Dark soliton oscillations in Bose–Einstein condensates with multi-body interactions

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Abstract

We consider the dynamics of dark matter wave solitons moving through non-uniform cigar-shaped Bose–Einstein condensates described by the mean field Gross–Pitaevskii equation with generalized nonlinearities, in the case when the condition for the modulation stability of the Bose–Einstein condensate is fulfilled. The analytical expression for the frequency of the oscillations of a deep dark soliton is derived for nonlinearities which are arbitrary functions of the density, while specific results are discussed for the physically relevant case of a cubic–quintic nonlinearity modelling two- and three-body interactions, respectively. Opposite to the usual (cubic) Gross–Pitaevskii equation for which the dark soliton effective mass is known to be constant (equal to 2), in the presence of a cubic–quintic nonlinearity we find that the effective mass depends on the product of the initial density background and the ratio between the coefficient of quintic and cubic nonlinearities, this leading to the interesting possibility of measuring three-body interactions directly from the dark soliton dynamics. A comparison between analytical results and direct numerical simulations of the cubic–quintic Gross–Pitaevskii equation shows good agreement between them which confirms the validity of our approach.

1. Introduction

Since the first experimental realization of Bose–Einstein condensates (BECs) of diluted atomic gases, the dynamics of matter waves in different trapping potentials has attracted the attention of many researchers and presently is a field of growing interest \[1, 2\]. The dispersive properties of BEC induced by the quantum properties of a slow atomic motion, on one side, and the intrinsic nonlinearity of the system induced by the interatomic interactions, on the other side, make BECs ideal systems for exploring competing effects of dispersion and nonlinearity. This results in a variety of spectacular phenomena ranging from the existences of matter wave vortices \[3\] and solitons \[4–6\] to dispersive shocks \[7, 8\].

In the standard mean-field approximation \[1\] the BEC dynamics is governed by the Gross–Pitaevskii (GP) equation

\[i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\mathbf{r})\psi + f(|\psi|^2)\psi\]  

(1)

for the condensate ‘wavefunction’ \(\psi\), where \(V(\mathbf{r})\) is the potential of the external forces trapping the condensate. In the standard GP approximation

\[f(|\psi|^2) = g|\psi|^2,\]  

(2)

where \(g = 4\pi\hbar^2a_s/m\) is the nonlinear interaction parameter expressed in terms of the \(s\)-wave scattering length \(a_s\) of atoms with mass \(m\). In this approximation, only two-body collisions are taken into account. The scattering length \(a_s\), however, can be varied and even made equal to zero by means of Feshbach resonances. Then contribution of multi-particle (three-body) collisions becomes comparable with the two-body contribution and a more complicated form of the nonlinear term, such as

\[f(|\psi|^2) = g|\psi|^2 + h|\psi|^4,\]  

(3)

arises in the GP equation. Equation (1) was also considered in \[9\] for the case of \(f(\rho) = \rho^\alpha\) as a model on BEC with attractive \(N\)-body \((N = \alpha + 1)\) interactions and in \[10\] for the case of a discrete version of it which support travelling solitons for arbitrarily high order nonlinearities.
Since the three-dimensional GP equation (1) is too complicated for analytical studies, in the following we restrict ourselves to experimental low-dimensional settings, e.g., we consider cigar-shaped traps with a strong radial confinement, for which the radial motion of the BEC is ‘frozen’ and equation (1) reduces to an effective 1D generalized nonlinear Schrödinger (NLS) equation [11],

$$i\psi_t = -\frac{1}{2}\psi_{xx} + V(x)\psi + f(|\psi|^2)\psi,$$  \hspace{1cm} (4)

(here standard non-dimensional variables have been introduced, holding the previous notation for convenience). In this reduction, the function $f(\rho) = |\psi|^2$, preserves its form (3) up to a renormalization of the constants $g$ and $h$. Also note that the conservation of the norm of the condensate wavefunction (normalized number of atoms) is guaranteed for arbitrary functions $f(\rho)$. More complicated form of the function $f(\rho)$ can arise if we take into account the radial motion of BEC (see, e.g., [12, 13]). Thus, we arrive at the problem of 1D dynamics of BEC with generalized form of the nonlinearity $f(\rho)$ and confined in the axial direction $x$ by the trap $V(x)$.

As is well known, in the case of $f(\rho)$ given by equation (2) with $g > 0$ (repulsive two-body interaction) and under the assumption of negligible changes of the potential $V(x)$ in the region of $x$ under consideration, the NLS equation (2), (4), has a dark soliton solution corresponding to the motion of a hole in the density distribution function, $\rho(x - v t)$, along a uniform distribution $\rho = \rho_0$ with constant velocity $v$ which depends on the depth of the soliton [14]. If the potential $V(x)$ is a slowly varying function at distances of the order of the soliton’s width, then the coordinate of the soliton is well defined and we can speak about value of the potential $V(x)$ at the ‘point’ $x$ of the soliton position. In this case we can formulate the problem of motion of a dark soliton ‘in the potential’ $V(x)$ in terms of its collective coordinate. This problem has been addressed in several papers (see, e.g., [15–21]) where it was shown that the dynamics of dark solitons is quite nontrivial. In particular, if a BEC described by the standard GP equation (2) is confined in a harmonic axial trap,

$$V(x) = \frac{1}{2}\omega_0 x^2,$$  \hspace{1cm} (5)

then the dark soliton oscillates inside the condensate with the frequency [15]

$$\omega = \frac{\omega_0}{\sqrt{2}}.$$  \hspace{1cm} (6)

This result is quite surprising because apparently it contradicts to a simple consequence of the Ehrenfest theorem which says that the centre of mass of the whole BEC always oscillates with the trap frequency $\omega_0$. The remarkable theoretical prediction (6) was confirmed in the experiment [22]. Equation (6) then implies that the motion of the dark soliton must be accompanied by a deformation of the density distribution which re-tunes the oscillation of the whole condensate to the trap frequency. In this situation the motion of the dark soliton becomes ‘decoupled’ from the motion of the centre of mass and the Ehrenfest theorem is not violated.

However, the result (6) is not universal. If the nonlinearity function is given by

$$f(\rho) = h\rho^2$$  \hspace{1cm} (7)

with $h > 0$, then the frequency $\omega$ of oscillations of a shallow dark soliton coincides with the trap frequency [16],

$$\omega = \omega_0$$  \hspace{1cm} (8)

and for deep solitons this frequency is given by a more complicated expression (see [18])

$$\omega = \left[1 + \frac{3}{2}\ln \frac{1 + h}{h}\right]^{-1/2} \omega_0 \approx 0.6572\omega_0.$$  \hspace{1cm} (9)

This observation poses the problem of the dark soliton motion in the case of a more general form of the nonlinearity.

The aim of the present paper is to address this problem by considering a non-uniform cigar-shaped BEC with nonlinear interactions modelled by an arbitrary function $f(\rho)$ of the density. In particular, we consider the physically relevant example of the cubic–quintic nonlinearity function in (3), modelling the mean field effect of the two-body and three-body collisions in the GP equation. In this respect it is worth noting that for appropriate values of density and scattering lengths, the three-body collisions could have an appreciable contribution even in a very dilute regime. This occurs when the so-called Efimov effect [23] becomes possible and the two-body scattering length becomes larger than the effective two-body interaction radius, this usually occurring near a two-body resonance. In this case a very large number of three-body bound states (so-called Efimov states) can be formed and the contribution of the three-body elastic collisions to the density energy may become comparable to the one arising from two-body interactions. In this situation the condensate may appear extremely dilute with respect to the two-body collisions but somewhat dense with respect to the three-body collisions [24]. The phenomena display a universal behaviour which is independent on the sign of the two-body scattering length in the sense that three-body Efimov states may form even if weakly bound dimer states cannot form due to a repulsive two-body interaction. Mostly important, one can change the strength and the sign of the three-body interaction by controlling the strength of the two-body interaction via a Feshbach resonance. When the two-body scattering length is tuned so to be just before (just after) the value for a three-body Efimov bound state formation, the three-body coefficient $h$ assumes large negative (large positive) values in perfect analogy with the behaviour of the two-body scattering amplitude near a Feshbach resonance [24–26]. Thus, using the bound state energy of an Efimov state it is possible in principle to change the sign of $h$ while keeping the sign of $g$ fixed. Although theoretical investigations on Efimov states have been largely developed in the past four decades, only very recently Efimov states have been experimentally observed both in the regime where weakly bound dimer states are absent [27, 28] and in the regime where they are present [29]. Theoretical estimates of the three-body scattering length associated with the quintic nonlinearity in the GPE were provided in [25, 26, 30, 31] (for rubidium atom three-body interactions are expected to be attractive [32]).
We also remark that together with an elastic three-body scattering there is also an inelastic contribution which becomes relevant at the threshold of the weakly three-body bound state formation. These dissipative effects, however, are expected to become negligible as the detuning from the exact three-body bound state resonance is increased. In the following we shall assume this condition always satisfied. In this case we find that the relative contribution of the two nonlinear terms to the soliton oscillation depends on the background density $\rho_0$: if a deep soliton oscillates in vicinity of the maximum $\rho_0$ of the density distribution, then for $|h|\rho_0 \ll g$ its frequency must be equal to (6), and for $|h|\rho_0 \gg g$ to (9). The transition from one limiting case to another is of considerable theoretical interest. We derive an analytical expression for the oscillation frequency of a deep dark soliton which reveals the explicit dependence of the oscillations frequency both on the background density and on the strength of the nonlinearity parameters. Our formulae reproduce the results considered in [17, 18] for the particular cases (2) and (7) of the nonlinearity for which the above dependence disappears.

We also consider the case of a quintic attractive nonlinearity ($h < 0$) with an overall repulsive interaction. In this case for the density $\rho > g/(2|h|)$ we get modulation instability of BEC, i.e., the sound velocity

$$c_s = \sqrt{\rho_0 f'(\rho_0)}$$

(10)

becomes imaginary so that the notion of ‘dark solitons’ loses its sense. We show that in this case the oscillation frequency $\omega$ as a function of $\rho_0$ becomes singular at $\rho_0 = g/(2|h|)$. The form of this singularity may provide further advances in the understanding of the dynamics of dark solitons in BEC.

Our theory is based on the elegant approach of [17, 18] where the dark soliton was considered as a ‘Landau quasi-particle’ whose motion in the stationary potential $V(x)$ is governed by the quasi-particle’s ‘energy conservation’ law in close analogy with the Newtonian particle motion. The generalization of this approach to arbitrary $f(\rho)$ allows us to derive the equation of motion of the dark soliton for the general case and to show the dependence of the soliton effective mass on the system parameters.

A comparison between analytical results and direct numerical simulations of the cubic–quintic GP equation shows a very good agreement which further confirms the validity of our analysis. In the case of the cubic–quintic interaction in (3) we find, quite remarkably, that independently on the sign of $h$ the effective mass of the dark soliton (e.g., the ratio of its oscillation frequency and the parabolic trap frequency) depends only on $|h|\rho_0/g$, where $\rho_0$ is the density of the soliton background. Since $g$ and $\rho_0$ can be measured very precisely we arrive at the conclusion that our theoretical study provides an interesting method to measure the three-body interaction by means of the measurement of the oscillation frequency of a dark soliton in a parabolic trap.

The paper is organized as follows. In section 2, we discuss the general theory of the dynamics of a dark soliton of the NLS equation with an arbitrary nonlinearity. The results will be used to study the cases of both deep and shallow soliton dynamics for arbitrary nonlinearity. In section 3, we consider oscillations of dark solitons of the GP equation with the cubic–quintic nonlinearity specific for BEC with two- and three-body interactions. The case of a repulsive cubic nonlinearity ($g > 0$) with repulsive ($h > 0$) and attractive ($h < 0$) three-body interactions will be considered in sections 3.1, 3.2, respectively, the last being done for parameters satisfying the condition of modulation stability of BEC. For both cases, an analytic expression of the dark soliton effective mass is derived and its dependence on parameters is investigated. The analytical results will also be compared with direct numerical simulations of the cubic–quintic GP equation for specific set of parameters. In the last section we draw our conclusions and summarize the main results of the paper.

2. Motion of a dark soliton in a trap: general theory

First, we have to find the soliton solution of the generalized NLS equation (4) with zero potential $V(x) \equiv 0$ but under the boundary condition that the condensate’s density $\rho(x)$ tends to the limiting background density $\rho_0$ as $|x| \to \infty$ and the condensate’s velocity vanishes at infinity. To this end, it is convenient to look for the solution in the form

$$\psi = \sqrt{\rho(\xi)} \exp(i\phi(\xi) - i\mu t),$$

(11)

where

$$\mu = f(\rho_0)$$

(12)

is the chemical potential of the condensate and its density $\rho$ and the local flow velocity $u = \psi_\xi$ depend only on the variable $\xi = x - vt$ with $v$ being the soliton velocity. The above formulated boundary conditions take the form

$$\rho \to \rho_0, \quad \psi_\xi \to 0, \quad \text{as} \quad |\xi| \to \infty.$$  

(13)

Substitution of (11) into (4) with $V(x) \equiv 0$ and separation of real and imaginary parts yields the system of two equations for $\rho(\xi)$ and $u(\xi) = \psi_\xi$. One of these equations can be integrated at once to give

$$u(\xi) = \psi_\xi = v\left(1 - \frac{\rho_0}{\rho}\right),$$

(14)

where the integration constant is chosen according to the conditions (13). Then the second equation can be transformed to

$$\frac{1}{2} \rho_0^2 - \frac{1}{2} \rho_0 \alpha + \rho_0^2 f(\rho) - \mu \rho^2 + \frac{1}{2} v^2 (\rho_0^2 - \rho^2) = 0$$

(15)

and this equation can also be integrated once. As a result we obtain the equation

$$\rho_0^2 = Q(\rho),$$

(16)

where

$$Q(\rho) = 8\rho \int_\rho^{\rho_0} (1 - f(\rho')) d\rho' - 4v^2 (\rho_0^2 - \rho^2)^2$$

(17)

and again the integration constant is chosen according to the condition (13). Thus, we have reduced the problem of finding the density profile to the inversion of the integral

$$\xi = \int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{Q(\rho)}}.$$  

(18)
where \( \rho_m \) is the minimal density at the centre \( \xi = 0 \) of the soliton. The function \( Q(\rho) \) has a double zero at \( \rho = \rho_0 \) and the condition \( dQ(\rho)/d\rho|_{\rho=\rho_0} = 0 \) reproduces the relation (12). At \( \rho = \rho_m \) the function \( Q(\rho) \) has a single zero, and this condition yields the relationship between \( \rho_m \) and the soliton velocity \( v \),

\[
v^2 = \frac{Q_0(\rho_m)}{4(\rho_0 - \rho_m)^2},
\]

where

\[
Q_0(\rho) = 8\rho \int_0^{\rho_m} \left[ f(\rho_0) - f(\rho') \right] d\rho'.
\]

is the function (17) with zero value of the velocity \( v \) (we have used here equation (12) for \( \mu \)).

Now we have to obtain the expression for the energy of the dark soliton. To this end, we note that the function \( \Psi = \sqrt{\rho} \exp(\psi) = \psi \cdot \exp(i\mu t) \) satisfies the equation

\[
i\Psi_t + \frac{1}{2} \Psi_{xx} + (\mu - f(|\Psi|^2))\Psi = 0
\]

which can be written in a Hamiltonian form

\[
i\Psi_t = \frac{\delta H}{\delta \Psi^*}
\]

with

\[
H = \int_{-\infty}^{\infty} \left[ \frac{1}{2} |\Psi|^2 + \int_0^{\rho_m} f(\rho) d\rho - \mu |\Psi|^2 \right] dx.
\]

To get the contribution of the dark soliton into the energy of BEC, we subtract the background energy \( H_0 = \int_{-\infty}^{\infty} \int_0^{\rho_m} f(\rho) d\rho - \mu |\Psi|^2 \) dx and express the resulting formula in terms of \( \rho \) and \( \varphi_x \),

\[
E = H - H_0 = \int_{-\infty}^{\infty} \left[ \frac{1}{2} |\Psi|^2 + \frac{\varphi_x^2}{4\rho} \right] \rho
\]

\[
+ \int_0^{\rho_m} [f(\rho_0) - f(\rho')] d\rho'.
\]

Substitution of expressions (14) and (16) for the dark soliton solution casts this formula into

\[
E = 2 \int_{-\infty}^{\infty} \int_0^{\rho_m} [f(\rho_0) - f(\rho')] d\rho' dx.
\]

At last, integration with respect to \( x \) can be transformed to integration with respect to \( \rho \) from \( \rho_m \) to \( \rho_0 \) with account of (18),

\[
E = 4 \int_{\rho_m}^{\rho_0} \frac{d\rho}{\sqrt{Q(\rho)}} \int_0^{\rho_m} [f(\rho_0) - f(\rho')] d\rho'
\]

\[
= \frac{1}{2} \int_{\rho_m}^{\rho_0} \frac{Q_0(\rho)}{\rho \sqrt{Q(\rho)}} d\rho.
\]

Here \( \rho_m \) can be considered according to (19) as a function of \( \rho_0 \) and \( v^2 \), hence the soliton energy is also a function of these two variables,

\[
E = E(\rho_0, v^2). \quad (27)
\]

Now we turn to the problem of motion of the dark soliton along a nonuniform condensate confined in a trap with the axial potential \( V(x) \). Then in the Thomas–Fermi (TF) approximation the density \( \rho(x) \) is a function of \( x \) determined implicitly by the equation

\[
\mu - V(x) = f(\rho(x)). \quad (28)
\]

When this TF distribution of the density \( \rho = \rho(x) \) is substituted into (27) instead of \( \rho_0 \), we obtain the energy of the soliton moving with velocity \( v \) and located at a given time \( t \) at the point \( x \). As was shown in [17], the velocity \( v \) changes during the motion of the dark soliton in such a way that the soliton energy (27) is conserved. If we denote the soliton coordinate as \( x = X(t) \), then \( v = X(t) \) and equation (27) converts into

\[
E(\rho(X), X^2) = E_0.
\]

where \( E_0 \) is the initial energy of the soliton. Since the function \( E(\rho, v^2) \) is known, equation (29) is the differential equation for finding the soliton coordinate \( X \) as a function of time \( t \). In general, this first order ordinary differential equation should be solved numerically. Here we shall consider the most interesting limiting cases admitting complete analytical study.

2.1. Small amplitude oscillations of a deep soliton

Let us consider oscillations of a deep soliton,

\[
\rho_m \ll \rho_0,
\]

where \( \rho_0 \) is the maximal density of the condensate at its ‘top’ when it is confined in a trap with the potential \( V(x) \) (see (28)). In this case the amplitude of oscillations is much less than the TF radius of the condensate and the velocity of its motion is much less than the local sound velocity

\[
v \ll c, \quad c_s = \sqrt{\mu f'(\rho_0)}.
\]

Therefore, if we expand the energy into series with respect to small amplitude and velocity, then the first nontrivial terms in (29) will give the expression

\[
\frac{1}{2} m_s X^2 + V(X) = \text{const},
\]

where \( V(X) \) changes little for the small amplitude vibrations. Hence, we get the ‘energy conservation law’ for a Newtonian particle moving in the potential \( V(X) \), where \( m_s \) is the effective mass of the dark soliton.

For small \( v \) we obtain from (19)

\[
\rho_m \approx \frac{E_0^2 v^2}{2a},
\]

where

\[
a = \int_0^{\rho_0} [f(\rho_0) - f(\rho)] d\rho.
\]

To estimate the first term of the series expansion of \( E \) with respect to \( v^2 \), we split the integral (26) into two parts (see [18]),

\[
E = E_1 + E_2,
\]

\[
E_1 = \frac{1}{2} \int_{\rho_m}^{\rho_0} \frac{Q_0(\rho)}{\rho \sqrt{Q(\rho)}} d\rho,
\]

\[
E_2 = \frac{1}{2} \int_{\rho_m}^{\rho_0} \frac{Q_0(\rho)}{\rho \sqrt{Q(\rho)}} d\rho,
\]

where we have introduced an intermediate integration limit \( \rho_1 \) so that

\[
\rho_m \ll \rho_1 \ll \rho_0.
\]

Dependence of \( E_1 \) on \( v^2 \) results mainly from the lower integration limit (33) and in the main approximation we
and elementary integration yield
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\[ \rho \]

As we see, it diverges in the limit \( \rho \to 0 \). To cancel this divergent term, we have to identify a similar term in the derivative \( \frac{dE_2}{dv^2} \) given by the expression
\[ \frac{dE_2}{dv^2} \bigg|_{v=0} = \int_{\rho_1}^{\rho_0} \frac{(\rho_0 - \rho)^2}{\sqrt{Q(\rho)}} \, d\rho. \]  

(41)

According to equation (40), the diverging term is proportional to \( \rho_1^{-1/2} \). Correspondingly, we transform the integral in (41) in the following way by integration by parts,
\[ \frac{dE_2}{dv^2} \bigg|_{v=0} \approx \frac{\rho_0^2}{\sqrt{2a_0 \rho_1}}. \]  

(40)

As we see, it diverges in the limit \( \rho_1 \to 0 \). We need the coefficient of \( v^2 \) in the expansion of \( E \) in powers of \( v^2 \) and the contribution of \( E_1 \) into this coefficient is equal to
\[ \frac{dE_1}{d\rho_1} \frac{d\rho}{d\rho_1} \left( \frac{\rho_0^2}{\sqrt{2a_0 \rho_1}} \right). \]  

(42)

The first term vanishes at the upper limit \( \rho = \rho_0 \) and at the lower limit \( \rho = \rho_1 \ll \rho_0 \) we can approximate \( Q_0(\rho) \) as \( Q_0(\rho) \approx 8a_0 \rho \) (see (38)). In the second term the integral is convergent and we can take here the lower integration limit equal to \( \rho_1 = 0 \). As a result we obtain
\[ \frac{dE_2}{dv^2} \bigg|_{v=0} \approx \frac{\rho_0^2}{\sqrt{2a_0 \rho_1}} + 2 \int_{\rho_1}^{\rho_0} \frac{d\rho}{\sqrt{\rho}} \frac{dP(\rho_0 - \rho)^2}{\sqrt{Q(\rho)}}. \]  

(43)

The first term cancels with the contribution (40) and we obtain the expression for the soliton mass
\[ m_s = 2 \frac{dE_2}{dv^2} \bigg|_{v=0} = 4 \int_{\rho_1}^{\rho_0} \frac{d\rho}{\sqrt{\rho}} \frac{dP(\rho_0 - \rho)^2}{\sqrt{Q(\rho)}}. \]  

(44)

For the motion of a dark soliton near the ‘top’ of the condensate.

The introduction of the potential \( V(x) \) leads to the shift of the chemical potential \( \mu \to \mu - V(x) \). Taking into account the thermodynamic formula
\[ N = -\frac{dE}{d\mu}, \]  

(45)

and the expression for the energy in the limit \( v^2 \to 0 \),
\[ E = E_0 = 4 \int_{\rho_1}^{\rho_0} \frac{d\rho}{\sqrt{Q_0(\rho)}} \int_{\rho}^{\rho_0} [\mu - f(\rho')] \, d\rho'. \]  

hence
\[ N = -2 \int_{\rho_0}^{\rho} \frac{(\rho_0 - \rho') \, d\rho'}{\sqrt{Q_0(\rho)}}, \]  

(46)

we find the expansion of the soliton energy as
\[ E = E_0 + \frac{dE}{dv^2} \cdot v^2 + \frac{dE}{d\mu} \cdot (-v(x)) \]  

(47)

and division by \( N \) yields the energy conservation law for the soliton’s motion
\[ m_s = 2 \frac{v^2 + V(x)}{2} = \text{const}, \]  

(48)

where
\[ m_s = \frac{m_s}{2}, \]  

(49)

is the effective mass of the soliton’s motion. Hence, in the case of a harmonic trap (5) we get the formula
\[ \omega = \frac{\omega_0}{m_s}, \]  

(50)

for the frequency of oscillations of deep solitons. For a usual GP equation we have
\[ f(\rho) = \rho, \]  

(51)

hence
\[ Q_0(\rho) = 4\rho(\rho_0 - \rho)^2, \]  

(52)

\[ m_s = -4\sqrt{\rho_0}, \]  

(53)

\[ N = -2\sqrt{\rho_0} \]  

so that (50) reproduces the well-known result.

2.2. Motion of a shallow soliton

In the shallow soliton limit
\[ \rho_0 - \rho \ll \rho_0 \]  

(54)

the series expansion of equation (19) gives
\[ v^2 \cong \rho_0 \left[ f'(\rho_0) - \frac{1}{2} f''(\rho_0) (\rho_0 - \rho_m) \right], \]  

(55)

or taking account of the expression (10) for the sound velocity we obtain
\[ \rho_m = \rho_0 - \frac{c_s^2 - v^2}{f'(\rho_0) + \frac{1}{2} f''(\rho_0)}. \]  

(56)

In this limit the dark soliton can be considered in the Korteweg–de Vries (KdV) approximation (see, e.g. [33]) for which the NLS equation with \( V(x) = 0 \) reduces to the equation
\[ \rho_i' + c_s^2 \rho_i' + \frac{3}{2c_s^4} f'(\rho_0) + \frac{1}{3} f''(\rho_0) f'(\rho_0) \rho_i' + \frac{1}{c_s^4} \rho_i' \frac{dV}{dx} + \frac{1}{8c_s^4} \rho_i'^2 + \rho_i'^4 = 0. \]  

(57)

for evolution of small deviations \( \rho' = \rho_0 - \rho \) of the density from the background value \( \rho_0 \). This equation has the soliton solution
\[ \rho' = \rho_0 - \rho = -\frac{c_s^2 - v^2}{f'(\rho_0) + \frac{1}{2} f''(\rho_0)} \cdot \frac{1}{\cosh^2[\kappa(x - vt)]} \]  

(58)
where
\[ \kappa = 2 \sqrt{\frac{c_s (c_s^2 - v^2)}{f'(\rho_0) + \frac{1}{2} \rho_0 f''(\rho_0)}} \] (59)
is the inverse soliton’s half-width and the minimal density \( \rho_m \) at the soliton’s centre \( x = vt \) is given by the expression (56).

In the same approximation
\[ Q(\rho) = 4 \left[ f'(\rho_0) + \frac{1}{2} \rho_0 f''(\rho_0) \right](\rho_0 - \rho)^2(\rho - \rho_m) \] (60)
and the calculation of the soliton energy (26) gives
\[ E = 4 \frac{f'(\rho_0)}{3 \left[ f'(\rho_0) + \frac{1}{2} \rho_0 f''(\rho_0) \right]} \left( c_s^2 - v^2 \right)^{3/2}. \] (61)

If the width \( 1/\kappa \) of the soliton is much smaller than the condensate’s size (the TF radius), then we can consider the motion of such a soliton in the trap by the replacement of \( \rho_0 \) by the solution \( \rho(x) \) of the equation (28) and of \( v \) by its variable velocity \( dX/dt \) to get the equation
\[ \left( \frac{dX}{dt} \right)^2 = \rho f'(\rho) - 3 \frac{E}{4 f'(\rho)} \left[ f'(\rho) + 1 \frac{f''(\rho)}{(\rho_0)} \right]^2, \] (62)
where \( \rho = \rho(X) \) is given by equation (28). For example, in the case of \( f = h\rho^2, h > 0 \), we have \( f'(\rho) = 2h \rho \) and \( \mu - V(X) = h\rho^2 \), hence (62) reduces to
\[ \left( \frac{dX}{dt} \right)^2 + V(X) + 2 \left( \frac{h E}{3} \right) \frac{\left( \mu - V(X) \right)^{1/3}}{h} = \mu. \] (63)

As we see, for a harmonic trap (5) the frequency of oscillations depends on the energy \( E \) of the soliton. However, if \( E \) is small enough,
\[ E \ll \frac{\mu}{\sqrt{h}} \] (64)
then the last term in the left-hand size can be neglected and we return to the usual energy conservation law for a Newtonian particle which for a harmonic trap gives equation (8) in agreement with [16, 18].

3. Oscillation of a dark soliton for the cubic–quintic nonlinearity

Now we shall consider in some detail the motion of a dark soliton described by the generalized NLS equation with \( f(\rho) \) given by equation (3), i.e.,
\[ f(\rho) = g\rho + h\rho^2. \] (65)

Conditions for realization of motion of a shallow soliton are very restrictive and we shall confine ourselves to the most practical case of small oscillations of a deep soliton (section 2.1).

Substitution of (65) into (20) gives
\[ Q_0(\rho) = 8\rho(\rho_0 - \rho)^2 \left[ 1 + 1 \frac{h(2\rho_0 + \rho)}{g} \right]. \] (66)

Situations with \( h > 0 \) and \( h < 0 \) should be considered separately.

3.1. Repulsive quintic interaction \((h > 0)\)

In this case elementary integration of equation (18) with \( Q(\rho) \) given by (66) yields the profile of the density in the dark solitons solution
\[ \rho(x) = \rho_0 - \frac{\rho_0 (\rho_0 - \rho_+) (\rho_0 - \rho_-)}{(\rho_0 - \rho_+) \cosh^2(\kappa \xi) + \rho_0 - \rho_+}. \] (67)

where
\[ \rho_\pm = \pm \sqrt{\frac{3g}{4h} + \rho_0^2} + \frac{3v^2}{2h} - \frac{3g}{4h} + \rho_0 \] (68)
and
\[ \kappa = \sqrt{2h(\rho_0 - \rho_+)(\rho_0 - \rho_-)/3}. \] (69)

Substitution of (66) into (46) and simple calculation give
\[ N = \sqrt{\frac{6}{h}} \ln \left( \frac{\sqrt{2h\rho_0}}{3g + 4h\rho_0} + \frac{3(3g + 2h\rho_0)}{3g + 4h\rho_0} \right). \] (70)

Similar calculation of (44) yields
\[ m_s = \sqrt{\frac{6h\rho_0(g + 2h\rho_0)}{(3g + 4h\rho_0)}}, \] (71)

Hence the effective mass of the soliton motion is equal to
\[ m_s = 1 + \frac{\sqrt{6h\rho_0(g + 2h\rho_0)}}{(3g + 4h\rho_0) \ln \left( \frac{\sqrt{2h\rho_0}}{3g + 4h\rho_0} + \frac{3(3g + 2h\rho_0)}{3g + 4h\rho_0} \right)} \] (72)

For \( h\rho_0 \ll g \) this formula reproduces
\[ m_s = 2, \quad h\rho_0 \ll g, \] (73)
in agreement with (6), and for \( h\rho_0 \gg g \) we obtain
\[ m_s = 1 + \frac{\sqrt{3}}{2 \ln 1 + 1 \sqrt{2}}, \quad h\rho_0 \gg g, \] (74)
in agreement with (9).

Note from equation (72) that \( m_s \) depends only on the combined parameter \( \rho_0/h \). The same is true also for the case of attractive three-body interactions (see equation (81) below). By knowing \( g \) and \( \rho_0 \) and measuring \( m_s \) through the dark soliton oscillation frequency, one can then determine \( h \) using formulae (72) and (81). In figure 1 the dependence of \( m_s \) on the three-body interaction strength is depicted for different values of the ratio \( \rho_0/m \) from which we see that the range of variation of \( m_s \) is larger for \( h < 0 \) and the effective mass becomes 2 exactly when the three-body interactions reduce to zero.

To check our theory, we have numerically integrated the GP equation using as initial condition the exact dark soliton solution of the cubic–quintic NLS in equation (1). The parabolic potential was introduced adiabatically by rising with time the frequency \( \omega_0 \) from zero (pure cubic-quintic solution) up to a desired value. Effects of adiabatic changes of parameters on BEC bright and dark solitons were considered in [34] where the effectiveness of the adiabatic method was
more extended comparison between the theory and numerical results is presented in the left panel of figure 3 in which we see that in general there is a good qualitative agreement which becomes also quantitatively good at higher values of soliton backgrounds. The deviation observed at lower values of $\rho_0$ can probably be ascribed to a large soliton comparable in width to the size of the whole condensate. In this situation the conditions of validity of the theory are not satisfied well enough.

3.2. Attractive quintic interaction ($h < 0$)

For $h < 0$ the integrand function in equation (46) is real for

$$\rho_0 < \frac{g}{2|h|}.$$  

(75)
and this inequality coincides with the condition of modulation stability of BEC, namely the condition that the sound velocity
\( c_s = \sqrt{\rho_0 f'(\rho_0)} \) is real. It is worth noticing that this condition is more restrictive than the condition \( \rho_0 < g/|h| \) for which the nonlinearity function is positive, \( f(\rho) = g\rho - |h|\rho^2 > 0 \). Obviously, we can discuss the soliton oscillations for BECs satisfying the condition (75) only.

In this case \( Q(\rho) \) has the double zero \( \rho_0 \) and the simple zeroes \( \rho_+ > \rho_0 \) and \( 0 < \rho_- < \rho_0 \) where
\[
\rho_{\pm} = \pm \sqrt{\left( \frac{3g}{4|h|} - \rho_0 \right)^2 - \frac{3\rho_0^2}{2|h|} + \left( \frac{3g}{4|h|} - \rho_0 \right)}.
\]
Hence there exist both dark and bright soliton solutions. We shall consider here motion only of the dark soliton with the density profile
\[
\rho(\xi) = \rho_0 - \frac{(\rho_0 - \rho_-)(\rho_+ - \rho_0)}{(\rho_+ - \rho_-) \cosh^2(\kappa \xi) - (\rho_0 - \rho_-)}.
\]
where
\[
\kappa = \sqrt{\frac{2|h|(|\rho_0 - \rho_-)(\rho_+ - \rho_0)/3}.}
\]
Calculation of the number of particles \( N \) and the soliton mass \( m_s \) gives
\[
N = -\frac{6}{|h|} \arcsin \frac{2|h|\rho_0}{3g - 4|h|\rho_0},
\]
and
\[
m_s = -\frac{6}{|h|} \arcsin \frac{2|h|\rho_0}{3g - 4|h|\rho_0} + \frac{2\sqrt{6|h|\rho_0(g - 2|h|\rho_0)}}{3g - 4|h|\rho_0}.
\]
Hence the effective mass of the soliton motion is equal to
\[
m_s = \frac{m_s}{N} = 1 + \frac{\sqrt{6|h|\rho_0(g - 2|h|\rho_0)}}{(3g - 4|h|\rho_0) \arcsin \sqrt{\frac{2|h|\rho_0}{3g - 4|h|\rho_0}}},
\]
The condition (75) means that in these expressions the argument of the arcsin function is less than unity. (Of course, expressions (79)–(81) can be obtained from (70)–(72) by means of their analytical continuation to negative values of \( h \).

For \( |h|\rho_0 \ll g \) we again reproduce (73), as it should be. For \( g - 2|h|\rho_0 \ll g \) we find the limiting behaviour
\[
m_s = 1 + \frac{2}{\pi \sqrt{8(g - 2|h|\rho_0)}} g - 2|h|\rho_0 \ll g.
\]
Thus, \( m_s \) as a function of \( \rho_0 \) has a branching point singularity at \( \rho_0 = g/(2|h|) \) where the system becomes modulationally unstable.

The plot of the ratio of the trap frequency to the soliton’s oscillations frequency is illustrated in figure 3 (right panel) and it demonstrates a very good agreement with the results of the numerical simulations.

4. Conclusion
In this paper we have considered the dynamics of a dark soliton inside a BEC described by the quasi-one-dimensional GP equation with a parabolic trapping potential. In particular, we have developed the theory which permits one to study the motion of a dark soliton through the stationary background state of BEC and to derive the formulae for the frequency of the oscillation of deep solitons which are very effective for arbitrary form of the nonlinearity function \( f(\rho) \). When the form of \( f(\rho) \) is appropriately specified, the formulae for the frequency of oscillations reproduce the results of all previously studied special cases. We applied our theory to the study of the practically interesting case of the cubic–quintic nonlinearity function \( f(\rho) = g\rho + h\rho^3 \) and we have demonstrated, as new effect, the dependence of the frequency of the oscillations (dark soliton effective mass) on the strength of the three-body interaction. In principle this effect can be used for the experimental study of higher nonlinearity on the BEC dynamics.

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