Hölder stability of quantitative photoacoustic tomography based on partial data

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Abstract

We consider the reconstruction of the diffusion and absorption coefficients of the diffusion equation from the internal information of the solution obtained from the first step of the inverse photoacoustic tomography (PAT). In practice, the internal information is only partially provided near the boundary due to the high absorption property of the medium and the limitation of the equipment. Our main contribution is to prove a Hölder stability of the inverse problem in a subregion where the internal information is reliably provided based on the stability estimation of a Cauchy problem satisfied by the diffusion coefficient. The exponent of the Hölder stability converges to a positive constant independent of the subregion as the subregion contracts towards the boundary. Numerical experiments demonstrates that it is possible to locally reconstruct the diffusion and absorption coefficients for smooth and even discontinuous media.

Keywords: Photoacoustic tomography, Hölder stability, Cauchy problem

1 Introduction

PAT is a hybrid medical imaging technique which combines the high contrast of optical parameters with the high resolution of ultrasonic waves [33, 6, 17, 26, 22]. In PAT, near infra-red (NIR) photons are sent into the biological tissue which is heated up due to the absorption of the energy. The heating then results in the expansion of the tissue which generates a pressure field. The measurement of the pressure field on the boundary is used to reconstruct the optical properties of the tissue.

The inverse problem of PAT can be decomposed into two steps. The first step is to reconstruct the absorbed radiation map $H(x)$ from the measurement of ultrasonic waves on the boundary [17, 1, 20, 23, 19, 31, 28]. The second step is to reconstruct the diffusion coefficient $D(x)$ and the absorption coefficient $\mu(x)$ through the internal data $H(x)$ obtained in the first step [16, 9, 24, 27, 29, 5, 4, 7, 2]. Let us consider the Dirichlet problem in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -\text{div}(D(x)\nabla u(x)) + \mu(x)u(x) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

We need to reconstruct $D(x)$ and $\mu(x)$ in (1.1) from the knowledge of the coefficients $D(x)$ and $\mu(x)$ on the boundary, the boundary condition $g(x)$, and the internal data $H(x) = \Gamma(x)\mu(x)u(x)$, where $\Gamma(x)$ is the coupling coefficient quantifying the amount of ultrasound generated by photons. It has been proved in [8] that it is impossible to reconstruct $(\Gamma, D, \mu)$ at the same time no matter how many sets of internal data are used for a fixed frequency. In this paper, we focus on the second step, and we further assume that $\Gamma \equiv 1$.

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PAT provides in theory images of optical contrasts and ultrasound resolution. However, in practice it has been observed in various experiments that the imaging depth, i.e., the maximal depth of the medium at which structures can be resolved at expected resolution, of PAT is still fairly limited, usually on the order of millimeters. This is mainly due to the fact that optical waves are significantly attenuated by absorption and scattering. In fact the generated optical signal decays very fast in the depth direction. This is indeed a well-known faced issue in optical tomography [33]. Recently in [30], assuming that the medium is layered, the authors derived a stability estimate showing that the reconstruction of the optical coefficients is stable in the region close to the optical illumination source and deteriorate exponentially far away. Due to the high absorption property of the tissue, the limitation of the equipment, etc., the boundary source \( g(x) \) is in practice confined near the impact zone of the near infra-red photons, and it is impossible to illuminate the whole tissue or to take the measurement on the whole boundary (see [11] and references therein). Therefore the data \( H(x) \) is only reliably provided near the boundary of measurement [12, 15]. To our best knowledge, the stability analysis based on partial data of \( H(x) \) has not been addressed, which is the motivation of this paper.

In this paper we first derive a Cauchy problem satisfied by \( \sqrt{D} \) whose coefficient and source term depend locally on \( H(x) \) in Section 2. We prove a Hölder stability of the Cauchy problem in a subregion near the boundary of measurement in Section 2.1, which results in a Hölder stability estimation for the reconstruction of \( D \) and \( \mu \) in Section 2.2. Actually there already exists a Hölder stability estimation for the Cauchy problem inside a subregion away from the boundary; see for example [21, 3, 13, 10]. The main drawback of the existing stability estimation is that the constant inside the upper bound tends to infinity while the distance between the subregion and the boundary goes to zero. We modify their method such that the constant becomes independent of the subregion. We also propose a choice of the exponent of the Hölder stability estimation which increasingly converges to a strictly positive constant independent of the subregion as the subregion contracts towards the boundary. That is, we improve the existing theory to handle a subregion including the boundary of measurement and prove that the stability increases as the subregion becomes smaller. The obtained stability results show that the resolution of PAT is better near the impact zone of the optical illumination sources, and deteriorates far away. At last several numerical experiments on smooth, discrete and realistic media are presented in Section 3. Our algorithm is able to reconstruct all the inhomogeneity accurately.

# 2 Local stability estimation

We consider the problem of reconstructing \( D(x) \) and \( \mu(x) \) in (1.1) from a set of internal data \( H_j(x) = \mu(x)u_j(x) \), \( j = 1, 2, \ldots, n+1 \), where \( u_j(x) \) is the solution to (1.1) corresponding to the boundary value \( g(x) = g_j(x) \). Assume that \( u_1(x) \) does not vanish inside \( \Omega \), then it is easy to verify that \( \frac{u_2}{u_1}, \ldots, \frac{u_{n+1}}{u_1} \) satisfy

\[
\begin{align*}
\text{div}(\sigma \frac{u_2}{u_1}) &= 0, \\
\vdots & \\
\text{div}(\sigma \frac{u_{n+1}}{u_1}) &= 0,
\end{align*}
\]

where \( \sigma = Du_1^2 \). Otherwise, if \( \sigma \) is smooth enough, we have

\[
M \cdot \nabla \ln \sigma = N, \quad M = \begin{pmatrix}
\nabla \frac{u_2}{u_1}^T \\
\vdots \\
\nabla \frac{u_{n+1}}{u_1}^T
\end{pmatrix}, \quad N = \begin{pmatrix}
\Delta \frac{u_2}{u_1} \\
\vdots \\
\Delta \frac{u_{n+1}}{u_1}
\end{pmatrix},
\]

where \((\cdot)^T\) denotes the transpose of a vector or a matrix. Since \( \frac{u_2}{u_1} = \frac{H_2}{H_1}, \ldots, \frac{u_{n+1}}{u_1} = \frac{H_{n+1}}{H_1} \), we are able to reconstruct \( \nabla \ln \sigma(x) \) locally by solving the linear system (2.1) if the matrix generated by the internal data is nonsingular at \( x \). In fact many internal measurements can be collected in a very short time, and considering \( M \) as an invertible matrix is indeed a realistic assumption. Notice that in theory, it is always possible to reconstruct \( \sigma(x) \) by solving only one linear steady state transport equation [11, 32]. However, in practice the measurements are noisy, and the transport speed can posses critical points with large multiplicity values which may generate severe instabilities in the inversion.

Here we set up a threshold based on the estimation of the noise level and we formulate the linear system (2.1) in the region where \( H_1(x) \) is larger than the threshold. Such region should be near the set where \( g_1(x) \) is large.
(the impact zone) which mathematically is a consequence of the maximum principle and Harnack’s inequality [18]. Indeed, we shall provide sufficient theoretical conditions that are at the same time consistent with experimental observations, to guarantee the existence of a subregion in which $M$ is nonsingular in Theorem 2.1.

**Theorem 2.1.** Let $\Omega$ be a $C^{2,1}$ domain, $x^* \in \partial \Omega$ and $(\gamma(x^*), \tau_1(x^*), \ldots, \tau_{n-1}(x^*))$ be the curvilinear coordinates at $x^*$. Let $g_1, \ldots, g_{n+1} \in C^{2,1}(\partial \Omega)$ be the boundary illuminations satisfying

- $\eta^{-1} < g_i(x)$ for all $x \in \partial \Omega$,
- $\|g_i\|_{C^{2,1}(\partial \Omega)} \leq \eta, j = 1, \ldots, n+1$.

Denote $h_j = g_{j+1}/g_1, j = 1, \ldots, n$. We further assume

- $h_1(x) < h_1(x^*)$ for all $x \in \partial \Omega \setminus \{x^*\}$,
- $\det(\nabla h_2(x^*), \ldots, \nabla h_n(x^*)) > \varepsilon$,

where $\varepsilon > 0$ is a fixed constant and $\nabla h(x^*) = (\nabla h(x^*) \cdot \tau_1(x^*), \ldots, \nabla h(x^*) \cdot \tau_{n-1}(x^*))^T$. Consider the set of coefficient

$$\mathcal{D} = \{ (D, \mu) | D \in C^2(\overline{\Omega}), \mu \in C^1(\overline{\Omega}), D(x) \geq K^{-1}, \mu(x) \geq K^{-1}, \|D\|_{C^2(\overline{\Omega})} \leq K, \|\mu\|_{C^1(\overline{\Omega})} \leq K \}$$

with a constant $K > 1$. Then for $u_j > \eta^{-1}, j = 1, \ldots n+1, in \Omega$, there exist constants $r_0 = r_0(\Omega, n, \mathcal{D}, \delta, \eta, \varepsilon) > 0$ and $C = C(\Omega, n, \mathcal{D}, \delta, \eta, \varepsilon) \geq 1$ such that $v_j = \frac{u_j}{u_1}, j = 1, \ldots, n$, satisfy

$$\det(\nabla v_1(x), \ldots, \nabla v_n(x)) \geq C^{-1}, \quad \|\nabla v_1(x), \ldots, \nabla v_n(x)\|^{-1}_F \leq C,$$

for all $x \in B_{r_0}(x^*) \cap \Omega$, where $\|\cdot\|_F$ denotes the Frobenius norm of the matrix.

**Proof.** We deduce from classical elliptic regularity that $u_j \in C^{2,1}(\overline{\Omega})[18, Theorem 6.14]$. The maximum principle implies that the minimum of $u_j(x)$ is achieved on $\partial \Omega$. That is $u_1 \geq \min_{\partial \Omega} g_1 > \eta^{-1}$ in $\Omega$. Since $v_1$ satisfies

$$\begin{cases} \text{div}(Dv_1^2 \nabla v_1) = 0 & \text{in } \Omega, \\ v_1 = h_1 & \text{on } \partial \Omega, \end{cases}$$

we again use the maximum principle to obtain $\max_{\Omega} v_1(x) = h_1(x^*)$ [18, 25]. On the other hand Hopf-Oleinik Lemma implies that $\nabla v_1(x^*) > 0$. Actually, one can show by contradiction and compactness arguments that there exists a constant $c_0 = c_0(\Omega, n, \mathcal{D}, \delta, \eta) > 0$ such that $\nabla v_1(x^*) \geq c_0$. Since $h_1(x)$ reaches its maximum at $x^*$, we have $\nabla v_1 h_1(x^*) = 0$, and therefore

$$f(x^*) = \det(\nabla v_1(x^*), \ldots, \nabla v_n(x^*)) = \det \begin{pmatrix} \nabla v_1(x^*) & \ldots & \nabla v_n(x^*) \\ \nabla v_1 h_1(x^*) & \ldots & \nabla v_n h_n(x^*) \end{pmatrix} = \nabla v_1 h_1(x^*) \det(\nabla v_2 h_2(x^*), \ldots, \nabla v_n h_n(x^*)) > c_0 \varepsilon.$$

Since $v \in C^{2,1}(\overline{\Omega})$, we have $f(x) \in C^{1,1}(\overline{\Omega})$. The strict positivity and differentiability of $f(x^*)$ imply the existence of the region $B_{r_0}(x^*) \cap \Omega$ where the inequality 2.2 is fulfilled.

**Note 1.** In Theorem (2.1), to guarantee the non-singularity of $M$ locally, we only impose conditions on $g_j$ and the coefficients. The subregion might be very small since we use the continuity and the non-singularity of $M$ on one point that belongs to the boundary. In practice, we setup a threshold and compare the condition number of $M$ with it to detect the non-singular region inside which we are able to do the reconstruction.
Substituting \( u_1 = \frac{v}{\sqrt{D}} \) into (1.1) results an equation for \( \sqrt{D} \)

\[
-\Delta \sqrt{D} + \frac{\Delta v}{\sqrt{\sigma}} \sqrt{D} = \frac{H_1}{\sqrt{\sigma}} \tag{2.4}
\]

To be able to handle this problem theoretically, we assume that we know \( \sqrt{D} \) and \( \partial_n \sqrt{D} \) on a part of the boundary. That is, we formulate a Cauchy problem for \( v/\sqrt{D} \). Numerically, we propose a much easier method. We complete the missing region with background value or averaging of existing value and solve (2.4) once to reconstruct \( \sqrt{D} \). This simple idea works very well for our numerical experiments and the results are very accurate with small relative errors; see Section 3 for more details.

2.1 Hölder stability of the Cauchy problem

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega \). Consider the operator \( L = -\text{div}(a \nabla v) + b \cdot v \) and the Cauchy problem

\[
\begin{align*}
Lv &= f \quad \text{in } \Omega, \\
v &= g \quad \text{on } \Gamma, \\
a \partial_n v &= h \quad \text{on } \Gamma,
\end{align*}
\]

where \( \Gamma \) is a subset of \( \partial \Omega \). We assume that there exists \( K > 0 \) so that

\[
a(x), b(x) > K^{-1} \quad \text{for all } x \in \Omega, \quad \text{and } \|a\|_{L^1(\Omega)}, \|b\|_{L^1(\Omega)} < K .
\]

Pick \( \psi \in C^2(\overline{\Omega}) \) without critical points in \( \Omega \) and let \( \varphi = e^{4\psi} \). Let us recall the Carleman estimate for elliptic operators [13, 14].

Proposition 2.1 (Carleman inequality). There exist three strictly positive constants \( C, \lambda_0 \) and \( \tau_0 \), which depend only on \( \psi, \Omega, \) and \( K \), so that

\[
C \int_{\Omega} \left( \lambda^3 \tau^3 \varphi^3 v^2 + \lambda^2 \tau \varphi |\nabla v|^2 \right) e^{2\tau \varphi} d x \leq \int_{\Omega} (Lv)^2 e^{2\tau \varphi} d x + \int_{\partial \Omega} \left( \lambda^3 \tau^3 \varphi^3 v^2 + \lambda \tau \varphi |\nabla v|^2 \right) e^{2\tau \varphi} d \sigma \tag{2.6}
\]

for all \( v \in H^2(\Omega), \lambda \geq \lambda_0 \) and \( \tau \geq \tau_0 \).

A Hölder stability of the Cauchy problem (2.5) has been proved in [3, 13]. It has been shown that the \( L_2 \) norm of the solution \( v \) in a closed subregion of \( \Omega \) can be bounded by a constant times terms corresponding to the Cauchy data, the source and the a prior estimation of the solution. The constant goes to infinity while the closed subregion approaches \( \Gamma \) which is a contradiction to intuition. In the following, we study the stability problem near the boundary \( \Gamma \).

We assume that the boundary \( \partial \Omega \) satisfies the uniform exterior sphere property (UESP), i.e., there exists \( \rho > 0 \) so that, for any \( \hat{x} \in \partial \Omega, \exists \hat{x}_0 \in \mathbb{R} \setminus \Omega \) satisfies

\[
B(\hat{x}_0, \rho) \cap \overline{\Omega} = \emptyset \quad \text{and } \overline{B}(\hat{x}_0, \rho) \cap \overline{\Omega} = \{ \hat{x} \}.
\]

From now on, we fix \( \hat{x} \) to be in the interior of \( \Gamma \). Let us denote \( \Omega(d) = B(\hat{x}_0, \rho + d) \cap \Omega \). The setup of the problem is demonstrated in Figure 2.1. Here \( \delta \in (0, 1) \) and \( \theta \in (0, 1) \) are constants independent of \( r \). Since \( \hat{x} \) is in the interior of \( \Gamma \), there exists a constant \( n_0 \) such that, for all \( r + \delta < n_0 \), we have \( \partial \Omega \cap \partial \Omega(r + \delta) = \hat{x} \). We will give an upper bound of the solution inside \( \Omega(\delta r) \) and study the asymptotic property when \( r \to 0 \) in Theorem 2.2. For the rest of the paper, we use C to denote a constant which may vary from formula to formula and we will clarify its dependence if necessary. We fix \( \lambda \equiv \lambda_0 > 1 \).

Theorem 2.2. There exist two constants \( C > 0 \) and \( 0 < \gamma(r) < 1 \) so that, for any \( v \in H^2(\Omega) \) satisfying (2.5) with the prior estimation

\[
\|v\|^2_{L^2(\Omega(r_0))} \leq K' \quad \text{and } \|f\|^2_{L^2(\Omega(r_0))} + \|v\|^2_{L^2(\Gamma)} + \|\nabla v\|^2_{L^2(\Gamma)} \leq K',
\]

we have

\[
C\delta^4 \|v\|^2_{L^2(\Omega(\delta r))} \leq \left( \|f\|^2_{L^2(\Omega(r_0))} + \|v\|^2_{L^2(\Gamma)} + \|\nabla v\|^2_{L^2(\Gamma)} \right)^{\gamma(r)},
\]
where \( C \) is independent of \( \delta \) and \( r \). Moreover, a possible choice of \( \gamma \) is

\[
\gamma = \frac{(\rho + r)^{2\lambda_0} - (\rho + \theta r)^{2\lambda_0}}{((\rho + r)^{2\lambda_0} - \rho^{2\lambda_0})(\rho + \theta r)^{2\lambda_0}},
\]

which is a decreasing function of \( r \) for

\[
r < \min \left\{ \frac{1}{\theta}, \frac{3(1 - \theta)}{2\lambda_0 - 1}\right\} \rho,
\]

and converges to \( 1 - \theta \) as \( r \to 0 \).

**Proof.** Define

\[
\psi(x) = \ln \left( \left( \rho + r_0 \right)^2 / |x - x_0|^2 \right).
\]

Then

\[
|\nabla \psi(x)| = \frac{2}{|x - x_0|} \geq \frac{2}{\rho + r_0} > 0, \quad \text{for all } x \in \Omega(r + \delta).
\]

That is to say, \( \Psi \) satisfies the non-critical-point condition.

Let \( \chi \in C^\infty(\Omega) \), \( \chi = 1 \) in \( \Omega(r) \) and \( \chi = 0 \) in \( \Omega \setminus \Omega(r + \delta) \). Therefore \( \partial^a \chi \leq K'' \delta^{-|\alpha|} \), \( |\alpha| \leq 2 \), where \( K'' \) is a constant independent of \( \delta \). Applying the Carleman inequality to \( u = \chi v \in \Omega(r + \delta) \), we obtain

\[
C \int_{\Omega(r+\delta)} |\nabla \psi|^2 e^{2\nabla \psi} \, dx \leq \int_{\Omega(r+\delta)} (L(\chi v))^2 e^{2\nabla \psi} \, dx + \int_{\Gamma} \left( (\nabla \chi)^2 + |\nabla v|^2 \right) e^{2\nabla \psi} \, d\sigma,
\]

where

\[
\varphi = \frac{(\rho + r_0)^{2\lambda_0}}{|x - x_0|^{2\lambda_0}}
\]

and \( C \) depends only on \( \Omega, K, \rho, r_0, \lambda_0 \) and \( r_0 \).

Using \( L(\chi v) = \text{div}(a \nabla \chi) v + 2 a \nabla \chi \cdot \nabla v + \chi f \) and the estimates on \( \chi \) and its derivatives, we obtain

\[
\int_{\Omega(r+\delta)} (L(\chi v))^2 e^{2\nabla \psi} \, dx \leq C \int_{\Omega(r)} f^2 e^{2\nabla \psi} \, dx + \frac{C}{\delta^4} \int_{\Omega(r+\delta) \setminus \Omega(r)} (v^2 + |\nabla v|^2) e^{2\nabla \psi} \, dx,
\]

where \( C \) depends only on \( K \) and \( K'' \). For the second term of the right-hand side of (2.9), we have

\[
\int_{\Gamma} (v^2 + |\nabla v|^2) e^{2\nabla \psi} d\sigma \leq \frac{C}{\delta^4} \int_{\Gamma} (v^2 + |\nabla v|^2) e^{2\nabla \psi} d\sigma,
\]

where \( C \) depends only on \( K'' \). Combining (2.9)-(2.11) results

\[
C \delta^4 \int_{\Omega(r+\delta)} |\nabla \psi|^2 e^{2\nabla \psi} \, dx \leq \int_{\Omega(r)} f^2 e^{2\nabla \psi} \, dx + \int_{\Gamma} (v^2 + |\nabla v|^2) e^{2\nabla \psi} d\sigma + \int_{\Omega(r+\delta) \setminus \Omega(r)} (v^2 + |\nabla v|^2) e^{2\nabla \psi} d\sigma,
\]

where \( C \) depends only on \( \Omega, K, K'', \rho, r_0, \lambda_0 \) and \( r_0 \).

Define
\[ \varphi_0 = \frac{(\rho + r_0)^{2}\lambda_0}{(\rho + \theta r)^{2}\lambda_0}, \quad \varphi_1 = \frac{(\rho + r_0)^{2}\lambda_0}{(\rho + r)^{2}\lambda_0} \quad \text{and} \quad \varphi_2 = \frac{(\rho + r_0)^{2}\lambda_0}{\rho^{2}\lambda_0}. \]

Then \( \varphi \geq \varphi_0 \) in \( \Omega(\theta r), \varphi \leq \varphi_1 \) in \( \Omega(r + \delta) \setminus \Omega(r) \) and \( \varphi \leq \varphi_2 \) in \( \overline{\Omega} \). Substituting these estimations into (2.12) results

\[
C \delta^4 \| v \|_{L^2(\Omega(\theta r))}^4 \leq \exp(\alpha(r; \theta)) \| v \|_{H^1(\Omega(r_0))}^2 + \exp(\beta(r; \theta)) \left( \| f \|_{L^2(\Omega(r_0))}^2 + \| \nabla v \|_{L^2(\Omega(r))}^2 \right),
\]

where

\[
\alpha(r; \theta) = 2(\varphi_0 - \varphi_1), \quad \beta(r; \theta) = 2(\varphi_2 - \varphi_0).
\]

For simplicity, let us denote

\[ \mathcal{A} = \| v \|_{H^1(\Omega(r_0))}^2, \quad \mathcal{B} = \| f \|_{L^2(\Omega(r_0))}^2 + \| \nabla v \|_{L^2(\Omega(r))}^2 \quad \text{and} \quad \mathcal{F}(r) = \exp(\alpha(r; \theta)) + \exp(\beta(r; \theta)). \]

By calculating the derivative, it is easy to show that \( \mathcal{F}(r) \) first decreases and then increases as \( r \) goes from 0 to infinity, and \( \mathcal{F} \) obtains its minimum at

\[ \tilde{r} = \frac{\ln \frac{\mathcal{A}}{\mathcal{B}}}{\alpha + \beta}. \]

If \( r_0 \leq \tilde{r} \), we can take \( r = \tilde{r} \), and in this case

\[
\mathcal{F}(\tilde{r}) = \left( \frac{\mathcal{A}}{\mathcal{B}} \right)^{\frac{\alpha}{\alpha + \beta}} + \left( \frac{\mathcal{B}}{\mathcal{A}} \right)^{\frac{\beta}{\alpha + \beta}} \mathcal{F}^{\frac{\alpha}{\alpha + \beta}} \mathcal{B}^{\frac{\beta}{\alpha + \beta}}.
\]

If \( r_0 > \tilde{r} \), that is, \( \exp(-r_0 \alpha) \mathcal{A} < \frac{\mathcal{B}}{\mathcal{A}} \exp(r_0 \beta) \mathcal{B} \), we have

\[
\mathcal{F}(r_0) \leq \left( 1 + \frac{\mathcal{B}}{\mathcal{A}} \right) \exp(r_0 \beta) \mathcal{B}^{\frac{\beta}{\alpha + \beta}} \mathcal{A}^{\frac{\alpha}{\alpha + \beta}}.
\]

To obtain a suitable upper bound of (2.14) and (2.15) independent of \( r \), we will study the monotonicity and bound of \( \beta, \frac{\alpha}{\beta} \) and \( \frac{\alpha}{\beta} \) as \( r \to 0 \).

By the mean value theorem, there exists \( \eta \in (0, 1) \) such that

\[ \beta = 2(\rho + r_0)^{2}\lambda_0 \left( 1 - \frac{1}{(\rho + \theta r)^{2}\lambda_0} \right) = \theta r \frac{4\lambda_0(\rho + r_0)^{2}\lambda_0}{(\rho + \theta(1 - \eta)r)^{2}\lambda_0 + 1}, \]

and therefore

\[ \beta \leq \theta r_0 \frac{4\lambda_0(\rho + r_0)^{2}\lambda_0}{\rho^{2}\lambda_0 + 1}. \]

Since

\[ \frac{\alpha}{\beta} = 1 - \frac{\varphi_1}{\varphi_2} = 1 - \frac{(\rho + \theta r)^{2}\lambda_0}{(\rho + \theta r)^{2}\lambda_0 + 1}, \]

it is easy to show

\[ \lim_{r \to 0} \frac{\alpha}{\beta} = \frac{1}{\theta} - 1. \]

The derivative of \( \frac{\alpha}{\beta} \) satisfies

\[
\left( \frac{\alpha}{\beta} \right)’ \propto 2\lambda_0 \left( \frac{\rho + \theta r}{\rho + r} \right)^{2\lambda_0 - 1} \left( \frac{1 - \theta}{\rho + \theta r} \right) - 2\lambda_0 \left( \frac{\rho + \theta r}{\rho + r} \right)^{2\lambda_0 - 1} \theta \left( 1 - \frac{\rho + \theta r}{\rho + r} \right)^{2\lambda_0}, \]

\[ \propto (\rho + \theta r)^{2\lambda_0 - 1}(1 - \theta) - (\rho + r)^{2\lambda_0 + 1} - (\rho + \theta r)^{2\lambda_0} (\rho + r) \theta. \]
By Taylor expansions, there exist $ξ_1, ξ_2, ξ_3 \in (0, 1)$ such that
\[
\begin{align*}
\rho r^{2λ_0} + 2λ_0 ρ^{2λ_0-1} \theta r + λ_0(2λ_0-1)ρ^{2λ_0-2}θ^2 r^2 \\
\rho r^{2λ_0+1} + (2λ_0 + 1)ρ^{2λ_0} r + (2λ_0 + 1)λ_0ρ^{2λ_0-1}r^2 \\
\rho r^{2λ_0} + 2λ_0 ρ^{2λ_0-1} \theta r + λ_0(2λ_0-1)(ρ + ξ_3 r)^{2λ_0-2}θ^2 r^2.
\end{align*}
\]
By substituting $ξ_2 = 0$ and $ξ_1 = ξ_3 = 1$, we obtain the following upper bound,
\[
\left(\frac{α}{β}\right)' \propto \left((ρ + θ r)^2λ_0 - ρ^{2λ_0+1}\right) + (ρ^{2λ_0+1} - (ρ + r)^{2λ_0+1} + (ρ + θ r)^{2λ_0})\theta,
\]
\[
< - (2λ_0 + 1)λ_0ρ^{2λ_0-1}(1 - θ)θ^r + \frac{1}{3}(2λ_0 + 1)λ_0(2λ_0 - 1)(ρ + θ r)^{2λ_0-2}θ^r r^3.
\]
To obtain $(\frac{α}{β})' < 0$, we only need
\[
\frac{1}{3}(2λ_0 + 1)λ_0(2λ_0 - 1)(ρ + θ r)^{2λ_0-2}θ^r r^3 < (2λ_0 + 1)λ_0ρ^{2λ_0-1}(1 - θ)θ^r r^2,
\]
which is
\[
r < \frac{3(1 - θ)ρ}{(2λ_0 - 1)[(1 + θ r)^{2λ_0-2} - 1]}. \tag{2.16}
\]
Assume that $r < \frac{ξ}{β}$. Then (2.16) is satisfied if
\[
r < \frac{3(1 - θ)}{(2λ_0 - 1)[(4λ_0-1 - 1)]} \rho.
\]
To summarize, for
\[
r < \min \left\{ \frac{1}{θ} \frac{3(1 - θ)}{(2λ_0 - 1)(4λ_0-1 - 1)} \rho \right\}
\]
\(
\frac{α}{β}
\)

is a decreasing function of $r$ and
\[
\lim_{r→0} \frac{α}{β} = \frac{1}{θ} = 1.
\]
Therefore, $\frac{α}{α + β}$ is also a decreasing function of $r$

and
\[
\lim_{r→0} \frac{α}{α + β} = 1 - θ.
\]
We have proved that $β$, $\frac{α}{β}$ and $\frac{α}{α + β}$ can be lower and upper bounded by positive constants depending only on $θ, ρ, K′, λ_0$ and $τ_0$ if $r$ satisfies (2.8). Combining (2.13)-(2.15), we have the conclusion
\[
Cβ \frac{∥v∥^2_{L^2(Ω)} + ∥v∥^2_{H^1(Ω)}}{∥v∥^2_{L^2(Ω)} + ∥∇v∥^2_{L^2(Γ)}} \gamma(r), \tag{2.17}
\]
where $C$ is independent of $r$ and $δ$. For $r$ satisfying (2.8), $γ(r) = \frac{α}{α + β}$ is a decreasing function of $r$, which converges to $1 - θ$ as $r → 0$. ■
2.2 Hölder stability to reconstruct \( D \) and \( \mu \)

Assume that \( \Omega \) is a bounded domain with \( C^{3,1} \) boundary. Let us consider a set of coefficients

\[
\mathcal{C} = \left\{ (D, \mu) \mid D(x), \mu(x) > \kappa^{-1} \text{ for all } x \in \Omega, \quad \|D\|_{C^{2,1}(\overline{\Omega})}, \|\mu\|_{C^{1}(\overline{\Omega})} < \kappa \right\}
\]

for a constant \( \kappa > 1 \), a set of boundary conditions

\[
\mathcal{G} = \{ g \mid g \in C^{3,1}(\partial \Omega), g(x) = 0 \text{ for all } x \in \partial \Omega \},
\]

and a subregion \( \Omega(r_0) \) defined in the previous section with \( \Gamma := \partial \Omega \cap \partial \Omega(r_0) \). For \( g \in \mathcal{G} \), we deduce from Shauder elliptic regularity [18, Theorem 6.14 and 6.19], that (1.1) has a unique solution \( u \in C^3(\overline{\Omega}) \).

To study the stability of the inverse problem, we choose \((D, \mu)\) and \((D', \mu')\) from \(\mathcal{C}\) and solve (1.1) with the same boundary conditions \(\{g_j\}_{j=1}^{n+1} \subset \mathcal{G}\) to obtain \(\{H_j\}_{j=1}^{n+1}\) and \(\{H_{j}^r\}_{j=1}^{n+1}\) respectively. Moreover, we assume that \(D \equiv D'\) and \(\partial_r D \equiv \partial_r D'\) on \(\Gamma\). The following lemma provides a piecewise stability estimation to reconstruct functions related to \(\sigma(x)\).

**Lemma 2.1.** Let \( c > 1 \) be fixed, and assume that \( H_1(x), H_1'(x) \geq c, M(x) \) and \( M'(x)\) defined in (2.1) are invertible, and \(\|M^{-1}(x)\|_F, \|M(x)\|_F \leq c\), for all \( x \in \Omega(r_0) \), where \(\|\cdot\|_F\) denotes the Frobenius norm of the matrix. Assume also that \( H_1(x), H_1'(x) > c > 0 \) for a constant \( c \) and for all \( x \in \Omega(r_0) \). Then there exists a strictly positive constant \( C \) such that

\[
C\|\nabla \ln \sigma(x) - \nabla \ln \sigma'(x)\|_2 \leq \sum_{j=1}^{n+1} \|H_j - H_j'\|_{C^0(\Omega(r_0))}^2 \]  (2.18)

\[
C\|\text{div} \{\nabla \ln \sigma(x) - \nabla \ln \sigma'(x)\}\|_2 \leq \sum_{j=1}^{n+1} \|H_j - H_j'\|_{C^0(\Omega(r_0))}^2 \]  (2.19)

Meanwhile, if

\[
\sum_{j=1}^{n+1} \|H_j - H_j'\|_{C^0(\Omega(r_0))}^2 \ll 1,
\]

we also have

\[
C\|\sigma(x) - \sigma'(x)\|_2 \leq \sum_{j=1}^{n+1} \|H_j - H_j'\|_{C^0(\Omega(r_0))}^2 \]  (2.20)

**Proof.** Since

\[
\begin{align*}
M(x) \cdot \nabla \ln \sigma(x) &= N(x), \\
M'(x) \cdot \nabla \ln \sigma'(x) &= N'(x),
\end{align*}
\]

we have

\[
\nabla \ln \sigma(x) - \nabla \ln \sigma'(x) = -M^{-1}(x)\{M(x) - M'(x)\}\nabla \ln \sigma'(x) + M^{-1}(x)\{N(x) - N'(x)\}
\]

and therefore

\[
C\|\nabla \ln \sigma(x) - \nabla \ln \sigma'(x)\|_2^2 \leq \|M(x) - M'(x)\|_F^2 + \|N(x) - N'(x)\|_2^2.
\]  (2.22)

Let us recall

\[
M(x) - M'(x) = \begin{pmatrix}
\nabla \left( \frac{H_2(x)}{H_1(x)} - \frac{H_2'(x)}{H_1'(x)} \right)^T \\
\vdots \\
\nabla \left( \frac{H_{n+1}(x)}{H_1(x)} - \frac{H_{n+1}'(x)}{H_1'(x)} \right)^T
\end{pmatrix}
\]

and

\[
N(x) - N'(x) = \begin{pmatrix}
\Delta \left( \frac{H_2(x)}{H_1(x)} - \frac{H_2'(x)}{H_1'(x)} \right) \\
\vdots \\
\Delta \left( \frac{H_{n+1}(x)}{H_1(x)} - \frac{H_{n+1}'(x)}{H_1'(x)} \right)
\end{pmatrix}.
\]

Therefore we have the following estimations.
Subtracting one equality by another results

Since

\[ \text{Theorem 2.3.} \]

This lemma leads to the main result of this paper.

Through integration along a curve \( l \in \Omega(r_0) \) connecting a boundary point \( x_0 \in \Gamma \) and \( x \), we obtain

\[ \begin{align*}
\ln\sigma(x) &= \int_l \nabla\ln\sigma(x) \cdot dl + \ln\sigma(x_0), \\
\ln\sigma'(x) &= \int_l \nabla\ln\sigma'(x) \cdot dl + \ln\sigma'(x_0).
\end{align*} \]

Subtracting one equality by another results

\[ C|\ln\sigma(x) - \ln\sigma'(x)|^2 \leq \sum_{j=1}^{n+1} \|H_j - H'_j\|_{C^1([\Omega(r_0)])}^2. \]

Since \( \sigma - \sigma' = \sigma(1 - e^{\ln\sigma' - \ln\sigma}) \), we can apply the Taylor expansion of \( e^x \) to obtain (2.20) if

\[ \sum_{j=1}^{n+1} \|H_j - H'_j\|_{C^1([\Omega(r_0)])}^2 \ll 1. \]

This lemma leads to the main result of this paper.

**Theorem 2.3.** Let us choose \((D, \mu)\) and \((D', \mu')\) from \( \mathcal{C} \) such that \( D \equiv D' \) and \( \partial_v D = \partial_v D' \) on \( \Gamma \). Assume that the data set \( \{H_j\}_{j=1}^{n+1} \) and \( \{H'_j\}_{j=1}^{n+1} \) satisfy all assumptions in Lemma 2.1. Then we have

\[ C\|D - D'\|_{L^2(\Omega(r_0))} \leq \left( \sum_{j=1}^{n+1} \|H_j - H'_j\|_{C^1([\Omega(r_0)])}^2 \right)^{\gamma(r)}, \quad (2.23) \]

where \( \gamma(r) \) is given by (2.7).

**Proof.** Since \( \sqrt{D} \) and \( \sqrt{D'} \) satisfy (2.4) and they have the same boundary value and normal derivative on \( \Gamma \), we can formulate the following Cauchy problem for \( \sqrt{D} - \sqrt{D'} \)

\[ \begin{align*}
-\Delta(\sqrt{D} - \sqrt{D'}) + \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} (\sqrt{D} - \sqrt{D'}) &= -\left( \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} - \frac{\Delta \sqrt{\sigma'}}{\sqrt{\sigma'}} \right) \sqrt{D'} + \frac{1}{\sqrt{\sigma}} (H_1 - H'_1) \quad &\text{in} \ \Omega, \\
\sqrt{D} - \sqrt{D'} &= 0 &\text{on} \ \Gamma, \\
\partial_v(\sqrt{D} - \sqrt{D'}) &= 0 &\text{on} \ \Gamma.
\end{align*} \]

Since

\[ \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} = \frac{1}{4} \|\nabla\ln\sigma\|^2 + \frac{1}{2} \text{div}(\nabla\ln\sigma), \]

applying Theorem 2.2 results

\[ C\|\sqrt{D} - \sqrt{D'}\|_{L^2(\Omega(r_0))}^2 \leq \left( \|H_1 - H'_1\|_{L^2(\Omega(r_0))}^2 + \|\nabla\ln\sigma - \nabla\ln\sigma'\|_{L^2(\Omega(r_0))}^2 \right)^{\gamma(r)}. \]

Integrating all inequalities in Lemma 2.1 over \( \Omega(r_0) \), we obtain the final estimation (2.23). \( \blacksquare \)

The stability estimation to reconstruct \( \mu \) is given by the next corollary.
Corollary 2.1. Assume that all assumptions in Theorem 2.3 are satisfied. Then we have the following Hölder stability estimation to reconstruct \( \mu \)

\[
C\| \mu - \mu' \|_{L^2(\Omega \setminus \partial \Omega)} \leq \left( \sum_{j=1}^{n+1} \| H_j - H_j' \|^2_{C^1(\Omega \setminus \partial \Omega)} \right)^{1/2}. \tag{2.24}
\]

Proof. Since \( u_1 = \frac{\sqrt{p}}{\sqrt{D}} \), combining estimations for \( D \) and \( \sigma \) results

\[
C\| u_1(x) - u_1'(x) \|_{L^2(\Omega \setminus \partial \Omega)} \leq \left( \sum_{j=1}^{n+1} \| H_j - H_j' \|^2_{C^1(\Omega \setminus \partial \Omega)} \right)^{1/2}. \]

Applying the same procedure on \( \mu = \frac{H_0}{n} \) gives the estimation (2.24).

Note 2. The obtained stability results in Theorem 2.3, and Corollary 2.1, indicate that the resolution of PAT is better near the impact zone of the optical illumination sources, and deteriorates far away. According to Theorem 2.1, it is possible to impose conditions only on the boundary data \( \{ \gamma_j \}_{j=1}^{n+1}, \{ \gamma'_j \}_{j=1}^{n+1} \subset \mathcal{G} \) in order to have all the assumptions in Theorem 2.3 being satisfied. Doing so, we can trace out the stability constants in Theorem 2.3, and Corollary 2.1 and show that they only depend on the boundary data, \( n \), and \( \Omega \).

3 Numerical experiments

In this section, we will present three numerical experiments, for which we choose \( \Omega \) to be the unit disc. To simulate the internal data \( H_j(x), j = 1, 2, 3 \), we solve (1.1) with three different boundary conditions, each of which is a normal distribution with the standard deviation 0.3 and the peak at the angle \( \frac{\pi}{2}, \frac{3}{2} \pi \) and \( \frac{5}{2} \pi \) respectively.

We take the gradient and Laplace of \( H_0 \) and \( H_3 \) to formulate the linear system (2.1) which is solved for each discrete position \( x \) whenever it is possible. Actually the non-singularity of the matrix is a quite weak constraint. At least for all our experiments, we never violate it. Note that we always smooth the data locally before taking the derivative to alleviate oscillation caused by noise or numerical discretization. By integration along a suitable curve from a known boundary point to the unknown point \( x \), we are able to obtain \( \ln \sigma(x) \) and hence \( \sigma(x) \). In practice, we choose to start from 10 different boundary points and average to stabilize the computation.

The next step is to compute the coefficient \( \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} \) of the equation (2.1). Since we have constructed \( \sigma \), we can calculate the coefficient directly by taking the Laplace of \( \sqrt{\sigma} \). This procedure is extremely unstable because the error from the numerical integration is dramatically amplified during calculating the derivative. Alternatively, we calculate the coefficient by the following equality

\[
\frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} = \frac{1}{4} |\nabla \ln \sigma|^2 + \frac{1}{2} \text{div(} \nabla \ln \sigma \text{)}. \]

Let us emphasize that we only do the computation in a suitable subregion where \( H(x) \) is larger than a threshold which depends on the estimation of the noise level.

Since the coefficient and source terms of (2.4) are partially reconstructed, theoretically to reconstruct \( D \) we need to solve the Cauchy problem corresponding to \( \sqrt{D} \). If the medium is homogeneous, we can verify that

\[
\frac{H}{\sqrt{\sigma}} = \frac{\mu}{\sqrt{D}} \quad \text{and} \quad \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} = \frac{\mu}{\sqrt{D}}. \]

Numerically we complete the missing region by a suitable constant, for example the background value or the average of the known value, and solve the partial differential equation (2.4) only once to reconstruct \( \sqrt{D} \). We find out that this strategy only affects the reconstruction in a small area close to the boundary of the completing region. In the following experiments, we only demonstrate reconstructions in a proper subregion.
3.1 Synthetic medium – smooth case

The medium is a unit disc with inhomogeneity composed of rectangles and discs of different size. The background truth of $D$ and $\mu$ are 0.2 and 20 respectively. The variation of $D$ (resp. $\mu$) varies from 0.1 to 0.35 (resp. 10 to 35). To obtain the smooth medium, we take the convolution of the piecewise constant medium from the next experiment with a Gaussian function. The positions of the inhomogeneity are almost the same for $D$ and $\mu$ except that we intentionally remove the rectangle on the top in $D$ and add a triangle in $\mu$; refer to Figure 3.1a and 3.1b.

We are able to correctly reconstruct positions and values of the inhomogeneity; refer to Figure 3.1c and 3.1d. The relative error in the region $y > 0.2$ is 3.42% for $D$ and 3.19% for $\mu$. We could notice from the color of Figure 3.1 that the reconstruction is a little lighter than the truth, which may be the result of the smoothing technique applied each time before calculating derivatives.

3.2 Synthetic medium – discontinuous case

To obtain the theoretical stability, it requires certain smoothness of the coefficients $D$ and $\mu$. In this experiment we try our reconstruction algorithm on a problem with piecewise constant value. We do not pay any additional attention to the discontinuity or use any special trick inside the code. The background truth and reconstructed results are demonstrated in Figure 3.2. All the embedded inclusions with different shapes are well reconstructed. The relative error is larger than the smooth case, 8.56% for $D$ and 4.70% for $\mu$ in the region $y > 0.2$.

Interestingly, due to the discontinuity of $D$ and $\mu$, we observe huge jumps in the reconstructed coefficient $\frac{\Delta y \sigma}{\Delta \sigma}$ on the boundary of the inclusions. The value of the jump depends on the smoothing technique. That’s to say, in the process of filtering out potential noise while computing derivatives, we decrease the true extreme value of the coefficient as well. But the huge error in formulating the coefficient and source of (2.4) does not affect the reconstruction of $D$ too much, especially qualitatively.

3.3 Blood vessel

We try our algorithm on a more realistic example – imaging the blood vessel of a piece of biological tissue. We assign the tissue with proper diffusion and absorption values; see Figure 3.3a and 3.3b. As demonstrated in Figure 3.3c and 3.3d, all the features are well characterized by our results. We only lose a little the contrast like the previous experiments. The relative error is 4.74% for $D$ and 2.47% for $\mu$. 

Figure 3.1: Local reconstructions of smooth $D$ and $\mu$. 

![Figure 3.1: Local reconstructions of smooth $D$ and $\mu$.](image)
Figure 3.2: Local reconstructions of discontinuous $D$ and $\mu$.

Figure 3.3: Local reconstructions of $D$ and $\mu$ of the blood vessel.
4 Conclusion

In this paper, we prove a Hölder stability of the quantitative PAT in a subregion where the internal information is reliably provided based on the stability estimation of a Cauchy problem satisfied by the diffusion coefficient. The exponent of the Hölder stability converges to a positive constant independent of the subregion as the subregion contracts towards the boundary.

Numerical experiments demonstrates that it is possible to locally and efficiently reconstruct the diffusion and absorption coefficients for smooth and even discontinuous media through the solution of an elliptic equation.

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