The Evolution of Beliefs over Signed Social Networks

Guodong Shi
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden
guodongs@kth.se

Alexandre Proutiere
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden
alepro@kth.se

Mikael Johansson
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden
mikaelj@kth.se

John S. Baras
Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742, USA
baras@umd.edu

Karl H. Johansson
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden
kallej@kth.se

We study the evolution of opinions (or beliefs) over a social network modeled as a signed graph. The sign attached to an edge in this graph characterizes whether the corresponding individuals or end nodes are friends (positive link) or enemies (negative link). Pairs of nodes are randomly selected to interact over time, and when two nodes interact, each of them updates her opinion based on the opinion of the other node in a manner dependent on the sign of the corresponding link. Our model for the opinion dynamics is essentially linear and generalizes DeGroot model to account for negative links – when two enemies interact, their opinions go in opposite directions. We provide conditions for convergence and divergence in expectation, in mean-square, and in almost sure sense, and exhibit phase transition phenomena for these notions of convergence depending on the parameters of our opinion update model and on the structure of the underlying graph. We establish a no-survivor theorem, stating that the difference in opinions of any two nodes diverges whenever opinions in the network diverge as a whole. We also prove a live-or-die lemma, indicating that almost surely, the opinions either converge to an agreement or diverge. Finally, we extend our analysis to cases where opinions have hard lower and upper limits. In these cases, we study when and how opinions may become asymptotically clustered, and highlight the impact of the structural properties (namely structural balance) of the underlying network on this clustering phenomenon.

**Key words:** opinion dynamics, signed graph, social networks, opinion clustering
1. Introduction

1.1. Motivation

We all form opinions about economical, political and social events that take place in society. These opinions can be binary (e.g., whether one supports a candidate in an election or not) or continuous (to what degree one expects a prosperous future economy). Our opinions are revised when we interact with each other in the social networks we live in. Characterizing the evolution of opinions and understanding the dynamic and asymptotic behavior of the social belief are fundamental challenges in the theoretical study of social networks.

Building a good model on how individuals interact and influence each other is essential for studying opinion dynamics. For example, a trusted friend has a different influence on our opinion than a dubious stranger. The observation that sentiment influences opinions can be traced back to the 1940’s when Heider (1946) introduced the theory of signed social networks, where each interaction link in the social network is associated with a sign (positive, or negative) indicating whether the two corresponding individuals or end nodes are friends or enemies. Efforts to understand the structural properties of signed social networks have led to the development of structure balance theory, with seminal early contributions by Cartwright and Harary (1956) and Davis (1963, 1967). A fundamental insight from these studies is that local structural properties imply hard global constraints on the entire social network, and is now referred as Harary’s theorem (Harary (1953)).

In this paper, we attempt to model the evolution of opinions in signed social networks when local hostile or antagonistic relations influence the global social belief. The relative strengths and structures of the positive and negative relations are shown to have an essential effect on the opinion convergence. In some cases, tight conditions for convergence and divergence can be established.

1.2. Related Work

The concept of signed social networks was introduced by Heider (1946). His objective was to formally distinguish between friendly (positive links) and hostile (negative links) relationships. The notion of structure balance was introduced to understand local interactions, and formalize intricate local scenarios (e.g., 'two of my friends are enemies'). A number of classical results on social balance have been established by Harary (1953), Cartwright and Harary (1956), Davis (1963, 1967), who
derived critical conditions on the global structure of the social network to ensure structural balance. Social balance theory has since then become an important topic in the study of social networks. Some recent works in this area are Facchetti et al. (2011) (who studied how to efficiently compute the degree of balance of a large network), and Marvel et al. (2011) (who analyzed continuous-time dynamics for signed networks and showed convergence to structural balance).

Opinion dynamics is another long-standing topic in the study of social networks, see Jackson (2008) and Easley and Kleinberg (2010) for recent textbooks. Following the survey by Acemoglu and Ozdaglar (2011), we classify opinion evolution models into Bayesian and non-Bayesian updating rules. The main difference between the two types of rule lies in whether each node has access and acts according to a global model. We refer to Banerjee (1992), Bikhchandani et al. (1992) and, more recent work Acemoglu et al. (2011) for Bayesian opinion dynamics. In non-Bayesian models, nodes follow simple updating strategies. DeGroot’s model (DeGroot 1974) is a classical non-Bayesian opinion dynamics model, where each node updates her belief as a convex combination of her neighbors’ beliefs, see e.g. DeMarzo et al. (2003), Golub and Jackson (2010), Blondel et al. (2009, 2010), Jadbabaie et al. (2012). Note that DeGroot’s models are related to averaging consensus processes, see e.g. Tsitsiklis (1984), Xiao and Boyd (2004), Boyd et al. (2006), Tahbaz-Salehi and Jadbabaie (2008), Fagnani and Zampieri (2008), Touri and Nedić (2011), Matei et al. (2013).

The influence of misbehaving nodes have been studied to some extent. For instance, in Acemoglu et al. (2010), a model of the spread of misinformation in large societies was discussed. There, some individuals are ”forceful,” meaning that they influence the beliefs of (some) of the other individuals they meet, but do not change their own opinion. In Acemoglu et al. (2013), the authors studied the propagation of opinion disagreement under DeGroot’s rule, when some nodes stick to their initial beliefs during the entire evolution. This idea was then extended to binary opinion dynamics under the voter model in Yildiz et al. (2013). In Altafini (2012, 2013), the authors propose and analyze a linear model for belief dynamics over signed graphs, that, a priori, seems close to our model. In Altafini (2013), it is shown that a bipartite agreement, i.e., clustering of opinions, is reached as long as the signed social graph is strongly balanced from the classical structural balance theory (Cartwright and Harary 1956), which presents a remarkable
link between opinion dynamics and structure balance. However, in the model studied in [Altafini (2012, 2013)], all beliefs converge to a common value, equal to zero, if the graph is not strongly balanced, and this seems to be difficult to interpret and justify from real-world observations. A game-theoretical approach was introduced in [Theodorakopoulos and Baras (2008)] for studying the interplay between prescribed good and bad players in collaborative networks.

1.3. Contribution

In this paper, we propose and analyze a model for belief dynamics over signed social networks. Nodes randomly interact pairwise and update their beliefs. In case of positive link (the two nodes are friends), the update follows DeGroot’s rule which drives the two beliefs closer to each other. On the contrary, in case of a negative link (the two nodes are enemies), the update is linear (in the previous beliefs), but tends to increase the difference between the two beliefs. Thus, two opposite types of opinion updates are defined, and the beliefs are driven not only by random node interactions but also by the type of relationship of the interacting nodes.

Under this simple attraction–repulsion model for opinions on signed social networks, we establish a number of fundamental results on belief convergence and divergence, and study the impact of the parameters of the update rules and of the network structure on the belief dynamics. We analyze various notions of convergence and divergence: in expectation, in mean-square, and almost sure.

- Using classical spectral methods, we derive conditions for mean and mean-square convergence and divergence of beliefs. We establish phase transition phenomena for these notions of convergence, and study how the thresholds depend on the parameters of our opinion update model and on the structure of the underlying graph.

- We derive phase-transition conditions for almost sure convergence or divergence of beliefs. The proof is built around what we call the Triangle lemma, which characterizes the evolution of the beliefs held by three different nodes, and leverages and combines probabilistic tools the Borel-Cantelli lemma, the Martingale convergence theorems, the strong law of large numbers, and sample-path arguments).

We establish two somewhat counter-intuitive results about the way beliefs evolve: (i) a no-survivor theorem which states that the difference in opinions of any two nodes tends to infinity almost surely (along a subsequence of instants) whenever the difference between the maximum
and the minimum beliefs in the network tends to infinity (along a subsequence of instants); (ii) a live-or-die lemma which demonstrates that almost surely, the opinions either converge to an agreement or diverge.

We also show that, essentially, networks whose positive component include an hypercube are (the only) robust networks in the sense that almost sure convergence of beliefs holds irrespective of the number of negative links, their positions in the network, and the strength of the negative update.

- Finally, we extend the results to cases where updates may be asymmetric (in the sense that when two nodes interact, only one of them may update her belief), and where beliefs have hard lower and upper constraints. In these cases, we study when and how beliefs may become asymptotically clustered, and highlight the impact of the structural properties (namely structural balance) of the underlying network on this clustering phenomenon. More precisely, we show that almost sure belief clustering is achieved if the social network is strongly balanced (or complete and weakly balanced) and the strength of the negative updates is sufficiently large. In absence of balanced structure, and if the positive graph is connected, we prove that the belief of each node oscillates between the lower and upper bounds and touches the two belief boundaries an infinite number of times.

The classical structure balance of a signed social network has a fundamental role for asymptotic formation of opinions. We believe our results provide some new insight and understanding on how opinions evolve on signed social networks.

1.4. Paper Organization

In Section 2, we present the signed social network model, specify the dynamics on positive and negative links, and define the problem of interest. Section 3 focuses on the mean and mean-square convergence and divergence analysis, and Section 4 on these properties in the almost sure sense. In Section 5, we study a model with hard lower and upper bounds and asymmetric update rules. It is shown how the structure balance determines the clustering of opinions. Finally some concluding remarks are given in Section 6.
Notation and Terminology

An undirected graph is denoted by $G = (V, E)$. Here $V = \{1, \ldots, n\}$ is a finite set of vertices (nodes). Each element in $E$ is an unordered pair of two distinct nodes in $V$, called an edge. The edge between nodes $i, j \in V$ is denoted by $\{i, j\}$. Let $V_* \subseteq V$ be a subset of nodes. The induced graph of $V_*$ on $G$, denoted $G_{V_*}$, is the graph $(V_*, E_{V_*})$ with $\{u, v\} \in E_{V_*}$, $u, v \in V_*$ if and only if $\{u, v\} \in E$. A path in $G$ with length $k$ is a sequence of distinct nodes, $v_1 v_2 \ldots v_{k+1}$, such that $\{v_m, v_{m+1}\} \in E$, $m = 1, \ldots, k$.

The length of a shortest path between two nodes $i$ and $j$ is called the distance between the nodes, denoted $d(i, j)$. The greatest length of all shortest paths is called the diameter of the graph, denoted $\text{diam}(G)$. The degree matrix of $G$, denoted $D(G)$, is the diagonal matrix $\text{diag}(d_1, \ldots, d_n)$ with $d_i$ denoting the number of nodes sharing an edge with $i, i \in V$. The adjacency matrix $A(G)$ is the symmetric $n \times n$ matrix such that $[A(G)]_{ij} = 1$ if $\{i, j\} \in E$ and $[A(G)]_{ij} = 0$ otherwise. The matrix $L(G) := D(G) - A(G)$ is called the Laplacian of $G$. Two graphs containing the same number of vertices are called isomorphic if they are identical subject to a permutation of vertex labels.

All vectors are column vectors and denoted by lower case letters. Matrices are denoted with upper case letters. Given a matrix $M$, $M'$ denotes its transpose and $M^k$ denotes the $k$-th power of $M$ when it is a square matrix. The $ij$-entry of a matrix $M$ is denoted $[M]_{ij}$. Given a matrix $M \in \mathbb{R}^{mn}$, the vectorization of $M$, denoted by $\text{vec}(M)$, is the $mn \times 1$ column vector $([M]_{11}, \ldots, [M]_{m1}, [M]_{12}, \ldots, [M]_{m2}, \ldots, [M]_{1n}, \ldots, [M]_{mn})'$. We have $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$ for all real matrices $A, B, C$ with $ABC$ well defined. With the universal set prescribed, the complement of a given set $S$ is denoted $S^c$. The orthogonal complement of a subspace $S$ in a vector space is denoted $S^\perp$. Depending on the argument, $|\cdot|$ stands for the absolute value of a real number, the Euclidean norm of a vector, and the cardinality of a set. Similarly with argument well defined, $\sigma(\cdot)$ represents the $\sigma$-algebra of a random variable (vector), or the spectrum of a matrix. The smallest integer no smaller than a given real number $a$ is denoted $\lceil a \rceil$. We use $P(\cdot)$ to denote the probability, $\mathbb{E}\{\cdot\}$ the expectation, $\mathbb{V}\{\cdot\}$ the variance of their arguments, respectively.

2. Signed Social Networks and Belief Dynamics

In this section, we present our model of interaction between nodes in a signed social network, and describe the resulting dynamics of the beliefs held at each node.
2.1. Node Pair Selection

We consider a social network with \( n \geq 3 \) members, each labeled by a unique integer in \( \{1, 2, \ldots, n\} \). The network is represented by an undirected graph \( G = (V, E) \) whose node set \( V = \{1, 2, \ldots, n\} \) corresponds to the members and whose edge set \( E \) describes potential interactions between the members. Actual interactions follow the model introduced in Boyd et al. (2006): each node initiates interactions at the instants of a rate-one Poisson process, and at each of these instants, picks a node at random to interact with. Under this model, at a given time, at most one node initiates an interaction. This allows us to order interaction events in time and to focus on modeling the node pair selection at interaction times.

The node selection process is characterized by an \( n \times n \) stochastic matrix \( P = [p_{ij}] \), where \( p_{ij} \geq 0 \) for all \( i, j = 1, \ldots, n \) and \( p_{ij} > 0 \) only if \( \{i, j\} \in E \). \( p_{ij} \) represents the probability that node \( i \) initiates an interaction with node \( j \). Without loss of generality we assume that \( p_{ii} = 0 \) for all \( i \). The node pair selection is then performed as follows.

**Definition 1 (Node Pair Selection).** At each interaction event \( k \geq 0 \),

(i) A node \( i \in V \) is drawn uniformly at random, i.e., with probability \( 1/n \);

(ii) Node \( i \) picks node \( j \) with probability \( p_{ij} \), in which case, we say that the unordered node pair \( \{i, j\} \) is selected.

The node pair selection process is assumed to be i.i.d., i.e., the nodes that initiate an interaction and the selected node pairs are identically distributed and independent over \( k \geq 0 \). Formally, the node selection process can be analyzed using the following probability spaces. Let \((E, \mathcal{S}, \mu)\) be the probability space, where \( \mathcal{S} \) is the discrete \( \sigma \)-algebra on \( E \), and \( \mu \) is the probability measure defined by \( \mu(\{i, j\}) = \frac{p_{ij} + p_{ji}}{n} \) for all \( \{i, j\} \in E \). The node selection process can then be seen as a random event in the product probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \Omega = E^N = \{\omega = (\omega_0, \omega_1, \ldots) : \forall k, \omega_k \in E\} \), where \( \mathcal{F} = \mathcal{S}^N \), and \( \mathbb{P} \) is the product probability measure (uniquely) defined by: for all finite subset \( K \subset \mathbb{N} \), \( \mathbb{P}( (\omega_k)_{k \in K} ) = \prod_{k \in K} \mu(\omega_k) \) for any \( (\omega_k)_{k \in K} \in E^{|K|} \). For any \( k \in \mathbb{N} \), we define the coordinate mapping \( G_k : \Omega \rightarrow E \) by \( G_k(\omega) = \omega_k \), for all \( \omega \in \Omega \) (note that \( \mathbb{P}(G_k = \omega_k) = \mu(\omega_k) \)), and we refer to \((G_k, k = 0, 1, \ldots)\) as the node pair selection process. We further refer to \( \mathcal{F}_k = \sigma(G_0, \ldots, G_k) \) as the \( \sigma \)-algebra capturing the \((k + 1)\) first interactions of the selection process.
2.2. Symmetric Attraction-Repulsion Dynamics over Signed Graphs

Each node maintains a scalar real-valued belief, which it updates whenever it interacts with other nodes. We let $x(k) \in \mathbb{R}^n$ denote the vector of the beliefs held by nodes at interaction event $k$.

The belief update depends on the relationship between the interacting nodes. In particular, each edge in $E$ is assigned a unique label, either $+$ or $-$. In classical social network theory, a $+$ label indicates a friend relation, while a $-$ label indicates an enemy relation (Heider (1946), Cartwright and Harary (1956)). The graph $G$ equipped with a sign on each edge is then called a signed graph. Let $E_{pst}$ and $E_{neg}$ be the collection of the positive and negative edges, respectively; clearly, $E_{pst} \cap E_{neg} = \emptyset$ and $E_{pst} \cup E_{neg} = E$. We call $G_{pst} = (V, E_{pst})$ and $G_{neg} = (V, E_{neg})$ the positive and the negative graph, respectively; see Figure 1 for an illustration.

Suppose that node pair $\{i, j\}$ is selected at time $k$. The nodes that are not selected keep their beliefs unchanged, whereas the beliefs held at nodes $i$ and $j$ are updated as follows:

- **(Positive Update)** If $\{i, j\} \in E_{pst}$, each node $m \in \{i, j\}$ updates its belief as

  $$x_m(k+1) = x_m(k) + \alpha (x_{-m}(k) - x_m(k)) = (1 - \alpha) x_m(k) + \alpha x_{-m}(k),$$  

  (1)

  where $-m \in \{i, j\} \setminus \{m\}$ and $0 \leq \alpha \leq 1$.

- **(Negative Update)** If $\{i, j\} \in E_{neg}$, each node $m \in \{i, j\}$ updates its belief as

  $$x_m(k+1) = x_m(k) - \beta (x_{-m}(k) - x_m(k)) = (1 + \beta) x_m(k) - \beta x_{-m}(k),$$  

(2)
where $\beta \geq 0$.

The positive update is consistent with the classical DeGroot model \cite{Degroot1974}, where each node iteratively updates its belief as a convex combination of the previous beliefs of its neighbors in the social graph. This update naturally reflects trustful or cooperative relationships. It is sometimes referred to as naïve learning in social networks, under which wisdom can be held by the crowds \cite{Golub2010}. The positive update tends to drive node beliefs closer to each other and can be thought of as the attraction of the beliefs.

The dynamics on the negative edges, on the other hand, is not yet universally agreed upon in the literature. Considerable efforts have been made to characterize these mistrustful or antagonistic relationships, which has led to a number of insightful models, e.g., \cite{Acemoglu2010, Acemoglu2013, Altafini2012, Altafini2013}. Our negative update rule enforces belief differences between interacting nodes, and is the opposite of the attraction of beliefs represented by the positive update.

Remark 1. In \cite{Altafini2013}, the author proposed a different update rule for two nodes sharing a negative link. The model \cite{Altafini2013} is written in continuous time (beliefs satisfy some ODE), and its corresponding discrete-time version on a negative link $\{i, j\} \in E_{\text{neg}}$ is:

$$
x_m(k+1) = x_m(k) - \beta (x_{-m}(k) + x_m(k)) = (1 - \beta)x_m(k) - \beta x_{-m}(k), \quad m \in \{i, j\},
$$

(3)

where $\beta \in (0,1)$ represents the negative strength. Under (3), the beliefs always remain bounded since $|x_m(k+1)| \leq \max \{|x_i(k)|, |x_j(k)|\}, m \in \{i, j\}$, i.e., non-expansiveness of the absolute value of opinions. This property explains the essential difference between the model studied in the current paper and the one investigated by Altafini.

Remark 2. In \cite{Shi2013}, a model was presented for studying the spread of agreement and disagreement in networks, with randomized attraction, neglect, and repulsion updates. Note that the current model is fundamentally different as the underlying network is given by a signed graph.

Without loss of generality, we adopt the following assumption throughout the paper.

Assumption 1. The underlying graph $G$ is connected, and the negative graph $G_{\text{neg}}$ is nonempty.
2.3. Convergence and Divergence Notions

Let \( x(k) = (x_1(k) \ldots x_n(k))' \), \( k = 0, 1, \ldots \) be the (random) vector of beliefs at time \( k \) resulting from the node interactions. The initial beliefs \( x(0) \), also denoted as \( x^0 \), is assumed to be deterministic. We study the dynamics of process \( (x(k), k \geq 0) \), and to this aim, we introduce various notions of convergence and divergence.

**Definition 2.** (i) Belief convergence is achieved

- **in expectation** if \( \lim_{k \to \infty} E\{x_i(k) - x_j(k)\} = 0 \) for all \( i \) and \( j \);
- **in mean square** if \( \lim_{k \to \infty} E\{(x_i(k) - x_j(k))^2\} = 0 \) for all \( i \) and \( j \);
- **almost surely** if \( P\{\lim_{k \to \infty} |x_i(k) - x_j(k)| = 0\} = 1 \) for all \( i \) and \( j \).

(ii) Belief divergence is achieved

- **in expectation** if \( \limsup_{k \to \infty} \max_{i, j} E\{|x_i(k) - x_j(k)|\} = \infty \);
- **in mean square** if \( \limsup_{k \to \infty} \max_{i, j} E\{(x_i(k) - x_j(k))^2\} = \infty \);
- **almost surely** if \( P\{\limsup_{k \to \infty} \max_{i, j} |x_i(k) - x_j(k)| = \infty\} = 1 \).

Basic probability theory tells us that mean-square belief convergence implies belief convergence in expectation (mean convergence), and similarly belief divergence in expectation implies belief divergence in mean square. However, in general there is no direct connection between almost sure convergence/divergence and mean or mean-square convergence/divergence. Finally observe that, a priori, it is not clear that either convergence or divergence should be achieved.

3. Mean and Mean-square Convergence and Divergence

The belief dynamics as described above can be written as:

\[
x(k + 1) = W(k)x(k),
\]

where \( W(k), k = 0, 1, \ldots \) are i.i.d. random matrices satisfying

\[
\begin{align*}
P\left(W(k) = W^+_{ij} := I - \alpha(e_i - e_j)(e_i - e_j)'ight) &= \frac{p_{ij} + p_{ji}}{n}, \quad \{i, j\} \in E_{\text{pst}}, \\
P\left(W(k) = W^-_{ij} := I + \beta(e_i - e_j)(e_i - e_j)'ight) &= \frac{p_{ij} + p_{ji}}{n}, \quad \{i, j\} \in E_{\text{neg}},
\end{align*}
\]

and \( e_m = (0 \ldots 0 1 0 \ldots 0)' \) is the \( n \)-dimensional unit vector whose \( m \)-th component is 1. In this section, we use spectral properties of the linear system \( (4) \) to study convergence and divergence in mean and mean-square. Our results can be seen as extensions of existing convergence results on deterministic consensus algorithms, e.g., Xiao and Boyd (2004).
3.1. Convergence in Mean

We first provide conditions for convergence and divergence in mean. We then exploit these conditions to establish the existence of a phase transition for convergence when the negative update parameter $\beta$ is increased. These results are illustrated at the end of this subsection.

3.1.1. Conditions for convergence and divergence

Denote $P^\dagger = (P + P')/n$. We write $P^\dagger = P^\dagger_{\text{pst}} + P^\dagger_{\text{neg}}$, where $P^\dagger_{\text{pst}}$ and $P^\dagger_{\text{neg}}$ correspond to the positive and negative graphs, respectively. Specifically, $[P^\dagger_{\text{pst}}]_{ij} = [P^\dagger]_{ij}$ if $\{i, j\} \in E_{\text{pst}}$ and 0 otherwise, while $[P^\dagger_{\text{neg}}]_{ij} = [P^\dagger]_{ij}$ if $\{i, j\} \in E_{\text{neg}}$ and 0 otherwise. We further introduce the degree matrix $D^\dagger_{\text{pst}} = \text{diag}(d^+_1 \ldots d^+_n)$ of the positive graph, where $d^+_i = \sum_{j=1}^n [P^\dagger_{\text{pst}}]_{ij}$. Similarly, the degree matrix of the negative graph is defined as $D^\dagger_{\text{neg}} = \text{diag}(d^-_1 \ldots d^-_n)$ with $d^-_i = \sum_{j=1}^n [P^\dagger_{\text{neg}}]_{ij}$. Then $L^\dagger_{\text{pst}} = D^\dagger_{\text{pst}} - P^\dagger_{\text{pst}}$ and $L^\dagger_{\text{neg}} = D^\dagger_{\text{neg}} - P^\dagger_{\text{neg}}$ represent the (weighted) Laplacian matrices of the positive graph $G_{\text{pst}}$ and negative graph $G_{\text{neg}}$, respectively.

It can be easily deduced from (5) that

$$
\mathbb{E}\{W(k)\} = I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}}. \tag{6}
$$

Clearly, $1^\prime \mathbb{E}\{W(k)\} = \mathbb{E}\{W(k)\} 1 = 1$ where $1 = (1 \ldots 1)'$ denotes the $n \times 1$ vector off all ones, but $\mathbb{E}\{W(k)\}$ is not necessarily a stochastic matrix since it may contain negative entries.

Introduce $y_i(k) = x_i(k) - \sum_{s=1}^n x_s(k)/n$ and let $y(k) = (y_1(k) \ldots y_n(k))'$. Define $U := 11'/n$ and note that $y(k) = (I - U)x(k)$; furthermore, $(I - U)W(k) = W(k)(I - U) = W(k) - U$ for all possible realizations of $W(k)$. Hence, the evolution of $\mathbb{E}\{y(k)\}$ is linear:

$$
\mathbb{E}\{y(k+1)\} = \mathbb{E}\{(I - U)W(k)x(k)\} = \mathbb{E}\{(I - U)W(k)(I - U)x(k)\} = (\mathbb{E}\{W(k)\} - U)\mathbb{E}\{y(k)\}.
$$

The following elementary inequalities

$$
|\mathbb{E}\{x_i(k) - x_j(k)\}| \leq |\mathbb{E}\{y_i(k)\}| + |\mathbb{E}\{y_j(k)\}|, \quad |\mathbb{E}\{y_i(k)\}| \leq \frac{1}{n} \sum_{s=1}^n |x_i(k) - x_s(k)| \tag{7}
$$

imply that belief convergence in expectation is equivalent to $\lim_{k \to \infty} |\mathbb{E}\{y(k)\}| = 0$, and belief divergence is equivalent to $\limsup_{k \to \infty} |\mathbb{E}\{y(k)\}| = \infty$. Belief convergence or divergence is hence determined by the spectral radius of $\mathbb{E}\{W(k)\} - U$.

Geršgorin’s Circle Theorem (see, e.g., Theorem 6.1.1 in Horn and Johnson (1985)) guarantees that each eigenvalue of $I - \alpha L^\dagger_{\text{pst}}$ is nonnegative. It then follows that each eigenvalue of $I - \alpha L^\dagger_{\text{pst}} - U$
is nonnegative since $L^\dagger_{\text{pst}} U = U L^\dagger_{\text{pst}} = 0$ and the two matrices $I - \alpha L^\dagger_{\text{pst}}$ and $U$ share the same eigenvector 1 for eigenvalue one. Moreover, it is well known in algebraic graph theory that $L^\dagger_{\text{pst}}$ and $L^\dagger_{\text{neg}}$ are positive semi-definite matrices. As a result, Weyl’s inequality (see Theorem 4.3.1 in Horn and Johnson (1985)) further ensures that each eigenvalue of $\mathbb{E}\{W(k)\} - U$ is also nonnegative. To summarize, we have shown that:

**Proposition 1.** Belief convergence is achieved in expectation for all initial values if $\lambda_{\max}(I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U) < 1$; belief divergence is achieved in expectation for almost all initial values if $\lambda_{\max}(I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U) > 1$.

In the above proposition and what follows, $\lambda_{\max}(M)$ denotes the largest eigenvalue of the real symmetric matrix $M$, and by “almost all initial conditions,” we mean that the property holds for any initial condition $y(0)$ except if $y(0)$ is perfectly orthogonal to the eigenspace of $\mathbb{E}\{W(k)\} - U$ corresponding to its maximal eigenvalue $\lambda_{\max}(I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U)$. Hence the set of initial conditions where the property does not hold has zero Lebesgue measure.

The Courant-Fischer Theorem (see Theorem 4.2.11 in Horn and Johnson (1985)) implies

$$\lambda_{\max}(I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U) = \sup_{|z|=1} z^\prime (I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U) z$$

$$= 1 + \sup_{|z|=1} \left[ -\alpha \sum_{\{i,j\} \in E_{\text{pst}}} [P^\dagger]_{ij} (z_i - z_j)^2 + \beta \sum_{\{i,j\} \in E_{\text{neg}}} [P^\dagger]_{ij} (z_i - z_j)^2 - \frac{1}{n} \left( \sum_{i=1}^{n} z_i \right)^2 \right]. \quad (8)$$

We see from (8) that the influence of $G_{\text{pst}}$ and $G_{\text{neg}}$ to the belief convergence/divergence in mean are separated: links in $E_{\text{pst}}$ contribute to belief convergence, while links in $E_{\text{neg}}$ contribute to belief divergence. As will be shown later on, this separation property no longer holds for mean-square convergence, and there may be a non-trivial correlation between the influence of $E_{\text{pst}}$ and that of $E_{\text{neg}}$.

### 3.1.2. Phase Transition

Next we study the impact of update parameters $\alpha$ and $\beta$ on the convergence in expectation. Define: $f(\alpha, \beta) := \lambda_{\max}(I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U)$. $f$ has the following properties:

(i) **Convexity** Since both $L^\dagger_{\text{pst}}$ and $L^\dagger_{\text{neg}}$ are symmetric, $f(\alpha, \beta)$ is the spectral norm of $I - \alpha L^\dagger_{\text{pst}} + \beta L^\dagger_{\text{neg}} - U$. As every matrix norm is convex, we have

$$f(\lambda(\alpha_1, \beta_1) + (1 - \lambda)(\alpha_2, \beta_2)) \leq \lambda f(\alpha_1, \beta_1) + (1 - \lambda) f(\alpha_2, \beta_2) \quad (9)$$
for all \( \lambda \in [0, 1] \) and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \). This implies that \( f(\alpha, \beta) \) is convex in \((\alpha, \beta)\).

(ii) *(Monotonicity)* From (8), \( f(\alpha, \beta) \) is non-increasing in \( \alpha \) for fixed \( \beta \), and non-decreasing in \( \beta \) for fixed \( \alpha \). As a result, setting \( \alpha = 1 \) provides the *fastest* convergence whenever belief convergence in expectation is achieved (for a given fixed \( \beta \)). Note that when \( \alpha = 1 \), when two nodes interact, they simply switch their beliefs, so almost sure belief convergence never happens as soon as at least two nodes initially hold different beliefs.

When \( G_{\text{pst}} \) is connected, the second largest eigenvalue of \( L_{\text{pst}} \), denoted by \( \lambda_2(L_{\text{pst}}) \), is positive. We can readily see that \( f(\alpha, 0) = 1 - \alpha \lambda_2(L_{\text{pst}}) < 1 \). From (8), we also have \( f(\alpha, \beta) \to \infty \) as \( \beta \to \infty \) provided that \( G_{\text{neg}} \) is nonempty. Combining these observations with the monotonicity of \( f \), we conclude that:

**Proposition 2.** Assume that \( G_{\text{pst}} \) is connected. Then for any fixed \( \alpha \in (0, 1] \), there exists a threshold value \( \beta^* > 0 \) (that depends on \( \alpha \)) such that

(i) Belief convergence in expectation is achieved for all initial values if \( 0 \leq \beta < \beta^* \);

(ii) Belief divergence in expectation is achieved for almost all initial values if \( \beta > \beta^* \).

We remark that belief divergence can only happen for almost all initial values since if the initial beliefs of all the nodes are identical, they do not evolve over time.

### 3.1.3. Examples

An interesting question is to determine how the phase transition threshold \( \beta^* \) scales with the network size. Answering this question seems challenging. However there are networks for which we can characterize \( \beta^* \) exactly. Next we derive explicit expressions for \( \beta^* \) when \( G \) is a complete graph and a ring graph, respectively. These two topologies represent the most dense and almost the most sparse structures for a connected network.

**Example 1 (Complete Graph).** Let \( G = K_n \), the complete graph with \( n \) nodes, and consider the node pair selection matrix \( P = \frac{1}{n-1}(11' - I) \). Let \( L(K_n) = nI - 11' \) be the Laplacian of \( K_n \). Then \( L(K_n) \) has eigenvalue 0 with multiplicity 1 and eigenvalue \( n \) with multiplicity \( n-1 \). Define \( L(G_{\text{neg}}) \) as the standard Laplacian of \( G_{\text{neg}} \). Observe that

\[
I - \alpha L_{\text{pst}} + \beta L_{\text{neg}} - U = I - \alpha(L_{\text{pst}} + L_{\text{neg}}) + (\alpha + \beta)L_{\text{neg}} - U
\]

\[
= I - \frac{2\alpha}{n(n-1)}L(K_n) + \frac{2(\alpha + \beta)}{n(n-1)}L(G_{\text{neg}}) - U. \tag{10}
\]
Also note that $L(G_{\text{neg}})L(K_n) = L(K_n)L(G_{\text{neg}}) = nL(G_{\text{neg}})$. From these observations, we can then readily conclude that:

$$\beta_* = \frac{n\alpha}{\lambda_{\max}(L(G_{\text{neg}}))} - \alpha. \quad (11)$$

**Example 2 (Erdős-Rényi Negative Graph over Complete Graph).** Let $G = K_n$. Let $G_{\text{neg}}$ be the Erdős-Rényi random graph (Erdős and Rényi (1960)) where for any $i, j \in V$, $\{i, j\} \in E_{\text{neg}}$ with probability $p$ (independently of other links). Note that since $G_{\text{neg}}$ is a random subgraph, the function $f(\alpha, \beta)$ becomes a random variable, and we denote by $P$ the probability measure related to the randomness of the graph in Erdős-Rényi’s model. Spectral theory for random graphs suggests that $(Ding and Jiang (2010))$

$$\frac{\lambda_{\max}(L(G_{\text{neg}}))}{pn} \to 1, \text{ as } n \to \infty. \quad (12)$$

in probability. Now for fixed $p$, we deduce from $(11)$ and $(12)$ that the threshold $\beta_*$ converges, as $n$ grows large, to $\alpha/p$ in probability. Now let us fix the update parameters $\alpha$ and $\beta$, and investigate the impact of the probability $p$ on the convergence in mean.

- If $p < \frac{\alpha}{\alpha + \beta}$, we show that $P[f(\alpha, \beta) < 1] \to 1$, when $n \to \infty$, i.e., when the network is large, we likely achieve convergence in mean. Let $\epsilon < \frac{\alpha}{(\alpha + \beta)p} - 1$. It follows from $(12)$ that

$$P(f(\alpha, \beta) < 1) = P\left(1 - \frac{2\alpha}{n(n-1)} + \frac{2(\alpha + \beta)}{n(n-1)}\lambda_{\max}(L(G_{\text{neg}})) < 1\right)$$

$$= P\left((\alpha + \beta)\lambda_{\max}(L(G_{\text{neg}})) < \alpha n\right)$$

$$= P\left(\frac{\lambda_{\max}(L(G_{\text{neg}}))}{pn} < \frac{\alpha}{(\alpha + \beta)p}\right)$$

$$\geq P\left(\left|\frac{\lambda_{\max}(L(G_{\text{neg}}))}{pn} - 1\right| < \epsilon\right) \to 1, \text{ as } n \to \infty. \quad (13)$$

- If $p > \frac{\alpha}{\alpha + \beta}$, we similarly establish that $P(f(\alpha, \beta) > 1) \to 1$, when $n \to \infty$, i.e., when the network is large, we observe divergence in mean with high probability.

Hence we have a sharp phase transition between convergence and divergence in mean when the proportion of negative links $p$ increases and goes above the threshold $p_* = \alpha/\alpha + \beta$.

**Example 3 (Ring Graph).** Denote $R_n$ as the ring graph with $n$ nodes. Let $A(R_n)$ and $L(R_n)$ be the adjacency and Laplacian matrices of $R_n$, respectively. Let the underlying graph $G = R_n$ with only one negative link (if one has more than two negative links, it is easy to see that divergence in expectation is achieved irrespective of $\beta > 0$). Take $P = A(R_n)/2$. We know that $L(R_n)$ has
eigenvalues $2 - 2 \cos(2\pi k/n)$, $0 \leq k \leq n/2$. Applying Weyl’s inequality we obtain $f(\alpha, \beta) \geq 1 + \frac{\beta - \alpha}{n}$. We conclude that $\beta* < \alpha$, irrespective of $n$.

3.2. Mean-square Convergence

We now turn our attention to the analysis of the mean-square convergence and divergence. Define:

$$
\mathbb{E}\{|y(k)|^2\} = \mathbb{E}\{x(k)'(I - U)x(k)\}
= x(0)'\mathbb{E}\{W(0)\ldots W(k-1)(I - U)W(k-1)\ldots W(0)\}x(0).
$$

(14)

Again based on inequalities (7), we see that belief convergence in mean square is equivalent to

$$
\lim_{k \to \infty} \mathbb{E}\{|y(k)|^2\} = 0,
$$

and belief divergence to

$$
\limsup_{k \to \infty} \mathbb{E}\{|y(k)|^2\} = \infty.
$$

Define:

$$
\Phi(k) = \begin{cases} 
\mathbb{E}\{W(0)\ldots W(k-1)(I - U)W(k-1)\ldots W(0)\}, & k \geq 1, \\
I - U, & k = 0.
\end{cases}
$$

(15)

Then, $\Phi(k)$ evolves as a linear dynamical system (Fagnani and Zampieri (2008))

$$
\Phi(k) = \mathbb{E}\{W(0)\ldots W(k-1)(I - U)W(k-1)\ldots W(0)\}
= \mathbb{E}\{W(0)(I - U)W(1)\ldots W(k-1)(I - U)W(k-1)\ldots W(1)(I - U)W(0)\}
= \mathbb{E}\{(W(k) - U)\Phi(k-1)(W(k) - U)\},
$$

(16)

where in the second equality we have used the facts that $(I - U)^2 = I - U$ and $(I - U)W(k) = W(k)(I - U) = W(k) - U$ for all possible realizations of $W(k)$, and the third equality is due to that $W(k)$ and $W(0)$ are i.i.d. We can rewrite (16) using an equivalent vector form:

$$
\text{vec}(\Phi(k)) = \Theta \text{vec}(\Phi(k-1)),
$$

(17)

where $\Theta$ is the matrix in $\mathbb{R}^{n^2 \times n^2}$ given by

$$
\Theta = \mathbb{E}\{(W(0) - U) \otimes (W(0) - U)\}
= \sum_{\{i,j\} \in G_{\text{pos}}} [P^i]_{ij} \left( (W_{ij}^+ - U) \otimes (W_{ij}^+ - U) \right) + \sum_{\{i,j\} \in G_{\text{neg}}} [P^i]_{ij} \left( (W_{ij}^- - U) \otimes (W_{ij}^- - U) \right).
$$

Let $S_{\lambda}$ be the eigenspace corresponding to an eigenvalue $\lambda$ of $\Theta$. Define

$$
\lambda* := \max\{\lambda \in \sigma(\Theta) : \text{vec}(I - U) \notin S_{\lambda}^\perp\},
$$
which denotes the spectral radius of \( \Theta \) restricted to the smallest invariant subspace containing \( \text{vec}(I - U) \), i.e., \( S := \text{span}\{\Theta^k \text{vec}(I - U), k = 0, 1, \ldots \} \). Then mean-square belief convergence/divergence is fully determined by \( \lambda_* \): convergence in mean square for all initial conditions is achieved if \( \lambda_* < 1 \), and divergence for almost all initial conditions is achieved if \( \lambda_* > 1 \).

Observing that \( \lambda \leq 1 \) for every \( \lambda \in \sigma(W_{ij}^+) \) and \( \lambda \geq 1 \) for every \( \lambda \in \sigma(W_{ij}^-) \), we can also conclude that each link in \( E_{pst} \) contributes positively to \( \lambda_{\max}(\Theta) \) and each link in \( E_{neg} \) contributes negatively to \( \lambda_{\max}(\Theta) \). However, unlike in the case of the analysis of convergence in expectation, although \( \lambda_* \) defines a precise threshold for the phase-transition between mean-square convergence and divergence, it is difficult to determine the influence \( E_{pst} \) and \( E_{neg} \) have on \( \lambda_* \). The reason is that they are coupled in a nontrivial manner for the invariant subspace \( S \). Nevertheless, we are still able to propose the following conditions for mean-square belief convergence and divergence:

**Proposition 3.** Belief convergence is achieved for all initial values in mean square if 
\[
\lambda_{\max}(I - 2\alpha(1 - \alpha)L_{pst}^\dagger + 2\beta(1 + \beta)L_{neg}^\dagger - U) < 1;
\]
belief divergence is achieved in mean square for almost all initial values if 
\[
\lambda_{\max}(I - \alpha L_{pst}^\dagger + \beta L_{neg}^\dagger - U) > 1 \text{ or } \lambda_{\min}(I - 2\alpha(1 - \alpha)L_{pst}^\dagger + 2\beta(1 + \beta)L_{neg}^\dagger - U) > 1.
\]

The condition \( \lambda_{\max}(I - \alpha L_{pst}^\dagger + \beta L_{neg}^\dagger - U) \) is sufficient for mean square divergence, in view of Proposition 4 and the fact that \( \mathcal{L}_1 \) divergence implies \( \mathcal{L}_p \) divergence for all \( p \geq 1 \). The other conditions are essentially consistent with the upper and lower bounds of \( \lambda_* \) established in Proposition 4.4 of Fagnani and Zampieri (2008). Proposition 3 is a consequence of Lemma 3 (see Appendix), as explained in Remark 4.

4. Almost Sure Convergence vs. Divergence

In this section, we explore the almost sure convergence of beliefs in signed social networks. While the analysis of the convergence of beliefs in mean and square-mean mainly relied on spectral arguments, we need more involved probabilistic methods (e.g., sample-path arguments, martingale convergence theorems) to study almost sure convergence or divergence. We first establish two insightful properties of the belief evolutions: (i) the no-survivor property stating that in case of almost sure divergence, the difference between the beliefs of any two nodes in the network tends to infinity (along a subsequence of instants); (ii) the live-or-die property which essentially states that the maximum difference between the beliefs of any two nodes either grows to infinity, or vanishes to
zero. We then show a zero-one law and a phase transition of almost sure convergence/divergence. Finally, we investigate the robustness of networks against negative links. More specifically, we show that when the graph $G_{\text{pos}}$ of positive links contains an hypercube, and when the positive updates are truly averaging, i.e., $\alpha = 1/2$, then almost sure belief convergence is reached in finite time, irrespective of the number of negative links, their positions in the network, and the negative update parameter $\beta$. We believe that these are the only networks enjoying this strong robustness property.

4.1. No-Survivor Theorem

The following theorem establishes that in case of almost sure divergence, there is no pair of nodes that can survive this divergence: for any two nodes, the difference in their beliefs grow arbitrarily large.

**Theorem 1.** (No-Survivor) Fix the initial condition and assume almost sure belief divergence. Then $P\left(\limsup_{k \to \infty} |x_i(k) - x_j(k)| = \infty\right) = 1$ for all $i \neq j \in V$.

Observe that the above result only holds for the almost sure divergence. We may easily build simple network examples where we have belief divergence in expectation (or mean square), but where some node pairs survive, in the sense that the difference in their beliefs vanishes (or at least bounded). The no-survivor theorem indicates that to check almost sure divergence, we may just observe the evolution of beliefs held at two arbitrary nodes in the network.

4.2. The Live-or-Die lemma and Zero-One Laws

Next we further classify the ways beliefs can evolve. Specifically, we study the following events: for any initial beliefs $x^0$,

$$C_{x^0} \equiv \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\}, \quad \mathcal{D}_{x^0} \equiv \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\},$$

$$C^*_{x^0} \equiv \left\{ \liminf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\}, \quad \mathcal{D}^*_{x^0} \equiv \left\{ \liminf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\},$$

and

$$C \equiv \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\} \text{ for all } x^0 \in \mathbb{R}^n,$$

$$\mathcal{D} \equiv \left\{ \exists \text{ (deterministic) } x^0 \in \mathbb{R}^n \text{ s.t. } \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\}.$$
We establish that the maximum difference between the beliefs of any two nodes either goes to \( \infty \), or to 0. This result is referred to as \textit{live-or-die} lemma:

**Lemma 1.** \((\text{Live-or-Die})\) Let \( \alpha \in (0,1) \) and \( \beta > 0 \). Suppose \( G_{\text{pst}} \) is connected. Then (i) \( \mathbb{P}(C_0^*) + \mathbb{P}(D_0^*) = 1 \); (ii) \( \mathbb{P}(C_0) + \mathbb{P}(D_0) = 1 \).

As a consequence, almost surely, one the following events happen:

\[
\{ \lim_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \}; \\
\{ \lim_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \}; \\
\{ \lim \inf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0; \lim \sup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \}.
\]

The Live-or-Die lemma deals with events where the initial beliefs have been fixed. We may prove stronger results on the probabilities of events that hold for \textit{any} initial condition, e.g., \( C \), or for at least one initial condition, e.g., \( D \):

**Theorem 2.** \((\text{Zero-One Law})\) Let \( \alpha \in [0,1] \) and \( \beta > 0 \). Both \( C \) and \( D \) are trivial events (i.e., each of them occurs with probability equal to either 1 or 0) and \( \mathbb{P}(C) + \mathbb{P}(D) = 1 \).

To prove this result, we show that \( C \) is a tail event, and hence trivial in view of Kolmogorov’s zero-one law (the same kind of arguments has been used by Tahbaz-Salehi and Jadbabaie (2008)). From the Live-or-Die lemma, we then simply deduce that \( D \) is also a trivial event. Note that \( C_0^* \) and \( D_0^* \) may not be trivial events. In fact, we can build examples where \( \mathbb{P}(C_0^*) = 1/2 \) and \( \mathbb{P}(D_0^*) = 1/2 \).

### 4.3. Phase Transition

As for the convergence in expectation, for fixed positive update parameter \( \alpha \), we are able to establish the existence of thresholds for the value \( \beta \) of the negative update parameter, which characterizes the almost sure belief convergence and divergence.

**Theorem 3.** \((\text{Phase Transition})\) Suppose \( G_{\text{pst}} \) is connected. Fix \( \alpha \in (0,1) \) with \( \alpha \neq 1/2 \). Then

(i) there exists \( \beta^\sharp(\alpha) > 0 \) such that \( \mathbb{P}^*(C) = 1 \) if \( 0 \leq \beta < \beta^\sharp \);

(ii) there exists \( \beta^\flat(\alpha) > 0 \) such that \( \mathbb{P}(\lim \inf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty) = 1 \) for almost all initial values if \( \beta > \beta^\flat \).
It should be observed that the divergence condition in (ii) is stronger than our notion of almost sure belief divergence (the maximum belief difference between two nodes diverge almost surely to $\infty$). Also note that $\beta^\circ \leq \beta^\#$, and we were not able to show that the gap between these two thresholds vanishes (as in the case of belief convergence in expectation or mean-square).

### 4.4. Robustness to Negative Links: the Hypercube

We have seen in Theorem 3 that when $\alpha \neq 1/2$, one single negative link is capable of driving the network beliefs to almost sure divergence as long as $\beta$ is sufficiently large. The following result shows that the evolution of the beliefs can be robust against negative links. This is the case when nodes can reach an agreement in finite time. In what follows, we provide conditions on $\alpha$ and the structure of the graph under which finite time belief convergence is reached.

**Proposition 4.** Suppose there exist an integer $T \geq 1$ and a finite sequence of node pairs $\{i_s, j_s\} \in G_{pst}$, $s = 1, 2, \ldots, T$ such that $W^+_{i_T,j_T} \cdots W^+_{i_1,j_1} = U$. Then $P(\mathcal{E}) = 1$ for all $\beta \geq 0$.

Proposition 4 is a direct consequence of the Borel-Cantelli Lemma. If there is a finite sequence of node pairs $\{i_s, j_s\} \in G_{pst}$, $s = 1, 2, \ldots, T$ such that $W^+_{i_T,j_T} \cdots W^+_{i_1,j_1} = U$, then

$$P\left(W(k+T) \cdots W(k+1) = U \right) \geq \left(\frac{p_*}{n}\right)^T,$$

for all $k \geq 0$, where $p_* = \min\{p_{ij} + p_{ji} : \{i, j\} \in E\}$. Noting that $UW(k) = W(k)U = U$ for all possible realizations of $W(k)$, the Borel-Cantelli Lemma guarantees that

$$P\left(\lim_{k \to \infty} W(k) \cdots W(0) = U \right) = 1$$

for all $\beta \geq 0$, or equivalently, $P(\mathcal{E}) = 1$ for all $\beta \geq 0$. This proves Proposition 4.

The existence of such finite sequence of node pairs under which the beliefs of the nodes in the network reach a common value in finite time is crucial (we believe that this condition is actually necessary) to ensure that the influence of $G_{neg}$ vanishes. It seems challenging to know whether this is at all possible. As it turns out, the structure of the positive graph plays a fundamental role. To see that, we first provide some definitions.

**Definition 3.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be a pair of graphs. The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \square G_2$, is defined by
Figure 2  The hypercubes $H^1$, $H^2$, and $H^3$.

(i) the vertex set of $G_1 \square G_2$ is $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of $V_1$ and $V_2$;
(ii) for any two vertices $(v_1, v_2), (u_1, u_2) \in V_1 \times V_2$, there is an edge between them in $G_1 \square G_2$ if and only if either $v_1 = u_1$ and $\{v_2, u_2\} \in E_2$, or $v_2 = u_2$ and $\{v_1, u_1\} \in E_1$.

Let $K_2$ be the complete graph with two nodes. The $m$-dimensional Hypercube $H^m$ is then defined as

$$H^m = K_2 \square K_2 \ldots \square K_2,$$  

$m$ times

An illustration of hypercubes is in Figure 2

The following result provides sufficient conditions to achieve finite-time convergence.

**Proposition 5.** If $\alpha = 1/2$, $n = 2^m$ for some integer $m > 0$, and $G_{psl}$ has a subgraph isomorphic with an $m$-dimensional hypercube, then there exists sequence of $(n \log_2 n)/2$ node pairs $\{i_s, j_s\} \in G_{psl}, s = 1, \ldots, (n \log_2 n)/2$ such that $W_{i_1 j_1}^+ \cdots W_{i_T j_T}^+ = U$.

Next we derive necessary conditions for finite time convergence. Let us first recall the following definition.

**Definition 4.** Let $G = (V, E)$ be a graph. A matching of $G$ is a set of pairwise non-adjacent edges in the sense that no two edges share a common vertex. A perfect matching of $G$ is a matching which matches all vertices.

**Proposition 6.** If there exist an integer $T \geq 1$ and a sequence of node pairs $\{i_s, j_s\} \in G_{psl}, s = 1, 2, \ldots, T$ such that $W_{i_T j_T}^+ \cdots W_{i_1 j_1}^+ = U$, then $\alpha = 1/2$, $n = 2^m$, and $G_{psl}$ has a perfect matching.
In fact, in the proof of Proposition 6, we show that if \( \mathbf{W} \cdot \mathbf{W}^+ \cdots \mathbf{W}^+ \mathbf{T} \mathbf{W} \mathbf{T}^{-1} \mathbf{T} \cdots \mathbf{W}^+ \mathbf{W} \mathbf{T} = \mathbf{U} \), then \( \{\{i_1, j_1\}, \ldots, \{i_T, j_T\}\} \) forms a perfect matching of \( G_{pst} \).

We have seen that the belief dynamics and convergence can be robust against negative links, but this robustness comes at the expense of strong conditions on the number of the nodes and the structure of the positive graph.

5. Asymmetric and Constrained Update: Belief Clustering

So far we have studied the belief dynamics when the node interactions are symmetric, and the values of beliefs are unconstrained. In this section we consider the case when these assumptions do not hold, that is:

- When \( \{i, j\} \) is selected, it might happen that only one of the two nodes in \( i \) and \( j \) updates its belief;
- There might be a hard constraint on beliefs: \( x_i(k) \in [-A, A] \) for all \( i \) and \( k \), and for some \( A > 0 \).

In this section, we consider the following model for the updates of the beliefs. Define:

\[
\mathbb{I}_A(z) = \begin{cases} 
-A, & \text{if } z < -A; \\
\text{z}, & \text{if } z \in [-A, A]; \\
A, & \text{if } z > A.
\end{cases}
\]  

(18)

Let \( a, b, c > 0 \) be three positive real numbers such that \( a + b + c = 1 \), and define the function \( \theta : E \to \mathbb{R} \) so that \( \theta(\{i, j\}) = \alpha \) if \( \{i, j\} \in E_{pst} \) and \( \theta(\{i, j\}) = -\beta \) if \( \{i, j\} \in E_{neg} \). Assume that node \( i \) interacts with node \( j \) at time \( k \). Nodes \( i \) and \( j \) update their beliefs as:

Asymmetric Constrained Model:

\[
x_i(k+1) = \mathbb{I}_A((1 + \theta)x_i(k) - \theta x_j(k)) \quad \text{and} \quad x_j(k+1) = x_j(k), \quad \text{with probability } a;
\]

\[
x_j(k+1) = \mathbb{I}_A((1 + \theta)x_j(k) - \theta x_i(k)) \quad \text{and} \quad x_i(k+1) = x_i(k), \quad \text{with probability } b;
\]

\[
x_m(k+1) = \mathbb{I}_A((1 + \theta)x_m(k) - \theta x_{-m}(k)) \quad \text{with probability } c.
\]

(19)

Under this model, the belief dynamics become nonlinear, which brings new challenges in the analysis. We continue to use \( \mathbb{P} \) to denote the overall probability measure capturing the randomness of the updates in the asymmetric constrained model.

We first study the belief dynamics in specific graphs, referred to as balanced graphs, and show that for these graphs, the beliefs become asymptotically clustered (the belief at a node converges
either to $A$ or $-A$) when the negative update parameter $\beta$ is large enough. Then, we investigate what can happen in absence of the balance structure.

### 5.1. Balanced Graphs and Clustering

Balanced graphs are defined as follows, for which we refer to Wasserman and Faust (1994) for a comprehensive discussion.

**Definition 5.** Let $G = (V, E)$ be a signed graph. Then

(i) $G$ is **weakly balanced** if there is an integer $k \geq 2$ and a partition of $V = V_1 \cup V_2 \cdots \cup V_k$, where $V_1, \ldots, V_k$ are nonempty and mutually disjoint, such that any edge between different $V_i$'s is negative, and any edge within each $V_i$ is positive.

(ii) $G$ is **strongly balanced** if it is weakly balanced with $k = 2$.

Harary's balance theorem states that a signed graph $G$ is strongly balanced if and only if there is no cycle with an odd number of negative edges in $G$ (Cartwright and Harary (1956)), while $G$ is weakly balanced if and only if no cycle has exactly one negative edge in $G$ (Davis (1967)).

In the case of strongly balanced graphs, we can show that beliefs are asymptotically clustered when $\beta$ is large enough, as stated in the following theorem.

**Theorem 4.** Assume that the graph is strongly balanced under partition $V = V_1 \cup V_2$, and that $G_{V_1}$ and $G_{V_2}$ are connected. For any $\alpha \in (0, 1) \setminus \{1/2\}$, when $\beta$ is sufficiently large, for almost all initial values, almost sure belief clustering is achieved under the update model (19). In other words, for almost all $x^0$, there are random variables $B_{1}^i(x^0)$ and $B_{2}^i(x^0)$, both taking values in $\{-A, A\}$, such that:

$$
P\left( \lim_{k \to \infty} x_i(k) = B_{1}^i(x^0), \forall i \in V_1; \lim_{k \to \infty} x_i(k) = B_{2}^i(x^0), \forall i \in V_2 \right) = 1. \quad (20)$$

We remark that $B_{1}^i(x^0) + B_{2}^i(x^0) = 0$ holds almost surely in Theorem 4. In other words, for weakly balanced social networks, beliefs are eventually polarized to the two opinion boundaries. The analysis of belief dynamics in weakly balanced graphs is more involved, and we restrict our attention to complete graphs. In social networks, this case means that everyone knows everyone else – which constitutes a suitable model for certain social groups of small sizes (a classroom, a
sport team, or the UN, see Easley and Kleinberg (2010). As stated in the following theorem, for weakly balanced complete graphs, beliefs are again clustered.

**Theorem 5.** Assume that \( G = (V, E) \) is a complete weakly balanced graph under the partition \( V = V_1 \cup V_2 \cdots \cup V_m \) with \( m \geq 2 \). Further assume that \( G_{V_j}, j = 1, \ldots, m \) are connected. For any \( \alpha \in (0, 1) \setminus \{1/2\} \), when \( \beta \) is sufficiently large, almost sure belief clustering is achieved for almost all initial values under (19), i.e., for almost all \( x^0 \), there are \( m \) random variables, \( B^1_j(x^0), \ldots, B^m_j(x^0) \), all taking values in \( \{-A, A\} \), such that:

\[
P\left( \lim_{k \to \infty} x_i(k) = B^j_i(x^0), \forall i \in V_j, j = 1, \ldots, m \right) = 1.
\]

(21)

**Remark 3.** Note that under the model (3), it can be shown, as in Altafini (2013), that if \( G \) is not strongly balanced, then

\[
P\left( \lim_{k \to \infty} x_i(k) = 0, i \in V \right) = 1.
\]

This almost sure convergence to zero may seem unrealistic in our real-world scenarios, and is difficult to interpret. Observe that under our model, Theorem 5 shows that nontrivial belief clustering occurs in weakly balanced graphs (and hence in some graphs that are not strongly balanced).

### 5.2. When Balance is Missing

In absence of any balance property for the underlying graph, belief clustering may not happen. However, we can establish that when the positive graph is connected, then clustering cannot be achieved when \( \beta \) is large enough. In fact, the belief of a given node touches the two boundaries \(-A\) and \( A\) an infinite number of times. Note that if the positive graph is connected, then the graph cannot be balanced.

**Theorem 6.** Assume that the positive graph \( G_{\text{pos}} \) is connected. For any \( \alpha \in (0, 1) \setminus \{1/2\} \), when \( \beta \) is sufficiently large, for almost all initial beliefs, under (19), we have: for all \( i \in V \),

\[
P\left( \lim_{k \to \infty} \inf x_i(k) = -A, \lim_{k \to \infty} \sup x_i(k) = A \right) = 1.
\]

(22)

### 6. Conclusions

The evolution of opinions over signed social networks was studied. Each link marking interpersonal interaction in the network was associated with a sign indicating friend or enemy relations.
The dynamics of opinions was defined along positive and negative links, respectively. We have presented a comprehensive analysis to the belief convergence and divergence under various modes: in expectation, in mean-square, and almost surely. Phase transitions were established with sharp thresholds for the mean and mean-square convergence. In the almost sure sense, some surprising results were presented. When opinions have hard lower and upper bounds with asymmetric updates, the classical structure balance properties were shown to play a key role in the belief clustering. We believe that these results have largely extended our understanding to how trustful and antagonistic relations shape social opinions.

Some interesting directions for future research include the following topics. Intuitively there is natural coupling between the structure dynamics and the opinion evolution for signed networks. How this coupling determines the formation of the social structure is an interesting question bridging the studies on the dynamics of signed graphs (e.g., Marvel et al. (2011)) and the opinion dynamics on signed social networks (e.g., Altafini (2012, 2013)). It will also be interesting to ask what might be a proper model, and what is the role of structure balance, for Bayesian opinion evolution on signed social networks (e.g., Bikhchandani et al. (1992)).

Appendix: Proofs of Statements

A. The Triangle Lemma

We establish a key technical lemma on the relative beliefs of three nodes in the network in the presence of at least one link among the three nodes. Denote \( J_{ab}(k) := |x_a(k) - x_b(k)| \) for \( a, b \in V \) and \( k \geq 0 \).

**Lemma 2.** Let \( i_0, i_1, i_2 \) be three different nodes in \( V \). Suppose \( \{i_0, i_1\} \in E \). There exist a positive number \( \delta > 0 \) and an integer \( Z > 0 \), such that

(i) there is a sequence of \( Z \) successive node pairs leading to \( J_{i_1 i_2}(Z) \geq \delta J_{i_0 i_1}(0) \);

(ii) there is a sequence of \( Z \) successive node pairs leading to \( J_{i_1 i_2}(Z) \geq \delta J_{i_0 i_2}(0) \).

Here \( \delta \) and \( Z \) are absolute constants in the sense that they do not depend on \( i_0, i_1, i_2 \), nor on the values held at these nodes.
Proof. We assume \( n \geq 5 \). Generality is not lost by making this assumption because for \( n = 3 \) and \( n = 4 \), some (tedious but straightforward) analysis on each possible \( G \) leads to the desired conclusion.

\((i)\). There are two cases: \( \{i_0, i_1\} \in E_{\text{pst}}, \) or \( \{i_0, i_1\} \in E_{\text{neg}} \). We prove the desired conclusion for each of the two cases. Without loss of generality, we assume that \( x_{i_0}(0) < x_{i_1}(0) \).

- Let \( \{i_0, i_1\} \in E_{\text{pst}} \). If \( x_{i_2}(0) \) \( \in \left[\frac{3}{4} x_{i_0}(0) + \frac{1}{4} x_{i_1}(0), \frac{1}{4} x_{i_0}(0) + \frac{3}{4} x_{i_1}(0)\right] \), we have \( J_{i_1 i_2}(0) \geq \frac{1}{4} J_{i_0 i_1}(0) \). Thus, the desired conclusion holds for \( \delta = \frac{1}{4} \), arbitrary \( Z > 0 \), and any node pair sequence over \( 0, 1, \ldots, Z - 1 \) for which \( i_0, i_1, i_2 \) are never selected.

On the other hand suppose \( x_{i_2}(0) \not\in \left[\frac{3}{4} x_{i_0}(0) + \frac{1}{4} x_{i_1}(0), \frac{1}{4} x_{i_0}(0) + \frac{3}{4} x_{i_1}(0)\right] \). Take

\[
d_+ = \begin{cases} \lceil \log_{1-2\alpha} \frac{1}{4} \rceil & \text{if } \alpha \neq \frac{1}{2}, \\ 1, & \text{if } \alpha = \frac{1}{2}. \end{cases}
\]

(23)

If \( \{i_0, i_1\} \) is selected for \( 0, 1, \ldots, d_+ - 1 \), we obtain \( J_{i_0 i_1}(d_+) \leq \frac{1}{4} J_{i_0 i_1}(0) \) which leads to

\[
x_{i_1}(d_+) \in \left[\frac{5}{8} x_{i_0}(0) + \frac{2}{8} x_{i_1}(0), \frac{3}{8} x_{i_0}(0) + \frac{5}{8} x_{i_1}(0)\right]; \quad x_{i_2}(d_+) = x_{i_2}(0).
\]

This gives us \( J_{i_1 i_2}(d_+) \geq \frac{1}{8} J_{i_0 i_1}(0) \).

- Let \( \{i_0, i_1\} \in E_{\text{neg}} \). If \( x_{i_2}(0) \not\in \left[\frac{1}{4} x_{i_0}(0) + \frac{1}{2} x_{i_1}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{4} x_{i_1}(0)\right] \), we have \( J_{i_1 i_2}(0) \geq \frac{1}{2} J_{i_0 i_1}(0) \). The conclusion holds for \( \delta = \frac{1}{2} \), arbitrary \( Z > 0 \), and any node pair sequence over \( 0, 1, \ldots, Z - 1 \) for which \( i_0, i_1, i_2 \) are never selected.

On the other hand let \( x_{i_2}(0) \) \( \in \left[\frac{1}{4} x_{i_0}(0) + \frac{1}{2} x_{i_1}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{4} x_{i_1}(0)\right] \). Take \( d^* = \lceil \log_{1+2\beta} 4 \rceil \). Let \( \{i_0, i_1\} \) be selected for \( 0, 1, \ldots, d^* - 1 \). In this case, \( x_{i_0}(s) \) and \( x_{i_1}(s) \) are symmetric with respect to their center \( \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_1}(0) \) for all \( s = 0, \ldots, d^* \), and \( J_{i_0 i_1}(d_+) \geq 4 J_{i_0 i_1}(0) \). Thus we have \( x_{i_2}(d^*) = x_{i_2}(0) \), and

\[
x_{i_1}(d^*) \geq \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_1}(0) + 2 (x_{i_1}(0) - x_{i_0}(0))
\]

\[
= \frac{3}{2} x_{i_0}(0) + \frac{5}{2} x_{i_1}(0).
\]

(24)

We can therefore conclude that \( J_{i_1 i_2}(d^*) \geq J_{i_0 i_1}(0) \).

In summary, the desired conclusion holds for \( \delta = \frac{1}{8} \) and

\[
Z = \begin{cases} \max\{\lceil \log_{1+2\beta} 4 \rceil, \lceil \log_{1-2\alpha} \frac{1}{4} \rceil\} & \text{if } \alpha \neq \frac{1}{2}, \\ \lceil \log_{1+2\beta} 4 \rceil, & \text{if } \alpha = \frac{1}{2}. \end{cases}
\]

(25)
(ii). We distinguish the cases \( \{i_0, i_1\} \in E_{\text{pst}} \) and \( \{i_0, i_1\} \in E_{\text{neg}} \). Without loss of generality, we assume that \( x_{i_0}(0) < x_{i_2}(0) \).

- Let \( \{i_0, i_1\} \in E_{\text{pst}} \). If \( x_{i_1}(0) \notin \left[ \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_2}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \right] \), we have \( J_{i_1i_2}(0) \geq \frac{1}{2} J_{i_0i_2}(0) \). The conclusion holds for \( \delta = \frac{1}{2} \), arbitrary \( Z > 0 \), and any node pair sequence \( 0, 1, \ldots, Z - 1 \) for which \( i_0, i_1, i_2 \) are never selected.

Now let \( x_{i_1}(0) \in \left[ \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_2}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \right] \). We write \( x_{i_1}(0) = (1 - \varsigma) x_{i_0}(0) + \varsigma x_{i_2}(0) \) with \( \varsigma \in [\frac{1}{2}, \frac{3}{2}] \). Let \( \{i_0, i_1\} \) be the node pair selected for \( 0, 1, \ldots, d_* - 1 \) with \( d_* \) defined by (23). Note that according to the structure of the update rule, \( x_{i_0}(s) \) and \( x_{i_1}(s) \) will be symmetric with respect to their center \( (1 - \frac{\varsigma}{2}) x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \) for all \( s = 0, \ldots, d_* \), and \( J_{i_0i_1}(d_*) \leq \frac{1}{2} J_{i_0i_1}(0) \). This gives us \( x_{i_2}(d_*) = x_{i_2}(0) \) and

\[
x_{i_1}(d_*) = \left[ (1 - \frac{\varsigma}{2}) x_{i_0}(0) + \frac{\varsigma}{2} x_{i_2}(0) - \frac{1}{8} (x_{i_1}(0) - x_{i_0}(0)),
\right.
\[
\left. (1 - \frac{\varsigma}{2}) x_{i_0}(0) + \frac{\varsigma}{2} x_{i_2}(0) + \frac{1}{8} (x_{i_1}(0) - x_{i_0}(0)) \right],
\]

which implies

\[
J_{i_1i_2}(d_*) \geq (1 - \frac{5\varsigma}{8}) J_{i_0i_2}(0) \geq \frac{1}{16} J_{i_0i_2}(0).
\]  

(27)

- Let \( \{i_0, i_1\} \in E_{\text{neg}} \). If \( x_{i_1}(0) \notin \left[ \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_2}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \right] \), the conclusion holds for the same reason as in the case where \( \{i_0, i_1\} \in E_{\text{pst}} \).

Now let \( x_{i_1}(0) \in \left[ \frac{1}{2} x_{i_0}(0) + \frac{1}{2} x_{i_2}(0), -\frac{1}{2} x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \right] \). We continue to use the notation \( x_{i_1}(0) = (1 - \varsigma) x_{i_0}(0) + \varsigma x_{i_2}(0) \) with \( \varsigma \in [\frac{1}{2}, \frac{3}{2}] \). Let \( \{i_0, i_1\} \) be the node pair selected for \( 0, 1, \ldots, d^* - 1 \) where \( d^* = \lceil \log_{1+2\beta} 4 \rceil \). In this case, \( x_{i_0}(s) \) and \( x_{i_1}(s) \) are still symmetric with respect to their center \( (1 - \frac{\varsigma}{2}) x_{i_0}(0) + \frac{3}{2} x_{i_2}(0) \) for all \( s = 0, 1, \ldots, d^* \), and \( J_{i_0i_1}(d_*) \geq 4 J_{i_0i_1}(0) \).

This gives us \( x_{i_2}(d_*) = x_{i_2}(0) \) and

\[
x_{i_1}(d_*) \geq (1 - \frac{\varsigma}{2}) x_{i_0}(0) + \frac{\varsigma}{2} x_{i_2}(0) + 2 (x_{i_1}(0) - x_{i_0}(0))
\]

\[
= (1 - \frac{5\varsigma}{2}) x_{i_0}(0) + \frac{5\varsigma}{2} x_{i_2}(0)
\]

which implies

\[
J_{i_1i_2}(d_*) \geq (\frac{5\varsigma}{2} - 1) J_{i_0i_2}(0) \geq \frac{1}{4} J_{i_0i_2}(0).
\]

(29)

In summary, the desired conclusion holds for \( \delta = \frac{1}{16} \) with \( Z \) defined in (23).
Introduce

$$X_{\min}(k) = \min_{i \in V} x_i(k); \quad X_{\max}(k) = \max_{i \in V} x_i(k).$$

We define $\mathcal{X}(k) = X_{\max}(k) - X_{\min}(k)$. Suppose belief divergence is achieved almost surely. Take a constant $N_0$ such that $N_0 > \mathcal{X}(0)$. Then almost surely,

$$K_1 := \inf_k \{\mathcal{X}(k) \geq N_0\}$$

is a finite number. Then $K_1$ is a stopping time for the node pair selection process $G_k, k = 0, 1, 2, \ldots$ since

$$\{K_1 = k\} \in \sigma(G_0, \ldots, G_{k-1})$$

for all $k = 1, 2, \ldots$ due to the fact that $\mathcal{X}(k)$ is, indeed, a function of $G_0, \ldots, G_{k-1}$. Strong Markov Property leads to: $G_{K_1}, G_{K_1+1}, \ldots$ are independent of $\mathcal{F}_{K_1-1}$, and they are i.i.d. with the same distribution as $G_0$ (e.g., Theorem 4.1.3 in Durrett (2010)).

Now take two different (deterministic) nodes $i_0$ and $j_0$. Since $\mathcal{X}(K_1) \geq N_0$, there must be two different (random) nodes $i_\ast$ and $j_\ast$ satisfying $x_{i_\ast}(K_1) < x_{j_\ast}(K_1)$ with $J_{i_\ast j_\ast}(K_1) \geq N_0$. We make the following claim.

**Claim.** There exist a positive number $\delta_0 > 0$ and an integer $Z_0 > 0$ ($\delta_0$ and $Z_0$ are deterministic constants) such that we can always select a sequence of node pairs for time steps $K_1, K_1 + 1, K_1 + Z_0 - 1$ which guarantees $J_{i_0 j_0}(K_1 + Z_0) \geq \delta_0 N_0$.

First of all note that $i_\ast$ and $j_\ast$ are independent with $G_{K_1}, G_{K_1+1}, \ldots$, since $i_\ast, j_\ast \in \mathcal{F}_{K_1-1}$. Therefore, we can treat $i_\ast$ and $j_\ast$ as deterministic and prove the claim for all choices of such $i_\ast$ and $j_\ast$ (because we can always carry out the analysis conditioned on different events $\{i_\ast = i, j_\ast = j\}$, $i, j \in V$). We proceed the proof recursively taking advantage of the Triangle Lemma.

Suppose $\{i_0, j_0\} = \{i_\ast, j_\ast\}$, the claim holds trivially. Now suppose $i_0 \notin \{i_\ast, j_\ast\}$. Either $J_{i_0 i_\ast}(K_1) \geq \frac{N_0}{2}$ or $J_{i_0 j_\ast}(K_1) \geq \frac{N_0}{2}$ must hold. Without loss of generality we assume $J_{i_0 i_\ast}(K_1) \geq \frac{N_0}{2}$. Since $G$ is connected, there is a path $i_0 i_1 \ldots i_\tau j_0$ in $G$ with $\tau \leq n - 2$.

Based on Lemma 2 there exist $\delta > 0$ and integer $Z > 0$ such that a selection of node pair sequence for $K_1, K_1 + 1, \ldots, K_1 + Z - 1$ leads to

$$J_{i_0 i_1}(K_1 + Z) \geq \delta J_{i_0 i_\ast}(K_1) \geq \frac{\delta N_0}{2}.$$
since \( \{i_0, i_1\} \in E \). Applying recursively the Triangle Lemma based on the fact that \( \{i_1, i_2\}, \ldots, \{i_\tau, j_0\} \in E \), we see that a selection of node pair sequence for \( K_1, K_1 + 1, \ldots, K_1 + (\tau + 1)Z - 1 \) will give us

\[
J_{i_0 j_0}(K_1 + (\tau + 1)Z) \geq \delta^{\tau+1} J_{i_0 i_*}(K_1) \geq \frac{\delta^{\tau+1} N_0}{2}.
\]

Since \( \tau \leq n - 2 \), the claim always holds for \( \delta_0 = \frac{\delta^{n-1}}{2} \) and \( Z_0 = (n - 1)Z \), independently of \( i_* \) and \( j_* \).

Therefore, denoting \( p_* = \min \{p_{ij} + p_{ji} : \{i, j\} \in E\} \), the claim we just proved yields that

\[
\mathbb{P}\left( J_{i_0 j_0}(K_1 + (n - 1)Z) \geq \frac{\delta^{n-1} N_0}{2} \right) \geq \left( \frac{p_*}{n} \right)^{(n-1)Z}.
\]

We proceed the analysis by recursively defining

\[
K_{m+1} := \inf \{ k \geq K_m + Z_0 : X(k) \geq N_0 \}, \quad m = 1, 2, \ldots.
\]

Given that belief divergence is achieved, \( K_m \) is finite for all \( m \geq 1 \) almost surely. Thus,

\[
\mathbb{P}\left( J_{i_0 j_0}(K_m + Z_0) \geq \frac{\delta^{n-1} N_0}{2} \right) \geq \left( \frac{p_*}{n} \right)^{Z_0},
\]

for all \( m = 1, 2, \ldots \). Moreover, the node pair sequence

\[
G_{K_1}, \ldots, G_{K_1+Z_0-1}; \ldots; G_{K_m}, \ldots, G_{K_m+Z_0-1}; \ldots.
\]

are independent and have the same distribution as \( G_0 \) (This is due to that \( \mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_1+1} \subseteq \cdots \subseteq \mathcal{F}_{K_1+Z_0-1} \subseteq \mathcal{F}_{K_2} \ldots \) (cf. Theorem 4.1.4 in Durrett (2010))).

Therefore, we can finally invoke the second Borel-Cantelli Lemma (cf. Theorem 2.3.6 in Durrett (2010)) to conclude that almost surely, there exists an infinite subsequence \( K_{m_s}, s = 1, 2, \ldots \), satisfying

\[
J_{i_0 j_0}(K_{m_s} + Z_0) \geq \frac{\delta^{n-1} N_0}{2}, \quad s = 1, 2, \ldots,
\]

conditioned on that belief divergence is achieved. Since \( \delta \) is a constant and \( N_0 \) is arbitrarily chosen, (32) is equivalent to \( \mathbb{P}\left( \limsup_{k \to \infty} |x_{i_0}(k) - x_{j_0}(k)| = \infty \right) = 1 \), which completes the proof.
C. Proof of Lemma \[1\]

(i). It suffices to show that \( \mathbb{P}\left( \limsup_{k \to \infty} \mathcal{X}(k) \in [a_*, b_*] \right) = 0 \) for all \( 0 < a_* < b_* \). We prove the statement by contradiction. Suppose \( \mathbb{P}\left( \limsup_{k \to \infty} \mathcal{X}(k) \in [a_*, b_*] \right) = p > 0 \) for some \( 0 < a_* < b_* \).

Take \( 0 < \varepsilon < 1 \) and define \( a = a_*(1 - \varepsilon), b = b_*(1 + \varepsilon) \). We introduce

\[
T_1 := \inf \{ k \in [a, b] \}.
\]

Then \( T_1 \) is finite with probability at least \( p \). \( T_1 \) is a stopping time. \( G_{T_1}, G_{T_1+1}, \ldots \) are independent of \( \mathcal{F}_{T_1-1} \), and they are i.i.d. with the same distribution as \( G_0 \).

Now since \( G_{\text{neg}} \) is nonempty, we take a link \( \{i_*, j_*\} \in E_{\text{neg}} \). Repeating the same analysis as the proof of Theorem \[1\] the following statement holds true conditioned on that \( T_1 \) is finite: there exist a positive number \( \delta_0 > 0 \) and an integer \( Z_0 > 0 \) (\( \delta_0 \) and \( Z_0 \) are deterministic constants) such that we can always select a sequence of node pairs for time steps \( T_1, T_1 + 1, T_1 + Z_0 - 1 \) which guarantees \( J_{i_*, j_*}(T_1 + Z_0) \geq \delta_0 a \).

Here \( \delta_0 \) and \( Z_0 \) follow from the same definition in the proof of Theorem \[1\]. Take

\[
m_0 = \left\lfloor \log_{2\beta+1} \frac{2b}{\delta_0 a} \right\rfloor
\]

and let \( \{i_*, j_*\} \) be selected for \( T_1 + Z_0, \ldots, T_1 + Z_0 + m_0 - 1 \). Then noting that \( \{i_*, j_*\} \in E_{\text{neg}} \), the choice of \( m_0 \) and the fact that \( J_{i_*, j_*}(s + 1) = (2\beta + 1)J_{i_*, j_*}(s), s = T_1 + Z_0, \ldots, T_1 + Z_0 + m_0 - 1 \) lead to

\[
\mathcal{X}(T_1 + Z_0 + m_0) \geq J_{i_*, j_*}(T_1 + Z_0 + m_0) \geq (2\beta + 1)^{m_0} \delta_0 a \geq 2b \geq 2b_*.
\]

We have proved that

\[
\mathbb{P}\left( \mathcal{X}(T_1 + Z_0 + m_0) \geq 2b_*, T_1 < \infty \right) \geq \left( \frac{p_*}{n} \right)^{Z_0 + m_0}. \tag{33}
\]

Similarly, we proceed the analysis by recursively defining

\[
T_{m+1} := \inf \{ k \geq T_m + Z_0 + m_0 : \mathcal{X}(k) \in [a, b] \}, \quad m = 1, 2, \ldots.
\]

Given \( \mathbb{P}\left( \limsup_{k \to \infty} \mathcal{X}(k) \in [a_*, b_*] \right) = p, \) \( T_m \) is finite for all \( m \geq 1 \) with probability at least \( p \). Thus, there holds

\[
\mathbb{P}\left( \mathcal{X}(T_m + Z_0 + m_0) \geq 2b_*, T_m < \infty \right) \geq \left( \frac{p_*}{n} \right)^{Z_0 + m_0}, \quad m = 1, 2, \ldots. \tag{34}
\]
The independence of

\[ G_{T_1}, \ldots, G_{T_1 + Z_0 + m_0 - 1}; \ldots; G_{T_m}, \ldots, G_{T_m + Z_0 + m_0 - 1}; \ldots \]

once again allows us to invoke the Borel-Cantelli Lemma to conclude that almost surely, there exists an infinite subsequence \( T_{m_s}, s = 1, 2, \ldots \), satisfying

\[
\mathcal{X}(T_{m_s} + Z_0 + m_0) \geq 2b_s, \quad s = 1, 2, \ldots,
\]

given that \( T_m, m = 1, 2, \ldots \), are finite. In other words, we have obtained that

\[
P\left( \limsup_{k \to \infty} \mathcal{X}(k) \geq 2b_s \mid \limsup_{k \to \infty} \mathcal{X}(k) \in [a_s, b_s] \right) = 1,
\]

which is impossible and the first part of the theorem has been proved.

(ii). It suffices to show that \( P\left( \liminf_{k \to \infty} \mathcal{X}(k) \in [a_s, b_s] \right) = 0 \) for all \( 0 < a_s < b_s \). The proof is again by contradiction. Assume that \( P\left( \liminf_{k \to \infty} \mathcal{X}(k) \in [a_s, b_s] \right) = q > 0 \). Let \( a, b \), and \( T_1 := \inf_k \{ \mathcal{X}(k) \in [a, b] \} \) as defined earlier. \( T_1 \) is finite with probability at least \( q \).

Let \( \ell_0 \in V \) satisfying \( x_{\ell_0}(T_1) = X_{\min}(T_1) \). There is a path from \( \{\ell_0\} \) to every other node in the network since \( G_{\text{pst}} \) is connected. We introduced

\[
V^*_t := \{ j : d(\ell_0, j) = t \text{ in } G_{\text{pst}} \}, \quad t = 0, \ldots, \text{diam}(G_{\text{pst}})
\]

as a partition of \( V \). We relabel the nodes in \( V \setminus \{\ell_0\} \) in the following manner.

\[
\ell_s \in V^*_1, s = 1, \ldots, |V^*_1|;
\]

\[
\ell_s \in V^*_2, s = |V^*_1| + 1, \ldots, |V^*_1| + |V^*_2|;
\]

\[ \ldots \]

\[
\ell_s \in V^*_{\text{diam}(G_{\text{pst}})} - 1, s = \sum_{t=1}^{\text{diam}(G_{\text{pst}}) - 1} |V^*_t|, \ldots, n - 1.
\]

Then the definition of \( V^*_t \) and the connectivity of \( G_{\text{pst}} \) allow us to select a sequence of node pairs in the form of

\[
G_{T_1 + s} = \{ \ell_\rho, \ell_{s+1} \}, \quad \{ \ell_\rho, \ell_{s+1} \} \in E_{\text{pst}} \text{ with } \rho \leq s,
\]

for \( s = 0, \ldots, n - 2 \). Next we give an estimation for \( \mathcal{X} \) under the selected sequence of node pairs.
• Since \( \{\ell_0, \ell_1\} \) is selected at time \( T_1 \), we have

\[
x_{\ell_0}(T_1 + 1) = (1 - \alpha)x_{\ell_0}(T_1) + \alpha x_{\ell_1}(T_1) \leq (1 - \alpha)X_{\min}(T_1) + \alpha X_{\max}(T_1);
\]

\[
x_{\ell_1}(T_1 + 1) = (1 - \alpha)x_{\ell_1}(T_1) + \alpha x_{\ell_0}(T_1) \leq (1 - \alpha)X_{\max}(T_1) + \alpha X_{\min}(T_1).
\]

This leads to \( x_{\ell_s}(T_1 + 1) \leq (1 - \alpha_s)X_{\min}(T_1) + \alpha X_{\max}(T_1), s = 0, 1 \), where \( \alpha_s = \max\{\alpha, 1 - \alpha\} \).

• Note that \( X_{\max}(T_1 + 1) = X_{\max}(T_1) \), and that either \( \{\ell_0, \ell_2\} \) or \( \{\ell_1, \ell_2\} \) is selected at time \( T_1 + 1 \). We deduce:

\[
x_{\ell_s}(T_1 + 2) \leq (1 - \alpha)[(1 - \alpha_s)X_{\min}(T_1) + \alpha X_{\max}(T_1)] + \alpha X_{\max}(T_1)
\]

\[
\leq (1 - \alpha_s)^2X_{\min}(T_1) + (1 - (1 - \alpha_s)^2)X_{\max}(T_1), \quad s = 0, 1;
\]

\[
x_{\ell_2}(T_1 + 2) \leq \alpha[(1 - \alpha_s)X_{\min}(T_1) + \alpha X_{\max}(T_1)] + (1 - \alpha)X_{\max}(T_1)
\]

\[
\leq (1 - \alpha_s)^2X_{\min}(T_1) + (1 - (1 - \alpha_s)^2)X_{\max}(T_1),
\]

Thus we obtain \( x_{\ell_s}(T_1 + 2) \leq (1 - \alpha_s)^2X_{\min}(T_1) + (1 - (1 - \alpha_s)^2)X_{\max}(T_1), s = 0, 1, 2 \).

• We carry on the analysis recursively, and finally get:

\[
x_{\ell_s}(T_1 + n - 1) \leq (1 - \alpha_s)^n - 1X_{\min}(T_1) + (1 - (1 - \alpha_s)^n - 1)X_{\max}(T_1),
\]

for \( s = 0, 1, 2, \ldots, n - 1 \). Equivalently:

\[
X_{\max}(T_1 + n - 1) \leq (1 - \alpha_s)^n - 1X_{\min}(T_1) + (1 - (1 - \alpha_s)^n - 1)X_{\max}(T_1).
\]

We conclude that:

\[
\mathcal{X}(T_1 + n - 1) = X_{\max}(T_1 + n - 1) - X_{\min}(T_1 + n - 1)
\]

\[
= X_{\max}(T_1 + n - 1) - X_{\min}(T_1)
\]

\[
\leq r_0 \mathcal{X}(T_1),
\]

where \( r_0 = 1 - (1 - \alpha_s)^n - 1 \) is a constant in \((0, 1)\).

With the above analysis taking

\[
L_0 = \left\lceil \log_{r_0} \frac{a}{2b} \right\rceil,
\]

and selecting the given pair sequence periodically for \( L_0 \) rounds, we obtain

\[
\mathcal{X}(T_1 + (n - 1)L_0) \leq r_0^{L_0} \mathcal{X}(T_1) \leq \frac{a}{2b} \leq \frac{a}{2} < \frac{a}{2}.
\]
In light of (41) and the selection of the node pair sequence, we have obtained that
\[ P \left( X(T_1 + (n - 1)L_0) \leq \frac{a_s}{2} \right) \geq \left( \frac{2b_s}{n} \right)^{(n-1)L_0} \] (42) given that \( T_1 \) is finite. We repeat the above argument for \( T_{m+1} \), \( m = 2, 3, \ldots \). Borel-Cantelli Lemma then implies
\[ P \left( \liminf_{k \to \infty} X(k) \leq \frac{a_s}{2} \right) \middle| \liminf_{k \to \infty} X(k) \in [a_s, b_s] \right) = 1, \] (43) which is impossible and completes the proof.

D. Proof of Theorem 2

Let \( \omega \notin \mathcal{C} \). Then there exists an initial value \( x^0 \in \mathbb{R}^n \) from which
\[ \limsup_{k \to \infty} X(k)(\omega) > 0. \] (44) According to Lemma 1, (44) implies that
\[ P \left( \limsup_{k \to \infty} X(k) = \infty \right) = P \left( \mathcal{D} \right) = 1, \] (45) which implies \( P(\mathcal{C}) + P(\mathcal{D}) = 1 \).

With \( P(\mathcal{C}) + P(\mathcal{D}) = 1 \), \( \mathcal{D} \) is a trivial event as long as \( \mathcal{C} \) is a trivial event. Therefore, for completing the proof we just need to verify that \( \mathcal{C} \) is a trivial event.

We first show that \( \mathcal{C} = \{ \lim_{k \to \infty} W_k \ldots W_0 = U \} \). In fact, if \( \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \) under \( x^0 \in \mathbb{R}^n \), then we have \( \lim_{k \to \infty} x(k) = \frac{1}{n} 11'x^0 \) because the sum of the beliefs is preserved. Therefore, we can restrict the analysis to \( x^0 = e_i, i = 1, \ldots, n \) and on can readily see that \( \mathcal{C} = \{ \lim_{k \to \infty} W_k \ldots W_0 = U \} \).

Next, we apply the argument, which was originally introduced in Tahbaz-Salehi and Jadbabaie (2008) for establishing the weak ergodicity of product of random stochastic matrices with positive diagonal terms, to conclude that \( \mathcal{C} \) is a trivial event. A more general treatment to zero-one laws of random averaging algorithms can be found in Touri and Nedić (2011). Define a sequence of event \( \mathcal{C}_s = \{ \lim_{k \to \infty} W_k \ldots W_s = U \} \) for \( s = 1, 2, \ldots \). We see that
- \( P(\mathcal{C}_s) = P(\mathcal{C}) \) for all \( s = 1, 2, \ldots \) since \( W_k, k = 0, 1, \ldots \), are i.i.d.
- \( \mathcal{C}_{s+1} \subseteq \mathcal{C}_s \) for all \( s = 1, 2, \ldots \) since \( \lim_{k \to \infty} W_k \ldots W_{k+1} = U \) implies \( \lim_{k \to \infty} W_k \ldots W_s = U \) due to the fact that \( UW_s \equiv U \).
Therefore, we have $\bigcap_{s=1}^{\infty} \mathcal{C}_s$ is a tail event within the tail $\sigma$-field $\bigcap_{s=1}^{\infty} \sigma(G_s, G_{s+1}, \ldots)$. By Kolmogorov’s zero-one law, $\bigcap_{s=1}^{\infty} \mathcal{C}_s$ is a trivial event. Hence $\mathbb{P}(\mathcal{C}) = \lim_{s \to \infty} \mathbb{P}(\mathcal{C}_s) = \mathbb{P}(\bigcap_{s=1}^{\infty} \mathcal{C}_s)$ is a trivial event, and the desired conclusion follows.

**E. Proof of Theorem 3**

Theorem 3 is a direct consequence of the following lemmas.

**Lemma 3.** Suppose $G_{pst}$ is connected. Then for every fixed $\alpha \in (0, 1)$, we have $\mathbb{P}(\mathcal{C}) = 1$ for all $0 \leq \beta < \beta^*$ with

$$\beta^* := \inf \left\{ \beta : \beta(1+\beta) < \frac{\lambda_{\max}(L_{neg})}{\lambda_2(L_{pst})} \alpha(1-\alpha) \right\}.$$  

**Proof.** Let $x_{ave} = \sum_{i \in V} x_i(0)/n$ be the average of the initial beliefs. We introduce $V(k) = \sum_{i=1}^{n} |x_i(k) - x_{ave}|^2 = |(I - U)x(k)|^2$. The evolution of $V(k)$ follows from

$$\mathbb{E}\left\{ V(k+1) | x(k) \right\} = \mathbb{E}\left\{ (x(k)+1)'(I-U)^2x(k+1) | x(k) \right\}$$

$$\overset{a)}{=} \mathbb{E}\left\{ x(k)'W(k)(I-U)W(k)x(k) | x(k) \right\}$$

$$\overset{b)}{=} \mathbb{E}\left\{ x(k)'(I-U) [W(k)(I-U)W(k)](I-U)x(k) | x(k) \right\}$$

$$\overset{c)}{\leq} \lambda_{\max}\left( \mathbb{E}\{W(k)(I-U)W(k)\} \right) |(I-U)x(k)|^2$$

$$\overset{d)}{=} \lambda_{\max}\left( \mathbb{E}\{W^2(k)\} - U \right) V(k),$$

where $a)$ is based on the facts that $W(k)$ is symmetric and the simple fact $(I-U)^2 = I - U$, $b)$ holds because $(I-U)W(k) = W(k)(I-U)$ always holds and again $(I-U)^2 = I - U$, $c)$ follows from Rayleigh-Ritz theorem (cf. Theorem 4.2.2 in **Horn and Johnson (1985)**) and the fact that $W(k)$ is independent of $x(k)$, $d)$ is based on simple algebra and $W(k)U = UW(k) = U$.

We now compute $\mathbb{E}(W^2(k))$. Note that

$$(I - \alpha(e_i - e_j)(e_i - e_j))^2 = I - 2\alpha(1-\alpha)(e_i - e_j)(e_i - e_j)'$$

$$(I + \beta(e_i - e_j)(e_i - e_j))^2 = I + 2\beta(1-\beta)(e_i - e_j)(e_i - e_j)'.$$  

This observation combined with 5 leads to

$$\mathbb{P}\left( W^2(k) = I - 2\alpha(1-\alpha)(e_i - e_j)(e_i - e_j)' \right) = \frac{p_{ij} + p_{ji}}{n}, \quad \{i, j\} \in E_{pst};$$

$$\mathbb{P}\left( W^2(k) = I + 2\beta(1-\beta)(e_i - e_j)(e_i - e_j)' \right) = \frac{p_{ij} + p_{ji}}{n}, \quad \{i, j\} \in E_{neg}.$$
As a result, we have
\[
E\{W^2(k)\} = I - 2\alpha(1 - \alpha)L_{\text{pst}}^\dagger + 2\beta(1 + \beta)L_{\text{neg}}^\dagger.
\] (48)

Consequently, we have \( 0 < \gamma := \lambda_{\text{max}}(E(W^2(k)) - U) < 1 \) for all \( \beta \) satisfying
\[
\beta(1 + \beta) < \frac{\lambda_{\text{max}}(L_{\text{neg}}^\dagger)}{\lambda_2(L_{\text{pst}}^\dagger)}(1 - \alpha).
\] (49)

Since \( g(\beta) = \beta(1 + \beta) \) is nondecreasing, we conclude from (46) that
\[
E\{V(k + 1) | x(k)\} < \gamma V(k)
\] (50)
with \( 0 < \gamma < 1 \) for all \( 0 \leq \beta < \beta^* \). This means that \( V(k) \) is a supermartingale as long as \( 0 \leq \gamma \leq 1 \) [Durrett, 2010], and \( V(k) \) converges to a limit almost surely by the martingale convergence theorem (Theorem 5.2.9, [Durrett, 2010]). Next we show that this limit is zero almost surely if \( 0 \leq \gamma < 1 \).

Let \( \epsilon > 0 \) and \( 0 \leq \gamma < 1 \). We have:
\[
P\left(V(k) > \epsilon \text{ infinitely often}\right) = P\left(\sum_{k=0}^{\infty} P(V(k + 1) > \epsilon | x(k)) = \infty\right)
\]
\[
\leq P\left(\frac{1}{\epsilon} \sum_{k=0}^{\infty} E\{V(k + 1) | x(k)\} = \infty\right)
\]
\[
\leq P\left(\frac{1}{\epsilon} \sum_{k=1}^{\infty} V(k) = \infty\right),
\] (51)
where \( a) \) is straightforward application of the Second Borel-Cantelli Lemma (Theorem 5.3.2, in Durrett, 2010), \( b) \) is from the Markov’s inequality, and \( c) \) holds directly from (50). Observing that
\[
\sum_{k=1}^{\infty} E\{V(k)\} \leq \sum_{k=1}^{\infty} \gamma^k V(0) \leq \frac{\gamma}{1 - \gamma} V(0) < \infty,
\] (52)
we obtain \( P\left(\sum_{k=1}^{\infty} V(k) = \infty\right) = 0 \). Therefore, we have proved that \( P(V(k) > \epsilon \text{ infinitely often}) = 0 \), or equivalently, \( P(\lim_{k \to \infty} V(k) = 0) = 1 \).

Finally, observe that:
\[
V(k) = \sum_{i=1}^{n} |x_i(k) - x_{\text{ave}}|^2 \geq |x_{\rho_1}(k) - x_{\text{ave}}|^2 + |x_{\rho_1}(k) - x_{\text{ave}}|^2 \geq \frac{1}{2} |x_{\rho_1}(k) - x_{\rho_2}(k)|^2 = \frac{1}{2} \chi^2(k),
\]
where \( \rho_1 \) and \( \rho_2 \) are chosen such that \( x_{\rho_1}(k) = X_{\text{min}}(k), x_{\rho_2}(k) = X_{\text{max}}(k) \). Hence \( P(\lim_{k \to \infty} V(k) = 0) = 1 \) implies \( P(\lim_{k \to \infty} \chi(k) = 0) = 1 \). This completes the proof. \( \square \)
Remark 4. We have shown that:

\[ E\{V(k+1)\} \leq \lambda_{\text{max}}(E\{W^2(k)\} - U)E\{V(k)\} \quad (53) \]

from (46). A symmetric analysis leads to:

\[ E\{V(k+1)\} \geq \lambda_{\text{min}}(E\{W^2(k)\} - U)E\{V(k)\} \quad (54) \]

Proposition 3 readily follows from these inequalities.

Lemma 4. Suppose \( \alpha \in [0,1] \) with \( \alpha \neq 1/2 \). There exists a constant \( \beta^* > 0 \) such that

\[ \mathbb{P}(\liminf_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty) = 1 \] for almost all initial beliefs if \( \beta > \beta^* \).

Proof. Suppose \( X(0) > 0 \). We have:

\[ J_{ij}(k+1) = \begin{cases} 2\alpha - 1 |J_{ij}(k)|, & \text{if } G_k = \{i,j\} \in E_{\text{pst}} \\ 2\beta + 1 |J_{ij}(k)|, & \text{if } G_k = \{i,j\} \in E_{\text{neg}}. \end{cases} \quad (55) \]

Thus, \( X(k) > 0 \) almost surely for all \( k \) as long as \( X(0) > 0 \). As a result, the following sequence of random variables is well defined:

\[ \zeta_k = \frac{X(k+1)}{X(k)}, \quad k = 0, 1, \ldots \quad (56) \]

The proof is based on the analysis of \( \zeta_k \). We proceed in three steps.

Step 1. In this step, we establish some natural upper and lower bounds for \( \zeta_k \). First of all, from (55), it is easy to see that:

\[ \mathbb{P}(\zeta_k = \frac{X(k+1)}{X(k)} \geq |2\alpha - 1|) = 1 \quad (57) \]

and \( \mathbb{P}(\zeta_k < 1) \leq \mathbb{P}(\text{one link in } E_{\text{pst}} \text{ is selected}) \).

On the other hand let \( \{i_0, j_0\} \in G_{\text{neg}} \). Suppose \( i_\ast \) and \( j_\ast \) are two nodes satisfying \( J_{i_\ast j_\ast} = X(0) \). Repeating the analysis in the proof of Theorem 1 by recursively applying the Triangle Lemma, we conclude that there is a sequence of node pairs for time steps \( 0, 1, \ldots, (n-1)Z - 1 \) which guarantees

\[ J_{i_0j_0}((n-1)Z - 1) \geq \frac{\delta^{n-1}}{2} X(0) \quad (58) \]

where \( \delta = 1/16 \) and \( Z = \max\{\lceil \log_{1+2\beta} 4 \rceil, \lceil \log_{|1-2\alpha|} 4 \rceil \} \) are defined in the Triangle Lemma. For the remaining of the proof we assume that \( \beta \) is sufficiently large so that \( \lceil \log_{1+2\beta} 4 \rceil \leq \lceil \log_{|1-2\alpha|} 4 \rceil \), which means that we can select \( Z = \lceil \log_{|1-2\alpha|} 4 \rceil \) independently of \( \beta \).
Now take an integer \( H_0 \geq 1 \). Continuing the previous node pair sequence, let \( \{i_0, j_0\} \) be selected at time steps \((n-1)Z, \ldots, (n-1)Z + H_0 - 1\). It then follows from (55) and (58) that

\[
X((n-1)Z + H_0) \geq J_{i_0,j_0}((n-1)Z + H_0) \geq \frac{(2\beta + 1)^{H_0}\delta^{n-1}}{2}X(0).
\]

Denote \( Z_{H_0} = (n-1)Z + H_0 \). This node sequence for \( 0, 1, \ldots, Z_{H_0} \), which leads to (59), is denoted \( S_{i_0,j_0}(0, Z_{H_0}) \).

Step 2. We now define a random variable \( Q_{Z_{H_0}}(0) \), associated with the node pair selection process in steps \( 0, \ldots, Z_{H_0} - 1 \), by

\[
Q_{Z_{H_0}}(0) = \begin{cases} 
|2\alpha - 1|^{Z_{H_0}}, & \text{if at least one link in } E_{pst} \text{ is selected in steps } 0, 1, \ldots, Z_{H_0} - 1; \\
\frac{(2\beta + 1)^{H_0}\delta^{n-1}}{2}, & \text{if node sequence } S_{i_0,j_0}(0, Z_{H_0}) \text{ is selected in steps } 0, 1, \ldots, Z_{H_0} - 1; \\
1, & \text{otherwise.}
\end{cases}
\]

From direct calculation based on the definition of \( Q_{Z_{H_0}}(0) \), we conclude that

\[
\mathbb{E}\left\{ \log Q_{Z_{H_0}}(0) \right\} \geq \left( \frac{p^*}{n} \right)^{Z_{H_0}} \log \left( \frac{2\beta + 1}{2} \right) + \left( 1 - \left( 1 - \frac{p^*}{n} \right)^{E_0Z_{H_0}} \right) \log |2\alpha - 1|^{Z_{H_0}} := C_{H_0}
\]

where \( p^* = \max\{p_{ij} + p_{ji} : \{i, j\} \in E\} \) and \( E_0 = |E_{pst}| \) denotes the number of positive links. Since \( Z \) does not depend on \( \beta \), we see from (62) that for any fixed \( H_0 \), there is a constant \( \beta^{\circ}(H_0) > 0 \) with

\[
[\log_{1+2\beta^{\circ}} 4] \leq [\log_{|1-2\alpha|} \frac{1}{4}] \text{ guaranteeing that}
\]

\[
\beta > \beta^{\circ}(H_0) \Rightarrow C_{H_0} > 0.
\]

Step 3. Recursively applying the analysis in the previous steps, node pair sequences \( S_{i_0,j_0}((sZ_{H_0}, (s+1)Z_{H_0})) \) can be found for \( s = 1, 2, \ldots \), and \( Q_{Z_{H_0}}(s), s = 1, 2, \ldots \) can be defined associated with the node pair selection process (following the same definition of \( Q_{Z_{H_0}}(0) \)). Since the node pair selection process is independent of time and node states, \( Q_{Z_{H_0}}(s), s = 0, 1, 2, \ldots \), are independent
random variables (not necessarily i.i.d since $S_{i0}(s) = (sZ_{H_0}, (s+1)Z_{H_0})$ may correspond to different pair sequences for different $s$.) The lower bound established in (62) holds for all $s$, i.e.,

$$\mathbb{E}\{\log Q_{Z_{H_0}}(s)\} \geq C_{H_0}, s = 0, 1, \ldots. \quad (63)$$

Moreover, we can prove as (61) was established that:

$$\mathbb{P}\left(\prod_{k=0}^{tZ_{H_0} - 1} \zeta_k = \frac{\mathcal{X}(tZ_{H_0})}{\mathcal{X}(0)} \geq \prod_{s=0}^{t-1} Q_{Z_{H_0}}(s), \ t = 0, 1, 2, \ldots\right) = 1. \quad (64)$$

It is straightforward to see that $\forall \{\log Q_{Z_{H_0}}(s)\}, s = 0, 1, \ldots$ is bounded uniformly in $s$. Kolmogorov’s strong law of large numbers (for a sequence of mutually independent random variables under Kolmogorov criterion, see Feller (1968)) implies that:

$$\mathbb{P}\left(\lim_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t} \left(\log Q_{Z_{H_0}}(s) - \mathbb{E}\{\log Q_{Z_{H_0}}(s)\}\right) = 0\right) = 1. \quad (65)$$

Using (63), (65) further implies that:

$$\mathbb{P}\left(\liminf_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t} \log Q_{Z_{H_0}}(s) \geq C_{H_0}\right) = 1. \quad (66)$$

The final part of the proof is based on (64). With the definition of $\zeta_k$, (64) yields:

$$\mathbb{P}\left(\log \mathcal{X}((t+1)Z_{H_0}) - \log \mathcal{X}(0) = \sum_{k=0}^{(t+1)Z_{H_0} - 1} \log \zeta_k \geq \sum_{s=0}^{t} \log Q_{Z_{H_0}}(s), \ t = 0, 1, 2, \ldots\right) = 1,$n

which together with (66) gives us:

$$\mathbb{P}\left(\liminf_{t \to \infty} \mathcal{X}((t+1)Z_{H_0}) = \infty\right) = 1. \quad (67)$$

We can further conclude that:

$$\mathbb{P}\left(\liminf_{k \to \infty} \mathcal{X}(k) = \infty\right) = 1 \quad (68)$$

since $\mathbb{P}(\mathcal{X}(k) \geq |2\alpha - 1|^Z_{H_0} \mathcal{X}(\lceil \frac{k}{Z_{H_0}} \rceil Z_{H_0})) = 1$ in view of (57).

Therefore, for any integer $H_0 \geq 1$, we have proved that belief divergence is achieved for all initial condition satisfying $\mathcal{X}(0) > 0$ if $\beta > \beta^\odot(H_0)$. Define

$$\beta^\sharp := \inf_{H_0 \geq 1} \beta^\odot(H_0).$$

With this choice of $\beta^\sharp$, the desired conclusion holds. □
F. Proof of Proposition 5

Note that there exist \( \{i_s, j_s\} \in G_{pst}, s = 1, 2, \ldots, T \) with \( T \geq 1 \) such that

\[
W_{i_T}^+ \cdots W_{i_1}^+ = U
\]  

(69)

if and only if for any \( y(0) = y^0 = (y_1^0 \ldots y_n^0)' \), the dynamical system

\[
y(k) = W_{i_k j_k}^+ y(k-1), \ k = 1, \ldots, T
\]  

(70)

drives \( y(k) = (y_1(k), \ldots, y_n(k))' \) to \( y(T) = \text{ave}(y(0))1 \) where \( \text{ave}(y(0)) = \sum_{i=1}^n y_i^0/n \). Thus we may study the matrix equality (69) through individual node dynamics, which we leverage in the proof.

The claim follows from an induction argument. Assume that the desired sequence of node pairs with length \( T_k = k2^{k-1} \) exists for \( m = k \). Assume that \( G_{pst} \) has a subgraph isomorphic to an \( m+1 \) dimensional hypercube. Without loss of generality we assume \( V \) has been rewritten as \( \{0,1\}^{k+1} \) following the definition of hypercube.

Now define

\[
V_0^k := \{i_1 \times \cdots \times i_{k+1} \in V : i_{k+1} = 0\}; \quad V_1^k := \{i_1 \times \cdots \times i_{k+1} \in V : i_{k+1} = 1\}.
\]

It is easy to see that each of the subgraphs \( G_{V_0^k} \) and \( G_{V_1^k} \) contains a positive subgraph isomorphic with an \( m \)-dimensional hypercube. Therefore, for any initial value of \( y(0) \), the nodes in each set \( G_{V_s^k}, s = 0, 1 \) can reach the same value, say \( C^0(y(0)) \) and \( C^1(y(0)) \), respectively. Then we select the following \( 2^k \) edges for updates from \( G \):

\[
\{i_1 \times \cdots \times i_k \times 0, i_1 \times \cdots \times i_k \times 1\} : i_s \in \{0,1\}, s = 1, \ldots, k.
\]

After these updates, all nodes reach the same value \( (C^0(y(0)) + C^1(y(0)))/2 \) which has to be \( \text{ave}(y(0)) \) since the sum of the node beliefs is constant during this process. Thus, the desired sequence of node pairs exists also for \( m = k + 1 \), with a length

\[
T_{k+1} = 2T_k + 2^k = 2k2^{k-1} + 2^k = (k + 1)2^k.
\]

This proves the desired conclusion.
G. Proof of Proposition 6

The requirement of \( \alpha = 1/2 \) is obvious since otherwise \( W_{ij}^+ \) is nonsingular for all \( \{i, j\} \in E_{pst} \), while \( \text{rank} U = 1 \). The necessity of \( m = 2^k \) for some \( k \geq 0 \) was proved in Shi et al. (2012) through an elementary number theory argument by constructing a particular initial value for which finite-time convergence can never be possible by pairwise averaging.

It remains to show that \( G_{pst} \) has a perfect matching. Now suppose Eq. (69) holds. Without loss of generality we assume that Eq. (69) is irreducible in the sense that the equality will no longer hold if any (one or more) matrices are removed from that sequence. The idea of the proof is to analyze the dynamical system (70) backwards from the final step. In this way we will recover a perfect matching from \( \{\{i_1, j_1\}, \ldots, \{i_T, j_T\}\} \). We divide the remaining of the proof into three steps.

Step 1. We first establish some property associated with \( \{i_T, j_T\} \). After the last step in (70), two nodes \( i_T \) and \( j_T \) reach the same value, \( \text{ave}(y^0) \), along with all the other nodes. We can consequently write

\[
y_{i_T}(T-1) = \text{ave}(y^0) + h_T(y^0), \quad y_{j_T}(T-1) = \text{ave}(y^0) - h_T(y^0),
\]

where \( h_T(\cdot) \) is a real-valued function marking the error between \( y_{i_T}(T-1) \), \( y_{j_T}(T-1) \) and the true average \( \text{ave}(y^0) \).

Indeed, the set \( \{y^0 : h_T(y^0) = 0\} \) is explicitly given by

\[
\left\{ y^0 : (0 \ldots 1 \ldots -1 \ldots 0)^{i_{T-1}}_{j_{T-1}} W^+_{i_{T-1}j_{T-1}} \ldots W^+_{i_1j_1} y^0 = 0 \right\},
\]

which is a linear subspace with dimension \( n - 1 \) (recall that the equation \( W^+_{i_Tj_T} \ldots W^+_{i_1j_1} = U \) is irreducible). Thus there must be \( h_T(y^0) \neq 0 \) for some initial value \( y^0 \).

Step 2. If there are only two nodes in the network, we are done. Otherwise \( \{i_{T-1}, j_{T-1}\} \neq \{i_T, j_T\} \). We make the following claim.

**Claim.** \( i_{T-1}, j_{T-1} \notin \{i_T, j_T\} \).

Suppose without loss of generality that \( i_{T-1} = i_T \). Then

\[
y_{j_{T-1}}(T) = y_{j_{T-1}}(T-1) = y_{i_{T-1}}(T-1) = y_{i_T}(T-1) = \text{ave}(y^0) + h_T(y^0).
\]

While on the other hand \( y_{j_{T-1}}(T) = \text{ave}(y^0) \) for all \( y^0 \). The claim holds observing that as we just established, \( \{h_T(y^0) \neq 0\} \) is a nonempty set.
We then write:

\[ y_{i_{T-1}}(T-2) = \text{ave}(y^0) + h_{T-1}(y^0), \quad y_{j_{T-1}}(T-2) = \text{ave}(y^0) - h_{T-1}(y^0) \]

where \( h_{T-1}(\cdot) \) is again a real-valued function and \( h_{T-1}(y^0) \neq 0 \) for some initial value \( y^0 \) (applying the same argument as for \( h_T(y^0) \neq 0 \)). Note that

\[ \{ y^0 : h_T(y^0) \neq 0 \} \cap \{ y^0 : h_{T-1}(y^0) \neq 0 \} = \left( \{ y^0 : h_T(y^0) = 0 \} \cup \{ y^0 : h_{T-1}(y^0) = 0 \} \right)^c \]

is nonempty because it is the complement of the union of two linear subspaces of dimension \( n - 1 \) in \( \mathbb{R}^n \).

Step 3. Again, if there are only four nodes in the network, we are done. Otherwise, we can define:

\[ T_* := \max \left\{ \tau : \{ i_{\tau}, j_{\tau} \} \notin \{ i_{T-1}, j_{T-1}, i_T, j_T \} \right\} \] (71)

We emphasize that \( T_* \) must exist since Eq. (69) holds. As before, we have \( i_{T_*}, j_{T_*} \notin \{ i_{T-1}, j_{T-1}, i_T, j_T \} \) and \( h_{T_*}(y^0) \) can be found with \( \{ h_{T_*}(y^0) = 0 \} \) being another \( (n-1) \)-dimensional subspace such that

\[ y_{i_{T_*}}(T_* - 1) = \text{ave}(y^0) + h_{T_*}(y^0), \quad y_{j_{T_*}}(T_* - 1) = \text{ave}(y^0) - h_{T_*}(y^0). \]

We thus conclude that this argument can be proceeded recursively until we have found a perfect matching of \( G_{\text{pst}} \) in \( \{ \{ i_1, j_1 \}, \ldots, \{ i_T, j_T \} \} \). We have now completed the proof.

**H. Proof of Theorem 4**

We first state and prove intermediate lemmas that will be useful for the proofs of Theorems 4, 5, and 6.

**Lemma 5.** Assume that \( \alpha \in (0, 1) \). Let \( i_1 \ldots i_k \) be a path in the positive graph, i.e., \( \{ i_s, i_{s+1} \} \in G_{\text{pst}}, s = 1, \ldots, k-1 \). Take a node \( i_s \in \{ i_1, \ldots, i_k \} \). Then for any \( \varepsilon > 0 \), there always exists an integer \( Z_*(\varepsilon) \geq 1 \), such that we can select a sequence of node pairs from \( \{ i_s, i_{s+1} \}, s = 1, \ldots, k-1 \) under asymmetric updates which guarantees

\[ J_{i_s,i_s}(Z_*) \leq 2A\varepsilon, \quad s \in \{ 1, \ldots, k \} \]

for all initial condition \( x_{i_s}(0), s = 1, \ldots, k \).
The proof is easy and an appropriate sequence of node pairs can be built just observing that $J_{i^*, i} \leq 2A$ for all $s \in \{1, \ldots, k\}$. 

**Lemma 6.** Fix $\alpha \in (0, 1)$ with $\alpha \neq 1/2$. Under belief dynamics (19), there exist an integer $Z_0 \geq 1$ and a constant $\vartheta_0 > 0$ such that

$$\mathbb{P}\left(\exists \{i^*, j^*\} \in E_{\text{neg}} \text{ s.t. } J_{i^*, j^*}(Z_0) \geq \frac{1}{2n} \mathcal{X}(0)\right) \geq \vartheta_0. \quad (72)$$

**Proof.** We can always uniquely divide $V$ into $m_0 \geq 1$ mutually disjoint sets $V_1, \ldots, V_{m_0}$ such that $G_{\text{pst}}(V_k), k = 1, \ldots, m_0$ are connected graphs, where $G_{\text{pst}}(V_k)$ is the induced graph of $G_{\text{pst}}$ by node set $V_k$. The idea is to treat each $G_{\text{pst}}(V_k)$ as a super node (an illustration of this partition is shown in Figure 3). Since $G$ is connected and $G_{\text{neg}}$ is nonempty, these super nodes form a connected graph whose edges are negative.

One can readily show that there exist two distinct nodes $\eta_1, \eta_2 \in V$ with $\eta_i \in V_{\nu_i}, i = 1, 2$ ($V_{\nu_1}$ and $V_{\nu_2}$ can be the same, of course) such that there is at least one negative edge between $V_{\nu_1}$ and $V_{\nu_2}$ and such that:

$$J_{\eta_1 \eta_2}(0) \geq \frac{1}{m_0} \mathcal{X}(0). \quad (73)$$

Now select $v_1 \in V_{\nu_1}$ and $v_2 \in V_{\nu_2}$ such that $\{v_1, v_2\} \in E_{\text{neg}}$. In view of Lemma 5 and observing that asymmetric updates happen with a strictly positive probability, we can always find $\vartheta_0 > 0$ and...
$Z_0 \geq 1$ (both functions of $(\alpha, n, a, b, c)$) such that:
\[
\mathbb{P}\left( x_{\nu_i}(Z_0) = x_{\nu_i}(0), \ J_{\nu_i\nu_j}(Z_0) \leq \frac{1}{4n}\mathcal{X}(0), \ i = 1, 2 \right) \geq \vartheta_0, \tag{74}
\]
(because $G_{\text{pst}}(V_{\nu_i}), i = 1, 2$ are connected graphs). (72) follows from (73) and (74) since $m_0 \leq n$. \hfill \Box

**Lemma 7.** Fix $\alpha \in (0,1)$ with $\alpha \neq 1/2$. Under belief dynamics (19), there exists $\beta^\circ(\alpha) > 0$ such that $\mathbb{P}(\limsup_{k \to \infty} \mathcal{X}(k) = 2A) = 1$ for almost all initial beliefs if $\beta > \beta^\circ$.

**Proof.** In view of Lemma 6, we have:
\[
\mathbb{P}\left( \mathcal{X}(Z_0 + t) \geq \min \left\{ \frac{1}{4n} \mathcal{X}(0), 2A \right\} \right) \geq \left( \frac{c_p^*}{n} \right)^t \vartheta_0, \quad t = 0, 1, \ldots. \tag{75}
\]
We can conclude that:
\[
\mathbb{P}(\limsup_{k \to \infty} \mathcal{X}(k) = 2A) + \mathbb{P}(\limsup_{k \to \infty} \mathcal{X}(k) = 0) = 1 \tag{76}
\]
as long as $\beta > 0$ using the same argument as that used in the proof of statement (i) in Lemma 1.

With (75), we have:
\[
\mathbb{P}(\mathcal{X}(Z_0 + 1) \geq \frac{\beta + 1}{2n} \mathcal{X}(0)) \geq \left( \frac{c_p^*}{n} \right) \vartheta_0 \tag{77}
\]
conditioned on $\mathcal{X}(0) \leq 4An/(1 + \beta)$. Moreover, (57) still holds for belief dynamics (19). Therefore, we can invoke exactly the same argument as that used in the proof of Lemma 4 to conclude that there exists $\beta^\circ(\alpha) > 0$ such that
\[
\mathbb{P}(\limsup_{k \to \infty} \mathcal{X}(k) \geq 4An/(1 + \beta)) = 1 \tag{78}
\]
for all $\beta > \beta^\circ(\alpha)$. Combining (76) and (78), we get the desired result. \hfill \Box

**Lemma 8.** Assume that the graph is strongly balanced under partition $V = V_1 \cup V_2$, and that $G_{V_1}$ and $G_{V_2}$ are connected. Let $\alpha \in (0,1) \setminus \{1/2\}$. Fix the initial beliefs $x^0$. Then under belief dynamics (19), there are two random variables, $B^1_1(x^0), B^1_2(x^0)$ both taking value in $\{-A, A\}$, such that
\[
\mathbb{P}\left( \lim_{k \to \infty} x_i(k) = B^1_1, \ i \in V_1; \lim_{k \to \infty} x_i(k) = B^1_2, \ i \in V_2 \middle| E_{\text{sep}}(\epsilon) \right) = 1 \tag{79}
\]
for all $\epsilon > 0$, where by definition, $E_{\text{sep}}(\epsilon)$ is the $\epsilon$-separation event:
\[
E_{\text{sep}}(\epsilon) := \left\{ \limsup_{k \to \infty} \max_{i \in V_1, j \in V_2} |x_i(k) - x_j(k)| \geq \epsilon \right\}.
\]
Proof. Suppose \( x_{i_1}(0) - x_{i_2}(0) \geq \varepsilon > 0 \) for \( i_1 \in V_1 \) and \( i_2 \in V_2 \). By assumption, \( G_{V_1} \) and \( G_{V_2} \) are connected. Thus, from Lemma 5, there exist an integer \( Z_1 \geq 1 \) and a constant \( \bar{p} \) (both depending on \( \varepsilon, n, \alpha, a, b \)) such that

\[
\min_{i \in V_1} x_i(Z_1) - \max_{i \in V_2} x_i(Z_1) \geq \frac{\varepsilon}{2}
\]

(80)

happens with probability at least \( \bar{p} \). Intuitively Eq. (80) characterizes the event where the beliefs in the two sets \( V_1 \) and \( V_2 \) are completely separated. Since all edges between the two sets are negative, conditioned on event (80), it is then straightforward to see that almost surely we have

\[
\lim_{k \to \infty} x_i(k) = A, \quad i \in V_1 \text{ and } \lim_{k \to \infty} x_i(k) = -A, \quad i \in V_2.
\]

Given \( E_{\text{sep}}(\varepsilon) \), \( \{ \exists i_1 \in V_1, i_2 \in V_2 \text{ s.t. } x_{i_1}(k) - x_{i_2}(k) \geq \varepsilon \text{ for infinitely many } k \} \) is an almost sure event. Based on our previous discussion and by a simple stopping time argument, the Borel-Cantelli Lemma implies that the complete separation event happens almost surely given \( E_{\text{sep}}(\varepsilon) \). This completes the proof. \( \square \)

**Lemma 9.** Assume that the graph is strongly balanced under partition \( V = V_1 \cup V_2 \), and that \( G_{V_1} \) and \( G_{V_2} \) are connected. Suppose \( \alpha \in (0, 1) \setminus \{1/2\} \). Then under dynamics (14), there exists \( \beta \) sufficiently large such that \( \mathbb{P}(E_{\text{sep}}(A/2)) = 1 \) for almost all initial beliefs.

Proof. Let us first focus on a fixed time instant \( k \). Suppose \( x_i(k) - x_j(k) \geq A \) for some \( i, j \in V \). If \( i \) and \( j \) belong to different sets \( V_1 \) and \( V_2 \), we already have \( \max_{i \in V_1, j \in V_2} |x_i(k) - x_j(k)| \geq A \). Otherwise, say \( i, j \in V_1 \). There must be another node \( l \in V_2 \). We have \( \max_{i \in V_1, j \in V_2} |x_i(k) - x_j(k)| \geq A/2 \) since either \( |x_i(k) - x_l(k)| \geq A/2 \) or \( |x_j(k) - x_l(k)| \geq A/2 \) must hold. Therefore, we conclude that

\[
\mathcal{X}(k) \geq A \implies \max_{i \in V_1, j \in V_2} |x_i(k) - x_j(k)| \geq A/2.
\]

(81)

Then the desired conclusion follows directly from Lemma 7. \( \square \)

Theorem 4 is a direct consequence of Lemmas 8 and 9.

**I. Proof of Theorem 5**

The proof is similar to that of Theorem 4. We just provide the main arguments.
First by Lemma 7 we have $\mathbb{P}(\limsup_{k \to \infty} X(k) = 2A) = 1$ for almost all initial values with sufficiently large $\beta$. Then as for (81), we have

$$X(k) \geq A \implies \max_{i \in V_s, j \in V_t, s \neq t \in \{1, \ldots, m\}} |x_i(k) - x_j(k)| \geq \frac{A}{m},$$

(82)

where $m \geq 2$ comes from the definition of weak balance. Therefore, introducing

$$E^{\ast}_{\text{sep}}(\epsilon) := \left\{ \limsup_{k \to \infty} \max_{i \in V_s, j \in V_t, s \neq t \in \{1, \ldots, m\}} |x_i(k) - x_j(k)| \geq \epsilon \right\},$$

we can show that $\mathbb{P}(E^{\ast}_{\text{sep}}(A/m)) = 1$ for almost all initial beliefs, for sufficiently large $\beta$.

Next, suppose there exist a constant $\eta > 0$ and two node sets $V_{i_1}$ and $V_{i_2}$ with $i_1, i_2 \in \{1, \ldots, m\}$ such that the complete separation event

$$\min_{i \in V_{i_1}} x_i(k) - \max_{i \in V_{i_2}} x_i(k) \geq \eta$$

(83)

happens. Recall that the underlying graph is complete. Then if $(\beta + 1)\eta \geq 2A$, we can always select $Z_s := |V_{i_1}| + |V_{i_2}|$ negative edges between nodes in the sets $V_{i_1}$ and $V_{i_2}$, so that after the corresponding updates:

$$x_i(k + Z_s) = A, i \in V_{i_1}, \quad x_i(k + Z_s) = -A, i \in V_{i_2}.$$  

(84)

One can easily see that we can continue to build the (finite) sequence of edges for updates, such that nodes in $V_k$ will hold the same belief in $\{-A, A\}$, for all $k = 1, \ldots, m$. After this sequence of updates, the beliefs held at the various nodes remain unchanged (two nodes with the same belief cannot influence each other, even in presence of a negative link; and two nodes with different beliefs are necessarily enemies). To summarize, conditioned on the complete separation event (83), we can select a sequence of node pairs under which belief clustering is reached, and this clustering state is an absorbing state.

Finally, the Borel-Cantelli Lemma and $\mathbb{P}(E^{\ast}_{\text{sep}}(A/m)) = 1$ guarantee that almost surely the complete separation event (83) happens an infinite number of times if $\eta = A/2m$ in view of Lemma 5. The end of the proof is then done as in that of Theorem 4.

**J. Proof of Theorem 6**

Again the result is obtained by combining Lemmas 5 and 7 with Borel-Cantelli lemma.
References

Acemoglu, D., Ozdaglar A., and ParandehGheibi A. (2010) Spread of (Mis)information in social networks. *Games and Economic Behavior* 70: 194-227.

Acemoglu, D., Dahleh, M.A., Lobel, I., and Ozdaglar, A. (2011) Bayesian learning in social networks. *Review of Economic Studies* 78: 1201-1236.

Acemoglu, D., Como, G., Fagnani F., and Ozdaglar A. (2013) Opinion fluctuations and disagreement in social networks. *Mathematics of Operations Research* 38: 1-27.

Acemoglu, D. and Ozdaglar, A. (2011) Opinion dynamics and learning in social networks. *Dynamic Games and Applications*, 1: 3-49.

Altafini, C. (2012) Dynamics of opinion forming in structurally balanced social networks. *PLoS ONE* 7(6): e38135.

Altafini, C. (2013) Consensus problems on networks with antagonistic interactions. *IEEE Trans. Automatic Control* 58:935-946.

Banerjee, A. (1992) A simple model of herd behavior. *The Quarterly Journal of Economics*, 107: 797-817.

Bikhchandani, S., Hirshleifer, D., and Welch, I. (1992) A theory of fads, fashion, custom, and cultural change as information cascades. *The Journal of Political Economy*, 100: 992-1026.

Boyd, S., Ghosh, A., Prabhakar B., and Shah, D. (2006) Randomized gossip algorithms. *IEEE Trans. Information Theory* 52: 2508-2530.

Blondel, V. D., Hendrickx, J. M., and Tsitsiklis, J. N. (2009) On Krause’s multi-agent consensus model with state-dependent connectivity. *IEEE Trans. Autom. Control*, 54: 2586-2597.

Blondel, V. D., Hendrickx, J. M., and Tsitsiklis, J. N. (2010) Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM J. Control and Optimization*, 48(8): 5214-5240.

Cartwright, D., and Harary, F. (1956) Structural balance: A generalization of Heider’s theory. *Psychol Rev.*, 63:277-293.

Davis, J. A. (1963) Structural balance, mechanical solidarity, and interpersonal relations. *American Journal of Sociology*, 68:444-462.

Davis, J. A. (1967) Clustering and structural balance in graphs. *Human Relations* 20: 181-187.

DeGroot, M. H. (1974) Reaching a consensus. *Journal of the American Statistical Association* 69: 118-121.

DeMarzo, Vayanos, and Zwiebel (2003) Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics* 118:909-968.

Ding, X. and Jiang, T. (2010) Spectral distributions of adjacency and Laplacian matrices of random graphs. *The Annals of Applied Probability* 20: 2086-2117.
Durrett, R. (2010) *Probability Theory: Theory and Examples*. 4th ed. Cambridge University Press: New York.

Easley, D. and Kleinberg, J. (2010) *Networks, Crowds, and Markets: Reasoning About a Highly Connected World*. Cambridge University Press.

Erdös, P. and Rényi, A. (1960) On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 17-61.

Facchetti, G., Iacono, G., and Altafini, C. (2011) Computing global structural balance in large-scale signed social networks. *PNAS* 108(52):20953-8.

Fagnani, F. and Zampieri, S. (2008) Randomized consensus algorithms over large scale networks. *IEEE J. Selected Areas in Communications* 26: 634-649.

Feller, W. (1968) *An Introduction to Probability Theory and Its Applications*. 3rd ed. New York: Wiley.

Golub, B. and Jackson, M. O. (2010) Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics* 2(1): 112-149.

Theodorakopoulos, G. and Baras, J. S. (2008) Game theoretic modeling of malicious users in collaborative networks. *IEEE J. Selected Areas in Communications*, 26: 1317-1327.

Heider, F. (1946) Attitudes and cognitive organization. *J Psychol* 21:107-112.

Harary, F. (1953) On the notion of balance of a signed graph. *Michigan Math. Journal*, 2(2):143-146.

Horn, R. A. and Johnson, C. R. (1985) *Matrix Analysis*. Cambridge University Press.

Jackson, M. O. (2008) *Social and Economic Networks*. 1st ed. Princeton University Press.

Jadbabaie, A., Molavi, P., Sandroni, A., and Tahbaz-Salehi, A. (2012) Non-Bayesian social learning. *Games and Economic Behavior* 76: 210-225.

Marvel, S. A., Kleinberg, J., Kleinberg, R. D., and Strogatz, S. H. (2011) Continuous-time model of structural balance. *PNAS* 108(5): 1751-1752.

Matei, I., Baras, J. S., and Somarakis C. (2013) Convergence results for the linear consensus problem under Markovian random graphs. *SIAM Journal on Control and Optimization* 51(2): 1574-1591.

Shi, G., Johansson, M., and Johansson, K. H. (2012) When do gossip algorithms converge in finite time? arXiv: 1206.0992.

Shi, G., Johansson, M., and Johansson, K. H. (2013) How agreement and disagreement evolve over random dynamic networks. *IEEE J. Selected Areas in Communications* 31: 1061-1071.

Tahbaz-Salehi, A. and Jadbabaie, A. (2008) A necessary and sufficient condition for consensus over random networks. *IEEE Trans. on Autom. Control* 53: 791-795.

Touri, B. and Nedić, A. (2011) On ergodicity, infinite flow and consensus in random models. *IEEE Trans. on Automatic Control*, 56: 1593-1605.
Tsitsiklis, J. N. (1984) *Problems in decentralized decision making and computation*. Ph.D. thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Boston, MA.

Wasserman, S. and Faust, K. (1994) *Social Network Analysis: Methods and Applications, Structural Analysis in the Social Sciences*. Cambridge Univ Press, New York.

Xiao, L., and Boyd, S. (2004) Fast linear iterations for distributed averaging. *Systems & Control Letters* **53**: 65-78.

Yildiz, E., Acemoglu, D., Ozdaglar, A., Saberi, A., and Scaglione, A. (2013) Binary opinion dynamics with stubborn agents. *ACM Trans. Econ. Comp.*, in press (MIT Technical Report LIDS 2858).