Quantized Normal Matrices: Some Exact Results and Collective Field Formulation

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Abstract

We formulate and study a class of $U(N)$-invariant quantum mechanical models of large normal matrices with arbitrary rotation-invariant matrix potentials. We concentrate on the $U(N)$ singlet sector of these models. In the particular case of quadratic matrix potential, the singlet sector can be mapped by a similarity transformation onto the two-dimensional Calogero-Marchioro-Sutherland model at specific couplings. For this quadratic case we were able to solve the $N$-body Schrödinger equation and obtain infinite sets of singlet eigenstates of the matrix model with given total angular momentum. Our main object in this paper is to study the singlet sector in the collective field formalism, in the large-$N$ limit. We obtain in this framework the ground state eigenvalue distribution and ground state energy for an arbitrary potential, and outline briefly the way to compute bona-fide quantum phase transitions in this class of models. As explicit examples, we analyze the models with quadratic and quartic potentials. In the quartic case, we also touch upon the disk-annulus quantum phase transition. In order to make our presentation self-contained, we also discuss, in a manner which is somewhat complementary to standard expositions, the theory of point canonical transformations in quantum mechanics for systems whose configuration space is endowed with non-euclidean metric, which is the basis for constructing the collective field theory.

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1 Introduction

Random normal matrix models were used recently in studying Laplacian growth processes, and in particular, the fingering instability of the boundary of various two-dimensional fluids [1], the theory of the $\tau$-function for analytic curves [2] and two-dimensional string theory [3].

An earlier important use of normal matrix models, closely related to the works mentioned above, was in quantum Hall physics [4], where it was shown that the partition function of the normal matrix model coincided with the zero-temperature partition function of two-dimensional electrons in strong (uniform, as well as varying) magnetic fields, and its identification as a complexified form of the Toda lattice $\tau$-function was established. (The electron quantum Hall liquid was one of the systems studied in [1].)

The structure of correlation functions in the normal matrix models was studied in detail in [5, 6] and later in [7].

Normal matrices were also mentioned recently in [8], as a specific kind of complex random matrices which evaded a certain generic geometrical constraint on the shape of the two-dimensional eigenvalue distribution of a large class of circularly symmetric probability ensembles of complex random matrices, known as the “Single Ring Theorem” [9].

All the studies mentioned above employed time independent normal matrices. An interesting quantum mechanical model of truly dynamical normal matrices, with quadratic matrix potential, was introduced in [10], in their study of the physics of two-dimensional long-range Bose liquids. It was pointed out in [10] that the $U(N)$ singlet sector of their normal matrix model was equivalent to a certain two-dimensional generalization of the Calogero-Sutherland model [11, 12], which contained long-range three-body interactions, in addition to the repulsive two-body interactions familiar from the one-dimensional model, at specific values of the couplings of those interactions. (For more details concerning this equivalence see section 2.2.) Under this equivalence, the $N$ complex eigenvalues are mapped onto the positions of the interact-
ing particles of the Calogero-Sutherland model in the plane, in a manner analogous to
the relation between the Dyson matrix ensembles and the one-dimensional Calogero-
Sutherland model [12]. (See also section 3 of [13].)

It should be noted that a three-dimensional version of the Calogero model, very
similar to the one studied in [10], was formulated and partly solved long ago in [14].
Thus, we shall refer to these higher-dimensional versions of the Calogero model (with
the Sutherland modification of the quadratic piece of the potential) more appropri-
ately as Calogero-Marchioro-Sutherland (CMS) models [15, 16]. We should also
mention the construction of multidimensional Calogero models and their relation to
matrix models in [17], as well as the generalized multidimensional Calogero models
discussed recently in [18].

In this paper we shall generalize the quantum mechanical normal matrix model
of [10] into a large class of models with \( U(N) \)- and rotation-invariant potentials.
We will then analyze the singlet sector of these models in the large-\( N \) limit, in the
framework of quantum collective field theory [19, 20]. We will focus on studying
the ground state of our generic quantum mechanical model. Thus, given an arbitrary
\( U(N) \)- and rotation-invariant potential, we will determine the ground state eigenvalue
distribution and ground state energy explicitly, in the large-\( N \) limit. In a separate
publication [21], we will construct and analyze the effective hamiltonian of small
fluctuations around the ground state configuration.

This paper is organized as follows. In the next section we discuss the quantum
mechanics of the normal matrix model, concentrating on the singlet sector. In par-
ticular, we solve the \( N \)-body Schrödinger equation with quadratic matrix potential
and present infinite sets of exact singlet eigenstates with given angular momentum.
We also discuss in this section the mapping of the singlet sector of the normal matrix
model onto the two-dimensional CMS model. In Section 3 we will construct the col-
lective field quantum hamiltonian corresponding to the singlet sector hamiltonian \( H_s \)
in (2.15). The standard construction of collective field hamiltonians, as presented in
[19, 20], is formulated for systems defined in a flat configuration space, endowed with
cartesian coordinates. Such coordinates are not explicitly available for our system of normal matrices, as will be explained in that section. In general, the transformation to the collective hamiltonian is achieved by a quantum mechanical point canonical transformation from the original dynamical variables to the collective coordinates. Thus, we begin section 3 with a general brief discussion of quantum mechanical point canonical transformations in a configuration space endowed with a non-euclidean metric. We believe our presentation and results in this part of section 3 supplement those presented in chapter 6 of [20]. We then use these results to construct the collective field theory of our quantum mechanical model of normal matrices. We show that the Heisenberg equations of motion (as well as the classical Hamilton equations of motion) of this model can be interpreted as the equations of motion of an eulerian fluid, similarly to an analogous interpretation of the equations of motion of the collective field theory of hermitean matrices [25], or its equivalent formulation as fermionic field theory [26].

Finally, in section 4 we find the ground state eigenvalue distribution and ground state energy of this collective hamiltonian (and thus, those of the original model) for an arbitrary matrix potential (2.14). We also outline briefly the way to compute bona-fide quantum phase transitions in this class of models. As explicit examples, we analyze the models with quadratic and quartic potentials. In the quartic case, we also touch upon the disk-annulus quantum phase transition.
2 The Quantum Mechanical Normal Matrix Model

Let us first recall some basic facts about normal matrices. Consider the $N \times N$ complex normal matrix $M$. Thus,

$$[M, M^\dagger] = 0, \quad (2.1)$$

which means $M$ and $M^\dagger$ are diagonalized by the same unitary matrix $U$. Consequently, any normal matrix $M$ can be decomposed as

$$M = U^\dagger ZU \quad (2.2)$$

where

$$Z = \text{diag} \ (z_1, \ldots, z_N) \quad (2.3)$$

are the complex eigenvalues. Note that $U$ in (2.2) is not unique, since there is freedom to multiply it on the left by a diagonal unitary matrix of $N$ arbitrary phases. Thus, the count of real independent degrees of freedom in $M$ is the $2N$ real and imaginary parts of the eigenvalues, plus the $N^2$ real independent parameters of $U$, less the $N$ arbitrary phases, adding up to $N(N + 1)$ real independent parameters.

The geometry of the manifold of normal matrices is defined by its embedding in the larger space of complex matrices, with the embedding effected by the constraint (2.1). Thus, its line element is inherited from the euclidean line element $ds^2 = \text{Tr} \ dM^\dagger dM$ of the latter, by substituting (2.2) for $M$. Therefore, writing

$$dM = U^\dagger (dZ + [dR, Z])U, \quad (2.4)$$

where

$$dR = UdU^\dagger, \quad dR^\dagger = -dR \quad (2.5)$$

is the right-invariant form (i.e., invariant under $U \rightarrow UV$ with a fixed unitary $V$), we obtain the normal matrix line element as

$$ds^2 = \sum_i |dz_i|^2 + 2 \sum_{i<j} dR^*_{ij} dR_{ij} |z_i - z_j|^2. \quad (2.6)$$
Note that this line element is independent of the diagonal elements $dR_{ii}$ (which are essentially the differentials of the $N$ arbitrary phases mentioned above). It depends only on the differentials of the real and imaginary parts of the $z_i$, and on the differentials of the real and imaginary parts of the elements in the lower-triangular part of $dR$, $N(N + 1)$ in all. Note also that there are no metric elements in (2.6) which mix $dz$’s and $dR$’s: motions tangential to eigenvalues and to the unitary group are orthogonal.

The laplacian associated with the metric (2.6) is thus the sum of two terms

$$\nabla^2 = \nabla^2_s + \nabla^2_{U(N)},$$ \hspace{1cm} (2.7)

where $\nabla^2_s$ arises from pure eigenvalue variations and $\nabla^2_{U(N)}$ arises from pure $U(N)$ rotations. In particular, $\nabla^2_{U(N)}$ annihilates $U(N)$ singlets. Thus, $\nabla^2 = \nabla^2_s$ in the $U(N)$ singlet sector. Clearly, $\nabla^2_s$ and $\nabla^2_{U(N)}$ are analogous, respectively, to the radial and to the $L^2/r^2$ parts of the laplacian in atomic physics.

The explicit expressions for $\nabla^2_s$ and $\nabla^2_{U(N)}$ can be computed from (2.6) in a straightforward manner. The angular part of the laplacian is

$$\nabla^2_{U(N)} = 2 \sum_{i < j} \frac{\partial R_{ij}}{\partial z_i} \frac{\partial R_{ij}}{\partial z_j},$$ \hspace{1cm} (2.8)

and the singlet part is

$$\nabla^2_s = \frac{1}{|\Delta|^2} \sum_i \left[ \frac{\partial}{\partial x_i} \left( |\Delta|^2 \frac{\partial}{\partial x_i} \right) + \frac{\partial}{\partial y_i} \left( |\Delta|^2 \frac{\partial}{\partial y_i} \right) \right],$$ \hspace{1cm} (2.9)

where

$$\Delta = \prod_{i > j} (z_i - z_j)$$ \hspace{1cm} (2.10)

is the Vandermonde determinant of eigenvalues, and $z_i = x_i + iy_i$. In terms of the complex derivatives

$$\partial_i = \frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right), \hspace{1cm} \partial_i^* = \frac{\partial}{\partial z_i^*} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right),$$ \hspace{1cm} (2.11)

we can write $\nabla^2_s$ more elegantly as

$$\nabla^2_s = 2 \sum_i \left( \frac{1}{\Delta^*} \partial_i^* \Delta \partial_i + \frac{1}{\Delta} \partial_i \Delta \partial_i^* \right).$$ \hspace{1cm} (2.12)
In this paper we study $U(N)$- and rotation-invariant normal matrix models defined by quantum hamiltonians of the general form

$$H = -\frac{1}{2} \nabla^2 + \text{Tr} V(M^\dagger M).$$  \hfill (2.13)

Here $V$ is a generic polynomial potential, whose couplings scale properly with $N$, in order to ensure a well-behaved large-$N$ limit. We will see later-on that the required $N$-dependence is

$$V(M^\dagger M) = \sum_{p \geq 1} g_p (M^\dagger M)^p N^{p-1},$$  \hfill (2.14)

(with $N$-independent $g_p$).

According to general principles, the ground state of (2.13) must be a $U(N)$ singlet. As was stated earlier, we will find this ground state explicitly in the large-$N$ limit. Thus, in this paper we shall focus exclusively on the $U(N)$-singlet sector of (2.13), in which this hamiltonian is reduced to

$$H_s = -\frac{1}{2} \nabla^2_s + \sum_i V(|z_i|^2),$$  \hfill (2.15)

which defines the dynamics of the $N$ complex eigenvalues $z_i$. It acts on singlet wave functions $\chi_s(z_1, z_1^*, \ldots, z_N, z_N^*)$ which are completely symmetric under eigenvalue permutations.

Note from (2.12) that $H_s$ is invariant under the interchange $z_i, \partial_i \leftrightarrow z_i^*, \partial_i^*$, i.e., under reflection with respect to the $x$-axis. This means that if $\chi_s$ is an eigenstate of $H_s$ with eigenvalue $E$, so is $\chi_s^*$:

$$H_s \chi_s = E \chi_s \iff H_s \chi_s^* = E \chi_s^*. \hfill (2.16)$$

Thus, if $\chi_s$ is complex (and not just a real wave function multiplied by a phase), its corresponding energy will be at least doubly degenerate.

It is clear from the metric (2.6) that $H_s$ is symmetric $\langle \chi_{s1} | H_s \chi_{s2} \rangle = \langle H_s \chi_{s1} | \chi_{s2} \rangle$ with respect to the measure

$$d\mu (\{z_i, z_i^*\}) = |\Delta|^2 \prod_{k=1}^N d^2 z_k.$$  \hfill (2.17)
and the inner product
\[ \langle \chi_{s1}|\chi_{s2}\rangle = \int d\mu \chi_{s1}^{\ast} \chi_{s2}. \] (2.18)

As is evident from (2.12) (or (2.9)), all interparticle interactions are lumped into the kinetic term of \( H_s \), through kind of a complex gauge field
\[ A_i (\{z_k\}) = \frac{\partial_i \Delta}{\Delta} = \sum_j' \frac{1}{z_i - z_j}. \] (2.19)

(where the prime indicates that the sum is over all \( j \neq i \)), and its complex conjugate field \( A_i^{\ast} = A_i (\{z_k^{\ast}\}) \), which multiply the first order derivatives \( \partial_i^{\ast} \) and \( \partial_i \), respectively,
\[ \nabla_s^2 = 2 \sum_i (2\partial_i^{\ast} \partial_i + A_i^{\ast} \partial_i + A_i \partial_i^{\ast}) . \] (2.20)

2.1 Exact Eigenstates of the Matrix Model With Quadratic Potential

Let us consider now (2.15) with quadratic potential
\[ \text{Tr}V(M^\dagger M) = \frac{1}{2} m^2 \text{Tr}M^\dagger M = \frac{1}{2} m^2 \sum_i |z_i|^2, \quad m^2 > 0 . \] (2.21)

Remarkably, infinite (albeit incomplete) sets of eigenstates of this singlet hamiltonian \( H_s \) can be found analytically, in a similar manner\(^1\) to [14].

2.1.1 Radial Eigenstates

It is possible to compute explicitly the ground state \( \chi_0 \) of this matrix model, and also an infinite set of radial excitations above it. To this end, define the radial combination
\[ \varrho = \sum_{i=1}^{N} |z_i|^2 , \] (2.22)

which is symmetric under permutation of eigenvalues, and look for eigenstates of \( H_s \) of the form \( \chi = F(\varrho) \). Using the identity
\[ \sum_{i=1}^{N} z_i \partial_i \log \Delta = \frac{N(N - 1)}{2} \] (2.23)

\(^1\)In contrast with (2.21), the quadratic potential term in [14] was \( \propto \sum_{i<j} |z_i - z_j|^2 \).
and a similar identity for $\Delta^*$, we can write the Schrödinger equation $H_s F(\varrho) = EF(\varrho)$ with potential (2.21) as

\[2\varrho \frac{d^2 F}{d\varrho^2} + N(N + 1) \frac{dF}{d\varrho} + \left(E - \frac{1}{2} m^2 \varrho\right) F = 0. \tag{2.24}\]

The eigenfunction solutions of (2.24) and their corresponding eigenvalues are readily found as

\[\chi_n(\varrho) = L_n^{(N+1/2)}(m \varrho) e^{-\frac{1}{2}m \varrho}, \quad E_n = \left(\frac{N(N + 1)}{2} + 2n\right) m, \tag{2.25}\]

\((n = 0, 1, \ldots),\) where

\[L_n^{(N+1/2)}(u) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(\frac{N(N+1)}{2} + n - 1)!}{(n-k)! \left(\frac{N(N+1)}{2} + k - 1\right)!} u^k \tag{2.26}\]

is a Laguerre polynomial.

In particular, the nodeless gaussian eigenfunction $\chi_0 = e^{-\frac{1}{2}m \varrho} = e^{-\frac{1}{2}m \sum_i |z_i|^2}$ is the ground state [15], with energy

\[E_0 = \frac{N(N + 1)}{2} m. \tag{2.27}\]

The radial states of higher energy are evenly spaced, with constant gap $2m$.

From (2.17) and (2.25), we obtain that the joint probability density of the $N$ eigenvalues in the ground state is proportional to

\[|\Delta \chi_0|^2 = \left|\prod_{i>j}(z_i - z_j)\right|^2 e^{-m \sum_i |z_i|^2}. \tag{2.28}\]

Starting from the two-dimensional CMS model (at couplings tuned to the point corresponding to the normal matrix model), it was shown in [15] (prior to [10]) that the ground state wave function of the CMS model was simply

\[\psi_0^{CMS} = \left|\prod_{i>j}(z_i - z_j)\right| \chi_0, \tag{2.29}\]

which thus led to a joint probability density for the positions of the $N$ particles, which was also proportional to (2.28). It was further pointed out in [15] that (2.28) was
also proportional to the joint probability density of the \( N \) eigenvalues in Ginibre’s ensemble [22] of gaussian random complex matrices, which also happens to coincide with the joint probability density of the gaussian (time independent) normal matrix model. This is, of course, in complete analogy with the one-dimensional case, in which the joint probability density for the positions of the \( N \) particles in the ground state of the Calogero-Sutherland model at specific values of the repulsive interaction, coincides with that of the appropriate Dyson random matrix ensemble [12].

It was noted in [16]\(^2\) that the eigenstates (2.25) could be also obtained using group theoretical considerations. As it turns out, the normal matrix model with quadratic potential\(^3\) (2.21) has dynamical symmetry \( O(2,1) \)

\[
[h, D] = ih, \quad [h, K] = 2iD, \quad \text{and} \quad [K, D] = -iK,
\]

generated by

\[
h = -\frac{1}{2} \nabla_s^2 = -\sum_i \left( \frac{1}{\Delta_i} \Delta_i^* \partial_i^* \partial_i + \frac{1}{\Delta_i} \partial_i \Delta_i^* \partial_i^* \right)
\]

\[
K = \frac{1}{2} \theta = \frac{1}{2} \sum_{i=1}^{N} |z_i|^2
\]

\[
D = \frac{i}{4} \sum_i (\{z_i, \partial_i\} + \{z_i^*, \partial_i^*\} + N - 1).
\]

The operator \( h \) is just the kinetic part of \( H_s \), \( K \) is the conformal generator, and \( D \) generates dilatations. Let us define the raising and lowering operators \( B^\pm \) as

\[
B^\pm = \frac{1}{2} (h - K \mp 2iD).
\]

These operators, together with the full hamiltonian \( H_s = h + K \) generate the \( SU(1,1) \) algebra

\[
[H_s, B^\pm] = \pm 2B^\pm, \quad [B^-, B^+] = H_s.
\]

\(^2\)This was done in section 2 of [16], in the context of the two-dimensional CMS model, and not directly in the normal matrix model. The adaptation of the discussion in [16] to the case of our normal matrix model is what follows.

\(^3\)In the following group theoretical discussion we will set \( m = 1 \) in (2.21) for convenience (but will reinstate it back following (2.34)).
Then, the states (2.25) are obtained by applying \( B^+ \) repeatedly to the ground state \( \chi_0 \), namely,
\[
\chi_n = (B^+)^n \chi_0 ,
\]
(2.34)
from which it is easy to check that indeed \( E_n = E_0 + 2mn \).

### 2.1.2 Eigenstates with Nonvanishing Angular Momentum

The eigenstates (2.25) are rotation invariant and thus do not carry angular momentum. Here we derive an infinite set of eigenstates which do carry angular momentum. The derivation of these states is similar to the derivation of the radial states.

We shall look here for eigenstates of the form \( F(\rho) \, g(z_1, \ldots z_N) \), with \( g(z_1, \ldots z_N) \) a totally symmetric holomorphic function. Using the identity (2.23) (and the corresponding one for \( \Delta^* \)), we can write the Schrödinger equation as
\[
\left[ 2\rho \frac{d^2 F}{d\rho^2} + N(N + 1) \frac{dF}{d\rho} + \left( E - \frac{1}{2} m^2 \rho \right) F \right] g + 2 \frac{dF}{d\rho} \sum_i (z_i \partial_i g) + F \sum_i \left( \partial_i g \frac{\partial_i \Delta^*}{\Delta^*} \right) = 0 .
\]
(2.35)
From the identity \( \sum_i \partial_i \Delta^* = 0 \) we see that if we choose the symmetric function \( g \) such that \( \partial_i g \) will be independent of the index \( i \), and furthermore, if we choose it to be homogeneous of some degree \( p \) such that \( \sum_i (z_i \partial_i g) = pg \), then (2.35) will imply that
\[
2\rho \frac{d^2 F}{d\rho^2} + (N(N + 1) + 2p) \frac{dF}{d\rho} + \left( E - \frac{1}{2} m^2 \rho \right) F = 0 ,
\]
(2.36)
which coincides with (2.24) up to shifting \( N(N + 1) \) by \( 2p \). The two restrictions on \( g \) uniquely fix it (up to a multiplicative constant) as \( g(z_1, \ldots z_N) = (\sum_i z_i)^p \). Thus, by comparing (2.36) and (2.24) we see that
\[
\chi_n^{(p)} = (z_1 + \ldots + z_N)^p L_n^{\left( \frac{N(N+1)}{2} + p - 1 \right)}(m \rho) e^{-\frac{1}{2} m \rho}
\]
(2.37)
is an eigenfunction of \( H_s \), with corresponding eigenvalue
\[
E_n^{(p)} = \left( \frac{N(N + 1)}{2} + p + 2n \right) m
\]
(2.38)
\((n = 0, 1, \ldots)\). The radial eigenstates (2.25) are just the states (2.37) with \( p = 0 \). Of course, \( \chi_n^{(p)*} \) is also an eigenstate, linearly independent of \( \chi_n^{(p)} \) (for \( p \neq 0 \)), and with the same eigenvalue \( E_n^{(p)} \), in accordance with (2.16).
By applying the angular momentum operator

\[ L_z = \sum_i (z_i \partial_i - z_i^* \partial_i^*) \]  

(2.39)
to \( \chi_n^{(p)} \) we see that it carries angular momentum \( p \). Thus, \( p \) is a nonnegative integer. Similarly, \( \chi_n^{(p)^*} \) carries angular momentum \(-p\). It is reasonable to expect that analogous states should exist also in the two-dimensional CMS model at arbitrary couplings.

The generalized multi-dimensional Calogero model discussed recently in [18] allows for \( N \) distinguishable particles in arbitrary number of dimensions. Part of the spectrum of that model was studied using group theoretical methods. In two dimensions, and for the special case of identical particles, the model in [18] can be mapped by a similarity transformation onto our singlet sector hamiltonian (2.15) with quadratic potential (2.21). (See the discussion following (2.46).) Our eigenstates (2.37) coincide with a certain subset of the eigenstates obtained in [18] upon tuning the couplings of the latter model to coincide with ours.

By comparing (2.38) and (2.25), we see that the pair of first excited states among the states we have constructed is the one corresponding to \( n = 0 \) and \( p = 1 \), with energy difference

\[ E_0^{(1)} - E_0^{(0)} = m \]  

(2.40)
above the ground state, which thus sets an upper bound on the gap of excitations in the spectrum of \( H_s \). This upper bound on the gap, obtained from exact singlet eigenstates, is half the gap computed recently in [23] in the large-\( N \) limit, using collective field analysis, for the equivalent CMS model at the normal matrix point in coupling space.

2.1.3 The Quantum Normal Matrix Model and Physics of Electrons in the Lowest Landau Level

The joint probability density (2.28) for \( N \) eigenvalues in the ground state \( \chi_0 \) coincides, upon identification of \( \frac{1}{\sqrt{2m}} \) as the magnetic length, with that of \( N \) noninteracting elec-
trons which occupy the lowest Landau Level (LLL) (at minimal angular momentum).
Indeed, $\Delta \chi_0$ can be identified with the $\nu = 1$ Laughlin wave function.

However, this relation between the ground state of $H_s$ with quadratic potential and many-body wave functions in the LLL, does not persist to the excited states $\chi_n (n \geq 1)$ and $\chi_n^{(p)*} (p \geq 1)$ in (2.25) and (2.37), for a very simple reason. Our quantum mechanical normal matrix model is invariant under reflection with respect to the $x$-axis, which leads to the degeneracy (2.16). In contrast, in the quantum Hall problem, $N$-electron states in the LLL (in the symmetric gauge) are in the form of an antisymmetric holomorphic function $f_A(z_1, \ldots, z_N)$ (multiplied by the gaussian $e^{-\frac{1}{2}m\varphi}$). Any dependence in $f_A$ on nonhomolorphic variables $z_i^*$ means that the corresponding state belongs to higher Landau levels$^4$. Thus, for example, antiholomorphic wave functions will correspond to higher energies than their holomorphic counterparts, unlike (2.16). As a particular example, consider the pairs of degenerate eigenstate $\chi_0^{(p)}, \chi_0^{(p)*}$ of $H_s$. The wave function $\Delta \chi_0^{(p)}$ is of course a linear combination of LLL states$^5$, but $\Delta \chi_0^{(p)*}$ is of higher energy in the Landau problem.

In very intense magnetic fields, which in our normal matrix model corresponds to large $m^2$, the gap from the LLL to the next Landau level becomes very large, and low energy physics can be described entirely in terms of antisymmetric holomorphic wave functions. This means that the number of degrees of freedom is cut in half: to each electron corresponds the combination $z_i = x_i + iy_i$ of position operators, and not $x_i$ and $y_i$ separately. However, in our investigation of the singlet sector of the normal matrix model in this paper we do not have any special interest in taking the limit $m \to \infty$ so as to make the gap to the excited states infinite. Thus, we will have to take into account all singlet states, holomorphic as well as nonholomorphic.

In fact, it might be the case that a version of the quantum normal matrix model

$^4$As is well-known, this happens because the Landau hamiltonian in the symmetric gauge is the sum of a rotation invariant piece and a piece proportional to $BLz = B \sum_i (z_i \partial_i - z_i^* \partial_i^*)$, which breaks the symmetry under interchanging $z_i, \partial_i \leftrightarrow z_i^*, \partial_i^*$.

$^5$The $p = 1$ case is special, since $\Delta \chi_0^{(1)}$ can be written as a single determinant (rather than a linear combination of determinants) of holomorphic monomials $z^n$, multiplied by gaussian factors, of the form $\tilde{\Delta} \chi_0$, where $\tilde{\Delta}$ differs from the Vandermonde determinant by having its last row entries $z_i^N$ instead $z_i^{N-1}$.
considered in this paper, in which a random piece is added to the potential \( V(|z|^2) \), could teach us something relevant for understanding the levitation (or floating) problem of the extended states away from the center of the Landau band in quantum Hall physics, at weak magnetic fields.

In order to relate our quantum normal matrix model and the time-independent normal matrix models [1, 4], however, the large \( m^2 \) limit is important, precisely because of the large energy gap, proportional to \( m \), separating the ground state \( \chi_0 \) and the excited states of \( H_s \). Thus, at large \( m^2 \), the low-energy physics of \( H_s \) (with \( V(|z|^2) = \frac{1}{2} m^2 |z|^2 \)) is captured entirely by the ground state. As was mentioned earlier, according to [4], the joint probability density of \( N \) noninteracting electrons, which occupy the LLL in a strong (and possibly nonuniform) magnetic field, coincides with the joint probability density of the eigenvalues of the zero-dimensional normal matrix model, which is given by

\[
P(z_1, \ldots, z_N) = \frac{1}{Z} \left| \prod_{i>j} (z_i - z_j) \right|^2 e^{-\frac{1}{2} \sum_i W(|z_i|^2)}, \tag{2.41}
\]

where \( W(|z|^2) \) is the magnetostatic potential, namely, \( B(z, z^*) \propto \partial \partial^* W(|z|^2) \). This result was obtained by observing that the \( N \)-electron wave function, with all spins parallel and with minimal angular momentum \( \left( \frac{N(N-1)}{2} \right) \), was simply

\[
\psi_{\text{LLL}} = \prod_{i>j} (z_i - z_j) e^{-\frac{1}{2} \sum_i W(|z_i|^2)}, \tag{2.42}
\]

i.e., a generalized form of the \( \nu = 1 \) Laughlin wave function. For a gaussian \( W = \frac{1}{2} m^2 |z|^2 \), which corresponds to a uniform magnetic field, \( \psi_{\text{LLL}} \) is just the function \( \Delta \chi_0 \), as we saw above.

Note, however, that for a polynomial \( W(|z|^2) \) of higher degree, \( \psi_{\text{LLL}} = e^{-\frac{1}{2} \sum_i W(|z_i|^2)} \) is not an eigenstate of \( H_s \) (2.15) with a local potential \( V(|z|^2) \). Thus, in nonuniform magnetic fields, we cannot relate \( \psi_{\text{LLL}} \) to the ground state of \( H_s \) with any reasonable potential.

As the final comment in this section, note that the many-body wave function \( \chi_n^{(p)} \) is somewhat reminiscent of the single particle wave function in the \( n \)th Landau
level and with angular momentum \( p \), which is given (in the symmetric gauge) by
\[ z^p L_n^p(m \rho)e^{-\frac{1}{2} m \rho} \] (where in the latter expression \( \rho = |z|^2 \)).

### 2.2 Relation to the Two-Dimensional Calogero-Marchioro-Sutherland Model

The hamiltonian of the CMS model is [14, 15]
\[
H_{CMS} = -\frac{1}{2} \sum_i \nabla^2_i + g \sum_{i<j} \frac{1}{r_{ij}^2} + G \sum_i \sum_{j<k} \frac{r_{ij} \cdot r_{ik}}{r_{ij}^2 r_{ik}^2} + \frac{m^2}{2} \sum_i r_i^2
\] (2.43)
where \( r_{ij} = r_i - r_j \). Thus, \( H_{CMS} \) contains a three-body long-range interaction, in addition to the repulsive two-body long-range interaction familiar from the one-dimensional Calogero-Sutherland model.

In order to see the equivalence between our normal matrix model and the two-dimensional CMS model (at the specific point in coupling space) we have to remove the “gauge fields” \( A_i \) and \( A_i^* \) (2.19) from (2.20). These fields can be gauged away by a nonunitary gauge transformation, or more precisely, by a (singular) similarity transformation. Indeed, it is a straightforward calculation to show that
\[
\nabla^2 s = \frac{1}{|\Delta|} \left( 4 \sum_i \partial^*_i \partial_i - \sum_i \frac{\partial_i \Delta}{|\Delta|^2} \frac{\partial_i \Delta^*}{|\Delta^*|^2} \right) |\Delta|.
\] (2.44)

Thus,
\[
H_s = \frac{1}{|\Delta|} \left( -\frac{1}{2} \sum_i \nabla^2_i + \sum_i V(|z_i|^2) + \frac{1}{2} \sum_i \frac{\partial_i \Delta}{|\Delta|^2} \frac{\partial_i \Delta^*}{|\Delta^*|^2} \right) |\Delta|,
\] (2.45)
where we used \( 4 \partial^*_i \partial_i = \frac{\partial^2}{\partial z_i^2} + \frac{\partial^2}{\partial \bar{z}_i^2} = \nabla^2_i \). The hamiltonian obtained from \( H_s \) after the similarity transformation,
\[
\mathcal{H}_s = |\Delta| H_s \frac{1}{|\Delta|} = -\frac{1}{2} \sum_i \nabla^2_i + \sum_i V(|z_i|^2) + \frac{1}{2} \sum_i \frac{\partial_i \Delta}{|\Delta|^2} \frac{\partial_i \Delta^*}{|\Delta^*|^2},
\] (2.46)
has the standard kinetic term. Thus, we have removed the gauge fields in (2.20) at the price of introducing the interaction terms \( \sum_i \frac{\partial_i \Delta}{|\Delta|^2} \frac{\partial_i \Delta^*}{|\Delta^*|^2} \).

In order to proceed we note, using (2.19), that
\[
\sum_i \frac{\partial_i \Delta}{|\Delta|^2} \frac{\partial_i \Delta^*}{|\Delta^*|^2} = \sum_i \sum_{j \neq i} \frac{1}{z_i - z_j} \sum_{k \neq i} \frac{1}{z_i^* - z_k^*}
\]

\[
\sum_{i<j} \frac{1}{|z_i - z_j|^2} + 2 \sum_i \sum_{j<k} \text{Re} \left( \frac{(z_i - z_j)^* (z_i - z_k)}{|z_i - z_j|^2 |z_i - z_k|^2} \right)
\]
\[
= 2 \sum_{i<j} \frac{1}{r_{ij}^2} + 2 \sum_i \sum_{j<k} \frac{r_{ij} \cdot r_{ik}}{r_{ij}^2 r_{ik}^2}, \tag{2.47}
\]

where we denoted the position operator of the \(i\)th eigenvalue \(z_i = x_i + iy_i\) in the plane by \(r_i = x_i \hat{x} + y_i \hat{y}\). Substituting (2.47) in (2.46) and comparing it with (2.43), we see that for the particular potential \(V(|z|^2) = \frac{1}{2} m^2 |z|^2\), the Hamiltonian \(H_s\) coincides with \(H_{CMS}\) at \(g = G = 1\) [10, 15, 16].

The CMS model (2.43), at the special point \(g = G = 1\), was studied recently in [23], in the collective field approach in the large-\(N\) limit, and it was found that the particles condensed into a uniform disk, in accordance with the findings of [15]. The authors of [23] went one step farther, and also computed fluctuations around the uniform ground state configuration, which required taking the first subleading term in the large-\(N\) expansion. They have found that the spectrum of those fluctuations had an energy gap equal to \(m\), corresponding to setting \(V(|z|^2) = \frac{1}{2} m^2 |z|^2\) in (2.46), as was already briefly mentioned following (2.40). In this work we extend the leading large-\(N\) results of [23] to the case of an arbitrary external potential.

The Hamiltonian \(H_s\) acts on singlet wave-functions \(\chi_s\), which are completely symmetric under permutations of the \(z_i\). The similarity transformation on \(H_s\) in (2.46) implies a similarity transformation \(\chi_s \rightarrow |\Delta| \chi_s (z_1, z_1^*, \ldots, z_N, z_N^*)\) into wave functions which are also completely symmetric under permutations of the eigenvalues. Thus, \(H_s\) in (2.46), with its conventional flat space kinetic term, corresponds to a system of interacting bosons in two dimensions. In contrast, in the corresponding quantum mechanical hermitean matrix models, the analog of \(H_s\) acts on one-dimensional non-interacting fermions, as was shown in [24].

Note, however, that the similarity trasformation (2.46), which maps the matrix model hamiltonian \(H_s\) onto the CMS hamiltonian \(H_s\) is highly singular, since \(|\Delta| = \left| \prod_{i>j} (z_i - z_j) \right|\) has branch points wherever two eigenvalues coincide. This might lead to some delicate issues in trying to decide whether the singlet sector of the normal matrix model and the CMS model are completely equivalent or not. There is no such
problem in the mapping of the singlet sector of the quantum mechanical hermitean matrix models onto one-dimensional noninteracting fermions.

We stress that our construction of the quantum collective field formulation of the singlet sector of the normal matrix model in section 2.2 is based entirely on the non-euclidean configuration space with metric (2.6) and the corresponding hamiltonian $H_s$. Nowhere in our construction do we resort to the singular similarity transformation into euclidean configuration space and CMS hamiltonian $\mathcal{H}_s$. Nevertheless, as we have already remarked above, the results of our collective field analysis (for the particular potential $V(|z|^2) = \frac{1}{2}m^2|z|^2$) agree (to leading order in $\frac{1}{N}$) with those of [23], which were obtained starting with $\mathcal{H}_s$.

2.3 A Concluding Remark Concerning Cartesian Coordinates and Feynman Diagrams

We end this section with the following comment: It is customary to refer to (2.2) as the polar decomposition of $M$. Thus, the $z_i$ and $R_{ij}$ ($i < j$) are referred to as the polar coordinates of $M$. The matrix elements $M_{ij}$ are known as the cartesian coordinates. Thus, (2.6), and (2.8), (2.9) express the metric and the laplacian in polar coordinates. There are no simple expressions for these objects of normal matrix theory in terms of the cartesian coordinates $M_{ij}$, since it is rather complicated to solve the constraint (2.1) in these coordinates explicitly. To this end one has to work in the bigger space of complex matrices and introduce a matrix Lagrange multiplier field to impose (2.1) on the dynamics, and then follow Dirac’s procedure for constrained quantization. Interestingly enough, these complications do not arise for hermitean matrices, which is of course a subclass of normal matrices. In this case, the linear constraint $M_{ij} = M_{ji}^*$ implies simply $\nabla^2 = \sum_{ij} \partial^2/(\partial M_{ij} \partial M_{ij}^*) = \text{Tr}(\partial^2/\partial M^2)$.

Having a lagrangian formulation of the normal matrix model in cartesian coordinates is necessary in order to formulate the large-$N$ “double-line” diagrammatic expansion of the model. In this respect, it should be noted that this expansion will be in powers of $\frac{1}{N}$ rather than powers of $\frac{1}{N^2}$, as should be clear from the exact ex-
pression (2.27) for the ground state energy in the case of a quadratic potential.
3 Construction of the Collective Hamiltonian

The transformation from the hamiltonian in its original form to its expression in terms of collective coordinates is achieved by a quantum mechanical point canonical transformation from the original dynamical variables to collective coordinates.

The configuration space of our quantum mechanical normal matrix model is parametrized by the polar coordinates \( z_i \) and \( R_{ij} (i < j) \), and is endowed with the non-euclidean metric (2.6).

Thus, in order to prepare the ground for the construction of the collective field hamiltonian for the normal matrix model, we will first recall some basic facts about point canonical transformations in configuration spaces whose geometry is defined by a non-euclidean metric, thus extending the discussion in chapter 6 of [20].

3.1 Point Canonical Transformations in Non-Euclidean Configuration Space

We shall start by stating some elementary facts, mainly in order to introduce notations. We make our points starting in Eq.(3.14).

Consider a quantum mechanical system, defined in a D-dimensional configuration space \( \mathcal{C} \), with coordinates \( q^a (a = 1, \ldots D) \) and line element \( ds^2 = g_{ab}(q) dq^a dq^b \). The hamiltonian is given by

\[
H_q = -\frac{\hbar^2}{2} \nabla_q^2 + V(q),
\]

where

\[
\nabla_q^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^a} \left( g^{ab} \sqrt{g} \frac{\partial}{\partial q^b} \right)
\]

is the invariant laplacian, expressed in terms of the coordinates \( q^a \). In (3.2) \( g^{ab} \) is the inverse metric and \( g = \det g_{ab} \). This laplacian (and the corresponding hamiltonian \( H_q \)) are symmetric \( \langle \psi_1 | H_q | \psi_2 \rangle = \langle H_q \psi_1 | \psi_2 \rangle \) with respect to the measure \( \sqrt{g} dq \) and the inner product \( \langle \psi_1 | \psi_2 \rangle = \int_{\mathcal{C}} \sqrt{g} dq \psi_1^*(q) \psi_2(q) \).

In this subsection we will write all factors of \( \hbar \) explicitly, in order to identify the purely quantum mechanical terms induced by the transformation in the effective hamiltonian, defined later in this section.
Let us consider a point canonical transformation

\[ q^a \rightarrow Q^a = Q^a(q), \]  

(3.3)

and assume its inverse exists as well:

\[ q^a = q^a(Q). \]  

(3.4)

We know from rudimentary differential geometry, that since the line element \( ds^2 \) is invariant under this transformation, the metric in the new coordinates is given by

\[ \Omega_{ab}(Q) = g_{mn}(q(Q)) \frac{\partial q^m}{\partial Q^a} \frac{\partial q^n}{\partial Q^b}, \]  

(3.5)

its inverse by

\[ \Omega^{ab}(Q) = g^{mn}(q(Q)) \frac{\partial Q^a}{\partial q^m} \frac{\partial Q^b}{\partial q^n}, \]  

(3.6)

and

\[ \Omega(Q) = \det \Omega_{ab} = g \left( \det \left( \frac{\partial q}{\partial Q} \right) \right)^2. \]  

(3.7)

Thus,

\[ \sqrt{\Omega} = \sqrt{g} J, \]  

(3.8)

where

\[ J = \det \left( \frac{\partial q}{\partial Q} \right) \]  

(3.9)

is the Jacobian of the transformation, rendering the measure invariant

\[ \sqrt{\Omega} dQ = \sqrt{g} dq. \]  

(3.10)

Finally, the Hamiltonian in the new coordinates is written

\[ H_Q = -\frac{\hbar^2}{2} \nabla_Q^2 + \tilde{V}(Q), \]  

(3.11)

where

\[ \nabla_Q^2 = \frac{1}{\sqrt{\Omega}} \frac{\partial}{\partial Q^a} \left( \Omega^{ab} \sqrt{\Omega} \frac{\partial}{\partial Q^b} \right) \]  

(3.12)

is the invariant Laplacian in the new coordinates \( Q \), and \( \tilde{V}(Q) = V(q(Q)) \) (i.e., the potential is a scalar function under (3.3)).
Wave functions also transform as scalars, \( \psi(q) \rightarrow \tilde{\psi}(Q) = \psi(q(Q)) \), since amplitudes must remain invariant against the coordinate transformations:

\[
\langle \psi_1 | \psi_2 \rangle = \int_C \sqrt{\Omega} \, dQ \, \psi_1^*(Q) \, \psi_2(Q) = \int_C \sqrt{\tilde{\Omega}} \, dq \, \psi_1^*(q) \, \psi_2(q) .
\]

(3.13)

The form (3.11) of the transformed hamiltonian, and the fact that wave functions are scalars, guarantee the invariance of matrix elements \( \langle \psi_1 | H | \psi_2 \rangle \) under coordinate transformations.

Let us assume that for one reason or another, the \( Q \)-coordinates represent the physical picture more transparently, and thus, working with \( H_Q \) has some extra merit. \( H_Q \) is symmetric with respect to the measure \( \sqrt{\Omega} dQ \). Thus, part of the price to be paid working with the \( Q \)-coordinates is the need to drag along that pesky measure everywhere. Life would be simpler if we could rid ourselves from that measure and map \( H_Q \) onto an effective hamiltonian \( H_{eff} \) which is symmetric with respect to the flat measure \( dQ \). This we can achieve by performing a similarity transformation.

Observing that

\[
\langle \psi_1 | H | \psi_2 \rangle = \int_C \sqrt{\Omega} \, dQ \, \psi_1^*(Q) \, H_Q \psi_2(Q) = \int_C dQ \left( \Omega^{\frac{1}{4}} \psi_1^*(Q) \right) \left( \Omega^{\frac{1}{4}} H_Q \Omega^{-\frac{1}{4}} \right) \left( \Omega^{\frac{1}{4}} \psi_2(Q) \right)
\]

it is obvious that required transformation is

\[
H_{eff} = \Omega^{\frac{1}{4}} H_Q \Omega^{-\frac{1}{4}}
\]

(3.14)

with the appropriate transformation

\[
\tilde{\psi}(Q) = \Omega^{\frac{1}{4}} \psi(Q)
\]

(3.15)

of wave functions. Let us now massage \( H_{eff} \) into a more transparent form. It is a matter of straightforward calculation to show that

\[
\Omega^{\frac{1}{4}} \nabla_Q^2 \Omega^{-\frac{1}{4}} = \frac{1}{2} \nabla_Q^2 \left( \Omega^{ab} \sqrt{\Omega} \frac{\partial}{\partial Q^a} \frac{\partial}{\partial Q^b} \right) \Omega^{-\frac{1}{4}}
\]

\[
= \left( \Omega^{\frac{1}{4}} \frac{\partial}{\partial Q^a} \Omega^{\frac{1}{4}} \right) \Omega^{ab} \left( \Omega^{\frac{1}{4}} \frac{\partial}{\partial Q^b} \Omega^{-\frac{1}{4}} \right)
\]

\[
= \frac{\partial}{\partial Q^a} \Omega^{ab} \frac{\partial}{\partial Q^b} - \left( \frac{\partial}{\partial Q^a} (\Omega^{ab} C_b) \right) - C_a \Omega^{ab} C_b
\]

(3.16)
where we have defined
\[ C_a = \frac{1}{4} (\log \Omega),_a \]  
(3.17)
and where \((\cdot),_a\) indicates a derivative with respect to \(Q^a\). The operator \(\Omega^{\frac{1}{2}} \nabla_Q^2 \Omega^{-\frac{1}{2}}\) is manifestly symmetric with respect to the flat measure \(dQ\), as is evident in each of the lines in (3.16). Thus, \(H_{\text{eff}}\) is indeed the desired hamiltonian we set out to find, which, following (3.16), we may write explicitly as
\[ H_{\text{eff}} = \frac{1}{2} P_a \Omega^{ab} P_b + \frac{\hbar^2}{2} C_a \Omega^{ab} C_b + \tilde{V}(Q) + \frac{\hbar^2}{2} \left( \Omega^{ab} C_b \right),_a \]  
(3.18)
where we introduced the momentum operators
\[ P_a = -i\hbar \frac{\partial}{\partial Q^a}. \]  
(3.19)
The terms in (3.18) quadratic in \(\hbar\) may be thought of as a generalization of the centrifugal barrier which arises in the radial hamiltonian in \(D\) dimensions\(^7\). Evidently, these terms are purely a quantum mechanical effect.\(^8\)

It is easy to see that
\[ C_a = \frac{1}{2} \Gamma^b_{ba} \]  
(3.20)
where \(\Gamma^a_{bc}\) is the second Christoffel symbol (i.e., the connection) of \(\Omega_{ab}\). However, sometimes a direct computation of the \(C_a\) from their definition (3.17), or from the identity (3.20), may be too difficult to carry in practice. Thus, in order to bypass these potential difficulties, we shall now derive an identity satisfied by the \(C_a\), from which we could compute them with somewhat less effort.

To this end we argue as follows: The invariant laplacian acting on a function which is a scalar under coordinate transformation produces yet another scalar function. Thus,
\[ \nabla_q^2 \psi(q) = \nabla_Q^2 \psi(q(Q)). \]  
(3.21)
\(^7\)The radial part of the \(D\)-dimensional laplacian \(\nabla^2_r = r^{-(D-1)} \partial_r (r^{D-1} \partial_r)\), defined with respect to the measure \(r^{D-1} dr\), may be transformed by a similarity transformation into \(\nabla^2_r = \frac{\partial^2}{\partial r^2} - \frac{(D-1)(D-3)}{4r^2} = \partial_r^2 - C_r \Omega^{rr} C_r\), which is defined with respect to the flat measure \(dr\).
\(^8\)As a side remark, we also mention that the last term in (3.18) may be written as \(\frac{\hbar^2}{2} \left( \Omega^{ab} C_b \right),_a = -\frac{\hbar^2}{4} \Omega^{\frac{1}{2}} \left( \nabla_Q^2 \Omega^{-\frac{1}{2}} \right)\).
In particular, the coordinate functions themselves are scalars (their differentials are one-forms). Let us define the quantities

$$\omega^a = -\hbar \nabla^2 Q^a = -\hbar \nabla^2 Q^a. \quad \text{(3.22)}$$

It follows from the definitions (3.22) and (3.17) that

$$\omega^a = -\hbar \nabla^2 Q^a = -\frac{\hbar}{\sqrt{\Omega}} \frac{\partial}{\partial Q^b} \left( \Omega^{ab} \sqrt{\Omega} \right) = -\frac{\hbar}{2} \Omega^{ab} (\log \Omega),_b - \hbar \Omega^{ab},_b$$
or

$$\omega^a + 2\hbar \Omega^{ab} C_b + \hbar \Omega^{ab},_b = 0, \quad \text{(3.23)}$$

which is the desired identity to determine the $C_a$. To simplify the computation, we are free to choose in (3.22) the coordinates $q^a$ in which the computation of $\omega^a = -\hbar \nabla^2 Q^a$ is as simple as possible.

This concludes our brief review and exposition of point canonical coordinate transformations in quantum mechanics, which we will use in the second part of this section to construct the collective hamiltonian of our normal matrix model.

The discussion of point canonical transformations in section 6.1 in [20] was presented\(^9\) for the special case of flat euclidean metric $g_{ab} = \delta_{ab}$. For this metric, it is a straightforward exercise to show that $\omega^a$, $C_a$ and $\Omega^{ab}$ which we defined here for an arbitrary metric, coincide with their counterparts in chapter 6 of [20]. In addition, in that metric, our expression (3.18) for $H_{\text{eff}}$ and the identity (3.23) reduce to their counterparts, Eqs.(6.21) and (6.20), respectively, in [20].

### 3.2 The Collective Field Hamiltonian for Normal Matrices

The ground state, as well as other $U(N)$-singlet eigenstates of the hamiltonian (2.13), are totally symmetric functions $\chi_s(z_1, z_1^*, \ldots, z_n, z_n^*)$ of the eigenvalues. As we saw in the introduction, these eigenvalues comprise a two-dimensional system of interacting bosons.

\(^9\)We should mention that there is a typographical error in Eq.(6.19) in [20]. Its correct form is the sum of anticommutators $H_{\text{eff}} - H_{\text{eff}}^\dagger = \frac{i}{2} \{ \omega^a + 2\hbar \Omega^{ab} C_b + \hbar \Omega^{ab},_b, P_a \}$. Also note, that the last (total divergence) term in our (3.18) is missing from the analogous equation, Eq.(6.21), in [20].
Let
\[ \mathbf{r}_i = x_i \hat{x} + y_i \hat{y} \] (3.24)
be the position operator of the \( i \)th eigenvalue \( z_i = x_i + i y_i \) in the plane. The density operators
\[ \phi(\mathbf{r}) = \frac{1}{N} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) , \] (3.25)
form a continuous basis (parametrized by \( \mathbf{r} \)) of commuting operators which are symmetric in the operators \( \mathbf{r}_i \). Clearly, any operator which is purely a symmetric combination of \( \mathbf{r}_i \)'s may be expressed in terms of the density operators (3.25). Thus, it is conceivable that any singlet wave function \( \chi_s \) in our model could be expressed as functional of the eigenvalues of these density operators, namely, as a functional of the density function, which we also denote by \( \phi(\mathbf{r}) \). The density function is evidently positive,
\[ \phi(\mathbf{r}) \geq 0 . \] (3.26)

This motivates us to apply the method of coordinate transformation which we derived in the previous subsection, and make a transformation from the eigenvalues \( z_i \) (or the corresponding commuting position operators \( \mathbf{r}_i \)), which live in a configuration space with the non-euclidean metric (2.6), to the set of commuting density operators. The latter are the collective variables for our system of bosons - they are the combinations of the original dynamical degrees of freedom, the \( \mathbf{r}_i \), which are invariant under the permutation symmetry of the problem. The transformation we will perform affects only the singlet degrees of freedom, and leaves the \( U(N) \)-angular variables \( R_{ij} \) unchanged, of course.

In this section we will make the transformation to these collective variables and study the system of bosons in the large-\( N \) limit. This method of transformation, which is commonly known as the collective field method, was introduced in [19] (for a review, see chapter 7 of [20]). It should be thought of as a natural extension of the Bohm-Pines theory of high density plasma oscillations [27]. An important application of this method in [19] was to study the singlet sector of the quantum
hermitean matrix model. As was mentioned earlier, the same problem was solved earlier in \cite{24}, by mapping it onto a one dimensional gas of noninteracting fermions. Here, we will extend these works to normal matrices.

The number of degrees of freedom in the singlet sector of (2.13) is $2N$. In contrast, there is a continuum of density operators $\phi(r)$, which are not all independent. For example, it is obvious that the constraint

$$\int dr \phi(r) = 1$$  \hspace{1cm} \text{(3.27)}

should hold. As our independent collective variables we choose the set of $2N$ Fourier modes

$$\phi_k = \int dr e^{-ik \cdot r} \phi(r) = \frac{1}{N} \sum_{i=1}^{N} e^{-i k \cdot r_i}$$  \hspace{1cm} \text{(3.28)}

cutoff by $k_{max}, |k| \leq k_{max}$, where $k$ is properly discretized, e.g., by putting the bosons in a large box of linear size $L$, and imposing appropriate boundary conditions. The details of this discretization are not important for our discussion of the large-$N$ limit. Next, we assume that the eigenvalues condense into a lump of linear size $L_c$, where $L_c << L$. The average bulk density of particles in the system is thus

$$\rho \sim \frac{N}{L_c^2},$$  \hspace{1cm} \text{(3.29)}

and the microscopic interparticle distance will be of the order $l \sim 1/\sqrt{\rho}$. Thus the maximal Fourier components relevant to our problem should be of the order $k_{max} = 1/l \sim \sqrt{\rho}$. We shall take the limit $N \to \infty$ together with $L \to \infty$ such that $\rho >> 1$, or more precisely, $L >> L_c >> l$. Thus, the high density limit makes $k_{max} \to \infty$, and letting $L \to \infty$ makes $k$ continuous.

In this limit, which we shall assume throughout our analysis, we can consider the continuum of $\phi(r)$s as independent variables, constrained by (3.27) (i.e., by $\phi_{k=0} = 1$).

To summarize, we wish to transform the normal matrix hamiltonian (2.13) from the coordinates $q^a = r_i, R_{ij}$ and metric $g_{ab}$ given by (2.6), to the coordinates $Q^a = \phi_k, R_{ij}$ and metric $\Omega_{ab}$ given by (3.5), and to construct the effective hamiltonian (3.18), according to the formalism developed in the previous subsection.
Since this transformation does not affect the $U(N)$-angular coordinates $R_{ij}$ and does not mix them with the eigenvalues, $\Omega^{ab}$ will have the same block-diagonal structure as $g^{ab}$, and of course, their pure $U(N)$-angular sectors will coincide. We need only compute the transform of the pure eigenvalue sector $\Omega^{\phi_k,\phi_{k'}}$.

We start by computing the latter, which we denote more conveniently by $\Omega(k, k'; [\phi])$, with the functional dependence on $\phi$ indicated explicitly. Since (2.6) is equivalent to
\[
ds^2 = \sum_i d\mathbf{r}_i^2 + 2 \sum_{i<j} dR^*_i dR_{ij} |\mathbf{r}_i - \mathbf{r}_j|^2,
\]
we obtain from (3.6) and (3.28) that
\[
\Omega(k, k'; [\phi]) = \sum_i \frac{\partial \phi_k}{\partial \mathbf{r}_i} \cdot \frac{\partial \phi_{k'}}{\partial \mathbf{r}_i} = -\frac{k \cdot k'}{N} \phi_{k+k'}.
\]
The effective Hamiltonian (3.18) contains also the quantities $C_a$, which we will determine from the identity (3.23). Thus, we have to compute $\omega^{\phi_k} = \omega(k; [\phi])$. From the definition (3.22) we obtain\(^{10}\)
\[
\omega(k; [\phi]) = -\nabla^2 \phi_k = -\nabla^2 \phi_k,
\]
where $\nabla^2$ was defined in (2.7). Thus, from (2.9), we find\(^{11}\)
\[
\omega(k; [\phi]) = -\frac{1}{|\Delta|^2} \sum_i \left[ \frac{\partial}{\partial x_i} \left( |\Delta|^2 \frac{\partial}{\partial x_i} \right) + \frac{\partial}{\partial y_i} \left( |\Delta|^2 \frac{\partial}{\partial y_i} \right) \right] \phi_k
\]
\[
= k^2 \phi_k + \frac{i k}{N} \sum_i e^{-ik \cdot \mathbf{r}_i} \frac{\partial \log |\Delta|^2}{\partial \mathbf{r}_i}.
\]
From (2.10) we obtain
\[
\frac{\partial \log |\Delta|^2}{\partial \mathbf{r}_i} = 2 \sum_j \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2).
\]
The latter sum has a simple physical interpretation in terms of two-dimensional electrostatics. If we think of the eigenvalues located at the points $\mathbf{r}_k$ in the plane as

\(^{10}\)From this point throughout the rest of the paper we set $\hbar = 1$.

\(^{11}\)This expression for $\omega(k; [\phi])$ is the two-dimensional analog of Eq.(7.55) in [20] and Eq.(5.6) in [19], namely, the expression $\omega(k; [\phi]) = N k^2 \int_0^1 d\alpha \phi_k \alpha \phi_k(1-\alpha) = k^2 \phi_k + \frac{2ik}{N} \sum_i e^{-ik \lambda_i} \sum_j \frac{1}{\lambda_i - \lambda_j}$, corresponding to the hermitean matrix model with eigenvalues $\lambda_i$.  

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positive point unit charges, then

$$E^{(i)}(r) = \sum_{j \neq i} \frac{r - r_j}{(r - r_j)^2}$$

(3.33)

is the electric field at $r$ due to all the other charges, except the one at $r_i$. Thus,

$$\omega(k; [\phi]) = k^2 \phi_k + \frac{2i k}{N} \sum_i e^{-ikr_i} E^{(i)}(r_i).$$

(3.34)

We would like to avoid the $i$-dependence of the vector field $E^{(i)}(r)$ in (3.34) in order to simplify the summation. The aim is to replace it by the full electric field

$$E(r) = \sum_j \frac{r - r_j}{(r - r_j)^2}.$$  

(3.35)

However, since $E(r)$ is singular at $r_i$ we have to replace $E^{(i)}(r)$ by a regulated form of $E(r)$, which will tend in the large density limit to the macroscopic smoothed electric field, which corresponds to the smoothed macroscopic density $\phi(r)$, namely,

$$E(r) = N \int dr' \frac{r - r'}{(r - r')^2} \phi(r'),$$

(3.36)

and satisfies Gauss’ law

$$\nabla \cdot E(r) = 2\pi N \phi(r).$$

(3.37)

That (3.36) satisfies (3.37) is a trivial consequence of the identity

$$\nabla \cdot \frac{r}{r^2} = 2\pi \delta(r).$$

(3.38)

To identify the regulated form, note first that

$$E_s^{(i)}(r) = \sum_{j \neq i} \frac{r - r_j}{(r - r_j)^2 + s^2},$$

(3.39)

with $s$ a small real parameter (which we shall assume is much smaller than the mean interparticle distance $\sim 1/\sqrt{\rho}$), should have a negligible effect on $E^{(i)}(r)$ in a typical configuration of the eigenvalues, in which it is most likely that $|r_i - r_j| \geq 1/\sqrt{\rho}$ for $j \neq i$, due to electrostatic repulsion. In $E_s^{(i)}(r_i)$ we can obviously relax the restriction $j \neq i$ in the summation. Thus, it is clear that

$$E_s(r) = \sum_j \frac{r - r_j}{(r - r_j)^2 + s^2}.$$  

(3.40)
is the desired regulated form of the electric field, with \( E_s^i(r_i) = E_s(r_i) \) to be substituted in (3.34).

The electric field \( E(r) = \frac{r}{r^2} \) of a point unit charge at the origin is regulated in this way to \( E_s(r) = \frac{r}{r^2 + s^2} \), such that \( \nabla \cdot E(r) = 2\pi \delta(r) \rightarrow \nabla \cdot E_s(r) = 2\pi \delta_s(r) \), where \( 2\pi \delta_s(r) = \frac{2s^2}{(r^2 + s^2)^2} \). As \( s \) tends to zero, \( \delta_s(r) \) tends to \( \delta(r) \). As another motivation for introducing the regularization (3.40), note that

\[
E_{sx} - iE_{sy} = \sum_j \frac{z^* - z_j^*}{|z - z_j|^2 + s^2} = \text{Tr} \left( \frac{(z - M)^\dagger}{(z - M)^\dagger(z - M) + s^2} \right),
\]

(3.41)

where \( M \) is a normal matrix with eigenvalues \( z_i \), is the regulated form of the Green’s function \( G(z, z^*) = \text{Tr} \frac{1}{z - M} \) one would obtain from the method of hermitization of [28].

\( E_s(r) \) is the interpolating regulated electric field which should tend, in the large density limit to a smooth asymptotic distribution, which we can obtain by solving (3.37) for \( E(r) \), with the corresponding limiting smooth distribution of \( \phi(r) \) as the source term. Thus, in the large density limit we can replace \( E^{(i)}(r_i) \) in (3.34) by \( E_s(r_i) \) and write

\[
\omega(k; [\phi]) = k^2 \phi_k + \frac{2ik}{N} \cdot \sum_i e^{-ik \cdot r_i} E_s(r_i).
\]

Henceforth we will suppress the index \( s \) in \( E_s(r) \), with the limit \( s \rightarrow 0 \) understood as the last limit taken.

Since \( E(r) \) is of order \( N \), as can be seen from (3.35) (or (3.36)), we should neglect the subleading \( O(N^0) \) term \( k^2 \phi_k \) in the last equation. Therefore, to leading order in \( \frac{1}{N} \),

\[
\omega(k; [\phi]) = \frac{2ik}{N} \cdot \sum_i e^{-ik \cdot r_i} E(r_i),
\]

(3.42)

which is a quantity of order \( N \).

Thus, we have determined \( \Omega(k, k'; [\phi]) \) and \( \omega(k; [\phi]) \). The last ingredient needed for determining \( C_{\phi_k} = C(k; [\phi]) \) from (3.23) is the divergence \( \Omega^a_b \). Happily enough, it vanishes

\[
\sum_{k'} \frac{\delta \Omega(k, k'; [\phi])}{\delta \phi_{k'}} = -\sum_{k'} \frac{k \cdot k'}{N} \delta(k) = 0,
\]

(3.43)
as can be seen from (3.31). Substituting (3.31), (3.42) and (3.43) in (3.23), we obtain the equation

$$\omega(k; [\phi]) + 2 \sum_{k'} \Omega(k, -k'; [\phi]) C(k'; [\phi]) = 0$$  \hspace{1cm} (3.44)

for $C(k; [\phi])$. In the combined large density and large volume limits discussed above, the $k$-sums tend to Fourier integrals. Thus, in the limit

$$\omega(k; [\phi]) + 2 \int \frac{dk'}{(2\pi)^2} \Omega(k, -k'; [\phi]) C(k'; [\phi]) = 0,$$  \hspace{1cm} (3.45)

which we will transform now to $r$-space.

To this end we need

$$\omega(r; [\phi]) = \int \frac{dk}{(2\pi)^2} e^{ik\cdot r} \omega(k; [\phi]) = 2 \nabla \cdot (\phi(r) \mathbf{E}(r)),$$  \hspace{1cm} (3.46)

where we used

$$\int \frac{dk}{(2\pi)^2} \frac{1}{N} \sum_i e^{ik(r-r_i)} \mathbf{E}(r_i) = \frac{1}{N} \sum_i \delta(r-r_i) \mathbf{E}(r) = \phi(r) \mathbf{E}(r).$$

We also need

$$\Omega(r, r'; [\phi]) = \int \frac{dkdk'}{(2\pi)^4} e^{ik\cdot r + k'\cdot r'} \Omega(k, k'; [\phi]) = \frac{1}{N} \nabla_r \cdot \nabla_{r'} \left( \phi(r) \delta(r-r') \right).$$  \hspace{1cm} (3.47)

Using (3.46) and (3.47) we Fourier transform (3.45) to $r$-space and obtain

$$\omega(r; [\phi]) - \frac{2}{N} \nabla \cdot (\phi(r) \nabla C(r; [\phi])) =$$

$$\frac{2}{N} \nabla \cdot \left[ \phi(r) \left( N \mathbf{E}(r) - \nabla C(r; [\phi]) \right) \right] = 0.$$  \hspace{1cm} (3.48)

The minimal solution of this equation is simply

$$\nabla C(r; [\phi]) = N \mathbf{E}(r).$$  \hspace{1cm} (3.49)

Thus, from (3.36), $C(r; [\phi]) = NU(r)$, where $U(r)$ is (minus) the electrostatic potential

$$U(r) = \frac{1}{2} \int d\mathbf{r}' \log (r - r')^2 \phi(r').$$  \hspace{1cm} (3.50)
∇C(r; [φ]) in (3.49) coincides with the leading large-N behavior of the analogous quantity in [23]. Also, it is the two-dimensional analog of the corresponding quantity obtained in the collective field formulation of the hermitean matrix model [19],

\[ \frac{\partial C(x; [φ])}{\partial x} = N \int \frac{φ(y)}{x-y} dy \]

which reproduces the results of [24] correctly.\(^\text{12}\)

In two dimensions, however, we have the freedom of adding a piece \( \frac{u(r)}{φ(r)} \) to the right hand side of (3.49), where \( u \) is an arbitrary divergence-free vector

\[ \nabla \cdot u = 0. \tag{3.51} \]

We shall not pursue this possibility here.

We now have all the ingredients needed for constructing the effective collective hamiltonian (3.18). From (3.47) and (3.49) we obtain the collective potential \( V_{\text{coll}} = \frac{1}{2} C_a \Omega^{ab} C_b \) as

\[
V_{\text{coll}} = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' C(\mathbf{r}) \Omega(\mathbf{r}, \mathbf{r}') C(\mathbf{r}') = \frac{1}{2N} \int d\mathbf{r} φ(\mathbf{r})(\nabla C(\mathbf{r}))^2 \\
= \frac{N}{2} \int d\mathbf{r} φ(\mathbf{r}) E(\mathbf{r})^2, \tag{3.52}
\]

which is a quantity of order \( N^3 \). In the collective field formulation of hermitean matrices, this term can be written equivalently as an integral over \( φ^3(x) \) [19, 20].

Similarly to (3.52), the kinetic term \( \frac{1}{2} P_a \Omega^{ab} P_b \) in our model is

\[
K = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' Π(\mathbf{r}) \Omega(\mathbf{r}, \mathbf{r}') Π(\mathbf{r}') = \frac{1}{2N} \int d\mathbf{r} \nabla Π(\mathbf{r}) \cdot φ(\mathbf{r}) \nabla Π(\mathbf{r}), \tag{3.53}
\]

where \( Π(\mathbf{r}) \) is the momentum operator conjugate to \( φ(\mathbf{r}) \). \( φ(\mathbf{r}) \) and \( Π(\mathbf{r}) \) satisfy the equal-time canonical commutation relation

\[
[φ(\mathbf{r}), Π(\mathbf{r}')] = i \left( δ(\mathbf{r} - \mathbf{r}') - \frac{1}{L^2} \right), \tag{3.54}
\]

\(^\text{12}\)In this context, we should also mention that Eq.(3.46) (which leads, together with (3.47), to (3.49)), is essentially the two-dimensional analog of of Eq.(7.58) in [20] and Eq.(5.9) in [19] for \( ω(x; [φ]) \) in the hermitean matrix model. In those references, the one-dimensional analog of of (3.46) was obtained directly from the expression for \( ω(k; [φ]) \), which contained the subleading \( k^2 φ_k \) term. However, the latter term was effectively lost by taking the continuum limit of the Fourier sum into a Fourier integral.

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where the inverse area subtraction arises because the zero-mode $\phi_{k=0}$ is non-dynamical, due to the constraint (3.27).

Next, we have the divergence term $\frac{1}{2} \left( \Omega^{ab} C_b \right)_{,a} = -\frac{1}{4} (\omega^a + \Omega^{ab})_{,a}$, where we used (3.23). Thus, from (3.46) and (3.43) we find it as

$$V_{\text{divergence}} = -\frac{1}{4} \int d\mathbf{r} \frac{\delta \omega(\mathbf{r}; \phi)}{\delta \phi(\mathbf{r})} - \frac{1}{2} \int d\mathbf{r} \nabla \cdot \left( \frac{\delta}{\delta \phi(\mathbf{r})} (\phi(\mathbf{r}) E(\mathbf{r})) \right).$$

(3.55)

It is evidently a singular boundary contribution. Assuming boundary conditions such that $\phi(\mathbf{r})$ vanishes at spatial infinity faster than $r^{-2}$, we obtain

$$V_{\text{divergence}} = -\frac{1}{4} \delta(\mathbf{r} = 0) \int d\mathbf{r} \nabla \cdot E(\mathbf{r}) = -N \pi \delta(\mathbf{r} = 0),$$

(3.56)

where we used (3.37). This result for $V_{\text{divergence}}$ is merely a universal constant shift of energy, independent of the external potential $V(\mathbf{r}^2)$. It is proportional to $N$, whereas the leading behavior of energy eigenvalues is $O(N^2)$, as we will see momentarily. Thus, to leading order in $\frac{1}{N}$, we shall neglect this term from now on.

We should mention that this result for $V_{\text{divergence}}$ agrees, to leading order in $\frac{1}{N}$, with the one obtained in Eq. (11) in [23] for the quadratic potential $V(\mathbf{r}^2) = \frac{m^2}{2} r^2$. It was shown in [23] that this term was canceled against the divergent zero-point energy of the collective field $\phi(\mathbf{r})$. This cancellation should occur for any $V(\mathbf{r}^2)$, since our result (3.56) is manifestly independent of the external potential.

Finally, the contribution of the external potential is

$$V_{\text{ext}} = N \int d\mathbf{r} \phi(\mathbf{r}) V(\mathbf{r}^2).$$

(3.57)

Gathering all terms together, we obtain the collective hamiltonian as

$$H_{\text{coll}} = \frac{1}{2N} \int d\mathbf{r} \nabla \Pi(\mathbf{r}) \cdot \phi(\mathbf{r}) \nabla \Pi(\mathbf{r}) + \frac{N}{2} \int d\mathbf{r} \phi(\mathbf{r}) E(\mathbf{r})^2 + N \int d\mathbf{r} \phi(\mathbf{r}) V(\mathbf{r}^2).$$

(3.58)

In order to display the large-$N$ behavior of this theory explicitly, we rescale the coordinates as $\mathbf{r} = \sqrt{N} \mathbf{x}$. This is just the statement that $\text{Tr} M^\dagger M \sim N \int r^2 \phi(\mathbf{r}) d\mathbf{r}$ is typically $O(N)$, assuming $\phi(\sqrt{N} \mathbf{x})$ is supported in a disk $|\mathbf{x}| < O(N^0)$. Consequently, $\phi(\mathbf{r}) = N^{-1} \varphi(\mathbf{x})$ (since $\phi(\mathbf{r}) d\mathbf{r}$ is invariant), and $E(\mathbf{r}) = \sqrt{N} \mathcal{E}(\mathbf{x})$ (from (3.35)), where

$$\mathcal{E}(\mathbf{x}) = \int d\mathbf{x'} \frac{\mathbf{x} - \mathbf{x'}}{(\mathbf{x} - \mathbf{x'})^2} \varphi(\mathbf{x'})$$

(3.59)
is a quantity of $O(N^0)$.

Also, from (2.14), $V(r^2) = V(Nx^2) = NV(x^2)$, where
\[ V(x^2) = \sum_p g_p(x^2)^p. \] (3.60)

Finally, in order that the momentum dependent terms scale as the rest, we must rescale $\Pi(r) = N^2\pi(x)$. The rescaled form of the hamiltonian (3.58) is thus
\[ H_{coll} = N^2 \left\{ \int dx \left( \frac{1}{2} \nabla \pi(x) \cdot \varphi(x) \nabla \pi(x) + \frac{1}{2} \varphi(x) \mathcal{E}(x)^2 + \varphi(x) \mathcal{V}(x^2) \right) \right. \\
\left. - \lambda \left( \int dx \varphi(x) - 1 \right) \right\}, \] (3.61)
where we have added a lagrange multiplier to impose the constraint (3.27). As expected, the hamiltonian (3.61) and its eigenvalues, the ground state energy in particular, scale as $N^2$.

The commutation relation of the rescaled canonically conjugate fields is
\[ [\varphi(x), \pi(x')] = \frac{i}{N^2} \left( \delta(x - x') - \frac{N}{L^2} \right), \] (3.62)
where we used (3.54). This commutation relation indicates that $\varphi(x), \pi(x)$ and the dynamics of (3.61) become classical in the large-$N$ limit.

### 3.2.1 The Hamiltonian Equations of Motion: Fluid Dynamics Interpretation

The Heisenberg equations of motion (as well as their classical analogs) resulting from (3.61) and (3.62) are easily found to be
\[ \dot{\varphi}(x, t) = -\frac{1}{2} \nabla \cdot \left( \{ \varphi(x, t), \nabla \pi(x, t) \} \right), \]
i.e.,
\[ \dot{\varphi}(x, t) = -\nabla \cdot (\varphi(x, t) \nabla \pi(x, t)), \] (3.63)
and
\[ \dot{\pi}(x, t) = -\frac{1}{2} (\nabla \pi(x, t))^2 - \frac{\delta}{\delta \varphi(x)} \int dx' \left[ \frac{1}{2} \varphi(x') \mathcal{E}(x')^2 + \varphi(x') (\mathcal{V}(x'^2) - \lambda) \right] + \frac{N}{L^2} A, \] (3.64)
where
\[ A = \int dx \left[ \frac{1}{2}(\nabla \pi(x))^2 + \int dy \frac{\delta}{\delta \varphi(y)} \left( \frac{1}{2} \varphi(x) E(x)^2 + \varphi(x) (V(x^2) - \lambda) \right) \right] \] (3.65)
is a constant. By assumption\(^{13}\), the coefficient \( \frac{N}{\tau} \) in front of \( A \) in (3.64) tends to zero, and we shall neglect it.

Eq. (3.63) just means that the constraint (3.27) is preserved by the dynamics.

Let us take the gradient of (3.64) so as to make the equations of motion more symmetric. The \( \pi \)-dependent terms in the resulting equation can be lumped together into the combination \( \partial_t \nabla \pi + (\partial_\mu \pi) \nabla (\partial_\mu \pi) = \partial_t \nabla \pi + (\nabla \pi) \cdot \nabla (\nabla \pi) \). Thus, the gradient of (3.64) may be written as
\[ \partial_t \nabla \pi + (\nabla \pi) \cdot \nabla (\nabla \pi) = -\nabla w \] (3.66)
where
\[ w(x; [\varphi]) = \frac{\delta}{\delta \varphi(x)} \int dx' \left( \frac{1}{2} \varphi(x') E(x')^2 + \varphi(x') V(x'^2) - \lambda \varphi(x') \right). \] (3.67)
Evidently, we should interpret the equations of motion as describing the isentropic flow of a two-dimensional eulerian fluid, with density \( \varphi(x, t) \), an irrotational velocity field \( \mathbf{v}(x, t) = \nabla \pi(x, t) \), and enthalpy density \( w(x; [\varphi]) \) given by (3.67). Eq.(3.63) is the continuity equation
\[ \dot{\varphi}(x, t) + \nabla \cdot (\varphi(x, t) \mathbf{v}(x, t)) = 0, \] (3.68)
and (3.66) is just Newton’s law
\[ \partial_t \mathbf{v}(x, t) + (\mathbf{v}(x, t) \cdot \nabla) \mathbf{v}(x, t) = -\nabla w \] (3.69)
for isentropic flow, since in such flows the adiabatic variation of the enthalpy is related to the variation of the pressure \( P \) according to \( dw_s = V dP = dP/\varphi \). The fluid dynamical interpretation of the collective field equations of motion (3.63) and (3.64) is by no means special to our model. As we have already mentioned in the introduction, there is an analogous interpretation of the collective field of hermitean matrices in terms of a one-dimensional fluid [25, 26]. This is certainly a generic feature of field theoretic hamiltonians with a kinetic term of the form \( \int dx \nabla \pi(x) \cdot \varphi(x) \nabla \pi(x) \).

\(^{13}\)See the discussion following (3.29).
4 The Ground State Energy and Distribution of Eigenvalues

As was discussed in the previous section, \( \varphi(x, t) \), \( \pi(x, t) \) and the dynamics governed by (3.61) become classical in the large-\( N \) limit. More precisely, on general theoretical grounds [29], the quantum theory defined by (2.13) (and (3.61)) contains a special set of generalized coherent states, which may be used to construct a classical phase space and to derive a classical hamiltonian, whose resulting classical dynamics is equivalent to the large-\( N \) limit of the original quantum theory. The large-\( N \) ground state is then found by minimizing this classical hamiltonian.

The solution of the classical equations of motion derived from (3.61) for \( \varphi(x, t) \) and \( \pi(x, t) \), which minimizes \( H_{coll} \), should be interpreted as the expectation values of these fields in the ground state of (3.61). Evidently, the ground state (or vacuum) expectation value of \( \pi(x, t) \) should be null, and that of \( \varphi(x, t) \) should be time independent. Thus, setting \( \pi_0(x, t) = \langle 0 | \pi(x, t) | 0 \rangle = 0 \) and \( \langle 0 | \varphi(x, t) | 0 \rangle = \varphi_0(x) \) into the classical equations of motion, we see that (3.63) is satisfied trivially, while (3.64) leads to the equation

\[
\delta \left( \frac{1}{2} \varphi(x') \mathcal{E}(x')^2 + \varphi(x') \mathcal{V}(x'^2) - \lambda \varphi(x') \right) \bigg|_{\varphi_0} = 0. \tag{4.1}
\]

Eq. (4.1) is just the statement that \( \varphi_0(x) \) is the minimum of the effective potential

\[
V_{eff} = \int dx \left( \frac{1}{2} \varphi(x) \mathcal{E}(x)^2 + \varphi(x) \mathcal{V}(x^2) - \lambda \varphi(x) \right). \tag{4.2}
\]

The external potential \( \mathcal{V}(x^2) \) in our model is rotation-invariant. Thus, we would expect that the eigenvalue distribution in the ground state should be rotation invariant as well. Furthermore, in a physically sensible model, the external potential \( \mathcal{V}(x^2) \) must be confining, in order to balance the electric repulsion of the eigenvalues. The compromise between repulsion and confinement should produce a rotation-invariant charge density which is non-vanishing only inside a disk of finite radius \( R \).

In other words, it is sufficient to look for the minimum of (4.2) in the class of normalized density functions \( \varphi \) which depend only on \( r = |x| \), and vanish outside a
disk of some radius $R$, which is fixed by the normalization condition

$$2\pi \int_0^R \varphi(r) r \, dr = 1. \quad (4.3)$$

A radial charge density $\varphi(r)$ is the source for a radial electrostatic field $E(x) = E(r)\hat{r}$ for which Gauss’ law reads

$$\nabla \cdot E = \frac{1}{r} \frac{\partial}{\partial r} (rE(r)) = 2\pi \varphi(r), \quad (4.4)$$

and from which it follows that

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\delta E(r)}{\delta \varphi(r')} \right) = 2\pi \delta(r - r'). \quad (4.5)$$

We stress that (4.5) holds only inside the support of $\varphi(r)$, i.e., only in the region $r \leq R$. For $r > R$, the radial field

$$E(r) = \frac{1}{r} \quad (4.6)$$

identically, for any $\varphi(r)$, rendering the variational derivative null in that region.

Upon integrating (4.5) we obtain

$$r \frac{\delta E(r)}{\delta \varphi(r')} = 2\pi r' \theta(r - r') \theta(R - r'), \quad (4.7)$$

where we used the fact that $E(r)$ arises only due to charges inside the disk of radius $r$ to fix the integration constant.

Thus, evaluating the effective potential (4.2) on the class of radially dependent densities, and taking the variation with respect to $\varphi(r)$, we obtain, using (4.7),

$$\frac{\delta V_{eff}}{\delta \varphi(r)} = 2\pi r \left[ \frac{1}{2} E^2(r) + \int_r^R 2\pi \varphi(r')E(r') \, dr' + V(r^2) - \lambda \right]. \quad (4.8)$$

Therefore, the desired minimum condition is

$$\frac{1}{2} E^2(r) + \int_r^R 2\pi \varphi(r')E(r') \, dr' + V(r^2) - \lambda = 0. \quad (4.9)$$
We can achieve considerable simplification of (4.9) by applying \( \partial_r \) to both its sides and then use Gauss’ law (4.4). We thus obtain the simple and elegant formula

\[
\mathcal{E}^2(r) = r \frac{\partial}{\partial r} \mathcal{V}(r^2), \quad r \leq R
\]

for the electric field inside the condensate of eigenvalues.

By substituting the left-hand side of (4.4) for \( 2\pi \varphi'(r') \) in (4.9), integrating, and using (4.10), we obtain an equation for the Lagrange multiplier \( \lambda \):

\[
\frac{1}{2} \mathcal{E}^2(R) + \mathcal{V}(R^2) - \lambda = 0.
\]

(4.11)

Note from (4.10) that since \( \mathcal{E}^2(r) \geq 0 \), the eigenvalues can condense only in regions where \( \mathcal{V}(r^2) \) is increasing (inside the disk \( r \leq R \)). This is obviously the result of electric repulsion among eigenvalues: in a region in which \( \mathcal{V}(r^2) \) is decreasing, the external force pushes the eigenvalues away from the origin towards larger radii, where they can keep farther apart from each other, until they reach a region in which \( \mathcal{V}(r^2) \) is increasing and the external force is pointed inwards, which eventually confines them. This effect is demonstrated most clearly in the model in which the eigenvalues are confined inside a rigid disk of radius \( R \), namely, \( \mathcal{V}(r^2) = 0 \) for \( r < R \) and infinite elsewhere. Evidently, in this case the eigenvalues are all stuck to the wall at \( r = R \).

A generic external potential \( \mathcal{V}(r^2) \) might have several local wells inside the disk \( r \leq R \), some (or all) of which might be occupied by eigenvalues. The eigenvalues will reside, of course, only in regions inside these wells in which \( \mathcal{V}(r^2) \) is increasing. Let us assume that \( \varphi_K(r) \) is an extremal eigenvalue density in which there are \( K \) occupied wells. Consider the \( k \)th well \((k = 1, \ldots, K)\). The eigenvalues will occupy in this well an annular region \( r_k^{\min} \leq r \leq r_k^{\max} \) (and of course, \( r_K^{\max} = R \)). Thus, the eigenvalues in such a case will condense into a system of \( K \) concentric annuli, with voids in between the annuli. Inside the \( k \)th annulus, the electric field is given by (4.10). In the void \( r_k^{\max} \leq r \leq r_{k+1}^{\min} \) between the \( k \)th and \( k + 1 \)st annuli, the electric field is obviously

\[
\mathcal{E}_k^{(\text{void})}(r) = \frac{\nu(r_k^{\max})}{r},
\]

(4.12)
where \( \nu(r_k^{\text{max}}) = 2\pi \int_0^{r_k^{\text{max}}} \varphi(r) r dr \) is the fraction of eigenvalues confined up to \( r_k^{\text{max}} \).

For \( k = K \), (4.12) coincides with (4.6). It follows from (4.4) that \( \mathcal{E}(r) \) is continuous wherever \( \varphi(r) \) is continuous. Thus, assuming \( \mathcal{V}(r^2) \) has continuous derivatives, the electric field \( \mathcal{E}(r) \) should be continuous throughout the plane. The \( K \)-annular extremal eigenvalue density \( \varphi(r) \) depends upon the \( 2K \) parameters \( r_k^{\text{min,max}} \), and the continuity conditions on the electric field at the \( 2K \) boundaries of the annuli provide the exact number of equations to determine these parameters. (The first of these conditions is of course \( \mathcal{E}(r_1^{\text{min}}) = 0 \), and the last one \( \mathcal{E}(r_K^{\text{max}} = R) = 1/R \) is equivalent to the overall normalization condition (4.3).)

Given the external potential mentioned above, the extremum condition (4.9) might have several solutions \( \varphi_K(r) \), depending on the number \( K \) of occupied wells. In order to decide which of these extrema is the actual ground state configuration, we should evaluate the effective potential (4.2) at each of these extremal solutions and pick the one with minimal energy. By tuning the couplings in \( \mathcal{V}(r^2) \) we might induce quantum phase transitions between these possible multi-annular extremal configurations.

In this paper we shall not pursue the generic case any further, with its multi-annular eigenvalue density configuration, and the fascinating possibility of inducing quantum phase transitions in this kind of matrix models. Instead, in what follows, we will solve for the ground state energy and eigenvalue distribution of the generic monotonically increasing external potentials, which necessarily yields a disk-like eigenvalue density configuration. However, in the explicit examples which will follow this exposition, we will compute the disk-annulus phase transition for the quartic potential.

Thus, assume \( \mathcal{V}(r^2) \) is such that \( \varphi_0(r) \) is voidless and supported in the disk \( r \leq R \). The eigenvalues behave like positive point charges in our electrostatic description of the matrix model. Thus, we should take the positive root of (4.10) for

\[
\mathcal{E}(r) = \sqrt{\frac{\partial}{\partial r} \mathcal{V}(r^2)}, \quad r \leq R.
\]

\(^{14}\)Note that once the \( 2K \) parameters \( r_k^{\text{min,max}} \) are set, all the \( \nu_k \) are determined, since inside the filled annuli \( \varphi(r) \) is given by (4.4) and \( \mathcal{E}(r) \) by (4.10).
From the continuity of $\mathcal{E}(r)$ at $r = R$ and (4.6), we obtain

$$RE(R) = 1,$$  \hspace{1cm} (4.14)

which together with (4.13) forms an equation for $R$. (Equivalently, we can derive this equation by substituting Gauss' law (4.4) into the normalization condition (4.3).)

Then, using (4.4), we can determine the eigenvalue density in the ground state as

$$\varphi_0(r) = \frac{\theta(R - r)}{2\pi r} \frac{\partial}{\partial r} \left( r^3 \frac{\partial}{\partial r} \mathcal{V}(r^2) \right)^{\frac{1}{2}}. \hspace{1cm} (4.15)$$

From (4.11) and (4.14) we obtain

$$\lambda = \frac{1}{2R^2} + \mathcal{V}(R^2) \hspace{1cm} (4.16)$$

for the Lagrange multiplier $\lambda$. Finally, the ground state energy $E_0$ is found by substituting (4.13)-(4.15) (and $\pi_0(x, t) = 0$) in the collective hamiltonian (3.61), and using the normalization condition (4.3). After some algebra, we obtain

$$E_0 = N^2 \int_0^R dr \left( \frac{\partial}{\partial r}(r\mathcal{E}(r)) \right) \left( \frac{1}{2} \frac{\partial \mathcal{V}(r^2)}{\partial r} + \mathcal{V}(r^2) \right)$$

$$= N^2 \int_0^{R^2} du \frac{\partial(u\mathcal{V}(u))}{\partial u} \frac{\partial}{\partial u} \left( 2u^2 \frac{\partial}{\partial u} \mathcal{V}(u) \right)^{\frac{1}{2}}. \hspace{1cm} (4.17)$$

### 4.1 Examples

We end this paper by applying the formalism developed in this section in two explicit examples.

#### 4.1.1 The Quadratic Potential

Consider the normal matrix model with quadratic potential (2.21), namely,

$$\mathcal{V}(r^2) = \frac{1}{2} m^2 r^2 \hspace{1cm} (4.18)$$
with \( m^2 > 0 \). This is essentially the model studied recently in detail in [23], and our purpose here is simply to show that the general formalism developed above reproduces the results of [23] to leading order in \( \frac{1}{N} \).

From (4.13) - (4.15) we obtain

\[
E(r) = mr
\]  

(4.19)

and

\[
\varphi_0(r) = \frac{m}{\pi} = \text{const.}
\]  

(4.20)

in a disk of radius

\[
R = \frac{1}{\sqrt{m}}.
\]  

(4.21)

Our result (4.20) is in agreement with Eq.(17) of [23] (with their \( \lambda = 1 \)). The ground state energy as found from (4.17) is

\[
E_0 = \frac{1}{2}N^2m,
\]  

(4.22)

in agreement with (2.27) and also with Eq.(15) of [23], to leading order in \( \frac{1}{N} \).

### 4.1.2 The Quartic Potential

Next we consider the quartic potential, which gives rise to a nonuniform ground state eigenvalue distribution \( \varphi_0(r) \). In this case the eigenvalues condense either into a disk or an annulus, depending on the sign of the coefficient of the quadratic term. The disk phase occurs for a positive coefficient of \( r^2 \), and the annular phase occurs for a negative coefficient.

#### (a) The Disk Phase

Consider the quartic potential

\[
\mathcal{V}(r^2) = \frac{1}{2}m^2r^2 + \frac{g}{4}r^4
\]  

(4.23)

with \( m^2, g > 0 \). From (4.10), the electric field is

\[
\mathcal{E}(r) = mr \left( 1 + \frac{gr^2}{m^2} \right)^{\frac{1}{2}}
\]  

(4.24)
and the density is

\[ \varphi_0(r) = \frac{m}{\pi} \frac{1 + \frac{3}{2} \frac{g r^2}{m^2}}{\sqrt{1 + \frac{g r^2}{m^2}}}, \]  

(4.25)
in a disk of radius \( R \), where \( R \) is obtained by solving (4.14). In the quartic case (4.14) leads to the cubic equation

\[ u^2(u + 1) = a^2 \]  

(4.26)
for the positive dimensionless quantities

\[ u = \frac{g R^2}{m^2} \quad \text{and} \quad a = \frac{g}{m^3}. \]  

(4.27)
The physical root of (4.26) should be chosen so as to have the correct value in the weak coupling and strong coupling limits.

In the weak coupling limit \( a = \frac{g}{m^3} \to 0 \), the electric field (4.24) and eigenvalue density (4.25) tend manifestly to the corresponding quantities in the quadratic case. The root of (4.26) which tends smoothly to (4.21) is \( u \approx a \ll 1 \).

The strong coupling limit \( a \to \infty \) also yields simple expressions. The electric field tends to

\[ \mathcal{E}(r) = \sqrt{g} r^2, \]  

(4.28)
the eigenvalue density tends to

\[ \varphi_0(r) = \frac{3\sqrt{g}}{2\pi} r, \]  

(4.29)
and the radius of the disk tends to

\[ R = g^{-\frac{1}{2}}. \]  

(4.30)
The corresponding root of (4.26) is \( u \approx a^2 \gg 1 \).

We shall not go into the solution of the cubic equation (4.26) for arbitrary coupling in any more detail. However, it is worth mentioning that the weak coupling and strong coupling branches of the solution correspond to two different roots of (4.26) each of which becomes real in its turn, which cross at \( a_* = \left( \frac{g}{m^3} \right)_* = \frac{2}{\sqrt{2}} \).
Finally, the ground state energy as found from (4.17) is
\[ E_0 = \frac{mN^2}{4a^2} \left[ \frac{9}{7} ((u + 1)^\frac{7}{2} - 1) - \frac{9}{5} ((u + 1)^\frac{5}{2} - 1) - \frac{1}{3} ((u + 1)^\frac{3}{2} - 1) + ((u + 1)^\frac{1}{2} - 1) \right]. \]  
(4.31)

It is easy to check that in the weak coupling limit \((a, u << 1)\) this expression tends to (4.22), as it should. In the strong coupling limit \((a, u >> 1)\), it tends to \[ E_0 = \frac{9N^2g^\frac{1}{2}}{28} \left( 1 + \mathcal{O}(a^{-\frac{1}{2}}) \right), \]  
(4.32)
which can be verified by plugging \(V(r^2) = \frac{g}{4}r^4\) directly into (4.17). The scaling of the ground state energy in the strong coupling limit as \(g^\frac{1}{2}\) is expected, of course, due to simple dimensional analysis.

(b) The Annular Phase

In the annular phase the quartic potential is
\[ V(r^2) = -\frac{1}{2} \mu^2 r^2 + \frac{g}{4} r^4, \]  
(4.33)
with \(\mu^2, g > 0\). Then, from (4.10), the electric field is
\[ \mathcal{E}(r) = \mu r \left( \frac{g r^2}{\mu^2} - 1 \right)^{\frac{1}{2}} \]  
(4.34)
and the density is
\[ \varphi_0(r) = \frac{\mu}{\pi} \sqrt{\frac{g r^2}{\mu^2} - 1}, \]  
(4.35)
in an annulus \(r_{min} \leq r \leq R\).

The radius of the inner boundary \[ r_{min}^2 = \frac{\mu^2}{g} \]  
(4.36)
is found from the requirement that \(\mathcal{E}^2(r)\) be positive. Note that near the inner boundary the density diverges as
\[ \varphi_0(r) = \frac{g^{-\frac{1}{2}}}{\pi} \left( \frac{\mu}{2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{r - r_{min}}} + \mathcal{O}(\sqrt{r - r_{min}}). \]  
(4.37)
The outer radius $R$ is obtained by solving (4.14). The latter leads, in this case, to the cubic equation
\begin{equation}
  u^2(u - 1) = a^2
\end{equation}
for the positive dimensionless quantities
\begin{align*}
  u &= \frac{g R^2}{\mu^2} \\
  a &= \frac{g}{\mu^3},
\end{align*}
in a similar manner to the disk phase.

As in the disk phase, the physical root of (4.38) should be chosen so as to have the correct value in the weak coupling and strong coupling limits.

In the annular phase, unlike the disk phase, the weak coupling limit $a = \frac{g}{\mu^3} \to 0$ is singular, since at $g = 0$, the potential (4.33) is unbounded from below and monotonically decreasing. The corresponding root of (4.38) in this limit is $u = 1 + a^2 + \mathcal{O}(a^4)$ which implies $R = r_{\text{min}}(1 + \frac{a^2}{2} + \mathcal{O}(a^4))$. Thus, in the weak coupling limit the eigenvalues condense into a narrow annulus of very large radius $\simeq r_{\text{min}} = \frac{1}{\sqrt{\mu a}}$ and width $r_{\text{min}}(\frac{a^2}{2} + \mathcal{O}(a^4)) \simeq \frac{1}{2} \left(\frac{a^2}{\mu}\right)^{\frac{3}{2}}$.

The strong coupling limit $a \to \infty$ is essentially the same as in the disk phase, as given by (4.28) - (4.30), and the corresponding root of (4.38) is $u \simeq a^{\frac{3}{4}} \gg 1$. In particular, note that in the strong coupling limit, the radius $r_{\text{min}}$ of the inner boundary shrinks like $g^{-\frac{2}{3}}a^{-\frac{2}{3}} = Ra^{-\frac{2}{3}}$, and the coefficient of the singularity (4.37) near the inner boundary diminishes like $g^{\frac{1}{4}}a^{-\frac{1}{2}}$, so that the voidless bounded density (4.29) will be obtained in the limit.

We shall not go into the solution of the cubic equation (4.26) for arbitrary coupling in any more detail, except for mentioning the fact that (4.38) has only one real root, which interpolates smoothly between the weak and strong coupling limits.

Finally, the ground state energy in the annular phase can be computed in a manner similar to the derivation of (4.17). One finds
\begin{equation}
  E_0 = N^2 \int_{r_{\text{min}}}^{R^2} du \frac{\partial (u \mathcal{V}(u))}{\partial u} \frac{\partial}{\partial u} \left(2u^2 \frac{\partial \mathcal{V}(u)}{\partial u}\right)^{\frac{1}{2}},
\end{equation}
which leads to the explicit expression

\[ E_0 = \frac{\mu N^2}{4a^2} \left[ \frac{9}{7}(u - 1)^{\frac{7}{2}} + \frac{9}{5}(u - 1)^{\frac{5}{2}} - \frac{1}{3}(u - 1)^{\frac{3}{2}} - (u - 1)^{\frac{1}{2}} \right]. \] (4.41)

In the strong coupling limit \((a, u >> 1)\), it tends to (4.32), namely the strong coupling limit of the disk phase.

In the weak coupling limit \((a << 1, u \simeq 1)\), on the other hand, (4.41) tends to the singular limit

\[ E_0 = -\frac{\mu N^2}{4a} \left( 1 + \mathcal{O}(a^2) \right) \simeq -\frac{N^2\mu^4}{4g}. \] (4.42)

Thus, in the weak coupling limit, \(E_0\) tends to \(-\infty\), symptomatic of the fact that the model (4.33) does not exist at \(g = 0\).

(c) A Few Comments on the Disk-Annulus Phase Transition

Starting from a given point \((m^2, g)\) in the disk phase, with the potential (4.23), we can induce a quantum phase transition into the annular phase, by holding \(g\) fixed and tuning \(m^2\) down all the way through \(m^2 = 0\) into negative values. The electric field \(E(r)\), eigenvalue density \(\varphi_0(r)\) and ground state energy \(E_0\) are all continuous through the transition point at \(m^2 = \mu^2 = 0\). At the transition, they are given by the strong coupling formulas (4.28)-(4.30) and (4.32). Moreover, the annular eigenvalue distribution starts with a vanishing inner radius \(r_{\text{min}}\) at the transition.

This behavior is also expected of the disk-annulus phase transition in the zero dimensional (i.e., time independent) quartic ensemble of normal matrices.

It is also interesting to compare the disk-annulus phase transition here with the corresponding transition in the zero dimensional quartic ensemble of complex matrices, studied in [8] and [9]. In the latter case, the Green’s function (the resolvent) for the complex matrix \(M\) (i.e., the electric field, see (3.41)), the eigenvalue distribution in the complex plane, the Green’s function for \(M^\dagger M\) and the density of singular values of \(M\) related to it, and thus the “free energy” of the Dyson gas of singular values, were all continuous through the transition. However, unlike the situation here, the disk phase there invaded well into the \(\mu^2 > 0\) region in parameter space, and moreover,
at the transition, the disk fragmented into an annulus of finite critical inner radius. (This occurred by having a central part of the disk, of finite size, progressively depleted, as the transition point was approached from the disk phase.) Also, the density of eigenvalues at the boundaries of either the disk or the annulus was always finite, unlike (4.37).
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