Two-temperature advective transonic accretion flows around black holes

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Abstract. We present here unique transonic two-temperature accretion solutions in presence of radiation processes. Unlike one-temperature flow, the set of equations governing two-temperature flow is not complete, since for the latter, the number of equations is less than the number of flow variables. Consequently, a large number of transonic solutions of the equations of motion exist, for a given set of constants of motion. We invoke the second law of thermodynamics and identified the solution with the maximum entropy as the physically possible solution. In this paper, we compare spherical and rotating accretion two-temperature flows. We also show that the luminosity, as well as radiative efficiency of rotating flows, is higher by an order of magnitude, as compared to spherical flows.

1. Introduction
Accretion is a process by which the gravity of a compact object like white dwarf, neutron star or a black hole (BH), accumulates matter onto it, from the surrounding environment. This mechanism powers the brightest sources in the Universe, which are Active Galactic Nuclei (AGN). Accretion model started with the seminal papers of Hoyle & Lyttleton\textsuperscript{[1]} (1939) and Bondi (1952)\textsuperscript{[2]} who gave the first accretion solutions around a Sun-like star and a compact object respectively. But it was until a decade later, with the discovery of quasars and X-ray binaries, that accretion as a mechanism to explain the huge amount of emission coming from these objects, was realised. In 1964, Salpeter\textsuperscript{[3]}, used the Bondi accretion model (available at that time) to explain the luminosities of quasars ($L \sim 10^{12} L_\odot$). But, he reported that this flow is too fast to explain it. Shakura & Sunyaev (1973)\textsuperscript{[4]}, on the other hand, considered a rotation dominated flow to allow maximum time before the flow enters into the BH. Such a flow has Keplerian azimuthal velocity and negligible radial velocity. The flow is cold with negligible pressure gradient force, geometrically thin and optically thick. These disks were famously known as Shakura-Sunyaev disks (SSD) or Keplerian Disks (KD). The overall spectral profile of a SSD is a modified multi-coloured blackbody. Although this model was successful in explaining the thermal component of X-ray binaries and AGNs, but it could not explain the non-thermal part present in the spectra of these objects. Thorne & Price in 1975\textsuperscript{[7]} argued that the instability present at the inner portion of the accretion disk could puff up the optically thick, radiation pressure dominated region into an optically thin, gas pressure dominated region. Using this assumption, Shapiro, Lightman & Eardley in 1976\textsuperscript{[6]}, put forward the first model of...
two-temperature theory. They assumed that the inner puffed up region consisted of protons and electrons described by two different temperature distributions and explained the hard X-rays coming from Cygnus X-1. From the above papers, a general conclusion can be drawn that accretion flow should be a combination of Keplerian and sub-Keplerian matter. It has to be transonic, since the matter crosses the horizon at the speed of light, while at the outer boundary it should be subsonic. In 1980, Liang & Thompson (LT80)[8] pointed out this fact and reported that rotating sub-Keplerian flows around BH possess multiple sonic points, due to the presence of gravity and angular momentum. Presence of relativistic effects induces the formation of another sonic point. It was later showed that rotating sub-Keplerian flows passing through the outer sonic point, may undergo shock transition and enter the BH through the inner sonic point (Fukue 1987[9], Chakrabarti 1989[10]). Sub-Keplerian flows also have significant advection and the flow is also hotter than SSD (Abramowicz et al. 1988[11], for a brief review see, Bisnovatyi-Kogan & Lovelace 2001[12]). Few works were also focussed to study radiative properties of these systems. Colpi, Marashi & Treves (1984)[13] studied two-temperature version of spherical flows around BHs, where they elaborately discussed the radiative mechanisms while Narayan & Yi (1995) (NY95)[14] considered rotating advection-dominated two-temperature flows in order to obtain the spectra of low accreting BH systems. But in NY95, self-similarity was assumed while obtaining solutions, which is valid only far away from the BH and cannot be assumed throughout the flow. Also electron energy equation was dealt locally, neglecting its advection. In 1996, Nakamura et al.[15], were the first to obtain global transonic two-temperature advection dominated accretion solutions. Manmoto et al.[16] in 1997, computed the spectra of such flows. Since then, works have been done in two-temperature advection dominated branch of solutions, notable being by Chakrabarti & Titarchuk (1995) [17], Bisnovatyi-Kogan & Lovelace (1997)[18] and Mandal & Chakrabarti (2005)[19]. Few works considered transonicity like Rajesh & Mukhopadhyay (2010)[20] and Dihingia et al.(2017)[21].

Flows around BH are trans-relativistic in nature varying from thermally non-relativistic, adiabatic index ($\Gamma \approx 5/3$), very far away from the central object to mildly-relativistic/relativistic, $\Gamma \rightarrow 4/3$, near the horizon, therefore a fixed $\Gamma$ equation of state (EoS) is untenable. However, most of the previous works in two-temperature model assumed so. In 2009, Chattopadhyay & Ryu[22], gave an approximate analytical form of EoS for multispecies flow, which is easy to handle and it matches perfectly well[23] with the relativistically perfect EoS[24].

It is worth noting that the number of hydrodynamic equations in one-temperature and two-temperature is same, but there is one more variable in the latter ($T_e$=electron and $T_p$=ion temperature). Number of unknowns is hence more than the number of equations. Or in other words the two-temperature model is degenerate. If we scan the parameter space of two-temperature transonic solutions we can obtain infinite solutions for the same constants of motion. This problem was never been reported and solved for, in literature known so far, although hint of this was given in LT80, where they parametrized $T_e$ and $T_p$ by a constant ratio. The standard Coulomb interaction between electrons and ions is weak, therefore the energy exchange is not substantial to establish a stronger correlation between $T_e$ and $T_p$. In absence of any other physical processes dictated by plasma physics, all other parametrization employed by various authors are arbitrary. In Sarkar & Chattopadhyay (2019) (SC19)[25], we gave a detailed methodology to obtain a unique transonic two-temperature accretion solution around BHs, albeit for spherical accretion. In SC19, we derived for the first time, an entropy measure form, close to the horizon. Second law of thermodynamics dictates that nature prefers a solution with maximum entropy. Hence, we select the maximum entropy solution as the unique transonic two-temperature solution for a given set of constants of motion. In this paper, we extend SC19 to rotating, conical accretion flow around BHs. Since, rotating flows possess multiple sonic points, complexity of the problem is increased further. In the methodology section, we will discuss in detail how SC19 is modified.
in the present paper. We have also used the variable Γ index CR EoS which removed the problem of specifying the adiabatic indices of the species.

The paper is divided into the following sections. In section (2), we will give a brief overview of the basic equations and assumptions used to model the flow. In section (3), we will discuss the methodology to find unique transonic two-temperature solutions, showing some important results in section (4) and would finally conclude in section (5).

2. Basic Equations and Assumptions

We consider here inviscid conical rotating axis-symmetric two-temperature accretion flows around BHs described by Schwarzschild metric. It is to be noted that qualitative feature of the flow is not hampered while neglecting viscosity. It has been observed in viscous, single temperature, rotating accretion flow that angular momentum remains almost constant till \( \sim \) few \( \times 100 \) Schwarzschild radius \( (r_g) \) [26, 27, 28]. Spherical coordinate system has been employed where, \( r, \theta, \phi \) have their usual meaning and \( t \) is the time coordinate. It is to be noted that throughout the paper, we have used a system of units where, \( G = M = c = 1 \), such that the unit of length, time and velocity are given as \( r_g = GM/c^2 \), \( r_g/c = GM/c^3 \) and \( c \), respectively, where \( G = \) Gravitational constant, \( M = \) mass of BH, \( c = \) speed of light. In the subsequent sections all the variables are in this unit system unless otherwise mentioned. The system is in steady state and is axis-symmetric. Additionally, all the quantities are averaged in the vertical direction. Therefore, \( \partial/\partial t = \partial/\partial \phi = \partial/ \partial \theta = 0 \).

The energy momentum tensor of the accreting matter is

\[
T_{\mu\nu} = (e + p)u^\mu u^\nu + pg_{\mu\nu} + t_{\text{rad}},
\]

where, \( e \) and \( p \) are the internal energy density and isotropic gas pressure, all measured in local fluid frame, \( u^\mu \)s are the components of four-velocity and \( t_{\text{rad}} \) is the radiation tensor. \( g^{\mu\nu} \)s are the inverse of metric tensor components, non-zero components of which are:

\[
\begin{align*}
g_{tt} &= -\left(1 - \frac{2}{r}\right); & g_{rr} &= \left(1 - \frac{2}{r}\right)^{-1}; & g_{\theta\theta} &= r^2; & g_{\phi\phi} &= r^2 \sin^2 \theta
\end{align*}
\]

Conservation law for :

1. Energy :

\[
T_{\mu\nu}^{\nu} = 0,
\]

2. particle density flux :

\[
(nu^\mu)_{,\nu} = 0 \quad \text{where,} \quad n = \text{particle number density. If we project the four-divergence of the energy-momentum tensor along the space direction, \( i.e. \) } h_{\mu}T_{\nu\mu}^{\nu} = 0, \quad \text{we get the Euler equation, radial component of which is :}
\]

\[
u^r \frac{du^r}{dr} + \frac{1}{r^2} - (r - 3)u^\phi u^\phi + (g^{rr} + u^r u^r) \frac{1}{e + p} \frac{dp}{dr} = 0.
\]

The azimuthal component is absent, since the angular momentum is a constant of motion in the absence of viscosity.

On integrating the equation for conservation of particle density flux, we get the form of mass accretion rate, which is another constant of motion:

\[
\dot{M} = 4\pi \rho u^r r^2 \cos \theta,
\]

where, \( \rho = n(m_p + m_e) = \) mass density, \( m_p \) and \( m_e \) being the mass of proton and electron respectively, \( \theta \) is the angle which the surface of the flow makes with the normal.

The first law of thermodynamics is \( u^\mu T_{\mu\nu}^{\nu} = 0 \) and can be simplified to:

\[
u^r \left[ \left(\frac{e + p}{\rho}\right) \rho_r - e_r \right] = \Delta Q,
\]

where, \( \Delta Q = Q^+ - Q^- \), \( Q^+ \) is the rate of heating present in the flow and \( Q^- \) gives the cooling rate. Generalised Bernoulli constant, a constant of motion, is obtained by integrating Eq.(2) with the help of Eq.(4) and is given by:

\[
E = -hu_t \exp(X_f),
\]
where, $h = (e + p)/\rho$, is the specific enthalpy and $X_f = \int \frac{\Delta Q_i + \Delta Q_i}{\rho u R} \, dr$. $E$ is constant even in the presence of dissipation. In case of no dissipation, $X_f = 0$ and $E \to E = -hu$, which is the canonical form of relativistic Bernoulli constant [26].

To close the equations of motion, we use CR EoS [22], which for electron-proton plasma, in geometric units, is given by,

$$e = n_e m_e \left(f_e + \frac{f_p}{\eta}\right) = \frac{\rho f}{K},$$

where, $K = 1 + 1/\eta$ and $\eta = m_e/m_p$. Here, $f_i = 1 + \Theta_i \left(\frac{\Theta_i + 2}{\Theta_i + 2}\right)$ and $\Theta_i = \frac{k T_i}{m_i c^2}$ is the non-dimensional conjugate of the dimensional temperature $(T_i)$, where $k$ is Boltzmann constant and $i \equiv$ electron or proton. Polytropic index and adiabatic index are defined as, $N_i = \frac{df_i}{d\eta}$ and $\Gamma_i = 1 + \frac{1}{N_i}$. If we rewrite the first law of thermodynamics Eq.(4), separately for protons and electrons using Eq.(6), we get the differential equation for their temperatures :

$$\frac{d\Theta_p}{dr} = -\frac{\Theta_p}{N_p} \left( A + \frac{1}{v(1 - v^2)} \frac{dv}{dr} \right) - \frac{\eta \eta}{v} \Theta_i$$

and

$$\frac{d\Theta_e}{dr} = -\frac{\Theta_e}{N_e} \left( A + \frac{1}{v(1 - v^2)} \frac{dv}{dr} \right) - \frac{\eta \eta}{v} \Theta_i$$

where, $A = -\frac{2v-3}{r(r-2)}$, $P = \frac{\Delta Q_p K}{N_p \rho}$, $E = \frac{\Delta Q_i K}{N_e \rho}$. These equations are not independent, but are coupled by a Coulomb coupling term. If we simplify Eq.(2), using Eqs.(3),(6-7), we obtain:

$$\frac{dv}{dr} = \frac{\eta}{D}$$

where, $v = \sqrt{u_r u_r / (1 + u_r u_r)}$, $N' = -\frac{1}{r(r-2)} + \frac{\lambda g_{ij}(r-3)}{r^4} + a^2 A + \frac{\Gamma_p N_p \rho + \Gamma_e N_e \rho}{h \chi} - \frac{\Delta Q}{\rho u^r}$ and $D = \frac{v}{1 - v^2} \left( 1 - \frac{a^2}{v^2} \right)$. The sound speed is defined as, $a = \sqrt{\phi/h \chi}$, where $\phi = \frac{\Gamma_p \Theta_p}{\eta} + \Gamma_e \Theta_e$.

2.1. Sonic point conditions and shock conditions:

The mathematical form of Eq.(8) suggests that there exists a point where the numerator vanishes. For smooth continuous flow, the denominator also has to vanish. Thus, $dv/dr$ has a 0/0 form and this point is called the sonic point of the flow. The sonic point conditions are : $N' = 0$ and $D = 0$. At the sonic point we need to employ L’Hospital rule in order to find $(dv/dr)_c$.

The relativistic shock conditions or the Rankine-Hugoniot conditions [29] are : (1) Conservation of mass flux : $[M] = 0$, (2) energy flux : $[E] = 0$ and (3) momentum flux : $[(e+p)u^r u^r + pg^{rr}] = 0$, where, the square brackets denote the difference of the quantities across the shock.

2.2. Radiative processes:

- Equipartition of magnetic energy density demands that magnetic field lines reconnect such that magnetic field is not frozen into the plasma [30, 31]. The heat dissipated [32] in such a way $(Q_B)$ is assumed to be absorbed by protons and electrons in equal portions ($\delta = 0.5$).
- Coulomb coupling $(Q_{ep})$ cause energy exchange between electrons and protons [33].
- Inverse bremsstrahlung $(Q_{ib})$ acts as a cooling mechanism for protons [34, 35].
- Electrons is cooled via three processes : bremsstrahlung $(Q_{br})$ [36], synchrotron $(Q_{syn})$ [37] and inverse-Comptonization $(Q_{ic})$ [37].
- If the temperature of the electrons is less than the energy of the interacting photons, then electrons can gain heat via Compton heating $(Q_{comp})$.

Thus to conclude we have, for protons : $Q_{p}^+ = \delta Q_B$ and $Q_{p}^- = Q_{ep} + Q_{ib}$, and for electrons: $Q_{e}^+ = (1 - \delta)Q_B + Q_{ep} + Q_{comp}$ and $Q_{e}^- = Q_{br} + Q_{syn} + Q_{ic}$. In addition, all the relativistic transformations have been considered to compute the actual energy loss.
2.3. Entropy accretion rate expression:
To obtain the expression for entropy, one needs to integrate the first law of thermodynamics by switching off explicit heating and cooling terms. Due to the presence of Coulomb interaction term, Eq.(4) is not analytically integrable. However, close to the BH horizon, gravity overpowers all other interactions. Therefore, at \( r = r_{\text{in}} \) (asymptotically close to the horizon), \( Q_{\text{ep}} \) can be neglected. Thus we obtain,

\[
\frac{d\Theta_p}{dr} = \frac{\Theta_p}{N_p} n_p \frac{dn_p}{dr} \quad \text{and} \quad \frac{d\Theta_e}{dr} = \frac{\Theta_e}{N_e} n_e \frac{dn_e}{dr} \tag{9}
\]

Hence, we can integrate equation (9) at \( r_{\text{in}} \) to obtain:

\[
n_{\text{ein}} = \kappa_1 \exp\left(\frac{f_{\text{ein}} - 1}{\Theta_{\text{ein}}}\right) \Theta_{\text{ein}}^\frac{2}{3}(3\Theta_{\text{ein}} + 2)^\frac{3}{2} \quad \text{and} \quad n_{\text{pin}} = \kappa_2 \exp\left(\frac{f_{\text{pin}} - 1}{\Theta_{\text{pin}}}\right) \Theta_{\text{pin}}^\frac{2}{3}(3\Theta_{\text{pin}} + 2)^\frac{3}{2} \tag{10}
\]

where, \( \kappa_1 \) and \( \kappa_2 \) are the integration constants which are measure of entropy. Subscript ‘in’ indicates quantities measured just outside the horizon. Due to charge neutrality in the flow \( n_{\text{ein}} = n_{\text{pin}} = n_{\text{in}} \). Therefore we can write,

\[
n_{\text{in}} = \sqrt{n_{\text{ein}} n_{\text{pin}}} = \kappa \sqrt{\exp\left(\frac{f_{\text{ein}} - 1}{\Theta_{\text{ein}}}\right) \exp\left(\frac{f_{\text{pin}} - 1}{\Theta_{\text{pin}}}\right) \Theta_{\text{ein}}^\frac{2}{3}(3\Theta_{\text{ein}} + 2)^\frac{3}{2} \Theta_{\text{pin}}^\frac{2}{3}(3\Theta_{\text{pin}} + 2)^\frac{3}{2}} \tag{11}
\]

where, \( \kappa = \sqrt{\kappa_1 \kappa_2} \). Thus, the expression of entropy accretion rate can be written as,

\[
\dot{M}_{\text{in}} = \frac{M}{4\pi \kappa (m_e + m_p)} = \sqrt{\exp\left(\frac{f_{\text{ein}} - 1}{\Theta_{\text{ein}}}\right) \exp\left(\frac{f_{\text{pin}} - 1}{\Theta_{\text{pin}}}\right) \Theta_{\text{ein}}^\frac{2}{3}(3\Theta_{\text{ein}} + 2)^\frac{3}{2} \Theta_{\text{pin}}^\frac{2}{3}(3\Theta_{\text{pin}} + 2)^\frac{3}{2}} u^r r^2 \cos(\theta) \tag{12}
\]

In the next section we will briefly discuss the significance of this formula and its use as a tool to remove degeneracy in two-temperature problem.

3. Methodology for obtaining unique transonic solutions
3.1. Finding of sonic points and obtaining a transonic solution:
This is the first step to find a general two-temperature accretion solution. Since the system is dissipative, sonic points are not known a priori. In order to compute the location of the sonic point/points, for a given set of constants of motion \((E, \dot{M} \text{ and } \lambda)\), we start, with integration from \( r_{\text{in}} = 2.001 \). At \( r_{\text{in}} \), \( X_f = 0 \) and from Eq.(5), \( E \simeq \mathcal{E} \). Simplifying this, we obtain an analytical form for \( v_{\text{in}} \) (which is a function of \( T_{\text{p, in}}, T_{\text{e, in}} \)). We assume a value of \( T_{\text{p, in}} \) and iterate \( T_{\text{e, in}} \) while simultaneously calculating \( v_{\text{in}} \), until both the sonic point conditions (see, section 2.1) are satisfied. But as discussed before, rotating flows can possess multiple sonic points. So we keep \( T_{\text{p, in}} \) same and reduce/increase \( T_{\text{e, in}} \) by a large factor, and iterate it again until the sonic point conditions are satisfied. However, multiple sonic points form only for a limited range of constants of motion. Once we get the sonic point, we integrate Eqs.(7) and (8) inwards and outwards from the sonic point, using 4th order Runge-Kutta method to obtain transonic solutions. At the sonic point, \( dv/dr = (dv/dr)_c \) (see section 2.1).

Fig.1(a1-a4), shows the Mach number \((M = u/a)\) plot vs log \( r \) plot for ‘spherical’ flows \((\lambda = 0)\) and in panel (b1-b6) solutions are for ‘rotating’ conical flows \((\lambda = 2.5)\) around BH. Parameters used are \( E = 1.002, \dot{M} = 0.05\dot{M}_{\text{edd}}, M_{\text{BH}} = 10M_\odot, \theta = 60^\circ \). Thus, the constants of motion are same in all the cases, but \( T_{\text{p, in}} \) is varied to get different solutions. The dotted solutions in every
case are mathematical transonic outflows. Solid lines denote the global accretion solutions since it connects BH horizon from infinity.

We can see that in case of spherical flows the global solution always passes through a single sonic point (a1-a4), while in case of rotating flows, the flow might pass through one or more sonic points (b1-b6). In (b1-b2), global solution passes through the outer sonic point only, while in (b6) it passes through the inner sonic point. In both the cases only one physical sonic point is present. In (b3-b6) solutions possess multiple sonic points, out of which in (b4), we see that the flow harbours shock. So the global solution first passes through the outer sonic point, becomes supersonic and then after going through a shock transition, jumps to the subsonic branch and enter the BH supersonically after passing through the inner sonic point.

![Figure 1](image_url)

**Figure 1:** Left : (a) $\mathcal{M}_{in}$ vs $T_{p\,in}$ plot for spherical flows. Single sonic point exist for every $T_{p\,in}$. Panels (a1-a4) $M$ vs log $r$ plot for various $T_{p\,in}$s marked with solid coloured dots in panel (a). The values of $T_{p\,in}$ are : (a1) $3.0 \times 10^{11} K$, (a2) $4.0 \times 10^{11} K$, (a3) $5.0 \times 10^{11} K$, (a4) $6.0 \times 10^{11} K$. Right : (b) Shows $\mathcal{M}_{in}$ vs $T_{p\,in}$ plot for rotating conical flows ($\lambda = 2.5$). Solid black curve is for the solutions passing through outer sonic point, while dotted black curve is for solutions passing through inner sonic points. Panels (b1-b4): Shows solutions for various $T_{p\,in}$s marked in panel (b), value of which are: (b1) $4.0 \times 10^{11} K$, (b2) $5.1 \times 10^{11} K$, (b3) $6.4 \times 10^{11} K$, (b4) $6.7 \times 10^{11} K$, (b5) $6.86 \times 10^{11} K$, (b6) $8.0 \times 10^{11} K$. Entropy maximises at a certain $T_{p\,in}$ in both the cases (green star). Parameters used are $E = 1.002$, $M = 0.05 \dot{M}_{edd}$, $M_{BH} = 10 M_{\odot}$, $\theta = 60^\circ$.

### 3.2. Selection of a unique solution:

A solution should be unique for a given set of constants of motion, but in Fig.1(a1-a4) and (b1-b6), it was noticed that a change in $T_{p\,in}$ gave solutions with completely new topology and sonic point properties. In Fig.1, we plotted the case of spherical flows, in order to contrast for the complexity that arises due to the presence of angular momentum. Any wrong choice of solution would give us a completely different information and hence a wrong spectrum. So it is necessary to select the correct solution. The only way to deal with this degeneracy is to have a measure of entropy of the system. This measure of entropy is plotted in panel (a) and (b) against the corresponding $T_{p\,in}$s, using the entropy form given by Eq.(12).

In case of spherical flows, which possess single sonic points, there is a clear entropy maxima at $T_{p\,in} = 5 \times 10^{11} K$ (green star). By employing the second law of thermodynamics, we select this solution (panel a2). Thus, we are able to obtain a unique transonic two-temperature solution for the given set of constants of motion [25].

In case of rotating flows, global solutions may pass through one or more sonic points. So the system is complicated unlike spherical flows. Dotted black curve in panel (b) is for solutions passing through inner sonic points while solid black curve is for solutions passing through outer sonic points. In a certain range of $T_{p\,in}$ both inner and outer sonic points exist (solution b3-b5).
We can see that the entropy maximises in the outer sonic point branch of solutions (green star). Since flow starts from infinity, it would select the maximum entropy solution. Then, it could pass through an inner sonic point only if the shock conditions are satisfied (see section 2.1). For the present case, $T_{p\text{ in}} = 5.1 \times 10^{11} K$ possess maximum entropy and the global solution passes through outer sonic point (solution b2). In the sections to follow, we use this method of maximising entropy to obtain global advective transonic two-temperature accretion solutions.

4. Results

4.1. General transonic two-temperature solution :

In Fig.(2), we plot a typical two-temperature transonic advective accretion disk solution with its corresponding flow variables. The parameters used are, $E = 1.00001$, $\lambda = 3.0$, $\dot{M} = 0.001 \dot{M}_{\text{edd}}$, $M_{\text{BH}} = 10 M_\odot$ and $\theta = 60^\circ$. In panel (a) we plot the global solution, which passes through multiple sonic points (green solid line). First the solution passes through an outer sonic point $r_{co} = 5935.59 r_g$ and then through an inner sonic point $r_{ci} = 6.29 r_g$ (denoted by black stars) through a shock transition at $r_{\text{shock}} = 257.09 r_g$. The infall speed of the supersonic matter after passing through the outer sonic point, is slowed down, due to the twin effect of centrifugal force and thermal pressure. This slowed down matter acts as a barrier to the matter coming from behind. This causes the formation of a shock. After the shock, matter becomes subsonic but it ultimately falls into the BH supersonically after passing through the inner sonic point. The radial three-velocity ($v$) in co-rotating frame and flow velocity in the azimuthal direction ($v_\phi$) is plotted in panel (b) (solid cyan and dotted blue respectively). The fate of $v$ and $v_\phi$ near the horizon is quite different, one approaching $c$ while the other going to 0, since near the BH, gravity is so strong that matter only has a radial velocity component and negligible azimuthal
component. \( T_p \) (solid, orange) and \( T_e \) (dotted, red) as a function of \( r \) is plotted in panel (c). At very large \( r \), \( T_p \approx T_e \). But as the matter flows inwards, cooling processes in electrons start to dominate. Thus, protons and electrons settle down into two different temperatures. They are however, coupled by a Coulomb coupling term which acts as an energy exchange term between the protons and electrons. But the term is weak, unlike in one-temperature case, where it is infinite. Panel (d), shows that gamma index of both protons (solid, orange) and electrons (dotted, red) varies with the flow. This justifies our use of CR EoS. \( \Gamma_p \) and \( \Gamma_e \sim 1.66 \) at very large distances, suggesting that both of them are thermally non-relativistic. When the flow nears the BH, \( \Gamma_e \) becomes mildly-relativistic and then relativistic with \( \Gamma_e \sim 1.33 \) near the horizon, though \( \Gamma_p \) never becomes truly relativistic. Panel (e), we have plotted number density (in units of \( \text{cm}^{-3} \)) vs \( \log r \). Since accretion is an example of convergent flow, the number density increases with decrease in radius. In panel (f), we prove that the generalised Bernoulli constant is a constant of motion throughout the flow, even in the presence of dissipation. In panel (g), we plot the spectrum for the accretion flow. It is plotted by summing up all the emissions coming from each radius. Spectrum of each emission process is also plotted separately. Bremsstrahlung is shown in dotted violet line, synchrotron by dashed yellow line and inverse-Comptonization by dotted-dashed grey line.

4.2. Luminosity and Efficiency of Bondi flows as compared to rotating flows:

In Fig.3, we have compared the change in luminosity with increase in accretion rate (\( \dot{M} \)), which is in terms of Eddington rate (\( \dot{M}_{\text{edd}} \)). Circled points are for spherically symmetric flows whereas diamond shaped points are for flows having angular momentum \( \lambda = 2.4 \). For both the flows parameters used are \( E = 1.001 \), \( M_{\text{BH}} = 10M_\odot \) and \( \theta = 60^\circ \). We can see that as the supplied matter is increased the luminosity increases, irrespective of the type of flow. It is apparent from the plot that Bondi flows are less luminous flows, about an order of magnitude lower than rotating flows. The presence of angular momentum slows down the matter giving it more time to radiate. The color bar plots the efficiency of the system, which is given by the formula, \( \tilde{\eta} = L/(\dot{M}c^2) \). The efficiency also have a similar trend, with the increase in accretion rate efficiency increases but flows with angular momentum have a higher efficiency than spherical flows, at a given an accretion rate.
5. Conclusions
In this paper we present our results in case of two-temperature rotating flows, that is flows having some angular momentum. Pure GR treatment was made, which helped us, model the strong gravity of BHs. In these flows, Coulomb coupling is never too strong, to make the flow attain a single temperature. We have also presented results of spherical flows in few cases in order to contrast the differences of these flows with flows having angular momentum.

Our focus in this paper is to obtain unique transonic solutions in two-temperature model. We, in SC19, have reported that two-temperature solutions are degenerate in nature. Infinite transonic solutions exist for the same constants of motion. We solved the problem with the help of the entropy measure (Eq.12) which was obtained from the first principles. In this paper we first demonstrated the use of this entropy form in case of spherical flows, since spherical flows are simple and possess single sonic points. We could see a clear maxima at a certain solution. We applied the same methodology in case of rotating flows as well. But these systems are complex. Still the entropy measure form of ours worked and we again selected the solution with maximum entropy. This helped us in selecting a unique solution from the infinite solutions obtained, for a given set of constants of motion, in case of rotating conical flows. This is the first time, to the best of our knowledge, that such work has been done. This methodology to remove the degeneracy in two-temperature flows is required since any wrong choice of solution would give us a wrong solution and hence an overall wrong information of the system.

In our previous paper SC19, although we have worked on spherical transonic two-temperature flows but we did not consider any GR and doppler effects. We also ignored any photon trapping effects by BH. This led us to high luminosity values in case of spherical flows. In this paper we took these corrections into account and compared the luminosities of these flows with rotating flows. We saw that at $1\dot{M}_{\text{edd}}$, Bondi flows have efficiencies $\approx 0.14\%$, while rotating flows have efficiencies $\approx 2.12\%$. Rotating flows were always an order of magnitude more luminous than spherical flows. Similar trend was also observed in efficiency.

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