Homogenization of Randomly Deformed Conductivity Resistant Membranes

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Abstract

We study the homogenization of a stationary conductivity problem in a random heterogeneous medium with highly oscillating conductivity coefficients and an ensemble of simply closed conductivity resistant membranes. This medium is randomly deformed and then rescaled from a periodic one with periodic membranes, in a manner similar to the random medium proposed by Blanc, Le Bris and Lions [14]. Across the membranes, the flux is continuous but the potential field itself undergoes a jump of Robin type. We prove that, for almost all realizations of the random deformation, as the small scale of variations of the medium goes to zero, the random conductivity problem is well approximated by that on an effective medium which has deterministic and constant coefficients and contains no membrane. The effective coefficients are explicitly represented. One of our main contributions is to provide a solution to the associated auxiliary problem that is posed on the whole domain with infinitely many interfaces, in a setting that is neither periodic nor stationary ergodic in the usual sense.

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1 Introduction

In this article, we investigate the stochastic homogenization problem for a second order elliptic equation of divergence form that is posed on domains separated by an ensemble of simply closed surfaces, with jump type transmission conditions across them. The surfaces that separate the spatial domain have length scale $\varepsilon \ll 1$ and they are realized as a random deformation from a periodic structure of surfaces. Our goal is to study the behavior, as $\varepsilon$ goes to zero, of the solution to this equation.

More precisely, let $D$ be an open bounded subset in $\mathbb{R}^d$, $d = 2, 3$. It is separated by a random ensemble of simply closed interfaces $\Gamma_\varepsilon$ into $D^+_{\varepsilon}$ and $D^-_{\varepsilon}$, where $D^+_{\varepsilon}$ denotes the union of the interiors enclosed by the interfaces in $\Gamma_\varepsilon$, and $D^+_{\varepsilon}$ denotes the rest of the domain. The small parameter $0 < \varepsilon \ll 1$ is the length scale of interfaces. We study the
behavior of \( u_\varepsilon = u^+_\varepsilon(x,\omega)\chi_{D^+_\varepsilon} + u^-_\varepsilon(x,\omega)\chi_{D^-_\varepsilon}, \) where \( \chi_U \) denotes the indicator function of an open set \( U, \) and \( u_\varepsilon \) solves the following problem:

\[
\begin{cases}
-\nabla \cdot (A^\varepsilon(x,\omega)\nabla u^\pm_\varepsilon(x,\omega)) = f(x), & \text{for } x \in D^\pm_\varepsilon, \\
\frac{\partial}{\partial \nu_{A^\varepsilon}} u^+_\varepsilon(x,\omega) = \frac{\partial}{\partial \nu_{A^\varepsilon}} u^-_\varepsilon(x,\omega), & \text{for } x \in \Gamma_\varepsilon, \\
u^+_\varepsilon(x,\omega) - u^-_\varepsilon(x,\omega) = \varepsilon \frac{\partial}{\partial \nu_{A^\varepsilon}} u^+_\varepsilon(x,\omega), & \text{for } x \in \Gamma_\varepsilon, \\
u^+_\varepsilon(x,\omega) = 0, & \text{for } x \in \partial D.
\end{cases}
\] (1.1)

The first line in (1.1) is to be understood as two equations for, respectively, \( u^+_\varepsilon \) on \( D^+_\varepsilon \) and \( u^-_\varepsilon \) on \( D^-_\varepsilon. \) The variable \( \omega \) denotes the realization of the random ensemble \( \Gamma_\varepsilon, \) which is obtained by a random deformation of a periodic ensemble followed by rescaling. Notice that \( \Gamma_\varepsilon, D^+_\varepsilon, D^-_\varepsilon \) and hence \( u_\varepsilon \) are all random.

The problem above models, among many other natural applications, the stationary conductivity of heat through a medium that contains heat resistant membranes \( \Gamma_\varepsilon. \) The anisotropic diffusion matrix \( A^\varepsilon = (a^\varepsilon_{ij}) \) is a \( d \times d \) matrix with entries

\[
a^\varepsilon_{ij}(x,\omega) = \tilde{a}_{ij}(\Phi^{-1}(\frac{x}{\varepsilon},\omega)),
\]

where \( \Phi(\cdot,\omega) \) is a random diffeomorphism on \( \mathbb{R}^d \) and \( \tilde{A}(y) = (\tilde{a}_{ij}(y)) \) is a \([0,1]^d \)-periodic uniformly elliptic matrix. For simplicity, we assume that \( (\tilde{a}_{ij}) \) is symmetric. The second and third equations in (1.1) are the transmission conditions across the membranes. There, we have defined the conormal derivative of \( u^\pm_\varepsilon, \) i.e. the normal flux, at \( x \in \Gamma_\varepsilon \) as

\[
\frac{\partial u^\pm_\varepsilon}{\partial \nu_{A^\varepsilon}} := \nu_x \cdot A^\varepsilon \nabla u^\pm_\varepsilon(x,\omega),
\] (1.2)

where \( \nu_x \) is the unit outer normal vector along the boundary \( \Gamma_\varepsilon \) of \( D^-_\varepsilon. \) The transmission conditions depict that the flux is continuous across the interface while the potential field \( u_\varepsilon \) itself has a jump which is proportional to the flux. This seemingly unusual transmission condition is due to the fact that the membranes serve as interfacial thermal barriers. We refer to Carslaw and Jaeger [17] for the physical justification of these conditions, and the book of Milton [35] for a comprehensive treatment of composite materials.

Equations with highly oscillating coefficients and/or highly oscillating domains arise naturally in many applications in physics and engineering. Due to the small scale variations, it is difficult to study such equations directly. For instance, straightforward numerical simulations of such equations become a daunting task when the scale is very small. It is hence plausible to seek for simplified equations which approximate the heterogeneous ones when the small scale tends to zero, under certain assumptions on the coefficients and the problem settings, e.g. periodicity or stationary ergodicity. This is the well known homogenization theory, which has a long history that dates back to the 70’s; see e.g. Bensoussan, Lions and Papanicolaou [13] and Tartar [39] for the periodic setting, and Papanicolaou and Varadhan [38] and Kozlov [31] for the random setting. We refer to the books of Zhikov, Kozlov and Oleıım [29] for a comprehensive treatment of homogenization theory.
In this paper we study homogenization of (1.1) where both the elliptic coefficients and the interfaces are random and vary on a scale of \( \varepsilon \). More precisely, our random setting is obtained by a random deformation from the corresponding periodic setting, in which the coefficients and the interfaces are periodic. Our idea takes inspiration from the random settings of Blanc, Le Bris and Lions [14, 15]. Details of the setting are in Section 2.1. The resulting medium is stationary ergodic, but in a different sense than the usual one.

Our main result shows that as \( \varepsilon \to 0 \), for almost all \( \omega \), the unique solution \( u_\varepsilon(\cdot, \omega) \) of (1.1) converges to the solution of the following deterministic equation

\[
\begin{aligned}
-\nabla \cdot A^0 \nabla u_0(x) &= f(x), \quad \text{for } x \in D, \\
u_0(x) &= 0, \quad \text{for } x \in \partial D.
\end{aligned}
\]

(1.3)

The precise meaning of convergence is stated in Theorem 2.3. We note that \( u_\varepsilon \) converges strongly in \( L^2(D) \) to \( u_0 \), and the flux \( \chi_{D^+} A^\varepsilon \nabla u_\varepsilon^+ + \chi_{D^-} A^\varepsilon \nabla u_\varepsilon^- \) converges weakly in \( [L^2(D)]^d \) to the homogenized flux \( A^0 \nabla u_0 \). The homogenized elliptic coefficients \( (A^0)_{ij} \) are deterministic constants, which are explicitly represented in (2.11). In particular, the effective medium does not contain any conductivity resistant membrane.

The homogenization problem of (1.1) in the periodic case was first studied by Monsurro [36]. Later, the parabolic version was studied by Donato and Monsurro [22], and the wave equation case was studied by Donato, Faella and Monsurro [21]. Since the interfaces divide the physical domain of the equation, homogenization of equations with interfaces is closely related to homogenization in perforated domains. The study of the latter problem goes back at least to Cioranescu and Saint Jean Paulin [19], and a general framework for periodic perforations was developed by Cioranescu and Murat [18]. The main tool in [19] was the construction of an operator that extends functions to interior of the perforations, which was used also in [36, 22, 21]. Allaire and Murat [3] studied homogenization of Neumann problem on perforated domains without using this extension operator. Recently, Allaire and Habibi [1, 2] studied the interface problem using the two-scale convergence method. In the random setting, homogenization in perforated domain was studied by Zhikov [40]. To our best knowledge, random homogenization of the interface problem (1.1) was first studied by the author with Ammari, Garnier, Giovangigli and Seo [4]. In that paper, we studied both the periodic and the random settings; the former cases was treated using two-scale convergence method, and in the random setting we constructed a random extension operator following the ideas of [19, 36]. The current paper concerns both random interfaces and random coefficients, and hence generalizes our previous result.

Since we have a linear homogenization problem for second order elliptic equations, it is natural to apply the standard oscillating test function method of Tartar; see e.g. Murat and Tartar [37]. The key step is to build oscillating test functions from the auxiliary equation: for any fixed nonzero \( p \in \mathbb{R}^d \), find \( w_p = (w_p^+, w_p^-) \) such that

\[
-\nabla \cdot A(y, \omega) \left( p + \nabla w_p^\pm(y, \omega) \right) = 0, \quad \text{on } \Phi(\mathbb{R}_d^\pm),
\]

(1.4)

with proper transmission conditions across \( \Gamma_d \), where \( \Phi \) is the random deformation and \( \mathbb{R}^d = \mathbb{R}_d^+ \cup \Gamma_d \cup \mathbb{R}_d^- \) represents the decomposition of \( \mathbb{R}^d \) (before the deformation) into the regions \( \mathbb{R}_d^- \) that are enclosed by the interfaces in \( \Gamma_d \) and the region \( \mathbb{R}_d^+ \) that is outside. See Theorem 2.2 for the details. In the periodic case, the above is usually called the
“cell problem” because by periodicity it reduces to a problem posed on the $d$-torus $\mathbb{T}^d$ which is compact, and the natural space for the solution is $H^1_{\text{per}}(Y^+) \times H^1(Y^-)$ which contains functions that are $H^1$ on both $Y^-$ and $Y^+$ and satisfy periodic conditions at the boundary of the unit cell $Y$. This space enjoys a Poincaré inequality and the existence and uniqueness of (2.9) are standard. In the general random case, the solution lives in $H^1_{\text{loc}}(\Phi(\mathbb{R}^d_+)) \times H^1_{\text{loc}}(\Phi(\mathbb{R}^d_-))$ which does not admit any Poincaré inequality. This loss of compactness is a main difficulty in stochastic homogenization. In the standard stationary ergodic setting, e.g. in Papanicolaou and Varadhan [38] and Kozlov [31], where a stationary process $f(x, \omega)$ can be represented as $\tilde{f}(\tau_x \omega)$ for certain measurable function $\tilde{f}$ on $\Omega$ and certain ergodic measure preserving group action $\{\tau_x \mid x \in \mathbb{R}^d\}$, it is possible to lift the auxiliary problem to the probability space, and use Weyl decomposition for $L^2$ vectors in $\Omega$ to solve the problem in probability space; then push this solution back to the spatial space and it solves the auxiliary problem. We refer to the aforementioned references for the details; see also Chapter 7 of [29].

The stationary ergodic setting in this paper, however, is not the standard one, and the program above breaks down or needs careful reformulation. Our strategy is as follows: instead of lifting (1.4) to the probability space, for each realization, we seek for a weak solution to (2.9). We first regularization the problem by adding a small zero-order absorption term, to remedy the lack of Poincaré inequality. Then we solve the regularized problem in the space of locally uniform Sobolev spaces; our method takes inspiration from the work of Gérard-Varet and Masmoudi [26] and that of Dalibard and Frange [20] where boundary layer systems of Navier-Stokes equations in unbounded channels were studied. Usage of locally uniform Sobolev spaces to study problems on unbounded domains was pioneered by Kato [30]. Next we establish averaged (in the probability space) estimates for the regularized solution that are uniform with respect to the regularization parameter. These estimates are obtained for a seemingly weaker formulation of the regularized problem which integrates the probability space, and which we show is equivalence with the former formulation. Sending the regularization to zero, we finally obtain the desired solution to (2.9). Once this is completed, the homogenization theory is proved essentially by the standard oscillating test function method.

The rest of this paper is organized as follows. In Section 2 we make precise the problem settings on the diffusion coefficients and the interfaces and state the main results. In Section 3 we record some preliminary results, which includes ergodic theorems for the special random setting of [14], the random extension operator of [4] and basic energy estimates for (1.1). The proof of our main theorem is detailed in Section 5 using the standard oscillating test function method of Murat and Tartar [37] while the key to the proof, i.e. the study of the auxiliary problem, is in Section 4. Finally, we conclude the paper in Section 6 by showing some properties of the homogenized problem and by discussing some possible further studies.

Notations. We use the standard notation $(\Omega, \mathcal{F}, \mathbb{P})$ for probability spaces: $\omega \in \Omega$ is a realization, and $\mathbb{E}$ is the mathematical expectation with respect to the probability measure $\mathbb{P}$. The standard notations $L^p$ and $W^{k,p}$ are used for the Lebesgue space and the Sobolev space respectively, and $H^k$ is used as a short-hand notation for $W^{k,2}$. The spatial domain for these functional spaces are usually specified. $W^k_{\text{loc}}$ and $W^k_{\text{uloc}}$ denote respectively, local Sobolev and locally uniform Sobolev spaces. When there is no risk of confusion, Einstein’s summation convention is adopted so repeated indices are summed over, e.g. $a_{ij} \xi_i \xi_j =$.
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The Euclidean norm and $|k|_\infty = \max_{1 \leq j \leq d} |k_j|$ denotes the supremum norm. We write $U \subset \subset V$ to mean $U$ is compactly contained in $V$. For any measurable subset $A$ of $\mathbb{R}^d$, $|A|$ denotes its Lebesgue measure. Finally, if $S$ is a smooth $d - 1$ dimensional surface in $\mathbb{R}^d$, $d\sigma$ denotes the standard induced Lebesgue measure on the surface.

2 Problem Settings and Main Results

In this section we first describe the random settings for the elliptic coefficients in $\mathbb{L}^1$ and the interfaces which divide the spatial domain. Then we state the main results of the paper.

2.1 Random ensemble of surfaces

The random medium of this paper, i.e. the random coefficients and the random interfaces in $\mathbb{L}^1$, is obtained as the image of a periodic medium with periodic coefficients and periodic interfaces under a random deformation followed by a rescaling. Hence, we describe the periodic setting first.

**Periodic setting.** Due to periodicity, the medium is determined on the unit cell. This cell is denoted by $Y = [0,1)^d$, the unit cube in $\mathbb{R}^d$. Let $Y^-$ be a simply connected open subset of $Y$ with smooth (say $C^2$) boundary $\partial Y^-$. Hence this boundary is the unit interface and is denoted as $\Gamma_0$. This interface decomposes the unit cell to $Y^-$ and $Y^+ := Y \setminus \overline{Y^-}$.

Therefore, $Y^-$, $\Gamma_0$ and $Y^+$ represent, respectively, the unit interior region, the separating surface and the outer environment. Set $\delta = \text{dist}(\partial Y, \Gamma_0)$ and assume that $\delta \lesssim 1$, that is $\delta$ is smaller than but comparable to one. To build a periodically structured, we set for all $k \in \mathbb{Z}^d$,

$$Y_k = Y + k, \quad Y^+_k = Y^+ + k, \quad Y^-_k = Y^- + k, \quad \Gamma_k = \Gamma_0 + k.$$  

The union of all separating surfaces is then written as $\Gamma_d = \bigcup_{k \in \mathbb{Z}^d} \Gamma_k$. The union of all interior regions enclosed by these surfaces is denoted by $\mathbb{R}^-_d = \bigcup_{k \in \mathbb{Z}^d} Y^-_k$ and the outer environment is characterized as $\mathbb{R}^+_d = \mathbb{R}^d \setminus (\Gamma_d \cup \mathbb{R}^-_d)$, or equivalently $\mathbb{R}^+_d = \bigcup_{k \in \mathbb{Z}^d} Y^+_k$. Clearly, $\Gamma_d$ and $\mathbb{R}^\pm_d$ are periodic. Further, $\mathbb{R}^+_d$ is connected while $\mathbb{R}^-_d$ has simply connected components that are separated by a distance that is at least $2\delta$. The geometry of this periodic structure is shown in Figure 1.

In addition to the geometry of the periodic medium, we specify its physical properties. We assume that the interior region $\mathbb{R}^-_d$ and the environment $\mathbb{R}^+_d$ are filled with a material whose conductivity is characterized by a matrix valued function $A(y) = (\overline{a}_{ij}(y))$. We assume further that $\overline{A}$ is $[0,1)^d$-periodic, $C^2$ and uniformly elliptic, that is

$$\overline{a}_{ij}(y + k) = \overline{a}_{ij}(y), \quad \text{for all } y \in \mathbb{R}^d, k \in \mathbb{Z}^d,$$  

and for some positive constants $\lambda \leq \Lambda$ it holds that

$$\lambda |\xi|_2^2 \leq \overline{a}_{ij}(y)\xi_i\xi_j \leq \Lambda |\xi|_2^2, \quad \text{for all } y \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$  

The surfaces $\Gamma_d$ separate the materials occupying $\mathbb{R}^+_d$ and $\mathbb{R}^-_d$, and the physical properties of the surfaces are described by the transmission condition for the potential fields in $\mathbb{R}^+_d$.
Figure 1: Left: the periodic medium; Middle: the deformed medium; Right: the medium modified near $\partial D$.

and $\mathbb{R}^-_d$ as seen in (1.1). The geometry and the physical properties together complete the periodic model medium with interfaces. This periodic medium is exactly the one studied by Monsurrò and her coauthors in [36, 22, 21].

**Random setting.** Following the idea of Blanc, Le Bris and Lions [14, 15], who considered random diffusive media where the conductivity tensor $A$ is obtained as the image of a periodic tensor $\tilde{A}$ under a random deformation, we construct our random medium, i.e. the conductivity tensor and the conductivity resistant interfaces, by randomly deforming a periodic one. Let $\Phi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be a random orientation preserving diffeomorphism on $\mathbb{R}^d$; that is for each $\omega \in \Omega$, $\Phi(\cdot, \omega)$ is a diffeomorphism on $\mathbb{R}^d$. Then, for the periodic structure $(\Gamma_d, \mathbb{R}^-_d)$ and $\tilde{A}$ defined above, under each realization $\Phi(\cdot, \omega)$, one obtains the deformed structure $\Phi(\mathbb{R}^+_d) \cup \Phi(\Gamma_d) \cup \Phi(\mathbb{R}^-_d)$ with conductivity coefficient $\tilde{A} \circ \Phi^{-1}$. Again, the physical importance of the interfaces will appear as a transmission condition for the potential fields across them. We refer to this medium as the reference random medium. Note that $\Phi(\mathbb{R}^+_d)$ remains connected, and $\Phi(\mathbb{R}^-_d)$ has connected components.

To model the heterogeneous medium whose structure and physical properties vary on a small scale of $\varepsilon$, $0 < \varepsilon \ll 1$, we rescale the reference random medium. This is done by using the scaling operator $\varepsilon Id : \mathbb{R}^d \to \mathbb{R}^d$ given by $y \mapsto \varepsilon y$. Consequently, we obtain a connected environment $\varepsilon \Phi(\mathbb{R}^+_d)$, the separating interfaces $\varepsilon \Phi(\Gamma_d)$ and the interior regions $\varepsilon \Phi(\mathbb{R}^-_d)$. Besides, the materials in $\varepsilon \Phi(\mathbb{R}^+_d)$ have conductivity coefficient $A^\varepsilon := \tilde{A} \circ \Phi^{-1}(\cdot/\varepsilon)$.

Finally, in the open bounded domain $D$ on which (1.1) is posed, we would like to set $D^\varepsilon_+ = D \cap \varepsilon \Phi(\mathbb{R}^+_d)$ and $\Gamma^\varepsilon = D \cap \varepsilon \Phi(\Gamma_d)$. However, $\partial D \cap \varepsilon \Phi(\Gamma_d)$ may not be empty. In other words, the boundary $\partial D$ may cut certain components of $\varepsilon \Phi(\Gamma_d)$. In [4], (1.1) models diffusion phenomena in a suspension of cells and we would like to avoid the cells being cut by the boundary of the domain. This requires a modification of the above proposal of $D^\varepsilon_+$ near the boundary of $D$. In this paper and with this biological application in mind, we keep this constraint though it can be removed as long as the intersection of $\partial D$ and the interfaces makes sense in the physical application. We provide the details of this modification in the next subsection under some assumptions on the diffeomorphism $\Phi(\cdot, \omega)$. 
2.2 Stationary and ergodic deformations

Let $\Phi$ be the aforementioned random orientation preserving diffeomorphism of $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathcal{F}, P)$. Throughout the paper, we assume that $\mathcal{F}$ is countably generated so that $L^2(\Omega)$ is separable. We assume further that the probability space has the following structure.

(S1) The group $(\mathbb{Z}^d, +)$ acts on $\Omega$ by some action $\{\tau_k : \Omega \to \Omega\}_{k \in \mathbb{Z}^d}$. For all $k \in \mathbb{Z}^d$, the map $\tau_k$ is $P$-preserving, i.e. $P(\tau_k A) = P(A)$ for all $A \in \mathcal{F}$.

(S2) The group action above is ergodic, i.e. $A \in \mathcal{F}$ and $\tau_k A = A$ for all $k \in \mathbb{Z}^d$ implies that $P(A) \in \{0, 1\}$.

In this paper, we say that a locally integrable random process $F \in L^1_{\text{loc}}(\mathbb{R}^d, L^1(\Omega))$ is stationary if for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$ we have

$$\forall k \in \mathbb{Z}^d, \quad F(x + k, \omega) = F(x, \tau_k \omega).$$

(2.3)

This notion of stationarity for random processes is different from the standard one e.g. in [29, 31, 38] where the action $\{\tau_z\}_{z \in \mathbb{R}^d}$ is used and (2.3) should be satisfied for all $k \in \mathbb{R}^d$. It is known that neither of the two notions is a special case of the other; nevertheless, both notions include periodic functions as a special case. Nevertheless, it is well known that, e.g. as shown in [14, 15], such stationary processes still enjoy certain types of ergodic theorems.

The main assumptions on the random diffeomorphism $\Phi$ are: for all $\omega \in \Omega$,

(A1) The random field $\nabla \Phi(y, \omega)$ is stationary.

(A2) There exists a constant $\mu$ such that $\inf_{y \in \mathbb{R}^d} \det(\nabla \Phi(y, \omega)) \geq \mu > 0$.

(A3) There exists a constant $M$ such that $\sup_{y \in \mathbb{R}^d} |\nabla \Phi(y, \omega)| \leq M < \infty$.

We call any $\Phi$ satisfying the above conditions a stationary random diffeomorphism. Let $\Psi$ be the inverse of $\Phi$. Then (A2) and (A3) implies that for all $\omega \in \Omega$,

$$\sup_{x \in \mathbb{R}^d} |\nabla \Psi(x, \omega)| \leq M' < \infty, \quad \inf_{x \in \mathbb{R}^d} \det(\nabla \Psi(x, \omega)) \geq \mu' > 0.$$  

(2.4)

Here, $\mu'$ and $M'$ are two constants depending on $\mu, M$ and the dimension $d$ but not on $\omega$.

By the assumption (A3), for any two points $y_1, y_2 \in \mathbb{R}^d$, we have

$$|\Phi(y_1) - \Phi(y_2)| = \left| \int_0^{y_1 - y_2} D\Phi \left( y_1 + s \frac{y_1 - y_2}{|y_1 - y_2|} \right) \right| ds \leq M |y_1 - y_2|,$$

and similarly by (2.4), we have

$$|\Phi(y_1) - \Phi(y_2)| \geq (M')^{-1} |y_1 - y_2|.$$

These estimates indicate that in the reference random medium, the interfaces are still well separated at least by a distance of $2\delta/M'$. After the rescaling, the interfaces in $\Gamma_{\varepsilon}$ are well separated by a distance of $2\varepsilon\delta/M'$. 

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Now we construct $\Gamma_\varepsilon$ more carefully so that $\partial D$ does not cut any component of $\Gamma_\varepsilon$. For each $\omega \in \Omega$, let $\overline{D}_{\varepsilon-1} = \Phi^{-1}([\varepsilon])$ be the preimage of $D$ under the map $\varepsilon \Phi(\cdot, \omega)$. Let $\overline{D}'_{\varepsilon-1}$ be its subset that is $\delta$ away from the boundary, i.e.

$$\overline{D}'_{\varepsilon-1} = \{ z \in \overline{D}_{\varepsilon-1} \mid \text{dist}(z, \partial \overline{D}_{\varepsilon-1}) \geq \delta \}.$$ 

Let $I_\varepsilon \subset \mathbb{Z}^d$ be the indices of the cubes inside $\overline{D}'_{\varepsilon-1}$, i.e. those of $\{Y_k \subset \overline{D}'_{\varepsilon-1}\}$. Then we set

$$\Gamma_\varepsilon = \sum_{k \in I_\varepsilon} \varepsilon \Phi(\Gamma_k), \quad D^-_\varepsilon = \sum_{k \in I_\varepsilon} \varepsilon \Phi(Y_k^\ast), \quad D^+_\varepsilon = D \setminus \overline{\Gamma}_\varepsilon. \quad (2.5)$$

That is, we only keep the deformed and rescaled cells that are inside $D$ and have a distance at least $\varepsilon \delta / M'$ away from the boundary. We also define the following two subsets of $D$:

$$E_\varepsilon = \sum_{k \in I_\varepsilon} \varepsilon \Phi(Y_k), \quad K_\varepsilon = D \setminus \overline{E}_\varepsilon. \quad (2.6)$$

The set $E_\varepsilon$ encloses all the $\varepsilon$-scale interfaces in $\Gamma_\varepsilon$, the region inside these surfaces, i.e. $D^-$ and their immediate surroundings $\cup_{k \in I_\varepsilon} \varepsilon \Phi(Y_k^\ast)$. The set $K_\varepsilon$ can be thought as a cushion layer close to the boundary that prevents the interfaces from touching the boundary. From the construction we verify that

$$\inf_{x \in D^-_\varepsilon} \text{dist}(x, \partial D) \geq \varepsilon \delta / M', \quad \text{and} \quad \sup_{x \in K_\varepsilon} \text{dist}(x, \partial D) \leq \varepsilon \delta \sqrt{dM}. \quad (2.7)$$

Hence, the interfaces $\Gamma_\varepsilon$ are separated from $\partial D$ and the cushion layer is restricted to a vicinity of $\partial D$ whose thickness is comparable to $\varepsilon$.

**Remark 2.1.** We provide some examples. First if $\Phi = \text{Id}$ is the identity operator, we recover the periodic setting. If $\Phi$ is a deterministic diffeomorphism, we obtain a deterministic medium. For a less trivial example, let $X = \{X_k \mid k \in \mathbb{Z}^d\}$ be the set of i.i.d. Bernoulli variables with indices in $\mathbb{Z}^d$, i.e. each $X_k$ is either 0 or 1 with probability $\frac{1}{2}$. Set the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be the canonical space for the random process $X$. That is, $\Omega = \{0, 1\}^{\mathbb{Z}^d}$; $\mathcal{F}$ is the Borel $\sigma$-algebra generated by finite dimensional cylindrical sets in $\Omega$ and $\mathbb{P}$ is defined by setting, for any $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \mathbb{P}_0\{X \in A\}$$

where $\mathbb{P}_0$ is the underlying probability measure associated to the Bernoulli sequence. We then check that the group $\{\tau_k \mid k \in \mathbb{Z}^d\}$ which acts on $\Omega$ by

$$\tau_kX = \tau_k\{X_\ell \mid \ell \in \mathbb{Z}^d\} = \{X_{\ell+k} \mid \ell \in \mathbb{Z}^d\}$$

is measure preserving and ergodic.

Now consider two $C^2$ functions $\Phi_{0,1} : Y \to Y$ given by: $\Phi_0 = \text{Id}$; $\Phi_1 - \text{Id} \neq 0$ and $\Phi_1 - \text{Id}$ is compactly supported in $Y$, $|\nabla \Phi_1| \leq M$ and $\det(\nabla \Phi_1) \geq \mu$. For each $\omega \in \Omega$, i.e. for each $X = \{X_k \mid k \in \mathbb{Z}^d\}$ and for each $x \in \mathbb{R}^d$, with $[x]$ denoting the unique number in $\mathbb{Z}^d$ such that $x - [x] \in [0, 1)^d$, we set

$$\Phi(x, \omega) = [x] + \Phi_{X_{[x]}}(x - [x]).$$

Then $\Phi(x, \omega)$ is a random diffeomorphism satisfying the aforementioned conditions. One checks that for each cube $Y_k$, $\Phi$ leaves its boundary unchanged and may or may not deform its interior according to the outcome of the Bernoulli variable $X_k$. 


2.3 The main results

Assumptions. Throughout the rest of this paper, we assume that the random coefficients $A^\varepsilon$, the random surfaces $\Gamma_\varepsilon(\omega)$, and the decomposition of $D$ into $D^+_\varepsilon$ and $D^-_\varepsilon$ in (1.1) are constructed as in Section 2.1. In particular, the unscaled coefficient $A$ and the unit interface $\Gamma_0$ are $C^2$. We assume that $f \in L^2(D)$ in (1.1). Further, the assumptions (S1)(S2) on the probability space, and the assumptions (A1)(A2)(A3) on the random diffeomorphism $\Phi$ are invoked.

Due to the jump type transmission condition across the interfaces $\Gamma_\varepsilon$, the solution $u_\varepsilon$ that solves (1.1) are piecewisely defined as $u^+_\varepsilon$ on $D^+_\varepsilon$ and $u^-_\varepsilon$ on each components of $D^-_\varepsilon$. A natural space for the solution is $H^1(D^+_\varepsilon(\omega)) \times H^1(D^-_\varepsilon(\omega))$. As a result, the functional space on which the solutions are defined depends on both $\varepsilon$ and $\omega$. This poses some difficulty on making sense of the convergence of $u^\pm_\varepsilon$. Hence for $u^+_\varepsilon$ we introduce certain extension $u^+_\varepsilon(\cdot,\omega) \mapsto u^{\text{ext}}_\varepsilon(\cdot,\omega)$ where the latter function belongs to $H^1(D)$ and agrees with $u^+_\varepsilon$ on $D^+_\varepsilon$; see Proposition 3.7 below. For $u^-_\varepsilon$, we take the trivial extension $u^-_\varepsilon(\cdot,\omega) \mapsto Q u^-_\varepsilon(\cdot,\omega)$ where the latter belongs to $L^2(D)$ and vanishes on $D^+_\varepsilon$.

The main result of this paper is the almost sure homogenization of the problem (1.1). We first introduce two quantities that appear in the presentation of the homogenized problem. They are $q$, the mean volume of the unit cube $Y$ after deformation, and $\theta$, the mean volume fraction of $Y^-$ after deformation. They are given by

\begin{align*}
q &= \mathbb{E} \int_Y \det \nabla \Phi(z) dz = \mathbb{E} |\Phi(Y,\omega)|, \\
\theta &= \frac{1}{q} \mathbb{E} \int_{Y^-} \det \nabla \Phi(z) dz = \frac{\mathbb{E} |\Phi(Y^-,\omega)|}{\mathbb{E} |\Phi(Y,\omega)|} 
\end{align*}

Due to the assumptions on $\Phi$, we verify that $0 < \theta < 1$.

Before stating the main homogenization theorem of (1.1), we present the key result that it relies on; namely the existence of a solution to the following auxiliary problem, which is the analog of the “cell problem” in the periodic case. We recall that from our construction, $A = \tilde{A} \circ \Psi$ where $\tilde{A}$ is the periodic coefficient, and $\Psi$ is the inverse of the random diffeomorphism. We have

**Theorem 2.2.** For a.e. $\omega \in \Omega$ and for each fixed $p \in \mathbb{R}^d$, there exists a function $w_p = \tilde{w}_p \circ \Psi$ with $\tilde{w}_p \in H^1_{\text{loc}}(\mathbb{R}^+_d) \times H^1_{\text{loc}}(\mathbb{R}^-_d)$, and $w_p$ is a solution to the following problem

\begin{align}
\begin{cases}
-\nabla \cdot (A(y,\omega) \nabla [w^+_p(y,\omega) + p \cdot y]) = 0, & \text{for } y \in \Phi(\mathbb{R}^+_d,\omega), \\
\frac{\partial}{\partial \nu_A} w^+_p(y,\omega) = \frac{\partial}{\partial \nu_A} w^-_p(y,\omega), & \text{for } y \in \Phi(\Gamma_d,\omega), \\
w^+_p(y,\omega) - w^-_p(y,\omega) = \frac{\partial}{\partial \nu_A} w^+_p(y,\omega) + \nu_y \cdot Ap, & \text{for } y \in \Phi(\Gamma_d,\omega).
\end{cases}
\end{align}

In addition, $\tilde{w}^+_p(\cdot,\omega)$ admits an extension $\tilde{w}^\text{ext}_p(\cdot,\omega) \in L^2(\Omega, H^1_{\text{loc}}(\mathbb{R}^d))$ which satisfies that $\nabla \tilde{w}^\text{ext}_p$ is stationary, $\nabla \tilde{w}^\text{ext}_p = P (\nabla \tilde{w}^+_p)$ where $P$ is the extension operator of Proposition 2.3, and

\begin{align*}
\mathbb{E} \int_Y \nabla \tilde{w}^\text{ext}_p(y,\omega) dy = 0.
\end{align*}

Moreover, the solution $w_p(\cdot,\omega)$ is unique up to an additive constant $C(\omega)$. 

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Note that this problem is posed on the whole space \( \Phi(\mathbb{R}^+, \omega) \cup \Phi(\Gamma_d, \omega) \cup \Phi(\mathbb{R}^-, \omega) \), and there are infinitely many interfaces in \( \Phi(\Gamma_d, \omega) \). As explained earlier, due to the lack of compactness, this auxiliary problem is arguably more difficult to deal with than the cell problem in the periodic setting.

Finally, our main theorem on the homogenization of the random interface problem is

**Theorem 2.3.** Let \( A^0 = (a^0_{ij}) \) be a deterministic constant matrix defined as

\[
a^0_{ij} := \frac{1}{\varrho} \mathbb{E} \left( \int_{\Phi(Y^+, \omega)} e_j \cdot A(e_i + \nabla w_{\varepsilon_1}^+) dx + \int_{\Phi(Y^-, \omega)} e_j \cdot A(e_i + \nabla w_{\varepsilon_1}^-) dx \right).
\]

Let \( u_0 \in H^1(D) \) be the unique solution to (1.3). There exists a subset \( \Omega_\ast \subset \Omega \) with full probability measure, and for each \( \omega \in \Omega_\ast \), the sequence of unique solutions \( u_\varepsilon(\cdot, \omega) \) to (1.1) satisfy that as \( \varepsilon \to 0 \),

(i) The function \( u_\varepsilon \) converges strongly in \( L^2(D) \) to \( u_0 \).

(ii) The extended function \( u_\varepsilon^{\text{ext}}(\cdot, \omega) \in H^1_0(D) \) of \( u_\varepsilon(\cdot, \omega) \) given by the extension operator of Proposition 3.8 converges weakly in \( H^1(D) \) to \( u_0 \).

(iii) The trivial extension \( Q u_\varepsilon^-(\cdot, \omega) \) of \( u_\varepsilon^-(\cdot, \omega) \) converges weakly in \( L^2(D) \) to \( \theta u_0 \).

(iv) The flux \( \chi_{D^+} A^0 \nabla u_\varepsilon^+ + \chi_{D^-} A^0 \nabla u_\varepsilon^- \) converges weakly in \( [L^2(D)]^d \) to the homogenized flux \( A^0 \nabla u_0 \).

The homogenized equation (1.3) indeed has unique solution; in fact one can verify that the homogenized conductivity \( A^0 \) is uniformly elliptic; see Section 6.

# 3 Preliminary Results

In this section, we present several preliminary results used later. These include properties of ergodic processes in the sense of (2.3), functional spaces defined on “perforated” domains and extension operators, and some basic energy estimates for (1.1).

## 3.1 Stationary ergodic random processes

Most results from this subsection are extracted from the works of Blanc, Le Bris and Lions [14, 15]. As mentioned earlier, the notion of stationarity in this paper is different from the standard one, e.g. in [38, 31, 29]. Nevertheless, the following version of ergodic theorems (see e.g. Dunford and Schwartz [23] and Krengel [33]) hold.

**Proposition 3.1.** (i) Let \( F \in L^\infty(\mathbb{R}^d, L^1(\Omega)) \) be a stationary random process. Then

\[
\frac{1}{(2N+1)^d} \sum_{|k|_{\infty} \leq N} F(\cdot, \tau_k \omega) \xrightarrow{L^\infty_{\mathbb{P}}} \mathbb{E} F(\cdot, \omega), \quad \text{for a.e. } \omega \in \Omega.
\]

This implies that

\[
F\left(\frac{z}{\varepsilon}, \omega\right) \xrightarrow{\varepsilon \to 0, \ L^\infty \text{ weak-*}} \mathbb{E} \left( \int_{Y} F(z, \cdot) dz \right), \quad \text{for a.e. } \omega \in \Omega.
\]

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(ii) For \( p \in (1, \infty) \), suppose \( F \in L^p_{\text{loc}}(\mathbb{R}^d, L^1(\Omega)) \) is a stationary random process, then
the above convergence results still hold if we replace \( L^\infty \) weak-* by \( L^p_{\text{loc}} \) weak.

Since we mainly deal with functions on the deformed space, we will encounter functions which are not stationary themselves but their preimage before the deformation is. Such a function can be written as \( g \circ \Psi(y, \omega) \) where \( g \) is stationary. We have the following result.

**Lemma 3.2.** Let \( g \in L^p_{\text{loc}}(\mathbb{R}^d, L^1(\Omega)) \), for some \( p \in (1, \infty) \), be a stationary process. Then we have
\[
g \left( \frac{x}{\varepsilon}, \omega \right) \xrightarrow{L^p_{\text{loc}}, \varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \left( \int_{\Phi(y, \cdot)} g \circ \Psi(y, \cdot) dy \right) \quad \text{for a.e. } \omega \in \Omega. \quad (3.3)
\]

When \( p = \infty \), the convergence above holds provided that \( L^p_{\text{loc}} \) is replaced by \( L^\infty \) and the weak convergence is replaced by weak-* convergence.

The case \( p = \infty \) was proved by Blanc, Le Bris and Lions [14]. The proof for the general case is essentially the same. We provide it here for the sake of completeness.

**Proof.** For any \( p \in (1, \infty] \), let \( p' \) be the Hölder conjugate of \( p \). In view of the density of simple functions in \( L^p \), and the regularity of measurable sets in \( \mathbb{R}^d \), it suffices to show that for any \( A \subset \subset \mathbb{R}^d \)
\[
\int_A g \circ \Psi \left( \frac{x}{\varepsilon}, \omega \right) dx = \int_{\varepsilon \Phi^{-1}(A)} g \left( \frac{z}{\varepsilon}, \omega \right) \det \left( \nabla \Phi \left( \frac{z}{\varepsilon}, \omega \right) \right) dx \xrightarrow{\varepsilon \to 0} \frac{|A|}{\varepsilon} \mathbb{E} \int_{\Phi(y)} g \circ \Psi(y, \omega) dy.
\]

It is proved in Lemma 2.1 of [14] that \( \varepsilon \Phi^{-1} \left( \frac{x}{\varepsilon}, \omega \right) \) converges to \( \mathbb{E} \left[ \int_Y \nabla \Phi(y, \cdot) dy \right]^{-1} x \) locally uniformly as \( \varepsilon \to 0 \), which implies that the indicator of the set \( \varepsilon \Phi^{-1} \left( \frac{x}{\varepsilon} \right) \) converges strongly in \( L^{p'} \) to that of \( \mathbb{E} \left[ \int_Y \nabla \Phi(y, \cdot) dy \right]^{-1} A \). Now Lemma 3.2 says
\[
g \left( \frac{z}{\varepsilon}, \omega \right) \det \left( \nabla \Phi \left( \frac{z}{\varepsilon}, \omega \right) \right) \xrightarrow{L^p \text{ weak}, \varepsilon \to 0} \mathbb{E} \int_Y g(z, \omega) \det(\nabla \Phi(z, \omega)) dz = \mathbb{E} \int_{\Phi(y)} g \circ \Psi(y, \omega) dy.
\]

When \( p = \infty \), the weak topology above should be replaced by weak-*. As a result, we have
\[
\int_A g \circ \Psi \left( \frac{x}{\varepsilon}, \omega \right) dx \xrightarrow{\varepsilon \to 0} \mathbb{E} \int_{\Phi(y)} g \circ \Psi(y, \omega) dy.
\]

In particular, if we set \( g \equiv 1 \), we get
\[
\left( \det \mathbb{E} \left[ \int_Y \nabla \Phi(y, \cdot) dy \right] \right)^{-1} \left( \mathbb{E} \int_{\Phi(y)} 1 dy \right) = \left( \det \mathbb{E} \left[ \int_Y \nabla \Phi(y, \cdot) dy \right] \right)^{-1} \varepsilon = 1.
\]

Substitute this relation to the preceding equation; we obtain the desired result. \( \Box \)
Another useful fact about random processes with stationary gradients is that they grow sublinearly at infinity. We state this result in the following lemma, which can be proved following the same argument of Lemma A.5 in [6]; see also Theorem 9 in [32].

Lemma 3.3. Suppose that \( w : \mathbb{R}^d \times \Omega \to \mathbb{R} \) and for almost every \( \omega \in \Omega \), \( G = Dw \) in the sense of distribution. Assume that for some \( \alpha > d \), \( w(\cdot, \omega) \) belongs to \( W^{1,\alpha}_\text{loc}(\mathbb{R}^d) \). Suppose \( G \) is stationary and satisfies

\[
\mathbb{E} \int_Y G(z, \cdot) dz = 0, \quad \text{and} \quad \mathbb{E} \int_Y |G(z, \cdot)|^\alpha dz < \infty. \tag{3.4}
\]

Then

\[
\lim_{|y| \to \infty} |y|^{-1} w(y, \omega) = 0, \quad \text{for a.e. } \omega \in \Omega. \tag{3.5}
\]

3.2 Extension Lemmas

We record in this section some extension operators for functions defined on \( \mathbb{R}^+ \times \mathbb{R}^- \), \( \Phi(\mathbb{R}^+_{d}) \times \Phi(\mathbb{R}^-_{d}) \), and \( \varepsilon \Phi(\mathbb{R}^+_{d}) \times \varepsilon \Phi(\mathbb{R}^-_{d}) \). The starting point is to introduce extension operators for functions defined on \( Y^+ \times Y^- \). We have

Theorem 3.4. Let \( Y^+, Y^- \) and \( \Gamma_0 \) be as defined in Section 2.1. Then there exists an extension operator \( P : W^{1,p}(Y^+) \to W^{1,p}(Y) \) for all \( p \geq 1 \), and a constant \( C = C(d, p, \Gamma_0) \) such that, for any \( p \geq 1 \) and any \( f \in W^{1,p}(Y^+) \), we have

\[
\| Pf \|_{L^p(Y)} \leq C \| f \|_{L^p(Y^+)}, \quad \| \nabla P f \|_{L^p(Y)} \leq C \| \nabla f \|_{L^p(Y^+)}. \tag{3.6}
\]

This theorem was first proved by Cioranescu and Saint Jean Paulin [19]; see also the book of Zhikov, Kozlov and Oleinik [29]. The extension operator \( P \) is given by

\[
P f = \frac{1}{|Y^+|} \int_{Y^+} f dx + E \left( f - \frac{1}{|Y^+|} \int_{Y^+} f dx \right), \tag{3.7}
\]

where \( E \) is the more standard extension operator for Sobolev functions on bounded domain; see e.g. Section 5.4 of [24]. The subtraction of the averaged value of \( f \) over \( Y^+ \) is needed to have the first inequality. In fact, for the standard extension operator \( E \), one only has \( \| E f \|_{W^{1,p}} \leq C \| f \|_{W^{1,p}} \), which is not good enough for scaling as we will do shortly.

For functions in \( W^{1,p}_\text{loc}(\mathbb{R}^+_{d}) \), we apply the extension operator \( P \) above in each cube \( Y_k \), \( k \in \mathbb{Z}^d \). Then we obtain an extension operator from \( W^{1,p}_\text{loc}(\mathbb{R}^+_{d}) \) to \( W^{1,p}(\mathbb{R}^d) \). Denote this extension operator still by \( P \). Evidently, we have

Proposition 3.5. Let \( P : W^{1,p}_\text{loc}(\mathbb{R}^+_{d}) \to W^{1,p}_\text{loc}(\mathbb{R}^d) \) be defined as above. Then for the same \( C \) as in Theorem 3.4 and for any \( K \subset \subset \mathbb{R}^d \), we have

\[
\| P f \|_{L^p(K)} \leq C \| f \|_{L^p(K \cap \mathbb{R}^+_{d})}, \quad \| \nabla P f \|_{L^p(K)} \leq C \| \nabla f \|_{L^p(K \cap \mathbb{R}^+_{d})}. \tag{3.8}
\]

Fix an \( \omega \in \Omega \), for any \( f(\cdot, \omega) \in W^{1,p}_\text{loc}(\Phi(\mathbb{R}^+_{d})) \), we define \( P_\omega f \) as

\[
(P_\omega f)(\cdot, \omega) = [P (f \circ \Phi)] \circ \Phi^{-1}(\cdot, \omega). \tag{3.9}
\]
Proposition 3.6. Let \( P_\omega : W^{1,p}_{\text{loc}}(\Phi(\mathbb{R}^d_\varepsilon, \omega)) \to W^{1,p}_{\text{loc}}(\mathbb{R}^d) \) be defined as above. Then there exists some \( C = C(d, p, Y^-, M, \mu) \), which is independent of \( \omega \), such that for any \( K \subset \subset \mathbb{R}^d \), we have
\[
\|P_\omega f\|_{L^p(\Phi(K, \omega))} \leq C\|f\|_{L^p(\Phi(K \cap \mathbb{R}^d_\varepsilon, \omega))},
\]
\[
\|\nabla P_\omega f\|_{L^p(\Phi(K, \omega))} \leq C\|\nabla f\|_{L^p(\Phi(K \cap \mathbb{R}^d_\varepsilon, \omega))}.
\] (3.10)

For a proof of this result, we refer to Appendix A of \([3]\). The constant \( C \) above can be made independent of \( \omega \) because the bounds in (A2)(A3) and (2.4) are uniform in \( \omega \). Next, we consider functions defined on the scaled space. For any \( \varepsilon \in \Omega \) and \( f \in \varepsilon \Phi(\mathbb{R}^d_\varepsilon) \), define
\[
(P_\omega^\varepsilon f)(\cdot, \omega) = (P_\omega f_\varepsilon(\cdot, \omega)) \left( \frac{\cdot}{\varepsilon} \right), \quad \text{where} \quad f_\varepsilon(x, \omega) = f(\varepsilon x, \omega).
\] (3.11)

Then we have

Proposition 3.7. Let \( P_\omega^\varepsilon : W^{1,p}_{\text{loc}}(\varepsilon \Phi(\mathbb{R}^d_\varepsilon, \omega)) \to W^{1,p}_{\text{loc}}(\mathbb{R}^d) \) be defined as above. Then there exists some \( C = C(d, p, Y^-, M, \mu) \), which is independent of \( \omega \), such that for any \( K \subset \subset \mathbb{R}^d \), we have
\[
\|P_\omega^\varepsilon f\|_{L^p(\varepsilon \Phi(K, \omega))} \leq C\|f\|_{L^p(\varepsilon \Phi(K \cap \mathbb{R}^d_\varepsilon, \omega))},
\]
\[
\|\nabla P_\omega^\varepsilon f\|_{L^p(\varepsilon \Phi(K, \omega))} \leq C\|\nabla f\|_{L^p(\varepsilon \Phi(K \cap \mathbb{R}^d_\varepsilon, \omega))}.
\] (3.12)

Finally, recall the decomposition of \( D \) in (2.5). We can extend a function \( f \in W^{1,p}(D^+_\varepsilon) \) to \( P_\omega^\varepsilon f \in W^{1,p}(D) \) by using the extension operator \( P_\omega^\varepsilon \) in (3.11) on each of the deformed and rescaled cubes \( \varepsilon \Phi(Y_k, \omega) \) in \( E_\varepsilon \), while leaving the function unchanged in the cushion layer \( K_\varepsilon \). Abusing notations, we denote this operator still by \( P_\omega^\varepsilon \). Then

Proposition 3.8. Let \( P_\omega^\varepsilon : W^{1,p}(D^+_\varepsilon) \to W^{1,p}(D) \) be as above. Then there exists some \( C = C(d, Y^-, M, \mu) \) such that, for all \( f \in W^{1,p}(D^+_\varepsilon) \),
\[
\|P_\omega^\varepsilon f\|_{L^p(D)} \leq C\|f\|_{L^p(D^+_\varepsilon)},
\] (3.13)

For the proofs of Propositions 3.7 and 3.8 we refer to Appendix A of \([3]\). The periodic versions of these propositions were developed by Monsurrò \([36]\).

### 3.3 Basic energy estimates

Here we record some basic energy estimates for the solutions of (1.1).

#### 3.3.1 Functional space on the perforated domain in \( D \)

Fix an \( \varepsilon > 0 \) and a realization \( \omega \in \Omega \), the natural functional space for (1.1) is
\[
W_\varepsilon := \{ u = u^+ \chi_\varepsilon^+ + u^- \chi_\varepsilon^- : u^+ \in H^1(D^+_\varepsilon), u^- \in H^1(D^-_\varepsilon), u|_{\partial D} = 0 \},
\] (3.14)
where \( \chi_\varepsilon^\pm \) denote the characteristic functions of the sets \( D^\pm_\varepsilon(\omega) \), and \( u|_{\partial D} \) is the trace of \( u \) on \( \partial D \). It is easy to verify that
\[
\|u\|_{W_\varepsilon} = \left(\|\nabla u^+\|^2_{L^2(D^+_\varepsilon)} + \|\nabla u^-\|^2_{L^2(D^-_\varepsilon)} + \varepsilon\|u^+ - u^-\|^2_{L^2(\Gamma_\varepsilon)}\right)^{\frac{1}{2}}
\] (3.15)
defines a norm on $\mathcal{W}_\varepsilon$. Abusing notations, we set $H^1_0(D_\varepsilon^+) := \{ w \in H^1(D_\varepsilon^+) | u|_{\partial D} = 0 \}$, that is $H^1$ functions on $D_\varepsilon^+$ that vanish at the boundary of $\partial D$. Then $\mathcal{W}_\varepsilon = H^1_0(D_\varepsilon^+) \times H^1(D_\varepsilon^-)$, and in view of the Poincaré inequality (3.18), the somewhat more standard norm for this space is given by

$$
\| u \|_{H^1_0(D_\varepsilon^+) \times H^1(D_\varepsilon^-)} = \left( \| \nabla u^+ \|_{L^2(D_\varepsilon^+)}^2 + \| \nabla u^- \|_{L^2(D_\varepsilon^-)}^2 + \| u^- \|_{L^2(D_\varepsilon^-)}^2 \right)^{\frac{1}{2}}. \tag{3.16}
$$

In fact, these two norms are equivalent:

**Proposition 3.9.** The norm $\| \cdot \|_{\mathcal{W}_\varepsilon}$ is equivalent with the standard norm in (3.16). Moreover, there exist positive constants $C_1 < C_2$, independent of $\varepsilon$ and $\omega$, such that for all $u \in \mathcal{W}_\varepsilon$, we have

$$
C_1 \| u \|_{\mathcal{W}_\varepsilon} \leq \| u \|_{H^1_0(D_\varepsilon^+) \times H^1(D_\varepsilon^-)} \leq C_2 \| u \|_{\mathcal{W}_\varepsilon}. \tag{3.17}
$$

This equivalence relation was established by Monsurrò [36] in the periodic setting, and in the random setting it was proved in [3]. Some additional properties of the functions in $\mathcal{W}_\varepsilon$ are recorded below; we refer to [3, Appendix C] for the proof, and to [36] for similar results in the periodic setting.

**Proposition 3.10.** There exists some constant $C > 0$, which is independent of $\varepsilon$ and $\omega$, such that for all $v \in \mathcal{W}_\varepsilon$, we have

$$
\begin{align*}
\| v^+ \|_{L^2(D_\varepsilon^+)} & \leq C \| \nabla v^+ \|_{L^2(D_\varepsilon^+)} , \tag{3.18} \\
\| v^- \|_{L^2(D_\varepsilon^-)} & \leq C \left( \sqrt{\varepsilon} \| v^- \|_{L^2(D_\varepsilon^-)} + \varepsilon \| \nabla v^- \|_{L^2(D_\varepsilon^-)} \right) , \tag{3.19} \\
\| v \|_{L^2(D)} & \leq C \| v \|_{\mathcal{W}_\varepsilon} . \tag{3.20}
\end{align*}
$$

3.3.2 The energy estimates

With $\varepsilon > 0$ and $\omega \in \Omega$ fixed as before, a function $u_\varepsilon \in \mathcal{W}_\varepsilon$ is said to be a weak solution to (1.1) if for all $v = v^+\chi_\varepsilon^+ + v^-\chi_\varepsilon^- \in \mathcal{W}_\varepsilon$, the following holds:

$$
\int_{D_\varepsilon^+} A^\varepsilon(x, \omega) \nabla u_\varepsilon^+(x) \cdot \nabla v^+(x) dx + \int_{D_\varepsilon^-} A^\varepsilon(x, \omega) \nabla u_\varepsilon^-(x) \cdot \nabla v^-(x) dx
\quad + \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} (u_\varepsilon^+ - u_\varepsilon^-)(v^+ - v^-) d\sigma(x) = \int_D f(x)v(x) dx. \tag{3.21}
$$

By the standard Lax-Milgram theorem, one obtains the existence and uniqueness of the weak solution to (1.1) and some basic energy estimates.

**Proposition 3.11.** Let $f \in L^2(D)$. There exists a unique weak solution $u_\varepsilon \in \mathcal{W}_\varepsilon$ for (1.1). Moreover, assume that $\varepsilon \leq 1/\sqrt{2}$ there exists a constant $C$, which is independent of $\varepsilon$ and $\omega$, such that

$$
\begin{align*}
\| \nabla u_\varepsilon^+ \|_{L^2(D_\varepsilon^+)} + \| \nabla u_\varepsilon^- \|_{L^2(D_\varepsilon^-)} & \leq C \| f \|_{L^2} , \tag{3.22} \\
\| u_\varepsilon^+ - u_\varepsilon^- \|_{L^2(\Gamma_\varepsilon)} & \leq C \sqrt{\varepsilon} \| f \|_{L^2} . \tag{3.23}
\end{align*}
$$
\textbf{Proof.} For the existence and uniqueness of weak solution, define the bilinear form $A^\varepsilon(\cdot,\cdot)$ on $W_\varepsilon \times W_\varepsilon$ and the linear form $\ell$ on $W_\varepsilon$ by

$$A^\varepsilon(u,v) := \int_{D^+_\varepsilon} A^\varepsilon \nabla u^+ \cdot \nabla v^+ dx + \int_{D^-_\varepsilon} A^\varepsilon \nabla u^- \cdot \nabla v^- dx + \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} (u^+ - u^-)(v^+ - v^-) d\sigma(x),$$

$$\ell(v) := \int_D f(x) v(x) dx.$$  

It is clear that $A^\varepsilon$ and $\ell$ are bounded operators. Moreover, due to (2.2), we have

$$A^\varepsilon(u,u) \geq \left( \int_{D^+_\varepsilon} \lambda |\nabla u^+|^2 dx + \int_{D^-_\varepsilon} \lambda |\nabla u^-|^2 dx \right) + \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} |u^+ - u^-|^2 d\sigma(x) \geq \min(\lambda,1) \|u\|^2_{W_\varepsilon}.$$  

This shows that $A^\varepsilon$ is coercive. By Lax–Milgram theorem there exists a unique $u_\varepsilon \in W_\varepsilon$ such that $A^\varepsilon(u_\varepsilon,v) = \ell(v)$ for all $v \in W_\varepsilon$, i.e. $u_\varepsilon$ is the weak solution to (1.1).

To obtain the energy estimates, we take $v = u_\varepsilon$ in (3.21). In view of the ellipticity of $A^\varepsilon$ and (3.20), we have

$$\lambda \left( \|\nabla u^+_\varepsilon\|_{L^2(D^+_\varepsilon)}^2 + \|\nabla u^-_\varepsilon\|_{L^2(D^-_\varepsilon)}^2 \right) + \frac{1}{\varepsilon} \|u^+_\varepsilon - u^-_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2 \leq \|f\|_{L^2(D)} \|u_\varepsilon\|_{L^2(D)}$$

$$\leq C \|f\|_{L^2(D)} \left( \|\nabla u^+_\varepsilon\|_{L^2(D^+_\varepsilon)}^2 + \|\nabla u^-_\varepsilon\|_{L^2(D^-_\varepsilon)}^2 \right) + \frac{1}{\varepsilon} \|u^+_\varepsilon - u^-_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2 \leq C \|f\|_{L^2(D)}.$$  

In the last line above, we used the inequality that $ab \leq \delta a^2 + \frac{1}{4\delta} b^2$. Since $\varepsilon \leq 1/\sqrt{2}$ is small, $\varepsilon^{-1} - \varepsilon \geq 1/(2\varepsilon)$, and we have

$$\frac{\lambda}{2} \left( \|\nabla u^+_\varepsilon\|_{L^2(D^+_\varepsilon)}^2 + \|\nabla u^-_\varepsilon\|_{L^2(D^-_\varepsilon)}^2 \right) + \frac{1}{2\varepsilon} \|u^+_\varepsilon - u^-_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2 \leq C \|f\|_{L^2(D)}^2.$$  

This yields (3.22) and (3.23). In particular, $C$ depends on $\lambda$ but not on $\varepsilon$ or $\omega$. \hfill \Box

Let $P^\varepsilon_\omega$ denote the extension operator of Proposition 3.8. Then we have the following estimates.

\textbf{Corollary 3.12.} Assume the same conditions of Proposition 3.11. Let $P^\varepsilon_\omega$ be the extension operator. Then there exists a constant $C$, which is independent of $\varepsilon$ and $\omega$, such that

$$\|P^\varepsilon_\omega u^+_\varepsilon\|_{H^1(D)} \leq C \|f\|_{L^2(D)},$$  

$$\|P^\varepsilon_\omega u^+_\varepsilon - u_\varepsilon\|_{L^2(D)} \leq C \varepsilon \|f\|_{L^2(D)}.$$  

\textbf{Proof.} The first inequality follows immediately from (3.22) and the Poincaré inequality (3.18). For the second inequality, in view of (3.19), we have

$$\|P^\varepsilon_\omega u^+_\varepsilon - u_\varepsilon\|_{L^2(D)} = \|P^\varepsilon_\omega u^+_\varepsilon - u^-_\varepsilon\|_{L^2(D^-_\varepsilon)} \leq C \varepsilon \|P^\varepsilon_\omega u^+_\varepsilon - u^-_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + C \varepsilon \|\nabla (P^\varepsilon_\omega u^+_\varepsilon - u^-_\varepsilon)\|_{L^2(D^-_\varepsilon)}.$$  

Thanks to (3.23), the first item on the right is bounded by $C \varepsilon \|f\|_{L^2}$. For the second term, we have

$$\varepsilon \|\nabla (P^\varepsilon_\omega u^+_\varepsilon - u^-_\varepsilon)\|_{L^2(D^-_\varepsilon)} \leq \varepsilon \|\nabla P^\varepsilon_\omega u^+_\varepsilon\|_{L^2(D)} + \varepsilon \|\nabla u^-_\varepsilon\|_{L^2(D^-_\varepsilon)},$$  

and in view of (3.23), it is bounded by $C \varepsilon \|f\|_{L^2(D)}$. \hfill \Box
4 The Auxiliary Problem

A standard approach to prove the homogenization result is the method of oscillating test functions, due to Tartar and Murat; see e.g. [37, 35, 38]. The key step is to solve the auxiliary problem (2.9) whose solution serves as building blocks of oscillating test functions. The natural functional space to seek a solution for (2.9) is

\[ H := \{ w = \tilde{w} \circ \Phi^{-1} \mid \tilde{w} \in \tilde{H} \} \quad \text{where} \quad \tilde{H} := L^2(\Omega, H^1_{\text{loc}}(\mathbb{R}^d)) \times H^1_{\text{loc}}(\mathbb{R}^d). \] (4.1)

We say that \( w_p \in H \) is a weak solution to (2.9) if \( \nabla \tilde{w}_p \) is stationary, and for any \( \varphi \) in \( H \) with support \( K \subset \subset \mathbb{R}^d \), it holds that

\[
\begin{align*}
\mathbb{E} \int_{K \cap \Phi(\mathbb{R}^d)} A(\nabla w^+_p + p) \cdot \nabla \varphi^+ \, dy + \mathbb{E} \int_{K \cap \Phi(\mathbb{R}^d)} A(\nabla w^-_p + p) \cdot \nabla \varphi^- \, dy \\
+ \mathbb{E} \int_{K \cap \Phi(\Gamma_d, \omega)} (w^+_p - w^-_p)(\varphi^+ - \varphi^-) \, d\sigma(y) = 0.
\end{align*}
\] (4.2)

Proposition 4.1. Suppose \( w_p \in H \) is a weak solution to (2.9) in the sense of (4.2). Then there exists a subspace \( \Omega_0 \) of \( \Omega \) with full measure \( \mathbb{P}(\Omega_0) = 1 \), such that for all \( \omega \in \Omega_0 \), for all \( \tilde{\phi} \in H^1_{\text{loc}}(\mathbb{R}^d) \times H^1_{\text{loc}}(\mathbb{R}^d) \), \( \tilde{\phi} \) being compactly supported in \( \mathbb{R}^d \) and \( \phi(\cdot, \omega) = \tilde{\phi} \circ \Phi^{-1}(\cdot, \omega) \), it holds that

\[
\begin{align*}
\int_{\Phi(K \cap \mathbb{R}^d)} A(\nabla w^+_p + p) \cdot \nabla \tilde{\phi}^+ \, dy + \int_{\Phi(K \cap \mathbb{R}^d)} A(\nabla w^-_p + p) \cdot \nabla \tilde{\phi}^- \, dy \\
+ \int_{\Phi(K \cap \Gamma_d, \omega)} (w^+_p - w^-_p)(\tilde{\phi}^+ - \tilde{\phi}^-) \, d\sigma(y) = 0.
\end{align*}
\] (4.3)

Remark 4.2. The above proposition says that the weak solution defined in (4.2), where the test functions are integrated against \( dx \times d\mathbb{P}(\omega) \), is in fact also a weak solution in the usual sense for almost all \( \omega \in \Omega \). The proof of this proposition is postponed to Appendix A.2.

As explained in the Introduction, due to the lack of compactness in (2.9) and moreover the lack of an isomorphism between \( H \) and some functional space on \( \Omega \) only, finding the solution of (2.9) is not an easy task. We follow the procedures of Papanicolaou and Varadhan [38] and of Kozlov [31]. We regularize the problem by adding a small zero-order term, and wish to obtain a meaningful limit when the regularization is sent to zero. This leads us to investigate the following regularized problem.

\[
\begin{align*}
- \nabla \cdot \left( A(y, \omega) \nabla \left[ w^+_p(y, \omega) + p \cdot y \right] \right) + \delta w^+_p & = 0, \quad \text{for} \quad y \in \Phi(\mathbb{R}^d), \\
\frac{\partial}{\partial \nu_A} w^+_p(y, \omega) & = \frac{\partial}{\partial \nu_A} w^-_p(y, \omega), \quad \text{for} \quad y \in \Phi(\Gamma_d), \\
w^+_p(y, \omega) - w^-_p(y, \omega) & = \frac{\partial}{\partial \nu_A} w^+_p(y, \omega) + \nu_y \cdot A p, \quad \text{for} \quad y \in \Phi(\Gamma_d), \\
w^+_p(y) & = \bar{w}^+_p(y), \quad \text{for} \quad y \in \Phi(\Gamma_d), \\
w^-_p(y) & = \bar{w}^-_p(y).
\end{align*}
\] (4.4)

The convenient functional space for the solution \( \bar{w}^\pm_{p, \delta} \) above is \( H^1_{\text{loc}}(\mathbb{R}^d) \times H^1_{\text{loc}}(\mathbb{R}^d) \), the space of functions which are, together with their first order derivatives, are locally
uniformly square integrable on $\mathbb{R}^+_d$ and $\mathbb{R}^-_d$. More precisely, functions in such spaces are of the form $\tilde{u} = \tilde{u}_1\chi_{\mathbb{R}^+_d} + \tilde{u}_2\chi_{\mathbb{R}^-_d}$ where $\tilde{u}_1 \in H^1_\text{uloc}(\mathbb{R}^+_d)$, $\tilde{u}_2 \in H^1_\text{uloc}(\mathbb{R}^-_d)$. The latter spaces are defined by

$$ W^{1,p}_\text{uloc}(\mathbb{R}^+_d) := \{ \tilde{v} \in W^{1,p}_\text{loc}(\mathbb{R}^+_d) \mid \sup_{k \in \mathbb{Z}^d} ||\tilde{v}||_{W^{1,p}(Y^+_k)} < \infty \}, $$

$$ W^{1,p}_\text{uloc}(\mathbb{R}^-_d) := \{ \tilde{v} \in W^{1,p}_\text{loc}(\mathbb{R}^-_d) \mid \sup_{k \in \mathbb{Z}^d} ||\tilde{v}||_{W^{1,p}(Y^-_k)} < \infty \}. $$

(4.5)

As usual, when $p = 2$, we write $W^{1,p}$ as $H^1$. Spaces of locally uniformly Sobolev functions are sometimes called Sobolev-Kato spaces and were introduced by Kato [30]. They are natural for problems that are posed on unbounded domains without decaying condition at infinity. The usage of locally uniform spaces in such situations was pioneered by Kato [30] and has found success in many settings, e.g. [31, 7].

Analogous to (4.2), we say $w_{p,\delta} \in \mathcal{H}$ is a weak solution to the regularized problem (4.4) if $\tilde{w}_{p,\delta}$ is stationary and for any $\varphi$ in $\mathcal{H}$ with support $K \subset \subset \mathbb{R}^d$, it holds that

$$ E \int_{\Phi(Y^+_\Gamma, \omega)} A(\nabla w_{p,\delta}^+ + p) \cdot \nabla \varphi^+ dy + E \int_{\Phi(Y^-_\Gamma, \omega)} A(\nabla w_{p,\delta}^- + p) \cdot \nabla \varphi^- dy + E \int_K w_{p,\delta} \varphi dy + E \int_{\Phi(Y_\Gamma, \omega)} (w_{p,\delta}^+ - w_{p,\delta}^-)(\varphi^+ - \varphi^-) d\sigma(y) = 0. $$

(4.6)

The statement of Proposition 4.4.1 can be adapted to the regularized problem also, and in particular a weak solution in the above sense qualifies, for almost all realizations, as a weak solution in the usual sense.

The key lemma below establishes the existence, uniqueness and uniform bounds of the weak solution to the regularized problem (4.4). The notation $\mathcal{H}_S$ in the sequel denotes the subspace of $\mathcal{H}$, the functions in which are stationary after composition with $\Phi$.

**Lemma 4.3.** For each $\omega \in \Omega$, for each fixed $p \in \mathbb{R}^d$ and $\delta > 0$, there exists $w_{p,\delta}(\cdot, \omega) \in \mathcal{H}_S$ which solves the regularized problem (4.4) in the sense of (4.6). In addition, there exists a constant $C$ which depends only on $d, M, \nu$ and $\Gamma_0$ such that

$$ E \int_{\Phi(Y_\Gamma, \omega)} |\nabla w_{p,\delta}^+(y, \cdot)|^2 dy \leq C, $$

$$ E \int_{\Phi(Y_\Gamma, \omega)} |\nabla w_{p,\delta}^-(y, \cdot)|^2 dy \leq C, $$

$$ E \int_{\Phi(Y_\Gamma, \omega)} |w_{p,\delta}(y, \cdot)|^2 dy \leq C. $$

(4.7)

Proof. Step one: Construction of a seemingly weaker solution. We first construct a solution in a sense that seems weaker than (4.6). We observe that $\mathcal{H}_S$, equipped with the inner product

$$ \langle u, v \rangle_{\mathcal{H}_S} := E \left[ \int_{Y^+_\Gamma} \nabla \tilde{u}^+ \cdot \nabla \tilde{v}^+(z, \cdot) dz + \int_{Y^-_\Gamma} \nabla \tilde{u}^- \cdot \nabla \tilde{v}^-(z, \cdot) dz + \int_{Y_\Gamma} \tilde{u} \tilde{v}(z, \cdot) dz \right], $$

(4.8)

is a Hilbert space. Above, $\tilde{u}(z, \cdot) = u \circ \Phi(z, \cdot)$. Let $A_\delta : \mathcal{H}_S \times \mathcal{H}_S \rightarrow \mathbb{R}$ be the bilinear form

$$ A_\delta(u, v) := E \left[ \int_{\Phi(Y_\Gamma, \omega)} A \nabla u^+ \cdot \nabla v^+(y, \cdot) dy + \int_{\Phi(Y^-_\Gamma, \omega)} A \nabla u^- \cdot \nabla v^-(y, \cdot) dy + \int_{\Phi(Y_\Gamma, \omega)} (u^+ - u^-)(v^+ - v^-)(y, \cdot) d\sigma(y) + \delta \int_{\Phi(Y_\Gamma, \omega)} uv(y, \cdot) dy \right]. $$

(4.9)
Let $F_p: \mathcal{H}_S \to \mathbb{R}$ be the linear form

$$F_p(v) = -E \left[ \int_{\Phi(Y,+)} \mathcal{A} \cdot \nabla v^+(y,\cdot) dy + \int_{\Phi(Y,-)} \mathcal{A} \cdot \nabla v^-(y,\cdot) dy \right]. \quad (4.10)$$

It is easy to check that, due to $[2,2]$ and the assumptions (A1)-(A3), for each fixed $\delta > 0$, $\mathcal{A}_\delta$ is bi-continuous and coercive on $\mathcal{H}_S$, and $F_p$ is continuous on $\mathcal{H}_S$. By the Lax-Milgram theorem, there exists a unique $w_{p,\delta} = (w_{p,\delta}^+, w_{p,\delta}^-) \in \mathcal{H}_S$ such that

$$\mathcal{A}_\delta(w_{p,\delta}, v) = F_p(v), \quad \text{for all } v \in \mathcal{H}_S. \quad (4.11)$$

By setting $v = w_{p,\delta}$ and use $[2,2]$ and the assumptions (A1)-(A3), we see that $w_{p,\delta}$ satisfies the estimates in $[4,7]$.

**Step two:** Construction of weak solution for fixed $\omega \in \Omega$. Here we fix a realization $\omega$, and prove that there exists a unique weak solution $W_{p,\delta} \in H^1_{uloc}(\Phi(\mathbb{R}^+_d)) \times H^1_{uloc}(\Phi(\mathbb{R}^-_d))$ in the usual sense. That is, for all $\phi(\cdot) \in H^1_{loc}(\Phi(\mathbb{R}^+_d)) \times H^1_{loc}(\Phi(\mathbb{R}^-_d))$ with support $K \subset \subset \mathbb{R}^d$, we have

$$\begin{align*}
&\int_{K \cap \Phi(\mathbb{R}^+_d)} A(\nabla W^+_{p,\delta} + p) \cdot \nabla \phi^+ d\gamma + \int_{K \cap \Phi(\mathbb{R}^-_d)} A(\nabla W^-_{p,\delta} + p) \cdot \nabla \phi^- d\gamma \\
&\quad + \delta \int_K W_{p,\delta} \phi d\gamma + \int_{K \cap \Phi(\mathbb{R}^-_d)} (W^+_{p,\delta} - W^-_{p,\delta})(\phi^+ - \phi^-) d\sigma(y) = 0.
\end{align*} \quad (4.12)$$

In the last step of the proof, we will show that this solution qualifies as a weak solution in the sense of $[4,11]$ and coincides with the seemingly weaker solution from step one.

Our strategy is as follows: for each positive integer $n \in \mathbb{N}$, we solve a truncated problem on the deformed cube $\Phi(Q_n, \omega)$, where $Q_n = (-n, n)^d$ is the cube with side length $2n$, and obtain a sequence of functions $W_{p,\delta,n} \in H^1_{uloc}(\Phi(\mathbb{R}^+_d)) \times H^1_{uloc}(\Phi(\mathbb{R}^-_d))$. We prove that the norm of $W_{p,\delta,n}$ in this space is uniformly bounded with respect to $n$. Consequently, a limit can be obtained through a converging subsequence, and it provides a weak solution of $[4,11]$ in the usual sense. Furthermore, we show that the solution of $[4,11]$ in $H^1_{uloc}(\Phi(\mathbb{R}^+_d)) \times H^1_{uloc}(\Phi(\mathbb{R}^-_d))$ is unique.

**Step two, part I:** Existence. For each fixed $n \in \mathbb{N}, n \geq 1$, we consider the truncated problem with Dirichlet boundary conditions on $\partial \Phi(Q_n)$:

$$\begin{align*}
-\nabla \cdot (A(y, \omega) \nabla \left[ W^+_{p,\delta,n}(y,\omega) + p \cdot y \right]) + \delta W^+_{p,\delta,n} &= 0, \quad \text{for } y \in \Phi(Q_n \cap \mathbb{R}^+_d), \\
\frac{\partial}{\partial \nu_{\mathbb{A}}} W^+_{p,\delta,n}(y,\omega) - \nu_{\mathbb{A}} \cdot \mathcal{A} W^+_{p,\delta,n}(y,\omega) &= 0, \quad \text{for } y \in \Phi(Q_n \cap \Gamma_d), \\
W^+_{p,\delta,n}(y,\omega) - W^-_{p,\delta,n}(y,\omega) &= \frac{\partial}{\partial \nu_{\mathbb{A}}} W^+_{p,\delta,n}(y,\omega) + \nu_{\mathbb{A}} \cdot \mathcal{A} p, \quad \text{for } y \in \Phi(Q_n \cap \Gamma_d), \\
W^+_{p,\delta,k}(y,\omega) &= 0, \quad \text{for } y \in \Phi(\partial Q_n). \quad (4.13)
\end{align*}$$

The weak solution to this problem is the unique function $(W^+_{p,\delta,n}, W^-_{p,\delta,n})$ in the space

$$\mathcal{W}_n := \{ W = \widetilde{W} \circ \Phi^{-1} \mid \widetilde{W} \in \mathcal{W}_n \}. $$
where
\[ \tilde{W}_n := \{ \tilde{W} \in H^1_{\text{loc}}(\mathbb{R}^d_+) \times H^1_{\text{loc}}(\mathbb{R}^d_-) \mid \tilde{W} = 0 \text{ on } \mathbb{R}^d \setminus Q_n \}, \]
such that for all \( \phi \in W_n \) we have
\[
\int_{\Phi(Q_n \cap \mathbb{R}^d_+)} A(y) \nabla W^+_{p,\delta,n} \cdot \nabla \phi^+ dy + \int_{\Phi(Q_n \cap \mathbb{R}^d_-)} A(y) \nabla W^-_{p,\delta,n} \cdot \nabla \phi^- dy \\
+ \delta \int_{\Phi(Q_n)} W_{p,\delta,n} \phi dy + \int_{\Phi(Q_n \cap \Gamma_d)} (W^+_{p,\delta,n} - W^-_{p,\delta,n}) (\phi^+ - \phi^-) d\sigma(y) \\
= - \int_{\Phi(Q_n \cap \mathbb{R}^d_+)} A(y)p \cdot \nabla \phi^+ dy - \int_{\Phi(Q_n \cap \mathbb{R}^d_-)} A(y)p \cdot \nabla \phi^- dy.
\]

Thanks to the regularization term, the existence and uniqueness of \( W_{p,\delta,n}^\pm \) follows directly from Lax-Milgram theorem. As a result, we obtain a sequence of functions \( \{W_{p,\delta,n}\}_n \) that belong to \( W := H^1_{\text{loc}}(\Phi(\mathbb{R}^d_+)) \times H^1_{\text{loc}}(\Phi(\mathbb{R}^d_-)) \) and solve (4.14).

Next we establish uniform (with respect to \( n \)) bounds for \( W_{p,\delta,n} \). In terms of \( \tilde{W}_{p,\delta,n} = W_{p,\delta,n} \circ \Phi \), we will show that for some \( C = C(\lambda, \Lambda, \mu, M, d, \delta) \), which is independent of \( n \), the quantities

\[
E_k = \int_{Q_k \cap \mathbb{R}^d_+} |\nabla \tilde{W}_{p,\delta,n}^+|^2 dx + \int_{Q_k \cap \mathbb{R}^d_-} |\nabla \tilde{W}_{p,\delta,n}^-|^2 dx + \delta \int_{Q_k} |\tilde{W}_{p,\delta,n}|^2 dx \\
+ \int_{Q_k \cap \Gamma_d} (\tilde{W}_{p,\delta,n}^+ - \tilde{W}_{p,\delta,n}^-)^2 d\sigma(x)
\]
satisfy the estimate \( E_k \leq Ck^d \) for all \( k \leq n \).

To simplify the presentation, for each \( k \in \mathbb{N} \), let \( Q_{k+1} \setminus Q_k \) be a short-hand notation for \( Q_{k+1} \setminus Q_k \). We can find a smooth cutoff function \( \chi_k = \bar{\chi}_k \circ \Phi^{-1} \) where \( \bar{\chi}_k \) is a nonnegative cutoff function supported on \( Q_{k+1} \) and equals one in \( Q_k \). These cutoff functions can be chosen such that \( \|\nabla \bar{\chi}_k\|_{L^\infty} \lesssim 1 \). We note also that \( \nabla \bar{\chi}_k \) is supported in \( Q_{k+1} \). Set \( \bar{\phi} = \bar{\chi}_k \tilde{W}_{p,\delta,n} \) and take \( \phi = \bar{\phi} \circ \Phi^{-1} \) in the weak formulation (4.14); we get

\[
\int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_+)} \chi_k A \nabla W_{p,\delta,n}^+ \cdot \nabla W_{p,\delta,n}^+ dy + \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_-)} \chi_k A \nabla W_{p,\delta,n}^- \cdot \nabla W_{p,\delta,n}^- dy \\
+ \int_{\Phi(Q_{k+1} \cap \Gamma_d)} \chi_k (W_{p,\delta,n}^+ - W_{p,\delta,n}^-)^2 d\sigma(y) + \delta \int_{\Phi(Q_{k+1})} \chi_k W_{p,\delta,n}^2 dy \\
+ \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_+)} W_{p,\delta,n}^+ A \nabla W_{p,\delta,n}^+ \cdot \nabla \chi_k dy + \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_-)} W_{p,\delta,n}^- A \nabla W_{p,\delta,n}^- \cdot \nabla \chi_k dy \\
= - \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_+)} \chi_k Ap \cdot \nabla W_{p,\delta,n}^+ dy - \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_-)} \chi_k Ap \cdot \nabla W_{p,\delta,n}^- dy \\
- \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_+)} W_{p,\delta,n}^+ Ap \cdot \nabla \chi_k dy + \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_-)} W_{p,\delta,n}^- Ap \cdot \nabla \chi_k dy.
\]

The first four terms on the left are good ones because they are positive. Let us denote the absolute value of the other terms by \( I_5, I_6, I_7, I_8, I_9 \) and \( I_{10} \), according to their order of
backward induction (see Lemma A.2 in the appendix), there exists another positive integer $n$ independent of $\lambda, \Lambda, \mu, M, p$ and $(k + 1)^d$ is the order of the volume of $\Phi(Q_k)$; $I_8$ shares the same estimate with $W_{p,\delta,n}^+$ replaced by $W_{p,\delta,n}^-$. The terms $I_5, I_6, I_9, I_{10}$ involve integrals over $\Phi(Q_{k,k+1})$ which is the space between $\Phi(Q_k)$ and $\Phi(Q_{k+1})$. Furthermore, they involve integrals of $W_{p,\delta,n}^\pm$, which appears as $\delta |W_{p,\delta,n}^\pm|^2$ in the definition of $E_k$. Therefore, we control them as follows:

$$I_5 \leq \frac{\lambda}{2} \int_{\Phi(Q_{k+1} \cap \mathbb{R}^d_+)} |\nabla W_{p,\delta,n}^+|^2 dy + C(k + 1)^d, \quad (4.17)$$

where $C = C(\lambda, \Lambda, \mu, M, p)$ and $(k + 1)^d$ is the order of the volume of $\Phi(Q_k)$; $I_8$ shares the same estimate with $W_{p,\delta,n}^+$ replaced by $W_{p,\delta,n}^-$. Evidently, $I_6$ satisfies the same estimate as long as $W_{p,\delta,n}^+$ is replaced by $W_{p,\delta,n}^-$. In the same manner, we have

$$I_9 \leq \delta \int_{\Phi(Q_{k,k+1})} |W_{p,\delta,n}^-|^2 dy + \frac{C}{\delta} (k + 1)^d,$$

and $I_{10}$ has a similar bound. In (4.17), we further divide the integral over $\Phi(Q_{k+1})$ into two pieces: one integral over $\Phi(Q_k)$ and another over $\Phi(Q_{k,k+1})$. We have deliberately made the coefficient in front of the $\Phi(Q_k)$ integrals less than the corresponding ones on the left hand side of (4.16). Hence

$$\frac{1}{2} \left[ \int_{\Phi(Q_k \cap \mathbb{R}^d_+)} A \nabla W_{p,\delta,n}^+ \cdot \nabla W_{p,\delta,n}^+ dy + \int_{\Phi(Q_k \cap \mathbb{R}^d_+)} A \nabla W_{p,\delta,n}^- \cdot \nabla W_{p,\delta,n}^- dy \right]$$

$$+ \delta \int_{\Phi(Q_k)} |W_{p,\delta,n}|^2 dy + \int_{\Phi(Q_{k,k+1} \cap \mathbb{R}^d_+)} (W_{p,\delta,n}^+ - W_{p,\delta,n}^-)^2 d\sigma(y)$$

$$\leq \frac{C}{\delta} \left[ \int_{\Phi(Q_{k,k+1} \cap \mathbb{R}^d_+)} |\nabla W_{p,\delta,n}^+|^2 dy + \int_{\Phi(Q_{k,k+1} \cap \mathbb{R}^d_+)} |\nabla W_{p,\delta,n}^-|^2 dy + (k + 1)^d \right]$$

$$+ \delta \int_{\Phi(Q_{k,k+1})} |W_{p,\delta,n}|^2 dy.$$

After changing variable $y$ to $\Phi(x)$ in the integrals above, we find that this inequality shows

$$E_k \leq C_\delta (E_{k+1} - E_k + (k + 1)^d), \quad \text{for all } k \leq n. \quad (4.18)$$

The constant $C_\delta$ depends on $\lambda, \Lambda, \mu, M, p, d, \delta$ and in particular it is of order $\delta^{-1}$, but it is independent of $n$ and $k$.

We observe that $E_n \leq C k^d$ for some $C(\lambda, \Lambda, \mu, M, p, d)$, which follows from (4.14). By a backward induction (see Lemma A.2 in the appendix), there exists another positive integer $C_\delta'$ such that

$$E_k \leq C_\delta' k^d, \quad \text{for all } k \leq n. \quad (4.19)$$
In particular, we have \( E_1 \leq C'_\delta \). By definition, \( \tilde{W}_{p,\delta,n}^\pm \) vanishes at \( \partial Q_n \). Hence, if we extend it by zero outside \( Q_n \), then \( \tilde{W}_{p,\delta,n}^\pm \in H^1_{uloc}(\mathbb{R}^d_+) \times H^1_{uloc}(\mathbb{R}^d_-) \). The bound in \((4.19)\) with \( k = 1 \) shows
\[
\|\nabla \tilde{W}_{p,\delta,n}^+\|_{L^2(Y^+)}^2 + \|\nabla \tilde{W}_{p,\delta,n}^-\|_{L^2(Y^-)}^2 + \|\tilde{W}_{p,\delta,n}^+ - \tilde{W}_{p,\delta,n}^-\|_{L^2(\Gamma_0)}^2 + \delta \|W_{p,\delta,n}\|_{L^2(Y)}^2 \leq C'_\delta.
\]

Examine the estimates on the items in \((4.16)\), and the estimate for \( E_n \), we observe that they are all translation invariant. Therefore, we get
\[
\sup_{k \in \mathbb{Z}^d} \left( \|W_{p,\delta,n}^+\|_{H^1(k+Y^+)}^2 + \|W_{p,\delta,n}^-\|_{H^1(k+Y^-)}^2 + \|\tilde{W}_{p,\delta,n}^+ - \tilde{W}_{p,\delta,n}^-\|_{L^2(k+\Gamma_0)}^2 \right) \leq C'_\delta.
\]

This is an \( H^1_{uloc}(\mathbb{R}^d_+) \times H^1_{uloc}(\mathbb{R}^d_-) \) bound for \( \{W_{p,\delta,n}\} \). As a result, we extract a subsequence of \( \tilde{W}_{p,\delta,n_k} \) that converges weakly to some \( W_{p,\delta} \in H^1_{uloc}(\mathbb{R}^d_+) \times H^1_{uloc}(\mathbb{R}^d_-) \). The function \( W_{p,\delta} = W_{p,\delta} \circ \Phi^{-1} \) then satisfies \((4.12)\), i.e. solving \((4.4)\) in the usual weak sense.

**Step two, part II: Uniqueness.** Given any two solutions \( W_{p,\delta}^{(1)} \) and \( W_{p,\delta}^{(2)} \) that satisfy \((4.12)\). Let \( V_{p,\delta} \) denote their difference. Then this function satisfies \((4.14)\) with \( W_{p,\delta,n} \) replaced by \( V_{p,\delta} \) and \( p = 0 \). The analysis that follows \((4.14)\) can be repeated, and in particular, the \((k+1)^d\) term in \((4.18)\) disappears and this yields
\[
E_k \leq C_\delta(E_{k+1} - E_k), \quad \forall k \leq n.
\]

Here, \( E_k \) is defined as in \((4.15)\) with \( \tilde{W}_{p,\delta,n} \) replaced by \( V_{p,\delta} \). The above inequality shows that \( E_k \leq \eta \delta E_{k+1} \) for some \( 0 < \eta < 1 \) for all \( n \) and all \( k \leq n \). Since \( V_{p,\delta} \in \tilde{W} \), we have
\[
E_n \leq C n^d \|V_{p,\delta}\|_{\tilde{W}}.
\]

By a simple backward induction, we get
\[
E_1 \leq C n^{n-1} \eta \delta - n^d \|V_{p,\delta}\|_{\tilde{W}}.
\]

Let \( n \to \infty \), we get \( E_1 = 0 \), which implies that \( V_{p,\delta} \equiv 0 \) in \( Y \). By translation invariance of the above argument, \( V_{p,\delta} = 0 \) over the whole space. This proves the uniqueness of the weak solution satisfying \((4.12)\).

**Step 3: Equivalence of weak solutions.** Let \( W_{p,\delta} \) and \( \tilde{W}_{p,\delta} \) be as defined above. We first observe that \( \tilde{W}_{p,\delta}(\cdot + k, \omega) = \tilde{W}_{p,\delta}(\cdot, \tau_k \omega) \) for all \( k \in \mathbb{Z}^d \) and \( \omega \in \Omega \). Indeed, due to the stationarity of the parameters in \((4.4)\) and the domain on which the problem is posed, we check directly that \( W_{p,\delta}(\cdot + k, \omega) \) is a weak solution to \((4.4)\), with realiztion \( \tau_k \omega \), in the sense of \((4.12)\). On the other hand, due to uniqueness, \( W_{p,\delta}(\cdot + k, \omega) \) has to agree with \( W_{p,\delta}(\cdot, \tau_k \omega) \). This shows that \( \tilde{W}_{p,\delta} \) is stationary, i.e. belonging to \( \tilde{H}_S \).

Recall that by assumption \( L^2(\Omega) \) is separable and admits a countable dense subset \( \{\psi_j\}_{j=1}^\infty \). For each \( j \), multiply \( \psi_j \) to both sides of \((4.12)\) and integrate over \( \Omega \); then the equality still holds. Hence \( W_{p,\delta} \) satisfies \((4.16)\) for test functions of the form \( \varphi = \psi_j(\omega)\phi(x) \) with \( \phi \) belongs to \( H^1_{uloc}(\mathbb{R}^d_+) \times H^1_{uloc}(\mathbb{R}^d_-) \) and having compact support. For general test functions, \((4.16)\) follows by density. Therefore, \( W_{p,\delta} \) is a weak solution to \((4.4)\) in the sense of \((4.6)\).
Finally, we show that $W_{p,\delta} = w_{p,\delta}$. For any $v \in \mathcal{H}_S$, since $W_{p,\delta}$ solves (2.9), integration by parts yields

$$\int_{\Phi(Y,\omega)} A \left( p + \nabla W_{p,\delta}^+ \right) \cdot \nabla v^+ \, dy + \int_{\Phi(Y,\omega)} A \left( p + \nabla W_{p,\delta}^- \right) \cdot \nabla v^- \, dy + \delta \int_{\Phi(Y)} W_{p,\delta} \, v \, dy$$

$$+ \int_{\Phi(Y,\omega)} (W_{p,\delta}^+ - W_{p,\delta}^-)(v^+ - v^-) \, d\sigma(y) = \int_{\Phi(Y,\omega)} v^+ \nu_y \cdot A \left( p + \nabla W_{p,\delta}^- \right) \, d\sigma(y).$$

After a change of variable and in light of the formulas (A.1) and (A.2), the last term can be written as

$$\int_{\partial Y} \tilde{v}^+ (D\Phi(x))^{-1} \nu_x \cdot \tilde{A} \left( p + (D\Phi(x))^{-1} \nabla \tilde{W}_{p,\delta}^+ \right) \det(D\Phi(x)) \, d\tilde{\sigma}(x).$$

The integration of this term over $\Omega$ vanishes because the functions in the integrand are stationary except that $\nu_x$ has opposite signs when evaluated at a pair of opposite sides of $Y$. Therefore, $W_{p,\delta}$ satisfies (4.11). By uniqueness of the solution to (4.11), we have $W_{p,\delta} = w_{p,\delta}$. The seemingly weaker solution $w_{p,\delta}$ from step one is in fact the weak solution that is sought after. This completes the proof of Lemma 4.3. □

### 4.1 Proof of Theorem 2.2

Now we construct a solution to the auxiliary problem (2.9) by sending the regularization parameter $\beta$ in the regularized problem (4.4) to zero.

**Proof of Theorem 2.2.** Existence. Let $\{w_{p,\delta} \in \mathcal{H}_S\}$ be as obtained in the previous section. Using the extension operator $P_\omega$ of Proposition 3.6, we obtain a family of functions $\{P_\omega w_{p,\delta}^+ \in L^2(\Omega, H^1_{\text{loc}}(\mathbb{R}^d))\}$. In the sequel, to simplify notations, we denote the extended function by $\tilde{w}_{p,\delta}$. Let $\tilde{w}_{p,\delta} = w_{p,\delta} \circ \Phi$. In view of (3.9), we check that $\tilde{w}_{p,\delta} = w_{p,\delta}$ and $\tilde{w}_{p,\delta}$ is stationary. By (4.1), we have, for any $K \subset \subset \mathbb{R}^d$,

$$\|\nabla \tilde{w}_{p,\delta} \|_{L^2(\Omega, L^2(K))} \leq C(K). \quad (4.21)$$

That is, $\nabla \tilde{w}_{p,\delta}$ is bounded in $L^2(\Omega, (L^2(K))^d)$. Hence, there exists a subsequence that converges weakly in $L^2(\Omega, (L^2(K))^d)$. Recall that $L^2(\Omega)$ is separable. By taking a sequence of $K$’s that exhaust $\mathbb{R}^d$, we obtain a vector field $\tilde{\xi}_p^1 \in L^2(\Omega, L^2_{\text{loc}}(\mathbb{R}^d))$ and a sequence of $\delta_k \to 0$, still denoted as $\delta$, such that $\nabla \tilde{w}_{p,\delta}$ converges weakly to $\tilde{\xi}_p^1$ in $L^2(\Omega, L^2_{\text{loc}}(\mathbb{R}^d))^d$. It is easy to check that $\tilde{\xi}_p^1$ inherits stationarity and remains a potential field. Therefore, we can find some $\tilde{w}_{p,\delta}^1 \in L^2(\Omega, H^1_{\text{loc}}(\mathbb{R}^d))$ such that $\tilde{\xi}_p^1 = \nabla \tilde{w}_{p,\delta}^1$. Moreover, because $\tilde{w}_{p,\delta}$ is stationary, we have

$$E \int_Y \nabla \tilde{w}_{p,\delta}^1(x,\omega) \, dx = 0.$$

After passing this equality to the limit, we obtain that

$$E \int_Y \tilde{\xi}_p^1 \, dx = E \int_Y \nabla \tilde{w}_{p,\delta}^1 \, dx = 0. \quad (4.22)$$
In a similar manner, from (4.7) we also have
\[
\|\nabla \tilde{w}_{p,\delta}^-\|_{L^2(\Omega, (L^2(Y_k^-))^d)} \leq C, \quad \|\tilde{w}_{p,\delta}^+ - \tilde{w}_{p,\delta}^-\|_{L^2(\Omega, L^2(\Gamma_k^\delta))} \leq C, \quad \forall k \in \mathbb{Z}^d. \tag{4.23}
\]

So there exists a vector field $\tilde{\xi}_p^-$ defined piecewisely on each $Y_k^-$, $k \in \mathbb{Z}^d$, such that along the same subsequence found above, $\nabla \tilde{w}_{p,\delta}^-$ converges weakly in $L^2(\Omega, (L^2(Y_k^-))^d)$ to $\tilde{\xi}_p^-$ for all $k \in \mathbb{Z}^d$. Again, $\tilde{\xi}_p^-$ is stationary and is potential on each $Y_k^-$. Hence we can find $\tilde{\xi}_p^+$ that is defined piecewisely on each $Y_k^-$ such that $\tilde{\xi}_p^- = \nabla \tilde{w}_{p,\delta}^-$. We define $\tilde{\xi}_p$ piecewisely as $\tilde{\xi}_p^1 \chi_{\mathbb{R}_d^+} + \tilde{\xi}_p^- \chi_{\mathbb{R}_d^-}$. Finally, due to the second inequality in (4.23) we have that $\tilde{w}_{p,\delta}^+ - \tilde{w}_{p,\delta}^-$ converges weakly to $\tilde{\xi}_p$ in $L^2(\Gamma_k)$ for all $k \in \mathbb{Z}^d$.

By a change of variable, set $\xi_p^1 = \tilde{\xi}_p^1 \circ \Phi^{-1}$, $\xi_p^- = \tilde{\xi}_p^- \circ \Phi^{-1}$ and $\zeta_p = \tilde{\xi}_p \circ \Phi^{-1}$. Then it is easy to check that $\nabla w_{p,\delta}^{ext}$ converges to $\xi_p^1$, similarly $\nabla w_{p,\delta}^-$ converges to $\tilde{\xi}_p^-$ and $w_{p,\delta}^+ - w_{p,\delta}^-$ converges to $\zeta_p$. Again, $\xi_p^1$ and $\tilde{\xi}_p^-$ are potential fields and there exist $w_p^1$ and $w_p^-$ such that $\xi_p^1 = \nabla w_p^1$ and $\xi_p^- = \nabla w_p^-$. Define $\xi_p$ piecewisely as $\xi_p^1 \chi_{\Phi(\mathbb{R}_d^+)} + \tilde{\xi}_p^- \chi_{\Phi(\mathbb{R}_d^-)}$; we check that
\[
\xi_p(y) = (D\Psi(y))^\dagger \tilde{\xi}_p(\Psi(y)), \tag{4.24}
\]
which follows by passing limit in the corresponding relation between $\nabla w_{p,\delta}^+$ and $\nabla \tilde{w}_{p,\delta}^+$. Therefore, it is possible to choose $w_p^1$ and $w_p^-$ so that $w_p^1 = \tilde{\xi}_p^1 \circ \Phi^{-1}$ and $w_p^- = \tilde{\xi}_p^- \circ \Phi^{-1}$.

Set $w_p^1 = w_p^1 \chi_{\Phi(\mathbb{R}_d^+)}$ and $w_p^- = (w_p^1, w_p^-)$. By passing to the limit in (4.6), we get for any $\varphi \in \mathcal{H}$ with support $K \subset \subset \mathbb{R}^d$,
\[
\begin{align*}
\mathbb{E} \int_{K \cap \Phi(\mathbb{R}_d^+, \omega)} A(\nabla w_p^1 + p) \cdot \nabla \varphi^+ dy + \mathbb{E} \int_{K \cap \Phi(\mathbb{R}_d^-, \omega)} A(\nabla w_p^- + p) \cdot \nabla \varphi^- dy \\
+ \mathbb{E} \int_{K \cap \Phi(\Gamma_d, \omega)} \zeta_p(\varphi^+ - \varphi^-) d\sigma(y) = 0.
\end{align*}
\tag{4.25}
\]

In particular, choose $\varphi$ compactly supported in $\Phi(\mathbb{R}_d^+)$ (respectively, in $\Phi(\mathbb{R}_d^-)$); then we verify that
\[-\nabla \cdot (A(p + \nabla w_p^+)) = 0 \quad \text{in} \quad \Phi(\mathbb{R}_d^+),
\]
(respectively, the same conclusion for $w_p^-$ on $\Phi(\mathbb{R}_d^-)$). From divergence theorem and by choosing $\varphi^- = 0$ (respectively, $\varphi^+ = 0$), we have
\[
\frac{\partial w_p^+}{\partial v_A} + \nu_y \cdot A p = \zeta_p, \quad \text{respectively,} \quad \frac{\partial w_p^-}{\partial v_A} + \nu_y \cdot A p = \zeta_p.
\]

Finally, for any $h \in C^\infty(\Phi(\Gamma_d))$ and $\int_{\Phi(\Gamma_d)} h = 0$, we can find $\varphi^\pm$ so that, for some fixed ball $B^\delta \subset \subset Y$ whose boundary is well separated from that of $Y^-$,
\[
\begin{cases}
-\nabla \cdot (A \nabla \varphi^\pm) = 0, & \text{in} \quad \Phi(Y_k^-) \text{ and } \Phi(k + B^\delta \setminus \Phi(Y_k^-)), \\
\nu \cdot A \nabla \varphi^+ = \nu \cdot A \nabla \varphi^-, & \text{on} \quad \Phi(\Gamma_k), \\
\varphi^+ = 0, & \text{on} \quad \Phi(k + \partial B^\delta).
\end{cases}
\]

Use $(\varphi^+, \varphi^-)$ constructed above as test functions in (4.25) and apply divergence theorem; then we verify that $w_p^+ - w_p^- = \zeta_p + C(\omega, k)$ on each $\Phi(\Gamma_k)$. By adding the constant to
$w_p^-$, we may set $\zeta = w_p^+ - w_p^-$ in (4.25) and verify that $(w_p^+, w_p^-)$ satisfies (4.22). We check that $\nabla \tilde{w}_p^+ = \tilde{\zeta} p_{LR_d}$ is stationary by construction. Moreover, $\tilde{w}_p^+$ is an extension of $\tilde{w}_p^+$ to the whole space, and (i) $\nabla \tilde{w}_p^1 = P(\nabla \tilde{w}_p^+)$ where $P$ is the extension operator of Proposition 3.3. This is verified by passing to limit in the relation $\nabla \tilde{w}_{p, \delta}^+ = P (\nabla \tilde{w}_p^+)$, (ii) $\nabla \tilde{w}_p^+$ satisfies (4.22), which verifies the normalization condition (2.10). To summarize, we have constructed a solution $(w_p^+, w_p^-)$ for (2.9) satisfying (2.10).

**Uniqueness** (of $\nabla w_p$). Suppose there are two solutions to the auxiliary problem. Let $v_p$ be their difference, then $v_p$ satisfies (2.9) with $p$ replaced by zero. Integrate this equation against $\tilde{v}_p$ over the deformed cube $\Phi(Q_N)$ where $Q_N = (-N, N)^d$. We get

$$\int_{\Phi(Q_N \cap \mathbb{R}^d_+)} \nabla v_p^+ \cdot A \nabla v_p^+ dx + \int_{\Phi(Q_N \cap \mathbb{R}^d_-)} \nabla v_p^- \cdot A \nabla v_p^- dx + \int_{\Phi(Q_N \cap \Gamma_d)} |v_p^+ - v_p^-|^2 d\sigma(x)$$

$$= \int_{\partial \Phi(Q_N)} v_p^+ \nu_x \cdot A \nabla v_p^+ d\sigma(x).$$

Since by assumption $A$ and $\Phi(\Gamma_k)$ are $C^2$ and $A$ is uniformly elliptic, by elliptic regularity, we have $|\nabla v_p| \leq C \|\nabla v_p\|_{L^2(\mathbb{R}^d_+)}$ for the deformed cubes along $\partial \Phi(Q_N)$. Let $K_{\partial Q_N}$ denote the indices of those cubes. By a change of variables and extension, we also have such bound of $\nabla \tilde{v}_p^+$. On the other hand, due to the assumption that $\nabla \tilde{v}_p^+$ is stationary and satisfies (2.10), in light of Lemma 3.3, we conclude that $\tilde{v}_p^+$ grows sublinearly almost surely. Consequently, the integral on the right is of order $o(N) \sum_{k \in K_{\partial Q_N}} \|\nabla v_p\|_{L^2(\Phi(\Gamma_k))}$. Divide the above equality by $(2N + 1)^d$, and change variable in the integrals, we have

$$\frac{1}{(2N + 1)^d} \left[ \int_{Q_N \cap \mathbb{R}^d_+} |\nabla v_p^+|^2 dx + \int_{Q_N \cap \mathbb{R}^d_-} |\nabla v_p^-|^2 dx + \int_{Q_N \cap \Gamma_d} |\tilde{\nu}^+ - \tilde{\nu}^-| d\sigma(x) \right]$$

$$= o\left( \frac{N}{(2N + 1)^d} \right) \sum_{k \in K_{\partial Q_N}} \|\nabla \tilde{v}_p^+\|_{L^2(\mathbb{R}^d_+)}.$$

Send $N$ to infinity and use the ergodic theorem; we conclude that

$$\mathbb{E} \left[ \int_{Y^+} |\nabla \tilde{v}_p^+|^2 dx + \int_{Y^-} |\nabla \tilde{v}_p^-|^2 dx + \int_{\Gamma_0} |\tilde{\nu}^+ - \tilde{\nu}^-| d\sigma(x) \right] = 0,$$

which implies that $\tilde{\nu}_p = v_p = C$, i.e. the two solutions are different by a constant. In particular, the gradient of the solution to (2.9) is unique. Note that since $\nabla \tilde{v}_p^+ = P(\nabla \tilde{v}_p^+) = 0$, we proved that the gradient of $\tilde{w}_p^+$ is also unique. □

**Remark 4.4.** It is clear from the construction that the functions $\nabla w_p^+, \nabla w_p^-$ and $w_p^+ - w_p^-$ inherit the estimates in (4.25). That is

$$\mathbb{E} \int_{\Phi(Y^+)} |\nabla w_p^+(y, \cdot)|^2 dy + \mathbb{E} \int_{\Phi(Y^-)} |\nabla w_p^-(y, \cdot)|^2 dy \leq C,$$

$$\mathbb{E} \int_{\Phi(\Gamma_0)} (w_p^+ - w_p^-)^2(y, \cdot) d\sigma(y) \leq C.$$

(4.26)
5 Proof of the Homogenization Result

5.1 Oscillating test functions

We start with the construction of oscillating test functions. For any \( p \in \mathbb{R}^d \), let \((w_p^+, w_p^-)\) be the solution to the auxiliary problem (2.9) provided by Theorem 2.2 and let \( w_p^{ext} \) be the corresponding extension of \( w_p^+ \). Define

\[
\begin{cases}
  w_{1p}^\varepsilon(x, \omega) = x \cdot p + \varepsilon w_p^{ext}(\frac{x}{\varepsilon}, \omega), & x \in \mathbb{R}^2, \\
  w_{2p}^\varepsilon(x, \omega) = x \cdot p + \varepsilon Q w_p^- (\frac{x}{\varepsilon}, \omega), & x \in \mathbb{R}^2.
\end{cases}
\]

(5.1)

Here and in the sequel, \( Q \) denotes the trivial extension operator which sets \( Qf = 0 \) outside the spatial support of \( f \). By scaling the auxiliary problem, we verify that \((w_p^{\varepsilon +}, w_p^{\varepsilon -})\), where \( w_p^{\varepsilon +} \) is the restriction of \( w_p^\varepsilon \) in \( \varepsilon \Phi(\mathbb{R}_d^2) \) and \( w_p^{\varepsilon -} \) is the restriction of \( w_p^\varepsilon \) in \( \varepsilon \Phi(\mathbb{R}_d^2) \), satisfy

\[
\begin{align*}
  -\nabla \cdot (A \nabla w_p^{\varepsilon +}) &= 0 & & \text{in } \varepsilon \Phi(\mathbb{R}_d^2), \\
  -\nabla \cdot (A \nabla w_p^{\varepsilon -}) &= 0 & & \text{in } \varepsilon \Phi(\mathbb{R}_d^2), \\
  \frac{\partial w_p^{\varepsilon +}}{\partial \nu_{A^\varepsilon}} &= \frac{\partial w_p^{\varepsilon -}}{\partial \nu_{A^\varepsilon}} & & \text{on } \varepsilon \Phi(\Gamma_d).
\end{align*}
\]

This means that for any test function \( \varphi = (\varphi^+, \varphi^-) \in \Phi \) with support \( K \subset \subset \mathbb{R}^d \), one has

\[
\begin{align*}
  \mathbb{E} \int_{K \cap \varepsilon \Phi(\mathbb{R}_d^2)} A \nabla w_p^{\varepsilon +} \cdot \nabla \varphi^+ dx + \mathbb{E} \int_{K \cap \varepsilon \Phi(\mathbb{R}_d^2)} A \nabla w_p^{\varepsilon -} \cdot \nabla \varphi^- dx & \\
  + \frac{1}{\varepsilon} \mathbb{E} \int_{K \cap \varepsilon \Phi(\Gamma_d)} (w_p^{\varepsilon +} - w_p^{\varepsilon -})(\varphi^+ - \varphi^-) d\sigma(x) = 0.
\end{align*}
\]

(5.2)

Clearly, this is the scaled version of (4.2). Similar to the conclusion of Proposition 4.1 there exists an \( \Omega_0 \subset \Omega \) with full probability measure, and the above weak formulation also holds pointwisely in \( \Omega_0 \). Set \( \eta_p^{\varepsilon \pm} = A \nabla w_p^{\varepsilon \pm} \). Then \( w_p^\varepsilon \) and \( \eta_p^{\varepsilon \pm} \) satisfy the following.

\[
\text{Lemma 5.1. Let } p, w_p^\varepsilon \text{ and } \eta_p^{\varepsilon \pm} \text{ be defined as above. There exists } \Omega_1 \subset \Omega \text{ with } \mathbb{P}(\Omega_1) = 1, \text{ such that for all } \omega \in \Omega_1, \text{ for any bounded open subset } \Theta \subset \mathbb{R}^d, \text{ we have}
\]

\[
\begin{align*}
  w_{1p}^\varepsilon &\to x \cdot p, & & \text{uniformly in } \Theta, \\
  w_{2p}^\varepsilon &\to x \cdot p, & & \text{in } L^2(\Theta), \\
  (Q \eta_p^{\varepsilon \pm}) &\to \frac{1}{\varepsilon} \mathbb{E} \int_{\Phi(\Gamma_d)} A(y, \omega) (p + \nabla w_p^{\varepsilon \pm}(x, \omega)) dy, & & \text{in } [L^2(\Theta)]^d.
\end{align*}
\]

(5.3)\hspace{1cm}(5.4)\hspace{1cm}(5.5)

\textbf{Proof.} To prove the first result, we recall that \((w_p^+, w_p^-)\) solves (2.9) and by the elliptic regularity theorem adapted to the space \( \mathcal{H} \) we have

\[
\mathbb{E} \int_{\Phi(\Gamma_d)} |\nabla w_p^+(x, \omega)|^s dx < \infty, \quad \text{which implies} \quad \mathbb{E} \int_Y |\nabla w_p^{ext}(y, \omega)|^s dy < \infty
\]

for some \( s > d \). In addition, \( \nabla w_p^{ext} \) is stationary and its integral over \( Y \) has expectation zero. By Birkhoff’s ergodic theorem, e.g. Theorem 9 of [32], there exists a subset \( \Omega_1 \) of \( \Omega \) with full measure such that, for any \( \omega \in \Omega_1 \) and for any compact set \( K \subset \mathbb{R}^2 \),

\[
\lim_{\varepsilon \to 0} \sup_{x \in K} \left| \varepsilon \nabla w_p^{ext}(\frac{x}{\varepsilon}, \omega) \right| = 0.
\]

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Since \( w^\text{ext}_p = \tilde{w}^\text{ext}_p \circ \Psi \), (5.3) then follows by a change of variable.

For the second convergence result, we first write

\[
\epsilon^2 \sum_{n \in L_\varepsilon} \int_{\Phi(Y_n^{-})} \left| \frac{w^\text{ext}_p(x)}{\varepsilon} - \frac{x}{\varepsilon} \right|^2 \, dx = \epsilon^{d+2} \sum_{n \in L_\varepsilon} \int_{\Phi(Y_n^{-})} \left| w^\text{ext}_p(x) - \tilde{w}^-_p(x) \right|^2 \, dx
\leq C \varepsilon^{d+2} \sum_{n \in L_\varepsilon} \int_{Y_n^{-}} \left| \tilde{w}^-_p(y) - \tilde{\tilde{w}}^-_p(y) \right|^2 \, dy.
\]

In view of (5.3), the second item on the right converges uniformly in \( \Theta \) to zero. Therefore, if \( J_\varepsilon \) denotes the \( L^2(\Theta) \) norm of the first item, then it suffices to show that \( J_\varepsilon \) converges to zero. Let \( I_\varepsilon \subset \mathbb{Z}^2 \) be the index of the deformed and scaled cubes that intersect with \( \Theta \). Then \( \Theta \subset \bigcup_{k \in I_\varepsilon} \varepsilon \Phi(Y_k) \) and \( |I_\varepsilon| \leq C(\Theta) \varepsilon^{-d} \), where the constant \( C \) can be chosen to be independent of \( \varepsilon \) and \( \omega \). We have, in view of (A2), (A3) and (2.4),

\[
J^2_\varepsilon \leq C \varepsilon^2 \left[ \frac{1}{|I_\varepsilon|} \sum_{n \in I_\varepsilon} \left( \int_{\Gamma_n} \left| \tilde{w}^-_p(y) - \tilde{\tilde{w}}^-_p(y) \right|^2 \, d\sigma(y) + \int_{Y_n^{-}} \left| \nabla \tilde{w}^-_p(y) - \nabla \tilde{\tilde{w}}^-_p(y) \right|^2 \, dy \right) \right].
\]

Note that the integrands above are stationary and the item inside the bracket is ready for applying ergodic theorem. Redefine \( \Omega_1 \) by intersecting with the previous \( \Omega_1 \) if necessary; then for all \( \omega \in \Omega_1 \), this item above converges to

\[
\mathbb{E} \int_{\Gamma_0} \left| \tilde{w}^-_p - \tilde{\tilde{w}}^-_p \right|^2(y) \, d\sigma(y) + \mathbb{E} \int_{Y^-} \left| \nabla \tilde{w}^-_p - \nabla \tilde{\tilde{w}}^-_p \right|^2 \, dy,
\]

which is bounded in light of (1.26). Consequently, \( \lim_{\varepsilon \to 0} J_\varepsilon = 0 \) and (5.4) is proved.

For the third convergence result, we observe that \( Q\eta_p^\pm(x, \omega) = (Q\eta_p^\pm)(\Phi^{-1}(x, \omega)) \) and

\[
Q\eta^\pm_p = \left( \tilde{A} \left\{ p + ((D\Phi)^{-1})^t \nabla \tilde{w}^\pm_p \right\} \chi_{\mathbb{R}^d} \right)|_{\Phi^{-1}(x, \omega)}.
\]

We verify that \( Q\eta^\pm_p \) are stationary and belong to \( L^2_{\text{loc}}(\mathbb{R}^d, L^1(\Omega)) \). In view of Lemma 3.2, we obtain (5.5) for all \( \omega \in \Omega_1 \).

5.1.1 Proof of the homogenization theorem

In this subsection we prove the homogenization theorem using the method of oscillating test functions. The strategy is as follows: In the first step, we recall the energy estimates for the solution \( u_\varepsilon \) to the problem (1.1) and extract a subsequence along which \( u^\text{ext}_\varepsilon \) converges weakly in \( H^1(D) \) to some \( u_0 \), and the trivially extended flux \( Q(A^\varepsilon \nabla u^\pm_\varepsilon) \) and \( Q(A^\varepsilon \nabla u^-_\varepsilon) \) have weak limits in \( [L^2(D)]^d \). In the second step, we use the oscillating test function \((\varphi u^\pm_{1p}, \varphi u^\pm_{2p})\) in (4.3), and the oscillating test function \((\varphi u^+_\varepsilon, \varphi u^-_\varepsilon)\) in the pointwise version of (4.25). Passing to limits, we obtain the equation with proper boundary conditions satisfied by \( u_0 \).
Finally, we show that the whole sequence $u_{\varepsilon}$ converges by proving that the solution to the equation of $u_0$ is unique. As a by-product, we also prove that the trivial extension $Qu_{\varepsilon}$ converges weakly in $L^2(D)$ to $\theta u_0$ for some constant $\theta$ strictly less than one.

**Proof of Theorem 2.3.** Step 1. A converging subsequence. In light of Theorem 2.2, Proposition 4.1 and Lemma 5.1, there exists a measurable subset $\Omega_{\ast} \subset \Omega$ with full measure such that for all $\omega \in \Omega_{\ast}$, the pointwise version of (5.2) is valid, together with the conclusions of Lemma 5.1. We henceforth fix any $\omega \in \Omega_{\ast}$ and omit the dependence of functions on $\omega$. Denote the vector fields $A\nabla u_{\varepsilon}^{\pm}$ by $\xi_{\varepsilon}^{\pm}$. From (3.26) and (3.22) we see that

$$
\|u_{\varepsilon}^{\pm}\|_{H^1(D)} + \|Q\xi_{\varepsilon}^{\pm}\|_{L^2(D)} + \|Q\xi_{\varepsilon}^{-}\|_{L^2(D)} \leq C.
$$

As a result, there exist $u_0 \in H^1(D)$, $\xi_1, \xi_2 \in [L^2(D)]^d$, and a subsequence of $u_{\varepsilon}$ still indexed by $\varepsilon$, such that

$$
\begin{align*}
    u_{\varepsilon}^{\pm} &\rightarrow u_0 \text{ weakly in } H^1(D), \\
    u_{\varepsilon}^{\pm} &\rightharpoonup u_0 \text{ strongly in } L^2(D); \\
    Q\xi_{\varepsilon}^{+} &\rightharpoonup \xi_1 \text{ weakly in } [L^2(D)]^d, \\
    Q\xi_{\varepsilon}^{-} &\rightharpoonup \xi_2 \text{ weakly in } [L^2(D)]^d.
\end{align*}
$$

(5.6)

In the proof of Proposition 3.12 we also proved that

$$
    u_{\varepsilon}^{\ast} \chi_{\varepsilon}^{\pm} - Qu_{\varepsilon}^{\pm} \rightarrow 0 \text{ strongly in } L^2(D),
$$

(5.7)

so in fact $u_{\varepsilon}$ converges to $u_0$ strongly in $L^2(D)$.

Step 2: Equation for $u_0$. Fix an arbitrary test function $\varphi \in C^\infty_0(D)$ with support $K \subset \subset D$. Take $(\varphi \chi_{\varepsilon}^{\pm}, \varphi \chi_{\varepsilon}^{-})$ as the test function in (3.21). Then the interface term disappears and

$$
\int_D (Q\xi_{\varepsilon}^{+}) \cdot \nabla \varphi dx + \int_D (Q\xi_{\varepsilon}^{-}) \cdot \nabla \varphi dx = \int_D f \varphi dx.
$$

Passing to the limit $\varepsilon \rightarrow 0$ along the chosen subsequence, one finds

$$
\int_\Omega (\xi_1 + \xi_2) \cdot \nabla \varphi dx = 0.
$$

(5.8)

In other words, $\xi_1 + \xi_2$ is a divergence free field.

Next, recall the definition of $D_{\varepsilon}^-, \Gamma_{\varepsilon}, K_{\varepsilon}$ and $E_{\varepsilon}$ in (2.5) and (2.6). For $\varepsilon$ sufficiently small, the function $\varphi$ is compactly supported in $E_{\varepsilon}$. In particular, we have $\varepsilon \Phi(\Gamma_{\varepsilon}) \cap K = \Gamma_{\varepsilon} \cap K$ where $K$ is the support of $\varphi$. Let $\{e_k : k = 1, \ldots, d\}$ denote the standard basis for $\mathbb{R}^d$. Let $w_{1_{e_k}}$ and $w_{2_{e_k}}$ be as defined in (5.1). Take $(\varphi u_{\varepsilon}^{\pm}, \varphi u_{\varepsilon}^{-})$ as the test function in pointwise version of (5.2); then for each $k = 1, 2, \ldots, d$, we get

$$
\int_D (Qw_{1_{e_k}}^{\varepsilon}) \cdot \nabla (\varphi u_{\varepsilon}^{\pm}) dx + \int_D (Qw_{2_{e_k}}^{\varepsilon}) \cdot \nabla (\varphi u_{\varepsilon}^{-}) dx + \frac{1}{\varepsilon} \int_{\Gamma_{\varepsilon}} (w_{1_{e_k}}^{\varepsilon} - w_{2_{e_k}}^{\varepsilon}) \varphi (u_{\varepsilon}^{\pm} - u_{\varepsilon}^{-}) ds = 0.
$$

Similarly, take $(\varphi w_{1_{e_k}}^{\varepsilon}, \varphi w_{2_{e_k}}^{\varepsilon})$ as the test function in (3.21); we get

$$
\int_D (Q\xi_{\varepsilon}^{+}) \cdot \nabla (\varphi w_{1_{e_k}}^{\varepsilon}) dx + \int_D (Q\xi_{\varepsilon}^{-}) \cdot \nabla (\varphi w_{2_{e_k}}^{\varepsilon}) dx + \frac{1}{\varepsilon} \int_{\Gamma_{\varepsilon}} (w_{1_{e_k}}^{\varepsilon} - w_{2_{e_k}}^{\varepsilon}) \varphi (u_{\varepsilon}^{\pm} - u_{\varepsilon}^{-}) ds = 0.
$$

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Subtract the second equality from the first one. Then the interface terms cancel out, and the terms in which the derivative does not land on $\varphi$ also cancel out. We obtain

$$
\left( \int_D (Q\eta_{\varepsilon}^+ \cdot \nabla \varphi u_{\varepsilon}^{\text{ext}} \, dx + \int_D (Q\eta_{\varepsilon}^- \cdot \nabla \varphi u_{\varepsilon}^{\text{ext}} \, dx + \int_D (Q\eta_{\varepsilon}^- \cdot \nabla \varphi (Q u_{\varepsilon}^- - u_{\varepsilon}^{\text{ext}}) \, dx \right)
- \left( \int_D (Q\xi_{\varepsilon}^+ \cdot \nabla \varphi u_{\varepsilon}^{\text{ext}} \, dx + \int_D (Q\xi_{\varepsilon}^- \cdot \nabla \varphi w_{\varepsilon}^k \, dx \right) = 0.
$$

In view of (5.5), (5.3), (5.4), (5.6) and (5.7), we observe that each integrand above is a product of a strong converging term with a weak converging one. We can pass to the limit above and get

$$
\int_D (\eta_{1\varepsilon} + \eta_{2\varepsilon}) u_0 \cdot \nabla \varphi \, dx = \int_D (\xi_1 + \xi_2) x_k \cdot \nabla \varphi \, dx,
$$

(5.9)

where $\eta_{1\varepsilon}$ and $\eta_{2\varepsilon}$ is defined to be the right hand side of (5.5), with $\eta_{1\varepsilon}$ corresponding to the positive sign and $\eta_{2\varepsilon}$ the negative sign. In view of (5.8), the last integral above can be written as

$$
\int_D (\xi_1 + \xi_2) \cdot (\nabla (\varphi x_k) - e_k \varphi) \, dx = - \int_D (\xi_1 + \xi_2) \cdot e_k \varphi \, dx,
$$

For the first integral in (5.9), from the definition of $\eta_{1\varepsilon}$ and $\eta_{2\varepsilon}$ and that of $A_0$ in (2.11), we first observe that

$$
(\eta_{1\varepsilon} + \eta_{2\varepsilon}) \cdot \nabla \varphi = \sum_{\ell=1}^d \frac{1}{\varrho} \left( \int_{\Phi(y^+)} a_{\ell k}(e_k + \nabla w_{\varepsilon}^k) \, dx + \int_{\Phi(y^-)} a_{\ell k}(e_k + \nabla w_{\varepsilon}^k) \, dx \right) \frac{\partial \varphi}{\partial x_\ell}
= \sum_{\ell=1}^d a_{0\ell k} \frac{\partial \varphi}{\partial x_\ell} = e_k \cdot A_0 \nabla \varphi.
$$

Using the above facts, we rewrite (5.9) as

$$
\int_D (u_0 e_k \cdot A_0) \cdot \nabla \varphi \, dx = - \int_D e_k \cdot (\xi_1 + \xi_2) \varphi \, dx,
$$

which yields that $(\xi_1 + \xi_2) \cdot e_k = \text{div}(u_0 e_k \cdot A_0) = e_k \cdot A_0 \nabla u_0$. Therefore, the vector field $\xi_1 + \xi_2$ coincides with $A_0 \nabla u_0$. In light of (5.8), we have

$$
- \nabla \cdot (A_0 \nabla u_0) = 0 \quad \text{in } D
$$

(5.10)

in the distributional sense.

Next we consider the boundary condition that is satisfied by $u_0$. Recall that $u_\varepsilon^{\text{ext}} \rightharpoonup u_0$ weakly in $H^1(D)$, $u_\varepsilon^{\text{ext}} |_{\partial D} = 0$ for all $\varepsilon$, and that the trace operator from $H^1(D)$ to $L^2(\partial D)$ is continuous with respect to the weak topology. Therefore,

$$
u_0 = 0 \quad \text{on } \partial D.
$$

(5.11)

Combining (5.10) and (5.11) together, we conclude that $u_0 \in H^1_0(D)$ and it solves the deterministic problem (1.3). Finally, provided that $A_0$ is uniformly elliptic which we prove
in Section 6 it is obvious that (1.3) has a unique solution in $H^1_0(D)$. As a result, the whole family $\{u^\text{ext}_\varepsilon\}$ converge strongly in $L^2(D)$ and weakly in $H^1_0(D)$ to $u_0$, the unique solution to (1.3). Items (i), (ii) and (iv) of Theorem 2.3 are proved.

Step 3: Convergence of $Qu^-\varepsilon$. We can write $Qu^-\varepsilon$ as $u^\text{ext}_\varepsilon \chi^-\varepsilon + (Qu^-\varepsilon - u^\text{ext}_\varepsilon \chi^-\varepsilon)$ where $\chi^-\varepsilon$ is the indicator function of $A$. Due to (5.7) and the fact that $u^\text{ext}_\varepsilon$ converges strongly to $u_0$, we only need to verify that $\chi^-\varepsilon$ converges weakly in $L^2(D)$ to $\theta$. For this purpose, fix an arbitrary open set $K \subset \subset D$. Then for sufficiently small $\varepsilon$, $K$ lies in $E_\varepsilon$ defined in (2.6). We have

$$\int_K \chi^-\varepsilon dx = \int_{K \cap \varepsilon\Phi(\mathbb{R}^d_\omega)} dx = \int_K \chi_{\varepsilon\Phi(\mathbb{R}^d_\omega)}(x) dx.$$ 

We observe that $x \in \varepsilon\Phi(\mathbb{R}^d_\omega)$ if and only if $\Phi^{-1}(\varepsilon \frac{x}{\varepsilon}) \in \mathbb{R}^d_\omega$, which yields

$$\chi_{\varepsilon\Phi(\mathbb{R}^d_\omega)}(x) = \chi_{\mathbb{R}^d_\omega}(\Phi^{-1}(\varepsilon \frac{x}{\varepsilon}, \omega)),$$

and apparently $\chi_{\mathbb{R}^d_\omega}(z)$ is periodic and hence stationary, and it is uniformly bounded. By Lemma 3.2, the above function converges in $L^\infty$ weak-$*$ topology. More precisely, in view of the definitions in (2.8), we have for almost all $\omega \in \Omega$,

$$\chi_{\varepsilon\Phi(\mathbb{R}^d_\omega)} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\theta} \int_{\Phi(Y^-)} dx = \theta. \quad (5.12)$$

Upon redefining $\Omega_\ast$ by intersection, we will assume that the above is valid for all $\omega \in \Omega_\ast$ that was chosen at the beginning of step one. As a result, we have that

$$\lim_{\varepsilon \rightarrow 0} \int_K \chi^-\varepsilon(x, \omega) dx = \int_K \theta dx = \theta |K|, \quad \forall K \subset \subset D.$$

By invoking the density of simple functions in $L^2(D)$, we conclude that $\chi^-\varepsilon$ converges weakly to $\theta$. Hence $Qu^-\varepsilon$ converges weakly in $L^2(D)$ to $\theta u_0$. This completes the proof of Theorem 2.3. □

6 Further Discussions

We first show that the homogenized coefficient $A^0$ defined by (2.11) is uniformly elliptic. $A^0$ is clearly bounded from above, so we concentrate on the coercivity of $A^0$. For any vector $\xi \in \mathbb{R}^d$, by the definition of $A^0$ and the linearity of $p \mapsto w_p$, we have

$$a^0_{ij}\xi_i\xi_j = \frac{1}{\theta} \mathbb{E} \left( \int_{\Phi(Y^+,\omega)} \xi \cdot A(\xi + \nabla w^+\xi) dx + \int_{\Phi(Y^-,\omega)} \xi \cdot A(\xi + \nabla w^-\xi) dx \right).$$

Take $p = \xi$ in the auxiliary problem (2.9) and take $w_\xi$ as the test function. By an argument that is similar to the uniqueness step in the proof of Theorem 2.3, we verify that

$$\mathbb{E} \left( \int_{\Phi(Y^+)} \nabla w^+\xi \cdot A(\xi + \nabla w^+\xi) dx + \int_{\Phi(Y^-)} \nabla w^-\xi \cdot A(\xi + \nabla w^-\xi) dx + \int_{\Phi(\Gamma_{\ast})} |w^+\xi - w^-\xi|^2 \right) = 0,$$
which yields
\[ a^0_{ij} = -\frac{1}{\varrho} \mathbb{E} \left( \int_{\Phi(Y_+,\omega)} (\xi + \nabla w^+_\xi) \cdot A(\xi + \nabla w^+_\xi) \, dx + \int_{\Phi(Y_-,\omega)} (\xi + \nabla w^-_\xi) \cdot A(\xi + \nabla w^-_\xi) \, dx \right) + \frac{1}{\varrho} \mathbb{E} \int_{\Phi(\Gamma_0,\omega)} |w^+_\xi - w^-_\xi|^2 \, d\sigma(x) \geq 0. \]

This shows that \( a^0_{ij} \) is positive semidefinite. Further, the inequality above becomes equality if and only if \( w_\xi \) has no jump across \( \Gamma_0 \) and \( \nabla w_\xi = -\xi \). Then \( \nabla \tilde{w}_\xi = -(D\Phi)^t\xi \). Recall that the integral of \( \nabla \tilde{w}_\xi \) over the unit cube has mean zero and \( D\Phi \) is non-degenerate; this forces \( \xi \) to be zero. It follows that \( A^0 \) is uniformly elliptic.

Our analysis applies to several variations of the problem (1.1). First, like in [36], instead of assuming that the materials across the interfaces are the same, we may consider two different materials. This amounts to using two different elliptic coefficients \( A^1 \) and \( A^2 \) in (2.1). Secondly, similar to [4], rather than considering the Dirichlet boundary condition at \( \partial D \), we may treat also Neumann or Robin type boundary conditions, and we may consider non-homogeneous boundary conditions as well. Finally, if the underlying application is not in biology, the modification near the boundary \( \partial D \) at the end of Section 2.2 is not necessary, since we see already that in the homogenization proof we only need to take test functions that are compactly supported in \( D \). These claims can be rigorously justified by examining our analysis and making slight modifications.

We conclude this paper by some interesting questions beyond homogenization which are out of the scope of this paper. The current article concerns only the homogenization of (1.1), and it is natural to ask about the convergence rate. Such quantitative estimates in stochastic homogenization is much more difficult, but there have been important progresses in several situations, e.g. [16, 28, 27, 5]. Once the convergence rate is clear, one may investigate further the detailed structures of the mean of the error and the distributions of the random error, and those in numerical homogenization schemes; see e.g. [25, 8, 10, 9]; see also [11, 12].

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A Appendix

A.1 Surface integrals under diffeomorphisms

We record a geometric fact which concerns surface integrals under a diffeomorphism of the ambient space.
Proposition A.1. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving diffeomorphism and its inverse is denoted by $\Psi$. Let $S$ be an $n-1$ dimensional smooth surface given by $S = g^{-1}\{0\}$ for some smooth function $g : \mathbb{R}^n \to \mathbb{R}$. Let $\tilde{S} = \Phi(S)$ be the image of $S$ under $\Phi$. Denote by $\nu_y$ and $\nu_x$ the outward unit normal vector of $S$ at $y$ and that of $\tilde{S}$ at $x$. We have

$$
\tilde{\nu}_x = \frac{\|\nabla g(\Phi(x))\|}{\| (D\Phi(x))^t \nabla g(\Phi(x))\|} (D\Phi(x))^t \nu_y,
$$

(A.1)

where $(D\Phi)^t$ is the transpose of the Jacobian matrix.

Let $d\sigma(y)$ and $d\tilde{\sigma}(x)$, where $y = \Phi(x)$, denote the surface measures on $S$ and $\tilde{S}$. Then for any integrable function $f$ on $S$, with $\tilde{f}$ denoting $f \circ \Phi$, we have

$$
\int_S f(y) d\sigma(y) = \int_{\tilde{S}} \tilde{f}(x) \frac{\| (D\Phi(x)^{-1})^t \nabla_x \tilde{g}(x)\| \det(D\Phi(x))}{\|\nabla_x \tilde{g}(x)\|} d\tilde{\sigma}(x).
$$

(A.2)

Proof. The first equality is a calculus fact and it follows from the relations

$$
\nu_y = \frac{\nabla_y g(y)}{\|\nabla_y g(y)\|}, \quad \nu_x = \frac{\nabla_x \tilde{g}(x)}{\|\nabla_x \tilde{g}(x)\|}, \quad \tilde{g}(x) = g(\Phi(x)),
$$

and an application of the chain rule. We verify the second equality from a geometric point of view. For this purpose, set $M = \mathbb{R}^n$ and $N = \Phi(\mathbb{R}^n)$ so that $\Phi : M \to N$ is a diffeomorphism between two manifolds. Recall that the volume form $\gamma_{n-1}$ on the surface $S$, which is a submanifold of $N$, is $\nu_{\Phi^{-1}(y)} \gamma_{n}$, i.e. the interior product of the vector $\nu_y$ with the volume form $\gamma_n$ of $N$. Similarly, the volume form $\tilde{\gamma}_{n-1}$ of $\tilde{S}$ is $\nu_x \tilde{\gamma}_n$. By the change of variable formula, we have

$$
\int_S f(y) d\sigma(y) = \int_{\tilde{S}} f_{\gamma_{n-1}} = \int_{\tilde{S}} \tilde{f}_{\Phi^*\gamma_{n-1}},
$$

where $\Phi^*\gamma_{n-1}$ is the pull-back of the volume form $\gamma_{n-1}$.

Next we aim to find the relation between $\Phi^*\gamma_{n-1}$ to $\tilde{\gamma}_{n-1}$. Let $h$ be the function in front of $(D\Phi(x))^t \nu_y$ in (A.1). We calculate

$$
\Phi^*\gamma_{n-1} = \Phi^* (\nu_y \gamma_n) = \iota_{(D\Phi)^{-1} \nu_y} \Phi^* \gamma_n = \det(D\Phi) (\iota_{(D\Phi)^{-1} \nu_y} \gamma_n)
$$

$$
= \frac{\det(D\Phi)}{h} (\iota_{(D\Phi)\iota_{\nu_x} \gamma_{n}}) = \frac{\det(D\Phi)}{h} \tilde{\gamma}_n((D\Phi)(D\Phi)^t \nu_x, X_2, \cdots, X_n) \nu_x \tilde{\gamma}_{n-1} = h \det(D\Phi) \tilde{\gamma}_{n-1}.
$$

In the second line above, $(\nu_x, X_2, \cdots, X_n)$ is chosen to be an orthonormal frame at $x \in \tilde{S}$. The second equality of the proposition follows from the above identity. □

A.2 Proof of Proposition 4.1

In this section, we prove Proposition 4.1. We recall that by assumption, the $\sigma$-algebra $\mathcal{F}$ is countably generated so that $L^2(\Omega)$ is separable.
Proposition 4.1 says that for almost all realization $\omega \in \Omega$, the weak solution to the auxiliary problem defined in (4.2), where both variables of $\Omega \times \mathbb{R}^d$ are integrated in the weak formulation, is also a weak solution in the usual sense, where only the spatial variable is integrated. Recall that the space of test functions in the usual weak formulation is the composition of
\[
C^\infty_c(\mathbb{R}_d^+ \times \mathbb{R}^-) := \{ \tilde{\phi} = (\tilde{\phi}^+, \tilde{\phi}^-) \mid \tilde{\phi}^+ \in C^\infty(\mathbb{R}_d^+), \tilde{\phi}^- \in C^\infty(\mathbb{R}^-), \text{supp}\tilde{\phi} \subset\subset \mathbb{R}^d \},
\]
with $\Phi^{-1}$. Due to the compact support, the space above is separable, and we can choose a countable dense subset of it denoted by $\{\tilde{\phi}_k\}_{k=1}^\infty$. For any $\omega \in \Omega$, set $\phi_k(\cdot, \omega) = \tilde{\phi}_k \circ \Phi^{-1}(\cdot, \omega)$.

**Proof of Proposition 4.1.** Let $w_p \in \mathcal{H}$ be a weak solution to (2.9) in the sense of (4.2). Fix a $\tilde{\phi}_k$ and let $K$ denote its support. Then for any $\psi \in L^2(\Omega)$, the function $\psi(\omega)\phi_k(x, \omega) \in \mathcal{H}$, and we have
\[
\int_\Omega \psi(\omega) \left( \int_{\Phi(K \cap \mathbb{R}_d^+) \omega} A(\nabla w^+_p + p) \cdot \nabla \phi_k^+ dy + \int_{\Phi(K \cap \mathbb{R}^-) \omega} A(\nabla w^-_p + p) \cdot \nabla \phi_k^- dy ight.
\]
\[
\left. + \int_{\Phi(K \cap \Gamma_d, \omega)} (w^+ - w^-)(\phi^+_k - \phi^-_k) d\sigma(y) \right) d\mathbb{P}(\omega) = 0.
\]
Since $\psi$ is arbitrary, the sum of the inner integrals must be zero almost surely. In other words, there exists a measurable subset $\Omega_k \subset \Omega$ with $\mathbb{P}(\Omega_k) = 1$ and for all $\omega \in \Omega_k$, we have
\[
\int_{\Phi(K \cap \mathbb{R}_d^+) \omega} A(\nabla w^+_p + p) \cdot \nabla \phi_k^+ dy + \int_{\Phi(K \cap \mathbb{R}^-) \omega} A(\nabla w^-_p + p) \cdot \nabla \phi_k^- dy 
\]
\[
+ \int_{\Phi(K \cap \Gamma_d, \omega)} (w^+ - w^-)(\phi^+_k - \phi^-_k) d\sigma(y) = 0. \tag{A.3}
\]
Set $\Omega_0 = \bigcap_{k=1}^\infty \Omega_k$. Then $\mathbb{P}(\Omega_0) = 1$ and (A.3) holds for all $\omega \in \Omega_0$ and all $k = 1, 2, \cdots$. The general case (4.3) follows by the standard density argument. This completes the proof of Proposition 4.1. □

### A.3 Backward induction

For the sake of completeness, we include here a proof of the backward induction, which was used in step two in the proof of Lemma 4.3.

**Lemma A.2 (Backward induction).** Let $E_1 \leq E_2 \leq \cdots \leq E_n$ be an increasing sequence of $n$ nonnegative real numbers. Suppose $E_n \leq Cn^d$ and
\[
E_k \leq C_1(E_{k+1} - E_k + (k + 1)^d), \quad k = 1, 2, \cdots, n - 1. \tag{A.4}
\]
Then there exists a constant $C' = C'(C, C_1, d)$ such that
\[
E_k \leq C'k^d, \quad k = 1, 2, \cdots, n. \tag{A.5}
\]
Proof. First, it is easy to see that the conclusion of the lemma follows if we could prove that there exist \( C_2 > 0, C_3 > 0 \) such that

\[
E_k \leq C_2 (k^d + C_3), \quad k = 1, 2, \ldots, n. \tag{A.6}
\]

Indeed, we can choose \( C' = C_2(C_3 + 1) \) in (A.5). So, we focus on the proof of (A.6). Let us choose \( C_2 = \beta C_1 \) for some \( \beta > 1 \) so that \( C_2 \geq C \). Then for any \( C_3 > 0 \), the inequality in (A.6) holds for \( k = n \). Suppose that this inequality holds for \( j = n, n - 1, \ldots, k + 1 \) but

\[
E_k > C_2 (k^d + C_3) = C_1 \beta (k^d + C_3). \tag{A.7}
\]

Then we will have

\[
E_{k+1} - E_k < C_2 ((k + 1)^d + C_3) - C_2 (k^d + C_3) = C_2 ((k + 1)^d - k^d).
\]

Substitute this relation into (A.4); we get

\[
E_k \leq C_1 ((C_2 + 1)(k + 1)^d - C_2 k^d).
\]

Comparing this inequality with (A.7). We see that

\[
0 > C_1 \beta \left[ C_3 + k^d \left( 1 + C_1 - \left( C_1 + \frac{1}{\beta} \right) \left( 1 + \frac{1}{k} \right)^d \right) \right]. \tag{A.8}
\]

Given \( C_1 > 0 \) and \( \beta > 1 \). There exists a \( k_0(C_1, \beta) \) such that

\[
\left( C_1 + \frac{1}{\beta} \right) \left( 1 + \frac{1}{k} \right)^d - (C_1 + 1) \begin{cases} 
\leq 0 & \text{if } k > k_0, \\
> 0 & \text{if } k \leq k_0.
\end{cases}
\]

Therefore, if we take

\[
C_3 = \max_{k \leq k_0} k^d \left( \left( C_1 + \frac{1}{\beta} \right) \left( 1 + \frac{1}{k} \right)^d - (C_1 + 1) \right),
\]

then (A.8) cannot happen. This means that (A.6) holds with \( C_2 = \beta C_1 \) and \( C_3 \) given above. This completes the proof of the backward induction. \( \square \)

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