POISSON-GERSTENHABER BRACKETS IN REPRESENTATION ALGEBRAS

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Abstract. We introduce cyclic bilinear forms on coalgebras and use them to generalize Van den Bergh’s Poisson brackets in representation algebras.

1. Introduction

Van den Bergh [8] introduced double Poisson algebras as a non-commutative version of Poisson geometry, see also Crawley-Boevey [2] for a related approach. A double Poisson algebra is an algebra \( A \) (possibly, non-unital) equipped with a double Poisson bracket defined as a linear map \( \{\{-, -\}\} : A \otimes A \to A \otimes A \) satisfying certain axioms. A double Poisson bracket in \( A \) induces a Poisson structure on the variety \( \text{Rep}(A, N) \) of \( N \)-dimensional representations of \( A \) for all \( N \geq 1 \). To give a more precise formulation, note that the pair \((A, N)\) determines a commutative algebra \( A_N \) and a homomorphism \( t_N \) from \( A \) to the algebra \( \text{Mat}_N(A_N) \) of \( N \times N \) matrices over \( A_N \) characterized by the following universal property: for any commutative algebra \( B \) and any homomorphism \( s : A \to \text{Mat}_N(B) \), there is a unique homomorphism \( r : A_N \to B \) such that \( s = \text{Mat}_N(r) t_N \) (see [7, 4]). One calls \( A_N \) the coordinate algebra of \( \text{Rep}(A, N) \). Van den Bergh showed that a double Poisson bracket in \( A \) induces a Poisson bracket in \( A_N \) for all \( N \geq 1 \).

The aim of this paper is to generalize Van den Bergh’s Poisson bracket in \( A_N \) to a wider class of commutative algebras associated with \( A \). Fix a commutative ring \( \mathbb{K} \) which will be the ground ring of all modules, algebras, and coalgebras. For an algebra \( A \) and a coalgebra \( M \) (possibly, non-counital), we define a commutative representation algebra \( A_M \). It is characterized by the universal property as above with \( \text{Mat}_N(B) \) replaced by the convolution algebra \( \text{Hom}_\mathbb{K}(M, B) \). The algebra \( A_M \) is described here via generators \( \{a_\alpha \mid a \in A, \alpha \in M\} \) and certain relations. For the coalgebra \( M = (\text{Mat}_N(\mathbb{K}))^* \) dual to \( \text{Mat}_N(\mathbb{K}) \), we recover \( A_N \) because

\[
\text{Hom}_\mathbb{K}(\text{Mat}_N(\mathbb{K}))^*, B) = B \otimes_\mathbb{K} \text{Mat}_N(\mathbb{K}) = \text{Mat}_N(B).
\]

With each bilinear form \( v : M \otimes M \to \mathbb{K} \) on a coalgebra \( M \) we associate a linear map \( \tilde{v} : M \otimes M \to M \otimes M \). We call \( v \) cyclic if \( \tilde{v} \) commutes with the permutation of the factors. Our main result is the following theorem.

**Theorem 1.1.** For a double Poisson algebra \((A, \{\{-, -\}\})\) and a cyclic bilinear form \( v \) on a coalgebra \( M \), there is a unique Poisson bracket \( \{\{-, -\}_v\} \) in the representation algebra \( A_M \) such that for any \( a, b \in A, \alpha, \beta \in M \), and any finite expansions

\[
\{a, b\} = \sum_i a_i \otimes b_i \in A \otimes A, \quad \tilde{v}(\alpha \otimes \beta) = \sum_j \alpha_j \otimes \beta_j \in M \otimes M,
\]
we have
\[ \{a_\alpha, b_\beta\}_v = \sum_{i,j} (a_i)_\alpha (b_i)_\beta \in A_M. \]

Theorem 1.1 derives a Poisson bracket in \( A_M \) from a double Poisson bracket in \( A \) and a cyclic bilinear form on \( M \). For constructions of double brackets, see [8, 9, 5, 6].

We give here two constructions of cyclic bilinear forms. One construction starts from a \( \mathbb{K} \)-valued conjugation-invariant function on a finite group \( G \) and produces a cyclic form on the coalgebra \( (\mathbb{K}[G])^\ast \) dual to the group algebra \( \mathbb{K}[G] \). In particular, this derives cyclic forms on \( (\mathbb{K}[G])^\ast \) from the traces of linear representations of \( G \) of finite rank. The second construction starts from a symmetric Frobenius algebra and produces a cyclic form on the dual coalgebra. For example, the algebra of matrices \( \text{Mat}_N(\mathbb{K}) \) with \( N \geq 1 \) is a symmetric Frobenius algebra with Frobenius pairing being the trace of the product of matrices. This determines a cyclic bilinear form \( v \) on \( (\text{Mat}_N(\mathbb{K}))^\ast \). The bracket \( \{-,-\}_v \) in \( A_N \) is the original Van den Bergh’s Poisson bracket. Note that symmetric Frobenius algebras naturally arise in Topological Quantum Field Theory, see [3].

We establish several properties of the bracket \( \{-,-\}_v \) in \( A_M \). If the coalgebra \( M \) is counital, then the group \( U(M^\ast) \) of invertible elements of the algebra \( M^\ast \) naturally acts on \( A_M \). The commutator \( [\varphi, \psi] = \varphi \psi - \psi \varphi \) turns \( M^\ast \) into a Lie algebra, \( M^\ast \), which also acts on \( A_M \). The actions of \( U(M^\ast) \) and \( M^\ast \) are compatible in an appropriate sense and preserve the bracket \( \{-,-\}_v \). For non-degenerate \( v \), we define a natural trace map \( A \to A_M \) and compute \( \{-,-\}_v \) on its image.

When the algebra \( A \) is unital, we define a quotient \( A_M^+ \) of \( A_M \) characterized by the universal property as above with \( B \) running over unital commutative algebras. The bracket \( \{-,-\}_v \) in \( A_M \) induces a bracket \( \{-,-\}_v^+ \) in \( A_M^+ \), and the actions of \( U(M^\ast) \) and \( M^\ast \) on \( A_M \) descend to \( A_M^+ \). We use the action of \( M^\ast \) to formulate Hamiltonian reduction in this setting and to study double quasi-Poisson algebras.

We also introduce an equivariant generalization of the bracket \( \{-,-\}_v \) and discuss connections with the topology of surfaces.

The main body of the paper is written in the more general setting of graded algebras. Instead of (double) Poisson brackets we consider their graded versions, the (double) Gerstenhaber brackets.

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2. Representation algebras

We begin with generalities on graded algebras and coalgebras and then introduce the representation algebras.

2.1. Graded algebras. By a module we mean a module over the fixed commutative ring \( \mathbb{K} \). By a graded module we mean a \( \mathbb{Z} \)-graded module \( A = \bigoplus_{p \in \mathbb{Z}} A^p \). An element \( a \) of \( A \) is homogeneous if \( a \in A^p \) for some \( p \); we call \( p \) the degree of \( a \) and write \( |a| = p \). For any integer \( n \), the \( n \)-degree \( |a|_n \) of a homogeneous element \( a \in A \) is defined by \( |a|_n = |a| + n \). Note that the \( n \)-degree of \( 0 \in A \) is an arbitrary integer.

An algebra is a module endowed with an associative bilinear multiplication. A graded algebra is a graded module \( A \) endowed with an associative bilinear multiplication such that \( A^p A^q \subset A^{p+q} \) for all \( p, q \in \mathbb{Z} \). Thus, \( |ab| = |a| + |b| \) for homogeneous \( a, b \in A \). A graded algebra \( A \) is unital if it has a two-sided unit \( 1_A \in A^0 \). Unless explicitly stated to the contrary, we do not require algebras to be unital.
For a graded algebra $A$, denote by $[A,A]$ the submodule of $A$ spanned by the vectors $ab - (-1)^{|a||b|}ba$ where $a,b$ run over all homogeneous elements of $A$. The graded algebra $A$ is \textit{commutative} if $[A,A] = 0$. Factoring an arbitrary graded algebra $A$ by the 2-sided ideal generated by $[A,A]$ we obtain a commutative graded algebra $\text{Com}(A)$.

Given graded algebras $A$ and $B$, a \textit{graded algebra homomorphism} $A \to B$ is an algebra homomorphism $A \to B$ carrying $A^p$ to $B^p$ for all $p \in \mathbb{Z}$. The graded algebras and graded algebra homomorphisms form a category denoted $\mathcal{GA}$.

Non-graded algebras will be identified with graded algebras concentrated in degree 0.

2.2. Coalgebras. A \textit{coalgebra} is a module $M$ endowed with a coassociative linear map $\mu : M \to M \otimes M$ called the \textit{comultiplication} (here and below $\otimes = \otimes_K$). We do not suppose $M$ to be graded. The image of any $\alpha \in M$ under $\mu$ expands (non-uniquely) as a sum $\sum_i \alpha_i^1 \otimes \alpha_i^2$ where $\alpha_i^1, \alpha_i^2 \in M$ and the index $i$ runs over a finite set. To shorten the formulas, we will drop the index $i$ and the summation sign and write simply $\mu(\alpha) = \alpha^1 \otimes \alpha^2$. The coassociativity of $\mu$ reads then $(\alpha^1)^1 \otimes (\alpha^1)^2 \otimes \alpha^2 = \alpha^1 \otimes (\alpha^2)^1 \otimes (\alpha^2)^2$.

A \textit{counit} of a coalgebra $M$ is a linear map $\varepsilon = \varepsilon_M : M \to K$ such that $\varepsilon(\alpha^1)\alpha^2 = \varepsilon(\alpha^2)\alpha^1 = \alpha$ for all $\alpha \in M$. If a counit exists, then it is unique. A coalgebra having a counit is \textit{counital}. Unless explicitly stated to the contrary, we do not require coalgebras to be counital.

2.3. The algebra $\tilde{A}_M$. From now on, the symbol $A$ denotes a graded algebra and the symbol $M$ denotes a coalgebra with comultiplication $\mu$. Consider the algebra $\tilde{A}_M$ with generators $\{a_\alpha\}$, where $a$ runs over $A$ and $\alpha$ runs over $M$, subject to the following two sets of relations:

(i) \text{ (the bilinearity relations) for all } k \in K, a, b \in A, \text{ and } \alpha, \beta \in M, \quad
k a_\alpha = (ka)_\alpha = a_{k\alpha}, \quad (a+b)_\alpha \equiv a_\alpha + b_\alpha, \quad a_{\alpha+\beta} = a_\alpha + a_\beta; \quad

(ii) \text{ (the multiplicativity relations) for all } a, b \in A \text{ and } \alpha \in M, \quad
(ab)_\alpha = a_{\alpha_1}b_{\alpha_2}. \quad

A fully developed version of the latter relation is $(ab)_\alpha = \sum_i a_{\alpha_i^1}b_{\alpha_i^2}$ for an expansion $\mu(\alpha) = \sum_i \alpha_i^1 \otimes \alpha_i^2$ with $\alpha_i^1, \alpha_i^2 \in M$. The bilinearity relations ensure that $a_{\alpha_1}b_{\alpha_2} = \sum_i a_{\alpha_i^1}b_{\alpha_i^2}$ does not depend on the choice of the expansion.

We turn $\tilde{A}_M$ into a graded algebra by declaring that the generator $a_\alpha$ is homogeneous of degree $|\alpha|$ for all homogeneous $a \in A$ and all $\alpha \in M$. A typical element of $\tilde{A}_M$ is represented by a non-commutative polynomial in the generators with zero free term. One can alternatively define $\tilde{A}_M$ as the tensor algebra $\oplus_{n \geq 1}(A \otimes M)^{\otimes n}$ quotiented by the relations $ab \otimes \alpha = (a \otimes \alpha^1)(b \otimes \alpha^2)$ for all $a, b \in A, \alpha \in M$.

The construction of $\tilde{A}_M$ is functorial: a graded algebra homomorphism $f : A \to B$ induces a graded algebra homomorphism $\tilde{f}_M : \tilde{A}_M \to \tilde{B}_M$ by $\tilde{f}_M(a_\alpha) = (f(a))_\alpha$ for all $a \in A, \alpha \in M$. For a fixed $M$, the formulas $A \mapsto \tilde{A}_M, f \mapsto \tilde{f}_M$ define an endofunctor of the category of graded algebras $\mathcal{GA}$.

2.4. The universal property. We now formulate the universal property of $\tilde{A}_M$. Note that the linear maps from $M$ to a graded algebra $B$ form a graded module

$$H_M(B) = \text{Hom}_K(M,B) = \oplus_p \text{Hom}_K(M,B^p).$$
The product of \( f_1, f_2 \in H_M(B) \) is the map \( m_B(f_1 \otimes f_2)\mu : M \to B \) where \( m_B : B \otimes B \to B \) is the multiplication in \( B \). This makes \( H_M(B) \) into a graded algebra called the convolution algebra. The map \( B \mapsto H_M(B) \) obviously extends to a functor \( H : \mathcal{G}A \to \mathcal{G}A \).

**Lemma 2.1.** Let \( A \) be a graded algebra and \( M \) be a coalgebra. For any graded algebra \( B \), there is a canonical bijection

\[
(2.4.1) \quad \text{Hom}_{\mathcal{G}A}(\bar{A}_M, B) \xrightarrow{\sim} \text{Hom}_{\mathcal{G}A}(A, H_M(B))
\]

which is natural in \( A \) and \( B \).

**Proof.** The map (2.4.1) carries a graded algebra homomorphism \( r : \bar{A}_M \to B \) to the linear map \( s = s_r : A \to H_M(B) \) defined by \( s(a)(\alpha) = r(a_\alpha) \) for all \( a \in A \) and \( \alpha \in M \). The map \( s \) is grading-preserving and multiplicative: for \( a,b \in A \),

\[
s(a) (b)(\alpha) = m_B(s(a) \otimes s(b))(\mu(\alpha)) = m_B(s(a) \otimes s(b))(\alpha^1 \otimes \alpha^2) = m_B(s(a)(\alpha^1) \otimes s(b)(\alpha^2)) = r(a_\alpha^1) r(b_\alpha^2) = r(a_\alpha b_\alpha) = s(ab)(\alpha).
\]

The map inverse to (2.4.1) carries a graded algebra homomorphism \( s : A \to H_M(B) \) to the algebra homomorphism \( r = r_s : \bar{A}_M \to B \) defined on the generators by \( r(a_\alpha) = s(a)(\alpha) \). That this rule is compatible with the bilinearity relations between the generators of \( \bar{A}_M \) follows from the linearity of \( s \). The compatibility with the multiplicativity relations:

\[
r((ab)\alpha) = s(ab)(\alpha) = s(a) s(b)(\alpha) = m_B(s(a) \otimes s(b))(\mu(\alpha)) = m_B(s(a) \otimes s(b))(\alpha^1 \otimes \alpha^2) = s(a)(\alpha^1) s(b)(\alpha^2) = r(a_{\alpha^1} b_{\alpha^2}) = r(a_\alpha b_\alpha).
\]

Clearly, \( r \) preserves the grading, and the maps \( s \mapsto r_s \) and \( r \mapsto s_r \) are mutually inverse. \( \square \)

Lemma 2.1 implies the following universal property of \( \bar{A}_M \): for any graded algebra \( B \) and any graded algebra homomorphism \( s : A \to H_M(B) \), there is a unique graded algebra homomorphism \( r : \bar{A}_M \to B \) such that the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & H_M(\bar{A}_M) \\
\downarrow{s} & & \downarrow{H_M(r)} \\
H_M(B) & \longrightarrow & H_M(B)
\end{array}
\]

commutes. Here the horizontal arrow is the graded algebra homomorphism carrying \( a \in A \) to the map \( M \to \bar{A}_M, \alpha \mapsto a_\alpha \). This homomorphism corresponds to the identity automorphism of \( \bar{A}_M \) via (2.4.1).

### 2.5. The algebra \( A_M \)

The commutative graded algebra \( A_M = \text{Com}(\bar{A}_M) \) is called the **representation algebra of \( A \ with respect to \( M \)**. This algebra is defined by the same generators and relations as \( A_M \) with additional relations \( a_\alpha b_\beta = (-1)^{[\alpha][\beta]} b_\beta a_\alpha \) for all homogeneous \( a,b \in A \) and \( \alpha, \beta \in M \). The construction of \( A_M \) is functorial: a morphism \( f : A \to B \) in \( \mathcal{G}A \) induces a morphism \( \bar{f}_M : \bar{A}_M \to \bar{B}_M \) in \( \mathcal{G}A \), which in its turn induces a morphism \( f_M : A_M \to B_M \) in the category of commutative graded algebras \( \mathcal{CG}A \). For any commutative graded algebra \( B \),

\[
\text{Hom}_{\mathcal{CG}A}(A_M, B) \simeq \text{Hom}_{\mathcal{G}A}(\bar{A}_M, B) \simeq \text{Hom}_{\mathcal{G}A}(A, H_M(B)).
\]
The algebra $A_M$ has the same universal property as $\tilde{A}_M$ with $B$ running over commutative graded algebras.

2.6. Examples. 1. Consider the coalgebra $M = (\text{Mat}_N(\mathbb{K}))^*$ dual to the matrix algebra $\text{Mat}_N(\mathbb{K})$ with $N \geq 1$. For $i, j \in \{1, ..., N\}$, let $\tau_{i,j} \in M$ be the linear functional on $\text{Mat}_N(\mathbb{K})$ carrying each matrix to its $(i,j)$-term. These functionals form a basis of $M$. The comultiplication in $M$ carries $\tau_{i,j}$ to $\sum_r \tau_{i,r} \otimes \tau_{r,j}$ for all $i, j$. Given a graded algebra $A$, we write $a_{i,j}$ for the generator $a_{(\tau_{i,j})}$ of $\tilde{A}_M$. The defining relations of $A_M$ in these generators are as follows:

$$(ka)_{i,j} = k a_{i,j}, \quad (a + b)_{i,j} = a_{i,j} + b_{i,j}, \quad (ab)_{i,j} = \sum_{r=1}^{N} a_{i,r} b_{r,j}$$

for all $a, b \in A, k \in \mathbb{K}, i, j \in \{1, ..., N\}$.

2. Consider the group algebra $A = \mathbb{K}[G]$ of a finite group $G$ and the dual coalgebra $M = A^*$. The underlying module of $M$ is free with basis $\{\delta_g\} g \in G$ dual to the basis $\{g\} g \in G$ of $A$. The comultiplication in $M$ is given by $\mu(\delta_g) = \sum_{x, y \in G} \delta_x \otimes \delta_y$ for any $g \in G$.

Given a graded algebra $A$, we write $a_g$ for the generator $a_{(\delta_g)}$ of $\tilde{A}_M$. The defining relations of $A_M$ in these generators are as follows:

$$(ka)_g = k a_g, \quad (a + b)_g = a_g + b_g, \quad (ab)_g = \sum_{x, y \in G} a_x b_y$$

for all $a, b \in A, k \in \mathbb{K}, g \in G$.

3. If, in the previous example, $A = \mathbb{K}[\pi]$ is the (non-graded) group ring of a group $\pi$, then the algebra $\tilde{A}_M$ is generated by the set $\{a_g\} a \in \pi, g \in G$ subject only to the third relation in (2.6.1) for all $a, b \in \pi, g \in G$.

3. Brackets and double brackets

We recall the notions of Gerstenhaber algebras and double brackets.

3.1. Gerstenhaber algebras. A bracket in a graded module $B$ is a linear map $B \otimes B \to B$. A bracket $\{-, -\}$ in $B$ is $n$-graded for $n \in \mathbb{Z}$ if $\{B^p, B^q\} \subset B^{p+q+n}$ for all $p, q \in \mathbb{Z}$. An $n$-graded bracket $\{-, -\}$ in $B$ determines a linear map $\{-, -\} : B \otimes B \to B$, called the Jacobi form, by

$$\{a, b, c\} = (-1)^{[a][b][c]} \{a, \{b, c\}\} + (-1)^{[b][a][c]} \{b, \{c, a\}\} + (-1)^{[c][a][b]} \{c, \{a, b\}\}$$

for any homogeneous $a, b, c \in B$. Note that $\{a, b, c\} = \{b, c, a\} = \{c, a, b\}$.

By an $n$-graded biderivation in a graded algebra $A$ we mean an $n$-graded bracket $\{-, -\}$ in $A$ such that for any homogeneous $a, b, c \in A$,

$$(3.1.1) \quad \{a, bc\} = \{a, b\} c + (-1)^{[a][b]} b \{a, c\},$$

$$(3.1.2) \quad \{ab, c\} = a \{b, c\} + (-1)^{[b][c]} c \{a, b\},$$

$$(3.1.3) \quad \{a, b\} = -(-1)^{[a][b]} \{b, a\}.$$
Formulas (3.1.1) and (3.1.2) are the \( n \)-graded Leibniz rules. Formula (3.1.3) is the \( n \)-graded antisymmetry. Clearly, each of the Leibniz rules follows from the other one and the antisymmetry.

An \( n \)-graded biderivation in a graded algebra is Gerstenhaber if its Jacobi form is equal to zero (this is the \( n \)-graded Jacobi identity). An \( n \)-graded Gerstenhaber biderivation is also called an \( n \)-graded Gerstenhaber bracket. A graded algebra equipped with an \( n \)-graded Gerstenhaber bracket is called an \( n \)-graded Gerstenhaber algebra. For non-graded algebras (i.e., for graded algebras concentrated in degree zero) and \( n = 0 \), we say “Poisson” instead of “Gerstenhaber”.

An example of a 0-graded Gerstenhaber algebra in provided by an arbitrary graded algebra with bracket \( \{ a, b \} = ab - (-1)^{|a||b|}ba \) for homogeneous \( a, b \).

### 3.2. Double brackets

Double brackets in algebras were introduced by Van den Bergh [8]. The exposition here follows [6]. We begin with notation.

The tensor product of \( m \geq 2 \) copies of a graded algebra \( A \) is denoted \( A^\otimes m \). For a permutation \((i_1, \ldots, i_m)\) in \( \{1, \ldots, m\} \) and \( n \in \mathbb{Z} \), the symbol \( P_{i_1 \cdots i_m, n} \) denotes the \( n \)-graded permutation \( A^\otimes m \to A^\otimes m \) carrying \( a_1 \otimes \cdots \otimes a_m \) with homogeneous \( a_1, \ldots, a_m \in A \) to \((-1)^{t}a_{i_1} \otimes \cdots \otimes a_{i_m} \) where \( t \in \mathbb{Z} \) is the sum of the products \(|a_{i_k}|_{n} |a_{i_l}|_{n} \) over all pairs of indices \( k < l \) such that \( i_k > i_l \). Set \( P_{i_1 \cdots i_m} = P_{i_1 \cdots i_m, 0} \).

We define two \( A \)-bimodule structures on \( A^\otimes 2 \) by \( a(x \otimes y)b = ax \otimes yb \) and

\[
a * (x \otimes y) * b = (-1)^{|a||x||b|}xb \otimes ay
\]

for any homogeneous \( a, b, x, y \in A \).

An \( n \)-graded double bracket in \( A \) with \( n \in \mathbb{Z} \) is a linear map \( \{ -,- \} \in \text{End}(A^\otimes 2) \) satisfying the following conditions:

(i) (the grading condition) for any integers \( p, q, \)

\[
\{ A^p, A^q \} \subset \bigoplus_{i+j=p+q+n} A^i \otimes A^j;
\]

(ii) (the \( n \)-graded antisymmetry) \( \{ -,- \} P_{21,n} = -P_{21} \{ -,- \} \);

(iii) (the \( n \)-graded Leibniz rules) for all homogeneous \( a, b, c \in A \),

\[
\{ a, bc \} = \{ a, b \} c + (-1)^{|a||b|}b \{ a, c \},
\]

\[
\{ ab, c \} = a * \{ b, c \} + (-1)^{|b||c|}c * \{ a, c \} b.
\]

Note that (3.2.2) is a consequence of (3.2.1) and the antisymmetry. Observe also that each \( x \in A \otimes A \) expands (non-uniquely) as a sum \( x = \sum x'_i \otimes x''_i \) where \( x'_i, x''_i \in A \) are homogeneous and the index \( i \) runs over a finite set. In the sequel, we will drop the index and the summation sign and write simply \( x = x' \otimes x'' \). In this notation,

\[
\{ a, b \} = \{ a, b \}' \otimes \{ a, b \}'',
\]

for any \( a, b \in A \). The \( n \)-graded antisymmetry condition may be rewritten as

\[
\{ b, a \} = -(-1)^{|a||b|_{n}} \{ a, b \}' \otimes \{ a, b \}'',
\]

for any homogeneous \( a, b \in A \).

An \( n \)-graded double bracket \( \{ -,- \} \) in \( A \) determines a linear endomorphism \( \{ -,-,- \} \) of \( A^\otimes 3 \), called the induced tribracket, by

\[
\{ -,-,- \} = \sum_{i=0}^{2} P_{312}^i (\{ -,- \} \otimes \text{id}_A)(\text{id}_A \otimes \{ -,- \})P_{312}^{-i}.
\]
The double bracket \(\{\cdot,\cdot\}\) is an \textit{n-graded double Gerstenhaber bracket} if the induced tribracket is equal to zero. The pair \((A,\{\cdot,\cdot\}\})\) is called then an \textit{n-graded double Gerstenhaber algebra}. For non-graded algebras and \(n = 0\), we say “Poisson” instead of “Gerstenhaber”.

4. Cyclic bilinear forms and induced brackets

We define cyclic bilinear forms on coalgebras and introduce the associated brackets in representation algebras.

4.1. The map \(\hat{v}\). The comultiplication \(\mu\) in a coalgebra \(M\) induces a linear map \(\mu^m : M \to M^\otimes m\) for all \(m \geq 2\). Namely, \(\mu^2 = \mu\) and inductively, for \(m \geq 3\),

\[
\mu^m = (\id_M \otimes \mu \otimes \id_M \otimes \ldots \otimes \id_M \otimes (\mu^{m-2}))\mu^{m-1} : M \to M^\otimes m
\]

where \(k\) is any integer between 0 and \(m - 2\). The coassociativity of \(\mu\) ensures that \(\mu^m\) does not depend on the choice of \(k\). For \(\alpha \in M\), we write \(\mu^m(\alpha) = \alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_m\) where \(\alpha_1, \alpha_2, \ldots, \alpha_m \in M\) depend on an index running over a finite set, and summation over this index is tacitly assumed.

We identify elements of \((M \otimes M)^* = \Hom_K(M^\otimes 2, K)\) with \(K\)-valued bilinear forms on \(M\). Each \(v \in (M \otimes M)^*\) determines a linear map \(\hat{v} : M^\otimes 2 \to M^\otimes 2\) by

\[
(4.1.1) \quad \hat{v}(\alpha \otimes \beta) = v(\alpha \otimes \beta^2)\beta^1 \otimes \beta^3
\]

for any \(\alpha, \beta \in M\).

For a permutation \((i_1, \ldots, i_m)\) of \((1, \ldots, m)\) with \(m \geq 2\), the symbol \(p_{i_1, \ldots, i_m}\) denotes the linear map \(M^\otimes m \to M^\otimes m\) carrying any vector \(\alpha_1 \otimes \cdots \otimes \alpha_m\) to \(\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_m}\). In particular, \(p_{21} : M^\otimes 2 \to M^\otimes 2\) is the flip of the tensor factors.

**Lemma 4.1.** For any \(v \in (M \otimes M)^*\), the following two diagrams commute:

\[
\begin{array}{ccc}
M \otimes M & \xrightarrow{\hat{v}} & M \otimes M \\
\id_M \otimes \mu \downarrow & & \downarrow \id_M \otimes \mu \\
M \otimes M \otimes M & \xrightarrow{\hat{v} \otimes \id_M} & M \otimes M \otimes M,
\end{array}
\]

\[
\begin{array}{ccc}
M \otimes M & \xrightarrow{\hat{v}} & M \otimes M \\
\id_M \otimes \mu \downarrow & & \downarrow \mu \otimes \id_M \\
M \otimes M \otimes M & \xrightarrow{p_{21} \otimes \id_M} & M \otimes M \otimes M.
\end{array}
\]

**Proof.** The commutativity of the first diagram: for any \(\alpha, \beta \in M\),

\[
(\hat{v} \otimes \id_M)(\id_M \otimes \mu)(\alpha \otimes \beta) = (\hat{v} \otimes \id_M)(\alpha \otimes \beta^1 \otimes \beta^2)
\]

\[
= v(\alpha \otimes \beta^2)\beta^1 \otimes \beta^3 \otimes \beta^4 = (\id_M \otimes \mu)(v(\alpha \otimes \beta^2)\beta^1 \otimes \beta^3) = (\id_M \otimes \mu)\hat{v}(\alpha \otimes \beta).
\]

The commutativity of the second diagram is verified similarly: for \(\alpha, \beta \in M\),

\[
(\id_M \otimes \hat{v})(p_{21} \otimes \id_M)(\id_M \otimes \mu)(\alpha \otimes \beta) = (\id_M \otimes \hat{v})(\beta^1 \otimes \alpha \otimes \beta^2)
\]

\[
= v(\alpha \otimes \beta^3)\beta^1 \otimes \beta^2 \otimes \beta^4 = v(\alpha \otimes \beta^2)\mu(\beta^1) \otimes \beta^3 = (\mu \otimes \id_M)\hat{v}(\alpha \otimes \beta). \quad \square
\]
Given $\alpha, \beta \in M$, we can expand $\tilde{\omega}(\alpha \otimes \beta)$ as a finite sum $\sum x_i \otimes y_i$ with $x_i, y_i \in M$. To shorten the formulas, we will suppress the index and the summation sign and write $\alpha_\beta$ for $x_i$ and $\beta_\alpha$ for $y_i$. Thus, we write $\tilde{\omega}(\alpha \otimes \beta) = \alpha_\beta \otimes \beta_\alpha$. In this notation, Lemma 4.1 says that for any $\alpha, \beta \in M$,

\begin{equation}
(\alpha_\beta)\otimes (\beta_\alpha)^2 = \alpha_{(\beta_1)} \otimes (\beta_1)^\alpha \otimes \beta_2, \tag{4.1.2}
\end{equation}

\begin{equation}
(\alpha_\beta)^1 \otimes (\beta_\alpha)^2 = \beta^1 \otimes \alpha_{(\beta_2)} \otimes (\beta_2)^\alpha. \tag{4.1.3}
\end{equation}

4.2. Cyclic bilinear forms. We call a bilinear form $v \in (M \otimes M)^*$ cyclic if the endomorphisms $\tilde{\omega}$ and $p_{21}$ of $M^G$ commute: $\tilde{\omega}p_{21} = p_{21}\tilde{\omega}$ or, equivalently, $\tilde{\omega} = p_{21}\tilde{\omega}p_{21}$. A bilinear form $v$ is cyclic if and only if for any $\alpha, \beta \in M$,

\begin{equation}
v(\alpha \otimes \beta^2)\beta^1 \otimes \beta^3 = v(\beta \otimes \alpha^2)\alpha^3 \otimes \alpha^1. \tag{4.2.1}
\end{equation}

The set of cyclic bilinear forms on $M$ is a submodule of $(M \otimes M)^*$. In particular, $v = 0$ is cyclic. More interesting examples will be given in Sections 5 and 6.

The following lemma is our main technical result. It derives a bracket in the representation algebra $A_M$ from a double bracket in $A$ and a cyclic form on $M$.

**Lemma 4.2.** Let $\{-, -\}$ be an $n$-graded double bracket in a graded algebra $A$ with $n \in \mathbb{Z}$. Let $v$ be a cyclic bilinear form on a coalgebra $M$. There is a unique $n$-graded biderivation $\{-, -\}_v$ in $A_M$ such that for any $a, b \in A$ and $\alpha, \beta \in M$,

\begin{equation}
\{a_\alpha, b_\beta\}_v = \{a, b\}_{(\alpha_\beta)} \{a, b\}_{(\beta_\alpha)} = v(\alpha \otimes \beta^2)\{a, b\}_{(\beta_\alpha)} \{a, b\}_{(\beta_\alpha)} \tag{4.2.2}
\end{equation}

where we expand $\{a, b\}_{(\alpha_\beta)} \{a, b\}_{(\beta_\alpha)}$ as in Section 3.2. The associated Jacobi form $\{-, -\}_v$ in $A_M$ is computed on the generators of $A_M$ as follows: for any homogeneous $a, b, c \in A$ and any $\alpha, \beta, \gamma \in M$,

\begin{equation}
\{a_\alpha, b_\beta, c_\gamma\}_v = Q - R \tag{4.2.3}
\end{equation}

\begin{align}
Q &= Q(a, b, c, \alpha, \beta, \gamma) = (-1)^{|a||b|} \{a, b, c\}_{(\alpha_\beta_\gamma)} \{a, b, c\}_{(\beta_\alpha_\gamma)} \{a, b, c\}_{(\gamma_\alpha_\beta)} \tag{4.2.4}
R &= R(a, b, c, \alpha, \beta, \gamma) = (-1)^{|a||b|} \{a, b, c\}_{(\alpha_\beta_\gamma)} \{a, b, c\}_{(\beta_\alpha_\gamma)} \{a, b, c\}_{(\gamma_\alpha_\beta)} \tag{4.2.5}
\end{align}

Proof. The uniqueness of $\{-, -\}_v$ is obvious because the symbols $a_\alpha$ generate $A_M$. To prove the existence, consider the graded algebra $A$ generated by the symbols $\{a_\alpha | a \in A, \alpha \in M\}$ with $|a_\alpha|$ = $|a|$ subject to the bilinearity relations. The kernel of the natural projection $A \to A_M$ is generated by the multiplicity relations and the commutativity relations. It is clear that there is a unique bilinear pairing $\{-, -\} : A \otimes A \to A_M$ given by (4.2.2) on the generators and satisfying the Leibniz rules. We claim that this pairing is antisymmetric and the kernel of the projection $A \to A_M$ lies in its annihilator. This will imply the existence of $\{-, -\}_v$.

Pick any homogeneous $a, b \in A$ and set $x = \{a, b\} \in A \otimes A$. For $\alpha, \beta \in M$,

\begin{align}
\{a_\alpha, b_\beta\}_v &= x^\prime_{\alpha_\beta} x^\prime_{\beta_\alpha} = x^\prime_{\alpha_\beta} x^\prime_{\beta_\alpha} = (-1)^{|x^\prime||x^\prime|} x^\prime_{\alpha_\beta} x^\prime_{\beta_\alpha} \\
&= (-1)^{|x||x|} x_{\alpha_\beta} x_{\beta_\alpha} = (-1)^{|x||x|} \{b_\beta, a_\alpha\}
\end{align}

where the penultimate equality follows from the antisymmetry of $\{-, -\}$. Therefore, the bracket $\{-, -\} : A \otimes A \to A_M$ is antisymmetric in the sense that $\{f, g\} = (-1)^{|f||g|} \{g, f\}$ for any homogeneous $f, g \in A$. 


Pick homogeneous $a, b, c \in A$ and set $x = \{a, b\}, y = \{a, c\}$. Then
\[
\{a, bc\} = xc + (-1)^{|a||b|}by = x' \otimes x''c + (-1)^{|a||b|}by' \otimes y''.
\]

For $\alpha, \beta \in M$,
\[
\{a_\alpha, (bc)_\beta\} = \{a, bc\}_{a_\alpha} \{a, bc\}_{bc}'' = x''_{a_\alpha}(x''c)_\beta + (-1)^{|a||b|}(by')_{a_\beta} y''_{\beta}.
\]

On the other hand,
\[
\{a_\alpha, b_{\beta 1}c_{\beta 2}\} = \{a_\alpha, b_{\beta 1}\} c_{\beta 2} + (-1)^{|a||b|}b_{\beta 1} \{a_\alpha, c_{\beta 2}\}
= x''_{a_{\beta 1}} x''_{(\beta 1)\alpha} c_{(\beta 2)} + (-1)^{|a||b|}b_{(\beta 1)} y''_{(\beta 2)\alpha}.
\]

Now, the identities (4.1.2) and (4.1.3) imply that $\{a_\alpha, (bc)_\beta\} = \{a_\alpha, b_{\beta 1}c_{\beta 2}\}$. Therefore, the bracket $\{\cdot, \cdot\} : A \otimes A \to A_M$ annihilates the multiplicativity relations. It also annihilates the commutativity relations:
\[
\{a_\alpha, b_{\beta 1}c_{\gamma}\} = \{a_\alpha, b_{\beta 1}\} c_{\gamma} + (-1)^{|a||b|}b_{\beta 1} \{a_\alpha, c_{\gamma}\}
= (-1)^{|a||b||c|}c_{\gamma} \{a_\alpha, b_{\beta 1}\} + (-1)^{|a||b||c|}b_{\beta 1} \{a_\alpha, c_{\gamma}\} = (-1)^{|b||c|} \{a_\alpha, c_{\beta 1}\} b_{\gamma} + (-1)^{|a||b||c|}c_{\gamma} \{a_\alpha, b_{\beta 1}\}
= \{a_\alpha, (-1)^{|b||c|}c_{\gamma} b_{\beta 1}\}.
\]

It remains only to prove (4.2.3). In the rest of the argument, we write $\{\cdot, \cdot\}$ for $\{\cdot, \cdot\}_\psi$. The identities $P_{312, n}^{-1} = P_{312, n}^2$ imply that
\[
\{a, b, c\} = \{a, \{b, c\}_\varphi\} \otimes \{b, c\}_\psi + (-1)^{|a||b||c|} P_{312} \left( \{b, \{c, a\}_\varphi\} \otimes \{c, a\}_\psi \right)
= \{a, \{b, c\}_\varphi\} \otimes \{a, \{b, c\}_\psi\} + \{b, \{c, a\}_\varphi\} \otimes \{c, a\}_\psi + (-1)^{|a||b||c|} P_{312} \left( \{b, \{c, a\}_\varphi\} \otimes \{c, a\}_\psi \right).
\]

Using the commutativity of $A_M$, we deduce that for any $\varphi, \psi, \rho \in M$,
\[
\{a, b, c\}_\varphi \otimes \{a, b, c\}_\psi = \{a, \{b, c\}_\varphi\} \otimes \{a, \{b, c\}_\psi\} + \{b, \{c, a\}_\varphi\} \otimes \{c, a\}_\psi + (-1)^{|a||b||c|} \{c, \{a, b\}_\varphi\} \otimes \{a, b\}_\psi.
\]

Setting $\varphi = a_{(\beta_1)}, \psi = (\beta_2), \rho = \gamma_\beta$ and multiplying by $(-1)^{|a||b||c|}$, we obtain
\[
Q = (-1)^{|a||b||c|} u_1 + (-1)^{|a||b||u_2} + (-1)^{|b||c||u_3}
\]
where
\[
u_1 = \{a, \{b, c\}_\varphi\}_{a_{(\beta_1)}} \{a, \{b, c\}_\psi\}_{(\beta_2)} \otimes \{b, c\}_\gamma,
\]
\[
u_2 = \{b, \{c, a\}_\varphi\}_{(\beta_2)} \{b, \{c, a\}_\psi\}_{a_{(\beta_1)}} \otimes \{c, a\}_\gamma,
\]
\[
u_3 = \{c, \{a, b\}_\varphi\}_{a_{(\beta_1)}} \{c, \{a, b\}_\psi\}_{(\beta_2)} \otimes \{a, b\}_\gamma.
\]
Where

\[ R = (-1)^{|ab| |c|} t_1 + (-1)^{|ac| |b|} t_2 + (-1)^{|bc| |a|} t_3 \]

where

\[ t_1 = \{ \{ a, \{ c, b \} \} \}_{\alpha(a)} \{ \{ a, \{ c, b \} \} \}_{\gamma}^{\beta} \{ c, b \}_{\beta}^{\gamma} \]
\[ t_2 = \{ \{ c, \{ b, a \} \} \}_{\gamma}^{\beta} \{ \{ c, \{ b, a \} \} \}_{\beta}^{\gamma} \{ b, a \}_{\beta}^{\gamma} \]
\[ t_3 = \{ \{ b, \{ a, c \} \} \}_{\beta}^{\gamma} \{ \{ b, \{ a, c \} \} \}_{\gamma}^{\beta} \{ a, c \}_{\gamma}^{\beta} \].

We next compute \{ a_\alpha, b_\beta, c_\gamma \}. Set \( x = \{ b, c \} \in A^\otimes 2 \) and observe that

\[ \{ a_\alpha, \{ b_\beta, c_\gamma \} \} = \{ a_\alpha, x_\beta', x_\gamma'' \} = \{ a_\alpha, x_\beta', x_\gamma'' \} + (-1)^{|a_\alpha| |x_\beta'|} \{ a_\alpha, x_\gamma'' \} \]
\[ = \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} x'' + (-1)^{|a| |x''|} \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} x'' \].

We rewrite the last summand as follows. Note that

\[ | \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} | = | \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} | = | a | + | x'' | + n = | a_\alpha | + | x'' |. \]

Also, by (3.23), \( x'' \otimes x'' = (-1)^{|b| |c| + |a| |b|} y'' \otimes y'' \) where \( y = \{ c, b \} \). These formulas and the commutativity of \( A_M \) imply that

\[ (-1)^{|a_\alpha| |x''|} x'' \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} = (-1)^{|b_\gamma| |x''|} x'' \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} \]
\[ = (-1)^{|b_\beta| |x''|} x'' \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} = (-1)^{|b_\gamma| |x''|} x'' \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} \]
\[ = (-1)^{|b_\beta| |x''|} x'' \{ a, x'' \}_{\alpha(a)} \{ a, x'' \}_{\gamma}^{\beta} \].

As a result, we obtain that

\[ (-1)^{|a_\alpha| |x''|} \{ a_\alpha, \{ b_\beta, c_\gamma \} \} = (-1)^{|a_\alpha| |x''|} \{ w_1, z_1 \} \]

where

\[ w_1 = \{ a, \{ b, c \} \} \}
\[ z_1 = \{ a, \{ c, b \} \} \}

Cyclically permuting \( a_\alpha, b_\beta, c_\gamma \), we obtain

\[ (-1)^{|a_\alpha| |x''|} \{ b_\beta, \{ c_\gamma, a_\alpha \} \} = (-1)^{|a_\alpha| |x''|} \{ w_2, z_2 \}, \]
\[ (-1)^{|b_\gamma| |x''|} \{ c_\gamma, \{ a_\alpha, b_\beta \} \} = (-1)^{|b_\gamma| |x''|} \{ w_3, z_3 \} \]

where

\[ w_2 = \{ b, \{ c, a \} \} \}
\[ z_2 = \{ b, \{ c, a \} \} \}
\[ w_3 = \{ c, \{ a, b \} \} \}
\[ z_3 = \{ c, \{ a, b \} \} \}

Below we prove that

\[ w_1 = u_1, \quad w_2 = u_2, \quad w_3 = u_3, \quad z_1 = t_1, \quad z_2 = t_3, \quad z_3 = t_2. \]
Therefore
\[ \{a_\alpha, b_\beta, c_\gamma\}_v = \]
\[ (-1)^{|a|_n c_{\gamma}} \{a_\alpha, \{b_\beta, c_\gamma\}\} + (-1)^{|b|_n \{a_\alpha, c_\gamma\}\} + (-1)^{|c|_n \{a_\alpha, b_\beta\}\} = \]
\[ = (-1)^{|a|_n c_{\gamma}} u_1 + (-1)^{|b|_n \{a_\alpha, c_\gamma\}\} u_2 + (-1)^{|c|_n \{a_\alpha, b_\beta\}\} =
\]
\[-(-1)^{|a|_n c_{\gamma}} t_1 - (-1)^{|b|_n \{a_\alpha, c_\gamma\}\} t_3 - (-1)^{|a|_n |b|_n \{c_\gamma, \{a_\alpha, b_\beta\}\}\} = Q - R.\]

The equalities \( u_1 = u_2 \) and \( z_1 = t_2 \) are tautological. The formula \( u_2 = u_2 \) would follow from the identity
\[ (4.2.7) \quad \beta(\gamma_\alpha) \otimes (\gamma_\alpha)^\beta \otimes \alpha^\gamma = (\beta_\gamma)^\alpha \otimes \gamma^\beta \otimes \alpha(\beta_\gamma).\]

Similarly, the formulas \( u_3 = u_3, z_2 = t_3, z_3 = t_2 \) would follow from the identities
\[ (4.2.8) \quad \gamma(\alpha_\beta) \otimes (\gamma_\beta)^\alpha \otimes \alpha^\beta = \gamma^\alpha \otimes \alpha(\beta_\gamma) \otimes (\beta_\gamma)^\alpha,\]
\[ (4.2.9) \quad \beta(\alpha_\gamma) \otimes (\alpha_\gamma)^\beta \otimes \gamma^\alpha = \beta^\gamma \otimes \alpha(\gamma_\alpha) \otimes (\gamma_\alpha)^\alpha,\]
\[ (4.2.10) \quad \gamma(\beta_\alpha) \otimes (\beta_\alpha)^\gamma \otimes \alpha^\beta = (\gamma_\alpha)^\beta \otimes \beta^\gamma \otimes \alpha(\beta_\alpha).\]

Formula (4.2.3) is deduced from (4.2.7) by replacing \( \alpha, \beta, \gamma \) with \( \gamma, \alpha, \beta \), respectively, and permuting the tensor factors. Formula (4.2.9) is deduced from (4.2.10) by replacing \( \alpha, \beta, \gamma \) with \( \beta, \gamma, \alpha \), respectively, and permuting the tensor factors. It remains to prove (4.2.7) and (4.2.10).

We rewrite (4.2.7) in the equivalent form
\[ (4.2.11) \quad \alpha^\gamma \otimes \beta(\gamma_\alpha) \otimes (\gamma_\alpha)^\beta = \alpha(\beta_\gamma) \otimes (\beta_\gamma)^\alpha \otimes \gamma^\beta.\]

Set
\[ X = h \otimes \text{id}_M \in \text{End} M^{\otimes 3} \quad \text{and} \quad v = \text{id}_M \otimes h \in \text{End} M^{\otimes 3}.\]
The sides of (4.2.11) are the images of \( \beta(\gamma) \otimes \alpha \in M^{\otimes 3} \) under \( p_{312}XY \) and \( XYp_{312} \). Thus, (4.2.11) follows from the equality
\[ (4.2.12) \quad p_{312}XY = XYp_{312} \]
which we now prove. For \( \alpha, \beta, \gamma \in M \),
\[ XY(\alpha \otimes \beta \otimes \gamma) = X(v(\beta \otimes \gamma^2)\alpha \otimes \gamma^1 \otimes \gamma^3) = v(\alpha \otimes \gamma^2)v(\beta \otimes \gamma^4)\gamma^1 \otimes \gamma^3 \otimes \gamma^5.\]

Hence,
\[ p_{312}XY(\alpha \otimes \beta \otimes \gamma) = v(\alpha \otimes \gamma^2)v(\beta \otimes \gamma^4)\gamma^5 \otimes \gamma^1 \otimes \gamma^3.\]

Similarly,
\[ XYp_{312}(\alpha \otimes \beta \otimes \gamma) = XY(\gamma \otimes \alpha \otimes \beta) = v(\alpha \otimes \gamma^2)v(\gamma \otimes \beta^2)\beta^1 \otimes \beta^3 \otimes \beta^5.\]

We must show that
\[ (4.2.13) \quad v(\alpha \otimes \beta^4)v(\gamma \otimes \beta^2)\beta^1 \otimes \beta^3 \otimes \beta^5 = v(\alpha \otimes \gamma^2)v(\beta \otimes \gamma^4)\gamma^5 \otimes \gamma^1 \otimes \gamma^3.\]

To this end, we take (4.2.1) and replace \( \alpha \) with \( \gamma \). This gives
\[ v(\gamma \otimes \beta^2)\beta^1 \otimes \beta^3 = v(\beta \otimes \gamma^2)\gamma^5 \otimes \gamma^1.\]

Applying \( \text{id}_M \otimes \mu^3 \) to both sides, we obtain that
\[ v(\gamma \otimes \beta^2)\beta^1 \otimes \beta^3 \otimes \beta^4 \otimes \beta^5 = v(\beta \otimes \gamma^4)\gamma^5 \otimes \gamma^1 \otimes \gamma^2 \otimes \gamma^3.\]
We apply to both sides the linear map $M^{\otimes 4} \to M^{\otimes 3}$ carrying any $x \otimes y \otimes z \otimes t$ to $v(\alpha \otimes z) x \otimes y \otimes t$. This gives (4.2.13) and completes the proof of (4.2.7).

To prove (4.2.10), we rewrite it in the following equivalent form:

$$\alpha \beta \otimes \gamma (\beta^\alpha)') = \alpha (\beta^\alpha) \otimes (\gamma^\beta) \otimes \beta^\gamma.$$  

The latter can be reformulated in terms of the maps $X, Y$ as the equality

$$(4.2.14)\quad Y p_{132} X = X p_{132} Y. $$

To prove (4.2.14), set $\sigma = p_{213}$ and $\tau = p_{132}$. Clearly $\sigma^2 = \tau^2 = 1$, $\sigma \tau = \tau \sigma$, $p_{312} = \sigma \tau$. It follows from the definition of $X$ and $Y$ that $Y = p_{312} X p_{312}^{-1} = \sigma X \tau \sigma$. Formula (4.2.12) may be rewritten in this notation as

$$\sigma \tau X \sigma \tau X \tau = X \sigma \tau X \tau \sigma \tau X = X \sigma \tau X.$$

The cyclicity of $v$ gives $\sigma X = X \sigma$ and $\tau Y = Y \tau$. Then

$$Y p_{132} X = \tau X \sigma \tau X = \sigma X \tau \sigma X = \sigma X \sigma \tau X \sigma \tau X \sigma \tau X \tau = \sigma X \sigma \tau X \sigma \tau X \tau X \sigma \tau X \tau = \sigma X \sigma \tau X \sigma X \tau,$$

We can now formulate the main result of this paper.

**Theorem 4.3.** For any $n$-graded double Gerstenhaber bracket $\{-, -\}$ in a graded algebra $A$ and any cyclic bilinear form $v$ on a coalgebra $M$, the $n$-graded biderivation $\{-, -\}_v$ in the representation algebra $A_M$ is an $n$-graded Gerstenhaber bracket.

**Proof.** We need to prove that the Jacobi form $\{-, -\}_v$ in $A_M$ associated with the bracket $\{-, -\}_v$ is equal to zero. Using the $n$-graded Leibniz rules and the antisymmetry for $\{-, -\}_v$, we easily compute that the Jacobi form satisfies the following Leibniz-type formula: for any $a_1, a_2, b, c \in A_M$.

$$(4.2.15) \quad \{a_1 a_2, b, c\}_v = (-1)^{|a_1||c|} a_1 \{a_2, b, c\}_v + (-1)^{|a_2||b|} \{a_1, b, c\}_v a_2.$$  

This and the cyclic invariance of the Jacobi form imply that if this form is zero on the generators of $A_M$, then it is zero on all elements of $A_M$. The theorem now follows from Lemma 4.2 and the assumption $\{-, -\} = 0$. \hfill $\square$

For non-graded algebras, Theorem 4.3 yields Theorem 1.1 of the introduction.

5. **Cyclic structures on algebras and coalgebras**

We reformulate cyclic bilinear forms on coalgebras in terms of so-called cyclic structures on coalgebras and algebras.

5.1. **Cyclic structures on coalgebras.** A *cyclic structure* on a coalgebra $M$ is a linear map $M \to M^*, \alpha \mapsto \bar{\alpha}$ such that for any $\alpha, \beta \in M$,

$$(5.1.1) \quad \bar{\alpha}(\beta^2) \beta^3 = \bar{\beta}(\alpha^2) \alpha^3 \otimes \alpha^1.$$  

**Lemma 5.1.** For $v \in (M \otimes M)^*$, we let $\text{ad}_v : M \to M^*$ be the left adjoint map carrying any $\alpha \in M$ to the linear map $M \to \mathbb{K}$, $\beta \mapsto v(\alpha \otimes \beta)$. Then:

(a) $v \in (M \otimes M)^*$ is cyclic if and only if $\text{ad}_v$ is a cyclic structure on $M$;

(b) the formula $v \mapsto \text{ad}_v$ establishes a bijection of the set of cyclic bilinear forms on $M$ onto the set of cyclic structures on $M$.  

Proof. For $\alpha, \beta \in M$,
\[ \tilde{v}p_{21}(\beta \otimes \alpha) = \tilde{v}(\alpha \otimes \beta) = v(\alpha \otimes \beta^2)\beta^1 \otimes \beta^3 = \text{ad}_v(\alpha)(\beta^2)\beta^1 \otimes \beta^3 \]
and
\[ p_{21} \tilde{v}(\beta \otimes \alpha) = p_{21}(v(\beta \otimes \alpha^2)\alpha^1 \otimes \alpha^3) = v(\beta \otimes \alpha^2)\alpha^3 \otimes \alpha^1 = \text{ad}_v(\beta)(\alpha^2)\alpha^3 \otimes \alpha^1. \]

Clearly, $\tilde{v}p_{21} = p_{21} \tilde{v}$ if and only if $\text{ad}_v$ is a cyclic structure on $M$. This proves (a). Claim (b) is obvious; the inverse bijection carries a cyclic structure $M \rightarrow M^*$, $\alpha \mapsto \overline{\alpha}$ to the bilinear form $\alpha \otimes \beta \mapsto \overline{\alpha}(\beta)$ on $M$. \qed

5.2. Cyclic structures on algebras. Dualizing cyclic structures on coalgebras we obtain the following notion. A cyclic structure on an algebra $A$ is a linear map $A^* \rightarrow A$, $\alpha \mapsto \overline{\alpha}$ such that $\alpha(a\overline{b}) = \beta(b\overline{a})$ for any $\alpha, \beta \in A^*$ and $a, b \in A$. Here and below $A^* = \text{Hom}_K(A, K)$.

Given a coalgebra $M$, a cyclic structure $M^{**} \rightarrow M^*$, $x \mapsto \overline{x}$ on the dual algebra $M^*$ induces (under additional assumptions) a cyclic structure on $M$. Recall that multiplication in $M^*$ is defined by $(ab)(\alpha) = a(\alpha^3)b(\alpha^2)$ for $a, b \in M^*$ and any $\alpha \in M$. Let $e : M \rightarrow M^{**}$ be the evaluation map carrying $\alpha \in M$ to the functional $M^{**} \rightarrow K, a \mapsto \alpha(a)$. For $\alpha, \beta \in M$, $a, b \in M^*$,
\[ a(\beta^1) \overline{e(\alpha)(\beta^2)b(\beta^3)} = (a \overline{e(\alpha)b}(\beta)) = \overline{e(\beta)}(a \overline{e(\alpha)b}) \]
and
\[ \overline{e(\alpha)(\beta^2)b(\beta^3)} = (b \overline{e(\alpha)a}(\beta)) = \overline{b(\alpha^1)} \overline{e(\beta)}(a^1) \overline{a(\alpha^3)}. \]

In other words, the evaluations of $a \otimes b$ on the vectors
\[ (5.2.1) \quad \overline{e(\alpha)(\beta^2)} \beta^1 \otimes \beta^3, \quad \overline{e(\beta)}(\alpha^2) \alpha^3 \otimes \alpha^1 \in M \otimes M \]
are equal. Since this holds for all $a, b \in M^*$, we can deduce the following: if the underlying module of $M$ is free, then the vectors (5.2.1) are equal for all $\alpha, \beta \in M$. This means that $\overline{\alpha} : M \rightarrow M^*$ is a cyclic structure on $M$.

We will often focus on algebras and coalgebras whose underlying modules are free of finite rank. Any such algebra $A$ is dual to a well defined coalgebra $(M = A^*, \mu)$ where $\mu : M \rightarrow M \otimes M$ is determined by the condition that the evaluation of $\mu(\alpha)$ on $a \otimes b$ is equal to $\alpha(ab)$ for any $\alpha \in M$, $a, b \in A$. The evaluation map $M \rightarrow M^{**}$ is an isomorphism and we use it to identify $M^{**}$ with $M$. The arguments above show that cyclic structures on $M$ and $A$ are the same maps $M \rightarrow A$. The cyclic bilinear form $v : M \otimes M \rightarrow K$ associated with a cyclic structure $M \rightarrow A, \alpha \mapsto \overline{\alpha}$ is computed by $v(\alpha \otimes \beta) = \beta(\overline{\alpha})$ for $\alpha, \beta \in M$.

5.3. Example. In Example 2.6.2, any conjugation-invariant function $F : G \rightarrow K$ determines a cyclic structure $M = \mathcal{A}^* \rightarrow \mathcal{A}$ on $\mathcal{A}$ and on $M$ by
\[ \overline{\alpha} = \sum_{z \in G} F(gz)z \in A \quad \text{for} \quad g \in G. \]

The cyclic identity is easily verified on the basis vectors: for any $a, b, g, h \in G$,
\[ \delta_g(a\overline{b}b) = \delta_g(\sum_{z \in G} F(hz)azb) = F(ha^{-1}gb^{-1}) \]
and
\[ F(gb^{-1}ha^{-1}) = \delta_h(\sum_{z \in G} F(gz)bza) = \delta_h(b\overline{gb}a). \]
The associated cyclic bilinear form \( v \) on \( M \) is computed by \( v(\delta_g \otimes \delta_h) = F(gh) \) for \( g, h \in G \). Given a graded algebra \( A \) with double bracket \( \langle - , - \rangle \), the bracket \( \{ - , - \}_v \) in \( A_M \) is computed from (4.2.2): for \( a, b \in A \) and \( g, h \in G \),

\[
\{ a_g, b_h \}_v = \sum_{x,y \in G} F(gx^{-1}hy^{-1}) \langle a, b \rangle_x^y^x_y.
\]

Note that a linear representation \( \rho : G \to GL_N(\mathbb{K}) \) with \( N \geq 1 \) determines a conjugation-invariant function \( G \to \mathbb{K}, g \to \text{tr}(\rho(g)) \). The corresponding cyclic bilinear form on \( M \) carries \( \delta_g \otimes \delta_h \) to \( \text{tr}(\rho(gh)) \) for any \( g, h \in G \).

6. Cyclic bilinear forms from Frobenius algebras

We begin by recalling basic definitions concerning Frobenius algebras.

6.1. Frobenius algebras. A \( \mathbb{K} \)-valued bilinear form is non-degenerate if its left adjacent map is an isomorphism. A Frobenius algebra is a pair consisting of an algebra \( A \) whose underlying module is free of finite rank and a non-degenerate bilinear form \( ( - , - ) : A \times A \to \mathbb{K} \) such that \( (ab, c) = (a, bc) \) for any \( a, b, c \in A \). The algebra \( A \) is not required to be graded or unital. The form \( ( - , - ) \) is called the Frobenius pairing.

A Frobenius algebra \( A \) is symmetric if the Frobenius pairing is symmetric, i.e., \( (a, b) = (b, a) \) for any \( a, b \in A \). An example of a symmetric Frobenius algebra is provided by the ring \( \mathbb{K} \) with the ring multiplication in the role of the Frobenius pairing. Any unital commutative Frobenius algebra \( A \) is symmetric: for \( a, b \in A \),

\[
(a, b) = (a, b1_A) = (ab, 1_A) = (ba, 1_A) = (b, a1_A) = (b, a).
\]

6.2. From Frobenius algebras to cyclic forms. The following theorem derives from every symmetric Frobenius algebra a cyclic bilinear form on the dual coalgebra.

**Theorem 6.1.** Let \( A \) be a symmetric Frobenius algebra with Frobenius pairing \( ( - , - ) \). For any \( \alpha \in A^* \), there is a unique \( \overline{\alpha} \in A \) such that \( \alpha(a) = (a, \overline{\alpha}) \) for all \( a \in A \). The map \( \alpha^* : A^* \to A, \alpha \mapsto \overline{\alpha} \) is a cyclic structure on \( A \) and on \( A^* \). The associated cyclic bilinear form \( v \) on \( A^* \) is given by \( v(\alpha \otimes \beta) = \beta(\overline{\alpha}) = (\overline{\alpha}, \beta) \) for any \( \alpha, \beta \in A^* \).

**Proof.** The existence and uniqueness of \( \overline{\alpha} \) follows from the non-degeneracy of the Frobenius pairing. For any \( \alpha, \beta \in A^* \) and \( a, b \in A \),

\[
\alpha(a\beta b) = (a\overline{\beta}b, \overline{\alpha}) = (a\overline{\beta}, b\overline{\alpha}) = (b\overline{\alpha}, a\overline{\beta}) = (b\overline{\alpha}a, \overline{\beta}) = \beta(b\overline{\alpha}a).
\]

The computation of \( v \) follows from the remark at the end of Section 5.2. \( \square \)

We will combine Theorem 6.1 with the following construction of symmetric Frobenius algebras. Let \( A \) be an algebra whose underlying module is free of finite rank. We say that \( \theta \in A^* \) is trace-like if the bilinear form \( ( - , - )_\theta \) on \( A \) defined by \( (a, b)_\theta = \theta(ab) \) for \( a, b \in A \) is symmetric and non-degenerate. Then the pair \( (A, ( - , - )_\theta) \) is a symmetric Frobenius algebra. The associated cyclic bilinear form on the coalgebra \( A^* \) will be denoted \( v_\theta \).
6.3. Examples. 1. The group algebra \( \mathcal{A} = \mathbb{K}[G] \) of a finite group \( G \) has a trace-like \( \theta \in \mathcal{A}^* \) which carries the neutral element of \( G \) to 1 and all other elements of \( G \) to 0. The cyclic form \( v_\theta \) on \( \mathcal{A}^* \) is the bilinear form of Section 5.3 determined by the function \( \theta |_G : G \to \mathbb{K} \).

2. In analogy with quaternions we define a unital algebra \( \mathcal{A} = \mathbb{K}1_\mathcal{A} \oplus \mathbb{K}i \oplus \mathbb{K}j \oplus \mathbb{K}k \) with unique (associative) multiplication such that \( i^2 = j^2 = k^2 = ijk = -1_\mathcal{A} \). The linear map \( \theta : \mathcal{A} \to \mathbb{K} \) carrying \( 1_\mathcal{A} \) to 1 and \( i, j, k \) to 0 is trace-like. This turns \( \mathcal{A} \) into a (non-commutative) symmetric Frobenius algebra and yields a cyclic bilinear form \( v_\theta \) on the dual coalgebra \( \mathcal{A}^* \).

3. For any \( n \geq 1 \), the truncated polynomial algebra \( \mathcal{A} = \mathbb{K}[x]/x^{n+1} \) has a trace-like \( \theta \in \mathcal{A} = \mathcal{A}^* \) defined by \( \theta(x^n) = 1 \) and \( \theta(x^i) = 0 \) for \( i = 0, 1, ..., n - 1 \). Let \( (u_i)_{i=0}^n \) be the basis of \( \mathcal{A} \) dual to the basis \( (x^i)_{i=0}^n \) of \( \mathcal{A} \). The comultiplication \( \mu \) in \( \mathcal{A} \), the cyclic structure \( M \to \mathcal{A}, \alpha \to \mathfrak{P} \), and the form \( v_\theta \) on \( M \) are computed by

\[
\mu(u_i) = \sum_{0 \leq k \leq i} u_k \otimes u_{i-k}, \quad \mathfrak{P}_i = x^{n-i}, \quad v_\theta(u_i \otimes u_j) = \delta_{i+j,n}
\]

for all \( i, j \in \{0, 1, ..., n\} \). Given a graded algebra \( \mathcal{A} \), we write \( a_i \) for the generator \( a_{u_i} \) of \( \mathcal{A}M \). The bracket \( \{-,-\}_v \) in \( \mathcal{A}M \) derived from a double bracket \( \{\} \{\}, \{\} \} \) in \( \mathcal{A} \) is computed from (12.22): for any \( a, b \in \mathcal{A} \) and \( i, j \in \{0, 1, ..., n\} \),

\[
\{a_i, b_j\}_v = \sum_{0 \leq k \leq i+j-n} \{a, b\}'_k \{a, b\}''_{i+j-n-k}.
\]

It is understood that if \( i + j - n < 0 \), then the right-hand side is equal to zero.

4. In Example 2.6.1, the trace of matrices \( \theta = \sum_i \tau_i \) is a trace-like element of \( M \). The associated cyclic form \( v_\theta \) on \( M \) is computed by \( v_\theta(\tau_{i,j} \otimes \tau_{k,l}) = \delta_{i,k}\delta_{j,l} \) for all \( i, j, k, l \in \{1, 1, ..., N\} \) where \( \delta \) is the Kronecker delta. Given a graded algebra \( \mathcal{A} \) with a double bracket \( \{\} \{\}, \{\} \} \), the bracket \( \{-,-\}_v \) in \( \mathcal{A}M \) is computed from (12.22): for any \( a, b \in \mathcal{A} \) and \( i, j, k, l \),

\[
\{a_{i,j}, b_{k,l}\}_v = \{a, b\}'_{k,j} \{a, b\}''_{i,l}.
\]

This bracket was first introduced by Van den Bergh [8].

5. In generalization of the previous example, consider a symmetric Frobenius algebra \( (\mathcal{A}, \{-,-\}) \) with cyclic structure \( \mathcal{A}^* \to \mathcal{A}, \alpha \to \mathfrak{P} \) and with cyclic bilinear form \( v \) on \( \mathcal{A}^* \) as in Theorem 6.1. For any integer \( N \geq 1 \), the matrix algebra \( \text{Mat}_N(\mathcal{A}) \) acquires a symmetric Frobenius pairing

\[
(a, b) = \sum_{i,j=1}^N (a_{i,j}, b_{ji}) \in \mathbb{K}
\]

where \( a = (a_{i,j})_{i,j} \), \( b = (b_{ij})_{i,j} \in \text{Mat}_N(\mathcal{A}) \). This determines a cyclic structure \( \alpha \to \mathfrak{P} \) and a cyclic bilinear form, \( v_N \), on the coalgebra \( M = (\text{Mat}_N(\mathcal{A}))^* \). We identify elements of \( M \) with matrices \( \alpha = (\alpha_{ij})_{i,j} \) where \( i, j \in \{1, 1, ..., N\} \), \( \alpha_{ij} \in \mathcal{A}^* \) for all \( i, j \), so that \( \alpha(a) = \sum_{i,j} \alpha_{ij}(a_{ij}) \) for any \( a = (a_{ij})_{i,j} \in \text{Mat}_N(\mathcal{A}) \). It is easy to see that \( \mathfrak{P} = (\mathfrak{P})_{i,j} \) and \( v_N(\alpha \otimes \beta) = \sum_{i,j} v(\alpha_{ij} \otimes \beta_{ji}) \) for any \( \alpha, \beta \in M \).

6.4. Remark. Each 2-dimensional Topological Quantum Field Theory (TQFT) gives rise to a unital commutative Frobenius algebra, see [3]. Combining with Theorem 6.1 we conclude that every 2-dimensional TQFT gives rise to a cyclic bilinear form on a coalgebra.
7. A GROUP ACTION ON $A_M$

We show that coalgebra automorphisms of $M$ act on $A_M$ by algebra automorphisms and study the behavior of the bracket \ref{eq:bracket} under this action. For counital $M$, we exhibit a group of automorphisms of $A_M$ preserving the bracket.

7.1. The group $\text{Aut}(M)$. A coalgebra automorphism of a coalgebra $M = (M, \mu)$ is an invertible linear map $\omega : M \to M$ such that $\mu \omega = (\omega \otimes \omega) \mu$. Let $\Omega = \text{Aut}(M)$ be the group of all coalgebra automorphisms of $M$ with multiplication $\omega \omega' = \omega \circ \omega'$ for $\omega, \omega' \in \Omega$.

**Lemma 7.1.** The group $\Omega$ acts on $(M \otimes M)^\ast$ on the right by $v \mapsto v^\omega = v(\omega \otimes \omega)$ for $\omega \in \Omega$ and $v \in (M \otimes M)^\ast$. This action preserves the set of cyclic bilinear forms.

**Proof.** The first claim is obvious. It is easy to check that for any $v \in (M \otimes M)^\ast$,

$$\tilde{\omega} = (\omega \otimes \omega)^{-1} \tilde{v}(\omega \otimes \omega) \in \text{End}(M \otimes M).$$

If $v$ is cyclic, then $p_{21}$ commutes with both $\tilde{\omega}$ and $\omega \otimes \omega$ and hence commutes with $\tilde{v}$. Hence, $v^\omega$ is cyclic. \hfill \Box

For any graded algebra $A$, there is a natural left action of $\Omega = \text{Aut}(M)$ on $A_M$. Any $\omega \in \Omega$ acts on the generators by $\omega a_\alpha = a_{\omega(\alpha)}$. This extends uniquely to a graded algebra automorphism of $A_M$. The compatibility with the bilinearity relations is obvious. The compatibility with the multiplicativity relations:

$$\omega(ab)_\alpha = (ab)_{\omega(\alpha)} = a_{\omega(\alpha)} b_{\omega(\alpha)} = a_{\omega(\alpha)} b_{\omega(\alpha)} = \omega a_\alpha \cdot \omega b_\alpha = \omega(a_\alpha b_\alpha)$$

where the third equality holds because $\mu \omega = (\omega \otimes \omega) \mu$. The action of $\Omega$ on $A_M$ induces an action of $\Omega$ on $A_M = \text{Com}(A_M)$ by graded algebra automorphisms.

**Lemma 7.2.** For any cyclic bilinear form $v$ on $M$, any $\omega \in \Omega$ and $x, y \in A_M$, we have $\{\omega x, \omega y\}_v = \omega \{x, y\}_v$. Therefore, the bracket $\{-, -\}_v$ in $A_M$ is preserved under the action of the isotropy group $\{\omega \in \Omega \mid v^\omega = v\}$ of $v$.

**Proof.** It is easy to see that if the identity $\{\omega x, \omega y\}_v = \omega \{x, y\}_v$ holds for generators of $A_M$, then it holds for any $x, y \in A_M$. Given $a, b \in A$ and $\alpha, \beta \in \Omega$,

$$\{\omega a_\alpha, \omega b_\beta\}_v = \{a_{\omega(\alpha)}, b_{\omega(\beta)}\}_v = v(\omega(\alpha) \otimes \omega(\beta)) \{a, b\}_{\omega(\alpha) \otimes \omega(\beta)} = v(\omega(\alpha) \otimes \omega(\beta)) \{a, b\}_{\omega(\alpha) \otimes \omega(\beta)} = \omega(\omega(\alpha) \otimes \omega(\beta)) \{a, b\}_{\omega(\alpha) \otimes \omega(\beta)} = \omega(\omega(\alpha) \otimes \omega(\beta)) \{a, b\}_{\omega(\alpha) \otimes \omega(\beta)} = \omega(\omega(\alpha) \otimes \omega(\beta)) \{a, b\}_{\omega(\alpha) \otimes \omega(\beta)} = \omega \{a_\alpha, b_\beta\}_v.$$ 

\hfill \Box

7.2. The counital case. For a counital coalgebra $M$, one defines inner automorphisms as follows. Note that the counit $\varepsilon = \varepsilon_M \in M^\ast$ is a two-sided unit of the algebra $M^\ast$ dual to $M$. Since $M^\ast$ has a unit, we can consider the group $U = U(M^\ast)$ of invertible elements of $M^\ast$. Each $u \in U$ acts on $M$ by the inner automorphism $\alpha \mapsto u^\alpha = u^{-1}(\alpha^1)\alpha^2(\alpha^3)$. This defines a left action of $U$ on $M$ by coalgebra automorphisms, that is a homomorphism $U \to \text{Aut}(M)$. Composing with the actions of $\text{Aut}(M)$ on $A_M$, $A_M$ we obtain actions of $U$ on $A_M$, $A_M$. 

Note for the record that the action of any \( u \in U \) on \( M \) is dual to the conjugation by \( u \) in \( M^* \). Indeed, for any \( a \in M^* \) and \( \alpha \in M \),

\[
(7.2.1) \quad a^{(\alpha)} = a(u^{-1}(\alpha^1)\alpha^2u(\alpha^3)) = u^{-1}(\alpha^1)a(\alpha^2)u(\alpha^3) = (u^{-1}a\epsilon)(\alpha).
\]

**Theorem 7.3.** Let \( v \in (M \otimes M)^* \) be a cyclic bilinear form on a counital coalgebra \( M \). Then

(i) \( v \) is symmetric;

(ii) the action of \( U = U(M^*) \) on \( A_M \) preserves the bracket \( \{-, -\}_v \);

(iii) the fixed point algebra

\[
A_M^v = \{ x \in A_M \mid ux = x \text{ for all } u \in U \}
\]

is closed under the bracket \( \{-, -\}_v \).

**Proof.** Let \( \alpha, \beta \in M \). Applying \( \epsilon \otimes \epsilon : M \otimes M \rightarrow \mathbb{K} \) to both sides of \((1.2.1)\), we obtain that \( v(\alpha \otimes \beta) = v(\beta \otimes \alpha) \). This equality may be rewritten as \( \overline{\alpha}\beta = \overline{\beta}\alpha \) where the overbar denotes the cyclic structure of \( M \rightarrow M^* \) determined by \( v \).

By Lemma 7.2 in order to prove (ii), it is enough to show that \( v^u = v \) for all \( u \in U \). We first verify that \( \overline{u\alpha} = u\overline{\alpha}u^{-1} \) for any \( \alpha \in M \). We have

\[
\overline{\alpha} = u^{-1}(\alpha^1)\alpha^2u(\alpha^3) = u^{-1}(\alpha^1)\alpha^2u(\alpha^3).
\]

Evaluating on any \( \beta \in M \), we obtain

\[
\overline{\alpha}(\beta) = u^{-1}(\alpha^1)\overline{\alpha^2}(\beta)u(\alpha^3)
\]

\[
= u^{-1}(\alpha^1)\overline{\beta}(\alpha^2)u(\alpha^3) = u(\beta^1)\overline{\alpha}(\beta^2)u^{-1}(\beta^3) = (u\overline{\alpha}u^{-1})(\beta)
\]

where the penultimate equality is obtained by evaluating \( u \otimes u^{-1} \) on both sides of \((5.1.1)\). We conclude that \( \overline{u\alpha} = u\overline{\alpha}u^{-1} \). Next, identifying \( u \in U \) with its image in \( \text{Aut}(M) \), we obtain for any \( \alpha, \beta \in M \),

\[
v^n(\alpha \otimes \beta) = v^n(\beta \otimes u\alpha) = n\overline{\alpha}(\beta) = (u\overline{\alpha}u^{-1})(\beta)
\]

\[
\overset{7.2.1}{=} (u^{-1}u\overline{\alpha}u^{-1}u)(\beta) = \overline{\alpha}(\beta) = v(\alpha \otimes \beta).
\]

We conclude that \( v^u = v \). This proves (ii). Clearly, (ii) implies (iii). \[\square\]

8. A Lie algebra action on \( A_M \)

We show that coderivations of \( M \) act on \( A_M \) by derivations and study the behavior of the bracket \((8.2.2)\) under this action. For counital \( M \), we exhibit a Lie algebra of derivations of \( A_M \) preserving the bracket.

8.1. Derivations. A (degree zero) **derivation** in a graded algebra \( A \) is a linear map \( \delta : A \rightarrow A \) such that \( \delta(AF) \subseteq AF \) for all \( p \in \mathbb{Z} \) and

\[
(8.1.1) \quad \delta(ab) = \delta(a)b + a\delta(b)
\]

for any \( a, b \in A \). The derivations of \( A \) form a module, \( \text{der}(A) \). We provide \( \text{der}(A) \) with the Lie bracket by \([\delta, \delta'] = \delta d - d\delta \) for any \( \delta, \delta' \in \text{der}(A) \). Any derivation of \( A \) carries \([A, A] \) into itself and induces a derivation of the algebra \( \text{Com}(A) \).

An **action** of a (non-graded) Lie algebra \( g \) on \( A \) is a Lie algebra homomorphism \( g \rightarrow \text{der}(A) \). Such an action induces an action of \( g \) on \( \text{Com}(A) \) in the obvious way.

A bracket \( \{-, -\} \) in \( A \) is **invariant** under a derivation \( \delta : A \rightarrow A \) if \( \delta\{a, b\} = \{\delta(a), b\} + \{a, \delta(b)\} \) for any \( a, b \in A \). A bracket \( \{-, -\} \) in \( A \) is invariant under an
action of a Lie algebra $\mathfrak{g}$ on $A$ if it is invariant under the action of all elements of $\mathfrak{g}$. Note that then $\{A^g, A^g\} \subset A^g$ where
\[(8.1.2) \quad A^g = \{a \in A | w(a) = 0 \text{ for all } w \in \mathfrak{g}\}.
\]

8.2. Coderivations. A coderivation in a coalgebra $M$ is a linear map $\delta : M \to M$ such that
\[(8.2.1) \quad \mu \delta = (\delta \otimes \text{id}_M + \text{id}_M \otimes \delta)\mu.
\]
This formula may be rewritten as the identity
$$\delta(\alpha)^1 \otimes \delta(\alpha)^2 = \delta(\alpha^1) \otimes \alpha^2 + \alpha^1 \otimes \delta(\alpha^2)$$
for any $\alpha \in M$. The coderivations of $M$ form a Lie algebra, coder$(M)$, with the Lie bracket $[\delta, d] = \delta d - d\delta$ for any $\delta, d \in \text{coder}(M)$.

The Lie algebra coder$(M)$ contains a Lie subalgebra of inner coderivations. Namely, consider the algebra $M^*$ dual to $M$, and let $M^*$ be the module $M^*$ equipped with the Lie bracket $[\varphi, \psi] = \varphi \psi - \psi \varphi$ for any $\varphi, \psi \in M^*$. Every $\varphi \in M^*$ determines an inner coderivation $\delta_{\varphi}$ of $M$ by $\delta_{\varphi}(\alpha) = \varphi(\alpha^2)(\alpha^1) - \varphi(\alpha^1)(\alpha^2)$ for $\alpha \in M$.

The map $M^* \to \text{coder}(M)$, $\varphi \mapsto \delta_{\varphi}$ is a Lie algebra homomorphism.

**Lemma 8.1.** For a graded algebra $A$ and a coalgebra $M$, there is a unique action of the Lie algebra coder$(M)$ on $\tilde{A}_M$ such that $\delta(a_\alpha) = a_{\delta(\alpha)}$ for any $\delta \in \text{coder}(M)$, $a \in A$, $\alpha \in M$.

**Proof.** The uniqueness is obvious, and we need only to construct the action in question. We derive from each $\delta \in \text{coder}(M)$ a derivation $\tilde{\delta}$ in $\tilde{A}_M$ such that $\tilde{\delta}(a_\alpha) = a_{\delta(\alpha)}$ for all generators $a_\alpha$ of $\tilde{A}_M$. We check the compatibility with the defining relations of $\tilde{A}_M$. The compatibility with the bilinearity relations is obvious. The compatibility with the multiplicativity relations:

$$\tilde{\delta}((ab)_{\alpha}) = (ab)_{\delta(\alpha)} = a_{\delta(\alpha)}b_{\delta(\alpha)}^2$$

$$= a_{\delta(\alpha)}b_{\alpha^2} + a_{\alpha^1}b_{\delta(\alpha^2)}$$

$$= \tilde{\delta}(a_{\alpha^1})b_{\alpha^2} + a_{\alpha^1}\tilde{\delta}(b_{\alpha^2}) = \tilde{\delta}(a_{\alpha^1}, b_{\alpha^2}).$$

The map coder$(M) \to \text{der}(\tilde{A}_M)$, $\delta \mapsto \tilde{\delta}$ is linear and preserves the Lie bracket: for $\delta, d \in \text{coder}(M)$, $a \in A$, $\alpha \in M$,
$$[\tilde{\delta}, \tilde{d}](a_\alpha) = \tilde{\delta}d(a_\alpha) - \tilde{d}\delta(a_\alpha) = a_{\delta d(\alpha)} - a_{d\delta(\alpha)} = a_{[\delta, d]_{\alpha}} = [\tilde{\delta}, \tilde{d}](a_\alpha).$$

Since $\{a_\alpha\}$ generate the algebra $\tilde{A}_M$, we have $[\tilde{\delta}, \tilde{d}] = [\tilde{\delta}, \tilde{d}]$. \hfill \Box

Composing the map $M^* \to \text{coder}(M)$, $\varphi \mapsto \delta_{\varphi}$ with the action of coder$(M)$ on $\tilde{A}_M$, we obtain an action of the Lie algebra $M^*$ on $\tilde{A}_M$. This induces an action of $M^*$ on $A_M = \text{Com}(\tilde{A}_M)$. We state an analogue of Theorem 7.3 in this context.

**Theorem 8.2.** For any double bracket $\{\{-, -\}\}$ in a graded algebra $A$ and any cyclic bilinear form $v$ on a counital coalgebra $M$, the induced bracket $\{\{-, -\}_v\}$ in $A_M$ is invariant under the action of $M^*$. As a consequence, the graded algebra
\[(8.2.2) \quad A^M_M = (A^M)_{M^*} = \{x \in A_M | \delta_{\varphi}(x) = 0 \text{ for all } \varphi \in M^*\}
\]
is closed under the bracket $\{\{-, -\}_v\}$.
On the other hand, we only need to prove that for all $\alpha, \beta \in M$. We have
\[
\{\delta(\alpha), \beta\} = \{a_\alpha, \delta(\beta)\} = \{a_\alpha, \beta\} + \{a_\alpha, \delta(\beta)\} = \{a_\alpha, \beta\} + \delta(\alpha) \delta(\beta).
\]

On the other hand, using these formulas and the identity $\delta(\alpha) \beta = \alpha \delta(\beta)$, we obtain
\[
\delta(\alpha) \beta = \alpha \delta(\beta).
\]

Hence, evaluating the left-hand side of (8.2.3) on $\alpha \otimes \beta \in M \otimes M$, we obtain
\[
v(\delta(\alpha) \beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta).
\]

Hence, evaluating the left-hand side of (8.2.3) on $\alpha \otimes \beta \in M \otimes M$, we obtain
\[
v(\delta(\alpha) \beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta).
\]

The first and third terms cancel out by the assumption $v(\delta \otimes \id_M) = -v(\id_M \otimes \delta)$. The remaining two terms give
\[
v(\delta(\alpha) \beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta) = (\delta(\alpha) \otimes \beta^1) \beta^1 + \beta^1 \otimes \delta(\beta).
\]

This proves (8.2.3).

To accomplish the proof, it remains to show that $v(\delta \otimes \id_M) = -v(\id_M \otimes \delta)$ for all $\varphi \in M^\times$. We evaluate both sides on any $\alpha \otimes \beta \in M \otimes M$. Recall the cyclic structure $M \to M^\times$, $\gamma \mapsto \overline{\gamma}$ determined by $v$. Applying $\id_M \otimes \epsilon_M$ and $\epsilon_M \otimes \id_M$ to both sides of (8.1.1), we obtain
\[
(8.2.4) \quad \overline{\beta}(\alpha^1) \alpha^2 = \overline{\alpha}(\beta^2) \beta^1 \quad \text{and} \quad \overline{\beta}(\alpha^2) \alpha^1 = \overline{\alpha}(\beta^1) \beta^2.
\]

Using these formulas and the identity $\overline{\gamma}(\beta) = \overline{\beta}(\gamma)$ for all $\gamma \in M$, we obtain
\[
v(\delta \otimes \id_M) = \overline{\varphi}(\delta(\alpha)) = \varphi(\alpha) \overline{\beta}(\gamma) = -v(\alpha \otimes \delta(\beta)).
\]

This proves (8.2.3).

8.3. Remark. Subtracting the first of the equalities (8.2.4) from the second one, we obtain the following useful identity: for all $\alpha, \beta \in M$,
\[
\delta(\alpha) = \overline{\delta}(\beta).
\]
8.4. **Compatibility.** To relate the actions of $\text{Aut}(M)$ and $\text{coder}(M)$ constructed above, we use the language of Lie pairs introduced in [5]. A Lie pair is a pair $(G, \mathfrak{g})$ where $G$ is a group and $\mathfrak{g}$ is a Lie algebra equipped with a (left) action of $G$ by Lie algebra automorphisms $w \to g^w$ where $w$ runs over $\mathfrak{g}$ and $g$ runs over $G$. A morphism of Lie pairs $(G', \mathfrak{g}') \to (G, \mathfrak{g})$ is a pair (a group homomorphism $F : G' \to G$, a Lie algebra homomorphism $f : \mathfrak{g}' \to \mathfrak{g}$) such that $f(g^w) = F(g)(f(w))$ for all $g \in G'$, $w \in \mathfrak{g}'$. An action of a Lie pair $(G, \mathfrak{g})$ on a graded algebra $A$ is a (a left) action of $G$ on $A$ by graded algebra automorphisms, an action of $\mathfrak{g}$ on $A$ such that $(g^w)x = gw^{-1}x$ for any $g \in G$, $w \in \mathfrak{g}$, $x \in A$. Note that then the algebra $A^\theta$ defined by (8.1.2) is $G$-invariant. Indeed, if $g \in G$ and $x \in A^\theta$, then $gx \in A^\theta$ because for all $w \in \mathfrak{g}$,

$$wgx = g(g^{-1}wg)x = g(g^{-1}w)x = g0 = 0.$$  

An action of $(G, \mathfrak{g})$ on $A$ composed with a morphism of Lie pairs $(G', \mathfrak{g}') \to (G, \mathfrak{g})$ yields an action of $(G', \mathfrak{g}')$ on $A$. An action of a Lie pair on $A$ induces an action of this Lie pair on $\text{Com}(A)$ in the obvious way.

Any coalgebra $M$ gives rise to a Lie pair $(G = \text{Aut}(M), \mathfrak{g} = \text{coder}(M))$ where $G$ acts on $\mathfrak{g}$ by $g^w = gw^{-1} : M \to M$ for $g \in G$ and $w \in \mathfrak{g}$. The above-defined actions of $G$ and $\mathfrak{g}$ on $\tilde{A}_M$ determine an action of the Lie pair $(G, \mathfrak{g})$ because $(g^w)x = gw^{-1}x$ for any $g \in G$, $w \in \mathfrak{g}$, $x \in A$. Since $g^w$ and $gw^{-1}$ act as derivations in $\tilde{A}_M$, it is enough to check this for the generators $x = a_\alpha \in \tilde{A}_M$ where $a \in A$, $\alpha \in M$. We have

$$g^w(a_\alpha) = a_{(\varphi w(\alpha))} = a_{gw^{-1}(\alpha)} = gw(a_{g^{-1}(\alpha)}) = gw^{-1}(a_\alpha).$$

A counital coalgebra $M$ gives rise to a Lie pair $(U, M^*)$ where $U = \text{U}(M^*)$ acts on $M^*$ by $^u\varphi = u\varphi u^{-1}$ for $u \in U$, $\varphi \in M^*$. It is easy to check that the above-defined homomorphisms $U \to \text{Aut}(M)$ and $M^* \to \text{coder}(M)$ determine a morphism of Lie pairs. Therefore the above-defined actions of $U$ and $M^*$ on $\tilde{A}_M$ determine an action of the Lie pair $(U, M^*)$ on $\tilde{A}_M$. Considering the induced action on $A_M = \text{Com}(\tilde{A}_M)$, we can conclude that, under the assumptions of Theorem 8.2, the graded algebra $A^M_M \subset A_M$ is $U$-invariant.

9. **The bracket $\langle - , - \rangle$ and the traces**

A double bracket in $A$ induces a bracket $\langle - , - \rangle$ in $\tilde{A} = A/[A, A]$, and we relate it to the bracket $\{ - , - \}_v$ in $A_M$.

9.1. **The bracket $\langle - , - \rangle$.** An $n$-graded double bracket $\{ - , - \}$ in a graded algebra $A$ induces a bilinear form $A \times A \to A, (a, b) \mapsto \llbracket a, b \rrbracket^\theta$. The latter descends to an $n$-graded antisymmetric bracket $\langle - , - \rangle : \tilde{A} \times \tilde{A} \to A$ in the graded module $\tilde{A} = A/[A, A]$, see [5]. If $\llbracket - , - \rrbracket$ is Gerstenhaber, then $\langle - , - \rangle$ is an $n$-graded Lie bracket, i.e., it is $n$-graded antisymmetric and satisfies the $n$-graded Jacobi identity.

9.2. **Traces.** Every element $\theta$ of a coalgebra $M$ determines a degree-preserving linear map $\text{tr}_\theta : A \to A_M$ by $\text{tr}_\theta(a) = a_\theta$ for $a \in A$. The subalgebra of $A_M$ generated by the set $\{ a_\theta \}_{a \in A}$ is denoted $A(\theta)$.

We call $\theta \in M$ symmetric if $\mu(\theta) \in M \otimes M$ is invariant under the permutation of the tensor factors, i.e., if $p_{21} \mu(\theta) = \mu(\theta)$.

**Lemma 9.1.** For a symmetric $\theta \in M$, we have $[A, A] \subset \text{Ker} \text{tr}_\theta$ and $A(\theta) \subset A^M_M$ where $A^M_M$ is the subalgebra of $A_M$ defined by (8.2.2).
Proof. For any homogeneous \(a, b \in A\),
\[
\text{tr}_{\theta}(ab - (-1)^{|a||b|}ba) = (ab)_{\theta} - (-1)^{|a||b|}(ba)_{\theta} = a_{\theta}b_{\theta} - (-1)^{|a||b|}b_{\theta}a_{\theta} = a_{\theta}b_{\theta} - a_{\theta}b_{\theta} = 0
\]
where the last equality holds because \(p_{21}\mu(\theta) = \mu(\theta)\). Thus, \(\text{tr}_{\theta}(\langle A, A \rangle) = 0\).

The equality \(p_{21}\mu(\theta) = \mu(\theta)\) implies that each inner coderivation \(\delta\) of \(M\) annihilates \(\theta\). Therefore \(\delta(a_{\theta}) = a_{\delta(\theta)} = 0\) for all \(a \in A\). Hence \(\text{tr}_{\theta}(A) \subset A^M_M\). \(\square\)

We say that \(\theta \in M\) is adjoint to a bilinear form \(v\) on \(M\) if the map \(M \rightarrow \mathbb{K}\), \(\alpha \mapsto v(\theta \otimes \alpha)\) is a counit of \(M\).

**Lemma 9.2.** Any \(\theta \in M\) adjoint to a cyclic bilinear form \(v\) on \(M\) is symmetric.

Proof. Since the map \(\alpha \mapsto v(\theta \otimes \alpha) = \varepsilon(\alpha)\) is the counit of \(M\),
\[
(9.2.1) \quad \hat{\varepsilon}(\theta \otimes \varepsilon) = v(\theta \otimes \theta^2)\theta^1 \otimes \theta^1 = \varepsilon(\theta^2)\theta^1 \otimes \theta^1 = \theta^1 \otimes \theta^2 = \mu(\theta).
\]
Therefore \(\theta\) is symmetric:
\[
p_{21}\mu(\theta) = p_{21}\hat{\varepsilon}(\theta \otimes \theta) = \hat{\varepsilon}p_{21}(\theta \otimes \theta) = \hat{\varepsilon}(\theta \otimes \theta) = \mu(\theta).
\]

By Lemma 9.1, the map \(\text{tr}_{\theta} : A \rightarrow A_M\) associated with a symmetric \(\theta \in M\) induces a linear map \(\hat{\theta} \rightarrow A_M\). It is called the trace and also denoted by \(\text{tr}_{\theta}\).

**Lemma 9.3.** For a double bracket \(\langle -,-\rangle\) in a graded algebra \(A\), a cyclic bilinear form \(v\) on a coalgebra \(M\), and any \(\theta \in M\) adjoint to \(v\), the trace \(\text{tr}_{\theta} : \hat{A} \rightarrow A_M\) carries the bracket \(\langle -,-\rangle\) in \(\hat{A}\) to the bracket \(\langle -,-\rangle_v\) in \(A_M\). Consequently, the algebra \(A(\theta)\) is closed under the bracket \(\langle -,-\rangle_v\).

Proof. Pick any \(a, b \in A\) and let \(\hat{a}, \hat{b}\) be their projections to \(\hat{A}\). Then
\[
\{\text{tr}_{\theta}(a), \text{tr}_{\theta}(b)\}_v = \{a_{\theta}, b_{\theta}\}_v
\]

The following lemma gives more information about \(A(\theta)\) for some \(\theta\).

**Lemma 9.4.** Any non-degenerate cyclic bilinear form \(v\) on a counital coalgebra \(M\) has a unique adjoint \(\theta \in M\) and \(A(\theta) \subset A_M^U \cap A_M^U\) where \(U = U(M^*)\).

Proof. The first claim is obvious. To prove the second claim, we must show that \(u(\theta) = a_\theta\) for all \(u \in U\) and \(a \in A\). Pick any \(\alpha \in M\) and observe that
\[
v(\theta \otimes \alpha) = v(u_\theta \otimes u^{-1}\alpha) = v(\theta \otimes u^{-1}\alpha)
\]
\[
= \varepsilon_M(u^{-1}\alpha) = u\varepsilon_M u^{-1}(\alpha) = \varepsilon_M(\alpha) = v(\theta \otimes \alpha)
\]
where we use that \(v^u = v\). The injectivity of \(\text{ad}_v\) implies that \(u\theta = \theta\). Hence, \(u(\theta) = a_{(u \theta)} = a_\theta\). \(\square\)

The next claim yields a rich source of adjoint elements of cyclic bilinear forms.

**Lemma 9.5.** Let \(A\) be a unital algebra whose underlying module is free of finite rank. Let \(M = A^*\) be the dual coalgebra and let \(\theta \in M\) be a trace-like element. Then \(\theta\) is adjoint to the cyclic bilinear form \(v_\theta\) on \(M\) defined in Section 6.2.
Proof. Consider the symmetric Frobenius algebra \((A, (-,-)_\theta)\) as in Section 6.2. The cyclic structure \(M \to A, \alpha \mapsto \bar{\alpha}\) defined in Theorem 6.1 carries \(\theta\) to \(1_A\). Indeed, for all \(a \in A\),

\[
\theta(a\bar{\theta}) = (a, \bar{\theta})_\theta = \theta(a) = \theta(a1_A)
\]

and so \(\bar{\theta} = 1_A\). For any \(\alpha \in M\), we have \(v_\theta(\alpha \otimes \alpha) = \alpha(\bar{\theta}) = \alpha(1_A)\). Therefore \(\theta\) is adjoint to \(v_\theta\).

\[\square\]

10. The Unital Case

For a unital graded algebra \(A\) and a counital coalgebra \(M\), we define unital versions \(\tilde{A}^+_M\), \(A^+_M\) of \(\tilde{A}_M\), \(A_M\). We also formulate Hamiltonian reduction and discuss double quasi-Poisson algebras.

10.1. Unital representation algebras. Unital graded algebras and graded algebra homomorphisms carrying 1 to 1 form a category \(\mathcal{G}A^+\). For a unital graded algebra \(A\) and a counital coalgebra \(M\) with counit \(\varepsilon = \varepsilon_M\), we define a unital graded algebra \(\tilde{A}_M^+\) as follows. First, we adjoin a two-sided unit to \(\tilde{A}_M\), that is consider the unital graded algebra \(\mathbb{K}e \oplus \tilde{A}_M\) with two sided unit \(e\) of degree zero. A typical element of \(\mathbb{K}e \oplus \tilde{A}_M\) is represented by a non-commutative polynomial in the generators \(\{a_\alpha \mid a \in A, \alpha \in M\}\). By definition, \(\tilde{A}_M^+\) is the quotient of the algebra \(\mathbb{K}e \oplus \tilde{A}_M\) by the normalization relations \(\{(1_A)_\alpha = \varepsilon(\alpha)e\}_{\alpha \in M}\). The construction of \(\tilde{A}_M^+\) obviously extends to a functor \(f \mapsto \tilde{f}_M^+: \mathcal{G}A^+ \to \mathcal{G}A^+\).

**Lemma 10.1.** For any unital graded algebra \(B\), the convolution algebra \(H_M(B)\) is unital with unit \(M \to B, \alpha \mapsto \varepsilon(\alpha)_1_B\). The bijection \((2.4.1)\) induces a bijection

\[
\text{Hom}_{\mathcal{G}A^+}(\tilde{A}_M^+, B) \xrightarrow{\sim} \text{Hom}_{\mathcal{G}A^+}(A, H_M(B))
\]

which is natural in \(A\) and \(B\).

**Proof.** The first claim is obvious. To prove the second claim, we apply the same constructions \(r \mapsto s_r\) and \(s \mapsto r_s\) as in the proof of Lemma 2.1. Observe that the map \(s_r: A \to H_{\mathbb{K}}(M, B)\) carries \(1_A\) to the map

\[
M \to B, \alpha \mapsto s((1_A)_{\alpha} = r(\varepsilon(\alpha)e) = \varepsilon(\alpha)r(e) = \varepsilon(\alpha)_1_B
\]

which is the unit of the algebra \(H_M(B)\). The map \(r = r_s\) is compatible with the normalization relations because

\[
r((1_A)_{\alpha}) = s(1_A)(\alpha) = \varepsilon(\alpha)_1_B = r(\varepsilon(\alpha)e).
\]

Replacing \(\tilde{A}_M^+\) by \(A_M^+ = \text{Com}(\tilde{A}_M^+)\) we obtain that for any unital commutative graded algebra \(B\),

\[
\text{Hom}_{c_{\mathcal{G}A^+}}(A_M^+, B) \simeq \text{Hom}_{\mathcal{G}A^+}(\tilde{A}_M^+, B) \simeq \text{Hom}_{\mathcal{G}A^+}(A, H_M(B))
\]

where \(c_{\mathcal{G}A^+}\) is the category of unital commutative graded algebras. The algebra \(A_M^+\) can be alternatively obtained by adjoining a two-sided unit \(e\) to \(A_M = \text{Com}(\tilde{A}_M)\) and then factoring \(\mathbb{K}e \oplus A_M\) by the relations \(\{(1_A)_{\alpha} = \varepsilon(\alpha)e\}_{\alpha \in M}\). The construction of \(A_M^+\) obviously extends to a functor \(f \mapsto f_M^+: \mathcal{G}A^+ \to c_{\mathcal{G}A^+}\).
10.2. The actions. Given a unital graded algebra \( A \) and a counital coalgebra \( M \), the action of \( \text{Aut}(M) \) on \( \tilde{A}_M \) defined in Section 7.1 induces an action of \( \text{Aut}(M) \) on \( \tilde{A}_M^+ \) by graded algebra automorphisms fixing the unit \( e \). We check the compatibility with the normalization relations: for \( g \in \text{Aut}(M) \), \( a \in M \),

\[
g(1_A)_\alpha = (1_A)_{g(\alpha)} = \varepsilon(g(\alpha))e = \varepsilon(\alpha)e = g(\varepsilon(\alpha)e).
\]

Similarly, the action of the Lie algebra \( \text{coder}(M) \) on \( \tilde{A}_M \) defined in Lemma 8.3 induces an action of \( \text{coder}(M) \) on \( \tilde{A}_M^+ \) annihilating \( e \). The compatibility with the normalization relations holds because \( \varepsilon \delta = 0 \) for all \( \delta \in \text{coder}(M) \); this equality is deduced from (8.2.1) by applying \( \varepsilon \otimes \varepsilon \) to both sides.

The actions of \( \text{Aut}(M) \) and \( \text{coder}(M) \) on \( \tilde{A}_M^+ \) form an action of the Lie pair \( (\text{Aut}(M), \text{coder}(M)) \) on \( \tilde{A}_M^+ \) and induce an action of the Lie pair \( (U(M^*), M^*) \) on \( \tilde{A}_M^+ \). As a consequence, the set

\[
E = (A_M^+)^{M^*} = \{ x \in A_M^+ | \delta_\varphi(x) = 0 \text{ for all } \varphi \in M^* \}
\]

is a \( U(M^*) \)-invariant subalgebra of \( A_M^+ \). Since the derivations \( \delta_\varphi \) of \( A_M^+ \) are grading-preserving, \( E = \oplus_{p \in \mathbb{Z}} E \cap (A_M^+)^p \). This makes \( E \) into a commutative graded algebra with unit \( 1_E = e \).

10.3. The bracket \( \{-,-\}_v^+ \) and Hamiltonian reduction. Let \( \{ -,- \}_v \) be an \( n \)-graded double bracket in a unital graded algebra \( A \) with \( n \in \mathbb{Z} \), and let \( v \) be a cyclic bilinear form on a counital coalgebra \( M \). The induced \( n \)-graded biderivation \( \{-,-\}_v \in A_M \) uniquely extends to a bracket in \( \mathbb{K}e \otimes A_M \) annihilating \( e \) both on the left and on the right. Since \( \{1_A, A\} = \{ A, 1_A \} = 0 \), the latter bracket annihilates \( (1_A)_\alpha - \varepsilon_M(\alpha)e \) for all \( \alpha \in M \) and descends to an \( n \)-graded biderivation \( \{-,-\}_v^+ \in A_M^+ \). The Jacobi form of \( \{-,-\}_v^+ \) can be computed from Lemma 4.2. Hence, if \( \{ -,- \} \) is Gerstenhaber, then so is \( \{-,-\}_v^+ \).

Theorem 8.2 implies that the bracket \( \{-,-\}_v^+ \) is \( M^* \)-invariant. As a consequence, the graded algebra \( E \subset A_M^+ \) defined by (10.2.1) satisfies \( \{ E, E \}_v^+ \subset E \). This graded algebra appears in Hamiltonian reduction as follows.

Theorem 10.2. Let \( B \) be a graded algebra and let \( p : A \rightarrow B \) be a graded algebra epimorphism whose kernel is generated, as a 2-sided ideal of \( A \), by a set \( \mathcal{M} \subset A^{-n} \) satisfying the following condition: for every \( \xi \in \mathcal{M} \) there is a scalar \( k_\xi \in \mathbb{K} \) such that for all \( a \in A \),

\[
\{ \xi, a \} \equiv k_\xi(a \otimes 1_A - 1_A \otimes a) \mod(A \otimes \text{Ker} p + \text{Ker} p \otimes A).
\]

Then the \( n \)-graded biderivation \( \{-,-\}_v^+ \) in \( E \) descends uniquely to an \( n \)-graded biderivation \( \{-,-\} \) in the graded algebra \( p_M^+(E) \subset B_M^+ \). If \( \{ -,- \} \) is Gerstenhaber, then so is \( \{-,-\} \).

Proof. The uniqueness of \( \{-,-\} \) is obvious, and we need only to prove the existence. We begin by considering some \( \xi \in A^{-n} \) and \( k \in \mathbb{K} \) such that for all \( a \in A \),

\[
\{ \xi, a \} = k(a \otimes 1_A - 1_A \otimes a).
\]

We claim that for any \( \alpha \in M, x \in A_M^+ \),

\[
\{ \xi_\alpha, x \}_v^+ = k \delta_\alpha(x)
\]

where the overbar stands for the cyclic structure \( M \rightarrow M^* \) determined by \( v \). Both sides of (10.3.1) are derivations in \( x \) (here we use that \( \xi \in A^{-n} \)). Hence, it is
enough to prove (10.3.1) for each generator $x = a_\beta \in A^+_M$ where $a \in A$, $\beta \in M$. We have

$$\{\xi, a_\beta\}^{+}_v = k \{\xi, a\}_b^{1}_a \{\xi, a\}^{2}_{\beta^0} = k(a_{\alpha_\beta}(1_A)\beta^0 - (1_A)a_{\alpha_\beta})$$

where $\varepsilon = \varepsilon_M : M \to K$ is the counit of $M$. Observe that

$$\varepsilon(\beta^0)\alpha - \varepsilon(\alpha \beta)\beta^0 = (\text{id}_M \otimes \varepsilon - \varepsilon \otimes \text{id}_M)(\alpha \otimes \beta^0)$$

$$= (\varepsilon \otimes \varepsilon)(\beta^0)\beta^1 - \varepsilon(\alpha \otimes \varepsilon(\beta^0))\beta^0$$

$$= \alpha \otimes \varepsilon(\beta^0)\beta^1 - \varepsilon(\alpha \otimes \beta^0)\beta^2 = \varepsilon(\beta^0)\beta^1 - \alpha \otimes \varepsilon(\beta^0)\beta^2 = \delta_{\varepsilon}(\beta).$$

Therefore $\{\xi, a_\beta\}^{+}_v = k\delta_{\varepsilon}(a_\beta)$. This proves (10.3.1).

The assumptions of the theorem imply that the kernel, $J$, of the homomorphism $p^+_M : A^+_M \to B^+_M$ is generated, as a two-sided ideal of $A^+_M$, by the set $\{\xi_\alpha \mid \xi \in \mathcal{M}, \alpha \in M\}$. We claim that for all the generators $\xi_\alpha$ and all $x \in A^+_M$,

$$\{\xi_\alpha, x\}^{+}_v \equiv k_\xi \delta_{\varepsilon}(x) \mod J.$$

As above, it suffices to verify (10.3.2) for $x = a_\beta$ of $A^+_M$ where $a \in A$, $\beta \in M$. By the assumptions, $\{\xi, a\}$ is a sum of $k_\xi (a \otimes 1_A - 1_A \otimes a)$ and a finite number of vectors $b \otimes c \in A \otimes A$ with $b$ or $c$ in $\ker p$. The contributions of these terms to $\{\xi_\alpha, a_\beta\}^{+}_v$ are equal respectively to $k_\xi \delta_{\varepsilon}(x)$ and $b_{\alpha_\beta}c_{\beta^0} \in J$. This yields (10.3.2).

Formula (10.3.2) implies that $\{\xi_\alpha, E\}^{+}_v \subset J$ for all $\xi \in \mathcal{M}, \alpha \in M$. Using the Leibniz rule in the first variable, we deduce that $\{J, E\}^{+}_v \subset J$. Therefore, $\{E \cap J, E\}^{+}_v \subset \{E, E\}^{+}_v \cap \{J, E\}^{+}_v \subset E \cap J$.

By the antisymmetry of $\{-, -\}^{+}_v$, also $\{E, E \cap J\}^{+}_v \subset E \cap J$. Hence, the bracket $\{-, -\}^{+}_v$ in $E$ descends to a bracket $\{-, -\}$ in $p^+_M(E) \subset E/(E \cap J)$. The properties of $\{-, -\}$ stated in the theorem follow from the properties of $\{-, -\}^{+}_v$. \qed

It is clear that in Theorem 10.2

$$p^+_M(E) \subset (B^+_M)^{\mathcal{M}^*} = \{x \in B^+_M \mid \delta_{\varphi}(x) = 0 \text{ for all } \varphi \in \mathcal{M}^*\}$$

and the map $p^+_M : A^+_M \to B^+_M$ is $U$-equivariant where $U = U(\mathcal{M}^*)$. The action of $U$ on $B^+_M$ restricts to an action of $U$ on $p^+_M(E)$. Theorem 10.2 implies that the bracket $\{-, -\}^{+}_v$ and the induced bracket in $p^+_M(E)$ are $U$-invariant.

Corollary 10.3. Let, under the assumptions of Theorem 10.2 $\theta \in M$ be adjoint to $v$ and let $B(\theta)^+$ be the unital subalgebra of $B^+_M$ generated by the set $\{b_0\}_{b \in B}$. There is a unique $n$-graded biderivation $\{-, -\}$ in $B(\theta)^+$ such that

$$(10.3.3) \quad \{a_\theta, b_0\} = (\{a, b\}' \{a, b\}''^\theta)_\theta$$

for any $a, b \in B$. If $\{\xi, a\}$ is Gerstenhaber, then so is $\{-, -\}$. \qed

Proof. Lemmas 9.1 and 9.2 imply that $B(\theta)^+ \subset p^+_M(E)$. Lemma 9.3 shows that the bracket in $p^+_M(E)$ provided by Theorem 10.2 evaluates on the generators of $B(\theta)^+$ via (10.3.3). Therefore $B(\theta)^+$ is closed under this bracket. \qed
10.4. Moment maps. To apply Theorem 10.2 one may pick any set $\mathcal{M} \subset A^{-n}$ and define $B$ to be the quotient of $A$ by the two-sided ideal generated by $\mathcal{M}$. The set $\mathcal{M}$ may consist of just one element $\xi \in A^{-n}$. The conditions of Theorem 10.2 are satisfied, for example, if $\{\xi, a\} = a \otimes 1_A - 1_A \otimes a$ for all $a \in A$. In this case, $\xi$ is called a moment map, cf. [8, 5]. For $n = 0$, one considers multiplicative moment maps $\eta \in A^0$ such that for all $a \in A$,

$$\{\eta, a\} = a \otimes \eta - \eta \otimes a + a \eta \otimes 1_A - 1_A \otimes \eta a,$$

cf. [8, 5]. Then the 1-element set $\mathcal{M} = \{\xi\}$ with $\xi = \eta - 1_A$ satisfies the conditions of Theorem 10.2 for $k_\xi = 2$.

10.5. Double quasi-Poisson algebras. In parallel with the notion of a quasi-Poisson bracket [1], Van den Bergh [8] introduced a class of double quasi-Poisson brackets in unital (non-graded) algebras. A double bracket $\{\cdot, \cdot\}$ in a unital algebra $A$ is quasi-Poisson if the induced tribracket in $A$ is computed by

$$\{a, b, c\} = c \otimes a \otimes b + 1_A \otimes ab \otimes c + a \otimes 1_A \otimes bc + ca \otimes b \otimes 1_A$$

for any $a, b, c \in A$. The pair $(A, \{\cdot, \cdot\})$ is then a double quasi-Poisson algebra. Note that a double quasi-Poisson bracket in our sense is 2 times a double quasi-Poisson bracket in the sense of [8].

**Theorem 10.4.** Let $(A, \{\cdot, \cdot\})$ be a double quasi-Poisson algebra, and let $v$ be a cyclic bilinear form on a counital coalgebra $M$. The restriction of the bracket $\{\cdot, \cdot\}_v^+$ in $A^+_M$ to the algebra $E \subset A^+_M$ defined by $10.2.1$ satisfies the Jacobi identity and makes $E$ into a Poisson algebra.

Proof. Let $\{\cdot, \cdot, \cdot\}_w : (A^+_M)^{\otimes 3} \to A^+_M$ be the Jacobi form of $(\cdot, \cdot, \cdot)_v^+$. We must prove that $\{E, E, E\} = 0$. We prove a stronger claim $\{E, A^+_M, A^+_M\} = 0$.

For $a, b, c \in A$ and $\alpha, \beta, \gamma \in M$, Lemma 10.2 yields $\{a_\alpha, b_\beta, c_\gamma\} = Q - R$ with $Q$ and $R$ given by $10.2.1$, $10.2.2$. Note that since $A$ is non-graded, the signs in the expressions for $Q$ and $R$ are equal to $+1$. The subindices appearing in $Q$ form a vector in $M \otimes M \otimes M$ equal to

$$\alpha(\beta_\gamma) \otimes (\beta_\gamma)^\alpha \otimes \beta_\gamma = (\hat{v} \otimes \text{id}_M)(\text{id}_M \otimes \hat{v})(\alpha \otimes \beta \otimes \gamma) = v(\alpha \otimes \gamma^2) v(\beta \otimes \gamma^4) \gamma^1 \otimes \gamma^3 \otimes \gamma^5 = \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) \gamma^1 \otimes \gamma^3 \otimes \gamma^5.
$$

Now, each term on the right-hand side of $10.5.1$ contributes a summand to $Q$. The resulting eight summands of $Q$ are equal to

\begin{align*}
x_1 &= \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) c_\gamma a_\gamma b_\gamma = a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_2 &= \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) a_\gamma b_\gamma c_\gamma = a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_3 &= \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) a_\gamma b_\gamma c_\gamma = a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_4 &= \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) c_\gamma a_\gamma b_\gamma = a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_5 &= -\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) a_\gamma b_\gamma c_\gamma = -a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_6 &= -\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) a_\gamma b_\gamma c_\gamma = -a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_7 &= -\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) c_\gamma a_\gamma b_\gamma = -a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma, \\
x_8 &= -\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^4) c_\gamma a_\gamma b_\gamma = -a_{\overline{\alpha}(\gamma^2)\gamma} b_{\overline{\beta}(\gamma^4)\gamma} c_\gamma.
\end{align*}
\[ x_8 = -\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^3) c_\gamma, a_\gamma b_\gamma = -a_\gamma \overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^3) c_\gamma. \]

Here, in the computation of \( x_2, x_3, \) etc., we use the relation \((1A)\alpha = \varepsilon(\alpha)1_{A^+_M}. \)

Using (8.3.1), we obtain that
\[
x_1 + x_7 = a_\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^3) c_\gamma, \\
= a_\overline{\alpha}(\gamma^2) \overline{\beta}(\gamma^3) c_\gamma.
\]

Similarly,
\[
x_2 + x_3 = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma, \\
x_3 + x_6 = -\overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma, \\
x_4 + x_8 = -\overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma.
\]

Analogous computations allow us to deduce further that
\[
x_1 + x_7 + x_4 + x_8 = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma, \\
x_2 + x_5 + x_3 + x_6 = -\overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma.
\]

Summing up, we obtain that
\[
(10.5.2) \quad Q = Q(a, b, c, \alpha, \beta, \gamma) = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma.
\]

Similarly, starting from the expression
\[
\alpha(\gamma^3) \otimes (\gamma^3) = (\nu \otimes \text{id}_M)p_{23}(\text{id}_M \otimes \nu)(\alpha \otimes \beta \otimes \gamma),
\]
\[
= v(\alpha \otimes \gamma^4) v(\beta \otimes \gamma^2) \gamma^3 \otimes \gamma^4 \gamma^1
\]
\[
= \overline{\alpha}(\gamma^4) \overline{\beta}(\gamma^3) \gamma^5 \otimes \gamma^1
\]

we obtain that
\[
(10.5.3) \quad R = R(a, b, c, \alpha, \beta, \gamma) = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma = \overline{\gamma}(a_\alpha) \overline{\beta}(b_\beta) c_\gamma.
\]

These expansions for \( Q \) and \( R \) yield an expansion of \( \{a_\alpha, b_\beta, c_\gamma\} = Q - R. \) Now, using the fact that the Jacobi form is a derivation in the first and second variables (cf. (12.2.15) we deduce that for any \( x, y \in A^+_M, \)
\[
\{x, y, c_\gamma\} = (\overline{\gamma}(x) \overline{\beta}(y) - \overline{\gamma}(x) \overline{\beta}(y))c_\gamma + (\overline{\gamma}(x) \overline{\beta}(y) - \overline{\gamma}(x) \overline{\beta}(y))c_\gamma.
\]

If \( x \in E, \) then the inner derivations annihilate \( x, \) and so \( \{x, A^+_M, c_\gamma\} = 0 \) for all the generators \( c_\gamma \) of \( A^+_M. \) Since the Jacobi form is a derivation in the third variable, we deduce that \( \{x, A^+_M, A^+_M\} = 0. \) Thus, \( E, A^+_M, A^+_M \) = 0.

**Corollary 10.5.** If, under the assumptions of Theorem 10.2, \( n = 0, A \) is concentrated in degree zero, and the double bracket in \( A \) is quasi-Poisson, then the induced bracket in \( p^+_M(E) \) is a Poisson bracket.

Theorem 10.3 and Corollary 10.5 are due to Van den Bergh for \( M \) and \( v \) as in Example 6.3.4.
10.6. **Group algebras.** We discuss in more detail the case where $A = \mathbb{K}[\pi]$ is the (non-graded) algebra of a group $\pi$. Consider a conitual coalgebra $M$ whose underlying module is free of finite rank and consider the dual algebra $M^*$. For any (non-graded) algebra $B$, we have $H_M(B) = B \otimes M^*$. If $B$ is unital, then so is the algebra $B \otimes M^*$, and we can identify the set $\text{Hom}_{G,A^*}(A, H_M(B))$ with the set of group homomorphisms from $\pi$ to the group $U(B \otimes M^*)$ of invertible elements of $B \otimes M^*$. The bijection \([10.1.2]\) exhibits $A_M^+$ as the coordinate algebra of the functor $B \mapsto \text{Hom}(\pi, U(B \otimes M^*))$ where $B$ runs over unital commutative (non-graded) algebras. For example, if $M = (\text{Mat}_N(A))^*$ where $N \geq 1$ and $A$ is a symmetric Frobenius algebra, then $U(B \otimes M^*) = GL_N(B \otimes A)$ is the group of invertible elements of the matrix algebra $B \otimes \text{Mat}_N(A) = \text{Mat}_N(B \otimes A)$. If $M = (\mathbb{K}[G])^*$ where $G$ is a finite group, then $U(B \otimes M^*)$ is the group of invertible elements of the group algebra $B \otimes \mathbb{K}[G] = B[G]$.

A double bracket $\{\quad,\quad\}$ in $A$ and a cyclic bilinear form $v$ on $M$ induce a (zero-graded) biderivation $\{-,-\}_v^+$ in $A_M^+$. If $\{\quad,\quad\}$ is Poisson, then so is $\{-,-\}_v^+$. If $\{\quad,\quad\}$ is quasi-Poisson, then the restriction of $\{-,-\}_v^+$ to the algebra $E \subset A_M^+$ defined by \([10.2.1]\) is a Poisson bracket.

These constructions apply to the fundamental group $\pi = \pi_1(\Sigma, \star)$ of an oriented surface $\Sigma$ with boundary and with base point $\star \in \partial \Sigma$. The algebra $A = \mathbb{K}[\pi]$ carries a canonical double quasi-Poisson bracket, see \([5]\). This induces a biderivation $\{-,-\}_v^+$ in $A_M^+$ which restricts to a Poisson bracket in $E \subset A_M^+$. When $\pi = \text{Mat}_N(\mathbb{R})^*$ and $\Sigma$ is compact, one can identify $A_M^+$ and $E$ with the algebras of regular functions on $\text{Hom}(\pi, GL_N(\mathbb{R}))$ and $\text{Hom}(\pi, GL_N(\mathbb{R}))/GL_N(\mathbb{R})$, respectively. In this case, the bracket $\{-,-\}_v^+$ in $E$ is well-known; its extension to $A_M^+$ was first constructed in \([1]\) up to isomorphism. For more on this, see \([5]\).

The Hamiltonian reduction has the following interpretation in the context of surfaces. Assume that $\partial \Sigma \cong S^1$. Attaching a 2-disk to $\Sigma$, we obtain a surface $\Sigma'$. The inclusion $\Sigma \to \Sigma'$ induces an epimorphism $A = \mathbb{K}[\pi] \to B = \mathbb{K}[\pi_1(\Sigma', \star)]$ with kernel $A(\eta - 1)A$ where $\eta \in \pi$ is represented by $\partial \Sigma$. The element $\eta$ is a multiplicative moment map. Corollary \([10.5]\) implies that the Poisson bracket $\{-,-\}_v^+$ in $E$ descends to a Poisson bracket in $p_M^*(E) \subset B_M^+$ which is again well-known when $\pi = \text{Mat}_N(\mathbb{R})^*$ and $\Sigma$ is compact.

11. **Equivariant representation algebras and brackets**

We introduce equivariant representation algebras and construct brackets in them.

11.1. **Algebras over enriched groups.** An *enriched group* $G$ is a group endowed with a distinguished subgroup $G_0$ of index 1 or 2. Given an enriched group $G$, we introduce graded $G$-algebras and $G$-coalgebras as follows. A graded $G$-algebra is a graded algebra $A$ equipped with a left action of $G$ such that the elements of $G_0$ act by graded algebra automorphisms of $A$ and the elements of $G \setminus G_0$ act by graded algebra antiautomorphisms of $A$. Here a graded algebra antiautomorphism of $A$ is a grading-preserving linear isomorphism $f : A \to A$ such that $f(ab) = (-1)^{|a||b|}f(b)f(a)$ for all homogeneous $a, b \in A$.

A map between graded $G$-algebras is *equivariant* if it commutes with the action of $G$. Graded $G$-algebras and equivariant graded algebra homomorphisms form a category denoted $G-\mathcal{G}A$. 
A G-coalgebra is a coalgebra $M$ with comultiplication $\mu$ equipped with a left action of $G$ such that the elements of $G_0$ act by coalgebra automorphisms of $M$ and the elements of $G \setminus G_0$ act by coalgebra antiautomorphisms of $M$. Here an antiautomorphism of $M$ is a linear isomorphism $h : M \to M$ such that $(h \otimes h)\mu = p_{21}h\mu$ where $p_{21} : M \otimes M \to M \otimes M$ is the permutation of the tensor factors.

Given a $G$-coalgebra $M$ and a graded algebra $B$, we provide the convolution algebra $H_M(B) = \text{Hom}_G(M, B)$ with the left action of $G$ by

$$gf = f \circ (g^{-1} : M \to M)$$

for $g \in G$, $f \in H_M(B)$. All $g \in G_0 \subset G$ act on $H_M(B)$ by algebra automorphisms:

$$g(f_1f_2) = (f_1f_2) \circ g^{-1} = m_B(f_1 \otimes f_2)\mu \circ g^{-1} = m_B(f_1 \otimes f_2) \circ (g^{-1} \otimes g^{-1})\mu$$

$$= m_B((f_1 \circ g^{-1}) \otimes (f_2 \circ g^{-1}))\mu = m_B(gf_1 \otimes gf_2)\mu = (gf_1)(gf_2)$$

where $f_1,f_2 \in H_M(B)$ and $m_B : B \otimes B \to B$ is the multiplication in $B$. If $B$ is commutative, then all $g \in G \setminus G_0$ act on $H_M(B)$ by algebra antiautomorphisms: for homogeneous $f_1,f_2 \in H_M(B)$,

$$g(f_1f_2) = m_B(f_1 \otimes f_2)\mu \circ g^{-1} = m_B(f_1 \otimes f_2) \circ p_{21}(g^{-1} \otimes g^{-1})\mu$$

$$= (-1)^{|f_1||f_2|} m_B(f_2 \otimes f_1) \circ (g^{-1} \otimes g^{-1})\mu = (-1)^{|f_1||f_2|}(gf_2)(gf_1).$$

Thus, in this case, $H_M(B)$ is a graded $G$-algebra.

11.2. Equivariant representation algebras. Fix an enriched group $G$, a graded $G$-algebra $A$, and a $G$-coalgebra $M$. Let $A_M^G$ be the commutative graded algebra obtained as the quotient of $A_M$ by the relations $\{(ga)_{\alpha} = a_{\alpha} | a \in A, \alpha \in M\}$. Note the equivalent system of relations $\{(ga)_{\alpha} = a_{g^{-1}\alpha} | a \in A, \alpha \in M\}$.

We call $A_M^G$ the equivariant representation algebra of $A$ with respect to $M$. The construction of $A_M^G$ is functorial: a morphism $f : A \to B$ in the category $G-\mathcal{G}A$ induces a graded algebra homomorphism $f_M^G : A_M^G \to B_M^G$ by $f_M^G(a_{\alpha}) = (f(a))_{\alpha}$ for all $a \in A, \alpha \in M$.

**Lemma 11.1.** For any commutative graded algebra $B$, there is a canonical bijection

$$(11.2.1) \quad \text{Hom}_{CGA}(A_M^G, B) \cong \text{Hom}_{G-\mathcal{G}A}(A, H_M(B))$$

which is natural in $A$ and $B$.

**Proof.** The map $f_M^G$ carries a graded algebra homomorphism $r : A_M^G \to B$ to the linear map $s = s_r : A \to H_M(B)$ defined by $s(a)(\alpha) = r(a_{\alpha})$ for all $a \in A$ and $\alpha \in M$. The proof of Lemma 2.1 shows that $s$ is a graded algebra homomorphism. It is equivariant: $s(ga) = g(s(a))$ for any $g \in G$ and $a \in A$. Indeed, for all $\alpha \in M$,

$$s(ga)(\alpha) = r((ga)_{\alpha}) = r(a_{g^{-1}\alpha}) = s(a)(g^{-1}\alpha) = g(s(a))(\alpha).$$

The map inverse to $f_M^G$ carries an equivariant graded algebra homomorphism $s : A \to H_M(B)$ to the algebra homomorphism $r = r_s : A_M^G \to B$ defined on the generators by $r(a_{\alpha}) = s(a)(\alpha)$. This rule is compatible with the defining relations in $A_M$, see the proof of Lemma 2.1 and with the relations $(ga)_{\alpha} = a_{\alpha}$:

$$r((ga)_{\alpha}) = s(ga)(\alpha) = (gs(a))(\alpha) = s(a)(g^{-1}ga) = s(a)(\alpha) = r(a_{\alpha}).$$

Clearly, the maps $s \mapsto r_s$ and $r \mapsto s_r$ are mutually inverse. \(\square\)
Lemma 11.2 yields the following universal property of $A^G_M$: for any commutative graded algebra $B$ and any equivariant graded algebra homomorphism $s : A \rightarrow H_M(B)$, there is a unique graded algebra homomorphism $r : A^G_M \rightarrow B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \longrightarrow & H_M(A^G_M) \\
\downarrow s & & \downarrow H_M(r) \\
H_M(B). & & 
\end{array}
$$

Here the horizontal arrow is the equivariant graded algebra homomorphism carrying $a \in A$ to the map $M \rightarrow A^G_M, \alpha \mapsto a_\alpha$.

11.3. A bracket in $A^G_M$. Let $G, A, M$ be as in Section 11.2. Under certain assumptions on a double bracket $\{ -, - \}$ in $A$ and a cyclic bilinear form $v$ on $M$, we derive from them a bracket in $A^G_M$. We say that $\{ -, - \}$ is **equivariant** if for all $g \in G, a, b \in A$,

$$\{ ga, gb \} = \left\{ \begin{array}{ll} (g \otimes g)(\{a, b\}) & \text{if } g \in G_0, \\ P_{21}(g \otimes g)(\{a, b\}) & \text{if } g \in G \setminus G_0 \end{array} \right.$$

where $P_{21}$ is the graded permutation in $A \otimes A$ defined in Section 3.2. We say that $v$ is **equivariant** if $v(ga \otimes gb) = v(\alpha \otimes \beta)$ for all $g \in G, \alpha, \beta \in M$.

**Lemma 11.2.** Let $\{ -, - \}_v$ be the $n$-graded biderivation in $A_M$ derived from an equivariant $n$-graded double bracket $\{ -, - \}$ in $A$ and an equivariant cyclic bilinear form $v$ on $M$. Let $q : A_M \rightarrow A^G_M$ be the natural projection. If $G$ is finite, then there is a unique $n$-graded biderivation $\{ -, - \}^G_v$ in $A^G_M$ such that for any $a, b \in A$, $\alpha, \beta \in M$,

$$\{ q(a_\alpha), q(b_\beta) \}^G_v = \sum_{\gamma \in G} q(\{ ga_\alpha, b_\beta \}_v).$$

If $\{ -, - \}$ is a double Gerstenhaber bracket, then $\{ -, - \}^G_v$ is a Gerstenhaber bracket.

**Proof.** The uniqueness of the bracket $\{ -, - \}_v^G$ is obvious, and we need only to prove the existence. We begin with a few computations in the $G$-coalgebra $M$. First of all, for any $g \in G_0$ and $\beta \in M$,

$$(g\beta)^1 \otimes (g\beta)^2 \otimes (g\beta)^3 = \mu^2(g\beta) = g(\beta^1) \otimes g(\beta^2) \otimes g(\beta^3)$$

Using the equivariance of $v$, we deduce that for any $\alpha, \beta \in M$,

$$\hat{v}(g\alpha \otimes g\beta) = v(g\alpha \otimes (g\beta)^2)(g\beta)^1 \otimes (g\beta)^3 = v(g\alpha \otimes g(\beta^2))g(\beta^1) \otimes g(\beta^3) = v(\alpha \otimes \beta^2)(g \otimes g) (\beta^1 \otimes \beta^3) = (g \otimes g) \hat{v}(\alpha \otimes \beta).$$

We rewrite this in the notation of Section 11.1

$$\{ ga, gb \} = \{ ga_\alpha, gb_\beta \} = \{ ga_\alpha, gb_\beta \}^{\beta(\alpha)} = \{ ga_\alpha, g(\beta^\alpha) \}.$$
In the notation of Section 11.1, this gives

(11.3.3) \((ga)_{g\beta} \otimes (g\beta)^{g\alpha} = g(\beta^\alpha) \otimes g(\alpha_\beta)\).  

Next, we define a left action of \(G\) on \(A_M\) by graded algebra automorphisms. Each \(g \in G\) acts on the generators by \(g a_\alpha = (ga)_{ga}\) for \(a \in A, \alpha \in M\). We check the compatibility with the defining relations of \(A_M\). The compatibility with the bilinearity relations is obvious. Consider the multiplicativity relation \((ab)_{\alpha} = a_a b_\beta\) with homogeneous \(a, b \in A\) and \(\alpha, \beta \in M\). If \(g \in G_0\), then

\[\begin{align*}
g(ab)_{\alpha} &= (g(ab))_{ga} = (g(a)g(b))_{ga} = (ga)_{ga} \otimes (gb)_{ga} = g(a_b)g(b_a) = g(a_b)\end{align*}\]

If \(g \in G \setminus G_0\), then

\[\begin{align*}
g(ab)_{\alpha} &= (g(ab))_{ga} = (-1)^{a|b}(gb)(a)_{ga} = (-1)^{a|b}(gb)(ga)^{ga_{ga}} = (ga)_{ga}(gb)_{ga} = g(a_b)g(b_a) = g(a_b)\end{align*}\]

The compatibility with the commutativity relation \(a_a b_\beta = (-1)^{a|b}b_\beta a_a\): for all \(g\),

\[g(a_b) = g(a_a)g(b_\beta) = (ga)_{ga}(gb)_{ga} = (-1)^{a|b}(gb)(ga)_{ga} = (-1)^{a|b}g(b_a a_a)\]

The bracket \((-,-)\) in \(A_M\) is invariant under the action of \(G\). Indeed, it is enough to check that \(\{g(a_{\alpha}), g(b_\beta)\} = g\{a_{\alpha}, b_\beta\}\) for any \(g \in G\), \(a, b \in A\), \(\alpha, \beta \in M\). We have

\[\begin{align*}
\{g(a_{\alpha}), g(b_\beta)\} &= \{(ga)_{ga}, (gb)_{ga}\} = (ga, gb)^{ga}_{ga} = (ga, gb)^{ga}_{ga}\end{align*}\]

Set \(x = x' \otimes x'' = \{a, b\}\). If \(g \in G_0\), then the equivariance of \(\{-,-\}\) and (11.3.2) imply that the right-hand side of (11.3.4) is equal to \((gx')_{g(\alpha)}(gx'')_{g(\beta)}\). If \(g \in G \setminus G_0\), then using (11.3.3) we compute the right-hand side of (11.3.4) to be

\[\begin{align*}
(-1)^{|x'||x''|}(gx'')_{g(\beta)}(gx')_{g(\alpha)} = (gx')_{g(\alpha)}(gx'')_{g(\beta)}\end{align*}\]

In both cases,

\[\begin{align*}
\{g(a_{\alpha}), g(b_\beta)\} &= (gx')_{g(\alpha)}(gx'')_{g(\beta)} = g(x'_{\alpha \beta})g(x''_{\beta \alpha}) = g(x'_{\alpha \beta}x''_{\beta \alpha}) = g\{a_{\alpha}, b_\beta\}\end{align*}\]

It is obvious that \(q(gz) = q(z)\) for all \(g \in G\) and \(z \in A_M\). The \(G\)-invariance of the bracket \((-,-)\) in \(A_M\) implies that for any \(x, y \in A_M\),

\[\begin{align*}
q(x, y) = q(gx, g^{-1}y) = q(x, g^{-1}y)\end{align*}\]

Summing up over all \(g \in G\), we obtain that

\[\begin{align*}
\sum_{g \in G} q(gx, y) = \sum_{g \in G} q(x, gy) \in A_M^G.
\end{align*}\]

Denote this sum by \([x, y]\). This defines a bilinear pairing \([-,-] : A_M \times A_M \to A_M^G\). The \(n\)-graded antisymmetry for \([-,-]\) implies that for homogeneous \(x, y \in A_M\),

\[\begin{align*}
[x, y] &= \sum_{g \in G} q(gx, y) = -(-1)^{|x||y|} \sum_{g \in G} q(y, gx) = -(-1)^{|x||y|} [y, x].
\end{align*}\]

The first Leibniz rule for \([-,-]\) implies that for any homogeneous \(x, y, z \in A_M\),

\[\begin{align*}
[x, yz] &= [x, y]q(z) + (-1)^{|x||y|} q(y)[x, z].
\end{align*}\]

Observe also that for any \(h \in G, a \in A, \alpha \in M\),

\[\begin{align*}
[x, (ha)_{\alpha} - a_\alpha] &= \sum_{g \in G} q((x, (gha)_{ga}) - \sum_{g \in G} q((x, (ga)_{ga}) = 0.
\end{align*}\]
These two formulas imply that $[A_M, \text{Ker } q] = 0$. The antisymmetry of $[-,-]$ yields $[\text{Ker } q, A_M] = 0$. Hence, the pairing $[-,-] : A_M \times A_M \to A_M^G$ descends to an $n$-graded biderivation $\{-,-\}_v^G$ in $A_M^G$, satisfying the conditions of the lemma.

The Jacobi form $\{-,-\}$ of $\{-,-\}_v^G$ can be computed on the generators using Lemma 11.2. For homogeneous $a, b, c \in A$ and any $\alpha, \beta, \gamma \in M$, we obtain

\begin{equation}
(11.3.5) \quad \{q(a_\alpha), q(b_\beta), q(c_\gamma)\} = \sum_{g,h \in G} q(Q(ga, hb, c, go, h\beta, \gamma)) - R(ga, hb, c, go, h\beta, \gamma))
\end{equation}

where $Q, R \in A_M$ are defined by (4.2.4), (4.2.5).

Therefore, if $\{-,-\}$ is Gerstenhaber, then so is $\{-,-\}_v^G$. □

11.4. **The unital case.** Suppose now that the graded $G$-algebra $A$ is unital and the $G$-coalggebra $M$ is counital. It is understood that the action of $G$ on $A$ fixes the unit $1_A$ and the action of $G$ on $M$ fixes the counit $\varepsilon = \varepsilon_M$. We construct a unital graded algebra $A_M^{G^+}$ by adjoining a two-sided unit $e$ to $A_M^G$ and quotienting the resulting algebra $\mathbb{K}e \oplus A_M^G$ by the relations $(1_A)_\alpha = \varepsilon(\alpha)e$ for all $\alpha \in M$. For any unital commutative graded algebra $B$, the bijection (11.2.1) induces a natural bijection

\begin{equation}
(11.4.1) \quad \text{Hom}_{\mathbb{CG}A^+}(A_M^{G^+}, B) \xrightarrow{\cong} \text{Hom}_{G-\mathbb{GA}^+}(A, H_M(B))
\end{equation}

where $\mathbb{CG}A^+$ is the category of unital commutative graded algebras and $G-\mathbb{GA}^+$ is the category of unital $G$-algebras and equivariant graded algebra homomorphisms carrying 1 to 1. The construction of $A_M^{G^+}$ obviously extends to a functor $f \mapsto f_M^{G^+} : G-\mathbb{GA}^+ \to \mathbb{CG}A^+$.

We introduce an equivariant analogue $\mathcal{E} \subset A_M^{G^+}$ of the algebra (10.2.1). To this end, we define a right action of $G$ on $M^*$ by $(\varphi(g))(\alpha) = \varphi(g\alpha)$ for $\varphi \in M^*, g \in G, \alpha \in M$. Let $L$ be the set of all $\varphi \in M^*$ such that $\varphi g = \varphi$ for all $g \in G_0$ and $\varphi g = -\varphi$ for all $g \in G \setminus G_0$. It is easy to see that $L$ is a Lie subalgebra of the Lie algebra $M^*$ and the coderivations $\{\delta_\varphi : M \to M\}_{\varphi \in L}$ are $G$-equivariant. The induced derivations $\{\delta_\varphi\}_{\varphi \in L}$ of $A_M$ commute with the action of $G$ on $A_M$ defined in the proof of Lemma 11.2 and induce an action of $L$ on $A_M^{G^+}$ by derivations. Then

\begin{equation}
\mathcal{E} = \{x \in A_M^{G^+} \mid \delta_\varphi(x) = 0 \text{ for all } \varphi \in L\}
\end{equation}

is a unital graded subalgebra of $A_M^{G^+}$.

Suppose now that $G$ is finite, $\mathcal{E}$ carries an equivariant $n$-graded double bracket, and $M$ carries an equivariant cyclic bilinear form $v$. The bracket $\{-,-\}_v^G$ in $A_M^G$ extends to a bracket in $\mathbb{K}e \oplus A_M$ annihilating $e$ on the left and on the right. The latter bracket annihilates $(1_A)_\alpha - \varepsilon(\alpha)e$ for all $\alpha \in M$ and descends to an $n$-graded biderivation $\{-,-\}_v^{G^+}$ in $A_M^{G^+}$ invariant under the action of the Lie algebra $L$. By definition, for any $a, b \in A, \alpha, \beta, \gamma \in M$,

\begin{equation}
\{q^+(a_\alpha), q^+(b_\beta)\}_v^{G^+} = \sum_{g \in G} q^+(\{(ga)_\alpha, b_\beta\}_v) = \sum_{g \in G} q^+(\{a_\alpha, (gb)_\beta\}_v)
\end{equation}

where $q^+$ is the natural projection $A_M \to A_M^{G^+}$. Moreover, $\{\mathcal{E}, \mathcal{E}\}_v^{G^+} \subset \mathcal{E}$. If $\{-,-\}$ is Gerstenhaber, then so is $\{-,-\}_v^{G^+}$.

We state equivariant versions of Theorems 10.2 and 10.3.
Theorem 11.3. (Equivariant Hamiltonian reduction) Under the assumptions above, let $B$ be a graded $G$-algebra and let $p : A \to B$ be an equivariant algebra epimorphism whose kernel satisfies the same condition as in Theorem 10.2. Then the bracket $\{\cdot, \cdot\}_v^G$ in $E$ descends to an $n$-graded Leibniz bracket in the graded algebra $p^+_M(E) \subset B^+_M$. If $\{\cdot, \cdot\}$ is Gerstenhaber, then so is the latter bracket.

Proof. We define a map $\ell : M^* \to L \subset M^*$ by

$$\ell(\varphi) = \sum_{g \in G_0} \varphi g - \sum_{g \in G \setminus G_0} \varphi g.$$  

We claim that for any $\varphi \in M^*$, $g \in G$, $b \in A$, $\beta \in M$

$$\sum_{g \in G} q^+ \delta \varphi ((gb)_{g\beta}) = q^+ \delta \ell(\varphi)(b\beta).$$  

Indeed, it follows from the definitions that

$$\delta \varphi (g\beta) = \begin{cases} g \delta \varphi (g\beta) & \text{if } g \in G_0 \\ -g \delta \varphi (\beta) & \text{if } g \in G \setminus G_0. \end{cases}$$  

Therefore

$$\sum_{g \in G} q^+ \delta \varphi ((gb)_{g\beta}) = q^+ \left( \sum_{g \in G_0} (gb)_{g\delta \varphi (g\beta)} - \sum_{g \in G \setminus G_0} (gb)_{g\delta \varphi (\beta)} \right)$$

$$= q^+ \left( \sum_{g \in G_0} b_{g\delta \varphi (g\beta)} - \sum_{g \in G \setminus G_0} b_{g\delta \varphi (\beta)} \right) = q^+ (b_{\delta \ell(\varphi)(\beta)}) = q^+ \delta \ell(\varphi)(b\beta).$$  

The rest of the argument follows the proof of Theorem 10.2 with (10.3.1) replaced by

$$\{q^+(\xi_\alpha), x\}_v^G = q^+ \delta \ell(\varphi)(x)$$  

for any $x \in A^G$. We check this equality for $x = q^+(b\beta)$ with $b \in A$, $\beta \in M$:

$$\{q^+(\xi_\alpha), q^+(b\beta)\}_v^G = \sum_{g \in G} q^+ \{\{\xi_\alpha, (gb)_{g\beta}\}_v\} = \sum_{g \in G} q^+ \delta \ell(\varphi)(b\beta).$$

Theorem 11.4. If, under the assumptions above, $A$ is concentrated in degree zero, $n = 0$, and the double bracket in $A$ is quasi-Poisson, then the restriction of the bracket $\{\cdot, \cdot\}_v^G$ to $E \subset A_M$ satisfies the Jacobi identity and makes $E$ into a Poisson algebra.

Proof. To compute the Jacobi form of the bracket $\{\cdot, \cdot\}_v^G$ we use the expansion (11.3.5) with $q$ replaced by $q^+$. To compute the terms of this expansion we use (10.5.2), (10.5.3) and to compute their sum over all $g, h \in G$ we use (11.4.2). The rest of the arguments is as in the proof of Theorem 10.4.
11.5. The involutive case. We focus now on the case where $G$ is a cyclic group of order two with generator $\iota$ and $G_0 \subset G$ is the trivial subgroup. To turn a unital graded algebra $A$ into a $G$-algebra, one needs only to fix an involutive graded antiautomorphism $\iota$ of $A$ such that $\iota(1_A) = 1_A$. An $n$-graded double bracket $\{ - , - \}$ in $A$ is equivariant if and only if for all $a,b \in A$,

\begin{equation}
\{ \iota(a), \iota(b) \} = P_{21}(\iota \otimes \iota)(\{a,b\}).
\end{equation}

Similarly, to turn a counital coalgebra $M$ into a $G$-coalgebra, one needs to fix an involutive antiautomorphism $\iota$ of $M$ such that $\varepsilon_M \iota = \varepsilon_M$. A cyclic bilinear form $v$ on $M$ is equivariant if and only if $v(\iota \otimes \iota) = v : M \otimes M \rightarrow \mathbb{K}$. By Section 11.4, an equivariant $n$-graded double bracket $\{ - , - \}$ in $A$ and an equivariant cyclic bilinear form $v$ on $M$ induce an $n$-graded biderivation $\{-,-\}^+_G$ in $A^G_M$.

Suppose that $A = \mathbb{K}[\pi]$ is the group algebra of a group $\pi$. We treat $A$ as a graded algebra concentrated in degree zero and equip it with the involutive antiautomorphism $\iota$ inverting all elements of $\pi$. Consider a counital $G$-coalgebra $(M,\iota : M \rightarrow M)$ whose underlying module is free of finite rank. For any (non-graded) unital algebra $B$, the convolution algebra $H_M(B) = B \otimes M^*$ is a unital $G$-algebra with involutive antiautomorphism $\text{id}_B \otimes \iota^*$. We identify the set $\text{Hom}_{G,A^*}(A,H_M(B))$ with the set of group homomorphisms from $\pi$ to the group $U_t(B \otimes M^*)$ consisting of all invertible $u \in B \otimes M^*$ such that $u^{-1} = (\text{id}_B \otimes \iota^*)(u)$. The bijection (11.4.1) exhibits $A^G_M$ as the coordinate algebra of the functor $B \mapsto \text{Hom}(\pi,U_t(B \otimes M^*))$ where $B$ runs over unital commutative algebras.

For example, consider a symmetric Frobenius algebra $\mathcal{A}$ equipped with an involutive algebra automorphism $\Delta$ preserving the Frobenius pairing. Pick an invertible matrix $J \in \text{Mat}_N(A)$ with $N \geq 1$ such that $J^t = \Delta(J)$ where $t$ is the matrix transposition and $\Delta$ applies to matrices entry-wise. (One can take $J$ to be the unit matrix or, more generally, a diagonal matrix whose diagonal entries are invertible and fixed by $\Delta$.) The antiautomorphism $X \mapsto J \Delta(X^t)J^{-1}$ of $\text{Mat}_N(A)$ induces an involutive coalgebra antiautomorphism $\iota$ of the coalgebra $M = (\text{Mat}_N(A))^*$ and makes $M$ into a counital $G$-colagebra. The cyclic form $v_N$ on $M$ defined in Example 11.3.5 is equivariant for any choice of $J$. This form can be used to derive an $n$-graded biderivation in $A^G_M$ from an equivariant double bracket in $A$. Note that for a unital commutative algebra $B$, the group $U_t(B \otimes M^*) \subset GL_N(B \otimes A)$ consists of all invertible $(N \times N)$-matrices $X$ over the ring $R_B = B \otimes A$ such that $J = XJ\Delta(X^t)$. The group $U_t(B \otimes M^*)$ is nothing but the group of automorphisms of the free $R_B$-module $(R_B)^N$ of rank $N$ preserving the $R_B$-valued sesquilinear form determined by $J$. For $\Delta = \text{id}_A$, we recover the group of invertible $(N \times N)$-matrices $X$ over $R_B$ such that $J = XJX^t$, i.e., the group of automorphisms of the module $(R_B)^N$ preserving the $R_B$-valued bilinear form determined by $J$.

These constructions apply to the fundamental group $\pi$ of an oriented surface with base point in the boundary. Indeed, the double quasi-Poisson bracket in $A = \mathbb{K}[\pi]$ constructed in [11] satisfies (11.5.1). This bracket and the cyclic form $v_N$ induce a biderivation in $A^G_M$ which restricts to a Poisson bracket in $\mathcal{E} \subset A^G_M$. As in Section 10.6 the equivariant Hamiltonian reduction yields a Poisson bracket in algebras associated with oriented surfaces with empty boundary.

References

[1] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, Quasi-Poisson manifolds. Canad. J. Math. 54 (2002), no. 1, 3–29.
[2] W. Crawley-Boevey, *Poisson structures on moduli spaces of representations*. J. Algebra 325 (2011), 205–215.

[3] J. Kock, Frobenius algebras and 2D topological quantum field theories. London Math. Soc. Student Texts, 59. Cambridge Univ. Press, Cambridge, 2004.

[4] L. Le Bruyn, G. Van de Weyer, *Formal structures and representation spaces*. J. Algebra 247 (2002), no. 2, 616–635.

[5] G. Massuyeau, V. Turaev, *Quasi-Poisson structures on representation spaces of surfaces*. Internat. Math. Research Notices (2012) doi: 10.1093/imrn/rns215.

[6] G. Massuyeau, V. Turaev, *Brackets in loop algebras of manifolds*, in preparation.

[7] C. Procesi, *A formal inverse to the Cayley-Hamilton theorem*. J. Algebra 107 (1987), no. 1, 63–74.

[8] M. Van den Bergh, *Double Poisson algebras*. Trans. Amer. Math. Soc. 360 (2008), no. 11, 5711–5769.

[9] G. Van de Weyer, *Double Poisson structures on finite dimensional semi-simple algebras*. Algebr. Represent. Theory 11 (2008), no. 5, 437–460.

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