Approximating Nash Equilibrium Via Multilinear Minimax

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Abstract

Nash equilibrium (NE) can be stated as a formal theorem on a multilinear form, free of game theory terminology. On the other hand, inspired by this formalism, we state and prove a multilinear minimax theorem, a generalization of von Neumann’s bilinear minimax theorem. As in the bilinear case, the proof is based on relating the underlying optimizations to a primal-dual pair of linear programming problems, albeit more complicated LPs. The theorem together with its proof is of independent interest. Next, we use the theorem to associate to a multilinear form in NE a multilinear minimax relaxation (MMR), where the primal-dual pair of solutions induce an approximate equilibrium point that provides a nontrivial upper bound on a convex combination of expected payoffs in any NE solution. In fact we show any positive probability vector associated to the players induces a corresponding diagonally-scaled MMR approximate equilibrium with its associated upper bound. By virtue of the proof of the multilinear minimax theorem, MMR solution can be computed in polynomial-time. On the other hand, it is known that even in bimatrix games NE is PPAD-complete, a complexity class in NP not known to be in P. The quality of MMR solution and the efficiency of solving the underlying LPs are the subject of further investigation. However, as shown in a separate article, for a large set of test problems in bimatrix games, not only the MMR payoffs for both players are better than any NE payoffs, so is the computing time of MMR in contrast with that of Lemke-Howsen algorithm. In large size problems the latter algorithm even fails to produce a Nash equilibrium. In summary, solving MMR provides a worthy approximation even if Nash equilibrium is shown to be computable in polynomial-time.

Keywords: Nash Equilibrium, von Neumann Minimax Theorem, Linear Programming, Matrix Scaling, NP, PPAD-complete.

1 Nash Equilibrium

Nash equilibrium, [8], is a fundamental result in game theory. The complexity of its computing has been investigated extensively, e.g. in [11, 14, 5]. In particular, it has been shown that even in bimatrix games Nash equilibrium is PPAD-complete, a complexity class in NP not known to be in P. In the bimatrix games while the Lemke-Howson algorithm [7] can compute a Nash equilibrium, the worst-case complexity of the algorithm is exponential [10]. It is thus worthwhile to consider polynomial-time approximations to Nash equilibrium.

In this note we first formally state Nash equilibrium as a theorem on a multidimensional matrix, independent of game theory terminology (Section 1). To do so we introduce the necessary notations and for the sake of completeness, using the notations, provide a proof of Nash equilibrium in an appendix (Appendix 1).

On the other hand, inspired by this formalism, we state and prove a multilinear minimax theorem, a generalization of von Neumann’s bilinear minimax theorem. Analogous to the bilinear case, we present a proof by relating the underlying optimizations to a primal-dual pair of linear programming problems, albeit more complicated LPs (Section 2). The theorem together with its proof is of independent interest.

Next, we use the theorem to associate to a given multilinear form in NE a multilinear minimax relaxation (MMR) whose solution induces an approximate equilibrium point that provides a nontrivial upper bound on a convex combination of expected payoffs in any solution of NE (Section 3).
In fact we show any positive probability vector associated to the players induces a corresponding diagonally-scaled MMR approximate equilibrium with its associated upper bound (Section 4). This implies we can provide many alternative approximate equilibria and corresponding upper bounds.

By virtue of the proof of the multilinear minimax, MMR solution can be computed in polynomial-time. The quality of MMR solution and the efficiency of solving the underlying LPs are the subject of further investigation. However, as we show in a separate article [6], for a large set of test problems in bimatrix games, not only the MMR payoffs for both players are better than any NE payoffs, so is the computing time of MMR in contrast with that of Lemke-Howson algorithm. In large size problems the latter algorithm even fails to produce a solution.

In summary, the proposed multilinear minimax relaxation to Nash equilibrium provides a polynomial-time computable approximate equilibrium with a nontrivial upper bound to a weighted expected payoffs in any Nash equilibrium in a game with any number of players. Indeed the very positive nature of our extensive computational results for the bimatrix games in [6] gives further incentive to investigate the efficiency and quality of MMR solutions in the case of three or more players in future work.

We now give some notations and definitions to be used throughout the article.

**Notation 1.** Given \( t \in \mathbb{N} \) (number of players), \( t \geq 2 \), let \( T = \{1, \ldots, t\} \) (set of players). Given \( n_i \in \mathbb{N} \), \( i \in T \), let \( N_i = \{1, \ldots, n_i\} \). Let \( N \) be the Cartesian product \( N_1 \times \cdots \times N_t \) and \( n = n_1 \times \cdots \times n_t \). A \( t \)-dimensional \( n \) matrix is a multidimensional array of real elements, written as \( A = (a_{ij}) \), where the index vector \( I = (i_1, \ldots, i_t) \) ranges in \( N = N_1 \times \cdots \times N_t \). For each \( i \in T \), let \( x^i = (x^i_1, \ldots, x^i_{n_i})^T \in \mathbb{R}^{n_i} \) be a vector of real variables. Set \( x = (x^1, \ldots, x^t) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_t} \). Given \( I = (i_1, \ldots, i_t) \in N \), we write \( x_I \) for the product \( x^1_{i_1} \times \cdots \times x^t_{i_t} \). Consider the multilinear form corresponding to \( A = (a_{ij}) \):

\[
A[x] = A[x^1, \ldots, x^t] = \sum_{I \in N} a_{I} x_I.
\]

For each \( i \in T \), let \( S_n = \{x \in \mathbb{R}^{n_i} : \sum_{j=1}^{n_i} x_j = 1, \; x \geq 0\} \) (the unit simplex in \( \mathbb{R}^{n_i} \)). Let

\[
\Delta_n = S_{n_1} \times \cdots \times S_{n_t}.
\]

Given \( p = (p^1, \ldots, p^t) \in \Delta_n \), \( A[p] \) is called the value of \( A \) at \( p \). For each \( x^i \in \mathbb{R}^{n_i}, i \in T \), define the linear form

\[
A[p|x^i] \equiv A[p^1, \ldots, p^{i-1}, x^i, p^{i+1}, \ldots, p^t].
\]

**Theorem 1.** (Nash Equilibrium) Let \( A_0 = (a_{ij}) \) be a \((t+1)\)-dimensional \( t \times n \) matrix, where \( n = n_1 \times \cdots \times n_t \), \( i \in T \) and \( I \in N \). For each \( i \in T \), consider the \( t \)-dimensional \( n \) submatrix of \( A_0 \), \( A_i = (a_{ij}) \). Then, there exists \( p_* = (p^1_*, \ldots, p^t_*) \in \Delta_n \) such that for each \( i \in N \),

\[
\max_{x^i \in S_{n_i}} A_i[p_*, x^i] = A_i[p_*]. \quad (1)
\]

For the sake of completeness in Appendix 1 we give the standard proof of the theorem but using the above notation.

### 1.1 Game Theory Terminology

Using the same notation as above, \( t \) is the number of players, \( T \) is the set of players. For each player \( i \in T \), \( N_i \) is its set of pure actions or pure strategies. Each \( I = (i_1, \ldots, i_t) \in N = N_1 \times \cdots \times N_t \) is a pure strategy profile. For each \( I \in N \), the matrix entry \( a_{i_1i_2} \) is payoff or utility for player \( i \) when pure strategy profile \( I \) is selected. Each player \( i \) may choose an action according to a mixed strategy with probability vector \( p^i \in S_{n_i} \). When a profile is selected with probability vector \( p = (p^1, \ldots, p^t) \in \Delta_n \), the expected utility or expected payoff of player \( i \) is \( A_i[p] \). Nash equilibrium states that there exists an equilibrium point \( p_* = (p^1_*, \ldots, p^t_*) \in \Delta_n \) such that if player \( k \) knows that each player \( i \neq k \) will use probability vectors \( p^k_* \), he cannot improve his expected payoff by selecting any other probability vector in \( S_{n_k} \) than \( p^k_* \). There is thus equilibrium.
The question of complexity of computing or approximating a Nash equilibrium point \( p^* \in \Delta_n \) has been studied extensively, [1], [4], [5]. It is not known if it can be computed or even approximated efficiently. One may ask: Can we compute upper bounds on \( A_i[p^*] \)? There are of course trivial upper bounds such as the maximum entry of the multidimensional matrix \( A_0 \), however we may ask: Can we compute nontrivial upper bounds on \( A_i[p^*] \)? Motivated by this question, in the next section (Section 2) we state and prove a multilinear form of von Neumann’s minimax theorem which is an interesting problem of independent interest. Then in Section 3 we show how an optimal solution of the minimax problem gives a bound on a convex combination of \( A_i[p^*] \)'s, their average, as well as an approximation to \( p^* \). In Section 4, we consider diagonal scaling to derive alternate bounds on a convex combination of \( A_i[p^*] \), as well as different approximations to \( p^* \). In Appendix 1 we provide a proof of Nash equilibrium, using the notations.

2 A Multilinear von Neumann Minimax Theorem

In this section we state and prove a multilinear minimax theorem. As defined before, given an integer \( t \geq 2 \), let \( A_0 = (a_{i,I}) \), \( i \in T = \{1, \ldots, t\} \), \( I = (i_1, \ldots, i_t) \in N = N_1 \times \cdots \times N_t \) be a \((t + 1)\)-dimensional matrix. Given \( x \in S_t \) and \( p \in \Delta_n \), consider the multilinear from

\[
A_0[x, p] = \sum_{i \in T} \sum_{I \in N} x_i a_{i,I} p_I,
\]

where \( p_I = p_{i_1}^1 \times \cdots \times p_{i_t}^t \). Clearly,

\[
A_0[x, p] = x_1 A_1[p] + \cdots + x_t A_t[p].
\]

**Theorem 2.** (Multilinear Minimax Theorem)

\[
\min_{x \in S_t} \max_{p \in \Delta_n} A_0[x, p] = \max_{p \in \Delta_n} \min_{x \in S_t} A_0[x, p].
\]

**Proof.** The proof of von Neumann minimax for bilinear can be established via linear programming duality theory, see e.g. [2], [3]. Analogously, we will represent the left-hand-side and right-hand-side of (4) as a pair of primal and dual linear programs. However, the proof is somewhat trickier than the bilinear case.

For each \( x \in S_t \), the maximum of \( A_0[x, p] \) over \( p \in \Delta_n \) is the maximum of \( \sum_{i \in T} a_{i,I} x_i \), as \( I \) ranges in \( N \). Thus the value of the left-hand-side in (4) is equivalent to the optimal value of the following primal LP:

\[
\min \delta
\]

\[
(LP) \quad \sum_{i \in T} a_{i,I} x_i \leq \delta, \quad \forall I \in N,
\]

\[
\sum_{i \in T} x_i = 1,
\]

\[
x_i \geq 0, \quad \forall i \in N.
\]

Next, we consider the right-hand-side in (4). For each \( p \in \Delta_n \), the minimum of \( A_0[x, p] \) over \( x \in S_t \) is the minimum of \( \sum_{i \in T} a_{i,I} x_i, \) as \( I \) ranges in \( T \). We can find the maximum of these over \( p \in \Delta_n \) by putting a lower bound of \( \lambda \) on each such term and then maximizing it over all \( I \) in \( N \). For each \( I \in N \) we introduce a variable \( q_I \) corresponding to the term \( p_I \). We claim the right-hand-side in (4) is equivalent to

\[
\max \lambda
\]

\[
(DLP) \quad \sum_{I \in N} a_{i,I} q_I \geq \lambda, \quad \forall i \in T,
\]

\[
\sum_{I \in N} q_I = 1,
\]

\[
q_I \geq 0, \quad \forall I \in N.
\]
Before proving the claim, first we prove (DLP) is the dual of (LP). Let $A$ be the $n \times t$ constraint matrix corresponding to the inequalities in (LP) (excluding $x \geq 0$). Let $e_i \in \mathbb{R}^t$ and $e_n \in \mathbb{R}^n$ be the vector of ones. Then (LP) is equivalent to

$$\min \{ \delta : Ax \leq \delta e_n, \ e_n^T x = 1, \ x \geq 0 \}. \quad (5)$$

We claim the dual of (5) is

$$\max \{ \lambda : A^T w \geq \lambda e_t, \ e_n^T w = 1, \ w \geq 0 \}. \quad (6)$$

To prove the claim we will use the well-known primal-dual relation for an LP in the standard form with a finite optimal objective (e.g. see [2]): Given compatible input data, matrix $A'$, and vectors $b', \ c'$ we have,

$$\min \{ c^T x : A'x = b', \ x \geq 0 \} = \max \{ b^T w : A^T w \leq c' \}. \quad (7)$$

Introducing slack variables $s$ in (5), the optimization problem is equivalent to:

$$\min \{ \delta : Ax - \delta e_n + s = 0, \ e_n^T x = 1, \ x \geq 0, \ s \geq 0 \}. \quad (8)$$

To turn (8) into a standard LP, without loss of generality we assume the optimal value of $\delta$ is nonnegative (when the optimal value is nonpositive, simply replacing $\delta$ with $-\delta$, similar arguments apply). The dual of (8), with the added assumption that $\delta \geq 0$, can then be written from (7) and shown to be

$$\max \{ \lambda : A^T w + \lambda e_t \leq 0, \ -e_n^T w \leq 1, \ w \leq 0 \}. \quad (9)$$

Replacing $w$ with $-w$, (9) reduces to

$$\max \{ \lambda : A^T w \geq \lambda e_t, \ e_n^T w \leq 1, \ w \geq 0 \}. \quad (10)$$

We argue that if $(w_*, \lambda_*)$ is an optimal solution of (10), $e_n^T w = 1$. Otherwise, we can scale $w_*$ by a number larger than one, thereby obtaining a better solution for (10). Thus the constraint $e_n^T w \leq 1$ in (10) can be replaced with $e_n^T w = 1$, proving that the dual of (8) is (10).

Now by identifying $w$ in (10) with the vector of variables corresponding to $q_I, I \in N$, it follows that (10) is equivalent to (DLP). Hence (LP) and (DLP) are primal-dual pair.

We now prove the claim that the right-hand-side in (11) is equivalent to (DLP). Let us consider the set of $q_I, I \in N$, satisfying the constraints $\sum_{I \in N} q_I = 1$ as an ordered vector, say $Q_N$ in $S_n$, the unit simplex in dimension $n$. Next, for each $Q_N \in S_n$, we construct a unique $p \in \Delta_n$, to be referred as its derived point: For each $i \in T$ and each $j \in N_i$, let $N_j^i$ be the set of all $I \in N$ whose $i$-th coordinate is $j$. Then set

$$p_j^i = \sum_{I \in N_j^i} q_I. \quad (11)$$

It is easy to check that the vector $p^i = (p_1^i, \ldots, p_n^i)^T$ lies in $S_n$. Thus setting $p = (p_1^i, \ldots, p_t^i)$, it follows that $p \in \Delta_n$. Hence the proof of theorem is complete.

**Corollary 1.** Let $x^*$ be an optimal solution of (LP). Let $Q_N^* = \{ q_I^*: I \in N \} \subseteq S_n$ be an optimal solution of (DLP) and let $p^* \in \Delta_n$ be the corresponding derived solution (see (11)). Then

$$\min \max_{x \in S_t} A_0(x, p) = \max_{p \in \Delta_n} \min_{x \in S_t} A_0(x, p) = A_0[x^*, p^*]. \quad (12)$$

Furthermore, for each $x \in S_t$ and each $p \in \Delta_n$ we have

$$A_0[x^*, p] \leq A_0[x^*, p^*] \leq A_0[x, p^*]. \quad (13)$$

**Proof.** By the fact that (LP) and (DLP) are primal and dual pair, it follows that if $\delta^*$ and $\lambda^*$ are the respective optimal objective values, then $\delta^* = \lambda^*$. 

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Given $Q_N \in S_n$, let $p \in \Delta_n$ be its derived point. Next we substitute $x^*$ for $x$ and $\delta^*$ for $\delta$ in (LP). Then for each $I \in N$ multiplying the corresponding inequality constraint by $q_I$, and then summing over these inequalities while replacing $Q_N$ by its derived point we get,

$$A_0[x^*, p] \leq \delta^* \sum_{I \in N} q_I = \delta^*. \tag{14}$$

In particular, substituting $Q^*_N$ for $Q_N$ and its corresponding derived point $p^*$ in (14) we get

$$A_0[x^*, p^*] \leq \delta^*. \tag{15}$$

Next substituting $q^*_I$ for $q_I$, $I \in N$ in the corresponding inequality in (DLP), then for $i \in T$, multiplying the $i$-th inequality constraint by $x_i$, replacing $Q^*_N$ by its derived point $p^*$ and finally summing these up we get

$$A_0[x, p^*] \geq \lambda^* \sum_{i \in T} x_i = \lambda^*. \tag{16}$$

In particular, substituting $x^*$ for $x$ in (16) we get

$$A_0[x^*, p^*] \geq \lambda^*. \tag{17}$$

Now from (15) and (17) and that $\delta^* = \lambda^*$ we get (12). Next from (14), (16) and (12) we get (13). □

### 3 Multilinear Minimax Relaxation to Nash Equilibrium

We now state a connection between Nash equilibrium and multilinear minimax solution.

**Theorem 3.** (Multidimensional Minimax Relaxation (MMR)) Given the $(t+1)$-dimensional matrix $A_0$ defined previously, let $p_*$ be a corresponding optimal Nash equilibrium point and $(x^*, p^*)$ an optimal pair of multilinear minimax. We have

$$A_0[x^*, p_*] \leq A_0[x^*, p^*] = \min_{i \in T} A_i[p^*]. \tag{18}$$

**Proof.** The inequality in (18) follows from (13) in Corollary 1, selecting $p = p_*$. To show the equality in (18), from (13) and selecting $x = e_i^t$, the $t$-vector with one in its $i$-th position and zero otherwise, it follows that $A_0[x^*, p^*] \leq \min \{ A_i[p^*] : i = 1, \ldots, t \}$. On the other hand,

$$A_0[x^*, p^*] = \sum_{i=1}^t x_i^* A_i[p^*] \geq \sum_{i=1}^t x_i^* \min_{i \in T} A_i[p^*] = \min_{i \in T} A_i[p^*] \sum_{i=1}^t x_i^* = \min_{i \in T} A_i[p^*].$$

□

Theorem 3 implies that by computing the multilinear minimax pair $(x^*, p^*)$ we can derive an upper bound on a convex combination of expected payoffs of players at any Nash equilibrium point. We can thus consider $p^*$ as an approximation to $p_*$. 

We state two corollaries.

**Corollary 2.** There exists $i = 1, \ldots, t$ such that $A_i[p_*] \leq A_i[p^*]$. 

**Proof.** Since $A_0[x^*, p^*] = \sum_{i=1}^t x_i^* A_i[p_*]$ and $x^* \in S_t$, if inequality does not hold (18) will be violated. □

By Corollary 2 when solving MMR we obtain an approximate equilibrium point where at least one of player has expected payoff better than the payoff in any Nash equilibrium. Indeed in many computational tests in bimatrix games (see [6]) in MMR solution both players end up with better expected payoffs.
Corollary 3. Consider \( p_\star \) and \((x^\star, p^\star)\) be as in Theorem 3. Let \( \sigma_t = \frac{1}{t} \sum_{i=1}^t A_i[p_\star]. \) Let \( x_{\min}^\star = \min\{x_i^\star : i = 1, \ldots, t\} \). If \( x_{\min}^\star > 0 \), and \( A_i[p_\star] \geq 0 \) for all \( i = 1, \ldots, t \), then

\[
\sigma_t \leq \frac{1}{t x_{\min}^\star} A_0[x^\star, p^\star].
\]  \hspace{1cm} (19)

Proof. Clearly, \( A_0[x^\star, p_\star] = \sum_{i=1}^t x_i^\star A_i[p_\star] \geq t x_{\min}^\star \sigma_t. \)

According to Corollary 3 if \( x^\star \) is positive then MMR solution provides an upper bound on average expected payoffs of Nash equilibrium. However, the quality of the bound may be poor. The quality of any approximation to \( p_\star \) can be quantified as follows.

Definition 1. We say \( p \in \Delta_n \) is an \( \alpha \)-approximate Nash equilibrium, if \( \alpha = \max\{\alpha_i : i = 1, \ldots, t\} \), where \( \alpha_i \) is defined as \( \max_{x^i \in S_{N_i}} A_i[p|x^i] = \alpha_i A_i[p] \).

Clearly, the closer \( \alpha \) is to one, the better \( p \) approximates \( p_\star \), or more precisely, \( A_i[p] \) approximates \( A_i[p_\star] \). An important question open to further investigation is the quality of \( p^\star \). Computationally in bamatrix games, \( \Box \), \( p^\star \) provides an excellent approximation that is also efficiently computable.

In the next section we consider alternate bounds by solving alternate MMR problems. Hence by solving several MMR problems we increase the chance of obtaining better approximations.

4 Alternate Solutions Via Diagonally Scaled Minimax Relaxations

In what follows we consider alternate minmax problems derived via diagonal scaling of the matrix \( A_0 = (a_{i,t}) \). Let

\[ S^0_t = \{x \in S_t : x > 0\}. \]

Given each \( d = (d_1, \ldots, d_t) \in S^0_t \), set

\[ A_0(d) = (d_i a_{i,t}). \]

This corresponds to scaling each submatrix \( A_i \) of \( A_0 \) by \( d_i \). Thus \( A_i(d) = d_i A_i \). We can view \( d \) as a vector of positive probabilities assigned to the set of \( t \) players. Next consider the optimal diagonally-scaled minmax pair of solutions for the matrix \( A_0(d) \), denote the pair by \((x^\star(d), p^\star(d))\). In particular, \((x^\star, p^\star)\) in the previous section corresponds to \( d = (1/t, \ldots, 1/t)^t \in \mathbb{R}^t \). As before, denoting \( p_\star \) as a Nash equilibrium point for \( A_0 \), note that it also remains an equilibrium point for \( A_0(d) \). However, from Theorem 3 we have

\[ A_0(d)[x^\star(d), p_\star] \leq A_0(d)[x^\star(d), p^\star(d)]. \]

Equivalently,

\[ \sum_{i \in T} d_i x^\star_i(d) A_i[p_\star] \leq \sum_{i \in T} d_i x^\star_i(d) A_i[p^\star(d)]. \]

Dividing both sides by \( \sigma = \sum_{i \in T} d_i x^\star_i(d) \) and letting \( d'_i = d_i x^\star_i(d)/\sigma \), from \( d \in S^0_t \) we get a new point \( d' \in S_t \) satisfying an alternate bound

\[ \sum_{i \in T} d'_i A_i[p_\star] \leq \sum_{i \in T} d'_i A_i[p^\star(d)]. \]

If \( d' > 0 \), we can replace \( d \) with \( d' \) and repeat the process. Replacing \( d \) with \( d' \) corresponds to a projective transformation. Some questions arise regarding the above bound for arbitrary \( d \in S^0_t \). What if we randomly select such \( d \) and compute the above? If \( d' \) is the vector with 1 in its \( i \)-th coordinate and zero otherwise then we will have a bound on \( A_i[p_\star] \). In other words we may be able to produce bounds on individual player payoffs. Another question is what will be the infimum of the above over \( d \in S^0_t \)? From the practical point of view, for each \( d \in S^0_t \) the computation of \( p^\star(d) \) gives rise to a \( t(d) \)-approximate solution (see Definition \[1\]).


5 Appendix 1. Proof of Nash Equilibrium

Proof. For each \( i \in T, j \in N_i \), let \( e^n_j \in \mathbb{R}^{n_i} \) be the vector with 1 in its \( j \)-th coordinate and zero otherwise. From linearity, for each \( i \in T \) we have

\[
A_i[p \cdot x^i] = \sum_{j=1}^{n_i} x^i_j A_i[p \cdot e^n_j].
\]

Thus, to prove (1) (Theorem 1) it is enough to prove there exists \( p_* \in \Delta_n \) such that

\[
A_i[p_* \cdot e^n_i] \leq A_i[p_*], \quad \forall i \in T, \quad \forall j \in N_i.
\] (20)

Employing (20), an iterative scheme will be defined that given \( p = (p^1, \ldots, p^t) \in \Delta_n \) that does not satisfy (20), attempts to find an improved point via the mapping \( f(p) = (\overrightarrow{p^1}, \ldots, \overrightarrow{p^t}) \in \Delta_n \), defined as follows: For each \( i \in T \) and \( j \in N_i \), let

\[
G^j_i(p) = \max \left\{ 0, A_i[p \cdot e^n_j] - A_i[p] \right\},
\] (21)

\[
g^j_i(p) = p^j + G^j_i(p), \quad p^j = \frac{g^j_i(p)}{\sum_{k \in N, g^j_k(p)}},
\] (22)

From (21), (22) it follows that \( f(p) \in \Delta_n \). Since \( f \) is continuous, by Brouwer fixed point theorem, there exists \( p = (p^1, \ldots, p^t) \in \Delta_n \) satisfying \( f(p) = p \). We show \( p \) satisfies

\[
G^j_i(p) = 0, \quad \forall i \in T, \quad \forall j \in N_i.
\] (23)

Clearly (23) implies (20). From (21) and (22), and that \( f(p) = p \), we have

\[
p^j = \frac{p^j + G^j_i(p)}{\sum_{k \in N_i} (p^j_k + G^j_k(p))} = \frac{p^j + G^j_i(p)}{1 + \sum_{k \in N_i} G^j_k(p)},
\] (24)

where the second equality uses that \( \sum_{k \in N, p^j_k} = 1 \). Let \( c_i = \sum_{k \in N_i} G^j_k(p) \). From (24) it follows that

\[
c_i p^j = G^j_i(p), \quad \forall j \in N_i.
\] (25)

Suppose \( p \) does not satisfy (1). Then there exists \( i \in T \) such that \( c_i > 0 \). From (21) and (25) we have

\[
c_i p^j = A_i[p \cdot e^n_i] - A_i[p], \quad \forall j \in N_i.
\] (26)

Multiplying both sides of (26) by \( p^j \), summing over \( j \in N_i \) and using linearity of \( A_i[p \cdot e^n_i] \), we get

\[
c_i \sum_{j \in N_i} (p^j)^2 = \sum_{j=1}^{n_i} p^j A_i[p \cdot e^n_i] - \sum_{j=1}^{n_i} p^j A_i[p] = A_i[p] - A_i[p] = 0.
\] (27)

But the left-hand-side of (27) is positive, a contradiction. Hence the proof of (20).

Final Remarks

Nash’s equilibrium theorem, despite its tremendous ingenuity and applications, is after all a formal theorem on multilinear forms. We have used this formal view to offer new insights on Nash equilibrium by first proving a multilinear minimax theorem as a generalization of von Neumann’s minimax and then associating a multilinear minimax relaxation (MMR) to a Nash equilibrium that is computable in polynomial-time. Approximation of Nash equilibrium is a worthy task, especially in view of the inherent difficulty in computing
NE, as revealed by tremendous research in game theory. Indeed, solving MMR provides a worthy approximation even if Nash equilibrium is shown to be computable in polynomial-time. This is because solving MMR could result in an approximate equilibrium with better expected payoffs than a Nash equilibrium for all players.

As shown here, by solving a primal-dual pair of linear programs, it is possible to obtain an approximate equilibrium point that provides a nontrivial upper bound on a weighted expected payoff in any Nash equilibrium. Furthermore, diagonal scaling of the multidimensional matrix gives rise to new upper bounds and approximate solutions. The fact that the multilinear minimax problem, considered as a relaxation of Nash equilibrium, is solvable in polynomial-time makes it tractable in theory. It is also an interesting problem in its own right. For instance, it is interesting to note that since (DLP) has \((t + 1)\) nontrivial constraints, in an optimal solution at most \((t + 1)\) of the \(q_i\)'s are positive (see (DLP) in Theorem \[5\]). Note that even solving the multilinear minimax as a relaxation of Nash equilibrium could be a difficult problem linear program when there are several players, each having, say \(O(m)\) strategies. This is because in such case the bound based on the complexity of polynomial-time LP algorithms is a high degree polynomial in \(m\). Nevertheless, having obtained the minimax pair of solutions \((x^*, p^*)\), we may consider \(p^*\) as a relaxation to \(p^\ast\) and measure its quality as an \(\alpha\)-approximation solution (see Definition \[1\]).

As shown in \[6\], for a large set of test problems in bimatrix games, not only the MMR payoffs for both players are better than any NE payoffs, so is the computing time of MMR in contrast with that of Lemke-Howson algorithm. In large size problems the latter algorithm even fails to produce a solution.

These results motivate further theoretical and practical investigations such as: (1) How efficiently can the minimax relaxations be solved? That is, is there a way to make use of their structure to give faster algorithms than the straightforward application of known LP algorithms? (2) Can we bound the quality of \(p^*\) as an \(\alpha\)-approximate solution to \(p^\ast\)? (3) Are there classes of multidimensional matrices for which \(p^*\) gives a good \(\alpha\)-approximate solution to \(p^\ast\)? (4) What is the quality of approximation under diagonal scalings? (5) Finally, can we derive lower bounds to Nash equilibrium expected payoffs? e.g. instead of solving a minimax problem, by solving a maximin problem over the corresponding domains.

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