ON THE PARTIAL SUMS OF WALSH-FOURIER SERIES

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ABSTRACT. In this paper we investigate some convergence and divergence of some specific subsequences of partial sums with respect to Walsh system on the martingale Hardy spaces. By using these results we obtain relationship of the ratio of convergence of the partial sums of the Walsh series with the modulus of continuity of martingale. These conditions are in a sense necessary and sufficient.

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1. INTRODUCTION

It is well-known that (see e.g. [2] and [24]) Walsh system does not form basis in the space $L_1$. Moreover, there exists a function in the dyadic Hardy space $H_1$, such that the partial sums of $f$ are not bounded in $L_1$-norm, but partial sums $S_n$ of the Walsh-Fourier series of a function $f \in L_1$ convergence in measure (see also [7] and [12]).

For Walsh-Fourier series Onneweer [16] showed that if modulus of continuity of $f \in L_1[0,1)$ satisfies the condition

$$\omega_1(\delta,f) = o\left(\frac{1}{\log (1/\delta)}\right) \quad \text{as} \quad \delta \to 0,$$

then its Walsh-Fourier series converges in $L_1$-norm. He also proved that condition (1) can not be improved.

It is also known that subsequence of partial sums $S_{m_k}$ is bounded from $L_1$ to $L_1$ if and only if $\{m_k : k \geq 0\}$ have uniformly bounded variations. In [24, Ch. 1] it was proved that if $f \in L_1(G)$ and $\{m_n : n \geq 1\}$ be subsequence of positive numbers $\mathbb{N}$, such that

$$\omega_1\left(1/m_n, f\right) = o\left(\frac{1}{L_S(m_n)}\right) \quad \text{as} \quad n \to \infty,$$

where the number $L_S(n)$ is $n$-th Lebesgue constant, then subsequence $S_{m_n}f$ of partial sums of Walsh-Fourier series converge in $L_1$-norm. Goginava and Tkebuchava [11] proved that the condition (2) can not be improved. Since

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\[
\frac{V(n)}{8} \leq L_S(n) \leq V(n)
\]
the condition (2) can be rewritten with the condition
\[
\omega_1\left(\frac{1}{m_n}, f\right) = o\left(\frac{1}{V(m_n)}\right) \text{ as } n \to \infty.
\]
In [20] it was proved that if \( F \in H_p \) and
\[
\omega_{H_p}\left(\frac{1}{2^n}, F\right) = o\left(\frac{1}{(n^{[p]}2^{(1/p-1)n})}\right) \text{ as } n \to \infty,
\]
where \( 0 < p \leq 1 \) and \([p]\) denotes integer part of \( p \), then \( S_nF \to F \) as \( n \to \infty \) in \( L_{p,\infty} \)-norm. Moreover, there was showed that condition (4) can not be improved.

Uniform and pointwise convergence and some approximation properties of partial sums in \( L_1 \) norms was investigate by Goginava [8] (see also [11], [9]), Nagy [15], Avdispahić and Memić [1]. Fine [4] obtained sufficient conditions for the uniform convergence which are complete analogy with the Dini-Lipschits conditions. Gulicćev [13] estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. These problems for Vilenkin groups were considered by Blahota [3], Fridlí [5] and Gát [6].

The main aim of this paper is to find characterizations of boundedness of the subsequence of partial sums of the Walsh series of \( H_p \) martingales in terms of measurable properties of a Dirichlet kernel corresponding to partial summing. As a consequence we get the corollaries about the convergence and divergence of some specific subsequences of partial sums. For \( p = 1 \) the simple numerical criterion for the index of partial sum in terms of its dyadic expansion is given which governs the convergence (or the ratio of divergence). Another type of results covered by the paper is the relationship of the ratio of convergence of the partial sums of the Walsh series with the modulus of continuity of martingale. The conditions given below are in a sense necessary and sufficient.

2. Preliminaries

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \). Denote by \( Z_2 \) the discrete cyclic group of order 2, that is \( Z_2 := \{0, 1\} \), where the group operation is the modulo 2 addition and every subset is open. The Haar measure on \( Z_2 \) is given so that the measure of a singleton is \( 1/2 \).

Define the group \( G \) as the complete direct product of the group \( Z_2 \), with the product of the discrete topologies of \( Z_2 \)’s. The elements of \( G \) are represented by sequences \( x := (x_0, x_1, \ldots, x_j, \ldots) \), where \( x_k = 0 \lor 1 \).
It is easy to give a base for the neighborhood of \( x \in G \)

\[
I_0 (x) := G, \ I_n (x) := \{ y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1} \} \ (n \in \mathbb{N}).
\]

Denote \( I_n := I_n (0), \ \overline{I}_n := G \setminus I_n \) and \( e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G, \) for \( n \in \mathbb{N} . \) Then it is easy to show that

\[
(5) \quad \overline{I}_M = \bigcup_{s=0}^{M-1} I_s \setminus I_{s+1}.
\]

If \( n \in \mathbb{N} , \) then every \( n \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j 2^j, \) where \( n_j \in \mathbb{Z}_2 \) \((j \in \mathbb{N})\) and only a finite numbers of \( n_j \) differ from zero.

Let

\[
\langle n \rangle := \min\{j \in \mathbb{N}, n_j \neq 0\} \quad \text{and} \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\},
\]

that is \( 2^{|n|} \leq n \leq 2^{|n|+1} . \) Set

\[
d (n) = |n| - \langle n \rangle , \quad \text{for all} \quad n \in \mathbb{N} .
\]

Define the variation of an \( n \in \mathbb{N} \) with binary coefficients \((n_k, k \in \mathbb{N})\) by

\[
V (n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}| .
\]

The norms (or quasi-norm) of the spaces \( L_p (G) \) and \( L_{p, \infty} (G) \) \((0 < p < \infty)\) are respectively defined by

\[
\|f\|_p^p := \int_G |f|^p \, d\mu, \quad \|f\|_{L_{p, \infty}}^p := \sup_{\lambda > 0} \lambda^p \mu (f > \lambda).
\]

The \( k \)-th Rademacher function is defined by

\[
r_k (x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}) .
\]

Now, define the Walsh system \( w := (w_n : n \in \mathbb{N}) \) on \( G \) as:

\[
w_n (x) := \prod_{k=0}^{\infty} r_k^{n_k} (x) = r_{|n|} (x) (1 - \sum_{k=0}^{\infty} n_k x_k) \quad (n \in \mathbb{N}) .
\]

The Walsh system is orthonormal and complete in \( L_2 (G) \) \((\text{see e.g. [18]})\). If \( f \in L_1 (G) \) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Walsh system in the usual manner:

\[
\hat{f} (k) := \int_G f w_k d\mu \quad (k \in \mathbb{N}) ,
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f} (k) w_k , \quad D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+) .
\]

Recall that

\[
(6) \quad D_{2n} (x) = \begin{cases} 2^n , & \text{if} \ x \in I_n \\ 0 , & \text{if} \ x \notin I_n \end{cases}
\]
and

\[ D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}), \text{ for } n = \sum_{i=0}^{\infty} n_i 2^i. \]

Let us denote \( n \)-th Lebesgue constant by

\[ L_S(n) := \| D_n \|_1. \]

The \( \sigma \)-algebra generated by the intervals \( \{ I_n(x) : x \in G \} \) will be denoted by \( \zeta_n (n \in \mathbb{N}) \). Denote by \( F = (F_n, n \in \mathbb{N}) \) the martingale with respect to \( F_n (n \in \mathbb{N}) \) (for details see e.g. \[22\]).

The maximal function of a martingale \( F \) is defined by

\[ F^* = \sup_{n \in \mathbb{N}} |F_n|. \]

In case \( f \in L_1(G) \), the maximal functions are also be given by

\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|. \]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p(G) \) consist all martingale for which

\[ \| F \|_{H_p} := \| F^* \|_p < \infty. \]

The best approximation of \( f \in L_p(G) \) \((1 \leq p \leq \infty)\) is defined as

\[ E_n(f, L_p) = \inf_{\psi \in p_n} \| f - \psi \|_p, \]

where \( p_n \) is set of all Walsh polynomials of order less than \( n \in \mathbb{N} \).

The integrated modulus of continuity of \( f \in L_p \) is defined by

\[ \omega_p \left( \frac{1}{2^n}, f \right) = \sup_{h \in I_n} \| f(\cdot + h) - f(\cdot) \|_p. \]

The concept of modulus of continuity in \( H_p(G) \) \((0 < p \leq 1)\) can be defined in following way

\[ \omega_{H_p} \left( \frac{1}{2^n}, F \right) := \| F - S_{2^n} F \|_{H_p}. \]

Watari \[21\] showed that there are strong connections between

\[ \omega_p \left( \frac{1}{2^n}, f \right), \quad E_{2^n}(f, L_p) \text{ and } \| f - S_{2^n} f \|_p, \quad p \geq 1, \quad n \in \mathbb{N}. \]

In particular,

\[ \frac{1}{2} \omega_p \left( \frac{1}{2^n}, f \right) \leq \| f - S_{2^n} f \|_p \leq \omega_p \left( \frac{1}{2^n}, f \right), \]

(8)
and
\[ \frac{1}{2} \| f - S_{2^n} f \|_p \leq E_{2^n} (f, L_p) \leq \| f - S_{2^n} f \|_p. \]

A bounded measurable function \( a \) is called \( p \)-atom, if there exist a dyadic interval \( I \), such that
\[ \int_I a \, d\mu = 0, \quad \| a \|_\infty \leq \mu (I)^{-1/p}, \quad \text{supp} (a) \subset I. \]

The dyadic Hardy martingale spaces \( H_p \) for \( 0 < p \leq 1 \) have an atomic characterization. Namely, the following theorem is true (see [19] and [23]):

**Theorem W:** A martingale \( F = (F_n, n \in \mathbb{N}) \) is in \( H_p \) (\( 0 < p \leq 1 \)) if and only if there exists a sequence \( (\mu_k, k \in \mathbb{N}) \) of real numbers such that for every \( n \in \mathbb{N} \)

\[ \sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty, \]

Moreover,
\[ \| F \|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]
where the infimum is taken over all decomposition of \( F \) of the form (9).

It is easy to check that for every martingale \( F = (F_n, n \in \mathbb{N}) \) and every \( k \in \mathbb{N} \) the limit

\[ \hat{F} (k) := \lim_{n \to \infty} \int_G F_n (x) w_k (x) d\mu (x) \]
exists and it is called the \( k \)-th Walsh-Fourier coefficients of \( F \).

If \( F := (E_n f : n \in \mathbb{N}) \) is regular martingale, generated by \( f \in L_1 (G) \), then \( \hat{F} (k) = \hat{f} (k), k \in \mathbb{N}. \)

For the martingale
\[ F = \sum_{n=0}^{\infty} (F_n - F_{n-1}) \]
the conjugate transforms are defined as
\[ \widehat{F}^{(t)} = \sum_{n=0}^{\infty} r_n (t) (F_n - F_{n-1}), \]
where \( t \in G \) is fixed. Note that \( \widehat{F}^{(0)} = F. \) As is well known (see e.g. [22])

\[ \| \widehat{F}^{(t)} \|_{H_p} = \| F \|_{H_p}, \quad \| F \|_{H_p}^p \sim \int_G \| \widehat{F}^{(t)} \|_p^p \, dt, \quad S_n \widehat{F}^{(t)} = S_n \widehat{F}^{(t)}. \]
3. Formulation of Main Results

**Theorem 1.** a) Let $0 < p < 1$ and $F \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that
\[
\|S_n F\|_{H_p} \leq c_p 2^{d(n)(1/p-1)} \|F\|_{H_p}.
\]
b) Let $0 < p < 1$, $\{m_k : k \geq 0\}$ be any increasing sequence of positive integers $\mathbb{N}_+$ such that
\[
\sup_{k \in \mathbb{N}} d(m_k) = \infty
\]
and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be any nondecreasing function, satisfying condition
\[
\lim_{k \to \infty} \frac{\Phi(m_k)}{\Phi(m_k)} = \infty.
\]
Then there exists a martingale $F \in H_p$, such that
\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} F}{\Phi(m_k)} \right\|_{L_{p, \infty}} = \infty.
\]

**Corollary 1.** a) Let $0 < p < 1$ and $F \in H_p$. Then there exists an absolute constant $c_p$, depending only on $p$, such that
\[
\|S_n F\|_{H_p} \leq c_p n \mu \{\supp(D_n)\}^{1/p-1} \|F\|_{H_p}.
\]
b) Let $0 < p < 1$, $\{m_k : k \geq 0\}$ be any increasing sequence of positive integers $\mathbb{N}_+$ such that
\[
\sup_{k \in \mathbb{N}} m_k \mu \{\supp(D_{m_k})\} = \infty
\]
and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be any nondecreasing function, satisfying condition
\[
\lim_{k \to \infty} \frac{m_k \mu \{\supp(D_{m_k})\}^{1/p-1}}{\Phi(m_k)} = \infty.
\]
Then there exists a martingale $F \in H_p$, such that
\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} F}{\Phi(m_k)} \right\|_{L_{p, \infty}} = \infty.
\]

**Corollary 2.** Let $n \in \mathbb{N}$ and $0 < p < 1$. Then there exists a martingale $F \in H_p$, such that
\[
\sup_{n \in \mathbb{N}} \left\| S_{2n+1} F \right\|_{L_{p, \infty}} = \infty.
\]

**Corollary 3.** Let $n \in \mathbb{N}$ and $0 < p \leq 1$ and $F \in H_p$. Then
\[
\|S_{2n+2n-1} F\|_{H_p} \leq c_p \|F\|_{H_p}.
\]
Theorem 2. a) Let \( n \in \mathbb{N}_+ \) and \( F \in H_1 \). Then there exists an absolute constant \( c \), such that
\[
\| S_n F \|_{H_1} \leq c V(n) \| F \|_{H_1}.
\]
b) Let \( \{ m_k : k \geq 0 \} \) be any increasing sequence of positive integers \( \mathbb{N}_+ \), such that
\[
\text{sup}_{k \in \mathbb{N}} V(m_k) = \infty
\]
and \( \Phi : \mathbb{N}_+ \to [1, \infty) \) be any nondecreasing function, satisfying condition
\[
\lim_{k \to \infty} \frac{V(m_k)}{\Phi(m_k)} = \infty.
\]
Then there exist a martingale \( F \in H_1 \), such that
\[
\sup_{k \in \mathbb{N}} \left\| \frac{S_{m_k} F}{\Phi(m_k)} \right\|_1 = \infty.
\]

Theorem 3. Let \( 2^k < n \leq 2^{k+1} \). Then there exist absolute constant \( c_p \), depending only on \( p \), such that
\[
\| S_n F - F \|_{H_p} \leq c_p 2^{d(n)(1/p-1)} \omega_{H_p} \left( \frac{1}{2^{k}}, F \right), \quad (0 < p < 1)
\]
and
\[
\| S_n F - F \|_{H_1} \leq c_1 V(n) \omega_{H_1} \left( \frac{1}{2^{k}}, F \right).
\]

Theorem 4. a) Let \( 0 < p < 1 \), \( F \in H_p \) and \( \{ m_k : k \geq 0 \} \) be a sequence of nonnegative integers, such that
\[
\omega_{H_p} \left( \frac{1}{2|m_k|}, F \right) = o \left( \frac{1}{2d(m_k)(1/p-1)} \right) \quad \text{as} \quad k \to \infty.
\]
Then
\[
\| S_{m_k} F - F \|_{H_p} \to 0 \quad \text{as} \quad k \to \infty.
\]
b) Let \( \{ m_k : k \geq 0 \} \) be any increasing sequence of positive integers \( \mathbb{N}_+ \), satisfying condition (12). Then there exists a martingale \( F \in H_p \) and subsequence \( \{ \alpha_k : k \geq 0 \} \subset \{ m_k : k \geq 0 \} \), for which
\[
\omega_{H_p} \left( \frac{1}{2|\alpha_k|}, F \right) = O \left( \frac{1}{2d(\alpha_k)(1/p-1)} \right) \quad \text{as} \quad k \to \infty
\]
and
\[
\limsup_{k \to \infty} \| S_{\alpha_k} F - F \|_{L_p, \infty} > c_p > 0 \quad \text{as} \quad k \to \infty,
\]
where \( c_p \) is an absolute constant depending only on \( p \).
Corollary 4. a) Let $0 < p < 1$, $F \in H_p$ and $\{m_k : k \geq 0\}$ be a sequence of nonnegative integers, such that

$$\omega_H \left( \frac{1}{2|m_k|}, F \right) = o \left( \frac{1}{(m_k \mu (\text{supp} D_{m_k}))^{1/p-1}} \right) \text{ as } k \to \infty. \tag{24}$$

Then (22) is satisfied.

b) Let $\{m_k : k \geq 0\}$ be any increasing sequence of positive integers $\mathbb{N}_+$, satisfying condition (14). Then there exists a martingale $F \in H_p$ and subsequence $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$, for which

$$\omega_H \left( \frac{1}{2|\alpha_k|}, F \right) = O \left( \frac{1}{(\alpha_k \mu (\text{supp} D_{\alpha_k}))^{1/p-1}} \right) \text{ as } k \to \infty. \tag{25}$$

and (23) is satisfied.

Theorem 5. a) Let $F \in H_1$ and $\{m_k : k \geq 0\}$ be a sequence of nonnegative integers, such that

$$\omega_H \left( \frac{1}{2|m_k|}, F \right) = o \left( \frac{1}{V(m_k)} \right) \text{ as } k \to \infty. \tag{26}$$

Then

$$\|S_{m_k} F - F\|_{H_1} \to 0 \text{ as } k \to \infty. \tag{27}$$

b) Let $\{m_k : k \geq 0\}$ be any increasing sequence of positive integers $\mathbb{N}_+$, satisfying condition (18). Then there exists a martingale $F \in H_1$ and $\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}$ for which

$$\omega_H \left( \frac{1}{2|\alpha_k|}, F \right) = O \left( \frac{1}{V(\alpha_k)} \right) \text{ as } k \to \infty. \tag{28}$$

and

$$\limsup_{k \to \infty} \|S_{\alpha_k} F - F\|_1 > c > 0 \text{ as } k \to \infty,$$

where $c$ is an absolute constant.

4. PROOF OF THE THEOREMS

Proof of Theorem 1. Suppose that

$$\left\| 2^{(1-1/p)d(n)} S_n F \right\|_p \leq c_p \|F\|_{H_p}. \tag{29}$$

By combining (1) and (29) we get that

$$\left\| 2^{(1-1/p)d(n)} S_n F \right\|_{H_p} \leq c_p \int_G \left\| 2^{(1-1/p)d(n)} S_n \widehat{F(t)} \right\|_p d\mu(t) \tag{30}$$

$$= c_p \int_G \left\| 2^{(1-1/p)d(n)} \widehat{S_n F(t)} \right\|_p d\mu(t) \leq c_p \int_G \left\| \widehat{F(t)} \right\|_{H_p} d\mu(t) \leq c_p \|F\|_{H_p}. \tag{31}$$
By using Theorem W, (29) the proof of Theorem 1 will be complete, if we show that

\[
\int_G \left| 2^{(1-1/p)d(n)} S_na \right|^p \, d\mu \leq c_p < \infty,
\]

for every p-atom \( a \), with support \( I \) and \( \mu (I) = 2^{-N} \).

We may assume that this arbitrary p-atom \( a \) has support \( I = I_M \). It is easy to see that \( S_n (a) = 0 \), when \( 2M > n \). Therefore, we can suppose that \( 2M < n \). Since \( \| a \|_\infty \leq 2^{M/p} \) we can write that

\[
2^{(1-1/p)d(n)} \left| S_n a (x) \right| \leq 2^{(1-1/p)d(n)} \| a \|_\infty \int_{I_M} |D_n (x + t)| \, d\mu (t)
\]

\[
\leq 2^{M/p} 2^{(1-1/p)d(n)} \int_{I_M} |D_n (x + t)| \, d\mu (t).
\]

Let \( x \in I_M \). Since \( V_n (n) \leq d_n \), by applying (3) we get that

\[
2^{(1-1/p)d(n)} \left| S_n a (x) \right| \leq 2^{M/p} 2^{(1-1/p)d(n)} \leq 2M/p \leq 2^{(1-1/p)d(n)}
\]

and

\[
\int_{I_M} \left| 2^{(1-1/p)d(n)} S_n a \right|^p \, d\mu \leq d(n) 2^{(1-1/p)d(n)} < c_p < \infty.
\]

Let \( t \in I_M \) and \( x \in I_s \setminus I_{s+1} \), \( 0 \leq s \leq M-1 < \langle n \rangle \) or \( 0 \leq s < \langle n \rangle \leq M-1 \). Then \( x + t \in I_s \setminus I_{s+1} \). By using (7) we get that \( D_n (x + t) = 0 \) and

\[
\left| 2^{(1-1/p)d(n)} S_n a (x) \right| = 0.
\]

Let \( x \in I_s \setminus I_{s+1} \), \( \langle n \rangle \leq s \leq M-1 \). Then \( x + t \in I_s \setminus I_{s+1} \), for \( t \in I_M \). By using (7) we can write that

\[
|D_n (x + t)| \leq \sum_{j=0}^s n_j 2^j \leq c 2^s.
\]

If we apply (31) we get that

\[
2^{(1-1/p)d(n)} \left| S_n a (x) \right| \leq \frac{2^{(1-1/p)d(n)} 2^{s}}{2M} = 2^{(n)(1/p-1)2^s}.
\]

By combining (5) and (33) we have

\[
\int_{I_M} \left| 2^{(1-1/p)d(n)} S_n a (x) \right|^p \, d\mu (x)
\]

\[
= \sum_{s=\langle n \rangle}^{M-1} \int_{I_s \setminus I_{s+1}} 2^{(n)(1/p-1)2^s} \, d\mu (x) \leq c \sum_{s=\langle n \rangle}^{M-1} \frac{2^{(n)(1-p)}}{2^{(1-1/p)2^s}} \leq c_p < \infty.
\]
Let prove second part of Theorem 1. Under condition (13), there exists sequence \( \{ \alpha_k : k \geq 0 \} \subset \{ m_k : k \geq 0 \} \), such that

\[
\sum_{\eta=0}^{\infty} \Phi^{p/2} (\alpha_\eta) 2^{d(\alpha_\eta)(1-p)/2} < \infty,
\]

(34)

Let

\[
F_n = \sum_{\{k : \| \alpha_k \| < n\}} \lambda_k \alpha_k,
\]

where

\[
\lambda_k = \frac{\Phi^{1/2} (\alpha_k)}{2^{d(\alpha_k)(1/p-1)/2}}, \quad \alpha_k = 2^{\| \alpha_k \| (1/p-1)} \left( D_{2^{\| \alpha_k \| +1}} - D_{2^{\| \alpha_k \|}} \right),
\]

(35)

By combining Theorem W and (34) we conclude that \( F = (F_n, n \in \mathbb{N}) \in H^p \).

By simple calculation we get that

\[
\hat{F}(j) = \begin{cases}
\Phi^{1/2} (\alpha_k) 2^{(\| \alpha_k \| + \langle \alpha_k \rangle)(1/p-1)/2} & \text{if } j \in \{ 2^{\| \alpha_k \|}, ..., 2^{\| \alpha_k \| +1} - 1 \}, \quad k = 0, 1, ... \\
0 & \text{if } j \notin \bigcup_{k=0}^{\infty} \{ 2^{\| \alpha_k \|}, ..., 2^{\| \alpha_k \| +1} - 1 \}.
\end{cases}
\]

(36)

Since

\[
D_{j+2^n} = D_{2^n} + w_{2^n} D_j, \quad \text{when } j \leq 2^n,
\]

by applying (36) we have

\[
\frac{S_{\alpha_k} F}{\Phi (\alpha_k)} = \frac{1}{\Phi (\alpha_k)} \sum_{\eta=0}^{k-1} \sum_{v=2^{\| \alpha_\eta \|}}^{2^{\| \alpha_\eta \| +1} - 1} \hat{F}(v) w_v + \frac{1}{\Phi (\alpha_k)} \sum_{v=2^{\| \alpha_\eta \|}} \hat{F}(v) w_v
\]

\[
= \frac{1}{\Phi (\alpha_k)} \sum_{\eta=0}^{k-1} \sum_{v=2^{\| \alpha_\eta \|}}^{2^{\| \alpha_\eta \| +1} - 1} \Phi^{1/2} (\alpha_\eta) 2^{(\| \alpha_\eta \| + \langle \alpha_\eta \rangle)(1/p-1)/2} w_v
\]

\[
+ \frac{1}{\Phi (\alpha_k)} \sum_{v=2^{\| \alpha_\eta \|}} \Phi^{1/2} (\alpha_k) 2^{(\| \alpha_k \| + \langle \alpha_k \rangle)(1/p-1)/2} w_v
\]

\[
= \frac{1}{\Phi (\alpha_k)} \sum_{\eta=0}^{k-1} \Phi^{1/2} (\alpha_\eta) 2^{(\| \alpha_\eta \| + \langle \alpha_\eta \rangle)(1/p-1)/2} \left( D_{2^{\| \alpha_\eta \| +1}} - D_{2^{\| \alpha_\eta \|}} \right)
\]

\[
+ \frac{2^{(\| \alpha_k \| + \langle \alpha_k \rangle)(1/p-1)/2} w_{2^{\| \alpha_k \|}} D_{\alpha_k - 2^{\| \alpha_k \|}}}{\Phi^{1/2} (\alpha_k)} := I + II.
\]

By using (34) for I we can write that

(39) \[ \| I \|_{L^p_{\alpha_k}} \]
It follows that
\[ |D_{\alpha_k - 2^{\alpha_k}}| \]
\[ = \left| D_{2^{\alpha_k}+1} - D_{2^\alpha_k} \right| + \sum_{j=(\alpha_k)+1}^{\alpha_k-1} (\alpha_k)_j \left( D_{2^j+1} - D_{2^j} \right) = \left| -D_{2^{\alpha_k}} \right| = 2^{\alpha_k} \]
and
\[ |II| = \frac{2^{(\alpha_k+1)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)} \left| D_{\alpha_k - 2^{\alpha_k}} \right| (x) \]
\[ = \frac{2^{(\alpha_k)(1/p-1)/2} 2^{\alpha_k}(1/p+1)/2}{\Phi^{1/2}(\alpha_k)}. \]

By using (39) we see that
\[ \|S_{\alpha_k}F\|_p \geq \|II\|_{L_p,\infty}^p - \|I\|_{L_p,\infty}^p \]
\[ \geq \frac{2^{(\alpha_k)(1/p-1)/2} 2^{\alpha_k}(1/p+1)/2}{\Phi^{1/2}(\alpha_k)} \mu \left\{ x \in G : |II| \geq \frac{2^{(\alpha_k)(1/p-1)/2} 2^{\alpha_k}(1/p+1)/2}{\Phi^{1/2}(\alpha_k)} \mu \left( \{I_{\alpha_k} \setminus I_{(\alpha_k)+1}\} \right) \right\}^{1/p} \]
\[ \geq \frac{2^{(\alpha_k)(1/p-1)/2} 2^{\alpha_k}(1/p+1)/2}{\Phi^{1/2}(\alpha_k)} \left( \mu \left( \{I_{\alpha_k} \setminus I_{(\alpha_k)+1}\} \right) \right)^{1/p} \]
\[ \geq c \frac{2^{(\alpha_k)(1/p-1)/2}}{\Phi^{1/2}(\alpha_k)} \to \infty, \quad \text{as} \quad k \to \infty. \]

Theorem 1 is proved.

**Proof of Corollaries 1-3.** By combining (6) and (7) we obtain
\[ I_{(n)} \setminus I_{(n)+1} \subset \text{supp} \{D_n\} \subset I_{(n)} \quad \text{and} \quad 2^{-(n)-1} \leq \mu \left( \text{supp} \{D_n\} \right) \leq 2^{-(n)}. \]

It follows that
\[ \frac{2^{d(n)(1/p-1)}}{4} \leq (n \mu \left( \text{supp} \{D_n\} \right))^{1/p-1} \leq 2^{d(n)(1/p-1)}. \]

Corollary 1 is proved.

To prove Corollary 2 we only have to calculate that
\[ |2^n + 1| = n, \quad |2^n + 1| = 0 \quad \text{and} \quad \rho (2^n + 1) = n. \]
By using the second part of Theorem 1 we see that there exists an martingale 
\( F = (F_n, n \in \mathbb{N}) \in H_p \) \((0 < p < 1)\), such that \( (16) \) is satisfied.

Let prove Corollary 3. Analogously to \((42)\) we can write that
\[
|2^n + 2^{n-1}| = n, \langle 2^n + 2^{n-1} \rangle = n - 1 \quad \text{and} \quad \rho(2^n + 2^{n-1}) = 1.
\]

By using the first part of Theorem 1 we immediately get inequality \((17)\), for all \(0 < p \leq 1\).

Corollaries 1-3 are proved. \(\square\)

**Proof of Theorem 2.** By using \((3)\) we have
\[
\left\| \frac{S_nF}{V(n)} \right\|_{1} \leq \|F\|_1 \leq \|F\|_{H_1}.
\]

By combining \((11)\) and \((43)\), after similar steps of \((29)\) we see that
\[
\left\| \frac{S_nF}{V(n)} \right\|_{H_1} \sim \int_{G} \left\| \frac{\tilde{S}_nF(t)}{V(n)} \right\|_{1} d\mu(t) \leq \|F\|_{H_1}.
\]

Now, we prove second part of Theorem 2. Let \(\{m_k : k \geq 0\}\) be subsequence of positive integers and function \(\Phi : \mathbb{N}_+ \to [1, \infty)\) satisfies conditions of Theorem 2. By using \((19)\) there exists an increasing sequence \(\{\alpha_k : k \geq 0\}\) of the positive integers such that
\[
\sum_{k=1}^{\infty} \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)} \leq \beta < \infty.
\]

Let
\[
F_n := \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k,
\]
where
\[
\lambda_k = \frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}, \quad a_k = D_{2^{\alpha_k} + 1} - D_{2^{\alpha_k}}.
\]

Analogously to Theorem 1 if we apply Theorem W and \((45)\) we conclude that \(F = (F_n, n \in \mathbb{N}) \in H_1\).

By simple calculation we get that
\[
\hat{F}(j) = \begin{cases} 
\frac{\Phi^{1/2}(\alpha_k)}{V^{1/2}(\alpha_k)}, & \text{if } j \in \{2^{\alpha_k}, ..., 2^{\alpha_k+1} - 1\}, \quad k = 0, 1, ... \\
0, & \text{if } j \notin \bigcup_{k=0}^{\infty} \{2^{\alpha_k}, ..., 2^{\alpha_k+1} - 1\}.
\end{cases}
\]

From \((37)\) and \((17)\) analogously to \((38)\) we obtain
\[
S_{\alpha_k}F = \sum_{\eta=0}^{k-1} \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \left(D_{2^{\alpha_\eta} + 1} - D_{2^{\alpha_\eta}}\right) + \frac{\Phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} w_{2^{\alpha_k}} D_{\alpha_k - 2^{\alpha_k}}.
\]
By combining (3) and (45) we have
\[
\left\| \frac{S_{\alpha_k}F}{\Phi(\alpha_k)} \right\|_1 \geq \frac{\phi^{1/2}(\alpha_k)}{\Phi(\alpha_k)V^{1/2}(\alpha_k)} \left\| D_{\alpha_k,2^{\alpha_k}} \right\|_1 - \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{k-1} \frac{\phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \geq \frac{V(\alpha_k - 2^{\alpha_k}) \phi^{1/2}(\alpha_k)}{8\Phi(\alpha_k)V^{1/2}(\alpha_k)} - \frac{1}{\Phi(\alpha_k)} \sum_{\eta=0}^{\infty} \frac{\phi^{1/2}(\alpha_\eta)}{V^{1/2}(\alpha_\eta)} \geq \frac{cV^{1/2}(\alpha_k)}{\phi^{1/2}(\alpha_k)} \to \infty, \text{ as } k \to \infty.
\]

Theorem 2 is proved.

**Proof of Theorem 3.** Let \(0 < p < 1\) and \(2^k < n \leq 2^{k+1}\). By using Theorem 1 we see that
\[(48) \quad \|S_nF - F\|_{H_p} \leq c_p \|S_nF - S_{2^k}F\|_{H_p} + c_p \|S_{2^k}F - F\|_{H_p} \leq c_p \left(1 + 2^{d(n)(1/p-1)}\right) \omega_{H_p}(\frac{1}{2^k}, F) \leq c_p 2^{d(n)(1/p-1)} \omega_{H_p}(\frac{1}{2^k}, F).
\]

The proof of estimate (20) is analogously to the proof of estimate (38). This completes the proof of theorem 3.

**Proof of Theorem 4.** Let \(0 < p < 1\), \(F \in H_p\) and \(\{m_k : k \geq 0\}\) be a sequence of nonnegative integers, satisfying condition (21). By using Theorem 3 we see that (22) holds.

Let proof second part of theorem 4. Under condition (12), there exists \(\{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\}\), such that
\[(49) \quad 2^{d(\alpha_k)} \uparrow \infty, \quad \text{as } k \to \infty, \quad 2^{(1/p-1)d(\alpha_k)} \leq 2^{(1/p-1)d(\alpha_{k+1})}.
\]

We set
\[F_n = \sum_{\{i : |\alpha_i| < n\}} \frac{a_i}{2^{(1/p-1)d(\alpha_i)}},
\]
where \(a_i\) is defined by (35). Since \(a_i\) is \(p\)-atom if we apply Theorem W and (49) we conclude that \(F \in H_p\). On the other hand
\[(50) \quad F - S_{2^n}F = \left(F(1) - S_{2^n}F(1), ..., F(n) - S_{2^n}F(n), ..., F(n+k) - S_{2^n}F(n+k)\right) = \left(0, ..., 0, F(n+1) - F(n), ..., F(n+k) - F(n), ...ight) = \left(0, ..., 0, \sum_{i=n}^{n+k} \frac{a_i}{2^{(1/p-1)d(\alpha_i)}}, ...ight), \quad k \in \mathbb{N}_+.
\]
is martingale. By combining (49) and Theorem W we get that
\begin{equation}
\omega_{H_p}(\frac{1}{2^{(\alpha_k)}}; F) \leq \sum_{i=k}^{\infty} \frac{1}{2^{(1/p-1)d(\alpha_i)}} = O\left(\frac{1}{2^{(1/p-1)d(\alpha_k)}}\right), \quad \text{as } n \to \infty.
\end{equation}

It is easy to show that
\begin{equation}
\hat{F}(j) = \begin{cases} 
2^{(1/p-1)(\alpha_k)}, & \text{if } j \in \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}, \ k = 0, 1, \ldots \\
0, & \text{if } j \notin \bigcup_{k=0}^{\infty} \{2^{\alpha_k}, \ldots, 2^{\alpha_k+1} - 1\}.
\end{cases}
\end{equation}

Analogously to (40) we can write that

\[ |D_{\alpha_k}| \geq 2^{(\alpha_k)}, \quad \text{for } I_{(\alpha_k)} \setminus I_{(\alpha_k)+1}. \]

Since
\[ \|D_{\alpha_k}\|_{L_{p,\infty}} \geq \left\|2^{(1/p-1)(\alpha_k)} \left(D_{2^{\alpha_k}} - D_{\alpha_k}\right)\right\|_{L_{p,\infty}} \]
\[ \geq \left(\mu \left(I_{(\alpha_k)} \setminus I_{(\alpha_k)+1}\right)\right)^{1/p} \]
by using (52) we have
\[ \|S_{\alpha_k} F - F\|_{L_{p,\infty}} \geq \left\|2^{(1/p-1)(\alpha_k)} \left(D_{2^{\alpha_k}} - D_{\alpha_k}\right)\right\|_{L_{p,\infty}} \]
\[ - \left\| \sum_{i=k+1}^{\infty} 2^{(1/p-1)(\alpha_i)} \left(D_{2^{\alpha_i}} - D_{\alpha_i}\right)\right\|_{L_{p,\infty}} \]
\[ = 2^{(1/p-1)(\alpha_k)} \|D_{\alpha_k}\|_{L_{p,\infty}} - 2^{(1/p-1)(\alpha_k)} \|D_{2^{\alpha_k}}\|_{L_{p,\infty}} \]
\[ - \sum_{i \geq k+1} \left|2^{(1/p-1)(\alpha_i)} \left(D_{2^{\alpha_i}} - D_{\alpha_i}\right)\right|_{L_{p,\infty}} \]
\[ \geq c - \frac{1}{2^{(1/p-1)d(\alpha_k)}} - \sum_{i \geq k+1} \frac{1}{2^{(1/p-1)d(\alpha_i)}} \geq c - \frac{2}{2^{(1/p-1)d(\alpha_k)}}. \]

This completes the proof of Theorem 4. \qed

**Proof of Theorem 5.** Let \( F \in H_1 \) and \( \{m_k : k \geq 0\} \) be a sequence of nonnegative integers, satisfying condition (25). By using Theorem 3 we see that (26) holds.

Let proof second part of theorem 5. Under conditions of second part of theorem 5, there exists sequence \( \{\alpha_k : k \geq 0\} \subset \{m_k : k \geq 0\} \) such that
\begin{equation}
V(\alpha_k) \uparrow \infty, \ k \to \infty \quad \text{and} \quad V^2(\alpha_k) \leq V(\alpha_{k+1}).
\end{equation}

We set
\[ F_n = \sum_{\{i : \alpha_i < n\}} \frac{a_i}{V(\alpha_i)}, \]
where \( a_i \) is defined by (46). Since \( a_i \) is a 1-atom if we apply Theorem W and (53) we conclude that \( F = (F_n, n \in \mathbb{N}) \in H_1. \)
Analogously to (50), (53) and Theorem W we can show that

\[ F - S_{2n}F = \left(0, \ldots, 0, \sum_{i=n}^{n+k} \frac{a_i}{V(\alpha_i)}, \ldots \right), \; k \in \mathbb{N}_+ \]

is martingale and

\[ \| F - S_{2n}F \|_{H_1} \leq \sum_{i=n+1}^{\infty} \frac{1}{V(\alpha_i)} = O\left(\frac{1}{V(\alpha_n)}\right) \quad \text{as} \quad n \to \infty. \]

It is easy to show that

\[ \hat{F}(j) = \begin{cases} \frac{1}{V(\alpha_k)}, & \text{if } j \in \left\{2^{\lfloor \alpha_k \rfloor}, ..., 2^{\lfloor \alpha_k \rfloor+1} - 1\right\}, k = 0, 1, \ldots \\ 0, & \text{if } j \notin \bigcup_{k=0}^{\infty} \left\{2^{\lfloor \alpha_k \rfloor}, ..., 2^{\lfloor \alpha_k \rfloor+1} - 1\right\}. \end{cases} \] 

By using (55) we have

\[ \| F - S_{\alpha_k}F \|_1 \geq \| \frac{D_{2^{\lfloor \alpha_k \rfloor+1}} - D_{\alpha_k}}{V(\alpha_k)} \|_1 + \sum_{i=k+1}^{\infty} \frac{D_{2^{\lfloor \alpha_i \rfloor+1}} - D_{2^{\lfloor \alpha_i \rfloor}}}{V(\alpha_i)} \]

\[ \geq \| \frac{D_{\alpha_k}}{V(\alpha_k)} \|_1 - \| \frac{D_{2^{\lfloor \alpha_k \rfloor+1}} - D_{\alpha_k}}{V(\alpha_k)} \|_1 - \| \sum_{i=k+1}^{\infty} \frac{D_{2^{\lfloor \alpha_i \rfloor+1}} - D_{2^{\lfloor \alpha_i \rfloor}}}{V(\alpha_i)} \|_1 \]

\[ \geq \frac{1}{8} - \frac{1}{V(\alpha_k)} - \sum_{i=k+1}^{\infty} \frac{1}{V(\alpha_i)} \geq \frac{1}{8} - \frac{2}{V(\alpha_k)}. \]

This completes the proof of Theorem 5. \(\square\)

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