Extension of the integrable, (1+1) Gross-Pitaevskii equation to chaotic behaviour and arbitrary dimensions

Bernhard Mieck

Abstract

The integrable, (1+1) Gross-Pitaevskii (GP-) equation with hermitian property is extended to chaotic behaviour as part of general complex fields within the sl(2, C) algebra for Lax pairs. Furthermore, we prove the involution property of conserved quantities in the case of GP-type equations with an arbitrary external potential. We solve the corresponding zero-curvature condition, following from compatibility of mixed partial derivatives with Lax pair matrices, in terms of the ’ad’-operators which are derived from a Cartan-Weyl basis of the general sl(n, C) algebra or one of its sub-algebras. A gauge invariance of the Lax pairs and its zero-curvature relation is proven so that one can reduce the total Cartan-Weyl basis for the spatial Lax matrix to the maximal commuting Cartan sub-algebra. This allows to attest the involution property of conserved quantities under very general conditions apart from the well-known, classical ’r’-matrix approach. We generalize the approach of Lax pair matrices to arbitrary spacetime dimensions and conclude for the type of nonlinear equations from the structure constants of the underlying algebra. One can also calculate conserved quantities from loops within the (N-1) dimensional base space and the mapping to the manifold of the general SL(n, C) group or a sub-group, provided that the resulting fibre space is of nontrivial homotopic kind. This condition is of crucial importance for proving the corresponding involution property and avoids possible contraction of the considered loops to trivial point mappings.

Keywords: Lax pairs, Liouville integrability, chaos, Cartan representation of algebras and groups, classical r-matrix.

PACS: 03.75.Nt, 02.30.Ik, 02.20.Sv, 02.20.Uw

\[\text{e-mail: "bjmeppstein@arcor.de";}
\text{freelance activity during 2007-2009; current location : Zum Kohlwaldfeld 16, D-65817 Eppstein, Germany.}\]
## Contents

1 Introduction ................................................. 3

2 Lax pairs for GP-type equations from the $\mathfrak{sl}(2, \mathbb{C})$ algebra
   2.1 Lie algebras of $\mathfrak{su}(2)$-, $\mathfrak{sp}(2, \mathbb{R})$- and $\mathfrak{sl}(2, \mathbb{C})$-Lax pairs for integrable and chaotic behaviour .................. 5
   2.2 Lax pairs of $\mathfrak{sl}(2, \mathbb{C})$ for an external potential ........................................... 10

3 The general $n \times n$ Lax pair matrices as $\mathfrak{sl}(n, \mathbb{C})$ algebras
   3.1 Solution of the zero-curvature condition ................................................................. 12
   3.2 Gauge invariance of the Lax pair and the zero-curvature condition ..................... 14

4 Determination and independence of conserved quantities
   4.1 Calculation of conserved quantities from the Lax pair ........................................... 16
   4.2 Involution of conserved quantities and the classical $\mathfrak{r}$-matrix .......................... 17

5 Extension of the zero-curvature condition beyond (1+1)-dimensions
   5.1 Determination of the nonlinear equations for the fields of relevant, physical observables .......................... 21
   5.2 Conserved quantities of monodromy matrix paths with nontrivial homotopy of fields .......................... 22
   5.3 Involution of conserved quantities and the classical $\mathfrak{r}$-matrix in arbitrary spacetime dimensions ........................................... 23

6 Summary and conclusion
   6.1 Lax pairs and chaotic behaviour of (1+1) GP-type equations ................................... 24
   6.2 Lax pair construction in arbitrary spacetime ......................................................... 25

A Reduction of $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{sl}(n, \mathbb{C})$ Lax pairs by separating the trivial trace parts ........................................... 25

B Involution of monodromy matrices for spatially constant $\mathfrak{r}$-matrix ........................................... 26
1 Introduction

In the study of nonlinear field equations there frequently occur various Lax pairs which allow to construct conserved quantities as part of an integrable system. Although it is generally difficult to determine for a given nonlinear field equation a corresponding Lax pair, the zero-curvature condition enables deep insight into integrability and soliton solutions by the inverse scattering and Bäcklund transformations for several, physically relevant equations, as e.g. the Gross-Pitaevskii (GP)-equation, the Korteweg-de Vries (KdV)-type or the Sine-Gordon-like equations [1]-[6]. However, this paper is initiated by a different notion of the integrable (1+1) GP-equation which can be summarized by the questions [7, 8, 9]:

- Is really any kind of the (1+1) GP-equation (1.1), e.g. with an external potential \( V(x, t) \), integrable in the sense of an infinite number of independent, conserved quantities? (cf. the definition of Liouville integrability in Ref. [10])

- Which conclusions can be achieved from the corresponding Lax pair with respect to the involution of the conserved quantities and the classical \( r \)-matrix?

\[
i \left( \partial_t \psi(x, t) \right) = - \left( \partial_x \partial_x \psi(x, t) \right) - 2 V(x, t) \psi(x, t) - 2 \left| \psi(x, t) \right|^2 \psi(x, t) . \tag{1.1}
\]

In particular we examine the (1+1) nonlinear GP-equation which has been intensively computed with an external potential, a so-called kick- or delta-function potential, for chaotic behaviour on a periodic circle \( \psi(x, t) = \psi(x + 2\pi, t) \) [11], [12]

\[
i \left( \partial_t \psi(x, t) \right) = - \left( \partial_x \partial_x \psi(x, t) \right) + K \cos(x) \sum_{n=-\infty}^{+\infty} \delta(t-n) \psi(x, t) + g \left| \psi(x, t) \right|^2 \psi(x, t) . \tag{1.2}
\]

In advance, we mention that the obvious, hermitian discretization (1.3) of the continuous (1+1) GP-equations (1.1, 1.2) fails to give a corresponding Lax pair whereas the peculiar, discrete, non-hermitian form (1.4) can be obtained from a Lax pair [11], [13]

\[
i \left( \partial_t \psi(x_n, t) \right) = - \frac{1}{(\Delta x)^2} \left( \psi(x_{n+1}, t) - 2 \psi(x_n, t) + \psi(x_{n-1}, t) \right) + \tag{1.3}
\]

\[
- 2 V(x_n, t) \psi(x_n, t) - 2 \left| \psi(x_n, t) \right|^2 \psi(x_n, t) ;
\]

\[
i \left( \partial_t \psi(x_n, t) \right) = - \frac{1}{(\Delta x)^2} \left( \psi(x_{n+1}, t) - 2 \psi(x_n, t) + \psi(x_{n-1}, t) \right) + \tag{1.4}
\]

\[
- \left( V(x_{n+1}, t) + V(x_n, t) \right) \psi(x_n, t) - \left| \psi(x_n, t) \right|^2 \left( \psi(x_{n+1}, t) + \psi(x_{n-1}, t) \right) .
\]

This suggests the preference of the discrete, non-hermitian kind (1.4) instead of the discrete, hermitian form (1.3) in order to prove an infinite number of conserved quantities from Lax pairs.

According to a mathematical view, it is necessary to apply the discrete, non-hermitian form (1.4) of the GP-equation in order to preserve exactly the Liouville integrability from the Lax pairs of the continuous kind (1.1, 1.2) of the GP-equation. Apart from the involution of the conserved quantities which is proven for more general cases in later sections, this can explain the numerically observed chaos of the kicked GP-equation which may be caused by a false representation of discrete grid points for the numerical integration despite of its hermitian form. Therefore, one should use the discrete kind (1.4) with non-hermitian property in order to keep the Liouville integrability from the Lax pairs for the continuous, kicked GP-equation (1.1, 1.2) without any chaos.
INTRODUCTION

From a physical point of view, one has to remark that any physical problem has a time-, length-scale or energy-, momentum-scale so that the application of delta-function-like time kicks within the (1+1) GP-equation has to be regarded as problematic because these time kicks involve any energies (from highly excited Rydberg-atoms to cosmic dimensions) \[14\]. Furthermore, it is not astonishing that different, discrete realizations of continuous nonlinear equations can cause different integrable or chaotic behaviour because of varying numerical properties; however, in order to maintain the Liouville integrability from Lax pairs of a equation as the kicked, continuous GP-equation \[1.2\], one has simply to introduce a finite time-, length-scale or broadened delta-function of time for the external kick-potential and has even more to include ”many more” (sufficient), discrete spacetime steps in the numerical integration so that one can trac the time development for the particular scale of the external potential without artifacts coming from below the physical scale.

In this paper we take the latter point or physical view and furthermore seek for a chaotic behaviour of the (1+1) GP-equations. Chaotic behaviour seems to be completely excluded because one can easily conclude for non-chaotic properties from the hermiticity and finite norm in any case of a (physical) external potential on a spatial, periodically confined circle. Therefore, any kind of GP-equation only seems to allow for integrable properties. However, we can also extend to a possible chaotic behaviour , as one considers the su(2)- or sp(2, \(R\))-Lax pairs (for an attractive or repulsive interaction) as sub-algebras of the general sl(2, \(C\)) algebra and Lax pairs. We specify the derivation of a Lax pair for an arbitrary external potential \(V(x, t)\) of the general sl(2, \(C\)) case so that this Lax pair can be used for the construction and involution of the conserved quantities following from the monodromy matrix.

In section \[3\] we further analyze the point that the construction of a Lax pair, as e.g. for the GP-equation, appears to be accidental as one chooses two matrix potentials \(\mathcal{L}(x, t)\), \(\mathfrak{M}(x, t)\) for a zero-curvature condition which finally determines the nonlinear equations. In appendix \[A\] we therefore demonstrate the reduction of the very general \(n \times n\), \(\text{gl}(n, \mathbb{C})\) Lax pair algebra to its nontrivial, traceless parts \(\text{sl}(n, \mathbb{C})\); furthermore, we prove a gauge invariance of the Lax pairs and their zero-curvature condition in section \[3.2\] provided that a Maurer-Cartan equation holds for the gauge matrices. The gauge invariance of the zero-curvature condition underlines that a Lax pair should not be assigned to a single, definite nonlinear equation (as e.g. the characteristic GP-case), but to a whole set of equivalent nonlinear equations. We can solve the zero-curvature condition for general \(\text{sl}(n, \mathbb{C})\) Lax pairs (or reduced to one of its sub-algebras as \(\text{su}(n)\), etc.) in terms of the ‘\(\text{ad}\)’-operator

\[
[k_i \hat{H}^i + p_\alpha(x, t) \hat{E}_+^\alpha + Q_\alpha(x, t) \hat{E}^-_\alpha, \ldots]_{-} \ldots ,
\]

for the Cartan-Weyl basis of the prevailing generators of a chosen, closed algebra \[15\]: this generally attests that the spatial- and time-development of (seemingly accidental) nonlinear equations as part of a whole set of equivalent equations is confined to the development within group manifolds, either by the general \(\text{sl}(n, \mathbb{C})\) Lax pair algebra or one of its possible sub-algebras as the \(\text{su}(n)\)-case. According to this property, it is more appropriate to classify the whole set of equivalent nonlinear equations as the GP-like case by the underlying, chosen sub-algebras \(\subseteq \text{sl}(n, \mathbb{C})\) for the Lax pairs.

Since the spatial and time development of nonlinear equations from Lax pairs is limited to that within group manifolds, it is not astonishing that one can confirm involution properties of the conserved quantities from the monodromy matrix under rather general assumptions by performing the possible gauge transformations to a diagonal, spatial Lax matrix \(\mathcal{L}(x, t) \rightarrow \mathcal{L}_\mu(x, t)\). This gauge transformation allows to detect conditions for the general independence of conserved quantities derived from the Lax pair and its zero-curvature relation apart from the introduction of classical \(\tau\)-matrices. Nevertheless, we also investigate the calculation of the classical \(\tau\)-matrix in the case of spatially-ultralocal Lax matrices \(\mathcal{L}(x, t)\) from the tensor product within canonical Poisson brackets (cf. section \[4.2\] and appendix \[B\])

\[
\{ \mathcal{L}(x_1, t) \otimes \hat{1} \circ \hat{1} \otimes \mathcal{L}(x_2, t) \} ,
\]
which can be separated into an eigenvalue part and a commutator of the Lax matrices $\mathcal{L}_1(x,t) = \mathcal{L}(x,t) \otimes \hat{1}$, $\mathcal{L}_2(x,t) = \hat{1} \otimes \mathcal{L}(x,t)$ with the $\tau_{12}, \tau_{21}$- matrices (concerning the general definitions of the tensor product space and the tensor product within Poisson brackets, etc., see Refs. [3, 10], appendix A and chap. 2.5, respectively).

In section 5 we extend the zero-curvature relation of Lax pairs beyond (1+1) dimensions and determine the nonlinear field equations which are also confined to the spacetime development within group manifolds by the "ao"- operators of the $\text{sl}(n, \mathbb{C})$ algebra or one of its sub-algebras. The calculation of conserved quantities and their involution as in the (1+1) dimensional case can be straightforwardly generalized for the corresponding Lax pairs within arbitrary spacetime dimensions; however, it has to be taken into account the following additional point:

As we specify closed loops within group manifolds for determining independent, conserved quantities, it has to be ascertained that these loops, contained within the mapping of n-1 spatial coordinates to the fields within the considered group manifold, possess a nontrivial homotopy in order to prevent possible contractions to trivial point mappings.

2 Lax pairs for GP-type equations from the $\text{sl}(2, \mathbb{C})$ algebra

2.1 Lie algebras of $\text{su}(2)$-, $\text{sp}(2, \mathbb{R})$- and $\text{sl}(2, \mathbb{C})$-Lax pairs for integrable and chaotic behaviour

In this section we follow Refs. [1, 2], but additionally emphasize the algebraic $\text{sl}(2, \mathbb{C})$ properties of the Lax pairs for the GP-equation. One starts out from two traceless $2 \times 2$ matrices $\mathcal{L}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k)$, $\mathcal{M}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k)$ (2.1-2.5) with complex-valued entries $\mathcal{Q}(x,t)$, $\mathcal{P}(x,t)$ for the physical wavefunctions and a complex-valued spectral parameter $k$ (2.3) which is included to obtain general $\text{sl}(2, \mathbb{C})$ matrices. In the context of eqs. (2.1-2.2), the Lax matrices with the physical fields $\mathcal{Q}(x,t)$, $\mathcal{P}(x,t)$ appear to be accidental, but separately determine the spatial and time evolution of a two component, complex-valued, (non-physical), auxiliary vector field $\Xi(x,t) = (\Xi_1(x,t), \Xi_2(x,t))^T$

$$\Xi_x = \begin{pmatrix} -ik & \mathcal{P} \\ \mathcal{Q} & ik \end{pmatrix} \Xi(x,t) = \mathcal{L}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k) \Xi(x,t); (2.1)$$

$$\Xi_t = \begin{pmatrix} 2k^2 - 2k \mathcal{P} - i \mathcal{P}_x \\ -2k \mathcal{Q} + i \mathcal{Q}_x \end{pmatrix} \Xi(x,t) = \mathcal{M}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k) \Xi(x,t); (2.2)$$

$$\Xi = \Xi(x,t) \in \mathbb{C}; \quad \mathcal{P} = \mathcal{P}(x,t) \in \mathbb{C}; \quad \mathcal{Q} = \mathcal{Q}(x,t) \in \mathbb{C}; \quad k \in \mathbb{C}; (2.3)$$

$$\mathcal{L}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k) = -(i k) (\hat{\tau}_3) + \mathcal{P}(x,t) (\hat{\tau}_+ + \mathcal{Q}(x,t) (\hat{\tau}_-) \implies \in \text{sl}(2, \mathbb{C}); (2.4)$$

$$\mathcal{M}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k) = (2ik^2 + \mathcal{P} \mathcal{Q}) (\hat{\tau}_3) + (-2k \mathcal{P} - i \mathcal{P}_x) (\hat{\tau}_+) + (-2k \mathcal{Q} + i \mathcal{Q}_x) (\hat{\tau}_-) \implies \in \text{sl}(2, \mathbb{C}); (2.5)$$

It is the separate evolution of space and time (2.1-2.2) for the auxiliary field $\Xi(x,t)$ which enforces the pairwise equivalence of the mixed and exchanged spacetime derivatives (2.6-2.7) for sufficiently continuous, (non-multivalued!) functions. After substitution of $(\partial_x \Xi)$ and $(\partial_t \Xi)$ by (2.1-2.2), one transfers the equivalence of mixed and exchanged derivatives of the auxiliary field $\Xi(x,t)$ to the Lax matrices $\mathcal{L}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k)$, $\mathcal{M}_{\text{sl}(2, \mathbb{C})}(\mathcal{Q}, \mathcal{P}; k)$ and attains the so-called zero-curvature relation (2.8) which restricts the physical fields $\mathcal{Q}(x,t)$, $\mathcal{P}(x,t)$ to a nonlinear field equation

zero-curvature condition
\[
(\partial_t \partial_x \Xi(x,t)) = (\partial_t \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) \Xi(x,t) + \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k)(\partial_t \Xi(x,t)) \\
= (\partial_t \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) \Xi(x,t) + \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k) \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k) \Xi(x,t) ;
\]

\[
(\partial_x \partial_t \Xi(x,t)) = (\partial_x \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) \Xi(x,t) + \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k)(\partial_x \Xi(x,t)) \\
= (\partial_x \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) \Xi(x,t) + \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k) \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k) \Xi(x,t) ;
\]

\[
\implies 0 = (\partial_t \mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) - (\partial_x \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k)) + [\mathcal{L}_{\text{sl}(2,\mathbb{C})}(Q,P;k), \mathcal{M}_{\text{sl}(2,\mathbb{C})}(Q,P;k)] .
\tag{2.8}
\]

As we calculate every matrix element of the \((2 \times 2)\) zero-curvature condition \((2.9)\), we (seemingly accidental) achieve two coupled, nonlinear equations \((2.10, 2.11)\) of \(\Omega(x,t), P(x,t)\) from the off-diagonal matrix entries of \((2.8)\) whereas the diagonal part completely adds to zero without a physical meaning as a wave equation. Since the trace of a commutator vanishes for finite dimensional matrices, the zero-curvature condition with the commutator between \(\mathcal{L}_{\text{sl}(2)}(Q,P;k)\) and \(\mathcal{M}_{\text{sl}(2)}(Q,P;k)\) always remains within the complex-valued, traceless Lie algebra generators \(\text{sl}(2,\mathbb{C})\). (In appendix \(A\) we demonstrate how to separate a trivial, diagonal, complex-valued unit part from the most general \(\text{gl}(n,\mathbb{C})\) Lax pair ansatz so that one has only to consider the traceless, \(\text{sl}(n,\mathbb{C})\) parts of Lax pairs and their zero-curvature relation for physical field equations which are indeed nonlinear and nontrivial.) It is possible to derive a continuity equation \((2.12)\) of the complex fields \(\Omega(x,t), P(x,t)\) from the two field equations \((2.10, 2.11)\) which allows to conclude for a time-like conserved quantity with complex constant \(c_0\) as we require spatially periodic boundary conditions \((2.10, 2.11)\) of the physical fields

\[
(\partial_t \mathcal{L}) = \left( \begin{array}{cc} 0 & \mathcal{P}_t \\ \mathcal{Q}_t & 0 \end{array} \right) - \left( \begin{array}{cc} \mathcal{P}_x & -2k \mathcal{Q}_x + i \mathcal{Q}_xx \\ -2k \mathcal{Q}_x + i \mathcal{Q}_xx & -i \mathcal{P}_x \end{array} \right) + \left( \begin{array}{cc} -2k \mathcal{P}_x - 2t \mathcal{P}^2 \mathcal{Q} \\ -2k \mathcal{Q}_x + 2t \mathcal{P}^2 \mathcal{Q} \end{array} \right) = \left( \begin{array}{cc} 0 & \mathcal{P}_t + i \mathcal{P}_xx - 2t \mathcal{P}^2 \mathcal{Q} \\ 0 & 0 \end{array} \right) ;
\tag{2.9}
\]

\[
\implies 0 = \partial_t \int_0^{2\pi} dx \ (\mathcal{P}(x,t) \mathcal{Q}(x,t)) = -i \int_0^{2\pi} dx \partial_t \mathcal{Q}(x,t) (\partial_x \mathcal{P}(x,t) - \mathcal{P}(x,t) \partial_x \mathcal{Q}(x,t)) = 0 ;
\tag{2.13}
\]

\[
\implies \int_0^{2\pi} dx \ (\mathcal{P}(x,t) \mathcal{Q}(x,t)) = \text{const.} = c_0 \in \mathbb{C} .
\tag{2.14}
\]

The conserved, complex-valued quantity \((2.14)\) refers to a generalized norm of \(\Omega(x,t), P(x,t)\), compared to the standard GP-equations, but lacks of its positive definiteness. It is the non-positive property of the generalized norm \((2.14)\) which will allow us to accomplish a possible chaotic behaviour within \(\text{sl}(2,\mathbb{C})\) Lax pairs of physical fields. The restriction of the \(\text{sl}(2,\mathbb{C})\) generators \(\mathcal{L}_{\text{sl}(2,\mathbb{C})}, \mathcal{M}_{\text{sl}(2,\mathbb{C})}\) to the sub-algebras \(\text{su}(2)\) and \(\text{sp}(2,\mathbb{R})\) leads to the well-known, \((1+1)\) GP-equations \((2.15, 2.16)\) for the attractive and repulsive interaction case. As we set in \((2.15)\) \(\Omega(x,t) = -\mathcal{P}^*(x,t) = \psi(x,t)\) to the physical wavefunction with periodic boundary conditions and with a \textit{real} spectral parameter \(k\), we acquire anti-hermitian Lax pairs \(\mathcal{L}_{\text{su}(2)}(Q,P;k), \mathcal{M}_{\text{su}(2)}(Q,P;k)\) whose zero-curvature condition determines the attractive case of interaction of the \((1+1)\) GP-equation

\[
\text{I)} \quad \text{attractive case, } \text{su}(2) \text{ sub-algebra of } \text{sl}(2,\mathbb{C})
\]

\[
\Omega(x,t) = -\mathcal{P}^*(x,t) = \psi(x,t) \in \mathbb{C}; \quad k \in \mathbb{R};
\]

\[
\partial_t \psi = -\partial_x \partial_x \psi - 2|\psi|^2 \psi; \quad \psi(x = 0,t) = \psi(x = 2\pi,t);
\]
2.1 Lie algebras of $\text{su}(2)$-, $\text{sp}(2,\mathbb{R})$- and $\text{sl}(2,\mathbb{C})$-Lax pairs for integrable and chaotic behaviour

\[
\mathfrak{L}_{\text{su}(2)}(Q, P; k) = \begin{pmatrix}
-\frac{k}{\psi} & -\psi^* \\
\psi & -\frac{k}{\psi}
\end{pmatrix};
\quad \mathfrak{M}_{\text{su}(2)}(Q, P; k) = \begin{pmatrix}
2\kappa k^2 - i |\psi|^2 & 2k \psi^* + i \psi_x^* \\
-2k \psi + i \psi_x & -2\kappa k^2 + i |\psi|^2
\end{pmatrix};
\]

\[
\mathfrak{L}_{\text{su}(2)}(Q, P; k) = -\left(\mathfrak{L}_{\text{su}(2)}(Q, P; k)\right)^\dagger;
\quad \mathfrak{M}_{\text{su}(2)}(Q, P; k) = -\left(\mathfrak{M}_{\text{su}(2)}(Q, P; k)\right)^\dagger.
\]

Similarly, one can choose the combination $(Q(x, t) = +\mathcal{P}^*(x, t) = \psi(x, t))$ for the physical wavefunction with a real spectral parameter $k$ in order to achieve Lax matrices within the symplectic algebra $\text{sp}(2, \mathbb{R})$, having the real parameters $\Re(\psi(x, t))$, $\Im(\psi(x, t))$, $k \in \mathbb{R}$; the zero-curvature relation then results in the GP-equation with a repulsive interaction

\[
\Pi) \quad \text{repulsive case, } \text{sp}(2, \mathbb{R}) \text{ sub-algebra of } \text{sl}(2, \mathbb{C})
\]

\[
\mathfrak{L}_{\text{sp}(2,\mathbb{R})}(Q, P; k) = \begin{pmatrix}
-k & \psi^* \\
\psi & -k
\end{pmatrix};
\quad \mathfrak{M}_{\text{sp}(2,\mathbb{R})}(Q, P; k) = \begin{pmatrix}
-2\kappa k^2 + i |\psi|^2 & 2k \psi^* - i \psi_x^* \\
2k \psi + i \psi_x & 2\kappa k^2 - i |\psi|^2
\end{pmatrix};
\]

Both cases, the $\text{su}(2)$- and $\text{sp}(2, \mathbb{R})$-sub-algebras for the attractive and repulsive interaction, have the identical continuity equation (2.17), due to the hermitian property of the corresponding Hamilton operator. The physical norm of the periodic fields on a circle is always positive definite and is set to one (or e.g. to a constant particle number $N_0$) for an interpretation of a true probability density. Apart from the normalization of wavefunctions at arbitrary time, we can apply the positive norm of constant value to estimate the maximal achievable value for the norm of the difference of two wavefunctions at different times. Since the absolute value of any wavefunction is restricted by the maximal probability density one at arbitrary times, the maximal achievable difference of the absolute values of two wavefunctions at two different time points is bounded by the value $8\pi$, due to the restricted spatial coordinate $x \in [0, 2\pi]$. Therefore, the hermitian property of Hamilton operators in the attractive $\text{su}(2)$- and repulsive $\text{sp}(2, \mathbb{R})$-cases only allows to assign a non-chaotic behaviour to the GP-equations (2.15, 2.16), due to the absence of an exponential-(or other type-)divergence at two arbitrary times

\[
\quad \partial_t \left(\psi^* \partial_x \psi - \psi \partial_x \psi^*\right);
\]

\[
\Rightarrow \partial_t \int_0^{2\pi} dx \left(\psi^*(x, t) \partial_x \psi(x, t) - \psi(x, t) \partial_x \psi^*(x, t)\right) = 0; \quad (2.18)
\]

\[
\Rightarrow \int_0^{2\pi} dx \left(\psi^*(x, t) \partial_x \psi(x, t)\right) = \text{const.} = r_0 \in \mathbb{R}; \quad r_0 = 1;
\]

\[
\Rightarrow \int_0^{2\pi} dx \left(\psi^*(x, t) \partial_x \psi(x, t)\right) = \int_0^{2\pi} dx \left(\psi^*(x, t) \psi(x, t) - \psi(x, t) \psi^*(x, t)\right) = 0.
\]

In section 4 we also prove the time-like conservation of traces from arbitrary powers of related monodromy matrices (2.21, 2.23) which even fulfill the involution property of the Poisson brackets. According to our sep-
arate consideration of the three algebraic cases, one can derive conserved quantities from the $\mathfrak{g}_\text{sl}(2, \mathbb{C})$ matrices for the rather general sl(2, C) case with independent fields $\Omega(x, t)$, $\mathcal{P}(x, t)$ or for the two special su(2)-, sp(2, R)-sub-algebras with the exponential matrices $\mathfrak{g}_{\text{su}}(2)(x, t; k)$, $\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k)$ of the spatial Lax generators $\mathfrak{g}_{\text{su}}(2)(\xi, t; k)$, $\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(\xi, t; k)$. The inequality \( 2.20 \), which restricts the wavefunctions to non-chaotic behaviour, complies with the infinite number of independently conserved quantities derived from the $\mathfrak{g}_{\text{su}}(2)(x, t; k)$, $\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k)$ matrices for the attractive and repulsive case, respectively (cf. section 4 and appendix B). According to the inequality \( 2.20 \) and the independently conserved quantities from the traces of powers of $\mathfrak{g}_{\text{su}}(2)(x, t; k)$, $\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k)$, it seems to be impossible to obtain any chaotic behaviour from the su(2)-, sp(2, R)- Lax pairs and their zero-curvature conditions. Chaotic behaviour from Lax pairs appears to be impossible in particular because the inequality and the independently conserved quantities from $\mathfrak{g}_{\text{su}}(2)(x, t; k)$, $\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k)$ can be simply generalized to the case with an arbitrary external potential in the GP-equations with attractive or repulsive interaction

$$\begin{align*}
\mathfrak{g}_{\text{sl}}(2, \mathbb{C})(x, t; k) &= \exp\left\{ \int_0^x d\xi \, \mathfrak{g}_{\text{sl}}(2, \mathbb{C})(\xi; t; k) \right\} 
&\implies C_{\text{sl}(2, \mathbb{C})}^{(n)}(t; k) = \text{Tr}\left[ (\mathfrak{g}_{\text{sl}}(2, \mathbb{C})(x = 2\pi, t; k))^n \right]; \\
1 &= \mathfrak{g}_{\text{sl}}^{-1}(2, \mathbb{C})(x, t; k) \mathfrak{g}_{\text{sl}}(2, \mathbb{C})(x, t; k); \\
\mathfrak{g}_{\text{su}}(2)(x, t; k) &= \exp\left\{ \int_0^x d\xi \, \mathfrak{g}_{\text{su}}(2)(\xi; t; k) \right\} 
&\implies C_{\text{su}(2)}^{(n)}(t; k) = \text{Tr}\left[ (\mathfrak{g}_{\text{su}}(2)(x = 2\pi, t; k))^n \right]; \\
1 &= \mathfrak{g}_{\text{su}}^{-1}(2)(x, t; k) \mathfrak{g}_{\text{su}}(2)(x, t; k); \\
\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k) &= \exp\left\{ \int_0^x d\xi \, \mathfrak{g}_{\text{sp}}(2, \mathbb{R})(\xi; t; k) \right\} 
&\implies C_{\text{sp}(2, \mathbb{R})}^{(n)}(t; k) = \text{Tr}\left[ (\mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x = 2\pi, t; k))^n \right]; \\
\left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) &= \mathfrak{g}_{\text{sp}}^{-1}(2, \mathbb{R})(x, t; k) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \mathfrak{g}_{\text{sp}}(2, \mathbb{R})(x, t; k); \\
\mathfrak{g}_{\text{sp}}^{-1}(2, \mathbb{R})(x, t; k) &= \exp\left\{ \int_0^x d\xi \, \mathfrak{g}_{\text{sp}}^{-1}(2, \mathbb{R})(\xi; t; k) \right\}. 
\end{align*}$$

Nevertheless, it is possible to compose the fields $\Omega(x, t)$, $\mathcal{P}(x, t)$ of the rather general sl(2, C) Lax pair algebra by the combination \( 2.23 \) of the physical wavefunctions $\psi_{\text{su}}(2)(x, t)$, $\psi_{\text{sp}}(2, \mathbb{R})(x, t)$ so that probability of su(2)- and sp(2, R)- sub-algebra matrices with their corresponding wavefunctions as parameters is allowed to flow and change within the total sl(2, C) Lax pair algebra; in consequence chaotic behaviour becomes possible despite of the formulation in terms of Lax pairs and their zero-curvature relations

$$\begin{align*}
c_0(\in \mathbb{C}) &= \int_0^{2\pi} dx \left( \mathcal{P}(x, t) \Omega(x, t) \right); \\
\Omega(x, t) &= \psi_{\text{su}}(2)(x, t) + \psi_{\text{sp}}(2, \mathbb{R})(x, t); \\
\mathcal{P}(x, t) &= -\psi_{\text{su}}^*(2)(x, t) + \psi_{\text{sp}}^*(2, \mathbb{R})(x, t). 
\end{align*}$$

We illustrate this possibility from the generalized conserved norm \( 2.21 \) of fields $\Omega(x, t)$, $\mathcal{P}(x, t)$ with the complex constant $c_0$. One has to compose the fields $\Omega(x, t)$, $\mathcal{P}(x, t)$ of sl(2, C) by the wavefunctions $\psi_{\text{su}}(2)(x, t)$, $\psi_{\text{sp}}(2, \mathbb{R})(x, t)$ of the attractive and repulsive interaction case of GP-equations according to eqs. \( 2.23 \) so that one can resolve from the real $\Re(c_0)$ and imaginary part $\Im(c_0)$ of the generalized norm \( 2.24 \) two independent conserved relations \( 2.29 \) in terms of the independent physical wavefunctions $\psi_{\text{su}}(2)(x, t)$, $\psi_{\text{sp}}(2, \mathbb{R})(x, t)$; in correspondence to the non-compact property of SL(2, C), we attain unbounded norms of
2.1 Lie algebras of $\text{su}(2)$-, $\text{sp}(2,\mathbb{R})$- and $\text{sl}(2,\mathbb{C})$-Lax pairs for integrable and chaotic behaviour

$\psi_{\text{su}(2)}(x,t), \psi_{\text{sp}(2,\mathbb{R})}(x,t)$, which only differ by the constant $\Re(c_0)$, and also acquire a restriction for a generalized angle between the wavefunction vectors of $\psi_{\text{su}(2)}(x,t), \psi_{\text{sp}(2,\mathbb{R})}(x,t)$ from the imaginary part (as one transfers the spatial argument "$x" to a discrete index). The decomposition of $Q(x,t), P(x,t)$ to the parameters $\psi_{\text{su}(2)}(x,t), \psi_{\text{sp}(2,\mathbb{R})}(x,t)$ of the sub-algebras $\text{su}(2), \text{sp}(2,\mathbb{R})$ within $\text{sl}(2,\mathbb{C})$ therefore removes the restriction to completely integrable behaviour (even for the cases with arbitrary external potential $V(x,t)$)

$$c_0(\in \mathbb{C}) = \int_0^{2\pi} dx \left(-\psi_{\text{su}(2)}^*(x,t) + \psi_{\text{sp}(2,\mathbb{R})}^*(x,t)\right) \left(\psi_{\text{su}(2)}(x,t) + \psi_{\text{sp}(2,\mathbb{R})}(x,t)\right); \quad (2.27)$$

$$\int_0^{2\pi} dx \left|\psi_{\text{sp}(2,\mathbb{R})}(x,t)\right|^2 = \int_0^{2\pi} dx \left|\psi_{\text{su}(2)}(x,t)\right|^2 + \Re(c_0) + i \left[\Im(c_0) - 2 \int_0^{2\pi} dx \Im \left(\psi_{\text{su}(2)}^*(x,t) \psi_{\text{sp}(2,\mathbb{R})}(x,t)\right)\right]; \quad (2.28)$$

$$\int_0^{2\pi} dx \Im \left(\psi_{\text{su}(2)}^*(x,t) \psi_{\text{sp}(2,\mathbb{R})}(x,t)\right) = \frac{1}{2} \Im(c_0). \quad (2.29)$$

After replacing the fields $Q(x,t), P(x,t)$ by \eqref{2.31} in the $\text{sl}(2,\mathbb{C})$ zero-curvature relation and field equations \eqref{2.31}, we attain two coupled GP-type equations with the attractive case of the $\text{su}(2)$-field $\psi_{\text{su}(2)}(x,t)$ and with the repulsive case of the $\text{sp}(2,\mathbb{R})$-field $\psi_{\text{sp}(2,\mathbb{R})}(x,t)$. Aside from the cubic, attractive interaction \(-2|\psi_{\text{su}(2)}(x,t)|^2 \psi_{\text{su}(2)}(x,t)\)' and the cubic, repulsive interaction \(2|\psi_{\text{sp}(2,\mathbb{R})}(x,t)|^2 \psi_{\text{sp}(2,\mathbb{R})}(x,t)\)', there appear the parts \(4|\psi_{\text{sp}(2,\mathbb{R})}(x,t)|^2 \psi_{\text{su}(2)}(x,t)'\) and \(4|\psi_{\text{su}(2)}(x,t)|^2 \psi_{\text{sp}(2,\mathbb{R})}(x,t)\)' acting as a repulsive and attractive external potential, respectively. Moreover, one has the two incoherent terms \(2\psi_{\text{su}(2)}^*(x,t) \psi_{\text{sp}(2,\mathbb{R})}(x,t)\)' and \(+2 \psi_{\text{sp}(2,\mathbb{R})}^*(x,t) \psi_{\text{su}(2)}(x,t)\)' which can give rise to a change of norm and integrated, total probability

$$i \left(\partial_t P\right) = P_{xx} - 2rP^2 \quad \Longrightarrow \quad (2.31)$$

$$i \left(\partial_t \psi_{\text{su}(2)}\right) = -\left(\partial_x \partial_x \psi_{\text{su}(2)}\right) - 2 \left(|\psi_{\text{su}(2)}|^2 - 2|\psi_{\text{sp}(2,\mathbb{R})}|^2\right) \psi_{\text{su}(2)} - 2 \psi_{\text{su}(2)}^* \left(\psi_{\text{sp}(2,\mathbb{R})}\right)^2; \quad (2.32)$$

$$i \left(\partial_t \psi_{\text{sp}(2,\mathbb{R})}\right) = -\left(\partial_x \partial_x \psi_{\text{sp}(2,\mathbb{R})}\right) + 2 \left(|\psi_{\text{sp}(2,\mathbb{R})}|^2 - 2|\psi_{\text{su}(2)}|^2\right) \psi_{\text{sp}(2,\mathbb{R})} + 2 \psi_{\text{sp}(2,\mathbb{R})}^* \left(\psi_{\text{su}(2)}\right)^2. \quad (2.33)$$

This becomes obvious as we derive continuity equations with the corresponding source parts

$$\left(\partial_t |\psi_{\text{su}(2)}(x,t)|^2\right)^2 = -2 \partial_x \Im \left(\psi_{\text{su}(2)}^*(x,t) \partial_x \psi_{\text{su}(2)}(x,t)\right) - 4 |\psi_{\text{su}(2)}(x,t)|^2 \Im \left(\psi_{\text{sp}(2,\mathbb{R})}^*(x,t)\right); \quad (2.34)$$

$$\left(\partial_t |\psi_{\text{sp}(2,\mathbb{R})}(x,t)|^2\right)^2 = -2 \partial_x \Im \left(\psi_{\text{sp}(2,\mathbb{R})}^*(x,t) \partial_x \psi_{\text{sp}(2,\mathbb{R})}(x,t)\right) + 4 |\psi_{\text{sp}(2,\mathbb{R})}(x,t)|^2 \Im \left(\psi_{\text{su}(2)}^*(x,t)\right). \quad (2.35)$$

According to the coupled GP-type equations \eqref{2.32}--\eqref{2.33}, one can distinguish three different, physical cases. If we set the two wavefunctions $\psi_{\text{su}(2)}(x,t), \psi_{\text{sp}(2,\mathbb{R})}(x,t)$ to an identical value $\psi_0(x,t)$, one just considers the trivial free propagation with the kinetic term \(-i(\partial_x \partial_x \psi_0)(x,t)\)' \eqref{2.36}. However, we can also take the attractive or repulsive case with the wavefunctions $\psi_{\text{su}(2)}(x,t), \psi_{\text{sp}(2,\mathbb{R})}(x,t)$ as the dominant term $\psi_0(x,t)$ in the two, coupled, nonlinear equations \eqref{2.32}--\eqref{2.33} and can then regard the other $\psi_{\text{sp}(2,\mathbb{R})}(x,t)$ or $\psi_{\text{su}(2)}(x,t)$ wave component as the corresponding perturbation $\Delta \psi(x,t)$ in order to compute for a possible chaotic behaviour of the $\psi_{\text{su}(2)}(x,t)$ and $\psi_{\text{sp}(2,\mathbb{R})}(x,t)$ wavefunctions, respectively

1) if $\psi_{\text{su}(2)} = \psi_{\text{sp}(2,\mathbb{R})} = \psi_0$; $\Delta \psi \equiv 0 \implies$ free propagation; $i (\partial_t \psi_0) = -i(\partial_x \partial_x \psi_0)$ \quad (2.36)
2 LAX PAIRS FOR GP-TYPE EQUATIONS FROM THE SL(2, C) ALGEBRA

2) if : $\psi_{su(2)} = \psi_0; \quad \psi_{sp(2, R)} = \Delta \psi$ (perturbation) $\implies$ chaotic behaviour of $\psi_{su(2)}$; \hfill (2.37)

3) if : $\psi_{sp(2, R)} = \psi_0; \quad \psi_{su(2)} = \Delta \psi$ (perturbation) $\implies$ chaotic behaviour of $\psi_{sp(2, R)}$. \hfill (2.38)

The latter two cases \textcolor{red}{(2.37, 2.38)} as sub-algebras of $sl(2, C)$ have an infinite number of independent, exactly conserved quantities which are derived from traces of powers of $T_{sl(2,C)}(x, t; k)$ monodromy matrices. These exactly conserved quantities from $T_{sl(2,C)}(x, t; k)$ contain both sub-groups, the SU(2)-group with $T_{su(2)}(x, t; k)$ matrices and the Sp(2, $R$)-group with the $T_{sp(2,R)}(x, t; k)$ matrices, so that one can consider perturbations of the dominantly conserved parts, which are given as traces of powers of $T_{su(2)}(x, t; k)$ or $T_{sp(2,R)}(x, t; k)$ with corresponding perturbative terms of $L_{sp(2,R)}(x, t; k)$ or $L_{su(2)}(x, t; k)$, respectively.

In this section we have limited discussion to sl(2, $C$) Lax pairs and zero-curvature relations with the su(2), sp(2, $R$) sub-algebras of the corresponding integrable, attractive or repulsive GP-equations. However, the coupled equations \textcolor{red}{(2.32, 2.33)} of $\psi_{su(2)}(x, t)$ and $\psi_{sp(2,R)}(x, t)$ wavefunctions within the non-compact sl(2, $C$) algebra can be generalized to arbitrary algebras sl($n, C$) with a chosen sub-algebra su($n$) so that one can achieve a possible chaotic behaviour for field equations following from sl($n, C$) Lax pairs and zero-curvature conditions. The rather general sl($n, C$) Lax pair has to be decomposed into sub-algebras as the compact su($n$) case of anti-hermitian Lax pairs and remaining algebra parts so that the norm of wavefunctions is not bounded by real, positive constants. This extension to chaotic behaviour, which follows from taking compact sub-algebras of the rather general sl($n, C$) algebra, is also possible for Lax pairs beyond (1+1) dimensions.

2.2 Lax pairs of sl(2, $C$) for an external potential

The derivations, concerning the integrable and chaotic behaviour in the previous section \textcolor{red}{2.1} can even be transferred to the case with an arbitrary external potential $V(x, t)$ in the attractive su(2) and repulsive sp(2, $R$) interaction case of GP-equations. In the following we construct sl(2, $C$) Lax matrices $L_{sl(2,C)}(Q, P; k)$, $M_{sl(2,C)}(A, B, C)$ so that the nonlinear field equations for $Q(x, t), P(x, t)$ of the zero-curvature relation allow under restriction to $\psi_{su(2)}(x, t)$, $\psi_{sp(2,R)}(x, t)$ fields and sub-algebras the GP-equations with the external potential $V(x, t)$. It suffices to derive the nonlinear equations

\begin{align*}
\tau P_t &= P_{xx} - 2 \left( Q P - V(x, t) \right) P; \hfill (2.39) \\
-\tau Q_t &= Q_{xx} - 2 \left( P Q - V(x, t) \right) Q, \hfill (2.40)
\end{align*}

so that the whole discussion of section \textcolor{red}{2.1} can be conveyed to the case with an external potential, especially the conclusions for integrable and chaotic behaviour. We point out that the spatial Lax matrix $L_{sl(2,C)}(Q, P; k)$ does not change aside from a possible spacetime dependence of the complex spectral parameter $k \rightarrow k(x, t) \in C$. Therefore, the conserved quantities, following from traces of powers of the monodromy matrix $T_{sl(2,C)}(x, t; k)$, are not essentially altered and the canonical variables $Q(x, t), P(x, t)$ within the Poisson brackets do not cause a different r-matrix which specifies the involution of the independently conserved quantities. (In the sequel one is even allowed to choose a constant spectral parameter $k \in C$, concerning the involution property, because the complex-valued, auxiliary field $W$ can always be adapted in such a manner so that an unwanted appearance of $V_x(x, t)$ can be absorbed by suitable dependence of $W_x$ in (2.59).)

In analogy to eqs. \textcolor{red}{(2.1, 2.5)}, we start out from traceless matrices $L_{sl(2,C)}(Q, P; k)$, $M_{sl(2,C)}(A, B, C)$ where the latter matrix for the time evolution of the auxiliary field $\Xi(x, t)$ is determined by an ansatz with the three complex-valued fields $A(P, Q; k), B(P, Q; k), C(P, Q; k)$

\begin{equation}
\Xi_x = \begin{pmatrix}
-\tau k & P \\
Q & \tau k
\end{pmatrix} \Xi(x, t) = L_{sl(2,C)}(Q, P; k) \Xi(x, t) \hfill (2.41)
\end{equation}
The zero-curvature condition \((2.47)\) results into the field equations \((2.49,2.50)\) of the fields \(\Omega(x,t)\), \(P(x,t)\). Aside from the external potential in the coefficient \(A(\Omega;Q;k)\) \((2.52)\), the ansatz \((2.52,2.54)\) is supplied with a sufficient number of complex-valued fields \(W(\Omega, Q, k; V), Y(\Omega, Q, k; V), Z(\Omega, Q, k; V)\) \((2.55)\) which have to be determined in such a manner that the nonlinear field equations \((2.57,2.58)\) of \(\Omega(x,t), P(x,t)\) only change by the external potential \(V(x,t)\). These conditions (cf. the two braces in eqs. \((2.57,2.58)\)) specify first order partial differential equations of \(Y_x(\Omega, Q, k; V), Z_x(\Omega, Q, k; V)\) whereas the additional equation \((2.59)\) of a (possibly chosen spacetime dependent !) spectral parameter \(k(x,t)\) results into a differential equation with the partial derivative \(W_x(\Omega, Q, k; V)\)

\[
\begin{align*}
\Xi_t &= \begin{pmatrix} A(\Omega;Q;k) & B(\Omega;Q;k) \\ C(\Omega;Q;k) & -A(\Omega;Q;k) \end{pmatrix} \Xi(x,t) = \mathcal{M}_{sl(2,\mathbb{C})}(A,B,C) \Xi(x,t) ; \\
\Xi &= \Xi(x,t) \in \mathbb{C} ; \ P = \mathcal{P}(x,t) \in \mathbb{C} ; \ Q = \Omega(x,t) \in \mathbb{C} ; \ k = k(x,t) \in \mathbb{C} ; \\
A &= A(\Omega;Q;k) \in \mathbb{C} ; \ B = B(\Omega;Q;k) \in \mathbb{C} ; \ C = C(\Omega;Q;k) \in \mathbb{C} ; \\
\mathcal{L}_{sl(2,\mathbb{C})}(\Omega;P;k) &= -\mathbb{I}(k) (\hat{\tau}_3) + \mathcal{P}(x,t) (\hat{\tau}_+ + \Omega(x,t) (\hat{\tau}_-) \rightarrow e \in sl(2,\mathbb{C}) ; \\
\mathcal{M}_{sl(2,\mathbb{C})}(A,B,C) &= A(\Omega;Q;k) (\hat{\tau}_3) + B(\Omega;Q;k) (\hat{\tau}_+) + C(\Omega;Q;k) (\hat{\tau}_-) \rightarrow e \in sl(2,\mathbb{C}) .
\end{align*}
\]
can be solved under rather general conditions with the appropriate ordering of exponential step operators and initial value \( \bar{\omega}(x = 0, t) \). Apart from the homogenous solution with latter field vector \( \bar{\omega}(x = 0, t) \), one has a particular solution with vector field \( \bar{\omega}(\mathcal{P}, \mathcal{Q}; k; V) \) which also contains the spatial derivative of the external potential \( V(x, t) \)

\[
\begin{align*}
\tau \mathcal{P}_t &= \mathcal{P}_{xx} - 2 (\mathcal{Q} \mathcal{P} - V(x, t)) \mathcal{P} ; \\
-\tau \mathcal{Q}_t &= \mathcal{Q}_{xx} - 2 (\mathcal{P} \mathcal{Q} - V(x, t)) \mathcal{Q} ; \\
\partial_x \begin{pmatrix} W \\ Y \\ Z \end{pmatrix} &= \begin{pmatrix} 0 & -\mathcal{Q} & \mathcal{P} \\ -2 \mathcal{P} & -2t & 0 \\ 2 \mathcal{Q} & 0 & 2t \end{pmatrix} \begin{pmatrix} W \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} \tau (V_x - k_t - 2 (k^2)_x) \\ 2k_x \mathcal{P} \\ 2k_x \mathcal{Q} \end{pmatrix}; \\
\partial_x \bar{\omega} &= \hat{\mathcal{W}}(\mathcal{P}, \mathcal{Q}; k) \bar{\omega} + \bar{\omega}(\mathcal{P}, \mathcal{Q}; k; V); \\
\bar{\omega}(x, t) &= \exp[\int_0^x d\xi \, \bar{\mathcal{W}}(\xi, t)] \bar{\omega}(x = 0, t) + \int_0^x dy \, \exp[\int_y^x d\xi \, \bar{\mathcal{W}}(\xi, t)] \bar{\omega}(y, t).
\end{align*}
\]

Therefore, one can construct Lax matrices for generalized GP-type equations \((2.60, 2.61)\) with the additional fields \( W(\mathcal{P}, \mathcal{Q}; k; V) \), \( Y(\mathcal{P}, \mathcal{Q}; k; V) \), \( Z(\mathcal{P}, \mathcal{Q}; k; V) \) \((2.62, 2.63)\) in such a manner that the zero-curvature relation \((2.47, 2.48)\) confines to the cases with an external potential \( V(x, t) \). As we have already mentioned at the beginning of this section \(2.2\) one can directly repeat considerations of section \(2.1\) in order to conclude for integrable or a possible chaotic behaviour by regarding the corresponding cases of \( \mathfrak{sl}(2, \mathbb{C}) \) Lax pairs with sub-algebras \( \mathfrak{su}(2) \) and \( \mathfrak{sp}(2, \mathbb{R}) \).

3 The general \( n \times n \) Lax pair matrices as \( \mathfrak{sl}(n, \mathbb{C}) \) algebras

3.1 Solution of the zero-curvature condition

The most general matrices for the Lax pairs \( \mathcal{L}(x, t) \), \( \mathcal{M}(x, t) \) are given by the complex-valued \( \mathfrak{gl}(n, \mathbb{C}) \) algebra with \( 2n^2 \) real parameters; however, due to the occurrence of the commutator \( [\mathcal{L}(x, t), \mathcal{M}(x, t)]_\cdot \), one can split a trivial, diagonal unity part from this most general ansatz with two complex-valued fields consisting of the sum of the diagonal entries from the \( \mathcal{L}(x, t) \), \( \mathcal{M}(x, t) \) matrices within \( \mathfrak{gl}(n, \mathbb{C}) \) (cf. appendix A). Therefore, the \( \mathfrak{sl}(n, \mathbb{C}) \) algebra with \( 2n^2 - 2 \) real parameters is only taken as the most general ansatz for Lax pairs \( \mathcal{L}(x, t) \), \( \mathcal{M}(x, t) \) so that, indeed, field equations remain with nontrivial, nonlinear properties.

In the following, we assume that the spatial Lax matrix \( \mathcal{L}(x, t) \) is chosen within a \( \mathfrak{sl}(n, \mathbb{C}) \) algebra or one of its sub-algebras as e.g. \( \mathfrak{su}(n) \), etc. Therefore, one can view the zero-curvature relation as a spatial evolution equation \((3.1)\) of \( \partial_t \mathcal{M} \) with the \( \omega \)-operator \( [\mathcal{L}(x, t), \ldots]_\cdot \) of a closed algebra acting onto \( \mathcal{M}(x, t) \) and an inhomogeneity \((\partial_t \mathcal{L}(x, t))\). The spatial evolution equation \((3.1)\) has a homogenous matrix solution \( \mathcal{M}_\text{hom}(x, t) \) \((3.2)\) with 'initial' matrix \( \mathcal{M}_{\text{init}}(x = 0, t) \) at the spatial origin which develops with spatial steps \( \Delta_x \cdot [\mathcal{L}(x, t), \ldots]_\cdot \) in exponential operators with appropriate spatial ordering from right to left \( \exp[\ldots] \). The action of the \( \omega \)-operator \( [\mathcal{L}(x, t), \ldots]_\cdot \) within the spatially ordered exponentials has the effect of left and right propagation for the initial matrix \( \mathcal{M}_{\text{init}}(x = 0, t) \) with exponents of \( \pm \Delta_x \cdot [\mathcal{L}(x, t); \ldots] \); this is reminiscent of the Heisenberg equation of motion in quantum mechanics or of the total development operator for the von-Neumann equation in statistical mechanics

\[
(\partial_t \mathcal{M}) = [\mathcal{L}, \mathcal{M}]_\cdot + (\partial_t \mathcal{L}) = [\mathcal{L}, \ldots]_\cdot \mathcal{M} + (\partial_t \mathcal{L});
\]
homogenous solution : \( \mathcal{M}_{\text{hom}}(x,t) \);

\[
\mathcal{M}_{\text{hom}}(x,t) = \exp \left\{ \int_0^x d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} \mathcal{M}_{\text{ini}(t)}(x = 0, t) ;
\]  \hspace{1cm} (3.2)

\[
\mathcal{M}_{\text{hom}}(x,t) = \exp \{ \Delta x \mathcal{L}(x,t) \} \exp \{ \Delta x \mathcal{L}(x-\Delta x,t) \} \exp \{ \Delta x \mathcal{L}(x-2\Delta x,t) \} \ldots \times \exp \{ \Delta x \mathcal{L}(2\Delta x,t) \} \ldots \times \exp \{ -\Delta x \mathcal{L}(x-2\Delta x,t) \} \exp \{ -\Delta x \mathcal{L}(x-\Delta x,t) \} \exp \{ -\Delta x \mathcal{L}(x,t) \} .
\]  \hspace{1cm} (3.3)

The particular solution of (3.1) similarly results as in the case of an ordinary, first order differential equation by a variational matrix-ansatz \( \mathcal{M}(x,t) \) (3.4) instead of the initial matrix \( \mathcal{M}_{\text{ini}}(x = 0, t) \). Straightforward transformations (3.5), (3.6) lead to the general solution (3.7) which consists of the homogenous part with matrix \( \mathcal{M}_{\text{ini}}(x = 0, t) \) and the particular part with the inhomogeneity \( (\partial_t \mathcal{L}(y,t)) \)

Variational ansatz for initial matrix \( \mathcal{M}_{\text{ini}}(x = 0, t) \) with \( \mathcal{M}(x,t) \)

\[
\mathcal{M}(x,t) = \exp \left\{ \int_0^x d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} \mathcal{M}_{\text{ini}}(x = 0, t) + \exp \left\{ \int_0^y d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} \mathcal{N}(x,t) ;
\]  \hspace{1cm} (3.4)

inserted into zero-curvature condition

\[
\partial_x \mathcal{M}(x,t) = (\partial_t \mathcal{L}(x,t)) ;
\]  \hspace{1cm} (3.5)

\[
\mathcal{M}(x,t) = \int_0^x dy \exp \left\{ - \int_0^y d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} (\partial_t \mathcal{L}(y,t)) ;
\]  \hspace{1cm} (3.6)

\[
\mathcal{M}(x,t) = \exp \left\{ \int_0^x d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} \mathcal{M}_{\text{ini}}(x = 0, t) + \int_0^x dy \exp \left\{ \int_y^x d\xi \left[ \mathcal{L}(\xi,t), \ldots \right] \right\} (\partial_t \mathcal{L}(y,t)).
\]  \hspace{1cm} (3.7)

The general solution (3.7) of the zero-curvature relation demonstrates that it suffices to choose a spatial Lax matrix \( \mathcal{L}(\xi,t) \) from a closed algebra as the general sl\((n,\mathbb{C})\) case or one of its sub-algebras as su\((n)\) in order to determine the matrix \( \mathcal{M}(x,t) \) or the physical, nonlinear equations wave equations following from it. After choosing the spatial matrix \( \mathcal{L}(\xi,t) \) and the initial matrix \( \mathcal{M}_{\text{ini}}(x = 0, t) \) within a closed Lie algebra of a Cartan-Weyl basis (3.8), one is constrained to the spectral development within the corresponding, closed Lie group, due to the action of the \( \mathfrak{d} \)-operator \( \left[ \mathcal{L}(\xi,t), \ldots \right] \) of the closed Lie algebra. This points to a classification of nonlinear equations by the underlying closed Lie algebra of Lax pairs (as the general sl\((n,\mathbb{C})\) case with its sub-algebras as e.g. su\((n)\) instead by the precise form of the physical, nonlinear equations (as e.g. the typical GP-equations). This becomes even more obvious as we can prove a general gauge invariance from a closed algebra of Lax pairs so that one has to regard a whole set of Lax pairs for a certain type of physical, nonlinear equations. Let us consider the Cartan-Weyl basis (3.8) for a closed Lie algebra \( [15] \)

\[
[\hat{H}^i, \hat{H}^j]_- = 0 ; \hspace{1cm} [\hat{E}^\alpha, \hat{E}^\beta]_- = N^{\alpha\beta} \hat{E}^\alpha+\beta ; \hspace{1cm} (\alpha + \beta \neq 0) ;
\]  \hspace{1cm} (3.8)

so that the spatial Lax matrix has the general, ultralocal form (3.9) with the fields \( \phi_\alpha(x,t) = (Q_\alpha(x,t), \mathcal{P}_\alpha(x,t)) \) for the ladder operators \( \hat{E}^\alpha = (\hat{E}^\alpha_-, \hat{E}^\alpha_+) \) and with the general, spectral parameters \( k_i(x,t) \) for the (maximal commuting, traceless) Cartan sub-algebra. According to the gauge invariance of Lax pairs (cf. next section 3.2), one can transform the general ansatz of \( \mathcal{L}(x,t;k) \) (3.9) with the ladder operators \( \hat{E}^\alpha_\pm \) to the (maximal commuting, traceless) Cartan sub-algebra generators \( \hat{H}^i \) where the corresponding fields \( \lambda^i_{\alpha H}(x,t;k) \) (3.10) follow from the fields \( \phi_\alpha(x,t) \) and the spectral parameters

\[
\mathcal{L}(x,t;k) = \hat{H}^i k_i(x,t) + \hat{E}^\alpha \phi_\alpha(x,t) ;
\]  \hspace{1cm} (3.9)
\[ \dot{E}^\alpha \phi_\alpha(x,t) = \dot{E}^-_\alpha \zeta_\alpha(x,t) + \dot{E}^+_\alpha \zeta_\alpha(x,t) ; \quad \phi_\alpha(x,t) = (\zeta_\alpha(x,t), \zeta_\alpha(x,t)) ; \]

\[ \mathcal{L}(x,t) = \mathcal{L}_\beta(x,t;k) = \hat{H}_t \lambda^{(R)}_1(\phi_\beta;k_j) = \hat{H}_t \lambda^{(R)}_1(x,t;k) ; \text{(by a gauge transformation)} ; (3.10) \]

\[ \mathcal{M}_{\text{ini}}(x=0,t) = \hat{H}^i A^{(\text{ini})}_i(x=0,t) + E^\alpha \lambda^{(\text{ini})}_i(x=0,t) ; \quad (3.11) \]

\[ \mathcal{M}(x,t) := \hat{H}^i a_i(\phi_\beta(x,t);k_j) + E^\alpha b_\alpha(\phi_\beta(x,t);k_j) = \hat{H}^i a_i(x,t) + E^\alpha b_\alpha(x,t) ; \quad (3.12) \]

\[ \mathcal{M}(x,t) = \hat{H}^i a_i(\phi_\beta(x,t);k_j) + E^\alpha b_\alpha(\phi_\beta(x,t);k_j) = \hat{H}^i a_i(x,t) + E^\alpha b_\alpha(x,t) \]

\[ = \hat{H}^i \left( A^{(\text{ini})}_i(x=0,t) + \int_0^x \, dy \, (\partial_t \lambda^{(R)}_j(y,t)) \right) + \sum_{\alpha \in \text{root}} \exp \left\{ \int_0^x \, d\xi \, \lambda^{(R)}_i(\xi,t) \alpha^i \right\} \hat{E}^\alpha B^{(\text{ini})}_\alpha(x=0,t) ; \]

\[ a_i(\phi_\beta(x,t);k_j) = a_i(x,t) = A^{(\text{ini})}_i(x=0,t) + \int_0^x \, dy \, (\partial_t \lambda^{(R)}_j(y,t)) ; \quad (3.16) \]

\[ b_\alpha(\phi_\beta(x,t);k_j) = b_\alpha(x,t) = B^{(\text{ini})}_\alpha(x=0,t) \times \exp \left\{ \int_0^x \, d\xi \, \lambda^{(R)}_i(\xi,t) \alpha^i \right\} . \quad (3.17) \]

Recursive application of the exponential operators with the \textit{ad}-operator \([\hat{H}^i, \ldots]_\alpha\) of the Cartan sub-algebra in (3.11) specifies the form of the time-like Lax matrix \(\mathcal{M}(x,t)\) by using the components \(\alpha^j\) of the root vectors \(\vec{\alpha}\) of corresponding ladder operators \(\hat{E}^\alpha\)

\[ \mathcal{M}(x,t) := \hat{H}^i a_i(x,t) + E^\alpha b_\alpha(x,t) = \hat{H}^i \left( A^{(\text{ini})}_i(x=0,t) + \int_0^x \, dy \, (\partial_t \lambda^{(R)}_j(y,t)) \right) + \sum_{\alpha \in \text{root}} \exp \left\{ \int_0^x \, d\xi \, \lambda^{(R)}_i(\xi,t) \alpha^i \right\} \hat{E}^\alpha B^{(\text{ini})}_\alpha(x=0,t) ; \]

We summarize the algebraic properties of the zero-curvature condition with following additional notes:

If \(\mathcal{L}(x,t)\) belongs to the generators of a closed algebra with the physical fields being the parameters or angles of the corresponding Lie group, one can generate \(\mathcal{M}(x,t)\) for the zero-curvature condition by (3.14,3.15) which then determines the nonlinear equations \((3.16,3.17)\) as GP-type equations, or other types corresponding to the chosen algebra of \(\mathcal{L}(x,t) \subseteq \mathfrak{s}l(n, \mathbb{C})\). One has to take into account that one does not obtain an overdetermined system which finally results with the achieved matrix \(\mathcal{M}(x,t)\) \((3.14,3.15)\) and its chosen parametric dependence \(a_i(\phi_\beta(x,t);k_j), b_\alpha(\phi_\beta(x,t);k_j)\) \((3.12)\) into contradictory equations constraining the solutions from the zero-curvature condition to fixed time- and spatial-terms without a physical time development.

\[ 3.2 \quad \text{Gauge invariance of the Lax pair and the zero-curvature condition} \]

It has already been stated that the Lax pair \(\mathcal{L}(x,t), \mathcal{M}(x,t)\) and its zero-curvature condition is by no means unique. If one restricts to transformations \((3.18)\) with a solely time dependent gauge matrix \(\theta_0 = \theta(t)\), one can immediately verify the invariance \((3.20)\) of the zero-curvature relation under the solely time dependent gauge
3.2 Gauge invariance of the Lax pair and the zero-curvature condition

transformation \( (3.19) \)

\[
\begin{align*}
G_0 & := G(t) ; \quad (\partial_x G_0) = (\partial_x G(t)) \equiv 0 ; \\
\mathcal{L} & \rightarrow \mathcal{L}' = G_0 \mathcal{L} G_0^{-1} ; \quad \mathcal{M} \rightarrow \mathcal{M}' = G_0 \mathcal{M} G_0^{-1} + (\partial_t G_0) G_0^{-1} ; \\
(\partial_t \mathcal{L}) - (\partial_t \mathcal{M}) + [\mathcal{L}, \mathcal{M}] & = 0 \iff (\partial_t \mathcal{L}') - (\partial_t \mathcal{M}') + [\mathcal{L}', \mathcal{M}'] = 0 .
\end{align*}
\]

The gauge invariance with \( G_0 = G(x,t) \) \( (3.21-3.28) \) can also be extended to a general spacetime dependence within the corresponding Lie group, following from the closed algebra \( \mathcal{L}(x,t) \), as e. g. in the Cartan-Weyl basis. As we begin with the transformations \( (3.23-3.25) \) of \( G_0 = G(x,t) \), acting onto the auxiliary field \( \Xi(x,t) \) and Lax pair \( \mathcal{L}(x,t) \), \( \mathcal{M}(x,t) \), we attain the invariance of the Lax pair equations \( (3.26) \) and its zero-curvature relation \( (3.27) \), provided that the Maurer-Cartan relation \( (3.28) \) is fulfilled by the gauge matrix \( G_0 = G(x,t) \)

Gauge invariance \( G_0 = G(x,t) \) with Lie group of the closed algebra from \( \mathcal{L}(x,t) \):

\[
(\partial_x \Xi) = \mathcal{L} \Xi ; \quad (\partial_t \Xi) = \mathcal{M} \Xi ;
\]

\[
\Xi \rightarrow \Xi' = G_0 \Xi ;
\]

\[
\mathcal{L} \rightarrow \mathcal{L}' = G_0 \mathcal{L} G_0^{-1} + (\partial_t G_0) G_0^{-1} ;
\]

\[
\mathcal{M} \rightarrow \mathcal{M}' = G_0 \mathcal{M} G_0^{-1} + (\partial_t G_0) G_0^{-1} ;
\]

\[
(\partial_x \Xi') = \mathcal{L}' \Xi' ; \quad (\partial_t \Xi') = \mathcal{M}' \Xi' ;
\]

\[
(\partial_t \mathcal{L}') - (\partial_t \mathcal{M}') + [\mathcal{L}', \mathcal{M}'] = 0 ;
\]

provided that:

\[
(\partial_t ((\partial_x G_0) G_0^{-1}) - (\partial_x ((\partial_t G_0) G_0^{-1} + [(\partial_x G_0) G_0^{-1}, (\partial_t G_0) G_0^{-1}]_\mathcal{L}) = 0 .
\]

On the condition of the general Maurer-Cartan relation \( (3.28) \), we can conclude for a whole set of equivalent Lax matrices \( (3.23-3.26) \) and their zero-curvature relations \( (3.27) \). However, it is in general even possible to transform the spatial Lax matrix \( \mathcal{L}(x,t) \) with a suitable gauge matrix \( G(x,t) \) to a diagonal form \( \mathcal{L}_\lambda(x,t) = \mathcal{H}_\lambda \lambda_\lambda^{(\mu)}(x,t) \) constrained to the commuting Cartan sub-algebra within a Cartan-Weyl basis of a Lie algebra. The diagonal form \( \mathcal{L}_\lambda(x,t) \) of the Lax matrix \( \mathcal{L}(x,t) \) has been used in section 3.1 to derive a general solution of the zero-curvature condition for the corresponding set of physical, nonlinear equations. In the following we construct the equations for the gauge matrix \( G(x,t) \) and its algebra \( \mathfrak{g}' \) with ‘rotation’ angles \( \vartheta_j(x,t) \) which transform the general Lax matrix \( \mathcal{L}(x,t) \) to a diagonal form \( \mathcal{L}_\lambda(x,t) = \mathcal{H}_\lambda \lambda_\lambda^{(\mu)}(x,t) \)

\[
\mathcal{L}_\lambda(x,t) = G(x,t) \mathcal{L}(x,t) G_0^{-1}(x,t) + (\partial_x G(x,t)) G_0^{-1}(x,t) ;
\]

\[
g(x,t) = \mathfrak{g}' \vartheta_j(x,t) ; \quad G(x,t) = \exp \{ g(x,t) \} .
\]

In order to simplify the Lie group current \( (\partial_x G(x,t)) G_0^{-1}(x,t) \) in \( (3.29) \), we apply relation \( (3.31) \) which allows to resolve the current into the action of the \( \mathfrak{d} \)-operator \( [g(x,t), \ldots]_\mathfrak{d} \) onto \( \partial g(x,t) / \partial x \) with the function \( (e^x - 1) / x \)

\[
(\partial_x G(x,t)) G_0^{-1}(x,t) = \left( \partial_x \exp \{ g(x,t) \} \right) \exp \{- g(x,t) \} = \int_0^1 dv \exp \{ v g(x,t) \} \frac{\partial g(x,t)}{\partial x} \exp \{- v g(x,t) \} = \int_0^1 dv \exp \{ v [g(x,t), \ldots]_\mathfrak{d} \} \frac{\partial g(x,t)}{\partial x}
\]

\[
= \left( \exp \{ [g(x,t), \ldots]_\mathfrak{d} \} - 1 \right) \frac{\partial g(x,t)}{\partial x} .
\]
After insertion of (3.31) into (5.29), we achieve eq. (3.32) for the transformation of \( \mathcal{L}(x, t) \rightarrow \mathcal{L}(\hat{t}, x, t) \) and \( \partial g(x, t) / \partial x \). Under the assumption of an invertible function (3.31) for the \( \mathfrak{g} \)-operator \( \mathfrak{g}(x, t), \ldots \)\( - \), one can finally derive a first order differential equation (3.33) for the Lie algebra \( \mathfrak{g}(x, t) = \mathfrak{g}^j \partial_j(x, t) \) of the gauge matrix \( \mathfrak{g}(x, t) \) so that the general spatial Lax matrix \( \mathcal{L}(x, t) \) can be converted to a diagonal form \( \mathcal{L}(\hat{t}, x, t) = \hat{H}^i \lambda_i^{(\hat{t})}(x, t) \) with a suitable angular dependence \( \theta_j(x, t) \) within the gauge transformation

\[
\mathcal{L}(\hat{t}, x, t) = \left( \exp \left\{ \frac{[\mathfrak{g}(x, t), \ldots]}{\partial x} \right\} \mathcal{L}(x, t) \right) + \left( \frac{\exp \left\{ \left[ \mathfrak{g}(x, t), \ldots \right] \right\} - 1}{\mathfrak{g}(x, t), \ldots} \frac{\partial g(x, t)}{\partial x} \right); \quad (3.32)
\]

\[
\frac{\partial g(x, t)}{\partial x} = \left[ \left( \frac{\exp \left\{ \left[ \mathfrak{g}(x, t), \ldots \right] \right\} - 1}{\mathfrak{g}(x, t), \ldots} \right)^{-1} \mathcal{L}(\hat{t}, x, t) - \left( \exp \left\{ \left[ \mathfrak{g}(x, t), \ldots \right] \right\} \mathcal{L}(x, t) \right) \right]; \quad (3.33)
\]

\[
e^{\frac{x - t}{x}} - 1 \rightarrow \left( \frac{e^{\frac{x - t}{x}} - 1}{x} \right)^{-1}. \quad (3.34)
\]

4 Determination and independence of conserved quantities

4.1 Calculation of conserved quantities from the Lax pair

The Lax pair \( \mathcal{L}(x, t), \mathcal{M}(x, t) \) and its zero-curvature condition, which specifies the nonlinear equations of physical fields, is accompanied by conserved, time-independent quantities. Let us consider the matrix \( \mathfrak{T}(x; t; k) \) (4.1) or the monodromy matrix \( \mathfrak{T}(t; k) \) (4.2) with periodic, spatial boundary conditions on a circle \( x \in [0, 2\pi) \)

\[
\mathfrak{T}(x; t; k) = \hat{\exp} \left\{ \int_0^x d\xi \mathcal{L}(\xi, t; k) \right\}; \quad (4.1)
\]

\[
\mathfrak{T}(t; k) = \hat{\exp} \left\{ \int_0^{2\pi} d\xi \mathcal{L}(\xi, t; k) \right\}, \quad (4.2)
\]

then the supposition, that all fields are periodic in \( x \) with period \( 2\pi \), implies that traces of powers of the monodromy matrix generate conserved quantities \( C^{(n)}(k) = C^{(n)}(t; k) \) independent of time

\[
C^{(n)}(t; k) = \text{Tr}[ \left( \mathfrak{T}(t; k) \right)^n ]. \quad (4.3)
\]

In order to attest this statement, we outline the defining, spatial ordering (4.4) of the exponential with generator \( \mathcal{L}(\xi, t; k) \) of the spatial Lax matrix

\[
\mathfrak{T}(t; k) = \hat{\exp} \left\{ \int_0^{2\pi} d\xi \mathcal{L}(\xi, t; k) \right\} = \exp \left\{ \Delta x L(\xi = 2\pi, t; k) \right\} \exp \left\{ \Delta x L(\xi = 2\pi - \Delta x, t; k) \right\} \ldots \times \quad (4.4)
\]

\[
\times \ldots \exp \left\{ \Delta x L(\xi = 2\Delta x, t; k) \right\} \exp \left\{ \Delta x L(\xi = \Delta x, t; k) \right\}.
\]

The time-like derivative \( (\partial_t \mathfrak{T}(t; k)) \) (4.5) of the monodromy matrix (4.2) involves the product rule with \( (\partial_t \mathcal{L}(y, t; k)) \) according to the spatial ordering of the exponentials (4.4). After substitution of \( (\partial_y \mathcal{L}(y, t; k)) \) by the zero-curvature condition, the integrand reduces to a total, spatial derivative \( \partial_y(\ldots) \) (4.5) within the integration boundaries \( y \in [0, 2\pi) \) of a circle. Hence, we can perform the spatial \( dy \)-integration along the circle and acquire a commutator (4.7) between the time-like Lax matrix \( \mathfrak{M}(x = 0; t, k) \) at the origin and the monodromy matrix \( \mathfrak{T}(t; k) \), due to the presupposed periodicity (4.6) on a spatial circle

\[
(\partial_t \mathfrak{T}(t; k)) = \int_0^{2\pi} dy \exp \left\{ \int_y^{2\pi} d\xi_2 \mathcal{L}(\xi_2, t; k) \right\} (\partial_y \mathcal{L}(y, t; k)) \exp \left\{ \int_0^y d\xi_1 \mathcal{L}(\xi_1, t; k) \right\} \quad (4.5)
\]
4.2 Involution of conserved quantities and the classical $r$-matrix

\[ \begin{align*}
&= \int_0^{2\pi} dy \exp \left\{ \int_y^{2\pi} d\xi_2 \mathfrak{L}(\xi_2, t; k) \right\} \left[ (\partial_y \mathcal{M}(y, t; k)) + [\mathcal{M}(y, t; k), \mathfrak{L}(y, t; k)] \right] \times \\
&\times \exp \left\{ \int_0^y d\xi_1 \mathfrak{L}(\xi_1, t; k) \right\} \\
&= \int_0^{2\pi} dy \partial_y \left( \exp \left\{ \int_y^{2\pi} d\xi_2 \mathfrak{L}(\xi_2, t; k) \right\} \mathcal{M}(y, t; k) \exp \left\{ \int_0^y d\xi_1 \mathfrak{L}(\xi_1, t; k) \right\} \right|_{y=0}^{y=2\pi} \\
&= \mathcal{M}(x = 2\pi, t; k) \mathfrak{I}(t; k) - \mathfrak{I}(t; k) \mathcal{M}(x = 0, t; k) ; \\
&\quad \Rightarrow \mathcal{M}(x = 0, t; k) , \mathfrak{I}(t; k) \right| . \tag{4.6}
\end{align*} \]

The conversion (4.7) of the time-like derivative ($\partial_t \mathfrak{I}(t; k)$) to a commutator allows to demonstrate the time independence of traces of arbitrary powers of the monodromy matrix $\mathfrak{I}(t; k)$ (4.2)

\[ \partial_t \text{Tr}[ (\mathfrak{I}(t; k))^n ] = n \text{Tr}[ (\partial_t \mathfrak{I}(t; k)) (\mathfrak{I}(t; k))^{n-1} ] = n \text{Tr}[ [\mathcal{M}(x = 0, t; k), \mathfrak{I}(t; k)] , (\mathfrak{I}(t; k))^{n-1} ] = 0 ; \\
\partial_t C^{(n)}(t; k) = \partial_t \text{Tr}[ (\mathfrak{I}(t; k))^n ] \equiv 0 ; \quad \Rightarrow C^{(n)}(t; k) = C^{(n)}(k) . \tag{4.8} \]

Therefore, the general time independence of $C^{(n)}(t; k) = \text{Tr}[ (\mathfrak{I}(t; k))^n ]$ is confirmed for arbitrary powers $n \in \mathbb{N}$ so that the $C^{(n)}(k) = C^{(n)}(t; k)$ (4.8) have to be regarded as the conserved quantities within a Liouville integrability of the corresponding nonlinear equations for the physical fields. Instead of the label '$n$' for powers of $\mathfrak{I}(t; k)$ (4.2), one can also conduct a power series expansion with the spectral parameters $k_j$ in order to generate conserved quantities from (orthogonal) polynomials of $k_j$. Note that in the described cases of the (1+1) GP-equations for the attractive and repulsive interactions with the $su(2)$ and $sp(2, \mathbb{R})$ algebra, respectively,

\[ \mathfrak{L}_{su(2)}(Q, P; k) = \begin{pmatrix} -i k & -\psi^* \\ \psi & i k \end{pmatrix} ; \quad (k \in \mathbb{R}) ; \quad \mathfrak{L}_{sp(2, \mathbb{R})}(Q, P; k) = \begin{pmatrix} -k & \psi^* \\ \psi & k \end{pmatrix} ; \quad (k \in \mathbb{R}) . \tag{4.9} \]

the spatial Lax matrices essentially stay unaltered as one introduces the external potential $V(x,t)$.

In order to prove the independence of the derived, conserved quantities $C^{(n)}(t; k)$ of previous section 4.1 one has to verify the involution from the Poisson brackets of the canonical fields (4.10). These have to be taken into account because the conserved quantities $C^{(n)}(t; k)$ are not affected by the inclusion of an external potential $V(x,t)$ into the GP-equations with a solely attractive ('$su(2)$') or repulsive ('$sp(2, \mathbb{R})$') interaction. Therefore, the independence of conserved quantities $C^{(n)}(t; k)$ can be directly transferred from the well-known case of the integrable, (1+1) GP-equations without any external potential to the more general GP-types with an arbitrary external potential $V(x,t)$.

4.2 Involution of conserved quantities and the classical $r$-matrix
\[
\frac{\partial}{\partial t_{n_1}} \ldots = \{C^{(n_1)}(t; k), \ldots\} ;
\]
\[
0 = \{C^{(n_1)}(t; k), \{C^{(n_2)}(t; k), \ldots\}\} + \n\]
\[
\{C^{(n_2)}(t; k), \ldots, C^{(n_1)}(t; k)\} + \ldots, \{C^{(n_1)}(t; k), \{C^{(n_2)}(t; k)\}\} ;
\]
\[
\iff 0 = \left(\frac{\partial}{\partial t_{n_1}} \ldots - \frac{\partial}{\partial t_{n_2}} \frac{\partial}{\partial t_{n_1}} \ldots\right) - \{C^{(n_1)}(t; k), C^{(n_2)}(t; k)\}, \ldots\right) .
\]

The involution of traces of powers of the monodromy matrix
\[
\{C^{(n_1)}(t; k), C^{(n_2)}(t; k)\} = 0 ,
\]
is usually investigated by the so-called $t$-matrix approach where one assumes the validity of a 'fundamental Poisson bracket relation' (4.15) between tensor products $L_1(x_1, t; k_1)$, $L_2(x_2, t; k_2)$ of spatial Lax matrices (cf. appendix A in [3] and chap. 2.5 in [10])
\[
\{L_1(\xi_1, t; k_1) \otimes L_2(\xi_2, t; k_2)\} = \left[r_{12}(k_1, k_2; t), L_1(x_1, t; k_1) + L_2(x_2, t; k_2)\right] - \delta(\xi_1 - \xi_2) ;
\]
\[
r_{12}(k_1, k_2; t) = -r_{21}(k_2, k_1; t) .
\]
The Poisson bracket $\{L_1(\xi_1, t; k_1) \otimes L_2(\xi_2, t; k_2)\}$ (4.15) is replaced by the commutator with the $r_{12}(k_1, k_2; t)$-matrix where we further require the 'ultralocal' form with the spatial delta function $\delta(\xi_1 - \xi_2)$. (The 'ultralocal' condition is in general obtained from spatial Lax matrices which only depend on the physical fields $\Omega_\alpha(x, t)$, $\mathcal{P}_\alpha(x, t)$ without any derivatives of these. This condition is hence fulfilled for the $(1+1)$ GP-equations.) As we consider more general types of $\Sigma(x, y; t; k)$ matrices (4.16) with initial coordinate $y$ and end point $x$ for the monodromy matrix (4.17) instead of $\Sigma(x; t; k)$ (4.11)
\[
\Sigma(x, y; t; k) = \exp\left\{\int_y^x d\xi \ L(\xi, t; k)\right\} ,
\]
we can start out from the general, tensorial Poisson bracket relation (4.17) whose right-hand side results from the validity of the Leibniz product rule for the Poisson bracket operation $\{\ldots \otimes \ldots\}$
\[
\{\Sigma_1(x_1, y_1; t; k_1) \otimes \Sigma_2(x_2, y_2; t; k_2)\} = \int_{y_1}^{x_1} d\xi_1 \int_{y_2}^{x_2} d\xi_2 \left[\Sigma_1(x_1, \xi_1; t; k_1) \Sigma_2(x_2, \xi_2; t; k_2) \times \left\{\Sigma_1(\xi_1, t; k_1) \otimes \Sigma_2(\xi_2, t; k_2)\right\}\right] \Sigma_1(\xi_1, y_1; t; k_1) \Sigma_2(\xi_2, y_2; t; k_2) .
\]

In appendix [12] we demonstrate according to Ref. [10] how to achieve the involution $\{C^{(n_1)}(t; k), C^{(n_2)}(t; k)\} = 0$ from the generally valid relation (4.17) of extended monodromy matrices $\Sigma(x, y; t; k)$ under the special assumption of the 'fundamental Poisson bracket relation' (4.15) with a spatially constant $r_{12}(k_1, k_2; t)$-matrix. Since the spatial Lax matrices $L_{su(2)}(Q, P; k)$, $L_{sp(2, \mathbb{R})}(Q, P; k)$ of the integrable, $(1+1)$ GP-equations do not change under the inclusion of an arbitrary external potential $V(x, t)$ (apart from a possibly chosen spacetime dependence of the spectral parameter $k \to k(x, t)$), the whole derivation of the involution of conserved quantities can be directly conveyed from the case without an external potential to the case with an arbitrary potential $V(x, t)$.

The given derivation of the involution property (4.10) of conserved quantities $C^{(n)}(t; k)$ in appendix [14] depends on the assumed fundamental Poisson bracket relation (4.15) with a spatially constant $r_{12}(k_1, k_2; t)$-matrix. In the following we attain more general statements as we begin from relation (4.17), caused by the
4.2 Involutions of conserved quantities and the classical \( r \)-matrix

Leibniz product rule of the tensorial Poisson brackets, and apply the gauge invariance of the Lax matrices which allows to transform to the diagonal, commuting Cartan sub-algebra. Further transformations of above relation \( 4.17 \) rely on simplification of the tensorial Poisson bracket \( \{ \ldots \circ \ldots \} \) of the spatial Lax matrices \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \). As we suppose a closed Lie algebra for \( \mathcal{L}(x, t; k) \), we can diagonalize or parametrize to \( \Lambda(x, t; k) \) (of the commuting Cartan sub-algebra) with the invertible 'eigenvector' matrix \( \Upsilon(x, t; k) \) which also contains the ladder operators \( \hat{E}_n^\pm \). This kind of parameters is taken within each part of the tensor product space \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \) of the spatial Lax matrix \( \mathcal{L}(x, t; k) \)

\[
\mathcal{L}(x, t; k) = \Upsilon(x, t; k) \Lambda(x, t; k) \Upsilon^{-1}(x, t; k); \quad \Lambda(x, t; k) = \hat{H}^i \lambda^j_i(x, t; k); \\
\mathcal{L}_1(x_1, t; k_1) = \Upsilon_1(x_1, t; k_1) \Lambda_1(x_1, t; k_1) \Upsilon_1^{-1}(x_1, t; k_1); \\
\mathcal{L}_2(x_2, t; k_2) = \Upsilon_2(x_2, t; k_2) \Lambda_2(x_2, t; k_2) \Upsilon_2^{-1}(x_2, t; k_2).
\]

(4.18)

(4.19)

(4.20)

On the condition of the "ultralocal" case of spatial Lax matrices, one can transfer the tensorial Poisson bracket \( \{ \ldots \circ \ldots \} \) of Lax matrices \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \) \( 4.19 \) to terms with their eigenvalues \( \Lambda_1(x_1, t; k_1) \), \( \Lambda_2(x_2, t; k_2) \) and 'eigenvector' matrices \( \Upsilon_1(x_1, t; k_1) \), \( \Upsilon_2(x_2, t; k_2) \) and a further part consisting of general spacetime dependent \( r_{12}(x_1, k_1; x_2, k_2) \), \( r_{21}(x_2, k_2; x_1, k_1) \)-matrices within commutators of \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \), respectively. The \( r_{12}(x_1, k_1; x_2, k_2) \), \( r_{21}(x_2, k_2; x_1, k_1) \)-matrices \( 4.22 \) are composed of further parts \( \Omega_{12} \) \( 4.23 \) which contain terms with the Poisson brackets among the eigenvalue matrices \( \Upsilon_1(x_1, t; k_1) \), \( \Upsilon_2(x_2, t; k_2) \) and between the eigenoperator matrix \( \Upsilon_1(x_1, t; k_1) \) and the diagonal Cartan sub-algebra part \( \Lambda_2(x_2, t; k_2) \). In summary the basic nine terms in \( 4.21 \) are achieved by the Leibniz product rule of the tensorial Poisson bracket of canonical fields and can be grouped into a first part with the Poisson bracket of the eigenvalues \( \{ \Lambda_1(x_1, t; k_1) \circ \cdots \circ \Lambda_2(x_2, t; k_2) \} \) and a second part with commutators between the \( r_{12}(x_1, k_1; x_2, k_2) \)-, \( r_{21}(x_2, k_2; x_1, k_1) \)-matrices and the corresponding spatial Lax matrices \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \) (in the ultralocal case '\( \delta(x_1 - x_2) \')

\[
\{ \mathcal{L}_1(x_1, t; k_1) \circ \mathcal{L}_2(x_2, t; k_2) \} = \\
\quad \quad \Upsilon_1(x_1, t; k_1) \Upsilon_2(x_2, t; k_2) \{ \Lambda_1(x_1, t; k_1) \circ \Lambda_2(x_2, t; k_2) \} \Upsilon_1^{-1}(x_1, t; k_1) \Upsilon_2^{-1}(x_2, t; k_2) + \\
\quad \frac{1}{2} \left[ \left( r_{12}(x_1, k_1; x_2, k_2) \right), \mathcal{L}_1(x_1, t; k_1) \mathcal{L}_2(x_2, t; k_2) \right] \delta(x_1 - x_2) ; \\
\mathcal{R}_{12} = \frac{1}{2} \left[ \left( r_{12}(x_1, k_1; x_2, k_2) \right), \mathcal{L}_1(x_1, t; k_1) \mathcal{L}_2(x_2, t; k_2) \right] ; \\
\mathcal{O}_{12} = \mathcal{L}_2(x_2, t; k_2) \{ \mathcal{L}_1(x_1, t; k_1) \circ \Lambda_2(x_2, t; k_2) \} \Upsilon_1^{-1}(x_1, t; k_1) \Upsilon_2^{-1}(x_2, t; k_2) \mathcal{L}_2(x_2, t; k_2) .
\]

(4.21)

(4.22)

(4.23)

(4.24)

The (spatially constant !) \( r \)-matrix approach, given in appendix \( B \) only allows to conclude for the involution of the conserved quantities \( C^{(m)}(t; k) \), provided that the eigenvalues \( \Lambda_1(x_1, t; k_1) \), \( \Lambda_2(x_2, t; k_2) \) of the spatial Lax matrices are in involution (or their tensorial Poisson brackets vanish completely) \( 4.25 \). This can be accomplished for a symmetrical dependence on canonical fields \( \phi_\alpha(x, t) = (\Omega_\alpha(x, t) \cdot P_\alpha(x, t)) \) within the eigenvalues \( \lambda^j_i(\alpha; \beta) \mathcal{L}^{(\alpha)}(\Omega_\alpha(x, t) \cdot P_\alpha(x, t)) \) which may originate from symmetrically chosen parameters \( \phi_\alpha(x, t) = (\Omega_\alpha(x, t) \cdot P_\alpha(x, t)) \) within the original Lax matrix \( \mathcal{L}(\Omega_\alpha(x), \beta) \) of a closed algebra determined by a Cartan-Weyl basis of ladder operators (e. g. \( \phi_\alpha E^\alpha = \Omega_\alpha E^\alpha + P_\alpha E^\alpha \))

\[
\{ \Lambda_1(x_1, t; k_1) \circ \Lambda_2(x_2, t; k_2) \} = 0 ; \quad \implies \text{(symmetric dependence on canonical fields } \phi_\alpha = (\Omega_\alpha(x, t) \cdot P_\alpha(x, t)) \text{ as e.g. } \lambda^j_i(\alpha; \beta) = \sqrt{k_j k_j + \Omega_\alpha \cdot P_\alpha} .
\]

(4.25)
Under the assumption of a symmetrical dependence on the canonical fields \( \Omega_\alpha(x,t) \), \( \mathcal{P}_\alpha(x,t) \) (4.25), the general tensorial Poisson bracket (4.21) reduces to the commutator (4.26) between the \( r_{12}(x_1, k_1; x_2, k_2; t) \)-matrices and the spatial Lax matrices \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \)

\[
\implies \{ \mathcal{L}_1(x_1, t; k_1) \otimes \mathcal{L}_2(x_2, t; k_2) \} = (4.26)
\]

Relation (4.26) is similar to the already assumed fundamental Poisson bracket eq. (4.15); however, the given \( r_{12}(x_1, k_1; x_2, k_2; t) \)-, \( r_{21}(x_2, k_2; x_1, k_1; t) \)-matrices have a priori no special symmetries, neither anti-symmetric nor symmetric. In order to be applicable for the derivation in appendix B we transform by a commutator between the spatial Lax matrix \( \mathcal{L}_1(x_1, t; k_1) \), \( \mathcal{L}_2(x_2, t; k_2) \) and an additional symmetric matrix \( \mathcal{H}_{12} = \mathcal{H}_{21} \) (4.27) which retains eq. (4.26) invariant. This allows to assign the anti-symmetry to the transformed \( r_{12} = -\mathcal{H}_{12} \) matrix (4.30) so that the derivation of appendix B can be performed with the further restriction to a spatially constant \( r_{12}(k_1, k_2; t) = -r_{21}(k_2, k_1; t) \) matrix (4.32).

Apart from the involution of conserved quantities according to the so-called \( r_{12} \)-matrix approach in appendix B and 10, we also suggest a different proof which relies on a gauge transformation of the spatial Lax matrix \( \mathcal{L}(x, t; k) \) to a completely diagonal form \( \mathcal{L}_0(x, t; k) = \hat{\Lambda}(x, t; k) \) given in section 3.2 and (4.38 4.39). This results in vanishing \( r_{12}(-,+,+,pp;+) \)-matrices within the basic tensorial Poisson bracket relation (4.21 4.22) so that the Poisson bracket of the diagonal Cartan sub-algebra matrices \( \mathcal{L}_0,1(x, t; k) = \hat{\Lambda}_1(x, t; k) \), \( \mathcal{L}_0,2(x, t; k) = \hat{\Lambda}_2(x, t; k) \) only remain because the diagonalizing matrices \( \Lambda_1(x_2, t; k) \), \( \Lambda_2(x_1, t; k) \) reduce to tensorial unit matrices (4.35). We remark again that \( \mathcal{L}(x, t; k) \) and its gauge transformed, diagonal form \( \mathcal{L}_0(x, t; k) = \hat{\Lambda}(x, t; k) \) represent the same nonlinear equations of fields \( \phi_\alpha(x, t) = (\Omega_\alpha(x, t), \mathcal{P}_\alpha(x, t)) \) within the general \( sl(n, \mathbb{C}) \) algebra or within one of its sub-algebras as e. g. \( su(n) \)

\[
\text{if } \mathcal{L}(x, t; k) \rightarrow \mathcal{L}_0(x, t; k) = \hat{\Lambda}(x, t; k) \text{ by a gauge transformation (4.33)}
\]

\[
\implies r_{12}(\#1; \#2; t) \equiv 0 \quad \land \quad r_{21}(\#2; \#1; t) \equiv 0 ;
\]

\[
\implies \{ \mathcal{L}_1(x_1, t; k_1) \otimes \mathcal{L}_2(x_2, t; k_2) \} \implies (4.35)
\]

As we repeat to choose symmetrical dependences (4.36) on canonical fields \( \phi_\alpha(x, t) = (\Omega_\alpha(x, t), \mathcal{P}_\alpha(x, t)) \) with diagonal, traceless Cartan sub-algebra matrices \( \mathcal{L}_0(x, t; k) = \hat{\Lambda}_0(x, t; k) \) of \( \lambda_\alpha(\{\mathcal{L}_0(x, t; k)\} \equiv \mathcal{P}_\alpha(x, t) \) ! (4.36)
electrodynamic case, one can introduce a matrix-valued field strength tensor instead of the single zero-curvature condition in previous sections for (1+1) dimensions. In analogy to the physical fields with corresponding (N-1) spatial Lax matrix components

\[ L_\alpha(x, t) = L_\alpha^i(x, t) \]

of the diagonal Lax matrices \( L_{\mu,1}(Q_\alpha, P_\alpha; k) \) and \( L_{\mu,2}(Q_\alpha, P_\alpha; k) \). In section 5.2 it has already been exemplified that the gauge transformation (4.38), which constrains the spatial Lax matrix tensorial Poisson bracket (4.37) of the diagonal Lax matrices

\[ \{ L_{\mu,1}(x_1, t; k_1) \otimes L_{\mu,2}(x_2, t; k_2) \} = 0 \quad (4.37) \]

we resolve the involution of the conserved quantities following from (4.21) by the remaining, vanishing tensorial Poisson bracket (4.37) of the diagonal Lax matrices \( L_{\mu,1}(Q_\alpha, P_\alpha; k) \) and \( L_{\mu,2}(Q_\alpha, P_\alpha; k) \). In section 5.2 it has already been exemplified that the gauge transformation (4.38), which constrains the spatial Lax matrix to the Cartan sub-algebra \( L_{\mu}(x, t; k) \) to the Cartan-Weyl basis (5.2)

\[ \{ \xi(x, t) \} \text{magnetic field strength } T_\mu(x, t) \]

In comparison to previous sections 2-4, we assign to the time-like matrix potential \( A_\mu(x, t) \) and to the (N-1) spatial matrix potential components \( A^\nu(x, t) \), \( \xi(x, t) \), \( \xi(x, t) \) the spatial Lax ”vector” \( L_\nu(x, t) \) with corresponding (N-1) spatial Lax matrix components \( L_\nu^i(x, t) \).

\[ L_\nu^i(x, t) = L_\nu^i(x, t) \]

In analogy to the electromagnetic theory, the Lax matrices \( L_\nu^i(x, t) \) are termed as matrix-potentials \( A_\mu(x, t) \) which depend on canonical fields \( \phi_\alpha^i(x, t) = \{ Q_\alpha^i(x, t), P_\alpha^i(x, t) \} \) and spectral parameters \( k_\nu^i(x, t) \) within a Cartan-Weyl basis (5.2) of the general sl(n, C) algebra or a chosen sub-algebra as e.g. su(n)

\[ \frac{\partial g(x, t)}{\partial x} = \left( \left( \exp \left\{ \left[ \left[ g(x, t), \ldots \right] \right] \right\} - 1 \right) \right)^{-1} \left( \hat{H}_i \right)_k \left( \lambda^\nu L_\nu^i(x, t) , P_\alpha(x, t) \right) + \partial p \right) \]

5 Extension of the zero-curvature condition beyond (1+1)-dimensions

5.1 Determination of the nonlinear equations for the fields of relevant, physical observables

In previous sections we have emphasized the algebraic properties of Lax pairs in (1+1) dimensions and have also considered cases where one can obtain chaotic behaviour within the general, non-compact sl(n, C) algebra by separating into sub-algebra parts as e.g. su(n) or sp(n, R), etc.

\[ \mathfrak{g}(x, t) = \exp \{ g(x, t) \} \text{ gauge transformation should exist} ! \quad (4.39) \]
\[ \mathfrak{A}^{\mu} = \partial^\nu \mathfrak{A} - \partial^\nu \mathfrak{A}^\mu = \mathfrak{A}^{\mu} = [\mathfrak{A}^\mu, \mathfrak{A}^\nu]_{-} . \]  

By using the Maurer-Cartan structure equations for the gauge matrices \( \mathfrak{g}_0 \)

\[ (\partial^\mu (\partial^\nu \mathfrak{g}_0) \mathfrak{g}^{-1}_0) - (\partial^\nu (\partial^\mu \mathfrak{g}_0) \mathfrak{g}^{-1}_0) + [(\partial^\nu \mathfrak{g}_0) \mathfrak{g}^{-1}_0, (\partial^\mu \mathfrak{g}_0) \mathfrak{g}^{-1}_0]_{-} = 0 , \]

we can straightforwardly derive a gauge invariance of the zero-curvature eqs. (5.4,5.5) under the gauge transformation (5.7) for the matrix potentials \( \mathfrak{A}^\nu \to \mathfrak{A}^\nu \)

\[ \mathfrak{A}^\nu \to \mathfrak{A}^\nu + \mathfrak{g}_0 \mathfrak{A}^\nu \mathfrak{g}^{-1}_0 + \partial^\nu \mathfrak{g}_0 / \partial^\nu + \partial^\nu \mathfrak{g}_0 / \partial^\nu , \mathfrak{A}^\nu \mathfrak{g}^{-1}_0 ; \mathfrak{g}_0 \in \text{sub-group} \subseteq \text{SL}(n, \mathbb{C}) . \]

In later sections we have to perform a gauge transformation in order to prove the involution of conserved quantities. One has to take a gauge transformation for the scalar product \( \mathfrak{L}(\mathfrak{x}(\zeta), t) = (d \mathfrak{x}(\zeta) / d \zeta) \cdot \mathfrak{L}(\mathfrak{x}(\zeta), t) \) along chosen 'loops' \( \mathfrak{x}(\zeta = 0) = \mathfrak{x}(\zeta = 2\pi) \) to the Cartan sub-algebra elements \( \mathfrak{L}_\alpha(\mathfrak{x}(\zeta), t) = \mathfrak{H}^l \mathfrak{L}^{(\alpha)}(\mathfrak{x}(\zeta), t) \cdot (d \mathfrak{x}(\zeta) / d \zeta) \).

The nonlinear equations from (5.8) can be transferred to the structure constants \( i c^{jk}_l \) of the underlying algebra from the matrix potentials \( \mathfrak{A}^\mu(\mathfrak{x}, t) \) which are defined with the generators \( \mathfrak{g}^j \) and corresponding fields \( \Psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k) \) (5.9) as an ansatz for nonlinear equations. As we resolve relations (5.4,5.10) in terms of the generator basis \( \mathfrak{g}^j \) and structure constants \( i c^{jk}_l \) (5.9), we finally attain the set of nonlinear equations (5.11) with the ansatz of the \( \Psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k) \) fields which can also be related to the more fundamental fields \( \Omega^\nu_\alpha(\mathfrak{x}, t), \mathfrak{T}^\nu_\alpha(\mathfrak{x}, t) \) in eq. (5.2)

\[ [\mathfrak{g}^j, \mathfrak{g}^k]_{-} = i c^{jk}_l \mathfrak{g}^l ; \mathfrak{A}^\mu(\mathfrak{x}, t) = \mathfrak{g}^j \Psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k) ; \]

\[ 0 = \left( \mathfrak{g}^l \left( (\partial^\mu \Psi^l) - (\partial^\mu \Psi^l) \right) + [\mathfrak{g}^j, \mathfrak{g}^k]_{-} \Psi^\nu_\alpha \Psi^\mu_j \right) ; \]

\[ 0 = (\partial^\mu \psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k)) - (\partial^\nu \psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k)) + i c^{jk}_l \psi^\nu_\alpha(\mathfrak{Q}, \mathfrak{P}; k) \psi^\mu_j(\mathfrak{Q}, \mathfrak{P}; k) . \]

### 5.2 Conserved quantities of monodromy matrix paths with nontrivial homotopy of fields

In the following we consider a closed 'loop' \( \mathfrak{x}(\xi) = \mathfrak{x}_\xi, \xi \in [0, 2\pi) \) (5.12) within the group manifold of the chosen Lie algebra (\( \text{sl}(n, \mathbb{C}) \)) or a corresponding sub-group) for the matrix potentials \( \mathfrak{A}^\nu(\mathfrak{M}, \mathfrak{L}) \) where a reference point \( \mathfrak{x}_{\mathfrak{P}_0} \) defines the beginning and the end of the loop

\[ \mathfrak{x}(\xi) = \mathfrak{x}_\xi ; \xi \in [0, 2\pi) ; \mathfrak{x}(\xi = 0) = \mathfrak{x}_{\mathfrak{P}_0} ; \mathfrak{x}(\xi = 2\pi) = \mathfrak{x}_{\mathfrak{P}_0} . \]

It is of crucial importance that the fibre space, given by the N-1 spatial coordinates for the base space and by the mapping to the group manifold of the generators \( \mathfrak{A}^i = \mathfrak{L}^i \), does not allow a continuous contraction of the loop to a trivial, single point as e. g. the reference point \( \mathfrak{x}_{\mathfrak{P}_0} \). This supposition yields a straightforward derivation of conserved quantities along nontrivial, non-contractable loops in analogy to section 4.11. One begins with the matrix path from the reference point \( \mathfrak{x}_{\mathfrak{P}_0} \) to a point \( \mathfrak{x}(\xi) = \mathfrak{x}_\xi \), which is specified by a parametrization \( \mathfrak{x}(\zeta), (\zeta \in [0, \xi), \xi \in [0, 2\pi)) \), and performs a spatial ordering of exponentials \( \exp \{ \Delta \zeta (d \mathfrak{x}_\zeta / d \zeta) \cdot \mathfrak{L}(\mathfrak{x}_\zeta, t; k) \} \) along a part of the loop

\[ \mathfrak{L}(\mathfrak{x}_\xi, \mathfrak{x}_{\mathfrak{P}_0}; t; k) = \exp \left\{ \int_0^\xi d \zeta \frac{d \mathfrak{x}_\zeta}{d \zeta} \cdot \mathfrak{L}(\mathfrak{x}_\zeta, t; k) \right\} . \]
In correspondence to section 4.1, we take the time derivative of (5.13) and have to regard the product rule following from the spatial ordering of the exponential step operators along the part $\zeta \in [0, \xi)$ of the loop $\vec{x}_\zeta = \vec{x}(\zeta)$

$$\partial_t \mathcal{E}(\vec{x}_\xi, \vec{x}_{P_0}; t; k) = \int_0^\xi d\zeta \exp \left\{ \int_\zeta^\xi d\zeta_1 \frac{d\vec{x}_{\zeta_1}}{d\zeta_1} \cdot \vec{L}(\vec{x}_{\zeta_1}, t; k) \right\} \times$$

$$\times \left( \frac{d\vec{x}_\xi}{d\zeta} \cdot (\partial_t \vec{L}(\vec{x}_\xi, t; k)) \right) \exp \left\{ \int_0^\xi d\zeta_2 \frac{d\vec{x}_{\zeta_2}}{d\zeta_2} \cdot \vec{L}(\vec{x}_{\zeta_2}, t; k) \right\}.$$  

(5.14)

We can replace the scalar product $(d\vec{x}_\zeta / d\zeta) \cdot (\partial_t \vec{L}(\vec{x}_\xi, t; k))$ in (5.14) by the zero-curvature relations (5.15,5.16)

$$\partial_t \vec{L} - \partial_\zeta \mathcal{M} = [\mathcal{M}, \vec{L}]_\zeta \implies \partial_t \vec{L} = \partial_\zeta \mathcal{M} + [\mathcal{M}, \vec{L}]_\zeta;$$

(5.15)

$$\frac{d\vec{x}_\xi}{d\zeta} \cdot \partial_t \vec{L} = \frac{d\vec{x}_\xi}{d\zeta} \cdot \partial_\zeta \mathcal{M} + \left[ \mathcal{M}, \frac{d\vec{x}_\xi}{d\zeta} \cdot \vec{L} \right]_\zeta,$$

(5.16)

in order to transform the integrand of (5.14) to a total derivative of the loop parameter $\zeta \in [0, \xi)$ (cf. section 4.1 eqs. (4.5,4.7))

$$\partial_t \mathcal{E}(\vec{x}_\xi, \vec{x}_{P_0}; t; k) = \int_0^\xi d\zeta \exp \left\{ \int_\zeta^\xi d\zeta_1 \frac{d\vec{x}_{\zeta_1}}{d\zeta_1} \cdot \vec{L}(\vec{x}_{\zeta_1}, t; k) \right\} \times$$

$$\times \left( \frac{d\vec{x}_\xi}{d\zeta} \cdot \partial_\zeta \mathcal{M} + \left[ \mathcal{M}, \frac{d\vec{x}_\xi}{d\zeta} \cdot \vec{L} \right]_\zeta \right) \exp \left\{ \int_0^\xi d\zeta_2 \frac{d\vec{x}_{\zeta_2}}{d\zeta_2} \cdot \vec{L}(\vec{x}_{\zeta_2}, t; k) \right\}$$

$$= \mathcal{M}(\vec{x}_\xi, t; k) \mathcal{E}(\vec{x}_\xi, \vec{x}_{P_0}; t; k) - \mathcal{E}(\vec{x}_\xi, \vec{x}_{P_0}; t; k) \mathcal{M}(\vec{x}_{P_0}, t; k).$$

(5.17)

As one takes $\xi = 2\pi$ so that $\vec{x}(\xi = 2\pi) = \vec{x}(\xi = 0) = \vec{x}_{P_0}$ are the beginning and the end of a closed loop '○' in the base space for a nontrivial, (non-contractable) homotopic mapping of the fields, one transforms the time derivative of monodromy matrix (5.19) to a commutator which finally gives the conserved quantities (5.20) of traces of powers of the monodromy matrix $\mathcal{E}(\circ, t; k)$

$$\mathcal{E}(\vec{x}(\xi = 2\pi), \vec{x}_{P_0}; t; k) = \mathcal{E}(\circ, t; k);$$

(5.18)

$$\partial_t \mathcal{E}(\circ, t; k) = [\mathcal{M}(\vec{x}_{P_0}, t; k), \mathcal{E}(\circ, t; k)]_\zeta;$$

(5.19)

$$\partial_t \text{Tr} [\mathcal{E}^n(\circ, t; k)] = (\partial_t C^{(n)}(\circ, t; k)) = n \text{ Tr} \left[ (\partial_t \mathcal{E}(\circ, t; k) ) \mathcal{E}^{n-1}(\circ, t; k) \right]$$

(5.20)

$$= n \text{ Tr} \left[ [\mathcal{M}(\vec{x}_{P_0}, t; k), \mathcal{E}(\circ, t; k)]_\zeta \mathcal{E}^{n-1}(\circ, t; k) \right] = 0.$$

5.3 Involution of conserved quantities and the classical $\mathfrak{t}$-matrix in arbitrary spacetime dimensions

As we have generalized the derivation of conserved quantities from a circle $x \in [0, 2\pi)$ within (1+1) dimensions to a closed loop $\vec{x}(\xi, \xi \in [0, 2\pi)$ within a (N-1) dimensional base space for a nontrivial, homotopic mapping of
a Lie group $SL(n, \mathbb{C})$ (or a sub-group as $SU(n)$, etc.), we can also extend the involution properties of section 4.2 for the (1+1) dimensional case to those of the ((N-1)+1) spacetime. One can repeat the calculations of section 4.2 by replacing the spatial Lax matrix $\mathcal{L}(x, t)$ by the scalar product $\tilde{\mathcal{L}}(\tilde{x}_\zeta, t; k) = \frac{d\tilde{x}_\zeta}{d\zeta} \cdot \mathcal{L}(\tilde{x}_\zeta, t; k)$ where the 'loop' parameter $\zeta \in [0, 2\pi)$ substitutes the spatial coordinate 'x' or 'ξ' of the circle for the (1+1) dimensional case in section 4.2.

Apply $\tilde{\mathcal{L}}(\tilde{x}_\zeta, t; k) = \frac{d\tilde{x}_\zeta}{d\zeta} \cdot \mathcal{L}(\tilde{x}_\zeta, t; k)$ instead of $\mathcal{L}(x, t; k)$ or $\mathcal{L}(\xi, t; k)$ in section 4.2 and repeat calculation of involution properties.

Diagonalize by a gauge transformation:

$$\tilde{\mathcal{L}}(\tilde{x}_\zeta, t; k) \rightarrow \tilde{\mathcal{L}}_R(\tilde{x}_\zeta, t; k) = \hat{H}^i \chi_i(\tilde{x}_\zeta, t; k) \cdot \frac{d\tilde{x}_\zeta}{d\zeta}. \tag{5.22}$$

Similarly, the classical τ-matrix approach can be conveyed from appendix B of the (1+1) dimensional case to ((N-1)+1) dimensions with the replacement (5.21) under restriction of spatially constant matrices $\tau(k_i^j, k_j^i; t)$.

However, we emphasize again that the analogous, corresponding transformations in place of the (1+1) dimensional case of section 4.2 and appendix B are only hold for non-contractable, nontrivial, homotopic mappings from the loop within the (N-1) dimensional base space to the chosen Lie group manifold as $SL(n, \mathbb{C})$ or one of its sub-groups.

6 Summary and conclusion

6.1 Lax pairs and chaotic behaviour of (1+1) GP-type equations

This article has been initiated by the notion whether any Lax pair construction can only lead to a completely integrable behaviour. As we have verified in sections 2.1 and 2.2 for Lax pairs of the (1+1) GP-equations as generators of the $sl(2, \mathbb{C})$ algebra, one can even determine Lax pairs for arbitrary external potentials $V(x, t)$ without changing the spatial Lax matrix component $\mathcal{L}(x, t)$. Since the conserved quantities and their involution only depend on the exponential step operators with the spatial Lax matrix $\mathcal{L}(\xi, t)$, $\xi \in [0, 2\pi)$, one can directly conclude for a Liouville integrability as in the cases without an external potential, either from a spatially constant $\tau_{12}$-matrix approach according to appendix B or from a gauge transformation to the diagonal, commuting Cartan sub-algebra elements as the eigenvalues of $\mathcal{L}(\xi, t)$ within the tensorial Poisson bracket relation (A symmetric dependence of the eigenvalues on the canonical fields within the Poisson bracket has to be presupposed.). As we reduce the generators of $\mathcal{L}(x, t)$, $\mathfrak{m}(x, t)$, either to the sub-algebra $su(2)$ or to $sl(2, \mathbb{R})$ (of the most general, nontrivial $sl(2, \mathbb{C})$ algebra, cf. appendix A) for an attractive or repulsive interaction, the hermitian property of the prevailing Hamiltonian, following from $su(2)$ or $sl(2, \mathbb{R})$ Lax pairs, prevents any chaotic behaviour, due to the chosen compactness with a spatial circle $x \in [0, 2\pi)$. However, as we combine the $\psi_{su(2)}(x, t)$ and $\psi_{sp(2,\mathbb{R})}(x, t)$ fields of the two integrable, (1+1) GP-equations with attractive and repulsive interaction to the complex-valued parameter fields within the most general, non-compact $sl(2, \mathbb{C})$ algebra, probability or density of the $\psi_{su(2)}(x, t)$ and $\psi_{sp(2,\mathbb{R})}(x, t)$ fields can flow and change between the coupled GP-equations which separately contain incoherent, non-hermitian terms for a chaotic behaviour. This chaotic behaviour from Lax pair construction is even possible for $sl(n > 2, \mathbb{C})$ algebras where one has to select a sub-algebra for a compact sub-group as $SU(n) \subset SL(n, \mathbb{C})$ so that coupled nonlinear equations of physical fields are also composed of incoherent terms, giving rise to unlimited increase of the $\psi_{su(n)}(x, t)$ fields and corresponding chaotic behaviour.
6.2 Lax pair construction in arbitrary spacetime

The given construction of Lax pairs for the (1+1) GP-equations as generators of sl(n, C) (or of a sub-algebra as su(n)) straightforwardly generalize to arbitrary spacetime dimensions. However, we emphasize again that this extension beyond (1+1) dimensions necessarily has to involve a nontrivial homotopic mapping from the loop within the (N-1) dimensional base space to the considered group manifold, following from the Lax pair generators. In absence of a nontrivial homotopy, it is possible to contract the loop within the fibre space to trivial point mappings so that the construction of the conserved quantities and their involution becomes trivial and meaningless. Therefore, Lax pair constructions for a Liouville integrability or a possible chaotic behaviour beyond (1+1) dimensions have to be accompanied by an investigation for a nontrivial homotopy of the underlying fibre space \([116]\).

A Reduction of gl(n, C) to sl(n, C) Lax pairs by separating the trivial trace parts

In this part \([A]\) of the appendix we assume that Lax matrices \(L(x,t), M(x,t)\) are not traceless and therefore belong to the gl(n, C) algebra as the most general case of \(n \times n\) matrices. The general Lax matrices \(L(x,t), M(x,t)\) are separated into diagonal unity parts \(\hat{1} \Delta L(x,t), \hat{1} \Delta M(x,t)\) and remaining traceless parts \(L_0(x,t), M_0(x,t)\) (A.1-A.2) of the sl(n, C) algebra \((n = N_2 = N_{\text{in}})\). This trace separation is also performed for the initial matrix \(M_{\text{ini}}(x=0,t)\) at the coordinate origin within the solution of the zero-curvature relation for \(M(x,t) = M_0(x,t) + \hat{1} \Delta M(x,t)\) where explicit use is made for the trace splitting of \(L(x,t) = L_0(x,t) + \hat{1} \Delta L(x,t)\) with the 'a\(\hat{a}\)'-operator

\[
\begin{align*}
L(x,t) & = L_0(x,t) + \frac{1}{N_2} \cdot \text{Tr}[L(x,t)] = L_0(x,t) + \hat{1} \Delta L(x,t); \quad \left(\text{Tr}[L_0(x,t)] \equiv 0\right); \quad (A.1) \\
M(x,t) & = M_0(x,t) + \frac{1}{N_2} \cdot \text{Tr}[M(x,t)] = M_0(x,t) + \hat{1} \Delta M(x,t); \quad \left(\text{Tr}[M_0(x,t)] \equiv 0\right); \quad (A.2) \\
M_{\text{ini}}(x=0,t) & = M_{\text{ini}}^{(0)}(x=0,t) + \hat{1} \Delta M_{\text{ini}}(x=0,t); \quad \left(\text{Tr}[M_{\text{ini}}^{(0)}(x=0,t)] = 0\right); \quad (A.3) \\
M(x,t) & = M_0(x,t) + \hat{1} \Delta M(x,t) = \text{exp}\left\{ \int_0^x d \xi \left[ L_0(\xi,t) + \hat{1} \Delta L(\xi,t), \ldots \right] \right\} \left( M_{\text{ini}}^{(0)}(x=0,t) + \hat{1} \Delta M_{\text{ini}}(x=0,t) \right) + \\
& + \int_0^x dy \text{exp}\left\{ \int_y^x d \xi \left[ L_0(\xi,t) + \hat{1} \Delta L(\xi,t), \ldots \right] \right\} \left( \partial_t L_0(y,t) \right) + \hat{1} \left( \partial_t \Delta L(y,t) \right) \right\}. \quad (A.4)
\end{align*}
\]

As we regard the complete vanishing of the 'a\(\hat{a}\)'-operator part \(\hat{1} \Delta L(\xi,t), \ldots\) in (A.4), one can conclude for the separation of the solution of the zero-curvature relation into the two independent parts (A.5-A.6) where equation (A.5) only consists of the total traceless sl(n, C) generators \(L_0(x,t), M_0(x,t), M_{\text{ini}}^{(0)}(x=0,t)\), and where equation (A.6) separately has the remaining fields \(\Delta L(x,t), \Delta M(x,t), \Delta M_{\text{ini}}(x=0,t)\) from the diagonal unity part \(\hat{1}\)

\[
\begin{align*}
M_0(x,t) & = \text{exp}\left\{ \int_0^x d \xi \left[ L_0(\xi,t), \ldots \right] \right\} M_{\text{ini}}^{(0)}(x=0,t) + \\
& + \int_0^x dy \text{exp}\left\{ \int_y^x d \xi \left[ L_0(\xi,t), \ldots \right] \right\} \left( \partial_t L_0(y,t) \right); \quad (A.5) \\
\Delta M(x,t) & = \Delta M_{\text{ini}}(x=0,t) + \int_0^x dy \left( \partial_t \Delta L(y,t) \right) \quad (A.6)
\end{align*}
\]
\[ \Delta \mathcal{M}_{ini}(x = 0, t) + \partial_t \left( \int_0^x dy \, \text{Tr} \left[ \mathcal{L}(y, t) \right] / N_{\mathcal{L}} \right) . \]

We can also verify from the zero-curvature relation (A.7) the separation property into traceless \( \mathfrak{sl}(n, \mathbb{C}) \) matrices and diagonal unit parts. Since the diagonal unit parts \( \Delta \mathcal{L}(x, t), \Delta \mathcal{M}(x, t) \) only cause vanishing commutators within the general zero-curvature condition, the latter relation splits into the solely traceless part of \( \mathfrak{sl}(n, \mathbb{C}) \) generators \( \mathcal{L}_0(x, t), \mathcal{M}_0(x, t) \) and a simple part for the completely diagonal terms with the fields \( \Delta \mathcal{L}(x, t), \Delta \mathcal{M}(x, t) \)

\[ (\partial_t \mathcal{L}) - (\partial_x \mathcal{M}) + [\mathcal{L}, \mathcal{M}] = (\partial_t \mathcal{L}_0) - (\partial_x \mathcal{M}_0) + [\mathcal{L}_0, \mathcal{M}_0] + \frac{1}{N_{\mathcal{L}}} \left[ \partial_t \left( \text{Tr} \left[ \mathcal{L}(x, t) \right] \right) - \partial_x \partial_t \left( \int_0^x dy \, \text{Tr} \left[ \mathcal{L}(y, t) \right] \right) \right] = 0 ; \quad (N_{\mathcal{L}} = N_{\mathcal{M}}) . \]

Therefore, the purely traceless generators with \( \mathcal{L}_0(x, t) \) (and the derived or traceless chosen generators for \( \mathcal{M}_0(x, t) \)) can give rise to nontrivial, nonlinear equations within a relevant Lax pair construction.

### B Involution of monodromy matrices for spatially constant r-matrix

A standard proof of conserved quantities from the spatial Lax matrix \( \mathcal{L}(x, t) \subseteq \mathfrak{sl}(n, \mathbb{C}) \) begins with the assumption of the fundamental, tensorial Poisson bracket relation (B.1), having a delta function \( \delta(x_1 - x_2) \) for the 'ultralocal' case of fields. We note that this relation transforms the quadratic term of \( \mathcal{L}(x, t) \) to a linear part within a commutator of the so-called '\( r_{12} \)'-matrix which is supposed to be independent on the space coordinate \( x' \)

\[ \{ \mathcal{L}_1(x_1, t; k_1) \otimes \mathcal{L}_2(x_2, t; k_2) \} = [r_{12}(k_1, k_2; t), \mathcal{L}_1(x_1, t; k_1) + \mathcal{L}_2(x_2, t; k_2)] - \delta(x_1 - x_2) . \quad (B.1) \]

We consider the spatial evolution eqs. (B.3,B.4) for the initial coordinate \( y \) and end point \( x' \) of the defined type (B.2) of monodromy matrix \( \mathcal{T}(x, y; t; k) \) which follows from subsequent spatial ordering of exponential step operators with Lax matrix \( \mathcal{L}(\xi, t; k) \). As one applies the Leibniz and product rule for the tensorial Poisson bracket (B.5) of two monodromy matrices (B.2), we accomplish the fundamental Poisson bracket (B.1,B.6) within a double spatial integral \( \xi_1 \in [y_1, x_1], \xi_2 \in [y_2, x_2] \)

\[ \mathcal{T}(x, y; t; k) = \exp \left\{ \int_y^x d\xi \, \mathcal{L}(\xi, t; k) \right\} ; \quad (B.2) \]

\[ (\partial_x \mathcal{T}(x, y; t; k)) = \mathcal{L}(x, t; k) \mathcal{T}(x, y; t; k) ; \quad (B.3) \]

\[ (\partial_y \mathcal{T}(x, y; t; k)) = -\mathcal{T}(x, y; t; k) \mathcal{L}(y, t; k) ; \quad (B.4) \]

\[ \{ \mathcal{T}_1(x_1, y_1; t; k_1) \otimes \mathcal{T}_2(x_2, y_2; t; k_2) \} = \int_{y_1}^{x_1} d\xi_1 \int_{y_2}^{x_2} d\xi_2 \, \mathcal{T}_1(x_1, \xi_1; t; k_1) \mathcal{T}_2(x_2, \xi_2; t; k_2) \times \]

\[ \times \left\{ \mathcal{L}_1(\xi_1, t; k_1) \otimes \mathcal{L}_2(\xi_2, t; k_2) \right\} - \{ \mathcal{L}_1(\xi_1, t; k_1), \mathcal{L}_2(\xi_2, t; k_2) \} - \delta(\xi_1 - \xi_2) . \quad (B.5) \]

After substitution and insertion of (B.1,B.6,B.3,B.4) into (B.5), the tensorial Poisson bracket of monodromy matrices reduces to simple integrals (B.7,B.8) because the corresponding integrands only consist of total derivatives \( \delta(\xi_1 - \xi_2) (\partial_{\xi_1} + \partial_{\xi_2}) \) and \( \partial_{\xi_1} \), respectively

\[ \{ \mathcal{T}_1(x_1, y_1; t; k_1) \otimes \mathcal{T}_2(x_2, y_2; t; k_2) \} = \mathcal{T}_1(x_1, x_{\text{min}}; t; k_1) \mathcal{T}_2(x_2, x_{\text{min}}; t; k_2) \times \quad (B.7) \]
\[
\times \int_{y_{\text{max}}}^{x_{\text{min}}} d\xi_1 d\xi_2 \delta(\xi_1 - \xi_2) \left[ \mathcal{T}_1(x_{\text{min}}, \xi_1; t; k_1) \mathcal{T}_2(x_{\text{min}}, \xi_2; t; k_2) \right. \\
\times \mathcal{r}_{12}(k_1, k_2; t) \left( (\partial_{\xi_1} \mathcal{T}_1(\xi_1, y_{\text{max}}; t; k_1)) + (\partial_{\xi_2} \mathcal{T}_2(\xi_2, y_{\text{max}}; t; k_2)) \right) + \\
+ \left( (\partial_{\xi_1} \mathcal{T}_1(x_{\text{min}}, \xi_1; t; k_1)) + (\partial_{\xi_2} \mathcal{T}_2(x_{\text{min}}, \xi_2; t; k_2)) \right) \mathcal{r}_{12}(k_1, k_2; t) \\
\times \mathcal{T}_1(\xi_1, y_{\text{max}}; t; k_1) \mathcal{T}_2(\xi_2, y_{\text{max}}; t; k_2) \\
= \mathcal{T}_1(x_1, x_{\text{min}}; t; k_1) \mathcal{T}_2(x_2, x_{\text{min}}; t; k_2) \int_{y_{\text{max}}}^{x_{\text{min}}} d\xi_1 d\xi_2 \delta(\xi_1 - \xi_2) \left( \partial_{\xi_1} + \partial_{\xi_2} \right) \\
\times \left[ \mathcal{T}_1(x_{\text{min}}, \xi_1; t; k_1) \mathcal{T}_2(x_{\text{min}}, \xi_2; t; k_2) \mathcal{r}_{12}(k_1, k_2; t) \mathcal{T}_1(\xi_1, y_{\text{max}}; t; k_1) \mathcal{T}_2(\xi_2, y_{\text{max}}; t; k_2) \right] \\
\times \mathcal{T}_1(y_{\text{max}}, y_1; t; k_1) \mathcal{T}_2(y_{\text{max}}, y_2; t; k_2) \\
\{ \mathcal{T}_1(x_1, y_1; t; k_1) \otimes \mathcal{T}_2(x_2, y_2; t; k_2) \} =
\] (B.8)

\[
\times \partial_{\xi} \left( \mathcal{T}_1(x_{\text{min}}, \xi; t; k_1) \mathcal{T}_2(x_{\text{min}}, \xi; t; k_2) \mathcal{r}_{12}(k_1, k_2; t) \mathcal{T}_1(\xi, y_{\text{max}}; t; k_1) \mathcal{T}_2(\xi, y_{\text{max}}; t; k_2) \right) \\
\times \mathcal{T}_1(y_{\text{max}}, y_1; t; k_1) \mathcal{T}_2(y_{\text{max}}, y_2; t; k_2).
\]

After performing the spatial integration in (B.8), we achieve the tensorial Poisson bracket (B.9) of evoluton matrices with the remaining integration boundaries for the spatial \(\xi\) integration. Let us assume the integer relation \((n_1 \geq n_2)\) so that taking the integration boundaries transforms the right-hand side of (B.9) to a commutator-like relation (B.10) with the classical \(\mathcal{r}_{12}\)-matrix

\[
\{ \mathcal{T}_1(2\pi n_1, 0; t; k_1) \otimes \mathcal{T}_2(2\pi n_2, 0; t; k_2) \} =
\] (B.9)

\[
= \mathcal{T}_1(2\pi n_1, \xi; t; k_1) \mathcal{T}_2(2\pi n_2, \xi; t; k_2) \mathcal{r}_{12}(k_1, k_2; t) \mathcal{T}_1(\xi, 0; t; k_1) \mathcal{T}_2(\xi, 0; t; k_2) \] \(\xi=2\pi M\text{(n}_1\text{n}_2)\)

\[
\{ \mathcal{T}_1(2\pi n_1, 0; t; k_1) \otimes \mathcal{T}_2(2\pi n_2, 0; t; k_2) \} =
\] (B.10)

\[
= \mathcal{T}_1(2\pi n_1, 2\pi n_2; t; k_1) \mathcal{r}_{12}(k_1, k_2; t) \mathcal{T}_1(2\pi n_2, 0; t; k_1) \mathcal{T}_2(2\pi n_2, 0; t; k_2) + \\
- \mathcal{T}_1(2\pi n_1, 0; t; k_1) \mathcal{T}_2(2\pi n_2, 0; t; k_2) \mathcal{r}_{12}(k_1, k_2; t). \\
\]

As we perform the total trace Tr\(_{12}\) in order to obtain the conserved quantities \(C^{(n_1)}(k_1), C^{(n_2)}(k_2)\) within the Poisson bracket of physical, canonical fields, the right-hand side of (B.10) yields the trace Tr\(_{12}\) of a commutator with the \(\mathcal{r}_{12}\)-matrix which results into zero and therefore demonstrates the independence of the conserved quantities \(C^{(n_1)}(k_1), C^{(n_2)}(k_2)\)

\[
\text{Tr}_{12} \left[ \{ \mathcal{T}_1(2\pi n_1, 0; t; k_1) \otimes \mathcal{T}_2(2\pi n_2, 0; t; k_2) \} \right] = \{ C^{(n_1)}(k_1), C^{(n_2)}(k_2) \} =
\] (B.11)

\[
= \text{Tr}_{12} \left[ \mathcal{T}_1(2\pi n_2, 0; t; k_1) \mathcal{T}_2(2\pi n_1, 2\pi n_2; t; k_1) \mathcal{T}_1(2\pi n_2, 0; t; k_2) \mathcal{T}_2(2\pi n_2, 0; t; k_2) \mathcal{r}_{12}(k_1, k_2; t) + \\
- \mathcal{T}_1(2\pi n_1, 0; t; k_1) \mathcal{T}_2(2\pi n_2, 0; t; k_2) \mathcal{r}_{12}(k_1, k_2; t) \right] \\
= \text{Tr}_{12} \left[ \mathcal{T}_1(2\pi n_1, 0; t; k_1) \mathcal{T}_2(2\pi n_2, 0; t; k_2), \mathcal{r}_{12}(k_1, k_2; t) \right] = 0.
\]
The given proof of this appendix relies on the spatial independence of the $r_{12}$-matrix, $r_{12}(k_1, k_2; t)$ with the ultralocal property $\delta(x_1 - x_2)$ in the assumed fundamental Poisson bracket relation (B.1-B.6). The spatial independence of $r_{12}$-matrix has to be incorporated in order to transform the spatial integrations in (B.7,B.8) to total spatial derivatives so that the integrands simplify to remaining integration boundaries in (B.9,B.10).

References

[1] M.J. Ablowitz, B. Prinari and A.D. Trubatch, "Discrete and Continuous Nonlinear Schrödinger Systems", (Cambridge University Press, "London Mathematical Society Lecture Note Series (No. 302)", London, 2003)

[2] M.A. Ablowitz and P.A. Clarkson, "Solitons, Nonlinear Evolution Equations and Inverse Scattering", (Cambridge University Press, "London Mathematical Society Lecture Note Series (No. 149)", London, 1991)

[3] A.R. Chowdhury and A.G. Choudhury, "Quantum Integrable Systems", (Chapman & Hall/CRC, Boca Raton, 2004)

[4] L.A. Dickey, "Soliton Equations and Hamiltonian Systems", (2nd. ed., Advanced series in Mathematical Physics, World Scientific, 2003)

[5] Y. Kosmann-Schwarzbach, B. Grammaticos and K.M. Tamizhmani (Eds.), "Integrability of Nonlinear Systems", (Springer Verlag, Berlin, Heidelberg, 2004)

[6] B. Grammaticos, Y. Kosmann-Schwarzbach and T. Tamizhmani (Eds.), "Discrete Integrable Systems", (Springer Verlag, Berlin, Heidelberg, 2004)

[7] B. Mieck and R. Graham, "Bose-Einstein condensate of kicked rotators", J. Phys. A : Math. Gen. 37 No44 (2004), L581-L588; (cond-mat/0405057)

[8] B. Mieck and R. Graham, "Bose-Einstein condensate of kicked rotators with time-dependent interaction", J. Phys. A : Math. Gen. 38 No7 (2005), L139-L144; (cond-mat/0411648)

[9] B. Mieck and R. Graham (private communication and talk)

[10] O. Babelon, D. Bernard and M. Talon, "Introduction to Classical Integrable Systems", (Cambridge University Press, Cambridge, 2003)

[11] C. Zhang, J. Liu, M.G. Raizen and Q. Niu, "Transition to Instability in a Kicked Bose-Einstein Condensate", Phys. Rev. Lett. 92 (2004), 054101

[12] W.P. Reinhardt and S.B. McKinney, "Dynamical and Wave Chaos in the Bose-Einstein Condensate" in "Quantum Chaos Y2K", (K.-F. Berggren and S. Aberg (eds.), Proceedings of Nobel Symposium 116, Bäckaskog Castle, Sweden 13-17 June 2000), (Physica Scripta, The Royal Swedish Academy of Sciences and World Scientific, 2001)

[13] Y.B. Suris, "The Problem of Integrable Discretization : Hamiltonian Approach", (Progress in Mathematics Vol. 219, Birkhäuser Verlag, 2003)
[14] G. Duffy, S. Parkins, T. Müller, M. Sadgrove, R. Leonhardt and A.C. Wilson, "Experimental investigation of early-time diffusion in the quantum kicked rotor using a Bose-Einstein condensate", Phys. Rev. E 70 (2004), 056206

[15] J. Fuchs and C. Schweigert, "Symmetries, Lie Algebras and Representations (A Graduate Course for Physicists)", (Cambridge University Press, 1997)

[16] B. Mieck, "Lax pair construction of nonlinear sigma models beyond (1+1) dimensions", in preparation