Limit cycles from a four-dimensional centre in $\mathbb{R}^m$ in resonance $p:q$

Luis Barreira$^a$, Jaume Llibre$^b$ and Claudia Valls$^a$

$^a$Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal; $^b$Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

(Received 9 December 2011; final version received 16 August 2012)

Given positive coprime integers $p$ and $q$, we consider the linear differential centre $\dot{x} = Ax$ in $\mathbb{R}^m$ with eigenvalues $\pm pi, \pm qi$ and 0 with multiplicity $m - 4$. We perturb this linear centre in the class of all polynomial differential systems of the form linear plus a homogeneous nonlinearity of degree $p + q - 1$, i.e. $\dot{x} = Ax + \varepsilon F(x)$, where every component of $F(x)$ is a linear polynomial plus a homogeneous polynomial of degree $p + q - 1$. When the displacement function of order $\varepsilon$ of the perturbed system is not identically zero, we study the maximal number of limit cycles that can bifurcate from the periodic orbits of the linear differential centre.

Keywords: periodic orbit; averaging theory; limit cycles; resonance $p:q$

AMS Subject Classifications: 34C29; 34C25; 47H11

1. Introduction

In the qualitative theory of polynomial differential systems, the study of their limit cycles and mainly the obtention of information on their number for a given polynomial differential system is one of the main topics. We recall that for a differential system a limit cycle is a periodic orbit isolated in the set of all its periodic orbits.

In dimension two, i.e. in the plane, the two main problems related with limit cycles are: first, the study of the number of limit cycles depending on the degree of the polynomial differential system. This is an old problem proposed by D. Hilbert in 1900, known as the 16th Hilbert problem (see the surveys [1,2] for details), and second the study of how many limit cycles emerge from the periodic orbits of a given centre when we perturb it inside a certain class of differential systems [3].

Since the study of limit cycles and mainly the obtention of information on their number for a given polynomial differential system is in general a very difficult problem (almost impossible), there are hundreds of papers trying to solve these questions in the plane for many particular families of polynomial systems, see the references quoted in the book [3] and in the surveys [1,2].

These problems have been studied intensively in dimension two, and unfortunately the results are far from being satisfactory. In fact, the Riemann conjecture and the 16th Hilbert problem are the two unique problems of the famous list of Hilbert which are not solved.
Our main aim is to extend these studies from dimension two to higher dimension, and to observe the differences which appear due to the increase of the dimension of the polynomial differential systems. Thus, we take a linear resonant centre \( p:q \) of dimension four living inside dimension \( m \geq 5 \) and study how many of the periodic orbits of the centre persist as limit cycles once this centre is perturbed inside a class of polynomial differential systems of degree \( p + q - 1 \). The interesting result obtained for this class of polynomial differential systems is that the number of limit cycles obtained are powers related to the dimension \( m \) having bases related to the degree of the perturbation \( p + q - 1 \) (for the precise result, see Theorem 1).

Here we study how many limit cycles emerge from the periodic orbits of a centre when we perturb it inside a given class of differential equations in dimension higher than four. More precisely, given \( m \geq 5 \), we consider the linear differential centre

\[
\frac{dx}{dt} = \dot{x} = Ax
\]

in \( \mathbb{R}^m \), where

\[
A = \begin{pmatrix}
0 & -p & 0 & 0 & 0 & \cdots & 0 \\
p & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -q & 0 & \cdots & 0 \\
0 & 0 & q & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

for some positive coprime integers \( p \) and \( q \). We perturb system (1) in the form

\[
\dot{x} = Ax + \varepsilon F(x),
\]

where \( \varepsilon \) is a small parameter, and where \( F : \mathbb{R}^m \to \mathbb{R}^m \) is a polynomial of the form

\[
F = (F_1 + F_N, \ldots, F_1 + F_N)
\]

with \( F^k_1 \) and \( F^k_N \) arbitrary homogeneous polynomials, respectively, of degrees 1 and \( N = p + q - 1 \) in the variables \( x = (x_1, \ldots, x_m) \) for \( k = 1, \ldots, m \), with the exception that \( F^k_1 = \lambda_k x_k \) for \( k = 5, \ldots, m \). We note that the polynomial perturbations \( F(x) \) of this form cover all polynomial perturbations of system (2) of degrees 2 and 3 (this follows from the theory of normal forms, see [4] for details).

For \( \varepsilon = 0 \) the differential system (2) has a four-dimensional centre in resonance \( p:q \). Without loss of generality we can assume that \( q > p \). We want to study how many limit cycles can bifurcate from the periodic orbits of this centre when we perturb it inside the class of polynomial vector fields of the linear form plus a homogeneous nonlinearity of degree \( p + q - 1 \). Our main result is the following theorem.

**Theorem 1:** Assume that \( p, q \geq 1 \) are coprime integers with \( q > p \) and that \( m \geq 5 \). If \( \varepsilon \neq 0 \) is sufficiently small and the displacement function of order \( \varepsilon \) (see (5)) is not identically zero, then the maximum number of limit cycles of the differential system (2) bifurcating from the periodic orbits of the four-dimensional linear differential centre (1) is at most

(a) \( 2^m + 2^{m-1}3^2 + 2^45^m - 4 \) if \( q = 2, p = 1, \) and

(b) \( 2pq(p + q - 1)^{m-3}(p + q)^2 + 2pq(p + q + 1)(p + q + 2)5^{m-4} \) if \( q \geq 3 \).
We refer to Section 2 for the definition of the displacement function of order $\varepsilon$. Theorem 1 is proved in Section 4 using the averaging theory described in Section 2. Indeed, Theorem 1 depends heavily on the computation of the averaged system associated to the differential system (2), because its singular points with Jacobian nonzero provide the limit cycles of the differential system (2) when the displacement function of order $\varepsilon$ is not identically zero. Theorem 1 improves and extends previous results for system (2) restricted to $\mathbb{R}^4$ (see [4,5]) and in $\mathbb{R}^m$ for $p = 1$ (see [6]).

When $p$, $q$ and $m$ are relatively small the averaged system can be computed explicitly, thus allowing one to improve the upper bound for the number of limit cycles given by Theorem 1. In particular, we have established the following result in [6].

**Theorem 2**: If $\varepsilon \neq 0$ is sufficiently small and the displacement function of order $\varepsilon$ is not identically zero, then the maximum number of limit cycles of the differential system (2) bifurcating from the periodic orbits of the four-dimensional linear differential centre (1) is at most

(a) 20 if $q = 2$, $p = 1$ and $m = 5$, and
(b) 46 if $q = 3$, $p = 1$ and $m = 5$.

We note that the corresponding upper bounds given by Theorem 1 are, respectively, 256 and 1044.

### 2. First-order averaging theory

The aim of this section is to present the first-order averaging method obtained in [7]. We first briefly recall the basic elements of averaging theory. Roughly speaking, the method gives a quantitative relation between the solutions of a nonautonomous periodic system and the solutions of its averaged system, which is autonomous. The following theorem provides a first-order approximation for periodic solutions of the original system.

We consider the differential system

$$
\dot{x}(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon),
$$

where $H : \mathbb{R} \times D \to \mathbb{R}^n$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, $T$-periodic in the first variable and where $D$ is an open subset of $\mathbb{R}^n$. We define $h : D \to \mathbb{R}^n$ by

$$
h(z) = \int_0^T H(s, z) \, ds,
$$

and we denote by $d_B(h, V, a)$ the Brouwer degree of $h$ at $a$ (see [8] for the definition).

**Theorem 3**: We assume that:

(i) $H$ and $R$ are locally Lipschitz with respect to $x$;

(ii) for $a \in D$ with $h(a) = 0$, there exists a neighbourhood $V$ of $a$ such that $h(z) \neq 0$ for all $z \in V \setminus \{a\}$ and $d_B(h, V, a) \neq 0$. Then for $\varepsilon \neq 0$ sufficiently small there exists an isolated $T$-periodic solution $\phi(\cdot, \varepsilon)$ of system (3) such that $\phi(a, 0) = a$.

The system $\dot{x} = \varepsilon h(x)$ is called the averaged system associated to system (3).
Lemma 4: Doing the change of variables from compact subset of the system \( h \) where Jacobian of \( 3 \). Averaged system that for every \( z \in D \) the following relations hold:

\[
\begin{align*}
\xi(T, z, \varepsilon) &= \xi(0, z, \varepsilon), \\
\xi(z, \varepsilon) &= \xi(0, z, \varepsilon), \quad \text{and} \quad \xi(z, \varepsilon) = \varepsilon h(z) + O(\varepsilon^2),
\end{align*}
\]

where \( h \) is given by (4) and where the symbol \( O(\varepsilon^2) \) denotes a function bounded on every compact subset of \( D \times (-\varepsilon_0, \varepsilon_0) \) multiplied by \( \varepsilon^2 \).

We note that in order to see that \( d_{\xi} h(V, a) \neq 0 \), it is sufficient to check that the Jacobian of \( D_{\xi} h(z) \) at \( z = a \) is not zero [8].

3. Averaged system

Writing

\[
F_1 = (F_1^1, F_1^2, F_1^3, F_1^4, 0, \ldots, 0), \quad F_N = (F_N^1, F_N^2, F_N^3, F_N^4, F_N^5, \ldots, F_N^m),
\]

system (2) becomes

\[
\begin{align*}
\dot{x}_1 &= -px_2 + \varepsilon(F_1^1(x) + F_N^1(x)), \\
\dot{x}_2 &= px_1 + \varepsilon(F_1^2(x) + F_N^2(x)), \\
\dot{x}_3 &= -qx_4 + \varepsilon(F_1^3(x) + F_N^3(x)), \\
\dot{x}_4 &= qx_3 + \varepsilon(F_1^4(x) + F_N^4(x)), \\
\dot{x}_k &= \varepsilon(\lambda_k x_k + F_N^k(x)), \quad k = 5, \ldots, m.
\end{align*}
\]

Lemma 4: Doing the change of variables from \((x_1, x_2, x_3, x_4, x_5, \ldots, x_m)\) to the new variables \((\theta, r, \rho, s, y_5, \ldots, y_m)\) given by

\[
x_1 = r \cos(\rho \theta), \quad x_2 = r \sin(\rho \theta), \quad x_3 = \rho \cos(q(\theta + s)), \quad x_4 = \rho \sin(q(\theta + s)), \quad x_k = y_k,
\]

for \( k = 5, \ldots, m \), and taking \( \theta \) as the new independent variable, system (6) is transformed into the system

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2), \\
\frac{d\rho}{d\theta} &= \varepsilon H_2(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2), \\
\frac{ds}{d\theta} &= \varepsilon H_3(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2), \\
\frac{dy_k}{d\theta} &= \varepsilon H_k(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2), \quad k = 5, \ldots, m.
\end{align*}
\]
where

\[ H_1 = (F_1^2 + F_N^2) \cos(p\theta) + (F_1^2 + F_N^2) \sin(p\theta), \]
\[ H_2 = (F_1^2 + F_N^2) \cos(q(\theta + s)) + (F_1^2 + F_N^2) \sin(q(\theta + s)), \]
\[ H_3 = \frac{1}{q^2} \left( (F_1^2 + F_N^2) \cos(q(\theta + s)) - (F_1^2 + F_N^2) \sin(q(\theta + s)) \right) \]
\[ - \frac{1}{p^2} \left( (F_1^2 + F_N^2) \cos(p\theta) - (F_1^2 + F_N^2) \sin(p\theta) \right), \]
\[ H_k = \lambda_k y_k + F^k_N. \]

**Proof:** In the variables \((\theta, r, \rho, s, y_5, \ldots, y_m)\) system (6) becomes

\[ \dot{\theta} = 1 + \frac{\varepsilon}{r} \left( \cos(p\theta)(F_1^2 + F_N^2) - \sin(p\theta)(F_1^2 + F_N^2) \right), \]
\[ \dot{r} = \varepsilon H_1(\theta, r, \rho, s, y_5, \ldots, y_m), \]
\[ \dot{\rho} = \varepsilon H_2(\theta, r, \rho, s, y_5, \ldots, y_m), \]
\[ \dot{s} = \varepsilon H_3(\theta, r, \rho, s, y_5, \ldots, y_m), \]
\[ \dot{y}_k = \varepsilon H_k(\theta, r, \rho, s, y_5, \ldots, y_m), \quad k = 5, \ldots, m. \]

For \(\varepsilon\) sufficiently small, \(\dot{\theta}(t) > 0\) for each \((t, (\theta, r, \rho, s, y_5, \ldots, y_m)) \in \mathbb{R} \times D\). Now we eliminate the variable \(t\) in the above system by considering \(\theta\) as the new independent variable. It is clear that the right-hand side of the new system is well-defined and continuous in \(\mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0)\), \(2\pi\)-periodic with respect to the independent variable \(\theta\) and locally Lipschitz with respect to \((r, \rho, s, y_5, \ldots, y_m)\). From (8), Equation (7) is obtained after an expansion with respect to the small parameter \(\varepsilon\).

We recall a technical result from [4] which we shall use later on.

**Lemma 5:** Let \(\alpha\) and \(\beta\) be real numbers. Given nonnegative integers \(i, j, k, l\), there exist constants \(c_{uv}\) and \(d_{uv}\) such that \(\cos^{i} \alpha \sin^{j} \alpha \cos^{k} \beta \sin^{l} \beta\) is equal to

\[ \sum_{u=0}^{[(i+j)/2]} \sum_{v=0}^{[(k+l)/2]} c_{uv} \cos \left( (i+j-2u)\alpha \pm (k+l-2v)\beta \right) \]

if \(j+l\) is even, and is equal to

\[ \sum_{u=0}^{[(i+j)/2]} \sum_{v=0}^{[(k+l)/2]} d_{uv} \sin \left( (i+j-2u)\alpha \pm (k+l-2v)\beta \right) \]

if \(j+l\) is odd. Here \([x]\) denotes the integer part function of \(x \in \mathbb{R}\).

Now we compute the corresponding averaged functions \(h_j(r, \rho, s, y_5, \ldots, y_m)\) for \(j = 1, \ldots, m\) of system (7) given in (4). We write

\[ F^j_1 = \sum_{j=1}^{m} a^j_x x_j \quad \text{and} \quad F^k_N = \sum_{i_1+i_2+\cdots+i_m=N} a^k_{i_1 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}. \]
for \( g = 1, \ldots, m \). We also write

\[
h_j(r, \rho, s, y_5, \ldots, y_m) = \int_{0}^{2\pi} H_j(\theta, r, \rho, s, y_5, \ldots, y_m) \, d\theta
\]

for \( j = 1, 2, 3, 5, \ldots, m \).

**Proposition 6:** We have

\[
h_1(r, \rho, s, y_5, \ldots, y_m) = a_1 r + r^{q-1} \rho^q (b_1 \sin(pqs) + c_1 \cos(pqs)) + \sum_{l=1}^{m} d_{l_1 \ldots l_m}^{1} \rho^q r^{l_1+1-2l-i_5-\ldots-i_m} \rho^{2l} y_5^{i_5} \cdots y_m^{i_m},
\]

for some constants \( a_1, b_1, c_1 \) and \( d_{l_1 \ldots l_m}^{1} \) depending on the coefficients of the perturbation.

**Proof:** We write the function \( H_1 \) as

\[
H_1 = H_1^1 + H_1^N = (F_1^1 \cos(p\theta) + F_1^2 \sin(p\theta)) + (F_N^1 \cos(p\theta) + F_N^2 \sin(p\theta)).
\]

Then

\[
h_1^1(r, \rho, s, y_5, \ldots, y_m) = \int_{0}^{2\pi} H_1^1(\theta, r, \rho, s, y_5, \ldots, y_m) \, d\theta = \sum_{j=1}^{m} \int_{0}^{2\pi} (a_j \cos(p\theta) + a_j^2 \sin(p\theta)) \, d\theta = \pi(a_1^2 + a_2^2) r,
\]

and

\[
h_1^N(r, \rho, s, y_5, \ldots, y_m) = \int_{0}^{2\pi} H_1^N(\theta, r, \rho, s, y_5, \ldots, y_m) \, d\theta
\]

\[
= \sum_{l_1 + \ldots + l_m = N} \int_{0}^{2\pi} \left( a_{l_1 \ldots l_m}^1 r^{l_1+1-2l-i_5-\ldots-i_m} \rho^{l_1+1} \cos(l_1+1) \sin^{l_1}(p\theta) \right) \sin^{l_5}(p\theta) \sin^{l_5}(q\theta+s) \ sin^{l_5}(q\theta+s) \ y_5^{i_5} \cdots y_m^{i_m} \, d\theta
\]

\[
+ \sum_{l_1 + \ldots + l_m = N} \int_{0}^{2\pi} a_{l_1 \ldots l_m}^2 r^{l_1+1-2l-i_5-\ldots-i_m} \rho^{l_1+1} \cos(l_1+1) \sin^{l_1}(p\theta) \sin^{l_1+1}(p\theta) \sin^{l_1}(q\theta+s) \ sin^{l_1}(q\theta+s) \ y_5^{i_5} \cdots y_m^{i_m} \, d\theta.
\]

By Lemma 5 we obtain

\[
h_1^N(r, \rho, s, y_5, \ldots, y_m) = \sum_{l_1 + \ldots + l_m = N} r^{l_1+1-2l-i_5-\ldots-i_m} \rho^{l_1+1} y_5^{i_5} \cdots y_m^{i_m} 
\]

\[
\cdot \int_{0}^{2\pi} \sum_{u=0}^{2\pi} \sum_{v=0}^{2\pi} C_{uv}^{l_1 \ldots l_m}(\theta) \, d\theta,
\]
where

\[ C_{i_1 \ldots i_m}^{i_1 \ldots i_m} = c_{i_1 \ldots i_m}^{i_1 \ldots i_m} \cos((i_1 + i_2 + 1 - 2u) \rho \theta \pm (i_3 + i_4 - 2v)q(\theta + s)) \]
\[ + d_{i_1 \ldots i_m}^{i_1 \ldots i_m} \sin((i_1 + i_2 + 1 - 2u) \rho \theta \pm (i_3 + i_4 - 2v)q(\theta + s)), \]

for some constants \(c_{i_1 \ldots i_m}^{i_1 \ldots i_m}\) and \(d_{i_1 \ldots i_m}^{i_1 \ldots i_m}\). Therefore all the integrals with respect to \(\theta\) are zero except possibly when

\[ p(i_1 + i_2 + 1 - 2u) = q(i_3 + i_4 - 2v). \]

Without loss of generality, we continue to assume that \(p < q\). Since \(p\) and \(q\) are coprime, there exists a nonnegative integer \(n\) such that \(i_1 + i_2 + 1 - 2u = nq\) and \(i_3 + i_4 - 2v = np\). Furthermore, since

\[ 0 \leq i_1 + i_2 + 1 - 2u \leq N + 1 = p + q, \]

we have that \(nq \leq p + q\), and thus \(n \leq (p + q)/q < 2\). So either \(n = 1\) or \(n = 0\), i.e. either \(i_3 + i_4 - 2v = p\) or \(i_3 + i_4 - 2v = 0\).

If \(i_3 + i_4 - 2v = p\), then \(i_1 + i_2 + 1 - 2u = q\), and it follows from (10) that

\[ i_5 + \cdots + i_m = N - (i_1 + i_2 + i_3 + i_4) = -2(u + v). \]

Therefore \(u = v = 0 = i_5 = \cdots = i_m = 0\), and hence \(i_1 + i_2 = q - 1\) and \(i_3 + i_4 = p\). This yields the term

\[ r^{q-1} \rho^p (b_1 \sin(pqs) + c_1 \cos(pqs)). \]

If \(i_3 + i_4 - 2v = 0\), then \(2v + i_5 + \cdots + i_m = N - i_1 - i_2\), and \(2v + i_5 + \cdots + i_m\) runs from 0 to \(N = p + q - 1\). This yields the terms

\[ \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d_{i_5 \ldots i_m}^{i_5 \ldots i_m} \rho^{p+q-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m}. \]

The proposition follows adding the terms from (9), (11) and (12).

\[ \square \]

**Proposition 7:** We have

\[ h_2(r, \rho, s, y_5, \ldots, y_m) = a_2 \rho + r^p \rho^{p-1} (b_2 \sin(pqs) + c_2 \cos(pqs)) \]
\[ + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d_{i_5 \ldots i_m}^{i_5 \ldots i_m} \rho^{p+q-2v-i_5-\cdots-i_m} \rho^{2v-1} y_5^{i_5} \cdots y_m^{i_m}, \]

for some constants \(a_2, b_2, c_2\) and \(d_{i_5 \ldots i_m}^{i_5 \ldots i_m}\) depending on the coefficients of the perturbation.

**Proof:** As in Proposition 6, we write the function \(H_2\) as

\[ H_2 = H_2^I + H_2^N = \left( F_1^1 \cos(q(\theta + s)) + F_1^1 \sin(q(\theta + s)) \right) \]
\[ + \left( F_N^1 \cos(q(\theta + s)) + F_N^1 \sin(q(\theta + s)) \right). \]
Then

\[ h_2^1(r, s, \rho, y_5, \ldots, y_m) = \int_0^{2\pi} H_2^1(\theta, r, s, \rho, y_5, \ldots, y_m) \, d\theta \]

\[ = \sum_{j=1}^m \int_0^{2\pi} (a_j^3 \cos(q(\theta + s)) + a_j^4 \sin(q(\theta + s))) x_j \, d\theta \]

\[ = \pi(a_3^3 + a_4^4) \rho, \]

and using Lemma 5 we obtain

\[ h_2^N(r, s, \rho, y_5, \ldots, y_m) = \int_0^{2\pi} H_2^N(\theta, r, s, \rho, y_5, \ldots, y_m) \, d\theta \]

\[ = \sum_{i_1 + \cdots + i_m = N} a_{i_1 \cdots i_m}^3 r_{i_1 + i_2} \rho_{i_3 + i_4} \cos^{i_1}(p\theta) \sin^{i_2}(p\theta) \]

\[ \cdot \cos^{i_3}(q(\theta + s)) \sin^{i_4}(q(\theta + s)) y_5^{i_5} \cdots y_m^{i_m} \, d\theta \]

\[ + \sum_{i_1 + \cdots + i_m = N} a_{i_1 \cdots i_m}^4 r_{i_1 + i_2} \rho_{i_3 + i_4} \cos^{i_1}(p\theta) \sin^{i_2}(p\theta) \]

\[ \cdot \cos^{i_3}(q(\theta + s)) \sin^{i_4}(q(\theta + s)) y_5^{i_5} \cdots y_m^{i_m} \, d\theta \]

\[ = \sum_{i_1 + \cdots + i_m = N} r_{i_1 + i_2} \rho_{i_3 + i_4} y_5^{i_5} \cdots y_m^{i_m} \]

\[ \cdot \int_0^{2\pi} [((i_1 + i_2)/2) [((i_3 + i_4)/2)] \]

\[ \cdot \int_0^{2\pi} D_{i_1 \cdots i_m}^{i_1 \cdots i_m}(\theta) \, d\theta, \]

where

\[ D_{i_1 \cdots i_m}^{i_1 \cdots i_m} = c_{i_1 \cdots i_m}^{i_1 \cdots i_m} \cos ((i_1 + i_2 - 2u) p\theta \pm (i_3 + i_4 + 1 - 2v) q(\theta + s)) \]

\[ + d_{i_1 \cdots i_m}^{i_1 \cdots i_m} \sin ((i_1 + i_2 - 2u) p\theta \pm (i_3 + i_4 + 1 - 2v) q(\theta + s)), \]

for some constants \( c_{i_1 \cdots i_m}^{i_1 \cdots i_m} \) and \( d_{i_1 \cdots i_m}^{i_1 \cdots i_m} \). All the integrals with respect to \( \theta \) are zero except possibly when

\[ p(i_1 + i_2 - 2u) = q(i_3 + i_4 + 1 - 2v). \]  

(14)

Since \( p \) and \( q \) are coprime, there exists a nonnegative integer \( u \) such that \( i_1 + i_2 - 2u = nq \) and \( i_3 + i_4 + 1 - 2v = np \). Furthermore, since

\[ 0 \leq i_3 + i_4 + 1 - 2v \leq N + 1 = p + q, \]

we have that \( np \leq p + q \), and thus \( n \leq (p + q)/q < 2 \). So either \( n = 1 \) or \( n = 0 \), i.e. either \( i_3 + i_4 + 1 - 2v = p \) or \( i_3 + i_4 + 1 - 2v = 0 \).

If \( i_3 + i_4 + 1 - 2v = p \), then by (14) we obtain that

\[ p + q - 1 - i_3 - i_4 - i_5 - \cdots - i_m - 2u = i_1 + i_2 - 2u = q, \]

and hence

\[ i_5 + \cdots + i_m + 2u + 2v = p - 1 - i_3 - i_4 + 2v = 0. \]
This implies that \( i_5 = \ldots = i_m = 0 \) and \( u = v = 0 \). Then \( i_3 + i_4 = p - 1 \) and \( i_1 + i_2 = q \), which yields the term

\[
r^q \rho^{p-1} \left( b_2 \sin(pqs) + c_2 \cos(pqs) \right).
\]

(15)

If \( i_3 + i_4 + 1 - 2v = 0 \), then

\[
2v + i_5 + \ldots + i_m - 1 = N - i_1 - i_2.
\]

Thus \( 2v + i_5 + \ldots + i_m \) runs from 1 to \( p + q \), yielding the terms

\[
\sum_{2v+i_5+\ldots+i_m=1}^{p+q} d^2_{i_5\ldots i_m} \rho^{p+q-2v-i_5-\ldots-i_m} \rho^{2v-1} y_2^{i_5} \ldots y_m^{i_m}.
\]

(16)

The proposition follows adding the terms of (13), (15) and (16).

\[ \square \]

**Proposition 8:** We have

\[
h_3(r, \rho, s, y_5, \ldots, y_m) = a_3 + r^{q-2} \rho^p \left( b_3 \sin(pqs) + c_3 \cos(pqs) \right)
\]

\[
+ r^q \rho^{p-1} \left( d_3 \sin(pqs) + e_3 \cos(pqs) \right)
\]

\[
+ \sum_{2v+i_5+\ldots+i_m=0}^{p+q-1} d^3_{i_5\ldots i_m} \rho^{p+q-2v-i_5-\ldots-i_m} \rho^{2v-1} y_2^{i_5} \ldots y_m^{i_m},
\]

\[
+ \sum_{2v+i_5+\ldots+i_m=1}^{p+q} d^4_{i_5\ldots i_m} \rho^{p+q-2v-i_5-\ldots-i_m} \rho^{2v-2} y_2^{i_5} \ldots y_m^{i_m},
\]

for some constants \( a_3, b_3, c_3, d_3, e_3, d^3_{i_5\ldots i_m} \) and \( d^4_{i_5\ldots i_m} \) depending on the coefficients of the perturbation.

**Proof:** We have \( H_3 = H^1_3 + H^N_3 \) where

\[
H^1_3 = \frac{1}{qr} \left( F^1_3 \cos(q(\theta + s)) + F^3_3 \sin(q(\theta + s)) \right) - \frac{1}{pr} \left( F^2_1 \cos(p\theta) - F^3_1 \sin(p\theta) \right),
\]

\[
H^N_3 = \frac{1}{qr} \left( F^N_3 \cos(q(\theta + s)) - F^3_3 \sin(q(\theta + s)) \right) - \frac{1}{pr} \left( F^2_1 \cos(p\theta) - F^3_1 \sin(p\theta) \right).
\]

Proceeding in a similar manner to the proofs of Propositions 6 and 7, we get

\[
h^1_3(r, \rho, s, y_5, \ldots, y_m) = \int_0^{2\pi} H^1_3(\theta, r, \rho, s, y_5, \ldots, y_m) \, d\theta
\]

\[
= \frac{\pi(a_3^2 - a_4^2)}{q} - \frac{\pi(a_3^2 - a_4^2)}{p}.
\]

(17)

Now we calculate

\[
h^N_3(r, \rho, s, y_5, \ldots, y_m) = \int_0^{2\pi} H^N_3(\theta, r, \rho, s, y_5, \ldots, y_m) \, d\theta.
\]
In a similar manner to the proofs of Propositions 6 and 7, we get

\[
\begin{align*}
h^N_{v}(r, \rho, s, y_5, \ldots, y_m) &= \frac{1}{q} \sum_{j_1+\cdots+j_m=N} r^{j_1+j_2+\cdots+j_m} \rho^{j_3+j_4} y_5^{j_5} \cdots y_m^{j_m} \\
&\cdot \int_0^{2\pi} \left[ \sum_{u=0}^2 \sum_{v=0}^{[j_5+j_4+1]/2} E_{uv}^{j_1\cdots j_m}(\theta) d\theta \right] \\
&- \frac{1}{p} \sum_{j_1+\cdots+j_m=N} r^{j_1+j_2-1} \rho^{j_3+j_4} y_5^{j_5} \cdots y_m^{j_m} \\
&\cdot \int_0^{2\pi} \left[ \sum_{u=0}^2 \sum_{v=0}^{[j_5+j_4+1]/2} F_{uv}^{j_1\cdots j_m}(\theta) d\theta \right],
\end{align*}
\]

where

\[
E_{uv}^{j_1\cdots j_m} = e_{uv}^{j_1\cdots j_m} \cos \left( (i_1 + i_2 - 2u) p\theta \pm (i_3 + i_4 + 1 - 2v) q(\theta + s) \right) \\
\quad + d_{uv}^{j_1\cdots j_m} \sin \left( (i_1 + i_2 - 2u) p\theta \pm (i_3 + i_4 + 1 - 2v) q(\theta + s) \right),
\]

and

\[
F_{uv}^{j_1\cdots j_m} = f_{uv}^{j_1\cdots j_m} \cos \left( (i_1 + i_2 + 1 - 2u) p\theta \pm (i_3 + i_4 - 2v) q(\theta + s) \right) \\
\quad + g_{uv}^{j_1\cdots j_m} \sin \left( (i_1 + i_2 + 1 - 2u) p\theta \pm (i_3 + i_4 - 2v) q(\theta + s) \right).
\]

The terms whose integrals need not be zero satisfy

\[
p(i_1 + i_2 - 2u) = q(i_3 + i_4 + 1 - 2v)
\]

in Equation (19), and

\[
p(i_1 + i_2 + 1 - 2u) = q(i_3 + i_4 - 2v)
\]

in Equation (20).

The arguments in the proof of Proposition 7 show that in (18) the terms that may remain in the first sum are

\[
r^q \rho^{p-q-2} \left( d_{v_1} \sin (pq_1) + e_3 \cos (pq_3) \right) \\
+ \sum_{2v_1+j_5+\cdots+j_m=1}^{p+q} d_{v_1j_5\cdots j_m}^{q} \rho^{p+q-2v_1-i_5-\cdots-i_m} y_5^{j_5} \cdots y_m^{j_m},
\]

and the arguments in the proof of Proposition 6 show that the terms that may remain in the second sum are

\[
r^{q-2} \rho^p \left( b_3 \sin (pq_3) + c_3 \cos (pq_3) \right) \\
+ \sum_{2v_1+j_5+\cdots+j_m=0}^{p+q-1} d_{v_1j_5\cdots j_m}^{q-2} \rho^{p+q-2v_1-i_5-\cdots-i_m} y_5^{j_5} \cdots y_m^{j_m},
\]

The proposition follows adding the terms in (17), (21) and (22).
Proposition 9: For $k = 5, \ldots, m$, we have

$$h_k(r, \rho, s, y_5, \ldots, y_m) = \lambda_k y_k + \sum_{2v + i_1 + \cdots + i_m = 0}^{p+q-1} d_{i_1 \cdots i_m}^k \rho^{p+1-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m},$$

for some constants $d_{i_1 \cdots i_m}^k$ depending on the coefficients of the perturbation.

Proof: As in the former proofs, we write $H_k = H_k^1 + H_k^N$ where $H_k^1 = \lambda_k y_k$ and $H_k^N = F_N^k$, and we compute the function

$$h_k^N(r, s, \rho, y_5, \ldots, y_m) = \int_0^{2\pi} H_k^N(\theta, r, s, \rho, y_5, \ldots, y_m) \, d\theta.$$

Proceeding as in the proofs of Propositions 6 or 7, we obtain

$$h_k^N(r, \rho, s, y_5, \ldots, y_m) = \sum_{i_1 + \cdots + i_m = N} \int_0^{2\pi} \sum_{u=0}^{i_1+i_2} \rho^{i_1+i_2} \rho^{i_3+i_4} \cos^i(p\theta) \sin^i(p\theta) \cdot \cos^i(q(\theta + s)) \sin^i(q(\theta + s)) y_5^{i_5} \cdots y_m^{i_m} \, d\theta$$

$$= \sum_{i_1 + \cdots + i_m = N} \rho^{i_1+i_2} \rho^{i_3+i_4} y_5^{i_5} \cdots y_m^{i_m} \cdot \int_0^{2\pi} \frac{[(i_1+i_2)/2][(i_3+i_4)/2]}{G_{i_1 \cdots i_m}^{i_5 \cdots i_m}(\theta) \, d\theta},$$

where

$$G_{i_1 \cdots i_m}^{i_5 \cdots i_m} = g_{i_1 \cdots i_m}^{i_5 \cdots i_m} \cos \left( (i_1 + i_2 - 2u) p \theta \pm (i_3 + i_4 - 2v) q(\theta + s) \right)$$

$$+ h_{i_1 \cdots i_m}^{i_5 \cdots i_m} \sin \left( (i_1 + i_2 - 2u) p \theta \pm (i_3 + i_4 - 2v) q(\theta + s) \right).$$

All the integrals with respect to $\theta$ are zero except possibly when

$$p(i_1 + i_2 - 2u) = q(i_3 + i_4 - 2v). \quad (23)$$

Proceeding as in the proof of Proposition 6, we find that either $i_3 + i_4 - 2v = p$ or $i_3 + i_4 - 2v = 0$.

If $i_3 + i_4 - 2v = p$, then by (23) we obtain

$$q - 1 - i_5 - \cdots - i_m - 2v - 2u = q, \quad \text{that is,} \quad -1 - i_5 - \cdots - i_m - 2v - 2u = 0,$$

which yields a contradiction. Therefore, this case does not occur.

If $i_3 + i_4 - 2v = 0$, then

$$2v + i_5 + \cdots + i_m = p + q - 1 - i_1 - i_2.$$

Hence $2v + i_5 + \cdots + i_m$ runs from 0 to $p + q - 1$, and we obtain the terms

$$\sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d_{i_1 \cdots i_m}^k \rho^{p+q-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m}.$$

This yields the desired statement.
4. Proof of Theorem 1
We recall a technical result proved in [5].

Lemma 10: If \( p, q, \alpha \) and \( \beta \) are nonnegative integers with \( \alpha + \beta = q - 1 \) and \( \gamma + \delta = p \), then

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \cos^{\alpha}(p\theta) \sin^{\beta}(p\theta) \cos^{\gamma}(q(\theta + s)) \sin^{\delta}(q(\theta + s)) \, d\theta = \begin{cases} 
\frac{(-1)^{q/2}}{2^{p+q-1}} \cos(pqs) & \text{if } \beta, \delta \text{ are even,} \\
\frac{(-1)^{(q+1)/2}}{2^{p+q-1}} \sin(pqs) & \text{if } \beta \text{ is even and } \delta \text{ is odd,} \\
\frac{(-1)^{(q-1)/2}}{2^{p+q-1}} \sin(pqs) & \text{if } \beta \text{ is odd and } \delta \text{ is even,} \\
\frac{-(-1)^{q/2}}{2^{p+q-1}} \cos(pqs) & \text{if } \beta, \delta \text{ are odd.} 
\end{cases}
\]

We will use the following proposition.

Proposition 11: The function \( h_3(r, \rho, s, y_5, \ldots, y_m) \) is given by

\[
h_3(r, \rho, s, y_5, \ldots, y_m) = a_3 + \frac{1}{p} r^{p-2} \rho^p \left( - c_1 \sin(pqs) + b_1 \cos(pqs) \right) + \frac{1}{q} r^q \rho^{q-2} \left( - c_2 \sin(pqs) + b_2 \cos(pqs) \right) + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^{(3)}_{i_1 \cdots i_m} r^{p+q-2-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m} + \sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^{(1)}_{i_1 \cdots i_m} r^{p+q-2-2v-i_5-\cdots-i_m} \rho^{2v-2} y_5^{i_5} \cdots y_m^{i_m},
\]

where \( b_1, c_1 \) are the constants in Proposition 6, and \( b_2, c_2 \) are the constants in Proposition 7.

Proof: Using the notation of Proposition 8, we shall prove that \( b_3 = -c_1/p, c_3 = b_1/p, d_3 = -c_2/q \) and \( e_3 = b_2/q \). To simplify the proof, let \( a^{(1)}_{i_1 \cdots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \) be a monomial in \( F^3_N \) such that \( i_1 + i_2 = q - 1, i_3 = 0, i_4 = p \) and \( i_5 = \cdots = i_m = 0 \). When we compute \( h_1 \) and \( h_3 \), this monomial appears in \( h_1 \) as

\[
\int_{0}^{2\pi} a^{(1)}_{i_1 \cdots i_m} \cos^{i_1+1}(p\theta) \sin^{i_2}(p\theta) \sin(q(\theta + s)) \, d\theta,
\]

and in \( h_3 \) as

\[
\frac{1}{p} \int_{0}^{2\pi} a^{(1)}_{i_1 \cdots i_m} \cos^{i_1}(p\theta) \sin^{i_2+1}(p\theta) \sin(q(\theta + s)) \, d\theta.
\]

By Lemma 10 the term in (24) is equal to

\[
\begin{cases} 
\frac{(-1)^{i_2/2}}{2^{p+q}} a^{(1)}_{i_1 \cdots i_m} \sin(pqs), & \text{if } i_2 \text{ is even,} \\
\frac{-(-1)^{(i_2+1)/2}}{2^{p+q}} a^{(1)}_{i_1 \cdots i_m} \cos(pqs), & \text{if } i_2 \text{ is odd,}
\end{cases}
\]
For $i_2$ even the coefficient of the monomial appears in a sum determining the coefficient of $\rho^{q-1} \rho^s \cos(pq)$ in $h_1$, and also appears in a sum determining the coefficient of $\rho^{q-2} \rho^s \sin(pq)$ in $h_3$ with the opposite sign. In a similar way for $i_2$ even the coefficient of the monomial appears in a sum determining the coefficient of $\rho^{q-1} \rho^s \sin(pq)$ in $h_1$, and appears in a sum determining the coefficient of $\rho^{q-2} \rho^s \cos(pq)$ in $h_3$ with the same sign.

We can do the same for all monomials in $F^2_N$, $F^3_N$ and $F^4_N$, and thus we conclude that $b_3 = -c_1/p$, $c_3 = b_1/p$, $d_3 = -c_2/q$ and $e_3 = b_2/q$.

Now we have all the ingredients to prove Theorem 1.

**Proof of Theorem 1:** It follows from Propositions 6, 7, 9 and 11 that

\[
h_1 = a_1 r + r^{q-1} \rho^s (b_1 \sin(pq) + c_1 \cos(pq)) + \sum_{2v+i_1, \ldots, i_m=0}^{p+q-1} d_{v_1 \ldots i_m}^q \rho^{q-1-2v-i_1-\ldots-i_m} \rho^{2v} y_5 \ldots y_m,
\]

\[
h_2 = a_2 \rho + r^{q} \rho^{q-1} (b_2 \sin(pq) + c_2 \cos(pq)) + \sum_{2v+i_1, \ldots, i_m=1}^{p+q} d_{v_1 \ldots i_m}^2 \rho^{q-2-2v-i_1-\ldots-i_m} \rho^{2v-1} y_5 \ldots y_m,
\]

\[
h_3 = a_3 + \frac{1}{p} r^{q-2} \rho^s (-c_1 \sin(pq) + b_1 \cos(pq)) + \frac{1}{q} r^{q} \rho^{q-2} (-c_2 \sin(pq) + b_2 \cos(pq)) + \sum_{2v+i_1, \ldots, i_m=0}^{p+q-1} d_{v_1 \ldots i_m}^3 \rho^{q-2-2v-i_1-\ldots-i_m} \rho^{2v-1} y_5 \ldots y_m
\]

\[
h_k = \lambda_k y_k + \sum_{2v+i_1, \ldots, i_m=0}^{p+q-1} d_{v_1 \ldots i_m}^k \rho^{q-1-2v-i_1-\ldots-i_m} \rho^{2v} y_5 \ldots y_m,
\]

where $h_j = h_j(r, \rho, s, y_5, \ldots, y_m)$.

According to the results of Section 2, we must study the real solutions of the system

\[
h_k(r, \rho, s, y_5, \ldots, y_m) = 0 \quad \text{for } k = 1, 2, 3, 5, \ldots, m
\]

that have nonzero Jacobian. In order that these solutions can provide limit cycles of system (2), we must look for those such that $r^2 + \rho^2 \neq 0$. We distinguish three cases.
Case 1: \( r = 0 \) and \( \rho \neq 0 \). If \( q > 2 \) then in system (26) the variable \( s \) does not appear. So the Jacobian of the system is always zero, and consequently the number of limit cycles of system (2) provided by the averaging theory is zero in this case.

In this case if \( q = 2 \), then \( p = 1 \), and it is easy to check that all the equations of system (26) (except the first one which is identically zero) are polynomial equations of degree two in the variables \( r, \rho, y_5, \ldots, y_m, \cos(2s) \) and \( \sin(2s) \). Therefore, adding to system (26) the equation \( \cos^2(2s) + \sin^2(2s) = 1 \) by the Bézout Theorem [9], the maximum number of limit cycles that can appear in this subcase is \( 2^m \). Since for each solution \( w_0 = \cos(2s) \) and \( z_0 = \sin(2s) \) of \( \cos^2(2s) + \sin^2(2s) = 1 \) we can find \( s_1, s_2 \in [0, 2\pi) \) such that \( \sin(2s_1) = z_0 \) and \( \cos(2s_2) = w_0 \) for \( i = 1, 2 \), we get that the total number of solutions of system (26) is at most \( 2^m \).

Case 2: \( b_2 = c_2 = 0, \rho = 0 \) and \( r \neq 0 \). Then the degree of the polynomial equations of system (26) in the variables \( r, \rho, y_5, \ldots, y_m, \cos(pqs) \) and \( \sin(pqs) \) are \( p + q - 1, p + q, p + q, p + q - 1, \ldots, p + q - 1 \) respectively. Therefore, adding to system (26) the equation \( \cos^2(pqs) + \sin^2(pqs) = 1 \) by the Bézout Theorem, the maximum number of limit cycles that can appear in this case is \( 2(p+q-1)^{m-3}(p+q)^2 \). Since for each solution \( w_0 = \cos(pqs) \) and \( z_0 = \sin(pqs) \) of \( \cos^2(pqs) + \sin^2(pqs) = 1 \) we can find \( s_1, \ldots, s_{pq} \in [0, 2\pi) \) such that \( \sin(pqs_1) = z_0 \) and \( \cos(pqs_2) = w_0 \) for \( i = 1, \ldots, pq \), we obtain that the total number of solutions of system (26) is at most \( 2pq(p+q-1)^{m-3}(p+q)^2 \).

Case 3: \( r \rho \neq 0 \). Now we perform the change of variables

\[
\rho^q \rho^p = B, \quad \rho/r = A, \quad \sin(pqs) = z, \quad \cos(pqs) = w, \quad y_k/r = C_k
\]

for \( k = 5, \ldots, m \). In the new variables the functions

\[
\tilde{h}_1 = h_1/r, \quad \tilde{h}_2 = h_2/r, \quad \tilde{h}_3 = \rho h_3/r, \quad \tilde{h}_4 = z^2 + w^2 - 1, \quad \tilde{h}_k = h_k/r
\]

for \( k = 5, \ldots, m \) are given by

\[
\begin{align*}
\tilde{h}_1 &= a_1 + AB(b_1z + c_1w) + A^{1-p}BP_1(A^2, C_5, \ldots, C_m), \\
\tilde{h}_2 &= a_2A + B(b_2z + c_2w) + A^{-p}BP_2(A^2, C_5, \ldots, C_m), \\
\tilde{h}_3 &= a_3A + 1/p A^2B(-c_1z + b_1w) + 1/q B(-c_2z + b_2w) \\
&\quad + A^{-p}BP_3(A^2, C_5, \ldots, C_m) + A^{-p}BP_4(A^2, C_5, \ldots, C_m), \\
\tilde{h}_4 &= z^2 + w^2 - 1, \\
\tilde{h}_k &= \lambda_k C_k + A^{1-p}BP_k(A^2, C_5, \ldots, C_m),
\end{align*}
\]

for \( k = 5, \ldots, m \), where

\[
P_i(A^2, C_5, \ldots, C_m) = \sum_{2l+i_5+\cdots+i_m=0}^{p+q-1} \delta_{i_5\ldots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}
\]

for \( i = 1, 3, k \) and

\[
P_i(A^2, C_5, \ldots, C_m) = \sum_{2l+i_5+\cdots+i_m=1}^{p+q} \delta_{i_5\ldots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}
\]

for \( i = 2, 4 \).
Solving \((\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = (0, 0, 0)\) we find the solution

\[
\begin{align*}
  z &= A^{-p} Z(A^2, C_5, \ldots, C_m), \\
  w &= A^{-p} W(A^2, C_5, \ldots, C_m), \\
  B &= A^{p-1} B(A^2, C_5, \ldots, C_m),
\end{align*}
\]

where

\[
\begin{align*}
  Z &= \frac{Z_1}{Z_2}, \\
  W &= \frac{W_1}{Z_2}, \quad \text{and} \quad B = \frac{B_1}{B_2},
\end{align*}
\]

with

\[
\begin{align*}
  Z_1 &= A^4 a_2 (b_1 P_1 - c_1 p P_3) q + a_1 p (-b_2 P_2 + c_2 P_4 q) + A^2 (a_2 b_2 p P_1 \\
  &- (a_1 b_1 P_2 + p (a_3 c_1 P_1 - a_3 c_1 P_2 - a_1 c_2 P_3 + a_2 c_1 P_4)) q), \\
  Z_2 &= a_1 (b_2^2 + c_2^2) p - A^4 a_2 (b_1^2 + c_1^2) q + A^2 (-a_2 b_1 b_2 + c_1 c_2) p \\
  &+ (a_1 b_1 b_2 + a_1 c_1 c_2 - a_3 b_2 c_1 p + a_3 b_2 c_1 p) q), \\
  W_1 &= A^4 a_2 (c_1 P_1 + b_1 P_3) q - a_1 p (c_2 P_2 + b_2 P_4 q) \\
  &- A^2 (-a_3 b_2 P_1 + a_1 c_1 P_2 + a_3 b_2 p P_3) q - a_2 p (c_2 P_1 + b_1 P_4 q), \\
  B_1 &= a_1 (b_2^2 + c_2^2) p - A^4 a_2 (b_1^2 + c_1^2) q + A^2 (-a_2 b_1 b_2 + c_1 c_2) p \\
  &+ (a_1 b_1 b_2 + a_1 c_1 c_2 - a_3 b_2 c_1 p + a_3 b_2 c_1 p) q), \\
  B_2 &= (b_1 b_2 + c_1 c_2) p P_2 - (b_2^2 + c_2^2) p P_1 + (A^2 b_1^2 P_2 + c_1 (-c_2 P_1 + c_1 P_2 + b_2 P_3) \\
  &- b_1 (b_2 P_1 + c_2 P_3) + (b_2 c_1 - b_1 c_2) P_4) q).
\end{align*}
\]

Therefore in the variables \((A^2, C_5, \ldots, C_m)\), \(\overline{B}\) is a quotient of a polynomial of degree two by a polynomial of degree \(p + q + 1\), \(Z\) is a quotient of a polynomial of degree \(p + q + 1\) by a polynomial of degree 2 and \(W\) is a quotient of a polynomial of degree \(p + q + 1\) by a polynomial of degree 2.

Substituting \(z\) and \(w\) in the equation \(\tilde{h}_k = 0\), we obtain a quotient of a polynomial of degree \(2(p + q + 1)\) by a polynomial of degree \(4 + p\) in the variables \((A^2, C_5, \ldots, C_m)\).

Substituting \(\overline{B}\) in the equations \(h_k = 0\), we obtain a quotient of a polynomial of degree \(p + q + 2\) by a polynomial of degree \(p + q + 1\) in the variables \((A^2, C_5, \ldots, C_m)\). Therefore, by applying Bézout’s theorem we have that the maximum number of possible roots \((\overline{h}_4, \overline{h}_5, \ldots, \overline{h}_m) = 0\) is given by \(2(p + q + 1)(p + q + 2)^{m-4}\). For each solution \((A^2_0, C_{50}, \ldots, C_{m0})\) we have at most one \(B_0 = B(A^2_0, C_{50}, \ldots, C_{m0})\) and one pair

\[
(z_0, w_0) = (z(A^2_0, C_{50}, \ldots, C_{m0}), w(A^2_0, C_{50}, \ldots, C_{m0})).
\]

For each pair \((z_0, w_0)\) we can find \(s_1, \ldots, s_{pq} \in [0, 2\pi)\) such that \(\sin(pqs_i) = z_0\) and \(\cos(pqs_i) = w_0\) for \(i = 1, \ldots, pq\). So in this case the maximum number of zeros of system (26) is at most \(2pq(p + q + 1)(p + q + 2)^{m-4}\).

Now we put together the results of the three cases. By Theorem 3 the maximum number of limit cycles obtained via averaging theory for system (2) is

\[
2^m + 2pq(p + q - 1)^{m-3}(p + q)^2 + 2pq(p + q + 1)(p + q + 2)^{m-4} = 2^m + 2^{m-1} 3^2 + 2^4 5^{m-4}
\]
\[ q = 2, \quad p = 1, \quad \text{or} \quad 2pq(p + q - 1)^{m-3}(p + q)^2 + 2pq(p + q + 1)(p + q + 2)^{m-4}, \]

if \( p + q > 3 \). This completes the proof of the theorem. \( \square \)

**Acknowledgments**

L. Barreira and C. Valls were partially supported by the FCT grant PTDC/MAT/117106/2010 and by FCT through CAMGSD, Lisbon. J. Llibre was supported by the grants MCYT/FEDER MTM 2008–03437, Generalitat de Catalunya 2009SGR410 and ICREA Academia.

**References**

[1] Yu. Ilyashenko, *Centennial history of Hilbert’s 16th problem*, Bull. Amer. Math. Soc. 39 (2002), pp. 301–354.

[2] J. Li, *Hilbert’s 16th problem and bifurcations of planar polynomial vector fields*, Int. J. Bifur. Chaos Appl. Sci. Eng. 13 (2003), pp. 47–106.

[3] C. Christopher and C. Li, *Limit Cycles of Differential Equations, Advanced Courses in Mathematics, CRM Barcelona*, Birkhäuser Verlag, Basel, 2007.

[4] C.A. Buzzi, J. Llibre, J.C. Medrado, and J. Torregrosa, *Bifurcation of limit cycles from a centre in \( \mathbb{R}^4 \) in resonance 1 : N*, Dyn. Syst. Int. J. 24 (2009), pp. 123–137.

[5] J. Llibre, A.C. Mereu, and M.A. Teixeira, *Limit cycles of resonant four-dimensional systems*, Dyn. Syst. Int. J. 24 (2010), pp. 146–148.

[6] L. Barreira, J. Llibre, and C. Valls, *Bifurcation of limit cycles from a four-dimensional centre in \( \mathbb{R}^m \) in resonance 1 : N*, J. Math. Anal. 389 (2012), pp. 754–768.

[7] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. 128 (2004), pp. 7–22.

[8] N.G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, New York, Melbourne, 1978.

[9] I.R. Shafarevich, *Basic Algebraic Geometry*, Springer, New York, 1974.