Reduced fidelity susceptibility and its finite-size scaling behaviors in the Lipkin-Meshkov-Glick Model

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We derive a general formula of the reduced fidelity susceptibility when the reduced density matrix is 2 × 2 block-diagonal. By using this result and the continuous unitary transformations, we study finite-size scaling of the reduced fidelity susceptibility in the Lipkin-Meshkov-Glick Model. It is found that it can be used to characterize quantum phase transitions, implying that we can extract information of quantum phase transitions only from the fidelity of a subsystem, which is of practical meaning in experiments.

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I. INTRODUCTION

During the past few years, some important concepts in quantum information theory have been introduced to characterize quantum phase transitions (QPTs). For example, entanglement, which is one of the central concepts in quantum information theory, has been investigated extensively in QPTs in various models, like Ising model and Lipkin-Meshkov-Glick (LMG) model. Recently, fidelity, which is another important quantum information concept, has also been applied in characterizing QPTs. The introducing of fidelity in QPTs is natural since it’s mathematically the overlap between two states, while QPTs are just dramatic changes in ground-state properties. However, the fidelity used in the study of QPTs depends computationally on an arbitrary yet finite small change of the driving parameter. To cancel the arbitrariness, Zanardi et al. introduced the Riemannian metric tensor, while You et al. suggested the fidelity susceptibility. The fidelity susceptibility then becomes an effective tool to study critical properties in many-body systems.

It’s noticed that all the above works are concentrated on the fidelity of the global ground states, and we may call this kind of fidelity susceptibility the global fidelity susceptibility. However, in experiments, one always probe the subsystem but not the whole system for practical convenience. Here we use the reduced fidelity [24] (also called partial fidelity in [25, 38]) susceptibility (RFS), which describes the fidelity susceptibility of a subsystem. In this work, first we derive a general formula of the reduced fidelity susceptibility when the reduced density matrix is 2 × 2 block-diagonal. Then, considering the LMG model, we show that the RFS can be used to characterize QPTs, and find that the scaling exponent is different from that of the global fidelity susceptibility.

This paper is organized as follows. In Sec. II, we briefly review the concept of fidelity susceptibility, and give a general formula of RFS for a special but interesting case that the density matrix is 2 × 2 block-diagonal. Then in Sec. III, we introduce the LMG model. In isotropic case, we find that the critical behavior of RFS χ in response to magnetic transverse field h as (hc − h)−1 in thermodynamic limit. While in the anisotropic case, by using the continuous unitary transformations (CUTs) [27–29], we find that the maximum of χ over h diverged as N2/3 for an N-spin system, and (hc − h)−1 in thermodynamic limit. Finally, we perform a numerical scaling analysis, and the results are well consistent with our theoretical ones.

II. REDUCED FIDELITY SUSCEPTIBILITY

We first give a brief review on the concept of fidelity susceptibility. The Hamiltonian of a quantum system undergoing QPTs can be written as

\[ H(h) = H_0 + hH_1, \]

where H1 is supposed to be the driving term with control parameter h. The global fidelity is defined as \( F(h, \delta) = \langle \varphi_0(h) | \varphi_0(h + \delta) \rangle \), where |\( \varphi_0(h) \rangle \) is the ground state of \( H(h) \), and \( \delta \) is a small quantity. The reduced fidelity is defined as the overlap between the reduced density matrix (RDM) \( \rho(h) \) of the ground state |\( \varphi_0(h) \rangle \). In the follows, we take \( \rho \equiv \rho(h) \) and \( \bar{\rho} \equiv \rho(h + \delta) \). Then the reduced fidelity is given by [30]

\[ F(h, \delta) = \text{tr} \sqrt{\rho^{1/2} \bar{\rho}^{1/2}}. \]

The corresponding fidelity susceptibility is defined as [7, 11]

\[ \chi = \lim_{\delta \to 0} \frac{-2 \ln F}{\delta^2}, \]

and then we could write \( F \simeq 1 - \chi \delta^2 / 2 \).
In this paper we consider that the RDM is block-diagonal,
\[ \rho = \bigoplus_{i=1}^{n} \rho_i, \]  
where \( \rho_i \)'s are 2 x 2 semi-positive definite Hermitian matrices, since \( \rho \) is a density matrix. Now we introduce some useful formulas at first. Let A and B are arbitrary 2 x 2 semi-positive definite matrices, then we have
\[ \text{tr} \sqrt{AB^2} = \sqrt{\text{tr}(AB)} + 2 \sqrt{\det(AB)}, \]  
and if A = B, it becomes
\[ \text{tr} (A^2) = (\text{tr}A)^2 - 2 \det A. \]  
Take derivations of the above equation with respect to some variable \( h \), we get
\[ \text{tr} (AA') = \text{tr} A \text{tr} A' - \partial_h (\det A), \]  
\[ \text{tr} (AA'') = \text{tr} A \text{tr} A'' - \partial_h^2 (\det A) + 2 \det A', \]  
where \( A' \equiv \partial_h A, A'' \equiv \partial_h^2 A \) and \( \partial_h \text{tr} A = \text{tr} (A') \). Now the fidelity can be written as
\[ F = \sum_{i=1}^{n} \sqrt{\text{tr} \rho_i^{1/2} \hat{\rho} \rho_i^{1/2}} = \sum_{i=1}^{n} \sqrt{\text{tr} \rho_i \hat{\rho}_i + 2 \sqrt{\det \rho_i \hat{\rho}_i}}, \]  
and recall that \( F \simeq 1 - \chi \delta^2 / 2 \), the susceptibility \( \chi = \sum_{i=1}^{n} \chi_i \), with \( \chi_i \) corresponds to the ‘susceptibility’ of the \( i \)-th block in Eq. (4). To obtain the susceptibility, we should expand the fidelity with respect to \( \delta \), and for \( \hat{\rho}_i \equiv \rho_i (h) + \rho_i' (h) \delta + \delta^2 \rho_i'' (h) / 2 + O (h^3) \), we have
\[ \begin{align*}
\text{tr} (\hat{\rho}) & \simeq \text{tr} (\rho^2) + \text{tr} (\rho \rho') \delta + \frac{\delta^2}{2} \text{tr} (\rho \rho''), \\
\det \hat{\rho} & \simeq \det \rho + \partial_h (\det \rho) \delta + \frac{\delta^2}{2} \partial_h^2 (\det \rho),
\end{align*} \]  
(10)
here we omit the subscript \( i \) for convenience.

In the case that \( \det \rho \neq 0 \), we have \( \text{tr} \rho \neq 0 \) since \( \rho \) is semi-positive definite. Then we get
\[ \sqrt{\det (\rho \rho')} \simeq \det \rho + \frac{\delta}{2} \partial_h \det \rho \\
+ \frac{\delta^2}{4} \left[ \partial_h^2 \det \rho - \frac{(\partial_h \det \rho)^2}{2 \det \rho} \right]. \]  
(11)

Take the above expression into Eq. (9) and with the help of Eqs. (6), (7) and (8) we obtain
\[ \begin{align*}
\text{tr} \mathcal{V}^{1/2} \rho_1^{1/2} \simeq & \text{tr} \rho \delta \rho + \delta \text{tr} \rho' + \frac{\delta^2}{4} \text{tr} \rho'' \\
+ & \frac{\delta^2}{8 \text{tr} \rho} \left[ 4 \det \rho' - (\text{tr} \rho')^2 - \frac{[\partial_h \det (\rho)]^2}{\det (\rho)} \right].
\end{align*} \]  
(12)
If \( \det \rho = 0 \) but \( \text{tr} \rho \neq 0 \), we have \( \det (\rho \rho') = 0 \). Moreover, since \( \rho \) is positive semi-definite, zero is the lower bound of \( \det \rho \), which requires \( \partial_h \det \rho = 0 \) and \( \partial_h^2 \det \rho > 0 \). Thus we have
\[ \begin{align*}
\text{tr} (\hat{\rho}) & \simeq (\text{tr} \rho)^2 + \text{tr} \rho \text{tr} \rho' \delta \\
& \quad + \frac{\delta^2}{2} \left[ \text{tr} \rho \text{tr} \rho'' - \partial_h^2 (\det \rho) + 2 \det \rho' \right],
\end{align*} \]  
and
\[ \begin{align*}
\text{tr} \mathcal{V}^{1/2} \rho_1^{1/2} \simeq & \text{tr} \rho \delta \rho + \frac{\delta}{2} \text{tr} \rho' + \frac{\delta^2}{4} \text{tr} \rho'' \\
+ & \frac{\delta^2}{8 \text{tr} \rho} \left[ 4 \det \rho' - (\text{tr} \rho')^2 - 2 \partial_h^2 (\det \rho) \right].
\end{align*} \]  
(14)
In the last case that \( \text{tr} \rho = 0 \), \( \rho \) is equivalent to a zero matrix, since \( \rho \) is Hermitian. Then \( \text{tr} (\hat{\rho}) = \sqrt{\det (\hat{\rho})} = 0 \), and \( F = 0 \).

Conclude the above three cases, we get the ‘susceptibility’ for block \( \rho_i \) as
\[ \chi_i = \begin{cases} 
\frac{1}{4 \text{tr} \rho_i} \left[ (\text{tr} \rho_i')^2 - 4 \text{det} \rho_i' + [\partial_h \det (\rho_i)]^2 / \det (\rho_i) \right] & \text{for } \text{tr} \rho_i \neq 0, \rho_i \neq 0, \\
\frac{1}{4 \text{tr} \rho_i} \left[ (\text{tr} \rho_i')^2 - 4 \text{det} \rho_i' + 2 \partial_h^2 (\det \rho_i) \right] & \text{for } \text{tr} \rho_i \neq 0, \rho_i = 0, \\
0 & \text{for } \text{tr} \rho_i = 0,
\end{cases} \]  
(15)
due to \( \text{tr} (\rho) \equiv 1 \), and \( \text{tr} (\rho') = \text{tr} (\rho'') = 0 \).

Finally, we consider a more special case that \( \rho \) is diag-
onal, the then susceptibility is obtained readily
\[
\chi = \sum_{i=1}^{n} \frac{(\lambda_i')^2}{4\lambda_i},
\]
where \(\lambda_i\)'s are the nonzero diagonal terms.

### III. The LMG Model and Its Scaling Exponents of RFS

#### A. The LMG model and RFS

The LMG model was introduced in nuclear physics to describe mutually interacting spin-1/2 particles, embedded in a transverse magnetic field. In the thermodynamic limit, it undergoes a QPT that is described by the mean field analysis \[31\]. Recently the finite-size scaling was studied by the \(1/N\) expansion in the Holstein-Primakoff single boson representation \[32\] and by the CUTs \[33, 34\].

The Hamiltonian of the LMG model reads
\[
H = H_0 + h H_I
\]
\[
= -\frac{\lambda}{N} (1 + \gamma) (S^2 - S_z^2 - N/2)
- \frac{\gamma}{2N} (1 - \gamma) (S_+^2 + S_-^2) - 2h S_z, \tag{17}
\]
where \(S_\alpha = \sum \sigma_\alpha / 2\), with \(\sigma_\alpha (\alpha = x, y, z)\) the Pauli matrices, and \(S_\pm = S_x \pm i S_y\). The prefactor \(1/N\) ensures finite energy per spin in the thermodynamic limit. In the context, we set the parameters: \(\lambda = 1, \gamma \leq 1, h \geq 0\). We take \(h \geq 0\) as the spectrum is invariant under the transformation \(h \leftrightarrow -h\). In addition, we only consider the maximum spin sector \(S = N/2\) in which the lowest energy state lies.

Now we consider a 2-body RDM of the LMG model \[35\]
\[
\rho_{ij} = \begin{pmatrix} v_+ & 0 & 0 & u \\ 0 & y & y & 0 \\ 0 & y & y & 0 \\ u & 0 & 0 & v_- \end{pmatrix}, \tag{18}
\]
in the standard basis \{\(|00\), |01\), |10\), |11\}\}, where \(\sigma_z|0\rangle = -|0\rangle\) and \(\sigma_z|1\rangle = |1\rangle\), while the nonzero matrix elements reads
\[
v_\pm = \frac{N^2 - 2N + 4 \langle S_z^2 \rangle \pm 4 \langle S_z \rangle (N - 1)}{4N (N - 1)},
\]
\[
y = \frac{N^2 - 4 \langle S_z^2 \rangle}{4N (N - 1)}, \quad u = \frac{\langle S_z^2 - S_\alpha^2 \rangle}{N (N - 1)}, \tag{19}
\]
where \([A, B]_+ = AB + BA\) is the anti-commutator for operators \(A\) and \(B\). The zero elements of \(\rho_{ij}\) result from the fact that the total spin and the parity are conserved quantities, i.e.,
\[
[H, S^2] = \left[ H, \prod_{i=1}^{N} \sigma_i \right] = 0. \tag{20}
\]

It’s noticed that \(\rho_{ij}\) is actually block-diagonal in the rearranged basis \{\(|00\), |11\), |01\), |10\}\}, and the two blocks are
\[
\varrho_1 = \begin{pmatrix} v_+ & u \\ u & v_- \end{pmatrix}, \quad \varrho_2 = \begin{pmatrix} y & y \\ y & y \end{pmatrix}. \tag{21}
\]

With the help of Eq. (15), we can give the RFS explicitly
\[
\chi = \frac{y^2}{2y} + \frac{1}{4(y_+ + y_-)} \left[ (y_+ - y_-)^2 + 4u^2 \right] \tag{22}
+ \left( \frac{(y_+ y_- + y_+ y_- - 2u' u)}{(y_+ y_- - u^2)} \right),
\]
here we consider the case that \(\varrho_1 \neq 0\), and the following computations are based on the above formula.

#### B. The isotropic case

Firstly, we consider the isotropic case, \(\gamma = 1\), and the Hamiltonian reads
\[
H = -\frac{2}{N} (S^2 - S_z^2 - N/2) - 2h S_z, \tag{23}
\]
which is diagonal in the standard eigenbasis \{\(|S, M\rangle\}\} of \(S^2\) and \(S_z\). For \(S = N/2\) the eigenstates are
\[
E(M, h) = \frac{2}{N} \left( M - \frac{hN}{2} \right)^2 - \frac{N}{2} (1 + h^2), \tag{24}
\]
and the ground state is readily obtained when
\[
M_0 = \begin{cases} \frac{N}{2} & \text{for } h \geq 1, \\
\frac{N}{2} - R[N(1-h)/2] & \text{for } 0 \leq h < 1, \tag{25}
\end{cases}
\]
where \(R(x) \equiv \text{round}(x)\). Then one can see level crossings exist at \(h = h_j\), where \(h_j = 1 - (2j + 1) / N\), between the two states \(|N/2, N/2 - j\rangle\) and \(|N/2, N/2 + j - 1\rangle\). In the thermodynamic limit, these critical points form a region of criticality.

The elements of the RDM in ground state are readily obtained as
\[
v_\pm = \frac{(N \pm 2M_0) (N - 2 \pm 2M_0)}{4N (N - 1)}, \quad y = \frac{(N^2 - 4M_0)}{4N (N - 1)}, \quad u = 0. \tag{26}
\]
As \(N\) is very large, \(M_0 (h < 1) \approx hN/2\). With Eq. (22), we obtain the susceptibility in thermodynamic limit
\[
\lim_{N \to \infty} \chi \left( h > \frac{1}{N} \right) \approx \frac{1}{2 (1 - h^2)}. \tag{27}
\]
Obviously, the asymptotic behavior of \(\chi\) as \(h \to 1\) is \(1/(1 - h)\). However, there is no QPT in its symmetric phase \(h \to 1\), because the ground state is independent of \(h\).
Here we firstly recall the CUTs introduced by Wegner [27] and independently by Glazek and Wilson [28, 29]. For a pedagogical introduction to this technique, one can see [36]. The main idea of CUTs is to diagonalize the Hamiltonian in a continuous way starting from the original Hamiltonian \( H = H (l = 0) \). A flowing Hamiltonian is then defined by

\[
H (l) = U^\dagger (l) H (0) U (l),
\]

where \( U (l) \) is unitary and \( l \) is a scaling parameter such that \( H (l = \infty) \) is diagonal. A derivation of the Eq. (28) with respect to \( l \) yields the flow equation

\[
\partial_l H (l) = [\eta (l), H (l)],
\]

where \( \eta (l) = -U^\dagger \partial_l U \) is an anti-Hermitian generator. To obtain the expectation value of any operator \( \Omega \) on an eigenstate \( |\psi\rangle \) of \( H \), one should follow the flow of the operator \( \Omega (l) = U^\dagger (l) H (0) U (l) \), by solving Eq. (29). Fortunately the results of the spin expectation values have been obtained by Dusuel and Vidal in [33, 34], and here we’ll compute the scaling behavior of the derivatives of these values.

Firstly, we consider the system size \( N \) is very large, and the matrix elements are rewritten as

\[
v_\pm = \frac{1}{4} + \frac{\langle S_z^2 \rangle}{N^2} \pm \frac{\langle S_z \rangle}{N},
\]

\[
y = \frac{1}{4} - \frac{\langle S_z^2 \rangle}{N^2}, \quad u = \frac{\langle S_z^2 \rangle - \langle S_y^2 \rangle}{N^2}.
\]

The spin expectation values appeared in the above expressions can be solved by the CUTs with \( 1/N \) expansion. For symmetry phase \((h > 1)\), we have

\[
\begin{align*}
\frac{2(S_z)}{N} & = 1 + \frac{1}{N} \left( \frac{P_{xx}^{(1)}}{G^{1/2}} + 2 \right) + \frac{1}{N^2} \left( \frac{P_{xx}^{(2)}}{G^2} + \frac{Q_{xx}^{(2)}}{G^{3/2}} + \frac{(1 - \gamma)^2}{N^3} \left( \frac{P_{xx}^{(3)}}{G^{7/2}} + \frac{Q_{xx}^{(3)}}{G^3} \right) + O \left( \frac{1}{N^4} \right) \right), \\
\frac{4(S_y^2)}{N^2} & = (h - \gamma) \left\{ \frac{1}{NG^{1/2}} + \frac{1}{N^2} \left( \frac{P_{yy}^{(2)}}{G^2} + \frac{Q_{yy}^{(2)}}{G^{3/2}} \right) + \frac{1}{N^3} \left( \frac{P_{yy}^{(3)}}{G^{7/2}} + \frac{Q_{yy}^{(3)}}{G^3} \right) \right\} + O \left( \frac{1}{N^4} \right), \\
\frac{4(S_z^2)}{N^2} & = \frac{1}{h - \gamma} \left\{ \frac{G^{1/2}}{N} + \frac{1}{N^2} \left( \frac{P_{zz}^{(2)}}{G^2} + \frac{Q_{zz}^{(2)}}{G^{3/2}} \right) + \frac{1}{N^3} \left( \frac{P_{zz}^{(3)}}{G^{7/2}} + \frac{Q_{zz}^{(3)}}{G^3} \right) \right\} + O \left( \frac{1}{N^4} \right), \\
\frac{4(S_y^2)}{N^2} & = 1 + \frac{1}{N} \left( \frac{P_{zz}^{(1)}}{G^{1/2}} + 2 \right) + \frac{1}{N^2} \left( \frac{P_{zz}^{(2)}}{G^2} + \frac{Q_{zz}^{(2)}}{G^{3/2}} \right) + \frac{1}{N^3} \left( \frac{P_{zz}^{(3)}}{G^{7/2}} + \frac{Q_{zz}^{(3)}}{G^3} \right) + O \left( \frac{1}{N^4} \right),
\end{align*}
\]

where \( G \equiv G (h, \gamma) = (h - 1) (h - \gamma) \). Here we do not present \( P_{xi}^{(i)} \equiv P_{xi}^{(i)} (h, \gamma) \) and \( Q_{xi}^{(i)} \equiv Q_{xi}^{(i)} (h, \gamma) \) \((i = 1, 2, 3 \text{ and } \xi = z, xx, yy, zz)\), which are polynomials of \( h \) and \( \gamma \), whereas of little meaning for computing the scaling exponents. For more details, you can refer to the appendix part of [34]. It’s noticed that, the above expressions can be written in the form

\[
\Phi_N (h, \gamma) = \Phi_N^{\text{reg}} (h, \gamma) + \Phi_N^{\text{sing}} (h, \gamma),
\]

where the superscripts ‘reg’ and ‘sing’ stand for regular
and singular respectively. A nonsingular contribution is understood to be a function of $h$ which is nonsingular at $h = 1$, as well as all its derivatives. Take $2 \langle S_z \rangle / N$ for example, the regular part is $1 + 1/N$ and the remaining forms the singular part. As $h$ approaches to 1, the terms involving $Q^{(i)}$’s are small compared to the terms involving $P^{(i)}$’s by a factor $G(h, \gamma)$, hence we could only consider the terms involving $P^{(i)}$’s.

2. Finite-size scaling

Here we show how to derive the finite-size scaling exponents of the spin expectation values and their derivatives, and take $2 \langle S_z \rangle / N$ for example,

$$\frac{2 \langle S_z \rangle}{N} = 1 + \frac{1}{N} + \frac{1}{NG^{1/2}} \left\{ P^{(1)} + \frac{(1-\gamma)^2 P^{(2)}}{NG^{3/2}} + \frac{(1-\gamma)^2 P^{(3)}}{NG^{3/2}^2} + O\left(\frac{1}{NG^{3/2}^3}\right) \right\},$$

where the singular part (terms after $1 + 1/N$) can be written in the form

$$\left( \frac{2 \langle S_z \rangle}{N} \right)_{\text{sing}} \sim \frac{1}{NG(h, \gamma)^{1/2}} \mathcal{F}_{S_z} \left[ NG(h, \gamma)^{3/2}, \gamma \right],$$

where $\mathcal{F}_{\Phi} (\Phi = S_z, S_x^2, S_y^2, S_z^2)$ is a scaling function for these spin expectation values. While in fact that there can be no singularity in any physical quantity in a finite-size system, and the critical point $h_c = 1$ only for thermodynamic limit $N \to \infty$. This implies that the singularity of $G(h, \gamma)^{-1/2}$ has to be canceled by the one of $\mathcal{F}_{S_z} \left[ NG(h, \gamma)^{3/2}, \gamma \right]$. Thus one must have $\mathcal{F}_{S_z}(x, \gamma) \sim x^{-1/3}$, which in turn implies the following finite size scaling:

$$\left. \frac{2 \langle S_z \rangle}{N} \right|_{h=1} \sim \frac{a_z(0)}{N^{2/3}}.$$

Immediately, one can obtain the asymptotic form of all the spin expectation values

$$\left. \frac{2 \langle S_z \rangle}{N} \right|_{h=1} \sim 1 + \frac{1}{N} + \frac{a_z(0)}{N^{2/3}}.$$

$$\left. \frac{4 \langle S_x^2 \rangle}{N^2} \right|_{h=1} \sim \frac{a_{xz}(0)}{N^{4/3}}.$$

$$\left. \frac{4 \langle S_y^2 \rangle}{N^2} \right|_{h=1} \sim \frac{a_{yu}(0)}{N^{4/3}}.$$

$$\left. \frac{4 \langle S_z^2 \rangle}{N^2} \right|_{h=1} \sim 1 + \frac{2}{N} + \frac{a_{zz}(0)}{N^{2/3}}.$$

where $a^{(0)}_\xi$’s ($\xi = z, xx, yy, zz$) are all constants depending on $\gamma$. Then take the first-order derivatives of Eq. (31)

$$\left. \left( \frac{\partial}{\partial h} \frac{2 \langle S_z \rangle}{N} \right)_{\text{sing}} \right|_{h=1} \sim \frac{1}{NG(h, \gamma)^{1/2}} \mathcal{G}_{S_z} \left[ NG(h, \gamma)^{3/2}, \gamma \right],$$

where $\mathcal{G}_{\Phi}$ is a scaling function for the derivatives of spin expectation values, and then we find the finite size scaling

$$\left. \left( \frac{\partial}{\partial h} \frac{2 \langle S_z \rangle}{N} \right) \right|_{h=1} \sim a_z^{(1)}.$$
The scaling form of the other derivatives are
\[ \frac{\partial}{\partial h} \frac{4(S^z_h)^2}{N^2} \bigg|_{h=1} \sim a^{(1)}_{z z}, \]
\[ \frac{\partial}{\partial h} \frac{4(S^y_h)^2}{N^2} \bigg|_{h=1} \sim a^{(1)}_{y y}, \]
\[ \frac{\partial}{\partial h} \frac{4(S^x_h)^2}{N^2} \bigg|_{h=1} \sim a^{(1)}_{x x}, \]
\[ (39) \]
where \( a^{(1)}_{\xi} \)'s (\( \xi = z, x, y, y \)) are constants depending on \( \gamma \). As we can see that, except for \( 4(S^z_h)^2/N^2 \), the other first-order derivatives are all independent of \( N \). Then with the help of Eq. (22), we find that the maximum RFS \( \chi_m \equiv \chi(h_m, N, \gamma) \) is
\[ \chi_m \sim -\frac{(a^{(2)}_{z z})^2 N}{a^{(0)}_{z z} N^{1/3} + 2}, \]
\[ (40) \]
for large \( N \), and here we just present the divergent term. It's noticed that \( a^{(0)}_{\xi} \) should be less than \(-2\) to ensure the matrix element \( y > 0 \), thus \( \chi_m > 0 \). Then we have
\[ \ln \chi = A_N \ln N + \text{const.}, \]
\[ (41) \]
where the constant only depends on \( \gamma \) and the scaling exponent \( A_N \) approaches to 2/3 as \( N \) increases, which is verified numerically, and \( A_N = 2/3 \) in thermodynamic limit. The numerical comparisons are shown in Fig. (2). While in the broken symmetric phase (\( 0 < h < 1 \)), we can derive the same scaling exponents [34]. However, for global fidelity susceptibility, the scaling exponent is 9/7 [19].

Then if we cancel \( N \) in Eq. (37), with similar steps, we can get the relation between the susceptibility \( \chi \) and \( \eta = h - h_c \) in thermodynamic limit,
\[ \ln \chi(h, \gamma) = A_h \ln |h - h_c| + \text{const.}, \]
\[ (42) \]
where \( A_h \) approaches to \(-1\) as \( h \) goes to \( h_c \), and the constant depends on \( \gamma \). Therefore we could take the form of the susceptibility for finite size as
\[ \chi(h, N, \gamma) = \frac{A}{N^{-2/3} + B(h - h_m)}, \]
\[ (43) \]
To study the critical behavior around the phase transition point, we could perform the finite scaling analysis. According to the scaling ansatz [37], the susceptibility is a function of \( N^\nu(h - h_m) \). In the case of logarithmic divergence, it behaves as
\[ \chi(h_m, N)/\chi(h, N) \sim Q[N^\nu(h - h_m)], \]
where the function \( Q(x) \approx \ln x \) for large \( x \) is universal and does not depend on system size \( N \). Hence with Eqs. (41) and (43), we determine the exponent \( \nu = 2/3 \), which is confirmed numerically, as shown in Fig. (3). However, the curves for different system sizes does not collapse to a single one exactly, since the system sizes are not large enough.

IV. CONCLUSION

In summary, we have investigated the RFS in the second order quantum phase transition of the LMG model. For the case that \( \rho \) is block-diagonal in \( 2 \times 2 \) matrices, we derive a general formula for RFS. Then with the CUTs and the scaling ansatz, the critical exponents, including the finite-size scaling exponents of the RFS are obtained analytically, and confirmed numerically. Our results show that, the RFS undergoes singularity around the critical point, indicating that the RFS can be used to characterize the QPTs. And it’s suggested that we can extract information of the QPTs only from the fidelity of a subsystem, without probing the global system, which is of practical significance in experiments. It is also interesting to study finite-size scaling of RFS in other models such as quantum Ising model, which is under consideration.

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