Abstract
There are numerous styles of planar graph drawings, notably straight-line drawings, poly-line drawings, orthogonal graph drawings and visibility representations. In this note, we show that many of these drawings can be transformed from one style to another without changing the height of the drawing. We then give some applications of these transformations.

Keywords: Graph drawing; straight-line drawing; orthogonal drawing; visibility representation; upward drawings.

1. Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. We assume that $G$ is planar, i.e., it can be drawn without crossing. In planar graph drawing, one aims to create a crossing-free picture of $G$. It was known for a long time that such drawings exist even with straight lines $[18, 8, 17]$, and even in an $O(n) \times O(n)$-grid $[10, 16]$. Many improvements have been developed since, see for example $[7, 15]$.

Formally, a drawing of a graph consists of assigning a point or an axis-aligned box to every vertex, and a curve between the points/boxes of $u$ and $v$ to every edge $(u, v)$. The drawing is called planar if no two elements of the drawing intersect unless the corresponding elements of the graph do. Thus no vertex points/boxes coincide, no edge curve self-intersects, no edge curves intersect each other (except at common endpoints), and no edge curve intersects a vertex point/box other than its endpoints. In this paper all drawings are required to be planar.

In a straight-line drawing vertices are represented by points and edges are drawn as straight-line segments. In a poly-line drawing, vertices are points and each edge curve is a contiguous sequence of line segments. The place where an edge curves changes direction is called a bend. An orthogonal drawing uses a box for every vertex, and requires that edges are drawn as poly-lines for which every line segment is either horizontal or vertical. A visibility representation is an orthogonal drawing without bends.

In this paper we sometimes restrict these drawings further. An orthogonal drawing is called flat if all boxes of vertices are degenerated into horizontal segments. We say that a drawing $\Gamma$ is $y$-monotone if for any edge $(v, w)$, the drawing of the edge in $\Gamma$ forms a $y$-monotone path. (Horizontal edge segments are allowed.) Any straight-line drawing and any visibility representation is automatically $y$-monotone. See also Fig. 1. (In our drawings, we thicken vertex boxes slightly, so that horizontal segments appear as boxes of small height.)

In all drawings, the defining elements (i.e., points of vertices, corners of boxes of vertices, bends, and attachment points of edges to vertex-boxes) must be placed at points with integer coordinates. A drawing is said to have width $w$ and height $h$ if (possibly after translation) all such points are placed on the $[1, w] \times [1, h]$-grid. The height is thus measured by the number of rows, i.e., horizontal lines with integer $y$-coordinates that are occupied by the drawing. After rotation, we may assume that the height is no larger than the width.

\footnote{If the maximum degree is 4, then one could additionally demand vertices to be points. We will not study this model in the current paper, so an orthogonal drawing always allows boxes.}

Preprint submitted to Elsevier May 11, 2014
In a previous paper [2], we studied transformations between these graph drawing styles that preserved the asymptotic area. In particular, we showed that if \( G \) has a visibility representation, then it has a poly-line drawing of asymptotically the same area. In this paper, we study such transformations with the goal of keeping the height of the drawing unchanged. Our results are illustrated in Figure 2. All our transformations have two additional properties that are useful in the proofs and some of the applications. First, the resulting drawing has the same \( y \)-coordinates, i.e., any vertex (and also any bend that is not removed) has the same \( y \)-coordinate in the new drawing as in the original one. (Since we give transformations only for flat orthogonal drawings, this concept makes sense even when transforming vertex-boxes into points.) Second, the resulting drawings have the same left-to-right order in each row, i.e., if vertices \( v \) and \( w \) had the same \( y \)-coordinate, with \( v \) left of \( w \), then the same also holds in the resulting drawing.

![Figure 1: Height-preserving transformations proved in this paper. Dashed arrows are trivial implications.](image)

We then study some applications of these results. Most importantly, they allow to derive some height-bounds for drawing styles for which we are not aware of any direct proof, and they allow to formulate some NP-hard graph drawing problems as integer programs.

2. Flat visibility representations to straight-line drawings

**Theorem 1.** Any flat visibility representation \( \Gamma \) can be transformed into a straight-line drawing \( \Gamma' \) with the same \( y \)-coordinates and the same left-to-right orders in each row.

**Proof:** For any vertex \( v \), use \( x_l(v), x_r(v) \) and \( y(v) \) to denote leftmost and rightmost \( x \)-coordinate and (unique) \( y \)-coordinate of the box that represents \( v \) in \( \Gamma \). Use \( X(v) \) and \( Y(v) \) to denote the (initially unknown) \( x \)-coordinate of \( v \) in \( \Gamma' \). For any vertex set \( Y(v) = y(v) \), hence \( y \)-coordinates are the same.

Let \( v_1, \ldots, v_n \) be the vertices sorted by \( x_l(.) \), breaking ties arbitrarily. The algorithm determines \( X(.) \) for each vertex by processing vertices in this order and expanding the drawing \( \Gamma'_{i-1} \) created for \( v_1, \ldots, v_{i-1} \) into a drawing \( \Gamma'_i \) of \( v_1, \ldots, v_i \), which has the same left-to-right orders.

Suppose \( X(v_g) \) has been computed for all \( g < i \) already. To find \( X(v_i) \), determine lower bounds for it by considering all predecessors of \( v_i \) and taking the maximum over all of them. (For each vertex \( v_i \), the predecessors of \( v_i \) are the neighbours of \( v_i \) that come earlier in the order \( v_1, \ldots, v_n \).) A first (trivial) lower bound for \( X(v_i) \) is that it needs to be to the right of anything in row \( y(v_i) \). Thus, if \( \Gamma'_{i-1} \) contains a vertex or part of an edge at point \( (X, y(v_i)) \), then \( X(v_i) \geq |X| + 1 \) is required.

Next consider any predecessor \( v_g \) of \( v_i \) with \( y(v_g) \neq y(v_i) \). Since \( v_g \) and \( v_i \) are not in the same row, they must see each other vertically in \( \Gamma \), which means that \( x_d(v_g) \geq x_l(v_i) \). See also Fig. 2. So if \( v_g \) has a neighbour \( v_k \) to its right in \( \Gamma \), then \( x_d(v_k) \geq x_r(v_g) \geq x_d(v_i) \), which implies that \( v_k \) has not yet been added to \( \Gamma'_{i-1} \). Since \( \Gamma'_{i-1} \) has the same left-to-right orders, \( v_g \) is hence the rightmost vertex in its row in \( \Gamma'_{i-1} \) and can see towards infinity on the right. But then \( v_g \) can also see the point \( (+\infty, y(v_i)) \), or in other words, there exists some \( X_g \) such that \( v_g \) can see all points \( (X, y(v_i)) \) for \( X \geq X_g \). Impose the lower bound \( X(v_i) \geq |X_g| + 1 \) on the \( x \)-coordinate of \( v_i \).
Now let $X(v_i)$ be the smallest value that satisfies the above lower bounds (from the row $y(v_i)$ and from all predecessors of $v_i$ in different rows.) Set $X(v_i) = 0$ if there were no such lower bounds.\footnote{To simplify the calculations below it helps to use 0 (as opposed to 1) for the leftmost $x$-coordinate.} Directly by construction, placing $v_i$ at $(X(v_i),y(v_i))$ allows it to be connected with straight-line segments to all its predecessors. This includes the predecessor (if any) that is in the same row as $v_i$, since there can be at most one in a flat visibility representation, and it is in the same row as $v_i$. This gives a drawing $\Gamma'_i$ of $v_1, \ldots, v_i$ as desired, and the result follows by induction.

2.1. Width considerations

While our transformation keeps the height intact, the width can increase dramatically. Fig. 3 shows a flat visibility representations of height $h$ and width $O(n)$ such that the transformation of Theorem 1 has width $\Omega((h-2)^{(n-3)}).$ Specifically, using induction one shows that vertex $i$ (for $i \geq 3$) is placed with $x$-coordinate $1 + (h-2) + \ldots + (h-2)^{i-3}$ and leaves an edge with slope $\pm 1/(1 + (h-2) + \ldots + (h-2)^{i-3}).$ But this is (asymptotically) the worst that can happen.

**Lemma 1.** For $h \geq 4$, the width of the drawing obtained with Theorem 1 is $O((h-2)^{n-3}).$

**Proof:** Define two recursive functions as $W(2) = W'(2) = 0$, $W'(i) = 1 + (h-2)W'(i-1)$ and $W(i) = 1 + (h-1)W'(i-1)$ for $i \geq 3$. We will show that for $i \geq 4$, any point $p$ of the drawing $\Gamma'_i$ has $x$-coordinate at most $W(i)$, and if $p$ is not on the first or last row, then it has $x$-coordinate at most $W'(i)$. Observe that $W'(i) = 1 + (h-2) + \ldots + (h-2)^{i-3} \in O((h-2)^{i-3})$ and $W(i) = W'(i-1) + W'(i) \leq 2W'(i)$; hence $W(n) \in O((h-2)^{n-3})$ as desired.

For the base case, we have two cases. If two of $\{v_1, v_2, v_3\}$ are in different rows, then two of $\{v_1, v_2, v_3\}$ have $x$-coordinate 0 and the third has $x$-coordinate at most 1, hence the claim holds for $\Gamma'_3$ since $W(3) = W'(3) = 1$. If $v_1, v_2, v_3$ are all in the same row, then they have $x$-coordinates 0,1,2. Vertex $v_4$ is either also placed in this row (and then has $x$-coordinate 3) or it is in a different row (and then has $x$-coordinate 0.) Either way, all points in $\Gamma'_3$ then have $x$-coordinate at most $3 \leq W'(4) \leq W(4)$. This shows the base case.

For the induction step, we distinguish cases on how the $x$-coordinate $X(v_i)$ of $v_i$ was determined:
• Assume first that \(X(v_i)\) was determined via \(X(v_i) = |X| + 1\), where \(X\) is the \(x\)-coordinate of some point \(p\) in \(\Gamma'_{i-1}\) in the row of \(v_i\). By induction we know that \(X \leq W(i-1)\), and so \(X(v_i) \leq 1 + X \leq 1 + W(i-1) \leq 1 + 2W'(i-1) \leq W(i)\) by \(h \geq 3\). If \(v_i\) is not in the first or last row, then neither is \(p\), so \(X \leq W'(i-1)\) and \(X(v_i) \leq 1 + W'(i-1) \leq W'(i)\).

• Assume now that \(X(v_i) = |X| + 1\), where \(X_g\) is the \(x\)-coordinate of the intersection of the row of \(v_i\) with a line \(\ell\) through some predecessor \(v_g\) of \(v_i\) and some point \(p\) of drawing \(\Gamma'_{i-1}\). Assume as in Figure 2 that \(y(v_g) \geq y(p)\); the other case is similar. This implies \(y(v_g) \geq y(v_i)\) as well, otherwise \(p\) would not have been an obstruction for the edge \((v_g, v_i)\).

If \(X(p) \leq X(v_g)\), then \(X_g \leq X(v_i)\), therefore \(X(v_i) \leq X(v_g) + 1\) satisfies the bound as in the first case. If \(p\) is in the bottommost row, then \(X_g \leq X(p)\), therefore \(X(v_i) \leq X(p) + 1\) satisfies the bound as in the first case. Finally assume \(X(p) > X(v_g)\) and \(p\) is not in the bottommost row, hence \(X(p) \leq W'(i-1)\).

Now

\[
X_g = X(v_g) + (y(v_g) - y(v_i)) \frac{X(p) - X(v_g)}{y(v_g) - y(p)} \leq (y(v_g) - y(v_i))X(p) \leq (y(v_g) - y(v_i))W'(i-1).
\]

If \(y(v_g) - y(v_i) \leq h - 2\) then \(X(v_i) \leq 1 + X_g \leq 1 + (h - 2)W'(i-1) = W'(i)\). Otherwise \(v_i\) is in the bottommost row (and \(v_g\) in the top row), and \(X(v_i) \leq 1 + X_g \leq 1 + (h - 1)W'(i-1) = W(i)\) as desired.

We note here that \(h \geq 4\) is needed only for \(1 + (h - 2) + \ldots + (h - 2)^{n-3} \in O((h - 2)^{n-3})\); much the same proof shows that the height is \(O(n)\) for \(h = 3\).

3. \(y\)-monotone flat orthogonal drawings to flat visibility representations

**Theorem 2.** Any flat \(y\)-monotone orthogonal drawing \(\Gamma\) can be transformed into a flat visibility representation \(\Gamma'\) with the same \(y\)-coordinates and the same left-to-right orders in each row.

**Proof:** First, expand every vertex \(v\) to the left and right until it covers all bends (if any) of edges that attach horizontally at \(v\). Since \(v\) has height 1, there is at most one edge \(e\) each on the left and right side of \(v\), and the expansion of \(v\) covers only space previously used by \(e\), hence creates no overlap.

Now we arrive at a drawing where all edges that have bends attach vertically at their endpoints. Let \(e\) be an edge with bends (if there is none we are done.) Since \(e\) is drawn \(y\)-monotone, it attaches at the top of one endpoint and the bottom of the other endpoint, and the only way it can have bends is to have a right turn followed by a left turn or vice versa. Thus, \(e\) has a “zig-zag”.

It is well known that such a zig-zag can be removed by transforming the drawing as follows (see also Figure 4): Extend the ends of the zig-zag upward and downward to infinity, and then shift the two sides of the resulting separation apart until the two rays of the zig-zag align. See for example [9] for details. This operation adds width, but no height. Applying this to all edges that have bends gives a visibility representation.

![Figure 4: Removing a zig-zag by shifting parts of the drawing rightwards.](image-url)
3.1. Width considerations

Our construction may increase the width quite a bit, but mostly with columns that are redundant: They contain neither a vertical edge, nor are they the only column of a vertex. A natural post-processing step is to remove redundant columns. We then obtain small width. In fact, the width is small for any visibility representation.

Lemma 2. Any visibility representation of a connected graph has width at most $\max\{m, n\}$ after deleting redundant columns.

Proof: Let $m_h$ and $m_v$ be the number of edges drawn horizontally and vertically. Let $V_h$ be the vertices without incident vertical edge. Then the width is at most $m_v + |V_h|$. This shows the claim if $m_v = 0$. If $m_v > 0$, then let $v$ be a vertex not in $V_h$, pick an arbitrary spanning tree $T$ and root it at $v$. For any vertex $w \in V_h$, the edge from $w$ to its parent in $T$ must be horizontal by definition of $V_h$. Hence there are at least $|V_h|$ horizontal edges, and the width is at most $m_v + m_h = m$. \hfill $\Box$

4. Poly-line drawing to flat orthogonal drawing

Theorem 3. Any poly-line drawing $\Gamma$ can be transformed into a flat orthogonal drawing $\Gamma'$ with the same $y$-coordinates and the same left-to-right orders in each row. $\Gamma'$ is $y$-monotone if $\Gamma$ was.

Proof: We first transform $\Gamma$ into an $h$-layer drawing, i.e., a straight-line drawing where all edges are horizontal or connect adjacent rows. We do this by inserting pseudo-vertices (i.e., subdivide edges) at bends and whenever a segment of an edge crosses a row. (We allow non-integral $x$-coordinates for pseudo-vertices.) For each row $r$, let $w_1, \ldots, w_k$ be the vertices (including pseudo-vertices) in $r$ in left-to-right order. In $\Gamma'$, replace each $w_i$ by a box of width $\max\{1, \deg^{up}(w_i), \deg^{down}(w_i)\}$, where $\deg^{up}(w_i)$ and $\deg^{down}(w_i)$ are the number of neighbours of $w_i$ with larger/smaller $y$-coordinate. Place these boxes in row $r$ in the same left-to-right order.

Each horizontal edge is drawn horizontally in $\Gamma'$ as well. Each non-horizontal edge connects two adjacent rows since we inserted pseudo-vertices. Connect the edges between two adjacent rows using VLSI channel routing (see e.g. [13]), using two bends per edge and lots of new rows (with non-integer $y$-coordinates) that contain horizontal edge segments and nothing else.

![Figure 5: Converting a poly-line drawing to an orthogonal drawing. Pseudo-vertices are white.](image)

Now bends only occur at zig-zags; remove these as in the proof of Theorem 2. This empties all rows except those with integer coordinates and gives the desired height and a flat visibility representation of the graph with pseudo-vertices. Any pseudo-vertex can now be removed and replaced by a bend if needed. \hfill $\Box$

Note that the visibility representation obtained as part of the proof has width at most $\max\{n + p, m + p\}$, where $p$ is the number of pseudo-vertices inserted. An even better bound can be obtained by observing that any pseudo-vertex that is not at a bend in $\Gamma$ will receive two incident vertical segments and hence can be removed in the visibility representation. So the width is at most $\max\{n + b, m + b\}$, where $b$ is the number of bends in $\Gamma$. 

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5. Flat orthogonal drawings to poly-line drawings

Combining the previous theorems, it is easy to see that any flat orthogonal drawing can be converted to a poly-line drawing of the same height: First convert it to a visibility representation, then convert it to a straight-line drawing, and then interpret the result as a poly-line drawing. However, since this involves Theorem 1, the width might grow exponentially. We now give a simple direct proof of this transformation that shows that it can be done while keeping the width small.

**Theorem 4.** Any flat orthogonal drawing $\Gamma$ can be transformed into a poly-line drawing $\Gamma'$ with the same $y$-coordinates and the same left-to-right orders in each row. Moreover, $\Gamma'$ has no more width than $\Gamma$, and it is $y$-monotone if $\Gamma$ was.

**Proof:** First subdivide edges at all bends and all vertical edge-segments with pseudo-vertices so that any vertical edge connects two vertices in adjacent layers. Apply the algorithm in Theorem 1 to find a straight-line drawing of the resulting graph; removing the pseudo-vertices then gives the desired poly-line drawing. All properties are easily verified, except for the width. Observe that when applying the construction of Theorem 1 for any vertex $v_i$ the predecessors are in the same or in adjacent rows. Hence all lines from $v_i$ to predecessors are unobstructed, and $v_i$ can simply be placed in the leftmost free position of its row. Hence the width of $\Gamma'$ is the maximal number of vertices or pseudo-vertices in a row, which is no more than the width of $\Gamma$ since $\Gamma$ is orthogonal.

6. Applications

We give a few applications of the results in this paper.

6.1. Drawing graphs with small height

The pathwidth $pw(G)$ of a graph $G$ is a graph parameter that is related to heights of planar graph drawing: any planar graph that has a straight-line drawing of height $h$ has pathwidth at least $h$ [9]. But not all graphs with pathwidth $h$ have a drawing of height $O(h)$. Our transformations show that such heights do exist for outer-planar graphs:

**Corollary 1.** Any outer-planar graph $G$ has a straight-line drawing of height $O(pw(G))$.

**Proof:** By a result of Babu et al. [1], we can add edges to $G$ to obtain a maximal outerplanar graph $G'$ with pathwidth in $O(pw(G))$. In particular, $G'$ is 2-connected and hence by [3] it has a flat visibility representation of height at most $4pw(G') \in O(pw(G))$. By Theorem 1 therefore $G'$ (and with it $G$) has a straight-line drawing of height $O(pw(G))$.

Recall that outerplanar graphs have constant treewidth and hence pathwidth $O(\log n)$, so any outerplanar graph has a straight-line drawing of height $O(\log n)$.

In a similar fashion, any graph drawing algorithm that produces drawings of small height in one of our models produces, with our transformations, graph drawings of small heights in all other models. We give one more example:

**Corollary 2.** Any 4-connected planar graph has a visibility representation of height at most $\lfloor n/2 \rfloor$.

**Proof:** It is known that any 4-connected planar graph has a straight-line drawing where the sum of the width and height is at most $n$ [14]. Therefore, after possible rotation, the height is at most $\lfloor n/2 \rfloor$, and with Theorem 2 and 3 we get a flat visibility representation of height $\lfloor n/2 \rfloor$.

The best previous bound on the height of visibility representations of 4-connected planar graphs was $\lceil 3n/4 \rceil$ [12].
6.2. Integer programming formulations

In a recent paper, we developed integer program (IP) formulations for many graph drawing problems where vertices and edges are represented by axis-aligned boxes [4]. By adding some constraints, one can force that edges degenerate to line segments and vertices to horizontal line segments. In particular, it is easy to create an IP that expresses “G is drawn as a flat visibility representation”, using $O(n^3)$ variables and constraints. With the transformations given in this paper, we can then use IPs for many other graph drawing problems. The following result (based on Theorem 1, 2, 3) is crucial:

**Corollary 3.** A planar graph $G$ has a planar straight-line drawing of height $h$ if and only if it has a flat visibility representation of height $h$.

It is very easy to encode the height in the IP formulations of [4]. We therefore have:

**Corollary 4.** There exists an integer program with $O(n^3)$ variables and constraints to find the minimum height of a planar straight-line drawing of a graph $G$.

A directed acyclic graph has an upward drawing if it has a planar straight-line drawing such that for any directed edge $v \rightarrow w$ the $y$-coordinate of $v$ is smaller than the $y$-coordinate of $w$. Testing whether a graph has an upward drawing is NP-hard [11]. There exists a way to formulate “$G$ has an upward drawing” as either IP or as a Satisfiability-problem, using partial orders on the edges and vertices [6]. Our transformations give a different way of testing this via IP:

**Lemma 3.** A directed acyclic graph has an upward planar drawing if and only if it has a visibility representation where all edges are vertical lines, with the head above the tail.

**Proof:** Given a straight-line upward drawing, we can transform it into the desired visibility representation using Theorems 2 and 3. Since $y$-coordinates are unchanged, any edge is necessarily drawn vertical with the head above the tail. Vice versa, given such a visibility representation, we can transform it into a flat visibility representation simply by replacing boxes of positive height by horizontal segments; this leads to no conflict since there are no horizontal edges. Then apply Theorem 1 this gives an upward drawing since $y$-coordinates are unchanged.

It is easy to express “edge $v \rightarrow w$ must be drawn vertically, with the head above the tail” as constraints in the IP for visibility representations defined in [4]. We therefore have:

**Corollary 5.** There exists an integer program with $O(n^3)$ variables and constraints to test whether a planar graph has an upward planar drawing. Moreover, the same integer program also finds the minimum-height upward drawing.

7. Conclusion and open problems

In this paper, we studied transformations between different types of graph drawings, in particular between straight-line drawings and flat visibility representations. We demonstrated applications of these results, especially for drawings of small heights, and upward drawings.

We have not been able to create transformations that start with an arbitrary (i.e., not necessarily flat) visibility representation and turn it into a straight-line drawing of approximately the same height. Does such a transformation exist?

Another open problem concerns the width, especially for the transformation from flat visibility representations to planar straight-line drawings. Is it possible to make the width polynomial if we may change the $y$-coordinates while keeping the height asymptotically the same?

**Acknowledgments**

Research partially supported by NSERC and by the Ross and Muriel Cheriton Fellowship. Some of the results in this paper appeared in [3].
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