COMPUTING HITTING TIMES VIA FLUID APPROXIMATION: 
APPLICATION TO THE COUPON COLLECTOR PROBLEM

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Abstract. In this paper, we show how to use stochastic approximation to 
compute hitting time of a stochastic process, based on the study of the time 
for a fluid approximation of this process to be at distance $1/N$ of its fixed 
point.

This approach is developed to study a generalized version of the coupon 
collector problem. The system is composed by $N$ independent identical Markov 
chains. At each time step, one Markov chain is picked at random and performs 
one transition. We show that the time at which all chains have hit the same 
state is bounded by $c_1 N \log N + c_2 N \log \log N + O(N)$ where $c_1$ and $c_2$ are 
two constants depending on eigenvalues of the Markov chain.

1. Introduction

The coupon collector is a classical problem in probability theory. There are $N$ 
types of coupons. Coupon are collected at random with replacement. The goal is to 
compute the number of coupons to be collected to have at least one coupon of each 
kind. This simple problem has a simple answer and a simple proof: on average, 
one has to buy $N/N + (N - 1)/N + \cdots + 1/N \approx N \log N$ coupons to complete 
a collection: if you already have $k$ different types of coupons, it takes in average 
$(N - k)/N$ to get a coupon of a new type.

Because of its simplicity, this problem has many applications, especially in com-
puter science where it often serves as a basic tool for computing the completion 
time of randomized algorithm [11, 8]. Many variants of it have been studied during 
the years. For example, the time needed to complete a $T$ collections of the same $N$ 
coupons is shown to be $N(\log N + (T - 1) \log \log N + O(1))$ in [13, 12]. The time to 
complete the first collection is $N \log N$. However, the time to complete each next 
collection is only $N \log \log N$.

However, even a slight modification such has obtaining $T$ different collections 
instead of one leads to much more complicated proofs. The approach taken in this 
paper aims at being more general but also at giving a new insight on the relation 
between hitting time and stochastic approximation.

Contributions. We develop an approach based on stochastic approximation to 
compute the hitting time of a stochastic process that has an absorbing state. The 
system is composed of $N$ identical Markov chains that have an absorbing state $0$. 
At each time step, one chain is picked at random and performs one transition. Our 
goal is to compute the number of steps until which all Markov chains are in their 
absorbing state. The coupon collector problem is a particular case of this problem
by considering $N$ deterministic Markov chains that have state 1 (no coupon of that type has been collected) or 0 (at least one coupon of type $i$ has been collected).

Using a classical stochastic approximation approach, like [2], one can show that if $N$ is large, the proportion of Markov chains that are in a given state can be approximated by a linear ordinary differential equation (ODE) $\dot{m} = mQ$. This ODE has a unique fixed point to which all trajectories converge exponentially fast, corresponding to a state where all chains are in state 0. However, this approximation is not enough accurate to bound the hitting time of the stochastic process. The time for this ODE to reach its equilibrium is infinite whereas the expected time for the stochastic system to hit this equilibrium is finite.

In this paper, we establish a relation between the expected time $E[T_N]$ for all the chains to be completed and the time $t_N$ for the ODE to be at distance $1/N$ of its equilibrium point. The main results of this paper are Theorem 1 and Theorem 2. We first show that $T_N$ is bounded by $N \cdot t_N$. Using this result, we derive the existence of constants $c_1, c_2$ that depend on the spectral properties of the original Markov chain such that

$$E[T_N] \leq N t_N + O(N) \leq c_1 N \log N + c_2 N \log \log N + O(N).$$

Applied to the time to complete $T$ collections, the allows to derive directly the results of [13].

We also study two particular cases for which we have simple closed-form bounds for $T_N$. In the more general case, if we only know that the expected hitting time of one Markov chain is bounded by $T$ starting from its initial state, then $T_N$ is only bounded by $N^2 T$. However, if the expected hitting for a Markov chain is bounded by $T$ independently of its initial state, then we show that $E[T_N] \leq NT \log N + O(N)$. We provide examples that show that these bounds are tight up to a linear term. Finally, we also show that this method can be applied to study the completion time of distributed algorithm.

We believe that the interest of this method is twofold. On the one hand, it gives a new insight on the coupon collector by providing a new proof of a more general results. We also think that this results could be adapted to more general stochastic approximation algorithms and that this could be helpful to understand the relationship between the extinction time of stochastic models and the time for a fluid approximation of it to get close to extinction.

**Related work.** Stochastic approximation algorithm have been introduced in [14] for solving root finding problems. Application of these methods are scattered on many fields, like economics [3] or computer science [2]. In all of these works, a first step is to show that the stochastic system can be approximated on any finite time interval by a fluid approximation—e.g. described by a differential equation. Then, this approximation is used to derive asymptotic properties such as characterizing the limiting dynamics [1], computing approximation of the steady-state distribution [3, 2] or proving stability properties [5, 7].

However, there are few results on the relation between the time for a stochastic process to escape a region and the behavior of a fluid approximation of it. In [6], the authors shows that if the time for the differential equation to escape a region is finite, then the time for the stochastic system to escape this region converges to the same value (Theorem 4.3 of [6]). When the differential equation stays far from
the absorbing boundary, the time for the stochastic system to reach its absorbing state can be bounded by large-deviation results, as in [9]. Our case of interest in this paper is that the deterministic system converges to a fixed point but does not reach this point in finite time while the stochastic process does hit this point in finite time.

Outline of the paper. The rest of the paper is organized as follows. In Section 2, we give a formal definition of the problem. Section 3 contains the main results of this paper. We show that $T_N$ is bounded by $N t_N + O(N)$ and derive an asymptotic development of $t_N$. Section 4 establishes a explicit link between $T_N$ and the average completion time of one algorithm. Finally, we show how this can be used to compute the completion time of randomized algorithms in Section 5.

2. Formal description and notations

Let $Y$ be a Markov chain on a finite state space $S = \{0 \ldots S\}$ and let denote $P$ its transition matrix. We assume that this chain has an absorbing state, denoted 0, all the other states being transient. If $I$ denotes the identity matrix, then

$$P - I = \begin{bmatrix} 0 & 0 \\ Q^0 & Q \end{bmatrix}$$

where $Q$ is a non-singular matrix such that for all $i, j$: $Q_{ii} < 0$, $Q_{ij} \geq 0$ and $\sum_j Q_{ij} \leq 0$. $Q^0$ is a vector such that for all $i$, $\sum_j Q_{ij} + Q^0_i = 0$.

We consider a Markov chain on $S^N$ composed by $N$ copies of the original Markov chain. Its state at time $t$ is denoted $(X_1(t), \ldots X_N(t))$. The evolution of the Markov chain is as follows:

1. at each time step, a chain $i \in \{1 \ldots N\}$ is picked uniformly at random
2. the $i$th chain $X_i(t)$ changes its state according to the transition matrix $P$.

The states of the other chains do not change.

Our goal is to compute the time for all chains to hit 0 starting from a state $(x_1 \ldots x_N) \in S^N$. We define $T_N$ this hitting time:

$$T_N \overset{\text{def}}{=} \inf \{t : (X_1(t), \ldots X_N(t)) = (0 \ldots 0)\}.$$

2.1. Notations. For a state $x \in S$, we denote by $e_x$ the line vector that has all coordinates equal to 0 except for the $x$th one which is equal to 1. The vector 1 denotes the column vector with all coordinates equal to 1. For a line vector $\alpha$ and a matrix $P$, $\alpha P$ denotes the classical matrix product. For example, if $P$ is a $S \times S$ matrix, then $\alpha P 1 = \sum_{i=1}^S \sum_{j=1}^S \alpha_i P_{ij}$.

For each state $x \in S$, we denote by $W(x)$ the hitting time of 0 starting from $x$: if $Y$ is a Markov chain of transition probability $P$, then:

$$W(x) \overset{\text{def}}{=} \mathbb{E} [\inf \{t : Y(t) = 0\} \mid Y(0) = x].$$

Using the vector notation above, we have $\mathbb{P} (\inf \{t : Y(t) = 0\} \geq 1 + i \mid Y(0) = x) = e_x Q^i 1$ for all $i \in \mathbb{N}$. Thus, $W$ can also be written $W(x) = \sum_{i=0}^\infty e_x Q^i 1$. 

3. Hitting time and fluid approximation

In this part, we bound the expectation of $T_N$ using a deterministic ordinary differential equation (ODE) approximation. In particular, we show that $\mathbb{E}[T_N]$ is bounded by $N$ times the time $t_N$ for the linear ODE (3) to be at distance $1/N$ from its fixed point plus a term of order $O(N)$. Moreover, this time $t_N$ is of order $\Omega(\log(N))$, showing that the term in $O(N)$ becomes negligible compared to $N t_N$ as $N$ grows.

3.1. A differential equation approximation. For any state $x \in S$ and any time step $k$, we define the quantity $\bar{M}_x^N(k)$ to be the proportion of Markov chains that are in state $x$ at time step $k$:

$$\bar{M}_x^N(k) = \frac{1}{N} \sum_{i=1}^{N} 1_{X_i(k)=x},$$

where $1_{X_i(k)=x}$ equals 1 if $X_i(k) = x$ and 0 otherwise. $\bar{M}^N(k)$ denotes the vector of all $\bar{M}_x^N(k)$ for $x \in S$: $\bar{M}_x^N(k) = \sum_{x \in S} \bar{M}_x^N(k) e_x$ where $e_x$ denotes a unit vector having its $x$th coordinate equal to 1 and the others 0. The process $\bar{M}_x^N(k)$ is a Markov chain: with probability $\bar{M}_i^N(k)$, a chain that is in state $i$ is chosen and goes with probability $P_{ij}$ in state $k$. This shows that the expected variation of $\bar{M}_x^N(k)$ during one time step is:

$$\mathbb{E} [\bar{M}_x^N(k+1) - \bar{M}_x^N(k) | \bar{M}_x^N(k)] = \sum_{x \in S} \sum_{j \neq i} \bar{M}_i^N(k) P_{ij} \frac{1}{N} (e_j - e_i)$$

$$= \frac{1}{N} \bar{M}_x^N(k) Q. \tag{2}$$

The function $f : m \mapsto mQ$ is called the drift of the system.

Let us consider the system of differential equation corresponding to the drift:

$$\dot{m}(t) = m(t) \cdot Q. \tag{3}$$

Equation (2) shows that $M^N(k)$ can be described by a stochastic approximation with constant step size $1/N$: it corresponds to a Euler discretization of the ODE (3) with a random noise $U$ (i.e. such that $\mathbb{E} [U^N(k+1) | \bar{M}_x^N(k)] = 0$)

$$\bar{M}_x^N(k+1) = \bar{M}_x^N(k) + \frac{1}{N} \left( \bar{f}(\bar{M}_x^N(k)) + U^N(k+1) \right). \tag{4}$$

Let us call $M^N(t)$ the state of the system when the time has been rescaled by $t$: $M^N(t) = M^N(\lfloor tN \rfloor)$. Using classical tools of stochastic approximation (Theorem 1 of [2] for example), one can show that if $M^N(0)$ converges in probability to $m(0)$, then $M^N(t)$ converges in probability to $m(t)$ uniformly on $[0;T]$:

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \|M^N(t) - m(t)\| = 0 \text{ in probability.}$$

However, when one wants to compute the hitting time of the 0 by $M^N(t)$, this approximation is not accurate enough and leads to overestimated bounds. In the following, we will see how to link the hitting time of $M^N(t)$ and the time for $m_0(t)$ to be greater than $1 - 1/N$. 

3.2. Link between the hitting time and the time to reach $1/N$. Let us now look at the quantity $M_0^N(t)$ which is the proportion of Markov chain in state 0. The quantity $T_N$ can be defined as

$$T_N = \inf\{t : M_0^N(t) = 1\}.$$

If $m(t)$ is the solution of the ODE (3), one clearly has $\lim_{N \to \infty} m_0(t) = 1$ but unless if $m(t)$ starts exactly with $m_0(t) = 1$, the time to reach $m_0(t) = 1$ is infinite: $\inf\{t : m_0(t) = 1\} = +\infty$.

Due to the discrete nature of $M_0^N(t)$, $M_0^N(t)$ takes values in $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\}$. Thus, when $M_0^N(t)$ is greater than $1 - 1/N$, it is equal to 1. This suggests to introduce $t_N$, the time for the ODE to be such that $m_0(t) \geq 1 - 1/N$:

$$t_N \overset{\text{def}}{=} \inf\{t : m_0(t) \geq 1 - \frac{1}{N}\}.$$ (5)

On Figure 1 are reported two simulations for the coupon collector with 2 cards. We compare the hitting time $t_N$ of the stochastic system for $N = 20$ and $N = 1000$ with the time $t_N$ for the ODE to reach $1 - 1/N$. The time of the stochastic system has been accelerated by $N$. This suggests that $t_N$ is indeed a good estimate of $T_N/N$.

![Figure 1](image.png)

**Figure 1.** Comparison of the hitting time of the stochastic of the system rescaled by $1/N$ and the time for the differential equation to reach $1 - 1/N$ for the coupon collector problem with 2 cards. The smooth curve represents the differential equation $m_0(t)$, the dotted line is the line $1 - 1/N$ and the curve with the jumps represents $M^N(Nt)$ for one sample of the simulation for $N = 20$ or $N = 1000$. For each curve, the hitting time of 1 for the stochastic system is close to the hitting time of $1 - 1/N$ for the deterministic ODE.

Classical stochastic approximation results show that the rate of convergence of $M^N(t)$ to $m(t)$ is of order $O(1/\sqrt{N})$. The bound is too loose to guarantee the convergence of $t_N$ to $T_N$. In the next Theorem 1, we use a slightly different approach to show that $Nt_N$ is indeed a very good approximation of $T_N$.

**Theorem 1.** Let $t_N$ be defined by Equation (5) with $m$ satisfying the differential equation (3) with initial condition $m(0) = \alpha \overset{\text{def}}{=} N^{-1} \sum_{i=1}^{N} e_{x_i}$. Then, hitting time
$T_N$ of $(0 \ldots 0)$ for the stochastic system composed of the $N$ chains starting from $(x_1 \ldots x_n)$ satisfies:

$$
\mathbb{E} [T_N] \leq N \left( t_N + \alpha (I - R)^{-1} 1 + 2 \max_{j,k} Q_{jk}^{-1} \right) = N t_N + O(N),
$$

where $R$ is the matrix defined by $R_{ii} = 0$ and $R_{ij} = -\frac{Q_{ij}}{Q_{ii}}$ for $i \neq j$.

**Proof.** The outline of the proof is as follows. The main idea is to write $T_N$ as the maximum of $N$ dependent random variables that correspond to the time for each chain to reach its absorbing state. Then we establish a relation between the expectation of this maximum and the tail behavior of the marginal distribution of each random variable. The marginal distribution for each chain follows a phase-type distribution. We show in Lemma 1 that its tail behavior can be approximated by the one of a continuous phase-type distribution that leads to the term $t_N$.

Let us pick a chain $i \in \{1 \ldots N\}$ at random. The distribution of the initial state of $i$ is the distribution $\alpha$. If $i$ was alone, the probability for this chain to be in the absorbing state 0 at time step $k$ starting from state $i$ would be $(P^k)_{i,0} = \alpha (I+Q)^k 1$.

When considering the system composed by the $N$ Markov chains, the Markov chain $i$ makes a transition with probability 1/$N$. Thus, the probability for this particular chain to be in its absorbing state 0 at time $k$ is $\alpha (I+N^{-1}Q)^k 1$. Therefore, the time $T_i$ at which the Markov chain $i$ has hit its absorbing state satisfies:

(6) $\mathbb{P} (T^i \geq k) = \alpha (1 + \frac{1}{N} Q)^k 1$.

If $i_1 \ldots i_N$ denotes a random permutation of $\{1 \ldots N\}$, then the time for all the Markov chains to have hit 0 is $T_N = \max_{1 \leq k \leq N} T^{i_k}$. The variables $T^{i_k}$ are identically distributed following the law given by Equation (6). However, these variables are not independent.

Using the union bound and the fact that $\mathbb{P} (T_N \geq k) \leq 1$, we have:

$$
\mathbb{P} (T_N \geq k) \leq \min \left( 1, \sum_{k=1}^{N} \mathbb{P} (T^{i_k} \geq k) \right) = \min \left( 1, N \alpha (1 + \frac{1}{N} Q)^k 1 \right).
$$

Therefore, the expectation of $T_N$ can be bounded by:

(7) $\mathbb{E} [T_N] = \sum_{k=1}^{\infty} \mathbb{P} (T_N \geq k) \leq \sum_{k=1}^{x_N^{-1}} \min \left( 1, N \alpha (1 + \frac{1}{N} Q)^k 1 \right) \leq \sum_{k=1}^{x_N^{-1}} 1 + \sum_{k=x_N}^{\infty} N \alpha (1 + \frac{1}{N} Q)^k 1$.
where $x_N = \min \{ k \in \mathbb{N} : \alpha(1+N^{-1}Q)^k \leq 2/N \}$. Moreover, using that $\sum_{k=x_N}^{\infty} (1+N^{-1}Q)^k = -(1+N^{-1}Q)^{x_N}NQ^{-1}$, we have
\[
\sum_{k=x_N}^{\infty} \alpha(1+N^{-1}Q)^k 1 = -\alpha(1+N^{-1}Q)^{x_N}NQ^{-1} 1 \leq \max_{j,k} (-Q_{j,k}^{-1}) N \sum_{j=1}^{S} \left( \alpha(1+N^{-1}Q)^{\alpha} \right)_j \\
\leq \max_{j,k} (-Q_{j,k}^{-1}) N \cdot \frac{2}{N} \leq 2 \max_{j,k} (-Q_{j,k}^{-1}),
\]
where the Inequality (8) comes from the definitions of $x_N$.

Combining this inequality and the Equation (7), we get:
\[
\mathbb{E}[T_N] \leq x_N + 2N \max_{j,k} (-Q_{j,k}^{-1}).
\]

The quantity $x_N$ is defined by $x_N = \min \{ k \in \mathbb{N} : \alpha(1+N^{-1}Q)^k \leq 2/N \}$. We show in Lemma 1 that:
\[
x_N \leq N \inf \{ t : \exp(tQ) \leq \frac{1}{N} \} + \alpha(I-R)^{-1} 1 = Nt_N + \alpha(I-R)^{-1} 1,
\]
where $R$ is a matrix defined by $R_{ii} = 0$ and $R_{ij} = -\frac{Q_{ij}}{Q_{ii}}$ for $i \neq j$. \qed

### 3.3. Discrete and continuous phase-type distribution.

A random variable such that $\mathbb{P}(X \geq t) = \alpha \exp(tQ) 1$ is said to have a continuous phase-type distribution of parameter $(Q,\alpha)$. Let $Y(t)$ be a Markov chain on $S$ such that the rate of transition from $i \neq 0$ to $j \neq i$ is $Q_{ij}$ and the rate of transition from 0 to $i \neq 0$ is zero. If $\alpha$ is the initial distribution of $Y(0)$, then the time for $Y(\cdot)$ to reach zero follows a phase-type distribution of parameters $(Q,\alpha)$. Similarly, a random variable such that $\mathbb{P}(X \geq k) = \alpha(1+Q/N)^k 1$ is said to have a discrete phase-type distribution of parameter $(1+Q/N,\alpha)$. This corresponds to the time for a discrete Markov chain of transition matrix $1+Q/N$ to reach zero. We refer to [10], Chapter 2 for more definitions and properties of phase-type distributions.

The next lemma shows the relation between the tail of a continuous phase-type distribution of parameter $(Q,\alpha)$ and the tail of a discrete phase-type distribution of parameter $(1+Q/N,\alpha)$.

### Lemma 1.

Let $x_N = \min \{ k \in \mathbb{N} : \alpha(1+N^{-1}Q)^k \leq 2/N \}$ and $t_N = N \min \{ t \in \mathbb{R} : \alpha \exp(Qt) 1 \leq 1/N \}$ be defined as in the proof of Theorem 1. Then:
\[
x_N \leq N \left( t_N + \alpha(1-R)^{-1} 1 \right),
\]
where $R$ is a matrix defined by $R_{ii} = 0$ and $R_{ij} = -Q_{ij}/Q_{ii}$ for $i \neq j$.

**Proof.** Let us consider the Markov chain $Y(\cdot)$ associated with the continuous phase-type distribution of parameter $(Q,\alpha)$. With probability $\alpha_i$, the Markov chain starts in state $i$. If after $k$ jumps, the Markov chain is in state $i$, it stays there for a time $T_{ki}$ exponentially distributed of parameter $-Q_{ii}$ and then jump to a state $j \neq i$ with probability $-Q_{ij}/Q_{ii}$. Thus, the probability of being in state $i$ after $k$ jumps
is \((\alpha R^k)\), where \(R\) denotes the matrix with \(R_{ij} = 0\) and \(R_{ij} = -Q_{ij}/Q_{ii}\) for \(i \neq j\). Therefore, if \(X^N\) is a continuous phase-type random variable of parameter \((\alpha, Q)\), \(X^N\) has the same distribution as:

\[
X = \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki},
\]

where \(U_{ki}\) are (dependent) Bernoulli variables of parameter \((\alpha R^k)\) and \(T_{ki}\) are (independent) exponentially distributed variable of parameter \(-Q_{ii}\).

Similarly, the quantity \(\alpha (1+N^{-1}Q)^k\) corresponds to the probability for a discrete phase-type random variable to be greater than \(k\) and a variable \(X\) with discrete phase-type distribution of parameter \((\alpha, I + Q/N)\) has the same distribution as:

\[
\sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki}^{(N)}
\]

where \(U_{ki}\) are the same variable as before and \(T_{ki}^{(N)}\) are independent geometric random variables of parameter \(-Q_{ii}/N\).

Since \(T_{ki}^{(N)}\) is a geometric random variable of parameter \(-Q_{ii}/N\), for all \(t \in \mathbb{R}^+\), we have:

\[
\mathbb{P} \left( T_{ki}^{(N)} \geq tN \right) = (1 - \frac{Q_{ii}}{N})^{[tN]}
\]

\[
\leq (1 - \frac{Q_{ii}}{N})^{tN + 1}
\]

\[
\leq \exp(-Q_{ii}(t + \frac{1}{N}))
\]

\[
= \mathbb{P} \left( T_{ki} \geq t + \frac{1}{N} \right).
\]

where the last inequality comes from the fact that \(\log(1 + x) \leq x\). This shows that \(N^{-1}T_{ki}^{(N)}\) is less than \(T_{ki} + \frac{1}{N}\) (for the stochastic order). As pointed out, all the \(T_{ki}^{(N)}\) and \(T_{ki}\) are independent in Equations (9) and (10). Therefore, we can assume that \(N^{-1}T_{ki}^{(N)} \leq T_{ki} + \frac{1}{N}\) almost surely. Using that for any positive random variable \(A, B\) and any \(t \in (0; \infty)\) and \(\ell \in [0; t]\), we have:

\[
\mathbb{P} \left( A + B \geq t \right) \leq \mathbb{P} ((A \geq \ell) \cup (B \geq t - \ell)) \leq \mathbb{P} (A \geq t - \ell) + \mathbb{P} (B \geq \ell),
\]

this shows that for any \(\ell:\)

\[
\mathbb{P} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki}^{(N)} \geq Nt \right) \leq \mathbb{P} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki} + \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} \frac{1}{N} \geq t \right)
\]

\[
\leq \mathbb{P} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki} \geq t - \ell \right) + \mathbb{P} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} \geq \ell \right).
\]

By Markov inequality, \(\mathbb{P} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} \geq N\ell \right) \leq \mathbb{E} \left[ \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} \right] / (N\ell),\) with \(\mathbb{E} \left[ \sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} \right] = \sum_k \alpha R^k \mathbf{1} = \alpha (1 - R)^{-1} \mathbf{1}.\) This shows that if \(\ell = \alpha (1 - R)^{-1} \mathbf{1},\) then the second part of (11) is less than \(1/N.\) Moreover, if \(t = \ell + t_N,\)
the first part of (11) is less than $1/N$. This shows that if $t \geq t_N + \alpha(1 - R)^{-1}1$, then:
\[
\mathbb{P}\left(\sum_{k=0}^{\infty} \sum_{i=1}^{S} U_{ki} T_{ki}^{(N)} \geq Nt\right) \leq \frac{2}{N}.
\]
Thus, this shows that $x \leq N(t_N + \alpha(1 - R)^{-1}1)$.

3.4. The logarithmic trend. The quantity $\alpha \exp(Qt)1$ is equal to one minus the cumulative distribution function $F()$ of a continuous phase-type random variable of parameter $(\alpha, Q)$. According to Theorem 2.7.2 of [10], there exist $\gamma > 0$ and $k \geq 0$ such that the density of this variable $f()$ satisfies:
(12) \[ f(t) = \gamma t^k \exp(-\nu t) + o(t^k \exp(-\nu t)), \]
where $-\nu$ is an eigenvalue of $Q$ such that $\nu > 0$ and $k + 1 \geq 1$ is the multiplicity of the eigenvalue $-\nu$.

Equation (12) leads to the logarithmic bound for $T_N$, expressed by the following theorem.

**Theorem 2.** Let $-\nu$ be the eigenvalue of $Q$ with the greatest real part and let $k + 1 \geq 1$ denotes its multiplicity. Then, $\nu$ is real and positive and $T_N$ satisfies:
\[
\mathbb{E}[T_N] \leq \frac{1}{\nu} N \log(N) + \frac{k}{\nu} N \log \log N + O(N).
\]

**Proof.** By Equation (12), the cumulative distribution function $F$ satisfies:
\[
\alpha \exp(Qt)1 = 1 - F(t) = \int_{t}^{\infty} f(s) ds = \gamma t^k \exp(-\nu t) + o(t^k \exp(-\nu t)).
\]
Let $s_N(x) \overset{def}{=} \nu^{-1}(\log(\gamma N) + k \log \log \gamma N - k \log \nu + x)$. For all fixed $x$, we have $s_N(x) = t_N + \nu^{-1}x + o(1)$. Using that $\exp(-\nu s_N(x)) = \frac{1}{\gamma N} (\log \gamma N)^{-k \nu^k} \exp(-x)$, the quantity $s_N(x)^k \exp(-\nu s_N(x))$ is equal to
\[
v^{-k}(\log(\gamma N) + k \log \log \gamma N - k \log \nu + x)^k \frac{1}{\gamma N} (\log \gamma N)^{-k \nu^k} \exp(-x)
\]
\[
= \frac{1}{\gamma N} \exp(-x)(1 + \frac{k \log \log \gamma N - k \log \nu + x}{\log \gamma N})^{-k}.
\]
The last factor of (13) goes to 1 as $N$ goes to infinity. Therefore, if $x > 0$ (or $x < 0$), then (13) is strictly less (or greater) than $1/\gamma N$ if $N$ is large enough. This shows that for all $\epsilon > 0$, if $N$ is large enough, we have
\[
1 - F(s_N(-\epsilon)) < 1/N < 1 - F(s_N(\epsilon)).
\]
This shows that the number $t_N$ such that $t \geq t_N$ implies $\alpha \exp(Qt)1 \leq \frac{1}{N}$, is equal to:
(14) \[ t_N = \frac{1}{\nu} (\log(\gamma N) + k \log \log N - k \log(\nu) + o(1)). \]

Combining (14) and Theorem 1 concludes the proof of the theorem. \qed
3.5. **Application to the coupon collector problem.** Let us consider the classical coupon collector problem: there are $N$ different types of coupon. At each time step, a coupon of type $i$ is picked at random where $i$ is drawn uniformly at random. It has been shown in [13] that the time to collect $T$ coupon of each type is bounded by $N(\log N + (T - 1) \log \log N + O(1))$. In this section, we show that our approach allows one to retrieve this result directly.

![Figure 2](image_url)

**Figure 2.** Markov chains used to represent the coupon collector problem: the state indicates the number of coupons of that remain to be collected.

Let us consider the Markov chain represented on Figure 2(a). Its state space is $\{0 \ldots T\}$. The initial state is $T$ and for all $0 < i \leq T$: $P_{i,i-1} = 1$. The matrix $Q$ corresponding to this Markov chain is a $T \times T$ matrix that has $-1$ on its diagonal and $1$ on its sub-diagonal. On Figure 2(b) is represented the particular case for $T = 1$.

The ODE corresponding to this system is:

\[
\begin{align*}
\dot{m}_T(t) &= -m_T(t) \\
\dot{m}_i(t) &= -m_i(t) + m_{i+1}(t) \quad \text{for } 0 < i < T
\end{align*}
\]

with $m_T(0) = 1$ and $m_i(t) = 0$ for $i \in \{0 \ldots T - 1\}$.

A direct computation shows that $m_0(t)$ is the cumulative distribution function of an Erlang variable of parameter $(T, 1)$ (i.e. the sum of $T$ i.i.d. exponential variable of parameter $1$) which can be written:

\[
m_0(t) = 1 - \sum_{k=0}^{T-1} \exp(-x)x^k/k!
\]

\[
= 1 - \exp(-x)x^{T-1}/(T-1)! + O(\exp(-x)x^{T-2}).
\]

Using Theorem 1, this shows that the time $T_N$ to collect $T$ cards of each type is bounded by

\[
\mathbb{E}[T_N] \leq \log N + (T - 1) \log \log N + (T + 2)N + O(1).
\]

where $T + 2$ comes from $Q_{ij}^{-1} = -1_{j \leq i}$ and $1 - R = -Q$.

4. **Explicit formula for two particular cases**

Theorem 2 gives a precise idea on the behavior of $T_N$ in the general case. However, the computation of the constants $\nu$, $k$ or $O(N)$ can be difficult when the state space of the original Markov chain is large. In this section, we derive explicit formulas for these constants assuming that the hitting time for one Markov chain is bounded by $T$.

We first show that if the hitting time of the absorbing state is bounded by $T$ for all single chain, then $T_N$ is less than $N^2T$ (Theorem 3), which is a loose bound in many cases. When the hitting time of the absorbing state is uniformly bounded
by $T$ for all initial states $x \in S$, then $\mathbb{E}[T_N]$ is bounded by $TN \log N + O(N)$ (Theorem 4). At the end of the section, we provide two examples that shows that these bounds are tight.

The results presented in this section remain valid if the state space of the chain is countable instead of finite.

4.1. Unbounded case. If $W(x)$ denotes the expected hitting time of 0 for a single Markov chain, then the following results hold.

**Theorem 3.** The time $T_N$ such that all the chains have reached 0 is bounded by:

$$\mathbb{E}[T_N] \leq N \sum_{i=1}^{N} W(X_i(0)).$$

In particular, if for all $i$, $W(X_i(0)) \leq T$, then

$$\mathbb{E}[T_N] \leq TN^2.$$  

**Proof.** For all $i \in \{1 \ldots N\}$, let us call $R_i$ the time at which the Markov chain $i$ reaches 0: $R_i = \inf\{t : X_i(t) = 0\}$. It should be clear that $T_N = \max_{1 \leq i \leq N} R_i \leq \sum_{i=1}^{N} R_i$. Moreover, the hitting time for just one Markov chain when it performs one transition at each time is $W(X_i(0))$. As the probability for a Markov chain to perform one transition during one time step is $1/N$, we have $\mathbb{E}[R_i] = NW(X_i(0))$. □

This result seems to be in contradiction with Theorem 2 that shows that if we fix a Markov chain, the expected hitting time of a system composed of $N$ of these Markov chains is bounded by $O(N \log N)$. However, the constant hidden in $O(N \log N)$ depends on the Markov chain and the trend in $N \log N$ is only valid when the number $N$ goes to infinity while the bound $TN^2$ only depends on $T$. At the end of the section, we provide an example that shows that this bound is tight.

The Markov chain used for this example depends on $N$.

4.2. Uniformly bounded case $\sup_{x \in S} W(x) \leq T$. On Figure 3(a), we present a Markov chain that the bound of the previous theorem is tight. This chain has a very particular shape: starting from the initial state, the hitting time of the absorbing state is 1 with a probability $1 - 1/N^2$. With probability $1/N^2$, the chain jumps into a state from which it takes $N^2(T - 1)$ steps to hit 0. This later causes the hitting time to be large when multiple chain.

In this section, we show that if there are no such problematic states, the bound on $T_N$ can be improved dramatically. More precisely, we show that if the hitting time of 0 is bounded by $T$ independently of the initial state -- $\sup_{x \in S} W(x) \leq T$, then $T_N$ is of order $TN \log N$.

**Theorem 4.** If the hitting time for one chain is uniformly bounded by $T$ (i.e. $\sup_{x \in S} W(x) \leq T$), then the time $T_N$ such that all the chains have reached 0 satisfies:

$$\mathbb{E}[T_N] \leq TN \log N + 2NT + 1$$

**Proof.** Let $\mathcal{F}_t$ denotes the filtration associated to the process $X(t)$ and let us define the potential of the system at time $t$, $\Phi_t$ by:

$$\Phi(t) \overset{def}{=} \frac{1}{T} \sum_{i=1}^{N} W(X_i(t)).$$
$T_N$ is the time at which all $X_i(t)$ are equal to 0 and can be written $T_N = \inf\{t : \Phi(t) = 0\}$. In the following, we first show that the time for $\Phi(t)$ to be lower than 1 is less than $NT \log N + NT + 1$ using [15]. Then, we use Theorem 3 to bound the remaining time by $NT$.

We say that a Markov chain is active if it did not reach 0. If an active Markov chain is picked in step 1, then the potential will decrease in expectation by $1/T$. Let $\alpha(t)$ denotes the number of active Markov chains at time $t$ (i.e. $\alpha(t) = \sum_{i=1}^{N} 1_{X_i(t) \neq 0}$). The probability of picking an active Markov chain is $\alpha(t)/N$. Therefore, the expected decrease of the potential between time $t$ and $t+1$ is:

\[
\mathbb{E}[\Phi(t+1) - \Phi(t) \mid \mathcal{F}_t] \leq -\frac{\alpha(t)}{TN}.
\]

By hypothesis, $\sup_{x \in S} W(x) \leq T$. Thus, an active processor contributes at most 1 to the potential and we have $\Phi(t) \leq \alpha(t)$. Combining this with (15), we get:

\[
\mathbb{E}[\Phi(t+1) \mid \mathcal{F}_t] \leq \Phi(t) \left(1 - \frac{1}{NT}\right).
\]

Because of Equation (16), our potential function satisfies the hypothesis of Theorem 1 of [15] with $m = 1$ and $h(r) = 1 - 1/(NT)$. According to this theorem, we have:

\[
\inf\{t : \Phi(t) < 1\} \leq \lambda(\log(\Phi(0)) + 1) + 1.
\]

where $\lambda = -1/\log(1 - 1/(TN)) \leq TN$ and $\Phi(0) \leq N$.

By Theorem 3, when $T_N$ is less than 1, $\sum_{i} W(X_i(t)) \leq T$. Therefore, the remaining time to hit 0 is bounded by $NT$.

\[\square\]

### 4.3. Comparison with previous bounds and tightness

Theorems 3 and 4 need stronger assumptions than Theorems 1 and 2 and are often less precise. However, their main advantage is to give explicit formulas for the hitting time, even if computing the time $T_N$ or the eigenvalue of the individual Markov chains is hard. This fact is important in practical situation where the Markov chains often have a complicated geometry. This is the case for the example of [4] presented in the next Section 5.

The loss of precision of these bounds are well illustrated by the coupon collector problem. Consider the Markov chain of Figure 2(a) that corresponds to the problem of collecting $T$ cards of each type. The hitting time of 0 from any state is clearly bounded by $T$. Therefore, using Theorem 4, one has $\mathbb{E}[T_N] \leq TN(\log N + 2) + 1$. This bounds is worse by a factor $T$ compared with the bound obtained by the ODE approach which was $N(\log N + (T - 1)\log \log N) + O(N)$. This is explained by the fact that Theorem 4 does not take into account the particular shape of the Markov chain of Figure 2(a): the bound of Equation (16) neglects the fact the hitting time starting from state $\{0 \ldots T - 1\}$ is strictly less than $T$.

### 4.3.1. Tightness of the bounds of Theorems 3 and 4

A Markov chain that shows the tightness of the bound of Theorem 3 is represented on Figure 3(a). The chain has $N + 2$ states, denoted $\{0, \ldots, N^2(T - 1)\} \cup \{i\}$. Its initial state is $i$. From $i$, the chain goes with probability $1/N^2$ to state $N^2(T - 1)$ and with probability $1/N^2$ in state 0. From any state $x \in \{1 \ldots N^2(T - 1)\}$, the chain goes to state $x - 1$ with probability 1. For any state $x \in \{0 \ldots N^2(T - 1)\}$, the expected hitting time of
the state 0 is $W(x) = x$. When starting in $i$, the expected hitting time of 0 is $T$.
Therefore, Theorem 3 shows that the expected hitting time of $(0 \ldots 0)$ for a system composed of $N$ of these chains starting in $(i \ldots i)$ is bounded by $TN^2$.

Let us compute a lower bound on the hitting time of $(0 \ldots 0)$ starting from $(i \ldots i)$. With probability $1 - (1 - 1/N^2)^N$, there will be at least one chain that needs $N^2(T - 1)$ transitions to converge. At each time step, this chain makes a transition with probability $1/N$. Thus, in average this chain will take $(T - 1)N^3$ time steps to converge. Since this happens with probability $1 - (1 - 1/N^2)^N$, a lower bound for the hitting time of $(0 \ldots 0)$ starting from everyone in state $i$ is
\[
E[T_N] \geq N^3(T - 1)(1 - (1 - 1/N^2)^N) = (T - 1)(N^2 + o(1)).
\]
This shows that the bound of Theorem 3 is almost tight up to an additive term of $N^2$.

To show that the bound of Theorem 4 cannot be improved much without further assumption, let us consider the Markov chain represented on Figure 3(b): it has two states and the probability of going to 1 to 0 is $1/T$. The expected hitting time of state 0 starting from state 0 is $T$. Theorem 4 implies that $T_N \leq NT(\log N + 2) + 1$. The exact value for $T_N$ is $NT \sum_{i=1}^{N} i^{-1} \approx NT \log N + \gamma NT + o(N)$ where $\gamma \approx .57$ is the Euler–Mascheroni constant. This is close to our theoretical bound up to an additive term of $(2 - \gamma)NT$.

5. Computing completion time of randomized algorithms

In this section, we show how these results can be applied to study the completion time of randomized algorithms and show how this can be used to design efficient distributed protocols.

5.1. Completion time of randomized algorithm. One motivation for this work comes from the study of the time for a set of $N$ distributed randomized algorithm to all finish, in a scenario similar to [4].

Let us consider that we want to solve a resource allocation problem among a population of agent. We assume that we have a randomized algorithm that converges to a stable allocation of the resource that is efficient but not fair among different agents. The final allocation might depend on the random choices done by the algorithm and each allocation favors a different group of agents. In order to improve the fairness of the equilibrium, we consider the following scenario. We execute $N$ independent copies of the algorithm. At each time step, we do a step of computation of one algorithm taken at random among the $N$ algorithms. After some time, the $N$ algorithms will have reached their stable allocation $S_1 \ldots S_N$. 

![Figure 3. Two Markov chains used to show that the bounds of Theorems 4 and 3 are tight.](image-url)
Since each allocation is efficient, the resulting allocation will also be efficient but the resulting allocation will be more fair since at each time step, one allocation is picked at random among $S_1 \ldots S_N$.

If the original algorithm uses bounded memory, it can be represented by a Markov chain with a finite state space. After some time, the Markov chain will reach an absorbing state representing the fact that the algorithm reached a stable allocation. The resulting algorithm can be represented by $N$ independent Markov chains. At each time step, one Markov chain is picked at random and performs one transition. Our framework, and in particular Theorems 4 and 3 can be used to compute the time to reach the final allocation if we know the time taken by a single algorithm to converge.

5.2. Correlated equilibria and distributed protocols. These ideas are applied in [4] to design a distributed algorithm that converges to a fair and efficient allocation of wireless radio channel to a set of user.

Their scenario is the following. There are $U$ users that want to share $C$ wireless channels. The time is slotted and at each time slot, each user can transmit data on one channel. If two or more users are transmitting on the same channel at the same time, there are interferences and no data are received. The only information available to a user before transmitting is whether a given channel was used by one or more users at the previous time slot. Authors of [4] proposed a distributed randomized algorithm that converges to a constant assignment of the $C$ channels to $C$ of these $U$ users while the others $U - C$ users do not transmit at all. This algorithm guarantees a 100% of utilization of the channels but is unfair since $U - C$ users are not transmitting at all. The time of convergence of the algorithm can be bounded by some constant $T$.

In order to improve the fairness of their algorithm, they introduced a centralized entity that sends a correlation signals. At each time step $t$, this entity sends a signal $n(t) \in \{1, \ldots, N\}$ where $N$ is a predefined constant (to be chosen by the entity). The signal $n(t)$ is picked uniformly at random each time. The users keep $N$ copies of the previous randomized algorithm. At time $t$, the users apply the algorithm number $n(t)$. Even if this algorithm is no more completely distributed, it is still scalable: the centralized entity has to broadcast a signal $n(t)$ at time $t$ but it does not have to gather any information from the agents. Moreover, this algorithm improves the fairness of the initial algorithm. As each algorithm is independent, if $N$ is large, each user will be assigned in average to $C/U$ channels. The more $N$ is large, the more fair will be the allocation. However, a large $N$ slows the convergence of the algorithm. An accurate bound on the time of convergence of this algorithm allows one to choose the right compromise between the speed of convergence and the performance of the algorithm.

As the convergence time of each copy of the algorithm is bounded in expectation by some $T$, this model satisfies the hypothesis of Theorem 4. Therefore, the convergence time of the whole algorithm is bounded by $NT \log N + 2NT + 1$.

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