Theoretical Limit for Radar Parameter Estimation

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Abstract

In this paper, we employ the thoughts and methodologies of Shannon’s information theory to solve the problem of the optimal radar parameter estimation. Based on a general radar system model, the \textit{a posteriori} probability density function of targets’ parameters is derived. Range information and entropy error (EE) are defined to evaluate the performance. It is proved that acquiring 1 bit of the range information is equivalent to reducing estimation deviation by half. The closed-form approximation for EE is deduced in all signal-to-noise ratio (SNR) regions, which demonstrates that EE degenerates to the mean square error (MSE) when the SNR is tending to infinity and EE is a generalization of MSE. Parameter estimation theorem is then proved. The achievability result states that sampling \textit{a posteriori} probability (SAP) is the optimal estimator and the empirical EE of SAP approaches the theoretical EE when the snapshot number tends to infinity. The converse claims that there exists no unbiased estimator whose empirical EE is smaller than the theoretical EE. Numerical simulations are carried out to verify achievability results. Simulation results also demonstrate that the theoretical EE is tighter than the commonly used Cramér-Rao bound and the Ziv-Zakai bound.

Index Terms

Parameter estimation, theoretical limit, optimal estimator, entropy error, sampling \textit{a posteriori} probability (SAP).

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I. Introduction

Starting from the pioneering work of Shannon [1], information theory has been widely applied in communication for more than half a century. Shannon’s information theory provides theoretical limits for communication systems, i.e., channel capacity is the limit for channel coding and rate-distortion function is the limit for lossy source coding. The application of information theory in radar signal processing traces back to the 1950s. Woodward and Davies [2] adopt the inverse probability principle to study the mutual information and obtain the approximate relationship among the range mutual information [3], the time-bandwidth product, and the signal-to-noise ratio (SNR) [4]. Bell presents adaptive waveform design algorithms based on a mutual information measure and shows that more series of information of the targets can be extracted from the received signals [5], [6].

The main goal of radar systems is to detect, localize, and track targets based on the reflected echoes [7]–[9]. The echoes can be exploited to estimate parameters of targets [10]–[12], including range, velocity, shape, and angular [13], [14]. To characterize the accuracy of estimators, lower bounds on the mean square error (MSE) are widely used when determining the exact MSE is difficult [15]. These bounds can be used as a fundamental limit of a parameter estimation problem, or as a benchmark for evaluating the performance of a specific estimator.

The most commonly used bounds include Cramér-Rao bound (CRB) [16], [17], Barankin bound (BB) [18], Ziv–Zakai bound (ZZB) [19]–[21], and Weiss–Weinstein bound (WWB) [22], [23]. It is known that any unbiased estimator’s variance is always lower bounded by CRB [24], [25], which is the ultimate limitation for resolving the signals, independent of the estimation method. Additionally, CRB can be achieved by maximum likelihood estimators (MLE) under standard regularity and high SNR conditions [26], [27]. BB is tighter than CRB, but the BB requires optimization over a set of test points, which is difficult to evaluate and of more computation burden. ZZB and WWB are Bayesian bounds based on the assumption that the parameter is a random variable with known a priori distribution. Based on the a priori probability density function (PDF), they provide bounds on the global MSE. However, they are
strongly affected by the performance at parameter values where the largest errors occur.

For a typical case in ranging estimation, commonly used bounds and MSE of MLE obtained from simulations are plotted in Fig. 1. The observation interval is limited to finite by physical considerations. It can be found that all bounds can closely predict the performance of MLE in the asymptotic region. However, CRB does not adequately characterize the performance of unbiased estimators outside of the asymptotic region. BB is tighter than CRB in the medium SNR, but BB exceeds the MSE of MLE in low SNR region. This phenomenon can be attributed to the lack of a priori information in the BB. Both ZZB and WWB are tight in all SNR regions. Nevertheless, ZZB needs to set the a priori threshold and the asymptotic threshold to derive its explicit expression. The evaluation of WWB involves choosing test points and inverting a matrix.

These bounds are proposed by providing bounds on MSE, but MSE is inherently flawed when applied to parameter estimation. The reason is that MSE, as a second moment of the error, has good evaluation performance when the a posteriori PDF of the estimated parameter obeys a Gaussian distribution. In low and medium SNR, the a posteriori PDF does not obey a Gaussian distribution and MSE no longer reflects the performance of the estimator accurately. Although ZZB introduces the a priori assumption of
uniform distribution to make it is achievable in low SNR region, the setting of \textit{a priori} threshold is neither sufficiently rigorous nor enough precise (e.g., the threshold is defined as the SNR where the bound drops approximately 3 dB below the \textit{a priori} performance level in [28]). Improperly designed thresholds and test points can have undesirable consequences on the accuracy of ZZB. More importantly, the achievability of ZZB cannot be proved mathematically and the optimal estimator cannot be found.

Theoretical limits proposed in this paper have been presented in our previous works. The concept of spatial information was proposed in 2017 [29], which unifies the range information (RI) and the scattering information in a unified framework. In 2019, the concept of entropy error (EE) was put forward as a metric for parameter estimation systems [30]. However, these works do not address the more important issue in parameter estimation, namely, the achievability of theoretical limits and the optimal estimator.

\textbf{A. Contributions and Organization}

Motivated by those facts, we propose achievable limits and an optimal estimator for radar parameter estimation, whose achievability and optimality can be mathematically proved in all SNR regions. On the basis of the thoughts of Shannon’s information theory, we derive the PDF of the received signal according to the statistical properties of noise and targets’ scattering, which is considered as the estimation channel. According to Bayes’ formula, the expression of the \textit{a posteriori} PDF of targets’ parameter conditioned on the received signal is derived. The main contributions of this paper are summarized as follows:

1) We define RI of targets as the mutual information between the received signal and range. It is proved that acquiring 1 bit of RI is equivalent to reducing the estimation deviation by half, which means that radar ranging can be quantified by bits.

2) We define EE as the entropy power of the range. The closed-form approximation for EE is derived in all SNR regions, which demonstrates that EE degenerates to MSE when the SNR tends to infinity and EE is a generalization of MSE.

3) We propose a stochastic estimator named sampling \textit{a posteriori} probability (SAP), which generates random numbers from the \textit{a posteriori} PDF as estimation results. The performance of SAP is entirely
determined by the theoretical *a posteriori* PDF of the radar system. Empirical EE is then defined to evaluate estimators and it is proved that the empirical EE of SAP is convergent as the snapshot number increases. In addition, simulation results show that SAP estimator has a better performance.

4) We prove parameter estimation theorem by the weak law of large numbers and the definition the joint typical sequence. The achievability result of the theorem states that the empirical EE of SAP approaches the theoretical EE when the snapshot number tends to infinity. The converse claims that there exists no unbiased estimator whose empirical EE is smaller than the theoretical EE. The theorem is analogous to the coding theorem in communication, i.e., SAP estimator is equivalent to the random coder and is the optimal estimator in terms of EE.

The rest of the manuscript is organized as follows. Section II establishes a radar system model, which is equivalent to a communication system with joint amplitude, phase, and time delay modulation. In Section III, range-scattering information and EE are proposed to evaluate the performance of estimator. Section IV proposes SAP estimator. In Section V, the parameter estimation theorem is proved. Simulation results are presented in Section VI. Finally, Section VII concludes our work.

### Notations:

Throughout the paper, $a$ denotes a scalar, $a$ denotes a vector, and $A$ denotes a set. The superscripts $\cdot^*$ denotes the conjugate, $\cdot'$ is the derivative of a function. $E(\cdot)$ is the expectation, $\text{Re}\{\cdot\}$ stands for the real part of a complex number. Operators $(\cdot)^{\text{T}}$ and $(\cdot)^{\text{H}}$ denote transpose and conjugate transpose of a vector. A Gaussian variable with expectation $a$ and variance $\sigma^2$ is denoted by $\mathcal{N}(a, \sigma^2)$.

### II. RADAR SYSTEM MODEL

In a radar system, the receiver collects some echoes when the transmitted signal is reflected by some unknown targets. There are two characteristics that we are interested in, i.e., the ranges $X$ between the targets and the receiver, and the scattering properties $S$ of different targets. To simplify the model, we have to make some assumptions.

A1: Targets are points in observe interval.
A2: Different targets are independent. e.g., the joint PDF of target 1 and target 2 satisfies $p(x_1, x_2, s_1, s_2) = p(x_1, s_1)p(x_2, s_2)$.

A3: In the observation interval, the signal attenuation with distance can be ignored and the SNR can be considered as an invariant.

Without loss of generality, let $s_k = \alpha_k e^{j\phi_k}$ denote the complex reflection coefficient of the $k$th target and $d_k$ denote the distance between the $k$th target and the receiver, for $k = 1, ..., K$. Down converting the received signal to base band, the radar system equation is shown as follows

$$y(t) = \sum_{k=1}^{K} s_k \psi(t - \tau_k) + w(t),$$

where $\psi(\cdot)$ denotes the real base band signal and carrier frequency is $f_c$. The phase of the transmitted signal can be expressed as $\phi_k = -2\pi f_c \tau_k + \phi_{k_0}$, where $\phi_{k_0}$ denotes the initial phase and $\tau_k = 2d_k/v$ denotes the time delay of the $k$th target. $w(t)$ is the complex additive white Gaussian noise (CAWGN) with mean zero and variance $N_0/2$ in its real and imaginary parts. The above equation describes a radar system model, which is equivalent to a communication system with joint amplitude, phase, and time delay modulation.

For the convenience of theoretical analysis, it is assumed that the reference point is located at the center of the observation interval and the observation range is $[-D/2, D/2)$, which is shown in Fig. 2(a). The time delay interval is $[-T/2, T/2)$, which is shown in Fig. 2(b). It is also assumed that the emitted baseband signal is an ideal low-pass signal with base band $B/2$, that is

$$\psi(t) = \text{sinc}(Bt) = \frac{\sin(\pi Bt)}{\pi Bt},$$

the corresponding spectrum is given by

$$\psi(f) = \begin{cases} \frac{1}{B}, & |f| \leq \frac{B}{2} \\ 0, & \text{others} \end{cases}$$

To facilitate the analysis, we give an assumption on the relationship of the signal and the observation interval.

A4: The observation interval is much wider than the main lobe width of the signal, i.e., $T \gg 1/B$. 
According to the assumption, signal energy is nearly entirely within the observation interval and satisfies

\[ E_s = \int_{-\tau/2}^{\tau/2} \psi^2(t) dt. \]  \hspace{1cm} (4)

Apply a low-pass filter to the received signal. According to the Nyquist-Shannon sampling theorem, \( y(t) \) can be sampled with a rate \( B \) to obtain a discrete form of (1), which is shown in Fig. 2(c)

\[ y \left( \frac{n}{B} \right) = \sum_{k=1}^{K} s_k \psi \left( \frac{n - B \tau_k}{B} \right) + w \left( \frac{n}{B} \right) , \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \]  \hspace{1cm} (5)

where \( N \) is the time bandwidth product (TBP) and satisfies \( N = TB \). Let \( x_k = B \tau_k \) represent the range, the discrete form system equation can be expressed as

\[ y(n) = \sum_{k=1}^{K} s_k \psi (n - x_k) + w(n) , \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \]  \hspace{1cm} (6)
where the auto-correlation function [2] of Gaussian noise is

\[ R(\tau) = \frac{N_0 B \sin \pi B \tau}{2\pi B \tau}. \] (7)

It can be observed from (7) that the discrete noise sample values \( w(n) \) obtained at the sampling rate \( B \) are uncorrelated. Since the \( w(n) \) are complex Gaussian random variables, they are independent of each other. For convenience, write (1) in vector form

\[ y = U(x)s + w, \] (8)

where \( y = [y(-N/2), \ldots, y(N/2-1)]^T \) denotes received signal. \( s = [s_1, \ldots, s_K]^T \) denotes target scattering vector and \( U(x) = [u(x_1), \ldots, u(x_K)]^T \) denotes matrix determined by the transmitted signal waveform and the range. \( u(x_l) = [\text{sinc}(-N/2-x_k), \ldots, \text{sinc}(N/2-1-x_k)]^T \) is the echo from the \( k \)th target, \( w = [w(-N/2), \ldots, w(N/2-1)]^T \) is the noise vector whose components are independent and identically distributed complex Gaussian random variables with mean value 0 and variance \( N_0 \).

### A. Statistical Model of Target and Channel

The statistical properties of target in the radar parameter estimation system is the joint PDF of range and scattering, which corresponds to the source and the channel in the communication system

\[ \pi(x, s) = \pi(x)\pi(s), \] (9)

where \( \pi(x) \) is the \textit{a priori} PDF of the range and \( \pi(s) \) is the \textit{a priori} PDF of the scattering signal. Without \textit{a priori} information [2], the range is assumed to obey uniformly distribution in the observation interval, i.e., \( \pi(x) = 1/N \).

For scattering signals, only two typical statistical models, i.e., constant modulus (Swerling 0) and complex Gaussian (Swerling 1) are considered. The statistical properties of two kinds of scattered signals can be expressed as

\[ \pi(s) = \pi(\alpha)\pi(\varphi) = \begin{cases} \frac{1}{2\pi} \delta(\alpha - \alpha_0) & \text{Swerling 0,} \\ \frac{1}{2\pi} \frac{\alpha}{\sigma^2_a} \exp\left(-\frac{\alpha^2}{2\sigma^2_a}\right) & \text{Swerling 1.} \end{cases} \] (10)
III. Performance Evaluation for Radar Parameter Estimation

A. A Posteriori PDF of Range and Scattering

The foundation of our metrics is the a posteriori PDF. For a single target with constant amplitude and CAWGN, the multi-dimensional PDF of the received signal $y$ is

$$p(y|x, \varphi) = \left(\frac{1}{\pi N_0}\right)^N \exp\left\{-\frac{1}{N_0} \sum_{n=-N/2}^{N/2-1} |y(n) - \alpha e^{j\varphi} \text{sinc}(n-x)|^2 \right\},$$

(11)
as $p(y|x) = \int_{0}^{2\pi} p(y|x, \varphi)p(\varphi)d\varphi$

$$p(y|x) = \left(\frac{1}{\pi N_0}\right)^N \exp\left\{-\frac{1}{N_0} \sum_{n=-N/2}^{N/2-1} |y(n)|^2 + \alpha^2 \right\} \frac{1}{2\pi} \int_{0}^{2\pi} \exp\left\{ \frac{2\alpha}{N_0} \text{Re}\left[e^{-j\varphi} \sum_{n=-N/2}^{N/2-1} |y(n)\text{sinc}(n-x)|^2 \right]\right\} d\varphi.$$  

(12)

Introduce the first kind zero-order modified Bessel function

$$p(y|x) \propto I_0 \left[ \frac{2\alpha}{N_0} \|U^H y\|^2_2 \right].$$

(14)

where $U^H = [\text{sinc}(N/2-x), \cdots, \text{sinc}(N/2-1-x)]$ and $y = [y(-N/2-x), \cdots, y(N/2-1-x)]$. The inside of the 2-norm is the output of $y$ through the matched filter. According to Bayes’ formula, the a posteriori PDF of $x$ is

$$p(x|y) = \frac{p(y|x)p(x)}{\int_{-N/2}^{N/2} p(y|x)p(x)dx} \propto p(x) I_0 \left[ \frac{2\alpha}{N_0} \|U^H y\|^2_2 \right].$$

(15)

For a snapshot, the range of the target is $x_0$ and the scattering signal is $\alpha e^{j\varphi_0}$. Substituting (6) to (15), the a posteriori PDF of $w$ is

$$p(x|w) = \frac{I_0 \left\{ \frac{2\alpha}{N_0} \sum_{n=-N/2}^{N/2-1} \left[ \alpha e^{j\varphi_0} \text{sinc}(n-x_0) \text{sinc}(n-x) + w_0(n)\text{sinc}(n-x) \right] \right\}}{Z(\alpha, n)},$$

(16)

where $Z(\alpha, n)$ equals the integral of the numerator on the range $x$. Extract the $\alpha e^{j\varphi_0}$ in the summation symbol

$$p(x|w) = \frac{I_0 \left\{ \frac{2\alpha}{N_0} \sum_{n=-N/2}^{N/2-1} \left[ \text{sinc}(n-x_0) \text{sinc}(n-x) + \frac{1}{\alpha} e^{-j\varphi_0} w_0(n)\text{sinc}(n-x) \right] \right\}}{Z(\alpha, n)}. $$

(17)
According to the assumption A4 and the property of sinc signal
\begin{equation}
\sum_{n=-N/2}^{N/2-1} \text{sinc}(n - x_0) \text{sinc}(n - x) = \text{sinc}(x - x_0),
\end{equation}
as $e^{-j\phi_0}$ in (17) does not affect the value in the absolute value sign, the *a posteriori* PDF can be simplified as
\begin{equation}
p(x|w) = \frac{I_0 \left\{ \rho^2 \left| \text{sinc}(x - x_0) + \frac{1}{\alpha} w(x) \right| \right\}}{\int_{-N/2}^{N/2-1} I_0 \left\{ \rho^2 \left| \text{sinc}(x - x_0) + \frac{1}{\alpha} w(x) \right| \right\} dx},
\end{equation}
where $\rho^2 = 2\alpha^2/N_0$ and $w(x)$ satisfies
\begin{equation}
w(x) = e^{-j\phi_0} \sum_{n=-N/2}^{N/2-1} w_0(n) \text{sinc}(n - x),
\end{equation}
w(x) is a Gaussian white noise process with zero mean and variance $N_0$. From the expression of $p(x|w)$, the *a posteriori* PDF is symmetric centered on the target location. The shape of the *a posteriori* PDF is determined by the numerator. Let $\frac{1}{\alpha} w(x) = \frac{\sqrt{2}}{\rho} \mu(x)$, another *a posteriori* PDF expression can be written as
\begin{equation}
p(x|\mu) = \frac{I_0 \left\{ \rho^2 \left| \text{sinc}(x - x_0) + \frac{\sqrt{2}}{\rho} \mu(x) \right| \right\}}{\int_{-N/2}^{N/2-1} I_0 \left\{ \rho^2 \left| \text{sinc}(x - x_0) + \frac{\sqrt{2}}{\rho} \mu(x) \right| \right\} dx},
\end{equation}
where $\mu(x)$ is a complex Gaussian stochastic process with zero mean and variance 1.

In high SNR region, $p(x|\mu)$ can be approximated to the form of the Gaussian distribution \[30\] by omitting the noise term and performing a Taylor expansion for the sinc $(x - x_0)$ in (21),
\begin{equation}
p(x|\mu) \approx \frac{\exp \left[ -\frac{1}{\alpha^2} \rho^2 \pi^2 (x - x_0)^2 \right]}{\int_{-N/2}^{N/2} \exp \left[ -\frac{1}{\alpha^2} \rho^2 \pi^2 (x - x_0)^2 \right] dx}
= \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{(x - x_0)^2}{2\sigma^2} \right],
\end{equation}
where $\sigma^2 = 1/\rho^2 \beta^2$. $\beta^2 = \pi^2/3$ denotes the normalized bandwidth. It is shown in (22) that $p(x|\mu)$ obeys a Gaussian distribution with the mean $x_0$ and the variance $\sigma^2 = 1/\rho^2 \beta^2$.

In Swerling 1 model, the *a posteriori* PDF \[31\] of $x$ can be represented as
\begin{equation}
p(x|y) = \frac{\exp \left[ \frac{1}{N_0(1+2\rho^2)} \left\| U^H y \right\|_2^2 \right]}{\int_{-N/2}^{N/2-1} \exp \left[ \frac{1}{N_0(1+2\rho^2)} \left\| U^H y \right\|_2^2 \right] dx},
\end{equation}
where $U^H y$ is also the output of $y$ through the matched filter and is similar to the inside of the 2-norm in Swerling 0 model.
B. Range-Scattering Information and Entropy Error

Based on the a posteriori PDF, the a posteriori entropy can be derived. The range-scattering information and EE are proposed to evaluate radar parameter estimation.

**Definition 1.** The range-scattering information of a target is defined as the joint mutual information $I(Y; X, S)$ between the received signal, the range, and the scattering signal. Range-scattering information satisfies

$$I(Y; X, S) = h(X, S) - h(X, S|Y),$$

(24)

when the bases of the logarithm are 2 and $e$, the units in the definition are bit and nat, respectively.

In the above equation, $h(X, S)$ is the a priori entropy of $X$ and $S$. $h(X, S|Y)$ is the a posteriori entropy of $X$ and $S$ conditioned on the received signal $Y$. The range-scattering information can also be calculated by

$$I(Y; X, S) = E \left[ \log \frac{p(y|x, s)}{p(y)} \right]$$

$$= E \left[ \log \frac{p(y|x) p(y|x, s)}{p(y)} \right]$$

$$= E \left[ \log \frac{p(y|x)}{p(y)} \right] + E \left[ \log \frac{p(y|x, s)}{p(y|x)} \right]$$

(25)

where $I(Y; X)$ is RI and $I(Y; S|X)$ is the scattering information conditioned on the known range. RI is equal to

$$I(Y; X) = h(X) - h(X|Y).$$

(26)

The a priori of $X$ obeys uniform distribution $\pi(x) = 1/N$ in the observation interval $N$. Assume the unit of the interval $N$ is m (meter in units of length), the unit of the $\pi(x)$ is 1/m and the unit of entropy is $\log_2$m. Meanwhile, the unit of RI is still bit or nat for the reason that the $\log_2$m is eliminated. RI can be calculated by
\( I(Y; X) = h(X) - \mathbb{E}_Y \left[ - \int_{-\infty}^{\infty} p(x|y) \ln p(x|y) dX \right] \), \hspace{1cm} (27)

where \( y \) is the received signal and \( \mathbb{E}_Y[\cdot] \) denotes the expectation of the sample space of \( Y \)

\[
\mathbb{E}_Y [h(x|y)] = \int p(y) h(x|y) dy.
\] \hspace{1cm} (28)

In our previous work [30], the upper bound of RI of a single target with constant-amplitude scattering is derived

\[
I(Y; X) \leq \log T \beta \rho \sqrt{\frac{2}{\pi e}},
\] \hspace{1cm} (29)

for a single target with complex-Gaussian scattering, the upper bound of RI is

\[
I(Y; X) \leq \log \frac{T \beta \rho}{\sqrt{2 \pi e}} - \frac{\gamma}{2 \ln 2},
\] \hspace{1cm} (30)

where \( \gamma \) denotes the Euler’s constant. The scattering information \( I(Y; S|X) \) is given by the following equation

\[
I(Y; S|X) = h(Y|X) - h(Y|X, S),
\] \hspace{1cm} (31)

and the scattering information satisfies

\[
I(Y; S|X) = \log \left( 1 + \frac{\rho^2}{2} \right),
\] \hspace{1cm} (32)

which is consistent with the Shannon’s channel capacity formula. As the above content shows, higher SNR and larger bandwidth lead to an increase in RI, which reflects that RI is a positive metric. Then, we propose EE as a negative metric.

**Definition 2.** EE is defined as the entropy power of the \( X \), which can be expressed as

\[
\sigma_{EE}^2(X, S|Y) = \frac{2^{2h(X, S|Y)}}{2\pi e}
\] \hspace{1cm} (33)

and the square root of EE is named as the entropy deviation \( \sigma_{EE}(X, S|Y) \).
According to the above equation, the unit of EE is \( \text{m}^2 \) and the unit of the entropy deviation is m, which means that EE has the same unit with MSE. EE of range-scattering can also be calculated by

\[
\sigma_{EE}^2 (X, S|Y) = \sigma_{EE}^2 (X|Y) \cdot \sigma_{EE}^2 (S|X, Y)
\]

(34)

where \( \sigma_{EE}^2 (X|Y) \) is EE of ranging and \( \sigma_{EE}^2 (S|X, Y) \) is EE of the scattering conditioned on the known range. The definition of EE of ranging is shown as follows

\[
\sigma_{EE}^2 (X|Y) = \frac{2h(X|Y)}{2\pi e}
\]

(35)

Without causing confusion, we abbreviate the theoretical EE of ranging as EE, which is independent of the specific estimator.

Fig. 3: The \textit{a posteriori} PDF of \( x \), RI, EE, and MSE in different SNR. (a) No signal (b) SNR = 5dB (c) SNR = 7.5dB (d) SNR = 9dB (e) SNR = 10.5dB (f) SNR = 12dB.

As is shown in (22), the \textit{a posteriori} PDF obeys a Gaussian distribution under high SNR and EE degenerates to MSE. To illustrate the relationship of EE and MSE under different SNR, numerical results
are provided in Fig. 3. For a single target with constant scattering coefficient, the actual range is set at $x_0 = 0$. The SNR of each sub-figure is set according to RI, i.e., Fig. 3(a) to Fig. 3(f) correspond the condition of RI = 0 bit to the condition of RI = 5 bit. As can be seen, the \textit{a posteriori} PDF roughly obeys a uniform distribution under low SNR. As the SNR increases, RI increases and the difference between EE and MSE reduces. When SNR = 12dB, the \textit{a posteriori} PDF roughly obeys a Gaussian distribution and EE equals MSE. Under low and medium SNR, the \textit{a posteriori} PDF is roughly uniform distributed or has several peaks. In these cases, MSE cannot adequately measure the performance, while EE can well characterize the performance.

For a single target with constant-amplitude scattering, a closed-form approximation for EE can be derived from (21), which is shown in proposition 1.

\textbf{Proposition 1.} For a single target scenario, the approximation for EE is

$$\sigma_{\text{EE}}^2 = \frac{1}{p^2 \beta^2 p_s^2},$$  \hspace{1cm} (36)

where $p_s$ satisfies

$$p_s = \frac{\exp\left(\rho^2/2 + 1\right)}{T \rho^2 \beta + \exp\left(\rho^2/2 + 1\right)},$$  \hspace{1cm} (37)

and converges to 1 under high SNR.

\textbf{Proof.} See Appendix. A. \hfill \Box

Furthermore, the relationship of RI and entropy deviation is stated in the Theorem 1.

\textbf{Theorem 1.} Let $\sigma_{\text{EE}}(X)$ denote the entropy deviation of the \textit{a priori} PDF and $\sigma_{\text{EE}}(X|Y)$ denote the entropy deviation of the \textit{a posteriori} PDF, we have

$$\frac{\sigma_{\text{EE}}(X|Y)}{\sigma_{\text{EE}}(X)} = 2^{h(X|Y) - h(X)} = 2^{-h(Y;X)},$$  \hspace{1cm} (38)

For radar parameter system, acquiring 1 bit RI is equivalent to reducing the entropy deviation by half.

The theorem indicates that RI represents the amount of information acquired and EE represents the accuracy of the parameter estimation, which are equivalent.
IV. Sampling a Posteriori Probability Estimation

In Swerling 0 model, the estimation value that maximizes \( p(x|y) \) in (15) is called maximum a posteriori probability (MAP) estimation of the range \( x \), which is denoted as \( \hat{x}_{\text{MAP}} \)

\[
\hat{x}_{\text{MAP}} = \arg\max_{x} \left \{ \pi(x) I_0 \left[ \frac{2\alpha}{N_0} \sum_{n=-N/2}^{N/2-1} y(n) \sin c(n-x) \right] \right \}. \tag{39}
\]

The estimation value \( \hat{x} \) that maximizes \( p(y|x) \) in (14) is called the maximum likelihood estimation of the range \( x \), which is denoted as \( \hat{x}_{\text{MLE}} \)

\[
\hat{x}_{\text{MLE}} = \arg\max_{x} \left \{ I_0 \left[ \frac{2\alpha}{N_0} \sum_{n=-N/2}^{N/2-1} y(n) \text{sinc}(n-x) \right] \right \}. \tag{40}
\]

If the range \( x \) follows uniform distribution in the observation interval \( N \), the \( \pi(x) \) in (15) can be omitted and MLE is equivalent to MAP estimator.

Based on the a posteriori PDF, we propose SAP estimator, which can be expressed as

\[
\hat{x}_{\text{SAP}} = \arg\max_{x} \left \{ p(x|y) \right \} = \arg\max_{x} \left \{ p(y|x)\pi(x) \right \}, \tag{41}
\]

where \( \arg\max_{x} \{ \cdot \} \) denotes the sampling operation. To illustrate the sampling operation, we provide a figure and some descriptions.

SAP estimator is analogous to random coder in communication, which can be considered as a random number generator. SAP generates \( M \) random numbers from the a posteriori PDF as \( M \) outputs. For the a posteriori PDF of a single target’s range \( x \), we illustrate estimation results of SAP and MAP in Fig. 4. It can be observed that MAP always outputs the \( \hat{x}_{\text{MAP}} \) corresponding to a peak, while SAP randomly outputs \( \hat{x}_{\text{SAP}} \) according to the PDF of the \( x \). Thus, MAP is a typical deterministic estimator, i.e., its outputs are fixed for the same received signal. While SAP is a stochastic estimator.

In the previous section, we defined the theoretical RI and the theoretical EE, which are related to the parameter estimation system. RI and EE associated with the specific estimator are named the empirical RI and the empirical EE, which are defined as follows.

**Definition 3.** The empirical entropy of estimator with \( M \) snapshots is defined as

\[
h^{(M)}(\hat{X}|Y) = -\frac{1}{M} \log p(\hat{X}^{(M)}|y^{(M)}), \tag{42}
\]
Fig. 4: Difference between estimation results of SAP and MAP for $x$ obeying standard normal distribution, (a)-(f) are estimation results.

and the empirical RI of estimator with $M$ snapshots is given by

$$I^{(M)}(\hat{X}|Y) = h^{(M)}(X) - h^{(M)}(\hat{X}|Y),$$

(43)

where $h(X)$ is the a priori entropy of $X$.

**Definition 4.** The empirical EE of estimator with $M$ snapshots is defined as

$$\sigma_{EE}^2(M) = \frac{1}{2\pi e} 2^{2h^{(M)}(\hat{X}|Y)},$$

(44)

Next, we demonstrate an important property of SAP, i.e., the empirical entropy of SAP is invariant with respect to the actual range $x_0$. According to the assumption A4, $x_0$ does not affect the shape of $p[x|y(x_0)]$ when the observation interval is much wider than the main lobe width of the signal.

In addition, estimation principle determines that the empirical entropy of SAP is convergent. The difference between theoretical entropy and the empirical entropy of SAP is plotted in Fig. 5, the SNR is
5dB and other parameter settings are the same as in Fig. 4. It is shown that the empirical entropy of SAP gradually converges to the theoretical entropy of $X$ as $M$ increases.

V. PARAMETER ESTIMATION THEOREM

In this section, we prove the parameter estimation theorem, which corresponds to the channel coding theorem in Shannon’s information theory. Specifically, the theoretical EE, the SAP estimator, and the snapshot number are equivalent to the channel capacity, the random coder, and the code length, respectively.

A. Parameter Estimation System

Fig. 6 shows a simplified radar ranging process. The estimator is a function $f(\cdot)$ of the received signal $y$ and outputs a range estimation $\hat{x}$. A range estimation process can be described as, a target generates a set of received sequences through the channel, and the estimator outputs $\hat{x}$ based on the received sequences,
such a process is called a snapshot. $M$ snapshots will generate memory-less extended targets $x^M$ and memory-less extended channel $p(y^M|x^M)$, which satisfies

$$p(y^M|x^M) = \prod_{m=1}^{M} p(y_m|x_m).$$  \hspace{1cm} (45)

The radar ranging system consists the range $X$ to be estimated, the a priori PDF $\pi(x)$ of $X$, the transition probability $p(y|x)$ of channel, estimator $f(\cdot)$, and the received signal $Y$. The $\pi(x)$ is considered to follow uniform distribution in the observation interval $N$, i.e., $\pi(x) = 1/N$. The joint target channel $p(y|x)$ determines the a posteriori PDF $p(x|y)$ and the theoretical a posteriori entropy $h(X|Y)$. Based on the $h(X|Y)$, RI and EE are the theoretical limits the parameter estimation system. Before the proof the theorem, some preparatory works are provided.

**B. Preparatory Works**

Firstly, we provide the definition of the achievability of EE.

**Definition 5.** EE is said to achievable if there exists an estimator whose empirical EE of $M$ snapshots satisfies

$$\lim_{M \to \infty} \sigma_{EE}^2(M) = \sigma_{EE}^2.$$  \hspace{1cm} (46)

The weak law of large numbers and the definition of jointly typical sequences are introduced to demonstrate the relationship of the theoretical entropy and the empirical entropy with $M$ snapshots.

**Lemma 1.** Weak law of large numbers

Let $Z_1, Z_2, \cdots, Z_M$ be a sequence of i.i.d. random variables with mean $\mu$ and variance $\sigma^2$. $\bar{Z}_M = \frac{1}{M} \sum_{i=1}^{M} Z_i$ is the sample mean and satisfies

$$\Pr \{|\bar{Z}_M - \mu| > \epsilon\} \leq \frac{\sigma^2}{M \epsilon^2}.$$  \hspace{1cm} (47)
Definition 6. The set $A_{\varepsilon}^{(M)}$ of jointly typical sequences $(x^M, y^M)$ with respect to the $p(x, y)$ is the set of $M$ sequences with empirical entropy $\varepsilon$ close to the true entropy, i.e.,

$$
A_{\varepsilon}^{(M)} = \left\{ (x^M, y^M) \in X^M \times Y^M : \left| -\frac{1}{M} \log p(x^M) - H(X) \right| < \varepsilon, \right.
\left. \left| -\frac{1}{M} \log p(y^M) - H(Y) \right| < \varepsilon, \right. \left. \left| -\frac{1}{M} \log p(x^M, y^M) - H(X, Y) \right| < \varepsilon \right\},
$$

(48)

where $H(X)$, $H(Y)$, and $H(X, Y)$ are the mean value of $h(x^M)$, $h(y^M)$, and $h(x^M, y^M)$, respectively. The joint PDF is

$$
p(x^M, y^M) = \prod_{m=1}^{M} p(x_m, y_m).
$$

(49)

The jointly typical sequence defined here is consistent with Shannon’s information theory, i.e., the input and output of the extended source channel constitute the joint typical sequence. According to the weak law of large numbers, when $M$ is large enough, the difference between the empirical entropy and the theoretical entropy is no greater than an arbitrarily small $\varepsilon$. On the basis of the definition of jointly typical sequence, we have the following lemma.

Lemma 2. For a memory-less snapshot channel $(X^M, p(y^M | x^M), Y^M)$, if $\hat{x}^M$ is the $M$ sampling estimates of a posteriori PDF $p(x|y)$, $\left(\hat{x}^M, y^M\right)$ is the joint typical sequence of $p(\hat{x}^M, y^M)$.

Proof. We denote the $M$ SAP estimation for the theoretical a posteriori PDF $p(x|y)$ by $\hat{x}^M$. Then, $p_{SAP}(\hat{x}^M|y^M)$ is the PDF generated from the $\hat{x}^M$, which satisfies

$$
p_{SAP}(\hat{x}^M|y^M) = p(\hat{x}^M|y^M).
$$

(50)

The joint PDF of the estimated value sequence and the received signal sequence satisfies

$$
p_{SAP}(\hat{x}^M, y^M) = p(y^M) p_{SAP}(\hat{x}^M|y^M) = p(y^M) p(\hat{x}^M|y^M) = p(\hat{x}^M, y^M).
$$

(51)

□
C. Content of Parameter Estimation Theorem

**Theorem 2.** EE is achievable, i.e., given that the estimator knows the joint source channel statistical properties, for any \( \varepsilon > 0 \), there exists an estimator whose empirical EE satisfies

\[
\sigma_{EE}^2 e^{-4\varepsilon} < \lim_{M \to \infty} \sigma_{EE}^2(M) < \sigma_{EE}^2 e^{4\varepsilon},
\]

and

\[
\lim_{M \to \infty} \sigma_{EE}^2(M) = \sigma_{EE}^2.
\]

Conversely, there exists no unbiased estimator whose empirical EE are great than EE, which can be represented as

\[
\sigma_{EE}^2(M) \geq \sigma_{EE}^2.
\]

**Proof.** See Appendix B.

**Corollary 1.** RI is achievable, that is

\[
\lim_{M \to \infty} I^M(\hat{X};X_0) = I[X;Y(X_0)],
\]

where \( X_0 \) is the actual range to be estimated and \( Y(X_0) \) is the corresponding received signals. Conversely, there exists no unbiased estimator whose empirical RI is smaller than RI.

**Proof.** See Appendix C.

The main challenge for radar systems is in the low to medium SNR, which means that it is of importance to precisely characterize the performance estimators outside the asymptotic region. CRB is commonly used, while it is quite loose in low and medium SNR. As is shown in theorem 2 and corollary 1, the achievability of RI and EE can be mathematically proved under all SNR conditions.
VI. Simulation Results

Numerical simulations are performed to verify the parameter estimation theorem and illustrate the superiority of proposed limits and estimator. In the following simulations, the range of a single target with constant scattering coefficient is set at \( x_0 = 0 \), the TBP is set as \( N = 16 \) and the independent simulation runs is 1500.

The relationship between EE, CRB, and ZZB is shown in Fig. 7. For a fair comparison, the \textit{a priori} threshold and the asymptotic threshold are set at 0dB and 15dB, respectively. In this case, the curve of ZZB almost coincides with that of EE under low and high SNR. In the asymptotic region, both ZZB and EE approach CRB. Under medium SNR, it can be observed that EE is tighter than ZZB. When \( \text{MSE}/\text{EE}=1 \), the gap between EE and the empirical EE of MLE is about 2dB, while the gap between ZZB and MSE of MLE is about 4dB. As such, ZZB can not adequately characterize the performance of MLE. The reason is that in the medium SNR, the shape of the \textit{a posteriori} PDF does not match the assumption in \textit{a priori} region or asymptotic region. While the empirical EE is based on the \textit{a posteriori} entropy and has a good evaluation performance in all SNR regions.

![Fig. 7: EE, CRB, ZZB, and MLE performance versus SNR.](image-url)
Fig. 8: RI, the empirical RI of SAP, and that of MLE versus SNR.

RI, the empirical RI of MLE, and that of SAP are compared in Fig. 8. To determine the a priori entropy, the range $X$ is assumed to obey a uniform distribution. It can be seen from (39) and (40) that MLE is equivalent to MAP estimator under the assumption. As can be seen, the curve of SAP overlaps with that of RI. Also, the curve of the MLE is lower than that of SAP in the medium and high SNR regions.

Fig. 9: Performance of MLE and SAP versus SNR.

Fig. 9 compares the performance of MLE and that of SAP. In terms of MSE, SAP has about 1dB
performance gain under medium SNR. In asymptotic region, both MSE of SAP and that of MLE converge to CRB. Similar conclusions can be obtained in terms of EE. Thus, the SAP has a better performance than MLE in range estimation.

![Graph showing EE and empirical EE of SAP with different snapshot number versus SNR.](image)

Fig. 10: EE and the empirical EE of SAP with different snapshot number versus SNR.

When the SNR is 5dB, we have illustrated that the empirical entropy of SAP approaches the theoretical entropy as the snapshot number tends to infinity. It is of interest to see the relationship between EE and the empirical EE of SAP in different SNR. As Fig. 10 shows, the deviation between the empirical EE of SAP and EE is large when the number of snapshots is small. When the number of snapshots increases, the empirical EE of SAP and EE gradually overlap, which verifies the results of Fig. 5.

VII. Conclusion

In this paper, RI and EE are presented as achievable limits and SAP is proposed as an optimal estimator. With RI, the radar ranging can be quantified by bits. The closed-form approximation for EE indicates that EE is a generalization of MSE and degenerates to MSE in high SNR. Parameter estimation theorem proves the achievability of EE and the optimality of SAP. Numerical simulations are conducted to compare EE and MSE, results show that EE can provide a tighter than MSE in low and medium SNR. Also, it is
shown that SAP estimator has better performance than MLE. It is of interest to analyze multiple targets scenarios and extensions of results here will be the topic of a subsequent article.

APPENDIX A

PROOF OF THE APPROXIMATION OF ENTROPY ERROR

As is derived in [31], the approximation for the *a posteriori* entropy of a single target is written as

\[ h(X|Y) = p_s H_s + (1 - p_s) H_w + H(p_s), \]  

(56)

where \( H_s \) is the normalized *a posteriori* entropy in high SNR region and can be represented as

\[ H_s = \frac{1}{2} \ln \left( 2\pi e \sigma^2 \right) = \ln \frac{B \sqrt{2\pi e}}{\rho \beta}, \]  

(57)

\( H_w \) is the normalized *a priori* entropy in low SNR region, which satisfies

\[ H_w = \ln \frac{N \rho \sqrt{2\pi}}{e^{\sigma^2/2}}, \]  

(58)

\( H(p_s) \) denotes the uncertainty of the target, which can be expressed as

\[ H(p_s) = -p_s \log p_s - (1 - p_s) \log (1 - p_s), \]  

(59)

where \( p_s \) is called ambiguity and denotes the probability that the target range is around \( x_0 \). \( p_s \) satisfies

\[ p_s = \frac{\exp\left(\rho^2/2 + 1\right)}{T \rho^2 + \exp\left(\rho^2/2 + 1\right)}. \]  

(60)

Substitute (57-60) into equation (56), the *a posteriori* entropy can be derived as

\[ h(X|Y) = p_s \log \frac{\sqrt{2\pi e}}{\beta \rho} + (1 - p_s) \log \frac{T \rho \sqrt{2\pi e}}{e^{\rho^2/2 + 1}} + H(p_s) \]  

(61a)

\[ = \log \sqrt{2\pi e} + \log \left( \frac{1}{\beta \rho} \right) + \log \left( \frac{T \rho}{e^{\rho^2/2 + 1}} \right)^{1-p_s} \left( \frac{1}{p_s} \right)^{p_s} \left( \frac{1}{1 - p_s} \right)^{1-p_s}, \]  

(61b)

\[ = \log \sqrt{2\pi e} + \log \left( \frac{1}{\beta \rho p_s} \right) + \log \left( \frac{T \rho}{e^{\rho^2/2 + 1}} \right)^{1-p_s} \left( \frac{1}{1 - p_s} \right)^{1-p_s}. \]  

(61c)

Substitute (60) into equation (61c)

\[ h(X|Y) = \log \sqrt{2\pi e} + \log \left( \frac{1}{\beta \rho p_s} \right)^{p_s} + \log \left( \frac{T \rho}{e^{\rho^2/2 + 1}} \right)^{1-p_s} \]  

(62a)
\[ \log \sqrt{2\pi e} + \log \left( \frac{1}{\rho \beta s} \right)^{1-p_s} \]  
\[ = \log \sqrt{2\pi e} + \log \left( \frac{1}{\rho \beta p_s} \right). \]  
(62b)

\[ \log \sqrt{2\pi e} + \log \left( \frac{1}{\rho \beta p_s} \right). \]  
(62c)

Substitute (62c) into (35), the approximation of EE can be obtained

\[ \sigma_{EE}^2 = \frac{1}{p^2 \beta^2 p_s^2}. \]  
(63)

**APPENDIX B**

**Proof of the Achievability of the Entropy Error and the Converse**

Consider the following sequence of events:

1) \( M \) extensions of the target \( x^M \) are generated independently according to the PDF of \( x \).

2) The receiving sequence is generated according to \( x^M \) and the conditional probability of \( M \) extension of the channel \( p(y|x) \) satisfies

\[ p(y^M|x^M) = \prod_{m=1}^{M} p(y_m|x_m). \]  
(64)

Introducing SAP estimator, assume \( \hat{x}^M \) is the \( M \) sampling estimation of memory-less snapshot channel. Then, \( (\hat{x}^M, y^M) \) is the jointly typical sequence with respect to the PDF \( p(\hat{x}^M, y^M) \). Based on the definition of the jointly typical sequence and the law of large numbers, when \( M \) is large enough, for any \( \varepsilon > 0 \)

\[ \left| \frac{1}{M} \log p[\hat{x}^M, y^M(x_0)] - h[X, Y(X_0)] \right| < \varepsilon \]  
(65a)

\[ \left| \frac{1}{M} \log p[y^M(x_0)] - h[Y(X_0)] \right| < \varepsilon \]  
(65b)

\[ \left| \frac{1}{M} \log p(\hat{x}^M) - h(X) \right| < \varepsilon. \]  
(65c)

According to Bayes’ formula

\[ \left| \frac{1}{M} \log p[\hat{x}^M|y^M(x_0)] - h[X|Y(X_0)] \right| < 2\varepsilon, \]  
(66)

based on the definition of the empirical EE

\[ \sigma_{EE}^2(M)[X; Y(X_0)] = \frac{1}{2\pi e} 2^{2h(M)(\hat{x})[Y]} = \frac{1}{2\pi e} 2^{-\frac{1}{2} \log p[\hat{x}^M|y^M(x_0)]}, \]  
(67)
therefore

\[
\sigma_{EE}^2 2^{-4\epsilon} < \lim_{M \to \infty} \sigma_{EE}^2 (M)[X; Y(X_0)] < \sigma_{EE}^2 2^{4\epsilon}.
\] (68)

On the basis of the translation invariance of the empirical entropy of SAP, \( h(\hat{X}_{\text{SAP}}|X_0) \) is equal to \( h[\hat{X}_{\text{SAP}}|Y(X_0)] \). Thus, EE is also independent of the \( x_0 \) and the \( \sigma_{EE}^2 (M)(\hat{X}_{\text{SAP}}; X_0) \) satisfies

\[
\sigma_{EE}^2 2^{-4\epsilon} < \sigma_{EE}^2 (M)(\hat{X}_{\text{SAP}}; X_0) < \sigma_{EE}^2 2^{4\epsilon},
\] (69)

according to the weak law of large numbers

\[
\lim_{M \to \infty} \sigma_{EE}^2 (M)(\hat{X}_{\text{SAP}}; X_0) = \sigma_{EE}^2.
\] (70)

Converse: Let \( \hat{x} = f(y) \) be an unbiased estimator, and the information obtained by the estimator with the actual \( x_0 \) and \( M \) snapshots is denoted as \( \sigma_{EE}^2 (M)(\hat{X}; X_0) \). As can be seen from Fig. 6, \((x^{M}, y^{M}, \hat{x}^{M})\) forms a Markov chain. Based on data processing theorem

\[
\sigma_{EE}^2 (M)(\hat{X}; X_0) \geq \sigma_{EE}^2.
\] (71)

APPENDIX C

PROOF OF THE ACHIEVABILITY OF RANGE INFORMATION

For the reason that the \textit{a priori} of range \( X \) follows uniform distribution in the observation interval \( N \), the \textit{a priori} entropy is \( \log N \). According to the definitions in (26) and (35), there is a one-to-one correspondence between RI and EE. The proof of EE can also be applied to RI.

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