MOTIVIC CHERN CLASSES OF CONFIGURATION SPACES

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Abstract. We calculate the equivariant motivic Chern class for configuration space of smooth variety and the space of vectors with different directions. We prove the formulas for generating series of these classes. We generalize the localization theorems results about BB-decomposition to acquire some stability for the motivic Chern classes of configuration spaces.

1. Introduction

We consider complex quasiprojective varieties. The motivic Chern class $mC$ introduced in [BSY10] (see also [SY07] for a survey) generalize previously defined characteristic classes of singular varieties (e.g. the Chern-Schwartz-MacPherson class [Mac74], [Ohm06] for the equivariant version). Lately its equivariant counterpart was defined in [FRW18b, AMSS19] and studied for Schubert varieties in flag varieties. Consider an algebraic torus $T$. The motivic Chern class $mC^T$ assigns to every $T$-equivariant map $f : X \to M$ of $T$-varieties an element of $G^T(M)[y]$, i.e. a polynomial in $y$ over the equivariant $K$-theory of coherent sheaves on $M$. It is uniquely defined by three properties (after [FRW18b], section 2.3):

1. Additivity: If $X = Y \sqcup U$, then
   \[ mC^T(X \to M) = mC^T(Y \to M) + mC^T(U \to M). \]

2. Functoriality: For a proper map $f : M \to M'$ we have
   \[ mC^T(X \xrightarrow{f} M') = f_* mC^T(X \xrightarrow{g} M). \]

3. Normalization: For smooth $M$ we have
   \[ mC^T(id_M) = \lambda_y(T^* M), \]
   where $\lambda_y$ is the Grothendieck $\lambda$ operation defined by equation (π).

The goal of this note is to compute and study behaviour of the torus equivariant motivic Chern classes of configuration spaces. We consider ordered configuration space of a smooth $T$-equivariant variety $B$

\[ \text{Conf}_k(B) = \{(x_1, ..., x_k) \in B^k | x_i \neq x_j \} \]

and the space of pairwise linearly independent nonzero vectors

\[ C_k(\mathbb{C}^n) := \{(v_1, \ldots, v_k) \in (\mathbb{C}^n)^k | \forall_i \forall_j v_i \neq 0, \text{ span}(v_i) \neq \text{ span}(v_j) \}. \]

Both of these spaces are open varieties with singular complement. We compute classes of inclusions

\[ \text{Conf}_k(B) \subset B^k \text{ and } C_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k. \]

We consider only ordered configurations. For classes of symmetric products see [CMS+17, Kom, MS11, Ohm08] and others. In the subsequent sections we establish generating series and search for stability results. In the section 3 we simplify generating series:
Proposition.

\[ 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot \frac{mC^T_\alpha(Conf_k(C^n) \subset (C^n)^k)}{eu^T_\alpha(\{0\} \subset (C^n)^k)} = \exp \left( \frac{\lambda_y(T^*C^n)}{\lambda_{-1}(T^*C^n)} \log(1 + t) \right) \]

Proposition.

\[ 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot \frac{mC^T_\alpha(C_k(C^n) \subset (C^n)^k)}{eu^T_\alpha(\{0\} \subset (C^n)^k)} = \prod_{i=1}^{n} \exp \left( \frac{\lambda_y(T^*P(C^n))|_{E_i}}{\lambda_{-1}(T^*P(C^n))|_{E_i}} \log \left( 1 + \frac{t(1 + y)}{\alpha_i - 1} \right) \right) \]

In the section 4 we show that there is a connection between class of configuration space of a smooth, projective variety and the configuration space of its fixed points component, critical in BB-decomposition \cite{BB73, BB74, BB76} for some one dimensional subtorus. Namely

**Proposition.** Let \( C^* \) be one dimensional subtorus of torus \( T \). Denote by \( T_1 \) quotient torus \( T/C^* \). Let \( B \) a be smooth, projective \( T \)-variety. Let \( F_1 \) be the sink of the BB decomposition according to torus \( C^* \). Then

\[ mC^{T_1}(Conf_k(F_1) \to F_1^k) = \lim_{t \to 0} \left( mC^T(Conf_k(B) \to B^k)|_{F_1} \right) \]

for appropriately defined limit map (definition \[4.1\]). Analogues result holds when \( F_1 \) is the source.

To prove this statement we generalize the Lefschetz-Riemann-Roch theorem and the results of \cite{Web17} connecting BB decomposition with localization theorems to the relative case.

**Tools and notations.** We consider only complex quasiprojective varieties. We work with the algebraic \( T \)-equivariant \( K \)-theory of vector bundles which we denote by \( K^T(X) \). By localized \( K \)-theory \( S^{-1}K^T(X) \) we mean \( K \)-theory localized in the multiplicative system \( K^T(pt) - \{0\} \). We use the multiplicative characteristic class \( \lambda_y \) in \( K \)-theory defined by:

\[(*) \quad \lambda_y(E) := \sum_{i=0}^{\text{rank} E} [\Lambda^i E]y^i.\]

The class \( \lambda_{-1}(E^*) \) is the Euler class in \( K \)-theory, we sometimes denote it by \( eu(E) \). Main advantage of working with torus equivariant case is the possibility to recover global invariant from its restriction to fixed points using the localization theorems and the Lefschetz-Riemann-Roch theorem.

**Theorem 1.1** (Lefschetz-Riemann-Roch, \cite{CG10} theorem 5.11.7). Assume that a torus \( T \) acts on smooth varieties \( X \) and \( Y \). Consider the multiplicative system \( S \) of nonzero elements in \( K^T(pt) \). Let \( F \subset Y^T \) be a component of fixed points. For any proper \( T \)-equivariant map \( f : X \to Y \) and element \( \alpha \in S^{-1}K^T(X) \) the pushforward \( f_*\alpha \) can be computed using an equality

\[ \frac{i_F^*f_*\alpha}{eu^T(\nu_F)} = \sum_{G \subset X^T \cap f^{-1}(F)} f|_G^* \frac{i_G^*e\alpha}{eu^T(\nu_G)}. \]

Where the sum is indexed by the fixed points components of \( X \) which lie in preimage of \( F \).
Remark 1. The Lefschetz-Riemann-Roch theorem is a consequence of the localization formula [AB66] [BV82]. Namely

\[ i_F^* \alpha = i_F^* f_* \sum_{G \subset X^T} i_G^* \frac{i_F^* \alpha}{\text{eu}^*(\nu_G)} = \sum_{G \subset X^T} i_F^* i_F^*(G) f_* i_G^* \frac{i_G^* \alpha}{\text{eu}^*(\nu_G)} = \text{eu}(\nu_F) \sum_{G \subset X^T \cap f^{-1}(F)} \frac{i_G^* \alpha}{\text{eu}^*(\nu_G)}, \]

where morphisms \( i \) are inclusions of fixed points set components. The first equality follows from the localization formula.

We consider the Grothendieck group \( K^T(Var/X) \) of \( T \)-equivariant morphisms from quasiprojective varieties to given \( T \)-variety \( X \) (cf. [Loo02] [AMSS19] [Bi04]). When we think of a morphism as an element of the Grothendieck group we put it in square brackets. We denote by \( \alpha_1, ..., \alpha_n \) weights of the diagonal torus in \( GL_n(\mathbb{C}) \).

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2. Computation of the motivic Chern classes

For a smooth variety \( B \) with an action of an algebraic torus \( T \) we consider the configuration space

\[ \text{Conf}_{B,k}(B) = \{ (x_1, ..., x_k) \in B^k | x_i \neq x_j \}, \]

with the natural action of the torus \( T \). We consider the space

\[ C_k(\mathbb{C}^n) := \{ (v_1, ..., v_k) \in (\mathbb{C}^n)^k | v_i \neq 0, \text{span}(v_i) \neq \text{span}(v_j) \} \]

of nonzero, pairwise linearly independent vectors (a bundle over the configuration space of projective space). This space admits the natural action of the group \( GL_n(\mathbb{C}) \times (\mathbb{C}^*)^k \). Denote by \( T_\alpha \) the diagonal torus in the group \( GL_n(\mathbb{C}) \), and by \( T_\beta \) the torus \( (\mathbb{C}^*)^k \). Denote by \( \alpha_1, ..., \alpha_n \) and \( \beta_1, ..., \beta_k \) weights of the tori \( T_\alpha \) and \( T_\beta \). Our goal in this section is computation of the classes \( mC^\mathcal{G} (\text{Conf}_{B,k}(B) \subset B^k) \) for any smooth \( T \)-variety \( B \) and \( mC^\mathcal{G}(\text{Conf}_{B,k}(\mathbb{C}^n)) \subset (\mathbb{C}^n)^k \). For simplicity we sometimes omit an ambient space in notation of the motivic Chern class.

Remark 2. Spaces \( C_k(\mathbb{C}^n) \) are orbit configuration spaces \( F_{C^*}(\mathbb{C}^n - \{0\}, k) \) (defined in [XM97] notation from [FZ02]) where the group \( C^* \) acts by homotheties. It might look more natural to consider the bigger space \( F_{C^*}(\mathbb{C}^n, k) \). Its class can be easily computed from the class of \( F_{C^*}(\mathbb{C}^n - \{0\}, k) \) using the equality:

\[ F_{C^*}(\mathbb{C}^n, k) = F_{C^*}(\mathbb{C}^n - \{0\}, k) \sqcup \bigcup_{i=1}^k F_{C^*}(\mathbb{C}^n - \{0\}, k-1). \]

2.1. Configuration space. Let \( [k] \) denote the set \( \{1, ..., k\} \). Define a partition of a set \( A \) as a set of nonempty, pairwise disjoint subsets, whose sum is the whole set \( A \). Let \( X_k \) denote the set of partitions of \( [k] \). For a given partition \( P \in X_k \) and \( i \in [k] \) denote by \( P(i) \) element of \( P \) which contain \( i \). For a partition \( P \in X_k \) consider the set

\[ B_P = \{ (x_1, ..., x_k) \in B^k | x_i = x_j \text{ when } P(i) = P(j) \}. \]

The configuration space \( \text{Conf}_{B,k}(B) \) is given by \( \binom{k}{2} \) conditions \( x_i \neq x_j \). The inclusion-exclusion formula implies formula (for combinatorial details see appendix)

\[ [\text{Conf}_{B,k}(B) \rightarrow B^k] = \sum_{P \in X_k} a(P) [B_P \rightarrow B^k], \]
where numbers \( a(P) \in \mathbb{Z} \) depends only on partition \( P \). Consider the set \( G(P) \) of graphs whose set of vertices is equal to \([k]\) and whose connected components induce partition \( P \) on the set \([k]\). For a given graph \( G \) let \( E(G) \) be its set of edges. Then (cf. lemma \( 4.1 \))

\[
a(P) = \sum_{G \in G(P)} (-1)^{|E(G)|} = \prod_{P_i \in P} (-1)^{|P_i| - 1}(|P_i| - 1!).
\]

Thus to compute the classes \( mC^T(Conf(f_k(B) \to B^k)) \) it is enough to compute the classes \( mC^T(B_P \to B^k) \). Let’s choose any order on a partition \( P \). The set \( B_P \) is a smooth subvariety isomorphic to \( B^{[P]} \). Its inclusion is a “diagonal” map

\[
i_P : B^{[P]} \to B^k,
\]

such that \( i_P(x_1, \ldots, x_{|P|})_j = x_s \) when \( j \in P_s \). It follows from the functorial properties of the motivic Chern classes that

\[
mC^T(Conf_k(X)) = \sum_{P \in P_k} \left( a(P)i_{P,\ast}(\lambda_y(TX^{[P]})) \right).
\]

**Example 1.** Consider the affine space \( \mathbb{C}^n \) with action of the diagonal torus. Denote by \( \alpha_1, \ldots, \alpha_n \) weights of the torus. Formula (1) implies that:

\[
mC^T(Conf_k(\mathbb{C}^n)) = \sum_{P \in P_k} \prod_{P_a \in P} (-1)^{|P_a| - 1}(|P_a| - 1)! \left( \prod_{j \neq i(P_a)} \left( 1 + \frac{y}{\alpha_j} \right) \left( 1 - \frac{1}{\alpha_j} \right)^{|P_a| - 1} \right)
\]

**Example 2.** Consider the projective space \( \mathbb{P}(\mathbb{C}^n) \) with action of the diagonal torus. Denote by \( \alpha_1, \ldots, \alpha_n \) weights of the torus. Denote by \( e_1, \ldots, e_n \) the fixed points of the action on \( \mathbb{P}(\mathbb{C}^n) \). Let \( e = (e_1, \ldots, e_n) \in \mathbb{P}(\mathbb{C}^n)^k \) be a fixed point. Such point induces partition \( P_e \) of the set \([k]\) such that \( x, y \) belong to the same element of the partition \( P_e \) if and only if \( \tau_x = \tau_y \). Consider the set \( X_e \) of partitions of the set \([k]\) which subdivide \( P_e \). For an element of such partition \( P_a \in P \in X_e \) let \( i(P_a) \) denote the number \( i_x \) for any \( x \in P_a \). Formula (1) implies that:

\[
mC^T(Conf_k(\mathbb{P}(\mathbb{C}^n)))|_e = \sum_{P \in X_e} \prod_{P_a \in P} (-1)^{|P_a| - 1}(|P_a| - 1)! \left( \prod_{j \neq i(P_a)} \left( 1 + \frac{y\alpha_i(P_a)}{\alpha_j} \right) \left( 1 - \frac{\alpha_i(P_a)}{\alpha_j} \right)^{|P_a| - 1} \right)
\]

### 2.2. Orbit Configuration space.

The case of an orbit configuration space is very similar to the case of classical configuration space. For a partition \( P \in X_k \)

\[
B_P = \{ (v_1, \ldots, v_k) \in (\mathbb{C}^n - \{0\})^k | \text{span}(v_i) = \text{span}(v_j) \text{ when } P(i) = P(j) \}
\]

The inclusion-exclusion formula once more implies the equality

\[
[C_k(\mathbb{C}^n) \to (\mathbb{C}^n)^k] = \sum_{P \in X_k} a(P)[B_P \to (\mathbb{C}^n)^k],
\]

where numbers \( a(P) \) are the same as in the previous subsection. Next step is computation of the classes \( mC^T \times T^T(B_P \to B^k) \). Motivic Chern classes are multiplicative with respect to the cartesian product of morphisms [AMSS19] remark 4.3 in the sense that:

\[
mC^T(f \times f' : X \times X' \to Y \times Y') = mC^T(f : X \times Y) \boxtimes mC^T(f' : X' \times Y').
\]

So it is enough to compute the class \( mC^T \times T^T(B_P \to B^k) \) for the partition with only one element. Namely

\[
B_k := B_{\{k\}} = \{ (v_1, \ldots, v_k) \in (\mathbb{C}^n - \{0\})^k | \text{dim span}(v_1, \ldots, v_k) = 1 \}.
\]
Unfortunately this variety doesn’t have a smooth closure in the affine space \((\mathbb{C}^n)^k\) so we have to resolve its singularities. Consider the variety
\[
\tilde{B}_k\{(v_1, \ldots, v_k, l) \in (\mathbb{C}^n - \{0\})^k \times \mathbb{P}(\mathbb{C}^n) | \forall_i v_i \in l \} \subset (\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n),
\]
It has a smooth closure in \((\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n)\) (its boundary is SNC divisor) and the diagram
\[
\begin{array}{ccc}
\tilde{B}_k & \to & (\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n) \\
\downarrow \simeq \ & & \downarrow p \\
B_k & \to & (\mathbb{C}^n)^k
\end{array}
\]
commutes. Moreover the vertical arrow \(\tilde{B}_k \to B_k\) is an isomorphism. It implies that
\[
mC_{T^\alpha \times T^\beta}(B_k \to (\mathbb{C}^n)^k) = p_\ast mC_{T^\alpha \times T^\beta}(\tilde{B}_k \to (\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n)).
\]
To compute the equivariant pushforward we use the Lefschetz-Riemann-Roch (1.1)
\[
mC_{T^\alpha \times T^\beta}(B_k \to (\mathbb{C}^n)^k) = \sum_{i=1}^n \left( \frac{mC_{T^\alpha \times T^\beta}(\tilde{B}_k \to (\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n))_{|0 \times e_i}}{\lambda_1(T^\ast \mathbb{P}(\mathbb{C}^n))_{|e_i}} \right).
\]
For a fixed point \(0 \times e_i\) consider standard affine chart \(U_i\) on \((\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n)\). The equivariant Verdier-Riemann-Roch theorem (AMSS19 theorem 4.2) for an open embedding implies that to compute the motivic Chern class of \(\tilde{B}_k\) at point \(e_i\) one can consider only affine patch \(U_i\)
\[
mC_{T^\alpha \times T^\beta}(\tilde{B}_k \to (\mathbb{C}^n)^k \times \mathbb{P}(\mathbb{C}^n))_{|0 \times e_i} = mC_{T^\alpha \times T^\beta}(\tilde{B}_k \cap U_i \to U_i)_{|0}.
\]
Consider coordinates \(\{x_{a,j}\}_{a \leq k, j \leq n}\) on \((\mathbb{C}^n)^k\) and projective coordinates \([y_j]\) on \(\mathbb{P}(\mathbb{C}^n)\). On the set \(U_i\) we have affine coordinates given by \(\{x_{a,j}\}\) with torus weights \(\beta_a + \alpha_j\) and \(\frac{y_j}{y_i}\) with torus weight \(\alpha_j - \alpha_i\). The variety \(\tilde{B}_k \cap U_i\) is given by equations:
\[
\begin{cases}
x_{a,i} \neq 0 & \text{for all } a \in [k], \\
x_{a,j} - x_{a,i} \frac{y_j}{y_i} = 0 & \text{for all } a \in [k], j \in [n], j \neq i.
\end{cases}
\]
We choose new coordinates on the affine space \(U_i\). Let variables \(\frac{y_j}{y_i}\) be the same as before and consider variables
\[
z_{a,j} = \begin{cases} x_{a,i} & \text{if } j = i, \\
x_{a,j} - x_{a,i} \frac{y_j}{y_i} & \text{if } j \neq i,
\end{cases}
\]
with torus weight \(\beta_a + \alpha_i\). Now the set \(\tilde{B}_k\) is given by coordinate equations so we can use the fundamental calculation from [FRW18b].

**Lemma 2.1** ([FRW18b] subsection 2.7). Let a torus \(T = (\mathbb{C}^*)^r\) act on \(\mathbb{C}\), and let the class of this equivariant line bundle over a point be \(\alpha \in K^2(pt)\). Using the additivity of the motivic Chern class we have
\[
mC(\{0\} \subset \mathbb{C}) = 1 - 1/\alpha, \quad mC(\mathbb{C} \subset \mathbb{C}) = 1 + y/\alpha,
\]
\[
mC(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + y/\alpha) - (1 - 1/\alpha) = (1 + y)/\alpha.
\]
To simplify notation consider function
\[
\psi_{i,j}(\theta) = \begin{cases} 
1 - \frac{1}{\theta} & \text{dla } i \neq j, \\
\frac{1}{\theta} & \text{dla } i = j.
\end{cases}
\]

Application of the lemma 2.1 and multiplicative property of the motivic Chern class leads to formula
\[
mC^{T_{\alpha} \times T_{\beta}} (\tilde{B}_k \cap U_i \subset U_i) |_{e_i} = \lambda_y(T^*\mathbb{P}(\mathbb{C}^n)) e_i \prod_{j=1}^{n} \prod_{a=1}^{k} \psi_{i,j}(\beta_a \alpha_j).
\]

Combining all the ingredients we obtain the final formula
\[
mC (C_k(\mathbb{C}^n)) = \sum_{P \in X} \prod_{P \in P} \left( \left(1 - \frac{1}{\theta} \right)^{|P_i| - 1} \left(|P_i| - 1\right) \right) \sum_{i=1}^{n} \prod_{j \neq i} \left( \frac{1 + \frac{y_i}{\alpha_j}}{1 - \frac{1}{\alpha_j}} \right) \prod_{j=1}^{n} \prod_{a \in P_a} \psi_{i,j}(\beta_a \alpha_j).
\]

Remark 3. The class \(mC(C_k(\mathbb{C}^n))\) can be computed in alternative way using the fact that space \(C_k(\mathbb{C}^n)\) is a multiple affine cone (without the vertex) over the configuration space \(Conf_k(\mathbb{P}(\mathbb{C}^n))\) and following the procedure given in [FRW18a], section 4.4.

3. GENERATING SERIES

It is a common practice to present calculation of genera, or characteristic classes of family of varieties in the form of a generating series (e.g. [Ohm08, CM S]).

Application of the lemma 2.1 and multiplicative property of the motivic Chern class leads to formula
\[
mC^{T_{\alpha} \times T_{\beta}} (\tilde{B}_k \cap U_i \subset U_i) |_{e_i} = \lambda_y(T^*\mathbb{P}(\mathbb{C}^n)) e_i \prod_{j=1}^{n} \prod_{a=1}^{k} \psi_{i,j}(\beta_a \alpha_j).
\]

Combining all the ingredients we obtain the final formula
\[
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It is a common practice to present calculation of genera, or characteristic classes of family of varieties in the form of a generating series (e.g. [Ohm08, CM S], [Kom] for symmetric products, [CMO+13, BNW07] for Hilbert schemes). In this section we aim to prove the equalities in the ring of formal power series \(S^{-1}K[T](pt)[y][[t]].\)

Proposition 3.1.
\[
1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot mC^{T_{\alpha}}(Conf_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k) = \exp \left( \frac{\lambda_y(T^*\mathbb{C}^n)}{\lambda_{-1}(T^*\mathbb{C}^n)} \log(1 + t) \right)
\]

Proposition 3.2.
\[
1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot mC^{T_{\alpha}}(Conf_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k) = \exp \left( \frac{\lambda_y(T^*\mathbb{C}^n)}{\lambda_{-1}(T^*\mathbb{C}^n)} \log(1 + t\lambda_{-1}(T^*\mathbb{C}^n)) \right)
\]

Proposition 3.3.
\[
1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot mC^{T_{\alpha}}(C_k(\mathbb{C}^n) \subset (\mathbb{C}^n)^k) = \prod_{i=1}^{n} \exp \left( \frac{\lambda_y(T^*\mathbb{P}(\mathbb{C}^n)) e_i}{\lambda_{-1}(T^*\mathbb{P}(\mathbb{C}^n)) e_i} \log \left( 1 + \frac{t(1 + y)}{\alpha_i - 1} \right) \right)
\]

Where the functions \(\exp(-)\) and \(\log(1 + (-))\) are defined as appropriate power series.

Remark 4. The formula from remark 2 implies that
\[
\frac{mC^{T_{\alpha}}(F_{C^i}(\mathbb{C}^n), k)}{eu^T_{\alpha} \{0\} \subset (\mathbb{C}^n)^k} = \frac{mC^{T_{\alpha}}(C_k(\mathbb{C}^n))}{eu^T_{\alpha} \{0\} \subset (\mathbb{C}^n)^k} + k \cdot \frac{mC^{T_{\alpha}}(C_{k-1}(\mathbb{C}^n))}{eu^T_{\alpha} \{0\} \subset (\mathbb{C}^n)^k}.
\]

If we denote by \(f\) the power series from the proposition 3.3 then
\[
1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \cdot mC^{T_{\alpha}}(F_{C^i}(\mathbb{C}^n), k) \left( \frac{e_i}{eu^T_{\alpha} \{0\} \subset (\mathbb{C}^n)^k} \right) = f(t) + tf'(t).
\]

Define an ordered partition of the set \([k]\) as a sequence of nonempty pairwise disjoint subsets of \([k]\) whose sum is the whole set \([k]\). Let \(Y_k\) be the set of ordered partitions of \([k]\). For a partition \(P \in Y_k\) we use notation \(P = [P_1, ..., P_{|P|}]\) and \(p_a = |P_a|\).
Lemma 3.4. For an $\mathbb{Q}$-algebra $R$ and an arbitrary function $f : \mathbb{N} \to R[[t]]$

$$1 + \sum_{k=1}^{\infty} \sum_{P \in Y_k} \frac{a(P)}{k!} \prod_{p_a \in P} f(|P_a|) = \exp \left( \sum_{u=1}^{\infty} \frac{(-1)^{u-1} f(u)}{u} \right),$$

when both series are well defined. If we assume that $t^n|f(n)$ for all natural numbers $n$, then both series are well defined.

Proof. Observe that

$$1 + \sum_{k=1}^{\infty} \sum_{P \in Y_k} \frac{a(P)}{k!|P|^2} \prod_{a=1}^{\frac{|P|}{k}} f(p_a) =$$

$$1 + \sum_{k=1}^{\infty} \sum_{P \in Y_k} \frac{1}{|P|^k \prod_{p_a \in P} p_a} \prod_{a=1}^{\frac{|P|}{k}} (-1)^{p_a} f(p_a) =$$

$$1 + \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{p_1, \ldots, p_s > 0} \frac{1}{s! \prod_{a=1}^{s}} \prod_{a=1}^{s} \frac{(-1)^{p_a} f(p_a)}{p_a} =$$

$$1 + \sum_{s=1}^{\infty} \frac{1}{s!} \left( \prod_{u=1}^{\infty} \frac{(-1)^{u-1} f(u)}{u} \right)^s = \exp \left( \sum_{u=1}^{\infty} \frac{(-1)^{u-1} f(u)}{u} \right).$$

Corollary 3.5. For the function $f : \mathbb{N} \to S^{-1}K^\mathbb{C}(pt)[y][[t]]$ given by

$$f(u) = t^u \prod_{j=1}^{n} \frac{1 + \frac{y}{\alpha_j}}{1 - \frac{y}{\alpha_j}} = t^u \frac{\lambda_y(T^*\mathbb{C}^n)}{\lambda_{-1}(T^*\mathbb{C}^n)},$$

we obtain the proof of proposition 3.1.

Corollary 3.6. For the function $f : \mathbb{N} \to S^{-1}K^\mathbb{C}(pt)[y][[t]]$ given by

$$f(u) = t^u \lambda_{-1}(T^*\mathbb{C}^n)^{u-1} \lambda_y(T^*\mathbb{C}^n),$$

we obtain the proof of proposition 3.2.

Corollary 3.7. For the function $f : \mathbb{N} \to S^{-1}K^\mathbb{C}(pt)[y][[t]]$ given by

$$f(u) = \sum_{i=1}^{n} \left[ \left( \frac{(1 + y)t}{(\alpha_i - 1)} \right)^u \prod_{j \neq i} \frac{1 + \frac{y}{\alpha_j}}{1 - \frac{y}{\alpha_j}} \right],$$

we obtain the proof of proposition 3.1.
Proof of corollary 3.7.

\[ 1 + \sum_{k=1}^{\infty} \frac{k!}{1 + k} \cdot mC_{k}(\mathbb{C}^{n}) \subset (\mathbb{C}^{n})^{k} \cdot t^{k} = \]

\[ = \exp \left( \sum_{u=1}^{\infty} t^{u}(1 + y)^{u}\frac{(-1)^{u-1}}{u} \cdot \left( \sum_{i=1}^{\infty} \frac{1}{(\alpha_{i} - 1)} \prod_{j \neq i} \frac{1}{\alpha_{j} - 1} \right) \right) = \]

\[ = \exp \left( \sum_{i=1}^{n} \left( \frac{1 + y_{\alpha_{i}}}{1 - \frac{1}{\alpha_{i}}} \right) \left( \sum_{u=1}^{\infty} (-1)^{u-1} \frac{t(1+y)^{u}}{(\alpha_{i} - 1)} \right) \right) = \]

\[ = \prod_{i=1}^{n} \exp \left( \log \left( 1 + \frac{t(1+y)}{\alpha_{i} - 1} \right) \prod_{j \neq i} \frac{1 + y_{\alpha_{j}}}{1 - \frac{1}{\alpha_{j}}} \right) \]

\[ \square \]

The expression in proposition 3.3 can be written in a different form using residues methods (See [Zie14] for cohomology and [WZ19] for K-theory). Consider the function

\[ F(z) = \frac{1}{z(1+y)} \log \left( 1 + \frac{t(1+y)}{z-1} \right) \prod_{i=1}^{n} \frac{1 + y_{\alpha_{i}}}{1 - \frac{1}{\alpha_{i}}} . \]

Remark 5. The logarithm inside of the function \( F \) is a formal power series, not a branch of the complex logarithm. We take residues in each gradation of the formal variable \( t \) separately.

The function \( F \) has residues in \( z \in \{ \alpha_{1}, ..., \alpha_{n}, 1, 0, \infty \} \). Residues in 0 and \( \infty \) are easily computable:

\[ \text{Res}_{z=0}F(z) = \frac{\log(1 - t(1+y))}{(1+y)} , \text{ Res}_{z=\infty}F(z) = 0 . \]

So the residue theorem implies that

\[ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot mC_{k}(\mathbb{C}^{n}) \subset (\mathbb{C}^{n})^{k} \cdot t^{k} = \]

\[ = \exp \left( - \sum_{i=1}^{n} \text{Res}_{z=\alpha_{i}}F(z) \right) = \exp \left( \text{Res}_{z=0}F(z) + \text{Res}_{z=1}F(z) + \text{Res}_{z=\infty}F(z) \right) = \]

\[ = \exp \left( \frac{\log(1 - t(1+y))}{(1+y)} + \text{Res}_{z=1} \left( \frac{\lambda_{yz}(T^{*}\mathbb{C}^{n})}{z(1+y)\lambda_{z}(T^{*}\mathbb{C}^{n})} \cdot \log \left( 1 + \frac{t(1+y)}{z-1} \right) \right) \right) = \]

\[ = (1 - t(1+y))^{1+\pi} \text{Res}_{z=1} \left( \frac{\lambda_{yz}(T^{*}\mathbb{C}^{n})}{z(1+y)\lambda_{z}(T^{*}\mathbb{C}^{n})} \cdot \log \left( 1 + \frac{t(1+y)}{z-1} \right) \right) . \]

4. Stability

4.1. Change of number of points. For any variety \( B \) there is a decomposition in the group \( K^{T}(Var/B^{k}) \)

\[ [Conf_{k+1}(B) \rightarrow B^{k+1}] = [Conf_{k}(B) \times B \rightarrow B^{k+1}] - \sum_{i=1}^{k} [Conf_{k}(B) \rightarrow B^{k} \mapsto B^{k+1}] , \]
where the map \( j_i \) is given by \( j_i(x_1, \ldots, x_k) = (x_1, \ldots, x_k, x_i) \). The maps \( j_i \) are proper, so this decomposition induces an equality

\[
mC^T(Conf_{k+1}(B)) = mC^T(Conf_k(B)) \boxtimes mC^T(B) - \sum_{i=1}^k j_i^*mC^T(Conf_k(B)).
\]

**Example 4.** Consider the projective space \( \mathbb{P}(\mathbb{C}^n) \) with the natural action of the diagonal torus \( T_\alpha \). Denote the fixed points by \( e_1, \ldots, e_n \) of the action on \( \mathbb{P}(\mathbb{C}^n) \). Choose a fixed point \( e = (e_{i_1}, \ldots, e_{i_{k+1}}) \in \mathbb{P}(\mathbb{C}^n)^{k+1} \) and let

\[
I = \{ j \leq k | t_j = t_{k+1} \}.
\]

Let \( \tilde{e} = (e_{i_1}, \ldots, e_{i_{k}}) \). Then

\[
mC^T(Conf_{k+1}(\mathbb{P}(\mathbb{C}^n)))|_{\tilde{e}} = mC^T(Conf_k(\mathbb{P}(\mathbb{C}^n)))|_{\tilde{e}} - \sum_{i \in I} mC^T(Conf_k(\mathbb{P}(\mathbb{C}^n)))|_{\tilde{e}} \nu(j_i) = mC^T(Conf_k(\mathbb{P}(\mathbb{C}^n)))|_{\tilde{e}} \nu(e_{i_{k+1}} \subset \mathbb{P}(\mathbb{C}^n))) = \frac{1}{1 + y_{i_{k+1}} - |I| \prod_{i \neq i_{k+1}} \left(1 - \frac{\alpha_{i_{k+1}}}{\alpha_i}\right)}.
\]

### 4.2. Connections with BB decomposition.

When one dimensional torus \( \mathbb{C}^* \) acts on a smooth, projective variety \( M \), there is the BB-decomposition \([BB73, BB74, BB76]\) (see also \([Bro05]\))

\[
M = \bigsqcup_{F \in M^{\mathbb{C}^*}} B_F^+,
\]

where

\[
B_F^+ = \{ x \in M | \lim_{t \to 0} tx \in F \}.
\]

Such decomposition induces a partial order on the fixed points components such that \( F_1 \geq F_2 \) if and only if the closure of the BB cell \( B_{F_1}^+ \) intersects \( F_2 \). We call the minimal and maximal elements of such decomposition sink and source respectively. Assume that a torus \( T \) with a chosen one dimensional subtorus \( \mathbb{C}^* \subset T \) acts on a smooth, projective variety \( M \). Consider the BB decomposition for the action of the one dimensional subtorus \( \mathbb{C}^* \). Let \( F_1 \subset M^{\mathbb{C}^*} \) be the sink (or the source) of this decomposition. It turns out that the motivic Chern class of the configuration space of \( M \) determines the class of the configuration space of \( F_1 \). To formulate this fact formally we define a limit map.

**Definition 4.1.** Let \( F \) be a \( T \) variety with the trivial action of a chosen subtorus \( \mathbb{C}^* \subset T \). Consider the quotient torus \( T_1 = T/\mathbb{C}^* \). Denote by \( t \) the character of torus \( \mathbb{C}^* \). Choose a split \( T_1 \times \mathbb{C}^* \simeq T \). Consider a subring

\[
K^{T_1}(F)[t] \subset K^{T_1}(F)[t, t^{-1}] \simeq K^{T_1 \times \mathbb{C}^*}(F) \simeq K^T(X),
\]

where the first isomorphism follow from the paragraph 5.2.1 \([CG10]\), and the second is given by the split. Define the map \( \lim_{t \to 0} \)

\[
K^{T_1}(F)[t] \to K^{T_1}(F)
\]
by killing all positive powers of \( t \). The subring \( K^{T_1}(F)[t] \subset K^T(F) \) and the map \( \lim_{t \to 0} \) are independent from the choice of splitting.

Consider a subring \( A \) of the localized \( K \)-theory defined by

\[
A = \{ f \in S^{-1}K^T(F) \mid f = \frac{a}{b}, a \in K^{T_1}(F)[t], b \in K^{T_1}(pt)[t], b \notin tK^{T_1}(pt)[t] \}.
\]

The map \( \lim_{t \to 0} \) extends to this ring by applying limit map to the numerator and the denominator separately. Analogously one can define the limit map \( \lim_{t^{-1} \to 0} \).

**Lemma 4.2.** The limit map is well defined.

**Proof.** We need to show that the limit map is independent from the choice of split and representation as fraction. Consider two split maps \( s_1, s_2 : T_1 \to T \). Let \( \alpha_1, \alpha_2 \) be isomorphisms

\[
K^{T_1}(F)[t, t^{-1}] \to K^T(F)
\]

induced by these maps. Note that the quotient \( \frac{a}{b} \) induce a group homomorphism \( h : T_1 \to C^* \). Direct calculation provide us with the formula

\[
\alpha_2^{-1}\alpha_1(Et^i) = E \otimes C_{h(i)} t^i
\]

for \( E \in K^{T_1}(F) \). It implies independence of the limit map from a split.

Now we need to check that for \( \alpha \in A \) the element \( \lim_{t \to 0} \alpha \) doesn’t depend on a choice of representation of \( \alpha \) as a fraction. Assume that we have two representations \( \alpha = \frac{a}{b} = \frac{c}{d} \) satisfying conditions from the definition of limit map. Namely

\[
a = \sum_{i \geq 0} a_i t^i, c = \sum_{i \geq 0} c_i t^i, b = \sum_{i \geq 0} b_i t^i, d = \sum_{i \geq 0} d_i t^i,
\]

where \( b_0 \) and \( d_0 \) are nonzero and all sums are finite. Our aim is to prove that

\[
\frac{a_0}{b_0} = \frac{c_0}{d_0}.
\]

The equality \( \frac{a}{b} = \frac{c}{d} \) implies that there exist a nonzero element

\[
s = \sum_{i \in \mathbb{Z}} s_i t^i \in K^T(pt),
\]

such that \( sb = sad \). Let \( i_{\min} \) denote the smallest number \( i \) such that \( s_i \neq 0 \). Then looking at coefficient \( t^{i_{\min}} \) we acquire equation

\[
s_{i_{\min}} c_0 b_0 = s_{i_{\min}} a_0 d_0.
\]

Which proves the lemma. \( \Box \)

**Remark 6.** Limit map can be defined geometrically in a split-free way. Consider an algebraic tours \( T \) with a chosen subtorus \( C^* \subset T \) and the quotient torus \( T_1 = T/C^* \). Choose a partial completion \( C^* \subset C \). Interpret \( R(T) \) as subring of functions on algebraic torus \( T \). We have inclusions

\[
T \subset T \times_{C^*} C \supset T \times_C \{0\} \simeq T_1.
\]

The limit map on \( R(T) \) is defined on the functions that extend to variety \( T \times_{C^*} C \) as a restriction to the zero fiber (which is equal to \( T_1 \)). Now assume that \( F \) is a \( T \)-variety with the trivial action of \( C^* \). There is natural morphism

\[
K^{T_1}(F) \otimes_{R(T)} R(T) \to K^T(F).
\]

Any choice of a split \( T \simeq T_1 \times C^* \) imply that this map is an isomorphism. The limit map is defined by applying the limit map to \( R(T) \) (on a subring where its defined). To
prove that this definition coincides with the one given in [4.1] note that the choice of split induces isomorphism $\mathbb{T} \times_{\mathbb{C}} \mathbb{C} \simeq \mathbb{T}_{1} \times \mathbb{C}$.

Let’s note some basic properties of the limit map.

**Proposition 4.3.** (1) The limit map is additive and multiplicative.

(2) It commutes with proper pushforwards (by maps between varieties with trivial $\mathbb{C}^{*}$ action).

(3) It commutes with pullbacks (by maps between varieties with trivial $\mathbb{C}^{*}$ action)

**Proof.** First point is obvious. Second and third follows from the fact that for any $\mathbb{T}$ varieties $X, Y$ with trivial $\mathbb{C}^{*}$ action and $\mathbb{T}$-equivariant map $f$ between them there are isomorphisms of cohomology rings:

$$K^{\mathbb{T}}(X) = K^{\mathbb{T}_{1}}(X)[t, t^{-1}], \quad K^{\mathbb{T}}(Y) = K^{\mathbb{T}_{1}}(Y)[t, t^{-1}]$$

and the maps $f^{\mathbb{T}}_{*}, f^{\mathbb{T}}_{1}$ are given by applying $f^{\mathbb{T}_{1}}_{*}, f^{\mathbb{T}_{1}}_{1}$ to the coefficients of Laurent polynomials.

□

Now we can formulate the theorem.

**Theorem 4.4.** Assume that a torus $\mathbb{T}$ acts on a smooth compact variety $B$. Consider one dimensional subtorus $\mathbb{C}^{*} \subset \mathbb{T}$. Denote by $\mathbb{T}_{1}$ the quotient torus $\mathbb{T}/\mathbb{C}^{*}$. Let $F$ be a component of the fixed points of the torus $\mathbb{C}^{*}$. Let $B_{F}^{+}$ be the positive BB cell of $F$. Then for any $\mathbb{T}$-equivariant map $f : X \to B$ the limit map is well defined on the element

$$mC^{\mathbb{T}}(f : X \to B)_{|F} \in S^{-1}K^{\mathbb{T}}(F).$$

Moreover

$$mC^{\mathbb{T}_{1}}(f_{F} : f^{-1}(B_{F}^{+}) \to F) = \lim_{t \to 0} \left( \frac{mC^{\mathbb{T}}(f : X \to B)_{|F}}{eu^{\mathbb{T}}(\nu_{F})} \right).$$

Analogous result is true for negative BB cells and the limit map $\lim_{t^{-1} \to 0}$.

**Remark 7.** The right hand side of formula (3) lives in the localized $K$-theory of a fixed points component. The Euler class $eu^{\mathbb{T}}(\nu_{F})$ is invertible in this ring (lemma 6.1).

This theorem is generalization of the theorem 10 from [Web17]. The proof is analogous but some formal modifications need to be done. It is the content of the next section.

**Remark 8.** In our case the limit map is simpler than the one in [Web17]. It is consequence of using the motivic Chern class in $K$-theory instead of the Hirzebruch class in cohomology. The equivariant Hirzebruch class lives in completed equivariant cohomology ring, so one needs to define limit of power series instead of Laurent polynomial.

**Corollary 4.5.** Let $F_{1}$ be the sink of the BB decomposition, then

$$mC^{\mathbb{T}_{1}}(Conf_{k}(F_{1}) \to F_{1}^{k}) = \lim_{t \to 0} \left( mC^{\mathbb{T}}(Conf_{k}(M) \to M^{k})_{|F_{1}^{+}} \right).$$

Analogues result hold when $F_{1}$ is the source.

**Proof.** In the theorem 4.4 choose $B := M^{k}$. Let $f$ be the inclusion of configuration space and $F := F_{1}^{k}$ product of the sinks in the BB decomposition of $M$. Then the cell $B_{F}^{+}$ is equal to $F$. Moreover the action of one dimensional torus $\mathbb{C}^{*}$ on the normal bundle $v_{F}$ has only positive weights. It follows that

$$\lim_{t \to 0} eu^{\mathbb{T}}(\nu_{F}^{+}) = 1.$$ 

Now the corollary follows from the theorem 4.4. □
Example 5. Consider the projective space $\mathbb{P}(\mathbb{C}^n)$ with action of the diagonal torus

$$\mathbb{T}_n = \mathbb{T}_{n-1} \times \mathbb{C}^* \alpha_n,$$

where the subtorus $\mathbb{T}_{n-1}$ acts on the first $n-1$ coordinates, and $\mathbb{C}^* \alpha_n$ on the last coordinate. Action of $\mathbb{C}^*$ has two fixed points components: the source is a projective space $\mathbb{P}(\mathbb{C}^{n-1})$ with natural action of $\mathbb{T}_{n-1}$ and the sink is a point. Choose a fixed point $e \in \mathbb{P}(\mathbb{C}^{n-1}) \subset \mathbb{P}(\mathbb{C}^n)$.

The direct computations (example [2]) shows that the class $mC^{\mathbb{T}_n}(\text{Conf}_k(\mathbb{P}(\mathbb{C}^n)))|_e$ belongs to subring $\mathbb{Z}[[\alpha_1^\pm, \ldots, \alpha_{n-1}^\pm, \alpha_n^{-1}]] \subset K^{\mathbb{T}_n}(e)$, moreover

$$\lim_{\alpha_n \to 0} \left( mC^{\mathbb{T}_n}(\text{Conf}_k(\mathbb{P}(\mathbb{C}^n)))|_e \right) = mC^{\mathbb{T}_{n-1}}(\text{Conf}_k(\mathbb{P}(\mathbb{C}^{n-1})))|_e.$$

As predicted by corollary 4.5

5. Proof of theorem 4.4

First step of the proof is reduction to the case when the map $f$ is identity on a smooth variety. Both sides of the formula (3) are additive with respect to addition in the Grothendieck group of varieties $K^\mathbb{G}(\text{Var}/\mathbb{B})$. Proper maps from smooth varieties generate Grothendieck group (cf. section 5, [Web17]), so we can assume that $f$ is proper and $X$ is smooth. Assume that the theorem holds for $id_B$. Then for any proper map from a smooth variety:

$$\lim_{t \to 0} \left( \frac{mC^t(f : X \to B)|_F}{e u^t(\nu_F)} \right) = \lim_{t \to 0} \left( \frac{f_* mC^t(id_X)|_F}{e u^t(\nu_F)} \right) = \lim_{t \to 0} \left( \sum_{G \subset X^c \cap f^{-1}(F)} f^!G \frac{mC^t(id_X)|_G}{e u^t(\nu_G)} \right) = \sum_{G \subset X^c \cap f^{-1}(F)} f^!G \lim_{t \to 0} \left( \frac{mC^t(id_X)|_G}{e u^t(\nu_G)} \right) = \sum_{G \subset X^c \cap f^{-1}(F)} \left( \frac{mC^t(id_X)|_G}{e u^t(\nu_G)} \right) = mC^{T^1}(X^+_G \to F) = mC^{T^1}(X^+_G \to F) = mC^{T^1}(J^1_B \to F)$$

We have used the relative version of Lefschetz-Riemann-Roch (theorem 6.4), commutation of the limit map with pushforwards and additivity of the limit map (proposition 4.3).

To finalize we prove the formula (3) for $f = id_B$. The Variety $B$ is smooth so there is an equality

$$\frac{mC^t(id_B)|_F}{e u^t(\nu_F)} = \lambda^T_y(T^*F) \frac{\lambda^T_y(\nu_F)}{\lambda^{T^1}_y(\nu_F)}.$$

The action of $\mathbb{C}^*$ on the vector bundle $T^*F$ is trivial thus

$$\lim_{t \to 0} \lambda^T_y(T^*F) = \lambda^{T^1}_y(T^*F).$$

Lemma 5.1 (cf. corollary 19 from [Web17]). Let $n^+$ be the number of positive weights of the torus $\mathbb{C}^*$ on the vector bundle $\nu_F$, then

$$\lim_{t \to 0} \frac{\lambda^T_y(\nu_F)}{\lambda^{T^1}_y(\nu_F)} = (-y)^{n^+}.$$
Lemma 6.1. Assume that a torus \( \mathbb{T} \) acts on a smooth variety \( X \). Consider a subtorus \( \mathbb{T}_1 \subset \mathbb{T} \). Let \( F \subset X^{\mathbb{T}_1} \) be a component of the fixed points set of the chosen subtorus. Then the Euler class of the normal bundle \( \mathbb{C}^* \) to the fixed point component. Consider the weight decomposition of \( \nu_F^* \) according to the torus \( \mathbb{C}^* \)

\[
\nu_F^* = \bigoplus_{\omega \in \mathbb{Z}, \omega \neq 0} E_\omega \otimes \mathbb{C}^{\mathbb{C}^*},
\]

where \( E_\omega \) are bundles over \( F \) with the trivial \( \mathbb{C}^* \) action. For \( \omega > 0 \)

\[
\lim_{t \to 0} \frac{\lambda_{y_1}^\omega(E_\omega \otimes \mathbb{C}^{\mathbb{C}^*})}{\lambda_{y_1}^\omega(E_\omega \otimes \mathbb{C}^{\mathbb{C}^*})} = \lim_{t \to 0} \frac{\sum_k y^k t^{k \omega} \Lambda^k E_\omega}{\sum_k (-1)^k t^{k \omega} \Lambda^k E_\omega} = 1.
\]

On the other hand for \( \omega < 0 \)

\[
\lim_{t \to 0} \frac{\sum_k y^k t^{k \omega} \Lambda^k E_\omega}{\sum_k (-1)^k t^{k \omega} \Lambda^k E_\omega} = \lim_{t \to 0} \frac{\sum_k y^k t^{(k-\dim(E_\omega)) \omega} \Lambda^k E_\omega}{\sum_k (-1)^k t^{(k-\dim(E_\omega)) \omega} \Lambda^k E_\omega} = \frac{y^{\dim(E_\omega)}}{(-1)^{\dim(E_\omega)}} \det(E_\omega) = (-y)^{\dim(E_\omega)}.
\]

It follows that

\[
\lim_{t \to 0} \frac{\lambda_{y_1}^\omega(\nu_F^*)}{\lambda_{y_1}^\omega(\nu_F^*)} = (-y)^{\sum_{\omega < 0} \dim(E_\omega)} = (-y)^n.
\]

Thus the right hand side of the formula (3) for \( f = id_B \) is equal to

\[
\lambda_{y_1}^\nu(T^*F)(-y)^{\dim B^+_F} = mC^\nu(1)F(-y)^{\dim B^+_F}.
\]

But \( B^+_F \) is a Zariski affine bundle over \( F \) (theorem 4.3 [BB73]) so

\[
mC^\nu(id_F)(-y)^{\dim B^+_F} = mC^\nu(B^+_F \to F).
\]

Which proves the theorem 4.4.

6. Appendix 1: Relative Localization Theorems

Lemma 6.1. Assume that a torus \( \mathbb{T} \) acts on a smooth variety \( X \). Consider a subtorus \( \mathbb{T}_1 \subset \mathbb{T} \). Let \( F \subset X^{\mathbb{T}_1} \) be a component of the fixed points set of the chosen subtorus. Then the Euler class of the normal bundle \( eu(\nu_F) \) is invertible in localized \( K \)-theory \( S^{-1}K^T(X) \).

Proof. The lemma follows from slightly more general fact. Namely for a smooth \( \mathbb{T} \)-variety \( Y \) element \( a \in S^{-1}K^T(Y) \) is invertible if and only if it is invertible after restriction to each fixed points component.

The "only if" part is trivial. Let’s prove the "if" part. Assume that for every component \( H \subset X^\mathbb{T} \) element \( i_H^* a \) has inverse \( a_H \). Consider the element

\[
\tilde{a} = \sum_{H \subset X^\mathbb{T}} i_H^* \frac{a_H}{eu^2(\nu_H)} \in S^{-1}K^T(Y).
\]

Then for every component \( G \subset X^\mathbb{T} \) restriction

\[
i^*_G(a \tilde{a}) = i^*_G(a) \sum_{H \subset X^\mathbb{T}} i^*_G \frac{a_H}{eu^2(\nu_H)} = i^*_G(a) a_G = 1.
\]

Then the localization theorem ([Tho92], theorem 2.1) implies that \( a \tilde{a} = 1 \).

To complete the proof of the lemma consider the element \( eu(\nu_F) \in S^{-1}K^T(F) \). Bundle \( \nu_F \) restricted to the fixed points set has all weights nontrivial. It is a classical fact that such bundle has invertible Euler class at fixed points (proposition 5.10.3 [CG10]). Thus \( eu(\nu_F) \) is invertible in \( S^{-1}K^T(F) \).
Theorem 6.2 (relative first localization theorem cf. [Tho92] theorem 2.1, or [Seg68] proposition 4.1 for topological case). Assume that a torus $\mathbb{T}$ acts on a smooth variety $X$. Let $X^{\mathbb{T}_1}$ denote fixed points of a subtorus $\mathbb{T}_1 \subset \mathbb{T}$. Then pullback map in localized $K$-theory

$$S^{-1}K^\mathbb{T}(X) \to S^{-1}K^\mathbb{T}(X^{\mathbb{T}_1})$$

is an isomorphism.

Proof. Observe that the diagram

$$
\begin{array}{ccc}
S^{-1}K^\mathbb{T}(X) & \longrightarrow & S^{-1}K^\mathbb{T}(X^{\mathbb{T}_1}) \\
\downarrow^{\cong} & & \downarrow^{\cong} \\
S^{-1}K^\mathbb{T}(X) & \cong & S^{-1}K^\mathbb{T}(X^{\mathbb{T}_1})
\end{array}
$$

commutes. From the first localization theorem two out of three maps on the diagram are isomorphisms. Thus third map is also isomorphism. □

Theorem 6.3 (relative localization formula cf. [AB66, BV82] for cohomology version). Assume that a torus $\mathbb{T}$ acts on a smooth variety $X$. Let $X^{\mathbb{T}_1}$ denote fixed points of subtorus $\mathbb{T}_1 \subset \mathbb{T}$. Then any element $\alpha \in S^{-1}K^\mathbb{T}(X)$ can be recovered from its restriction to the fixed points set using formula

$$\alpha = \sum_{F \subset X^{\mathbb{T}_1}} i_*^F \frac{i_*^F \ell_*^\mathbb{T}_1 \alpha}{eu^\mathbb{T}(\nu_F)}.$$

Proof. Note that Euler class $eu^\mathbb{T}(\nu_F)$ is invertible in the localized $K$-theory $S^{-1}K^\mathbb{T}(F)$ (lemma [6.1]). We use the same reasoning as in the classical localization formula. Namely, both sides of the formula are equal after restriction to the fixed points set $X^{\mathbb{T}_1}$. The claim follows from relative first localization theorem. □

Theorem 6.4 (relative Lefschetz-Riemann-Roch 1.1, cf. theorem 5.11.7 [CG10]). Assume that a torus $\mathbb{T}$ acts on smooth varieties $X$ and $Y$. Consider a subtorus $\mathbb{T}_1 \subset \mathbb{T}$. Let $F \subset Y^{\mathbb{T}_1}$ be a component of the fixed points of the torus $\mathbb{T}_1$. For any proper $\mathbb{T}$-equivariant map $f : X \to Y$ and element $\alpha \in S^{-1}K^\mathbb{T}(X)$ pushforward $f_*\alpha$ can be computed using an equality

$$
\frac{i_*^F f_* \alpha}{eu^\mathbb{T}(\nu_F)} = \sum_{G \subset X^{\mathbb{T}_1} \cap f^{-1}(F)} f|_G^* \frac{i_*^G \alpha}{eu^\mathbb{T}(\nu_G)},
$$

where the sum is indexed by the $\mathbb{T}_1$-fixed points components of $X$ which lie in the preimage of $F$.

Proof. We use the same reasoning as in LRR theorem (remark [6]). Namely, for any $\alpha \in S^{-1}K^\mathbb{T}(X)$

$$
i^F f_* \alpha = i^F f_* \sum_{G \subset X^{\mathbb{T}_1}} i^G_* \frac{i^G_* \alpha}{eu^\mathbb{T}(\nu_G)} = \sum_{G \subset X^{\mathbb{T}_1}} i^F_* f|_G^* \frac{i^G_* \alpha}{eu^\mathbb{T}(\nu_G)} = eu(\nu_F) \sum_{G \subset X^{\mathbb{T}_1} \cap f^{-1}(F)} \frac{i^G_* \alpha}{eu^\mathbb{T}(\nu_G)}.
$$

After dividing both sides by $eu(\nu_F)$ we obtain the proof of the theorem. □
7. Appendix 2: Combinatorics

Consider a finite set $\Omega$ and its subsets $X_1, X_2, \ldots, X_m$. Denote by $P([m])$ the power set of the set $[m]$. For $A \in P([m])$ let

$$X_A = \begin{cases} \bigcap_{i \in A} X_i & \text{when } A \neq \emptyset, \\ \Omega & \text{when } A = \emptyset. \end{cases}$$

The Inclusion-Exclusion formula states that

$$|\bigcup_{i=1}^m X_i| = \sum_{i} |X_i| - \sum_{i \neq j} |X_i \cap X_j| + \sum_{i \neq j, j \neq k, k \neq i} |X_i \cap X_j \cap X_k| - \cdots = \sum_{A \in P([m]), A \neq \emptyset} (-1)^{|A|-1}|X_A|.$$ 

It implies that

$$|\Omega - \bigcup_{i=1}^m X_i| = \sum_{A \in P([m])} (-1)^{|A|}|X_A|.$$ 

This formula has its motivic counterpart. Consider an algebraic $\mathbb{T}$-variety $\Omega$ and closed $\mathbb{T}$-subvarieties $X_1, X_2, \ldots, X_m$. For an element $A \in P([m])$ consider subvarieties $X_A$ defined as above. Then we have an equality in the Grothendieck group $K^T(Var/\Omega)$

$$[\Omega - \bigcup_{i=1}^m X_i \subset \Omega] = \sum_{A \in P([m])} (-1)^{|A|}X_A \subset \Omega].$$

For a smooth $\mathbb{T}$ variety $B$ consider $\Omega = B^k$. Denote by $[[k]]$ the set of unordered pairs $(i, j)$ such that $i, j \in [k], i \neq j$. Consider closed subvarieties

$$X_{ij} = \{(x_1, \ldots, x_k) \in B^k | x_i = x_j \} \subset B^k$$

for every pair $(i, j) \in [[k]]$. The Inclusion-exclusion formula implies that

$$[Conf_k(B) \subset B^k] = [B^k - \bigcup X_{ij} \subset B^k] = \sum_{A \in P([k])} (-1)^{|A|}X_A \subset B^k].$$

A subset $A \subset [[k]]$ can be visualised as a graph $G_A$ whose set of vertices is $[k]$ and whose set of edges is $A \subset [[k]]$. After such interpretation the subvariety $X_A$ consists of tuples $(x_1, \ldots, x_k) \in B^k$ such that $x_i = x_j$ when vertices $i, j$ of the graph $G_A$ are connected by an edge. The equality relation is transitive so the set $X_A$ depends only on the partition induced on the set $[k]$ by connected components of the graph $G_A$. For a partition $P$ of the set $[k]$ consider subvariety

$$B_P = \{(x_1, \ldots, x_k) \in B^k | x_i = x_j \text{ when } P(i) = P(j) \}.$$ 

Moreover denote by $G(P)$ the set of graphs with the set of vertices equal to the set $[k]$ whose connected components induce partition $P$ on the set $[k]$. Define

$$a(P) = \sum_{G \in G(P)} (-1)^{|E(G)|}.$$ 

It follows that

$$[Conf_k(B) \subset B^k] = \sum_{A \in P([k])} (-1)^{|A|}X_A \subset B^k = \sum_{P \in X_k} a(P)|B_P \to B^k].$$

The numbers $a(P)$ can be computed using the following simple formula.
Lemma 7.1.

\[ a(P) = \prod_{P_i \in P} (-1)^{|P_i|-1}(|P_i| - 1)! \]

It is well known fact for specialists but for completeness we give a proof.

**Proof.** When a given partitions \( P \in X_k \) and \( Q \in X_l \) and bijection \([k] \sqcup [l] \to [k + l] \) we may consider disjoint sum partition \( P \sqcup Q \in X_{k+l} \). Both sides of the desired formula are multiplicative with respect to the disjoint sum of partitions. Thus it is enough to prove the lemma for the partition with only one element. Namely we need to show that

\[ (-1)^k(k-1)! = \sum_{G \text{ connected graph}} |E(G)|. \]

Denote the left hand side by \( b_k \). We proceed by induction.

For \( k = 1 \)

\[ b_1 = 1 = (-1)^k(k-1)!. \]

Assume \( k \geq 2 \). There is a bijection between the set of connected graphs which don’t contain the edge \([1, 2] \) and the set of connected graphs which contain the edge \([1, 2] \) and removal of this edge won’t split graph. This bijection (adding/removing edge) changes parity of \(|E(G)|\). So we can sum over the set of connected graphs which contain the edge \([1, 2] \) and removal of this edge splits graph. Such graph can be divided into two disjoint connected subgraphs (one containing the vertex \( 1 \), and the other containing the vertex \( 2 \)) connected by only one edge \([1, 2] \). Such reasoning leads us to the recursive formula for \( b_k \)

\[ b_k = (-1)^{k-1}\sum_{i=1}^{k-1} \binom{k-1}{i} b_ib_{k-i}, \]

where \( i \) corresponds to the number of vertices which are in the connected component of the vertex \( 1 \) after removing the edge \([1, 2] \). The lemma follows from this formula by induction. \( \square \)

**Example 6.** Consider a configuration space \( Conf_3(B) \subset B^3 \). Denote coordinates in \( B^3 \) by \( x, y, z \). Then

\[ [Conf_3(B)] = [B^3] - [x = y] - [x = z] - [z = x] + 3[x = y = z] - [x = y = z] = [B_{\{1\},\{2\},\{3\}}] - [B_{\{1,2\},\{3\}}] - [B_{\{1,3\},\{2\}}] - [B_{\{1\},\{2,3\}}] + 2[B_{\{1,2,3\}}]. \]

The term \( 3[x = y = z] \) corresponds to imposing two equality conditions on three elements and the term \((-1)[x = y = z]\) to imposing all three equality conditions.

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