The \( \omega \)-Vaught’s Conjecture

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The main result

- There is a "purely structural" strengthening of Vaught’s conjecture named the $\omega$-Vaught’s conjecture ($\omega$-VC).
- Linear orders satisfy the $\omega$-Vaught’s conjecture.
Summary of the talk

1. Vaught’s conjecture and the Morley analysis
2. $\omega$-VC
3. Selected points from the proof for linear orders
Vaught’s Conjecture

**Conjecture:** [Vaught 61] Given a first order theory over a countable vocabulary, the number of countable models of the theory is either countable or continuum.

**Conjecture (infinitary version):** [Vaught 61] Given a formula $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over a countable vocabulary, the number of countable models of $\varphi$ is either countable or continuum.

- $\mathcal{L}_{\omega_1,\omega}$ is infinitary logic; it extends first order logic by allowing countable conjunctions and disjunctions.
- Under CH the conjecture trivially holds. You can replace "continuum" with "perfectly many" to get a statement independent of set theoretic considerations.
Conjecture: [Martin] Given a complete, consistent first order theory $T$ over a countable vocabulary, add a predicate for every type to create $T_1$. If $T$ has fewer than $2^{\aleph_0}$ many models, then any model of $T$ is $\aleph_0$–categorical in its $T_1$ theory.

Conjecture: [Becker-Kechris] For any continuous action of a Polish group on a Polish space, there are either countable or continuum many orbits.

Theorem: [Becker] One of the following holds for any complete, left invariant Polish $G$-space $X$:

- $X$ has perfectly many orbits.
- Every orbit of $X$ is $\Pi^0_\omega$. 
Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

- $\varphi \in \mathcal{L}_{\omega_1,\omega}$ is in $\Sigma^i_0 = \Pi^i_0$ if it is quantifier free and has no infinitary disjunctions or conjunctions.
- For $\alpha \in \omega_1$, $\varphi$ is $\Sigma^i_\alpha$ if $\varphi = \bigwedge_i \exists (\bar{x}) \psi_i(\bar{x})$ for $\psi_i \in \Pi^i_\beta$ with $\beta < \alpha$.
- For $\alpha \in \omega_1$, $\varphi$ is $\Pi^i_\alpha$ if $\varphi = \bigwedge_i \forall (\bar{x}) \psi_i(\bar{x})$ for $\psi_i \in \Sigma^i_\beta$ with $\beta < \alpha$. 
Complexity of $\mathcal{L}_{\omega_1,\omega}$ formulas

- For two models $M, N$ we say $M \leq_\alpha N$ if $\Pi^{in}_\alpha - \text{Th}(M) \subseteq \Pi^{in}_\alpha - \text{Th}(N)$.
- Note that $M \geq_\alpha N$ if and only if $\Sigma^{in}_\alpha - \text{Th}(M) \subseteq \Sigma^{in}_\alpha - \text{Th}(N)$.
- We put $M \equiv_\alpha N$ if both of the above hold.

**Fact:** The $\equiv_\alpha$ are Borel equivalence relations.

**Theorem:** [Silver 80] Borel equivalence relations have either countable or continuum many equivalence classes.
Scott rank

**Theorem:** [Scott] For every countable structure $M$ there is a sentence $\varphi \in L_{\omega_1,\omega}$ such that $N \cong M \iff N \models \varphi$.

**Corollary:** On countable structures,

$$\cong = \bigcap_{\alpha \in \omega_1} \equiv_{\alpha}$$

**Definition:** A $\varphi$ as in the theorem statement is called a *Scott sentence*.

**Definition:** [Montalbán] The (parametrized) *Scott rank* of $M$ is the least $\alpha \in \omega_1$ such that $M$ has a $\Sigma^{in}_{\alpha+2}$ Scott sentence. We write $\text{SR}(M) = \alpha$. 
The Morley analysis

**Theorem:** [Morley] Given a formula $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over a countable vocabulary, the number of countable models of $\varphi$ is either countable, continuum, or $\aleph_1$.

**Proof (Sketch):** Let $SS(\varphi) := \{\alpha \in \omega_1 | \exists M, M \models \varphi \land SR(M) = \alpha\}$ and consider cases:

1. For some $\beta < \omega_1$ there are continuum many $\equiv \beta$ classes.
2. $SS(\varphi)$ is bounded below some $\beta < \omega_1$. In this case, $\approx$ is $\equiv_{\beta+2}$ so is Borel. If we are not in case 1, there are only $\aleph_0$ many models.
3. $SS(\varphi)$ is cofinal in $\omega_1$ and for all $\beta < \omega_1$ there are countably many $\equiv_\beta$ classes. This means there are $\aleph_1$ many models.
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The Vaught ordinal

Given a $\varphi \in \mathcal{L}_{\omega_1,\omega}$ we define the Vaught ordinal, written $\text{vo}(\varphi)$ as the least $\beta$ such that either

- there are continuum many models of $\varphi$ up to $\equiv_\beta$ equivalence,
- or there are only countably many models of $\varphi$ and they all have Scott rank less than $\beta$.

Vaught’s conjecture holds if and only if $\text{vo}(\varphi)$ is well defined for all $\mathcal{L}_{\omega_1,\omega}$ sentences $\varphi$. 
Vaught ordinal examples

- Linear orders: a $\Pi^i_1$ sentence with $\text{vo}(\varphi) = 3$ as there are uncountably many $\equiv_3$ classes.
- If $\psi \in \Sigma^i_{\alpha+2}$ is a Scott sentence then $\text{vo}(\psi) = \alpha + 1$.
- Both $\mathbb{Q}$-vector spaces and algebraically closed fields: a $\Pi^i_2$ sentence with $\text{vo}(\chi) = 3$ as they always have $\text{SR}(M) < 3$.
- Boolean algebras: a $\Pi^i_2$ sentence with $\text{vo}(\theta) = \omega$ as there are uncountably many $\equiv_\omega$ classes but only countably many $\equiv_n$ classes for $n \in \omega$. 
The ω-Vaught’s conjecture

**Conjecture:** Given a formula $\varphi \in \Pi^i_\alpha$ over a countable vocabulary,

$$\text{vo}(\varphi) \leq \alpha + \omega.$$ 

- Because of the example of Boolean algebras, this is the best possible general bound.
- It is similar to Martin’s conjecture in that we are allowing and additional $\omega$ many quantifiers to classify models.
- It is different in that it is essentially infinitary and more precisely tied to the Morley analysis and computable structure theory.
- It also gives more precise information in the ”continuum case” about where the continuum is witnessed.
- It is unknown if one implies the other.
Linear orders

**Theorem:** [Steel 78] For any $\varphi \in \mathcal{L}_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders,

$$\text{vo}(\varphi) \leq \omega^{\varphi}_1.$$ 

Note that $\omega^{\varphi}_1$

- is really, really big,
- is dependant on more than the complexity of the formula $\varphi$,
- uses notions from higher recursion theory.
Steel: \( \text{vo}(\varphi) \leq \omega^\varphi_1 \).

Where does he need such a large ordinal?

**Definition:** For any \( \alpha \in \omega_1 \) and \( x, y \in L \) a countable linear order, say

\[
x \sim_\alpha y \iff \text{SR}((x, y)_L) < \alpha.
\]

**Lemma:** If \( \text{SR}(L) \geq \omega^\varphi_1 \), then \( L/ \sim_\omega^\varphi_1 \) is a dense linear order.

Proof uses \( \Sigma^1_1 \) bounding; is not true at non-admissible ordinals (e.g. your ordering is itself an ordinal).

A better bound requires a finer combinatorial analysis of \( L/ \sim_\alpha \) for smaller \( \alpha \).
The Vaught ordinal for linear orders

Over time we preformed this analysis for $\varphi \in \Pi^\infty_\alpha$ whose models are linear orders:

- **Version 1:** $\text{vo}(\varphi) \leq (\alpha + \omega)^\omega$
- **Version 2:** $\text{vo}(\varphi) \leq \alpha \cdot \omega^2 + \omega + 5$
- **Version 3:** $\text{vo}(\varphi) \leq (\alpha + \omega) \cdot 5 + \omega \cdot 5$
- **Version 4:** $\text{vo}(\varphi) \leq \alpha + \omega \cdot 3$
- **Version 5:** $\text{vo}(\varphi) \leq \alpha + \omega + 25$
- **Final version:** $\text{vo}(\varphi) \leq \alpha + \omega$

**Theorem:** [G., Montalbán] For any $\varphi \in L_{\omega_1,\omega}$ over $\{\leq\}$ that implies all models are linear orders, $\varphi$ satisfies $\omega$-VC.
The main lemma

**Definition:** A structure $M$ is $(\beta, \beta + \omega)$-small if for all $n \in \omega$

$$\left| \{ B \mid B \equiv_\beta A \} / \equiv_{\beta+n} \right| \leq \aleph_0.$$

**Lemma:** The following are equivalent for $\varphi \in \Pi^\infty_\alpha$:

1. Every $\psi$ that implies $\varphi$ satisfies $\omega - \text{VC}$.
2. For every $\beta \geq \alpha$ and $(\beta, \beta + \omega)$-small $A$ with $A \models \varphi$ and $\text{SR}(A) \geq \beta + \omega$, there is a $B \equiv_\beta A$ with $\text{SR}(B) \geq \beta + \omega$ and $B \not\equiv_{\beta+\omega} A$.

Proof idea for (2) implies (1): Assume there is some $(\alpha, \alpha + \omega)$-small model of $\varphi$ with a large Scott rank. Build a perfect binary tree of $\equiv_\alpha$ structures that are not $\equiv_{\alpha+\omega}$ at a given height. The set of limit structures at each path witness distinct $\equiv_{\alpha+\omega}$ classes.
The objective: Given a \((\beta, \beta + \omega)\)-small \(A\) with \(\text{SR}(A) \geq \beta + \omega\), explore the space of \(B\) that have \(B \equiv^\beta A\). Try to find a transformation of \(A\) into a \(B\) that satisfies the two competing goals:

1. The Scott rank of \(B\) stays at at least \(\beta + \omega\),
2. \(B\) disagrees with \(A\) on some \(\Pi^\in_{\beta+n}\) formula.
The replacement lemma

**Lemma:** There is a non-decreasing function $f : \omega \rightarrow \omega$ which, given an $(\alpha, \alpha + \omega)$-small structure $L$ with $\text{SR}(L) \geq \alpha + n$, guarantees that there is a structure $P$ with

$$L \equiv_{\alpha+n} P \text{ and } \alpha + n \leq \text{SR}(P) \leq \alpha + f(n).$$

Idea: Apply this lemma to intervals inside of a linear ordering to control the Scott ranks of end segments.
Lemma: For a fixed vocabulary, given any ordinal $\alpha$, there is a $\Pi^1_{2\alpha+3}$ sentence $\rho_\alpha$ such that

$$\mathcal{A} \models \rho_\alpha \iff \text{SR}(\mathcal{A}) \geq \alpha.$$ 

We use this idea to define $\psi_{\leq,i} := \exists x \text{SR}(L_{\leq x}) = \alpha + i$ of quantifier rank less than $\alpha + \omega$ and an analogous $\psi_{\geq,i}$.

In nearly all cases considered we construct models that disagree on some Boolean combination of the $\psi_{\leq,i}$ and $\psi_{\geq,i}$.
To apply the replacement lemmas effectively we need to understand how the Scott rank of suborders relate to the Scott rank of the orders they comprise.

**Lemma:** For any linear orderings $A, B$

\[
SR(A + B) \leq \max(SR(A), SR(B)) + 4.
\]

**Lemma:** For any linear ordering $A$ with $SR(A_{\leq x}) \leq \beta$ for all $x \in A$,\n
\[
SR(A) \leq \beta + 4.
\]
The rest of the proof is purely about the combinatorics of linear orderings.

One big idea: Steel used that $L/\not\sim_{\omega_1^\varphi}$ is a dense linear order. We can reduce to the case that for some $n$, $L/\sim_{\alpha+n}$ with a suitable application of the replacement lemma.

While ordinals are descriptively complicated they are actually quite combinatorially simple; this is quite an important reduction.
The main takeaways

- The new result involves only relatively low-level definability and structural information about orderings.
- The use of higher recursion theory or descriptive set theory is not needed to prove VC for linear orders.
- A purely structural proof of Vaught’s conjecture for other structures may be possible via $\omega$-VC.
- Vaught’s conjecture is only the beginning.
- If you think this is a straw-man, please tear it down!
Thank you!
References

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