SUGIHARA ALGEBRAS AND MONOIDS:
MULTISORTED DUALITIES

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Abstract. The authors developed in a recent paper natural dualities for finitely generated quasivarieties of Sugihara algebras. They thereby identified the admissibility algebras for these quasivarieties which, via the Test Spaces Method devised by Cabrer et al., give access to a viable method for studying admissible rules within relevance logic, specifically for extensions of the deductive system $R$-mingle.

This paper builds on the work already done on the theory of natural dualities for Sugihara algebras. Its purpose is to provide an integrated suite of multisorted duality theorems of a uniform type, encompassing finitely generated quasivarieties and varieties of both Sugihara algebras and Sugihara monoids, and embracing both the odd and the even cases. The overarching theoretical framework of multisorted duality theory developed here leads on to amenable representations of free algebras, More widely, it provides a springboard to further applications.

1. Introduction

Sugihara monoids and algebras, particularly in the odd case, have attracted interest on a number of fronts. This has stemmed in part from investigations of the rich class of residuated lattices and of associated substructural logics. See for example the introductory survey [3] and the comprehensive monograph [25] for the general theory and context and [22–24, 28, 29] for indications of where the special class of Sugihara monoids fits into a bigger algebraic picture.

From the perspective of logic, the motivation for studying Sugihara monoids originates in the work of Dunn [21] and Anderson and Belnap [1]; see also the contextual comments in [29]. Sugihara algebras provide complete algebraic semantics for $R$, the algebraizable deductive system $R$-mingle. Sugihara monoids include a constant in their algebraic language so as to model $RM^1$, viz. $R$-mingle afforded with Ackermann’s constant.

Our introductory remarks above indicate that effective mathematical tools for studying Sugihara algebras and monoids would be of interest and value both as regards these algebraic structures per se and for the algebraic and relational semantics for related deductive systems. Our objective in this paper is to enlarge the armoury of such tools, as a basis for further theory. Our longer-term aim is for the present work to lead on to a full exploration of alternative relational models, and to a structural analysis of the classes $\mathcal{SA}$ of Sugihara algebras and $\mathcal{SM}$ of Sugihara monoids based on these models. The material we present here, and specifically our theorems in Sections 4 and 5, may be seen in part as a stepping stone along the

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way to our ultimate goal. Paper [9], devoted to free algebras, provides a foretaste of what our methods can achieve.

Our work on multisorted dualities has the longer-term goal of leading to a full exploration of alternative relational models for finitely generated quasivarieties and varieties of $\mathcal{A}$ and of $\mathcal{M}$, and to a structural analysis based on these models. (Our appendix hints at such possibilities.) However this goal cannot be reached within a single paper. Accordingly, our objectives in this paper are more limited, and the material we present, and specifically our theorems in Sections 4 and 5, may be seen in part as stepping stones along the way to our ultimate goal.

We note the recent study by Fussner and Galatos [22] of the algebraic structure of (odd) Sugihara monoids and of relational models for them. However there is little overlap between our work and theirs in that the novelty of our approach rests on the use of multisorted natural dualities whereas they make only very limited use of natural duality theory and then only in a single-sorted form.

Our previous paper [4] was motivated by a very specific problem: to find a computationally feasible method for studying admissible rules for certain extensions of RM. See [27] and [8] and the references therein for background and possible general methodologies. Strong evidence for the power of the duality-based approach in [8] given in [4, Section 8]. There are significant differences between RM and RM$^t$ as regards structural completeness properties, with strong assertions being available for the latter. This led us in [4] to focus on the variety $\mathcal{A}$ and specifically on its finitely generated quasivarieties. The generators for these quasivarieties are the algebras $Z_k$ for $k \geq 1$, where $Z_k$ has a lattice reduct which is a chain with $\frac{k+1}{2}$ elements. Analogously, the finitely generated quasivarieties of $\mathcal{M}$ have generators $W_k$, where $W_k$ has $\mathcal{A}$-reduct $Z_k$ and has $t$ interpreted as 0 if $k$ is odd and as 1 if $k$ is even. (Details of the definitions are recalled in Section 3.) A Sugihara algebra or monoid is odd if its involution $\neg$ has a (necessarily unique) fixed point, and even otherwise. Thus $Z_k$ and $W_k$ are odd if and only if $k$ is odd.

The general structure theory of Sugihara monoids, and of their relational models, is much more highly developed in the odd case than in the even case; see in particular [22–24, 29]. In [4] we partially redressed the balance by providing strong dualities for all the quasivarieties $\mathcal{A}_k := \text{ISP}(Z_k)$ of $\mathcal{A}$. Our representation theorems did take different forms in the odd and even cases, with [4, Theorem 6.4 (even case)] looking more complicated than [4, Theorem 4.3 (odd case)]. At the heart of this dichotomy is the fact that the quasivarieties $\text{ISP}(Z_{2m-1})$ are varieties whereas $\text{ISP}(Z_{2m}) \subsetneq \text{HSP}(Z_{2m})$; the variety generated by $Z_{2m}$ is $\text{ISP}(Z_{2m}, Z_{2m-1})$. (Here we adopt the customary notation for class operators.) Details of these claims, and the analogues for Sugihara monoids, with $Z_k$ replaced by $W_k$, are given in Section 3. These observations suggest that we should switch attention from quasivarieties to the varieties they generate. We shall reveal that the machinery of multisorted dualities allows us to do this, and brings other benefits too.

Our objective in using duality theory is to be able to study a class of algebras $\mathcal{A}$ by setting up a well-behaved dual category equivalence between $\mathcal{A}$ and a category $\mathcal{K}$ so that problems about $\mathcal{A}$ can be faithfully translated into problems about $\mathcal{K}$ which one anticipates will be more tractable. Our approach to the admissible rules problem originating in [8] and exploited in [4] relies crucially on the use of strong dualities. As the term is used in natural duality theory [11], a strong duality for a finitely generated quasivariety $\mathcal{A} = \text{ISP}(M)$ sets up a dual equivalence between $\mathcal{A}$
and a category $\mathcal{X}$ of topological relational structures, generated as a topological quasivariety by $M$, a compatible ‘alter ego’ for $M$ which is injective in $\mathcal{X}$ (the technical details need not concern us here). In [4], and equally in this paper and its successors, The existence of dualities will be guaranteed by general theory and our assumptions. But more is at stake: we seek dualities which are based on economical, and so amenable, alter egos. We achieved this in [4] by using the method of generalised piggybacking which was introduced in [16] and is based on a theory of multisorted natural dualities. But in [4] multisortedness was kept covert. Our dualities there used dual categories whose objects are traditional single-sorted structures; the alter egos determining the dual categories were obtained by a multi-carrier version of the piggyback method and strongness can also be engineered; see [4, Theorem 3.3].

In this paper we extol the virtues of employing multisorted duality theory in its full-blown form. Now the dual categories have objects which are multisorted topological structures. As a byproduct, this gives us the freedom to seek a duality for a class $\text{ISP}(M)$, where $M$ is a finite set of finite algebras over a common language. This is exactly what we shall do for $\text{HSP}(\mathbb{Z}_{2m}) = \text{ISP}(\mathbb{Z}_{2m}, \mathbb{Z}_{2m-1})$ in Theorem 4.8. This new duality is three-sorted. As compared with its single-sorted counterpart for $\text{ISP}(\mathbb{Z}_{2m})$ the new duality is appealingly simple. We also provide a two-sorted duality for $\text{ISP}(\mathbb{Z}_{2m-1})$, equivalent to [4, Theorem 4.3]. It is of particular interest here that the passage from a quasivariety $\text{ISP}(M)$ to the variety $\text{HSP}(M)$ can lead to a duality which is simpler to work with. This phenomenon promises to be valuable for applications, especially if, as seems likely, it is not confined to the Sugihara example. (Of course, free algebras in a quasivariety $\text{ISP}(M)$ exist, and can equally well be calculated in $\text{HSP}(M)$.)

In this paper we shall assume that the reader is familiar with the basic theory of natural dualities, as summarised in black box fashion in [4, Section 3]. Section 2 below provides a brief formal introduction to the use of multisorted dual categories, with the theory tailored to our needs. We are mindful that multisorted duality theory is less well known than it deserves to be. Accordingly, by way of salesmanship, we include a short appendix to our paper to set in context our main results in Sections 4 and 5. The appendix, aimed at those new to multisorted dualities, illustrates the key ideas using two simple examples: Kleene algebras and Kleene lattices and shows how a multisorted perspective facilitates making connections between natural dualities and alternative, Priestley-style, representations.

Once Section 2’s theoretical foundations are in place, we focus on Sugihara algebras and monoids. Section 3 recalls facts from [4] on Sugihara algebras and assembles the information we need, likewise, about Sugihara monoids. Section 4 begins with Theorem 4.4, the translation into two-sorted form of [4, Theorem 4.3]. We highlight the similarities and the differences between what are essentially two formulations of the same result. By contrast, Theorem 4.8 enters new territory. It supplies a three-sorted strong duality for $\mathcal{V}(\mathbb{Z}_{2m}) = \text{HSP}(\mathbb{Z}_{2m})$, the variety generated by $\mathbb{Z}_{2m}$. This duality is easier to work with than the single-sorted one with $2m - 1$ carriers for the quasivariety $\text{ISP}(\mathbb{Z}_{2m})$ [4, Theorem 6.4]. In Section 5 we capitalise on our work in the algebras case to exhibit Theorems 5.2 and 5.3, the corresponding duality theorems for the monoids case. We stress two points. Firstly, the transition from algebras to monoids involves very little work. Secondly, in both
the odd and even cases, comparison of the theorems reveal that the differences between the results are limited and localised in nature, so that the effect of including or omitting the truth constant from the language feeds through in a transparent way to our representation theorems.

Finally in this introduction we draw attention to an important limitation on the applicability of natural duality theory to Sugihara algebras and monoids. Each of the varieties $\mathcal{IA}$ and $\mathcal{IM}$ is generated by a single algebra $\mathcal{Z}$, having the integers as its lattice reduct. There are instances of natural dualities based on infinite alter egos, for example those for abelian groups (the famous Pontryagin duality) and Ockham algebras. However the requirements for an infinite generating algebra to have a compatible alter ego are stringent (see the general discussion in the varieties $\mathcal{IA}$ and $\mathcal{IM}$ cannot be brought under the natural duality umbrella this way. We argue that, nevertheless, finitely generated subquasivarieties and subvarieties of a variety which is not finitely generated can carry valuable information, especially for locally finite varieties. For example, every finitely generated free algebra in $\mathcal{IA}$ or $\mathcal{IM}$ already belongs to some finitely generated subquasivariety and can be analysed there; see [9] and, for an application, [4, Section 8], concerning admissible rules for $R$-mingle. In addition, information revealed by applying duality methods to sub(quasi)varieties of a variety may provide pointers to algebraic features of the variety as a whole.

2. The framework of multisorted natural dualities

By way of background, we remark that researchers who have driven natural duality theory forward over the past twenty years have concentrated on the single-sorted case. They were aware that extensions to the multisorted case would be possible, and often straightforward, but that the heavier notation involved in working in maximum generality could obscure the underlying ideas; [15], a recent major review of single-sorted piggybacking, is a case in point. Moreover, the multisorted theory has largely evolved on a ‘need-to-know’ basis, as worthwhile potential applications have emerged.

The treatment of multisorted piggybacking and strong dualities in [11, Chapter 7] omits details and proofs. It does anyway not quite meet our needs. We shall instead draw on the paper [17] by Davey and Tahkuder. This includes a clear summary of the basic theory and also the technical result on strongness that we shall require. (Furthermore, [17] studies dualities for finitely generated varieties of Heyting algebras, so the algebras involved there have a connection with Sugihara algebras and monoids, though this relationship is not pursued in the present paper.)

We do not need the theory in the most general form possible. The classes of algebras we wish to consider will be of the form $\mathcal{V}(M) = \mathcal{HSP}(M)$ or $\mathcal{Q}(M)$, where $M$ is finite and has a distributive lattice reduct; here $\mathcal{V}(M) = \mathcal{HSP}(M)$ and $\mathcal{Q}(M) = \mathcal{ISP}(M)$ are, respectively, the variety and quasivariety generated by $M$. On the universal algebra front, these assumptions ensure that $\mathcal{V}(M)$ is congruence distributive and that Jónsson’s Lemma applies. This means that $\mathcal{V}(M)$ will be expressible as $\mathcal{ISP}(\mathcal{M})$, where $\mathcal{M} = \{M_1, \ldots, M_N\}$ is finite and each $M_i \in \mathcal{HS}$. As regards duality theory, we will have access to both the Multisorted Piggyback Duality Theorem (for piggybacking over the variety $D$ of all distributive lattices) [11, Theorem 7.2.1] and the Multisorted NU Strong Duality Theorem, making use of the fact that we are dealing with lattice-based algebras, [11, Theorem 7.1.2].
So consider a class \( \mathcal{A} = \mathbb{ISP}(\mathcal{M}) \), where \( \mathcal{M} \) is a finite set of finite algebras over a common language and having reducts in \( D_u \). We regard \( \mathcal{A} \) as a category, in which the morphisms are all homomorphisms. We seek a dual category \( \mathcal{E} \) whose objects are multisorted topological structures and whose morphisms are maps which preserve the sorts and are continuous and structure-preserving. A background reference for multisorted structures is [26]. We want to construct \( \mathcal{E} \) in such a way that there are functors \( D: \mathcal{A} \to \mathcal{E} \) and \( E: \mathcal{E} \to \mathcal{A} \) setting up a dual equivalence. This needs to be done in a very specific way, so that \( D \) and \( E \) are given by appropriately defined hom-functors.

We now explain what constitutes an admissible alter ego \( \mathcal{M} \) for \( \mathcal{M} \), how to construct the dual category \( \mathcal{E} \) of multisorted structures generated by \( \mathcal{M} \) and set up a dual adjunction between \( \mathcal{A} \) and \( \mathcal{E} \). Our alter ego for \( \mathcal{M} \) takes the form \( \mathcal{M} = (M_1 \cup \cdots \cup M_N; G, H, K, R, T) \). Here \( R \) is a set of relations each of which is a subalgebra of some \( M_i \times M_j \), where \( i, j \in \{1, \ldots, N\} \). The sets \( G \) and \( H \) consist, respectively, of homomorphisms and (non-total) partial homomorphisms, each from some \( M_i \) into some \( M_j \). The elements of \( K \) are the one-element subalgebras of the members of \( \mathcal{M} \). The alter ego \( \mathcal{M} \) carries the disjoint union topology derived from the discrete topology on the individual sorts \( M_i \). (These assumptions parallel those in the single-sorted case; see [4, Section 3].)

We form multisorted topological \( \mathcal{M} \)-structures \( X = X_1 \cup \cdots \cup X_N \) where each of the sorts \( X_i \) is a Boolean space, \( X \) is equipped with the disjoint union topology and regarded as a structure, \( X \) is of the same type as \( \mathcal{M} \). Thus \( X \) is equipped with a set \( R^X \) of relations \( r^X \); if \( r \subseteq M_i \times M_j \), then \( r^X \subseteq X_i \times X_j \); and similar statements apply to \( G^X, H^X \) and \( K^X \). Given \( \mathcal{M} \)-structures \( X \) and \( Y \), a morphism \( \phi: X \to Y \) is defined to be a continuous map preserving the sorts, so that \( \phi(X_i) \subseteq Y_i \), and \( \phi \) preserves the structure. The terms isomorphism, embedding, etc., are then defined appropriately. As in the single-sorted case, care needs to be taken with embeddings when \( H \neq \emptyset \); see [4, Section 3].

We define our dual category \( \mathcal{E} \) to have as objects those \( \mathcal{M} \)-structures \( X \) which belong to a class of topological structures which we shall denote by \( \mathbb{ISP}^+(\mathcal{M}) \). Specifically, \( \mathcal{E} \) consists of isomorphic copies of closed substructures of powers of \( \mathcal{M} \). Here powers are formed ‘by sorts’: given a non-empty set \( S \), the underlying set of \( \mathcal{M}^S \) is the union of disjoint copies of \( M_i \), for \( M \in \mathcal{M} \), equipped with the disjoint union topology obtained when each \( M_i \) is given the product topology. The structure given by \( R, G, H \) and \( K \) is lifted pointwise to substructures of such powers. The superscript + indicates that the empty structure is included in \( \mathcal{E} \).

We now set up hom-functors \( D: \mathcal{A} \to \mathcal{E} \) and \( E: \mathcal{E} \to \mathcal{A} \) using \( \mathcal{M} \) and its alter ego \( \mathcal{M} \):

\[
D(A) = \mathcal{A}(A, M_1) \cup \cdots \cup \mathcal{A}(A, M_N), \quad D(f) = - \circ f;
\]
\[
E(X) = \mathcal{E}(X, \mathcal{M}), \quad E(\phi) = - \circ \phi.
\]

Here the disjoint union \( \mathcal{A}(A, M_1) \cup \cdots \cup \mathcal{A}(A, M_N) \) is a (necessarily closed) substructure of \( M_1 \cup \cdots \cup M_N \), and so a member of \( \mathbb{ISP}^+(\mathcal{M}) \). We recall from above that \( \mathcal{E}(X, \mathcal{M}) \), as a set, is the collection of continuous structure-preserving maps \( \phi: X \to \mathcal{M} \) which are such that \( \phi(X_i) \subseteq M_i \) for \( 1 \leq i \leq N \). This set acquires the structure of a member of \( \mathcal{A} \) by virtue of viewing it as a subalgebra of the power
Theorem 2.1. The well-definedness of the functors $D$ and $E$ is of central importance to our enterprise. It hinges on the algebraic assumptions we have made on $G, H, K, R$ and the requirement that each $M_i$ is finite and carries the discrete topology. Moreover, $D$ and $E$ set up a dual adjunction, $(D, E, e, ε)$ in which the unit and counit maps are evaluation maps, and these evaluations are embeddings.

We say $M$ yields a multisorted duality if, for each $A ∈ 𝒜$, the evaluation map $e_A : A → ED(A)$ is an isomorphism. The duality is full if, for each $X ∈ 𝒫$, the evaluation map $ε_X : X → DE(X)$ is an isomorphism. Thus a duality provides a concrete representation $ED(A)$ of $A ∈ 𝒜$. If in addition the duality is full, we also know that every $X ∈ 𝒫$ arises, up to isomorphism, as a topological structure $D(A)$, for some $A ∈ 𝒜$. We record an important fact, true for any multisorted duality, and adding weight to the duality’s claim to be called ‘natural’. In $𝒜 = ISP(M)$, the free algebra $F_𝒜(s)$ on $s$ generators is isomorphic to $E(𝒜^S)$ [11, Lemma 2.2.1 and Section 7.1].

For many applications we require a duality which is full. In practice, fullness of a duality is normally obtained at second hand by showing that the duality is strong. Strongness of a single-sorted duality can be defined in several equivalent ways and the same can be expected of a multisorted duality (compare [17] and [14]). We follow the former but, in accordance with our policy of a black-box treatment of duality theory, we suppress the details as far as possible. In particular we omit the statement of the hom-closure condition for a multisorted duality to be strong. This can be found in [17, Section 4]. In addition, [17, Theorem 4.1] shows that a multisorted strong duality is full. In certain applications—and this was crucial in the TSM method for testing admissibility [4, 8]—consequences of strongness are required, whereby each of the functors setting up a strong duality converts an embedding to a surjection and a surjection to an embedding.

The multisorted piggyback duality theorem originated in [16, Theorem 2.2] and thereafter various specialisations have been well documented, as for example in [11, Theorem 7.2.1]. Most previous applications in the literature piggyback over $𝔻$, bounded distributive lattices with $0, 1$ included in the language. However, as in [4], we shall, except briefly in the appendix, piggyback over $𝔻_u$. The upgrade to a strong duality was not discussed in [16]. For this see for example see [11, Theorem 7.1.2] (the Multisorted NU Strong Duality Theorem and [17, Theorem 2.1].

We now state the multisorted duality theorem that we shall apply to quasivarieties and varieties of Sugihara algebras and monoids. The reason for including the condition on subalgebras being subdirectly irreducible is that it ensures that partial operations of arity $> 1$ are not needed; cf. [11, Section 7.1]. A discussion of the role of the set $K$ of one-element subalgebras of members of $ℳ$ and the need, for the application of the Multisorted NU Strong Duality Theorem, to include these in $ℳ$, can be found in [17, Lemma 4.5].

**Theorem 2.1** (Multisorted Piggyback Strong Duality Theorem, for distributive-lattice-based algebras). Let $𝒜 = ISP(M)$, where $ℳ = \{M_1, \ldots, M_N\}$ is a finite set of pairwise disjoint finite algebras in $𝒜$ and assume that there is a forgetful functor $U$ from $𝒜$ into $𝔻_u$ such that $U(A)$ is a reduct of $A$ for each $A ∈ 𝒜$. Assume in addition that for each $i = 1, \ldots, N$ every non-trivial subalgebra of $M_i$ is subdirectly irreducible.

For each $M_i ∈ ℳ$ let $Ω_{M_i}$ be a (possibly empty) subset of $𝔻_u(U(M_i), 2)$, where 2 denotes the 2-element lattice in $𝔻_u$. 


Let $\mathcal{M}$ be the discretely topologised relational structure with universe $M_0 := M_1 \cup \cdots \cup M_N$ and equipped with

1. $G$, a subset of $\bigcup \{ \mathcal{A}(M_i, M_j) \mid M_i, M_j \in \mathcal{M} \}$ satisfying the following separation condition: for all $M_i \in \mathcal{M}$, given $a, b \in M_i$ with $a \neq b$, there exist $M_j \in \mathcal{M}$ and $u \in \mathcal{A}(M_i, M_j) \cap G$ and $\omega_j \in \Omega_{M_j}$ such that $\omega_j(u(a)) \neq \omega_j(u(b))$;
2. $R$, the collection of maximal $\mathcal{A}$-subalgebras of sublattices of the form $(\omega_i, \omega_j)^{-1}(\leq) := \{ (a, b) \in M_i \times M_j \mid \omega_i(a) \leq \omega_j(b) \}$, for which $\omega_i \in \Omega_{M_i}, \omega_j \in \Omega_{M_j}$ (where $M_i, M_j$ range over $\mathcal{M}$).

Then $\mathcal{M}$ yields a duality on $\mathcal{A}$.

If the structure of the alter ego is augmented so as to include

3. $H$, the set of all homomorphisms $h$ whose domain is a subalgebra of some $M_i$ and whose image is a subalgebra of some $M_j$;
4. $K$, the set of elements which are universes of one-element subalgebras in some member of $\mathcal{M}$, viewed as nullary operations of $\mathcal{M}$.

then $\mathcal{M}$ yields a strong and hence full duality on $\mathcal{A}$.

Some comments are in order here. Recall that Theorem 2.1 is derived by pasting together sufficient conditions for a reasonably economical duality (by multisorted piggybacking) and for an upgrade to a strong duality (Multisorted NU Strong Duality Theorem). There is no a priori reason to expect that there will be a close tie-up between the relations in (ii) and the operations and partial operations in (iii).

The theorem ensures that, when conditions (i)–(iv) are satisfied that a strong duality is available. But we would be likely to want to massage the alter ego so delivered in order to arrive at an alter ego which is easier to work with but which still strongly dualises $\mathcal{A}$. A result that facilitates the streamlining of an alter ego in the single-sorted case is Clark and Davey’s $M$-Shift Strong Duality Lemma, of which we presented a simplified for in [4, Lemma 3.4]. We now present the multisorted version we shall need in this paper.

**Lemma 2.2** (Multisorted $M$-Shift Strong Duality Lemma). Assume that a finitely generated quasivariety $\mathcal{A} = \text{ISP}(\mathcal{M})$ is strongly dualised by $\mathcal{M} = (M_0; G, H, K, R, T)$. Then an alter ego $\mathcal{M}' = (M_0; G', H', K', R', T)$ will also yield a strong duality if any of the following applies:

1. $R' \supset R$;
2. $H'$ is obtained by deleting from $H$ any element expressible as a composition of the elements that remain;
3. $\mathcal{M}'$ yields a duality on $\mathcal{A}$ and is obtained from $\mathcal{M}$ by deleting members of $R \cup K$;
4. $\mathcal{M}'$ yields a duality on $\mathcal{A}$ and is obtained by deleting from $H$ any element which has a proper extension belonging to $G \cup H$.

**Proof.** As in the single-sorted case the proofs of the duality claims in (a) and (b) are bookkeeping exercises with evaluation maps; cf. [10, Lemmas 1.4 and 3.1]. The part of the proof of [10, Lemma 3.1] that deals with strongness relies on the fact that a closed substructure of a non-zero power of $\mathcal{M}$ in the dual category is hom-closed. The multisorted version of this assertion is stated and proved in [17, Lemma 4.3]. The proofs of (b)–(d) then proceed as in the single-sorted case.
(Here (b) is a restricted form of hom-entailment, sufficient for our purposes; see [11, Section 9.4].)

The Shift Lemma, items (b) and (d), promises to be valuable for eliminating unnecessary maps from the set stipulated in Theorem 2.1(iv). Moreover, in our applications, the piggyback relations in (ii) will turn out to be graphs of partial endomorphisms of sorts or of partial homomorphisms between sorts, or converses of such graphs. But, assuming we want our duality to be strong, we do need to include partial operations in the alter ego rather than their graphs.

We make some further comments on these matters in relation to our dualities at the end of Section 5, once all the dualities are in place.

3. Sugihara algebras and monoids, odd and even cases

Given the freestanding treatment of the class $\mathcal{A}$ of Sugihara algebras available in [4] our aim as far as possible is to draw on the Sugihara algebra results to arrive at corresponding results for monoids, to avoid starting afresh. Therefore we shall regard the class $\mathcal{M}$ of Sugihara monoids as being obtained by enriching the language of $\mathcal{A}$, instead of going in the opposite direction.

We recall that the variety $\mathcal{A}$ of Sugihara algebras is generated by the algebra $Z$ whose underlying lattice is the chain of integers. The negation and implication are given, respectively, by $\neg a = -a$ and $a \to b = \begin{cases} (-a) \lor b & \text{if } a \leq b, \\ (-a) \land b & \text{otherwise.} \end{cases}$

Then the algebras $Z_k$ are the subalgebras of $Z$ for which $Z_{2n+1}$ has universe $\{ a \in Z \mid -n \leq a \leq n \} (n \geq 0)$ and $Z_{2n} = Z_{2n+1} \setminus \{0\} (n \geq 1)$. A finite Sugihara algebra is subdirectly irreducible if and only if it is (isomorphic to) some algebra $Z_k (k \geq 2)$. This result dates back to Blok and Dzobiak [2]; see also [4, Section 2].

The variety $\mathcal{M}$ of Sugihara monoids consists of algebras $A = (A; \land, \lor, \to, \neg, t)$ such that the reduct $(A; \land, \lor, \to, \neg)$ belongs to $\mathcal{A}$ and $t$ is a constant which is an identity for the derived fusion operation given by $a \cdot b = -(a \to \neg b)$. It is useful to denote the derived constant $-t$ by $f$. When $k$ is odd, the Sugihara monoid $W_k$ associated with the Sugihara algebra $Z_k$ has the constants $t$ and $f$ interpreted as 0. When $k$ is even, $t$ is interpreted as 1 and $f$ as $-1$. It has long been known (see [21] and also [1, pp. 422–423]) that the finite subdirectly irreducible members of $\mathcal{M}$ are the algebras $W_k$ for $k > 1$. We let $\mathcal{M}_k := ISP(W_k)$.

We next assemble information about finitely generated quasivarieties and varieties in $\mathcal{A}$ and $\mathcal{M}$, drawing on some fundamental results from universal algebra. Suppose we have a variety $\mathcal{A} := V(M)$, where $M$ is some finite algebra $M$. Let $S := si(\mathcal{A})$ be the set of subdirectly irreducible algebras in $\mathcal{A}$, up to isomorphism. Then Birkhoff’s Subdirect Product Theorem tells us that $S$ is finite and that $HSP(M) = ISP(S)$. We shall be dealing with the situation in which $\mathcal{A}$ is congruence-distributive. A corollary of Jónsson’s Lemma asserts that in such a variety, every subdirectly irreducible algebra is a homomorphic image of a subalgebra of $M$. In addition, the lattice of subvarieties of $\mathcal{A}$ is a finite distributive lattice. It follows that of subvarieties of $\mathcal{A}$ is isomorphic to the lattice of down-sets of $S$, where $A \leq B$ in $S$ if and only if $A \in HSP(B)$ (see for example [20]).
Take \( m \geq 2 \). We note that \( \mathbb{Z}_{2m-1} \) is a homomorphic image of \( \mathbb{Z}_{2m} \) via the homomorphism \( u \) given by

\[
u(a) = \begin{cases} a - 1 & \text{if } 0 < a \leq m, \\ a + 1 & \text{if } -m \leq a < 0. \end{cases}
\]

Proposition 3.1 leads to the diagram in [4, Figure 1] for the lattice of finitely generated subquasivarieties of \( \mathcal{A} \). We omitted the justification there since we did not explicitly need the result. In this paper we so need it.

In [4] we considered the chains of quasivarieties \( \{ \mathcal{A}_{2n+1} \}_{n \geq 1} \) (odd case) and \( \{ \mathcal{A}_{2n} \}_{n \geq 1} \) (even case) and presented dualities for \( \mathcal{A}_{2n+1} \) and \( \mathcal{A}_{2n} \), for general \( n \). In Proposition 3.1 we have changed labels from \( n \) to \( m \). Part (ii) explains why. In order to provide a multisorted duality for \( \mathcal{V}(\mathbb{Z}_n) \) we would expect to draw on ingredients from the dualities for \( \mathbb{Z}_n \) and \( \mathbb{Z}_{2n-1} \) at the same time and it is convenient subsequently to label the odd case quasivarieties as \( \{ \mathcal{A}_{2n-1} \}_{m \geq 2} \) and the even case quasivarieties as \( \{ \mathcal{A}_{2m} \}_{m \geq 2} \). (The class \( \mathcal{A} \), the trivial class and \( \mathcal{A}_2 \) is term-equivalent to the variety of Boolean algebras, and we have nothing to say about either class.)

**Proposition 3.1** (Sugihara algebras: quasivarieties and varieties).

(i) \( \text{ISP}(\mathbb{Z}_{2m-1}) = \text{HSP}(\mathbb{Z}_{2m-1}) \), and so is a variety.

(ii) \( \text{ISP}(\mathbb{Z}_{2m}) \subseteq \text{HSP}(\mathbb{Z}_{2m}) \) and \( \text{HSP}(\mathbb{Z}_{2m}) = \text{ISP}(\mathbb{Z}_{2m}, \mathbb{Z}_{2m-1}) \).

(iii) \( \text{HSP}(\mathbb{Z}_1) \subseteq \text{HSP}(\mathbb{Z}_2) \subseteq \text{HSP}(\mathbb{Z}_3) \subseteq \cdots \subseteq \mathcal{A} \).

**Proof.** To apply Jónsson’s Lemma to the variety \( \mathcal{V}(\mathbb{Z}_k) \) we need to investigate \( \mathbb{H}(\mathbb{Z}_k) \). By [4, Proposition 2.1], any given subalgebra of \( \mathbb{Z}_k \) is isomorphic to some \( \mathbb{Z}_r \), where \( r \leq k \) and \( r \) is even if \( k \) is even. Now consider a homomorphism \( h \) with domain \( \mathbb{Z}_r \). Its image \( im h \) is isomorphic to a quotient \( \mathbb{Z}_r/\theta \), where \( \theta \) is a congruence on \( \mathbb{Z}_r \). According to [4, Proposition 2.4], there exists \( p < \left[ \frac{r+1}{2} \right] \) such that \( a \theta b \) if and only if \( a = b \) or \( |a|, |b| \leq p \). Thus \( \mathbb{Z}_r/\theta \cong \mathbb{Z}_{2(\frac{r-1}{2})+1} \). We now separate the odd and even cases.

Assume \( k = 2m-1 \). In this case \( s \leq p \). It follows that \( \mathbb{Z}_{2(\frac{s-1}{2})+1} \in \text{ISP}(\mathbb{Z}_k) \). This implies that every subdirectly irreducible algebra in \( \text{HSP}(\mathbb{Z}_{2m-1}) \) belongs to \( \text{ISP}(\mathbb{Z}_{2m-1}) \) and hence that \( \text{HSP}(\mathbb{Z}_{2m-1}) \subseteq \text{ISP}(\mathbb{Z}_{2m-1}) \). Hence the two classes coincide. We have proved (i).

Assume \( k = 2m \). Any member of \( \mathbb{H}(\mathbb{Z}_{2m}) \) which is already in \( \mathbb{S}(\mathbb{Z}_{2m}) \) certainly belongs to \( \text{ISP}(\mathbb{Z}_{2m}, \mathbb{Z}_{2m-1}) \). Now suppose that we have a non-trivial congruence \( \theta \) on some subalgebra \( \mathbb{Z}_r \), with \( r \) even and \( r \leq 2m \). From above, \( \mathbb{Z}_r/\theta \cong \mathbb{Z}_{2(\frac{s-1}{2})+1} \). Since \( s = r/2 \leq m \) and \( p \geq 1 \), the quotient is isomorphic to a subalgebra of \( \mathbb{Z}_{2m-1} \) and so in \( \text{ISP}(\mathbb{Z}_{2m}, \mathbb{Z}_{2m-1}) \). Clearly \( \text{ISP}(\mathbb{Z}_{2m}) \) is not a variety for \( m > 1 \) since it does not contain \( \text{im } u = \mathbb{Z}_{2m-1} \). We deduce that (i) holds.

Finally (iii) is now immediate from (i) and (ii).

The following proposition parallels the content of Proposition 3.1 and can be proved in the same way. Note that \( W_{2m} \) has \( W_{2m-1} \) as an \( \mathcal{A} \)-morphic image: the surjective \( \mathcal{A} \)-morphism \( u: \mathbb{Z}_{2m} \rightarrow \mathbb{Z}_{2m-1} \) defined above is an \( \mathcal{A} \)-morphism.

**Proposition 3.2** (Sugihara monoids: quasivarieties and varieties).

(i) \( \text{ISP}(W_{2m-1}) = \text{HSP}(W_{2m-1}) \), and so is a variety.

(ii) \( \text{ISP}(W_{2m}) \subseteq \text{HSP}(W_{2m}) \) and \( \text{HSP}(W_{2m}) = \text{ISP}(W_{2m}, W_{2m-1}) \).
(iii) The inclusion orderings of the subquasivarieties \( \{\text{HSP}(W_k)\}_{k \geq 1} \) of \( \mathcal{M} \) and of the subvarieties \( \{\text{HSP}(W_k)\}_{k \geq 1} \) of \( \mathcal{M} \) are as shown in Figure 1.

Proof. (i) and (ii) can be proved in the same way as the corresponding claims for Sugihara algebras (see also for example [29, Section 3]). We may also imitate the proof in Proposition 3.1 to deduce that the finitely generated subquasivarieties of \( \mathcal{M} \) are ordered in exactly the same way as those of \( \mathcal{A} \) are. It is not true that \( W_{2m} \) belongs to \( \text{HSP}(W_{2m+1}) \). If it did, \( W_{2n} \) would have to be a homomorphic image of a subalgebra of \( W_{2m+1} \). But this is impossible since the constants \( t \) and \( f \) are distinct in \( W_{2m} \) but coincide in any member of \( \text{HSP}(W_{2m+1}) \). □

The cornerstone for our development of amenable single-sorted dualities for the quasivarieties \( \mathcal{A}_k \), for \( k \) both odd and even, was our analysis of the partial endomorphisms of the algebras \( Z_k \). The results in [4, Section 2] are equally important in the multisorted case and we shall want their analogues for Sugihara monoids too.

We recall for reference the definitions of the maps from \( Z_{2m-1} \) to itself that [4, Proposition 2.9] shows constitute a generating set for \( \text{End}_p(Z_{2m-1}) \):

partial endomorphisms:

\[
\begin{align*}
  f_0 &: Z_{2m-1} \setminus \{0\} \to Z_{2m-1}, & f_0(a) &= a \text{ for } a \neq 0; \\
  f_1 &: Z_{2m-1} \setminus \{1, -1\} \to Z_{2m-1}, & f_1(a) &= a \text{ for } a \neq \pm 1; \\
  \text{and for } 1 < i < m, \\
  f_i &: Z_{2m-1} \setminus \{i, -i\} \to Z_{2m-1}, & f_i(a) &= \begin{cases} 
  i & \text{if } a = i - 1, \\
  -i & \text{if } a = -(i - 1), \\
  a & \text{otherwise};
  \end{cases}
\end{align*}
\]

endomorphism:

\[
g &: Z_{2m-1} \to Z_{2m-1}, & g(a) &= \begin{cases} 
  a - 1 & \text{if } a > 0, \\
  a + 1 & \text{if } a < 0, \\
  0 & \text{if } x = 0.
  \end{cases}
\]

We remark that, of course, any morphism between Sugihara monoids necessarily includes the constants \( t \) and \( f \) in its domain and image. This ensures that the family of partial endomorphisms of a Sugihara monoid forms a monoid under composition. (Cf. the comments in [4] for the algebra case.)
We shall adopt the convention that the same symbol is used for two morphisms, one of Sugihara algebras and the other of Sugihara monoids, when they have the same underlying set map. However we shall distinguish sets of (partial) endomorphisms and, later, (partial) homomorphisms of Sugihara monoids from their Sugihara algebra counterparts by including a superscript \(^t\). This usage accords with that used for R-mingle logics. For partial endomorphisms of Sugihara monoids we have the following analogue of [4, Propositions 2.8 and 2.9].

**Proposition 3.3** (partial endomorphism of Sugihara monoids, odd case).

(i) The endomorphisms of \(W_{2m-1}\) are the same as those of \(Z_{2m-1}\).

(ii) \(\text{End}^t_p(W_{2m-1})\) is generated by \(\{f_1, \ldots, f_{m-1}, g\}\).

*Proof.* (i) Since 0 is the unique \(\neg\)-fixed point in \(Z_{2m-1}\), every endomorphism of \(Z_{2m-1}\) is also an endomorphism of \(W_{2m-1}\).

For (ii) we first note that \(0\) is in the domain of any element \(h\) of \(\text{End}_p^t(W_{2m-1})\) and \(h(0) = 0\) since \(h\) preserves \(\neg\). The only member of our standard generating set for \(\text{End}_p^t(Z_{2m-1})\) whose domain omits \(0\) is \(f_0\).

We aim to show that any finite composition of partial endomorphisms of \(Z_{2m-1}\) which includes one or more occurrences of \(f_0\) fails to include \(0\) in its domain and so cannot be a member of \(\text{End}_p^t(W_{2m-1})\). Consider \(p \circ f_0 \circ q\), where \(p \in \text{End}_p^t(W_{2m-1})\) abs \(q\) is a composition not including \(f_0\). Then \(q(0)\) is defined and \(q(0) = 0\). But \(p(f_0(y))\) is undefined when \(y = q(0)\). Hence \(p \circ f_0 \circ q \notin \text{End}_p^t(W_{2m-1})\). When \(p\) and/or \(q\) is absent the argument is even simpler.

Conversely, every element of \(\{f_1, \ldots, f_{m-1}, g\}\) preserves \(0\) and so any composition of maps from this set belongs to \(\text{End}_p^t(W_{2m-1})\).

We now turn to the even case. Here [4, Proposition 2.8] tells us that \(\text{End}_p(Z_{2m})\) is generated by the following maps: \(h_i: Z_{2m} \setminus \{i, -i\} \to Z_{2m}\) is defined by

\[
h_i(a) = \begin{cases} 
i & \text{if } a = i - 1, \\
-i & \text{if } a = -(i - 1), \\
da & \text{otherwise} \end{cases}
\]

and \(j: Z_{2m} \setminus \{1, -1\} \to Z_{2m}\) by

\[
j(a) = \begin{cases} a - 1 & \text{if } a > 0, \\
+a & \text{otherwise.} \end{cases}
\]

We have used different symbols here for the maps from those used in [4] to avoid a conflict of notation when we work with the variety generated by \(Z_{2m}\) and shall need to consider \(Z_{2m}\) and \(Z_{2m-1}\) at the same time.

Each partial endomorphism of \(W_{2m}\) must include \(\pm 1\) in its domain and must fix these points. Assume \(m \geq 3\). For each element \(e \in \text{End}_p(Z_{2(m-1)})\) we define \(\overline{e}\) as follows:

\[
\overline{e}(a) = \begin{cases} a & \text{if } a = \pm 1, \\
e(a - 1) + 1 & \text{if } a > 1 \text{ and } a - 1 \in \text{dom } e, \\
e(a + 1) - 1 & \text{if } a < -1 \text{ and } a + 1 \in \text{dom } e. \end{cases}
\]

**Proposition 3.4** (partial endomorphisms of Sugihara monoids, even case).

(i) The only endomorphism of \(W_{2m}\) is the identity map.
(ii) Let \( e \in \text{End}_p(Z_{2(m-1)}) \) and define \( \tau \) as above. Then \( \kappa : e \mapsto \tau \) sets up a bijection between \( \text{End}_p(Z_{2(m-1)}) \) and \( \text{End}_p^t(W_{2m}) \). Moreover, \( \text{End}_p^t(W_{2m}) \) is generated by \( \{h_2, \ldots, h_{m-1}\} \).

Proof. Certainly \( \tau \) is a partial endomorphism of \( W_{2m} \) for each \( e \in \text{End}_p(Z_{2(m-1)}) \). Also, every element of \( \text{End}_p(Z_{2m}) \) is injective, so the same must be true of elements of \( \text{End}_p^t(W_{2m}) \). It follows from this that \( \kappa \) is surjective. The map \( \lambda \) inverse to \( \kappa \) acts by restricting \( h \in \text{End}_p^t(W_{2m}) \) to \( \text{dom } h \setminus \{\pm 1\} \), relabelling domain and image in the obvious way to realise this as an element of \( \text{End}_p(Z_{2(m-1)}) \).

The claim concerning a generating set for \( \text{End}_p^t(W_{2m}) \) follows from the corresponding result for \( \text{End}_p(Z_{2(m-1)}) \) (see [4, Proposition 2.8]). \( \square \)

4. Multisorted dualities for Sugihara algebras

As promised in Section 1 We shall present multisorted dualities for the classes \( \text{HSP}(Z_{2m-1}) = \text{ISP}(Z_{2m-1}) \) (the odd case) and \( \text{HSP}(Z_{2m}) = \text{ISP}(Z_{2m}, Z_{2m-1}) \) (even case). In the odd case our sole purpose here is to recast our Piggyback Strong Duality Theorem [4, Theorem 4.3] in two-sorted form, with a view to future applications. The process we use to convert our earlier two-carrier duality for \( \mathcal{I} \mathcal{A}_{2m-1} \) could be used more generally to split a sort with more than one carrier map into separate sorts, with suitable adaptations being necessary to the alter ego.

Our first—and principal—task is to establish appropriate notation. Fix \( m \geq 2 \). We create two disjoint copies of \( Z_{2m-1} \) and call these \( P^- \) and \( P^+ \). Let \( \text{id}_{-+} \) and \( \text{id}_{+-} \) denote, respectively, the natural isomorphisms from \( P^- \) to \( P^+ \) and from \( P^+ \) to \( P^- \). When working with \( P^- \) and \( P^+ \) individually we shall often think of each of them as equal to \( Z_{2m-1} \). When working with both sorts at the same time, we shall use superscripts to indicate interpretations on \( P^- \) and \( P^+ \). For example, \( g^- \) and \( g^+ \) denote the interpretations on \( P^+ \) and \( P^- \) of the endomorphism \( g \) of \( Z_{2m-1} \).

We now set up carrier maps from \( P^- \) and \( P^+ \) into \( 2 \). Define \( \Omega_{P-} = \{\delta^-\} \) and \( \Omega_{P+} = \{\delta^+\} \), where

\[
\delta^+(a) = 1 \iff a \geq 1 \quad \text{and} \quad \delta^-(a) = 1 \iff a \geq 0
\]

(cf. [4, Section 4]). (Because we are working here with \( Z_{2m-1} \) and shall later bring \( Z_{2m+1} \) into the picture as well, we reserve the symbols \( \alpha^\pm \) as used with \( Z_{2m+1} \) in [4, Section 4] to use for carrier maps on \( Z_{2m+1} \).)

To illustrate the use of multisorted structures we shall present in full the check of the separation property needed in our application of Theorem 2.1; compare the following lemma with the single-sorted version, [4, Lemma 4.1].

**Lemma 4.1** (separation lemma for two-sorted duality for \( \mathcal{I} \mathcal{A}_{2m-1} \)). Let \( M \in \{P^+, P^-\} \). Let \( a, b \in M \) with \( a \neq b \). Then there exists \( M' \in \{P^+, P^-\} \) and a homomorphism \( \zeta : M \to M' \) such that \( \omega_{M'}(\zeta(a)) \neq \omega_M(\zeta(b)) \). Here \( \zeta \) is either the identity map on one of the sorts or is a composite of maps drawn from \( \{g^-, \text{id}_{-+}, \text{id}_{+-}\} \).

Proof. We may without loss of generality assume that \( M = P^- \).

Suppose first that \( a < 0 \leq b \). Let \( M' = P^- \) and \( \zeta = \text{id}_{+-} \). Next suppose that \( a < b \leq 0 \). Let \( M' = P^- \) and \( \zeta = (g^-)^{-b} \). Then \( \delta^-(\zeta(a)) = 0 \neq 1 = \delta^-(\zeta(b)) \).

Now take \( a \leq 0 < b \). Let \( M' = P^+ \) and \( \zeta = \text{id}_{-+} \). Then \( \delta^+(\text{id}_{-+}(a)) = 0 \neq 1 = \delta^+(\text{id}_{-+}(b)) \). Finally suppose \( 0 \leq a < b \). Again let \( M' = P^+ \) and let \( \zeta = (g^+)^a \circ \text{id}_{-+} \). Then \( \delta^+((\zeta(a)) = 0 \neq 1 = \delta^-(\zeta(b)) \).

\( \square \)
Our next task is to identify the piggyback subalgebras. In [4, Proposition 4.2], for the single-sorted case, we showed that every piggyback relation (maximal or not) is the graph of a partial endomorphism of $\mathbb{Z}_{2m-1}$ or the converse of such a graph. Switching to the two-sorted version, nothing changes except that we replace our piggyback subalgebras of $\mathbb{Z}_{2m-1}^2$ by the corresponding subalgebras of $M \times M'$, where $M, M' \in \{P^-, P^+\}$, with the appropriate carrier map acting on each coordinate. For that we need to set the following notation, given $M, M' \in \{P^-, P^+\}$ we let $\text{Hom}_p(M, M')$ denote the set of $\mathcal{A}$-morphisms $h$ with $\text{dom} h \subseteq M$ and $\text{im} h \subseteq M'$.

**Proposition 4.2** (multisorted piggyback relations for $\mathcal{A}_{2m-1}$).

(i) A subalgebra of $(\delta^-, \delta^-)^{-1}(\leq)$ is the graph of some $h \in \text{End}_p(P^-)$.

(ii) A subalgebra of $(\delta^+, \delta^+)^{-1}(\leq)$ is the converse of the graph of some $k \in \text{End}_p(P^+)$.

(iii) A subalgebra of $(\delta^-, \delta^+)^{-1}(\leq)$ is the graph of some $h \in \text{Hom}_p(P^-, P^+)$ for which $0 \notin \text{dom} h$ and $0 \notin \text{im} h$.

(iv) A subalgebra of $(\delta^+, \delta^-)^{-1}(\leq)$ is the graph of some $h \in \text{Hom}_p(P^+, P^-)$ or is the converse of the graph of some $k \in \text{Hom}_p(P^-, P^+)$.

Proposition 4.3 will allow us to avoid ‘doubling up’ of mirror-image structure, since we have included the linking isomorphisms between the sorts. This idea will enable us in due course to streamline our multisorted alter ego for $\mathbb{Z}_{2m-1}$. We opt to let the sort $P^-$ bear the brunt of structuring the alter ego. But there is complete symmetry between the sorts and we could equally well have prioritised $P^+$.

**Proposition 4.3** (entailment of partial endomorphisms and homomorphisms, odd case). Any map in

(i) $\text{End}_p(P^-)$, (ii) $\text{End}_p(P^+)$, (iii) $\text{Hom}_p(P^-, P^+)$, (iv) $\text{Hom}_p(P^+, P^-)$

is obtained by composition by a generating set for $\text{End}_p(\mathbb{Z}_{2m-1})$, whose members are interpreted on $P^-$, together with $\{\text{id}^+, \text{id}^+_-\}$.

**Proof.** The claim in the proposition for a map of type (i) is immediate. A map of type (ii), (iii) or (iv) is, respectively, expressible in the form $\text{id}^+_- \circ e^- \circ \text{id}^+_-$, $\text{id}^+_- \circ e^+ \circ \text{id}^+_-$, or $e^- \circ \text{id}^+_-$, where $e^- \in \text{End}_p(P^-)$.

We now present our promised two-sorted strong duality theorem for $\mathcal{A}_{2m-1}$. As anticipated, it has a clear relationship to its single-sorted, multi-carrier, counterpart. In particular we incorporate into the alter ego (a copy of) the same generating set for $\text{End}_p(\mathbb{Z}_{2m-1})$ as we used in [4, Theorem 4.3]; see Section 3, recalling [4, Proposition 2.9].

**Theorem 4.4** (Two-sorted Piggyback Strong Duality Theorem for $\mathcal{A}_{2m-1}$). Fix $m$ with $m \geq 2$. Take disjoint sorts $P^-$ and $P^+$ each isomorphic to $\mathbb{Z}_{2m-1}$. Let $\Omega_{P^-} = \{\delta^-\}$ and $\Omega_{P^+} = \{\delta^+\}$. Consider the two-sorted structure based on $\mathcal{M} := \{P^-, P^+\}$ having universe $M_0 := P^- \cup P^+$. Let

$$G = \{g, \text{id}^+_-, \text{id}^+\}, \quad H = \{f_0, f_1, f_2, \ldots, f_{m-2}\}, \quad K = \{0_{P^{-}}\}.$$

(Here $g, f_0, \ldots, f_{m-2}$ are to be interpreted on $P^-$.) Then $\mathcal{M} := (M_0; G, H, K, T)$ yields a strong and hence full duality on $\mathcal{A}_{2m-1}$.

**Proof.** We proceed in the same way as in the proof of [4, Theorem 4.3], but invoking Theorem 2.1 instead of the single-sorted, multi-carrier Piggyback Strong Duality Theorem given in [4, Theorem 3.3]. We know that every non-trivial subalgebra of
each of $P^+$ and $P^-$ is subdirectly irreducible. Lemma 4.1 established the separation condition. Finally we want to invoke the Shift Lemma as given in 2.2 to confirm that our chosen, slimmed down, alter ego strongly dualises $\mathcal{SA}_{2m-1}$. Propositions 4.2 and 4.3 provide all the facts we need.

We next provide a three-sorted duality for the variety $V(Z_{2m}) = ISP(Z_{2m}, Z_{2m-1})$ for $m \geq 2$. Formally our sorts will have disjoint(ified) universes. We include two copies, $P^-$ and $P^+$, of $Z_{2m-1}$ and a single copy $Q$ of $Z_{2m}$. We shall sometimes treat the sorts as though identified with the appropriate $Z_k$, being more explicit when this is warranted.

In the odd case we introduced the maps $id_-$ and $id_+$ so as to avoid including in our two-sorted alter ego sets of partial homomorphisms $Hom_p(M, M')$ for all four choices of $M, M'$ from $\{P^-, P^+\}$. Likewise, with the third sort $Q$ now in play we want to use linking maps between $Q$ and $P^\pm$ to avoid our alter ego needing to include partial homomorphisms going between $Q$ and the other sorts. We already in Section 3 made use of the surjective homomorphism $u: Z_{2m} \to Z_{2m-1}$. This gives rise to surjective homomorphisms $u^\pm: Q \to P^\pm$. When we restrict $u^-$ to $Q \setminus \{\pm 1\}$ we obtain a bijective homomorphism mapping onto $P^- \setminus \{0\}$. We denote its inverse by $v^-$. Corresponding claims hold for $u^+$. See Figure 2.

![Figure 2. Linking maps between sorts](image)

We now set up the carrier maps we shall need in order to satisfy the separation condition in the Piggyback Theorem. As for the odd case,

- from $U(P^-) \to 2$: $\delta^+(a) = 1 \iff a \geq 1$;
- from $U(P^+) \to 2$: $\delta^-(a) = 1 \iff a \geq 0$;

and, from the even case as analysed in [4],

- from $U(Q) \to 2$: $\beta(a) = 1 \iff a > 0$.

We take $\Omega_{P^-} = \{\delta^-\}$, $\Omega_{P^+} = \{\delta^+\}$, and $\Omega_Q = \{\beta\}$. For $M \in \{P^-, P^+, Q\}$, we write $\omega_M$ for the unique element of $\Omega_M$.

**Lemma 4.5** (separation lemma for $V(Z_{2m})$). Let $S = \{g^-, id_-, id_-, u^-\}$, where the members of $S$ are as defined above.

Let $M \in \{P^-, P^+, Q\}$ and $a \neq b$ in $M$. Then there exists $M' \in \{P^-, P^+, Q\}$ and $\zeta \in Hom(M, M')$ such that $\omega_M(\zeta(a)) \neq \omega_{M'}(\zeta(b))$, where $\zeta$ is an identity map on one of the sorts or is a composite of maps drawn from $\{g^-, id_-, id_+, u^-\}$. 

Proof. Lemma 4.1 covers the cases in which $M \in \{P^-, P^+\}$. So let $c \neq d$ in $Q$. If $c$ and $d$ are of opposite sign then $\beta(c) \neq \beta(d)$ and we take $M' = Q$ and $\zeta = id_Q$. Now take $c < d \leq -1$ in $Q$. Then $u^-(c) < u^-(d) \leq 0$ in $P^-$. Let $a := u^- (c)$ and $b := u^-(d)$ and proceed as in the proof of Lemma 4.1. Similarly, if $c > d \geq 1$ in $Q$ then $u^+(c) > u^+(d) \geq 0$ in $P^+$. Again we can now argue as in the proof of Lemma 4.1. □

We must now describe the multisorted piggyback relations for $\mathcal{V}(\mathbb{Z}_{2m})$. Given $M, M' \in \{P^-, P^+, Q\}$, we shall write $R_{M,M'}$ for the set of universes $r$ of subalgebras $r$ of $M \times M'$ such that $r \subseteq (\omega_M, \omega_{M'})^{-1}(\leq)$ and $r$ is maximal with respect to this containment.

We already know that every piggyback relation in the odd case (maximal or not) is the graph of a partial homomorphism or the converse of such a graph. In addition, every relation in $R_{Q,Q}$ is the graph of a member of $\text{End}_p(Q)$, by [4, Proposition 6.3]. We shall subsequently want to present a duality for the Sugihara monoid variety $\mathcal{V}(\mathbb{W}_{2m})$ by making small adaptations to that for $\mathcal{V}(\mathbb{Z}_{2m})$. This means that it is expedient when we encounter graphs of $\mathcal{MS}$-morphisms in the algebra setting we should be sufficiently explicit to be able detect easily whether $\mathcal{MS}$-morphisms are also available. Furthermore our structural analysis of Sugihara algebras and monoids in succeeding papers requires detailed information on piggyback relations and it is convenient to record this now.

|       | $P^-$ | $Q$ | $P^+$ |
|-------|-------|-----|-------|
| $P^-$ | graph $h$ | graph $h$ | graph $h$ with $0 \notin \text{dom} \, h$, $0 \notin \text{im} \, h$ |
| $Q$   | graph $h$ | graph $h$ | graph $h$, with $0 \notin \text{im} \, h$ |
| $P^+$ | graph $h$ or graph $k^-$ | graph $k^-$, with $0 \notin \text{dom} \, k$ | graph $k^-$ |

Table 1. Piggyback relations for $\mathcal{V}(\mathbb{Z}_{2m})$

**Proposition 4.6** (multisorted piggyback relations for $\mathcal{V}(\mathbb{Z}_{2m})$). Let $m \geq 2$. For each choice of $M, M' \in \{P^-, P^+, Q\}$, the entry in the $M, M'$-cell in Table 1 shows the form that a member $S$ of $R_{M,M'}$ must take. If graph $h$ appears in that cell it is to be assumed that $h$ is a partial homomorphism with $\text{dom} \, h \subseteq M$ and $\text{im} \, h \subseteq M'$, with additional properties as stipulated. In cases in which graph $k^-$ appears then $k$ is to be taken to be a homomorphism with $\text{dom} \, k \subseteq M$ and $\text{im} \, k \subseteq M$. Necessary restrictions arising from $0 \notin Q$ are left implicit.

Proof. The entries in the four corner cells in the table deal with the forms taken by the piggyback relations in the sets $R_{M,M'}$ for which neither $M$ nor $M'$ is $Q$; see Proposition 4.2. The results originate in [4, Proposition 4.2], applied with $n = m - 1$. For the entry in the bottom left corner the dichotomy is explained in item (iv) of that proposition. To describe the members of $R_{Q,Q}$ we call on [4, Proposition 6.3]: any (maximal) piggyback subalgebra is the graph of a (necessarily invertible) partial endomorphism.

We now deal with the four remaining cases, in which we need to consider sorts of different parities.
Throughout we work with $S \in R_{M,M'}$ initially viewed as a subalgebra of $\mathbb{Z}_{2m+1}^2$, with $Q$ identified with $\mathbb{Z}_{2m}$ and whichever of $P^-$ or $P^+$ is in play in a particular item identified with $\mathbb{Z}_{2m-1}$; in this setting we regard $\mathbb{Z}_{2m}$ and $\mathbb{Z}_{2m-1}$ as subalgebras of $\mathbb{Z}_{2m+1}$. We may then regard $\omega_{P^+} = \delta^-$ and $\omega_{P^-} = \delta^+$, as restrictions to $\mathbb{Z}_{2m-1}$ of the maps $\alpha^-$ and $\alpha^+$ used as carrier maps for the duality for $\mathcal{M}_{2m+1}$ as $\alpha^+(a) = 1 \iff a \geq 1$ and $\alpha^-(a) = 1 \iff a \geq 0$. o our strategy for classifying piggyback relations linking $Q$ with the sorts $P^-$ and $P^+$ will be first to realise them, up to isomorphism, as subalgebras of $\mathbb{Z}_{2m+1}^2$ associated with appropriate piggyback relations for the odd case. We then wish to assert that the piggyback relations from which we started are graphs of partial homomorphisms or converses thereof, where We take account of any constraints inherent in the various cases as regards domain and image of the partial maps which can arise. To justify our claims two approaches are available. We can appeal directly to [4, Proposition 4.2] and then switch into the language of sorts or alternatively we can translate to sorts first and then call on Proposition 4.2. The approaches are equivalent but, either way, some obvious re-alignment of notation is required.

$R_{P^-,Q}$: Consider a subalgebra $S$ of $\mathbb{Z}_{2m-1} \times \mathbb{Z}_{2m}$ contained in $(\delta^-,\beta)^{-1}(\subseteq)$. Then $(a,b) \in S$ and $\delta^-(a) = 1$ together imply $\beta(b) = 1$. This implies that $\alpha^-(a) = 1$ forces $\alpha^-(b) = 1$ since $b = 0$ does not arise. Then $S$ is the graph of a partial endomorphism $e$ of $\mathbb{Z}_{2m+1}$, where dom $e$ omits $\pm n$ and im $e$ omits $0$. Translating into the language of sorts, we obtain the characterization of $R_{P^-,Q}$ shown in the table.

$R_{Q,P^-}$: Let $S$ be a subalgebra of $\mathbb{Z}_{2m} \times \mathbb{Z}_{2m-1}$ for which $S \subseteq (\beta, \delta^-)^{-1}(\subseteq)$. This time we may view $S$ as a subalgebra of $\mathbb{Z}_{2m+1}^2$ within $(\alpha^+, \alpha^-)^{-1}(\subseteq)$ and for which, necessarily, $S \cap (\{0\} \times \mathbb{Z}_{2m}) = \emptyset$: We deduce that $S \subseteq (\alpha^-, \alpha^-)^{-1}(\subseteq)$ and as such is the graph of a member of End$_p(\mathbb{Z}_{2m+1})$. Restating in terms of sorts, we obtain the required result.

$R_{Q,P^+}$: Let $S$ be a subalgebra of $\mathbb{Z}_{2m} \times \mathbb{Z}_{2m-1}$ for which $S \subseteq (\beta, \delta^+)^{-1}(\subseteq)$. This time $(a,b) \in S$ implies $a \neq 0$ so that $S$, regarded as a subalgebra of $\mathbb{Z}_{2m+1}^2$, is contained in $(\alpha^-, \alpha^+)^{-1}(\subseteq)$. Any allowable subalgebra is the graph of a partial endomorphism $e$ of $\mathbb{Z}_{2m+1}$ which excludes $0$ from both domain and image. In terms of sorts, this gives the graph of a partial homomorphism whose image excludes $0$.

$R_{P^-,P}$: Let $S$ be a subalgebra of $\mathbb{Z}_{2m-1} \times \mathbb{Z}_{2m}$ contained in $(\delta^+, \beta)^{-1}(\subseteq)$. View $S$ as a subalgebra of $\mathbb{Z}_{2m+1}^2$. Here $(a,b) \in S$ implies $a \neq \pm m$, $b \neq 0$ and $\delta^+(a) = 1$ forces $\beta(b) = 1$. But for such $a,b$ this last condition is the same as $\alpha^+(a) = 1$ forces $\alpha^+(b) = 1$. Hence $S^\omega$ is the graph of a partial endomorphism $k$ of $\mathbb{Z}_{2m+1}$, by [4, Proposition 4.2(ii)]. We require $0 \notin \text{dom } h$ (this is of course necessary) and $h \in \mathbb{Z}_{2m-1}$. So $k$ can be viewed as a partial homomorphism from $Q$ into $P^+$.

A few additional remarks on the entries in Table 1 are in order. First of all there is more symmetry than may at first sight appear. The elements of $R_{Q,P}$ are graphs of partial endomorphisms $h$ of $Q$. Any such $h$ is invertible and $k := h^{-1}$ is such that graph $k^\omega = \text{graph } h$. When we translate from $\mathbb{Z}_{2m+1}^2$ into partial maps between sorts, presence or absence of $0$ from domain or image is automatic.

We now derive an analogue of of Proposition 4.3 but now dealing with entailment of partial homomorphisms between non-isomorphic sorts. Refer back to the $M$-Shift
Proposition 4.7 (partial homomorphisms between sorts).

(i) Any \( h \in \text{Hom}_p(\mathbf{P}^-, \mathbf{Q}) \) is expressible as a composite of maps drawn from \( \text{End}_p(\mathbf{Q}) \cup \{ v^- \} \).

Any \( h' \in \text{Hom}_p(\mathbf{P}^+, \mathbf{Q}) \) is expressible as a composite of maps drawn from \( \text{End}_p(\mathbf{Q}) \cup \{ v^- \} \cup \{ \text{id}_{+-} \} \).

(ii) Any \( h \in \text{Hom}_p(\mathbf{Q}, \mathbf{P}^-) \) expressible as the restriction of a composition of maps drawn from \( \text{End}_p(\mathbf{P}^-) \cup \{ u^- \} \).

Any \( h' \in \text{Hom}_p(\mathbf{Q}, \mathbf{P}^+) \) is expressible as the restriction of a composition of maps drawn from \( \text{End}_p(\mathbf{P}^-) \cup \{ u^- \} \cup \{ \text{id}_{+-} \} \).

Proof. In [4, Proposition 2.6] it was shown that for any \( p \leq n \) and for \( 0 < a_1 < \cdots < a_p \) and \( 0 < b_1 < \cdots < b_p \) there exists \( e \in \text{End}_p(\mathbf{Z}_{2n}) \) such that \( e(a_i) = b_i \) for \( 1 \leq i \leq p \). Moreover \( e \) extends to a partial endomorphism of \( \mathbf{Z}_{2n+1} \) which sends 0 to 0. Necessarily \( a \in \text{dom} \ e \) if and only if \( -a \in \text{dom} \ e \). We may assume that \( |\text{dom} \ e| = 2p \) in the odd case and \( 2p + 1 \) in the even case.

Consider (i). Take \( h \in \text{Hom}_p(\mathbf{P}^-, \mathbf{Q}) \). Necessarily \( 0 \notin \text{dom} \ h \) (since \( \mathbf{Q} \) has no fixed point). Then \( h \) is injective and so \( |\text{dom} \ h| = |\text{im} \ h| \). (See [4, Proposition 2.2].) But \( v^- (\text{dom} \ h) \) is also a subalgebra of \( \mathbf{Q} \) of cardinality \( |\text{dom} \ h| \). Moreover, both \( h \) and \( v^- \) are strictly monotonic. Hence there exists \( e \in \text{End}_p(\mathbf{Q}) \) with \( h(a) = e(v^-(a)) \) for \( a \in \text{dom} \ h \) and \( \text{dom} (e \circ v^-) = \text{dom} \ h \). Therefore \( h = e \circ v^- \). This proves the first claim in (i).

Consider \( h' \in \text{Hom}_p(\mathbf{P}^+, \mathbf{Q}) \). From above, \( h' \circ \text{id}_{+-} = e \circ v^- \) for some \( e \in \text{End}_p(\mathbf{Q}) \). Now observe that \( h' = e \circ v^- \circ \text{id}_{+-} \). This completes the proof of (i).

Now consider (ii). Take \( h \in \text{Hom}_p(\mathbf{Q}, \mathbf{P}^-) \). First assume \( 0 \notin \text{im} \ h \). Then \( h \) maps \( \text{dom} \ h \) into \( \mathbf{P}^- \setminus \{0\} \) and [4, Proposition 2.2] implies that \( h \) is injective and \( u^-|_{\text{dom} \ h} \) is also injective. Then there exists \( e \in \text{End}_p(\mathbf{P}^-) \) such that \( \text{dom} \ e = u^- (\text{dom} \ h) \) and \( h(a) = e(u^-(a)) \) for all \( a \in \text{dom} \ h \). Thus \( h = (e \circ u^-)|_{\text{dom} \ h} \).

Now assume that \( 0 \in \text{im} \ h \). Then there exists a minimal \( s > 0 \) such that \( h^{-1}(0) \subseteq [s, s] = (u^-)^{-1}(0) \). Let \( h_s := (g^s)^{s-1} \circ u^- \). Then \( h_s \) is a homomorphism from \( \mathbf{Q} \) into \( \mathbf{P}^- \). Both \( h \) and \( h_s \) are strictly monotonic on \( \{s + 1, \ldots, m\} \). There exists a partial endomorphism \( e \) of \( \mathbf{P}^- \) with \( \text{dom} \ e = \text{im} \ h \) and \( h(a) = e(h_s(a)) \) for \( a \in \text{dom} \ h \) and \( a > s \). Since \( \text{dom} \ h \cap (h_s)^{-1}(0) \neq \emptyset \) and \( 0 \in \text{dom} \ e \) the maps \( e \circ h_s \) and \( h \) agree on the domain of \( h \). This completes the proof of the first assertion in (ii).

For the second assertion we consider \( \text{id}_{+-} \circ h' \).

We have now assembled all the ingredients for our duality theorem for \( \mathcal{V}(\mathbf{Z}_{2m}) \). Table 2 indicates the maps we shall include in our multisorted alter ego. In the table all undecorated maps are to be viewed as being interpreted as maps between the indicated sorts. That is, we have omitted the superscripts previously used to indicate the sorts when interpretations of \( \mathcal{K} \)-morphisms between \( \mathbf{Z}_{2m}, \mathbf{Z}_{2m-1} \), and their subalgebras, are involved. (The position of each entry dictates the intended interpretations of the maps.)

Our duality theorem is proved in essentially the same way as that for the odd case, and we need only give a sketch of the proof.
Lemma 4.5. Let \( P^- \), \( P^+ \) and \( Q \) be disjointified copies of \( \mathbb{Z}_2^{m-1} \), \( \mathbb{Z}_2^{m-1} \) and \( \mathbb{Z}_2^m \), respectively, with carrier maps \( \delta^- \), \( \delta^+ \) and \( \beta \), respectively. Let \( M = \{ P^-, P^+, Q \} \) and let \( T \) be the disjoint union topology on \( M_0 := P^- \cup P^+ \cup Q \) derived from the discrete topology on each sort. Let \( M \hat{=} (M_0; G, H, K, T) \), where the elements of \( G \cup H \) are the maps presented in Table 2 and \( K = \{0_{P^-}\} \). Then \( M \hat{=} \) strongly dualises \( \mathcal{V}(\mathbb{Z}_2^m) \).

Proof. With our selected sorts and carrier maps, the proposed alter ego ensures that the separation condition is satisfied; see Lemma 4.5. We now consider an enlarged alter ego in which we include all the structure that Theorem 2.1 tells us will give a strong duality, then we can discard from this relational structure any operations, partial operations, relations or constants which are entailed by the structure we retain. Proposition 4.6 implies that all the piggyback relations are entailed by the full sets of partial endomorphisms and partial homomorphisms which we have included in our enlarged alter ego to guarantee strongness. Observe that \( f_0: P^- \setminus \{0\} \rightarrow P^- \setminus \{0\} \) coincides with \( v^- \circ h_m \circ \cdots \circ h_2 \circ u^- \). Hence, by [4, Propositions 2.8 and 2.9], every element of \( \text{End}_p(P^-) \) and \( \text{End}_p(Q) \) can be obtain as a composition of maps in \( G \cup H \). Finally we call on Proposition 4.3 and Proposition 4.7. We conclude that, with \( G \cup H \cup K \) as in our proposed alter ego, we indeed obtain a strong duality for \( \mathcal{V}(\mathbb{Z}_2^m) \).

We have made our duality economical at the price of a loss of symmetry as regards \( P^- \) and \( P^+ \). When working with the duality it is likely to be convenient to think in terms of a symmetric formulation. However it is worth noting that we cannot thereby dispense with the linking maps \( \text{id}_{-+} \) and \( \text{id}_{+-} \) since these are needed for the separation lemma. The same comment applies to the duality for \( \mathcal{V}(\mathbb{Z}_2^{m-1}) \).

We should expect some asymmetry in relation to the piggyback relations, in Theorems 4.4 and 4.8, and in their single-sorted counterparts in [4]. This stems from the way the carrier maps operate. Hints of this feature will be visible in our discussion of Kleene algebras in Section 6 and it will come to the fore when we later use our duality to describe free algebras [9].

We draw attention to the omission of \( f_0 \) from \( \text{End}_p(\mathbb{Z}_2^{m-1}) \) that was made possible by the presence of the (necessary) map \( v^- \) which links \( P^- \) and \( Q \). This aside, the duality theorem for \( \mathcal{V}(\mathbb{Z}_2^m) \) encodes within it all the information for the duality theorem for \( \mathcal{V}(\mathbb{Z}_2^{m-1}) \). We just delete all reference to the sort

| \( P^- \) | \( Q \) | \( P^+ \) |
|---|---|---|
| \( f_1, \ldots, f_{m-2}; g \) | \( v \) | \( \text{id}_{-+} \) |
| [as per odd case] | [linking partial isomorphism] | [linking isomorphism] |
| \( u \) | \( h_2, \ldots, h_m, j \) | |
| [surjective homomorphism] | | [as per even case] |
| \( \text{id}_{+-} \) | | [linking isomorphism] |

Table 2. Maps to be included in the alter ego for \( \mathcal{V}(\mathbb{Z}_2^m) \)
denote the natural forgetful functor. Thus $V$ and $3.4$.

$Q$ for Theorem 4.4. We shall head straight for a multisorted duality for the variety $V_{2m}$; including $J\mathcal{M}_{2m-1}$ along the way rather than separating it out. In all other respects, we shall structure our argument in the same way as the corresponding argument in Section 4: sorts and linking maps; entailment of maps on and between sorts; carrier maps and separation; piggyback relations; the strong duality theorem for $V_{2m}$. Throughout we highlight where adaptations do and do not need to be made to transition from $J\mathcal{A}$ to $J\mathcal{M}$. Let $V: J\mathcal{M} \to J\mathcal{A}$ denote the natural forgetful functor. Thus $V(W_k) = Z_k$ for each $k$.

We replace the three sorts $P$, $S^-$, and $T$. The latter will be disjointified copies of $W_{2m-1}$, $W_{2m-1}$ and $W_{2m}$, respectively. We may assume that $V(S^+) = P^+$ and $V(T) = Q$.

Sugihara monoid homomorphisms must preserve $t$ (and also $f = \neg t$). The isomorphisms $id_{-}$ and $id_{+}$ are $J\mathcal{M}$-morphisms between $S^-$ and $S^+$ and Proposition 4.3 carries over, mutatis mutandis, to the monoid case.

Since $0$ in any Sugihara monoid is a fixed point for $\neg$, there are no monoid (partial) homomorphisms from $W_{2m-1}$ to any even Sugihara monoid. Any $J\mathcal{M}$-morphism from $W_{2m}$ to any $W_{2k-1}$ must send $\pm 1$ to $0$.

The previously-defined map $u: Z_{2m} \to Z_{2m-1}$ sends $\pm 1$ to $0$ and so also serves as an $J\mathcal{M}$-morphism from $W_{2m}$ onto $W_{2m-1}$. We define $u^\pm$ in just the same way as before, but now using the monoid sorts.

We shall need an $J\mathcal{M}$ analogue of Proposition 4.7, whereby partial homomorphisms between sorts are derived from composition by link maps between sorts and generating sets for partial endomorphism monoids. We also still have available the endomorphism $g$ which played an important role in the proof of Proposition 4.7(ii). A key difference however emerges with (i). The distinguished constant(s) must be in the domain of any monoid (partial) homomorphism. Consequently the sets $\text{Hom}^t_p(S^\pm, T)$ are empty, since any member of $\text{Hom}^t_p(P^\pm, Q)$ excludes $0$ from its domain. Hence entailment does not arise. Note that analogues of the maps $v, v^\pm$ do not exist, but are not needed since in Proposition 4.7 $v^\neg$ was used only to handle $\text{Hom}_p(P^\pm, Q)$.

When applying Proposition 5.1 later we shall take advantage of Propositions 3.3 and 3.4.

**Proposition 5.1** (partial endomorphisms and homomorphisms).

(i) Any member of $\text{End}^t_p(S^-)$, $\text{End}^t_p(S^+)$, $\text{Hom}^t_p(S^-, S^+)$ or $\text{Hom}^t_p(S^+, S^-)$ is expressible as the composition drawn from a set obtained by adding $id_{-}$ and $id_{+}$ to a generating set for $\text{End}^t_p(W_{2m-1})$ whose members are interpreted on $S^-$.

(ii) Any $h \in \text{End}^t_p(T)$ is entailed by a generating set for $\text{End}^t_p(W_{2m})$ interpreted on $T$.

(iii) Any $h \in \text{Hom}^t_p(T, S^-)$ is expressible as a restriction of a composition of maps drawn from a set obtained by adding $u^-$ to a generating set for $\text{End}^t_p(W_{2m})$ interpreted on $S^-$. 

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Any $h' \in \text{Hom}_{M}^1(T, S^+)$ is expressible as a restriction of a composition of maps drawn from a set obtained by adding $u^-$ and $\text{id}_{-}$ to a generating set for $\text{End}_{p}^1(W_{2m})$ interpreted on $S^-$. Carrier maps act on the lattice reducts of the sorts and no change is required when we pass to Sugihara monoids. So we take $\omega_S^{-} = \delta^-$, $\omega_S^+ = \delta^+$ and $\omega_T = \beta$. The proof of the separation condition given for the algebra case in Lemma 4.5 carries over to the monoid case: all the maps used there are $\mathcal{IM}$-homomorphism too.

We want to identify the forms taken by (maximal) $\mathcal{IM}$-subalgebras of sublattices $(\omega_M, \omega_M')^{-1}(\leq)$ of $M$, $M'$, where $M, M' \in \{S^-, S^+, T\}$.

Consider first the choices of $M, M'$ for which a characterisation is available of all $\mathcal{IA}$-subalgebras of $(\omega_M, \omega_M')^{-1}(\leq)$, where $M, M' \in \{P^-, P^+, Q\}$, as a graph of an $\mathcal{IA}$-partial homomorphism or converse of such a graph. The only situation in which we identified only the maximal $\mathcal{IA}$-subalgebras of $(\omega_M, \omega_M')^{-1}(\leq)$ was that in which $M = M' = Q$. So suppose $S$ is an $\mathcal{IM}$-subalgebra of $(\omega_M, \omega_M')^{-1}(\leq)$, where $M, M' \in \{S^-, S^+, T\}$ and $M, M'$ are not both $T$. Then $V(S)$ takes the form shown in the $(V(M), V(M'))$ cell in Table 1. This tells us that $V(S)$, or its converse, is the graph of an $\mathcal{IA}$-morphism, possibly with additional restrictions on domain and/or image of the map. We must then insist that the morphism must be an $\mathcal{IM}$-morphism, and consider whether any restrictions on domain or image impose further constraints. Specifically, let $M \in \{S^-, S^+, T\}$, so $t^{S^\pm} = 0$ and $t^{T} = 1$. For sorts $M, M'$ and $h \in \text{Hom}_{p}^1(M, M')$, necessarily $t^M \in \text{dom} \, h$ and $t^{M'} = h(t^M)$.

In addition, a subalgebra of a product $M \times M'$ of sorts contains $(t^M, t^{M'})$. We note in particular that $(\pm 1, 0)$ belongs to any $\mathcal{IM}$-subalgebra of $T \times S^\pm$. Since $(1, 0) \not\in (\delta^-, \delta^+)^{-1}(\leq)$ and $(1, 0) \not\in (\delta^-, \beta)^{-1}(\leq)$ we see immediately that there are no $\mathcal{IM}$-subalgebras of $S^- \times S^+$ or of $T \times S^+$. This signals the key differences between the dualities for Sugihara algebras and monoids, in both odd and even cases.

The $\mathcal{IM}$-subalgebras of $(\beta, \beta)^{-1}(\leq)$ require special consideration. We know that a maximal $\mathcal{IA}$-subalgebra of $(\beta, \beta)^{-1}(\leq)$ is the graph of an $\mathcal{IA}$-homomorphism. But there may be fewer $\mathcal{IM}$-subalgebras of $(\beta, \beta)^{-1}(\leq)$ than $\mathcal{IA}$-subalgebras, and we have no guarantee that a maximal element in $\mathcal{R}(Q, Q)$ will be the graph of an $\mathcal{IM}$-homomorphism. Hence we need to re-run the proof of the $\mathcal{IA}$ result in [4, Proposition 6.3 (residual case (a))] to check it works analogously when applied in the $\mathcal{IM}$ setting. The following table parallels that in Table 1.

|       | $S^-$ | $T$  | $S^+$ |
|-------|-------|------|-------|
| $S^-$ | graph $h$ | —    | —     |
| $T$   | graph $h$ | graph $h$ | —     |
| $S^+$ | graph $h$ or graph $k^-$ | —    | graph $k^-$ |

Table 3. Piggyback relations for $V(W_{2m})$

We identified generating sets for the partial endomorphisms of $W_{2m-1}$ and of $W_{2m}$ in Lemmas 3.3 and 3.4 and we incorporate these into out alter ego for $V(W_{2m})$. 
We now present without further comment our duality theorems for \( \mathcal{SM}_{2m-1} \) and for \( \mathcal{V}(W_{2m}) \). All the requisite facts have already been established and no new features arise.

**Theorem 5.2 (Two-sorted Piggyback Strong Duality Theorem for \( \mathcal{SM}_{2m-1} \)).** Fix \( m \) with \( m \geq 2 \). Take disjoint sorts \( S^- \) and \( S^+ \) each isomorphic to \( W_{2m-1} \).

Let \( \Omega_{S^-} = \{ \delta^- \} \) and \( \Omega_{S^+} = \{ \delta^+ \} \). Consider the two-sorted structure based on \( M := \{ S^-, S^+ \} \) having universe \( M_0 := S^- \cup S^+ \). Let

\[
G = \{ g, \text{id}_{-+}, \text{id}_{+-} \}, \quad H = \{ f_1, f_2, \ldots, f_{m-2} \}, \quad K = \{ 0_{S^-} \}.
\]

(Here \( g, f_0, \ldots, f_{m-2} \) are to be interpreted on \( P^- \).) Then \( \mathcal{M} := (M_0; G, H, K, T) \) strongly dualises \( \mathcal{SM}_{2m-1} \).

**Theorem 5.3 (Three-sorted Piggyback Strong Duality Theorem for \( \mathcal{V}(W_{2m}) \)).** Let \( S^- \), \( S^+ \) and \( T \) be disjointified copies of \( W_{2m-1} \), \( W_{2m-1} \) and \( W_{2m} \), respectively, with carrier maps \( \delta^- \), \( \delta^+ \) and \( \beta \), respectively. Let \( \mathcal{M} = \{ S^-, S^+, T \} \) and let \( T \) be the disjoint union topology on \( M_0 := S^- \cup S^+ \cup T \) derived from the discrete topology on each sort. Let \( \mathcal{M} = (M_0; G, H, K, T) \), where the elements of \( G \cup H \) are the maps presented in Table 4 and \( K = \{ 0_{S^-} \} \). Then \( \mathcal{M} \) strongly dualises \( \mathcal{V}(W_{2m}) \).

Observe how our elimination of the partial endomorphism \( f_0 \) from our alter ego for the duality for \( \mathcal{V}(Z_{2m-1}) \) allows better alignment between the duality theorems for \( \mathcal{V}(Z_{2m}) \) and \( \mathcal{V}(W_{2m}) \) than if we had retained it.

We have provided four multisorted strong duality theorems: algebras/monoids; odd/even. We now take stock of our results, with a view to how these might be used, and make some comments on what the theorems already reveal about the relationships between the the classes of algebras we have worked with.

Our statement of the Multisorted Piggyback Strong Duality Theorem explicitly kept separate the sufficient conditions for a dualising alter ego and for an alter ego upgraded to yield a strong (and hence full) duality. There are three levels at which we may want to operate in applications. For the admissible rules problem we addressed in [4] it was essential to have a strong duality: the Test Spaces Method relied on our being able to translate injectivity and surjectivity of homomorphisms into equivalent conditions of surjectivity and embedding on their dual morphisms. Fullness is wanted if we need a dual equivalence for an intended application, and we did exploit this feature when we identified admissibility algebras in [4, Sections 5 and 7] by applying the functors \( \mathbf{E} \) to appropriate dual structures.
On the other hand, there are situations in which a duality suffices. This happens, for example, for the important problem of describing free algebras, of which the dual spaces are powers of a dualising alter ego. Moreover, as noted earlier, we can work within a quasivariety or within the variety it generates, as expedient. Information about the free algebras themselves may best be sought by translating from a multisorted piggyback duality to a restricted Priestley duality (as defined in [13]). This process can be based on the results in [5, Section 2], which ensure that, at the least, the lattice reducts of finitely generated free algebras can be described. We apply this methodology Sugihara algebras and monoids in [9]. See Section 6 for a taster example.

Given that we do not always require a piggyback duality to be strong, or even full, to be useful in applications, it is natural to ask when a piggyback duality is automatically strong, once small technical changes are made. In each of our alter egos the core structure comprises a generating set for one partial endomorphism monoid (odd case) or generating sets for two partial endomorphism monoids (even case); in addition, we need to include and at most six linking maps. Consider the generating sets we have identified in [4, Propositions 2.8 and 2.9] and Propositions 3.3 and 3.4. Our choice of carrier map $\alpha^-$ ensures that for each of the maps $f_i$ is such that $a \leq f_i(a)$ for $a \geq 0$ and hence graph $f_i$ in $R_P \rightarrow P$. Also graph $g_\sim \in R_P \rightarrow P$. Similarly, graph $h_i \in R_Q \rightarrow Q$, for each $i$, and graph $j \in R_Q \rightarrow Q$. Since $w_Q = \beta$. Consider a piggyback duality based on conditions (i) and (ii) in Theorem 2.1. We arrive at a strong duality if we include the partial operations and the one-element subalgebras specified in (iii) and (iv). To satisfy (iii) we do not need to add partial operations unconnected to piggybacking. Moreover, for $\mathcal{J}_{2m-1}, \mathcal{V}(Z_{2m})$, $\mathcal{M}_{2m-1}$ or $\mathcal{V}(W_{2m})$ we obtain a piggyback duality by satisfying the separation condition and also including in the alter ego those piggyback relations whose graphs or converses belong to the sets identified in whichever of Propositions 4.3, 4.7 and 5.1, as appropriate.

6. Appendix: The case for multisorted dualities

Here we give a brief contextual appraisal of the role of multisorted duality theory, as it applies to distributive-lattice-based algebras and illustrated by Kleene algebras and lattices. We do not include in this appendix any results not previously known. However the material is scattered across a number of sources and the methods we want to discuss are often applied to families of classes of algebras, and this can obscure the ideas. Our purpose here is to exemplify through our running examples how a multisorted approach can be seen as a facilitator, allowing different types of dual representation to operate collaboratively. We shall take these ideas forward for Sugihara monoids and algebras in subsequent papers.

Quasivarieties generated by small finite distributive lattices with additional operations, with or without distinguished constants, provided a rich source of examples on which Davey and Werner were able to test the power of the foundational theory. The emphasis here is on small. Davey and Werner’s method relied on their NU Duality Theorem: for a lattice-based quasivariety $\mathbb{Q}(M)$ this yields a dualising alter ego $(M; \mathcal{S}(M^2), \mathcal{T})$. Taking this forward to arrive at a workable duality necessitated describing all subalgebras of $M^2$ and then using entailment constructs to weed out superfluous relations.
Example 6.1 (Kleene algebras and lattices: hand-crafted single-sorted dualities).

Let $\mathbf{3} = \{(0, a, 1), \wedge, \vee, \neg, 0, 1\}$ whose reduct $\langle\{0, a, 1\}, \wedge, \vee\rangle$ is the distributive lattice with underlying order $0 < a < 1$ and $\neg$ is the involution for which $\neg 0 = 1$, $\neg 1 = 0$ and $\neg a = a$. The class $\mathcal{K}$ of Kleene algebras is $\mathcal{V}(\mathbf{3})$; it coincides with $\text{ISP}(\mathbf{3})$. Moreover, $\mathcal{K}$ is the unique proper subvariety of the variety $\mathcal{DM}$ of De Morgan algebras which covers Boolean algebras. Specifically, $\mathcal{K}$ is characterised within $\mathcal{DM}$ by the law $(a \wedge \neg a) \wedge (b \vee \neg b) = a \wedge \neg a$ (which can equivalently be captured by $a \wedge \neg a \leq b \vee \neg b$). The class of Kleene lattices is $\mathcal{KL} := \mathcal{V}(\mathbf{3}_u) = \text{ISP}(\mathbf{3}_u)$, where $\mathbf{3}_u = \langle\{0, a, 1\}, \wedge, \vee, \neg\rangle$ is obtained by omitting the constants from the language.

Davey and Werner’s duality for $\mathcal{K}$ [18] (or see [11, 4.3.10]) has alter ego $\mathbf{3} = \langle\langle 0, a, 1\rangle; \leq, \neg, K_0, T \rangle$, where $\leq$ is the partial order with $() < a$, $1 < a$ and $0$ and $1$ incomparable, $\neg$ is $\{0, a, 1\}^2 \setminus \{(0, 1), (1, 0)\}$ and $K_0 = \{0, 1\}$. This is proved by listing all $11$ subalgebras of $\mathbf{3}$ and whittling down the list down by exhibiting entailments.

A minor adaptation yields an alter ego $\mathbf{3}_u$ for $\mathbf{3}_u$, viz. we add the one-element subalgebra $a$, regarded as a nullary operation, to $\mathbf{3}$. (A detailed account is given in [22, Section 4]. See also Example 6.3 below.)

The number $3$ is small. The ad hoc method in Example 6.1 will rarely be viable for quasivarieties $\text{ISP}(\mathbf{M})$ with $|\mathbf{M}|$ significantly bigger, either by hand or even with computer assistance: the number of subalgebras of $\mathbf{M}^2$ is liable to be unmanageably large. This led Davey and Werner to develop their piggyback method [19]. It applies to certain quasivarieties $\text{ISP}(\mathbf{M})$, where $\mathbf{M}$ is finite, with a distributive lattice reduct (with or without bounds). The idea is to exploit Priestley duality as it applies to the reducts to identify a dualising alter ego more refined than that delivered by the NU Duality Theorem. But in its original form piggybacking is contingent on there being a single carrier map (a homomorphism from the reduct of $\mathbf{M}$ into $\mathbf{2}$) which, composed with the operations of $\mathbf{M}$, separates the points of $\mathbf{M}$. The separation condition fails for Kleene algebras, as explained in the introduction to citeDP87. It was their analysis of this ‘rogue’ example which led Davey and Priestley to introduce multisorted dualities and to develop a far-reaching generalisation of the piggyback method. The latter is applicable to any quasivariety or variety generated by a finite distributive-lattice-based algebra $\mathbf{M}$. This subsumes a brute force version for a quasivariety $\text{ISP}(\mathbf{M})$ whereby all homomorphisms from (the reduct of) $\mathbf{M}$ into $\mathbf{2}$ are employed as carrier maps. This was what we used in [4, Theorem 6.4].

Example 6.2 (Kleene algebras and lattices: piggyback dualities with two carrier maps). Consider $\mathcal{KL}$. We let $\mathbf{3}$ be the algebra defined in Example 6.1 and let $U(\mathbf{3})$ be its reduct in $\mathcal{D}$. Let $\beta^-$, respectively $\beta^+$, the $\mathcal{D}$-homomorphism from $U(\mathbf{3})$ to $\mathbf{2}$ sending $a$ to $1$, respectively to $0$. Then

\[
\begin{align*}
(\beta^-, \beta^-)^{-1}(\leq) \\
(\beta^+, \beta^+)^{-1}(\leq) \\
(\beta^-, \beta^+)^{-1}(\leq) \\
(\beta^+, \beta^-)^{-1}(\leq)
\end{align*}
\]

has a unique maximal subalgebra which is
We obtain a dualising alter ego $\mathfrak{3}$ for $\mathfrak{3}$ by taking the first three of these relations and the discrete topology. This duality is already strong. We thereby recapture Davey and Werner’s original duality for $\mathcal{K}A$ as described in Example 6.1.

For Kleene lattices, the Piggyback Strong Duality Theorem as formulated in [4, Theorem 3.3], for single-sorted piggybacking over $\mathcal{D}_n$, applies directly (or see [22, Section 4] for a treatment based on the NU Strong Duality Theorem).

The dualities for $\mathcal{K}A$ and $\mathcal{K}A_u$ recalled above 6.1 have disadvantages. The roles of the elements $\preceq$, $\sim$ and $K_0$ in the alter ego are far from obvious, so the dualities lack intuitive appeal, and the axiomatisation of the dual categories (see [11, 4.3.9] and [22, Proposition 4.3]) does little to assist. In addition, obtaining $F_{\mathcal{K}A}(s)$, the free Kleene algebra on $s$ generators, from its natural dual $\mathfrak{3}^S$ is not the transparent process we could wish.

Example 6.3 (Kleene algebras: a rival duality). Cornish and Fowler [12] proved that the category of Kleene algebras is dually equivalent to the category $\mathcal{Y}$ of Kleene spaces. This provides what would now be referred to as a restricted Priestley duality; see [13]. Here the objects of $\mathcal{Y}$ are topologised structures $\mathcal{Y} = (Y; \preceq, g; \mathcal{T})$ where $(Y; \preceq, \mathcal{T})$ is a Priestley space and $g$ is a continuous order-reversing involution on $Y$ such that $\forall y (y \geq g(y) \text{ or } y \leq g(y))$; and the morphisms are continuous maps preserving $\preceq$ and $g$. It is an easy exercise in Priestley duality to show that the existence of a map $g$, as in the definition of a Kleene space, is exactly what is needed to capture the Kleene negation on the Priestley second dual of the reduct of a Kleene algebra. The objective of Cornish and Fowler was to describe coproducts, and in particular free algebras, in $\mathcal{K}A$ by using the duality. This was successful, but the construction is convoluted.

A version of Cornish and Fowler’s duality exists for Kleene lattices, in which the dual category involves pointed Priestley spaces.

Restricted Priestley dualities have the advantage of providing a set-based, topologico-relational, representation, akin to Kripke relational semantics. But, as might be expected of second-hand dualities, restricted Priestley dualities often lack many of the good features of a natural duality, such as direct access to free algebras, whose natural dual spaces are given by concrete pos. By contrast, Cornish and Fowler’s use of their duality to describe free Kleene algebras is far from transparent. For a finitely generated variety or quasivariety of distributive-lattice-based algebras, a natural duality and a restricted Priestley duality each has a role to play. But for duality methods to be at their most powerful we need to be able translate backwards and forwards between the two dual categories.

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1. The nomenclature is a little ambiguous and the term ‘enriched’ is sometimes used instead of ‘restricted’. Often both adjectives have a valid connotation, with ‘restricted’ referring to the forgetful functor to Priestley spaces not in general being surjective, whereas ‘enriched’ refers to the dual objects being Priestley spaces with extra structure, usually relations or operations, with appropriate topological conditions imposed.
We now play the multisortedness trump card. Moving to a dual category of multisorted structures will make it clear how to reconcile the single-sorted natural dualities in 6.2 and the restricted Priestley dualities in 6.3 once the former are recast in terms of dual categories of multisorted structures.

**Example 6.4.** Kleene algebras: two-sorted duality

Take disjoint copies of $\mathfrak{3}$, denoted $\mathfrak{3}^-$ and $\mathfrak{3}^+$. Define carrier maps as before, allocating $\beta^-$ to $\mathfrak{3}^-$ and $\beta^+$ to $\mathfrak{3}^+$. For temporary convenience, we denote the universe of $\mathfrak{3}^\pm$ by $M^\pm := \{0^\pm, a^\pm, 1^\pm\}$. We form an alter ego $\mathfrak{3}^-\sim\mathfrak{3}^+$ based on the set $\{0^-, a^-, 1^-\} \cup \{0^+, a^+, 1^+\}$. We equip this with the relations which are maximal subalgebras of sublattices $(\gamma, \delta)^{-1}(\leq)$, where $\gamma, \delta$ range over $\{\beta^-, \beta^+\}$. These binary relations are the same as those in 6.2 except that the coordinates now need to be tagged to indicate the sorts to which these refer.

![Figure 3.](image)

Figure 3 depicts the alter ego, and its relations.

A key feature of the multisorted approach is that duals of free algebras are calculated ‘by sorts’. In the case of $\mathcal{K}A$ this means that the dual of $F_{\mathcal{K}A}(s)$, the free Kleene algebra on $s$ generators, has universe $(M^-)^s \cup (M^+)^s$, with $\leq$, $-$ and $K_0$ lifted pointwise.

Davey and Priestley in [16, Theorem 3.8] presented the first systematic translation process for toggling between natural dualities and their restricted Priestley duality counterparts. The context there, and in [11, Section 7.6] too, was Ockham algebras. This means that, within a much wider framework, Kleene algebras provide the simplest non-trivial example. The results given in [16, Section 3] apply to Ockham algebra quasi-varietyes, They are heavy on notation and hide the simplicity of the ideas.

**Example 6.5** (Reconciliations: the best of both worlds). We don’t give a full translation, preferring to illustrate the process as it applies to $F_{\mathcal{K}A}(1)$ and $F_{\mathcal{K}A}(2)$. The dual spaces of these algebras are, respectively, $\mathcal{M}$ and $\mathcal{M}^2$. We may view the union of the multisorted piggyback relations,

$$\leq^- \cup \geq^+ \cup \{0^-, 0^+\} \cup \{1^-, 1^+\} \cup (M^+ \times M^-) \setminus \{(0^+, 1^-), (1^+, 0^-)\}$$

as a preorder on $M^- \cup M^+$. When we quotient by $\leq \cap \geq$ to obtain a partial order, the each of the pairs $0^+, 0^-$ and $1^+, 1^-$ collapses to a point. On the resulting poset, $2^2$, the linking maps $\text{id}_{-+}$ and $\text{id}_{+-}$ induce a well-defined order-reversing involution $g$. We have obtained the Kleene space which is the Cornish–Fowler dual of $F_{\mathcal{K}A}(1)$. 

With $M^2$ we proceed similarly, remembering that the power is formed ‘by sorts’; see Figure 4. The natural dual of $F_{\mathcal{X}}(2)$ is shown in Figure 4(a). Quotienting, we obtain the ordered set $Y$ shown in (b). The isomorphisms $id_{++}$ and $id_{+-}$ linking the two sorts encode, after the quotienting, the map $g$ in the restricted Priestley duality. The Kleene example is particularly simple because the map $g$ is determined by the order structure.

Denote the quotienting map by $q$. Then, in the quotient, $\{ y \mid y \geq g(y) \} = \beta^{-}(3^{-})$ and $\{ y \mid y \leq g(y) \} = \beta^{+}(3^{+})$. We are looking here at the dual spaces of a very particular algebra in $\mathcal{X}$, but this behaviour is exhibited quite generally. This reflects the fact that single-carrier, single-sorted piggybacking does not work. See [11, Section 7.2], [16] and [5, Section 3] for detailed explanations.

Likewise to find the cardinality and the structure of a free Kleene algebra $F_{\mathcal{X}}(s)$, a good way to proceed is to form its 2-sorted natural dual and then to pass to the restricted Priestley dual.

For $\mathcal{X}_u$, the translation works in the same way, but now piggybacking is over $D_u$; when recapturing the reduct of an algebra from its $D_u$ the duality used is that between $D_u$ and doubly pointed Priestley spaces, as in [11, Section 1.2 and Theorem 4.3.2]. The lattice reduct of $F_{\mathcal{X}}(s)$ has a join-irreducible top and meet-irreducible bottom. Simply delete these elements to obtain $F_{\mathcal{X}_u}(s)$.

Further examples of translation between dualities of different types are given in [6, 7, 30] and a systematic account of this process, for the underlying lattices, is presented in [5]. However one would not expect straightforward, or uniform, procedure for capturing algebraic operations of arity $>1$ in dual terms, and for translating between different types of dual representations.

Discussion of this important aspect of our methodology, and its application to finitely generated free Sugihara algebras and monoids, is deferred to a later paper.

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