Continued Fraction Expansion as Isometry
The Law of the Iterated Logarithm for
Linear, Jump, and 2–Adic Complexity

Michael Vielhaber, Member, IEEE \(^1\)

Instituto de Matemáticas
Universidad Austral de Chile
Casilla 567, Valdivia
uach@gmx.net

Abstract — In the cryptanalysis of stream ciphers and pseudorandom sequences, the notions of linear, jump, and 2–adic complexity arise naturally to measure the (non)randomness of a given string. We define an isometry \(K\) on \(\mathbb{F}_q^\infty\) that is the precise equivalent to Euclid’s algorithm over the reals to calculate the continued fraction expansion of a formal power series. The continued fraction expansion allows to deduce the linear and jump complexity profiles of the input sequence. Since \(K\) is an isometry, the resulting \(\mathbb{F}_q^\infty\)–sequence is i.i.d. for i.i.d. input. Hence the linear and jump complexity profiles may be modelled via Bernoulli experiments (for \(\mathbb{F}_2\): coin tossing), and we can apply the very precise bounds as collected by Révész, among others the Law of the Iterated Logarithm.

The second topic is the 2–adic span and complexity, as defined by Goresky and Klapper. We derive again an isometry, this time on the dyadic integers \(\mathbb{Z}_2\) which induces an isometry \(A\) on \(\mathbb{F}_2^\infty\). The corresponding jump complexity behaves on average exactly like coin tossing.

Index terms — Formal power series, isometry, linear complexity, jump complexity, 2–adic complexity, 2–adic span, law of the iterated logarithm, Lévy classes, stream ciphers, pseudorandom sequences

\(^1\)Supported by Project FONDECYT 2001, No. 1010533 of CONICYT, Chile
I. Introduction

For some prime power \( q \), let \( \mathbb{F}_q \) be the finite field with \( q \) elements \( [17] \). We consider the set \( \mathbb{F}_q^\infty \) of infinite sequences over \( \mathbb{F}_q \) as our starting point. To assess the randomness of such \( \mathbb{F}_q^\infty \)-sequences one computes their linear and jump complexity profiles. These profiles as well as the 2-adic complexity should behave well in the sense that no large jumps occur and on average, for linear or 2-adic complexity, resp., \( q - 1 \) out of every \( 2^q \) resp. \( q \) symbols should lead to a jump.

We shall occasionally employ the set of finite words over \( \mathbb{F}_q \) which we denote as \( \mathbb{F}_q^* \) (we do not use the multiplicative group of \( \mathbb{F}_q \), so no confusion should arise). In particular, the empty word \( \varepsilon \) is in \( \mathbb{F}_q^* \).

Dealing with linear and jump complexity, a sequence \( a = (a_1, a_2, \ldots) \in \mathbb{F}_q^\infty \) is considered as coefficient sequence of its generating function, the formal power series

\[
G: \mathbb{F}_q^\infty \to \mathbb{S}_q \quad (a_i) \mapsto \sum_{i=1}^{\infty} a_i x^{-i}
\]

Here \( \mathbb{S}_q = \{ f \in \mathbb{F}_q[[x^{-1}]] : f = \sum_{i=1}^{\infty} a_i x^{-i}, a_i \in \mathbb{F}_q \} \) is a subring (without unity) of the ring \( \mathbb{F}_q[[x^{-1}]] \) of formal power series. The \( f \in \mathbb{S}_q \) have negative degree.

Furthermore, we use the polynomial ring \( \mathbb{F}_q[x] \) and its field of fractions \( \mathbb{F}_q(x) \). Then, \( \mathbb{S}_q \cap \mathbb{F}_q(x) \) denotes the image of the ultimately periodic sequences in \( \mathbb{F}_q^\infty \) under \( G \).

We define the leading coefficient \( lc(f) \) of a formal power series as

\[
lc(0) = 0 \text{ and } lc(\sum_{k=i}^{\infty} a_k x^{-k}) = a_i \in \mathbb{F}_q \setminus \{0\},
\]

and the degree as

\[
|0| = -\infty \text{ and } |\sum_{k=i}^{\infty} a_k x^{-k}| = -i \text{ (with } a_i \neq 0\).
\]

The analogue for polynomials is the usual degree \(|\sum_{k=0}^{d} a_k x^k| := d\), where the leading coefficient \( lc(\sum_{k=0}^{d} a_k x^k) := a_d \) again is assumed nonzero. The degree fulfills the ultrametric inequality \(|f \pm g| \leq \max\{|f|, |g|\}\) and equality \(|f| \neq |g| \Rightarrow |f \pm g| = \max\{|f|, |g|\}\) (we shall not use the associated norm).

A function \( f: \mathbb{F}_q^\infty \to \mathbb{F}_q^\infty \) is then called an isometry, if it preserves distance.
that is for all \( a, b \in \mathbb{F}_q^\infty: \|a - b\| = |f(a) - f(b)| \) (coordinate-wise subtraction in \( \mathbb{F}_q \)).

The outline of the paper is as follows:

The next section introduces an isometry \( K \) on \( \mathbb{F}_q^\infty \), where \( K(a) \) describes the partial denominators of the continued fraction expansion of \( G(a) \). As isometry, \( K \) is information preserving and we shall see, how each symbol of \( K(a) \) describes precisely what can be said at that moment about the partial denominators.

Section III makes the connection to linear and jump complexity and gives the wellknown Euclid–Lagrange–Berlekamp–Massey–Dornstetter algorithm in a form that delivers precisely \( K(a) \) as discrepancy sequence.

The section IV is a compilation of consequences of \( K \) being an isometry. Some of these results are already known but the proofs using \( K \) are shorter.

The main point of the paper is stated in section V. Since \( K \) is an isometry, an i.i.d. sample \( a \in \mathbb{F}_q^\infty \) leads to an i.i.d. resulting \( K(a) \), and thus on average, \( K(a) \) can be described by a Bernoulli experiment (in the binary case \( q = 2 \) just coin tossing) and we may apply the sharp known bounds (Lévy classes) for coin tossing to linear and jump complexity.

Section VI changes the focus to the 2–adic complexity as defined by Klapper and Goresky. Again, there is an isometry on \( \mathbb{F}_q^\infty \) associated to 2–adic complexity and the induced jump complexity turns out to behave on average just like the “linear” jump complexity.

We state an open problem and finish with the conclusion.

II. The Isometry \( K \)

In this section we will obtain, for a sequence \( a \in \mathbb{F}_q^\infty \), its generating function, the expansion of the latter into a continued fraction, and finally an encoding of the partial denominators of the continued fraction. The whole process will turn out to define an isometry on \( \mathbb{F}_q^\infty \), which we call \( K \) (“Kettenbruch”).

A. Continued Fractions of Formal Power Series

We start with the continued fraction expansion of a formal power series \( G(a) \in S \setminus \{0\} \). For \( \xi \in \mathbb{F}_q[[x^{-1}]] \) (not only in \( S \)), \( \xi = \sum_{k=n}^{\infty} a_k x^{-k} \), we define the integral part as \( [\xi] := \sum_{k=n}^{0} a_k x^{-k} \). \( [\xi] \) is zero for \( n \geq 1 \) and a polynomial of degree \( -n \) for \( n \leq 0 \).

Given \( \xi_0 := G(a) \in S \setminus \{0\} \), we now proceed iteratively to obtain \( A_i(x) = [\xi_i] \in \mathbb{F}_q[x] \) and \( \xi_{i+1} := (\xi_i - A_i(x))^{-1} \in S \). We discard \( A_0(x) \) which is
always zero, and obtain a sequence \((A_1(x), A_2(x), \ldots)\) of polynomials with positive degree which are called the partial denominators of the continued fraction expansion. This sequence is finite, if \(G(a) \in \mathbb{F}_q(x)\) (that is, if \(a\) is ultimately periodic), it is infinite otherwise. More on continued fractions can be found in Perron [30] over the reals, and in de Mathan [20] ch. IV or Artin [11 §12 f.] for formal power series.

We can now write the formal power series \(G(a)\) as

\[
G(a) = \sum_{i=1}^{\infty} a_i x^{-i} = \frac{1}{A_1(x) + \frac{1}{A_2(x) + \frac{1}{A_3(x) + \ldots}}} =: \frac{1}{|A_1(x)| + \frac{1}{|A_2(x)| + \frac{1}{|A_3(x)| + \ldots}}
\]

More formally, we state this map from \(S\setminus\{0\}\) to sequences of polynomials with positive degree in \((\mathbb{F}_q[x]\setminus\mathbb{F}_q)^* \cup (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^\infty\) as operator \(K\):

\[
\begin{align*}
\mathcal{K}: \quad & (S\setminus\{0\}) \setminus \mathbb{F}_q(x) / \{0\} \quad \mapsto \quad (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^* \\
\sum_{i=1}^{\infty} a_i x^{-i} = \frac{1}{|A_1(x)|} + \frac{1}{|A_2(x)|} + \ldots + \frac{1}{|A_k(x)|} \quad & \mapsto \quad (A_1(x))_{i=1}^k
\end{align*}
\]

\[
\begin{align*}
\mathcal{K}: \quad & \mathbb{F}_q(x) \setminus \{0\} \quad \mapsto \quad (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^\infty \\
\sum_{i=1}^{\infty} a_i x^{-i} = \frac{1}{|A_1(x)|} + \frac{1}{|A_2(x)|} + \frac{1}{|A_3(x)|} + \ldots \quad & \mapsto \quad (A_1(x))_{i=1}^\infty
\end{align*}
\]

We further define \((0^{\infty})^G \xrightarrow{\mathcal{K}} 0 \xrightarrow{\mathcal{K}} \varepsilon \in (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^*\), the empty sequence of (no) polynomials.

The next step gets us back to \(\mathbb{F}_q^\infty\). We encode the polynomials with positive degree by words from \(\mathbb{F}_q^*\), where a polynomial of degree \(d \geq 1\) is encoded in \(2d\) symbols from \(\mathbb{F}_q\), the first \(d\) symbols determining the degree and the second half determining the coefficients. Let

\[
\Pi_q := \{(a_1, \ldots, a_n) \in \mathbb{F}_q^* \mid \exists d \in \mathbb{N} : n = 2d, a_1 = \ldots = a_{d-1} = 0, a_d \neq 0\}
\]

be the set of all allowed encodings of polynomials. Then we define the encoding function \(\pi\) and the subdivision into degree and coefficient part as:

\[
\begin{align*}
\pi: \mathbb{F}_q[x]\setminus\mathbb{F}_q & \quad \rightarrow \quad \Pi_q \subset \mathbb{F}_q^*, \quad \sum_{i=0}^{d} a_i x^i \quad \mapsto \quad 0^{d-1} a_d a_{d-1} \ldots a_1 a_0 \in \mathbb{F}_q^{2d} \\
\pi_D: \mathbb{F}_q[x]\setminus\mathbb{F}_q & \quad \rightarrow \quad \mathbb{F}_q^*, \quad \sum_{i=0}^{d} a_i x^i \quad \mapsto \quad 0^{d-1} a_d \in \mathbb{F}_q^{d} \\
\pi_C: \mathbb{F}_q[x]\setminus\mathbb{F}_q & \quad \rightarrow \quad \mathbb{F}_q^*, \quad \sum_{i=0}^{d} a_i x^i \quad \mapsto \quad a_{d-1} \ldots a_1 a_0 \in \mathbb{F}_q^d
\end{align*}
\]

\((a_d \neq 0)\)

Hence \(\pi\) induces a function \(\pi^\infty\) on the set \((\mathbb{F}_q[x]\setminus\mathbb{F}_q)^* \cup (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^\infty\) of finite or infinite sequences of polynomials with positive degree as

\[
\begin{align*}
\pi^\infty: (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^* & \quad \rightarrow \quad \mathbb{F}_q^\infty, \quad (A_1)^{k-1} \quad \mapsto \quad \pi(A_1) \ldots \pi(A_k)|0^\infty \\
\pi^\infty: (\mathbb{F}_q[x]\setminus\mathbb{F}_q)^\infty & \quad \rightarrow \quad \mathbb{F}_q^\infty, \quad (A_1)^{k-1} \quad \mapsto \quad \pi(A_1)\pi(A_2)\ldots
\end{align*}
\]

where \(|\) indicates concatenation of elements from \(\mathbb{F}_q^*\).
In a similar way, \( \pi_D^\infty \) and \( \pi_C^\infty \) are built up from the \( \pi_D \) and \( \pi_C \).

The set \( \Pi'_q := \Pi_q \cup \{0^\infty\} \) is a complete prefix code for \( F_q^\infty \), i.e. every sequence \( a \in F_q^\infty \) can be decomposed in exactly one way into elements from \( \Pi'_q \) and hence \( \pi^-\infty : F_q^\infty \to (F_q[x]\backslash F_q)^* \cup (F_q[x]\backslash F_q)^\infty \) is bijective.

B. The Continued Fraction Operator \( K : F_q^\infty \to F_q^\infty \)

We thus map the continued fraction expansion of a generating function back into the space \( F_q^\infty \) and define the Continued Fraction Operator \( K \) on \( F_q^\infty \) as

\[
K : F_q^\infty \to F_q^\infty, \quad K = \pi^\infty \circ K \circ G.
\]

We define \( K \) for finite (prefix) words by

\[
\forall a \in F_q^* : \quad K(a) := K(a[0^\infty])_{i=1}
\]

(we will see in Theorem 5 that the continuation \( 0^\infty \) after \( a \) is irrelevant) and we obtain the inverse operator as

\[
K^{-1} = G^{-1} \circ K^{-1} \circ \pi^-\infty.
\]

\( K_D \) and \( K_C \) are not injective and thus \( K_D^{-1}, K_C^{-1} \) do not make sense)

We give two examples for \( K, K^{-1} \):

(i) Let \( a = (a_i)_{i=1}^\infty = 1(110)^\infty \in F_2^\infty \), then

\[
G(a)(x) = x^{-1} + x^{-2} + x^{-3} + x^{-5} + x^{-6} + x^{-8} + x^{-9} + \ldots
\]

\[
= \frac{1}{x} + \frac{x^{-2} + x^{-3}}{1 + x^{-3}} = \frac{x^2 + 1}{x^3 + x^2 + x} \cdot \frac{x + 1}{x + 1 + \frac{1}{x^2 + 1}},
\]

from \( x^3 + x^2 + x = (x^2 + 1)(x+1) + 1 \). Thus \( \mathcal{K}(G(a)) = (x+1, x^2 + 1) \in F_2[x]^2 \) and \( K(a) = \pi^\infty \circ K \circ G(a) = 1101010^\infty \in F_q^\infty \), where \( 11 = \pi(x+1), \ 0101 = \pi(x^2 + 1) \).

(ii) Let \( a = 11011001001011 \ldots \in F_2^\infty \). By repeatedly taking the integral part and inverting, we obtain

\[
G(a)(x) = \frac{1}{|x + 1|} + \frac{1}{|x|} + \frac{1}{|x^3 + x + 1|} + \frac{1}{|x + 1|} + \ldots
\]
The encodings of the partial denominators are $\pi(x + 1) = 11$, $\pi(x) = 10$, $\pi(x^3 + x + 1) = 001011$, and again $\pi(x + 1) = 11$, hence $K(a) = 11100101111 \ldots$

Let $A_i$ be the partial denominators of $G(a)$. Then we iteratively obtain convergents $P_k/Q_k$ to $G(a)$ using $P_i := A_i \cdot P_{i-1} + P_{i-2}$ and $Q_i := A_i \cdot Q_{i-1} + Q_{i-2}$ with the initial conditions $P_{-2} = Q_{-2} = 0$, $P_{-1} = Q_{-1} = 1$.

The recursion for the $P_i, Q_i$ leads us to

**Theorem 1** \[ P_n \cdot Q_{n-1} - P_{n-1} \cdot Q_n = (-1)^{n-1} \text{ for } n \geq -1 \]

**Proof.** By induction on $n$. See also [26 §6].

The next theorem gives a bound for the precision of the approximation of $G(a)$ by $P_k/Q_k$.

**Theorem 2** Let $P_k/Q_k$ be a convergent to $G \in \mathbb{S}$ with $G \neq P_k/Q_k$. Then

(i) \[ |G - P_k/Q_k| = -|Q_k| - |Q_{k+1}| < -2 \cdot |Q_k| \]

(ii) For $k \in \mathbb{N}$ and all $Z, N \in \mathbb{F}_q[x]$ with $0 \leq |N| < |Q_{k+1}|$ we have:

\[ |G - Z/N| \geq |G - P_k/Q_k| = -|Q_k| - |Q_{k+1}|. \]

**Proof.** (compare [20 p. 69–74])

(i) Let $\xi_i, A_i$ as before. By induction on $i$ we obtain the equation

\[ \forall i \in \mathbb{N}_0 : G = \frac{\xi_i \cdot P_{i-1} + P_{i-2}}{\xi_i \cdot Q_{i-1} + Q_{i-2}} \]

and hence

\[
\begin{align*}
|G - P_k/Q_k| &= \left| \frac{\xi_{i+1} \cdot P_i + P_{i-1}}{\xi_{i+1} \cdot Q_i + Q_{i-1}} - \frac{P_i}{Q_i} \right| \\
&= \left| \frac{\xi_{i+1} \cdot (P_i Q_i - P_{i-1} Q_i) + (P_{i-1} Q_i - P_i Q_{i-1})}{\xi_{i+1} \cdot Q_i^2 + Q_i Q_{i-1}} \right| \\
&= \left| (-1)^i - 2|Q_i| - |\xi_{i+1}| \right| \\
&= -|Q_{i+1}| - |Q_i|,
\end{align*}
\]

where we have used Theorem 1 and $|\xi_{i+1}| = |A_{i+1}| = |Q_{i+1}| - |Q_i|$.

(ii) See [26 p.221, Th. B.1].

The approximations $P_k/Q_k$ are called *convergents*. Furthermore, there may occur intermediate results [20 p. 71], which we shall call *subconvergents*: Let $A_i(x) = \sum_{j=0}^{\lfloor |A_i| \rfloor} a_j^{(i)} \cdot x^j$ be a partial denominator, that is $P_i = A_i \cdot P_{i-1} + P_{i-2}$.
and \( Q_i = A_i \cdot Q_{i-1} + Q_{i-2} \), resp., a (main) numerator and denominator, resp. Then for each \( k = |A_i|, |A_i| - 1, \ldots, 2, 1 \) we can define an auxiliary partial denominator \( A_i^{(k)}(x) := \sum_{j=k}^{|A_i|} a_j^{(i)} \cdot x^j \) that in turn defines a subconvergent

\[
\frac{P_i^{(k)}(x)}{Q_i^{(k)}(x)} := \frac{A_i^{(k)} \cdot P_{i-1} + P_{i-2}}{A_i^{(k)} \cdot Q_{i-1} + Q_{i-2}}.
\]

In the preceding example, \( A_2(x) = x^2 + x + 1 \) and \( \frac{P_2(x)}{Q_2(x)} = \frac{x^2 + x + 1}{x^4 + x^2} \). From \( A_2^{(2)}(x) = x^2 \) and \( A_2^{(1)}(x) = x^2 + x \) we obtain the subconvergents

\[
\begin{align*}
P_2^{(2)}(x) &= \frac{x^2 \cdot 1 + 0}{x^2 \cdot (x^2 + x + 1) + 1} = \frac{x^2}{x^4 + x^3 + x^2 + 1} \\
P_2^{(1)}(x) &= \frac{(x^2 + x) \cdot 1 + 0}{(x^2 + x) \cdot (x^2 + x + 1) + 1} = \frac{x^2 + x}{x^4 + x + 1}.
\end{align*}
\]

We will see in the proof of Theorem 5 that the subconvergents \( \frac{P_i^{(k)}}{Q_i^{(k)}} \) have a precision between that of the convergents \( \frac{P_{i-1}}{Q_{i-1}} \) and \( \frac{P_i}{Q_i} \). All convergents including the subconvergents will be obtained during the calculation via Euclid’s algorithm.

Of course, Theorem 1 is also valid for subconvergents

\[
P_i^{(k)} \cdot Q_{i-1} - P_{i-1} \cdot Q_i^{(k)} = (-1)^{i-1},
\]

(again by induction, on \( k \), for fixed \( i \)) see also Carter [3, Lemma 4.2.1].

C. \textbf{K is an Isometry}

In the sequel, we will see that the \( n \)-th symbol \( K(a)_n \) depends exactly on \( a_1, \ldots, a_n \), but not on \( a_{n+1}, \ldots \). In particular, this means that at the end of \( K(a) \) for \( a \in \mathbb{F}_q^* \) we might have an incomplete encoding. This then describes exactly what is known at that position about the partial denominator.

In the next theorem, we show that \( K^{-1} \) is an isometry on \( \mathbb{F}_q^\infty \) that is, two inputs differing for the first time in position \( n \) lead to outputs also differing here, but not before.

\textbf{Proposition 3 “The Ultrametric Square”} Let \( P, Q, R, S \in \mathbb{F}_q^\infty \) be four “points” with \( |P - Q|, |R - S| < |P - R| \). Then \( |Q - S| = |P - R| \).

\textbf{Proof.} By assumption and the ultrametric inequality \( |P| \neq |Q| \Rightarrow |P \pm Q| = \max\{|P|, |Q|\} \), we have \( |P - S| = |(P - R) + (R - S)| = \max\{|P -
We have \( |R|, |R - S| = |P - R| \). By the same reasoning \( |Q - S| = |(Q - P) + (P - S)| = |P - R| \). \(\square\)

**Lemma 4** Let \( c \in \mathbb{F}_q^\infty \) with \( K^{-1}(c) = a, n \in \mathbb{N} \), and \( d \in \mathbb{F}_q^\infty \) with \( d_i = c_i \) for \( i \neq n \), \( d_n \in \mathbb{F}_q \{ c_n \} \), and \( b = K^{-1}(d) \in \mathbb{F}_q^\infty \). Then \( |a - b| = -n \).

**Proof.** Let the continued fraction of \( G(a) \) be \( G(a) = \frac{1}{A_1(x)} + \frac{1}{B_1(x)} + \frac{1}{A_2(x)} + \frac{1}{B_2(x)} + \ldots \). Then we have \( c = \pi(A_1)\pi(A_2)\pi(A_3)\ldots \). Similarly, let \( G(b) = \frac{1}{A_1(x)} + \frac{1}{B_1(x)} + \frac{1}{A_2(x)} + \frac{1}{B_2(x)} + \ldots \) and thus \( d = \pi(B_1)\pi(B_2)\pi(B_3)\ldots \). Now let \( k \in \mathbb{N} \) be such that \( A_i = B_i \) for \( i < k \) and \( A_k \neq B_k \). This implies \( |Q_{k-1}| + |Q_k| \leq n \leq 2|Q_k| \) \((c_n \text{ is } lc(A_k) \text{ or part of } \pi_c(A_k))\). W.l.o.g. let \( |A_k| \leq |B_k| \).

We have \( |G(a) - \frac{P_k}{Q_k}| = -|Q_k| - |Q_{k+1}| < -2|Q_k| \leq -n \) by Theorem 2\( (i) \). We consider the ultrametric square made up of \( G(a), G(b) \) and their convergents \( \frac{P_k}{Q_k}, \frac{P_{k+1}}{Q_{k+1}} \). First we treat the case \( |A_k| = |B_k| \). Then \( B_k = A_k + (d_n - c_n)x^d \) with \( g = 2|Q_k| - n \) \((g \text{ more symbols after } c_n \text{ until the end of } \pi(A_k))\).

Then \( \left| G(b) - \frac{P_k}{Q_k} \right| = \left| G(b) - \frac{P_k + x^gP_{k-1}}{Q_k + x^gQ_{k-1}} \right| = -|Q_k| - |Q_{k+1}| < -2|Q_k| \leq -n \)

and \( \left| \frac{P_k + x^gP_{k-1}}{Q_k + x^gQ_{k-1}} - \frac{P_k}{Q_k} \right| = \left| \frac{P_k(x+g)Q_k - P_kQ_{k-1} - P_kQ_kx^g}{Q_k(x+g)Q_{k-1}} \right| = \left| \frac{x^gQ_{k-1}}{Q_k(x+g)Q_{k-1}} \right| = |g - 2|Q_k| = n. \)

Altogether we have an ultrametric square where \( |G(a) - \frac{P_k}{Q_k}| < n, \; |G(b) - \frac{P_k}{Q_k}| < n, \; |\frac{P_k}{Q_k} - \frac{P_{k+1}}{Q_{k+1}}| = -n \), and with Proposition 3 now follows

\[ |G(a) - G(b)| = |a - b| = -n = |c - d| = |K(a) - K(b)|. \]

Let now \( |A_k| < |B_k| \). In this case \( c_n = lc(A_k), d_n = 0 \) and \( n = |Q_{k-1}| + |Q_k| \).

Then \( \left| G(a) - \frac{P_{k+1}}{Q_{k+1}} \right| = -|Q_k| - |Q_{k-1}| = -n \) and \( \left| G(b) - \frac{P_{k+1}}{Q_{k+1}} \right| = -|Q_{k-1}| - |Q_k| \leq -n \) and thus as before \( |G(a) - G(b)| = -n. \) \(\square\)

In the sequel we shall say that we are in case A, B, C, resp., if \( c_n \) is A) a zero of some \( \pi_D \) \((i) \) of the above example for \( n = 5, 6 \), B) the last coefficient \((n = 2, 4, 10, 12) \), C) the leading or some other coefficient except the last \((n = 1, 3, 7 \ldots 9, 11) \).

**Theorem 5** Let \( n \in \mathbb{N} \) and \( c, d \in \mathbb{F}_q^\infty \) with \( d_i = c_i \) for \( i < n \), \( c_n \neq d_n \).

Let \( a = K^{-1}(c) \in \mathbb{F}_q^\infty \) and \( b = K^{-1}(d) \in \mathbb{F}_q^\infty \). Then

\[ |a - b| = |K(a) - K(b)| = -n. \]

**Proof.** We shall change from \( (c_i) \) to \( (d_i) \) one symbol at a time, using

Lemma 4. Let \( d_i^{(k)} = \begin{cases} d_i, & i \leq k \\ c_i, & i > k \end{cases} \) Then \( d^{(0)} = c \) and \( \lim_{k \to \infty} d^{(k)} = d \). Let
\[ b^{(k)} = K^{-1}(d^{(k)}) \]. Then \( |a - b^{(0)}| = -\infty \), since \( a = b^{(0)} \). Furthermore, \( |b^{(i-1)} - b^{(i)}| = \begin{cases} -\infty, & c_i = d_i \\ -i, & c_i \neq d_i \end{cases} \) from \( d^{(i-1)} = d^{(i)} \) and by Lemma 4, respectively.

Hence \( |a - b^{(n)}| = -n \) (first difference between \( c \) and \( d \)), and \( |b^{(n+k)} - b^{(n+k+1)}| \leq -n - k - 1 < -n \) for \( k \in \mathbb{N}_0 \).

By the ultrametric equality, we thus have \( |a - b^{(n+k)}| = |(a - b^{(n)}) + (b^{(n)} - b^{(n+1)}) + \ldots + (b^{(n+k-1)} - b^{(n+k)})| = -n \) for all \( k \in \mathbb{N} \), that is after the first impact of changing \( c_n \) into \( d_n \), the further changes are “absorbed” by the ultrametric. In the limit \( k \to \infty \), we obtain \( |a - b| = -n = |K(a) - K(b)| \). □

In summary, the first \( n \) symbols of \( a \) determine the first \( n \) symbols of \( K(a) \) and vice versa. At each step, the resulting symbols from \( \mathbb{F}_q \) encode all information available at that point about the continued fraction expansion: In case B, we just have the complete last determined partial denominator, in case A, the next partial denominator \( A_k \) will have degree at least \( n - 2|Q_{k-1}| \) and that is all we can say at the moment, and in case C, the missing coefficients of the current partial denominator are as yet unknown and all possible values will be assumed for a suitable continuation of \( a \).

We have thus obtained the fundamental result of the paper:

**Theorem 6**

The function \( K: \mathbb{F}_q^\infty \to \mathbb{F}_q^\infty \) is an isometry on \( \mathbb{F}_q^\infty \), for every finite field \( \mathbb{F}_q \).

**Remark** We call the domain of \( K \) (range of \( K^{-1} \)) coefficient space and the range of \( K \) (domain of \( K^{-1} \)) discrepancy space (as in the Berlekamp–Massey–Algorithm).

### III. Linear Complexity, Euclid’s Algorithm

We recall the notion of linear complexity \( L \), give the connection between \( K \) and \( L \), and adapt the Berlekamp–Massey algorithm to produce precisely \( b = K(a) \) as discrepancy sequence on input \( a \). Furthermore, we describe an alternative method, due to Niederreiter and the author, to calculate \( K(a) \) by means of the shift commutator \([K^{-1}, \sigma]\).

#### A. Linear Complexity

We define the linear complexity profile of a sequence \( a \in \mathbb{F}_q^\infty \) as \( L_a: \mathbb{N}_0 \to \mathbb{N}_0 \) with \( L_a(0) = 0 \) and for \( n \geq 1 \) let \( L_a(n) = \sum_{i=1}^k |A_i| \), where \( A_k \) is the last
partial denominator whose leading coefficient is encoded in $K(a)_{i,i=1...n}$.

The sequence $(L_a(n))_{n \geq 0}$ is called the linear complexity profile of $a$. $(L_a(n))$ is monotonously increasing and jumps, $L_a(i) > L_a(i - 1)$, where $K(a)_i$ encodes a leading coefficient. We next show that the usual definition in linear feedback shift register (LFSR) theory is equivalent:

**Theorem 7** $L_a(n)$ denotes the length of a shortest LFSR, that produces $a_1...a_n$.

**Proof.** Given the formal power series $G(a)$ with the convergent numerators and denominators $(P_k)$ and $(Q_k)$, resp., we consider the LFSR with normalized feedback polynomial $(lc(Q_k))^{-1}Q_k$ of length $|Q_k| = \sum_{i=1}^k |A_i| = L_a(n)$. This LFSR produces, for a suitable initial content, a sequence $b$ with $G(b) = \frac{P_k}{Q_k}$. From $|G(a) - \frac{P_k}{Q_k}| = -|Q_{k+1}| - |Q_k| < -n$ now follows that this LFSR will produce $a$ (at least) up to $a_n$. On the other hand,

$$\left|G(a) - \frac{P_{k-1}}{Q_{k-1}}\right| = -|Q_k| - |Q_{k-1}| = -2|Q_{k-1}| - |A_k| \geq -n,$$

since the leading coefficient of $A_k$ lies in $K(a)_{i,i=1...n}$. Hence the previous convergent $\frac{P_k}{Q_k}$ is not sufficiently precise and by Theorem 2(ii) also no intermediate LFSR length will do. □

We see that typically $2 \cdot L_a(n) \approx n$, since with $L_a(n) = \sum_{i=1}^k |A_i|$ we need just $2 \cdot L_a(n)$ symbols to encode $A_1,...,A_k$, hence the deviation from the sequence length $n$ consists only in the length of the last incomplete encoding of a partial denominator (missing coefficients in case C of Lemma 4, exceeding zeroes in case A). Hence we define:

Given a sequence $a$ with linear complexity profile $(L_a(n))$ we define the linear complexity deviation of $a$ at $n$ as

$$m_a(n) := 2 \cdot L_a(n) - n \in \mathbb{Z}, \quad n \geq 0.$$

**Remark** Comparing with the three cases of Lemma 4, we have: $m_a(n) < 0$ in case A, $m_a(n) = 0$ in case B, $m_a(n) > 0$ in case C.

**B. Euclid’s Algorithm**

**History** Euclid (“Elements”, 300 BC) invents his algorithm to calculate the gcd of two natural numbers. Lagrange (1770 AD) has our Theorems 1 and 2, all this over the reals.

Berlekamp [2] describes in 1967 a decoding method for BCH–codes over arbitrary finite fields. In 1969, Massey [19] uses this method to obtain a
shortest LFSR producing a given sequence. In 1979 Welch and Scholtz \[37\] make the connection with continued fractions and they show that the main convergents appear as polynomials in the BMA.

In 1987 Dornstetter \[8\] gives the precise equivalence between the algorithms of Berlekamp and Euclid, already mentioning the subconvergents.

During 1988–1991 Niederreiter \[21\][22][23][25] as well as Dai and Zeng \[5\] then show the detailed connection between linear complexity ($L_a$) and the continued fraction expansion of $G(a)$: A jump by $k$ in the profile corresponds to a partial denominator of degree $k$. Furthermore, in \[5\] a connection between subconvergents and discrepancies is given for the case $F_2$.

Before giving the implementation in a form that exactly delivers $b = K(a)$ as discrepancy sequence, we shall recompile the changes necessary in comparison with \[19\]:

(i) The main convergents have to start with $P_{-2} = 0$, $P_{-1} = 1$, $Q_{-2} = 1$, $Q_{-1} = 0$ as is the case over $\mathbb{R}$ \[30\]. From $A_0 = 0$ we also have $P_0 = 0$, $Q_0 = 1$,

(ii) we use the feedback polynomial, not its reciprocal, the “connection polynomial”, and

(iii) the feedback polynomial will not be normalized.

In this way we will obtain all partial denominators and all convergents and subconvergents and the discrepancy sequence is the encoding $K(a)$ of the partial denominators.

\textbf{Euclid–Lagrange–Berlekamp–Massey–Dornstetter–Algorithm}

START

Input $a(1,...)$ // elements of $\mathbb{F}_q$

\( P = 0 \), \( AP = 1 \) // numerator

\( Q = 1 \), \( AQ = 0 \) // here $Q=Q_0$, $AQ = Q_{-1}$ as initial values

\( d = 0 \), \( m = 0 \), \( j = 0 \), \( r = -1 \)

//loop invariant : $m = 2 \cdot d - j$, $d = \left| Q \right|$

DO \( j = 1, \ldots \)

\( b(j) = \sum_{i=0}^{d} Q(i) \cdot a(j + i - d) \)  //equals $\sum Q(d - i) \cdot a(j - i)$

CASE ($b(j), m$) IS

(0, ·) :

\( m = m - 1 \)

(\( \neq 0, > 0 \)) :

\( m = m - 1 \)

\( \tilde{b} = \overline{r} \cdot b(j) \)
\[ P = P + \tilde{b} \cdot x^m \cdot AP \ // \text{numerator} \]
\[ Q = Q + \tilde{b} \cdot x^m \cdot AQ \]
\((\neq 0, \leq 0) : \]
\[ m = -(m - 1) \]
\[ \tilde{b} = r \]
\[ r = b(j) \]
\[ \tau = -r^{-1} \]
\[ \tilde{b} = \tilde{b} \cdot \tau \]
\[ P_{\text{tmp}} = AP, \ AP = P, \ P = P_{\text{tmp}} // \text{numerator} \]
\[ P = P + \tilde{b} \cdot x^m \cdot AP // \text{numerator} \]
\[ Q_{\text{tmp}} = AQ, \ AQ = Q, \ Q = Q_{\text{tmp}} \]
\[ Q = Q + \tilde{b} \cdot x^m \cdot AQ \]
\[ d = d + m \]

END CASE

//loop invariant: \( m = 2 \cdot d - j, \ d = |Q| \)
Output \( b(j), P, Q \)

END DO

\( P/Q \) is the actual (sub)convergent, \( AP/AQ \) the previous convergent, \((b(j)) = K(a(j))\). The lines with comment “/numerator” are not necessary to obtain \((b_j)\). For \( F_2 \) we have \( \tilde{b} = b(j), b(j) \neq 0 \Rightarrow b(j) = 1, \) and \( r = \tau = 1, \) which simplifies the program.

C. The Shift Commutator of \( K \)

Another method to actually compute \( K \) uses the shift commutator \n
\[ [K^{-1}, \sigma] = K \circ \sigma^{-1} \circ K^{-1} \circ \sigma. \]

Let some \( w = K(v) \) be given, e.g. \( 0^\infty = K(0^\infty) \), then \( K(av) = [K^{-1}, \sigma](aw) \) for \( a \in F_q \), since

\[ aw \xrightarrow{\sigma} w \xrightarrow{K^{-1}} v \xrightarrow{\sigma^{-1}} av \xrightarrow{K} K(av) = [K^{-1}, \sigma](aw) \]

\([K^{-1}, \sigma]\) can be computed by a transducer with finite state space and an up-down-counter in amortized linear time (for a detailed description see [27] for the case \( F_2 \) and [28] for general finite fields). Hence we can compute \( K \) in amortized quadratic time by repeated application of \([K^{-1}, \sigma]\) via the above formula. We thereby obtain with no additional cost all continued fractions of all shifted sequences \((a_1, \ldots, a_n), (a_2, \ldots, a_n) \ldots (a_{n-1}, a_n)\) as well.
IV. Consequences for Linear and Jump Complexity of \( K \) being Isometry

We derive the partition of \( K \) into \( K_D \) and \( K_C \) from the behaviour of the algorithm in III.B. Furthermore, we derive from \( K_D \) the linear and jump complexity profiles and some combinatorial results.

A. Translation Theorem

Theorem 8 For every length \( n \in \mathbb{N} \) and every sequence prefix \( a \in \mathbb{F}_q^n \) we have:

(i) \( m_a(n) > 0 \) : \( m_a|_{a_{n+1}}(n + 1) = m_a(n) - 1 \).

(ii) \( m_a(n) \leq 0 \) :
\[ \exists_1 \alpha \in \mathbb{F}_q : m_a|_{\alpha}(n + 1) = m_a(n) - 1, \quad \forall \alpha' \neq \alpha : m_a|_{\alpha'}(n + 1) = 1 - m_a(n). \]

(iii) \( L_a(n) > n/2 \) : \( L_a|_{a_{n+1}}(n + 1) = L_a(n) \).

(iv) \( L_a(n) \leq n/2 \) :
\[ \exists_1 \alpha \in \mathbb{F}_q : L_a|_{\alpha}(n + 1) = L_a(n), \quad \forall \alpha' \neq \alpha : L_a|_{\alpha'}(n + 1) = n + 1 - L_a(n). \]

(In \( \mathbb{F}_2 \) obviously \( \forall \alpha' \neq \alpha \Leftrightarrow \alpha' = \alpha + 1 \))

Proof. (i) Since \( m_a(n) > 0 \), when running the algorithm from III.B, only the cases \( (b, m) = (0, \cdot) \) or \( (b, m) = (\neq 0, > 0) \) may appear. Hence \( m(n + 1) = m(n) - 1 \).

(ii) For exactly one choice \( \alpha \in \mathbb{F}_q \) as \( a_{n+1} = 0 \) we have \( b_{n+1} = 0 \), hence the case \( (b, m) = (0, \cdot) \) and \( m(n + 1) = m(n) - 1 \). All other \( \alpha' \neq \alpha \) lead to \( b_{n+1} \neq 0 \) and \( (b, m) = (\neq 0, \leq 0) \), thus \( m(n + 1) = 1 - m(n) \).

(iii), (iv) are equivalent to (i), (ii) by the definition of \( m_a \), see also [33] p. 34.

For the following theorem and again in Section V, we need a measure on \( \mathbb{F}_q^\infty \). We define the measure \( \mu \) on \( \mathbb{F}_q \) as equidistribution, \( \mu(a) = \frac{1}{q}, \quad \forall a \in \mathbb{F}_q \).

Taking the infinite product Haar measure of \( \mu \), we obtain the measure \( \mu^\infty \) on \( \mathbb{F}_q^\infty \). A set \( A = \{a \in \mathbb{F}_q^\infty \mid a_i = b_i, 1 \leq i \leq k\} \) for fixed \( b_i \in \mathbb{F}_q \) is called a cylinder set and it has measure \( \mu^\infty(A) = q^{-k} \). The set of ultimately periodic (rational) sequences in \( \mathbb{F}_q^\infty \) is countable and thus has measure zero.

Theorem 9 Translation Theorem

Let \( \alpha = (\alpha_1, \ldots, \alpha_k) \) and \( \beta = (\beta_1, \ldots, \beta_l) \) be two sequences from \( \mathbb{F}_q^* \) with \( m_\alpha(k) = m_\beta(l) \). Furthermore, let \( A = \alpha|_{\mathbb{F}_q^\infty} \) and \( B = \beta|_{\mathbb{F}_q^\infty} \) be the cylinder sets of the sequences in \( \mathbb{F}_q^\infty \) starting with \( \alpha \) or \( \beta \), resp. Then we have for all \( t \in \mathbb{N}_0 \) and for all \( d \in \mathbb{Z} \):
\[(i) \quad \{a \in \mathbb{F}_q^{k+t} \mid a_i = \alpha_i, \ i \leq k, \ m_\alpha(k+t) = d\} \]

\[= \{b \in \mathbb{F}_q^{l+t} \mid b_i = \beta_i, \ i \leq l, \ m_\beta(l+t) = d\} \]

\[(ii) \quad \frac{\mu^\infty(\{a \in A \mid m_\alpha(k+t) = d\})}{\mu^\infty(A)} = \frac{\mu^\infty(\{b \in B \mid m_\beta(l+t) = d\})}{\mu^\infty(B)}.\]

In other words: The distribution of deviations \(m\) after a given prefix depends only on the \(m\) at the end of that prefix; it does not depend on the length or the particular symbols of this prefix.

Proof.

\(i\) By induction on \(t\): For \(t = 0\) both sets contain exactly one element for \(d = m_\alpha(k)\) (which is \(\alpha\) resp. \(\beta\) itself) and for \(d \neq m_\alpha(k)\) both sets are empty. The step from \(t\) to \(t+1\) follows from Theorem 8(ii, iv).

\(ii\) This is just a measure theoretic reformulation of the result in \(i\). \(\square\)

Corollary 10 Let \(m_\alpha(2k) = 0\) for a sequence \(s\) of length \(2k\) (thus \(L_s(2k) = k\)). Then the distribution of linear complexity deviations on the cylinder set \(s|F^\infty_q\) equals that on \(F^\infty_q\).

Example Extending \(s = \varepsilon\) (empty sequence) and \(s = 10\), resp., by 3 bits:

| \(s_1\) | \(s_2\) | \(s_3\) | \(m(3)\) | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(m(5)\) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | -3 | 1 | 0 | 0 | 0 | 0 | -3 |
| 0 | 0 | 1 | 3 | 1 | 0 | 0 | 0 | 1 | 3 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | -1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | -1 |
| 1 | 1 | 1 | -1 | 1 | 0 | 1 | 1 | 1 | 1 |

B. Partition of \(K\) into \(K_D\) and \(K_C\)

We need a slight modification of \(m\) to be able to relate the positions in \(K\) to those in \(K_D, K_C\). Let

\[m'(n) = m(n-1) - 1, n > 0 \text{ and } m'(0) = 0\]

\(m'\) is equal to \(m\), except for the positions of the leading coefficients in \(K\), since here \(m(n) = d\), but \(m'(n) = -d\), with \(d\) the degree of the current partial denominator).

Theorem 11 Using \(m'\) we have the following connection between the linear complexity deviation and the distribution of \(K\) onto \(K_D\) and \(K_C\):

\[m'(n) < 0 : K(a)_n = K_D(a)_{(n-m'(n))/2}\]

\[m'(n) \geq 0 : K(a)_n = K_C(a)_{(n-m'(n))/2}\]
Proof. By induction on the codings \( \pi \): Let some \( \pi \) end at the (even) position \( 2k \). Then \( K_D \) and \( K_C \) both contain \( k \) of the \( 2k \) symbols \( K(a)_{i,i=1...2k} \) up to that position and \( K(a)_{2k} = K_C(a)_k \). Also, before the first coding we have \( k = 0 \) and \( K_D = K_C = \varepsilon \) empty up to now.

The next \( d \) symbols (\( d \) the degree of the next possibly incomplete next partial denominator or the length of the terminating zero run) lie in \( \pi_D \) and yield \( m'(n) = m'(2k + i) = -i < 0, \ 1 \leq i \leq d \), hence \( K(a)_{2k+i} = K_D(a)_{k+i} = K_D(a)_{2k+i} = K_D(a)_{(n-m'(n))/2} \) for \( 1 \leq i \leq d \).

For \( 2k+d < n \leq 2k+2d \), now follows the \( K_C \) part (\( K(a)_{2k+d} \) was a leading coefficient), and the positions \( n = 2k + d + i, 1 \leq i \leq d \) (as far as present) have \( m'(n) = m'(2k + d + i) = d - i \geq 0 \) and \( K(a)_{2k+d+i} = K_C(a)_{k+i} = K_C(a)_{2k+d+i-(d-i))/2} = K_C(a)_{(n-m'(n))/2} \) for \( 1 \leq i \leq d \). \( \square \)

C. \( K_D \) and the Linear Complexity Profile

Theorem 12 The distribution of \( m(a,t) \) on the set \( \mathbb{F}_q[t] \) of all prefixes of length \( t > 0 \) can be calculated via \( K \) and \( K_D \), by regarding \( \mathbb{F}_q[t] \) as discrepancy space.

(i) For even \( t > 0 \) there are \( (q-1) \cdot q^{t-1} \) sequences with \( m(t) = 0 \).

(ii) For \( 0 > m(t) \equiv t \mod 2 \) with \( |m(t)| \leq t \) there are \( (q-1) \cdot q^{t-1+m(t)} \) sequences with \( m(t) > -t \) and one sequence with \( m(t) = -t \).

(iii) For \( 0 < m(t) \equiv t \mod 2 \) with \( |m(t)| \leq t \) there are \( (q-1) \cdot q^{t-m(t)} \) sequences with \( m(t) = t \).

Proof. The set \( \mathbb{F}_q[t] \) of all sequence prefixes of length \( t \) yields also \( \mathbb{F}_q[t] \) as set of all discrepancy prefixes, since \( K \) is an isometry.

(i) Since \( m(t) = 0 \), both \( K_D \) and \( K_C \) have \( \frac{t}{2} \) symbols up to now, which (only) requires \( K_D(a)_{t/2} \neq 0 \) to be a leading coefficient. The other \( t - 1 \) symbols are arbitrary, hence we get \( (q-1) \cdot q^{t-1} \) possible sequences.

(ii) With \( m(t) < 0, \ m(t) = m'(t) \). For \( m(t) \neq -t, \ K(a)_{i,i=t-m(t)-1...t} = K_D(a)_{i,i=(t+m(t))/2...(t-m(t))/2} = lc(0^{m(t)}), \) for some \( lc \neq 0, \) thus \( (q-1) \cdot 1^{m(t)} \).

\( q^t \cdot |m(t)| \) cases, and for \( m(t) = -t, \ K(a)_{i,i=1...t} = K_D(a)_{i,i=1...t} = 0^t \) (1 case).

(iii) Since \( m(t) > 0 \), the last \( K_D \) has encoded a polynomial of degree \( m(t) \) or higher and thus ended in \( 0^{m(t)-1}\alpha, \alpha \neq 0 \). The initial part of \( K_D \) and all of \( K_C \) are irrelevant for this \( m(t) \), and we have \( 1^{m(t)-1} \cdot (q-1) \cdot q^{t-m(t)} \) possible sequences. \( \square \)

Corollary 13 (see also Gustavson [13]) Let \( N(t,m(t)) \) be the number of sequences with a given length \( t \) and value
Then for all \( t \in \mathbb{N}_0 \) and \( m(t) \) with \(-t \leq m(t) \leq t \) and \( m(t) \equiv t \pmod{2} \)

\[
N(t, m(t)) = \begin{cases} 
1, & m(t) = -t \\
(q - 1) \cdot q^{\frac{-|m(t) - \frac{t}{2}|}{2}}, & \text{otherwise}
\end{cases}
\]

The following theorem will finally be the foundation for Theorem 16 and the following section. We shall see that global statements about the behaviour of \( m, L \) and \( J \) are considered advantageously in the discrepancy space. For equidistributed \( a \) the resulting \( K_D(a) \) in the discrepancy space are also equidistributed by the following theorem (with respect to \( K(a) \) we have equidistribution anyway, since \( K \) is isometry).

Theorem 14

\[
\forall n \in \mathbb{N}_0, \forall b \in \mathbb{F}_q^n : \{ a \in \mathbb{F}_q^{2n} | K_D(a)_{i=1...n} = b \} = q^n
\]

The \( q^{2n} \) sequences \( a \) are thus equidistributed concerning the \( K_D \) part of their partial denominators.

Proof. Instead of \( a \in \mathbb{F}_q^{2n} \) we consider the sequence \( a' = K(a) \in \mathbb{F}_q^{2n} \) in the discrepancy space. Since \( K \) is an isometry, every \( a' \in \mathbb{F}_q^{2n} \) occurs exactly once for an \( a \) and the theorem is equivalent to

\[
\forall n \in \mathbb{N}_0, \forall b \in \mathbb{F}_q^n : \{ a' = K(a)_{i=1...2n} \in \mathbb{F}_q^{2n} | K_D(a)_{i=1...n} = b \} = q^n.
\]

We thus have to consider only the partition of \( K(a) \) into the parts \( K_D \) and \( K_C \). First of all we have \( |K_D(a)| \geq |K_C(a)| \), such that \( 2n \) symbols of \( K(a) \) include at least \( n \) symbols of \( K_D(a) \).

Now let \( b = K_D(a) \) be given. Then, by Theorem 11 the indices \( t_1, \ldots, t_n \) with \( K(a)_{t_i} = K_D(a)_i = b_i, 1 \leq i \leq n \) are fixed, and the other symbols \( K(a)_s, s \neq t_i, \forall i, \) can be chosen arbitrarily, that is in \( q^n \) ways, without affecting \( K_D(a)_{i=1...n} \). This proves the theorem. \( \square \)

Theorem 15

(i) \( K_D(a)_n \neq 0 \) if and only if the linear complexity profile assumes the value \( n \).

(ii) We can infer the whole linear complexity profile already from \( K_D \) alone (compare Wang [36, Th. 2.4]).

Proof.

(i) The linear complexity profile assumes the value \( n \), if one of the sums \( l_k := \sum_{i=1}^k |A_i| \) is equal to \( n \). In this case \( K_D \) starts with

\[
0^{|A_1|-1}lc_10^{|A_2|-1}lc_2 \ldots 0^{|A_k|-1}lc_k,
\]
where $lc_i = lc(A_i) \in \mathbb{F}_q \setminus \{0\}$ are the leading coefficients, and thus $K_D(a)_n = lc_k \neq 0$.

On the other hand, all indices $n$ with $l_k < n < l_{k+1}$ for some $k$ lead to $K_D(a)_n = 0$, since this element is in the $0^{l_{k+1}}$ part.

(iii) By (i) we know from $K_D$, which linear complexities $l_1 < l_2 < \ldots \in \mathbb{N}$ occur at all, and we put $l_0 = 0$.

At the end of each $\pi$, $m$ is zero, hence $\forall i \in \mathbb{N}_0 : m_{a}(2 \cdot l_i) = 0$ and $L_{a}(2 \cdot l_i) = l_i$. For those $n \in \mathbb{N}$ with $n \neq l_i$ for all $i \in \mathbb{N}_o$ there is a $k$ with $2 \cdot l_k < n < 2 \cdot l_{k+1}$ and then $L_{a}(n) = \{ l_k, k < l_k + l_{k+1}, k \geq l_k + l_{k+1} \}$ (where $K(a)_{l_k + l_{k+1}} = lc(A_{k+1})$).

In the next theorem we shall obtain the number of occurrences of values $m(t)$ by counting strings in $K_D$.

Theorem 16

(i) A linear complexity deviation $m_a = k \leq 0$ occurs wherever $K_D$ contains a string $\alpha 0^{k} | \alpha \neq 0$, or if $a$ begins with $0^{k}$.

(ii) A linear complexity deviation $m_a = k > 0$ occurs wherever $K_D$ contains a string $0^{k-1} \alpha, \alpha \neq 0$.

Proof.

(i) The case $a = 0^k \ldots$ is obvious. When $K_D(a)$ contains an $\alpha 0^{k}$, then $\alpha$ terminates a $\pi_D$ and $0^{k}$ leads to a complexity deviation $m = k \leq 0$.

On the other hand, by Theorem 9 this $m_a$ must result from a $\pi_D$, which begins with $0^{k}$, thus either $K_D = 0^{k} |$ (at the beginning of $a$) or $K_D$ contains the string $\alpha 0^{k}$.

(ii) A complexity deviation $m_a = k > 0$ occurs exactly after a jump to $k$ or higher, hence in a coding $\pi(A_i) = 0^{d-1}c_dc_{d-1} \ldots c_0$ with $d \geq k$, at $c_d$. □

D. Iterated Application of $K$

Pseudorandom sequences should avoid easily guessable patterns, for instance neither $a$, nor $K(a)$ should terminate in $0^{\infty}$. Since $K$ is an isometry, we can apply $K$ again on $K(a)$ to obtain $K^2(a)$ which ends in $0^{\infty}$ exactly for quadratic–algebraic $a$ (with rational $K(a)$) and also is far from random. In fact, for every exponent $k$, the sequence $K^k(a)$ should be well–behaved, that is look random.

Hence, defining $K^\infty$ as $(K^\infty)(a)_k := (K^k)(a)_k$, also $K^\infty$ should behave well.

Conjecture Whenever $K^i(a)$ is rational for some $i$, $K^\infty(a)$ is algebraic.

Then by a result of Christol et al. [4], $K^\infty(a)$ has finite tree complexity.
[29], which can serve as a means to determine this \(a\) as nonrandom.

E. Jump Complexity

We represent the jump complexity \(J_\alpha(t)\), as introduced by Carter [3] and Wang [35] via \(K\) by the number of nonzero symbols in \(K_D(a)\). The jump heights then will be lengths of zero runs in \(K_D(a)\). This prepares the global results on \(J, m\) and \(L\) in the next section. The Jump Complexity

\[ J_\alpha(n) := | \{ k \mid 1 \leq k \leq n \land L_\alpha(k-1) < L_\alpha(k) \} |, \quad n \in \mathbb{N}_0 \]

counts the number of jumps in the linear complexity profile of the sequence \((a_1, \ldots, a_n)\) (see Carter [3]).

**Theorem 17** Jump complexity and the discrepancy space

(i) The distribution of jump heights corresponds to the distribution of zero runs in \(K_D\): every zero run of length \(k-1, k \geq 1\) corresponds to a jump by \(k\).

(ii) The average distribution of jump heights on \(F_\infty^\infty\) can be modelled by the distribution of zero runs in \(K_D\). Hence a jump by \(k\) occurs with probability \(p(k) = (q-1)/q^k\).

(iii) \(J_\alpha(2n) = |\{ i \mid K_D(a)_i \neq 0, 1 \leq i \leq n \}| + \delta, \) with \(\delta = 1\) for \(m_\alpha(2n) > 0\) and \(\delta = 0\) otherwise.

(iv) We set

\[ J'_\alpha(2n) = \begin{cases} 
J_\alpha(2n), & \text{if } m_\alpha(2n) \leq 0 \\
J_\alpha(2n) - 1, & \text{if } m_\alpha(2n) > 0 
\end{cases} \]

Then \(|\{ a \in F_\infty^{2n} \mid J'_\alpha(2n) = j \}| = |\{ b \in F_\infty^n \mid b \text{ contains exactly } j \text{ nonzeroes} \}| = \binom{n}{j} \cdot (q-1)^j \cdot \binom{q}{j}.

Proof.

(i) Every jump by \(d\) corresponds to a \(\pi_D\) of \(0^{d-1}\alpha, \alpha \in F_\infty\{0\}\) that is a sequence \(\beta 0^{d-1}\alpha, \alpha, \beta \in F_\infty\{0\}\) in \(K_D\) (the first \(\pi_D\) is not preceded by a \(\beta\)).

(ii) This follows as a global statement from part (i), see also Selmer [31 VI,4] (Golomb’s Theorem).

(iii) Every jump produces exactly one nonzero symbol in \(K_D\). In the case \(m_\alpha(2n) > 0\), the nonzero corresponding to the last jump is \(K_D(a)_{n+m(2n)/2}\) after \(K_D(a)_{i,i=1\ldots n}\) and hence must be accounted for by adding \(\delta = 1\).

(iv) The statement follows from (iii) and Theorem 14 (see also [3] 5.3). □

**Theorem 18** (Carter [3] for \(F_2\), Niederreiter [25] for \(F_q, \forall q\))

The expectation for the jump complexity is
(i) for even \( t \)
\[
J(t) = \frac{t}{2} \cdot \frac{q-1}{q} + \frac{1}{q+1} - \frac{1}{(q+1) \cdot q^t},
\]

(ii) and for odd \( t \)
\[
J(t) = \frac{t}{2} \cdot \frac{q-1}{q} + \frac{q^2+1}{2q \cdot (q+1)} - \frac{1}{(q+1) \cdot q^t}.
\]

**Proof.** (i) The first \( t \) bits of \( K(a) \) contain at least \( \frac{t}{2} \) bits from \( K_D(a) \), of whose on average \( \frac{t}{2} \cdot \frac{q-1}{q} \) are nonzero and thus mark a jump. Furthermore, if \( m_a(t) > 0 \) there was another jump in \( K_D(a) \) after \( \frac{t}{2} \). Since \( t \) is even, we obtain with Corollary 13 that
\[
\sum_{m=2,4,6,...t} (q-1) \cdot q^{t-m} = (q-1) \cdot \sum_{m'=0}^{t/2-1} (q^2)^{m'} = (q-1) \cdot \frac{(q^2)^{t/2} - 1}{q^2 - 1} = \frac{q^t-1}{q+1}
\]
of all \( q^t \) sequences in \( \mathbb{F}_q^t \) will deliver one more jump, thus in total
\[
J(t) = \frac{t}{2} \cdot \frac{q-1}{q} + \frac{1}{q^t \cdot (q+1)} - \frac{1}{(q+1) \cdot q^t}
\]

(ii) Similar to (i) we get \( \frac{t+1}{2} \cdot \frac{q-1}{q} \) jumps on average and one more jump for positive odd \( m_a(t) \). With
\[
\sum_{m=3,5,7,...t} (q-1) \cdot q^{t-m} = (q-1) \cdot \sum_{m'=0}^{(t-3)/2} (q^2)^{m'} = (q-1) \cdot \frac{(q^2)^{(t-1)/2} - 1}{q^2 - 1} = \frac{q^{t-1}-1}{q+1}
\]
we thus obtain
\[
J(t) = \frac{t+1}{2} \cdot \frac{q-1}{q} + \frac{1}{q^{t-1} \cdot (q+1)} = \frac{t}{2} \cdot \frac{q-1}{q} + \frac{q-1}{2q} + \frac{1}{q(q+1)} \cdot \frac{1}{(q+1)q^t}
\]
\( \square \)

F. Recurrence Times, \( m \)-Pattern Frequencies

For \( k, l \in \mathbb{Z} \) let
\[
\Delta(k,l) = \sum_{\tau=1}^{\infty} \tau \cdot p \left( m(t+\tau) = l \mid m(t) = k \land m(t+1), \ldots, m(t+\tau-1) \neq l \right)
\]
denote the average recurrence time to go from \( m(t) = k \) to \( m(t + \tau) = l \) (by the Translation Theorem 9 the probabilities are independent of \( t \)).

**Theorem 19**

\[
\begin{align*}
(i) \quad \Delta(k, k) &= \frac{1}{q-1} \cdot 2 \cdot q^{k-\frac{1}{2}|l+\frac{1}{2}|}, \quad k \in \mathbb{Z} \\
(ii) \quad \Delta(k, l) &= k - l, \quad k > l \geq 0 \\
(iii) \quad \Delta(l, k) &= \Delta(k, k) - \Delta(k, l), \quad k > l \geq 0 \\
(iv) \quad \Delta(-k, 0) &= k + 2 \cdot q/(q - 1), \quad k \in \mathbb{N}_0 \\
(v) \quad \Delta(-k, l) &= \Delta(-k, 0) - l, \quad 0 < l \leq k \\
(vi) \quad \Delta(k, -k) &= 2 \cdot q \cdot (q^k - 1)/(q - 1) \quad k \in \mathbb{N} \\
(vii) \quad \Delta(k, -l) &= \Delta(l, -l) + k - l, \quad k, l \in \mathbb{N}_0 \\
(viii) \quad \Delta(-k, -l) &= \Delta(-l, -l) + k - l, \quad 0 < l \leq k \\
(ix) \quad \Delta(-k, l) &= \Delta(0, l) - \Delta(0, -k), \quad 0 < k < |l| \\
(x) \quad p(m(t) = k) &= \Delta(k, k)^{-1}
\end{align*}
\]

**Proof.**

(i) To obtain an \( m(t) = k > 0 \), a polynomial of degree \( d \geq k \) must occur, hence \( K_D \) must include a pattern \( 0^{k-1} \alpha, \alpha \neq 0 \) (see 16(ii)). This pattern has probability \( \frac{q-1}{q^k} \), its recurrence time in \( K_D \) is thus \( \frac{q^k}{q-1} \). Since an equally large part in \( K_C \) must be passed, we have \( \Delta(k, k) = 2q^k/(q - 1) \) for \( k \in \mathbb{N} \). Negative \( m(t) = -k \leq 0 \) similarly require the pattern \( \alpha 0^k \), whence \( \Delta(-k, -k) = 2q^{k+1}/(q - 1) \) for \( k \in \mathbb{N}_0 \).

(ii) From \( m(t) = k > 0 \) the first \( k - l \) symbols lead to \( m(t + k - l) = l \geq 0 \).

(iii) Every path from \( k \) to \( k \) passes \( l \), hence \( \Delta(k, k) = \Delta(k, l) + \Delta(l, k) \).

(iv) We jump after \( n \) steps with probability \( \frac{q-1}{q^n} \) and then reach \( m = 0 \) after a total of \( k + 2n \) steps. Hence \( \Delta(-k, 0) = \sum_{n=1}^{\infty} \frac{q-1}{q^n} \cdot (k + 2n) = k + 2q \).

(v) is a consequence of (iv) and (ii), since \( \Delta(-k, 0) = \Delta(-k, l) + \Delta(l, 0) \) for \( 1 \leq l \leq k \).

(vi) We have \( \Delta(k, -k) = k + \Delta(0, -k) \) and \( \Delta(-k, -k) = \Delta(-k, 0) + \Delta(0, -k) \), hence \( \Delta(k, -k) = k + \Delta(-k, -k) - \Delta(-k, 0) = k + \frac{2q^{k+1}}{q-1} - (k + \frac{2q}{q-1}) \).

(vii) For \( k \geq l \) we have to add \( k - l \) steps before \( \Delta(l, -l) \), for \( k < l \) the first \( l - k \) steps have to be omitted.

(viii) Since \( | -k| \geq | -l| \), the path leads through zero, hence \( \Delta(-k, -l) = \Delta(-k, 0) + \Delta(0, -l) \). The result now follows from (iv) and (vii).

(ix) To get from \( m = 0 \) to \( m = l \in \mathbb{Z} \), we have to pass the values \( m = -1, -2, \ldots, -|l| + 1 \), hence \( \Delta(0, l) = \Delta(0, -k) + \Delta(-k, l) \) für \( 1 \leq k < |l| \).

(x) This follows from

\[
p(m(t) = k) = \lim_{n \to \infty} \frac{1}{n} \cdot \{ t \mid m(t) = k, 1 \leq k \leq n \} = \Delta(k, k)^{-1} \quad \square
\]
Corollary 20  In the binary case the formulae can be considerably simplified. Remarkably, over \( \mathbb{F}_2 \) the value \( \Delta(k,l) \) is integral for all \( k,l \in \mathbb{Z} \).

(i) \( \Delta_2(k,k) = 2^{\lfloor k-\frac{1}{2} \rfloor + \frac{k}{2}} \), \( k \in \mathbb{Z} \)

(ii) \( \Delta_2(k,l) = k - l, \quad k > l \geq 0 \)

(iii) \( \Delta_2(l,k) = 2^{k+1} + k - l, \quad k > l \geq 0 \)

(iv) \( \Delta_2(-k,0) = k + 4, \quad k \in \mathbb{N}_0 \)

(v) \( \Delta_2(-k,l) = k - l + 4, \quad 0 < l \leq k \)

(vi) \( \Delta_2(k,-k) = 2^{k+2} - 4, \quad k \in \mathbb{N} \)

(vii) \( \Delta_2(k,-l) = 2^{l+2} + k - l - 4, \quad k,l \in \mathbb{N}_0 \)

(viii) \( \Delta_2(-k,-l) = 2^{l+2} + k - l, \quad 0 < l \leq k \)

(ix) \( \Delta_2(-k,l) = \Delta_2(0,l) - \Delta_2(0,-k), \quad 0 < k < |l| \)

Niederreiter defines in [25, Sect. 5] formulae for the average frequency of \( L(t) = \frac{t+c_0}{2} \wedge L(t+1) = \frac{t+c_1}{2} \) in the sequence \( L(i)_{i=1}^n \) for \( n \to \infty \). Using \( m \)-notation this corresponds to \( m(t) = c_0 \) and \( m(t+1) = c_1 - 1 \). We give a formula for arbitrary patterns \( m(t), m(t+1), \ldots, m(t+k-1) \).

Theorem 21  Let an \( m \)-pattern \( (m_0,m_1,\ldots,m_k) \) be given with \( m_{i+1} \in \{m_i - 1, 1 - m_i\} \) and \( m_i > 0 \Rightarrow m_{i+1} = m_i - 1 \). Let

\[ \#(1) := \{|m_i| m_i \leq 0 \wedge m_{i+1} = m_i - 1, 0 \leq i \leq k - 1\} \]

and

\[ \#(1) := \{|m_i| m_i \leq 0 \wedge m_{i+1} = 1 - m_i, 0 \leq i \leq k - 1\} \]

The probability of occurrence for the pattern \( (m_0,m_1,\ldots,m_k) \), that is

\[ \lim_{N \to \infty} \frac{1}{N} \cdot |\{t \mid 1 \leq t \leq N, m(t+j) = m_j, 0 \leq j \leq k\}|, \]

is

\[ p(m_0,m_1,\ldots,m_k) = \frac{1}{\Delta(m_0,m_0)} \cdot \frac{(q-1)^{\#(1)}}{q^{\#(1) + \#(1)}} \]

Proof.  The conditions make the pattern feasible (all other patterns have probability zero). The pattern occurs, whenever \( m_0 \) occurs with \( p(m_0) = \Delta(m_0,m_0)^{-1} \), and furthermore, if for \( m_i \leq 0 \) the linear complexity deviation jumps \#(1) times (with probability \( p = \frac{q-1}{q} \)) and is decremented \#(1) times (with probability \( p = \frac{1}{q} \)). \( \square \)

V. LÉVY CLASSES AND SHARP ASYMPTOTIC BOUNDS FOR J AND M

As we have seen, the linear complexity deviation and the jump complexity of a sequence \( a \) can be described immediately in terms of \( \mathbf{K}_D(a) \). Fur-
thermore, for each length \( t \) and any given sequence \( b \in \mathbb{F}_q^t \) the probability 
\( p(K_D(a), i=1 \ldots t = b) = q^{-t} \), hence equidistributed. We may thus convert theorems about the behaviour of Bernoulli sequences, like the Law of the Iterated Logarithm, directly into corollaries about the asymptotic behaviour of \( m_a(t) \) and \( J_a(t) \).

The theorems used here are compiled in Révész [31]. Since Révész only treats the case \( \mathbb{F}_2 \), we also refrain from utmost generality and consider only binary sequences, anyway the most important case from a practical point of view. An exception is the Law of the Iterated Logarithm for the jump complexity which we show for \( \mathbb{F}_q^\infty \), \( q \) an arbitrary prime power.

A. Lévy Classes

We shall use repeatedly the notation \( (\forall \mu \ a \in \mathbb{F}_q^\infty \ldots) \) or \( (\mu–almost all a \in \mathbb{F}_q^\infty \ldots) \) to imply that the statement \ldots is valid on a subset \( A \subseteq \mathbb{F}_q^\infty \) of measure \( \mu^\infty(A) = 1 \), hence false at most on a set of measure zero.

The functions \( m_a, L_a, J_a \) and similar are defined on \( \mathbb{N} \) (we ignore the value at zero). When we vary \( a \) on \( \mathbb{F}_q^\infty \), we obtain functions on \( \mathbb{F}_q^\infty \times \mathbb{N} \). Let us first examine the partition into an \( a \)-invariant part and the oscillation that depends on \( a \). Given a function \( f : \mathbb{F}_q^\infty \times \mathbb{N} \rightarrow \mathbb{R} \), we define its Lévy classes, four classes of functions \( (i.e. \ classes \ of \ real–valued \ sequences) \) that describe the asymptotic behaviour of \( f \) (upper and lower class, resp.):

\[
\begin{align*}
(i) \quad & UUC(f) = \{ \alpha \in \mathbb{R}^\mathbb{N} | \forall \mu \ a, \exists t_0 \in \mathbb{N}, \forall t > t_0 : f(a, t) < \alpha(t) \} \\
(ii) \quad & ULC(f) = \{ \alpha \in \mathbb{R}^\mathbb{N} | \forall \mu \ a, \forall t_0 \in \mathbb{N}, \exists t > t_0 : f(a, t) \geq \alpha(t) \} \\
(iii) \quad & LUC(f) = \{ \alpha \in \mathbb{R}^\mathbb{N} | \forall \mu \ a, \forall t_0 \in \mathbb{N}, \exists t > t_0 : f(a, t) \leq \alpha(t) \} \\
(iv) \quad & LLC(f) = \{ \alpha \in \mathbb{R}^\mathbb{N} | \forall \mu \ a, \exists t_0 \in \mathbb{N}, \forall t > t_0 : f(a, t) > \alpha(t) \}
\end{align*}
\]

Thus for all choices \( \alpha_1 \in LLC(f), \alpha_2 \in LUC(f), \alpha_3 \in ULC(f), \alpha_4 \in UUC(f) \) and for almost all sequences \( a \in A^\infty \), we have \( \alpha_1 < f(a) < \alpha_4 \) asymptotically, but \( \mu \)-almost all sequences will make \( f \) oscillate so much as to repeatedly leave the interval \( (\alpha_2, \alpha_3) \) of unavoidable oscillation.

In the sequel we will use the following typical examples for functions \( f \): maximum length of runs of zeroes (that is jump height, degree of partial denominators, deviation \( m \)) in \( K_D \), as well as deviations \( |J_a(t) - \frac{1}{2} \cdot \frac{q-1}{q}| \). For the case \( q = 2 \) there exist very precise estimates for the Lévy classes.

The model used by Révész is a discrete Brownian motion on \( \mathbb{Z} \): Let \( X_i \in \{-1, +1\} \) be random variables with \( p(X_i = +1) = p(X_i = -1) = \frac{1}{2} \). We

22
start at time $t = 0$ at zero. Given a sequence $b \in \mathbb{F}_2^\infty$, let

$$S_b(n) = \sum_{t=1}^{n} (2 \cdot b_t - 1) = -n + 2 \cdot \sum_{t=1}^{n} b_t, \ n \in \mathbb{N}_0$$

be the deviation from zero after $n$ moves, where $b_t = 0$ corresponds to $X_t = -1$, and $b_t = 1$ to $X_t = +1$.

B. Jump Complexity

The jump complexity counts the number of partial denominators in the encoding $K(a) = \pi(A_1)\pi(A_2)\ldots$, which is equivalent to the number of nonzeros in $K_D(a) = \pi_D(A_1)\pi_D(A_2)\ldots = 0\delta_{1}^{d_1-1}\delta_{2}^{d_2-1}\delta_{3}^{d_3-1}\ldots$ (using twice as many symbols in $K$ as in $K_D$). Hence we have the following model for $J$ in terms of $S$:

Theorem 22 For every sequence $a \in \mathbb{F}_2^\infty$, $b = K(a)$, and every length $2 \cdot t, t \in \mathbb{N}_0$ we have:

$$J_a(2 \cdot t) = \frac{S_b(t) + t}{2} + \delta, \ \delta = \begin{cases} 0, & m_a(2 \cdot t) \leq 0 \\ 1, & m_a(2 \cdot t) > 0 \end{cases}$$

Proof. In Theorem 17(iii) we saw $J_a(2 \cdot t) = \sum_{i=1}^{t} K_D(a)_i + \delta$ (we may just sum up instead of counting nonzeros since we work over $\mathbb{F}_2$). Now, identifying $K_D(a)$ with $(b_k)$, the theorem follows from $\frac{S_a(t)+t}{2} = \sum_{k=1}^{t} b_k$. □

Theorem 23 Law of the Iterated Logarithm for tossing a fair coin

$$f(t) \in \mathcal{UUC}(S_a(t)/\sqrt{t}) \iff \sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot e^{-\frac{f(n)^2}{2}} < \infty$$

$$f(t) \in \mathcal{ULC}(S_a(t)/\sqrt{t}) \iff \sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot e^{-\frac{f(n)^2}{2}} = \infty$$

$$f(t) \in \mathcal{LUC}(S_a(t)/\sqrt{t}) \iff -f(t) \in \mathcal{ULC}(S_a(t)/\sqrt{t})$$

$$f(t) \in \mathcal{LLC}(S_a(t)/\sqrt{t}) \iff -f(t) \in \mathcal{UUC}(S_a(t)/\sqrt{t})$$

Proof. The proof goes back to Erdős [11], Feller [12], and Kolmogoroff [16]. The theorem is [5.2] of Révész [31]. □

Some example functions bounding $S_a(t)/\sqrt{t}$ show that we can not avoid oscillations on the order of the “iterated logarithm” $\log \log(t)$.

Example For all $\varepsilon > 0$ we have:

$$(2 \cdot \log \log(t) + (3 + \varepsilon) \cdot \log \log \log t)^{1/2} \in \mathcal{UUC}(S_a(t)/\sqrt{t})$$

$$(2 \cdot \log \log(t) + (3 + \varepsilon) \cdot \log \log \log t)^{1/2} \in \mathcal{ULC}(S_a(t)/\sqrt{t})$$

$$-(2 \cdot \log \log(t) + (3 + \varepsilon) \cdot \log \log \log t)^{1/2} \in \mathcal{LUC}(S_a(t)/\sqrt{t})$$

$$-(2 \cdot \log \log(t) + (3 + \varepsilon) \cdot \log \log \log t)^{1/2} \in \mathcal{LLC}(S_a(t)/\sqrt{t})$$
From Theorem 23 we now infer a Law of the Iterated Logarithm for the Jump Complexity in the binary case:

**Theorem 24** For $\mu$-almost all $a \in \mathbb{F}_2^\infty$ with $b = K(a)$ the jump complexity $J_a(2t)$ observes:

\[
\begin{align*}
f(t) \in UUC(S_b(t)/\sqrt{t}) & \Rightarrow (\sqrt{t} \cdot f(t) + t)/2 + 1 \in UUC(J_a(2t)) \\
f(t) \in UL(C(S_b(t)/\sqrt{t}) & \Rightarrow (\sqrt{t} \cdot f(t) + t)/2 \in UL(C(J_a(2t)) \\
f(t) \in LUC(S_b(t)/\sqrt{t}) & \Rightarrow (\sqrt{t} \cdot f(t) + t)/2 + 1 \in LUC(J_a(2t)) \\
f(t) \in LLC(S_b(t)/\sqrt{t}) & \Rightarrow (\sqrt{t} \cdot f(t) + t)/2 \in LLC(J_a(2t))
\end{align*}
\]

In particular, the classes contain the following functions:

\[
\begin{align*}
t/4 + 1 + \sqrt{t} \cdot \sqrt{\log \log(t)/4 + (3/8 + \varepsilon) \cdot \log \log \log t} & \in UUC(J_a(t)) \\
t/4 + 1 - \sqrt{t} \cdot \sqrt{\log \log(t)/4 + 1/8 \cdot \log \log \log t} & \in UL(C(J_a(t)) \\
t/4 - \sqrt{t} \cdot \sqrt{\log \log(t)/4 + (3/8 + \varepsilon) \cdot \log \log \log t} & \in LUC(J_a(t)) \\
t/4 - \sqrt{t} \cdot \sqrt{\log \log(t)/4 + 1/8 \cdot \log \log \log t} & \in LLC(J_a(t))
\end{align*}
\]

**Proof.** By Theorem 22, we can replace $J_a$ by $\frac{S_b(t) + t}{2} + \delta$ and thus obtain with Theorem 23 and the above example, resp. the statements. See also the next theorem for arbitrary finite fields $\mathbb{F}_q$. \hfill $\square$

**Theorem 25** The Law of the Iterated Logarithm for the Jump Complexity over arbitrary finite fields $\mathbb{F}_q$

\[
(i) \lim_{n \to \infty} (J_a(n) - \frac{n}{2} \cdot \frac{q-1}{q}) / \sqrt{\frac{q-1}{q^2} n \cdot \log \log n} = +1 \quad \mu - \text{a.e.}
\]

\[
(ii) \lim_{n \to \infty} (J_a(n) - \frac{n}{2} \cdot \frac{q-1}{q}) / \sqrt{\frac{q-1}{q^2} n \cdot \log \log n} = -1 \quad \mu - \text{a.e.}
\]

**Proof.** Theorem 22 can be immediately generalized to all prime powers $q$. Thus we can apply the Law of the Iterated Logarithm for Bernoulli sequences with $p = \frac{1}{q}$ as proportion of zeroes (we do not discriminate between nonzero symbols) (see Feller [11]) to obtain (i) and (ii). \hfill $\square$

**C. Linear Complexity Deviation $m$**

We now consider the linear complexity deviation $m$. $m$ takes on its maximum values at a jump (except for rational $a$, but we can ignore the set $K^{-1}(S \cap \mathbb{F}_q[x])$ of measure zero). Hence the length of zero runs in $K_D(a)$ is of importance, since every $\pi_D$ consists of a zero run (of length $d - 1 \in \mathbb{N}_0$) and the leading coefficient that makes $m$ jump. We first define the length of the largest uninterrupted sequence of $-1$ in $(X_i)_{i=1,...,n}$ or of zeroes in $b_1, \ldots, b_n$, resp. as

\[
Z_b(n) := \max_{0 \leq r \leq n} \left\{ r = \max_{0 \leq k \leq n-r} \left( S_b(k) - S_b(k+r) \right) \right\}
\]
Theorem 26  Let $a \in F_2^\infty$ with $b = K_D(a)$ and $m_a(2n) = 0$, then:

$$\max_{t \leq 2n} |m_a(t)| = Z_b(n) + 1$$

Proof. The largest value $|m_a(t)|$ occurs after the longest zero run in $K_D$, which is of length $\max_{t \leq 2n} |m_a(t)| - 1$ and terminates with a leading coefficient, since $m_a(2n) = 0$. $Z_b$ just gives that longest run length, and at $t = 2n$, $K_D$ contains $n$ symbols. □

Theorem 27  Lévy classes for $Z(n)$

$$\sum_{n=1}^{\infty} 2^{-f(n)} < \infty \iff f(n) \in UUC(Z(n))$$

$$\sum_{n=1}^{\infty} 2^{-f(n)} = \infty \iff f(n) \in ULC(Z(n))$$

$f(n) = [\log_2(n) - \log_2 \log_2 \log_2(n) + \log_2 \log_2(e) - 1 + \varepsilon] \in LUC(Z(n)), \forall \varepsilon > 0$

$f(n) = [\log_2(n) - \log_2 \log_2 \log_2(n) + \log_2 \log_2(e) - 2 - \varepsilon] \in LLC(Z(n)), \forall \varepsilon > 0$

Proof. See Erdős and Révész [10], Révész [32]. □

Theorem 28  Lévy classes for $m^+(n) := \max_{t \leq n} |m(t)|$

$$\sum_{n=1}^{\infty} 2^{-f(n)} < \infty \iff f(n) \in UUC(m^+(n))$$

$$\sum_{n=1}^{\infty} 2^{-f(n)} = \infty \iff f(n) \in ULC(m^+(n))$$

$$f(n) = 1 + [\log_2\left(\frac{n}{2}\right) - \log_2 \log_2 \log_2\left(\frac{n}{2}\right) + \log_2 \log_2(e) + \varepsilon]$$

$$\in LUC(m^+(n)), \forall \varepsilon > 0$$

$$f(n) = 1 + [\log_2\left(\frac{n}{2}\right) - \log_2 \log_2 \log_2\left(\frac{n}{2}\right) + \log_2 \log_2(e) - 2 - \varepsilon]$$

$$\in LLC(m^+(n)), \forall \varepsilon > 0$$

Proof. The theorem follows from Theorems 26 and 27, where the convergence of the sums does not depend on the constant +1 or the fact that the function $f(n/2)$ is sampled twice as often (both supply only a factor $2^{\pm 1}$). The statements about $UUC$ and $ULC$ already appear as [22, Th. 8, 9]. □
Theorem 28 allows us to show that a fixed bound on \( m \) generally is too restrictive, but logarithmic growth is feasible for \( m \). We therefore define the notions of \emph{perfect} and \emph{good} profiles.

A sequence \( a \in \mathbb{F}_q^\infty \) is called \emph{\( d \)-perfect} for a \( d \in \mathbb{N} \) if \( \forall t \in \mathbb{N} : \ |m_a(t)| \leq d \).

A sequence \( a \) has a \emph{good linear complexity profile}, if

\[
\exists \ C \in \mathbb{R}, \forall n \in \mathbb{N} : \ |m_a(n)| \leq 1 + C \cdot \log n.
\]

**Corollary 29**

(i) There are \( \mu \)-almost no sequences with \( d \)-perfect linear complexity profile.

(ii) \( \mu \)-almost all sequences have a good linear complexity profile

**Proof.** See Niederreiter \cite{24} Th. 2.

**Remark** Theorems 27 and 30 together show: The largest linear complexity deviation ever occurring is of the order \( \log f(\lim_{t \to t_0} g(n)) = \infty \) for almost all sequences — however, for all sequences from a fixed \( n_0 \) onwards the linear complexity deviation is only of the order \( f(n) := \log_{2}(k) n \) for arbitrarily large \( k \).

**Theorem 31**

(i) For the lengths \( Z_2(n), Z_3(n), \ldots \) of the second, third … largest run of zeroes Deheuvels \cite{6} has found the following functions (where \( \log^{(j)} := \frac{1}{\log_2(\log^{(j-1)} x)} \) with \( \log_2(x) = 0 \) for \( x < 1 \), \( \log_2(x) = \log_2(x) \) for \( x \geq 1 \).

For all \( k \in \mathbb{N}, r \geq 2, \varepsilon > 0 \) we have:

\[
\begin{align*}
f(n) &= \log_2(n) + \frac{1}{k} \cdot (\log_2^{(2)}(n) + \ldots + \log_2^{(r-1)}(n) + (1 + \varepsilon) \log_2^{(r)}(n)) \in UUC(Z_k(n)) \\
f(n) &= \log_2(n) + \frac{1}{k} \cdot (\log_2^{(2)}(n) + \ldots + \log_2^{(r-1)}(n) + \log_2^{(r)}(n)) \in ULC(Z_k(n)) \\
f(n) &= \lfloor \log_2(n) - \log_2 \log_2 \log_2(n) + \log_2 \log_2(e) + \varepsilon \rfloor \in LUC(Z_k(n)) \\
f(n) &= \lfloor \log_2(n) - \log_2 \log_2 \log_2(n) + \log_2 \log_2(e) - 2 - \varepsilon \rfloor \in LLC(Z_k(n))
\end{align*}
\]
(ii) Hence for the second etc. largest degree $d_k(t)$ in the continued fraction expansion, we obtain:

$$f(t) = 1 + \log_2(\frac{t}{2}) + \frac{1}{k} \cdot (\log_2(\frac{t}{2}) + \ldots + \log_2^{(e-1)}(\frac{t}{2}) + (1 + \varepsilon) \log_2^{(e)}(\frac{t}{2})) \in UUC(d_k(t))$$

$$f(t) = 1 + \log_2(\frac{t}{2}) + \frac{1}{k} \cdot (\log_2(\frac{t}{2}) + \ldots + \log_2^{(e-1)}(\frac{t}{2}) + \log_2^{(e)}(\frac{t}{2})) \in ULC(d_k(t))$$

$$f(t) = 1 + [\log_2(\frac{t}{2}) - \log_2 \log_2 \log_2(\frac{t}{2}) + \log_2 \log_2(\varepsilon) + \varepsilon] \in LUC(d_k(t))$$

$$f(t) = 1 + [\log_2(\frac{t}{2}) - \log_2 \log_2 \log_2(\frac{t}{2}) + \log_2 \log_2(\varepsilon) - 2 - \varepsilon] \in LLC(d_k(t))$$

Proof. (i) See Deheuvels [6] and Révész [31, S. 61]. Observe that $LUC, LLC$ do not depend on $k$.

(ii) follows from (i) with Theorem 26, compare the proof to Theorem 28. □

VI. 2–adic Span and Complexity

Klapper and Goresky [14] [15] introduced another measure to assess the (non–)randomness of a bit string, the representation of $(a_1, a_2, a_3, \ldots)$ as a 2–adic integer $a = \sum_{i=1}^{\infty} a_i2^{-i-1} \in \mathbb{Z}_2$ and its approximations by rational numbers from $\mathbb{Q}$, as given in Mahler [18] and De Weger [7].

Klapper and Goresky view a sequence $(a_1, a_2, \ldots) \in \mathbb{F}_2^\infty$ as a more and more precise description of a 2–adic integer in $\mathbb{Z}_2$ (the base field now must be prime, not just a prime power, and we shall treat only $p = 2$). For every finite prefix $(a_1, \ldots, a_k)$, we obtain the number $a^{(k)} := \sum_{i=1}^{k} a_i2^{-i-1} \in \mathbb{Z} \subset \mathbb{Z}_2$. We are now interested in describing the number $a^{(k)}$ by a fraction $\frac{p_k}{q_k}$ with the condition $q_k \cdot a^{(k)} \equiv p_k \mod 2^k$.

We need some definitions:

For $k \in \mathbb{N}, a_1, \ldots, a_k \in \mathbb{F}_2$, let $a^{(k)} = \sum_{i=1}^{k} a_i2^{i-1} \in \mathbb{Z}$. We define the lattice (or $\mathbb{Z}$–module) $\mathcal{L}_a(k) = \{(p_k, q_k) \in \mathbb{Z}^2 \mid q_k \cdot a^{(k)} \equiv p_k \mod 2^k\}$ and set $\mathcal{L}_a'(k) = \mathcal{L}_a(k) \setminus \{(0, 0)\}$.

For $(p, q) \in \mathbb{Z}^2$, let $\Phi(p, q) = \max(|p|, |q|)$.

Let $(c_k, d_k)$ be a minimal pair from $\mathcal{L}_a(k)$ in the sense of $\Phi((c_k, d_k)) \leq \Phi((p_k, q_k))$ for all $(p_k, q_k) \in \mathcal{L}_a'(k)$.

We call the sequence $(c_k, d_k)$ the minimal approximating sequence of $a$.

The 2–adic complexity now is defined as $\phi_2(a, k) := \log_2(\Phi(c_k, d_k)) \in \mathbb{R}$
in [14]. However, we are interested here in an isometric model of 2–adic approximation. Thus we define $A: F_2^\infty \to F_2^\infty$ as

$$A(a_1, a_2, \ldots)_k = \begin{cases} 0, & \text{if } (c_k, d_k) = (c_{k-1}, d_{k-1}) \\ 1, & \text{if } (c_k, d_k) \neq (c_{k-1}, d_{k-1}) \end{cases}$$

for $k \in \mathbb{N}$, where we set $(c_0, d_0) := (0, 1)$.

**Theorem 32** $A$ is an isometry on $F_2^\infty$.

**Proof.** Let $a, b \in F_2^\infty$ with $a_i = b_i, 1 \leq i < k$ and $a_k = 1 - b_k$. Then $A(a)_i = A(b)_i$ for $1 \leq i < k$, since both sequences lead to the same sequence $(c_i, d_i)$ of rational approximations.

However, from $d_{k-1} \cdot \sum_{i=1}^{k-1} a_i 2^{i-1} \equiv c_{k-1} \mod 2^{k-1}$, we may infer $d_{k-1} \cdot \sum_{i=1}^{k-1} a_i 2^{i-1} + \delta \cdot 2^{k-1} \equiv c_{k-1} \mod 2^k$, for a $\delta \in \{0, 1\}$ and thus $d_{k-1} \cdot \sum_{i=1}^{k-1} a_i 2^{i-1} + (1 - \delta) \cdot 2^{k-1} \neq c_{k-1} \mod 2^k$.

Thus $A(a_1, \ldots, a_{k-1}, \delta, \ldots)_k = 0$ and $A(a_1, \ldots, a_{k-1}, 1 - \delta, \ldots)_k = 1$.

One of these corresponds to $A(a)_k$, the other to $A(b)_k$, hence $|A(a) - A(b)| = -k = |a - b|$ and $A$ is an isometry. □

We further define the 2–adic jump complexity $J_A$ to count the number of changes in the 2–adic complexity profile: $J_A(a)(n) := \sum_{i=1}^{n} A(a)_i$, which should behave like $J_A(n) \approx \frac{n}{2}$. In analogy to $m$, we thus define the 2–adic jump complexity deviation $m_A(n) := 2 \cdot J_A(n) - n \in \mathbb{Z}$.

**Theorem 33** $m_A(t)$ and $S_b(t)$ have the same average and asymptotic behaviour, precisely

$f \in LLC \ldots UUC(m_A(t)) \iff f \in LLC \ldots UUC(S_b(t))$

**Proof.** $J_A(t)$ and the sum $\sum_{i=1}^{t} b_i$ of the coin tossing sequence $b_t$ both represent the same behaviour, summing up a $p = \frac{1}{2}$ Bernoulli experiment. With $m_A(n) = 2 \cdot J_A(n) - n$ and $S_B(n) = 2 \cdot \sum_{i=1}^{n} b_i - n$ the result follows. □

We have thus

**Corollary 34** Law of the Iterated Logarithm for the 2–adic Jump Complexity Deviation

(i) $\lim_{n \to \infty} m_A(n) / (\sqrt{2n \cdot \log \log n}) = +1 \quad \mu - \text{a.e.}$

(ii) $\lim_{n \to \infty} m_A(n) / (\sqrt{2n \cdot \log \log n}) = -1 \quad \mu - \text{a.e.}$

**Remark** Whereas the original 2–adic complexity $\phi$ is not usually integral, $J_A$ and $m_A$ put us again in the world of coin tossing. However, in order to
compute \( A \), until now we have to iteratively calculate \( \Phi \) and derive \( A \) from the sequence of convergents.

**Open Problem**

Can we calculate \( A(a)_{i,i=1...n} \) with bit complexity \( O(n^2) \)?

Can we even calculate the shifted 2–adic profiles \( A(a_1...a_n), A(a_2...a_n) \) \( ... A(a_{n-1}...a_n) \) together in \( O(n^2) \) time?

**VII. Conclusion**

We have shown that both the expansion of formal power series into their continued fraction expansion and the approximation of 2–adic numbers by rationals induce an isometry on \( \mathbb{F}_2^\infty, K \) and \( A \), resp.

We modelled linear and jump complexity as well as 2–adic jump complexity via Bernoulli experiments, and applied the known general bounds to this model to derive sharp bounds (Lévy classes) for \( J, m, \) and \( J_A \).

We gave an adaptation of the Berlekamp–Massey–algorithm that implements the continued fraction expansion exactly as for the reals to obtain \( K \).

**Acknowledgement**

I want to heartfully thank Harald Niederreiter of the Austrian Academy of Sciences and National University of Singapore for introducing me to the subject and many stimulating and fruitful discussions, since accepting me as a PhD student a decade ago.

**References**

[1] E. Artin, “Quadratische Körper im Gebiete der höheren Kongruenzen. I.” *Math. Zeitschrift* **19** (1924), 153 – 206.

[2] E. R. Berlekamp, *Algebraic Coding Theory*, McGraw–Hill, NY, 1968.

[3] G. D. Carter, *Aspects of Local Randomness*, Ph. D. Thesis, Royal Holloway & Bedford New College, London, 1988.

[4] G. Christol, T. Kamae, M. Mendès–France, G. Rauzy, “Suites algébriques, automates et substitutions”, *Bull. Soc. Math. France* **108** (1980), 401 – 419.
[5] Z.D. Dai, K. C. Zeng, “Continued Fractions and the Berlekamp–Massey–Algorithm”, in: Advances in cryptology – AUSCRYPT ’90, (J. Seberry, J. Pieprzyk, Eds.), LNCS 453, 24 – 31, Springer, Berlin, 1990.

[6] P. Deheuvels, “On the Erdős–Rényi theorem for random fields and sequences and its relationship with the theory of runs and spacings”, Z. Wahrschein. verw. Gebiete 70 (1985), 91 – 115.

[7] B. M. M. de Weger, “Approximation lattices of $p$–adic integers”, J. Numb. Theory, 24 (1986), 70 – 88.

[8] J. L. Dornstetter, “On the equivalence between Berlekamp’s and Euclid’s algorithm” IEEE Trans. Inform. Th. 33(3) 1987, 428 – 431.

[9] P. Erdős, “On the law of the iterated logarithm”, Ann. of Math. (2) 43 (1942), 419 – 436.

[10] P. Erdős, P. Révész, “On the length of the longest head–run”, in: Colloq. Math. Soc. J. Bolyai, vol. 16, Topics in Information Theory, 219 – 228, Keszthely, 1975.

[11] W. Feller, An Introduction to Probability Theory and Its Applications I, 3rd Ed., John Wiley & Sons, New York, 1968.

[12] W. Feller, “The general form of the so–called law of the iterated logarithm”, Trans. Amer. Math. Soc. 51 (1943), 373 – 402.

[13] F. G. Gustavson, “Analysis of the Berlekamp–Massey linear feedback shift–register synthesis algorithm”, IBM J. Res. Develop. 20 (1976), 204 – 212.

[14] A. Klapper, M. Goresky, “2–adic shift registers”, in: Fast Software Encryption, Cambridge Security Workshop Proc., Springer, 1994.

[15] A. Klapper, M. Goresky, “Feedback Shift Registers, Combiners with Memory, and 2–adic Span”, J. Cryptology 10(2) (1997) 111–147.

[16] A. Kolmogoroff, “Über das Gesetz des iterierten Logarithmus”, Math.-Ann. 101 (1929), 126 – 135.

[17] R. Lidl, H. Niederreiter, Introduction to Finite Fields and their Applications, Cambridge University Press, 1986
[18] K. Mahler, “On a geometric representation of \( p \)-adic numbers”, Ann. Math., 41 (1940), 8 – 56.

[19] J. L. Massey, “Shift–Register Synthesis and BCH Decoding”, IEEE Trans. Inform. Th. IT–15 (1969), 122 – 127.

[20] B. de Mathan, “Approximations diophantiennes dans un corps local”, Bull. Soc. Math. France, Suppl. Mémoire 21 (1970).

[21] H. Niederreiter, “Sequences with almost perfect linear complexity profile”, in: Advances in cryptology – EUROCRYPT ’87, (D. Chaum, W.L. Price, Eds.), LNCS 304, Springer, Berlin, 1988.

[22] H. Niederreiter, “The probabilistic theory of linear complexity”, in: Advances in Cryptology – EUROCRYPT ’88 (C.G. Günther, Ed.), LNCS 330, 191 – 209, Springer, Berlin, 1988.

[23] H. Niederreiter, “Keystream sequences with a good linear complexity profile for every starting point”, in: Advances in cryptology – EUROCRYPT ’89, (J. J. Quisquater, J. Vandewalle, Eds.), LNCS 434, 523 – 532, Springer, Berlin, 1990.

[24] H. Niederreiter, “A Combinatorial Approach to Probabilistic Results on the Linear–Complexity Profile of Random Sequences”, in: Journal of Cryptology 2 (1990), 105 – 112.

[25] H. Niederreiter, “The Linear Complexity Profile and the Jump Complexity of Keystream Sequences”, in: Advances in Cryptology – EUROCRYPT ’90 (I. Damgård, Ed.), LNCS 473, 174 – 188, Springer, Berlin, 1991.

[26] H. Niederreiter, Random Number Generation and Quasi–Monte Carlo Methods, SIAM, Philadelphia, 1992.

[27] H. Niederreiter, M. Vielhaber “Simultaneous shifted continued fraction expansions in quadratic time”, Appl. Alg. Eng. Commun. Comp. 9, (2), 125 – 138, 1998.

[28] H. Niederreiter, M. Vielhaber, “An algorithm for shifted continued fraction expansions in parallel linear time”, Theor. Comp. Sc. 226, 93-104, 1999.

[29] H. Niederreiter, M. Vielhaber Tree complexity and a doubly exponential gap between structured and random sequences, J. of Complexity 12, 3, 187 – 198, 1996.
[30] O. Perron, *Die Lehre von den Kettenbrüchen I*, Teubner, Stuttgart, 1954/1977.

[31] P. Révész, *Random walk in random and non–random environments*, World Scientific, Singapore, 1990.

[32] P. Révész, “Strong theorems on coin tossing”, *Proc. Int. Cong. of Mathematicians*, 749 – 754, Helsinki, 1978.

[33] R.A. Rueppel, *Analysis and Design of Stream Ciphers*, Springer, Berlin, 1986.

[34] E. S. Selmer, *Linear recurrence relations over finite fields*, Dept. of Math., Univ. of Bergen, 1966.

[35] M. Wang, “Linear Complexity Profiles and Continued Fractions”, in: *Advances in cryptology – EUROCRYPT*’89, (J. J. Quisquater, J. Vandewalle, Eds.), LNCS 434, 571 – 585, Springer, Berlin, 1990.

[36] M. Wang, *Cryptographic Aspects of Sequence Complexity Measures*, ETH Diss. No. 8723, Zürich, 1988.

[37] L.R. Welch, R.A. Scholtz, “Continued Fractions and Berlekamp’s Algorithm”, *IEEE Trans. Inform. Th.* IT–25 (1979), 19 – 27.