THE LORENTZ-DIRAC EQUATION AND THE STRUCTURES OF SPACETIME

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ABSTRACT

A new interpretation of the causality implementation in the Lienard-Wiechert solution raises new doubts against the validity of the Lorentz-Dirac equation and the limits of validity of the Minkowski structure of spacetime.

1 INTRODUCTION

Finding the correct equation of motion for a pointlike charged classical particle was, early in this century, a major problem in theoretical physics. The proposed third-order Lorentz-Dirac equation could not be accepted because of its numerous problems. These problems have not been solved but just forgotten since with the advent of quantum mechanics came also the hope that they could be properly understood in the scope of a quantum theory. This represented, in the point of view of this paper, a bad cornerstone for theoretical physics: for not solving them, one has failed on seeing that the Minkowski space is not the appropriate underlying geometric structure for the description of close interacting fields. The solution to these problems is still of great relevance since it may signal steering corrections one has to make in field theory for avoiding old problems of QED and the stalling situations found today in, for example, quantum gravity and QCD.

In modern field theories Poincaré invariance is imposed, and then the Minkowski space-time is taken as the appropriate scenario for describing non-gravitational phenomena. For electromagnetic fields in vacuum, far from charges, this has received confirmation from a solid experimental basis, but not for fields in a close vicinity of their sources. Even from a theoretical
viewpoint the question is not so clear: the problems quantum field theories face for dealing with fields defined in close neighboring points are well known. These difficulties are generally taken as indications of some failures in the quantum basis of the theories or of an scale on its limit of validity. In this letter we want first to emphasize that this is the same problem that occurs in classical physics disguised on this old controversy about the correct equation of motion for the classical electron. Having inherited the same spacetime structure of their classical predecessors, it is not a surprise that the quantum theories also face a similar problem for defining fields in a too close vicinity. Therefore, the roots of this problem must be searched at deeper grounds, in the very foundation of the assumed structures of the space-time continuum.

A classical spinless point charge in an isotropic and homogenous Poincaré invariant space-time and the validity of energy momentum conservation lead unequivocally to the Lorentz-Dirac equation,

\[ ma = e F_{\text{ext}} \cdot V + \frac{2e^2}{3}(\dot{a} - a^2 V). \] (1)

This equation is written in a context where the electron world-line, parameterized by its proper time \( \tau \), is a known function, \( z(\tau) \). Then, the electron velocity is \( V = dz/d\tau a = dV/d\tau \), and \( \dot{a} \equiv da/d\tau \). \( e F_{\text{ext}} \) is the exterior force driving the electron, which, if taken as of electromagnetic origin, is put as \( F_{\mu} = F_{\mu\nu} V_{\nu} \). \( m \) and \( e \) are the electron mass and charge, respectively. \( c \) is the speed of light. The presence of the Schott term, \( \frac{2e^2}{3} \dot{a} \), is the cause of some pathological features of (1), like microscopic non-causality, runaway solutions, preacceleration, and other bizarre effects\(^{(4)}\). The adoption of an integral equation with a convenient choice of limits can avoid either one of these two last problems, but not both. On the other hand the presence of this term is necessary for the maintenance of energy-momentum conservation; without it it would be required a null radiance for an accelerated charge. The argument, although correct, that such causality violations are not observable because they are outside the scope of classical physics\(^{(5)}\) and are blurred\(^{(6)}\) by quantum-mechanics effects is not enough compelling, because these same problems remain in a quantum formalism, just disguised in other apparently distinct problems.

It must be added that the inclusion of spin and some extension or structure for the electron would be just a complication without a changing in the essence of the problem. Taking a spinless particle is a valid simplifying hy-
pothesis since the point at stake is not that one must consider every property of the physical electron, but why one gets physically non-acceptable results if one starts from apparently good premises and uses only mathematically sound procedures? It can only mean that something in the premises or in the procedures is not as good as one thinks. The problems that appear in both classical and quantum theories when one has to consider the limit situation of two objects, the electron and its electromagnetic field for example, defined in a same point are the crux of the question. For the classical electron the picture is quite clear: the energy-momentum conservation produces sound physical results in any region around the charge except at the position of the charge. This, it will be argued in the following, is a strong indication of the breaking down of the validity of some accepted premises about the structure of the space-time: electron and photon require different local space-time structure. The failure of recognizing this results in equation (1). It amounts to requiring that the propagation of a massive object (the electron) attend the same constraint of the photon, a massless object.

The geometrization of a physical principle is a very useful tool because it assures its automatic implementation and allows that we concentrate our attention on other aspects of the problem we are studying. The Minkowski spacetime represents a geometrization of the relativistic requirement that the velocity of light be a universal constant. Despite its undisputed success through the Theory of Special Relativity, in the interface between a field and its source, it produces a causality violating description. Revisiting the Lienard-Wiechert solution we show that its implicit causality implementation can be also geometrized; it implies a structure of spacetime more complex than the Minkowskian one and its light-cone structure. This suggested new model of spacetime requires a revision of our concepts of field theories of interacting massive and massless fields, a discussion to be started in a subsequent paper, and shows the weak points in the demonstrations of the Lorentz-Dirac equation, which resumes our goals here.

2 THE LORENTZ-DIRAC EQUATION

The derivation of the Lorentz-Dirac equation, with the use of techniques of distribution theory, can be roughly schematized in the following way. The electromagnetic field $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, with $\partial.A \equiv \partial_\mu A^\mu = 0$, satisfies the
Maxwell’s equations, $\Box F = 4\pi J$, where $J$, given by,

$$J(x) = e \int d\tau V \delta^4[x - z(\tau)],$$

(2)

is the current for a point particle with electric charge $e$ and four-velocity $V$. The Lienard-Wiechert solution\(^{(3,5,7)}\),

$$A = \frac{eV}{\rho}, \quad \rho > 0,$$

(3)

in terms of retarded coordinates, by which any spacetime point $x$ is constrained with a particle world-line point $z(\tau)$ by

$$R^2 = 0$$

(4)

and $R^0 > 0$, with $R \equiv x - z(\tau), \quad \rho \equiv -V\eta.R$, where $\eta$ is the Minkowski metric tensor,(with signature $+2$). $\rho$ is the spatial distance between the point $x$ where the electromagnetic field is observed and the point $z(\tau)$, position of the charge, in the charge rest frame at its retarded time. The total particle and field energy momentum tensor, $T = T_m + \Theta$, consists of

$$T_m = m \int d\tau V V \delta^4[x - z(\tau)]$$

$$\Theta^{\mu\nu} = \frac{1}{4\pi} (F^{\mu\alpha} F_{\alpha}^\nu - \frac{\eta^{\mu\nu}}{4} F^{\alpha\beta} F_{\beta\alpha}),$$

where $F = F_{\text{ret}} + F_{\text{ext}}$ is the retarded field added to any external electromagnetic field acting on the charge. It induces $\Theta = \Theta_{\text{ret}} + \Theta_{\text{mix}} + \Theta_{\text{ext}}$. Mention to some messy calculations\(^1\) related to the highly non integrable parts of $\Theta_{\text{ret}}$, which requires some renormalizations of $\Theta$ on the charge worldline, are being omitted here.

The required momentum conservation,

$$T^{\mu\nu}_{\ \ \nu} = 0,$$

(5)

is satisfied without any problem at any point except at $\rho = 0, \quad (x = z)$, where $T$ is not defined. In order to handle the singularity at $\rho = 0$, $T$ must be treated not as just a function defined only at $\rho > 0$ but as a distribution defined everywhere. Then, (3) is replaced by

$$\int dx^4 T^{\mu\nu}_{\ \ \nu} \phi(x) = - \int dx^4 T^{\mu\nu}_{\ \ \nu} \phi(x) = - \lim_{\varepsilon \to 0} \int dx^4 T^{\mu\nu} \phi(x) \theta(\rho - \varepsilon) = 0,$$

(6)
where $\phi(x)$ is an arbitrary differentiable function with a compact support and $\theta(x)$ is the Heaviside function, $\theta(x > 0) = 1$, $\theta(x < 0) = 0$. Another integration by parts gives

$$\lim_{\varepsilon \to 0} \int dx^4 \rho, \nu T^{\mu\nu} \phi(x) \delta(\rho - \varepsilon) = 0,$$

which, after integration, produces, in the limit, the Lorentz-Dirac equation (7). We want to pinpoint a crucial passage (common to most derivation of this kind) in this procedure for posterior careful analysis: the limit $\varepsilon \to 0$, which represents a change from a point $x$ where there is only electromagnetic field and no electric charge, $\rho > 0$, to a point $z(\tau)$, instantaneous location of the electron, $\rho = 0$.

## 3 GEOMETRY OF CAUSALITY

There is a beautiful and physically meaningful underlying geometry describing the structure of causality in relativistic classical electrodynamics, of which we will give here just a brief description. The Lienard-Wiechert solution (3) is an explicit function of $x$ and of $\tau$, the retarded proper time of its source, solution of the constraint (4). When taking derivatives of functions of retarded coordinates, like (3), the differentiation of the constraint (4) must be considered and it implies (7.9) on $R.dR = 0$, or

$$d\tau + K.dx = 0$$

where $K \equiv R/\rho = -\partial \tau / \partial x$. The effects of this constraints on derivatives of functions of retarded time, like $A$, can be automatically accounted for if each derivative is replaced by a directional derivative,

$$\partial_{\mu} \longrightarrow \nabla_{\mu} \equiv \partial_{\mu} - K_{\mu} \partial / \partial \tau$$

The constraints (4) and (8) have a clear geometrical and physical meaning: the electron and its electromagnetic field must belong to and remain in a same lightcone; they represent, respectively, a global and a local implementation of the relativistic causality. In the standard formalism, which we are reviewing, there is a clear distinction between the treatment given to the electron and
the one given to its electromagnetic field. It is now convenient to adopt a notation where these distinctions are reduced to the essentially necessary. So, we change the notation, replacing $R$ by $\Delta x$ and extending its meaning to be a change in the location of a physical object (particles, fields, etc). Therefore, the constraint $R^2 = 0$ is replaced by

$$\Delta x.\eta.\Delta x = 0,$$

(10)

showing, in an explicit way, that the electromagnetic field, as a massless field, propagates keeping constant its propertime, $\Delta \tau = 0$. K as a null vector, $K^2 = 0$, represents a lightcone generator, the direction of propagation of the electromagnetic field. Equation (4) or (10) can be seen as a restriction on the set of solutions of (8), and both (10) and (8), are constraints to be imposed on the propagation of massless objects. They make sense for the electromagnetic field and as such they have accordingly been used in section 2 for $\rho > 0$, but they cannot be extended to the propagation of a massive object, like an electron. The appropriate constraint, equivalent to (10) for an electron, has to be

$$-(\Delta \tau)^2 = \Delta x.\eta.\Delta x,$$

(11)

where $\Delta \tau$ is the variation of the electron propertime during its propagation along a distance $\Delta x$; likewise the constraint (8) must be replaced by

$$d\tau + V.d\mathbf{x} = 0;$$

(12)

and, similarly, a directional derivative corresponding to (9) is defined, replacing $K$ by $V$:

$$\partial_{\mu} \rightarrow \nabla_{\mu} \equiv \partial_{\mu} - V_{\mu}\partial/\partial\tau$$

(13)

The differences between (9) and (13) just reflects the distinct constraints on the propagation of massive and of massless physical objects.

We are now in condition to define the unifying geometric background that underlies equations (9-13). Consider all the physical objects (electrons, electromagnetic fields, etc) immersed in a flat 5-dimensional space, $R_5 \equiv R_4 \otimes R_1$, whose line elements are defined by

$$(\Delta S_5)^2 = \Delta x^M\eta_{MN}\Delta x^N = (\Delta S_4)^2 - (\Delta x^5)^2 = \Delta x.\eta.\Delta x - (\Delta x^5)^2,$$

(14)

where $M, N = 1$ to 5. Immersed in this larger space, every physical object is restricted to a 4-dimensional submanifold, its SPACETIME, by

$$-(\Delta x^5)^2 = \Delta x.\eta.\Delta x.$$

(15)
Figure 1:

\[(\Delta S_5)^2 = -2(\Delta \tau)^2\] for a physical object, always. In other words, the range of \(x^5\) of a physical object is restricted to the range of its very propertime, \(\Delta x^5 = \Delta \tau\). This is a causality condition, standing for both (10) and (11). So, the constraints on the propagation of physical objects become restrictions on their allowed domain in \(R_5\), that is in the definition of their allowed spacetime. (15) may be written, in an obvious notation, as

\[(\Delta t)^2 = (\Delta \tau)^2 + (\Delta \vec{x})^2,\] which defines a 4-dimensional hypercone in the local tangent space of \(R_5\). See figure 1. It is a CAUSALITY-CONE, a generalization of the Minkowski lightcone.

A lightcone, the domain of a massless physical object, is an intersection of a causality-cone and a 4-dimensional hyperplane of RELATIVISTIC ABSO-
LUTE SIMULTANEITY, defined by: \( x^5 = const \). The interior of a lightcone is the projection of a causality-cone on such a \( (x^5 = const) \)-hyperplane. Each observer perceives an strictly \((1+3)\)-dimensional world and his \( \Delta x^5 \) coincides with the elapsed time measured on his own clock, as required by special relativity; it represents his aging, according to his own clock.

This is in contradistinction to Kaluza-Klein type of theory for unification of fields, which uses a spacelike fifth dimension and then needs a compactification mechanism to justify the non observability of \( x^5 \). The use of a timelike fifth coordinate is, of course, not new in physics. See for example the references [8,9] and the references therein.

A subtle detail must be observed. It is not correct that we are interpreting \( x^5 \) as a proper time; it is \( \Delta x^5 \), the variation of \( x^5 \) of a physical object, that is interpreted as the variation of its proper time, its aging. The propagation of physical objects, in this geometric setting, is restricted by the differential of \( \Delta \tau d\tau + \Delta x dx = 0 \), or

\[
d\tau + f dx = 0, \tag{17}
\]

\( f dx = f_\mu dx^\mu \), where \( f = \frac{\Delta x}{\Delta \tau} \), and \( f \) is a timelike 4-vector if \( d\tau \neq 0 \), or (extending \( \Delta \tau d\tau + \Delta x dx = 0 \) to include) a light-like 4-vector if \( d\tau = 0 \). \( \Delta \tau d\tau + \Delta x dx = 0 \) defines a family of 4-dimensional hyperplanes parameterized by \( f \) and enclosed by the causality-cone \( \Delta \tau d\tau \). \( \Delta \tau d\tau + \Delta x dx = 0 \) defines a causality-cone generator whose tangent, projected on a \( (\Delta x^5 = constant) - \text{hyperplane} \), is \( f \). A lightlike \( f \) corresponds to \( K \) of \( \Delta x^5 \) while a timelike \( f \) stands for \( V \) of \( \Delta \tau d\tau + \Delta x dx = 0 \).

Let us consider the figure 2 in order to have a clear understanding of the meaning of \( x^5 \) of a physical object as its aging.

This figure may represent a vain physicist looking himself at a mirror, or the limiting case \( v \approx c \) of the twin paradox in Special Relativity. \( PR' \) and \( PQ \) belong to a same causality-cone. \( PQ \) belongs to the light-cone (taking \( v \approx c \)). \( PR' = (0,0,0,\Delta \tau, \Delta x^5) \) with \( \Delta t = \Delta \tau \), while \( PQ = (\Delta \vec{r}, \frac{\Delta \vec{r}}{2},0) \) and \( QR = (-\Delta \vec{r}, \frac{\Delta \vec{r}}{2},0) \). \( PR' \) is the physicist world-line on his rest frame. \( \vec{r} \) is the physicist image reflected (back to him) at \( Q \), or his twin brother returning from a trip to \( Q \). They meet again at the time \( t = t_R > 0 \), at the same space point \( \vec{r} = 0 \), from where they had departed from each other, but now with distinct fifth coordinates, \( x^5_R = 0 \) and \( x^5_{R'} > 0 \), that represent their distinct agings.

Let us mention now a rich and interesting point of this geometry. Observe the difference between \( \tau \) and \( t \) in \( \Delta \tau d\tau + \Delta x dx = 0 \): they are invariant under different
Figure 2: The twin paradox.
subgroups of isometry — $SO(3, 1)$ and $O_4$, respectively — of the causality-cone. Both sides of
\[(\Delta \tau)^2 = (\Delta t)^2 + (\Delta \vec{x})^2\]
are invariant under transformation of the $SO(1, 3)$ group, that is, rotation in
a Minkowski spacetime $(\vec{x}, t)$; but in
\[(\Delta t)^2 = (\Delta \tau)^2 + (\Delta \vec{x})^2,\]
both sides are invariant under $O_4$, the rotation group in Euclidian 4-dimensional spacetime $(\vec{x}, \tau)$.

The use of $t$ in the place of $\tau$ as the invariant corresponds to Wick rotation without the need of analytic continuation\(^{(10,11)}\), $t \rightarrow it$, and lends to it a clear physical and geometrical interpretation. Physically it means that, for an Euclidian 4-dimensional spacetime, events should be labelled not by the time measured in the observer’s clock, but with their local proper time, read on their local clocks. $O_4$ is the invariance group of the causality-cone for rotations around its $t$-axis. Care must be taken with the interpretation of the $O_4$ sub-groups involving $\tau$, as they correspond to Lorentz and conformal transformations.

For those unaccustomed to the idea of extra-dimensions, we remind again that the Minkowski spacetime represents the geometrization of an experimentally founded physical principle: the constancy of the speed of light. It requires that the time (up to then, just a parameter) be treated as the fourth coordinate of a 4-dimensional manifold, the spacetime. We are doing here something very similar: the geometrization of causality, embodied in the relations (15) or (16). It requires a fifth coordinate with the role of a propertime.

We can now return to our initial problem, which in this geometry is pictured by an electron and its electromagnetic field in a same causality-cone, each along its own cone-generator. Up to here this is just another picturization of this problem without any real change in its usual dynamics. The first real and fundamental change appears when one considers the metric structure for the spacetime of each physical object. The metric induced by (17) on the spacetime of a physical object, its 4-dimensional submanifold, $(dS_5)^2 = dx.\eta.dx - (f.dx)^2 = dx.(\eta - ff).dx$, is given by $g_{\mu\nu} = \eta_{\mu\nu}$ for a massless field (since then $d\tau = -f.dx = 0$), and by $g_{\mu\nu} = \eta_{\mu\nu} - f.\mu f.\nu$, and $g^{\mu\nu} = \eta^{\mu\nu} + \frac{f.\mu f.\nu}{2}$ for massive physical objects. The distinct causality
requirements of massive and of massless fields and particles are, therefore, represented by immersions with distinct metric structures. They can both be written in a single expression (using either $f^2 = 0$ or $f^2 = -1$ for, respectively massless and massive objects):

\begin{align}
  g_{\mu\nu} &= \eta_{\mu\nu} + f^2 f_\mu f_\nu, \\
  g^{\mu\nu} &= \eta^{\mu\nu} - f^2 \frac{f^\mu f^\nu}{1 + (f^2)^2}.
\end{align}

(18)  

(19)

At this point we can understand that a plus sign in front of $(\Delta \tau)^2$ of the line element (14) would imply on $\Delta S_5 \equiv 0$ for all physical object and would induce $g_{\mu\nu} = \eta_{\mu\nu} + f_\mu f_\nu$, as a metric on a causality-cone of a massive object, which would not be consistent because then, $g_{\mu\nu} f^{\mu} \equiv 0$.

The existence of two distinct metric structures for a massive and a massless field invalidates (1) as the result of (6). While $T^{\mu\nu}_{\rho\nu} = 0$ for $\rho > 0$ remains valid in this new picture, its limit when $\rho \to 0$ is not as simple as described before because it involves now a local change of manifolds with different metric structure ($\eta \to \eta - ff$). Physically it only makes sense! For $\rho > 0$ one is dealing with electromagnetic fields (photons) for which (8,10) represent the causality requirement that $A(x,t)$ and $z(\tau_{ret})$ remain on a same light-cone, but for $\rho = 0$ one has an electron, a massive particle, which must attend a completely different causality relation (11,12). As a matter of fact, the limit of $K$ when $\rho$ tends to zero is an indeterminacy of the type 0/0 that can be resolved with a derivative $d/d\tau$ and the L'Hospital rule:

\begin{equation}
  \lim_{\rho \to 0} K_\mu = \lim_{\rho \to 0} f_\mu = V^\nu \eta_{\nu\mu} \equiv V_\mu.
\end{equation}

(20)

This is coherent with (8) and (13). The geometric meaning of (20) can easily be understood if we remind that K, as a 4-vector tangent to the lightcone, can be written as $K = V + N$, with $V.N = 0$. This change of K to V was not considered in the limit passage of (7), as also, of course, the change of metric required by (19). This procedure extends, in fact, the photon causality constraint (10) to the electron; it corresponds to treating the electron as if it were a massless object. This new vision of spacetime requires a revision not only of the Lorentz-Dirac equation but of any theory of interacting fields. This will be done in a subsequent paper. Our immediate goal has been attained with the stressing of the connections among causality violation in the Lorentz-Dirac equation and the spacetime structure.
4 SUMMARY AND CONCLUSIONS

The Lienard-Wiechert solutions are closely related to the Lorentz-Dirac equation, but while the first have a well drawn picture of causality preservation, based on the light-cone structure of the Minkowski spacetime, the second is, nonetheless, well known for its problematic causality violating solutions. For this reason, this equation has always been accompanied by many doubts about its validity. But it has been obtained from the most diverse approaches and its uniqueness has been scrutinized and proved under very generic and acceptable conditions\(^{(1,2)}\). However, we do not endorse the apparently most accepted view that this is, after all, the correct equation and that its problems appear only when we stretch its application to situations when a quantum theory should be used instead.

With the strategy of geometrizing the Principle of Causality, that is, of transferring its implementation to the background spacetime structure, we find that a model of spacetime, more complex than the Minkowski’s one, is required. It makes clear that the weak point common to all demonstrations of the Lorentz-Dirac equation is the extrapolation for the electron of restrictions that are valid only for its electromagnetic field. The Minkowski spacetime represents just a geometrization of the Einstein postulates of Special Relativity, and so, it does not contemplate the difference in the metric structure required for a geometric implementation of causality. Therefore, the Lorentz-Dirac equation is the result of imposing to the electron a causality behaviour that is valid only for its electromagnetic field. Our next step is to show that if these distinct metric structures are taken into consideration, the Schott term does not appear in the equation of motion of a classical point! electron without any jeopardy to energy conservation. But this belongs to a subsequent paper.

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