T)}$ resp. $\psi$ (these properties imply that $\chi$ is a homotopy equivalence).

If $Y \subseteq X$ is an inclusion of topological spaces and $f, g : X \to X$ are maps preserving $Y$, satisfying $f|_Y = g|_Y$, and $f, g$ are homotopic rel. $Y$, then there is a continuous map $\epsilon : T(f) \to T(g)$ whose restriction to $T(f|_Y)$ is the natural identification between $T(f|_Y)$ and $T(g|_Y)$. Indeed, suppose that $H : X \times I \to X$ satisfies $H(x, 0) = g(x)$, $H(x, 1) = f(x)$ and $H(y, t) = f(y) = g(y)$ for every $x \in X$, $y \in Y$, $t \in [0, 1]$. Then $\epsilon$ is defined by the map $\bar{\epsilon} : X \times [0, 1] \to X \times [0, 1]$ given by

$$\bar{\epsilon}(x, t) = \begin{cases} (x, 2t) & \text{if } t \in [0, 1/2], \\ (H(x, 2t - 1), 0) & \text{if } t \in [1/2, 1]. \end{cases}$$

Using the previous principle, and the facts that $h_C$ and $h_T \times \text{Id}_{0,1}$ are homotopic rel. $C_0$ and $h_C$ and $h_T \times \text{Id}_{0,1}$ are homotopic rel. $C_1$ (which can be proved by lifting the maps to $C^d$ as in the proof of Proposition [15, 1] and interpolating linearly), we deduce the existence of maps $\chi_0 : T(h_C) \to T(h_T) \times [0, 1]$ and $\chi_1 : T(h_C) \to T(h_T) \times [0, 1]$ such that $\chi_i$ restricted to $T_i(h_C)$ is the identity for $i = 0, 1$. Furthermore, $\chi_0$ and $\psi \times \text{Id}_{0,1} \circ \chi_1$ are homotopic.

The universal cover of $T(h_C)$ can be identified with $\mathbb{R}^n \times [0, 1] \times \mathbb{R}$, and that of $T(h_T) \times [0, 1]$ with $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$. Fix a lift $h_T^2 : \mathbb{R}^n \to \mathbb{R}^n$ of $h_T : T^n \to T^n$. Crucially, $h_T^2$ is an affine isomorphism. The group $\Gamma = \pi_1(T(h_T) \times [0, 1]) \simeq \pi_1(T(h_T)) \simeq \mathbb{Z}^n \rtimes \mathbb{Z}$ acts on $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$ preserving the affine structure induced by the inclusion $\mathbb{R}^n \times \mathbb{R} \times [0, 1] \subseteq \mathbb{R}^{n+2}$: the factor $\mathbb{Z}^n$ adds by addition on the first factor of $\mathbb{R}^n \times \mathbb{R} \times [0, 1]$, and the action of the second factor is generated by the transformation $(z, t, s) \mapsto (h_T^2(z), t - 1, s)$. 


Choose lifts of $\chi_0$ and $(\psi \times \text{Id}_{[0,1]}) \circ \chi_1$ to the universal coverings, and call them $\theta_0$ and $\theta_1$ respectively, so that $\theta_i : \mathbb{R}^n \times [0,1] \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times [0,1]$. Since $\theta_0$ and $\theta_1$ are homotopic there exists a morphism of groups $\rho : \pi_1(T(h_C)) \to \pi_1(T(h_T) \times [0,1])$ such that both $\theta_0$ and $\theta_1$ are $\rho$-equivariant, meaning that $\theta_i(\gamma \cdot w) = \rho(\gamma) \cdot \theta_i(w)$ for every $\gamma \in \pi_1(T(h_C))$ and $w \in \mathbb{R}^n \times [0,1] \times \mathbb{R}$. Define $\theta : \mathbb{R}^n \times [0,1] \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times [0,1]$ as $\theta(p,s,t) = (1-s)\theta_0(p,s,t) + s\theta_1(p,s,t)$. Then $\theta$ satisfies the same $\rho$-equivariance property as $\theta_i$, because the action of $\pi_1(T(h_T) \times [0,1])$ on $\mathbb{R}^n \times \mathbb{R} \times [0,1]$ is affine. This implies that $\theta$ descends to a map $\chi : T(h_C) \to T(h_T) \times [0,1]$, and by construction the restriction of $\chi$ to $T_0(h_C)$ resp. $T_1(h_C)$ coincides with $\text{Id}_{T(h_T)}$ resp. $\psi$. This finishes the proof of the theorem.

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