DEFORMATIONS AND COHOMOLOGY THEORY OF ROTA-BAXTER SYSTEMS

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ABSTRACT. Inspired by the work of Wang and Zhou [4] for Rota-Baxter algebras, we develop a cohomology theory of Rota-Baxter systems and justify it by interpreting the lower degree cohomology groups as formal deformations and as abelian extensions of Rota-Baxter systems. A further study on an $L_\infty$-algebra structure associated to this cohomology theory will be given in a subsequent paper.

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1. INTRODUCTION

By a general philosophy, the deformation theory of any given mathematical object can be described by a certain differential graded (=dg) Lie algebra or more generally a $L_\infty$-algebra associated to the mathematical object. Therefore it is an important question to construct explicitly this dg Lie algebra or $L_\infty$-algebra governing deformation theory of this mathematical object. Another important question about algebraic structures is to study their homotopy versions, just like $A_\infty$-algebras

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for usual associative algebras. The nicest result would be providing a minimal model of the operad governing an algebraic structure. When this operad is Koszul, there exists a general theory, the so-called Koszul duality for operads, which defines a homotopy version of this algebraic structure via the cobar construction of the Koszul dual cooperad, which, in this case, is a minimal model. However, when the operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out. These two questions, say, describing controlling $L_\infty$-algebras and constructing homotopy versions, are closed related. In fact, given a cofibrant resolution, in particular, a minimal model, of the operad in question, one can form the deformation complex of the algebraic structure and construct its $L_\infty$-structure, see [4, Introduction] for more explanation on this method.

Recently Wang and Zhou developed an ad hoc method in [4] to deal with the above two questions. Surprisingly, this method works well for many individual nonKoszul algebra structures. The idea of this method is as follows. Given an algebraic structure on a space $V$ realised as an algebra over an operad, by considering the formal deformations of this algebraic structure, we first construct the deformation complex and find an $L_\infty$-structure on the underlying graded space of this complex such that the Maurer-Cartan elements are in bijection with the algebraic structures on $V$. When $V$ is graded, we define a homotopy version of this algebraic structure as Maurer-Cartan elements in the $L_\infty$-algebra constructed above. Finally under suitable conditions, we could show that the operad governing the homotopy version is a minimal model of the original operad. Using their method, Wang and Zhou successfully find a minimal model of the operad governing Rota-Baxter algebra structures in [4]. Recall that an associative algebra $A$ over a field $K$ is called a Rota-Baxter algebra of weight $\lambda$ (where $\lambda \in K$) if there is a $K$-linear operator $R : A \to A$ (called a Rota-Baxter operator of weight $\lambda$) satisfying the equation $R(a)R(b) = R(R(a)b + aR(b) + \lambda ab)$ for all $a, b \in A$.

In the present paper, we discuss the formal deformations and cohomology theory of Rota-Baxter systems. Rota-Baxter system is a generalisation of the notion of Rota–Baxter algebra introduced by Brzeziński [1]. This generalisation consists of two operators $R, S$ acting on an associative algebra $A$ and satisfying equations similar to the Rota–Baxter equation. Since the two operators $R, S$ are wrapped up in each other, in order to develop a cohomology theory of Rota-Baxter systems, we have to modify the construction of the cohomology cochain complex in [4] for Rota-Baxter algebras, see Section 4 for the details.

In a subsequent paper, we will discuss an $L_\infty$-algebra structure over the cochain complex of the Rota-Baxter system and realize the Rota-Baxter system structures as the Maurer-Cartan elements of this $L_\infty$-algebra, parallel to the work on Rota-Baxter algebras in [4, Section 8].

This paper is organised as follows. Section 2 contains a quick review on formal deformations and Hochschild cohomology of associative algebras. In Section 3 we give basic definitions and facts about Rota-Baxter systems and Rota-Baxter system bimodules. A cohomology cochain complex of Rota-Baxter system operators, and with the help of the usual Hochschild cocohain complex, a cochain complex, whose cohomology groups should control deformation theory of Rota-Baxter systems, is exhibited in Section 4. We justify this cohomology theory by interpreting lower degree cohomology groups as formal deformations (Section 5) and as abelian extensions of Rota-Baxter systems (Section 6).
2. A quick review on formal deformations and Hochschild cohomology of associative algebras

Throughout this paper, let $\mathbb{K}$ be a field of arbitrary characteristic. All vector spaces are defined over $\mathbb{K}$, all tensor products and Hom-spaces are taken over $\mathbb{K}$, unless otherwise stated. Besides, all algebras considered in this paper are associative (but not necessarily unital) over $\mathbb{K}$. We say that an algebra $A$ is non-degenerate provided that for any $b \in A$, $ba = 0$ or $ab = 0$ for all $a \in A$ implies that $b = 0$. Obviously, any unital algebra is non-degenerate. We denote by $\mathbb{K}[[t]]$ the power series ring in one variable $t$ over the field $\mathbb{K}$.

In this section, we give a quick review on formal deformations and Hochschild cohomology of associative algebras. For backgrounds and more details on these subjects, we refer to [2], [3], and [4].

2.1. Hochschild cohomology of associative algebras. Let $(A, \mu)$ be an associative $\mathbb{K}$-algebra. We often write $\mu(a \otimes b) = a \cdot b = ab$ for any $a, b \in A$, and for $a_1, \ldots, a_n \in A$, we write $a_{1,n} := a_1 \otimes \cdots \otimes a_n \in A^\otimes n$. Let $M$ be a bimodule over $A$. Recall that the Hochschild cochain complex of $A$ with coefficients in $M$ is

$$C^\bullet_{\text{Alg}}(A, M) := \bigoplus_{n=0}^{\infty} C^n_{\text{Alg}}(A, M),$$

where $C^n_{\text{Alg}}(A, M) = \text{Hom}(A^\otimes n, M)$ and the differential $\delta^n : C^n_{\text{Alg}}(A, M) \to C^{n+1}_{\text{Alg}}(A, M)$ is defined as:

$$\delta^n(f)(a_{1,n+1}) = (-1)^{n+1}a_1f(a_{2,n+1}) + \sum_{i=1}^{n}(-1)^{n-i+1}f(a_{1,i-1} \otimes a_i \cdot a_{i+1} \otimes a_{i+2,n+1}) + f(a_{1,n})a_{n+1}$$

for all $f \in C^n_{\text{Alg}}(A, M), a_1, \ldots, a_{n+1} \in A$.

The cohomology of the Hochschild cochain complex $C^\bullet_{\text{Alg}}(A, M)$ is called the Hochschild cohomology of $A$ with coefficients in $M$, denoted by $\text{HH}^\bullet(A, M)$. When the bimodule $M$ is the regular bimodule $A$ itself, we just denote $C^\bullet_{\text{Alg}}(A, A)$ by $C^\bullet_{\text{Alg}}(A)$ and call it the Hochschild cochain complex of the associative algebra $(A, \mu)$. Denote the cohomology $\text{HH}^\bullet(A, A)$ by $\text{HH}^\bullet(A)$, called the Hochschild cohomology of the associative algebra $(A, \mu)$.

2.2. Formal deformations of associative algebras. Given an associative $\mathbb{K}$-algebra $(A, \mu)$, consider $\mathbb{K}[[t]]$-bilinear associative multiplications on

$$A[[t]] = \{ \sum_{i=0}^{\infty} a_it^i | a_i \in A, \forall i \geq 0 \}.$$

Such a multiplication is determined by

$$\mu_t = \sum_{i=0}^{\infty} \mu_it^i : A \otimes A \to A[[t]],$$

where for all $i \geq 0$, $\mu_i : A \otimes A \to A$ are $k$-linear maps. When $\mu_0 = \mu$, we say that $\mu_t$ is a formal deformation of $\mu$ and $\mu_1$ is called the infinitesimal of the formal deformation $\mu_t$. The only constraint is the associativity of $\mu_t$:

$$\mu_t(a \otimes \mu_t(b \otimes c)) = \mu_t(\mu_t(a \otimes b) \otimes c), \forall a, b, c \in A,$$
which is equivalent to the following family of equations:

\[ \sum_{i+j=n} \mu_i(\mu_j(a \otimes b) \otimes c) - \mu_i(a \otimes \mu_j(b \otimes c)) = 0, \forall a, b, c \in A, n \geq 0. \]

Looking closely at the cases \( n = 0 \) and \( n = 1 \), one obtains:

(i) when \( n = 0 \),

\[ (a \cdot b) \cdot c = a \cdot (b \cdot c), \forall a, b, c \in A, \]

which is exactly the associativity of \( \mu \);

(ii) when \( n = 1 \),

\[ a\mu_1(b \otimes c) - \mu_1(ab \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c = 0, \forall a, b, c \in A, \]

which says that the infinitesimal \( \mu_1 \) is a 2-cocycle in the Hochschild cochain complex \( C^*_\text{Alg}(A) \).

### 3. Rota-Baxter systems and Rota-Baxter system bimodules

In this section, we first recall the definition of Rota-Baxter systems and define the Rota-Baxter system bimodules, and then we give several interesting observations about them, following similar ideas from [1, Definition 4.3].

**Definition 3.1.** (see [1, Definition 2.1]) A triple \((A, R, S)\) consisting of an associative algebra \(A\) and two \(\mathbb{K}\)-linear operators \(R, S : A \to A\) is called a Rota-Baxter system if, for all \(a, b \in A\),

\[ R(a)R(b) = R(R(a)b + aS(b)), \]
\[ S(a)S(b) = S(R(a)b + aS(b)). \]

In this case, \((R, S)\) are called Rota-Baxter system operators.

**Remark 3.2.** (see [1, Lemma 2.2]) Let \(A\) be an algebra. If \(R\) is a Rota-Baxter operator of weight \(\lambda\) on \(A\), then \((A, R, R + \lambda \text{id})\) and \((A, R + \lambda \text{id}, R)\) are Rota-Baxter systems.

**Remark 3.3.** (see [1, Lemma 2.3]) Let \(A\) be an algebra. Let \(R : A \to A\) be a left \(A\)-linear map and \(S : A \to A\) be a right \(A\)-linear map. Then \((A, R, S)\) is a Rota-Baxter system if and only if, for all \(a, b \in A\),

\[ aR \circ S(b) = 0 = S \circ R(a)b. \]

In particular, if \(A\) is a non-degenerate algebra, then \((A, R, S)\) is a Rota-Baxter system if and only if \(R\) and \(S\) satisfy the orthogonality condition

\[ R \circ S = S \circ R = 0. \]

**Remark 3.4.** (see [1, Remark 2.6]) A morphism of Rota-Baxter systems from \((A, R_A, S_A)\) to \((B, R_B, S_B)\) is an algebra map \(f : A \to B\) rendering the following diagrams commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{R_A} & A \\
\downarrow{f} & & \downarrow{f} \\
B & \xrightarrow{R_B} & B \\
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{S_A} & A \\
\downarrow{f} & & \downarrow{f} \\
B & \xrightarrow{S_B} & B \\
\end{array}
\]
Definition 3.5. Let \((A, R, S)\) be a Rota-Baxter system and \(M\) be a bimodule over the associative algebra \(A\). We say that \(M\) is a bimodule over Rota-Baxter system \((A, R, S)\) or a Rota-Baxter system bimodule if \(M\) is endowed with two linear operators \(R_M, S_M : M \to M\) such that the following equations hold for any \(a \in A\) and \(m \in M\):

\[
\begin{align*}
R(a)R_M(m) &= R_M(R(a)m + aS_M(m)), \\
R_M(m)R(a) &= R_M(R_M(m)a + mS(a)), \\
S(a)S_M(m) &= S_M(R(a)m + aS_M(m)), \\
S_M(m)S(a) &= S_M(R_M(m)a + mS(a)).
\end{align*}
\]

Of course, \((A, R, S)\) itself is a bimodule over the Rota-Baxter system \((A, R, S)\), called the regular Rota-Baxter system bimodule.

Proposition 3.6. (Compare to [4, Proposition 4.4]) Let \((A, R, S)\) be a Rota-Baxter system and \(M\) be a bimodule over associative algebra \(A\). It is well known that \(A \oplus M\) becomes an associative algebra whose multiplication is

\[
(a, m)(b, n) = (ab, an + mb).
\]

Write \(\iota : A \to A \oplus M, a \mapsto (a, 0)\) and \(\pi : A \oplus M \to A, (a, m) \mapsto a\). Then \(A \oplus M\) is a Rota-Baxter system such that \(\iota\) and \(\pi\) are both morphisms of Rota-Baxter systems if and only if \(M\) is a Rota-Baxter system bimodule over \(A\). This new Rota-Baxter system will be denoted by \(A \ltimes M\), called the semi-direct product (or trivial extension) of \(A\) by \(M\).

Proof. First we notice that the associative algebra \(A \oplus M\) is a Rota-Baxter system such that \(\iota\) and \(\pi\) are both morphisms of Rota-Baxter systems if and only if there exist two \(\mathbb{K}\)-linear operators \(R', S' : A \oplus M \to A \oplus M\) such that the following equations hold for any \(a_1, a_2, a \in A\) and \(m_1, m_2, m \in M\):

\[
\begin{align*}
R'(a_1, m_1)R'(a_2, m_2) &= R'(R'(a_1, m_1)(a_2, m_2) + (a_1, m_1)S'(a_2, m_2)), \\
S'(a_1, m_1)S'(a_2, m_2) &= S'(S'(a_1, m_1)(a_2, m_2) + (a_1, m_1)S'(a_2, m_2)), \\
R'(a_1, m_1)(a_2, m_2) &= (R(a_1)m_1 + a_1S(a_2), m_2) + (a_1, m_1)(R(a_2), 0), \\
S'(a_1, m_1)(a_2, m_2) &= (S(a_1)m_1 + a_1R(a_2), m_2) + (a_1, m_1)(S(a_2), 0),
\end{align*}
\]

where the last four equations \((13)-(16)\) come from the definition of morphism of Rota-Baxter systems (cf. Remark 3.4) and are equivalent to the following equations

\[
\begin{align*}
R'(a, 0) &= (R(a), 0), \\
R'(0, m) &= (0, R'_2(0, m)), \\
S'(a, 0) &= (S(a), 0), \\
S'(0, m) &= (0, S'_2(0, m)).
\end{align*}
\]
Now we assume that the latter part of the above equivalent conditions holds and write \( R_M(m) := R'_2(0, m) \) and \( S_M(m) := S'_2(0, m) \). Then \( R_M, S_M : M \to M \) become two linear operators, and we have
\[
R'(a, m) = R'(a, 0) + R'(0, m) = (R(a), 0) + (0, R_M(m)) = (R(a), R_M(m)),
\]
\[
S'(a, m) = S'(a, 0) + S'(0, m) = (S(a), 0) + (0, S_M(m)) = (S(a), S_M(m)).
\]
Furthermore, Equation (11) becomes
\[
\left( R(a_1)R(a_2), R(a_1)R_M(m_2) + R_M(m_1)R(a_2) \right)
= \left( R(a_1), R_M(m_1) \right) \left( R(a_2), R_M(m_2) \right)
= R' \left( (R(a_1), R_M(m_1))(a_2, m_2) + (a_1, m_1)(S(a_2), S_M(m_2)) \right)
= R' \left( (R(a_1)a_2, R(a_1)m_2 + R_M(m_1)a_2) + (a_1S(a_2), a_1S_M(m_2) + m_1S(a_2)) \right)
= R' \left( R(a_1)a_2 + a_1S(a_2), (R(a_1)m_2 + a_1S_M(m_2)) + (R_M(m_1)a_2 + m_1S(a_2)) \right)
= \left( R(a_1)R(a_2), R_M(R(a_1)m_2 + a_1S_M(m_2)) + R_M(R_M(m_1)a_2 + m_1S(a_2)) \right),
\]
and Equation (12) becomes
\[
\left( S(a_1)S(a_2), S(a_1)S_M(m_2) + S_M(m_1)S(a_2) \right)
= \left( S(a_1), S_M(m_1) \right) \left( S(a_2), S_M(m_2) \right)
= S' \left( (R(a_1), R_M(m_1))(a_2, m_2) + (a_1, m_1)(S(a_2), S_M(m_2)) \right)
= S' \left( (R(a_1)a_2, R(a_1)m_2 + R_M(m_1)a_2) + (a_1S(a_2), a_1S_M(m_2) + m_1S(a_2)) \right)
= S' \left( R(a_1)a_2 + a_1S(a_2), (R(a_1)m_2 + a_1S_M(m_2)) + (R_M(m_1)a_2 + m_1S(a_2)) \right)
= \left( S(a_1)S(a_2), S_M(R(a_1)m_2 + a_1S_M(m_2)) + S_M(R_M(m_1)a_2 + m_1S(a_2)) \right).
\]
Let \( m_1 = 0 \), we get the following equations
\[
R(a_1)R_M(m_2) = R_M(R(a_1)m_2 + a_1S_M(m_2)),
\]
\[
S(a_1)S_M(m_2) = S_M(R(a_1)m_2 + a_1S_M(m_2)).
\]
Similarly, let \( m_2 = 0 \), we get
\[
R_M(m_1)R(a_2) = R_M(R_M(m_1)a_2 + m_1S(a_2)),
\]
\[
S_M(m_1)S(a_2) = S_M(R_M(m_1)a_2 + m_1S(a_2)).
\]
The above four equations show that \( M \) is a Rota-Baxter system bimodule over \( A \).

Finally, given a Rota-Baxter system bimodule \( M \) over \( A \) with two \( \mathbb{K} \)-linear operators \( R_M, S_M \), we define two \( \mathbb{K} \)-linear operators \( R', S' : A \oplus M \to A \oplus M \) by
\[
R'(a, m) = (R(a), R_M(m)), \quad S'(a, m) = (S(a), S_M(m)).
\]
Then it is easy to verify that \( R' \) and \( S' \) satisfy Equations (11)-(16). This finishes the proof of our proposition. \( \square \)
Proposition 3.7. (see [1 Corollary 2.7]) Let \((A, \mu, R, S)\) be a Rota-Baxter system. Define a new binary operation over \(A\) as

\[
a \star b := R(a) \cdot b + a \cdot S(b)
\]

for any \(a, b \in A\). Then the operation \(\star\) is associative and \((A, \star)\) is a new associative algebra and we denote it by \(A_\star\).

Remark 3.8. Let \((A, \mu, R, S)\) be a Rota-Baxter system. If \(R\) and \(S\) satisfy \(R \circ S = S \circ R\), then one can verify that \((A, \star, R, S)\) is also a Rota-Baxter system.

One can construct new bimodules over the associative algebra \(A_\star\) from Rota-Baxter system bimodules over \((A, \mu, R, S)\).

Proposition 3.9. Let \((A, \mu, R, S)\) be a Rota-Baxter system and \((M, R_M, S_M)\) be a Rota-Baxter system bimodule over it. We define a left action “\(\triangleright\)”, and a right action “\(\triangleleft\)” of \(A\) on the space \(M \oplus M\) as follows: for any \(a \in A, m_1, m_2 \in M\),

\[
a \triangleright (m_1, m_2) := (R(a)m_1 - R_M(am_2), S(a)m_2 - S_M(am_2)),
\]

\[
(m_1, m_2) \triangleleft a := (m_1R(a) - R_M(m_1a), m_2S(a) - S_M(m_1a)).
\]

Then these actions make \(M \oplus M\) into a bimodule over \(A_\star\), and we denote this new bimodule by \(\triangleright \mathcal{D}(M) \triangleleft\).

Proof. Firstly, we show that \((M \oplus M, \triangleright)\) is a left module over \(A_\star\), that is

\[
a \triangleright (b \triangleright (m_1, m_2)) = (a \star b) \triangleright (m_1, m_2).
\]

On the one hand,

\[
a \triangleright (b \triangleright (m_1, m_2))
\]

\[
= a \triangleright (R(b)m_1 - R_M(bm_2), S(b)m_2 - S_M(bm_2))
\]

\[
= R(a)(R(b)m_1 - R_M(bm_2)) - R_M(aS(b)m_2 - aS_M(bm_2)),
\]

\[
S(a)(S(b)m_2 - S_M(bm_2)) - S_M(aS(b)m_2 - aS_M(bm_2))
\]

\[
= (R(a)R(b)m_1 - R_M(R(a)bm_2 + aS(b)m_2),
\]

\[
S(a)S(b)m_2 - S_M(R(a)bm_2 + aS(b)m_2))
\]

On the other hand,

\[
(a \star b) \triangleright (m_1, m_2)
\]

\[
= (R(a \star b)m_1 - R_M((a \star b)m_2), S(a \star b)m_2 - S_M((a \star b)m_2))
\]

\[
= (R(a)R(b)m_1 - R_M(R(a)bm_2 + aS(b)m_2),
\]

\[
S(a)S(b)m_2 - S_M(R(a)bm_2 + aS(b)m_2))
\]

Next one can similarly check that the operation \(\triangleleft\) defines a right module structure on \(M \oplus M\) over \(A_\star\).
Finally, we have the equations:

\[
\begin{align*}
  (a \triangleright (m_1, m_2)) \triangleleft b &= \left( R(a)m_1 - R_M(am_2), S(a)m_2 - S_M(am_2) \right) \triangleleft b \\
  &= \left( (R(a)m_1 - R_M(am_2))R(b) - R_M(R(a)m_1b - R_M(am_2)b), \\
  &\quad (S(a)m_2 - S_M(am_2))S(b) - S_M(R(a)m_1b - R_M(am_2)b) \right) \\
  &= \left( R(a)m_1R(b) - R_M(R(a)m_1b + am_2S(b)), \\
  &\quad S(a)m_2S(b) - S_M(R(a)m_1b + am_2S(b)) \right),
\end{align*}
\]

which give the compatibility of operations \(\triangleright\) and \(\triangleleft\):

\[
(a \triangleright (m_1, m_2)) \triangleleft b = a \triangleright ((m_1, m_2) \triangleleft b).
\]

\[\square\]

**Remark 3.10.** (1) Unlike the Rota-Baxter algebra situation as in [4, Proposition 4.7], in order to define the new bimodule \(\mathcal{D}(M)\) over the associative algebra \(A_\star\), we have changed the space from \(M\) to \(M \oplus M\).

(2) If \(R\) and \(S\) satisfy \(R \circ S = S \circ R\), then \((A, \star, R, S)\) is also a Rota-Baxter system (cf. Remark 3.8). In this case, we can define a Rota-Baxter system bimodule \((\mathcal{D}(M), R_{\mathcal{D}(M)}, S_{\mathcal{D}(M)})\) over \((A, \star, R, S)\) by letting

\[
\begin{align*}
  R_{\mathcal{D}(M)} : M \oplus M &\to M \oplus M, (m_1, m_2) \mapsto (R_M(m_1), R_M(m_2)), \\
  S_{\mathcal{D}(M)} : M \oplus M &\to M \oplus M, (m_1, m_2) \mapsto (S_M(m_1), S_M(m_2)).
\end{align*}
\]

4. Cohomology theory of Rota-Baxter systems

In this section, we will define a cohomology theory for Rota-Baxter systems following (and modifying) the ideas from [4, Section 5].
4.1. Cohomology of Rota-Baxter system operators. Firstly, let’s study the cohomology of Rota-Baxter system operators. Let \((A, \mu, R, S)\) be a Rota-Baxter system and \((M, R_M, S_M)\) be a Rota-Baxter system bimodule over it. Recall that Proposition 3.7 and Proposition 3.9 give a new \(A, \mu, R, S\) Rota-Baxter system operators. Let \((\ast)\) be defined by the following formula:

\[
C_{\text{Alg}}(A, \ast) := \bigoplus_{n=0}^{\infty} C_{\text{Alg}}(A, \ast). \quad \quad \quad (4.1)
\]

More precisely, for \(n \geq 0\),

\[
C_{\text{Alg}}(A, \ast) = \text{Hom}(A \otimes_n M \oplus M) \cong \text{Hom}(A \otimes_n M) \oplus \text{Hom}(A \otimes_n M)
\]

and its differential

\[
\partial^n : C_{\text{Alg}}(A, \ast) \to C_{\text{Alg}}(A, \ast)
\]

is defined by the following formula:

\[
\partial^n(f, g)(a) = (-1)^{n+1} a_1 \ast (x, y)(a_{2, n+1}) + \sum_{i=1}^{n} (-1)^{n-i+1} (x, y)(a_{1, i-1} \ast a_{i+1} \ast a_{i+2, n+1}) + (x, y)(a_{1, n}) < a_{n+1} = \left((-1)^{n+1} R(a_1)x(a_{2, n+1}) - (-1)^{n+1} R_M(a_1y(a_{2, n+1})) + \sum_{i=1}^{n} (-1)^{n-i+1} R(a_1)x(a_{1, i-1} \ast a_{i+1} \ast a_{i+2, n+1}) + x(a_{1, n})R(a_{n+1}) - R_M(x(a_{1, n})a_{n+1}) + a_i S(a_{i+1} \ast a_{i+2, n+1}) + (-1)^{n+1} S(a_1y(a_{2, n+1})) + \sum_{i=1}^{n} (-1)^{n-i+1} R(a_1)x(a_{1, i-1} \ast a_{i+1} \ast a_{i+2, n+1}) + y(a_{1, n})S(a_{n+1}) - S_M(x(a_{1, n})a_{n+1}) \right)
\]

for \(n \geq 1\), \(x, y \in C_{\text{Alg}}(A, M)\) and \(a_1, \ldots, a_{n+1} \in A\).

**Definition 4.1.** Let \((A, \mu, R, S)\) be a Rota-Baxter system and \((M, R_M, S_M)\) be a Rota-Baxter system bimodule over it. Then the Hochschild cochain complex \(\left(C_{\text{Alg}}(A, \ast), \partial^n\right)\) is called the cochain complex of Rota-Baxter system operators \((R, S)\) with coefficients in \((M, R_M, S_M)\).
denoted by $C_{\text{RBSO}}^\bullet(A, M)$. The cohomology of $C_{\text{RBSO}}^\bullet(A, M)$, denoted by $H_{\text{RBSO}}^\bullet(A, M)$, is called the cohomology of Rota-Baxter system operators $(R, S)$ with coefficients in $(M, R_M, S_M)$.

When $(M, R_M, S_M)$ is the regular Rota-Baxter system bimodule $(A, R, S)$, we denote $C_{\text{RBSO}}^\bullet(A, A)$ by $C_{\text{RBSO}}^\bullet(A)$ and call it the cochain complex of Rota-Baxter system operators $(R, S)$, and denoted $H_{\text{RBSO}}^\bullet(A, A)$ by $H_{\text{RBSO}}^\bullet(A)$ and call it the cohomology of Rota-Baxter system operators $(R, S)$.

### 4.2. Cohomology of Rota-Baxter systems

In this subsection, we will combine the Hochschild cochain complex of associative algebra $(A, \mu)$ and the cochain complex of Rota-Baxter system operators $(R, S)$ to define a cohomology theory for Rota-Baxter system $(A, \mu, R, S)$.

Let $M = (M, R_M, S_M)$ be a Rota-Baxter system bimodule over a Rota-Baxter system $A = (A, \mu, R, S)$. Let $C_{\text{Alg}}^\bullet(A, M)$ be the Hochschild cochain complex of $(A, \mu)$ with coefficients in $M$ and $C_{\text{RBSO}}^\bullet(A, M)$ be the cochain complex of Rota-Baxter system operators $(R, S)$ with coefficients in $(M, R_M, S_M)$. We now define a chain map $\Phi^\bullet : C_{\text{Alg}}^\bullet(A, M) \to C_{\text{RBSO}}^\bullet(A, M)$ as follows.

Define $\Phi^0 = (\Phi_R^0, \Phi_S^0) : C_{\text{Alg}}^0(A, M) = \text{Hom}(\mathbb{K}, M) \to C_{\text{RBSO}}^0(A, M) \cong \text{Hom}(\mathbb{K}, M) \oplus \text{Hom}(\mathbb{K}, M)$ by

$$\Phi_R^0(f)(a_1 \cdot \cdot \cdot a_n) = f(R(a_1) \otimes \cdot \cdot \cdot \otimes R(a_n))$$

and for $n > 1$, define

$$\Phi_R^n(f)(a_1 \cdot \cdot \cdot a_n) = f(R(a_1) \otimes \cdot \cdot \cdot \otimes R(a_{i-1}) \otimes a_i \otimes S(a_{i+1}) \otimes \cdot \cdot \cdot \otimes S(a_n)),$$

$$\Phi_S^n(f)(a_1 \cdot \cdot \cdot a_n) = f(S(a_1) \otimes \cdot \cdot \cdot \otimes S(a_n))$$

$$- S_M \circ \sum_{i=1}^{n} f(R(a_1) \otimes \cdot \cdot \cdot \otimes R(a_{i-1}) \otimes a_i \otimes S(a_{i+1}) \otimes \cdot \cdot \cdot \otimes S(a_n))$$

for $f \in C_{\text{Alg}}^n(A, M)$.

**Proposition 4.2.** The map $\Phi^\bullet : C_{\text{Alg}}^\bullet(A, M) \to C_{\text{RBSO}}^\bullet(A, M)$ is a chain map, that is, the following diagram commutes:

$$
\begin{array}{ccccccccc}
C_{\text{Alg}}^0(A, M) & \xrightarrow{\partial^0} & C_{\text{Alg}}^1(A, M) & \cdots & & & & & & C_{\text{Alg}}^n(A, M) & \xrightarrow{\partial^n} & C_{\text{Alg}}^{n+1}(A, M) \\
\downarrow{\Phi^0} & & \downarrow{\Phi^1} & & & & & & \downarrow{\Phi^n} & & \downarrow{\Phi^{n+1}} \\
C_{\text{RBSO}}^0(A, M) & \xrightarrow{\partial^0} & C_{\text{RBSO}}^1(A, M) & \cdots & & & & & & C_{\text{RBSO}}^n(A, M) & \xrightarrow{\partial^n} & C_{\text{RBSO}}^{n+1}(A, M) \\
\end{array}
$$

**Proof.** We just need to prove $\partial^n \circ \Phi^n(f) = \Phi^{n+1} \circ \delta^n(f)$ for any $n \geq 0$ and for any $f \in C_{\text{Alg}}^n(A, M)$. 

When $n = 0, f \in \text{Hom}(\mathbb{K}, M), a \in A$, we have
\[
\delta^0 \circ \Phi^0(f)(a) \\
= \delta^0(f, f)(a) \\
= \left( -R(a)f(1_\mathbb{K}) + f(1_\mathbb{K})R(a) - R_M(f(1_\mathbb{K})a) + R_M(af(1_\mathbb{K})) \right), \\
= -S(a)f(1_\mathbb{K}) + f(1_\mathbb{K})S(a) - S_M(f(1_\mathbb{K})a) + S_M(af(1_\mathbb{K})) \right);
\]
on the other hand, we have
\[
\Phi^1 \circ \delta^0(f)(a) \\
= \left( \delta^0(f)(R(a)) - R_M(\delta^0(f)(a)), \delta^0(f)(S(a)) - S_M(\delta^0(f)(a)) \right) \\
= \left( -R(a)f(1_\mathbb{K}) + f(1_\mathbb{K})R(a) + R_M(af(1_\mathbb{K}) - f(1_\mathbb{K})a), \\
= -S(a)f(1_\mathbb{K}) + f(1_\mathbb{K})S(a) + S_M(af(1_\mathbb{K}) - f(1_\mathbb{K})a) \right).
\]
This proves $\delta^0 \circ \Phi^0 = \Phi^1 \circ \delta^0$.

When $n \geq 1$, for $f \in C^n_{\text{Alg}}(A, M), a_1, \ldots, a_{n+1} \in A$, we have (here we write $R \cdot f(a_1, n+1) := R(a_1)f(a_2, n+1)$)
\[
\partial^n \circ \Phi^n(f) \\
= \partial^n \circ (\Phi^n_R, \Phi^n_M)(f) \\
= \partial^n \left( f(R^n) - R_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}), f(S^n) - S_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}) \right) \\
= \left( (1)^{n+1}R \cdot \left( f(R^n) - R_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}) \right) \\
- (1)^{n+1}R_M \left\{ \text{Id} \cdot \left( f(S^n) - S_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}) \right) \right\} \\
+ \sum_{i=1}^n (-1)^{n-i+1} f \left( R^{\otimes i-1} \otimes R(\text{Id} + \text{Id} \cdot S) \otimes R^{\otimes n-i} \right) \\
- R_M \sum_{i=1}^n \left\{ \sum_{1 \leq j \leq i-1} (1)^{n-j+1} f \left( R^{\otimes j-1} \otimes R(\text{Id} + \text{Id} \cdot S) \otimes R^{\otimes i-j-2} \otimes \text{Id} \otimes S^{\otimes n-i+1} \right) \\
+ (-1)^{n-i+1} f(R^{\otimes i-1} \otimes (\text{Id} + \text{Id} \cdot S) \otimes S^{\otimes n-i}) \\
+ \sum_{i+1 \leq j \leq n} (1)^{n-j+1} f \left( R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes j-i-1} \otimes S(\text{Id} + \text{Id} \cdot S) \otimes S^{\otimes n-j} \right) \right\} \\
+ f(R^n) - R_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}) \right) \cdot R \\
- R_M \left\{ f(R^n) - R_M \sum_{i=1}^n f(R^{\otimes i-1} \otimes \text{Id} \otimes S^{\otimes n-i}) \right\} \cdot \text{Id} \right),
\]
\[
(-1)^{n+1} \cdot \left( f(S^{\otimes n}) - S_M \sum_{i=1}^{n} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i}) \right) \\
-(-1)^{n+1} S_M \left\{ Id \cdot \left( f(S^{\otimes n}) - S_M \sum_{i=1}^{n} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i}) \right) \right\} \\
+ \sum_{i=1}^{n} (-1)^{n-i+1} f \left( S^{\otimes i-1} \otimes S(R \cdot Id + Id \cdot S) \otimes S^{\otimes n-i} \right) \\
-S_M \sum_{i=1}^{n} \left\{ \sum_{1 \leq j \leq i-1} (-1)^{n-j+1} f \left( R^{\otimes j-1} \otimes R(R \cdot Id + Id \cdot S) \otimes R^{\otimes i-j-2} \otimes Id \otimes S^{\otimes n-i+1} \right) \\
+(-1)^{n-i+1} f(R^{\otimes i-1} \otimes (R \cdot Id + Id \cdot S) \otimes S^{\otimes n-i}) \\
+ \sum_{i+1 \leq j \leq n} (-1)^{n-j+1} f \left( R^{\otimes i-1} \otimes R \otimes S^{\otimes j-i-1} \otimes S(R \cdot Id + Id \cdot S) \otimes S^{\otimes n-j} \right) \right\} \\
+ \left( f(S^{\otimes n}) - S_M \sum_{i=1}^{n} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i}) \right) \cdot S \\
-S_M \left\{ \left( f(R^{\otimes n}) - R_M \sum_{i=1}^{n} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i}) \right) \cdot Id \right\},
\]

and

\[
\Phi^{n+1} \circ \delta^n(f) \\
= \left( \Phi_R^{n+1}, \Phi_S^{n+1} \right) \circ \delta^n(f) \\
= \left( \delta^n(f)(R^{\otimes n+1}) - R_M \sum_{i=1}^{n+1} \delta^n(f)(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i+1}), \right) \\
\delta^n(f)(S^{\otimes n+1}) - S_M \sum_{i=1}^{n+1} \delta^n(f)(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i+1}) \right) \\
= \left( (-1)^{n+1} \cdot f(R^{\otimes n}) + \sum_{i=1}^{n} (-1)^{n-i+1} f(R^{\otimes i-1} \otimes R \cdot R \otimes R^{\otimes n-i}) + f(R^{\otimes n}) \cdot R \\
- R_M \left\{ (-1)^{n+1} Id \cdot f(S^{\otimes n}) + (-1)^{n+1} \sum_{i=2}^{n+1} R \cdot f(R^{\otimes i-2} \otimes Id \otimes S^{\otimes n-i+1}) \right\} \\
+ \sum_{i=1}^{n+1} \left\{ \sum_{1 \leq j \leq i-1} (-1)^{n-j+1} f(R^{\otimes j-1} \otimes R \cdot R \otimes R^{\otimes i-j-2} \otimes Id \otimes S^{\otimes n-i+1}) \\
+(-1)^{n-i+2} f(R^{\otimes i-2} \otimes R \cdot Id \otimes S^{\otimes n-i}) + (-1)^{n-i+1} f(R^{\otimes i-1} \otimes Id \cdot S \otimes S^{\otimes n-i}) \\
+ \sum_{i+1 \leq j \leq n} (-1)^{n-j+1} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes j-i-1} \otimes S \cdot S \otimes S^{\otimes n-j}) \right\} \\
+ \sum_{i=1}^{n} f(R^{\otimes i-1} \otimes Id \otimes S^{\otimes n-i}) \cdot S + f(R^{\otimes n}) \cdot Id \right\},
\]
\((-1)^n f(S^\bigotimes n) + \sum_{i=1}^{n} (-1)^{n-i+1} f(S^\bigotimes i - 1 \otimes S \otimes S^\bigotimes n-i) + f(S^\bigotimes n) \cdot S\)

\(-S_M \left\{ (-1)^{n+1} \text{Id} \cdot f(S^\bigotimes n) + (-1)^{n+1} \sum_{i=2}^{n+1} R \cdot f(R^\bigotimes i-2 \otimes \text{Id} \otimes S^\bigotimes n-i+1) \right\}

\begin{align*}
&+ \sum_{i=1}^{n+1} \sum_{1 \leq j \leq i-2} (-1)^{n-j+1} f(R^\bigotimes j-1 \otimes R \otimes R^\bigotimes i-j-2 \otimes \text{Id} \otimes S^\bigotimes n-i+1) \\
&+ (-1)^{n-i+2} f(R^\bigotimes i-2 \otimes \text{Id} \otimes S^\bigotimes n-i+1) + (-1)^{n-i+1} f(R^\bigotimes i-1 \otimes \text{Id} \cdot S \otimes S^\bigotimes n-i) \\
&+ \sum_{i+1 \leq j \leq n} (-1)^{n-j+1} f(R^\bigotimes i-1 \otimes \text{Id} \otimes S^\bigotimes j-i-1 \otimes S \otimes S^\bigotimes n-j) \\
&+ \sum_{i=1}^{n} f(R^\bigotimes i-1 \otimes \text{Id} \otimes S^\bigotimes n-i) \cdot S + f(R^\bigotimes n) \cdot \text{Id} \right) \\
\end{align*}

Comparing the above two equations we obtain \(\partial^n \circ \Phi^n(f) = \Phi^{n+1} \circ \delta^n(f)\). Hence, \(\Phi^*\) is a chain map.

**Definition 4.3.** Let \(M = (M, R_M, S_M)\) be a Rota-Baxter system bimodule over a Rota-Baxter system \(A = (A, R, S)\). We define the cochain complex \((C^\bullet_{RBS}(A, M), d^\bullet)\) of Rota-Baxter system \((A, R, S)\) with coefficients in \(M\) to be the negative shift of the mapping cone of \(\Phi^*\), that is, we have

\[
\begin{align*}
C^0_{RBS}(A, M) &= C^0_{Alg}(A, M), \\
C^n_{RBS}(A, M) &= C^n_{Alg}(A, M) \oplus C^{n-1}_{RBSO}(A, M), \forall n \geq 1,
\end{align*}
\]

and the differential \(d^n : C^n_{RBS}(A, M) \to C^{n+1}_{RBS}(A, M)\) is given by

\[
d^n(f, (x, y)) = (\delta^n(f), -\partial^{n-1}(x, y) - \Phi^n(f))
\]

for any \(f \in C^n_{Alg}(A, M), x, y \in C^{n-1}_{Alg}(A, M)\). The cohomology of \((C^\bullet_{RBS}(A, M), d^\bullet)\), denoted by \(H^\bullet_{RBS}(A, M)\), is called the cohomology of Rota-Baxter system \((A, R, S)\) with coefficients in \(M\).

When \((M, R_M, S_M) = (A, R, S)\), we just denote \(C^\bullet_{RBS}(A, A), H^\bullet_{RBS}(A, A)\) by \(C^\bullet_{RBS}(A), H^\bullet_{RBS}(A)\) respectively, and call them the cochain complex, the cohomology of Rota-Baxter system \((A, R, S)\) respectively.

There is an obvious short exact sequence of complexes:

\[
0 \longrightarrow sC^\bullet_{RBSO}(A, M) \longrightarrow C^\bullet_{RBS}(A, M) \longrightarrow C^\bullet_{Alg}(A, M) \longrightarrow 0
\]

which induces a long exact sequence of cohomology groups

\[
\begin{align*}
0 \longrightarrow H^0_{RBS}(A, M) \longrightarrow HH^0(A, M) \longrightarrow H^0_{RBSO}(A, M) \longrightarrow H^1_{RBS}(A, M) \longrightarrow \cdots \\
\cdots \longrightarrow HH^p(A, M) \longrightarrow H^p_{RBS}(A, M) \longrightarrow H^{p+1}_{RBS}(A, M) \longrightarrow \cdots
\end{align*}
\]

**Remark 4.4.** Let \((A, R, \lambda)\) be a Rote-Baxter algebra of weight \(\lambda\). Then \((A, R, R + \lambda \text{id})\) is a Rota-Baxter system (cf. Remark 3.2). We observe that there is a monomorphism from the cochain
complex $(C^\bullet_{RBA}(A), d^\bullet)$ of Rota-Baxter algebra $(A, R, \lambda)$ defined in [4] to the cochain complex $(C^\bullet_{RBS}(A), d^\bullet)$ of Rota-Baxter system $(A, R, R + \text{id})$:

$$\psi^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} : C^n_{RBA}(A) = C^n_{\text{Alg}}(A) \oplus C^{n-1}_{\text{Alg}}(A) \to C^n_{RBS}(A) \cong C^n_{\text{Alg}}(A) \oplus C^{n-1}_{\text{Alg}}(A) \oplus C^{n-1}_{\text{Alg}}(A)$$

for all $n \geq 0$ and with $\text{Coker}\psi^n = C^{n-1}_{\text{Alg}}(A) = \text{Hom}(A^{-n-1}, A)$. Furthermore, there is a short exact sequence of complexes:

$$0 \longrightarrow C^\bullet_{RBA}(A) \xrightarrow{\psi} C^\bullet_{RBS}(A) \longrightarrow \text{Coker}\psi \longrightarrow 0,$$

where the differential $\bar{d}^n : \text{Coker}\psi^n = \text{Hom}(A^{-n-1}, A) \to \text{Coker}\psi^{n+1} = \text{Hom}(A^{-n}, A)$ of the complex $\text{Coker}\psi$ is given by

$$\bar{d}^n(h)(a_{1,n}) = (-1)^{n-1}R(a_1)h(a_{2,n}) + \sum_{i=1}^{n-1}(-1)^{n-i-1}h(a_{i,n}) + (R(a_i)a_{i+1} + a(R + \lambda)(a_{i+1})) \otimes a_{i+2,n}$$

$$- h(a_{i,n})(R + \lambda)(a_n).$$

5. Formal deformations of Rota-Baxter systems and cohomological interpretation

In this section, we will study formal deformations of Rota-Baxter systems and interpret them via lower degree cohomology groups of Rota-Baxter systems defined in last section.

5.1. Formal deformations of Rota-Baxter systems. Let $(A, \mu, R, S)$ be a Rota-Baxter system. Consider a 1-parameter family:

$$\mu_t = \sum_{i=0}^{\infty} \mu_i t^i, \quad \mu_i \in C^2_{\text{Alg}}(A),$$

$$R_t = \sum_{i=0}^{\infty} R_i t^i, \quad S_t = \sum_{i=0}^{\infty} S_i t^i, \quad R_i, S_i \in C^1_{\text{Alg}}(A).$$

**Definition 5.1.** A 1-parameter formal deformation of Rota-Baxter system $(A, \mu, R, S)$ is a triple $(\mu_t, R_t, S_t)$ which endows the $k[[t]]$-module $A[[t]]$ with a Rota-Baxter system structure over $k[[t]]$ such that $((\mu_0, R_0, S_0) = (\mu, R, S)$.

Power series $\mu_t$, $R_t$ and $S_t$ determine a 1-parameter formal deformation of Rota-Baxter system $(A, \mu, R, S)$ if and only if for any $a, b, c \in A$, the following equations hold:

$$\mu_t(a \otimes \mu_t(b \otimes c)) = \mu_t(\mu_t(a \otimes b) \otimes c),$$

$$\mu_t(R_t(a) \otimes R_t(b)) = R_t(\mu_t(R_t(a) \otimes b) + \mu_t(a \otimes S_t(b))),$$

$$\mu_t(S_t(a) \otimes S_t(b)) = S_t(\mu_t(R_t(a) \otimes b) + \mu_t(a \otimes S_t(b))).$$
By expanding these equations and comparing the coefficients of $t^n$, we obtain that \( \{\mu_i\}_{i \geq 0} \), \( \{R_i\}_{i \geq 0} \) and \( \{S_i\}_{i \geq 0} \) have to satisfy: for any \( n \geq 0 \),

\[
\sum_{i=0}^{n} \mu_i \circ (\mu_{n-i} \otimes \text{Id}) = \sum_{i=0}^{n} \mu_i \circ (\text{Id} \otimes \mu_{n-i}),
\]

(22)

\[
\sum_{i+j+k=n, i,j,k \geq 0} \mu_i \circ (R_j \otimes R_k) = \sum_{i+j+k=n, i,j,k \geq 0} R_i \circ \mu_j \circ (R_k \otimes \text{Id}) + \sum_{i+j+k=n, i,j,k \geq 0} R_i \circ \mu_j \circ (\text{Id} \otimes S_k),
\]

(23)

\[
\sum_{i+j+k=n, i,j,k \geq 0} \mu_i \circ (S_j \otimes S_k) = \sum_{i+j+k=n, i,j,k \geq 0} S_i \circ \mu_j \circ (R_k \otimes \text{Id}) + \sum_{i+j+k=n, i,j,k \geq 0} S_i \circ \mu_j \circ (\text{Id} \otimes S_k).
\]

(24)

Obviously, when \( n = 0 \), the above conditions are exactly the associativity of \( \mu = \mu_0 \) and Equations (2) (3) which are the defining relations of Rota-Baxter system operators \( R_0 = R, S_0 = S \) respectively.

Recall that we have constructed a cochain complex \( C^\bullet_{\text{RBS}}(A) \) of Rota-Baxter system \( (A, R, S) \) in last section.

**Proposition 5.2.** (Compare to [4, Proposition 6.2]) Let \( (A[[t]], \mu_t, R_t, S_t) \) be a 1-parameter formal deformation of Rota-Baxter system \( (A, \mu, R, S) \). Then \( (\mu_1, R_1, S_1) \) is a 2-cocycle in the cochain complex \( C^\bullet_{\text{RBS}}(A) \) of Rota-Baxter system \( (A, R, S) \).

**Proof.** When \( n = 1 \), Equation (22) becomes

\[
\mu_1 \circ (\mu \otimes \text{Id}) + \mu \circ (\mu_1 \otimes \text{Id}) = \mu_1 \circ (\text{Id} \otimes \mu) + \mu \circ (\text{Id} \otimes \mu_1),
\]

and Equations (23) (24) become

\[
\mu_1 \circ (R \otimes R) - \{R \circ \mu_1 \circ (R \otimes \text{Id}) + R \circ \mu_1 \circ (\text{Id} \otimes S)\}
= - \{\mu \circ (R \otimes R_1) - R \circ \mu \circ (\text{Id} \otimes S_1)\} + \{R_1 \circ \mu \circ (R \otimes \text{Id}) + R_1 \circ \mu \circ (\text{Id} \otimes S)\}
- \{\mu \circ (R_1 \otimes R) - R \circ \mu \circ (R_1 \otimes \text{Id})\},
\]

\[
\mu_1 \circ (S \otimes S) - \{S \circ \mu_1 \circ (R \otimes \text{Id}) + S \circ \mu_1 \circ (\text{Id} \otimes S)\}
= - \{\mu \circ (S \otimes S_1) - S \circ \mu \circ (\text{Id} \otimes S_1)\} + \{S_1 \circ \mu \circ (R \otimes \text{Id}) + S_1 \circ \mu \circ (\text{Id} \otimes S)\}
- \{\mu \circ (S_1 \otimes S) - S \circ \mu \circ (R_1 \otimes \text{Id})\}.
\]

Note that the first equation is exactly \( \delta^2(\mu_1) = 0 \in C^\bullet_{\text{Alg}}(A) \) and the second and the third equations show

\[
\Phi^2(\mu_1) = -\partial^1(R_1, S_1) \in C^\bullet_{\text{RBSO}}(A).
\]

So \( (\mu_1, R_1, S_1) \) is a 2-cocycle in \( C^\bullet_{\text{RBS}}(A) \) by the formula (20) of the differential \( d \).

**Definition 5.3.** The 2-cocycle \( (\mu_1, R_1, S_1) \) is called the infinitesimal of the 1-parameter formal deformation \( (A[[t]], \mu_t, R_t, S_t) \) of Rota-Baxter system \( (A, \mu, R, S) \).

**Definition 5.4.** Let \( (A[[t]], \mu_t, R_t, S_t) \) and \( (A[[t]], \mu'_t, R'_t, S'_t) \) be two 1-parameter formal deformations of Rota-Baxter system \( (A, \mu, R, S) \). A formal isomorphism from the formal deformation
\((A[[t]], \mu'^1, R'_t, S'_t)\) to \((A[[t]], \mu_t, R_t, S_t)\) is a power series \(\Psi_t = \sum_{i=0}^\infty \Psi_i t^i : A[[t]] \to A[[t]]\), where \(\Psi_i : A \to A\) are linear maps with \(\Psi_0 = \text{Id}_A\), such that:

\[
\begin{align*}
\Psi_t \circ \mu'_1 &= \mu_t \circ (\Psi_t \otimes \Psi_t), \\
\Psi_t \circ R'_t &= R_t \circ \Psi_t, \\
\Psi_t \circ S'_t &= S_t \circ \Psi_t.
\end{align*}
\]

In this case, we say that the two formal deformations \((A[[t]], \mu_t, R_t, S_t)\) and \((A[[t]], \mu'_1, R'_t, S'_t)\) are equivalent.

Theorem 5.5. The infinitesimals of two equivalent 1-parameter formal deformations of \((A, \mu, R, S)\) are in the same cohomology class in \(H^2_{\text{RBS}}(A)\).

Proof. Let \(\Psi_t : (A[[t]], \mu_t, R'_t, S'_t) \to (A[[t]], \mu_t, R_t, S_t)\) be a formal isomorphism. Expanding the identities and collecting coefficients of \(t\), we get from Equations (25)-(27)

\[
\begin{align*}
\mu'_1 &= \mu_1 + \mu \circ (\text{Id} \otimes \Psi_1) - \Psi_1 \circ \mu + \mu \circ (\Psi_1 \otimes \text{Id}), \\
R'_1 &= R_1 + R \circ \Psi_1 - \Psi_1 \circ R, \\
S'_1 &= S_1 + S \circ \Psi_1 - \Psi_1 \circ S.
\end{align*}
\]

That is, we have

\[
(\mu'_1, R'_1, S'_1) - (\mu_1, R_1, S_1) = (\delta^1 (\Psi_1), -\Phi^1 (\Psi_1)) = d^1 (\Psi_1, 0, 0) \in C^2_{\text{RBS}}(A).
\]

Definition 5.6. A Rota-Baxter system \((A, \mu, R, S)\) is said to be rigid if every 1-parameter formal deformation is trivial.

Theorem 5.7. Let \((A, \mu, R, S)\) be a Rota-Baxter system. If \(H^2_{\text{RBS}}(A) = 0\), then \((A, \mu, R, S)\) is rigid.

Proof. Let \((A[[t]], \mu_t, R_t, S_t)\) be a 1-parameter formal deformation of \((A, \mu, R, S)\). By Proposition 5.2, \((\mu_1, R_1, S_1)\) is a 2-cocycle. By \(H^2_{\text{RBS}}(A) = 0\), there exists a 1-cochain

\[
(\Psi'_1, f, f) \in C^1_{\text{RBS}}(A) = C^1_{\text{Alg}}(A) \oplus (\text{Hom}(K, A) \oplus \text{Hom}(K, A))
\]

such that \((\mu_1, R_1, S_1) = d^1 (\Psi'_1, f, f)\), that is, \(\mu_1 = \delta^1 (\Psi'_1)\) and \((R_1, S_1) = -\partial^0 (f, f) - \Phi^1 (\Psi'_1)\).

Let \(\Psi_1 = \Psi'_1 + \delta^0 (f)\). Since \(\Phi^1 (\delta^0 (f)) = \partial^0 (\Phi^0 (f)) = \partial^0 (f, f)\), then

\[
\mu_1 = \delta^1 (\Psi_1) = \mu \circ (\text{Id} \otimes \Psi_1) - \Psi_1 \circ \mu + \mu \circ (\Psi_1 \otimes \text{Id}),
\]

and

\[
(R_1, S_1) = -\Phi^1 (\Psi_1) = (\Psi_1 \circ R + R \circ \Psi_1, -\Psi_1 \circ S + S \circ \Psi_1).
\]
Setting $\Psi_t^{(1)} = Id_A - \Psi_1 t : A[[t]] \to A[[t]]$, then $(\Psi_t^{(1)})^{-1} = Id_A + \Psi_1 t + \Psi_2 t^2 + \cdots$, and we have a deformation $(A[[t]], \mu_t^{(1)}, R_t^{(1)}, S_t^{(1)})$, where

$$
\mu_t^{(1)} = (\Psi_t^{(1)})^{-1} \circ \mu_t \circ (\Psi_t^{(1)} \otimes \Psi_t^{(1)}),
\quad R_t^{(1)} = (\Psi_t^{(1)})^{-1} \circ R_t \circ \Psi_t^{(1)},
\quad S_t^{(1)} = (\Psi_t^{(1)})^{-1} \circ S_t \circ \Psi_t^{(1)}.
$$

So we have

$$
\mu_1^{(1)} = \Psi_1 \circ \mu + \mu_1 + \mu(-\Psi_1 \otimes Id) + \mu(Id \otimes (-\Psi_1)) = 0,
\quad R_1^{(1)} = \Psi_1 \circ R + R_1 + R \circ (-\Psi_1) = 0,
\quad S_1^{(1)} = \Psi_1 \circ S + S_1 + S \circ (-\Psi_1) = 0.
$$

Then

$$
\mu_t^{(1)} = \mu + \mu_2^{(1)} t^2 + \cdots,
\quad R_t^{(1)} = R + R_2^{(1)} t^2 + \cdots,
\quad S_t^{(1)} = S + S_2^{(1)} t^2 + \cdots.
$$

Assume there exists $\Psi_t^{(1)} = Id_A - \Psi_1 t, \cdots, \Psi_t^{(n)} = Id_A - \Psi_n t^n$ such that for

$$
\mu_t^{(n)} = (\Psi_t^{(n)})^{-1} \circ \cdots \circ (\Psi_t^{(1)})^{-1} \circ \mu_t \circ \left((\Psi_t^{(1)} \circ \cdots \circ \Psi_t^{(n)}) \otimes (\Psi_t^{(1)} \circ \cdots \circ \Psi_t^{(n)})\right),
\quad R_t^{(n)} = (\Psi_t^{(n)})^{-1} \circ \cdots \circ (\Psi_t^{(1)})^{-1} \circ R_t \circ (\Psi_t^{(1)} \circ \cdots \circ \Psi_t^{(n)}),
\quad S_t^{(n)} = (\Psi_t^{(n)})^{-1} \circ \cdots \circ (\Psi_t^{(1)})^{-1} \circ S_t \circ (\Psi_t^{(1)} \circ \cdots \circ \Psi_t^{(n)}),
$$

we have

$$
\mu_1^{(n)} = \cdots = \mu_n^{(n)} = 0,
\quad R_1^{(n)} = \cdots = R_n^{(n)} = 0,
\quad S_1^{(n)} = \cdots = S_n^{(n)} = 0.
$$

By Equations [23] and [24], $(\mu_{n+1}^{(n+1)}, R_{n+1}^{(n+1)}, S_{n+1}^{(n+1)})$ is a 2-cocycle in $C^*_\text{RBS}(A)$. Then there exists $\Psi_{n+1} \in \text{Hom}(A, A)$ such that $d^1 \left(\Psi_{n+1}, (0, 0)\right) = (\mu_{n+1}^{(n+1)}, R_{n+1}^{(n+1)}, S_{n+1}^{(n+1)})$. Let $\Psi_{n+1}^{(n+1)} = Id_A - \Psi_{n+1} t^{n+1}$,

$$
\mu_t^{(n+1)} = (\Psi_t^{(n+1)})^{-1} \circ \mu_t \circ (\Psi_t^{(n+1)} \otimes \Psi_t^{(n+1)}),
\quad R_t^{(n+1)} = (\Psi_t^{(n+1)})^{-1} \circ R_t \circ \Psi_t^{(n+1)},
\quad S_t^{(n+1)} = (\Psi_t^{(n+1)})^{-1} \circ S_t \circ \Psi_t^{(n+1)}.
$$
then
\[ \mu_1^{(n+1)} = \cdots = \mu_{n+1}^{(n+1)} = 0, \]
\[ R_1^{(n+1)} = \cdots = R_{n+1}^{(n+1)} = 0, \]
\[ S_1^{(n+1)} = \cdots = S_{n+1}^{(n+1)} = 0. \]

Define \( \Psi_t = \Psi_t^{(1)} \circ \Psi_t^{(2)} \circ \cdots \circ \Psi_t^{(n)} \circ \cdots : A[[t]] \to A[[t]], \) and
\[ \mu_t' = \Psi_t^{-1} \circ \mu_t \circ (\Psi_t \otimes \Psi_t), \]
\[ R_t' = \Psi_t^{-1} \circ R_t \circ \Psi_t, \]
\[ S_t' = \Psi_t^{-1} \circ S_t \circ \Psi_t. \]

Since
\[ \Psi_t = \Psi_t^{(1)} \circ \Psi_t^{(2)} \circ \cdots \circ \Psi_t^{(n)} \mod(t^{n+1}), \]
\[ (\Psi_t)^{-1} = (\Psi_t^{(n)})^{-1} \circ \cdots \circ (\Psi_t^{(1)})^{-1} \mod(t^{n+1}), \]
\[ \mu_t' = \mu_t^{(n)} \mod(t^{n+1}), \]
\[ R_t' = R_t^{(n)} \mod(t^{n+1}), \]
\[ S_t' = S_t^{(n)} \mod(t^{n+1}), \]
we have
\[ \mu_t' = \cdots = \mu_{n+1}' = 0, \]
\[ R_t' = \cdots = R_{n+1}' = 0, \]
\[ S_t' = \cdots = S_{n+1}' = 0, \]
for all \( n \geq 1. \)

Hence \( \mu_t' = \mu, R_t' = R \) and \( S_t' = S. \) So \((A[[t]], \mu_t, R_t, S_t)\) is equivalent to \((A[[t]], \mu, R, S),\) that is, \((A, \mu, R, S)\) is rigid. \( \square \)

5.2. Formal deformations of Rota-Baxter system operators with multiplication fixed.
Let \((A, \mu, R, S)\) be a Rota-Baxter system. Let us consider the case where we only deform the Rota-Baxter system operators with the multiplication fixed. So \( A[[t]] = \{ \sum_{i=0}^{\infty} a_i t^i | a_i \in A, \forall i \geq 0 \} \) is endowed with the multiplication induced from that of \( A, \) say,
\[ \left( \sum_{i=0}^{\infty} a_i t^i \right) \left( \sum_{j=0}^{\infty} b_j t^j \right) = \sum_{n=0}^{\infty} \left( \sum_{i+j=n, i,j \geq 0} a_i b_j \right) t^n. \]

Then \( A[[t]] \) becomes a free \((???)\) \( K[[t]]\)-algebra, whose multiplication is still denoted by \( \mu. \) In this case, a 1-parameter formal deformation \((\mu_t, R_t, S_t)\) of Rota-Baxter system \((A, \mu, R, S)\) satisfies \( \mu_i = 0, \forall i \geq 1. \) So Equation \( \{22\} \) degenerates and Equations \( \{23\} \) become
\[ \mu \circ (R_t \otimes R_t) = R_t \circ \mu(R_t \otimes Id + Id \otimes S_t), \]
\[ \mu \circ (S_t \otimes S_t) = S_t \circ \mu(R_t \otimes Id + Id \otimes S_t). \]
Expanding these equations and comparing the coefficients of \( t^n \), we obtain that \( \{R_i\}_{i \geq 0}, \{S_i\}_{i \geq 0} \) have to satisfy: for any \( n \geq 0 \),
\[
\sum_{i+j=n}^{i,j \geq 0} \mu \circ (R_i \otimes R_j) = \sum_{i,j \geq 0}^{i+j=n} R_i \circ \mu \circ (R_j \otimes Id) + \sum_{i,j \geq 0}^{i+j=n} R_i \circ \mu \circ (Id \otimes S_j),
\]
(28)
\[
\sum_{i+j=n}^{i,j \geq 0} \mu \circ (S_i \otimes S_j) = \sum_{i,j \geq 0}^{i+j=n} S_i \circ \mu \circ (R_j \otimes Id) + \sum_{i,j \geq 0}^{i+j=n} S_i \circ \mu \circ (Id \otimes S_j).
\]
(29)

Obviously, when \( n = 0 \), Equations (28) and (29) become exactly Equations (2) defining Rota-Baxter system operators \( (R,S) \).

When \( n = 1 \), Equations (28) and (29) have the form
\[
\mu \circ (R \otimes R_1 + R_1 \otimes R) = R \circ \mu \circ (R_1 \otimes Id) + R_1 \circ \mu \circ (R \otimes Id + Id \otimes S) + R \circ \mu \circ (Id \otimes S_1),
\]
\[
\mu \circ (S \otimes S_1 + S_1 \otimes S) = S \circ \mu \circ (R_1 \otimes Id) + S_1 \circ \mu \circ (R \otimes Id + Id \otimes S) + S \circ \mu \circ (Id \otimes S_1),
\]
which say exactly that \( \partial^1(R_1, S_1) = (0, 0) \in C^1_{RBSO}(A) \), where \( C^\bullet_{RBSO}(A) \) is the cochain complex of Rota-Baxter system operators \( (R,S) \) defined in last section. This proves the following result:

**Proposition 5.8.** Let \( R_t, S_t \) be a 1-parameter formal deformation of Rota-Baxter system operators \( (R,S) \). Then \( (R_1, S_1) \) is a 1-cocycle in the cochain complex \( C^\bullet_{RBSO}(A) \).

This means that the cochain complex \( C^\bullet_{RBSO}(A) \) controls formal deformations of Rota-Baxter system operators.

### 6. Abelian extensions of Rota-Baxter systems

In this section, we study abelian extensions of Rota-Baxter systems and show that they are classified by the second cohomology, as one would expect of a good cohomology theory. Notice that a vector space \( M \) together with two linear transformations \( R_M, S_M : M \to M \) is naturally a Rota-Baxter system where the multiplication on \( M \) is defined to be \( uv = 0 \) for all \( u, v \in M \).

**Definition 6.1.** An abelian extension of Rota-Baxter systems is a short exact sequence of morphisms of Rota-Baxter systems
\[
0 \to (M, R_M, S_M) \xrightarrow{i} (\hat{A}, \hat{R}, \hat{S}) \xrightarrow{p} (A, R, S) \to 0,
\]
(30)
that is, there exist two commutative diagrams:
\[
\begin{array}{ccc}
0 \to M & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \to 0 \\
R_M \downarrow & & \hat{R} & \downarrow & R & \\
0 \to M & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \to 0
\end{array}
\]
\[
\begin{array}{ccc}
0 \to M & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \to 0, \\
S_M \downarrow & & \hat{S} & \downarrow & S & \\
0 \to M & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \to 0,
\end{array}
\]

where the Rota-Baxter system \( (M, R_M, S_M) \) satisfies \( uv = 0 \) for all \( u, v \in M \).

We will call \( (\hat{A}, \hat{R}, \hat{S}) \) an abelian extension of \( (A, R, S) \) by \( (M, R_M, S_M) \).
Definition 6.2. Let \((\hat{A}_1, \hat{R}_1, \hat{S}_1)\) and \((\hat{A}_2, \hat{R}_2, \hat{S}_2)\) be two abelian extensions of \((A, R, S)\) by \((M, R_M, S_M)\). They are said to be isomorphic if there exists an isomorphism of Rota-Baxter systems \(\zeta : (\hat{A}_1, \hat{R}_1, \hat{S}_1) \to (\hat{A}_2, \hat{R}_2, \hat{S}_2)\) such that the following commutative diagram holds:

\[
\begin{array}{ccc}
0 & \longrightarrow & (M, R_M, S_M) \\
\downarrow & & \downarrow \zeta \\
0 & \longrightarrow & (A, R, S) \\
0 & \longrightarrow & (A, R, S) \\
\end{array}
\]

\[(31)\]

A section of an abelian extension \((A, \hat{R}, \hat{S})\) of \((A, R, S)\) by \((M, R_M, S_M)\) is a linear map \(t : A \to \hat{A}\) such that \(p \circ t = \text{Id}_A\).

We will show that isomorphism classes of abelian extensions of \((A, R, S)\) by \((M, R_M, S_M)\) are in bijection with the second cohomology group \(H^2_{\text{RBS}}(A, M)\).

Let \((\hat{A}, \hat{R}, \hat{S})\) be an abelian extension of \((A, R, S)\) by \((M, R_M, S_M)\) having the form Equation (30). Choose a section \(t : A \to \hat{A}\). We define

\[am := t(a)m, \quad ma := mt(a), \quad \forall a \in A, m \in M.\]

Proposition 6.3. With the above notations, \((M, R_M, S_M)\) is a Rota-Baxter system bimodule over \((A, R, S)\).

Proof. For any \(a, b \in A, m \in M\), since \(t(ab) - t(a)t(b) \in M\) implies \(t(ab)m = t(a)t(b)m\), we have

\[(ab)m = t(ab)m = t(a)t(b)m = a(bm).\]

Hence, this gives a left \(A\)-module structure and the case of right module structure is similar.

Moreover, \(\hat{R}(t(a)) - t(R(a)) \in M\) means that \(\hat{R}(t(a))m = t(R(a))m\). Thus we have

\[R(a)R_M(m) = t(R(a))R_M(m) = \hat{R}(t(a))R_M(m) = R_M(\hat{R}(t(a))m + t(a)S_M(m)) = R_M(R(a)m + aS_M(m)).\]

It is similar to see

\[R_M(m)R(a) = R_M(R_M(m)a + mS(a)), \quad S(a)S_M(m) = S_M(R(a)m + aS_M(m)), \quad S_M(m)S(a) = S_M(R_M(m)a + mS(a)).\]

Hence, \((M, R_M, S_M)\) is a Rota-Baxter system bimodule over \((A, R, S)\). \(\square\)

We further define linear maps \(\Psi : A \otimes A \to M\) and \(\chi_R, \chi_S : A \to M\) respectively by

\[\Psi(a \otimes b) = t(a)t(b) - t(ab), \quad \forall a, b \in A,\]

\[\chi_R(a) = \hat{R}(t(a)) - t(R(a)), \quad \forall a \in A,\]

\[\chi_S(a) = \hat{S}(t(a)) - t(S(a)), \quad \forall a \in A.\]

Proposition 6.4. The triple \((\Psi, \chi_R, \chi_S)\) is a 2-cocycle of Rota-Baxter system \((A, R, S)\) with coefficients in the Rota-Baxter system bimodule \((M, R_M, S_M)\) introduced in Proposition 6.3.
Proof. Since \( d(\Psi, \chi_R, \chi_S) = (\delta^2(\Psi), -\Phi^2(\Psi) - \partial^1(\chi_R, \chi_S)) \), we just need to prove

\[
\delta^2(\Psi) = 0, \quad \Phi^2(\Psi) + \partial^1(\chi_R, \chi_S) = 0.
\]

On the other hand, since \( t \circ R - \hat{R} \circ t, t \circ S - \hat{S} \circ t \in M \), we have

\[
\begin{align*}
\quad & t(R(a)) - R_M(t(a) \hat{S}(t(b)) + \hat{R}(t(a))t(b)) + (\hat{R}(t(a)) - t(R(a)))t(R(b)) \\
& = (\hat{R}(t(a)) - t(R(a)))(t(R(b)) - \hat{R}(t(b))) = 0, \\
\end{align*}
\]

\[
\begin{align*}
& t(S(a)) - S_M(t(a) \hat{S}(t(b)) + \hat{R}(t(a))t(b)) + (\hat{S}(t(a)) - t(S(a)))t(S(b)) \\
& = (\hat{S}(t(a)) - t(S(a)))(t(S(b)) - \hat{S}(t(b))) = 0.
\end{align*}
\]

Furthermore, we have

\[
\begin{align*}
\delta^2(\Psi)(a \otimes b \otimes c) \\
& = -a \Psi(b \otimes c) + \Psi(ab \otimes c) - \Psi(a \otimes bc) + \Psi(a \otimes b)c \\
& = -t(a)(t(b)t(c) - t(bc)) + (t(ab)t(c) - t(abc)) \\
& - (t(a)t(bc) - t(abc)) + (t(a)t(b) - t(ab))t(c) \\
& = 0,
\end{align*}
\]

\[
\Phi^2(\Psi)(a \otimes b)
\]

\[
\begin{align*}
& = \left( \Psi(R(a) \otimes R(b)) - R_M(\Psi(R(a) \otimes b + a \otimes S(b))) \\
& - S_M(\Psi(R(a) \otimes b + a \otimes S(b))) \right) \\
& = (t(R(a))t(R(b)) - t(R(a)R(b)) - R_M(t(R(a))t(b) + t(a)t(S(b)) - t(R(a)b + aS(b))), \\
& t(S(a))t(S(b)) - t(S(a)S(b)) - S_M(t(R(a))t(b) + t(a)t(S(b)) - t(R(a)b + aS(b)));
\end{align*}
\]

\[
\partial^1(\chi_R, \chi_S)(a \otimes b)
\]

\[
\begin{align*}
& = \left( R(a)\chi_R(b) - R_M(a\chi_S(b)) - \chi_R(R(a)b + aS(b)) + \chi_R(a)R(b) - R_M(\chi_R(a)b), \\
& S(a)\chi_S(b) - S_M(a\chi_S(b)) - \chi_S(R(a)b + aS(b)) + \chi_S(a)S(b) - S_M(\chi_R(a)b) \right) \\
& = (R(a)(\hat{R}(t(b)) - t(R(b))) - R_M(a(\hat{S}(t(b)) - t(S(b)))) - \hat{R}(t(R(a)b + aS(b))) \\
& - t(R(a)R(b)) + (\hat{R}(t(a)) - t(R(a)))(R(b) - R_M((\hat{R}(t(a)) - t(R(a))))b) \\
& - S(a)(\hat{S}(t(b)) - t(S(b))) - S_M(a(\hat{S}(t(b)) - t(S(b)))) - \hat{S}(t(R(a)b + aS(b))) \\
& - t(S(a)S(b)) + (\hat{S}(t(a)) - t(S(a)))(S(b) - S_M((\hat{R}(t(a)) - t(R(a)))b)) \\
& = - t(R(a))t(R(b)) - t(R(a)R(b)) - R_M(t(R(a))t(b) + t(a)t(S(b)) - t(R(a)b + aS(b))), \\
& t(S(a))t(S(b)) - t(S(a)S(b)) - S_M(t(R(a))t(b) + t(a)t(S(b)) - t(R(a)b + aS(b)));
\end{align*}
\]

This finishes the proof of our proposition. \( \square \)
The choice of the section $t$ in fact determines a splitting

$$0 \to M \xrightarrow{i} \hat{A} \xrightarrow{p} A \to 0$$

subject to $s \circ i = \text{Id}_M, s \circ t = 0$ and $is + tp = \text{Id}_{\hat{A}}$. Then there is an induced isomorphism of vector spaces

$$\left( \begin{array}{c} p \\ s \end{array} \right) : \hat{A} \cong A \oplus M : \left( \begin{array}{c} t \\ i \end{array} \right).$$

We can transfer the Rota-Baxter system structure on $\hat{A}$ to $A \oplus M$ via this isomorphism. It is direct to verify that this endows $A \oplus M$ with a multiplication $\cdot \Psi$ and two operators $R_\chi, S_\chi$ defined by

$$\begin{align*}
(a, m) \cdot \Psi (b, n) &= (ab, an + mb + \Psi(a, b)), \quad \forall a, b \in A, m, n \in M, \\
R_\chi (a, m) &= (R(a), \chi_R(a) + R_M(m)), \quad \forall a \in A, m \in M, \\
S_\chi (a, m) &= (S(a), \chi_S(a) + S_M(m)), \quad \forall a \in A, m \in M.
\end{align*}$$

Moreover, we get an abelian extension

$$0 \to (M, R_M, S_M) \xrightarrow{(\text{Id}_M, 0)} (A \oplus M, R_\chi, S_\chi) \xrightarrow{(\text{Id}_A, 0)} (A, R, S) \to 0$$

which is easily seen to be isomorphic to the original one.

**Proposition 6.5.** (i) Different choices of the section $t$ give the same Rota-Baxter system bimodule structures on $(M, R_M, S_M)$;

(ii) the cohomological class of $(\Psi, \chi_R, \chi_S)$ does not depend on the choice of sections.

*Proof.* Let $t_1$ and $t_2$ be two distinct sections of $p$. We define $\gamma : A \to M$ by $\gamma(a) = t_1(a) - t_2(a)$.

Since the Rota-Baxter system $(M, R_M, S_M)$ satisfies $uv = 0$ for all $u, v \in M$, we have

$$t_1(a)m = t_2(a)m + \gamma(a)m = t_2(a)m.$$

So different choices of the section $t$ give the same Rota-Baxter system bimodule structures on $(M, R_M, S_M)$.

Now, we show that the cohomological class of $(\Psi, \chi_R, \chi_S)$ does not depend on the choice of sections.

$$\Psi_1(a \otimes b) = t_1(a)t_1(b) - t_1(ab) = (t_2(a) + \gamma(a))(t_2(b) + \gamma(b)) - (t_2(ab) + \gamma(ab)) = (t_2(a)t_2(b) - t_2(ab)) + t_2(a)\gamma(b) + \gamma(a)t_2(b) - \gamma(ab) = (t_2(a)t_2(b) - t_2(ab)) + a\gamma(b) + \gamma(a)b - \gamma(ab) = \Psi_2(a \otimes b) + \delta^1(\gamma)(a \otimes b),$$
\[ \chi_{R_1}(a) = \hat{R}(t_1(a)) - t_1(R(a)) \]
\[ = \hat{R}(t_2(a) + \gamma(a)) - (t_2(R(a)) + \gamma(R(a))) \]
\[ = (\hat{R}(t_2(a)) - t_2(R(a))) + \hat{R}(\gamma(a)) - \gamma(R(a)) \]
\[ = \chi_{R_2}(a) + R_M(\gamma(a)) - \gamma(R(a)) \]
\[ = \chi_{R_2}(a) - \Phi_H^1(\gamma)(a), \]
\[ \chi_{S_1}(a) = \hat{S}(t_1(a)) - t_1(S(a)) \]
\[ = \hat{S}(t_2(a) + \gamma(a)) - (t_2(S(a)) + \gamma(S(a))) \]
\[ = (\hat{S}(t_2(a)) - t_2(S(a))) + \hat{S}(\gamma(a)) - \gamma(S(a)) \]
\[ = \chi_{S_2}(a) + S_M(\gamma(a)) - \gamma(S(a)) \]
\[ = \chi_{S_2}(a) - \Phi_H^2(\gamma)(a). \]

That is, \((\Psi_1, \chi_{R_1}, \chi_{S_1}) = (\Psi_2, \chi_{R_2}, \chi_{S_2}) + d^1(\gamma).\) Thus \((\Psi_1, \chi_{R_1}, \chi_{S_1})\) and \((\Psi_2, \chi_{R_2}, \chi_{S_2})\) form the same cohomological class in \(H^2_{RBS}(A, M)\).

**Proposition 6.6.** Let \(M\) be a vector space and \(R_M, S_M \in \text{End}_K(M)\). Then \((M, R_M, S_M)\) is a Rota-Baxter system with trivial multiplication. Let \((A, R, S)\) be a Rota-Baxter system. Two isomorphic abelian extensions of Rota-Baxter system \((A, R, S)\) by \((M, R_M, S_M)\) give rise to the same cohomology class in \(H^2_{RBS}(A, M)\).

**Proof.** Suppose \((\hat{A}_1, \hat{R}_1, \hat{S}_1)\) and \((\hat{A}_2, \hat{R}_2, \hat{S}_2)\) are two abelian extensions of \((A, R, S)\) by \((M, R_M, S_M)\) as is given in (31). Let \(t_1\) be a section of \((\hat{A}_1, \hat{R}_1, \hat{S}_1)\). As \(p_2 \circ \zeta = p_1\), we have \(p_2 \circ (\zeta \circ t_1) = p_1 \circ t_1 = \text{Id}_A\).

Therefore, \(\zeta \circ t_1\) is a section of \((\hat{A}_2, \hat{R}_2, \hat{S}_2)\). Denote \(t_2 = \zeta \circ t_1\). Since \(\zeta\) is a homomorphism of Rota-Baxter systems such that \(\zeta|_M = \text{Id}_M, \zeta(am) = \zeta(t_1(a)m) = t_2(a)m = am,\) so \(\zeta|_M : M \to M\) is compatible with the induced Rota-Baxter system bimodule structures. We have

\[ \Psi_2(a \otimes b) = t_2(a)t_2(b) - t_2(ab) \]
\[ = \zeta(t_1(a))\zeta(t_1(b)) - \zeta(t_1(ab)) \]
\[ = \zeta(t_1(a)t_1(b) - t_1(ab)) \]
\[ = \zeta(\Psi_1(a \otimes b)) \]
\[ = \Psi_1(a \otimes b), \]

\[ \chi_{R_2}(a) = (\hat{R}_2(t_2(a)) - t_2(R(a))) \]
\[ = \hat{R}_2(\zeta(t_1(a))) - \zeta(t_1(R(a))) \]
\[ = \zeta(\hat{R}_1(t_1(a)) - t_1(R(a))) \]
\[ = \zeta(\chi_{R_1}(a)) \]
\[ = \chi_{R_1}(a), \]
\[\chi_{S2}(a) = (\hat{S}_2(t_2(a)) - t_2(S(a)))
\begin{align*}
&= \hat{S}_2(\zeta(t_1(a))) - \zeta(t_1(S(a))) \\
&= \zeta(\hat{S}_1(t_1(a)) - t_1(S(a))) \\
&= \zeta(\chi_{S1}(a)) \\
&= \chi_{S1}(a).
\end{align*}\]

Hence, two isomorphic abelian extensions give rise to the same cohomology class in \(H^2_{RBS}(A, M)\). □

Now we consider the reverse direction. Let \((M, R_M, S_M)\) be a Rota-Baxter system bimodule over Rota-Baxter system \((A, R, S)\), given three linear maps \(\Psi : A \otimes A \to M\) and \(\chi_R, \chi_S : A \to M\), one can define a multiplication \(\cdot_{\Psi}\) and two operators \(R_{\chi}, S_{\chi}\) on \(A \oplus M\) by Equations \((32)\) and \((33)\) \((34)\). The following fact is important:

**Proposition 6.7.*** The quadruple \((A \oplus M, \cdot_{\Psi}, R_{\chi}, S_{\chi})\) is a Rota-Baxter system if and only if \((\Psi, \chi_R, \chi_S)\) is a 2-cocycle in the cochain complex of the Rota-Baxter system \((A, R, S)\) with coefficients in the Rota-Baxter system bimodule \((M, R_M, S_M)\). In this case, we obtain an abelian extension

\[
0 \longrightarrow (M, R_M, S_M) \xrightarrow{(\iota_M^0)} (A \oplus M, R_{\chi}, S_{\chi}) \xrightarrow{(\text{Id}_A \ 0)} (A, R, S) \longrightarrow 0
\]

and the canonical section \(t = (\text{Id}_A \ 0) : (A, R, S) \to (A \oplus M, R_{\chi}, S_{\chi})\) endows \(M\) with the original Rota-Baxter system bimodule structure.

*Proof.* If \((A \oplus M, \cdot_{\Psi}, R_{\chi}, S_{\chi})\) is a Rota-Baxter system, then the associativity of \(\cdot_{\Psi}\) implies

\[
a\Psi(b \otimes c) - \Psi(ab \otimes c) + \Psi(a \otimes bc) - \Psi(a \otimes b)c = 0,
\]

which means \(\delta^2(\Psi) = 0\) in \(C^*(A, M)\). Since \((R_{\chi}, S_{\chi})\) are Rota-Baxter system operators, for any \(a, b \in A, m, n \in M\), we have

\[
\begin{align*}
R_{\chi}((a, m)) \cdot_{\Psi} R_{\chi}((b, n)) &= R_{\chi}(R_{\chi}(a, m) \cdot_{\Psi} (b, n) + (a, m) \cdot_{\Psi} S_{\chi}(b, n)), \\
S_{\chi}((a, m)) \cdot_{\Psi} S_{\chi}((b, n)) &= S_{\chi}(R_{\chi}(a, m) \cdot_{\Psi} (b, n) + (a, m) \cdot_{\Psi} S_{\chi}(b, n)).
\end{align*}
\]

Then \(\chi_R, \chi_S, \Psi\) satisfy the following equations:

\[
\begin{align*}
R(a) \chi_R(b) + \chi_R(a)R(b) + \Psi(R(a) \otimes R(b))
&= R_M(\chi_R(a)b) + R_M(a \chi_S(b)) + \chi_R(R(a)b + aS(b)) \\
&+ R_M\left(\Psi(R(a) \otimes b + a \otimes S(b))\right), \\
S(a) \chi_S(b) + \chi_S(a)S(b) + \Psi(S(a) \otimes S(b))
&= S_M(\chi_R(a)b) + S_M(a \chi_S(b)) + \chi_S(R(a)b + aS(b)) \\
&+ S_M\left(\Psi(R(a) \otimes b + a \otimes S(b))\right),
\end{align*}
\]

That is,

\[
\partial^1(\chi_R, \chi_S) + \Phi^2(\Psi) = 0.
\]
Hence, \((\Psi, \chi_R, \chi_S)\) is a 2-cocycle.

Conversely, if \((\Psi, \chi_R, \chi_S)\) is a 2-cocycle, one can easily check that \((A \oplus M, \cdot \Psi, R_\chi, S_\chi)\) is a Rota-Baxter system.

The last statement is clear. □

Finally, we show the following result:

**Proposition 6.8.** Two cohomologous 2-cocycles give rise to isomorphic abelian extensions.

**Proof.** Given two 2-cocycles \((\Psi_1, \chi_{R1}, \chi_{S1})\) and \((\Psi_2, \chi_{R2}, \chi_{S2})\), we can construct two abelian extensions \((A \oplus M, \cdot \Psi_{1,2}, R_{\chi_{1,2}}, S_{\chi_{1,2}})\) via Equations (32) and (33). If they represent the same cohomology class in \(H^2_{\text{RBS}}(A, M)\), then there exists two linear maps \(\gamma_0: \mathbb{K} \to M, \gamma_1: A \to M\) such that

\[
\begin{align*}
(\Psi_1, \chi_{R1}, \chi_{S1}) = (\Psi_2, \chi_{R2}, \chi_{S2}) + (\delta^1(\gamma_1), -\Phi^1(\gamma_1) - \partial^0(\gamma_0, \gamma_0)).
\end{align*}
\]

Notice that \(\partial^0(\gamma_0, \gamma_0) = \partial^0 \circ \Phi^0(\gamma_0) = \Phi^1 \circ \delta^0(\gamma_0)\). Define \(\gamma: A \to M\) to be \(\gamma_1 + \delta^0(\gamma_0)\). Then \(\gamma\) satisfies

\[
(\Psi_1, \chi_{R1}, \chi_{S1}) = (\Psi_2, \chi_{R2}, \chi_{S2}) + (\delta^1(\gamma), -\Phi^1(\gamma)).
\]

Define \(\zeta: A \oplus M \to A \oplus M\) by

\[
\zeta(a, m) := (a, -\gamma(a) + m).
\]

Then \(\zeta\) is an isomorphism of \((A \oplus M, \cdot \Psi_{1,2}, R_{\chi_{1,2}}, S_{\chi_{1,2}})\). □

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