ON A CONJECTURE OF CHEEGER

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Abstract. This note details how a recent structure theorem for normal 1-currents proved by the first and third author allows to prove a conjecture of Cheeger concerning the structure of Lipschitz differentiability spaces. More precisely, we show that the push-forward of the measure from a Lipschitz differentiability space under a chart is absolutely continuous with respect to Lebesgue measure.

Keywords: Lipschitz differentiability space, Cheeger’s conjecture, Alberti representation, metric measure space.

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1. Introduction

In [Che99] Cheeger proved that in every doubling metric measure space $(X, \rho, \mu)$ satisfying a Poincaré inequality, Lipschitz functions are differentiable $\mu$-almost everywhere. More precisely, he showed the existence of a family $\{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ of Borel charts (that is, $U_i \subset X$ is a Borel set, $X = \bigcup_i U_i$ up to a $\mu$-negligible set, and $\varphi_i : X \to \mathbb{R}^{d(i)}$ is Lipschitz) such that for every Lipschitz map $f : X \to \mathbb{R}$ at $\mu$-almost every $x_0 \in U_i$ there exists a unique (co-)vector $df(x_0) \in \mathbb{R}^{d(i)}$ with

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0.$$  

This fact was later axiomatized by Keith [Kei04], leading to the notion of Lipschitz differentiability space, see Section 2 below.

Cheeger also conjectured that the push-forward of the reference measure $\mu$ under every chart $\varphi_i$ has to be absolutely continuous with respect to the Lebesgue measure, that is,

$$(\varphi_i)_\#(\mu \res U_i) \ll \mathcal{L}^{d(i)},$$

see [Che99, Conjecture 4.63]. Some consequences of this fact concerning existence of bi-Lipschitz embeddings of $X$ into some $\mathbb{R}^N$ are detailed in [Che99, Section 14], also see [CK06, CK09].

Let us assume that $(X, \rho, \mu) = (\mathbb{R}^d, \rho_E, \nu)$ with $\rho_E$ the Euclidean distance and $\nu$ a positive Radon measure, is a Lipschitz differentiability space when equipped with the (single) identity chart (note that it follows a-posteriori from the validity of Cheeger’s conjecture that no mapping into a higher-dimensional space can be a chart in a Lipschitz differentiability structure of $\mathbb{R}^d$). In this case the validity of Cheeger’s conjecture reduces to the validity of the (weak) converse of Rademacher’s theorem, which states that a positive Radon measure
ν on \( \mathbb{R}^d \) with the property that all Lipschitz functions are differentiable \( \nu \)-almost everywhere must be absolutely continuous with respect to \( L^d \). Actually, it is well known to experts that this converse of Rademacher’s theorem implies Cheeger’s conjecture in any metric space, see for instance [Kei04, Section 2.4], [Bat15, Remark 6.11], and [Gon12].

The (strong) converse of Rademacher’s theorem has been known to be true in \( \mathbb{R} \) since the work of Zähorski [Zah46], where he characterized the sets \( E \subset \mathbb{R} \) that are sets of non-differentiability points of some Lipschitz function. In particular, he proved that for every Lebesgue negligible set \( E \subset \mathbb{R} \) there exists a Lipschitz function which is nowhere differentiable on \( E \).

The same result for maps \( f: \mathbb{R}^d \to \mathbb{R}^d \) has been proved by Alberti, Csörnyei & Preiss for \( d = 2 \) as a consequence of a deep structural result for negligible sets in the plane [ACP05, ACP10]. In 2011, Csörnyei & Jones [Jon11] announced the extension of the above result to every Euclidean space. For Lipschitz maps \( f: \mathbb{R}^d \to \mathbb{R}^n \) with \( n < d \) the situation is fundamentally different and there exists a null set such that every Lipschitz function is differentiable at at least one point from that set, see [Pre90, PS15]. We finally remark that the weak converse of Rademacher’s theorem in \( \mathbb{R}^2 \) can also be obtained by combining the results of [Alb93] and [AM16], see [AM16, Remark 6.2 (iv)].

Recently, a result concerning the singular structure of measures satisfying a differential constraint was proved in [DR16]. When combined with the main result of [AM16], this proves the weak converse of Rademacher’s theorem in any dimension, see [DR16, Theorem 1.14].

In this note we detail how the results in [AM16, DR16] in conjunction with Bate’s result on the existence of a sufficient number of independent Alberti representations in a Lipschitz differentiability space [Bat15] imply Cheeger’s conjecture; see Section 2 for the relevant definitions.

**Theorem 1.1.** Let \( (X, \rho, \mu) \) be a Lipschitz differentiability space and let \((U, \varphi) \) be a \( d \)-dimensional chart. Then, \( \varphi_\#(\mu \llcorner U) \ll L^d \).

Note that by the same arguments of this paper Cheeger’s conjecture would also follow from the results announced in [ACP05] and [Jon11].

After we finished writing this note we learned that similar results have been proved by Kell and Mondino [KM16] and by Gigli and Pasqualetto [GP16].

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2. Setup

2.1. Lipschitz differentiability spaces. In the sequel, the triple \( (X, \rho, \mu) \) will always denote a metric measure space, that is, \( (X, \rho) \) is a separable, complete metric space and \( \mu \in \mathcal{M}_+(X) \) is a positive Radon measure on \( X \).
We call a pair \((U, \varphi)\) such that \(U \subset X\) is a Borel set and \(\varphi : X \to \mathbb{R}^d\) is Lipschitz, a \(d\)-dimensional chart, or simply a \(d\)-chart. A function \(f : X \to \mathbb{R}\) is said to be differentiable with respect to a \(d\)-chart \((U, \varphi)\) at \(x_0 \in U\) if there exists a unique (co-)vector \(df(x_0) \in \mathbb{R}^d\) such that
\[
\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0.
\]

We call a metric measure space \((X, \rho, \mu)\) a Lipschitz differentiability space (also called a metric measure space that admits a measurable differentiable structure) if there exists a countable family of \(d\)-(i)-charts \((U_i, \varphi_i)\) \((i \in \mathbb{N})\) such that \(X = \bigcup U_i\) and any Lipschitz map \(f : X \to \mathbb{R}\) is differentiable with respect to every \((U_i, \varphi_i)\) at \(\mu\)-almost every point \(x_0 \in U_i\).

2.2. Alberti representations. We denote by \(\Gamma(X)\) the set of curves in \(X\), that is, the set of all Lipschitz maps \(\gamma : \text{Dom} \gamma \to X\), for which the domain \(\text{Dom} \gamma \subset \mathbb{R}\) is non-empty and compact. Note that we are not requiring \(\text{Dom} \gamma\) to be an interval and thus the set \(\Gamma(X)\) is sometimes also called the set of curve fragments on \(X\). We equip \(\Gamma(X)\) with the Hausdorff metric \(\text{dist}_H\) on graphs and we consider it as a subspace of the Polish space \(K = \{K \subset \mathbb{R} \times X : K \text{ compact}\}\), endowed with the Hausdorff metric. Moreover, by arguing as in [Sch16, Lemma 2.20], it is easy to see that \(\Gamma(X)\) is an \(F_\sigma\)-subset of \(K\), i.e. a countable union of closed sets.

The decomposition of a measure into a family of 1-dimensional Hausdorff measures supported on curves leads to the notion of Alberti representation. First introduced in [Alb93] for the study of the rank-one property of BV-derivatives, this decomposition has turned out to be a key tool in the study of differentiability properties of Lipschitz functions, see for instance [ACP05, ACP10, AM16, Bat15].

**Definition 2.1.** Let \((X, \rho, \mu)\) be a metric measure space. An Alberti representation of \(\mu\) on a \(\mu\)-measurable set \(A \subset X\) is a parametrized family \((\mu_\gamma)_{\gamma \in \Gamma(X)}\) of positive Borel measures \(\mu_\gamma \in \mathcal{M}_+(X)\) with
\[
\mu_\gamma \ll \mathcal{H}^1 \big| \text{Im} \gamma,
\]
together with a Borel probability measure \(\pi \in \mathcal{P}(\Gamma(X))\) such that
\[
\mu(B) = \int \mu_\gamma(B) \, d\pi(\gamma) \quad \text{for all Borel sets } B \subset A. \tag{2.2}
\]
Here, the measurability of the integrand is part of the requirement of being an Alberti representation.

**Remark 2.2.** Note that this definition is slightly different from the one in [Bat15, Definition 2.2] since there the set \(\Gamma(X)\) consist of bi-Lipschitz curves. Clearly, the existence of a representation in the sense of [Bat15] implies the existence of a representation in our sense and this will suffice for our purposes. Let us, however, point out that the converse holds true as well. Indeed, the
part of $\gamma$ that contributes to the integral in (2.2) can be decomposed into countably many bi-Lipschitz pieces, see [Sch16, Remark 2.17].

We will further need the notion of independent Alberti-representations of a measure. Let $C \subset \mathbb{R}^d$ be a closed, convex, one-sided cone, i.e. a set of the form

$$C := \{ v \in \mathbb{R}^d : v \cdot w \geq (1 - \theta) \|v\| \}$$

for some $w \in S^{d-1}$ and $\theta \in (0,1)$. With a Lipschitz map $\varphi: X \to \mathbb{R}^d$, we say that an Alberti representation $\int \nu_\gamma \ d\pi(\gamma)$ has $\varphi$-directions in $C$ if

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\} \quad \text{for} \ \pi\text{-a.e. curve } \gamma \text{ and } H^1\text{-a.e. } t \in \text{Dom } \gamma.$$

A number of $m$ Alberti representations of $\mu$ are $\varphi$-independent if there are linearly independent cones $C_1, \ldots, C_m$ such that the $i$'th Alberti representation has $\varphi$-directions in $C_i$. Here, linear independence of the cones $C_1, \ldots, C_m$ means that any collection of vectors $v_i \in C_i \setminus \{0\}$ is linearly independent. In the case $X = \mathbb{R}^d$ we will always consider $\varphi = \text{Id}$.

One of the main results of [Bat15] asserts that a Lipschitz differentiability space necessarily admits many independent Alberti representations, also cf. [AM16, Theorem 1.1]. Recall that according to Remark 2.2 any representation in the sense of [Bat15] is also a representation in the sense of Definition 2.1.

**Theorem 2.3.** Let $(X,\rho,\mu)$ be a Lipschitz differentiability space with a $d$-chart $(U,\varphi)$. Then, there exists a countable decomposition

$$U = \bigcup_{k \in \mathbb{N}} U_k, \quad U_k \subset U \text{ Borel sets},$$

such that every $\mu \ll U_k$ has $d$ $\varphi$-independent Alberti representations.

A proof of this theorem can be found in [Bat15, Theorem 6.6].

### 2.3. One-dimensional currents.

In order to use the results of [DR16] we need a link between Alberti representation and 1-dimensional currents. Recall that a 1-dimensional current $T$ in $\mathbb{R}^d$ is a continuous linear functional on the space of smooth and compactly supported differential 1-forms on $\mathbb{R}^d$. The boundary of $T$, $\partial T$ is the distribution (0-current) defined via $\langle \partial T, f \rangle := \langle T, df \rangle$ for every smooth and compactly supported function $f: \mathbb{R}^d \to \mathbb{R}$. The mass of $T$, denoted by $M(T)$, is the supremum of $\langle T, \omega \rangle$ over all 1-forms $\omega$ such that $|\omega| \leq 1$ everywhere. In particular, finite-mass currents can be naturally identified with $\mathbb{R}^d$-valued Radon measures. A current $T$ is called normal if both $T$ and $\partial T$ have finite mass; we denote the set of normal 1-currents by $N_1(\mathbb{R}^d)$.

By the Radon–Nikodým theorem, a 1-dimensional current $T$ with finite mass can be written in the form $T = \int \langle \omega(x), \vec{T}(x) \rangle \ d\|T\|(x)$.

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By the Radon–Nikodým theorem, a 1-dimensional current $T$ with finite mass can be written in the form $T = \int \langle \omega(x), \vec{T}(x) \rangle \ d\|T\|(x)$.
An integer-multiplicity rectifiable 1-current (in the following called simply rectifiable 1-current) $T = \|E, \tau, m\|$ is a 1-current which acts on 1-forms $\omega$ as

$$\langle T, \omega \rangle = \int_E \langle \omega(x), \tau(x) \rangle m(x) \, d\mathcal{H}^1(x),$$

where $E$ is a 1-rectifiable set, $\tau(x)$ is a unit vector spanning the approximate tangent space $\text{Tan}(E, x)$ and $m$ is an integer-valued function such that $\int_E m \, d\mathcal{H}^1 < \infty$. More information on currents can be found in [Fed69].

The relation between Alberti representations and normal 1-currents is partially encoded in the following decomposition theorem, due to Smirnov [Smi93].

**Theorem 2.4.** Let $T = \vec{T} \|T\| \in N_1(\mathbb{R}^d)$ be a normal 1-current with $|\vec{T}(x)| = 1$ for $\|T\|$-almost every $x$. Then, there exists a family of rectifiable 1-currents

$$T_\gamma = [E_\gamma, \tau_\gamma, 1], \quad \gamma \in \Gamma,$$

where $\Gamma$ is a measure space endowed with a finite positive Borel measure $\pi \in \mathcal{M}_+(\Gamma)$, such that the following assertions hold:

(i) $T$ can be decomposed as

$$T = \int_{\Gamma} T_\gamma \, d\pi(\gamma)$$

and

$$\mathcal{M}(T) = \int_{\Gamma} \mathcal{M}(T_\gamma) \, d\pi(\gamma) = \int_{\Gamma} \mathcal{H}^1(E_\gamma) \, d\pi(\gamma);$$

(ii) $\tau_\gamma(x) = \vec{T}(x)$ for $\mathcal{H}^1$-almost every $x \in E_\gamma$ and for $\pi$-almost every $\gamma \in \Gamma$;

(iii) $\|T\|$ can be decomposed as

$$\|T\| = \int_{\Gamma} \mu_\gamma \, d\pi(\gamma),$$

where each $\mu_\gamma$ is the restriction of $\mathcal{H}^1$ to the 1-rectifiable set $E_\gamma$.

An Alberti representation of a Euclidean measure splits it into measures concentrated on “fragments” of curves. In general, these fragments cannot be glued together to obtain a 1-dimensional normal current since the boundary may have infinite mass. Nevertheless, the “holes” of every curve appearing in an Alberti representation of a measure $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ can be “filled” in such a way as to produce a normal 1-current $T$ with $\nu \ll \|T\|$. Moreover, if the representation has directions in a cone $C$, then the constructed normal current $T$ has orienting vector $\vec{T}$ in $C \setminus \{0\}$ almost everywhere (with respect to $\|T\|$). Indeed, we have the following lemma, which is essentially [AM16, Corollary 6.5]; it can be interpreted as a partial converse to Theorem 2.4.

**Lemma 2.5.** Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a finite Radon measure. If there is an Alberti representation $\nu = \int \nu_\gamma \, d\pi(\gamma)$ with directions in a cone $C$, then there exists a normal 1-current $T \in N_1(\mathbb{R}^d)$ such that $\vec{T}(x) \in C \setminus \{0\}$ for $\|T\|$-almost every $x \in \mathbb{R}^d$ and $\nu \ll \|T\|$.
Proof. For the purpose of illustration we sketch the proof.

Step 1. Given $\nu$ as in the statement, we claim that there exists a normal 1-current $T = \bar{T}||T||$ with $\mathcal{M}(T) \leq 1$ and $\mathcal{M}(\partial T) \leq 2$ such that $\bar{T}(x) \in C$, for $||T||$-almost every $x$ and that $\nu$ is not singular with respect to $||T||$.

The claim follows from the proof of [AM16, Lemma 6.12]. For the sake of completeness let us present the main line of reasoning. By arguing as in Step 1 of the proof of [AM16, Lemma 6.12], to every $\gamma \in \Gamma(\mathbb{R}^d)$ with $\gamma'(t) \in C$ and a Borel measure $\nu_{\gamma} \ll \mathcal{H}^1 \text{Im} \gamma$, we can associate a 1-Lipschitz map $\psi_{\nu_{\gamma}} : [0,1] \to \mathbb{R}^d$ satisfying

$$\nu_{\gamma}(\text{Im}(\psi_{\nu_{\gamma}})) > 0 \quad \text{and} \quad \psi'_{\nu_{\gamma}}(t) \in C \setminus \{0\} \quad \text{for} \quad \mathcal{H}^1\text{-a.e.} \quad t \in [0,1].$$

This map can moreover be chosen such that $\gamma \mapsto \psi_{\nu_{\gamma}}$ coincides with a Borel measurable map $\pi$-almost everywhere once we endow the set of curves with the topology of uniform convergence, see Step 3 in the proof of [AM16, Lemma 6.12].

Let $T_{\nu_{\gamma}} := [\text{Im} \psi_{\nu_{\gamma}}, \tau_{\psi_{\nu_{\gamma}}}, 1]$ be the rectifiable 1-current associated to $\psi_{\nu_{\gamma}}$ and set

$$T := \int T_{\nu_{\gamma}} \, d\pi(\gamma).$$

Since $\psi_{\nu_{\gamma}}$ is 1-Lipschitz, $\mathcal{H}^1(\text{Im} \psi_{\nu_{\gamma}}) \leq 1$ and thus $\mathcal{M}(T) \leq 1$. Moreover, for all smooth compactly supported functions $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\langle \partial T, f \rangle = \langle T, df \rangle = \int f(\psi_{\nu_{\gamma}}(1)) - f(\psi_{\nu_{\gamma}}(0)) \, d\pi(\gamma),$$

so that $\mathcal{M}(\partial T) \leq 2$.

By assumption, $\bar{T}(x) \in C \setminus \{0\}$ for $||T||$-almost every $x \in \mathbb{R}^d$. To show that $||T||$ and $\nu$ are not mutually singular, for $\pi$-almost every $\gamma$ set

$$\nu'_{\gamma} := \nu_{\gamma} \ll \text{Im} \psi_{\nu_{\gamma}} \quad \text{and} \quad \nu' := \int \nu'_{\gamma} \, d\pi(\gamma),$$

so that $\nu' \neq 0$ and $\nu' \leq \nu$. We will now establish that $\nu' \ll ||T||$, for which we will prove that $\nu$ and $||T||$ are not mutually singular. Let $E \subset \mathbb{R}^d$ be such that $||T||(E) = 0$. Using

$$T = \int [\text{Im} \psi_{\nu_{\gamma}}, \tau_{\psi_{\nu_{\gamma}}}, 1] \, d\pi(\gamma) \quad \text{with} \quad \tau_{\psi_{\nu_{\gamma}}} = \frac{\psi'_{\nu_{\gamma}}}{|\psi'_{\nu_{\gamma}}|} \in C,$$

we get

$$\mathcal{H}^1(\text{Im} \psi_{\nu_{\gamma}} \cap E) = 0 \quad \text{for} \quad \pi\text{-a.e.} \quad \gamma.$$ 

Since by definition $\nu_{\gamma} \ll \mathcal{H}^1 \text{Im} \gamma$, we have that $\nu'_{\gamma} \ll \mathcal{H}^1 \text{Im} \psi_{\nu_{\gamma}}$. Thus, $\nu'(E) = 0$.

Step 2. Let us define

$$\mathcal{T} := \{ T \in \mathcal{N}_1(\mathbb{R}^d) : \mathcal{M}(T) \leq 1, \mathcal{M}(\partial T) \leq 2 \text{ and } \bar{T} \in C \ ||T||\text{-a.e.} \}$$

and

$$\mathcal{T}_\nu := \{ T \in \mathcal{T} : \nu \text{ and } T \text{ are not singular} \}.$$
Note that if \( C = \{ v \in \mathbb{R}^d : v \cdot w \geq (1 - \theta)\|v\| \} \) for some \( w \in S^{d-1}, \theta \in (0,1) \), then \( \bar{T} \in C \) almost everywhere implies that
\[
\|T\| \geq T : w \geq (1 - \theta)\|T\|
\] (2.3)
as measures (here we are identifying \( T \) with an \( \mathbb{R}^d \)-valued Radon measure and use the pointwise scalar product). Moreover, as a consequence of the Radon–Nikodým theorem, for every \( T \in \mathcal{T}_\nu \) we may write
\[
\nu = g_{\|T\|}\|T\| + \nu^\perp_{\|T\|} \quad \text{with} \quad \nu^\perp_{\|T\|} \perp \|T\|, \quad \int g_{\|T\|} \, d\|T\| > 0.
\]
Let us set \( M := \sup_{T \in \mathcal{T}_\nu} \int g_{\|T\|} \, d\|T\| > 0 \) and let \( T_k \in \mathcal{T}_\nu \) be a sequence with
\[
\int g_{\|T_k\|} \, d\|T_k\| \to M.
\]
Define
\[
T := \sum_k 2^{-k}T_k
\]
and note that \( T \in \mathcal{T} \). Moreover, by (2.3), \( \|T_k\| \ll \|T\| \) for all \( k \in \mathbb{N} \), so that there exist \( h_k : \mathbb{R}^d \to \mathbb{R} \) with
\[
\int_E h_k \, d\|T\| = \int_E g_{\|T_k\|} \, d\|T_k\| \leq \nu(E) \quad \text{for all Borel sets } E \subset \mathbb{R}^d.
\]
In particular, \( T \in \mathcal{T}_\nu \) and \( h_k \leq g_{\|T\|} \). Set \( m_k = \max_{1 \leq j \leq k} h_j \). By the monotone convergence theorem, \( m_k \to m_\infty \leq g_{\|T\|} \) in \( L^1(\mathbb{R}^d, \|T\|) \) and
\[
M \leq \lim_{k \to \infty} \int m_k \, d\|T\| = \int m_\infty \, d\|T\| \leq \int g_{\|T\|} \, d\|T\| \leq M.
\]
Hence, \( M \) is actually a maximum and it is attained by \( T \).

We now claim that \( \nu \ll \|T\| \). Indeed, assume by contradiction that \( \nu = g_{\|T\|} \, d\|T\| + \nu^\perp_{\|T\|}, \nu^\perp_{\|T\|} \neq 0 \). Since the Alberti representation of \( \nu \) induces an Alberti representation of \( \nu^\perp_{\|T\|} \), we can apply Step 1 to find a normal 1-current
\[
S \in \mathcal{T}_{\nu^\perp_{\|T\|}} \subset \mathcal{T}_\nu
\]
such that \( \nu^\perp_{\|T\|} \) and \( \|S\| \) are not mutually singular. In particular, if \( \nu = g_{\|S\|} \, d\|S\| + \nu^\perp_{\|S\|} \), then there exists a Borel set \( F \subset \mathbb{R}^d \) such that
\[
\|T\|(F) = 0 \quad \text{and} \quad \int_F g_{\|S\|} \, d\|S\| > 0. \quad \text{(2.4)}
\]
Let us define \( W := (T + S)/2 \) and note that by (2.3) it holds that \( \|T\|, \|S\| \ll \|W\| \) so that \( W \in \mathcal{T}_\nu \). Moreover, there are functions \( h_T, h_S \leq g_{\|W\|} \) such that
\[
\int_E h_T \, d\|W\| = \int_E g_{\|T\|} \, d\|T\|, \quad \int_E h_S \, d\|W\| = \int_E g_{\|S\|} \, d\|S\|
\]
for all Borel sets \( E \). However, for \( F \) as in (2.4) we obtain
\[
M \geq \int_{\mathbb{R}^d} g_{\|W\|} \, d\|W\| \geq \int_{\mathbb{R}^d} g_{\|T\|} \, d\|T\| + \int_F g_{\|S\|} \, d\|S\| > M,
\]
a contradiction. \( \square \)
3. Proof of Cheeger’s conjecture

The key tool to prove Cheeger’s conjecture is the following result from [DR16, Corollary 1.12]:

**Theorem 3.1.** Let $T_1 = T_1\|T_1\|, \ldots, T_d = T_d\|T_d\| \in N_1(\mathbb{R}^d)$ be 1-dimensional normal currents. Let $\nu \in M_+(\mathbb{R}^d)$ be a positive Radon measure such that

(i) $\nu \ll \|T_i\|$ for $i = 1, \ldots, d$, and

(ii) $\text{span}\{T_1(x), \ldots, T_d(x)\} = \mathbb{R}^d$ for $\nu$-almost every $x$.

Then, $\nu \ll L^d$.

Combining the above result with Lemma 2.5 we immediately get the following:

**Lemma 3.2.** Let $\nu \in M_+(\mathbb{R}^d)$ have $d$ independent Alberti representations. Then, $\nu \ll L^d$.

**Proof.** Denote by $C_1, \ldots, C_d$ independent cones such that there are $d$ Alberti representations having directions in these cones. By Lemma 2.5 there are $d$ normal 1-dimensional currents $T_1 = T_1\|T_1\|, \ldots, T_d = T_d\|T_d\| \in N_1(\mathbb{R}^d)$ such that $\nu \ll \|T_i\|$ for $i = 1, \ldots, d$, and $T_i(x) \in C_i$ for $\nu$-almost every $x \in \mathbb{R}^d$. By the independence of the cones, $\text{span}\{T_1(x), \ldots, T_d(x)\} = \mathbb{R}^d$ for $\nu$-a.e. $x \in \mathbb{R}^d$.

This implies $\nu \ll L^d$ via Theorem 3.1. \qed

In order to use the above result to prove Theorem 1.1 one further needs the following “push-forward lemma”.

**Lemma 3.3.** Let $(X, \rho, \mu)$ be a Lipschitz differentiability space with a $d$-chart $(U, \varphi)$. If $\mu \ll U$ has $d$ $\varphi$-independent Alberti representations, then also the push-forward $\varphi_#(\mu \ll U) \in M_+(\mathbb{R}^d)$ has $d$ independent Alberti representations.

**Proof.** It is enough to show that if there exists a representation of the form $\mu \ll U = \int \mu_{\gamma} d\pi(\gamma)$ with $\varphi$-directions in a cone $C$ (i.e. such that $(\varphi \circ \gamma)'(t) \in C \setminus \{0\}$ for almost all $t \in \text{Dom } \gamma$ and for $\pi$-almost every $\gamma$), then we can build an Alberti representation

$$\varphi_#(\mu \ll U) = \int \nu_{\tilde{\gamma}} d\tilde{\pi}(\tilde{\gamma}) \quad \text{with} \quad \tilde{\pi} \in \mathcal{P}(\Gamma(\mathbb{R}^d)).$$

with $\tilde{\gamma}'(t) \in C \setminus \{0\}$ for $\tilde{\pi}$-almost every $\tilde{\gamma}$ and almost every $t \in \text{Dom } \tilde{\gamma}$. To this end consider the map $\Phi: \Gamma(X) \rightarrow \Gamma(\mathbb{R}^d)$ given by $\Phi(\gamma) := \varphi \circ \gamma$ and let $\tilde{\pi} := \Phi_# \pi \in M_+(\Gamma(\mathbb{R}^d))$. Note that, by the very definition of the push-forward measure, for $\tilde{\pi}$-almost every $\tilde{\gamma}$, it holds that $\tilde{\gamma} = \varphi \circ \gamma$ for some $\gamma \in \Gamma(X)$.

By considering $\pi$ as a probability measure defined on the Polish space $\mathcal{K}$ defined in (2.1), and noting that $\pi$ is concentrated on $\Gamma(X)$, we can apply the disintegration theorem for measures [AGS05, Theorem 5.3.1] to show that for
π-almost every $\bar{\gamma}$, there exists a Borel probability measure $\eta_{\bar{\gamma}}$ concentrated on $\Phi^{-1}(\bar{\gamma})$ and such that

$$\pi(A) = \int \eta_{\bar{\gamma}}(A) \, d\bar{\pi}(\bar{\gamma})$$

for all Borel sets $A \subset \Gamma(X)$. Note also that, by the disintegration theorem, the map $\bar{\gamma} \mapsto \eta_{\bar{\gamma}}$ is Borel measurable. Let us now set

$$\nu_{\bar{\gamma}} := \int_{\Phi^{-1}(\bar{\gamma})} \varphi_#(\mu_{\gamma}) \, d\eta_{\bar{\gamma}}(\gamma).$$

Clearly, we have the representation

$$\varphi_#(\mu_{\sqcup U}) = \int \nu_{\bar{\gamma}} \, d\bar{\pi}(\bar{\gamma})$$

and $\varphi'(t) = (\varphi \circ \gamma)'(t) \in C \setminus \{0\}$ for π-almost every $\bar{\gamma}$ and almost every $t \in \text{Dom } \gamma$. Hence, to conclude we only have to show that

$$\nu_{\bar{\gamma}} \ll \mathcal{H}^1 \sqcup \text{Im } \bar{\gamma}$$

for π-a.e. $\bar{\gamma}$.

Let $E$ be a set with $\mathcal{H}^1(E \cap \text{Im } \bar{\gamma}) = 0$. Since $\varphi'(t) \neq 0$ for almost every $t \in \text{Dom } \gamma$, the area formula implies that $\mathcal{L}^1(\varphi^{-1}(E)) = 0$. If $\gamma \in \Phi^{-1}(\bar{\gamma})$, say $\bar{\gamma} = \varphi \circ \gamma$, then

$$\mathcal{H}^1(\varphi^{-1}(E) \cap \text{Im } \gamma) \leq \mathcal{H}^1(\gamma(\varphi^{-1}(E))) = 0$$

for all $\gamma \in \Phi^{-1}(\bar{\gamma})$.

Hence, $\mu_{\gamma}(\varphi^{-1}(E)) = 0$ for all $\gamma \in \Phi^{-1}(\bar{\gamma})$, which immediately gives

$$\nu_{\bar{\gamma}}(E) = \int_{\Phi^{-1}(\bar{\gamma})} \mu_{\gamma}(\varphi^{-1}(E)) \, d\eta_{\bar{\gamma}}(\gamma) = 0.$$  

This concludes the proof.

**Proof of Theorem 1.1.** Let $(U, \varphi)$ be a $d$-chart. By Theorem 2.3 there are $d\varphi$-independent Alberti representations of $\mu_{\sqcup U_k}$, where $U = \bigcup_{k \in \mathbb{N}} U_k$ is the decomposition from Bate’s theorem. Then, via Lemma 3.3 the push-forward $\varphi_#(\mu_{\sqcup U_k})$ also has $d$ independent Alberti representations. Finally, Lemma 3.2 yields $\varphi_#(\mu_{\sqcup U_k}) \ll \mathcal{L}^d$ and this concludes the proof.

**References**

[ACP05] G. Alberti, M. Csörgyi, and D. Preiss. Structure of null sets in the plane and applications. In *Proceedings of the Fourth European Congress of Mathematics (Stockholm, 2004)*, pages 3–22. European Mathematical Society, 2005.

[ACP10] G. Alberti, M. Csörgyi, and D. Preiss. Differentiability of lipschitz functions, structure of null sets, and other problems. In *Proceedings of the International Congress of Mathematicians 2010 (Hyderabad 2010)*, pages 1379–1394. European Mathematical Society, 2010.

[AGS05] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser, 2005.

[Alb93] G. Alberti. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123:239–274, 1993.

[AM16] G. Alberti and A. Marchese. On the differentiability of lipschitz functions with respect to measures in the Euclidean space. *Geom. Funct. Anal.*, 26:1–66, 2016.

[Bat15] D. Bate. Structure of measures in Lipschitz differentiability spaces. *J. Amer. Math. Soc.*, 28:421–482, 2015.
J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9:428–517, 1999.

J. Cheeger and B. Kleiner. On the differentiability of Lipschitz maps from metric measure spaces to Banach spaces. In *Inspired by S. S. Chern*, volume 11 of *Nankai Tracts Math.*, pages 129–152. World Scientific, 2006.

J. Cheeger and B. Kleiner. Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon-Nikodým property. *Geom. Funct. Anal.*, 19:1017–1028, 2009.

G. De Philippis and F. Rindler. On the structure of $\mathcal{A}$-free measures and applications. *Ann. of Math.*, 2016. to appear, arXiv:1601.06543.

H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

J. Gong. Rigidity of derivations in the plane and in metric measure spaces. *Illinois J. Math.*, 56:1109–1147, 2012.

N. Gigli and E. Pasqualetto. Behaviour of the reference measure on RCD spaces under charts. arXiv:1607.05188, 2016.

P. Jones. Product formulas for measures and applications to analysis and geometry, 2011. Talk given at the conference “Geometric and algebraic structures in mathematics”, Stony Brook University, May 2011, http://www.math.sunysb.edu/Videos/dennisfest/.

S. Keith. A differentiable structure for metric measure spaces. *Adv. Math.*, 183:271–315, 2004.

M. Kell and A. Mondino. On the volume measure of non-smooth spaces with Ricci curvature bounded below. arXiv:1607.02036, 2016.

D. Preiss. Differentiability of lipschitz functions on banach spaces. *J. Funct. Anal.*, 91:312–345, 1990.

D. Preiss and G. Speight. Differentiability of Lipschitz functions in Lebesgue null sets. *Invent. Math.*, 199:517–559, 2015.

A. Schioppa. Derivations and Alberti representations. *Adv. Math.*, 293:436–528, 2016.

S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows. *Algebra i Analiz*, 5:206–238, 1993, translation in St. Petersburg Math. J. 5 (1994), 841–867.

Z. Zahorski. Sur l’ensemble des points de non-dérivabilité d’une fonction continue. *Bull. Soc. Math. France*, 74:147–178, 1946.

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