On higher Poisson and Koszul–Schouten brackets

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Abstract

In this note we show how to construct a homotopy BV-algebra on the algebra of differential forms over a higher Poisson manifold. The Lie derivative along the higher Poisson structure provides the generating operator.

Keywords: Strong homotopy Lie algebra, derived bracket, higher Poisson bracket, higher Koszul–Schouten bracket, Batalin–Vilkoviski algebra, odd symplectic geometry.

1 Introduction and Background

Recall that a higher Poisson manifold is the pair \((M, P)\), where \(M\) is a supermanifold and the higher Poisson structure \(P \in \mathfrak{X}^*(M) (= C^\infty(\Pi T^*M))\) is an even parity (but otherwise arbitrary) multivector field that satisfies the “classical master equation”; \([ P, P ] = 0\). Here the bracket is the canonical Schouten–Nijenhuis bracket on \(\Pi T^*M\). The original formulation of higher Poisson manifolds was laid down by Voronov [23, 24]. A similar notion was put forward by de Azcárraga et.al [7, 8].

Associated with the higher Poisson structure is a homotopy Poisson algebra on \(C^\infty(M)\). That is an \(L_\infty\)-algebra in the sense of Lada & Stasheff [19] (suitably “superised”) such that the series of brackets are multi-derivations over the (supercommutative) product of functions. The series of higher Poisson brackets are given by;

\[
\{f_1, f_2, \cdots, f_r\}_P = [\cdots [[[P, f_1], f_2], \cdots, f_r]]_M ,
\]

where \(f_i \in C^\infty(M)\). The “classical master equation” is directly equivalent to the higher Poisson brackets satisfying the so-called Jacobiators. This is fundamental as without this condition the series of higher brackets would not form an \(L_\infty\)-algebra.

In this note we show how to construct a series of higher brackets on the algebra of differential forms\(^1\) on a higher Poisson manifold. We do this by following Koszul [18] (also see [11, 6, 23]), as

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\(^1\)we define the algebra of differential forms over a supermanifold to be \(\Omega^*(M) := C^\infty(\Pi TM)\). As we will not delve into the theory of integration over supermanifolds, such a definition is perfectly adequate for our purposes.
higher antibrackets generated by the Lie derivative along the higher Poisson structure. The \( n+1 \)-th bracket in essence measures the failure of \( n \)-th bracket to be a multi-derivation. As the Poisson structure is an even multivector field, the Lie derivative is an odd operator acting on differential forms and thus the series of brackets are odd. The “classical master equation” guarantees that the Lie derivative along the higher Poisson structure is nilpotent. Such a series of brackets forms an \( L_\infty \)-algebra commonly referred to as a homotopy BV-algebra.

We will refer to this series of brackets as higher Koszul–Schouten brackets. These brackets are considered as the natural analogues of the classical Koszul–Schouten bracket [16, 17, 18]. We show that the exterior derivative acts as a differential over the higher Koszul–Schouten brackets, just as it does in the classical case. Thus we have what we will call a differential homotopy BV-algebra.

It must be noted that Khudaverdian & Voronov [13] constructed a homotopy Schouten algebra on the algebra of differential forms over a higher Poisson manifold. That is they have an odd analogue of a homotopy Poisson algebra constructed from the higher Poisson structure. Importantly, the higher Schouten brackets are multi-derivations over the algebra of differential forms. This is in stark contrast to the higher Koszul–Schouten brackets.

Thus, there is at least two ways to equip the algebra of differential forms over a higher Poisson manifold with the structure of an \( L_\infty \)-algebra. However, these two constructions are not completely independent. Due to ideas present in [23], the higher Schouten brackets can be thought of as the “classical limit” of the Koszul–Schouten brackets. It is not surprising that some relation exists between the two classes of higher bracket as both are build from the initial geometric data of a higher Poisson structure.

The distinction between Koszul–Schouten and Schouten brackets in the case of classical (binary) Poisson manifolds is non-existent. Simply put, there is no tri-bracket or higher and the distinction is not present.

Although we will not present details here, it is clear that the constructions presented in this note carry over to higher Poisson structures on Lie algebroids.

However, it is clear that a direct analogue for higher Schouten manifolds does not exist. Associated to such a structure is a homotopy Poisson algebra on differential forms over the supermanifold. One can construct the higher Poisson structure using arguments almost identical to that found in [13]. As such brackets are not all odd, no operator can generate something resembling a homotopy Poisson algebra in some “classical limit”.

As is customary in super-mathematics, the prefix “super” will generally be omitted. For example, by manifold we explicitly mean supermanifold. We will denote the parity of an object \( A \), by \( \tilde{A} \in \mathbb{Z}_2 \). By even or odd we will be explicitly referring to parity. By a \( Q \)-manifold will mean the pair \( (M, Q) \) where \( M \) is a supermanifold and \( Q \in \text{Vect}(M) \) is an odd vector field known as a homological vector field satisfying \( [Q, Q] = 0 \).

The notion of homotopy Poisson/Schouten/BV algebra as used here is much more restrictive that that found elsewhere in the literature [9, 10]. We make no use of the theory of operads. However, the notion used throughout this work seems very well suited to geometric considerations and suits the purposes explored here.
2 Higher Koszul–Schouten brackets

Differential forms on a manifold \( M \), are understood as functions on antitangent bundle \( \Pi T^*M \). In natural local coordinates \( \{x^A, dx^A\} \) on \( \Pi T^*M \) an arbitrary differential form is locally given by \( \alpha(x, dx) \in \Omega^*(M) = C^\infty(\Pi T^*M) \). The parities of the local coordinates are \( \tilde{x}^A = \tilde{A} \) and \( d\tilde{x}^A = (\tilde{A} + 1) \). Under transformations induced by coordinate changes on the base the fibre coordinates transform as \( d\tilde{x}^A = dx^B \left( \frac{\partial x^B}{\partial \tilde{x}^A} \right) \).

Similarly, multivector fields on \( M \), are identified with functions on the anticotangent bundle \( \Pi T^*M \). We pick natural local coordinates \( \{x^A, x^*_A\} \), such that \( \tilde{x}^*_A = (\tilde{A} + 1) \). Multivector fields on \( M \) are locally described by \( X(x, x^*) \in \mathfrak{X}^*(M) = C^\infty(\Pi T^*M) \). The fibre coordinates transform as \( x^*_A = \left( \frac{\partial x^*_B}{\partial x^A} \right) x^*_B \) under transformations induced by coordinate changes on the base. A filtration can be defined on multivectors (and differential forms) that are polynomial in fibre coordinates. We will refer to this as multivector degree. Where no confusion will arise we will simply use degree for multivector degree. We will call a multivector field an \( r \)-vector if it is a monomial of degree \( r \) in the fibre coordinates.

**Warning:** The parity of an object is generally independent of the degree. By odd and even we will be explicitly referring to parity and not weight.

Importantly, the anticotangent bundle comes equipped with a canonical odd symplectic structure. That is we have an odd Poisson bracket on \( \mathfrak{X}^*(M) \). This bracket is the Schouten–Nijenhuis bracket.

From an arbitrary multivector field \( X \in \mathfrak{X}^*(M) \) we can associate a differential operator acting on differential forms on \( M \) known as the Lie derivative:

\[
X \leadsto L_X = [d, i_X],
\]

where \( d = dx^A \frac{\partial}{\partial x^A} \) is the exterior derivative and \( i_X = (-1)^{\tilde{X}} X(x, \partial/\partial dx) \) is the interior product. As a differential operator, the Lie derivative is of order equal to that of the degree of the multivector field (assuming it is defined).

The Lie derivative is clearly not a derivation on the algebra of differential forms, apart form the isolated case of vector fields. Instead of the derivation property we have;

\[
L_{[X,Y]} = [L_X, L_Y].
\]

Thus we see directly that for a higher Poisson structure \( P \), the Lie derivative is nilpotent;

\[
(L_P)^2 = \frac{1}{2} [L_P, L_P] = \frac{1}{2} L_{[P,P]} = 0.
\]

A further property needed later on is that the exterior derivative is natural with respect to the Lie derivative, that is;

\[
[d, L_X] = 0.
\]

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2Generically it will in fact be a pseudo-differential operator if we do not restrict attention to multivectors that are polynomial in \( x^* \).
Remark The Lie derivative of a differential form along a multivector field should be understood as the infinitesimal action of $\text{Can}(\Pi^* T^* M)$ on differential forms on $M$ (functions on $\Pi^* T M$). See \cite{12, 14, 15, 21} for more details.

Definition 2.1 Let $(M, P)$ be a higher Poisson manifold. The higher Koszul–Schouten brackets on $\Omega^*(M)$ are the higher derived brackets generated by the operator $L_P = [d, i_P]$

$$[\alpha_1, \alpha_2, \cdots, \alpha_r]_P := \cdots [[L_P, \alpha_1], \alpha_2] \cdots , \alpha_r]_1, \quad (2.5)$$

with $\alpha_i \in \Omega^*(M)$.

Here $\mathbb{I}$ is the constant value 1 zero form. That is the identity element in $\Omega^*(M)$. As the Lie derivative along the higher Poisson structure is odd and nilpotent the series on higher Koszul–Schouten brackets form a homotopy BV-algebra.

We know the Lie derivative $L_P$ is a differential operator of order equal to the degree (should it be defined) of $P$. The order of $L_P$ is at most $k$ if $\Phi^r_P$ vanishes (for all $\alpha_i$) for $r \geq k + 1$. Thus for any multivector of (finite) degree $k$, the above series of brackets terminates after the $k$ place. The top non-zero bracket is a multi-derivation. The $(r + 1)$ bracket is the obstruction to the Leibnitz rule for the $r$ bracket. However, it should be noted that we do not require polynomial multivector fields in order to define the higher Koszul–Schouten brackets.

Proposition 2.2 The exterior derivative satisfies a derivation rule over the Koszul brackets. Specifically,

$$d[\alpha_1, \cdots, \alpha_r]_P + \sum_{i=1}^{r} (-1)^{\epsilon_i}[\alpha_1, \cdots, d\alpha_i, \cdots \alpha_r]_P = 0, \quad (2.6)$$

where

$$\epsilon_j = \begin{cases} 
0 & j = 1 \\
\sum_{k=1}^{j-1} \tilde{\alpha}_k & j > 1.
\end{cases} \quad (2.7)$$

Proof The above proposition follows directly from the naturality of the exterior derivative with respect to the generalised Lie derivative: $[d, L_P] = 0$.

We take the attitude that the higher Koszul–Schouten brackets give a clear geometric example of "higher antibrackets", \cite{11, 2, 5, 6}. Slightly more than this, we have a "differential homotopy BV-algebra" with the differential being provided by the exterior derivative.

3 Relations between higher Poisson and higher Koszul–Schouten brackets

In section we will generalise the known relations between the classical (binary) Poisson and Koszul–Schouten brackets. In order to achieve this we need to restrict our attention to finite degree Poisson structures. This is clear as although the higher Poisson brackets rely only on the Taylor expansion of the higher Poisson structure near $M \subset \Pi^* T^* M$ the higher Koszul–Schouten brackets require the full Poisson structure. First we need a lemma.
Lemma 3.1 Let $P \in \mathfrak{X}^*(M)$ be a higher Poisson structure of degree $k$. Then the $r$-th higher Poisson bracket is given by
\[
\{f_1, f_2, \ldots, f_r\}_P = -L^r_P(f_1 df_2 \cdots df_r) = i^r_P(df_1 df_2 \cdots df_r).
\] (3.1)

Proof The above can be proved via direct computation.

Proposition 3.2 Let $P \in \mathfrak{X}^*(M)$ be a higher Poisson structure of degree $k$. Then the higher Koszul–Schouten brackets satisfy the following:

1. $[f_1, df_2 \cdots, df_r]_P = -\{f_1, f_2, \cdots, f_r\}_P$,
2. $[f_1, f_2, \cdots, f_r]_P = 0$ for $r > 1$,
3. $[\emptyset]_P = d\{\emptyset\}_P$,
4. $[df_1, df_2, \cdots, df_r]_P = d\{f_1, f_2, \cdots, f_r\}_P$,

with $f_I \in C^\infty(M)$.

Proof By counting the (form) degree of $[f_1, df_2 \cdots, df_r]_P$ it is clear that the only contribution is from the $s \leq r$ components of $P$. Furthermore, we know that the only contribution to the $r$-bracket from a degree $r$ multivector is the top component. By expanding out the definition of the higher Koszul–Schouten brackets we see that we have the simplification
\[
[f_1, df_2 \cdots, df_r]_P = L^r_P(f_1 df_2 \cdots df_r),
\] (3.2)
and via Lemma 3.1 we obtain 1. As the higher Koszul–Schouten bracket carries form degree $(1 - r)$ it is clear that for $r > 1$ 2. holds. (The $r = 1$ case is contained within 1.). For $r = 0$ we have 3. which can easily be seen using local expressions. 4. follows directly from 1. and the derivation property of the exterior derivative.

The statements 3. and 4. in Proposition 3.2 show that the exterior derivative provides an morphism as $L_\infty$-algebras between the higher Poisson and higher Koszul–Schouten brackets.

Remark The above relations between the higher Poisson and higher Koszul–Schouten brackets reduce to the case of classical Poisson structures for 2-Poisson structures, up to conventions.

If we have a classical Poisson manifold, then it is well known that the Poisson anchor provides a homeomorphism between the (binary) Koszul–Schouten bracket and the Schouten–Nijenhuis bracket as
\[
\phi^*_P[\alpha, \beta]_P = [[\phi^*_P \alpha, \phi^*_P \beta]].
\] (3.3)
However, it is not at all obvious how this relation generalises to higher Poisson manifolds. In particular, the higher Koszul–Schouten brackets are a series of higher brackets as where there is only one (binary) Schouten–Nijenhuis. Thus the higher Poisson anchor cannot be a simple map between the brackets. It is expected that this could be formulated in terms of an $L_\infty$-algebra morphism.
4 From higher Koszul–Schouten brackets to higher Schouten brackets

Khudaverdian & Voronov [13] define a higher Schouten algebra on $\Omega^*(M)$. At first their constructions seem unrelated to constructions presented here. The Koszul–Schouten brackets defined in this work provide the algebra of differential forms with the structure on a homotopy BV algebra. That is the $n + 1$ Koszul–Schouten bracket can be defined recursively as the failure of $n$ Koszul–Schouten bracket to be a multi-derivation. In the case of higher Schouten brackets one loses this recursive definition and replaces it with a strict Leibnitz rule. That is the higher Schouten brackets form a higher anaologue of an odd Poisson algebra. We shall see that the higher Schouten brackets of Khudaverdian & Voronov are the classical limit [23] of the higher Koszul–Schouten brackets. To show this we need to “deform” the higher Poisson structure as

$$P \rightsquigarrow P[\hbar] = \sum_{i=0}^{\infty} \frac{(\hbar)^i}{i!} P^{A_1 \cdots A_i} \partial x^{A_1} \cdots \partial x^{A_i},$$

(4.1)

where $\hbar$ is an even formal deformation parameter. In essence it counts the degree of the components of the multivector field. As usual, the infinite sum is understood formally.

**Definition 4.1** The higher Schouten brackets associated with the higher Poisson structure $P \in \mathfrak{x}^*(M)$ are defined as

$$(\alpha_1, \alpha_2, \cdots, \alpha_r)_P = \lim_{\hbar \to 0} \left( \frac{1}{\hbar} \right)^r [\alpha_1, \alpha_2, \cdots, \alpha_r]_{P[\hbar]},$$

(4.2)

with $\alpha_I \in \Omega^*(M)$.

As this classical limit does not effect the symmetries or the Jacobitators, the Schouten brackets do form a genuine $L_\infty$-algebra. The multi-derivation property becomes

$$(\alpha_1, \cdots, \alpha_r, \alpha_{r+1})_P = (\alpha_1, \cdots, \alpha_r)_{P} \alpha_{r+1} \pm \alpha_r (\alpha_1, \cdots, \alpha_{r+1})_P \pm \lim_{\hbar \to 0} \hbar (\alpha_1, \cdots, \alpha_r, \alpha_{r+1})_P,$$

(4.3)

which clearly gives the strict Leibnitz rule.

**Theorem 4.2** The Schouten brackets can be cast into the form

$$(\alpha_1, \alpha_2, \cdots, \alpha_r)_P = [\alpha_1, \alpha_2, \cdots, \alpha_r]_P = \{\cdots \{K_P, \alpha_1\}, \alpha_2\}, \cdots, \alpha_r\}|_{TM},$$

(4.4)

where $K_P \in C^\infty(T^*(\Pi TM))$ is a higher Schouten structure, i.e. it is odd and $\{K_P, K_P\} = 0$. Here $\{\cdot, \cdot\}$ is the canonical Poisson bracket on $T^*(\Pi TM)$.  


**Proof** Let us work with natural local coordinates \( \{ x^A, dx^A, p_A, \pi_A \} \) on \( T^*(\Pi T M) \). We define the total symbol of the Lie derivative along the higher Poisson structure as

\[
\sigma L_P = K_P = L_P(x, dx, p, \pi) \in C^\infty(T^*(\Pi T M)),
\]

(4.5)

via \( \frac{\partial}{\partial x^A} \leftrightarrow p_A \) and \( \frac{\partial}{\partial x^A} \leftrightarrow \pi_A \). As the Lie derivative transforms as a tensor under morphisms induced by \( \text{Diff}(M) \) the function \( K_P \) is well defined between natural local coordinates.

From the general theory of differential operators and microlocal analysis, see for example Hörmander \[11\] we know that the symbol map takes commutators of operators to Poisson brackets of functions. In the current situation this extends to the total symbol due to the tensor nature of the Lie derivative between natural local coordinates. Then we have

\[
\sigma[L_P, L_P] = \{ K_P, K_P \} = 0,
\]

(4.6)

thus \( K_P \in C^\infty(T^*(\Pi T M)) \) is a higher Schouten structure.

Then consider

\[
\lim_{h \to 0} [\cdots [[L_P[h], \alpha_1], \alpha_2], \cdots, \alpha_r] \mathds{1} = [\cdots [[[L_{\tilde{P}}[h], \alpha_1], \alpha_2], \cdots, \alpha_r] \mathds{1}.
\]

As the above is a differential form, the action on the unit form is via multiplication and as such it can be dropped. Then taking the total symbol gives

\[
[\cdots [[[L_{\tilde{P}}, \alpha_1], \alpha_2], \cdots, \alpha_r] = [\cdots [\sigma L_{\tilde{P}}, \alpha_1], \alpha_2], \cdots, \alpha_r],
\]

(4.7)

Putting this together we see that

\[
\lim_{h \to 0} [\alpha_1, \alpha_2, \cdots, \alpha_r]_{|_{\Pi T M \subset T^*(\Pi T M)}} = [\cdots [\{ K_P, \alpha_1 \}, \alpha_2], \cdots, \alpha_r],
\]

(4.8)

and the result is established.

Via these constructions it is straight forward to see that

\[
K_P = \sum_{r=0}^{\infty} \left( \frac{1}{r!} dx^B \frac{\partial P_{A_1 \cdots A_r}}{\partial x^B} \pi_{A_1} \cdots \pi_{A_r} - \frac{1}{(r-1)!} P_{A_1 \cdots A_r} \pi_{A_r} \cdots \pi_{A_2} p_{A_1} \right),
\]

(4.9)

This function can then be pulled-back to \( T^*(\Pi T^* M) \), via the canonical double vector bundle morphism \( R : T^*(\Pi T^* M) \to T^*(\Pi T M) \). Via natural local coordinates \( \{ x^A, x_A^*, p_A, \pi_A \} \) on \( T^*(\Pi T^* M) \), the canonical morphism is given by \( R^*(dx^A) = (-1)^{\tilde{A}} \pi^A \) and \( R^*(p_A) = x_A^* \). Thus we have

\[
R^*K_P = \left( \sum_{r=0}^{\infty} \frac{(-1)^{\tilde{B}}}{r!} dx_{A_1}^* \cdots dx_{A_r}^* \frac{\partial P_{A_1 \cdots A_r}}{\partial x^B} \pi^B \right) - \left( \sum_{r=0}^{\infty} \frac{1}{(r-1)!} P_{A_1 \cdots A_r} x_{A_r}^* \cdots x_{A_2}^* \right) p_B,
\]

(4.10)

which is immediately recognised as the linear Hamiltonian associated with the homological vector field

\[
Q_P = -[\Pi P, \bullet] \in \text{Vect}(\Pi T^* M).
\]

(4.11)
Recall that an $L_\infty$-algebroid structure on a vector bundle $E \to M$ is a homological vector field on $\Pi E$. Thus, there is an $L_\infty$-algebroid structure on $T^*M$ provided by the above homological vector field (this first appears in [13]). This is the higher analogue of the Lie algebroid associated with a Poisson manifold.

In essence, Khudaverdian & Voronov reverse this process and define a map $P \hookrightarrow K_P$ and use this in the definition of the Schouten brackets viz Eqn. (4.4). We see that the Schouten brackets of Khudaverdian & Voronov are in correspondence with the higher Koszul–Schouten brackets.

**Remark** Theorem (3.2) carries over directly to the Schouten brackets. This can be understood by counting form degrees and realising that the Koszul–Schouten and Schouten brackets coincide when restricted to functions and exterior derivatives of functions as required by the theorem. Then (up to conventions) this theorem is identical to a theorem found in [13].

### 5 Concluding Remarks

The construction of higher Poisson and higher Koszul–Schouten brackets represent a geometric generalisation of the classical brackets found in Poisson geometry and classical mechanics. That is we consider the natural generalisation of a higher order multivector field that Schouten–Nijenhuis self-commutes and derive the theory from this starting place. Laying behind this is the algebraic theory of higher derived brackets of Voronov [23] [24] which leads us to the theory of $L_\infty$-algebras viz Lada & Stasheff [19].

The classical master equation is a “differential geometric condition”. Thus, from a geometric point of view higher Poisson structures are quite natural, as compared to say Nambu–Poisson structures [20] [22]. Despite this, the physical applications of higher Poisson structures as a generalised setting for classical mechanics is unclear. This is before one even begins to consider quantisation, which if possible would require some deviation from established methods.

In this note we showed that the Koszul–Schouten bracket found in classical Poisson geometry generalises to the higher case, including the morphism between the Poisson and Koszul–Schouten brackets provided by the exterior derivative. We showed how they relate to the Schouten brackets found in [13].

What other aspects of classical Poisson geometry carry over to the higher case awaits to be explored. It should be possible to consider a generalised version of Poisson homology and cohomology for higher Poisson manifolds, for example. For the homogenous case initial work was presented by de Azcárraga et.al [8].

The structure of a higher Poisson manifold can be found lying behind the classical Batalin-Vilkovisky antifield formulism [3] [4], modulo extra gradings. Indeed, the initial motivation for this work lies in wanting a deeper understanding of the BV-formulism and the relation with classical differential geometry.

Thus, we see that the space of field-antifield valued differential forms comes equipped with the structures of a homotopy BV-algebra, or equivalently a homotopy Schouten algebra. However, the role that such higher brackets play in field theory awaits to be explored.
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