Tomography for amplitudes of hard exclusive processes

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Abstract

We discuss which part of information about hadron structure encoded in the Generalized Parton Distributions (GPDs) can be restored from the known amplitude of a hard exclusive process. The physics content of this partial image is analyzed. Among other things, we show that this partial image contains direct information about how the target hadron responds to the (string) quark-antiquark operator of arbitrary spin $J$. Explicit equations relating physics content of the partial image of GPDs directly to the data are derived. Also some new results concerning the dual parametrization of GPDs are presented.

Introduction

Measurements of hard exclusive processes can provide us with rich information about hadron structure encoded in generalized parton distributions (GPDs) [1] (for recent reviews of GPDs see Refs. [2, 3, 4]). Frequently one speaks about “hadron tomography”, “3D images of proton”, or “fempto-hologram”, etc. Indeed, if we would know GPD as a functions of all its variable we would determine, among many other things, the distributions of hadron’s constituents (quarks and gluons) in three dimensional space. The amplitudes of hard exclusive processes, which we consider as direct observables, are given by the convolution of GPDs with perturbative kernels. It means that measurements of the amplitudes provide us with some kind of sectional images of GPDs – the convolution integral “projects out” one of variables in GPDs. This is a typical problem of tomography which is usually solved with help of Radon transformation [5]. Recently the Radon transformation was applied in Ref. [6] in order to restore the double distributions from known GPDs.

In the present paper we address the question: What part of the hadron image provided by GPDs can be reconstructed if we know the amplitude of hard exclusive process? We give explicit formulae which relate the forward-like parton distributions of dual parametrization of GPDs [9] to the amplitude of hard exclusive processes. We shall see that at fixed renormalization scale it is impossible to restore the complete GPD image, however, part of the total image carries valuable information about hadron structure. Furthermore, studying of the scaling violation in hard exclusive processes would allow, in principle, to restore total GPD. We suggest a practical way to determine GPDs from data on hard exclusive processes.
We restrict ourselves to hard exclusive processes at the leading order (LO) of perturbative expansion as this order provides the bulk information about the hadron structure. The perturbative corrections can be included into our analysis. Recently similar problems, from other point of view, were addressed in Refs. [7, 8] including the NLO correction.

Basics of the dual parametrization of GPDs

For our analysis we employ dual parametrization of GPDs suggested in Ref. [9]. This parametrization is based on representation of parton distributions as an infinite series of t-channel exchanges [10]. The dual parametrization has been already used to describe the data on deeply virtual Compton scattering in Ref. [11]. In this section we remind the basics of the dual parametrization and present also some new results.

In the dual parametrization GPDs $H(x, \xi, t)$ is expressed in terms of set of functions $Q_{2\nu}(x, t) \ (\nu = 0, 1, 2, \ldots)$ as [9]:

$$
H(x, \xi, t) = \sum_{\nu=0}^{\infty} \left\{ \frac{\xi^{2\nu}}{2} \left[ H^{(\nu)}(x, \xi, t) - H^{(\nu)}(-x, \xi, t) \right] - \left(1 - \frac{x^2}{\xi^2}\right) \theta(\xi - |x|) \sum_{l=1 \text{ odd}}^{2\nu-3} C_{2\nu-l-2}^{3/2} \left(\frac{x}{\xi}\right) P_l \left(\frac{1}{\xi}\right) \int_0^1 dy \ y^{2\nu-l-2} Q_{2\nu}(y, t) \right\},
$$

where the function $H^{(\nu)}(x, \xi, t)$ is defined on the interval $-\xi \leq x \leq 1$ and it is given as the following integral transformation:

$$
H^{(\nu)}(x, \xi, t) = \theta(x > \xi) \frac{1}{\pi} \int_{y_0}^1 dy \ y \left[ \left(1 - y \frac{\partial}{\partial y}\right) Q_{2\nu}(y, t) \right] \int_{s_1}^{s_2} ds \frac{x_s^{1-2\nu}}{\sqrt{2x_s - x_s^2 - \xi^2}} + \theta(x < \xi) \frac{1}{\pi} \int_0^1 dy \ y \left[ \left(1 - y \frac{\partial}{\partial y}\right) Q_{2\nu}(y, t) \right] \int_{s_1}^{s_3} ds \frac{x_s^{1-2\nu}}{\sqrt{2x_s - x_s^2 - \xi^2}}.
$$

Here $x_s = 2 \frac{x - s \xi}{(1 - s^2)^{1/2}}$ and integration limits $s_1, s_2, s_3$ and $y_0$ are given by the following expressions:

$$
\begin{align*}
\frac{1}{y_0} & = \left[ 1 - \sqrt{1 - \xi^2} - \sqrt{2 \left(1 - xy\right) \left(1 - \sqrt{1 - \xi^2}\right) - \xi^2 \left(1 - y^2\right)} \right], \\
\frac{1}{y_0} & = \left[ 1 - \sqrt{1 - \xi^2} + \sqrt{2 \left(1 - xy\right) \left(1 - \sqrt{1 - \xi^2}\right) - \xi^2 \left(1 - y^2\right)} \right], \\
\frac{1}{y_0} & = \left[ 1 + \sqrt{1 - \xi^2} - \sqrt{2 \left(1 - xy\right) \left(1 + \sqrt{1 - \xi^2}\right) - \xi^2 \left(1 - y^2\right)} \right], \\
y_0 & = \frac{1}{\xi^2} \left[ x \left(1 - \sqrt{1 - \xi^2}\right) + \sqrt{\left(x^2 - \xi^2\right) \left(2 \left(1 - \sqrt{1 - \xi^2}\right) - \xi^2\right)} \right].
\end{align*}
$$

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The resulting GPD automatically satisfy polynomiality property, all sum rules and have correct scale dependence.

Let us remind the main properties of the forward-like functions $Q_{2\nu}(x, t)$ *:

- At the LO scale dependence of functions $Q_{2\nu}(x, t)$ is given by the standard DGLAP evolution equation, so that these functions behave as usual parton distributions under QCD evolution.

- The function $Q_0(x, t)$ at $t = 0$ is related to the forward distribution $q(x)$ as:

$$Q_0(x, t = 0) = \left[(q(x) + \bar{q}(x)) - \frac{x}{2} \int_x^1 \frac{dz}{z^2} (q(z) + \bar{q}(z))\right]. \quad (4)$$

The inverse transformation reads:

$$q(x) + \bar{q}(x) = Q_0(x) + \frac{\sqrt{x}}{2} \int_x^1 \frac{dy}{y^{3/2}} Q_0(y). \quad (5)$$

- The Mellin moments of functions $Q_{2\nu}(x, t)$ give the form factors of the twist-2 local operators with fixed conformal spin, see details in Ref. [9]. In particular, the $x$-moment of the lowest functions $Q_0(x, t)$ and $Q_2(x, t)$ gives the form factors of the quark energy-momentum tensor:

$$\int_0^1 dx \, x \, Q_0(x, t) = \frac{5}{6} M_2^Q(t), \quad \int_0^1 dx \, x \, Q_2(x, t) = d_1(t) + \frac{5}{12} M_2^Q(t). \quad (6)$$

Here $M_2^Q(t)$, $d_1(t)$ are form factors of the quark part of the energy momentum tensor in notations of Ref. [13]. $M_2^Q(0)$ is the fraction of the hadron momentum carried by quarks, $d_1(t)$ is the leading coefficient in the Gegenbauer expansion of the D-term [12] and this form factor can be related to the spatial distribution of strong forces inside the hadron [13]. In the case of spin-1/2 hadron the above sum rule contains one additional form factor $J^Q(t)$, with $J^Q(0)$-the angular momentum in the hadron carried by the quarks.

- The coefficients $d_n(t)$ of the Gegenbauer expansion of the D-term

$$D(z, t) = (1 - z^2) \sum_{n=1}^{\infty} d_n(t) C_n^{(3/2)}(z),$$

can be computed with help of the following generating function:

$$\sum_{n=1}^{\infty} d_n(t) \, \alpha^n = \frac{1}{\alpha} \int_0^1 \frac{dz}{z} \sum_{\nu=0}^{\infty} (\alpha \, z)^{2\nu} Q_{2\nu}(z, t) \left(\frac{1}{\sqrt{1 + \alpha^2 z^2}} - \delta_{\nu,0}\right). \quad (8)$$

*We restrict ourselves to the case the spinless hadron and singlet distributions. Generalization for other cases is trivial*
Now we give couple of new results concerning the dual parametrization. Using the expression (1) we can write the expansion of the GPD $H(x, \xi, t)$ around the point $\xi = 0$ with fixed $x$ ($x > 0$) to the order $\xi^2$ as follows:

$$H(x, \xi, t) \sim \frac{1}{2} Q_0(x, t) + \frac{\sqrt{x}}{4} \int_x^1 \frac{dy}{y^{1/2}} Q_0(y, t)$$

$$+ \frac{\xi^2}{8} \left[ -\frac{1-x^2}{x} \frac{\partial}{\partial x} Q_0(x, t) + \frac{1}{8} \int_x^1 \frac{dy}{y^2} Q_0(y, t) \left( 3 \sqrt{\frac{y}{x}} - \left( \frac{y}{x} \right)^{3/2} \right) \right]$$

$$+ \frac{3}{8} \int_x^1 \frac{dy}{y} Q_0(y, t) \left( \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right)$$

$$+ Q_2(x, t) + \frac{3}{8} \int_x^1 \frac{dy}{y} Q_2(y, t) \left( \frac{1}{2} \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} + 5 \left( \frac{y}{x} \right)^{3/2} \right) \right] + O(\xi^4). \quad (9)$$

It is very instructive expression, we see that the expansion of the GPD in small $\xi$ to the order $\xi^{2\nu}$ involves only finite number of functions $Q_{2\mu}(x, t)$ with $\mu \leq \nu$ (to the order $\xi^2$ these are only $Q_0$ and $Q_2$). If the expansion of GPD around $\xi = 0$ is known (for example if the GPD is computed in a model), than it can be used to determine the functions $Q_{2\nu}(x, t)$ performing the small $\xi$ expansion order by order. This justifies the naming the functions $Q_{2\nu}(x, t)$--forward-like parton distributions.

Another limiting case, when the Eq. (2) simplifies is the limit $\xi \to 1$. In this limit GPD has all properties of distribution amplitudes. This limit can be used to constrain functions $Q_{2\nu}(x, t)$ because sometimes one can derive the form of the GPD in this limit, for example pion GPDs are constrained by the soft pion theorems [10].

$$H(x, \xi = 1, t) = \frac{1}{2} (1 - x^2) \sum_{\nu = 0}^{\infty} \int_0^1 dy \frac{1}{y} \left[ \left( 1 - 2xy + y^2 \right)^{3/2} - \frac{1}{(1 + 2xy + y^2)^{3/2}} \right]$$

$$- \sum_{k=1, \text{odd}}^{2\nu-3} y^k C_k^{(3/2)}(x) \right] Q_{2\nu}(y, t). \quad (10)$$

Expanding the integrand in the above equation in variable $y$ we obtain the Gegenbauer series for $H(x, \xi = 1, t)$:

$$H(x, \xi = 1, t) = (1 - x^2) \sum_{k=1, \text{odd}}^{\infty} h_k(t) C_k^{(3/2)}(x), \quad (11)$$

with Gegenbauer coefficients computed as:

$$h_k(t) = \int_0^1 dx \ x^k \sum_{\nu = 0}^{k+1} Q_{2\nu}(x, t). \quad (12)$$

In particular:
\[ h_1(t) = \int_0^1 dx \ x \ (Q_0(x, t) + Q_2(x, t)) = \frac{5}{4} M_2^0(t) + d_1(t), \]  

(13)

i.e. this coefficient is fixed completely in terms of form factors of the energy-momentum tensor. This, actually, is not surprising as it is a consequence of the polynomiality condition.

Finally, we note that to ensure the finiteness of the D-term and existence of the Mellin moments of GPDs the small \( x \) behaviour of the function \( Q_{2\nu}(x, t) \sim \frac{1}{x^\lambda} \) should be with \( \lambda < 2\nu \) (for \( \nu \geq 1 \)) and \( \lambda < 2 \) for \( \nu = 0 \).

**Forward-like distributions in terms of amplitudes: the inversion problem**

The knowledge of GPD as a function of three variables is equivalent to the knowledge of set of forward like parton distributions \( Q_{2\nu}(x, t) \). In this section we attempt to express the forward-like parton distributions directly in terms of the amplitudes of hard exclusive processes.

The leading order amplitude of hard exclusive reactions is expressed in terms of the following elementary amplitude\(^\dagger\):

\[ A(\xi, t) = \int_0^1 dx \ H(x, \xi, t) \left[ \frac{1}{\xi - x - i0} - \frac{1}{\xi + x - i0} \right]. \]  

(14)

We see that the amplitude is given by the convolution integral in which dependence of GPDs on variable \( x \) is "integrated out". We shall see below, that mathematically from the equations (14) one can not completely restore\(^\ddagger\) the GPD \( H(x, \xi, t) \). So we are not able to perform "complete imaging" of the target hadron from the knowledge of the amplitude. The key question is: what part of the "complete image" can be restored from the known amplitude? What is the physics content of the restorable part of the complete image? Below we address this question in terms of dual parametrization.

We can express the amplitudes in terms of forward-like functions \( Q_{2\nu}(x, t) \) as following [9]:

\[
\begin{align*}
\text{Im } A(\xi, t) &= \int_{\frac{1}{\sqrt{1-\xi^2}}}^{\frac{1}{\sqrt{1-\xi^2}}} \frac{dx}{x} N(x, t) \left[ \frac{1}{\sqrt{\frac{2}{\xi} - x^2 - 1}} \right], \\
\text{Re } A(\xi, t) &= \int_0^{\frac{1}{\sqrt{1-\xi^2}}} \frac{dx}{x} N(x, t) \left[ \frac{1}{\sqrt{1 - \frac{2x}{\xi} + x^2}} + \frac{1}{\sqrt{1 + \frac{2x}{\xi} + x^2}} - \frac{2}{\sqrt{1 + x^2}} \right] \\
&\quad + \int_{\frac{1}{\sqrt{1-\xi^2}}}^{1} \frac{dx}{x} N(x, t) \left[ \frac{1}{\sqrt{1 + \frac{2x}{\xi} + x^2}} - \frac{2}{\sqrt{1 + x^2}} \right] + 2D(t). \quad (15)
\end{align*}
\]

\(^\dagger\)We restrict ourselves to the singlet (even signature) amplitudes, generalization for odd signature amplitudes is trivial.

\(^\ddagger\)We consider the inversion problem at a single value of the large photon virtuality.
Here we introduced the function:

\[ N(x, t) = \sum_{\nu=0}^{\infty} x^{2\nu} Q_{2\nu}(x, t) , \]  

(16)

and the D-form factor:

\[ D(t) = \sum_{n=1}^{\infty} d_n(t) = \frac{1}{2} \int_{-1}^{1} dz \frac{D(z, t)}{1-z} . \]  

(17)

Here \( D(z, t) \) is the D-term [12].

Now we clearly see that the knowledge of the LO amplitude is equivalent to the knowledge of the function \( N(x, t) \) and D-form factor \( D(t) \). Moreover the D-form factor can be computed in terms of \( N(x, t) \) and \( Q_0(x, t) \). Note that the latter function is to great extend is fixed by the forward parton distributions, see Eq. (4). Indeed, if we use Eq. (8) at \( \alpha = 1 \), we can write:

\[ D(t) = \int_{-1}^{1} \frac{dz}{z} Q_0(z, t) \left( \frac{1}{\sqrt{1+z^2}} - 1 \right) + \int_{0}^{1} \frac{dz}{z} [N(z, t) - Q_0(z, t)] \frac{1}{\sqrt{1+z^2}} \]  

(18)

Note that in this equation all integrals are convergent if the functions \( Q_{2\nu}(x, t) \) satisfy the small \( x \) behaviour discussed at the end of the second section. In Ref. [16] the following representation for the D-form factor in terms of GPD has been suggested:

\[ D(t) = \int_{1}^{1} \frac{dx}{x} [H(x, x, t) - H(x, 0, t)] . \]  

(19)

Unfortunately, this expression is divergent, as \( H(x, x) \) and \( H(x, 0) \) have different coefficients in front of leading small-\( x \) asymptotic, see Refs. [14, 9]. The reason for this divergency is that the above expression was obtained by small \( \xi \) expansion of GPDs, which explodes if \( x \sim \xi \).

Another remarkable physics feature of the function \( N(x, t) \) is that its Mellin moments are related to the contributions of states with fixed angular momentum in the \( t \)-channel:

\[ \int_{0}^{1} dx \ x^{t-1} N(x, t) = \frac{1}{2} \int_{-1}^{1} dz \frac{\Phi_{J}(z, t)}{1-z} , \]  

(20)

where \( \Phi_{J}(z, t) \) is the distribution amplitude corresponding to two quark exchange in the \( t \)-channel with fixed angular momentum \( J \). The quantity on RHS of Eq. (20) carries valuable information about the hadron structure – it tells how the target nucleon responses to the well defined quark-antiquark probe of arbitrary spin \( J \).

Now we turn to the inversion problem: how to obtain the function \( N(x, t) \) if we know the amplitude of a hard exclusive process. This problem is the central for the physics of hard exclusive processes. To solve the problem we start with the expression (15) for the imaginary part of the amplitude. It is useful to make the following substitution for the integration variable \( x \):

\[ \frac{1}{w} = \frac{1}{2} \left( x + \frac{1}{x} \right) . \]
This substitution correspond to famous Joukowski conformal map [15], which historically was used to understand some principles of aerofoil design. After this change of variables the expression for the imaginary part of the amplitude gets the form:

\[
\text{Im } A(\xi, t) = \int_{\xi}^{1} \frac{dw}{w} M(w, t) \frac{1}{\sqrt{\xi - 1}},
\]

where the function \( M(w, t) \) is related to the function \( N(x, t) \) as:

\[
M(w, t) = N\left(\frac{1 - \sqrt{1 - w^2}}{w}, t\right) \frac{w}{\sqrt{2(1 - w^2)\sqrt{1 - \sqrt{1 - w^2}}}}.
\]

Now we can easily invert Eq. (21), this can be done in many ways. One of them is to view the problem as the tomography problem and solve it with help of inverse Radon transformation [5]. The solution of the integral equation (21) is:

\[
M(w, t) = \frac{w}{\pi} \int_{w}^{1} \frac{d\xi}{\sqrt{\xi - w}} \left\{ \frac{1}{2} \text{Im } A(\xi, t) - \xi \frac{d}{d\xi} \text{Im } A(\xi, t) \right\}
\]

Performing trivial change of variables we arrive at the final result for the function \( N(x, t) \):

\[
N(x, t) = \frac{2}{\pi} \frac{x(1 - x^2)}{(1 + x^2)^{3/2}} \int_{\frac{2x}{1 + x^2}}^{1} \frac{d\xi}{\xi^{3/2}} \frac{1}{\sqrt{\xi - \frac{2x}{1 + x^2}}} \left\{ \frac{1}{2} \text{Im } A(\xi, t) - \xi \frac{d}{d\xi} \text{Im } A(\xi, t) \right\}
\]

This remarkable formula allows to restore the function \( N(x, t) \) from the measured imaginary part of the amplitude. Note that the inversion formula contains the amplitude only in the physical region. At \( \xi \to 1 \) the imaginary part of the amplitude should go to zero. Let us assume that \( \text{Im } A(\xi, t) \sim (1 - \xi)^{\beta} \) as \( \xi \to 1 \), than from Eq. (24) one can easily obtain that

\[
N(x, t) \sim \frac{1}{2^\beta \cdot \pi} \frac{\beta}{\beta - 1/2} (1 - x)^{2\beta}
\]

as \( x \to 1 \).

As to small \( x \) behaviour, the Regge like asymptotic of the imaginary part of the amplitude \( \text{Im } A(\xi, t) \sim 1/\xi^\alpha \), according to Eq. (24), corresponds to the following small \( x \) behaviour of \( N(x, t) \):

\[
N(x, t) \sim \frac{1}{2^\alpha} \frac{\Gamma(1 + \alpha)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \alpha\right)} \frac{1}{x^\alpha}.
\]

Note that the forward-like function \( Q_{2\nu}(x, t) \) enter the function \( N(x, t) \) with the weight \( x^{2\nu} \). Given the same small \( x \) behaviour of these functions, the contribution of higher functions to the small \( \xi \) (large energies) behaviour of the amplitude is suppressed by \( \xi^{2\nu} \). This resembles the contribution of daughter Regge trajectories of dual resonance models.
(DRMs) for hadron interactions. The relations of the dual parametrization of GPDs to DRMs will be discussed elsewhere.

Let us give also the expression for the Mellin moments of the function $N(x, t)$ which, as we discussed, are related to the quark exchanges in the $t$–channel with fixed angular momentum, in terms of the amplitude. The corresponding expression has the form:

$$
\int_0^1 dx x^{J-1} N(x, t) = \frac{1}{\pi} \int_0^{\pi} \frac{d\xi}{\xi^{3/2}} \left\{ \frac{1}{2} \text{Im} A(\xi, t) - \xi \frac{d}{d\xi} \text{Im} A(\xi, t) \right\} R_J(\xi),
$$

(25)

with the function $R_J(\xi)$ given by the following integral:

$$
R_J(\xi) = \int_0^\xi \frac{dw}{\sqrt{w}} \left( \frac{1 - \sqrt{1-w^2}}{w} \right)^{J+\frac{1}{2}} \frac{1}{\sqrt{\xi - w}}.
$$

(26)

Its small $\xi$ behaviour has the form:

$$
R_J(\xi) \sim \left( \frac{\xi}{2} \right)^{J+\frac{1}{2}} \frac{\Gamma(J+1)\Gamma(\frac{1}{2})}{\Gamma(J+\frac{3}{2})}.
$$

(27)

Finally, we can in principle compute the D-form factor with help of Eq. (18). For that calculation we need additional information – the function $Q_0(x, t)$. Fortunately this function can be fixed (up to $t$-dependence) with help of Eq. (4). We do not give the complete expression for the D-form factor, but rather the key integral in Eq. (18):

$$
\int_{\epsilon}^1 \frac{dx}{x} \frac{1}{\sqrt{1+x^2}} N(x, t) = \sqrt{2} \frac{1}{\pi} \int_{2\epsilon}^{1} \frac{d\xi}{\xi} \left\{ \frac{1}{2} \text{Im} A(\xi, t) - \xi \frac{d}{d\xi} \text{Im} A(\xi, t) \right\},
$$

(28)

where we introduced infinitesimally small regulating parameter $\epsilon$ to cut possible divergencies at small $x$. In principle, from the measurements of the imaginary part of the amplitude we can determine the function $N(x, t)$ and from measurements of the real part the D-form factor $D(t)$. These two measurements would allow to constrain considerably the function $Q_0(x, t)$ which gives directly the probability density of partons in 3D space [19].

To summarize this section: all above formulae give us the possibility access the wide set of hadrons properties directly in terms of amplitudes of the hard exclusive processes. Still we are not able to restore the complete GPD as for this we need to determine not only the function $N(x, t)$ but rather the whole set of functions $Q_{2\nu}(x, t)$. This can be achieved if we would know more general function:

$$
N_\alpha(x, t) = \sum_{\nu=0}^\infty (\alpha x)^{2\nu} Q_{2\nu}(x, t).
$$

(29)

In principle, this function can be determined if we would measure the logarithmic scaling violation for the amplitude. Probably, in practice, it is difficult problem and more practical way is to model the function $N_\alpha(x, t)$ using the non-perturbative information about hadron structure. We shall discuss possible ways for such modelling after we derive dispersion relations and crossing in the next section.
Dispersion relations and crossing

Up to now we made use of only the imaginary part of the amplitude, let us study the real part of the amplitude. For this we take the expression (24) for the \( N(x,t) \) in terms of \( \text{Im}A(\xi,t) \) and substitute it into Eq. (15). After some simple calculations we arrive to the following expression for real part of the amplitude:

\[
\text{Re}A(\xi,t) = 2D(t) + \frac{1}{\pi} \nu p \int_0^1 d\zeta \ \text{Im}A(\zeta,t) \left( \frac{1}{\xi - \zeta} - \frac{1}{\xi + \zeta} \right),
\]

(30)
in which we can immediately recognize the dispersion relation for the amplitude with one subtraction at non-physical point \( \xi = \infty \) (corresponding to \( \nu = (s - u)/4m = 0 \)). The D-form factor is the corresponding subtraction constant. This result was obtained recently in Refs. [16, 17] by independent methods. We see that the dual parametrization automatically ensures the dispersion relations for the amplitudes.

The very idea of the dual parametrization of GPDs in terms of \( t \)-channel exchanges was motivated by the crossing relations [10] between GPDs and generalized distribution amplitude [18]. The later enter the description of the hard exclusive processes in the cross channel, like \( \gamma^* + \gamma \to h + \bar{h} \). In the LO the amplitude of the cross process can be expressed in terms of the function \( N(x,t) \) analytically continued to time-like \( t \) (\( t > 0 \))§:

\[
\mathcal{A}^{\text{cross}}(\eta,t) = \int_0^1 \frac{dx}{x} N(x,t) \left[ \frac{1}{\sqrt{1 - 2x\eta + x^2}} + \frac{1}{\sqrt{1 + 2x\eta + x^2}} - \frac{2}{\sqrt{1 + x^2}} \right] + 2D(t).
\]

(31)

Here \( -\eta \) is directly related to the \( \cos(\theta_{cm}) \) — cosine of scattering angle in centre of mass system, see for details Refs. [18, 10]. Now substituting our inversion formula (24) into this expression we obtain, rather simple result:

\[
\mathcal{A}^{\text{cross}}(\eta,t) = \frac{2}{\pi} \int_0^{\mid\eta\mid} d\xi \frac{\xi}{1 - \xi^2} \ \text{Im}A \left( \frac{\xi}{\mid\eta\mid},t \right) + 2 \ D(t).
\]

(32)

Actually this equation is the consequence of the dispersion relation (30).

A way to model \( Q_{2\nu}(x,t) \)

As we mentioned above the complete knowledge of GPD is equivalent to the knowledge of function (29). In this section we consider possible ways to model this function. For simplicity we do not consider the \( t \)– dependence and that why we do not write the corresponding argument. Without loss of generality we can represent the function \( Q_{2\nu} \) as the Mellin convolution:

\[
Q_{2\nu}(x) = \int_x^1 \frac{dz}{z} \ Q_0(z) \ P_{\nu} \left( \frac{x}{z} \right),
\]

(33)

§Such continuation can be performed with help of dispersion relations in variable \( t \), see e.g. Ref. [10]
then the function $N_\alpha(x)$ can be written as the integral transform of the $Q_0(x)$:

$$N_\alpha(x) = \int_x^1 \frac{dz}{z} K\left(\alpha x, \frac{x}{z}\right) Q_0(z),$$  \hspace{1cm} (34)

where the integral kernel is:

$$K(x, y) = \sum_{\nu=0} x^{2\nu} P_\nu(y).$$  \hspace{1cm} (35)

The presentation of $Q_{2\nu}$ as the Mellin convolution (33) has an advantage as the functions $P_\nu(y)$ and hence the integral kernel (35) are scale independent at LO. It is little known about the kernel (35), its small $x$ expansion has the form:

$$K(x, y) \sim \delta(1 - y) + x^2 P_1(y) + \ldots.$$  \hspace{1cm} (36)

We make the following ansatz for the integral kernel $K(x, y)$:

$$K(x, y) = (1 + A(x))\delta(1 - y) + B(x).$$  \hspace{1cm} (37)

Surely, this ansatz is very simple, but still it gives enough freedom in modelling of the function $N_\alpha(x, t)$. Let us assume that apart from the knowledge of the amplitude we have an additional non-perturbative information. For instance, one possess the knowledge of the D-term. Using the generating function (8) we easily obtain its relation to functions $A(x)$ and $B(x)$ in our ansatz (37):

$$\sum_{n=1}^\infty \alpha^n d_n = \frac{1}{\alpha} \int_0^1 \frac{dx}{x} Q_0(x) \left\{ \frac{1 + A(\alpha x)}{\sqrt{1 + \alpha^2 x^2}} - 1 + \int_0^x \frac{dz}{z} \frac{B(\alpha z)}{\sqrt{1 + \alpha^2 z^2}} \right\}.$$

If the functions $A(x)$ and $B(x)$ are related to each other by the following equation:

$$(1 + x^2)(x A'(x) + B(x)) = x^2 (1 + A(x)),$$  \hspace{1cm} (39)

the corresponding D-term is zero. This relation can be used for modelling of the GPDs for which the D-term is identically zero, e.g. the combination $H(x, \xi) + E(x, \xi)$ of the nucleon GPDs. For general case we can use the following ansatz:

$$A(x) = C(x) + A^{(0)}(x), \quad B(x) = B^{(0)}(x),$$  \hspace{1cm} (40)

where $A^{(0)}(x)$ and $B^{(0)}(x)$ satisfy Eq. (39). The function $C(x)$ can be determined from known coefficients $d_n$ and known function $Q_0(x)$ through the following relation:

$$\sum_{n=1}^\infty \alpha^n d_n = \frac{1}{\alpha} \int_0^1 \frac{dx}{x} Q_0(x) \left( \frac{1 + C(\alpha x)}{\sqrt{1 + \alpha^2 x^2}} - 1 \right).$$  \hspace{1cm} (41)
If we look for the solution of Eq. (39) in class of polynomials of finite order $2N$, the resulting polynomials $A^{(0)}(x)$ and $B^{(0)}(x)$ depend on $N - 1$ free parameters. For example the solution in the space of the 4th order polynomials depends on one free parameter $a$ and has the form:

$$A^{(0)}(x) = ax^2 + (1 - a)x^4, \quad B^{(0)}(x) = (1 - 2a)x^2 - 3(1 - a)x^4.$$  \hfill (42)

The expression for the function $N_\alpha(x)$ in this case has the form:

$$N_\alpha^{(0)}(x) = [1 + a (\alpha x)^2 + (1 - a) (\alpha x)^4] Q_0(x) + [(1 - 2a) (\alpha x)^2 - 3(1 - a) (\alpha x)^4] \int_x^1 \frac{dz}{z} Q_0(z). \hfill (43)$$

Allowing solutions in the class of higher order polynomials we introduce new free parameters which can be fitted to presumably known function $N(x)$, the latter can be obtained from the data with help of Eq. (24).

**Conclusions**

In the framework of dual parametrization of GPDs we derive explicit equation (24) which expresses particular combination (16) of forward-like functions $Q_{2\nu}(x,t)$ in terms of the imaginary part of the process amplitude. The real part of the amplitude contains one additional constant – the D-form factor, which is actually the subtraction constant of the corresponding dispersion relation. We show that the D-form factor can be computed in terms of the function $N(x,t)$ and the forward-like function $Q_0(x,t)$ which is at $t = 0$ is related to forward parton distributions by Eq. (4). It means that the amplitudes of hard exclusive processes (at fixed hard scale) provide us only with partial image of total GPDs. This partial image is usefully encoded in the function $N(x,t)$ and D-form factor $D(t)$ [equivalently in $N(x,t)$ and forward-like function $Q_0(x,t)$]. The function $N(x,t)$ contains direct information about the two quark exchange amplitude in t-channel with fixed angular momentum. We showed that the $J$’s Mellin moments of $N(x,t)$ gives the response of the target hadron to the (string) quark-antiquark operator with spin $J$.

**Acknowledgements**

We are thankful to N. Kivel and D. Müller for many valuable discussions. The work is supported by the Sofja Kovalevskaja Programme of the Alexander von Humboldt Foundation, the Federal Ministry of Education and Research and the Programme for Investment in the Future of German Government.

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