A case for DOT: Theoretical Foundations for Objects with Pattern Matching and GADT-Style Reasoning

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Many programming languages in the OO tradition now support pattern matching in some form. Historical examples include Scala and Ceylon, with the more recent additions of Java, Kotlin, TypeScript, and Flow. But pattern matching on generic class hierarchies currently results in puzzling type errors in most of these languages. Yet this combination of features occurs naturally in many scenarios, such as when manipulating typed ASTs. To support it properly, compilers need to implement a form of subtyping reconstruction: the ability to reconstruct subtyping information uncovered at runtime during pattern matching. We introduce cDOT, a new calculus in the family of Dependent Object Types (DOT) intended to serve as a formal foundation for subtyping reconstruction. Being descended from pDOT, itself a formal foundation for Scala, cDOT can be used to encode advanced object-oriented features such as generic inheritance, type constructor variance, F-bounded polymorphism, and first-class recursive modules. We demonstrate that subtyping reconstruction subsumes GADTs by encoding $\lambda_{\mathbb{C}}G\mu$, a classical constraint-based GADT calculus, into cDOT.

CCS Concepts: • Software and its engineering → Patterns; Classes and objects; Object oriented languages; • Theory of computation → Type structures.

Additional Key Words and Phrases: DOT, pattern matching, GADT, classes, type systems

1 INTRODUCTION

In recent years, many programming languages in the object-oriented (OO) tradition started incorporating support for functional programming idioms, such as lambda expressions and pattern matching. Notably, Java gained a simple form of pattern matching (limited to single `instanceof` type patterns) in JDK 16, and work on extending this feature to more full-fledged pattern matching support is ongoing, with an implementation available as a feature preview in JDK 17. Version 7.0 of C# introduced support for type patterns in `switch` statements, and version 8.0 introduced `switch` expressions. Kotlin similarly supports a restricted form of pattern matching through its `when` block feature. Other widely-used statically-typed languages such as TypeScript and Flow support a variant of pattern matching called flow-sensitive or occurrence typing. Finally, languages like Scala and Ceylon, which merge the object-oriented and functional paradigms in a single language, have supported pattern matching from the start.

These new pattern matching approaches all rely on the idea to statically refine the type of a “scrutinee” (the object which is being pattern-matched) based on its runtime shape. Typically, they allow one to test whether a given value is an instance of some class C, and in the branch where it is known to be, use the value as such, without requiring an unsafe cast to C.

But combining generic classes with this form of refining type tests leads to puzzling cases where code that should seemingly compile is rejected. Consider the example in Figure 1, where we use Kotlin to define `Expr`, a class that represents simple arithmetic expressions, and `eval`, a function which evaluates said expressions. `eval` takes an argument of type `Expr<T>` and returns a `T.

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sealed class Expr<A>()
class IntLit (val value: Int) : Expr<Int>()
class ...

// other subclasses

fun <T>eval(expr: Expr<T>): T =
when (expr) {
  is IntLit -> expr.value
  is ... -> // other cases
}

Fig. 1. Example: evaluating simple conditional expressions in Kotlin

Intuitively, the single case shown in Figure 1 is correct. Yet, the latest version of the Kotlin compiler to date (as of April 2022) rejects it, saying that the type of expr.value is Int and not T. But our intuition is not wrong: we have checked that expr is an instance of IntLit, which means that it is simultaneously an Expr<Int> and an Expr<T>, which in turn means that T is Int.\(^1\) In other words, pattern matching on a value of type Expr<T> lets us discover that T is the same type as Int in the IntLit branch, which should let us conclude that eval is well-typed.

Generic class hierarchies [Bracha et al. 1998] have become pervasive in real-world code bases in Java [Parnin et al. 2013] and many other OO languages. Therefore, it seems inevitable that the adoption of new pattern-matching capabilities in these languages will lead to problems like the one outlined above. Yet, until now, this important topic saw surprisingly little formal or informal investigation. To the best of our knowledge, Scala is the only object-oriented language whose implementation currently attempts to allow such pattern matches, but formal investigations into how it does this have remained preliminary [Parreaux et al. 2019; Xu et al. 2021], and previous work on the topic [Emir et al. 2006, 2007; Kennedy and Russo 2005] was restricted to simple type systems. Furthermore, none of the previous approaches were mechanically verified. This is a problematic state of affairs because such reasoning turns out to be very subtle and hard to implement correctly [Giarrusso 2013; Parreaux et al. 2019].

The specific example of Figure 1 will appear familiar to readers already aware of generalized algebraic data types (GADTs), available in languages such as Haskell and OCaml, which also allow discovering hidden types and type relationships by matching on values. However, traditional GADTs operate in very different type systems than the ones of object-oriented languages. Traditional GADT constructors do not introduce a type of their own, and are only attached to a single parent type, forming a flat and closed hierarchy, contrasting with OOP’s nested and possibly open class hierarchies. Moreover, traditional GADT implementations only reason about type equations,\(^2\) whereas inheritance hierarchies, particularly when they involve type constructor variance, require us to reason about type inequations (i.e., subtyping relationships).

In this paper, we lay out formal foundations for GADT-style reasoning in object-oriented programming languages. We introduce subtyping reconstruction, a technique that allows pattern-matching functions like eval to be recognized as well-typed. We base our approach on a new and improved version of DOT (the Dependant Object Types calculus) extended with a case construct for pattern matching, and thus named cDOT. The expressiveness of this foundational calculus

\(^1\)This reasoning is based on the fact that in languages like Kotlin, Java, Scala, C\(^++\), etc. a derived class may inherit from a given base class at most once, with specific type arguments. It would not work in C\(^++\), where a class could inherit from two instantiations of the same class, such as Expr<Int> and Expr<bool>, which C\(^++\) considers to be two unrelated classes.

\(^2\)Interestingly, only reasoning about type equations leads to some paradoxes in languages like OCaml which do support (an explicit form of) subtyping, which was previously investigated by Scherer and Rémy [2013].
allows us to establish our reasoning principles in the presence of advanced object-oriented type system features, such as generic inheritance, type constructor variance, F-bounded polymorphism, and first-class recursive module types. This is important because such features are often present in practical programming languages like Java, C♯, and Scala, and they significantly complicate the task at hand. We also rigorously explore the connection between our GADT-style reasoning and traditional GADTs, formally demonstrating that our approach encompasses traditional GADT reasoning, since the latter can be encoded in the former.

We make the following specific contributions:

- We introduce cDOT, a formal calculus to serve as a foundation for subtyping reconstruction (Section 3 and Section 4). cDOT is based on pDOT [Rapoport and Lhoták 2019], which is itself an evolution of the original DOT [Amin et al. 2016], which has been proposed as a foundation for statically-typed object-oriented programming [Martres 2022]. We provide a mechanized proof of soundness for cDOT.
- We propose a variant of the $\lambda_{2G?}$ calculus formalizing traditional GADTs [Xi et al. 2003] (Section 5.2). Our variant removes some idiosyncrasies present in the original formulation of $\lambda_{2G?}$, notably by making its operational semantics deterministic and by making progress hold. We also provide a mechanized soundness proof for this modified calculus.
- We develop an encoding of our variant of $\lambda_{2G?}$ into cDOT (Section 5.4) and show that this encoding preserves typing (Section 5.5), which demonstrates that cDOT can express traditional GADT reasoning. This goes to show that subtyping reconstruction subsumes (i.e., is at least as powerful as) traditional GADT reasoning.

2 STATIC TYPING OF CLASS-BASED PATTERN MATCHING

In this section, we present the problem of objects with pattern-matching and GADT-style reasoning, as well as our proposed solution to it, from a high-level and intuition-focused point of view.

2.1 A Core View on Pattern Matching

We define pattern-matching loosely as any conditional expression form which allows learning about the runtime shape of a value and consequently extracting information from this shape in a statically type-safe way. For instance, Java’s original instanceof construct does not constitute proper pattern matching because making use of the information it returns (a boolean value) requires the use of an unsafe cast, as in ‘if (x instanceof IntLit) return ((IntLit)x).value’. The problem is that the consistency between the runtime class instance test and the cast to IntLit is not statically checked, and programmers could (and often do) get it wrong, for example if they had written ‘if (some_condition || x instanceof IntLit) return ((IntLit)x).value’ instead, possibly resulting in runtime ClassCastException crashes. On the other hand, Java’s newer construct ‘if (x instanceof IntLit y) return y.value’ is a proper pattern matching implementation, statically known to be safe.

Many pattern matching implementations allow complex nested patterns, but these can usually be treated as “syntactic sugar” over a core representation of pattern matching that proceeds one level at a time. In an object-oriented language, this core construct would be matching on the class of a given instance,3 similar to Java’s new instanceof construct and to Kotlin’s ‘when’.

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3Indeed, the Scala compiler essentially desugars its complex pattern matching syntax into a succession of instance tests.
2.2 Uncovering Types and Subtype Relationships

Consider the Expr/eval example from the introduction again, with more cases filled in:

```kotlin
sealed class Expr<A>()

class IntLit (val value: Int) : Expr<Int>()
class MkPair<B,C>(val lhs: Expr<B>, val rhs: Expr<C>) : Expr<Pair<B,C>>()
class First <B,C>(val pair: Expr<Pair<B,C>>) : Expr<B>()
class Second <B,C>(val pair: Expr<Pair<B,C>>) : Expr<C>()

fun <T> eval (expr: Expr<T>): T =
    when (expr) {
        is IntLit -> expr.value
        is MkPair<*<*> -> Pair (eval(expr.lhs), eval(expr.rhs))
        is First <T<*> -> eval(expr.pair).first
        is Second<*<T> -> eval(expr.pair).second
    }
```

As we saw, the IntLit branch of this example should compile even though expr.value returns an Int where a value of type T is expected: when we enter the IntLit branch, we discover that expr is simultaneously an Expr<T> and an Expr<Int>, and since Expr is invariant, this can only hold if T and Int are the same type. While this particular case appears simple, in general this form of reasoning is non-trivial.

Consider the other cases of eval, and notice the use of * to fill in those type parameters we locally do not know. For instance, having an Expr<T> that is also a First<B,C> tells us that B is T, because First<B,C> extends Expr<B>; however, it tells us nothing about C, which could be Int, Boolean, or any other type. Accordingly, C needs to be treated as an unknown. Just like for traditional GADTs, matching on generic classes may uncover unknown types, which are thus said to be existentially quantified. To handle these types adequately, the compiler must treat them as unspecified abstract types, locally considered to be distinct from all other types in the program (sometimes referred to as a “skolem”). Here again the Kotlin compiler fails us: it disregards these types, widening the corresponding pattern class types to Any and Nothing.

Typing eval also necessitates relating types that are only partially known, containing references to locally-uncovered type arguments. This is the case in the MkPair branch: the result of the branch should have type Pair<?B, ?C>, where ?B and ?C are the locally-uncovered unknown types associated with MkPair<?, ?,?>, and a compiler implementing GADT-style reasoning should recognize that in this branch, T is the same as Pair<?, ?>, allowing the expression to type check.

The addition of bounded polymorphism [Cardelli and Wegner 1985], and F-bounded polymorphism [Canning et al. 1989] in particular, which are supported by many object-oriented programming languages (including Java) and whereby every type parameter may be associated with possibly-cyclic bounds, complicates the picture further. Essentially, it mean that we should not treat locally-uncovered types as complete unknowns, but rather as bounded, possibly-recursive abstract types. This is already a strong justification to reach for the expressive power of the Dependent Object Types calculus, which features abstract types with recursive bounds.

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4 In Kotlin, the star symbol * is used as a placeholder when a type argument is unknown.
5 A standard example is a polymorphic function working over any type T where T is assumed to be a subtype of Comparable<T>, which means that T has a cyclic bound – i.e., T’s bound refers to T itself.
2.3 Type Constructor Variance

Languages like Kotlin, Scala, and C♯ support a concept known as (declaration-site) type constructor variance, which allows more flexibility in the way generic types can be related with each other. Declaring a class such as Expr to be covariant, which is done in Kotlin with the syntax `class Expr<out T>`, means that if Apple is a subtype of Fruit (written Apple <: Fruit), then Expr<Apple> is itself considered a subtype of Expr<Fruit> (i.e., Expr<Apple> <: Expr<Fruit>). How does defining Expr as covariant impact eval?

It turns out eval should still type check even when Expr is made covariant, but the reasoning becomes more subtle. For example, in the IntLit branch, T is no longer the same as Int. Indeed, the Expr<Int> we got in parameter may have been widened to any supertype, such as Expr<Any> (since Int is a subtype of Any) at the time it was passed to eval, so T = Any is another possibility we now have to account for. So all we can assume here is that Int is a subtype of T. Fortunately, this is enough for this branch to type check. Moreover, similar subtype-based reasoning can be applied to the other branches, allowing eval to type check as a whole.

As another example, if we discover through pattern matching that some value v is simultaneously a Comparable<T> and a Comparable<Int>, we cannot actually conclude anything about T and Int. Since Comparable is contravariant (meaning Comparable<S> <: Comparable<T> if T <: S), v could for example very well be a Comparable<Any>, since we have both T <: Any and Int <: Any, without T and Int being related in any way. This case differs from the IntLit case of the eval function described above because in the latter we know what is the precise type A that IntLit inherits Expr<A> with. Assuming a covariant Expr, we would be similarly stuck if all we had was the information that the scrutinee is both of type Expr<T> and of type Expr<Int> — we would then be unable to related T and Int.

In general, type constructor variance makes GADT-style reasoning for pattern matching in object-oriented languages a lot more difficult. Indeed, when a pattern class is covariant or contravariant, we can learn strictly less about the way its type parameters relate with the type of the scrutinee than in the invariant case. To make matters worse, some languages like Scala (but unlike Kotlin) support a feature called variant inheritance, whereby a class may inherit from a variant base several times and at different type arguments. For example, the following Kotlin code is illegal, but its equivalent in Scala is legal and is treated as Derived2 inheriting from both Base<Any> and Base<String>:

```kotlin
interface Base<out T>
open class Derived1: Base<Any>()
class Derived2: Derived1(), Base<String>
```

This has led to paradoxes related to pattern matching [Giarrusso 2013; Parreaux et al. 2019].

These considerations show that the study of subtyping reconstruction diverges significantly from that of traditional GADT reasoning.

2.4 A Guiding Reasoning Principle

The previous subsection seems to point us in an important direction: our reasoning should somehow be based on the existence of a “most precise” type argument used when inheriting from the scrutinee’s class. When we create a new instance of a class, we need to pick specific type arguments for every type parameter of every inherited class. But the type arguments we later ascribe to this instance may become less precise due to variance, and it is crucial to take that into account.

Let us inspect another example. We define a SUB<S, T> class that is contravariant in S and covariant in T and that works as runtime evidence that S is a subtype of T [Yallop and Dolan 2019]:
sealed class SUB<in S, out T>()
class Refl<U>() : SUB<U, U>()

To illustrate its use, we define a function to convert between two seemingly unrelated types:

fun <T> convert(t: T, ev: SUB<T, Int>): Int =
  when (ev) { is Refl <*> -> t }

When we check that ev is an instance the Refl class, we discover is that there was some type U which was used as a type argument to Refl when creating ev. This type is both a supertype of T and a subtype of Int, so it can only exist if T <: Int (otherwise the bounds of U would be inconsistent), which is what we need to type check convert.

Let us inspect one final example:

fun <T> convert2(t: T, ev: SUB<Expr<T>, Expr<Int>>): Int =
  when (ev) { is Refl <*> -> t }

By analogous reasoning, we discover that Expr<T> <: Expr<Int>. However, by itself this is not enough to type check convert2. Since Expr is covariant, we need to infer that T must necessarily be a subtype of Int for this relationship to hold, which exemplifies that subtyping reconstruction needs to “invert” subtyping relationships, inferring things about the arguments of related types.

Given a value x of some covariant class type Expr<T> (declared as class Expr<out A>), let us denote by x.A the precise argument type used to construct this instance, regardless of any widening that may have happened afterwards. So we must have that x.A <: T. The fact that this notation looks like a reference to a member A in x is not accidental: our main idea is to represent these “most precise types” as type members. Type members live inside class instances, which thus in a sense behave like first-class modules. This notation allows us to conveniently make explicit how a function like convert above could type check, by rephrasing it as follows:

fun <T> convert(t: T, ev: SUB<T, Int>): Int =
  when (ev) { is Refl <*> -> t : ev.U }

In the above, we added a type annotation to t which shows that it has type ev.U, since T <: ev.U by contravariance, which allows the term to type check since ev.U <: Int by covariance.

Reasoning about most-precise type arguments in terms of type members allows us to clarify and resolve subtleties and paradoxes related to pattern matching and inheritance. Going back to the Derived2 example from Section 2.3, we can now explain how Kotlin and Scala have two different notion of inheritance: when seeing an inheritance clause like 'Base<Int>', Kotlin implicitly assumes this.T = Int, whereas Scala only assumes this.T <: Int because T is covariant in Base. In fact, in Scala, it is even possible to inherit (indirectly) from both Base<Int> and Base<String>, which will result in this.T <: Int & String (where & denotes an intersection type). Both choices are sound and have pros and cons. They also lead to different subtyping reconstruction capabilities.

2.5 Type Parameters as Type Members, Classes as Runtime Tags

As we shall see in the next section, we go even further and completely do away with type parameters. Indeed, it turns out that type parameters themselves, along with type constructor variance, can be encoded using type members, intersections, and structural types, which are all part of our DOT-based calculus. Doing so allows us to work on a unified representation of all these concepts.

In fact, in Scala makes an exception for final and case classes, which use invariant inheritance. This is because case classes are usually intended for pattern matching, so it is better to recover more information from them in such use cases.
greatly simplifying the specification and soundness proof of our subtyping reconstruction principles.

Similarly, classes themselves are not a “core” concept of the various DOT calculi, and are normally encoded using DOT’s expressive recursive object type system. Indeed, it is possible to encode nominal classes by using bounded type members and by defining libraries as abstract modules that hide their implementations. This encoding of the static typing aspect of classes is not new [Martres 2022]. What is new in this paper is that we associate runtime “tags” to such encoded classes, which can be matched against through a case construct to recover some static type information about the corresponding instance, as we shall explained in detail in the next section.

Many of the ideas presented in this paper are already well-known. In fact, they are essentially how the problem of type checking pattern matching was reasoned about while implementing the Scala 3 compiler, after a long history of unsound and limited GADT support in Scala 2. But this paper is the first to rigorously formalize the system and derive (mechanized) proofs of its soundness.

2.6 Real-World Justification

Our experience with Scala is that GADT-style reasoning is pervasive in code bases that make advanced use of the type system to enforce compile-time guarantee about their programs. Classes like Expr naturally occur in programs that manipulate typed abstract syntax trees, for instance in database libraries, which need to manipulate query representations. Indeed, one of our motivating examples has been the design of a query compiler called dbStage using the Squid type-safe metaprogramming framework [Parreaux 2020] that relies very heavily on GADT-style reasoning. As another example, Eisenberg [2020] recently described Stitch, an interpreter and type checker similarly making heavy use of GADTs. Being able to discover type relationships by pattern matching also allows design patterns that were not possible or convenient before, for example using subtyping evidence [Yallop and Dolan 2019] as described in Section 2.4; the Scala standard library itself makes heavy use of similar subtyping evidence types.

3 INFORMAL INTRODUCTION TO CDOT

In this section, we informally present cDOT and discuss how it allows pattern matching to uncover hidden types and subtyping relationships.

cDOT belongs to the DOT family of systems, which were originally intended as formal foundations for Scala. The base DOT calculus was designed to explain surface type features of Scala using as few core features as possible [Amin et al. 2016]. Importantly for our purposes, there are no classes nor type parameters in DOT, as both can be encoded through type members [Amin et al. 2016; Martres 2022; Rapoport and Lhoták 2019]. This incidentally demonstrates that a language does not need to feature nominality in order to support subtyping reconstruction.

We now gradually introduce the concepts of cDOT by using Scala as the surface syntax. Our first goal is to encode IntLit from Figure 1. In cDOT, objects are created via object literals, which are typed with structural types, as in val e = new { val x = 0 } : { val x: Int }.

Types are compared based on their structure, so to encode IntLit, we need to first recover nominality. To do this, we rely on the fact that in cDOT, as well as in Scala, an object instance can contain associated types — called type members. For instance:

type Animal = { type Food; def eat(food: this.Food): Unit }

7The Scala versions of all examples we have shown so far type check correctly in Scala 3.
8Early prototypes of dbStage can be found at https://github.com/epfldata/dbstage.
9For example, the unzip method on List<T> requires an implicit parameter evidence that T <: (A, B) for some A, B.
def feed(ani: Animal, food: ani.Food) = ani.eat(food)

In the above example, Food is a type member of the object type Animal. Given a value of type Animal, for instance ani, we can reference the type member that lives in that instance using a path-dependent type, in this case ani.Food. This is analogous to defining Animal as a generic class and adding a type parameter to feed. The distinction between the above type and one like Animal<Food> is that a type member is existential by default. For instance, we can refer to a List<Animal>, a list in which every animal may have its own distinct type of food.

Like type parameters, type members can have (upper and lower) bounds, and in particular they can also be equal to some type (which simply means that their lower and upper bounds are the same). Much like instantiating a generic class requires specific type arguments for all type parameters of the class (they are often inferred, but are there nonetheless), when we create an object in cDOT, all of its type members must be defined to be equal to some existing types. This idea allows cDOT to capture our intuition from before about the “most precise” type arguments of object instances. To illustrate, we can create a value of type Animal as follows:

val goat: Animal = new { type Food = Any; def eat(food: Any) = () }

Given the above val, goat.Food refers to the precise food type that this specific animal eats, even though it is an abstract type and we know nothing about it due to the Animal type ascription.

Using bounded type members and structural types, we can recover nominal classes. To encode IntLit, we create a "package" object g with a type member IntLit that defines the members of the class. We also define a method newIntLit which allows us to create instances of said type member (i.e., a class constructor). We annotate the package object with a type that only leaves an upper bound on g.IntLit. Taking everything together, our g looks as follows:

val g: { type IntLit <: { type A }; def newIntLit(i: Int): IntLit } = new { type IntLit = { val value: Int }; def newIntLit(i: Int): IntLit = new { val value = i } }

Outside of g, we only have an upper bound on g.IntLit. As a result, with the above definition the only way to create an object of type g.IntLit is by calling g.newIntLit. In particular, a value that has the same shape as IntLit but does not statically have g.IntLit as part of its type will not be usable as a g.IntLit. In this way, by hiding the exact type g.IntLit stands for, we have encoded nominality.

We now take a separate look at encoding generic classes based on the example of Expr<A>. The class itself is encoded as a type member: type Expr <: { type A }; notice that its type parameter turns into a type member. An applied type like Expr<Int> is translated to an intersection type Expr & { type A = Int }. This translation is variance-sensitive; if Expr were covariant, its type argument would be translated using an upper bound instead, as Expr & { type A <: Int }; likewise, contravariant type arguments correspond to lower bounds in applied types.

Let us now come back to our package object g, which we extend with Expr<A>, with which IntExpr is now related through an intersection type:

val g: { type Expr <: { type A }; type IntLit <: Expr & { type A = Int; val value: Int }; def newIntLit(i: Int): IntLit }
Again, g’s annotation leaves only upper bounds for g.Expr and g.IntLit, to encode nominality. In cDOT, structural types with multiple members are represented via single-member structural type intersections. For instance, 

\[
\{ \text{type } A = \text{Int} ; \text{val } value : \text{Int} \}
\]

is represented as 

\[
\{ \text{type } A = \text{Int} \} \& \{ \text{val } value : \text{Int} \}.
\]

We use the former as a shorthand for the latter.

Function type parameters are also encoded using type members. A generic function in cDOT takes an additional argument with one type member per original type parameter. To illustrate, we can now take a look at an encoded version of eval from Figure 1:

```haskell
def eval(tp : { type T }, e: g.Expr & { type A = tp.T }): tp.T =
e match { case e1: g.IntLit => e1.value }
```

As expected, this example is well-typed in cDOT; let us inspect it in detail. The type of e1 is e.type & g.IntLit, with e.type being the singleton type of e. Singleton types are, conceptually, only inhabited by a single object instance (here e). Subtyping interacts with intersection types as expected: we can type e1 both as e.type and as g.IntLit. Since we have e1 : e.type, we know that e1 is the same object as e, i.e. it is an alias. However, because of the other part of the intersection type, the type of e1 is more precise than g.Expr & { type A = tp.T }, the type of e. In this sense, e1 makes the type of e more precise, which is what we need to type the example.

Since we have both e1 : g.IntLit and e1 : e.type, we respectively know that Int <: e1.A and e1.A <: tp.T. We conclude that Int <: tp.T by transitivity of subtyping. So we can return e1.value (whose type is Int) from eval (whose result type is tp.T).

One problem with eval so far is that the match expression cannot succeed at runtime, since our encoding of IntLit so far lacks a runtime tag. Implementations of OO languages typically tag every object value with its runtime class, which enables dynamic type checks and casts. To account for this, we actually encode each class C as both a type member (as seen before) and a tag value:

```haskell
type C <: { val tag : g.C_tag.type; ... }; val C_tag = g.freshTag()
```

where each call to g.freshTag() creates a unique tag value.\(^{10}\) The representation of tag values at runtime does not matter as long as we can compare them. E.g., we could use new empty objects, compared by object identity. Accordingly, to match on an object we now inspect its tag:

```haskell
x match { case e1: g.C => ... }
// is encoded as:
x.tag match { case g.C_tag => ... }
```

By construction of the encoding, we know that only objects created through the constructor of g.C will be associated with g.C_tag at runtime, which lets us type x in the branch body as x.type & g.C. (Note that this reasoning is not supported by Scala itself.) Finally, to encode class hierarchies more than one level deep, we would need to add multiple tags to each object.

\(^{10}\)This concept of unique tag is akin to TypeScript’s unique symbol feature (https://www.typescriptlang.org/docs/handbook/release-notes/typescript-2.7.html), although the latter only works with static paths, while we allow path-dependent tags.
This concludes our informal explanation of cDOT’s main ideas. As we shall see, in the actual formal calculus we strive for simplicity: objects may only have one tag and all type members are associated with a runtime tag, even those not meant to represent classes. Extending the system so that objects can have multiple tags is straightforward and so is left out of cDOT. Associating all type members with a tag is not a problem because when translating a program to cDOT all tags used in objects correspond to actual classes, so non-class type member tags could be erased.

4 FORMAL PRESENTATION OF CDOT

We now present the cDOT calculus. cDOT extends pDOT [Rapoport and Lhoták 2019], which itself is a generalization of DOT [Amin et al. 2016] that allows arbitrary path lengths in path-dependent types: both $x$.$\text{type}$ and $x$.$y$.$\text{type}$ are permitted. Importantly for us, it also formalizes singleton types.

All cDOT terms are in a variant of A-normal form, or ANF [Sabry and Felleisen 1993]. For instance, to apply one term to another, we must first bind the expressions to variables:

$$\text{let } x = t \text{ in let } y = s \text{ in } x y$$

This does not lead to any loss of expressivity, since a simple syntactic translation can change regular terms to ANF. Essentially, ANF gives us a name for every value, which is important when we allow function results to depend on their arguments with path-dependent types. Other DOT systems did not require ANF [Rompf and Amin 2016], but at the cost of having two rules for typing application: one for value arguments and another for variable arguments.

Since function results can depend on their arguments, in cDOT they are typed with path-dependent function types. The identity function is typed as follows:

$$\vdash \lambda(x : \top) x : \forall(x : \top) x.$\text{type}$$

If the result of a function type $\forall(x : S) T$ does not depend on its argument (i.e. if we have $x \not\in \text{fv } T$), we spell it as $S \rightarrow T$ as a shorthand.

---

$\text{x, y, z}$  
$\text{Variable}$  
$s := \text{Stable Term}$  
$s := p | v$  
$\text{path or value}$  

$\text{a, b, c}$  
$\text{Term member}$  
$d := \text{Definition}$  
$d := \{a = s\}$  
$\text{field definition}$  

$\text{A, B, C}$  
$\text{Type member}$  
$\{A = T\}$  
$\text{type definition}$  

$p, q, r := \text{Path}$  
$x := \text{variable}$  
$d \land d$  
$\text{aggregate definition}$  

$t, u := \text{Term}$  
$s := \text{stable term}$  
$S, T, U, V := \text{Type}$  
$T$  
$\text{top type}$  

$p q := \text{application}$  
$\bot$  
$\text{bottom type}$  

$\text{let } x = t \text{ in } u$  
$\text{let binding}$  
$\{a : T\}$  
$\text{field declaration}$  

$\text{case } p \text{ of } y : q.A \Rightarrow t_1 \text{ else } \Rightarrow t_2$  
$\text{case matching}$  
$\{A : S.T\}$  
$\text{type declaration}$  

$\nu := \text{Value}$  
$\nu(x : T)[p.A]d$  
$S \land T$  
$\text{intersection}$  

$\lambda(x : T) t$  
$\mu(x : T)$  
$\forall(x : S) T$  
$\text{recursive type}$  

$\text{p.A}$  
$p.A$  
$\text{type projection}$  
$\text{singleton type}$
So far, we have been simplifying one aspect of the syntax. Whereas one would typically use variables to reference let-bound values, in cDOT we can use *paths*. A path \( x.a_1\cdots.a_n \) is a chain of field selections \( .a_i \) starting from a variable \( x \). Like in pDOT, a cDOT path also represents the *identity* of an object, an important notion for a path-dependent type system. In other DOT systems, objects were bound in a store and referenced by simple variables, whereas in pDOT and cDOT objects may also be nested inside other objects, justifying the use of paths for object identity. The typing judgment \( \Gamma \vdash p : q.type \) means that \( p \) and \( q \) have the same identity. Like in pDOT, a path is never directly substituted for a value in cDOT, since doing so would strip an object of its identity.

In cDOT, objects are tagged in order to allow pattern matching. An object literal in cDOT looks as follows, where \( v \) is a binder that stands for the usual \texttt{new} in Scala:

\[
v(x : T)[p.A]d
\]

Let us inspect it from the right. \( d \) is the body of the object; it can contain fields and type members, whose names are case-sensitive: we use \( A, B, C \) for type members and \( a, b, c \) for term members. Object definitions in cDOT can be circular: their fields can freely reference other fields, regardless of the definition order. In order to ensure this does not lead to initialization problems, all fields must be initialized with *stable* terms \( s \) which are either values or paths. Methods are represented with fields bound to lambda abstractions. The “tag” \( p.A \) is the part of the object value that enables pattern matching. In cDOT, every object is tagged with a *type member* − recall that type members are how we formally represent classes. The *self reference* \( x \) (usually known as \texttt{this}) is explicitly bound in the syntax and annotated with a type \( T \), which can be used to specify the type that this object instance should have.

As an illustration, the body of the \texttt{newIntLit} function from before would look as follows:

\[
\lambda(i : \text{Int}) \, v(x : \{A = \text{Int}\} \land \{\text{value} : \text{Int}\})[g.\text{IntLit}] \{A = \text{Int}\} \land \{\text{value} = i\}
\]

Finally, the case form \textbf{case} \( p \) of \( y : q.A \Rightarrow t_1 \textbf{else} \Rightarrow t_2 \) allows pattern-matching on arbitrary values. There are two branches in this form: if \( p \) resolves to an object whose tag conforms to \( q.A \), we bind \( y \) to \( p \) and enter \( t_1 \), otherwise we enter \( t_2 \).

We will now inspect the typing and subtyping rules of cDOT. Most of the rules are the same as in pDOT; we have highlighted all the changes in grey.

### 4.1 Typing

We first go through the typing and subtyping rules of cDOT one by one. Figure 3 presents the typing rules of cDOT. We start with term typing rules, followed by definition typing, and finally explain the subtyping rules.

**Term typing rules.** The \texttt{VAR}, \texttt{LET} and \texttt{SUB} rules for typing variables, let forms and subsumption are as expected. Term abstractions are typed with dependent function types by \texttt{ALL-L}. Rule \texttt{ALL-E} assigns term applications the function result type, with the term argument replacing the function parameter. For example, if \( \Gamma \vdash p : \forall(x : T) \, x.type \) and \( q \) is a typeable path, then \( p\,q \) is typed as \( q.type \). The \texttt{{}-I} rule types the tagged objects. The body is typed with definition typing, explained later. The object type also must conform to its self-reference type and its tag.

Rules \texttt{SNGL-TRANS}, \texttt{SNGL-SELF} and \texttt{SNGL-E} deal with singleton types. If \( p \) is typeable, it can be typed as \( p.type \) with \texttt{SNGL-SELF}. This rule not only lets cDOT model Scala’s type system more closely than pDOT (which surprisingly lacks it), but also turns out to seriously impact both soundness (Section 4.4) and subtyping (Section 4.2). If \( \Gamma \vdash p : q.type \), we say that \( p \) aliases \( q \). In such a case, we can always assign \( p \) the same type as \( q \) using \texttt{SNGL-TRANS}. Aliasing is propagated through field selection with \texttt{SNGL-E}: \( p.a \) aliases \( q.a \) if \( q \) has a field \( a \) and \( p \) aliases \( q \).
Term typing

\[ \Gamma \vdash t : T \]

\[
\begin{align*}
\Gamma(x) &= T & (\text{VAR}) \\
\Gamma \vdash x : T
\end{align*}
\]

\[
\begin{align*}
\Gamma, x : T & \vdash t : U & (\text{ALL-I}) \\
\Gamma & \vdash \lambda(x : T) t : \forall(x : T) U \\
\Gamma & \vdash p : \{a : T\} & (\text{FLD-I}) \\
\Gamma & \vdash p : \{a : T\} & (\text{SNGL-SELF})
\end{align*}
\]

Field selections are typed with \( \Gamma / u1D45D./u1D44E \{ \}

\[
\begin{align*}
\Gamma & \vdash p : q.A & (\text{SNGL-E}) \\
\Gamma & \vdash p : q.A & (\text{SNGL-E}) \\
\Gamma & \vdash p : q.A & (\text{SNGL-E})
\end{align*}
\]

Definition typing

\[ p; \Gamma \vdash d : T \]

\[
\begin{align*}
\Gamma & \vdash \{A = T\} : \{A : T\} & (\text{DEF-TYP}) \\
\Gamma & \vdash \lambda(x : T) t : \forall(x : T) U & (\text{DEF-ALL}) \\
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\} \\
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\}
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\} \\
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\}
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\} \\
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\}
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\} \\
\Gamma & \vdash p : \{a = \lambda(x : T) t\} : \{a : \forall(x : T) U\}
\end{align*}
\]

Typeable paths

\[ \Gamma \vdash p \]

\[
\begin{align*}
\Gamma & \vdash p : T & (\text{WR}) \\
\Gamma & \vdash p : T
\end{align*}
\]

Tight bounds

\[ \text{tight } T = \begin{cases} 
U = V & \text{if } T = \{A : U..V\} \\
tight U & \text{if } T = \mu(x : U) \text{ or } \{a : U\} \\
tight U \text{ and tight } V & \text{if } T = U \land V \\
\text{true} & \text{otherwise}
\end{cases} \]

Fig. 3. cDOT typing rules

Field selections are typed with FLD-E. The FLD-I rule introduces field types when the field selection is typeable. This rule increases the expressiveness of the type system [Rapoport and Lhoták 2019]. Given \( \Gamma \vdash p : \mu(x : \{a : q.type\}) \) and \( \Gamma \vdash q : U \), we cannot derive that \( \Gamma \vdash q.type : < U \). However, with FLD-I we still can type \( p \) as \( \{a : U\} \).

The Rec-I and Rec-E rules respectively introduce and eliminate recursive types. Recursive types do not participate in cDOT subtyping; instead, we unwrap recursive types with Rec-E and reintroduce them with Rec-I. For example, the following derivation tree shows how we can derive
We now inspect the subtyping rules defined in Figure 4. Subtyping is made reflexive and transitive with Refl and Trans. Rules for top, bottom and intersection types are typical (respectively Top, Bot and \(\text{AND}_1<:; \text{AND}_2<:; <:-\text{AND}\)). Rules \(\text{FLD}<:-\text{FLD}\), \(\text{TYP}<:-\text{TYP}\) and \(\text{ALL}<:-\text{ALL}\) derive the standard subtyping relationships between object fields, type members and functions. Rules \(<:-\text{SEL}\) and \(\text{SEL}<:-\) relate a path-dependent type and its bounds. Importantly, these rules also allow path-dependent types to introduce new subtyping relationships. For example, if we have \(\Gamma \vdash p: \{A: \text{Int}.q.T\}\), then by Trans and relating \(q.T\) with its bounds we also can derive \(\Gamma \vdash \text{Int} <: q.T\).

The \(\text{\&}_I\) rule is the introduction rule for intersection types. One may think that this rule is redundant because we have the SUB and \(<:-\text{AND}\) rules. However, this rule does add expressiveness when interacting with recursive types [Amin et al. 2016].

The Case rule types case forms case \(p\) of \(y: q.A \Rightarrow t_1\) else \(\Rightarrow t_2\). Typing the else branch \(t_2\) is straightforward. When typing \(t_1\), we extend the environment with \(y: p.\text{type} \land q.A\), which essentially introduces a witness that the path \(p\) can be typed as \(q.A\). We see in Section 4.2 that together with subtyping inversion, such a witness allows reconstruction of subtype relationships.

**Definition typing.** The definition typing rules in Figure 3 are used to type object bodies. In pDOT and cDOT, only function values, object values and paths can be used to initialize a field. The Def-ALL, Def-NEW and Def-PATH rules type object fields as their precise type. The tight \(T\) condition in Def-NEW ensures all type members in \(T\) have equal bounds. In addition to type checking the definition, the Def-NEW rule also checks the conformance between the object type and the type tag. The Def-Typ rule types the type member definition. The AndDef-I rule type multiple member definitions as an intersection type. The prefix \(p\) in definition typing keeps track of the identity of the currently typing object, since the definition typing rules can be used to type nested object definitions.

**Tagged objects.** An object form \(v(y: T)[x.A]d\) must itself be typeable as its tag \(x.A\); this condition is checked by the \(\{\}\)I and Def-NEW rules. It may appear problematic to require all object to refer to another object’s type member in its tag: after all we need a tag to type check the very first object in a cDOT program, but tags themselves come from objects! This is not actually a problem because in cDOT objects can be typed recursively: we can type check a self-tagged object let \(x = v(x: \{A = T\})[x.A]\{A = T\}\), whose type member \(x.A\) may later be used as a “top tag” that can tag any object. In later examples we will sometimes skip the tag of an object; we understand such objects to be tagged with this “top tag”. Notice that our tag system does not support inheritance: in order to encode class hierarchies more than one level deep, we would need to tag objects with several class tags. For simplicity, we leave this straightforward extension out of cDOT.

### 4.2 Subtyping

We now inspect the subtyping rules defined in Figure 4. Subtyping is made reflexive and transitive with Refl and Trans. Rules for top, bottom and intersection types are typical (respectively Top, Bot and \(\text{AND}_1<:; \text{AND}_2<; <:-\text{AND}\)). Rules \(\text{FLD}<:-\text{FLD}\), \(\text{TYP}<:-\text{TYP}\) and \(\text{ALL}<:-\text{ALL}\) derive the standard subtyping relationships between object fields, type members and functions. Rules \(<:-\text{SEL}\) and \(\text{SEL}<:-\) relate a path-dependent type and its bounds. Importantly, these rules also allow path-dependent types to introduce new subtyping relationships. For example, if we have \(\Gamma \vdash p: \{A: \text{Int}.q.T\}\), then by Trans and relating \(q.T\) with its bounds we also can derive \(\Gamma \vdash \text{Int} <: q.T\).
Subtyping \( \Gamma \vdash S <: T \)

\[ \begin{array}{ll}
\Gamma \vdash T <: \top & \text{(Top)} \\
\Gamma \vdash \bot <: T & \text{(Bot)} \\
\Gamma \vdash T :: T & \text{(REFL)} \\
\Gamma \vdash S :: T & \text{(TRANS)} \\
\Gamma \vdash T \& U ::= T & \Gamma \vdash T \& U ::= U & \text{(AND)} \\
\Gamma \vdash S :: T & \Gamma \vdash S :: U & \text{(<:<AND)} \\
\Gamma \vdash T :: U & \Gamma \vdash \{a :: T\} ::= \{a :: U\} & \text{(FLD::<:<FLD)} \\
\Gamma \vdash p ::= \{d :: S,T\} & \Gamma \vdash S ::= S,A & \text{(<::SEL)} \\
\Gamma \vdash T ::= T & \Gamma \vdash \{q \& p\} & \text{(<::<::<::TYPI)} \\
\Gamma \vdash p ::= q.fulltype & \Gamma \vdash q & \text{(<::<::<::SGLq}<:<::q)} \\
\Gamma \vdash T ::= T & \Gamma \vdash \{A :: S,T\} & \text{(<::<::<::SEL)} \\
\Gamma \vdash p ::= p.fulltype & \Gamma \vdash \{A :: S,T\} & \text{(<::<::<::SGLq}<:<::q)} \\
\Gamma \vdash T ::= T & \Gamma \vdash \forall (x :: S,T) & \text{(<::<::<::All)} \\
\end{array} \]

Unique membership \( S \setminus T \)

\[ \begin{array}{ll}
\Gamma \vdash T :: S \setminus U & \text{(UNIQUE-<:<LABEL)} \\
\end{array} \]

Unique membership with label \( S \setminus U :: L \)

\[ \begin{array}{ll}
\{A :: S,T\} \setminus \{A :: S,T\} :: \{A\} & \text{(ONE-TYP)} \\
\{a :: T\} \setminus \{a :: T\} :: \{a\} & \text{(ONE-FLD)} \\
\mu(x :: T) \setminus \mu(x :: T) :: \emptyset & \text{(ONE-RECE)} \\
\end{array} \]

Aliasing paths. Rules SGLpq<:< and SGLq<:< establish equivalence of aliasing paths: if we can derive \( \Gamma \vdash p ::= q.fulltype \), then with subtyping we can freely substitute \( p \) for \( q \) in types and vice versa. The expressivity of these rules is augmented by SGL-SEL. Consider the following example:

**case p of y: LitInt \Rightarrow p else \Rightarrow \ldots**

In the matched branch we should be able to type \( p \) as \( q.fulltype \). However, this is not possible without the SGL-SEL rule. When typing the type case, the environment is extended with \( y ::= p.fulltype \& \text{LitInt} \). In pDOT, we can only type \( \Gamma \vdash y ::= p.fulltype \), but not the other direction due to the asymmetric nature of pDOT path aliasing: the SGL-TRANS rule only allows the propagation of typing from \( p \) to \( y \), but not the reverse. This means that, although \( y \) should witness that \( p \) can be typed as \( q.fulltype \), there is no way to assign the witnessed type to \( p \). The SGL-SEL rule mitigates the limitation by making path aliasing symmetric (i.e. we can derive \( \Gamma \vdash p ::= q.fulltype \) from \( \Gamma \vdash q ::= p.fulltype \)).
SNGL-TRANS can type \( p \) as \( \text{gLitLit} \). The following derivation tree illustrates how SNGL-SELF enables us to type path aliasing symmetrically. Here \( \Gamma' \) denotes \( \Gamma, y : p.\text{type} \land \text{gLitLit} \).

\[
\begin{align*}
\Gamma' \vdash p & \quad \text{SNGL-SELF} \\
\Gamma' \vdash p : p.\text{type} & \quad \text{SNGL-SELF} \\
\Gamma' \vdash y : p.\text{type} & \quad \text{y<:SNGL-qp:<:} \\
\Gamma' \vdash p.\text{type} \land \Gamma' \vdash y.\text{type} & \quad \text{SUB} \\
\Gamma' \vdash p : y.\text{type} & \quad \text{SUB}
\end{align*}
\]

**Subtyping inversion.** Finally, we can reconstruct premises of other subtyping rules with \text{Fld-<:-Fld-INV}, \text{Typ-<:-Typ-INV}_1, and \text{Typ-<:-Typ-INV}_2, the inversion rules. For instance, using \text{Fld-<:-Fld-INV}, we can derive \( \Gamma \vdash \text{Int} < x.\text{T} \) from \( \Gamma \vdash \{a : \text{Int}\} < \{a : x.\text{T}\} \), respectively the premise and conclusion of \text{Fld-<:-Fld}. The inversion rules are only useful if the bindings in the context introduce some subtyping relationships as assumptions. Consider the following example:

\[
\lambda(x : (T : \text{Int})) \lambda(y : [A : \text{Int}].\{a : x.\text{T}\}) \ x 0
\]

To type it, we need to derive that \( \text{Int} < x.\text{T} \). Based on the bounds of \( y.A \), we can derive that \( \{a : \text{Int}\} < \{a : x.\text{T}\} \); in other words, this relationship is an assumption introduced by \( y \). Intuitively, in any context in which we can construct a value for \( y \), we can also derive that \( \text{Int} < x.\text{T} \); using bounds of \( y \) and subtyping inversion, we can also derive that if we only have \( y \):

\[
\begin{align*}
\Gamma \vdash \{a : \text{Int}\} < : y.A & \quad \text{<:-SEL} \\
\Gamma \vdash y.A < : \{a : x.\text{T}\} & \quad \text{SEL-<:} \\
\Gamma \vdash \{a : \text{Int}\} < : \{a : x.\text{T}\} & \quad \text{TRANS} \\
\Gamma \vdash \text{Int} < : x.\text{T} & \quad \text{Fld-<:-Fld-INV}
\end{align*}
\]

The combination of bindings that introduce subtyping assumptions and inversion rules that allow deriving premises of assumptions is subtyping reconstruction: the ability to use a binding to derive subtyping relationships necessary to construct a value which could stand for the binding.

The inversion rules use the unique membership relation \( T \setminus U \), defined in Figure 4. It allows extracting a component \( U \) out of an intersection type \( T \), which allows inverting relationships involving intersection types, for instance:

\[
\begin{align*}
\Gamma \vdash \{a : S\} \land \{b : U\} < : \{a : T\} & \quad \text{Fld-<:-Fld-INV} \\
\Gamma \vdash S < : T
\end{align*}
\]

In order to invert relationships that involve an intersection type on the RHS, we can use \text{TRANS}. For instance, in the above derivation, the inverted subtyping relation could be derived like this:

\[
\begin{align*}
\Gamma \vdash \{a : S\} \land \{b : U\} < : \{a : T\} & \quad \text{TRANS} \\
\Gamma \vdash \{a : S\} \land \{b : U\} < : \{a : T\}
\end{align*}
\]

In order for subtyping inversion to be sound, all the labels of \( T \) in \( T \setminus U \) must be unique. To see why, consider this relationship:

\[
\Gamma \vdash \{a : T_1\} \land \{a : T_2\} < : \{a : U\}
\]

The original premise used to derive this relationship may have been either \( \Gamma \vdash T_1 < : U \) or \( \Gamma \vdash T_2 < : U \). Since we cannot know which one it was, we can derive neither with the inversion rules.

### 4.3 Evaluation

The operational semantics of cDOT is presented in Figure 5. We define \( \sim^* \) as the transitive, reflexive closure of \( \sim \). Since our reduction rules are significantly different from pDOT, we do not highlight the differences.

**Store.** Similarly to pDOT, cDOT reduction is store-based. During evaluation, we still need to keep track of object identities; to do so, the store \( y \) binds variables to values. Note that objects still contains nested objects during evaluation: in general, the identity of an object may be a path.
Path resolution. Compared to pDOT, the most significant change in cDOT is that the Resolve rule allows reducing aliasing paths: given $\gamma \vdash p \leadsto q$, we can reduce $\gamma \vdash p \mapsto \gamma \mid q$. We call a path $p$ that directly resolves to a value in $\gamma$ a resolved path $\rho^\gamma$ (see Figure 5). Resolved paths cannot be reduced any further; much like singleton types capture object identities on the type level, on the term level a resolved path directly corresponds to the runtime identity of an object. Importantly, as a consequence of modelling recursive objects and modules, it is possible to create objects with circular fields. References to such fields results in infinite loops. A circular object may be defined like this:

$$v(x : \{a : x.b.type \land \{b : x.a.type\}\} \{a = x.b\} \land \{b = x.a\})$$

In pDOT, it is impossible to reduce paths, and even resolving path aliases breaks preservation. For example, if $p$ can be typed as $q.type$ in $\Gamma$ and it looks up to $q$ in $\gamma$, there is still no guarantee that $q$ can be typed as $q.type$ in $\Gamma$ and so allowing $\gamma \mid p \mapsto \gamma \mid q$ would violate type preservation. In cDOT, we can always type $q$ as $q.type$ thanks to the SnGL-SELF rule and so reducing paths preserves their types.

$$\gamma \mid \text{let } x = p \text{ in } t \mapsto \gamma \mid t[p/x]$$
$$\gamma \mid \text{let } x = \rho^\gamma \text{ in } t \mapsto \gamma \mid t[\rho^\gamma/x]$$

Fig. 6. The LET-PATH (top) and APPLY (bottom) rules of pDOT (left) and cDOT (right).
The reduction of paths also brings significant changes to the reduction of let bindings and lambda applications, as we can see on Figure 6. In pDOT, a let-bound path is reduced by substituting the path into the let body; in cDOT, we first reduce the path until it is fully resolved.

Case term reduction. Both pDOT and cDOT ensure that paths typeable with function and object types always resolve to a value, which is necessary for soundness [Rapoport and Lhoták 2019]. Figure 6 shows that pDOT transitively looks up paths when reducing apply forms \( q \circ p \); being unable to look up \( q \) would result in a stuck term. However, cDOT case forms allow scrutinees of arbitrary types and such scrutinees may refer to a circularly-defined field. To avoid transitive lookups of such fields, cDOT allows reducing paths, letting case forms with such scrutinees loop endlessly. This mirrors how looking up lazily defined, circular fields in Scala also results in endless loops.

### 4.4 Metatheory

We provide a mechanized soundness proof for cDOT, basing on soundness proof for pDOT [Rapoport and Lhoták 2019], at its core formulated in the standard progress-and-preservation style [Wright and Felleisen 1994]. Our Coq proof scripts have \( \approx 9800 \) lines in the pDOT soundness proof. We attach the proof as an artifact.

Evaluating any well-typed cDOT program either diverges or results in a normal form.

**Theorem 4.1 (Type Safety).** If \( \vdash t : T \), then the evaluation of \( t \) either diverges, or \( \emptyset \vdash t \mapsto \gamma \mid u \) such that (a) \( u \) is a normal form under \( \gamma \); (b) \( \vdash u : T \) for some \( \Gamma \) and (c) \( \gamma : \Gamma \).

\( \gamma : \Gamma \) means that \( \gamma \) conforms to \( \Gamma \), i.e. for any variable \( x_i \in \text{dom} \Gamma \), if \( \Gamma(x_i) = T_i \) and \( \gamma(x_i) = v_i \), then \( \Gamma \vdash v_i : T_i \). Normal forms are values and resolved paths.

**4.4.1 Soundness Proof: From pDOT to cDOT.** Our soundness proof extensively reuses the infrastructure and toolkits provided by pDOT. The strategy central to the proof of pDOT is to stratify typing rules into seven levels [Rapoport and Lhoták 2019], as illustrated in the following diagram:

\[
\begin{align*}
\text{General} & \rightarrow \text{Tight} & \rightarrow \text{Introduction-qp} & \rightarrow \text{Introduction-pq} & \rightarrow \text{Elim-III} & \rightarrow \text{Elim-II} & \rightarrow \text{Elim-I} & \rightarrow \text{Elim} (\tau_e) \\
\end{align*}
\]

The stratified typing level hierarchy tackles the bad bounds problem [Amin et al. 2016] and eliminates cycles in type derivation [Rapoport et al. 2017; Rapoport and Lhoták 2019]. Innermost typing levels assign precise typing information to paths directly followed from the environment and path aliasing. It is easy to do induction and reason about typing judgments on these levels. With a series of transformation lemmas converting judgments between levels, the general recipe of pDOT soundness proof is to transform the surface typing level (general typing \( \vdash \)) to inner typing levels and then reason about the typing judgments. (See Appendix B for a detailed description.)

The soundness proof of cDOT follows the stratified typing levels of the pDOT soundness proof. We describe our extensions in detail in Section 4.4.2. We state Preservation and Progress as follows:

**Theorem 4.2 (Preservation).** Let \( \Gamma \) be an inert, well-formed typing environment and let \( \gamma : \Gamma \). If \( \gamma \mid t \mapsto \gamma' \mid t' \) and \( \Gamma \vdash t : T \), then there exists an inert and well-formed typing environment \( \Gamma' \) such that \( \Gamma' : \gamma' \) and \( \Gamma' \vdash t' : T \).

**Theorem 4.3 (Progress).** Let \( \Gamma \) be an inert, well-formed typing environment and let \( \gamma : \Gamma \). If \( \Gamma \vdash t : T \), then either \( t \) is a normal form, or there exists \( \gamma' \) and \( t' \), such that \( \gamma \mid t \mapsto \gamma' \mid t' \).

An environment is inert if it only contains inert types, i.e. types that cannot introduce subtyping assumptions. A type is inert if it is a lambda type or an object types with tight type members (see Appendix B). An environment \( \Gamma \) is well-formed if every path mentioned in \( \Gamma \) is typeable in \( \Gamma \).
We assume that both \( p \) and \( q \) are resolved, the pattern is matched and \( t_2 \) is a term that can be typed as \( p . \text{type} \). We can type the case term \( t \) as \( p . \text{type} \) with the Case rule. Then, when reducing the case term, we apply the Case-Then rule and reduce the term to \( y [p / \gamma] = p \). Here, to prove preservation, \( \Gamma \vdash p : p . \text{type} \) must be derivable, which is only made possible by the SnGL-Self rule.

The Progress proof in the case for terms of the form \( \text{case } p \text{ of } y : q . A \Rightarrow y \text{ else } \Rightarrow t_2 \) relies on Lemma 4.1, which shows if \( p \) resolves to an object \( \nu (x : T)[r . A]d \) then \( r \) also can be resolved. Otherwise, neither Case-Then nor Case-Else would apply, making the term stuck.

**Lemma 4.1 (Tag Resolution).** Let \( \Gamma \) be inert and well formed. If \( p \) is typeable in \( \Gamma, \gamma : \Gamma \vdash p \leadsto \nu (x : T)[r . A]d \), then there exists \( \rho^\nu \) such that \( \gamma \vdash r \leadsto^* \rho^\nu \).

### 4.4.2 Proving Inversion Rules with Invertible Subtyping

When implementing the soundness proof, we add inversion rules only to the surface typing level, and keep the inner levels unchanged. Then, we only need to re-prove the transformation lemma from general typing to the tight level. This is to prove that the inversion rules are admissible in tight typing when the environment is inert, which is stated in the following two lemmas.

**Lemma 4.2 (Field Inversion).** In an inert environment \( \Gamma \), if \( \Gamma \vdash _\downarrow U <: \{ a : T_2 \} \) and \( U \downarrow \{ a : T_1 \} \), then \( \Gamma \vdash _\downarrow T_1 <: T_2 \).

**Lemma 4.3 (Type Member Inversion).** In an inert environment \( \Gamma \), if \( \Gamma \vdash _\downarrow U <: \{ A : S_2 .. T_2 \} \) and \( U \downarrow \{ A : S_1 .. T_1 \} \), then \( \Gamma \vdash _\downarrow S_2 <: S_1 \) and \( \Gamma \vdash _\downarrow T_1 <: T_2 \).

\( \Gamma \vdash _\downarrow S <: T \) denotes tight subtyping, a notion inherited from pDOT (see Appendix B). Tight typing and inertness forbid the derivation of subtyping relations from type member bounds. They can rule out absurd subtyping relations, such as \( \{ a : \nu (x : S T) \} <: \{ a : \bot \} \), which cannot be inverted.

The major obstacle when proving subtyping inversion is Trans. For instance, if we try to invert the tight subtyping relation \( \Gamma \vdash _\downarrow \{ a : S \} <: \{ a : T \} \) to get \( \Gamma \vdash _\downarrow S <: T \) by induction on its derivation, we become stuck at the Trans case. There, we have \( \Gamma \vdash _\downarrow \{ a : S \} <: U \) and \( \Gamma \vdash _\downarrow U <: \{ a : T \} \) as premises and the inductive hypothesis is not applicable since the shape of \( U \) is unknown.

To tackle this problem, we add invertible subtyping as an additional layer (see Figure 7). In this layer, we remove Trans and instead inline it in the path replacement and type member subtyping rules. We have four rules for path replacement in invertible subtyping, compared to two in general and tight subtyping. This is because general subtyping relies on transitivity to replace aliasing.
paths on the left hand side. If $\Gamma \vdash S <: T$ and $p$ aliases $q$, to derive $\Gamma \vdash S [q/p] <: T$ we first need to derive $\Gamma \vdash S [q/p] <: S$ by $\text{Sgl}q\cdot<:a$ and then derive $\Gamma \vdash S [q/p] <: T$ by $\text{Trans}$.

Invertible subtyping can easily be inverted and inductively reasoned about. Similarly to typing, we prove that tight subtyping can be transformed to invertible subtyping.

**Lemma 4.4** ($\lhd_0$ to $\lhd_\#$). If $\Gamma$ is an inert typing environment and $\Gamma \vdash S <: T$, then $\Gamma \vdash_\# S <: T$.

And it is easy to prove that we can transform invertible subtyping to tight subtyping.

**Lemma 4.5** ($\lhd_\#$ to $\lhd_0$). For any environment $\Gamma$, if $\Gamma \vdash_\# S <: T$ then $\Gamma \vdash S <: T$.

**Lemma 4.6** (Field Inversion in $\lhd_\#$). If $\Gamma \vdash_\# \{a: S\} <: \{a: T\}$, then $\Gamma \vdash_\# S <: T$.

**Proof sketch.** We can prove the theorem by induction on the derivation of the judgment. The $\text{Refl}_\#$ and $\text{Fld}<:-\text{Fld}_\#$ cases are trivial. In the first case we get $S = T$. The goal can be proven by applying the reflexivity rule. In the second case the goal is already proven. The other four cases from path replacement can be proven with the inductive hypothesis and the typing rules.

## 5 TRANSLATING CONSTRAINT-BASED GADTS INTO CDOT

In this section, we demonstrate the expressiveness of subtyping reconstruction by proving that it subsumes generalised algebraic data types (GADTs).

### 5.1 Introducing GADTs

Algebraic Data Types (ADTs) are a feature of ML-like languages which allow representing data types that can be created with a set of constructors known in advance. They are similar to class hierarchies one level deep, like $\text{Expr}$ from Figure 1; ADT constructors correspond to subclasses of $\text{Expr}$. However, every constructor of a polymorphic ADT must have the exact same type parameters as the ADT itself. To correspond to an ADT, each subclass of $\text{Expr}$ must have the form $\text{Expr} <\Lambda$.

11 Pattern matching on GADTs extends $\Delta$ with additional information. As long as the constraint entails equivalence between some types (denoted as $\Delta \models \tau_1 \equiv \tau_2$), they are interchangeable to the typing judgment. For instance, in the above eval example, the constraint $\Delta$ used to type the $\text{IntLit}$ branch would contain $\alpha \equiv \text{Int}$, which we could use to type $i$ as $\alpha$ instead of Int. The entailment relationship in $\lambda_{2,G\mu}$ is based on substitution: we have $\Delta \models \tau_1 \equiv \tau_2$ if for any $\theta$ that solves $\Delta$, we also have $\theta \tau_1 \equiv \theta \tau_2$. To illustrate, given List $\alpha \equiv \text{Int List} \in \Delta$, we would also have $\Delta \models \alpha \equiv \text{Int}$.

Our notion of subtyping reconstruction subsumes GADTs from $\lambda_{2,G\mu}$: it supports subtyping and class hierarchies of arbitrary depth. cDOT does not need an additional context $\Delta$, since its bindings can introduce subtyping assumptions which later can be inverted (Section 4.2). In order to make this claim precise and validate it, we show an encoding of a variant of $\lambda_{2,G\mu}$ into cDOT.

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11 Xi et al. used $\Gamma$ for their environments as expected. We use $\Pi$ instead, to avoid reusing the $\Gamma$ metavariable.
5.2 A Variant of $\lambda_{2,G\mu}$

We have adjusted $\lambda_{2,G\mu}$ by removing those features which are not relevant for the purpose of demonstrating that cDOT can encode GADTs. In order of significance, the changes are:

*Adjusted constraint reasoning.* Besides the constraint-based entailment, $\lambda_{2,G\mu}$ also features an equivalent relation $\Delta \vdash \tau_1 \equiv \tau_2$ defined syntactically [Xi et al. 2003, Fig. 7]. Our variant of $\lambda_{2,G\mu}$ uses a restricted version of this relation, which we denote as $\vdash'$. Our changes are limited to removing rules that allow ex-falso reasoning, as well as reasoning about function and universal types. We give the exact rules in the appendix.

Ex-falso rules allow deriving arbitrary type equivalences from a contradictory constraint. Branches which introduce contradictory constraints would never be entered at runtime, since they correspond to patterns that cannot match. Such cases could in principle be handled while preserving their semantics by replacing the branch bodies with a loop, for instance. We excluded these rules, as programming languages that support GADTs (for instance OCaml, Haskell, Agda) typically do not allow typing arbitrary code if a branch cannot be entered.

Similarly, we have excluded the rules that allow reasoning based on equality of function and universal types. Again, such reasoning is not core to GADTs and not all implementations of GADTs allow it (for instance, Agda and Idris don’t invert function types and Haskell does not invert polymorphic types). We still allow inverting equalities of GADTs and tuples, for example given $\alpha \neq \beta \equiv \gamma \neq \delta \in \Delta$ we can infer $\Delta \vdash' \alpha \equiv \gamma$.

To show what the rules defining $\vdash'$ look like, we present two of them below. One allows replacing a type with a variable equal to it, another allows inverting GADT equality.

$$\frac{\overline{a}, \bar{\Delta} \vdash \alpha \mapsto \tau \quad \vdash' \tau_1 \equiv \tau_2[\alpha \mapsto \tau]}{\overline{a}, \tau \equiv \alpha, \bar{\Delta} \vdash' \tau_1 \equiv \tau_2} \quad \frac{\overline{a}, \bar{\Delta} \equiv \bar{\tau}_A \vdash \bar{\tau}_B \vdash' \tau_1 \equiv \tau_2}{\overline{a}, (\bar{\tau}_A)\bar{T} \equiv (\bar{\tau}_B)\bar{T} \vdash' \tau_1 \equiv \tau_2}$$

*Deterministic reduction.* $\lambda_{2,G\mu}$ reduces case forms non-deterministically if multiple patterns match, which is a strange choice of evaluation semantics. Handling non-determinism in cDOT would be quite involved, as we would either need to modify the evaluation rules of pDOT to be non-deterministic or encode non-determinism. While either choice is an interesting problem, non-determinism is orthogonal to GADTs and reduction is deterministic in our variant of $\lambda_{2,G\mu}$.

*Exhaustive and not-nested patterns.* In contrast to Xi et al., our variant of $\lambda_{2,G\mu}$ disallows nested patterns and requires case forms to match on all GADT constructors. Nested patterns can be desugared to remove the nesting. Many languages, like Coq for example [Inria, CNRS and contributors 2021], desugar such matches to make compilation into primitive operations easier. See [Maranget 2008] for an efficient algorithm converting nested pattern matches into decision trees.

Original $\lambda_{2,G\mu}$ allowed inexhaustive pattern matches, which were allowed to get stuck. We instead require patterns to be exhaustive; verifying this is simple without nested patterns.

5.3 Metatheory

In order to validate our variant of $\lambda_{2,G\mu}$, we provide a mechanised proof of soundness, expressed as standard progress and preservation theorems. Thanks to the additional exhaustivity requirement we added to pattern matching, well-typed terms never get stuck: we can prove progress, which did not hold in the original formulation. The mechanized proof was originally described by Waśko [2021] and is attached as one of the artifacts of this paper.

**Theorem 5.1 (Preservation).** For any well typed term $\Delta; \Pi \vdash e : \tau$, if $e \rightarrow e'$, then the other term is also well-typed and has the same type: $\Delta; \Pi \vdash e' : \tau$. 

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A case for DOT
Technical Report, 2022, arXiv

\[ \texttt{let lib = } v(\text{lib})[\text{lib.A}] \{
\begin{align*}
A &= \top; \\
\text{Expr} &= \mu(x : \{X : \bot..\top\} \land \{\text{pmatch : } \forall(r : \{R : \bot..\top\})\forall(\text{litIntCase : lib.IntLit } \to \text{r.R}) \text{r.R}\}); \\
\text{IntLit} &= \mu( \\
&\quad x : \{X : \text{Int}\} \land \{\text{data : } \lambda(_.) \, x.X\} \\
&\quad \land \{\text{pmatch : } \forall(r : \{R : \bot..\top\})\forall(\text{litIntCase : lib.IntLit } \to \text{r.R}) \text{r.R}\} \\
\}; \\
\text{mkIntLit} &= \lambda(x : \text{Int})v(y)[\text{lib.IntLit}] \{
\begin{align*}
X &= \text{Int}; \text{data : } \forall(\_ \, y.X); \text{data = } \lambda(_.) \, x; \\
\text{pmatch : } \forall(r : \{R : \bot..\top\})\forall(\text{litIntCase : lib.IntLit } \to \text{r.R}) \text{r.R} \\
\text{pmatch} &= \lambda(r : \{R : \bot..\top\})\lambda(\text{litIntCase : lib.IntLit } \to \text{r.R}) \, \text{litIntCase y} \\
\}; \\
\text{eval} &= \forall(x : \text{lib.Expr}) \, x.X; \\
\text{eval} &= \lambda(x : \text{lib.Expr}) \, \text{let } y = \lambda(y : \text{lib.IntLit}) \, y.\text{data in x.pmatch y} \\
\} \text{ in } \cdots
\]
\]

Fig. 8. Translating \text{Expr} from \(\lambda_2.\mu\) into cDOT

**Theorem 5.2 (Progress).** For any well typed term \(\Delta ; \Pi \vdash e : \tau\), the term is either a value or it can be reduced further: there exists a term \(e'\) such that \(e \rightarrow e'\).

### 5.4 Encoding

To encode \(\lambda_2.\mu\) in cDOT, we emulate GADTs using object types equipped with type members which will serve the purpose of encoding type parameters and evidence for type equalities together with a visitor function (analogous to Scott-encoding [Stump 2009, Sec. 5.2]) which will allow to emulate exhaustive pattern matching together with refinement of type equalities. As our encoding does not depend on case forms, it remains applicable to other systems in the DOT family which may potentially support subtyping reconstruction, such as gDOT [Giarrusso et al. 2020].

We denote an encoding of a term \(e\) as \(E(e, \Theta)\) and an encoding of a type \(\tau\) as \(T(\tau, \Theta)\). Both definitions depend on a substitution \(\Theta\) mapping \(\lambda_2.\mu\) type names \(\alpha\) to type projections \(p.\alpha\) and \(\lambda_2.\mu\) variable names \(x\) to cDOT expressions. For example, we may have: \(\Theta = [\alpha \mapsto x_\alpha : T, x \mapsto y]\).

In order to encode a complete \(\lambda_2.\mu\) program, we define a function to encode the signature \(\Sigma\) and a term \(11b\) representing basic primitives like the unit and tuple types. The signature \(\Sigma\) defines what GADTs definitions a program may use: it maps GADT constructors to their signatures, for example: \(\Sigma(c_j) = \forall \beta_j, \tau_j \rightarrow (\sigma_j) T\). We show a full definition in the appendix. Overall, encoding of a \(\lambda_2.\mu\) program \(e\) under a signature \(\Sigma\) is defined as follows:

\[ \text{EncodeProgram}(e, \Sigma) = \]

\[ \begin{align*}
\text{let } \text{lib} &= \ldots \text{ in let } \text{env} &= \text{EncodeSigma}(\Sigma) \text{ in } E(e, \emptyset) \\
\end{align*} \]

Figure 8 shows how we would translate a \(\lambda_2.\mu\) version of \text{Expr} from Figure 1 into cDOT.

### 5.5 Typing Preservation

To prove typing preservation, we first define a notion of correspondence between environments in \(\lambda_2.\mu\) and cDOT. We define cDOT type equality as follows:
We gather both notions in a single property: we conjecture that our encoding preserves the operational semantics of the original λ₂,Gₘ program. To prove this, we define a property specifying that all equations from the original environment Λ are also analogously emulated in the encoded environment Γ.

**Definition 5.5 (Equational correspondence).** A cDOT environment Γ together with a substitution Θ satisfies equations of Λ if the following conditions are met:

1. For each α ∈ Λ, for Θ(α) = p:A, Γ ⊢ p : {A : ⊥..Γ}.
2. For each τ₁ ≡ τ₂ ∈ Λ, Γ ⊢ T(τ₁, Θ) ≜ T(τ₂, Θ).
3. Γ contains the bindings lib and env as defined in S(Σ).

Now we can define a property specifying that all equations from the original environment Λ are also analogously emulated in the encoded environment Γ.

**Definition 5.6 (Environment correspondence).** A cDOT environment Γ corresponds to λ₂,Gₘ contexts Λ and Π under a substitution Θ if Ψ(Λ; Θ; Γ) and for each xf such that Π(xf) = τ, we have Γ ⊢ Θ(xf) : T(τ, Θ).

With all that, we can state the main theorem.

**Theorem 5.7 (Type Preservation).** For every Π, Λ, e, τ, any Θ such that fv(e) ∪ fv(τ) ⊆ dom(Θ), any Γ such that Φ(Δ; Π; Θ; Γ), we have Γ ⊢ E(e, Θ) : T(τ, Θ).

Based on this theorem and correctness of the env and lib encodings (shown in the appendix), we can prove that for any well-typed λ₂,Gₘ program, the result of EncodeProgram is also a well-typed program in cDOT.

Apart from other more technical lemmas, an important lemma that we are using in the proof is one that specifies that under our modified notion of constraint reasoning, if an equation is derivable using this reasoning, then an analogous equation is also derivable in cDOT starting from analogous assumptions:

**Lemma 5.8 (Equation emulation).** For any Δ ⊢ τ₁ ≡ τ₂ and any Θ and Γ, if Ψ(Δ; Θ; Γ), then Γ ⊢ T(τ₁, Θ) ≜ T(τ₂, Θ).

### 5.6 Preservation of Operational Semantics

We conjecture that our encoding preserves the operational semantics of λ₂,Gₘ in the sense that if e, a λ₂,Gₘ term, reduces to some e′, then Enc(e) →* t′ such that Enc(e′) corresponds to t′. This is illustrated by the following diagram, where ≃ is an equivalence relation described below:
The main technicality which would be involved in the proof of the above property is that reduction in $\lambda_{2.G\mu}$ relies on substitution as opposed to a store like cDOT. We envision the gap between the two can be bridged by defining an equivalence relation $\equiv$ on cDOT terms. Two terms should be considered equivalent if they are the same modulo $\alpha$-equivalence as well as order of let- and store-bindings and presence of unused bindings. In particular, this equivalence makes it possible to deal with the fact that reduction in $\lambda_{2.G\mu}$ inherently may duplicate a value, while reduction in cDOT only duplicates references to a store-binding.

Aside from this problem, the proof should be rather straightforward: most forms from $\lambda_{2.G\mu}$ are translated to their direct cDOT equivalents, and our encodings of type abstraction and tuples are standard [Amin et al. 2016]. The most involved case of the proof will be correctness of the pattern matching translation, which relies on encoding case forms into calls to pmatch like in Scott encoding [Stump 2009, Sec. 5.2]. In particular, the proof will have to demonstrate that replacing contradictory branches by infinite loops in the encoded term preserves the operational semantics; intuitively, this is because a contradictory branch cannot be entered, which is precisely why $\lambda_{2.G\mu}$ allows typing any term as any type in such branches in the first place.

6 IMPLEMENTING SUBTYPING RECONSTRUCTION

How can we implement subtyping reconstruction? We begin by inspecting the basic problem with writing an algorithm that can use subtyping relationships reconstructed by pattern matching.

6.1 Towards Formal Decidable Subtyping

Recall that cDOT term variables can introduce subtyping assumptions to the context (see subtyping inversion in Section 4.2). As a concrete example, consider $\Gamma = x : \{A : \bot..\top\}, y : \{B : \bot..\top\}, z : \{C : x.A..y.B\}$. We can derive $\Gamma \vdash x.A < : y.B$ thanks to the bounds of $z.C$.

The question is how to design an algorithmic subtyping judgment which is a subset of the declarative subtyping rules from Figure 4. More specifically, how do we decide what subtyping assumptions are introduced by bindings and when to use the inversion rules? Kennedy and Russo [2005] show C$^\#$ minor, a formalisation of C$^\#$ which allows pattern matching much like cDOT. The declarative subtyping judgment in C$^\#$ minor has the form $\Delta \vdash S < : T$, where $\Delta$ is a list of subtyping assumptions. This judgment features inversion rules which allow recovering the premises of assumptions, again much like cDOT. In their system, a branch of a pattern match may introduce subtyping assumptions; specifically, if $Q$ is the type of the value being matched and $P$ is the type of values matched by the pattern, then the subtyping assumptions introduced by the branch can be found by finding a class $K$ such that $Q < : K[\overline{T}]$ and $P < : K[\overline{S}]$, and then relating $S_t$ to $T_t$ based on type parameter variance. Conceptually, the problem is much simpler in cDOT: the assumptions introduced by a binding can be found by simply listing all of its type members and then relating each member’s upper and lower bounds. cDOT also clearly points to the reason we can relate $S_t$ to $T_t$: if the branch is entered, we have a value $v$ which can be typed as both $Q$ and $P$, and this value’s type member (representing the most precise type argument to $K$) is related to both $S_t$ and $T_t$.

This leaves the question of deciding when to use the inversion rules. Emir et al. [2006] solve the problem for C$^\#$ minor: they show an algorithmic subtyping judgment $\Psi \vdash S < : T$, with $\Psi$ being a list of upper and lower bounds on type variables of the form $X < : T$ and $T < : X$. This judgment no longer needs inversion rules: the assumptions in $\Psi$ are as simple as possible and cannot be inverted. $\Delta$ and $\Psi$ are related: any subtyping assumption $S < : T$ can be decomposed into...
bounds by repeatedly applying appropriate inversion rules; we denote the result of doing so as \( \text{Dec}(S <: T) \). Any \( \Delta \) can be converted to an equivalent \( \Psi \) by first decomposing assumptions in \( \Delta \) and then repeatedly simplifying the assumptions transitively introduced by the bounds, i.e. by reaching a fixpoint of the following sequence:

\[
\Psi_1 = \bigcup \{ \text{Dec}(S <: T) : S <: T \in \Delta \} \\
\Psi_{i+1} = \bigcup \{ \text{Dec}(S <: T) : \{ S <: X, X <: T \} \subset \Psi_i \} \cup \Psi_i
\]

The algorithmic and declarative subtyping judgments are equivalent; subtyping in \( \text{C}^\# \) minor is decidable only if we restrict contravariant classes [Kennedy and Pierce 2007].

Intuitively, we should be able to define an algorithmic subtyping judgment for cDOT with a similar approach, though we leave the details of this process to future work. This hypothetical subtyping would have the form \( \Gamma ; \hat{\Psi} \vdash S <: T \), with \( \hat{\Psi} \) being a list of upper and lower bounds on abstract types \( \hat{\Lambda} \): type projections \( p.A \) and singleton types \( p.type \). However, cDOT features intersection types and dependent function types (both absent from \( \text{C}^\# \) minor), which make the problem more complex. For instance, it is not possible in general to decompose \( \hat{\Lambda} \land \hat{B} <: T \) into bounds. These difficulties would be encountered when implementing subtyping reconstruction for any sufficiently advanced type system, such as the one in Scala or TypeScript. Still, it is possible to have an algorithm which gives an incomplete approximation of the declarative subtyping. For the particular case of cDOT, we are unlikely to do better. We can encode System \( \text{F}_\subset \) in cDOT [Amin et al. 2014], making it likely that cDOT subtyping is undecidable [Hu and Lhoták 2019].

6.2 Outline of the Scala Implementation

The first author of this paper was at the center of the effort to implement subtyping reconstruction for Scala 3. Based on this experience, we outline key aspects of the implementation and relate them to the formalism developed in the present paper.

The sketch from the previous section gives us the two basic problems the implementation needs to solve: finding subtyping assumptions introduced by patterns, and decomposing them into bounds so that they can be used for type checking. In the implementation, the abstract types \( \hat{\Lambda} \) for which bounds can be reconstructed include type projections \( p.A \) and singleton types \( p.type \), as well as type parameters and pattern-bound types,\(^{12}\) both denoted as \( X \). As the assumptions following from a pattern are decomposed into bounds, we store the latter in a map from abstract types to pairs of types: upper and lower bounds of the abstract types.\(^{13}\) This map will later be used for type checking the branch associated with the pattern, making it part of the context used to type check Scala code. Before we describe how we find assumptions and decompose them, we should note that the implementation can be incomplete: it is always sound to reconstruct less precise bounds than the ones permitted by the formalism. Being able to do so is helpful when dealing with a language with a complex type system. The Scala implementation at first only supported reconstructing bounds on function type parameters and finding assumptions following from class\(^{14}\) types; it was then progressively expanded to support bounds on class type parameters, type projections and singleton types, as well as assumptions following from structural, intersection and union types.\(^{15}\)

We now inspect the two problems in a simplified setting. The bounds are reconstructed based on the \( \text{scrutinee} \) type \( Q \), i.e. the type of the value being matched, and the \( \text{pattern} \) type \( P \), i.e. the

---

\(^{12}\)In Scala, types starting with a lower-case letter mentioned in a pattern are \textit{bound} by the pattern, c.f. Section 2.2.

\(^{13}\)The particular data structures of the Scala implementation were described by Xu et al. [2021].

\(^{14}\)When discussing Scala, we follow Martres [2022] and say “class” to mean either a \textit{proper} class or a trait.

\(^{15}\)The support for type projections, singleton types, and structural types is currently under review and not yet merged.
type of values matched by the pattern. We initially disregard Scala’s variant refinement, as well as intersection, union, and higher-kindred types. We only consider two forms of pattern types: either $P = K$, where $K$ is a class, or $P = K[\overline{X}]$, where $\overline{X}$ are all pattern-bound types, i.e. fresh abstract types. To illustrate, only the first two patterns in the following example would be allowed:

```java
def eval[T](expr: Expr[T]): T = expr match
  case e : IntLit => e.value
  case e : First[b,c] => eval[b](e.pair)._1
  case e : Second[Int,c] => // pattern type disallowed
```

Observe how naming type arguments in the above example mirrors how in cDOT we need to use type projections $x.A$ to reference type arguments of a pattern type representing a generic class. To illustrate, we show a cDOT term that formally models the two first cases (assuming a simple generalisation of case forms). In particular, note how we reference the type argument of the encoded version of First:

```java
eval = λ(tl: {T: ⊥..T}) λ(expr: g.Expr ∨ {A : tl.T..tl.T})
  case expr of e : g.IntLit ⇒ e.value
  | e : g.First ⇒ eval(ν(s)(T = e.B))(e.pair)._1
```

Essentially, there are three procedures involved in reconstructing bounds. First, $\text{ASSUM}(Q,P)$ lists subtyping assumptions following a value of type $Q$ being also of type $P$. Second, $\text{DEC}(S < : T)$ decomposes an assumption into a list of bounds. Third, $\text{BOUNDS}(Q,P) = \text{ASSUM}(Q,P).\text{map}(\text{DEC})$ reconstructs bounds following from $Q$ and $P$. These definitions should be understood as pseudocode rather than the precise implementation.

To find subtyping assumptions based on scrutinee and pattern types $Q$ and $P$, we look for classes $K$ such that $Q < : K[\overline{T}]$ and $P < : K[\overline{S}]$: such classes can be found by walking the inheritance hierarchy and in general there might be more than one such class. The relationships between the type arguments of each such $K$ are what we are looking for: if the $i$-th parameter of $C$ is covariant or invariant, we must have $S_i < : T_i$, and likewise for contravariance. For instance, if $Q = \text{Expr}[X]$ and $P = \text{IntLit}$, we have $Q < : \text{Expr}[X]$ and $P < : \text{Expr}[\text{Int}]$ and we return $\text{Int} < : X, X < : \text{Int}$.

Then, each assumption we find is decomposed into bounds and stored in the aforementioned map. Decomposing an assumption $S < : T$ is implemented similarly to checking if $S$ is a subtype of $T$. The main difference is that if we encounter a (sub)goal of the form $\hat{A} < : T$ or $T < : \hat{A}$ which cannot be concluded based on current bounds, we add the relationship to the map of reconstructed bounds. For instance, when decomposing List[X] < : List[Int], we encounter the subgoal $X < : \text{Int}$ and store the relationship. Much like in the formal sketch, adding a new bound may result in new transitive subtyping assumptions: as a simple example, if we extend List[X] < : Y with $Y < : \text{List}[\text{Int}]$, it now follows that List[X] < : List[Int]. To handle such cases, we first decompose these transitive assumptions and then return to decomposing the original assumption $S < : T$.

What we described so far is very close to the algorithm sketched by Kennedy and Russo [2005] and presented there more formally. We will now briefly sketch how we support more complex type system features. The implementation supports pattern types containing subelements of the form $K[\overline{T}]$ (generic classes applied to arbitrary types, not only pattern-bound fresh abstract types) by behaving as though all type arguments were fresh abstract types, then checking if the concrete arguments conform to the bounds of the corresponding abstract type. Due to higher-kindred types (HKTs), we need to consider the injectivity of type operators when decomposing assumptions. Specifically, if we encounter a (sub)goal of the form $F[\overline{S}] < : T$ or $S < : F[\overline{T}]$, we proceed only if $F$ is an alias of a class type, and therefore is known to be injective. This correctly handles situations where one can conclude that $F[\overline{S}] < : F[\overline{T}]$ with $\overline{S}$ and $\overline{T}$ being unrelated, for instance when we
define type F[X] = Int. While we could support annotating abstract types with their injectivity [Stolarek et al. 2015], so far we did not find the complexity that this would introduce worth the corresponding gains. Scala’s variant inheritance\(^\text{16}\) forces adjustments to listing subtyping assumptions \(\text{ASSUM}(Q, P)\). If \(P\) is an (applied) class type using variant inheritance and \(Q <: K[\overline{T}]\) as well as \(P <: K[\overline{S}]\), we only take the relationships between invariant type arguments as assumptions. The support for intersection and union types is the most complex extension and relies on seeing reconstructed bounds \(\Psi\) as constraints on abstract types; we unfortunately lack the space for an appropriately detailed description.\(^\text{17}\)

While developing the implementation, we always used a translation from class types into structural types as a guide for what bounds to reconstruct. Contrary to expectations, correctly handling various features of Scala’s type system required quite sophisticated notions. Based on our findings, we believe that the complexity of the implementation may be reduced if we instead based it directly on cDOT concepts. While we leave developing this alternative implementation as future work, we envision it as follows. To find subtyping assumptions based on pattern and scrutinee types \(P, Q\), we would first translate them to structural types \(\overline{P}, \overline{Q}\) and then return the relationships between upper and lower bounds of the type members of \(\overline{P} \land \overline{Q}\). For example, \(\text{Expr}[X]\) and \(\text{IntLit}\) would be translated to \{type \(A = X\)\} and \{type \(A = \text{Int}\)\} and we would return \(\text{Int} <: X, X <: \text{Int}\), like the current implementation. Likewise, to decompose an assumption \(S <: T\) into bounds, we would first translate \(S\) and \(T\) into structural types \(\overline{S}, \overline{T}\) and then decompose \(\overline{S} <: \overline{T}\) into bounds according to cDOT inversion rules. For example, to decompose \(\text{List}[X] <: \text{List}[\text{Int}]\), we would translate it to \{type \(A <: X\)\} <: \{type \(A <: \text{Int}\)\}, which can be inverted into \(X <: \text{Int}\). We believe that this algorithm would at least match the precision of the current implementation, would require less special cases and would potentially naturally extend to intersections and unions without resolving to constraint-like reasoning. To illustrate, consider that the current implementation needs special reasoning to handle a covariant \(\text{Expr}\) and \(\text{IntLit}\) using invariant refinement, while the alternative implementation would simply translate \(\text{Expr}\) to \{type \(A <: X\)\} and \(\text{IntLit}\) to \{type \(A = \text{Int}\)\}.

7 RELATED WORK

The earliest implementation of pattern matching based on runtime subtype checks we are aware of is that of Modula-3 [Böszörményi and Weich 2012] with its TYPECASE statements,\(^\text{18}\) allowing type-safe conditional refinement based on the runtime type of a scrutinee.

Subtyping in DOT systems is believed to be undecidable, as we can encode \(F_<\) bounded polymorphism in DOT. Hu and Lhoták [2019] show that this reasoning does not constitute a proof and demonstrate \(D_<\), an undecidable fragment of DOT without self-references and intersection types. \(D_<\) subtyping itself has two fragments which have subtyping algorithms. However, undecidability of cDOT subtyping is unlikely to be a problem in practice. Specific class hierarchies (allowed in Scala) on their own make subtyping undecidable [Kennedy and Pierce 2007]. Similarly, Java generics are also Turing-complete, as one can reduce a Turing machine to a fragment of Java [Grigore 2017].

Parreaux et al. [2019] describe that DOT type members can be used to encode GADTs. They observe that doing so solves the soundness problem of variant inheritance [Giarrusso 2013].

\(^{16}\)Recall that in Scala, only final and case classes invariantly inherit from their parents (see Section 2.3).

\(^{17}\)The most important aspect is that we need to consider constraint disjunctions (see, for instance, Petrucciani [2019]) in \(\text{ASSUM}(Q, P)\) when either type is a union type and in \(\text{DEC}(S <: T)\) when \(S\) is an intersection type or \(T\) is a union type. Since calculating exact disjunctions is too costly, we approximate them such that the exact constraint entails the approximation.

\(^{18}\)https://www.cs.purdue.edu/homes/hosking/m3/reference/typecase.html
GADTs are well-known and widely implemented [Peyton Jones et al. 2006]. Xi et al. [2003] may have been the first to study them in an ML-like setting with $\lambda_{ZG\mu}$. The original OCaml GADT implementation was based on a type-and-constraint system like $\lambda_{ZG\mu}$ [Garrigue and Normand 2011]. Sulzmann et al. [2007] present System $F_C$, an intermediate representation in the Haskell compiler which supports GADTs (and other features) with type equality witnesses and coercions [Mitchell 1984]. The Haskell implementation was previously based on a type-and-constraint system, which can be encoded into System $F_C$ [Sulzmann et al. 2007]. In System $F_C$ the IntLit constructor of Expr $\alpha$ contains a witness $\text{co : } \alpha \sim \text{Int}$. Pattern matching on Expr $\alpha$ binds this witness, which then can be used to type 0 as Int with an explicit coercion: $0 \triangleright \text{sym co}$. Sulzmann et al. claim that explicit casts make System $F_C$ code more robust to transformations by ensuring that they do not violate witness scoping. cDOT objects also act like witnesses of subtyping assumptions, but they only participate implicitly in subtyping. It would be interesting to see a variant of cDOT where objects participate in subtyping through explicit coercions, like System $F_C$ witnesses.

Scherer and Rémy [2013] study the combination of GADTs and variance in the OCaml setting. They demonstrate that associating the constructors of a GADT such as $\text{Expr}$ with type equality constraints prevents the GADT from being covariant. The problem does not arise in our setting, where constructors are, essentially, associated with subtyping constraints.

GADTs present a unique challenge for checking pattern exhaustivity: based on a value’s type, we can rule out that it was created using particular constructors. For instance, a value of type $\text{Expr < Int >}$ cannot be an instance of MkPair. Without special support for GADTs, checking pattern exhaustivity emits spurious inexhaustivity warnings, for instance that a pattern for MkPair is missing when matching on an $\text{Expr < Int >}$. Karachalias et al. [2015] show a GADT-aware algorithm implemented in GHC, the Haskell compiler. Boruch-Gruszecki [2017] builds on their work to propose an analogous algorithm for Scala.

Recently, Martres [2022] has shown a complete version of the class encoding we sketched in Section 3. In order to preserve subtyping relationships in general, this encoding relies on subtyping between recursive types using rules from an understudied branch of the DOT family [Rompf and Amin 2016]. Such rules are unnecessary to encode specific class hierarchies (like in our sketch); Martres conjectures that not relying on them is almost always possible if encoding a class type accounts for generic classes it extends. For instance, an applied type like $\text{MkPair[T1, T2]}$ would be encoded as $g.\text{MkPair} \land \{B <: \overline{T_1}; C <: \overline{T_2}; A <: (\overline{T_1}, \overline{T_2})\}$, where $\overline{T}$ is the encoded version of $T$.

Intensional type analysis allows performing runtime type analysis on values. Harper and Morrisett [1995] develop $\lambda^\text{ML}$ to allow efficient universal representations of values in polymorphic contexts. Their calculus was later generalised by Trifonov et al. [2000], arriving at a fully reflexive version: one that can analyse the runtime type of any value. cDOT supports a limited version of runtime type analysis with its tagged objects and case forms. Unlike true intensional type analysis, we cannot analyze arbitrary runtime type representations, but on the other hand our system admits efficient implementation strategies (like the runtime-class-based design it formalizes).

Bindings in DOT calculi, including cDOT, can introduce subtyping assumptions. This was noticed early in the study of DOT systems, as it leads to the crucial bad bounds problem [Amin et al. 2014; Rompf and Amin 2016]. Hu and Lhoták [2019] frame this idea as subtyping reflection: the notion that the context may contain evidence of subtyping, which can be reflected into the subtyping judgment. This name hints at the analogy to equality reflection from extensional type theories, where the context can contain evidence of equality which can be reflected into the typing judgment.

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19See, for instance, Nordström et al. [1990].
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DATA-AVAILABILITY STATEMENT
The mechanized soundness proofs of cDOT and our variant of $\lambda_{2,G\mu}$, as well as some lemmas related to encoding $\lambda_{2,G\mu}$ into cDOT, are available online [Boruch-Gruszecki et al. 2022b].

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A SUBSUMING GADTS

A.1 Modifications to $\lambda_{2,G}\mu$

In this section, we cover those details of our modified version of $\lambda_{2,G}\mu$ which are only relevant to the encoding. The variant is exhaustively described in [Waśko 2021].

Type annotations. We will want to encode $\lambda_{2,G}\mu$ into cDOT. In some places this encoding will need to state the types of some objects in the syntax of the target calculus. To ensure that the encoding can be defined just by looking at the syntax of the input terms, we add the necessary annotations to the syntax of the calculus. These annotations do not change semantics or expressivity. In fact, a well-typed program without these annotations can be easily converted into a program with the annotations, just by appending the necessary types coming from the typing derivation. We actually have mechanized two variants of the calculus: one with these annotations and another without them.

The only modifications are in the tuple syntax, where $\langle e_1, e_2 \rangle$ becomes $\langle e_1 : \tau_1, e_2 : \tau_2 \rangle$ and in the pattern matching syntax, explained in the next paragraph. This modification does not affect the reduction rules (they can simply ignore the annotations). In the typing rules, we simply verify that the type in the annotation matches the actual type stemming from the typing judgment. For example, below we show the old and new versions of the $\text{ty-tup}$ rule. The change is purely technical in that the type now not only appears in the judgment, but also at the syntax level.

$$\begin{align*}
\frac{\Delta; \Pi \vdash e_1 : \tau_1 \quad \Delta; \Pi \vdash e_2 : \tau_2}{\Delta; \Pi \vdash \langle e_1, e_2 \rangle : \tau_1 \ast \tau_2} & \quad (\text{old ty-tup}) \\
\frac{\Delta; \Pi \vdash e_1 : \tau_1 \quad \Delta; \Pi \vdash e_2 : \tau_2}{\Delta; \Pi \vdash \langle e_1 : \tau_1, e_2 : \tau_2 \rangle : \tau_1 \ast \tau_2} & \quad (\text{new ty-tup})
\end{align*}$$

Case forms. As we mentioned before, our variant disallows nested patterns and requires exhaustive matches. Without nesting, matching on tuples (which can be unpacked using the $\text{fst}$ and $\text{snd}$ operations) and unit is not really useful, so we remove it from our variant.

Since our pattern matching requires all branches to be of the same constructor, we actually know statically which GADT is being matched by a particular match. To simplify the encoding, we can also include this information at the syntax level. Just like the type annotations, this can be trivially reconstructed from an unannotated variant coupled with a typing derivation. The match form that used to be $\text{case e of ms}$ now becomes $\text{matchgadt e as T returning } \tau_{\text{ret}} \text{ with ms}$, where $T$ is the name of the matched GADT and $\tau_{\text{ret}}$ is the type that is being returned by the whole match.

Because we have removed the nesting of pattern matches and the matching of tuples and unit, the only rule left from the $\text{pat-}$ rules is $\text{pat-cons}$. When removing nesting we actually need to merge it with $\text{pat-var}$, thus we create the following new rule:

$$\begin{align*}
\Delta; \Pi \vdash e : \tau \\
\Delta; \Pi \vdash \langle e_1 : \tau_1, e_2 : \tau_2 \rangle : \tau_1 \ast \tau_2
\end{align*}$$
\[ \Sigma(c) = \forall \overline{a}. \tau \rightarrow (\overline{r_1})T \]
\[ \Delta_0, \overline{a}, \overline{r_1} \equiv \overline{r_2} \vdash \tau : * \]  
\hfill (pat-cons')

Due to our syntactic changes, the `ty-case` rule becomes:
\[ \Delta; \Gamma \vdash e : (\overline{\sigma})T \quad \Delta; \Gamma \vdash ms : (\overline{\sigma})T \Rightarrow \tau_2 \]
\[ \Delta; \Gamma \vdash \text{matchgadt } e \text{ as } T \text{ returning } \tau_2 \text{ with } ms : \tau_2 \]  
\hfill (ty-case”)

The rule used in the mechanization of the calculus goes one step further and inlines `pat-cons’ in `ty-case’ to create a single big rule. This helps with the mechanized proof, but would decrease the clarity on paper, so we will use the rules as shown above. The rule used in the mechanization is however equivalent to this pair of rules.

### A.2 Encoding definition

In this section, we present the translation from $\lambda_{2,Gd}$ into cDOT. The translation was originally presented by Waśko [2021] and is presented here with some minor modifications (mostly notational).

Moreover, we add yet one more modification to the typing relation: we rely on a weaker $\vdash$ notion which is a strict weakening of the original one. The paper introduced a relation $\vdash'$ defined by a set of rules (see Figure 7) and proved that it is equivalent to the original $\vdash$. We weaken the relation by removing some of the rules. We will call this new relation $\vdash$ and use it in place of $\vdash$. We remove the ex falso-like rules and the ones allowing to infer equalities from equality of type lambdas and arrows. The remaining rules will be enumerated as part of proof of lemma 5.8.

For conciseness, we will use some shorthand conventions when working with cDOT terms. For example, instead of writing $\{a\} \land \{b\}$ we will sometimes write $\{a;b\}$. Also to avoid repetition, we will use $\{A = T\}$ as a shorthand for $\{A : T_.T\}$. We will also use a more concise syntax for defining objects: instead of writing $\nu(s : \{A : T..T; x : U\})\{p.B\}\{A = T; x = u\}$ we write:
\[ \nu(s)[p.B]\{A = T; x : U; x = u\} \]

Whenever we use some variables that do not appear in the inputs of the encoding functions, we mean that these are some freshly generated identifiers that do not appear anywhere else in the program. We use the underscore for names which are not referred to anywhere.

Now we can show the encoding of types:

\[
\begin{align*}
T(a, \Theta) &= \Theta(a) \\
T(1, \Theta) &= \text{lib.Unit} \\
T(r_1 \ast r_2, \Theta) &= \text{lib.Tuple} \land \{T_1 = T(r_1, \Theta)\} \land \{T_1 = T(r_2, \Theta)\} \\
T(r_1 \rightarrow r_2, \Theta) &= \forall (\text{arg} : T(r_1, \Theta)). T(r_2, \Theta) \\
T((\tau_1, \ldots, \tau_n) T, \Theta) &= \text{env.T} \land \{A_1 = T(\tau_1, \Theta)\} \land \cdots \land \{A_n = T(\tau_n, \Theta)\} \\
T(\forall \alpha, x, \Theta) &= \forall (\alpha : \{\text{T : } \perp..\perp\}). T(\tau, \Theta[\alpha \mapsto A.T])
\end{align*}
\]

Then we introduce the term for encoding tuples and unit (the `lib`) and the construction of env which encodes the signature $\Sigma$.

```latex
\text{let lib} = \nu[\text{lib.Any}](\text{lib:} \{
\text{Any} : \perp..\perp \\
\text{Unit} = \{U : \perp..\perp\} \\
\text{unit} : \text{lib.Unit} \\
\text{Tuple} = \mu(s : \{T_1 : \perp..\perp; T_2 : \perp..\perp; \text{fst: } s.T_1; \text{snd: } s.T_2\}) \\
\text{tuple} : \forall(tl : \{T_1 : \perp..\perp; T_2 : \perp..\perp\})
\} \}
\```
∀ (x1: tl . T1) ∀ (x2: tl . T2)
lib . Tuple ∧ {T1 = tl . T1} ∧ {T2 = tl . T2}

}) {
   Any = ⊤
Unit = {U: ⊥ .. ⊤}
unit = ∀[lib . Unit](s: {U= ⊤})
Tuple = ∀(s: {T1: ⊥ .. ⊤; T2: ⊥ .. ⊤; fst: s . T1; snd: s . T2})
tuple = ∀(x1: tl . T1) ∀(x2: tl . T2)
   ∀[lib . Tuple](s: {T1 = tl . T1; T2 = tl . T2; fst = x1; snd = x2})
}

We assume that the provided Σ is described by the concrete syntax as described in Section 2.2 of the paper [Xi et al. 2003], and consists of a list of definitions (Σ = T1, ..., Tn) where Ti has form the following form:

(type, ..., type) T = {β1i}. (σ1i) c1 of τi
    | ...
    | {βni}. (σni) cn of τn

The type symbol is repeated m-times and indicates the cardinality of a given type constructor T. Each ci denotes a name of one of its n constructors. In each jth constructor, βji denotes the list of its type parameter names, τj indicates the type of the constructor value-level parameter and σji indicates to what type each of the type constructor parameters is instantiated in a given case. Each of σji can reference the parameters βji.

We also use Σ as a mapping of GADT constructor signatures. With a definition like the above, we will have Σ(cj) = ∀βji.τj → (σji)T.

For example, the definition of a GADT emulating equality would look like following:

(type, type) T = {β}. (β, β) refl of 1

The function EncodeSigma that we define below takes a list of type definitions (T_i) and returns an env object encoding each Ti as a base type and its associated constructors.

EncodeSigma (T1, ..., Tn) =
   ∀[lib . Any](env: {
   EncodeGADT (T1)
   ...
   EncodeGADT (Tn)
   })

To make the definitions more concise, we define a scheme for helper substitutions:

θ(x)_{βi} = [βi,1_1 → x . B_{i,1}, ..., βi,m_i → x . B_{i,m_i}] \tag{1}

With these tools, we can now define the EncodeGADT function, as shown on Figure 9.
To avoid name clashes we assume that all constructor and type names in Σ are unique.
And finally the encoding of terms. Note that the xf ligature is used to denote that this rule applies both to x-variables and f-variables.

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\[E(xf, \Theta) = \Theta(xf)\]
\[E(c_i)[\tau](e, \Theta) = \text{let } ts = v[\text{lib.Any}](ts : \{B_{i,1} = T(r_1, \Theta); \ldots; B_{i,m_i} = T(r_{m_i}, \Theta)\}) \text{ in}
\]
\[\text{env}.c_i \text{ ts } v\]
\[E(\text{()}, \Theta) = \text{lib.unit}\]
\[E((e_1 : r_1, e_2 : r_2), \Theta) = \text{let } v_1 = E(e_1, \Theta) \text{ in}
\]
\[\text{let } v_2 = E(e_2, \Theta) \text{ in}
\]
\[\text{lib.tuple } v[\text{lib.Any}](\text{() : } T_1 = T(r_1, \Theta); T_2 = T(r_2, \Theta)) \text{ } v_1 \text{ } v_2\]
\[E(\text{fst}(e), \Theta) = \text{let } v = E(e, \Theta) \text{ in } v.\text{fst}\]
\[E(\text{snd}(e), \Theta) = \text{let } v = E(e, \Theta) \text{ in } v.\text{snd}\]
\[E(\lambda x : r_1.e, \Theta) = \lambda (x_0 : T_1) E(e, \Theta[x \mapsto x_0])\]
\[E(e_1(e_2), \Theta) = \text{let } v_1 = E(e_1, \Theta) \text{ in}
\]
\[\text{let } v_2 = E(e_2, \Theta) \text{ in}
\]
\[v_1 \text{ } v_2\]
\[E(\Lambda a.e, \Theta) = \lambda (x_a : \{T : \bot \ldots T\}) E(e, \Theta[a \mapsto x_a.T])\]
\[E(e[r_1], \Theta) = \text{let } tl = v[\text{lib.Any}](\text{() : } T = T(r_1, \Theta)) \text{ in } E(e, \Theta) \text{ } tl\]
\[E(\text{fix } f : \tau.e, \Theta) = \text{let } hlpObj = v[\text{lib.Any}](\text{self : }\{
\]
\[\text{fix} = \lambda (\text{() : lib.Unit}) E(e, \Theta[f \mapsto \text{self.fix } \text{lib.unit}])\]
\[\}) \text{ in } hlpObj.\text{fix } \text{lib.unit}\]
\[E(\text{let } x = e_1 \text{ in } e_2 \text{ end}, \Theta) = \text{let } x = E(e_1, \Theta) \text{ in } E(e_2, \Theta[x \mapsto x])\]

\[E(\text{matchgadt } e \text{ as } T \text{ returning } \tau_{\text{ret}} \text{ with}
\]
\[| c_1[\beta_{1,1}, \ldots, \beta_{1,m_1}](x_1) \Rightarrow e_1 \ldots
\]
\[| c_n[\beta_{n,1}, \ldots, \beta_{n,m_n}](x_n) \Rightarrow e_n, \Theta) =
\]
\[\text{let } tl = v[\text{lib.Any}](\text{() : } T = T(r_{\text{ret}}, \Theta)) \text{ in}
\]
\[\text{let } v = E(e, \Theta) \text{ in}
\]
\[\text{let } \text{case}_{c_1} = \lambda (arg_1 : \text{env}.T_{C_1} \land v.\text{type})
\]
\[\text{let } x'_1 = arg_1.\text{data } \text{ in}
\]
\[E(e_1, \Theta[x_1 \mapsto x'_1, \beta_{1,1} \mapsto arg_1.B_{1,1}, \ldots, \beta_{1,m_1} \mapsto arg_1.B_{1,m_1}]) \text{ in}
\]
\[\ldots
\]
\[\text{let } \text{case}_{c_n} = \lambda (arg_n : \text{env}.T_{C_n} \land v.\text{type})
\]
\[\text{let } x'_n = arg_n.\text{data } \text{ in}
\]
\[E(e_n, \Theta[x_n \mapsto x'_n, \beta_{n,1} \mapsto arg_n.B_{n,1}, \ldots, \beta_{n,m_n} \mapsto arg_n.B_{n,m_n}]) \text{ in}
\]
\[v.\text{pmatch } tl \text{ case}_{c_1} \ldots \text{ case}_{c_n}\]
A.3 Correctness proof for the GADT encoding

Our proof will rely on many simple properties of cDOT derived from cDOT and of the $\equiv$ relation. While they are rather trivial, a more thorough discussion of them can be found in section 4.5 of [Waśko 2021]. One notable example that we will use quite a lot is the following lemma:

**Lemma A.1** (Lemma 6 from [Waśko 2021]). *For any types $A$, $B$, $C$ and $D$, if $\Gamma \vdash A \equiv B$ and $\Gamma \vdash C \equiv D$, then $\Gamma \vdash A \land C \equiv B \land D$.**
Lemma A.2. The term for $\text{lib}$ from the encoding and the ones generated by $\text{EncodeGADT}$ are well-typed and satisfy the requirements stated in the Definition 5.4.

Proof. The typing proof of $\text{lib}$ has been mechanized\footnote{A mechanized proof can be found in the attached artifacts, in translation/Helpers.v under the name eq_and_merge.} and quite simple so we skip it here and refer to the mechanization for details.

The $\text{env}$ contains necessary definitions to encode each GADT from $\Sigma$. Let’s consider a single GADT definition and show that the term returned by $\text{EncodeGADT}$ admits the expected type. We start by $\{\}-\text{I}$, then the type definitions are trivially checked by the Def-Typ rule. The non-trivial part are the constructors $c_i$. We start with the Def-ALL rule, followed by two rather standard applications of All-I rule.

Now, we need to show that the new object created in the body of the lambda is itself well-typed and satisfies the expected type. As always, we start with $\{\}-\text{I}$ and will assign to the result the type that directly stems from the definition — we will generalize it to the desired result type and to show that it fits the ascribed tag a few lines later. This “primitive” result type will be\footnote{See translation/Library.v, $v$ within attached artifacts.}:

$$\mu(s : \{B_{i,1} = ts.B_{i,1}\} \land \ldots \land \{A_1 = T(\sigma_{i,1}, \theta^{(s)}_{\vec{\beta}_i})\} \land \ldots \land \{\text{data : v.type} \land \{\text{pmatch} : \ldots\} \})$$

(we omit the pmatch type as it is very long but it is exactly the same as inside of env.$T$).

The types can be checked trivially with Def-Typ, the data field gets ascribed type $v$.type by Def-Path. Finally, the hardest part is the pmatch lambda. We use Def-ALL followed by All-I, in a rather standard way. Inside the body of the lambda we construct a value $h$ of type $\{z : s$.type$\}$ (checked by $\{\}-\text{I}$, Def-Path and Rec-E) and use it to apply $h.z$ to case$_{c_i}$. To typecheck this application we need to show that $h.z$ fits the type env.$T_{c_i} \land s$.type. We have $h.z : s$.type trivially by Fld-E. We proceed to show $h.z : \text{env}.T_{c_i}$. We have $h.z : s$.type, and by SNGL-Trans so now all it remains to be shown is that $s : \text{env}.T_{c_i}$. But from the assumption of $\{\}-\text{I}$ we know that $s$ admits the “primitive” type described earlier. We use Rec-E, then split the big intersection type into parts using Sub with and$_\text{D}_1$-<: and and$_\text{D}_2$-<:. After we generalize each type to fit env.$T_{c_i}$ (shown in the next paragraph), we can rebuild the type back using $\triangledown$-I and ‘close’ it with Rec-I.

We generalize the type of data using Fld-E, SNGL-Trans again with $v$.type and then get back with Fld-I. $v$ itself has type $T(\tau_i, \theta^{(ts)}_{\vec{\beta}_i})$ and we need $T(\tau_i, \theta^{(s)}_{\vec{\beta}_i})$. We can get from $\theta^{(ts)}_{\vec{\beta}_i}$ to $\theta^{(s)}_{\vec{\beta}_i}$, because they differ only in mapping each $\beta_{i,j}$ to $ts.B_{i,j}$ and $s.B_{i,j}$ respectively. But by definition of $s, s.B_{i,j} =:= ts.B_{i,j}$, so we can simply perform the replacement by Lemma A.4.

We generalize the types of $B_{i,j}$ from $\{B_{i,j} = ts.B_{i,j}\}$ to $\{B_{i,j} : \{\ldots\}$ by Sub with Typ-<:-Typ.

The types $A_j$ stay as-is. This makes us fit the requirements of env.$T_{c_i}$ apart from one — its first requirement is for the overall type to also fit env.$T$. But that is easy — the pmatch function already has the right type and we simply generalize types $A_j$ in the same way as we did with $B_{i,j}$. Then we collect the parts with $\triangledown$-I and Rec-I, in a standard way.

All this allows us to derive both $s : \text{env}.T$ and finally $s : \text{env}.T_{c_i}$, thus giving us $h.z : \text{env}.T_{c_i} \land s$.type. Thus the application case$_{c_i} h.z$ is valid — its result is $r.R$, like we want.

We have shown that the constructor constructs a valid object, with the primitive type described earlier. All that remains is to generalize it. Firstly, we need to remember that the newly created $v(\ldots)$ object is bound to a path thanks to the expression \textbf{let} $s = v[\text{env}.T_{c_i}](\ldots)$ \textbf{in} $s$. We have that $s : \mu(s : \ldots)$ where $\ldots$ is the primitive type mentioned before. We now need to show that
We use exactly the same reasoning that we did inside of \( \Gamma \). We have some \( \theta^{(ts)}_{\beta_i} \) and \( \theta^{(s)}_{\beta_i} \) map each \( \beta_{i,k} \) to \( ts.B_{i,k} \) and \( s.B_{i,k} \) respectively, and from the primitive type we also have \( \{ s.B_{i,k} = ts.B_{i,k} \} \), giving us \( s.B_{i,k} =: ts.B_{i,k} \) by \(<\cdot,\text{Sel} \text{ and } \text{Sel-}\cdot>\) semantics.

Before we can prove the preservation of types, we need to show that we can ‘emulate’ all equalities of our modified entailment relation \( \vdash' \) from \( \lambda_{2,Gp} \) in cDOT. To do so we prove the following lemma:

**Lemma 5.8** (Equation emulation). For any \( \Delta \vdash' \tau_1 \equiv \tau_2 \) and any \( \Theta \) and \( \Gamma \), if \( \Psi(\Delta; \Theta; \Gamma) \), then \( \Gamma \vdash \tau(\tau_1, \Theta) ::= \tau(\tau_2, \Theta). \)

**Proof.** The proof proceeds by induction on the derivation of the judgment \( \vdash' \). We inspect each case.

\[
\frac{\vec{a} \vdash' \tau : *}{\vec{a} \vdash \tau \equiv \tau}
\]

This case is trivially handled by the 'Refl' rule.

\[
\frac{\vec{a}, \Delta \vdash' \tau_1 \equiv \tau_2}{\vec{a}, \alpha \equiv \alpha, \Delta \vdash' \tau_1 \equiv \tau_2}
\]

We have some \( \Gamma \) and \( \Theta \) where \( \Psi(\vec{a}, \alpha \equiv \alpha, \Delta; \Theta; \Gamma) \) holds and want to prove that \( \Gamma \vdash \tau(\tau_1, \Theta) ::= \tau(\tau_2, \Theta) \). We can just use the inductive hypothesis: we just need to show that \( \Psi(\vec{a}, \Delta; \Theta; \Gamma) \) holds. But that is a weaker requirement\(^{23} \), so it holds trivially.

\[
\frac{\vec{a}, \Delta[\alpha \mapsto \tau] \vdash' \tau_1[\alpha \mapsto \tau] \equiv \tau_2[\alpha \mapsto \tau]}{\alpha \text{ has no free occurrences in } \tau}
\]

\[
\frac{\vec{a}, \alpha \equiv \alpha, \Delta \vdash' \tau_1 \equiv \tau_2}{\vec{a}, \alpha \equiv \alpha, \Delta \vdash' \tau_1 \equiv \tau_2}
\]

The two rules are symmetric so we will describe the proof for the second one, the first one is completely analogous.

We want to show that \( \Gamma \vdash \tau(\tau_1, \Theta) ::= \tau(\tau_2, \Theta) \). We know that \( \Psi(\vec{a}, \alpha \equiv \tau, \Delta; \Theta; \Gamma) \) gives us \( \Gamma \vdash \tau(\alpha, \Theta) ::= \tau(\tau, \Theta) \), equivalently \( \Gamma \vdash \Theta(\alpha) ::= \tau(\tau, \Theta) \).

From the inductive hypothesis we know that \( \Gamma \vdash \tau(\tau_1[\alpha \mapsto \tau], \Theta) ::= \tau(\tau_2[\alpha \mapsto \tau], \Theta) \) if only we can show that \( \Psi(\vec{a}, \alpha \equiv \tau, \Delta; \Theta; \Gamma) \) we have that for any \( \tau_1' \equiv \tau_2' \in \Delta[\alpha \mapsto \tau] \), we have \( \Gamma \vdash \tau(\tau_1', \Theta) ::= \tau(\tau_2', \Theta) \). Now, we need to show the same thing for any \( \tau_1'' \equiv \tau_2'' \in \Delta[\alpha \mapsto \tau] \). But for any \( \tau_1'' \equiv \tau_2'' \in \Delta[\alpha \mapsto \tau] \), we know that there are some \( \tau_1'' \equiv \tau_2'' \in \Delta[\alpha \mapsto \tau] \) such that \( \tau_1'' = \tau_1'[\alpha \mapsto \tau] \) and \( \tau_2'' = \tau_2'[\alpha \mapsto \tau] \). Thus we know that \( \Gamma \vdash \tau(\tau_1', \Theta) ::= \tau(\tau_2', \Theta) \) and just need to show \( \Gamma \vdash \tau(\tau_1'[\alpha \mapsto \tau], \Theta) ::= \tau(\tau_2'[\alpha \mapsto \tau], \Theta) \).

\(^{23}\text{Technically it is equivalent because } \alpha \equiv \alpha \text{ does not introduce any new information – it holds in every environment.}\)
We will get $\Gamma \vdash T(\tau'_1[\alpha \mapsto \tau], \Theta) ::= T(\tau'_1, \Theta)$ by Lemma A.5 and analogously for $\tau_2$ and get the desired conclusion by transitivity of $=:=$.

We have proven $\Psi(\bar{a}, \Delta[\alpha \mapsto \tau]; \Theta; \Gamma)$, which gives us $\Gamma \vdash T(\tau_1[\alpha \mapsto \tau], \Theta) ::= T(\tau_2[\alpha \mapsto \tau], \Theta)$. Again, in the same way we use Lemma A.5 to show that $\Gamma \vdash T(\tau_1[\alpha \mapsto \tau], \Theta) ::= T(\tau_1, \Theta)$ and also analogously for $\tau_2$ — then the conclusion follows trivially from transitivity of $=:=$.

In the last rule we handle the $T$ type constructor could stand for the GADT type constructors but also tuple and arrow types. However, due to our restriction the arrow rule is removed, so we stay with the following two rules: one for GADTs and one for tuples.

\[
\frac{\bar{a}, \tau_A \equiv \bar{\tau}_B \vdash \tau_1 \equiv \tau_2}{\bar{a}, (\tau_A)T \equiv (\tau_B)T \vdash \tau_1 \equiv \tau_2}
\]

\[
\frac{\bar{a}, \tau_{A_1} \equiv \tau_{B_1}, \tau_{A_2} \equiv \tau_{B_2} \vdash \tau_1 \equiv \tau_2}{\bar{a}, \tau_{A_1} \times \tau_{A_2} \equiv \tau_{B_1} \times \tau_{B_2} \vdash \tau_1 \equiv \tau_2}
\]

Let’s analyze the tuple example first.

To get the desired equation we can just apply the inductive hypothesis, so it suffices that we show $\Psi(\bar{a}, \tau_{A_1} \equiv \tau_{B_1}, \tau_{A_2} \equiv \tau_{B_2}; \Theta; \Gamma)$.

From our assumptions, we have $\Psi(\bar{a}, \tau_{A_1} \times \tau_{A_2} \equiv \tau_{B_1} \times \tau_{B_2}; \Theta; \Gamma)$. Unpacking the definition of $\Psi$, we know that $\Gamma \vdash T(\tau_{A_1} \times \tau_{A_2}, \Theta) ::= T(\tau_{B_1} \times \tau_{B_2}, \Theta)$. Let’s introduce shorthands for the tuple types:

\[
U_1 \triangleq T(\tau_{A_1} \times \tau_{A_2}, \Theta)
\]

\[
U_2 \triangleq T(\tau_{B_1} \times \tau_{B_2}, \Theta)
\]

Unfolding $T$, we get

\[
U_1 = \text{lib.Tuple} \land \{T_1 = T(\tau_{A_1}, \Theta)\} \land \{T_2 = T(\tau_{A_2}, \Theta)\}
\]

\[
U_2 = \text{lib.Tuple} \land \{T_1 = T(\tau_{B_1}, \Theta)\} \land \{T_2 = T(\tau_{B_2}, \Theta)\}
\]

We will want to use Typ-<;-Typ-INV1 and Typ-<;-Typ-INV2 to get $\Gamma \vdash T(\tau_{A_1}, \Theta) ::= T(\tau_{B_1}, \Theta)$. However, we cannot do this straight away, as the lib.Tuple is not acceptable in the $\bot$ judgment. Instead, let’s use the judgment $\Gamma \vdash U_1$ lib.Tuple := TupleDef where TupleDef stands for the tuple type definition $\mu(...)$ found in lib. By Lemma A.1, we can replace the reference with the expanded definition to get $U_1' = \text{TupleDef} \land \{T_1 = T(\tau_{B_1}, \Theta)\} \land \{T_2 = T(\tau_{B_2}, \Theta)\}$ with an equation $\Gamma \vdash U_1' ::= U_1$, and analogously for $U_2'$. Now it suffices to show $\Gamma \vdash U_1' ::= U_2'$ and through the helper equations and transitivity we will get the desired equation. Because TupleDef = $\mu(...)$, we can now derive $U_1' \bot \{T_1 = T(\tau_{A_1}, \Theta)\}$ and analogously $U_2' \bot \{T_1 = T(\tau_{B_1}, \Theta)\}$, thus using the inversion rules we get $\Gamma \vdash T(\tau_{A_1}, \Theta) ::= T(\tau_{B_1}, \Theta)$. We have shown that indeed $\Gamma$ with $\Theta$ satisfies the equation $\tau_{A_1} \equiv \tau_{B_1}$. We show $\tau_{A_2} \equiv \tau_{B_2}$ completely analogously.

The case for GADTs is also analogous — we have an equality env.$T \land \{A_1 = T(\tau_{A_1}, \Theta)\} \land \ldots \land \text{dom} \land \{A_1 = T(\tau_{B_1}, \Theta)\} \land \ldots$ and using the same technique we invert it to get $T(\tau_{A_1}, \Theta) ::= T(\tau_{B_1}, \Theta)$ and the analogous for all other pairs of corresponding type parameters. This way, we show $\Psi(\bar{a}, \tau_{A_1} \equiv \bar{\tau}_B; \Theta; \Gamma)$, which (as above) gives us the desired equation by the inductive hypothesis.

\[\text{Theorem 5.7 (Type Preservation). For every } \Pi, \Delta, e, \tau, \Theta \text{ such that } \text{fv}(e) \cup \text{fv}(\tau) \subseteq \text{dom}(\Theta), \text{any } \Gamma \text{ such that } \Phi(\Delta; \Pi; \Theta; \Gamma), \text{we have } \Gamma \vdash E(e, \Theta) : T(\tau, \Theta).\]

\[\text{Proof. We prove the theorem by induction on the derivation of the typing judgment.}\]

\[\text{A mechanized version of this part of the proof (inverting the tuple equality) is attached with the paper. See translation/DestructTupleLemma.v.}\]
- **(ty-unit)** We get $\Gamma \vdash \text{lib.unit} : \text{lib.Unit}$ by weakening and from the trivial conclusion that $\Gamma \vdash \text{lib} : \{	ext{unit} : \text{lib.Unit}\}$ because $\Gamma$ is defined to contain the bindings of $S(\Sigma)$.

- **(ty-var)** From the assumption we know that $\Pi(xf) = \tau$, since $E(xf, \Theta) = \Theta(xf)$, we get the result from $\Phi(\Delta; \Pi; \Theta; \Gamma)$.

- **(ty-tup)** Let’s introduce aliases for shortening the notation:

  $$\text{tupleApp} \triangleq \text{lib.tuple} \nu([T_1 = T(\tau_1, \Theta); T_2 = T(\tau_2, \Theta))] x_1 x_2$$

  $$\text{tupleTyp} \triangleq \text{lib.Tuple} \{T_1 = T(\tau_1, \Theta)\} \wedge \{T_2 = T(\tau_2, \Theta)\}$$

  We want to prove that

  $$\Gamma \vdash \text{let } x_1 = E(e_1, \Theta) \text{ in let } x_2 = E(e_2, \Theta) \text{ in } \text{tupleApp} : \text{tupleTyp}$$

  From the inductive hypotheses we have $\Gamma \vdash E(e_1, \Theta) : T(\tau_1, \Theta)$ and analogously for $e_2$ with $\tau_2$. Let’s define

  $$\Gamma' \triangleq \Gamma, x_1 : T(\tau_1, \Theta), x_2 : T(\tau_2, \Theta)$$

  We can use the LET rule to get to $\Gamma' \vdash \text{tupleApp} : \text{tupleTyp}$. There we can just use the standard application rules and the definition of $\text{lib.tuple}$.

- **(ty-fst)** We need to prove that $\Gamma \vdash \text{let } x = E(e, \Theta) \text{ in } x.\text{fst} : T(\tau_1, \Theta)$. From the inductive hypothesis, we have $\Gamma \vdash E(e, \Theta) : \text{lib.tuple} \wedge \{T_1 = T(\tau_1, \Theta)\} \wedge \{T_2 = T(\tau_2, \Theta)\}$, we apply the LET rule and then we can derive

  $$\Gamma \vdash E(e, \Theta) \rightarrow E(e, \Theta)$$

  for $\tau_2$ and analogously for $\text{fst}$.

- **(ty-snd)** Analogously as above.

- **(ty-lam)** Let’s allocate a fresh $x_0 \notin \text{dom}(\Gamma)$. We can easily show that since $\Phi(\Delta; \Pi; \Theta; \Gamma)$, then also $\Phi(\Delta; \Pi; x_0 : \tau_1 ; \Theta[x \mapsto x_0]; \Gamma, x_0 : T(\tau_1, \Theta))$. This allows us to get from the inductive hypothesis that

  $$\Gamma, x_0 : T(\tau_1, \Theta) \vdash E(e, \Theta[x \mapsto x_0]) : T(\tau_2, (\Theta[x \mapsto x_0]))$$

  Then, with the ALL-I rule we prove that

  $$\Gamma \vdash \lambda(x_0 : T(\tau_1, \Theta)) E(e, \Theta[x \mapsto x_0]) : \forall(x_0 : T(\tau_1, \Theta)) T(\tau_2, \Theta)$$

- **(ty-app)** From the inductive hypotheses we have (we pick a fresh $x_0$ and $\alpha$-rename the type properly):

  $$\Gamma \vdash E(e_1, \Theta) : \forall(x_0 : T(\tau_1, \Theta)) T(\tau_2, \Theta)$$

  and

  $$\Gamma \vdash E(e_2, \Theta) : T(\tau_1, \Theta)$$

  We can then use the LET rule twice to get from the goal of

  $$\Gamma \vdash \text{let } x_1 = E(e_1, \Theta) \text{ in let } x_2 = E(e_2, \Theta) \text{ in } x_1 x_2 : T(\tau_2, \Theta)$$

  to

  $$\Gamma, x_1 : \forall(x_0 : T(\tau_1, \Theta)) T(\tau_2, \Theta), x_2 : T(\tau_1, \Theta) \vdash x_1 x_2 : T(\tau_2, \Theta)$$

  and finish with ALL-E by noticing that $T(\tau_2, \Theta)[x_2/x_0] = T(\tau_2, \Theta)$, because from construction $x_0 \notin \text{fv}(T(\tau_2, \Theta))$.

---

25In short, from $x : \text{lib.tuple}$ we get $x : \{\text{fst} : x.\text{T}_1\}$ and then from $x : \{T_1 = T(\tau_1, \Theta)\}$ we get $x.\text{T}_1 : \text{T}(\tau_1, \Theta)$ allowing us to derive the conclusion.
- **(ty-tlam)** We need to show

\[ \Gamma \vdash E(\lambda \alpha. e, \Theta) : \mathcal{T}(\forall \alpha. \tau, \Theta) \]

Unfolding the encoding (assuming that \( x_{\alpha} \) is some fresh name) that is:

\[ \Gamma \vdash \lambda(x_{\alpha} : \{ T : \bot \implies \}) E(e, \Theta[\alpha \mapsto x_{\alpha}.T]) : \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T]) \]

Since we have \( \Phi(\Delta; \Pi; \Theta; \Gamma) \) then clearly we also have

\[ \Phi(\Delta, \alpha; \Pi; \Theta[\alpha \mapsto x_{\alpha}.T]; \Gamma, x_{\alpha} : \{ T : \bot \implies \}) \]

Thus from the inductive hypothesis, we get

\[ \Gamma, x_{\alpha} : \{ T : \bot \implies \} \vdash E(e, \Theta[\alpha \mapsto x_{\alpha}.T]) : \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T]) \]

Clearly, \( x_{\alpha} \notin \text{fv}(\{ T : \bot \implies \}) \). Thus we can use the All-I rule to get what we needed:

\[ \Gamma \vdash \lambda(x_{\alpha} : \{ T : \bot \implies \}) E(e, \Theta[\alpha \mapsto x_{\alpha}.T]) : \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T]) \]

- **(ty-tapp)** We want to show that

\[ \Gamma \vdash E(e[\tau_1], \Theta) : \mathcal{T}(\tau[\alpha \mapsto \tau_1], \Theta) \]

Unfolding the \( E \), that is

\[ \Gamma \vdash \text{let } x_1 = E(e, \Theta) \text{ in let } tl = v(\_ : \{ T = \mathcal{T}(\tau_1, \Theta) \}) \text{ in } x_1, \text{ tl} : \mathcal{T}(\tau[\alpha \mapsto \tau_1], \Theta) \]

From the inductive hypothesis we know that, for some fresh \( x_{\alpha} \):

\[ \Gamma \vdash E(e, \Theta) : \forall(x_{\alpha} : \{ T : \bot \implies \}) \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T]) \]

Thus by applying the Let rule twice\(^{26}\), let’s define

\[ \Gamma' \triangleq \Gamma, x_1 : \forall(x_{\alpha} : \{ T : \bot \implies \}) \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T]), \text{ tl} : \{ T = \mathcal{T}(\tau_1, \Theta) \} \]

We can then transform our goal into

\[ \Gamma' \vdash x_1, \text{ tl} : \mathcal{T}(\tau[\alpha \mapsto \tau_1], \Theta) \]

We can use Sub with Typ-\(-\llbracket-\rrbracket\)Typ to get \( \Gamma' \vdash \text{tl} : \{ T : \bot \implies \} \). Thus, we can use the All-E rule to get

\[ \Gamma' \vdash x_1, \text{ tl} : \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T])[\text{tl}/x_{\alpha}] \]

Now, knowing that \( \Gamma' \vdash \text{tl.T} := \mathcal{T}(\tau_1, \Theta) \), we need to use Lemma A.3 (proven just below) to get

\[ \Gamma' \vdash \mathcal{T}(\tau, \Theta[\alpha \mapsto x_{\alpha}.T])[\text{tl}/x_{\alpha}] := \mathcal{T}(\tau[\alpha \mapsto \tau_1], \Theta) \]

With that we can conclude by using the Sub rule and replacing the type we have with the desired type.

- **(ty-fix)** We want to prove

\[ \Gamma \vdash \text{let } h = v[\text{lib.Any}](\text{self} : \{ \text{fix} = \lambda(x : \text{lib.Unit}) E(e, \Theta[f \mapsto \text{self}.\text{fix lib.unit}]) \}) \text{ in } h.\text{fix lib.unit} : \mathcal{T}(\tau, \Theta) \]

Let’s define

\[ \Gamma' \triangleq \Gamma, \text{self} : \mu(\text{self} : \{ \text{fix} : \forall(x : \text{lib.Unit}) \mathcal{T}(\tau, \Theta) \}) \]

First, we show that \( \Phi(\Delta; \Pi, f : \tau; \Gamma'; \Theta[f \mapsto \text{self}.\text{fix lib.unit}]) \). This goes easily by Rec-E, Fld-E and All-E.

This allows us to use the inductive hypothesis to get

\[ \Gamma' \vdash E(e, \Theta[f \mapsto \text{self}.\text{fix lib.unit}]) : \mathcal{T}(\tau, \Theta) \]

\(^{26}\)We also use \(-I\) and a trivial instantiation Rec-E to typecheck \( tl \), but it’s completely standard.
This allows us to typecheck the \( \nu \) in \( h = \mu (\text{self} : \{ \text{fix} : \forall x : \text{lib.Unit} \Rightarrow (\tau, \Theta) \}) \) (we also need to show that \text{self} types to \text{lib.Any}, but that is trivial since \text{lib.Any} \text{Any} \lhd \tau).

Then we proceed with the \text{LET} rule followed by \text{Rec-E}, which transforms our \( h \) to have type \( \{ \text{fix} : \forall x : \text{lib.Unit} \Rightarrow (\tau, \Theta) \} \) and then we can easily finish by \text{All-E}.

- (\text{ty-let}) Follows directly by the \text{LET} rule. Clearly \( \Theta_x (x) \not\in \nu (\text{fix} (\tau_1, \Theta)) \) as \( \lambda_2 \cdot G \mu \) does not allow variable references in types.

- (\text{ty-eq}) From the inductive hypothesis we have \( \Gamma \vdash E (e, \Theta) : \tau (\tau_1, \Theta) \). We also know that \( \Delta \vdash r_1 \equiv r_2 \), so by Lemma 5.8, we have \( \Gamma \vdash \tau (\tau_1, \Theta) \equiv \tau (\tau_2, \Theta) \), thus by \text{SUB} we get the desired result.

- (\text{ty-cons}) To avoid getting lost in identifier names, let’s rewrite the rule to the following:

\[
\frac{\Sigma (c_i) = \forall \vec{\tau} . \tau_A \rightarrow \tau_B \quad \Delta \vdash \vec{\tau} : \vec{\pi} \quad \Delta ; \Pi \vdash e : \tau_A [\vec{a} \mapsto \vec{\tau}] }{\Delta ; \Pi \vdash c_i [\vec{\tau}] (e) : \tau_B [\vec{a} \mapsto \vec{\tau}]} \quad (\text{ty-cons})
\]

and let \( \vec{\tau} \triangleq (\tau_1, ..., \tau_m) \), \( \vec{\alpha} \triangleq (\alpha_1, ..., \alpha_m) \).

We need to prove that

\[
\Gamma \vdash \text{let} \; ts = \nu (ts : \{ B_i, 1 = \tau (\tau_1, \Theta); \ldots \}) \; \text{in} \; \\
\text{let} \; x = E (e, \Theta) \; \text{in} \; \\
\text{let} \; tmp = \text{env} . c_i \; \text{ts} \; \text{in} \; \text{tmp} \; x \; \\
: \tau (\tau_B [\vec{a} \mapsto \vec{\tau}], \Theta)
\]

From the assumption about \( \Sigma (c_i) \) we know that \text{env} . c_i has the type:

\[
\forall (ts : \{ B_i, 1 = \tau (\tau_1, \Theta); \ldots \}) \; \forall (v : \tau (\tau_A , \theta (ts), \vec{\alpha})) \; \tau (\tau_B , \theta (ts), \vec{\alpha})
\]

where \( \theta (ts) \) is defined as before.

First we prove the typing of \( ts \) by applying \( \emptyset - \text{I} \) and then numerously applying \text{DEF-TYP}. From the inductive hypothesis we get \( \Gamma \vdash E (e, \Theta) : \tau (\tau_A [\vec{a} \mapsto \vec{\tau}], \Theta) \), so also \( x \) will have that type.

Now we will use \text{LET} and \text{All-E} twice to finish the derivation.

First, we can type \( \text{tmp} \) as \( \forall (v : \tau (\tau_A , \theta (ts), \vec{\alpha})) \; \tau (\tau_B , \theta (ts), \vec{\alpha}) \) \( [ts \mapsto ts] \).

Because \( \tau_A \) and \( \tau_B \) should not have any free variables apart from \( \vec{\alpha} \), we can type \( \text{tmp} \) as \( \forall (v : \tau (\tau_A , \Theta \cup \theta (ts), \vec{\alpha})) \; \tau (\tau_B , \Theta \cup \theta (ts), \vec{\alpha}) \) \( [ts \mapsto ts] \).

Let \( \Gamma' \) be the context \( \Gamma \) extended with \( ts , x \) and \( \text{tmp} \) with types as shown above. We have \( \Gamma' \vdash \tau (\tau_B , \Theta \cup \theta (ts), \vec{\alpha}) \) \( [ts \mapsto ts] \).

Thus, we can use the Lemma A.3 repeatedly to get

\[
\Gamma' \vdash \tau (\tau_A , \Theta \cup \theta (ts), \vec{\alpha}) \; [ts \mapsto ts] \; =: \tau (\tau_B , \vec{\tau} , \Theta)
\]

and the same for \( \tau_B \). Thus, using the \text{SUB} rule, we show that indeed the argument \( x \) fits the expected type in \( \text{tmp} \) and the return type of the application \( \text{tmp} \; x \) can indeed be coerced to what we wanted in the first place.

- (\text{ty-case}) We need to typecheck the following term:
let \( tl = v(tl : \{ R = \mathcal{T}(\tau_2, \Theta) \}) \) in
let \( v = \mathcal{E}(e, \Theta) \) in
let \( \text{case}_{c_1} = (\lambda(\text{arg}_1 : \text{env}.T_{c_1} \land \text{v.type}) \) \\
\text{let } x_1 = \text{arg}_1.\text{data} \text{ in} \) \\
\( \mathcal{E}(e_1, \Theta[\beta_{1,1} \mapsto \text{arg}.B_{1,1}, \ldots]) \) \\
\) in
...
\( \text{v.pmatch } tl \text{ case}_{c_1} \ldots \text{ case}_{c_n} \)

as having type \( \mathcal{T}(\tau_2, \Theta) \).
The last application is a series of multiple applications so technically, it should be replaced with a stack of let bindings with temporary names for intermediate step — we skip that this time for clarity, as the application itself is pretty routine.

From the inductive hypothesis we get that \( \Gamma \vdash \mathcal{E}(e, \Theta) : \mathcal{T}(\tau_1, \Theta) \).

From the restrictions placed on our variant of the calculus, we know that all pattern matching is only performed on GADTs, so \( \tau_1 = (\delta_1, \ldots, \delta_m)T \) for some types \( \vec{\delta} \) and some GADT name \( T \).

Thus, we have \( \Gamma \vdash \mathcal{E}(e, \Theta) : \text{env}.T \land \{ A_1 = \mathcal{T}(\delta_1, \Theta) \} \land \ldots \).

From that and through routine usage of \( \text{LET} \) we have \( v : \text{env}.T \land \{ A_1 = \mathcal{T}(\delta_1, \Theta) \} \land \ldots. \)

Thus, \( \text{v.pmatch} \) has type \( \forall (r : \{ \text{R} : \bot \ldots \text{R} \}) \forall (\text{case}_{c_1} : \forall (\text{arg} : \text{env}.T_{c_1} \land \text{s.type})\text{R}) \ldots \text{R}. \)

So by rather standard application of \( \text{ALL-E} \) we will have

\( \text{v.pmatch} tl : (\forall (\text{case}_{c_1} : (\forall (\text{arg} : \text{env}.T_{c_1} \land \text{s.type})tl.R) \ldots tl.R) \)

Thus, by continuously using \( \text{ALL-E} \), assuming that each \( \text{case}_{c_1} \) has the expected type (we will show that in a moment), we get that the whole application types to \( tl.R. \)

Since we have

\( tl.R := \mathcal{T}(\tau_2, \Theta) \) by the \( \text{SEL} \) rules, we get exactly what we needed.

Now we just need to verify that for each branch, the generated lambda has the correct type.

Let’s consider the branch \( i \).

From the assumptions of \( \text{ty-case} \), we have \( \Delta; \Pi \vdash (p_i \Rightarrow e_i) : (\tau_1 \Rightarrow \tau_2) \) which means in particular that \( \Delta \vdash p \downarrow \tau_1 \Rightarrow (\Delta', \Pi') \) and that \( \Delta, \Delta'; \Pi, \Pi' \vdash e_i : \tau_2. \)

From the restrictions, we know that \( p_i \) must have the form \( c_1[\beta_{i,1}, \ldots, \beta_{i,m_i}](x_i) \) and so that the first judgment must have been derived by \( \text{pat-cons} \).

From that, we get \( \Sigma(c_i) = \forall \vec{\beta}_i, \tau_C \rightarrow ((\sigma_{i,1}, \ldots, \sigma_{i,m_i})T). \)

Note that the \( \sigma_{i,j} \) here are the same as in the definition of \( \text{env}.T_{c_i} \), this will be important in a moment. More importantly, because we were matching type \( (\vec{\delta})T \), we know that \( \Delta' = \vec{\tilde{\beta}}_i, \vec{\tilde{\sigma}}_i \equiv \vec{\delta} \) and \( \Pi' = x_i : \tau_C \).

Thus the second judgment has form \( \Delta, \vec{\tilde{\beta}}_i, \vec{\tilde{\sigma}}_i \equiv \vec{\delta}, \Pi, x_i : \tau_C \vdash e_i : \tau_2. \)

We want to show that

\( \lambda(\text{arg}_i : \text{env}.T_{c_i} \land \text{v.type}) \text{ let } x_i' = \text{arg}_i.\text{data} \text{ in} \mathcal{E}(e_i, \Theta[x_i \mapsto x_i', \beta_{i,1} \mapsto \text{arg}.B_{i,1}, \ldots]) \)

types to \( \mathcal{T}(\tau_2, \Theta) \). We can apply \( \text{ALL-I} \) followed by \( \text{LET} \), knowing that \( \text{arg}_i.\text{data} : (\tau_C, \hat{\Theta}) \) where

\( \hat{\Theta} \triangleq [\beta_{i,1} \mapsto \text{arg}_i.B_{i,1}, \ldots] \)

to get to the goal\(^{27}\)

\( \Gamma' \vdash \mathcal{E}(e_i, \Theta') : \mathcal{T}(\tau_2, \Theta) \)

\(^{27}\)Technically in the meantime our environment accumulated some more identifiers, for example earlier case branch definitions are visible in the latter ones - we can ignore them though because independent branches do not rely on them and can be typechecked without them, and then add them using the weakening lemma.
Lemma A.3

\[ \Gamma' \triangleq \Gamma, v : \text{env.T} \land \{ A_1 = \mathcal{T}(\delta_1, \Theta) \} \land \ldots, \text{arg}_i : \text{env.T}_i \land \text{v.type}, x'_i : \mathcal{T}(\tau_C, \check{\Theta}) \]

\[ \Theta' \triangleq \Theta[x_i \mapsto x'_i, \beta_{i,1} \mapsto \text{arg}_i.B_{i,1}, \ldots] \]

Now, as long as we can ensure that \( \Phi(\Delta, \check{\beta}_i, \check{\sigma}_i \equiv \check{\delta}; \Pi, x_i : \tau_C; \Theta'; \Gamma') \) we can finish by applying the inductive hypothesis from the assumption

\[ \Delta, \check{\beta}_i, \check{\sigma}_i \equiv \check{\delta}; \Pi, x_i : \tau_C + e_i : \tau_2 \]

giving us

\[ \Gamma' \vdash E(e_i, \Theta') : \mathcal{T}(\tau_2, \Theta') \]

Trivially, \( \mathcal{T}(\tau_2, \Theta') = \mathcal{T}(\tau_2, \Theta) \), because we can assume all the added identifiers \( \beta_{i,j} \) are fresh in \( \tau_2 \).

So now we just need to show \( \Phi(\Delta, \check{\beta}_i, \check{\sigma}_i \equiv \check{\delta}; \Pi, x_i : \tau_C; \Theta'; \Gamma') \) to finish the proof.

We already know that \( \Phi(\Delta; \Pi; \Theta; \Gamma) \). What remains to be shown is that the conditions of \( \Phi \) hold for the added parts:

- For each \( \beta_{i,j} \) we know that \( \Theta'(\beta_{i,j}) = \text{arg}_i.B_{i,j} \) and since \( \text{arg}_i : \text{env.T}_i \), we can get\(^{28} \)

\[ \text{arg}_i : \{ B_{i,j} : \perp \ldots \tau \} \]

- We need to show that \( \Gamma' + \Theta'(x_i) : \mathcal{T}(\tau_C, \Theta') \) which is equivalent to \( \Gamma' + x'_i : \mathcal{T}(\tau_C, \Theta') \) but since \( \mathcal{T}(\tau_C, \Theta') = \mathcal{T}(\tau_C, \check{\Theta})^{29} \), it follows directly from VAR.

- For each \( \sigma_{i,j} \equiv \delta_j \), we need to show that \( \Gamma' + \mathcal{T}(\sigma_{i,j}, \Theta') \equiv \mathcal{T}(\delta_j, \Theta') \). For the same reason as previously, \( \mathcal{T}(\sigma_{i,j}, \Theta') = \mathcal{T}(\sigma_{i,j}, \check{\Theta}) \), now, because \( \text{arg}_i : \text{env.T}_i \), we have \( \text{arg}_i : \{ A_j = \mathcal{T}(\sigma_{i,j}, \check{\Theta}) \} \). We also have already mentioned that \( v : \{ A_j = \mathcal{T}(\delta_j, \check{\Theta}) \} \).

With that, we can prove the desired equality:

\[ \mathcal{T}(\sigma_{i,j}, \Theta') = \mathcal{T}(\sigma_{i,j}, \check{\Theta}) = \equiv \text{arg}_i.A_j = \equiv v.A_j = \equiv \mathcal{T}(\delta_j, \Theta) \]

The two outer equalities rely on the \( \text{Sel} \) rules and the inner equality is derived by the two \( \text{Sngl} \) rules, because we have \( \text{arg}_i : \text{v.type} \).

\[ \square \]

**Lemma A.3** (Substitution through translation). For every \( \Delta, \alpha, \tau_1 \) and \( \tau_2 \), if \( \Delta \vdash \tau_1 : * \) and \( \Delta, \alpha \vdash \tau_2 : * \), then for every \( \Gamma \) and \( \Theta \) such that \( \Psi(\Delta; \Theta; \Gamma) \), if \( \Gamma \vdash a.T := \mathcal{T}(\tau_1, \Theta) \) for some \( a \), then for every fresh \( x_\alpha \), we have \( \Gamma \vdash \mathcal{T}(\tau_2, \Theta[\alpha \mapsto x_\alpha.T])[a/x_\alpha] := \mathcal{T}(\tau_2[\alpha \mapsto \tau_1], \Theta) \).

**Proof.** The proof goes by induction on the structure of \( \tau_2 \).

- \( \tau_2 = 1 \): the equality stems trivially from the \( \text{Refl} \) rule, because \( \mathcal{T}(1, \ldots) = \text{lib.Unit} \), so both sides are syntactically equal.

- \( \tau_2 = \tau_A * \tau_B \): from the inductive hypotheses we have \( \Gamma \vdash \mathcal{T}(\tau_A, \Theta[\alpha \mapsto x_\alpha.T])[a/x_\alpha] = \equiv \mathcal{T}(\tau_A[\alpha \mapsto \tau_1], \Theta) \) and the same for \( \tau_B \).

We can transform both sides of the equation:

\[ \mathcal{T}(\tau_A * \tau_B, \Theta[\alpha \mapsto x_\alpha.T])[a/x_\alpha] = \]

\[ (\text{lib.Tuple} \land \{ T_1 = \mathcal{T}(\tau_A, \Theta[\alpha \mapsto x_\alpha.T])[a/x_\alpha] \} \land \{ T_2 = \mathcal{T}(\tau_B, \Theta[\alpha \mapsto x_\alpha.T])[a/x_\alpha] \}) \]

\[ \mathcal{T}((\tau_A * \tau_B)[\alpha \mapsto \tau_1], \Theta) = \]

\[ \mathcal{T}(\tau_A[\alpha \mapsto \tau_1] * \tau_B[\alpha \mapsto \tau_1], \Theta) = \]

\[ \text{lib.Tuple} \land \{ T_1 = \mathcal{T}(\tau_A[\alpha \mapsto \tau_1], \Theta) \} \land \{ T_2 = \mathcal{T}(\tau_B[\alpha \mapsto \tau_1], \Theta) \} \]

\[ ^{28} \text{We do this based on the definition of env.T}_i \text{ in } \Gamma' \text{; we need to use REC-E and AND}_{i \leftarrow} \text{: to get there.} \]

\[ ^{29} \text{Because } \text{fv}(\tau_C) = \{ \beta_{i,1}, \ldots, \beta_{i,m_i} \} \text{ and for each } j, \Theta'(\beta_{i,j}) = \check{\Theta}(\beta_{i,j}). \]
By using the rules \( \text{Typ}\rightarrow\cdot\text{-Typ} \), from the inductive hypotheses we get

\[
\Gamma \vdash \{ T_1 = T(\tau_A[\alpha \mapsto \tau_1], \Theta) \} =: \{ T_1 = T(\tau_A, \Theta[\alpha \mapsto x_{\alpha}.T])[a/x_{\alpha}] \}
\]

and the same for \( T_B \), then by Lemma A.1 (applied twice), we get the desired result.

- The case \( T_2 = (\overline{\tau})T \) goes analogously as for the tuples: we get inductive hypotheses for each type in \( \overline{\tau} \), we can push the substitution down the structure and get the final equality by applying Lemma A.1 enough times.

- \( \alpha' = \alpha \): In this case the lhs further simplifies\(^{30}\) to \( \forall (x_{\alpha'} : \{ T : \bot \ldots \top \}) \forall (\tau, \Theta[\alpha \mapsto x_{\alpha}.T]) \).

The rhs can be unfolded\(^{31}\) to \( T(\forall \alpha.\tau, \Theta) \), and from the definition of \( T \), this is exactly the same as the transformed lhs.

- \( \alpha' \neq \alpha \):

In the rhs we can push the substitution deeper:

\[
T((\forall \alpha'.\tau)[\alpha \mapsto \tau_1], \Theta) = T(\forall \alpha'.(\tau[\alpha \mapsto \tau_1]), \Theta) = \forall (x_{\alpha'} : \{ T : \bot \ldots \top \}) T(\tau, \Theta[\alpha \mapsto x_{\alpha'.T}])
\]

We can reorder the substitution in the lhs to get:

\[
\forall (x_{\alpha'} : \{ T : \bot \ldots \top \}) T(\tau, \Theta[\alpha' \mapsto x_{\alpha'.T}, \alpha \mapsto x_{\alpha}.T])[a/x_{\alpha}]
\]

After applying the \( \text{ALL}\rightarrow\cdot\text{-ALL} \) rule to get the subtyping in both directions, it suffices to prove

\[
\Gamma, x_{\alpha'} : \{ T : \bot \ldots \top \} \vdash T(\tau[\alpha \mapsto \tau_1], \Theta') =: T(\tau, \Theta'[\alpha \mapsto x_{\alpha}.T])[a/x_{\alpha}]
\]

where

\[
\Theta' \triangleq \Theta[\alpha' \mapsto x_{\alpha'.T}]
\]

Clearly, \( \forall (\Delta, \alpha'; \Theta'; \Gamma, x_{\alpha'} : \{ T : \bot \ldots \top \}) \) so we get the desired equality by the inductive hypothesis.

- Finally, \( \tau_2 = \alpha' \). Let’s consider two cases:

  - \( \alpha' = \alpha \): The lhs is \( T(\alpha, \Theta[\alpha \mapsto x_{\alpha}.T])[a/x_{\alpha}] = (x_{\alpha}.T)[a/x_{\alpha}] = a.T \) and the rhs is \( T(\alpha[\alpha \mapsto \tau_1], \Theta) = T(\tau_1, \Theta) \), so we just use the assumption \( \Gamma \vdash a.T ::= T(\tau_1, \Theta) \).

  - \( \alpha' \neq \alpha \): The lhs is \( T(\alpha', \Theta[\alpha \mapsto x_{\alpha}.T])[a/x_{\alpha}] = \Theta(\alpha')[a/x_{\alpha}] = \Theta(\alpha') \) (we can do the last transformation, because we know that \( x_{\alpha} \) was fresh for \( \Theta \)). The rhs is \( T(\alpha'[\alpha \mapsto \tau_1], \Theta) = \Theta(\alpha'), \) so both sides are equal: we get the type equality by \( \text{REFL} \).

\[\square\]

**Lemma A.4.** For any type \( \tau \) and any substitutions \( \Theta_1 \) and \( \Theta_2 \), if \( \forall \alpha(\tau) \subseteq \text{dom}(\Theta_1) = \text{dom}(\Theta_2) \) and for every \( x \in \text{dom}(\Theta_1), \Gamma \vdash \Theta_1(x) ::= \Theta_2(x) \), then also \( \Gamma \vdash T(\tau, \Theta_1) ::= T(\tau, \Theta_2) \).

**Proof.** Easy induction on the structure of \( \tau \). The base case is for variables and it goes directly from the assumption \( \Gamma \vdash \Theta_1(x) ::= \Theta_2(x) \). All the other cases are trivial or can be handled by constructing equations on terms based on equations on the sub-terms derived from the inductive hypotheses, in the same way as in Lemma A.3.

\[\square\]

\(^{30}\)We can get rid of the \([a/x_{\alpha}]\) substitution at the end, because if the \( \mapsto x_{\alpha}.T \) disappeared, then from the freshness of \( x_{\alpha} \) for \( \Theta \), we know that this substitution will act as identity.

\(^{31}\)Again, we are free to remove the substitution \([\alpha \mapsto \tau_1] \) inside of \( T \), because the name is aliased by the same name in \( \forall \alpha \).
**Lemma A.5.** For any types $\tau_A$ and $\tau_B$, any variable $\alpha$, substitution $\Theta$ and environment $\Gamma$, if $\Gamma \vdash \Theta(\alpha) \iff T(\tau_A, \Theta)$ then $\Gamma \vdash T(\tau_B, \Theta) \iff T(\tau_B[\alpha \mapsto \tau_A], \Theta)$.

**Proof.** Easy induction on the structure of $\tau_B$.

The base case for the variable $\alpha$ goes directly from the assumption $\Gamma \vdash \Theta(\alpha) \iff T(\tau_A, \Theta)$. For any $\alpha' \neq \alpha$, we have $T(\alpha', \Theta) = T(\alpha'[\alpha \mapsto \tau_A], \Theta)$, so it goes by Refl.

All the other cases are trivial or can be handled by constructing equations on terms based on equations on the sub-terms derived from the inductive hypotheses, in the same way as in Lemma A.3.

For example in the case of tuples, i.e. if $\tau_B = \tau_1 \times \tau_2$, from the inductive hypotheses we have

$$\Gamma \vdash T(\tau_1, \Theta) \iff T(\tau_1[\alpha \mapsto \tau_A], \Theta)$$

and same for $\tau_2$.

$$T(\tau_1 \times \tau_2, \Theta) = \text{lib}.\text{Tuple} \land \{T_1 = T(\tau_1, \Theta)\} \land \{T_2 = T(\tau_2, \Theta)\}$$

and

$$T(\tau_1[\alpha \mapsto \tau_A], \Theta) = \text{lib}.\text{Tuple} \land \{T_1 = T(\tau_1, \Theta)[\alpha \mapsto \tau_A]\} \land \{T_2 = T(\tau_2, \Theta)[\alpha \mapsto \tau_A]\}$$

and so we construct the necessary equation using Typ- <- Typ and Lemma A.1.

One more interesting case that may be worth showing is $\tau_B = \forall \alpha'. \tau$. Let’s see two cases:

- if $\alpha = \alpha'$, then $(\forall \alpha. \tau)[\alpha \mapsto \tau_A] = \forall \alpha. \tau$ so the equality follows trivially from Refl.
- $\alpha \neq \alpha'$, then

$$T((\forall \alpha'. \tau)[\alpha \mapsto \tau_A], \Theta) = (\forall (x_{\alpha'} : \{T : \bot..\tau\})T(\tau, \Theta[\alpha' \mapsto x_{\alpha'}.T])$$

and

$$T((\forall \alpha'. \tau)[\alpha \mapsto \tau_A], \Theta) = (\forall (x_{\alpha'} : \{T : \bot..\tau\})T(\tau[\alpha \mapsto \tau_A], \Theta[\alpha' \mapsto x_{\alpha'}.T])$$

Of course, if $\Gamma \vdash \Theta(\alpha) \iff T(\tau_A, \Theta)$, then by weakening also

$$\Gamma, x_{\alpha'} : \{T : \bot..\tau\} \vdash \Theta(\alpha) \iff T(\tau_A, \Theta)$$

and because $\alpha'$ is fresh for $\tau_A$, $T(\tau_A, \Theta) = T(\tau_A[\alpha' \mapsto x_{\alpha'}.T])$ and $(\Theta[\alpha' \mapsto x_{\alpha'}.T])(\alpha) = \Theta(\alpha)$, so from the inductive hypothesis we get

$$\Gamma, x_{\alpha'} : \{T : \bot..\tau\} \vdash T(\tau_1, \Theta[\alpha' \mapsto x_{\alpha'}.T]) \iff T(\tau_1[\alpha \mapsto \tau_A], \Theta[\alpha' \mapsto x_{\alpha'}.T])$$

Now we use All- <- All to get the desired equation.

\[\square\]

Using the Theorem 5.7 together with Lemma A.2 we finally can prove that the result of EncodeProgram is well-typed.

**B. Soundness Proof of PDOT**

Rapoport and Lhoták introduces the seven-level stratified typing rules in the soundness of pDOT.

\[\begin{align*}
\text{General (}\tau\text{)} & \rightarrow \text{Tight (}\tau\#\text{)} \rightarrow \text{Introduction-qp (}\tau_{q1p}\text{)} \rightarrow \text{Introduction-pq (}\tau_{p1q}\text{)} \rightarrow \text{Elim-III (}\tau_{e3}\text{)} \rightarrow \text{Elim-II (}\tau_{e2}\text{)} \rightarrow \text{Elim-I (}\tau_{e1}\text{)}
\end{align*}\]

These levels in pDOT are designed and organized to tackle two key problems in the metatheory: bad bounds and cycles in typing derivations. We will explain the two problems along with pDOT’s solutions to them one by one.
When typing the body of the above function, the absurd subtyping judgment with a concrete type, serving as the proof of the subtyping relation implied by its bounds. Therefore, soundness. Specifically, when evaluating the program, the type member derivable in the environment. DOT calculi are designed in a way that bad bounds will not break expressiveness in an inert environment.

Lemma B.1 (\(\vdash_t\) to \(\vdash_{\ell}\)). If \(\Gamma\) is inert and \(\Gamma \vdash t : T\), then \(\Gamma \vdash_{\ell} t : T\).

Cycles in typing derivation. pDOT typing rules can result in cycles in typing derivation. For example, we can apply Rec-I and Rec-E repeatedly in the derivation tree. And we can similarly produce cycles with the Fld-I, Fld-E, Sngpq-<-; and Sngq-<-; rules. Such cycles hinder the inductive reasoning of typing judgments. pDOT deals with the cycles by separating each pair of the symmetric typing rules into different levels. For instance, the Sngpq-<-; and Sngq-<-; rules are inlined in
Introduction-qp \((r_{qp})\) Introduction-pq \((r_{pq})\) levels respectively. Starting from the Introduction-qp level, pDOT separates the reasoning about values and paths. Therefore, we have two sets of typing rules, one for the values and another for the paths on these levels. Transformation lemmas are proven for them respectively.