THE SUBGROUP DETERMINED BY A CERTAIN IDEAL IN A FREE GROUP RING

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Abstract. For normal subgroups $R$ and $S$ of a free group $F$, an identification of the subgroup $F \cap (1 + rfs)$ is derived, and it is shown that the quotient $\frac{F \cap (1 + rfs)}{R \cap S}$ is, in general, non-trivial.

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1. Introduction

Every two-sided ideal $a$ in the integral group ring $\mathbb{Z}[F]$ of a free group $F$ determines a normal subgroup $F \cap (1 + a)$ of $F$. Identification of such subgroups is a fundamental problem in the theory of group rings ([5], [9]). Let $R$ and $S$ be normal subgroups of $F$. In this paper we examine the subgroup $F \cap (1 + rfs)$, where, for a normal subgroup $G$ of $F$, $g$ denotes the two-sided ideal of $\mathbb{Z}[F]$ generated by $G - 1$. This subgroup has been studied by C. K. Gupta [4] (see also [7]). It is easy to check that $[R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S', R \cap S'] \subseteq F \cap (1 + rfs)$, where $R'$ (resp. $S'$) is the derived subgroup of $R$ (resp. $S$). Whereas the identification given in [4], namely that the preceding inclusion is an equality, holds up to torsion, our investigation shows that, $\frac{F \cap (1 + rfs)}{R \cap S} \approx L_1 SP^2 \left( \frac{R \cap S}{(R \cap S')(R \cap S')} \right)$, and is, in general, non-identity; here $L_1 SP^2$ is the first derived functor of the second symmetric power functor.

2. The subgroup $F \cap (1 + rfs)$

Let $F$ be a free group and $R$, $S$ its normal subgroups with bases, as free groups, $\{r_i\}_{i \in I}$ and $\{s_j\}_{j \in J}$ respectively. Then the ideal $r$ is a free right $\mathbb{Z}[F]$-module with basis $\{r_i - 1 | i \in I\}$ and the ideal $s$ is a free left $\mathbb{Z}[F]$-module with basis $\{s_j - 1 | j \in J\}$ ([3], Theorem 1, p. 32). Further, recall that $R/R' \approx \frac{r}{r'f}$ (resp. $rR' \mapsto r - 1 + fr$).
From these observations it immediately follows that we can make the following identification
\[ R/R' \otimes S/S' \cong \frac{r}{r'} \otimes \frac{s}{s'} = \frac{r}{r'} \otimes_{\mathbb{Z}[F]} \frac{s}{s'} = \frac{rs}{r'} \frac{r}{s}. \quad (2.1) \]

Here \( \otimes \) is tensor product over \( \mathbb{Z} \) which we can replace by \( \otimes_{\mathbb{Z}[F]} \) since the action of \( \mathbb{Z}[F] \) on components of the tensor product is trivial.

**Theorem 2.1.** If \( R \) and \( S \) are normal subgroups of a free group \( F \), then there is a natural isomorphism
\[ F \cap (1 + rs) \cong L_1 \text{SP}^2 \left( \frac{R \cap S}{(R' \cap S)(R \cap S')} \right). \]

**Proof.** Let us set
\[ Q := \frac{R \cap S}{R' \cap S'}, \quad U := \frac{R' \cap S}{R' \cap S'}, \quad V := \frac{R \cap S'}{R' \cap S'}. \quad (2.2) \]

The group \( Q \) is free abelian because it injects into \( R/R' \oplus S/S' \), and so are \( U, V \) both being subgroups of \( Q \). Observe that \( Q/U \) is also free abelian, since it is isomorphic to the subgroup \( \frac{R \cap S}{R' \cap S'} \) of \( R/R' \).

For an abelian group \( A \), we denote by \( \text{SP}^2(A) \) its symmetric square, defined as the quotient \( \text{SP}^2(A) := A \otimes A / \langle a \otimes b - b \otimes a, \ | a, b \in A \rangle \) and by \( \Lambda^2(A) \) its exterior square \( \Lambda^2(A) := A \otimes A / \langle a \otimes a \ | a \in A \rangle \). Recall (see [8]) that, for any free resolution
\[ 0 \to C \to B \to A \to 0 \]
of \( A \), the so-called Koszul complex
\[ 0 \to \Lambda^2(C) \to C \otimes B \to \text{SP}^2(B) \]
represents the object \( L\text{SP}^2(A) \) of the derived category of abelian groups; in particular, its zeroth (resp. first) homology is equal to the zeroth (resp. first) derived functor of \( \text{SP}^2 \) applied to \( A \).

Consider the natural commutative diagram with exact rows and columns which contains maps between quadratic Koszul complexes:
\[
\Lambda^2(U) \longrightarrow U \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q/U \otimes Q \\
\Lambda^2(U) \longrightarrow U \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q/U \otimes Q \\
\Lambda^2(U) \longrightarrow U \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q \otimes Q \longrightarrow \text{SP}^2(Q) \\
\Lambda^2(Q) \longrightarrow Q/U \otimes Q
Since the middle horizontal complex is acyclic, the homology of the lower complex are the same as of the upper complex shifted by one. That is, there exists a short exact sequence

\[ 0 \rightarrow \frac{\Lambda^2(Q)}{\Lambda^2(U)} \rightarrow Q/U \otimes Q \rightarrow SP^2(Q/U) \rightarrow 0 \]

which can be naturally extended to the following diagram:

\[
\begin{array}{c}
K \\
\downarrow \\
Q/U \otimes V \\
\downarrow \\
\frac{\Lambda^2(Q)}{\Lambda^2(U)} \\
\downarrow \\
Q/U \otimes Q \\
\downarrow \\
SP^2(Q/U) \\
\downarrow \\
Q/U \otimes Q/V
\end{array}
\]

Here \( K \) is, by definition, the kernel of the lower horizontal map. By Snake Lemma, \( K \) is isomorphic to the kernel of the right hand vertical map \( Q/U \otimes V \rightarrow SP^2(Q/U) \) in the diagram. Observe that this map is part of the Koszul complex

\[ 0 \rightarrow \Lambda^2(VU/U) \rightarrow Q/U \otimes V \rightarrow SP^2(Q/U) \]

which represents the object \( LSP^2(Q/UV) \) of the derived category of abelian groups. Here we have used the fact that \( V = VU/U = V/(V \cap U) \), since \( V \cap U \) is the zero subgroup of \( Q \). The homology groups of the above Koszul complex are the derived functor evaluations \( L_iSP^2(Q/UV) \), \( i = 1, 2 \) (see [8]). Therefore, we get the following short exact sequence:

\[ 0 \rightarrow \Lambda^2(V) \rightarrow K \rightarrow L_1SP^2(Q/UV) \rightarrow 0. \]

Consequently the lower sequence of the diagram (2.3), yields the following exact sequence:

\[ 0 \rightarrow L_1SP^2(Q/UV) \rightarrow \frac{\Lambda^2(Q)}{\Lambda^2(U) + \Lambda^2(V)} \rightarrow Q/U \otimes Q/V \]  

(2.4)

We next observe that there are natural isomorphisms

\[
\Lambda^2(Q) \cong \frac{\gamma_2(R \cap S)}{[R' \cap S', R \cap S]}
\]

\[
\frac{\Lambda^2(Q)}{\Lambda^2(U) + \Lambda^2(V)} \cong \frac{\gamma_2(R \cap S)}{[R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S', R \cap S']}
\]

and natural monomorphisms \( Q/U \rightarrow R/R', Q/V \rightarrow S/S' \). The exact sequence (2.4) thus implies that there is an exact sequence

\[ 0 \rightarrow L_1SP^2(Q/UV) \rightarrow \frac{\gamma_2(R \cap S)}{[R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S', R \cap S']} \rightarrow R/R' \otimes S/S'. \]  

(2.5)
The statement of the theorem follows from the fact (see [2]) that

\[ F \cap (1 + rs) = \gamma_2(R \cap S) \]

and the identification (2.1). \(\square\)

For an abelian group \(A\), a description of the group \(L_1SP^2(A)\) is available in many papers on polynomial functors; for example, see [1] or ([6], Theorem 2.2.5). Recall the main properties of \(L_1SP^2(A)\). For any abelian group \(A\), \(L_1SP^2(A)\) is a natural quotient of the group \(\text{Tor}(A, A)\) by diagonal elements. We have

\[ L_1SP^2(\mathbb{Z}/m\mathbb{Z}) = L_1SP^2(\mathbb{Z}) = 0, \]

for all natural numbers \(m\), and, for all abelian groups \(A, B\), there is a (bi)natural isomorphism

\[ \text{Tor}(A, B) = \text{Ker}\{L_1SP^2(A \oplus B) \rightarrow L_1SP^2(A) \oplus L_1SP^2(B)\}. \]

For a free abelian group \(A\) and a natural number \(m \geq 1\), there is a natural isomorphism

\[ L_1SP^2(A \otimes \mathbb{Z}/m\mathbb{Z}) \simeq \Lambda^2(A \otimes \mathbb{Z}/m\mathbb{Z}). \]

Observe also that, the functor \(L_1SP^2\) is related to the homology of the Eilenberg-MacLane spaces \(K(-, 2)\). Namely, for any abelian group \(A\), there is a natural short exact sequence

\[ 0 \rightarrow L_1SP^2(A) \rightarrow H_5K(A, 2) \rightarrow \text{Tor}(A, \mathbb{Z}/2\mathbb{Z}) \rightarrow 0. \]

Invoking this description for \(L_1SP^2(Q/UV)\), we have the following identification of the subgroup \(F \cap (1 + rs)\):

**Theorem 2.2.**

\[ F \cap (1 + rs) = [R' \cap S', R \cap S][R' \cap S, R' \cap S][R \cap S, R \cap S']W, \]

where \(W\) is the subgroup of \(F\) generated by elements\(^1\)

\[ [x_1, y][x, y_2]^{-1}, \]

such that

\[ x, y \in R \cap S, \ m \geq 2, \]

\[ x^m = x_1x_2, \ y^m = y_1y_2, \]

\[ x_1, y_1 \in R' \cap S, \]

\[ x_2, y_2 \in R \cap S'. \]

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\(^1\)For an elements \(g, h\) of a group, we use the standard commutator notation \([g, h] := g^{-1}h^{-1}gh\).
Proof. Consider the generating elements from $W$, as in the Theorem. Modulo $rfs$, we have

$$[x_1, y][x_2, y_2]^{-1} - 1 \equiv [x^m, y][x_2, y_2]^{-1}[x, y_2]^{-1} - 1$$

$$(x^m - 1)(y - 1) - (y^m - 1)(x - 1)$$

$$- (x_2 - 1)(y - 1) + (y - 1)(x_2 - 1)$$

$$- (x - 1)(y_2 - 1) + (y_2 - 1)(x - 1)$$

$$\equiv (x_1 - 1)(y - 1) - (y_1 - 1)(x - 1) +$$

$$(y - 1)(x_2 - 1) - (x - 1)(y_2 - 1).$$

All four products $(x_1 - 1)(y - 1), (y_1 - 1)(x - 1), (y - 1)(x_2 - 1), (x - 1)(y_2 - 1)$ lie in $rfs$. The subgroup $W$ is chosen as a subgroup of representatives of $L_1\text{SP}^2\left(\frac{R \otimes S}{(R \cap S)(R \cap S')}\right)$ in $F \cap (1 + rfs)$.

Consider generators of $L_1\text{SP}^2\left(\frac{R \otimes S}{(R \cap S)(R \cap S')}\right)$ viewed as a natural quotient of the group $\text{Tor}\left(\frac{R \otimes S}{(R \cap S)(R \cap S')}, \frac{R \otimes S}{(R \cap S)(R \cap S')}\right)$. The generators are given as pairs of elements $(x, y), x, y \in R \cap S$, with the property that, there exists $m \geq 2$, such that $x^m, y^m \in (R' \cap S)(R \cap S')$. Consider now the diagram (2.3) and find the image of the pair $(x, y)$ in the quotient $Q/U \otimes Q/\Lambda^2(V)$ (here we use the notation (2.2)) and choose its representative in $Q/U \otimes V$. It is given as

$$(x.U) \otimes y_2.(R' \cap S') - (y.U) \otimes x_2.(R' \cap S'),$$

where $x_2, y_2$ are defined in the formulation of the Theorem. Going further in the diagram (2.3), we find a representative of the element (2.6) in $\Lambda^2(Q)/\Lambda^2(V)$, given as

$$(x \wedge y_2) + (x_2 \wedge y) - (x^m \wedge y) + \Lambda^2(V).$$

Indeed, the natural map $\Lambda^2(Q) \to Q/U \otimes Q$ sends (we omit the notation $-(R' \cap S')$ for the elements from $Q$ for the sake of simplification of notations)

$$x \wedge y_2 + (x_2 \wedge y) - (x^m \wedge y)$$

In the free group $F$, this element is represented as a product of commutators

$$[x, y_2][x_2, y][x^m, y]^{-1}.$$

Since modulo $rfs$,

$$[x_1, y][x_2, y_2]^{-1} - 1 \equiv [x^m, y][x_2, y_2]^{-1}[x, y_2]^{-1} - 1 \equiv ([x, y_2][x_2, y][x^m, y]^{-1})^{-1} - 1$$

we get the asserted description of the set $W$. □
Remark. Since the groups $F/\gamma_3(R \cap S), R/R', S/S'$ are always torsion-free, the sequence (2.5) implies that there is the following identification

$$L_1\text{SP}^2 \left( \frac{R \cap S}{(R' \cap S)(R \cap S')} \right) \cong \text{torsion of} \quad \frac{F}{[R' \cap S', R \cap S][R' \cap S, R \cap S'][R \cap S', R \cap S']}.$$  

3. Example

Finally, let us give an example of subgroups $R, S$ in a free group $F$, such that $L_1\text{SP}^2 \left( \frac{R \cap S}{(R' \cap S)(R \cap S')} \right) \neq 0$.

Let $F = F(a_1, \ldots, a_n, b)$, $n \geq 2$,

$$R = \langle a_1, \ldots, a_n, [F, F]^F \rangle,$$

$$S = \langle a_1^2, \ldots, a_n^2, b, [F, F]^F \rangle.$$

Since $[F, F] \subset R, [F, F] \subset S$,

$$(R' \cap S)(R \cap S') = R'S'.$$

For every $i = 1, \ldots, n$, the element $[a_i, b]$ lies in $R \cap S$. Observe that,

$$[a_i^2, b] = [a_i, b][[a_i, b], a_i][a_i, b]$$

Therefore,

$$[a_i, b]^2 \in R'S'.$$

Since $R'S' = \langle [a_i, a_j], [a_i, b]^2, \gamma_3(F) \rangle$, the elements $[a_i, b], i = 1, \ldots, n$ form an abelian subgroup of $\frac{R'S'}{(R \cap S)(R \cap S')}$ isomorphic to $(\mathbb{Z}/2)^{3n}$. For $n \geq 2$, the first derived functor of $\text{SP}^2$ of such group is non-zero.

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