The geometry of a light wavefront evolving from a flat wavefront under the action of weak gravity field in the 3-space associated to a post-Newtonian relativistic spacetime, is studied numerically by means of the ray tracing method.

**1. Introduction**

The curvature of initially plane light wavefronts by a gravity field is a purely general relativistic effect that has no special relativistic analogue. In order to obtain an experimental measurement of the curvature of a light wavefront, Samuel recently proposed a method based on the relation between the differences of arrival time recorded at four points on the Earth and the volume of a parallelepiped determined by four points in the curved wavefront surface, see Ref. 1. In this work and in Ref. 2 with more detail, we study a discretized model of the wavefront surface by means of a regular triangulation for the study of the curvature(s) (mean and relative, see Ref. 3) of this surface.

**2. Light propagation in a weak gravitational field**

Let us consider a spacetime \((M,g)\) corresponding to a weak gravitational field determined by a metric tensor given in a global coordinate system \((z,ct)\) by \(g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\), with \(\eta_{\alpha\beta} = \text{diag}(1,1,1,1)\). Where the coordinate components of the metric perturbation \(h_{\alpha\beta}\) are:

\[
\begin{align*}
    h_{ab} &= 2c^{-2}\kappa \|z\|^{-1} \delta_{ab}, \\
    h_{a4} &= -4c^{-3}\kappa \|z\|^{-1} \dot{z}_a, \\
    h_{44} &= 2c^{-2}\kappa \|z\|^{-1},
\end{align*}
\]

(1)

here \(\kappa := GM\) represents the gravitational constant of the Sun, located at \(Z^a(t)\), and \(c\) represents the vacuum light speed. The null geodesics, \(z(t) = (z(t), t)\) satisfy the following equations, see Ref. 4:

\[
\begin{align*}
    \ddot{z}^a &= \frac{1}{2} c^2 h_{44,a} - [\frac{1}{2} h_{44,t} \delta^a_k + h_{ak,t} + c(h_{4a,k} - h_{4k,a})] \dot{z}^k \\
    &- (h_{44,t} \delta^a_t + h_{ak,t} - \frac{1}{2} h_{kl,a}) \dot{z}^k \dot{z}^l \\
    &- (c^{-1} h_{4k,j} - \frac{1}{2} c^{-2} h_{jk,t}) \dot{z}^j \dot{z}^k \dot{z}^a, \\
    0 &= g_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta.
\end{align*}
\]

The second equation is the isotropy constraint satisfied by the null geodesics.
3. Local approximation of the wavefront

Let be $S_0$ a flat initial surface far from the Sun, formed by points $(z_1, z_2, -\zeta)$ (with $\zeta > 0$) in an asymptotically Cartesian coordinate system $\{z\}$. For the discretization of $S_0$ a triangulation is constructed in such a form that each vertex is represented by a complex number of the set:

$$\bar{V} := \{z = a_1 + a_2 \omega + a_3 \omega^2 \mid a_1, a_2, a_3 \in \mathcal{A}, \omega := \exp(2\pi i/3)\},$$

(2)

The initial triangulation by $V$ induces a triangulation on the final wavefront $S_t$. The evolution of a photon $z_0 := z(0) \in V$ with velocity $\dot{z}_0 := (0, 0, c)$ in phase space $u = (z, \dot{z})$ may be written as a first order differential system $\dot{u} = F(u, t)$. This determines a flow, $z(t) = \varphi_t(z_0, \dot{z}_0)$, in the 3-dimensional curved quotient space of $\mathcal{M}$ by the global timelike vector field $\partial_t$ associated to the global coordinate system used in the post-Newtonian formalism. For each time $t$, the flow $\varphi_t$ determines a 2-dimensional curved wavefront $S_t$.

To compute the curvatures of the wavefront surface corresponding to the mesh $V$ at each inner vertex, we consider a 1–ring formed by the six vertices closest. For each 1–ring, one obtains on the mesh $V$ the image under the flow $\varphi_t$. In a neighbourhood of the image point the wavefront can be approximated by a least-squares fitting of the data obtained as the quadric:

$$y^3 = f(y^1, y^2) := \frac{1}{2} a_1 (y^1)^2 + a_2 y^1 y^2 + \frac{1}{2} a_3 (y^2)^2.$$

(3)

using adapted normal coordinates $\{y^i\}$.

4. Numerical integrator

We apply the ray tracing method, see Refs.2,5, to a tubular region of light wavefront region supposing a gravitational model generated by a static Sun, considered as a point. The mean and relative total curvatures, defined in Refs.2,3, are computed at each inner vertex of mesh on the light wavefront surface $S_t$ in the vicinity of the Sun, by the implementation of the following pseudocode:

```python
Data: u^*_n := (z^*_n, \dot{z}^*_n), n = 1, \ldots N
for n = 1 \ldots N do
    u_n := Taylor(t, u^*_n)
    y_n := NormalCoordinates(z_n)
    for i = 0 \ldots 6 do
        y_{ni} := Ring(y_n)
    end
    (a_1, a_2, a_3) := LeastSquares(y_{ni})
    \gamma_{AB}(x_n) := Metric(a_1, a_2, a_3, x_n)
    B = SecondFundamentalForm(x_n)
    (\lambda_1, \lambda_2) = Diagonalize(B)
    (K_{rel}, H) = Curvature(\lambda_1, \lambda_2)
end
```
Figure 1. Wavefront surface and relative curvature (gray scale) deformed by a spherical gravitational field (a different scale is used for the vertical axis).

In Figure 1, the surface $S_T$ at the time when the wavefront arrives at the Earth is shown using a gray-scale to represent the relative curvature (note we have used a different scale on the $Oz^3$-axis). One sees in this figure that the absolute value of the relative curvature defined on $S_T$ increases as the distance between the photon and the $Oz^3$-axis, where the Sun is located, decreases.

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