Even $A$-cycles have the edge-Erdős-Pósa property

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Abstract

I prove that even $A$-cycles have the edge-Erdős-Pósa property.

1 Introduction

Recently, it has become ever more clear that there is a significant difference between the ordinary Erdős-Pósa property and its edge-version. Initially it might have seen as if packing and covering was largely the same in the vertex world and in the edge world: as Erdős and Pósa [8] showed every graph either has $k$ vertex-disjoint cycles or a vertex set of size $O(k \log k)$ meeting all cycles, and similarly, there are always either $k$ edge-disjoint cycles or an edge set of size $O(k \log k)$ meeting all cycles. The same is true for many other classes of target objects.

Say that a class $F$ of graphs has the (ordinary) Erdős-Pósa property (resp. the edge-Erdős-Pósa property) if there is a function $f$ such that for every positive integer $k$, every graph $G$ either contains $k$ disjoint (resp. edge-disjoint) subgraphs each isomorphic to some graph in $F$, or if there is a vertex set $X$ (resp. an edge set) of size $|X| \leq f(k)$ such that $G - X$ is devoid of subgraphs from $F$. Then not only have cycles both the ordinary and the edge-Erdős-Pósa property but this is also true for even cycles [6, 14, 4]; for $A$-cycles [10, 11], cycles that each contain at least one vertex from some fixed set $A$; long cycles [13, 2], cycles that have a certain pre-fixed minimum length; $K_4$-subdivisions [13, 1]; and many other classes of graphs.

Recently, however, differences have been discovered. While I do not know of any class that has the edge-Erdős-Pósa property but not the ordinary property, the converse does exist: even $A$-paths have the ordinary property but not the edge-property [4]. The same is true for subdivisions of subcubic trees of large pathwidth [3]. I will present a class that sits, in some sense, right at the edge: even $A$-cycles turn out to have both properties, the ordinary and the edge-Erdős-Pósa property, but slightly modifying the constraints breaks the edge-property but not the ordinary one.

That even $A$-cycles have the ordinary Erdős-Pósa property is a special case of a much more general result due to Kakimura and Kawarabayashi:

Theorem 1 (Kakimura and Kawarabayashi [9]). Let $m > 1$ be an integer. Then $A$-cycles of length divisible by $m$ have the Erdős-Pósa property.
I will prove:

**Theorem 2.** Even $A$-cycles have the edge-Erdős-Pósa property.

Allowing more complicated constraints on the length of the cycles, as in the theorem of Kakimura and Kawarabayashi, breaks the edge-property. $A$-cycles of a length divisible by 3, for example, do not have the edge-Erdős-Pósa property; see Section 2. We can also insist on a certain minimum length, i.e. only consider even $A$-cycles of a length at least $\ell$. Imposing a minimum length never seems to affect the ordinary Erdős-Pósa property, and indeed long even $A$-cycles still have the ordinary property [5]. In contrast, the edge-property is lost; see Section 2.

How large is the hitting set in Theorem 2? (The hitting set is the vertex or edge set that meets all subgraphs from the target class.) Unfortunately, the size depends on the vertex hitting set of Theorem 1 and Kakimura and Kawarabayashi do not give any explicit bound on their hitting set. The bound is likely very large as Kakimura and Kawarabayashi extensively use the techniques of minor theory, where bounds tend to become huge. It is possible, however, to say what the proof of Theorem 2 adds. If $g(k)$ is an upper bound on the size of a vertex hitting set for even $A$-cycles then the edge hitting set of Theorem 2 has size at most $10^5 \cdot g(k)$. Clearly, this is far from optimal.

There is an extensively list of result on the ordinary Erdős-Pósa property, see for instance [12] and [4] for references. There is far less known about the edge-property. Even cycles [4] as well as $A$-cycles [11] were known to have the edge-Erdős-Pósa property. For other simple classes, such as the subdivisions of a ladder graph with four rungs or of the binary tree of height 3, it is, on the other hand, still open whether they possess the edge-Erdős-Pósa property.

## 2 More constraints break the edge-property

Before proving the main theorem I will first briefly discuss how extensions of Theorem 2 fail. In the notation I follow Diestel [7], where also basic graph-theoretic concepts may be found. All graphs will be simple and finite.

**Proposition 3.** Let $\ell \geq 5$ be an integer. Even $A$-cycles of length at least $\ell$ do not have the edge-Erdős-Pósa property.

Fix an $\ell \geq 5$, and call a cycle of length at least $\ell$ long. We construct for every integer $h > 0$ a graph $G_h$ that does not contain two edge-disjoint long even $A$-cycles and that does not admit an edge set of size at most $h$ that meets all long even $A$-cycles. This then shows that long even $A$-cycles do not have the edge-Erdős-Pósa property.

To construct $G_h$ we start with an (elementary) wall $W$ of size $10h \times 10h$. (See for instance [4] for a formal definition of a wall, or Figure 11 for a picture of a $6 \times 6$-wall.) Let $U, V$ be the two bipartition classes of the bipartite wall $W$. We add a complete bipartite graph with one bipartition class consisting of two vertices, $u$ and $v$, and the other consisting of the vertex set $A$ of size $10h$. We link $u$ to the left-most vertex in every second row of the wall $W$, and $v$ to the right-most vertex in every second row such that the neighbours of $u$ in $W$ are all in $U$ and the neighbours of $v$ in $W$ are all in $V$. Finally, we suppress the vertices of $V$ in the top row such that only the vertices of $V$ remain in the top
row; see Figure 1. (Suppressing a vertex of degree 2 means to replace it by an edge between its neighbours.)

We first observe that any path in $W$ between a neighbour of $u$ and a neighbour of $v$ has odd length, unless it passes through an odd number of edges from the top row. Thus, any even $A$-cycle that meets the wall needs to traverse the wall from left to right, from a neighbour of $u$ to a neighbour of $v$, and to pass through at least one edge from the top row. Clearly, there cannot be two such edge-disjoint cycles. On the other hand, any cycle of length at least 5 has to meet the wall.

To see that there is no set of at most $h$ edges that meets all long even $A$-cycles, it suffices to observe that $u$ and $v$ are well connected to the wall (each with $5h$ edges), that the wall is sufficiently large and that it contains many parity breaking edges, the edges in the top row.

Theorem 1 of Kakimura and Kawarabayashi covers quite general modularity constraints on the length of the $A$-cycles. All of these, except for requiring the cycles to be even, break the edge-Erdős-Pósa property:

**Proposition 4.** For integers $m > 2$, the class of $A$-cycles of length $\equiv 0 \pmod{m}$ does not have the edge-Erdős-Pósa property.

The construction is quite similar to the one for long even $A$-cycles; it suffices to subdivide the edges in the graph $G_h$ shown in Figure 1 in a suitable way: Subdivide all edges between $u$ and $A$ such that the resulting paths have length $m - 2$, leave the edges between $v$ and $A$ and the edges in the top row (in grey) as they are, but subdivide all other edges so that they become paths of length $m$. Then, $A$-cycles of length $\equiv 0 \pmod{m}$ will have the same form as before: In particular, they will contain a subpath that traverses the wall from left to right and passes through an edge in the top row in between. Here, we use that $m \neq 2$ to exclude that the subdivision of the complete bipartite graph between $A$ and $\{u, v\}$ contains an $A$-cycle of length $\equiv 0 \pmod{m}$. Any cycle there has length $2(m - 2) + 2 \cdot 1 = 2m - 2 \neq 0$ whenever $m \neq 2$.
3 Basic definitions and lemmas

In the remainder of the article I will prove Theorem 2. In this section I collect simple lemmas and a little bit of notation. In particular, we will use Diestel’s [7] path notation. That is, if \( P \) is a path that contains the vertices \( u \) and \( v \) then \( uPv \) is the subpath of \( P \) with endvertices \( u \) and \( v \).

A second central notion is that of a block-tree of a connected graph \( G \). If \( B \) is the set of blocks of \( G \) and \( C \) the set of cutvertices of \( G \) then the block-tree of \( G \) is defined on \( B \cup C \) as vertex set such that \( Bc \) with \( B \in B \) and \( c \in C \) is an edge precisely when \( c \) lies in \( B \).

A theta graph is a subdivision of the multigraph consisting of two vertices joined by three parallel edges. As we deal with (simple) graphs we insist that two of the parallel edges need to be subdivided. The two vertices of degree 3 in a theta graph are its branch vertices, the three internally disjoint paths between the branch vertices are its subdivided edges.

We need a very simple lemma that nevertheless is key to the proof. I contend that, in some sense, the reason why even \( A \)-cycles have the edge-Erdős-Pósa property but not \( A \)-cycles of a length divisible by 3, say, lies in the fact that theta graphs always contain an even cycle but not necessarily one of a length divisible by 3.

\[ \text{Lemma 5.} \quad \text{Let } \theta \text{ be a theta graph such that two of its subdivided edges meet a vertex set } A. \text{ Then } \theta \text{ contains an even } A\text{-cycle.} \]

\[ \text{Proof.} \quad \text{Let } P_1, P_2, P_3 \text{ be the subdivided edges of } \theta. \text{ Then each of } P_1 \cup P_2, P_2 \cup P_3 \text{ and } P_3 \cup P_1 \text{ is an } A\text{-cycle, and at least one of these is even.} \]

A second observation shows that we can exploit the fact that the ordinary Erdős-Pósa property is already known to hold for even \( A \)-cycles. Here, we call the upper bound on a hitting set, the function \( f \) of the definition of the (ordinary or edge-) Erdős-Pósa property, a bounding function.

\[ \text{Lemma 6 (Bruhn and Heinlein [1]).} \quad \text{Let } F \text{ be a class of graphs that has the Erdős-Pósa property with bounding function } g. \text{ Let } h : \mathbb{N} \to \mathbb{R} \text{ be a function such that for every } k \text{ and for every graph } G \text{ that has a vertex } z \text{ such that } G \setminus z \text{ does not contain any subgraph of } F \text{ it holds that: either } G \text{ contains } k \text{ edge-disjoint subgraphs from } F \text{ or there is an edge set } F \text{ of size } |F| \leq h(k) \text{ that meets every subgraph from } F. \text{ Then } F \text{ has the edge-Erdős-Pósa property with bounding function } f = gh. \]

We note that, although not formulated in this way, the lemma and indeed its proof still hold true for classes of graphs with labels, such as even \( A \)-cycles.

We need a couple of more simple lemmas. Let \( a, b, c \) be three distinct vertices. An \( a-b-c \) path is a path that starts in \( a \), passes through \( b \) and ends in \( c \).
Lemma 7. Let \( a, b, c \) be distinct vertices in a graph \( G \). Then \( G \) contains an \( a-b-c \) path if and only if no single vertex, except for \( b \), separates \( b \) from \( \{a, c\} \).

Proof. As no single vertex separates \( b \) from \( \{a, c\} \) there are two internally disjoint \( b-\{a, c\} \) paths \( P_1, P_2 \). Clearly, we are done if \( P_1 \) and \( P_2 \) have different endvertices. Thus, we assume that \( P_1 \) and \( P_2 \) end in \( c \). Again, as no single vertex separates \( b \) from \( \{a, c\} \) there is a path \( Q \) from \( b \) to \( a \) that avoids \( c \). Viewed from \( a \) let \( z \) be the first vertex on \( Q \) in \( P_1 \cup P_2 \). Let us say that \( z \in V(P_1) \). Then \( aQzP_1bP_2c \) is an \( a-b-c \) path. \( \blacksquare \)

The diameter of a tree is the length of a longest path in the tree.

Lemma 8. Every tree with \( s \geq 2 \) leaves and diameter \( d \) has at most \( \frac{s d}{2} + 1 \) vertices.

Proof. Let \( T \) be a tree with \( s \) leaves and diameter \( d \). For \( s = 2 \) the tree \( T \) is a path of length \( d \) and thus has \( d + 1 = \frac{s d}{2} + 1 \) many vertices. For \( s \geq 3 \), pick a longest path \( P \) and a leaf \( \ell \) not on \( P \), and let \( Q \) be the path that starts in \( \ell \), ends in a vertex \( t \) of degree at least 3 and that has no internal vertex of degree 3 or more. In particular, except for possibly \( t \), no vertex in \( Q \) lies in the longest path \( P \). Moreover, we note that the length of \( Q \) is at most \( d/2 \).

Let \( T' \) be the tree obtained from deleting all of \( Q \) except for \( t \). Then \( T' \) has \( s - 1 \) leaves and at most \( d/2 \) fewer vertices than \( T \). Induction on \( s \) now finishes the proof. \( \blacksquare \)

Let \( T \) be a rooted tree with root \( r \). For any vertex \( v \) we denote by \( T_v \) the subtree of \( T \) on all vertices \( u \) for which the \( w-r \) path in \( T \) passes through \( v \).

Lemma 9. Every tree \( T \) with \( \Delta(T) \geq 3 \) and with \( s \) leaves contains \( \lfloor s/(2\Delta(T)) \rfloor \) disjoint subtrees that each contain three leaves of \( T \).

Proof. Pick a root \( r \) and consider \( T \) as a rooted tree. In \( T \) choose a vertex \( v \) that is farthest from \( r \) such that the subtree \( T_v \) contains at least three leaves. Then the components of \( T_v - v \) each contain at most two leaves of \( T \), and there are at most \( \Delta(T) \) such components. Therefore, \( T_v \) contains at most \( 2\Delta(T) \) many leaves of \( T \) (this is also true if \( v \) is a leaf itself). We now delete \( T_v \) from \( T \), and then delete iteratively each leaf that is not a leaf from \( T \). Then all leaves of the resulting tree \( T' \) are leaves of \( T \), and \( T' \) has at least \( s - 2\Delta(T) \) many leaves. Induction on the number of leaves now yields the desired result. \( \blacksquare \)

Lemma 10. Let \( \alpha, \beta, \gamma \) be positive integers. For every tree \( T \) on at least \( \alpha \beta \gamma \) vertices at least one of the following statements is true:

(i) \( \Delta(T) \geq \alpha \); or

(ii) \( T \) contains \( \beta \) disjoint subtrees that each contain three leaves of \( T \); or

(iii) there is a path in \( T \) of length at least \( \gamma \).

Proof. Assume \( T \) to be a tree that satisfies neither (i), (ii) nor (iii). We will bound the number \( n \) of its vertices. By Lemma 9 it follows for the number \( s \) of leaves of \( T \) that

\[
\left\lfloor \frac{s}{2\Delta(T)} \right\rfloor \leq \beta - 1 \quad \Rightarrow \quad \frac{s}{2\Delta(T)} \leq \beta,
\]
which implies $s \leq 2(\alpha - 1)\beta \leq 2\alpha \beta - 2$.

Lemma 8 yields $n \leq \frac{1}{2}s(\gamma - 1) + 1 < \alpha \beta \gamma$.

The following lemma is a more general form of a lemma in [4], which, in turn, is the finite special case of a lemma of Thomassen [15]. It can be proved with a simple greedy argument.

**Lemma 11.** Let $s$ be a positive integer, and let $Z$ be a vertex set in a tree $T$. Then $T$ contains $\lfloor |Z|/s \rfloor$ edge-disjoint subtrees that each contain at least $s$ vertices from $Z$.

4 Proof of main result

In the course of this section we will prove that for all positive integers $k$, every graph $G$ with a single vertex hitting set either contains $k$ edge-disjoint even $A$-cycles or admits an edge set $F$ of size $|F| \leq 1080k^5$ that meets all even $A$-cycles. Lemma 6, together with Theorem 1, then finishes the proof of Theorem 2.

We fix throughout this section a graph $G$, a vertex set $A \subseteq V(G)$, and an integer $k \geq 2$. For the proof we can make three assumptions:

1. There is a vertex $z$ that meets every even $A$-cycle of $G$.
2. $G$ does not contain $k$ edge-disjoint even $A$-cycles.
3. $G$ is 2-connected.

The last assumption is justified because every even $A$-cycle lies in a block of $G$.

**Lemma 12.** If $z \in A$ then there is an edge set $F$ of size $|F| \leq 4k$ that meets all even $A$-cycles of $G$.

**Proof.** Suppose that $z$ has degree at least $4k$. Let $G'$ be the graph arising from subdividing every edge incident with $z$ once. By (3), $G$ is 2-connected, and $G'$ therefore too. Thus, there is a tree $T'$ in $G' - z$ that contains all neighbours of $z$. Of these at least $4k$ neighbours, at least $2k$ lie in the same partition class of the bipartite graph $T'$; let the set of these neighbours be $Z$. Lemma 11 yields a set $P'$ of $k$ edge-disjoint $Z$-paths in $T'$, each of which has even length by choice of $Z$. Moreover, since the vertices in $Z$ have degree 1 in $G'$, no vertex in $Z$ lies in two paths of $P'$. Now, by identifying the endvertices of the paths in $P'$ with $z$, we obtain $k$ edge-disjoint even cycles in $G$ that each pass through $z$. As $z \in A$ we have thus found $k$ edge-disjoint even $A$-cycles, which is impossible by (2).

Therefore $z$ has degree less than $4k$. By (1), the set $F$ of edges incident with $z$ meets all even $A$-cycles.

Thus, from now on, we may assume that $z \notin A$.

**Lemma 13.** Let $B$ be a block of $G - z$, and let $a \in A \cap V(B)$. Then $a$ has at most two neighbours in $B$.

**Proof.** Suppose that $a$ has three neighbours $b_1, b_2, b_3$ in $B$. Pick a smallest tree $T$ in $B - a$ that contains $b_1, b_2, b_3$. Then $T$ together with $a$ and the edges $ab_1, ab_2, ab_3$ forms a theta graph $\theta$ with branch vertex $a$. By Lemma 5, $\theta$ contains an even $A$-cycle that then is disjoint from $z$, a contradiction to (1).
For a subgraph $H$ of $G$, an even cycle is $H$-heavy if it passes through an edge of $H$ that is incident with a vertex in $A$. Observe that, since $z \notin A$ by (4), every even $A$-cycle needs to be $B$-heavy for some block $B$ of $G - z$.

By (3), $G$ is 2-connected, which means that $G - z$ is connected and therefore admits a block-tree. It also follows that

\[
\text{every leaf-block of the block-tree of } G - z \text{ contains a neighbour of } z. \tag{5}
\]

**Lemma 14.** For every block $B$ of $G - z$ there is a set $F_B \subseteq E(G)$ of size $|F_B| \leq 12k$ that meets every $B$-heavy even cycle.

**Proof.** Assume first that $B$ contains at most two vertices from $A$. Then let $F_B$ be the set of edges of $B$ that are incident with vertices from $A$. By Lemma 13, we have $|F_B| \leq 4$. Clearly, $F_B$ meets every even $A$-cycle that is $B$-heavy.

Thus, we assume now that $B$ contains at least three vertices from $A$. Consider a cycle $C \subseteq B$ through at least one vertex of $A$. Then, every vertex in $A \cap V(B)$ lies in $C$; otherwise we could find a $C$-path $P$ in $B$ that passes through some vertex $a' \in V(B - C)$, which would imply with Lemma 5 that the theta graph $C \cup P$ contained an even $A$-cycle that avoids $z$, which is impossible by (1).

Let $a_0, a_1, \ldots, a_{\ell - 1}$ be an enumeration of $A \cap V(B)$ in the order in which the vertices appear on $C$. Note that $\ell \geq 2$. Then, $C$ splits into edge-disjoint subpaths $P_0, \ldots, P_{\ell - 1}$ such that $P_i$ has endvertices $a_i$ and $a_{i+1}$, where indices are taken mod $\ell$. For $i = 0, \ldots, \ell - 1$ define $S_i$ to be the union of $P_i$ with all $C$-paths with both endvertices in $P_i$.

We prove:

(i) $B = \bigcup_{i=0}^{\ell-1} S_i$;

(ii) $S_i \cap S_j = \emptyset$ whenever $i + 1 < j$; and

(iii) $S_i \cap S_{i+1} = \{a_{i+1}\}$ for every $i = 0, \ldots, \ell - 1$ (indices taken mod $\ell$).

Indeed, for (i) observe that no $C$-path in $B$ may meet $a_0, \ldots, a_{\ell - 1}$ as these vertices have degree $\leq 2$ in $B$, by Lemma 13 and each of the vertices has already degree 2 in $C$. Thus, any $C$-path $Q$ with its endvertices not both in some $P_i$ has its endvertices in the interior of two distinct $P_i$, $P_j$ with $i < j$. Then $C \cup Q$ is a theta graph such that two of its subdivided edges meet one of $a_i, a_{i+1}$. Lemma 5 yields again an even $A$-cycle that avoids $z$, which is impossible by (1).
For (ii and iii), suppose there were $i < j$ and a $C$-path $Q_i$ with its endvertices in $P_i$, and a $C$-path $Q_j$ with its endvertices in $P_j$ such that $Q_i$ and $Q_j$ meet, in a vertex $v$ say. Then $v$ must be an interior vertex of $Q_i$ (and of $Q_j$). Thus, $Q_iQ_j$ is a $C$-path with its endvertices in distinct paths of $P_0, \ldots, P_{k-1}$, which is impossible by (i).

Next:

Fewer than $3k$ of the graphs $S_0 = a_1, \ldots, S_{k-1} = a_k$ contain a cutvertex of $G - z$ or a neighbour of $z$. \hfill (6)

Suppose that for $i_0 < \ldots < i_{3k-1}$ each of $S_{i_j} = a_{i_j+1}$ contains a cutvertex of $G - z$ or a neighbour of $z$. In each case, we find, by (iv), a $C-z$ path $Q_{i_j}$ that starts in $P_{i_j}$ and meets $B$ only in $S_{i_j}$. In particular, the paths $Q_{i_j}$ pairwise meet only in $z$. Let $q_{i_j}$ be the endvertex of $Q_{i_j}$ on $C$, and denote by $R_{i_j}$ the $q_{i_j}q_{i_j+1}$ subpath of $C$ that contains $a_{i_j+1}$. Then for $t = 0, \ldots, 3k - 1$

$$\theta_t = Q_{i_t} \cup R_{i_t} \cup Q_{i_{t+1}} \cup R_{i_{t+1}} \cup Q_{i_{t-2}}$$

is a theta graph such that two of its subdivided edges, namely $Q_{i_t} \cup R_{i_t}$ and $R_{i_{t+1}} \cup Q_{i_{t+2}}$, meet $A$ (in $a_{i_{t+1}}$, resp. in $a_{i_{t+1}+1}$). Thus, $\theta_0, \ldots, \theta_{k-1}$ are edge-disjoint graphs that each, by Lemma 5, contain an even $A$-cycle in contradiction of (ii). This proves (i).

Let $i_0 < \ldots < i_s$ be the sequence of all indices $i_j$ such that $S_{i_j} = a_{i_j+1}$ contains a cutvertex of $G - z$ or a neighbour of $z$; and let $F_B$ be the set of edges of $B$ incident with $\{a_{i_j}, a_{i_j+1} : j = 0, \ldots, s\}$. Then, as each $a_{i_j}$ has degree 2 in $B$, it follows from (i) that $|F_B| \leq 2 \cdot 2 \cdot 3k$.

Consider a $B$-heavy even $A$-cycle $D$. Then, $D$ contains an edge $ab \in E(B)$ such that $a \in A$. As $D$ passes through $z$, by (ii), it also contains a subpath $R$ that is completely contained in $B$, that starts in a cutvertex of $G - z$ or in a neighbour of $z$ in $B$ and that contains $ab$. Let the endvertex $r \neq a$ of $R$ be in $S_{i_j}$. Now either $a \in \{a_{i_j}, a_{i_j+1}\}$, in which case it holds that $ab \in F_B$, or $a$ lies outside $S_{i_j}$. Then $R$ needs to pass through one of $\{a_{i_j}, a_{i_j+1}\}$ and thus through an edge in $F_B$. In every case, $D$ meets $F_B$.

Consider a path $S = b_0b_1 \ldots b_{k-1}$ in the block-tree of $G - z$, where $B_0, \ldots, B_{k-1}$ are blocks and $b_0, \ldots, b_k$ are cutvertices of $G - z$. The path $S$ is a string if

- $z$ does not have any neighbour in $\bigcup_{i=1}^{k} B_i$;
- every $B_i$ and every $b_i$ has degree 2 in the block-tree;
- $B_1$ and $B_{k-1}$ contain a vertex from $A$; and
- the path $S$ is maximal subject to the other conditions.

I will sometimes treat a string $S$ also as a subgraph of $G - z$, that is, I will speak of a string $S = b_0B_1 \ldots B_{k-1}b_k$ but mean the subgraph $\bigcup_{i=1}^{k} B_i$. In particular, when I consider an even $A$-cycle and it is $S$-heavy, I will mean that the cycle is $\bigcup_{i=1}^{k} B_i$-heavy.

**Lemma 15.** For every string $S$ there is an edge set $F_S \subseteq E(G)$ with $|F_S| \leq 24k^2$ such that $F_S$ meets every $S$-heavy even $A$-cycle.
Proof. Let $S = b_0B_1\ldots B_{t-b_0}$. First assume that fewer than $2k$ of the blocks $B_i$ contain a vertex from $A$. For each such block $B_i$ that meets $A$, let $F_{B_i}$ be as in Lemma 14, i.e. an edge set of at most $12k$ edges that meets every $B_i$-heavy even $A$-cycle. For a block $B_i$ that is disjoint from $A$ put $F_{B_i} = \emptyset$. Then

$$F_S = \bigcup_{i=1}^t F_{B_i}$$

meets every $S$-heavy even $A$-cycle. Moreover, it has size $|F_S| \leq 2k \cdot 12k = 24k^2$.

Thus, assume that at least $2k$ of the blocks $B_i$ contain a vertex from $A$. Next, if possible, choose an $F_S \subseteq E(G)$ of size $|F_S| \leq 10k$ such that

- $F_S$ separates $z$ from $b_0$ in $G - (S - b_0)$; or
- $F_S$ separates $z$ from $b_\ell$ in $G - (S - b_\ell)$; or
- $F_S$ separates $b_0$ from $b_\ell$ in $S$.

As any $S$-heavy even $A$-cycle would pass through any such separator, we would be done if there was such an $F_S$. Thus, we may assume that is not the case. Then, in particular, there is a set $C$ of $10k$ edge-disjoint cycles that pass through $z$, through each of $b_0,b_1,\ldots,b_\ell$ and through an edge from each of $B_1,\ldots,B_\ell$.

Pick indices $i_1 < \ldots < i_{2k}$ such that the blocks $B_{i_j}$ each contain a vertex $a_{i_j} \in A$. Next, we note that, because, by Lemma 13, the vertices $a_{i_j}$ have degree at most (in fact, exactly) 2 in $B_{i_j}$,

$$\text{each } a_{i_j} \text{ may lie in at most one of the cycles in } C. \quad (7)$$

![Figure 4: $C_1 \cup C_2$ as in the proof of Lemma 15 contains a theta graph](image)

Next, consider an $a_{i_j}$ that does not lie in any cycle of $\mathcal{C}$. Then there is a path $P \subseteq B_{i_j}$ through $a_{i_j}$ that starts in in some cycle $C \in \mathcal{C}$, ends in some (possibly different) cycle $C' \in \mathcal{C}$ and is otherwise edge-disjoint from all cycles in $\mathcal{C}$. With Lemma 7 we see that there is a $b_{i_j-1} - a_{i_j} - b_{i_j}$ path in $C \cup C' \cup P$. Using this path to replace $C\cap B_{i_j}$ in $C$, we obtain a new cycle $\hat{C}$ that passes through $a_{i_j}$ and that is edge-disjoint from every cycle in $\mathcal{C}$, except for $C$ and $C'$. If $C'$ is disjoint from $a_{i_1},\ldots,a_{i_{2k}}$ then remove $C,C'$ from $\mathcal{C}$ and add $\hat{C}$. If $C'$ already meets one of $a_{i_1},\ldots,a_{i_{2k}}$ then choose some cycle $D \in \mathcal{C}$ that is disjoint from $a_{i_1},\ldots,a_{i_{2k}}$ (this is possible, by (7) and $|\mathcal{C}| \geq 10k$), and use $D \cap B_{i_j}$ to replace $C'\cap B_{i_j}$ in $C'$. We add the resulting cycle to $\mathcal{C}$ as well as $\hat{C}$, and remove $C,C'$ and $D$ from $\mathcal{C}$. In both cases, the new set $\mathcal{C}$ will consist of pairwise edge-disjoint cycles of a size one less than before. Moreover, one more of $a_{i_1},\ldots,a_{i_{2k}}$ will
lie in a cycle of $C$. By repeating this process, we will find a set $C_1, \ldots, C_{2k}$ of pairwise disjoint cycles each passing through $z$ such that each of $a_{i1}, \ldots, a_{i2k}$ lies in exactly one cycle of $C$ (we use (7) here).

Then, however, each of $C_1 \cup C_2, C_3 \cup C_4, \ldots, C_{2k-1} \cup C_{2k}$ contains a theta graph such that in each at least two subdivided edges meet $A$; see Figure 4. Again, we find with Lemma (5) $k$ edge-disjoint even $A$-cycles, which is impossible by (2).

Let $S$ be the set of strings. Let $A_S$ be the set of blocks of $G - z$ that meet $A$ and that are not contained in any string. We note that:

\[
\text{every even } A\text{-cycle is } Y\text{-heavy for some } Y \in S \cup A_S.
\] (8)

Indeed, every $A$-cycle $C$ is $B$-heavy for some block $B$ of $G - z$. If $B \notin A_S$ then $B$ lies in some string $S$, which implies that $C$ is $S$-heavy.

For each string $S \in S$, pick a block in it that meets $A$ and denote by $A_S$ the set of all the chosen blocks. Note that

\[|S| = |A_S|\]

as strings are disjoint.

Define $T'$ to be the tree obtained from the block-tree of $G - z$ by iteratively deleting all leaves that are not in $A_S \cup A_{\overline{S}}$ and by iteratively suppressing vertices of degree 2 that are not in $A_S \cup A_{\overline{S}}$. Clearly, $T'$ contains all of $A_S \cup A_{\overline{S}}$, and every leaf of $T'$ and every vertex of degree 2 lies in $A_S \cup A_{\overline{S}}$.

We will now go through the different outcomes of Lemma 10.

**Lemma 16.** $T'$ does not contain $k$ disjoint subtrees that each contain three leaves of $T'$.

**Proof.** Suppose there are such subtrees. Then there are also $k$ disjoint subtrees $T_1, \ldots, T_k$ of the block-tree of $G - z$ such that:

(i) $T_i$ has exactly three leaves $L_1, L_2, L_3$, and these are leaf-blocks of the block-tree of $G - z$; and

(ii) if $B_i$ is the degree 3-vertex in $T_i$ then each of the $B_i - L_j$ paths in $T_i$ contains a block, not $B_i$, that contains a vertex from $A$.

For (i), note that the leaves of $T'$ lie in $A_S \cup A_{\overline{S}}$.

Consider a $T_i$. Then, for $j = 1, 2, 3$, there is, by (i), a $B_i - z$ path $Q_j \subseteq G$ contained in the union of $z$ with the blocks of $G - z$ in the $B_i - L_j$ path of $T_i$. Moreover, because of (ii) and Lemma 7 we can ensure that each $Q_j$ passes through some vertex in $A$. Let $S$ be a tree in $B_i$ that contains the endvertices of $Q_1, Q_2, Q_3$. Then $S \cup Q_1 \cup Q_2 \cup Q_3$ contains a theta graph $\theta_i$ such that each of its subdivided edges passes through some vertex in $A$ (as $Q_1, Q_2, Q_3$ pass through $A$). As $\theta_1, \ldots, \theta_k$ are pairwise edge-disjoint and as, by Lemma 5 each $\theta_i$ contains an even $A$-cycle, we have again obtained $k$ edge-disjoint even $A$-cycles in contradiction to (2).

**Lemma 17.** $T'$ does not contain any path of length $15k$. 


Proof. Suppose that \( T' \) contains a path \( P' \) of length at least \( 15k \). We first treat the case when \( P' \) contains at least \( 3k \) vertices that have degree at least \( 3 \) in \( T' \). By always grouping three consecutive such vertices together with their branches hanging off \( P' \) we find \( k \) disjoint subtrees of \( T' \) that each contain three leaves of \( T' \). This, however, is impossible by Lemma 16.

Therefore, \( P' \) needs to contain at least \( 12k \) vertices of degree \( 2 \). By definition of \( T' \) each such vertex lies in \( A_S \cup B_S \). Observe that then there is also a path \( P \) in the block-tree of \( G - z \) that contains \( 12k \) vertices that lie in \( A_S \cup B_S \). We partition \( P \) into edge-disjoint subpaths \( P_1, \ldots, P_{3k} \) such that each subpath \( P_i \) contains at least four vertices from \( A_S \cup B_S \).

Consider an arbitrary subpath \( P_s = b_0b_1\ldots b_\ell b_\ell \). Let \( B_{i_1}, B_{i_2}, B_{i_3}, B_i \) with \( i_1 < i_2 < i_3 < i_4 \) be blocks in \( A_S \cup B_S \). Pick a \( b_0-b_\ell \) path \( P \) that passes through every block \( B_i \). Moreover, we can ensure that \( P \) passes through a vertex of \( A \) in \( B_{i_1} \) and in \( B_{i_4} \) (since the \( B_i \) are either a single edge or \( 2 \)-connected).

Note that \( B_{i_2} \) and \( B_{i_3} \) do not lie in a common string: indeed, if \( B_{i_2}, B_{i_3} \in A_S \) then they lie in distinct strings, and distinct strings are disjoint. As a consequence, the subpath \( b_{i_2-1}B_{i_2}\ldots B_{i_3}b_{i_3} \) of \( P_s \) must contain either a vertex that has degree \( \geq 3 \) in the block-tree or some neighbour of \( z \). Thus, there is, by (3), a \( P-z \) path \( Q \) that starts in some vertex \( x \in \bigcup_{j=i_2}^{i_3} B_{j} \). Note that each subpath \( b_0Px \) and \( xPb_\ell \) contains a vertex of \( A \), namely some vertex in \( B_{i_1} \) and in \( B_{i_4} \). Denote \( P \cup Q \) by \( R_s \), and note that \( R_1, \ldots, R_{3k} \) are pairwise edge-disjoint.

Now, we observe that each of \( R_1 \cup R_2 \cup R_3 \ldots, R_{3k-2} \cup R_{3k-1} \cup R_{3k} \) contains a theta graph such that two of its subdivided edges pass through \( A \). By Lemma 5 we thus find \( k \) edge-disjoint even \( A \)-cycles, which we had excluded (2).

**Lemma 18.** \( T' \) does not contain a vertex of degree at least \( 3k \).

Proof. Suppose there is such a vertex. Then, clearly, there is also a vertex \( X \) in the block-tree of \( G - z \) of degree at least \( 3k \). This implies that there are \( 3k \) paths \( P_1, \ldots, P_{3k} \) in the block-tree between \( X \) and the leaves of the block-tree such that pairwise the paths meet exactly in \( X \). Moreover, as the leaves of \( T' \) are a subset of \( A_S \cup B_S \), we can choose the paths \( P_i \) such that each contains a block different from \( X \) that contains a vertex from \( A \).

We now pick for each \( i \) a \( X-z \) path \( P_i \subseteq G \) such that \( P_i - z \) is contained in \( \bigcup_{B_i \in P_i} B_i \); this is possible because of (3). Since \( P_i \) contains a block different from \( X \) that meets \( A \), we can furthermore ensure that \( P_i \) passes through \( A \) (using Lemma 4 if necessary).
Seen in $G - z$, the vertex $X$ in the block-tree is either a block or a cutvertex. Extend $X$ to a graph $X'$ by adding a new vertex $p_i$ for each $P_i$ and by connecting it to the endvertex of $P_i$ in $X$. We apply Lemma 11 to a spanning tree of $X'$, with the set $\{p_1, \ldots, p_{3k}\}$ in the role of $Z$. Thus, there are $k$ edge-disjoint trees $S_1, \ldots, S_k \subseteq X'$ that each contain three of the vertices in $\{p_1, \ldots, p_{3k}\}$. By changing the enumeration, we may assume that $S_1$ contains $p_1, p_2, p_3$, that $S_2$ contains $p_4, p_5, p_6$ and so on.

Now, for $i = 1, \ldots, k$, the graph $S_i - \{p_{3i-2}, p_{3i-1}, p_{3i}\} \cup P_{3i-2} \cup P_{3i-1} \cup P_{3i}$ contains a theta graph $\theta_i$ such that all three of its subdivided edges pass through a vertex from $A$. The $\theta_i$ are pairwise edge-disjoint and each contains, by Lemma 5, an even $A$-cycle — this is again a contradiction to (2).

Proof of Theorem 2. Suppose first that $|A S \cup A S| \geq 45k^3$. Since $V(A S \cup A S) \subseteq V(T')$ we may apply Lemma 10 with $\alpha = 3k$, $\beta = k$ and $\gamma = 15k$. Each of the outcomes of Lemma 10 however, leads to a contradiction as Lemmas 16, 17 and 18 demonstrate.

We conclude that $|A S \cup A S| < 45k^3$. Then $|S \cup A S| < 45k^3$. For each $Y \in S \cup A S$ we pick $F_Y$ as in Lemma 14 or 15 depending on whether $Y \in S$ or $Y \in A S$. In each case $|F_Y| \leq 24k^2$, which means that

$$F = \bigcup_{Y \in S \cup A S} F_Y \text{ has size } |F| \leq 45k^3 \cdot 24k^2 = 1080k^5$$

Consider some even $A$-cycle $D$. Then, by (8), $C$ must be $Y$-heavy for some $Y \in S \cup A S$ and thus meets $F_Y \subseteq F$. This shows that $F$ meets every even $A$-cycle. 

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