Forbidden pairs for equality of edge-connectivity and minimum degree

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Abstract

Let $\mathcal{H}$ be a class of given graphs. A graph $G$ is said to be $\mathcal{H}$-free if $G$ contains no induced copies of $H$ for any $H \in \mathcal{H}$. In this article, we characterize all pairs $\{R, S\}$ of graphs such that every connected $\{R, S\}$-free graph has the same edge-connectivity and minimum degree.

Keywords: forbidden subgraph; edge-connectivity; minimum degree

1 Introduction

We use Bondy and Murty [1] for terminology and notations not defined here and consider finite simple graphs only.

Let $G = (V(G), E(G))$ be a connected graph. We use $n(G), e(G), \kappa(G), \kappa'(G)$ and $\delta(G)$ to denote the order, size, connectivity, edge-connectivity and minimum degree of $G$, respectively. Let $u$ be a vertex of $G$. We use $N_G(u)$ to denote the set of vertices which is adjacent with $u$ (also called the neighbors of $u$) in the graph $G$. Let $S$ be a subset of $V(G)$ (or $E(G)$). The induced subgraph of $G$ is denoted by $G[S]$. Furthermore, we use $G - S$ to denote the subgraph $G[V(G) \backslash S]$ (or $G[E(G) \backslash S]$), respectively. For $x, y \in V(G)$, the length of a shortest path joining $x$ and $y$ is called the distance between $x$ and $y$ and denoted by $d_G(x, y)$. The diameter of a graph $G$, denoted by $\text{dim}(G)$, is the greatest distance between two vertices of $G$. 
Let $H$ be a given graph. A graph $G$ is said to be $H$-free if $G$ contains no induced copies of $H$. If $G$ is $H$-free, then $H$ is called a forbidden subgraph of $G$. Note that if $H_1$ is an induced subgraph of $H_2$, then every $H_1$-free graph is also $H_2$-free. For a class of graphs $\mathcal{H}$, the graph $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph $H_2 \in \mathcal{H}_2$, there exists a graph $H_1 \in \mathcal{H}_1$ such that $H_1$ is an induced subgraph of $H_2$. If $\mathcal{H}_1 \preceq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

As usual, we use $K_n$ to denote the complete graph of order $n$, and $K_{m,n}$ to denote the complete bipartite graph with partition sets of size $m$ and $n$. So the $K_1$ is a vertex, $K_3$ is a triangle, $K_{1,r}$ is a star (the $K_{1,3}$ is also called a claw). The clique $C$ is a subgraph of a graph $G$ such that $G[V(C)]$ is a complete graph, and the clique number $\omega(G)$ of a graph $G$ is the maximum cardinality of a clique of $G$. Then we will show some special graphs which are needed: (see Figure 1).

- $P_i$, the path with $i$ vertices (note that $P_1 = K_1$ and $P_2 = K_2$);
- $Z_i$, a graph obtained by identifying a vertex of a $K_3$ with an end-vertex of a $P_{i+1}$;
- $H_1$, a graph obtained by identifying a vertex of a $K_3$ with the one-degree vertex of a $Z_1$;
- $T_{i,j,k}$, a graph consisting of three paths $P_{i+1}, P_{j+1}$ and $P_{k+1}$ with the common starting vertex.

![Figure 1: Some special graphs: $P_i, Z_i, H_1$ and $T_{i,j,k}$.](image)

Let $X$ and $Y$ be nonempty subsets of $V(G)$, we denote by $E[X,Y]$ the set of edges of $G$ with one end in $X$ and the other end in $Y$, and by $e(X,Y)$ their number. When $Y = V(G) \setminus X$, the set $E[X,Y]$ is called the edge cut of $G$ associated with $X$. The edge cut set $S$ with the minimum number of edges is called the minimum edge cut. It is well-known that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. In [9], Wang, Tsuchiya and Xiong characterize all the pairs $R, S$ such that every connected $\{R, S\}$-free graph $G$ has $\kappa(G) = \kappa'(G)$.

**Theorem 1.** (Wang, Tsuchiya and Xiong [9]) Let $S$ be a connected graph. Then $G$ being a connected $S$-free graph implies $\kappa(G) = \kappa'(G)$ if and only if $S$ is an induced subgraph of $P_3$.

**Theorem 2.** (Wang, Tsuchiya and Xiong [9]) Let $\mathcal{H} = \{R, S\}$ be a set of two connected graphs such that $R, S \neq P_3$. Then $G$ being a connected $\mathcal{H}$-free graph implies $\kappa(G) = \kappa'(G)$ if and only if $\mathcal{H} \preceq \{Z_1, P_3\}$, $\mathcal{H} \preceq \{Z_1, K_1, 4\}$, $\mathcal{H} \preceq \{Z_1, T_{1,1,2}\}$, $\mathcal{H} \preceq \{P_1, H_0\}$ or $\mathcal{H} \preceq \{K_{1,3}, H_0\}$, where $H_0$ is the unique simple graph with degree sequence $4, 2, 2, 2, 2$.  

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In [4], Hellwig and Volkmann introduce a lot of sufficient conditions for $\kappa'(G) = \delta(G)$.

**Theorem 3.** Let $G$ be a connected graph satisfying the one of following conditions:

1. (Chartrand [3]) $n(G) \leq 2\delta(G) + 1$,
2. (Lesniak [6]) $d_G(u) + d_G(v) \geq n(G) - 1$ for all pairs $u, v$ of nonadjacent vertices,
3. (Plesnık [8]) $\dim(G) = 2$,
4. (Volkmann [8]) $G$ is bipartite and $n(G) \leq 4\delta(G) - 1$,
5. (Plesnık and Znám [7]) there are no four vertices $u_1, u_2, v_1, v_2$ with
   $d_G(u_1, u_2), d_G(u_1, v_2), d_G(v_1, u_2), d_G(v_1, v_2) \geq 3$,
6. (Plesnık and Znám [7]) $G$ is bipartite and $\dim(G) = 3$,
7. (Xu [10]) there exist $\lceil n(G)/2 \rceil$ pairs $(u_i, v_i)$ of vertices such that $d_G(u_i) + d_G(v_i) \geq n(G)$ for $i = 1, 2, \cdots, \lceil n(G)/2 \rceil$,
8. (Dankelmann and Volkmann [3]) $\omega(G) \leq p$ and $n(G) \leq 2(p\delta(G)/p - 1) - 1$.

Then $\kappa'(G) = \delta(G)$.

In this paper, we also consider and characterize the forbidden subgraphs for $\kappa'(G) = \delta(G)$.

**Theorem 4.** Let $S$ be a connected graph. Then $G$ being a connected $S$-free graph implies $\kappa'(G) = \delta(G)$ if and only if $S$ is an induced subgraph of $P_4$.

**Theorem 5.** Let $\mathcal{H} = \{R, S\}$ be a set of two connected graphs such that $R$ and $S$ are not an induced subgraph of $P_4$. Then $G$ being a connected $\mathcal{H}$-free graph implies $\kappa'(G) = \delta(G)$ if and only if $\mathcal{H} \subseteq \{H_1, P_5\}$, $\mathcal{H} \subseteq \{Z_2, P_6\}$, or $\mathcal{H} \subseteq \{Z_2, T_{1,1,3}\}$.

Note that all families of connected graphs satisfies $\kappa(G) < \kappa'(G)$ or $\kappa'(G) < \delta(G)$ should be $\kappa(G) < \delta(G)$ by Theorems 1 and 2 we may get the following corollaries.

**Corollary 6.** Let $S$ be a connected graph. Then $G$ being a connected $S$-free graph implies $\kappa(G) = \delta(G)$ if and only if $S$ is an induced subgraph of $P_3$.

**Corollary 7.** Let $\mathcal{H} = \{R, S\}$ be a set of two connected graphs such that $R$ and $S$ are not an induced subgraph of $P_3$. Then $G$ being a connected $\mathcal{H}$-free graph implies $\kappa(G) = \delta(G)$ if and only if $\mathcal{H} \subseteq \{H_0, P_4\}$, $\mathcal{H} \subseteq \{Z_1, P_5\}$, or $\mathcal{H} \subseteq \{Z_1, T_{1,1,2}\}$.

In fact, we also present a general result as follow. Now Corollaries 6 and 7 follow easily from Theorems 1, 2, 3, 4, 5, and 8. Note that $P_4$ may be one of the pair of forbidden subgraphs, see Theorem 5, while $P_4$ is the forbidden subgraph from Theorem 4, this means that the other subgraph may be any subgraph of $G$ when $P_4$ is one of a pair of forbidden subgraphs.
Theorem 8. Let \( G \) be a connected graph, and \( f(G), g(G), t(G) \) are three invariants of \( G \) with \( f(G) \leq g(G) \leq t(G) \). If the following statements hold:

1. \( G \) is \( H \)-free implies \( f(G) = g(G) \) if and only if \( H \in H_1 \);
2. \( G \) is \( H \)-free implies \( g(G) = t(G) \) if and only if \( H \in H_2 \),

then \( G \) is \( H \)-free implies \( f(G) = t(G) \) if and only if \( H \in H_1 \cap H_2 \). Here \( H_i \) is the set of class of given graphs, i.e., each element of \( H_i \) is a class of given graphs \( H \), for \( i \in \{1, 2\} \).

\( H_1 \cap H_2 := \{ H_1 \cap H_2 | H_1 \in H_1, H_2 \in H_2, \text{and } |H_1| = |H_2| \} \), and \( H_1 \cap H_2 \) is the set with order \( |H_1| \), which each element is the common induced subgraph of one graph in \( H_1 \) and one graph in \( H_2 \), respectively.

Proof. First suppose \( G \) is \( H \)-free and \( H \in H_1 \cap H_2 \), then \( H \in H_1 \) and \( H \in H_2 \). By (1) and (2), \( f(G) = g(G) \) and \( g(G) = t(G) \). It means that \( f(G) = g(G) = t(G) \). This completes the sufficiency.

Now we prove the necessity. Suppose that \( f(G) = t(G) \). Then \( f(G) = g(G) = t(G) \) since \( f(G) \leq g(G) \leq t(G) \). Therefore, both \( H \in H_1 \) and \( H \in H_2 \) must hold, by (1) and (2). It means that \( H \in H_1 \cap H_2 \). This completes the proof. \( \square \)

2 The necessity part of Theorems 4 and 5

We first construct some families of connected graphs \( G_i, i = 1, \cdots, 7 \) (see Figure 2). It is easy to see that each \( G \in G_i \) satisfies \( 1 = \kappa'(G) < \delta(G) = 2 \).

![Figure 2: Some classes of graphs satisfies 1 = κ'(G) < δ(G) = 2.](image)

The necessity part of Theorem 4. Let \( S \) be a graph such that every connected \( S \)-free graph is \( \kappa'(G) = \delta(G) \). Then \( S \) is an induced subgraph of all graphs in \( G_i, i = 1, \cdots, 7 \).

Note that the common induced subgraph of the graphs in \( G_1 \) and \( G_2 \) is a path. Since the largest induced path of the graphs in \( G_1 \) is \( P_4 \), \( S \) must be an induced subgraph of \( P_4 \). This completes the proof of the necessity part of Theorem 4. \( \square \)
The sufficiency part of Theorem 5. Let $R$ and $S$ are not an induced subgraph of $P_3$ graphs such that every connected $\{R, S\}$-free graph is $\kappa'(G) = \delta(G)$. Then all graphs in $G_i, i = 1, \cdot \cdot \cdot , 7$ should contain either $R$ or $S$ as an induced subgraph. Without loss of generality, we may assume that $R$ is an induced subgraph of all graphs in $G_1$. Note that all graphs in $G_1$ contain no induced cycle with length at least 4 as an induced subgraph, so we need to consider the following four cases.

**Case 1.** $R$ contain a clique $K_t$ with $t \geq 4$. It means that, for $i \in \{2, 3, 4, 5, 6, 7\}$, all graphs in $G_i$ are $R$-free, and should contain $S$ as an induced subgraph. Note that all graphs in $G_2$ are $K_3$-free, and all graphs in $G_3$ are $K_{1,3}$-free, so $S$ should be a path. Since the largest induced path of the graphs in $G_4$ is $P_4$, $S$ should be an induced subgraph of $P_4$, a contradiction.

**Case 2.** $R$ don’t contain the clique $K_t$ with $t \geq 4$, but contain two $K_3$. Since $R$ is an induced subgraph of all graphs in $G_1$, $R$ should be $H_1$. It means that, for $i \in \{2, 3, 5, 6, 7\}$, all graphs in $G_i$ are $R$-free, and should contain $S$ as an induced subgraph. Note that all graphs in $G_2$ are $K_3$-free, and all graphs in $G_3$ are $K_{1,3}$-free, so $S$ should be a path. Since the largest induced path of the graphs in $G_5$ is $P_5$, $S$ should be an induced subgraph of $P_5$. So $H = \{R, S\} \leq \{H_1, P_3\}$.

**Case 3.** $R$ don’t contain the clique $K_t$ with $t \geq 4$, but contain exactly one $K_3$. Since $R$ is an induced subgraph of all graphs in $G_1$, $R$ should be an induced subgraph of $Z_2$. It means that, for $i \in \{2, 6, 7\}$, all graphs in $G_i$ are $R$-free, and should contain $S$ as an induced subgraph. Note that the common induced subgraph of all graphs in $G_2$ and $G_7$ are a tree with the maximum degree 3 or a path. If $S$ is a tree with the maximum degree 3, since the common induced tree with the maximum degree 3 of all graphs in $G_6$ and $G_7$ are $T_{1,1,3}$, $S$ should be an induced subgraph of $T_{1,1,3}$. So $H = \{R, S\} \leq \{Z_2, T_{1,1,3}\}$. If $S$ is a path. Since the largest induced path of the graphs in $G_6$ is $P_6$, $S$ should be an induced subgraph of $P_6$. So $H = \{R, S\} \leq \{Z_2, P_6\}$.

**Case 4.** $R$ is a tree.

Since all graphs in $G_1$ are $K_{1,3}$-free, $R$ should be a path. Note that the largest induced path of the graphs in $G_1$ is $P_4$, so $R$ should be an induced subgraph of $P_4$, a contradiction. From the proofs above, we have that $H \leq \{H_1, P_3\}, H \leq \{Z_2, P_6\},$ or $H \leq \{Z_2, T_{1,1,3}\}$. This completes the proof of the necessity part of Theorem 5. □

### 3 The sufficiency part of Theorems 4 and 5

**The sufficiency part of Theorem 4.** Let $G$ be a connected $P_4$-free graph. Then $\text{dim}(G) \leq 2$. If $\text{dim}(G) = 1$, $G$ must be a complete graph and $\kappa'(G) = \delta(G) = n - 1$. If $\text{dim}(G) = 2$, by Theorem 3 (3), $\kappa'(G) = \delta(G)$. This completes the proof of the sufficiency part of Theorem 4. □

**The sufficiency part of Theorem 5.** Let $G$ be a connected $\mathcal{H}$-free graph such that $\kappa'(G) < \delta(G)$, where $\mathcal{H} \leq \{H_1, P_3\}, \{Z_2, P_6\}$, or $\{Z_2, T_{1,1,3}\}$. Then there must exists a minimum edge cut, say $M$, such that $|M| = \kappa'(G) < \delta(G)$. Let $G_1$ and $G_2$ are the
components of \( G - M \), and let \( S_i = V(G_i) \cap V(M) \), \( i \in \{1, 2\} \). Then \( |S_i| \leq |M| = \kappa'(G) < \delta(G) \), say \( |S_i| = s_i \), \( i \in \{1, 2\} \).

Claim 1. For \( i \in \{1, 2\} \), \( V(G_i - S_i) \neq \emptyset \). Moreover, for any \( x \in V(G_i - S_i) \), \( N_G(x) \cap V(G_i - S_i) \neq \emptyset \).

Proof. We will count the number of edges of \( G_i \) for \( i \in \{1, 2\} \).

\[
|E(G_i)| = \frac{1}{2} \left( \sum_{x \in V(G_i)} d_G(x) - \kappa'(G) \right) \\
\geq \frac{1}{2} \left( \delta(G)|V(G_i)| - \kappa'(G) \right) \\
\geq \frac{1}{2} \left( \delta(G)s_i - \kappa'(G) \right) \\
> \frac{1}{2} \kappa'(G) (s_i - 1) \\
\geq \frac{1}{2} s_i (s_i - 1)
\]

Note that the complete graph \( K_{s_i} \) has \( \frac{1}{2} s_i (s_i - 1) \) edges. It means that \( |V(G_i)| > s_i \), i.e., \( V(G_i - S_i) \neq \emptyset \).

Moreover, for any \( x \in V(G_i - S_i) \), since \( d_G(x) \geq \delta(G) > \kappa'(G) \geq s_i \), \( N_G(x) \cap V(G_i - S_i) \neq \emptyset \). This completes the proof of Claim 1. \( \square \)

Now we will distinguish the following two cases to complete our proof.

Case 1. \( G \) contains a \( P_4 = x_0x_1x_2x_3 \) with \( x_0 \in V(G_1 - S_1) \), \( x_1 \in S_1 \), \( x_2 \in S_2 \), and \( x_3 \in V(G_2 - S_2) \).

Subcase 1.1. \( \mathcal{H} \preceq \{H_1, P_5\} \).

By Claim 1, there exist two vertices \( x_0' \in V(G_1 - S_1) \) and \( x_3' \in V(G_2 - S_2) \) such that \( x_0x_0', x_3x_3' \in E(G) \). Then \( G[x_0', x_0, x_1, x_2, x_3, x_3'] \cong H_1 \) (if \( x_1x_0', x_2x_3' \in E(G) \)), or \( G[x_0', x_0, x_1, x_2, x_3] \cong P_5 \) (if \( x_1x_0' \notin E(G) \)), or \( G[x_0', x_0, x_1, x_2, x_3] \cong P_5 \) (if \( x_2x_3' \notin E(G) \)), a contradiction.

Subcase 1.2. \( \mathcal{H} \preceq \{Z_2, P_5\} \).

By Claim 1, there exist two vertices \( x_0' \in V(G_1 - S_1) \) and \( x_3' \in V(G_2 - S_2) \) such that \( x_0x_0', x_3x_3' \in E(G) \). Then \( G[x_0', x_0, x_1, x_2, x_3, x_3'] \cong P_5 \) (if \( x_1x_0', x_2x_3' \notin E(G) \)), or \( G[x_0', x_0, x_1, x_2, x_3] \cong Z_2 \) (if \( x_1x_0' \in E(G) \)), or \( G[x_0', x_0, x_1, x_2, x_3] \cong Z_2 \) (if \( x_2x_3' \in E(G) \)), a contradiction.

Subcase 1.3. \( \mathcal{H} \preceq \{Z_2, T_{1,1,3}\} \).

By Claim 1, \( N_G(x_0) \cap V(G_1 - S_1) \neq \emptyset \) and \( N_G(x_3) \cap V(G_2 - S_2) \neq \emptyset \).

Suppose that \( |N_G(x_0) \cap V(G_1 - S_1)| \geq 2 \) or \( |N_G(x_3) \cap V(G_2 - S_2)| \geq 2 \). Without loss of generality, we may assume that \( |N_G(x_0) \cap V(G_1 - S_1)| \geq 2 \), it means there exist two vertices \( x_0', x_0'' \in V(G_1 - S_1) \) such that \( x_0x_0', x_0x_0'' \in E(G) \). Then \( G[x_0', x_0'', x_0, x_1, x_2, x_3] \cong T_{1,1,3} \) (if \( x_0x_0', x_0'x_1, x_0''x_1 \notin E(G) \)), or \( G[x_0', x_0'', x_0, x_1, x_2] \cong Z_2 \) (if \( x_0x_0' \in E(G) \) and \( x_0'x_1, x_0''x_1 \notin E(G) \)), or \( G[x_0', x_0, x_1, x_2, x_3] \cong Z_2 \) (if \( x_0'x_1 \in E(G) \)), or \( G[x_0'', x_0, x_1, x_2, x_3] \cong Z_2 \) (if \( x_0''x_1 \in E(G) \)), a contradiction.
Suppose that \( N_G(x_0) \cap V(G_1 - S_1) = \{ x'_0 \} \) and \( N_G(x_3) \cap V(G_2 - S_2) = \{ x'_3 \} \). Note that \( N_G(x_0) \subseteq \{ x'_0 \} \cup S_1 \) and \( N_G(x_3) \subseteq \{ x'_3 \} \cup S_2 \). Then \( d_G(x_0) \leq s_1 + 1 \) and \( d_G(x_3) \leq s_2 + 1 \). Since \( d_G(x_0) \geq \delta(G) > \kappa'(G) \geq s_1 + 1 \) and \( d_G(x_3) \geq \delta(G) > \kappa'(G) \geq s_2 + 1 \) and \( d_G(x_3) \geq s_2 + 1 \). Therefore \( d_G(x_0) = s_1 + 1 \) and \( d_G(x_3) = s_2 + 1 \). It means that \( N_G(x_0) = S_1 \cup \{ x'_0 \} \), \( N_G(x_3) = S_2 \cup \{ x'_3 \} \), and \( s_1 = s_2 = \kappa'(G) \). Since \( |M| = \kappa'(G) = s_1 = s_2 \), each vertex in \( S_1 \) is just adjacent to exactly one vertex which is in \( S_2 \), and vice versa. Suppose \( s_1 \geq 2 \). Then there exists a vertex \( x'_i \in S_1 \) such that \( x'_i \neq x_1 \). Therefore \( G[\{ x'_0, x'_0, x'_1, x_1, x_2, x_3 \}] \cong T_{1,1,3} \) (if \( x_0x'_1, x_0x_1, x'_1x_1 \notin E(G) \)), or \( G[\{ x'_0, x'_0, x_0, x_1, x_2 \}] \cong Z_2 \) (if \( x'_0x'_1 \in E(G) \) and \( x'_0x_1, x'_1x_1 \notin E(G) \)), or \( G[\{ x'_0, x_0, x_1, x_2, x_3 \}] \cong Z_2 \) (if \( x'_0x_1 \in E(G) \)), or \( G[\{ x'_1, x_0, x_1, x_2, x_3 \}] \cong Z_2 \) (if \( x'_1x_1 \in E(G) \)), a contradiction. Suppose \( s_1 = 1 \). Then \( s_2 = \kappa'(G) = 1 \) and \( \delta(G) = 2 \). Assume \( d_G(x_1) \geq 3 \). Then there exists a vertex \( x'_i \in V(G_1 - S_1) \), such that \( x'_i, x_i \in E(G) \) and \( x'_i \neq x_0 \). Therefore \( G[\{ x_0, x'_0, x'_1, x_1, x_2, x_3 \}] \cong T_{1,1,3} \) (if \( x_0x'_1, x_0x_1, x'_1x_1 \notin E(G) \)), or \( G[\{ x_0, x'_0, x_1, x_2, x_3 \}] \cong Z_2 \) (if \( x_0x'_1 \in E(G) \)), or \( G[\{ x_0, x_0, x_1, x_2, x_3 \}] \cong Z_2 \) (if \( x_0x_1 \in E(G) \)), or \( G[\{ x_1, x_0, x_1, x_2, x_3 \}] \cong Z_2 \) (if \( x'_1x_1 \in E(G) \)), a contradiction. Assume \( d_G(x_1) = 2 \). Then it means that \( N_G(x_1) = \{ x_0, x_2 \} \) and \( d_G(x) = d_G(x) \) for any \( x \in V(G_1 - \{ x_0, x_1 \}) \). Since \( \delta(G) = 2 \) and \( d_G(x) \geq 1 \), there exist some vertices in \( V(G_1 - S_1) \) such that their degree in \( G \) are at least 3. Then we choose a vertex \( y_0 \in V(G_1 - S_1) \), such that \( d_G(y_0) \geq 3 \) and \( d_G(y_0, x_1) \) as small as possible. Let \( P' \) is the shortest path between \( x_1 \) and \( y_0 \). Then all inner vertices of \( P' \) should have degree two. Let \( y_1, y_2 \in N_G(y) \) and \( y_1, y_2 \notin V(P') \). Then \( G[\{ y_1, y_2, x_2, x_3 \} \cup V(P')] \) contains an induced \( T_{1,1,3} \) (if \( y_1y_2 \notin E(G) \)), or \( G[\{ y_1, y_2, x_2 \} \cup V(P')] \) contains an induced \( Z_2 \) (if \( y_1y_2 \notin E(G) \)), a contradiction.

**Case 2.** \( G \) contains no \( P_4 = x_0x_1x_2x_3 \) with \( x_0 \in V(G_1 - S_1) \), \( x_1 \in S_1 \), \( x_2 \in S_2 \), and \( x_3 \in V(G_2 - S_2) \).

Let \( S'_1 = \{ x \in S_1 : N_G(x) \cap V(G_i - S_i) \neq \emptyset \} \), and \( S'_2 = S_i - S'_i \) for \( i = 1, 2 \). Then \( S'_1 \neq \emptyset \) and \( E(S'_1, S'_2) = \emptyset \). By the minimality of \( M \) and the definition of \( S_i \), \( E(S'_1, S'_2), E(S'_1, S'_3), E(S'_2, S'_3) \neq \emptyset \). Now we choose a path \( P_0 \) between \( x_1 \) and \( x_2 \), such that \( x_1 \in S'_1 \) and \( x_2 \in S'_2 \), and the length of path as small as possible. Then \( |V(P_0)| \geq 3 \) and all inner vertices of \( P_0 \) must be in \( S'_1 \). Let \( x_0 \in V(G_1 - S_1) \) and \( x_3 \in V(G_2 - S_2) \), such that \( x_0x_1, x_2x_3 \in E(G) \). Then \( G[V(P_0) \cup x_0, x_3] \) is an induced path with at least 5 vertices.

**Subcase 2.1.** \( H \preceq \{ H_1, P_0 \} \).

\( P_0 \) is an induced path with at least 5 vertices, a contradiction.

**Subcase 2.2.** \( H \preceq \{ Z_2, P_0 \} \).

By Claim 1 there exist a vertex \( x'_0 \in V(G_1 - S_1) \) such that \( x_0x'_0 \in E(G) \). Then \( G[\{ x'_0 \} \cup V(P_1)] \) contains an induced \( P_6 \) (if \( x_1x'_0 \notin E(G) \)), or an induced \( Z_2 \) (if \( x_1x'_0 \in E(G) \)), a contradiction.

**Subcase 2.3.** \( H \preceq \{ Z_2, T_{1,1,3} \} \).

By Claim 1 and \( |S'_1| < s_1 < \delta(G) \), there exist two vertices \( x'_0, x'_0 \in V(G_1 - S_1) \) such that \( x_0x'_0, x_0x'_0 \in E(G) \). Then \( G[\{ x'_0 \} \cup V(P_1)] \) contains an induced \( T_{1,1,3} \) (if \( x'_0x'_0, x_0x_1, x'_0x_1 \notin E(G) \)), or an induced \( Z_2 \) (if \( x'_0x'_0 \in E(G) \) and \( x'_0x_0, x'_0x_1 \notin E(G) \)), or an induced \( Z_2 \) (if \( x'_0x_1 \in E(G) \) and \( x'_0x_1 \in E(G) \)), a contradiction.

This completes the proof of the sufficiency part of Theorem 5. □
4 Concluding remark

In this paper, we give a completely characterzation of all pairs \( \{R, S\} \) of graphs such that every connected \( \{R, S\}\)-free graph has the same edge-connectivity and minimum degree. All graphs in Figure 2 have edge-connectivity one, we also can construct some graphs for arbitrarily large edge-connectivity to show that Theorem 4 also hold. But for forbidden pairs \( \mathcal{H} = \{R, S\} \), we have not enough graphs to see that whether could get more wide forbidden pairs to guarantee the graphs having the same edge-connectivity and minimum degree, when we increase the edge-connectivity.

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