Quantum computation, discreteness, and contextuality

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Abstract

We establish a link between contextuality of quantum mechanics and quantum-mechanical computation. Specifically, we show that no deterministic measurement-based quantum computation evaluating a non-linear function can be described by a non-contextual hidden-variable model. We give examples for such computations derived from quantum codes with suitable transversality properties, and from a counterexample to the LU-LC conjecture.

1 Introduction

When confronted with the mysteries of the quantum world, one often calls for classical analogues. Quantum information is no exception. For example, it has been asked whether quantum computation is a quantum analogue to classical digital or to classical analog computation. In this regard, it was pointed out that classical analog computation cannot be stabilized against error while quantum computation can [1, 2]. It is therefore hard to perceive the two as mutual analogues.

Here we explore the remaining possibility, namely that quantum computation is digital. We pursue this position not as an end, but rather as a means to inquire about the foundations of quantum computation, and to aim at a different approach to quantum algorithms.

The established concepts of quantum interference, phase estimation [3] and amplitude amplification [4] provide a framework within which the most powerful known quantum algorithms, such as Shor’s factoring algorithm [5] and Grover’s search algorithm [6], are conveniently explained. These concepts therefore shape our thinking about quantum algorithms and the origin of the quantum speed-up. In the search for novel quantum algorithms and their construction principles, we may therefore begin with an alternative concept of the above general kind. Our pick is the digital character of quantum computation, or discreteness. We take the position that both classical and quantum computation are built on discrete data structures; and this is the respect in which they are mutual analogues. Is such a position justifiable? What can be deduced from this starting point?

The elementary unit of information in classical computation is the ‘bit’. Its quantum counterpart is usually taken to be the qubit—the merger of the classical bit with the superposition principle of quantum mechanics. But the qubit is not discrete: in any neighborhood, however small, of a valid one-qubit quantum state there is another such state.

To begin, we therefore need to identify discrete structures in Hilbert space that underlie algorithmic procedures. Such structures exist, and we provide examples for them in this paper. To summarize, our program is as follows: (1) Identify discrete structures in finite-dimensional Hilbert spaces, and relate them to quantum mechanical computation, (2) Show that quantum mechanics is indeed harnessed in these computations, (3) Identify key features required for a speed-up to emerge and construct families of algorithms that possess these features. Repeat until an example algorithm with a significant quantum speed-up is found. (4) Develop a theory for discrete quantum computations.
In the present paper, we complete steps 1 and 2 of the above program. We derive examples for discrete quantum algorithms from quantum codes with non-Clifford transversal gates. That is, we sever quantum codes from their originally intended use of correcting errors, and instead let them compute. A further discrete quantum algorithm is derived from a counterexample \[11\] to the recently refuted LU-LC conjecture \[12, 13\].

A suitable setting for our discussion is deterministic quantum computation \[10\], where the computational result is obtained with unit probability. Thus, voluntarily and emphatically we restrict to quantum phenomena in which the general probabilistic character of quantum-mechanical predictions does not come into play. Thereby, arguably, one of the most prominent features of quantum mechanics is left out of the picture. We therefore need to ask: What is quantum mechanical about deterministic quantum algorithms?

We address this question in Section 4. We prove that no deterministic quantum computation which computes a non-linear function can, in its measurement-based version, be described by a non-contextual hidden-variable model. In other words, every such computation implies a proof of the Kochen-Specker theorem\[8, 9\]. Also note that efficient quantum algorithms remain in our class. Specifically, a deterministic version of the quantum algorithm solving discrete log exists \[10\]. It is closely related to Shor’s factoring algorithm \[5\], and breaks the Diffie-Hellmann public key exchange.

We point out that the simple deterministic quantum computations discussed as examples in this paper do not yet lead to significant quantum speed-ups. We discuss them nonetheless, in order to demonstrate first of all that there are discrete structures in Hilbert space that give rise to computations. The functions computed in these examples are non-linear. The ‘linear vs. non-linear’ separation is important and natural for the one-way quantum computer \[15\]. Also, it is the separation between measurement-based quantum computations which can and which cannot be described by a non-contextual hidden variable model.

## 2 The setting of deterministic quantum computation

A setting conducive to our study is deterministic quantum computation \[10\]. Specifically, we restrict to operation on finite-dimensional Hilbert spaces, and require

**Criterion 1.** For deterministic quantum computations,

\[
\text{Input and output are classical bit strings;} \tag{1a}
\]

\[
\text{The computation succeeds with certainty.} \tag{1b}
\]

Property \[1a\] excludes quantum algorithms in which the output is presented as the real-valued expectation value of an observable.

We consider deterministic quantum computation in both the circuit and the measurement-based model. The former requires less preparation, but in the latter the connection with contextuality of quantum mechanics is easier to make.

### 2.1 The circuit model

We consider a quantum register of two sets \(I\) and \(A\) of qubits, with \(B_I\) the computational basis of \(\mathcal{H}_I\), and a unitary circuit \(C\) such that

\[
C (|i\rangle_I \otimes |0\rangle_A) = |o\rangle_I \otimes |\psi(i)\rangle_A, \text{ with } |o\rangle_I \in B_I \forall |i\rangle_I \in B_I. \tag{2}
\]

We regard \(i\) as the input and \(o\) as the output of the computation, and are interested in the function \(o = f(i)\).

\[1\] The Hilbert spaces in these proofs have dimension larger than 3, though.
2.2 Measurement-based model

We now translate the circuits (2) into their measurement-based versions, for which the connection with contextuality of quantum mechanics will be easier to make. Specifically, we consider the one-way quantum computer [15] in which quantum computation is driven by local measurements on an entangled quantum resource state $|\Psi\rangle$.

For all qubits $k$ in the support $\Omega = \{1, 2, ..., n\}$ of $|\Psi\rangle$ the measured local observables take the form

$$O_k[q_k] = \cos \phi_k X_k + (-1)^{q_k} \sin \phi_k Y_k.$$  

We call the $\phi_k$, $k \in \Omega$, the measurement angles. The measurement outcomes corresponding to the observables $O_k[q_k]$ are denoted as $s_k = [s]_k \in \mathbb{Z}_2$, with $(-1)^{s_k}$ being the measured eigenvalue. The relations between measurement outcomes $s$ and computational output $o$, and between input $i$, measurement outcomes and basis choice $q$ are all linear [16]. Specifically,

$$
\begin{pmatrix}
q \\
o \\
i
\end{pmatrix} = \begin{pmatrix}
Q & T \\
0 & Z
\end{pmatrix} \begin{pmatrix}
i \\
s
\end{pmatrix}.
$$  

Therein, all addition is modulo 2. It may appear that through the matrix representation in [4] a need to perform multiplication enters into the classical processing. This is not the case. The binary-valued matrices $Q$, $T$ and $Z$ are known in advance and are invariable. Therefore, the individual components of $o$ and $q$ can be expressed as sums $o_i = o_{J_i} = \sum_{l \in J_i} s_l \mod 2$, $q_i = q_{K_i} = \sum_{l \in K_i} s_l \mod 2$, with fixed sets $J_i, K_i$. Thus, addition mod 2 is sufficient for the classical processing.

For a (partial) ordering among the measurements to exist we require that if the measurement event at $k$ is influenced by the measurement at $l$, then the event at $l$ cannot be influenced by the event at $k$. That is, $[T]_{kl} [T]_{lk} = 0$, $\forall k, l \in \Omega$. To summarize, a deterministic quantum algorithm is in its measurement-based version specified by the quantum resource state $|\Psi\rangle$, the set of measurement angles $\{\phi_k, k \in \Omega\}$, and the matrices $Q, Z, T$. See Fig. for an illustration.

All examples for deterministic quantum computations given in the subsequent sections pertain to the special case of flat temporal order, $T = 0$. Then, the equations [4] for $q$ and $o$ decouple,

$$q = Qi,$$

$$o = Zs.$$  

In particular, the choice of measurement bases $q$ is a linear function of the input $i$ only. For flat temporal order, the quantum mechanical observables which bit-wise correspond to the computational output take a very simple form. Denote by $z^{(r)}$ the vector corresponding to the $r$-th row of $Z$, and define the correlation operator

$$O_{z^{(r)}}[q] = \bigotimes_{j | [z^{(r)}]_j = 1} O_j[q_j]$$  

of the local operators $O_j[q_j]$. Given the promise of deterministic computation, $\langle \Psi | O_{z^{(r)}}[q] | \Psi \rangle = \pm 1$, $\forall r$, the relation between the correlation operator $O_{z^{(r)}}[q]$ and the $r$-th output bit, $o_r$, is

$$(-1)^{o_r} = \langle \Psi | O_{z^{(r)}}[q] | \Psi \rangle.$$  

With Eqs. [6] and [7], it is easily verified that the linear relation [5b] holds.

The harder part is to identify a set of measurement angles $\{\phi_j, j \in \Omega\}$ and a matrix $Q$, c.f. Eq. [5a], such that the promise $\langle \Psi | O_{z^{(r)}}[q] | \Psi \rangle = \pm 1$, $\forall r$, is satisfied. In this paper, we are interested in quantum computations outside the Clifford group. For the measurement-based version, since the $|\Psi\rangle$ will be taken stabilizer states, we require non-Pauli local measurements. That is, $O_j[q_j] \neq$
Figure 1: Input and output in measurement-based quantum computation. Each bit of the computational output is given by the extremal expectation value of a correlation operator, and is extracted as a parity of local measurement outcomes. Through the relation $q = Q_i$, the input $i$ selects the correlations to be measured. For each qubit site $a$, the component $q_a$ of the vector $q$ specifies the local measurement basis at $a$.

$X, Y, Z$, for all $j \in \Omega$. In this situation, satisfying the promise $\langle \Psi | O_{z(r)}[q] | \Psi \rangle = \pm 1$, $\forall r$, even for a single vector $q$, at least requires a careful adjustment of the measurement angles. Even then, satisfying the promise is the exception rather than the rule. But we require more. The number of possible evaluations of the function $o = g(i)$ equals the number of admissible vectors $q$. Thus, for fixed measurement angles, we require the set of admissible $q$ to contain two or more elements. On top of that, we require this set to be a vector space. Only then is the computed function $o = g(i)$ total; c.f. relation (5a). If, against many odds, for a suitable resource $|\Psi\rangle$ all these requirements are met, we can finally ask the question: "Is the computed function $o = g(i)$ any interesting?"

2.3 Why non-linear functions

In the present context, we are first of all interested in the phenomenology of deterministic quantum computations, and would like to come up with examples of small size. We require a suitable criterion for what should constitute a sufficiently interesting such example. For the confines of this paper, we choose the following

**Criterion 2.** *A proper deterministic quantum computation evaluates a non-linear Boolean function.*

A Boolean function $f$ is non-linear if it cannot be written as $f(x) = Ax + b$, with $x \in \mathbb{Z}_2^n$, $b \in \mathbb{Z}_2^n$ and $A$ a binary-valued $n \times n$ matrix.

One could certainly place the bar much higher than in Criterion 2—Ultimately, we are interested in quantum algorithms that yield a significant speed-up. However, our present much weaker criterion seems adequate for the task of gathering initial phenomenology. There is three-fold justification for placing a separation line between linear and non-linear functions,

1. The separation between linear vs. non-linear functions is natural in measurement-based quantum computation with cluster states [15]. There, the computational capability of the classical control device is restricted to addition modulo 2. This limited device can evaluate all linear functions by itself; for non-linear functions it requires access to a quantum resource.
2. The capability to compute one arbitrary non-linear function, on top of the linear functions, implies classical universality. 

3. Deterministic measurement-based algorithms which compute a non-linear function cannot be described by a non-contextual hidden-variable model, and are thus non-classical. See Section 3.2.

The separation between linear vs. non-linear functions in measurement-based quantum computation (containing one-way quantum computation as a special case) has recently been emphasized in [7]. Therein, a particularly simple measurement-based realization of the “and”-gate was presented, using a GHZ-state as quantum resource. If supplemented by GHZ-states and the capability to measure them locally, a classical control computer restricted to additions mod 2 becomes classically universal. For this GHZ example, a relationship between computational power and the violation of local realistic theories was established.

In this paper, we will only consider deterministic quantum computations which, in addition to obeying Criterion 2, contain non-Clifford operations. The reason is that such computations are not amenable to efficient classical simulation by the stabilizer formalism [17].

Equivalence of non-linear functions. In the scenario introduced above, the ability to compute non-linear Boolean functions is a resource whereas computing linear functions is for free. We therefore call two functions \( f, g \) equivalent if there exist linear functions \( L_1, L_2, L_3, L'_1, L'_2, L'_3 \) such that

\[
g \equiv L_3(f, L_2)L_1, \quad \text{and} \quad f \equiv L'_3(g, L'_2)L'_1. \tag{8}
\]

Therein, \((m, n)\) denotes the binary vector obtained by concatenating \(m\) and \(n\).

3 Example 1

This section introduces, through a simple example, the two main themes of this paper: identifying discrete structures in Hilbert space for computation, and the relation of these structures to foundational aspects of quantum mechanics.

3.1 Deterministic quantum computation with a 4-particle GHZ-state

We consider the deterministic measurement-based computation with a four-qubit Greenberger-Horne-Zeilinger (GHZ)-state, \(|GHZ_4\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}\), as computational resource. The choice for the locally measured observables is between

\[
L := O[1] = \frac{X - Y}{\sqrt{2}}, \quad \text{and} \quad R := O[0] = \frac{X + Y}{\sqrt{2}}. \tag{9}
\]

Furthermore, the linear relations (5) between input \(i\) and measurement bases specified by \(q\), and between measurement results \(s\) and computational output \(o\) are

\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
i_1 \\
i_2 \\
i_3 \\
\end{pmatrix}, \quad \begin{pmatrix}
o_1 \\
o_2 \\
o_3 \\
o_4 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4 \\
\end{pmatrix}. \tag{10}
\]

---

2 Computability of any non-linear function on top of linear functions implies the “and”-gate: Wlog consider a nonlinear function \(f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2\). Then, there exist vectors \(a, b, c \in \mathbb{Z}_2^n\) such that \(f(c) \oplus f(c \oplus a) \oplus f(c \oplus b) \oplus f(c \oplus a \oplus b) = 1\). For this triple of vectors \(a, b, c\), and for \(r, s \in \{0, 1\}\), define \(g(r, s) := f(c \oplus ra \oplus sb) \oplus f(c \oplus ra) \oplus f(c \oplus sb) \oplus f(c)\). The function \(g\) can be computed by querying \(f\) and addition mod 2. Also, it is easily verified that \(g(r, s) \equiv rs = r \land s\). Addition mod 2 plus the ability to perform “and”-gates yields universal computation.
As we now demonstrate, this procedure allows to compute a non-linear function equivalent to the Toffoli gate. It is easily verified that $|\text{GHZ}_4\rangle$ is an eigenstate with eigenvalue 1 of the following tensor product operators:

$$
-L_1L_2L_3L_4, \quad L_1L_2R_3R_4, \quad L_1R_2L_3R_4, \quad L_1R_2R_3L_4, \quad L_1R_3R_2L_4,
$$

$$
R_1L_2L_3R_4, \quad R_1L_2R_3L_4, \quad R_1R_2L_3L_4, \quad -R_1R_2R_3R_4.
$$

Note that two of these operators, namely those with all $L$ or all $R$, carry a minus sign.

Now, if $(t_1,t_2,t_3) = (1,1,1)$ then $(q_1,q_2,q_3,q_4) = (1,1,1,1)$, and the local measured observables are $L_1, L_2, L_3, L_4$. Then, with Eqs. (9), (11), $o_1 = \sum_{i=1}^{4} s_i \mod 2 = 1$. If $(t_1,t_2,t_3) = (0,0,0)$ then $(q_1,q_2,q_3,q_4) = (0,0,0,0)$, and the local measured observables are $R_1, R_2, R_3, R_4$. Then, $o_1 = \sum_{i=1}^{4} s_i \mod 2 = 0$. Thus, the computed function is

$$
o_1 \equiv f(i) = \delta(i_1,i_2)\delta(i_2,i_3).
$$

With the equivalence relation (8), i.e., modulo linear functions, the function $f \equiv o_1$ computed by measurement-based computation on the resource $|\text{GHZ}_4\rangle$ is equivalent to a Toffoli gate $g : (t,c_1,c_2) \rightarrow \text{Toff}(t,c_1,c_2,t) = (t \oplus c_1c_2,c_1,c_2)$. In (8), let $L_1(x) = A_1x + b_1$, $L_2(y) = A_2y + b_2$, $L_3(z) = A_3z$, and

$$
A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

Similar expressions hold for $L_1', L_2', L_3'$.

### 3.2 Computation and contextuality

Certainly, it does not require a quantum system to efficiently implement a classical Toffoli gate. Therefore, for our quantum-mechanical way of implementing it, we need to ask: *Do we really harness quantum mechanics in this computation?* The answer is ‘Yes’. Our procedure is quantum, or at least non-classical, because it cannot be described by a non-contextual hidden-variable model.

Hidden-variable models (HVMs) were spurred by Einstein, Podolsky and Rosen’s observation [31] that, given a set of assumptions about what should constitute ‘physical reality’, quantum mechanics could not be considered a complete description of the physical world. Contrary to a quantum-mechanical description of a physical system, according to which the outcome of a measurement is brought into existence by the act of measurement, in a hidden-variable model the outcome is merely revealed by the measurement and was a property of the measured system all the time before. Specifically, for each member in an ensemble of identically prepared states, values $v_A, v_B, v_C, \ldots$ are assigned as pre-existing measurement outcomes to observables $A, B, C, \ldots$. The sets of values may differ from copy to copy in the ensemble, and are distributed in such a way as to reproduce the quantum-mechanical predictions for the measurement of a set of commuting observables.

With no additional assumptions made, hidden-variable models cannot be ruled out as descriptions of physical reality; See [32] for an example. This changes, however, with the seemingly most innocent additional assumptions. One such assumption is *non-contextuality*. It asserts that, for commuting observables, the values $v$ only depend upon the observable they are assigned to, but not on the observables measured in conjunction; i.e, $v_A = v(A), v_B = v(B)$, etc. With this additional assumption, HVMs are ruled out by the Bell-Kochen-Specker theorem [8, 9].

Versions of the Bell-Kochen-Specker (BKS) theorem have been derived for different settings, and the original proofs have been simplified [33] - [37]. In the context of quantum information, the BKS theorem has been related to quantum codes [38, 39], quantum cryptography [40], two-party
secure computation [41] and to quantum games [42, 43]. Also, the BKS theorem has recently been tested experimentally [44].

Of particular interest to us are Mermin’s proofs [34] of the BKS theorem, one applying to a Hilbert space of dimension \(d \geq 4\) and one to \(d \geq 8\). The latter of the two admits the foundation of non-contextuality in locality. Also, the argument becomes state-dependent, with the state in question being a three-party Greenberger-Horne-Zeilinger state (GHZ-state) [29]. This state-dependence may appear as a drawback from some perspective, but for us it provides a link to computation: as has been shown by Anders and Browne [7], accessing GHZ-states by local measurement renders a very limited classical control computer classically universal.

Since the present computational resource is a four-particle GHZ-state, our discussion of contextuality will closely parallel Mermin’s argument [34] for a three-particle GHZ-state. We start from the assumption that a non-contextual hidden variable model describing measurement-based computation on \(|\text{GHZ}_4\rangle\) exists, and derive a contradiction. We consider the nonlocal observables from the first row of Eq. (11) and the local observables \(L_i, R_i\), for \(i = 1..4\), whose measurement drives the computation. An HVM must assign values

\[
v(L_1 L_2 L_3 L_4), v(L_1 L_2 R_3 R_4), v(L_1 R_2 L_3 R_4), v(L_1 R_2 R_3 L_4),
\]

\[
v(L_1), v(L_2), v(L_3), v(L_4),
\]

\[
v(R_2), v(R_3), v(R_4),
\]

(13)

to the respective observables such that its predictions are compatible with quantum mechanics. From Eq. (11) then follows that \(v(L_1 L_2 L_3 L_4) = -1, v(L_1 L_2 R_3 R_4) = v(L_1 R_2 L_3 R_4) = v(L_1 R_2 R_3 L_4) = 1\). Also, in order to agree with quantum-mechanics, for a set of mutually commuting operators the corresponding values \(v\) must obey the same identities as the operators do. Thus, \((L_1 L_2 L_3 L_4) L_1 L_2 L_3 L_4 = I\) implies \(v(L_1 L_2 L_3 L_4) v(L_1) v(L_2) v(L_3) v(L_4) = 1\), etc, such that

\[
v(L_1) v(L_2) v(L_3) v(L_4) = -1,
\]

\[
v(L_1) v(L_2) v(R_3) v(R_4) = 1,
\]

\[
v(L_1) v(R_2) v(L_3) v(R_4) = 1,
\]

\[
v(L_1) v(R_2) v(R_3) v(L_4) = 1.
\]

(14)

No assignment to local values \(v(\cdot) = \pm 1\) can satisfy the set of constraints (14). The product \(\Pi\) of all \(v\) as they appear in (14) can be evaluated in two ways, namely (a) by first multiplying within the rows and (b) by first multiplying within the columns. In (a) one obtains \(\Pi = -1\), as is evident from the right column in (14). Regarding (b), note that every \(v(\cdot)\) appears an even number of times in the table, hence \(\Pi = 1\). Contradiction. Thus, the statistics from measurement of observables (9) on a four-particle GHZ-state does not permit a non-contextual HVM description.

4 Measurement-based quantum computation is contextual

The connection between deterministic quantum computation and contextuality of quantum mechanics is not limited to a few examples such as the one discussed in the previous section. Rather, it generalizes to a large class of deterministic measurement-based quantum computations. In this regard, note the following

**Theorem 1.** Consider measurement-based quantum computation by local projective measurements on a multi-qubit quantum state, with at most two choices for the measurement basis at each qubit location. If such a computation deterministically computes a non-linear Boolean function then it cannot be described by a non-contextual hidden-variable model.

In other words, every proper deterministic quantum computation by measurement implies a proof of the Bell-Kochen-Specker Theorem, albeit in higher dimensions than three.
Proof. We assume a non-contextual HVM for measurements on the computational resource $|\Psi\rangle$ assigning pre-existing values $v(O_i[q_i]) = (-1)^s_i[q_i]$ to the local observables $O_i[q_i]$, $q_i \in \{0, 1\}$, $\forall i$, and pre-existing values $v(O_J[q]) = (-1)^{a_J[q]}$ to the non-local observables $O_J[q]$, for all allowed supports $J$. Recall that, for each $J$, $\langle O_i[q_i]\rangle$ yields one bit of output, and $q$ comprises the input to the computation. Let $K$ be the index set for an ensemble of identically prepared quantum states $|\Psi_k\rangle \equiv |\Psi\rangle$, for all $k = 1..|K|$. Within the HVM, the assigned values $s_i$ may depend on $k \in K$; and

$$s_i^{(k)} = c_i^{(k)} \oplus d_i^{(k)} q_i,$$

(15)

There is no loss of generality in assuming a linear relation between $s_i$ and $q_i$, since there are only two data points to match, $q_i = 0$ and $q_i = 1$. The parameters $c_i^{(k)}$ and $d_i^{(k)}$ contain exactly the same information as $s_i^{(k)}[q_i = 0]$ and $s_i^{(k)}[q_i = 1]$.

In an HVM description, the expectation for the observable $O_J$ is $\langle O_J[q]\rangle = \frac{1}{|K|} \sum_{k=1}^{|K|} (-1)^{s_j^{(k)}[q]}$. Since $O_J[q]$ and the local $O_J[q_j]$ are mutually commuting, in order for the HVM to agree with quantum mechanics, we must require that $a_j^{(k)}[q] = \sum_{j \in J} s_j^{(k)}[q_j] \mod 2$, for all $k = 1..|K|$, and therefore

$$\langle O_J[q]\rangle \equiv \langle \Psi|O_J[q]|\Psi\rangle = \frac{1}{|K|} \sum_{k=1}^{|K|} (-1)^{\sum_{j \in J} s_j^{(k)}[q_j]}.$$

(16)

Since the computation is deterministic, $\langle O_J[q]\rangle = \pm 1$. There are $|K|$ terms in the sum over $k$ on the rhs of (16), each having a weight of $1/|K|$. Therefore, the expectation value can only be extremal if each term in the $k$-sum has the same sign, i.e., $\sum_{j \in J} s_j^{(k)}[q_j] = \sum_{j \in J} s_j^{(1)}[q_j]$ for all $k = 1..|K|$. Thus, with (15),

$$a_j^{(k)}[q] = \sum_{j \in J} s_j^{(k)}[q_j] = \sum_{j \in J} c_j^{(1)} \oplus d_j^{(1)} q_j \mod 2 =: a_j[q], \ \forall k = 1..|K|.$$

(17)

For all valid input vectors $q$, the output $a_j[q]$ of the computation is linear in $q$. Thus, a description of a deterministic MQC by a non-contextual HVM implies that only linear functions can be computed. Negation of this statement yields Theorem 1.

5 Computing with codes and LU-LC counterexamples

Next, beyond the initial example given in Section 3, we identify a range of examples for discrete structures in Hilbert space that give rise to computation. So, where shall we look? As noted in the introduction, elements of discreteness are revealed in the techniques of quantum coding and error-correction. Among them are error-discretization by measurement [13], discrete sets of transversal encoded gates, and the Solovay-Kitaev construction [19, 20] for approximating arbitrary gates by sequences of gates from a fixed set. We pick the discreteness of the transversal gate set as our starting point. For any quantum code a certain group of encoded gates can be implemented transversally. This group varies from code to code, but it is always finite. For a given quantum code, within a sufficiently small neighborhood of a transversal gate there is no other.

A second place to look for discrete structures in quantum computation are counterexamples to the recently refuted [11] LU-LC conjecture. The LU-LC conjecture says that whenever two stabilizer states are local unitary equivalent they are also local Clifford equivalent. Were the LU-LC conjecture true, it would imply a computationally efficient test for local unitary equivalence among stabilizer states, because local Clifford equivalence can be efficiently tested [13]. Such a test is desirable for a number of reasons: Local unitary equivalent stabilizer states contain the same amount and kind of entanglement and—as resources for measurement based quantum computation—have the same computational power. Furthermore, local unitary equivalent stabilizer codes offer the same
protection against local error. But, as has recently been demonstrated by a counterexample \cite{11}, the LU-LC conjecture is false. Due to the above motivations for the conjecture, its breakdown is easily seen through a negative lens. But there is a positive angle to it as well: Hilbert space is more intricate and more intriguing than sometimes expected!

The unifying element among the above quantum codes and LU-LC counterexamples is the interconversion between stabilizer states by non-Clifford local unitaries. The existence of such transformations is a remarkable property of Hilbert space. Here, we relate this property to computation.

5.1 From quantum error-correction to quantum algorithms

Consider a family of quantum state \( \{ |\psi_\alpha \rangle := \bigotimes_{i=1}^n \exp(i\alpha Z_i)|\psi_0 \rangle, \alpha \in \mathbb{R} \} \), where \( |\psi_0 \rangle \) is in the code space \( \mathcal{H}_{\text{code}} \) of an \( n \)-qubit stabilizer code. For all except a finite set of angles \( \alpha_k = k\alpha_1, k \in \mathbb{Z} \), the state \( |\psi_\alpha \rangle \) is not in the code space. As \( \alpha \) is increased from 0, the trajectory of \( |\psi_\alpha \rangle \) departs from the code space but then intersects the code space again at a finite angle \( \alpha_1 \). \( |\psi_{\alpha_1} \rangle \) is related to \( |\psi_0 \rangle \) by an encoded unitary transformation \( U \). E.g. for the Steane code, \( \alpha_1 = \pi/4 \) and \( U = \exp(-i\pi/4 Z) \) is a transversal gate. A 15-qubit CSS code based on the punctured Reed-Muller code \( \mathcal{R}^*(1, 4) \) has a non-Clifford transversal gate \cite{26}. For this reason, we will discuss the Reed-Muller codes in greater detail below.

Our task now is to make the transition from error-correction to algorithmic procedures. To this end, we sever quantum codes from their native context of error-correction and devise a method to employ them as computational resources. We consider an \( [n, 1, d] \)-stabilizer code with stabilizer generators \( \{g_i, i = 1, \ldots, n-1\} \), for which an encoded gate \( U = \exp(i\beta Z) \) is transversal, i.e. \( \exists V = \exp(i\alpha Z) \) such that, on the code space,

\[
U \equiv \bigotimes_{i=1}^n V_i. \tag{18}
\]

We invoke this code into a quantum circuit consisting of the following steps:

1. Preparation of a state \( |\Psi_{\text{in}}(\mathbf{p}) \rangle \) with stabilizer generated by \( \{(-1)^{\rho_i} g_i, i = 1, \ldots, n - 1\} \cup \{(-1)^{\rho_i} X\} \),

2. Applying the transversal unitary \( \bigotimes_{i=1}^n V_i \), followed by \( U^{-1} \),

3. Measurement of the observables \( g_i \) and \( X \), yielding outcomes \( (-1)^{\alpha_i}, i = 1..n \).
A graphic representation of this circuit is shown in Fig 2b. The circuit \[ (19) \] is a special instance of the general setting \[ (2) \], with the ancilla set \( A = \emptyset \), if one adds encoding and decoding circuits to transform between the computational and the stabilizer basis.

**How does the circuit \[ (19) \] compute?** The binary string \( p = (p_1, \ldots, p_n) \) encodes the input to the computation, and \( o = (o_1, \ldots, o_n) \) is its output. First, consider the special case of \( p_i = 0 \) for all \( 1 \leq i \leq n-1 \). That is, \( |\Psi_{\text{in}}(p)\rangle \) is in the code space and only the eigenvalue \((-1)^{p_n}\) of \( X \) is allowed to vary. Then, the unitaries in step 2 of the protocol \[ (19) \] mutually cancel, and the computed function is the identity \( \text{Id} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \). Next, consider more general inputs \( p \) for which \( |\Psi_{\text{in}}(p)\rangle \) is perpendicular to the code space. Then, the transversality identity \[ (18) \] no longer applies. The expectation values of the observables measured in step 3 of \[ (19) \] will in general not be extremal, violating the determinism condition \[ (1a) \]. Thus, we restrict the binary vectors \( p \) to a set \( \mathcal{P} \), such that \[ (1a) \] remains satisfied for all \( p \in \mathcal{P} \). For those \( p \), we are interested in the function \( o = f(p) \) computed by the circuit \[ (19) \]. We seek quantum codes for which the function \( f \) is non-linear.

We have, from the present perspective, no guarantee that such quantum codes exist. Furthermore, if they do, we do not know whether there is additional structure to the corresponding sets \( \mathcal{P} \) of admissible input vectors. It is desirable for \( \mathcal{P} \) to be a vector space, such that the computed function \( f \) is total. To get a first glimpse at the situation, we try a few examples.

### 5.2 Example 2: the punctured Reed-Muller code \( R^*(1, 5) \).

We now specialize the circuit \[ (19) \] to an instance where a non-linear function is indeed computed. Left unspecified in the circuit \[ (19) \] is the code upon which the incoming quantum state \( |\Psi_{\text{in}}(p)\rangle \) is based. Of our interest are circuits with non-Clifford gates, and we thus require quantum codes for which such gates are transversal. One such family are CSS codes \[ [21, 22] \] based on punctured Reed-Muller codes \( R^*(1, m) \) \[ [23, 24] \]. The resulting quantum codes \( R^*_Q(1, m) \) require \( 2^m - 1 \) qubits and have a transversal non-Clifford gate \( U = \exp((i\pi/8)Z) \), if \( m \geq 4 \) \[ [26] \].

We begin with a definition for Reed-Muller codes \[ [23, 24] \]; also see \[ [25] \]. Consider a Boolean function \( h(v_1, \ldots, v_m) = h(v) \), and denote by \( h \) the \( 2^m \)-component vector obtained from all the evaluations of \( h \), \( [h]_v = h(v) \). Then,

**Definition 1.** The \( r \)-th order binary Reed-Muller code \( R(r, m) \) of length \( n = 2^m \), for \( 0 \leq r \leq m \), is the set of all vectors \( h \), where \( h(v_1, \ldots, v_m) \) is a Boolean function which is a polynomial of degree at most \( r \). For \( 0 \leq r \leq m-1 \), the punctured Reed-Muller code \( R^*(r, m) \) is obtained by puncturing (or deleting) the coordinate corresponding to \( v_1 = v_2 = \ldots = v_m = 0 \) from all codewords of \( R(r, m) \).

Each code \( R^*(r, m) \) has a specific basis \( B^*(r, m) = \{b^{(1)}, \ldots, b^{(d)}\} \), where the basis vectors \( b^{(i)} \), for \( 1 \leq i \leq \dim(R^*(r, m)) \), are obtained, respectively, from the polynomials \( 1, v_1, v_2, \ldots, v_m, v_1v_2, \ldots, v_{m-1}v_m, v_1v_2v_3, \ldots, v_{m-r+1}v_{m-r+2} \cdots v_m \) of degree \( 1 \cdots r \). In particular, \( b^{(1)} = (1, 1, \ldots, 1) \). Denote by \( [B^*(r, m)] \) the binary matrix with the elements of \( B^*(r, m) \) as its rows. Further, let the binary matrices \( G_X, G_Z \) specify the stabilizer generator matrix for a CSS code in standard form, and \( X \) and \( Z \) be the characteristic vectors of the support of the encoded Pauli operators \( X \) and \( Z \). Then, the CSS code \( R^*_Q(r, m) \) based on the punctured Reed-Muller code \( R^*(r, m) \) is defined by

\[
\begin{pmatrix}
X \\
G_X \\
Z \\
G_Z
\end{pmatrix} = \begin{pmatrix}
B^*(r, m) \\
B^*(m - r - 1, m)
\end{pmatrix}.
\]  

(20)

Before we study the circuits \[ (19) \] numerically, we note that whenever the quantum states \( |\Psi_{\text{in}}(p)\rangle \) entering the circuit are CSS states, and \( p \in \mathcal{P} \), the computed function takes the form

\[
\begin{pmatrix}
o_X \\
o_Z
\end{pmatrix} = \begin{pmatrix}p_X \oplus \hat{f}(p_Z) \\
p_Z
\end{pmatrix}.
\]  

(21)
Therein, \( p = (p_X, p_Z) \) and \( o = (o_X, o_Z) \). Thus, since we are after non-linear functions, we may focus on \( o_X \), and furthermore set \( p_X = 0 \).

We now perform a numerical study of the circuit \((19)\), with the initial states \( |\Psi_{in}(p)\rangle \) related to the CSS code \( R_Q^+(1, m) \) as follows: \( |\Psi_{in}(0)\rangle = |+\rangle \) is in the code space of \( R_Q^+(1, m) \), and the states \( |\Psi_{in}(p)\rangle \), for all \( p \neq 0 \), have the same stabilizer as \( |\Psi_{in}(0)\rangle \) except that some stabilizer eigenvalues, as specified by \( p \), are flipped.

The resulting circuit is, for moderate values of \( m \), amenable to classical simulation using weight enumerators. The smallest value of \( m \) which permits a non-Clifford transversal gate is \( m = 4 \). For this case, however, the computed function turns out to be \( \tilde{f}(p_Z) \equiv 0 \), which is not useful.

Next, we try \( m = 5 \) which results in a quantum code on 31-qubits. This time, the computed function \( \tilde{f}(p_Z) \) is non-constant. Also, it turns out that the set \( P \) of admissible \( p_Z \) is a vector space. We may thus write \( p_Z = P_i \), where the columns of \( P \) form a basis of \( P \). The computed function is thus total. Furthermore, in each component corresponding to the measurement of an \( X \)-stabilizer generator, the output value 0 is found more often than the output value 1. The relative frequencies for finding these outcomes have a ratio of 9:7. Since any linear function is constant or balanced, the function \( o_X(i) = \tilde{f}(P_i) \) computed here must be non-linear. We have thus found an example for a proper deterministic quantum computation.

### 5.3 Example 3: Measurement-based computation with Reed-Muller codes

It is now natural to ask “Are the examples for deterministic computation from the previous sections isolated ones, or are they members of large families?” In this section we show that the two examples discussed previously each belong to a large family, and, in fact, the same family. We provide a criterion for when measurement-based computation with Reed-Muller codes deterministically evaluates a nonlinear function, and give a closed-form expression for what these functions are.

The previous example 1 was within the framework of measurement-based computation, and example 2 within the circuit model. To facilitate their grouping together, we translate the latter into the measurement-based model. Under this translation, the “Reed-Muller-ness” is moved into the computational resource \( |\Psi\rangle \). We make the following

**Definition 2.** A Reed-Muller quantum state \( |R(r, m)\rangle \) is

\[
|R(r, m)\rangle = \frac{1}{2^{d/2}} \sum_{h \in \mathcal{R}(r, m)} |h\rangle,
\]

with \( d \) the dimension of \( \mathcal{R}(r, m) \).

If a circuit \((19)\) is based on a CSS-code \( R_Q^+(r, m) \), then the corresponding measurement-based computation uses a resource state \( |R(r, m)\rangle \). We do not provide a detailed proof of this statement.

The key observation is that \( |R(r, m)\rangle \) is a Bell state among a bare qubit and a qubit encoded with \( R_Q^+(r, m) \). The remainder of the argument consists of re-ordering operations.

The \( |R(r, m)\rangle \) are stabilizer states, and are highly symmetric under permutations of their respective qubits. This symmetry is revealed when placing the qubits of \( |R(r, m)\rangle \) on the sites of an \( m \)-dimensional hypercube; See Table 1 for a graphic representation.

To completely characterize a measurement-based based quantum computation, we need more than the resource state. We also need to specify the measurement angles \( \{\phi_k, k \in \Omega\} \) for the locally measured observables, as well as the linear relations between computational input and choice of

---

\[3\]A simple consistency check is as follows. The computational resource \( |\Psi\rangle \) of the measurement-based version must have as many qubits as the circuit \((19)\) has non-Clifford gates. Now, the circuit based on \( R_Q^+(r, m) \) has \( 2^m - 1 \) qubits and \( 2^m \) non-Clifford gates. Thus, in the measurement-based version, the resource state \( |R(r, m)\rangle \) must have \( 2^m \) qubits, which is confirmed by Definition 2.
Table 1: Small computational resources $|\mathcal{R}(r,m)|$, for parameters $0 \leq m \leq 4$, $0 \leq r \leq 2$. The Reed-Muller codes $\mathcal{R}(r,m)$ are associated with Euclidean geometries $EG(m,2)$, i.e., $m$-dimensional hypercubes. The qubits of $|\mathcal{R}(m,r)|$ are located on the vertices of the hypercube, and the $X$ and $Z$-stabilizer generators have support on the $m-r$ and $r+1$-dimensional sub-cubes, respectively. The support for $Z$-stabilizer generators (edge, face or volume) is indicated in red.

measurement bases, and between local measurement outcomes and result of the computation. For this purpose, we make the following

**Definition 3.** $Q(r,t,m,\chi)$ is a measurement-based quantum computation on a state $|\mathcal{R}(r,m)|$, specified by the measurement angle $\phi_j = \phi$ for all $j \in \Omega$,

$$\phi = \frac{\pi}{2\chi}, \; \chi \in \mathbb{N},$$

and

$$Q^T = [B(\mathcal{R}(t,m))],$$

$$Z = [B(\mathcal{R}(r,m))],$$

$$T = 0.$$  \hspace{1cm} (24)

Recall that the matrices $Q$, $Z$ and $T$ govern the linear relations (4) between input $i$ and choice $q$ of measurement bases, and between measurement outcomes $s$ and computational output $o$. As before, $[B(\mathcal{R}(t,m))]$ denotes the matrix with the elements of the basis $B(\mathcal{R}(t,m))$ as its rows. The angles $\phi_j \equiv \phi$ specify the local measured observables [3]. The choice (23) reflects the transversality property (18) of CSS codes based on Reed-Muller codes $\mathcal{R}_q^*(r,m)$.

With definition [3], $Q(1,2,5,2)$ is the measurement-based equivalent to Example 2 of Section 5.2. Also, it is easily verified that $Q(0,1,2,2)$ is Example 1 of Section 3.

The question now is for which sets of parameters $r,t,m,\chi$ the measurement-based quantum computation $Q(r,t,m,\chi)$ is deterministic and computes a non-linear function. This question is answered by

**Theorem 2.** $Q(r,t,m,\chi)$, with parameters $m,r,t \geq 0$, $\chi \geq 1$ is deterministic only if $m > 2r$ and deterministically computes a non-linear function only if $m > 3r$. $Q(r,t,m,\chi)$ deterministically
computes a non-linear function if $\chi \geq 2$ and

$$(\chi - 1)(r + t) < m - r \leq \chi t.$$  

This condition can be satisfied if $m > 3r + 1$.

The next question is after the functions that are being computed. It is answered by

**Theorem 3.** For $Q(r, t, m, \chi)$, under the condition $\chi \geq 2$, all inputs $q = Q1 \mod 2 \in R(t, m)$ satisfy the promise $Zq \mod 2^{\chi - 1} = 0$, and $Z (1, 1, \ldots, 1)^T \mod 2^{\chi} = 0$. The function computed by $Q(r, t, m, \chi)$ is $o[i] = \frac{Z(1, 1, \ldots, 1)^T \mod 2^{\chi} + o_0}{2^{\chi}}$, with $o_0 = (Z (1, 1, \ldots, 1)^T \mod 2^{\chi+1}) / 2^{\chi}$.

Note that each component of $o[q]$ can only equal 0 or 1. We obtain one bit of information per component of output, compatible with the information that can be gained in the measurement of an observable whose expectation value is promised to be extremal. The non-linearity in the function $o[i]$ arises through the mismatch of the involved additions mod 2 and mod $2^{\chi}$. The proofs of Theorems 2 and 3 are given in Appendix A.

The constraint (25) gives rise to the phase diagram for the deterministic computations with computational resources $|R(r, m))$, displayed in Fig. 3. With (25), if $m > 3r + 1$ then deterministic computation of non-linear functions is possible with a resource state $|R(r, m))$. To the left of the transversality line (26), i.e., $m < 3r + 1$ deterministic computation requires $\chi = 1$ which can only yield linear functions. The only region in the $r/m$-plane where we cannot conclude whether or not deterministic computation of a non-linear function is possible with a resource $|R(r, m))$ is the transversality line $m = 3r + 1, r \geq 1$.

To summarize, we have described and characterized a family of deterministic measurement-based quantum computations based on Reed-Muller states $\{|R(r, m))\} \mid r, m \in \mathbb{N}, r \leq m\}$. Theorem 2 provides a sufficient criterion for when these computations are non-linear, and Theorem 3 a closed-form expression for the computed function. It turns out that, for the discussed family of resource
states, the computed functions are rather simple. To their defense, let us point out that in a scenario of distributed computation a similar protocol for integer addition, using GHZ-states and local measurements, leads to exponential savings in the communication cost \([30]\).

### 5.4 Example 4: Letting LU-LC counterexamples compute

We will provide one more individual example for deterministic computation of a non-linear Boolean function, in the measurement-based setting. It is derived from a counterexample to the LU-LC conjecture. The counterexample is obtained using the software supplementing \([31]\). We consider the two stabilizer states

\[
|\Phi_1\rangle = \sum_{x \in S} |x\rangle, \quad |\Phi_2\rangle = \sum_{x \in S} (-1)^{Q(x)} |x\rangle.
\]  

(26)

|\Phi_1\rangle, |\Phi_2\rangle are quantum states on 35 qubits. \(S\) is a vector space spanned by the six binary vectors

\[
\begin{align*}
\xi^1 &= (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0), \\
\xi^2 &= (0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
\xi^3 &= (0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
\xi^4 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
\xi^5 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\
\xi^6 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).
\end{align*}
\]  

(27)

and, for \(x \in S\), \(|x\rangle\) is the computational basis state specified by \(x\). \(Q\) is a quadratic form given by

\[
Q(x) = x_1 x_{17} + x_2 x_{32} + x_1 x_{22} + x_2 x_9 + x_1 x_{25} + x_2 x_{10} + x_1 x_{27} + x_2 x_{21} + x_1 x_{23} + x_1 x_{16} + x_1 x_{18} + x_2 x_{23} + x_3 x_{25} + x_1 x_{13} + x_2 x_{11} + x_1 x_{26} + x_1 x_{21} + x_1 x_{20} + x_1 x_{24} + x_2 x_{22} + x_1 x_{12} + x_2 x_{13} + x_2 x_{20} + x_2 x_{33} + x_2 x_{24} + x_3 x_{10} + x_3 x_{11} + x_3 x_{17} + x_3 x_{19} + x_3 x_{24} + x_4 x_{16} + x_4 x_{21} + x_4 x_{28} + x_4 x_{31} + x_5 x_{14} + x_5 x_{30} + x_6 x_{29}.
\]  

(28)

We define the local unitary

\[
U(\epsilon) = \bigotimes_{j=1}^{35} \exp \left( i \frac{\pi}{8} \epsilon_j Z_j \right).
\]  

(29)

Then, for

\[
\epsilon = (3, 3, 7, 1, 1, 1, 3, 3, 3, 3, 3, 3, 7, 3, 1, 7, 7, 5, 5, 3, 5, 5, 7, 7, 5, 3, 5, 3, 3, 3, 5, 5, 5, 5, 1),
\]  

(30)

\(|\Phi_2\rangle = U(\epsilon)|\Phi_1\rangle\). \(U(\epsilon)\) is a non-Clifford local unitary. In contrast, there are no Clifford local unitaries that relate the two states. The pair of states \(|\Phi_1\rangle, |\Phi_2\rangle\) thus represents a counterexample to the LU-LC conjecture. Furthermore, it is easily verified that

\[
|\Phi_2\rangle = U(\epsilon + a k^{(1)} + b k^{(2)}) \mod 8 |\Phi_1\rangle.
\]  

(31)

for all \(a, b \in \mathbb{Z}_2\), where

\[
\begin{align*}
k^{(1)} &= (4, 4, 0, 4, 0, 2, 0, 4, 2, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 2, 0, 2, 0), \\
k^{(2)} &= (4, 0, 0, 2, 2, 2, 0, 0, 0, 2, 4, 0, 0, 2, 0, 0, 0, 2, 4, 0, 0, 2, 0, 0, 0, 2, 2, 0, 2, 0, 2, 0, 2, 0).
\end{align*}
\]  

(32)

The local non-Clifford unitary relating \(|\Phi_1\rangle, |\Phi_2\rangle\) is thus not unique. This fact is central for the conversion of the above LU-LC counterexample into an example for measurement-based quantum computation of non-linear functions. In accordance with Eq. \((31)\), we subsequently restrict to \(k = a k^{(1)} + b k^{(2)} \mod 8\).

The following procedure computes the AND-gate, \(a \text{ AND } b \equiv ab\):
1. Classical pre-processing: From the input $a, b$ and the known vectors $k^{(1)}/2, k^{(2)}/2$ compute $q = a k^{(1)}/2 + b k^{(2)}/2$ mod 2.

2. Put in place the stabilizer state state $|\Phi_2\rangle$ as computational resource.

3. Measure the local observables
   \[ O_j(q_j) = \cos(\pi e_j/4) X_j + (-1)^{q_j} \sin(\pi e_j/4) Y_j, \]
   with $e_j := [e]_j$ as specified in Eq. (30). Obtain the measurement outcomes $s_j \in \mathbb{Z}_2$.

4. Classical post-processing: Compute the parities $o_l = \sum_{j|\xi^l_j|} s_j$, for $l = 1..6$, and $\xi^l$ as given in Eq. (27). The output of the computation is $o = (o_1, ..., o_6)$.

For fixed and known vectors $k^{(1)}/2, k^{(2)}/2$, the classical processing in steps 1 and 4 can be done with addition mod 2 alone. Regarding step 1, each component of $q$ is linear in $a$ and $b$, with fixed known coefficients.

The first output bit of the above protocol is $o_1(a, b) = a \text{ AND } b$. Let us briefly explain why this holds. First note that, for all $e_j \in \{1, 3, 5, 7\}$ and $k_j \in \{0, 2, 4, 6\}$,

\[ e_j + k_j \mod 8 = (-1)^{q_j} e_j + 4v(e_j, k_j) \mod 8, \]
\[ q_j = k_j/2 \mod 2, \]
\[ v(e_j, k_j) = k_j(e_j + k_j/2)/4 \mod 2. \]  

(34)

Thus, up to a global phase, $U(e + k) = V(k, e)W(k)$, where $V(k, e) := \bigotimes_{j=1}^{35} (Z_j)^{e_j(k_j)}$, and $W(k) := \bigotimes_{j=1}^{35} \exp(i \frac{\pi}{8} (-1)^{k_j/2} Z_j)$.

By (26), the operators $X(l) := \bigotimes_{j|\xi^l_j|} X_j$ are in the stabilizer of $|\Phi_1\rangle$. Then, with the relations in hand, we find

\[ 1 = \langle \Phi_1 | X(l) | \Phi_1 \rangle = \langle \Phi_2 | U(e + k)X(l)U(-e - k) | \Phi_2 \rangle = \langle \Phi_2 | V(e, k)W(k)X(l)W(-k)V(-e, -k) | \Phi_2 \rangle = (-1)^{\eta_l(e, k)} \bigotimes_{j|\xi^l_j|} O_j(q_j) | \Phi_2 \rangle, \]

where

\[ \eta_l(e, k) = \sum_{j|\xi^l_j|} v(e_j, k_j) \mod 2. \]

(35)

The above protocol extracts the expectation values $\langle \Phi_2 | \bigotimes_{j|\xi^l_j|} O_j(q_j, \phi_j) \rangle | \Phi_2 \rangle =: (-1)^{o_l}$, for $l = 1..6$ (after the classical post-processing of the local measurement outcomes), and thus $o_l = \eta_l(e, k)$.

Recall that we have specialized to $k = ak^{(1)} + bk^{(2)}$. Then, with Eqs. (27) and (35), we obtain the logical table of an AND-gate for $o_1(a, b) = \eta_1(e, ak^{(1)} + bk^{(2)})$; i.e. $o_1(0, 0) = o_1(0, 1) = o_1(1, 1) = 1$. The AND-gate is a non-linear Boolean function.

6 Conclusion and outlook

In this paper, we have explored elements of discreteness in quantum computation. To start out, we posed two questions: 1) “Can discrete structures in Hilbert space be linked to algorithmic tasks?”, and 2) “If so, what is quantum about the resulting computations?”. We have identified discrete Hilbert space structures in quantum codes and a LU-LC counterexample, and turned them into simple computations. Thus we answered the first question to the affirmative. Regarding the second
question, for measurement-based quantum computations obeying the discreteness conditions [1], we have identified contextuality as an element of non-classicality.

Continuing to explore the link between quantum-mechanical computation and contextuality of quantum mechanics, two further questions now become pertinent. First, above we proved that whenever a measurement-based quantum computation deterministically evaluates a non-linear Boolean function it implies a proof of the Kochen-Specker theorem. It is natural to inquire about the reverse direction. Do the known proofs of the Kochen-Specker theorem lend themselves to measurement-based computations? Which functions are being computed? Is there additional structure to the set of admissible inputs? Second, Is the property of contextuality in the discussed computations model-independent? In particular, the measurement-based examples discussed in Sections 5.3 and 5.4 have all counterparts in the circuit model. Thus, it is worth testing whether these circuit counterparts also exhibit contextuality. The tools for such an analysis are provided in [45], where the notion of contextuality has been extended to state preparation and evolution.

The long-term goal of the present work is to discover novel efficient quantum algorithms, through the classification of discrete structures in Hilbert space. At present, such a classification does not appear within range, and intermediate steps need to be identified. Two such steps are (a) The classification of deterministic quantum algorithms with flat temporal order, and (b) Finding and analyzing examples for deterministic quantum algorithms with non-flat temporal order. It is expected that non-flat temporal order is a key ingredient for deterministic computations with strong algorithmic applications.

Regarding (a), the classification of temporally flat deterministic computation will draw from coding theory, and could also benefit from a classification of the counterexamples to the recently refuted [11] LU-LC conjecture. The unifying element among quantum codes with non-Clifford transversal gates and LU-LC counterexamples is the interconversion between stabilizer states by non-Clifford local unitaries. The existence of such conversions is a remarkable and intriguing property of Hilbert space. As we have shown in this paper, this property can be put to work for algorithmic uses.

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A Proof of Theorems 2 and 3

Proof of Theorem 2[

Every observable $O_z[q] = \bigotimes_{j \in \Omega} O_j[q_j]$ yields one bit $o_z$ of output, through the relation $\langle R(m, r) | O_z[q] | R(m, r) \rangle = (-1)^{o_z[q]}$. The binary vector $z$ specifies the support of $O_z[q]$, and the binary vector $q$ specifies the local observables $O_j[q_j]$ occurring in $O_z[q]$. We consider $z \in R(m, r)$, $q \in R(t, m)$. Further, denote $X[z] := \bigotimes_{j \in \Omega} z_j^{q_j} X_j$. Then,

\[
(-1)^{o_z[q]} = \langle R(m, r) | O_z[q] | R(m, r) \rangle = \langle R(m, r) | \bigotimes_{i \in \Omega} e^{-i \frac{\phi}{2}} (-1)^{m_i} Z_i X[z] \bigotimes_{i \in \Omega} e^{i \frac{\phi}{2}} (-1)^{m_i} Z_i | R(m, r) \rangle
\]

\[
= \langle R(m, r) | X[z] \bigotimes_{i \in \Omega} e^{i \frac{\phi}{2}} Z_i X[z] \bigotimes_{i \in \Omega} e^{-i \frac{\phi}{2}} Z_i | R(m, r) \rangle.
\]

Denote $|\mathcal{R}[q]| := X[z] | R(m, r) \rangle$. Then, using $X[z] | R(m, r) \rangle = | R(m, r) \rangle$,

\[
\langle R(m, r) | O_z[q] | R(m, r) \rangle = \langle \mathcal{R}[q] \bigotimes_{j \in \Omega} \exp (i \phi Z_j) | \mathcal{R}[q] \rangle
\]

From its above definition, $|\mathcal{R}[q]| = \frac{1}{\sqrt{|\mathcal{R}(m, r)|}} \sum_{c \in \mathcal{R}(m, r)} |c \oplus q\rangle$, such that

\[
\langle R(m, r) | O_z[q] | R(m, r) \rangle = \frac{e^{i \phi |z|}}{|\mathcal{R}(m, r)|} \sum_{c \in \mathcal{R}(m, r)} \exp \left(-2i \phi \sum_{j \in \Omega} c_j \oplus q_j\right) = \frac{e^{i \phi |z|}}{|\mathcal{R}(m, r)|} \sum_{c \in \mathcal{R}(m, r)} \exp \left(-2i \phi \sum_{j \in \Omega} z_j c_j - 2z_j c_j q_j\right).
\]

Since there are $|\mathcal{R}(m, r)|$ terms in the $c$-sum on the rhs of (37), each of weight $1/|\mathcal{R}(m, r)|$, the expectation value $\langle R(m, r) | O_z[q] | R(m, r) \rangle$ can have unit modulus if and only if all terms are equal. Since $c = 0 \in \mathcal{R}(m, r)$, each term in the $c$-sum on the rhs of (37) must evaluate to 1, i.e.,

\[
\exp \left(-2i \phi \sum_{j \in \Omega} z_j c_j \right) = e^{i \alpha(z, c)}, \quad \exp \left(-2i \phi \sum_{j \in \Omega} z_j c_j q_j\right) = e^{-i \alpha(z, c)}, \quad \text{with } \alpha(z, c) = \text{const}(q) \in \mathbb{R}.
\]

Since $q = 0 \in \mathcal{R}(t, m)$, $\exp (4i \phi \sum_{j \in \Omega} z_j c_j q_j) = \exp \left(-2i \phi \sum_{j \in \Omega} z_j c_j\right) = 1$ for all $z, q, c$. We now introduce the coordinate-wise product $ab$ of two codewords $a, b$, i.e., $[ab]_j = [a]_j [b]_j$. With this notation, keeping in mind that $\phi = \pi/2$ (c.f. Eq. (23)), the above conditions read

\[
|cqz| \mod 2^{x-1} = 0, \quad \text{Eq. (38a)}
\]

\[
|cz| \mod 2^x = 0. \quad \text{Eq. (38b)}
\]

Eq. (38) is the necessary and sufficient condition for the MQC-computation on a Reed-Muller state $|\mathcal{R}(m, r)\rangle$ being deterministic. Note that $c_0 = (1, 1, \ldots, 1) \in \mathcal{R}(r, m)$ for all $r \leq m$. Thus, if the determinism conditions (38) are satisfied, the following promise holds

\[
|cq| \mod 2^{x-1} = 0 \text{ and } |z| \mod 2^x = 0, \quad \forall c, z \in \mathcal{R}(m, r), q \in \mathcal{R}(t, m).
\]

Then, with Eqs. (23), (36), (37), each output bit $o_z[q]$ is given by $(-1)^{o_z[q]} = e^{i \phi |z|} e^{-2i \phi z \cdot q} = \pm 1$ in (37). Also, it guarantees that the output $o_z[q]$ in Eq. (40) is binary-valued. □
Proof of Theorem 3. The determinism constraints of Eq. (38) are a precondition for the input-output relation Eq. (40). - But for which tuples \((r, t, m, \chi)\) do they hold? To answer this question, we need to further analyze the codewords appearing in Eq. (38). For this purpose, we choose a specific basis \(B(r, m)\) of \(\mathcal{R}(r, m)\), namely

\[
B(r, m) = \left\{ z^\alpha : [z^\alpha]_j = \prod_{k \in \alpha} j_k, \forall \alpha \subset \{1, 2, \ldots, m\} \text{ with } |\alpha| \leq r, \forall j \right\}.
\]

By the definition of Reed-Muller codes, the components \(c_j := [c]_j\) of the codeword \(c \in \mathcal{R}(r, m)\) are \(c_j = P(j_1, \ldots, j_m)\), where \(P\) is a polynomial of degree \(\leq r\), and \((j_1, j_2, \ldots, j_m)\) is the binary string representing \(j\). Now denote by \(J_{\alpha}\) the set with characteristic vector \(z^\alpha\), i.e., \(J_{\alpha} = \{ j \in \Omega | [z^\alpha]_j = 1 \}\). By our choice (41) for the basis of \(\mathcal{R}(r, m)\), \(j \in J_{\alpha}\) if and only if \(j_k = 1\) \(\forall k \in \alpha\), and thus

\[
[cz^\alpha]_j = \left\{ \begin{array}{ll}
0, & \text{if } j \in J_{\alpha} \\
1, & \text{if } j \notin J_{\alpha}.
\end{array} \right.
\]

Therefore, \(|cz^\alpha| \equiv c|_{J_{\alpha}}\), and \(\{c|_{J_{\alpha}} : c \in \mathcal{R}(r, m)\}\) is again a Reed-Muller code. Specifically,

\[
\{c|_{J_{\alpha}} : c \in \mathcal{R}(r, m)\} = \mathcal{R}(\min(r, m - |\alpha|), m - |\alpha|), \forall z^\alpha \in B(r, m).
\]

Similarly, \(|qz^\alpha| \equiv q|_{J_{\alpha}}\), \(|czq^\alpha| \equiv cq|_{J_{\alpha}}\), and

\[
\{q|_{J_{\alpha}} : q \in \mathcal{R}(t, m)\} = \mathcal{R}(\min(t, m - |\alpha|), m - |\alpha|), \forall z^\alpha \in B(r, m),
\]

\[
\{cq|_{J_{\alpha}} : c \in \mathcal{R}(r, m), q \in \mathcal{R}(t, m)\} \subseteq \mathcal{R}(\min(r + t, m - |\alpha|), m - |\alpha|), \forall z^\alpha \in B(r, m).
\]

In (44b) there is no equality because, for \(c \in \mathcal{R}(r, m), q \in \mathcal{R}(t, m)\), the codewords \(cq\) do not necessarily fill the entire space \(\mathcal{R}(r + t, m)\).

Main case: \(r, t > 0\). We now use the Theorem of Ax [40], as stated in [37]:

Theorem 4 (Ax). The \(r\)-th order generalized Reed-Muller code \(\mathcal{R}_q(r, m)\) over the field \(GF(q)\) is divisible by \(q^{\lceil m/r \rceil - 1}\). Moreover, if \(p\) is the prime dividing \(q\), this divisor is the highest power of \(p\) that divides the code.

Here, we specialize to \(p = q = 2\), and apply theorem 4 to the codewords \(cz^\alpha\). Then, Eq. (43) implies that \(\forall \alpha \in \{1, 2, \ldots, m\}, |\alpha| \leq r, \forall c \in \mathcal{R}(r, m), q \in \mathcal{R}(t, m)\)

\[
|cz^\alpha| \mod \max\left(2^{\left\lceil \frac{m - |\alpha|}{r} \right\rceil - 1}, 1\right) = 0.
\]

Furthermore, for a given \(\alpha\), \(\max\left(2^{\left\lceil \frac{m - |\alpha|}{r} \right\rceil - 1}, 1\right)\) is the largest power of 2 for which this relation holds. Now, by comparison between Eq. (38b) and Eq. (45) we obtain constraints on \(\chi\), namely \(\chi \leq \max\left(\left\lceil \frac{m - |\alpha|}{r} \right\rceil - 1, 0\right)\), \(\forall |\alpha| \leq r\). The strongest of these constraints arises for \(|\alpha| = r\), i.e.,

\[
\chi \leq \max\left(\left\lceil \frac{m}{r} \right\rceil - 2, 0\right).
\]

This condition is necessary for \(Q(r, t, m, \chi)\) to be deterministic.

Case A: \(r = m\). The inequality (46) simplifies to \(\chi \leq 0\), which contradicts the assumption \(\chi \geq 1\) of the theorem.

Case B: \(r < m\). The determinism condition (46) simplifies to \(\chi < \frac{m}{r} - 1\). This, together with the assumption \(\chi \leq 1\), implies \(m > 2r\) as a necessary condition for a deterministic computation \(Q(r, m, t, \chi)\).

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Furthermore, \( Q(r, m, t, \chi) \) can only compute a non-linear function if \( \chi \geq 2 \); see Eq. (40). Thus, \( 2 \leq \chi < \frac{m}{r} - 1 \) is the stated necessary condition for deterministic computation of a non-linear function. This concludes the proof of the first part of Theorem 2 concerning necessary conditions.

We now turn to proving the sufficient condition (25) for \( Q(r, m, t, \chi) \) deterministically computing a non-linear function. This will be done in two steps. First, we provide a sufficient condition for \( Q(r, m, t, \chi) \) to compute a non-constant function. Second, we show that for \( \chi \geq 2 \), non-constant implies non-linear.

**Step 1.** The output \( o_\alpha \) of the computation can be a non-constant function only if the phase factor \( e^{-2i\alpha q} \) in Eq. (37) equals -1 at least for one \( z_0 \in \mathcal{R}(r, m) \) and one \( q_0 \in \mathcal{R}(t, m) \). Thus we require, in addition to the determinism constraints Eq. (37), that \( |q_0 z_0| \mod 2^\chi = 2^{\chi - 1} \). (47)

Let us now consider what the theorem of Ax says about the codeword weights appearing in Eqs. (38) and (47). We can rule out \( m = r \) as before, and only consider the remaining case, \( r < m \). Then, by applying the first part of Theorem 4 to Eqs. (44a) it follows that \( \forall \alpha \in \{1, 2, \ldots, m\} \) with \( |\alpha| \leq r \), \( \forall c \in \mathcal{R}(r, m) \), \( q \in \mathcal{R}(t, m) \), \( \forall z^\alpha \in \mathcal{B}(r, m) \)

\[
|cqz^\alpha| \mod 2^\left\lfloor \frac{m-|\alpha|}{r} \right\rfloor = 0, \quad (48a)
\]
\[
|cz^\alpha| \mod 2^\left\lfloor \frac{m-|\alpha|}{r} \right\rfloor = 0. \quad (48b)
\]

In addition, by applying the second part of Theorem 4 to Eq. (44a), there exist \( z^\alpha \in \mathcal{B}(r, m) \), \( q_0 \in \mathcal{R}(t, m) \) for which

\[
|q_0 z^\alpha| \mod 2^\left\lfloor \frac{m-|\alpha|}{r} \right\rfloor = 2^\left\lfloor \frac{m-|\alpha|}{r} \right\rfloor - 1. \quad (49)
\]

Now, the conditions (38) for determinism and (47) for the computed function being non-constant are satisfied if there exists an \( |\alpha| \leq r \) such that the following conditions hold

\[
\chi \leq \left\lfloor \frac{m}{r} \right\rfloor - 2, \quad (50a)
\]
\[
\chi \leq \left\lfloor \frac{m-r}{r+t} \right\rfloor, \quad (50b)
\]
\[
\chi = \left\lfloor \frac{m-|\alpha|}{t} \right\rfloor. \quad (50c)
\]

To see this, note that substituting (50a) and \( \chi \geq 1 \) into Eq. (45) yields \( |cz^\alpha| \mod 2^\chi = 0 \), implying the determinism constraint (38b) for all \( z^\alpha \in \mathcal{B}(r, m) \). Similarly, substituting (50b) into (48b) implies the determinism condition (38a) for all \( z^\alpha \in \mathcal{B}(r, m) \). Thus, the necessary and sufficient determinism condition (38) is satisfied for all \( z \), i.e., \( \langle \mathcal{R}(r, m)|O_{z^\alpha} |q| \mathcal{R}(r, m) \rangle = \pm 1 \) for all \( z^\alpha \in \mathcal{B}(r, m) \). Now note that if \( \langle \mathcal{R}(r, m)|O_{z^\alpha} |q| \mathcal{R}(r, m) \rangle = \pm 1 \) then \( \langle \mathcal{R}(r, m)|O_{z^\prime z^\alpha} |q| \mathcal{R}(r, m) \rangle = \pm 1 \). Therefore, if the condition (38) is obeyed for all \( z^\alpha \in \mathcal{B}(r, m) \) then it is also obeyed for all \( z \in \mathcal{R}(r, m) \). Finally, substituting (50c) into (49) implies the condition (47). There are no further constraints on \( r, m, t, \chi \) to satisfy.

We rewrite the sufficient conditions (50) as linear inequalities. That is, we require there exists an \( |\alpha| \leq r \) such that

\[
(\chi + 1)r < m, \quad (51a)
\]
\[
\chi r + (\chi - 1)t < m, \quad (51b)
\]
\[
(\chi - 1)t + |\alpha| < m \leq \chi t + |\alpha|. \quad (51c)
\]
Recall that the sufficient condition in Theorem 2 includes $\chi \geq 2$. If $\chi \geq 2$, the left inequality in (51c) is implied by (51b). The only remaining place where $|\alpha|$ appears is the right inequality of (51c). Thus, the constraints (51) can be satisfied if and only if they are satisfied for $|\alpha| = r$. Then, (51b) and the right inequality in (51c) imply

$$t > (\chi - 1)r, \quad \text{for } \chi \geq 2.$$  

(52)

Therefore, (51a) is also dependent, and the sufficient conditions (51) for deterministically computing a non-constant function simplify to

$$(\chi - 1)(r + t) < m - r \leq \chi t, \quad \text{for } \chi \geq 2.$$  

(53)

This is the condition (25) appearing in Theorem 2 but not yet for computing a non-linear function.

**Step 2: Non-constant implies non-linear for $\chi \geq 2$.** We start from the assumption that $o_{x^\alpha}[q]$ is linear in $q$, i.e.,

$$o_{x^\alpha}[0] + o_{x^\alpha}[p] + o_{x^\alpha}[q] + o_{x^\alpha}[p + q] = 0, \quad \forall p, q \in \mathcal{R}(t, m), \forall z^\alpha \in B(r, m),$$  

(54)

and demonstrate that this assumption always leads to a contradiction. To this end, we pick parameters $r, t, m, \chi$ such that the conditions (51) for the output being deterministic and non-constant are satisfiable for some set $\alpha$. Then, for any $\alpha$ satisfying (51) we show that we can always find vectors $p, q \in \mathcal{R}(t, m)$ that violate the linearity condition (54). To begin, we choose a specific basis for $\{q\} = \mathcal{R}(t, m)$, namely $\{q^\beta\} = B(t, m)$, c.f. (41).

**Sub-case A:** $m - |\alpha| - t = 1$. Recall that the assumption of the main case is $t \geq 1$. Then, from (50c) and the sub-case assumption, $\chi = \lceil \frac{t+1}{t} \rceil = 2$. To specify $q^\beta$ we choose sets $\beta$ with $|\beta| = t$ and $\alpha \cap \beta = \emptyset$. Then, the codewords $z^\alpha q^\beta$ depend on a single parameter $\kappa \in \{1, 2, \ldots, m\}$ and are of the form

$$[z^\alpha q^\beta(\kappa)]_j = \prod_{k \in \{1, \ldots, m\} - \kappa} j_k.$$  

(55)

That is, $[z^\alpha q^\beta(\kappa)]_j = 1$ if and only if $j \in \{2^m - 1, 2^{m-2^\kappa-1}\}$. For a fixed $\kappa$, there are $t+1$ choices for $\kappa$. Thus, there are always at least two choices, $\kappa_1$ and $\kappa_2 \neq \kappa_1$. For those, $[z^\alpha q^\beta(\kappa_1) \oplus z^\alpha q^\beta(\kappa_2)]_j = 1$ if and only if $j \in \{2^m - 2^\kappa_1 - 1, 2^m - 2^\kappa_2 - 1\}$. Thus,

$$[z^\alpha q^\beta(\kappa_1)]_j = [z^\alpha q^\beta(\kappa_2)]_j = [z^\alpha q^\beta(\kappa_1) \oplus z^\alpha q^\beta(\kappa_2)]_j = 2.$$  

(56)

Now, set $\beta_1 := \{1, \ldots, m\} - \alpha - \kappa_1, \beta_2 := \{1, \ldots, m\} - \alpha - \kappa_2$. Then, with $\chi = 2$, Eqs. (40) and (56), $o_{x^\alpha}[q^\beta_1] = \frac{|z^\alpha q^\beta_1|}{2} \mod 4 \oplus o_{x^\alpha}[0] = 1 \oplus o_{x^\alpha}[0]$. Analogously, $o_{x^\alpha}[q^\beta_2] = 1 \oplus o_{x^\alpha}[0]$ and $o_{x^\alpha}[q^\beta_1 \oplus q^\beta_2] = 1 \oplus o_{x^\alpha}[0]$. Therefore, the choices $p = q^\beta_1, q = q^\beta_2$ yield a contradiction in (54). The output $o_{x^\alpha}[q]$ is thus non-linear in $q = Qi \mod 2$, and hence in $i$.

**Sub-case B:** $m - |\alpha| - t \geq 2$. Note that $|z^\alpha q^\beta| = 2^m - |\alpha \cup \beta|$ and $m - |\alpha \cup \beta| \geq m - |\alpha| - t$. Thus,

$$|z^\alpha q^\beta| \mod 2^m - |\alpha| - t = 0.$$  

(57)

The assumption of the main case, $r \geq 1$, and $\chi \geq 2$ in Eq. (52) imply $t \geq 2$. Then, with Eq. (50c), $\chi - 1 = \left\lceil \frac{m - |\alpha| - t}{t} \right\rceil < m - |\alpha| - t$, and thus $\chi \leq m - |\alpha| - t$. Combining this with (57), we find

$$|z^\alpha q^\beta| \mod 2^\chi = 0, \quad \forall z^\alpha \in B(r, m), q^\beta \in B(t, m).$$  

(58)

However, with (50c) in (49), there exists a $z^\alpha \in B(r, m)$ and a $q_0 \in \mathcal{R}(t, m)$ such that

$$|z^\alpha q_0| \mod 2^\chi = 2^{\chi-1}.$$  

(59)
Now, expand all $q_0$ satisfying (59) as $q_0 = \sum i_\beta q^\beta$. Among those, choose one $q_0$ with minimal support wrt. $i$. Because of (58), $|i| \geq 2$. Now choose one $\hat{\beta}$ for which $i_{\hat{\beta}} = 1$. Then, with (40), (58) and (59),
\[
oz[\hat{\beta}] = \oz[0], \oz[q_0] = \oz[0] \oplus 1, \oz[q_{\hat{\beta}} \oplus q_0] = \oz[0]. \tag{60}
\]
The choice $p = q_{\hat{\beta}}$, $q = q_0$ thus leads to a contradiction in the linearity condition (54). The output $\oz[q]$ is therefore a non-linear function in $q = Qi \mod 2$, and hence in the input $i$.

**Remaining cases:** $r = 0$ or $t = 0$.  
**Case A:** $t = 0$. There is only one bit of input, hence the computed function cannot be non-linear. Correspondingly, $t = 0$ is excluded by (25) for $\chi \geq 2$.

**Case B:** $r = 0$, $t > 0$. If $c \in R(0,m)$ then Eqs. (48), (47) are replaced by $|c| = n2^m$, with $n \in \{0, 1\}$. Then we find the sufficient condition $(\chi - 1)t \leq m \leq \chi t$, $\chi \geq 2$ for $Q(0,t,m,\chi)$ to compute a non-linear function. It coincides with condition (25) for $r = 0$.

Finally, we show that for $m > 3r + 1$, one can always find parameters $\chi \geq 2$, $t \geq 0$ such that the condition (25) can be satisfied. To this end, we set $\chi = 2$ and $m = 3r + c$. For these choices, (25) becomes $r + \frac{c}{2} \leq t < r + c$. With $r$ and $t$ being integers, a solution for $t$ exists if $c > 1$. □