Computing Clique Cover with Structural Parameterization

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An abundance of real-world problems manifest as covering edges and/or vertices of a graph with cliques that are optimized for some objectives. We consider different structural parameters of graph, and design fixed-parameter tractable algorithms for a number of clique cover problems. Using a set representation of graph, we introduce a framework for computing clique cover with different objectives. We demonstrate use of the framework for a variety of clique cover problems. Our results include a number of new algorithms with exponential to double exponential improvements in the running time.

1 Introduction

A set of cliques is an edge (resp. vertex) clique cover of a graph if every edge (resp. vertex) is contained in at least one of the cliques in the set. In a clique cover problem, we seek to compute a clique cover that is optimized for some objectives such as minimizing number of cliques in the cover. The clique cover problems we are pursuing in this study are NP-hard, and inapproximable within any constant factor unless \( P = NP \). For applications of clique cover problems, see [1, 2, 7, 14, 17, 23, 26, 30, 33, 37–39, 43].

One of the ways of tackling intractability of NP-hard problems is to study the problems through the lens of parameterized complexity [8, 13, 19]. In parameterized complexity, a problem is called fixed-parameter tractable (FPT) with respect to a parameter \( p \) if problem instance of size \( n \) can be solved in \( f(p)n^{O(1)} \) time, where \( f(p) \) is computable and independent of \( n \), but can grow arbitrarily with \( p \). In terms of applicability, the hope is that for small values of parameter an FPT problem would be solvable in a reasonable amount of time.

1.1 Motivations

A number of clique cover problems have been shown to be FPT with respect to the number of cliques in a solution [2,17,22]. These FPT running times have super exponential dependence on the number of cliques, making them unlikely to be useful for graphs which are not sufficiently dense. For example, deciding whether edges of a graph can be covered with at most \( k \) cliques is solvable in \( 2^{O(k)}n^{O(1)} \) time, where \( f(p) \) is computable and independent of \( n \), but can grow arbitrarily with \( p \). In terms of applicability, the hope is that for small values of parameter an FPT problem would be solvable in a reasonable amount of time.

Known frameworks of computing clique covers are based on enumeration of binary matrices of dimension \( O(k) \times O(k) \) [7,17,22] (\( k \) is the number of cliques in a solution), or enumeration of maximal cliques of different subgraphs [14,22]. Clearly, enumeration of binary matrices of dimension \( O(k) \times O(k) \) disregards any sparsity measures of graph. Also, time and space complexities of algorithm designed using maximal clique enumeration may have prohibitively large dependence on input size. For example, [14] have described an algorithm for the problem of covering the edges of a graph with cliques such that the number of individual assignments of vertices to the cliques is minimum. For a graph with \( n \) vertices, the algorithm of [14] takes \( 2^{2^{O(n)}} \) time and \( 2^{O(n)} \) space. We investigate amenability of different frameworks for designing better algorithms that also take sparsity of graph into account.

Many clique cover problems are unlikely to be FPT with respect to certain choices of parameters. For example, covering vertices of a graph with at most \( k \) cliques is not FPT with respect to \( k \) unless \( P = NP \). We investigate whether structural parameters of graph would be helpful for designing FPT algorithms for these problems (and corresponding graph coloring problems).

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1Throughout the paper we assume graphs are finite, undirected, and simple, i.e., contains no self-loop and parallel edges.

2See Section 2.3 for a brief introduction of relevant terms.
1.2 Problems

We describe the clique cover problems we have studied in this paper.

Covering the edges of a graph with at most \( k \) cliques is an \textit{NP-complete} problem \([30,36]\). The problem has many applications \([39]\), and appears in different guises such as keyword conflict \([30]\), intersection graph basis \([21]\), indeterminate string \([26]\). For ease of reference, the problem is outlined as follows.

**Edge Clique Cover (ECC)**

\textbf{Input:} A graph \( G = (V,E) \), and a nonnegative integer \( k \).

\textbf{Output:} If one exists, a set of at most \( k \) cliques of \( G \) such that every edge of \( G \) is contained in at least one of the cliques in the set; otherwise \textit{NO}.

Motivated by applications from applied statistics \([23,37]\), \([14]\) have studied the \textit{assignment-minimum clique cover} problem that seeks to cover edges of a graph with cliques such that the number of individual assignments of vertices to cliques is minimum. \([14]\) have demonstrated that minimizing number of cliques and number of individual assignments of vertices simultaneously is not always possible. Therefore, number of cliques is not an appropriate choice for \textit{natural parameterization} of the problem. We introduce the following parameterized problem for \textit{assignment-minimum clique cover}.

**Assignment Clique Cover (ACC)**

\textbf{Input:} A graph \( G = (V,E) \), and a nonnegative integer \( t \).

\textbf{Output:} If one exists, a clique cover \( C \) of \( G \) such that (1) every edge of \( G \) is contained in at least one of the cliques in \( C \), and (2) \( \sum_{C_i \in C} |C_i| \leq t \); otherwise \textit{NO}.

\([17]\) have introduced the following parameterized clique cover problem.

**Weighted Edge Clique Partition (WECP)**

\textbf{Input:} A graph \( G = (V,E) \), a weight function \( w : E \rightarrow \mathbb{Z}_{>0} \), and a nonnegative integer \( k \).

\textbf{Output:} If one exists, a clique cover \( C \) of \( G \) such that (1) \( |C| \leq k \), and (2) each edge \( e \in E \) appears in exactly \( w(e) \) cliques of \( C \); otherwise \textit{NO}.

Motivated by applications from identification of gene co-expression modules, \([17]\) have introduced a generalization of \textit{WECP} problem as follows.

**Exact Weighted Clique Decomposition (EWCD)**

\textbf{Input:} A graph \( G = (V,E) \), a weight function \( w : E \rightarrow \mathbb{R}_{>0} \), and a nonnegative integer \( k \).

\textbf{Output:} If one exists, a clique cover \( C \) of \( G \) and \( \gamma_i \in \mathbb{R}_{>0} \) for all \( C_i \in C \) such that (1) \( |C| \leq k \), (2) and for each edge \( e = \{u,v\} \in E \), \( \sum_{\{u,v\} \in C_i} \gamma_i = w(e) \); otherwise \textit{NO}.

The \textit{vertex clique cover} problem (abbreviated \textit{VCC}) is the problem of deciding whether vertices of a graph can be covered (or partitioned) using at most \( k \) cliques. \textit{VCC} is \textit{NP-complete} \([28]\). In parameterized complexity, many variants of the \textit{NP-complete} problems from \([28]\) had been extensively studied such as \textit{vertex cover} \([24]\), \textit{graph coloring} \([18,27]\). \textit{VCC} in this regard had been totally unexplored. We introduce following generalization of \textit{VCC}.

**Link Respected Clique Cover (LRCC)**

\textbf{Input:} A graph \( G = (V,E) \), a nonnegative integer \( k \), and a set of edges \( E^* \subseteq E \).

\textbf{Output:} If one exists, a set of cliques \( C \) of \( G \) such that (1) \( |C| \leq k \), (2) every vertex of \( G \) is contained in at least one of the cliques in \( C \), (3) and every edge \( \{x,y\} \in E^* \) is contained in at least one of the cliques in \( C \); otherwise \textit{NO}.

\textit{LRCC} can be seen as the problem of covering vertices of a network with \( k \) communities such that certain set of links (edges) are preserved in the cover. Note that if \( E^* = E \), then \textit{LRCC} is equivalent to \textit{ECC}. On the other hand, if \( E^* = \emptyset \), then \textit{LRCC} reduces to \textit{VCC}.

\textit{Colorability} is the problem of coloring vertices of a graph such that each pair of adjacent vertices receive different colors. \textit{VCC} is equivalent to \textit{Colorability} in the complement graph. The following problem is equivalent to \textit{LRCC} in the complement graph.
**Theorem 1.2.** ECC

Theorem 1.1 by a factor of ACC

Theorem 1.3.

proof of this claim in the literature. We fill this gap by proving the following claim.

is bounded or not.

For \( \alpha \) of Theorem 1.1 is faster than the algorithm of \([22]\) regardless of whether the

Framework-2 obtained using Corollary 1.5.

Theorem 1.4.

Framework-2 2 the factor of

Theorem 1.6.

WECP not noted Framework-1

the degeneracy

and we want a conflict free assignments of vertices into \( k \) slots such that each pair of vertices in \( F \) has at least one common slot. If \( F = \emptyset \), then \( \text{ACC} \) reduces to \( \text{Colorability} \).

From the structural parameters that capture graph sparsity, we focus on degeneracy and clique number of graph. Many structural parameters (such as arboricity, thickness, treewidth, vertex cover number) are either within constant factors of or upper bounds on our chosen structural parameters. Therefore, our algorithms are also FPT with respect to many structural parameters that we do not explicitly focus on.

1.3 Our Results

For a graph \( G = (V, E) \), let \( n = |V| \) denote number of vertices, \( m = |E| \) denote number of edges, \( d \) denote the degeneracy, \( \beta \) denote the clique number, and \( \alpha \) denote the independence number.

Our results are obtained from two different algorithmic frameworks: one described in Section 3 (denoted Framework-1), and the other described in Section 4 and Section 5 (denoted Framework-2).

Using Framework-1, we obtain the following result for ECC.

**Theorem 1.1.** ECC parameterized by \( d \) and \( k \) has an FPT algorithm running in \( 1.4423^{dk}n^{O(1)} \) time.

With kernelization, we achieve a factor of \( 2^{O(\alpha^2)} \) times faster algorithm than the algorithm of \([22]\).

For \( \alpha = \Omega(n) \), the improvement in the running time amounts to a factor of \( 2^{o(n)} \). We point out that the factor of \( 2^{O(\alpha^2)} \) improvement in the running time is irrespective of graph density, i.e., the algorithm of Theorem 1.1 is faster than the algorithm of \([22]\) regardless of whether the degeneracy of input graph is bounded or not.

Using Framework-2, we obtain the following result for ECC.

**Theorem 1.2.** ECC parameterized by \( \beta \) and \( k \) has an FPT algorithm running in \( 2^\beta \log \beta nk^{O(1)} \) time.

For \( \beta = o(d/ \log d) \), the algorithm of Theorem 1.2 is asymptotically faster than the algorithm of Theorem 1.1 by a factor \( 2^{O(dk)} \). For many instances of ECC, \( \beta = o(d/ \log d) \) is a very mild requirement: \( d \) can grow linearly with the input size, while \( \beta \) remains constant.

[14] have claimed that the assignment-minimum clique cover is NP-hard; but we have not found any proof of this claim in the literature. We fill this gap by proving the following claim.

**Theorem 1.3.** ACC is NP-complete.

We describe two FPT algorithms for ACC, one is obtained using Framework-1, and the other is obtained using Framework-2. The following is our best FPT running time for ACC that we obtain using Framework-2.

**Theorem 1.4.** ACC has an FPT algorithm running in \( 4^t \log n^{O(1)} \) time.

In contrast to \( 2^{O(n)} \) running time and \( 2^{O(n)} \) space complexity of the algorithm described by [14], we obtain the following result from Theorem 1.3.

**Corollary 1.5.** An assignment-minimum clique cover of a graph \( G \) can be computed in \( 2^{O(m \log n)} \) time, using \( O(mn) \) space.

Rest of our results are obtained using Framework-2. For WECP, we have the following result.

**Theorem 1.6.** WECP parameterized by \( \beta \) and \( k \) has an FPT algorithm running in \( 2^{\beta \log \beta nk^{O(1)}} \) time.

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3See Section 2.2 for a brief description of different parameters and their relationships.

4Unless otherwise specified, base of logarithms in this paper is two.
For WECP, \[17\] have described an FPT algorithm with \(2^{O(k^{3/2} w^{1/2} \log(k/w))} n^{O(1)} \) running time, where \(w\) is the maximum edge weight. For \(\beta = o((\sqrt{k/w} \log(k/w))/\log k\), the algorithm of Theorem 1.6 improves the running time by a factor of \(2^{O(k^{3/2} w^{1/2} \log(k/w))}\). This is a significant improvement, considering the fact that \(\beta\) is bounded by the graph size, while \(k\) and \(w\) could be arbitrarily large. For nontrivial instances of WECP, \(k\) could be up to (but not including) \(mw\). On the other hand, if the maximum edge weight \(w\) is bounded by some constant, then for \(\beta = o(\sqrt{k})\), the algorithm of Theorem 1.6 is \(2^{O(k^{3/2} \log k)}\) times faster than the algorithm of \[17\].

For EWCD, we have the following result.

**Theorem 1.7.** EWCD parameterized by \(\beta\) and \(k\) has an FPT algorithm running in \(2^{O(\beta k \log k)} n^{O(1)} L\) time, where \(L\) is the number of bits required for input representation\[9\].

For EWCD, \[7\] have described an FPT algorithm with running time \(O(4^{k^2 k^3 (32^k + k^3 L)}\), where \(L\) is the number of bits required for input representation. For \(\beta = o(k/\log k)\), the algorithm of Theorem 1.7 improves the running time by a factor of \(2^{O(k^3)}\). For the cases when all weights are restricted to integers, \[7\] have described an algorithm with running time \(O(4^{k^3 32^k w^k k})\), where \(w\) is the maximum edge weight. For these cases, if \(\beta = o(k/\log k)\), then an adaptation of the algorithm of Theorem 1.7 also improves the running time by a factor of \(2^{O(k^3)}\).

For LRCC, we have the following result.

**Theorem 1.8.** LRCC parameterized by \(\beta\) and \(k\) has an FPT algorithm running in \(2^{O(k \log k)} n^{O(1)}\) time.

Through a parameterized reduction, we obtain the following result from Theorem 1.8.

**Corollary 1.9.** PMC parameterized by \(\alpha\) and \(k\) has an FPT algorithm running in \(2^{O(k \log k)} n^{O(1)}\) time.

### 1.4 Overview of Our Techniques

Our algorithms are based on combination of data reduction rules and bounded search tree algorithm. We use two different frameworks for designing bounded search tree algorithms. We make sure that all of our search tree algorithms are compatible with the existing data reduction rules (and conceivably with many rules yet to be found). For an instance of WECP (or EWCD), kernelization produces an instance of a more general problem. For these cases, to make use of the kernelization, we describe search tree algorithms for the general problems.

Our first framework (Section 3) for search tree algorithms is based on enumeration of cliques of subgraphs, introduced by [22]. With an edge selection rule, [22] have attempted to bound the branching factors of a search tree. But the rule itself does not lead to any provable bound on the branching factors. As a result, the branching factors of a search tree of [22] is only bounded by a function of \(k\) that is double exponential in \(k\) (obtained through kernelization). We enumerate cliques of subgraphs such that the size of each subgraph is bounded by a polynomial function of degeneracy. This allows us to bound the branching factors of a search tree with a single exponential function of degeneracy (or other parameters).

We introduce a general way of computing clique cover based on two concepts that we call *locally minimal clique cover* and *implicit set representation*. The concepts may be of independent interest, and we devote a fair amount of exposition highlighting their characterizations and relationships (Section 4). We use the concepts to demonstrate a general framework for designing search tree algorithms for clique cover problems (Section 4).

The concept of *locally minimal clique cover* is based on a relaxation of a global minimality of clique cover. With this relaxation, systematic exploration of clique covers of a graph becomes straightforward. Every graph corresponds to some family of sets called *set representation* of the graph [35]. The union of the sets in a *set representation* of a graph corresponds to an *edge clique cover* of the graph; but this correspondence does not immediately lead to efficient construction of *edge clique cover*. *Implicit set representation* of a graph is a relaxation of *set representation* of the graph that helps efficient construction of *edge clique cover* of a graph. The two concepts, *locally minimal clique cover* and *implicit set representation*, nicely fit together, and give us a general way of computing clique cover with different objectives.

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\[9\]More precisely, \(L\) is the number of bits required to encode the input in a linear program (see (3)).
1.5 Related Work

[22] have described the first FPT algorithm for ECC. The algorithm of [22] is based on a kernel with $2^k$ vertices and a search tree algorithm that takes $2^{O(k)}$ time. Unless the polynomial hierarchy collapses, no polynomial kernel exists for ECC [9]. The FPT running time of [22] is essentially optimal with respect to k [10], [2] have described an FPT algorithm for ECC parameterized by treewidth, and for planar graphs an FPT algorithm for ECC parameterized by branchwidth.

The algorithm of [14] is based on the observation that an edge clique cover of a graph consisting of all maximal cliques of the graph must contain an assignment-minimum clique cover of the graph. The algorithm starts with all maximal cliques of a graph, and enumerates all possible individual assignments of vertices that can be removed, maintaining an edge clique cover. The requirement of having (simultaneously) all maximal cliques of a graph with n vertices results in $O(n^{3/2}) = 2^{O(n)}$ space requirement, and enumeration of the choices of removing individual assignments of vertices results in $2^{2^{O(n)}}$ running time.

[17] have described an FPT algorithm with $2^{O(k/2 \cdot 2^{k/2}}\log(k/w))nO(1)$ running time, where w is the maximum edge weight. The FPT algorithm of [17] is based on a bi-kernel with $4^k$ vertices, and a search tree algorithm that enumerates matrices of $\{0, 1\}^{k \times k}$ such that the dot product of any pair of rows in a matrix is bounded by w.

Using the bi-kernel of [17] and a search tree algorithm that enumerates matrices of $\{0, 1\}^{2k \times k}$, [7] have described two FPT algorithms for EWCD. For arbitrary positive weights, [7] have described a linear program based search tree algorithm with running time $O(4^k \cdot 2^{k(2k + k^3)}L))$, where L is the number of bits required for input representation. For the cases when all weights are restricted to integers, [7] have described an integer partitioning based algorithm with running time $O(4^k \cdot 2^{k-3}w^k)$, where w is the maximum edge weight.

In the literature, we have not found any study on parameterizations of VCC. On the other hand, Colorability parameterized by treewidth is known to be FPT [5, 19]. Preceding result is not useful for obtaining FPT algorithm for VCC, since no parameterized reduction is known from any parameterization of VCC to Colorability parameterized by treewidth. Similarly, no parameterized reduction is known from any parameterization of VCC to other parameterizations of Colorability that are known to be FPT [27].

1.6 Paper Organization

In Section 2 we provide a brief description of relevant structural parameters and parameterized complexity terms. In Section 3 we describe FPT algorithms for ECC, ACC, and a proof of NP-completeness of ACC. The building blocks of our new framework are described in Section 4. In Section 5 using our new framework, we describe a new set of algorithms for the problems listed in Section 1.2. We conclude with a discussion on implementations and open problems in Section 6.

2 Preliminaries

In this section, we describe relevant structural parameters and their relationships. We also provide a brief description of the relevant terms from parameterized complexity.

2.1 Notations

We describe commonly used notations used in the paper. Additional notations are introduced throughout the paper where they are appropriate for introduction.

For a vertex $x \in V$, let $N(x) = \{y | \{x, y\} \in E\}$ denote the neighbourhood of x in G, and let $N[x] = N(x) \cup \{x\}$ denote the closed neighbourhood of x in G. A vertex x is called an isolated vertex if $N(x) = \emptyset$. Let $\Delta$ denote the maximum degree of G, i.e., $\Delta = \max_{x \in V} |N(x)|$. If the end points of an edge $e \in E$ are $x$ and $y$, then we denote the edge e by $\{x, y\}$; it would be clear from the context whether $\{x, y\}$ denote an edge or an arbitrary pair of vertices.

We call $H = (V_H, E_H)$ a subgraph of G if $V_H \subseteq V$ and $E_H$ is a subset of edges of $E$ that have both end points in $V_H$. We call $H$ an induced subgraph of G if $E_H$ is the set all edges of $E$ that have both end points in $V_H$. If $H$ is an induced subgraph of G, then we say $V_H$ induces the subgraph $H$ in G and denote the subgraph by $G[V_H]$. 
2.2 Structural Parameters

Definition 2.1 (Degeneracy \[^{[31]}\]). The degeneracy of a graph \(G\) is the smallest value \(d\) such that every nonempty subgraph of \(G\) contains a vertex that has at most \(d\) adjacent vertices in the subgraph.

Following Definition 2.1 we can find a vertex with at most \(d\) neighbours and delete the vertex and its incident edges. Repeating this process on the resulting subgraphs would lead to the empty graph. Therefore, in linear time \[^{[34]}\], we can compute a permutation of vertices of \(V\) defined as follows.

Definition 2.2 (Degeneracy Ordering). The degeneracy ordering of a graph \(G = (V, E)\) is an ordering of vertices in \(V\) such that each vertex has at most \(d\) neighbours that come later in the ordering.

The following two propositions ensue from the preceding definitions.

Proposition 2.3. \(d + 1 \leq n\)

Proposition 2.4. \(m < nd\)

From Proposition 2.3 and the fact that \(\sum_{x \in V} |N(x)| = 2m\), we get the following result.

Corollary 2.5. Average degree of \(G\) is at most \(2d\).

Degeneracy is within a constant factors of many sparsity measures of graph such as arboricity \[^{[4]}\] and thickness \[^{[11]}\]. Degeneracy is also known as the \(d\)-core number \[^{[11]}\], width \[^{[20]}\], linkage \[^{[29]}\], and is equivalent to the coloring number \[^{[16]}\].

Definition 2.6 (Vertex cover number). A set of vertices \(S \subset V\) is called a vertex cover of \(G\) if for every edge \(\{x, y\} \in E\), \(S\) contains at least one end point of \(\{x, y\}\). The vertex cover number \(\tau\) is the size of a smallest vertex cover of \(G\).

Definition 2.7 (Independence number). A set of vertices \(S \subset V\) is called an independent set of \(G\) if for every pair of distinct vertices \(\{x, y\} \in S\), \(\{x, y\} \notin E\). The independence number \(\alpha\) is the size of a largest independent set of \(G\).

Definition 2.8 (Treewidth \[^{[10]}\]). A tree decomposition of a graph \(G = (V, E)\) is a pair \((T, \{X_t\}_{t \in V_T})\) where \(T\) is a tree whose node set is \(V_T\) and each \(X_t \subseteq V\) such that following conditions hold: (1) \(\bigcup_{t \in V_T} X_t = V\), (2) for every edge \(\{x, y\} \in E\) there exist \(t \in V_T\) such that \(\{x, y\} \in X_t\), (3) for each \(x \in V\) the set of nodes \(\{t \in V_T | x \in X_t\}\) induces a connected subtree of \(T\).

The width of a tree decomposition \((T, \{X_t\}_{t \in V_T})\) is \(\max_{t \in V_T} |X_t| - 1\). The treewidth of a graph \(G\) is the smallest width of a tree decomposition over all tree decompositions of \(G\).

A graph \(G\) with treewidth \(k\) is also a subgraph of a \(k\)-tree (see Definition 10.2.1 of \[^{[13]}\]). \(k\)-trees are the chordal graphs all of whose maximal cliques are the same size \(k + 1\). Since any chordal graph admits an ordering of vertices such that a vertex \(x\) and neighbours of \(x\) that come later in the ordering form a clique (also known as the perfect elimination ordering), degeneracy of \(G\) is at most the treewidth of the graph. Also, treewidth of a graph is at most the vertex cover number of the graph \[^{[18]}\]. Therefore, \(d \leq \tau\).

From the fact that if \(S\) is a vertex cover of a graph, then \(V \setminus S\) is an independent set of the graph, we have \(\tau + \alpha = n\). Thus we have the following.

Proposition 2.9. \(d + \alpha \leq n\)

Remark 2.10. Degeneracy is a better measure of graph sparsity than many other parameters. To see this, consider the bipartite graph \(K_{1,n-1}\) whose degeneracy is 1, but maximum degree is \(n - 1\). A popular measure of graph sparsity is treewidth. A graph may have bounded degeneracy, but arbitrarily large treewidth: from any graph \(G\) with treewidth \(tw\), we can subdivide the edges (replace edge \(\{x, y\}\) of \(G\) with two edges \(\{x, z\}\) and \(\{z, y\}\)) and obtain a graph whose degeneracy is at most two, but whose treewidth remains \(tw\). Even extremely sparse graphs can have large treewidth: the degeneracy of an \(n \times n\) grid graph is at most three, whereas its treewidth is \(\sqrt{n}\).

Definition 2.11 (Clique number). We call a set of vertices \(C\) a clique of \(G\) if \(C\) induces a complete subgraph in \(G\). The clique number \(\beta\) of \(G\) is the size of a largest clique of \(G\).

If degree of every vertex in a subgraph of \(G\) is more than \(d\) then it would contradict Definition 2.1. Thus we have the following.
Proposition 2.12. \( \beta \leq d + 1 \).

Remark 2.13. Unlike degeneracy, computing the clique number of a graph is an NP-hard problem. But, clique number may be a better parameter for bounding running time of algorithm, if possible. To see this, consider complete bipartite graph \( K_{p,q} \), where \( p + q = n \). Degeneracy of \( K_{p,q} \) is \( \min\{p, q\} \leq n/2 \), but clique number is only 2.

The following bound on degeneracy is easy to show (for a proof see Appendix A).

Lemma 2.14. If \( G \) contains no isolated vertices, then \( d + 1 \leq 2\sqrt{m} \).

Corollary 2.15. If \( G \) contains no isolated vertices, then \( \beta \leq 2\sqrt{m} \).

If the number of cliques in a vertex clique cover of a graph \( G \) is \( k \), then the independence number of \( G \) is always a lower bound on \( k \). Assume \( G \) contains no isolated vertices. Then, any edge clique cover of \( G \) is also a vertex clique cover of \( G \). Thus we have the following.

Proposition 2.16. Let \( G \) be a graph with no isolated vertices. If \( C = \{C_1, C_2, \ldots, C_k\} \) is an edge clique cover of \( G \), then \( \alpha \leq k \).

Throughout the paper we use the following notations. Let \( \langle u_1, u_2, \ldots, u_n \rangle \) be a degeneracy ordering of \( V \). For each \( u_i \in V \), let \( N_d(u_i) \) denote the neighbours of \( u_i \) in \( G \) that comes later in the degeneracy ordering, i.e., \( N_d(u_i) = N(u_i) \cap \{u_{i+1}, \ldots, u_n\} \). Let \( N_d[u_i] = N_d(u_i) \cup \{u_i\} \). We define an ordering of edges (abbreviated DEP for degeneracy edge permutation) as follows.

Definition 2.17 (DEP). Let \( E_d(x) = \{\langle x, y \rangle \mid y \in N_d(x)\} \), i.e., the set of edges incident on \( x \) from vertices that come after \( x \) in the degeneracy ordering. Let \( \pi^E_i = \langle e_1, e_2, \ldots, e_q \rangle \) be a permutation of a set of edges \( E^* \). We say \( \pi^E \) a degeneracy edge permutation of \( G \) if \( \pi^E \) is a concatenation of permutations \( \pi^{E_d(u_i)} \) in increasing values of \( i \), i.e., \( \pi^E = (\pi^{E_d(u_1)}, \pi^{E_d(u_2)}), \ldots, \pi^{E_d(u_{n-1})}) \).

2.3 Parameterized Complexity

We provide a brief overview of fixed-parameter tractability, kernelization, bounded search tree, and other relevant terms. For a comprehensive description, see [S13][I19].

Definition 2.18 (Parameterized problem). A parameterized problem is a language \( L \subseteq \Sigma^* \times \mathbb{N} \), where \( \Sigma \) is a fixed and finite alphabet. For an instance \( (x, p) \in \Sigma^* \times \mathbb{N} \), \( p \) is called the parameter.

Definition 2.19 (Fixed-parameter tractability). A parameterized problem \( L \subseteq \Sigma^* \times \mathbb{N} \) is called fixed-parameter tractable (FPT), if for any \( (x, p) \in \Sigma^* \times \mathbb{N} \), we can decide whether \( (x, p) \in L \) in time \( f(p)\langle (x, p) \rangle^{O(1)} \), where \( f(p) \) is computable and is independent of \( x \), but can grow arbitrarily with \( p \). An algorithm is called a fixed-parameter tractable algorithm, if it decides whether \( (x, p) \in L \) in time bounded by \( f(p)\langle (x, p) \rangle^{O(1)} \).

In a discrete optimization problem, we are interested in finding optimal value of a function \( f(x) \) given that \( x \) is an instance of a language \( X \). Each such optimization problem corresponds to a decision problem \( X_D \) that asks for existence of \( x \in X \) such that \( f(x) \leq p \) or \( f(x) \geq p \), for a nonnegative integer \( p \). A parameterized decision problem considers a decision problem \( X_D \) with respect to a parameter \( p^* \) such that \( \mathbb{N} \) is the domain of \( p^* \). If \( p^* \) and \( p \) are the same, i.e., the parameter is the threshold of \( f(x) \) in a decision problem \( X_D \) that corresponds to an optimization problem, then we call \( p^* \) the natural parameterization of \( X_D \).

For convenience, the definitions of parameterized problems and fixed-parameter tractability are often described in terms of a parameter whose domain is the set of natural numbers. Without loss of generality, the definitions allow much more broader notions of parameters such as graphs, algebraic structures, vectors in \( \mathbb{N}^c \) for some fixed constant \( c \). An equivalent notion of FPT is as follows (for a proof, see Appendix A).

Proposition 2.20. An instance \( (x, p) \) of a parameterized problem is decidable in \( f(p)\langle (x, p) \rangle^{O(1)} \) time if and only if \( (x, p) \) is decidable in \( f'(p) + \langle (x, p) \rangle^{O(1)} \) time.

Definition 2.21 (Equivalent instance). Two instances \( (x, p) \) and \( (x^*, p^*) \) of a parameterized problem \( L \) are called equivalent if \( (x, p) \in L \) if and only if \( (x^*, p^*) \in L \).
A natural consequence of Proposition 2.20 is the notion of kernelization also known as data reduction or prepossessing.

**Definition 2.22 (Kernelization).** Given an instance \((x, p)\) of a parameterized problem \(L\), kernelization is an algorithm that in \(\lvert (x, p)\rvert^{O(1)}\) time reduces the instance \((x, p)\) to an equivalent instance \((x^*, p^*)\) of \(L\) such that \(\lvert x^*\rvert \leq f(p)\) and \(p^* \leq g(p)\).

We call the reduced instance \((x^*, p^*)\) a kernel. If \(f(p)\) is a polynomial (resp. exponential) function of \(p\) then we say \(L\) admits a polynomial (resp. exponential) kernel. If the reduction is to an instance of a different parameterized problem then we call the reduced instance a compression or bi-kernel. Kernelization often consists of a set of reduction rules such that each rule produces an equivalent instance. We call an instance \((x, p)\) reduced with respect to a set of rules if the rules are not applicable to \((x, p)\).

The following equivalence is straightforward to show (see Proposition 4.8.1 of [13]), which is the basis of many FPT algorithms.

**Proposition 2.23.** A parameterized problem \(L\) is FPT if and only if \(L\) admits a kernelization.

Similar to polynomial time many-one reductions (also known as Karp reductions), we have the following type of reductions for parameterized problems.

**Definition 2.24 (Parameterized reduction [8]).** Let \(L, L^* \subseteq \Sigma^* \times \mathbb{N}\) be two parameterized problems. A parameterized reduction from \(L\) to \(L^*\) is an algorithm that, given an instance \((x, p)\) of \(A\), returns an instance \((x^*, p^*)\) of \(L^*\) such that (1) \((x, p)\) is a YES instance of \(L\) if and only if \((x^*, p^*)\) is a YES instance of \(L^*\), (2) \(p^* \leq g(p)\) for some computable function \(g\), and (3) runs in \(f(p)\lvert x\rvert^{O(1)}\) time for some computable function \(f\).

From Definition 2.21 if there exists a parameterized reduction from \(L\) to \(L^*\) and \(L^*\) has an FPT algorithm with respect to \(p^*\), then \(L\) also has an FPT algorithm with respect to \(p\).

**Bounded search tree.** Bounded search tree is one of the most commonly used techniques for designing FPT algorithms that often produces provably better FPT algorithms than many other techniques. The idea is to treat the sequence of decisions made by an algorithm as a tree \(T\). Each node \(u\) of \(T\) corresponds to a choice made by the algorithm such as deciding to include an element in a partial solution. Children of a node \(u\) correspond to subsequent choices that arise as ramifications of the choice made at node \(u\). Each node \(u\) corresponds to a branch of the tree rooted at node \(u\), and the number of children of \(u\) is called the branching factor of node \(u\). An algorithm systematically explores the ramifications of choices made at node \(u\) by recursively visiting nodes of all branches rooted at children of \(u\) until a solution is found. While visiting a child node of \(u\), the algorithm makes substantial progress such as increasing the size of a partial solution. By a recurrence relation, time required in a subtree \(T_u\) rooted at node \(u\) can be bounded in terms of time required in the subtrees rooted at the children of \(u\) and time required at node \(u\) alone. The recurrence relation can be solved to provide an upper bound on the time requirement of the algorithm. A simple strategy often suffices for bounding time requirement of a search tree algorithm. Let the depth or the height of a tree \(T\) be bounded by \(h(p)\), and let the branching factor of any node of \(T\) be bounded by \(b(p)\). Then the number of nodes in \(T\) is bounded by \(b(p)^{h(p)}\). If time spent at each node is bounded by a polynomial in \(\lvert (x, p)\rvert\), then total time required by the algorithm is bounded by \(b(p)^{h(p)}\lvert (x, p)\rvert^{O(1)}\), which is an FPT running time.

## 3 Enumerating Cliques of Restricted Subgraph

In this section, we describe FPT algorithms for \(ECC\) and \(ACC\) (which can be extended to design algorithms for other clique cover problems such as the problems described in Section 1.2). Our focus is on bounded search tree algorithms for the problems, but we touch upon a number of relevant data reduction rules. The general theme of these search tree algorithms is as follows. In each node of a search tree, the corresponding algorithm branches on a set of cliques constructed from a restricted subgraph. The restriction is justified with a goal to bound the branching factors in terms of degeneracy or other parameters. A proof of \(NP\)-completeness of \(ACC\) is included at the end of this section.

### 3.1 Edge Clique Cover

The exponential kernel of [22] is based on a result of [25]. We use the result of [25] and include an exposition of a proof of the result adapted from [22].
Definition 3.1 (Equivalent Vertices). Let \( \{x, y\} \) be an edge of \( G \). \( x \) and \( y \) are equivalent vertices if the vertices have the same closed neighbourhood, i.e., \( N[x] = N[y] \).

Lemma 3.2 ([24]). Let \( G \) be a graph with \( n \) vertices such that \( G \) contains neither isolated vertices nor equivalent vertices. If \( C = \{C_1, C_2, \ldots, C_k\} \) is an edge clique cover of \( G \), then \( n < 2^k \).

Proof. For the sake of contradiction suppose \( n \geq 2^k \). Let \( B \in \{0, 1\}^{n \times k} \) be a matrix such that \( B[x, l] = 1 \) if vertex \( x \) is contained in clique \( C_l \), otherwise, \( B[x, l] = 0 \). The number of distinct rows in \( B \) is at most \( 2^k \). If it is \( 2^k \), then included among these rows is a row \( x \) such that \( B[x, l] = 0 \) for all \( l \in [k] \). If the number of distinct rows is less than \( 2^k \), then, since \( n \geq 2^k \), there exist two distinct rows \( x \) and \( y \) such that \( B[x, l] = B[y, l] \) for all \( l \in [k] \). The first condition leads to an isolated vertex, and the second condition leads to a pair of equivalent vertices. Both conditions lead to contradiction, and the claim follows.

An immediate consequence of Proposition 3.2 and Lemma 3.2 is as follows.

Corollary 3.3. Let \( G \) be a graph with degeneracy \( \alpha \) and independence number \( \beta \) such that \( G \) contains neither isolated vertices nor equivalent vertices. If \( C = \{C_1, C_2, \ldots, C_k\} \) is an edge clique cover of \( G \), then \( \alpha + \beta < 2^k \).

Let \( (G, k) \) be an instance of ECC. The following kernelization rules ensue from Lemma 3.2.

Rule 3.4. If \( x \in V \) is an isolated vertex, then \( (G - \{x\}, k) \) and \( (G, k) \) are equivalent instances. For a solution \( C \) of \( (G - \{x\}, k) \), report \( C \) as the solution of \( (G, k) \).

Rule 3.5. If \( \{x, y\} \in E \) and \( N[x] = N[y] \), then \( (G - \{x\}, k) \) and \( (G, k) \) are equivalent instances. For a solution \( C \) of \( (G - \{x\}, k) \), report \( (C \setminus \{y \in C_i \}) \cup \{C_i \cup \{x\} \} y \in C_i \) as the solution of \( (G, k) \).

Let \( (G, k) \) be a reduced instance with respect to Rules 3.4 and 3.5. We apply the following rule on \( (G, k) \).

Rule 3.6. If \( n > (d + 1)k \), then report \( (G, k) \) is a NO instance of ECC.

The correctness of Rule 3.6 follows from the fact that with \( k \) cliques and clique number bounded by \( d + 1 \), we can cover at most \( (d + 1)k \) vertices. Since \( (G, k) \) is reduced with respect to Rule 3.4, an edge clique cover must include all the vertices of \( G \).

Let \( (G, k) \) be a reduced instance with respect to Rules 3.4, 3.5, and 3.6. We have a kernel with \( (d + 1)k \) vertices. Note that only Rules 3.4 and 3.5 are sufficient to get the kernel. We need Rule 3.6 to show a better search tree algorithm.

The search tree algorithm of [22] (henceforth referred as ECCG) works by enumerating all maximal cliques that contain an uncovered edge selected at every node of a search tree. For analysis of FPT algorithm, it is sufficient to consider number of maximal cliques in a subgraph, since each such clique can be listed spending polynomial time per clique [22].

Lemma 3.7 ([15]). The number of maximal cliques in a graph with \( n \) vertices is at most \( 3^{n/3} \).

Using Lemma 3.7, we can obtain a bounded search tree algorithm with depth at most \( k \) and branching factor at most \( 3^{(d+1)k/3} \), i.e., an algorithm \( A \) that takes \( 3^{(d+1)k/3}n^{O(1)} \) time. By Lemma 3.7, ECCG takes \( 3^{2d/3}n^{O(1)}n^{O(1)} \) time. The running time of \( A \) would be better (or competitive) than the running time of ECCG if \( d = O(2^k/k) \). Next, we describe a search tree algorithm with (unconditionally) better running time than ECCG.

Figure 1 shows a bounded search tree algorithm for ECC, henceforth referred as ECCS. We assume ECCS has access to a degeneracy edge permutation of \( G \). At step 3, we select the first uncovered edge \( \{x, y\} \) from the degeneracy edge permutation such that \( y \in N_d(x) \). At step 4, we restrict the search for maximal cliques in a subgraph induced by vertices of \( N_d[x] \cap N[y] \) in \( G \). In each iteration of step 4, we select a maximal clique from the restricted search space, and cover the edge \( \{x, y\} \) with the clique and recurse. This process is repeated until a solution is found in a branch (step 4b), or all branches are exhausted without any solution (step 5).

We point out the crucial differences between ECCG and ECCS. At step 3, ECCG uses a heuristic to select an edge, for which ECCG has to enumerate maximal cliques of the subgraph \( G[N[x] \cap N[y]] \), in contrast to enumerating maximal cliques of a restricted subgraph such as \( G[N_d[x] \cap N[y]] \). Consequently,
Definition 2.17 of cliques for an edge \( E \) following invariant maintained by \( \text{ECC} \):

1. if \( \mathcal{C} \) covers edges of \( G \), then return \( \mathcal{C} \)
2. if \( k < 0 \), then return \( \emptyset \)
3. select the first uncovered edge \( \{x, y\} \) from the \( \text{DEP} \) of \( G \) where \( y \in N_d(x) \) // Definition 2.17
4. for each maximal clique \( Z \) containing \( \{x, y\} \) in the subgraph \( G[N_d[x] \cap N[y]] \) do
   (a) \( Q \leftarrow \text{ECC}(G, k-1, \mathcal{C} \cup \{Z\}) \)
   (b) if \( Q \neq \emptyset \), then return \( Q \)
5. return \( \emptyset \)

Figure 1: A bounded search tree algorithm for \( \text{ECC} \), denoted \( \text{ECCS} \).

\( \text{ECCS} \) can only provide an upper bound of \( 3^{2k/3} \) on the number of maximal cliques enumerated at step 4: \textit{kernelization}, based on Lemma 3.2.

Next, we describe correctness and running time of \( \text{ECCS} \). From the choices made at step 3, we observe following invariant maintained by \( \text{ECCS} \).

Proposition 3.8. Let \( \langle u_1, u_2, \ldots, u_n \rangle \) be a degeneracy ordering of \( V \). At any node of a search tree of \( \text{ECCS} \), if \( x = u_i \), then all edges incident on vertices of \( \{u_1, u_2, \ldots, u_{i-1}\} \) are covered.

We also note a property that holds for induced subgraphs, but does not necessarily hold for arbitrary subgraphs.

Proposition 3.9. For a graph \( G = (V, E) \), let \( X \subseteq V \) be a set of vertices. If \( (G, k) \) is a YES instance of \( \text{ECC} \), then \( (G[X], k) \) is a YES instance of \( \text{ECC} \).

Lemma 3.10. Algorithm \( \text{ECCS} \) correctly solves the parameterized problem \( \text{ECC} \).

Proof. Let \( \mathcal{A} \) be a family of algorithms that consider uncovered edges \( \{x, y\} \) in arbitrary order in step 3 of \( \text{ECCS} \), and in step 4 branch into each of the choices of maximal cliques containing \( \{x, y\} \) in the subgraph induced by \( N[x] \cap N[y] \).

Consider any algorithm \( A_i \in \mathcal{A} \). For a search tree \( T \) of \( A_i \), let us define a node \( u^T \) to be a YES node if \( A_i \) returns at step 1 of \( u^T \). We claim that \( (G, k) \) is a YES instance of \( \text{ECC} \) if and only if \( A_i \) returns from a YES node. If \( A_i \) returns from a YES node, then clearly \( (G, k) \) is a YES instance of \( \text{ECC} \). It remains to show that if \( (G, k) \) is a YES instance of \( \text{ECC} \), then \( A_i \) would return from a YES node. This can be seen as follows. \( A_i \) considers all possible ways to cover every uncovered edge: \( \{x, y\} \) can be covered with one of the maximal cliques containing \( \{x, y\} \) in the subgraph induced by \( N[x] \cap N[y] \). \( A_i \) considers all such possible maximal cliques, and allows a search tree containing at most \( k \) cliques in any leaves of the search tree.

\( \text{ECCS} \) deviates from the family of algorithms \( \mathcal{A} \). Next, we show that the deviations are safe. Clearly, if \( \text{ECCS} \) returns from a node at step 1, then \( (G, k) \) is a YES instance of \( \text{ECC} \). Therefore, we only need to show that if any algorithm \( A_i \in \mathcal{A} \) returns from a YES node, then \( \text{ECCS} \) would also return from a node at step 1.

At each node of the search tree of \( \text{ECCS} \), \( y \in N_d[x] \), and \( N_d[x] \subseteq \{u_1, u_{i+1}, \ldots, u_n\} \). Therefore, \( N_d[x] \cap N[y] \subseteq \{u_1, u_2, \ldots, u_n\} \). In the enumeration of maximal cliques, vertices that are not considered by \( \text{ECCS} \) are \( (N[x] \cap N[y]) \setminus (N_d[x] \cap N[y]) = (N(x) \setminus N_d(x)) \cap N(y) \subseteq N(x) \setminus N_d(x) \subseteq \{u_1, u_2, \ldots, u_{i-1}\} \).

By Proposition 3.8, all edges incident on vertices of \( \{u_1, u_2, \ldots, u_{i-1}\} \) are covered when we are considering maximal cliques for an edge \( \{x, y\} \). By Proposition 3.9, if \( (G[N[x] \cap N[y]], k^*) \) is a YES instance of \( \text{ECC} \), then \( (G[N_d[x] \cap N[y]], k^*) \) is a YES instance of \( \text{ECC} \). Therefore, it is safe to consider maximal cliques for an edge \( \{x, y\} \) only in the subgraph \( G[N_d[x] \cap N[y]] \). It follows that if any algorithm \( A_i \in \mathcal{A} \) returns from a YES node, then \( \text{ECCS} \) would also return from a node at step 1. \qed
Lemma 3.11. The number of nodes in a search tree of \( \text{ECCS} \) is at most \( 1.4423^{dk} \).

Proof. The number of vertices in a subgraph induced by vertices of \( N_d[x] \cap N[y] \) is at most \( d + 1 \), i.e., \( |N_d[x] \cap N[y]| \leq d + 1 \). Since the vertices \( x \) and \( y \) are fixed, we only need to enumerate maximal cliques of the subgraph \( G[N_d(x) \cap N(y)] \), containing at most \( d - 1 \) vertices.

By Lemma 3.7, at each node of the search tree, the number of maximal cliques containing \( x \) and \( y \) in the subgraph induced by vertices of \( N_d[x] \cap N[y] \) is at most \( 3^{(d-1)/3} < 3^{d/3} \). Therefore, the total number of nodes in the search tree is at most \( 3^{dk/3} \leq 1.4423^{dk} \). \( \Box \)

From Figure 1 it is evident that the time spent at every node of a search tree of \( \text{ECCS} \) is bounded by a polynomial of input size. Considering the time needed for data reduction, we obtain Theorem 1.1 from Lemma 3.11 and Proposition 2.20.

Next, we discuss a number of consequences of \( \text{ECCS} \).

Using Lemma 3.7, the number of nodes is a search tree of \( \text{ECCG} \) is bounded by \( 3^{2^k/3} \). By Corollary 3.3, \( 2^k > d + \alpha \). Therefore, we have a factor of \( 3^{2^k/3} > 3^{(d+\alpha)/3} = 3^{k/3} = 2^{(k/3)} = 2^{\Omega(k\alpha^3)} \) improvement over the bound on search tree size. The last equality follows from Lemma 2.10. The improvement becomes significant for sparse graphs, such as when \( d = o(2^k) \) or \( \alpha = \Omega(n) \), and we may obtain a factor of \( 2^{2^{\Omega(k\alpha^3)}} \) reduction on the number of nodes in the search tree of \( \text{ECCS} \) compared to \( \text{ECCG} \). Note that with kernelization we have \( n < 2^k \). Therefore, preceding improvement carry over to the overall running time.

We point out that single exponential dependence on \( k \), with low degree polynomial of \( k \) in the exponent, suffices for sparse graphs.

Corollary 3.12. For \( d = O(k) \), \( \text{ECC} \) has an FPT algorithm running in \( 2^{O(k^3)} n^{O(1)} \) time.

Corollary 3.13. For \( k = \Omega(\sqrt{m}) \), \( \text{ECC} \) has an FPT algorithm running in \( 2^{O(k^3)} n^{O(1)} \) time.

Corollary 3.13 follows from Lemma 2.14.

From the preceding two observations, it is obvious that the inherent difficulty of solving \( \text{ECC} \) on sparse graphs is poorly captured by parameter \( k \) alone. Next, we demonstrate that this difficulty is indeed better captured by degeneracy.

Corollary 3.14. If \( m \geq \lfloor n^2/4 \rfloor \), then running time of \( \text{ECCS} \) is \( 2^{O(d^3)} n^{O(1)} \).

Proof. For \( \text{ECC} \), we can assume \( k \leq \lfloor n^2/4 \rfloor \). Also, by Proposition 2.4, we can assume \( k \leq m < nd \). Since \( \lfloor n^2/4 \rfloor \leq m \), combining preceding inequalities, we have \( k < 4d^2 \). Therefore, \( \text{ECCS} \) takes \( 2^{O(d^3)} n^{O(1)} \) time.

By algebraic manipulation similar to the preceding proof, we can show the following.

Corollary 3.15. For \( c > 0, \epsilon > 0, \) if \( m \geq cn^{1+\epsilon} \), then running time of \( \text{ECCS} \) is \( 2^{O(d^3/d^{1-\epsilon})} n^{O(1)} \).

Corollary 3.16. For \( c > 0, \epsilon > 0, \) if \( m \geq cn(\log n)^\epsilon \), then running time of \( \text{ECCS} \) is \( 2^{O(d^2/d^{1-\epsilon})} n^{O(1)} \).

The preceding observations show that the dependence of the running time of \( \text{ECCS} \) on degeneracy shifts from single exponential to double exponential as graph becomes sparser. Unlike double exponential dependence on \( k \), this shift captures what we expect for sparse graphs.

\( \text{ECC} \) is \( NP \)-complete on planar graphs \([19]\). Degeneracy of planar graph is at most 5. Thus we have the following.

Corollary 3.17. For planar graphs, \( \text{ECC} \) has an FPT algorithm running in \( 2^{O(k)} n^{O(1)} \) time.

Note that for planar graphs we can easily obtain a kernel with \( 4k \) vertices, since clique number of planar graph is at most 4. Then, instead of running time with exponent linear in \( k \) (as in Corollary 3.13), we would get running time with exponent quadratic in \( k \).
Assignment-Clique-Cover-Search($G, t, C$):

```cpp
// Abbreviated ACCS($G, t, C$)
1. if $C$ covers edges of $G$, then return $C$
2. if $t < 2$, then return $\emptyset$
3. select the first uncovered edge $\{x, y\}$ from the DEP of $G$ where $y \in N_d(x)$ // Definition 2.17
4. for each clique $C$ in the subgraph $G[N_d[x] \cap N[y]]$ do
   (a) $Q \leftarrow \emptyset$
   (b) if $t - |Z| \geq 0$, then $Q \leftarrow$ ACCS($G, t - |Z|, C \cup \{Z\}$)
   (c) if $Q \neq \emptyset$, then return $Q$
5. return $\emptyset$
```

Figure 2: A bounded search tree algorithm for ACC, denoted ACCS.

### 3.2 Assignment Clique Cover

We describe an FPT algorithm for ACC. First, we describe a set of simple data reduction rules that reduce a given instance $(G, t)$ of ACC. We note that many data reduction rules for ECC may not be directly applicable for ACC, for example Rule 3.5 (we expand on this later).

**Rule 3.18.** If $x \in V$ is an isolated vertex or a vertex with no uncovered incident edges, then $(G - \{x\}, t)$ and $(G, t)$ are equivalent instances. For a solution $C$ of $(G - \{x\}, t)$, report $C$ as the solution of $(G, t)$.

**Rule 3.19.** Let $x$ be an uncovered vertex with all its incident edges uncovered, and $N[x]$ induces a clique $C$ in $G$. Mark all edges of $C$ as covered. Then $(G - \{x\}, t - |N[x]|)$ and $(G, t)$ are equivalent instances. For a solution $C^*$ of $(G - \{x\}, t - |N[x]|)$, report $C^* \cup \{C\}$ as the solution of $(G, t)$.

Rule 3.19 is correct. Every edge $\{x, y\} \in E$ has to be covered by a clique. Therefore, covering the edge $\{x, y\}$ with a clique other than the clique induced by $N[x]$ would increase number of assignments of $x$.

**Rule 3.20.** Let $e = \{x, y\}$ be an uncovered edge, and $N(x) \cap N(y) = \emptyset$. Mark the edge $e$ as covered. Then $(G - \{e\}, t - 2)$ and $(G, t)$ are equivalent instances. For a solution $C^*$ of $(G - \{e\}, t - 2)$, report $C^* \cup \{\{x, y\}\}$ as the solution of $(G, t)$.

Rule 3.20 is correct: edge $\{x, y\}$ can only be covered by a clique $C = \{x, y\}$. Let $(G, t)$ be a reduced instance with respect to Rules 3.18 3.19 and 3.20. We apply following rule on $(G, t)$.

**Rule 3.21.** If $n > t$, then report $(G, t)$ is a NO instance of ACC.

Since $(G, t)$ is reduced with respect to Rule 3.18 an assignment clique cover must include all the vertices of $G$. Therefore, Rule 3.20 is correct. Next, we describe a bounded search tree algorithm for ACC.

Figure 2 shows a bounded search tree algorithm for ACC; henceforth referred as ACCS. ACCS uses the same strategy as ECCS to select uncovered edge at step 3. Unlike ECCS, ACCS searches for cliques rather than maximal cliques at step 4. But, the restriction of search space for cliques is same as ECCS: subgraph induced by vertices of $N_d[x] \cap N[y]$ in $G$.

The need for enumerating all cliques in step 4, instead of only maximal cliques, follows from the fact that a maximal clique may include additional vertices not required for an edge clique cover, and the corresponding branch of a search tree may miss a solution due to excess decrement of the value of parameter $t$. The correctness of subgraph restriction is similar to what has been described in the proof of Lemma 3.10. Therefore, we conclude the following.

**Lemma 3.22.** Algorithm ACCS correctly solves the parameterized problem ACC.
Theorem 3.23. ACC has an FPT algorithm running in $1.4143^2 n^{O(1)}$ time.

Proof. As we have shown in the proof of Lemma 3.11 at each node of a search tree of ACCS, we only need to enumerate cliques of the subgraph $G[N_d(x) \cap N[y]]$ with at most $d-1$ vertices. At each node of a search tree, the number of cliques containing the edge $\{x, y\}$ in the subgraph $G[N_d[x] \cap N[y]]$ is at most $\sum_{s=1}^{d-1} (d-1)^{s-1}$. The depth of a search tree is bounded by $t/2$. Therefore, the total number of nodes in a search tree is bounded by $2^{dt/2} \leq 1.4143^2$, since by Rule 3.21 $d < n \leq t$. The claim follows from Proposition 2.20 considering time needed for data reduction.

Remark 3.24. For a reduced instance of ACC with respect to Rule 3.78, $n \leq t$. Therefore, to achieve a running time with respect to parameter $t$, it would suffice to select edges arbitrarily at step 3 of ACCS, and enumerate cliques of subgraph $G[N[x] \cap N[y]]$ at step 4. The restriction of search for cliques in a subgraph $G[N_d[x] \cap N[y]]$ is what makes the algorithm attuned to sparse graphs.

Corollary 3.25. An assignment-minimum clique cover of a graph $G$ can be found in $2^{O(mn)}$ time using $O(m^{3/2})$ space.

Proof. We can invoke the FPT algorithm for ACC that we just have described with values of $t$ in the range $[1, 2n]$ (by linearly increasing values of $t$ or doing a binary search in the range) until a solution is found. Total running time would be $\sum_{t \in [2m]} 2^{O(dt)} n^{O(1)} = 2^{O(dm)} = 2^{O(mn)}$.

The depth a search tree of ACCS is at most $m$, and each such clique would take $O(d)$ space. Also, enumeration of cliques in step 4 can be done using a binary vector of size $O(d)$. Therefore, the total space usage of ACCS is $O(dm) = O(m^{3/2})$. The last equality follows from Lemma 2.19.

We have noted that Rule 3.5 is not directly applicable for ACC, but we can use rules like that in a broader context. The following two data reduction rules are applicable for computing an assignment-minimum clique cover of a graph $G$.

Rule 3.26. Let $\{x, y\}$ be an edge of $G$ and $N[x] = N[y]$. If $C$ is an assignment-minimum clique cover of $G \setminus \{x\}$, then $(C \setminus \{C \cap y \in C\}) \cup \{C \cup \{x\} \setminus \{y \in C\} \}$ is an assignment-minimum clique cover of $G$.

Rule 3.27. Let $x$ and $y$ be distinct vertices of $G$ such that $\{x, y\} \notin E$ and $N(x) = N(y)$. If $C$ is an assignment-minimum clique cover of $G \setminus \{x\}$, then $C \cup \{C \setminus \{y\} \cup \{x\} \setminus \{y \in C\}, C \cap \{x\} \subseteq C \}$ is an assignment-minimum clique cover of $G$.

Since a corresponding correct parameter $t$ for $G \setminus \{x\}$ cannot be known beforehand, there is no equivalent of Rule 3.20 or Rule 3.27 for ACC. It is straightforward to see that the preceding two rules are correct.

3.3 Other Clique Covers

What we have shown in the preceding discussion for ECC and ACC can be extended to design algorithms for other clique cover problems, namely WECP, EWCD, and LRVCC. We refrain from going into the details of these extensions. Instead, we describe algorithms for these problems using a different framework in Section 5 one may utilize the descriptions in Section 5 to design full-fledged algorithms based on enumeration of cliques of restricted subgraph.

3.4 NP-completeness of ACC

We conclude this section with a proof of Theorem 1.3. We use a construction used by [30] (Proposition 2).

Proof of Theorem 1.3. ACC is in NP: given an instance $(G, t)$ and a certificate $C$, in polynomial time we can verify that $C$ covers edges of $G = (V, E)$, and $\sum_{C \cap C} |C| \leq t$.

Let $(G^*, k)$ be an instance of VCC. We construct an instance $(G, t)$ of ACC from $(G^*, k)$, and show that $(G^*, k)$ is a YES instance of VCC if and only if $(G, t)$ is a YES instance of ACC.

Let $G^* = (V^*, E^*)$ has $n$ vertices and $m$ edges. To construct $G$, we start with an empty graph and include all the vertices and edges of $G^*$. Then, we include a set of additional vertices $X = \{x_1, x_2, \ldots, x_q\}$ with $q \geq 2m + 1$. Then, we connect each pair of vertices $x_i \in X$ and $v \in V^*$ with an edge. Resulting
graph $G = (V, E)$ has vertices $V = V^* \cup X$ and edges $E = E^* \cup \{x_i, v\} | x_i \in X, v \in V^*$. We complete the construction by setting $t = (n + k)q + 2m$.

**Only if.** Let $(G^*, k)$ be a YES instance of VCC, and $C^* = \{C^*_1, C^*_2, \ldots, C^*_n\}$ be a corresponding vertex clique cover of $G^*$. Since each vertex $x_i$ is connected to all the vertices of $V^*$ and no pair of vertices in $X$ are connected by edge, we can cover all the edges incident to $x_i$ with a set of cliques $C(x_i) = \{C^*_i \cup \{x_i\} | l \in [k]\}$. Since the number of individual assignments of vertices to cliques in $C^*$ is $n$, the number of individual assignments of vertices to cliques in $C(x_i)$ is $n + k$. Therefore, to cover the edges incident on a vertex $x_i$, we would need at most $\sum_{C_i \in C(x_i)} |C| \leq (n + k)q$ individual assignments of vertices to cliques. To cover all the edges incident on the vertices of $X$, we would need at most $\sum_{x_i \in X} \sum_{C_i \in C(x_i)} |C| \leq (n + k)q$ individual assignments of vertices to cliques. Finally, to cover $m$ edges of $G^*$, we would need at most $2m$ individual assignments of vertices to cliques. Therefore, in polynomial time, we can construct an edge clique cover $\mathcal{C}$ of $G$ such that $\sum_{C_i \in \mathcal{C}} |C| \leq (n + k)q + 2m = t$. It follows that $(G, t)$ is a YES instance of ACC.

**If.** Let $(G, t)$ be a YES instance of ACC, and $\mathcal{C}$ be a corresponding assignment clique cover of $G$. Let $\mathcal{C}(x_i) = \{C_i | x_i \in C_i, C_i \in \mathcal{C}\}$. For any $i \in [q]$, the set of cliques $\mathcal{C}(x_i)$ with $x_i$ removed is a vertex clique cover of $G^*$. We need to choose $i \in [q]$ such that $|\mathcal{C}(x_i)| \leq k$. Note that for any pair of distinct vertices $x_i$ and $x_j$ of $X$, $\mathcal{C}(x_i)$ and $\mathcal{C}(x_j)$ are disjoint, since $\{x_i, x_j\} \notin E$. To show $(G^*, |\mathcal{C}(x_i)|)$ is a YES instance of VCC, choosing $\mathcal{C}(x_i)$ such that $\sum_{C_i \in \mathcal{C}(x_i)} |C|_i$ is minimum suffices. We have

$$\sum_{C_i \in \mathcal{C}(x_i)} |C_i| = \min_{j \in [q]} \sum_{C_i \in C(x_j)} |C| \leq \frac{\sum_{j \in [q]} \sum_{C_i \in \mathcal{C}(x_j)} |C|}{q} \leq \frac{\sum_{C_i \in \mathcal{C}} |C|}{q} \leq \frac{t}{q} = \frac{(n + k)q + 2m}{q} = n + k + \frac{2m}{q} \leq n + k.$$

The last inequality follows from the choice of $q \geq 2m + 1$. For the sake of contradiction, assume $|\mathcal{C}(x_i)| = k^* > k$. Then, by the definition of $\mathcal{C}(x_i)$, $x_i$ appears in $k^*$ times in $\mathcal{C}(x_i)$. Since $x_i$ is connected to all $n$ vertices of $V^*$, $\sum_{C_i \in \mathcal{C}(x_i)} |C_i| \geq n + k^* > n + k$, which contradicts the fact that $\sum_{C_i \in \mathcal{C}(x_i)} |C_i| \leq n + k$. Therefore, $|\mathcal{C}(x_i)| \leq k$, i.e., $(G^*, |\mathcal{C}(x_i)|)$ is a YES instance of VCC.

## 4 New Framework: Building Blocks

We describe underlying concepts of a new framework that we use in Section 5 for algorithm design. Our framework is based on a relaxation of a global minimality of clique cover and a set representation of graph. We include a number of characterizations which may be of independent interest (could be used in other contexts such as designing polynomial time algorithm for specific class of graphs). We start with a discussion of a number of pitfalls that we want to deal with the new framework.

### 4.1 Pitfalls Addressed

In terms of exact algorithm design, we have found two major ways that have been adopted to systematically explore search space of clique covers of a graph. One way is to explore matrices $B \in \{0, 1\}^{n \times m}$ such that $BB^T = A$ where $A$ is the adjacency matrix of graph. The matrix multiplication is defined with addition rule $1 + 1 = 1$, and diagonal entries of $A$ are all assumed to be $1$. Several FPT algorithms \cite{17, 18} are based on exploration of this type of search space, where binary matrices of dimension $O(k) \times O(k)$ are enumerated from a search space of size $2^{O(k^2 \log k)}$. Consequently, this type of algorithms does not have any regard for sparsity of graph, making them unsuitable even for moderately small values of parameters.

The other major way is to explore search space consisting of maximal cliques of (sub)graph. Depending on objective of clique cover, exploring search space of maximal cliques could become infeasible, even for moderately small input size. An example is the search tree algorithm of \cite{14} for computing assignment-minimum clique cover of a graph. Search tree size of \cite{13} is double exponential in input size in the worst case, and the algorithm also requires space exponential in input size in the worst case.

Algorithms described in Section 3 explore search spaces that are structurally dependent on the degeneracy of graph. We ask whether a more robust parameter than degeneracy can be used to restrict search space further. More specifically, we ask whether it is possible to restrict search space structurally with
Definition 4.1 (Minimum Clique Cover). A minimum clique cover of a graph $G$ is an edge clique cover of $G$ with smallest number of cliques.

Definition 4.2 (Minimal Clique Cover). A minimal clique cover $C$ of a graph $G$ is an edge clique cover of $G$ such that no other edge clique cover of $G$ is contained in $C$, i.e., $C$ an inclusion-wise minimal set of cliques that covers all the edges of $G$.

It is evident from the preceding two definitions that a minimum clique cover is also a minimal clique cover. Therefore, it is natural to ask whether we can find a minimum clique cover by systematically exploring a search space of minimal clique covers. We face with a number of difficulties.

First, note that the complete graph $K_n$ has a minimal clique cover of size $\binom{n}{2}$, clearly indicating a difficult search space to explore. Second, it is not clear how to systematically explore the search space of minimal clique covers. Third, any conceivable way to systematically explore the set of minimal clique covers of a graph would require a massive amount of tests for minimality and provision for removing and restoring cliques, i.e., enumerating minimal clique covers of a graph would be prohibitive in terms of time and space. Next, we introduce a relaxation on global minimality to avoid these pitfalls.

4.2 Locally Minimal Clique Cover

Definition 4.3 (Locally Minimal Clique Cover). Let $H = (V_H, E_H)$ be a proper subgraph of a graph $G = (V, E)$, and $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of $H$. Let $\{x, y\}$ be an uncovered edge of $G$, and $H^*$ be the subgraph induced by vertices of $V_H \cup \{x, y\}$ in $G$. Let $C^*$ be an edge clique cover of a subgraph of $H^*$ constructed as follows.

(i) If the edge $\{x, y\}$ can be covered in a clique $C_i \in C$, then we cover $\{x, y\}$ with exactly one such clique of $C$ by letting $C_i^* = C_i \cup \{x, y\}$, and we set $C^*$ to be $(C \setminus \{C_i\}) \cup \{C_i^*\}$.

(ii) Otherwise, we create a new clique $C_{k+1} = \{x, y\}$, and set $C^*$ to be $C \cup \{C_{k+1}\}$.

We call an edge clique cover $C$ locally minimal if $C$ is obtained from an empty clique cover using aforesaid construction; i.e., for every expansion of the cliques contained in $C$, we either have used (i) whenever applicable, or (ii) otherwise.

The bottom-up constructive clique cover in Definition 4.3 is simple but has far-reaching consequences. First, search space of locally minimal clique cover is easy to explore systematically. Second, the search space is much more compact as it does not try to construct maximal cliques, let alone all maximal cliques of graph. Third, we will be able to make the search space exploration efficient, with the help of a set representation introduced later. Forth, it admits a number of desirable characterizations. We elaborate on these in subsequent discussion.

Definition 4.4 ($E_{\pi(i)}$). Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$. Let $\pi$ be a permutation of $[k]$. $E_{\pi(i)} = \{\{x, y\} \in C_{\pi(i)} | \{x, y\} \notin C_{\pi(j)}, j < i\}$, i.e., the set of edges exclusive to $C_{\pi(i)}$ with respect to the cliques $\{C_{\pi(1)}, C_{\pi(2)}, \ldots, C_{\pi(i-1)}\}$.

An immediate observation for locally minimal clique cover is as follows.

Proposition 4.5. If $C = \{C_1, C_2, \ldots, C_k\}$ is a locally minimal clique cover of $G$, then there exists a permutation $\pi$ such that $E_{\pi(i)} \neq \emptyset$, for all $i \in [k]$.

The following is a characterization that minimal clique cover does not admit, but locally minimal clique cover does. Recall the minimal clique cover of $K_n$ with $\binom{n}{2}$ cliques, which arises precisely due to the fact that minimal clique cover lacks the following characterization.

Proposition 4.6. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of $G = (V, E)$. If $C$ is locally minimal, then for each pair of distinct cliques $\{C_a, C_b\}$ in $C$, there exist $x \in C_a$, $y \in C_b$ such that $x \neq y$ and $\{x, y\} \notin E$. 

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Proof. We prove the contrapositive, i.e., if for some pairs of distinct cliques \( \{C_a, C_b\} \) in \( \mathcal{C} \) it holds that for all \( x \in C_a \) and for all \( y \in C_b, x \neq y, \{x, y\} \notin E \), then \( \mathcal{C} \) is not a locally minimal clique cover of \( G \).

Let \( \pi \) be a permutation of \([k]\) and let \( \pi(i) = a \) and \( \pi(j) = b \). WLOG assume \( i < j \), and consider an edge \( \{y, z\} \in E_{\pi(j)} \). If no such edge \( \{y, z\} \) exists for all \( \pi \), then, by Proposition 4.6, \( \mathcal{C} \) is not locally minimal. By our assumption, for all \( x \in C_a \setminus \{y, z\}, \{x, y\} \in E \) and \( \{x, y\} \notin E \). Clearly, rule (i) of the construction of locally minimal clique cover is applicable to the edge \( \{y, z\} \), but rule (ii) of the construction is being applied. Since preceding argument holds every permutation \( \pi \), it follows that \( \mathcal{C} \) is not locally minimal.

The converse of Proposition 4.6 does not hold. But, we can establish a close equivalent of the converse. This will be crucial for showing sufficiency of search for different clique covers in a search space of locally minimal clique covers.

**Proposition 4.7.** Let \( \mathcal{C} = \{C_1, C_2, \ldots, C_k\} \) be an edge clique cover of \( G = (V, E) \). If for each pair of distinct cliques \( \{C_a, C_b\} \) in \( \mathcal{C} \), there exists \( x \in C_a, y \in C_b \) such that \( x \neq y \) and \( \{x, y\} \notin E \), then there exists a permutation \( \pi \) of \([1, \ldots, k]\) such that \( E_{\pi(i)} \neq \emptyset \), for all \( i \in [k] \).

Proof. Let \( P_1 \) denote the property that for each pair of distinct cliques \( \{C_a, C_b\} \) in \( \mathcal{C} \), there exists \( x \in C_a, y \in C_b \) such that \( x \neq y \) and \( \{x, y\} \notin E \). Let \( P_2 \) denote the property that for all permutation \( \pi \) of \([1, 2, \ldots, k]\) there exists \( i \in [k] \) such that \( E_{\pi(i)} = \emptyset \). We want to show \( P_1 \implies \neg P_2 \), equivalently \( P_2 \implies \neg P_1 \).

Consider any \( C_j \) such that \( \pi(k) = j \) and \( E_{\pi(k)} \neq \emptyset \). We remove \( C_j \) from further consideration by considering a permutation \( \pi \) such that \( \pi(k) = j \). We restrict our permutation space to \( \{1, \ldots, k-1\} \) from \([1, \ldots, k]\) as depicted below.

\[
\begin{array}{cccc}
1 & 2 & \ldots & k-1 & k \\
C_{\pi(1)} & C_{\pi(2)} & \ldots & C_{\pi(k-1)} & C_j \\
\end{array}
\]

Repeatedly removing all such \( C_j \) depicted above from further consideration, we would be able to restrict our permutation space to the permutations of \([1, \ldots, l]\) such that \( E_{\pi(i)} = \emptyset \), for any permutation \( \pi \) of \([1, \ldots, l]\). Note that \( l > 1 \) since \( E_{\pi(1)} \neq \emptyset \) for all \( \pi \). Following two cases completely characterize the restricted permutation space.

1. For \( i, j \in [l] \) there exists a pair of distinct cliques \( \{C_i, C_j\} \) such that \( C_i = C_j \).

2. A least one non-trivial covering of some clique \( C_g \) of \( G \) exists in \( \{C_1, \ldots, C_l\} \) (a covering of a clique is trivial if it consists of a single clique).

It is straightforward to see that if (1) holds, then \( P_1 \) is false. If (2) holds, then any pairs of cliques in the non-trivial covering makes \( P_1 \) false.

Next characterization shows a close equivalent of the converse of Proposition 4.6.

**Proposition 4.8.** Let \( \mathcal{C} = \{C_1, C_2, \ldots, C_k\} \) be an edge clique cover of \( G = (V, E) \). If for each pair of distinct cliques \( \{C_a, C_b\} \) in \( \mathcal{C} \), there exists \( x \in C_a, y \in C_b \) such that \( x \neq y \) and \( \{x, y\} \notin E \), then there exists a locally minimal clique cover \( \mathcal{C}' \) of \( G \) such that \( |\mathcal{C}'| = |\mathcal{C}^*| \) and \( \mathcal{C}' \subseteq C_l \) for all \( l \in [k] \).

Proof. From Proposition 4.7, we have a permutation \( \pi \) of \([1, \ldots, k]\) such that \( E_{\pi(i)} \neq \emptyset \) for all \( i \in [k] \). Consider any such \( \pi \), and let \( C_{\pi(i)} = \{x \in E_{\pi(i)} \} \). Clearly, \( C_{\pi(i)} \subseteq C_{\pi(i)} \) for all \( i \in [k] \), and by definition of \( E_{\pi(i)} \), \( \mathcal{C}' \) is an edge clique cover of \( G \) with size \( |\mathcal{C}| \).

Following characterization shows that it is sufficient to search for minimum clique cover in a search space of locally minimal clique cover.

**Proposition 4.9.** Let \( \mathcal{C} = \{C_1, C_2, \ldots, C_k\} \) be an edge clique cover of a graph \( G = (V, E) \). If \( \mathcal{C} \) is a minimum clique cover of \( G \), then there exists a locally minimal clique cover \( \mathcal{C}' \) of \( G \) such that \( |\mathcal{C}'| = |\mathcal{C}| \) and \( \mathcal{C}' \subseteq C_l \) for all \( l \in [k] \).

Proof. Any minimum clique cover satisfies the property that for each pair of distinct cliques \( \{C_i, C_j\} \in \mathcal{C} \), there exists \( x \in C_i, y \in C_j \) such that \( x \neq y \) and \( \{x, y\} \notin E \). Therefore, the claim follows from Proposition 4.8.
In fact, a much stronger characterization exists for assignment-minimum clique cover.

**Proposition 4.10.** Let \( C = \{ C_1, C_2, \ldots, C_k \} \) be an edge clique cover of a graph \( G = (V, E) \). If \( C \) is an assignment-minimum clique cover of \( G \), then there exists a locally minimal clique cover \( C^* \) of \( G \) such that \( |C^*| = |C| \) and \( C_l^* = C_l \) for all \( l \in [k] \).

**Proof.** Any assignment-minimum clique cover satisfies the property that for each pair of distinct cliques \( \{C_i, C_j\} \subset C \), there exists \( x \in C_i, y \in C_j \) such that \( x \neq y \) and \( \{x, y\} \notin E \). Therefore, we can use the construction described in Proposition 4.8 to get a clique cover \( C^* \) such that \( |C^*| = |C| \). For assignment-minimum clique cover, it must be the case that \( C_l^* = C_l \) for all \( l \in [k] \); otherwise that would contradict the minimality of number of individual assignments of vertices to cliques in \( C \). \( \square \)

A characterization that holds for both minimal clique cover and locally minimal clique cover is as follows.

**Proposition 4.11.** Let \( C = \{ C_1, C_2, \ldots, C_k \} \) be an edge clique cover of a graph \( G = (V, E) \). If \( C \) is locally minimal, then for each pair of distinct cliques \( \{C_i, C_j\} \subset C \), \( C_i \nsubseteq C_j \).

**Proof.** For the sake of contradiction assume \( C \) is locally minimal and there exists a pair of distinct cliques \( \{C_i, C_j\} \subset C \) such that \( C_i \subseteq C_j \). This implies there exists a pair of distinct cliques \( \{C_i, C_j\} \subset C \) such that for all \( x \in C_i \), and for all \( y \in C_j \), \( x \neq y \), \( \{x, y\} \in E \). By Proposition 4.6, we have a contradiction to the assumption that \( C \) is locally minimal. \( \square \)

Note that Proposition 4.10 implies Proposition 4.11 but the converse does not hold. Our next characterization shows that a tight upper bound on the size of minimum clique cover (Theorem 2 [15]) also holds for locally minimal clique cover.

**Proposition 4.12.** Let \( C = \{ C_1, C_2, \ldots, C_k \} \) be an edge clique cover of a graph \( G = (V, E) \) with \( n \) vertices. If \( C \) is locally minimal, then \( k \leq \left\lfloor \frac{n^2}{4} \right\rfloor \).

**Proof.** We induct on \( n \). It is straightforward to see that the claim holds for \( n = 2 \) and \( n = 3 \). Assume that the claim holds for all \( n \leq m \leq n \).

Let \( C = \{ C_1, C_2, \ldots, C_k \} \) be a locally minimal clique cover of a graph \( G = (V, E) \) with \( n + 2 \) vertices. Let \( \{x, y\} \in E \) be an edge of \( G \), and \( E_{xy} \) be the set of edges incident to \( x \) or \( y \), including the edge \( \{x, y\} \), i.e., \( E_{xy} = \{ \{w, z\} \in E \mid z \in \{x, y\} \} \). Let \( H = (V_H, E_H) \) be the subgraph induced by vertices of \( V \setminus \{x, y\} \) in \( G \). Note that \( E_H = E \setminus E_{xy} \).

Let \( C^* = \{ C_1, C_2, \ldots, C_k^* \} \) be a locally minimal clique cover of \( H \). Since \( |V_H| = n \), by inductive hypothesis, \( k^* \leq \left\lfloor \frac{n^2}{4} \right\rfloor \).

Number of edges in \( E_{xy} \) is at most \( 2n + 1 \). By the construction of locally minimal clique cover, number of additional cliques needed to cover edges of \( E_{xy} \) is at most \( n + 1 \). This can be seen as follows.

Let \( z \) be any vertex of \( V_H \). Assume \( \{x, z\} \) and \( \{y, z\} \) exist in \( G \). WLOG, assume we cover the edge \( \{x, z\} \) first, either with a clique of \( C^* \) or a new clique \( C_{k^*+1} = \{x, z\} \). Next, we may be able to cover the edge \( \{y, z\} \) with a clique of \( C^* \); if not, then the clique \( C_{k^*+1} = \{x, z\} \), we just have created, is sufficient to cover the edge \( \{y, z\} \), since the edge \( \{x, y\} \) exists in \( G \). In any of the cases, to cover edges incident from \( x \) and \( y \) to \( z \), at most one additional clique would be needed (this also holds if any of the edges \( \{x, z\} \) and \( \{y, z\} \) does not exist in \( G \)). Therefore, at most \( n + 1 \) additional cliques would be needed, considering all \( n \) vertices of \( V_H \) and the edge \( \{x, y\} \).

Note that the ordering of covering edges of \( E_{xy} \) does not affect the preceding argument. To see this, let \( z_1 \) and \( z_2 \) be two distinct vertices of \( V_H \). If the edges \( \{x, z_1\}, \{x, z_2\}, \) and \( \{z_1, z_2\} \) exist in \( G \), then we may cover these edges with a clique \( \{x, z_1, z_2\} \). If the edge \( \{y, z_1\} \) and \( \{y, z_2\} \) both exist in \( G \), then we would not need any additional cliques, since the clique \( \{x, y, z_1, z_2\} \) would cover all the edges. Now, if only one of \( \{y, z_1\} \) and \( \{y, z_2\} \) exists in \( G \), then we may need one additional clique. Therefore, to cover edges incident on \( z_1 \) and \( z_2 \) from \( x \) and \( y \), at most two additional cliques would be needed. Generalizing this argument, we see that any \( \{z_1, z_2, \ldots, z_q\} \subseteq V_H \), we would need at most \( q \) additional cliques to cover all edges incident from \( x \) and \( y \) to vertices of \( \{z_1, z_2, \ldots, z_q\} \).

We conclude that \( k \leq k^* + n + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + n + 1 = \left\lfloor \frac{(n+2)^2}{4} \right\rfloor \).\( \square \)

Proposition 4.12 gives us a characterization of compactness of locally minimal clique cover in terms of number of cliques. In subsequent characterizations, we show compactness of locally minimal clique cover in terms of vertex assignments and edge assignments.
Proposition 4.13. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$, and $N(x)$ denote the neighbours of a vertex $x$. If $C$ is locally minimal, then each vertex $x$ appears in at most $|N(x)|$ distinct cliques of $C$.

Proof. Let $C^*$ be a locally minimal clique cover of a subgraph of $G$, and let $\{x, y\}$ be an edge not included in the cliques of $C^*$. Now, consider the expansion of $C^*$ for $\{x, y\}$. Before the expansion, none of the cliques in $C^*$ contained both $x$ and $y$. After the expansion, $\{x, y\}$ would not get selected for expansion in subsequent choices. The number of uncovered edges $\{x, y\}$ incident on a vertex $x$ is at most $|N(x)|$. Therefore, the number of such expansions of clique covers for a vertex $x$ is at most $|N(x)|$, i.e., $x$ appears at most $|N(x)|$ times in $C$.

Proposition 4.14. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$ with $m$ edges. If $C$ is locally minimal, then the number of individual assignments of vertices to cliques of $C$ is at most $2m$.

Proof. Since $C$ is locally minimal, by Proposition 4.13, each vertex appears in at most $|N(x)|$ distinct cliques of $C$. Therefore, $\sum_{l \in [k]} |C_l| \leq \sum_{x \in V} |N(x)| = 2m$.

Next characterization follows from Proposition 4.14 and Proposition 2.4.

Proposition 4.15. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$ with degeneracy $d$. If $C$ is locally minimal, then on average a vertex is included in at most $2d$ cliques of $C$, i.e., $\frac{\sum_{C \in C}|C|}{m} < 2d$.

Proposition 4.16. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$ with $m$ edges and $\Delta$ maximum degree. If $C$ is locally minimal, then the number of individual assignments of edges to cliques of $C$ is at most $m\Delta$.

Proof. Consider any edge $\{x, y\} \in E$. Since $C$ is locally minimal, by Proposition 4.13, any vertex $x \in V$ appears in at most $|N(x)|$ distinct cliques of $C$. Both $x$ and $y$ can appear in at most $\min\{|N(x)|, |N(y)|\} \leq \Delta$ distinct cliques of $C$. Therefore, $\sum_{(x, y) \in E} |\{C_l \mid \{x, y\} \in C_l, C_l \in C\}| \leq m\Delta$.

4.3 Implicit Set Representation

From the definition of locally minimal clique cover, a natural question emerges: from an edge clique cover of a subgraph how would one quickly find a clique that can cover an edge $\{x, y\}$ or report that none exists. Obviously, scanning the entire set of cliques of the clique cover is an inefficient method. We address this problem by introducing a set representation of graph connected to intersection graph theory.

Definition 4.17 (Intersection Graph). Let $F = \{F_1, F_2, \ldots, F_n\}$ be a family of sets. The intersection graph of $F$ is a graph that has $F$ as the vertex set, and an edge for each pair of distinct sets $F_x$ and $F_y$ if and only if $F_x \cap F_y \neq \emptyset$.

On the other hand, every graph is an intersection graph of some family of sets [15], which leads us to the following definition.

Definition 4.18 (Set Representation). Let $G = (V, E)$ be a graph with $n$ vertices. A family of sets $F = \{F_1, F_2, \ldots, F_n\}$ is called a set representation of $G$ if $\{x, y\} \in E$ if and only if $F_x \cap F_y \neq \emptyset$.

Note that intersection graph of a family of sets is unique, but a graph can have many set representations.

Let $U_F = \cup_{x \in [n]} F_x$. Every element $l \in U_F$ corresponds to a clique of $G$ such that the set $\{F_x \mid l \in F_x\}$ corresponds to the vertex set of the clique. Each set $F_x$ corresponds to a set of cliques of $G$ that contain the vertex $x$. By definition of set representation, for every edge $\{x, y\} \in E$, there exists $l \in U_F$ such that $l \in F_x \cap F_y$. Therefore, the set $U_F$ corresponds to an edge clique cover of $G$.

Definition 4.19 (Intersection Graph Basis [21]). Let $G = (V, E)$ be a graph with $n$ vertices and $F = \{F_1, F_2, \ldots, F_n\}$ be a set representation of $G$. Let $U_F = \cup_{x \in [n]} F_x$. If $|U_F|$ is minimum over all set representations of $G$, then $U_F$ is called an intersection graph basis of $G$.
Figure 3: An example graph $G$ showing $R_x \cap R_y \neq \emptyset$ does not imply $(x, y) \in E$. A minimum clique cover $C$ of $G$ consists of following cliques: $C_1 = \{x, a, z\}$, $C_2 = \{x, b, w\}$, $C_3 = \{y, c, w\}$, $C_4 = \{y, d, z\}$, $C_5 = \{w, z\}$. For all $v \in \{a, b, c, d, z, w\}$, set representation and implicit set representation of $G$ have the same sets for $F_v$ and $R_v$ respectively. But, set representation has $F_x = \{1, 2\}$, $F_y = \{3, 4\}$, whereas implicit set representation has $R_x = \{1, 2, 5\}$, $R_y = \{3, 4, 5\}$.

As a consequence, computing a minimum clique cover of $G$ is equivalent to computing an intersection graph basis of $G$ (see Theorem 1.6 of [35] for a proof). Although every set representation of a graph is in one-to-one correspondence with an edge clique cover of the graph, that does not give us an efficient way to construct either of them from scratch. Next, we introduce a constructive way of representing graph that implicitly contains a set representation.

Definition 4.20 (Representative Set). Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a subgraph of $G = (V, E)$. We call a clique $C_i \in C$ a representative of a vertex $x \in V$ if either (1) $x \in C_i$ or (2) $x \notin C_i$ and $C_i \subseteq N(x)$. A representative set $R_x$ of a vertex $x$ is the set of (indices of) representatives of $x$ in $C$, i.e., $R_x = \{i \in [k] | x \in C_i \text{ or } (x \notin C_i \text{ and } C_i \subseteq N(x))\}$.

Definition 4.21 (Implicit Set Representation). Let $C$ be an edge clique cover of graph $G = (V, E)$ and $R_x$ be the corresponding representative set for a vertex $x$. We call the family of sets $R = \{R_1, R_2, \ldots, R_n\}$ an implicit set representation of $G$.

Note that set representation requires biconditional for every edge: $\{x, y\} \in E \iff F_x \cap F_y \neq \emptyset$. Implicit set representation can only ensure one way implication: $\{x, y\} \in E \implies R_x \cap R_y \neq \emptyset$. Figure 3 shows an example where $R_x \cap R_y \neq \emptyset$ does not imply $(x, y) \in E$. It would be evident from following characterization that an implicit set representation of a graph always contains a set representation of the graph.

Proposition 4.22. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G$ with $n$ vertices, and $R$ be the corresponding implicit set representation of $G$. Let $U_R = \bigcup_{x \in V} R_x$. If $|U_R|$ is minimum over all implicit set representations of $G$, then $C$ is a minimum clique cover of $G$.

Proof. Let $F = \{F_1, F_2, \ldots, F_n\}$ be a family of sets where $F_x = R_x \setminus \{i \in [k] | x \notin C_i\}$. Let $U_F = \bigcup_{x \in V} F_x$. Note that $F$ is a set representation of $G$, since $F_x = \{i | x \in C_i\}$. Moreover, $U_F = U_R$. $|U_F|$ is minimum over all set representation of $G$. To see this, suppose $|U_F|$ is not minimum, i.e., there exists $l \in U_F$ such that $C_l$ can be removed from $C$, which contradicts the assumption that $|U_R|$ is minimum. Therefore, $U_F$ is an intersection graph basis of $G$. Since $U_F$ is an intersection graph basis of $G$, $C$ is a minimum clique cover of $G$.

As in Proposition 4.22, we can obtain similar characterizations for assignment-minimum clique cover.

Proposition 4.23. Let $C = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G$ with $n$ vertices, and $R$ be the corresponding implicit set representation of $G$. If $\sum_{x \in V} |R_x \setminus \{i \in [k] | x \notin C_i\}|$ is minimum over all implicit set representations of $G$, then $C$ is an assignment-minimum clique cover of $G$.

Definition 4.20 captures the idea that a representative of a vertex $x$ corresponds to a clique that may be used to cover an edge incident on $x$, regardless of whether $x$ is contained in the clique. So far it may not be evident how Definition 4.20 resolves the problem of efficiently constructing a set representation from scratch. In following characterization we provide a clarification.
Proposition 4.24. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a subgraph of $G = (V, E)$, and $R_x$ be the corresponding representative set of a vertex $x$. Let $\{x, y\} \in E$ be an uncovered edge. There exists representative set $l \in R_x \cap R_y$ if and only if $\{x, y\}$ can be covered by a clique $C_i \in \mathcal{C}$.

Proof. Only if. Since $\{x, y\}$ is an uncovered edge, both $x$ and $y$ are not contained in any of the cliques of $\mathcal{C}$. By the definition of representative set, one of the following holds for $C_i$: (i) $x \in C_i$, $y \notin C_i$, $C_i \subseteq N(y)$, (ii) $y \in C_i$, $x \notin C_i$, $C_i \subseteq N(x)$, (iii) $x \notin C_i$, $y \notin C_i$, $C_i \subseteq N(x)$, $C_i \subseteq N(y)$. Each of the three cases, both $x$ and $y$ can be included in $C_i$ and thus $\{x, y\}$ can be covered by $C_i$.

If. Since the edge $\{x, y\}$ is uncovered, if $x$ is contained in $C_i$, then it must be the case $y \notin C_i$ and $C_i \subseteq N(y)$. It follows that if $x$ is contained in $C_i$, then $l \in R_x$ and $l \in R_y$. Similarly, $l \in R_x$ and $l \in R_y$ if $y$ is contained in $C_i$ or both $x$ and $y$ are not contained in $C_i$. Therefore, $l \in R_x \cap R_y$.

Proposition 4.24 immediately connects us to the construction of locally minimal clique cover. To cover an edge $\{x, y\}$, we can simply select a clique $C_i \in \mathcal{C}$ if $l$ is contained in $R_x \cap R_y$. Each such choice to cover an edge may require update of some sets in $\mathcal{R}$ to conform to Definition 4.20. We describe the updates in the context of algorithm design (Section 5). Our next characterization provides a bound on the space requirement of implicit set representation for constructing a locally minimal clique cover.

Proposition 4.25. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be an edge clique cover of a graph $G = (V, E)$ with $m$ edges, and $\mathcal{R}$ be the corresponding implicit set representation of $G$. If $\mathcal{C}$ is locally minimal, then $\sum_{x \in V} |R_x| < 2m + k\Delta$.

Proof. From the definition of representative set, we have
\[
\sum_{x \in V} |R_x| = \sum_{x \in V} |\{l| x \in C_l\}| + \sum_{x \notin V} |\{l| x \notin C_l, C_l \subseteq N(x)\}|.
\]

For the first sum on the right hand side, by Proposition 4.14 we have
\[
\sum_{x \in V} |\{l| x \in C_l\}| = \sum_{C_l \in \mathcal{C}} |C_l| \leq 2m.
\]

It remains to show a bound for the second sum of the right hand side.

Since $\mathcal{C}$ is locally minimal, there exists a permutation $\pi$ of $\{1, \ldots, k\}$ such that $C_{\pi(i)}$ contains an edge $\{x, y\}$ not contained in any $C_{\pi(j)}, j < i$. For the permutation $\pi$, existence of $C_{\pi(i)}$ is necessary for covering $\{x, y\}$. WLOG we can assume that in the locally minimal construction of $\mathcal{C}$, the edge $\{x, y\}$ triggered the creation of $i^{th}$ clique in $\mathcal{C}$. Immediately after including $\{x, y\}$ in the $i^{th}$ clique of $\mathcal{C}$, we would have $\pi(i) \in R_x$ for all $z \in N[x] \cap N[y]$ (from the definition of representative set).

For any subsequent edge $\{x, y\}$ added to the $i^{th}$ clique of $\mathcal{C}$ (for locally minimal construction), $\pi(i) \in R_x \cap R_y$ holds by Proposition 4.24. Therefore, all representative sets that include $\pi(i)$ have already been accounted for, when we created (by locally minimal construction) the $i^{th}$ clique of $\mathcal{C}$.

Any vertex $z$ such that $z \notin C_{\pi(i)}$, but $C_{\pi(i)} \subseteq N(z)$, must be a common neighbour of $x$ and $y$ such that the edge $\{x, y\}$ required the creation of $C_{\pi(i)}$. And $|\{z| z \notin C_{\pi(i)}, C_{\pi(i)} \subseteq N(z)\}| \leq |N(x) \cap N(y)| < \Delta$. Summing over all cliques we get,
\[
\sum_{C_l \in \mathcal{C}} |\{x| x \notin C_l, C_l \subseteq N(x)\}| < k\Delta.
\]

Consider a bipartite graph $B = (V, \mathcal{C}, E_B)$ where $E_B$ is the relation of vertices and cliques from the second condition of representative: $E_B = \{(x, C_l)| x \in V, C_l \in \mathcal{C}, x \notin C_l, C_l \subseteq N(x)\}$. Sum of the degrees of the vertices on the $V$ side of $B$ is equal to the sum of the degrees of the vertices on the $\mathcal{C}$ side of $B$. Therefore,
\[
\sum_{x \in V} |\{l| x \notin C_l, C_l \subseteq N(x)\}| = \sum_{C_l \in \mathcal{C}} |\{x| x \notin C_l, C_l \subseteq N(x)\}| < k\Delta.
\]

\[\square\]

Remark 4.26. A tighter bound than Proposition 4.25 is possible if we construct locally minimal clique cover by choosing the edges in the order of DEP of $G$, and keep maintaining only necessary representative sets. At any stage of the construction of locally minimal clique cover, let $\{x, y\}$ be the first uncovered edge of DEP where $y \in N_d(x)$. Note that if $x = u_i$, then all edges incident on $u_j$ such that $j < i$ are
5 New Framework: Algorithms

In this section, we describe a new set of bounded search tree algorithms for the problems listed in Section 4.2. The algorithms are based on the concepts introduced in Section 4. The general theme of these search tree algorithms is as follows. At each node of a search tree, the corresponding algorithm branches on the number of cliques in a clique cover or other parameters. Most notably, we would be able to show that the size of a search tree is structurally dependent on the clique number of graph, a feature lacking in the algorithms we have presented in Section 4.

5.1 Edge Clique Cover

We will continue to assume that data reduction rules are applied on an instance \((G, k)\) of ECC, including the rules described in Section 3.1. Figure 4 shows our new bounded search tree algorithm for ECC, henceforth referred as \(ECCS2\). For better readability, we describe the details of steps 3a, 3d, 4a, and 4d separately as subroutines in Figure 5.

**ECCS2 works as follows.** At every node of the search tree, it selects the last uncovered edge \(\{x, y\}\) from the DEP of \(G\) such that \(y \in N_d(x)\). Then, it branches on each of the choices of representatives \(l \in R_x \cap R_y\), by covering the edge \(\{x, y\}\) with the clique \(C_l\) (steps 3a-3d). If all of these branches fail and the parameter \(k\) permits, then it creates an additional branch (steps 4a-4d), by covering the edge \(\{x, y\}\) with a new clique.

Algorithm \(ECCS2\) works by modifying a single edge clique cover \(C\) and a single implicit set representation \(R\) at every node of a search tree. This makes the algorithm space efficient. Now, to conform to the definition of representative set, we need to update \(R\) in response to changes in the cliques of \(C\) (steps 3a, 3d), or inclusion/exclusion of clique in \(C\) (steps 4a, 4d). Next, we expand on these updates, included in the subroutines of \(ECCS2\) in Figure 5.

We use a data structure \(D\) to help the updates of sets in \(R\). A set \(D_l \in D\) contains a set of vertices that have clique \(C_l\) in their representative sets, i.e., \(D_l = \{x \in V | l \in R_x, C_l \in C\}\). The sets in \(R\) contain a mapping of vertices to the representatives (cliques), whereas the sets in \(D\) contain the corresponding inverse mapping of the representatives (cliques) to the vertices.

In the subroutine for step 4a, a new clique, with index \(q\), is included in the solution. By the definition of representative set, this prompts \(q\) to be included in the representative sets of all vertices in \(N[x] \cap N[y]\). Similarly, in the subroutine for step 4d, the clique \(q\) is excluded from the clique cover, which prompts \(q\) to be excluded from the representative sets of all vertices in \(N[x] \cap N[y]\).

In the subroutine for step 3a, since the edge \(\{x, y\}\) to be included in \(C_l\), any vertex \(z\) such that \(l \in R_z\) must have both \(x\) and \(y\) in the neighbourhood of \(z\). The set of vertices that violates this requirement is collected in a set \(U\) (a local data structure allocated for every node in a search tree), and corresponding representative sets of these vertices are updated. Indicator variable \(x_l\) (resp. \(y_l\)) is set that would denote whether the vertex \(x\) (resp. \(y\)) is already contained in \(C_l\). The subroutine for step 3d simply undoes the updates of \(R\) and \(D\) using the set \(U\). The indicator variables are used to restore the clique \(C_l\) to the state prior to including the edge \(\{x, y\}\) in \(C_l\). Each of the subroutines of Figure 5 takes \(O(\Delta)\) time.

The definition of locally minimal construction is oblivious of edge permutation. An exact algorithm that strictly adheres to the definition of locally minimal construction would potentially need to try all permutations of edges (for \(ECC\), this would be equivalent to trying all possible choices of uncovered edge at step 2). Consequently, we would have algorithms with large dependency on parameters (for example, we would need \(2^{O(3k^2)}n^{O(1)}\) or \(2^{O(k^2(n + k)\log k)}n^{O(1)}\) running time for \(ECC\)). Furthermore, we want to impose permutations of edges (such as step 2 of \(ECCS2\)) so that we can bound the branching factors of a search tree with certain choices of parameters (such as degeneracy). We address these issues by judicious relaxation of locally minimal construction.
Edge-Clique-Cover-Search2(G, k, C, R):

// Abbreviated ECCS2(G, k, C, R)
1. if C covers edges of G, then return C
2. select the last uncovered edge \{x, y\} from the DEP of G where y ∈ N_d(x)
   // cf. step 3 of ECCS in Figure 4
3. for each l ∈ R_x ∩ R_y do
   (a) cover the edge \{x, y\} with the clique C_l and update R // Figure 5 (a)
   (b) Q ← ECCS2(G, k, C, R)
   (c) if Q ≠ ∅, then return Q
   (d) undo changes done to C and R at step 3a // Figure 5 (b)
4. if k > 0, then
   (a) set C to C ∪ \{x, y\} and update R // Figure 5 (c)
   (b) Q ← ECCS2(G, k − 1, C, R)
   (c) if Q ≠ ∅, then return Q
   (d) undo changes done to C and R at step 4a // Figure 5 (d)
5. return ∅
The space use of $ECCS_2$ can be bounded as follows (see Appendix B for a proof).

**Lemma 5.2.** In a search tree with at most $k$ cliques, $ECCS_2$ takes $O(m + k\Delta)$ space.

Next, we describe bounds on the depth and the branching factors of a search tree of $ECCS_2$.

**Lemma 5.3.** The depth of a search tree of $ECCS_2$ is at most $\beta k$.

**Proof.** Every new clique starts with two vertices (step 4a), and each expansion of a clique (step 3a) with an uncovered edge $\{x, y\}$ will include at least one new vertex to the clique. Therefore, number of total possible expansions of any clique is at most $\beta - 2$. With at most $k$ cliques allowed in any branches, the depth of a search tree is at most $\beta k$.

Recall by $\langle u_1, u_2, \ldots, u_n \rangle$ we denote degeneracy ordering of $V$. Since for any $x = u_i$, we only consider $y \in N_d(x)$, any vertex $z \in N(x) \setminus N_d(x)$ would not be included in any clique $C_l \in C$ until we process a node such that $x = z$. Therefore, by our choice of uncovered edge $\{x, y\}$ for each node of the search tree of $ECCS_2$, the following invariant holds.

**Proposition 5.4.** At any node of a search tree of $ECCS$, if $x = u_i$, then for all $C_l \in C, C_l \subseteq \{u_i, u_{i+1}, \ldots, u_n\}$.

**Lemma 5.5.** The number of branches at any node of a search tree of $ECCS_2$ is at most $\min\{k, \binom{d+1}{2}\}$.

**Proof.** Consider an uncovered edge $\{x, y\}$ at step 2 of $ECCS_2$. For the branches of step 3, by Proposition 5.4 we only need to consider cliques created for the subgraph induced by vertices of $\{u_i, u_{i+1}, \ldots, u_n\}$ in $G$. 

---

**Figure 5:** Subroutines of $ECCS_2$ shown in Figure 4.
We will continue to assume that data reduction rules are applied on an instance (WECP)

Lemma 5.6. The number of nodes in a search tree of ECCS2 is at most $2\beta k \log k$.

Proof. From Lemma 5.3, the depth of a search tree of ECCS2 is at most $\beta k$. From Lemma 5.5, the branching factors of a search tree of ECCS2 are bounded by $\min\{k, \left(\frac{d+1}{2}\right)\} \leq k$. Therefore, the number of nodes in a search tree of ECCS is at most $k^{\beta k} = 2^{\beta k \log k}$.

Considering time needed for data reduction, Theorem 1.2 follows from Lemma 5.6 and Proposition 2.20.

We have shown in Section 3 that the algorithm ECCS has consistently better running time than ECCG [22]. For a large class of graphs, ECCS2 further beats ECCS in the running time. This can be seen as follows. From Lemma 5.3, we have $\min\{k, \left(\frac{d+1}{2}\right)\} \leq \left(\frac{d+1}{2}\right) < (d+1)^2$. Therefore, the number of nodes in a search tree of ECCS2 is at most $2^{O(\beta k \log d)}$. For $\beta = o(d/\log d)$, ECCS2 improves the bound on search tree size by a factor of $\frac{\log \beta k}{\log \frac{d}{\beta k}} = 2^{O(k(d-\beta \log d))} = 2^{O(dk)}$, an exponential improvement over the bound on search tree size. For many instances of ECC, $\beta = o(d/\log d)$ is a very mild requirement, considering that $d$ can grow linearly with the input size, while $\beta$ remains constant (Remark 2.13).

5.2 Assignment Clique Cover

We will continue to assume that data reduction rules are applied on an instance $(G, t)$ of ACC, including the rules described in Section 3.2. Figure 6 shows our new bounded search tree algorithm for ACC; henceforth referred as ACCS2. Steps 4c, 4f, 5a, 5d of ACCS2 are identical to the steps 3a, 3d, 4a, 4d of ECCS2 described in Figure 3.

In a search tree of ACCS2, there is no depth bound for the number of cliques. ACCS will create as many cliques as permitted by the parameter $t$, until it finds a clique cover of $G$ at step 1. At step 4a of ACCS2, we count out of two vertices $x$ and $y$ how many are missing in $C_i$: it could only be one or two. Only scenario for which we cannot execute steps 4c-4f would be when $s = 2$ but $t = 1$. We are reusing the subroutines of ECCS2 for ACCS2, and rest of the steps of ACCS2 are fairly straightforward adaptation of ECCS2. We can adapt the proof of Lemma 5.4 to show correctness of ACCS2 (see Appendix B). Thus we conclude the following.

Lemma 5.8. Algorithm ACCS2 correctly solves the parameterized problem ACC.

5.3 Weighted Edge Clique Partition

We describe a bounded search tree algorithm for the WECP problem, considering that a given instance of WECP would be reduced with respect to the data reduction rules described by [17]. For WECP [17] have developed a bi-kernel of size at most $4^k$. The bi-kernel is based on a general problem that considers a subset of vertices annotated with integer weights as input, in addition to the input of WECP. The general problem is as follows.

For a large class of graphs, ACCS2 further beats ECCS in the running time. This can be seen as follows. From Lemma 5.3, we have $\min\{k, \left(\frac{d+1}{2}\right)\} \leq \left(\frac{d+1}{2}\right) < (d+1)^2$. Therefore, the number of nodes in a search tree of ECCS2 is at most $2^{O(\beta k \log d)}$. For $\beta = o(d/\log d)$, ECCS2 improves the bound on search tree size by a factor of $\frac{\log \beta k}{\log \frac{d}{\beta k}} = 2^{O(k(d-\beta \log d))} = 2^{O(dk)}$, an exponential improvement over the bound on search tree size. For many instances of ECC, $\beta = o(d/\log d)$ is a very mild requirement, considering that $d$ can grow linearly with the input size, while $\beta$ remains constant (Remark 2.13).

We have shown in Section 3 that the algorithm ECCS has consistently better running time than ECGG [22]. For a large class of graphs, ECCS2 further beats ECCS in the running time. This can be seen as follows. From Lemma 5.3, we have $\min\{k, \left(\frac{d+1}{2}\right)\} \leq \left(\frac{d+1}{2}\right) < (d+1)^2$. Therefore, the number of nodes in a search tree of ECCS2 is at most $2^{O(\beta k \log d)}$. For $\beta = o(d/\log d)$, ECCS2 improves the bound on search tree size by a factor of $\frac{\log \beta k}{\log \frac{d}{\beta k}} = 2^{O(k(d-\beta \log d))} = 2^{O(dk)}$, an exponential improvement over the bound on search tree size. For many instances of ECC, $\beta = o(d/\log d)$ is a very mild requirement, considering that $d$ can grow linearly with the input size, while $\beta$ remains constant (Remark 2.13).

We will continue to assume that data reduction rules are applied on an instance $(G, t)$ of ACC, including the rules described in Section 3.2. Figure 6 shows our new bounded search tree algorithm for ACC; henceforth referred as ACCS2. Steps 4c, 4f, 5a, 5d of ACCS2 are identical to the steps 3a, 3d, 4a, 4d of ECCS2 described in Figure 3.

In a search tree of ACCS2, there is no depth bound for the number of cliques. ACCS will create as many cliques as permitted by the parameter $t$, until it finds a clique cover of $G$ at step 1. At step 4a of ACCS2, we count out of two vertices $x$ and $y$ how many are missing in $C_i$: it could only be one or two. Only scenario for which we cannot execute steps 4c-4f would be when $s = 2$ but $t = 1$. We are reusing the subroutines of ECCS2 for ACCS2, and rest of the steps of ACCS2 are fairly straightforward adaptation of ECCS2. We can adapt the proof of Lemma 5.4 to show correctness of ACCS2 (see Appendix B). Thus we conclude the following.

Algorithm 5.8. Algorithm ACCS2 correctly solves the parameterized problem ACC.

The branching factors of ACCS2 are bounded by $\left(\frac{d+1}{2}\right)$; this follows from the first part of the proof of Lemma 5.6. By Rule 3.21, $d + 1 \leq n \leq t$. The depth of a search tree of ACCS2 is at most $t$. Therefore, the number of nodes in a search tree of ACCS2 is at most $t^t < t^{2t} = 4^{t \log t}$.

Theorem 1.3 follows from Lemma 5.8 and Proposition 2.20 considering time needed for data reduction. The running time of Corollary 1.5 follows from similar arguments presented for Corollary 3.25. The space bound of Corollary 1.5 follows from Lemma 5.2 since for an assignment-minimum clique cover $k = m$ suffices.

We describe a bounded search tree algorithm for the WECP problem, considering that a given instance of WECP would be reduced with respect to the data reduction rules described by [17]. For WECP [17] have developed a bi-kernel of size at most $4^k$. The bi-kernel is based on a general problem that considers a subset of vertices annotated with integer weights as input, in addition to the input of WECP. The general problem is as follows.

For a large class of graphs, ACCS2 further beats ECCS in the running time. This can be seen as follows. From Lemma 5.3, we have $\min\{k, \left(\frac{d+1}{2}\right)\} \leq \left(\frac{d+1}{2}\right) < (d+1)^2$. Therefore, the number of nodes in a search tree of ECCS2 is at most $2^{O(\beta k \log d)}$. For $\beta = o(d/\log d)$, ECCS2 improves the bound on search tree size by a factor of $\frac{\log \beta k}{\log \frac{d}{\beta k}} = 2^{O(k(d-\beta \log d))} = 2^{O(dk)}$, an exponential improvement over the bound on search tree size. For many instances of ECC, $\beta = o(d/\log d)$ is a very mild requirement, considering that $d$ can grow linearly with the input size, while $\beta$ remains constant (Remark 2.13).
Assignment-Clique-Cover-Search2\((G, t, C, R)\):

// Abbreviated ACCS2\((G, t, C, R)\)

1. if \(C\) covers edges of \(G\), then return \(C\)
2. if \(t \leq 0\) return \(\emptyset\)
3. select the last uncovered edge \(\{x, y\}\) from the \(DEP\) of \(G\) where \(y \in N_d(x)\)
   // cf. step 3 of ACCS in Figure 2
4. for each \(l \in R_x \cap R_y\) do
   (a) \(s \leftarrow |\{x, y\} \setminus (C_l \cap \{x, y\})|\)
   (b) if \(t < s\), then go to the next iteration of step 4
   (c) cover the edge \(\{x, y\}\) with the clique \(C_l\) and update \(R\) // Figure 3 (a)
   (d) \(Q \leftarrow ACCS2(G, t - s, C, R)\)
   (e) if \(Q \neq \emptyset\), then return \(Q\)
   (f) undo changes done to \(C\) and \(R\) at step 4c // Figure 3 (b)
5. if \(t \geq 2\), then
   (a) set \(C\) to \(C \cup \{x, y\}\) and update \(R\) // Figure 3 (c)
   (b) \(Q \leftarrow ACCS2(G, t - 2, C, R)\)
   (c) if \(Q \neq \emptyset\), then return \(Q\)
   (d) undo changes done to \(C\) and \(R\) at step 5a // Figure 3 (d)
6. return \(\emptyset\)

Figure 6: A bounded search tree algorithm for ACC, denoted ACCS2.

Annotated weighted edge clique partition (AWECP)

Input: A graph \(G = (V, E)\), a weight function on edges \(w^E : E \rightarrow \mathbb{Z}_{\geq 0}\), a nonnegative integer \(k\), a set of vertices \(S \subseteq V\), and a weight function \(w^S : S \rightarrow \mathbb{Z}_{\geq 0}\).

Output: If one exists, a clique cover \(C\) of \(G\) such that (1) \(|C| \leq k\), (2) each edge \(e \in E\) appears in exactly \(w^E(e)\) cliques of \(C\), and (3) each vertex \(x \in S\) appears in exactly \(w^S(x)\) cliques of \(C\); otherwise report \(NO\).

When \(S = \emptyset\), AWECPS reduces to the special case WEC. For the data reduction rules that give rise to an instance of AWECPS from an instance of WEC, see Section 2 of [17]. We point out that a reduced instance of AWECPS with respect to the data reduction rules of [17] does not contain any isolated vertices.

Note that for any vertex \(x \in S\), if the weight on the vertex \(x\) is smaller than the weight of all the edges incident on \(x\), i.e., \(w^S(x) < w^E(e)\) for all \(e = \{x, y\} \in E\), then we immediately know that a given instance of AWECPS is a \(NO\) instance. Also, for any vertex \(x \in S\), if the weight on the vertex \(x\) is larger than the sum of the weights of all the edges incident on \(x\), i.e., \(w^S(x) > \sum_{e = \{x, y\} \in E} w^E(e)\), then we immediately know that a given instance of AWECPS is a \(NO\) instance.

Figure 7 shows a bounded search tree algorithm for AWECPS, henceforth referred as AWECPS. The subroutines for steps 3a, 3b, 3c, 4a, 4d of AWECPS are described in Figure 8. AWECPS extends the mechanism of building clique cover employed by ECCS2. At every node of a search tree, AWECPS extends a clique cover if and only if the extension is (locally) consistent with the weight functions \(w^E\) and \(w^S\).

AWECPS treats the weight functions \(w^E\) and \(w^S\) as budgets for the edges of \(E\) and vertices of \(S\) respectively. At every node of a search tree, every edge \(e\) has a budget \(w^E(e)\), denoting how many more cliques the edge \(e\) can be included. Similarly, at every node of a search tree, every vertex \(z \in S\) also has a budget \(w^S(z)\), denoting how many more cliques the vertex \(z\) can be included. For AWECPS, we maintain the notions of including an edge (or a vertex) in a clique (in this context covering an edge \(e\) by
Annotated-Weighted-Edge-Clique-Partition-Search($G, k, w^E, S, w^S, C, \mathcal{R}$):

// Abbreviated AWECPS($G, k, w^E, S, w^S, C, \mathcal{R}$)

1. if $w^E(e) = 0$ for all $e \in E$ and $w^S(z) = 0$ for all $z \in S$, then return $C$
   // cf. step 1 ECCS2 Figure 4
2. select the last edge $e = \{x, y\}$ from the DEP of $G$ where $y \in N_d(x)$ and $w^E(e) > 0$
   // cf. step 2 ECCS2 Figure 4
3. for each $l \in R_x \cap R_y$ such that $\{x, y\} \not\subseteq C_l$ do
   (a) if ‘$e$ cannot be included’ in $C_l$, then go to the next iteration of step 3 // Figure 5 (a)
   (b) include the edge $\{x, y\}$ in the clique $C_l$, and update $\mathcal{R}$, $w^E$, $w^S$ // Figure 5 (b)
   (c) $Q \leftarrow$ AWECPS($G, k, w^E, S, w^S, C, \mathcal{R}$)
   (d) if $Q \neq \emptyset$, then return $Q$
   (e) undo changes done to $C$, $\mathcal{R}$, $w^E$, and $w^S$ at step 3b // Figure 5 (c)
4. if $k > 0$, then
   (a) if ($x \in S$ and $w^S(x) = 0$) or ($y \in S$ and $w^S(y) = 0$), then return $\emptyset$
   (b) set $C$ to $C \cup \{\{x, y\}\}$, and update $\mathcal{R}$, $w^E$, $w^S$ // Figure 5 (d)
   (c) $Q \leftarrow$ AWECPS($G, k-1, w^E, S, w^S, C, \mathcal{R}$)
   (d) if $Q \neq \emptyset$, then return $Q$
   (e) undo changes done to $C$, $\mathcal{R}$, $w^E$, and $w^S$ at step 4b // Figure 5 (e)
5. return $\emptyset$

Figure 7: A bounded search tree algorithm for AWECP, denoted AWECPS.

clique could be used to denote including the edge $e$ in exactly $w^E(e)$ distinct cliques.

Since both $w^E$ and $w^S$ are positive integer weight functions, if budgets of all edges of $E$ and budgets of all vertices of $S$ can be spent exactly at a node of a search tree of AWECPS with at most $k$ cliques, then the given instance of AWECP is a YES instance, and step 1 would return a corresponding clique cover. Step 2 of AWECPS essentially select the same edge $e = \{x, y\}$ as would be selected by step 2 of ECCS2, except AWECPS selects the same edge $e$ repeatedly until its budget is entirely exhausted, i.e., $w^E(e) = 0$. Steps 3a and 4a enforce that including an edge $e = \{x, y\}$ to a clique must be within the budgets of the corresponding edges of $E$ and within the budgets of corresponding vertices of $S$. Rest of the steps of AWECPS are self-explanatory.

Turning to the subroutines of AWECPS in Figure 5, we see that the subroutine for step 3a simply makes sure that every new vertex or new edge that would be included in the clique $C_l$ (as a result of including the edge $e = \{x, y\}$ in the clique $C_l$) has the budget to do so. Rest of the subroutines make sure that the budgets of the edges of $E$ and the vertices of $S$ are updated accordingly when we include (resp. exclude) an edge $e$ in (resp. from) a clique. Note that (for brevity) the subroutines of ECCS2 are invoked from the subroutines of AWECPS.

The proof correctness of ECCS2 (Lemma 5.1) can be adapted to obtain a proof of correctness for AWECPS (see Appendix B). Thus we conclude the following.

**Lemma 5.9.** Algorithm AWECPS correctly solves the parameterized problem AWECP.

**Lemma 5.10.** The number of nodes in a search tree of AWECPS is at most $2^{\beta k \log k}$.

*Proof.* It is straightforward to see that the depth of a search tree of AWECPS is bounded by $\beta k$. Let $w$ be the maximum edge weight in an instance of AWECP, i.e., $w = \max_{e \in E} \{w^E(e)\}$. Then, number of branches at any node of a search tree of AWECPS is bounded by $\min\{k, (\tfrac{\beta k + 1}{2})w\} \leq k$: this follows from...
Step 3a:
1. if \( x \in S \) and \( x \not\in C \) and \( w^S(x) = 0 \), then report ‘\( e \) cannot be included’
2. if \( y \in S \) and \( y \not\in C \) and \( w^S(y) = 0 \), then report ‘\( e \) cannot be included’
3. for each \( z \in C \setminus \{x, y\} \) do
   - Let \( e_1 = \{x, z\} \) and \( e_2 = \{y, z\} \)
   (a) if \( (x \not\in C \) and \( w^E(e_1) = 0 \) \) or \( (y \not\in C \) and \( w^E(e_2) = 0 \) \), then report ‘\( e \) cannot be included’
4. report ‘\( e \) can be included’

Step 3b:
1. execute the subroutine for step 3a of ECCS2 from Figure 7
2. for each \( z \in C \setminus \{x, y\} \) do
   (a) if \( x_1 = 0 \), then decrement \( w^E(e_1) \) by 1, where \( e_1 = \{x, z\} \)
   (b) if \( y_1 = 0 \), then decrement \( w^E(e_2) \) by 1, where \( e_2 = \{y, z\} \)
3. decrement \( w^E(e) \) by 1
4. if \( x_1 = 0 \) and \( x \in S \), then decrement \( w^S(x) \) by 1
5. if \( y_1 = 0 \) and \( y \in S \), then decrement \( w^S(y) \) by 1

Step 3c:
1. execute the subroutine for step 3b of ECCS2 from Figure 7
2. for each \( z \in C \setminus \{x, y\} \) do
   (a) if \( x_1 = 0 \), then increment \( w^E(e_1) \) by 1, where \( e_1 = \{x, z\} \)
   (b) if \( y_1 = 0 \), then increment \( w^E(e_2) \) by 1, where \( e_2 = \{y, z\} \)
3. increment \( w^E(e) \) by 1
4. if \( x_1 = 0 \) and \( x \in S \), then increment \( w^S(x) \) by 1
5. if \( y_1 = 0 \) and \( y \in S \), then increment \( w^S(y) \) by 1

Step 4b:
1. execute the subroutine for step 4a of ECCS2 from Figure 7
2. decrement \( w^E(e) \) by 1
3. if \( x \in S \), then decrement \( w^S(x) \) by 1
4. if \( y \in S \), then decrement \( w^S(y) \) by 1

Step 4c:
1. execute the subroutine for step 4d of ECCS2 from Figure 7
2. increment \( w^E(e) \) by 1
3. if \( x \in S \), then increment \( w^S(x) \) by 1
4. if \( y \in S \), then increment \( w^S(y) \) by 1

Step 4d:
1. execute the subroutine for step 4e of ECCS2 from Figure 7
2. report \( y \)
3. if \( \beta \) is 1, then increment \( w^S(x) \) by 1
4. if \( \beta \) is 1, then increment \( w^S(y) \) by 1

Figure 8: Subroutines of AWECPS shown in Figure 7

similar arguments presented for ECCS2 in Lemma 5.5. Therefore, the number of nodes in a search tree of AWECPS is at most \( k^{3k} = 2^{\beta k \log k} \).

Considering time needed for data reduction, Theorem 1.6 follows from Lemma 5.10 and Proposition 2.20.

For an instance of WECP, the algorithm of [17] takes \( 2^{O(k^{3/2}w^{1/2} \log (k/w))}O(k^{O(1)}) \) time, where \( w \) is the maximum edge weight of the instance. For \( \beta = \alpha(\sqrt{k/w \log (k/w)}) / \log k \), AWECPS improves the running time by a factor of \( \frac{2^{O(k^{3/2}w^{1/2} \log (k/w))}}{2^{O(k^{3/2}w^{1/2} \log (k/w)-\beta \log k)}} = 2^{O(k^{3/2}w^{1/2} \log (k/w))} \). This is a significant improvement, considering the fact that \( \beta \) is bounded by the graph size, while \( k \) and \( w \) could be arbitrarily large. For nontrivial instances of WECP, \( k \) could be up to (but not including) \( mw \).
On the other hand, if the maximum edge weight \( w \) is bounded by some constant, then for \( \beta = o(\sqrt{k}) \), \( \text{AWECPS} \) is \( 2^{O(k^{3/2} \log k)} \) times faster than the algorithm of [17].

### 5.4 Exact Weighted Clique Decomposition

The data reduction rules of \( \text{WECP} \) described by [17] are also applicable to any instance of \( \text{EWCD} \). Consequently, we have a bi-kernel for \( \text{EWCD} \) with \( 4^k \) vertices. The bi-kernel is based on a general problem that considers a subset of vertices annotated with positive weights as input, in addition to the input of \( \text{EWCD} \). The general problem is as follows.

**Annotated exact weighted clique decomposition (AEWCD)**

**Input:** A graph \( G = (V, E) \), a weight function on edges \( w^E : E \rightarrow \mathbb{R}_{>0} \), a nonnegative integer \( k \), a set of vertices \( S \subseteq V \), and a weight function \( w^S : S \rightarrow \mathbb{R}_{>0} \).

**Output:** If one exists, a clique cover \( C \) of \( G \) and positive weight \( \gamma_i \) for every clique \( C_i \in C \) such that (1) \( |C| \leq k \), (2) for each edge \( e = \{x, y\} \in E \), \( \sum_{\{x, y\} \in C_i, C_i \in C} \gamma_i = w^E(e) \), and (3) for each vertex \( x \in S \), \( \sum_{x \in C_i, C_i \in C} \gamma_i = w^S(x) \); otherwise report NO.

Note that when \( S = \emptyset \), \( \text{AEWCD} \) reduces to the special case \( \text{EWCD} \). If the weight functions are integer valued and weight on each of the cliques are forced to be 1, then \( \text{AEWCD} \) is equivalent to \( \text{AWECPS} \).

Let \( C \) be a clique cover of \( G \). If \( C \) contains at most \( k \) cliques, then, for an instance \( (G, k, w^E, S, w^S) \) of \( \text{AEWCD} \), the following linear program (LP) would compute weights of the cliques \( \gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{|C|}\} \) in \( C \) (if the LP is feasible). Note that the objective function of the LP is a constant, i.e., an LP solver would only need to find a feasible solution \( \gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{|C|}\} \), if one exists.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i : C_i \in C} 0 \times \gamma_i \\
\text{subject to} & \quad \sum_{i : \{x, y\} \in C_i, C_i \in C} \gamma_i = w^E(e), \quad \text{for all } e = \{x, y\} \in E \\
& \quad \sum_{i : x \in C_i, C_i \in C} \gamma_i = w^S(x), \quad \text{for all } x \in S \\
& \quad \gamma_i \geq 0, \quad \text{for all } i : C_i \in C
\end{align*}
\]

Figure 9 shows a bounded search tree algorithm for \( \text{AEWCD} \), henceforth referred to as \( \text{AEWCDS} \). We reuse the subroutines of \( \text{ECCS2} \) for \( \text{AEWCDS} \): steps 3a, 3h, 4a, 4h of \( \text{AEWCDS} \) are identical to the steps 3a, 3d, 4a, 4d of \( \text{ECCS2} \) described in Figure 5. We assume \( \text{AEWCD} \) has access to the weight functions at step 1 of any node of a search tree. Whenever a clique cover \( C \) of \( G \) is found at a node, we check the feasibility of \( C \) for the instance \( (G, k, w^E, S, w^S) \), by solving the LP (A). If (A) is feasible, then the clique cover \( C \) and a corresponding solution \( \gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{|C|}\} \) of (A) is returned at step 1.

For \( \text{AEWCD} \), we explicitly distinguish between covering an edge by a clique and including an edge in a clique. An edge can be included in many cliques, without being identified as covered. If an edge \( \{x, y\} \) is marked as covered (steps 3b, 4b) at a node \( u^T \) of a search tree \( T \), then the corresponding descendent nodes (steps 3c, 4c) of \( u^T \) in \( T \) would not select the edge \( \{x, y\} \) as uncovered at step 2. On the other hand, if an edge \( \{x, y\} \) is marked as uncovered (steps 3e, 4e) at a node \( u^T \) of a search tree \( T \), then the corresponding descendent nodes (steps 3f, 4f) of \( u^T \) in \( T \) may select the edge \( \{x, y\} \) as uncovered at step 2.

For \( \text{AEWCDS} \), the branches at steps 3e-3g and 4e-4g are crucial: an edge \( e = \{x, y\} \) may need to be included in many cliques so that the weight \( w(e) \) can be matched with the weights of the neighbouring edges of \( e \). To see this, consider the complete graph \( K_4 \) and weights on the edges as shown in Figure 10. For \( k = 3 \), the instance of \( \text{AEWCD} \) is a \( \text{YES} \) instance. A corresponding clique cover \( C \) of \( G \) consists of following cliques: \( C_1 = \{a, b, c, d\} \), \( C_2 = \{a, b, e\} \), \( C_3 = \{a, b\} \). Corresponding feasible weights on the cliques are \( \gamma_1 = 1 \), \( \gamma_2 = 1 \), and \( \gamma_3 = 99 \). WLOG, we can assume that the edge \( e = \{a, b\} \) is the last edge in the \( \text{DEP} \) of \( G \). The edge \( e \) needs to appear in all three cliques of \( C \). Therefore, we need to make sure that an edge is included in a sufficient number of cliques (at most \( k \)), so that we have correct combination of cliques for the edges that are chosen later from the edge permutation.
Annotated-Exact-Weighted-Clique-Decomposition-Search\((G, k, C, R)\):

// Abbreviated $\text{AEWCDS}(G, k, C, R)$

1. if $C$ covers edges of $G$, then
   (a) for $C$ solve the LP \((A)\)
   (b) if \((A)\) is feasible, then return \((C, \gamma)\)
   (c) else return \((\emptyset, \emptyset)\)

2. select the last uncovered edge \(\{x, y\}\) from the DEP of $G$ where $y \in N_d(x)$

3. for each \(l \in R_x \cap R_y\) such that \(\{x, y\} \not\subseteq C\) do
   (a) include the edge \(\{x, y\}\) in the clique $C_l$ and update $R$ // Figure 5 (a)
   (b) mark the edge \(\{x, y\}\) as covered
   (c) \((Q, \gamma) \leftarrow \text{AEWCDS}(G, k, C, R)\)
   (d) if \(Q \neq \emptyset\), then return \((Q, \gamma)\)
   (e) mark the edge \(\{x, y\}\) as uncovered
   (f) \((Q, \gamma) \leftarrow \text{AEWCDS}(G, k, C, R)\)
   (g) if \(Q \neq \emptyset\), then return \((Q, \gamma)\)
   (h) undo changes done to $C$ and $R$ at step 3a // Figure 5 (b)

4. if $k > 0$, then
   (a) set $C$ to $C \cup \{\{x, y\}\}$ and update $R$ // Figure 5 (c)
   (b) mark the edge \(\{x, y\}\) as covered
   (c) \((Q, \gamma) \leftarrow \text{AEWCDS}(G, k-1, C, R)\)
   (d) if \(Q \neq \emptyset\), then return \((Q, \gamma)\)
   (e) mark the edge \(\{x, y\}\) as uncovered
   (f) \((Q, \gamma) \leftarrow \text{AEWCDS}(G, k-1, C, R)\)
   (g) if \(Q \neq \emptyset\), then return \((Q, \gamma)\)
   (h) undo changes done to $C$ and $R$ at step 4a // Figure 5 (d)

5. return \((\emptyset, \emptyset)\)

Figure 9: A bounded search tree algorithm for \textit{AEWCD}, denoted \textit{AEWCDS}.

The way we have described \textit{AEWCD} (in Figure 9) is merely for ease of reading. An equivalent (in terms of correctness) description would have a subroutine consisting of step 3 and step 4, excluding steps 3b, 3e-3g, 4b, 4e-4g. Now, instead of step 3 and 4, we can simply call the subroutine once, after marking the edge \(\{x, y\}\) as covered, and if the call fails, then we can call the subroutine once, after marking the edge \(\{x, y\}\) as uncovered. The search trees corresponding to the equivalent description are likely to be smaller for many instances of \textit{AEWCD}.

[7] have described an LP based algorithm for \textit{AEWCD}, called \textit{CliqueDecomp-LP}. The LP used by \textit{CliqueDecomp-LP} has \(k\) variables and at most \(4k^2\) constraints. In contrast, the LP \((A)\) has at most \(k\) variables and at most \(m + |S| \leq m + n\) constraints. At every iteration, \textit{CliqueDecomp-LP} selects a permutation matrix \(P \in \{0, 1\}^{2k \times k}\). For every such permutation matrix \(P\), \textit{CliqueDecomp-LP} needs to solve the corresponding LP, regardless of whether \(P\) correspond to a clique cover of the graph or not. For a single permutation matrix \(P\), \textit{CliqueDecomp-LP} may need to solve the LP up to \(2k^2\) times. Therefore, \textit{CliqueDecomp-LP} may need to solve the LP up to \(2^{2k^2} \times 2k\) times, in the worst case. In contrast, \textit{AEWCDS} solves the LP \((A)\) (at step 1) only when \(C\) is a clique cover of the graph. In most cases, for
YES instances of AEWCD, the number of clique covers of a graph with at most \( k \) cliques is significantly smaller than \( 2^{k^2} \times 2k \).

A correctness proof for AEWCDs can be obtained similarly as the correctness proof for AWECPs (Lemma 5.20). A minor modification would be required for the base case of induction. In the base case, i.e., \( k = 1 \), for some constant \( q \), we would have \( w^k(e) = q \) for all \( e \in E \) and \( w^k(x) = q \) for all \( x \in S \). Rest of the proof would be analogous. Thus we have the following.

**Lemma 5.11.** Algorithm AEWCDs correctly solves the parameterized problem AEWCD.

**Lemma 5.12.** The number of nodes in a search tree of AEWCDs is at most \( 2^{βk(1+\log k)} \).

**Proof.** The depth of a search tree of AEWCDs is bounded by \( βk \). At step 3 of any node of the search tree \( R_x \cap R_y \leq k \). If \( R_x \cap R_y = k \), then there is no branch at step 4 of the node. Therefore, number of branches at any node of a search tree of AEWCDs is at most \( 2k \). It follows that the number of nodes in a search tree of AEWCDs is at most \( (2k)^{βk} = 2^{βk(1+\log k)} \). 

For AEWCD, the LP (A) would have at most \( k \) variables, and at most \( m+n \) constraints. For nontrivial instances of AEWCD, we can assume \( k < m \). Therefore, (A) is solvable in \( O(k^3 L) \) time, where \( L \) is the number of bits needed to encode the LP [10][44]. Since \( n \leq 4^k \), running time of AEWCDs is \( 2^{O(βk \log k)} \). Now, considering time needed for data reduction, Theorem 1.7 follows from Proposition 2.20.

For the cases when all the weights are restricted to integers (including the weights on the cliques), [7] have described an integer partitioning based algorithm called CliqueDecomp-IP. CliqueDecomp-IP requires maintaining \((w+1)^k\) weight matrices in the worst case, where \( w \) is the maximum edge weight. Consequently, space requirement of CliqueDecomp-IP could be prohibitive for many AEWCD instances even if the instances are solvable in a reasonable amount of time. For these cases, we point out that instead of solving the LP (A), at step 1 of AEWCDs, one can search for feasible set of weights \( γ \), by enumerating at most \( w^k = 2^{k \log w} \) choices. Therefore, AEWCD instances restricted to integer weights are solvable in \( 2^{O(k(β \log k + \log w))} \) time, using \( O(m + kΔ) \) space (Lemma 5.2).

CliqueDecomp-LP and CliqueDecomp-IP both require enumerating \( 2^{2k^2} \) permutation matrices in the worst case. For \( β = o(k/\log k) \), AEWCD improves the running time by a factor of \( 2^{O(\alpha^2)} \) = \( 2^{O(k(k−β \log k))} = 2^{O(k^2)} \).

### 5.5 Generalized Vertex Clique Cover and Colorability

We conclude our demonstration of applicability of our new framework with description of a bounded search tree algorithm for LRCC. This serves several purposes. First, this highlights the natural bridges that exist between clique cover problems and corresponding graph coloring problems in the complement graph. Second, even though many variants of graph coloring problems had been studied in parameterized
complexity (see [18, 27]), the results are not useful for obtaining FPT algorithms for the corresponding vertex clique cover problems (since no parameterized reductions are known from parameterization of the vertex clique cover problems to the parameterizations of graph coloring problems that are known to be FPT). Third, we are able to show that instances of PMC can be solved by a parameterized reduction from PMC to LRCC.

The equivalence of VCC and Colorability stems from the fact that a clique in a graph corresponds to a partition of vertices into independent sets in the complement graph. Since Colorability is NP-complete for \( k \geq 3 \) [12], the existence of an \( f(k)n^{O(1)} \) time algorithm for VCC would imply \( P = NP \). Therefore, LRCC is not FPT with respect to \( k \) unless \( P = NP \). We show that LRCC is FPT with respect to \( \beta \) and \( k \).

To solve an instance \( I^* = (G, k, E^*) \) of LRCC such that the number of edges in \( E^* \) is small (far from being an instance of ECC), one may think of solving an instance \( I = (G, k) \) of VCC, and then somehow convert the solution of \( I \) to a solution of \( I^* \). It is unlikely that there is any feasible approach to do that. To see this, consider the example shown in Figure 11. A solution of \( (G, k) \) may miss many edges in \( E^* \), even when the solutions of \( I \) and \( I^* \) need same number of cliques. Therefore, we turn to other viable approaches.

One approach to solve LRCC is to construct an edge clique cover of a subgraph required to cover the edges of \( E^* \), and then extend the edge clique cover to a solution of \( I^* \). This would only allow us to bound the branching factors of a search tree with \( k \). We use a different approach so that we can preserve the bound on the branching factors of a search tree that we have obtained for ECCS2 (Lemma 5.5). Our approach is to interleave the tasks of covering the edges of \( E^* \) and covering the vertices of \( G \). Next, we elaborate on this approach.

Figure 12 shows our bounded search tree algorithm for LRCC, henceforth referred as LRCCS. LRCCS achieves our goal of bounding the branching factors by selecting the first vertex \( x \) in the reverse degeneracy ordering of \( V \) such that either the vertex \( x \) is uncovered or an edge \( \{x, y\} \in E^* \) is uncovered where \( y \in N_d(x) \). At any node of a search tree, LRCCS have two possible choices to consider: an uncovered edge \( e = \{x, y\} \in E^* \) (step 2) or an uncovered vertex \( u \) (step 3). The two possible choices give rise to four different cases: (1) \( e = \emptyset \), (2) \( u = \emptyset \), (3) \( e \neq \emptyset \), \( u \neq \emptyset \), and \( x \) precedes \( u \) in the degeneracy ordering of \( V \), (4) \( e \neq \emptyset \), \( u \neq \emptyset \), and \( x \) does not precede \( u \) in the degeneracy ordering of \( V \). Note that step 4 makes the recognition of cases (1) and (3) identical for the later steps.

For cases (1) and (3), the uncovered vertex \( u \) needs to be covered (to be consistent with our goal). In these cases LRCC chooses the representatives in \( R_u \) (step 6), to enumerate at step 7. For cases (2) and (4), the uncovered edge \( e \) needs to be covered (to be consistent with our goal). In these cases LRCC chooses the representatives in \( R_e \) (step 5), to enumerate at step 7. For cases (1) and (3), step 6a (resp. step 7a) converts the task of covering the vertex \( x \) to covering an edge \( \{x, y\} \) (resp. covering \( \{x\} \): this allows LRCCS to reuse the subroutines of ECCS2 from Figure 1.

A proof of correctness of LRCC can be obtained analogously as the proof of correctness shown for ECCS2 (Lemma 5.1). Thus we have the following.

**Lemma 5.13.** Algorithm LRCCS correctly solves the parameterized problem LRCC.
Lemma 5.14. The number of nodes in a search tree of LRCC is at most $2^{\beta k \log k}$.

Proof. For an edge $e \neq \emptyset$ selected at step 2, the branching factors of a search tree of LRCC is bounded by $\min\{k, \binom{d+1}{2}\}$: this follows from the same argument as presented for ECCS2 in Lemma 5.5. For a vertex $u \neq \emptyset$ selected at step 3, $N_d(u) \leq d$. Therefore, $|R_u| \leq \min\{k, \binom{d+1}{2}\} \leq k$. It is straightforward to see that the depth of a search tree of LRCCS is bounded by $\beta k$. Therefore, the number of nodes in a search tree of LRCC is at most $k^{3k} = 2^{\beta k \log k}$.

Since time spent at any node of a search tree of LRCCS is bounded by a polynomial in $n$, Theorem 5.14 follows from Lemma 5.14. We obtain a proof of Corollary 5.14 using a parameterized reduction from PMC to LRCC.

Proof of Corollary 5.14. For an instance $(G, k, F)$ of PMC, in polynomial time we can construct the complement graph $\overline{G}$. Now, the non-edges of $G$ in $F$ becomes edges of $\overline{G}$. Let $\alpha$ be the independence number of $G$, and $\beta$ be the clique number of $G$. Note that $\alpha = \beta$. By the equivalence of PMC and LRCC in the complement graph, $(\overline{G}, k, F)$ is a YES instance of PMC if and only if $(G, k, F)$ is a YES instance of LRCC. The instance $(G, k, F)$ can be solved with the algorithm LRCCS in $2^{\beta k \log k} n^{O(1)}$ time. Thus we can obtain a solution of $(G, k, F)$ in $n^{O(1)} + 2^{\beta k \log k} n^{O(1)} = 2^{\alpha k \log k} n^{O(1)}$ time.
6 Implementations and Open Problems

To solve large real-world instances of a clique cover problem exactly, one may need to employ two things: effective data reduction rules and efficient search tree algorithm. In this article, utilizing a few data reduction rules, our focus has been on efficient search tree algorithms. Although, lower bounds on kernelization (such as non-existence of polynomial kernel for ECC with respect to clique cover size [9]) restrict design of provably smaller kernels, such restrictions are barely impediment for designing data reduction rules that are effective in practice. An example of this is the work of [13] on vertex clique cover. Vertex clique cover does not admit any kernel with respect to clique cover size (assuming \( P \neq NP \)), but [13] have designed effective data reduction rules for vertex clique cover that can solve large real-world instances. Therefore, designing effective data reduction rules for the clique cover problems we have studied is a direction that needs attention from future research.

To see efficacy of our proposed search tree algorithms in practice, we have implemented ECCG, ECCS, and ECCS2, and have compared performance of the algorithms. Given the lack of effective data reduction rules for large real-world graphs, we have limited comparisons of the algorithms on random instances that ECCG can solve within a few hours. The results are remarkable and demonstrate that improvements in our analyses do carry over into the implementations (code and details of the comparisons are available here: https://drive.google.com/drive/folders/11Sa9PX0m5TcKFSwoNQnXzyq63CZLMoZ). On our test instances, ECCG took approximately 2000-5200 seconds, whereas ECCS and ECCS2 took approximately 1-26 seconds only. Within a span of few hours, ECCS has been able to produce solutions using search trees that are orders of magnitude smaller than the corresponding search trees of ECCG. On our test instances, the search trees of ECCG have approximately 781-6200 million nodes, whereas the search trees of ECCS have approximately 14-111 million nodes only.

We conclude by highlighting a few problems that we think are important to resolve.

Using Corollary[3,14] Corollary[3,15] and Corollary[3,16] we have shown that for sparse graphs running time of ECC is better captured by degeneracy, instead of clique cover size. Thus we ask the following.

Problem 6.1. Does ECC parameterized by \( d \) has an FPT algorithm running in \( f(d)n^{O(1)} \) time?

For planar graphs, we have shown that ECC is solvable in \( 2^{O(k)}n^{O(1)} \) time (Corollary[3,17]). For many NP-complete problems on planar graphs, the bidimensionality theory [12] has led to FPT algorithms with sub-exponential dependence on parameter. Thus a natural question to ask is the following.

Problem 6.2. Does ECC on planar graphs has an FPT algorithm running in \( 2^{o(k)}n^{O(1)} \) time?

From the proof of Lemma[5,23] (or Theorem[5,24]), it is obvious that on planar graphs ACC is solvable in \( 2^{O(1)}n^{O(1)} \) time. Although, we have not discussed the algorithms for WECF, EWCD, and LRVC using the framework of enumerating cliques of restricted subgraphs, each of these problems are solvable in \( 2^{O(d)}n^{O(1)} \) time. Therefore, assuming the problems remain NP-complete on planar graphs (complexity of these problems on planar graphs are unresolved), questions similar to Problem 6.2 exist for these problems.

For ACC, we have shown an FPT algorithm whose exponent of the running time has quadratic dependency on parameter \( t \) (Section[5,22]). With our new framework, we have shown an FPT algorithm whose exponent of the running time has quasilinear dependency on parameter \( t \) (Section[5,22]). Thus an important question in this regard is the following.

Problem 6.3. Does ACC has an FPT algorithm running in \( 2^{O(1)}n^{O(1)} \) time?

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A Proofs omitted in Section 2

Using relationship between degeneracy and arboricity, we can obtain a proof for Lemma 2.14 as follows.

Definition A.1 (Arboricity). Arboricity of a graph is the minimum number of spanning forests that cover all the edges of the graph.

Lemma A.2. If \( a \) is the arboricity of \( G \), then \( d \leq 2a - 1 \).

Proof. Let \( H = (V_H, E_H) \) be a subgraph of \( G \) with \( |V_H| = n_H \) and \( |E_H| = m_H \). From Definition 2.1, we have,

\[
d = \max_{V_H \subseteq V} \min_{x \in V_H} |\{x, y\} \in E_H|.
\]

Since the number of spanning forests needed to cover \( G \) is \( a \), the number of spanning forests needed to cover \( H \) is at most \( a \), i.e., \( m_H \leq a(n_H - 1) \). For the average degree of vertices in \( H \), we have

\[
\frac{2m_H}{n_H} \leq \frac{2a(n_H - 1)}{n_H} < 2a. \tag{1}
\]

Therefore, the minimum degree of vertices in \( H \) is at most \( 2a - 1 \), i.e.,

\[
\min_{x \in V_H} |\{x, y\} \in E_H| \leq 2a - 1.
\]

The claim follows, since \( H \) is an arbitrary subgraph of \( G \).

Using an equivalent definition of degeneracy in terms of acyclic orientation of graph, an exposition of the preceding proof can be found in [3].

Lemma A.3 (Lemma 1 [4]). If \( a \) is the arboricity of \( G \), then \( a \leq \lfloor (\sqrt{2m + n})/2 \rfloor \).

Lemma 2.14. If \( G \) contains no isolated vertices, then \( d + 1 \leq 2\sqrt{m} \).

Proof. Let \( G \) has \( q \) connected components with \( n_1, n_2, \ldots, n_q \) vertices respectively. \( m \geq \sum_{i=1}^{q} n_i - 1 = n - q \). Since \( G \) does not contain any isolated vertices \( q \leq \lfloor \frac{n}{2} \rfloor \). Therefore, \( m \geq n - \lfloor \frac{n}{2} \rfloor \geq \frac{n}{2} \).

Combining Lemma A.2 and Lemma A.3 we have,

\[
d \leq 2a - 1 \leq 2\lfloor (\sqrt{2m + n})/2 \rfloor - 1 \leq 2\lfloor (\sqrt{2m + 2m})/2 \rfloor - 1 = 2\sqrt{m} - 1.
\]

B Proofs omitted in Section 5

Lemma 5.1. Algorithm ECCS2 correctly solves the parameterized problem ECC.

Proof. Let \( A \) denote the family of algorithms that consider an arbitrary uncovered edge \( \{x, y\} \) at step 2 of ECCS2. Clearly, ECCS2 \( \in A \).

Consider any algorithm \( A_i \in A \). For a fixed value of \( k \), consider the family of instances of ECC, \( \mathcal{G}_k = \{(G, k)\} \). For a search tree \( T \) of \( A_i \), let us define a node \( u^T \) of \( T \) to be a YES node if \( A_i \) returns at step 1 of \( u^T \). We claim that for any \( (G, k) \in \mathcal{G}_k \), \( (G, k) \) is a YES instance of ECC if and only if \( A_i \)
would return from a YES node. If \( A_i \) returns from a YES node, then clearly \((G,k)\) is a YES instance of ECC. It remains to show that for any \((G,k) \in \mathcal{G}_k\), if \((G,k)\) is a YES instance of ECC, then \( A_i \) would return from a YES node.

We induct on \( k \). For the base case, we have \( k=1\), i.e., \( G \) is a complete graph. In this case, \( A_i \) would create a clique \( C_1 \) at step 4, only for the very first edge selected at step 2. Any subsequent edges, chosen at step 2 by \( A_i \), would only be covered at step 3: this follows from Proposition 4.24, since \( G \) is a complete graph. Therefore, the claim holds for all instances of ECC in \( \mathcal{G}_1 \).

For \( k>1 \), assume the claim holds for all instances of ECC in \( \mathcal{G}_{k-1} \) such that \( k^*<k \). Since \((G,k)\) is a YES instance of ECC, let \( C = \{C_1,C_2,\ldots,C_k\} \) be a corresponding clique cover. Fix a permutation \( \pi \) of \( \{1,2,\ldots,k\} \). From Definition 4.3, \( E_{\pi(i)} = \{(x,y) \in C_{\pi(i)} \mid \{x,y\} \notin C_{\pi(j)}, j<i\} \), i.e., the set of edges exclusive to \( C_{\pi(i)} \) with respect to the cliques \( \{C_{\pi(1)},C_{\pi(2)},\ldots,C_{\pi(i-1)}\} \). Let \( G' = G \setminus E_{\pi(k)} \). Since \((G,k)\) is a YES instance of ECC, clearly, \((G^*,k-1)\) is a YES instance of ECC. By inductive hypothesis, for \((G^*,k-1)\), \( A_i \) would return from a YES node. Let \( u^T \) be such a YES node.

Note, in a search tree \( T \) of \((G,k)\), \( A_i \) can cover the edges of \( E_{\pi(k)} \) in the descendent nodes of \( u^T \).

Lemma 5.2. In a search tree with at most \( k \) cliques, \( \text{ECCS}_2 \) takes \( O(m + k\Delta) \) space.

**Proof.** For a search tree \( T \) of \( \text{ECCS}_2 \), let \( u^T \) be a node corresponding to an uncovered edge \( \{x,y\} \), selected at step 2. If at node \( u^T \) we need to create a new clique to cover the edge \( \{x,y\}\), then it must be the case that all branches of step 3 have failed. Any node \( v^T \) such that \( u^T \) is an ancestor of \( v^T \) in the search tree \( T \) would not select the edge \( \{x,y\} \) as uncovered. Therefore any \( x \in V \) would appear at most \( |N(x)| \) times in \( \mathcal{C} \) and \( \sum_{x \in V} |\{l \mid x \notin C_l\}| = O(m) \). Also, using the same argument as in the proof of Proposition 4.24 for a search tree with at most \( k \) cliques, \( \sum_{x \in V} |\{l \mid x \notin C_l, C_l \subseteq N(x)\}| = O(k\Delta) \).

Therefore, \( \sum_{x \in V} |R_x| = O(m + k\Delta) \) holds for \( \text{ECCS}_2 \).

In aggregate, the sets in \( \mathcal{D} \) and the sets in \( \mathcal{R} \) hold same information of the relationship among vertices and cliques. To see this, consider the bipartite graph \( B = (V,\mathcal{C},\mathcal{E}_B) \), where \( \mathcal{E}_B = \{(x,C) \mid x \in V \text{ or } x \notin C_l, C_l \subseteq N(x)\} \). In \( B \), the sum of the degrees of vertices of the \( V \) side must be equal to the sum of the degrees of vertices of the \( \mathcal{C} \) side. Therefore, \( \sum_{D \in \mathcal{D}} |D| = \sum_{x \in V} |R_x| = O(m + k\Delta) \).

At each node in the subroutine for step 3a, a vertex of \( D_l \) is either kept in \( D_l \) or moved to \( U \). Therefore, space usage of \( U \) data structures in total is already accounted for.

Lemma 5.7. Algorithm \( \text{ACCS}_2 \) correctly solves the parameterized problem ACC.

**Proof.** Let \( A \) denote the family of algorithms that consider an arbitrary uncovered edge \( \{x,y\} \) at step 2 of \( \text{ACCS}_2 \). Clearly, \( \text{ACCS}_2 \in A \).

Consider any algorithm \( A_i \in A \). For a fixed value of \( t \), consider the family of instances of ACC, \( \mathcal{G}_t = \{(G,t)\} \). For a search tree \( T \) of \( A_i \), let us define a node \( u^T \) of \( T \) to be a YES node if \( A_i \) returns at step 1 of \( u^T \). We claim that for any \((G,t) \in \mathcal{G}_t\), \((G,t)\) is a YES instance of ACC if and only if \( A_i \) would return from a YES node. If \( A_i \) returns from a YES node, then clearly \((G,t)\) is a YES instance of ACC.

It remains to show that for any \((G,t) \in \mathcal{G}_t\), if \((G,t)\) is a YES instance of ACC, then \( A_i \) would return from a YES node.

We induct on \( t \). For the (non-trivial) base case, we have \( t=2 \), i.e., \( G \) is the graph containing a single edge. In this case, \( A_i \) would simply create a clique \( C_1 \) at step 4 for the only edge of \( G \). Therefore, the claim holds for all instances of ACC in \( \mathcal{G}_2 \) such that \( t \leq 2 \).

For \( t > 2 \), assume the claim holds for all instances of ACC in \( \mathcal{G}_t \) such that \( t^* < t \). Since \((G,t)\) is a YES instance of ACC, let \( \mathcal{C} = \{C_1,C_2,\ldots,C_k\} \) be a corresponding clique cover. Fix a permutation \( \pi \) of \( \{1,2,\ldots,k\} \). From Definition 4.3, \( E_{\pi(i)} = \{(x,y) \in C_{\pi(i)} \mid \{x,y\} \notin C_{\pi(j)}, j<i\} \), i.e., the set of edges exclusive to \( C_{\pi(i)} \) with respect to the cliques \( \{C_{\pi(1)},C_{\pi(2)},\ldots,C_{\pi(i-1)}\} \). Let
Lemma 5.9. Algorithm AWECP correctly solves the parameterized problem AWECP.

Proof. Let \( A \) denote the family of algorithms that consider an arbitrary edge \( e = \{x, y\} \) at step 2 of AWECS such that \( w^E(e) > 0 \). Clearly, AWECS \( \in A \).

Consider any algorithm \( A_i \in A \). For a fixed value of \( k \), consider the family of instances of AWECP, \( G_k = \{(G, k, w, S, w^S)\} \). For a search tree \( T \) of \( A_i \), let us define a node \( u \) to be a YES node if \( A_i \) returns at step 1 of \( u \). We claim that for any \( I \in G_k \), \( G, k \) is a YES instance of AWECP if and only if \( A_i \) returns a YES node. If \( A_i \) returns a YES node, then clearly \( I \) is a YES instance of AWECP. It remains to show that for any \( I \in G_k \), if \( I \) is a YES instance of AWECP, then \( A_i \) would return from a YES node.

We induct on \( k \). For the base case, we have \( k = 1 \), i.e., \( G = (V, E) \) is a complete graph, \( w^E(e) = 1 \) for all \( e \in E \), and \( w^S(x) = 1 \) for all \( x \in S \). In this case, \( A_i \) would create a clique \( C_1 \) at step 4, only for the very first edge selected at step 2. It is straightforward to see that any subsequent edges chosen at step 2 by \( A_i \) would only be included in \( C_1 \) at step 3. Therefore, the claim holds for all instances of AWECP in \( G_1 \).

For \( k > 1 \), assume the claim holds for all instances of AWECP in \( G_{k_1} \), such that \( k^* < k \). Since \( I = (G, k, w, S, w^S) \) is a YES instance of AWECP, let \( C = \{C_1, C_2, \ldots, C_k\} \) be a corresponding clique cover. We decompose the instance \( I = (G, k, w, S, w^S) \) into two instances \( I_1 = (G^*, k - 1, w^*, S^*, w^{S*}) \) and \( I_2 = (H, 1, w^H, S_H, w^{S_H}) \) such that \( I_1 \in \text{AWECP} \) and \( I_2 \in \text{AWECP} \).

Construction of \( I_2 \): Let \( H = (V_H, E_H) \) be the subgraph induced by \( C \in C \) and \( G = (G[\bar{C}]) \). Let \( w^H(e) = 1 \) for all \( e \in E_H \). For all \( e \in S_H \), let \( w^{S_H}(x) = 1 \) for all \( x \in S_H \).

Construction of \( I_1 \): Let \( G^* = (V^*, E^*) \) where \( E^* = E \setminus \{e \in E \mid w^E(e) = 1\} \) and \( V^* = \{x \mid (x, y) \in E^*\} \). For all \( e \in E^* \), let \( w^*(e) = w^E(e) - 1 \), and for all \( e \in E^* \), let \( w^*(e) = w^E(e) \). Let \( S^* = S \setminus \{x \in S \cap V_H \mid w^S(x) = 1\} \). For all \( x \in S^* \cap V_H \), let \( w^{S*}(x) = w^S(x) - 1 \), and for all \( s \in S^* \), let \( w^{S*}(x) = w^S(x) \).

It is straightforward to verify that the composition (union of the graphs and addition of the parameters following the decomposition) of \( I_1 \) and \( I_2 \) is exactly the instance \( I = (G, k, w^E, S, w^S) \). Since \( I \) is a YES instance of AWECP, both \( I_1 \) and \( I_2 \) are YES instances of AWECP. By inductive hypothesis, for both \( I_1 \) and \( I_2 \), \( A_i \) would return from a YES node. Let \( u^* \) be a YES node for \( I_1 \).

Now, in a search tree \( T \) of \( A_i \), for \( I_1 \), \( A_i \) can construct the solution for \( I_2 \) in the descendent nodes of \( u^* \), and append the solution of \( I_2 \) to the solution of \( I_1 \). Note that at node \( u^* \) in \( T \), \( A_i \) would use at most \( k - 1 \) cliques. In the descendent nodes of \( u^* \) of \( T \), the solution for \( I_2 \) may be exclusively constructed at step 3 without creating any new clique. In the worst case, at most one additional clique would be needed to construct a solution for \( I_2 = (H, 1, w^H, S_H, w^{S_H}) \), \( I_2 \) is a YES instance of AWECP.

Note that the permutation of cliques in \( C \) does not affect the preceding arguments, and the arguments hold for any clique \( C_i \in C \). Therefore, it follows that for the instance \( (G, k, w^E, S, w^S) \), \( A_i \) would return from a YES node. This concludes our inductive step.