Exact duality transformations for sigma models and gauge theories

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1 April 2003

Abstract

We present an exact duality transformation in the framework of Statistical Mechanics for various lattice models with non-Abelian global or local symmetries. The transformation applies to sigma models with variables in a compact Lie group $G$ with global $G \times G$-symmetry (the chiral model) and with variables in coset spaces $G/H$ and a global $G$-symmetry (for example, the non-linear $O(N)$ or $\mathbb{R}P^N$ models) in any dimension $d \geq 1$. It is also available for lattice gauge theories with local gauge symmetry in dimensions $d \geq 2$ and for the models obtained from minimally coupling a sigma model of the type mentioned above to a gauge theory. The duality transformation maps the strong coupling regime of the original model to the weak coupling regime of the dual model. Transformations are available for the partition function, for expectation values of fundamental variables (correlators and generalized Wilson loops) and for expectation values in the dual model which correspond in the original formulation to certain ratios of partition functions (free energies of dislocations, vortices or monopoles). Whereas the original models are formulated in terms of compact Lie groups $G$ and $H$, coset spaces $G/H$ and integrals over them, the configurations of the dual model are given in terms of representations and intertwiners of $G$ and $H$. They are spin networks and spin foams. The partition function of the dual model describes the group theoretic aspects of the strong coupling expansion in a closed form.

PACS: 05.20.-y, 11.15.Ha, 11.15.Me
key words: High temperature expansion, strong coupling expansion, duality transformation, sigma model, lattice gauge theory

1 Introduction

The most prominent example of an exact duality transformation in Statistical Mechanics is the transformation for the two-dimensional Ising model [1]. It is an exact transformation

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which changes the variables of the full partition function of the model and maps the high temperature regime of the original model to the low temperature regime of the dual model and conversely (for the Ising model, the original and the dual model coincide).

In the following, we make use of the correspondence of Quantum Field Theory in the Euclidean (imaginary time) formulation in \( d \) space dimensions plus time with Equilibrium Statistical Mechanics in \( d+1 \) dimensions and often use the words path integral, action and coupling for partition function, energy and temperature, respectively.

The duality transformation of the Ising model was subsequently generalized to more general lattice systems with \( \mathbb{Z}_2 \) symmetries \([2]\), namely systems in \( d \)-dimensions whose variables are \( \mathbb{Z}_2 \)-valued \( k \)-forms, \( 0 \leq k \leq d \), i.e. spin models with global \( \mathbb{Z}_2 \) symmetry, pure lattice gauge theories with local \( \mathbb{Z}_2 \) gauge symmetry, theories for \( \mathbb{Z}_2 \)-valued antisymmetric tensor fields and so on, and to their counterparts with \( U(1) \)-symmetries \([3]\), in particular to the XY-model and pure \( U(1) \) gauge theory on the lattice \([3,4]\). For lattice models with Abelian symmetries, there exists an essentially complete picture \([5]\), and the systems to which the duality transformation applies, include even some Higgs models \([6]\), namely \( U(1) \)-lattice gauge theory minimally coupled to a \( U(1) \)-valued scalar field, i.e. a Higgs field with frozen radial component.

All these examples of the Abelian duality transformation have some features in common. They map the strong coupling regime of the original model to the weak coupling regime of the dual model. This is a consequence of the change of variables employed in the transformation which essentially involves the Fourier decomposition of the interaction terms \( e^{-S} \) for some action \( S \). For example, this replaces \( U(1) \)-variables by integers \( \mathbb{Z} \) and maps Boltzmann weights with narrow peaks to weights with wide peaks and conversely. The structure of the dual model can be sketched as follows. If, say in a sigma model, the variables are originally associated with the points of the lattice and the interaction terms with the bonds, then the variables of the dual model which are introduced by the Fourier expansion, are located at the bonds. In a second step one removes the old variables by performing the relevant sums or integrals which yields additional Boltzmann weights, often constraints, for each point.

As a consequence of the strong-weak nature of the duality transformation, the dual partition function contains essential information on the strong coupling expansion of the original model. In fact, if one understands how the Fourier coefficients of the Boltzmann weight depend on the coupling, the summands of the dual partition function are precisely the terms of the strong coupling expansion and can be sorted by the magnitude of their contribution at strong coupling.

A systematic generalization of these transformations to systems with non-Abelian symmetries proved to be difficult. The calculation of strong coupling expansions of pure non-Abelian lattice gauge theory (see, for example \([7]\) already exhibits some features of the dual model which one wishes to construct. Fourier expansion is generalized to character decomposition, and the dual variables are irreducible representations of the symmetry group, generalizing the wave numbers of the Fourier series. The main technical difficulties are firstly to solve the integrals over the variables of the original model in a systematic way, and secondly to disentangle the lattice combinatorics in order to make the structure of the dual model transparent. Both problems can be overcome if one deals with the non-Abelian group variables at a sufficiently abstract level and if one uses an efficient diagrammatic notation.

The first examples of non-Abelian generalizations were an explicit calculation for pure \( SU(2) \) lattice gauge theory in \( d = 3 \) dimensions \([8]\) and, much less obviously, the equivalence of lattice BF-theory (similar to pure lattice Yang–Mills theory, but with \( \delta \)-functions as the
Boltzmann weights) to certain topological state sum models [9]. This correspondence was developed in a non-perturbative approach to quantum gravity. For review articles, see, for example [10, 11]. The approach to quantum gravity by quantizing a discrete version of the gauge theory formulation of general relativity, has lead to the definition of spin foams [12, 13, 14]. A spin foam is an abstract two-complex, consisting of vertices, edges and faces whose faces are labelled with irreducible representations of some symmetry group while the edges are labelled with compatible intertwiners. Spin foams can be understood as a higher dimensional analogue of spin networks. A spin network is a graph whose edges are labelled with representations and whose vertices are labelled with compatible intertwiners (precise definitions of spin networks and spin foams are given below in Section 3.2).

Spin foams provide the appropriate language for a generalization of the exact duality transformation to pure non-Abelian lattice gauge theory in arbitrary dimension \( d \geq 2 \) whose gauge group is a generic compact Lie group \( G \). See [15, 16] for lattice gauge theory on hypercubic lattices and [17, 18] for the generalization to more general lattices and quantum groups rather than Lie groups.

In this article, we extend the non-Abelian duality transformation to a large class of sigma models\(^1\) whose variables take values in \( G \) or \( G/H \), \( G \) a compact Lie group and \( H \) a Lie subgroup, and which have certain global or local symmetries. This includes, for example, the chiral, the \( O(N) \) and \( \mathbb{R}P^N \) models, and the models that are obtained from minimally coupling such a sigma model to a non-Abelian lattice gauge theory, for example, some generalized Higgs models with frozen radial degree of freedom.

The duality transformation retains its key properties, namely that it provides a strong-weak relation, that it yields a closed form of the strong coupling expansion of the original model and that it maps expectation values of the dual model to ratios of partition functions (free energies of dislocations, vortices or monopoles) in the original formulation and conversely. It therefore relates the fundamental variables of one formulation with some topological defects (collective properties) in the other.

The transformation maps the original model which is formulated in terms of compact Lie groups \( G \) and \( H \), functions on \( G \) and integrals over \( G \) or \( G/H \), to the dual model which is given in terms of the irreducible representations and intertwiners of \( G \) and \( H \). The transformation can be understood as a particular application of a Tannaka–Krein like duality relating groups to their representation categories. That these categories will appear in the dual formulation, had already been proposed in [19]. The dual model can be formulated using merely the language of category theory. In the simplest case, it uses the category of finite-dimensional representations of the symmetry group \( G \). This can be extended to more general categories that do not arise as the representation categories of compact Lie groups. The generalization of lattice gauge theory to quantum groups [17, 18] is one example. For more details on the relation of groups and quantum groups with certain tensor categories, see, for example [20]. In this article, we do not explicitly use the language of category theory, but rather present diagrams in addition to the explicit formulas so that one can easily infer the categorial formulation from these diagrams.

While the configurations of the model dual to lattice gauge theory are spin foams [15], one obtains spin networks as the configurations of the model dual to a sigma model. We thus call the dual models spin foam models and spin network models, respectively. As the notions of spin networks and spin foams have been developed in an approach to quantum gravity, but

\(^1\)The author is grateful to Alan Sokal who suggested to study this generalization.
might not be familiar to the reader working on Statistical Mechanics, we try to make this article self-contained and therefore review all relevant definitions and also some background material on the representation theory of compact Lie groups.

The present article is organized as follows. In Section 2 we summarize some background material on the representation theory of compact Lie groups and introduce a convenient diagrammatical notation. In Section 3 we present our notation for the lattices we use, namely graphs and abstract two-complexes, and we recall the definitions of spin networks and spin foams. In Section 4, we present the duality transformation for the lattice chiral model with symmetry group $G$. This is generalized in Section 5 to the non-linear sigma model with variables in coset spaces $G/H$ and in Section 6 to the non-linear sigma model for $G/H$ coupled to a lattice gauge theory with gauge group $G$. We conclude in Section 7 where we discuss applications, directions for future research and open questions.

2 Mathematical background

In this section, we review some basic concepts and results from the representation theory of compact Lie groups. The material presented here is largely textbook knowledge, see, for example [21, 22] where most of the proofs can be found. The purely algebraic evaluation of the group integrals was first given in [15], our diagrammatic language follows [17, 18].

2.1 Representation functions

Let $G$ be a compact Lie group. This notion includes in particular any finite group (with the discrete topology). We denote finite-dimensional complex vector spaces on which $G$ is represented by $V_\rho$ and by $\rho : G \to \text{Aut} V_\rho$ the corresponding group homomorphisms. Since each finite-dimensional complex representation of $G$ is equivalent to a unitary representation, we select a set $\tilde{R}_G$ containing one unitary representation of $G$ for each equivalence class of finite-dimensional representations. The tensor product, the direct sum and taking the dual are supposed to be closed operations on this set. This amounts to a particular choice of representation isomorphisms $\rho_1 \otimes \rho_2 \leftrightarrow \rho_3$ etc., $\rho_j \in \tilde{R}_G$, which is implicit in our formulas.

We furthermore denote by $R_G \subseteq \tilde{R}_G$ the subset of irreducible representations.

For a representation $\rho \in \tilde{R}_G$, the dual representation is denoted by $\rho^*$, and the dual vector space of $V_\rho$ by $V_\rho^*$. The dual representation is given by $\rho^* : G \mapsto \text{Aut} V_\rho^*$, where

$$\rho^*(g) : V_\rho^* \to V_\rho^*, \quad \eta \mapsto \eta \circ \rho(g^{-1}),$$

i.e. $(\rho^*(g)\eta)(v) = \eta(\rho(g^{-1})v)$ for all $v \in V_\rho$. There exists a one-dimensional ‘trivial’ representation of $G$ which is isomorphic to $\mathbb{C}$.

For the unitary representations $V_\rho$, $\rho \in \tilde{R}_G$, we have standard (sesquilinear) scalar products $\langle \cdot ; \cdot \rangle$ and orthonormal bases $\{ e_j \}_j$. Therefore, we can define a bijective antilinear map $\ast : V_\rho \to V_\rho^*$ induced by the scalar product,

$$\ast(v) := (w \mapsto \langle v; w \rangle), \quad v \in V_\rho,$$

and construct the dual bases $\{ \eta^j \}_j$ by $\eta^j := \ast(e_j)$. Identifying $(V_\rho^*)^* = V_\rho$, this yields $\langle e_j; e_k \rangle = \eta^j(e_k) = \delta_{jk}$ and furthermore induces a scalar product on $V_\rho^*$, namely $\langle \eta^j; \eta^k \rangle = \eta^k(e_j)$, $1 \leq j, k \leq \dim V_\rho$. 


The matrix elements of the representation matrices $\rho(g)$ define complex valued functions,

$$t^{(\rho)}_{jk}(g) = \eta^j(\rho(g)e_k) = \langle \rho(g) \rangle_{jk},$$

where $\rho \in \tilde{\mathcal{R}}_G$, $1 \leq j, k \leq \dim V_\rho$. They are called representation functions of $G$ and form a commutative and associative unital algebra over $\mathbb{C}$,

$$C_{\text{alg}}(G) := \{t^{(\rho)}_{jk} : \rho \in \tilde{\mathcal{R}}_G, 1 \leq j, k \leq \dim V_\rho \},$$

whose product is given by the matrix elements of the tensor product of representations,

$$(t^{(\rho)}_{jk} \cdot t^{(\sigma)}_{\ell m})(g) := t^{(\rho \otimes \sigma)}_{j\ell,km}(g),$$

where $\rho, \sigma \in \tilde{\mathcal{R}}_G$, $1 \leq j, k \leq \dim V_\rho$ and $1 \leq \ell, m \leq \dim V_\sigma$.

We find the following expressions involving the group unit $e \in G$,

$$t^{(\rho)}_{jk}(e) = \delta_{jk},$$

products of group elements,

$$t^{(\rho)}_{jk}(g \cdot h) = \sum_{\ell=1}^{\dim V_\rho} t^{(\rho)}_{j\ell}(g) \cdot t^{(\rho)}_{\ell k}(h),$$

and inverse group elements,

$$t^{(\rho)}_{jk}(g^{-1}) = (\rho(g)^{-1})_{jk} = \overline{(\rho(g))_{kj}} = t^{(\rho^*)}_{kj}(g),$$

as well as,

$$t^{(\rho)}_{jk}(g^{-1}) = \eta^j(\rho(g)^{-1}e_k) = (\rho^*(g)\eta^j)(e_k) = \langle \eta^j; \rho^*(g)\eta^j \rangle = t^{(\rho^*)}_{kj}(g),$$

so that for unitary representations, the dual representation is just the conjugate one. The bar denotes complex conjugation.

### 2.2 Peter–Weyl decomposition and theorem

The structure of the algebra $C_{\text{alg}}(G)$ can be understood if $C_{\text{alg}}(G)$ is considered as a representation of $G \times G$ by combined left and right translation of the function argument,

$$(G \times G) \times C_{\text{alg}}(G) \rightarrow C_{\text{alg}}(G), \quad ((g_1, g_2), f) \mapsto (h \mapsto f(g_1^{-1}hg_2)).$$

It can then be decomposed into its irreducible components as a representation of $G \times G$.

**Theorem 2.1 (Peter–Weyl decomposition).** Let $G$ be a compact Lie group.

1. There is an isomorphism

$$C_{\text{alg}}(G) \cong \bigoplus_{\rho \in \mathcal{R}_G} (V_\rho \otimes V_\rho^*),$$

of representations of $G \times G$. Here the direct sum runs over the equivalence classes of finite-dimensional irreducible representations of $G$. The direct summands $V_\rho \otimes V_\rho^*$ are irreducible as representations of $G \times G$. 
2. The direct sum in (2.11) is orthogonal with respect to the $L^2$-scalar product on $C_{\text{alg}}(G)$ which is formed using the Haar measure of $G$ on the left hand side, and using the standard scalar products on the right hand side,
\[
\langle t_{jk}^{(\rho)} \cdot t_{lm}^{(\sigma)} \rangle_{L^2} := \int_G t_{jk}^{(\rho)}(g) \cdot t_{lm}^{(\sigma)}(g) \, dg = \frac{1}{\dim V_\rho} \delta_{\rho \sigma} \delta_j \delta_k \delta_{lm},
\] (2.12)

where $\rho, \sigma \in \mathcal{R}_G$ are irreducible. The Haar measure is denoted by $\int_G$ and normalized so that $\int_G dg = 1$.

If $G$ is finite, the Haar measure coincides with the normalized summation over all group elements. The decomposition (2.11) directly corresponds to our notation of the representation functions $t_{jk}^{(\rho)}$ for irreducible $\rho \in \mathcal{R}_G$.

**Corollary 2.2.** Each representation function $f \in C_{\text{alg}}(G)$ can be decomposed according to (2.11),
\[
f(g) = \sum_{\rho \in \mathcal{R}_G} \sum_{j,k=1}^{\dim V_\rho} \hat{f}_{jk}^{(\rho)} t_{jk}^{(\rho)}(g), \quad \text{where} \quad \hat{f}_{jk}^{(\rho)} = \dim V_\rho \int_G t_{jk}^{(\rho)}(g) f(g) \, dg.
\] (2.13)

For any algebraic $f \in C_{\text{alg}}(G)$, all except finitely many coefficients $f_{jk}^{(\rho)}$ are zero. The analytical aspects of $C_{\text{alg}}(G)$ are given by the Peter–Weyl theorem.

**Theorem 2.3 (Peter–Weyl Theorem).** Let $G$ be a compact Lie group. Then $C_{\text{alg}}(G)$ is dense in $L^2(G)$ with respect to the $L^2$-norm.

We use the Peter–Weyl theorem in order to complete $C_{\text{alg}}(G)$ with respect to the $L^2$-norm to $L^2(G)$. Functions $f \in L^2(G)$ then correspond to square summable series in (2.13). These series are invariant under a reordering of summands, and their limits commute with group integrations. We make use of these invariances in the duality transformation. If $G$ is a finite group, $C_{\text{alg}}(G)$ is a finite-dimensional vector space so that the corresponding results hold trivially.

We can summarize these ideas and state that the algebraic structure of $C_{\text{alg}}(G)$ is sufficient to determine the structure of the larger function space $L^2(G)$.

### 2.3 Character decomposition

The **characters** of $G$ are the algebraic class functions, i.e. those functions $f \in C_{\text{alg}}(G)$ that satisfy $f(hgh^{-1}) = f(g)$ for all $g, h \in G$.

**Proposition 2.4.** For class functions $f \in C_{\text{alg}}(G)$, the Peter–Weyl decomposition (2.13) specializes to the **character decomposition**
\[
f(g) = \sum_{\rho \in \mathcal{R}_G} \hat{f}_\rho \chi^{(\rho)}(g), \quad \text{where} \quad \hat{f}_\rho = \int_G \chi^{(\rho)}(g) f(g) \, dg.
\] (2.14)

Here
\[
\chi^{(\rho)} := \sum_{j=1}^{\dim V_\rho} t_{jj}^{(\rho)}
\] (2.15)
denotes the character of the representation $\rho \in \tilde{\mathcal{R}}_G$. For irreducible $\rho, \sigma \in \mathcal{R}_G$, the orthogonality relation \((2.12)\) implies,
\[
\langle \chi^{(\rho)}; \chi^{(\sigma)} \rangle_{L^2} = \int_G \chi^{(\rho)}(g) \chi^{(\sigma)}(g) \, dg = \delta_{\rho\sigma}.
\] \tag{2.16}

### 2.4 Algebraic evaluation of group integrals

For the duality transformation, it is important to understand the Haar measure of $G$ in the picture of the Peter–Weyl decomposition \((2.11)\). First we decompose a generic representation function into representation functions of irreducible representations.

**Proposition 2.5.** Let $G$ be a compact Lie group and $\rho \in \tilde{\mathcal{R}}_G$ be a finite-dimensional unitary representation of $G$ with the complete decomposition
\[
V_\rho \cong \bigoplus_{j=1}^k V_{\tau_j}, \quad \tau_j \in \mathcal{R}_G, k \in \mathbb{N},
\] \tag{2.17}
into irreducible components $\tau_j$. Let $P^{(j)}: V_\rho \to V_{\tau_j} \subseteq V_\rho$ be the $G$-invariant orthogonal projectors associated with the above decomposition. Then
\[
t^{(\rho)}_{mn}(g) = \sum_{j=1}^k \sum_{p,q=1}^{\dim V_{\tau_j}} P^{(j)}_{pq} t^{(\tau_j)}_{pq}(g) P^{(j)}_{qn},
\] \tag{2.18}
where $P^{(j)}_{qn} = \langle w^{(j)}_q; v_n \rangle$. Here $\{v_i\}_i$ denotes an orthonormal basis of $V_\rho$ and $\{w^{(j)}_i\}_i$ an orthonormal basis of $V_{\tau_j} \subseteq V_\rho$.

**Proof.** The representation function is Peter–Weyl decomposed by inserting $1 = \sum_{j=1}^k P^{(j)}$ twice into the right hand side of $t^{(\rho)}_{mn}(g) = \langle v_m; \rho(g)v_n \rangle$. We use $G$-invariance $[P^{(j)}; \rho(g)] = 0$ and transversality $P^{(i)} P^{(j)} = \delta_{ij} P^{(j)}$ in order to obtain
\[
t^{(\rho)}_{mn}(g) = \sum_{j=1}^k \langle v_m; P^{(j)} \rho(g) P^{(j)} v_n \rangle.
\] \tag{2.19}
Here $\rho(g) P^{(j)} = \tau_j(g) P^{(j)}$ and
\[
P^{(j)} = \sum_{p=1}^{\dim V_{\tau_j}} w^{(j)}_p \cdot \varrho^{(j)}_p,
\] \tag{2.20}
where $\\{\varrho^{(j)}_i\}_i$ denotes a basis dual to $\{w^{(j)}_i\}_i$. Inserting \((2.20)\) into \((2.19)\), we obtain \((2.18)\). \qed

For representation functions of an irreducible representation $\rho \in \mathcal{R}_G$, the Haar measure is
\[
\int_G t^{(\rho)}_{j_{nk}}(g) \, dg = \begin{cases} 1, & \text{if } \rho \text{ is trivial}, \\ 0, & \text{otherwise}, \end{cases}
\] \tag{2.21}
as a consequence of its left-right translation invariance. This can be applied to \((2.18)\) in order to derive an entirely algebraic expression for the Haar measure.
Corollary 2.6. Let $G$ be a compact Lie group and $\rho \in \hat{\mathcal{R}}_G$ be a finite-dimensional unitary representation of $G$ with the decomposition (2.17). Assume that precisely the first $\ell$ components $\tau_1, \ldots, \tau_\ell$, $0 \leq \ell \leq k$, are equivalent to the trivial representation. Then the Haar measure of a representation function $t^{(\rho)}_{mn}$, $1 \leq m, n \leq \dim V_\rho$, is given by

$$\int_G t^{(\rho)}_{mn}(g) \, dg = \sum_{j=1}^\ell P^{(j)}_m P^{(j)}_n. \quad (2.22)$$

Here we have omitted the vector indices corresponding to the one-dimensional representations.

In our calculations, we will refer to Corollary 2.6 in a context in which the integrand is a product of representation functions of irreducible representations. This motivates the following definition.

Definition 2.7. Let $G$ be a compact Lie group and $\rho_1, \ldots, \rho_r \in \mathcal{R}_G$, $r \in \mathbb{N}$, be finite-dimensional irreducible representations of $G$. The Haar intertwiner

$$T: \bigotimes_{\ell=1}^r V_{\rho_\ell} \to \bigotimes_{\ell=1}^r V_{\rho_\ell}, \quad (2.23a)$$

is the linear map defined by its matrix elements

$$T_{m_1m_2\ldots m_r;n_1n_2\ldots n_r} := \int_G t^{(\rho_1)}_{m_1n_1}(g) t^{(\rho_2)}_{m_2n_2}(g) \cdots t^{(\rho_r)}_{m rnn_r}(g) \, dg. \quad (2.23b)$$

Proposition 2.8. The Haar intertwiner $T$ of (2.23) satisfies

$$T_{m_1m_2\ldots m_r;n_1n_2\ldots n_r} = \sum_j P^{(j)}_{m_1m_2\ldots m_r} P^{(j)}_{n_1n_2\ldots n_r}, \quad (2.24)$$

for the projectors

$$P^{(j)}_{n_1n_2\ldots n_r} := \langle w^{(j)}; e^{(\rho_1)}_{n_1} \otimes e^{(\rho_2)}_{n_2} \otimes \cdots \otimes e^{(\rho_r)}_{n_r} \rangle, \quad (2.25)$$

with the definitions of Proposition 2.5 as well as for all $h \in G$,

$$T = (\rho_1(h) \otimes \cdots \otimes \rho_r(h)) \circ T = T \circ (\rho_1(h) \otimes \cdots \otimes \rho_r(h)), \quad (2.26)$$

$$T \circ T = T. \quad (2.27)$$

The first equation (2.24) is a consequence of Corollary 2.6 while (2.26) and (2.27) follow from the translation invariance of the Haar measure. In particular, $T$ forms a morphism of representations (intertwiner) of $G$. The map $T$ has been studied in a more general context in [18].

In the subsequent sections, we will apply Corollary 2.6 in rather complicated calculations. It is therefore convenient to introduce diagrams which visualize the structure of the indices in these formulas (Figure 1).

The diagrams are read from top to bottom. We draw directed lines which are labelled with representations $\rho \in \hat{\mathcal{R}}_G$ of $G$. If the arrow points down, the line denotes the identity map of $V_\rho$, Figure 1(a). If the arrow points up, it refers to the identity map of the dual representation $V_\rho^*$. A representation function $t^{(\rho)}_{mn}$ is denoted by a box with an incoming and an outgoing line (b), and a product of representation functions by boxes placed next to each other (c). The Haar intertwiner is visualized by the box labelled $T$ in (d), and the calculation of $T$ given by (2.23b) is shown in diagram (e) where the full dots represent the projectors, and the dotted line indicates the simultaneous summation over them.
2.5 Coset spaces and spherical functions

In the study of coset spaces $G/H$, we allow $H$ to be any Lie subgroup of $G$. First we recall some basic definitions.

Definition 2.9. Let $G$ be a compact Lie group and $H \leq G$ be a Lie subgroup.

1. A finite-dimensional irreducible representation $V_\rho$ of $G$ is said to be of class-1 with respect to $H$ if $V_\rho$ contains an $H$-invariant vector $0 \neq v_0 \in V_\rho$, i.e. $\rho(h)v_0 = v_0$ for all $h \in H$. The subset $\mathcal{R}_H^G \subseteq \mathcal{R}_G$ denotes the set of class-1 representations of $G$ with respect to $H$.

2. $H$ is called a massive subgroup of $G$ if for each class-1 representation $\rho \in \mathcal{R}_H^G$, the subspace of $H$-invariant vectors,

$$\text{Inv}^\rho_H = \{ v \in V_\rho : \rho(h)v = v \text{ for all } h \in H \},$$

is one-dimensional.

Proposition 2.10. Let $G$ be a compact Lie group, $H \leq G$ a Lie subgroup and $\rho \in \mathcal{R}_G$ a finite-dimensional irreducible representation of $G$. The subspace of $H$-invariant vectors in $V_\rho$ is spanned by the (not necessarily linearly independent) vectors

$$v^{(k)} := \sum_{j=1}^{\dim V_\rho} v^{(k)}_j e_j, \quad \text{where} \quad v^{(k)}_j := \int_H t^{(\rho)}_{jk}(h) \, dh. \quad (2.29)$$

Here $\{e_j\}_j$ denotes the standard orthonormal basis of $V_\rho$, and $1 \leq k \leq \dim V_\rho$.

The motivation for studying $H$-invariant vectors is given by the following result which allows us to construct functions on the space $G/H$ of left cosets, i.e. functions on $G$ that are constant on the cosets $gH$. 

Figure 1: Diagrams to visualize the index structure in the calculation of group integrals. (a) The identity map of $V_\rho$; (b) a representation function $t^{(\rho)}_{mn}$; (c) a product of representation functions $t^{(\rho_1)}_{m_1n_1} \cdots t^{(\rho_r)}_{m_rn_r}$; (d) the Haar intertwiner; and (e) the calculation of the Haar intertwiner (2.23b).
Proposition 2.11. Let \( 0 \neq v^{(k)} \in V_\rho \) be an \( H \)-invariant vector. Then the generalized spherical functions
\[
H_{jk}^{(\rho)}(g) := \sum_{\ell=1}^{\dim V_\rho} t_{j\ell}^{(\rho)}(g) v_{\ell}^{(k)}, \tag{2.30}
\]
1 \( \leq j \leq \dim V_\rho \), are constant on the cosets \( gH \in G/H \) and therefore induce functions \( H_{jk}^{(\rho)} : G/H \to \mathbb{C}, \ x \mapsto H_{jk}^{(\rho)}(x) := H_{jk}^{(\rho)}(g_x) \), where \( g_x \in G \) is a representative of the coset \( x = g_x H \in G/H \).

Combining the Peter–Weyl decomposition (2.13) of \( C_{\text{alg}}(G) \) with the above ideas, we can construct the algebraic functions \( C_{\text{alg}}(G/H) \) on the coset space.

**Corollary 2.12.** Let \( G \) be a compact Lie group and \( H \leq G \) be a Lie subgroup. Denote the dimensions of the \( H \)-invariant subspaces by \( \kappa_\rho := \dim \text{Inv}_H^{(\rho)} \) and choose the orthonormal basis \( \{e_j\}_j \) of each \( V_\rho \) so that precisely \( e_1, \ldots, e_{\kappa_\rho} \) are \( H \)-invariant. Then the functions
\[
H_{jk}^{(\rho)} : G/H \to \mathbb{C}, \quad x \mapsto H_{jk}^{(\rho)}(x) := t_{jk}^{(\rho)}(g_x), \tag{2.31}
\]
\( \rho \in \mathcal{R}_G^H, \ 1 \leq j \leq \dim V_\rho, \ 1 \leq k \leq \kappa_\rho \), form a basis of \( C_{\text{alg}}(G/H) \). These functions satisfy the orthogonality relation,
\[
\langle H_{jk}^{(\rho)}; H_{\ell m}^{(\sigma)} \rangle_{L^2} = \int_{G/H} \overline{H_{jk}^{(\rho)}(x)} H_{\ell m}^{(\sigma)}(x) \, dx = \frac{1}{\dim V_\rho} \delta_{\rho\sigma} \delta_{j\ell} \delta_{km}. \tag{2.32}
\]

**Remark 2.13.**

1. Spherical functions exist only for class-1 representations as \( \kappa_\rho \neq 0 \) only there.

2. In the case of a massive subgroup \( H \), there is \( \kappa_\rho = 1 \) for all class-1 representations \( \rho \in \mathcal{R}_G^H \). The second index of the spherical functions can thus be omitted, i.e.
\[
H_j^{(\rho)} : G/H \to \mathbb{C}, \quad x \mapsto H_j^{(\rho)}(x) := t_j^{(\rho)}(g_x), \tag{2.33}
\]
where \( 1 \leq j \leq \dim V_\rho \).

3. If \( H \leq G \) is a normal subgroup, there is \( \kappa_\rho = \dim V_\rho \) for all class-1 representations. In other words, for a given irreducible representation \( \rho \in \mathcal{R}_G \) of \( G \), either all representation functions \( t_{jk}^{(\rho)}, \ 1 \leq j, k \leq \dim V_\rho \), are spherical functions, or none of them is.

**Example 2.14.**

1. The spheres \( S^N \cong SO(N+1)/SO(N) \) or \( S^N \cong O(N+1)/O(N) \) are formed using massive subgroups.

2. Odd spheres can alternatively be obtained from \( S^{2N+1} \cong SU(N+1)/SU(N) \) or \( S^{2N+1} \cong U(N+1)/U(N) \), in particular \( S^3 \cong SU(2) \). The spherical functions of \( S^3 \) can thus be constructed either as functions on \( SO(4)/SO(3) \) using the construction sketched above or from the identification \( S^3 \cong SU(2) \). For the latter approach, see the introductory part of [23].

3. Other coset spaces which are of interest in the context of sigma models, are \( \mathbb{RP}^{N-1} \cong O(N)/\{O(N-1) \times O(1)\} \) as a special case of the Grassmanian \( G_{KN}^\mathbb{R} \cong O(N)/(O(N-k) \times O(k)) \) and their complex counterparts \( \mathbb{CP}^{N-1} \cong U(N)/(U(N-1) \times U(1)) \) and \( G_{KN}^\mathbb{C} \cong U(N)/(U(N-k) \times U(k)) \).
Figure 2: (a) A spherical function $H^{(\rho)}_{\text{spin}}(x)$; (b) the coset space Haar map and (c) its calculation in terms of $G$-invariant projectors \((2.36)\), see Proposition \(2.17\).

Remark 2.15. Any function $f: G/H \to \mathbb{C}$ naturally extends to a function $\tilde{f}: G \to \mathbb{C}$ which is constant on the cosets, i.e. $f(gh) = \tilde{f}(g)$ for all $g \in G$, $h \in H$. Obviously $f(x) = \tilde{f}(g_x)$ for all $x \in G/H$ and an arbitrary representative $g_x \in G$ of $x$. Integrals over $G/H$ can thus be evaluated using integrals over $G$, 

$$\int_{G/H} f(x) \, dx = \int_{G} \tilde{f}(g) \, dg. \quad (2.34)$$

As the context is usually clear, we omit the tilde ($\tilde{}$) from now on.

The analogue of the Haar intertwiner \((2.23)\) for coset spaces can be defined as follows.

Definition 2.16. Let $G$ be a compact Lie group, $H \leq G$ be a Lie subgroup and $\rho_1, \ldots, \rho_r \in \mathcal{R}_H$, $r \in \mathbb{N}$, be of class-1 with respect to $H$. The coset space Haar map, 

$$I: \bigotimes_{\ell=1}^{r} V_{\rho_\ell} \to \bigotimes_{\ell=1}^{r} V_{\rho_\ell}, \quad (2.35a)$$

is the linear map defined by its matrix elements 

$$I_{m_1m_2\ldots m_r, n_1n_2\ldots n_r} := \int_{G/H} H^{(\rho_1)}_{m_1n_1}(x)H^{(\rho_2)}_{m_2n_2}(x)\cdots H^{(\rho_r)}_{m_rn_r}(x) \, dx, \quad (2.35b)$$

where $1 \leq m_\ell \leq \dim V_{\rho_\ell}$ and $1 \leq n_\ell \leq \kappa_{\rho_\ell}$.

Proposition 2.17. The coset space Haar map \((2.35)\) satisfies 

$$I_{m_1m_2\ldots m_r, n_1n_2\ldots n_r} = \sum_j P^{(j)}_{m_1m_2\ldots m_r} P^{(j)}_{n_1n_2\ldots n_r}, \quad (2.36)$$
MATHEMATICAL BACKGROUND

Figure 3: The specialization of Figure 2 to the case of a massive subgroup $H \leq G$, see Corollary 2.18

with the notation of Proposition 2.8 as well as for all $h \in G$,

$$I = (\rho_1(h) \otimes \cdots \otimes \rho_r(h)) \circ I,$$

which makes use of the left action of $G$ on $G/H$. If in addition $H \leq G$ is a normal subgroup, then $I$ satisfies for all $y \in G/H$,

$$I = I \circ (\rho_1(y) \otimes \cdots \otimes \rho_r(y)),$$

using the notation $\rho_\ell(y) := \rho_\ell(g_y)$ for any representative $g_y$ of $y$.

Proof. Equations (2.36) and (2.37) follow from Proposition 2.8 and Remark 2.15. The same is true for (2.38) if we write it for a representative of the coset $y \in G/H$. □

Observe that the coset space Haar map is in general not an intertwiner of $G$. However, for any fixed choice of indices $n_\ell \in \{1, \ldots, \kappa_\rho\}$, it defines a $G$-invariant vector $I_{n_1 \cdots n_r} \in \bigotimes_{\ell=1}^r V_{\rho_\ell}$.

We visualize spherical functions $H_{mn}(x)$ and the coset space Haar map as in Figure 2. The contraction of indices whose range is restricted to $\{1, \ldots, \kappa_\rho\}$, is represented by dashed lines. They do not correspond to representations of $G$ and are therefore not labelled with any symbol such as $\rho$. A thick line in the box for $H_{mn}^{(\rho)}$ and in the coset space Haar map $I$ indicates that the indices on this side are special. Figure 2 (c) shows the calculation (2.36).

If $H \leq G$ is a normal subgroup, $\kappa_\rho = \dim V_\rho$ for all class-1 representations so that the dashed lines become solid again as they do correspond to representations of $G$. In particular Definition 2.16 and Proposition 2.17 restrict to Definition 2.7 and Proposition 2.8 respectively, if $H = \{e\}$ is the trivial group. The special case when $H$ is a massive subgroup, is also of interest.

**Corollary 2.18.** If $H \leq G$ is a massive subgroup, then $\kappa_\rho = 1$ for the class-1 representations. Therefore all indices $n_\ell$ can be omitted from the expressions so that the Haar map reduces to a map

$$I: \mathbb{C} \to \bigotimes_{\ell=1}^r V_{\rho_\ell},$$

(2.39)
2.6 The centre of the group

defined by its matrix elements

\[ I_{m_1m_2...m_r} := \int_{G/H} H^{(\rho_1)}(x) H^{(\rho_2)}(x) \cdots H^{(\rho_r)}(x) \, dx. \]  

(2.40)

Equation (2.36) specializes to

\[ I_{m_1m_2...m_r} = \sum_j P_j^{(\rho)} m_1 m_2 \cdots m_r, \]  

(2.41)

and (2.37) indicates that \( I \) defines a \( G \)-invariant vector \( I \in \bigotimes_{\ell=1}^r V_{\rho_\ell} \).

The situation for massive subgroups is illustrated in Figure 3. Further properties of the diagrams used in Figure 3 can be deduced as in the introductory section of [23].

2.6 The centre of the group

If representation functions are restricted to the centre \( Z(G) \), we obtain representation functions of the Abelian group \( Z(G) \).

Lemma 2.19. Let \( G \) be a compact Lie group and \( X \in Z(G) \). Then for any irreducible unitary representation \( \rho \) of \( G \) and \( 1 \leq i, j \leq \dim V_\rho \),

\[ t^{(\rho)}_{ij}(X) = \delta_{ij} \cdot \overline{t}(\rho)(X), \]  

(2.42)

where \( \overline{t}(\rho) : Z(G) \rightarrow \mathbb{C} \) is a representation of the centre \( Z(G) \) which is induced by \( \rho \).

Proof. By Schur’s lemma, the centre is mapped to multiples of the unit matrix. \( \square \)

2.7 Special properties of some groups

For \( G = U(1) \), all finite-dimensional irreducible representations are one-dimensional. They are denoted by \( V_k \cong \mathbb{C} \) and characterized by integers \( k \in \mathbb{Z} \) (wave numbers of the Fourier series). In the unitary case, their representation functions are given by \( t^{(k)}(g) = g^k, g \in U(1) \), and their duals by \( \overline{t}^{(k^*)}(g) = g^{-k} \). All representation functions are characters, \( \chi^{(k)}(g) = t^{(k)}(g) = e^{ik\phi} \), where we write \( g = e^{i\phi} \in U(1) \).

From the representation functions and the definition (2.25), we can calculate the tensor product which is again one-dimensional,

\[ V_{k_1} \otimes \cdots \otimes V_{k_n} \cong \bigotimes_{\ell=1}^n V_{k_\ell}. \]  

(2.43)

It is isomorphic to the trivial representation if and only if

\[ \sum_{\ell=1}^n k_\ell = 0. \]  

(2.44)

Since all irreducible representations are one-dimensional, the Haar intertwiner (2.23),

\[ T : \bigotimes_{\ell=1}^r V_{k_\ell} \rightarrow \bigotimes_{\ell=1}^r V_{k_\ell}, \]  

(2.45)
Figure 4: (a) Simplification of (2.24) for $G = SU(2)$ in the case of three tensor factors; (b)–(d) Even if we do not write the dotted line anymore, this does not mean that any conceivable symmetry holds.

is just multiplication by a number. We have $T = 1$ if (2.44) holds and $T = 0$ otherwise. The sum over projectors (2.24) is either empty or contains a single unique term.

We write the elements of the cyclic groups $G = \mathbb{Z}_N$ as roots of unity, $e^{2\pi i \ell/N}$, $\ell \in \{0, \ldots, N - 1\}$, and parameterize their finite-dimensional irreducible unitary representations $V_k \cong \mathbb{C}$ by $k \in \{0, \ldots, N - 1\}$. Representation functions and characters are $t^{(k)}(g) = g^k$, $t^{(k^*)}(g) = g^{-k}$, $\chi^{(k)} = t^{(k)}$, and (2.43) and (2.44) still hold if the sums are taken modulo $N$.

For $G = SU(2)$, we characterize the finite-dimensional irreducible representations $V_j$, $\dim V_j = 2j + 1$, by non-negative half-integers $j \in \frac{1}{2} \mathbb{N}_0$. Parameterizing elements of $SU(2)$ by

$$g(\vartheta, \underline{n}) = \mathbf{1} \cos \frac{\vartheta}{2} + i \underline{n} \cdot \underline{\sigma} \sin \frac{\vartheta}{2},$$

where $\vartheta \in [0, 4\pi)$, $\underline{n} \in S^2 \subseteq \mathbb{R}^3$ and $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, the characters are given by

$$\chi^{(j)}(g) = \frac{\sin((2j + 1)\frac{\vartheta}{2})}{\sin \frac{\vartheta}{2}},$$

in particular for the fundamental representation $\chi^{(\frac{1}{2})}(g) = \cos \frac{\vartheta}{2}$.

Since for $SU(2)$, there are no higher multiplicities in the decomposition of $V_{j_1} \otimes V_{j_2}$, the space of invariant projectors $V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \rightarrow \mathbb{C}$ has a dimension of at most one. For three irreducible representations, we can therefore omit the summation over projectors from (2.24) as is illustrated in Figure 4(a) and impose the conditions $|j_1 - j_2| \leq j_3 \leq j_1 + j_2$, etc. instead. This provides a substantial simplification. However, the three-valent vertex that appears there, has in general only a cyclic, but not a full symmetry (Figure 4(b–d)) so that one still has to take the ordering of the tensor factors into account. Neglecting this subtlety is a common mistake.
2.8 Some character decompositions

For the duality transformation, we will apply the character decomposition to the Boltzmann weight \( f(g) = \exp(-s(g)) \) whose local action \( s: G \to \mathbb{R} \) is an \( L^2 \)-integrable class function that is bounded below. The most common example is the Wilson action,

\[
s(g) = -\frac{\beta}{2 \dim V_\rho} (\chi^{(\rho)}(g) + \chi^{(\overline{\rho})}(g)),
\]

where \( \rho \) denotes the fundamental representation of \( G \) and \( \beta \) is the inverse temperature or inverse coupling constant.

For \( G = U(1) \), the Wilson action reads \( s(g) = -\beta \cos \varphi \), \( g = e^{i\varphi} \). The character decomposition coincides with the Fourier series,

\[
f(g) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\varphi}, \quad \hat{f}_k = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ik\varphi} \exp(-s(e^{i\varphi})) d\varphi = I_k(\beta),
\]

where \( I_k(\beta) \) denote modified Bessel functions.

For \( G = \mathbb{Z}_N, g = e^{2\pi i \ell/N} \), we write this decomposition as

\[
f(g) = \sum_{k=1}^{N-1} \hat{f}_k e^{2\pi i k\ell/N}, \quad \hat{f}_k = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{-2\pi i k\ell/N} \exp(-s(e^{2\pi i \ell/N})).
\]

For \( G = SU(2) \), we have the Wilson action \( s(g) = -\beta \cos^{\frac{3}{2}} \), and the character expansion of \( f(g) = \exp(-s(g)) \) is given by

\[
f(g) = \sum_{j \in \frac{1}{2} \mathbb{N}_0} \hat{f}_j \frac{\sin((2j+1)\frac{\beta}{2})}{\sin \frac{\beta}{2}}, \quad \hat{f}_j = \frac{2j+1}{\beta} I_{2j+1}(\beta).
\]

Another common action is the heat kernel or generalized Villain action which is given for any compact Lie group in terms of the character decomposition of the corresponding Boltzmann weight,

\[
f(g) = \sum_{\rho \in \mathcal{R}_G} \hat{f}_\rho \chi^{(\rho)}(g), \quad \hat{f}_\rho = \dim V_\rho \cdot \exp\left(-\frac{C_\rho}{2\beta}\right),
\]

where \( C_\rho \) denotes the eigenvalue of a quadratic Casimir operator in the representation \( \rho \). For example, we have for \( G = U(1) \),

\[
\hat{f}_k = \exp\left(-\frac{k^2}{2\beta}\right),
\]

and for \( G = SU(2) \),

\[
\hat{f}_j = (2j+1) \exp\left(-\frac{j(j+1)}{2\beta}\right).
\]

In all these cases, the Boltzmann weight \( f(g) = \exp(-s(g)) \) has a sharp peak at the group unit if \( \beta \) is large (weak coupling, low temperature) which facilitates a perturbative treatment whereas the peak is very wide for small \( \beta \) (strong coupling, high temperature). For small \( \beta \), however, the dominant contribution to the character expansions listed above originates from
3 Notation and definitions

3.1 Graphs and abstract 2-complexes

In order to formulate sigma models and gauge theories on very general lattices, it is sufficient to focus on the combinatorial structure of the lattices rather than on the details of their embedding into some space or space-time manifold. Therefore we employ the notions of graphs and abstract two-complexes. Sigma models are defined on graphs so that we obtain the same expressions for their partition function in any dimension \( d \geq 1 \). Similarly, gauge theories are defined on abstract two-complexes, and we obtain a uniform description of gauge theories valid in any dimension \( d \geq 2 \).

The following definitions differ slightly from what is standard, but will prove most convenient for the purpose of the duality transformation.

**Definition 3.1.** An oriented (or directed) graph \((V, E)\) consists of finite sets \(V\) (vertices) and \(E\) (edges) together with maps

\[
\begin{align*}
\partial_+ : E &\to V, \quad \text{(end point of an edge)} \tag{3.1a} \\
\partial_- : E &\to V, \quad \text{(starting point of an edge)} \tag{3.1b}
\end{align*}
\]

The notion of an abstract two-complex extends this definition and also includes faces whose boundary consists of a sequence of edges.

**Definition 3.2.** An oriented two-complex \((V, E, F)\) is an oriented graph \((V, E)\) together with a finite set \(F\) (faces) and maps

\[
\begin{align*}
N : F &\to \mathbb{N}, \quad \text{(number of edges in the boundary of a face)} \tag{3.2a} \\
\partial_j : F &\to E, \quad \text{(the } j\text{-th edge in the boundary of a face)} \tag{3.2b} \\
\varepsilon_j : F &\to \{-1, +1\}, \quad \text{(its orientation)} \tag{3.2c}
\end{align*}
\]

Figure 5: The maps \(\partial_\pm, \partial_j\) and \(\varepsilon_j\) and the conditions. Here \(N(f) = 3\), \(\varepsilon_1 f = +1\), \(\varepsilon_2 f = +1\) and \(\varepsilon_3 f = (-1)\).

the ‘small’ representations of \(G\). An expansion in terms of irreducible representations of \(G\) therefore provides us with a strong coupling expansion. This is most obvious for the heat kernel action where at small \(\beta\) the representations with small Casimir eigenvalue dominate.

More details on strong coupling expansion techniques can be found in [7]. For spherical functions, see [22] and in particular for \(S^N\) also [24].
such that
\[
\partial_{-\varepsilon_j} f \partial_j f = \partial_{\varepsilon_{j+1}} f \partial_{j+1} f, \quad 1 \leq j \leq N(f) - 1, \quad (3.3a)
\]
\[
\partial_{-\varepsilon_{N(f)}} f \partial_{N(f)} f = \partial_{\varepsilon_1} f \partial_1 f, \quad (3.3b)
\]
for all \( f \in F \).

The conditions (3.3) state that the edges in the boundary of a face \( f \in F \) are in cyclic ordering from \( \partial_{N(f)} f \) to \( \partial_1 f \) where one encounters the edges with the orientation given by \( \varepsilon_j f \), see Figure 5. Observe that (3.3) contains combinatorial information similar to the condition \( \partial \circ \partial = 0 \) on the boundary operator \( \partial \) in Abelian simplicial homology.

In the subsequent calculations, it is convenient to use the following abbreviations.

**Definition 3.3.** Let \((V, E, F)\) denote an oriented two-complex. For a given edge \( e \in E \), the sets
\[
e_+ := \{ f \in F : e = \partial_j f, \ \varepsilon_j f = (+1) \ \text{for some } j, \ 1 \leq j \leq N(f) \}, \quad (3.4a)
\]
\[
e_- := \{ f \in F : e = \partial_j f, \ \varepsilon_j f = (-1) \ \text{for some } j, \ 1 \leq j \leq N(f) \}, \quad (3.4b)
\]
contain all faces that have the edge \( e \) in their boundary with positive (+) or negative (−) orientation, and we write \( \delta e := e_+ \cup e_- \) for the coboundary of an edge \( e \in E \). For a given face \( f \in F \), the set
\[
f_0 := \{ v \in V : v = \partial_{-\partial_j} f \ \text{for some } j, \ 1 \leq j \leq N(f) \}, \quad (3.5)
\]
denotes all vertices that belong to the boundary of the face \( f \). Finally, the sets
\[
f_+ := \{ e \in E : e = \partial_j f, \ \varepsilon_j f = (+1) \ \text{for some } j, \ 1 \leq j \leq N(f) \}, \quad (3.6a)
\]
\[
f_- := \{ e \in E : e = \partial_j f, \ \varepsilon_j f = (-1) \ \text{for some } j, \ 1 \leq j \leq N(f) \}, \quad (3.6b)
\]
contain all edges in the boundary of the face \( f \) that occur with positive (+) or negative (−) orientation, and \( \partial f := f_+ \cup f_- \) denotes the full boundary of \( f \in F \).

We have formulated our definitions of graphs and two-complexes so that they have only a finite number of vertices, edges and faces. This condition ensures that our partition functions are well defined. The collections of points, links and plaquettes of standard hyper-cubic lattices form a special case of oriented two-complexes in the obvious manner.

### 3.2 Spin networks and spin foams

Spin networks were introduced by Penrose [25] in the context of a quantization of space-time geometry. A spin network with symmetry group \( G \) is a graph together with a colouring of its edges with irreducible representations of \( G \) and a colouring of its vertices with compatible intertwiners (representation morphisms). For the subsequent calculations it is most convenient to separate the notions of graph and colouring and to speak of a spin network that lives on a graph.

**Definition 3.4.** Let \( G \) be a compact Lie group and \((V, E)\) be an oriented graph. A spin network \((\tau, Q)\) with symmetry group \( G \) on \((V, E)\) is a colouring of the edges with irreducible representations of \( G \),
\[
\tau : E \to \mathcal{R}_G, \quad e \mapsto \tau_e, \quad (3.7a)
\]
together with a colouring of the vertices $v \in V$ with compatible intertwiners,

$$Q^{(v)} \in \text{Hom}_G\left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e}, \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right). \quad (3.7b)$$

The tensor product in the domain is over the ‘incoming’ edges and that in the image over the ‘outgoing’ edges.

Spin networks appeared first in the context of quantum gravity. There they define, for example, the physical states in the loop formulation of gauge theories and the kinematical states of loop quantum gravity [26]. The observables of non-Abelian lattice gauge theory can also be constructed from spin networks [15,16]. They are given by the spin network functions (Definition 6.3 below).

The concept of a spin network can be generalized by introducing additional representations at the vertices, called charges, and by modifying the compatibility condition (3.7b) accordingly.

**Definition 3.5.** Let $G$ be a compact Lie group, $(V, E)$ an oriented graph, and $\rho: V \to \mathcal{R}_G, \ v \mapsto \rho_v$ assign an irreducible representation of $G$ to each vertex. A spin network $(\tau, Q, \rho)$ with charges $\rho$ is a colouring of the edges with irreducible representations,

$$\tau: E \to \mathcal{R}_G, \ e \mapsto \tau_e, \quad (3.8a)$$

together with a colouring of the vertices $v \in V$ with compatible intertwiners,

$$Q^{(v)} \in \text{Hom}_G\left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e}, \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right). \quad (3.8b)$$

We show in this article that spin networks with charges appear as the configurations in the dual expression for correlators in sigma models, and that they characterize the observables of generalized Higgs models.

A higher dimensional analogue of spin networks is the concept of spin foams. Spin foams also appeared first in the context of quantum gravity [12,13,14].

**Definition 3.6.** Let $G$ be a compact Lie group and $(V, E, F)$ be an oriented two-complex. A spin foam $(\rho, P)$ with symmetry group $G$ on $(V, E, F)$ is a colouring of the faces with finite-dimensional irreducible representations of $G$,

$$\rho: F \to \mathcal{R}_G, \ f \mapsto \rho_f, \quad (3.9a)$$

together with a colouring of the edges $e \in E$ with compatible intertwiners,

$$P^{(e)} \in \text{Hom}_G\left( \bigotimes_{f \in e^+} \rho_f, \bigotimes_{f \in e^-} \rho_f \right). \quad (3.9b)$$

The tensor product in the domain is over the representations at the ‘incoming’ faces, that in the image over the ‘outgoing’ ones. Incoming and outgoing are here defined by the relative orientations of the edges and faces.
These spin foams are often called closed spin foams as opposed to open spin foams which are bounded by a spin network. Open spin foams can be understood as the higher dimensional analogue of spin networks with charges and are defined as follows.

**Definition 3.7.** Let $G$ be a compact Lie group, $(V, E, F)$ define an oriented two-complex and $(\tau, Q)$ be a spin network on $(V, E)$. A spin foam $(\rho, P, \tau, Q)$ bounded by the spin network $(\tau, Q)$ is a colouring of the faces with finite-dimensional irreducible representations, 

$$\rho: F \rightarrow \mathcal{R}_G, \quad f \mapsto \rho_f,$$

(3.10a)

together with a colouring of the edges $e \in E$ with compatible intertwiners,

$$P^{(e)} \in \text{Hom}_G\left(\left(\bigotimes_{f \in e_+} V_{\rho_f}\right) \otimes V_{\tau_e}, \bigotimes_{f \in e_-} V_{\rho_f}\right).$$

(3.10b)

## 4 The chiral model

In this section, we develop the duality transformation for the chiral model with a symmetry group $G$ that is a compact Lie group. This model forms the basis for the generalizations to the non-linear sigma model with variables in a coset space $G/H$ and to the generalized Higgs models in which the chiral model or the non-linear sigma model is coupled to a lattice gauge theory.

### 4.1 Partition function

**Definition 4.1.** Let $G$ be a compact Lie group and $(V, E)$ be an oriented graph. Let $s: G \rightarrow \mathbb{R}$ be an $L^2$-integrable and bounded class function that satisfies $s(g^{-1}) = s(g)$. The lattice chiral model with action $s$ is defined by the partition function

$$Z = \left(\prod_{v \in V_G} \int d g_v\right) \prod_{e \in E} w(g_{\partial_+ e}^{-1} g_{\partial_- e}),$$

(4.1)

whose Boltzmann weight is given by $w(g) = \exp(-s(g))$.

**Remark 4.2.**

1. The set of configurations is the product $G^V := G \times \cdots \times G$ of one copy of $G$ per vertex $v \in V$. The partition sum is just the Haar measure of $G^V$. There is an interaction term for each edge $e \in E$ relating the variables at the two end points, $g_{\partial_+ e}$ and $g_{\partial_- e}$.

2. It is possible to choose different actions $s_e: G \rightarrow \mathbb{R}$ for each edge $e \in E$ so that one obtains Boltzmann weights $w_e(g) = \exp(-s_e(g))$. This is useful, for example, if one wishes to study inhomogeneous or anisotropic systems or non-regular lattices for which one would introduce geometric factors in order to compensate for the different lengths of the various edges. All calculations presented below generalize to this case, too, but we try to keep the notation simple and do not write down the additional index $e$ in the following sections.

**Lemma 4.3.** Orientation reversal of any edge $e \in E$ is a symmetry of the lattice chiral model. The model therefore depends only on the unoriented graph. Orientation reversal of an edge $e \in E$ maps $g_{\partial_- e} \mapsto g_{\partial_+ e}$ and conversely, which leaves the action invariant since $s(g^{-1}) = s(g)$.
Our subsequent calculations are most transparent for the generic case in which the partition function can depend on the orientations even though this generality is not required for the common examples.

**Proposition 4.4.** The lattice chiral model (4.1) has got a global left-right $G \times G$-symmetry. Let $h, \tilde{h} \in G$, then the transformation

$$g_v \mapsto h \cdot g_v \cdot \tilde{h}^{-1},$$

(4.2)

for all $v \in V$, is a symmetry of the action $s(g_{\partial^+e} \cdot g_{\partial^-e}^{-1})$ for each edge $e \in E$ because $s$ is a class function.

### 4.2 Expectation values

The observables of the lattice chiral model are all possible expectation values of functions $G^V \to \mathbb{C}$ that are compatible with the symmetries. With the help of the Peter–Weyl decomposition, it is possible to calculate the generic form of these observables. For the chiral model, one obtains the well-known $n$-point functions. We present the full calculation here because it illustrates the method and this method generalizes to the more complicated models for which we derive new results in the subsequent sections.

**Theorem 4.5.** Each algebraic function $f: G^V \to \mathbb{C}$ that is compatible with the global $G \times G$-symmetry (4.2), is a linear combination of functions (observables) of the following type,

$$f_{\rho,P,Q}(\{g_v\}_{v \in V}) = \left(\prod_{v \in V} \sum_{\ell_v,m_v = 1}^{\dim V_{\rho_v}} P_{\ell_v...} Q_{m_v...} \prod_{v \in V} P_{\ell_v m_v} (g_v).\right)$$

(4.3)

Here

$$\rho: V \to \mathcal{R}_G, \quad v \mapsto \rho_v$$

(4.4)

associates an irreducible representation of $G$ with each vertex, and

$$P: \bigotimes_{v \in V} V_{\rho_v} \to \mathbb{C}, \quad Q: \bigotimes_{v \in V} V^*_{\rho_v} \to \mathbb{C},$$

(4.5)

are intertwiners of $G$.

**Remark 4.6.** 1. By the notation

$$\prod_{v \in V} \sum_{\ell_v,m_v = 1}^{\dim V_{\rho_v}} \sum_{\ell_v,m_v = 1}^{\dim V_{\rho_v}} \sum_{v \in V}$$

(4.6)

we mean that there is one sum over $\ell_v$ and $m_v$ for each vertex $v \in V$. Similarly,

$$P_{\ell_v...}$$

(4.7)

indicates that the symbol $P$ has got one index $\ell_v$ for each $v \in V$. It is also understood that the ordering of tensor factors in (4.3) corresponds to the ordering of indices of $P$ and $Q$ in (4.3). We use this notation frequently in the subsequent calculations.
4.2 Expectation values

Figure 6: (a) An oriented graph with vertices \( u, v, w \) and edges \( e, f, h \). (b) The structure of the observable (4.3) of the lattice chiral model on that graph. (c) The two-point function (4.8).

2. The structure of the observable (4.3) is illustrated in Figure 6(b).

3. The irreducible representations \( \rho_v \) describe the charges that are located at the vertices \( v \in V \). If there are precisely \( k \) vertices whose \( \rho_v \) is non-trivial, the normalized expectation value of the observable is a \( k \)-point function. For each \( v \in V \), there is a representation function \( t_{\ell_v m_v}^{(\rho_v)} \) that describes the \( G \)-dependence of the observable, and the intertwiners \( P \) and \( Q \) involve its indices \( \ell_v \) and \( m_v \) and are used in order to obtain a globally \( G \times G \)-invariant expression. The well-known 2-point function for two vertices \( v, w \in V \) is the normalized expectation value of

\[
\chi(\rho)(g_v \cdot g_w^{-1}) = \sum_{\ell_v, m_v=1}^{\dim V_\rho} t_{\ell_v m_v}^{(\rho_v)}(g_v) \cdot t_{\ell_w m_w}^{(\rho_w^*)}(g_w) \cdot \delta_{\ell_v \epsilon_v} \delta_{m_v m_w},
\]

(Figure 6(c)). It forms a special case of (4.3) in which the only non-trivial representations are a charge \( \rho \) at \( v \) and an anti-charge \( \rho^* \) at \( w \), and the intertwiners are trivial, \( P_{\ell_v \epsilon_w} = \delta_{\ell_v \epsilon_w} \), etc..

4. As a consequence of the Peter–Weyl theorem, the \( L^2 \)-integrable functions that are compatible with the global \( G \times G \)-symmetry, are in the closure of the set of all algebraic \( f_{\rho, P, Q} \), i.e. they can be obtained as limits of square summable series of functions \( f_{\rho, P, Q} \).

Proof of Theorem 4.5. Algebraic functions \( f: G^V \to \mathbb{C} \) are elements \( f \in \bigotimes_{v \in V} C_{\text{alg}}(G) \) and therefore have the Peter–Weyl decomposition

\[
f(\{g_v\}_{v \in V}) = \left( \prod_{v \in V} \sum_{\rho_v \in R_G} \right) \left( \prod_{v \in V} \sum_{j_v, k_v=1}^{\dim V_{\rho_v}} \right) \left( \prod_{j_v \cdots k_v \cdots} \right) \prod_{v \in V} f_{j_v \cdots k_v}(g_v).
\]
If \( f \) satisfies the global \( G \times G \)-symmetry, we can apply (4.2) for each vertex, and as this holds for arbitrary \( h, \tilde{h} \in G \), we can integrate the result over \( h \) and \( \tilde{h} \),

\[
f(\{ g_v \}_{v \in V}) = \int_{G \times G} \, dh \, d\tilde{h} \left( \prod_{v \in V} \sum_{\rho_v \in R_G} \sum_{j_v, k_v = 1}^{\dim V_{\rho_v}} \check{f}_{j_v, \ldots, k_v}^{(\rho_v, \ldots)} \right) \prod_{v \in V} t_{\ell_v, m_v}^{(\rho_v)} (g_v) t_{m_v k_v, m_v}^{(\rho_v)} (\tilde{h}^{-1}).
\]

(4.10)

We apply (2.9), writing \( t_{m_v k_v}^{(\rho_v)} (\tilde{h}^{-1}) = t_{k_v m_v}^{(\rho_v)} (\tilde{h}) \), and move all summations to the front of the expression. Then we sort the products by the arguments \( g_v, h \) and \( \tilde{h} \),

\[
f(\{ g_v \}_{v \in V}) = \left( \prod_{v \in V} \sum_{\rho_v \in R_G} \sum_{j_v, k_v = 1}^{\dim V_{\rho_v}} \check{f}_{j_v, \ldots, k_v}^{(\rho_v, \ldots)} \right) \prod_{v \in V} t_{\ell_v, m_v}^{(\rho_v)} (g_v) \times \left( \int_{G} \prod_{v \in V} t_{j_v, \ell_v}^{(\rho_v)} (h) \, dh \right) \left( \int_{G} \prod_{v \in V} t_{k_v m_v}^{(\rho_v)} (\tilde{h}) \, d\tilde{h} \right).
\]

(4.11)

The integrals over \( G \) can be evaluated using (2.21) so that

\[
f(\{ g_v \}_{v \in V}) = \left( \prod_{v \in V} \sum_{\rho_v \in R_G} \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \prod_{v \in V} \sum_{\ell_v, m_v = 1}^{\dim V_{\rho_v}} \check{f}_{j_v, \ldots, k_v}^{(\rho_v, \ldots)} \right) \prod_{v \in V} t_{\ell_v, m_v}^{(\rho_v)} (g_v) \times \left( \prod_{v \in V} t_{\ell_v, m_v}^{(\rho_v)} (g_v) \right),
\]

(4.12)

where

\[
\check{f}_{j_v, \ldots, k_v}^{(\rho_v, \ldots)} = \left( \prod_{v \in V} \sum_{j_v, k_v = 1}^{\dim V_{\rho_v}} \check{f}_{j_v, \ldots, k_v}^{(\rho_v, \ldots)} \right) \prod_{v \in V} t_{j_v, \ell_v}^{(\rho_v)} (g_v) t_{\ell_v, m_v}^{(\rho_v)} (g_v).
\]

(4.13)

Here \( \mathcal{P} \) denotes a basis of the space of \( G \)-invariant projectors

\[
\bigotimes_{v \in V} V_{\rho_v} \rightarrow \mathbb{C},
\]

(4.14)

whose elements \( P \in \mathcal{P} \) are normalized so that \( P^2 = P \) where the trivial representation is embedded as \( \mathbb{C} \subseteq \bigotimes_{v \in V} V_{\rho_v} \). Similarly, \( \mathcal{Q} \) denotes a basis of \( G \)-invariant projectors

\[
\bigotimes_{v \in V} V_{\rho_v}^* \rightarrow \mathbb{C},
\]

(4.15)

with the analogous normalization. The expression (4.12) is a linear combination of observables of the form (4.3).

**Remark 4.7.** The global \( G \times G \)-symmetry can be realized as the translation symmetry of the multiple Haar measure because for each \( v \in V \), \( h, \tilde{h} \in G \) and any function \( u \in C_{\text{alg}}(G) \),

\[
\int_{G} u(g_v) \, dg_v = \int_{G} u(h \cdot g_v \cdot \tilde{h}^{-1}) \, dg_v.
\]

(4.16)
As the Boltzmann weight is invariant, the expectation value of any non-invariant function $f': G^V \to \mathbb{C}$ under the partition function vanishes. Note that this holds for any finite graph $(V, E)$.

Similarly, the expectation value vanishes for any function that is not invariant under simultaneous orientation reversal of all edges which corresponds to taking the dual of all representations and which is realized by the inversion symmetry of the Haar measure,

$$
\int_G u(g_v) \, dg_v = \int_G u(g_v^{-1}) \, dg_v.
$$

(4.17)

Therefore all interesting observables are functions $G^V \to \mathbb{R}$.

### 4.3 Duality transformation

The duality transformation consists of two steps. First, we character expand the Boltzmann weight of the original partition function (4.1) of the lattice chiral model. This introduces sums over all irreducible representations of $G$ for each edge as the new dual variables. Furthermore, this step is responsible for the strong-weak or high temperature-low temperature relation of the duality transformation as we have explained in Section 2.8.

The second step is to employ the methods outlined in Section 2.4 in order to solve all integrals over $G$ and therefore to remove the old variables from the partition function.

We start with the partition function (4.1) and insert the character expansion (2.14) of the Boltzmann weight for each edge $e \in E$,

$$
Z = \left( \prod_{v \in V} \int_G dg_v \right) \prod_{e \in E} \sum_{\tau_e \in R_G} \tilde{w}_{\tau_e} \chi^{(\tau_e)}(g_{\partial_+ e} \cdot g_{\partial_- e}^{-1}).
$$

(4.18)

The character can be simplified using (2.7) and (2.9),

$$
\chi^{(\tau_e)}(g_{\partial_+ e} \cdot g_{\partial_- e}^{-1}) = \sum_{p_e, q_e=1}^{\dim V_{\tau_e}} t_{p_e q_e}^{(\tau_e)}(g_{\partial_+ e}) t_{p_e q_e}^{(\tau_e^*)}(g_{\partial_- e}).
$$

(4.19)

The sums are moved to the front of the expression, and we sort the product of representation functions by the vertex $v \in V$ of their arguments $g_v$,

$$
Z = \left( \prod_{e \in E} \sum_{\tau_e \in R_G} \right) \left( \prod_{e \in E} \tilde{w}_{\tau_e} \right) \left( \prod_{e \in E} \sum_{p_e, q_e=1}^{\dim V_{\tau_e}} \right)
\times \prod_{v \in V} \int_G dg_v \left( \prod_{e \in E: v=\partial_+ e} t_{p_e q_e}^{(\tau_e)}(g_v) \right) \left( \prod_{e \in E: v=\partial_- e} t_{p_e q_e}^{(\tau_e^*)}(g_v) \right).
$$

(4.20)

Here the last two products are over all edges $e \in E$ that have $v = \partial_{\pm} e$. The integrals over $G$ can be evaluated using (2.24),

$$
Z = \left( \prod_{e \in E} \sum_{\tau_e \in R_G} \right) \left( \prod_{e \in E} \tilde{w}_{\tau_e} \right) \left( \prod_{e \in E} \sum_{p_e, q_e=1}^{\dim V_{\tau_e}} \right)
\times \prod_{v \in V} \sum_{S^{(v)} \in S^{(v)}} \left( S^{(v)}_{p_e e} \right) \left( S^{(v)}_{q_e e} \right) \left( S^{(v)}_{p_e e} \right) \left( S^{(v)}_{q_e e} \right).
$$

(4.21)
Figure 7: (a) The spin network that appears in the dual partition function \((4.23)\) of the lattice chiral model on the graph of Figure 6(a). (b) The alternative expression \((4.25)\) for the same graph.

where \(S^{(v)}, v \in V\), denotes a basis of \(G\)-invariant projectors

\[
\left( \bigotimes_{v=\partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{v=\partial_- e} V_{\tau_e}^* \right) \to \mathbb{C},
\]

which are normalized so that \(S^{(v)}^2 = S^{(v)}\) if the trivial representation \(\mathbb{C}\) is embedded in the big tensor product. We obtain the following result.

**Theorem 4.8 (Dual partition function).** Let \(G\) be a compact Lie group and \((V, E)\) denote an oriented graph. The partition function of the lattice chiral model \((4.1)\) is equal to

\[
Z = \left( \prod_{e \in E} \sum_{\tau_e \in R_G} \right) \left( \prod_{v \in V} \sum_{S^{(v)} \in S^{(v)}} \right) \left( \prod_{e \in E} \hat{w}_{\tau_e} \right) \left( \prod_{e \in E} \sum_{p_e, q_e=1}^{\dim V_{\tau_e}} \right)
\]

\[
\times \prod_{v \in V} S^{(v)}_{p_v \ldots p_v} S^{(v)}_{q_v \ldots q_v} \quad \text{\(e \in E:\)}
\]

\[
\varepsilon = \partial_+ e \quad \varepsilon = \partial_- e \quad v = \partial_+ e \quad v = \partial_- e
\]

\[(4.23)\]

where \(S^{(v)}\) is a basis of \(G\)-invariant projectors \((4.22)\).

**Remark 4.9.** 1. This dual partition function can be described in words as follows. The partition sum consists of a sum over all colourings of the edges \(e \in E\) with irreducible
4.3 Duality transformation

representations $\tau_e$ of $G$ and over all colourings of the vertices $v \in V$ with compatible intertwiners $S^{(v)}$. Compatible here means that each $S^{(v)}$ corresponds to a map from the tensor product of the representations at the incoming edges $e \in E: v = \partial_+ e$ to the tensor product of the outgoing edges $e \in E: v = \partial_- e$,

$$S^{(v)}: \left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \rightarrow \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right). \quad (4.24)$$

Indeed, such an $S^{(v)}$ is related to the one appearing in (4.22) by the canonical isomorphisms $\text{Hom}_G(V \otimes W^*, \mathbb{C}) \cong \text{Hom}_G(V, W)$. The Boltzmann weight of the dual partition function consists of the character expansion coefficients $\hat{\omega}_{\tau_e}$ for each edge $e \in E$ and of a spin network given by the $S^{(v)}$ whose indices are contracted by the summations over $p_e$ and $q_e$. This is illustrated in Figure 7(a).

2. The dual partition function of the lattice chiral model is therefore given by a sum over spin networks. We call such a model a spin network model in analogy to the spin foam models which arise as the dual formulation of lattice gauge theory. The two layers of Figure 7(a) reflect the chiral structure given by the two-fold global $G$-symmetry. The fact that the spin networks extend over the entire graph is a consequence of the global nature of the symmetry.

3. We comment on the Abelian special case below in Section 4.4.

4. There is an alternative form of the dual partition function which uses a diagrammatical formulation similar to that developed in [18] for lattice gauge theory. This result is given in the following corollary and illustrated in Figure 7(b).

**Corollary 4.10.** From the intermediate step (4.20) of the proof, we obtain the following slightly different expression which involves the Haar intertwiner (2.23) instead of the sum over projectors $S^{(v)}$,

$$Z = \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_G} \right) \left( \prod_{e \in E} \hat{\omega}_{\tau_e} \right) \left( \prod_{v \in V} \dim V_{\tau_v} \right) \prod_{v \in V} \prod_{e \in E: v = \partial_+ e} T^{(v)}_{p_e ... p_e ... p_e ... q_e ... q_e ...}, \quad (4.25)$$

where for each $v \in V$, the Haar intertwiner $T^{(v)}$ is a map

$$T^{(v)}: \left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e}^* \right) \rightarrow \left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e}^* \right). \quad (4.26)$$

The next step is the generalization of Theorem 4.8 to the expectation value of the observable (4.3),

$$\langle f_{\rho, P, Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \int \mathcal{D}g_v \right) \left( \prod_{e \in E} w(g_{\partial_+ e}, g_{\partial_- e}^{-1}) \right) \times \left( \prod_{v \in V} \sum_{\ell_v, m_v = 1}^{\dim V_{\tau_v}} P_{\ell_v ... \ell_v} Q_{m_v ... m_v} \prod_{v \in V} f_{\rho_v}^{(\rho_v)}(g_v). \right) \quad (4.27)$$
Again, we character expand the Boltzmann weight, simplify the characters that occur in the expression and reorganize everything. The step that generalizes (4.20) then reads

$$
\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{e \in E} \sum_{\tau_e \in G} \right) \left( \prod_{e \in E} \sum_{\tau_e} \right) \left( \prod_{v \in V} \sum_{\ell_v,m_v=1} \right) \left( \prod_{v \in V} \tau_v \rho_v \right) \left( \prod_{v \in V} \tau_v \rho_v \right) \left( \prod_{e \in E} \sum_{p_e,q_e=1} \right) \left( \prod_{e \in E} \sum_{p_e,q_e} \right) \left( \prod_{e \in E} \right) \int_{v \in V} dg_v \left( \prod_{e \in E:v=\partial_+ e} t^{(\tau_e)}_{p_e,q_e}(g_v) \right) \left( \prod_{e \in E:v=\partial_+ e} t^{(\tau_e)}_{p_e,q_e}(g_v) \right) \left( \prod_{e \in E} \right) \int_{v \in V} dg_v \left( \prod_{e \in E:v=\partial_- e} t^{(\tau_e)}_{p_e,q_e}(g_v) \right) \left( \prod_{e \in E:v=\partial_- e} t^{(\tau_e)}_{p_e,q_e}(g_v) \right).
$$

(4.28)

Compared with (4.20), there is an additional factor $t^{(\rho_v)}_{\ell_v,m_v}$ for each $v \in V$ under the integral. Solving the integrals, we obtain the following result.

**Theorem 4.11 (Dual observable).** Let $G$ be a compact Lie group, $(V,E)$ an oriented graph and $f_{\rho,P,Q}$ denote an observable of the form (4.3). Then the expectation value (4.27)
of $f_{\rho,P,Q}$ in the lattice chiral model is equal to

$$
\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \sum_{\ell_v,m_v=1}^{\dim V_{\ell_v}} P_{\ell_v,m_v} \right) \left( \prod_{v \in V} \sum_{\ell_v}^{\dim V_{\ell_v}} Q_{\ell_v} \right) \left( \prod_{e \in E} \sum_{\ell_e}^{\dim V_{\ell_e}} \tilde{\omega}_{\ell_e} \right) \left( \prod_{e \in E} \sum_{\nu_e,\mu_e=1}^{\dim V_{\nu_e}} \right)
$$

$$
\times \left( \prod_{e \in E} \sum_{\nu_e}^{\dim V_{\nu_e}} \right) \left( \prod_{v \in V} \sum_{\ell_v}^{\dim V_{\ell_v}} S_{\rho,P,Q}^{(v)} \right)
$$

$$
\times \prod_{e \in E} S_{\rho,P,Q}^{(v)} \prod_{v \in V} S_{\rho,P,Q}^{(v)}
$$

(4.29)

For each vertex $v \in V$, $S_{\rho,P,Q}^{(v)}$ denotes a basis of $G$-invariant projectors

$$
\left( \bigotimes_{e \in E} V_{\ell_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\nu_e} \right) \otimes V_{\rho_v} \to \mathbb{C}^1
$$

(4.30)

with the usual normalization.

**Remark 4.12.**

1. Compared with the dual partition function (4.23), the new features are the sums over the $\ell_v$ and $m_v$ and the matrix elements of $P$ and $Q$ from the definition of the observable (4.3). The remainder of the expression has the same structure as the dual partition function except for the fact that the intertwiners $S_{\rho,P,Q}^{(v)}$ have changed. They now include the charges $\rho_v$, $v \in V$, of the observable in the compatibility condition (4.30), and the spin networks of the dual partition function are coupled to these charges — the $\ell_v$ and $m_v$ appear as additional indices of the $S_{\rho,P,Q}^{(v)}$. The numerator of the dual expression is therefore given by a sum over spin networks with charges (Definition 3.5). This is illustrated in Figure 3a).

2. Equation (4.29) shows that an expectation value of an observable is mapped to a ratio of partition functions in the dual picture, say $\langle f_{\rho,P,Q} \rangle = Z(\rho)/Z$. The numerator $Z(\rho)$ is similar to the partition function, but the spin networks appearing there are now coupled to the charges of the observable, i.e. the numerator corresponds to the partition function in the presence of 'background charges'.

3. Again there exists an alternative formulation based on the intermediate step (4.28) and involving the Haar intertwiner. This is stated in the following corollary and shown in Figure 3b).

**Corollary 4.13.** From the intermediate step (4.28) of the proof, we obtain,

$$
\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \sum_{\ell_v,m_v=1}^{\dim V_{\ell_v}} P_{\ell_v,m_v} \right) \left( \prod_{v \in V} \sum_{\ell_v}^{\dim V_{\ell_v}} Q_{\ell_v} \right) \left( \prod_{e \in E} \sum_{\ell_e}^{\dim V_{\ell_e}} \tilde{\omega}_{\ell_e} \right) \left( \prod_{e \in E} \sum_{\nu_e,\mu_e=1}^{\dim V_{\nu_e}} \right)
$$

$$
\times \left( \prod_{e \in E} \sum_{\nu_e}^{\dim V_{\nu_e}} \right) \left( \prod_{v \in V} \sum_{\ell_v}^{\dim V_{\ell_v}} T_{\rho,P,Q}^{(v)} \right)
$$

(4.31)

where the Haar intertwiner $T_{\rho,P,Q}^{(v)}$ for any given $v \in V$ is a map

$$
T_{\rho,P,Q}^{(v)}: \left( \bigotimes_{e \in E} V_{\ell_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\nu_e} \right) \otimes V_{\rho_v} \to \left( \bigotimes_{e \in E} V_{\ell_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\nu_e} \right) \otimes V_{\rho_v}.
$$

(4.32)
4.4 The Abelian special case

In this section, we illustrate the specialization to the Abelian case in detail.

For $G = U(1)$, the partition function reads

$$Z = \left( \prod_{v \in V} \frac{1}{2\pi} \int_0^{2\pi} d\varphi_v \right) \prod_{e \in E} \exp \left( -s(e^{i(\varphi_{v_1} - \varphi_{v_2})}) \right)$$

for some action $s: U(1) \to \mathbb{R}$. For $G = \mathbb{Z}_N$ we have

$$Z = \left( \prod_{v \in V} \frac{1}{N} \sum_{\ell_v = 0}^{N-1} \right) \prod_{e \in E} \exp \left( -s(e^{2\pi i (\ell_{v_1} - \ell_{v_2})/N}) \right),$$

i.e. the chiral model restricts to the $XY$-model if $G = U(1)$, to the $\mathbb{Z}_N$-vector Potts model if $G = \mathbb{Z}_N$ and in particular to the Ising model if $G = \mathbb{Z}_2$. The dual partition function contains a sum over irreducible representations for each edge which becomes in the Abelian case a sum over $\mathbb{Z}$ or $\mathbb{Z}_N$ (Section 2.7).

As all irreducible representations are one-dimensional, the indices of $S^{(v)}$ in (4.23) are absent, and the sum over projectors restricts to the constraint (2.44), therefore for $G = U(1)$,

$$Z = \left( \prod_{e \in E} \sum_{k_e = -\infty}^{\infty} \right) \left( \prod_{v \in V} \delta( \sum_{e \in E: v \in \partial^+_e} k_e - \sum_{e \in E: v \in \partial^-_e} k_e ) \right) \left( \prod_{e \in E} \hat{w}_{k_e} \right),$$

where we write $\delta(x)$ for the constraint that $x = 0$. For $G = \mathbb{Z}_N$, the sum over the $k_e$ is over $\{0, \ldots, N - 1\}$ and all arithmetic involving the $k_e$ is modulo $N$. The coefficients $\hat{w}_k$ are the character expansion coefficients of the Boltzmann weight (see (2.49) and (2.51)).

Equation (4.35) is the well-known dual expression of the partition function at a stage before the constraints are solved, see, for example [1, 2]. The solution of these constraints then depends on the dimension and on the topology of the lattice. For $G = U(1)$ one obtains the solid-on-solid model in $d = 2$ and $\mathbb{Z}$-lattice gauge theory in $d = 3$, whereas for $G = \mathbb{Z}_N$, one finds again the $\mathbb{Z}_N$-vector Potts model in $d = 2$ with the self-duality of the Ising model [1] as a special case for $N = 2$, and a $\mathbb{Z}_N$-lattice gauge theory in $d = 3$ whose $N = 2$ case was studied in [2].

In the Abelian situation, the dual partition function (4.35) contains only a colouring at one level, namely the sum over all colourings of edges with irreducible representations (wave numbers). The generalization to non-Abelian symmetry groups introduces as a second level the sum over all colourings of the vertices with compatible intertwiners. This second colouring restricts to the familiar constraint of the form (2.44) if $G$ is Abelian.

The symmetry compatible functions (4.3) read in the Abelian case $G = U(1)$ [or $G = \mathbb{Z}_N$],

$$f_{\ell_v}(\{g_v\}_{v \in V}) = \prod_{v \in V} g_v^{\ell_v},$$

where the $\ell_v \in \mathbb{Z}$ [or $\ell_v \in \{0, \ldots, N - 1\}$] specify the charges located at the vertices. The dual of the expectation value (4.29) is then given by

$$\langle f_{\ell_v} \rangle = \frac{1}{Z} \left( \prod_{e \in E} \sum_{k_e = -\infty}^{\infty} \right) \left( \prod_{v \in V} \delta( \sum_{e \in E: v \in \partial^+_e} k_e - \sum_{e \in E: v \in \partial^-_e} k_e + \ell_v ) \right) \left( \prod_{e \in E} \hat{w}_{k_e} \right)$$

(4.37)
for $G = U(1)$ and the obvious analogue for $G = \mathbb{Z}_N$.

Already in the Abelian case, the duality transformation maps the expectation value to a ratio of partition functions whose numerator is a modification of the partition function in which the presence of background charges has modified the constraints or compatibility conditions.

### 4.5 Expectation values of the dual model

As Theorem 4.11 shows, the dual expression for the expectation value of an observable is given by a ratio of partition functions. In particular, this dual expression does not coincide with any expectation value under the dual partition function.

It is therefore an interesting problem to study the natural observables of the dual partition function and to transform these expressions back to the original formulation. From the Abelian special case it is familiar that the transformation maps expectation values to ratios of partition functions and therefore correlators constructed from fundamental variables to free energies of collective excitations and conversely, see, for example [5, 27].

For lattice gauge theory with gauge group $G = U(1)$ in $d = 4$ dimensions, for example, there exist particular expectation values of the dual partition function which describe the correlators of world-lines of magnetic monopoles [27]. If one transforms these expressions back to the original picture, one obtains ratios of partition functions $Z(X)/Z$. The numerator can be understood as the partition function of the model in the presence of a background magnetic field probing monopoles, and the ratio $Z(X)/Z = e^{-F}$ is related to the free energy $F$ of this monopole configuration. A first natural generalization to the non-Abelian case was given by the correlation functions of centre monopoles in [16], expressions which have been studied in lattice gauge theory for some time, but which have not been seen in the context of the duality transformation.

In the Abelian sigma models, the analogue of the magnetic monopoles is given by dislocations, vortices or world lines of vortices, depending on the dimension and on the precise model. In the following, we present the analogous definition for the lattice chiral model with non-Abelian symmetry group $G$ which we call centre dislocations as it uses the centre $Z(G)$ of the symmetry group $G$ similarly to the centre monopoles in order to parameterize the observables and because it specializes to the dislocations studied in [2] in the case $G = \mathbb{Z}_2$.

**Definition 4.14.** Let $G$ be a compact Lie group, $(V, E)$ be an oriented graph and $X: E \to Z(G), e \mapsto X_e$ assign an element of the centre $Z(G)$ to each edge $e \in E$. The centre dislocation is the following function $O_X: (\mathcal{R}_G)^V \to \mathbb{C}$ of the configurations of the dual partition function $\tilde{\tau}$,

$$O_X(\{\tau_e\}_{e \in E}) := \prod_{e \in E}\tilde{\tau}(\tau_e)(X_e), \quad (4.38)$$

where $\tilde{\tau}(\tau_e)$ denotes the representation functions of $Z(G)$ induced from the representation $\tau_e \in \mathcal{R}_G$ (Lemma 2.19).

We can now employ the techniques of Section 4.3 in order to transform the expectation value of the centre dislocation back to the original picture.

**Theorem 4.15.** The normalized expectation value of the centre dislocation $(4.38)$ under the
dual partition function (4.23) is equal to
\[ \langle O_X \rangle_{\text{dual}} = \frac{1}{Z} \left( \prod_{v \in V} \int_G dg_v \right) \prod_{e \in E} w(g_{\partial_+ e} \cdot g_{\partial_- e}^{-1} \cdot X_e). \]  
(4.39)

**Proof.** Start from (4.39), insert the expansion of \( w(g) \) and apply Lemma 2.19. The proof is entirely analogous to that of Theorem 4.8 with one additional factor \( \tilde{t}(\tau_e)(X_e) \) for each edge \( e \in E \) in the integrand. \( \square \)

**Remark 4.16.** The expectation value of the dual observable takes the form of a ratio of partition functions in the original formulation. This is essentially the converse of Theorem 4.11. The numerator can again be viewed as the partition function in the presence of a background field \( X \).

In the Abelian case, we have \( Z(G) = G \). The possible choices for fields \( X \) depend on the particular group and on the dimension and topology of the lattice. They have been carefully studied for several models.

1. If \( G = U(1) \) and \( (V, E) \) is a two-dimensional cubic lattice, then the disorder parameter of the \( XY \)-model which is related to the free energy of a vortex-antivortex pair, is of the form (4.39). In higher dimensions, this generalizes to vortex strings, vortex sheets, etc.

2. For \( G = \mathbb{Z}_2 \) we obtain the dislocations of [2] as the simplest dual observables. Their expectation value is again related to their free energies.

There are more general functions \( (\mathcal{R}_G)^V \rightarrow \mathbb{C} \) whose expectation value under the dual partition function can be calculated. Let \( e_0 \in E \) be an edge and \( \sigma \in \mathcal{R}_G \) an irreducible representation of \( G \). Then we can study the indicator function,
\[ O_{e_0,\sigma}(\{\tau_e\}_{e \in E}) = \delta_{\tau_{e_0}} \sigma, \]  
(4.40)

which probes whether the representation \( \sigma \) is assigned to the edge \( e_0 \). The centre dislocations can be expressed as linear combinations of these indicator functions,
\[ O_X(\{\tau_e\}_{e \in E}) = \left( \prod_{e \in E} \sum_{\sigma_e \in \mathcal{R}_G} \right) \prod_{e \in E} O_{e,\sigma_e} \tilde{t}(\tau_e)(X_e). \]  
(4.41)

The expectation value of an indicator function (4.40) under the dual partition function (4.23) is then equal to
\[ \langle O_{e_0,\sigma} \rangle_{\text{dual}} = \frac{1}{Z} \left( \prod_{v \in V} \int_G dg_v \right) \prod_{e \in E} \tilde{w}_{e_0,\sigma}(g_{\partial_+ e} \cdot g_{\partial_- e}^{-1}), \]  
(4.42)

where the Boltzmann weight \( w(g) \) is modified at the edge \( e_0 \),
\[ \tilde{w}_{e_0,\sigma}(g) = \begin{cases} w(g), & \text{if } e \neq e_0, \\ \sum_{\rho \in \mathcal{R}_G} \delta_{\rho,\sigma} \tilde{w}_\rho \chi^{(\rho)}(g), & \text{if } e = e_0, \end{cases} \]  
(4.43)

In general, a function involving the indicator functions in the dual formulation leads to a convolution of the Boltzmann weight in the original picture.
4.6 The strong-weak relation

Remark 4.17. 1. The definition of dual expectation values presented here is restricted to functions of the irreducible representations at the edges. It is also conceivable to make use of functions of the intertwiners at the vertices.

2. Indicator functions similar to (4.40) have been used to construct geometrical observables in the spin foam model of three-dimensional quantum gravity [28].

4.6 The strong-weak relation

The dual partition function (4.23) of the lattice chiral model is strong-weak dual to the original formulation (4.1). This follows from the properties of the character expansion of the Boltzmann weight and is most transparent for the heat kernel action (2.52). The only $\beta$-dependent term of the dual partition function is the product

$$\prod_{e \in E} \tilde{w}_{\tau_e} = \exp\left(-\frac{1}{2\beta} \sum_{e \in E} C^{(2)}_{\tau_e}\right),$$

(4.44)

where the inverse temperature $\beta$ appears in the denominator! The result for the Wilson action of $G = U(1)$ or $G = SU(2)$ looks more complicated and involves modified Bessel functions, but it is qualitatively quite similar. In all these cases, the term corresponding to (4.44) has a sharp peak as a function of the $C^{(2)}_\tau$ if $\beta$ is small.

The $\beta$-dependence (4.44) of the dual partition function also encodes essential information on the strong coupling expansion of the lattice chiral model. For small $\beta$, the dominant contribution to (4.44) comes from spin networks (assignments of representations to the edges of the graph) whose sum of the quadratic Casimir eigenvalues over all edges is very small. It is now possible to sort them by the value of this sum so that the configurations of the dual partition function are precisely the terms of the strong coupling expansion!

5 The non-linear sigma model

We construct the lattice non-linear sigma model with variables in some coset space $G/H$, where $H \leq G$ is a Lie subgroup of $G$, starting from the chiral model. One half of the $G \times G$-symmetry of the chiral model is used to couple elements $h \in H$ to the action term. Integration over $h$ then makes sure that the action is constant on the cosets $gH$ and therefore defines a model with variables in $G/H$.

5.1 Partition function

Lemma 5.1. Let $G$ be a compact Lie group and $H \leq G$ be a Lie subgroup. Let $f \in L^2(G)$ be a class function of $G$ with character expansion

$$f(g) = \sum_{\rho \in \mathcal{R}_G} \widehat{f}_\rho \chi^{(\rho)}(g).$$

(5.1)

1. For any $g_1, g_2 \in G$,

$$\int_H f(g_1 \cdot h \cdot g_2^{-1}) \, dh = \sum_{\rho \in \mathcal{R}_G^0} \widehat{f}_\rho \sum_{j=1}^{\dim V_\rho} \sum_{k=1}^{n_\rho} \bar{H}_{jk}^{(\rho)}(g_1) H_{jk}^{(\rho^*)}(g_2),$$

(5.2)

using the conventions of Section 2.5.
2. The function $f$ defines a map $\tilde{f}: G/H \times G/H \to \mathbb{C}$,

$$\tilde{f}(x, y) := \int_H f(g_x \cdot h \cdot g_y^{-1}) dh,$$

where $g_x, g_y \in G$ denote representatives of the cosets $x, y \in G/H$.

3. The function $\tilde{f}(x, y)$ has a global left $G$-symmetry, i.e. for any $g \in G$, $x, y \in G/H$,

$$\tilde{f}(g \cdot x, g \cdot y) = \tilde{f}(x, y).$$

4. If in addition $f(g^{-1}) = f(g)$, then $\tilde{f}(x, y) = \tilde{f}(y, x)$.

**Remark 5.2.** If $H$ is a massive subgroup of $G$, then $\kappa_p = 1$ for the class-1 representations. In this case, any $L^2$-function $G/H \times G/H \to \mathbb{C}$ with the symmetry (5.5) is of the form (5.2). This statement does, however, not extend to the case of generic Lie subgroups $H \subseteq G$. We define the lattice non-linear sigma model for Boltzmann weights of the special form (5.2).

**Definition 5.3.** Let $G$ be a compact Lie group, $H \subseteq G$ be a Lie subgroup and $(V, E)$ denote an oriented graph. Let $s: G \to \mathbb{R}$ be an $L^2$-integrable and bounded class function that satisfies $s(g^{-1}) = s(g)$. Construct $\tilde{w}: G/H \times G/H \to \mathbb{R}$ from $w(g) = \exp(-s(g))$ as in Lemma 5.1. The **lattice non-linear sigma model** is defined by the partition function

$$Z = \left( \prod_{v \in V} \int_{G/H} dx_v \right) \prod_{e \in E} \tilde{w}(x_{\partial_+ e}, x_{\partial_- e}).$$

**Proposition 5.4.** The lattice non-linear sigma model has got a global left-$G$ symmetry. For any fixed $g \in G$, the transformation

$$x_v \mapsto g \cdot x_v,$$

for all $v \in V$, is a symmetry of the weight $\tilde{w}(x_{\partial_+ e}, x_{\partial_- e})$. In the special case in which $H \subseteq G$ is a normal subgroup, there is also a global right-$G/H$ symmetry. Let $y \in G/H$. Then the transformation

$$x_v \mapsto x_v \cdot y^{-1},$$

for all $v \in V$, is also a symmetry of the weight.

**Example 5.5.** The Boltzmann weights $\tilde{w}(x, y) := \exp(-\tilde{s}(x, y))$ of the lattice $N$-vector model (the $O(N)$ non-linear sigma model) and of the $\mathbb{R}P^{N-1}$-model are of the type of Lemma 5.1. For the $N$-vector model, $G = O(N)$, $H = O(N-1)$ and

$$\tilde{s}(x, y) = -\beta x \cdot y,$$

where $x, y \in S^{N-1} \subseteq \mathbb{R}^N$, and the dot denotes the standard scalar product. For the $\mathbb{R}P^{N-1}$-model, $G = O(N)$, $H = O(N-1) \times \mathbb{Z}_2$, and

$$\tilde{s}(x, y) = -\frac{q}{2}(x \cdot y)^2,$$

for representatives $x, y$ of classes in $\mathbb{R}P^{N-1} \cong S^{N-1}/\mathbb{Z}_2$. On cubic lattices, there exists in both cases a suitable naïve continuum (or weak field) limit in which the lattice constant tends to zero and the lattice action towards the action of the corresponding continuum model.

**Remark 5.6.**

1. The partition function again depends only on the unoriented graph.

2. If $H = \{ e \}$ is the trivial group, then any representation function is a generalized spherical function. The non-linear sigma model for $G/H$ coincides in this case with the chiral model for $G$, and the global $G \times G$-symmetry is restored.
5.2 Expectation values

The observables of the lattice non-linear sigma model can be found by the same methods as for the chiral model (Section 4.2). The calculation is very similar so that we just state the results.

**Theorem 5.7.** Each algebraic function \( (G/H)^V \to \mathbb{C} \) that is compatible with the global left-\( G \) symmetry (5.6), is a linear combination of observables of the following type,

\[
f_{\rho,P,k_v\ldots}(\{x_v\}_{v \in V}) = \left( \prod_{v \in V} \dim V_{\rho_v} \sum_{\ell_v = 1} \right) P_{\ell_v\ldots} \prod_{v \in V} H_{\ell_v,k_v}^{\rho_v}(x_v),
\]

where

\[
\rho: V \to \mathbb{R}^G, \quad v \mapsto \rho_v,
\]

assigns a class-1 representation of \( G \) with respect to \( H \) to each vertex; \( k_v \in \{1, \ldots, \kappa_{\rho_v}\} \) for all \( v \in V \), and

\[
P: \bigotimes_{v \in V} V_{\rho_v} \to \mathbb{C},
\]

is an intertwiner of \( G \).

**Remark 5.8.**

1. The structure of the function (5.10) is illustrated in Figure 9(a).

2. The well-known two-point function for a charge-anticharge pair \( \rho, \rho^* \) at \( v, w \in V \), is a special case,

\[
f_{k_v,k_w}(x_v,x_w) = \sum_{j_v = 1}^{\dim V_{\rho_v}} H_{j_v,k_v}^{\rho_v}(x_v)H_{j_w,k_w}^{\rho^*}(x_w)\delta_{j_v,j_w},
\]

for fixed \( k_v, k_w \in \{1, \ldots, \kappa_{\rho}\} \).
3. If $H$ is a massive subgroup of $G$, we have $\kappa_\rho = 1$ for the class-1 representations so that the indices $k_v$ can be omitted from all expressions (Figure 9(b)).

**Theorem 5.9.** If in addition $H \leq G$ is a normal subgroup, then the algebraic functions $(G/H)^V \to \mathbb{C}$ that are compatible with both the global left-$G$ and the global right-$G/H$ symmetry, are linear combinations of observables of the following form,

$$f_{\rho,P,Q}(\{x_v\}_{v \in V}) = \left( \prod_{v \in V} \dim V_{\rho_v} \kappa_{\rho_v} \right) \left( \prod_{v \in V} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} P_{\ell_v,\ldots,\ell_v} \cdot Q_{k_v,\ldots,k_v} \cdot \prod_{v \in V} H_{\ell_v,k_v}(x_v) \right). \quad (5.14)$$

Here

$$\rho: V \to \mathcal{R}_H, \quad v \mapsto \rho_v, \quad (5.15)$$

assigns a class-1 representation of $G$ with respect to $H$ to each vertex and

$$P: \bigotimes_{v \in V} V_{\rho_v} \to \mathbb{C}, \quad Q: \bigotimes_{v \in V} V_{\rho_v}^* \to \mathbb{C}, \quad (5.16)$$

are intertwiners of $G$.

**Remark 5.10.**

1. Figure 9(c) illustrates the structure of the observables (5.14) if $H \leq G$ is a normal subgroup. Here the indices $k_v$ of (5.10) are no longer independent, but rather exhibit a $G/H$-symmetry under which invariance is required. Therefore we need the second intertwiner $Q$. Furthermore, $\kappa_\rho = \dim V_\rho$ for all class-1 representations so that the dashed lines have become solid.

2. In particular for $H = \{e\}$, we recover the observable (4.3) of the chiral model.

3. In order to have non-vanishing expectation values, the observable not only has to be invariant under the symmetries (5.6) and (5.7) (if applicable), but also under orientation reversal (Remark 4.7).

### 5.3 Duality transformation

The duality transformation for the non-linear sigma model is very similar to that of the chiral model. We summarize the main steps which differ from the calculation for the chiral model and focus directly on the most general case, the dual of an expectation value, from which the transformation of the partition function can be easily inferred.

We start with an observable $f_{\rho,P,Q}$ of the form (5.14). If $H \leq G$ is a normal subgroup, then $Q$ is an intertwiner of $G$. Otherwise, $Q$ is arbitrary so that we obtain the function (5.10) for generic $k_v \in \{1,\ldots,\kappa_{\rho_v}\}$, $v \in V$.

We start with the expectation value of the observable (5.14) under the partition function (5.5),

$$\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \int_{G/H} dx_v \right) \left( \prod_{e \in E} \tilde{w}(x_{\partial_v+e}, x_{\partial_v-e}) \right) \times \left( \prod_{v \in V} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} P_{\ell_v,\ldots,\ell_v} \cdot \sum_{k_v=1}^{\kappa_{\rho_v}} Q_{k_v,\ldots,k_v} \cdot \prod_{v \in V} H_{\ell_v,k_v}(x_v) \right), \quad (5.17)$$
5.3 Duality transformation

and insert for each $e \in E$ the expansion of Lemma 5.1

$$\hat{w}(x_{\partial_e}, x_{\bar{\partial}_e}) = \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} H_{j_e m_e}^{(\tau_e)}(x_{\partial_e}) H_{j_e m_e}^{(\tau_e)}(x_{\bar{\partial}_e}), \quad (5.18)$$

where the $\hat{w}_{\tau_e}$ are the character expansion coefficients of the function $w(g) = \exp(-s(g))$ of Definition 5.3. The reorganized expression then reads

$$\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} \left[ \prod_{e=1}^{\ell_v} \prod_{k_v=1}^{\kappa_{\rho_v}} \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right) \right] \right) \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right), \quad (5.19)$$

so that we can evaluate the integrals over $G/H$ using (5.36).

$$\int_{G/H} dx_v \left( \cdots \right) = \sum_{S^{(v)} \in \mathcal{S}^{(v)}} S^{(v)} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} \left( \prod_{e \in E} \sum_{v=\partial e}^{\partial_e} S^{(v)} \right) \left( \prod_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right), \quad (5.20)$$

Here $\mathcal{S}^{(v)}$, $v \in V$, denotes a basis of $G$-invariant projectors

$$\left( \bigotimes_{e \in E} V_{\tau_e} \right) \otimes \left( \bigotimes_{v=\partial e}^{\partial_e} V_{\tau_v} \right) \otimes V_{\rho_v} \rightarrow \mathbb{C} \quad (5.21)$$

with the usual normalization. We obtain the following result.

**Theorem 5.11 (Dual observable).** Let $G$ be a compact Lie group, $H \leq G$ a Lie subgroup and $(V, E)$ denote an oriented graph. The expectation value (5.17) of the observable of the lattice non-linear sigma model is equal to the expressions

$$\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} \left[ \prod_{e=1}^{\ell_v} \prod_{k_v=1}^{\kappa_{\rho_v}} \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right) \right] \right) \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right), \quad (5.22)$$

$$\langle f_{\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{v \in V} \sum_{\ell_v=1}^{\dim V_{\rho_v}} \sum_{k_v=1}^{\kappa_{\rho_v}} \left[ \prod_{e=1}^{\ell_v} \prod_{k_v=1}^{\kappa_{\rho_v}} \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right) \right] \right) \left( \prod_{e \in E} \sum_{\tau_e \in \mathcal{R}_H^G} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\dim V_{\tau_e}} \hat{w}_{\tau_e} \right), \quad (5.23)$$
Figure 10: (a) The structure of the dual form \((5.22)\) for the expectation value of an observable of the lattice non-linear sigma model on the graph of Figure 6(a). (b) The special case of a massive subgroup \(H \leq G\). (c) The situation for a normal subgroup \(H \triangleleft G\).

Here \(\tilde{S}(v), v \in V\), denotes a basis of \(G\)-invariant projectors \((5.21)\), and the \(\hat{w}_v\) are the character expansion coefficients of the function \(w(g) = \exp(-s(g))\) where \(s(g)\) is the class function of Definition 5.3. The coset space Haar map \(I(v), v \in V\), in \((5.22)\) is a map

\[
\bigotimes_{e \in E: v = \partial_e}; V_{\tau_e} \otimes \bigotimes_{e \in E: v = \partial_e}; V_{\rho_v} \rightarrow \bigotimes_{e \in E: v = \partial_e}; V_{\tau_e} \otimes \bigotimes_{e \in E: v = \partial_e}; V_{\rho_v}.
\]

Remark 5.12. 1. The dual expression \((5.22)\) for the observable of the non-linear sigma model is very similar to the dual observable of the chiral model in Theorem 4.11. The differences are the ranges of the indices which follow from the choice of the subgroup \(H \leq G\). The structure of the dual observable is illustrated in Figure 10(a) if \(H \leq G\) is a generic, non-normal subgroup, in (b) if \(H\) is a massive subgroup and in (c) for the case of a normal subgroup \(H \triangleleft G\). Figure 10(a-c) correspond to \((5.22)\). The diagrams for the other formulation \((5.23)\) are obtained by the replacements shown in Figure 2(c) or 3(c).
5.3 Duality transformation

Figure 11: (a) The structure of the dual partition function (5.25) of the lattice non-linear sigma model on the graph of Figure 6. (b) The special case of a massive subgroup $H \leq G$.

2. Again, the dual expression for the observable of the chiral model can be obtained from (5.23) for a trivial subgroup $H = \{e\}$.

3. If one seeks a purely categorial picture of the dual non-linear sigma model, one should generally view all representations as representations of $H$. Otherwise the integrals over $H$ which are still implicitly present in the spherical functions, would not be honest intertwiners. The dashed lines with open ends labelled $k_v$ then enumerate different trivial representations of $H$. The special cases of massive and normal subgroups, however, are easier and can be handled already in the context of the representations of $G$.

The dual expression for the partition function can be calculated by specializing the numerator of (5.22) to the trivial observable. This result is given in the following corollary and visualized in Figure 11.

**Corollary 5.13 (Dual partition function).** Let $G$ be a compact Lie group with a Lie subgroup $H \leq G$ and $(V, E)$ be an oriented graph. The partition function (5.5) of the lattice
A non-linear sigma model is equal to

$$Z = \left( \prod_{e \in E} \sum_{\tau_e \in R^G_H} \right) \left( \prod_{v \in V} \bar{w}_{\tau_e} \right) \left( \prod_{v \in V} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\kappa_{\tau_e}} \prod_{e \in E} I^{(v)}_{j_e \ldots j_e \ldots m_e \ldots m_e \ldots} \right) \prod_{e \in E} S^{(v)}_{j_e \ldots j_e \ldots m_e \ldots m_e \ldots} \cdot \prod_{e \in E} S^{(v)}_{j_e \ldots j_e \ldots m_e \ldots m_e \ldots} \cdot$$

$$= \left( \prod_{e \in E} \sum_{\tau_e \in R^G_H} \right) \left( \prod_{v \in V} \sum_{j_e=1}^{\dim V_{\tau_e}} \sum_{m_e=1}^{\kappa_{\tau_e}} \prod_{e \in E} S^{(v)}_{j_e \ldots j_e \ldots m_e \ldots m_e \ldots} \cdot \prod_{e \in E} S^{(v)}_{j_e \ldots j_e \ldots m_e \ldots m_e \ldots} \cdot$$

(5.26)

Here $S^{(v)}$, $v \in V$, denotes a basis of $G$-invariant projectors

$$\left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right) \rightarrow \mathbb{C}$$

(5.27)

with the usual normalization, and the coset space Haar map $I^{(v)}$ is a linear map

$$\left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right) \rightarrow \left( \bigotimes_{e \in E: v = \partial_+ e} V_{\tau_e} \right) \otimes \left( \bigotimes_{e \in E: v = \partial_- e} V_{\tau_e} \right).$$

(5.28)

### 5.4 Expectation values of the dual model

If the natural observables of the dual partition function are again constructed from the labelling of the edges with representations, the result is the same as for the lattice chiral model in Section 4.5 restricted to the class-1 representations. The analogue of (4.39) is then

$$\langle O_X \rangle_{\text{dual}} = \frac{1}{Z} \left( \prod_{v \in V_{G/H}} \int dx_v \right) \prod_{e \in E} \bar{w}(X_e \cdot x_{\partial_+ e}; x_{\partial_- e}).$$

(5.29)

### 6 The generalized Higgs models

In this section, we couple the chiral model and the non-linear sigma model to lattice gauge theory. In some particular cases, this yields certain Higgs models with frozen radial component which motivates the title of this section. Before we study the coupled models, it is useful to summarize the results of the duality transformation for lattice gauge theory \[15,17,16\] in the language of the present article.

#### 6.1 Lattice gauge theory

**Definition 6.1.** Let $G$ be a compact Lie group, $(V, E, F)$ be an oriented two-complex and $s: G \rightarrow \mathbb{R}$ be an $L^2$-integrable class function of $G$ that is bounded below and satisfies $s(g^{-1}) = s(g)$ for all $g \in G$. The partition function of a lattice gauge theory with gauge group $G$ is defined by

$$Z = \left( \prod_{e \in E} \int dg_e \right) \prod_{f \in F} u(g_f), \quad g_f := g_{\partial_1 f} \cdots g_{\partial_{N(f)} f},$$

(6.1)
6.1 Lattice gauge theory

where $u(g) = \exp(-s(g))$.

The set of configurations of lattice gauge theory is the product $G^E$ of one copy of $G$ for each edge $e \in E$. The ordered product of group elements attached to the edges in the boundary of the face $f \in F$ is denoted by $g_f$. The Boltzmann weight exhibits a local gauge symmetry.

**Proposition 6.2.** Let $h: V \to G, v \mapsto h_v$ associate a group element with each vertex. The Boltzmann weight $u(g_f) = \exp(-s(g_f))$ in (6.1) is invariant under the local gauge transformations

$$g_e \mapsto h_{\partial_+ e} \cdot g_e \cdot h_{\partial_- e}^{-1},$$

for all $e \in E$.

This definition of lattice gauge theory is motivated by the fact that on regular hypercubic lattices, the Wilson action tends towards the continuum Yang–Mills action in the weak field limit of vanishing lattice constant. The group elements $g_e$ attached to the edges of the lattice correspond to the parallel transports of the gauge connection along these edges. For more details on lattice gauge theory, see, for example [29, 30].

The most general observable of lattice gauge theory whose expectation value under the partition function (6.1) can be calculated, is constructed from spin networks. Each algebraic function $G^E \to \mathbb{C}$ that is invariant under the transformation (6.2), is a linear combination of spin network functions. They generalize the notion of Wilson loops and are defined as follows.

**Definition 6.3.** Let $G$ be a compact Lie group, $(V, E, F)$ be an oriented two-complex and $(\sigma, Q)$ be a spin network (Definition 3.4). The **spin network function** of $(\sigma, Q)$ associates with each configuration a complex number,

$$W_{\sigma,Q}(\{g_e\}_{e \in E}) := \left( \prod_{e \in E}^{\dim V_{\sigma_e}} \sum_{k_e,\ell_e=1}^{1} \right) \left( \prod_{e \in E} \ell_{(\sigma_e)}(g_e) \right) \left( \prod_{v \in V} Q_{(v)}^{(k_{e_1}...k_{e_2})} \right).$$

(6.3)

**Remark 6.4.**

1. The above definition uses the spin network $(\sigma, Q)$ to label edges with representations and vertices with intertwiners, and then employs a representation function for each edge in order to obtain a function $G^E \to \mathbb{C}$.

2. All edges $e \in E$ for which $V_{\sigma_e} \cong \mathbb{C}$ is the trivial representation, contribute only a factor 1 to the expression (6.3). For an ordinary Wilson loop, for example, all edges are labelled with the trivial representation except for those edges that are part of the loop. These are labelled with the fundamental representation of $G$. The intertwiners $Q^{(v)}$ (if non-vanishing) are in this case uniquely determined up to normalization.

3. The spin network function (6.3) can be evaluated by putting $g_e = e$ (group unit) for all edges $e \in E$. The result is an invariant of $G$ which is often called the **value** of the spin network $(\sigma, Q)$.

4. If $G$ is Abelian, then the set $R_G$ of irreducible representations forms an additive group, and all irreducible representations are one-dimensional. Thus $W_{\sigma,Q}$ can be decomposed into a sum of products of ordinary Wilson loops.

We have the following dual expressions for the partition function and the expectation value of a spin network function [15, 17].
Figure 12: (a) A two-complex with a vertex \( v \) attached to four edges. There are six faces, one between each pair of edges. (b) The spin network \( C(v) \) of (6.6) that appears in the dual partition function of lattice gauge theory and (c) the spin network (6.14) from the dual of an expectation value.

**Theorem 6.5 (Dual partition function).** Let \( G \) be a compact Lie group. The partition function (6.1) of lattice gauge theory is equal to the expression

\[
Z = \left( \prod_{f \in F} \sum_{\tau_{f} \in \mathcal{R}_{G}} \right) \left( \prod_{e \in E} \sum_{U_{\tau_{e}}(v) \in \mathcal{U}(v)} \right) \left( \prod_{v \in V} \hat{u}_{\tau}(v) \right) \left( \prod_{v \in V} C(v) \right).
\]

(6.4)

Here \( \mathcal{U}(v), e \in E \), denotes a basis of \( G \)-invariant projectors

\[
(\bigotimes_{f \in e_{+}} V_{\tau_{f}}) \otimes (\bigotimes_{f \in e_{-}} V_{\tau_{f}}) \to \mathbb{C}.
\]

(6.5)

The \( \hat{u}_{\tau} \) are the coefficients of the character expansion of the Boltzmann weight \( u(g) \). The weights per vertex \( C(v) \) are given by a trace involving representations and projectors in the neighbourhood of the vertex \( v \in V \),

\[
C(v) = \left( \prod_{f \in F: v \in f_{0}}^{\dim V_{\tau_{f}}} \sum_{n_{f} = 1}^{\dim V_{\tau_{f}}} \right) \left( \prod_{e \in E: v = \partial_{+} e}^{\dim V_{\tau_{e}}} \sum_{n_{e} = 1}^{\dim V_{\tau_{e}}} \right) \left( \prod_{e \in E: v = \partial_{-} e} \right) \left( \prod_{e \in E: v = \partial_{+} e} \right) U_{\tau_{e}}(v).
\]

(6.6)

Here the range \( f \in F: v \in f_{0} \) of the first product refers to all faces \( f \in F \) that contain the vertex \( v \) in their boundary, the second product is over the range \( e \in E: v = \partial_{+} e \) of all edges that have \( v \) as their endpoint, etc., see Section 3.1.

**Remark 6.6.** 1. For each edge \( e \in E \), the projectors (6.5) are related by natural isomorphisms to intertwiners

\[
\bigotimes_{f \in e_{+}} \rightarrow \bigotimes_{f \in e_{-}} V_{\tau_{f}}.
\]

(6.7)

from the tensor product of the representations at the ‘incoming’ faces to the tensor product at the ‘outgoing’ ones.
Figure 13: An edge \( e \in E \) in the boundary of three faces, two triangles and one quadrilateral. (a) The structure of the spin networks \( C(v) \) in the dual partition function of lattice gauge theory (6.4). (b) The alternative formulation (6.8) using the Haar intertwiner. We have omitted labels and arrows in both diagrams.

2. The dual partition function (6.4) labels the faces with irreducible representations of \( G \) and the edges with compatible intertwiners in the sense of (6.7). The configurations of the dual partition function are therefore spin foams (Definition 3.6) so that the dual model is a spin foam model. Compared with the situation for the sigma models, all the labels appear one level ‘higher’, i.e. at the faces rather than at the edges, and at the edges rather than the vertices.

3. The expression \( C(v) \) for given projectors \( U^{(e)} \) is itself a spin network. Figure 12(b) visualizes it for a vertex with four edges attached. In particular, for \( G = SU(2) \), the \( C(v) \) are the 6j-symbols of \( SU(2) \). The collection of all \( C(v) \) in a two-complex is illustrated in Figure 13(a).

4. The spin networks of the dual partition function for lattice gauge theory decompose into one independent \( C(v) \) for each vertex. This is a consequence of the local \( G \)-symmetry and is in contrast to the chiral model whose dual partition function involves two spin networks that extend over the entire graph, reflecting the two-fold global \( G \)-symmetry. For the non-linear sigma model with a massive subgroup \( H \leq G \), the dual partition function still contains one spin network that extends over the entire graph which corresponds to a single global \( G \)-symmetry.

5. Again there exists an alternative formulation using the Haar intertwiner rather than the sum over projectors which is stated in the following corollary. This result agrees with the purely diagrammatical picture of \( 18 \) and is illustrated in Figure 13(b). Upon use of (2.24), we recover (6.4) and Figure 13(a).

**Corollary 6.7.** Let \( G \) be a compact Lie group and \((V, E, F)\) denote an oriented two-complex.
The partition function of lattice gauge theory (6.4) is equal to

\[ Z = \left( \prod_{f \in F} \sum_{\tau_f \in R_G} \right) \left( \prod_{f \in F} \bar{u}_{\tau_f} \right) \left( \prod_{f \in F} \sum_{v \in f_0 \cap n(f,v)=1} \right)^{\dim V_{\tau_f}} \times \prod_{e \in E} T^{(e)}_{n(f,\partial_+ e) \ldots n(f,\partial_+ e) \ldots n(f,\partial_- e) \ldots n(f,\partial_- e) \ldots} \]  

(6.8)

where \( T^{(e)} \), \( e \in E \), denotes the Haar intertwiner (2.23) for the following representations,

\[ T^{(e)} : \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V^*_{\tau_f} \right) \rightarrow \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V^*_{\tau_f} \right). \]  

(6.9)

Finally, the analogous statements are available for expectation values of spin network functions.

**Theorem 6.8 (Dual observable).** Let \( G \) be a compact Lie group, \( (V, E, F) \) be an oriented two-complex and \( Z \) denote the partition function (6.1) of lattice gauge theory. The expectation value of the spin network function (6.3),

\[ \langle W_{\sigma,Q} \rangle = \frac{1}{Z} \left( \prod_{e \in E} \int_{G} d\mathbf{g}_e \right) \prod_{f \in F} u(\varepsilon_{-1} f) \ldots \varepsilon_{N(f) f}) \times \left( \prod_{e \in E} \sum_{k_e,\ell_e=1} \right) \left( \prod_{e \in E} \mathbf{J}_{\alpha_e} (g_e) \right) \left( \prod_{v \in V} Q^{(v)}_{\ell_e} \ldots \kappa_e \ldots \right), \]  

(6.10)

is equal to the following expressions,

\[ \langle W_{\sigma,Q} \rangle = \frac{1}{Z} \left( \prod_{f \in F} \sum_{\tau_f \in R_G} \right) \left( \prod_{f \in F} \bar{u}_{\tau_f} \right) \left( \prod_{f \in F} \sum_{v \in f_0 \cap n(f,v)=1} \right)^{\dim V_{\tau_f}} \times \left( \prod_{v \in V} Q^{(v)}_{\ell_e} \ldots \kappa_e \ldots \right) \prod_{e \in E} T^{(e)}_{n(f,\partial_+ e) \ldots n(f,\partial_+ e) \ldots k_e \ldots n(f,\partial_- e) \ldots n(f,\partial_- e) \ldots \ell_e} \]  

(6.11)

\[ = \frac{1}{Z} \left( \prod_{f \in F} \sum_{\tau_f \in R_G} \right) \left( \prod_{e \in E} \mathbf{J}_{\alpha_e} (g_e) \right) \left( \prod_{f \in F} \bar{u}_{\tau_f} \right) \left( \prod_{v \in V} C^{(v)} \right). \]  

(6.12)

Here \( \mathbf{U}^{(e)} \), \( e \in E \), denotes a basis of \( G \)-invariant projectors

\[ \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V^*_{\tau_f} \right) \otimes \mathbf{V}_{\alpha_e} \rightarrow \mathbb{C}. \]  

(6.13)

The weights per vertex \( C^{(v)} \) are given by a trace involving representations and projectors in
the neighbourhood of the vertex \( v \in V \),

\[
\overline{C}(v) = \left( \prod_{e \in E: v = \partial_+e} \dim V_{\sigma_e} \right) \left( \prod_{e \in E: v = \partial_-e} \dim V_{\sigma_e} \right) Q^{(v)} \left( \sum_{k_e=1}^{\dim V_{\sigma_e}} \right) \left( \sum_{\ell_e=1}^{\dim V_{\sigma_e}} \right) \left( \prod_{f \in F: v \in \partial_0f} \dim V_{\sigma_f} \right) \left( \sum_{n_f=1}^{\dim V_{\sigma_f}} \right)
\]
\[
\times \left( \prod_{e \in E: v = \partial_+e} U^{(e)}_{\tau^*_f n_f ... n_f \tau^*_f \tau^*_f} \right) \left( \prod_{e \in E: v = \partial_-e} U^{(e)}_{\tau^*_f n_f ... n_f \tau^*_f \tau^*_f} \right).
\]

(6.14)

The Haar intertwiner \( T^{(e)} \), \( e \in E \), in (6.11) is a map

\[
T^{(e)}: \left( \bigotimes_{f \in e_+} V_{\tau^*_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau^*_f} \right) \otimes V_{\sigma_e} \rightarrow \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f} \right) \otimes V_{\sigma_e}.
\]

(6.15)

**Remark 6.9.**

1. The general pattern is already familiar: The dual of the expectation value is given by a ratio of partition functions whose numerator is a modification of the partition function, here given by the background spin network \((\sigma, Q)\) to which the spin foams couple. The structure remains unchanged, just the compatibility condition is modified so that the numerator of the dual expectation value is given by a sum over all spin foams bounded by the spin network \((\sigma, Q)\) (Definition 3.7).

2. The spin networks \( \overline{C}(v) \) of (6.14) are shown in Figure 12(c). Compared with (b), there is in addition a piece of the spin network \((\sigma, Q)\) in the middle of the diagram.

Similarly to the sigma models, we can again ask what are the natural functions whose expectation value under the dual partition function we can study. A construction using the centre \( Z(G) \) which is essentially analogous to Section 4.5 was given in 16. In the language of the present article, it reads as follows.

**Definition 6.10.** Let \( G \) be a compact Lie group, \( (V, E, F) \) be an oriented two-complex and \( X: F \rightarrow Z(G) \), \( f \mapsto X_f \) assign an element of the centre to each face \( f \in F \). The centre monopole correlator is is the following function \( \mathcal{O}_X: (\mathcal{R}_G)^E \rightarrow \mathbb{C} \) of the configurations of the dual partition function (6.4),

\[
\mathcal{O}_X(\{\tau_f\}_{f \in F}) := \prod_{f \in F} \overline{t}(\tau_f)(X_f),
\]

(6.16)

where \( \overline{t}(\tau_f) \) denotes the representation functions of \( Z(G) \) of Lemma 2.19.

**Theorem 6.11.** The expectation value of the centre monopole correlator under the dual partition function (6.4) reads in the original formulation

\[
\langle \mathcal{O}_X \rangle_{\text{dual}} = \frac{1}{Z} \left( \prod_{e \in E \setminus G} \int dg_e \right) \prod_{f \in F} u(g_f \cdot X_f).
\]

(6.17)

For a deliberate choice of \( X \), this expression restricts to the monopole correlator 27 of \( U(1) \)-lattice gauge theory in \( d = 4 \) and coincides with the \( \mathbb{Z}_N \) centre monopoles and vortices which are being studied in \( SU(N) \)-lattice gauge theory.

A construction using indicator functions in the dual formulation which probe whether a particular face \( f \in F \) is assigned a given representation \( \tau_f \in \mathcal{R}_G \) results in a convolution of the Boltzmann weight in the original formulation. This construction proceeds in complete analogy to Section 4.5.
6.2 The generalized Higgs model

In this section, we study the models that can be obtained by coupling a non-linear sigma model with variables in $G/H$ to a lattice gauge theory with gauge group $G$. When we study these models, we keep a particular Abelian special case in mind, namely the $U(1)$-Higgs model with frozen radial component for which Einhorn and Savit [6] have developed a duality transformation. In all the following steps, the lattice chiral model will be contained as a special case of the non-linear sigma model for the choice $H = \{e\}$.

If we wish to couple a lattice gauge theory to the non-linear sigma model, we have to make use of the left-action of $G$ on $G/H$. A similar coupling has already been performed when we passed from the chiral model to the non-linear sigma model. In Lemma 5.1, we have used the action of $H$ by right-multiplication on $G$ in order to couple one variable $h \in H$ for each edge to the variables of the chiral model. The collection of all the integrals over $H$ for each edge just describes a lattice gauge theory with gauge group $H$ and zero action for the gauge fields. Therefore we have coupled the chiral model with symmetry group $G$ to a lattice gauge theory with gauge group $H$. The result of this ‘non-dynamical’ gauge field is merely to average over the cosets and therefore to give rise to a model with variables in $G/H$.

In this section, we couple a ‘second’ gauge field with gauge group $G$ to the chiral model which is dynamical and which realizes a lattice gauge theory as described in the previous section.

Definition 6.12. Let $G$ be a compact Lie group, $H \leq G$ be a Lie subgroup and $(V, E, F)$ denote an oriented two-complex. Let $s_s, s_g: G \to \mathbb{R}$ be $L^2$-integrable class functions that are bounded below and satisfy $s_s(g^{-1}) = s_s(g)$, $s_g(g^{-1}) = s_g(g)$. The function $s_g$ is called the gauge action and $s_s$ the sigma model action. Define furthermore the Boltzmann weight $u(g) = \exp(-s_g(g))$ and, using Lemma 5.1, a function $\tilde{w}: G/H \times G/H \to \mathbb{R}$ from $u(g) = \exp(-s_s(g))$. Then the generalized lattice Higgs model is given by the partition function

$$Z = \left( \prod_{e \in E_G} \int_{G} \, dg_e \right) \left( \prod_{v \in V} \int_{G/H} \, dx_v \right) \left( \prod_{f \in F} \, u(g_{\partial_1 f} \cdots g_{\partial_{N(f)} f}) \right) \left( \prod_{e \in E} \tilde{w}(g_e^{-1} \cdot x_{\partial_+_e}, x_{\partial_- e}) \right). \quad (6.18)$$

Remark 6.13. 1. This definition combines the partition sum of gauge theory, integration over $G$ for each edge, with that of the non-linear sigma model, integration over $G/H$ for each vertex. The configurations of the partition function are elements of $G^E \times (G/H)^V$. The Boltzmann weight $u(g_f)$ of lattice gauge theory is unchanged whereas the Boltzmann weight of the non-linear sigma model $\tilde{w}(x, y)$ is modified to $\tilde{w}(g^{-1} \cdot x, y)$ in order to implement the minimal coupling. We use $g^{-1}$ rather than $g$ here so that the subsequent results are consistent with the left-cosets which we have chosen for the non-linear sigma model and with the notation established in the previous section for gauge theory.

2. The expression does again not depend on the orientations as $\tilde{w}(g^{-1} \cdot x, y) = \tilde{w}(x, g \cdot y)$. Also we could choose different Boltzmann weights $u_f(g)$ for each face $f \in F$ and $\tilde{w}_e(x, y)$ for each edge $e \in E$.

3. Many Higgs models with frozen radial modes appear as special cases of Definition 6.12, see, for example [7].

Proposition 6.14. The total Boltzmann weight of the generalized lattice Higgs model (6.18) has got a local left-$G$ symmetry. For each function $h: V \to G$, $v \mapsto h_v$, which assigns a group
6.3 Expectation values

Using similar methods as in the previous sections, one can calculate all functions $G^E \times (G/H)^V \rightarrow \mathbb{C}$ that are compatible with these symmetries and therefore determine all observables whose expectation value under the partition function can be calculated.

**Theorem 6.15.** Any algebraic function $G^E \times (G/H)^V \rightarrow \mathbb{C}$ that is invariant under the transformations (6.19), is a linear combination of functions of the form

$$f_{\sigma,\rho,P,k_v,...}(\{g_e\}_{e \in E},\{x_v\}_{v \in V}) = \left( \prod_{e \in E} \sum_{\rho_e = 1}^{\dim V_{\rho_e}} \right) \left( \prod_{v \in V} \sum_{j_v = 1}^{\dim V_{\rho_v}} \right) \left( \prod_{v \in V} P_{\rho_v}^{(v)} \prod_{e \in E} g_e \prod_{v = \partial_+ e} H_{j_v}^{(\rho_v)}(x_v) \right) \left( \prod_{e \in E} t_{\rho_e}(g_e) \right) \prod_{v \in V} \frac{\partial v}{\partial_+ e} \frac{\partial_+ e}{\partial_+ e} \frac{\partial_+ e}{\partial_+ e}.$$  

(6.21)
Here $\sigma: E \to R_G$, $e \mapsto \sigma_e$ assigns an irreducible representation of $G$ to each edge $e \in E$, and $\rho: V \to R_H^G$, $v \mapsto \rho_v$ assigns a class-1 representation to each vertex $v \in V$. There are intertwiners of $G$,

$$P^{(v)} \in \text{Hom}_G \left( \bigotimes_{e \in E: v = \partial_+ e} V_{\sigma_e} \otimes \bigotimes_{e \in E: v = \partial_- e} V_{\sigma_e}^* \otimes V_{\rho_v}, \mathbb{C} \right),$$

for each vertex, and the indices $k_v$ are arbitrary, $k_v \in \{1, \ldots, \kappa_{\rho_v}\}$. If in addition $H \leq G$ is a normal subgroup, then the invariant functions are of the form

$$f_{\sigma, \rho, P, Q}(\{g_e\}_{e \in E}, \{x_v\}_{v \in V}) = \left( \prod_{v \in V} P_{e \in E^v \backslash \{e\}} \right) \left( \prod_{v \in V} \sum_{k_v = 1}^{\kappa_{\rho_v}} Q_{k_v} \right) \left( \prod_{v \in V} \sum_{j_v = 1}^{\kappa_{\rho_v}} \sum_{k_v = 1}^{\kappa_{\rho_v}} \right),$$

where $\sigma$, $\rho$ and $P$ are as above, and $Q$ is an intertwiner of $G$,

$$\bigotimes_{v \in V} V_{\rho_v}^* \to \mathbb{C}. \quad (6.24)$$

**Remark 6.16.** 1. These functions combine a spin network function of the type (6.3) given by the spin network $(\sigma, P)$ with an observable of the type (5.10) specified by $\rho$ and by the $k_v$ or by $\rho$ and $Q$, respectively. They are characterized by a spin network with charges $(\sigma, P, \rho)$ (Definition 3.5). The fact that the local gauge transformation (6.19) also affects the variables $x_v$ of the sigma model does not only fix the structure of the minimal coupling term, but also enforces the compatibility condition (6.22) between the spin network function and the sigma model observables. The structure of the functions (6.21) and (6.23) is illustrated in Figure 14(a–c) for the generic case, for a massive and for a normal subgroup.

2. The chiral model coupled to a lattice gauge theory is contained as the special case for $H = \{e\}$. In this case, all dashed lines in Figure 14(b) become solid.

### 6.4 Duality transformation

The duality transformation for the partition function (6.18) and for the expectation values of the functions (6.21) and (6.23) are straightforward using the methods established in the preceding sections. Since the expressions become very long, we only quote the results. As the very number of sum and product signs is probably deterring at first sight, we carefully comment on the meaning of the various terms and refer to the figures for illustration.

**Theorem 6.17 (Dual partition function).** Let $G$ be a compact Lie group, $H \leq G$ a Lie subgroup and $(V, E, F)$ denote an oriented two-complex. The partition function of the
6.4 Duality transformation

generalized lattice Higgs model (6.18) is equal to the following expressions,

\[ Z = \left( \prod_{e \in E} \sum_{\eta_e \in \mathcal{R}_H^G} \left( \prod_{f \in F} \sum_{\tau_f \in \mathcal{R}_G} \hat{w}_{\eta_e} \right) \prod_{f \in F} \hat{u}_{\tau_f} \right) \times \left( \prod_{f \in F} \prod_{v \in V_0} \sum_{n_f = 1}^{\text{dim } V_{\tau_f}} \left( \prod_{e \in \partial_v^+} \kappa_{\eta_e} \right) \left( \prod_{e \in \partial_v^-} \kappa_{\eta_e} \right) \left( \prod_{f \in F} \sum_{\tau_f \in \mathcal{R}_G} \hat{w}_{\eta_e} \right) \right) \times \left( \prod_{e \in E} \prod_{f \in F} \sum_{\tau_f \in \mathcal{R}_G} \hat{u}_{\tau_f} \right) \times \left( \prod_{e \in E} \prod_{v \in V} \sum_{m_v = 1}^{\text{dim } V_{\eta_v}} \left( \prod_{f \in F} \sum_{\tau_f \in \mathcal{R}_G} \hat{w}_{\eta_v} \right) \right) \times \left( \prod_{f \in F} \prod_{v \in V} \sum_{m_v = 1}^{\text{dim } V_{\eta_v}} \left( \prod_{e \in \partial_v^+} \kappa_{\eta_v} \right) \left( \prod_{e \in \partial_v^-} \kappa_{\eta_v} \right) \right) \prod_{v \in V} (6.26) \]

where for each \( v \in V \),

\[ D(v) = \left( \prod_{f \in F} \sum_{n_f = 1}^{\text{dim } V_{\tau_f}} \left( \prod_{e \in \partial_v^+} \kappa_{\eta_e} \right) \right) \times \left( \prod_{e \in \partial_v^-} \kappa_{\eta_e} \right) \prod_{v \in V} (6.27) \]

Here \( \hat{u}_\tau \) and \( \hat{w}_\eta \) denote the character expansion coefficients of the functions \( u(g) \) and \( w(g) \) of Definition 6.12. For each edge \( e \in E \), \( U^{(e)} \) is a basis of \( G \)-invariant projectors

\[ \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f}^* \right) \otimes V_{\eta_e}^* \rightarrow \mathbb{C}, \quad (6.28) \]

and for each vertex \( v \in V \), \( S^{(v)} \) denotes a basis of \( G \)-invariant projectors

\[ \left( \bigotimes_{e \in \partial_v} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in \partial_v} V_{\eta_e}^* \right) \rightarrow \mathbb{C}. \quad (6.29) \]

The coset space Haar map \( I^{(v)} \), \( v \in V \), in (6.25) is a map

\[ \left( \bigotimes_{e \in E} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\eta_e}^* \right) \rightarrow \left( \bigotimes_{e \in E} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\eta_e}^* \right), \quad (6.30) \]

while the Haar intertwiner \( T^{(e)} \), \( e \in E \), maps

\[ \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f}^* \right) \otimes V_{\eta_e}^* \rightarrow \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f}^* \right) \otimes V_{\eta_e}^*. \quad (6.31) \]
Figure 15: (a) The dual partition function \((6.26)\) of the generalized Higgs model in the neighbourhood of a vertex with a spin network \(D(v)\) of \((6.27)\). (b) The analogous diagram for the dual expression \((6.33)\) of the expectation value of an observable \((6.21)\).

Remark 6.18. 1. We first comment on the dual partition function in the form \((6.26)\). The dual partition sum comprises the partition sums of both the non-linear sigma model and of lattice gauge theory. For the non-linear sigma model, we have a sum over all colourings of the edges with class-1 representations \(\eta_e\) and a sum over all colourings of the vertices with compatible intertwiners \(S(v)\) where the compatibility condition \((6.29)\) is the same as for the non-linear sigma model. For lattice gauge theory, there are additional sums over all colourings of the faces with irreducible representations \(\tau_f\) and of the edges with compatible intertwiners \(U^{(e)}\). This compatibility condition \((6.28)\) is, however, not the same as in lattice gauge theory. The minimal coupling term has modified this condition so that each spin foam appearing in the dual of the gauge theory sector is bounded by the spin network that occurs in the dual of the non-linear sigma model. In other words, the spin network diagrams of the high temperature expansion of the non-linear sigma model appear as spin network functions whose expectation value is calculated under the partition function of gauge theory. The minimal coupling term of the generalized Higgs model could have been found from this entirely dual point of view.

2. In addition to the character expansion coefficients, we find under the dual partition sum several spin networks. There is one would-be spin network from the non-linear sigma model, given by the representations \(V_{\eta_e}\) and by the intertwiners \(S^{(e)}\) which extends over the entire graph. It does not form a proper spin network because the summation over the indices \(m_e\) extends only over \(1, \ldots, \kappa_{\eta_e}\) i.e. over the \(H\)-invariant subspaces of the representations. This is the same type of network that is usually denoted by dashed
6.4 Duality transformation

lines and has already appeared in the dual partition function of the non-linear sigma model, see the top layer of Figure 11(a).

3. Under the partition sum, there are furthermore the spin networks denoted by \( D(v) \) for each vertex. They are similar to the spin networks \( C(v) \) from the dual partition function of lattice gauge theory (6.6), but include in addition a part of the spin network given by the representations \( V_{\eta} \) and the intertwiners \( S(v) \). The difference between the \( C(v) \) of lattice gauge theory and the \( D(v) \) appearing here is essentially the same as that of the \( C(v) \) and the \( \tilde{C}(v) \), cf. Figure 12(b) and (c). The neighbourhood of a vertex with the spin network \( D(v) \) and the dashed lines of the would-be spin network is shown in Figure 15(a).

4. The structure of the dual partition function (6.26) of the generalized Higgs model can be explained in other words starting from the corresponding expression of the chiral model (Figure 7(a)). First, we are concerned with the non-linear sigma model rather than with the chiral model. This was implemented by coupling elements \( h \in H \) to one chiral half of the model which corresponds to the top layer in Figure 7(a), and then by averaging over the subgroup in Lemma 5.1. This averaging is the reason why the top layer of Figure 11(a) consists of dashed lines (‘would-be spin network’). Then we have minimally coupled lattice gauge theory to the other chiral half which corresponds to the spin network in the bottom layer of Figure 7(a). The effect of the minimal coupling term is that lattice gauge theory just considers this spin network as an observable to which it couples its spin foams. The bottom layer of Figure 7(a) is therefore treated as the spin network function in the expectation value of lattice gauge theory, and becomes disconnected, leading to Figure 12(c) for lattice gauge theory and to Figure 15(a) for the generalized Higgs model.

5. As usual, there is an alternative formulation of the dual partition function which uses the Haar intertwiners and Haar maps rather than sums over projectors. This version is given in the first equation (6.25).

6. As \( G \) acts transitively on \( G/H \), one can easily fix a ‘unitary’ gauge by choosing \( h_v := g_{x_v}^{-1} \) in (6.19) where \( g_{x_v} \) is a representative of \( x_v \). This step is often convenient because it removes the scalar degrees of freedom from the model. For the duality transformation it is, however, pointless because the corresponding symmetry is already manifest in the dual picture.

Finally, the duality transformation is also available for the expectation value of the observable (6.23). The result is stated in the following theorem which contains the most complicated formulas we are going to present. We formulate the result for the correlator in the form (6.23). If \( H \leq G \) is a non-normal subgroup, then the requirement that \( Q \) is \( G \)-invariant can be dropped so that one recovers the expression (6.21) for generic \( k_v \in \{1, \ldots, \kappa_{v\rho} \} \).

**Theorem 6.19 (Dual observable).** Let \( G \) be a compact Lie group, \( H \leq G \) a Lie subgroup and \( (V, E, F) \) denote an oriented two-complex. The expectation value of the function (6.23)
under the partition function of the generalized Higgs model is equal to

\[
\langle f_{\sigma,\rho,P,Q} \rangle = \frac{1}{Z} \left( \prod_{e \in E} \sum_{\kappa_{\rho,\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \sum_{\kappa_{\rho,\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \sum_{\kappa_{\rho,\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \sum_{\kappa_{\rho,\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \sum_{\kappa_{\rho,\psi} = 1}^{\dim V_{\eta_e}} \right),
\]

where

\[
\bar{D}(v) = \left( \prod_{f \in F} \sum_{n_f = 1}^{\dim V_{\eta_f}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right) \left( \prod_{e \in E} \sum_{\kappa_{\psi} = 1}^{\dim V_{\eta_e}} \right).
\]

For each edge \( e \in E \), \( \bar{U}^{(e)} \) denotes a basis of \( G \)-invariant projectors

\[
\left( \bigotimes_{f \in F} V_{\eta_f} \right) \otimes \left( \bigotimes_{f \in F} V_{\eta_f}^{*} \right) \otimes V_{\eta_e} \otimes V_{\eta_e} \rightarrow \mathbb{C},
\]

and for each vertex \( v \in V \), \( \bar{S}^{(v)} \) is a basis of \( G \)-invariant projectors

\[
\left( \bigotimes_{e \in E} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\eta_e}^{*} \right) \otimes V_{\rho_v} \rightarrow \mathbb{C}.
\]

The coset space Haar map \( I^{(v)} \), \( v \in V \), in (6.32) is a map

\[
\left( \bigotimes_{e \in E} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\eta_e}^{*} \right) \otimes V_{\rho_v} \rightarrow \left( \bigotimes_{e \in E} V_{\eta_e} \right) \otimes \left( \bigotimes_{e \in E} V_{\eta_e}^{*} \right) \otimes V_{\rho_v}.
\]
while the Haar intertwiner \( T(e), e \in E \), maps
\[
\left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f}^* \right) \otimes V_{\eta_e} \otimes V_{\sigma_e} \rightarrow \left( \bigotimes_{f \in e_+} V_{\tau_f} \right) \otimes \left( \bigotimes_{f \in e_-} V_{\tau_f}^* \right) \otimes V_{\eta_e} \otimes V_{\sigma_e}.
\] (6.38)

**Remark 6.20.**

1. The features that are new in the dual expectation value (6.32) compared with the dual partition function (6.25), are first the sums and intertwiners from the definition (6.23). The presence of the spherical functions \( H_{j,v}^{(\rho_v)} \) for each vertex \( v \in V \) has lead to a an additional representation \( V_{\rho_v} \) in the coset space Haar map (6.37) and thus to a modification of the compatibility condition (6.36). The presence of the representation function \( t(\sigma_e) \) has resulted in an additional representation \( V_{\sigma_e} \) of the Haar intertwiner (6.38) and thus in a modification of the compatibility condition (6.35). The correlator (6.23) which is given by a spin network with charges, has modified the numerator of (6.23) so that the configurations of the dual picture, spin foams bounded by spin networks, are now themselves bounded by the given spin network with charges. The structure of (6.38) is illustrated in Figure 15(b) which shows the spin network \( \tilde{D}(v) \) in the neighbourhood of a vertex.

2. For the special cases in which \( H \) is normal or massive, the situation is completely analogous to the non-linear sigma model. The only changes in these cases apply to the open ends of the dashed lines labelled \( k_v \).

### 6.5 Expectation values of the dual model

It is possible to construct natural observables for the dual partition function of the generalized Higgs model in the same way as for the non-linear sigma model and for lattice gauge theory. If these observables only probe the representations \( \eta_e \) assigned to the edges and \( \tau_f \) assigned to the faces, the result is the product of a dual observable of the non-linear sigma model and one of lattice gauge theory, both independent of each other.

### 6.6 The Abelian special case

In analogy to Section 4.4, we show the Abelian special case of the generalized Higgs model for \( G = U(1), H = \{ e \} \), in greater detail.

We write \( e^{i\varphi_e} \in U(1), e \in E \), for the variables of lattice gauge theory and \( e^{i\vartheta_v}, v \in V \), for the sigma model. The partition function (6.18) then reads
\[
Z = \left( \prod_{v \in V} \frac{1}{2\pi} \int_0^{2\pi} d\vartheta_v \right) \left( \prod_{e \in E} \frac{1}{2\pi} \int_0^{2\pi} d\varphi_e \right) \left( \prod_{f \in F} \exp(-s_g(e^{i\sum_{j=1}^{N(f)} (\varepsilon_j f) \cdot \varphi_{\partial j f}})) \right) \times \left( \prod_{e \in E} \exp(-s_{\delta}(e^{i(\vartheta_{\partial + e} - \vartheta_{\partial - e} + \varphi_e)})) \right).
\] (6.39)

This is the \( U(1) \)-Higgs model studied by Einhorn and Savit [6]. The dual expression for the
partition function, equation (6.26), specializes to

$$Z = \left( \prod_{e \in E} \sum_{\ell_e = -\infty}^{\infty} \right) \left( \prod_{f \in F} \sum_{k_f = -\infty}^{\infty} \right) \left( \prod_{e \in E} \hat{w}_{\ell_e} \right) \left( \prod_{f \in F} \hat{u}_{k_f} \right) \times \left( \prod_{v \in V} \delta \left( \sum_{e \in E: v = \partial_+ e} \ell_e - \sum_{e \in E: v = \partial_- e} \ell_e \right) \right) \left( \prod_{f \in F} \delta \left( \sum_{k_f - \sum_{f \in e_-}} k_f + \ell_e \right) \right),$$

(6.40)

where $\hat{w}_t$ and $\hat{u}_k$ are the Fourier coefficients of $w(g) = \exp(-s_s(g))$ and $u(g) = \exp(-s_g(g))$, $g \in U(1)$, respectively. This expression combines the dual partition function of the $XY$-model with that of $U(1)$-lattice gauge theory and implements the minimal coupling by the compatibility condition encoded in the constraint. It agrees with the result of [6] before the constraint is integrated.

Since the labellings of the edges with integers $\ell_e$ and of the faces with integers $k_f$ are Abelian, we can visualize (6.40) as a sum over all closed lines living on the edges together with a sum over all closed surfaces living on the faces where each surface is either closed or bounded by one of the lines.

If we use the Villain action for both the sigma model and gauge theory, i.e. $\hat{w}_t = e^{-\ell^2/2\beta_1}$ and $\hat{u}_k = e^{-k^2/2\beta_2}$, then the total exponent of the dual Boltzmann weight is the length of the lines weighted with $1/\beta_1$ plus the area of the surfaces weighted with $1/\beta_2$. This is the effective (open) string model for the strong coupling regime of the $U(1)$-Higgs model.

The observables (6.23) reduce to functions,

$$f_{p_v, q_e, \ldots}(\{\theta_v\}_{v \in V}, \{\varphi_e\}_{e \in E}) := \left( \prod_{v \in V} e^{ip_v \theta_v} \right) \left( \prod_{e \in E} e^{iq_e \varphi_e} \right),$$

(6.41)

which describe charges $p_v \in \mathbb{Z}$ at the vertices $v \in V$ and Wilson loops $q_e \in \mathbb{Z}$ at the edges $e \in E$ provided that for each $v \in V$, the following compatibility condition holds,

$$\sum_{e \in E: v = \partial_+ e} q_e - \sum_{e \in E: v = \partial_- e} q_e + p_v = 0.$$  

(6.42)

The dual of the expectation value then reads,

$$\langle f_{p_v, q_e, \ldots} \rangle = \frac{1}{Z} \left( \prod_{e \in E} \sum_{\ell_e = -\infty}^{\infty} \right) \left( \prod_{f \in F} \sum_{k_f = -\infty}^{\infty} \right) \left( \prod_{e \in E} \hat{w}_{\ell_e} \right) \left( \prod_{f \in F} \hat{u}_{k_f} \right) \times \left( \prod_{v \in V} \delta \left( \sum_{e \in E: v = \partial_+ e} \ell_e - \sum_{e \in E: v = \partial_- e} \ell_e + p_v \right) \right) \left( \prod_{f \in F} \delta \left( \sum_{k_f - \sum_{f \in e_-}} k_f + \ell_e + q_e \right) \right),$$

(6.43)

i.e. the closed lines of (6.40) now couple to the charges $p_v, v \in V$, and can thus end at one of these charges while the surfaces are either closed or bounded by the lines or by the background Wilson loop $q_e, e \in E$.

This is the picture which is generalized to sums over spin networks and spin foams in the non-Abelian case.
7 Discussion

We have presented an exact duality transformation for the partition functions and expectation values of observables of the lattice chiral model, of the lattice non-linear sigma model and of a class of generalized Higgs models. We conclude with various miscellaneous comments on applications, limitations and open questions.

Throughout the present article, we have chosen ultra-local actions, i.e. the action is a sum over all edges [or faces] and can be calculated independently for each edge [or face]. A generalization to more complicated, less local, actions is straightforward. Observe that the character expansion of the Boltzmann weight is always a series of charges [or spin network functions] and that we can perform the duality transformation for generic expectation values of these charges [or spin network functions].

The dual form of the partition function can be used for numerical studies. From the Abelian special case it is familiar (see, for example [31]) that for some observables the original model is much easier to simulate whereas for others the simulations are much more efficient in the dual model. At present, algorithms are being developed for pure $SU(2)$-lattice gauge theory in three dimensions [32] and for a technically closely related model [33] in the context of quantum gravity.

If one wishes to implement Monte Carlo algorithms for the dual model, one has to make sure that the importance sampling is applied to a positive measure. While the character expansion coefficients of the common Boltzmann weights are positive, the situation is less clear for the spin networks (such as the $C(v)$ of (6.6)) which appear under the dual partition sum. At least for the $O(4)$-symmetric non-linear sigma model and for the $SU(2)$-symmetric chiral model, these spin networks have non-negative real values [34]. Should there be alternating signs in other models, one has to associate the sign with the observable which is measured while the modulus can be dealt with by the importance sampling. This is familiar, for example, from the sign problems in the simulation of fermionic systems.

It might finally be more than a mere coincidence that the dual partition function resembles a cluster decomposition. The lack of efficient cluster algorithms for gauge theories may have a natural explanation in the dual picture where the weights $C(v)$ of lattice gauge theory are localized at the vertices as opposed to the spin network which appears in the dual sigma model and which extends over the entire lattice.

We emphasize that there are intermediate steps in the duality transformation, for example (4.20) and (5.19), in which both the old and the new variables are present and which resemble an extended ‘phase space’ path integral whose weight, however, does not have any obvious positivity properties. Upon solving all sums, one recovers the original partition function with positive Boltzmann weights while performing the integrals, one obtains the dual expression, again with positive weights (at least in some cases which we have listed above).

In the Abelian case, there are higher level generalizations of sigma models and gauge theories in which the fundamental variables are located not at vertices or edges, but rather at higher level, e.g. at cubes, hypercubes, etc., and described by discretized $k$-forms [2, 3]. This construction does not have any obvious generalization to the non-Abelian case. Any such model would make use of a suitable definition of non-Abelian cohomology.

We also stress that the non-Abelian generalization of the duality transformation parallels the Abelian special case only up to the point where one solves the constraints. In the non-Abelian situation, there are no longer just constraints, but rather sums over compatible intertwiners so that there exists no obvious step which generalizes the integration of the
constraints. This restricts us to the original lattice as opposed to the Abelian case in which one usually passes to a suitable ‘dual’ lattice. This can, however, also be seen as an advantage because our generalization is therefore independent of the topology of the lattice. The case of non-trivial topology in Abelian systems was studied in [35].

An interesting generalization of lattice gauge theory is available in $d \leq 4$ dimensions in the dual formulation where one can replace the gauge group by a quantum group [17, 18]. This includes in particular supergroups as the gauge groups. Similar constructions in which the category of representations of a compact Lie group in the dual formulation is replaced by more general categories, have already been known from the definition of topological invariants and from Topological Quantum Field Theory, see, for example [36, 37]. From the formulas stated in the present article, one obtains at least a formal topological invariant from the partition functions if the Boltzmann weights, say, $w(g)$, are replaced by $\delta$-functions $w(g) = \delta(g)$ and similarly $\hat{w}_\rho = \dim V_\rho$ in the dual picture. Non-compact Lie groups have recently attracted attention in the context of quantum gravity, see, for example [38].

What has been missing so far is firstly a generalization which includes fermions (this is mainly due to the still rather limited understanding of fermions in a non-perturbative formulation) and secondly an analogue of the vortex - spin wave decomposition of [39, 40].

The present article is entirely written in the Lagrangian language of path integrals and expectation values. All results are in one-to-one correspondence to the analogous statements in the Hamiltonian formulation which involves the quantum statistical operator $e^{-H}$. Matrix elements of this operator can be calculated in the dual picture from sums over spin networks and spin foams.

As far as the strong-weak relation of the duality transformation is concerned, we stress that the dual partition function provides a closed form for the strong coupling expansion which makes it possible to separate the group combinatorics from the lattice combinatorics. This has been advocated in the context of high order strong coupling expansions, see, for example [7, 41]. The key to the duality transformation was to abstract from a particular group and to focus on the structures that are common to all compact Lie groups. It remains a considerable challenge to evaluate the dual expressions for particular groups, Boltzmann weights and shapes of the lattice.

As far as the construction of strong coupling expansions in gauge theories is concerned, it is interesting to note that there exists an effective string model which describes the strong coupling regime of Abelian lattice gauge theories. In the non-Abelian case, however, mere strings are insufficient, and the world-sheets of the strings should rather be allowed to branch according to the combinatorics of the representation theory. A familiar example is the strong coupling calculation of the static three-quark potential in QCD. The lack of branchings of the world-sheets causes the string picture to break down when spin foams appear as the fundamental non-perturbative structure.

Acknowledgements

The author is grateful to Emmanuel College, Cambridge, for a Research Fellowship. I thank Alan Sokal who suggested to extend the techniques of the duality transformation to sigma models. I am also grateful to John Barrett, Alan Macfarlane, Shahn Majid, Robert Oeckl.

\footnote{A detailed study is in preparation.}
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Daniele Oriti, Arttu Rajantie, Nuno Romão, Tony Sudbury and Toby Wiseman for valuable discussions and for comments on the relevant literature.

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