INTEGRAL OPERATORS INDUCED BY THE FOCK KERNEL

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ABSTRACT. We study the $L^p$ boundedness and find the norm of a class of integral operators induced by the reproducing kernel of Fock spaces over $\mathbb{C}^n$.

1. INTRODUCTION

Our analysis will take place in the $n$-dimensional complex Euclidean space $\mathbb{C}^n$. For any two points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$ we write

$$\langle z, w \rangle = z_1 w_1 + \cdots + z_n w_n,$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

For any $t > 0$ we consider the Gaussian probability measure

$$dv_t(z) = \left( \frac{t}{\pi} \right)^n e^{-t|z|^2} dv(z)$$

on $\mathbb{C}^n$, where $dv$ is ordinary Lebesgue volume measure on $\mathbb{C}^n$. Let $H(\mathbb{C}^n)$ denote the space of all entire functions on $\mathbb{C}^n$. We then define

$$F^p_t = L^p(\mathbb{C}^n, dv_t) \cap H(\mathbb{C}^n)$$

for $0 < p < \infty$. These spaces are often called Fock spaces, or Segal-Bargman spaces, over $\mathbb{C}^n$. See [1][2][3] [7][9][11][16][18].

For $p > 0$ and $t > 0$ we are going to write

$$\|f\|_{t,p} = \left[ \int_{\mathbb{C}^n} |f(z)|^p dv_t(z) \right]^{\frac{1}{p}},$$

and

$$\langle f, g \rangle_t = \int_{\mathbb{C}^n} f(z) \overline{g(z)} dv_t(z).$$

Date: December 20, 2006.

2000 Mathematics Subject Classification. 32A36 and 32A15.

Key words and phrases. Fock spaces, Gaussian measure, integral operators.

Dostanić is partially supported by MNZZS Grant $N^o$ON144010.

Zhu is partially supported by the US National Science Foundation.
It is well known that each Fock space $F^p_t$ is a closed linear subspace of $L^p(\mathbb{C}^n, dv_t)$. In particular, in the Hilbert space setting of $L^2$, there exists a unique orthogonal projection

$$P_t : L^2(\mathbb{C}^n, dv_t) \rightarrow F^2_t.$$  

Furthermore, this projection coincides with the restriction of the following integral operator to $L^2(\mathbb{C}^n, dv_t)$:

$$S_t f(z) = \int_{\mathbb{C}^n} e^{t(z,w)} f(w) dv_t(w). \quad (1)$$

The integral kernel above,

$$K_t(z, w) = e^{t(z,w)},$$

is the reproducing kernel of $F^2_t$.

The purpose of this paper is to study the action of the operator $S_t$ on the spaces $L^p(\mathbb{C}^n, dv_s)$, where $s > 0$. We also consider the closely related integral operator

$$T_t f(z) = \int_{\mathbb{C}^n} |K_t(z, w)| f(w) dv_t(w),$$

or more explicitly,

$$T_t f(z) = \int_{\mathbb{C}^n} |e^{t(z,w)}| f(w) dv_t(w). \quad (2)$$

The main result of the paper is the following.

**Main Theorem.** Suppose $t > 0$, $s > 0$, and $p \geq 1$. Then the following conditions are equivalent.

(a) $T_t$ is bounded on $L^p(\mathbb{C}^n, dv_s)$.
(b) $S_t$ is bounded on $L^p(\mathbb{C}^n, dv_s)$.
(c) $pt = 2s$.  

Furthermore, the norms of $T_t$ and $S_t$ on $L^p(\mathbb{C}^n, dv_s)$ satisfy

$$\|S_t\| \leq \|T_t\| = 2^n$$

whenever $pt = 2s$.

The equivalence of (a), (b), and (c) is not new; it is implicit in [11] for example. So our main contribution here is the identity $\|T_t\| = 2^n$. The accurate calculation of the norm of an integral operator is an interesting but often difficult problem. We mention a few successful examples in the literature: the norm of the Cauchy projection on $L^p$ of the unit circle is determined in [13], the norm of the Cauchy projection on $L^p$ spaces of more general domains is estimated in [5], an asymptotic formula for the norm of
the Bergman projection on $L^p$ spaces of the unit ball is given in [21], and
the norm of the Berezin transform on the unit disk is calculated in [6].
As a consequence of the theorem above, we see that the densely defined
operator
\[
P_t : L^p(\mathbb{C}^n, dv_t) \to L^p(\mathbb{C}^n, dv_t)
\]
is unbounded for any $p \neq 2$. This is in sharp contrast to the theory of Hardy
spaces and the theory of Bergman spaces. For example, if $P$ is the Bergman
projection for the open unit ball $B_n$, that is, if $P$ is the orthogonal projection
\[
P : L^2(B_n, dv) \to L^2(B_n, dv) \cap H(B_n),
\]
where $H(B_n)$ is the space of holomorphic functions in $B_n$, then
\[
P : L^p(B_n, dv) \to L^p(B_n, dv)
\]
is bounded for every $p > 1$. A similar result holds for the Cauchy-Sz"ego
projection in the theory of Hardy spaces. See [14] and [20].
A more general class of integral operators induced by the Bergman kernel
on the unit ball $B_n$ have been studied in [8][12][19].
We wish to thank Peter Duren and James Tung for bringing to our attention
the references [11] and [15].

2. Preliminaries

For an $n$-tuple $m = (m_1, \ldots, m_n)$ of nonnegative integers we are going
to write
\[
|m| = m_1 + \cdots + m_n, \quad m! = m_1! \cdots m_n!.
\]
If $z \in \mathbb{C}^n$, we also write
\[
z^m = z_1^{m_1} \cdots z_n^{m_n}.
\]
When the dimension $n$ is 1, we use $dA$ instead of $dv$, and $dA_t$ instead of
$dv_t$. Thus for $t > 0$ and $z \in \mathbb{C}$, we have
\[
dA_t(z) = \frac{t}{\pi} e^{-t|z|^2} dA(z),
\]
where $dA$ is ordinary area measure on the complex plane $\mathbb{C}$.

**Lemma 1.** Let $m = (m_1, \ldots, m_n)$ be an $n$-tuple of nonnegative integers. For any $t > 0$ and $p > 0$ we have
\[
\int_{\mathbb{C}^n} |z^m|^p dv_t(z) = \frac{\prod_{k=1}^n \Gamma((pm_k/2) + 1)}{t^{pm_k/2}}.
\]
In particular,
\[
\int_{\mathbb{C}^n} |z^m|^2 dv_t(z) = \frac{m!}{t^{\frac{1}{2}|m|}}.
\]
Proof. We evaluate the integral in polar coordinates.

\[
\int_{\mathbb{C}^n} |z|^p \, d\nu_t(z) = \prod_{k=1}^n \int_{\mathbb{C}} |z_k|^{pm_k} \, dA_t(z_k)
\]
\[
= \prod_{k=1}^n \frac{t}{\pi} \int_{\mathbb{C}} |z_k|^{pm_k} e^{-t|z_k|^2} \, dA(z_k)
\]
\[
= \prod_{k=1}^n 2t \int_0^\infty r^{pm_k+1} e^{-tr^2} \, dr
\]
\[
= \prod_{k=1}^n t \int_0^\infty r^{pm_k/2} e^{-tr} \, dr
\]
\[
= \prod_{k=1}^n \Gamma((pm_k/2) + 1)
\]

The second integral is obviously a special case of the first one. \(\square\)

Recall that the restriction of the operator \(S_t\) to \(L^2(\mathbb{C}^n, d\nu_t)\) is nothing but the orthogonal projection onto \(F_t^2\). Consequently, we have the following reproducing formula.

**Lemma 2.** If \(f\) is in \(F_t^2\), then \(S_t f = f\), that is,

\[
f(a) = \int_{\mathbb{C}^n} e^{t\langle a, z \rangle} f(z) \, d\nu_t(z)
\]

for all \(a \in \mathbb{C}^n\).

A special case of the reproducing formula above is the following:

\[
K_t(a, a) = \int_{\mathbb{C}^n} |K_t(a, z)|^2 \, d\nu_t(z), \quad a \in \mathbb{C}^n.
\] (3)

As an application of this identity, we obtain the following fundamental integrals for powers of kernel functions in Fock spaces.

**Lemma 3.** Suppose \(t > 0\) and \(s\) is real. Then

\[
\int_{\mathbb{C}^n} |e^{s\langle z, a \rangle}| \, d\nu_t(z) = e^{s^2|a|^2/4t}
\]

for all \(a \in \mathbb{C}^n\).
Proof. It follows from (3) that
\[
\int_{C^n} |e^{s\langle z,a \rangle}| \, dv_t(z) = \int_{C^n} |e^{t(sa/2t, z)}|^2 \, dv_t(z) = \int_{C^n} |K_t(sa/2t, z)|^2 \, dv_t(z) = K_t(sa/2t, sa/2t) = e^{s^2|a|^2/4t}.
\]
This proves the desired identity. □

We need two well-known results from the theory of integral operators. The first one concerns the adjoint of a bounded integral operator.

**Lemma 4.** Suppose \(1 \leq p < \infty\) and \(1/p + 1/q = 1\). If an integral operator
\[
Tf(x) = \int_X K(x, y) f(y) \, d\mu(y)
\]
is bounded on \(L^p(X, d\mu)\), then its adjoint
\[
T^*: L^q(X, d\mu) \to L^q(X, d\mu)
\]
is the integral operator given by
\[
T^*f(x) = \int_X \overline{K(y, x)} f(y) \, d\mu(y).
\]

Proof. See [10] for example. □

The second result is a useful criterion for the boundedness of integral operators on \(L^p\) spaces, usually referred to as Schur’s test.

**Lemma 5.** Suppose \(H(x, y)\) is a positive kernel and
\[
Tf(x) = \int_X H(x, y) f(y) \, d\mu(y)
\]
is the associated integral operator. Let \(1 < p < \infty\) with \(1/p + 1/q = 1\). If there exists a positive function \(h(x)\) and positive constants \(C_1\) and \(C_2\) such that
\[
\int_X H(x, y) h(y)^q \, d\mu(y) \leq C_1 h(x)^q, \quad x \in X,
\]
and
\[
\int_X H(x, y) h(x)^p \, d\mu(x) \leq C_2 h(y)^p, \quad y \in X,
\]
then the operator \(T\) is bounded on \(L^p(X, d\mu)\). Moreover, the norm of \(T\) on \(L^p(X, d\mu)\) does not exceed \(C_1^{1/q} C_2^{1/p}\).

Proof. See [20] for example. □
3. Integral Operators Induced by the Fock Kernel

For any $s > 0$ we rewrite the integral operators $S_t$ and $T_t$ defined in (1) and (2) as follows.

$$S_t f(z) = \left( \frac{t}{s} \right)^n \int_{\mathbb{C}^n} e^{t(z,w) + s|w|^2 - t|w|^2} f(w) \, dv_s(w),$$

and

$$T_t f(z) = \left( \frac{t}{s} \right)^n \int_{\mathbb{C}^n} |e^{t(z,w) + s|w|^2 - t|w|^2}| f(w) \, dv_s(w).$$

It follows from Lemma 4 that the adjoint of $S_t$ and $T_t$ with respect to the integral pairing

$$\langle f, g \rangle_s = \int_{\mathbb{C}^n} f(z) \overline{g(z)} \, dv_s(z)$$

is given respectively by

$$S_t^* f(z) = \left( \frac{t}{s} \right)^n e^{(s-t)|z|^2} \int_{\mathbb{C}^n} e^{t(z,w)} f(w) \, dv_s(w),$$

and

$$T_t^* f(z) = \left( \frac{t}{s} \right)^n e^{(s-t)|z|^2} \int_{\mathbb{C}^n} |e^{t(z,w)}| f(w) \, dv_s(w).$$

We first prove several necessary conditions for the operator $S_t$ to be bounded on $L^p(\mathbb{C}^n, dv_s)$.

**Lemma 6.** Suppose $0 < p < \infty$, $t > 0$, and $s > 0$. If $S_t$ is bounded on $L^p(\mathbb{C}^n, dv_s)$, then $pt \leq 2s$.

**Proof.** Consider functions of the following form:

$$f_{x,k}(z) = e^{-x|z|^2} z_1^k, \quad z \in \mathbb{C}^n,$$

where $x > 0$ and $k$ is a positive integer.

We first use Lemma 1 to calculate the norm of $f_{x,k}$ in $L^p(\mathbb{C}^n, dv_s)$.

$$\int_{\mathbb{C}^n} |f_{x,k}|^p \, dv_s = \left( \frac{s}{p} \right)^n \int_{\mathbb{C}^n} |z_1|^{pk} e^{-(px+s)|z|^2} \, dv(z)$$

$$= \left( \frac{s}{px + s} \right)^n \int_{\mathbb{C}^n} |z_1|^p \, dv_{px+s}(z)$$

$$= \left( \frac{s}{px + s} \right)^n \Gamma((pk/2) + 1) \frac{(px + s)^{pk/2}}{(px + s)^{pk/2}}.$$
We then calculate the closed form of \( S_t(f_{x,k}) \) using the reproducing formula from Lemma 2.

\[
S_t(f_{x,k})(z) = \left( \frac{t}{t+x} \right)^n \int_{\mathbb{C}^n} e^{t(z|w)w_1^k} e^{-(t+x)|w|^2} dv(w)
\]

\[
= \left( \frac{t}{t+x} \right)^n \int_{\mathbb{C}^n} e^{(t+x)(tz/(t+x),w_1^k)} w_1^k dv_{t+x}(w)
\]

\[
= \left( \frac{t}{t+x} \right)^n \left( \frac{t}{t+x} \right)^k
\]

\[
= \left( \frac{t}{t+x} \right)^{n+k} z_1^k.
\]

We next calculate the norm of \( S_t(f_{x,k}) \) in \( L^p(\mathbb{C}^n, dv_s) \) with the help of Lemma 1 again.

\[
\int_{\mathbb{C}^n} |S_t(f_{x,k})|^p dv_s = \left( \frac{t}{t+x} \right)^{p(n+k)} \int_{\mathbb{C}^n} |z_1|^p dv_s(z)
\]

\[
= \left( \frac{t}{t+x} \right)^{p(n+k)} \frac{\Gamma((pk/2) + 1)}{s^{pk/2}}.
\]

Now if the integral operator \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \), then there exists a positive constant \( C \) (independent of \( x \) and \( k \)) such that

\[
\left( \frac{t}{t+x} \right)^{p(n+k)} \frac{\Gamma((pk/2) + 1)}{s^{pk/2}} \leq C \left( \frac{s}{px + s} \right)^n \frac{\Gamma((pk/2) + 1)}{(px + s)^{pk/2}},
\]

or

\[
\left( \frac{t}{t+x} \right)^{p(n+k)} \leq C \left( \frac{s}{s + px} \right)^{n+(pk/2)}.
\]

Fix any \( x > 0 \) and look at what happens in the above inequality when \( k \to \infty \). We deduce that

\[
\left( \frac{t}{t+x} \right)^2 \leq \frac{s}{s + px}.
\]

Cross multiply the two sides of the inequality above and simplify. The result is

\[
pt^2 \leq 2st + sx.
\]

Let \( x \to 0 \). Then \( pt^2 \leq 2st \), or \( pt \leq 2s \). This completes the proof of the lemma.

\[\square\]

**Lemma 7.** Suppose \( 1 < p < \infty \) and \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \). Then \( pt > s \).
Proof. If \( p > 1 \) and \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \), then \( S_t^* \) is bounded on \( L^q(\mathbb{C}^n, dv_s) \), where \( 1/p + 1/q = 1 \). Applying the formula for \( S_t^* \) from (4) to the constant function \( f = 1 \) shows that the function \( e^{(s-t)|z|^2} \) is in \( L^q(\mathbb{C}^n, dv_s) \). From this we deduce that

\[
q(s - t) < s,
\]

which is easily seen to be equivalent to \( s < pt \).

Lemma 8. If \( S_t \) is bounded on \( L^1(\mathbb{C}^n, dv_s) \), then \( t = 2s \).

Proof. Fix any \( a \in \mathbb{C}^n \) and consider the function

\[
f_a(z) = \frac{e^{t(z,a)}}{|e^{t(z,a)}|}, \quad z \in \mathbb{C}^n.
\]

Obviously, \( \|f_a\|_\infty = 1 \) for every \( a \in \mathbb{C}^n \). On the other hand, it follows from (4) and Lemma 3 that

\[
S_t^*(f_a)(a) = \left( \frac{t}{s} \right)^n e^{(s-t)|a|^2} \int_{\mathbb{C}^n} |e^{t(w,a)}| dv_s(w)
\]

\[
= \left( \frac{t}{s} \right)^n e^{(s-t)|a|^2} e^{t^2|a|^2/(4s)}.
\]

Since \( S_t^* \) is bounded on \( L^\infty(\mathbb{C}^n) \), there exists a positive constant \( C \) such that

\[
\left( \frac{t}{s} \right)^n e^{(s-t)|a|^2} e^{t^2|a|^2/(4s)} \leq \|S_t^*(f_a)\|_\infty \leq C \|f_a\|_\infty = C
\]

for all \( a \in \mathbb{C}^n \). This clearly implies that

\[
s - t + \frac{t^2}{4s} \leq 0,
\]

which is equivalent to

\[
(2s - t)^2 \leq 0.
\]

Therefore, we have \( t = 2s \).

Lemma 9. Suppose \( 1 < p \leq 2 \) and \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \). Then \( pt = 2s \).

Proof. Once again, we consider functions of the form

\[
f_{x,k}(z) = e^{-x|z|^2} z^k, \quad z \in \mathbb{C}^n,
\]
where \( x > 0 \) and \( k \) is a positive integer. It follows from (4) and Lemma 2 that

\[
S_t^*(f_{x,k})(z) = \left( \frac{t}{s + x} \right)^n e^{(s-t)|z|^2} \int_{\mathbb{C}^n} e^{t(z,w)} w_1^k e^{-(s+x)|w|^2} \, dv(w)
\]

\[
= \left( \frac{t}{s + x} \right)^n e^{(s-t)|z|^2} \int_{\mathbb{C}^n} e^{(s+x)(tz/(s+x),w)} w_1^k dv_{s+x}(w)
\]

\[
= \left( \frac{t}{s + x} \right)^n e^{(s-t)|z|^2} \left( \frac{tz_1}{s + x} \right)^k
\]

\[
= \left( \frac{t}{s + x} \right)^{n+k} e^{(s-t)|z|^2} z_1^k.
\]

Suppose \( 1 < p \leq 2 \) and \( 1/p + 1/q = 1 \). If the operator \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \), then the operator \( S_t^* \) is bounded on \( L^q(\mathbb{C}^n, dv_s) \). So there exists a positive constant \( C \), independent of \( x \) and \( k \), such that

\[
\int_{\mathbb{C}^n} |S_t^*(f_{x,k})|^q \, dv_s \leq C \int_{\mathbb{C}^n} |f_{x,k}|^q \, dv_s.
\]

It follows from the proof of Lemma 1 that

\[
\int_{\mathbb{C}^n} |f_{x,k}|^q \, dv_s = \left( \frac{s}{qx + s} \right)^n \frac{\Gamma((qk/2) + 1)}{(qx + s)^{qk/2}}.
\]

On the other hand, it follows from Lemma 7 and its proof that \( s - q(s-t) > 0 \), so the integral

\[
I = \int_{\mathbb{C}^n} |S_t^*(f_{x,k})|^q \, dv_s
\]

can be evaluated with the help of Lemma 1 and its proof as follows.

\[
I = \left( \frac{t}{s + x} \right)^{(n+k)} \left( \frac{S}{\pi} \right)^n \int_{\mathbb{C}^n} |z_1|^q e^{-(s-q(s-t))|z|^2} \, dv(z)
\]

\[
= \left( \frac{t}{s + x} \right)^{(n+k)} \left( \frac{s}{s - q(s-t)} \right)^n \int_{\mathbb{C}^n} |z_1|^q \, dv_{s-q(s-t)}(z)
\]

\[
= \left( \frac{t}{s + x} \right)^{(n+k)} \left( \frac{s}{s - q(s-t)} \right)^n \frac{\Gamma((qk/2) + 1)}{(s - q(s-t))^{qk/2}}.
\]

Therefore,

\[
\left( \frac{t}{s + x} \right)^{(n+k)} \left( \frac{s}{s - q(s-t)} \right)^n \frac{\Gamma((qk/2) + 1)}{(s - q(s-t))^{qk/2}}
\]

is less than or equal to

\[
C \left( \frac{s}{qx + s} \right)^n \frac{\Gamma((qk/2) + 1)}{(qx + s)^{qk/2}},
\]
which easily reduces to
\[
\left( \frac{t}{s + x} \right)^{q(n+k)} \leq C \left( \frac{s - q(s - t)}{s + qx} \right)^{n+(qk/2)}.
\]
Once again, fix \( x > 0 \) and let \( k \to \infty \). We find out that
\[
\left( \frac{t}{s + x} \right)^2 \leq \frac{s - q(s - t)}{s + qx}.
\]
Using the relation \( 1/p + 1/q = 1 \), we can change the right-hand side above to
\[
\frac{pt - s}{(p-1)s + px}.
\]
It follows that
\[
t^2(p - 1)s + t^2px \leq (pt - s)(s^2 + 2sx + x^2),
\]
which can be written as
\[
(pt - s)x^2 + [2s(pt - s) - t^2p]x + s^2(pt - s) - t^2(p - 1)s \geq 0.
\]
Let \( q(x) \) denote the quadratic function on the left-hand side of the above inequality. Since \( pt - s > 0 \) by Lemma 7, the function \( q(x) \) attains its minimum value at
\[
x_0 = \frac{pt^2 - 2s(pt - s)}{2(pt - s)}.
\]
Since \( 2 \geq p \), the numerator above is greater than or equal to
\[
pt^2 - 2ptx + ps^2 = (t - s)^2.
\]
It follows that \( x_0 \geq 0 \) and so \( h(x) \geq h(x_0) \geq 0 \) for all real \( x \) (not just nonnegative \( x \)). From this we deduce that the discriminant of \( h(x) \) cannot be positive. Therefore,
\[
[2s(pt - s) - pt^2]^2 - 4(pt - s)[s^2(pt - s) - t^2(p - 1)s] \leq 0.
\]
Elementary calculations reveal that the above inequality is equivalent to
\[
(pt - 2s)^2 \leq 0.
\]
Therefore, \( pt = 2s \).

\[\square\]

Lemma 10. Suppose \( 2 < p < \infty \) and \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \). Then \( pt = 2s \).

Proof. If \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \), then \( S_t^* \) is bounded on \( L^q(\mathbb{C}^n, dv_s) \), where \( 1 < q < 2 \) and \( 1/p + 1/q = 1 \). It follows from (4) that there exists a positive constant \( C \), independent of \( f \), such that
\[
\int_{\mathbb{C}^n} \left| e^{(s-t)|z|^2} \int_{\mathbb{C}^n} e^{i(z,w)} \left[ f(w) e^{(t-s)|w|^2} \right] dv_t(w) \right|^q dv_s(z)
\]
is less than or equal to
\[ C \int_{\mathbb{C}^n} |f(w)|^q \, dv_s(w), \]
where \( f \) is any function in \( L^q(\mathbb{C}^n, dv_s) \). Let
\[ f(z) = g(z)e^{(s-t)|z|^2}, \]
where \( g \in L^q(\mathbb{C}^n, dv_{s-q(s-t)}) \) (recall from Lemma 7 that \( s - q(s - t) > 0 \)). We obtain another positive constant \( C \) (independent of \( g \)) such that
\[ \int_{\mathbb{C}^n} |S_t g|^q \, dv_{s-q(s-t)} \leq C \int_{\mathbb{C}^n} |g|^q \, dv_{s-q(s-t)} \]
for all \( g \in L^q(\mathbb{C}^n, dv_{s-q(s-t)}) \). Therefore, the operator \( S_t \) is bounded on \( L^q(\mathbb{C}^n, dv_{s-q(s-t)}) \). Since 1 < \( q < 2 \), it follows from Lemma 9 that
\[ qt = 2[s - q(s - t)]. \]
It is easy to check that this is equivalent to \( pt = 2s \).  

We now complete the proof of the first part of the main theorem. As was pointed out in the introduction, this part of the theorem is known before. We included a full proof here for two purposes. First, this gives a different and self-contained approach. Second, as a by-product of this different approach, we are going to obtain the inequality \( \|T_t\| \leq 2^n \), which is one half of the identity \( \|T_t\| = 2^n \).

**Theorem 11.** Suppose \( t > 0, s > 0, \) and \( p \geq 1 \). Then the following conditions are equivalent.

(a) The operator \( T_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \).
(b) The operator \( S_t \) is bounded on \( L^p(\mathbb{C}^n, dv_s) \).
(c) The weight parameters satisfy \( pt = 2s \).

**Proof.** When \( p = 1 \), that (b) implies (c) follows from Lemma 8, that (c) implies (a) follows from Fubini’s theorem and Lemma 5, and that (a) implies (b) is obvious.

When \( 1 < p < \infty \), that (b) implies (c) follows from Lemmas 9 and 10, and that (a) implies (b) is still obvious.

So we assume \( 1 < p < \infty \) and proceed to show that condition (c) implies (a). We do this with the help of Schur’s test (Lemma 5).

Let \( 1/p + 1/q = 1 \) and consider the positive function
\[ h(z) = e^{\lambda|z|^2}, \quad z \in \mathbb{C}^n, \]
where \( \lambda \) is a constant to be specified later.

Recall that
\[ T_t f(z) = \int_{\mathbb{C}^n} H(z, w) f(w) \, dv_s(w), \]
where
\[ H(z, w) = \left( \frac{t}{s} \right)^n |e^{t(z,w)}e^{(s-t)|w|^2}| \]
is a positive kernel. We first consider the integrals
\[ I(z) = \int_{\mathbb{C}^n} H(z, w)h(w)^q \, dv_s(w), \quad z \in \mathbb{C}^n. \]
If \( \lambda \) satisfies
\[ t > q\lambda, \]
then it follows from Lemma 3 that
\[ I(z) = (\frac{t}{\pi})^n \int_{\mathbb{C}^n} |e^{t(z,w)}| e^{-(t-q\lambda)|w|^2} \, dv(w) \]
\[ = (\frac{t}{t-q\lambda})^n \int_{\mathbb{C}^n} |e^{t(z,w)}| \, dv_{t-q\lambda}(w) \]
\[ = (\frac{t}{t-q\lambda})^n e^{2|z|^2/4(t-q\lambda)}. \]
If we choose \( \lambda \) so that
\[ \frac{t^2}{4(t-q\lambda)} = q\lambda, \]
then we obtain
\[ \int_{\mathbb{C}^n} H(z, w)h(w)^q \, dv_s(w) \leq \left( \frac{t}{t-q\lambda} \right)^n h(z)^q \]
for all \( z \in \mathbb{C}^n. \)
We now consider the integrals
\[ J(w) = \int_{\mathbb{C}^n} H(z, w)h(z)^p \, dv_s(z), \quad w \in \mathbb{C}^n. \]
If \( \lambda \) satisfies
\[ s - p\lambda > 0, \]
then it follows from Lemma 3 that
\[ J(w) = (\frac{t}{s})^n \int_{\mathbb{C}^n} e^{t(z,w)}e^{(s-t)|w|^2} \, h(z)^p \, dv_s(z) \]
\[ = (\frac{t}{\pi})^n e^{(s-t)|w|^2} \int_{\mathbb{C}^n} |e^{t(z,w)}| e^{-(s-p\lambda)|z|^2} \, dv(z) \]
\[ = (\frac{t}{s-p\lambda})^n e^{(s-t)|w|^2} e^{2|w|^2/4(s-p\lambda)} \]
\[ = (\frac{t}{s-p\lambda})^n e^{[(s-t)+t^2/4(s-p\lambda)]|w|^2}. \]
If we choose $\lambda$ so that
\[
s - t + \frac{t^2}{4(s - p\lambda)} = p\lambda, \tag{10}
\]
then we obtain
\[
\int_{\mathbb{C}^n} H(z, w)h(z)^p \, dv_s(z) \leq \left( \frac{t}{s - p\lambda} \right)^n h(w)^p \tag{11}
\]
for all $w \in \mathbb{C}^n$. In view of Schur’s test and the estimates in (8) and (11), we conclude that the operator $T_t$ would be bounded on $L^p(\mathbb{C}^n, dv_s)$ provided that we could choose a real $\lambda$ to satisfy conditions (6), (7), (9), and (10) simultaneously.

Under our assumption that $pt = 2s$ it is easy to verify that condition (7) is the same as condition (10). In fact, we can explicitly solve for $q\lambda$ and $p\lambda$ in (7) and (10), respectively, to obtain
\[
q\lambda = \frac{t}{2}, \quad p\lambda = \frac{2s - t}{2}.
\]
The relations $pt = 2s$ and $1/p + 1/q = 1$ clearly imply that the two resulting $\lambda$’s above are consistent, namely,
\[
\lambda = \frac{2s - t}{2q} = \frac{2s - t}{2p}. \tag{12}
\]
Also, it is easy to see that the above choice of $\lambda$ satisfies both (6) and (9). This completes the proof of the theorem. \hfill \Box

**Theorem 12.** If $1 \leq p < \infty$ and $pt = 2s$, then
\[
\int_{\mathbb{C}^n} |S_t f|^p \, dv_s \leq \int_{\mathbb{C}^n} |T_t f|^p \, dv_s \leq 2^{np} \int_{\mathbb{C}^n} |f|^p \, dv_s
\]
for all $f \in L^p(\mathbb{C}^n, dv_s)$.

**Proof.** With the choice of $\lambda$ in (12), the constants in (8) and (11) both reduce to $2^n$. Therefore, Schur’s test tells us that, in the case when $1 < p < \infty$, the norm of $T_t$ on $L^p(\mathbb{C}^n, dv_s)$ does not exceed $2^n$.

When $p = 1$, the desired estimate follows from Fubini’s theorem and Lemma 3. \hfill \Box

Theorem 12 above can be stated as $\|S_t\| \leq \|T_t\| \leq 2^n$, with $S_t$ and $T_t$ considered as operators on $L^p(\mathbb{C}^n, dv_s)$. We now proceed to the proof of the inequality $\|T_t\| \geq 2^n$. Several lemmas are needed for this estimate.

**Lemma 13.** For $c > 0$ and $p \geq 1$ we have
\[
\lim_{h \to 0^+} \int_c^\infty \left[ \int_c^\infty (uv)^{-\frac{1}{2}} \exp \left( \sqrt{uv} - \frac{u + v}{2} - \frac{hv}{p} \right) \, dv \right]^p \, du = \left( 2\sqrt{2\pi} \right)^p.
\]
Proof. We begin with the inner integral
\[ I(u) = \int_{c}^{\infty} (uv)^{-\frac{1}{4}} \exp \left( \sqrt{uv} - \frac{u + v}{2} - \frac{hv}{p} \right) dv. \]
Let \( a = \frac{1}{2} + \frac{h}{p} \) and change variables according to \( v = t^2 \). Then
\[ I(u) = 2u^{-\frac{1}{4}} e^{-\frac{u}{2}} \int_{c}^{\infty} \sqrt{t} \exp(-at^2 + \sqrt{ut}) dt. \]
Write
\[ -at^2 + \sqrt{ut} = -a \left( t - \frac{\sqrt{u}}{2a} \right)^2 + \frac{u}{4a}, \]
make a change of variables according to \( x = t - (\sqrt{u}/2a) \), and simplify the result. We obtain
\[ I(u) = 2u^{-\frac{1}{4}} e^{-\frac{u}{2}} \int_{\sqrt{c} - \sqrt{u}/2a}^{\infty} \sqrt{x + \frac{\sqrt{u}}{2a}} e^{-ax^2} dx. \]
It is then clear that we can rewrite \( I(u) \) as follows.
\[ I(u) = \varphi_1(u) + \varphi_2(u), \]
where
\[ \varphi_1(u) = \frac{2}{\sqrt{2a}} e^{-\frac{u}{2}} \int_{\sqrt{c} - \frac{\sqrt{u}}{2a}}^{\infty} e^{-ax^2} dx, \]
and
\[ \varphi_2(u) = \frac{2 e^{-\frac{u}{2ap}}}{u^\frac{1}{4}} \int_{\sqrt{c} - \frac{\sqrt{u}}{2a}}^{\infty} -\sqrt{2a} e^{-ax^2} \left( \sqrt{x + \frac{\sqrt{u}}{2a}} - \sqrt{\frac{\sqrt{u}}{2a}} \right) dx. \]
For the function \( \varphi_2 \) we rationalize the numerator in its integrand to obtain
\[ |\varphi_2(u)| \leq \frac{2}{u^\frac{1}{4}} e^{-\frac{u}{2ap}} \int_{\sqrt{c} - \frac{\sqrt{u}}{2a}}^{\infty} \frac{|x|}{\sqrt{\frac{\sqrt{u}}{2a}}} e^{-ax^2} dx \leq \frac{2\sqrt{2a}}{u^\frac{1}{4}} e^{-\frac{u}{2ap}} \int_{-\infty}^{+\infty} |x| e^{-ax^2} dx. \]
A simple calculation of the last integral above then gives
\[ |\varphi_2(u)| \leq \frac{4}{\sqrt{2a}} u^{-\frac{1}{4}} e^{-\frac{u}{2ap}}. \quad (13) \]
Similarly, we have
\[ |\varphi_1(u)| \leq \frac{2}{\sqrt{2a}} e^{-\frac{u}{2ap}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\sqrt{2\pi}}{a} e^{-\frac{u}{2ap}}. \quad (14) \]
We now use the above estimates to show that
\[
\lim_{h \to 0^+} h \int_c^\infty \varphi_1(u)^{p-1} |\varphi_2(u)| \, du = 0, \tag{15}
\]
and
\[
\lim_{h \to 0^+} h \int_c^\infty |\varphi_2(u)|^p \, du = 0. \tag{16}
\]
In fact, according to (13) and (14),
\[
h \varphi_1(u)^{p-1} |\varphi_2(u)| \leq \left( \frac{\sqrt{2\pi}}{a} \right)^{p-1} \frac{4}{\sqrt{2a}} h u^{-\frac{1}{2}} e^{-\frac{uh}{2a}},
\]
from which we derive that
\[
h \int_c^\infty \varphi_1(u)^{p-1} |\varphi_2(u)| \, du \leq \left( \frac{\sqrt{2\pi}}{a} \right)^{p-1} \frac{4}{\sqrt{2a}} h \int_c^\infty u^{-\frac{1}{2}} e^{-\frac{uh}{2a}} \, du = 4 \left( \frac{\sqrt{2\pi}}{a} \right)^{p-1} \sqrt{h} \int_{\frac{\sqrt{2a}}{2a}}^{+\infty} \omega^{-\frac{1}{2}} e^{-\omega} \, d\omega.
\]
Since \(a \to \frac{1}{2}\) as \(h \to 0^+\), we obtain (15). On the other hand, it follows from (13) that
\[
|\varphi_2(u)|^p \leq \left( \frac{4}{\sqrt{2a}} \right)^p u^{-\frac{p}{2}} e^{-\frac{uh}{2a}},
\]
so
\[
h \int_c^\infty |\varphi_2(u)|^p \, du \leq \left( \frac{4}{\sqrt{2a}} \right)^p h \int_c^\infty u^{-\frac{p}{2}} e^{-\frac{uh}{2a}} \, du = 2^{p-1} a^{-p/2} h^{p/2} \int_{\frac{\sqrt{2a}}{2a}}^{+\infty} \omega^{-\frac{p}{2}} e^{-\omega} \, d\omega.
\]
Let \(h \to 0^+\) and use the fact that \(a \to \frac{1}{2}\) as \(h \to 0^+\). We obtain (16). By the change of variables \(s = uh/2a\), we have
\[
h \int_c^\infty \varphi_1(u)^p \, du = h \int_c^\infty \left( \frac{2}{\sqrt{2a}} \right)^p e^{-\frac{uh}{2a}} \left( \int_{\sqrt{c} - \frac{\sqrt{2a}}{2a}}^{+\infty} e^{-ax^2} \, dx \right)^p \, du = \frac{2^{p+1} a}{(\sqrt{2a})^p} \int_{\frac{\sqrt{2a}}{2a}}^{+\infty} e^{-s} \left( \int_{\sqrt{c} - \frac{\sqrt{2a}}{2a}}^{+\infty} e^{-ax^2} \, dx \right)^p \, ds.
\]
Let \(h \to 0^+\), notice that \(a \to \frac{1}{2}\) as \(h \to 0^+\), and use Lebesgue’s dominated convergence theorem. We get
\[
\lim_{h \to 0^+} h \int_c^\infty \varphi_1^p \, du = 2^p \int_0^{+\infty} e^{-s} \left( \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \, dx \right)^p \, ds = \left( 2\sqrt{2\pi} \right)^p. \tag{17}
\]
If $p \geq 1$, it is easy to see that the function
\[ g(z) = \frac{|1 + z|^p - |z|^p}{1 + |z|^{p-1}} \]
is continuous and bounded on $\mathbb{C}$. Replacing $z$ by $z/w$, we see that
\[ |z + w|^p - |z|^p \leq C(|z|^{p-1}|w| + |w|^p) \]
for all $z$ and $w$, where $C$ is a positive constant that only depends on $p$. This along with (15) and (16) shows that
\[
\lim_{h \to 0^+} h \int_0^\infty |(\varphi_1(u) + \varphi_2(u))^p - \varphi_1(u)^p| \, du = 0.
\]
Combining this with (17), we conclude that
\[
\lim_{h \to 0^+} h \int_0^\infty (\varphi_1(u) + \varphi_2(u))^p \, du = \left(2\sqrt{2\pi}\right)^p.
\]
This proves the desired result. \qed

**Lemma 14.** Let
\[
\mathcal{K}(x, y) = \sum_{n=0}^{\infty} \frac{t^{2n}x^ny^n}{4^n(n!)^2}
\]
and define an integral operator
\[
A : L^p(0, \infty) \to L^p(0, \infty)
\]
by
\[
Af(x) = \int_0^\infty te^{-\frac{t}{2}(x+y)}\mathcal{K}(x, y)f(y) \, dy,
\]
where $p \geq 1$ and $t$ is any fixed positive constant. Then the norm of $A$ on $L^p(0, \infty)$ satisfies $\|A\| \geq 2$.

**Proof.** It follows from the asymptotic behavior of the Bessel function $J_0(x)$ (see page 199 of [17] for example) that
\[
\sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2} \sim \frac{e^z}{\sqrt{2\pi z}}
\]
as $z \to \infty$. Thus
\[
\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sim \frac{e^{2\sqrt{\pi}}}{\sqrt{4\pi u^2}}
\]
as $u \to \infty$. Fix an arbitrary $\eta > 0$ and choose some $u_0 > 0$ such that
\[
\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} > (1 - \eta) \frac{e^{2\sqrt{\pi}}}{\sqrt{4\pi u^2}} \quad (18)
\]
for every $u \geq u_0$. 
Let $\delta_0 = 2\sqrt{u_0}/t$. It follows from (18) that

$$K(x, y) > (1 - \eta) \frac{e^{-\frac{xy}{2}}}{(xy)^{\frac{1}{4}} \sqrt{2\pi t}}$$

(19)

for all $x \geq \delta_0$ and all $y \geq \delta_0$. Fix a positive number $\varepsilon$ and let

$$f_\varepsilon(x) = \exp \left(-\frac{\varepsilon x}{p}\right).$$

Then

$$\|f_\varepsilon\| = \left[ \int_0^\infty |f_\varepsilon(x)|^p \, dx \right]^{1/p} = \varepsilon^{-\frac{1}{p}},$$

and so

$$\|A\| \geq \varepsilon^{\frac{1}{p}} \|Af_\varepsilon\|.$$ (20)

On the other hand, it follows from (19) that

$$\|Af_\varepsilon\| \geq \left( \int_{\delta_0}^\infty |Af_\varepsilon(x)|^p \, dx \right)^{1/p}$$

$$\geq \left[ \int_{\delta_0}^\infty dx \left( \int_{\delta_0}^\infty te^{-\frac{t}{2}(x+y)}K(x, y)e^{-\frac{\varepsilon y}{p}} \, dy \right)^p \right]^{\frac{1}{p}}$$

$$\geq \frac{(1 - \eta)t}{\sqrt{2\pi t}} \left[ \int_{\delta_0}^\infty dx \left( \int_{\delta_0}^\infty e^{-\frac{t}{2}(x+y)+t\sqrt{xy}-\varepsilon y - \frac{\varepsilon y}{p}} \, dy \right) \right]^{\frac{1}{p}}.$$

Combining this with (20), we obtain

$$\|A\| \geq \frac{(1 - \eta)t}{\sqrt{2\pi}} \left[ \varepsilon \int_{\delta_0}^\infty dx \left( \int_{\delta_0}^\infty e^{-\frac{t}{2}(x+y)+t\sqrt{xy}-\frac{\varepsilon y}{p}} \, dy \right) \right]^{\frac{1}{p}}.$$ (21)

After the change of variables $xt = u$ and $yt = v$ we obtain

$$\|A\| \geq \frac{1 - \eta}{\sqrt{2\pi}} \left[ \varepsilon \int_{\delta_0}^\infty du \left( \int_{\delta_0}^\infty e^{-\frac{1}{2}(u+v)+\sqrt{uv}-\frac{\varepsilon v}{p}} \, dv \right) \right]^{\frac{1}{p}}.$$ (21)

Let $t\delta_0 = c$ and $\varepsilon/t = h$. Clearly, $h \to 0^+$ when $\varepsilon \to 0^+$. Let $\varepsilon \to 0^+$ in (21) and apply Lemma [13]. We obtain

$$\|A\| \geq \frac{1 - \eta}{\sqrt{2\pi}} \cdot 2\sqrt{2\pi} = 2(1 - \eta).$$

Since $\eta > 0$ is arbitrary, we obtain $\|A\| \geq 2$, and the proof of Lemma [14] is complete.$\square$

We can now prove the main result of the paper.
Theorem 15. If $1 \leq p < \infty$ and $pt = 2s$, then the norm of $T_t$ on $L^p(\mathbb{C}^n, dv_s)$ is given by $\|T_t\| = 2^n$.

Proof. In view of Theorem 12 it is enough for us to prove the inequality $\|T_t\| \geq 2^n$.

Recall that when $n = 1$, we use the notation $dA_s$ instead of $dv_s$. For $f \in L^p(\mathbb{C}, dA_s)$ we consider

$$\Phi(z_1, \ldots, z_n) = f(z_1) \cdots f(z_n).$$

Then $\Phi \in L^p(\mathbb{C}^n, dv_s)$ and we have

$$\|T_t\|^p \geq \frac{\|T_t \Phi\|^p}{\|\Phi\|^p} = \left[ \frac{\int_{\mathbb{C}} \left| e^{tz} \int_{\mathbb{C}} |f(\zeta)| dA_t(\zeta) \right|^p dA_s(z)}{\int_{\mathbb{C}} |f(\zeta)|^p dA_s(\zeta)} \right]^n.$$

When $f$ runs over all unit vectors in $L^p(\mathbb{C}, dA_s)$, the supremum of the quotient inside the brackets above is exactly the $p$th power of the norm of the operator $T_t$ on $L^p(\mathbb{C}, dA_s)$. So we only need to prove the inequality $\|T_t\| \geq 2^n$ for $n = 1$.

Now we assume $n = 1$, $p \geq 1$, and let

$$T_t : L^p(\mathbb{C}, dA_s) \to L^p(\mathbb{C}, dA_s)$$

be the integral operator defined by

$$T_t f(z) = \int_{\mathbb{C}} |e^{tz} \cdot f(\zeta) dA_t(\zeta).$$

To obtain a lower estimate of the norm of $T_t$ on $L^p(\mathbb{C}, dA_s)$, we apply $T_t$ to a family of special functions. More specifically, we consider functions of the form

$$f(z) = G(|z|^2)e^{d|z|^2/2}, \quad z \in \mathbb{C},$$

where $G$ is any unit vector in $L^p(0, \infty)$. It follows from polar coordinates and the assumption $pt = 2s$ that

$$\|f\|^p = \int_{\mathbb{C}} |f|^p dA_s = s \int_0^\infty |G(x)|^p dx = s. \quad (22)$$
On the other hand,

\[ \int_{C} |e^{tz|w|} G(|w|^2) e^{t|w|^2/2} dA_t(w) \]

\[ = \int_{C} \left| \sum_{n=0}^{\infty} \frac{(tz|w|/2)^n}{n!} G(|w|^2) e^{t|w|^2/2} dA_t(w) \right|^2 \]

\[ = \int_{C} \left| \sum_{n=0}^{\infty} \frac{t^{2n}|z|^{2n}}{4^n(n!)^2} \int_{C} |w|^{2n} G(|w|^2) e^{t|w|^2/2} dA_t(w) \right|^2 \]

\[ = \sum_{n=0}^{\infty} \frac{t^{2n}|z|^{2n}}{4^n(n!)^2} \int_{0}^{\infty} y^n G(y) e^{-ty/2} dy \]

\[ = \int_{0}^{\infty} t e^{-ty/2} K(|z|^2, y) G(y) dy \]

\[ = e^{t|z|^2/2} AG(|z|^2), \]

where the kernel \( K \) and the operator \( A \) are from Lemma 14. Using polar coordinates and the assumption \( pt = 2s \) one more time, we obtain

\[ \| T_t f \|^p = \int_{C} |T_t f(z)|^p dA_s(z) = s \int_{0}^{\infty} |AG(x)|^p dx. \quad (23) \]

By (22) and (23) we must have

\[ \| T_t \|^p \geq \frac{\| T_t f \|^p}{\| f \|^p} = \int_{0}^{\infty} |AG(x)|^p dx. \quad (24) \]

Take the supremum over \( G \) and apply Lemma 14. The result is

\[ \| T_t \|^p \geq \| A \|^p \geq 2^p. \]

This completes the proof of the theorem. \( \square \)

We conclude the paper with two corollaries.

**Corollary 16.** For any \( s > 0 \) and \( p \geq 1 \) the Fock space \( F^p_s \) is a complemented subspace of \( L^p(\mathbb{C}^n, dv_s) \), that is, there exists a closed subspace \( X^p_s \) of \( L^p(\mathbb{C}^n, dv_s) \) such that

\[ L^p(\mathbb{C}^n, dv_s) = F^p_s \oplus X^p_s, \]

where \( \oplus \) denotes the direct sum of two subspaces.

**Proof.** Choose \( t > 0 \) such that \( pt = 2s \). Then by Theorem 11, the operator \( S_t \) is a bounded projection from \( L^p(\mathbb{C}^n, dv_s) \) onto \( F^p_s \). This shows that \( F^p_s \) is complemented in \( L^p(\mathbb{C}^n, dv_s) \). \( \square \)
The following result is obviously a generalization of Theorem 11, but it is also a direct consequence of Theorem 11.

**Corollary 17.** Suppose $a > 0$, $b > 0$, $s > 0$, and $p \geq 1$. Then the following conditions are equivalent.

(a) The integral operator
\[
T_{a,b} f(z) = \int_{\mathbb{C}^n} |e^{-a|z|^2 + (a+b)(z,w) - b|w|^2} f(w) dv(w)
\]
is bounded on $L^p(\mathbb{C}^n, dv_s)$.

(b) The integral operator
\[
S_{a,b} f(z) = \int_{\mathbb{C}^n} e^{-a|z|^2 + (a+b)(z,w) - b|w|^2} f(w) dv(w)
\]
is bounded on $L^p(\mathbb{C}^n, dv_s)$.

(c) The parameters satisfy $p(a + b) = 2(s + pa)$.

**Proof.** The boundedness of $S_{a,b}$ on $L^p(\mathbb{C}^n, dv_s)$ is equivalent to the existence of a positive constant $C$, independent of $f$, such that
\[
\int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} e^{(a+b)(z,w) - b|w|^2} f(w) dv(w) \right)^p dv_s+pa(z)
\]
is less than or equal to
\[
C \int_{\mathbb{C}^n} |f(z) e^{a|z|^2}|^p dv_s+pa(z).
\]
Replacing $f(z)$ by $f(z) e^{-a|z|^2}$, we see that the above condition is equivalent to
\[
\int_{\mathbb{C}^n} \left( \int_{\mathbb{C}^n} e^{(a+b)(z,w)} f(w) dv_{a+b}(w) \right)^p dv_s+pa(z) \leq C \int_{\mathbb{C}^n} |f|^p dv_s+pa.
\]
This is clearly equivalent to the boundedness of $S_{a+b}$ on $L^p(\mathbb{C}^n, dv_s+pa)$, which, according to Theorem 11, is equivalent to $p(a + b) = 2(s + pa)$. Therefore, conditions (b) and (c) are equivalent. The equivalence of (a) and (c) is proved in exactly the same way. \(\square\)

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