

CURVATURE OF $R^{n-p} f_* \Omega^p_{X/S}(K_{X/S}^\otimes m)$
AND APPLICATIONS

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ABSTRACT. Given an effectively parameterized family $f : X \to S$ of canonically polarized manifolds, the Kähler-Einstein metrics on the fibers induce a hermitian metric on the relative canonical bundle $K_{X/S}$. We use a global elliptic equation to show that this metric is strictly positive everywhere and give estimates.

The direct images $R^{n-p} f_* \Omega^p_{X/S}(K_{X/S}^\otimes m)$, $m > 0$, carry induced natural hermitian metrics. We prove an explicit formula for the curvature tensor of these direct images. The formula for $p = n$, implies that $f_*(K_{X/S}^\otimes (m+1))$ is Nakano-positive with estimates (for effectively parameterized families). We apply it to the morphisms $S^p T_S \to R^p f_* T_{X/S}$ induced by the Kodaira-Spencer map and obtain a differential geometric proof for hyperbolicity properties of $\mathcal{M}_{\text{can}}$. Similar results hold for families of polarized Ricci-flat manifolds. These will appear elsewhere.

1. Introduction

For any holomorphic family $f : X \to S$ of canonically polarized, complex manifolds, the unique Kähler-Einstein metrics on the fibers define an intrinsic metric on the relative canonical bundle $K_{X/S}$. The construction is functorial in the sense of compatibility with base changes. By definition, its curvature form has at least as many positive eigenvalues as the dimension of the fibers indicates. It turned out that it is strictly positive, provided the induced deformation is not infinitesimally trivial at the corresponding point of the base.

Actually the first variation of the metric tensor in a family of compact Kähler-Einstein manifolds contains the information about the induced deformation, more precisely, it contains the harmonic representatives $A_s = A^a_{\alpha \beta} \partial_\alpha dz^\beta$ of the Kodaira-Spencer classes $\rho(\partial/\partial s)$. The positivity of the hermitian metric will be measured in terms of a certain global function. Essential is an elliptic equation on the fibers, which relates this function to the pointwise norm of the harmonic Kodaira-Spencer

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forms. The "strict" positivity of the corresponding (fiberwise) operator $(\Box + id)^{-1}$, where $\Box$ is the complex Laplacian, can be seen in a direct way (cf. [SCH4]). For families of compact Riemann surfaces the elliptic equation was previously derived in terms of automorphic forms by Wolpert [WO]. Later in higher dimensions a similar equation arose in the work of Siu [SIU1] for families of canonical polarized manifolds.

In this article, we will reduce estimates for the positivity of the curvature of $\mathcal{K}_{\mathcal{X}/\mathcal{S}}$ on $\mathcal{X}$ to estimates of the resolvent kernel of the above integral operator, whose positivity was already shown by Yosida in [YO]. Finally estimates for the resolvent kernel follow from the estimates for the heat kernel, which were achieved by Cheeger and Yau in [C-Y].

The positivity of the relative canonical bundle is important, when estimating the curvature of the direct image sheaves $R^{n-p}f_*\Omega^p_{\mathcal{X}/\mathcal{S}}(\mathcal{K}^{\otimes m}_{\mathcal{X}/\mathcal{S}})$. These are equipped with a natural hermitian metric that is induced by the $L^2$-inner product of harmonic tensors on the fibers of $f$. We will give an explicit formula for the curvature tensor.

A main motivation of our approach is the study of the curvature of the classical Weil-Petersson metric, in particular the computation of the curvature tensor by Tromba [TR] and Wolpert [WO]. It immediately implies the hyperbolicity of the classical Teichmüller space. The curvature tensor of the generalized Weil-Petersson metric for families of metrics with constant negative Ricci curvature was explicitly computed by Siu in [SIU1]; in [SCH2] a formula in terms of elliptic operators and harmonic Kodaira-Spencer tensors was derived. However, the curvature of the generalized Weil-Petersson metric seems not to satisfy any negativity condition. This difficulty was overcome in the work of Viehweg and Zuo in [V-Z1]. Their approach to hyperbolicity makes use of the period map in the sense of Griffiths.

On the other hand our results are motivated by Berndtsson’s result on the Nakano-positivity for certain direct images. In [SCH4] we showed that the positivity of $f_*\mathcal{K}^{\otimes 2}_{\mathcal{X}/\mathcal{S}}$ together with the curvature of the generalized Weil-Petersson metric is sufficient to imply a hyperbolicity result for moduli of canonically polarized complex manifolds in dimension two.

For ample $K_X$ the cohomology groups $H^{n-p}(X, \Omega^p_X(K_X^{\otimes m}))$ are critical with respect to the Kodaira-Nakano vanishing theorem. The understanding of this situation is the other main motivation. We will consider the relative case. Let $A_m^{\alpha\beta}(z, s)\partial_\alpha\partial_\beta\mathcal{F}$ be a harmonic Kodaira-Spencer form. Then for $s \in \mathcal{S}$ the cup product together with the
contraction defines

\[ A^0_{\overrightarrow{\partial_a}d l^n} \cup \omega : A^{0,n-p}(X_s, \Omega^p_{X_s}(K_{X_s}^{\otimes m})) \to A^{0,n-p+1}(X_s, \Omega^p_{X_s}(K_{X_s}^{\otimes (m+1)})) \]

\[ A^{0,1}_{\overrightarrow{\partial_a}d l^n} \cup \omega : A^{0,n-p}(X_s, \Omega^p_{X_s}(K_{X_s}^{\otimes m})) \to A^{0,n-p-1}(X_s, \Omega^p_{X_s}(K_{X_s}^{\otimes (m+1)})). \]

We will apply the above product to harmonic \((0, n-p)\)-forms. In general the result is not harmonic. We denote the pointwise \(L^2\) inner product by a dot.

**Main Theorem I.** The curvature tensor for \(R^{n-p}f_*\Omega^p_{X/S}(K_{X/S}^{\otimes m})\) is given by

\[
R_{\overrightarrow{\partial_a}d l^n}(s) = m \int_{X_s} (\Box + 1)^{-1} (A_i \cdot A_j) \cdot (\psi^k \cdot \psi^l) g dV + m \int_{X_s} (\Box - 1)^{-1} (A_i \wedge \nu^k) \cdot (A_j \wedge \nu^l) g dV.
\]

The potentially negative third term is not present for \(p = n\), i.e. for \(f_*K_{X/S}^{\otimes (m+1)}\). From the main theorem we get immediately a fact which is contained in Berndtsson’s theorem [B]:

*The locally free sheaf \(f_*K_{X/S}^{\otimes (m+1)}\) is Nakano-positive.*

(Very recently Berndtsson considered the curvature of \(f_*(K_{X/S}^{\otimes m} \otimes \mathcal{L}), [B_1]\).)

We prove the estimate

\[
R(A, \overrightarrow{A}, \psi, \overrightarrow{\psi}) \geq P_n(d(X_s)) \cdot \|A\|^2 \cdot \|\psi\|^2,
\]

where the coefficient \(P_n(d(X_s)) > 0\) is an explicit function depending on the dimension and the diameter of the fibers.

By Serre duality the theorem yields the following version, which contains for \(p = 1\) the curvature formula for the generalized Weil-Petersson metric. Again a tangent vector of the base is identified with a harmonic Kodaira-Spencer form \(A_i\), and \(\nu_k\) stands for a section of the relevant sheaf:

**Main Theorem II.** The curvature for \(R^p f_*\Lambda^p T_{X/S}\) equals

\[
R_{\overrightarrow{\partial_a}d l^n}(s) = - \int_{X_s} (\Box + 1)^{-1} (A_i \cdot A_j) \cdot (\nu^k \cdot \nu^l) g dV
- \int_{X_s} (\Box + 1)^{-1} (A_i \wedge \nu^l) \cdot (A_j \wedge \nu^k) g dV
- \int_{X_s} (\Box - 1)^{-1} (A_i \wedge \nu^k) \cdot (A_j \wedge \nu^l) g dV.
\]
In order to prove a result about hyperbolicity of moduli spaces we use Demailly’s approach. An upper semi-continuous Finsler metric of negative holomorphic curvature on a relatively compact subspace of the moduli stack of canonically polarized varieties can be constructed so that any such space is hyperbolic with respect to the orbifold structure.

We get immediately the following fact related to Shafarevich’s hyperbolicity conjecture for higher dimensions solved by Migliorini [M], Kovács [KV1, KV2, KV3], Bedulev-Viehweg [B-V], and Viehweg-Zuo [V-Z1, V-Z2].

**Application.** Let $\mathcal{X} \to C$ be a non-isotrivial holomorphic family of canonically polarized manifolds over a curve. Then $g(C) > 1$.

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2. **Fiber integration and Lie derivatives**

2.1. **Definition of fiber integrals and basic properties.** Let denote $\{\mathcal{X}_s\}_{s \in S}$ a holomorphic family of compact complex manifolds $\mathcal{X}_s$ of dimension $n > 0$ parameterized by a reduced complex space $S$. By definition, it is given by a proper holomorphic submersion $f : \mathcal{X} \to S$, such that the $\mathcal{X}_s = f^{-1}(s)$ for $s \in S$. In case of a smooth space $S$, if $\eta$ is a differential form of class $C^\infty$ of degree $2n + r$ the fiber integral

$$\int_{\mathcal{X}/S} \eta$$

is a differential form of degree $r$ on $S$. It can be defined as follows: Fix a point $s_0 \in S$ and denote by $X = \mathcal{X}_{s_0}$ the fiber. Let $U \subset S$ be an open neighborhood of $s_0$ such that there exists a $C^\infty$ trivialization of the family:

$$f^{-1}U \xrightarrow{\Phi} X \times U$$

Let $z = (z^1, \ldots, z^n)$ and $s = (s^1, \ldots, s^k)$ denote local (holomorphic) coordinates on $X$ and $S$ resp.

The pull-back $\Phi^* \eta$ possesses a summand $\eta'$, which is of the form

$$\sum \eta_k(z, s)dV_z \wedge d\sigma^{k_1} \wedge \ldots \wedge d\sigma^{k_r},$$

where the $\sigma^k$ run through the real
and complex parts of $s^{j}$, and where $dV_z$ denotes the relative Euclidean volume element. Now

$$\int_{X/S} \eta := \int_{X \times S/S} \Phi^* \eta := \sum_{k=(k_1, \ldots, k_r)} \left( \int_{X_z} \eta_k(z, s) dV_z \right) d\sigma^{k_1} \wedge \ldots \wedge d\sigma^{k_r}. $$

The definition is independent of the choice of coordinates and differentiable trivializations. The fiber integral coincides with the push-forward of the corresponding current. Hence, if $\eta$ is a differentiable form of type $(n + r, n + s)$, then the fiber integral is of type $(r, s)$.

Singular base spaces are treated as follows: Using deformation theory, we can assume that $S \subset W$ is a closed subspace of some open set $W \subset \mathbb{C}^N$, and that an almost complex structure is defined on $X \times S$ so that $\mathcal{X}$ is the integrable locus. Then by definition, a differential form of class $C^\infty$ on $\mathcal{X}$ is given on the whole ambient space $X \times W$ (with a type decomposition defined on $\mathcal{X}$).

Fiber integration commutes with taking exterior derivatives:

(1) \[ d \int_{X/S} \eta = \int_{X/S} d\eta, \]

and since it preserves the type (or to be seen explicitly in local holomorphic coordinates), the same equation holds true for $\partial$ and $\bar{\partial}$ instead of $d$.

A Kähler form $\omega_X$ on a singular space, by definition is a form that possesses locally a $\partial \bar{\partial}$-potential, which is the restriction of a $C^\infty$ function on a smooth ambient space. It follows from the above facts that given a Kähler form $\omega_X$ on the total space, the fiber integral

(2) \[ \int_{X/S} \omega_X^{n+1} \]

is a Kähler form on the base space $S$, which possesses locally a smooth $\partial \bar{\partial}$-potential, even if the base space of the smooth family is singular.

For the actual computation of exterior derivatives of fiber integrals (1), in particular of functions, given by integrals of $(n, n)$-forms on the fibers, the above definition seems to be less suitable. Instead the problem is reduced to derivatives of the form

(3) \[ \frac{\partial}{\partial s^{i}} \int_{X_z} \eta, \]

where only the vertical components of $\eta$ contribute to the integral. Here and later we will always use the summation convention.
Lemma 1. Let
\[ w_i = \left( \frac{\partial}{\partial s^i} + b_i^\alpha (z, s) \frac{\partial}{\partial z^\alpha} + c_i^\beta (z, s) \frac{\partial}{\partial z^\beta} \right) \bigg|_s \]
be differentiable vector fields, whose projection to \( \mathcal{S} \) equal \( \frac{\partial}{\partial s^i} \). Then
\[ \frac{\partial}{\partial s^i} \int_{X_s} \eta = \int_{X_s} L_{w_i}(\eta), \]
where \( L_{w_i} \) denotes the Lie derivative.

Concerning singular base spaces, observe that it is sufficient that the above equation is given on the first infinitesimal neighborhood of \( s \) in \( \mathcal{S} \).

Proof. Because of linearity, one may consider the real and imaginary parts of \( \partial/\partial s^i \) and \( w_i \) resp. separately.

Let \( \partial/\partial t \) stand for \( Re(\partial/\partial s^i) \), and let \( \Phi_t : X \to X_t \) be the one parameter family of diffeomorphisms generated by \( Re(w_i) \). Then
\[ \frac{d}{dt} \int_{X_s} \eta = \int_X \frac{d}{dt} \Phi_t^* \eta = \int_X L_{Re(w_i)}(\eta). \]

It is known that the vector fields \( Re(w_i) \) and \( Im(w_i) \) need not commute. \( \square \)

In our applications, the form \( \eta \) will typically consist of inner products of differential forms with values in hermitian vector bundles, whose factors need to be treated separately. This will be achieved by the Lie derivatives. In this context, we will have to use covariant derivatives with respect to the given hermitian vector bundle on the total space and to the Kähler metrics on the fibers.

2.2. Direct images and differential forms. Let \( (\mathcal{E}, h) \) be a hermitian, holomorphic vector bundle on \( \mathcal{X} \), whose direct image \( R^q f_* \mathcal{E} \) is locally free. Furthermore we assume that for all \( s \in \mathcal{S} \) the cohomology \( H^{q+1}(\mathcal{X}_s, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_s}) \) vanishes. Then the statement of the Grothendieck-Grauert comparison theorem holds for \( R^q f_* \mathcal{E} \), in particular \( R^q f_* \mathcal{E} \otimes_{\mathcal{O}_S} \mathbb{C}(s) \) can be identified with \( H^q(\mathcal{X}_s, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}_s}) \).

For simplicity, we assume that he base space \( S \) is smooth. Locally, after replacing \( S \) by a neighborhood of any given point, we can represent sections of the q-th direct image sheaf in terms of Dolbeault cohomology by \( \bar{\partial} \)-closed \((0, q)\)-forms. On the other hand, fiberwise, we have harmonic representatives of cohomology classes with respect to the Kähler form and hermitian metric on the fibers. The following fact will be essential.
Lemma 2. Let $\tilde{\psi} \in R^{q} f_{*} \mathcal{E}(S)$ be a section. Let $\psi_{s} \in \mathcal{A}^{0,q}(\mathcal{X}_{s}, \mathcal{E}_{s})$ be the harmonic representatives of the cohomology classes $\tilde{\psi}|_{\mathcal{X}_{s}}$.

Then locally with respect to $S$ there exists a $\overline{\partial}$-closed form $\psi \in \mathcal{A}^{0,q}(\mathcal{X}, \mathcal{E})$, which represents $\tilde{\psi}$, and whose restrictions to the fibers $\mathcal{X}_{s}$ equal $\psi_{s}$.

We omit the simple proof.

In this way, the relative Serre duality can be treated in terms of such differential forms. Let $\mathcal{E}^{\vee} = \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{O}_{\mathcal{X}})$. Then

$$R^{p} f_{*} \mathcal{E} \otimes_{\mathcal{O}_{S}} R^{n-p} f_{*} (\mathcal{E}^{\vee} \otimes_{\mathcal{O}_{X}} \mathcal{K}_{X/S}) \rightarrow \mathcal{O}_{S}$$

is given by the fiber integral of the wedge product of $\overline{\partial}$-closed differential forms in the sense of Lemma 2. By (1) (for the operator $\overline{\partial}$), the result is a $\overline{\partial}$-closed 0-form i.e. holomorphic function.

3. Estimates for resolvent and heat kernel

Let $(X, \omega_{X})$ be a compact Kähler manifold. The Laplace operator for differentiable functions is given by $\Box = \partial \partial^{*} + \partial^{*} \partial$, where the adjoint $\partial^{*}$ is the formal adjoint operator. The Laplacian is self-adjoint with non-negative eigenvalues.

The corresponding resolvent operator $(id + \Box)^{-1}$ is defined on the space of continuous functions and bounded.

First, we observe that the resolvent operator is positive: If $\chi \geq 0$ everywhere on $X$, then the function given by $(\Box + id)^{-1}\chi$ is non-negative. This fact follows immediately from the minimum principle applied to the elliptic equation

$$\Box \phi + \phi = \chi.$$ 

So the integral kernel $P(z, w)$ must be non-negative for all $z$ and $w$.

For any function $\chi(z)$

$$(\Box + id)^{-1}(\chi)(z) = \int_{X} P(z, w)\chi(w)g(w)dV_{w}$$

holds. In a similar way we denote by $P(t, z, w)$ the integral kernel for the heat operator

$$\frac{d}{dt} + \Box$$

so that the solution of the heat equation with initial function $\chi(z)$ for $t = 0$ is given by

$$\int_{X} P(t, z, w)\chi(w)g(w)dV_{w}.$$
The explicit representation of the above operators in terms of eigenfunctions of the Laplacian yields the following relation.

**Lemma 3.** Let \( P(z, w) \) be the integral kernel of the resolvent operator and denote by \( P(t, z, w) \) the heat kernel. Then

\[
P(z, w) = \int_0^\infty e^{-t} P(t, z, w) dt.
\]

**Proof.** Let \( \{\chi_\nu\} \) be a set of eigenfunctions of the Laplacian with eigenvalues \( \lambda_\nu \) so that

\[
P(z, w) = \sum \frac{1}{1 + \lambda_\nu} \chi_\nu(z) \chi_\nu(w)
\]

and

\[
P(t, z, w) = \sum e^{-t\lambda_\nu} \chi_\nu(z) \chi_\nu(w).
\]

Then, since the eigenvalues are non-negative,

\[
\int_0^\infty e^{-t(\lambda+1)} dt = \frac{1}{1 + \lambda}
\]

implies (4), (cf. also [C-Y, (3.13)]). \( \square \)

We now apply the lower estimates for the heat kernel by Cheeger and Yau [C-Y] to the resolvent kernel. Assuming constant negative Ricci curvature \(-1\), we use the estimates from [ST, (4.3) Corollary].

\[
P(t, z, w) \geq Q_n(t, r(z, w)) := \frac{1}{(2\pi t)^n} e^{-\frac{r^2(z, w)}{4t}} e^{-\frac{2n-4}{4} t},
\]

Where \( r = r(z, w) \) denotes the geodesic distance (and \( n = \dim X \)).

Let

\[
P_n(r) = \int_0^\infty e^{-t} Q_n(t, r) dt > 0.
\]

Using Lemma 3 and (5) we get

\[
P(z, w) \geq P_n(r(z, w)) \geq P_n(d(X)),
\]

where \( d(X) \) denotes the diameter of \( X \). However, \( \lim_{r \to \infty} P_n(r) = 0 \).

**Proposition 1.** Let \( (X, \omega_X) \) be a Kähler-Einstein manifold of constant Ricci curvature \(-1\) with volume element \( gdV \) and diameter \( d(X) \). Let \( \chi \) be a non-negative continuous function. Let

\[
(1 + \Box) \phi = \chi.
\]

Then

\[
\phi(z) \geq P_n(d(X)) \cdot \int_X \chi gdV
\]
for all $z \in X$.

Conversely let for all solutions of (8) an estimate $\phi(z) \geq P \cdot \int_X \chi g dV$ hold. for some number $P$. Then $P \leq \inf P(z, w)$ follows immediately.

We mention that symbolic integration of (5) with (6) yields an explicit estimate.

Let $X$ be a canonically polarized manifold of dimension $n$, equipped with a Kähler-Einstein metric $\omega_X$. In terms of local holomorphic coordinates $(z^1, \ldots, z^n)$ we write

$$\omega_X = \sqrt{-1} g_{\alpha\beta}(z) \, dz^\alpha \wedge d\bar{z}^\beta$$

so that the Kähler-Einstein equation reads

$$\omega_X = -\text{Ric}(\omega_X), \text{ i.e. } \omega_X = \sqrt{-1} \partial \bar{\partial} \log g(z),$$

where $g := \det g_{\alpha\beta}$. We consider $g$ as a hermitian metric on the anti-canonical bundle $K_X^{-1}$.

For any holomorphic family of compact, canonically polarized manifolds $f : X \rightarrow S$ of dimension $n$ with fibers $X_s$ for $s \in S$ the Kähler-Einstein forms $\omega_{X_s}$ depend differentiably on the parameter $s$. The resulting relative Kähler form will be denoted by

$$\omega_{X/S} = \sqrt{-1} g_{\alpha\beta}(z, s) \, dz^\alpha \wedge d\bar{z}^\beta.$$
Both the statement of the Theorem and the methods are valid for smooth, proper families of singular (even non-reduced) complex spaces (for the necessary theory cf. \[F-S\]).

It is sufficient to prove the theorem for base spaces of dimension one assuming \(S \subset \mathbb{C}\). (In order to treat singular base spaces, the claim can be reduced to the case where the base is a double point \((0, \mathbb{C}[s]/(s^2))\). The arguments below will still be meaningful and can be applied literally.)

We denote the Kodaira-Spencer map for the family \(f : \mathcal{X} \to S\) at a given point \(s_0 \in S\) by
\[
\rho_{s_0} : T_{s_0} \to H^1(X, T_X)
\]
where \(X = \mathcal{X}_{s_0}\). The family is called \textit{effectively parameterized} at \(s_0\), if \(\rho_{s_0}\) is injective. The Kodaira-Spencer map is induced as edge homomorphism by the short exact sequence
\[
0 \to T_{\mathcal{X}/S} \to T_X \to f^* T_S \to 0.
\]

If \(v \in T_{s_0} S\) is a tangent vector, say \(v = \frac{\partial}{\partial s}|_{s_0}\) and \(\frac{\partial}{\partial s} + b^\alpha \frac{\partial}{\partial z^\alpha}\) is any lift of class \(C^\infty\) to \(\mathcal{X}\) along \(X\), then
\[
\bar{\partial} \left( \frac{\partial}{\partial s} + b^\alpha (z) \frac{\partial}{\partial z^\alpha} \right) = \frac{\partial b^\alpha (z)}{\partial \bar{z}^\beta} \frac{\partial}{\partial z^\alpha} dz^\beta
\]
is a \(\bar{\partial}\)-closed form on \(X\), which represents \(\rho_{s_0}(\partial/\partial s)\). Observe that \(b^\alpha\) cannot be a tensor on \(X\), unless the family is infinitesimally trivial.

We will use the semi-colon notation as well as raising and lowering of indices for covariant derivatives with respect to the \(\mathbb{K}\)-\(\alpha\)\(\mathbb{E}\)\(\i\)\(t\)\(\i\)\(n\)\(e\)\(s\)\(\i\)\(n\)\(e\)\(s\) metrics on the fibers. The \(s\)-direction will be indicated by the index \(s\). In this sense the coefficients of \(\omega_X\) will be denoted by \(g_{s\bar{s}}, g_{a\bar{s}}, g_{a\bar{a}}\) etc.

Next, we define \textit{canonical lifts} of tangent vectors of \(S\) as differentiable vector fields on \(\mathcal{X}\) along the fibers of \(f\) in the sense of Siu [SIU1]. By definition these satisfy the property that the induced representative of the Kodaira-Spencer class is \textit{harmonic}.

Since the form \(\omega_X\) is positive, when restricted to fibers, \textit{horizontal lifts} of tangent vectors with respect to the pointwise sesquilinear form \(\langle -, - \rangle_{\omega_X}\) are well-defined (cf. also [SCH2]).

\textbf{Lemma 4.} The horizontal lift of \(\partial/\partial s\) equals
\[
v = \partial_s + a^\alpha_s \partial_\alpha,
\]
where
\[
a^\alpha_s = -g^{\beta\alpha} g_{s\bar{a}}.
\]

\textbf{Proposition 2.} The horizontal lift induces the harmonic representative of \(\rho_{s_0}(\partial/\partial s)\).
Proof. The Kodaira-Spencer form of the tangent vector \( \partial / \partial s \) is given by \( \partial v / \partial s = a^\alpha_{s\beta} \partial_\alpha dz^\beta \). We consider the tensor

\[
A_{s\beta}^\alpha := \left( a^\alpha_{s\beta} \right) |_{X_s}
\]
on \( X \). Then

\[
g^{\gamma\delta} A_{s\beta\gamma}^\alpha = -g^{\gamma\delta} g^\alpha_{s\beta\gamma} g^{s\delta\beta} = -g^{\gamma\delta} g^\alpha_{s\beta\gamma} = -g^{\gamma\delta} g^\alpha_{s\beta\gamma} \left( g^{s\delta\beta} - g^{s\tau} R_{s\beta\gamma}^\tau \right) = -g^{\delta\alpha} \left( (\partial \log g / \partial s)^\beta_{s\beta} + g_{s\tau} R_{\beta\delta}^\tau \right) = 0.
\]

\[\square\]

It follows immediately from the proposition that the harmonic Kodaira-Spencer forms induce symmetric tensors. This fact reflects the close relationship between the Kodaira-Spencer tensors and the Kähler-Einstein metrics.

Corollary 1. Let \( A_{s\beta\gamma} = g_{s\beta\gamma} A_{s\delta}^\alpha \). Then

\[
A_{s\beta\gamma} = A_{s\delta}^\alpha A_{s\beta}^\alpha
\]

Next, we introduce a global function \( \varphi(z, s) \), which is by definition the pointwise inner product of the canonical lift \( v \) of \( \partial / \partial s \) at \( s \in S \) with itself with respect to \( \omega_X \).

Definition 1.

\[
\varphi(z, s) := \langle \partial_s + a^\alpha_{s\beta} \partial_\alpha, \partial_s + a^\beta_{s\delta} \partial_\beta \rangle_{\omega_X}
\]

Since \( \omega_X \) is not known to be positive definite in all directions, \( \varphi \geq 0 \) is not known at this point.

Lemma 5.

\[
\varphi = g_{s\tau} - g_{s\beta} g_{s\beta} g^\tau_{s\alpha} g^{s\alpha}
\]

Proof. The proof follows from Lemma 4 and

\[
\varphi = g_{s\tau} + g_{s\beta} g^\beta_{s\tau} + a^\alpha_{s\beta} g_{s\tau} + a^\beta_{s\delta} g_{s\tau}
\]

\[\square\]

Denote by \( \omega_X^{n+1} \) the \( (n+1) \)-fold exterior product, divided by \( (n+1)! \) and by \( dV \) the Euclidean volume element in fiber direction. Then the global real function \( \varphi \) satisfies the following property:

Lemma 6.

\[
\omega_X^{n+1} = \varphi \cdot g \cdot dV \sqrt{-1} ds \wedge \overline{ds}.
\]
Proof. Compute the following \((n + 1) \times (n + 1)\)-determinant
\[
\det \begin{pmatrix} g_{\alpha \bar{\beta}} & g_{\alpha \bar{\gamma}} \\ g_{\beta \gamma} & g_{\beta \bar{\gamma}} \end{pmatrix},
\]
where \(\alpha, \beta = 1, \ldots, n\).

So far we are looking at local computations, which essentially only involve derivatives of certain tensors. The only global ingredient is the fact that we are given global solutions of the Kähler-Einstein equation.

The key quantity is the differentiable function \(\varphi\) on \(\mathcal{X}\). Restricted to any fiber it ties together the yet to be proven positivity of the hermitian metric on the relative canonical bundle and the canonical lift of tangent vectors, which is related to the harmonic Kodaira-Spencer forms.

We use the Laplacian operators \(\Box_{g,s}\) with non-negative eigenvalues on the fibers \(\mathcal{X}_s\) so that for a real valued function \(\chi\) the Laplacian equals \(\Box_{g,s} \chi = -g^{\alpha \bar{\beta}} \partial_{\bar{s}} \partial_{s} \chi_{\alpha \beta}\).

**Proposition 3.** The following elliptic equation holds fiberwise:
\[
(\Box_{g,s} + \text{id}) \varphi(z, s) = \|A_s(z, s)\|^2,
\]
where
\[
A_s = A_{s \bar{\alpha}} \partial_{\bar{s}} \partial_{s} \varphi(z, s).
\]
is the harmonic representative of the Kodaira-Spencer class \(\rho_s(\frac{\partial}{\partial s})\) as above.

**Proof.** The essence to prove an elliptic equation for \(\varphi\) involving tensors on the fibers is to eliminate the second order derivatives with respect to the base parameter. This is achieved by the left hand side of (14).

First,
\[
g^{\bar{\gamma} \bar{\delta}} g_{\bar{\sigma} \gamma} \partial_{s} \partial_{\bar{s}} g_{\bar{\sigma} \bar{\delta}} = \partial_{s} (g^{\bar{\gamma} \bar{\delta}} \partial_{s} g_{\bar{\sigma} \bar{\delta}}) - a_{s \bar{\gamma} \bar{\delta}} \partial_{s} g_{\bar{\sigma} \bar{\delta}}
\]
\[
= \partial_{s} \partial_{\bar{s}} \log g + a_{s \bar{\gamma} \bar{\delta}} a_{\bar{\sigma} \gamma \bar{\delta}}
\]
\[
= g_{s \bar{s}} + a_{s \bar{\gamma} \bar{\delta}} a_{\bar{\sigma} \gamma \bar{\delta}} g^{\bar{\gamma} \bar{\delta}}.
\]

Next
\[
(a_{s \bar{\gamma} \bar{\delta}} a_{\bar{\sigma} \gamma \bar{\delta}}) g^{\bar{\gamma} \bar{\delta}} = (a_{s \bar{\gamma} \bar{\delta}} a_{\bar{\sigma} \gamma \bar{\delta}} + A_{s \bar{\delta}} a_{\bar{\delta} \bar{\gamma} \bar{\sigma}} + a_{s \bar{\gamma} \bar{\delta}} a_{\bar{\sigma} \gamma \bar{\delta}} + a_{s \bar{\gamma} \bar{\delta}} A_{s \bar{\sigma} \gamma \bar{\delta}}) g^{\bar{\gamma} \bar{\delta}}.
\]
The last term vanishes because of the harmonicity of \(A_s\), and
\[
a_{s \bar{\gamma} \bar{\delta}} g^{\bar{\gamma} \bar{\delta}} = A_{s \bar{\delta}} g^{\bar{\gamma} \bar{\delta}} + a_{s \bar{\gamma} \bar{\delta}} R_{\gamma \bar{\delta} \gamma}^{\bar{\lambda}} g^{\bar{\gamma} \bar{\delta}}
\]
\[
= 0 - a_{s \bar{\gamma} \bar{\delta}} R_{\gamma \bar{\delta} \gamma}^{\bar{\lambda}}
\]
\[
= a_{s \bar{\gamma} \bar{\delta}}.
\]
Definition 2. The Weil-Petersson hermitian product on $T_s S$ is given by the $L^2$-inner product of harmonic Kodaira-Spencer forms:

\[
\| \frac{\partial}{\partial s} \|_{WP}^2 := \int_{X_s} A_{\alpha \beta}^\alpha A_{\gamma \delta}^\gamma g_{\alpha \delta} \bar{g}^{\gamma \beta} g dV = \int_{X_s} A_{\alpha \beta}^\alpha A_{\gamma \delta}^\gamma \bar{g}^{\gamma \beta} g dV
\]

If $\frac{\partial}{\partial s} \in T_s S$ are part of a basis, we denote by $G_{WP}^i(s)$ the inner product, and set

\[
\omega_{WP} := \sqrt{-1} G_{WP}^i ds^i \wedge ds^j
\]

Observe that the generalized Weil-Petersson form is equal to a fiber integral:

Proposition 4 (cf. [F-S]).

\[
\omega_{WP} = \int_{X/S} \omega_{X}^{n+1}.
\]

The proposition implies the Kähler property of $\omega_{WP}$ immediately. The proof follows from Lemma 6 and Proposition 3.

5. CURVATURE OF $R^{n-p} f_* \Omega^p_{X/S}(\mathcal{K}_{X/S}^{\otimes m})$ — STATEMENT OF THE THEOREM AND APPLICATIONS

5.1. Statement of the theorem. We consider an effectively parameterized family $X \rightarrow S$ of canonically polarized manifolds, equipped with Kähler-Einstein metrics of constant Ricci curvature $-1$. For any $m > 0$ the direct image sheaves $f_* \mathcal{K}_{X/S}^{\otimes (m+1)} = f_* \mathcal{O}_{X/S}(\mathcal{K}_{X/S}^{\otimes m})$ are locally free. For values of $p$ other than $n$ we assume local freeness of $R^{n-p} f_* \mathcal{O}_{X/S}(\mathcal{K}_{X/S}^{\otimes m})$.

The assumptions of Section 2.2 are satisfied by Kodaira-Nakano vanishing so that we can apply Lemma 2. If necessary, we replace $S$ by a (Stein) open subset, such that the direct image is actually free, and denote by $\{\psi^1, \ldots, \psi^r\} \subset R^{n-p} f_* \Omega^p_{X/S}(\mathcal{K}_{X/S}^{\otimes m})(S)$ a basis of the corresponding free $\mathcal{O}_S$-module, and at a given point $s \in S$ we denote by $\{(\partial/\partial s_i)|_s; i = 1, \ldots, M\}$ a basis of the complex tangent space $T_s S$ of $S$ over $\mathbb{C}$, where the $s_i$ are holomorphic coordinate functions of a minimal smooth ambient space $U \subset \mathbb{C}^M$.

Let $A_{\alpha \beta}^\alpha(z, s) \partial_\alpha dz^\beta$ be a harmonic Kodaira-Spencer form. Then for $s \in S$ the cup product together with the contraction defines

\[
A_{\alpha \beta}^\alpha(z, s) \partial_\alpha dz^\beta \cup_\omega A^{0, n-p}(X_s, \mathcal{O}_{X_s}^p(\mathcal{K}_{X_s}^{\otimes m})) \rightarrow A^{0, n-p+1}(X_s, \mathcal{O}_{X_s}^{p+1}(\mathcal{K}_{X_s}^{\otimes m}))
\]

\[
A_{\alpha \beta}^\alpha(z, s) \partial_\alpha dz^\beta \cup_\omega A^{0, n-p}(X_s, \mathcal{O}_{X_s}^p(\mathcal{K}_{X_s}^{\otimes m})) \rightarrow A^{0, n-p-1}(X_s, \mathcal{O}_{X_s}^{p-1}(\mathcal{K}_{X_s}^{\otimes m})).
\]
We will apply the above product to harmonic \((0, n - p)\)-forms. In general the result is not harmonic. We use the notation \(\tilde{\psi} := \overline{\psi}^T\) for sections \(\psi_k\) (and a notation of similar type for tensors on the fibers):

**Theorem 2.** The curvature tensor for \(R^{n-p}f_*\Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S}^{\otimes m})\) is given by

\[
R_{\tilde{\sigma}k}^{\ell} (s) = m \int_{\mathcal{X}_s} (\Box + 1)^{-1} (A_i \cdot A_T) \cdot (\psi^k \cdot \tilde{\psi}) g dV \\
+ m \int_{\mathcal{X}_s} (\Box + m)^{-1} (A_i \cup \psi^k) \cdot (A_T \cup \tilde{\psi}) g dV \\
+ m \int_{\mathcal{X}_s} (\Box - m)^{-1} (A_i \cup \psi^T) \cdot (A_T \cup \psi^k) g dV.
\]

The only contribution in \((19)\), which may be negative, originates from the harmonic parts in the third term. It equals

\[
- \int_{\mathcal{X}_s} H(A_i \cup \psi^T) \overline{H(A_j \cup \psi^k)} g dV.
\]

Concerning the third term, the theorem contains the fact that the positive eigenvalues of the Laplacian are larger than \(m\).

Now the pointwise estimate \((7)\) of the resolvent kernel (cf. also Proposition \((1)\)) translates into an estimate.

**Proposition 5.** Let \(f : \mathcal{X} \to S\) be a family of canonically polarized manifolds, and \(s \in S\). Let a tangent vector of \(S\) at \(s\) be given by a harmonic Kodaira-Spencer form \(A\) and let \(\psi\) be a harmonic \((p, n - p)\)-form on \(\mathcal{X}_s\) with values in the \(m\)-canonical bundle. Then

\[
R(A, \overline{A}, \psi, \overline{\psi}) \geq P_n(d(\mathcal{X}_s)) \cdot \|A\|^2 \cdot \|\psi\|^2 - \|H(A \cup \overline{\psi})\|^2.
\]

For \(p = n\) we obtain the following result.

**Corollary 2.** For \(f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes (m+1)}\) the curvature equals

\[
R_{\tilde{\sigma}k}^{\ell} (s) = m \int_{\mathcal{X}_s} (\Box + m)^{-1} (A_i \cup \psi^k) \cdot (A_T \cup \tilde{\psi}) g dV \\
+ m \int_{\mathcal{X}_s} (\Box + 1)^{-1} (A_i \cdot A_T) \cdot (\psi^k \cdot \tilde{\psi}) g dV.
\]

The first term in \((21)\) yields immediately Nakano semipositivity, since the operator \((\Box + m)^{-1}\) is positive on the respective tensors. In fact more can be shown for the curvature of the direct image of relative \(m\)-canonical bundles.

Let

\[
H^{\tilde{\sigma}k} = \int_{\mathcal{X}_s} \psi^k \cdot \tilde{\psi} g dV.
\]
Corollary 3. Let $s \in S$ be any point. Let $\xi^i_k \in \mathbb{C}$. Then

$$R_{\jmath k}(s)\xi^i_k \xi^j_l \geq m \cdot P_n(d(X_s)) \cdot G_{\jmath k} \cdot H_{\jmath k} \cdot \xi^i_k \xi^j_l.$$  

In particular the curvature is strictly Nakano-positive with the above estimate.

Next, we set $m = 1$ and take a dual basis $\{\nu_i\} \subset R^p f_* \Lambda^p T_{X/S}(S)$ of the $\{\psi^k\}$ and normal coordinates at a given point $s_0 \in S$. Observing that the role of conjugate and non-conjugate tensors is being switched, we compute the curvature as follows.

Theorem 3. The curvature of $R^p f_* \Lambda^p T_{X/S}$ equals

$$R_{\jmath k}(s) = -\int_{X_s} (\Box + 1)^{-1} (A_i \cdot A_j) \cdot (\nu_k \cdot \nu_l) gdV$$

$$-\int_{X_s} (\Box + 1)^{-1} (A_i \wedge \nu_l) \cdot (A_j \wedge \nu_k) gdV$$

$$-\int_{X_s} (\Box - 1)^{-1} (A_i \wedge \nu_k) \cdot (A_j \wedge \nu_l) gdV.$$

The only possible positive contribution arises from

$$\int_{X_s} H(A_i \wedge \nu_k)H(A_j \wedge \nu_l) gdV.$$  

We observe that for $n = 1$ the third term in (24) is not present and we have the formula for the classical Weil-Petersson metric on Teichmüller space: It is known from the results of Wolpert that the classical Weil-Petersson metric for families of Riemann surfaces of genus larger than one has negative curvature: According to [WO] the sectional curvature is negative, and the holomorphic sectional curvature is bounded from above by a negative constant. A stronger curvature property, which is related to strong rigidity, was shown in [SCH1]. The strongest result on curvature by Liu, Sun, and Yau now follows immediately from Corollary 2.

Corollary 4 ([L-S-Y]). The Weil-Petersson metric on the Teichmüller space of Riemann surfaces of genus $p > 1$ is dual Nakano negative.

Proof. Observe that for a universal family $f : \mathcal{X} \to S$ the classical Weil-Petersson metric on $R^1 f_* T_{\mathcal{X}/S}$ corresponds to the $L^2$ metric on its dual bundle $f_*(\mathcal{K}_{\mathcal{X}/S}^\otimes 2)$, which is Nakano positive according to Corollary 8. \qed
For \( p = 1 \) we obtain the curvature for the generalized Weil-Petersson metric from [SCH2], (cf. [SIU1]). Again we can estimate the curvature like in Proposition 5.

The following case is of particular interest.

**Proposition 6.** Let \( f : \mathcal{X} \to S \) be a family of canonically polarized manifolds and \( s \in S \). Let a tangent vectors of \( S \) at \( s \) be given by harmonic Kodaira-Spencer forms \( A, A_1, \ldots, A_p \) on \( \mathcal{X}_s \). Let \( R \) denote the curvature tensor for \( R^p f_* \Lambda^p T_{\mathcal{X}/S} \). Then we have in terms of the Weil-Petersson norms:

\[
R(A, \overline{A}, H(A_1 \wedge \ldots \wedge A_p), H(A_1 \wedge \ldots \wedge A_p)) \leq -P_n(d(X_s)) \cdot \|A\|_2 \cdot \|H(A_1 \wedge \ldots \wedge A_p)\|_2^2 + \|H(A \wedge A_1 \wedge \ldots \wedge A_p)\|_2^2.
\]

(25)

**Proof.** Since the \( A \) and \( A_i \) are \( \overline{\partial} \)-closed forms, we have \( H(A \wedge H(A_1 \wedge \ldots \wedge A_p)) = H(A \wedge A_1 \wedge \ldots \wedge A_p) \). \( \square \)

Next, we define higher Kodaira-Spencer maps defined on the symmetric powers of the tangent bundle of the base. For \( p > 0 \) we let the morphism

\[
\rho^p : S^p T_S \to R^p f_* \Lambda^p T_{\mathcal{X}/S}
\]

send a symmetric power

\[
\frac{\partial}{\partial s_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial s_{i_p}}
\]

to the class of

\[
A_{i_1} \wedge \ldots \wedge A_{i_p} := A_{i_1}^{\alpha_1} \partial_{\alpha_1} d\overline{z}_{i_1} \wedge \ldots \wedge A_{i_p}^{\alpha_p} \partial_{\alpha_p} d\overline{z}_{i_p}.
\]

**Definition 3.** Let the tangent vector \( \partial/\partial s \) correspond to the harmonic Kodaira-Spencer tensor \( A_s \). The generalized Weil-Petersson function of degree \( p \) on the tangent space is

\[
\|\partial/\partial s\|_{WP}^p := \|A_s\|_p := \|\sum_{i=p} A_s \wedge \ldots \wedge A_s\|_p^{1/p} = \left( \int_{\mathcal{X}_s} H(A_s \wedge \ldots \wedge A_s) \cdot H(A_s \wedge \ldots \wedge A_s) g dV \right)^{1/2p}.
\]

For the computation of the curvature, we assumed that the sheaves \( R^p \Lambda^p T_{\mathcal{X}/S} \) are locally free.

Observe that the coefficient \( P_n(d(\mathcal{X}_s)) \) in (6) remains bounded as long as the fibers are smooth independent of the local freeness of the direct image sheaves.
Given a family over a curve, the $p$-th Weil-Petersson function of tangent vectors defines a hermitian (pseudo)metric on the curve, which we denote by $G_p$.

**Lemma 7.** The curvature $K_{G_p}$ of $G_p$, at points with $G_p(s) \neq 0$ satisfies

\[
K_{G_p} \leq \frac{1}{p} \left( -\frac{1}{c_{p,n}} P_n(d(\mathcal{X}_s)) + \max_{A \neq 0} \{ \|A\|_{p+1}^2/\|A\|_p^{2p+2} \} \right) \quad \text{for} \quad p < n
\]

\[
K_{G_p} \leq -\frac{1}{p \cdot c_{p,n}} P_n(d(\mathcal{X}_s)) \quad \text{for} \quad p = n, \text{ or if } G_{p+1} \equiv 0
\]

for some $c_{p,n} > 0$. These estimates are uniform on any relatively compact subset of the moduli space.

**Proof.** Let $A^p$ be the harmonic projection of the $p$-fold exterior product of $A_s$. Then the curvature tensor for $R^p f_s A^p \mathcal{T}_{\mathcal{X}/C}$ satisfies

\[
R(\partial_s, \partial_s, A^p, \overline{A^p}) \geq -\frac{\partial^2}{\partial s^2} \log(G_p^p) \cdot (G_p^p) \cdot A^p \overline{A^p} =
-\frac{p}{\partial s} \frac{\partial^2}{\partial s^2} \log(G_p) \cdot \|A\|_p^{2p} = pG_p K_{G_p} \|A\|_p^{2p}.
\]

Here $R(\partial_s, \partial_s, \omega)$ is the curvature form applied to the tangent vectors $\partial/\partial s$ and $\partial/\partial s$ resp. With respect to $G_p$, we identify $G_p = \|\partial/\partial s\|_p^2 = \|A_s\|_p^2$ so that

\[
R(A_s, A_s, A^p, \overline{A^p}) \geq pK_{G_p} \|A\|_p^{2p+2}.
\]

Now the estimate of Proposition 6 implies

\[
K_{G_p} \leq \frac{1}{p} \left( -P_n(d(\mathcal{X}_s))\|A\|_p^2 \|A\|_p^{2p} + \|A\|_{p+1}^{2(p+1)} \right)/\|A\|_p^{2(p+1)}.
\]

The second term is not present for $p = n$.

Now the proof follows from Lemma 8 below. \hfill \Box

**Lemma 8.**

\[
\|A\|_p \leq c_{p,n} \|A\|_1.
\]

for some $c_{p,n} > 0$.

**Proof of Lemma 8.** Since $A_s$ is harmonic and $\|A_s \wedge \ldots \wedge A_s\|^2 \geq \|H(A_s \wedge \ldots \wedge A_s)\|^2$, it is sufficient to show the pointwise estimate (up to a constant that only depends upon the dimension and degree)

\[
\|A_s \cdot A_s\|^p(z) \gtrsim \|(A_s \wedge \ldots \wedge A_s) \cdot (A_s \wedge \ldots \wedge A_s)\|.(z).
\]
We set
\[ B_{\gamma} = A_{s\beta}(z)A_{s\gamma}(z) \]
and use the symmetry \( A_{s\gamma} \). It follows that
\[ \|(A_s \wedge \ldots \wedge A_s) \cdot (A_s \wedge \ldots \wedge A_s)\| (z) \]
equals the determinant type expression
\[ B(z) = \sum_{\sigma \in S \epsilon} B_{\alpha_1} \ldots \cdot B_{\alpha_p} \]
where the inner summations take place with respect to the indices \( \alpha_1, \ldots, \alpha_p \). At the given point \( z \) we may assume that \( B_{\gamma} \) is diagonal with non-negative entries \( \lambda_1, \ldots, \lambda_n \). Now it is easy to see that \( B(z) \) equals
\[ \sum_{1 \leq i_1 < \ldots < i_p \leq n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_p} \]
up to a numerical constant depending on \( p \) and \( n \). Again, up to a numerical constant this can be estimated from below by
\[ \left( \sum_{i=1}^{n} \lambda_i \right)^p = \|A_s \cdot A_s\|^p (z). \]

Lemma 9. For any family over a base space \( S \), which is mapped to a relatively compact subset of the moduli space,
\[ \max_{A \neq 0} \{\|A\|_{p+1}/\|A\|_p\} \]
is finite.

Proof. First we take a modification \( \bar{S} \to S \) such that the pull backs of the direct image sheaves \( R^p f_{*} \Lambda^p T_{X/S} \) modulo torsion are locally free. Then, by continuity the above quotients are bounded from above. Next, we restrict the original family to the image of the locus of torsion, i.e. to the support of the annihilator ideal, and repeat the process. \( \square \)

5.2. Hyperbolicity conjecture of Shafarevich. In [DE, 3.2] Demailly gives a proof of the Ahlfors lemma for hermitian metrics of negative curvature in the context of currents using an approximation argument. Our argument depends upon the following special case:

Proposition 7 (Demailly). Let \( \gamma = \gamma(s)\sqrt{-1}ds \wedge \overline{ds} \), \( \gamma(s) \geq 0 \) be given on an open disk \( \Delta_R = \{|s| < R\} \), where \( \log \gamma(s) \) is a subharmonic function such that \( \sqrt{-1}\partial \overline{\partial} (\log \gamma) \geq A \gamma \) in the sense of currents for
some $A > 0$. Let $\rho$ denote the Poincaré metric on $\Delta_R$. Then $\gamma \leq \rho/A$ holds.

Before we construct a Finsler metric of negative holomorphic curvature, we will treat families over curves directly.

Let $C$ be a compact smooth complex curve and $f : \mathcal{X} \to C$ a holomorphic family of smooth canonically polarized varieties. Let $0 \neq \partial/\partial s$ be a local coordinate vector field on $C$ and $A_s$ the corresponding family of harmonic Kodaira-Spencer forms. Then we set

$$p_0 := \max\{p; H(A_s \wedge \ldots \wedge A_s) \neq 0 \text{ on some open } U \subset C\}.$$ 

Observe that the zero set of $H(A_s \wedge \ldots \wedge A_s)$ is analytic.

We do not have to assume that the direct images $R^p f_* \Omega^p_{\mathcal{X}/C}$ are locally free. This is the case over the complement $C' = C \setminus \{c_1, \ldots, c_k\}$ of a finite set of points. Near points $c_j$ we can compute Kodaira-Spencer forms $B_s$, $s \in C$ in terms of a differential trivialization so that the $L^2$-norms of the $B_s$ are bounded near the critical points. Hence the norms of the harmonic representatives $A_s$ are bounded near $c_j$. The same holds for the wedge product of these. Because of the boundedness $\log G_{p_0}$ is subharmonic on all of $C$.

**Proposition 8.** Let $f : \mathcal{X} \to C$ a non-isotrivial holomorphic family of smooth canonically polarized varieties over a curve, and let $0 < p_0 \leq n$ be chosen as above. Then $\log G_{p_0}$ is subharmonic, and for the curvature

$$K_{G_{p_0}} \leq -\frac{1}{p \cdot c_{p,n}} P_n(d(\mathcal{X}_s))$$

holds so that the curvature current is negative.

As long as the fibers are smooth, the diameters $d(\mathcal{X}_s)$ are bounded from above. The proof of the estimate follows from Lemma 7 at points, where the direct image of order $p_0$ is locally free.

At points with singular fibers, $\log G_{p_0}$ need no longer be subharmonic, unless current of integration is added. An elementary (counter-)example in fiber dimension zero is the map from a hyperelliptic to a rational curve.

By Proposition 7 the genus of the base must be larger than one. This gives a short proof of the following version of Shafarevich’s hyperbolicity conjecture for canonically polarized varieties [B-V, KE-KO, KO-KO, KV1, KV2, M, V-Z1, V-Z2].

**Application.** If a compact curve $C$ parameterizes a non-isotrivial family of canonically polarized manifolds, its genus must be greater than one.
5.3. **Finsler metric on the moduli stack.** Different notions are common. We do not assume the triangle inequality/convexity. Such metrics are also called *pseudo-metrics* (cf. [KB]).

**Definition 4.** Let $Z$ be a reduced complex space and $TZ$ it Zariski tangent fiber bundle. An upper semi-continuous function

$$ F : TZ \to [0, \infty) $$

is called Finsler pseudo-metric (or pseudo-length function), if

$$ F(av) = |a|F(v) \text{ for all } a \in \mathbb{C}, v \in TZ. $$

The triangle inequality on the fibers not required for the definition of the "holomorphic" (or "sectional") curvature.

All metrics $G_p$ from Section 5.1 are (upper semi-continuous) Finsler pseudo-metrics.

A pseudometric $\gamma$ for a curve $C$ like in Proposition 7 and 8 may have isolated zeroes.

We will use the fact that the holomorphic curvature of a Finsler metric at a certain point $p$ in the direction of a tangent vector $v$ is the supremum of the curvatures of the pull-back of the given Finsler metric to a holomorphic disk through $p$ and tangent to $v$ (cf. [A-P]). (For a hermitian metric, the holomorphic curvature is known to be equal to the holomorphic sectional curvature.) In view of Demailly’s theorem, a Finsler metric may be defined in the above sense (as long as the fibers are smooth, which is always the case. Furthermore, any convex sum $G = \sum_j a_j G_j$, $a_j > 0$ is upper semi-continuous and has the property that $\log G$ restricted to a curve is subharmonic.

**Lemma 10** (cf. [SCH3] Lemma 3). Let $C$ be a complex curve and $G_j$ a collection of pseudo-metrics of bounded curvature, whose sum has no common zero. Then the curvatures $K$ satisfy the following equation.

$$ K_{\sum_{j=1}^k G_j} \leq \sum_{j=1}^k \frac{G_j^2}{(\sum_{i=1}^k G_i)^2} K_{G_j}. $$

Now with Lemma 10 and Lemma 7 we can construct convex sums of the metrics $G_p$ with negative holomorphic curvature. In this way we arrive at a (upper semi-continuous) Finsler metric rather than a pseudo-metric. The convex sum accounts both cases where some $G_p$ vanishes or not. The metric is primarily given on local universal families, but intrinsically given. It descends to the coarse moduli space in the orbifold sense.
Theorem 4. On any relatively compact subset of the moduli space of cano-
ically polarized manifolds there exists a Finsler orbifold metric, i.e. a Finsler
metric on the moduli stack, whose holomorphic curvature is bounded from above by a negative constant.

6. Computation of the curvature

We know from Lemma 2 that the metric tensor for $R^{n-p}f_*\Omega^p_{\mathcal{X}/S}(\mathcal{K}_{\mathcal{X}/S})$
on the base space $S$ is given in terms of an integral which involves harmonic representatives of certain cohomology classes and that these are the restrictions of certain $\bar{\partial}$-closed differential forms on the total space. We already saw that these give rise to fiber integrals. When we actually compute derivatives with respect to the base, we will apply Lie derivatives with respect to horizontal lifts of tangent vectors of the base. At this point we need to take into account that the exterior derivatives $\partial$ has to be taken with respect to the hermitian metric on the relative canonical line bundle. Here covariant derivatives with respect to the total space occur (at least in an implicit way). Since we are dealing with alternating forms we may use covariant derivatives with respect to the Kähler structure on the fibers, which is necessary to somewhat simplify the computations. Again, we will use the semi-colon notation for covariant derivatives and use a $|$-symbol for ordinary derivatives, if necessary. Greek indices are being used for fiber coordinates, Latin indices indicate the base direction. Dealing with alternating forms, for instance of degree $(p,q)$, extra coefficients of the form $1/p!q!$ are sometimes customary; these play a role, when the coefficients of an alternating form are turned into skew-symmetric tensors by taking the average. However, for the sake of a halfway simple notation, we follow the better part of the literature and leave these to the reader.

6.1. Setup. As above, we denote by $f : \mathcal{X} \to S$ a smooth family of
canonicaly polarized manifolds and we pick up the notation from Section 4. The fiber coordinates were denoted by $z^\alpha$ and the coordinates of the base by $s^i$. We set $\partial_i = \partial/\partial s^i$, $\partial_\alpha = \partial/\partial z^\alpha$.

Again we have horizontal lifts of tangent vectors and coordinate vec-
tor fields on the base

$$v_i = \partial_i + a_i^\alpha \partial_\alpha.$$ 

As above we have the corresponding harmonic representatives

$$A_i = A_{i\overline{\beta}}^\alpha \partial_\alpha dz^\beta$$

of the Kodaira-Spencer classes $\rho(\partial_i|_{s_0})$. 
For the computation of the curvature it is sufficient to treat the case where \( \dim S = 1 \). We set \( s = s_1 \) and \( v_s = v_1 \) etc. In this case we write \( s \) and \( \mathbf{s} \) for the indices 1 and \( \mathbf{1} \) so that

\[
v_s = \partial_s + a_s^\alpha \partial_\alpha
\]

etc.

Sections of \( R^{n-p} f_* \Omega^p_X(V^0) \) will be denoted by letters like \( \psi \).

\[
\psi|_s = \psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n} \, dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_p} \wedge d\mathbf{z}_p^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n}
\]

where \( A_p = (\alpha_1, \ldots, \alpha_p) \) and \( \mathbf{B}_{n-p} = (\beta_{p+1}, \ldots, \beta_n) \). The further component of \( \psi \) is

\[
\psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} = \psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} \, \nu_{\gamma_1, \ldots, \gamma_n} \, d\mathbf{z}_p^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n} \wedge d\mathbf{z}_1^{\gamma_1} \wedge \ldots \wedge d\mathbf{z}_n^{\gamma_n}.
\]

Now Lemma 2 implies

\[
(32) \quad \psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} = \sum_{j=p+1}^{n} (-1)^{n-j} \psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_j, \ldots, \beta_n, \nu} \, \nu_{\gamma_1, \ldots, \gamma_n} \wedge d\mathbf{z}_p^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n} \wedge d\mathbf{z}_1^{\gamma_1} \wedge \ldots \wedge d\mathbf{z}_n^{\gamma_n}.
\]

Since these are the coefficients of alternating forms, on the right-hand side, we may also take the covariant derivatives with respect to the given structure on the fibers

\[
\psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} = \psi_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} \, \nu_{\gamma_1, \ldots, \gamma_n} \, d\mathbf{z}_p^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n} \wedge d\mathbf{z}_1^{\gamma_1} \wedge \ldots \wedge d\mathbf{z}_n^{\gamma_n}.
\]

6.2. Cup-Product. We define the cup-product of a differential form with values in the relative holomorphic tangent bundle and an (line bundle valued) differential form now in terms of local coordinates.

**Definition 5.** Let

\[
\mu = \mu_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} \, d\mathbf{z}_p^{\alpha_1} \wedge \ldots \wedge d\mathbf{z}_n^{\alpha_p} \wedge d\mathbf{z}_1^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n},
\]

and

\[
\nu = \nu_{\gamma_1, \ldots, \gamma_a, \delta_{a+1}, \ldots, \delta_b} \, d\mathbf{z}_p^{\gamma_1} \wedge \ldots \wedge d\mathbf{z}_n^{\gamma_a} \wedge d\mathbf{z}_1^{\delta_{a+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\delta_b}
\]

Then

\[
(33) \quad \mu \cup \nu := \mu_{\alpha_1, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n, \nu} \, \nu_{\gamma_1, \ldots, \gamma_a, \delta_{a+1}, \ldots, \delta_b} \, d\mathbf{z}_p^{\alpha_1} \wedge \ldots \wedge d\mathbf{z}_n^{\alpha_p} \wedge d\mathbf{z}_1^{\beta_{p+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\beta_n} \wedge d\mathbf{z}_1^{\gamma_1} \wedge \ldots \wedge d\mathbf{z}_n^{\gamma_a} \wedge d\mathbf{z}_1^{\delta_{a+1}} \wedge \ldots \wedge d\mathbf{z}_n^{\delta_b}
\]
6.3. Lie derivatives. Let again the base be smooth, dim $S = 1$ with local coordinate $s$. Then the induced metric on $R^{n-p} f_* \Omega^p_X(S K_X)\otimes S \otimes (K_X)_{X/S}$ is given by \ref{22}, where the pointwise inner product equals

$$\psi^k \cdot \psi^l g \, dV = (\sqrt{-1})^n (-1)^{n(n-p)} \frac{1}{g^m} \psi^k \wedge \psi^l,$$

and where $1/g^m$ stands for the hermitian metric on the $m$-th canonical bundle on the fibers.

**Lemma 11.**

$$\frac{\partial}{\partial s} H^{\gamma k} = \int_{\chi_s} L_v(\psi^k \cdot \psi^l) g \, dV = \langle L_v \psi^k, \psi^l \rangle + \langle \psi^k, L_v \psi^l \rangle,$$

where $L_v$ denotes the Lie derivative with respect to the canonical lift $v$ of the coordinate vector field $\partial/\partial s$.

**Proof.** Taking the Lie derivative is not type-preserving. We need the $(1,1)$-component: $L_v(g_{\alpha \beta}) = \left[ \partial_s + a^{\alpha}_s \partial_{\alpha}, g_{\alpha \beta} \right]_{\alpha \beta} = g_{\alpha \beta} + a^{\gamma}_s g_{\alpha \beta ; \gamma} + a_{s,\alpha} g_{\beta \gamma} = -a_{s,\alpha} + a^{\gamma}_s g_{\gamma \beta} = 0.$ So $L_v(\det(g_{\alpha \beta})) = 0.$ \hfill \Box

(34) \quad L_v \psi = L_v \psi' + L_v \psi'',$

where $L_v \psi'$ is of type $(p, n-p)$ and $L_v \psi''$ is of type $(p-1, n-p+1)$. We have

$$L_v \psi' = \left[ \partial_s + a^{\alpha}_s \partial_{\alpha}, \psi_{A_p} \pi_{n-p} d z^{A_p} d z^{\pi_{n-q}} \right]_{(p, n-p)}$$

(35) \quad = (\psi, s + a^{\alpha}_s \psi_{\alpha} + \sum_{j=1}^{p} \pi_{n-p} d z^{A_p} d z^{\pi_{n-q}})_{(p-1, n-p+1)}$$

$$L_v \psi'' = \left[ \partial_s + a^{\alpha}_s \partial_{\alpha}, \psi_{A_p} \pi_{n-p} d z^{A_p} d z^{\pi_{n-q}} \right]_{(p-1, n-p+1)}$$

$$= \sum_{j=1}^{p} A^{\alpha}_p \psi_{\alpha, \pi_{n-p}} d z^{A_p} d z^{\pi_{n-q}}$$

(36) \quad d z^{\alpha_1} \wedge \ldots \wedge d z^{\beta_k} \wedge \ldots \wedge d z^{\alpha_p} \wedge d z^{\beta_{k+1}} \wedge \ldots \wedge d z^{\pi_n}$
We also note the values for the derivatives with respect to $\overline{v}$.

\[
L_{\overline{v}}\psi' = \left[ \partial_{\pi} + a^\overline{\pi}_{\beta} \partial_{\beta}, \psi_{A_p,\overline{\pi}_{n-p}} \ dz^A_p \ dz^{\overline{B}_{n-q}} \right]_{(p,n-p)}
\]

\[
(37) \quad = (\psi_{\pi} + a^\overline{\pi}_{\beta} \psi_{\overline{\beta}} + \sum_{j=1}^p a^\overline{\pi}_{\beta_j} \psi_{A_p,\overline{\pi}_{p+1},\ldots,\overline{\beta}_j}) \ dz^A_p \ dz^{\overline{B}_{n-p}}
\]

\[
L_{\overline{v}}\psi'' = \left[ \partial_{\pi} + a^\overline{\pi}_{\beta} \partial_{\beta}, \psi_{A_p,\overline{\pi}_{n-p}} \ dz^A_p \ dz^{\overline{B}_{n-q}} \right]_{(p+1,n-p-1)}
\]

\[
(38) \quad = \sum_{j=p+1}^n A_{\overline{\pi}_{p+j}} (A_{A_p,\ldots,}\overline{\beta}_{p+1},\ldots,\overline{\beta}_n) \ dz^{\alpha_1} \wedge \ldots \wedge d z^{\alpha_p} \wedge d z^{\overline{\beta}_1} \wedge \ldots \wedge d z^{\overline{\beta}_n}
\]

Lemma 12.

\[
(39) \quad (L_{\overline{v}}\psi^k)'' = A_p \cup \psi^k
\]

\[
(40) \quad (L_{\overline{v}}\psi^k)'' = (-1)^p A_p \cup \psi^k
\]

Proof of (39). By (36) we have

\[
L_{\overline{v}}\psi'' =
\]

\[
= \sum_{j=1}^p A_{\alpha_{p+j}} (A_{A_p,\ldots,}\overline{\beta}_{p+1},\ldots,\overline{\beta}_n) \ dz^{\alpha_1} \wedge \ldots \wedge d z^{\alpha_p} \wedge d z^{\overline{\beta}_1} \wedge \ldots \wedge d z^{\overline{\beta}_n}
\]

\[
= (-1)^{p-1} \sum_{j=1}^p A_{\alpha_{p+j}} (A_{A_p,\ldots,}\overline{\beta}_{p+1},\ldots,\overline{\beta}_n) \ dz^{\alpha_1} \wedge \ldots \wedge d z^{\alpha_{p-1}} \wedge d z^{\overline{\beta}_1} \wedge \ldots \wedge d z^{\overline{\beta}_n}.
\]

Proof of (40). The claim follows in a similar way from (38).

The situation is not quite symmetric because of Lemma 2, which implies that the contraction of the global $(0, n-p)$-form $\psi$ with values in $\Omega^p_{\overline{A}/S}(\overline{K}/S)$ is well-defined. Like in Definition 5 we have a cup-product on the total space (restricted to the fibers).

\[
\overline{v} \cup \psi = (\partial_{\pi} + a^\overline{\pi}_{\beta} \partial_{\beta}) \cup \psi
\]

\[
= \psi_{A_p,\overline{\pi}_{p+1},\ldots,\overline{\beta}_{n-1}} + a^\overline{\pi}_{\beta} \psi_{A_p,\overline{\pi}_{p+1},\ldots,\overline{\beta}_{n-1}}
\]

Lemma 13.

\[
(41) \quad L_{\overline{v}}\psi' = (-1)^p \overline{\pi} \partial (\overline{v} \cup \psi).
\]
Proof. The proof follows from the fact that, according to Lemma 2, \( \psi \) is given by a \( \overline{\partial} \)-closed \((0, n - p)\)-form on the total space \( X \) with values in a certain holomorphic vector bundle.

We will need that the forms \( \psi \) on the fibers are also harmonic with respect to \( \partial \) (which was defined as the connection of the line bundle \( k_X \otimes m \)). First, we note the following fact, which immediately follows from the fact that the curvature of \((k_X, g^{-1})\) equals \(-\omega_X\). We will need this fact for both the total space and the restriction to fibers.

**Lemma 14.**

\[
\sqrt{-1} [\overline{\partial}, \partial] = -mL_X,
\]

where \( L_X \) denotes the multiplication with \( \omega_X \).

Now:

**Lemma 15.** The following equation holds on \( A^{(p,q)}(k_X^m) \).

\[
\Box_{\partial} = \Box_{\overline{\partial}} + m \cdot (n - p - q) \cdot \text{id}.
\]

In particular, the harmonic forms \( \psi \in A^{(p,n-q)}(k_X) \) are also harmonic with respect to \( \partial \).

**Proof.** We use the formulas

\[
\sqrt{-1} \overline{\partial}^* = [\Lambda, \partial]
\quad \text{and} \quad
-\sqrt{-1} \partial^* = [\Lambda, \overline{\partial}],
\]

where \( \Lambda \) denotes the adjoint operator to \( L \). Then

\[
\Box_{\partial} - \Box_{\overline{\partial}} = [\Lambda, \sqrt{-1}(\overline{\partial} \partial + \partial \overline{\partial})] = [\Lambda, m \cdot \omega_X] = m \cdot (n - p - q) \cdot \text{id}.
\]

Now we compute the curvature in the following way. Because of (11)

\[
\langle \psi^k, L_{\pi}(\psi^\ell)' \rangle = 0
\]

holds for all \( s \in S \) so that by Lemma 11

\[
\frac{\partial}{\partial s} H_{X}^{7k} = \langle L_v \psi^k, \psi^\ell \rangle + \langle \psi^k, L_{\pi} \psi^\ell \rangle = \langle (L_v \psi^k)' , \psi^\ell \rangle + \langle \psi^k, (L_{\pi} \psi^\ell)' \rangle
\]

\[
= \langle (L_v \psi^k)' , \psi^\ell \rangle.
\]

Later in the computation we will use normal coordinates (of the second kind) at a given point \( s_0 \in S \). The condition \( (\partial/\partial s) H_{X}^{7k}|_{s_0} = 0 \) for all \( k, \ell \) means that for \( s = s_0 \) the harmonic projection

\[
H((L_v \psi^k)') = 0
\]

vanishes for all \( k \).
In order to compute the second order derivative of $H^k$ we begin with
\begin{equation}
\frac{\partial}{\partial s} H^k = \langle L_v \psi^k, \psi^f \rangle.
\end{equation}
which contains both $(L_v \psi^k)'$ and $(L_v \psi^k)''$. Now
\begin{align*}
\partial \partial_s \langle \psi^k, \psi^f \rangle &= \langle L \partial \partial_s \psi^k, \psi^f \rangle + \langle L_v \psi^k, L_v \psi^f \rangle \\
&= \langle L \tau \partial_s \psi^k, \psi^f \rangle + \langle L_v \psi^k, \psi^f \rangle + \langle L_v \psi^k, L_v \psi^f \rangle \\
&= \langle L \tau \partial_s \psi^k, \psi^f \rangle + \partial_s \langle L \tau \psi^k, \psi^f \rangle - \langle L \tau \psi^k, L \tau \psi^f \rangle + \langle L_v \psi^k, L_v \psi^f \rangle
\end{align*}
We just saw that $\langle L \tau \psi^k, \psi^f \rangle \equiv 0$. Hence for all $s \in S$
\begin{equation}
\partial \partial_s \langle \psi^k, \psi^f \rangle = \langle L \tau \partial_s \psi^k, \psi^f \rangle - \langle L \tau \psi^k, L \tau \psi^f \rangle + \langle L_v \psi^k, L_v \psi^f \rangle
\end{equation}
The fact that we are computing Lie-derivatives of $n$-forms (with values in some line bundle) implies that
\begin{align*}
\langle L_v \psi^k, L_v \psi^f \rangle &= \langle (L_v \psi^k)', (L_v \psi^f)' \rangle - \langle (L_v \psi^k)'', (L_v \psi^f)'' \rangle,
\end{align*}
and
\begin{align*}
\langle L \tau \psi^k, L \tau \psi^f \rangle &= \langle (L \tau \psi^k)', (L \tau \psi^f)' \rangle - \langle (L \tau \psi^k)'', (L \tau \psi^f)'' \rangle.
\end{align*}

**Lemma 16.** Restricted to the fibers $X_s$ the following equation holds for $L \tau v$ applied to $K^{\otimes m}_{X/S}$-valued functions and differential forms resp.
\begin{equation}
L \tau v = \left[ - \varphi_\alpha \partial_\alpha + \varphi_{\overline{\beta}} \partial_{\overline{\beta}}, \varphi \right] - m \cdot \varphi \cdot \text{id}
\end{equation}
Prove. We first compute the vector field $[\overline{v}, v]$ on the fibers:
\begin{align*}
[\overline{v}, v] &= \left[ \partial_{\overline{\alpha}} + a_{\overline{\sigma}} \partial_{\overline{\alpha}}, \partial_s + a_s \partial_\alpha \right] \\
&= \left( \partial_{\overline{\alpha}}(a_{\overline{\sigma}}) + a_{\overline{\sigma}} a_s \partial_{\overline{\alpha}} \right) \partial_\alpha - \left( \partial_s(a_{\overline{\sigma}}) + a_s a_{\overline{\sigma}} \partial_\alpha \right) \partial_{\overline{\beta}}.
\end{align*}
Now
\begin{align*}
\partial_\beta (a_s) &= -\partial_\beta (g^\beta_\alpha g_{s,\beta}) \\
&= g^\beta_\sigma g_{\sigma,\pi} g^\pi_\alpha g_{s,\beta} - g^\beta_\sigma g_{s,\beta,\pi} \\
&= g^\beta_\sigma a_{\sigma,\pi} g^\pi_\alpha a_{s,\beta} - g^\beta_\sigma g_{s,\beta,\pi}
\end{align*}
Now (13) implies that the coefficient of $\partial_\alpha$ is $-\varphi_\alpha$. In the same way the coefficient of $\partial_{\overline{\beta}}$ is computed.

Next, we compute the contribution of the connection on $K^{\otimes m}_{X/S}$ which we denote by $[\overline{v}, v]_{K^{\otimes m}_{X/S}}$. We use (42):
\begin{align*}
[\partial_{\overline{\alpha}} + a_{\overline{\sigma}} \partial_{\overline{\alpha}}, \partial_s + a_s \partial_\alpha]_{K^{\otimes m}_{X/S}}
&= -m \left( g_{s,\beta} + a_{\overline{\sigma}} g_{s,\beta} + a_\alpha g_{s,\beta} + a_{\overline{\sigma}} a_\alpha g_{s,\beta} \right) = -m \varphi.
\end{align*}
Lemma 17.

\begin{equation}
\langle L[v, v]^{\psi^k}, \psi^\ell \rangle = -m \langle \varphi \psi^k, \psi^\ell \rangle = -m \int_{X_s} (\Box + 1) (A_s \cdot A) \psi^k \psi^\ell g \, dV
\end{equation}

Proof. The \( \partial \)-closedness of the \( \psi^k \) can be read as

\[ \psi^k = \sum_{j=1}^{p} \psi_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p} \overline{B}_{n-p} ; \alpha_j . \]

Hence

\[ [\varphi^{\alpha} \partial_{\alpha}, \psi^k_{A_p \overline{B}_{n-p}}]' = \varphi^{\alpha} \psi_{\alpha} + \sum_{j=1}^{p} \varphi^{\alpha_j} \psi^k_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p \overline{B}_{n-p}} \]

\[ = \sum_{j=1}^{p} (\varphi^{\alpha_j} \psi^k_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p \overline{B}_{n-p}}) ; \alpha_j \]

\[ = \partial (\varphi^{\alpha} \partial_{\alpha} \cup \psi^k) . \]

Now

\[ \langle [\varphi^{\alpha} \partial_{\alpha}, \psi^k_{A_p \overline{B}_{n-p}}], \psi^\ell \rangle = \langle [\varphi^{\alpha} \partial_{\alpha}, \psi^k_{A_p \overline{B}_{n-p}}]', \psi^\ell \rangle \]

\[ = \langle \partial (\varphi^{\alpha} \partial_{\alpha} \cup \psi^k), \psi^\ell \rangle = \langle \varphi^{\alpha} \partial_{\alpha} \cup \psi^k, \partial^* \psi^\ell \rangle = 0. \]

In the same way we get

\[ \langle [\varphi^{\alpha} \partial_{\alpha}, \psi^k_{A_p \overline{B}_{n-p}}], \psi^\ell \rangle = 0, \]

and, according to Lemma [16], we are left with the desired term. \( \square \)

Proposition 9. In view of (17) and (18) we have

\begin{align}
(49) & \quad \overline{\partial} (L[v, v]^{\psi^k})' = \partial (A_s \cup \psi^k) \\
(50) & \quad \overline{\partial}' (L[v, v]^{\psi^k}) = 0 \\
(51) & \quad \partial^* (A_s \cup \psi^k) = 0 \\
(52) & \quad \overline{\partial}' (L[v, v]^{\psi^k})' = \partial^* (A_s \cup \psi^k) \\
(53) & \quad \overline{\partial} (L[v, v]^{\psi^k})' = 0 \\
(54) & \quad \overline{\partial}' (A_s \cup \psi^k) = 0
\end{align}

The proof of the above proposition is the technical part of this article and will be given at the end of the manuscript.

Proof of Theorem 2. Again, we may set \( i = j = s \) and use normal coordinates at a given point \( s_0 \in S \).

We continue with (46) and apply (18). Let \( G_{\partial} \) and \( G_{\overline{\partial}} \) denote the Green’s operators on the spaces of differentiable \( K_{\chi^*} \)-valued \((p, q)\)-forms
on the fibers with respect to $\Box_\partial$ and $\Box_{\overline{\partial}}$ resp. We know from Lemma 15 that for $p + q = n$ the Green’s operators $G_\partial$ and $G_{\overline{\partial}}$ coincide.

We compute $\langle (L_v \psi^k)' , L_v \psi^\ell \rangle$:

Since the harmonic projection

$H((L_v \psi^k)') = 0$ vanishes for $s = s_0$, we have

$$(L_v \psi^k)' = G_\partial \Box_\partial (L_v \psi^k)' = G_{\overline{\partial}} \overline{\partial} (L_v \psi^k)' = \overline{\partial} G_{\overline{\partial}} \partial (A_s \cup \psi^k)$$

by (50) and (49). The form $\overline{\partial} (L_v \psi^k)' = \partial (A_s \cup \psi^k)$ is of type $(p, n - p + 1)$ so that by Lemma 15 on this space of such forms $G_{\overline{\partial}} = (\Box_\partial + m)^{-1}$ holds.

Now

$$\langle (L_v \psi^k)', (L_v \psi^\ell)' \rangle = \langle \overline{\partial} G_{\overline{\partial}} \partial (A_s \cup \psi^k), (L_v \psi^\ell)' \rangle$$

$$= \langle G_{\overline{\partial}} \partial (A_s \cup \psi^k), \partial (A_s \cup \psi^\ell) \rangle = \langle (\Box_\partial + m)^{-1} \partial (A_s \cup \psi^k), \partial (A_s \cup \psi^\ell) \rangle$$

$$= \langle \partial^* (\Box_\partial + m)^{-1} \partial (A_s \cup \psi^k), A_s \cup \psi^\ell \rangle.$$  

Because of (51)

$$\langle (L_v \psi^k)', (L_v \psi^\ell)' \rangle = \langle (\Box_\partial + m)^{-1} \Box_\partial (A_s \cup \psi^k), A_s \cup \psi^\ell \rangle$$

$$= \langle A_s \cup \psi^k, A_s \cup \psi^\ell \rangle - m \langle (\Box_\partial + m)^{-1} (A_s \cup \psi^k), A_s \cup \psi^\ell \rangle.$$  

(For $(p - 1, n - p + 1)$-forms, we write $\Box = \Box_\partial = \Box_{\overline{\partial}}$) Altogether we have

$$\langle L_v \psi^k, L_v \psi^\ell \rangle|_{s_0} = -m \int_{X_s} (\Box_\partial + m)^{-1} (A_s \cup \psi^k) \cdot (A_\overline{s} \cup \psi^\ell) \, g \, dv.$$

Finally we need to compute $\langle L_\overline{\partial} \psi^k, L_\overline{\partial} \psi^\ell \rangle$.

By equation (40) we have that $\langle (L_\overline{\partial} \psi^k)', (L_\overline{\partial} \psi^\ell)' \rangle = \langle A_\overline{s} \cup \psi^k, A_\overline{s} \cup \psi^\ell \rangle$. Now Lemma 13 implies that the harmonic projections of the $(L_\overline{\partial} \psi^k)'$ vanish for all parameters $s$. So

$$\langle (L_\overline{\partial} \psi^k)', (L_\overline{\partial} \psi^\ell)' \rangle = \langle G_{\overline{\partial}} \Box_\partial (L_\overline{\partial} \psi^k)', (L_\overline{\partial} \psi^\ell)' \rangle$$

$$= \langle G_{\overline{\partial}} \Box_\partial \overline{\partial} (L_\overline{\partial} \psi^k)', (L_\overline{\partial} \psi^\ell)' \rangle = \langle G_{\overline{\partial}} \Box_\partial (L_\overline{\partial} \psi^k)', \overline{\partial} (L_\overline{\partial} \psi^\ell)' \rangle$$

$$= \langle G_{\overline{\partial}} \partial^* (A_\overline{s} \cup \psi^k), \partial^* (A_\overline{s} \cup \psi^\ell) \rangle.$$  

Now the $(p + 1, n - p)$-form $\overline{\partial} (L_\overline{\partial} \psi^k)' = \partial^* (A_\overline{s} \cup \psi^k)$ is orthogonal to both the spaces of $\Box_\partial$- and $\partial$-harmonic forms. On these, we have by Lemma 15

$\Box_{\overline{\partial}} = \Box_\partial - m \cdot \text{id}.$

We see that all eigenvalues of $\Box_\partial$ are larger or equal to $m$ for $(p, n - p - 1)$-forms.

**Claim.** Let $\sum \lambda_\nu \rho_\nu$ be the eigenfunction decomposition of $A_\overline{s} \cup \psi^k$. Then all $\lambda_\nu > m$ or $\lambda_0 = 0$. In particular $(\Box_\partial - m)^{-1} (A_\overline{s} \cup \psi^k)$ exists.
In order to verify the claim, we consider \( \partial^*(A \cup \psi^k) = \sum_{\nu} \partial^*(\rho_{\nu}) \)
with
\[
\Box_{\theta} \partial^*(\rho_{\nu}) = \lambda_{\nu} \partial^*(\rho_{\nu}) = \Box_{\theta} \partial^*(\rho_{\nu}) + m \cdot \partial^*(\rho_{\nu}).
\]
This fact implies that \( \sum_{\nu} \partial^*(\rho_{\nu}) \) is also the eigenfunction expansion
with respect to \( \Box_{\theta} \) and eigenvalues \( \lambda_{\nu} - m \geq 0 \) of \( \partial^*(A \cup \psi^k) = \Box_{\theta} (L_{\psi^k}) \).
The latter is orthogonal to the space of \( \Box_{\theta} \)-harmonic functions so that \( \lambda_{\nu} - m = 0 \) does not occur.
(The harmonic part of \( A \cup \psi^k \) may be present though.) This shows the claim.

Now
\[
G_{\theta} \partial^*(A \cup \psi^k) = (\Box_{\theta} - m)^{-1} \partial^*(A \cup \psi^k)
\]
so that (54) implies
\[
\langle (L_{\psi^k}), (L_{\psi^k}) \rangle = \langle (\Box_{\theta} - m)^{-1} \partial_{\theta} (A \cup \psi^k), A \cup \psi^k \rangle
\]
\[
= \langle A \cup \psi^k, A \cup \psi^k \rangle + m \cdot \langle (\Box_{\theta} - m)^{-1} (A \cup \psi^k), A \cup \psi^k \rangle.
\]
Now (40) yields the final equation (again with \( \Box_{\theta} = \Box_{\theta} = \Box \) for \( (p + 1, n - p - 1) \)-forms)
\[
(56) \quad \langle L_{\psi^k}, L_{\psi^k} \rangle = m \int_{X_s} (\Box - m)^{-1} (A \cup \psi^k) \cdot (A \cup \psi^k) g dV.
\]
The main theorem follows from (48), (55), (46), and (56).

**Proof of Proposition 9.** We verify (49): We will need various identities. For simplicity, we drop the superscript \( k \).
The tensors below are meant to be coefficients of alternating forms on the fibers, i.e. skew-symmetrized.

\[
(57) \quad \psi_{\sigma_{n+1}} = \psi_{\sigma_{n+1}} - m \cdot g_{\sigma_{n+1}} \psi = m \cdot a_{\sigma_{n+1}}
\]

\[
(58) \quad \psi_{\alpha \beta_{n+1}^{(p)}} = \psi_{\alpha \beta_{n+1}^{(p)}} - m \cdot g_{\alpha \beta_{n+1}^{(p)}} \psi
\]
\[
- \sum_{j=1}^p \psi_{\alpha_{j+1}, \sigma_{j+1}, \ldots, \sigma_p} B_{\sigma_{j+1}, \ldots, \sigma_p \beta_{n+1}^{(j)}} R_{\beta_{j+1}^{(j)}}^{\sigma_{j+1}} \alpha_{j+1} \beta_{n+1}^{(j)}
\]
\[
= -m \cdot a_{\alpha \beta_{n+1}^{(p)}} \psi - \sum_{j=1}^p a_{\alpha \sigma_{j+1}}^{(p)} \psi_{\alpha_{j+1}, \sigma_{j+1}, \ldots, \sigma_p} B_{\sigma_{j+1}, \ldots, \sigma_p \beta_{n+1}^{(j)}} R_{\beta_{j+1}^{(j)}}^{\sigma_{j+1}} \alpha_{j+1} \
(59) \quad a_{\alpha \beta_{n+1}^{(p)}} = A_{\alpha \beta_{n+1}^{(p)}} + a_{\sigma j} R_{\alpha \beta_{n+1}^{(p)}}^{\sigma j} 
\]
Now, starting from (35) we get, using (57), (58), and (59),

$$\overline{\partial} L_v \psi' = \left( \psi_{s;\gamma}^{n+1} + A^\alpha_{s;\gamma} \psi_\alpha + a^\alpha_{s;\gamma} \psi_\alpha \psi_{n+1} + \sum_{j=1}^{p} A^\alpha_{s;\alpha_j \gamma} \psi_\alpha \psi_{n+1} \right)\left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right)$$

$$+ \sum_{j=1}^{p} A^\alpha_{s;\alpha_j \gamma} \psi_\alpha \psi_{n+1} \left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right)$$

$$= \left( A^\alpha_{s;\gamma} \psi_\alpha + \sum_{j=1}^{p} A^\alpha_{s;\alpha_j \gamma} \psi_\alpha \psi_{n+1} \right)\left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right)$$

Because of the fiberwise $\partial$-closedness of $\psi$ this equals

$$\sum_{j=1}^{p} \left( A^\alpha_{s;\gamma} \psi_\alpha \psi_{n+1} \right)\left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right)$$

$$= (-1)^n \sum_{j=1}^{p} \left( A^\alpha_{s;\gamma} \psi_\alpha \psi_{n+1} \right)\left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right)$$

$$= \partial \left( (-1)^n A^\alpha_{s;\gamma} \psi_\alpha \psi_{n+1} \right)\left( d\overline{\partial}^{n+1} \wedge d\overline{\partial}^p \wedge d\overline{\partial}^{n-p} \right) = \partial \left( A_s \cup \psi \right).$$

This shows (49).

Next, we prove (50). We begin with (37). We first note

$$\partial_s (\Gamma^\sigma_{\alpha \gamma}) = -a^\sigma_{s;\alpha \gamma}$$

which follows in a straightforward way. Now this equation implies

$$\partial_s (\psi^g_{\gamma}) = \psi^g_{s;\gamma} - \sum_{j=1}^{p} a^\sigma_{s;\alpha_j \gamma} \psi_{s;\alpha_j \gamma}$$

so that (with $g^{\overline{n};\gamma} \psi^g_{\gamma} = 0$ and $\partial_s g^{\overline{n};\gamma} = g^{\overline{n};\alpha} a^\alpha_{s;\gamma}$)

$$g^{\overline{n};\gamma} \psi^g_{s;\gamma} = -\psi^g_{s;\gamma} g^{\overline{n};\gamma} a^\alpha_{s;\gamma} + \sum_{j=1}^{p} g^{\overline{n};\gamma} a^\alpha_{s;\alpha_j \gamma} \psi_{s;\alpha_j \gamma}$$

(60) follows. Next, since fiberwise $\psi$ is $\overline{\partial}^+$-closed,

$$g^{\overline{n};\gamma} (a^\alpha_{s;\gamma} \psi^g_{s;\gamma}) = g^{\overline{n};\gamma} a^\alpha_{s;\gamma} \psi^g_{s;\gamma},$$

and with the same argument

$$g^{\overline{n};\gamma} \left( \sum_{j=1}^{p} a^\alpha_{s;\alpha_j \gamma} \psi_{s;\alpha_j \gamma} \right) = g^{\overline{n};\gamma} \sum_{j=1}^{p} a^\alpha_{s;\alpha_j \gamma} \psi_{s;\alpha_j \gamma}$$

(62) follows. Now $\overline{\partial}^+ (L_v \psi') = 0$ follows from (60), (61), and (62).
We come to the $\partial^*$-closedness \((51)\) of $A_s \cup \psi$. We need to show that
\[
\left( A^{\alpha}_{s \beta_{n+1}} \psi_{\alpha, \alpha_2, \ldots, \alpha_p, \beta_{p+1}, \ldots, \beta_n} \right) \overline{\delta} \overline{\alpha}_p \nabla
\]
vanishes. Since $\partial^* \psi = 0$ the above quantity equals
\[
A^{\alpha}_{s \beta_{n+1}} \psi_{\alpha, \alpha_2, \ldots, \alpha_p, \beta_{n-1}} \overline{\delta} \overline{\alpha}_p.
\]
Because of the $\overline{\partial}$-closedness of $A_s$ this equals
\[
(A^{\alpha \alpha_p})_{s \beta_{n+1}} \psi_{\alpha, \alpha_2, \ldots, \alpha_p, \beta_{n-1}} \overline{\delta} \overline{\alpha}_p.
\]
However,
\[
A^{\alpha \alpha_p} = A^{\alpha_p \alpha}
\]
whereas $\psi$ is skew-symmetric so that also this contribution vanishes.

The proof of \((52)\), \((53)\), and \((54)\) is similar, we remark that \((53)\) follows from Lemma \([13]\). □

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CURVATURE OF $R^{n-p}f_*\Omega^p_{X/S}(\mathcal{K}_X^{\otimes m})$ AND APPLICATIONS

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