BOUNDS ON THE GLOBAL DIMENSION OF CERTAIN PIECEWISE HEREDITARY CATEGORIES

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Abstract. We give bounds on the global dimension of a finite length, piecewise hereditary category in terms of quantitative connectivity properties of its graph of indecomposables.

We use this to show that the global dimension of a finite dimensional, piecewise hereditary algebra \( A \) cannot exceed 3 if \( A \) is an incidence algebra of a finite poset or more generally, a sincere algebra. This bound is tight.

1. Introduction

Let \( A \) be an abelian category and denote by \( \mathcal{D}^b(A) \) its bounded derived category. \( A \) is called piecewise hereditary if there exist an abelian hereditary category \( \mathcal{H} \) and a triangulated equivalence \( \mathcal{D}^b(A) \cong \mathcal{D}^b(\mathcal{H}) \). Piecewise hereditary categories of modules over finite dimensional algebras have been studied in the past, especially in the context of tilting theory, see [1, 2, 3].

It is known [2, (1.2)] that if \( A \) is a finite length, piecewise hereditary category with \( n \) non-isomorphic simple objects, then its global dimension satisfies \( \text{gl.dim} \ A \leq n \). Moreover, this bound is almost sharp, as there are examples [5] where \( A \) has \( n \) simples and \( \text{gl.dim} \ A = n - 1 \).

In this note we show how rather simple arguments can yield effective bounds on the global dimension of such a category \( A \), in terms of quantitative connectivity conditions on the graph of its indecomposables, regardless of the number of simple objects.

Let \( G(A) \) be the directed graph whose vertices are the isomorphism classes of indecomposables of \( A \), where two vertices \( Q, Q' \) are joined by an edge \( Q \to Q' \) if \( \text{Hom}_A(Q, Q') \neq 0 \).

Let \( r \geq 1 \) and let \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_{r-1}) \) be a sequence in \( \{+1, -1\}^r \). An \( \varepsilon \)-path from \( Q \) to \( Q' \) is a sequence of vertices \( Q_0 = Q, Q_1, \ldots, Q_r = Q' \) such that \( Q_i \to Q_{i+1} \) in \( G(A) \) if \( \varepsilon_i = +1 \) and \( Q_{i+1} \to Q_i \) if \( \varepsilon_i = -1 \).

For an object \( Q \) of \( A \), let \( \text{pd}_A Q = \sup \{d : \text{Ext}^d_A(Q, Q') \neq 0 \text{ for some } Q'\} \) and \( \text{id}_A Q = \sup \{d : \text{Ext}^d_A(Q', Q) \neq 0 \text{ for some } Q'\} \) be the projective and injective dimensions of \( Q \), so that \( \text{gl.dim} \ A = \sup_Q \text{pd}_A Q \).

Theorem 1.1. Let \( A \) be a finite length, piecewise hereditary category. Assume that there exist \( r \geq 1, \varepsilon \in \{1, -1\}^r \) and an indecomposable \( Q_0 \) such that for any indecomposable \( Q \) there exists an \( \varepsilon \)-path from \( Q_0 \) to \( Q \).

Then \( \text{gl.dim} \ A \leq r + 1 \) and \( \text{pd}_A Q + \text{id}_A Q \leq r + 2 \) for any indecomposable \( Q \).

We give two applications of this result for finite dimensional algebras.
Let $A$ be a finite dimensional algebra over a field $k$, and denote by mod $A$ the category of finite dimensional right $A$-modules. Recall that a module $M$ in mod $A$ is sincere if all the simple modules occur as composition factors of $M$. The algebra $A$ is called sincere if there exists a sincere indecomposable module.

**Corollary 1.2.** Let $A$ be a finite dimensional, piecewise hereditary, sincere algebra. Then $\text{gl.dim } A \leq 3$ and $\text{pd } Q + \text{id } Q \leq 4$ for any indecomposable module $Q$ in mod $A$.

Let $X$ be a finite partially ordered set (poset) and let $k$ be a field. The incidence algebra $kX$ is the $k$-algebra spanned by the elements $e_{xy}$ for the pairs $x \leq y$ in $X$, with the multiplication defined by setting $e_{xy}e_{y'z} = e_{xz}$ when $y = y'$ and zero otherwise.

**Corollary 1.3.** Let $X$ be a finite poset. If the incidence algebra $kX$ is piecewise hereditary, then $\text{gl.dim } kX \leq 3$ and $\text{pd } Q + \text{id } Q \leq 4$ for any indecomposable $kX$-module $Q$.

The bounds in Corollaries 1.2 and 1.3 are sharp, see Examples 3.2 and 3.3.

The paper is organized as follows. In Section 2 we give the proofs of the above results. Examples demonstrating various aspects of these results are given in Section 3.

## 2. The proofs

### 2.1. Preliminaries

Let $\mathcal{A}$ be an abelian category. If $X$ is an object of $\mathcal{A}$, denote by $X[n]$ the complex in $D^b(\mathcal{A})$ with $X$ at position $-n$ and 0 elsewhere. Denote by $\text{ind } \mathcal{A}$, $\text{ind } D^b(\mathcal{A})$ the sets of isomorphism classes of indecomposable objects of $\mathcal{A}$ and $D^b(\mathcal{A})$, respectively. The map $X \mapsto X[0]$ is a fully faithful functor $\mathcal{A} \to D^b(\mathcal{A})$ which induces an embedding $\text{ind } \mathcal{A} \to \text{ind } D^b(\mathcal{A})$.

Assume that there exists a triangulated equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{H})$ with $\mathcal{H}$ hereditary. Then $F$ induces a bijection $\text{ind } D^b(\mathcal{A}) \simeq \text{ind } D^b(\mathcal{H})$, and we denote by $\varphi_F : \text{ind } \mathcal{A} \to \text{ind } \mathcal{H} \times \mathbb{Z}$ the composition

$$
\text{ind } \mathcal{A} \hookrightarrow \text{ind } D^b(\mathcal{A}) \xrightarrow{\sim} \text{ind } D^b(\mathcal{H}) = \text{ind } \mathcal{H} \times \mathbb{Z}
$$

where the last equality follows from [4 (2.5)].

If $Q$ is an indecomposable of $\mathcal{A}$, write $\varphi_F(Q) = (f_F(Q), n_F(Q))$ where $f_F(Q) \in \text{ind } \mathcal{H}$ and $n_F(Q) \in \mathbb{Z}$, so that $F(Q[0]) \simeq f_F(Q)[n_F(Q)]$ in $D^b(\mathcal{H})$. From now on we fix the equivalence $F$, and omit the subscript $F$.

**Lemma 2.1.** The map $f : \text{ind } \mathcal{A} \to \text{ind } \mathcal{H}$ is one-to-one.

**Proof.** If $Q, Q'$ are two indecomposables of $\mathcal{A}$ such that $f(Q), f(Q')$ are isomorphic in $\mathcal{H}$, then $Q[n(Q') - n(Q)] \simeq Q'[0]$ in $D^b(\mathcal{A})$, hence $n(Q) = n(Q')$, and $Q \simeq Q'$ in $\mathcal{A}$.

As a corollary, note that if $A$ and $H$ are two finite dimensional algebras such that $D^b(\text{mod } A) \simeq D^b(\text{mod } H)$ and $H$ is hereditary, then the representation type of $H$ dominates that of $A$.

We recall the following three results, which were introduced in [1 (IV,1)] when $\mathcal{H}$ is the category of representations of a quiver.
Lemma 2.2. Let $Q, Q'$ be two indecomposables of $A$, then

$$\text{Ext}^1_A(Q, Q') \simeq \text{Ext}^{1+n(Q')-n(Q)}_D(f(Q), f(Q'))$$

Corollary 2.3. Let $Q, Q'$ be two indecomposables of $A$ with $\text{Hom}_A(Q, Q') \neq 0$. Then $n(Q') - n(Q) \in \{0, 1\}$.

Lemma 2.4. Assume that $A$ is of finite length and there exist integers $n_0, d$ such that $n_0 \leq n(P) < n_0 + d$ for every indecomposable $P$ of $A$.

If $Q$ is indecomposable, then $\text{pd}_A Q \leq n(Q) - n_0 + 1$ and $\text{id}_A Q \leq n_0 + d - n(Q)$. In particular, $\text{gl.dim} A \leq d$.

Proof. See [1, IV, p.158] or [2, (1.2)]. □

2.2. Proof of Theorem 1.1. Let $r \geq 1$, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{r-1})$ and $Q_0$ be as in the Theorem. Denote by $r_+$ the number of positive $\varepsilon_i$, and by $r_-$ the number of negative ones. Let $F : \mathcal{D}^b(A) \to \mathcal{D}^b(H)$ be a triangulated equivalence and write $f = f_F$, $n = n_F$.

Let $Q$ be any indecomposable of $A$. By assumption, there exists an $\varepsilon$-path $Q_0, Q_1, \ldots, Q_r = Q$, so by Corollary 2.3 $n(Q_{i+1}) - n(Q_i) \in \{0, \varepsilon_i\}$ for all $0 \leq i < r$. It follows that $n(Q) - n(Q_0) = \sum_{i=0}^{r-1} \alpha_i \varepsilon_i$ for some $\alpha_i \in \{0, 1\}$, hence

$$n(Q_0) - r_- \leq n(Q) \leq n(Q_0) + r_+$$

and the result follows from Lemma 2.4 with $d = r + 1$ and $n_0 = n(Q_0) - r_-$.

2.3. Variations and comments.

Remark 2.5. The assumption in Theorem 1.1 that any indecomposable $Q$ is the end of an $\varepsilon$-path from $Q_0$ can replaced by the weaker assumption that any simple object is the end of such a path.

Proof. Assume that $\varepsilon_{r-1} = 1$ and let $Q$ be indecomposable. Since $Q$ has finite length, we can find a simple object $S$ with $g : S \to Q$. Let $Q_0, Q_1, \ldots, Q_{r-1}, S$ be an $\varepsilon$-path from $Q_0$ to $S$ with $f_{r-1} : Q_{r-1} \to S$. Replacing $S$ by $Q$ and $f_{r-1}$ by $g f_{r-1} \neq 0$ gives an $\varepsilon$-path from $Q_0$ to $Q$.

The case $\varepsilon_{r-1} = -1$ is similar. □

Remark 2.6. Let $\tilde{G}(A)$ be the undirected graph obtained from $G(A)$ by forgetting the directions of the edges. The distance between two indecomposables $Q$ and $Q'$, denoted $d(Q, Q')$, is defined as the length of the shortest path in $G(A)$ between them (or $+\infty$ if there is no such path).

The same proof gives that $|n(Q) - n(Q')| \leq d(Q, Q')$ for any two indecomposables $Q$ and $Q'$. Let $d = \sup_{Q, Q'} d(Q, Q')$ be the diameter of $\tilde{G}(A)$. When $d < \infty$, $\inf_Q n(Q)$ and $\sup_Q n(Q)$ are finite, and by Lemma 2.4 $\text{gl.dim} A \leq d + 1$ and $\text{pd}_A Q + \text{id}_A Q \leq d + 2$ for any indecomposable $Q$.

Remark 2.7. The conclusion of Theorem 1.1 (or Remark 2.6) is still true under the slightly weaker assumption that $A$ is a finite length, piecewise hereditary and $A = \bigoplus_{i=1}^r A_i$ is a direct sum of abelian full subcategories such that each graph $G(A_i)$ satisfies the corresponding connectivity condition.
2.4. **Proof of Corollary 1.2** Let $A$ be sincere, and let $S_1, \ldots, S_n$ be the representatives of the isomorphism classes of simple modules in $\text{mod } A$. Let $P_1, \ldots, P_n$ be the corresponding indecomposable projectives and finally let $M$ be an indecomposable, sincere module.

Take $r = 2$ and $\varepsilon = (-1, +1)$. Now observe that any simple $S_i$ is the end of an $\varepsilon$-path from $M$, as we have a path of nonzero morphisms $M \leftarrow P_i \rightarrow S_i$ since $M$ is sincere. The result now follows by Theorem 1.1 and Remark 2.5.

2.5. **Proof of Corollary 1.3** Let $X$ be a poset and $k$ a field. A $k$-diagram $\mathcal{F}$ is the data consisting of finite dimensional $k$-vector spaces $\mathcal{F}(x)$ for $x \in X$, together with linear transformations $r_{xx'} : \mathcal{F}(x) \to \mathcal{F}(x')$ for all $x \leq x'$, satisfying the conditions $r_{xx} = 1_{\mathcal{F}(x)}$ and $r_{xx''} = r_{x'x''}r_{xx'}$ for all $x \leq x' \leq x''$.

The category of finite dimensional right modules over $kX$ can be identified with the category of $k$-diagrams over $X$, see [6]. A complete set of representatives of isomorphism classes of simple modules over $kX$ is given by the diagrams $S_x$ for $x \in X$, defined by

$$S_x(y) = \begin{cases} k & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

with $r_{yy'} = 0$ for all $y < y'$. A module $\mathcal{F}$ is sincere if and only if $\mathcal{F}(x) \neq 0$ for all $x \in X$.

The poset $X$ is **connected** if for any $x, y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that for all $0 \leq i < n$ either $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$.

**Lemma 2.8.** If $X$ is connected then the incidence algebra $kX$ is sincere.

**Proof.** Let $kX$ be the diagram defined by $kX(x) = k$ for all $x \in X$ and $r_{xx'} = 1_k$ for all $x \leq x'$. Obviously $kX$ is sincere. Moreover, $kX$ is indecomposable by a standard connectivity argument; if $kX = \mathcal{F} \oplus \mathcal{F}'$, write $V = \{x \in X : \mathcal{F}(x) \neq 0\}$ and assume that $V$ not empty. If $x \in V$ and $x < y$, then $y \in V$, otherwise we would get a zero map $k \oplus 0 \to 0 \oplus k$ and not an identity map. Similarly, if $y < x$ then $y \in V$. By connectivity, $V = X$ and $\mathcal{F} = kX$. □

If $X$ is connected, Corollary 1.3 now follows from Corollary 1.2 and Lemma 2.8. For general $X$, observe that if $\{X_i\}_{i=1}^r$ are the connected components of $X$, then the category $\text{mod } kX$ decomposes as the direct sum of the categories $\text{mod } kX_i$, and the result follows from Remark 2.7.

**Corollary 2.9.** Let $X$ and $Y$ be posets such that $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$ and $\text{gl.dim } kY > 3$. Then $kX$ is not piecewise hereditary.

3. **Examples**

We give a few examples that demonstrate various aspects of global dimensions of piecewise hereditary algebras. In these examples, $k$ denotes a field and all posets are represented by their Hasse diagrams.

**Example 3.1** ([5]). Let $n \geq 2$, $Q^{(n)}$ the quiver

$$0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_n} n$$


and $I^{(n)}$ be the ideal (in the path algebra $kQ^{(n)}$) generated by the paths $\alpha_i\alpha_{i+1}$ for $1 \leq i < n$. By \cite[(IV, 6.7)]{1}, the algebra $A^{(n)} = kQ^{(n)}/I^{(n)}$ is piecewise hereditary of Dynkin type $A_{n+1}$.

For a vertex $0 \leq i \leq n$, let $S_i, P_i, I_i$ be the simple, indecomposable projective and indecomposable injective corresponding to $i$. Then one has $P_n = S_n$, $I_0 = S_0$ and for $0 \leq i < n$, $P_i = I_{i+1}$ with a short exact sequence $0 \to S_{i+1} \to P_i \to S_i \to 0$.

The graph $G(\text{mod } A^{(n)})$ is shown below (ignoring the self loops around each vertex).

![Graph](image)

Regarding dimensions, we have $\text{pd} S_i = n - i$, $\text{id} S_i = i$ for $0 \leq i \leq n$, and $\text{pd} P_i = \text{id} P_i = 0$ for $0 \leq i < n$, so that $\text{gl.dim } A^{(n)} = n$ and $\text{pd } Q + \text{id } Q \leq n$ for every indecomposable $Q$. The diameter of $\tilde{G}(\text{mod } A^{(n)})$ is $n + 1$.

The following two examples show that the bounds given in Corollary 1.3 are sharp.

**Example 3.2.** A poset $X$ with $kX$ piecewise hereditary and $\text{gl.dim } kX = 3$.

Let $X, Y$ be the following two posets:

![Poset X](image) ![Poset Y](image)

Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\text{gl.dim } kX = 3$, $\text{gl.dim } kY = 1$.

**Example 3.3.** A poset $X$ with $kX$ piecewise hereditary and an indecomposable $\mathcal{F}$ such that $\text{pd}_k X \mathcal{F} + \text{id}_k \mathcal{F} = 4$.

Let $X, Y$ be the following two posets:

![Poset X](image) ![Poset Y](image)

Then $\mathcal{D}^b(kX) \simeq \mathcal{D}^b(kY)$, $\text{gl.dim } kX = 2$, $\text{gl.dim } kY = 1$ and for the simple $S_x$ we have $\text{pd}_k X S_x = \text{id}_k X S_x = 2$.

We conclude by giving two examples of posets whose incidence algebras are not piecewise hereditary.

**Example 3.4.** A product of two trees whose incidence algebra is not piecewise hereditary.
By specifying an orientation $\omega$ on the edges of a (finite) tree $T$, one gets a finite quiver without oriented cycles whose path algebra is isomorphic to the incidence algebra of the poset $X_{T,\omega}$ defined on the set of vertices of $T$ by saying that $x \leq y$ for two vertices $x$ and $y$ if there is an oriented path from $x$ to $y$.

A poset of the form $X_{T,\omega}$ is called a tree. Equivalently, a poset is a tree if and only if the underlying graph of its Hasse diagram is a tree. Obviously, $\text{gl.dim } kX_{T,\omega} = 1$, so that $kX_{T,\omega}$ is trivially piecewise hereditary. Moreover, while the poset $X_{T,\omega}$ may depend on the orientation $\omega$ chosen, its derived equivalence class depends only on $T$.

Given two posets $X$ and $Y$, their product, denoted $X \times Y$, is the poset whose underlying set is $X \times Y$ and $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$ where $x, x' \in X$ and $y, y' \in Y$. It may happen that the incidence algebra of a product of two trees, although not being hereditary, is piecewise hereditary. Two notable examples are the product of the Dynkin types $A_2 \times A_2$, which is piecewise hereditary of type $D_4$, and the product $A_2 \times A_3$ which is piecewise hereditary of type $E_6$.

Consider $X = A_2 \times A_2$ and $Y = D_4$ with the orientations given below.

Then $\text{gl.dim } kX = 2$, $\text{gl.dim } kY = 1$ and $D^b(kX) \simeq D^b(kY)$, hence $D^b(k(X \times X)) \simeq D^b(k(Y \times Y))$. But $\text{gl.dim } k(X \times X) = 4$, so by Corollary 2.9, $Y \times Y$ is a product of two trees of type $D_4$ whose incidence algebra is not piecewise hereditary.

**Example 3.5.** The converse to Corollary 1.3 is false.

Let $X$ be the poset

Then $\text{gl.dim } kX = 2$, hence $\text{pd}_{kX} F \leq 2$, $\text{id}_{kX} F \leq 2$ for any indecomposable $F$, so that $X$ satisfies the conclusion of Corollary 1.3. However, $kX$ is not piecewise hereditary since $\text{Ext}^2_X(k_X, k_X) = k$ does not vanish (see [1] (IV, 1.9)). Note that $X$ is the smallest poset whose incidence algebra is not piecewise hereditary.

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