THE $A_f$ CONDITION AND RELATIVE CONORMAL SPACES FOR FUNCTIONS WITH NON-VANISHING DERIVATIVE

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Abstract. We introduce a join construction as a way of completing the description of the relative conormal space of function, then apply a recent result of the second author to deduce a numerical criterion for the $A_f$ condition for the case when the function has non-vanishing derivative at the origin.

1. Introduction

The $A_f$ condition is a relative stratification condition defined by Thom, for the study of functions and mappings on stratified sets. It plays an important role in Thom’s second isotopy theorem, and provides a transversality condition in the development of the Milnor fibration.

In this paper we work in the setting of a complex analytic family $(X,0) \to (Y,0)$ with $X \subset \mathbb{C}^k \times \mathbb{C}^n, (0,0)$ assuming that the parameter space $Y$ is embedded in the family $X$ as $\mathbb{C}^k \times 0$. Let $f$ be a function defined on $X$ with $f$ vanishing on $Y$. Suppose $X$ is given as $X = G(y,z)^{-1}(0)$ where $G : (\mathbb{C}^k \times \mathbb{C}^n,0) \to (\mathbb{C}^p,0)$.

Thom’s original definition of the $A_f$ condition ([T69]) was in terms of tangent planes to the fibers of $f$. Using hyperplanes tangent to the fibers of $f$, however, has many advantages. As we will see, this definition connects directly with the derivative of $(f,G)$, and has a nice algebraic description, which lends itself to description by analytic invariants.

The definition we will use of the $A_f$ condition is that it holds for $f$ and $X$ at $(0,0)$ if each limit of tangent hyperplanes to the fibers of $f$ at $(0,0)$ contains $Y$. The condition holds along $Y$ if it holds at every point of $Y$.

The closure of the set of tangent hyperplanes to the smooth part of $X$ is called the conormal space of $X$, denoted $C(X)$, while the closure of the set of tangent hyperplanes to the fibers of $f$ at smooth points of $f$ is called the relative conormal space which we denote by $C(X,f)$. Then the $A_f$ condition holds along $Y$ if $C(X,f)|_Y \subset C(Y)$.

The $A_f$ condition is known to hold generically by a result of Hironaka [H76], and is a necessary condition for showing that families of functions on analytic sets are Whitney equisingular. So it is important to understand the fiber of $C(X,f)$ over the origin and its relation to $C(X,f)|_Y$ ([H76], Thm. 4.2 and Cor. 4.3 in [M00]). Now, suppose $f \in m_Y^2$ and $f \notin m_Y^3$ and let $f'$ be an extension of $f$ to $\mathbb{C}^k \times \mathbb{C}^n$. Then there are some components of the fiber of

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T. Gaffney was partially supported by PVE-CNPq Proc. 401565/2014-9.
$C(X, f)$ over the origin which are the join of $d\tilde{f}(0)$ with components of the fiber of $C(X)$ over the origin. As we discuss below, these components may be hard to control, because their dimension may be small. By numerical control we can ensure that those components of the fiber of $C(X)$ over the origin, which are limits of tangent hyperplanes to $X$ along $X_0$ are in $C(Y)$. The result of the second author ensures that these small components of the fiber of $C(X)$ over the origin are in fact limits along $X_0$.

2. Relative Conormal Spaces

We begin with some constructions and notation. Let $(X, 0)$ be a complex analytic germ in $(\mathbb{C}^n, 0)$. Assume $X$ is contained in an open set $U$ of $\mathbb{C}^n$. Denote by $T_X^*U$ the space obtained by taking the closure of the conormal vectors to the smooth part of $X$ in $\mathbb{C}^n \times \mathbb{C}^{m*}$. As the fibers of $T_X^*U$ over points in $X$ are invariant under multiplication by elements from $\mathbb{C}^*$, we may projectivize $T_X^*U$ with respect to vertical homotheties of $T_X^*U$ and work with $\mathbb{P}(T_X^*U)$. This is precisely the conormal space $C(X)$ described in the introduction.

Suppose $f$ is a function on $X$ and $\tilde{f}$ is an extension of $f$ to $\mathbb{C}^n$. The relative conormal space $C(X, f)$ of $X$ with respect to $f$ as defined in the introduction can be obtained as follows. Let $T_f^*U$ be the closure of all $(x, \eta)$ in $T_X^*U$ where $x$ is a smooth point in $X$ and $\eta(T_xX \cap \ker(d\tilde{f})) = 0$. Then $C(X, f)$ is the projectivization of $T_f^*U$. Note that $C(X, f)$ does not depend on the choice of extension $\tilde{f}$ of $f$ (cf. Sect. 5 in [GK98]). Denote by $c: C(X, f) \to X$ the structure morphism.

The differential $d\tilde{f}$ of $\tilde{f}$ defines an embedding of $X$ in $\mathbb{C}^n \times \mathbb{C}^{m*}$ by the graph map. Let $z_1, \ldots, z_m$ be coordinates on $U$ and $w_1, \ldots, w_n$ be the cotangent coordinates. Then the blowup of $T_X^*U$ along the image of the graph map is the blowup of $T_X^*U$ by the ideal $(w_1 - \frac{\partial \tilde{f}}{\partial z_1}, \ldots, w_n - \frac{\partial \tilde{f}}{\partial z_m})$ in $T_X^*U$. We denote this blowup by $\text{Bl}_{\text{im}(d\tilde{f})}T_X^*U$. Thus, the blowup is contained in $X \times \mathbb{C}^{m*} \times \mathbb{P}^{m-1}$. Denote the exceptional divisor of this blowup by $E_f$. The projection of this exceptional divisor to $X$ is the singular set of $f$ on $X$ denoted $S(f)$.

Let $\pi: X \times \mathbb{C}^{m*} \times \mathbb{P}^{m-1} \to X \times \mathbb{P}^{m-1}$ denote the projection to projective space. Then $\pi(E_f)$ is independent of the extension of $\tilde{f}$ of $f$ by Cor. 2.12 in [M00]. The following result describes the relation between $C(X, f)$ and $\text{Bl}_{\text{im}(d\tilde{f})}T_X^*U$.

Lemma 2.1. The following holds.

(i) $E_f \cong \pi(E_f)$.

(ii) Suppose that the locus where $f$ fails to be a submersion is contained in $X_{\text{sing}}$. Then $\pi(E_f) \subset c^{-1}(X_{\text{sing}})$.

Proof. Part (i) is due to Massey (see the paragraph preceding Lemma 2.6 in [M00]). If $(x, w, \eta)$ is a point in $E_f$, then $w = d\tilde{f}(x)$, so $\pi$ induces an isomorphism between $E_f$ and $\pi(E_f)$.

Consider (ii). By Lemma 2.6 in [M00] (see also the proof of Prop. 2.2 below) it follows that $\pi(E_f) \subset C(X, f)$. But $E_f$ is supported over points
We can compute $G_t$ leading term of $O_g k$ if $O_G$ of $V C$ family of lines parameterized by $C, x$ is a curve on $V$ is independent of the choice of extension of $O$ so is the join, for we can view the join as the inverse image of the projection of $V$ to $\mathbb{P}^{m-2}$ from the point $a$. We denote the join of $a$ and $V$ by $a \ast V$. If $C, x$ is a curve on $X, x$, and $D_i, i = 1, 2$ lifts to $\mathbb{P}^{m-1}$, $D_{1,p} \neq D_{2,p}$, where $D_{1,p}$ is the fiber of $D_1$ over $p \in C, p$ near $x$, then let $(D_1 \ast D_2)C$ denote the family of lines parameterized by $C$ whose fiber over $p$ is $D_{1,p} \ast D_{2,p}$.

Let $x$ be a point in $X$. Suppose $f \notin m^2_{X,x}$. Denote by $\langle d\hat{f}(x) \rangle$ the point of $\mathbb{P}^{m-1}$ determined by $d\hat{f}(x)$. Denote the join of $\langle d\hat{f}(x) \rangle$ and a subset $V$ of $\mathbb{P}^{m-1}$ by $d\hat{f}(x) \ast V$ as well. It is easy to check that $d\hat{f}(x) \ast C(X)_x$ is independent of the choice of extension of $f$ to the ambient space. If $f \in m^2_{X,x}$, then by convention $d\hat{f}(x) \ast C(X)_x$ is empty.

In the next theorem we will be working with limits along curves, so we discuss this a little. Given $G \in \mathcal{O}_C$, $G(t) \neq 0$, for $t \neq 0$, the projective limit of $G$ at $t = 0$ is $\lim_{t \to 0}(G(t))$, which is a point of $\mathbb{P}^{m-1}$.

We can find the projective limit of $G$ by working directly with $G$ as follows. For any $g \in \mathcal{O}_C$ denote by $o(g(t))$ the order of $t$ in $g(t)$. If $G(t) \in \mathcal{O}_C$, then $o(G(t))$ is the minimum of the orders of the component functions $g_i$ of $G(t)$. If $o(G(t)) = k$, then the $p$-tuple whose entries are the coefficients of the degree $k$ terms of the $g_i$ is the leading term of $G$. If we denote the the leading term of $G$ by $L(G)$ then $\langle L(G) \rangle$ is the projective limit of $G$ at $t = 0$. We can compute $L(G)$ as

$$\lim_{t \to 0} \frac{1}{t^k}(G(t)),$$

where $k$ is the order of $G$.

The next proposition is the key to the description here of the relative conormal space. It grew out of an attempt to improve on some work of Massey (cf. Thm. 3.11 in [M07]). In particular, the idea of using the blow-up of the graph of the differential of $f$ to study the relative conormal space is an idea we learned from him.

**Theorem 2.2.** Suppose $(X, 0)$ is the germ of an analytic set and $f : X \to \mathbb{C}$ is non-singular on a Zariski open and dense subset of $X$. Then for each point $x \in X$ we have set-theoretic equality

$$C(X, f)_x = (\pi(E_f))_x \cup d\hat{f}(x) \ast C(X)_x$$

**Proof.** The case $f \in m^2_{X,x}$ is part of Thm. 4.2 in [M00]. Our proof and that of Massey both make use of curves.
We first show that \(C(X,f)_x\) contains \(df(x) \ast C(X)_x\). Suppose \(d\tilde{f}(x) \neq 0\). Suppose \(H \in C(X)_x\) and \(H \neq \langle d\tilde{f}(x) \rangle\). There exists a curve 

\[
\phi = (\phi_1, \phi_2): (\mathbb{C}, 0) \to (X \times \mathbb{P}^{m-1}, x \times H),
\]

such that the hyperplane \(\phi_2(t)\) is tangent to \(X\) at \(\phi_1(t)\), and \(\phi_1(t) \in X - X_{\text{sing}} - S(f)\) for \(t \neq 0\), where \(S(f)\) is the critical locus of \(f\). Denote the image of \(\phi_1\) by \(C\). For \(t\) sufficiently small, \(t \neq 0\), we can assume that the augmented Jacobian module, which is the module generated over \(\mathcal{O}_{X,x}\) by the columns of \((G, \tilde{f})\), has maximal rank because \(\phi_1(t) \in X - X_{\text{sing}} - S(f)\) for \(t \neq 0\). Then \(\langle d\tilde{f}(\phi_1(t)) \rangle\) and \(\phi_2(t)\) give two lifts of \(C\) to \(\mathbb{P}^{m-1}\) and since the rank of the augmented Jacobian module is maximal along \(C\), then \(\langle d\tilde{f} \ast \phi_2 \rangle_C\) is well defined. Since both \(C(X,f)\) and \(\langle d\tilde{f} \ast \phi_2 \rangle\) are Zariski closed and a Zariski open subset of the second lies in the first, then the second lies in the first as well. This implies that \(d\tilde{f}(x) \ast H\) is in \(C(X,f)_x\).

The rest of the proof is related to the fibers of \((C(X,f)\) or \(\text{Bl}_{\text{min}} d\tilde{f} T^*_X U\), so it is convenient to work along curves and take limits. Giving a curve on \(C(X,f)\) at smooth points of \(f\) on \(X\) amounts to giving smoothly varying linear combinations of the rows of the Jacobian matrix of \(F = (G, \tilde{f})\) where \(X = G^{-1}(0)\) and projectivising. We can make this precise as follows: if \((x, H) \in C(X,f)_x\) then there exist curves \(\phi, \psi\) such that \(\phi: (\mathbb{C}, 0) \to (X, x)\), and 

\[
\psi = (\psi_1, \psi_2): (\mathbb{C}, 0) \to \mathbb{C}^p \times \mathbb{C}
\]

where \(p\) is the number of equations that cut out \((X, 0)\) in \((\mathbb{C}^m, 0)\). Then \(H\) is the projective limit of the curve 

\[
(1) \quad \psi_1(t) \cdot DG(\phi(t)) - \psi_2(t) d\tilde{f}(\phi(t))
\]

where \(\psi_2(t)\) is taken with a minus sign for convenience of comparison with the blow-up construction.

We can do something similar for \(l \in \pi(E_j)_x\). Namely, we can use \(\phi\) and \(\psi\) as before with \(\psi_2 = 1\). Then \((x, l)\) is a point of \(\pi(E_j)_x\) if and only we can find curves \(\phi, \psi\) such that \(\phi(0) = x\) and \(l\) is the projective limit of the curve 

\[
\psi_1(t) \cdot DG(\phi(t)) - d\tilde{f}(\phi(t)).
\]

This description shows that \(\pi(E_j)_x\) is also contained in \(C(X,f)_x\).

Now suppose \((x, H) \in C(X,f)_x\). Then there exist curves \(\phi, \psi\) such that \(H\) is the projective limit of a curve of type \(\mathbb{I}\).

We deal separately with the cases where \(f \in m^2_{X,x}\) and \(f \notin m^2_{X,x}\). Assume \(f \in m^2_{X,x}\) and that \(\tilde{f}\) is chosen in such a way so that \(d\tilde{f}(x) = 0\). If 

\[
o(\psi_2(t)) < o(\psi_1(t) \cdot DG(\phi(t))),
\]

then \(\frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t))\) gives a lift of \(\phi\) to \(T^*_X U\) and the projective limit of \(\frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t)) - d\tilde{f}(\phi(t)) = H\), showing that \(H\) lies in \(\pi(E_j)_x\).

We have \(o(d\tilde{f}(\phi(t))) \geq 1\), because \(f \in m^2_{X,x}\). Suppose 

\[
o(\psi_2(t)) \geq o(\psi_1(t) \cdot DG(\phi(t))).
\]

Then 

\[
o(\psi_2(t)d\tilde{f}(\phi(t))) > o(\psi_1(t) \cdot DG(\phi(t))),
\]
so $H$ is the projective limit of $\psi_1(t) \cdot DG(\phi(t))$. Hence $H$ is a limiting tangent hyperplane to $X$.

There are two subcases, depending on the order $d\bar{f}(\phi)$.

If the order of the components of $d\bar{f}(\phi)$ is greater than 1, then we use the same $\bar{f}$ but replace $\psi_1(t)$ by $\psi_1(t)/t^k$ where $k$ is chosen so that the order of $(\psi_1(t)/t^k) \cdot DG(\phi(t))$ is greater than 0, but less than the order of $d\bar{f}(\phi(t))$. We may take $k = \text{max}\{0, o(\psi_1(t) \cdot DG(\phi(t)) - o(d\bar{f}(\phi(t)) - 1, 0)\}$. Then again $(\psi_1(t)/t^k) \cdot DG(\phi(t))$ provides a lift of $\phi$ to $T^*_XU$, and the projective limit of $(\psi_1(t)/t^k) \cdot DG(\phi(t)) - d\bar{f}(\phi(t))$ is again $H$. If the order of the components of $d\bar{f}(\phi(t))$ is 1, then we re-parameterize $\phi(t)$ so that the order of $d\bar{f}(\phi(t))$ is again greater than 1 and repeat the argument. This finishes the first case.

Now suppose $f \notin m^2_{X,x}$. If $o(\psi_2(t)) < o(\psi_1(t) \cdot DG(\phi(t)))$, then $H = \langle d\bar{f}(x) \rangle$. If the order relation is reversed, then $H$ is in $C(X)_x$. In either case $H \in d\bar{f}(x)^* C(X)_x$, unless $C(X)_x = \langle d\bar{f}(x) \rangle$.

If $o(\psi_2(t)) = o(\psi_1(t) \cdot DG(\phi(t)))$, then $H \in d\bar{f}(x)^* C(X)_x$ unless

$$\lim_{t \to 0} \frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t)) = d\bar{f}(x).$$

In this case we can again get a lift of $\phi$ to $T^*_XU$, using $\frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t))$, so again $H$ is in $\pi(E^*_f)_x$.

It remains to deal with the case where $C(X)_x = \langle d\bar{f}(x) \rangle$. We need to show that $\langle d\bar{f}(x) \rangle$ lies in $\pi(E^*_f)_x$. Since the dimension of $C(X)_x$ is zero, $X$ must be a hypersurface, and by a change of coordinates we may assume $f$ is a linear form. There exists $\phi$ such that the projective limits of $DG(\phi(t))$ and of $tDG(\phi(t))$ are both $\langle d\bar{f}(x) \rangle$. Let $k = o(DG(\phi(t)))$. Then

$$\lim_{t \to 0} \left(1 + \frac{t}{t^k} DG(\phi(t)) - d\bar{f}(\phi(t))\right) = \langle d\bar{f}(x) \rangle$$

which completes the proof. \hfill $\square$

In checking the $A_f$ condition at the origin in a family, we need to show that $C(X, f)_0 \subset U^*_X$ consists of hyperplanes which contain the tangent plane to $Y$ at the origin. The previous theorem shows that the components of $C(X, f)_0$ are of two types—blowup components and join components. Blowup components have large dimension, and can be detected and controlled numerically. However, $C(X)_0$ may contribute small components of join type when $f \notin m^2_{X,0}$. In the next section we prove a theorem which shows that these join components can be controlled using the fiber $X_0$.

3. Fibers of Generalized Conormal Spaces

Let $h : (X, 0) \to (Y, 0)$ be a complex analytic family such that $X$ is equidimensional and for each closed point $y \in Y$ the fibers $X_y$ are equidimensional of positive dimension $d$. Suppose $Y$ is irreducible and Cohen–Macaulay.

The purpose of this section is to understand the relation between the closed subscheme of the conormal $C(X)$ which set-theoretically consists of limits of hyperplanes through points of $X_0$, and the fiber $C(X)_0$ over $0 \in Y$ of the conormal $C(X)$. Our treatment is more general. The conormal $C(X)$ is the Projan of the Rees algebra of the Jacobian module of $X$ (cf. Sect. 1.5
in \[\text{KT00}\). Instead of working with conormal spaces, we work below with Projs of Rees algebras of modules.

Let \(\mathcal{M}\) be an \(\mathcal{O}_X\)-module contained in a free module \(\mathcal{F} := \mathcal{O}_X^n\) such that \(\mathcal{M}\) is free of rank \(e\) off a closed subset \(S\) of \(X\). Further, assume \(S\) is finite over \(Y\). Set \(r := d + e - 1\).

Form the symmetric algebra \(\text{Sym}(\mathcal{F})\) of \(\mathcal{F}\) and the Rees algebra \(\mathcal{R}(\mathcal{M})\) of \(\mathcal{M}\) which is the subalgebra of \(\text{Sym}(\mathcal{F})\) generated by \(\mathcal{M}\) placed in degree 1. Denote the \(k\)th graded components of these algebras by \(\mathcal{F}^k\) and \(\mathcal{M}^k\) respectively. Given a closed point \(y \in Y\) denote by \(\mathcal{M}^k(y)\) the image of \(\mathcal{M}^k\) in the free \(\mathcal{O}_{X_y}\)-module \(\mathcal{F}^k(y)\).

Set \(C := \text{Proj}(\mathcal{R}(\mathcal{M}))\). Denote by \(c: C \rightarrow X\) be the structure morphism. Let \(y\) be a closed point in \(Y\). Set \(C(y) := \text{Proj}(\mathcal{R}(\mathcal{M}(y)))\) and denote by \(C_y\) the fiber of \(h \circ c\) over \(y \in Y\). For an irreducible component \(V\) of \(C_y\) we say it is \(\text{horizontal}\) if it surjects onto an irreducible component of \(X_y\) or we say it is \(\text{vertical}\) otherwise.

**Theorem 3.1.** Suppose \(\dim c^{-1} 10 < r\). Then there exists a Zariski open neighborhood \(U\) of 0 in \(Y\) such that for each \(y \in U\) the irreducible components of \(C_y\) are horizontal. Furthermore, if \(\mathcal{M}\) is a direct summand of \(\mathcal{F}\) locally off \(S\), then we have an equality of fundamental cycles

\[
[C_y] = [C(y)].
\]

**Proof.** Because \(h\) has equidimensional fibers of dimension \(d\) and \(X\) is equidimensional, then \(\dim X = d + \dim Y\). Also, by assumption \(d > 0\) and \(S\) is finite over \(Y\). Thus \(S\) is nowhere dense in \(X\). Let \(x\) be a point in \(X\) with \(x \notin S\). Because the formation of Rees algebra commutes with flat base change we have \(\mathcal{R}(\mathcal{M})(x) = \mathcal{R}(\mathcal{M}_x)\). But \(\mathcal{M}_x\) is free of rank \(e\) because \(x \notin S\). Thus \(\mathcal{R}(\mathcal{M}_x) = \text{Sym}(\mathcal{O}_{X,x}^e)\) whence \(\dim c^{-1} x = e - 1\). The dimension formula applied for each irreducible component of \(C\) yields that \(C\) is equidimensional and \(\dim C = r + \dim Y\).

Let \(C'_y\) be an irreducible component of \(C_y\). Because \(Y\) is Cohen–Macaulay locally at each closed point \(y \in Y\), then by Krull’s height theorem \(C'_y\) is of codimension at most \(\dim Y\). Because \(C\) is of finite type over a field we get

\[
\dim C'_y \geq r.
\]

Replace \(X\) with Zariski neighborhood of 0 so that \(\dim c^{-1} x < r\) for each point \(x \in X\). Let \(U\) be a Zariski open subset of the image of \(h\) that contains \(0 \in Y\). Let \(y\) be a point in \(U\). Suppose that \(c\) maps \(C'_y\) to a point \(\zeta \in X_y\). Then \(C'_y \subset c^{-1} \zeta\). But \(\dim c^{-1} \zeta < r\) which contradicts with (3). But \(S_y\) is zero-dimensional. Thus there exists a Zariski open dense subset \(Z_y\) of \(C'_y\) whose image under \(c\) misses \(S_y\).

Let \(\zeta \in c(Z_y)\). Then \(\mathcal{M}_\zeta\) is free of rank \(e\). Thus \(\dim c^{-1} \zeta = e - 1\). By the dimension formula

\[
\dim C'_y = \dim c(C'_y) + e - 1.
\]

But \(\dim C'_y \geq r\). So \(\dim c(C'_y) \geq d\). Because \(c(C'_y) \subset X_y\) and \(\dim X_y = d\), then \(c(C'_y)\) is an irreducible component of \(X_y\) which proves the first claim of the theorem.
Next, assume $\mathcal{M}$ is locally a direct summand of $\mathcal{F}$ off $S$. Set $X = \text{Spec}(R)$ and $Y = \text{Spec}(Q)$. Then morphism $h$ induces a ring homomorphism $h^\#: Q \to R$. Denote by $\mathfrak{n}_y$ the image under $h^\#$ of the ideal of $y$ in $Q$. Consider the homomorphism

$$\phi_y : \mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M}) \to \text{Sym}(\mathcal{F}(y))$$

Denote its kernel by $I_{\phi_y}$. Observe that

$$(\mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M}))/I_{\phi_y} = \mathcal{R}(\mathcal{M}(y)).$$

Let’s identify $I_{\phi_y}$. Consider the homomorphism

$$\tilde{\phi}_y : \mathcal{R}(\mathcal{M}) \to \text{Sym}(\mathcal{F}(y)).$$

We have $\text{Ker}(\tilde{\phi}_y) = \mathfrak{n}_y\text{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M})$. By definition $I_{\phi_y}$ is the kernel of $\phi_y$. As $\tilde{\phi}_y$ factors through $\phi_y$ we get

$$I_{\phi_y} = (\mathfrak{n}_y\text{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M}))/\mathfrak{n}_y\mathcal{R}(\mathcal{M}).$$

Because the source of $\phi_y$ is $X_y$, then $\text{Supp}(I_{\phi_y}) \subset X_y$. Let $x \in X_y$ with $x \notin S_y$. Then $\mathcal{M}$ is locally a direct summand of $\mathcal{F}$ at $x$. Write $\mathcal{F}_x = \mathcal{M}_x \oplus L(x)$. The formation of symmetric and Rees algebras commutes with localization, hence

$$(\mathfrak{n}_y\text{Sym}(\mathcal{F}))_x = \mathfrak{n}_y\text{Sym}(\mathcal{F}_x) \text{ and } \mathcal{R}(\mathcal{M})_x = \mathcal{R}(\mathcal{M})_x.$$  

On the other hand,

$$\mathfrak{n}_y\text{Sym}(\mathcal{F}_x) = \mathfrak{n}_y\text{Sym}(\mathcal{M}_x) \otimes \mathfrak{n}_y\text{Sym}(L(x)).$$

Because $\mathcal{M}_x$ is free we have $\text{Sym}(\mathcal{M}_x) = \mathcal{R}(\mathcal{M}_x)$. Hence, locally at $x$ the ideals $\mathfrak{n}_y\mathcal{R}(\mathcal{M})$ and $\mathfrak{n}_y\text{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M})$ agree. Finally, we obtain that if $I_{\phi_y}$ is nonzero, then it is supported at points from $S_y$ only. In particular, $I_{\phi_y}$ vanishes locally at the minimal primes of $\mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M})$ because as we showed above each of these minimal primes contracts to a minimal prime of $X_y$. Therefore, $C(y)$ and $C_y$ differ by vertical embedded components supported over $S_y$. This proves $[2]$. \hfill \Box

**Remark 3.2.** Note that in general without assuming the bound on the dimension of $c^{-1}0$, the proof above shows that $C(y) = (C_y - W)^-$ for any $y \in Y$ where $W$ is the union of irreducible components of $C_y$ surjecting on $S_y$.

A more general version of the theorem above without assuming that $S$ is finite over $Y$ can be derived using Bertini’s theorem for extreme morphisms from [RIS]. The direct summand assumption can be relaxed at the expense of mild hypothesis on $X$ as remarked at the end of Sect. 2 in [RIS].

### 4. The $A_f$ Condition and the Main Result

Let $(X, 0)$ be a complex analytic set germ with $X = G^{-1}(0)$ where $G : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^p, 0)$. Assume $Y \subset X$ is smooth of dimension $k$. The Jacobian module $JM(X)$ of $X$ is the submodule of $\mathcal{O}_X^p$ generated by the partial derivatives of $G$. It is a direct summand of $\mathcal{O}_X^p$ locally off the singular locus of $X$. Denote the smooth part of $X$ by $X_{\text{sm}}$. Suppose $f$ is a function on $X$ and $\tilde{f}$ is an extension of $f$ to the ambient space. Define $H = (G, \tilde{f})$, and let $JM(H)$ denote the $\mathcal{O}_{X,x}$-module defined by the partial derivatives
of $H$. Note that $JM(H)$ is independent of the choice of extension of $f$ by the discussion in the beginning of Sct. 5 in [GK98]. Finally, denote by $c$ the structure morphism $c : C(X, f) \to X$ and by $C(Y)$ the conormal space of $Y$ in $\mathbb{C}^{n+k}$.

We review briefly the connection between the theory of integral closure of modules and the $A_f$ condition.

Recall that given a submodule $M$ of a free $O_{X,0}$ module $F$, we say that $u \in F$ is strictly dependent on $M$ and we write $u \in M^\dagger$, if for all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$, $\phi^* u$ is contained in $\phi^* (M)m_1$, where $m_1$ is the maximal ideal of $O_{\mathbb{C},0}$.

**Proposition 4.1.** Let $f$ be a function on $X$ such that $Y \subset f^{-1}(0)$. Then the following are equivalent

1) The $A_f$ condition holds for the pair $X_{\text{sm}}, Y$ at 0.

2) $c^{-1}(Y) \subset C(Y)$.

3) $\frac{\partial H}{\partial y_j} \in JM(H)^\dagger$ for all $j = 1, \ldots, k$.

**Proof.** The equivalence of i) and ii) is obvious; the equivalence of i) and iii) is Lemma 5.1 of [GK98].

A similar result holds for the Whitney $A$ condition. The condition we need for our main result is a much weaker version of Whitney $A$.

**Definition 4.2.** Given a family of spaces $(X, 0) \to (Y, 0)$ with $Y \subset X$ and $X = G^{-1}(0)$, then the infinitesimal Whitney $A$ fiber condition holds at $(0,0)$ if $\frac{\partial G}{\partial y_j} \in JM(X_0)^\dagger$ for all $j = 1, \ldots, k$.

This condition is equivalent to asking that limiting tangent hyperplanes to $X$ along curves on $X_0$ contain the tangent space to $Y$ (cf. Lemma 4.1 in [GK98]). So it is much weaker than asking that Whitney $A$ hold for the pair $(X_{\text{sm}}, Y)$ at $(0,0)$, which would require looking at all curves on $X$ passing through the origin.

We show how weak the infinitesimal Whitney $A$ condition is by considering a family of examples due to Trotman (see Prop. 5.1, p. 147 in [Tr86]). In these examples, the members of the families are the same, but the total space is different. The examples were used to show that a necessary and sufficient fiberwise condition for Whitney $A$ was impossible.

**Example 4.3.** Consider the family of plane curves with parameter $y$ given by $w^a - y^b v^c - v^d = 0$, so $X_0$ is the curve defined by $w^a - v^d = 0$ and $y = 0$. Then the infinitesimal Whitney $A$ fiber condition holds at $(0,0)$ if $b > 1$, for all $a, c, d$, because on $X_0$ we have $\frac{\partial G}{\partial y} = 0$. If $b = 1$, the condition holds if $c > \min\{d - 1, d - d/a\}$. This follows because it is only necessary to check the condition on the normalization of the fiber over zero, which is weighted homogeneous.

**Theorem 4.4.** Suppose $Y \subset X \subset Y \times \mathbb{C}^n$ is an equidimensional family with equidimensional fibers with isolated singular locus $Y$. Suppose $f$ is a family of functions with $Y \subset f^{-1}(0)$ and $f$ non-singular on $X - Y$. Suppose $\dim C(X, f)_0 < n$, and the infinitesimal Whitney $A$ fiber condition holds at $(0,0)$. Then $A_f$ holds for the pair $(X_{\text{sm}}, Y)$ at $(0,0)$. 

Proof. We need to show that $C(X, f)_0 \subset C(Y)$. By Theorem 2.2 we know the components of $C(X, f)_0$ are of two types: the blow-up components $\pi(E_f)_0$ and the join components $d\tilde{f}(0) \ast C(X)_0$ if $d\tilde{f}(0) \neq 0$. We will show that $\pi(E_f)_0$ surjects onto $Y$, while the join components are controlled by the infinitesimal Whitney A fiber condition.

We claim that the dimension of $\pi(E_f)_0$ is $\dim \text{Bl}_{\text{im}} d\tilde{f}^* U - 1 = n + k - 1$, where $U$ is a neighborhood of 0 in $\mathbb{C}^{n+k}$ that contains $X$. Indeed, by Lemma 2.1 (i) $\pi(E_f)$ is isomorphic to $E_f$, and $E_f$ is of pure dimension $n + k - 1$.

Now, by Lemma 2.1 (ii) $\pi(E_f)_0$ is supported over $Y$. By assumption, $\dim \pi(E_f)_0 \leq n - 1$. Hence, by upper semi-continuity $\dim \pi(E_f)_y \leq n - 1$ for each $y$ in a neighborhood of 0. But $\dim Y = k$. Thus the dimension formula implies that the irreducible components of $\pi(E_f)_0$ surject onto $Y$. Since the $A_f$ condition holds generically on $Y$, each irreducible component of $\pi(E_f)_0$ is generically contained in $C(Y)$. Hence each such component lies in $C(Y)$. In particular, $\pi(E_f)_0$ is contained in $C(Y)$.

Now we turn to the join components. Consider a component $V$ of the fiber of $C(X)_0$. Since $\dim d\tilde{f}(0) \ast V < n$, then $\dim V < n - 1$. Since $Y \subset f^{-1}(0)$, then $d\tilde{f}(0) \in C(Y)$. So it suffices to show that $V \subset C(Y)$. Apply Theorem 3.1 with $M := JM(X)$. Then $C$ is the conormal space $C(X)$ of $\pi_y$. Also, $C_0$ is the fiber of $C(X)$ over $0 \in Y$ and $C(0)$ is the closed subscheme of $C(X)$ that consists of all limits of tangent hyperplanes through points of $X_0$ only, and $r = n - 1$. Since the dimension of $V$ is less than $n - 1$ then by Theorem 3.1 it follows that $V$ consists of limits of tangent hyperplanes at 0 along curves on $X_0$. Thus the infinitesimal Whitney A fiber condition implies that $V$ is in $C(Y)$.

The usefulness of the last theorem rests on our ability to control the dimension of $\dim C(X, f)_0$ by numerical means. We give an example improving a result from [GR16] that shows how this works. For another example see Thm. 1.8.2 in [R17]. Recall that we say $X \subset \mathbb{C}^{n+k}$ is a determinantal singularity if the ideal of $X$ is generated by the minors of fixed size of an $(l+q) \times l$ matrix with entries in $\mathcal{O}_{n+k}$, and $X$ has the expected codimension.

The matrix is called the presentation matrix of $X$ and is denoted $M_X$.

**Theorem 4.5.** Suppose $(X, 0) \subset (\mathbb{C}^{n+k}, 0)$, is a family of determinantal singularities with presentation matrix $M_X : \mathbb{C}^n \rightarrow \text{Hom}(\mathbb{C}^l, \mathbb{C}^{l+q})$, defined by the maximal minors of $M_X$. Suppose $X = G^{-1}(0)$, where $G : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^p, 0)$ with $Y$ a smooth subset of $X$, coordinates chosen so that $\mathbb{C}^k \times 0 = Y$. Assume $X$ is equidimensional with equidimensional fibers of the expected dimension and $X$ is reduced.

Suppose $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ and set $Z = f^{-1}(0)$. Suppose the the infinitesimal Whitney A fiber condition holds at $(0, 0)$ if $f \notin m_Y^2$.

A) Suppose $X_y$ and $Z_y$ are isolated singularities, suppose the singular set of $f$ is $Y$. Suppose $c_Y(JM(G_y; f_y), \mathcal{O}_{n+k} \oplus N_Y(y))$ is independent of $y$. Then the union of the singular points of $f_y$ is $Y$, and the pair of strata $(X - Y, Y)$ satisfies Thom’s $A_f$ condition.
B) Suppose the critical locus of \( f \) is \( Y \) or is empty, and the pair \((X - Y, Y)\) satisfies Thom’s \( A_f \) condition. Then \( e_\Gamma(JM(G; f_y), \mathcal{O}_{n+k} \oplus N_D(y)) \) is independent of \( y \).

The invariant \( e_\Gamma(JM(G; f_y), \mathcal{O}_{n+k} \oplus N_D(y)) \) is calculated as follows. The module \( JM(G; f_y) \) is the restriction to the fiber \( X_y \) of the augmented Jacobian module \( H \) as defined in the beginning of the section. The module \( N_D(y) \) is the module of first order infinitesimal deformations of \( X_y \) coming from the deformations of the presentation matrix \( M_{X_y} \). Then the multiplicity of the pair of modules \((JM(G; f_y), \mathcal{O}_{l+q} \oplus M_{X_y}^\ast(JM(\Sigma_l)))\) and the intersection number of the image of \( M_{X_y} \) with a polar of \( \Sigma_l \) of complementary dimension to \( n \) sum to \( e_\Gamma(JM(G; f_y), \mathcal{O}_{n+k} \oplus N_D(y)) \). Here \( \Sigma_l \) is the \((l + q) \times l\) matrices of kernel rank \( l \). In [GR16], it is shown that the independence from parameter of this invariant implies that \( H \) has no polar variety of dimension \( k \). In turn this implies \( C(X, f)_0 \) has no component of dimension \( n \) or more. If \( f \in m_Y^2 \), this implies the result of [GR16]. If \( f \notin m_Y^2 \), then it allows us to use Thm. 4.4 to prove the above strengthening of the result of [GR16]. In a similar way, Theorems 5.3, 5.4 of [G09], and Theorem 5.6 of [G02] can be strengthened, dropping the hypothesis of \( f \in m_Y^2 \).

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