THE DIRECT IMAGE OF
A FLAT FIBRATION WITH COMPLEX FIBERS

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ABSTRACT. We consider a proper flat fibration with real base and complex fibers. First we construct odd characteristic classes for such fibrations by a method that generalizes constructions of Bismut-Lott [BL95]. Then we consider the direct image of a fiberwise holomorphic vector bundle, which is a flat vector bundle on the base. We give a Riemann-Roch-Grothendieck theorem calculating the odd real characteristic classes of this flat vector bundle.

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0. Introduction

We consider a compact real manifold $X$ equipped with a flat vector bundle $F$. We equip $X$ with a Riemannian metric $g^{TX}$. We equip $F$ with a Hermitian metric $g^F$. The real analytic torsion \cite{RS71} is a spectral invariant of the Hodge Laplacian associated with the de Rham complex $(\Omega^\cdot (X, F), d^F)$. Let $\det H^\cdot (X, F)$ be the determinant line of the de Rham cohomology $H^\cdot (X, F)$. The Ray-Singer metric is the product of the real analytic torsion and of the $L^2$-metric on $\det H^\cdot (X, F)$.

Cheeger \cite{Ch79} and Müller \cite{Mu78} proved independently that, if $g^F$ is flat, then the Ray-Singer metric is independent of $g^{TX}$. Müller \cite{Mu93} also extended this result to the unimodular case, i.e., the induced metric on $\det F$ is flat. In the general case, the dependence of the Ray-Singer metric on the metrics was calculated by Bismut-Zhang \cite{BZ92}. They also established an extension of the Cheeger-Müller theorem in the general case.

Now let $\pi : M \to S$ be a real smooth fibration with compact fiber $X$. Let $F$ be a complex flat vector bundle over $M$. Bismut and Lott \cite{BL95} established Riemann-Roch-Grothendieck formulas, which calculate the odd Chern classes of $R^\pi_* F$ in terms of the Euler class of the relative tangent bundle $TX$ and the corresponding odd Chern classes of $F$. When equipping the considered vector bundles with metrics, these classes can be represented by explicit differential forms. By transgressing the equality of cohomology classes at the level of differential forms, they also obtained even analytic torsion forms on $S$, whose coboundary is equal to the difference between the differential forms appearing on the left and right hand side of the R.R.G. formula.

In complex geometry, the objects parallel to the real analytic torsion and the Ray-Singer metric are known as the holomorphic analytic torsion and the Quillen metric, which were also introduced by Ray-Singer \cite{RS73}. These notions were extended to holomorphic fibrations by Bismut-Gillet-Soulé \cite{BGS88a} and Bismut-Köhler \cite{BK92}.

In this paper, we consider a flat fibration $q : N \to M$ with complex fiber $N$. We equip $N$ with a complex vector bundle $E$, which is holomorphic along $N$ and flat along horizontal directions in $N$. Then $R^\pi_* E$ is a flat vector bundle over $M$. We give a R.R.G. formula for the odd Chern classes of $R^\pi_* E$ in terms of the Todd class of the relative tangent bundle and of the Chern classes of $E$. Moreover, by equipping the various vector bundles with Hermitian metrics, we construct even analytic torsion forms on $M$, which transgress the equality of the corresponding cohomology classes. The results contained in this paper were announced in \cite{Zh16}.

Let us now give more detail on the content of this paper.

0.1. Chern-Weil theory and its extensions.

Let $M$ be a smooth manifold. Let $E$ be a complex vector bundle of rank $r$ over $M$. Let $\nabla^E$ be a connection on $E$. Let $P$ be an invariant polynomial on $\mathfrak{gl}(r, \mathbb{C})$. Chern-Weil theory assigns a closed differential form

\begin{equation}
P(E, \nabla^E) \in \Omega^\text{even}(M).
\end{equation}

Its cohomology class $[P(E, \nabla^E)] \in H^\text{even}(M)$ is independent of $\nabla^E$, and will be denoted by $P(E)$. This theory will be referred to as the even Chern-Weil theory. If $\nabla^E$ is a flat connection, i.e., $\nabla^E,^2 = 0$, $P(E, \nabla^E)$ is just a constant function on $M$. 


A Chern-Weil theory for flat vector bundles was developed by Bismut-Lott [BL95, §1]. Let \((E, \nabla^E)\) be a flat complex vector bundle over \(M\). Let \(g^E\) be a Hermitian metric on \(E\). Let \(f\) be an odd polynomial. Bismut and Lott assigned a closed differential form
\[
(0.2) \quad f(E, \nabla^E, g^E) \in \Omega^\text{odd}(M).
\]
Its cohomology class \([f(E, \nabla^E, g^E)]\) is independent of \(g^E\), and will be denoted by \(f(E, \nabla^E)\). This theory will be referred to as the odd Chern-Weil theory.

In this paper, we will construct characteristic classes for flat fibrations with complex fibers. Our construction is a mixture of the even and odd Chern-Weil theory.

Now we construct our flat fibration. Let \(G\) be a Lie group. Let \(p : P_G \to M\) be a flat \(G\)-principal bundle. Let \(N\) be a compact complex manifold. We assume that \(G\) acts holomorphically on \(N\). Set
\[
(0.3) \quad \mathcal{N} = P_G \times_G N.
\]
Let
\[
(0.4) \quad q : \mathcal{N} \to M
\]
be the canonical projection. Then \(q\) defines a flat fibration with canonical fiber \(N\). Let \(E_0\) be a holomorphic vector bundle over \(N\). We assume that the action of \(G\) lifts holomorphically to \(E_0\). Set
\[
(0.5) \quad E = P_G \times_G E_0.
\]
Then \(E\) is a complex vector bundle over \(\mathcal{N}\).

In §2 we construct odd characteristic forms as follows. We denote
\[
(0.6) \quad \Omega^\text{odd}(M) = \mathcal{C}^{\infty}(M, \Lambda^\text{odd}(T^*M)), \quad \Omega^\text{odd}(N, E) = \mathcal{C}^{\infty}(\mathcal{N}, \Lambda^\text{odd}(T^*\mathcal{N}) \otimes E).
\]
Let \(d_M\) be the de Rham operator on \(\Omega^\text{odd}(M)\). Let \(d^E_M\) be the lift of \(d_M\) to \(\Omega^\text{odd}(N, E)\). Let \(g^E\) be a Hermitian metric on \(E\). Set
\[
(0.7) \quad \omega^E = (g^E)^{-1}d^E_M g^E \in \Omega^1(\mathcal{N}, \text{End}(E)).
\]
Let \(\nabla^E_N\) be the fiberwise Chern connection on \((E, g^E)\). Let \(A^E\) be the unitary connection on \(E\) defined by
\[
(0.8) \quad A^E = \nabla^E_N + d^E_M + \frac{1}{2} \omega^E.
\]
Let \(r\) be the rank of \(E\). Let \(\mathfrak{gl}(r, \mathbb{C})\) be the Lie algebra of \(GL(r, \mathbb{C})\). Let \(P\) be an invariant polynomial on \(\mathfrak{gl}(r, \mathbb{C})\). For \(a, b \in \mathfrak{gl}(r, \mathbb{C})\), we use the following notation
\[
(0.9) \quad \langle P'(a), b \rangle = \frac{\partial}{\partial t}P(a + tb)_{t=0}.
\]
Let \(N^\Lambda(T^*\mathcal{N})\) be the number operator of \(\Lambda^\text{odd}(T^*\mathcal{N})\), i.e., for \(\alpha \in \Lambda^k(T^*\mathcal{N})\), \(N^\Lambda(T^*\mathcal{N})\alpha = k\alpha\). Put
\[
(0.10) \quad P(E, g^E) = (2\pi i)^{-\frac{1}{2}N^\Lambda(T^*\mathcal{N})} P(-A^E, 2) \in \Omega^\text{even}(\mathcal{N}),
\]
\[
\tilde{P}(E, g^E) = (2\pi i)^{\frac{1}{2} - \frac{1}{2}N^\Lambda(T^*\mathcal{N})} \langle P(-A^E, 2), \omega^E \rangle \in \Omega^\text{odd}(\mathcal{N}).
\]
Theorem 0.1. The differential form
\[(0.11) \quad q_\ast [P(E, g^E)] \in \Omega^{\text{even}}(M)\]
is a constant function.

The differential form
\[(0.12) \quad q_\ast [\tilde{P}(E, g^E)] \in \Omega^{\text{odd}}(M)\]
is closed. Its cohomology class
\[(0.13) \quad [q_\ast [\tilde{P}(E, g^E)]] \in H^{\text{odd}}(M)\]
is independent of \(g^E\).

In the sequel, we use the notation
\[(0.14) \quad q_\ast [\tilde{P}(E)] = [q_\ast [\tilde{P}(E, g^E)]] \in H^{\text{odd}}(M).\]

Now let \(F\) be another vector bundle (of rank \(r'\)) over \(N\) satisfying the same properties as \(E\). Let \(g^F\) be a Hermitian metric on \(F\). Let \(Q\) be an invariant polynomial on \(\mathfrak{gl}(r', \mathbb{C})\). The natural product on the forms \(\tilde{P}(E, g^E)\) and \(\tilde{Q}(F, g^F)\) is given by
\[(0.15) \quad \tilde{P}(E, g^E) \ast \tilde{Q}(F, g^F) = \tilde{P}(E, g^E)Q(F, g^F) + Q(E, g^E)\tilde{Q}(F, g^F).\]

0.2. A R.R.G. theorem for flat fibrations with complex fibers.

In the rest of the introduction, we suppose that \(N\) is a Kähler manifold. Let \(H^*(N, E)\) be the fiberwise Dolbeault cohomology group of \(E\) along \(N\). Then \(H^*(N, E)\) is a graded flat vector bundle over \(M\). Let \(\nabla^{H^*(N, E)}\) be its flat connection. Set \(f(x) = x \exp(x^2)\). Let
\[(0.16) \quad f(H^*(N, E), \nabla^{H^*(N, E)}) \in H^{\text{odd}}(M, \mathbb{R})\]
be the Bismut-Lott odd characteristic class [BL95, §1].

Theorem 0.2. We have
\[(0.17) \quad f(H^*(N, E), \nabla^{H^*(N, E)}) = q_\ast [\tilde{Td}(TN) \ast \tilde{ch}(E)] \in H^{\text{odd}}(M, \mathbb{R}).\]

Here \(\tilde{Td}(TN) \ast \tilde{ch}(E)\) is defined by \((0.10)\) and \((0.15)\).

Now we explain the idea of the proof. We will use the superconnection formalism [BL95, §2]. Put
\[(0.18) \quad \mathcal{E} = \mathcal{C}^\infty(N, \Lambda(T^*N) \otimes E),\]
which is an infinite dimensional flat vector bundle over \(M\). Let \(d^E_M\) be its flat connection. Let \(\overline{\partial}^E_N\) be the Dolbeault operator on \(\mathcal{E}\). Set
\[(0.19) \quad A^E = \overline{\partial}^E_N + d^E_M,\]
which acts on \(\Omega(M, \mathcal{E})\). Here \(A^E\) is a flat superconnection on \(\mathcal{E}\) in the sense of Bismut-Lott [BL95, Definition 1.1].

Let \(g^{TN}\) be a fiberwise Kähler metric on \(TN\). Let \(g^E\) be a Hermitian metric on \(E\). Let \(g^E\) be the induced \(L^2\)-metric on \(\mathcal{E}\). Let \(A^{E,*}\) be the adjoint superconnection of \(A^E\) in the sense of Bismut-Lott [BL95, Definition 1.6].
Let $N^\Lambda(T^*M)$ be the number operator of $\Lambda^r(T^*M)$. Set

$$D^\phi = 2^{-N^\Lambda(T^*M)}(A^\phi, - A^\phi)2^{N^\Lambda(T^*M)} \in \Omega(M, \text{End}(\phi)) \ .$$

For $t > 0$, let $D_t^\phi$ be the operator $D^\phi$ associated with the rescaled metric $\frac{1}{t}g^TN$. Following Bismut-Lott [BL95, (2.22), (2.23)], we define

$$\alpha_t = (2\pi i)\frac{1}{2} - \frac{1}{2}N^\Lambda(T^*M) \text{Tr}_s \left[ D_t^\phi \exp(D_t^\phi)^2 \right],$$

$$\beta_t = (2\pi i)^{-\frac{1}{2}}N^\Lambda(T^*M) \text{Tr}_s \left[ \frac{N^\Lambda(T^*N)}{2}(1 + 2D_t^\phi)^2 \exp(D_t^\phi)^2 \right].$$

We have

$$d_M\alpha_t = 0, \quad \frac{\partial}{\partial t} \alpha_t = \frac{1}{t}d_M\beta_t .$$

Let $g^{H(N,E)}$ be the metric on $H(N,E)$ induced by the $L^2$-metric on $\mathcal{E}$ via the Hodge theorem. Let

$$f(H(N,E), \nabla^{H(N,E)}, g^{H(N,E)}) \in \Omega^{\text{odd}}(M)$$

be the Bismut-Lott odd characteristic form [BL95, Definition 1.7].

**Theorem 0.3.** As $t \to \infty$,

$$\alpha_t = f(H(N,E), \nabla^{H(N,E)}, g^{H(N,E)}) + \mathcal{O}(\frac{1}{\sqrt{t}}) .$$

As $t \to 0$,

$$\alpha_t = q_* \left[ \tilde{\text{Td}}(TN, g^TN) * \tilde{\text{ch}}(E, g^E) \right] + \frac{\text{a fixed exact form}}{t} + \mathcal{O}(\sqrt{t}) .$$

**0.3. Analytic torsion forms.**

In the same way as in (0.24) and (0.25), we also obtain an asymptotic estimate for $\beta_t$ as $t \to \infty$ and $t \to 0$. We construct explicitly an analytic torsion form

$$\mathcal{T}(g^{TN}, g^E) \in \Omega^{\text{even}}(M) ,$$

which is defined by subtracting the singularities of the following integral

$$- \int_0^\infty \beta_t \frac{dt}{t} .$$

By (0.22), (0.24) and (0.25), we have

$$d_M\mathcal{T}(g^{TN}, g^E) = \frac{\text{a fixed exact form}}{t} + \mathcal{O}(\sqrt{t}) .$$

Moreover, we show that the degree zero component of $\mathcal{T}(g^{TN}, g^E)$ is the Ray-Singer holomorphic torsion [RS73] associated with $(N, g^{TN}, E, g^E)$.

This paper is organized as follows.

In §1 we recall several standard constructions and known results. Most of them can be found in [BeGV04] and [BL95, §1].

In §2 we construct characteristic classes for flat fibrations and prove Theorem 0.1.
In §3 we prove Theorem 0.3. As a consequence, we establish Theorem 0.2. We also construct the analytic torsion form $\mathcal{T}(g^{TN}, g^E)$.

**Acknowledgment**

This paper is part of the author’s PhD thesis. The author would like to thank his advisor Professor Jean-Michel Bismut for his guidance. The research leading to the results contained in this paper has received funding from the European Research Council (E.R.C.) under European Union’s Seventh Framework Program (FP7/2007-2013)/ERC grant agreement No. 291060.

1. **Preliminaries**

This section is organized as follows.

In §1.1 we introduce the superalgebra formalism.

In §1.2 we introduce the Clifford algebra.

In §1.3 we introduce the Chern-Weil theory.

In §1.4 we introduce several objects associated with a smooth fibration.

The constructions and results contained in this section can be found in [BeGV04, §1], [B86, §1], [BL95, §1].

1.1. **Superalgebras.**

In the sequel, all the algebras will be over $\mathbb{R}$ or $\mathbb{C}$.

**Definition 1.1.** A superalgebra is an algebra $A$ equipped with a $\mathbb{Z}_2$-grading $A = A^+ \oplus A^-$ such that

\begin{equation}
A^+ A^\pm \subseteq A^\pm, \quad A^- A^\pm \subseteq A^\mp.
\end{equation}

Let $A$ be a superalgebra. An element $a \in A$ is said to be homogeneous if $a \in A^\pm$. We denote $\deg a = 0$ (resp. $\deg a = 1$) if $a \in A^+$ (resp. $a \in A^-$).

The supercommutator of two homogeneous elements $a, b \in A$ is defined by

\begin{equation}
[a, b] = ab - (-1)^{\deg a \deg b} ba.
\end{equation}

Also $[\cdot, \cdot]$ extends by linearity to the whole superalgebra $A$.

**Definition 1.2.** Let $A$ and $B$ be two superalgebras. The $\mathbb{Z}_2$-graded tensor product $A \hat{\otimes} B$ is identified with $A \otimes B$ as vector spaces, and the multiplication is given by

\begin{equation}
(a_1 \otimes b_2) \cdot (a_2 \otimes b_2) = (-1)^{\deg a_2 \deg b_1} a_1 a_2 \otimes b_1 b_2.
\end{equation}

**Definition 1.3.** Let $A$ be a superalgebra. A super $A$-module is a $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$ equipped with an action of $A$ such that

\begin{equation}
A^+ V^\pm \subseteq V^\pm, \quad A^- V^\pm \subseteq V^\mp.
\end{equation}

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}_2$-graded vector space. Set

\begin{equation}
\tau = \text{id}_{V^+} - \text{id}_{V^-} \in \text{End}(V),
\end{equation}

and

\begin{equation}
\text{End}^\pm(V) = \left\{ a \in \text{End}(V) : \tau a = \pm a \tau \right\}.
\end{equation}
Then $\text{End}(V) = \text{End}^+(V) \oplus \text{End}^-(V)$ is a superalgebra, and $V$ is a super $\text{End}(V)$-module.

For $a \in \text{End}(V)$, its supertrace is defined by

\[(1.7) \quad \text{Tr}_s [a] = \text{Tr} [\tau a].\]

For $a, b \in \text{End}(V)$, we have

\[(1.8) \quad \text{Tr}_s [[a, b]] = 0.\]

In this paper, we will apply the superalgebra formalism to the following setting. Let $M$ be a smooth manifold. We denote by $\Omega(M)$ be the algebra of differential forms on $M$. We always equip $\Omega(M)$ with the $\mathbb{Z}_2$-grading $\Omega^{\text{even/odd}}(M)$. Then $\Omega(M)$ is a supercommutative superalgebra, i.e., $[\alpha_1, \alpha_2] = 0$ for $\alpha_1, \alpha_2 \in \Omega(M)$. Let $F$ be a complex vector bundle over $M$. We denote by $\Omega \cdot (M, F)$ the vector space of differential forms on $M$ with values in $F$. We equip $\Omega \cdot (M, F)$ with the $\mathbb{Z}_2$-grading $\Omega^{\text{even/odd}}(M, F)$. Then $\Omega(M, F)$ is a super $\Omega(M)$-module.

1.2. Clifford algebras.

Let $V$ be a real vector space. Let $g_V$ be an Euclidean metric on $V$. Let

\[(1.9) \quad \bigotimes V := \bigoplus_{j=0}^{\infty} V^\otimes j\]

be the tensor algebra of $V$.

**Definition 1.4.** Let $I \subseteq \bigotimes V$ be the bi-ideal generated by

\[(1.10) \quad u \otimes v + v \otimes u + 2g_V(u, v), \quad u, v \in V.\]

Set

\[(1.11) \quad C(V, g_V) = (\bigotimes V)/I,\]

called the Clifford algebra associated with $(V, g_V)$.

Let

\[(1.12) \quad c : V \rightarrow C(V, g_V)\]

be the map induced by the canonical injection $V \rightarrow \bigotimes V$. For $u, v \in V$, we have

\[(1.13) \quad c(u)c(v) + c(v)c(u) = -2g_V(u, v).\]

Let $e_1, \cdots, e_n \in V$ be an orthogonal basis of $V$. Then

\[(1.14) \quad c(e_{j_1})c(e_{j_2})\cdots c(e_{j_r}), \quad 0 \leq r \leq n, \quad j_1 < j_2 < \cdots < j_r,\]

is a basis of $C(V, g_V)$. Let $C^\pm(V, g_V) \subseteq C(V, g_V)$ be the vector subspace spanned by the terms in (1.14) with $r$ even/odd. Then $C(V, g_V)$ becomes a superalgebra.

Now we suppose that $V$ is equipped with a complex structure $J \in \text{End}(V)$ and that $g_V$ is $J$-invariant, i.e., $g_V(\cdot, \cdot) = g_V(J\cdot, J\cdot)$. Set

\[(1.15) \quad V_C = V \otimes_{\mathbb{R}} \mathbb{C}.\]

The action of $J$ extends $\mathbb{C}$-linearly to $V_C$. The Euclidean metric $g_V$ extends to a $\mathbb{C}$-bilinear form on $V_C$. 


Set
\[ V_{C}^{1,0} = \{ v \in V_{C} : Jv = iv \}, \quad V_{C}^{0,1} = \{ v \in V_{C} : Jv = -iv \}. \]

We have
\[ V_{C} = V_{C}^{1,0} \oplus V_{C}^{0,1}. \]

For \( v \in V_{C} \), let \( v^{(1,0)} \) (resp. \( v^{(0,1)} \)) be its component in \( V_{C}^{1,0} \) (resp. \( V_{C}^{0,1} \)).

Let \( V_{C}^{*} \) be the vector space of \( \mathbb{R} \)-linear forms on \( V_{C} \). For \( v \in V_{C} \), let \( v^{*} \in V_{C}^{*} \) be its dual (with respect to \( g^{V} \)).

Set
\[ V_{C}^{*,1,0} = \{ f \in V_{C}^{*} : f \circ J = if \}, \quad V_{C}^{*,0,1} = \{ f \in V_{C}^{*} : f \circ J = -if \}. \]

For \( v \in V_{C}^{1,0} \) (resp. \( v \in V_{C}^{0,1} \)), we have \( v^{*} \in V_{C}^{*,0,1} \) (resp. \( v^{*} \in V_{C}^{*,1,0} \)).

For \( v \in V_{C} \), we define the product operator
\[ v^{*} \wedge : \Lambda^{k} V_{C}^{*} \rightarrow \Lambda^{k+1} V_{C}^{*}, \]
\[ \alpha \mapsto v^{*} \wedge \alpha, \]
and the contraction operator
\[ i_{v} : \Lambda^{k} V_{C}^{*} \rightarrow \Lambda^{k-1} V_{C}^{*}, \]
\[ \alpha \mapsto \left( (u_{1}, \ldots , u_{k-1}) \mapsto \alpha(v, u_{1}, \ldots , u_{k-1}) \right). \]

Set
\[ c : V \rightarrow \text{End} \left( \Lambda^{*}(V_{C}^{*,0,1}) \right), \]
\[ v \mapsto v^{(1,0)*} \wedge -i_{v^{(0,1)}}. \]

For \( u, v \in V \), we have
\[ c(u)c(v) + c(v)c(u) + g^{V}(u, v) = 0. \]

Thus \( c \) extends to a representation
\[ c : C \left( V, \frac{1}{2} g^{V} \right) \rightarrow \text{End} \left( \Lambda^{*}(V_{C}^{*,0,1}) \right). \]

1.3. **Even/odd characteristic classes.**

Let \( M \) be a smooth manifold. Let \( F \) be a complex vector bundle over \( M \) of rank \( r \).

Let \( \nabla^{F} \) be a connection on \( F \). Then \( \nabla^{F} \) induces a differential operator
\[ \nabla^{F} : \Omega^{k}(M, F) \rightarrow \Omega^{k+1}(M, F). \]

Let
\[ \nabla^{F,2} \in \Omega^{2}(M, \text{End}(F)) \]
be the curvature of \( \nabla^{F} \).

For \( \omega \in \Omega^{k}(M) \), put
\[ \varphi \omega = (2\pi i)^{-k/2} \omega. \]

Let \( \text{Tr} [ \cdot ] : \text{End}(F) \rightarrow \mathbb{C} \) be the trace map, which extends to
\[ \text{Tr} [ \cdot ] : \Omega^{k}(M, \text{End}(F)) \rightarrow \Omega^{k}(M) \]
such that for $\alpha \in \Omega(M), A \in \mathcal{C}^\infty(M, \text{End}(F))$,

\[(1.28) \quad \text{Tr}[\omega A] = \omega \text{Tr}[A].\]

Let $P$ be an invariant polynomial on $\text{gl}(r, \mathbb{C})$.

**Theorem 1.5** (Chern-Weil). The differential form

\[(1.29) \quad \varphi P(-\nabla^{F,2}) \in \Omega^\text{even}(M)\]

is closed. The cohomology class

\[(1.30) \quad P(F) := [\varphi P(-\nabla^{F,2})] \in H^\text{even}(M)\]

is independent of $\nabla^F$.

Now we assume that $\nabla^F$ is a flat connection, i.e., $\nabla^{F,2} = 0$. Then

\[(1.31) \quad \varphi P(-\nabla^{F,2}) = P(0)\]

is just a constant function on $M$.

For flat vector bundles, there are non trivial characteristic classes of odd degree. We will follow the construction of Bismut-Lott [BL95, §1].

Let $g^F$ be a Hermitian metric on $F$. Let $\nabla^{F,*}$ be the adjoint connection, i.e., for $\sigma_1, \sigma_2 \in \mathcal{C}^\infty(M, F)$ and $U \in \mathcal{C}^\infty(M, TM)$, we have

\[(1.32) \quad g^F(\nabla^F_{\sigma_1} \sigma_2) + g^F(\sigma_1, \nabla^F_{\sigma_2} \sigma_2) = U g^F(\sigma_1, \sigma_2).\]

Then $\nabla^{F,*2} = 0$.

Set

\[(1.33) \quad \omega^F = \nabla^{F,*} - \nabla^F \in \Omega^1(M, \text{End}(F)).\]

Let $f$ be an odd polynomial in one variable with complex coefficients. Set

\[(1.34) \quad f(F, \nabla^F, g^F) = \sqrt{2\pi i} \varphi \text{Tr} \left[ f(\omega^F/2) \right] \in \Omega^\text{odd}(M).\]

The following theorem was established by Bismut-Lott [BL95, Theorem 1.8].

**Theorem 1.6.** The differential form

\[(1.35) \quad f(F, \nabla^F, g^F) \in \Omega^\text{odd}(M)\]

is closed. The cohomology class

\[(1.36) \quad f(F, \nabla^F) := [f(F, \nabla^F, g^F)] \in H^\text{odd}(M)\]

is independent of $g^F$.

**Remark 1.7.** If $f$ is an even polynomial, by [BL95, Proposition 1.3], we have

\[(1.37) \quad \text{Tr} \left[ f(\omega^F) \right] = f(0) r.\]
1.4. Fibrations equipped with a connection and a fiberwise metric.

Let $\pi : \mathcal{N} \to M$ be a smooth fibration with compact fiber $N$.

Let $TN$ be the relative tangent bundle of the fibration. We equip the fibration with a connection, i.e., a smooth splitting
\begin{equation}
TN = T^H\mathcal{N} \oplus TN.
\end{equation}
Then $T^H\mathcal{N} \cong \pi^*TM$. Let
\begin{equation}
P^TN : TN \to TN, \quad P^T^H\mathcal{N} : TN \to T^H\mathcal{N}
\end{equation}
be the projections. For $U \in TM$, let $U^H \in T^H\mathcal{N}$ be the lift of $U$, i.e., $\pi_*U^H = U$.

For $U, V$ vector fields on $M$, set
\begin{equation}
\langle T(U, V) = [U, V]^H - [U^H, V]^H \rangle.
\end{equation}
We have $T \in \Omega^2(M, C^\infty(N, TN))$. We call $T$ the curvature of the fibration.

We equip $TM$ and $TN$ with Riemannian metrics $g^{TM}$ and $g^{TN}$. Let $\pi^*g^{TM}$ be the induced metric on $T^H\mathcal{N}$. Set
\begin{equation}
g^{TN} = \pi^*g^{TM} \oplus g^{TN},
\end{equation}
which is a Riemannian metric on $g^{TN}$. Let $\langle \cdot, \cdot \rangle$ be the corresponding scalar product.

Let $\nabla^{TN}$ be the Levi-Civita connection on $TN$ associated with $g^{TN}$.

Definition 1.8. Let $\nabla^{TN}$ be the connection on $TN$ defined by
\begin{equation}
\nabla^{TN} = P^{TN}\nabla^{TN}P^{TN}.
\end{equation}
Then $\nabla^{TN}$ is independent of $g^{TM}$ (cf. [B86] §1(c)).

Let $L$ be the Lie derivative. For $U$ a vector field on $M$, set
\begin{equation}
\omega^{TN}(U) = (g^{TN})^{-1}L_Ug^{TN} \in C^\infty(\mathcal{N}, \text{End}(TN)).
\end{equation}
If $V \in TN$, then $\nabla^{TN}_{V}V$ coincides with the usual Levi-Civita connection along the fiber $N$. If $U \in TM$, then (cf. [B86] §1(c))
\begin{equation}
\nabla^{TN}_{U^H} = L_{U^H} + \frac{1}{2}\omega^{TN}(U).
\end{equation}

Put
\begin{equation}
\nabla^{TN,\oplus} = P^{TN}\nabla^{TN}P^{TN} \oplus P^{T^H\mathcal{N}}\nabla^{TN}P^{T^H\mathcal{N}}.
\end{equation}

Definition 1.9. For $U \in TN$, set
\begin{equation}
S^{TN}(U) = \nabla^{TN}_U - \nabla^{TN,\oplus}_U \in C^\infty(\mathcal{N}, \text{End}(TN)).
\end{equation}
Then $\langle S^{TN}(\cdot), \cdot \rangle$ is independent of $g^{TM}$ (cf. [B86] §1(c)).

2. THE CHERN-WEIL THEORY OF A FLAT FIBRATION

The purpose of this section is to construct characteristic classes and characteristic forms on the total space of a flat fibration with compact complex fibers. This section is organized as follows.

In §2.1 we state a consequence of the Chern-Weil theory, which will be of constant use in the rest of this section.

In §2.2 we define a flat fibration with complex fibers.

In §2.3 we construct a complex vector bundle $E$ over the total space of fibration.
In [2.4] we construct connections on \( E \). In particular, given a Hermitian metric on \( E \), we construct a unitary connection on \( E \) and show that the integral along the fiber of the usual Chern-Weil forms associated with this connection vanishes in positive degree.

In [2.5] we construct odd characteristic forms. These characteristic forms will appear on the right-hand side of the Riemann-Roch-Grothendieck formula in §3.

In [2.6] we construct a natural multiplication of the odd characteristic forms.

### 2.1. A consequence of Chern-Weil theory

Let \( N \) be a smooth compact oriented manifold. Let \((\Omega(N), d_N)\) be the de Rham complex of smooth differential forms on \( N \). We denote by \( H^\bullet(N) \) its cohomology.

Let \( V \) be a finite dimensional real vector space.

We will replace the de Rham complex \((\Omega(N), d_N)\) by the twisted de Rham complex \((\Omega(N, \Lambda(V^*)), d_N)\). Its cohomology is equal to \( H^\bullet(N) \hat{\otimes} \Lambda(V^*) \).

Let \((\Omega(N \times V), d_{N \times V})\) be the de Rham complex of \( N \times V \). Then \((\Omega(N, \Lambda(V^*)), d_N)\) can be identified with the subcomplex of \((\Omega(N \times V), d_{N \times V})\) that consists of forms which are constant along \( V \).

Let \( p : N \times V \to N \) and \( q : N \times V \to V \) be the natural projections. Let \( q_* \) denote the integral along the fiber \( N \), i.e., for \( \alpha \in \Omega(V) \) and \( \beta \in \Omega(N) \),

\[
q_*[\alpha \wedge \beta] = \alpha \int_N \beta,
\]

By restricting \( q_* \) to forms which are constant along \( V \), we get a map

\[
q_* : \Omega(N, \Lambda(V^*)) \to \Lambda(V^*).
\]

Let \( E \) be a complex vector bundle of rank \( r \) over \( N \). Let \( \nabla^E \) be a connection on \( E \). Its curvature \( \nabla^{E,2} \) is a smooth section of \( \Lambda^2(T^*N) \otimes \text{End}(E) \). The vector bundle \( E \) lifts to the vector bundle \( p^*E \) on \( N \times V \), and \( \nabla^E \) lifts to a connection on \( p^*E \), which is still denoted by \( \nabla^E \). Let \( S \) be a smooth section on \( N \) of \( V^* \otimes \text{End}(E) \). We can view \( S \) as a section of \( V^* \otimes \text{End}(E) \) on \( N \times V \), which is constant along \( V \). Then \( \nabla^E + S \) is also a connection on \( p^*E \). Its curvature \( (\nabla^E + S)^2 \) is a smooth section of \( \Lambda^2(T^*N) \hat{\otimes} \Lambda(V^*) \) \( \otimes \text{End}(E) \) over \( N \times V \), which is constant along \( V \).

The following proposition is a direct consequence of Chern-Weil theory.

**Proposition 2.1.** For any invariant complex polynomial \( P \) on \( \mathfrak{gl}(r, \mathbb{C}) \),

\[
P(- (\nabla^E + S)^2) \in \Omega(N, \Lambda(V^*))
\]

is closed. Its cohomology class

\[
[P(- (\nabla^E + S)^2)] \in H^\bullet(N) \hat{\otimes} \Lambda(V^*)
\]

is independent of \( \nabla^E \) and \( S \). In particular,

\[
[P(- (\nabla^E + S)^2)] \in H^\bullet(N) \subseteq H^\bullet(N) \hat{\otimes} \Lambda(V^*)
\]
2.2. A flat complex fibration.

Let $G$ be a Lie group. Let $N$ be a compact complex manifold of dimension $n$. We assume that $G$ acts holomorphically on $N$. Let $M$ be a real manifold. Let $p : PG \to M$ be a principal $G$-bundle equipped with a connection. Set

$$\mathcal{N} = PG \times_G N.$$  

Let $q : \mathcal{N} \to M$ be the natural projection, which defines a fibration with canonical fiber $N$.

Let $T_R N$ be the real tangent bundle of $N$. Set $T_R N = T_C N \otimes_R \mathbb{C}$.

The connection on $PG$ induces a connection on the fibration $q : \mathcal{N} \to M$, i.e., a splitting

$$TN = T_R N \oplus T^H N.$$  

Then $T^H N \cong q^* TM$. The splitting (2.7) induces the following identification

$$\Lambda (T^*_C N) = \Lambda (T^*_C N) \hat{\otimes} q^* \Lambda (T^*_C M).$$  

Let $TN$ be the holomorphic tangent bundle of $N$. Using the splitting $T_C N = TN \oplus \overline{TN}$, we get a further splitting

$$\Lambda (T^*_C N) = \Lambda (T^*_N) \hat{\otimes} \Lambda (\overline{T^*_N}) \hat{\otimes} q^* \Lambda (T^*_C M).$$  

Put

$$\Omega^{(p,q,r)}(N) = \mathcal{C}^\infty (N, \Lambda^p (T^*_N) \hat{\otimes} \Lambda^q (\overline{T^*_N}) \hat{\otimes} q^* \Lambda^r (T^*_C M)).$$  

Then

$$\Omega^k (\mathcal{N}) = \bigoplus_{p+q+r=k} \Omega^{(p,q,r)}(N).$$  

In the sequel, we assume that the connection on $PG$ is flat. Then $q : \mathcal{N} \to M$ is a flat fibration, i.e., its curvature $T = 0$ (cf. (1.40)).

Let $d_N$ be the de Rham operator on $\Omega (N)$. Let $d_M$ be the de Rham operator on $\Omega (M)$, which lifts to $\Omega (\mathcal{N})$ in the following sense: let $(f_\alpha)$ be a basis of $TM$, let $(f^\alpha)$ be the dual basis of $T^* M$. then

$$d_M = \sum_{\alpha} (q^* f^\alpha) \wedge L_{f^\alpha}.$$  

Let $d_N$ be the de Rham operator on $\mathcal{N}$. Since $T = 0$, by [BL95, Proposition 3.4], we get

$$d_N = d_N + d_M.$$  

Let $\partial_N$ (resp. $\overline{\partial}_N$) be the holomorphic (resp. anti-holomorphic) Dolbeault operator on $N$. We have

$$d_N = \partial_N + \overline{\partial}_N.$$  

By (2.13) and (2.14), we get

$$d_N = \partial_N + \overline{\partial}_N + d_M.$$
The following relations hold,

\[ d^2_M = d^2_N = \partial^2_N = \bar{\partial}^2_N = 0, \]
\[ [d_M, d_N] = [d_M, \partial_N] = [d_M, \bar{\partial}_N] = [d_N, \partial_N] = [d_N, \bar{\partial}_N] = [\partial_N, \bar{\partial}_N] = 0. \]

2.3. A fiberwise holomorphic vector bundle.

Let \( E_0 \) be a holomorphic vector bundle over \( N \) of rank \( r \). We assume that the action of \( G \) on \( N \) lifts to a holomorphic action on \( E_0 \). Set

\[ E = P_G \times_G E_0, \]

which is a complex vector bundle over \( N \). Furthermore, \( E \) is holomorphic along \( N \).

Let \( \bar{\partial}^E_N \) be the fiberwise holomorphic structure of \( E \). Let \( d^E_M \) be the lift of the de Rham operator on \( M \) to \( \Omega^\cdot (N, E) \). We have

\[ \bar{\partial}^E_N = d^E_M, \]
\[ [\partial^E_N, d^E_M] = [\partial^E_N, \bar{\partial}^E_N] = [\partial^E_N, d^E_M] = [\partial^E_N, \bar{\partial}^E_N] = 0. \]

2.4. Connections.

Set

\[ A^{E''} = \bar{\partial}^E_N + d^E_M, \]

which acts on \( \Omega^\cdot (N, E) \). By (2.18), we have

\[ (A^{E''})^2 = 0. \]

Let \( \bar{E}^* \) be the anti-dual vector bundle of \( E \). When replacing the complex structure of \( N \) by the conjugate complex structure, \( \bar{E}^* \) enjoys exactly the same properties as \( E \). We construct \( \bar{\partial}^E_N, d^E_M, \) and \( A^{E''} \) in the same way as \( \partial^E_N, d^E_M, \) and \( A^{E''} \). In particular,

\[ A^{E''} = \bar{\partial}^E_N + d^E_M. \]

Proceeding in the same way as in (2.18) and (2.20), we have

\[ \bar{\partial}^{E''}_N = d^{E''}_M, \]
\[ [\partial^{E''}_N, d^{E''}_M] = [\partial^{E''}_N, \bar{\partial}^{E''}_N] = [\partial^{E''}_N, d^{E''}_M] = [\partial^{E''}_N, \bar{\partial}^{E''}_N] = 0. \]
\[ (A^{E''})^2 = 0. \]

Let \( g^E \) be a Hermitian metric on \( E \). Then \( g^E \) defines an isomorphism \( g^E : E \to \bar{E}^* \).

Set

\[ \partial^E_N = (g^E)^{-1} \partial^E_N g^E, \]
\[ d^{E*}_M = (g^E)^{-1} d^{E*}_M g^E, \]

which act on \( \Omega^\cdot (N, E) \). By (2.22) and (2.24), we have

\[ \partial^E_N = d^{E*}_M, \]
\[ [\partial^E_N, d^{E*}_M] = [\partial^E_N, \bar{\partial}^{E*}_N] = [\partial^E_N, d^{E*}_M] = [\partial^E_N, \bar{\partial}^{E*}_N] = 0. \]

Set

\[ A^{E'} = (g^E)^{-1} A^{E''} g^E = \bar{\partial}^E_N + d^{E*}_M. \]

Then, by (2.25), we have

\[ (A^{E'})^2 = 0. \]

Let \( N^\Lambda (T^* M) \) be the number operator of \( \Lambda \cdot (T^* M) \).
Definition 2.2. Set

\[
A \in = 2^{-N \Lambda (T^* M)} \left( A^{E'} + A^{E''} \right) 2^{N \Lambda (T^* M)},
\]

\[
B \in = 2^{-N \Lambda (T^* M)} \left( A^{E'} - A^{E''} \right) 2^{N \Lambda (T^* M)}.
\]

By (2.20) and (2.27), we have

\[
A \in, 2 = 2^{-N \Lambda (T^* M)} \left[ A^{E'}, A^{E''} \right] 2^{N \Lambda (T^* M)} = -B \in, 2.
\]

By (2.33), we have

\[
U \in - 1 B \in \in = d \in_{M} + \frac{1}{2} \omega \in.
\]

Thus \( A \in \) is a Hermitian connection on \( E \) over \( N \).

Set

\[
\omega \in = d_{M}^{E, *} - d_{M}^{E} = (e^{E})^{-1} d_{M}^{E} g^{E} \in \mathcal{C}^{\infty} (N, T^* M \otimes \mathbb{R} \text{End}(E)).
\]

Then

\[
B \in = d \in_{N} - \overline{\partial} \in + \frac{1}{2} \omega \in.
\]

Thus \( B \in \in \in \Omega (M, \text{End}(\Omega (N, E))) \).

Proposition 2.3. For any invariant polynomial \( P \) on \( \mathfrak{gl}(r, \mathbb{C}) \), we have

\[
(\partial_{N} - \overline{\partial}_{N}) P (- A \in, 2) = 0.
\]

Also

\[
P (- A \in, 2) - P (- d_{N} \in) \in \text{Im}(\partial_{N} - \overline{\partial}_{N}).
\]

As a consequence, we have

\[
q_{*} [P (- A \in, 2)] = q_{*} [P (- d_{N} \in)],
\]

which is a constant function on \( M \).

Proof. Let \( N \Lambda (T^* N) \) be the number operator of \( \Lambda^\cdot (T^* N) \). Set \( U = (-1)^{N \Lambda (T^* N)} \).

To establish (2.34) and (2.35), we only need to show that

\[
d_{N} U P (- A \in, 2) = 0,
\]

and

\[
U P (- A \in, 2) - U P (- d_{N} \in) \in \text{Im}(d_{N}).
\]

By (2.33), we have

\[
U^{-1} B \in \in = d \in_{N} + \frac{1}{2} \omega \in.
\]

Now, applying (2.29), we get

\[
U^{-1} A \in, 2 U = -U^{-1} B \in, 2 U = - \left( d \in_{N} + \frac{1}{2} \omega \in \right)^{2}.
\]
We may and we will assume that $P$ is homogeneous. By (2.40), we have

\[(2.41)\quad UP(-A^{E,2}) = (-1)^{\deg P} P\left(-\left(d^E_N + \frac{1}{2} \omega^E\right)^2\right).\]

Applying Proposition 2.1 to the right-hand side of (2.41), we get (2.37).

We decompose (2.41) according to (2.11). By extracting the components which are of positive degree along $M$, we get

\[(2.42)\quad UP(-A^{E,2}) - UP(-d^E_N) \quad = \quad \left(-\deg P\right) P\left(-\left(d^E_N + \frac{1}{2} \omega^E\right)^2\right) - \left(-\deg P\right) P\left(-d^E_N\right).\]

Applying Proposition 2.1 to the right-hand side of (2.42), we get (2.38).

Taking the integral of (2.35) along $N$, we get (2.36). □

For $t \in \mathbb{R}$, set

\[(2.43)\quad A^E_t = d^E_N + td^E_M + (1 - t)d^E_M^*.\]

In particular,

\[(2.44)\quad A^E_{1/2} = A^E.\]

Set

\[(2.45)\quad V_t = (2 - 2t)^{N^\vee (T^* N)} (2t)^{N^\vee (T^* N)}.\]

Lemma 2.4. For $t \neq 0, 1$, we have

\[(2.46)\quad A^E_{t,2} = 4t(1 - t)V_t^{-1} A^{E,2} V_t.\]

Proof. By (2.19) and (2.26), we have

\[(2.47)\quad 2tV_t^{-1}A^{E,2} V_t = \partial^E_N + td^E_M,\]

\[(2.48)\quad (2 - 2t)V_t^{-1}A^{E,2} V_t = \partial^E_N + (1 - t)d^E_M^*.\]

By (2.18), (2.25), (2.29) and (2.47), we have

\[(2.49)\quad 4t(1 - t)V_t^{-1} A^{E,2} V_t = \left[(2 - 2t)V_t^{-1}A^{E,2} V_t, 2tV_t^{-1}A^{E,2} V_t\right]^2 = A^E_{t,2}.\]

Now we extend Proposition 2.3 by considering the extra parameter $t$.

Theorem 2.5. For any invariant polynomial $P$ on $\mathfrak{gl}(r, \mathbb{C})$ and $t \in \mathbb{R}$, we have

\[(2.50)\quad q_* \left[ P\left(-A^E_{t,2}\right) \right] = q_* \left[ P\left(-d^E_N\right) \right],\]

which is a constant function on $M$. □
Proof. Since \( q_*[P( - A_{t}^{E,2})] \) is polynomial on \( t \), it is sufficient to consider the case \( t \neq 0, 1 \).

We may suppose that \( P \) is homogeneous. By \((2.46)\), we have

\[
q_*[P( - A_{t}^{E,2})] = (4t(1-t))^{\deg P} q_*[V_t^{-1}P( - A_{t}^{E,2})].
\]

Applying Proposition \(2.3\) to the right-hand side of \((2.50)\), we get

\[
q_*[P( - A_{t}^{E,2})] = (4t(1-t))^{\deg P} q_*[V_t^{-1}P( - d_{N}^{E,2})].
\]

Since \( P( - d_{N}^{E,2}) \) is a \((\deg P, \deg P)\)-form on \( N \), we have

\[
V_t^{-1}P( - d_{N}^{E,2}) = (4t(1-t))^{-\deg P} P( - d_{N}^{E,2}).
\]

By \((2.51)\) and \((2.52)\), we get \((2.49)\). \(\square\)

2.5. The odd characteristic forms.

Set \( \varphi = (2\pi i)^{-\frac{3}{2}} N^A \cdot (T^*N) \).

Let \( P \) be an invariant polynomial on \( \mathfrak{gl}(r, \mathbb{C}) \).

Definition 2.6. For \( t \in \mathbb{R} \), set

\[
\tilde{P}_t(E, g^E) = \sqrt{2\pi i} \varphi \left\langle P'( - A_{t}^{E,2}), \frac{\omega^E}{2} \right\rangle \in \Omega^{\text{odd}}(N).
\]

Here the notation \( \left\langle P'(\cdot), \cdot \right\rangle \) was defined in \((0.9)\).

Proposition 2.7. For \( t \in \mathbb{R} \), the differential form

\[
q_*[\tilde{P}_t(E, g^E)] \in \Omega^{\text{odd}}(M)
\]

is closed. Its cohomology class

\[
[q_*[\tilde{P}_t(E, g^E)]] \in H^\cdot(M)
\]

is independent of \( g^E \).

Proof. We have (cf. \([\text{BeGV04}, \S 1.4]\))

\[
\sqrt{2\pi i} \varphi \frac{\partial}{\partial t} P( - A_{t}^{E,2}) = -\sqrt{2\pi i} \varphi \left\langle P'( - A_{t}^{E,2}), \left[ A_{t}^{E}, \frac{\partial}{\partial t} A_{t}^{E} \right] \right\rangle
\]

\[
= -\sqrt{2\pi i} \varphi d_N \left\langle P'( - A_{t}^{E,2}), \frac{\partial}{\partial t} A_{t}^{E} \right\rangle
\]

\[
= -d_N \varphi \left\langle P'( - A_{t}^{E,2}), \frac{\partial}{\partial t} A_{t}^{E} \right\rangle.
\]

Since

\[
\frac{\partial}{\partial t} A_{t}^{E} = d_{M}^{E} - d_{M}^{E,*} = -\omega^E,
\]

we have

\[
\sqrt{2\pi i} \varphi \frac{\partial}{\partial t} P( - A_{t}^{E,2}) = 2d_N \tilde{P}_t(E, g^E).
\]
By Proposition 2.5, we get
\[ \frac{\partial}{\partial t} q_* \left[ P(-A^E,\omega_E^2) \right] = 0. \]

By (2.58) and (2.59), we get
\[ d_M q_* \left[ \tilde{P}_t(E, g_E) \right] = q_* \left[ d_N \tilde{P}_t(E, g_E) \right] = 0. \]

Thus \( q_* \left[ \tilde{P}_t(E, g_E) \right] \) is closed.

The fact that \( q_* \left[ \tilde{P}_t(E, g_E) \right] \in H^*(M) \) is independent of \( g_E \) comes from the functoriality of our construction (cf. [BeGV04, §1.5]). \( \square \)

Now we study the dependence of \( \tilde{P}_t(E, g_E) \) on \( t \).

Recall that \( V_t \) was defined in (2.45).

Proposition 2.8. If \( P \) is homogeneous, for \( t \in \mathbb{R} \), we have
\[ \tilde{P}_t(E, g_E) = (4t(1-t))^{\deg P-1} V_t^{-1} \tilde{P}_1(E, g_E). \]

In particular,
\[ q_* \left[ \tilde{P}_t(E, g_E) \right] = (4t(1-t))^{\deg P-n-1} q_* \left[ \tilde{P}_1(E, g_E) \right]. \]

Proof. Since (2.61) is a rational function of \( t \), it is sufficient to consider the case \( t \neq 0, 1 \).

By (2.46), we have
\[ \left\langle P'(-A^E,\omega_E^2), \frac{\omega_E}{2} \right\rangle = \left\langle P'(-4t(1-t)V_t^{-1} A^E,\omega_E^2), \frac{\omega_E}{2} \right\rangle \]
\[ = (4t(1-t))^{\deg P'} V_t^{-1} \left\langle P'(-A^E,\omega_E^2), \frac{\omega_E}{2} \right\rangle \]
\[ = (4t(1-t))^{\deg P-1} V_t^{-1} \left\langle P'(-A^E,\omega_E^2), \frac{\omega_E}{2} \right\rangle, \]
which is equivalent to (2.61). \( \square \)

In the sequel, we use the convention
\[ \tilde{P}(E, g_E) = \tilde{P}_1(E, g_E). \]

The following proposition is a refinement of Proposition 2.7.

Proposition 2.9. We have
\[ d_N \tilde{P}(E, g_E) \]
\[ = \sqrt{2\pi i} \frac{\varphi}{2} \left( \frac{\partial}{\partial t} V_t^{-1} \right)_{t=1} \int_0^1 \left\langle P'(\frac{\partial N - \partial N}{2} + \frac{s\omega_E}{2}), \frac{\omega_E}{2} \right\rangle ds. \]

In particular, for \( p = 0, \cdots, n \), we have
\[ \left\{ d_N \tilde{P}(E, g_E) \right\}^{(p,p)} = 0. \]
Proof. By (2.46), we have

\[
\frac{\partial}{\partial t} \left\{ \sqrt{2\pi i \varphi} \ P(-A^E_{t,2}) \right\}_{t=\frac{1}{2}}
\]

(2.67)

\[
= \frac{\partial}{\partial t} \left\{ \sqrt{2\pi i \varphi} \ (4t(1-t))^{\text{deg} P} V^{-1}_t P(-A^E_{t,2}) \right\}_{t=\frac{1}{2}}.
\]

By (2.52) and (2.67), we have

\[
\frac{\partial}{\partial t} \left\{ \sqrt{2\pi i \varphi} \ P(-A^E_{t,2}) \right\}_{t=\frac{1}{2}} = \frac{\partial}{\partial t} \left\{ \sqrt{2\pi i \varphi} \ (4t(1-t))^{\text{deg} P} V^{-1}_t \left( P(-A^E_{t,2}) - P(-d^E_{N,2}) \right) \right\}_{t=\frac{1}{2}}.
\]

By (2.40), we have

\[
P(-A^E_{t,2}) - P(-d^E_{N,2}) = U \left( P\left( \left( d^E_N + \frac{\omega^E}{2} \right)^2 \right) - P\left( d^E_{N,2} \right) \right).
\]

(2.69)

As a consequence of Proposition 2.1 (cf. [BeGV04 §1.5]), we get

\[
P\left( \left( d^E_N + \frac{\omega^E}{2} \right)^2 \right) - P\left( d^E_{N,2} \right) = d_N \int_0^1 \left\langle P'\left( \left( d^E_N + \frac{\omega^E}{2} \right)^2 \right), \frac{\omega^E}{2} \right\rangle ds.
\]

Then

\[
U \left( P\left( \left( d^E_N + \frac{\omega^E}{2} \right)^2 \right) - P\left( d^E_{N,2} \right) \right) = (\partial_N - \bar{\partial}_N) \int_0^1 \left\langle P'\left( \left( d^E_N + \frac{\omega^E}{2} \right)^2 \right), \frac{\omega^E}{2} \right\rangle ds.
\]

(2.71)

By (2.58), (2.68), (2.69) and (2.71), we get (2.65).

For \( p = 0, \ldots, n \), we have

\[
V^{-1}_t|_{\Omega^{(p,p)}} = (4t(1-t))^{-p},
\]

whose derivative at \( t = \frac{1}{2} \) is zero. This proves (2.66). \( \square \)

2.6. Multiplication of odd characteristic forms.

Put

\[
P(E, g^E) = \varphi P(-A^E_{\frac{1}{2}}).
\]

Proposition 2.10. Let \( P, Q \) be two invariant polynomials. The following identity holds

\[
\tilde{P}Q(E, g^E) = \tilde{P}(E, g^E) \wedge Q(E, g^E) + P(E, g^E) \wedge \tilde{Q}(E, g^E).
\]

(2.74)

Proof. We have

\[
\left\langle (PQ)'(-A^E_{t,2}), \frac{\omega^E}{2} \right\rangle
\]

(2.75)

\[
= \left\langle P'(-A^E_{t,2}), \frac{\omega^E}{2} \right\rangle \wedge Q(-A^E_{t,2}) + P(-A^E_{t,2}) \wedge \left\langle Q'(-A^E_{t,2}), \frac{\omega^E}{2} \right\rangle,
\]

which implies (2.74). \( \square \)
For \((\alpha, \tilde{\alpha}), (\beta, \tilde{\beta}) \in \Omega^{\text{even}}(\mathcal{N}) \times \Omega^{\text{odd}}(\mathcal{N})\), put

\begin{equation}
(\alpha, \tilde{\alpha}) \cdot (\beta, \tilde{\beta}) = (\alpha \wedge \beta, \tilde{\alpha} \wedge \beta + \alpha \wedge \tilde{\beta}) .
\end{equation}

Then \((\Omega^{\text{even}}(\mathcal{N}) \times \Omega^{\text{odd}}(\mathcal{N}), +, \cdot)\) is a commutative ring.

Let \((\mathbb{C} [\mathfrak{gl}(r, \mathbb{C})])^{\text{GL}(r, \mathbb{C})} \) be the ring of invariant polynomials on \(\mathfrak{gl}(r, \mathbb{C})\).

**Proposition 2.11.** The following map is a ring homomorphism.

\begin{equation}
P \mapsto \left( P(E, g^E), \tilde{P}(E, g^E) \right) .
\end{equation}

**Proof.** This is a direct consequence of Proposition 2.10. \(\square\)

Let \(F\) be another complex vector bundle over \(\mathcal{N}\) satisfying the same properties as \(E\). Let \(r'\) be the rank of \(F\). Let \(g^F\) be a Hermitian metric on \(F\). Let \(Q\) be an invariant polynomial on \(\mathfrak{gl}(r', \mathbb{C})\).

**Definition 2.12.** We define

\begin{equation}
\tilde{P}(E, g^E) \ast \tilde{Q}(F, g^F) = \tilde{P}(E, g^E)Q(F, g^F) + P(E, g^E)\tilde{Q}(F, g^F) .
\end{equation}

**Proposition 2.13.** The differential form

\begin{equation}
q_+ [\tilde{P}(E, g^E) \ast \tilde{Q}(F, g^F)] \in \Omega^{\text{odd}}(M)
\end{equation}

is closed. Its cohomology class is independent of \(g^E\) and \(g^F\).

**Proof.** The argument leading to Proposition 2.7 still works: the key step is to show that

\begin{equation}
2d_N \tilde{P}(E, g^E) \ast \tilde{Q}(F, g^F) = \sqrt{2 \pi i \varphi} \frac{\partial}{\partial t} \left( P(-A_t^{E,2})Q(-A_t^{F,2}) \right)_{t=1/2} .
\end{equation}

\(\square\)

### 3. A Riemann-Roch-Grothendieck Formula

The purpose of this section is to establish a Riemann-Roch-Grothendieck formula, that express the odd Chern classes associated with the flat vector bundle \(H \cdot (N, E)\) in terms of the exotic characteristic classes that were defined in §2.5. This section is organized as follows.

In §3.1, we introduce the infinite dimensional flat vector bundle \(\mathcal{E} = \Omega^{(0,1)}(N, E)\).

In §3.2, we equip \(TN\) with a fiberwise Kähler metric, \(E\) with a Hermitian metric.

In §3.3, we introduce the Levi-Civita superconnection on \(\mathcal{E}\).

In §3.4, we define the index bundle, which is the fiberwise Dolbeault cohomology of \(E\). We show that the even characteristic form of the index bundle is a constant function on \(M\).

In §3.5, we construct differential forms \(\alpha_t, \beta_t\) in the same way as [BL95, §3(h)]. We state explicit formulas calculating their asymptotics as \(t \to \infty\) and \(t \to 0\). As a consequence of these formulas, we obtain a Riemann-Roch-Grothendieck formula.

In §3.6, we prove the asymptotics of \(\alpha_t, \beta_t\) stated in §3.5. The techniques applied in the proof were initiated by Bismut-Gillet-Soulé [BGS88b, §1(h)] and Bismut-Köhler
The key idea is a Lichnerowicz formula involving additional Grassmannian variables $da, d\bar{a}$.

In §3.7 following [BL95 §3(j)], we construct analytic torsion forms on $M$, that transgress the R.R.G. formula at the level of differential forms.

3.1. A flat superconnection and its dual.

Set
\begin{equation}
\mathcal{E}^q = \mathcal{C}^\infty(N, \Lambda^q(T^*N) \otimes E), \quad \mathcal{E} = \bigoplus_q \mathcal{E}^q.
\end{equation}
Then $\mathcal{E}$ is an infinite dimensional flat vector bundle over $M$. By (2.10), we have the identification
\begin{equation}
\Omega(M, \mathcal{E}) = \Omega^{(0\cdots)}(N, E).
\end{equation}

Let $\nabla^\mathcal{E}$ be the restriction of $d_M^E$ to $\Omega(M, \mathcal{E})$. Then $\nabla^\mathcal{E}$ is the canonical flat connection on $\mathcal{E}$.

Set
\begin{equation}
A^\mathcal{E} = \overline{\nabla}^\mathcal{E} + \nabla^\mathcal{E},
\end{equation}
which acts on $\Omega(M, \mathcal{E})$. Then $A^\mathcal{E}$ is a superconnection on $\mathcal{E}$.

Recall that the operator $A^{E''}$ on $\Omega(N, E)$ was defined in (2.19). We have
\begin{equation}
A^\mathcal{E} = A^{E''}|_{\Omega(0\cdots)(N,E)}.
\end{equation}
Then, by (2.20), we have
\begin{equation}
A^{\mathcal{E},2} = 0,
\end{equation}
which is equivalent to the following identities
\begin{equation}
\overline{\nabla}^\mathcal{E},2 = \nabla^\mathcal{E},2 = [\overline{\nabla}^\mathcal{E}, \nabla^\mathcal{E}] = 0.
\end{equation}

Set
\begin{equation}
\overline{\mathcal{E}}^* = \mathcal{C}^\infty(N, \Lambda^{n}(T^*N) \otimes \Lambda^n(T^*N) \otimes \overline{E}^*)
\end{equation}
Then $\overline{\mathcal{E}}^*$ is an infinite dimensional flat vector bundle over $M$. We have the identification
\begin{equation}
\Omega(M, \overline{\mathcal{E}}^*) = \Omega^{(\cdots n\cdots)}(N, \overline{E}^*)
\end{equation}

Let $\nabla^{\overline{\mathcal{E}}^*}$ be the restriction of $d_M^E$ to $\Omega(M, \overline{\mathcal{E}}^*)$. Then $\nabla^{\overline{\mathcal{E}}^*}$ is the canonical flat connection on $\overline{\mathcal{E}}^*$. Set
\begin{equation}
A^{\overline{\mathcal{E}}^*} = \overline{\nabla}^{\overline{\mathcal{E}}^*} + \nabla^{\overline{\mathcal{E}}^*},
\end{equation}
which acts on $\Omega(M, \overline{\mathcal{E}}^*)$. Then $A^{\overline{\mathcal{E}}^*}$ is a superconnection on $\overline{\mathcal{E}}^*$.

Recall that the operator $A^{E''}$ on $\Omega(N, \overline{E}^*)$ was defined in (2.21). We have
\begin{equation}
A^{\overline{\mathcal{E}}^*} = A^{E''}|_{\Omega(\cdots)(N, \overline{E}^*)}.
\end{equation}
Then, by (2.23), we have
\begin{equation}
A^{\overline{\mathcal{E}}^*,2} = 0.
\end{equation}
Let
\[(3.12)\quad (\cdot, \cdot)_E : \overline{E}^* \times E \to \mathbb{C}\]
be the canonical sesquilinear form, which extends to
\[(3.13)\quad (\cdot, \cdot)_E : (\Lambda^p(T^*N) \otimes \Lambda^n(T^*N) \otimes \overline{E}^*) \times (\Lambda^q(T^*N) \otimes E) \to \Lambda^{p+q}(T^*N) \otimes \Lambda^n(T^*N).
\]
We define
\[(3.14)\quad (\cdot, \cdot)_E : \overline{E}^* \times E \to \mathbb{C}
\]
\[(\alpha, \beta) \mapsto \int_N (\alpha, \beta)_E.
\]
Thus \(\overline{E}^*\) is formally the anti-dual of \(E\). For \(\alpha \in \Omega(M, \overline{E}^*)\) and \(\beta \in \Omega(M, E)\), the following identities hold
\[(3.15)\quad (\partial_N^E \alpha, \beta)_E + (-1)^{\deg \alpha} (\alpha, \partial_N^E \beta)_E = 0,
\]
\[(\nabla_N^E \alpha, \beta)_E + (-1)^{\deg \alpha} (\alpha, \nabla_E^E \beta)_E = d_M(\alpha, \beta)_E.
\]
By (3.3), (3.9) and (3.15), we get
\[(3.16)\quad (A^E \alpha, \beta)_E + (-1)^{\deg \alpha} (\alpha, A^E \beta)_E = d_M(\alpha, \beta)_E,
\]
i.e., \(A^E\) is the dual superconnection of \(A^\overline{E}\) in the sense of [BL95, Definition 1.5].

### 3.2. Hermitian metrics and connections on \(TN\) and \(E\)

From now on, we will assume that \(N\) is a Kähler manifold.

Let \(J : T_R N \to T_R N\) be the complex structure of \(N\).

**Proposition 3.1.** There exists a fiberwise Kähler metric \(g^{TN}\) on \(TN\), i.e., a Hermitian metric on \(TN\) whose restriction to each fiber \(N\) is a Kähler metric.

**Proof.** Let \((U_i)\) be a locally finite cover of \(M\) by open balls. Let \((f_i : U_i \to \mathbb{R})\) be a partition of unity. For each \(U_i\), we have the trivialization \(\varphi_i : q^{-1}(U_i) \to N \times U_i\) as flat fibration. Let \(p_i : N \times U_i \to N\) be the canonical projection. Let \(g_0^{TN}\) be a Kähler metric on \(TN_0\). Set
\[(3.17)\quad g^{TN} = \sum_i (q_i^* f_i)(\varphi_i^* p_i^* g_0^{TN}).
\]
Then \(g^{TN}\) satisfies the desired properties. \(\square\)

Let \(g^{TN}\) be a fiberwise Kähler metric on \(TN\). Let
\[(3.18)\quad \omega \in \mathcal{C}^\infty(N, T^*N \otimes T^*N)
\]
be the associated fiberwise Kähler form. Let
\[(3.19)\quad dv_N = \frac{\omega^n}{n!} \in \mathcal{C}^\infty(N, \Lambda^{2n}(T^*_N))
\]
be the induced fiberwise volume form.

Let \(g^{TN}\) and \(g^\Lambda(T^*N)\) be the Hermitian metrics on \(TN\) and \(\Lambda^*(T^*N)\) induced by \(g^{TN}\).

Let \(g_T^{TN}\) be the Riemannian metric on \(T_R N\) induced by \(g^{TN}\).
Let $\nabla^{T_N}$ be the connection on $T_N$ associated with $(g^{T_N}, T^H N)$ in the same way as in [1.4]. Recall that the connection $A^{TN}$ on $TN$ is defined by (2.28). In the sequel, we change the notation as follows

$$\nabla^{TN} = A^{TN}.$$  

Since the metric $g^{TN}$ is fiberwise Kähler, the connection on $T_N$ induced by $\nabla^{TN}$ along the fibre $N$ coincides with $\nabla^{T_N}$. Moreover, the complex structure of $T_N$ is flat with respect to the flat connection on $N$. By (1.44), (2.30) and (2.32), these two connections also coincide in horizontal directions. The conclusion is that the connection $\nabla^{T_N}$ preserves the complex structure $J$, and induces the connection $\nabla^{TN}$ on $TN$.

Let $\nabla^{TN}$ and $\nabla^{\Lambda(T^*N)}$ be the connections on $T^N$ and $\Lambda(T^*N)$ induced by $\nabla^{TN}$. Let $g^{E}$ be a Hermitian metric of $E$. Let $\nabla^{E}$ be the connection on $E$ defined by (2.28). Let $g^{\Lambda(T^*_CN)}$ be the $\mathbb{C}$-bilinear form on $\Lambda(T^*_CN)$ induced by $g^{TN}$. Let

$$\ast : \Lambda(T^*_CN) \to \Lambda^{2n-}(T^*_CN)$$

be the usual Hodge operator acting on $\Lambda(T^*_CN)$, i.e., for $\alpha, \beta \in \Lambda(T^*_CN)$,

$$g^{\Lambda(T^*_CN)}(\alpha, \beta)dv_N = \alpha \wedge \ast \beta .$$

In particular, $\ast$ maps $\Lambda(T^*N)$ to $\Lambda^n(T^*N) \otimes \Lambda^{n-}(T^*N)$.

The Hermitian metric $g^{E}$ induces an identification $g^{E} : E \to \overline{E}$. The Hodge operator $\ast$ extends to

$$\ast^E : \Lambda(T^*N) \otimes E \to \Lambda^n(T^*N) \otimes \Lambda^{n-}(T^*N) \otimes \overline{E}.$$ 

Let $g^{\mathcal{E}}$ be a Hermitian metric on $\mathcal{E}$, such that for $\alpha, \beta \in \mathcal{E}$,

$$g^{\mathcal{E}}(\alpha, \beta) = \frac{1}{(2\pi)^n} \int_{N} (g^{\Lambda(T^*N)} \otimes g^{E})(\alpha, \beta)dv_N = \frac{(-1)^{\deg \alpha \deg \beta}}{(2\pi)^n} (\ast^E \alpha, \beta)_{\mathcal{E}} .$$

Set

$$\omega^{\mathcal{E}} = (g^{\mathcal{E}})^{-1} \nabla^{\mathcal{E}} g^{\mathcal{E}} \in \mathcal{C}^{\infty}(M, T^*M \otimes \mathrm{End}(\mathcal{E}))$$

and

$$k_N = (dv_N)^{-1} d_M dv_N \in \mathcal{C}^{\infty}(N, T^*M) .$$

We define $\omega^{TN}$ in the same way as in (2.32). Let $\omega^{\Lambda(T^*N)}$ be the induced action of $\omega^{TN}$ on $\Lambda(T^*N)$. Then $\omega^{\Lambda(T^*N)}$ is just the horizontal variation of the metric $g^{\Lambda(T^*N)}$ on $\Lambda(T^*N)$ with respect to the flat connection. We have

$$\omega^{\mathcal{E}} = \omega^{\Lambda(T^*N)} + \omega^{E} + k_N .$$

3.3. The Levi-Civita superconnection.

Recall that $A^{\mathcal{E}}$ and $A^{\mathcal{E}}_*$ were defined in (3.3) and (3.9).

Set

$$A^{\mathcal{E}} \ast = (\ast^{E})^{-1} A^{\mathcal{E}} \ast^{E} ,$$

which acts on $\Omega(M, \mathcal{E})$. Then $A^{\mathcal{E}} \ast$ is the adjoint superconnection of $A^{\mathcal{E}}$ (with respect to $g^{\mathcal{E}}$) in the sense of [BL95, Definition 1.6].
By (3.11), we have

\[(3.28)\quad A^{\delta, *, 2} = 0 .\]

Set

\[(3.29)\quad C^\delta = 2^{-N^\Delta (T^* M)} \left( A^{\delta, *} + A^\delta \right) 2^{N^\Delta (T^* M)} , \quad D^\delta = 2^{-N^\Delta (T^* M)} \left( A^{\delta, *} - A^\delta \right) 2^{N^\Delta (T^* M)} .\]

By (3.5) and (3.28), we have

\[(3.30)\quad C^{\delta, 2} = D^{\delta, 2} = 2^{-N^\Delta (T^* M)} \left[ A^{\delta, *}, A^{\delta, *} \right] 2^{N^\Delta (T^* M)} , \quad [C^\delta, D^\delta] = 0 .\]

Let \(\partial^{E,*}_N\) be the formal adjoint of \(\partial^E_N\) with respect to \(g^E\). Set

\[(3.31)\quad D^E_N = \partial^E_N + \partial^{E,*}_N ,\]

which acts on \(\mathcal{E}\). Then \(D^E_N\) is the fiberwise spin\(^c\)-Dirac operator associated with \(g^{TN}/2\).

We recall that \(\nabla^E\) is defined in \(\S 3.1\). Let \(\nabla^{\delta,*}\) be the adjoint connection. Then

\[(3.32)\quad \nabla^{\delta,*} = \nabla^\delta + \omega^\delta .\]

Set

\[(3.33)\quad \nabla^{\delta,u} = \frac{1}{2} \left( \nabla^{\delta,*} + \nabla^\delta \right) = \nabla^\delta + \frac{1}{2} \omega^\delta ,\]

which is a unitary connection on \(\mathcal{E}\).

We have

\[(3.34)\quad C^\delta = D^E_N + \nabla^{\delta,u} , \quad D^\delta = \overline{\partial}^{E,*}_N - \overline{\partial}^E_N + \frac{1}{2} \omega^\delta .\]

Recall that the Levi-Civita superconnection was introduced in [B86].

**Proposition 3.2.** The superconnection \(C^\delta\) is the Levi-Civita superconnection associated with \((T^HN, g^{TN}, g^E)\).

**Proof.** Since the metric \(g^{TN}\) is fibrewise Kähler, up to the constant \(\sqrt{2}\), the operator \(D^E_N\) is the standard spin\(^c\)-Dirac operator along the fiber \(N\). As we saw in \(\S 3.2\), the connection \(\nabla^{TN}\) induced by \(\nabla^{TN}\) is exactly the connection that was considered in [B86]. Finally, since our fibration is flat, the term in the Levi-Civita superconnection that contains the curvature of our fibration vanishes identically. This completes the proof. \(\square\)

For \(t > 0\), let \(C^\delta_t, D^\delta_t\) be \(C^\delta, D^\delta\) associated with the rescaled metric \(g^{TN}/t\). By (3.34), we have

\[(3.35)\quad C^\delta_t = t\overline{\partial}^{E,*}_N + \overline{\partial}^E_N + \nabla^{\delta,u}, \quad D^\delta_t = t\overline{\partial}^{E,*}_N - \overline{\partial}^E_N + \frac{1}{2} \omega^\delta .\]
3.4. The index bundle and its characteristic classes.

Let \( H(N, E_0) \) be the Dolbeault cohomology of \( E_0 \). The action of \( G \) on \( E_0 \) induces an action of \( G \) on \( H(N, E_0) \). Set

\[
H(N, E) = P_G \times_G H(N, E_0).
\]

Let \( \nabla^H(N,E) \) be the flat connection on \( H(N, E) \) induced by the flat connection on \( P_G \). For \( s \in C^\infty(M, \mathcal{E}) \) satisfying \( \overline{\partial}_N s = 0 \), let

\[
[s] \in C^\infty(M, H(N, E))
\]

be the corresponding cohomology class. Then

\[
\nabla^H(N,E)[s] = [\nabla^E s] \in \Omega^1(M, H(N, E)).
\]

By Hodge theory, there is a canonical identification

\[
H(N, E) \cong \ker D^E_N \subseteq \mathcal{E}.
\]

Let \( g^H(N,E) \) be the metric on \( H(N, E) \) induced by \( g^E \) via the identification (3.39). Let \( \nabla^H(N,E, \ast) \) be the adjoint connection of \( \nabla^H(N,E) \) with respect to \( g^H(N,E) \). Set

\[
\nabla^H(N,E,u) = \frac{1}{2}(\nabla^H(N,E, \ast) + \nabla^H(N,E)),
\]

\[
\omega^H(N,E) = \nabla^H(N,E, \ast) - \nabla^H(N,E).
\]

Then \( \nabla^H(N,E,u) \) is a unitary connection and \( \omega^H(N,E) \in C^\infty(M, \text{End}(H(N, E))) \).

Put

\[
\chi(N, E) = \sum_p (-1)^p \dim H^p(N, E).
\]

**Proposition 3.3.** For \( t > 0 \), we have

\[
\varphi \text{Tr}_s \left[ \exp(D^E_t) \right] = \chi(N, E).
\]

**Proof.** By the local families index theorem [B86], as \( t \to 0 \),

\[
\varphi \text{Tr}_s \left[ \exp(D^E_t) \right] = q_s \left[ \text{Td}(TN, \nabla^TN) \text{ch}(E, \nabla^E) \right] + \mathcal{O}(\sqrt{t}).
\]

Furthermore,

\[
\frac{\partial}{\partial t} \text{Tr}_s \left[ \exp(D^E_t) \right] = \text{Tr}_s \left[ \left[ D^E_t, \frac{\partial}{\partial t} D^E_t \right] \exp(D^E_t) \right] = 0.
\]

By Proposition 2.5 and the Riemann-Roch-Hirzebruch formula, we have

\[
q_s \left[ \text{Td}(TN, \nabla^TN) \text{ch}(E, \nabla^E) \right] = \chi(N, E).
\]

Then (3.42) follows from (3.43)-(3.45). \( \square \)
3.5. A Riemann-Roch-Grothendieck formula.

For \( t > 0 \), set

\[
\alpha_t = \sqrt{2\pi i} \varphi \text{Tr}_s \left[ D_t^\varphi \exp(D_t^{\varphi,2}) \right] \in \Omega^\text{odd}(M),
\]

(3.46)

\[
\beta_t = \varphi \text{Tr}_s \left[ \frac{N^\Lambda(TN)}{2} (1 + 2D_t^{\varphi,2}) \exp(D_t^{\varphi,2}) \right] \in \Omega^\text{even}(M).
\]

Proposition 3.4. For \( t > 0 \), the differential form \( \alpha_t \) is closed. Its cohomology class is independent of \( g^{TN}, g^E \) and \( t \).

Proof. By (3.30), we have

(3.47)

\[
d_M \sqrt{2\pi i} \varphi \text{Tr}_s \left[ D_t^\varphi \exp(D_t^{\varphi,2}) \right] = \varphi \text{Tr}_s \left[ [C_t^\varphi, D_t^\varphi \exp(D_t^{\varphi,2})] \right] = 0,
\]

which proves the closeness. Then, by the functoriality of our constructions, \([\alpha_t] \in H^\cdot(M)\) is independent of the metrics. In particular, it is independent of \( t \). \( \square \)

Proposition 3.5. For \( t > 0 \), the following identity holds:

(3.48)

\[
\frac{\partial}{\partial t} \alpha_t = \frac{1}{t} d_M \beta_t.
\]

Proof. Set

(3.49)

\( \mathcal{N}_+ = \mathcal{N} \times \mathbb{R}_+, \quad M_+ = M \times \mathbb{R}_+ \).

Let

(3.50)

\( q_+ = q \oplus \text{id}_{\mathbb{R}_+} : \mathcal{N}_+ \to M_+ \)

be the obvious projection. Let \( t \) be the coordinate on \( \mathbb{R}_+ \).

We equip \( TN \) with the metric \( \frac{1}{2} g^{TN} \). Let \( \varphi^+, \omega^+, C^\varphi^+, D^\varphi^+ \) be the corresponding objects associated with the new fibration. The following identities hold (cf. (3.24))

(3.51)

\[
d_{M_+} = d_M + dt \wedge \frac{\partial}{\partial t},
\]

\[
\omega^\varphi^+ = \omega^\varphi + \frac{1}{t} dt \wedge (N^\Lambda(TN) - n).
\]

Then, by (3.34) and (3.35), we get

(3.52)

\[
C^\varphi^+ = C_t^\varphi + dt \wedge \frac{\partial}{\partial t} + \frac{1}{2t} dt \wedge (N^\Lambda(TN) - n),
\]

\[
D^\varphi^+ = D_t^\varphi + \frac{1}{2t} dt \wedge (N^\Lambda(TN) - n).
\]
Thus
\[\sqrt{2\pi i} \varphi \, \text{Tr}_s \left[ D^\varphi \exp(D^\varphi,2) \right]\]
\[= \sqrt{2\pi i} \varphi \, \text{Tr}_s \left[ D^\varphi \exp(D^\varphi,2) \right] + \frac{1}{2t} dt \wedge \varphi \, \text{Tr}_s \left[ \left( N \Lambda \left( T^N \right) - n \right) \exp(D^\varphi,2) \right] + \frac{1}{2t} dt \wedge \varphi \, \text{Tr}_s \left[ D^\varphi \exp \left( D^\varphi,2 + \frac{1}{2t} dt \wedge N \Lambda \left( T^N \right) \right) \right]\]
\[= \alpha_t + \frac{1}{2t} dt \wedge \varphi \, \text{Tr}_s \left[ N \Lambda \left( T^N \right) \exp(D^\varphi,2) \right] - \chi(N, E) \frac{n}{2t} dt + \frac{1}{2t} dt \wedge \varphi \, \text{Tr}_s \left[ D^\varphi \left[ D^\varphi \exp \left( D^\varphi,2 + \frac{1}{2t} dt \wedge N \Lambda \left( T^N \right) \right) \right] \right]\]
\[= \alpha_t + \frac{1}{2t} dt \wedge \varphi \, \text{Tr}_s \left[ D^\varphi \left[ D^\varphi \exp \left( D^\varphi,2 + \frac{1}{2t} dt \wedge N \Lambda \left( T^N \right) \right) \right] \right] \]
\[= \alpha_t + \frac{1}{2t} dt \wedge \beta_t - \chi(N, E) \frac{n}{2t} dt \in \Omega(M_+) . \]

By Proposition 3.4, we have
\[d_{M_+} \sqrt{2\pi i} \varphi \, \text{Tr}_s \left[ D^\varphi \exp(D^\varphi,2) \right] = 0 . \]

By (3.51), (3.53) and (3.54), we get (3.48). □

Set \( f(x) = xe^{x^2} \). Following [BL95, Definition 1.7], we define the odd characteristic form
\[f(H(N, E), \nabla^H(N, E), g^H(N, E)) = \sqrt{2\pi i} \varphi \, \text{Tr}_s \left[ f(\omega^H(N, E)/2) \right] \in \Omega^{\text{odd}}(M) . \]

Put
\[\chi'(N, E) = \sum_p (-1)^p p \dim H^p(N, E) . \]

Now we state the central result in this section. Its proof will be delayed to \[3.6\]

**Theorem 3.6.** As \( t \to +\infty \),
\[\alpha_t = f(H(N, E), \nabla^H(N, E), g^H(N, E)) + O\left( \frac{1}{\sqrt{t}} \right) , \]
\[\beta_t = \frac{1}{2} \chi'(N, E) + O\left( \frac{1}{\sqrt{t}} \right) . \]

As \( t \to 0 \),
\[\alpha_t = q_s \left[ \widetilde{Td}(TN, g^TN) \ast \text{ch}(E, g^E) \right]
+ \frac{1}{2t} d_{M_+} q_s \left[ \frac{\omega}{2\pi} \widetilde{Td}(TN, g^TN) \text{ch}(E, \nabla^E) \right] + O\left( \sqrt{t} \right) , \]
\[\beta_t = -\frac{1}{2} q_s \left[ \text{Td}'(TN, \nabla^TN) \text{ch}(E, \nabla^E) \right] + \frac{n}{2} \chi(N, E)
- \frac{1}{2t} q_s \left[ \frac{\omega}{2\pi} \text{Td}(TN, \nabla^TN) \text{ch}(E, \nabla^E) \right] + O\left( \sqrt{t} \right) . \]
Remark 3.7. By Proposition 2.3, we have

\[ q_* \left[ \frac{\omega}{2\pi} \text{Td}(TN, \nabla^{TN}) \text{ch}(E, \nabla^E) \right] \in \mathcal{C}^{\infty}(M). \]

By Proposition 3.4 and Theorem 3.6, we get the following R.R.G. formula.

**Theorem 3.8.** We have

\[ \left[ f(H(N, E), \nabla^H(N,E), g^H(N,E)) \right] \]
\[ = \left[ q_* \left[ \tilde{\text{Td}}(TN, g^{TN}) \ast \tilde{\text{ch}}(E, g^E) \right] \right] \in H^{\text{odd}}(M, \mathbb{R}). \]

3.6. Several intermediate results and the proof of Theorem 3.6.

We will now introduce various new odd Grassmann variables in order to be able to compute exactly the asymptotics of certain superconnection forms as \( t \to 0 \), and also to overcome the divergence of certain expressions. Our methods are closely related to the methods of [BGS88a, BGS88b, BK92], where similar difficulties also appeared.

Let \( a \) be an additional complex coordinate, \( \epsilon \) be an auxiliary odd Grassmann variable. For

\[ u, v \in \{ 1, da, d\bar{a}, dada, \epsilon, eda, ed\bar{a}, edad\bar{a} \} \]

and \( \sigma \in \Omega(M) \), we denote

\[ (v \land \sigma)^u = \begin{cases} \sigma & \text{if } u = v, \\ 0 & \text{else}. \end{cases} \]

**Lemma 3.9.** The following identity holds

\[ \text{Tr}_s \left[ D^\epsilon \exp \left( D^{\epsilon,2} \right) \right] \]
\[ = \text{Tr}_s \left[ \exp \left( - C^{\epsilon,2} - da \frac{1}{2} (\delta_N^E + \delta_N^{E,*)} \right. \right. \]
\[ \left. - d\bar{a} \left[ \delta_N^E + \delta_N^{E,*)} \frac{1}{2} \omega^\epsilon \right] + dada \left[ \frac{1}{2} \omega^\epsilon \right] \right] \]
\[ + d_M \text{Tr}_s \left[ \frac{1}{2} N^\Lambda(T^{TN}) \exp \left( D^{\epsilon,2} \right) \right]. \]

**Proof.** By (3.30) and (3.34), we have

\[ [N^\Lambda(T^{TN}), C^{\epsilon,2}] = - [N^\Lambda(T^{TN}), D^{\epsilon,2}] \]
\[ = - [N^\Lambda(T^{TN}), \left[ \delta_N^{E,*} - \delta_N^E, \frac{1}{2} \omega^\epsilon \right]] \]
\[ = \left[ \delta_N^E + \delta_N^{E,*} \frac{1}{2} \omega^\epsilon \right], \]
which implies

\[
\text{Tr}_s \left[ \exp \left( - C^{\varphi,2} - da \frac{1}{2} (\partial_N^E + \partial_N^{E,*}) - d\tilde{a} \left[ \partial_N^E + \partial_N^{E,*}, \frac{1}{2} \omega^\varphi \right] \right) \right]^{\text{coda}}
= \frac{\partial}{\partial b} \text{Tr}_s \left[ - \frac{1}{2} (\partial_N^E + \partial_N^{E,*}) \exp \left( - C^{\varphi,2} + b [N^\Lambda (\mathcal{T}^*N), C^{\varphi,2}] \right) \right]_{b=0}
= \frac{\partial}{\partial b} \text{Tr}_s \left[ - \frac{1}{2} (\partial_N^E + \partial_N^{E,*}) \exp \left( - C^{\varphi,2} + b [N^\Lambda (\mathcal{T}^*N), \exp (- C^{\varphi,2})] \right) \right]
= \text{Tr}_s \left[ - \frac{1}{2} [N^\Lambda (\mathcal{T}^*N), \partial_N^E + \partial_N^{E,*}] \exp (- C^{\varphi,2}) \right]
= \text{Tr}_s \frac{1}{2} (\partial_N^{E,*} - \partial_N^E) \exp (D^{\varphi,2}) .
\]

Then

\[
\begin{align*}
\text{Tr}_s \left[ \exp \left( - C^{\varphi,2} - da \frac{1}{2} (\partial_N^E + \partial_N^{E,*}) 
- d\tilde{a} \left[ \partial_N^E + \partial_N^{E,*}, \frac{1}{2} \omega^\varphi \right] + d\tilde{a} d\tilde{a} \frac{1}{2} \omega^\varphi \right) \right]^{\text{coda}}
&= \text{Tr}_s \left[ \exp \left( - C^{\varphi,2} - da \frac{1}{2} (\partial_N^E + \partial_N^{E,*}) - d\tilde{a} \left[ \partial_N^E + \partial_N^{E,*}, \frac{1}{2} \omega^\varphi \right] \right) \right]^{\text{coda}}
+ \text{Tr}_s \left[ \frac{1}{2} \omega^\varphi \exp (D^{\varphi,2}) \right] \\
&= \text{Tr}_s \frac{1}{2} (\partial_N^{E,*} - \partial_N^E + \omega^\varphi) \exp (D^{\varphi,2}) \\
&= \text{Tr}_s \left[ (\partial_N^{E,*} - \partial_N^E + \frac{1}{2} \omega^\varphi) \exp (D^{\varphi,2}) \right] - \text{Tr}_s \left[ \frac{1}{2} (\partial_N^{E,*} - \partial_N^E) \exp (D^{\varphi,2}) \right] \\
&= \text{Tr}_s \left[ D^\varphi \exp (D^{\varphi,2}) \right] - \text{Tr}_s \left[ [C^\varphi, \frac{1}{2} N^\Lambda (\mathcal{T}^*N)] \exp (D^{\varphi,2}) \right] \\
&= \text{Tr}_s \left[ D^\varphi \exp (D^{\varphi,2}) \right] - d_M \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\mathcal{T}^*N) \exp (D^{\varphi,2}) \right] .
\end{align*}
\]

The last equation is just what we needed to prove. □

Let \( N_+, M_+, q_+, \vartheta^+, \omega^+, C^\varphi_+ \) and \( D^{\varphi,+} \) be the same as in the proof of Proposition 3.35.

Lemma 3.10. For \( t > 0 \), the following identity holds:

\[
\begin{align*}
(N^\Lambda (\mathcal{T}^*M) + 1 + t \frac{\partial}{\partial t}) \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\mathcal{T}^*N) \exp (D^{\varphi,2}_t) \right]
&= \text{Tr}_s \left[ \exp \left( - C^{\varphi,2} - da \frac{1}{2} (\partial_N^E + t\partial_N^{E,*}) 
- d\tilde{a} \left[ \partial_N^E + t\partial_N^{E,*}, \frac{1}{2} \omega^\varphi \right] + d\tilde{a} d\tilde{a} \frac{1}{2} \omega^\varphi \right) \right]^{\text{coda}}
+ \text{closed form} .
\end{align*}
\]
Proof. By (3.63), we get

\[
\text{Tr}_s \left[ D^E \exp \left( D^{E,2} \right) \right] \\
= \text{Tr}_s \left[ \exp \left( - C^E + \frac{1}{2} \left( D^E_N + t D^E_N \right) \right) - d\bar{a} \left[ D^E_N + t D^E_N + \frac{e}{2} \omega \right] + d\bar{a} \left( \frac{e}{2} \omega \right) \right] \\
+ dM \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\frac{T^N}{\Lambda}) \exp \left( D^{E,2} \right) \right].
\]

(3.68)

Taking the \( dt \) component, we get

\[
\text{Tr}_s \left[ \frac{1}{2t} \left( N^\Lambda (\frac{T^N}{\Lambda}) - n \right) \exp \left( D^{E,2} \right) \right] \\
+ \text{Tr}_s \left[ D^E \exp \left( \left( D^E_N + \frac{1}{2} \frac{n^N}{\Lambda}^N \right) dt \right) \right] \\
= \text{Tr}_s \left[ \exp \left( - C^E + \frac{1}{2} \left( D^E_N + t D^E_N \right) \right) - d\bar{a} \left[ D^E_N + t D^E_N + \frac{e}{2} \omega \right] + d\bar{a} \left( \frac{e}{2} \omega \right) \right] \\
- dM \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\frac{T^N}{\Lambda}) \exp \left( \left( D^E_N + \frac{1}{2} \frac{n^N}{\Lambda}^N \right)^2 \right) \right] \\
+ \frac{\partial}{\partial t} \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\frac{T^N}{\Lambda}) \exp \left( D^{E,2} \right) \right].
\]

(3.69)

We multiply (3.69) by \( t \) and subtract the closed forms. Since \( dt \) supercommutes with \( N^\Lambda (\frac{T^N}{\Lambda}) \) and \( D^E_N \), By Proposition 3.3 3.4, we can delete \( \frac{n^N}{\Lambda}^N \) and \( \frac{1}{2} \frac{n^N}{\Lambda}^N \) on the left-hand side of (3.69). We obtain

\[
\text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\frac{T^N}{\Lambda}) \exp \left( D^{E,2} \right) \right] \\
+ \text{Tr}_s \left[ D^E \exp \left( \left( D^E_N + \frac{1}{2} \frac{n^N}{\Lambda}^N \right)^2 \right) \right] \\
= \text{Tr}_s \left[ \exp \left( - C^E + \frac{1}{2} \left( D^E_N + t D^E_N \right) \right) - d\bar{a} \left[ D^E_N + t D^E_N + \frac{e}{2} \omega \right] + d\bar{a} \left( \frac{e}{2} \omega \right) \right] \\
+ \frac{\partial}{\partial t} \text{Tr}_s \left[ \frac{1}{2} N^\Lambda (\frac{T^N}{\Lambda}) \exp \left( D^{E,2} \right) \right] + \text{closed form}.
\]

(3.70)
We have
\[
\begin{align*}
     d_M \text{Tr}_s & \left[ D_t^\ell \exp \left( \left( D_t^\ell + dt \frac{1}{2} N^A (T N) \right)^2 \right) \right]^{dt} \\
     & = \text{Tr}_s \left[ C_t^\ell , D_t^\ell \exp \left( \left( D_t^\ell + dt \frac{1}{2} N^A (T N) \right)^2 \right) \right]^{dt} \\
     & = - \text{Tr}_s \left[ D_t^\ell \exp \left( D_t^{\ell,2} + [C_t^\ell , D_t^\ell , dt \frac{1}{2} N^A (T N)] \right) \right]^{dt} \\
     & = \text{Tr}_s \left[ D_t^\ell \exp \left( D_t^{\ell,2} + [C_t^\ell , \left. D_t^\ell , dt \frac{1}{2} N^A (T N) \right) \right] \right]^{dt} \\
     & = \text{Tr}_s \left[ D_t^\ell \exp \left( D_t^{\ell,2} + [C_t^\ell , \left. D_t^\ell , dt \frac{1}{2} N^A (T N) \right) \right] \right]^{dt} \\
     & = \left( \begin{array}{c}
     \text{Tr}_s \left[ D_t^\ell \exp \left( D_t^{\ell,2} + dt \frac{1}{2} N^A (T N) \right) \right]^{dt} \\
\end{array} \right).
\end{align*}
\]

Thus
\[
\begin{align*}
     \text{Tr}_s & \left[ D_t^\ell \exp \left( \left( D_t^\ell + dt \frac{1}{2} N^A (T N) \right)^2 \right) \right]^{dt} \\
     & = \text{Tr}_s \left[ 2 D_t^{\ell,2} \exp \left( D_t^{\ell,2} + dt \frac{1}{2} N^A (T N) \right) \right]^{dt} + \text{closed form} \\
     & = \frac{\partial}{\partial b} \text{Tr}_s \left[ N^A (T N) \exp \left( (1 + b) D_t^{\ell,2} \right) \right]_{b=0} + \text{closed form} \\
     & = \frac{\partial}{\partial b} \text{Tr}_s \left[ N^A (T N) \exp \left( (1 + b) \frac{1}{2} N^A (T^* M) D_t^{\ell,2} (1 + b) \frac{1}{2} N^A (T^* M) \right) \right]_{b=0} + \text{closed form} \\
     & = t \frac{\partial}{\partial t} \text{Tr}_s \left[ N^A (T N) \exp \left( D_t^{\ell,2} \right) \right] + \frac{1}{2} N^A (T^* M) \text{Tr}_s \left[ N^A (T N) \exp \left( D_t^{\ell,2} \right) \right] + \text{closed form} .
\end{align*}
\]

By (3.70) and (3.72), we get (3.67). \(\square\)

Let \(r^N\) be the scalar curvature of \((N, g^TN)\). Let
\[
\begin{align*}
     R^E & = \nabla^{E,2}, \\
     R^{TN} & = \nabla^{TN,2}
\end{align*}
\]

be the curvatures of \(\nabla^E\) and \(\nabla^{TN}\) on \(E\) and \(TN\) over \(\mathcal{N}\). Let \(\mathcal{N}^A(T N)\) be the connection \(\mathcal{N}^A(T N)\), which is induced by \(\nabla^{TN}\). Then its curvature is \(\text{Tr}[R^{TN}]\).

Recall that \(S^{TN}_{2R} \) was defined in Definition 1.9. Since our fibration is flat, it follows from [B86, (1.28)] that, for \(U \in T_{R^N} \mathcal{N} \) and \(V, W \in T^H \mathcal{N}\),
\[
\langle S^{TN}_{2R} (U) V, W \rangle = \langle U, T(V, W) \rangle = 0 .
\]

Let \(\nabla^{\mathcal{N}^A(T N) \otimes E}\) be the connection on \(\mathcal{N}^A(T N) \otimes E\) induced by \(\nabla^{\mathcal{N}^A(T N)}\) and \(E\).
Recall that $\omega$ is the fiberwise Kähler form, $\omega^{TN}$ and $\omega^E$ are the variation of metrics on $TN$ and $E$. We also recall that $c(\cdot)$ is the Clifford action (cf. §(1.21)) associated with $g^{TN}/2$.

Let $(e_i)_{1\leq i \leq 2n}$ be an orthonormal basis of $T_R N$, let $(e^i)_{1\leq i \leq 2n}$ be the corresponding dual basis. Let $(f_\alpha)_{1\leq \alpha \leq m}$ a basis of $TM$. We identify the $f_\alpha$ with their horizontal lifts in $T^H N$. Let $(f^\alpha)_{1\leq \alpha \leq m}$ be the corresponding dual basis.

To interpret properly the formula that follows, we need to extend the basis $e_i$ to a parallel basis of $T^R N$ near the point $x$ which is considered. Moreover, we may suppose that $\nabla^T_R N e_i = 0$ at the point $x$.

**Proposition 3.11.** The following identity holds:

$$
= \frac{1}{2} \left( \nabla^\Lambda_{e_i} (T^N)^{\otimes E} + \langle S_{T^R N}(e_i) e_j, f_\alpha \rangle c(e_j) f^\alpha \right.
- \frac{1}{2} \left( \nabla^\Lambda_{e_i} (T^N)^{\otimes E} + \langle S_{T^R N}(e_i) e_j, f_\alpha \rangle c(e_j) f^\alpha \right.
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, e_j)\right) c(e_i) c(e_j)
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, f_\alpha) c(e_j) f^\alpha
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (f_\alpha, f_\beta) f^\alpha f^\beta - \frac{1}{8} r^N.
\]

**Proof.** Applying [B86, Theorem 3.5] with $t = 1/\sqrt{2}$ and (3.74), we have

$$
= \frac{1}{2} \left( \nabla^\Lambda_{e_i} (T^N)^{\otimes E} + \langle S_{T^R N}(e_i) e_j, f_\alpha \rangle c(e_j) f^\alpha \right.
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, e_j)\right) c(e_i) c(e_j)
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, f_\alpha) c(e_j) f^\alpha
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (f_\alpha, f_\beta) f^\alpha f^\beta - \frac{1}{8} r^N.
\]

Taking the degree 0 part of (3.76), we get

$$
= \frac{1}{2} \left( \nabla^\Lambda_{e_i} (T^N)^{\otimes E} + \langle S_{T^R N}(e_i) e_j, f_\alpha \rangle c(e_j) f^\alpha \right.
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, e_j)\right) c(e_i) c(e_j)
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (e_i, f_\alpha) c(e_j) f^\alpha
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right) (f_\alpha, f_\beta) f^\alpha f^\beta - \frac{1}{8} r^N.
\]

By [BGS88b, Proposition 1.19] and by (3.26), we get

$$
\omega^\delta = -\frac{\sqrt{-1}}{2} (d_M \omega)(e_i, e_j) c(e_i) c(e_j) - \frac{1}{4} (d_M \omega)(e_i, J e_i) + \omega^E.
\]
Since \( d_N \omega = 0 \) and \([d_N, d_M] = 0\), we have \( d_N d_M \omega = 0 \). Therefore
\[
\left[ \partial^E_N + \partial^{E,*}_N, -\frac{\sqrt{-1}}{4} (d_M \omega)(e_i, e_j)c(e_i)c(e_j) \right]
\]
(3.79)
\[
= \frac{\sqrt{-1}}{4} \left[ (c(e_k) \nabla^\Lambda (T^N) \otimes E, (d_M \omega)(e_i, e_j)c(e_i)c(e_j) \right]
\]
\[
= \frac{\sqrt{-1}}{4} \left[ \nabla^\Lambda (T^N) \otimes E (d_M \omega)(e_i, e_j)c(e_j) + (d_M \omega)(e_i, e_j)c(e_j) \nabla^\Lambda (T^N) \otimes E \right].
\]
By (3.77), (3.78) and (3.79), we get
\[
- \left( \partial^E_N + \partial^{E,*}_N \right)^2 - \frac{d}{2} \left( \partial^E_N + \partial^{E,*}_N, \frac{\sqrt{-1}}{2} \omega^\epsilon \right) + \partial \partial^* \frac{\sqrt{-1}}{2} (d_M \omega)(e_i, e_j)c(e_j)
\]
(3.80)
\[
- \partial \partial^* \left[ \nabla^\Lambda (T^N) \otimes E, \frac{\sqrt{-1}}{2} \omega^E - \frac{1}{8} (d_M \omega T^N)(e_j, J e_j)c(e_i) \right]
\]
\[
+ \partial \partial^* \left( \frac{1}{2} \omega^E - \frac{1}{8} (d_M \omega)(e_j, J e_j) \right)
\]
\[
- \frac{1}{2} \left( R^E + \frac{1}{2} \text{Tr}[R^T N] \right)(e_i, e_j)c(e_i)c(e_j) - \frac{1}{8} r^N.
\]
Comparing (3.76), (3.77), (3.80) with (3.75), it only remains to show that
\[
\sum_{i \neq j} \langle S_{T^N}(e_i)e_j, f_\alpha \rangle f^\alpha c(e_i)c(e_j) = 0,
\]
(3.81)
\[
\sum_i \sum_{j \neq k} (d_M \omega)(e_i, e_j)\langle S_{T^N}(e_i)e_k, f_\alpha \rangle f^\alpha c(e_i)c(e_k) = 0.
\]
By [B86 §1(c)], if \( U, V \in T^N \), then \( S_{T^N}(U)V - S_{T^N}(V)U \in T^N \). Thus
(3.82)
\[
\langle S_{T^N}(e_i)e_j, f_\alpha \rangle = \langle S_{T^N}(e_i)e_i, f_\alpha \rangle.
\]
By (3.82), we get the first identity in (3.81).

Now we prove the second identity in (3.81). For simplicity, we introduce the following notation
\[
\nabla_{f_\alpha} = i_{f_\alpha} d_M.
\]
By [B97 (1.5)], we have
\[
\langle S_{T^N}(e_i)e_k, f_\alpha \rangle = -\frac{1}{2} \left( (g_{T^N})^{-1}_N \nabla_{f_\alpha} g_{T^N}(e_i), e_k \right) = -\frac{1}{2} (\nabla_{f_\alpha} \omega)(e_i, J e_k).
\]
Therefore the second identity in (3.81) is equivalent to the follows:
\[
\sum_i \sum_{j \neq k} (\nabla_{f_\alpha}) (e_i, e_j)(\nabla_{f_\alpha}) (e_i, J e_k) f^\alpha f^\beta c(e_i)c(e_k) = 0.
\]
Since \((J e_i)_{1 \leq i \leq n}\) is also an orthogonal basis of \( T_{R}N \), using the fact that \( \omega \) and \( d_M \omega \) are \( J \)-invariant, we get
\[
\sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (e_i, J e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

\[
= \frac{1}{2} \sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (e_i, J e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

\[
+ \frac{1}{2} \sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (J e_i, J e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

\[
= \frac{1}{2} \sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (e_i, J e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

\[
- \frac{1}{2} \sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, J e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (e_i, e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

(3.86)

Exchanging the roles of \( j, k \) and of \( \alpha, \beta \), we obtain

\[
\sum \sum (\nabla f_{i\alpha} \nabla f_{\beta}) (e_i, J e_j) (\nabla f_{j\gamma} \nabla f_{\delta}) (e_i, e_k) f^\alpha f^\beta c(e_j) c(e_k)
\]

(3.87)

By (3.86) and (3.87), we get (3.85).

**Proof of Theorem 3.6** The proof of (3.57) follows the same argument as [BL95, Theorem 3.16].

Now we prove the first formula in (3.58). By Lemma 3.9, it is sufficient to establish the asymptotics of the following terms as \( t \to 0 \):

\[
\text{Tr}_s \left[ \exp \left( -C_t^{\alpha, \beta} - da \frac{1}{2} \left( \tilde{\nabla}_N + t \tilde{\nabla}_N^{E,*} \right) \right)
\]

\[
- d \tilde{a} \left[ \tilde{\nabla}_N + t \tilde{\nabla}_N^{E,*} , \frac{\epsilon}{2} \omega^\rho \right] + d d \tilde{a} \frac{\epsilon}{2} \omega^\rho \right] \right] c d d \tilde{a},
\]

\[
d_M \text{Tr}_s \left[ \frac{1}{2} N^A (T_N) \exp \left( D_t^{\alpha, \beta} \right) \right].
\]

(3.88)

As \( t \to 0 \), we claim that we can use equation (3.75) exactly as in Bismut-Köhler [BK92, Theorem 3.22]. The main difference is that in [BK92], the space of variations of the metrics is 1-dimensional, while here it is the full basis \( M \). By proceeding as in this reference, we get

\[
\sqrt{2\pi i \varphi} \text{Tr}_s \left[ \exp \left( -C_t^{\alpha, \beta} - da \frac{1}{2} \left( \tilde{\nabla}_N + t \tilde{\nabla}_N^{E,*} \right) \right)
\]

\[
- d \tilde{a} \left[ \tilde{\nabla}_N + t \tilde{\nabla}_N^{E,*} , \frac{\epsilon}{2} \omega^\rho \right] + d d \tilde{a} \frac{\epsilon}{2} \omega^\rho \right] \right] c d d \tilde{a}
\]

\[
= q_s \left[ \tilde{\text{Td}}(TN, g^{TN}) * \tilde{\text{ch}}(E, g^E) \right] + O(t).
\]

(3.89)

This gives the asymptotics of the first term in (3.88).
We turn to study the second term in \((3.88)\). As \(t \to 0\), by the local families index theorem technique \([B86]\), we get
\begin{equation}
\varphi \operatorname{Tr}_s \left[ t^N \exp \left( D_t^{\phi,2} \right) \right] = q_* \left[ \frac{\omega}{2\pi} Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] + O(\sqrt{t}) .
\end{equation}

Furthermore, by \([BGS88a]\, Theorems 2.11, 2.16\], the asymptotics of \(\operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right] \) is given by a Laurent series. By \((3.90)\), we get
\begin{equation}
\varphi \operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right] = C_{-1} t^{-1} + C_0 + O(t) ,
\end{equation}
with
\begin{equation}
C_{-1} = q_* \left[ \frac{\omega}{2\pi} Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] .
\end{equation}

Let \(C_{-1}^{(p)}\) (resp. \(C_{0}^{(p)}\)) be the component of degree \(p\) of \(C_{-1}\) (resp. \(C_0\)). By Remark \(3.7\) for \(p > 0\), \(C_{-1}^{(p)} = 0\). Then
\begin{equation}
(1 + N^A(T^{*}M) + t \frac{\partial}{\partial t}) \operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right] = \sum_p \left( (p + 1)C_{0}^{(p)} \right) + O(t) .
\end{equation}

Applying \((3.89)\) with \(E\) replaced by \(E_+\) (see the proof of Proposition \(3.5\)) and taking the \(dt\) component, we get
\begin{equation}
\varphi \operatorname{Tr}_s \left[ \exp \left( - C_{0}^{\phi,2} - da \frac{1}{2} (\nabla_N^{E_+} + t \nabla_N^{E_+}) \right) \right.
\end{equation}
\begin{equation}
\left. - d\bar{a} \left[ \nabla_N^{E_+} + t \nabla_N^{E_+}, \frac{et}{2} \omega^{E_+} \right] + d\bar{a} \frac{et}{2} \omega^{E_+} \right]^{ed\bar{a}dt} = - \frac{1}{2} q_* \left[ Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] + O(t) .
\end{equation}

By Theorem \(2.5\), Lemma \(3.10\) and \((3.94)\), we have
\begin{equation}
(1 + N^A(T^{*}M) + t \frac{\partial}{\partial t}) \operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right] = \text{closed form} + O(t) .
\end{equation}

By \((3.93)\) and \((3.95)\), we have
\begin{equation}
d_M C_{0} = 0 .
\end{equation}

By \((3.91)\), \((3.92)\) and \((3.96)\), as \(t \to 0\), we have
\begin{equation}
\sqrt{2\pi i} \varphi d_M \operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right] = d_M \varphi \operatorname{Tr}_s \left[ N^A(T^{*}N) \exp \left( D_t^{\phi,2} \right) \right]
\end{equation}
\begin{equation}
= \frac{t}{d} d_M q_* \left[ \frac{\omega}{2\pi} Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] + O(\sqrt{t}) .
\end{equation}

This gives the asymptotics of the second term in \((3.88)\).

The first formula in \((3.58)\) follows from Lemma \(3.9\), \((3.89)\) and \((3.97)\).

The second formula in \((3.58)\) may be proved as a consequence of the first one by applying the same technique as in the proof of Proposition \(3.5\). 
\(\square\)
3.7. Analytic torsion forms.

We choose \( g_1, g_2 \in C^\infty(\mathbb{R}_+, \mathbb{R}) \) satisfying

\[
\begin{align*}
(3.98) \\
g_1(t) &= 1 + \mathcal{O}(t), \quad g_2(t) = 1 + \mathcal{O}(t^2), \quad \text{as } t \to 0, \\
(3.99) \\
g_1(t) &= \mathcal{O}(t), \quad g_2(t) = \mathcal{O}(t^3), \quad \text{as } t \to +\infty,
\end{align*}
\]

and

\[
\begin{align*}
(3.100) \\
\int_0^1 g_1(t) - \frac{1}{t} \, dt + \int_1^{+\infty} \frac{g_1(t)}{t} \, dt &= \Gamma'(1) - 2, \\
\int_0^1 g_2(t) - \frac{1}{t^2} \, dt + \int_1^{+\infty} \frac{g_2(t)}{t^2} \, dt &= 1.
\end{align*}
\]

Using Mellin transformation, (3.100) can be reformulated as follows

\[
\begin{align*}
(3.101) \\
\left( \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} g_1(t) dt \right)_{s=0} &= -2, \\
\left( \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-2} g_2(t) dt \right)_{s=0} &= 0.
\end{align*}
\]

**Definition 3.12.** The analytic torsion forms \( \mathcal{T}(g^{TN}, g^E) \in \Omega^{even}(M) \) are defined by

\[
\mathcal{T}(g^{TN}, g^E) = -\int_0^{+\infty} \left\{ \beta t + \frac{g_1(t) - \frac{1}{2}}{2} \chi'(N, E) - \frac{g_1(t)}{2} n\chi(N, E) \\
+ \frac{g_2(t)}{2} q_s \left[ Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] \\
+ \frac{g_2(t)}{2t} q_s \left[ \omega \frac{\omega}{2\pi} Td(TN, \nabla^{TN})ch(E, \nabla^E) \right] \right\} dt.
\]

By Theorem 3.6, \( \mathcal{T}(g^{TN}, g^E) \) is well-defined. Here we remark that \( \mathcal{T}(g^{TN}, g^E) \) is independent of \( g_1 \) and \( g_2 \).

**Proposition 3.13.** We have

\[
(3.102)
\]

\[
d_M \mathcal{T}(g^{TN}, g^E) = q_s \left[ Td(TN, g^{TN}) * \tilde{ch}(E, g^E) \right] \\
- f(H(N, E), \nabla^{H(N,E), g^{H(N,E)}}) .
\]

**Proof.** By Theorem 2.5, \( q_s \left[ Td'(TN, \nabla^{TN})ch(E,\nabla^E) \right] \) is a constant function on \( M \). Then, by Proposition 3.5, we get

\[
(3.103)
\]

\[
d_M \mathcal{T}(g^{TN}, g^E)
\]

\[
= -\int_0^{+\infty} \left\{ d_M \beta t + \frac{g_2(t)}{2t} \right\} dt
+ \frac{\omega}{2\pi} Td(TN, \nabla^{TN})ch(E, \nabla^E) \right\} dt.
\]

By Theorem 3.6, (3.100) and (3.103), we get (3.103). \( \square \)
Proceeding in the same way as in [BL95 Theorem 3.16], we get

\begin{equation}
\text{Tr}_s \left[ N^\Lambda (T^\dagger N) \exp \left( - t D_{N}^{E,2} \right) \right] = \chi'(N, E) + \mathcal{O}(t^{-1}) , \quad \text{as } t \to +\infty .
\end{equation}

For \( s \in \mathbb{C} \) with \( \text{Re}(s) > n \), we define

\begin{equation}
\theta(s) = -\frac{1}{\Gamma(s)} \int_{0}^{+\infty} t^{s-1} \left[ \text{Tr}_s \left[ N^\Lambda (T^\dagger N) \exp \left( - t D_{N}^{E,2} \right) \right] - \chi'(N, E) \right] dt .
\end{equation}

By [Se67], the function \( \theta(s) \) admits a meromorphic continuation to the whole complex plane, which is regular at \( 0 \in \mathbb{C} \).

Let \( \mathcal{F}^{[0]}(g^{TN}, g^{E}) \) be the component of \( \mathcal{F}(g^{TN}, g^{E}) \) of degree zero.

**Proposition 3.14.** We have

\begin{equation}
\mathcal{F}^{[0]}(g^{TN}, g^{E}) = \frac{1}{2} \theta'(0) .
\end{equation}

**Proof.** By (3.35) and (3.46), we get

\begin{equation}
\beta_t^{[0]} = \text{Tr}_s \left[ \frac{N^\Lambda (T^\dagger N)}{2} \left( 1 - 2t D_{N}^{E,2} \right) \exp \left( - t D_{N}^{E,2} \right) \right] \\
= \frac{1}{2} \left( 1 + 2t \frac{\partial}{\partial t} \right) \text{Tr}_s \left[ N^\Lambda (T^\dagger N) \exp \left( - t D_{N}^{E,2} \right) \right].
\end{equation}

By (3.91), there exist \( a_0, a_1 \in \mathbb{C} \) such that, as \( t \to 0 \),

\begin{equation}
\text{Tr}_s \left[ N^\Lambda (T^\dagger N) \exp \left( - t D_{N}^{E,2} \right) \right] = a_{-1} t^{-1} + a_0 + \mathcal{O}(\sqrt{t}) .
\end{equation}

By (3.58), (3.108) and (3.109), we get

\begin{equation}
a_0 = -q_* \left[ Td'(TN, \nabla^{TN}) \text{ch}(E, \nabla^{E}) \right] + n \chi(N, E) .
\end{equation}

By (3.106), (3.109) and (3.110), we get

\begin{equation}
\theta(0) = q_* \left[ Td'(TN, \nabla^{TN}) \text{ch}(E, \nabla^{E}) \right] - n \chi(N, E) + \chi'(N, E) .
\end{equation}
By Definition 3.12, (3.101), (3.106), and (3.108), we have
\begin{align*}
\mathcal{S}^{[0]}(g_{TN}, g_E) &= - \int_0^{+\infty} \left( \frac{1}{2} (1 + 2t \frac{\partial}{\partial t}) \text{Tr}_s \left[ N^A(T^{TN}) \exp \left( - tD_{N,2}^E \right) \right] - \frac{1}{2} \chi'(N, E) \right) dt \\
&+ \frac{g_1(t)}{2} \left( q_s \left[ \text{Td}'(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] - n\chi(N, E) + \chi'(N, E) \right) \\
&+ \frac{g_2(t)}{2t} q_s \left[ \frac{\omega}{2\pi} \text{Td}(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] dt \\
= - \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left( 1 \Gamma(s) \right) \int_0^{+\infty} t^{s-1} \left( 1 + 2t \frac{\partial}{\partial t} \right) \left\{ \text{Tr}_s \left[ N^A(T^{TN}) \exp \left( - tD_{N,2}^E \right) \right] - \chi'(N, E) \right\} dt \\
&= - \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left( 1 \Gamma(s) \right) \int_0^{+\infty} t^{s-1} \left( q_s \left[ \text{Td}'(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] - n\chi(N, E) + \chi'(N, E) \right) dt \\
&= - \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} \left( 1 \Gamma(s) \right) \int_0^{+\infty} t^{s-2} \left( g_2(t) dt q_s \left[ \frac{\omega}{2\pi} \text{Td}(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] \right) \\
&= \frac{d}{ds} \bigg|_{s=0} \left( \frac{1 - 2s}{2} \theta(s) + q_s \left[ \text{Td}'(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] - n\chi(N, E) + \chi'(N, E) \right) \\
&= \frac{1}{2} \theta'(0) - \theta(0) + q_s \left[ \text{Td}'(TN, \nabla^{TN}) \text{ch} (E, \nabla^E) \right] - \chi(N, E) + \chi'(N, E).
\end{align*}

By (3.111) and (3.112), we obtain (3.107). \hfill \Box

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