Fully Constrained Mass Matrix: Can Symmetries alone determine the Flavon Vacuum Alignments?

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1 Introduction

More than two decades [1, 2] of experiments in neutrino oscillations have provided us with measurements of the neutrino mixing angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ as well as the mass-squared differences, $\Delta m^2_{21}$, $\Delta m^2_{31}$ [3, 4]. Yet, several features of neutrinos remain a mystery. Ordering of neutrino masses, CP violation in neutrino sector, nature of neutrinos (Majorana or Dirac), existence of sterile neutrinos etc. are some of them. Parameters such as the light neutrino mass and the complex phases in the mixing matrix also need to be measured. Many of these questions are expected to be resolved by future experiments in the coming decades [5, 6, 7, 8, 9, 10, 11].

The initial measurements of large solar ($\theta_{12}$) and atmospheric ($\theta_{23}$) mixing angles stimulated the theoretical study of flavour symmetries in neutrino sector based on discrete finite groups [12, 13, 14, 15]. Tribimaximal mixing [16] with $\theta_{12} = \sin^{-1}(1/\sqrt{3})$, $\theta_{23} = \pi/4$, $\theta_{13} = 0$ was widely used as a template for building models in neutrino sector. With the measurement of the non-zero reactor mixing angle inconsistent with tribimaximal mixing, theorists have turned to alternative mixing schemes. A natural approach is to extend tribimaximal mixing with one or more free parameters [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. One such ansatz called tri-phi-maximal mixing (T$\phi$M) [27], leads to a mixing matrix of the form

$$U_{T\phi M} = \begin{pmatrix} \frac{\sqrt{2}}{3} \cos \phi & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{3} \sin \phi \\ \frac{\cos \phi}{\sqrt{6}} - \frac{\sin \phi}{\sqrt{2}} & \frac{\cos \phi}{\sqrt{2}} + \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{2}} - \frac{\sin \phi}{\sqrt{6}} \\ \frac{\cos \phi}{\sqrt{6}} + \frac{\sin \phi}{\sqrt{2}} & \frac{\cos \phi}{\sqrt{2}} - \frac{\sin \phi}{\sqrt{3}} & \frac{\cos \phi}{\sqrt{2}} + \frac{\sin \phi}{\sqrt{6}} \end{pmatrix}. \quad (1)$$

The angle $\phi$ parametrises the non-zero reactor mixing angle. Like tribimaximal mixing, tri-phi-maximal mixing also has a trimaximal second column and is CP conserving.

An ansatz of Majorana neutrino mass matrices,

$$M_{\text{Maj}} \propto \begin{pmatrix} i + \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ 1 - \frac{1}{\sqrt{2}} & 0 & i + \frac{1}{\sqrt{2}} \end{pmatrix} \propto \begin{pmatrix} -i + \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ 1 - \frac{1}{\sqrt{2}} & 0 & i + \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (2)$$

which leads to $T\phi M$ with $\phi = \pm \pi/16$ was proposed [28] shortly after the discovery of non-zero $\theta_{13}$ by Daya Bay experiment. These matrices are fully constrained, in the sense that they do not contain free parameters. Hence, they also provided the neutrino mass ratios.

$$m_1 : m_2 : m_3 = \sqrt{2} \tan \left(\frac{3\pi}{16}\right) : 1 : \sqrt{2} \tan \left(\frac{5\pi}{16}\right). \quad (3)$$

These are consistent with the measured neutrino mass-squared differences and also predict the experimentally

$\dagger$ Given as $\frac{(2+\sqrt{7})}{1+\sqrt{2}(2+\sqrt{7})} : 1 : \frac{(2+\sqrt{7})}{1-\sqrt{2}(2+\sqrt{7})}$ in Ref. [28]
In Fig. 1, we compare these ratios with the experimental representation, we have namely $C$ can be used to model such fully constrained mass matrices. For the three-dimensional basis where the non-zero coefficients are given by

$$C_{\alpha \beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}.$$  

The tensor product expansion of two triplets of this group is given by

$$3 \otimes 3 = \bar{6} \oplus \bar{3},$$  

$$\bar{3} \otimes \bar{3} = 6 \oplus \bar{3}.$$  

$\Sigma(72 \times 3)$ is the smallest group which produces a complex sextet from the tensor product of two identical triplets as shown in Eqs. (5, 6). Note that the triplets of the continuous group $SU(3)$ also have the same tensor product expansion. In Ref. [29], we assigned the right-handed neutrinos to be a conjugate triplet,

$$\nu_R = (\nu_R1, \nu_R2, \nu_R3)^T \equiv 3.$$  

In the Majorana mass term, two of these conjugate triplets couple to produce a conjugate sextet,

$$\sum_{jk} S_{ijk} \nu_{Rj} \nu_{Rk} \equiv \bar{6}_i,$$  

where $\nu_{Rj}, \nu_{Rk}$ are the Lorentz invariant products of the right-handed neutrino Weyl spinors. $S_{ijk}$ are the familiar Clebsch-Gordan coefficients for the symmetric tensor product of two triplets of $SU(3)$. We use the conventional basis where the non-zero coefficients are given by

$$S_{111} = S_{222} = S_{333} = 1,$$

$$S_{441} = S_{432} = S_{531} = S_{513} = S_{612} = S_{621} = \frac{1}{\sqrt{2}}.$$  

We also introduced a flavon sextet,

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)^T \equiv \bar{6},$$  

which couples with the conjugate sextet, Eq. (8), to produce the $\Sigma(72 \times 3)$-invariant mass term,

$$\sum_{ijk} S_{ijk} \xi_i \nu_{Rj} \nu_{Rk}$$

$$= \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix}^T \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix} \cdot \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix}.$$  

The flavon sextet acquires a Vacuum Expectation Value (VEV) through Spontaneous Symmetry Breaking (SSB) and this VEV determines the structure of the mass matrix. Comparing Eq. (10) with Eq. (11), it is clear that there is a one-to-one correspondence between the components of the sextet and the elements of the $3 \times 3$ complex-symmetric Majorana mass matrix. A specific VEV of the sextet fully constrains the mass matrix. The VEVs which correspond to the Majorana mass matrices, Eqs. (2), are

$$\langle \xi \rangle \propto \begin{pmatrix} i + 1 - i \\ \sqrt{2} \\ 1, -i + 1 + i \\ \sqrt{2} \end{pmatrix}, 0, (\sqrt{2} - 1), 0 \rangle^T,$$  

$$\langle \xi \rangle \propto \begin{pmatrix} -i + 1 - i \\ \sqrt{2} \\ 1, i + 1 + i \\ \sqrt{2} \end{pmatrix}, 0, (\sqrt{2} - 1), 0 \rangle^T.$$  

In Ref. [29] we constructed flavon potentials which, through SSB, resulted in these VEVs and this reproduced the mass matrices, Eqs. (2). These mass matrices are diagonalised by $2 \times 2$ unitary matrices. The mixing matrix of the form $\text{ToM}(\phi = \pm \frac{\pi}{3})$ is obtained as the product of a $3 \times 3$ tri-maximal contribution from the charged-lepton sector and
the above mentioned $2 \times 2$ contribution from the neutrino sector. The mixing angles extracted from $T\phi M(\phi = \pm \pi/16)$ are quite close to the experimental values. We used higher order corrections in the charged-lepton sector to account for the small discrepancy between the $T\phi M(\phi = \pm \pi/16)$ and the experimental values.

2 Vacuum Alignment in Flavour Space

In this section, we briefly review the salient features of model building using flavons. Specifically we discuss the construction of the Majorana mass matrix involving three families of right-handed neutrinos\footnote{The principles discussed here are applicable for charged-leptons and quarks as well.}. The three neutrino states are assumed to form a triplet under a discrete flavour group, in general a subgroup of the continuous group, $U(3)$. Since the neutrinos are assumed to transform as a triplet under the flavour group, we calculate the tensor product expansion of two such triplets. This expansion gives rise to several neutrino-neutrino terms which transform as multiplets under the flavour group. Flavons also transform as multiplets under the flavour group. The neutrino-neutrino multiplets and the corresponding flavon multiplets (conjugates) couple, leading to flavour group invariant mass terms. In model building, once we settle on a suitable flavour group and a set of flavon multiplets, there are two questions that determine the mass matrix. What are the values of the coupling constants that appear along side the invariant mass terms and how are the flavon VEVs aligned? We discuss these aspects in the rest of this section.

Corresponding to each of the invariant mass term we associate a coupling constant. These coupling constants are independent of each other and their values are arbitrary from a theoretical point of view, i.e. neither the flavour symmetries nor the features of the flavons can be used to predict them. However, the presence of coupling constants could be an advantage when the aim of the model is to explain only certain features of the mass matrix while leaving other features untouched, i.e. the coupling constants in the model allows us to leave a few degrees of freedom within the mass matrix unconstrained by the theory. If the unconstrained features are experimentally known, we can fit our model with the data and subsequently determine the values of the coupling constants. Else, the values of these constants are left unknown.

The flavour structure of a model is also determined by the relative orientation between the fermion flavour eigenstates and the flavon VEVs. Assigning the three flavours (families) of fermions as a triplet under the flavour group implies that they are aligned along the basis states of the representation. To obtain the alignment of flavon VEVs, we construct a flavon potential invariant under the flavour group. This potential will have a discrete set of extrema points. Through the mechanism of SSB, the flavon acquires a VEV which corresponds to one of these extrema. The fact that the symmetry is discrete, limits the extrema points to a finite set. SSB randomly chooses one among these extrema as the vacuum alignment. By changing the nature of the flavon potential we may alter the set of extrema points and thus change the possible vacuum alignments. The flavon VEVs form the building blocks of the mass matrix, so the alignment of the VEVs in flavour space has important consequences for the structure of the mass matrix. We expect that a given alignment has specific symmetry properties under the flavour group, which in turn impart specific features to the mass matrix.

\[ x \quad y \quad z \]

\[ x \quad y \quad z \]

Fig. 2. The dots on the cubes represent extrema of two cases of flavon potentials which has $S_4$ symmetry. In the left figure, the extrema are on the face-centres of the cube. The VEV, denoted by the red dot, is aligned along one of the axes of symmetries of the cube. In the right figure, the $S_4$ symmetry of the potential results in 24 extrema points positioned symmetrically around the cube. However, the VEV (the red dot) is not aligned along any special direction with respect to the cube.

Let us use the discrete group $S_4$ as an example to demonstrate the alignment of states in the flavour space\footnote{For a detailed discussion of the representations of $S_4$, the construction of flavon potentials, the symmetries of flavon VEVs and the resulting mass matrices, refer to Appendix.}. The triplet representation $(3)$ of $S_4$ corresponds to the 24 proper rotations in the three dimensional real space that leaves a cube invariant. When we assign the fermions as a triplet, we are assigning its components, i.e. the three flavour states, as the basis states of the triplet representation. In the widely used basis of $(3)$, the basis states are aligned along the face centres of cube, i.e. the $x$, $y$ and $z$ axes as shown in Fig. 2. The cube remains invariant under rotations about these axes by multiples of $\pi/2$. Therefore, the three fermion flavour states correspond to the three cyclic $C_4$ subgroups of $S_4$.

To couple with two triplets $(3)$ of fermions, we introduce a flavon which transforms as a $3'$. The representation $3'$ also corresponds to 24 rotational symmetries of a cube (12 proper and 12 improper rotations). We may construct a potential whose extremum correspond to flavon orientations directed towards the face-centres of the cube, as shown in Fig. 2 (left). Each of these extrema remains invariant under a $C_2 \times C_2$ subgroup of $S_4$ consisting of 4

The principles discussed here are applicable for charged-leptons and quarks as well.
proper and improper rotations. Six such subgroups exist and the number of extrema orientations (the number of faces of the cube) is simply $24/4 = 6$. When the flavour symmetry group $S_4$ is broken by SSB, the resulting vacuum alignment will be along one of these extrema. So the symmetry breaking is not complete, i.e. a $C_2 \times C_2$ subgroup remains as the residual symmetry.

It is also possible to construct a potential whose extrema are oriented in directions with no symmetry properties. Such a potential will have 24 distinct extrema as shown in Fig. 2 (right). A VEV along one of these extrema breaks $S_4$ completely so that there remain no residual symmetries. By appropriately tuning the potential, we will be able to orient the extrema and the resulting VEV in any direction we may want. This seems to be true for all discrete groups, not just $S_4$. We note that a considerable number of publications rely almost entirely on flavon potentials to determine their vacuum alignments. Authors utilise quite complicated potentials to obtain VEVs which are phenomenologically viable, but they fail to provide a justification for these VEVs in terms of the symmetries of the flavour group. Even though this procedure is technically valid, we argue that it goes against the very spirit of using the properties of discrete groups for determining the flavour structure. If the VEV is made to orient in an arbitrary direction with no apparent connection to the original symmetry, the whole purpose of using discrete symmetries can be called into question. We argue that the orientation of the fermion basis states as well as the flavon VEVs should be determined by their symmetries alone. These symmetries are nothing but the subgroups of the discrete flavour group. The mathematical elegance of the subgroup structure of the flavour group should manifest as the strictiveness of the orientations of the flavour states and thus the predictiveness of the flavour model.

When the flavon vacuum alignments, Eqs. (12, 13), for the sextet of $\Sigma(72 \times 3)$ were proposed [29], they were not completely justified with the help of their symmetry properties alone. In this paper, we combine $\Sigma(72 \times 3)$ with a new discrete symmetry group which we call $X_{24}$. We introduce flavons which transform under both $\Sigma(72 \times 3)$ and $X_{24}$. Our flavon VEVs uniquely break the combined flavour group into its subgroups. In other words, the VEVs are completely determined by their symmetries alone. These flavons are coupled together to obtain the sextet of $\Sigma(72 \times 3)$. This sextet in turn couples with the neutrino triplets resulting in the Majorana mass term.

### 3 The Discrete Group $X_{24}$

We construct discrete group, $X_{24}$, using the following generators:

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$B = \begin{pmatrix}
\omega & 0 & 0 & 0 & 0 & 0 \\
0 & \tau & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{\tau} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{\tau} & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & \omega
\end{pmatrix},$$

where $\omega = e^{i\frac{2\pi}{3}}$, $\bar{\omega} = e^{-i\frac{2\pi}{3}}$ are the cube roots of unity and $\tau = e^{i\frac{2\pi}{3}}$, $\bar{\tau} = e^{-i\frac{2\pi}{3}}$ are the eighth roots of unity. The largest cyclic subgroup of this group is $C_{24}$, generated by $\omega\tau$ and hence the subscript $24$ in $X_{24}$. These generators, Eq. (14), are selected so that the group constructed from them helps to uniquely define the required flavon VEVs. The rest of this section covers the mathematical study of the properties of this group. A reader who is more inclined towards applying the group theoretical results for the construction of the VEVs and the mass matrix may skip over to section 4 and may revert to this section when it is deemed necessary.

As the first step in analysing $X_{24}$, we construct the group elements,

$$C(\tau_1) = (A^2 B A^3 B A^4)^9 = \text{Diag}(1, \bar{\tau}, \tau, 1, 1, 1),$$

$$C(\tau_2) = (A^2 B A^3 B A^4)^9 = \text{Diag}(1, 1, 1, \tau, \bar{\tau}, 1).$$

Using $C(\tau_1)$, $C(\tau_2)$ and $B$ we obtain the group element,

$$[B] = C(\tau_1), C(\tau_2), B^3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

$A$ and $[B]$ generate the group $S_3 \times S_3$ which forms a subgroup of $X_{24}$. To show this we obtain,

$$D_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$E_i = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},$$

where

$$D_1 = A^2 (A | B|^3)^3, E_i = A^2, D_2 = A^3, E_2 = (A | B|^4)^2.$$  

$D_1$, $E_1$ and $D_2$, $E_2$ in Eqs. (18, 19) separately form the generators of the group $S_3$, because they satisfy the following group presentation,

$$\langle D_i, E_i | D_i^2 = E_i^3 = (D_i E_i)^2 = 1 \rangle$$

along with the relationship,

$$E_i D_i = D_i E_i^2$$

for $i = 1, 2$.  

$S_3$ group elements, $g_1$ and $g_2$, generated by $D_1, E_1$ and $D_2, E_2$ respectively, can be expressed as

$$g_1 = D_1^{i_1} E_1^{j_1}, \quad g_2 = D_2^{i_2} E_2^{j_2},$$
where \( i_1, i_2 \in \{1, 2\} \) and \( j_1, j_2 \in \{1, 2, 3\} \). The first set of generators, Eq. (18), commute with the second set, Eq. (19), i.e.

\[
[D_i, D_j] = [E_i, E_j] = [D_i, E_j] = 0 \quad \text{for} \quad i \neq j,
\]

so that we obtain the direct group product of two \( S_3 \) groups. Thus we show that \( A \) and \( B \) generate the group \( S_3 \times S_3 \) with the total number of elements equal to \( 2 \times 3 \times 2 \times 3 \).

Note that the elements of \( S_3 \times S_3 \) in the basis given by, Eq. (18, 19), are matrices with ‘1’s and ‘0’s only. \( C(\tau_1) \) and \( C(\tau_2) \), Eqs. (15, 16), individually generate the cyclic group \( C_8 \). In \( X_{24} \), we can find two more such generators of \( C_8 \),

\[
\begin{align*}
C(\tau_3) &= E_1 C(\tau_1) E_1^2 = \text{Diag}(\tau, 1, \tau, 1, 1, 1), \\
C(\tau_4) &= E_2 C(\tau_1) E_2^2 = \text{Diag}(1, 1, 1, \tau, 1, 1, 1).
\end{align*}
\]

Four elements, similar to Eqs. (15, 16, 26, 27), which individually generate the cyclic group \( C_3 \) can also be found,

\[
\begin{align*}
C(\omega_1) &= D_1 B^3 D_1 B^2 E_1^2 = \text{Diag}(1, \bar{\omega}, \omega, 1, 1, 1), \\
C(\omega_2) &= E_1 C(\omega_1) D_1 = \text{Diag}(1, 1, 1, \omega, \bar{\omega}, 1), \\
C(\omega_3) &= E_2 C(\omega_1) E_1^2 = \text{Diag}(\omega, 1, \bar{\omega}, 1, 1, 1), \\
C(\omega_4) &= E_2 C(\omega_2) E_2^2 = \text{Diag}(1, 1, 1, \bar{\omega}, \omega, 1).
\end{align*}
\]

We also find a fifth independent \( C_3 \) generator,

\[
C(\omega)_5 = B^2 = \text{Diag}(\bar{\omega}, 1, 1, 1, 1, 1).
\]

Using \( 6 \times 6 \) special unitary diagonal matrices, the maximum number of independent \( C_8 \) generators that can be constructed is five and in Eqs. (28 - 32) we have listed all of them for \( C_3 \). For the case of the diagonal \( C_8 \) subgroups of \( X_{24} \), it so happens that the upper and the lower \( 3 \times 3 \) diagonal matrices are individually special unitary. This additional constraint limits the total number of independent generators to four, i.e. Eqs. (15, 16, 26, 27). Eqs. (15, 16, 26, 27, 28 - 32) constitute an exhaustive list of generators producing all the diagonal elements within \( X_{24} \). These elements form the subgroup \( C_8 \times C_8 \times C_8 \times C_8 \times C_3 \times C_3 \times C_3 \times C_3 \times C_3 \) of \( X_{24} \). The diagonal elements commute with each other, so they also form the centre (largest abelian subgroup) of \( X_{24} \). Note that \( 3 \) and \( 8 \) are co-prime numbers which implies \( C_8 \times C_3 \) is \( C_{24} \). This can also be inferred from the multiplication of \( C_8 \) and \( C_3 \) generators, for example,

\[
C(\tau_1) C(\omega_1) = \text{Diag}(1, \bar{\tau}, \omega, \bar{\tau}, 1, 1, 1) = C(\tau_1 \omega_1).
\]

In other words, the group \( C_{24} \times C_{24} \times C_{24} \times C_{24} \times C_3 \) forms the centre of \( X_{24} \).

Every representation matrix of \( X_{24} \) is of the form of a representation matrix of \( S_3 \times S_3 \) with phases replacing certain number of ‘1’s in the \( S_3 \times S_3 \) matrix. These phases can be extracted out using a diagonal phase matrix, i.e. an element of the centre of the group. In other words, any element of \( X_{24} \) can be obtained by left multiplying (or right multiplying) the corresponding element of \( S_3 \times S_3 \) with an appropriate diagonal phase matrix. Therefore, \( C_{24} \times C_{24} \times C_{24} \times C_{24} \times C_3 \) and \( S_3 \times S_3 \) form a normal subgroup and the associated quotient group respectively of \( X_{24} \). Using this information, we may express \( X_{24} \) as a semidirect product,

\[
X_{24} = (C_{24} \times C_{24} \times C_{24} \times C_{24} \times C_3) \times (S_3 \times S_3).
\]

Any element of \( X_{24} \) can be uniquely expressed as

\[
g = C(\tau_1)^{m_1} C(\tau_2)^{m_2} C(\tau_3)^{m_4} C(\tau_4) \omega_{n_1} \omega_{n_2} \omega_{n_3} \omega_{n_4} \omega_{n_5} \omega_{n_6} \omega_{n_7} \omega_{n_8},
\]

\[
D_i^j E_j D_j^k E_k,
\]

where \( m_1, ..., m_4 \in \{1, ..., 8\} \), \( n_1, ..., n_5 \in \{1, 2, 3\} \), \( i_1, i_2 \in \{1, 2\} \), \( j_1, j_2 \in \{1, 2, 3\} \). So it is clear that the order of the group \( X_{24} \) is \( 8^4 \times 3^2 \times 2^3 \). We used the group theory package GAP and verified that the group generated by \( A, B \) has this order, thus confirming our calculations. We also verified that the nextet representation, Eqs. (14), is irreducible. We note that \( X_{24} \) is not a subgroup of \( U(3) \).

### 4 The Model

In this section, we discuss the construction of the Majorana mass term and the resulting mass matrix of the form, Eq. (2), with the help of flavons transforming under the flavour groups \( \Sigma(72 \times 3) \) and \( X_{24} \). This term involves the couplings among the right-handed neutrinos. The discussion of the Dirac mass term for the neutrinos (couplings between the right-handed neutrinos and the left-handed lepton doublets) as well as for the charged-leptons (couplings between the right-handed charged-leptons and the left-handed lepton doublets) are omitted here. We assume that the Dirac sector follows the details as given in Ref. [29] in using \( \Sigma(72 \times 3) \) to obtain the relevant mass matrices.

The complete flavour group for our model is \( \Sigma(72 \times 3) \times X_{24} \times X_{24} \). Why we have used two copies of \( X_{24} \) will become apparent during the course of this section. Table 1 shows how the right-handed neutrinos (\( \nu_R \)) and the flavons (\( \phi, \bar{\phi}, \Delta \)) transform under the flavour group. In this paper, we use Latin and Greek letters to denote the indices which transform under \( \Sigma(72 \times 3) \) and \( X_{24} \), respectively, i.e. \( \nu_R \), \( \phi_{\alpha}, \bar{\phi}_{\alpha}, \Delta_{\alpha, \beta} \).

Using the C-G coefficients, Eqs. (9), and considering the transformation properties given in Table 1, we construct the invariant term in the Majorana sector,

\[
\mathcal{T}_{\text{Maj}} = \sum_{i,j,k} S_{i,j,k} \text{Sim}_\alpha \phi_{\alpha \mu} \bar{\phi}_{\beta \mu} \Delta_{\alpha, \beta} \nu_R^j \nu_R^k,
\]

where the summation is over all repeated indices. Comparing this invariant with Eq. (11), we obtain

\[
\xi_I = \sum_{i,j,k} S_{i,j,k} \phi_{\alpha \mu} \bar{\phi}_{\beta \mu} \Delta_{\alpha, \beta}.
\]

The flavon \( \phi \) (and \( \bar{\phi} \)) can be considered as a set of six \( \Sigma(72 \times 3) \)-triplets. In Eq. (37), we have a composite system
of these triplets coupled together with $\Delta$ to obtain $\xi$ which is a sextet under $\Sigma(72 \times 3)$ and an invariant singlet under $X_{24}$.

The flavons $\phi$, $\bar{\phi}$ and $\Delta$ acquire VEVs through SSB. Let these vacuum alignments to be

$$
\langle \phi \rangle = \langle \bar{\phi} \rangle = \begin{pmatrix} \tau & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \langle \Delta \rangle = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.
$$

Here we have listed the components of the flavons with the help of matrices. The rows and the columns of $\Delta$ denote the indices of the first and the second copies of $X_{24}$ in the flavour group. The rows and the columns of $\phi$ ($\bar{\phi}$) denote the first (second) $X_{24}$-index and the $\Sigma(72 \times 3)$-index respectively. Substituting the values of the VEVs, Eqs. (38), in the expression for $\xi$, Eq. (37), we obtain $\langle \xi \rangle$ as given in Eq. (12). On the other hand, if we use the conjugates of Eqs. (38) we obtain $\langle \xi \rangle$ as given in Eq. (13). The mass term, Eq. (36), can be written as a matrix equation,

$$
T_{\text{Maj}} = \nu_R^2 \frac{1}{2} \left( \phi^T \Delta \bar{\phi} + \bar{\phi}^T \Delta^T \phi \right) \nu_R.
$$

Consequently, the expression for the $3 \times 3$ Majorana mass matrix becomes

$$
M_{\text{Maj}} = \frac{1}{2} \left( \langle \phi \rangle^T \langle \Delta \rangle \langle \bar{\phi} \rangle + \langle \bar{\phi} \rangle^T \langle \Delta \rangle^T \langle \phi \rangle \right).
$$

Substituting the flavon VEVs, Eqs. (38), (or their conjugates) in Eq. (40), we obtain the Majorana mass matrices, Eqs. (2).

### 5 The Flavon Vacuum Alignments

In this section we show that the VEVs, Eqs. (38), can be uniquely defined by their symmetries. More concretely, we show that they can be expressed as unique and simultaneous invariant eigenstates of a set of group elements. These elements constitute a subgroup of the flavour group, $\Sigma(72 \times 3) \times X_{24} \times X_{24}$. Since the VEVs remain invariant under the action of these elements, they break the flavour group into this subgroup.

First we consider the flavon $\Delta$. Consider the group element

$$
O_{C,\Delta} = C(\omega)_1 C(\omega)_2 \times C(\omega)_1^2 C(\omega)_2^2 C(\omega)_1^3
$$

acting on the flavon VEV ($\Delta$), Eq. (38). The direct product, Eq. (41), (corresponding to $X_{24} \times X_{24}$) acts on the two indices of $\langle \Delta \rangle$. As a matrix equation, the operation of this group element on the VEV can be written as

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

where the rows and columns of $\langle \Delta \rangle$ corresponds to the first and the second $X_{24}$ in the direct product. It is clear that, this operation multiplies all the vanishing elements in the VEVs with $\omega$ or $\bar{\omega}$. Therefore, invariance of the VEV under $O_{C,\Delta}$ ensures that these elements vanish.

Consider the group element

$$
O_{D,\Delta} = I \times D_2
$$

where $I$ is the identity. As a matrix equation, the operation of this element on the VEV can be written as

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

This operation interchanges the columns 1, 2, 3 of the VEV with the columns 4, 5, 6 respectively. Invariance under this operation ensures that the columns 1, 2, 3 become equal to the columns 4, 5, 6 respectively. This condition is satisfied by our VEV.

Now consider the group element

$$
O_{D,\Delta} = D_1 \times I.
$$

As a matrix equation, its operation on the VEV can be written as

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

This operation interchanges the rows 1, 2, 3 of the VEV with the rows 6, 5, 4 respectively. Invariance under this operation ensures that the rows 1, 2, 3 become equal to the rows 6, 5, 4 respectively. This condition is also satisfied by our VEV.

Finally we consider the group element

$$
O_{E,\Delta} = E_2 \times E_1.
$$
As a matrix equation, its operation on the VEV is
\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^T
\]

This operation cycles various sets of three elements of the VEV. There are twelve such sets. These include (11, 22, 33), (14, 25, 36), (61, 52, 43), (64, 55, 46) where the pairs denote the indices of \(\langle \Delta \rangle\). Invariance under this operation ensures that the elements within a set are equal to one another. Our VEV satisfy this condition also.

The flavon, \(\Delta\), transforms in a vector space on which the flavour group acts. In a vector space, a state which does not change under the operation of a group element is an invariant eigenstate of the element. \(\langle \Delta \rangle\), Eq. (38), is an invariant eigenstate of four group elements, Eqs. (41, 43, 45, 47). Invoking the condition that the VEV is an invariant eigenstate of all these four elements, fixes the orientation of the VEV in the vector space ensuring that the VEV is proportional to \(\langle \Delta \rangle\), Eqs. (38).

The group elements \(O_{C\Delta}\) and \(O_{F\Delta}\), Eq. (41, 47), generate two \(C_3\) groups. This is evident by inspecting the corresponding matrix operations in Eqs. (42, 48). Similarly the group elements \(O_{D\Delta}\) and \(O_{D\bar{\Delta}}\), Eq. (43, 45), generate two \(C_2\) groups, as clear from the corresponding matrix operations, Eqs. (44, 46). To denote the action of a direct product element, we used left and right multiplications with the corresponding matrices. To represent a direct product element using a single matrix, we need to obtain the Kronecker product of the left and the right matrices. It can be shown that, the four Kronecker product matrices, corresponding to the four direct product elements, Eqs. (41, 43, 45, 47), commute with each other, i.e. they generate the subgroup \(C_3 \rtimes C_2 \times C_2 \times C_3\). Therefore, the flavon VEV \(\langle \Delta \rangle\) breaks \(X_{24} \times X_{24}\) into \(C_3 \times C_2 \times C_3 \times C_2 = C_6 \times C_6\). To summarise, the \(C_6 \times C_6\) subgroup generated by \(O_{C\Delta}, O_{D\Delta}, O_{D\bar{\Delta}}\) and \(O_{F\Delta}\) remains as the residual symmetry of the VEV and it uniquely defines the VEV (up to multiplication by an overall complex constant).

Now we turn our attention to the flavons, \(\phi\) and \(\bar{\phi}\). Consider the group element
\[
O_{C\phi} = C(\omega)C(\omega)_2C(\omega)_3C(\epsilon) \times C
\]
operating on the flavon VEVs \(\langle \phi \rangle\) or \(\langle \bar{\phi} \rangle\), Eqs. (38). Note that \(C\) is an element of \(\Sigma(72 \times 3)\), Eqs. (4). The \(X_{24}\) and \(\Sigma(72 \times 9)\) parts of \(O_{C\phi}\) act on the corresponding indices of the flavon. As a matrix equation, the operation of \(O_{C\phi}\)

\[\text{It is clear that, this operation multiplies all the vanishing elements in the VEV with } \omega \text{ or } \bar{\omega}. \text{ Therefore, invariance of the VEV under } O_{C\phi} \text{ ensures that these elements vanish.}

Consider the group element
\[
O_{D\phi} = C(\omega)_1C(\omega)_2C(\omega)_3C(\tau)_2C(\tau)_3C(\tau)_4D_2 \times I.
\]
As a matrix equation, its operation on the VEV is
\[
\begin{pmatrix}
\tau & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \tau
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \bar{\omega}
\end{pmatrix}^T
\]

This operation interchanges the rows, 1, 2, 3, of the VEV with the rows, 4, 5, 6 respectively, along with multiplication of these rows with certain specific values of phases. Invariance under this operation ensures that the elements in the upper and the corresponding lower rows in the VEV have the same magnitude, but differ by specific phases. Our VEV satisfies this condition.

Finally consider the group element
\[
O_{E\phi} = C(\omega)_2C(\omega)^2C(\omega)_3C(\tau)_2C(\tau)^2C(\tau)_3C(\tau)_4E_2 \times E
\]
As a matrix equation, its operation on the VEV is
\[
\begin{pmatrix}
0 & \bar{\tau} & 0 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{i} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{i} & 0 \\
0 & 0 & 0 & 0 & 0 & \omega
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^T
\]

This operation cycles various sets of three elements of the VEV, along with multiplying these elements with specific phases. There are six such sets in the VEV. These include (11, 22, 33), (41, 52, 63), where the pairs denote the indices of the VEV. Invariance under this operation ensures that the elements within a set are equal to one another in magnitude, but differ by specific phases. Our VEV satisfies this condition also.

Invoking the condition that the VEVs of the flavons \(\phi\) and \(\bar{\phi}\), Eq. (38), are invariant under \(O_{C\phi}, O_{D\phi}\) and \(O_{E\phi}\) uniquely defines the orientation of the VEVs in the flavour space. The group elements \(O_{C\phi}, O_{D\phi}\) and \(O_{E\phi}\), Eqs. (49, 51, 53), generate \(C_3, C_2\) and \(C_3\) groups respectively. This is evident by inspecting the corresponding matrix operations in Eqs. (50, 52, 54). These three elements also commute with each other, so that they generate the subgroup.
$C_3 \times C_2 \times C_3 = C_6 \times C_3$. To prove that they commute we need to calculate the Kronecker product matrices, as we discussed in the case of the $\Delta$ flavon. To summarise, the $C_3 \times C_3$ subgroup generated by $O_{C, \phi}$, $O_{D, \phi}$ and $O_{E, \phi}$ remains as the residual symmetry of $\langle \phi \rangle$ and $\langle \phi \rangle$ after SSB and it uniquely defines these VEVs (up to multiplication by an overall complex constant).

Since the neutrinos, $\nu_R$, form a triplet under $\Sigma(\times) \times \Sigma(\times)$, the individual states, $\nu_{R1}$, $\nu_{R2}$ and $\nu_{R3}$ correspond to the flavour basis states, $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ respectively. These states are the invariant eigenstates of the group elements $C$, $E^2CE$ and $ECE^2$ respectively where the group generators are given in Eqs. (4). Individually, the above mentioned group elements form $C_3$ subgroups of $\Sigma(\times) \times \Sigma(\times)$. To summarise, we have shown that the flavon VEVs as well as the neutrino states can be uniquely defined in terms of their symmetry properties. They are expressed as the invariant eigenstates of specific group elements which form specific subgroups of the flavour symmetry group. Thus the flavour structure of our model is entirely determined by the subgroup structure of the flavour symmetry group. It should be noted that, even though we have used matrix representations in convenient bases, our formalism is manifestly basis independent, i.e. expressible in terms of the abstract group generators.

6 Summary

Fermions and flavons transform as multiplets of a discrete group in the flavour space. The structure of the fermion mass matrix is determined by the relative orientation of the flavon VEVs with respect to the fermion flavour eigenstates. Therefore, fixing the vacuum alignments is central to the flavour problem. The canonical formalism involves constructing a flavon potential, extremising it and obtaining the VEV through spontaneous symmetry breaking. However, by carefully adjusting the potential we may obtain any arbitrary vacuum alignment. We argue that such a procedure goes against the spirit of using discrete symmetries to explain the flavour structure. The vacuum alignment should not be determined by the structure of the potential, but rather by its symmetries. In this paper we adopt a formalism in which the VEV is fully determined in terms of its residual symmetry, i.e. the unbroken part of the original discrete group. This constrains the possible orientations of the VEV, in relation to the fermion flavour eigenstates, to a unique and finite set. Thus we get rid of the arbitrariness of the vacuum alignment which could arise when using a potential.

In an earlier publication, we showed that a fully constrained Majorana mass matrix can be constructed using a sextet of $\Sigma(\times) \times \Sigma(\times)$. A specific VEV for this sextet leads to $\Phi$M-mixing with $\phi = \pi/16$ and neutrino mass ratios, Eq. (3). In this paper, we obtain this VEV using the formalism of residual symmetries. To achieve it, we propose a new discrete symmetry group, $X_{24}$. The flavons $\phi$, $\Delta$ and $\phi$ which transform under the expanded flavour group $\Sigma(\times) \times X_{24} \times X_{24}$ are introduced. The VEV of each of these flavons is uniquely identified as an invariant eigenstate of several elements of the flavour group. The VEV remains invariant under the residual group generated by these elements, i.e. each VEV is determined by a particular subgroup of $\Sigma(\times) \times X_{24} \times X_{24}$. The flavons $\phi$, $\Delta$ and $\phi$ are coupled together to obtain the required sextet of $\Sigma(\times) \times \Sigma(\times)$. By imposing the condition that the VEVs of the constituent flavons are invariant eigenstates under the simultaneous action of $\Sigma(\times) \times \Sigma(\times)$ and $X_{24}$, we make the VEV of the sextet of $\Sigma(\times) \times \Sigma(\times)$ implicitly dependent on $X_{24}$.

Because the fermions exist in three families, only the discrete subgroups of $U(3)$ have been used as flavour symmetry groups in literature so far. In this paper, we broadened the application of discrete groups in model building by utilising a group which is no longer required to be a $U(3)$ subgroup. In a general framework, we may express the flavour group, $G_f$, as a direct product, $G_f = G_{U(3)} \times G_X$. In this paper, we have $G_{U(3)} = \Sigma(\times) \times \Sigma(\times)$ and $G_X = X_{24} \times X_{24}$. But in general, $G_{U(3)}$ can be any discrete subgroup of $U(3)$ such as $A_4$, $S_4$, $A_5$, $\Delta(3n^2)$ and $\Delta(6n^2)$. On the other hand, $G_X$, which we call the “auxiliary group” can be any discrete group. We hope that this newly introduced framework will stimulate further research involving other choices of auxiliary groups combined with the commonly studied subgroups of $U(3)$. This may lead to novel choices of vacuum alignments for the $G_{U(3)}$ flavons and new textures of mass matrices.

I would like to thank Paul Harrison and Bill Scott for the helpful discussions and Aidan Wiederhold for helping with making the plot. I acknowledge the support from the University of Warwick and the hospitality of the Particle Physics Department at the Rutherford Appleton Laboratory. I thank the management of the School of the Good Shepherd, Thiruvananthapuram, for providing a convenient and flexible working arrangement conducive to recherche.

Appendix

Here we use the group $S_4$ to construct a couple of toy models for mass matrices. We investigate the flavon potentials and show that they may or may not lead to VEVs with specific symmetry properties. The representation $3$ of $S_4$ is generated using the matrices

$$E \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

(55)

where we have adopted the commonly used basis in literature. $3$ consists 24 proper rotations which include 9 rotations about the axes passing through face centres, 8 rotations about the axes passing through vertices, 6 rotations about the axes passing through edge centres and the identity element. We assume that the right-handed neutrinos transform as a $3$,

$$3 \equiv \nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})^T.$$

(56)
The individual fermion states correspond to the basis states of the representation, for example \( \nu_r \) corresponds to \( (1, 0, 0)^T \). This state remains invariant under the action of the \( C_4 \) subgroup of \( S_4 \) generated by
\[
FE \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\] (57)

This subgroup consists of rotations by \( n\pi/2 \) about the axis passing through \((1, 0, 0)^T \). Similarly we have two more \( C_4 \) subgroups in relation to the states \( \nu_{r2} \) and \( \nu_{r3} \).

The tensor product expansion of two triplets \( (3') \) is given by
\[
3 \otimes 3 = 1 \oplus 2 \oplus 3' \oplus 3.
\] (58)

Therefore, by coupling the neutrinos we obtain
\[
1 \equiv (\nu_r \nu_R)_s = \nu_{r1} \nu_{R1} + \nu_{r2} \nu_{R2} + \nu_{r3} \nu_{R3},
\] (59)
\[
2 \equiv (\nu_r \nu_R)_d = (2\nu_{r1} \nu_{R1} - \nu_{r2} \nu_{R2} - \nu_{r3} \nu_{R3}, \sqrt{3} \nu_{r2} \nu_{R2} - \sqrt{3} \nu_{r3} \nu_{R3})^T, \]
(60)
\[
3' \equiv (\nu_r \nu_R)_t = (\nu_{r2} \nu_{R3} + \nu_{r3} \nu_{R2}, \nu_{R3}, \nu_{r1} \nu_{R3}, \nu_{r1} \nu_{R1}, \nu_{r3} \nu_{R3}, \nu_{r3} \nu_{R1}, \nu_{r2} \nu_{R1})^T,
\] (61)

The generators corresponding to Eqs. (55), for the doublet (2) and the triplet (3') are
\[
E \equiv \begin{pmatrix} -1/2, \sqrt{3}/2 \end{pmatrix}, \quad F \equiv \begin{pmatrix} -1/2, -\sqrt{3}/2 \end{pmatrix}
\] (62)
and
\[
E \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\] (63)
respectively where the basis adopted is as per Eqs. (60, 61).

The triplet \( 3' \) also represents the symmetries of a cube. It consists of 12 proper rotations (3 rotations by an angle \( \pi \) about the axes passing through face centres, 8 rotations about the axes passing through vertices and the identity element) and 12 improper rotations (6 rotations by angles \( \pm \pi/2 \) about the axes passing through face centres and 6 rotations about the axes passing through edge centres combined with the reflection about the origin).

The doublet representation (2) is not faithful. The \( 2 \times 2 \) matrices, Eqs. (62), generate the dihedral group \( D_6 \) which forms a subgroup of \( S_4 \). \( D_6 \) represents the rotation as well as the reflection symmetries of an equilateral triangle as shown in the Fig. 3.

Let us define singlet \((\phi_s)\), doublet \((\phi_d)\) and triplet \((\phi_t)\) flavons which transform as \( 1 \) (invariant), \( 2 \) and \( 3' \) respectively. They couple with the neutrino multiplets, Eqs. (59-61), to produce the \( S_4 \) invariant mass term,
\[
k_s(\nu_R \nu_R) \phi_s + k_d(\nu_R \nu_R)^T \phi_d + k_t(\nu_R \nu_R)^T \phi_t,
\] (64)

where \( k_s, k_d \) and \( k_t \) are the coupling constants. The flavons and the coupling constants in Eq. (64) can be written in a matrix form,
\[
k_s, k_d I + \begin{pmatrix} k_{d1} & k_{t1} & k_{t2} \\ k_{t1} & k_{d1} & k_{t2} \\ k_{t2} & k_{t1} & k_{d1} \end{pmatrix} \begin{pmatrix} \phi_{d1} \\ \phi_{d2} \\ \phi_{d3} \end{pmatrix},
\] (65)

where we have expressed the doublet and the triplet flavons in terms of their components, i.e. \( \phi_d = (\phi_{d1}, \phi_{d2})^T \) and \( \phi_t = (\phi_{t1}, \phi_{t2}, \phi_{t3})^T \). Substituting a specific vacuum alignment for the flavons in Eq. (65) produces the mass matrix. In the rest of the Appendix, two examples are provided where we minimise flavon potentials to obtain the VEVs and the corresponding mass matrices. In Example 1 the VEVs can also be defined in terms of their symmetries while in Example 2 they do not have such symmetry properties.

**Example 1**

It is straightforward to write a potential for the invariant flavon \( \phi_s \)
\[
(\phi_s - 1)^2.
\] (66)

Extremising this potential leads to the VEV
\[
\langle \phi_s \rangle = 1.
\] (67)

In order to construct a potential for the doublet flavon, we first consider the tensor product of two doublets. It can be shown that the tensor product leads to another doublet,
\[
\langle \phi_d \phi_d \rangle_d = \begin{pmatrix} \phi_{d1}^2 - \phi_{d2}^2, -2\phi_{d1}\phi_{d2} \end{pmatrix}^T.
\] (68)

Now we construct the potential,
\[
|\langle \phi_d \phi_d \rangle - \phi_d|^2,
\] (69)
where the operator $|\rangle^2$ represents $(\cdot)^T$. This potential has three minima: $\phi_d = (1, 0), (-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. They form the vertices of the equilateral triangle as shown in Fig. 3 (left). We assume that the flavon acquires one of these minima as its VEV,

$$\langle \phi_d \rangle = (1, 0)^T.$$  

(70)

This VEV breaks $D_6$ to one of its subgroups, $C_2$, generated by

$$FE \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  

(71)

where $E, F$ are given in Eqs. (62). $C_2$ represents the reflection symmetry of the triangle which keeps $(1, 0)^T$ invariant. Conversely, the VEV can be uniquely identified by this residual $C_2$ symmetry.

Now we construct a potential for the triplet flavon, $\phi_t$, which transforms as a $3'$ from the tensor product of two $\phi_t$ triplets, we obtain the second order triplet,

$$(\phi_t \phi_t^T) = (t_2 \phi_1 t_3, t_3 \phi_1 t_2, \phi_1 t_2 t_3)^T,$$  

(72)

similar to Eq. (61). Using $\phi_t$ and $(\phi_t \phi_t^T)$, we construct the potential,

$$(|\phi_t |^2 - 1)^2 + |(\phi_t \phi_t^T)|^2.$$  

(73)

This potential has six minima $\phi_t = (\pm 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$. These are the face centres of the cube shown in Fig. 2 (left). We assume that the flavon acquires one of these minima as its VEV,

$$\langle \phi_t \rangle = (1, 0, 0)^T.$$  

(74)

This VEV breaks $S_4$ to one of its subgroups, $C_2 \times C_2$, generated by

$$EF E^2 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F E^2 F E F \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  

(75)

where $E, F$ are given in Eqs. (63). The generators, Eqs. (75) represent two improper rotations of the cube. Apart from these two elements, $C_2 \times C_2$ also consists of a proper rotation (Diag(1, -1, -1)) and the identity element. The VEV, $(1, 0, 0)^T$, remains invariant under the action of these elements. Conversely, the VEV can be uniquely identified by this residual $C_2 \times C_2$ symmetry.

Substituting the VEVs, $\langle \phi_s \rangle$, $\langle \phi_d \rangle$ and $\langle \phi_t \rangle$ in Eq. (65), we obtain the mass matrix,

$$\begin{pmatrix} k_s + 2k_d & 0 & 0 \\ 0 & k_s - k_d & k_t \\ 0 & k_t & k_s - k_d \end{pmatrix}. $$  

(76)

This matrix is diagonalised using the unitary matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, $$  

(77)

which provides a bimaximal contribution to mixing. By a suitable selection of the coupling constants, $k_s, k_d, k_t$, we can obtain any set of values for the masses without affecting the mixing part.

**Example 2**

In this example we construct potentials for the flavons $\phi_d$ and $\phi_t$ leading to VEVs which leave no residual symmetries. For $\phi_d$, we use the potential

$$(|\phi_d |^2 - 1)^2 + \frac{1}{\Lambda^2} \langle \phi_d^T (\phi_d \phi_d) d \rangle^2,$$  

(78)

where the scale $\Lambda$ is added with the higher dimensional term. This potential has six minima which are of the form

$$\phi_d = g_i (0, 1)^T,$$  

(79)

where $g_i$ are the six elements of the group $D_6$. These minima are shown in Fig. 3 (right). We assume that flavon acquires one of the minima,

$$\langle \phi_d \rangle = (0, 1)^T.$$  

(80)

as its VEV. This VEV does not possess any residual symmetry of $D_6$.

For constructing the potential for the triplet flavon, we first obtain a doublet from the tensor product of two triplets. Similar to Eq. (60), we obtain

$$\langle \phi_t \phi_t^T \rangle = \begin{pmatrix} 2\phi_1^2 - \phi_2^2 - \phi_3^2, \sqrt{3}\phi_1 \phi_2 - \sqrt{3}\phi_1 \phi_3 \end{pmatrix}^T.$$  

(81)

We construct the potential as

$$(|\phi_t |^2 - 1 + \kappa_1^2 + \kappa_2^2)^2 + (|\phi_t \phi_t^T| - \sqrt{3}(\kappa_1^2 - \kappa_2^2)\phi_d + (2 - \kappa_1^2 - \kappa_2^2)(\phi_d \phi_d)),$$  

(82)

where $\kappa_1$ and $\kappa_2$ are arbitrary constants. In Eq. (82) we have coupled the doublet flavon, $\phi_d$, with the triplet flavon, $\phi_t$. Therefore, we should extremise Eq. (82), together with the potential for the doublet flavon, Eq. (78). If we substitute $\phi_d = (0, 1)^T$ and $\phi_t = (1, \kappa_1, \kappa_2)^T$, both terms in Eq. (82) as well as in Eq. (78) vanish, indicating that these states of the flavons constitute a minimum of the potential. By transforming these flavon states under the action of $S_4$ we obtain further minima forming a discrete set. For $\phi_t$, these minima are shown in Fig. 2 (right)\(^6\). We select one of these minima

$$\langle \phi_t \rangle = (1, \kappa_1, \kappa_2)^T$$  

(83)

as the VEV. As mentioned previously, this VEV does not possess any residual symmetry of $S_4$. Using the VEVs of the doublet and the triplet flavons we obtain the mass matrix,

$$\begin{pmatrix} k_s & k_t \kappa_2 & k_t \kappa_1 \\ k_t \kappa_2 & k_s + \sqrt{3}k_d & k_t \kappa_1 \\ k_t \kappa_1 & k_t & k_s - \sqrt{3}k_d \end{pmatrix},$$  

(84)

which has more degrees of freedom compared to the previous case, Eq. (76). By suitably tuning these free parameters, we can ensure that this mass matrix is consistent

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\(^6\) In the figure, we have used $\kappa_1 = \kappa_2 = 0.75$. 

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R. Krishnan: Fully Constrained Mass Matrix: Can Symmetries alone determine the Flavon Vacuum Alignments?
References

1. S. Bilenky, Neutrino oscillations: From a historical perspective to the present status, Nuclear Physics B 908 (2016) 2–13, [arXiv:1602.0170].
2. G. Rajasekaran, The Story of the Neutrino, arXiv:1606.8715.
3. C. Giganti, S. Lavignac, and M. Zito, Neutrino oscillations: The rise of the PMNS paradigm, Progress in Particle and Nuclear Physics 98 (2018) 1–54, [arXiv:1710.0715].
4. I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, and T. Schwetz, Updated fit to three neutrino mixing: exploring the accelerator-reactor complementarity, Journal of High Energy Physics 2017 (2017), no. 1, 87, [arXiv:1611.1514].
5. F. Capozzi, E. Lisi, A. Marrone, and A. Palazzo, Current unknowns in the three neutrino framework, Progress in Particle and Nuclear Physics (2018) arXiv:1804.9678.
6. P. F. de Salas, D. V. Forero, C. A. Ternes, M. Tortola, and J. W. F. Valle, Status of neutrino oscillations 2018: 3e hint for normal mass ordering and improved CP sensitivity, Physics Letters B 782 (2018) 633–640, [arXiv:1708.1195].
7. M. Lattanzi and M. Gerbino, Status of neutrino properties and future prospects - Cosmological and astrophysical constraints, Frontiers in Physics 5 (2018) 070, [arXiv:1712.7109].
8. Y. Kwednko and D. Wark, Neutrino oscillations: status and prospects of accelerator and reactor experiments, Journal of Physics: Conference Series 934 (2017), no. 012001.
9. S. S. Chatterjee, P. Pasquini, and J. W. F. Valle, Probing atmospheric mixing and leptonic CP violation in current and future long baseline oscillation experiments, Physics Letters B 771 (2017) 524–531, [arXiv:1702.3160].
10. S. D. Donizzi, Future prospects for neutrinoless double-beta decay, NUPHYS2016-DIDOMIZIO (2016) arXiv:1705.3935.
11. L. J. Wen, J. Cao, and Y. F. Wang, Reactor Neutrino Experiments: Present and Future, Annual Review of Nuclear and Particle Science 67 (2017) 183–211, [arXiv:1803.1016].
12. G. Altarelli and F. Feruglio, Discrete Flavor Symmetries and Models of Neutrino Mixing, Reviews of Modern Physics 82 (2010), no. 2701 [arXiv:1002.0211].

with the current neutrino masses and mixing data. However, we argue that since the VEVs, Eqs. (80, 83), have no apparent connection with the original flavour symmetry (Sf), we cannot claim that the texture of the resulting mass matrix has its origin in the aforementioned symmetry.

13. W. Grimus, Theory of Neutrino Masses and Mixing, Physics of Particles and Nuclei 42 (2011), no. 566 [arXiv:1101.0137].
14. S. F. King and C. Luhn, Neutrino Mass and Mixing with Discrete Symmetry, Reports on Progress in Physics 76 (2013), no. 5 056201, [arXiv:1301.1340].
15. D. Meloni, GUT and flavor models for neutrino masses and mixing, Frontiers in Physics 5 (2017), no. 43 [arXiv:1709.2662].
16. P. F. Harrison, D. H. Perkins, and W. G. Scott, Tri-Bimaximal Mixing and the Neutrino Oscillation Data, Physics Letters B 530 (2002) 167–173, [hep-ph/0202074].
17. X. He and A. Zee, Minimal Modification to Tri-bimaximal Mixing, Physical Review D 84 (2011), no. 53004 [arXiv:1106.4359].
18. S. Gupta, A. S. Joshipura, and K. M. Patel, Minimal extension of tri-bimaximal mixing and generalized Z2 × Z2 symmetries, Physical Review D 85 (2012), no. 031903(R) [arXiv:1112.6113].
19. S. K. Garg and S. Gupta, Corrections for tribimaximal, bimaximal and democratic neutrino mixing matrices, Journal of High Energy Physics 128 (2013) [arXiv:1308.3054].
20. S. Dev, D. Raj, and R. R. Gautam, Deviations in tribimaximal mixing from sterile neutrino sector, Nuclear Physics B 911 (2016) 744–753, [arXiv:1607.08051].
21. E. Ma, New Lepton Family Symmetry and Neutrino Tribimaximal Mixing, Europhysics Letters 79 (2007), no. 6 [hep-ph/0701016].
22. S. Pakvasa, V. Rodejohann, and T. J. Weiler, TriMinimal Parametrization of the Neutrino Mixing Matrix, Physical Review Letters 100 (2008), no. 111801 [arXiv:0711.0052].
23. S. F. King, Parametrizing the lepton mixing matrix in terms of deviations from tri-bimaximal mixing, Physics Letters B 659 (2008) 244–251, [arXiv:0710.0530].
24. F. Plentinger and W. Rodejohann, Deviations from Tribimaximal Neutrino Mixing, Physics Letters B 625 (2005) 264–276, [hep-ph/0507143].
25. S. Antusch, C. Goss, V. Maurer, and C. Sluka, A flavour GUT model with $\theta_{13}^{PMNS} = \theta_C/\sqrt{2}$, Nuclear Physics B 877 (2013) 772–791, [arXiv:1305.6612].
26. S. Antusch, Models for Neutrino Masses and Mixings, Nuclear Physics B - Proceedings Supplements 235–236 (2013) 303–309, [arXiv:1301.5511].
27. P. F. Harrison and W. G. Scott, Symmetries and Generalisations of Tri-Bimaximal Neutrino Mixing, Physics Letters B 535 (2002) 163–169, [hep-ph/0203209].
28. R. Krishnan, A Model for Large $\theta_{13}$ Constructed using the Eigenvectors of the $S_3$ Rotation Matrices, Journal of Physics: Conference Series 447 (2013) 012043, [arXiv:1211.3364].
29. R. Krishnan, P. F. Harrison, and W. G. Scott, Fully Constrained Majorana Neutrino Mass Matrices
Using $\Sigma(72 \times 3)$, *The European Physical Journal C* **78** (2018), no. 74 [arXiv:1801.1019].

30. W. Grimus and P. O. Ludl, *Principal series of finite subgroups of SU(3)*, *Journal of Physics A: Mathematical and Theoretical* **43** (2010) 445209, [arXiv:1006.0098].