RENORMALIZABILITY OF SEMIQUANTIZED FIELDS

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ABSTRACT

A definition is given, in the framework of stochastic quantization, for the dynamics of a system composed of classical and quantum degrees of freedom mutually interacting. It is found that the theory breaks reflection positivity, and hence it is unphysical. The Feynman rules for the Euclidean vacuum expectation values are derived and the perturbative renormalizability of the theory is analyzed. Contrary to the naive expectation, the semiquantized theory turns out to be less renormalizable, in general, than the corresponding completely quantized theory.

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1. Introduction

The study of the dynamics of systems mixing quantum and classical degrees of freedom, has attracted some interest in the past related to the interplay of gravity and quantization. Indeed Einstein’s field theory is notoriously difficult to quantize due to its lack of renormalizability [1,2]. This problem has been dealt with in the literature by considering quantum gravity as an effective theory [3,4].

Alternatively, under the generic name of semiclassical gravity, there has been a number of proposals to couple quantum matter fields to a classical (or quasi-classical) gravitational field. One of the hopes in this kind of approach was that it would reduce the ultraviolet divergences which plague quantum gravity. Classical field here does not mean background. The classical field is dynamical, yet it is not quantized. The precise meaning of this varies in the different approaches found in the literature. In the older and more extended formulation [5-7], the classical field couples to the expectation value of the energy-momentum tensor of the quantum matter field, thus it behaves deterministically. In this line, corrections to the expectation value to include quantum fluctuations has been considered in [8] using the influence functional formalism.

More recently an interesting proposal was made in [9]. The mixed system is described in the canonical formalism by a density matrix containing both classical and quantum variables. The density matrix evolves according to a linear equation of motion which reduces to the commutator or to the Poisson bracket with the Hamiltonian in the limits of purely quantum or purely classical cases. Furthermore the evolution preserves the hermiticity and the trace of the density matrix, and it is invariant under classical canonical transformations and quantum unitary transformations. However, the requirement of positivity of the density matrix cannot be maintained in general. A similar approach is that of ref. [10] although this is formulated directly in terms of the classical and quantum variables evolving in Heisenberg picture and also differs in details from [9] (see also [10a,10b]). In these approaches the classical field does not evolve deterministically, rather it inherits fluctuations from its coupling to the quantum variables.

In this work we shall not consider gravitation specifically. In fact, we regard any semiclassical treatment not as fundamental but at best as an approximation to the fully quantum-mechanical theory, including gravitation. Rather, we study the dynamics of the mixed classical-quantum system by itself. This is done by giving a new definition of this kind of systems which allows their field theoretical aspects to be emphasized. In particular we consider renormalizability, which is precisely the original motivation which gave rise to the studies mentioned above. The stochastic quantization formalism is used only because our definition is very natural within it. It is an open problem to establish the translation of our prescription to the canonical formalism, and also its relation to the definition of [9]. Similarly to that reference, we also find a positivity problem in our formulation (cf. section 3).

2. Semiquantized dynamics

For simplicity, we restrict ourselves to the case of real bosonic fields. In order to avoid mathematical subtleties, present both in stochastic and path integral quantizations, we
shall consider a system described by a strictly finite number of degrees of freedom, as it is the case of field theories regularized in a finite lattice [14], and the Euclidean formulation of the theory. Any given configuration of the system is therefore described by the set of real variables $\phi_i$, $i = 1, \ldots, N$, where the discrete label $i$ contains all the coordinates of the fields including (discrete) spacetime, spin, flavor, type of particle and so on. To recover a true mechanical system (with continuous time coordinate) or a true field theory, one must take the infrared and ultraviolet limits. This can spoil some of the arguments valid in the finite case. To determine which of the assertions survive as the number of degrees of freedom diverge is not a trivial matter in general and each case requires a separate study. Nevertheless, we shall make some comments below about the important issue of the renormalizability of the theory. Moreover, we shall assume that the Euclidean action $S(\phi)$ is real and that it is a polynomial strictly increasing at infinity. This is sufficient to guarantee that the action is mathematically well-behaved. In particular, it will be bounded from below. A further physical requirement is that the action must be reflection positive, hence defining a positive definite measure for observables in Minkowskian space.

The dynamical problem is considered solved if the vacuum expectation values of arbitrary polynomial observables $O(\phi)$ are known. These expectation values have the form

$$\langle O(\phi) \rangle = \int O(\phi) P(\phi) [d\phi]$$

where $[d\phi]$ is the Lebesge measure over $\mathbb{R}^N$, and $P(\phi)$ is a real and positive probability density, normalized to unity, which in general can be a distribution. This approach unifies the various dynamics to be considered below, namely, classical, quantized or semiquantized. The particular distribution function $P(\phi)$ distinguishes the various dynamics and theories (actions). In the quantum theory $P(\phi)$ is given by the Boltzmann weight [14-16],

$$P_Q(\phi) = e^{-S(\phi)}$$

$P_Q(\phi)$ will be normalized by adding a suitable constant term to the action $S(\phi)$. On the other hand, in the classical dynamics, $P(\phi)$ is a Dirac delta distribution localized at the minimum of the action.

As we said in the introduction, there is a natural definition of the semiquantized dynamics within the stochastic quantization approach [13]. For convenience in the presentation, we sketch here the main ideas underlying this approach. Excellent reviews on this subject can be found in [11,12]. In the stochastic quantization formulation, the field configuration describes a continuous random walk in configuration space, $\phi_i(t)$, where $t$ denotes the simulation or Langevin time, not to be confused with the physical time. The equation governing the random walk is suitably chosen so that $P(\phi)$ is obtained as the stable stationary probability distribution. This implies that vacuum expectation values coincide, for arbitrary initial conditions, with a simultaneous stochastic and temporal average

$$\langle O(\phi) \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \langle O(\phi(t)) \rangle dt$$

Here $\langle \rangle$ is the average over the random noise.
To reproduce the probability distribution of the quantum dynamics, $P_Q(\phi)$, the field configuration is let to evolve according to the following Langevin equation:

$$\partial_t \phi_i(t) = -\partial_i S(\phi(t)) + \eta_i(t), \quad i = 1, 2, \ldots$$  

(2.4)

Here $\eta_i(t)$ are independent stochastic Gaussian variables normalized to $\langle \langle \eta_i(t)\eta_j(t') \rangle \rangle = 2\delta_{ij}\delta(t - t')$. The Langevin equation is a stochastic differential equation which is to be understood in Itô’s sense, that is, as the limit of the Markovian process $[16]$

$$\phi_{i,n+1} = \phi_{i,n} - \partial_i S(\phi_{i,n})\epsilon + \eta_{i,n}\sqrt{2\epsilon}$$  

(2.5)

where $\eta_{i,n}$ are independent normal Gaussian variables and the limit is taken as $\epsilon \to 0^+$ keeping fixed the Langevin time. Note that the $\phi_i(t)$ are not operators but stochastic c-number variables.

The stochastic term $\eta_i(t)$ introduces the quantum fluctuations which distinguishes the quantum dynamics from the classical one. This is more clearly seen by introducing the standard bookkeeping positive parameter $\hbar$, i.e., by rescaling $S \to S/\hbar$. For convenience we also rescale the Langevin time $t \to \hbar t$ (and thus $\eta_i \to \sqrt{\hbar}\eta_i$), so that the Langevin equation becomes

$$\partial_t \phi_i(t) = -\partial_i S(\phi(t)) + \sqrt{\hbar}\eta_i(t), \quad i = 1, 2, \ldots$$  

(2.6)

In the so-called classical limit, $\hbar \to 0^+$, the quantum fluctuations are switched off, this equation becomes deterministic and the equilibrium is attained when $\phi_i$ is a stable solution of the classical equations of motion

$$0 = \partial_i S(\phi),$$  

(2.7)

that is, when $\phi_i$ is the classical vacuum of the theory, or more generally, one of the minima of the Euclidean action. This minimum will vary in the presence of external currents added to the action, hence allowing to pick up classical excited states as well. This point will be further taken in Section 6, when we discuss the effective action.

Let us note that the related Minkowskian problem formally satisfies the Wick rotated equations. Such equations can directly be used in perturbation theory or in simple cases where an exact analytical solution is available (see Section 3 below). In general they can not be solved directly by the Langevin algorithm since the Boltzmann weight is complex in this case. They can be solved indirectly by using the so-called complex Langevin algorithm, but it does not always work properly even for well-behaved actions [17-19]. Another remark is that at any moment, we shall be able to discuss the classical limit of any of the formulae below by following the same procedure of introducing the parameter $\hbar$.

To show that indeed $P_Q(\phi)$ is the equilibrium distribution of the Langevin equation, consider the instantaneous probability density $P(\phi, t) = \langle \delta(\phi - \phi(t)) \rangle$, which depends on the initial conditions. It satisfies the following Fokker-Planck equation, as it is readily shown [20]:

$$\partial_t P(\phi, t) = \partial_i((\partial_i S(\phi))P(\phi, t) + \partial_i P(\phi, t))$$  

(2.8)
It is immediate to check that $P_Q(\phi)$ is a stationary solution. Furthermore, it is the unique stable solution. Indeed, let $\Psi(\phi,t) = \exp\left(\frac{1}{2}S(\phi)\right)P(\phi,t)$; this quantity satisfies the following Euclidean Schrödinger-like equation [20a]

$$-\partial_t \Psi(\phi,t) = H_{FP} \Psi(\phi,t)$$

$$H_{FP} = -\partial_i^2 + V(\phi)$$

$$V(\phi) = \frac{1}{4}(\partial_i S)^2 - \frac{1}{2}\partial_i^2 S$$

(2.9)

The potential $V(\phi)$ is also polynomial and strictly increasing at infinity, hence it has a unique ground state, namely, $\Psi_0(\phi) = \exp(-\frac{1}{2} S(\phi))$, with zero energy. All the other eigenfunctions have strictly positive eigenvalues and hence $P_Q(\phi)$ is a stable fixed point of the Fokker-Planck equation. That $H_{FP}$ is semidefinite positive can also be seen by rewriting it in the form

$$H_{FP} = Q_i^\dagger Q_i, \quad Q_i = \partial_i + \frac{1}{2}\partial_i S(\phi)$$

(2.10)

In passing, it has been noted that $H_{FP} \geq 0$ even if the action is unbounded from below, hence providing a prescription for the quantization of bottomless actions [20b].

It will be important later to note that in fact $P_Q(\phi)$ is also a solution of the set of equations

$$0 = (\partial_i S(\phi)) P(\phi) + \partial_i P(\phi), \quad i = 1, 2, \ldots$$

(2.11)

Again one can see from these equations that in the classical limit $P_Q(\phi)$ collapses to a delta function localized at the classical vacuum.

The Langevin equation (2.6) suggests itself a natural definition for the dynamics of mixed systems with classical and quantum mechanical degrees of freedom, namely, to modify the equation by switching off the quantum fluctuation terms corresponding to the classical fields. More explicitly, let us consider the set of equations

$$\partial_t \phi_i(t) = -\partial_i S(\phi(t)) + \sqrt{\lambda_i} \eta_i(t), \quad i = 1, 2, \ldots$$

(2.12)

where $\lambda_i = 0$ for classical degrees of freedom and $\lambda_i = 1$ for quantum ones. Note that the same dynamics is obtained by taking the latter $\lambda_i$ equal to any other common positive number, instead of unity, since this amounts to a redefinition of the Langevin time scale. For convenience, we shall keep $\lambda_i$ as free real and non-negative parameters in the formulae. Let us stress that here the classical fields are dynamic, not to be confused with background or external fields which are frozen and sometimes are also referred to as classical fields in the literature [21]. In the semiquantized system the classical fields do not behave deterministically, because they acquire stochastic fluctuations through their coupling to the quantum fields. This was to be expected. A similar phenomenon occurs in the canonical approach proposals of refs. [9a,9,10], where the coordinates of the classical particles depend dynamically on those of the quantum particles and vice versa.

The Fokker-Planck equation is modified to

$$\partial_t P = \partial_i((\partial_i S)P + \lambda_i \partial_i P).$$

(2.13)
Though the modification introduced by the parameters $\lambda_i$ looks fairly innocent, in fact they make this equation much more difficult to treat, even for the stationary solution. Indeed the set of equations, similar to (2.11),

$$0 = (\partial_i S)P + \lambda_i \partial_i P, \quad i = 1, 2, \ldots$$

(2.14)

is not consistent for generic actions, unless all the $\lambda_i$ are equal, that is, the completely classical case ($\lambda_i = 0, \forall i$) or the completely quantized case ($\lambda_i = 1, \forall i$). Therefore, we must look directly for normalizable solutions of the stationary Fokker-Planck equation:

$$0 = \partial_i ((\partial_i S)P + \lambda_i \partial_i P).$$

(2.15)

An important issue in our formulation is the stability of the random walk, i.e., whether for large enough Langevin times, all normalizable probability distributions $P(\phi, t)$ evolve to the same normalizable stationary solution. In principle, to decide about the stability of the semiquantized system, one would have to consider the eigenvalue problem corresponding to the Fokker-Planck eq. (2.13), and check that there is one non degenerated zero eigenvalue, whereas all other eigenvalues have a strictly negative real part. This is not obvious since the construction in eqs. (2.9,10) no longer works, and the spectrum may spread over the complex plane. For arbitrary well-behaved actions, there are at least three types of possible instabilities, not mutually exclusive:

1. The spectrum is continuous, hence there is no normalizable stable distribution (for instance, a Brownian motion, if the action vanishes). This possibility can be ruled out, since the action is a strictly increasing polynomial at infinity and will confine the random walk in the finite configuration space.

2. There is a zero eigenvalue but it is degenerated. This is the situation of the classical dynamics if the action have more than one minimum. Each of them is a stable fixed point of the Fokker-Planck equation. The same is true in the semiquantized case if the classical and quantum sectors are uncoupled. This instability is unlikely to show up if the classical and quantum sectors are coupled, due to tunneling.

3. The spectrum contains conjugate pairs of purely imaginary eigenvalues. This would give rise to stable orbits, rather than points, in the space of normalizable probability distributions.

We have been able to prove stability of the semiquantized system if the action is at most quadratic in the fields, as shown in Section 3 below.

In order to compute the vacuum expectation values defined in eq. (2.1), we follow the standard procedure of introducing a generating functional $Z(J)$ [21,15],

$$Z(J) = \langle e^{J_i \phi_i} \rangle = \int e^{J_i \phi_i} P(\phi)[d\phi]$$

$$= \sum_{n \geq 0} \frac{1}{n!} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle J_{i_1} \cdots J_{i_n}$$

(2.16)

and similarly, a generating functional for the connected vacuum expectation values, $W(J)$

$$W(J) = \log Z(J) = \sum_{n \geq 0} \frac{1}{n!} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle_c J_{i_1} \cdots J_{i_n}$$

(2.17)
They are normalized as $Z(0) = 1$, and $W(0) = 0$. By taking the Laplace transform of the stationary Fokker-Planck, eq. (2.15), one finds

$$J_i(\lambda_i J_i - (\partial_i S)(\frac{\partial}{\partial J})) Z(J) = 0$$

(2.18)

which is a linear differential equation for $Z(J)$, since the action is polynomial. On the other hand, the connected generating functional $W(J)$ satisfies a non-linear differential equation.

Before proceeding, let us discuss an important technical point in our prescription. To quantize a theory with action $S(\phi)/\hbar$ one can make use of the following generalized Langevin equation (again in Itô’s sense):

$$\partial_t \phi_i = -g^{ij} \partial_j S + \hbar \partial_j g^{ij} + \sqrt{\hbar} v^i_\alpha \eta_\alpha, \quad i = 1, 2, \ldots$$

(2.19)

where $v^i_\alpha$ can depend on $\phi$ in general and $g^{ij} = v^i_\alpha v^j_\alpha$ is known as the kernel of the equation [22,23,19]. The standard equation corresponds to take $v^i_\alpha = \delta_i \alpha$ and thus $g^{ij} = \delta_{ij}$. The generalized equation reproduces the same equilibrium distribution $P_Q(\phi)$ as the standard one for arbitrary non-singular kernel. This can be seen from the associated Fokker-Planck equation

$$\partial_t P = \partial_i (g^{ij} ((\partial_j S)P + \hbar \partial_j P))$$

(2.20)

which also allows for a generalization of the construction in eqs. (2.9,10), [19]. The spectrum of the Fokker-Planck Hamiltonian, and hence the rate to which the random walk thermalizes to its equilibrium distribution, does depend on the kernel (in fact this one of the reasons to consider kernelled Langevin equations in practice), but not the equilibrium distribution itself. Also the classical limit is independent of the kernel used. However a semiquantization based on the generalized equation will depend on the kernel chosen. Indeed, if we make the replacements $\sqrt{\hbar} \rightarrow \sqrt{\lambda_i}$ and $\hbar \rightarrow \sqrt{\lambda_i \lambda_j}$ in the Langevin equation, the same replacement will result in the Fokker-Planck equation. It is immediate to check that, unless all the $\lambda_i$ coincide, the kernel does not factor out, even at equilibrium. In other words, we can choose various inequivalent ways to define the semiquantized theory, all of them interpolating between the same quantum and classical theories.

It is evident that the standard Langevin equation is not invariant under changes of coordinates on the configuration manifold $\mathbb{R}^N$. On the other hand, the generalized eq. (2.19) is covariant under such transformations, with $v^i_\alpha$ transforming as the contravariant components of a tetrad field and $g^{ij}$ as the contravariant components of a metric tensor [19]. In other words, the stochastic quantization formulation requires to choose a proper Riemannian metric on $\mathbb{R}^N$, $ds^2 = g_{ij} d\phi_i d\phi_j$. The metric is irrelevant at equilibrium in the quantum and classical theories, but not in the semiquantized theory. Note that the classical limit, on the other hand, depends on a choice of coordinate system since the action, and hence its minimum, is not a scalar under general coordinate transformations.

In what follows we shall consider only the trivial kernel $g^{ij} = \delta_{ij}$. However there are cases in which a kernel is absolutely needed. For instance, if the fields $\phi_i$ have different scale dimensions, the standard metric $ds^2 = d\phi_i d\phi_i$, and thus the standard Langevin equation, violates scale invariance. In the case of relativistic fermion fields interacting with bosons, a
kernel is needed, and in fact there is a standard choice which reestablishes scale invariance, although it is introduced for different reasons [11].

3. Quadratic actions

Let us consider an action of the form

\[ S(\phi) = c - h_i \phi_i + \frac{1}{2} m_{ij} \phi_i \phi_j \]  

(3.1)

with \( m_{ij} \) symmetric and strictly positive definite. The differential equation for the connected generating functional \( W(J) \) (similar to eq. (2.18)) takes the form

\[ J_i (\lambda_i J_i + h_i - m_{ij} \partial_j W(J)) = 0 \]  

(3.2)

where \( \partial_j \) refers to partial derivative with respect \( J_j \). It is immediate to check that the most general solution which is analytic at \( J = 0 \) is a polynomial of second degree in \( J_i \)

\[ W(J) = w_i J_i + \frac{1}{2} w_{ij} J_i J_j \]  

(3.3)

By definition of \( W(J) \), \( w_i = \langle \phi_i \rangle \) and \( w_{ij} = \langle \phi_i \phi_j \rangle \). This implies that the dynamics is Gaussian, since higher order connected expectation values vanish. Substituting in the equation one finds the conditions

\[ m_{ij} w_j = h_i, \quad i = 1, \ldots, N \]  

(3.4)

\[ \lambda_i \delta_{ij} = \frac{1}{2} (m_{ik} w_{kj} + w_{ik} m_{kj}), \quad i, j = 1, \ldots, N \]  

(3.5)

The first equation expresses that \( \langle \phi_i \rangle \) satisfies the classical equations of motion independently of the choice of \( \lambda_i \). In the quantum case, this is a consequence of Ehrenfest theorem. The second equation is linear in the unknowns \( w_{ij} \), hence it can be solved in practice, although we have not found a nice closed expression for the general solution. Nevertheless, some statements can be made:

1. The linear system in eq. (3.5) is non-singular, since \( m_{ij} \) is strictly positive definite. Hence, the stationary Gaussian solution (3.3) exists and is unique. Furthermore, the symmetric matrix \( w_{ij} \) is non-negative: let \( \{ |n\rangle, \quad n = 1, \ldots, N \} \) be its complete set of eigenvectors, hence from eq. (3.5), the eigenvalues are given by \( \langle n| \lambda |n\rangle / \langle n| m |n\rangle \), where \( \lambda \) is the matrix \( \lambda_i \delta_{ij} \) and \( m \) the matrix \( m_{ij} \), and the \( \lambda_i \) non-negative. In fact, \( w_{ij} \) is a positive definite matrix unless there are uncoupled classical degrees of freedom.

2. The Gaussian solution (3.3) is in fact the unique fixed point of the time dependent Fokker-Planck equation. Indeed, let \( \Delta W(\phi, t) \) be the difference between any other solution and the stationary Gaussian solution. This difference satisfies the homogeneous equation \((\partial_t + m_{ij} J_i \partial_j) \Delta W = 0\). This equation holds also for the classical or quantum dynamics, which are stable and hence \( \lim_{t \to +\infty} \Delta W(\phi, t) = 0 \).

3. In the purely classical dynamics all the \( \lambda_i \) vanish, and \( w_{ij} \) vanish as well, i.e., there are no fluctuations in the variables \( \phi_i \). Of course, this is true for non quadratic actions too, since in this case \( W(J) = \log Z(J) = w_i J_i \) is the solution of eq. (2.18).
4. In the purely quantum dynamics all the $\lambda_i$ equal unity, and $w_{ij} = (m^{-1})_{ij}$, i.e., $w_{ij}$ is the usual propagator [15]. Comparing with eq. (3.4), we find the fluctuation-dissipation theorem, satisfied by the quantum dynamics, namely, the fluctuation $\langle \phi_i \phi_j \rangle_c$ coincides with the susceptibility, $\partial \langle \phi_i \rangle / \partial h_j$. This holds too for generic actions, since

$$\frac{\partial \langle \phi_i \rangle_J}{\partial J_j} = \frac{\partial^2 W(J)}{\partial J_i \partial J_j} = \langle \phi_i \phi_j \rangle_c, J \quad (3.6)$$

but it is violated by the classical and the semiquantized dynamics.

5. If the quantum and classical degrees of freedom are uncoupled, the solution is also straightforward, $w_{ij}$ being block diagonal, and vanishing in the classical sector. A similar statement is also true for general actions.

6. Since the semiquantized dynamics is Gaussian, it describes a system of non-interacting phonons, as in the quantum case.

As an example, consider a system containing two relativistic fields, $\phi_{1,2}(x)$, with Euclidean action

$$S(\phi_1, \phi_2) = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} m_1^2 \phi_1^2 + \frac{1}{2} (\partial \phi_2)^2 + \frac{1}{2} m_1^2 \phi_2^2 + g \phi_1 \phi_2 \quad (3.7)$$

In momentum representation, the mass matrix $m_{ij}$, is

$$m(k) = \begin{pmatrix} k^2 + m_1^2 & g \\ g & k^2 + m_2^2 \end{pmatrix} \quad (3.8)$$

We assume, $m_1^2, m_2^2 > 0$ and $m_1^2 m_2^2 > g^2$, so that $m(k)$ is positive definite. The equation for the semiquantized connected two point function, $w_{ij}$, can be solved to give

$$w_{SQ}(k) = \frac{\lambda_2 (k^2 + m_1^2) + \lambda_1 (k^2 + m_2^2)}{(k^2 + m_1^2) + (k^2 + m_2^2)} w_Q(k) + \frac{\lambda_1 - \lambda_2}{(k^2 + m_1^2) + (k^2 + m_2^2)} \sigma_3 \quad (3.9)$$

where $\sigma_3$ is the $z$-component Pauli matrix and $w_Q(k)$ is the inverse matrix of $m(k)$, i.e., the quantum propagator. It can be shown explicitly that $w_{SQ}(k)$ is positive definite, as also follows from the general argument above. By construction

$$\langle T \phi_i(y) \phi_j(x) \rangle_c^{SQ} = \int \frac{d^4 k}{(2\pi)^4} \exp(-ik(y - x)) w_{SQ}(k) \quad (3.10)$$

The large momentum limit gives

$$w_{SQ}(k) \rightarrow \frac{1}{k^2} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + O\left(\frac{1}{k^4}\right) \quad (3.11)$$

This implies, from eq. (3.10), that

$$\delta(y^0 - x^0) \langle [\phi_i(y), \partial_0 \phi_j(x)] \rangle = \lambda_i \delta_{ij} \delta(x - y) \quad (3.12)$$
that is, the rescaled equal-time quantum commutation relations. Note that the disconnected part does not contribute to the commutator. It is very instructive to study the Lehmann representation. For the purely quantum case we have

$$w_Q(k) = \frac{P_+}{k^2 + m_+^2} + \frac{P_-}{k^2 + m_-^2}$$  \hspace{1cm} (3.13)$$

where, $m_\pm = \frac{1}{2}(m_1^2 + m_2^2 \pm R)$, $R = \sqrt{(m_1^2 - m_2^2)^2 + 4g^2}$, are the normal masses, and $P_{\pm}$ are the two orthogonal projectors onto the normal modes. For the semiquantized case, one finds instead

$$w_{SQ}(k) = \frac{Q_+}{k^2 + m_+^2} + \frac{Q_-}{k^2 + m_-^2} + \frac{Q_3}{k^2 + m_3^2}$$  \hspace{1cm} (3.14)$$

where, $m_3^2 = \frac{1}{2}(m_1^2 + m_2^2)$, and

$$Q_\pm = \left(\frac{\lambda_1 + \lambda_2}{2} \pm \frac{(\lambda_1 - \lambda_2)(m_1^2 - m_2^2)}{2R}\right) P_{\pm}$$

$$Q_3 = \frac{\lambda_1 - \lambda_2}{2} \left(\sigma_3 - \frac{m_1^2 - m_2^2}{R}(P_+ - P_-)\right)$$  \hspace{1cm} (3.15)$$

One can see that there is an extra mode, namely, $m_3^2$. Unfortunately, whereas $Q_{\pm}$ are non-negative, $Q_3$ is not, since $\text{tr}(Q_3) = 0$. This means that the covariance matrix $w_{SQ}(k)$ is positive but not reflection positive, (except in the trivial cases $\lambda_1 = \lambda_2$ of $g = 0$). As a consequence this theory does not define a Hilbert space with positive definite metric, i.e., it does not define a positive physical measure, and for instance, one can construct operators with negative variance. In other words, the probabilistic interpretation (of which the classical case is a limit) breaks down, and the theory must be rejected (or work with a restricted set of observables, which in this context is ad hoc). This is a direct consequence of the commutation relations eq. (3.12).

### 4. Feynman rules

In order to set up a perturbative calculation let us consider an action of the form

$$S(\phi) = c - h_i \phi_i + \frac{1}{2!} m_{ij} \phi_i \phi_j + \frac{1}{3!} g_{ijk} \phi_i \phi_j \phi_k$$  \hspace{1cm} (4.1)$$

For simplicity, we do not include a quartic term in the action, which would be required to guarantee stability. The associated Langevin equation can be brought to the form

$$\phi_t = (\partial_t + m)^{-1}(\sqrt{\lambda_i} \eta_i + h_i - \frac{1}{2!} g_{ijk} \phi_j \phi_k)$$  \hspace{1cm} (4.2)$$

Because by assumption there is stable equilibrium probability density, in this equation we can take $(\partial_t + m)^{-1}$ as the retarded propagator and $\phi_t = 0$ at $t \to -\infty$ as boundary condition [11,12]. The equation and its iterative solution is represented in fig. 1 by means of tree diagrams. There, the crosses represent quantum fluctuations $\eta_i(t)$ which act as sources
for the fields and the dots are the background sources \( h_i \) in the Lagrangian. Algebraically it is cumbersome but straightforward to solve the equation iteratively and compute the \( n \)-point Euclidean Green function \( \langle \cdots \phi_{i_1}(t_1)\phi_{i_2}(t_2)\cdots \phi_{i_n}(t_n) \rangle \) by contracting the \( \eta_i(t) \) with the rules

\[
\langle \eta_i(t) \eta_j(t') \rangle_c = 2\delta_{ij} \delta(t-t') \\
\langle \eta_i(t_1) \eta_j(t_2) \cdots \eta_n(t_n) \rangle_c = 0 \quad \text{(for } n \neq 2) \quad (4.3)
\]

The subindex \( c \) stands for the connected part of the expectation value. Diagrammatically such an expectation value corresponds to combine \( n \) copies of the tree graphs of fig. 1, by contracting all the crosses pairwise in all possible forms. In this form the so-called stochastic diagrams are obtained [11,12]. An example is shown in fig. 2 for the two-point function. (Note that the meaning of the blobs in both figures is not exactly the same).

In general there are several stochastic diagrams corresponding to each standard Feynman graph, differing by the positions of the crosses in them. It is noteworthy that if \( \lambda_i = 0 \), thus removing the quantum fluctuations, only tree level graphs remain. This corresponds to the classical approximation. Moreover, because \( \lambda_i \) is a kind of \( \hbar \) parameter, we find a selection rule similar to that existing for Feynman graphs [24,21], namely the number of loops plus the number of external lines in a stochastic diagram equals the number of crosses plus the number of connected subgraphs. There is another structural property which is relevant to select the possible stochastic diagrams: by construction, if the graphs are cut by the crosses the resulting disconnected pieces are tree graphs containing exactly one external line.

Actually we only want to compute the \( n \)-point functions at equilibrium, and for this purpose it is more convenient to use directly the stationary Fokker-Planck eq. (2.15). The probability density itself is quite singular perturbatively because it is a delta function at zeroth order in the classical sector, thus we shall use instead the generating functional \( Z(J) \), eq. (2.16). To simplify, let us remove the external sources, \( h_i = 0 \), and assume a diagonal representation for the quadratic part, \( m_{ij} = s_i^{-1} \delta_{ij} \). The eq. (2.18) takes the form

\[
0 = -s_i^{-1} J_i \partial_i Z - \frac{1}{2} g_{ijk} J_i \partial_j \partial_k Z + \lambda_i J_i^2 Z 
\]

On the other hand, the connected generated functional \( W(J) \), satisfies the non linear master equation

\[
0 = s_i^{-1} J_i \partial_i W + \frac{1}{2} g_{ijk} J_i (\partial_j \partial_k W + \partial_j W \partial_k W) - \lambda_i J_i^2 
\]

In the purely quantum case \( \lambda_i = 1 \), this equation is equivalent to the set of equations

\[
\partial_i W = - \frac{1}{2} g_{ijk} s_i (\partial_j \partial_k W + \partial_j W \partial_k W) + s_i J_i, \quad i = 1, 2, \ldots 
\]

which is an alternative to the usual construct

\[
Z(J) = \exp\left(-\frac{1}{3!} g_{ijk} \partial_i \partial_j \partial_k \right) \exp\left(\frac{1}{2} s_i J_i^2 \right) 
\]
For unequal $\lambda_i$ only the master equation (4.5) is valid. Rather than solving this equation perturbatively directly, it is better to rewrite it as a set of exact identities among connected Green functions. Let $w_{i_1i_2\ldots i_n} = \langle \phi_{i_1} \cdots \phi_{i_n} \rangle_c$. Substituting the expansion of $W(J)$ in its master equation, we find the following hierarchy of identities

\begin{align*}
    w_a &= -\frac{1}{2}g_{aij} s_a(w_{ij} + w_iw_j) \\
    w_{ab} &= -2s_{ab}[[g_{aij}(\frac{1}{2}w_{bij} + w_jw_{bi})]] + \lambda_a s_a \delta_{ab} \\
    w_{abc} &= -3s_{abc}[[g_{aij}(\frac{1}{2}w_{bcij} + w_iw_{bcj} + w_{bi}w_{cj})]] \\
    &\vdots
\end{align*}

(4.8)

where $s_{ab\ldots} = (s_a^{-1} + s_b^{-1} + \cdots)^{-1}$, and the symbol $[[\cdots]]$ means permutation symmetric average on the free indices. It is worth noting that these identities are exactly the same as in the quantum theory except the second one, which depends on $\lambda_i$. These equations can be solved iteratively up to any order in perturbation theory (we disregard possible non perturbative solutions). Diagrammatically they are represented in fig. 3 by skeleton equations. By expanding iteratively these equations, we obtain again the stochastic diagrams (though they now stand for contributions to vacuum expectation values, i.e., at equilibrium). Their Feynman rules, which can be read from eq. (4.8), are more complicated than those corresponding to the usual Feynman diagrams. They are the following:

1. Draw all topologically distinct connected labeled stochastic graphs and apply the following rules to each one of them.
2. If the graph is the free two-point graph, apply rule 5. Otherwise, identify the graph in the skeleton equations in fig. 3. If this can be done in more than one way, all of them should be added. Note that the crosses are hidden inside the blobs and that all but one (uncrossed) line (say, with label $a$) and one three-point vertex (say, with labels $a, i, j$) will be inside the blobs.
3. The value of the graph picks up a factor $s_{abc\ldots}$ from the average propagator of the external lines. This substitutes the usual factor $s_a$ in the Feynman rules of standard Feynman graphs. Also add a factor $-g_{aij}$ from the three-point vertex and a standard symmetry factor from the structure of the skeleton equation.
4. Apply rules 2 and 3 to each of the subdiagram which were hidden in the blobs in previous steps.
5. The free two-point graph (or subgraph) gives a contribution $\lambda_a s_a$
6. Sum over internal indices and add the contributions of different graphs.

As an illustration, the values of all connected stochastic graphs up to second order are given in fig. 4. As a check it can be shown that in the particular case $\lambda_i = 1$, the stochastic graphs add up to the usual Feynman graphs. In the general case however the average propagators $s_{ijk\ldots}$ remain and this emphasizes clearly the need for a kernel if the fields have different dimensions.

It is noteworthy that the semiquantized stochastic graphs do not satisfy the usual reducibility rules. This is because the presence of the averaged propagators $s_{ijk\ldots}$ in the
Feynman rules prevents the required factorization properties to occur. This makes the Feynman rules more involved than in the purely quantum case and it will be relevant in the next section when we discuss the renormalizability of the theory.

5. Perturbative renormalizability

The original interest of studying systems with mixed quantum and classical degrees of freedom, is the possibility of obtaining finite results in physically relevant theories which are non renormalizable at the quantum level, such as gravitation. The idea is that, given that the classical theory contains no loops and is finite, a mixed theory would be more ultraviolet convergent than the completely quantum case.

In order to clarify this issue we can study a Lagrangian such as

\[ \mathcal{L}(x) = \mathcal{L}_{\text{KG}}(\phi) + \mathcal{L}_{\text{KG}}(\psi) + \frac{1}{2}g\phi(x)\psi^2(x) + \frac{1}{4!}\gamma\phi^4(x) \] (5.1)

\( \mathcal{L}_{\text{KG}} \) being the Klein-Gordon Lagrangian. In six space-time dimensions this theory is not renormalizable unless the parameter \( \gamma \) vanishes [26]. The interesting semiquantized case is when \( \psi(x) \) is a quantum field and \( \phi(x) \) is classical. The other way around gives a trivial theory with no loops. For our discussion it will be enough to consider the case \( \gamma = 0 \) in six dimensions. This semiquantized theory turns out to be non renormalizable. To see this, consider the stochastic graph in fig. 5a. It is the lowest order divergent graph for the two-point Green function of the field \( \phi(x) \). In the quantum theory such a divergence would be renormalized by redefining the mass and wavefunction of the field. The counterterm is shown in fig. 5b. In the semiquantized theory, however, there is no such contribution because \( \lambda \phi \) vanishes. Note that other second order stochastic graphs analogous to that in fig. 5a but differing in the position of the crosses also vanishes for the same reason. Similarly, the lowest order divergent vertex correction, shown in fig. 6a, cannot be canceled by the usual counterterm, fig. 6b, which does not exists in the semiquantized theory. The conclusion is that the semiquantized version of the original renormalizable quantum theory is non renormalizable, at least perturbatively. Clearly the same conclusion will hold for more complicated theories too (for instance letting \( \gamma \neq 0 \)).

A possible way out would be to take \( \lambda \phi \) as a free parameter with the prescription \( \lambda \phi \rightarrow 0^+ \) at the end. This would allow a mass and wavefunction counterterm of order \( g^2\lambda^{-1} \) to cancel the mass and wavefunction correction divergent graph, and similarly for the vertex correction. The analysis of this procedure at higher orders is very involved. In particular note that in the quantum theory, reducible graphs are automatically renormalized once irreducible graphs are, but this does not hold in the semiquantized case. This implies that for instance the two bubble graph in fig. 7 requires a further mass and wavefunction counterterm of order \( g^4\lambda^{-2} \). On the other hand it is not clear whether such a prescription is effectively cancelling the limit \( \lambda \phi \rightarrow 0 \) and thus going over to the full quantum theory.

The problem with the renormalizability is closely related to the fact that the stochastic diagrams lack good reducibility properties and hence it is more a problem of the stochastic approach than of the semiquantization itself. Indeed by taking arbitrary \( \lambda_i \) we are weighting differently the various stochastic graphs. What we see is that they are not renormalizable independently. Likely, this implies that even in the standard quantum
In the quantum case, the variables \( J_i \) are identified with currents in the sense that they create excited states on the vacuum. However, this is not true classically. For instance, if the classical Euclidean vacuum corresponds to \( \phi_i = 0 \), \( Z(J) = \langle e^{J_i \phi_i} \rangle \) is identically equal to unity, since there are no fluctuations. The correct way to introduce a current \( h_i \) to pick up excited states, valid both in the quantum and classical cases, is through the action, i.e., by considering the family of actions \( S(\phi) - h_i \phi_i \). Hence one must consider rather \( Z(J,h) = \langle e^{J_i \phi_i} \rangle \) as the generator of expectations values in the presence of external currents; in the quantum case \( Z(J,h) \) depends on \( J_i + h_i \) only. In order to define the effective action, let \( \phi_i(h) = \langle \phi_i \rangle_h \) be the so-called classical field, and let \( \Omega(h) \) its connected generator, i.e.,

\[
\phi_i(h) = \frac{\partial \Omega(h)}{\partial h_i}
\]

by definition. First, one must show that \( \Omega(h) \) exists, that is, that the integrability conditions of eq. (6.1) are met. This is true in the quantum case (\( \Omega \) is just \( W \)), and in the classical case: \( \Omega \) is the Legendre transform of the action. The proof follows from the Langevin equation,

\[
\partial_t \phi_i(t,h) = -\partial_i S + h_i - \sqrt{\lambda_i} \eta_i
\]

Applying \( \partial/\partial h_j \) one finds

\[
\frac{\partial \phi_i(t,h)}{\partial h_j} = (\partial_t + \partial^2 S)^{-1}_{ij}
\]

independently of \( \lambda_i \). Since the right hand side is manifestly symmetric in the indices \( ij \), this is also true after average over \( \eta_i \) and in the limit of large Langevin time, i.e., the matrix \( \partial \phi_i(h)/\partial h_j \) is also symmetric and \( \Omega \) exists in the semiquantized case. Note that in general \( \partial \phi_i(h)/\partial h_j \) does not coincide with \( \langle \phi_i \phi_j \rangle_{h,c} \). They coincide in the quantum case, but in the classical case the latter expression vanishes. The effective action is defined by the Legendre transform of \( \Omega(h) \),

\[
\Gamma(\phi) = -\Omega(h) + h_i \phi_i,
\]

This definition is the usual effective action in the quantum case and is the action \( S(\phi) \) in classical case. The best way to compute it is by solving

\[
h_i = \frac{\partial \Gamma(\phi)}{\partial \phi_i}
\]

For the action \( S(\phi) = \frac{1}{2} s_i^{-1} \phi_i^2 + \frac{1}{3!} g_{ijk} \phi_i \phi_j \phi_k \), one finds perturbatively

\[
\Gamma(\phi) = \frac{1}{2} s_i^{-1} \phi_i^2 + \frac{1}{3!} g_{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} g_{ijj} s_j \lambda_j \phi_i - \frac{1}{2} g_{ijj} g_{\ell jk} s_j s_k \lambda_j \phi_i \phi_\ell + O(g^3)
\]
corresponding respectively to the diagrams in fig. 8. It is interesting to note that the effective action only contains one particle irreducible graphs and it is regular as $\lambda_i \to 0$. This is unlike the Legendre transform of $W(J)$. It implies that the effective action is renormalizable if the action is renormalizable. However, recall that the effective action only generates the expectation value of $\phi_i$, not the two-point function, etc.

7. Conclusions

We have defined and studied some aspects of the dynamics of semiquantized fields. These systems interpolate between the completely quantum and the completely classical theories. The semiquantization is not uniquely defined, however. As pointed out above the choice of a concrete kernel is also needed in the stochastic quantization itself, and it is the origin of quantum anomalies in this formulation. The study shows that this kind of theories present two flaws. First, they break positivity: they do not define a positive measure for the Minkowskian expectation values. The other conflict comes from their renormalizability. Here we found a surprise, namely, the removal of quantum fluctuations in a subset of the degrees of freedom does not necessarily make the theory more ultraviolet convergent. On the contrary, in general (and in fact in the interesting cases), the renormalizability is spoiled by the semiquantization. This is because in the semiquantized theory we are setting to zero a subset of (stochastic) Feynman graphs and the remaining diagrams no longer form a closed set under renormalization. There are not enough counterterms left to cancel all the divergences. Less technically, the lesson from fig. 5 or 6 is that the missing intrinsic fluctuations of the classical field are needed to compensate the fluctuations induced through the coupling to the quantum field. In fact the problem with positivity, renormalizability, as well as the lack of uniqueness and of good reducibility properties of the Green functions, is a manifestation of the fact that the equations of the semiquantized theory are considerably less symmetric than those of the quantum or classical theories. We think that this kind of problems will appear also in the other approaches to semiquantization existing in the literature, and in our view this means that the concept of semiquantization is rather unnatural.

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**Figure captions**

Fig. 1: Diagrammatic representation of the Langevin equation (4.2) and its iterative solution.

Fig. 2: Typical stochastic graph for the two-point function. Fig. 3: Diagrammatic representation of the hierarchy of identities in eq. (4.8)

Fig. 4: Connected stochastic graphs and their values up to second order in perturbation theory.

Fig. 5: Two-point divergent stochastic graph at lowest order (a) and its counterterm graph (b).

Fig. 6: Three-point divergent stochastic graph at lowest order (a) and its counterterm graph (b).

Fig. 7: Divergent reducible two loop graph.

Fig. 8: Effective action graphs up to second order.
\[ \phi = \eta + h + \ldots \]

Fig. 1.

\[ \ldots + \quad \ldots \]

Fig. 2.

\[ \ldots + \quad \ldots \]
Fig. 3.
\[ s_a^6 a b \lambda_a \]

\[ -\frac{1}{2} g_{a i i} s_i s a i \lambda_i \]

\[ -g_{a b c} s_{a b c} s_b s_c \lambda_b \lambda_c \]

\[ \frac{1}{2} g_{a b i j} g_{i j} s^2 a b s_i s_j \lambda_b \lambda_j \]

\[ g_{a i j} g_{b i j} s^2 a i s_{a i} s_{a b} \lambda_a \lambda_i \]

\[ \frac{1}{2} g_{a i j} g_{b i j} s_{a i} s_{a i j} s_{ab} \lambda_a \lambda_i \]

\[ g_{a b i} g_{c d i} s_{a b c d} s_b s_{c d i} s_{c i} s_{cd} \lambda_b \lambda_c \lambda_d \]

\[ g_{a b i} g_{c d i} s_{a b c} s_{b i} s_{a c i} s_{c d} (s_{abi} + s_{c i}) \lambda_b \lambda_c \lambda_i \]

Fig. 4.
Fig. 5.

Fig. 6.
Fig. 7.

Fig. 8.