Integrality Gaps of Linear and Semi-definite Programming Relaxations for Knapsack

Anna R. Karlin* Claire Mathieu† C. Thach Nguyen*

Abstract

In this paper, we study the integrality gap of the Knapsack linear program in the Sherali-Adams and Lasserre hierarchies. First, we show that an integrality gap of $2 - \epsilon$ persists up to a linear number of rounds of Sherali-Adams, despite the fact that Knapsack admits a fully polynomial time approximation scheme [27, 33]. Second, we show that the Lasserre hierarchy closes the gap quickly. Specifically, after $t$ rounds of Lasserre, the integrality gap decreases to $t/(t - 1)$. To the best of our knowledge, this is the first positive result that uses more than a small number of rounds in the Lasserre hierarchy. Our proof uses a decomposition theorem for the Lasserre hierarchy, which may be of independent interest.

1 Introduction

Many approximation algorithms work in two phases: first, solve a linear programming (LP) or semi-definite programming (SDP) relaxation; then, round the fractional solution to obtain a feasible integer solution to the original problem. This paradigm is amazingly powerful; in particular, under the unique game conjecture, it yields the best possible ratio for MaxCut and a wide variety of other problems, see e.g. [37].

However, these algorithms have a limitation. Since they are usually analyzed by comparing the value of the output to that of the fractional solution, we cannot generally hope to get a better approximation ratio than the integrality gap of the relaxation. Furthermore, for any given combinatorial optimization problem, there are many possible LP/SDP relaxations, and it is difficult to determine which relaxations have the best integrality gaps.

This has lead to efforts to provide systematic procedures for constructing a sequence of increasingly tight mathematical programming relaxations for 0-1 optimization problems. A number of different procedures of this type have been proposed: by Lovász and Schrijver [34], Sherali and Adams [41], Balas, Ceria and Cornuejols [6], Lasserre [30,31] and others. While they differ in the details, they all operate in a series of rounds starting from an LP or SDP relaxation, eventually ending with an exact integer formulation. The strengthened relaxation after $t$ rounds can typically be solved in $n^{O(t)}$ time and, roughly, satisfies the property that the values of any $t$ variables in the original relaxation can be expressed as the projection of a convex combination of integer solutions.

A major line of research in this area has focused on understanding the strengths and limitations of these procedures. Of particular interest to our community is the question of how the integrality gaps for interesting combinatorial optimization problems evolve through a series of rounds of one of these procedures. On the one hand, if the integrality gaps of successive relaxations drop sufficiently

*University of Washington, email: {karlin,ncthach}@cs.washington.edu. Supported by NSF Grant CCF-0635147 and a Yahoo! Faculty Research Grant.
†Brown University, email: claire@cs.brown.edu
fast, there is the potential for an improved approximation algorithm (see [8,11,17,18] for example). On the other hand, a large integrality gap persisting for a large, say logarithmic, number of rounds rules out (unconditionally) a very wide class of efficient approximation algorithms, namely those whose output is analyzed by comparing it to the value of a class of LP/SDP relaxations. This implicitly contains most known sophisticated approximation algorithms for many problems including \textbf{SparsestCut} and \textbf{MaximumSatisfiability}. Indeed, several very strong negative results of this type have been obtained (see [1, 9, 12, 21, 22, 36, 38–40, 42] and others). These are also viewed as lower bounds of approximability in certain restricted models of computation.

How strong are these restricted models of computation? In other words, how much do lower bounds in these models tell us about the intrinsic hardness of the problems studied? To explore this question, we focus on one problem that is well-known to be “easy” from the viewpoint of approximability: \textbf{Knapsack}. We obtain the following results:

- We show that an integrality gap close to 2 persists up to a linear number of rounds of Sherali-Adams. (The integrality gap of the natural LP is 2.) This is interesting since \textbf{Knapsack} has a fully polynomial time approximation scheme [27,33]. This confirms and amplifies what has already been observed in other contexts (e.g [14]: the Sherali-Adams restricted model of computation has serious weaknesses: a lower bound in this model does not necessarily imply that it is difficult to get a good approximation algorithm.

- We show that Lasserre’s hierarchy closes the gap quickly. Specifically, after \( t \) rounds of Lasserre, the integrality gap decreases to \( t/(t − 1) \).

It is known that a few rounds of Lasserre can yield better relaxations. For example, two rounds of Lasserre applied to the \textbf{MaxCut} LP yields an SDP that is at least as strong as that used by Goemans and Williamson to get the best known approximation algorithm, and the SDP in [2] which leads to the best known approximation algorithm for \textbf{SparsestCut} can be obtained by three rounds of Lasserre. However, to the best of our knowledge, this is the first positive result that utilizes more than a small constant number of rounds in the Lasserre hierarchy.

1.1 Related Work

Many known approximation algorithms can be recognized in hindsight as starting from a natural relaxation and strengthening it using a couple of levels of lift-and-project. The original hope [2] had been to use lift and project systems as a systematic approach to designing novel algorithms with better approximation ratios. Instead, the last few years have mostly seen the emergence of a multitude of lower bounds. Indeed, lift and project systems have been studied mostly for well known difficult problems: \textbf{MaxCut} [14,19,40], \textbf{SparsestCut}, [14,15] \textbf{VertexCover} [13,18,21,22,26,40,42], \textbf{HypergraphVertexCover}, \textbf{TSP} [16], \textbf{MaximumAcyclicSubgraph} [14], \textbf{CSP} [39,43], and more.

The \textbf{Knapsack} problem [28,35] has a fully polynomial time approximation scheme [27,33]. The natural LP relaxation (to be stated in full detail in the next section) has an integrality gap of \( 2 − \epsilon \) [28]. Although we are not aware of previous work on using the lift and project systems for \textbf{Knapsack}, the problem of strengthening the LP relaxation via addition of well-chosen inequalities has been much the object of much interest in the past in the mathematical programming community, as stronger LP relaxations are extremely useful to speed up branch-and-bound heuristics. The knapsack polytope was studied in detail by Weismantel [14]. Valid inequalities were studied in [5,7,23,24,45]. In particular, whenever \( S \) is a minimal set (w.r.to inclusion) that does not fit in the knapsack, then \( \sum_{S \cup \{j: \forall i \in S, w_j \geq w_i\}} x_j \leq |S| − 1 \) is a valid inequality. Generalizations and
variations were also studied in [20,25,46]. Thus, in spite of the existence of a dynamic program to solve the problem, Knapsack is fundamental enough that understanding the polytope (and its lifted tightenings) is of intrinsic interest. In [10], Bienstock formulated LP with arbitrary small integrality gaps for Knapsack using “structural disjunctions”, and asked if the popular hierarchies reduce the gap of the Knapsack linear program. Our results give a negative answer for Sherali-Adams and a strong affirmative one for Lasserre.

Our results confirm the indication from [29,38] for example that the Sherali-Adams lift and project is not powerful enough to be an indicator of the hardness of problems. However, it should be noted that if the problem was phrased as a decision problem and the objective function was replaced by an additional constraint of the constraint polytope, then Sherali-Adams would succeed in reducing the integrality gap; thus the choice of the initial LP formulation is critical. On the other hand, little is know about the Lasserre hierarchy, as the first negative results were about \(k\)-CSP [39,43]. Our positive result leaves open the possibility that the Lasserre hierarchy may have promise as a tool to capture the intrinsic difficulty of problems.

2 Preliminaries

2.1 The Knapsack problem

Our focus in this paper is on the Knapsack problem. In the Knapsack problem, we are given a set of \(n\) objects \(V = [n]\) with sizes \(c_1, c_2, \ldots, c_n\), values \(v_1, v_2, \ldots, v_n\), and a capacity \(C\). We assume that for every \(i\), \(c_i \leq C\). The objective is to select a subset of objects of maximum total value such that the total size of the objects selected does not exceed \(C\).

The standard linear programming (LP) relaxation [28] for Knapsack is given by:

\[
\max \sum_{i \in V} v_i x_i \\
\text{s.t.} \begin{cases} 
\sum_{i \in V} c_i x_i \leq C \\
0 \leq x_i \leq 1 \quad \forall i \in V
\end{cases}
\]  

The intended interpretation of an integral solution of this LP is obvious: \(x_i = 1\) means the object \(i\) is selected, and \(x_i = 0\) means it is not. The constraint can be written as \(g(x) = C - \sum_i c_i x_i \geq 0\).

Let Greedy denote the algorithm that puts objects in the knapsack by order of decreasing ratio \(v_i/c_i\), stopping as soon as the next object would exceed the capacity. The following lemma is folklore.

**Lemma 1** Consider an instance \((C, V)\) of Knapsack and its LP relaxation \(K\) given by [7]. Then

\[
\text{Value}(K) \leq \text{Value}(\text{Greedy}(C, V)) + \max_{i \in V} v_i.
\]

2.2 The Sherali-Adams and Lasserre hierarchies

We next review the lift-and-project hierarchies that we will use in this paper. The descriptions we give here assume that the base program is linear and mostly use the notation given in the survey paper by Laurent [32]. To see that these hierarchies apply at a much greater level of generality we refer the reader to Laurent’s paper [32].
Let $K$ be a polytope defined by a set of linear constraints $g_1, g_2, \ldots, g_m$:

$$K = \{ x \in [0,1]^n | g_\ell(x) \geq 0 \text{ for } \ell = 1,2,\ldots,m \}.$$  

We are interested in optimizing a linear objective function $f$ over the convex hull $P = \text{conv}(K \cap \{0,1\}^n)$ of integral points in $K$. Here, $P$ is the set of convex combinations of all integral solutions of the given combinatorial problem and $K$ is the set of solutions to its linear relaxation. For example, if $K$ is defined by $V$, then $P$ is the set of convex combinations of valid integer solutions to Knapsack.

If all vertices of $K$ are integral then $P = K$ and we are done. Otherwise, we would like to strengthen the relaxation $K$ by adding additional valid constraints. The Sherali-Adams (SA) and Lasserre hierarchies are two different systematic ways to construct these additional constraints. In the SA hierarchy, all the constraints added are linear, whereas Lasserre’s hierarchy is stronger and introduces a set of positive semi-definite constraints. However, for consistency, we will describe both hierarchies as requiring certain submatrices to be positive semi-definite (readers who are not familiar with the following formulation of SA are referred to Appendix B for a linear formulation of the hierarchy.)

To this end, we first state some notation. Throughout this paper we will use $\mathcal{P}(V)$ to denote the power set of $V$, and $\mathcal{P}_t(V)$ to denote the collection of all subsets of $V$ whose sizes are at most $t$. Also, given two sets of coordinates $T$ and $S, T \subseteq S$ and $y \in \mathbb{R}^S$, by $y|_T$ we denote the projection of $y$ onto $T$.

Next, we review the definition of the shift operator between two vectors $x, y \in \mathbb{R}^{\mathcal{P}(V)}$: $x * y$ is a vector in $\mathbb{R}^{\mathcal{P}(V)}$ such that

$$(x * y)_I = \sum_{J \subseteq V} x_J y_{I \cup J}.$$  

**Lemma 2** (32) The shift operator is commutative: for any vectors $x, y, z \in \mathbb{R}^{\mathcal{P}(V)}$, we have $x * (y * z) = y * (x * z)$.

A polynomial $P(x) = \sum_{I \subseteq V} a_I \prod_{i \in I} x_i$ can also be viewed as a vector indexed by subsets of $V$. We define the vector $P * y$ accordingly: $(P * y)_I = \sum_{J \subseteq V} a_J y_{I \cup J}$.

Finally, let $\mathcal{T}$ be a collection of subsets of $V$ and $y$ be a vector in $\mathbb{R}^\mathcal{T}$. We denote by $M_{\mathcal{T}}(y)$ the matrix whose rows and columns are indexed by elements of $\mathcal{T}$ such that

$$(M_{\mathcal{T}}(y))_{I,J} = y_{I \cup J}.$$  

The main observation is that if $x \in K \cap \{0,1\}^n$ then $(y_I) = (\prod_{i \in I} x_i)$ satisfies $M_{\mathcal{P}(V)}(y) = y y^T \succeq 0$ and $M_{\mathcal{P}(V)}(g_\ell * y) = g_\ell(x) y y^T \succeq 0$ for all constraints $g_\ell$. Thus requiring principal submatrices of these two matrices to be positive semi-definite yields a relaxation.

**Definition 3** For any $1 \leq t \leq n$, the $t$-th Sherali-Adams lifted polytope $\text{SA}^t(K)$ is the set of vectors $y \in [0,1]^{\mathcal{P}(V)}$ such that $y_0 = 1$, $M_{\mathcal{P}(|U|)}(y) \succeq 0$ and $M_{\mathcal{P}(|W|)}(g_\ell * y) \succeq 0$ for all $\ell$ and subsets $U, W \subseteq V$ such that $|U| \leq t$ and $|W| \leq n - t - 1$.

We say that a point $x \in [0,1]^n$ belongs to the $t$-th Sherali-Adams polytope $\text{SA}^t(K)$ iff there exists a $y \in \text{SA}^t(K)$ such that $y_{\{i\}} = x_i$ for all $i \in [n]$.

**Definition 4** For any $1 \leq t \leq n$, the $t$-th Lasserre lifted polytope $\text{La}^t(K)$ is the set of vectors $y \in [0,1]^{\mathcal{P}_t(V)}$ such that $y_0 = 1$, $M_{\mathcal{P}_t(|U|)}(y) \succeq 0$ and $M_{\mathcal{P}_{t-1}(|W|)}(g_\ell * y) \succeq 0$ for all $\ell$.

We say that a point $x \in [0,1]^n$ belongs to the $t$-th Lasserre polytope $\text{La}^t(K)$ iff there exists a $y \in \text{La}^t(K)$ such that $y_{\{i\}} = x_i$ for all $i \in V$. 

1
Note that $M_{P(U)}(y)$ has at most $2^t$ rows and columns, which is constant for $t$ constant, whereas $M_{P_t(V)}(y)$ has $\binom{n+1}{t+1}$ rows and columns.

It is immediate from the definitions that $sa^{t+1}(K) \subseteq sa^t(K)$, and $la^{t+1}(K) \subseteq la^t(K)$ for all $1 \leq t \leq n-1$. Sherali and Adams [41] show that $sa^n(K) = P$, and Lasserre [30,31] show that $la^n(K) = P$. Thus, the sequences

$$K \supseteq sa^1(K) \supseteq sa^2(K) \supseteq \cdots \supseteq sa^n(K) = P$$

$$K \supseteq la^1(K) \supseteq la^2(K) \supseteq \cdots \supseteq la^n(K) = P$$

define hierarchies of polytopes that converge to $P$. Furthermore, the Lasserre hierarchy is stronger than the Sherali-Adams hierarchy: $la^n(K) \subseteq sa^n(K)$. In this paper, we show that for the Knapsack problem, the Lasserre hierarchy is strictly stronger.

### 3 Lower bound for the Sherali-Adams hierarchy for Knapsack

In this section, we show that the integrality gap of the $t$-th level of the Sherali-Adams hierarchy for Knapsack is close to 2. This lower bound even holds for the uniform Knapsack problem, in which $v_i = c_i = 1$ for all $i$.

**Theorem 5** For every $\epsilon, \delta > 0$, the integrality gap at the $t$-th level of the Sherali-Adams hierarchy for Knapsack where $t \leq \delta n$ is at least $(2 - \epsilon)(1/(1 + \delta))$.

**Proof.** (Sketch - for full proof see Appendix A) Consider the instance $K$ of Knapsack with $n$ objects where $c_i = v_i = 1$ for all $i \in V$ and capacity $C = 2(1 - \epsilon)$. Then the optimal integer value is 1. On the other hand, we claim that the vector $y$ where $y_0 = 1$, $y_{\{i\}} = C/(n + (t - 1)(1 - \epsilon))$ and $y_I = 0$ for all $|I| > 1$ is in $SA^t(K)$. Thus, the integrality gap of the $t$th round of Sherali-Adams is at least $Cn/(n + (t - 1)(1 - \epsilon))$, which is at least $(2 - \epsilon)(1/(1 + \delta))$ when $t \leq \delta n$.

### 4 A decomposition theorem for the Lasserre hierarchy

In this section, we develop the machinery we will need for our Lasserre upper bounds. It turns out that it is more convenient to work with families $(z^X)$ of characteristic vectors rather than directly with $y$. We begin with some definitions and basic properties.

**Definition 6 (extension)** Let $T$ be a collection of subsets of $V$ and let $y$ be a vector indexed by sets of $T$. We define the extension of $y$ to be the vector $y'$, indexed by all subsets of $V$, such that $y'_I$ equals $y_I$ if $I \in T$ and equals 0 otherwise.

**Definition 7 (characteristic polynomial)** Let $S$ be a subset of $V$ and $X$ a subset of $S$. We define the characteristic polynomial $P^X$ of $X$ with respect to $S$ as

$$P^X(x) = \prod_{i \in X} x_i \prod_{j \in S \setminus X} (1 - x_j) = \sum_{J:X \subseteq J \subseteq S} (-1)^{|J \setminus X|} \prod_{i \in J} x_i.$$
Lemma 8 (inversion formula) Let \( y' \) be a vector indexed by all subsets of \( V \). Let \( S \) be a subset of \( V \) and, for each \( X \) subset of \( S \), let \( z^X = P^X * y' \):

\[
z^X_I = \sum_{J: X \subseteq J \subseteq S} (-1)^{|J| - |X|} y'_{I \cup J}.
\]

Then \( y' = \sum_{X \subseteq S} z^X \).

**Proof.** Fix a subset \( I \) of \( V \). Substituting the definition of \( z^X_I \) in \( \sum_{X \subseteq S} z^X_I \), and changing the index of summation, we get

\[
\sum_{X \subseteq S} z^X_I = \sum_{A \subseteq S} \sum_{J \subseteq A} (-1)^{|J|} y'_{I \cup A}.
\]

For \( A \neq \emptyset \) the inner sum is 0, so only the term for \( A = \emptyset \), which equals \( y'_I \), remains. \( \blacksquare \)

Lemma 9 Let \( y' \) be a vector indexed by all subsets of \( V \), \( S \) be a subset of \( V \) and \( X \) be a subset of \( S \). Then

\[
\begin{cases}
  z^X_I = z^X_{I \cap X} & \text{for all } I \\
  z^X_I = z^X_{\emptyset} & \text{if } I \subseteq X \\
  z^X_I = 0 & \text{if } I \cap (S \setminus X) \neq \emptyset
\end{cases}
\]

**Proof.** Let \( I' = I \setminus X \) and \( I'' = I \cap X \). Using the definition of \( z^X_I \) and noticing that \( X \cup I'' = X \) yields \( z^X_I = z^X_{I''} \). This immediately implies that for \( I \subseteq X \), \( z^X_I = z^X_{\emptyset} \).

Finally, consider a set \( I \) that intersects \( S \setminus X \) and let \( i \in I \cap (S \setminus X) \). In the definition of \( z^X_I \), we group the terms of the sum into pairs consisting of \( J \) such that \( i \notin J \) and of \( J \cup \{i\} \). Since \( I = I \cup \{i\} \), we obtain:

\[
\sum_{J: X \subseteq J \subseteq S} (-1)^{|J| - |X|} y'_{I \cup J} = \sum_{J: X \subseteq J \subseteq S \setminus \{i\}} (-1)^{|J| - |X|} + (-1)^{|J \setminus X| + 1}) y'_{I \cup J} = 0.
\]

\( \blacksquare \)

Corollary 10 Let \( y' \) be a vector indexed by all subsets of \( V \), \( S \) be a subset of \( V \) and \( X \) be a subset of \( S \). Let \( w^X \) be defined as \( z^X/z^X_\emptyset \) if \( z^X_\emptyset \neq 0 \) and defined as 0 otherwise. Then, if \( z^X_\emptyset \neq 0 \), then \( w^X_{\{i\}} \) equals 1 for elements of \( X \) and 0 for elements of \( S \setminus X \).

Definition 11 (closed under shifting) Let \( S \) be an arbitrary subset of \( V \) and \( \mathcal{T} \) be a collection of subsets of \( V \). We say that \( \mathcal{T} \) is closed under shifting by \( S \) if

\[
Y \in \mathcal{T} \implies \forall X \subseteq S, \ X \cup Y \in \mathcal{T}.
\]

The following lemma generalizes Lemma 5 in [32]. It proves that the positive-semidefinite property carries over from \( y \) to \( (z^X) \).

Lemma 12 Let \( S \) be an arbitrary subset of \( V \) and \( \mathcal{T} \) be a collection of subsets of \( V \) that is closed under shifting by \( S \). Let \( y \) be a vector indexed by sets of \( \mathcal{T} \). Then

\[
M_\mathcal{T}(y) \succeq 0 \implies \forall X \subseteq S, \ M_\mathcal{T}(z^X) \succeq 0.
\]
Proof. Since $M_T(y) \geq 0$, there exist vectors $v_I$, $I \in T$, such that $\langle v_I, v_J \rangle = y_{I \cup J}$. Fix a subset $X$ of $S$. For each $I \in T$, let

$$w_I = \sum_{H \subseteq S \setminus X} (-1)^{|H|}v_{I \cup X \cup H},$$

which is well-defined since $T$ is closed under shifting by $S$.

Let $I, J \in T$. It is easy to check that $\langle w_I, w_J \rangle = (z^X)_{I \cup J}$. Indeed,

$$\langle w_I, w_J \rangle = \sum_{H \subseteq S \setminus X} \sum_{L \subseteq S \setminus X} (-1)^{|H|+|L|} \langle v_{I \cup X \cup H}, v_{J \cup X \cup L} \rangle = \sum_{H \subseteq S \setminus X} \sum_{L \subseteq S \setminus X} (-1)^{|H|+|L|} y_{I \cup J \cup X \cup H \cup L}$$

by definition of $v_I, v_J$ and since $T$ is closed under shifting by $S$ (so that this is well-defined). Consider a non-empty subset $H$ of $S \setminus X$ and let $i \notin H$. We group the terms of the inner sum into pairs consisting of $L$ such that $i \notin L$ and of $L \cup \{i\}$. Since $H = H \cup \{i\}$, we obtain:

$$\sum_{L \subseteq S \setminus X} (-1)^{|H|+|L|} y_{I \cup J \cup X \cup H \cup L} = \sum_{L \subseteq (S \setminus X) \setminus \{i\}} (-1)^{|H|+|L|} + (-1)^{|H|+|L|+1} y_{I \cup J \cup X \cup H \cup L} = 0.$$

Thus, the expression in (4) becomes

$$\langle w_I, w_J \rangle = \sum_{L \subseteq S \setminus X} (-1)^{|L|} y_{I \cup J \cup X \cup L} = (z^X)_{I \cup J}.$$

This implies that $M_T(z^X) \geq 0$.

In the rest of the section, we prove a decomposition theorem for the Lasserre hierarchy, which allows us to “divide” the action of the hierarchy and think of it as using the first few rounds on some subset of variables, and the other rounds on the rest. We will use this theorem to prove that the Lasserre hierarchy closes the gap for the Knapsack problem in the next section.

**Theorem 13** Let $t > 1$ and $y \in \text{La}^t(K)$. Let $k < t$ and $S$ be a subset of $V$ and such that

$$|I \cap S| \geq k \implies y_I = 0.$$  \hspace{1cm} (5)

Consider the projection $y|_{\text{La}^{2t-2k}(V)}$ of $y$ to the coordinates corresponding to subsets of size at most $2t - 2k$ of $V$. Then there exist subsets $X_1, X_2, \ldots, X_m$ of $S$ such that $y|_{\text{La}^{2t-2k}(V)}$ is a convex combination of vectors $w^{X_i}$ with the following properties:

- $w^{X_i}_{\{j\}} = \begin{cases} 1 & \text{if } j \in X_i \\ 0 & \text{if } j \in S \setminus X_i \end{cases}$;
- $w^{X_i} \in \text{La}^{t-k}(K)$; and
- if $K_i$ is obtained from $K$ by setting $x_j = w^{X_i}_{\{j\}}$ for $j \in S$, then $w^{X_i}|_{\text{La}^{2t-2k}(V \setminus S)} \in \text{La}^{t-k}(K_i)$.

To prove Theorem 13 we will need a couple more lemmas. In the first one, using assumption (5), we extend the positive semi-definite properties from $y$ to $y'$, and then, using Lemma 12 from $y'$ to $z^X$. 


Lemma 14 Let \( t, y, S, k \) be defined as in Theorem 13 and \( y' \) be the extension of \( y \). Let \( T_1 = \{ A \text{ such that } |A\setminus S| \leq t - k \} \) and \( T_2 = \{ B \text{ such that } |B\setminus S| < t - k \} \). Then for all \( X \subseteq S \), \( M_{T_1}(z^X) \succeq 0 \) and, for all \( \ell \), \( M_{T_2}(g_\ell \ast z^X) \succeq 0 \).

Proof. We will first prove that \( M_{T_1}(y') \succeq 0 \) and, for all \( \ell \), \( M_{T_2}(g_\ell \ast y') \succeq 0 \). Order the columns and rows of \( M_{T_1}(y') \) by subsets of non-decreasing size. By definition of \( T_1 \), any \( I \in T_1 \) of size at least \( t \) must have \(|I \cap S| \geq k\), and so \( y'_I = 0 \). Thus

\[
M_{T_1}(y') = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},
\]
where \( M \) is a principal submatrix of \( M_{P_\ell(V)}(y) \). Thus \( M \succeq 0 \), and so \( M_{T_1}(y') \succeq 0 \).

Similarly, any \( J \in T_2 \) of size at least \( t - 1 \) must have \(|J \cup \{ i \} \cap S| \geq k\) for every \( i \) as well as \(|J \cap S| \geq k\), and so, by definition of \( g_\ell \ast y' \) we must have \((g_\ell \ast y')_J = 0 \). Thus

\[
M_{T_2}(g_\ell \ast y') = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix},
\]
where \( N \) is a principal submatrix of \( M_{T_2-1}(V)(g_\ell \ast y) \). Thus \( N \succeq 0 \), and so \( M_{T_2}(g_\ell \ast y') \succeq 0 \).

Observe that \( T_1 \) is closed under shifting by \( S \). By definition of \( z^X \) and Lemma 12 we thus get \( M_{T_1}(z^X) \succeq 0 \).

Similarly, observe that \( T_2 \) is also closed under shifting by \( S \). By Lemma 2, we have \( g_\ell \ast (P^X \ast y') = P^X \ast (g_\ell \ast y') \), and so by Lemma 12 again we get \( M_{T_2}(g_\ell \ast z^X) \succeq 0 \).

Lemma 15 Let \( t, y, S, k \) be defined as in Theorem 13 and \( y' \) be the extension of \( y \). Then for any \( X \subseteq S \):

1. \( z^X_0 \succeq 0 \).
2. If \( z^X_0 = 0 \) then \( z^X_I = 0 \) for all \( |I| \leq 2t - 2k \).

Proof. Let \( T_1 \) be defined as in Lemma 14. By Lemma 14 \( M_{T_1}(z^X) \succeq 0 \) and \( z^X_0 \) is a diagonal element of this matrix, hence \( z^X_0 \succeq 0 \).

For the second part, start by considering \( J \subseteq V \) of size at most \( t - k \). Then \( J \in T_1 \), and so the matrix \( M_{\{\emptyset, J\}}(z^X) \) is a principal submatrix of \( M_{T_1}(z^X) \), hence is also positive semidefinite. Since \( z^X_0 = 0 \),

\[
M_{\{\emptyset, J\}}(z^X) = \begin{pmatrix} 0 & z^X_J \\ z^X_J & z^X_J \end{pmatrix} \succeq 0,
\]
hence \( z^X_J = 0 \).

Now consider any \( I \subseteq V \) such that \( |I| \leq 2t - 2k \), and write \( I = I_1 \cup I_2 \) where \( |I_1| \leq t - k \) and \( |I_2| \leq t - k \). \( M_{\{I_1, I_2\}}(z^X) \) is a principal submatrix of \( M_{T_1}(z^X) \), hence is also positive semidefinite. Since \( z^X_0 = z^X_J = 0 \), since

\[
M_{\{I_1, I_2\}}(z^X) = \begin{pmatrix} 0 & z^X_I \\ z^X_I & 0 \end{pmatrix} \succeq 0,
\]
hence \( z^X_I = 0 \).

We now have what we need to prove Theorem 13.

Proof of Theorem 13. By definition, Lemma 8 and the second part of Lemma 15 we have

\[
y|_{P_{2t-2k}(V)} = y'|_{P_{2t-2k}(V)} = \sum_{X \subseteq S} z^X |_{P_{2t-2k}(V)} = \sum_{X \subseteq S} z^X_0 w^X |_{P_{2t-2k}(V)}.
\]
By Lemma 8 and by definition of $y$, we have $\sum_{x \in S} z^X_0 = y_0 = 1$, and the terms are non-negative by the first part of Lemma 15, so $y|_{P_{2t - 2k}(V)}$ is a convex combination of $w^X$'s, as desired.

Consider $X \subseteq S$ such that $z^X_0 \neq 0$. By Lemma 14 $M_{T_1}(z^X) \succeq 0$ and $M_{T_2}(g_t \ast z^X) \succeq 0$ for all $\ell$, and so this also holds for their principal submatrices $M_{P_{t-k}(V)}(z^X)$ and $M_{P_{t-k-1}(V)}(g_t \ast z^X)$. Scaling by the positive quantity $z^X_0$, by definition of $w^X$ this also holds for $M_{P_{t-k}(V)}(w^X)$ and $M_{P_{t-k-1}(V)}(g_t \ast w^X)$. In other words, $w^X|_{P_{2t - 2k}(V)} \in \text{La}^{t-k} (K)$.

Since $M_{P_{t-k}(V)}(w^{X_i}) \succeq 0$, by taking a principal submatrix, we infer $M_{P_{t-k}(V \setminus S)}(w^{X_i}) \succeq 0$. Similarly, $M_{P_{t-k}(V)}(g_t \ast w^{X_i}) \succeq 0$ and so $M_{P_{t-k}(V \setminus S)}(g_t \ast w^{X_i}) \succeq 0$. Let $g'_t$ be the constraint of $K$, obtained from $g_t$ by setting $x_j = w^{X_i}_j$ for all $j \in S$. We claim that for any $I \subseteq V \setminus S$, $(g'_t \ast z^X)_I = (g_t \ast z^X)_I$; scaling implies that $M_{P_{t-k}(V \setminus S)}(g'_t \ast w^{X_i}) = M_{P_{t-k}(V \setminus S)}(g_t \ast w^{X_i})$ and we are done.

To prove the claim, let $g_t(x) = \sum_{j \in V} a_j x_j + b$. Then, by Corollary 10 $g'_t = \sum_{j \in V \setminus S} a_j x_j + (b + \sum_{j \in X_i} a_j)$. Let $I \subseteq V \setminus S$. We see that

$$(g_t \ast w^{X_i})_I - (g'_t \ast w^{X_i})_I = \sum_{j \in X_i} a_j w^{X_i}_{I \cup \{j\}} + \sum_{j \in S \setminus X_i} a_j w^{X_i}_{I \cup \{j\}} - \sum_{j \in X_i} a_j w^{X_i}_I.$$ 

By Lemma 9 $w^{X_i}_{I \cup \{j\}} = w^{X_i}_I$ for $j \in X_i$ and $w^{X_i}_{I \cup \{j\}} = 0$ for $j \in S \setminus X_i$. The claim follows.

## 5 Upper bound for the Lasserre hierarchy for Knapsack

In this section, we use Theorem 13 to prove that for the Knapsack problem the gap of $\text{La}^t (K)$ approaches 1 quickly as $t$ grows, where $K$ is the LP relaxation of $\text{La}$. First, we show that there is a set $S$ such that every feasible solution in $\text{La}^t (K)$ satisfies the condition of the Theorem.

Given an instance $(C, V)$ of Knapsack, let $OPT(C, V)$ denote the value of the optimal integral solution.

**Lemma 16** Consider an instance $(C, V)$ of Knapsack and its linear programming relaxation $K$ given by (7). Let $t > 1$ and $y \in \text{La}^t (K)$. Let $k < t$ and $S = \{i \in V \mid v_i > OPT(C, V)/k\}$. Then:

$$\sum_{i \in I \cap S} c_i > C \implies y_I = 0.$$

**Proof.** There are three cases depending on the size of $I$:

1. $|I| \leq t - 1$. Recall the capacity constraint $g(x) = C - \sum_{i \in V} c_i x_i \geq 0$. On the one hand, since $M_{P_{t-1}(V)}(g \ast y) \succeq 0$, the diagonal entry $(g \ast y)_I$ must be non-negative. On the other hand, writing out the definition of $(g \ast y)_I$ and noting that the coefficients $c_i$ are all non-negative, we infer $(g \ast y)_I \leq Cy_I - (\sum_{i \in I} c_i) y_I$. But by assumption, $\sum_{i \in I} c_i > C$. Thus we must have $y_I = 0$.

2. $t \leq |I| \leq 2t - 2$. Write $I = I_1 \cup I_2 = I$ with $|I_1|, |I_2| \leq t - 1$ and $|I_1 \cap S| \geq k$. Then $y_{I_1} = 0$. Since $M_{P_{t}(y)} \succeq 0$, its 2-by-2 principal submatrix $M_{\{I_1, I_2\}}(y)$ must also be positive semi-definite.

$$M_{\{I_1, I_2\}}(y) = \begin{pmatrix} 0 & y_{I_1} \\ y_{I_1} & y_{I_2} \end{pmatrix},$$

and it is easy to check that we must then have $y_{I_2} = 0$.
3. $2t - 1 \leq |I| \leq 2t$. Write $I = I_1 \cup I_2 = I$ with $|I_1|, |I_2| \leq t$ and $|I_1 \cap S| \geq k$. Then $y_{I_1} = 0$ since $t \leq 2t - 2$ for all $t \geq 2$. By the same argument as in the previous case, we must then have $y_I = 0$.

The following theorem shows that the integrality gap of the $t^{th}$ level of the Lasserre hierarchy for Knapsack reduces quickly when $t$ increases.

**Theorem 17** Consider an instance $(C, V)$ of Knapsack and its LP relaxation $K$ given by (1). Let $t \geq 2$. Then

$$Value(\text{La}^t(K)) \leq (1 + \frac{1}{t-1})OPT,$$

and so the integrality gap at the $t$-th level of the Lasserre hierarchy is at most $1 + 1/(t-1)$.

**Proof.** Let $S = \{i \in V | v_i > OPT(C, V)/(t-1)\}$. Let $y \in \text{La}^t(K)$. If $|I \cap S|$ have total value greater than $OPT(C, V)$, so they must not be able to fit in the knapsack: their total capacity exceeds $C$, and so by Lemma 16 we have $y_I = 0$. Thus the condition of Theorem 13 holds for $k = t - 1$.

Therefore, $y|_{\text{La}(V)}$ is a convex combination of $w^{X_i}$ with $X_i \subseteq S$, thus $Value(y) \leq \max_i Value(w^{X_i})$. By the first and third properties of the Theorem, we have:

$$Value(w^{X_i}) \leq \sum_{j \in X_i} v_j + Value(\text{La}^t(K_i)).$$

By the nesting property of the Lasserre hierarchy, Lemma 11 and definition of $S$,

$$Value(\text{La}^1(K_i)) \leq Value(K_i) \leq OPT(C - Cost(X_i), V \setminus S) + OPT(C, V)/(t-1).$$

By the second property of the Theorem, $w^{X_i}$ is in $\text{La}^{t-k}(K) \subseteq K$, so it must satisfy the capacity constraint, so $\sum_{i \in X_i} c_i \leq \sum_{i \in I} c_i \leq C$, so $X_i$ is feasible. Thus:

$$Value(y) \leq \max_{\text{feasible } X \subseteq S} \left( \sum_{j \in X} v_j + OPT(C - Cost(X), V \setminus S) \right) + OPT(C, V)/(t-1)$$

The first expression in the right hand side is equal to $OPT(C, V)$, hence the Theorem.

### 6 Conclusion

We have shown that for Knapsack, an integrality gap of $2 - \epsilon$ persists up to a linear number of rounds in the Sherali-Adams hierarchy. This broadens the class of problems for which Sherali-Adams is not strong enough to capture the intrinsic difficulty of problems.

On the other hand, our positive result for Lasserre opens the possibility that lower bounds in the Lasserre hierarchy good indicators of the intrinsic difculty of the problem, thus encourages more investigation on the effect of the hierarchy on “easy” problems (Spanning Tree, Bin Packing, etc.)

One obstacle along this line is the fact that the second positive semidefinite constraint of the hierarchy $(M_{P(\ell)}(g_\ell * y) \succeq 0)$ is notoriously hard to deal with, especially when $g_\ell$ contains many variables (in the lowerbounds for $k$-CSPs [39, 43], the authors are able to get around this by constructing vectors for only valid assignments, an approach that is possible only when all the constraints are “small”). Clearly, both lower bounds and upper bounds for the Lasserre hierarchy for problems with large constraints remain interesting to pursue.
Acknowledgement

Clare Mathieu would like to thank Eden Chlamtac for stimulating discussions.

References

[1] M. Alekhnovich, S. Arora, and I. Tourlakis. Towards strong non-approximability results in the Lovász-Schrijver hierarchy. In *Proceedings of the 37th ACM Symposium on Theory of Computing*, pages 294–303, 2005.

[2] S. Arora, B. Bollobás, and L. Lovász. Proving integrality gaps without knowing the linear program. pages 313–322, 2002.

[3] S. Arora, B. Bollobás, L. Lovász, and I. Tourlakis. Proving integrality gaps without knowing the linear program. *Theory of Computing*, 2:19–51, 2006.

[4] Sanjeev Arora, Satish Rao, and Umesh V. Vazirani. Expander flows, geometric embeddings and graph partitioning. *J. ACM*, 56(2), 2009.

[5] E. Balas. *Facets of the Knapsack Polytope*. 1998.

[6] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993.

[7] E. Balas and E. Zemel. Facets of the knapsack polytope from minimal covers. *SIAM Journal on Applied Mathematics*, 34:119–148, 1978.

[8] Mohammad Hossein Bateni, Moses Charikar, and Venkatesan Guruswami. MaxMin allocation via degree lower-bounded arborescences. In *Proceedings of the ACM Symposium on Theory of Computing*, 2009.

[9] Siavosh Benabbas and Avner Magen. Extending sdp integrality gaps to sherali-adams with applications to quadratic programming and maxcutgain. In *IPCO*, 2020.

[10] Daniel Bienstock. Approximate formulations for 0-1 knapsack sets. *Oper. Res. Lett.*, 36(3):317–320, 2008.

[11] Daniel Bienstock and Nuri Ozbay. Tree-width and the Sherali-Adams operator. *Discrete Optimization*, 1(1):13–21, 2004.

[12] J. Buresh-Oppenheim, N. Galesi, S. Hoory, A. Magen, and T. Pitassi. Rank bounds and integrality gaps for cutting plane procedures. In *Proceedings of the 44th IEEE Symposium on Foundations of Computer Science*, pages 318–327, 2003.

[13] M. Charikar. On semidefinite programming relaxations for graph coloring and vertex cover. In *Proceedings of the 13th ACM-SIAM Symposium on Discrete Algorithms*, pages 616–620, 2002.

[14] M. Charikar, K. Makarychev, and Y. Makarychev. Integrality gaps for Sherali-Adams relaxations. In *Proceedings of the ACM Symposium on Theory of Computing*, 2009.

[15] Jeff Cheeger, Bruce Kleiner, and Assaf Naor. A \((\log n)^{\Omega(1)}\) integrality gap for the Sparsest Cut SDP. In *Proceedings of the IEEE Symposium on Foundations of Computer Science*, 2009.
[16] Kevin K. H. Cheung. On Lovasz-Schrijver lift-and-project procedures on the Dantzig-Fulkerson-Johnson relaxation of the TSP. SIAM Journal on Optimization, Volume, 16(2):380–399, 2005.

[17] E. Chlamtac. Approximation algorithms using hierarchies of semidefinite programming relaxations. In Proceedings of the 48th IEEE Symposium on Foundations of Computer Science, pages 691–701, 2007.

[18] E. Chlamtac and G. Singh. Improved approximation guarantees through higher levels of SDP hierarchies. In Proceedings of the 11th International Workshop, APPROX, pages 46–62, 2008.

[19] W. Fernandez de la Vega and C. Kenyon-Mathieu. Linear programming relaxations of MaxCut. In Proceedings of the 18th ACM-SIAM Symposium on Discrete Algorithms, 2007.

[20] Laureano F. Escudero and Araceli Garn. An o(n log n) procedure for identifying facets of the knapsack polytope. Operational Research Letters, 31(3):211–218, 2003.

[21] K. Georgiou, A. Magen, T. Pitassi, and I. Tourlakis. Integrality gaps of 2 − o(1) for vertex cover SDPs in the Lovász-Schrijver hierarchy. In Proceedings of the 48th IEEE Symposium on Foundations of Computer Science, 2007.

[22] K. Georgiou, A. Magen, and I. Tourlakis. Vertex cover resists SDPs tightened by local hypermetric inequalities. In Proceedings of the 13th Conference on Integer Programming and Combinatorial Optimization (IPCO), 2008.

[23] P. L. Hammer, E. L. Johnson, and Uri N. Peled. Facets of regular 0-1 polytopes. Math. Programming, 8:179–206, 1975.

[24] P. L. Hammer and Uri N. Peled. Computing low capacity 0-1 knapsack polytopes, z. Oper. Res. Ser. A-B, 26:243–249, 1982.

[25] D. Hartvigsen and E. Zemel. On the computational complexity of facets and valid inequalities for the knapsack problem. Discrete Applied Math, 39:113–123, 1992.

[26] H. Hatami, A. Magen, and E. Markakis. Integrality gaps of semidefinite programs for vertex cover and relations to $\ell_1$ embeddability of negative type metrics. In Proceedings of APPROX, pages 164–179, 2007.

[27] O. H. Ibarra and C. E. Kim. Fast approximation algorithms for the knapsack and sum of subset problems. Journal of the ACM, 1975.

[28] Hans Kellerer, Ulrich Pferschy, and David Pisinger. Knapsack problems. Springer, 2003.

[29] Subhash Khot and Rishi Saket. SDP integrality gaps with local $\ell_1$-embeddability. In Proceedings of the IEEE Symposium on Foundations of Computer Science, 2009.

[30] J. B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. In Proceedings of the 8th International Conference on Integer Programming and Combinatorial Optimization, volume 2081, pages 293–303, 2001.

[31] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM Journal on Optimization, 11:796–817, 2001.
[32] M. Laurent. A comparison of the Sherali-Adams, Lovász-schrijver and Lasserre relaxations for 0-1 programming. Mathematics of Operations Research, 28:470–496, 2003.

[33] E L Lawler. Fast approximation algorithms for knapsack problems. In Proceedings of 18th Annual Symposium on Foundations of Computer Science, 1997.

[34] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. SIAM Journal on Optimization, 1:166–190, 1991.

[35] Silvano Martello and P. Toth. Knapsack Problems: Algorithms and Computer Implementations. John Wiley & Sons, Chichester, New York, 1990.

[36] T. Pitassi and N. Segerlind. Exponential lower bounds and integrality gaps for tree-like Lovász-Schrijver procedures, 2009.

[37] Prasad Raghavendra. Optimal algorithms and inapproximability results for every CSP? In Proceedings of the ACM Symposium on Theory of Computing, pages 245–254, 2008.

[38] Prasad Raghavendra and David Steurer. Integrality gaps for strong SDP relaxations of unique games. In Proceedings of the IEEE Symposium on Foundations of Computer Science, 2009.

[39] G. Schoenebeck. Linear level Lasserre lower bounds for certain k-csps. In Proceedings of the 49th IEEE Symposium on Foundations of Computer Science, 2008.

[40] G. Schoenebeck, L. Trevisan, and M. Tulsiani. Tight integrality gaps for Lovász-Schrijver SDP relaxations of vertex cover and max cut. In Proceedings of the 39th ACM Symposium on Theory of Computing, pages 302–310, 2007.

[41] H. D. Sherali and W. P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM Journal on Discrete Mathematics, 3:411–430, 1990.

[42] I. Tourlakis. New lower bounds for vertex cover in the Lovász-Schrijver hierarchy. In Proceedings of the 21st IEEE Conference on Computational Complexity, 2006.

[43] Madhur Tulsiani. CSP gaps and reductions in the Lasserre hierarchy. In Proceedings of the ACM Symposium on Theory of Computing, 2009.

[44] Robert Weismantel. On the 0/1 knapsack polytope. Math. Programming, 77:49–68, 1997.

[45] Laurence A. Wolsey. Valid inequalities for 0-1 knapsacks and mips with generalised upper bound constraints. Discrete Applied Mathematics, 29(2-3):251–261, 1990.

[46] E. Zemel. Easily computable facets of the knapsack problem. Mathematics of Operations Research, 14:760–774, 1989.

Appendix

A Full proof of Theorem 5

**Proof.** Let \( t \geq 2 \). Consider the instance \( K \) of Knapsack with \( n \) objects where \( c_i = v_i = 1 \) for all \( i \in V \) and capacity \( C = 2(1 - \epsilon) \). Let \( \alpha = C/(n + (t - 1)(1 - \epsilon)) \) and consider the vector
\[ y \in [0,1]^{P_t(V)} \text{ defined by} \]
\[
\begin{cases}
    y_0 = 1 \\
y_{(i)} = \alpha \\
y_I = 0 \text{ if } |I| > 1
\end{cases}
\]

We claim that \( y \in \text{SA}^t(K) \). Consider any subset \( U \subseteq V \) such that \(|U| \leq t\). We have
\[
M_{P(U)}(y) = \begin{pmatrix} M_{P_1(U)}(y) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{with } M_{P_1(U)}(y) = \begin{pmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\
\alpha & \alpha & 0 & \cdots & 0 \\
\alpha & 0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha & 0 & 0 & \cdots & \alpha \end{pmatrix}.
\]

Since \(|U| \leq t < n, |U|\alpha \leq 1\, and it is easy to see that this implies \( M_{P_1(U)}(y) \geq 0\), and so \( M_{P(U)}(y) \geq 0\).

Next, let \( g(x) = C - \sum_{i \in V} c_i x_i \) and consider any subset \( W \subseteq V \) such that \(|W| \leq t - 1\). Again, we have
\[
M_{P(W)}(g \ast y) = \begin{pmatrix} M_{P_1(W)}(g \ast y) & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{P_1(W)}(g \ast y) = \begin{pmatrix} C - n\alpha & (C - 1)\alpha & (C - 1)\alpha & \cdots & (C - 1)\alpha \\
(C - 1)\alpha & (C - 1)\alpha & 0 & \cdots & 0 \\
(C - 1)\alpha & 0 & (C - 1)\alpha & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(C - 1)\alpha & 0 & 0 & \cdots & (C - 1)\alpha \end{pmatrix}.
\]

Since \(|W| \leq t - 1\, by definition of \( \alpha \) we have \(|W|(C - 1)\alpha \leq C - n\alpha\, and it is easy to see that this implies \( M_{P_1(W)}(g \ast y) \geq 0\), and so \( M_{P(W)}(g \ast y) \geq 0\). Thus \( y \in \text{SA}^t(K)\).

The integer optimum has value 1, so the integrality gap is at least the value of \( y\), which is \( n\alpha = 2(1-\epsilon)/(1 + (t - 1)(1 - 2\epsilon)/n)\). The supremum over all \( \epsilon\) is \( 2/(1 + (t - 1)/n)\), and the supremum of that over all \( n\) is 2, so the integrality gap is at least 2.

On the other hand, it is well-known that the base linear program \( K\) has value at most \( 2\text{OPT} \) (that is an immediate consequence of Lemma 1), hence, by the nesting property, every linear program in the hierarchy has integrality gap exactly equal to 2.

\[ \Box \]

**B A linear formulation of the Sherali-Adams hierarchy**

For any constraint \( g_\ell(x) \geq 0 \) in the definition of the base polytope \( K \) and any subsets \( I, J \subseteq V \), the following constraint is a consequence of the fact that \( x \in [0,1]^n\):
\[
g_\ell(x) \prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) \geq 0. \tag{6}
\]

If \( x\) is indeed integral, then \( x_i^k = x_i \) for any \( k \geq 1\). Thus, the constraint obtained by expanding (6) and replacing \( x_i^k \) by \( x_i \) holds in \( P \) and can be added to strengthen the relaxation. However, this constraint is not linear. To preserve the linearity of the system, each product \( \prod_{i \in I} x_i \) is replaced by a variable \( y_I \).

In addition, to keep the number of variables from growing exponentially, we restrict ourselves to only variables \( y_I \) such that \(|I| \leq t\). By this, we “lift” the polytope \( K \) to a polytope \( \text{SA}^t(K) \subseteq [0,1]^{P_t(V)}\).
Definition 18 Let $K$ be a polytope defined as in equation 2. For any $1 \leq t \leq n$, the $t$-th Sherali-Adams lifted polytope $\text{SA}^t(K)$ is defined by

$$\text{SA}^t(K) = \{y \in P_t([n])| y_0 = 1, \text{ and } g_{\ell,I,J}(y) \geq 0 \text{ for any } \ell \text{ and } I, J \subseteq V \text{ s.t. } |I \cup J| \leq t - 1\}$$

where $g_{\ell,I,J}(y)$ is obtained by:

1. multiplying $g_{\ell}(x)$ by $\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j)$;
2. expanding the result and replacing each $x^k_i$ by $x_i$; and
3. replacing each $\prod_{i \in S} x_i$ by $y_S$.

We say that a point $x \in [0,1]^V$ belongs to the $t$-th Sherali-Adams polytope $\text{SA}^t(K)$ iff there exists a $y \in \text{SA}^t(K)$ such that $y_{\{i\}} = x_i$ for all $i \in V$.

In particular, in the case of Knapsack, $\text{SA}^t(K)$ is the set of all points in $[0,1]^{P_t(V)}$ that satisfy the following constraints for any $I, J \subseteq V$ such that $I \cap J = \emptyset$ and $|I| + |J| \leq t - 1$:

$$\sum_{i=1}^{n} c_i \sum_{L \subseteq J} (-1)^{|L|} y_{I \cup J \cup \{i\}} \leq C \sum_{L \subseteq J} (-1)^{|L|} y_{I \cup J}, \quad (7)$$

and

$$0 \leq \sum_{L \subseteq J} (-1)^{|L|} y_{I \cup J \cup \{i\}} \leq \sum_{L \subseteq J} (-1)^{|L|} y_{I \cup J}, \quad \forall i \in V.$$

For a proof that this definition is equivalent to Definition 3 we refer the reader to Laurent’s paper [32].