Lanczos potentials for linearly perturbed FLRW spacetimes

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Abstract. We study the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes.

1. Introduction
Penrose [17] conjectured that the gravitational entropy should be related to the clumping of matter and therefore associated with the Weyl or conformal curvature. Specifically, Penrose suggested that a measure of the gravitational entropy should involve an integral of a quantity derived from the Weyl tensor, and that the particle number operator for a linear spin-2 massless quantized free-field might provide some clues, since the entropy measure could be taken as an estimate of the ‘number of gravitons’ [17]. Since then, there have been several attempts to construct gravitational entropy measures using polynomial invariants of the Weyl tensor (see e.g. [8, 4, 16]) as well as density contrast functions [13, 10].

We have used Penrose’s conjecture and the particle number from linear theory in flat space to motivate a definition of gravitational entropy in curved space [14]. In order to do that we required a potential for the Weyl tensor which we took to be the Lanczos potential [12]. Illge [11] has shown that any spinor field with the symmetries of the Weyl spinor locally has a Lanczos potential which is determined by its value at a space-like hypersurface. Furthermore, for a vacuum spacetime there exists a potential for the Lanczos potential, i.e. a superpotential for the Weyl spinor [11] (see also [1]).

Apart from Illge’s result, which is difficult to apply, there is no general prescription for obtaining a Lanczos potential for a given spacetime. A general expression for a Lanczos potential in the case of perfect fluid spacetimes with zero shear and vorticity was given in [15]. More recently, this result has been extended by Holgersson [9] to Bianchi I perfect-fluid spacetimes. There are also several examples of Lanczos potentials for particular exact solutions, including Gödel, Schwarzschild, Taub and Kerr [3, 15, 5, 6].

In this short note, we consider the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes, which we then use to propose a new measure of the gravitational entropy in [14].
2. The perturbed FLRW model

We consider a spacetime with a distinguished time-like direction given by the velocity vector field \( u^a \) of the fluid, and use the formalism of [7, 18], with the projected metric

\[
h_{ab} = g_{ab} + u_a u_b,
\]

which is orthogonal to \( u^a \). The covariant derivative of \( u_a \) can be written as

\[
\nabla_b u_a = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b
\]

where

\[
\sigma_{ab} = \sigma_{(ab)}; \quad \sigma_a^a = 0; \quad \sigma_a^b u^b = 0; \quad \omega_{ab} = \omega_{[ab]}; \quad \omega_{ab} u^b = 0.
\]

Then \( \dot{u}^a \) is the acceleration (so that the overdot is \( u^a \nabla_a \)), \( \omega_{ab} \) is the vorticity tensor, \( \sigma_{ab} \) the shear, and \( \theta \) the expansion. The stress–energy tensor for perfect fluids is

\[
T_{ab} = \rho u_a u_b + p h_{ab},
\]

where \( \rho \) is the energy density and \( p \) the isotropic pressure of the fluid.

The Weyl tensor can be decomposed into its electric and magnetic parts, \( E_{ab} \) and \( H_{ab} \) relative to the velocity vector \( u^a \) as

\[
E_{ab} = C_{abcd} u^c u^d, \quad H_{ab} = C^*_{abcd} u^c u^d,
\]

where \( C^*_{abcd} = \frac{1}{2} \eta_{acbd} C_{abcd} \). An FLRW background is conformally-flat with the fluid-flow being geodesic, shear-free and twist-free so that \( u_a = \omega_{ab} = \sigma_{ab} = 0 = E_{ab} = H_{ab} \).

We shall now consider the FLRW metric \( g_{ab} \) with linear perturbations \( \delta g_{ab} = \Phi_{ab} \) such that

\[
\Phi_{ab} u^b = \Phi^a_a = \nabla^a \Phi_{ab} = 0.
\]

The perturbation is characterised as purely gravitational by requiring:

\[
\delta R^b_a = 0.
\]

This implies that \( \delta \rho = \delta p = 0 \), and with the gauge conditions (1) also \( \delta u^a = \delta u_a = 0 \), so that \( \delta T^b_a = 0 \) and \( \delta \theta = 0 = \delta \omega_{ab} = \delta \dot{u}_a \), while for the shear we introduce the notation:

\[
\Sigma_{ab} := \delta \sigma_{ab} = \frac{1}{2} \dot{\Phi}_{ab}.
\]

For the Weyl tensor, which is zero in the background, we find

\[
E_{ab} = -\dot{\Sigma}_{ab} - \frac{2}{3} \theta \Sigma_{ab},
\]

\[
H_{ab} = \text{curl} \Sigma_{ab},
\]

with

\[
\text{curl} X^{ab} := (\text{curl} X)^{ab} := \eta^{cd(a} D_c X^{b)}_d,
\]

where \( D_c \) is the covariant derivative on hypersurfaces orthogonal to \( u^a \). \( \eta_{abc} = \eta_{abcd} u^d \) is the hypersurface volume form and \( \eta_{abcd} \) the space-time volume form. Now, the field equation (2) reduces to

\[
\square \Phi_{ab} = \frac{2}{3} \rho \Phi_{ab}
\]

and from (3) and (2) we get

\[
\square \Sigma_{ab} = \frac{2}{3} \theta \Sigma_{ab} + \left( \frac{1}{6} \rho - \frac{3}{2} \rho + \frac{1}{3} \theta^2 \right) \Sigma_{ab}.
\]
3. The Lanczos potential

The Lanczos potential is a tensor $L_{abc} = -L_{bac}$ such that:

$$C_{cd}^{ab} = -\nabla [c L_{ab}^{d} a] - \nabla [a L_{cd}^{b} b] - 2\delta_{a}^{c} \nabla [e L_{be}^{d} d],$$

in the Lanczos gauge:

$$L_{ab}^{a} = 0 = \eta_{abcd} L_{abc} ; \nabla_{c} L_{ab}^{c} = 0.$$

Holgersson [9] gave a useful decomposition of the Lanczos potential into irreducible parts as:

$$L_{abc} = 2u^{a} A_{b}^{c} - A_{a}^{d} B_{bc} + \eta_{ab}^{d} S_{dc} + u^{a} \eta_{bc}^{d} P_{d} - u^{c} \eta_{ab}^{d} P_{d},$$

(7)

where $A_{a}$ and $P_{a}$ are orthogonal to $u^{a}$ and $S_{ab}$ and $C_{ab}$ are trace-free, symmetric and orthogonal to $u^{a}$.

Since the FLRW perturbation is trace-free, symmetric and orthogonal to $u^{a}$, we seek a Lanczos potential as in (7) with $A_{a} = P_{a} = 0$. Then from (7) and (3) we find the following expressions for $E_{ab}$ and $H_{ab}$:

$$E_{ab} = \frac{1}{2}(\text{curl } S_{ab} - \dot{C}_{ab}),$$

(8)

$$H_{ab} = \frac{1}{2}(\text{curl } C_{ab} + \dot{S}_{ab}).$$

(9)

which equated to (4) and (5) give the expressions for $C_{ab}$ and $S_{ab}$.

Now, suppose a superpotential $\phi_{ab}$ existed for all times with

$$L_{abc} = \nabla [a \phi_{b}^{c}],$$

(10)

then from (7), we get expressions for $C_{ab}$ and $S_{ab}$ as:

$$C_{ab} = \frac{1}{2}(\dot{\phi}_{ab} + \frac{\theta}{3}\phi_{ab}),$$

$$S_{ab} = \frac{1}{2}\text{curl } \phi_{ab},$$

which turn out to be incompatible with the Bianchi identities [14] (as is to be expected, since the superpotential should not exist for non-vacuum). However, this procedure suggests the ansatz:

$$C_{ab} = \frac{1}{2}(\psi_{ab} + \frac{\theta}{3}\phi_{ab})$$

$$S_{ab} = \frac{1}{2}\text{curl } \phi_{ab}$$

(11)

in terms of another unknown tensor $\psi_{ab}$. So, we find from (9) and (11)

$$H_{ab} = \frac{1}{4}\text{curl } (\dot{\phi} + \psi)_{ab}.$$  

Comparing this equation with (5) we can choose

$$\Sigma_{ab} = \frac{1}{4}(\dot{\phi}_{ab} + \psi_{ab}),$$

(12)

so that $\psi_{ab}$ is known once $\phi_{ab}$ has been found. Then, from (8)

$$E_{ab} = \frac{1}{4}(- \dot{\psi}_{ab} + \frac{\theta}{3}\dot{\phi}_{ab} - \frac{\theta}{3}\dot{\phi}_{ab} + \text{curl } \phi_{ab}),$$
and combining this with (4) and (12) we get
\[ \Box \phi_{ab} + \frac{4}{3} \theta \dot{\phi}_{ab} + \left( \frac{\dot{\theta}}{3} + \frac{\theta^2}{9} - \rho \right) \phi_{ab} = \frac{8}{3} \Theta_{ab}, \]  
which is a wave equation for $\phi_{ab}$. We therefore have a complete prescription to determine a unique $L_{abc}$ for linearly perturbed FLRW, subject to choice of initial data. We can achieve (10), at a given instant $t_0$ by choosing the data for (13) to be
\[ \phi_{ab}(x, t_0) = \Phi_{ab}(x, t_0), \]
\[ \dot{\phi}_{ab}(x, t_0) = \dot{\Phi}_{ab}(x, t_0). \]

We summarize our results in the following proposition:

**Proposition** Given a perturbed FLRW spacetime and a choice of time $t_0$, a Lanczos potential $L_{abc}$, in the Lanczos gauge, may be uniquely specified by (7) with (11), (12) and (13), subject to the data (14). We may define a superpotential $\phi_{ab}$ such that (10) holds at $t_0$ but this will not hold at other times.

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