A data-driven method for computing polyhedral invariant sets of black-box switched linear systems

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Abstract—In this paper, we consider the problem of invariant set computation for black-box switched linear systems using merely a finite set of observations of system simulations. In particular, this paper focuses on polyhedral invariant sets. We propose a data-driven method based on the one step forward reachable set. For formal verification of the proposed method, we introduce the concept of almost-invariant sets for switched linear systems. The convexity-preserving property of switched linear systems allows us to conduct contraction analysis on almost-invariant sets and derive an a priori probabilistic guarantee. In the spirit of non-convex scenario optimization, we also establish a posteriori the level of violation on the computed set. The performance of our method is then illustrated by a switched system under arbitrary switching between two modes.

I. INTRODUCTION

Switched linear systems consist of a finite set of linear dynamics (called modes) and a switching rule that indicates the current active mode of the system. They constitute an important family of hybrid systems. While the system is governed by linear dynamics dwelling in the same mode, the jump from one mode to another causes interesting hybrid phenomena distinct from the behaviors of the individual linear dynamics. For instance, despite the simplicity of the dynamics, stability analysis for a switched linear system is still complicated due to the switching signal, see [1] and the references therein.

Invariant set theory is widely used in system analysis and has been successfully generalized to study the properties of switched systems, see, e.g., [2]. One typical technique for invariant set characterization is to construct Lyapunov functions of the switched system, see, e.g., [1], [3], [4]. In the presence of state constraints, more complications arise because invariant sets have to be constraint admissible, see [5] for the case of polyhedral constraints. While handling general nonlinear constraints is still an open problem, there exist algorithms for computing invariant sets for certain classes of nonlinear constraints, see, e.g., [6], [7]. In [8], combinatorial methods have been introduced for switched systems where the switching signals are restricted by a labeled directed graph or an automaton.

The aforementioned algorithms are all based on the knowledge of a hybrid model of the switched system, which is usually obtained by hybrid system identification [9]. However, except for simple systems with very low dimensions, hybrid system identification is often computationally demanding. In fact, identifying a switched linear system is known to be NP-hard [10]. Data-driven analysis under the framework of black-box systems has been an active area of research in recent years, see [11]–[13]. For instance, probabilistic stability guarantees are provided in [12] for black-box switched linear systems, based merely on a finite number of observations of trajectories. Data-driven analysis also allows us to study set invariance for black-box switched linear systems without performing hybrid system identification. The data-driven stability analysis technique in [12] essentially attempts to compute an invariant ellipsoid. However, ellipsoidal invariant sets are often conservative for switched linear systems, because they rely on a common quadratic Lyapunov function, which may not exist even if the system is stable, see [1]. In this paper, we consider the computation of polyhedral invariant sets of switched linear systems under arbitrary switching. Our goal is to develop a data-driven method for computing polyhedral invariant sets in the spirit of the scenario optimization approach [14]. The contributions of this paper are threefold. First, inspired by [15], we propose a geometric algorithm based on a finite set of snapshot pairs of the states. Second, we introduce the concept of almost-invariant sets for switched linear systems and show their connections to \( \lambda \)-contractive sets via contraction analysis. Third, we derive a priori and a posteriori probabilistic guarantees for the proposed geometric algorithm.

The rest of the paper is organized as follows. This section ends with the notation, followed by the next section on the review of preliminary results on invariant sets and switched linear systems. Section III presents the proposed data-driven method. In Section IV probabilistic guarantees of the proposed method are discussed. Numerical results are provided in Section V.

Notation. The non-negative integer set is indicated by \( \mathbb{Z}^+ \). For a square matrix \( Q, Q \succeq 0 \) means \( Q \) is positive definite (semi-definite). \( \mathbb{S}_{n-1} \) and \( \mathbb{B}_n \) are the unit sphere and unit ball respectively in \( \mathbb{R}^n \). Let \( \mu(\cdot) \) denote the uniform spherical measure on \( \mathbb{S}_{n-1} \) with \( \mu(\mathbb{S}_{n-1}) = 1 \). For any symmetric matrix \( P \succ 0 \), we define \( \|x\|_P := \sqrt{x^TPx} \). Given any set \( S \subseteq \mathbb{R}^n \), \( \text{conv}(S) \) is the convex hull of \( S \) and let \( \|x\|_S := \min\{\lambda \geq 0 : x \in \lambda S\} \) for any \( x \in \mathbb{R}^n \). A (bounded) polytope \( S \) is called a C-polytope if it is convex and contains the origin in its interior. For any C-polytope \( S \), let \( V(S) \) denote the set of vertices and \( F(S) \) denote
the set of facets. Given any \( u \in \mathbb{R}^n \) and \( \theta \in [0, \pi/2] \), let \( \text{Cap}(u, \theta) := \{ v \in \mathbb{S}^{n-1} : u^T v \geq \|u\| \cos(\theta) \} \) denote the spherical cap with the direction \( u \) and the angle \( \theta \).

II. PRELIMINARIES AND PROBLEM STATEMENT

Switched linear systems are described below:

\[
x(t + 1) = A_{\sigma(t)}x(t), \quad t \in \mathbb{Z}^+ 
\]

where \( \sigma(t) : \mathbb{Z}^+ \to \mathcal{M} := \{1, 2, \cdots, M\} \) a time-dependent switching signal that indicates the current active mode of the system among \( M \) possible modes in \( \mathcal{A} := \{A_1, A_2, \cdots, A_M\} \). For any given switching sequence \( \sigma \), let

\[
A_{\sigma(k)} := A_{\sigma(k-1)} \cdots A_{\sigma(1)}A_{\sigma(0)}, \quad k \in \mathbb{Z}^+ 
\]

with \( \sigma(k) := \{\sigma(k-1), \cdots, \sigma(1), \sigma(0)\} \), \( \sigma(0) = \emptyset \), and \( A_{\sigma(0)} = I_n \). The stability of System (1) can be described by the joint spectral radius (JSR) of the matrix set \( \mathcal{A} \) defined by [16]

\[
\rho(\mathcal{A}) := \lim_{k \to \infty} \max_{\sigma(k) \in \mathcal{M}^n} \|A_{\sigma(k)}\|^{1/k} 
\]

Throughout the paper, we assume that \( \rho(\mathcal{A}) < 1 \). We focus on the computation of invariant sets of System (1) under arbitrary switching, which are formally defined below.

**Definition 1:** A nonempty set \( Z \subseteq \mathbb{R}^n \) is an invariant set for System (1) if \( x \in Z \) implies that \( Ax \in Z \) for any \( A \in \mathcal{A} \).

From the definition above, invariant sets are inherently related with the stability of System (1). For instance, the level set of a common quadratic Lyapunov function, which can be efficiently computed via semidefinite programming when it exists and the dynamics matrices \( \mathcal{A} \) are known, see, e.g., [1], is an ellipsoidal invariant set. In this paper, we focus on polyhedral invariant sets. Under the assumption that \( \rho(\mathcal{A}) < 1 \), the existence of a polyhedral invariant set is guaranteed, while an ellipsoidal invariant set may not exist because a common quadratic Lyapunov function does not necessarily exist. This is one of the reasons why polyhedral invariant sets are often more appealing for switched linear systems, even though the computation may be more expensive.

A necessary and sufficient condition for set invariance in the polyhedral case is given below.

**Proposition 1:** A C-polytope \( S \subseteq \mathbb{R}^n \) is an invariant set for System (1) if and only if

\[
\|A_{\sigma}x\|_S \leq \|x\|_S, \quad \forall x \in \mathbb{S}_{n-1}, \forall \sigma \in \mathcal{M}, \quad (4) 
\]

Proof: This proposition is a direct consequence of the homogeneity property, i.e., for any \( \gamma > 0 \), \( \|\gamma x\|_S = \gamma \|x\|_S \) and \( \|A_{\sigma} \gamma x\|_S = \gamma \|A_{\sigma} x\|_S \). \( \square \)

When the dynamics matrices \( \mathcal{A} \) are known, classical algorithms based on iterative linear programming exist, see, e.g., [5], [15], allowing to compute such a set efficiently. However, as we have mentioned above, in many cases, approximating the model of a switched system is computationally demanding, let alone identifying it exactly. This paper considers the case where the dynamics matrices \( \mathcal{A} \) are unknown. We call such systems black-box switched linear systems.

In the black-box case, we sample a finite set of the initial states and the switching modes. More precisely, we randomly and uniformly generate \( N \) initial states on \( \mathbb{S}_{n-1} \) and \( N \) modes in \( \mathcal{M} \), which are denoted by \( \omega_N := \{(x_i, \sigma_i) \in \mathbb{S}_{n-1} \times \mathcal{M} : i = 1, 2, \cdots, N\} \). From this random sampling, we observe the data set \( \{(x_i, A_{\sigma_i} x_i) : i = 1, 2, \cdots, N\} \), where \( A_{\sigma_i} x_i \) is the successor of the initial state \( x_i \). Note that the switching signal does not have to be observable.

For the given data set \( \omega_N \) (or \( \{(x_i, A_{\sigma} x_i)\}_{i=1}^N \)), we define the following sampled problem:

\[
\text{find } S \text{ s.t. } \|A_{\sigma} x\|_S \leq \|x\|_S, \forall (x, \sigma) \in \omega_N \quad (5)
\]

where \( S \) is a C-polytope. As we assume asymptotic stability under arbitrary switching, we are interested in invariant sets that contain the origin in their interiors. For this reason, \( S \) in (5) is restricted to be a C-polytope. In this paper, we attempt to solve this sampled problem (5) using a geometric algorithm by scaling the sampled points and computing the convex hull of the scaled points iteratively. We will show that convergence of this algorithm is guaranteed under the assumption that \( \rho(\mathcal{A}) < 1 \).

III. DATA-DRIVEN COMPUTATION OF POLYHEDRAL INvariant SETS

This section presents the proposed data-driven method for computing polyhedral invariant sets of black-box switched linear systems.

A. A geometric algorithm

We first present a geometric algorithm for computing invariant sets for the case where the matrices \( \mathcal{A} \) are known. This geometric algorithm is based on the one step forward reachable set [2], [15]. Given an initial C-polytope \( X \), let us define:

\[
R_{k+1} = \text{conv}(R_k \bigcup_{\sigma \in \mathcal{M}} A_{\sigma} R_k), \quad R_0 = X, k \in \mathbb{Z}^+. \quad (6)
\]

The properties of the algorithm above are stated in the following proposition.

**Proposition 2 ([15]):** Suppose \( \rho(\mathcal{A}) < 1 \), let us define \( R_k \) as in (6) for all \( k \in \mathbb{Z}^+ \) with an initial C-polytope \( X \). Then, the following results hold. (i) There exists a finite \( k \) such that \( R_{k+1} = R_k = R_\infty \). (ii) The set \( R_\infty \) is the smallest invariant set that contains \( X \).

Proof: A sketch of the proof is given here. We refer the readers to [15] for the detailed proof. (i) It can be shown by induction that, \( \forall k \in \mathbb{Z}^+ \),

\[
R_k = \text{conv}(X \bigcup_{\sigma \in \mathcal{M}} A_{\sigma} X \bigcup \cdots \bigcup A_{\sigma} X) \quad (7)
\]

where \( A_{\sigma} \) is defined in (2). Since \( \rho(\mathcal{A}) < 1 \) and \( X \) is bounded and contains the origin in the interior, there always exists a \( k \) such that \( A_{\sigma} X \subseteq X \) for all \( \sigma \in \mathcal{M}^{k+1} \), which implies that \( R_{k+1} = R_k = R_\infty \). (ii) For any invariant set \( S \) containing \( X \), from set invariance, it can be shown that \( R_k \subseteq S \) for all \( k \in \mathbb{Z}^+ \). \( \square \)
B. The proposed data-driven method

With the sample $\omega_N$ and an initial C-polytope $X$, we now present a data-driven version of the geometric algorithm (9):

$$\hat{R}_{k+1}(\omega_N) = \text{conv}(\hat{R}_k(\omega_N) \cup \Omega_k(\omega_N)), \quad \forall k \in \mathbb{Z}^+ \quad (8)$$

where $\hat{R}_0(\omega_N) = X$ and

$$\Omega_k(\omega_N) := \{ \frac{A_x x}{\|x\|_{R_k(\omega_N)}} : (x, \sigma) \in \omega_N \}$$

$$\cup \{ \frac{-A_x x}{\|x\|_{R_k(\omega_N)}} : (x, \sigma) \in \omega_N \}. \quad (9)$$

The convergence of the data-driven geometric algorithm is stated in the following lemma.

**Theorem 1:** Suppose $\rho(A) < 1$. Given a sample of $N$ points in $\mathbb{S}_{n-1} \times \mathcal{M}$, denoted by $\omega_N$, let $R_0$ and $\hat{R}_k(\omega_N)$ be defined as in (9) and (8) respectively for all $k \in \mathbb{Z}^+$ with the same initial C-polytope $X$. Then, the following results hold. (i) For any $k \in \mathbb{Z}^+$, $\hat{R}_k(\omega_N) \subseteq R_k$. (ii) The sequence $(\hat{R}_k(\omega_N))_{k \in \mathbb{Z}^+}$ is convergent. (iii) $\hat{R}_\infty(\omega_N)$ is a feasible solution to Problem (5).

**Proof:** (i) The proof goes by induction. Suppose $\hat{R}_k(\omega_N) \subseteq R_k$ for some $k \in \mathbb{Z}^+$. From the definition in (9), it holds that $\Omega_k(\omega_N) \subseteq \bigcup_{x \in \mathcal{M}} A_x R_k$. Hence, $\hat{R}_{k+1}(\omega_N) \subseteq R_{k+1}$. Thus, the statement is true as $\hat{R}_0(\omega_N) \subseteq R_0$. (ii) The convergence of $(\hat{R}_k(\omega_N))_{k \in \mathbb{Z}^+}$ is a direct consequence of (i). (iii) From (8), when $(\hat{R}_k(\omega_N))_{k \in \mathbb{Z}^+}$ converges, $\Omega_\infty(\omega_N) \subseteq \hat{R}_\infty(\omega_N)$, which implies that $\|A_x x\|_{\hat{R}_\infty(\omega_N)} \leq \|x\|_{\hat{R}_\infty(\omega_N)}$ for any $(x, \sigma) \in \omega_N$. Hence, $\|A_x x\|_{\hat{R}_\infty(\omega_N)} \leq \|x\|_{\hat{R}_\infty(\omega_N)}$ for any $(x, \sigma) \in \omega_N$. \hfill \square

The theorem above shows that $(\hat{R}_k(\omega_N))_{k \in \mathbb{Z}^+}$ eventually converges to a feasible solution of the sampled problem (5). However, finite-time convergence of (5) may not be preserved. For the practical implementation, we use a stopping criterion as shown in Algorithm 1.

**Algorithm 1** Data-driven computation of polyhedral invariant sets

**Input:** $X, \omega_N$ and some tolerance $\epsilon > 0$

**Output:** $\hat{R}_k(\omega_N)$

**Initialization:** Let $k \leftarrow 0$ and $\hat{R}_k(\omega_N) \leftarrow X$

1: Obtain $\Omega_k(\omega_N)$ from (9);
2: if $\Omega_k(\omega_N) \subseteq (1 + \epsilon)\hat{R}_k(\omega_N)$ then
3: Terminate;
4: else
5: Compute $\hat{R}_{k+1}(\omega_N)$ from (8);
6: Let $k \leftarrow k + 1$ and go to Step 1.
7: end if

IV. PROBABILISTIC SET INVARIANCE GUARANTEES

In this section, we formally discuss probabilistic guarantees on the data-driven method proposed in Section [11]

A. Contraction analysis

We begin by generalizing the definition of invariant sets for switched linear systems. In this paper, we consider sets that are invariant almost everywhere except in an arbitrarily small subset. Such a set is referred to as an almost-invariant set, which is formally defined below, adapted from [17].

**Definition 2:** Given $\epsilon \in (0, 1)$, a set $S \subseteq \mathbb{R}^n$ is an $\epsilon$ almost-invariant set for System (1) if $\mu(\{x \in \mathbb{S}_{n-1} : \|Ax\| \geq \|x\|, \forall A \in \mathcal{A}\}) \geq 1 - \epsilon$, where $\mu(\cdot)$ denotes the uniform spherical measure.

For switched linear systems, we can establish a contraction property for almost-invariant sets. To show this, we formalize the notion of $\lambda$-contractive sets, as stated below.

**Definition 3:** Given $\lambda \geq 0$, a set $S \subseteq \mathbb{R}^n$ is a $\lambda$-contractive set for System (1) if $x \in S$ implies that $Ax \in AS$ for any $A \in \mathcal{A}$. When $\lambda > 1$, the set can be in fact expansive, but we still call it a $\lambda$-contractive set to be consistent.

The contraction property becomes obvious when a contraction rate is computed. To obtain a tight contraction rate, we introduce additional definitions as follows. For any $\epsilon \in (0, 1/2)$, let

$$\delta(\epsilon) := \sqrt{1 - \frac{1}{2^{n-1}}}, \quad (10)$$

$$\theta(\epsilon) := \cos^{-1}(\delta(\epsilon)), \quad (11)$$

where $\mathcal{I}(x; a, b)$ is the regularized incomplete beta function (see, e.g., [12]) defined as

$$\mathcal{I}(x; a, b) := \int_0^x t^{a-1} (1-t)^{b-1} dt. \quad (12)$$

For any given C-polytope $S \subseteq \mathbb{R}^n$ and $u \in \mathcal{V}(S)$, let

$$\gamma(u, S, \epsilon) := \max_{\alpha \geq 0} \{ \alpha : \alpha u \in \mathcal{V}(\mathcal{S} \bigcap \mathcal{C}(u, \theta(\epsilon))) \} \quad (13)$$

where $\epsilon \in (0, 1/2)$ and $\mathcal{C}(u, \theta)$ is given by

$$\mathcal{C}(u, \theta) := \{ x \in \mathbb{R}^n : u^T x \leq \|x\| \|u\| \cos(\theta) \}, \quad (14)$$

which is the closure of the complement of the cone $C(u, \theta)$ with the direction $u$ and the angle $\theta$:

$$\mathcal{C}(u, \theta) := \{ x \in \mathbb{R}^n : u^T x \geq \|x\| \|u\| \cos(\theta) \}. \quad (15)$$

The geometric interpretation of the definition in (13) is illustrated in Figure (1) Let us define:

$$\gamma_{\min}(S, \epsilon) := \min_{u \in \mathcal{V}(S)} \gamma(u, S, \epsilon). \quad (16)$$

With these definitions, we state the contraction property of almost-invariant sets in the following proposition.

**Proposition 3:** Given $\epsilon \in (0, 1/2)$, suppose a C-polytope $S \subseteq \mathbb{R}^n$ is an $\epsilon$ almost-invariant set of System (1). Let $\gamma_{\min}(S, \epsilon)$ be defined as in (16). Then, $S$ is a $\lambda$-contractive set of System (1) with $\lambda = 1/\gamma_{\min}(S, \epsilon)$.

The proof of Proposition 3 is given in the appendix. From the definition in (16), to obtain $\gamma_{\min}(S, \epsilon)$, we need to compute $\gamma(u, S, \epsilon)$, which requires the computation of $\mathcal{V}(\{ x \in S : u^T x \leq \|x\| \|u\| \cos(\theta) \})$ for all $u \in \mathcal{V}(S)$. In general, computing the convex hull of a nonlinear constraint
set is a difficult problem, see [18]. For this reason, we formulate a relaxation of (16), which yields a computational tractable lower bound which can be computed by solving convex problems. Given a C-polytope $S$ and $\epsilon \in (0, 1/2)$, we define:

$$
\gamma(S, \epsilon) := \min_{u \in \mathcal{V}(S)} \delta(u, S, \epsilon) / \|u\| \tag{17}
$$

where $\delta(u)$ is defined in (10), and

$$
d_{\min}(u, S, \epsilon) := \min_{x \in \partial S} \{\|x\| : x \in \mathcal{C}(u, \theta(\epsilon))\}. \tag{18}
$$

The properties of $\gamma(S, \epsilon)$ are given in the following lemma.

**Lemma 1:** Given any C-polytope $S \subseteq \mathbb{R}^n$ and any $\epsilon \in (0, 1/2)$, let us define $\gamma_{\min}(S, \epsilon)$ and $\gamma(S, \epsilon)$ in (16) and (17) respectively. Then, $\gamma_{\min}(S, \epsilon) \geq \gamma(S, \epsilon)$.

The proof of Lemma 1 is given in the appendix. We then show that $\gamma(S, \epsilon)$ can be computed by solving a set of convex optimization problems. Given any C-polytope, let us define the following problem, for any $u \in \mathcal{V}(S)$ and $f \in \mathcal{F}(S)$ that satisfy $f \cap \mathcal{C}(u, \theta(\epsilon)) \neq \emptyset$,

$$
d_{\min}(u, S, \epsilon) := \min_{x \in \partial S} \{\|x\| : u^T x \geq \delta(u, \epsilon) \|u\|\} \tag{19}
$$

This is a convex problem and can be efficiently solved by classic solvers, see [19]. The following lemma shows that $\gamma(S, \epsilon)$ defined in (18) can be computed by solving (19).

**Lemma 2:** Given any $\epsilon \in (0, 1/2)$, C-polytope $S \subseteq \mathbb{R}^n$, and $u \in \mathcal{V}(S)$, one has:

$$
d_{\min}(u, S, \epsilon) = \min_{f \in \mathcal{F}(S)} d_{\min}^f(u, S, \epsilon), \tag{20}
$$

where $d_{\min}(u, S, \epsilon)$ and $d_{\min}^f(u, S, \epsilon)$ are defined in (18) and (19) respectively.

**Proof:** To compute $d_{\min}(u, S, \epsilon)$, we need to check all the points on $\partial S \cap \mathcal{C}(u, \theta(\epsilon))$. This can be equivalently done by checking all the facets of $S$ and solving Problem (19). □

**Remark 1:** Suppose $\bar{S} = \{x \in S_{n-1} : \|Ax\|_S \leq \|x\|_S, \forall A \in \mathcal{A}\}$ From the proof of Proposition 3 the results above also hold for the case where the violating subset $S_{n-1} \setminus \bar{S}$ is the union of a group of disjoint sets whose measures are bounded by $\epsilon$.

**B. A priori guarantee**

Now, we conduct a priori analysis to obtain a formal guarantee in which the level of confidence is computed a priori. Let us recall the notions of covering and packing numbers, see, e.g., Chapter 27 of [20].

**Definition 4:** Given $\epsilon \in (0, 1/2)$, a set $U \subseteq S_{n-1}$ is called an $\epsilon$-covering of $S_{n-1}$ if, for any $x \in S_{n-1}$, there exists $u \in U$ such that $u^T x \geq \delta(u, \epsilon)$. The covering number $N_c(\epsilon)$ is the minimal cardinality of an $\epsilon$-covering of $S_{n-1}$.

**Definition 5:** Given $\epsilon \in (0, 1/2)$, a set $U \subseteq S_{n-1}$ is called an $\epsilon$-packing of $S_{n-1}$ if, for any $u, v \in U$, $u^T v > \delta(u, \epsilon)$. The packing number $N_p(\epsilon)$ is the maximal cardinality of an $\epsilon$-packing of $S_{n-1}$.

With these two notions, the following lemma is obtained.

**Lemma 3:** For any $\epsilon \in (0, 1/2)$, let $\delta(\epsilon)$ and $\theta(\epsilon)$ be defined in (10) and (11) respectively. Then,

$$
N_c(\epsilon) \leq N_p(\epsilon) \leq \frac{2}{I(\sin^2(\theta(\epsilon)))} \frac{n-1}{\epsilon^2}. \tag{21}
$$

**Proof:** Suppose $U$ is the $\epsilon$-packing with the maximal cardinality. The first inequality follows from the fact that $U$ is also an $\epsilon$-covering. For any direction $u \in S_{n-1}$ and any angle $\theta \in [0, \pi/2]$, the spherical cap $C(u, \theta)$ has a measure of $\frac{n-1}{2} I(\sin^2(\theta); \frac{n-1}{2}, \frac{1}{2})$ (see [12] for details). From the definition of an $\epsilon$-packing, the spherical caps $\{C(u, \theta(\epsilon)/2)\}_{u \in U}$ are disjoint. Hence, $\sum_{u \in U} \mu(C(u, \theta(\epsilon)/2)) \leq 1$, which leads to the second inequality. □

The a priori probabilistic guarantee is then stated in the following theorem. Recall that $M$ is the number of modes in $\mathcal{M}$ (or $\mathcal{A}$).

**Theorem 2:** Suppose the same conditions as in Theorem 3 hold. With the sample $\omega_N$ and an initial C-polytope $X$, the set $\bar{R}_\infty(\omega_N)$ is defined as in (8). For any $\epsilon \in (0, 1/2)$, let

$$
B(\epsilon; N) = \frac{2M(1 - \frac{\pi}{2})^N}{I(\sin^2(\theta(\epsilon)))} \frac{n-1}{\epsilon^2}. \tag{22}
$$

where $\theta(\epsilon)$ is defined in (11). Then, given any $\epsilon \in (0, 1/2)$, with probability no smaller than $1 - B(\epsilon; N)$, $\bar{R}_\infty(\omega_N)$ is a $\lambda$-contractive set of System (1) with $\lambda = 1/\gamma(\bar{R}_\infty(\omega_N), I(\sin^2(\theta(\epsilon)); \frac{n-1}{2}, \frac{1}{2})/2)$, where $\gamma(\cdot, \cdot)$ is defined in (17).

**Proof:** Consider the maximal $\epsilon$-packing $U$ with the cardinality $N_p$. From Lemma 3 $\{C(u, \theta(\epsilon))\}_{u \in U}$ covers $S_{n-1}$. Suppose $\omega_N$ is sampled randomly according to the uniform distribution, then the probability that each spherical cap in $\{C(u, \theta(\epsilon))\}_{u \in U}$ contains $M$ points with $M$ different modes is no smaller than $1 - N_p M(1 - \frac{\pi}{2})^N \geq B(\epsilon; N)$. Hence, the angle of the largest spherical cap that violates the condition $\|Ax\|_\bar{R}_\infty(\omega_N) \leq \|x\|_\bar{R}_\infty(\omega_N)$, $\forall A \in \mathcal{A}$, is bounded by $\theta(\epsilon)$. Thus, the measure of the largest violating spherical cap is bounded by $I(\sin^2(\theta(\epsilon)); \frac{n-1}{2}, \frac{1}{2})/2$. From Proposition 3 and Lemmas 1 & 2 $\bar{R}_\infty(\omega_N)$ is a $\lambda$-contractive set with the rate of $1/\gamma(\bar{R}_\infty(\omega_N), I(\sin^2(\theta(\epsilon)); \frac{n-1}{2}, \frac{1}{2})/2)$.

**Remark 2:** As the dimension increases, the number of vertices of $\bar{R}_\infty(\omega_N)$ also increases. Thus, the computation
of $\gamma(\hat{R}_\infty(\omega_N), M_\varepsilon(s(\omega_N)))$ becomes expensive for high-dimensional systems.

C. A posteriori guarantee

We then use the chance-constraint theorem in [14] to derive a posteriori guarantees in which the level of confidence is computed a posteriori. The following definition is also needed.

**Definition 6:** Consider a sample of $N$ points in $\mathbb{S}_{n-1} \times \mathcal{M}$, denoted by $\omega_N$, and the iteration (8) with an initial C-polytope $X$, $(x, \sigma) \in \omega_N$ is called a supporting point, if $\hat{R}_\infty(\omega_N \setminus \{(x, \sigma)\}) \neq \hat{R}_\infty(\omega_N)$. Let $s(\omega_N)$ denote the number of supporting points in $\omega_N$.

With the definitions above, we obtain the following theorem, adapted from Theorem 1 in [14].

**Theorem 3:** Suppose $\rho(A) < 1$. Given $N \in \mathbb{Z}^+$, let $\omega_N$ be i.i.d. with respect to the uniform distribution $\mathbb{P}$ over $\mathbb{S}_{n-1} \times \mathcal{M}$. $\mathbb{P}^N$ denotes the probability measure in the $N$-Cartesian product of $\mathbb{S}_{n-1} \times \mathcal{M}$. Let $\hat{R}_\infty(\omega_N)$ be obtained from (8) with an initial C-polytope. Then, for any $\beta \in (0, 1)$

$$\mathbb{P}^N(\{ \omega_N : \mathcal{P}(V(\hat{R}_\infty(\omega_N))) > \varepsilon(s(\omega_N)) \}) \leq \beta$$

(23)

where $V(\hat{R}_\infty(\omega_N)) := \{(x, \sigma) : \|Ax\|_{\hat{R}_\infty(\omega_N)} > \|x\|_{\hat{R}_\infty(\omega_N)}\}$, $s(\omega_N)$ is the number of supporting points as defined in Definition 6 and $\varepsilon : \{0, 1, \cdots, N\} \to [0, 1]$ is a function defined as:

$$\varepsilon(k) := \begin{cases} 1 & \text{if } k = N; \\ 1 - \frac{N-k}{N} & 0 \leq k < N. \end{cases}$$

(24)

Since it is a simple adaptation of Theorem 1 in [14], the proof is omitted. Indeed, this bound is established a posteriori because it is based on the measured data $\omega_N$.

With Theorem 3 in hand, we can derive a probabilistic guarantee on set invariance in the following corollary.

**Corollary 1:** Suppose the same conditions as in Theorem 3 hold. Let $\hat{R}_\infty(\omega_N)$ be the solution obtained from (8) with an initial C-polytope $X$ and $s(\omega_N)$ be defined in Definition 6. Then, with probability no smaller than $1 - \beta$, $\hat{R}_\infty(\omega_N)$ is $M_\varepsilon(s(\omega_N))$ almost-invariant set for System (1), where $s(\omega_N)$ is defined in (24).

Proof: From Theorem 3 with probability no smaller than $1 - \beta$, $\mathbb{P}(V(\hat{R}_\infty(\omega_N))) \leq \varepsilon(s(\omega_N))$. Since $\{x \in \mathbb{S}_{n-1} : \|Ax\|_{\hat{R}_\infty(\omega_N)} > \|x\|_{\hat{R}_\infty(\omega_N)}\} = \{x \in \mathbb{S}_{n-1} : \exists \sigma \in \mathcal{M} : (x, \sigma) \in V(\hat{R}_\infty(\omega_N))\}$. Hence, $\mu(x \in \mathbb{S}_{n-1} : \|Ax\|_{\hat{R}_\infty(\omega_N)} > \|x\|_{\hat{R}_\infty(\omega_N)}) \leq M\mathbb{P}(V(\hat{R}_\infty(\omega_N))) \leq M\mathbb{P}(\varepsilon(s(\omega_N)))$. □

Similarly, we can also state the contraction property.

**Corollary 2:** Suppose the same conditions as in Theorem 3 hold. Let $\hat{R}_\infty(\omega_N)$ be obtained from (8). Consider the number of supporting points $s(\omega_N)$ and the function $\varepsilon : \{0, 1, \cdots, N\} \to [0, 1]$ as defined in Definition 6 and (24) respectively. Then, for any $\beta \in (0, 1)$, with probability no smaller than $1 - \beta$, $\hat{R}_\infty(\omega_N)$ is a $\lambda$-contractive set of System (1) with $\lambda = 1/\gamma(\hat{R}_\infty(\omega_N), M_\varepsilon(s(\omega_N)))$, where $\gamma(\hat{R}_\infty(\omega_N), M_\varepsilon(s(\omega_N)))$ is defined in (17).

Proof: This result is a direct consequence of Proposition 3 and Lemmas 1 & 2. □

**Remark 3:** When the initial C-polytope $X$ is symmetric, it can be shown that $\hat{R}_\infty(\omega_N)$ is also symmetric. In this case, the contraction bound in Corollary 2 becomes $\lambda = 1/\gamma(\hat{R}_\infty(\omega_N), M_\varepsilon(s(\omega_N)))/2$.

V. Numerical example

We consider a switched linear system with $A = \{A_1, A_2\}$, $A_1 = [0.4 - 0.8; 0.5 1.2], A_2 = [-1.1 - 0.3; 0.7 0.4]$. The initial C-polytope is set to be $X = \{x : \|x\|_\infty \leq 1\}$ and 400 points are sampled randomly and uniformly on the unit sphere. We then use Algorithm 1 to compute $\hat{R}_\infty(\omega_N)$ with a tolerance of $10^{-8}$. The results are given in Figure 2. While the matrices $A$ are unknown, we still show $R_\infty$ for reference. In order to evaluate the difference between $\hat{R}_\infty(\omega_N)$ and $R_\infty$, we compute their volumes, denoted by $vol(\hat{R}_\infty(\omega_N))$ and $vol(R_\infty)$, and show $vol(\hat{R}_\infty(\omega_N))/vol(R_\infty)$ for different sizes of the sample. From Figure 2 we can see that $\hat{R}_\infty(\omega_N)$ is already very close to $R_\infty$ with 400 sampled points. For rigorous verification, we compute the probabilistic bounds derived in Section IV. Let the confidence level $\beta = 0.01$. For different values of $\epsilon$, let $N_\epsilon$ be the $N$ such that $B(\epsilon, N) = \beta = 0.01$. The curve of $N_\epsilon$ is shown in Figure 3a. We also compute a posteriori the bounds $\varepsilon(s(\omega_N))$ and $\gamma(\hat{R}_\infty(\omega_N), M_\varepsilon(s(\omega_N)))$ as stated in Theorem 3 and Corollary 2 with the confidence level $\beta = 0.01$, and the results are shown in Figure 3b. Note that $\epsilon$ in Figure 3b is not the measure of the violating subset but the measure of the largest disjoint set contained in the violating subset, see Remark 1. With the a posteriori analysis, we gain additional information on the measure of the whole violating subset.
VI. CONCLUSIONS

We have presented a data-driven method for computing polyhedral invariant sets for black-box switched linear systems based on the one step forward reachable set. The convergence of this method is guaranteed under the stability assumption. Almost-invariant sets have been introduced for switched linear systems. The convexity-preserving property of switched linear systems allowed us to establish a probabilistic guarantee a priori via contraction analysis. With the chance-constraint theorem for nonconvex problems, we have also derived an a posteriori guarantee which provides a bound on the level of violation of the computed set. Finally, a numerical example was given to illustrate the performance of the proposed method.

APPENDIX

Proof of Proposition 3

Let \( S = \{ x \in S_{n-1} : \| Ax \| \leq \| x \|, \forall A \in A \} \) and \( \alpha^* = \max_{\alpha \geq 0} \{ \alpha : \alpha S \subseteq \text{conv}(\{ x \in S : x/\| x \| \in S_{n-1}\}) \} \). For any \( x \in S : x/\| x \| \in S_{n-1} \) and \( A \in A, Ax \in S \), which implies that \( \text{conv}(\{ x \in S : x/\| x \| \in S_{n-1}\}) \subseteq S \) for any \( A \in A \). Hence, \( \alpha^* A S \subseteq \text{conv}(\{ x \in S : x/\| x \| \in S_{n-1}\}) \subseteq S \) for any \( A \in A \). That is, \( S \) is a \( \frac{1}{\alpha^*} \)-contractive set. Now, it suffices to show that \( \gamma_{\min}(S, \epsilon) \) is a lower bound of \( \gamma^* \). For any \( u \in V(S) \), let \( \alpha(u) = \max_{\alpha \in \gamma(S)}(\alpha \in \{ x \in S \} ) \). Then, it holds that \( \alpha^* = \min_{u \in \gamma(S)}(\alpha(u)). \) In the rest of the proof, we aim to show that \( \alpha(u) \geq \gamma(u, S, \epsilon) \) for any \( u \in \gamma(S). \) Suppose \( \theta \) is the smallest \( \alpha \) such that the set \( \{ \alpha \geq 0 : \alpha u \in \text{conv}(\{ x \in \partial S : x/\| x \| \in S \cap \text{Cap}(u, \theta)) \} \) is non-empty. Let \( \alpha (u) = \max_{\alpha \in \gamma(S)}(\alpha \in \text{conv}(\{ x \in \partial S : x/\| x \| \in S \cap \text{Cap}(u, \theta)) \}). \) Since \( \{ x \in \partial S : x/\| x \| \in S \} \subseteq \text{conv}(\{ x \in S : x/\| x \| \in S \cap \text{Cap}(u, \theta)) \}, \alpha(u) \geq \gamma(u, \theta)) \). The value of \( \alpha(u) \) depends on the shape of the violating set \( S_{n-1} \cap S \). Note that the set \( (S_{n-1} \cap S) \) \( \partial S \) does not affect the value of \( \alpha(u) \). From this observation and the relation between the angel and the measure of the spherical cap (see [12] for details), we can see that \( \alpha(u) \) reaches the minimal when \( \text{Cap}(u, \theta) = S_{n-1} \cap S \) with \( \theta = \delta(\epsilon) \) as defined in [10]. It can be verified that \( \alpha(u) \) becomes \( \gamma(u, S, \epsilon) \) in this case. Therefore, \( \alpha(u) \geq \gamma(u, S, \epsilon) \) and thus \( \alpha^* \geq \gamma_{\min}(S, \epsilon). \)

Proof of Lemma 1

(i) Since \( \gamma(\theta) \in (0, \pi/2), \text{conv}(S \cap \text{Cap}(u, \theta)) \)} = \text{conv}(\partial S \cap \text{Cap}(u, \theta)) \) for any \( u \in \gamma(S). \) It is obvious that \( \partial S = \partial(S \cap \text{Cap}(u, \theta)) \)} \cup \partial(S \cap \text{Cap}(u, \theta)) \)} \). Taking convex hull of both sides yields \( S \subseteq \text{conv}(\partial S \cap \text{Cap}(u, \theta)) \)} \cup \partial(S \cap \text{Cap}(u, \theta)) \)} \), which implies that \( \text{conv}(S \cap \text{Cap}(u, \theta)) \)} \subseteq S \cap \text{conv}(\partial S \cap \text{Cap}(u, \theta)) \). Thus,
\[
\gamma(u, S, \epsilon) \leq \sup_{0 \leq \theta \leq 1} \{ \alpha : \alpha u \in S \} \cap \partial(S \cap \text{Cap}(u, \theta)) \}
\]
\[
= \sup_{0 \leq \theta \leq 1} \{ \alpha : \alpha u \not\in \text{conv}(S \cap \text{Cap}(u, \theta)) \}
\]
From the definition in [15], it can be verified that
\[
\partial(S \cap \text{Cap}(u, \theta)) = \{ x \in \partial(S : u^T x \geq \| x \| \| u \| \delta(\epsilon)) \}
\]
\[
\subseteq \{ x \in \partial(S : u^T x \geq \| u \| \| u \| \delta(\epsilon)) \}
\]
where \( d_{\min}(v, S, \epsilon) \) is defined as in [18]. Observe that
\[
\text{conv}(\{ x \in \partial S : u^T x \geq \| u \| \| u \| \delta(\epsilon)) \} = \{ x \in S : u^T x \geq \| u \| \| u \| \delta(\epsilon)) \}
\]
is the smallest \( \delta(\epsilon) \) for any \( u \in S \). Therefore, \( \gamma(u, S, \epsilon) \geq \delta(\epsilon) \| u \| \leq d_{\min}(u, S, \epsilon) \). This completes the proof. \( \Box \)

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