HARMONIC FUNCTIONS IN UNION OF CHAMBERS.

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Abstract. We characterize the set of harmonic functions with Dirichlet boundary conditions in unbounded domains which are union of several different chambers. We analyze the asymptotic behavior of the solutions in connection with the changes in the domain’s geometry. Finally we classify all (possibly sign-changing) infinite energy solutions having given asymptotic frequency at the infinite ends of the domain.

1. Introduction

In this paper we are concerned with solutions to the following problem

\[
\begin{align*}
\Delta u &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where $\Omega$ is a particular unbounded domain defined as the union of two or more infinite cylinders. In this context the term chamber stands exactly for cylinder. We became interested in these issues in connection with the problem of the interplay of the geometry of the domain with the transmission of frequencies of solutions, as it will appear in the sequel. As a matter of facts, problems of type (1) may arise, for example, from a blow-up analysis for eigenvalues equations in bounded domains with varying geometries. This type of equations may describe the possible transmission of frequency from a chamber to another one, when passing through a certain number of other chambers, connected by thin tubes (whose section is negligible with respect to its own length), (see e.g. [7, 2]).

As this is the simplest case where the domain presents a sensitive change of geometry, one may expect the domain’s geometry and solutions’ shape to be strictly related to each other. We mean that such geometric changes in the domain affects the solutions’ shape as well as, from the opposite point of view, that solutions may carry some information about the domain’s geometry.

We are now going to specify the context and the notation that we will use throughout the paper. Let $U^R$ and $U^L$ two open regular connected domains in $\mathbb{R}^{N-1}$ for $N \geq 2$, possibly unbounded and let

\[
\begin{align*}
C^R := \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \text{ s.t. } x > 0 \text{ and } y \in U^R\}; \\
C^L := \{(x,y) \in \mathbb{R} \times \mathbb{R}^{N-1} \text{ s.t. } x < 0 \text{ and } y \in U^L\}; \\
\Omega := C^R \cup C^L \cup \Gamma \text{ being } \Gamma := \partial C^R \cap \partial C^L.
\end{align*}
\]

We stress that in our setting positive solutions can not have finite energy at both ends of the domain. In the same way, uniqueness of solutions of inhomogeneous Laplace equations does not hold, unless the energy is supposed to be finite. Therefore, in order to classify solutions of (1), we need to waive the energy boundedness and to allow infinite energy solutions. As it will appear in the proofs, we can handle...
infinite energy solutions by imposing suitable thresholds to the so-called Almgren quotient. Being \( v \) any solution of (1) we define its Almgren frequency function:

\[
N(v)(x) := \frac{\int_{\Omega_x} |\nabla v|^2}{\int_{\Gamma_x} v^2},
\]

where \( \Gamma_x := \{ (x, y) : y \in U^R \} \), \( \Omega_x := \{ (\xi, \eta) \in \Omega : \xi \in (0, x) \} \) if \( x > 0 \) whereas \( \Omega_x := \{ (\xi, \eta) \in \Omega : \xi \in (x, 0) \} \) if \( x < 0 \). We set \( N(0)(x) \equiv 0 \) for the trivial solution to (1). The ratio in (2) is acquired from the well-known Almgren frequencies, which were introduced by Almgren in the '70s to study certain properties of harmonic functions and from then on they were employed in many other branches of the analysis of pdes.

On the connecting hyperplane between the two chambers, we are considering the following eigenvalue problems:

\[
\begin{cases}
-\Delta \psi^L_k = \lambda^L_k \psi^L_k & \text{on } U^L, \\
\psi^L_k = 0 & \text{on } \partial U^L, \\
-\Delta \psi^R_k = \lambda^R_k \psi^R_k & \text{on } U^R, \\
\psi^R_k = 0 & \text{on } \partial U^R;
\end{cases}
\]

here \( \Delta \) denotes the \((N-1)\)-Laplacian over \( U^L \) and \( U^R \) respectively.

For what concerns our first aim about possible characterization of solutions, our main result relies on the following lemma

**Lemma 1.1.** Let \( u \) be any nontrivial solution to the problem (1). Then there exist

\[
\begin{align*}
\lim_{x \to +\infty} N(x) &= l^R \in \left\{ \sqrt{\lambda^R_j} \right\}_{j=1}^{+\infty} \cup \{+\infty\}, \\
\lim_{x \to -\infty} N(x) &= l^L \in \left\{ \sqrt{\lambda^L_j} \right\}_{j=1}^{+\infty} \cup \{+\infty\}.
\end{align*}
\]

We fix two numbers \( d^R \in \{ \lambda^R_j \}_{j=1}^{+\infty} \) and \( d^L \in \{ \lambda^L_j \}_{j=1}^{+\infty} \) and define the following set

\[
S = \left\{ u \in C^1 \text{ solution to (1)} \right\} \text{ such that if } \int_{C^m,L} |\nabla u|^2 = +\infty \text{ then } l^{R,L} \leq \sqrt{d^{R,L}} \right\}.
\]

Now we can state our main result as follows

**Theorem 1.2.** The set \( S \) defined in (4) is a linear space of dimension

\[
\dim S = m(d^R) + m(d^L)
\]

where \( m \) denotes the Morse index.

In particular, if we restrict to those solutions with finite energy on one hand of the domain, namely \( C^L \), the set

\[
S_L = \left\{ u \in C^1 \text{ solution to (1)} \text{ with } \int_{C^L,L} |\nabla u|^2 < +\infty \text{ and } l^L \leq \sqrt{d^L} \right\}
\]

is a linear space of dimension \( m(d) \), where \( m \) denotes again the Morse index.

For the reader’s convenience we recall that in this context the Morse index of the eigenvalue \( \lambda^L_k \) is the sum of the multiplicity of the eigenvalues \( \lambda^L_j \) with \( j \leq k \).

If we focus our attention just on positive solutions, as a byproduct of the previous result we obtain the following theorem, which corresponds to the particular case \( d^R = \lambda^R_1 \) in Equation (5):

**Theorem 1.3.** There exists a unique (up to a multiplicative constant) positive \( C^1 \) solution \( v^L \) to the problem (1), provided

\[
\int_{C^L} |\nabla v^L|^2 < \infty.
\]
Moreover,

- if \( U^R \) is bounded, then \( v^L \) is asymptotic to \( e^{\sqrt{\lambda_1} x} \psi_1(y) \) as \( x \to \infty \) uniformly with respect to \( y \in U^R \), being \( \psi_1 \) and \( \lambda_1 \) the first eigenfunction and eigenvalue respectively to the problem (ii) in Equation \( \mathcal{H} \) with a little abuse of notation;
- if \( C^R \) is the whole right halfspace of \( \mathbb{R}^N \), then \( v^L \) is asymptotic to \( x \) as \( x \to \infty \) uniformly with respect to \( y \in \mathbb{R}^{N-1} \).

An analogous statement defines \( v^R \). Finally, all positive solutions to problem \( \mathcal{H} \) are positive convex combinations of \( v^L \) and \( v^R \).

**Remark 1.4.** As appearing in Theorem 1.3, the dimension of the space \( S_L \) even in Theorem 1.2 is related to the multiplicity of the first eigenvalue \( \lambda_1^R \). Therefore, we stress the uniqueness stated in Theorem 1.3 relies essentially on the assumption \( U^R \) is a connected domain in \( \mathbb{R}^{N-1} \).

In the framework of positive solutions, Theorem 1.3 is not a brand new result, as we can find it within the so-called General Martin Theory. This is a quite general theory which provides a one-to-one correspondence between regular positive solutions and the points of the so-called minimal Martin boundary by means of finite harmonic measures supported on the minimal Martin boundary. We then foresee that the Martin boundary is useful to gain some information about the number of (linearly independent) positive solutions to a differential equation, whenever the differential operator satisfies several minimal assumptions. We then find worthwhile recalling some general concepts of the General Martin Theory. To this aim we refer to the book by Ross Pinsky [17], chapter 7.

In order to state the known results, let us consider a quite general differential operator \( L \) on a domain \( D \subseteq \mathbb{R}^N \) satisfying the following

**Assumptions 1.5.** For any \( D' \subseteq D \) the operator \( L \) is of the form \( L = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla + V \), with \( a_{i,j}, b_i \in C^{1,\alpha}(D') \), \( V \in C^\alpha(D') \) and \( \sum_{i,j}(x)v_i v_j > 0 \) for all \( v \in \mathbb{R}^N \setminus \{0\} \) and for all \( x \in D' \).

It is defined

\[
\mathcal{P}_L(D) = \{ u \in C^{2,\alpha}(D) : Lu = 0 \text{ and } u > 0 \text{ in } D \}
\]

the set of all regular positive solutions; and, fixed a point \( x_0 \in D \) and, denoting \( G \) the Green’s function, define the Martin kernel as

\[
k(x,y) = \begin{cases} 
\frac{G(x,y)}{G(x_0,y)} & y \neq x, \ y \neq x_0 \\
0 & y = x_0, \ x \neq x_0 \\
1 & y = x = x_0.
\end{cases}
\]

**Definition 1.6.** A sequence \( \{y_n\}_n \subset D \) for which the limit \( \lim_{n \to \infty} k(x,y_n) \in \mathcal{P}_L(D) \) is called a Martin sequence. Two Martin sequences which have the same limit are called equivalent. The collection of such equivalence classes is called the Martin boundary for \( L \) on \( D \).

We briefly mention that the Martin boundary does not depend on the choice of the fixed point \( x_0 \) in the Martin kernel and it can be endowed with a suitable topology; we do not enter into the details, since they go beyond our specific aim. More related to our work, we find the following

**Definition 1.7.** A function \( u \in \mathcal{P}_L(D) \) is called minimal if whenever \( v \in \mathcal{P}_L(D) \) and \( v \leq u \) then in fact \( v = cu \) for some constant \( c \in (0,1] \).
Given a point $\xi$ of the Martin boundary, the notation $k(x;\xi)$ means that, up to positive multiples, 
$$k(x;\xi) = \lim_{n \to +\infty} k(x;y_n)$$
where $y_n$ is any representative of the equivalence class $\xi$.

A point $\xi$ on the Martin boundary is called a minimal Martin boundary point if $k(x,\xi)$ is minimal.

**Theorem 1.8** (Martin Representation Theorem). Let $L$ satisfy the Assumptions 1.5 on a domain $D \subseteq \mathbb{R}^N$ and assume that $L$ is subcritical. Then for each $u \in \mathcal{P}_L(D)$ there exists a unique finite measure $\mu_u$ supported on the minimal Martin boundary $\Lambda_0$ such that 
$$u(x) = \int_{\Lambda_0} k(x,\xi) \mu_u(d\xi).$$

Conversely, for each finite measure $\mu$ supported on the minimal Martin boundary $\Lambda_0$, 
$$u(x) := \int_{\Lambda_0} k(x,\xi) \mu_u(d\xi) \in \mathcal{P}_L(D).$$

As already mentioned this is the basic theorem in order to state a one-to-one correspondence between the elements of the set $\mathcal{P}_L(D)$ and the points of the minimal Martin boundary by means of finite harmonic measures supported on the minimal Martin boundary.

The General Martin Theory covers even our case of domains formed by different chambers by means of the following theorems

**Theorem 1.9** (Theorem 6.6 in [17]). Let $D$ be a non-compact open $N$-dimensional $C^{2,\alpha}$-Riemannian manifold with $m$ ends, i.e. it can be represented in the form $D = F \cup E_1 \cup \ldots \cup E_m$ with $F$ is bounded and closed, $E_i$ is open, $\bar{E}_i \cap \bar{E}_j = \emptyset$ for $i \neq j$ and $F \cap \bar{E}_i = \emptyset$. Let $L$ on $D$ satisfy the assumptions 1.5 and be subcritical. Then the Martin boundary for $L$ decomposes into $m$ components in the following sense: if $\{x_n\}_n \subset D$ is a Martin sequence, then all but a finite number of its terms lie in $E_i$ for some $i = 1, \ldots, m$.

**Corollary 1.10** (Corollary 6.7 in [17]). Let $L$ satisfy assumptions 1.5 and be subcritical on the domain $D = (\alpha,\beta)$ where $-\infty \leq \alpha < \beta \leq +\infty$. Then the Martin boundary of $L$ on $D$ consists of two points. More specifically, a sequence $\{x_n\}_n$ with no accumulation points in $D$ is a Martin sequence if and only if $\lim_{n \to \infty} x_n = \alpha$ or $\lim_{n \to \infty} x_n = \beta$.

Moreover (see Proposition 5.1.3 in [17]), in this case $\mathcal{P}_L(D)$ is 2-dimensional, which means that, via the Martin Representation Theorem, the minimal Martin boundary consists exactly of two points.

In particular, a suitable $N$-dimensional generalization of the previous Corollary covers the case of Theorem 1.3 in the present paper. More precisely, the Martin boundary of our domain $\Omega$ consists in its topological boundary together with the union of two points, which can be identified, roughly speaking, with the two ends of the domain. Taking into account Dirichlet boundary conditions in our problem (1), this means that problem (1) has got exactly two linearly independent positive solutions, as we are able to show, too. Thus, our Theorem 1.3 does not provide any additional information to the known results provided by the General Martin Theory, except maybe by gaining greater understanding of the positive solutions’ space $\mathcal{P}_L(D)$: under Dirichlet boundary conditions, a basis of $\mathcal{P}_L(D)$ is formed by two positive regular functions which have finite energy on one end of the domain, whereas on the other end they tend to infinity. Further, we are able to describe exactly the divergent behavior, as well as to prove every element in $\mathcal{P}_L(D)$ to
be minimal according to Definition 1.7. Then, our original contribution does no longer refer strictly to the result, but rather to the method: an upper bound for the Almgren quotient of possible solutions is the key ingredient for existence of solutions. In the case of positive solutions the specific threshold for the Almgren frequency is set by the positivity assumption of the solutions, but the method can be extended even to sign-changing solutions, which are not included in the General Martin Theory, providing a stronger result that is Theorem 1.2.

As already mentioned, we are enforced to consider infinite energy solutions. As a second point of our work, we follow the idea that normalizing their necessary divergent asymptotic behavior, we force the asymptotic (vanishing) behavior of the solution even at the other hand of the domain. For a pair of cylinders, the rate of growth at $+\infty$ can be related with the rate of vanishing at $-\infty$ by means of the evaluation of a transfer operator (see Section §4). As remarked in §4.1 composition of such transfer operators can be useful to handle a concatenation of many cylinders. Finally, we shall classify all positive solutions and all infinite energy solutions having the smallest possible growth at infinity.

The paper is organized as follows: in Section 2 we examine the existence of positive solutions to problem (1) when $\Omega$ is the cylinder $C^R$ either when its section is bounded or when it is a whole hyperplane, and we investigate their possible behavior at infinity; in Section 3 we collect the previous results in order to prove Theorem 1.3 (see also Theorem 3.2) and Theorem 1.2. In Section 4 we study the relation between the asymptotic behavior of positive solutions at $+\infty$ and $-\infty$, generalizing our results to domains which are union of more than two chambers in the very last subsection.

2. Existence and uniqueness of a positive harmonic function on $C^R$.

We claim the following

**Theorem 2.1.** There exists a unique (up to a multiplicative constant) positive solution $u$ to the problem

\[
\begin{align*}
\Delta v &= 0 \quad \text{in } C^R \\
v &= 0 \quad \text{on } \partial C^R
\end{align*}
\]

if $U_R$ is bounded or it is a whole hyperplane. In the first case it will be

\[
v(x, y) = \left(e^{\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_1}x}\right)\psi_1(y),
\]

being $\lambda_1$ and $\psi_1$ the first eigenvalue and the first eigenfunction respectively of the problem (3) item (ii); whereas in the second case it will be

\[
v(x, y) = x
\]

denoting $x$ the first variable in $\mathbb{R}^N$.

**Remark 2.2.** We stress the aforementioned solutions have an infinite energy.

In order to prove this theorem, we will study the two cases separately.

### 2.1. The case $U_R$ bounded.

It is quite simple to prove that the function $v$ defined in (3) is a solution to the problem (7). Moreover, we stress it is asymptotic to $e^{\sqrt{\lambda_1}x}\psi_1(y)$ as $x \to \infty$. We aim to prove it is in fact the unique solution.

**Proposition 2.3.** The function $v$ defined in (3) is the unique solution up to multiplications by constants.
The proof relies essentially on three different tools: the so-called “Phragmén-Lindelöf Principle”, which may be read as a comparison principle on unbounded domains, a boundary version of the Harnack inequality, and an Almgren-type argument. For similar arguments, see [14, 16].

Let us recall the well-known Phragmén-Lindelöf Principle stated for the Laplace operator:

**Theorem 2.4** (Phragmén-Lindelöf Principle, [18]). Let $D$ be a domain, bounded or unbounded, and let $u$ satisfy

$$-\Delta u \leq 0 \text{ in } D,$$
$$u \leq 0 \text{ on } \Gamma,$$

where $\Gamma$ is a subset of $\partial D$. Suppose that there is an increasing sequence of bounded domains $D_1 \subset D_2 \subset \cdots \subset D_k \subset \cdots$ with properties

1. each $D_k$ is contained in $D$; for each point $x \in D$ there is an integer $N$ such that $x \in D_N$;
2. the boundary of each $D_k$ consists in two parts $\Gamma_k$ and $\Gamma_k'$ where $\Gamma_k$ is a subset of $\Gamma$ and $\Gamma_k'$ is a subset of $D$.

Further, suppose there exists a sequence $\{w_k\}$ which satisfies

$$w_k(x) > 0 \text{ on } D_k \cup \partial D_k,$$
$$-\Delta w_k \geq 0 \text{ in } D_k.$$

Assume there is a function $w(x)$ with the property that at each point $x \in D$ the inequality

$$w_k(x) < w(x)$$

holds for all $k$ above a certain integer $N_x$. If $u$ satisfies the growth condition

$$\liminf_{k \to \infty} \left\{ \sup_{\Gamma_k'} u(x,y) w_k(x,y) \right\} \leq 0$$

then

$$u \leq 0 \text{ in } D.$$

**Lemma 2.5** (Boundary Harnack inequality, [10]). Let $D \subset \mathbb{R}^N$, $N \geq 2$, be a Lipschitz domain and let $V$ an open set such that $V \cap \partial D \neq \emptyset$. Suppose $W$ is a domain such that $W \subset D$, $\overline{W} \subset V$ and let $P_0$ be a point in $W$. Then there is a constant $C > 0$ such that if $u$ and $v$ are nonnegative harmonic functions in $D$ which vanish on $V \cap \partial D$ and satisfy $u(P_0) \leq v(P_0)$ then $u(P) \leq C v(P)$ for all $P \in W$.

Thanks to these two preliminary results, we can state

**Proposition 2.6.** Let $u$ and $v$ be two different positive solutions to the problem

Then $u = O(v)$.

**Proof.** According to the notation in Theorem 2.4, let $D_k$ denotes the rectangle $\{(x,y) \in C^R, \ k-1 < x < k \text{ and } y \in U^R\}$ and $\Gamma_k := \{(k-1,y), y \in U^R\} \cup \{(k,y), y \in U^R\}$. We can claim that

$$\liminf_{k \to \infty} \sup_{\Gamma_k'} u(x,y) v(x,y) > 0.$$

If not, Theorem 2.4 would apply with $w_k = v\chi_{D_k} + \varepsilon$ where $\varepsilon$ is any positive constant. Thus, we would obtain $u \leq 0$ and then $u = 0$, a contradiction.

We define

$$b_k = \max_{\Gamma_k} \frac{u(x,y)}{v(x,y)} \quad a_k = \min_{\Gamma_k} \frac{u(x,y)}{v(x,y)}.$$
Then, Equation (10) implies \( b_k \geq C > 0 \) for \( k \) large enough and then, by Lemma (2.5) we have \( |u(x_1,y_1) - u(x_2,y_2)| \leq C |v(x_1,y_1) - v(x_2,y_2)| \) for any \((x_1,y_1), (x_2,y_2) \in C^R\) in order to obtain

\[
(11) \quad 1 \leq b_k \leq C.
\]

This means that the two sequences \( a_k \) and \( b_k \) share the same asymptotic behavior. Moreover, their divergence to \( \infty \) cannot occur. If they diverged to \( +\infty \), then the inverse quotient \( \frac{u}{v} \) would be uniformly convergent to zero as \( x \to +\infty \), Theorem (2.4) would apply and provide the contradiction \( v \equiv 0 \).

In this way, they both cannot converge to zero, otherwise (10) would be violated, and then

\[
(12) \quad C_1 \leq \frac{u}{v} \leq C_2
\]

for some positive constants \( C_1 \) and \( C_2 \).

**Proposition 2.7.** Any solution to (7) is asymptotic to \( e^{\sqrt{\lambda}x} \psi_k(y) \) as \( x \to \infty \) uniformly with respect to \( y \in U^R \) for some \( k \in \mathbb{N} \), where \( \psi_k \) denotes one of eigenfunctions relative to the \( k \)-th eigenvalue of the problem

\[
\begin{align*}
\Delta \psi_k &= \lambda_k \psi_k & \text{in} & U^R \\
\psi_k &= 0 & \text{on} & \partial U^R.
\end{align*}
\]

To prove this last step we need several preliminary results, which are stated in Lemma (2.8), Lemma (2.11) and Lemma (2.10).

Being \( v \) any solution to (7), we recall the Almgren frequency function:

\[
N(v)(x) := \frac{\int_{\Omega_x} |\nabla v|^2}{\int_{\Gamma_x} v^2},
\]

where \( \Omega_x := \{(\xi,\eta) \in \Omega : 0 < \xi < x \} \) and \( \Gamma_x := \{(x,y) : y \in U^R \} \).

**Lemma 2.8.** Given a solution \( v \) to (7), the function \( N(v)(x) - \frac{C}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}x} \) is monotone increasing with respect to \( x \).

**Proof.** It is simple to see that

\[
D'(x) = \int_{\Gamma_x} |\nabla v|^2,
\]

\[
H'(x) = \int_{\Gamma_x} 2 v v_x.
\]

Multiplying the equation by \( v_x \) and integrating by parts we obtain

\[
\int_{\Omega_x} \nabla v \nabla v_x = \int_{\Gamma_x} v_x^2 - \int_{\Gamma_0} v_x^2;
\]

whereas differentiating it and multiplying it by \( v \) we obtain

\[
\int_{\Omega_x} \nabla v \nabla v_x = \int_{\Gamma_x} v v_{xx} = -\int_{\Gamma_x} v \Delta_y v = \int_{\Gamma_x} v_y^2;
\]

from which

\[
\int_{\Gamma_x} v_x^2 = \int_{\Gamma_x} v_y^2 + \int_{\Gamma_0} v_x^2.
\]
Let us compute the derivative
\[
\frac{d}{dx} N(x) = \frac{\int_{\Gamma_x} v_x^2 + v_y^2}{\int_{\Gamma_x} v^2} - 2 \left( \frac{\int_{\Gamma_x} v v_x}{\int_{\Gamma_x} v^2} \right)^2 \geq -\frac{\int_{\Gamma_x} v_x^2}{\int_{\Gamma_x} v^2} \geq -\frac{C}{e^{\sqrt{\lambda_1} x}}
\]
for some positive \( C \); the first inequality is given by the Hölder inequality and the second one is implied by Proposition (2.6).

**Remark 2.9.** Under our hypothesis we can claim \( N(v)(x) \) admits a finite limit as \( x \to \infty \). Indeed, it admits a limit in view of Lemma (2.8), and such a limit is finite since \( v \) is \( O(e^{\sqrt{\lambda_1} x} \psi_1(y)) \) from Proposition (2.6), so that \( N(v) \) is a bounded function from above.

In order to detect \( \lim_{x \to +\infty} N(v)(x) \) we introduce the sequence of normalized functions \( v_\xi(x, y) := \frac{v(x + \xi, y)}{\left( \int_{\Gamma_\xi} v^2(\xi, y) \right)^{1/2}} \) for \( \xi \in \mathbb{R} \), \( x \in (0, 1) \), \( y \in U^R \).

**Lemma 2.10.** As \( \xi \to \infty \) the sequence \( \{v_\xi\}_\xi \) converges \( C^1 \)-uniformly on compact sets of the cylinder \( \{(x, y) \in \mathbb{R}^N : x \in \mathbb{R} \text{ and } y \in U^R \} \) to a function harmonic on the cylinder whose \( N(x) \) is identically constant.

**Proof.** First we observe \( N(v_\xi)(x) = N(v)(x + \xi) \leq N \) for all \( x \in (0, 1) \) and for all \( \xi \in \mathbb{R} \), thanks to the definition of \( v_\xi \) and to Remark (2.6). Thus, \( \int_{\Omega_x} |\nabla v_\xi|^2 \leq \frac{N}{\int_{\Gamma_\xi} v^2} \) where we recall
\[
\int_{\Gamma_x} v_\xi^2 = \frac{\int_{\Gamma_{x+\xi}} v^2(x + \xi, y) \, dy}{\int_{\Gamma_x} v^2(\xi, y) \, dy}.
\]
Via Harnack inequality, if \( x \) ranges in a compact set, the previous ratio is bounded from above by a fixed constant, then also the \( H^1 \)-norm is uniformly bounded from above. Thus, there exists a subsequence at least \( C^1 \)-uniformly convergent to a function \( w \) which is harmonic on the whole cylinder. It holds for any fixed \( x \in \mathbb{R} \)
\[
N(v_\xi)(x) = N(v)(x + \xi) \to N \text{ as } \xi \to \infty, \text{ and then}
\lim_{\xi \to \infty} N(v_\xi)(x) = N \quad \forall \ x \in \mathbb{R}.
\]
Moreover this happens for any convergent subsequence. Then we can conclude the whole sequence \( v_\xi \) is \( C^1 \)-uniformly convergent to a function \( w \) which is harmonic on the whole cylinder and has \( N(x) \) identically constant.  

**Lemma 2.11.** Let \( w \) be a solution to
\[
\begin{cases}
\Delta w = 0 & \text{ on } \{(x, y) \in \mathbb{R}^N : x \in \mathbb{R} \text{ and } y \in U^R \} \\
w = 0 & \text{ if } y \in \partial U^R
\end{cases}
\]
with \( \int_{\mathbb{R}^n} |\nabla w|^2 < \infty \) for all \( \tau \). Then \( N(w)(x) \) is identically constant in \( x \) if and only if \( w(x, y) = e^{\sqrt{\lambda_k}} \psi_k(y) \) for some \( k \in \mathbb{N} \), being \( \lambda_k \) the \( k \)-th eigenvalue of problem (13) and \( \psi_k \) one of its relative eigenfunctions.

**Proof.** Note for such solutions it holds \( \int_{\Gamma_x} w_x^2 = \int_{\Gamma_x} w_y^2 \), so that

\[
\frac{d}{dx} N(w)(x) = 2 \int_{\Gamma_x} \frac{w_x^2}{w^2} \left( 1 - \frac{\left( \int_{\Gamma_x} w w_x \right)^2}{\|w\|_{L^2(\Gamma_x)}^2 \|w_x\|_{L^2(\Gamma_x)}^2} \right).
\]

Thus, \( N \) is identically constant in \( x \) if and only if we have an equality in the Hölder inequality, that is

\[
\left( \int_{\Gamma_x} w w_x \right)^2 = \int_{\Gamma_x} w^2 \int_{\Gamma_x} w_x^2.
\]

This happens if and only if \( w_x(x, y) = \lambda(x) w(0, y) \), which leads to

\[
w(x, y) = w(0, y) \left\{ 1 + \int_0^x \lambda(t) \, dt \right\}.
\]

If we substitute this expression in \( N(w)(x) \equiv N \) we obtain

\[
\lambda(x) = N \left\{ 1 + \int_0^x \lambda(t) \, dt \right\}
\]

which is a differential equation whose solution is \( \lambda(x) = N e^{\sqrt{\lambda_k}} x \); then \( w(x, y) = e^{N x} w(0, y) \), from which \( w(x, y) = e^{\sqrt{\lambda_k}} \psi_k(y) \) imposing \( w \) is harmonic and zero on the boundary. \( \square \)

**Proof of Proposition 2.7** We exploit the following chain of equalities:

\[
\lim_{x \to +\infty} N(v)(x) = \lim_{\xi \to +\infty} N(v)(x + \xi) = \lim_{\xi \to +\infty} N(v_\xi)(x) = N(w)(x) \equiv \sqrt{\lambda_k}.
\]

Therefore Lemma 2.11 gives immediately the proof. \( \square \)

**Proof of Proposition 2.5** By Remark 2.9 we need to prove \( A = B \). This is a straightforward consequence of Proposition 2.7 where positivity of solutions forces \( \lambda_k = \lambda_1 \). \( \square \)

### 2.2. The case \( U^R \) hyperplane.

The existence of a positive solution in this case is immediately proved by considering the function \( \bar{v}(x, y) := x \), where we recall \( x \) denotes the first variable in \( \mathbb{R}^N \).

We aim to prove this is in fact the unique solution to the problem (7) when \( U^R \) is a whole hyperplane of \( \mathbb{R}^N \), namely \( \{ x = 0 \} \). To do this, we follow the same outline as before.

Let \( B_r \) be the ball in \( \mathbb{R}^N \) centered in the origin with radius \( r \), we denote

\[
C_r := C^R \cap B_r \quad \text{and} \quad \Gamma_r := \partial B_r \cap C^R.
\]

**Proposition 2.12.** Any positive solution to the problem (7) is \( O(x) \) as \( x \to \infty \) uniformly with respect to \( y \).

The proof of this proposition is essentially the same as in the previous case, provided the domains \( D_k \) are now defined as \( C_k \).

**Proposition 2.13.** Any solution to (7) is asymptotic to \( r^{-N} v(1, \theta) \) as \( r \to \infty \) uniformly with respect to \( \theta \in \mathbb{S}^{N-1} \) in such a way that \( N(N-1) + \overline{N(N-1)} \) is an eigenvalue for the spherical Laplacian and \( v(1, \theta) \) is one of its relative eigenfunctions.
To prove this last step we need several preliminary results, which we state in the Lemma (2.14), Lemma (2.16) and Lemma (2.15).

We aim to pursue again an Almgren-type argument on the domains $C_r$. Being $v$ any solution to (7), let us introduce the following Almgren-type quotient

$$N(v)(r) := \frac{r^{2-N} \int_{C_r} |\nabla v|^2}{r^{1-N} \int_{\Gamma_r} v^2} =: \frac{D(r)}{H(r)},$$

**Lemma 2.14.** Given a solution $v$ to (7), the quotient $N(v)(r)$ is monotone increasing with respect to $r$.

**Proof.** It is quite simple to see

$$H'(r) = 2r^{1-N} \int_{\Gamma_r} v v_r.$$ Testing the equation by $v$ we obtain

$$H'(r) = 2r^{1-N} \int_{C_r} |\nabla v|^2 = \frac{2}{r} D(r);$$

from which $D(r) = (r/2)H'(r)$.

On the other hand we claim

$$D'(r) = 2r^{2-N} \int_{\Gamma_r} v v_r^2.$$ Indeed,

$$D'(r) = (2 - N)r^{1-N} \int_{C_r} |\nabla v|^2 + r^{2-N} \int_{\Gamma_r} |\nabla v|^2;$$

testing the equation with $\nabla v \cdot (x,y)$ and integrating by parts we obtain

$$\int_{C_r} \nabla v \cdot \nabla (\nabla v \cdot (x,y)) = r \int_{\Gamma_r} v v_r^2,$$

which is in fact

$$\int_{C_r} \nabla v \cdot \nabla (\nabla v \cdot (x,y)) = -\frac{N - 2}{2} \int_{C_r} |\nabla v|^2 + \frac{r}{2} \int_{\Gamma_r} |\nabla v|^2$$

via integration by parts. From (19), (20) and (21) we immediately obtain (18).

Now, the derivative of $N$ is of course $N'(r) = \frac{D'(r) H(r) - D(r) H'(r)}{H^2(r)}$, and we recall that $D(r) H'(r) = (r/2)(H'(r))^2$, so that

$$N'(r) = \frac{2r^{3-2N}}{H^2(r)} \left\{ \int_{\Gamma_r} v v_r^2 - \left( \int_{\Gamma_r} v r v \right)^2 \right\} \geq 0$$

gthanks to the Hölder inequality. \hfill \Box

Now we introduce the sequence of normalized functions

$$v_r(x,y) := \frac{v(rx,ry)}{\left( \int_{\Gamma_{r/2}} v^2(rx,ry) \right)^{1/2}}$$

for $r > 1$.

**Lemma 2.15.** As $r \to \infty$ the sequence $\{v_r\}_r$ converges $C^1$-uniformly on $C_1$ to a function which is harmonic on the whole halfspace and whose $N(x)$ is identically constant.

**Proof.** Here the proof is essentially the same as in Lemma (2.10). \hfill \Box
Lemma 2.16. Let $v$ any non-trivial solution to the problem (7). Then its Almgren’s frequency function is identically constant equal to $N$ if and only if

$$v(r, \theta) = r^N v(1, \theta)$$

in such a way that $[N(N-1)+N(N-1)]$ is an eigenvalue for the spherical Laplacian and $v(1, \theta)$ is one of its relative eigenfunctions.

Proof. If the derivative of the frequency function is identically zero, then an equality must hold in the Hölder inequality, so that

$$v_r(r, \theta) = \lambda(r) v(1, \theta),$$

that is

$$v(r, \theta) = v(1, \theta) \{1 + \int_1^r \lambda(t)dt\}.$$ 

Imposing $D(r)/H(r) = (r/2)(H'(r)/H(r)) = \overline{N}$ we obtain

$$N = \frac{r \int_{\Gamma_r} v v_r}{\int_{\Gamma_r} v^2} = \frac{r \int_{\Gamma_r} v^2(1, \theta) \lambda(r) \left(1 + \int_1^r \lambda(t)dt\right) d\theta}{\int_1^r \lambda(t)dt} = \frac{r \lambda(r)}{1 + \int_1^r \lambda(t)dt}.$$ 

The solution of the ordinary differential equation

$$r \lambda(r) = N \left(1 + \int_1^r \lambda(t)dt\right)$$

is indeed \( \int_1^r \lambda(t)dt = r^N - 1 \), which leads to $v(r, \theta) = r^N v(1, \theta)$. Imposing $v$ is harmonic on the whole halfspace, we deduce the conditions on $N$ and $v(1, \theta)$. □

Corollary 2.17. The solution $\bar{v}$ defined in (9) is the unique positive solution to the problem (7) up to multiplication by constants.

Proof. Positivity assumption forces $N = 1$ in Proposition (2.13). This homogeneity degree together with $v(0, y) = 0$ implies $v(x, y) = x$. □

3. Solutions on $\Omega$

3.1. Positive solutions on $\Omega$ with finite energy on $C^L$. The following proposition can be easily proved.

Proposition 3.1. Let us consider the case $\Omega := C^L \cup C^R$ where $U^R$ is the hyperplane $\{x = 0\}$. Let $\Phi$ be unique normalized positive solution of (7), extended as vanishing outside the semicylinder. There exists a unique positive solution $v$ to problem (7) such that $u = v - \Phi$ has finite energy on $\Omega$: it is the solution of the minimum problem

$$\min_{u \in D^{1,2}(\Omega)} \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_{\Gamma} \frac{\partial \Phi}{\partial x} |_{x=0} u.$$ 

We note that the minimizer $u$ is not a $C^1$ solution. Indeed, on one hand for every $\varphi \in D^{1,2}(\Omega)$ we have

$$\int_\Omega \nabla u \nabla \varphi = \int_{\Gamma} \frac{\partial \Phi}{\partial x} |_{x=0} \varphi;$$

whereas on the other hand, multiplying the equation by $\varphi$ nd integrating by parts over $C^L$ and $C^R$ we obtain

$$\int_\Omega \nabla u \nabla \varphi = \int_{C^L \cup C^R} \nabla u \nabla \varphi = \int_{\Gamma} \varphi \left( - \frac{\partial u^R}{\partial x} |_{x=0} + \frac{\partial u^L}{\partial x} |_{x=0} \right).$$
where \( u^L := u_{\chi_{C^L}} \) and \( u^R \) is defined similarly. Thus,

\[
\frac{\partial u^L}{\partial x} \bigg|_{x=0} = \frac{\partial u^R}{\partial x} \bigg|_{x=0} + \frac{\partial \Phi}{\partial x} \bigg|_{x=0},
\]

in the sense that must be specified yet (see Section 4). In order to obtain a \( C^1 \) solution, we need to consider the sum \( v = u + \Phi \) instead of \( u \).

Furthermore, if the test function \( \varphi \) has compact support far away from \( \Gamma \), Equation (23) shows that the minimum is a harmonic function in \( \Omega \setminus \Gamma \). In this way, if we are looking for a harmonic function \( u + \Phi \) on the whole \( \Omega \), \( \Phi \) must be the unique (up to multiplication by constants) solution to the problem (7) (see the previous section). In other words, given the function \( \Phi \) solution to the problem (7), the function \( u + \Phi \) is the unique solution to the problem (11) with finite energy on the left. Furthermore, it is possible to prove that any positive solution to the problem (11) with finite energy on the left takes the form \( u + \Phi \) for a certain \( \Phi \) solution to the problem (7), in order to state the following

**Theorem 3.2.** There exists a unique (up to multiplicative constants) solution to the problem (11) having finite energy on \( C^L \) and satisfying

\[
\lim_{x \to +\infty} N(x) = \sqrt{\lambda_1^R}.
\]

It is asymptotic to a multiple of (8) if \( U^R \) is bounded, whereas it is asymptotic to a multiple of (9) if \( U^R \) is a whole hyperplane.

**Proof.** The proof follows the same outline as the proof of Theorem (2.1). Propositions (2.6) and (2.12) can be stated and proved in the same way choosing \( D_k = \{ (x,y) \in \Omega : -k < x < k \} \) in the first case and \( D_k = \{ (x,y) \in \Omega : -k < x \leq 0 \} \cup C_k \) in the second case.

We conclude the proof throughout an Almgren type argument on the domains \( \Omega_x = \{ (\xi,\eta) \in \Omega : \xi < x \} \) (but now \( \Gamma_0 = \{ x = 0 \} \cap \partial \Omega \) in the first case and \( \Omega_x = \{ (x,y) \in \Omega : x \leq 0 \} \cup C_r \) in the second case. In both cases the computations are the same. \( \square \)

**Remark 3.3.** As already highlighted in [1], the minimum in Equation (22) is strictly related to the concept of compliance. We define

\[
\mathcal{C}(\Gamma) := \max_{w \in D^{1,2}(\Omega)} \left( 2 \int_\Gamma \frac{\partial \Phi}{\partial x} \bigg|_{x=0} w - \int_\Omega |\nabla w|^2 \, dx \right)
\]

the compliance functional associated to a force concentrated on the section \( \Gamma \) in the flavor of [4, 5]. In general, the compliance functional measures the rigidity of a membrane subject to a given (vertical) force: the maximal rigidity is obtained by minimizing the compliance functional \( \mathcal{C}(\Gamma) \) in a certain class of admissible regions \( \Gamma \).

### 3.2. Infinite energy solutions.

Up to now, we have proved that given a positive profile \( \phi \) on \( U^R \), there exist at least two positive solutions to the problem

\[
\begin{align*}
\Delta w &= 0, & \text{in } C^R, \\
w &= \phi, & \text{on } U^R, \\
w &= 0, & \text{on } \partial C^R \setminus U^R.
\end{align*}
\]

Indeed, one has finite energy and it is the minimum of the Dirichlet realization on \( C^R \); we name it \( u \); whereas the second one is obtained from the previous simply adding a multiple of the solution \( v \) of the Theorem (2.1).

**Theorem 3.4.** Any positive solution to the problem (26) is a linear combination \( u + cv \) with \( c \geq 0 \), being \( u \) and \( v \) as mentioned above.
Proof. Let $w > 0$ be a solution to the problem (28). If its energy is finite, then it coincides with $u$ since in this case we have uniqueness of solution.

If $w$ has an infinite energy, consider the difference $w - u$. Then, we can immediately state that

$$\liminf_{x \to + \infty} \sup_{\Gamma_x} \frac{w - u}{v} > 0$$

since if not, the Phragmén-Lindelöf Theorem would imply $w - u \leq 0$, a contradiction. As in the proof of Proposition (26) we obtain

$$c_1 \leq \frac{w - u}{v} \leq c_2.$$  \hspace{1cm} (27)

We follow the same outline as before and study the Almgren quotient $N(x)$ on $\Omega^0 := \{(\xi, \eta) \in \mathbb{R}^N : \xi \in (0, x) \}, \eta \in U^R$. As before, $N(x) = \frac{D(x)}{H^2(x)}$ where $D(x) = \int_{\partial \Omega^0} |\nabla w|^2$ and $H(x) = \int_{\Gamma_x} w^2$ being $\Gamma_x = \{(x, \eta) : \eta \in U^R\}$. Multiplying the Laplace equation by $w$ itself, we obtain

$$\int_{\Omega^0} |\nabla w|^2 = \int_{\Gamma_x} w w_x - \int_{\Gamma_0} w w_x.$$  

Multiplying the Laplace equation by $w_x$ we obtain

$$\int_{\Omega^0} \nabla w \cdot \nabla w_x = \int_{\Gamma_x} w_x^2 - \int_{\Gamma_0} w_x^2$$

where

$$\int_{\Omega^0} \nabla w \cdot \nabla w_x = \int_{\partial \Omega^0} \frac{1}{2} |\nabla w|^2 \nu \cdot e_1 = \int_{\Gamma_x} \frac{1}{2} |\nabla w|^2 - \int_{\Gamma_0} \frac{1}{2} |\nabla w|^2$$

so that

$$\int_{\Gamma_x} |\nabla w|^2 = \int_{\Gamma_0} |\nabla w|^2 + 2 \int_{\Gamma_x} w_x^2 - 2 \int_{\Gamma_0} w_x^2.$$  

Thus, the derivative

$$N'(x) = \frac{D'(x)H(x) - D(x)H'(x)}{H^2(x)}$$

$$= \frac{\left( \int_{\Gamma_0} |\nabla w|^2 + 2 \int_{\Gamma_x} w_x^2 - 2 \int_{\Gamma_0} w_x^2 \right) \int_{\Gamma_x} w^2 - \left( \int_{\Gamma_x} w_x^2 - \int_{\Gamma_0} w_x^2 \right) \int_{\Gamma_x} 2w w_x}{\left( \int_{\Gamma_x} w^2 \right)^2}$$

$$= 2 \left\{ \int_{\Gamma_x} w_x^2 \int_{\Gamma_x} w^2 - \left( \int_{\Gamma_x} w w_x \right)^2 \right\} + \int_{\Gamma_0} w_y^2 \int_{\Gamma_x} w^2 - \int_{\Gamma_x} w_x^2 \int_{\Gamma_x} w^2 + 2 \int_{\Gamma_0} w w_x \int_{\Gamma_x} w w_x$$

$$= \int_{\Gamma_x} w_y^2 - w_x^2 \int_{\Gamma_x} w^2 \int_{\Gamma_0} w w_x \int_{\Gamma_x} w w_x$$

via Hölder inequality. Thanks to the estimate (27) the function

$$\int_{\Gamma_0} w_y^2 - w_x^2 \int_{\Gamma_x} w^2 \int_{\Gamma_0} w w_x \int_{\Gamma_x} w w_x \in L^1(0, +\infty),$$
so that $N(x)$ admits a limit as $x \to +\infty$. Moreover, such a limit is finite since the quantities $a_k$ and $b_k$ cannot diverge to infinity via Lemma 2.5 and Theorem 2.3 as in the proof of Proposition 2.6. We conclude the proof invoking Proposition 2.7.

**Theorem 3.5.** Any positive solution to the problem (1) is a linear combination $c^L v^L + c^R v^R$ with $c^L, c^R \geq 0$ (at least one of the two constants must be different from zero), where $v^L$ and $v^R$ are the solutions in the Theorem 3.2 with finite energy on $C^L$ and $C^R$ respectively.

*Proof.* The proof relies essentially on the Phragmen-Lindelöf Principle. Let $w > 0$ be a solution to the problem (1). We simply apply the aforementioned principle on $w = (c^L v^L + c^R v^R) \cap C^L$ and let us denote $c^L = c$, $c^R = \frac{c}{c^L} + 1$. In this case we choose the sequence of domains $D_k$ as the union $\{(\xi, \eta) : \xi \in (0, k) \eta \in U^R \} \cup \{(\xi, \eta) : \xi \in (-k, 0) \eta \in U^L \}$ whenever $U^R$ is bounded, whereas $\{(\xi, \eta) : \xi \in (-k, 0) \eta \in U^L \} \cup (C^R \cap \{s > \xi \})$ where $s$ is the junction point between $C^L$ and $C^R$ whenever $U^R$ is the whole hyperplane.

We stress that such solutions have $\lim_{x \to -\infty} N(x)$ lowest as possible in order to be nontrivial, that is $\sqrt{\lambda_1^R}$ and $\sqrt{\lambda_2^R}$ respectively. We note that $v^L$ is asymptotic to a multiple of $e^{-\sqrt{\lambda_1^R}x}v^L_1$ as $x \to -\infty$. Does the reverse implication hold true? Not exactly, but we can state

**Theorem 3.6.** The function set

$$S := \left\{ w \text{ solution to (1)} \text{ s.t. } \lim_{x \to +\infty} N(x) \leq \sqrt{\lambda_1^R} \text{ or } \lim_{x \to -\infty} N(x) \leq \sqrt{\lambda_2^R} \right\}$$

is a linear space of dimension 2 and $\{v^L, v^R\}$ is a basis, being $v^L, v^R$ as in the previous theorem.

We remark that in this case no positivity assumption can be made on solutions, but we can state that they change their sign at most one time.

**Remark 3.7.** The procedure presented up to now works even in the case the upper bound for the Almgren frequency is set to be a $k$-th eigenvalue of the problem (13) with $k \geq 2$, up to minor modifications. This allows us to extend Theorem 3.2 providing the following

**Theorem 3.8.** Let $\lambda^R_k$ be the $k$-th eigenvalue of the problem (13) and let us denote $m^R_k$ its multiplicity. Then, there exist exactly $m^R_k$ linearly independent solutions to the problem (1) having finite energy on $C^L$ and satisfying

$$\lim_{x \to +\infty} N(x) = \sqrt{\lambda^R_k}.$$ 

Each of them is asymptotic to a multiple of $e^{\sqrt{\lambda^R_k}x}v^R_k(y)$, being $v^R_k$ one the eigenfunctions relative to $\lambda^R_k$, if $U^R$ is bounded, whereas each of them is asymptotic to a multiple of $r^N v^R(1, \theta)$ if $U^R$ is a whole hyperplane, in such a way that $[N(N - 1) + N(N - 1)]$ is an eigenvalue for the spherical Laplacian and $v(1, \theta)$ is one of its relative eigenfunctions.

Thus, Theorem 3.8 is finally proved.

4. Frequency transfer from two consecutive cylinders

Let us focus our attention on the unique solution which has finite energy at $-\infty$. We are talking about $u + \Phi$, where $u$ is the minimum of (22) and $\Phi$ the unique solution of the problem (7). Thanks to the uniqueness of such a solution, whenever
we impose the exact behavior at \( x \to +\infty \), the asymptotic behavior for \( x \to -\infty \) is determined. We aim to investigate how such a fact occurs.

**Remark 4.1.** Via the Phragmén-Lindelöf Theorem, the restrictions \( u^L := u|_{\Omega^L} \) and \( u^R := u|_{\Omega^R} \) are \( u^L(x,y) = O(e^{\sqrt{\lambda_j}x}\varphi_k^L(y)) \) whereas \( u^R(x,y) = O(e^{-\sqrt{\lambda_j}x}\varphi_k^R(y)) \). Indeed, given the particular domain’s geometry, \( u^L \) and \( u^R \) can be written as \( \sum_k c_k^L \varphi_k^L(y) \) and \( \sum_k c_k^R \varphi_k^R(y) \) respectively. Then, imposing that \( \Delta u^i = 0 \) for \( i = L, R \) and that their energy is finite, they take the form

\[
(28) \quad u^L(x,y) = \sum_k \alpha_k e^{\sqrt{\lambda_j}x} \varphi_k^L(y) \quad u^R(x,y) = \sum_k \beta_k e^{-\sqrt{\lambda_j}x} \varphi_k^R(y)
\]

where the eigenfunctions \( \{\varphi_k^L\} \) and \( \{\varphi_k^R\} \) are basis for \( L^2(U^L) \) and \( L^2(U^R) \) respectively.

The key points for this analysis are Equation (24) together with the fact that the two profiles of \( u^L \) and \( u^R \) coincides on the boundary \( \{x,y \in \Omega, \ x = 0\} \).

In particular, Equation (24) makes sense in a distributional sense, so that it should be read in the dual space \( H^{-1/2}(U^L) \). Indeed, both \( u^L \) and \( u^R \) are \( D^{1,2} \) functions on \( C^L \) and \( C^R \) respectively, then their traces on \( \{x = 0\} \) are \( H^{1/2} \) functions and then their partial derivatives on \( \{x = 0\} \) are in \( H^{-1/2}(U^L) \) and \( H^{-1/2}(U^R) \) respectively. In order to specify these concepts, we introduce the following spaces

\[
\begin{align*}
\hat{H}^{1/2}_L &:= \{(\alpha_j), \text{s.t.} \sum_j (\lambda_j^L)^{1/2} \alpha_j^2 < +\infty\}, \\
\hat{H}^{1/2}_R &:= \{(\alpha_j), \text{s.t.} \sum_j (\lambda_j^R)^{1/2} \alpha_j^2 < +\infty\},
\end{align*}
\]

being \( \lambda_j^L \) and \( \lambda_j^R \) the eigenvalues of \( \Delta_{N-1} \) on \( U^L \) and \( U^R \) respectively, and operators

\[
\begin{align*}
\hat{U} : \hat{H}^{1/2}_L &\to \hat{H}^{1/2}_R, \\
\alpha &= (\alpha_j)_j \mapsto (\hat{U}(\alpha))_j = U^L_j \alpha_j \\
\hat{U} : H^{1/2}(U^L) &\to H^{1/2}(U^R), \\
u &= \alpha^j \varphi_j^L \mapsto \hat{U}u = (U^L_j \alpha_j) \varphi_j^R.
\end{align*}
\]

Moreover, \( \hat{U}^* : H^{-1/2}(U^R) \to H^{-1/2}(U^L) \) will be the adjoint operator.

These mean that Equation (24) is correctly read as

\[
(29) \quad \frac{\partial u^L}{\partial x} \bigg|_{x=0} = \hat{U}^* \left( \frac{\partial u^R}{\partial x} \bigg|_{x=0} \right) + \hat{U}^* \left( \frac{\partial \varphi}{\partial x} \bigg|_{x=0} \right) \quad \text{in} \quad H^{-1/2}(U^L),
\]

which is

\[
(30) \quad \alpha_j \sqrt{\lambda_j^L} \varphi_j^L - \hat{U}^* \left( \beta_k \sqrt{\lambda_k^R} \varphi_k^R \right) = \hat{U}^* \left( \gamma_k \varphi_k^R \right)
\]

where \( \gamma_k \) are the coefficients of \( \frac{\partial \varphi}{\partial x} \bigg|_{x=0} \). Thus the equation for the coefficients becomes

\[
\begin{align*}
\alpha_j \sqrt{\lambda_j^L} - \hat{U}^* \left( \beta_k \sqrt{\lambda_k^R} \right) &= \hat{U}^* \left( \gamma_k \right) \\
\alpha_j \sqrt{\lambda_j^L} - \hat{U}^* \left( \sqrt{\lambda_k^R} \beta_k^j \alpha_j \right) &= \hat{U}^* \left( \gamma_k \right)
\end{align*}
\]

(31) for \( \beta_k = \alpha_j U^L_j \) from the fact \( u^L(0,y) = u^R(0,y) = \sum_k \beta_k \varphi_k^R(y) \).
Equation (31) becomes
\begin{align}
\Lambda^L \alpha - \mathcal{U}^* \Lambda R \mathcal{U} \alpha &= \alpha_0 \\
(\Lambda^L - \mathcal{U}^* \Lambda R \mathcal{U}) \alpha &= \alpha_0 \\
(\mathbb{I} - (\Lambda^L)^{-1} \mathcal{U}^* \Lambda R \mathcal{U}) \alpha &= \alpha_0
\end{align}
(32)
where \( \alpha_0 = (\Lambda^L)^{-1} \mathcal{U}^* (\gamma_k) \), \( \Lambda^R \) the diagonal operator between \( h_R^{1/2} \) and \( h_R^{-1/2} \) which multiplies by the square root of the eigenvalues \( \sqrt{\lambda_j} \), which is in fact an isometry between those two spaces, whereas \( (\Lambda^L)^{-1} \) is analogously an isometry from \( h_L^{-1/2} \) into \( h_L^{1/2} \).

**Proposition 4.2.** The operator \( T = (\Lambda^L)^{-1} \mathcal{U}^* \Lambda R \mathcal{U} \) is a contraction on \( h_L^{1/2} \).

**Proof.** Proving that \( \mathcal{U}^* \Lambda R \mathcal{U} \) has got the same eigenvalues of \( \Lambda^R \) will be sufficient to our aim. Once we have that, we apply the well-known Weyl’s law: being \( \lambda_j \) the \( j \)-th eigenvalue of the Laplacian on a bounded regular domain \( \Omega \) of dimension \( n \), the following asymptotic behavior holds \( \lambda_j \sim C_n j^{2/n} |\Omega|^{-2/n} \) as \( j \to +\infty \) and \( C_n \) is a constant depending only on the dimension \( n \). Then, not only the ratio \( \frac{\lambda_R}{\lambda_L} < 1 \) and then \( T \) is a contraction at every point, but also the ratio is uniformly far away from 1, so that \( T \) is a contraction on the whole space \( h_L^{1/2} \).

Let us study the eigenvalues of \( \mathcal{U}^* \Lambda R \mathcal{U} \). First of all we note that \( \mathcal{U} \) is a bounded operator from \( h_L^{1/2} \) into \( h_R^{1/2} \) with operator norm less or equal to 1. In fact, \( \mathcal{U} \) is an isometry from \( L^2(U^L) \) into \( L^2(U^R) \) as well as from \( H^1_0(U^L) \) into \( H^1_0(U^R) \). Being \( H^{1/2}(U^L) \) and \( H^{1/2}(U^R) \) intermediate spaces \([L^2(U^L), H^1(U^L)][1/2] \) and \([L^2(U^R), H^1(U^R)][1/2]\) respectively, the operator \( \tilde{\mathcal{U}} : H^{1/2}(U^L) \to H^{1/2}(U^R) \) has operator norm
\[ \| \tilde{\mathcal{U}} \| \leq \| \tilde{\mathcal{U}} \|_{L^2(U^L), H^1(U^L)} \cdot \| \tilde{\mathcal{U}} \|_{L^2(U^R), H^1(U^R)} \leq 1 \]
(see [11]).

Secondly, \( h_L^{1/2} \subset h_R^{1/2} \) thanks to the relation between the eigenvalues mentioned above.

Then, \( \mathcal{U} \) is a *partially isometric operator* from \( h_R^{1/2} \) into \( h_R^{1/2} \), since it is an isometry on the subspace \( h_L^{1/2} \). So, \( \mathcal{U} \mathcal{U}^* = \mathbb{I} \) on \( h_L^{1/2} \) (see [11]), and multiplying the eigenvalue equation \( (\mathcal{U}^* \Lambda R \mathcal{U}) \alpha = \mu \alpha \) by \( \mathcal{U} \) we obtain
\[ \Lambda^R \mathcal{U} \alpha = \mathcal{U} \mu \alpha = \mu \mathcal{U} \alpha, \]
the thesis. \( \Box \)

Thanks to the previous proposition, Equation (32) has a unique solution which is nontrivial since \( \alpha_0 \neq 0 \).

We note that whenever \( \Phi \) is the solution to the problem (7), then the first component \( \alpha_1 \) of the solution \( \alpha \) to Equation (32) is for sure different from zero. This is implied by the uniqueness of a positive solution to the problem (11). Moreover, from Remark (11) it describes the asymptotic behavior of \( u^L \) for \( x \to -\infty \).

**4.1. Generalization to union of many chambers.** Let us consider a domain which is a union of several different chambers, such that the width of each chamber is negligible with respect to the corresponding length. We mean \( \Omega = C^1 \cup \ldots \cup C^N \).

The previous case \( \Omega = C^L \cup C^R \) is obviously covered by this type of domains. The proof of existence and uniqueness of a \( C^1 \) positive harmonic function in such a domain is a straightforward consequence of Theorem (13). As a matter of fact,
we can merely iterate its proof \( N - 1 \) times with the suitable (slight) modifications, where \( N \) denotes the number of the chambers.

Moreover, suppose not to know the number of the chambers, but rather the asymptotic behavior of the solution for \( x \to -\infty \), that is

\[
u(x, y) \xrightarrow{x \to -\infty} \kappa \sqrt{\lambda_1} \varphi_1(y)
\]

where \( \lambda_1 \) denotes the first eigenvalue for \( \Delta^{N-1} \) for the first chamber and \( \varphi_1(y) \) its relative eigenfunction. Then it will be

\[
k = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{N-1},
\]

where \( \alpha_i \) are the analogues of \( \alpha_1 \) in Equation (28) for the couple of chambers \((C_j, C_{j+1})\). In this way we can deduce the number of the chambers from \( \kappa \), i.e. from the solution’s asymptotic behavior at \( -\infty \).

Conversely, if the domain consists in the union of \( N \) chambers, we can immediately state that the asymptotic behavior of the unique \( C^1 \) positive harmonic function for \( x \to -\infty \) is (33) with \( \kappa \) given by (34).

**References**

[1] L. Abatangelo, V. Felli, S. Terracini, On the sharp effect of attaching a thin handle on the spectral rate of convergence. Journal of Functional Analysis 266 (2014), 3632–3684.

[2] L. Abatangelo, V. Felli, S. Terracini, Singularity of eigenfunctions at the junction of shrinking tubes. Part II. To appear in Journal of Differential Equations.

[3] R. A. Adams, J. J. F. Fournier, Sobolev Spaces, Academic Press 2003.

[4] G. Buttazzo, F. Santambrogio, Asymptotical compliance optimization for connected networks, Netw. Heterog. Media 2 (2007), no. 4, 761–777.

[5] G. Buttazzo, F. Santambrogio, N. Varchon. Asymptotics of an optimal compliance-location problem, ESAIM Control Optim. Calc. Var. 12 (2006), no. 4, 752–769.

[6] B. E. J. Dahlberg, Estimates for harmonic measure, Arch. Rational Mech. Anal. 65 (1977), no. 3, 275–288.

[7] V. Felli, S. Terracini, Singularity of eigenfunctions at the junction of shrinking tubes. Part I, J. Differential Equations, 255 (2013), n.4, 633-700.

[8] Gilbarg, Trudinger. Elliptic partial differential equations, Springer 2001.

[9] V. Isakov, Inverse problems for Partial Differential Equations, Springer 2006.

[10] D. S. Jerison, C. E. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains. Adv. in Math. 46 (1982), no. 1, 80–147.

[11] T. Kato, Perturbation theory for linear operators, Springer-Verlag 1995.

[12] A. Kirsch, An Introduction to the Mathematical Theory of Inverse Problems, Springer, 1996.

[13] M. Murata, On construction of Martin boundaries for second order elliptic equations, Publ. Res. Inst. Math. Sci. 26 (1990), no. 4, 385627.

[14] Y. Pinchover, On positive solutions of second-order elliptic equations, stability results, and classification, Duke Math. J., 57 (1988), pp. 955-980.

[15] Y. Pinchover, On positive solutions of elliptic equations with periodic coefficients in unbounded domains, Maximum principles and eigenvalue problems in partial differential equations (Knoxville, TN, 1987), 218230, Pitman Res. Notes Math. Ser., 175, Longman Sci. Tech., Harlow, 1988

[16] Y. Pinchover, On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators, Ann. Inst.H. Poincaré Anal. Non Linéaire 11 (1994), no. 3, 313-341.

[17] R. G. Pinsky, Positive harmonic functions and diffusion, Cambridge Studies in Advanced Mathematics, 45. Cambridge University Press, Cambridge, 1995.

[18] M. Protter, H. Weinberger, Maximum Principles in Differential Equations, Springer, 1984.

[19] W. Rudin, Real and Complex Analysis, McGraw-Hill, 1987.

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