The quantum theory of the free Maxwell field on the de Sitter expanding universe

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Abstract

The theory of the free Maxwell field in two moving frames on the de Sitter spacetime is investigated pointing out that the conserved momentum and Hamiltonian operators do not commute with each other. This forces us to derive a new set of plane waves solutions of the Maxwell equation constituting the energy basis which should complete the usual approach in which one knows plane wave solutions of given momentum. The energy basis can be obtained grace to our new time-evolution picture we used already in the cases of the scalar and Dirac fields. Finally, using both these bases, it is shown that the second quantization of the free electromagnetic potential can be done in a canonical manner as in special relativity. The principal conserved one-particle operators associated to Killing vectors are derived, focusing on the Hamiltonian, momentum and total angular momentum operators.

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1 Introduction

The quantum theory of fields on curved spacetimes studies quantum systems in the presence of gravitation but without to affect the background geometry. Of a special interest in cosmology is the de Sitter (dS) expanding universe carrying fields variously coupled to gravity among them the free fields (minimally coupled) may be the principal ingredients in calculating scattering amplitudes using perturbations.

Recently we proposed a new approach to the relativistic quantum theory of the free fields on moving local charts of the de Sitter (dS) spacetimes, pointing out that the momentum and Hamiltonian operators are conserved (commuting with the operator of the field equation \([1, 2]\)) but do not commute with each other \([3]\). In these circumstances, the well-known solutions of the free field equations of given momentum \([4, 5]\) are not enough and, therefore, we needed to look for particular solutions having a well-determined energy. Some technical difficulties were avoided introducing a new time-evolution picture, called the Schrödinger picture, where new quantum modes of known energy and momentum direction can be found \([6]\) for scalar \([7]\) and Dirac fields \([8]\). We obtained thus two different bases, the momentum basis and energy one, which have suitable orthogonality and completeness properties. In this framework, the second quantization of the scalar and Dirac fields on dS manifolds can be done in canonical manner \([7, 3]\) obtaining a new Dirac propagator \([3]\) but the well-known one of the scalar field \([9]\). However, the principal benefit of our approach is the possibility to calculate the conserved one-particle operators corresponding to the Killing vectors of the dS geometry, either in the momentum basis or in the energy one. The momentum and Hamiltonian operators are crucial for understanding how can be measured the photon momentum and energy on the dS expanding universe \([7]\).

In the present paper we should like to study the free Maxwell field on dS spacetime in the same approach. Our main purpose is to find the conserved one-particle operators, focusing on the Hamiltonian and momentum ones which do not commute among themselves. In this case we have the advantage of the Maxwell equations which are conformal invariant such that in the moving chart with conformal flat line element one recovers many results of special relativity \([4]\). In this familiar conjecture we build the quantum theory of the free Maxwell field on the moving dS charts either with proper time or with conformal one \([4]\). We derive the new quantum modes of well-determined energy, momentum direction and polarization and perform the
canonical quantization in Coulomb gauge, deducing the form of the principal one-particle operators namely, the Hamiltonian, momentum, total angular momentum and polarization ones. In what concerns the problem of propagators (or two-point functions [9, 10]) which actually is of interest [11], we bring nothing new since the Maxwell propagator in the chart with conformal time coincides to that of the special relativity.

We start in the second section with a brief review of the theory of the free Maxwell field on the mentioned moving dS charts, introducing the principal conserved operators and defining the Schrödinger time-evolution picture. This allows us to find in the next section the new quantum modes of the energy basis and the transition coefficient between this basis and the traditional momentum basis. Section 4 is devoted to the canonical quantization of the Maxwell field in Coulomb gauge which provide us with well-known Green functions. We write down the desired one-particle operators in Section 5.

2 The Maxwell free field on de Sitter spacetimes

Let $M$ be a curved spacetime and $\{x\}$ a local chart of coordinates $x^\mu (\mu, \nu, ... = 0, 1, 2, 3)$ and the line element

$$ds^2 = g_{\mu \nu}dx^\mu dx^\nu,$$

(1)

defined by the metric tensor $g_{\mu \nu}$. We denote by $A_\mu$ the covariant components of the free electromagnetic potential $A$, minimally coupled to gravity, whose action reads

$$S[A] = \int d^4x \sqrt{g} \mathcal{L} = -\frac{1}{4} \int d^4x \sqrt{g} F_{\mu \nu} F^{\mu \nu},$$

(2)

where $g = |\det(g_{\mu \nu})|$ and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. From this action one derives the field equation

$$\partial_\nu (\sqrt{g} g^{\nu \alpha} g^{\mu \beta} F_{\alpha \beta}) = 0,$$

(3)

which is invariant under conformal transformations. Indeed, given another chart $\{x\}'$ with the line element $ds'^2 = \Omega(x)ds^2$ we find that the field $A'$ in this chart has the components

$$A'_\mu = A_\mu \quad A'^\mu = \Omega^{-1}A^\mu.$$

(4)
On the other hand, the whole theory must remain invariant under symmetry transformation. Since \( A \) is a real field there are no internal symmetries remaining only with the isometries related to the Killing vectors of \( M \). Given an isometry transformation \( x \to x' = \phi_\xi(x) \) depending on the group parameter \( \xi \) there exists an associated Killing vector field, \( K = \partial_\xi \phi_\xi|_{\xi=0} \) (which satisfy the Killing equation \( K^{\mu;\nu} + K^{\nu;\mu} = 0 \)). The vector field \( A \) transforms under this isometry as \( A \to A' = T_\xi A \), according to the operator-valued representation \( \xi \to T_\xi \) of the isometry group defined by the rule

\[
\frac{\partial \phi_\xi^\nu(x)}{\partial x_\mu}(T_\xi A)_\nu [\phi(x)] = A_\mu(x) . \tag{5}
\]

The corresponding generator, \( X_K = i \partial_\xi T_\xi|_{\xi=0} \), has the action

\[
(X_K A)_\mu = -i (K^\nu A_{\mu;\nu} + K^\nu_{\mu;\nu} A_\nu) . \tag{6}
\]

We say that these generators are conserved operators since they commute with the operator of the field equation \[2\]. Moreover, from the Noether theorem it results that each Killing vector \( K \) give rise to the conserved quantity,

\[
C[K] = -\frac{i}{2} \int_\Sigma d\sigma^\mu \sqrt{g} g^{\alpha\beta} \left[ A_\alpha \partial_\mu (X_K A_\beta) \right] , \tag{7}
\]

on a given hypersurface \( \Sigma \subset M \).

In what follows \( M \) will be the dS spacetime defined as a hyperboloid of radius \( 1/\omega \) in a five-dimensional pseudo-Euclidean manifold, \( M^5 \), of coordinates \( z^A \) labeled by the indices \( A, B, ..., = 0, 1, 2, 3, 5 \). The local charts of coordinates \( \{x\} \) on \( M^5 \) can be easily introduced giving the functions \( z^A(x) \). We use here either the chart \( \{t, x\} \) with the proper time, \( t \), Cartesian coordinates and the Robertson-Walker line element

\[
ds^2 = dt^2 - e^{2\omega t}(d\mathbf{x} \cdot d\mathbf{x}) , \tag{8}
\]

or the chart \( \{t_c, x\} \) with the conformal time \( t_c \), defined as \( \omega t_c = -e^{-\omega t} \), where the line element

\[
ds^2 = \frac{1}{(\omega t_c)^2} \left(dt_c^2 - d\mathbf{x} \cdot d\mathbf{x} \right) , \tag{9}
\]

is the conformal transformation of the Minkowski one \[4\].

The \( SO(4,1) \) group of the pseudo-orthogonal transformations in \( M^5 \) constitutes the isometry group of \( M \). For this reason, the basis-generator
of the $so(4,1)$ algebra are associated to ten independent Killing vectors, $K_{(AB)} = -K_{(BA)}$, which give rise to the basis-generators $X_{(AB)}$ of the vector representation of the $SO(4,1)$ group carried by the space of the vector potential, $A$. Among them we focus here on the Hamiltonian operator $H = \omega X_{(05)}$, the momentum components $P^i = \omega (X_{(5i)} - X_{(0i)})$ and those of the total angular momentum $J_i = \frac{1}{2} \varepsilon_{ijk} X_{(jk)} (i, j, ... = 1, 2, 3)$ \[3\]. The action of these operators can be calculated according to Eq. (6) using the concrete form of the corresponding Killing vectors which in the chart $\{t, x\}$ have the components \[2\], $\omega K^0_{(05)} = (-1, x^1, x^2, x^3)$ and

\[
\begin{align*}
K^0_{(5i)} - K^0_{(0i)} &= 0, & K^j_{(5i)} - K^j_{(0i)} &= \frac{1}{\omega} \delta_{ij}, \\
K^0_{(ij)} &= 0, & K^k_{(ij)} &= \delta_{ki} x^j - \delta_{kj} x^i.
\end{align*}
\]

The Hamiltonian and momentum operators do not have spin parts, acting as

\[
\begin{align*}
(H A)_\mu (t, x) &= i(\partial_\mu - \omega - \omega x^i \partial_i)A_\mu (t, x), \\
(P^i A)_\mu (t, x) &= -i \partial_\mu A^i (t, x),
\end{align*}
\]

while the action of the total angular momentum reads

\[
\begin{align*}
(J_i A)_j (t, x) &= (L_i A)_j (t, x) - i \varepsilon_{ijk} A^k (t, x), \\
(J_i A)_0 (t, x) &= (L_i A)_0 (t, x),
\end{align*}
\]

where $L = x \times P$ is the usual angular momentum operator. In addition, we define the Pauli-Lubanski operator $W = P \cdot J$ whose action depends only on the spin parts,

\[
\begin{align*}
(W A)_i (t, x) &= \varepsilon_{ijk} \partial_j A_k (t, x), & (W A)_0 (t, x) &= 0.
\end{align*}
\]

This operator will help us to define the circular polarization as in special relativity.

The other chart, $\{t_c, x\}$, is suitable for calculating the conserved quantities defined by Eq. \[7\]. Indeed, after a little calculation we obtain the compact form

\[
C[K_{(AB)}] = \frac{1}{2} \left\{ \delta_{ij} \left( A_i, [X_{(AB)} A]_j \right) - \left( A_0, [X_{(AB)} A]_0 \right) \right\}
\]
using the new notation
\[
\langle f, g \rangle = i \int d^3 x \, f^*(t_c, x) \, \vec{\partial}_{t_c} \, g(t_c, x),
\]  
(18)
where \( f \, \vec{\partial} \, g = f(\partial g) - g(\partial f) \). The integral (18) defines a Hermitian form which has to play a similar role as the relativistic scalar products of the Dirac [3] and scalar charged fields [7].

We have shown that the relativistic quantum mechanics in the chart \( \{ t, x \} \) can be formulated at least in two useful time-evolution pictures. The first one is the natural picture (NP) which is the genuine theory as it results from the action (2). The second picture, we called the Schrödinger picture (SP), is derived from the NP using the transformation \( A(x) \rightarrow A^S(x) = U(x)A(x) \) produced by the operator of time dependent dilatations [6]
\[
U(x) = \exp \left[ -\omega t(x^i \partial_i) \right],
\]  
(19)
which has the following convenient action
\[
U(x)F(x')U^{-1}(x) = F \left( e^{-\omega t x^i} \right), \quad U(x)G(\partial_i)U^{-1}(x) = G \left( e^{\omega t} \partial_i \right),
\]  
(20)
upon any analytical functions \( F \) and \( G \). In this new picture the operators \( H_S = U(x)HU(x)^{-1} \) and \( P^S_i = U(x)P^iU(x)^{-1} \) have the action
\[
(H_S A^S)_\mu(t, x) = i(\partial_t - \omega)A^S_\mu(t, x), \quad (P^S_i A^S)_\mu(t, x) = -ie^{\omega t} \partial_i A^S_\mu(t, x),
\]  
(21)
(22)
while the Pauli-Lubanski operator remains unchanged since it commutes with \( U \). We shall show that the SP is the suitable framework we need for deriving the solutions of energy basis in the same manner as in Refs. [3] and [7].

### 3 Polarized plane wave solutions

The specific feature of the quantum mechanics on \( M \) is that the Hamiltonian operator (12) does not commute with the momentum operators (13). This means that there are no particular solutions of the field equation with well-determined energy and momentum and, consequently, we can not speak about mass-shells. This forces us to consider different sets of plane waves solutions, depending either on momentum or on energy and momentum direction, determining thus two different bases, namely the momentum and energy ones. For both these bases we adopt the Coulomb gauge, with \( A_0 = 0 \) and \( (A^i)_j = 0 \), assuming that the polarization is circular.
3.1 The momentum basis

In the conformal flat chart, \( \{ t_c, x \} \), and Coulomb gauge the field equation is the same as in the Minkowski spacetime, i.e. \((\partial^2_{t_c} - \Delta)A_i = 0\), since, as mentioned, the Maxwell equations are conformal invariant. Therefore, the non-vanishing components of \( A \) can be expanded as

\[
A_i(x) = A_i^{(+)}(x) + A_i^{(-)}(x)
\]

in terms of wave functions in momentum representation, \( a(k, \lambda) \), polarization vectors, \( e_i(n_k, \lambda) \), and fundamental solutions, of the d’Alambert equation,

\[
f_k(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k}} e^{-ikt_c + ik \cdot x},
\]

where \( k = kn_k \) is the momentum vector and \( k = |k| \). We note that these solutions satisfy the orthonormalization relations

\[
\langle f_k, f_{k'} \rangle = - \langle f_k^*, f_{k'}^* \rangle = \delta^3(k - k'),
\]

and the completeness condition

\[
i \int d^3k f_k^*(t_c, x) \overleftarrow{\partial_{t_c}} f_k(t_c, x') = \delta^3(x - x'),
\]

with respect to the Hermitian form \( \langle \rangle \) which plays thus the role of a generalized scalar product as we observed before. For this reason we say that the set of the fundamental solutions \( f_k \) defines the momentum basis.

The polarization vectors \( e(n_k, \lambda) \) in Coulomb gauge must be orthogonal to the momentum direction,

\[
k \cdot e(n_k, \lambda) = 0,
\]

for any polarization \( \lambda \). We remind the reader that the polarization can be defined in different manners independent on the form of the fundamental solutions \( f_k \). In general, the polarization vectors have c-number components which must satisfy \( [12] \)

\[
e(n_k, \lambda) \cdot e(n_k, \lambda')^* = \delta_{\lambda\lambda'},
\]

\[
\sum_{\lambda} e_i(n_k, \lambda) e_j(n_k, \lambda)^* = \delta_{ij} - \frac{k_i k_j}{k^2}.
\]
Here we restrict ourselves to consider only the circular polarization for which the plane waves \( w_{i(k,\lambda)} = e_i(n_k, \lambda)f_k \) are common eigenfunctions of the complete set of commuting operators \( \{P^i, W\} \) corresponding to the eigenvalues \( \{k^i, k\lambda\} \) where \( \lambda = \pm 1 \).

### 3.2 The energy basis

Let us consider now the electromagnetic potential in the chart \( \{t, x\} \) where its components in Coulomb gauge, \( A_i \), remain unchanged since the space coordinates are the same in both the chart used here. Then, the field equation of the NP,

\[
\partial_t^2 A_i + \omega \partial_t A_i - e^{-2\omega t} \Delta A_i = 0,
\]

has to be transformed by the operator (19) into the field equation of the SP,

\[
\left[ (\partial_t + \omega x^i \partial_i)^2 + \omega (\partial_t + \omega x^i \partial_i) - \Delta \right] A_i^S = 0,
\]

which does not depend explicitly on time. We show that this equation has particular solutions representing plane waves of given energy, momentum direction and polarization.

We start assuming that the electromagnetic potential can be expanded in the SP as

\[
A_i^S(x) = A_i^{S(+)}(x) + A_i^{S(-)}(x) = \int_0^\infty dE \int d^3q \left[ \hat{A}_i^{S(+)}(E, q)e^{-i(Et - q \cdot x)} + \hat{A}_i^{S(-)}(E, q)e^{i(Et - q \cdot x)} \right] e^{\omega t}
\]

where \( E \) is the energy defined as the eigenvalue of the operator \( H_S \) which acts as in Eq. (21). Whenever the fields \( \hat{A}_i^{S(\pm)} \) behave as tempered distributions on the domain \( \mathbb{R}^3_q \), the Green theorem may be used and we can replace the momentum operators \( P^i_S \) by \( q^i \) and the coordinates \( x^i \) by \( i\partial_{q_i} \) obtaining the field equation of the SP in momentum representation,

\[
\left\{ \left[ \pm iE + \omega \left( q^i \partial_{q_i} + 2 \right) \right]^2 - \omega \left[ \pm iE + \omega \left( q^i \partial_{q_i} + 2 \right) \right] + q^2 \right\} \hat{A}_i^S(\pm)(E, q) = 0.
\]

The energy \( E \) is a conserved quantity but the momentum \( q \) does not have this property since this is no longer a basis generator produced by a Killing vector.
More specific, only the scalar momentum \( q = |\mathbf{q}| \) is not conserved while the momentum direction is conserved since the operator (22) is parallel with the conserved momentum \( \mathbf{P} \). For this reason we denote \( \mathbf{q} = q \mathbf{n} \) observing that the differential operator of Eq. (34) is of radial type and reads \( q^i \partial_{q_i} = q \partial_q \). Consequently, this operator acts only on the functions depending on \( q \) while the functions which depend on the momentum direction \( \mathbf{n} \) behave as constants. Therefore, we have to look for solutions of the form

\[
\hat{A}_i^S(\pm)(E, \mathbf{q}) = [\hat{A}_i^S(-)(E, \mathbf{q})]^* = h_S(E, q) e_i(\mathbf{n}, \lambda) a(E, \mathbf{n}, \lambda),
\]

where the function \( h_S \) satisfies an equation derived from Eq. (34) that can be written simply in the new variable \( s = q/\omega \) and using the notation \( \epsilon = E/\omega \). This equation

\[
\left[ \frac{d^2}{ds^2} + \frac{2i\epsilon + 4}{s} \frac{d}{ds} + \frac{3i\epsilon + 2 - \epsilon^2}{s^2} + 1 \right] h_S(\epsilon, s) = 0
\]

has solutions of the form \( h_S(\epsilon, s) = \text{const} \ s^{-i\epsilon-2} e^{is} \). Collecting all the above results we derive the final expression of the field (33) as

\[
\hat{A}_i^S(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_\lambda \left\{ e_i(\mathbf{n}, \lambda) f_{E,n}^S(x) a(E, \mathbf{n}, \lambda) + [e_i(\mathbf{n}, \lambda) f_{E,n}^S(x)]^* a^*(E, \mathbf{n}, \lambda) \right\},
\]

where the integration covers the sphere \( S^2 \subset \mathbb{R}_p^3 \). The fundamental solutions \( f_{E,n}^S \) of positive frequencies, with energy \( E \) and momentum direction \( \mathbf{n} \) result to have the integral representation

\[
f_{E,n}^S(x) = N e^{-iEt+\omega t} \int_0^\infty ds e^{is+is\mathbf{n} \cdot \mathbf{x} - i\epsilon \ln s},
\]

where \( N \) is a normalization constant. Notice that these functions can be expressed in terms of Euler’s \( \Gamma \) functions [13] but we prefer to work with their integral representation.

The physical meaning of this result may be pointed out turning back to the NP where the electromagnetic potential

\[
A_i(x) = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_\lambda \left\{ e_i(\mathbf{n}, \lambda) f_{E,n}(x) a(E, \mathbf{n}, \lambda) + [e_i(\mathbf{n}, \lambda) f_{E,n}(x)]^* a^*(E, \mathbf{n}, \lambda) \right\},
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\]
is expressed in terms of the solutions of NP which can be calculated as

\[ f_{E, n}(x) = U^{-1}(x) f^S_{E, n}(x) = N e^{-iEt + \omega t} \int_0^\infty ds \ e^{i\omega s n \cdot x_t - i\epsilon \ln s}, \quad (40) \]

where \( x_t = e^{\omega t} x \). Finally, changing the integration variable, \( e^{\omega t} s \to s \), we obtain the definitive integral representation

\[ f_{E, n}(t_c, x) = N \int_0^\infty ds \ e^{i\omega s (n \cdot x - t_c) - i\epsilon \ln s} = N \left( 1 - \epsilon \right) (i t_c - i n \cdot x)^{i\epsilon - 1}, \quad (41) \]

but in the chart of the conformal time, \( t_c \). Using the Hermitian form (18) we can show (as in Appendix) that the normalization constant

\[ N = \frac{1}{2} \sqrt{\frac{\omega}{\pi}} \frac{1}{(2\pi)^{3/2}} \quad (42) \]

(defined up to a phase factor) assures the desired orthonormalization relations,

\[ \langle f_{E, n}, f^*_{E', n'} \rangle = -\langle f^*_{E, n}, f_{E', n'} \rangle = \delta(E - E') \delta^2(n - n'), \quad (43) \]

\[ \langle f_{E, n}, f^*_{E, n'} \rangle = 0, \quad (44) \]

and the completeness condition

\[ i \int_0^\infty dE \int_{S^2} d\Omega_n \left\{ [f_{E, n}(t_c, x)]^* \ \tilde{\partial}_{t_c} f_{E, n}(t_c, x') \right\} = \delta^3(x - x'). \]

We say that the set of functions \( \{ f_{E, n} | E \in \mathbb{R}^+, n \in S^2 \} \) constitutes the complete system of fundamental solutions of the energy basis of the NP. The plane waves \( w_i(E, n, \lambda) = e_i(n, \lambda) f_{E, n} \) depend on the energy \( E \) (representing the eigenvalue of \( H \)) and on the direction \( n \) and the polarization \( \lambda \) which are no longer eigenvalues of differential operators. This is why the complete set of commuting operators determining the energy basis can be defined only at the level of quantum field theory.

Working simultaneously with two bases we need to know the transition coefficients which can be calculated straightforwardly as

\[ \langle f_k, f_{E, n} \rangle = \langle f_{E, n}, f_k^* \rangle = \frac{k^{-3/2}}{\sqrt{2\pi\omega}} \delta^2(n - n_k) e^{-i\frac{E \ln k}{\omega}}, \quad (46) \]
where \( n_k = k/k \). With their help we deduce the transformations

\[
a(k, \lambda) = \int_0^\infty dE \int_{S^2} d\Omega_n \langle f_k, f_{E,n} \rangle a(E, n, \lambda) \\
= \frac{k^{-3/2}}{\sqrt{2\pi \omega}} \int_0^\infty dE e^{-iE \ln k} a(E, n_k, \lambda), \tag{47}
\]

\[
a(E, n, \lambda) = \int d^3k \langle f_{E,n}, f_k \rangle a(k, \lambda) \\
= \frac{1}{\sqrt{2\pi \omega}} \int_0^\infty dk \sqrt{k} e^{iE \ln k} a(k, n_k, \lambda), \tag{48}
\]

which are similar with those found for the scalar field [7]. These relations will help us to perform the second quantization in canonical manner using both the bases defined above.

### 4 Quantization in Coulomb gauge

The quantization in Coulomb gauge has to be performed in canonical manner considering that the wave functions \( a \) of the fields (23) and (39) become field operators (such that \( a^* \to a^\dagger \)) [12]. These operators must fulfill the standard commutation relations in the momentum basis from which the non-vanishing ones are

\[
[a(k, \lambda), a^\dagger(k', \lambda')] = \delta_{\lambda\lambda'} \delta^3(k - k'). \tag{49}
\]

Then, from Eq. (17) it results that the field operators of the energy basis satisfy

\[
[a(E, n, \lambda), a^\dagger(E', n', \lambda')] = \delta_{\lambda\lambda'} \delta(E - E') \delta^2(n - n'), \tag{50}
\]

and

\[
[a(p, \lambda), a^\dagger(E, n, \lambda)] = \langle f_p, f_{E,n} \rangle, \tag{51}
\]

while other commutators are vanishing. In this way the Hermitian field \( A = A^\dagger \) is correctly quantized according to the canonical rule

\[
[A_i(t_c, x), \pi^j(t_c, x')] = [A_i(t_c, x), \partial_{t_c} A_j(t_c, x')] = i \delta^r_{ij}(x - x'), \tag{52}
\]

where

\[
\pi^j = \sqrt{g} \frac{\delta L}{\delta (\partial_{t_c} A_j)} = \partial_{t_c} A_j \tag{53}
\]

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is the momentum density derived from the action \( \text{(2)} \) and

\[
\delta_{ij}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3q \left( \delta_{ij} - \frac{q^iq^j}{q^2} \right) e^{i\mathbf{q} \cdot \mathbf{x}}
\]  

(54)
is the well-known transversal \( \delta \)-function \( [12] \) arising from Eq. \( \text{(30)} \).

The field operators act on the Fock space supposed to have an unique vacuum state \( |0\rangle \) accomplishing

\[
a(\mathbf{k}, \lambda) |0\rangle = 0, \quad \langle 0 | a^\dagger(\mathbf{k}, \lambda) = 0,
\]

(55)
and similarly for the energy basis. The sectors with a given number of particles have to be constructed using the standard methods, obtaining thus the generalized bases of momentum or energy.

In the quantum theory of fields it is important to study the Green functions related to the partial commutator functions (of positive or negative frequencies) defined as

\[
D_{ij}^{(\pm)}(x - x') = i[A_i^{(\pm)}(x), A_j^{(\pm)\dagger}(x')]
\]

(56)
and the total one, \( D_{ij} = D_{ij}^{(+)} + D_{ij}^{(-)} \). These function are solutions of the field equation with vanishing divergences in both the sets of variables and obey \([D_{ij}^{(\pm)}]^* = D_{ij}^{(\mp)} \) such that \( D_{ij} \) results to be a real function. Thus it is enough to focus only on the functions of positive frequencies,

\[
D_{ij}^{(+)}(x - x') = i \int d^3k f_k(x)f_k(x')^* \left( \delta_{ij} - \frac{k^ik^j}{k^2} \right)
\]

\[
= i \int_0^\infty dE \int_{S_2} d\Omega_n f_{E,n}(x)f_{E,n}(x')^* \left( \delta_{ij} - n^in^j \right),
\]

(57)
resulted from Eqs. \( \text{(23)} \), \( \text{(39)} \) and \( \text{(30)} \). Both these versions lead to the final expression

\[
D_{ij}^{(+)}(x - x') = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k} \left( \delta_{ij} - \frac{k^ik^j}{k^2} \right) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - ik(t_c - t'_{c})}
\]

(58)
from which we deduce what happens at equal time,

\[
\partial_{t_c} D_{ij}^{(+)}(t_c - t'_c, \mathbf{x} - \mathbf{x}') \big|_{t'_{c} = t_c} = \frac{1}{2} \delta_{ij}^{tr}(\mathbf{x} - \mathbf{x}').
\]

(59)
These functions help one to write the transversal Green functions, \( G_{ij}(x) = G_{ji}(x) \), which obeys
\[
\left( \partial_{t_c}^2 - \Delta_x \right) G_{ij}(x - x') = \delta(t_c - t'_c)\delta_{ij}(x - x')
\] (60)
and \( \partial_t G_{ij}^{\mu}(x) = 0 \). Of a special interest are the retarded, \( D_R^{ij}(x) = \theta(t_c)D_{ij}(x) \), and advanced, \( D_A^{ij}(x) = -\theta(-t_c)D_{ij}(x) \), transversal Green functions. The transversal Feynman propagator,
\[
D_F^{ij}(x - x') = i\langle 0 | T[A_i(x)A_j(x')] | 0 \rangle = \theta(t_c - t'_c)D_{ij}(x - x') - \theta(t'_c - t_c)D_{ij}^{(-)}(x - x'),
\] (61)
is defined as a causal Green function. It is not difficult to verify that all these functions satisfy Eq. (60) if one uses the identity \( \partial_t^2[\theta(t)f(t)] = \delta(t)\partial_t f(t) \)
and Eq. (59).

The conclusion is that in the chart \( \{t_c, x\} \) the Green functions have the same forms and properties as those of the Maxwell theory in Minkowski flat spacetime, including the representation in the complex \( k_0 \)-plane. The difference is that here the particular value \( k_0 = k \) is no more the photon energy since there is no mass-shell.

5 One-particle operators

The one-particle operators corresponding to the conserved quantities (17) can be calculated in the Coulomb gauge as
\[
\mathcal{X} = \frac{1}{2} \delta_{ij} : \langle A_i, (X A)_{ij} \rangle :
\] (62)
respecting the normal ordering of the operator products [12]. The obvious algebraic properties
\[
[\mathcal{X}, A_i(x)] = -(X A)_i(x), \quad [\mathcal{X}, \mathcal{Y}] = \frac{1}{2} \delta_{ij} : \langle A_i, ([X, Y] A)_{ij} \rangle :
\] (63)
are due to the canonical quantization adopted here. However, there are many other conserved operators which do not have corresponding differential operators at the level of the relativistic quantum mechanics. The simplest example is the operator of the number of particles,
\[
\mathcal{N} = \int d^3k \sum_\lambda a^\dagger_\lambda(k, \lambda)a(k, \lambda) = \int_0^\infty dE \int d\Omega_\epsilon \sum_\lambda a^\dagger_\lambda(E, \epsilon, \lambda)a(E, \epsilon, \lambda),
\] (64)
The principal conserved one-particle operators are the components of momentum operator,

\[ P^i = \frac{1}{2} \delta_{ij} : \langle A_i, (P^j A) \rangle := \int d^3k \sum_{\lambda} a^\dagger(k, \lambda) a(k, \lambda), \tag{65} \]

and the Pauli-Lubanski operator,

\[ W = \frac{1}{2} \delta_{ij} : \langle A_i, (W A)_j \rangle := \int d^3k \sum_{\lambda} \lambda a^\dagger(k, \lambda) a(k, \lambda), \tag{66} \]

which are diagonal in the momentum basis as well as the Hamiltonian operator,

\[ H = \frac{1}{2} \delta_{ij} : \langle A_i, (HA)_j \rangle := \int_0^\infty dE \int_{S^2} d\Omega_n \sum_{\lambda} a^\dagger(E, n, \lambda) a(E, n, \lambda), \tag{67} \]

expanded in the energy basis. More interesting are the operators \( \tilde{P}^i \) of the momentum direction since they do not come from differential operators and, therefore, must be defined directly as

\[ \tilde{P}^i = \int_0^\infty dE \int_{S^2} d\Omega_n n^i \sum_{\lambda} a^\dagger(E, n, \lambda) a(E, n, \lambda). \tag{68} \]

Similarly we define the new normalized Pauli-Lubanski operator

\[ \tilde{W} = \frac{1}{2} \delta_{ij} : \langle A_i, (WA)_j \rangle := \int d^3k \sum_{\lambda} \lambda a^\dagger(k, \lambda) a(k, \lambda) \]

\[ = \int_0^\infty dE \int_{S^2} d\Omega_n \sum_{\lambda} \lambda a^\dagger(E, n, \lambda) a(E, n, \lambda), \tag{69} \]

which is diagonal in both our bases. The above operators which satisfy simple commutation relations,

\[ [\mathcal{H}, P^i] = i\omega P^i, \quad [\mathcal{H}, W] = i\omega W, \quad [\mathcal{H}, \tilde{P}^i] = [\mathcal{H}, \tilde{W}] = 0, \tag{70} \]
\[ [W, P^i] = [W, \tilde{P}^i] = [\tilde{W}, P^i] = [\tilde{W}, \tilde{P}^i] = 0, \tag{71} \]

determine the momentum and energy bases as common eigenvectors of the sets of commuting operators \( \{P^i, W\} \) and respectively \( \{\mathcal{H}, \tilde{P}^i, \tilde{W}\} \).

It is worth pointing out that the transition coefficients (46) can be used for finding closed expressions for conserved one-particle operators in bases in
which these are not diagonal. For example, we can calculate the Hamiltonian operator in the momentum basis either starting with the identity

$$(H f_k)(x) = -i\omega \left( k^i \partial_{k_i} + \frac{3}{2} \right) f_k(x)$$  \hspace{1cm} (72)

or using directly Eq. (48). The final result,

$$\mathcal{H} = \frac{i\omega}{2} \int d^3 k \sum_{\lambda} a^\dagger(k,\lambda) \tilde{\partial}_{k_i} a(k,\lambda),$$  \hspace{1cm} (73)

is similar with those obtained for the scalar [7] and Dirac [3] fields on $M$.

The components of total angular momentum are not diagonal in the above considered bases but can be easily represented in both these bases. Thus, according to Eqs. (6) and (14), we find the following expansion in the momentum basis:

$$J_l = -\frac{i}{2} \varepsilon_{lij} \int d^3 k \left[ k^i \sum_{\lambda} a^\dagger(k,\lambda) \tilde{\partial}_{k_j} a(k,\lambda) 
+ \sum_{\lambda\lambda'} \vartheta_{ij\lambda\lambda'}(k) a^\dagger(k,\lambda)a(k,\lambda') \right],$$  \hspace{1cm} (74)

where

$$\vartheta_{ij\lambda\lambda'}(k) = 2e_i(k,\lambda)^* e_j(k,\lambda') + k^l \sum_{\lambda} e_l(k,\lambda)^* \tilde{\partial}_{k_j} e_l(k,\lambda'),$$  \hspace{1cm} (75)

and we recover the identity $\mathcal{W} = \sum_i \mathcal{P}_i J_i$. Similar formulas can be written in the energy basis.

6 Concluding remarks

We presented here the complete quantum theory of the Maxwell field minimally coupled with the gravitation of the dS expanding universe. The main points of our approach are the method of constructing conserved operators and our new Schrödinger time-evolution picture. This helped us the find the new set of fundamental solutions of the energy basis that completes the framework of the quantum theory of the Maxwell free field on dS backgrounds. Moreover, since the results obtained here are very similar with
those derived for the scalar and Dirac fields we can say that our quantum theory is coherent.

In the Maxwell case the theory was considered in natural frames since this is in the spirit of general relativity as long as there are involved only vectors and tensors. However, in theories where the Maxwell field is coupled with spinor (Dirac or Majorana) fields, one must construct the entire theory in local frames, as a tetrad-gauge covariant theory. This can be achieved simply rewriting all the above results in local frames using tetrad fields. The advantage is that in local frames we have a simple and effective theory of external symmetries \[2\] providing us with the conserved operators we need for determining quantum modes.

Finally, we remark that the canonical quantization of all the fields we worked out so far on dS manifolds leads to quantum fields which can be manipulated similarly as those of special relativity. This indicates that our approach could be the starting point for building a simple version of perturbation theory of the interacting quantum fields on the dS expanding universe.

Appendix: Normalization integrals

In spherical coordinates of the momentum space, \(\mathbf{n} \sim (\theta_n, \phi_n)\), and the notation \(\mathbf{q} = \omega s \mathbf{n}\), we have \(d^3q = q^2 dq d\Omega_n = \omega^3 s^2 ds d\Omega_n = d(\cos \theta_n) d\phi_n\). Moreover, we can write

\[
\delta^3(\mathbf{q} - \mathbf{q}') = \frac{1}{q^2} \delta(q - q') \delta^2(\mathbf{n} - \mathbf{n}') = \frac{1}{\omega^3 s^2} \delta(s - s') \delta^2(\mathbf{n} - \mathbf{n}') , \tag{76}
\]

where we denoted \(\delta^2(\mathbf{n} - \mathbf{n}') = \delta(\cos \theta_n - \cos \theta_n') \delta(\phi_n - \phi_n')\).

The normalization integrals can be calculated in NP starting with the Hermitian form \(18\). According to Eqs. \(11\) and \(76\), this yields

\[
\langle f_{E,n}, f_{E',n'} \rangle = \frac{2N^2(2\pi)^3}{\omega^2} \delta^2(\mathbf{n} - \mathbf{n}') \int_0^\infty \frac{ds}{s} e^{i(E-E') \ln s} . \tag{77}
\]

Finally, using the identity

\[
\frac{1}{2\pi \omega} \int_0^\infty \frac{ds}{s} e^{i(E-E') \ln s} = \delta(E - E') , \tag{78}
\]

we find the value of the normalization constant \(42\).
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