SOME ENDMORPHISMS OF THE HYPERFINITE $II_1$ FACTOR

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Abstract. For any finite dimensional $C^*$-algebra $A$ with any trace vector $\vec{s}$ whose components are rational numbers, we give an endomorphism $\Phi$ of the hyperfinite $II_1$ factor $R$ such that:

$$\forall k \in \mathbb{N}, \Phi^k(R)' \cap R = \otimes^k A.$$ 

The canonical trace $\tau$ on $R$ extends the trace vector $\vec{s}$ on $A$.

As a corollary, we construct a one-parameter family of inclusions of hyperfinite $II_1$ factors $N^\lambda \subset M^\lambda$ with trivial relative commutant $(N^\lambda)' \cap M^\lambda = \mathbb{C}$ and with the Jones index

$$[M^\lambda : N^\lambda] = \lambda^{-1} \in (4, \infty) \cap \mathbb{Q}$$

This partially solves the problem of finding all possible values of indices of subfactors with trivial relative commutant in the hyperfinite $II_1$ factor, by showing that any rational number $\lambda^{-1} > 4$ can occur.

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1. INTRODUCTION

Subfactor theory [3] is to describe the position of a subfactor $N$ in an ambient factor $M$. The standard invariant associated with Jones...
is a complete invariant in the amenable case [8]. To classify the standard invariant is the central topic ever since V. Jones founded the subfactor theory.

For a hyperfinite II$_1$ subfactor of finite Jones index, it is equipped with an extra structure: an endomorphism $\Phi$, sending the ambient factor $M$ onto the subfactor $N$. Therefore it is only natural to investigate the role of the endomorphism.

A well-known example of endomorphisms is the canonical shift in a strongly amenable inclusion. Another surprising example is the binary shift [5] which gives rise to a counterexample that fails the tensor product formula for entropy [4]. Via the Cuntz algebra, a lot of endomorphisms have been manufactured.

The Jones index $[M : \Phi(M)]$ is an outer-conjugacy invariant for endomorphisms. In the case of finite Jones index, there is a distinguished outer-conjugacy invariant: the tower of inclusions of finite dimensional $C^*$ algebras, $\{A_k = \Phi^k(M)' \cap M\}_{k=1}^\infty$. The main part of the paper is to investigate the above invariant.

We prove that for any finite dimensional $C^*$-algebra $A$ with any trace vector $\vec{s}$ whose components are rational numbers, there exists an endomorphism $\Phi$ of the hyperfinite II$_1$ factor $R$ such that:

$$\forall k \in \mathbb{N}, \Phi^k(R)' \cap R = \otimes^k A.$$ 

The canonical trace $\tau$ on $R$ extends the trace vector $\vec{s}$ on $A$.

Due to the idiopathic behavior of Powers’ binary shift, our main result has an unexpected feedback to its origin: the classification of hyperfinite II$_1$ subfactors. In short, there is an analogy between Powers’ binary shift and free product with amalgamation.

As an application, we partially solve the problem of finding all possible values of indices of subfactors with trivial relative commutant in the hyperfinite II$_1$ factor, by showing that any rational number $\lambda^{-1} > 4$ can occur.

2. Preliminaries

Let $M$ be a II$_1$ factor with the canonical trace $\tau$. Denote the set of unital *-endomorphisms of $M$ by $\text{End}(M, \tau)$. Then $\Phi \in \text{End}(M, \tau)$ preserves the trace and $\Phi$ is injective. $\Phi(M)$ is a subfactor of $M$. If there exists a $\sigma \in \text{Aut}(M)$ with $\Phi_1 \cdot \sigma = \sigma \cdot \Phi_2$ for $\Phi_i \in \text{End}(M, \tau)$
(i = 1, 2) then \( \Phi_1 \) and \( \Phi_2 \) are said to be conjugate. If there exists a \( \sigma \in \text{Aut}(M) \) and a unitary \( u \in M \) such that \( Adu \cdot \Phi_1 \cdot \sigma = \sigma \cdot \Phi_2 \), then \( \Phi_1 \) and \( \Phi_2 \) are outer conjugate.

The Jones index \([M : \Phi(M)]\) is an outer-conjugacy invariant. We consider only the finite index case unless otherwise stated. In such case, there is a distinguished outer-conjugacy invariant: the tower of inclusions of finite dimensional \( C^* \) algebras, \( \{A_k = \Phi^k(M)' \cap M\}_{k=1}^\infty \).

**Lemma 1.** \( A_k = \Phi^k(M)' \cap M \) contains a subalgebra that is isomorphic to \( \bigotimes_{i=1}^k A_i \), the \( k \)-th tensor power of \( A_1 \), where \( A_1 = \Phi(M)' \cap M \) as denoted.

**Proof.** We collect some facts here, for \( 1 \leq i < k \):

1. \( \Phi^i(A_1) \) is isomorphic to \( A_1 \), since \( \Phi^i \) is injective.
2. \( A_1 \cap \Phi^i(A_1) \subseteq \Phi(M)' \cap \Phi(M) = \mathbb{C} \).
3. \( [A_1, \Phi^i(A_1)] = 0 \) by the definition of \( A_1 \).
4. \( \Phi^i(A_1) \subseteq A_{i+1} \subseteq A_k \).

**Remark.** The dimension of the relative commutant \( \Phi^k(M)' \cap M \) is known to be bounded above by the Jones index \([M : \Phi(M)]^k\). The above lemma provides the lower bound for the growth estimate.

A good example is the canonical shift [1] on the tower of higher relative commutants for a strongly amenable inclusion of \( II_1 \) factors of finite index. The ascending union of higher relative commutants gives the hyperfinite \( II_1 \) factor, and the canonical shift can be viewed as a \( * \)-endomorphism on the hyperfinite factor. Lemma 1 is nothing but the commutation relations in S.Popă's \( \lambda \)-lattice axioms [9].

A natural question arises with the above observation: for any finite dimensional \( C^* \)-algebra \( A \), can we find a \( II_1 \) factor \( M \) and a \( \Phi \in \text{End}(M, \tau) \) such that for all \( k \in \mathbb{N} \),

\[
\Phi^k(M)' \cap M \simeq \bigotimes_{i=1}^k A_i, \text{ where } A_i \simeq A?
\]

The answer is positive and furthermore we can choose \( M \) to be the hyperfinite \( II_1 \) factor \( R \). We give the construction in the next section. The main technical tool in the construction is [5] R.Powers' binary shifts. We provide here the details of \( n \)-unitary shifts generalized by [2] M.Choda for the convenience of the reader.

Let \( n \) be a positive integer. We treat a pair of sets \( Q \) and \( S \) of integers satisfying the following condition \((*)\) for some integer \( m \):

\[
(*) \begin{align*}
Q &= (i(1), i(2), \cdots, i(m)), \quad 0 \leq i(1) < i(2) < \cdots < i(m), \\
S &= (j(1), j(2), \cdots, j(m)), \quad j(l) \in \mathbb{N}, \quad j(l) \leq n - 1 \\
\text{for } l = 1, 2, \cdots, m.
\end{align*}
\]
Definition 1. A unital $*$-endomorphism $\Psi$ of $R$ is called an $n$-unitary shift of $R$ if there is a unitary $u \in R$ satisfying the following:

1. $u^n = 1$;
2. $R$ is generated by $\{u, \Psi(u), \Psi^2(u), \ldots\}$;
3. $\Psi^k(u)u = u\Psi^k(u)$ or $\Psi^k(u)u = \gamma u\Psi^k(u)$ for all $k = 1, 2, \ldots$, where $\gamma = \exp(2\pi \sqrt{-1}/n)$.
4. for each $(Q, S)$ satisfying $\ast$, there are an integer $k (\geq 0)$ and a nontrivial $\lambda \in \mathbb{T} = \{\mu \in \mathbb{C}; |\mu| = 1\}$ such that

   $$\Psi^k(u)u(Q, S) = \lambda u(Q, S)\Psi^k(u),$$

where $u(Q, S)$ is defined by

   $$u(Q, S) = \Psi^{i(1)}(u)^{j(1)}\Psi^{i(2)}(u)^{j(2)}\ldots\Psi^{i(m)}(u)^{j(m)}.$$

The unitary $u$ is called a generator of $\Psi$. Put $S(\Psi; u) = \{k; \Psi^k(u)u = \gamma u\Psi^k(u)\}$. Note that the above condition (2) gives some rigidity on $S(\Psi; u)$. The Jones index $[R : \Psi(R)]$ is $n$.

One interesting example of $S_1 = S(\Psi_1; u_1) = \{1, 3, 6, 10, 15, \ldots, \frac{1}{2}i(i + 1), \ldots\}_{i \in \mathbb{N}}$, which corresponds to the $n$-stream $\{0101001000100001000001 \ldots\}$. It is pointed out in [2] that the relative commutant $\Psi_1^k(R)' \cap R$ is always trivial for all $k!$ That is, our question for $A = \mathbb{C}$ is answered by this example.

3. Main Theorem

Theorem 1. For any finite dimensional $C^*$-algebra $A$ with any trace vector $\vec{s}$ whose components are rational numbers, we give an endomorphism $\Phi$ of the hyperfinite $II_1$ factor $R$ such that:

$$\forall k \in \mathbb{N}, \Phi^k(R)' \cap R = \otimes^k A.$$

The canonical trace $\tau$ on $R$ extends the trace vector $\vec{s}$ on $A$.

$A$ is characterized by its trace vector $\vec{s}$ and its dimension vector $\vec{t}$.

$$\vec{s} = \left[\begin{array}{c} b_1 \\ c_1 \\
\vdots \\
 b_n \\ c_n \end{array}\right], \quad \vec{t} = \left[\begin{array}{c} a_1 \\ a_2 \\
\vdots \\
 a_n \end{array}\right]$$

$$\frac{b_1}{c_1}a_1 + \frac{b_2}{c_2}a_2 + \cdots + \frac{b_n}{c_n}a_n = 1$$

$$b_1, c_1, a_1, b_2, c_2, a_2, \ldots, b_n, c_n, a_n \in \mathbb{N}$$
Put \( d = c_1c_2 \cdots c_n \). We can embed \( A \) into \( B \subset M_d(\mathbb{C}) \) via

\[
A \simeq M_{a_1}(\mathbb{C}) \oplus M_{a_2}(\mathbb{C}) \oplus \cdots \oplus M_{a_n}(\mathbb{C}) \\
M_{a_1}(\mathbb{C}) \otimes M_{d_1}(\mathbb{C}) \oplus M_{a_2}(\mathbb{C}) \otimes M_{d_2}(\mathbb{C}) \oplus \cdots \oplus M_{a_n}(\mathbb{C}) \otimes M_{d_n}(\mathbb{C})
\]

\[
= B \subset M_d(\mathbb{C})
\]

where \( d_1 = \frac{db_1}{c_1}, d_2 = \frac{db_2}{c_2}, \ldots, d_n = \frac{db_n}{c_n} \).

For each \( i, 1 \leq i \leq n, M_{a_i}(\mathbb{C}) \subset A \subset B \) (the former being not a unital embedding) is generated by \( p_i, q_i \in \mathcal{U}(\mathbb{C}^{a_i}) \) with:

\[
p_i^{a_i} = q_i^{d_i} = 1_{M_{a_i}(\mathbb{C})}; \quad \gamma_i = \exp(2\pi \sqrt{-1}/a_i), \quad p_i q_i = \gamma_i q_i p_i
\]

where \( p_i \) is the diagonal matrix in \( M_{a_i}(\mathbb{C}) \), \( [1 \gamma_i \gamma_i^2 \cdots \gamma_i^{a_i-1}] \), and \( q_i \) is the permutation matrix in \( M_{a_i}(\mathbb{C}) \), \( (1 2 3 \cdots a_i) \).

For each \( i, 1 \leq i \leq n, M_{d_i}(\mathbb{C}) \subset A' \cap B \subset B \) (the former being not a unital embedding) is generated by \( p_i, q_i \in \mathcal{U}(\mathbb{C}^{d_i}) \) with:

\[
p_i^{d_i} = q_i^{d_i} = 1_{M_{d_i}(\mathbb{C})}; \quad \rho_i = \exp(2\pi \sqrt{-1}/d_i), \quad p_i q_i = \rho_i q_i p_i
\]

where \( p_i \) is the diagonal matrix in \( M_{d_i}(\mathbb{C}) \), \( [1 \rho_i \rho_i^2 \cdots \rho_i^{d_i-1}] \), and \( q_i \) is the permutation matrix in \( M_{d_i}(\mathbb{C}) \), \( (1 2 3 \cdots d_i) \).

Define \( v \in M_d(\mathbb{C}) \) to be the permutation matrix:

\[
v = (a_1 d_1) (a_1 d_1 + a_2 d_2) \cdots (a_1 d_1 + a_2 d_2 + \cdots + a_n d_n)
\]

Then \( v^n = 1 \). \( B \) and \( v \) generate \( M_d(\mathbb{C}) \).

Define \( r := sv s \), while \( s \) is a diagonal matrix in \( M_d(\mathbb{C}) \),

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 1_{a_1 d_1} & 0 & 0 & \cdots & 0 & 1_{a_1 d_1 + a_2 d_2} & \cdots & 0 & \cdots & 0 & 1_{a_1 d_1 + a_2 d_2 + \cdots + a_n d_n}
\end{bmatrix}
\]

\( s \) lies in \( B \) and \( r^n = s \). Since \( v = 1 - s + r \), \( B \) and \( r \) generates \( M_d(\mathbb{C}) \).

**Lemma 2.** \( < B, r > \simeq M_d(\mathbb{C}) \) is of the form:

\[
B + BrB + Br^2B + \cdots + Br^{n-1}B.
\]

**Proof.** It suffices to observe that \( sBs \) is abelian and \( Adv \) sends \( sBs \) onto itself.

\[
[s, v] = [s, r] = 0, \quad r = sv s = vs = sv
\]

\[
r^2 = (svs)(svs) = sv^2s, \quad r^* = sv^*s = sv^{n-1}s = r^{n-1}
\]

\[
rBr = (svs)B(svs) = sv(sBs)vs = sv^2(sBs)s \subset r^2B
\]

\[
r^*r = rr^* = r^n = s \in B
\]

□
Define \( w = \sum_{i=1}^{n} \gamma^{-i} 1_{M_{\alpha_i}(\mathbb{C})} \otimes 1_{M_{\delta_i}(\mathbb{C})} \), where

\[
\gamma = \exp(2\pi \sqrt{-1}/n), \quad w^n = 1_{M_{\delta}(\mathbb{C})}.
\]

Note that \( w \) is in the center of \( B \). Observe that:

1. \( Adw \) acts trivially on \( B \).
2. \( Adw(r) = \gamma r \).

Now we construct a tower of inclusions of finite dimensional \( C^* \)-algebras \( M_k \) with a trace \( \tau \). The ascending union \( M = \cup_{k \in \mathbb{N}} M_k \) contains infinitely many copies of \( M_{\delta}(\mathbb{C}) \) and thus, infinitely many copies of \( B \). Number them respectively by \( r_1, A_1 \subset B_1, w_1, r_2, A_2 \subset B_2, w_2, r_3, A_3 \subset B_3, w_3, \ldots \).

We endow on the union \( M \) the following properties:

\[
[r_l, B_m] = 0, \text{ if } m < l \\
r_l r_m = \gamma r_m r_l \text{ if } l - m \in S_1 = \{1, 3, 6, 10, 15, \ldots\} \\
r_l r_m = r_m r_l, \text{ if } l - m \notin S_1
\]

There is no twist in the relation between \( A_m \) and \( A_l, m < l \), where:

\[
A_m = \bigoplus_{i=1}^{n} < (p_i)_m, (q_i)_m > \\
(p_i)_m(q_i)_m = \gamma_i (q_i)_m(p_i)_m \\
A_l = \bigoplus_{j=1}^{n} < (p_j)_l, (q_j)_l > \\
(p_j)_l(q_j)_l = \gamma_j (q_j)_l(p_j)_l \\
(p_j)_l(p_i)_m = (p_i)_m(p_j)_l \\
(p_j)_l(q_i)_m = (q_i)_m(p_j)_l \\
(q_j)_l(q_i)_m = (q_i)_m(q_j)_l
\]

Define

\[
S_2 := \{ \frac{1}{2} i(i + 1) \mid i = 1 \mod 3 \}, \quad S_3 := \{ \frac{1}{2} i(i + 1) \mid i = 2 \mod 3 \}
\]
We add a twist in the relation between \((A' \cap B)_m\) and \((A' \cap B)_l\), where:

\[
(A' \cap B)_m = \bigoplus_{i=1}^{n} < (p_i)_m, (q_i)_m > \\
(p_i)_m(q_i)_m = \rho_i(q_i)_m(p_i)_m \\
(A' \cap B)_l = \bigoplus_{j=1}^{n} < (p_j)_l, (q_j)_l > \\
(p_j)_l(q_j)_l = \rho_j(q_j)_l(p_j)_l \\
(p_j)_l(p_i)_m = (p_i)_m(p_j)_l \\
(p_j)_l(q_i)_m = (q_i)_m(p_j)_l
\]

If \(l - m \in S_2\) and \(l > m^2\), then \((q_j)_l(p_i)_m = \rho_i^{-\delta_{ij}}(p_i)_m(q_j)_l\).

If \(l - m \notin S_2\) and \(l > m^2\),

\[
(q_j)_l(p_i)_m = (p_i)_m(q_j)_l
\]

If \(l - m \in S_3\) and \(l > m^2\),

\[
(q_j)_l(q_i)_m = \rho_j^{\delta_{ij}}(q_i)_m(q_j)_l
\]

If \(l - m \notin S_3\) and \(l > m^2\),

\[
(q_j)_l(q_i)_m = (q_i)_m(q_j)_l
\]

The construction is an induction process. We embed \(M_1 \simeq M_d(\mathbb{C})\) into \(\otimes^2 M_d(\mathbb{C})\), by sending any element \(x \in M_1\) to \(x \otimes 1\). \(M_1\) is equipped with the trace \(\frac{1}{d} Tr\).

Observe that \(2 - 1 = 1 \in S_2\). There is a twist in the relation of \(B_1\) and \(B_2\), where \(B_2\) is generated by \((p_j)_2\), \((q_j)_2\), \((p_j)_2\), and \((q_j)_2\), \(1 \leq j \leq n\).

Put

\[
(p_j)_2 = 1 \otimes p_i \in \otimes^2 M_d(\mathbb{C}) \\
(q_j)_2 = 1 \otimes q_i \in \otimes^2 M_d(\mathbb{C}) \\
(p_j)_2 = 1 \otimes p_j \in \otimes^2 M_d(\mathbb{C}) \\
(q_j)_2 = (q_j + 1 - 1_{M_d(\mathbb{C})} \otimes q_j) \in \otimes^2 M_d(\mathbb{C})
\]
Note that \( q_j + 1 - 1_{M_{d_j}}(\mathbb{C}) \in \mathcal{U}(\mathbb{C}^d) \). We have:

\[
\begin{align*}
(p_j)_{2j}^{d_j} &= (q_j)_{2j}^{d_j} = 1 \otimes 1_{M_{d_j}}(\mathbb{C}) \\
(p_j)_{2}(q_j)_{2} &= \rho_j(q_j)_{2}(p_j)_{2} \\
(p_j)_{2}(p_i)_{1} &= (p_i)_{1}(p_j)_{2} \\
(p_j)_{2}(q_i)_{2} &= (q_i)_{1}(p_j)_{2} \\
(q_j)_{2}(p_i)_{1} &= \rho_i^{-b_{ij}}(p_i)_{2}(q_j)_{2} \\
(q_j)_{2}(q_i)_{2} &= (q_i)_{2}(q_j)_{2}
\end{align*}
\]

Thus

\[ B_2 \simeq B = \oplus_{i=1}^{n} M_{a_i}(\mathbb{C}) \otimes M_{d_i}(\mathbb{C}) \subset M_d(\mathbb{C}) \]

Observe \( 2 - 1 = 1 \in S_1 \). Define \( r_2 := w \otimes r \). We have the following properties:

\[
\begin{align*}
[w, q_j + 1 - 1_{M_{d_j}}(\mathbb{C})] &= 0 \\
< B_2, r_2 > &\simeq M_d(\mathbb{C}) \\
[B_1, r_2] &= 0 \quad r_1r_2 = \gamma r_2r_1 \\
M_2 &= < B_1, r_1, B_2, r_2 >= \otimes^2 M_d(\mathbb{C})
\end{align*}
\]

\( M_2 \) is equipped with a unique normalized trace \( \tau \).

Assume that we have obtained \( M_k = < B_1, r_1, B_2, r_2, \ldots, B_k, r_k > \)

isomorphic to \( \otimes^k M_d(\mathbb{C}) \) with the trace \( \tau \). We embed \( M_k \) into \( M_k \otimes M_d(\mathbb{C}) \) by sending \( x \in M_k \) to \( x \otimes 1_{M_d(\mathbb{C})} \).

Define \( B_{k+1} \) by its generators: \((p_j)_{k+1}, (q_j)_{k+1}, (p_j)_{k+1}, (q_j)_{k+1}\),

where \( 1 \leq j \leq n \):

\[
\begin{align*}
(p_j)_{k+1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p_j \\
(q_j)_{k+1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes q_j \\
(p_j)_{k+1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p_j \\
(q_j)_{k+1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p_j \\
[(q_j + 1 - 1_{M_{d_j}}(\mathbb{C}))^{\beta_j}] \otimes \cdots \otimes [(q_j + 1 - 1_{M_{d_j}}(\mathbb{C}))^{\beta_k}] \\
[(p_j + 1 - 1_{M_{d_j}}(\mathbb{C}))^{\beta_{k+1}}] \otimes \cdots \otimes (p_j + 1 - 1_{M_{d_j}}(\mathbb{C}))^{\beta_{2k}} \otimes q_j
\end{align*}
\]

Where \( 1 \leq i \leq k \):

\[
\beta_i = 1, \quad \text{if } k + 1 - i \in S_2; \quad \beta_i = 0, \quad \text{if } k + 1 - i \notin S_2 \\
\beta_{k+i} = 1, \quad \text{if } k + 1 - i \in S_3; \quad \beta_{k+i} = 0, \quad \text{if } k + i - 1 \notin S_3
\]
We have:

\[(p_j)_{k+1}^{d_j} = (q_j)_{k+1}^{d_j} = \otimes^k 1 \otimes 1_{M_{d_j}}(\mathbb{C})\]
\[(p_j)_{k+1}(q_j)_{k+1} = \gamma_j(q_j)_{k+1}(p_j)_{k+1}\]
\[(p_j)_{k+1}^{d_j} = (q_j)_{k+1}^{d_j} = \otimes^k 1 \otimes 1_{M_{d_j}}(\mathbb{C})\]
\[(p_j)_{k+1}(q_j)_{k+1} = \rho_j(q_j)_{k+1}(p_j)_{k+1}\]

Therefore \(B_{k+1}\) is isomorphic to \(B\).

The commutation relations are given below.

\[l < k + 1:\]
\[[(p_j)_{k+1}, (p_i)_l] = 0\]
\[[(p_j)_{k+1}, (q_i)_l] = 0\]
\[[(q_j)_{k+1}, (p_i)_l] = 0\]
\[[(q_j)_{k+1}, (q_i)_l] = 0\]
\[A_{k+1} \cdot A_l = A_l \cdot A_{k+1}\]
\[[(p_j)_{k+1}, (p_i)_l] = 0\]
\[[(p_j)_{k+1}, (q_i)_l] = 0\]

The anti-commutation relations are given below.

If \(k + 1 - l \in S_2\) and \(k + 1 > l^2\),
\[(q_j)_{k+1}(p_i)_l = \rho_i^{-\delta_{ij}}(p_i)_l(q_j)_{k+1}\]
If \(k + 1 - l \notin S_2\) and \(k + 1 > l^2\),
\[(q_j)_{k+1}(p_i)_l = (p_i)_l(q_j)_{k+1}\]
If \(k + 1 - l \in S_3\) and \(k + 1 > l^2\),
\[(q_j)_{k+1}(q_i)_l = \rho_i^{\delta_{ij}}(q_i)_l(q_j)_{k+1}\]
If \(k + 1 - l \notin S_3\) and \(k + 1 > l^2\),
\[(q_j)_{k+1}(q_i)_l = (q_i)_l(q_j)_{k+1}\]

Define

\[r_{k+1} := w^{\alpha_1} \otimes w^{\alpha_2} \otimes \cdots \otimes w^{\alpha_k} \otimes r\]
\[\alpha_i = 1, \text{ if } k + 1 - i \in S_1; \quad \alpha_i = 0, \text{ otherwise.}\]
We have the following properties:

\[
[w, q_j + 1 - 1_{M_d}(C)] = [w, p_j + 1 - 1_{M_d}(C)] = 0
\]
\[
< B_{k+1}, r_{k+1} > \simeq M_n(C)
\]
\[
[B_j, r_{k+1}] = 0 \quad 1 \leq j \leq k
\]
\[
r_{k+1}r_j = \gamma r_j r_{k+1} \quad \text{if } k + 1 - j \in S_1
\]
\[
r_{k+1}r_j = r_j r_{k+1} \quad \text{if } k + 1 - j \notin S_1
\]
\[
M_{k+1} = < M_k, B_{k+1}, r_{k+1} > \simeq \otimes^{k+1} M_d(C)
\]

There is a unique normalized trace \(\tau\) on \(M_{k+1}\).

By induction we have constructed the ascending tower of inclusions of finite dimensional \(C^*\)-algebras with the desired properties.

We now explore some useful properties of the finite dimensional \(C^*\)-algebra, \(M_k\).

**Lemma 3.** For all \(k\), \(M_k\) is the linear span of the words, \(x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_k\), where \(x_j \in (M_d(C))_j = < B_j, r_j >\).

**Proof.** It suffices to prove \(x_j \cdot x_i\) is in \(M_i\), \(< B_j, r_j >= M_i \cdot (M_d(C))_j\), where \(i < j\) and \(M_i = \otimes^i M_d(C)\).

\[
r_j \cdot B_i = B_i \cdot r_j
\]
Either \(r_j \cdot r_i = r_i \cdot r_j\) or \(r_j \cdot r_i = \gamma r_i \cdot r_j\)

\(A_j \cdot A_i = A_i \cdot A_j \subset M_i \cdot A_j\)

\(A_j \cdot (A' \cap B)_i = (A' \cap B)_i \cdot A_j \subset M_i \cdot A_j\)

\((A' \cap B)_j \cdot A_i = A_i \cdot (A' \cap B)_j \subset M_i \cdot A_j\)

Note that :
\[
Ad(p_i + 1 - 1_{M_d}(C))(q_i) = \rho_i q_i
\]
\[
Ad(q_i + 1 - 1_{M_d}(C))(p_i) = \rho_i^{-1} p_i
\]
\[
Ad(p_i + 1 - 1_{M_d}(C))(q_i + 1 - 1_{M_d}(C)) = \rho_i q_i + 1 - 1_{M_d}(C)
\]
\[
Ad(q_i + 1 - 1_{M_d}(C))(p_i + 1 - 1_{M_d}(C)) = \rho_i^{-1} p_i + 1 - 1_{M_d}(C)
\]
\((A' \cap B)_j \cdot (A' \cap B)_i \subset M_i \cdot (A' \cap B)_j\)

In short, \(B_j \cdot B_i \subset M_i \cdot B_j\).
Note that:
\[ \text{Ad}(p_i + 1 - 1_{M_d(C)})(w) = w \]
\[ \text{Ad}(q_i + 1 - 1_{M_d(C)})(w) = w \]
\[ \text{Ad}(p_i + 1 - 1_{M_d(C)})(r) \in M_d(C) \]
\[ \text{Ad}(q_i + 1 - 1_{M_d(C)})(r) \in M_d(C) \]
In short, \( B_j \cdot r_i \subset M_i \cdot B_j \).

\[ \square \]

**Lemma 4.** Consider the GNS-construction of the pair \((M, \tau)\) described above. The weak closure of \( M \) is the hyperfinite \( II_1 \) factor.

**Proof.** There is a unique tracial state on \( M_k = \otimes^k M_d(C) \) for all \( k \in \mathbb{N} \), and hence a unique tracial state on \( M \), a \( d^\infty \) UHF-algebra. \[ \square \]

Define a unital \(*\)-endomorphism, \( \Phi \), on \( R \) by sending \( B_k \) to \( B_{k+1} \) and sending \( r_k \) to \( r_{k+1} \):

\[ 1 \leq i \leq n : \]
\[ \Phi((p_i)_k) = (p_i)_{k+1} \]
\[ \Phi((q_i)_k) = (q_i)_{k+1} \]
\[ \Phi((p_i)_k) = (p_i)_{k+1} \]
\[ \Phi((q_i)_k) = (q_i)_{k+1} \]
\[ \Phi(r_k) = r_{k+1} \]

We observe that \( \Phi(R) \) is a hyperfinite \( II_1 \) factor and
\[ [R : \Phi(R)] = d^2 < \infty. \]

**Lemma 5.** The relative commutant \( \Phi^k(R)' \cap R \) is exactly \( \otimes_{i=1}^k A \), on which the trace of \( R \) is the product trace given by the vector \( \vec{s} \).

**Proof.** Because of our decomposition in Lemma 2 and Lemma 3, \( R \) can be written as
\[ \left( \sum_{i=0}^{n-1} B_1 r_1^i B_1 \right) \cdot \left( \sum_{i=0}^{n-1} B_2 r_2^i B_2 \right) \cdots \left( \sum_{i=0}^{n-1} B_k r_k^i B_k \right) \cdot \Phi^k(R). \]
Assume \( x \in \Phi^k(R)' \cap R \). Let \( \vec{\alpha} = (i_1, i_2, \ldots, i_k) \) be a multi-index.

\[
0 \leq i_1, i_2, \ldots, i_k \leq n - 1
\]

\[
x = \sum_{\vec{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{i_1} r_1^{\vec{\alpha}} z_1^{i_1} y_2^{i_2} r_2^{\vec{\alpha}} z_2^{i_2} \cdots y_k^{i_k} r_k^{\vec{\alpha}} z_k^{i_k} \cdot y^\vec{\alpha}
\]

\[
y_1^{\vec{\alpha}}, z_1^{\vec{\alpha}} \in B_1
\]

\[
y_2^{\vec{\alpha}}, z_2^{\vec{\alpha}} \in B_2
\]

\[
\ldots
\]

\[
y_k^{\vec{\alpha}}, z_k^{\vec{\alpha}} \in B_k
\]

\[
y^\vec{\alpha} \in \Phi^k(R)
\]

Note that \( \Phi^k(R) \) is the weak closure of \( \{\Phi^k(M_j)\}_{j=1}^\infty \).

For every \( \epsilon > 0 \), there exists an integer \( j \in \mathbb{N} \) such that

\[
\forall \vec{\alpha} \quad \exists z^{\vec{\alpha}} \in \Phi^k(M_j) \subset < B_{k+1}, r_{k+1}, \ldots, B_{k+j}, r_{k+j}>
\]

\[
\|x - \sum_{\vec{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\vec{j}_1} r_1^{\vec{\alpha}} z_1^{\vec{j}_1} y_2^{\vec{ar{j}}_2} r_2^{\vec{\alpha}} z_2^{\vec{ar{j}}_2} \cdots y_k^{\vec{j}_k} r_k^{\vec{\alpha}} z_k^{\vec{j}_k} \cdot z^{\vec{\alpha}}\|_2 < \delta
\]

\[
\delta = (\sqrt{\frac{n}{d}})^k \epsilon
\]

Put \( L = l(l+1)/2+1 \) for some integer \( l > 2(k+j) \) and \( l = 0 \mod 3 \). We have the following properties:

\[
[r_L, B_1] = [r_L, B_2] = \cdots = [r_L, B_{k+j}] = 0
\]

\[
[r_L, r_2] = [r_L, r_3] = \cdots = [r_L, r_{k+j}] = 0
\]

\[
r_{LR_1} = \gamma^{j_1} r_{L_1}, \quad r_{LR_1^{j_1}} = \gamma^{j_1} r_{L_1}
\]

\[
r_{LR_1^*} = r_{L_1^*} r_{L_1} = s_L
\]

\[
r_{LR_1 r_{L_1^*}} = \gamma r_{L_1^*} s_L, \quad r_{LR_1^{j_1} r_{L_1^*}} = \gamma^{j_1} r_{L_1^*} s_L
\]

for \( 0 \leq m \leq n - 1 \), \( r_{L_1^*} = r_L^{m_1} r_{L_1}^{m_1} = \gamma^{j_1 m} r_{L_1^*} s_L \)

\[
[s_L, B_1] = [s_L, B_2] = \cdots = [s_L, B_{k+j}] = 0
\]

\[
[s_L, r_1] = [s_L, r_2] = \cdots = [s_L, r_{k+j}] = 0
\]

\[
[r_{L_1^*} s_L, B_1] = [r_{L_1^*} s_L, B_2] = \cdots = [r_{L_1^*} s_L, B_{k+j}] = 0
\]

\[
[r_{L_1^*} s_L, r_1] = [r_{L_1^*} s_L, r_2] = \cdots = [r_{L_1^*} s_L, r_{k+j}] = 0
\]
Therefore we claim:
\[
\|x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} s_L \|_{2,\tau} =
\]
\[
\|x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} s_L \|_{2,\tau} = \frac{1}{n} \sum_{m=0}^{n-1} r_{m}^* r_{m} \|_{2,\tau} =
\]
\[
\frac{1}{n} \left\| \sum_{\tilde{\alpha}} \sum_{m} (r_{m} x_{m} r_{m}^*) - y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} s_L \right\|_{2,\tau} =
\]
\[
\left\| x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k, j_1=0} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} s_L \right\|_{2,\tau} =
\]
\[
\sqrt{\frac{2}{n}} \left\| x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k, g_1=0} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} \right\|_{2,\tau}
\]

Since
\[
\{x, y_1^{\tilde{\alpha}}, z_1^{\tilde{\alpha}}, y_2^{\tilde{\alpha}}, z_2^{\tilde{\alpha}}, y_j^{\tilde{\alpha}}, z_j^{\tilde{\alpha}}, \ldots, y_k^{\tilde{\alpha}}, z_k^{\tilde{\alpha}}\} \subset \{s_L, r_L, B_L\}
\]
and \(\tau(s_L) = \frac{n}{d}\).

Note that \(\{s_L, r_L, B_L\}'' = M_d(\mathbb{C})_L\) is a type I factor [6].

By induction,
\[
\|x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k, j_1=0} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} \|_{2,\tau} < \sqrt{\frac{d}{n} \delta}
\]
\[
\|x - \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k, j_1=j_2=0} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} \|_{2,\tau} < (\sqrt{\frac{d}{n}})^2 \delta
\]
\[
\|x - \sum_{\tilde{\alpha} \in \{0\}^k} y_1^{\tilde{\alpha}} z_1^{\tilde{\alpha}} y_2 z_2^{\tilde{\alpha}} \cdots y_k z_k^{\tilde{\alpha}} \|_{2,\tau} < (\sqrt{\frac{d}{n}})^k \delta = \epsilon
\]
Put $L_1 = (l_1 + 1)/2 + 1$ for some integer $l_1 > 2(k + j)$ and $l_1 = 1 \mod 3$. We have the following properties:

$$U_{L_1} := \sum_{m_1=0}^{n-1} (q_{m_1})_{L_1}$$

$$U_{L_1}^d = 1$$

$$[U_{L_1}, \Phi(M_{k+j-1})] = 0$$

Similarly, put $L_2 = (l_2 + 1)/2 + 1$ for some integer $l_2 > 2(k + j)$ and $l_2 = 2 \mod 3$. We have the following properties:

$$U_{L_2} := \sum_{m_2=0}^{n-1} (q_{m_2})_{L_2}$$

$$U_{L_2}^d = 1$$

$$[U_{L_2}, \Phi(M_{k+j-1})] = 0$$

Therefore

$$\|x - \sum_{\bar{a} \in \{0\}^k} \alpha_{1_1} \alpha_{2_2} \cdots \alpha_{k_1} \cdot z^{\bar{a}}\|_{2,T} =$$

$$\|x - \sum_{\bar{a} \in \{0\}^k} y_{\bar{a}} z_{\bar{a}} y_{\bar{a}} z_{\bar{a}} \cdots y_{\bar{a}} z_{\bar{a}} \cdot z^{\bar{a}}\|_{2,T} =$$

$$\|x - \sum_{\bar{a} \in \{0\}^k} \frac{1}{d} \sum_{m_3=0}^{d-1} U_{L_1}^{m_3} U_{L_1}^{m_3} \cdots U_{L_2}^{m_4} U_{L_2}^{m_4} \|_{2,T} =$$

$$\|x - \sum_{\bar{a} \in \{0\}^k} \frac{1}{d^2} \sum_{m_3,m_4=0}^{d-1} U_{L_1}^{m_3} U_{L_1}^{m_3} \cdots U_{L_2}^{m_4} U_{L_2}^{m_4} \|_{2,T} =$$

Observe that

$$x^{\bar{a}} := \frac{1}{d^2} \sum_{m_3,m_4=0}^{d-1} U_{L_2}^{m_4} U_{L_1}^{m_3} (y_{\bar{a}} z_{\bar{a}}) U_{L_1}^{m_3} U_{L_2}^{m_4}$$

is the trace-preserving conditional expectation of $y_{\bar{a}} z_{\bar{a}} \in B_1$ onto $A_1$. 

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By induction,
\[
\|x - \sum_{\vec{\alpha} \in \{0\}^k} x_1^\vec{\alpha} (y_2^\vec{\alpha} z_2^\vec{\alpha}) \cdots y_k^\vec{\alpha} z_k^\vec{\alpha}\|_{2,\tau} = \\
\|x - \sum_{\vec{\alpha} \in \{0\}^k} x_1^\vec{\alpha} x_2^\vec{\alpha} (y_3^\vec{\alpha} z_3^\vec{\alpha}) \cdots y_k^\vec{\alpha} z_k^\vec{\alpha}\|_{2,\tau} = \\
\ldots \\
\|x - \sum_{\vec{\alpha} \in \{0\}^k} x_1^\vec{\alpha} x_2^\vec{\alpha} \cdots x_k^\vec{\alpha} \|_{2,\tau} < \epsilon
\]
where \(x_1^\vec{\alpha} \in A_1, x_2^\vec{\alpha} \in A_2, \ldots, x_k^\vec{\alpha} \in A_k\).

Note that the von Neumann algebra \(\{x, x_1^\vec{\alpha}, x_2^\vec{\alpha}, \ldots, x_k^\vec{\alpha}\}\) commutes with \(\Phi^k(M)\), which is a \(II_1\) factor. Any element in the former von Neumann algebra has a scalar conditional expectation onto \(\Phi^k(M)\). In short, the former von Neumann algebra and \(\Phi^k(M)\) are mutually orthogonal [6].

According to the Cauchy-Schwartz inequality
\[
\|z^\vec{\alpha}\|_{2,\tau}^2 \geq |\tau(z^\vec{\alpha})|^2
\]
we have:
\[
\|x - \sum_{\vec{\alpha} \in \{0\}^k} x_1^\vec{\alpha} x_2^\vec{\alpha} \cdots x_k^\vec{\alpha} \cdot \tau(z^\vec{\alpha})\|_{2,\tau} < \epsilon
\]
\[
\sum_{\vec{\alpha} \in \{0\}^k} x_1^\vec{\alpha} x_2^\vec{\alpha} \cdots x_k^\vec{\alpha} \cdot \tau(z^\vec{\alpha}) \in A_1 \cdot A_2 \cdots A_k = \otimes^k A
\]

\[\square\]

4. Application

The Temperley-Lieb algebra [3] is generated by projections \(e_i, i \in \mathbb{N}\) such that:
\[
[e_i, e_j] = 0 \quad \text{if } |i - j| \geq 2 \\
e_i e_{i \pm 1} e_i = \lambda e_i
\]

We are interested in the case that \(\lambda\) is a rational number and \(\lambda^{-1} > 4\). Put
\[
\lambda = \frac{p}{q} \quad p, q \in \mathbb{N}
\]

Take \(m \in \mathbb{N}\) and \(m \geq 3\). Define
\[
A_{1,m} := vN\{e_3, e_4, \ldots, e_m\} \quad A_{0,m} := vN\{e_2, e_3, e_4, \ldots, e_m\}
\]
It is known [3] that

\[ A_{1,m} \subset A_{0,m} \subset M_{q_{m/2}}(\mathbb{C}) \]

By the main theorem, we have an endomorphism \( \Phi \) of the hyperfinite \( II_1 \) factor \( R \) such that

\[ \Phi(R)' \cap R = A_{1,m} \]

Note that \( Q := \Phi(R) \) is the hyperfinite \( II_1 \) factor. Define

\[ M_1 = Q \quad \text{and} \quad M_l := vN\{Q, e_2, e_3, \ldots, e_l\} \quad 2 \leq l \leq m \]

A corollary to the main theorem is:

\[ M_l' \cap M_m = \{e_2, e_3, \ldots, e_l\}' \cap A_{1,m} = vN\{e_{l+2}, e_{l+3}, \ldots, e_m\} \]

Though \( M_{m-1} \subset M_m \) is an irreducible inclusion of hyperfinite \( II_1 \) factors, we do not know a priori whether \( M_{m-1} \neq M_m \). Nor do we know in general whether \( M_l \) is a factor.

**Lemma 6.** For an element \( x \in Q \),

\[ \tau(e_2 x) = \lambda \tau(x) \]

**Proof.** For every \( \epsilon > 0 \), we can find \( x_k \) in \( M_k = \otimes^k M_{q_{m/2}}(\mathbb{C}) \) such that \( \|x - \Phi(x_k)\|_{2,r} < \epsilon \). Let \( \tilde{\alpha} = (i_1, i_2, \ldots, i_k) \) be a multi-index.

\[ 0 \leq i_1, i_2, \ldots, i_k \leq n - 1 \]

\[ x_k = \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\tilde{\alpha}_1} z_1^{\tilde{\alpha}_1} y_2^{\tilde{\alpha}_2} z_2^{\tilde{\alpha}_2} \cdots y_k^{i_k} z_k^{i_k} \]

The right-side equation is:

\[ \tau(\Phi(x_k)) = \tau(x_k) \]

\[ = \tau\left( \sum_{\tilde{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\tilde{\alpha}_1} z_1^{\tilde{\alpha}_1} y_2^{\tilde{\alpha}_2} z_2^{\tilde{\alpha}_2} \cdots y_k^{i_k} z_k^{i_k} \right) \]

\[ = \tau\left( \sum_{\tilde{\alpha} \in \{0\}^k} y_1^{\tilde{\alpha}_1} z_1^{\tilde{\alpha}_1} y_2^{\tilde{\alpha}_2} z_2^{\tilde{\alpha}_2} \cdots y_k^{\tilde{\alpha}_k} \right) \]

\[ = \tau\left( \sum_{\tilde{\alpha} \in \{0\}^k} \frac{1}{d^2} \sum_{m_1, m_2} U_{L_2}^{m_1} U_{L_1}^{m_2} (y_1^{\tilde{\alpha}_1} z_1^{\tilde{\alpha}_1}) U_{L_1}^{m_3} U_{L_2}^{m_4} (y_2^{\tilde{\alpha}_2} z_2^{\tilde{\alpha}_2}) \cdots (y_k^{\tilde{\alpha}_k} z_k^{\tilde{\alpha}_k}) \right) \]

\[ = \tau\left( \sum_{\tilde{\alpha} \in \{0\}^k} x_1^{\tilde{\alpha}_1} y_2^{\tilde{\alpha}_2} \cdots y_k^{\tilde{\alpha}_k} \right) \]
Where

\[
L_1 = l_1(l_1 + 1)/2 + 1 \quad l_1 > 2(k + 1) \quad l_1 = 1 \mod 3 \\
U_{L_1} := \sum_{m_1=0}^{n-1} (q_{m_1})_{L_1} \\
[U_{L_1}, \Phi^2(M_{k-1})] = 0 \\
[\Phi(U_{L_1}), e_2] = 0 \\
L_2 = l_2(l_2 + 1)/2 + 1 \quad l_2 > 2(k + 1) \quad l_2 = 2 \mod 3 \\
U_{L_2} := \sum_{m_2=0}^{n-1} (q_{m_2})_{L_2} \\
[U_{L_2}, \Phi^2(M_{k-1})] = 0 \\
[\Phi(U_{L_2}), e_2] = 0
\]

Observe that

\[
x_1^{\vec{\alpha}} := \frac{1}{d^2} \sum_{m_3,m_4=0}^{d-1} U_{L_2}^{m_4} U_{L_1}^{m_3} (y_1^{\vec{\beta}} z_1^{\vec{\alpha}}) U_{L_1}^{* m_3} U_{L_2}^{* m_4}
\]

is the trace-preserving conditional expectation of \(y_1^{\vec{\beta}} z_1^{\vec{\alpha}} \in B_1\) onto \(A_1\).

By induction,

\[
\tau(\Phi(x_k)) = \tau(\sum_{\vec{\alpha} \in \{0\}^k} x_1^{\vec{\alpha}} x_2^{\vec{\alpha}} \cdots x_k^{\vec{\alpha}}) = \tau(\sum_{\vec{\alpha} \in \{0\}^k} \Phi(x_1^{\vec{\alpha}}) \cdot \Phi(x_2^{\vec{\alpha}}) \cdots \Phi(x_k^{\vec{\alpha}}))
\]

where \(x_j^{\vec{\alpha}}\) is the trace-preserving conditional expectation of \(y_j^{\vec{\beta}} z_j^{\vec{\alpha}} \in B_j\) onto \(A_j\).
The left-side equation is:
\[
\tau(e_2 \Phi(x_k)) = \tau(e_2 \Phi(\sum_{\vec{\alpha} \in \{0, 1, \ldots, n-1\}^k} y_1^{\vec{\alpha}} y_2^{\vec{\alpha}} y_k^{\vec{\alpha}})) = \tau(e_2 \Phi(\sum_{\vec{\alpha} \in \{0\}^k} y_1^{\vec{\alpha}} y_2^{\vec{\alpha}} y_k^{\vec{\alpha}}))
\]
\[
= \tau(e_2 \Phi(\sum_{\vec{\alpha} \in \{0\}^k} x_1^{\vec{\alpha}} x_2^{\vec{\alpha}} x_k^{\vec{\alpha}})) = \ldots
\]
\[
= \tau\left(\sum_{\vec{\alpha} \in \{0\}^k} \Phi(x_1^{\vec{\alpha}}) \cdot \Phi(x_2^{\vec{\alpha}}) \cdots \Phi(x_k^{\vec{\alpha}})\right)
\]
\[
= \lambda \tau\left(\sum_{\vec{\alpha} \in \{0\}^k} \Phi(x_1^{\vec{\alpha}}) \cdot \Phi(x_2^{\vec{\alpha}}) \cdots \Phi(x_k^{\vec{\alpha}})\right)
\]
\[
\square
\]

Remark. The trace \(\tau\) on \(\mathcal{M}_m\) gives a hyperfinite Markov trace on the universal Jones algebra associated to \(A_{1,m} \subset A_{0,m}\) [7].

4.1. The \(m = 3\) Case.
\[
A_{1,3} = \{e_3\}''
\]
\[
= \mathbb{C}e_3 \oplus \mathbb{C}(1 - e_3)
\]
\[
\subset \mathbb{C} \otimes M_p(\mathbb{C}) \oplus \mathbb{C} \otimes M_{q-p}(\mathbb{C}) \subset M_q(\mathbb{C})
\]

There is an intermediate subalgebra,
\[
A_{0,3} = \{e_2, e_3\}''
\]
\[
= M_2(\mathbb{C}) \oplus \mathbb{C}
\]
\[
\subset M_2(\mathbb{C}) \otimes M_p(\mathbb{C}) \oplus \mathbb{C} \otimes M_{q-2p}(\mathbb{C}) \subset M_q(\mathbb{C})
\]

By the above theorem, there is an endomorphism of \(\Phi\) of the hyperfinite \(II_1\) factor such that:
\[
\Phi^k(R) \cap R = \otimes^k A_{1,3}
\]
Take $Q = \Phi(R)$, we have a tower of inclusions of hyperfinite $II_1$ factors with the trace $\tau$:

$$Q \subset <Q, e_2> \subset <Q, e_2, e_3>$$

The problem is to determine whether $<Q, e_2>$ is equal to $<Q, e_2, e_3>$ or not.

Note that $[Q, e_3] = 0$. Let $\theta$ be the isomorphism of $Qe_3$ onto $Q(1-e_3)$. Then any element in $Q$ can be decomposed as:

$$x_1 \oplus \theta(x_1), \ x_1 \in Qe_3$$

In the matrix form, we can write

$$e_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} \lambda & \lambda \sqrt{(1-\lambda)} & 0 \\ \sqrt{(1-\lambda)} & 1-\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For an element $x \in Q$, we can write

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, \ \theta(x_1) = \begin{bmatrix} x_2 \\ x_3 \\ x_2 \end{bmatrix}, \ x_1 = e_3 \cdot x \cdot e_3$$

$$x_2 = (e_2 \lor e_3 - e_3) \cdot x \cdot (e_2 \lor e_3 - e_3)$$

$$x_23 = (e_2 \lor e_3 - e_3) \cdot x \cdot (1 - e_2 \lor e_3)$$

$$x_32 = (1 - e_2 \lor e_3) \cdot x \cdot (e_2 \lor e_3 - e_3)$$

$$x_3 = (1 - e_2 \lor e_3) \cdot x \cdot (1 - e_2 \lor e_3)$$

The matrix calculation shows:

$$e_2xe_2 = e_2 \cdot \begin{bmatrix} \lambda x_1 + (1-\lambda)x_2 & 0 & 0 \\ 0 & \lambda x_1 + (1-\lambda)x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

$$e_3e_2xe_2x'e_2e_3 = \lambda e_3 \cdot \begin{bmatrix} \lambda x_1 + (1-\lambda)x_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda x_1' + (1-\lambda)x'_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
Note that
\[ e_3 e_2 x e_2 e_3 = \lambda e_3 \cdot \begin{bmatrix} \lambda x_1 + (1 - \lambda) x_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ = \lambda e_3 \cdot [(\lambda x_1 + (1 - \lambda) x_2) \oplus \theta(\lambda x_1 + (1 - \lambda) x_2)] \in e_3 Q \]

To summarize, \( e_3 \) implements a conditional expectation of \( \langle Q, e_2 \rangle \) onto \( Q \).

We collect some useful facts.

\[ \forall x \in Q, \ [x, e_3] = 0 \]
\[ \tau(e_3 x) = \tau(E^\tau_{Q}(e_3) x) = \lambda \tau(x) \]
\[ \tau(x_1) = \lambda \tau(x) \]
\[ = \lambda \tau(x_1) + \lambda \tau(x_2) + \lambda \tau(x_3) \]
\[ \tau(e_2 x) = \lambda \tau(x) \]
\[ \lambda \tau(x_1) + (1 - \lambda) \tau(x_2) = \tau(x_1) \]
\[ \tau(x_1) = \tau(x_2) \]
\[ (1 - 2\lambda) \tau(x_1) = \lambda \tau(x_3) \]

By the above, we have the important identity:
\[ E^\tau_{\langle Q, e_2 \rangle}(e_3) = \lambda \]

We also proved that the conditional expectation implemented by \( e_3 \) is trace-preserving.

We establish that \( Q \subset \langle Q, e_2 \rangle \subset \langle Q, e_2, e_3 \rangle \) is a Jones basic construction with:
\[ [\langle Q, e_2, e_3 \rangle : \langle Q, e_2 \rangle] = \lambda^{-1} \]
\[ < Q, e_2 >' \cap < Q, e_2, e_3 > = \mathbb{C} \]

4.2. The \( m = 4 \) Case.

\[ A_{1,4} = \{ e_3, e_4 \}'' \]
\[ = M_2(\mathbb{C}) \oplus \mathbb{C} \]
\[ \subset M_2(\mathbb{C}) \otimes M_{pq}(\mathbb{C}) \oplus \mathbb{C} \otimes M_{q^2 - 2pq}(\mathbb{C}) \]
\[ \subset M_{q^2}(\mathbb{C}) \]
There is an intermediate subalgebra,
\[ A_{0,3} = \{e_2, e_3, e_4\}'' \]
\[ = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \]
\[ \subset M_2(\mathbb{C}) \otimes M_{p^2}(\mathbb{C}) \oplus M_3(\mathbb{C}) \otimes M_{pq-p^2}(\mathbb{C}) \oplus \mathbb{C} \otimes M_{q^2-3pq+p^2}(\mathbb{C}) \]
\[ \subset M_{q^2}(\mathbb{C}) \]

By the main theorem, there is an endomorphism of \( \Phi \) of the hyperfinite \( \text{II}_1 \) factor such that:
\[ \Phi^k(R)' \cap R = \otimes^k A_{1,4} \]

Take \( Q = \Phi(R) \), we have a tower of inclusions of hyperfinite \( \text{II}_1 \) factors with the trace \( \tau \):
\[ Q \subset < Q, e_2 > \subset < Q, e_2, e_3 > \subset < Q, e_2, e_3, e_4 > \]

The problem is to determine whether \( < Q, e_2, e_3, e_4 > \) properly contains \( < Q, e_2, e_3 > \). Nor is known whether \( < Q, e_2 > \) is a factor or not.

In the matrix form, we can write
\[
e_4 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
e_3 = \begin{bmatrix}
\lambda & \sqrt{\lambda(1-\lambda)} & 0 & 0 & 0 & 0 \\
\sqrt{\lambda(1-\lambda)} & 1-\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & \sqrt{\lambda(1-\lambda)} & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda(1-\lambda)} & 1-\lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
e_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\lambda}{1-\lambda} & \sqrt{\frac{\lambda(1-2\lambda)}{1-\lambda}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{\lambda(1-2\lambda)}{1-\lambda}} & \frac{1-2\lambda}{1-\lambda} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
For an element \( x \in Q \), we can write
\[
x = \begin{bmatrix}
x_1 & 0 & x_{12} & 0 & 0 \\
0 & x_1 & 0 & x_{12} & 0 \\
x_{21} & 0 & x_2 & 0 & 0 \\
0 & x_{21} & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & x_3 \\
0 & 0 & 0 & 0 & x_{43} \\
0 & 0 & 0 & 0 & x_4
\end{bmatrix}
\]

The matrix calculation shows:
\[
e_2 x e_2 = \begin{bmatrix}
x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\lambda}{1-\lambda} x_2 + \frac{1-2\lambda}{1-\lambda} x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\lambda}{1-\lambda} x_2 + \frac{1-2\lambda}{1-\lambda} x_3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Take \((q_1)_4\) as an example.
\[
(q_1)_4 =
\begin{bmatrix}
x_1 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & 0 & 0 \\
0 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & x_3 \\
0 & 0 & 0 & 0 & x_4
\end{bmatrix} \otimes 1 \otimes 1 \otimes q_1
\]
\[
x_1 = \begin{bmatrix} 1 \rho_1^1 \rho_1^2 \cdots \rho_1^{p^2-1} \end{bmatrix} \in M_{p^2}(\mathbb{C})
\]
\[
x_2 = \begin{bmatrix} \rho_1^{p^2} \rho_1^{p^2+1} \rho_1^{p^2+2} \cdots \rho_1^{p^2-1} \end{bmatrix} \in M_{p^2-1}(\mathbb{C})
\]
\[
x_3 = 1_{M_{pq-p^2}(\mathbb{C})}
\]
\[
x_4 = 1_{M_{q^2-3pq+p^2}(\mathbb{C})}
\]

Note that
\[
E_{A_{1,4}}(e_2(q_1)_4 e_2(q_1)_4^* e_2)
\]
\[
= 2\lambda \begin{bmatrix} \lambda & 0 & 0 \\
0 & \kappa & 0 \\
0 & 0 & \kappa
\end{bmatrix}
\]
\[
= 2\lambda^2 e_4 + 2\lambda\kappa(1 - e_4)
\]

where \( \kappa < \lambda(1 - \lambda) \)
By the relative Dixmier property [10]:

\[ E_{\tau, A_{1,4}}(e_2(q_1) e_2(q_1)^* e_2) = E_{\tau, Q \cap R}(e_2(q_1) e_2(q_1)^* e_2) \in < e_2, Q > \]

Therefore

\[ e_4 \in < Q, e_2 > \]

\[ Z(< Q, e_2 >) = C e_4 \oplus C(1 - e_4) \]

\[ < Q, e_2, e_3 >= < Q, e_2, e_3, e_4 > \]

However we can consider the inclusion of hyperfinite II$_1$ factors:

\[ Q e_4 \subset < Q, e_2 > e_4 \]

An element in $Q e_4$ can be written as

\[
\begin{bmatrix}
  x_1 & x_{12} \\
  x_{21} & x_2
\end{bmatrix}
\]

Since

\[ e_2 e_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \]

$< Q, e_2 > e_4$ is generated by:

\[
\begin{bmatrix}
  x_1 & 0 \\
  0 & 0
\end{bmatrix} \cdot \begin{bmatrix} 0 & x_{12} \\ x_{21} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix}
\]

$e_2 e_4$ implements a conditional expectation from $Q e_4$ onto $\{e_2 e_4\}' \cap Q e_4$ by

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix} \cdot \begin{bmatrix} x_1 & x_{12} \\ x_{21} & x_2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix}
\]

Note that

\[
\begin{bmatrix}
  x_1 & x_{12} \\
  x_{21} & x_2
\end{bmatrix} \in Q e_4 \Rightarrow \begin{bmatrix} x_1 & 0 \\
  0 & x_2 \end{bmatrix} \in Q e_4
\]

because of the Dixmier property.

By the above, we have the important identity:

\[ E_{Q e_4}(e_2 e_4) = \lambda e_4 \]

We also proved that the conditional expectation implemented by $e_2 e_4$ is trace-preserving.

We establish that

\[ \{e_2 e_4\}' \cap Q e_4 \subset Q e_4 \subset < Q, e_2 > e_4 \]

is a Jones basic construction with:

\[
[< Q, e_2 > e_4 : Qe_4] = \lambda^{-1}
\]
\[
Qe_4' \cap < Q, e_2 > e_4 = C e_4
\]

5. ANALOGY

In this section, we construct inclusions of non-hyperfinite II\(_1\)-factors via free product with amalgamation as a comparison to the main theorem.

**Theorem 2.** For any finite dimensional \(C^*\)-algebra \(A\) with any trace vector \(\vec{s}\) whose components are rational numbers, there exists a tower of inclusions of \(II_1\)-factors, \(M \subset M_1 \subset M_2 \subset M_3 \subset \cdots\), with the trace \(\tau\) such that

\[
M' \cap M_k = \bigotimes_{i=1}^k A.
\]

The canonical trace \(\tau\) on \(M_k\) extends the trace vector \(\vec{s}\) on \(A\).

The main tool is the relative commutant theorem by S. Popa [7].

**Lemma 7.** [7] Let \((P_1, \tau_1), (P_2, \tau_2)\) be two finite von Neumann algebras with a common von Neumann subalgebra \(B \subset P_1, B \subset P_2\), such that \(P_1 = Q \otimes B\) where \(Q\) is a nonatomic finite von Neumann algebra. If \((P, \tau)\) denotes the amalgamated free product \((P_1, \tau_1) *_B (P_2, \tau_1)\) then \(Q_0' \cap P = (Q_0' \cap Q) \otimes B\) for any nonatomic von Neumann subalgebra \(Q_0 \subset Q\).

Assume that \(L \subset P\) is a von Neumann subalgebra satisfying the properties:

(1) \(Q_0 \subset L\).

(2) \(L \cap P_2\) contains an element \(y \neq 0\) orthogonal to \(B\), i.e., \(E_B(y) = 0\) and with \(E_B(y^*y) \in C1\).

Then \(L' \cap P = L' \cap B\). If in addition \(L \cap B = C\) then \(L\) is a type \(II_1\) factor.

We can embed \(A\) in the full matrix algebra \(M_d(\mathbb{C})\). Consider the tensor product of \(M_d(\mathbb{C}) \otimes M_2(\mathbb{C})\). Identify \(x \in A\) as \(x \otimes 1_{M_2(\mathbb{C})}\). Take the element \(y\):

\[
y = 1_{M_d(\mathbb{C})} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

In \(M_d(\mathbb{C}) \otimes M_2(\mathbb{C})\), \(y \neq 0\) is an element orthogonal to \(A\), i.e., \(E_A(y) = 0\) and with \(E_A(y^*y) = 1\).

Let \(M\) be a \(II_1\)-factor. We construct \(M_1\) via the following map \(\Gamma\):

\[
M_1 = \Gamma(M) = (M \otimes A) *_{A} M_2n(\mathbb{C}).
\]

The trace \(\tau\) on \(M\) can be extended to \(M_1\).
Put
\[ P_1 = M \otimes A \]
\[ P_2 = M_{2d}(\mathbb{C}) = M_d(\mathbb{C}) \otimes M_2(\mathbb{C}) \]
\[ L = P = M_1 \]

We get:
\[ M_1' \cap M_1 = M_1' \cap A \subseteq M_{2d}(\mathbb{C})' \cap A = \mathbb{C} \]
That is, \( M_1 \) is a nonhyperfinite \( II_1 \) factor.

The relative commutant
\[ M' \cap M_1 = (M' \cap M) \otimes A = A \]

Viewing \( \Gamma \) as a machine producing nonhyperfinite \( II_1 \) factors, we get an ascending towers of \( II_1 \) factors:
\[ M \subset \Gamma(M) = M_1 \subset \cdots \subset M_i \subset \Gamma(M_i) = M_{i+1} \cdots . \]

There is a unique trace \( \tau \) on every \( M_i \).

We calculate the relative commutant \( M' \cap M_k \) by induction. Assume
\[ M' \cap M_k = \otimes^k A. \]

By the above lemma,
\[ M' \cap M_{k+1} = (M' \cap M_k) \otimes A \]
\[ = (\otimes^k A) \otimes A = \otimes^{k+1} A \]

In the end, we would boldly suggest an analogy between binary shifts and free product with amalgamation.

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