Supergravity on the Brane

A. Chamblin** & G.W. Gibbons

DAMTP, Silver Street, Cambridge, CB3 9EW, England
(October 2, 2018)

We show that smooth domain wall spacetimes supported by a scalar field separating two anti-de-Sitter like regions admit a single graviton bound state. Our analysis yields a fully non-linear supergravity treatment of the Randall-Sundrum model. Our solutions describe a pp-wave propagating in the domain wall background spacetime. If the latter is BPS, our solutions retain some supersymmetry. Nevertheless, the Kaluza-Klein modes generate “pp curvature” singularities in the bulk located where the horizon of AdS would ordinarily be.

12.10.-g, 11.10.Kk, 11.25.M, 04.50.+h

I. INTRODUCTION

It has long been thought that any attempt to model the Universe as a single brane embedded in a higher-dimensional bulk spacetime must inevitably fail because the gravitational forces experienced by matter on the brane, being mediated by gravitons travelling in the bulk, are those appropriate to the higher dimensional spacetime rather than the lower dimensional brane. Recently however, Randall and Sundrum have argued that there are circumstances under which this need not be so. Their model involves a thin “distributional” static flat domain wall or three-brane separating two regions of five-dimensional anti-de-Sitter spacetime. They solve for the linearized graviton perturbations and find a square integrable bound state representing a gravitational wave confined to the domain wall. They also found the linearized bulk or “Kaluza-Klein” graviton modes. They argue that the latter decouple from the brane and make negligible contribution to the force between two sources in the brane, so that this force is due primarily to the bound state. In this way we get an inverse square law attraction rather than the inverse cube law one might naively have anticipated (see [3] for a related discussion).

This result is rather striking and raises various questions. For example one would like to know how general the effect is. Is it just an effect of the linearized perturbations or does it persist when non-linearities are taken into account? One would expect to get only one massless spin two bound state if the effective theory on the brane is to be general relativity. In their derivation a crucial role is played by a delta-function in the linearized graviton equation of motion. This is responsible for the unique bound state. It also seems that the effect will only work for domain walls and not for other branes. However the full dynamics of the domain wall is not treated in detail in the Randall-Sundrum model. In fact gravitating domain walls have a drastic effect on the curvature of the ambient spacetime and it is not obvious that a simple model involving a single collective coordinate representing the transverse displacement of the domain wall is valid.

For these reasons it seems desirable to have a simple non-singular model which is exactly solvable. It is the purpose of this note to provide that.

II. THICK DOMAIN WALLS IN ADS

We first seek a static domain wall solution of of the d-dimensional Einstein equations

$$R_{mn} - \frac{1}{2} g_{mn} = \partial_m \Phi \cdot \partial_n \Phi$$

$$-g_{mn} \left( \frac{1}{2} \partial_a \Phi \cdot \partial_b \Phi g^{ab} + V(\Phi) \right)$$

where $a, b = 0, 1, 2, \ldots, d-1$. The right hand side of (2.1) is the energy momentum tensor of one or more scalar fields $\Phi$ with potential $V(\Phi)$ whose kinetic energy term may contain a non-trivial metric on the scalar field manifold. The metric is assumed to be of the form:

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu \nu} dx^\mu dx^\nu,$$

where $\mu, \nu = 0, 1, 2, \ldots, d - 2$ and $\eta_{\mu \nu}$ is the flat Minkowski metric. The scalar field is assumed to depend only on the transverse coordinate $r$ and if $t$ denotes differentiation with respect to $r$ then the Einstein equations require

$$-\Phi'' \cdot \Phi' = (d-2) A'',$$

$$\left( \frac{1}{2} \Phi' \cdot \Phi' - V \right) = \frac{(d-2)(d-1)}{2} (A')^2.$$
These two equations imply the scalar field equation:

$$\Phi'' + (d-1)\Phi' A' = \frac{\partial V}{\partial \Phi},$$

(2.4)

If there is a non-trivial covariant metric on the scalar field manifold the right hand side of (2.4) includes the contravariant metric.

A domain wall solution separating two anti-de-Sitter domains with the same cosmological constant would have has $A \approx -|r|/a$ as $|r| \to \infty$.

If the potential $V$ has the special form

$$V = \frac{1}{2} \partial W \cdot \partial W - \frac{d-1}{d-2} W^2,$$

(2.5)

where $W = W(\Phi)$ is a suitable superpotential then Einstein equations (2.3) and the scalar equation (2.4) are solved by solutions of the first order Bogomol’nyi equations:

$$\Phi' = \frac{\partial W}{\partial \Phi}, \quad A' = \frac{1}{d-2} W,$$

(2.6)

Note that the spacetime is uniquely specified by giving a solution of (2.6) which is the same as the equation for a domain wall in the absence of gravity. One then obtains $A$ by quadratures. The vacua correspond to critical points of the superpotential $W$. At these points the potential $V$ is negative, and so one is in an anti-de-Sitter phase. Recently, there has been a lot of interest in the possibility of obtaining such potentials within the context of $d = 5$ gauged supergravity models (18, 19, 22, 23). At present no superpotential with the correct properties derived from a supergravity model has yet been found. However a solution was exhibited in 24 which is not derived from a supergravity model. We will return to this point in the last section. We will now show, without assuming that it is supersymmetric or satisfies the first order equations, how to superpose a smooth domain wall background with plane-fronted gravitational waves moving in the anti-de-Sitter background.

III. PP-WAVES ON THE BRANE: THE BOUND STATE

An exact solution of Einstein’s equations representing a gravitational wave moving at the speed of light in the $x^1$ direction is given by retaining the form $\Phi(r)$ and $A(r)$ but modifying the metric (2.2) to take the form:

$$ds^2 = dr^2 + e^{2A(r)}(-dudv + H(u, r, x^i_1)du^2 + dx^i_1 dx^i_1),$$

(3.1)

with $u = t - x^1$, $v = t + x^1$, $i = 2, \ldots, d - 3$ and where the $u$ dependence of $H$ is arbitrary but it’s dependence upon $r$ and $x^i_1$ is governed by $H'' + (d-1)H' A' + e^{-2A} \nabla^2 \Phi = 0,$

(3.2)

where $\nabla^2$ is the flat Laplace operator in the coordinates $x^i_1$. This will have half as much supersymmetry as the domain wall background. One may further generalize this solution by replacing the flat metric $dx^i_1 dx^j_1$ by an arbitrary $(d-3)$-dimensional Ricci flat metric $g_{ij}$. If $g_{ij}$ admits covariantly constant spinors, then the background will still admit some supersymmetry.

If $g_{ij}$ is flat space, solutions of (3.2) propagate in surfaces of constant $r$ at the speed of light in the (arbitrarily chosen) $x^1$ direction with an amplitude depending upon $r$. Fourier analyzing in the $x^1$ direction gives $H \propto e^{ik\cdot x^i_1}$, where $k$ could in principle depend upon $u$. If $k$ is real, solutions would propagate faster than light in a given $r =$constant surface, and would appear as tachyons to an observer on the brane. On the other hand, solutions for which $k$ is pure imaginary propagate on the brane like Kaluza-Klein modes. Thus, if $k^2 = -m^2$, i.e., $\nabla^2 H = m^2 H$, we are led to the equation

$$H'' + (d-1)H' A' + e^{-2A} m^2 H = 0.$$
IV. PP-WAVES IN THE BULK: BLUESHIFT AND CURVATURE SINGULARITIES

Our spacetimes are timelike and lightlike geodesically incomplete as \(|r| \to \infty\). In the absence of gravitational waves, i.e., \(H = 0\), \(r = \infty\) corresponds to a regular Cauchy horizon, and the solution may be extended through the horizon (see for example [17]). If \(H \neq 0\) however, the solutions will generically become singular as \(|r| \to \infty\), and will not admit an extension. The nature of this singularity is most easily studied when the background is taken to be exactly \(AdS_d\). If we let \(z = ae^{r/a}\) then the metric (3.1) can be recast in so-called ‘Siklos’ coordinates [15]:

\[
ds^2 = \frac{a^2}{z^2}(dz^2 - du dv + H du^2 + dx_1^2 dx_2^2),
\]

where \(H\) now satisfies the generalized Siklos equation

\[
z^{(d-2)} \frac{\partial}{\partial z} \left[ \frac{1}{z^{d-2}} \frac{\partial H}{\partial z} \right] + \nabla^2 H = 0.
\]

Because all invariants formed from the Weyl tensor of (3.3) necessarily vanish, it is not possible to detect curvature singularities directly by calculating invariants. However, the necessary condition that one may extend through the singularity in the metric at \(z = \infty\) is that the components of the Riemann tensor in an orthonormal frame which has been parallelly propagated along every timelike geodesic are finite. This requirement arises because freely falling observers move along timelike geodesics, and the components of the curvature tensor will measure the tidal forces which these observers experience. Following the demonstration in [15], one may calculate these terms explicitly for the Siklos metrics. One finds that certain frame components of the Riemann tensor generically assume the form

\[
R_{(a)(b)(a)(b)} = \frac{\Lambda}{d-1} z^{d-5} \frac{\partial H}{\partial z},
\]

where we have suppressed various constants which are irrelevant to this discussion. It follows that any solution with \(z\)-dependence cannot be extended, and hence is singular. One sees that the \(z\)-dependent piece of (4.2) is the contribution from the Weyl tensor. It would therefore seem that the gravitons will be heavily ‘blueshifted’ as we move towards large values of \(z\).

If \(\nabla_\perp^2 H = m^2 H\), the Siklos equation has solutions of the form

\[
H = z^{d-1} e^{k \cdot x_\perp} \left[ D_1 J_{d-1}(mz) + D_2 Y_{d-1}(mz) \right],
\]

where \(J_n(x)\) and \(Y_n(x)\) are Bessel functions, and \(D_1, D_2\) are some constants. The \(z\)-dependence of \(H\) has the same form as the Kaluza-Klein modes of (4.1). The behaviour near \(z = \infty\) shows that these are singular on the Cauchy horizon.

In order to get a better feel for the singular nature of these spacetimes, it is useful to focus on a specific example of a Siklos-type metric where the \(z\)-dependence is non-trivial. The simplest example is the higher dimensional generalization [18] of Kaigorodov’s spacetime [19], for which \(H\) is

\[
H(z) = z^{d-1}.
\]

The Kaigorodov metric is

\[
ds^2 = \frac{a^2}{z^2} (-d^2 - 2z^2 dt^2 - 2z^d dt dx_1 + (1 + z^d)(dx_1)^2 + dx_2^2 + dx_3^2).
\]

This is the \(AdS_d\) analogue of the simplest vacuum pp-wave, namely, the homogeneous pp-wave in flat space. It has \(d-1\) obvious translational Killing vectors, and is also invariant under the \(R^+\)-action:

\[
(z, u, v) \to (\lambda z, \lambda^3 u, \lambda^5 v).
\]

This action, combined with translations in \(u\) and \(v\), generates a three-dimensional group of Bianchi Type \(VJ_b\), where \(h = (d-1)^{-1}\). Therefore, the Kaigorodov isometry group contains a simply transitive subgroup which takes every point with \(z\) positive to any other point with \(z\) positive. A similar \(d\)-dimensional simply transitive group exists in the \(AdS_d\) case, for which the \(R^+\) action is simply \(z \to \lambda z\). In the \(AdS_d\) case, we can extend beyond the reach of the group, in the Kaigorodov case we cannot.

Clearly, freely falling timelike observers (who can cross the surface \(z = \infty\) after a finite period of affine parameter time [15]) will see infinite tidal forces in this region. This shows that there are naked curvature singularities at the points \(z = \infty\). Given our discussion in the previous section, where we saw that generic \(z\)-dependent graviton perturbations will diverge at large \(z\), it is clear that we should regard these singularities as a generic feature of Siklos spacetimes.

V. DISCUSSION

We have shown that it is possible to include a non-linear gravitational wave on a thick domain wall background, in such a way that one may recover the Randall-Sundrum bound state. Given the formal Witten style stability proofs in [8], which work as long as one has a solution of the first order equations, one might have thought that this would ensure that the Randall-Sundrum scenario could be perturbed in this way without problems. However somewhat to our surprise, we have found that generically gravitons propagating in the bulk become singular on what is a Cauchy horizon in the unperturbed spacetime. These singularities are somewhat unusual, in
that scalar invariants formed from the curvature tensor do not blow up but rather the components of the curvature in a parallelly propagated frame along a timelike geodesic do blow up. Such singularities are called “pp curvature singularities” [15].

One might worry that these singularities signal a breakdown in our ability to make unitary predictions. However, any statements about unitarity should be restricted to physics on the brane at $z = \text{constant}$. Any pathological effects which may emerge from the singularity will be heavily red-shifted by the time they reach the brane. Consequently, the extent to which these singularities signal a pathology of the theory is at present unclear.

Interestingly, if one considers massless $z$-independent pp-waves (these would correspond to the Randall-Sundrum zero mode bound state), one finds that the components of the curvature do not blow up, and presumably the spacetime has a non-singular extension.

To conclude we would like to return to the question of whether a suitable super-potential exists which can be derived from a supergravity model. The results of [5] and [7] show that for the simplest case of a single scalar field in models of the type studied in [13] they do not. In fact one may show quite generally that for the models in [13] with an arbitrary number scalar fields they do not. The same is true for the models considered in [8]. It therefore remains an important open problem to find a suitable supergravity model or prove that no such model exists.

The authors thank M. Bañados, J. Harvey, S. Hawking, N. Lambert, R. Myers, M. Porrati, H. Reall and S. Siklos for useful conversations and correspondence. A.C. was supported by Pembroke College, Cambridge.

References:

[1] Lisa Randall and Raman Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, hep-ph/9905221; Joseph Lykken and Lisa Randall, *The Shape of Gravity*, hep-th/9908079.

[2] Lisa Randall and Raman Sundrum, *An Alternative to Compactification*, hep-th/9906064.

[3] M. Gogberashvili, *Gravitational Trapping for Extended Extra Dimension*, hep-ph/9908347.

[4] O DeWolfe, D Z Freedman, S.S. Gubser and A Karch, *Modelling the fifth dimension with scalars and gravity*, hep-th/9909134.

[5] Klaus Behrndt and Mirjam Cvetic, *Supersymmetric Domain-Wall World from D=5 Simple Gauged Supergravity*, hep-th/9909058.

[6] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization Group Flows from Holography–Supersymmetry and a c-Theorem*, hep-th/9904017.

[7] R. Kallosh, A. Linde and M. Shmakova, *Supersymmetric Multiple Basin Attractors*, hep-th/9910021.

[8] L. Girardello, M. Petrini, M. Porrati and A Zaffaroni, *The Supergravity Dual of N=1 Super Yang-Mills theory*, hep-th/9909047.

[9] S.T.C. Siklos, in: Galaxies, axisymmetric systems and relativity, ed. M.A.H. MacCallum, Cambridge University Press, Cambridge (1985).

[10] G.W. Gibbons and P.J. Ruback, *Classical gravitons and their stability in higher dimensions*, Phys. Lett. 171B, 390-395, (1986).

[11] G.W. Gibbons Vacuua and Solitons in Extended Supergravity, in: Relativity, Cosmology, Topological Mass and Supergravity, ed. C. Aragone, pages 163 - 177, World Scientific (1983).

[12] K. Skenderis and P.K. Townsend, *Gravitational Stability and Renormalization-Group Flow*, hep-th/9909070.

[13] M. Gunaydin, G. Sierra, and P.K. Townsend, *More on D=5 Maxwell-Einstein supergravity: Symmetric spaces and kinks*, Class. Quant. Grav. 3: 763, (1986).

[14] G.W. Gibbons, *Quantized fields propagating in plane-wave spacetimes*, Comm. Math. Phys. 45, 191-202 (1975).

[15] J. Podolsky, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe*, Class. Quant. Grav. 15, 719-733, (1998); gr-qc/9801052.

[16] J. Bicak and J. Podolsky, *Gravitational waves in vacuum spacetimes with cosmological constant. II. Derivation of geodesics and interpretation of non-twisting type N solutions*, gr-qc/9907049.

[17] G.W. Gibbons, *Global structure of supergravity domain walls spacetimes*, Nucl. Phys. B394, 3 (1993).

[18] M. Cvetic, H. Lu and C.N. Pope, *Spacetimes of Boosted p-branes, and CFT in Infinite-momentum Frame*, Nucl. Phys. B545, 309-339, (1999); hep-th/9810123.

[19] V.R. Kaigorodov, Sov. Phys. Doklady. 7, 803 (1963).

[20] J. Ehlers and W. Kundt, *Gravitation: An Introduction to Current Research*, ed. L. Witten (New York: Wiley) (1962).