ON SEPARABILITY PROBLEM FOR CIRCULANT S-RINGS

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Abstract. A Schur ring (S-ring) over a group $G$ is called separable if every of its similarities is induced by isomorphism. We establish a criterion for an S-ring to be separable in the case when the group $G$ is cyclic. Using this criterion, we prove that any S-ring over a cyclic $p$-group is separable and that the class of separable circulant S-rings is closed with respect to duality.

1. Introduction

A Schur ring or S-ring over a finite group $G$ can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free $\mathbb{Z}$-module spanned by a partition of $G$ closed under taking inverse and containing $\{1_G\}$ as a class (see [8] for details). Here a class of the partition or basic set, is also treated as the sum of its elements in the group ring. A Cayley isomorphism of two S-rings is defined as an isomorphism of the underlying groups that induces a ring isomorphism of them; a similarity of two S-rings is defined as a ring isomorphism that induces a bijection between their basic sets. Any Cayley isomorphism induces in a natural way a similarity, whereas not every similarity is induced by a Cayley isomorphism.

The third type of S-ring isomorphisms comes from graph theory. Namely, under isomorphism of S-rings $A$ and $A'$ over groups $G$ and $G'$ respectively, we mean a bijection $f : G \to G'$ such that for any basic set $X$ of $A$ the image of the Cayley graph $\text{Cay}(X,G)$ with respect to $f$, is the Cayley graph $\text{Cay}(X',G')$ where $X'$ is a basic set of $A'$. One can prove that the bijection $X \mapsto X'$ induces by linearity a similarity $\varphi$ from $A$ onto $A'$ such that

$$f(Xy) = X' \varphi f(y),$$

for all basic sets $X$ of $A$ and $y \in G$ where $X' = X'^{\varphi}$ In this case, we say that $\varphi$ is induced by $f$. For a fixed similarity $\varphi$, the set of all such isomorphisms $f$ is denoted by $\text{Iso}(A,A',\varphi)$. Every Cayley isomorphism is also an isomorphism, but the converse statement is not true.

The separability problem for S-rings can be formulated as follows: given S-rings $A$ and $A'$, and a similarity $\varphi : A \to A'$ check whether

$$\text{Iso}(A,A',\varphi) \neq \emptyset.$$ 

From the computational complexity point of view this problem is polynomial-time equivalent to the isomorphism problem for Cayley graphs [11]. One of the most interesting questions connected with the separability problem is to identify the separable S-rings $A$, i.e. those for which the set $\text{Iso}(A,A',\varphi)$ is not empty for all $\varphi$. 

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1In fact, the isomorphism of S-rings can be defined by means of [11], see [8].
In the present paper we do this for circulant S-rings (i.e. when the groups \( G \) and \( G' \) are cyclic).

The importance of separable S-rings comes from the observation that any separable S-ring is determined up to isomorphism by the array of structure constants (with respect to the basis corresponding to the partition of the underlying group). However, even for circulant S-rings the situation is quite complicated: there are infinitely many both separable and non-separable circulant S-rings (see [2, Theorem 6.6] and [1, Theorem 1.1]). To characterize the separable circulant S-rings, the theory of circulant S-rings developed by the authors in a series of papers will be considerably used. Let us briefly discuss its relevant notions.

Let \( A \) be an S-ring over a cyclic group \( G \). In proving that \( A \) is separable, without loss of generality we can assume that \( G' = G \), and also that \( A' = A \) [9]. Next, for technical reasons it is convenient to restrict oneself to the case when \( A \) is a quasidense S-ring, i.e. when it has no rank 2 sections of composite order. This restriction is not essential, because every S-ring over \( G \) is separable if and only if so is every quasidense S-ring over \( G \) (Theorem 4.1).

The proofs in this paper are heavily based on the theory of coset S-rings developed in [8]; recall that an S-ring over \( G \) is a coset one if any of its basic sets is a coset of a subgroup in \( G \). It was proved there that any circulant coset S-ring is schurian and separable. Our first theorem gives a criterion for a quasidense S-ring \( A \) to be separable, in terms of its coset closure \( A_0 \) which is by definition the intersection of all coset S-rings containing \( A \) (it should be noted that in our case \( A_0 \) is itself a coset S-ring).

Below we denote by \( \Phi(A) \) and \( \Phi_\infty(A) \) the group of all similarities of \( A \) and the group of all \( \phi \in \Phi(A) \) with Iso\( (A, \phi) \neq \emptyset \) where Iso\( (A, \phi) = \text{Iso}(A, A, \phi) \).

**Theorem 1.1.** Let \( A \) be a quasidense circulant S-ring and \( A_0 \) its coset closure. Then

\[
\Phi_\infty(A) = \Phi(A_0)^A
\]

where \( \Phi(A_0)^A \) is the subgroup of \( \Phi(A) \) induced by the action of \( \Phi(A_0) \) on \( A \). In particular, \( A \) is separable if and only if \( \Phi(A) = \Phi(A_0)^A \).

To apply the separability criterion given in Theorem 1.1 one has to have on hand the S-ring \( A_0 \) and the groups \( \Phi(A) \) and \( \Phi(A_0) \). However, they cannot in general be easily found. That is why we would like to simplify this criterion by using the observation that every similarity \( \varphi \in \Phi(A) \) is uniquely determined by its restrictions to principal \( A \)-sections, and hence to sections belonging to the class \( \mathcal{S}_0(A) \) of all \( A \)-sections of \( G \) that are projectively equivalent to subsections of principal \( A \)-sections. This enables us to replace \( \varphi \) by the family of these restrictions, and eventually to get an explicit form of the criterion that is in principle can be reduced to the compatibility of a modular linear system as in [3].

Let \( A \) be a circulant S-ring. Suppose that for every section \( S \in \mathcal{S}_0(A) \) we are given an automorphism \( \sigma_S \) of the group \( S \). Then the family

\[
\Sigma = \{ \sigma_S \}_{S \in \mathcal{S}_0(A)}
\]

is called a \( A \)-multiplier if for any sections \( S, T \in \mathcal{S}_0(A) \) such that \( T \) is projectively equivalent to a subsection of \( S \), the automorphisms \( \sigma_T \in \text{Aut}(T) \) and \( \sigma_S \in \text{Aut}(S) \)
are induced by raising to the same power (see Definition 5.1). The set of all of them forms a subgroup $\text{Mult}(\mathcal{A})$ of the direct product $\prod_{S \in \mathcal{S}} \text{Aut}(S)$.

For an $\mathcal{A}$-section $S$ denote by $\text{Aut}_\mathcal{A}(S)$ the group of Cayley automorphisms of the restriction $\mathcal{A}_S$ of the S-ring $\mathcal{A}$ to $S$. Then every $\mathcal{A}$-multiplier $\Sigma$ produces a family

$$\Sigma' = \{C_S\}_{S \in \mathcal{S}_0(\mathcal{A})}$$

where $C_S = \text{Aut}_\mathcal{A}(S)\sigma_S$. For this family the compatibility condition similar to that as for a $\mathcal{A}$-multiplier, is obviously satisfied. Therefore in the sense of Definition [3.1] the family $\Sigma'$ is an outer $\mathcal{A}$-multiplier. It follows that the mapping

$$\theta : \text{Mult}(\mathcal{A}) \to \text{OMult}(\mathcal{A}), \quad \Sigma \mapsto \Sigma'$$

is a group homomorphism where $\text{OMult}(\mathcal{A})$ is the group of all outer $\mathcal{A}$-multipliers. In these terms the explicit form of the criterion looks as follows.

**Theorem 1.2.** A quasidense circulant S-ring $\mathcal{A}$ is separable if and only if the homomorphism $\theta$ is surjective.

From the Burnside theorem on permutation groups of prime degree it follows that every S-ring over a group of prime order $p_i$ is normal or of rank 2. In both cases it is separable. Therefore all S-rings over such a group are separable. Using Theorem 1.2 we show that any cyclic $p$-group has this property.

**Theorem 1.3.** Any S-ring over a cyclic $p$-group is separable.

There is a deep relationship between the schurity and separability problems (see e.g. [1]). For circulant S-rings the first problem was solved in [5]. The schurity criterion obtained there implies that the class of schurian circulant S-rings is closed with respect to duality. The analog of this result for separable circulant S-rings is also true.

**Theorem 1.4.** A circulant S-ring is separable if and only if so is the dual S-ring.

We complete the introduction by remarking that this paper is closely related to paper [8] by techniques, results and spirit. Therefore here we follow the notation and terminology of that paper. When referring to it, we keep only the number of the statement, preceding it by the letter A (e.g. instead of [8, Theorem 4.2] we write Theorem A4.2).

2. Auxiliary statements on S-rings

2.1. Similarities and isomorphisms. The basics on isomorphisms and similarities of S-rings can be found in Section A3. Below we are mainly interested in the question how to extend them.

**Lemma 2.1.** Let $\mathcal{A}_i$ be an S-ring over a group $G_i$ and let $\mathcal{A}'_i$ be the minimal S-ring over $G_i$ that contains $\mathcal{A}_i$ and an element $\xi_i \in \mathbb{Z}G_i$, $i = 1, 2$. Then given a similarity $\varphi \in \Phi(\mathcal{A}_1, \mathcal{A}_2)$ there is at most one similarity $\varphi' \in \Phi(\mathcal{A}'_1, \mathcal{A}'_2)$ extending $\varphi$ and such that $\varphi'(\xi_1) = \xi_2$.

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We recall that any automorphism of a finite cyclic group is induced by raising to a power coprime to the order of this group.

The multipliers of $\mathcal{A}$ defined in [8] form a subgroup of the group $\text{Mult}(\mathcal{A})$; it seems reasonable to call them inner multipliers.
Proof. Every element of $A'_1$ can be obtained from the elements of $A_i$ and $\xi_i$ by taking $\mathbb{Q}$-linear combinations and products (ordinary and entrywise), $i = 1, 2$. Since the $\mathbb{Q}$-linear extension $\psi'$ of $\varphi'$ must preserve all these operations, it is uniquely determined (if exists). This implies the required statement because $\psi$ and $\varphi'$ coincide on $A'_1$.

It follows from equality (A12) that given two isomorphic S-rings one can choose their isomorphism to be normalized, i.e. preserving the neutral elements (any bijection between two groups that has this property, is also called normalized). The normalized isomorphisms admit a more natural description than in [1] (see Section A3.2).

Lemma 2.2. Let $A_1$ and $A_2$ be S-rings over groups $G_1$ and $G_2$ respectively, and let $f : G_1 \rightarrow G_2$ be a normalized bijection. Suppose that $f$ preserves the basic sets. Then $f \in \text{Iso}(A_1, A_2)$ if and only if $f(xy) = f(x)f(y)$ for all $x \in S(A_1)$ and $y \in G_1$.

Any isomorphism $f$ of two association schemes is also isomorphism from the extension of the first scheme by a relation $R$ onto the extension of the second scheme by the relation $R'$. A similar statement does not hold in general for Cayley schemes, and hence for S-rings. However, it becomes true under some extra condition.

Lemma 2.3. Let $A_i$ be an S-ring over a group $G_i$ and $X_i \subset G_i$, $i = 1, 2$. Suppose that $f \in \text{Iso}(A_1, A_2)$ is a normalized isomorphism such that $f(X_1) = X_2$ and $f(X_1y) = X_2f(y)$ for all $y \in G_1$. Then $f \in \text{Iso}(A'_1, A'_2)$ where $A'_i$ is the minimal S-ring over $G_i$ that contains $A_i$ and $X_i$.

Proof. Let us extend $f$ to a linear isomorphism from $QG_1$ to $QG_2$; it will be denoted by the same letter. Set $M$ to be the set of all elements $\xi \in QG_1$ such that $f(\xi x) = f(\xi)f(x)$ for all $x \in G_1$. Then $A_1 \subset M$ by Lemma 2.2 and $\sum_{x \in X_1} x \in M$ by lemma hypothesis. Besides, obviously $M$ is a $Q$-module such that $M \circ M \subset M$. Furthermore, if $M$ contains both $\xi$ and $\eta = \sum_{x \in G_1} c_x x$, then

$$f(\xi \eta) = \sum_{x \in G_1} c_x f(\xi x) = \sum_{x \in G_1} c_x f(\xi) f(x) = f(\xi) \sum_{x \in G_1} c_x f(x) = f(\xi) f(\eta) = f(\xi) f(\eta) f(y) = (\sum_{x \in G_1} c_x f(\xi) f(x)) f(y) =$$

$$= f(\xi) f(\eta) f(y)$$

for all $y \in G_1$. Thus $M M \subset M$. Since every element of $A'_1$ can be obtained from $\sum_{x \in X_1} x$ and the elements of $A_i$ by taking $\mathbb{Q}$-linear combinations and products (ordinary and entrywise), we conclude that $A'_1 \subset M$. On the other hand, $f(A'_i)$ is the $\mathbb{Z}$-module associated with the partition $f(S(A_i))$ of the group $G_2$ where $S(A_i)$ is the partition of $G_1$ into the basic sets of $A_i$. Besides, $f(A'_i)$ is closed with respect to multiplication because $A'_i \subset M$. So this module is an S-ring over $G_2$. However, it contains both $A_2 = f(A_1)$ and $\sum_{x \in X_2} x = f(\sum_{x \in X_1} x)$. Thus by the minimality of $A'_2$ we have $f(A'_1) \supset A'_2$. Similarly, interchanging $A_1$ and $A_2$ and taking $f^{-1}$ instead of $f$, one can prove that $f^{-1}(A'_2) \supset A'_1$. Therefore $f(A'_1) = A'_2$, and hence $f$ is a linear isomorphism from $A'_1$ onto $A'_2$. Thus $f \in \text{Iso}(A'_1, A'_2)$, as required.
Any section $U/L$ of a group $G$ naturally acts on the right $L$-cosets in $G$, namely $Lx \cdot Ly := Lxy$ for $x \in U$ and $y \in G$. If $U/L$ is a section of an $S$-ring $A$ over $G$, then a normalized isomorphism $f \in \text{Iso}(A)$ takes this section to another $A$-section, but in general does not preserve the above action. Condition 6 in the lemma below, means that $f$ does preserve it.

**Corollary 2.4.** Let $A$ be an $S$-ring over a group $G$, $S = U/L$ an $A$-section and let $f \in \text{Iso}(A)$ be a normalized isomorphism. Suppose that

\begin{equation}
(3) \quad f(Lxy) = f(Lx)f(Ly)
\end{equation}

for all $x \in U$ and $y \in G$. Then $f \in \text{Iso}(A')$ where $A'$ is the minimal $S$-ring over $G$ that contains $A$ and for which $(A')_S = \mathbb{Z}S$.

**Proof.** The $S$-ring $A'$ is obtained from $A$ by subsequent extensions of $A$ by means of $L$-cosets in $U$. Therefore it suffices to verify that given $x \in U$, $f$ is an isomorphism of the minimal $S$-ring over $G$ that contains $A$ and $Lx$. However, this follows from Lemma 2.3 for $A_1 = A_2 = A$ and $X_1 = Lx$, because by the hypothesis on $f$ we have

\[
 f(Lxy) = f(Lx)f(Ly) = f(Lx)f(L)y = f(Lx)f(y),
\]

here we used the fact that $f(Ly)$ equals the $f(L)$-coset containing $f(y)$.

2.2. **Projective equivalence.** Let $S = U/L$ and $S' = U'/L'$ be sections of a group $G$. Suppose that $S'$ is a multiple of $S$, i.e. that $UL' = U'$ and $U \cap L' = L$. Set $f_{S,S'} : xL \mapsto xL'$ to be the canonical projective isomorphism from $S$ to $S'$ defined in [3 Section 3.1]; we also set $f_{S,S'} := (f_{S',S})^{-1}$ when $S$ is a multiple of $S'$. Then the following statement obviously holds.

**Lemma 2.5.** Suppose that $S$ is a multiple of both sections $S'$ and $S''$ one of which is a multiple of the other. Then

\[
 f_{S,S'}f_{S',S''} = f_{S,S''}.
\]

Let now the sections $S$ and $S'$ be projectively equivalent. Then there exist sections $S = S_1,S_2,\ldots,S_{k-1},S_k = S'$ such that for all $i = 1,\ldots,k-1$ one of the sections $S_i$, $S_{i+1}$ is a multiple of the other. The mapping $f : S \to S'$ defined by

\begin{equation}
(4) \quad f = f_{S_1,S_2}\cdots f_{S_{k-1},S_k},
\end{equation}

is called a projective isomorphism from $S$ onto $S'$.

**Theorem 2.6.** A projective isomorphism between two projectively equivalent sections of a cyclic group is uniquely determined.

**Proof.** Let $S$ and $S'$ be projectively equivalent sections and let $f : S \to S'$ be the projective isomorphism 1. Denote by $S_0$ the largest section in the class of projectively equivalent sections that contains $S_1$ (apply Theorem A5.4 for the group ring). Then $S_0$ is a multiple of $S_i$ for all $i$. By Lemma 2.5 this implies that $f_{S_0,S_i}f_{S_i,S_{i+1}} = f_{S_0,S_{i+1}}$ for all $i = 1,\ldots,k-1$. Thus from (4) it follows that

\[
 f = f_{S_1,S_0}f_{S_0,S_k}.
\]

This proves the required statement.
Corollary 2.7. Let $S$ and $T$ be projectively equivalent sections of a cyclic group $G$. Then any section $S' \succeq S$ is projectively equivalent to the section $T' = (S')^{f_{S,T}}$ where $f_{S,T}$ is the projective isomorphism from $S$ onto $T$. Moreover, the restriction of $f_{S,T}$ to $S'$ equals $f_{S',T'}$.

Proof. The statement immediately follows from Theorem 2.6 when $T$ is a multiple of $S$. In a general case, we are done by induction.

Let $A$ be an $S$-ring over a group $G$. Under the projective equivalence "〜" on the set $\mathcal{S}(A)$ we mean the transitive closure of the relation "to be a multiple" on this set. Clearly, any two projectively equivalent $A$-sections are projectively equivalent as sections of $G$. The converse statement is not true in general, but holds when the group $G$ is cyclic.

Lemma 2.8. Let $A$ be a circulant $S$-ring, $S$ and $T$ projectively equivalent $A$-sections and $f_{S,T}$ the projective isomorphism from $S$ to $T$. Then

1. if $\varphi$ is a similarity of $A$, then $f_{S,T}$ takes $\varphi S$ to $\varphi T$,
2. if $f$ is a normalized isomorphism of $A$, then $f_{S,T}$ takes $fS$ to $fT$.

Proof. By the definition of projective isomorphism it suffices to verify statement (1) in the case when $T = U_2/L_2$ is a multiple of $S = U_1/L_1$. Without loss of generality we can assume that $U_2 = G$ and $L_1 = 1$. By [5, Theorem 3.2] the projective isomorphism $f_{S,T}$ is a Cayley isomorphism from $AS$ onto $AT$. Denote by $\psi$ the induced similarity. Then obviously $X^\psi = \pi(X)$ for any $X \in S(AS_1)$ where $\pi$ is the quotient epimorphism from $U_2$ onto $T$. Therefore

$$(X^{\varphi S})^\psi = \pi(X^{\varphi S}) = \pi(X^{\varphi}) = \pi(X)^{\varphi T} = (X^\psi)^{\varphi T}$$

and we are done. Statement (2) immediately follows from [5, Lemma 3.1] with $\Gamma = \text{Iso}(A)$.

2.3. Duality. Let $G$ be an abelian group and $\hat{G}$ its dual group, i.e. the group of complex characters of $G$. Given $\sigma \in \text{Aut}(G)$ denote by $\hat{\sigma}$ the automorphism of $\hat{G}$ such that $\chi^{\hat{\sigma}}(g) = \chi(g^{\sigma})$ for all $g \in G$ and $\chi \in \hat{G}$. Clearly, the mapping $\sigma \mapsto \hat{\sigma}$ induces an isomorphism from $\text{Aut}(G)$ onto $\text{Aut}(\hat{G})$.

Lemma 2.9. Let $G$ be a cyclic group. Then given sections $S, T \in \mathcal{S}(G)$ such that $S \succeq T$ we have $\hat{\sigma}S = \hat{S}$ for all $\sigma \in \text{Aut}(T)$ where $\hat{S}$ is the section of $\hat{G}$ dual to $S$.

Proof. We observe that $\hat{S} \succeq \hat{T}$ by statement (3) of Lemma A4.4. Since $G$ and $\hat{G}$ are cyclic groups, there are restriction epimorphisms $\pi : \text{Aut}(T) \to \text{Aut}(S)$ and $\hat{\pi} : \text{Aut}(\hat{T}) \to \text{Aut}(\hat{S})$, and the diagram

$$
\begin{array}{ccc}
\text{Aut}(T) & \xrightarrow{\pi} & \text{Aut}(S) \\
\downarrow & & \downarrow \\
\text{Aut}(\hat{T}) & \xrightarrow{\hat{\pi}} & \text{Aut}(\hat{S})
\end{array}
$$

is commutative where the vertical arrows mean the isomorphisms $\sigma \mapsto \hat{\sigma}$. Therefore for all $\sigma \in \text{Aut}(T)$, $g \in S$ and $\chi \in \hat{S}$ we have

$$\chi(g^{\pi(\sigma)}) = \hat{\pi}(\hat{\sigma})(\chi(g)) = \hat{\pi}(\hat{\sigma})(\chi(g)) = \chi^{\hat{\sigma}(\hat{\sigma})}(g).$$

We recall that notation $S' \succeq S$ means that $S'$ is a subsection of $S$. 

It follows that
\[ \widetilde{\chi^S}(g) = \chi(g^{\sigma^S}) = \chi(g^{\pi(\sigma)}) = \chi^S(g) = \chi^\tilde{S}(g), \]
as required.

**Lemma 2.10.** Let \( S \) and \( T \) be projectively equivalent sections of a cyclic group \( G \), and \( f := f_{S,T} \) and \( \tilde{f} := f_{\tilde{S},\tilde{T}} \) projective isomorphisms. Then \( \tilde{\sigma} \tilde{f} = \tilde{f} \sigma \) for all \( \sigma \in \text{Aut}(S) \).

**Proof.** We have to verify that the following diagram is commutative
\[
\begin{array}{ccc}
\text{Aut}(S) & \xrightarrow{\sigma \mapsto f} & \text{Aut}(T) \\
\downarrow & & \downarrow \\
\text{Aut}(S) & \xrightarrow{\tilde{\sigma} \mapsto \tilde{f}} & \text{Aut}(T)
\end{array}
\]
However, this is true because all isomorphisms of this diagram take a given automorphism induced by raising to a certain power to the automorphism induced by raising to the same power.

**3. Outer multipliers**

Let \( \mathcal{A} \) be an S-ring over a cyclic group \( G \). Suppose that for every section \( S \in \mathcal{S}_0(\mathcal{A}) \) we are given a coset \( C_S \subset \text{Aut}(S) \) of the group \( \text{Aut}(\mathcal{A})(S) = \text{Aut}(\mathcal{A}_S) \cap \text{Aut}(S) \).

**Definition 3.1.** The family \( \Sigma = \{C_S\}_{S \in \mathcal{S}_0(\mathcal{A})} \) is called an outer \( \mathcal{A} \)-multiplier if the following two conditions are satisfied for all sections \( S_1, S_2 \in \mathcal{S}_0(\mathcal{A}) \):

(\( M1 \)) if \( S_1 \succeq S_2 \), then \( (C_{S_1})^{S_2} = C_{S_2} \),
(\( M2 \)) if \( S_1 \sim S_2 \), then \( (C_{S_1})^{f_{S_1,S_2}} = C_{S_2} \)
where \( f_{S_1,S_2} \) is the projective isomorphism from \( S_1 \) onto \( S_2 \).

Clearly, the product of two \( \text{Aut}(\mathcal{A}) \)-cosets is also \( \text{Aut}(\mathcal{A}) \)-coset. This enables us to define the componentwise product of any two outer \( \mathcal{A} \)-multipliers. Since for the corresponding family, conditions (\( M1 \)) and (\( M2 \)) are obviously satisfied, the set of all outer \( \mathcal{A} \)-multipliers forms a group; we denote it by \( \text{OMult}(\mathcal{A}) \).

Let \( \Sigma \in \text{OMult}(\mathcal{A}) \) and \( T \in \mathcal{S}(\mathcal{A}) \). Then obviously the class \( \mathcal{C} := \mathcal{S}_0(\mathcal{A}_T) \) is contained in \( \mathcal{S}_0(\mathcal{A}) \), and conditions (\( M1 \)) and (\( M2 \)) are satisfied for the family \( \Sigma^T = \{C_S\}_{S \in \mathcal{C}} \). This proves the following statement.

**Lemma 3.2.** The family \( \Sigma^T \) is an outer \( \mathcal{A}_T \)-multiplier.

Let \( \mathcal{A} \) be a quasidense circulant S-ring and let \( \varphi \in \Phi(\mathcal{A}) \) be a similarity. Then for any section \( S \in \mathcal{S}_0(\mathcal{A}) \) the similarity \( \varphi_S \) is induced by an automorphism \( \sigma_S \in \text{Aut}(S) \). Indeed, if \( S \) is a principal section, then the S-ring \( \mathcal{A}_S \) has trivial radical, and this follows from Corollary 6.4 and statement (1) of Theorem 6.6 of \([2]\); if \( S \) is a subsection of a principal section, then \( \sigma_S \) is obtained by restriction; if \( S \) is projectively equivalent to a subsection of a principal section, then \( \sigma_S \) is obtained by means of projective isomorphism (Theorem A4.2). Set
\[ \Sigma(\varphi) = \{C_S(\varphi)\}_{S \in \mathcal{S}_0(\mathcal{A})}, \]
where \( C_S(\varphi) = \text{Aut}(\mathcal{A}(S))\sigma_S \).
Lemma 3.3. The family $\Sigma(\varphi)$ is an outer $A$-multiplier. Moreover, if $T \in \mathcal{G}(A)$, then $\Sigma(\varphi)^T = \Sigma(\varphi_T)$.

**Proof.** To prove the first statement, let $S_1, S_2 \in \mathcal{S}_0(A)$. Then the similarity $\varphi_{S_i}$ is induced by an automorphism $\sigma_{S_i}$ for $i = 1, 2$. Therefore if $S_1 \preceq S_2$, then $\varphi_{S_2}$ is induced by $(\sigma_{S_1})^{S_2}$. It follows that

$$(\sigma_{S_2})^{-1}(\sigma_{S_1})^{S_2} \in \text{Aut}_A(S_2).$$

Thus $C_{S_2}(\varphi) = C_{S_1}(\varphi)^{S_2}$ and condition (M1) is satisfied. Since condition (M2) follows from statement (1) of Lemma 2.8, $\Sigma(\varphi)$ is an outer $A$-multiplier as required. The second statement of the lemma is obvious. $\blacksquare$

The following theorem is the main result of this subsection.

**Theorem 3.4.** Let $A$ be an quasidense circulant $S$-ring. Then the mapping

$$\Phi(A) \rightarrow \text{OMult}(A), \quad \varphi \mapsto \Sigma(\varphi)$$

is a group isomorphism.

**Proof.** Mapping (5) is obviously a group homomorphism. To prove its injectivity suppose that $\Sigma(\varphi) = \Sigma(\psi)$ for some similarities $\varphi, \psi \in \Phi(A)$. Then obviously $\varphi_S = \psi_S$ for all sections $S \in \mathcal{S}_0(A)$. Therefore $\varphi$ and $\psi$ are equal on all principal $A$-sections. Thus $\varphi = \psi$ by Lemma 3.1.

Let us prove the surjectivity of homomorphism (5) by induction on the size of the group $G$ underlying $A$. Let $\Sigma = \{C_{S_i}\}_{S_i \in \mathcal{S}_0(A)}$ be an outer $A$-multiplier. First, suppose that $\text{rad}(A) = 1$. Then $G \in \mathcal{S}_0(A)$, the $S$-ring $A$ is cyclotomic, and any $\sigma \in C_G$ is a Cayley isomorphism of $A$ (Theorem 2.4). Clearly, the similarity induced by $\sigma$, does not depend on the choice of it. Denote this similarity by $\varphi$. Then $\varphi_S$ is induced by $\sigma^S$ for all $S \in \mathcal{S}_0(A)$. Now condition (M1) implies that

$$C_{S_2}(\varphi) = \text{Aut}_A(S)\sigma^S \circ (\text{Aut}_A(G)\sigma)^S = (C_G)^S = C_{S_2}.$$}

Thus $\Sigma(\varphi) = \Sigma$.

Now, assume that $\text{rad}(A) \neq 1$. Then by Theorem 5.2 the $S$-ring $A$ is a proper $U/L$-wreath product. By the inductive hypothesis applied to the $S$-ring $A_U$ and its outer multiplier $\Sigma^U$, as well as to the $S$-ring $A_{G/L}$ and its outer multiplier $\Sigma^{G/L}$, there exist similarities $\varphi_1 \in \Phi(A_U)$ and $\varphi_2 \in \Phi(A_{G/L})$ such that

$$\Sigma^U = \Sigma(\varphi_1) \quad \text{and} \quad \Sigma^{G/L} = \Sigma(\varphi_2).$$

By the first equality in (6) and Lemma 3.3 applied to $A = A_U$, $\varphi = \varphi_1$ and $T = U/L$, we have

$$\Sigma((\varphi_1)_{U/L}) = \Sigma(\varphi_1)^{U/L} = (\Sigma^U)^{U/L} = \Sigma^{U/L}.$$}

Similarly, by the second equality in (6) and Lemma 3.3 applied to $A = A_{G/L}$, $\varphi = \varphi_2$ and $T = U/L$ we have

$$\Sigma((\varphi_2)_{U/L}) = \Sigma(\varphi_2)^{U/L} = (\Sigma^{G/L})^{U/L} = \Sigma^{U/L}.$$}

Thus $\Sigma((\varphi_1)_{U/L}) = \Sigma((\varphi_2)_{U/L})$ and by the injectivity statement we have

$$(\varphi_1)_{U/L} = (\varphi_2)_{U/L}.$$
Thus by statement (2) of Theorem A3.3 there exists a unique similarity $\varphi \in \Phi(A)$ such that

$\varphi_U = \varphi_1$ and $\varphi_{G/L} = \varphi_2$.

To complete the proof let us verify that $\Sigma(\varphi) = \Sigma$. Let $S \in \mathcal{S}_0(A)$. Suppose first that $S$ is a principal section. Then it is of trivial radical. Therefore $S$ is either an $A_{U'}$- or $A_{G/L}$-section. Since $\Sigma(\varphi)^U = \Sigma^U$ and $\Sigma(\varphi)^{G/L} = \Sigma^{G/L}$ (see (6) and (7)), we conclude that

$C_S(\varphi) = C_S$.

Let now $S$ be a subsection of a principal section $T$. Then by above and condition (M1) we obtain that

$C_S(\varphi) = C_T(\varphi)^S = (C_T)^S = C_S$,

and (8) holds. Finally, let $S$ be a projectively equivalent to a subprincipal $A$-section $T$. By what we just proved, statement (1) of Lemma 2.8 and condition (M2), we obtain that

$C_S(\varphi) = C_T(\varphi)^{fr,s} = (C_T)^{fr,s} = C_S$

as required.

4. Reduction to quasidense S-rings

The main result in this section is the following theorem reducing the separability problem for circulant S-rings to the quasidense case.

**Theorem 4.1.** Let $G$ be a cyclic group. Then every S-ring over $G$ is separable if and only if so is every quasidense S-ring over $G$.

Theorem 4.1 will be deduced from Theorem 4.4 for the proof of which we recall, as in [8], some facts from paper [5]. Let $\mathcal{A}$ be an S-ring over a cyclic group $G$. A class $C$ of projectively equivalent $\mathcal{A}$-sections is called singular if its rank is 2, its order is greater than 2 and it contains two sections $L_1/L_0$ and $U_1/U_0$ such that the second is a multiple of the first and the following two conditions are satisfied:

1. $\mathcal{A}$ is both the $U_0/L_0$- and $U_1/L_1$-wreath product,
2. $\mathcal{A}_{U_1/L_0} = \mathcal{A}_{L_1/L_0} \otimes \mathcal{A}_{U_0/L_0}$.

By [5] Lemma 6.2 the above two sections are necessarily the smallest and largest $\mathcal{A}$-sections of $C$. Moreover, from Theorem 4.6 ibid., it follows that any rank 2 class of composite order belonging to $\mathcal{P}(\mathcal{A})$ is singular; in particular, the S-ring $\mathcal{A}$ is quasidense if and only if no class in $\mathcal{P}(\mathcal{A})$ is singular. It is worth mentioning that once $\mathcal{A}$ has a singular class, it admits a lot of non-trivial automorphisms; more exactly, the following statement is implied by Lemma 4.3 of paper [3].

**Lemma 4.2.** Let $\mathcal{A}$ be an S-ring over a cyclic group $G$ and let $U/L$ be the largest section in a singular class of $\mathcal{A}$. Then

$$\text{Aut}(\mathcal{A})^{G/L} \geq \prod_{X \in \mathcal{G}/\mathcal{U}} \text{Sym}(X/L).$$

For an $\mathcal{A}$-section $S$ we define the $S$-extension of $\mathcal{A}$ to be the smallest S-ring $\mathcal{A}' \geq \mathcal{A}$ such that $\mathcal{A}'_S = \mathbb{Z}S$. From Theorem A4.2 it follows that $\mathcal{A}'$ does not depend on the choice of $S$ in the class of $\mathcal{A}$-sections projectively equivalent to $S$. 
The following statement follows from Lemma A13.1 and statements (E1), (E2) inside it.

**Lemma 4.3.** Let $A$ be a circulant $S$-ring, $C \in \mathcal{P}(A)$ a singular class, $S = L_1/L_0$ the smallest section in $C$ and $A'$ the $S$-extension of $A$. Then

1. $\text{rk}(A') > \text{rk}(A)$,
2. $A'$ is both the $U_0/L_0$- and $U_1/L_1$-wreath product,
3. $A'_0 = A_0$, $A'_G/L_1 = A'_G \otimes L_1$, and $A'_{U_1/L_0} = ZS \otimes A_{U_0/L_0}$.

Now we are ready to prove the basic statement on extending similarities to $S$-extensions where $S$ belongs to a singular class.

**Theorem 4.4.** Let $A$ be a circulant $S$-ring, $C \in \mathcal{P}(A)$ a singular class, $S \in C$ and $A'$ the $S$-extension of $A$. Then given similarities $\varphi \in \Phi(A)$ and $\psi \in \Phi(ZS)$ there exists a unique similarity $\varphi' \in A'$ extending $\varphi$ and such that $\varphi'_S = \psi$. Moreover, if $\varphi \in \Phi_\infty(A)$, then $\varphi' \in \Phi_\infty(A')$.

**Proof.** Without loss of generality we can assume that $S = L_1/L_0$ is the smallest section in $C$. To prove the first statement let $\varphi \in \Phi(A)$ and $\psi \in \Phi(ZS)$. Then from statement (3) of Lemma 4.3 it follows that

$$\varphi_T \in \Phi(A'_T), \quad T \in \{U_0, U_0/L_0, G/L_1\}.$$ 

Furthermore, since $\text{rk}(A_S) = 2$, the similarity $\psi$ extends $\varphi_S$. However, $A'_S = ZS$. Thus, again by statement (3) of Lemma 4.3 the mapping $\psi \otimes \varphi_{U_0/L_0}$ is a similarity of $A'_{U_1/L_0}$ that extends $\varphi_{U_1/L_0}$.

By statement (2) of Lemma 4.3 the $S$-ring $A'_{U_1}$ is the $U_0/L_0$-wreath product. Since the restrictions of $\varphi_{U_0}$ and $\psi \otimes \varphi_{U_0/L_0}$ to $U_0/L_0$ coincide, we find by statement (2) of Theorem A3.3 a similarity $\psi' \in \Phi(A_{U_1})$ such that

$$\psi'_{U_0} = \varphi_{U_0} \quad \text{and} \quad \psi'_{U_1/L_0} = \psi \otimes \varphi_{U_0/L_0}.$$ 

Clearly, $\psi'$ extends $\varphi_{U_1}$ and $\psi'_{U_1/L_1} = \varphi'_{U_1/L_1}$. Furthermore, $A'$ is the $U_1/L_1$-wreath product by statement (2) of Lemma 4.3. Therefore in the same way as above, we find a similarity $\varphi' \in \Phi(A')$ such that $\varphi'_{U_1} = \psi'$ and $\varphi'_{G/L_1} = \varphi_{G/L_1}$. Since obviously $\varphi'$ extends $\varphi$, we are done. The uniqueness is clear by Lemma 2.1 and the definition of $S$-extension.

To prove the second statement let $\varphi \in \Phi_\infty(A)$. Then there exists a normalized isomorphism $f \in \text{Iso}(A, \varphi)$. Without loss of generality we can assume that $S = U/L$ is the largest section in the class $C$. For each $Y \in G/U$ fix a permutation $h_Y \in G_{Y \rightarrow U}$ with $h_U = \text{id}$. Besides, by Lemma A3.2 there exists an automorphism $\sigma \in \text{Aut}(S)$ belonging to $\text{Iso}(ZS, \psi)$. Then using Lemma 4.2 one can choose the isomorphism $f$ so that

$$f^{Y/L} = h_Y^{V/L} \cdot \sigma \cdot (h')^{U/L}, \quad Y \in G/U$$

where $h = h_Y$ and $h' = (h_Y)^{-1}$. However, the bijections $h$ and $h'$ are induced by multiplications by elements of $G$, say $a$ and $a'$, respectively. Thus the hypothesis of Corollary 2.4 is satisfied: indeed, if $x \in U$ and $y \in Y$, then $ay \in U$ and $(Lx)^a = f(Lx)$ (because $h_U = \text{id}$), and hence

$$f(Lxy) = f^{Y/L}(Lx) = (Laxy)^a = (Lx)^a(Lay)^a = f(Lx)f^{Y/L}(Ly) = f(Lx)f(Ly).$$
Thus \( f \in \text{Iso}(A') \). By the uniqueness of \( \varphi' \) we conclude that \( f \in \text{Iso}(A', \varphi') \) as required.

The first of two statements below immediately follows from Theorem 4.4 and is used in the proof of Theorem 4.1; the second one will be used in subsequent sections.

**Corollary 4.5.** In the notation of Theorem 4.4 the S-ring \( A \) is separable if and only if so is \( A' \).

**Corollary 4.6.** Let \( A \) be a circulant S-ring. Then given a similarity \( \varphi \in \Phi_\infty(A) \) there exists a normalized isomorphism \( f \in \text{Iso}(A, \varphi) \) such that \( f^S \in \text{Aut}(S) \) for all \( S \in S_0(A) \).

**Proof.** Let \( \varphi \in \Phi_\infty(A) \). Then \( \varphi \) is induced by a normalized isomorphism \( f \).

Besides, by Theorem 4.4 without loss of generality we can assume that there are no singular classes of \( A \).

**Lemma 4.7.** Let \( A \) be a circulant S-ring without singular classes, and \( f \in \text{Iso}(A) \). Then \( f^S \in \text{Aut}(S) \) for all \( S \in S_0(A) \).

**Proof.** Let \( S \in S_0(A) \). By statement (2) of Lemma 2.8 we can assume that \( S \) is principal, and hence \( \text{rad}(A_S) = 1 \). By [5, Theorem 4.1] in this case \( S = S_0 \times \cdots \times S_k \) where \( S_i \in S(A) \) for all \( i \), the S-ring \( A_{S_0} \) is normal and \( A_{S_i} \) is an S-ring of rank 2 and prime degree for \( i \geq 1 \). Then \( f^{S_0} \leq \text{Aut}(S_0) \) by normality and so it suffices to verify that

\[
(9) \quad f^{S_i} \leq \text{Aut}(S_i), \quad i \geq 1.
\]

However, since \( A \) has no singular classes, from [3, Theorem 5.1] it follows that any primitive \( A \)-section is projectively equivalent to a subnormal \( A \)-section. But for \( i \geq 1 \) the section \( S_i \) has prime degree, and hence is primitive. Thus any such section is projectively equivalent to a subnormal \( A \)-section. Now, (9) follows from statement (2) of Lemma 2.8.

**Proof of Theorem 4.1.** The ”only if” part is obvious. To prove the ”if” part suppose that every quasidense S-ring over \( G \) is separable. Let \( A \) be an arbitrary S-ring over \( G \). First, we observe that given a section \( S \in S(A) \) belonging to a singular class, the S-extension of \( A \) is separable if and only if so is \( A \) (Corollary 4.3). However, by statement (1) of Lemma 4.3 the rank of this extension is less than the rank of \( A \). So without loss of generality we can assume that the S-ring \( A \) has no singular classes, or equivalently that \( A \) is a quasidense S-ring. Thus \( A \) is separable by the assumption.

**5. Proof of Theorems 1.1 and 1.2**

Before proving the theorems we give the exact definition of \( A \)-multipliers discussed in Introduction, and study them in some details. Let \( A \) be an S-ring over a cyclic group \( G \). Suppose that for every section \( S \in S_0(A) \) we are given an automorphism \( \sigma_S \in \text{Aut}(S) \).

**Definition 5.1.** The family \( \Sigma = \{\sigma_S\}_{S \in S_0(A)} \) is called an \( A \)-multiplier if the following two conditions are satisfied for all sections \( S_1, S_2 \in S_0(A) \):

- **(SM1)** if \( S_1 \succeq S_2 \), then \((\sigma_{S_1})^{S_2} = \sigma_{S_2}\),
(SM2) if \( S_1 \sim S_2 \), then \((\sigma_{S_1})^{f_{S_1,S_2}} = \sigma_{S_2} \) where \( f_{S_1,S_2} \) is the projective isomorphism from \( S_1 \) onto \( S_2 \).^5

Clearly, the set of all multipliers of \( \mathcal{A} \) forms a group; we denote it by \( \text{Mult}(\mathcal{A}) \).

The restriction \( \Sigma^T \) of a multiplier \( \Sigma \in \text{Mult}(\mathcal{A}) \) to a section \( T \in \mathcal{S}(\mathcal{A}) \), is defined in the same way as for outer multipliers.

When \( \mathcal{A} \) is a coset \( S \)-ring, the sets \( \text{Mult}(\mathcal{A}) \) and \( \text{OMult}(\mathcal{A}) \) are closely related. Namely, in this case \( \mathcal{A}_S = ZS \) for all \( S \in \mathcal{S}_0(\mathcal{A}) \) (Theorem A8.3), and hence \( \text{Aut}_\mathcal{A}(S) = 1 \) for such \( S \). Therefore if \( \Sigma = \{ C_S \} \) is an outer \( \mathcal{A} \)-multiplier, then \( C_S \) is a singleton consisting of an automorphism \( \sigma_S \in \text{Aut}(S) \) for all \( S \), and hence \( \Sigma' := \{ \sigma_S \} \) is an \( \mathcal{A} \)-multiplier. Clearly, the mapping \( \Sigma \mapsto \Sigma' \) is a group isomorphism from \( \text{OMult}(\mathcal{A}) \) onto \( \text{Mult}(\mathcal{A}) \).

Let now \( \mathcal{A} \) be a quasidense \( S \)-ring. Then \( \mathcal{S}_0(\mathcal{A}_0) \supset \mathcal{S}_0(\mathcal{A}) \) where \( \mathcal{A}_0 \) is the coset closure of \( \mathcal{A} \). Since \( \mathcal{A}_0 \) is a coset \( S \)-ring, the discussion in the previous paragraph shows that every \( \mathcal{A}_0 \)-multiplier \( \Sigma_0 \) induces an ordinary \( \mathcal{A}_0 \)-multiplier \( \Sigma' \). The restriction of it to \( \mathcal{S}_0(\mathcal{A}) \) produces a family \( \Sigma \) which is obviously an outer \( \mathcal{A} \)-multiplier. The mapping from \( \text{OMult}(\mathcal{A}_0) \) to \( \text{Mult}(\mathcal{A}) \) that takes \( \Sigma_0 \) to \( \Sigma \), is denoted by \( \eta \).

**Lemma 5.2.** The mapping \( \eta : \text{OMult}(\mathcal{A}_0) \to \text{Mult}(\mathcal{A}) \) is a group isomorphism.

**Proof.** Let \( S \in \mathcal{S}_0 \) where \( \mathcal{S}_0 = \mathcal{S}_0(\mathcal{A}_0) \). Then \( S_p \in \mathcal{S}_0(\mathcal{A}) \) for all prime \( p \) dividing \( |\mathcal{S}| \) (Corollary A10.10) where \( S_p \) is the Sylow \( p \)-subgroup of \( S \). Since any element of \( \text{Aut}(S) \) is uniquely determined by its projections to \( \text{Aut}(S_p) \), an outer \( \mathcal{A}_0 \)-multiplier \( \Sigma_0 \) is uniquely determined by the \( \mathcal{A} \)-multiplier \( \eta(\Sigma_0) \). This shows that the mapping \( \eta \) is injective. To prove that it is surjective, let

\[
\Sigma = \{ \sigma_S' \}_{S \in \mathcal{S}_0(\mathcal{A})}
\]

be an \( \mathcal{A} \)-multiplier. Given \( S \in \mathcal{S}_0 \) set \( \sigma_S := \prod_p \sigma_{S_p}' \) (here \( S_p \in \mathcal{S}_0(\mathcal{A}) \)). Then \( \sigma_S \in \text{Aut}(S) \) for all \( S \in \mathcal{S}_0 \), and \( \sigma_S = \sigma_S' \) for \( S \in \mathcal{S}_0(\mathcal{A}) \). Let us check that conditions (M1) and (M2) are satisfied for the family

\[
\Sigma_0 = \{ C_S \}_{S \in \mathcal{S}_0}
\]

where \( C_S = \{ \sigma_S \} \).

Suppose first that \( S,T \in \mathcal{S}_0 \) and \( T \preceq S \). Then \( T_p \preceq S_p \) for all \( p \). By condition (SM1) for \( \Sigma \) this implies that

\[
\sigma_T = \prod_p \sigma_{T_p}' = \prod_p (\sigma_S')_{T_p} = (\sigma_S)'^T,
\]

as required. To verify (M2) let \( S,T \in \mathcal{S}_0 \) and \( S \sim T \). Then \( S_p \sim T_p \) for all \( p \). Therefore by Corollary 2.7 we have \( T_p = (S_p)^{f_{S_p,T_p}} \) and the restriction of \( f_{S,T} \) to \( S_p \) equals \( f_{S_p,T_p} \). By condition (SM2) for \( \Sigma \) this implies that

\[
\sigma_T = \prod_p \sigma_{T_p}' = \prod_p (\sigma_S')_{T_p} = (\sigma_S)'^{f_{S,T}}.
\]

Thus \( \Sigma_0 \) is an outer \( \mathcal{A}_0 \)-multiplier. Since obviously \( \eta(\Sigma_0) = \Sigma \), we are done. \( \blacksquare \)

**Proof of Theorem 1.1.** The \( S \)-ring \( \mathcal{A}_0 \) is a coset one, and so separable (Theorem A9.1). Therefore \( \Phi(\mathcal{A}_0)^d \) is a subgroup of the group \( \Phi_{\infty}(\mathcal{A}) \). Conversely,
let \( \varphi \in \Phi_\infty(A) \). Then by Corollary 4.6 there exists a normalized isomorphism \( f \in \text{Iso}(A, \varphi) \) such that \( f^S \in \text{Aut}(S) \) for all \( S \in \mathcal{S}_0(A) \). Set \( \Sigma = \{ \sigma_S \}_{S \in \mathcal{S}_0(A)} \) where \( \sigma_S = f^S \). Then condition (SM1) is trivially satisfied whereas condition (SM2) follows from statement (2) of Lemma 2.8. Thus \( \Sigma \) is an \( A \)-multiplier. By Lemma 6.2 it can be extended to an outer \( A_0 \)-multiplier \( \Sigma_0 \). Denote by \( \varphi_0 \) the similarity of the \( S \)-ring \( A_0 \) that corresponds to \( \Sigma_0 \) (Theorem 3.4). Since \( \Sigma_0 \) extends \( \Sigma \), the restriction of \( (\varphi_0)_S \) to \( A_S \) equals the similarity \( \varphi_S \) for all \( S \in \mathcal{S}_0(A) \). Therefore the similarities \( (\varphi_0)^A \) and \( \varphi \) coincide on all principal \( A \)-sections. By Lemma A3.1 this implies that \( (\varphi_0)^A = \varphi \) as required.

**Proof of Theorem 1.2.** To prove the "only if" part, suppose that the \( S \)-ring \( A \) is separable. Let \( \Sigma' \in \text{OMult}(A) \). Then by Theorem 3.4 there exists \( \varphi \in \Phi(A) \) such that \( \Sigma' = \Sigma(\varphi) \). So by Theorem 1.1 there exists \( \varphi_0 \in \Phi(A_0) \) such that \( \varphi = (\varphi_0)^A \).

Again by Theorem 5.4 applied this time to \( A_0 \), we find an outer \( A_0 \)-multiplier \( \Sigma_0 = \{ C_S \}_{S \in \mathcal{S}_0} \) corresponding to \( \varphi_0 \). Since \( A_0 \) is a coset \( S \)-ring, the coset \( C_S \) is a singleton consisting of an automorphism \( \sigma_S \in \text{Aut}(S) \). Therefore \( \Sigma' = \theta(\Sigma) \) where \( \Sigma = \{ \sigma_S \}_{S \in \mathcal{S}_0(A)} \). Since obviously \( \Sigma \in \text{Mult}(A) \), we are done.

To prove the "if" part, suppose that the homomorphism \( \theta \) is surjective. To check that the \( S \)-ring \( A \) is separable, let \( \varphi \in \Phi(A) \). Then by the hypothesis there exists an \( A \)-multiplier \( \Sigma \) such that \( \theta(\Sigma) = \Sigma(\varphi) \). Set

\[
\Sigma_0 = \eta^{-1}(\Sigma)
\]

where \( \eta \) is the isomorphism in Lemma 5.2. By Theorem 5.4 there exists \( \varphi_0 \in \Phi(A_0) \) such that \( \Sigma_0 = \Sigma(\varphi_0) \). Now, since the \( S \)-ring \( A_0 \) is separable (Theorem A9.1), the similarity \( \varphi_0 \) is induced by an isomorphism \( f \in \text{Iso}(A_0) \). Since obviously \( \varphi = (\varphi_0)^A \), we have \( f \in \text{Iso}(A, \varphi) \), as required.

**6. Proof of Theorem 1.3.**

By Theorem 4.1 we can restrict ourselves to quasiidence \( S \)-rings. To prove that every quasiidence \( S \)-ring \( A \) over a cyclic \( p \)-group \( G \) is separable, we use induction on \( |G| \). If \( \text{rad}(A) = 1 \), then \( A \) is the tensor product of \( S \)-rings of rank 2 and a normal \( S \)-ring [5, Theorem 4.1]. Since \( G \) is a cyclic \( p \)-group, this implies that \( \text{rk}(A) = 2 \) or \( A \) is normal. Thus \( A \) is separable: in the former case this is obvious whereas in the latter one this follows from [2, Theorem 6.6]. Let now \( \text{rad}(A) \neq 1 \).

**Lemma 6.1.** Let \( A \) be an \( S \)-ring over a cyclic \( p \)-group. Suppose that \( \text{rad}(A) \neq 1 \). Then \( A \) is a proper \( S \)-wreath product such that \( \text{rad}(A_S) = 1 \) or \( |S| = 4 \).

**Proof.** By Theorem A5.2 \( A \) is a proper \( S \)-wreath product where \( S = U/L \). Without loss of generality we assume that the section \( S \) is of minimal possible order. Then there exists \( X \in \mathcal{S}(A)_{G/U} \) such that \( \text{rad}(X) = L \) (here we use that the set of \( A \)-groups is linearly ordered by inclusion). Therefore the \( A \)-section \( T = \langle X \rangle / \text{rad}(X) \) is of trivial radical and contains \( S \) as a subsection. This implies that \( \text{rk}(A_T) = 2 \) or \( A_T \) is normal (see above). In the former case, obviously \( \text{rad}(A_S) = 1 \). In the latter case, \( A_T \) is a cyclotomic \( S \)-ring with trivial radical [2, Lemma 7.1]. However, from [6, Theorem 7.3] it follows that in a cyclotomic \( S \)-ring with trivial radical over a cyclic \( p \)-group, any of its subsections of order \( \neq 4 \) is of trivial radical. Thus \( \text{rad}(A_S) = 1 \) or \( |S| = 4 \). as required.

By Lemma 6.1 the \( S \)-ring \( A \) is a proper \( S \)-wreath product where the section \( S \) is normal or of trivial radical. By Theorem 1.2 it suffices to verify that given an outer
\(\mathcal{A}\)-multiplier \(\Sigma\) there exists an \(\mathcal{A}\)-multiplier \(\Sigma'\) such that \(\theta(\Sigma') = \Sigma\). To do this let \(S = U/L\). Denote by \(\varphi\) the similarity of \(\mathcal{A}\) corresponding to \(\Sigma\) (Theorem 3.4). Then by Lemma 3.3 we have

\[
\Sigma(\varphi_1) = \Sigma^U \quad \text{and} \quad \Sigma(\varphi_0) = \Sigma^{G/L}
\]

where \(\varphi_1 = \varphi_U\) and \(\varphi_0 = \varphi_{G/L}\). Since \(\mathcal{A}_U\) and \(\mathcal{A}_{G/L}\) are separable (the inductive hypothesis), these similarities belong to \(\Phi_\infty(\mathcal{A}_U)\) and \(\Phi_\infty(\mathcal{A}_{G/L})\) respectively. So by Corollary 4.6 there exist normalized isomorphisms \(f_1 \in \text{Iso}(\mathcal{A}_U, \varphi_1)\) and \(f_0 \in \text{Iso}(\mathcal{A}_{G/L}, \varphi_0)\) such that \((f_1)^T \in \text{Aut}(T)\) for all \(T \in \mathfrak{S}_0(\mathcal{A}_U)\) and \((f_0)^T \in \text{Aut}(T)\) for all \(T \in \mathfrak{S}_0(\mathcal{A}_{G/L})\). Then the families

\[
\Sigma_1 = \{(f_1)^T\}_{T \in \mathfrak{S}_0(\mathcal{A}_U)} \quad \text{and} \quad \Sigma_0 = \{(f_0)^T\}_{T \in \mathfrak{S}_0(\mathcal{A}_{G/L})}
\]

are \(\mathcal{A}_U\)- and \(\mathcal{A}_{G/L}\)-multipliers respectively (see the proof of Theorem 1.1). Moreover, by (10), they extend the multipliers \(\Sigma^U\) and \(\Sigma^{G/L}\).

Without loss of generality we can assume that \((\Sigma_1)^S = (\Sigma_0)^S\), or equivalently

\[
(f_1)^S = (f_0)^S.
\]

To see this, we first observe that the permutation \(\sigma := (f_1)^S \cdot (f_0)^S\) induce the trivial similarity of the S-ring \(\mathcal{A}_S\), because both \((f_1)^S\) and \((f_0)^S\) induce \(\varphi_S\). So \(\sigma\) belongs to \(\text{Aut}(\mathcal{A}_S)\), and by the choice of \(S\) also to \(\text{Aut}_\mathcal{A}(S)\). Then there exists \(f \in \text{Aut}(\mathcal{A}_U)\) such that

\[
f^S = \sigma \quad \text{and} \quad f^T \in \text{Aut}_\mathcal{A}(T) \quad \text{for all} \quad T \in \mathfrak{S}_0(\mathcal{A}_U).
\]

Indeed, the S-ring \(\mathcal{A}_U\) is quasidense and schurian. Therefore by [6, Theorem 3.5 and Remark 3.6] there exists \(f \in \text{Aut}(\mathcal{A}_U)\) such that \(f^S = \sigma\) and \(f^T \in \text{Aut}_\mathcal{A}(T)\) for any \(\mathcal{A}\)-section \(T\) of trivial radical. It follows that \(f^T \in \text{Aut}_\mathcal{A}(T)\) for all subprincipal sections \(T\). Since each nontrivial class of projective equivalence in a cyclic \(p\)-group is a singleton, this is true also for all quasisubprincipal sections. Thus, we found an automorphism \(f\) satisfying (10). Now, equality (12) holds with \(f_1\) replaced by \(f_1f\).

To complete the proof, let \(T \in \mathfrak{S}_0(\mathcal{A})\). Then \(T\) belongs to \(\mathfrak{S}_0(\mathcal{A}_U)\) or \(\mathfrak{S}_0(\mathcal{A}_{G/L})\); indeed, this is true if \(T\) is a subprincipal section, and hence for any \(T\) because each nontrivial class of projective equivalence in a cyclic \(p\)-group is a singleton. Moreover, by (12) we have \((f_1)^T = (f_0)^T\) if \(T \leq S\). Thus, the following element of the group \(\text{Aut}(T)\) is well defined:

\[
\sigma^T_T := \begin{cases} (f_1)^T, & \text{if } T \in \mathfrak{S}_0(\mathcal{A}_U), \\ (f_0)^T, & \text{if } T \in \mathfrak{S}_0(\mathcal{A}_{G/L}). \end{cases}
\]

Then the family \(\Sigma' := \{\sigma^T_T\}_{T \in \mathfrak{S}_0(\mathcal{A})}\) is obviously an \(\mathcal{A}\)-multiplier. Furthermore, from (10) and (11) it follows that the multipliers \(\theta((\Sigma')^U)\) and \(\theta((\Sigma')^{G/L})\) correspond to the similarities \((\varphi_1)_U\) and \((\varphi_0)_{G/L}\) respectively. Therefore \(\theta(\Sigma')\) corresponds to \(\varphi\), and hence \(\theta(\Sigma') = \Sigma\), as required.\[6\]
7. Proof of Theorem 1.4

Let \( \mathcal{A} \) be a circulant S-ring. By Theorem 1.4 we can assume that it is quasidense. Below for \( S \in \mathcal{G}(G) \) and \( C \subset \text{Aut}(S) \) we set \( C = \{ \sigma : \sigma \in C \} \). For the reader’s convenience we cite here Lemma A13.3.

**Lemma 7.1.** Let \( \mathcal{A} \) be a quasidense circulant S-ring. Then

1. \( \mathcal{G}_0(\mathcal{A}) = \{ \tilde{S} : S \in \mathcal{G}_0(\mathcal{A}) \} \).
2. \( \tilde{\mathcal{A}}_0 = \mathcal{A}_0 \) and \( \mathcal{G}_0(\tilde{\mathcal{A}}_0) = \{ \tilde{S} : S \in \mathcal{G}_0(\mathcal{A}_0) \} \).
3. \( \text{Aut}_{\tilde{\mathcal{A}}}(\tilde{S}) = \text{Aut}_{\mathcal{A}}(S) \) for all \( S \in \mathcal{G}(\mathcal{A}) \).

For a family \( \Sigma = \{ C_S \}_{S \in I} \) of sets \( C_S \subset \text{Aut}(S) \) where \( I \subset \mathcal{G}(G) \), let us define the family \( \tilde{\Sigma} = \{ \tilde{C}_S \}_{S' \in \tilde{I}} \) of sets \( C_{S'} \subset \text{Aut}(S') \), by \( C_{S'} = \tilde{C}_S \) with \( \tilde{S} = S' \) where \( \tilde{I} = \{ \tilde{S} : S \in I \} \).

**Corollary 7.2.** Let \( \mathcal{A} \) be a quasidense circulant S-ring and \( \Sigma \) an outer \( \mathcal{A} \)-multiplier. Then the family \( \tilde{\Sigma} \) is an outer \( \tilde{\mathcal{A}} \)-multiplier.

**Proof.** Let \( \Sigma = \{ C_S \}_{S \in \mathcal{G}_0(\mathcal{A})} \). Then from statement (3) of Lemma 7.1 it follows that \( \tilde{C}_S \) is a coset of the subgroup \( \text{Aut}_{\mathcal{A}}(\tilde{S}) \) in the group \( \text{Aut}(\tilde{S}) \) for all \( S \). Moreover, by Lemmas 2.9 and 2.10, conditions (M1) and (M2) are satisfied for the family \( \tilde{\Sigma} \). Thus \( \tilde{\Sigma} \) is an outer \( \tilde{\mathcal{A}} \)-multiplier.

Let us come back to the proof of Theorem 1.4. Let \( \mathcal{A} \) be a separable quasidense circulant S-ring. By duality we only have to prove that the S-ring \( \tilde{\mathcal{A}} \) is separable. However, the latter ring is quasidense by statement (2) of Theorem A5.9. Therefore by Theorem 1.2 and Lemma 8.2 it suffices to verify that the homomorphism \( \tilde{\eta} \tilde{\theta} \) is surjective, where \( \tilde{\eta} \) and \( \tilde{\theta} \) are defined for the S-ring \( \tilde{\mathcal{A}} \) as in Lemma 5.2 and in formula (2), respectively.

Let \( \Sigma' = \{ C_{S'} \}_{S' \in \mathcal{G}_0(\tilde{\mathcal{A}})} \) be an outer \( \tilde{\mathcal{A}} \)-multiplier. Then by Corollary 7.2 applied to \( \tilde{\mathcal{A}} \) and \( \Sigma' \), the family \( \Sigma := \{ C_S \}_{S \in \mathcal{G}_0(\mathcal{A})} \) is an outer \( \mathcal{A} \)-multiplier where \( C_{S'} \) is defined by \( \tilde{C}_{S'} = C_S \) with \( S' = \tilde{S} \) (here we identify the duals to \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{A}} \) with \( \mathcal{A} \) and \( \mathcal{A} \) respectively). Since the S-ring \( \tilde{\mathcal{A}} \) is separable, by Corollary 7.2 and Lemma 8.2 there exists an outer \( \mathcal{A}_0 \)-multiplier \( \Sigma_0 \) such that \( (\Sigma_0)^{\tilde{\mathcal{G}}_0} = \Sigma \). By Corollary 7.2 applied to \( \mathcal{A}_0 \) and \( \Sigma_0 \), the family \( \tilde{\Sigma}_0 \) is an outer \( \tilde{\mathcal{A}}_0 \)-multiplier, and hence an outer \( \tilde{\mathcal{A}}_0 \)-multiplier (statement (2) of Lemma 1.14). Thus the surjectivity statement is true because

\[
(\Sigma_0)^{\tilde{\mathcal{G}}_0} = ((\Sigma_0)^{\mathcal{G}})^{\tilde{\mathcal{G}}} = (\Sigma_0)^{\mathcal{G}} = \tilde{\Sigma} = \Sigma'.
\]

Here the first equality follows from statement (1) of Lemma 7.1 and the definition of \( \tilde{\eta} \), whereas the third and fourth equalities follow from the definitions of \( \Sigma_0 \) and \( \Sigma \) respectively. To prove the second equality let \( S' \in \mathcal{G}_0(\tilde{\mathcal{A}}) \), and let \( S \in \mathcal{G}_0(\mathcal{A}) \) be such that \( \tilde{S} = S' \). Denote the \( S' \)-component of \( (\Sigma_0)^{\mathcal{G}} \) by \( \sigma_{S'} \). Then by statement (1) of Lemma 7.1 the \( S' \)-component of \( (\Sigma_0)^{\mathcal{G}} \) equals

\[
\text{Aut}_{\mathcal{G}}(S') \sigma_{S'} = \text{Aut}_{\mathcal{A}}(S) \sigma_{S'}
\]

which is the \( S' \)-component of \( (\Sigma_0)^{\mathcal{G}} \) as required.
References

[1] S. Evdokimov, I. Ponomarenko, On a family of Schur rings over a finite cyclic group, Algebra and Analysis, 13 (2001), 3, 139–154.
[2] S. Evdokimov, I. Ponomarenko, Characterization of cyclotomic schemes and normal Schur rings over a cyclic group, Algebra and Analysis, 14 (2002), 2, 11–55.
[3] S. Evdokimov, I. Ponomarenko, Recognizing and isomorphism testing circulant graphs in polynomial time, Algebra and Analysis, 15 (2003), 6, 1–34.
[4] S. Evdokimov, I. Ponomarenko, Permutation group approach to association schemes, European Journal of Combinatorics, 30 (2009), 6, 1456–1476.
[5] S. Evdokimov, I. Ponomarenko, Schurity of S-rings over a cyclic group and generalized wreath product of permutation groups, Algebra and Analysis, 24 (2012), 3, 84–127.
[6] S. Evdokimov, I. Kovács, I. Ponomarenko, Characterization of cyclic Schur groups, Algebra and Analysis, 25 (2013), 5, Algebra and Analysis, 61–85.
[7] S. Evdokimov, I. Kovács, I. Ponomarenko, On schurity of finite abelian groups, arXiv:1309.0989 [math.GR] (2013), 1–20 (accepted to Comm. in Algebra).
[8] S. Evdokimov, I. Ponomarenko, On coset closure of a circulant S-ring and schurity problem, arXiv:1404.5826 [math.CO] (2014), 1–42.
[9] M. Muzychuk, On the structure of basic sets of Schur rings over cyclic groups, J. Algebra, 169 (1994), 655–678.

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