A simulation study: new optimal estimators for population mean by using dual auxiliary information in stratified random sampling

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ABSTRACT
Recently, Haq et al. [A new estimator of finite population mean based on the dual use of the auxiliary information. Commun Stat Theory Methods. 2017;46(9):4425–4436] utilized the dual auxiliary information under simple random sampling only. Motivated by their idea, we initiated the dual use of auxiliary variable under a stratified random sampling scheme. Dual use of auxiliary variable consists: (1) the original auxiliary information and (2) the ranked auxiliary information. We proposed new optimal exponential-type estimators for the estimation of the finite population mean. Mathematical properties such as bias and mean squared error of the proposed estimators are derived. Monte Carlo simulation studies are included to successfully validate the theoretical results. Moreover, the applicability of the proposed estimators is highlighted through empirical interpretation with the help of a real-life data set. It is clearly identified from the numerical results that our proposed estimators are more efficient over the competitors.

ARTICLE HISTORY
Received 11 September 2019
Revised 27 March 2020
Accepted 30 March 2020

KEYWORDS
Auxiliary variable; bias; mean squared error; ranked auxiliary variable; stratified random sampling

1. Introduction
One of the objectives of sample survey theory is to estimate the unknown population parameters of the study variable such as population total, mean, proportion, ratio and variance etc. A procedure is desirable that provides a precise estimator of the parameter of interest by surveying a suitably chosen sample of individuals. Supplementary/additional information provided by an auxiliary variable which is correlated with the study variable enhances the precision of the estimators. Survey statisticians take advantage of this information whenever it is available to explore the efficient estimators. Ratio, product, regression and their modified estimators are best examples in this regard.

An elaborate literature has grown for identifying more efficient estimators under different sampling designs, e.g. simple random sampling, stratified random sampling, cluster sampling, systematic sampling and etc. Simple random sampling does not produce administrative convenience and representative sample for a heterogeneous population. As it does not capture the diversity which is likely to be mined through stratified random sampling. Stratified random sampling is one of the possible ways to increase the precision of the estimates. It is a powerful and flexible method that is widely used in practice. Many researchers, such as Kadilar and Cingi [1,2], Koyuncu and Kadilar [3,4], Singh and Vishwakarma [5], Shabbir and Gupta [6], Haq and Shabbir [7], Singh and Solanki [8], Yadav et al. [9], Solanki and Singh [10,11], Aslam [12], Bhatti et al. [13], Javed et al. [14], Marin et al. [15–17], etc. have contributed to estimate the finite population mean under stratified random sampling scheme. All these contributions and alike published work under a stratified random sampling scheme are based on only the utilization of original auxiliary information. None of them tried the dual use of auxiliary information to enhance the estimation procedure.

Recently, Haq et al. [18] used an additional information of the auxiliary variable called ranked auxiliary variable to develop efficient estimators for the estimation of mean. These estimators are developed only to cope with the simple random sampling scheme.

Here, comes a new challenge/idea for exploring more optimal estimators using dual use of auxiliary information to deal with the stratified random sampling scheme. This challenge is successfully meet and new optimal estimators for finite population mean are developed under a stratified random sampling scheme in this article.

The remaining part of the paper is organized as follows: In section 2, procedures, notations and various estimators under stratified random sampling are introduced. In section 3, proposed estimators for estimating finite population mean using the original and ranked auxiliary information are defined. In section 4, an empirical study is carried out to evaluate the performance of the proposed estimators. Monte Carlo simulation

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studies are included to successfully validate the theoretical results in section 5. Finally, concluding remarks are enclosed in the last section.

2. Procedure, notations and review of literature

Consider \( U = \{U_1, U_2, U_3, \ldots, U_N\} \) be a finite population of size \( N \) and is divided into \( L \) homogenous strata with \( h \)th stratum containing \( N_h \) (\( h = 1, 2, \ldots, L \)) units with the condition that \( \sum_{h=1}^{L} N_h = N \). Under the condition \( \sum_{h=1}^{L} n_h = n \), a sample of size \( n_h \) is drawn under simple random sampling without replacement (SRSWOR) from \( h \)th stratum. Let

\[ \bar{Y} = \bar{Y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h \] be the population mean of the study variable (\( Y \))

\[ \bar{X} = \bar{X}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h \] be the population mean of the auxiliary variable (\( X \))

\[ \bar{Z} = \bar{Z}_{st} = \sum_{h=1}^{L} W_h \bar{z}_h \] be the population mean of the ranked auxiliary variable (\( Z \))

\[ \bar{Y}_h = \sum_{i=1}^{N_h} \left( \frac{Y_{hi}}{N_h} \right) \] be the population mean (\( h \)th stratum) of the \( Y \)

\[ \bar{x}_h = \sum_{i=1}^{n_h} \left( \frac{x_{hi}}{n_h} \right) \] be the sample mean (\( h \)th stratum) of the \( X \)

\[ \bar{z}_h = \sum_{i=1}^{n_h} \left( \frac{Z_{hi}}{n_h} \right) \] be the sample mean (\( h \)th stratum) of the \( Z \)

We define the following relative error terms and their expectations to drive the expressions for bias, MSE and minimum MSE of the proposed estimators.

\[ \xi_0 = \frac{\bar{y}_st - \bar{y}}{\bar{y}}, \xi_1 = \frac{\bar{z}_st - \bar{z}}{\bar{z}} \text{ and } \xi_2 = \frac{\bar{y}_st - \bar{x}}{\bar{x}}, \]

such that

\[ E(\xi_0) = E(\xi_1) = E(\xi_2) = 0, \]

Let us define,

\[ V_{abc} = \sum_{h=1}^{L} W_h^{a+b+c} \frac{E[(\bar{y}_st - \bar{y}_h)\bar{y}_h - \bar{x}_h \bar{z}_h \bar{y}_h]}{\bar{y}^2 \bar{z}^2 \bar{x}^2}. \]

Using (2.1), we can write as:

\[ E(\xi_0^2) = \frac{\sum_{h=1}^{L} W_h^2 \psi_h \bar{S}^2_{yh}}{\bar{y}^2} = V_{200}, \]

\[ E(\xi_1^2) = \frac{\sum_{h=1}^{L} W_h^2 \psi_h \bar{S}^2_{zh}}{\bar{z}^2} = V_{202}, \]

\[ E(\xi_2^2) = \frac{\sum_{h=1}^{L} W_h^2 \psi_h \bar{S}^2_{xh}}{\bar{x}^2} = V_{002}. \]

and

\[ E(\xi_0 \xi_1) = \frac{\sum_{h=1}^{L} W_h \psi_h \bar{S} Y Z}{\bar{y} \bar{z}} = V_{110}, \]

\[ E(\xi_0 \xi_2) = \frac{\sum_{h=1}^{L} W_h \psi_h \bar{S} X Z}{\bar{y} \bar{x}} = V_{010}, \]

\[ E(\xi_1 \xi_2) = \frac{\sum_{h=1}^{L} W_h \psi_h \bar{S} x z h}{\bar{z} \bar{x}} = V_{101}. \]

where

\[ \psi_h = \left(1 - \frac{f_h}{n_h} \right) \] be the finite population correction factor

\[ f_h = \frac{n_h}{N_h} \] be the sampling fraction for the stratum

\[ \bar{S}^2_{yh} = \frac{\sum_{i=1}^{N_h} (Y_{hi} - \bar{y}_h)^2}{N_h - 1} \] be the population variance (\( h \)th stratum) of \( Y \)

\[ \bar{S}^2_{xh} = \frac{\sum_{i=1}^{N_h} (x_{hi} - \bar{x}_h)^2}{N_h - 1} \] the population variance.
**2.1. Usual unbiased, combined ratio and combined regression estimators are detailed below**

\[
\hat{y}_{st} = \sum_{h=1}^{l} W_h\tilde{y}_h
\]

\[\hat{y}_{CR} = \tilde{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right)\]

\[\hat{y}_{CReg} = [\bar{y}_{st} + b(\bar{X} - \bar{x}_{st})], \text{ where } b = \frac{\sum_{h=1}^{l} W_h^2\psi_h S_{xyh}}{\sum_{h=1}^{l} W_h^2\psi_h S_{xh}^2}\]

**2.2. Haq and Shabbir [7] proposed two exponential ratio-type families of estimators detailed below**

\[\hat{y}_{HS1} = [\mu_1\bar{y}_{st} + \mu_2(\bar{X} - \bar{x}_{st})]\exp\left(\frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st}) - 1}\right)\]

\[\hat{y}_{HS2} = [\mu_3\bar{y}_{st} + \mu_4(\bar{X} - \bar{x}_{st})]\exp\left(\frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st}) - 1}\right)\times \left\{\frac{1}{2}\left[\frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})}ight] + \frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})}\right\}^2\]

where \(\eta\) is the suitable constant, \(a_{st}(\neq 0)\) and \(b_{st}\) are either real numbers or functions of known parameters of the auxiliary variable.

**2.3. Singh and Solanki [8] proposed a family of estimators as given below**

\[\hat{y}_{SS1} = \left[\mu_5\tilde{y}_{st}\left(\frac{\eta(a_{st}\tilde{x}_{st} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})}{(a_{st}\bar{X} + b_{st})}\right)^{\omega_1}\right] + \mu_6\tilde{y}_{st}\left(\frac{(a_{st}\bar{X} + b_{st})}{\eta(a_{st}\tilde{x}_{st} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})}\right)^{\omega_2}\]

where \(\eta, a_{st}(\neq 0)\) and \(b_{st}\) are defined earlier.
Remark 2.1: \( \hat{\gamma}_{SSS1} \) reduces to the ratio-type \( \hat{\gamma}_{SSSR} \), product-type \( \hat{\gamma}_{SSSP} \) and ratio-cum-product-type \( \hat{\gamma}_{SSSPP} \) estimators by placing the suitable values of the constants as: \((\omega_1 = 0, \omega_2 = 1, \eta = 1)\), \((\omega_1 = 0, \omega_2 = -1, \eta = 1)\) and \((\omega_1 = 1, \omega_2 = 1, \eta = 1)\), respectively.

2.4. Given below is the class of estimators suggested by Solanki and Singh [9]

\[
\hat{\gamma}_{SSS} = \left[ \omega_3 \hat{Y}_{st} - \frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})} \right]^{\omega_3} + \mu_8 \bar{Y}_{st} \exp \left\{ \frac{\omega_4(\omega_3(a_{st}\bar{X} + b_{st}) + (a_{st}\bar{X} + b_{st}))}{(\omega_3(a_{st}\bar{X} + b_{st}) + (a_{st}\bar{X} + b_{st}))} \right\},
\]

(2.10)

where \(\eta, a_{st}(\neq 0)\) and \(b_{st}\) are defined earlier.

Remark 2.2: \( \hat{\gamma}_{SSS} \) reduces the following different estimators by placing different values of \(\omega_3, \omega_4\) and \(\omega_8\) in (2.10) as:

(i) \( \hat{\gamma}_{SSSR} \) for \((\eta = 1, \omega_3 = 0, \omega_4 = 1)\).
(ii) \( \hat{\gamma}_{SSSP} \) for \((\eta = 1, \omega_3 = 0, \omega_4 = -1)\).
(iii) \( \hat{\gamma}_{SSSRP} \) for \((\eta = 1, \omega_3 = 1, \omega_4 = 1)\).
(iv) \( \hat{\gamma}_{SSSPP} \) for \((\eta = 1, \omega_3 = -1, \omega_4 = 1)\).
(v) \( \hat{\gamma}_{SSSPP} \) for \((\eta = 1, \omega_3 = 1, \omega_4 = 1)\).
(vi) \( \hat{\gamma}_{SSSPP} \) for \((\eta = 1, \omega_3 = -1, \omega_4 = 1)\).

2.5. Recently, Solanki and Singh [10] defined an improved estimation given as

\[
\hat{\gamma}_{SSS} = \left[ \mu_3 \bar{Y}_{st} - \frac{a_{st}\bar{X} + b_{st}}{\eta(a_{st}\bar{X} + b_{st}) + (1 - \eta)(a_{st}\bar{X} + b_{st})} \right]^{\omega_5} \exp \left\{ \frac{\omega_6(\omega_5(a_{st}\bar{X} + b_{st}) + (a_{st}\bar{X} + b_{st}))}{(\omega_5(a_{st}\bar{X} + b_{st}) + (a_{st}\bar{X} + b_{st}))} \right\},
\]

(2.11)

where \(\bar{X}_{st} = \frac{\sum_{h=1}^{l} W_h(a_{st}\bar{X}_h + b_{st})}{\sum_{h=1}^{l} W_h(a_{st}\bar{X}_h + b_{st})}\) and \(a_{st}(\neq 0), b_{st}\) are any real number to parameters related to auxiliary variate \(X\).

Remark 2.3: For obtaining different class of estimators \( \hat{\gamma}_{SSS} \), assume the different values of the constants \(\omega_5, \omega_6, \omega_7\) and \(\omega_8\). In Equation (2.11) as:

(i) \( \hat{\gamma}_{S S S 1} \) for \(\omega_5 = -1, \omega_6 = 1, \omega_7 = 1\) and \(\omega_8 = 1\).
(ii) \( \hat{\gamma}_{S S S 2} \) for \(\omega_5 = -1, \omega_6 = 1, \omega_7 = 1\) and \(\omega_8 = 0\).
(iii) \( \hat{\gamma}_{S S S 3} \) for \(\omega_5 = +1, \omega_6 = 1, \omega_7 = 1\) and \(\omega_8 = 1\).

Remark 2.4: The optimal weights of \(\mu_1, \mu_2, \mu_3, \ldots, \mu_{10}\) are determined for minimizing the MSE’s of estimators mentioned in (2.7)-(2.11).

\[
\mu_1 = \frac{V_{002}(2 - \delta^2 V_{002})}{2(-V_{101}^2 + V_{002}(1 + V_{200}))},
\]

\[
\mu_2 = \frac{\delta V_{002}(2 + \delta V_{101})}{2(X - V_{101}^2 + V_{002}(1 + V_{200}))},
\]

\[
\mu_3 = \frac{V_{002}(2 + \delta^2 V_{002})}{2(-V_{101}^2 + V_{002}(1 + 2\delta^2 V_{002} + V_{200}))},
\]

\[
\mu_4 = \frac{\delta^3 V_{002}^2 - 2 V_{101}^2(-1 + \delta V_{101}) + \delta V_{002}(-2 + \delta V_{101} + 2 V_{200})}{2\chi(-V_{101}^2 + V_{002}(1 + 2\delta^2 V_{002} + V_{200}))},
\]

\[
\mu_5 = \frac{A_2 A_4 - A_2 A_3}{A_2 A_1 - A_3^2}, \quad \mu_6 = \frac{A_1 A_5 - A_2 A_3}{A_2 A_1 - A_3^2},
\]

\[
\mu_7 = \frac{B_2 B_4 - B_2 B_3}{B_2 B_1 - B_3^2}, \quad \mu_8 = \frac{B_1 B_5 - B_2 B_3}{B_2 B_1 - B_3^2},
\]

\[
\mu_9 = \frac{C_2 C_4 - C_2 C_3}{C_2 C_1 - C_3^2}, \quad \mu_{10} = \frac{C_1 C_5 - C_4 C_3}{C_2 C_1 - C_3^2},
\]

where

\[
\tau = \frac{a_{st}\bar{X} + b_{st}}{\eta}, \quad \delta = \eta \tau,
\]

\[
A_1 = [1 + V_{200} + 4\eta\omega_1 \tau V_{101} + \omega_3(2\omega_2 - 1)\eta^2 \tau^2 V_{002}]
\]

\[
A_2 = [1 + V_{200} - 4\eta\omega_2 \tau V_{101} + \omega_2(2\omega_2 + 1)\eta^2 \tau^2 V_{002}]
\]

\[
A_3 = [1 + V_{200} + 2\eta(\omega_1 - \omega_2) \tau V_{101} + \omega_3(2\omega_2 - 1)\eta^2 \tau^2 V_{002}],
\]

\[
A_4 = [1 + \eta\omega_1 \tau V_{101} + \omega_3(2\omega_2 - 1)\eta^2 \tau^2 V_{002}],
\]

\[
A_5 = [1 + \eta\omega_2 \tau V_{101} + \omega_2(2\omega_2 + 1)\eta^2 \tau^2 V_{002}],
\]

\[
B_1 = [1 + V_{200} + \eta^2 \tau^2(2\omega_2^2 + \omega_3) V_{002} - 4\eta\omega_3 \tau V_{101}],
\]

\[
B_2 = [1 + V_{200} + \tau^2(\omega_2^2 + \omega_3) V_{002} - 2\omega_4 \tau V_{101}],
\]

\[
B_3 = \frac{(2\eta^2 \tau^2 + \omega_3^2) V_{002}}{2(2\eta^2 \tau^2 + \omega_3^2)}
\]

\[
\tau(2\eta\omega_3 + \omega_4) V_{101}
\]

\[
B_4 = \left[ 1 + \frac{\eta^2 \tau^2(\omega_2^2 + \omega_3) V_{002} - \omega_4 \tau V_{101}}{2} \right],
\]

\[
B_5 = \left[ 1 + \frac{\tau^2(\omega_2 + \omega_4) V_{002}}{2(\omega_2 + \omega_4)} \right],
\]

\[
C_1 = 1 + \frac{1}{\sqrt{2}} \sum_{h=1}^{l} W_h^2 \psi_h \left( S_{yh}^2 - 2k_1 a_{th} R^2 b_{yh} ight)
\]

\[
+ \frac{k_1(k_1 + 1)}{2} a_{th}^2 R^2 s_{yh}^2
\]
By placing the suitable weights in corresponding estimators, we have the following minimum MSE’s of above-said estimators.

\[ C_2 = 1 + \frac{1}{2\gamma^2} \sum_{h=1}^{l} W_{y^2}^{2} \left\{ S_{y^2}^{2} + 2\kappa^2 \alpha^2 \gamma^2 S_{X^2}^{2} \right\}, \]

\[ C_3 = 1 + \frac{1}{2\gamma^2} \sum_{h=1}^{l} W_{y^2}^{2} \left\{ S_{y^2}^{2} + (k_2 - k_2) \alpha^2 \gamma^2 S_{X^2}^{2} \right\}, \]

\[ C_4 = 1 - \frac{k_1}{2\gamma^2} \sum_{h=1}^{l} W_{y^2}^{2} \left\{ R^2 \alpha^2 \gamma^2 S_{X^2}^{2} - \frac{(k_1 + 2)}{4} \alpha^2 \gamma^2 S_{X^2}^{2} \right\}, \]

\[ C_5 = 1 + \frac{k_2}{2\gamma^2} \sum_{h=1}^{l} W_{y^2}^{2} \left\{ R^2 \alpha^2 \gamma^2 S_{X^2}^{2} + \frac{(k_2 - 2)}{4} \alpha^2 \gamma^2 S_{X^2}^{2} \right\}, \]

\[ k_1 = (2\omega_5 + \omega_6), \quad k_2 = (2\omega_7 + \omega_8), \]

\[ k_3 = \frac{S_{X^2}}{S_{y^2}}, \quad R^2 = \left( \frac{\bar{X}}{X^2} \right). \]

**Remark 2.5:** By placing the suitable weights in corresponding estimators, we have the following minimum MSE’s of above-said estimators.

\[ \text{MSE}(\hat{Y}_{st}) = \text{Var}(\hat{Y}_{st}) = \gamma^2 V_{200}, \quad (2.12) \]

\[ \text{MSE}(\hat{Y}_{CR}) = \gamma^2 (V_{200} + V_{202} - 2V_{101}), \quad (2.13) \]

\[ \text{MSE}(\hat{Y}_{CReg}) = \gamma^2 V_{200} (1 - \mu_{X^2}), \quad (2.14) \]

\[ \text{MSE}_{min}(\hat{Y}_{HS1}) \approx \gamma^2 \left[ \frac{4V_{101}^2 + V_{202}^2 [4\delta^2 V_{202}^2 - 4\delta^2 V_{101}^2 + 4(-1 + \delta^2 V_{200}^2)]}{4(V_{101}^2 - V_{202}^2 + V_{200}^2)} \right], \quad (2.15) \]

\[ \text{MSE}_{min}(\hat{Y}_{HS2}) \approx \gamma^2 \left[ \frac{4V_{101}^2 + V_{202}^2 [4\delta^2 V_{202}^2 - 4\delta^2 V_{101}^2 + 4(-1 + \delta^2 V_{200}^2)]}{4(V_{101}^2 - V_{202}^2 + V_{200}^2)} \right], \quad (2.16) \]

\[ \text{MSE}_{min}(\hat{Y}_{SS1}) \approx \gamma^2 \left[ 1 - \frac{A_2 A_4 - 2A_4 A_4 A_5 + A_5^2 A_1}{A_2 A_1 - A_3^2} \right], \quad (2.17) \]

\[ \text{MSE}_{min}(\hat{Y}_{SS2}) \approx \gamma^2 \left[ 1 - \frac{B_2 B_4 - 2B_4 B_5 B_8 + B_8^2 B_1}{B_2 B_1 - B_3^2} \right], \quad (2.18) \]

\[ \text{MSE}_{min}(\hat{Y}_{SS3}) \approx \gamma^2 \left[ 1 - \frac{C_2 C_4 - 2C_4 C_5 C_5 + C_5^2 C_2}{C_2 C_1 - C_3^2} \right]. \quad (2.19) \]

### 3. Proposed estimators

In this section, two new exponential-type estimators are proposed for the estimation of population mean using dual auxiliary information in stratified random sampling. Dual auxiliary information refers to the double use of auxiliary variable (i) the original/actual measurements of the auxiliary variable and (ii) the use of ranks of the auxiliary variable. Mathematical properties such as bias and mean square error (MSE) of the proposed estimators are derived up to first order of approximation. The bias of an estimator is the difference between the estimator’s expected value and the true value of the parameter being estimated i.e. \( \text{Bias}(\hat{Y}) = E(\hat{Y} - \bar{Y}) \) and MSE can be defined as the divergence of the estimator values from the true parameter value i.e. \( \text{MSE}(\hat{Y}) = E(\hat{Y} - \bar{Y})^2 \).

#### 3.1. First proposed estimator

\[ \hat{Y}_{p1} = \frac{\mu_{11}}{2} \left( \frac{\bar{X} + \bar{Y}_{st}}{\bar{X}} \right) \hat{Y}_{Ist, Avg} + \mu_{12}(\bar{Z} - \bar{Z}_{st}) \]

\[ + \mu_{13}(\bar{X} - \bar{X}_{st}) \exp \left( \frac{\bar{X} - \bar{X}_{st}}{\bar{X} + \bar{X}_{st}} \right), \quad (3.1) \]

where \( \mu_{11}, \mu_{12} \) and \( \mu_{13} \) are the suitably chosen weights. The bias and MSE of \( \hat{Y}_{p1} \) are given below

\[ \text{Bias}(\hat{Y}_{p1}) \approx (\mu_{11} - 1) \hat{Y} + \frac{V_{202}}{2} \left( \mu_{13} \bar{X} + \frac{5\mu_{11} \bar{Y}}{4} \right), \quad (3.2) \]

and

\[ \text{MSE}(\hat{Y}_{p1}) \approx \gamma^2 \left[ 1 + \mu_{11}^2 \left( 1 + V_{202} + \frac{5}{4} V_{200} \right) \right] + \mu_{12}^2 V_{200} + \mu_{13}^2 V_{200}, \]

\[ - \mu_{11}^2 \left( 2 + \frac{5}{4} V_{200} \right) - 2 \mu_{12} R \left( \mu_{13} V_{011} - \mu_{11} V_{110} \right) - \mu_{13} V_{200} \left( 2\bar{X} - \mu_{11} + 1 \right), \quad (3.3) \]

where \( \kappa = \rho_{yx} \frac{C_{st}}{C_{ist}}, \quad \bar{R} = \frac{\bar{X}}{X} \) and \( \tilde{R} = \frac{\bar{X}}{X} \).

The optimal weights \( \mu_{11}, \mu_{12} \) and \( \mu_{13} \) are obtained by minimizing Equation (3.3), so

\[ \mu_{11(\text{opt})} = \frac{E_1 E_2 - 2V_{002} V_{200} E_3}{E_2 E_4 - 2E_3^2}, \]

\[ \mu_{12(\text{opt})} = \frac{2V_{101} E_1 E_2 - 2V_{002} V_{200} E_3}{V_{011} E_1 E_3 - V_{002} V_{200} E_4} \]

\[ + \frac{2V_{101} E_1 E_2 - 2V_{002} V_{200} E_3}{2V_{011} E_1 E_3 - V_{002} V_{200} E_4} \]

\[ \mu_{13(\text{opt})} = \frac{V_{002} V_{200} E_4 - E_1 E_3}{2R(E_2 E_4 - 2E_3^2)}. \]

Inserting optimal weights of \( \mu_{11}, \mu_{12} \) and \( \mu_{13} \) in Equation (3.3), the minimum MSE of the proposed
The estimator is

\[ MSE_{\text{min}}(\hat{Y}_{P1(st)}) = \frac{\bar{V}^2}{4V_{020}F_1^2} \left( 4V_{020}F_1^2 + (4V_{020} + 4V_{020}V_{200} - 4V_{110} + 5V_{002}V_{020})F_2^2 + (V_{002}V_{020} - V_{011})F_3^2 \right) - (2(\bar{X}_{1010}V_{002} - V_{020}V_{002})F_1F_2 \cdot 2V_{002}V_{020}F_1F_3 \cdot \left( 2 \kappa' - 1 \right) \right) F_2F_3 - 2V_{002}V_{020}F_1F_3 \],

where

\begin{align*}
E_1 &= 8V_{020} + 5V_{002}V_{020}, \\
E_2 &= V_{002}V_{020} - V_{011}, \\
E_3 &= 2V_{110}V_{011} - V_{020}V_{002}(2\kappa' - 1), \\
E_4 &= 8V_{020}(1 + V_{200}) + 10V_{002}V_{020} - 8V_{110}F_1F_4 - 2E_2^3, \\
F_1 &= E_2E_4 - 2E_2^2, \\
F_2 &= E_1E_2 - 2V_{002}V_{020}E_3, \\
F_3 &= V_{002}V_{020}E_4 - E_1E_3
\end{align*}

(3.4)

### 3.2. Second proposed estimator

\[ \hat{Y}_{P2} = \mu_{14}\bar{Y}_{st} + \mu_{15}(\bar{Z} - \bar{Z}_{st}) + \mu_{16}(\bar{X} - \bar{X}_{st})\exp \left( \frac{2(\bar{X} - \bar{X}_{st})}{\bar{X} + \bar{X}_{st}} \right), \]

(3.5)

where \( \mu_{14}, \mu_{15} \) and \( \mu_{16} \) are the suitably chosen weights. The bias and MSE of \( \hat{Y}_{P2} \) are given below

\[ \text{Bias}(\hat{Y}_{P2}) \approx \bar{V} \left[ (\mu_{14} - 1) + \mu_{16}V_{002}\bar{R} \right], \]

and

\[ MSE(\hat{Y}_{P2}) \approx \bar{V}^2 \left[ 1 + \mu_{14}^2(1 + V_{200}) + \mu_{15}^2\bar{R}^2V_{020} + \mu_{16}^2\bar{R}^2V_{002} - 2\mu_{14} + 2\mu_{16}\bar{R}V_{002} - 2\mu_{14}V_{110}\bar{R}V_{002}(\kappa' - 1) + \mu_{15}V_{110}\bar{R}V_{011} - 2\mu_{14}\mu_{15}\bar{R}V_{110} \right]. \]

(3.6)

By minimizing Equation (3.7), the optimal weights \( \mu_{14}, \mu_{15} \) and \( \mu_{16} \) are as under:

\begin{align*}
\mu_{14(\text{opt})} &= \frac{(1 + V_{200})V_{020}E_5 - E_6(V_{011} + V_{011}V_{200} + V_{110}E_2)}{(1 + V_{200})E_5E_6 - E_6^2}, \\
\mu_{15(\text{opt})} &= \frac{V_{110}E_5 - E_6E_7}{R(E_5E_6 - E_6^2)}
\end{align*}

Inserting optimal weights of \( \mu_{14}, \mu_{15} \) and \( \mu_{16} \) in Equation (3.7), the minimum MSE of the proposed estimator is

\[ MSE_{\text{min}}(\hat{Y}_{P2(st)}) \approx \frac{\bar{V}^2}{(1 + V_{200})^2} \left[ (1 + V_{200})F_2^2 + V_{002}(1 + V_{200})F_2^2 + V_{020}(1 + V_{200})F_5^2 \right]
\]

\[ - 2(V_{002}F_4 - V_{011}F_6)(1 + V_{200})F_5 \]

\[ - 2 \left( V_{110}F_5 + V_{002}(\kappa' - 1)F_5 + F_4 \right) F_7 \]

(3.8)

where

\begin{align*}
E_5 &= V_{002}(1 + V_{200}) - V_{002}(\kappa' - 1)^2, \\
E_6 &= V_{011}(1 + V_{200}) - V_{110}V_{002}(\kappa' - 1), \\
E_7 &= V_{002}(1 + V_{200}) + V_{002}(\kappa' - 1), \\
E_8 &= V_{020}(1 + V_{200}) - V_{110}, \\
F_4 &= E_5E_8 - E_6, \\
F_5 &= E_7E_8 - V_{110}E_6, \\
F_6 &= V_{110}E_5 - E_6E_7, \\
F_7 &= V_{020}(1 + V_{200})E_5 - V_{011}(1 + V_{200})E_6 - V_{110}E_5 - V_{110}E_7 + V_{002}(\kappa' - 1)E_7E_8.
\end{align*}

### 4. Application on a real data

In this section, we compare the performance of newly proposed estimators with the traditional unbiased, combined ratio and combined regression estimators and existing estimators, i.e. Haq and Shabbir [7], Singh and Solanki [8] and Solanki and Singh [10,11]. We considered a real-life data set of Turkey (2007) used by Koyuncu and Kadilar [3]. For the remaining characteristics of the data set, interested readers may refer to Koyuncu and Kadilar [3]. Necessary data statistics are given in Table 1.

We calculated the MSEs of the proposed and competing exponential-type estimators and are presented in Table A1. Table A1 reveals that the proposed estimators have smaller MSE values i.e. (57.0590 and 67.9338) among all the reviewed exponential-type estimators i.e. \( \hat{Y}_{H51}, \hat{Y}_{H52}, \hat{Y}_{S110}, \hat{Y}_{S112}, \hat{Y}_{S11R}, \hat{Y}_{S22R}, \hat{Y}_{SS2R}, \hat{Y}_{SS2PR}, \hat{Y}_{SS2RR}, \hat{Y}_{SS2RP}, \hat{Y}_{SS31}, \hat{Y}_{SS32} \) and \( \hat{Y}_{SS33} \).

### 5. Simulation study based on real data

In the previous section, it is clearly observed that proposed estimators are efficient over the competing estimators. In addition, this superiority is assessed through a Monte Carlo simulation study using R software. Again, the real population presented in Table 1 is used. We considered different sample sizes (n = 180, 230 and 280).
through the proportional allocation method. The steps of a simulation study to find the average MSE of an estimator are as follows:

**Step 1:** Select a bivariate stratified sample of size \(n\) using SRSWOR from the bivariate stratified population.

**Step 2:** Use sample data from step 1 to find the MSE of all the estimators under study.

**Step 3:** The whole procedure is repeated 30,000 times and obtain 30,000 values i.e. \(\bar{y}\) for MSEs.

**Step 4:** Average MSE of each estimator is calculated as: \[
\text{MSE} = \frac{1}{30000} \sum_{i=1}^{30000} (\bar{y}_i - \bar{y})^2.
\]

Tables A2–A4 present the minimum mean square errors provided by the simulation study. It is quite obvious, as in the previous section, that the proposed estimators \(\hat{Y}_1\) and \(\hat{Y}_2\) have the least MSEs over all the competing estimators under study in different sample sizes i.e. \(n = 180, 230\) and 280. The sequel of the above findings, the performance of the proposed estimators \(\hat{Y}_1\) and \(\hat{Y}_2\) is the best among all the reviewed estimators under study.

6. Concluding remarks

Several estimators for the estimation of finite population mean based only on original auxiliary information under stratified random sampling are available in the literature. Haq et al. [18] built up a family of estimators for evaluating the population mean under simple random sampling scheme by using additional information of the auxiliary variable called ranked auxiliary variable. First time in this manuscript, new optimal estimators are suggested for the estimation of population mean by using the original and the ranked auxiliary information under a stratified random sampling scheme. Mathematical properties such as bias, mean square error (MSE) and minimum MSE of the proposed estimators are derived up to the first degree of approximation. Both real-life applications and simulation studies are performed to access the potentiality of the proposed estimators over the competitors. Numerical findings confirmed that the proposed estimators have the minimum mean square errors than all the other existing estimators such as usual unbiased, combined ratio, combined regression, Haq and Shabbir [7], Singh and Solanki [8] and Solanki and Singh [10,11]. Therefore, new proposed estimators under stratified random sampling are very attractive to the survey statisticians.

The possible extension of this current work to estimate the: (1) finite population mean under other sampling designs like stratified double sampling and different rank set sampling schemes, etc.; (2) other unknown finite population parameters including median, variance, interquartile range and proportions, etc.; (3) population mean of a sensitive variable in the presence of sensitive and non-sensitive auxiliary information.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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## Appendix

### Table A1. Minimum MSEs of different estimators based on a real population.

| $a_0$ | $b_{10}$ | $\hat{Y}_{H1}$ | $\hat{Y}_{H2}$ | $\hat{Y}_{SS1}$ | $\hat{Y}_{SS1P}$ | $\hat{Y}_{SS1PP}$ | $\hat{Y}_{SS2}$ | $\hat{Y}_{SS2P}$ |
|-------|---------|----------------|----------------|-----------------|-----------------|-----------------|----------------|----------------|
| 1     | 0       | 183.482        | 115.870        | 193.418         | 175.925         | 192.847         | 194.156        | 185.521        |
| 1     | $\sum_{h=1}^{L} W_h p_h$ | 183.485        | 115.895        | 193.419         | 175.925         | 192.848         | 194.156        | 185.520         |
| 1     | $\sum_{h=1}^{L} W_h c_{ah}$ | 183.488        | 115.914        | 193.420         | 175.925         | 192.849         | 194.156        | 185.519         |
| 1     | $\sum_{h=1}^{L} W_h \beta_{2ah}$ | 183.536        | 116.292        | 193.432         | 175.925         | 192.875         | 194.152        | 185.509         |
| 1     | $\sum_{h=1}^{L} W_h \beta_{1ah}$ | 183.494        | 115.964        | 193.421         | 175.925         | 192.853         | 194.155        | 185.518         |
| $\sum_{h=1}^{L} W_h c_{ah}$ | $\sum_{h=1}^{L} W_h \beta_{2ah}$ | 183.514        | 116.118        | 193.426         | 175.925         | 192.863         | 194.154        | 185.514         |
| $\sum_{h=1}^{L} W_h \beta_{1ah}$ | $\sum_{h=1}^{L} W_h c_{ah}$ | 183.484        | 115.882        | 193.418         | 175.925         | 192.848         | 194.156        | 185.521         |
Table A2. Minimum MSEs of different estimators based on a simulation study ($n = 180$).

| $a_{\alpha}$ | $b_{\beta}$ | $\bar{\hat{Y}}_{H1}$ | $\bar{\hat{Y}}_{H2}$ | $\bar{\hat{Y}}_{H3}$ | $\bar{\hat{Y}}_{H4}$ | $\bar{\hat{Y}}_{H5}$ | $\bar{\hat{Y}}_{H6}$ | $\bar{\hat{Y}}_{H7}$ | $\bar{\hat{Y}}_{H8}$ |
|-------------|-------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 1           | 0           | 172.008             | 104.345             | 181.844             | 163.382             | 181.228             | 182.417             | 173.304             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 171.478             | 103.978             | 181.294             | 162.897             | 180.683             | 181.871             | 172.900             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 172.610             | 104.599             | 182.500             | 163.926             | 181.885             | 183.076             | 173.070             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 171.054             | 104.128             | 180.793             | 162.449             | 180.194             | 181.350             | 172.295             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 171.047             | 103.276             | 180.894             | 162.377             | 180.282             | 181.465             | 172.325             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 172.405             | 104.791             | 182.243             | 163.748             | 181.635             | 182.811             | 173.681             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 172.502             | 104.963             | 182.322             | 163.991             | 181.701             | 182.909             | 173.861             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 172.031             | 104.454             | 181.851             | 163.484             | 181.230             | 182.433             | 173.367             |
| $a_{\alpha}$ | $b_{\beta}$ | $\bar{\hat{Y}}_{SSR}$ | $\bar{\hat{Y}}_{SSRP}$ | $\bar{\hat{Y}}_{SSRP}$ | $\bar{\hat{Y}}_{SSRP}$ | $\bar{\hat{Y}}_{SSSP}$ | $\bar{\hat{Y}}_{SSSP}$ | $\bar{\hat{Y}}_{SSSP}$ | $\bar{\hat{Y}}_{SSSP}$ |
| 1           | 0           | 182.084             | 130.465             | 181.559             | 179.980             | 94.757              | 130.465             | 160.744             | 52.425             | 63.241             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 181.534             | 130.064             | 181.011             | 179.447             | 94.430              | 130.059             | 160.228             | 52.425             | 63.241             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 182.741             | 130.814             | 182.216             | 180.624             | 94.870              | 130.805             | 161.282             | 52.425             | 63.241             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 181.028             | 129.823             | 180.516             | 178.921             | 94.240              | 129.733             | 159.780             | 52.425             | 63.241             |
| 1           | $\sum_{h=1}^{L} W_h \rho_h$ | 181.133             | 129.374             | 180.611             | 179.022             | 93.526              | 129.354             | 159.726             | 52.425             | 63.241             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 182.481             | 130.814             | 181.961             | 180.366             | 94.973              | 130.761             | 161.075             | 52.425             | 63.241             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 182.565             | 131.214             | 182.034             | 180.498             | 95.642              | 131.212             | 161.205             | 52.425             | 63.241             |
| $\sum_{h=1}^{L} W_h \rho_h$ | $\sum_{h=1}^{L} W_h \rho_h$ | 182.094             | 130.0               | 181.563             | 180.015             | 95.064              | 130.668             | 160.734             | 52.425             | 63.241             |
Table A3. Minimum MSEs of different estimators based on a simulation study ($n = 230$).

| $a_x$ | $b_x$ | $\bar{y}_{f1}$ | $\bar{y}_{f2}$ | $\bar{y}_{s1}$ | $\bar{y}_{s2}$ | $\bar{y}_{s31}$ | $\bar{y}_{s32}$ | $\bar{y}_{s33}$ | $\bar{y}_{p1}$ | $\bar{y}_{p2}$ |
|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1     | 0     | 128.404        | 93.426         | 133.472        | 124.181        | 133.169        | 133.784        | 129.180        |                  |                |
| 1     | $\sum_{h=1}^{L} W_h f_{h}$ | 128.481 | 93.299 | 133.575 | 124.241 | 133.268 | 133.889 | 129.264 |                  |                |
| 1     | $\sum_{h=1}^{L} W_h C_{h}$ | 127.542 | 92.471 | 132.617 | 123.297 | 132.311 | 132.927 | 128.310 |                  |                |
| 1     | $\sum_{h=1}^{L} W_h f_{z2h}$ | 127.622 | 92.729 | 132.682 | 123.301 | 132.387 | 132.976 | 128.328 |                  |                |
| 1     | $\sum_{h=1}^{L} W_h f_{1h}$ | 127.589 | 92.586 | 132.658 | 123.338 | 132.355 | 132.966 | 128.348 |                  |                |
| $\sum_{h=1}^{L} W_h C_{h}$ | $\sum_{h=1}^{L} W_h f_{z2h}$ | 128.157 | 93.117 | 133.235 | 123.888 | 132.934 | 133.541 | 128.910 |                  |                |
| $\sum_{h=1}^{L} W_h f_{1h}$ | $\sum_{h=1}^{L} W_h C_{h}$ | 128.108 | 92.972 | 133.199 | 123.843 | 132.895 | 133.508 | 128.873 |                  |                |
| $\sum_{h=1}^{L} W_h f_{z2h}$ | $\sum_{h=1}^{L} W_h f_{1h}$ | 128.172 | 93.044 | 133.259 | 123.915 | 132.955 | 133.569 | 128.940 |                  |                |

| $a_x$ | $b_x$ | $\bar{y}_{s222}$ | $\bar{y}_{s22p}$ | $\bar{y}_{s22P}$ | $\bar{y}_{s31}$ | $\bar{y}_{s32}$ | $\bar{y}_{s33}$ | $\bar{y}_{p1}$ | $\bar{y}_{p2}$ |
|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1     | 0     | 133.592        | 107.684        | 133.332        | 132.562        | 89.970        | 107.684        | 122.608        | 50.152        | 55.855        |
| 1     | $\sum_{h=1}^{L} W_h f_{h}$ | 133.696 | 107.656 | 133.432 | 132.663 | 89.842 | 107.653 | 122.630 | 50.152 | 55.855 |
| 1     | $\sum_{h=1}^{L} W_h C_{h}$ | 132.737 | 106.751 | 132.475 | 131.702 | 88.977 | 107.477 | 121.706 | 50.152 | 55.855 |
| 1     | $\sum_{h=1}^{L} W_h f_{z2h}$ | 132.798 | 152 | 132.545 | 131.731 | 88.894 | 107.707 | 121.801 | 50.152 | 55.855 |
| 1     | $\sum_{h=1}^{L} W_h f_{1h}$ | 132.778 | 106.810 | 132.517 | 131.738 | 89.043 | 106.800 | 121.769 | 50.152 | 55.855 |
| $\sum_{h=1}^{L} W_h C_{h}$ | $\sum_{h=1}^{L} W_h f_{z2h}$ | 133.354 | 107.328 | 133.095 | 132.308 | 89.493 | 107.302 | 122.309 | 50.152 | 55.855 |
| $\sum_{h=1}^{L} W_h f_{1h}$ | $\sum_{h=1}^{L} W_h C_{h}$ | 133.318 | 107.248 | 133.058 | 132.276 | 89.429 | 107.247 | 122.292 | 50.152 | 55.855 |
| $\sum_{h=1}^{L} W_h f_{z2h}$ | $\sum_{h=1}^{L} W_h f_{1h}$ | 133.379 | 107.335 | 133.118 | 132.340 | 89.532 | 107.334 | 122.350 | 50.152 | 55.855 |
Table A4. Minimum MSES of different estimators based on a simulation study \((n = 280)\).

| \(a_{\alpha} \) | \(b_{\alpha} \) | \(\hat{Y}_{151} \) | \(\hat{Y}_{152} \) | \(\hat{Y}_{153} \) | \(\hat{Y}_{154} \) | \(\hat{Y}_{155} \) | \(\hat{Y}_{156} \) | \(\hat{Y}_{157} \) | \(\hat{Y}_{158} \) |
|-----------------|-----------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1               | 0               | 99.582        | 79.330        | 102.510       | 97.205        | 102.340       | 102.693       | 100.059       |
| \(\sum_{h=1}^{L} W_{h} \rho_{h} \) | 99.737         | 79.475        | 102.668       | 97.347        | 102.499       | 102.850       | 100.208       |
| 1               | \(\sum_{h=1}^{L} W_{h} C_{h} \) | 99.387        | 79.074        | 102.323       | 96.990        | 102.154       | 102.505       | 99.857        |
| 1               | \(\sum_{h=1}^{L} W_{h} \beta_{1h} \) | 100.067       | 79.711        | 103.011       | 97.679        | 102.840       | 103.195       | 100.548       |
| \(\sum_{h=1}^{L} W_{h} \beta_{1h} \) | 99.407         | 79.147        | 102.337       | 97.014        | 102.168       | 102.518       | 99.875        |
| 1               | \(\sum_{h=1}^{L} W_{h} \beta_{1h} \) | 102.735       | 87.946        | 102.241       | 101.996       | 77.804        | 87.820        | 96.240        | 46.135        | 49.474        |
| \(\sum_{h=1}^{L} W_{h} \rho_{h} \) | 102.390        | 87.566        | 102.244       | 101.804       | 77.503        | 87.568        | 96.034        | 46.135        | 49.474        |
| 1               | \(\sum_{h=1}^{L} W_{h} \beta_{2h} \) | 102.762        | 87.899        | 102.618       | 102.167       | 77.771        | 87.873        | 96.370        | 46.135        | 49.474        |
| \(\sum_{h=1}^{L} W_{h} \beta_{2h} \) | 102.896        | 88.013        | 102.751       | 102.302       | 77.911        | 88.007        | 96.535        | 46.135        | 49.474        |
| 1               | \(\sum_{h=1}^{L} W_{h} \beta_{1h} \) | 102.625        | 87.898        | 102.479       | 102.044       | 77.874        | 87.883        | 96.274        | 46.135        | 49.474        |
| \(\sum_{h=1}^{L} W_{h} \beta_{2h} \) | 103.079        | 88.244        | 102.931       | 102.495       | 78.173        | 88.243        | 96.708        | 46.135        | 49.474        |
| 1               | \(\sum_{h=1}^{L} W_{h} \beta_{1h} \) | 102.404        | 87.611        | 102.258       | 101.817       | 77.578        | 87.611        | 96.072        | 46.135        | 49.474        |