Self-duality and shock dynamics in the $n$-component priority ASEP

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June 16, 2016

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Abstract

We study the $n$-component priority asymmetric simple exclusion process ($n$-ASEP) with reflecting boundaries. We obtain all invariant measures in explicit form and prove reversibility. Using the symmetry of the generator of the process under the quantum algebra $U_q[\mathfrak{g}(n + 1)]$ we construct duality functions with respect to which the $n$-ASEP is self-dual, both for the finite and the infinite integer lattice. For the $n$-ASEP on the infinite lattice we use self-duality to derive in explicit form the time evolution of a family of measures with $K$ shocks in terms of the transition probability of $K$ coloured particles in a shock exclusion process with particle-dependent hopping rates and nearest-neighbour colour exchange. This process is a gas of particles that forms a bound state, corresponding to shock coalescence on macroscopic scale.

Keywords: Asymmetric simple exclusion process, Duality, Quantum algebras, Shocks

AMS 2000 subject classifications: 60K35. Secondary: 82C20, 82C23
1 Introduction

1.1 Informal overview: Process, main tools, and results

We consider an asymmetric simple exclusion process (ASEP) with \( n \) species of particles and a priority jump rule where particles of a species \( \alpha \in S_{0,n} := \{0, \ldots, n\} \) “see” particles of a lower species \( \beta < \alpha \) as vacant sites [1, 2, 5, 88]. The Markov dynamics of this process local occupation variables \( \eta_k \in S_{0,n} \) on a finite one-dimensional integer lattice \( \Lambda^\pm = [L^-, L^+] \cap \mathbb{Z} \) with \( L = L^+ - L^- + 1 \) sites can be described informally as follows. Each site \( k \in \Lambda^\pm \) can be either empty (denoted by 0) or occupied by at most one particle of species \( \alpha \) with \( 1 \leq \alpha \leq n \). Each bond \((k,k+1)\) of \( \Lambda^\pm \), \( L^- \leq k \leq L^+ - 1 \) carries a clock \( \omega_k \). If \( \eta_k \neq \eta_{k+1} \) the clock \( \omega_k \) rings independently of all other clocks after an exponentially distributed random time with parameter \( \omega_k \) where \( \omega_k = wq > 0 \) if \( \eta_k > \eta_{k+1} \) and \( \omega_k = wq^{-1} > 0 \) if \( \eta_k < \eta_{k+1} \). When the clock rings the particle occupation variables are interchanged and the clocks \( k-1, k \) and \( k+1 \) instantly acquire the corresponding new parameter. If \( \eta_k = \eta_{k+1} \) then nothing happens, corresponding to parameter \( \omega_k = 0 \). We shall refer to this process as \( n \)-priority ASEP. We consider (i) the finite system with reflecting boundary conditions, which means that no jumps from the left boundary site \( L^- \) to the left and no jumps from the right boundary site \( L^+ \) to the right are allowed, (ii) the semi-infinite system with \( L^+ \to \infty \) or \( L^- \to -\infty \), and (iii) the infinite system defined on \( \mathbb{Z} \).

Besides standard probabilistic tools for stochastic interacting particle systems a convenient method to discuss this process is the so-called quantum Hamiltonian approach to interacting particle systems, described in probabilistic terms by Sudbury et al. in [85, 62]. This method reveals [1] that the generator of the \( n \)-species priority ASEP is related to the Hamiltonian operator of an integrable quantum spin system, viz. the Perk-Schultz chain [68], which is symmetric under the action of the quantum algebra \( U_q[gl(n+1)] \), i.e., the \( q \)-deformed universal enveloping algebra of the Lie algebra \( gl(n+1) \) [48, 49].

The three main results that we obtain from this approach are the following. (i) Reversibility: We present in explicit form all invariant measures for the finite system (Theorem (3.1)). We prove reversibility using the detailed balance condition and obtain the normalization factor from combinatorial arguments. We also construct some blocking measures for the infinite system. (ii) Self-duality: We construct by a ground state transformation the representation matrices of \( U_q[gl(n+1)] \) that commute with the generator of the process. Using general probabilistic arguments relating reversibility, symmetry and self-duality [79, 33] we obtain from these self-duality functions (Theorem (3.5)). The proof is constructive and can be generalized to obtain other duality functions. Also duality functions for the infinite systems, where some convergence issues need to be taken into account, are derived.

(iii) Microscopic structure of shocks: This is the main application of duality in this work. We first describe informally some features of the process in the hydrodynamic limit, particularly the appearance of shocks which until now have remained elusive in the rigorous treatment of stochastic particle systems with more than one conservation law due to the lack of attractiveness, see e.g. [86, 30, 87, 71]. For the \( n \)-priority ASEP on \( \mathbb{Z} \) we then define shock measures with \( K \) consecutive shocks of species \( n \), marked microscopically by \( K \) particles of species \( \alpha < n \). We prove that the time evolution of these shock measures can be expressed in terms of the transition probabilities of a different exclusion process on \( \mathbb{Z} \) with only \( K \) particles and particle-dependent hopping rates which can be interpreted
from a physics perspective as random walkers with different masses and on-site repulsion in a constant gravitational field. The $K$-particle transition probabilities of this “shock exclusion process” can be computed from the nested Bethe ansatz \cite{89, 5, 88, 41}. A single shock, in particular, performs a biased random walk. The stationary microscopic distances between the shock markers are independent geometrically distributed random variables, thus elucidating the microscopic meaning of macroscopic coalescence of shocks \cite{34}. Moreover, the result exposes a link between the time evolution of shocks, current fluctuations in the ASEP \cite{46, 20}, and bound states in quantum spin systems \cite{41}.

1.2 Setting of the problem

This work makes use of connections between probability theory, non-equilibrium statistical mechanics and integrability that have been known for a long time, but which have been explored more intensely only recently. In order to place our results relating the concepts of duality, symmetry and shocks into this context we first mention that the $n$-priority ASEP is a natural generalization of the standard ASEP ($n = 1$) \cite{60, 61, 81} to several conserved species of particles. The 2-priority ASEP is the ASEP with second-class particles going back to \cite{59} and studied recently in the context of duality in \cite{11, 12, 54} for finite lattices with reflecting boundary conditions. During the final stages of this work we were notified of a closely related result where a duality function of an $n$-priority ASEP with up to $2j$ particles per site is derived \cite{55}. For $j = 1/2$ this duality function is related to the duality function of Theorem (3.5) but slightly different, see Remark (3.6).

It should be noted that reflecting boundary conditions lead to rather different properties of the process than periodic boundary conditions where the invariant measure, which can be expressed in a matrix product form in the totally asymmetric limit $q \to \infty$ \cite{63, 35, 30, 72}, is not reversible for $q \neq 1$. Moreover, the $U_q[gl(n+1)]$-symmetry is broken for $q \neq 1$, except possibly for an unexplored residual property known for the simplest case $n = 1$ \cite{67, 82}. The invariant measures that we obtain for reflecting boundary conditions turn out to have the peculiar property of long-range interactions despite local nearest-neighbour dynamics, reminiscent of a similar phenomenon found in another exclusion process with periodic boundary conditions \cite{29, 23, 7, 15}. An intriguing question is whether the matrix product measures for the periodic system are related to (and can perhaps be constructed from) some residual quantum algebra symmetry. Also self-duality for open boundaries where first-class particles are injected and extracted but second-class particles are reflected \cite{53, 25, 26, 64} is an open problem.

The idea of using non-Abelian symmetries of the generator for deriving duality relations for stochastic interacting particle systems goes back to \cite{79, 80} where it was shown that duality functions arise from representations of the symmetry algebra. The range of models that can be treated in this fashion is large since non-Abelian symmetries, in particular Lie algebras and their quantum deformations \cite{49}, appear frequently in integrable quantum systems and some of their non-integrable generalizations, many of which are related to generators of stochastic interacting particle systems \cite{1, 81, 43}. Thus, given a symmetry, the derivation of duality functions reduces to finding those representations of the symmetry algebra that commute with the generator of the stochastic interacting particle system and to computing the matrix elements of representations of the symmetry operator. This approach was brought into a neat and systematic form by Giardinà et al. \cite{43} and Jansen and Kurt \cite{47} and was applied to various interacting particle systems to study current
fluctuations, shock motion, heat conduction and other properties of these systems [42, 65, 46, 18, 16, 11, 24, 20].

The two-species priority ASEP on $\mathbb{Z}$ has been used to study microscopic properties of shocks [31, 28, 32, 33]. Particles of species 2 are then the so-called first-class particles and particles of species 1 (called second-class particles) serve to define the microscopic positions of shocks which become macroscopic density discontinuities in the large-scale behaviour of usual ASEP ($n = 1$). An intriguing property is the fact that the time-evolution of certain shock measures with $k$ shocks for configurations with an arbitrary number of particles is given by the transition probability of $K$ exclusion particles [9, 8]. This fact, which appears to be related to the existence of bound states in the associated quantum system, provides detailed information microscopic properties of these shocks. The present work demonstrates that the $K$-particle property of these shock measures for the case of $n = 1$ and $n = 2$ arise from self-duality and shows that a similar property holds for general $n$. Conversely, random walk properties of shocks have been proved for various other processes [66, 74, 8]. This may allow for finding dualities and non-Abelian symmetries in these processes. Our results also indicate a link between current fluctuations [78, 73, 3, 46, 58, 40, 20] and the dynamics of shocks [69, 9, 8, 10, 22] via self-duality since for both problems the same duality functions are used.

1.3 Structure of the paper

In Sec. 2 we define the generator of the process and collect some known facts required to state and prove the main results, which are presented in Sec. 3 along with some further remarks on properties of the $n$-component priority ASEP. In Sec. 4 we present the proofs.

2 The $n$-component priority ASEP

2.1 Definitions and notation

Here we fix notation necessary to define the $n$-priority ASEP. Various other conventions and frequently used formulas are collected in Appendix (A).

2.1.1 State space and configurations

It is expedient to employ two distinct representations of the configurations of the the $n$-priority ASEP that we shall call the occupation variable representation $\eta$, which specifies the type of particle $\eta(k)$ on any given lattice site $k$, and the coordinate representation which specifies the positions $x_i$ and species $\alpha_i$ of the particles which are all uniquely tagged with a label $i$. To make these representations and their relation with each other precise we introduce first the lattices and the sets of particle species that we shall work with.

**Definition 2.1** For integers $L^\pm$ with $L := L^+ + 1 - L^- \geq 2$ we define four integer lattices $\Lambda$ by the sets

\[ \mathbb{Z}; \quad \Lambda_+ := \{ k \in \mathbb{Z} : k \geq L^- \}; \quad \Lambda_- := \{ k \in \mathbb{Z} : k \leq L^+ \}; \quad \Lambda_L := \Lambda_- \cap \Lambda_+. \]  

An element $k \in \Lambda$ is referred to as a site with coordinate $k$. Elements of the sets of integers

\[ S_n := \{1, \ldots, n\}, \quad S_{0,n} := \{0, 1, \ldots, n\} \]  

are called particle species.
A) Occupation variable representation:

For a given lattice \( \Lambda \) we denote by \( \eta = (\eta_k)_{k \in \Lambda} \) the configuration of the particle system where lattice site \( k \in \Lambda \) is occupied by a particle of species \( \eta_k \in \mathbb{S}_{0,n} \). The set of configurations is denoted \( S_{\Lambda}^{0,n} \) with the short-hands \( \Lambda \rightarrow L \) for \( \Lambda = \Lambda_L \) and \( \Lambda \rightarrow \pm \) for \( \Lambda = \Lambda_{\pm} \). We refer to particles of species 0 also as vacancies.

Next we define the following subsets of \( S_{\Lambda}^{0,n} \).

Definition 2.2 Let \( \Lambda \) be one of the lattices defined in (2.1) and define for \( \eta \in S_{\Lambda}^{0,n} \) the particle numbers

\[
N^\alpha(\eta) := \sum_{k \in \Lambda} \delta_{\eta_k,\alpha}, \quad M^\alpha(\eta) := \sum_{\beta=\alpha}^{n} N^\beta(\eta), \quad \alpha \in \mathbb{S}_{0,n}
\]

and the projectors

\[
\wp_{N}^{\alpha}(\eta) := \prod_{\beta=\alpha}^{n} \delta_{N^\alpha(\eta), N^\beta}
\]

on particle numbers \( N^\beta \) with \( \beta \geq \alpha \) and denote by \( N(\eta) := M^1(\eta) \) the total particle number and \( \mathcal{P}_{N}(\eta) := \wp_{N}^{1}(\eta) \) the projector on particle numbers \( N^1, \ldots, N^n \).

(a) For specified particle number \( N^\alpha \geq 0 \) the subset

\[
S_{N^\alpha}^{\Lambda} := \{ \eta \in S_{\Lambda}^{0,n} : N^\alpha(\eta) = N^\alpha \}
\]

of \( S_{\Lambda}^{0,n} \) is called the set of configurations with particle number \( N^\alpha \) of species \( \alpha \) and for specified particle numbers \( \bar{N} = (N^1, \ldots, N^n) \) with \( N^\alpha \geq 0 \) the subset

\[
S_{\bar{N}}^{\Lambda} := \{ \eta \in S_{\Lambda}^{0,n} : \mathcal{P}_{\bar{N}}(\eta) = 1 \}
\]

of \( S_{\Lambda}^{0,n} \) is called the set of configurations with particle numbers \( N^\alpha \) of species \( \alpha \geq 1 \).

(b) The sets of configurations which are right- or left-asymptotically fully occupied by particle of species \( \alpha \in S_{0,n} \) are defined by

\[
S_{\alpha}^{\Lambda>} := \{ \eta \in S_{0,n}^{\bar{N}} : \sum_{k=1}^{\infty} (1 - \delta_{\eta_k,\alpha}) < \infty \}
\]

\[
S_{\alpha<} := \{ \eta \in S_{0,n}^{\bar{N}} : \sum_{k=-\infty}^{0} (1 - \delta_{\eta_k,\alpha}) < \infty \}.
\]

For the finite lattice \( \Lambda_L \) one has trivially \( S_{\alpha>}^{\Lambda_L} = S_{\alpha<}^{\Lambda_L} = S_{\alpha}^{0,n} \) and for the semi-infinite lattices \( \Lambda_{\pm} \) one has \( S_{\alpha<}^{\Lambda_{\pm}} = S_{\alpha>}^{\Lambda_{\pm}} = S_{\alpha}^{0,n} \) for any \( \alpha \).

B) Coordinate representation:

Configurations in \( S_{\alpha<}^{0,n} \) which are left-asymptotically vacant can be specified in an alternative way by consecutively indexing each particle of species \( \alpha \geq 1 \) with an integer \( i \)
such that the left-most particle (which may be of any species $\alpha \geq 1$) is assigned the index 1. The sites occupied by particles of species $\alpha \geq 1$ are denoted $x_i^\alpha$ with $1 \leq i \leq N^\alpha$ and $\{x^\alpha\}$ is the set of these sites. The set of all sites occupied by a particle of species $\alpha \geq 1$ is $\{x_1,\ldots,x_N\} = \bigcup_{\alpha>0}\{x^\alpha\}$. The colour of particle $i$ is denoted by $\alpha_i$ with $1 \leq i \leq N$.

Many applications, such as the Bethe ansatz, are based on this coordinate representation, which is also frequently used below. The $n$-species priority ASEP becomes in this language an exclusion process which can be informally described as follows: Particles carry a “colour” index $\alpha \in \mathcal{S}_n$, corresponding to species $\alpha \geq 1$ in the language of the $n$-priority ASEP. If site $x_i \pm 1$ is empty, particle $i$ jumps after an exponentially distributed random time with parameter $w_q^{\pm 1}$. If two particles $i,i+1$ are nearest neighbours they do not jump but exchange their colour after an exponential random time with parameter $q$ if $n \geq \alpha_j > \alpha_{j+1} \geq 1$, and with parameter $q^{-1}$ if $1 \leq \alpha_j < \alpha_{j+1} \leq n$. Notice that the order $x_{j+1} > x_j$ of the particle coordinates remains preserved.

To define the state space in the coordinate representation formally we introduce some more notions.

**Definition 2.3**

a) (Weyl alcove of type $\tilde{C}_N$) Let $\Lambda$ be one of the lattices defined in (2.1). For a strictly positive integer $N$ we define the (shifted and scaled) Weyl alcove $W_N^\Lambda$ by the set of coordinate vectors $\bar{x} := (x_1,\ldots,x_N) \in \Lambda^N$ satisfying $L^- \leq x_1 < x_2 < \cdots < x_N \leq L^+$. For $N = 0$ we define $W_0^\Lambda := \emptyset$. We also define

$$W^\Lambda = \bigcup_{N \geq 0} W_N^\Lambda.$$  \hfill (9)

and for any $N \geq 1$ the coordinate set $\{\bar{x}\} = \{x_1,\ldots,x_N\} \subset \Lambda$

b) (Colour array) For $\alpha_i \in \mathcal{S}_n$ we define the $N$-particle colour array as $\bar{\alpha} := (\alpha_1,\ldots,\alpha_N) \in \mathcal{S}_n^N$. We define also the sets

$$V_N^\Delta := W_N^\Lambda \times \mathcal{S}_n^N, \quad V^\Lambda = \bigcup_{N \geq 0} V_N^\Delta$$  \hfill (10)

and denote elements of $V_N^\Delta$ by the pair $\bar{x} = (\bar{x},\bar{\alpha})$.

For a specific lattice $\Lambda$ the corresponding Weyl alcoves are denoted by $W_N^L$, $W_N^+$, and $W_N^\infty$ respectively and similarly for $V^\Lambda$.

**Relation between occupation variable and coordinate representations:**

We have an isomorphism between configurations $\bar{x} \in V_N^\Lambda$ and $\bar{\eta} \in \mathcal{S}_N^\Lambda$ through the bijection $\bar{x} \leftrightarrow \bar{\eta}$ where

$$\eta_k(\bar{x}) = \sum_{i=1}^{N(\bar{x})} \alpha_i \delta_{k,x_i}, \quad x_i(\bar{\eta}) = \min_k \left( \sum_{l=L^-}^{k} (1 - \delta_{\eta_l,0}) = i \right), \quad \alpha_i(\bar{\eta}) = \eta_{x_i}$$  \hfill (11)

Defining $\bar{0} = (\ldots,0,0,0,\ldots) \in \mathcal{S}_n^\Delta$ as representing configuration corresponding to the empty lattice this bijection together with the bijection $\bar{x}(\bar{0}) = \emptyset$ and $\bar{\eta}(\bar{0}) = \bar{0}$ yields also an isomorphism between $\bar{x} \in W^\Lambda \times \mathcal{S}_n^N$ and $\bar{\eta} \in \mathcal{S}_0^\Delta$ for infinite $N$. 

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In some instances it will indeed be convenient to view the particle positions as a function of a configuration \( \eta \) and vice versa. The function \( x : S_{0,0}^\Lambda \rightarrow V^\Lambda \) is then denoted by \( x(\eta) \). Conversely \( \eta(x) \) is the occupation variable representation interpreted as a function \( \eta : V^\Lambda \rightarrow S_{0,0}^\Lambda \) of a particle configuration \( x \) into the occupation variable representation \( \eta(x) \). We shall also use the function \( \vec{x}^\alpha : S_{0,n}^\Lambda \rightarrow W_{N,n}^\Lambda \) which gives only coordinates \( \vec{x}^\alpha(\eta) \) of particles of species \( \alpha \) and the function \( x_i^\alpha : S_{0,0}^\Lambda \rightarrow \Lambda \) gives the particle position \( x_i^\alpha(\eta) \) of the \( i \)th particle of species \( \alpha \).

Unless stated otherwise, \( \sum \eta \) will always be understood as \( \sum_{\eta \in S_{0,0}^\Lambda} \) and correspondingly \( \sum_x \equiv \sum_{x \in V^\Lambda} \) and similar for products. A sum over an empty index set is defined to be zero and a product over an empty index set is defined to be equal to 1.

### 2.1.2 Functions of the configurations

Several other functions of the configurations \( \eta \in S_{0,n}^\Lambda \) will play a role. Through the bijection (11) these induce analogous mappings in the coordinate representations that we do not all write explicitly. Generally, however, one has from (11)

\[
\delta_{\eta_k,\alpha} = \delta_{\alpha,0} + \sum_{i=1}^{N(x)} \delta_{x_i,k}(\delta_{\alpha,\alpha_i} - \delta_{\alpha,0})
\]

and therefore for any function \( f(\eta_k) \) of the local occupation variable

\[
f(\eta_k) = f(0) + \sum_{i=1}^{N(x)} \delta_{x_i,k}(f(\alpha_i) - f(0))
\]

where the r.h.s. is expressed in terms of the coordinate representation \( x(\eta) \). By iteration one obtains form this formula analogous expressions for arbitrary cylinder functions.

**Definition 2.4** Let \( \eta \in S_{0,n}^\Lambda \) where \( \Lambda \) is one of the four lattices defined in (2.1).

a) For \( k \in \Lambda \) the local cyclic flip operation \( \gamma_k(\eta) \) is defined by

\[
(\gamma_k(\eta))_l = \begin{cases} 
\eta_k + 1 \mod (n + 1) & l = k \\
\eta_l & l \neq k
\end{cases}
\]

and \( \Gamma := \gamma_{L^-} \circ \cdots \circ \gamma_{L^+} \) is called cyclic flip. For a given \( \eta \) we write \( \eta^p := (\gamma_k)^p(\eta) \) and \( \eta^p := \Gamma^p(\eta) \) for the \( p \)-fold action of \( \gamma_k \) and \( \Gamma \) resp.

b) For \( k < L^+ \) the local permutation is defined by

\[
(\pi^{k+1}(\eta))_l = \begin{cases} 
\eta_{k+1} & l = k \\
\eta_k & l = k + 1 \\
\eta_l & \text{else}
\end{cases}
\]

For a given \( \eta \) we use the notation \( \eta^{k+1} := \pi^{k+1}(\eta) \).

These mappings are invertible and one has \( (\gamma_k)^{-1} = (\gamma_k)^n \). As an array of occupation variables we can write

\[
\gamma_k(\eta) = (\ldots, \eta_{k-1}, (\eta_k + 1) \mod (n + 1), \eta_{k+1}, \ldots)
\]
\[ \pi^{kk+1}(\eta) = (\ldots, \eta_{k-1}, \eta_k, \eta_{k+1}, \eta_{k+2}, \ldots). \] (17)

Next we define various functions characterizing the occupation variables of a configuration \( \eta \). As a first step we consider a “lattice” with only a single site.

**Definition 2.5** The indicator functions \( n^\alpha : S_{0,n} \to \{0,1\} \) and \( m^\alpha : S_{0,n} \to \{0,1\} \) are defined by

\[ n^\alpha(\eta) := \delta_{\eta, \alpha}, \quad m^\alpha(\eta) := \sum_{\beta=\alpha}^{n} n^\beta(\eta) \] (18)

for \( 0 \leq \alpha \leq n \).

From these we construct the following indicator functions for general lattices \( \Lambda \).

**Definition 2.6** The local occupation numbers at site \( k \in \Lambda \) of a configuration \( \eta \in S^\Lambda_{0,n} \) are defined by

\[ n^\alpha_k(\eta) := n^\alpha(\eta_k), \quad m^\alpha_k(\eta) := \sum_{\beta=\alpha}^{n} n^\beta_k(\eta) \] (19)

for \( 0 \leq \alpha \leq n \).

By construction \( m^0_k(\eta) = n^0_k(\eta) \) and \( m^n_k(\eta) = 1 \). Notice also that from (19) one has \( n^0_k(\eta) = \delta_{\eta_k, \alpha} \) which yields the expressions

\[ N^\alpha(\eta) = \sum_{k \in \Lambda} n^\alpha_k(\eta), \quad M^\alpha(\eta) = \sum_{k \in \Lambda} m^\alpha_k(\eta), \quad \alpha \in S_{0,n}. \] (20)

for the particle numbers \( \sum_{\alpha} N^\alpha(\eta) \). We note the trivial, but frequently used identities \( M^0(\eta) = L, \ M^1(\eta) = N(\eta), \ M^n(\eta) = N^n(\eta), \) and

\[ n^0_k(\eta) = \delta_{\eta_k, \alpha} = \sum_{i=1}^{N^\alpha(\eta)} \delta_{x_i(\eta),k}. \] (21)

Using the representation (273) in Appendix A of the integer Theta-function this identity allows us to write

\[ \Theta(\eta_k - \eta_l) = \sum_{\alpha=1}^{n} \sum_{\beta=0}^{\alpha-1} n^\alpha_k(\eta)n^\beta_l(\eta) \] (22)

\[ = n^0_l(\eta)(1 - n^0_k(\eta)) + \sum_{\alpha=2}^{n} n^\alpha_k(\eta)n^\beta_l(\eta) \] (23)

\[ = \sum_{i=1}^{N(x)} \delta_{x_i,k} \left( 1 - \sum_{i=1}^{N(x)} \delta_{x_i,l} \right) + \sum_{i=1}^{N(x)} \sum_{j=1}^{N(x)} \delta_{x_i,k} \delta_{x_j,l} \Theta(\alpha_i - \alpha_j). \] (24)

Thus we can express the two-particle sign function

\[ \text{sgn}(\eta_k - \eta_l) = \sum_{\alpha=1}^{n} \sum_{\beta=0}^{\alpha-1} \left( n^\alpha_k(\eta)n^\beta_l(\eta) - n^\alpha_l(\eta)n^\beta_k(\eta) \right) \] (25)
\[
= n_0^0(\eta) - n_0^0(\eta) + \sum_{\alpha=2}^{n-1} \left(n_k^\alpha(\eta)n_1^\beta(\eta) - n_1^\alpha(\eta)n_k^\beta(\eta)\right) \tag{26}
\]
\[
= \sum_{i=1}^{N(x)} (\delta_{x_i,k} - \delta_{x_i,l}) + \sum_{i=1}^{N(x)} \sum_{j=1}^{N(x)} \delta_{x_i,k} \delta_{x_j,l} \sigma(\alpha_i - \alpha_j) \tag{27}
\]
in terms of occupation numbers \(n_k^\alpha\) and also in the coordinate representation.

In order to characterize a configuration globally we introduce the following quantities.

**Definition 2.7** Let \(\Lambda\) be one of the four lattices defined in (2.1) and \(\eta \in S_{0,n}^\Lambda\). The particle balances at site \(k\) are defined by

\[
N_k^\alpha(\eta) := \sum_{t=L^-}^{k-1} n_t^\alpha(\eta) - \sum_{t=k+1}^{L^+} n_t^\alpha(\eta), \quad M_k^\alpha(\eta) := \sum_{\beta=\alpha}^{n} N_k^\beta(\eta). \tag{28}
\]

for \(0 \leq \alpha \leq n\).

By construction \(M_k^0(\eta) = N_k^0(\eta)\). For \(\alpha = 0\) we have \(M_k^0(\eta) = 2k - L^+ - L^-\).

### 2.2 Generator for the \(n\)-priority ASEP

#### 2.2.1 The Markov generator and its general matrix formulation

We recall the definition of a Markov process \(\omega_t\) with state space \(\Omega\) and transition rates \(g_{\omega'\omega}\) from a configuration \(\omega\) to a configuration \(\omega'\) in terms of a generator \(\mathcal{G}\) acting on suitably chosen functions \(f(\omega)\) through the relation

\[
\mathcal{G}f(\omega) = \sum_{\omega' \in \Omega \setminus \{\omega\}} g_{\omega'\omega} [f(\omega') - f(\omega)]. \tag{29}
\]

The \(n\)-priority ASEP described informally in the introduction can thus be defined as follows.

**Definition 2.8** \((n\text{-ASEP})\) Let \(q^{-1} \in (0, 1]\) and \(\eta \in S_{0,n}^\Lambda\) for \(\Lambda\) as defined in (2.1) and let

\[
w_{q}^{kk+1}(\eta) = w \left(q^{\text{sgn}(\eta_k - \eta_{k+1})} - \delta_{\eta_k,\eta_{k+1}}\right), \tag{30}
\]

be the bond hopping rates for the transition rates

\[
w_{\eta'\eta}(q) = \sum_{k=L^-}^{L^+} w_{q}^{kk+1}(\eta) \delta_{\eta',\eta^{k+1}} \tag{31}
\]

from \(\eta\) to a locally permuted configuration \(\eta^{kk+1} = \pi^{kk+1}(\eta)\) defined in (13). Then the \(n\)-priority ASEP on \(\Lambda\) is the Markov process defined by the generator

\[
\mathcal{L}f(\eta) = \sum_{k=L^-}^{L^+} \mathcal{L}_{k,k+1}f(\eta) \tag{32}
\]
with local generators

\[ \mathcal{L}_{k,k+1}f(\eta) = w_q^{kk+1}(\eta)[f(\eta^{kk+1}) - f(\eta)] \]  

(33)

with the convention that \( L^+ = \infty \) for \( \Lambda = \Lambda_+ \), \( L^- = -\infty \) for \( \Lambda = \Lambda_- \), and \( L^\pm = \pm\infty \) for \( \Lambda = \mathbb{Z} \).

The configuration at time \( t \) is represented by \( \eta_t \).

In the semi-infinite and infinite cases some care needs to be taken regarding the class of functions to which \( f \) belongs as one needs to ensure that (32) converges. Following Liggett [60, 61] we note that this is the case for cylinder functions on \( \mathbb{S}_{0,n}^\mathbb{Z} \), i.e., functions that depend on only finitely many coordinates. Going beyond cylinder functions we note the following lemma.

**Lemma 2.9** Let \( g \) be a cylinder function on \( \mathbb{S}_{0,n}^\mathbb{Z} \), and let \( V_r^>(\eta) := \sum_{i=r}^{\infty}(1 - n_i^r(\eta)) \), \( V_r^<(\eta) := \sum_{i=-\infty}^{r}(1 - n_i^r(\eta)) \) for \( r \in \mathbb{Z} \). Then for the process defined on \( \Lambda = \mathbb{Z} \) one has for \( \alpha \geq 1 \) and \( a \in \mathbb{R} \):

a) \( \mathcal{L}f(\eta) < \infty \) for \( f(\eta) = e^{aV_r^>(\eta)}g(\eta) \) and \( \eta \in \mathbb{S}_{\alpha}^\mathbb{Z} \),

b) \( \mathcal{L}f(\eta) < \infty \) for \( f(\eta) = e^{aV_r^<(\eta)}g(\eta) \) and \( \eta \in \mathbb{S}_0^\Lambda \).

**Proof:** Consider only case (a), case (b) is similar: By definition (3) we have that \( e^{aV_r^>(\eta)} < \infty \) for any \( \eta \in \mathbb{S}_{\alpha}^\mathbb{Z} \), which means that \( f(\eta) \) is well-defined on \( \mathbb{S}_{\alpha}^\mathbb{Z} \). We write \( f(\eta) = e^{aV_r^>(\eta)}g(\eta) =: e^{aV_r^>(\eta)}g_x \) where \( x \) is the largest coordinate on which \( g \) depends. Since \( g \) is a cylinder function one has \( x < \infty \) which implies that also \( g_x \) is a cylinder function. Because of particle number conservation one has \( \mathcal{L}_{k,k+1}e^{a(n_k^\pm + n_{k+1}^\pm)} = 0 \) for any \( a \in \mathbb{R} \) and therefore \( \mathcal{L}_{k,k+1}f(\eta) = 0 \) for \( k > x \). Thus the sum (32) with \( L^\pm = \pm\infty \) contains only finitely many terms and is therefore finite.

Now we focus on the finite lattice \( \Lambda_L \) and write the action of the generator (32) in the so-called quantum Hamiltonian form [55, 51], i.e., in terms of the continuous-time transition matrix \( H \) defined by the matrix elements

\[ H_{\eta'\eta} = \begin{cases} -w_{\eta'\eta}(q) & \text{for } \eta' \neq \eta \\ \sum_{\eta' \in \mathbb{S}_{0,n}^L \setminus \eta} w_{\eta'\eta}(q) & \text{for } \eta' = \eta. \end{cases} \]

(34)

This is a square matrix of dimension \( d_{n,L} := |\mathbb{S}_{0,n}^L| = (n+1)^L \) where by definition all off-diagonal elements (the negative transition rates) are non-positive, the diagonal elements are all non-negative and in each column the sum of all matrix elements is equal to 0, expressing probability conservation \( \mathcal{L}f = 0 \) for the identity function \( f(\eta) = 1 \).

In terms of the matrix elements \( H_{\eta'\eta} \) the defining equation (29) then has the form

\[ \mathcal{L}f(\eta) = -\sum_{\eta' \in \mathbb{S}_{0,n}^L} f(\eta')H_{\eta'\eta} \]

(35)

which can be interpreted as a matrix multiplication \(-fH\) where according to standard convention for matrix multiplication \( f \) is understood as a row vector with entries \( f(\eta) \).
With (32), (33) we can write (35) as
\[
\mathcal{L} f(\eta) = - \sum_{k=L}^{L+1} \sum_{\eta' \in S_{0,n}^{L}} f(\eta')(h_{k,k+1})\eta'\eta
\]
with the local hopping matrices \( h_{k,k+1} \) which are the continuous-time transition matrices of the process restricted to the bond \((k,k+1)\). In slight abuse of language we shall call also
\[
H = \sum_{k=L}^{L+1} h_{k,k+1}
\]
the generator of the process.

### 2.2.2 The tensor basis

In order to write \( H \) explicitly it is natural to choose the canonical basis \( \mathcal{B} = \{ b(i), 1 \leq i \leq d_{n,L} \} \) which spans the vector space \( \mathbb{C}^{d_{n,L}} \). The sum (35) does not uniquely define the matrix \( H \) as the arrangement of the elements \( f(\eta) \) in a vector \( \mathbf{f} \) is not specified by this sum. One has still the freedom to define the mapping \( \iota : S_{0,n}^{L} \rightarrow \{1, \ldots, d_{n,L}\} \) that specifies which canonical basis vector \( b(\iota(\eta)) \) corresponds to a given configuration \( \eta \). We use the natural quantum Hamiltonian form \([85, 81]\) where the ordering of the basis is given by the numerical \( n \)-ary representation of a configuration \( \eta \) defined as follows.

**Definition 2.10** (Basis order) Let \( \mathbb{C}^{d_{n,L}} \) be the \( d_{n,L} \)-dimensional vector space over \( \mathbb{C} \) with canonical basis vectors \( b(i) = (0, \ldots, 0, 1, 0, \ldots, 0) \) in row form with entry 1 for component \( i \) with \( 1 \leq i \leq d_{n,L} \) and zero else. The basis \( \mathcal{E} = \{ \langle \eta |, \eta \in S_{0,n}^{L} \} \) of \( \mathbb{C}^{d_{n,L}} \) is defined by the row vectors \( \langle \eta | = b(\iota(\eta)) \) with
\[
\iota(\eta) = 1 + \sum_{k=L}^{L+1} \eta_k (n+1)^{k-L}.
\]

The dual basis of column vectors is given by \( |\eta\rangle := \langle \eta |^T \) where the superscript \( T \) denotes transposition.

The term “quantum Hamiltonian formalism”, which also motivates the use of the bra \( \langle \cdot | \) and ket \( | \cdot \rangle \) symbols for vectors, will become clear below. A general row vector with entries \( f(\eta) \) is denoted by \( \langle f | \) and a column vector with entries \( g(\eta) \) is denoted by \( | g \rangle \). The ordering of the basis induced by the numerical presentation of the configurations induces a tensor structure which is defined by the Kronecker product \([A.2]\) defined in the Appendix. Interpreting a row vector of dimension \( d \) as a \( 1 \times d \)-matrix we then have:

**Proposition 2.11** (Tensor basis) For \( \eta \in S_{0,n} \) let \( \langle \eta | \) be the canonical \((n+1)\)-dimensional basis vectors of \( \mathbb{C}^{n+1} \) with component 1 at position \( 1+\eta \) and zero else and let \( |\eta\rangle = \langle \eta |^T \) be the dual basis vector. Then one has
\[
\langle \eta | = (\eta_{L-} | \otimes (\eta_{L^-+1} | \otimes \cdots \otimes (\eta_{L+}|, \quad |\eta\rangle = |\eta_{L-}\rangle \otimes |\eta_{L^-+1}\rangle \otimes \cdots \otimes |\eta_{L+}\rangle
\]
with \( \langle \eta | \) and \( |\eta\rangle \) as in Definition (2.10).
Proof: This is an immediate consequence of the definition (A.2) of the Kronecker product and the \( n \)-ary representation (38) of a configuration \( \eta \). □

Following quantum mechanical convention we omit the tensor symbol \( \otimes \) in the Kronecker product of bra and ket vectors. In particular, we use

\[
|g\rangle\langle f| \equiv |g\rangle \otimes \langle f|.
\]

By Definition (A.2) this is \( d_{n,L} \times d_{n,L} \)-matrix \( C \) with matrix elements \( C_{i,j} = g_i f_j \). Specifically, we have the representation

\[
1 = \sum_{\eta \in S_{L}^0} |\eta\rangle\langle \eta|.
\]

of the \( d_{n,L} \)-dimensional unit matrix and

\[
\hat{\varphi}^\alpha = \sum_{\beta = \alpha}^{n} \sum_{\eta \in S_{L}^\alpha_\beta} |\eta\rangle\langle \eta|
\]

of the projector matrix with the property \( \langle \eta| \hat{\varphi}^\alpha = \langle \eta| \hat{\varphi}^\alpha_N(\eta) \) derived from the projector definition (11).

In addition to the tensor product we need an inner product from \( \mathbb{C}^{d_{n,L}} \) and its dual to \( \mathbb{C} \).

**Definition 2.12** The inner product for row vectors \( \langle f | = \sum_{\eta} f(\eta) |\eta| \) and column vectors \( |g\rangle = \sum_{\eta} g(\eta) |\eta\rangle \) is defined by

\[
\langle f | g \rangle := \sum_{\eta \in S_{L}^0} f(\eta) g(\eta).
\]

This implies the biorthogonality relation \( \langle \eta' | \eta \rangle = \delta_{\eta',\eta} \). Next we state some general rules of multilinear algebra that motivate the omission of the tensor symbol in (40), highlight a factorization property of the tensor basis under the inner product, and illustrates the use of the representation (41) of the unit matrix.

**Lemma 2.13** a) Let \( C = |g\rangle\langle f| \) be a tensor matrix according to definition (40). Then the inner product with vectors \( \langle a | \) and \( |b\rangle \) is given by

\[
\langle a | C | b \rangle \equiv \langle a | (|g\rangle \otimes \langle f|) | b \rangle \equiv \langle a | (|g\rangle \langle f|)(|b\rangle b) \rangle = \langle a | g \rangle \cdot \langle f | b \rangle
\]

where \( \cdot \) denotes ordinary multiplication in \( \mathbb{C} \).

b) Let the vectors \( |f\rangle = (f_{L^-}^L \otimes (f_{L^-} + 1)^L \cdots \otimes (f_{L^+}^L \rangle and |g\rangle = (g_{L^-}^L \otimes (g_{L^-} + 1)^L \cdots \otimes (g_{L^+}^L | be tensor products. Then

\[
\langle f | g \rangle = \prod_{k=L^-}^{L^+} (f_k | g_k) = \prod_{k=L^-}^{L^+} \left( \sum_{\alpha = 0}^{n} f_k(\alpha) g_k(\alpha) \right)
\]

(45)

c) For any pair of functions \( f \) and \( g \) the inner product can be expanded in a complete basis as

\[
\langle f | g \rangle = \sum_{\eta \in S_{L}^0} \langle f | \eta \rangle \langle \eta | g \rangle
\]

(46)
Remark 2.14 The expectation $E^\mu f := \sum_\eta f(\eta)\mu(\eta)$ of a function $f(\eta)$ under a probability measure $\mu(\eta)$ can be written as the inner product $E^\mu f = \langle f | \mu \rangle$. Defining the summation vector
\[
\langle s | := \sum_{\eta \in S_{0,n}} \langle \eta |
\]
where all entries are equal to 1 and for a function $f(\eta)$ the diagonal matrix
\[
\hat{f} := \sum_{\eta \in S_{0,n}} f(\eta)\langle \eta |\langle \eta |
\]
one can write $\langle f | = \langle s | \hat{f}$ and therefore $E^\mu f = \langle s | \hat{f} | \mu \rangle$.

This remark highlights the role of the representation of a function $f$ as a diagonal matrix which we shall generally denote by the circumflex (\^)-symbol.

We also use a diagonal matrix representation of a probability measure.

Proposition 2.15 Let $\mu > 0$ be a strictly positive reversible measure for a process with generator $H$. Then with the diagonal matrix representation
\[
\hat{\mu} := \sum_{\eta} \mu(\eta)\langle \eta |\langle \eta |
\]
of $\mu$ the transformation property
\[
\hat{\mu}^{-1}H\hat{\mu} = H^T
\]
is the condition of detailed balance.

Proof: “Sandwiching” (50) with $\langle \eta' |$ and $| \eta \rangle$ one finds
\[
\mu(\eta')w_{\eta'\eta} = \mu(\eta)w_{\eta\eta'}.
\]
which is indeed the detailed-balance condition for reversible measures.

2.2.3 Explicit construction of $H$

In the quantum Hamiltonian formalism the functions of configurations defined in (2.4) turn into endomorphisms on the vector space $C_{d,n}^L$, represented by matrices. In order to construct $H$ explicitly we first define for $1 \leq \alpha \leq n$ the following single-site matrices of dimension $n + 1$:

Definition 2.16 For $1 \leq \alpha \leq n$ the single-site raising and lowering operators $\sigma^{\alpha,\pm}$ and for $0 \leq \alpha \leq n$ the single-site projectors $\hat{n}^\alpha$ are defined by
\[
\sigma^{\alpha,+} := |\alpha - 1)(\alpha|, \quad \sigma^{\alpha,-} := |\alpha)(\alpha - 1|, \quad \hat{n}^\alpha = |\alpha)(\alpha|.
\]
The cyclic flip operator $\gamma$ and the species flip $\sigma^{\alpha\beta}$ are defined by
\[
\gamma := |n)(0| + \sum_{\alpha=1}^n \sigma^{\alpha,+}, \quad \sigma^{\alpha\beta} := |\alpha)(\beta|.
\]
Notice the cyclic property $\gamma^n = \gamma^{-1}$ and the representation

$$\sigma^{\alpha\beta} = \hat{n}^{\alpha} \gamma^{\beta-\alpha} = \gamma^{\beta-\alpha} \hat{n}^{\beta}$$

of the species flip operator. We denote the unit matrix of dimension $n + 1$ by $1$.

From these matrices we construct local operators acting non-trivially on the configuration at site $k \in \Lambda_L$ as follows.

**Definition 2.17** Let $a$ be a matrix of dimension $n + 1$. For $k \in \Lambda$ the local operator $a_k$ of dimension $d_{n,L}$ is defined by

$$a_k = 1^{\otimes (k-L^-)} \otimes a \otimes 1^{\otimes (L^+-k)}$$

with the convention that $1^{\otimes 0} = 1$.

Notice that for any pair of matrices $a, b$ one has the commutator property

$$[a_k, a_l] = 0$$

and for any pair of tensor products $A = \prod_{k\in \Lambda} a_k$ and $B = \prod_{k\in \Lambda} b_k$ one has the tensor factorization property

$$AB = \prod_{k\in \Lambda} (ab)_k.$$ 

Both these properties, which arise from the multilinearity of the tensor product, will be used throughout this work.

With this construction we obtain local operators $\gamma_k$ with the properties $\langle \eta | \gamma_k = \langle \eta^+_k |$ and $\gamma_k | \eta \rangle = \langle \eta^-_k |$. The global cyclic flip operator is then given by

$$\Gamma := \prod_{k=L}^{L^+} \gamma_k$$

and has the properties $\langle \eta | \Gamma = \langle \eta^+ |$ and $\Gamma | \eta \rangle = \langle \eta^- |$.

We are now in a position to write the generator $H$ in explicit form, using the fact that the inner product implies $H \eta' \eta = \langle \eta' | H | \eta \rangle$.

**Proposition 2.18** Define the single-bond hopping matrices

$$h_{k,k+1} := -w \sum_{\alpha=1}^{n-1} q_{k}^{\alpha} \sigma_{k+1}^{\alpha\beta} + q_{k}^{-1} \sigma_{k}^{\alpha\beta} + \hat{w}_{k,k+1}$$

where

$$\hat{w}_{k,k+1} = w \sum_{\alpha=1}^{n-1} q_{k}^{\alpha} \hat{n}_{k+1}^{\alpha} + q_{k}^{-1} \hat{n}_{k}^{\alpha} \hat{n}_{k+1}^{\alpha}.$$ 

The generator $H$ of the $n$-priority ASEP on $\Lambda_L$ defined by (32) and (33) is given in quantum Hamiltonian form by the matrix (37) with local transition matrices (59).
Proof: Since \( \langle \eta' | \gamma_k^{-1} | \eta \rangle = n_k^\alpha(\eta) \langle \eta | \eta \rangle \) the diagonal part, which yields the negative contribution to (33), follows from the expression (25) of the sign-function in terms of the single-site projectors and the projector Lemma (A.3). For the off-diagonal part notice that

\[
\langle \eta' | \gamma_k^{-1} | \eta \rangle = \delta_{\eta', \eta^{k+}}
\]

with \( \eta^{k+} = \gamma_k(\eta) \) defined in (14) and therefore

\[
\langle \eta' | \sigma_k \eta \rangle = n_k^\alpha(\eta') \delta_{\eta', \eta^{k+}} = \delta_{\eta', \eta^{k+}}.
\]

This yields

\[
\langle \eta' | \sigma_k \sigma_{k+1} \eta \rangle = n_k^\alpha(\eta') \delta_{\eta', \eta^{k+}} = n_k^\alpha(\eta) \delta_{\eta', \eta^{k+}}
\]

with the local permutation \( \eta^{k+} \) (15). Taking the summation over \( \alpha \) and \( \beta \) and using (22) yields the off-diagonal part of (33).

\[ \square \]

Remark 2.19 The generator \( H \) has a structure reminiscent of the quantum Hamiltonian of the Perk-Schultz quantum chain \[68\]

\[ H_{PS} = \sum_{k=L^-}^{L^+} h_{k,k+1} \]

with the single-bond matrices

\[
h_{k,k+1} = -w \sum_{\alpha=0}^{n-1} \left( \sigma_{k+1}^{\beta,\alpha} \sigma_{k}^{\alpha,\beta} + \sigma_{k+1}^{\alpha,\beta} \sigma_{k}^{\beta,\alpha} \right) + \tilde{w}_{k,k+1}.
\]

Notice that unlike the generator (37) the Hamiltonian \( H_{PS} \) is symmetric (and hence Hermitian as should be the case for a quantum system).

2.3 The quantum algebra \( U_q[\mathfrak{gl}(n+1)] \)

The Perk-Schultz quantum chain is an integrable model which brings in the notion of quantum algebras. We first introduce the quantum algebra \( U_q[\mathfrak{gl}(n+1)] \) [48, 49] in terms of abstract generators and then give representation matrices that satisfy its defining relations.

Definition 2.20 The quantum algebra \( U_q[\mathfrak{gl}(n+1)] \) [48, 49] is the associative algebra over \( \mathbb{C} \) generated by \( L_{\alpha}^{\pm}, \alpha = 0, \ldots, n \) and \( X_{\alpha}^{\pm}, \alpha = 1, \ldots, n \) and unit \( I \) with relations

\[
L_{\alpha}^{\mp}L_{\beta}^{\pm} = I
\]

\[
[L_{\alpha}, L_{\beta}] = 0
\]

\[
L_{\alpha}X_{\beta}^{\pm} = q^{\pm(\delta_{\alpha,\beta} - \delta_{\alpha,\beta})/2}X_{\beta}^{\mp}L_{\alpha}
\]

\[
[X_{\alpha}^{+}, X_{\beta}^{-}] = \delta_{\alpha,\beta} \frac{(L_{\alpha-1}L_{\alpha}) - (L_{\alpha-1}L_{\alpha})^{-2}}{q - q^{-1}}
\]

\[
[X_{\alpha}^{+}, X_{\beta}^{+}] = 0 \quad |\alpha - \beta| \neq 1,
\]

\[
(X_{\alpha}^{+})^2X_{\beta}^{+} - (2\eta\gamma_{\alpha}^{+}X_{\beta}^{+}X_{\alpha}^{+} + X_{\beta}^{+}(X_{\alpha}^{+})^2 = 0 \quad |\alpha - \beta| = 1.
\]
2.3.1 Fundamental representation of $\mathfrak{gl}(n+1)$

We define for $0 \leq \alpha \leq n$

$$L_{\alpha} = q^{\frac{1}{2}N_{\alpha}}. \quad (72)$$

In terms of the $N_{\alpha}$ the defining relations of the Lie algebra $\mathfrak{gl}(n+1)$ are given by the limit $q \to 1$ of (67) - (71):

$$[N_{\alpha}, N_{\beta}] = 0 \quad (73)$$

$$[N_{\alpha}, X_{\beta}^\pm] = \pm(\delta_{\alpha,\beta-1} - \delta_{\alpha,\beta})X_{\beta}^\pm \quad (74)$$

$$[X_{\alpha}^+, X_{\beta}^-] = \delta_{\alpha,\beta} (N_{\alpha-1} - N_{\alpha}) \quad (75)$$

$$[X_{\alpha}^\pm, X_{\beta}^\pm] = 0 \quad |\alpha - \beta| \neq 1, \quad (76)$$

$$[X_{\alpha}^\pm, [X_{\alpha}^\pm, X_{\beta}^\pm]] = 0 \quad |\alpha - \beta| = 1. \quad (77)$$

It is well-known (and easy to verify) that the matrices (52) form the fundamental representation of the Lie algebra $\mathfrak{gl}(n+1)$ (73) - (77) via the algebra homomorphism $X^\pm \mapsto \sigma^{\alpha,\pm}$, $N_{\alpha} \mapsto \hat{n}_{\alpha}$. It has been pointed out [49] that with (72) these matrices then also form a representation of the quantum algebra $U_q[\mathfrak{gl}(n+1)]$ (66) - (71). We remark that $\sigma^{\alpha,\pm}$ are nilpotent of degree 2, i.e., $(\sigma^{\alpha,\pm})^2 = 0$.

2.3.2 Relation between $U_q[\mathfrak{gl}(n+1)]$ and $U_q[\mathfrak{sl}(n+1)]$

**Definition 2.21** Let $A$ be the Cartan matrix of simple Lie algebras of type $A_{n+1}$ with matrix elements $A_{\alpha,\beta} = 2\delta_{\alpha,\beta} - \delta_{\alpha,\beta-1}\delta_{\alpha,\beta+1}$ and

$$H_{\alpha} = N_{\alpha-1} - N_{\alpha}, \quad 1 \leq \alpha \leq n. \quad (78)$$

Then the quantum algebra $U_q[\mathfrak{sl}(n+1)]$ is the subalgebra of $U_q[\mathfrak{gl}(n+1)]$ generated by $q^{A_{\alpha}/2}$ and $X_{\alpha}^\pm$ with relations

$$q^{H_{\alpha}/2}q^{-H_{\alpha}/2} = q^{-H_{\alpha}/2}q^{H_{\alpha}/2} = I \quad (79)$$

$$q^{H_{\alpha}/2}q^{H_{\beta}/2} = q^{H_{\beta}/2}q^{H_{\alpha}/2} \quad (80)$$

$$q^{H_{\alpha}}X_{\beta}^+q^{-H_{\alpha}} = q^{\pm A_{\alpha,\beta}}X_{\beta}^+ \quad (81)$$

$$[X_{\alpha}^+, X_{\beta}^-] = \delta_{\alpha,\beta} [H_{\alpha}]_q. \quad (82)$$

and (70), (72).

The fact that $U_q[\mathfrak{sl}(n+1)]$ is a subalgebra of $U_q[\mathfrak{gl}(n+1)]$ can be seen by noticing that $\prod_{\alpha=0}^n L_{\alpha}$ belongs to the center of $U_q[\mathfrak{gl}(n+1)]$. The fundamental representation of both $\mathfrak{sl}(n+1)$ and $U_q[\mathfrak{sl}(n+1)]$ is formed by the set of matrices $\sigma^{\alpha,\pm}$ defined in (52) and

$$\hat{h}_{\alpha} = \hat{n}_{\alpha-1} - \hat{n}_{\alpha} \quad (83)$$

with $1 \leq \alpha \leq n$. 

\[16\]
2.3.3 Coproduct representation of $U_q[\mathfrak{gl}(n+1)]$

The coproduct is an algebra homomorphism defined by

$$
\Delta(\sigma^{\alpha,\pm}) = \sigma^{\alpha,\pm} \otimes q^{-\frac{1}{2}h^\alpha} + q^{\frac{1}{2}h^\alpha} \otimes \sigma^{\alpha,\pm}, \quad \Delta(\hat{n}^\alpha) = \hat{n}^\alpha \otimes 1 + 1 \otimes \hat{n}^\alpha.
$$

(84)

Iteration yields the representation matrices

$$
X^{\alpha,\pm}_\alpha = \sum_{k=L^-}^{L^+} X^{\alpha,\pm}_\alpha(k)
$$

(85)

$$
\hat{N}^\alpha = \sum_{k=L^-}^{L^+} \hat{n}^\alpha_k
$$

(86)

with

$$
X^{\alpha,\pm}_\alpha(k) = \left( q^{\frac{1}{2}h^\alpha} \right)^{(k-L^-)} \otimes \sigma^{\alpha,\pm} \otimes \left( q^{-\frac{1}{2}h^\alpha} \right)^{(L^+-k)}
$$

(87)

$$
= q^{\frac{1}{2} \sum_{l=L^-}^{L^+} h^\alpha_k} \sigma^{\alpha,\pm}_k \otimes q^{-\frac{1}{2} \sum_{l=L^-}^{L^+} h^\alpha_l}
$$

(88)

The unit $I$ is represented by the $d_{n,L}$-dimensional unit matrix

$$
1 := 1 \otimes L.
$$

(91)

The crucial property of this representation are the commutation relations

$$
[H^{PS}, X^{\alpha,\pm}] = [H^{PS}, N^\alpha] = 0
$$

(92)

which express the symmetry of the Perk-Schultz Hamiltonian $H^{PS}$ under the action of the quantum algebra $U_q[\mathfrak{gl}(n+1)]$.

Notice that for the symmetric case $q = 1$ we shall denote the representations $X^{\alpha,\pm}_\alpha$ by $S^{\alpha,\pm}_\alpha$. Together with the diagonal matrices $N^\alpha$ they form a tensor representation of $\mathfrak{gl}(n+1)$ as defined in (73) - (77).

2.4 Remarks on the hydrodynamic limit

We prove some simple results on locally conserved currents that suggest certain properties of the hydrodynamic limit of the $n$-priority ASEP.

**Proposition 2.22** Let $\mathcal{L}$ be the generator of the $n$-species priority ASEP on the finite lattice $\Lambda_L$. The local indicators $n_k^\alpha(\eta)$ satisfy the discrete continuity equation

$$
\mathcal{L}n_k^\alpha = j_{k-1}^\alpha - j_k^\alpha, \quad L^- \leq k \leq L^+
$$

(93)

with the locally conserved instantaneous currents

$$
\dot{j}_k^\alpha = w \left( \sum_{\beta=0}^{\alpha-1} (qn_k^\alpha n_{k+1}^\beta - q^{-1}n_k^\beta n_{k+1}^\alpha) - \sum_{\beta=\alpha+1}^{n} (qn_k^\beta n_{k+1}^\alpha - q^{-1}n_k^\alpha n_{k+1}^\beta) \right).
$$

(94)

for $L^- \leq k \leq L^+$ and $j_{L^-}^\alpha = j_{L^+}^\alpha = 0$ for all $\alpha \in S_{0,n}$.  

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For $q=1$ this reduces to the linear diffusive current $j_k^\alpha = n_k^\alpha - n_{k+1}^\alpha$ which means that the symmetric $n$-component exclusion process is a decoupled gradient system. For $q \neq 1$ a naive guess (based on the coupling to indicators $\beta \neq \alpha$ in the expressions for the microscopic currents $j^\alpha$) seems to suggest that the hydrodynamic limit of the model would correspond to coupled Burgers equations similar to those treated in [84]. However, as demonstrated by the next result, these equations can be decoupled very simply.

**Proposition 2.23** Let $\mathcal{L}$ be the generator of the $n$-species priority ASEP on the finite lattice $\Lambda_L$. The local indicators $m_k^\alpha(\eta)$ satisfy the discrete continuity equation

$$\mathcal{L}m_k^\alpha = \tilde{j}_{k-1}^\alpha - \tilde{j}_k^\alpha, \quad L^- \leq k \leq L^+$$

with the locally conserved instantaneous currents

$$\tilde{j}_k^\alpha = w (qm_k^\alpha (1 - m_{k+1}^\alpha) - q^{-1} (1 - m_k^\alpha) m_{k+1}^\alpha)$$

for $L^- \leq k \leq L^+$ and $\tilde{j}_{L^-+1}^\alpha = 0$ for all $\alpha \in \mathbb{S}_{0,n}$.\[100\]

**Proof:** Straightforward computation yields $\tilde{j}_k^\alpha = -w_q \tilde{j}_{k+1}^\alpha (m_{k+1}^\alpha - m_k^\alpha)$ and after some algebra involving reshuffling of indices one gets from (100) the expression (101).

**Remark 2.24** Even for $q \neq 1$ there is no cross-coupling between different indices $\alpha, \beta$ in the currents (101). This stems from the fact that the process for the occupation variables $m_k^\alpha$ is the same as for a first-class particle $n$ that sees all other particles as vacancies and thus does not depend on $\alpha$. 

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This fact provides some intuition on the hydrodynamic limit \[83, 51\] of the process. We shall assume that each species $\alpha$ is represented by $\rho^\alpha := N^\alpha/L$ particles such that in the hydrodynamic limit $L \to \infty$ the densities $\rho^\alpha$ remain non-zero. For our purposes an informal discussion is sufficient.

The Markov projection on the microscopic occupation numbers $m_k$ and the resulting decoupling \[99\] shows that on macroscopic scale the $n$-species the priority ASEP with $q \neq 1$ is governed by a system of decoupled inviscid Burgers equations

$$\partial_t \sigma^\alpha(x, t) + w(q - q^{-1})\partial_x [\sigma^\alpha(x, t)(1 - \sigma^\alpha(x, t))] = 0 \quad (101)$$

for the local densities $\sigma^\alpha(x, t) := \sum_{\beta=\alpha}^n \rho^\beta(x, t)$, in complete analogy to the usual ASEP \[76\]. In infinite volume the Riemann problem can be solved with the method of characteristics and shock stability \[56, 57\]. In particular, for any fixed $\alpha$ there exists a shock solution with density

$$\sigma^\alpha(x, t) = \begin{cases} \sigma_-^\alpha & x < x^\alpha_s(t) \\ \sigma_+^\alpha & x > x^\alpha_s(t) \end{cases} \quad (102)$$

and deterministically moving shock position

$$x^\alpha_s(t) = x^\alpha_s(0) + v^\alpha_s t \quad (103)$$

with shock velocity

$$v^\alpha_s = w(q - q^{-1})(1 - \sigma_+^\alpha - \sigma_-^\alpha) \quad (104)$$

arising from the Rankine-Hugoniot condition \[56\]

$$v^\alpha_s(c_+, c_-) = \frac{j^+_\alpha - j^-_\alpha}{c_+ - c_-} \quad (105)$$

for discontinuities from $c_-$ to $c_+$ and currents $j^\pm_\alpha$ in the two branches of the shock, which in the present case are given by $j^\pm_\alpha = w(q - q^{-1})\sigma_\pm^\alpha(1 - \sigma_\pm^\alpha)$ for the shock densities $\sigma_\pm^\alpha$. Since by \[104\] $v^\alpha_s - v^\beta_s > 0$ for $\alpha > \beta$ the shock positions for the individual modes $\alpha$ satisfy $x^\alpha_s(t) > x^{\alpha+1}_s(t) > \cdots > x^1_s(t)$ at all times $t \geq 0$.

More complex weak solutions allow for consecutive shocks in each mode at positions $x^{\alpha,i}_s(t)$. According to \[104\] the corresponding shock velocities satisfy $v^{\alpha,i}_s > v^{\alpha,i+1}_s$ for all $i$, so that after a finite macroscopic time neighboring shocks coalesce \[34\] and eventually only a single shock of type $\alpha$ remains. The question then arises what these macroscopic discontinuities look like on microscopic scale, see \[28\] for the stationary limit of a single travelling shock in the standard ASEP and for the time-dependent case see \[9, 8\]. This issue will be further addressed below.

For the finite system with reflecting boundaries we scale the lattice edges $L^\pm = x^\pm L$ such that for $L \to \infty$ macroscopic volume $\ell = x^+ - x^-$ remains finite. The reflecting boundary conditions correspond on macroscopic scale to the evolution of $\sigma^\alpha(x, t)$ on the interval $[x^-, x^+] \subset \mathbb{R}$ with boundary conditions $j(x^-, t) = j(x^+, t) = 0$ for all $t \geq 0$. The microscopic conservation law for $M^\alpha := \bar{\sigma}^\alpha L$ translates into $\int_{x^-}^{x^+} dx \, \sigma^\alpha(x, t) = \bar{\sigma}^\alpha$. Defining $\bar{\sigma}^{n+1} := 0$ and the points $x^\alpha := x^+ (1 - \bar{\sigma}^\alpha) - x^- \bar{\sigma}^\alpha$ one finds on $[x^-, x^+]$ the static (weak) solution $\sigma^\alpha_{\text{stat}}(x) = \Theta(x - x^\alpha)$ which one expects as final state of the evolution of the shock solutions described above.

This implies for the individual static densities $0 \leq \rho^\alpha_{\text{stat}}(x) \leq 1$

$$\rho^\alpha_{\text{stat}}(x) = \Theta(x - x^\alpha)\Theta(x^{\alpha+1} - x) = \begin{cases} 1 & x^\alpha < x < x^{\alpha+1} \\ 0 & \text{else.} \end{cases} \quad (106)$$
This is a phase-separated state with successive blocks fully occupied by particles of species \( \alpha \) in increasing order, reminiscent of phase separation in a related two-species exclusion processes \([29, 6, 23, 7, 15]\). Also this feature will be further elucidated below.

3 Results

3.1 Reversible measures

All invariant measures of the \( n \)-priority ASEP on the finite lattice \( \Lambda^L \) are obtained in explicit form and blocking measures for the infinite system are presented.

Theorem 3.1 (Reversible measures) For \( \Lambda = \Lambda_L \) and for \( \eta \in S_{0,n}^L \) let

\[
E(\eta) := - \sum_{k=L^-}^{L^+} \sum_{l=L^-}^{k-1} \text{sgn}(\eta_k - \eta_l)
\]

be the “energy” of a configuration \( \eta \). Then for

\[
\pi(\eta) := q^{-E(\eta)}.
\]

we have:

(i) \( \pi \) is a reversible measure for the \( n \)-priority ASEP \([42]\) on \( \Lambda_L \).

(ii) With particle numbers \( \vec{N} = \{N^1, \ldots, N^n\} \) for each species \( \alpha \geq 1 \) the canonical measure

\[
\pi^c_N(\eta) = \pi(\eta) \frac{P_N^c(\eta)}{C_L(\vec{N})}
\]

with the canonical partition function

\[
C_L(\vec{N}) = \frac{[L]_q!}{\prod_{\alpha=0}^n [N^\alpha]_q!}
\]

is the unique invariant measure for the process \([42]\) on the subset \( S_{\vec{N}}^L \).

(iii) With chemical potentials \( \mu_\alpha \in \mathbb{R} \) and \( \vec{\mu} := (\mu_1, \ldots, \mu_n) \) the grand-canonical family

\[
\pi^g_{\vec{\mu}}(\eta) = e^{\sum_{\alpha=1}^n \mu_\alpha N^\alpha(\eta)} Z_L(\vec{\mu}) \pi(\eta)
\]

are invariant measures with the homogeneous multivariate Rogers-Szegő polynomial \([42]\)

\[
Z_L(\vec{\mu}) = \sum_{\vec{N}} e^{\sum_{\alpha=1}^n \mu_\alpha N^\alpha} C_L(\vec{N})
\]

as grand-canonical partition function.
Remark 3.2 For \( n = 1 \) one has \( Z_L(\mu) = \prod_{k=L^-}^{L^+} (1 + e^{\mu} q^{2k-L^+-L^-}) \) and the grand-canonical invariant measure (111) becomes a product measure with marginal densities \( \rho_k = e^{\mu} q^{2k-L^+-L^-}/(1 + e^{\mu} q^{2k-L^+-L^-}) \) which is the blocking measure of (60) for the ASEP restricted to the subset \( \Lambda_L \) of \( \mathbb{Z} \). The parameter \( \mu \) fixes the center of the shock, which is the lattice point \( k^* := \min k (\rho_k > 1/2) \).

We point out alternative forms of writing the reversible measure. From (25) we define the partial energies

\[
E^{\alpha\beta}(\eta) := - \sum_{k=L^-}^{L^+} \sum_{i=L^-}^{k-1} (n_i^\alpha(\eta)n_i^\beta(\eta) - n_i^\alpha(\eta)n_k^\beta(\eta)).
\]  

By sum rules (277) and (280) we have

\[
E^{\alpha\beta}(\eta) = - \sum_{k=L^-}^{L^+} n_k^\alpha(\eta)N_k^\beta(\eta) = \sum_{k=L^-}^{L^+} n_k^\beta(\eta)N_k^\alpha(\eta).
\]  

We decompose the energy as

\[
E(\eta) = \sum_{\alpha=1}^{n} \sum_{\beta=0}^{\alpha-1} E^{\alpha\beta}(\eta) = E^0(\eta) + \bar{E}(\eta)
\]  

where, both in occupation variable and coordinate representation,

\[
E^0(\eta) := - \sum_{k=L^-}^{L^+} N_k^0(\eta) = - \sum_{i=1}^{N^0(\eta)} (L^+ + L^- - 2x_i^0(\eta))
\]  

and

\[
E^0(x) = - \sum_{i=1}^{N(x)} (2x_i - L^+ - L^-)
\]  

This decomposition leads to a factorization of the reversible measure

\[
\pi(\eta) = \pi^0(\eta)\bar{\pi}(\eta) = \pi^0(\bar{x})\bar{\pi}(\bar{\alpha})
\]  

where the reduced measure

\[
\bar{\pi}(\eta) := q^{-\bar{E}(\eta)}
\]  

does not depend on the vacancy projectors.

Notice that \(-L^2 n/(2n + 2) \leq E \leq L^2 n/(2n + 2)\). For fixed particle numbers \( N^\alpha \) the minimal energy \( E^\text{min}_N = - \sum_{\alpha=1}^{n} \sum_{\beta=0}^{\alpha-1} N^\alpha N^\beta \) is achieved when all two-particle signs are
positive, which is the case for the block configuration $\eta^{\min}$ with all vacancies to the left, followed by blocks of species $\alpha$ in increasing order such that

$$\eta^{\min}_k = \alpha \text{ for } L^+ - M^\alpha < k \leq L^+ - M^\alpha + 1$$

(122)

with the convention $M^{n+1} = 0$. This is a microscopic realization of the phase-separated macroscopic stationary density profile [106]. The local energy associated with site $k$ is long-ranged even though the stochastic dynamics are local. A similar phenomenon was found in the ABC-model [29, 23, 7, 15] which also exhibits phase separation.

Indeed, the discussion of the hydrodynamic limit suggests that on microscopic scale the invariant infinite volume measure inside the blocks, i.e., on points $k = [x]$ with $x^\alpha < x < x^{\alpha+1}$ is concentrated on the configurations with all sites occupied by a particle of species $\alpha$. In the (lattice) vicinity of the phase boundaries $k^\alpha := [x^\alpha]$ one expects the invariant infinite-volume measure to be blocking measures with particles of species $\alpha$ and $\alpha+1$. This intuition is well borne out by the form of the blocking measure for the standard ASEP ($n=1$) [60] and by the fact that the $n$-species priority ASEP evolving on a subset of $S_{0,n}$ with only two species $\alpha, \beta \in S_{0,n}$ of any kind an be identified with the standard ASEP, see Remark (2.24). Thus one obtains infinite-volume blocking measures of the $n$-species priority ASEP:

**Theorem 3.3** (Blocking measures) Let $\alpha, \beta \in S_{n+1}$ be two different particles species of any kind with $\alpha < \beta$ and fix a strictly positive constant $\lambda \in \mathbb{R}_+$. Then the product measures

$$\pi_{\alpha \beta}^\gamma = \frac{1}{1+\lambda q^{2k_0}} \frac{\lambda q^{2k}}{1+\lambda q^{2k}}$$

for $\gamma = \alpha$

$$\rho^\beta_k := \text{Prob}[n^\beta_k = 1] = \begin{cases} \frac{1}{1+\lambda q^{2k}} & \text{for } \gamma = \alpha \\ \frac{\lambda q^{2k}}{1+\lambda q^{2k}} & \text{for } \gamma = \beta \\ 0 & \text{else} \end{cases}$$

are invariant measures for the $n$-species priority ASEP on $\mathbb{Z}$.

**Remark 3.4** The particle density $\rho^\beta_k$ of species $\beta$ has the shape of a discretized shifted hyperbolic tangent with $\lim_{k \to -\infty} \rho^\beta_k = 0$ and $\lim_{k \to \infty} \rho^\beta_k = 1$. The parameter $\lambda$ determines the lattice point $k$ where the local particle density $\rho^\beta_k$ comes closest to $1/2$. Specifically, for $\lambda = q^{-2k_0}$ one has $\rho^\alpha_{k_0} = \rho^\beta_{k_0} = 1/2$.

### 3.2 Duality

We establish that the $n$-priority ASEP is self-dual w.r.t. a family of duality functions which arise from the symmetry of the generator of the process under the quantum algebra $U_q[gl(n+1)]$.

**Theorem 3.5** (Self-duality) Fix arbitrary parameters $c_\alpha \in \mathbb{C}$, $\alpha \in S_{0,n}$ and let $\eta$ and $\zeta$ be two configurations of the $n$-component priority ASEP defined by (32) on the finite lattice $\Lambda_L$. The process is self-dual with respect to the duality function

$$D(\zeta, \eta; \bar{c}) = \prod_{k=L^-}^{L^+} \prod_{\alpha=0}^{n} (Q^\alpha_k(\eta)q^{c_\alpha(M^\alpha(\eta)-L)})^{n^\alpha_k(\zeta)}$$

(124)
where
\[ Q_k^0(\eta) = m_k^0(\eta) q^{M_k^0(\eta)} \] (125)

with the indicator functions \( n_k^0(\cdot), m_k^0(\cdot) \) [12] and the particle balance \( M_k^0(\cdot) \) [28].

**Remark 3.6** The duality function \( \tilde{D}(\zeta, \eta; \bar{c}) \) is neither the duality function of [12] for \( n = 2 \) nor the duality function of [54, 55] but a new duality function for the ASEP with second-class particles. A complete classification of duality functions is not the purpose of this work. However, the algebraic methods employed in the proof (which is constructive) exhibit quite explicitly how other duality functions can be computed, including the duality functions of [12] and [54, 55], see Remark [12] for further details.

We point out alternative forms of writing the duality function and the corresponding duality matrix. They follow from [121] by using \( m_k^0 = 1 \) which gives \( Q_k^0(\eta) = q^{2k - L^+ - L^-} \), the sum rules (275) and (279) which yield
\[
\prod_{k=L^-}^{L^+} q^{2k - L^+ - L^-} = 1, \quad \prod_{k=L^-}^{L^+} q^{(2k - L^+ - L^-) n_k^0} = \prod_{k=L^-}^{L^+} q^{-N_k^0},
\] (126)
and the projector property of Lemma (A.3) which implies \( m_k^0 q^{m_k^0} = q m_k^0 \). Thus one has
\[
\tilde{D}(\zeta, \eta; \bar{c}) = \prod_{k=L^-}^{L^+} q^{-N_k^0(\zeta)} \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} (Q_k^0(\eta) q^{c_{\alpha}(M^\alpha(\eta) - L)}) \alpha_k(\zeta)
\] (127)
\[
= \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} (\tilde{Q}_k^{\alpha}(\eta; c_{\alpha})) \alpha_k(\zeta)
\] (128)

where
\[
\tilde{Q}_k^{\alpha}(\eta; c_{\alpha}) = q^{-(1+c_{\alpha}) \sum_{i=L^-}^{L^+} (1-m_i^0(\eta)) + (1-c_{\alpha}) \sum_{i=k+1}^{L^+} (1-m_i^0(\eta)) m_k^0(\eta).}
\] (129)

Specifically when \( c_{\alpha} = c_n \) for all \( \alpha \) and and \( c_n \) takes values 0 or \( \pm 1 \) we denote \( \tilde{Q}_k^0(\eta) := \tilde{Q}_k^{0}(\eta; 0), \tilde{D}(\zeta, \eta) := \tilde{D}(\zeta, \eta; 0), \tilde{D}^\pm(\zeta, \eta) := \tilde{D}(\zeta, \eta; \pm 1) \).

In coordinate representation one has for a configuration \( \mathbf{x} \in \mathbb{Z}^2_N \) with \( N(\mathbf{x}) \) particles
\[
D(\mathbf{x}, \eta; \bar{c}) = \prod_{i=1}^{N(\mathbf{x})} \tilde{Q}_{x_i}^{\alpha_i}(\eta; c_{\alpha_i}) =: \tilde{Q}_x^\mathbf{c}(\eta)
\] (130)
with the particle coordinates \( x_i \) and species \( \alpha_i \) and therefore
\[
D(\mathbf{x}, \eta) = \prod_{i=1}^{N(\mathbf{x})} \tilde{Q}_{x_i}^{\alpha_i}(\eta) =: Q_x(\eta)
\] (131)
\[
D^+(\mathbf{x}, \eta) = \prod_{i=1}^{N(\mathbf{x})} q^{-2 \sum_{i=L^-}^{L^+} (1-m_i^{\alpha_i}(\eta)) m_{x_i}^{\alpha_i}(\eta)} =: Q_x^+(\eta)
\] (132)
\[
D^-(\mathbf{x}, \eta) = \prod_{i=1}^{N(\mathbf{x})} q^{2 \sum_{i=L^-}^{L^+} (1-m_i^{\alpha_i}(\eta)) m_{x_i}^{\alpha_i}(\eta)} =: Q_x^-(\eta)
\] (133)
Notice that in the duality functions $D^\pm$ the lattice enters through the position of only one its edges $L^\pm$ in the exponential factors. This observation together with Lemma (2.9) and the monotonicity property $1 - m^\alpha_l(\eta) \leq 1 - n^\alpha_l(\eta)$ for all $\alpha$ and all $l$ yields duality functions for the semi-infinite lattices $\Lambda^\pm$ and the infinite integer lattice $\mathbb{Z}$:

**Corollary 3.7** Let $x$ be the coordinate representation of a configuration with a finite number $N$ of particles of each species $\alpha \geq 1$ and let $\Lambda$ be a lattice defined in (2.1). Then:

a) The $n$-priority ASEP on $\Lambda^\pm$ is self-dual w.r.t. the duality function $D^\pm(x, \eta)$ (132) resp. for all $\eta \in \mathcal{S}_{L^\pm}^\pm$.

b) The $n$-priority ASEP on $\mathbb{Z}$ is self-dual w.r.t. the duality function $D^+(x, \eta)$ (132) resp. for all $\eta \in \mathcal{S}_{\mathbb{Z}_+}^\mathbb{Z}$.

Interpreting the duality function $Q^\alpha_x(\eta)$ as a function of the configurations $\eta$ indexed by $x$ (without thinking of $x$ as representing another particle configuration) the self-duality relation (124) reads

$$E^\eta Q^\alpha_x(\eta_t) = \sum_{y \in V_\Lambda^{\Lambda}(x)} Q^\alpha_y(\eta) P(y, t|x, 0).$$

with the transition probability $P(y, t|x, 0)$ from configuration $x$ at time $t = 0$ to $y$ at time $t \geq 0$. This transition probability can be computed explicitly from the nested coordinate Bethe ansatz [89, 41] for the $n$-component priority ASEP, see [27, 70, 17, 21] for work on $n = 2$ and [51, 88] for general $n$. Specifically for $N(x) = 1$ this observation leads to

**Corollary 3.8** Let $x$ be the configuration with a single particle of some species $\alpha \geq 1$ at site $k$, $\eta \in \mathcal{S}_{0,n}^L$. Then

$$\mathcal{L} Q^\alpha_k(\eta) = J^\alpha_{k-1}(\eta) - J^\alpha_k(\eta), \quad L^- \leq k \leq L^+$$

with the locally conserved instantaneous currents

$$J^\alpha_k(\eta) = w \left( Q^\alpha_k(\eta) - Q^\alpha_{k+1}(\eta) \right).$$

for $L^- \leq k \leq L^+$ and $J^\alpha_{L^-} = J^\alpha_{L^+} = 0$ for all $\alpha \in \mathcal{S}_{0,n}$.

The linearity of the locally conserved currents (136) corresponds to a discrete biased diffusion equation for the expectation of $Q^\alpha_k$ for arbitrary initial configurations $\eta$. One can interpret the substitution $m^\alpha_k(\eta) \rightarrow Q^\alpha_k(\eta)$ as the lattice analogue of the Cole-Hopf transformation for the Kardar-Parisi-Zhang equation [50, 78, 3], thus generalizing an analogous earlier observation for $n = 1$ [14, 80].

### 3.3 Microscopic structure of shocks

Here we show, using self-duality, that on microscopic scale shocks in the $n$-priority ASEP on $\mathbb{Z}$, or more precisely microscopic shock markers that indicate the positions of shocks on macroscopic scale, perform a random motion that we call the shock exclusion process.

The shock exclusion process, defined formally below, has jumps like the $n$-priority ASEP, but each particle has its individual jump rate. Thus the process has a natural
description in terms of coordinate representation: If site \( x_i \pm 1 \) next to particle \( i \) is empty, then particle \( i \) jumps after an exponentially distributed random time with parameter \( w_{i}^{\pm 1} \) defined below in (137). If two particles \( i, i+1 \) are nearest neighbours they exchange their colour after an exponential random time with parameter (a) \( w_{q}^{-1} \) if \( \alpha_i > \alpha_{i+1} \), (b) \( q \) if \( 0 < \alpha_i < \alpha_{i+1} \), (c) \( w_{q} \) if \( \alpha_{i+1} = 0 \), and (d) \( q^{-1} \) if \( \alpha_i = 0 \). Notice that for colours \( \alpha \in \{1, \ldots, n-1\} \) the hopping bias \( q \) is inverted compared to the \( n \)-priority ASEP.

**Definition 3.9** (Shock exclusion process) Let \( x = (\vec{x}, \vec{\alpha}) \in \mathbb{Z}^N \) be a configuration with \( N \) particles and let for a parameter \( \rho_0 \in (0,1) \) the functions \( v_i \), \( 1 \leq i \leq N \), be defined by

\[
v_i := \frac{(q - q^{-1})\rho_i(1 - \rho_i)}{\rho_i - \rho_{i-1}}, \quad v_i^{-1} = \frac{(q - q^{-1})\rho_{i-1}(1 - \rho_{i-1})}{\rho_i - \rho_{i-1}}. \tag{137}
\]

The shock exclusion process is defined by the generator

\[
\mathcal{M} f(x) = \sum_{i=1}^{N} D_i f(x) + \sum_{i=1}^{N-1} C_i f(x) \tag{138}
\]

with the single-particle hopping and the two-particle colour exchange generators

\[
D_i f(x) = \sum_{\sigma = \pm} w_{i}^\sigma (x) (f(x_i^\sigma) - f(x)), \quad C_i f(x) = c_i^{ii+1}(x)(f(x_i^{ii+1}) - f(x)) \tag{139}
\]

with the jump rates

\[
w_i^{\pm}(x) := w_{i}^{\pm 1}(1 - \delta_{x_i, x_i^{\pm 1}}), \quad 1 \leq i \leq N, \tag{140}
\]

and for \( 1 \leq i \leq N - 1 \) the colour exchange rates

\[
c_i^{ii+1}(x) := \left[ w \left( q^{\text{sgn}(\alpha_i, \alpha_{i+1})} - \delta_{\alpha_i, \alpha_{i+1}} \right) (1 - \delta_{\alpha_i, 0})(1 - \delta_{\alpha_{i+1}, 0}) + w q(1 - \delta_{\alpha_i, 0})\delta_{\alpha_{i+1}, 0} + w q^{-1}\delta_{\alpha_i, 0}(1 - \delta_{\alpha_{i+1}, 0}) \right] \delta_{x_i, x_{i+1}} \tag{141}
\]

where \( x_i^\sigma \) and \( x_i^{ii+1} \) are defined by

\[
(x_i^{\pm})_j = x_j \pm \delta_{j, i}, \quad \vec{\alpha}^{\pm} = \vec{\alpha}\]

\[
(x_i^{ii+1})_j = x_j \quad \vec{\alpha}^{ii+1} = \pi_i^{ii+1}(\vec{\alpha}) \tag{142}
\]

with the conventions \( x_0 := -\infty, x_{N+1} := +\infty \), and the colour permutation \( \pi_i^{ii+1}(\cdot) \) defined analogously to \( \pi_1 \).

**Remark 3.10** The definition of \( v_i \) implies that

\[
\frac{\rho_i(1 - \rho_{i-1})}{\rho_{i-1}(1 - \rho_i)} = q^2, \quad \forall i \in \{1, \ldots, N\}. \tag{143}
\]

and also \( w_i^{+}w_i^{-} = w_i^{2} \) as one has for the single-particle hopping rates \( w_{i}^{\pm} = w_{q}^{\pm 1} \) of the \( n \)-priority ASEP. In fact, one can show (see Proposition 4.12 below) that the shock exclusion process and the \( n \)-priority ASEP on the finite lattice \( \Lambda_L \) are related by a similarity transformation, up to a boundary term.
For $N = 1$ the shock exclusion process reduces to a simple random walk of a particle that moves with average speed
\[
v := \mathbb{E}^0 \frac{x(t)}{t} = w(q - q^{-1})(1 - \rho_1 - \rho_0)
\]
and diffusion coefficient
\[
D := \mathbb{E}^0 \frac{(x(t) - vt)^2}{t} = \frac{w}{2} (q - q^{-1}) \frac{\rho_1 (1 - \rho_1) + \rho_0 (1 - \rho_0)}{\rho_1 - \rho_0}
\]
and Gaussian fluctuations on coarse-grained diffusive scale. For $N$ particles the shock exclusion process can be seen on large scales as a gas of particles with different masses (or friction coefficients) that drift diffusively in a gravitational field with mean velocities $v_i > v_{i+1}$ until they collide, with $v_i$ and diffusion coefficients $D_i$ given by
\[
v_i = \frac{w(q - q^{-1}) \rho_i (1 - \rho_i)}{\rho_i - \rho_{i-1}} - \frac{w(q - q^{-1}) \rho_{i-1} (1 - \rho_{i-1})}{\rho_i - \rho_{i-1}},
\]
\[
D_i = \frac{w}{2} (q - q^{-1}) \frac{\rho_i (1 - \rho_i) + \rho_{i-1} (1 - \rho_{i-1})}{\rho_i - \rho_{i-1}}.
\]

**Definition 3.11** (Shock measure) Let $\mathbf{x} \in V^Z_K$ be a configuration with $K$ particles at positions $x_j \in \mathbb{Z}$, $x_{j+1} > x_j$ for all $j \in \{1, \ldots, K\}$, with $\alpha(j) = \eta_{x_j}(\mathbf{x}) \in \mathbb{S}_n$ specifying the particle species at position $x_j$ and define
\[
\rho_j(\lambda) := \frac{q^{2j-K+\lambda}}{1 + q^{2j-K+\lambda}}, \quad 0 \leq j \leq K
\]
with parameter $\lambda \in \mathbb{R}$. The product measures $\nu_{\mathbf{x}}$ on $\mathbb{S}_0^n$ indexed by $\mathbf{x}$ with parameter $\lambda$ and marginals $\nu^k_{\mathbf{x}}(\eta_k)$ for $\eta_k \in \mathbb{S}_0^n$ at site $k \in \mathbb{Z}$ given by
\[
\nu^k_{\mathbf{x}}(\eta_k) = \begin{cases}
\delta_{\eta_k, \alpha(j)-1} & k = x_j \\
\rho_0(\lambda) \delta_{x_k, 0} + (1 - \rho_0(\lambda)) \delta_{\eta_k, 0} & k < x_1 \\
\rho_j(\lambda) \delta_{x_k, n} + (1 - \rho_j(\lambda)) \delta_{\eta_k, 0} & x_j < k < x_{j+1}, 1 \leq j \leq N - 1 \\
\rho_N(\lambda) \delta_{x_k, n} + (1 - \rho_K(\lambda)) \delta_{\eta_k, 0} & k > x_K
\end{cases}
\]
are called shock measures for the $n$-priority ASEP on $\mathbb{Z}$ with $N^\alpha$ microscopic shock markers of type $\alpha(j)$ at positions $x_j$ and $K = \sum_{\alpha=0}^{n-1} N^\alpha$ shocks. The restriction to $\Lambda_L$ for $\mathbf{x} \in V^Z_N$,
\[
\mu^L_{\mathbf{x}}(\eta) := \prod_{k=L^-}^{L^+} \nu^k_{\mathbf{x}}(\eta_k)
\]
is also called shock measure.

We can identify the positions $\mathbf{x}$ and types $\alpha$ of the shockmarkers in (150) with a configuration $\mathbf{x} = (\mathbf{x}, \alpha^T) \in V^Z_N$, and similarly with $\mu^L_{\mathbf{x}}$ for the finite lattice $\Lambda_L$. We shall call this identification canonical.

Now we are in a position to state the main result of this subsection.
Theorem 3.12  Let \( \nu_x(t) \) denote the distribution at time \( t \) of the \( n \)-priority ASEP, starting from an \( N \)-particle shock measure \( \nu_x \). Then, for any \( x \in \mathbb{Z} \)

\[
\nu_x(t) = \sum_{y \in V_N} P(y, t|x, 0) \nu_y
\]

where \( P(y, t|x, 0) \) is the transition probability of the shock exclusion process (3.9) with the canonical identification of the shock markers in \( \nu_x \) with configurations in \( x \in V_N \).

Corollary 3.13  For \( N \) shock markers the invariant distribution of the distances \( r_i := x_{i+1} - x_i \) is a product distribution with geometric marginals

\[
\text{Prob} \left[ (r_i = k) \right] = \frac{(\rho_N - \rho_i)(\rho_i - \rho_0)}{\rho_i(1 - \rho_i)} \left( 1 - \frac{(\rho_N - \rho_i)(\rho_i - \rho_0)}{\rho_i(1 - \rho_i)} \right)^k
\]

We call this stationary limit a bound state of \( N \) shocks, which on macroscopic scale is a single shock with a jump discontinuity in the density from \( \rho_0 \) to \( \rho_N \).

This follows by straightforward computation from the standard mapping of the ASEP to the zero-range process where the distance process \( r_i(t) := x_{i+1}(t) - x_i(t) \) between \( N - 1 \) neighbouring particles in the shock exclusion process is a zero-range process on \( N - 1 \) sites with site-dependent hopping rates [13]. With \( z = \rho_0/(1 - \rho_0) \) we can write

\[
\text{Prob} \left[ (r_i = k) \right] = 1 - \frac{(q^{2N} - q^{2k})(q^{2k} - 1)}{q^{2k}(1 + z)(1 + q^{2N}z)}.
\]

With \( z = q^{-N-2p} \) the \( [(N/2+p)]^{th} \) microscopic shock marks the center of the macroscopic shock, i.e. \( N/2 + p = \min k \left[ \rho_k > 1/2 \right] \).

Several remarks are in place.

Remark 3.14  1) The proof of the theorem uses a specific duality function. Similar statements for other shock measures can be obtained for other choices of duality functions. It is not the purpose of this work to explore this fact in more detail.

2) One recognizes in the microscopic shock velocities (146) the Rankine-Hugoniot speeds (105) since \( j_i := w(q - q^{-1})\rho_i(1 - \rho_i) \) is the expectation of the particle current to the right of shock \( i \) and \( j_i \) is the current to the left of shock \( i \) in the ASEP. As seen from (147) they coincide with the shock velocity (104) for macroscopic shocks of type \( \alpha = n \). The shock diffusion coefficients (148) are consistent with the general result of [32] on shock motion in the ASEP on diffusive scale, see also [9] for \( n = 1 \) and [8] for \( n = 2 \). The fact that the shocks can be marked with particles of arbitrary colour of lower priority and the ensuing colour exchange process between shock markers is a new insight as well as the point that the shock exclusion process has its origin in duality.

3) The transition probability of a generalized shock exclusion process for \( n = 1 \) and any \( N \) has been calculated from Bethe ansatz in the limit \( w \to 0 \) and \( q \to \infty \) such that \( wq = 1 \) for arbitrary limiting rates \( \nu_i \) and can be expressed in determinantal form [75].
4) On macroscopic scale the final stage of the time evolution of the shock measure can be interpreted as coalescence of shocks \[23\]. In the associated (non-Hermitian) quantum problem this invariant state is a many-body bound state of (coloured) magnons in the XXZ-quantum chain \((n = 1)\) or their higher rank analogues in the Perk-Schultz chain resp. \((n > 1)\). This observation suggests that many-body bound states of other quantum systems, that are known to exist from Bethe-ansatz, might have a classical analog as shock coalescence or related phase separation phenomena.

4 Proofs

4.1 Proof of Theorem (3.1)

(i) Reversibility is guaranteed by the detailed balance condition \[61\] which here reads

\[ w^{r+1}_q(\eta^{r+1}) = w^r_q(\eta) q^{-E(\eta)+E(\eta^{r+1})} \]

for all \(r\) with the total energy \[107\]. We observe that since \(\eta\) and \(\eta^{r+1}\) differ only at sites \(r\) and \(r + 1\) by a permutation one has

\[ E(\eta^{r+1}) - E(\eta) = 2\text{sgn}(\eta_{r+1} - \eta_r). \]

The detailed balance relation then follows from

\[ w^r_q(\eta^{r+1}) = w(q^{\text{sgn}(\eta_{r+1} - \eta_r - \delta_{r+1}, \eta_r}) \text{ and } w^{r+1}_q(\eta) = w(q^{-\text{sgn}(\eta_{r+1} - \eta_r - \delta_{r+1}, \eta_r}). \]

(ii) From (i) and particle number conservation it follows that the canonical measure \[109\] is an invariant measure for the process \(\eta^r\). Uniqueness follows from ergodicity for fixed particle numbers which itself a consequence of the fact that the process is a sequence of permutations. It remains to prove normalization \(\sum_\eta \pi^c_N(\eta) = 1\). We work with the coordinate representation and recall the decomposition \[115\] of the total energy and the corresponding decomposition \[120\] of the invariant measure. For fixed particle numbers \(N^\alpha\) for all species \(\alpha \geq 1\) (and hence also for the vacancies \(\alpha = 0\) with \(N^0 = L - M^1\)) we first compute, using \[116\] and \(M^1 = N\),

\[ \sum_{x \in W^L_N} \pi(x) = \sum_{x^0 \in W^L_{N^0}} q^{-\sum_{i=1}^{N^0}(2x^0_i - L^- - L^-)} \pi(\bar{\alpha}) = \left( \frac{L}{N^0} \right)_q. \] \(155\)

The second equality, which is trivially identical to the sum over the positions of all particles in the Weyl alcove \(W^L_N\), was obtained in \[77\], but actually goes back to a classical result from the theory of partitions \[4\].

Now we observe that the colour array \(\bar{\alpha}\) is a configuration of a \((n-1)\)-priority ASEP on a finite lattice with \(L^- = 1\) and \(L^+ = N = L - N^0\) and by \[108\] \(\pi(\bar{\alpha})\) is a reversible measure for this process. Thus we can iterate the decomposition \[115\] with particles of species 1 playing the role of the vacancies. This yields

\[ \sum_{\bar{\alpha}^1 \in W^L_{N^1} - N^0} \pi(\bar{\alpha}) = \sum_{\bar{\alpha}^1 \in W^L_{N^1} - N^0} q^{-\sum_{i=1}^{N^1}(2x^1_i - (L - N^0) - 1)} \pi(\bar{\alpha}) = \left( \frac{L - N^0}{N^1} \right)_q. \] \(156\)

By further iteration one gets

\[ \sum_{\eta \in \mathbb{S}^L_N} \pi(\eta) = \prod_{\alpha=0}^{n-1} \left( \frac{M^\alpha}{N^\alpha} \right)_q \] \(157\)

which is equal to the normalization factor \[110\].
(iii) With (ii) we have \( \exp \left( \sum_{\alpha=1}^{n} \mu_{\alpha} N_{\alpha} \right) C_{L}(\bar{N}) \pi_{N}^{\alpha}(\eta) = \exp \left[ \sum_{\alpha=1}^{n} \mu_{\alpha} N_{\alpha}(\eta) \right] \pi(\eta) \mathcal{P}_{\bar{N}}(\eta) \) and therefore
\[
\sum_{\bar{N}} \sigma_{\alpha}^{\bar{N}} C_{L}(\bar{N}) \pi_{\bar{N}}^{\alpha}(\eta) = e^{\sum_{\alpha=1}^{n} \mu_{\alpha} N_{\alpha}(\eta) \pi(\eta)}.
\] (158)
Since \( \pi_{\bar{N}}^{\alpha} \) is normalized taking the sum over all \( \eta \) yields the sum (112).

4.2 Proof of Theorem (3.5)

The central ingredients in the proof of self-duality are the quantum algebra symmetry of the generator and its reversibility. The concrete form of the duality function follows from certain matrix elements of the symmetry operators of \( U_{q}[\mathfrak{gl}(n+1)] \). We begin with establishing the quantum algebra symmetry of \( H \).

4.2.1 Quantum algebra symmetry

**Lemma 4.1** Let \( \hat{\pi} \) be the matrix form (49) of the reversible measure (108) and fix a parameter \( c \in \mathbb{C} \). The representation matrices \( \sigma_{k}^{\alpha,+} \) of \( \mathfrak{gl}(n+1) \) acting non-trivially on site \( k \) transform under the diagonal similarity transformation

\[
\hat{\sigma}_{k}^{\alpha,\pm}(c) := \hat{\pi}^{-c} \sigma_{k}^{\alpha,\pm} \hat{\pi}^{c}, \quad 1 \leq \alpha \leq n,
\] (159)
as follows

\[
\hat{\sigma}_{k}^{\alpha,\pm}(c) = q^{c(\bar{N}^{\alpha}(k)+\bar{N}^{\alpha-1}(k))} \sigma_{k}^{\alpha,\pm}.
\] (160)

**Proof:** For the matrix elements \( \langle \hat{\sigma}_{k}^{\alpha,+}(c) | \hat{\eta} \rangle \) we obtain

\[
\langle \hat{\sigma}_{k}^{\alpha,+}(c) | \hat{\eta} \rangle = \frac{\pi^{c}(\eta)}{\pi^{c}(\bar{c})} \langle \chi | \sigma_{k}^{\alpha,+} | \eta \rangle = \frac{\pi^{c}(\eta)}{\pi^{c}(\bar{c})} n_{k}^{\alpha}(\eta) \delta_{\chi,\eta^{k,-}}.
\] (161)

Since \( \sigma(n \pm k) - \sigma(n) = \pm (\delta_{n,0} + \delta_{n,\mp k}) \pm 2 \sum_{l=1}^{k-1} \delta_{n,\mp l} \) we have that

\[
\sigma(n^{k,-}_{l} - n^{k,-}_{m}) = \sigma(n^{k,-}_{l} - n^{k,-}_{m}) - \delta_{l,k} (\delta_{m,0} + \delta_{m,\pm 1}) + \delta_{m,k} (\delta_{m,0} + \delta_{m,\pm 1}).
\] (162)

Specifically, for \( \eta_{k} = \alpha \) this yields

\[
\sigma(n^{k,-}_{l} - n^{k,-}_{m}) = \sigma(n^{k,-}_{l} - n^{k,-}_{m}) - \delta_{l,k} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta)) + \delta_{m,k} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta)).
\] (163)

Therefore

\[
-E(\eta) + E(\eta^{k,-}) = \sum_{l=L^{-}+1}^{L^{+}} \sum_{m=L^{-}}^{l-1} \delta_{l,k} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta)) - \delta_{m,k} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta))
\]
\[
= \sum_{m=L^{-}}^{k-1} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta)) - \sum_{m=k+1}^{L^{+}} (\alpha_{m}^{\alpha}(\eta) + \alpha_{m}^{\alpha-1}(\eta))
\] (164)

which yields (150) for positive sign. Going through similar steps one verifies (160) for \( \hat{\sigma}_{k}^{\alpha,-}(c) \).
Proposition 4.2 Let $\hat{\pi}$ be the diagonal matrix form of the reversible measure (108). The generator $H$ (37) of the $n$-species priority ASEP is related to the Perk-Schultz-Hamiltonian $H_{PS}$ (64) by the ground state transformation

$$H_{PS} = (\hat{\pi})^{-1/2}H(\hat{\pi})^{1/2}$$

and satisfies the commutation relations

$$[H, \hat{N}^\alpha] = [H, Y_{\alpha}^\pm] = 0$$

with the representation matrices of $U_q[\mathfrak{gl}(n+1)]$ given by $\hat{N}^\alpha$ (86) and $Y_{\alpha}^\pm$ defined by

$$Y_{\alpha}^\pm := \sum_{k=L^-}^{L^+} Y_{\alpha,\pm}(k)$$

with

$$Y_{\alpha,+}(k) = q^{\sum_{l=L^-}^{k-1} \hat{n}_l^\alpha} \sigma_k^\alpha + q^{\sum_{l=k+1}^{L^+} \hat{n}_l^\alpha}$$

$$Y_{\alpha,-}(k) = q^{\sum_{l=L^-}^{k-1} \hat{n}_l^\alpha - 1} \sigma_k^\alpha - q^{-\sum_{l=k+1}^{L^+} \hat{n}_l^\alpha - 1}.$$ 

and the fundamental representation $\hat{n}_\alpha^\alpha, \sigma_{\alpha,\pm}$ (52) of $\mathfrak{gl}(n+1)$.

Remark 4.3 The result implies that the coproduct defined via (86) and (167) is an algebra homomorphism for $U_q[\mathfrak{gl}(n+1)]$. Lemma (4.1) implies $(\hat{\pi}^{-1}Y_{\alpha}^\pm \hat{\pi})^T = Y_{\alpha}^\pm$.

Proof of Proposition (4.2): (i) Since $H_{PS}$ is symmetric the claim (165) is equivalent to

$$H^T = \hat{\pi}^{-1}H\hat{\pi},$$

which is the matrix form (50) of reversibility established in Theorem (3.1).

(ii) In order to prove (166) we assert that

$$\hat{N}^\alpha = \hat{\pi}^{1/2} \hat{N}^\alpha \hat{\pi}^{-1/2}, \quad Y_{\alpha}^\pm = \hat{\pi}^{1/2} X_{\alpha}^\pm \hat{\pi}^{-1/2}.$$ 

The first transformation property is trivial since both $\hat{N}^\alpha$ and $\hat{\pi}$ are diagonal matrices. The second transformation property follows from Lemma (4.1) with $c = -1/2$ and the definition (85) of $X_{\alpha}^\pm$. The statement (166) then follows from the symmetry (92) of $H_{PS}$.

Next we state a result for $n = 1$ which is a much simplified proof of the duality relation of [80]. The main interest in the present context is the subsequent corollary for general $n$. In order to avoid excessive indexing we shall omit the superscript 1 for species 1 throughout the proposition and its proof on all quantities $Q, n, N$ etc. related to the configurations $\eta, \zeta \in S_{0,1}^L$.

Proposition 4.4 Fix $n = 1$ and let $Y_1^+$ be the generator (167) of $U_q[\mathfrak{gl}(2)]$. Then

$$Y_1^+ = \sum_{r=0}^{L} \frac{(Y_1^+)^r}{r! q^r}$$

has matrix elements

$$\langle \zeta | Y_1^+ | \eta \rangle = \prod_{k=L^-}^{L^+} (Q_k(\eta))^{n_k(\zeta)}$$

with $Q_k(\eta)$ defined in (123).
Proof: Using the explicit form

\[
\prod_{k=L^{-}}^{L^{+}} (Q_k(\eta))^{n_k(\zeta)} = \prod_{k=L^{-}}^{L^{+}} \left[ (n_k(\eta))^{n_k(\zeta)} q^{n_k(\zeta)N_k(\eta)} \right]
\]  

we can write (172) in coordinate representation \( x = \zeta \) as

\[
\langle x | T_{1}^{+} | \eta \rangle = \prod_{i=1}^{N(x)} Q_{x_{i}}(\eta) = \prod_{i=1}^{N(x)} q^{N_{x_{i}}(\eta)} n_{x_{i}}(\eta).
\]  

In order to prove this we introduce \( \hat{\bar{\bar{Y}}}^{1,+}(k) = q^{N_{k}^{1,+}} \). From the coproduct (168) and the fundamental representation (52) one finds \( [Y^{1,+}(k), \hat{Y}^{1,+}(l)] = 0 \) for all \( k, l \in \Lambda_{L} \) which implies \( [Y_{1}^{+}, \hat{Y}^{1,+}(l)] = 0 \) for all \( l \in \Lambda_{L} \).

We also note that one has \( \langle s | = \langle 0 | T_{1}^{+} \) and \( \langle s | \sigma_{k}^{\alpha,+} = \langle s | \hat{n}_{k}^{\alpha} \) [80]. Thus for \( \bar{x} \in W_{N(x)}^{L} \) we can write in terms of ordered products

\[
\langle s | \prod_{i=1}^{N(x)} \hat{Q}_{x_{i}} = q^{N_{s}(s)-1} \langle s | \prod_{i=1}^{N(x)} \hat{Y}^{1,+}(x_{i}) = q^{N_{s}(s)-1} \langle 0 | T_{1}^{+} \prod_{i=1}^{N_{s}(s)} \hat{Y}^{1,+}(x_{i})
\]  

\[
= q^{N_{s}(s)-1} \langle 0 | \prod_{i=1}^{N_{s}(s)} \hat{Y}^{1,+}(x_{i}) T_{1}^{+} = \langle x | T_{1}^{+}.
\]  

Taking the inner product with \( | \eta \rangle \) proves (174) since \( \hat{Q}_{x_{i}} | \eta \rangle = Q_{x_{i}}(\eta) | \eta \rangle \). \( \square \)

Lifting Proposition (4.4) to \( \mathbb{C}^{d_{a,L}} \) we note the following corollary.

**Corollary 4.5** Define the matrix sums

\[
\Theta_{\alpha}^{\pm} := \sum_{l=0}^{L} \frac{\left(Y_{\alpha}^{\pm}ight)^{l}}{[l]_{q^{\pm}}^{\alpha}}.
\]  

For \( 1 \leq \alpha \leq n \) and all \( \zeta, \eta \in S_{0,n}^{L} \) we have

\[
\langle \xi^{r} | T_{1}^{+} | \xi^{s} \rangle = \prod_{k=L^{-}}^{L^{+}} A_{k}(q; \xi^{r}, \xi^{s}) B_{k}(\xi^{r}, \xi^{s}) \Delta_{k}^{\alpha}(\xi^{r}, \xi^{s})
\]  

with the functions

\[
A_{k}(q; \xi^{r}, \xi^{s}) := q^{N_{k}(\xi^{r})N_{k}(\xi^{s})}, \quad B_{k}(\xi^{r}, \xi^{s}) := (n_{k}(\xi^{s}))^{n_{k}(\xi^{r})}
\]  

\[
\Delta_{k}^{\alpha}(\xi^{r}, \xi^{s}) := \prod_{\beta \neq \alpha - 1, \alpha} \delta_{n_{k}^{\beta}(\xi^{r}), n_{k}^{\beta}(\xi^{s})}.
\]
4.2.2 Symmetry and reversibility

Self-duality stated in matrix form reads

\[ DH = H^T D \]  \hspace{1cm} (180)

where the duality function \( D(\zeta, \eta) \) is the matrix element \( \langle \zeta \mid D \mid \eta \rangle \). A symmetry of \( H \) is defined by the commutation relation

\[ SH = HS \]  \hspace{1cm} (181)

with symmetry operator \( S \). This, along with reversibility \( (50) \) in matrix form, shows that \( \hat{\pi}^{-1} S \) is a self-duality matrix, see Theorem 2.6 of [43]. According to Lemma \( (166) \) \( H \) commutes with the matrix sums \( (176) \) and with the number operators \( N^{\alpha} \) and products of these matrices in arbitrary order. Thus we obtain

**Proposition 4.6** Define

\[ \Upsilon^+ := \Upsilon_1^+ \Upsilon_2^+ \ldots \Upsilon_n^+ \]  \hspace{1cm} (182)

The matrix

\[ D = \hat{\pi}^{-1} \Upsilon^+ \]  \hspace{1cm} (183)

is a duality matrix for the \( n \)-component ASEP defined by the generator \( (32) \).

**Remark 4.7** The duality matrix is not unique as any other linear combination of products of the symmetry operators gives rise to some duality matrix. For \( n = 2 \) one can show that the duality matrix in Theorem 3.3 in [12] is \( \hat{\pi}^{-1} \Upsilon_2^+ \Upsilon_1^- \) while the duality matrix of Theorem 2.2 in [54] is obtained from \( \hat{\pi}^{-1} \Upsilon_2^- \Upsilon_1^- \).

Observe that \( D_c(\zeta, \eta) \) \( (124) \) for \( c_\alpha \neq 0 \) can be decomposed

\[ D_c(\zeta, \eta) = \prod_{\alpha=0}^{n} q^{c_\alpha N^{\alpha}(\zeta)(M^{\alpha}(\eta) - L)} D(\zeta, \eta) \]  \hspace{1cm} (184)

with \( D(\zeta, \eta) \) given in \( (131) \). Assume now that

\[ D(\zeta, \eta) = \langle \zeta \mid \hat{\pi}^{-1} \Upsilon^+ \mid \eta \rangle \]  \hspace{1cm} (185)

Then for a general choice of parameters \( c_\alpha \) the Theorem follows directly from Proposition \( (4.6) \) and the preceding discussion since particle number conservation is also a symmetry and one can therefore construct from \( (185) \) further duality functions which are arbitrary functions of the conserved particle numbers \( N^{\alpha}(\zeta) \) and \( N^{\alpha}(\eta) \). Therefore it remains to prove \( (185) \).

4.2.3 Proof of \( (185) \) for \( q = 1 \)

First we consider the symmetric case \( q = 1 \) where \( H = H^{PS} \) commutes with tensor representation \( S^{\alpha, \pm} = \sum_{k=L}^L \sigma_k^{\alpha, \pm} \) and \( N^{\alpha} \) obtained from the coproduct \( (84) \) for \( q = 1 \), corresponding to symmetry under the Lie algebra \( gl(n + 1) \). We generalize the approach of [79] for \( n = 1 \) to general \( n \).
Lemma 4.8 Let $\sigma^{\alpha,+}$ be the fundamental representation matrix of $\mathfrak{gl}(n+1)$ defined in (52) and let $(\zeta | \in \mathbb{C}^{n+1} (| \eta ) \in \mathbb{C}^{n+1})$ be the canonical row (column) basis vector with component 1 at position $\zeta (| \eta )$ and let $S^{\alpha,+} = \sum_{k=L}^{L^+} \sigma^{\alpha,+}_k$ be the tensor representation obtained from the coproduct (84) for $q = 1$. Then the following factorization properties hold:

$$e^{S^{\alpha,+}} = \prod_{k=L^-}^{L^+} (1 + \sigma^{\alpha,+}_k)$$  \hspace{1cm} (186)

$$\langle \zeta | \prod_{\alpha=1}^{n} e^{S^{\alpha,+}} | \eta \rangle = \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} (m^{\alpha}_k(\eta))^{n^{\alpha}_k(\zeta)} = \prod_{k=L^-}^{L^+} m^{\zeta^k}_k(\eta).$$  \hspace{1cm} (187)

for the ordered product with the lowest index on the left and with the indicator functions $n^{\alpha}_k(\cdot)$ and $m^{\alpha}_k(\cdot)$ (17).

Proof: The commutation relations (56) imply $\exp (S^{\alpha,+}) = \prod_{k=L^-}^{L^+} \exp (\sigma^{\alpha,+}_k)$. Observing that $\sigma^{\alpha,+}_k$ (and hence $\sigma^{\alpha,+}$) is nilpotent of degree 2 leads to the first property (186). From (45) in Lemma (2.13) we have the factorization property

$$\langle \zeta | \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} (1 + \sigma^{\alpha,+}_k) | \eta \rangle = \prod_{k=L^-}^{L^+} \langle \zeta | \prod_{\alpha=1}^{n} (1 + \sigma^{\alpha,+}_k) | \eta_k \rangle$$  \hspace{1cm} (188)

Using $n^{\alpha}_k(\zeta) = \delta_{\zeta_k,\alpha}$ (21) it is therefore sufficient to prove the lemma for a single fixed $k$. Dropping the subscript $k$ one finds from the explicit form $\sigma^{\alpha,+} = |\alpha - 1)(\alpha|$ |

$$\langle \zeta | \prod_{\alpha=1}^{n} (1 + \sigma^{\alpha,+}) | \eta \rangle = \sum_{\alpha=1}^{n} (\alpha | =: \prod_{\alpha=1}^{n} (m^{\alpha}_k(\eta))^{n^{\alpha}_k(\zeta)}$$  \hspace{1cm} (189)

and with the definitions (19) the inner product with $| \eta \rangle$ gives

$$\langle \zeta | \prod_{\alpha=1}^{n} (1 + \sigma^{\alpha,+}) | \eta \rangle = m^{\zeta}(\eta).$$  \hspace{1cm} (190)

On the other hand, again by definition (19)

$$\prod_{\alpha=1}^{n} (m^{\alpha}(\eta))^{n^{\alpha}(\zeta)} = \prod_{\alpha=1}^{n} (m^{\alpha}(\eta))^{\delta_{\alpha,\zeta}}$$  \hspace{1cm} (191)

which is trivially equal to $m^{\zeta}(\eta)$.

With this result established we return to Proposition (4.6) and note that for $q = 1$ one has $\pi(\eta) = 1$. Thus Theorem (3.5) reduces to

$$D^{0}(\zeta, \eta) = \prod_{k=L^-}^{L^+} \prod_{\alpha=0}^{n} (m^{\alpha}_k(\eta))^{n^{\alpha}_k(\zeta)} =: D_{\text{sym}}(\zeta, \eta)$$  \hspace{1cm} (192)

and we need to prove only $D_{\text{sym}}(\zeta, \eta) = \langle \zeta | \Upsilon^+_1 \Upsilon^+_2 \ldots \Upsilon^+_n | \eta \rangle$ for $q = 1$. Since by definition $\Upsilon^+_\alpha = e^{\sigma^+_\alpha}$ for $q = 1$ this follows from Lemma (4.8). \hfill \Box
Before proceeding to the general case $q \neq 1$ we note the following alternative representation of the duality function based on the expansion

$$\langle \zeta | \Upsilon^+ | \eta \rangle = \sum_{\xi^1} \cdots \sum_{\xi^{n-1}} \langle \zeta | \Upsilon_1^+ | \xi^1 \rangle \langle \xi^1 | \Upsilon_2^+ | \xi^2 \rangle \cdots \langle \xi^{n-1} | \Upsilon_n^+ | \eta \rangle$$

for general $q \neq 0$. Defining for $1 \leq \alpha \leq n$

$$\Delta_k(\xi^0, \ldots, \xi^n) := \prod_{\alpha=1}^{n} \Delta_k^\alpha(\xi^{\alpha-1}, \xi^{\alpha}) = \prod_{\alpha=1}^{n} \prod_{\beta=0}^{\alpha-1} \delta_{n_k^{\beta}(\xi^{\alpha-1}), n_k^{\beta}(\xi^{\alpha})}$$

$$\bar{B}_k(\xi^0, \ldots, \xi^n) := \prod_{\alpha=1}^{n} \bar{B}_k^\alpha(\xi^0, \ldots, \xi^n) = \prod_{\alpha=1}^{n} (n_k^{\alpha}(\xi^{\alpha}))_{n_k^{\alpha}(\xi^{\alpha-1})}$$

we have from Corollary (4.5) for $q = 1$

$$D_{\text{sym}}(\zeta, \eta) = \sum_{\xi^1} \cdots \sum_{\xi^{n-1}} \prod_{k=L^-}^{L^+} \bar{B}_k(\xi^0, \ldots, \xi^n) \Delta_k(\xi^0, \ldots, \xi^n).$$

with $\zeta = \xi^0$ and $\eta = \xi^n$.

4.2.4 Proof of (185) for $q \neq 1$

- **Step 1:** (Vacancy contribution) We treat the $Q^0$-contribution to the duality function separately, using the factorization (120) of the reversible measure. From (127) and (116) we have $D(\zeta, \eta) = q^{E_0}(\zeta) \bar{D}(\zeta, \eta)$ with

$$\bar{D}(\zeta, \eta) = \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} (Q_k^\alpha(\eta))_{n_k^\alpha(\xi^{\alpha})}.$$ 

We conclude that (185) is equivalent to

$$\langle \zeta | \Upsilon^+ | \eta \rangle = \bar{\pi}(\zeta)\bar{D}(\zeta, \eta)$$

which we set out to prove in the next two steps.

- **Step 2:** (Matrix elements of symmetry generators) Before dealing with (198) we prove a technical lemma involving matrix elements of the generators of $U_q[\mathfrak{gl}(n+1)]$.

**Lemma 4.9** Define for $1 \leq \alpha \leq n$

$$\bar{A}_k(q; \xi^0, \ldots, \xi^n) := \prod_{\alpha=1}^{n} \bar{A}_k^\alpha(q; \xi^{\alpha-1}, \xi^{\alpha})$$

$$\bar{F}(q; \xi^0, \ldots, \xi^n) := \prod_{k=L^-}^{L^+} \bar{A}_k(q; \xi^0, \ldots, \xi^n) \Delta_k(\xi^0, \ldots, \xi^n).$$
Then we have that

\[ F(q; \xi^0, \ldots, \xi^n) = C(q; \xi^0, \xi^n) \prod_{k=L^+}^{L^-} \Delta_k(\xi^0, \ldots, \xi^n). \]  

(201)

with

\[ C(q; \zeta, \eta) = \check{\pi}(\zeta) \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^n_k(\zeta^M(\bar{\mu}_k)(n). \]  

(202)

Proof: We have to take care of the Kronecker-symbols in \( \Delta_k(\cdot) \) to substitute the arguments of the particle number functions \( \bar{A}_k(\cdot) \) in (200). We decompose

\[ \prod_{\beta=0}^{\alpha-2} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) = \prod_{\beta=0}^{\alpha-1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) \prod_{\beta=0}^{\alpha-1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha). \]  

(203)

Thus

\[ \prod_{\alpha=1}^{n} \prod_{\beta=0}^{n-2} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) = \prod_{\alpha=2}^{n-1} \prod_{\beta=0}^{\alpha-1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) = \prod_{\beta=0}^{\alpha=\alpha+1} \prod_{\beta=0}^{\alpha=\alpha+1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) \]  

(204)

which implies

\[ n_k^\beta(\xi^\alpha) = n_k^\beta(\xi^\alpha) \] for \( 0 \leq \beta \leq n-2 \) and \( \beta < \alpha < n \)

(205)

in the \( q \)-dependent prefactor \( \bar{A}_k(q; \xi^0, \ldots, \xi^n) \) of (200). Similarly we note

\[ \prod_{\alpha=1}^{n} \prod_{\beta=0}^{\alpha-1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) = \prod_{\alpha=1}^{n} \prod_{\beta=0}^{\alpha-1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) = \prod_{\beta=0}^{\alpha=\alpha+1} \prod_{\beta=0}^{\alpha=\alpha+1} \delta_n^\beta(\xi^{\alpha-1}, n_k^\beta(\xi^\alpha) ) \]  

(206)

which implies

\[ n_k^\beta(\xi^\alpha) = n_k^\beta(\xi^0) \] for \( 2 \leq \beta \leq n \) and \( 0 < \alpha < \beta \).

(207)

Therefore, defining \( m^{n+1}(\cdot) = 0 \), and using the shorthand \( \bar{F} \equiv F(q; \xi^0, \ldots, \xi^n) \) in the following chain of equations, we have the chains of equations

\[ \bar{F} \equiv \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^n_k(\xi^0) N_k^\alpha(\xi^\alpha) \Delta_k(\xi^0, \ldots, \xi^n) \]  

(208)

\[ \bar{F} \equiv \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^n_k(\xi^0) N_k^\alpha(\xi^\alpha) \Delta_k(\xi^0, \ldots, \xi^n) \]  

(209)

\[ \bar{F} \equiv \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^{-n_k^\alpha}(\xi^0) N_k^\alpha(\xi^\alpha) \Delta_k(\xi^0, \ldots, \xi^n) \]  

(210)

\[ \bar{F} \equiv \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^{\sum_{\beta=1}^{\alpha-1} n_k^\beta(\xi^\alpha)-1} N_k^\alpha(\xi^0) + m^{n+1}(\xi^\alpha) N_k^\alpha(\xi^\alpha) \Delta_k(\xi^0, \ldots, \xi^n) \]  

(211)

\[ \bar{F} \equiv \prod_{k=L^-}^{L^-} \prod_{\alpha=1}^{n} q^{\sum_{\beta=0}^{\alpha-1} n_k^\beta(\xi^\alpha)-1} N_k^\alpha(\xi^0) + m^{n+1}(\xi^\alpha) N_k^\alpha(\xi^\alpha) \Delta_k(\xi^0, \ldots, \xi^n) \]  

(212)
Now we compute the product over the lattice in \( C(q; \zeta, \eta) \). By sum rule (217) and definition (118) one has

\[
\sum_{k=L^-}^{L^+} \sum_{a=1}^{n-1} \sum_{\beta=\alpha+1}^{n} n^\beta_k (\xi^a_0) N^\beta_k (\xi^0) = -\bar{E}(\xi^0)
\]

With \( \xi^0 = \zeta \) and \( \xi^n = \eta \) we thus obtain (202).

\[ \Box \]

- **Step 3:** (Product expansion of symmetry generators) Now we are in a position to prove (198) using the expansion (193) for \( q \neq 1 \). We find from Corollary (4.5)

\[
\langle \xi^\alpha | \Upsilon^+_{\alpha} | \xi^\alpha \rangle = \prod_{k=L^-}^{L^+} A^\alpha_k (q; \xi^{\alpha-1}_{-1}, \xi^\alpha) \Delta^\alpha_k (\xi^{\alpha-1}_{-1}, \xi^\alpha) \tilde{B}^\alpha_k (\xi^{\alpha-1}_{-1}, \xi^\alpha)
\]

and therefore

\[
\prod_{\alpha=1}^{n} \langle \xi^{\alpha-1} | \Upsilon^+_{\alpha} | \xi^\alpha \rangle = \prod_{k=L^-}^{L^+} A_k (q; \xi^0, \ldots, \xi^n) \tilde{B}_k (\xi^0, \ldots, \xi^n) \Delta_k (\xi^0, \ldots, \xi^n)
\]

Observing the projector property \((\Delta_k (\xi^0, \ldots, \xi^n))^2 = \Delta_k (\xi^0, \ldots, \xi^n)\) we can use Lemma (4.9) and write with \( \xi^0 = \zeta \) and \( \xi^n = \eta \)

\[
\langle \xi | \Upsilon^+ | \eta \rangle = \sum_{\xi^1 \ldots \xi^{n-1}} \bar{F}(q; \xi^0, \ldots, \xi^n) \prod_{k=L^-}^{L^+} \tilde{B}_k (\xi^0, \ldots, \xi^n) \Delta_k (\xi^0, \ldots, \xi^n)
\]

\[ \equiv \]

\[
\bar{C}(q; \zeta, \eta) \sum_{\xi^1} \cdots \sum_{\xi^{n-1}} \prod_{k=L^-}^{L^+} \tilde{B}_k (\xi^0, \ldots, \xi^n) \Delta_k (\xi^0, \ldots, \xi^n)
\]

\[ \equiv \]

\[
\bar{\pi}(\zeta) \prod_{k=L^-}^{L^+} \prod_{\alpha=1}^{n} \left( q^{M^\alpha_k (\eta)} \right) \bar{D}_{sym}(\zeta, \eta)
\]

where \( \bar{D}_{sym}(\zeta, \eta) \) is the reduced duality function for \( q = 1 \). This proves (198) and hence concludes the proof of Theorem (3.5).

\[ \Box \]

4.3 Proof of Theorem (3.12)

4.3.1 Preliminary remarks

In [9] we proved for the standard ASEP \( n = 1 \) a statement analogous to Theorem (3.12) for a certain family of shock measures for the process defined on \( \mathbb{Z} \). The proof of [9] consists in three steps: (i) One proves, using the quantum algebra symmetry, that the shock measures
at time \( t \) defined on a finite lattice with \( L^- = \lfloor L/2 \rfloor + 1 \) and \( L^+ = \lceil L/2 \rceil \) satisfy for all configurations \( x \) whose coordinates \( x_i \) exclude the boundary sites \( L^- \) and \( L^+ \) a linear evolution equation which, in the notation of the present work, can be written in the form

\[
\frac{d}{dt} \langle \mu^L_x(t) \rangle = - \sum_y G_{yx} \langle \mu^L_y(t) \rangle + \langle \tilde{B} - b \rangle \langle \mu^L_x(t) \rangle.
\]  

(219)

where \( \mu^L_x \) is a shock measure for the finite system with \( L \) sites, \( G_{yx} \) are the transition rates of an associated shock exclusion process that has the same non-zero rates as the original ASEP but particle-dependent hopping rates and inverse hopping ratio \( q \), boundary term \( \tilde{B} = w(q - q^{-1})(\hat{n}_{L^+} - \hat{n}_{L^-}) \) and constant \( b = \lim_{L \to \infty} E^{\mu^L_x} B \). (ii) Then one uses a convergence argument based on the coupling and the convergence theorems in [60] to show that the contribution of the boundary term to the evolution of the sequence of shock measures \( \mu^L_x(t) \) vanishes, at fixed \( t \), in the thermodynamic limit \( L \to \infty \). (iii) Finally, standard arguments from the theory of linear ordinary differential equations allow for integration of (219) to yield the time evolution of the shock measure in infinite volume according to Theorem 2 of [9] which is analogous to Theorem (3.12).

Steps (ii) and (iii) employ standard tools that do not rely on the specific form of the renormalized hopping rates and which are independent of \( n \). Hence they can be adapted straightforwardly to the present case and are therefore not repeated here. It remains only to prove (i), which, generally stated, follows from the following ingredients: (a) A similarity transformation \( U^n \) that relates the generator to itself plus some boundary term \( B^n \) (Proposition (4.12)), (b) an expression for suitably defined shock measures \( \mu^L_x \) in terms of a duality function \( D^* \) and the similarity transformation \( U \) (Proposition (4.13)), (c) a proof that \( b = \lim_{L \to \infty} E^{\mu^L_x} B \) does not depend on \( x \) and that the matrix \( G \) has positive transition rates and conserves probability (for those initial configurations \( x \) whose coordinates \( x_i \) exclude the boundary sites \( L^- \) and \( L^+ \) (Proposition (4.14)), and (d) a relation analogous to (219) via duality (Proposition (4.15)).

Items (b) - (d) are all generic in the sense that they can be adapted quite straightforwardly to other dualities for interacting particle systems. In fact, the proposition that yields item (b) provides a “recipe” for the construction of shock measures since \( D^* \) and \( U^n \) defined below are not unique. Item (c) includes the explicit computation of the matrix elements \( G_{yx} \) which are the transition rates for the shock exclusion process (3.3).

4.3.2 Auxiliary results

We first prove some transformation properties.

**Lemma 4.10** Let \( \sigma^\alpha_k \sigma^\beta_{k+1} \) and \( \hat{n}^\alpha_k \) be the matrices defined in (2.10) and (2.17) and let \( \hat{\pi} \) be the matrix form of the reversible measure (108). Under the transformations

\[
\hat{V}^\alpha := \prod_{k=L^-}^{L^+} q^N_k, \quad \Gamma := \prod_{k=L^-}^{L^+} \gamma_k
\]  

(220)

with the diagonal particle balance operators derived from (28) and the cyclic flip matrices \( \gamma \) defined in (78) one has

\[
\hat{V}^\gamma \sigma^\beta_k \sigma^\alpha_{k+1} (\hat{V}^\gamma)^{-1} = \begin{cases} 
\sigma^\beta_k \sigma^\alpha_{k+1} & \gamma \neq \alpha, \beta \\
q^2 \sigma^\beta_k \sigma^\alpha_{k+1} & \gamma = \beta \\
q^{-2} \sigma^\beta_k \sigma^\alpha_{k+1} & \gamma = \alpha
\end{cases}
\]  

(221)
and
\[ \Gamma^{-1}\sigma_k^{\alpha\beta} \Gamma = \sigma_k^{\alpha+1,\beta+1}, \quad \Gamma^{-1}\hat{n}_k^{\alpha} \Gamma = \hat{n}_k^\alpha + 1, \quad \Gamma^{-1}\hat{\pi} \Gamma = (\hat{V}^0)^{-2}\hat{\pi}, \]
(222)
with the species indices \( \alpha, \beta \) understood mod \((n+1)\).

**Proof:** From the factorization (57) of the tensor product one has
\[ \hat{V}^\gamma \sigma_k^{\alpha\beta} (\hat{V}^\gamma)^{-1} = \begin{cases} \sigma_k^{\alpha\beta} & \gamma \neq \alpha, \beta \\ q^{\gamma(L^+ + L^- - 2k)}\sigma_k^{\alpha\beta} & \gamma = \alpha \\ q^{-(L^+ + L^- - 2k)}\sigma_k^{\alpha\beta} & \gamma = \beta \end{cases} \]
(223)
This yields (221). From the definition (53) one finds
\[ \gamma^{-1}\sigma^\alpha \gamma = \left[ |0\rangle(n) + \sum_{\nu=1}^n |\nu\rangle(\nu - 1) \right] |\alpha\rangle(\beta|\gamma \]
\[ = |\alpha + 1\rangle(\beta) \left[ n|0\rangle + \sum_{\nu=1}^n |\nu - 1\rangle(\nu) \right] = |\alpha + 1\rangle(\beta + 1). \]
(224)
Similarly, the definition (54) yields \( \hat{n}^\alpha \gamma = \gamma \hat{n}^{\alpha+1} \) from which the first two equalities in (222) follow.

In order to prove the third equality we define
\[ \hat{E}_{kl} := -n^{-1} \sum_{\alpha=1}^{n-1} \sum_{\beta=0}^{\alpha-1} \left( \hat{n}_k^\alpha \hat{n}_l^\beta - \hat{n}_l^\alpha \hat{n}_k^\beta \right). \]
(225)
From Lemma (222) we obtain
\[ \Gamma^{-1}\hat{E}_{kl} \Gamma = \hat{E}_{k,l} + \sum_{\alpha=0}^n \hat{n}_k^\alpha \hat{n}_l^0 - \sum_{\alpha=0}^n \hat{n}_l^\alpha \hat{n}_k^0 + (k \leftrightarrow l) = \hat{E}_{k,l} - 2(\hat{n}_k^0 - \hat{n}_l^0). \]
(226)
This, using (118) and the resummation formula (280), yields the transformed energy (107)
\[ \Gamma^{-1}\hat{E} \Gamma = \hat{E} - 2\hat{E}^0. \] For the transformed measure this implies \( \Gamma^{-1}\hat{\pi} \Gamma = q^{-\hat{E} + 2\hat{E}^0} \) and with the factorization (120) the claim follows. \( \square \)

The following Lemma for finite state space establishes a relation between duality functions and measures that is useful when the function \( B \) appearing in the Lemma becomes irrelevant (in some sense) in the limit of infinite state space. Below we shall use it to construct the shock exclusion process.

**Lemma 4.11** Let \( H \) be the generator of a process \( \omega_t \) defined on a finite state space \( \Omega \) which is self-dual w.r.t. a duality function \( D(\xi, \omega) \) and which satisfies the intertwining relation \( UD^T H = (H + B)UD^T \) for a pair of matrices \( U, B \) such that the duality matrix \( D \) and the transformation matrix \( U \) satisfy \( \Phi(\omega) := \langle s | UD^T | \omega \rangle > 0 \) for all \( \omega \in \Omega \). Define the family of measures \( \mu_\omega^D(\xi) := (\Phi(\omega))^{-1} \langle \xi | UD^T | \omega \rangle \) indexed by \( \omega \) and the functions
\[ b(\omega) := \langle s | \hat{\Phi} H \hat{\Phi}^{-1} | \omega \rangle, \quad B(\omega) := \sum_{\xi \in \Omega} B_{\xi, \omega}. \]
(227)
One has
\[ UHT = (H + B)U, \quad E^\mu \omega B = b(\omega) \]
(228)
for all \( \omega \in \Omega \).
Proof: The first equality follows directly from self-duality \( H^T D^T = D^T H \) and the intertwining relation. A duality matrix can always be written in the form \( D = \sum_{\omega \in \Omega} |\omega\rangle \langle s| \hat{D}_\omega \) with diagonal matrix \( \hat{D}_\omega \) that has diagonal elements \( (\hat{D}_\omega)_{\xi\xi} = D(\xi, \omega) \). Also, \( (\Phi(\omega))^{-1} > 0 \) and \( \sum_{\xi \in \Omega} \mu^\omega \xi (\xi) = 1 \) so that the measures \( \mu^\omega \xi \) are well-defined for all \( \omega \in \Omega \). Observe that \( \langle s| B = \langle s| \hat{B} \) for the diagonal matrix representation \( \hat{B} \) of the function \( B(\omega) \). The representation \( \hat{D}_\omega \) of the unit matrix and conservation of probability \( \langle s| H = 0 \) then leads to the chain of equations

\[
\begin{align*}
    b(\omega) & = \Phi^{-1}(\omega) \langle s| \Phi H| \omega \rangle \\
    & = \Phi^{-1}(\omega) \sum_{\xi} \Phi(\xi) \langle \xi| H| \omega \rangle \\
    & = \Phi^{-1}(\omega) \sum_{\xi} \langle s| UD^T| \xi \rangle \langle \xi| H| \omega \rangle \\
    & = \Phi^{-1}(\omega) \langle s| UD^T H| \omega \rangle \\
    & = \Phi^{-1}(\omega) \langle s| (H + B) UD^T| \omega \rangle \\
    & = \Phi^{-1}(\omega) \langle s| \hat{B} UD^T| \omega \rangle \\
    & = \langle s| \hat{B}| \mu^\omega \rangle = \mathbf{E} \mu^\omega B
\end{align*}
\]

which is the assertion of the Lemma. \( \square \)

4.3.3 Main part of the proof

Items (a) - (d) form the following main part of the proof.

Proposition 4.12 (Item (a)) For \( \gamma \in S_{0,n} \) define the diagonal boundary matrix

\[
\hat{B}^\gamma := w (q - q^{-1}) (\hat{n}_L^+ - \hat{n}_L^-).
\]

and let \( \hat{V}^n \) and \( \Gamma \) be the transformation matrices \( \hat{V}^n \). Then under the composite transformation

\[
U^n = \hat{\pi} \hat{V}^n \Gamma
\]

the generator \( H \) on the \( n \)-species priority ASEP satisfies

\[
H^T = (U^n)^{-1} \left( H + \hat{B}^n \right) U^n.
\]

Proof: The matrices \( \hat{\pi}, \hat{V}^n \) and \( \hat{B}^n \) are all diagonal. Therefore \( (U^n)^{-1} \hat{B}^n U^n = \Gamma^{-1} \hat{B}^n \Gamma \). Then time reversal via the diagonal matrix form \( \hat{\pi} \) of the reversible measure \( \hat{V}^n \) and transposition reduces \( \hat{B}^n \) to

\[
H = \Gamma^{-1} \left( \hat{V}^n H (\hat{V}^n)^{-1} + \hat{B}^n \right) \Gamma.
\]

which we now prove with the help of the the decompositions \( H = H^f + H^{(n-1)} = H^0 + \hat{H} \) with the corresponding bond hopping matrices

\[
h^f_{k,k+1} = -w \sum_{\beta=0}^{n-1} q \left( \sigma_k^\beta \sigma_{k+1}^{n\beta} - \hat{n}_k^\beta \hat{n}_{k+1}^{n\beta} \right) + q^{-1} \left( \sigma_k^{n\beta} \sigma_{k+1}^\beta - \hat{n}_k^{n\beta} \hat{n}_{k+1}^{\beta} \right)
\]

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for the first-class particles (species \( n \)) and
\[
h^{(n-1)}_{k,k+1} = -w \sum_{\alpha=1}^{n-1} \sum_{\beta=0}^{\alpha-1} \left[ q(\sigma_k^{\alpha\beta} \sigma_k^{\alpha\beta} - \hat{n}_k^{\alpha\beta} \hat{n}_k^{\alpha\beta}) + q^{-1}(\sigma_k^{\alpha\beta} \sigma_k^{\alpha\beta} - \hat{n}_k^{\alpha\beta} \hat{n}_k^{\alpha\beta}) \right]
\]  \hspace{1cm} (235)
for the species of lower class and similarly
\[
h^0_{k,k+1} = -w \sum_{\alpha=1}^{n} \left[ q(\sigma_k^{0\alpha} \sigma_k^{0\alpha} - \hat{n}_k^{\alpha} \hat{n}_k^{\alpha}) + q^{-1}(\sigma_k^{0\alpha} \sigma_k^{0\alpha} - \hat{n}_k^{\alpha} \hat{n}_k^{\alpha}) \right]
\]  \hspace{1cm} (236)
for the vacancies (species 0) and
\[
\bar{h}_{k,k+1} = -w \sum_{\alpha=2}^{n} \sum_{\beta=1}^{\alpha-1} \left[ q(\sigma_k^{\alpha\beta} \sigma_k^{\alpha\beta} - \hat{n}_k^{\alpha\beta} \hat{n}_k^{\alpha\beta}) + q^{-1}(\sigma_k^{\alpha\beta} \sigma_k^{\alpha\beta} - \hat{n}_k^{\alpha\beta} \hat{n}_k^{\alpha\beta}) \right].
\]  \hspace{1cm} (237)

From Lemma 4.10 one has \( \hat{V}^n h^{(n-1)}_{k,k+1}(q)(\hat{V}^n)^{-1} = h^{(n-1)}_{k,k+1}(q) \) and \( \Gamma^{-1} h^{(n-1)}_{k,k+1}(q) \Gamma = \bar{h}_{k,k+1}(q) \) and therefore \( \Gamma^{-1} \hat{V}^n H^{(n-1)}(\hat{V}^n)^{-1} \Gamma = \bar{H} \).

Now we compute the transformation of the first-class part. In the following computations we write the \( q \)-dependence of the generator explicitly and define the transformed generator \( \bar{H}^I(q) := \hat{V}^n H^I(q)(\hat{V}^n)^{-1} \). Using again Lemma 4.10 yields
\[
\bar{h}^f_{k,k+1}(q) = -w \sum_{\beta=0}^{n-1} \left[ q(\sigma_k^{n\beta} \sigma_k^{n\beta} - \hat{n}_k^{n\beta} \hat{n}_k^{n\beta}) + q^{-1}(\sigma_k^{n\beta} \sigma_k^{n\beta} - \hat{n}_k^{n\beta} \hat{n}_k^{n\beta}) \right]
\]  \hspace{1cm} (238)
which is not stochastic. Observe, however, that we can write \( \bar{h}^f_{k,k+1}(q) = h^f_{k,k+1}(q^{-1}) - w(q - q^{-1})(\hat{n}_k^{n\beta} - \hat{n}_k^{n\beta}) \). Using the telescopic property of the sum this yields \( \bar{H}^I(q) = H^I(q^{-1}) - \bar{B}^n \). Next we apply the transformation \( \Gamma \). Lemma 4.10 yields \( \Gamma^{-1} H^I(q^{-1}) \Gamma = H^0(q) \) and therefore \( \Gamma^{-1} \hat{V}^n H^I(\hat{V}^n)^{-1} \Gamma = H^0 - \Gamma^{-1} \bar{B}^n \Gamma \) which proves \( (233) \).

Next we express the shock measure in terms of the duality function and the similarity transformation \( U^n \).

**Proposition 4.13** (Item (b)) Let \( D^*(x, \eta) = \prod_{\alpha=2}^{n} \delta_{N^n(\eta), N^n(x)} q^{\lambda_{N^n}(\eta)} D(x, \eta) \) for \( x \in V^L_N \), \( \eta \in S_{0,n}^L \) be the duality function with \( D(x, \eta) \) given by \( (131) \). Then with the transformation \( (231) \) and the normalization constant
\[
\Phi(x) := \langle s | U^n(D^*)^T | x \rangle
\]  \hspace{1cm} (239)
the shockmeasure \( (151) \) can be written as
\[
| \mu^L_x \rangle = \Phi(x)^{-1} U^n(D^*)^T | x \rangle.
\]  \hspace{1cm} (240)
for any \( L \).

**Proof:** It is convenient to express the transformation \( (231) \) in the alternative form
\[
U^n = \Gamma \pi(\hat{V}^0)^{-1} = \Gamma \overline{\pi}
\]  \hspace{1cm} (241)
where the first equality follows from Lemma 4.10 and the second from the definition (118) and the factorization (120). The duality matrix reads \( D^* = \sum \langle x | s \rangle \tilde{Q}_x \hat{\omega}^2 y^{\Lambda^0} \) which gives \( (D^*)^T | x \rangle = Q_x \hat{\omega}^2 y^{\Lambda^0} | s \rangle \).

Now we define the subsets \( \omega_\alpha^\beta := \bigcup_{\beta \in \{\alpha, \ldots, n\}} \{\bar{x}^\beta\} \subset \Lambda^L \) of all particle coordinates of species \( \beta \geq \alpha \) and their complements \( \bar{\omega}_\alpha^\beta := \Lambda^L \setminus \omega_\alpha^\beta \). We also define the (unnormalized) product measures

\[
|s_{x,\alpha,n}^{0,1}\rangle := \prod_{k \in \bar{\omega}_\alpha^\beta} (\tilde{n}_k^0 + \tilde{n}_k^1) \prod_{i=1}^n \tilde{n}_x^\beta_i |s\rangle
\]

with the conventions \(|s_{0,1}^{0,1}\rangle := \prod_{k \in \Lambda} (\tilde{n}_k^0 + \tilde{n}_k^1) |s\rangle\), \(|s_{x,\alpha,n}^{0,1}\rangle := |s_{x,\alpha,n}^{0,1}\rangle\) and note that

\[
|s_{x,\alpha,n}^{0,1}\rangle = \prod_{\beta = \alpha}^n \prod_{i=1}^n \sigma_{\alpha}^{\beta i} |s_{x,\alpha,n}^{0,1}\rangle
\]

with the raising operators \( \sigma_{\alpha}^{\beta i} \) defined in (533). Dividing by a normalization factor \((1/2)^{|\bar{\omega}_\alpha^\beta|}\) these are product measures with marginals \( \delta_{\bar{n}_k^\alpha \bar{n}_k^\beta} \) for the sites \( k \in \omega_\alpha^\beta \subset \Lambda^L \) occupied by particles of species \( \beta \geq \alpha \) and marginals \( (\delta_{\bar{n}_k^0,0} + \delta_{\bar{n}_k^1,0})/2 \) for the remaining sites \( \omega_\alpha^\beta \subset \Lambda^L \).

The first observation is that

\[
\prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} |s\rangle |s_{x,2,n}^{0,1}\rangle = \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s\rangle
\]

which is a consequence of the projector property of \( \tilde{m}_{x_i}^{\alpha} \) and the projection on \( N^\alpha(x) \) particles for \( 2 \leq \alpha \leq n \). Therefore with \( (D^*)^T | x \rangle = Q_x \hat{\omega}^2 y^{\Lambda^0} | s \rangle \) we have

\[
(D^*)^T | x \rangle = q^{\lambda N_0} \prod_{i=1}^{N^1(x)} q^{-1 \sum_{l=L}^{i-1} (n_l^0 + 1 - m_l^0)} m_{x_i}^{L^0} \times \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s_{x,2,n}^{0,1}\rangle
\]

\[
= q^{\lambda N_0} \prod_{i=1}^{N^1(x)} q^{-1 \sum_{l=L}^{i-1} (n_l^0 + 1 - m_l^0)} m_{x_i}^{L^0} \times \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s_{x,2,n}^{0,1}\rangle
\]

\[
= q^{\lambda N_0} \prod_{i=1}^{N^1(x)} q^{-1 \sum_{l=L}^{i-1} (n_l^0 + 1 - m_l^0)} m_{x_i}^{L^0} \times \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s_{x,2,n}^{0,1}\rangle
\]

\[
= q^{\lambda N_0} \prod_{i=1}^{N^1(x)} q^{-1 \sum_{l=L}^{i-1} (n_l^0 + 1 - m_l^0)} m_{x_i}^{L^0} \times \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s_{x,2,n}^{0,1}\rangle
\]

\[
= q^{\lambda N_0} \prod_{i=1}^{N^1(x)} q^{-1 \sum_{l=L}^{i-1} (n_l^0 + 1 - m_l^0)} m_{x_i}^{L^0} \times \prod_{\alpha = 2}^{n} \prod_{i=1}^{n} \tilde{m}_{x_i}^{\alpha} \hat{\omega}^2 y^{\Lambda^0} |s_{x,2,n}^{0,1}\rangle
\]
\[
\times \prod_{k \in \omega^1_X} q^{-N^1_k} \prod_{\alpha=2}^n q^{-c^{\alpha} N^{\alpha}(x) N^1} | s^0_{x}^{1} \rangle.
\] (245)

The substitution of \( \hat{m}^1 \) by \( \hat{n}^1 \) in the second equality comes from the fact that \( \{ \bar{x}^1 \} \) is in the complement \( \bar{\omega}^2_X \) where one has \( \hat{N}^0_{\bar{x}^1} | s^0_{x} \rangle = 0 \) for \( 2 \leq \alpha \leq n \). In the last equality we have used that by construction \( \prod_{i=1}^n N^1(x) \hat{n}^1_{x_i} | s^0_{x,x,n} \rangle = | s^0_{x} \rangle \).

The next step is to compute \( U^n(D^*)^T | x \rangle \) using (241). To this end we make a decomposition of \( \bar{E}() \) (113) as follows. From (113) we construct the doubly reduced energy

\[
\bar{E}(\cdot) := \sum_{\alpha=3}^n \sum_{\beta=2}^{\alpha-1} E^{\alpha\beta}(\cdot).
\] (246)

The representation (114) of the partial energies allows us to write

\[
\bar{E}(\cdot) = \bar{E}(\cdot) + \sum_{\alpha=2}^n E^{1\alpha}(\cdot) = \bar{E}(\cdot) - \sum_{k=L^-}^{L^+} m^2_k(\cdot) \bar{N}^1_k(\cdot)
\] (247)

with the corresponding decomposition of the matrix form of the reduced measure (121)

\[
\hat{\pi} = \frac{\hat{\pi}}{q} \pi^{-L^+_{k=L^-}} \bar{m}^2 \bar{N}^1_k.
\] (248)

Since \( \frac{\hat{\pi}}{q} | s^0_{x,x,n} \rangle = \bar{x}(x) \pi s^0_{x} \rangle \) and likewise \( \frac{\hat{\pi}}{q} | s^0_{x} \rangle = \bar{\pi}(x) \pi s^0_{x} \rangle \) we find

\[
\hat{\mathcal{P}}_x \pi | s^0_{x} \rangle = \bar{\pi}(x) \prod_{k \in \omega^1_X} q^{-N^1_k} | s^0_{x} \rangle
\] (249)

which comes from the fact that \( m^2_k(x) = \sum_{x \in \omega^2_X} \delta_{k,x} \). Therefore we arrive at the intermediate result

\[
\hat{\pi}(D^*)^T | x \rangle = \bar{\pi}(x) q^{N^0} \prod_{\alpha=1}^n \prod_{i=1}^{N^\alpha(x)} q^{-(1+c^{\alpha}) \sum_{i=L^-}^{L^+} \hat{n}^0_{x_i} + (1-c^{\alpha}) \sum_{i=x_{l+1}} L^+} \hat{n}^0_{x_i}
\]

\[
\times \prod_{\alpha=2}^n q^{-c^{\alpha} N^{\alpha}(x) N^1} | s^0_{x} \rangle.
\] (250)

Now we consider the transformation \( \Gamma \). With the definition \( | s^{n,0} \rangle := \prod_{k=L^-}^{L^+} (\hat{n}^0_k + \hat{n}^1_k) s \) we obtain from (213)

\[
| s^{n,0} \rangle := \Gamma | s^0_{x} \rangle = \prod_{i=1}^{N^1(x)} \hat{n}^0_{x_i} \prod_{\alpha=2}^n \prod_{i=1}^{N^\alpha(x)} \sigma^{\alpha-1,+}_{x_i} | s^{n,0} \rangle
\] (251)

with the representation matrices \( \sigma^{\alpha,+} \) (52). Notice that this is a (unnormalized) product measure with particles of type \( \alpha - 1 \) at the positions \( x_{l+1}^\alpha \) with probability 1, corresponding to shock markers of type \( \alpha \) at these positions. Putting these results together yields

\[
U^n(D^*)^T | x \rangle = \bar{\pi}(x) \Gamma q^{N^0} \prod_{\alpha=1}^n \prod_{i=1}^{N^\alpha(x)} q^{-(1+c^{\alpha}) \sum_{i=L^-}^{L^+} \hat{n}^0_{x_i} + (1-c^{\alpha}) \sum_{i=x_{l+1}} L^+} \hat{n}^0_{x_i} \Gamma^{-1}
\]

42
\begin{align*}
\times \Gamma \prod_{\alpha=2}^{n} q^{-c_{\alpha} N^\alpha(x)} & \Gamma^{-1} | s_{x-1}^{n,0} \rangle \\
= \frac{\pi(x)}{q^L N} \prod_{\alpha=1}^{N} q^{-1+ c_{\alpha}} \left( q^{L} \right)^{\sum_{i=x_{i}^{\alpha}+1}^{L} n_{i}^{\alpha}} \prod_{i=1}^{L} q^{(-1+c_{\alpha}) \sum_{i=L}^{L} n_{i}^{\alpha} + (1-c_{\alpha}) \sum_{i=x_{i}^{\alpha}+1}^{L} n_{i}^{\alpha}} \\
& \times \prod_{\alpha=2}^{n} q^{-c_{\alpha} N^\alpha(x)} | s_{x-1}^{n,0} \rangle. 
\end{align*}

(252)

Finally we set \( c_{\alpha} = 0 \) for all \( \alpha \) which leads to

\[ U^n(D^*)^T | x \rangle = \frac{\pi(x)}{q^L N} \prod_{k=1}^{N(x)} q^{-\hat{N}^n_k} | s_{x-1}^{n,0} \rangle. \]

(253)

and thus proves that \( U^n(D^*)^T | x \rangle / \Phi(x) \) with \( \Phi(x) = \langle s | U^n(D^*)^T | x \rangle \) is a product measure with shock markers of type \( \alpha \) at positions \( x_{i}^{\alpha} \) (which are particles of species \( \alpha - 1 \)).

In order to identify this measure with the shock measure (151) we compute the normalization \( \Phi(x) \). Fixing \( K = N(x) \) we have

\[ \prod_{k=1}^{N(x)} q^{-\hat{N}^n_k} = \prod_{k=L^{-}}^{L^+} q^{-\hat{N}^n_k} = \prod_{k=x_{i}^{\alpha}+1}^{x_{j+1}^{\alpha}} q^{-(K-1)+1} \hat{N}^n_k \prod_{k=x_{i}^{\alpha}+1}^{x_{j+1}^{\alpha}} q^{(2i-K)} \hat{N}^n_k \prod_{k=x_{i}^{\alpha}+1}^{x_{j+1}^{\alpha}} q^{2i-K} \hat{N}^n_k \]

(254)

with the conventions \( x_{0} = L^{-} - 1 \) and \( x_{K+1} = L^+ + 1 \).

Using the product structure of \( | s_{x-1}^{n,0} \rangle \) one thus gets the normalization

\[ \Phi(x) = \bar{\pi}(x) \prod_{i=0}^{K} (1 + q^{2i-K+\lambda})^{x_{i+1}^{\alpha} - x_{i}^{\alpha} - 1}. \]

(255)

and the marginal densities

\[ \rho_{i} = \frac{q^{2i-K+\lambda}}{1 + q^{2i-K+\lambda}} \]

(256)

for the sites \( x_{i} < k < x_{i+1} \) between the shock markers \( i \) and \( i + 1 \) which are those defined in (149).

The third ingredient (c) establishes the link between the matrix elements \( \hat{\Phi} H \hat{\Phi}^{-1} \) and the shock exclusion process.

**Proposition 4.14** *(Item (c))* Let \( \hat{V}_{N}^{L} := \{ x \in V_{N}^{L} \mid L^{-} < x_{i}^{\alpha} < L^+ \ \forall \ x_{i}^{\alpha} \in \{ x \} \} \) be the set of configurations with all particle positions restricted to the segment \([L^{-} + 1, L^+ - 1]\) of \( \Lambda_{L} \) and define for \( x \in V_{N}^{L} \) the function \( b(x) := E_{x}^{L} B_{x} \) for the shock measures \( \mu_{x}^{L} \) (151) and boundary matrix (230). The following holds:
(a) \( b(x) = b \) for all \( x \in \tilde{V}^L_N \) with \( b = w(q - q^{-1})(\rho^+ - \rho^-) \) and \( \rho^\pm = E_{\mu^k} n_{L^\pm} \).

(b) Define the matrix

\[ G := \Phi H \Phi^{-1} - b \]  

with the shock normalization (256). Then for the negative off-diagonal matrix elements \( w_{yx} := -G_{yx} \) one has positivity \( w_{yx} \geq 0 \) for all \( x, y \in V^L_N \), \( y \neq x \) and conservation of probability \( \sum_{y \in V^L_N} G_{yx} = 0 \) for \( x \in V^L_N \).

(c) The negative off-diagonal matrix elements \( w_{yx} \) are the shock transition rates (140).

Proof: Part (a) is trivial for the product measure (151) since for any \( x \) with coordinates \( x_i^\alpha \in [L^- + 1, L^+ - 1] \) its boundary marginals and hence the expectation \( E_{\mu^k} B^\gamma \) does not depend on \( x \) for any \( \gamma \in S_{0,n} \). The value of \( b \) follows directly from the definition of the boundary matrix \( B^n \) (250).

In order to prove part (b) we first note that positivity is trivial since by construction \( \Phi(x) > 0 \) and \( H_{yx} \leq 0 \) for all non-equal pairs \( x, y \in V^L_N \). Applying Lemma (4.11) to the present setting then yields conservation of probability for configurations \( x \in \tilde{V}^L_N \).

The last item (c) to be proved is the identification of \( -G_{yx} = -\Phi(y) H_{yx} (\Phi(x))^{-1} \) with the shock transition rates (140). We recall that due to the transformation \( \Gamma \) in the definition of the shock measure a configuration \( x = (\vec{x}, \vec{\alpha}) \) of the \( n \)-species priority ASEP corresponds to a configuration of shock markers of type \( \alpha_i - 1 \) at the sites \( x_i \). Observing that \( G_{yx} = 0 \iff H_{yx} = 0 \) we need to consider only configurations \( y \) that differ from \( x \) either by a single unit displacement (i.e., \( y_i^\alpha = x_i^\alpha \pm 1 \) for some specific \( x_i^\alpha \) and \( y_j^\beta = x_j^\beta \) for all other coordinates) or by an interchange of color (when \( y_k^\beta = x_i^\alpha = x + 1 \)).

We define

\[ \Psi(x) := \prod_{j=0}^{K} \left( 1 + q^{2j-K+\lambda} x_{j+1} x_{j-1} \right) \]  

(258)

which allows us to split the normalization (255) into two parts \( \Phi(x) = \hat{\pi}(x) \Psi(x) \). Thus the matrix element \( G_{yx} \) becomes a product of three terms

\[ G_{yx} = \frac{\hat{\pi}(y)}{\pi(x)} \times \frac{\Psi(y)}{\Psi(x)} \times H_{yx}. \]  

(259)

First we consider the case of a jump \( x_{j+1}^x - x_j^x > 1 \). Then \( H_{yx} = q^{\pm 1} \) and \( \hat{\pi}(y) / \hat{\pi}(x) = 1 \). From (255) and (256) one finds

\[ \frac{\Psi(y)}{\Psi(x)} = \left( 1 + q^{2(j-1)-K+\lambda} \right)^{\pm 1}, \]  

(260)

From (256) one also finds

\[ \frac{\rho_j}{1 - \rho_j} = q^{2j-K+\lambda}. \]  

(261)

Following [9] we note that (143) then yields

\[ (q - q^{-1}) \rho_j (1 - \rho_j) = q (\rho_j - \rho_{j-1}) \frac{1 - \rho_j}{1 - \rho_{j-1}} \]  

(262)
\((q - q^{-1})\rho_{j-1}(1 - \rho_{j-1}) = q^{-1}(\rho_j - \rho_{j-1}) \frac{1 - \rho_{j-1}}{1 - \rho_j}. \) (263)

Therefore \(G_{yx} = -wq^{\pm 1} [(1 - \rho_j)/(1 - \rho_{j-1})]^{\pm 1}\) which gives the shock hopping rates

\[ w_j^+ = wq \left( \frac{1 - \rho_j}{1 - \rho_{j-1}} \right) = \frac{w(q - q^{-1})\rho_j(1 - \rho_j)}{\rho_j - \rho_{j-1}} \] (264)

\[ w_j^- = wq^{-1} \frac{1 - \rho_{j-1}}{1 - \rho_j} = \frac{w(q - q^{-1})\rho_{j-1}(1 - \rho_{j-1})}{\rho_j - \rho_{j-1}} \] (265)

in agreement with the definition (140) of the shock exclusion process.

Next we consider the case of color exchange. In this case \(\Phi(y)/\Phi(x) = 1\) and proof reduces to calculating \(G_{yx} = H_{yx}\tilde{\pi}(y)/\tilde{\pi}(x)\). For \(2 \leq \alpha \leq n\) reversibility yields \(G_{yx} = H_{xy}\). For \(\alpha = 1\) one has \(\tilde{\pi}(y)/\tilde{\pi}(x) = 1\) and therefore \(G_{yx} = H_{yx}\). Both cases are in accordance with the colour exchange rates (141) of the shock exclusion process. □

The final building block in the proof of Theorem (3.12) reads:

**Proposition 4.15** (Item (d)) Let \(\mu_L^x\) be the shock measure defined in (151) and let \(H^B\) be the evolution operator

\[ H^B := H + \tilde{B}^n - b. \] (266)

Then

\[ H^B|\mu_L^x\rangle = \sum_y G_{yx}|\mu_L^y\rangle \] (267)

with the transition matrix elements \(G_{yx} = \langle y | G | x \rangle\).

**Proof:** From selfduality (180) with the duality matrix of Theorem (3.5), Proposition (4.12), and Proposition (4.13) we have the intertwining relation \((H + \tilde{B}^n - b)U^nD^T = U^nD^T(H - b)\). Thus with the representation (41) of the unit matrix and (240) of the shock measure

\[ (H + \tilde{B}^n - b)|\mu_L^x\rangle = \langle \Phi(x) \rangle^{-1}U^nD^T(H - b)|x\rangle \] (268)

\[ = \sum_{y} \frac{\Phi(y)}{\Phi(x)} |\mu_y\rangle \langle y |(H - b)|x\rangle \] (269)

\[ = \sum_{y} |\mu_y\rangle \langle y |(\tilde{\Phi}H\tilde{\Phi}^{-1} - b)|x\rangle. \] (270)

With the definition (257) of \(G\) this proves Proposition (4.15). □

As detailed in the preliminary remarks this completes the proof of Theorem (3.12).

**Acknowledgements**

GMS thanks the Institute of Mathematics and Statistics at the University of São Paulo for kind hospitality and L.R.G. Fontes for stimulating comments. This work was supported by FAPESP (2015/15258-9), CNPq (307347/2013-3) and DFG (SCHU 827/9-1).
A Some conventions and useful formulas

The Kronecker-$\delta$ is defined by

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

for $x, y$ from any set. For $x \in \mathbb{R}$ we define

$$\Theta(x) := \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}, \quad \sigma(x) := \Theta(x) - \Theta(-x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

Then one has for $k, l, m, n \in \mathbb{Z}$

$$\sum_{m=k}^{l-1} \delta_{m,n} = \Theta(l - n) - \Theta(k - n)$$

which we take as the definition of a summation when the upper summation index is smaller than the lower summation index. In particular, one has for any summable object $f_n, n \in \mathbb{Z}$,

$$\sum_{n=k}^{l-1} f_n = \begin{cases} 0 & l = k \\ -\sum_{n=l}^{k-1} f_n & l < k. \end{cases}$$

This implies analogous relations for products of $f_n$ when the upper product index is smaller than the lower product index through the formal identity $\prod_{n=k}^{l-1} = \exp \left( \sum_{n=k}^{l-1} \ln (f_n) \right)$. For $k = l$ we define $\prod_{n=k}^{k-1} f_n := 1$ even if $f_k = 0$, consistent with the convention $0^0 = 1$.

We also note various sum rules that are used in several places in the proofs. For numbers or matrices $a_k, b_l$ one has

$$\sum_{k=L^-}^{L^+} \sum_{l=L^-}^{k-1} (a_k - a_l) = \sum_{k=L^-}^{L^+} (2k - L^+ - L^-)a_k$$

$$\sum_{k=L^-}^{L^+} a_k \left( \sum_{l=L^-}^{k-1} b_l - \sum_{l=k+1}^{L^+} b_l \right) = -\sum_{k=L^-}^{L^+} b_k \left( \sum_{l=L^-}^{k-1} a_l - \sum_{l=k+1}^{L^+} a_l \right)$$

As a simple consequence of these sum rules we have

**Lemma A.1** (Resummation) For numbers or matrices $a_k, b_l$ define

$$A_k := \left( \sum_{l=L^-}^{k-1} a_l - \sum_{l=k+1}^{L^+} a_l \right), \quad B_k := \left( \sum_{l=L^-}^{k-1} b_l - \sum_{l=k+1}^{L^+} b_l \right).$$
Then one has
\[ \sum_{k=L^-}^{L^+} A_k = \sum_{k=L^-}^{L^+} (L^+ + L^- - 2k) a_k = \sum_{k=L^-}^{L^+} k = (L^+ + L^- - 2k) a_k \] (279)
and
\[ \sum_{k=L^-}^{L^+} a_k B_k = -\sum_{k=L^-}^{L^+} b_k A_k. \] (280)

For \( c \in \mathbb{C} \) and \( q, q^{-1} \in \mathbb{C} \setminus 0 \) we define the symmetric \( q \)-number by
\[
[c]_q := \frac{q^c - q^{-c}}{q - q^{-1}}. \tag{281}
\]

For integers \( n \in \mathbb{N} \) the \( q \)-factorial and the \( q \)-binomial coefficient are defined by
\[
[n]_q! := \prod_{k=1}^{n} [k]_q, \quad \binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-q]_q!}. \tag{282}
\]
and \([0]_q! := 1\). For finite-dimensional square matrices \( A \) the expression \([A]_q\) is defined analogously to (281) through the Taylor expansion of the exponential.

For two endomorphisms on some vector space represented by square matrices \( A \) and \( B \) we define the commutator symbol \( [A, B] := AB - BA \) with the matrix product \((AB)_{mn} = \sum_k A_{mk} B_{kn}\). The Kronecker product \( A \otimes B \) is defined for arbitrary rectangular matrices as follows.

**Definition A.2 (Kronecker product)** Let \( A \) and \( B \) be two matrices with \( m_A \) (\( m_B \)) rows and \( n_A \) (\( n_B \)) columns with matrix elements \( A_{ij}, 1 \leq i \leq m_A, 1 \leq j \leq n_A \) and \( B_{ij}, 1 \leq i \leq m_B, 1 \leq j \leq n_B \) respectively. The Kronecker product \( A \otimes B \) is a \( m_A m_B \times n_A n_B \)-matrix \( C \) with matrix elements \( C_{kl} = A_{pq} B_{kl} \) for \((p-1)n_B + 1 \leq l \leq pn_B, (q-1)m_B + 1 \leq k \leq qm_B\) where \( 1 \leq p \leq n_A \) and \( 1 \leq q \leq m_A \).

A matrix is called nilpotent of degree \( k \) if \( A^k = 0 \). Here \( 0 \) represents the matrix with all matrix elements \( A_{mn} \) are equal to \( 0 \). A matrix \( A \) is called a projector if \( A^2 = A \). We call a matrix \( A \) satisfying \( A^3 = A \) a signed projector.

We mention the following simple projector lemma:

**Lemma A.3 (Exponential of projectors)** (a) Let \( P \) be a projector. Then for \( c \in \mathbb{C} \) one has \( c^P = 1 + (c - 1)P \).
(b) Let \( Q \) be a signed projector. Then for \( c \in \mathbb{C} \setminus 0 \) one has \( c^Q = 1 + \frac{1}{2} (c - c^{-1}) Q + \frac{1}{2} (c + c^{-1} - 2) Q^2 \).

This is an immediate consequence of the Taylor expansion of the exponential and the projector property. In particular, we note that \( \Theta^2(n) = \Theta(n) \) and \( \sigma^3(n) = \sigma(n) \) so that Lemma A.3 can be applied to exponentials of these functions.
References

[1] F.C. Alcaraz and V. Rittenberg, Reaction-diffusion processes as physical realizations of Hecke algebras, Phys. Lett. B 314, 377–380 (1993).

[2] F.C. Alcaraz and R.Z. Bariev, Exact solution of asymmetric diffusion with N classes of particles of arbitrary size and hierarchical order, Braz. J. Phys. 30, 655–666 (2000).

[3] G. Amir, I. Corwin, and J. Quastel, Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions, Commun. Pure Appl. Math. 64, 466–537 (2011).

[4] G.E. Andrews, The Theory of Partitions, Encyclopedia of Math. and its Appl. 2, London, Addison Wesley (1976).

[5] C. Arita, A. Kuniba, K. Sakai, and T. Sawabe, Spectrum of a multi-species asymmetric simple exclusion process on a ring, J. Phys. A: Math. Theor. 42, 345002 (2009).

[6] P.F. Arndt, T. Heinzl, and V. Rittenberg, Spontaneous breaking of translational invariance in one-dimensional stationary states on a ring, J. Phys. A: Math. Gen. 31, L45–L51 (1998).

[7] A. Ayer, E. Carlen, J.L. Lebowitz, P. Mohanty, D. Mukamel, and E. Speer, Phase diagram of the ABC model on an interval. J. Stat. Phys. 137, 1166–1204 (2009).

[8] M. Balázs, G. Farkas, P. Kovács, and A. Rákos, Random walk of second class particles in product shock measures, J. Stat. Phys. 139(2), 252–279 (2010).

[9] V. Belitsky and G.M. Schütz, Diffusion and coalescence of shocks in the partially asymmetric exclusion process, Electron. J. Probab. 7, Paper No. 11, 1–21 (2002).

[10] V. Belitsky and G.M. Schütz, Microscopic position and structure of a shock in CA 184, J. Phys. A: Math. Theor. 44, 445003 (2011).

[11] V. Belitsky and G.M. Schütz, Quantum algebra symmetry of the ASEP with second-class particles, J. Stat. Phys. 161(4), 821-842 (2015).

[12] V. Belitsky and G.M. Schütz, Self-Duality for the Two-Component Asymmetric Simple Exclusion Process. J. Math. Phys. 56, 083302 (2015).

[13] I. Benjamini, P.A. Ferrari, and C. Landim, Asymmetric conservative processes with random rates, Stoc. Proc. Appl. 61(2), 181-204 (1996).

[14] L. Bertini and G. Giacomin, Stochastic Burgers and KPZ equations from particle systems, Commun. Math. Phys. 183, 571–607 (1997).

[15] T. Bodineau and B. Derrida, Phase Fluctuations in the ABC Model, J. Stat. Phys. 145, 745–762 (2011).

[16] A. Borodin, I. Corwin, and T. Sasamoto, From Duality to Determinants for Q-TASEP and ASEP, Ann. Probab. 42, 2314–2382 (2014).
[17] L. Cantini, Algebraic Bethe ansatz for the two species ASEP with different hopping rates, J. Phys. A: Math. Theor. 41, 095001 (2008).

[18] G. Carinci, C. Giardina, C. Giberti, and F. Redig, Duality for Stochastic Models of Transport, J. Stat. Phys. 152(4), 657–697 (2013).

[19] G. Carinci, C. Giardina, C. Giberti, and F. Redig, Dualities in population genetics: A fresh look with new dualities, Stoc. Proc. Appl. 125(3), 941–969 (2015).

[20] G. Carinci, C. Giardina, F. Redig, and T. Sasamoto, A generalized Asymmetric Exclusion Process with $U_q(sl_2)$ stochastic duality, Probab. Theory Rel. Fields, DOI 10.1007/s00440-015-0674-0 (2015).

[21] S. Chatterjee and G. M. Schütz, Determinant representation for some transition probabilities in the TASEP with second class particles, J. Stat. Phys. 140, 900–916 (2010).

[22] J. Cividini, H.J. Hilhorst, and C. Appert-Rolland, Exact domain wall theory for deterministic TASEP with parallel update. J. Phys. A: Math. Theor. 47, 222001 (2014).

[23] M. Clincy, and M.R. Evans, Phase transition in the ABC model. Phys. Rev. E 67, 066115 (2003).

[24] I. Corwin and L. Petrov, Stochastic higher spin vertex models on the line. [http://arxiv.org/abs/1502.07374], Cited 13 Mar 2015.

[25] N. Crampe, K. Mallick, E. Ragoucy, and M. Vanicat, Open two-species exclusion processes with integrable boundaries, J. Phys. A: Math. Theor. 48, 175002 (2015).

[26] N. Crampe, C. Finn, E. Ragoucy, and M. Vanicat, Integrable boundary conditions for multi-species ASEP, [arXiv:1606.01018v1 [math-ph]], Cited 10 Jun 2016.

[27] S.R. Dahmen, Reaction Diffusion Processes described by 3-State Quantum Chains, J. Phys. A: Math. Gen. 28, 905–922 (1995).

[28] B. Derrida, S.A. Janowsky, J.L. Lebowitz, and E.R. Speer, Exact solution of the totally asymmetric simple exclusion process: Shock profiles, J. Stat. Phys. 73, 813–842 (1993).

[29] M.R. Evans, Y. Kafri, H.M. Koduvely, and D. Mukamel, Phase separation and coarsening in one-dimensional driven diffusive systems: Local dynamics leading to long-range Hamiltonians, Phys. Rev. E 58 2764–2778 (1998).

[30] M.R. Evans, P.A. Ferrari, and K. Mallick, Matrix representation of the stationary measure for the multispecies TASEP, J. Stat. Phys. 135(2), 217–239 (2009).

[31] P.A. Ferrari, C. Kipnis, and E. Saada, Microscopic Structure of Travelling Waves in the Asymmetric Simple Exclusion Process, Ann. Probab. 19(1), 226–244 (1991).

[32] P.A. Ferrari and L.R.G. Fontes, Shock fluctuations in the asymmetric simple exclusion process, Probab. Theory Relat. Fields 99, 305–319 (1994).
[33] P.A. Ferrari, L.R.G. Fontes, and Y. Kohayakawa, Invariant measures for a two-species asymmetric process. J. Stat. Phys. 76, 1153–1177 (1994)

[34] P.A. Ferrari, L.R.G. Fontes, and M.E. Vares, The asymmetric simple exclusion model with multiple shocks, Ann. Inst. H. Poincaré Probab. Stat. 36(2), 109–126 (2000)

[35] P.A. Ferrari and J.B. Martin, Stationary distributions of multi-type totally asymmetric exclusion processes, Ann. Probab. 35(3), 807–832 (2007).

[36] P.L. Ferrari, T. Sasamoto and H. Spohn, Coupled Kardar-Parisi-Zhang equations in one dimension, J. Stat. Phys. 153, 377–399 (2013).

[37] J. Fritz, An Introduction to the Theory of Hydrodynamic Limits, Lectures in Mathematical Sciences 18. Graduate School of Mathematics, Univ. Tokyo (2001).

[38] J. Fritz and B. Toth, Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas, Commun. Math. Phys. 249, 1-27 (2004).

[39] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of Math. and its Appl. 96 (2nd ed.), Cambridge, Cambridge University Press (2004).

[40] M. Gorissen, A. Lazarescu, K. Mallick, and C. Vanderzande, Exact Current Statistics of the Asymmetric Simple Exclusion Process with Open Boundaries, Phys. Rev. Lett. 109, 170601 (2012).

[41] M. Gaudin, The Bethe wave function (Cambridge University Press, Cambridge, 2014).

[42] C. Giardinà, J. Kurchan, and F. Redig, Duality and exact correlations for a model of heat conduction, J. Math. Phys. 48(3), 033301 (2007).

[43] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi, Duality and Hidden Symmetries in Interacting Particle Systems, J. Stat. Phys. 135, 25–55 (2009).

[44] D. J. Grabiner, Brownian motion in a Weyl chamber, non-colliding particles, and random matrices, Ann. Inst. H. Poincaré Probab. Statist. 35, 177–204 (1999).

[45] D. J. Grabiner, Random Walk in an Alcove of an Affine Weyl Group, and Non-colliding Random Walks on an Interval, J. Combin. Theory, Ser. A 97, 285–306 (2002).

[46] T. Imamura and T. Sasamoto, Current Moments of 1D ASEP by Duality, J. Stat. Phys. 142(5), 919–930 (2011).

[47] S. Jansen and N. Kurt, On the notion(s) of duality for Markov processes, Prob. Surveys 11, 59–120 (2014).

[48] M. Jimbo, A \(q\)-difference analogue of \(U(g)\) and the Yang-Baxter equation, Lett. Math. Phys. 10, 63–69 (1985)

[49] M. Jimbo, A \(q\)-Analogue of \(U(gl(N + 1))\), Hecke Algebra, and the Yang-Baxter Equation, Lett. Math. Phys. 11, 247–252 (1986).

[50] M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic scaling of growing interfaces. Phys. Rev. Lett. 56, 889–892 (1986).
[51] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems* (Springer, Berlin, 1999)

[52] C. Krattenthaler, Asymptotics for random walks in alcoves of affine Weyl groups, Séminaire Lotharingien Combin. 52, B52i (2007).

[53] K. Krebs, F.H. Jafarpour, and G.M. Schütz, Microscopic structure of travelling wave solutions in a class of stochastic interacting particle systems, N. J. Phys. 5, 145.1–145.14 (2003).

[54] J. Kuan, Stochastic duality of ASEP with two particle types via symmetry of quantum groups of rank two, J. Phys. A: Math. Theor. 49, 115002 (2016)

[55] J Kuan, A Multi-species ASEP(q,j) and q-TAZRP with Stochastic Duality, arXiv:1605.00691v1, Cited 10 June 2016

[56] P.D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, CBMS 11, (Philadelphia: SIAM, 1973)

[57] P. D. Lax, Hyperbolic Partial Differential Equations, Courant Lecture Notes in Mathematics, vol. 14, New York, (2006),

[58] A. Lazarescu and K. Mallick, An exact formula for the statistics of the current in the TASEP with open boundaries J. Phys. A: Math. Theor. 44, 315001 (2011).

[59] T.M. Liggett, Coupling the simple exclusion process. Ann. Probab. 4, 339–356 (1976).

[60] T.M. Liggett, *Interacting particle systems* Springer, Berlin, (1985).

[61] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* Springer, Berlin (1999).

[62] P. Lloyd, A. Sudbury, and P. Donnelly, Quantum operators in classical probability theory: I. “Quantum spin” techniques and the exclusion model of diffusion, Stoch. Proc. Appl. 61(2), 205–221 (1996).

[63] K. Mallick, S. Mallick, and N. Rajewsky, Exact solution of an exclusion process with three classes of particles and vacancies. J. Phys. A: Math. Gen. 32, 8399–8410 (1999).

[64] O. Mandelshtam, Matrix ansatz and combinatorics of the $k$-species PASEP. arXiv:1508.04115v1 (2015).

[65] J. Ohkubo, Duality in interacting particle systems and boson representation, J. Stat. Phys. 139, 454–465 (2010).

[66] M. Paessens and G.M. Schütz, Reaction fronts in stochastic exclusion models with three-site interactions, New J. Phys. 6, 120 (2004).

[67] V. Pasquier and H. Saleur, Common structures between finite systems and conformal field theories through quantum groups, Nucl. Phys. B 330, 523–556 (1990).

[68] J.H.H. Perk and C.L. Schultz, New families of commuting transfer matrices in $q$-state vertex models, Phys. Lett. 84A, 407–410 (1981).
[69] C. Pigorsch and G.M. Schütz: Shocks in the asymmetric simple exclusion process in a discrete-time update, J. Phys. A: Math. Gen. 33, 7919–7935 (2000).

[70] V. Popkov, E. Fouladvand and G.M. Schütz: A sufficient integrability criterion for interacting particle systems and quantum spin chains. J. Phys. A: Math. Gen. 35, 7187–7204 (2002).

[71] V. Popkov and G.M. Schütz, Unusual shock wave in two-species driven systems with an umbilic point, Phys. Rev. E 67, 031139 (2012).

[72] S. Prolhac, M.R. Evans, K. Mallick, Matrix product solution of the multispecies partially asymmetric exclusion process, J. Phys. A: Math. Gen. 42, 165004 (2009).

[73] S. Prolhac, Tree structures for the current fluctuations in the exclusion process, J. Phys. A: Math. Theor. 43, 105002 (2010).

[74] A. Rákos, G.M. Schütz, Exact shock measures and steady-state selection in a driven diffusive system with two conserved densities, J. Stat. Phys. 117(1-2), 55–76 (2004).

[75] A. Rákos, G.M. Schütz, Bethe Ansatz and Current Distribution for the TASEP with Particle-Dependent Hopping Rates, Markov Proc. Relat. Fields 12, 323–334 (2006).

[76] F. Rezakhanlou, Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. Commun. Math. Phys. 140, 417–448 (1991).

[77] S. Sandow and G.M. Schütz, On $U_q[SU(2)]$-Symmetric Driven Diffusion, Europhys. Lett. 27, 7–12 (1994).

[78] T. Sasamoto and H. Spohn, The one-dimensional KPZ equation: an exact solution and its universality, Phys. Rev. Lett. 104, 230602 (2010).

[79] G. Schütz and S. Sandow, Non-abelian symmetries of stochastic processes: derivation of correlation functions for random vertex models and disordered interacting many-particle systems. Phys. Rev. E 49, 2726–2744 (1994).

[80] G.M. Schütz, Duality relations for asymmetric exclusion processes, J. Stat. Phys. 86(5/6), 1265–1287 (1997)

[81] G.M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in: Phase Transitions and Critical Phenomena. Vol. 19, C. Domb and J. Lebowitz (eds.), Academic Press, London (2001).

[82] G.M. Schütz, Duality relations for the periodic ASEP conditioned on a low current, In: Gonçalves, P., Soares, A.J. (eds.) From Particle Systems To Partial Differential Equations IV, Springer, Cham (in press)

[83] H. Spohn, Large Scale Dynamics of Interacting Particles, Springer, Berlin (1991)

[84] H. Spohn, Nonlinear Fluctuating hydrodynamics for anharmonic chains, J. Stat. Phys. 154, 1191–1227 (2014).

[85] A. Sudbury and P. Lloyd, Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems, Ann. Probab. 23(4), 1816–1830 (1995).
[86] B. Tóth and B. Valkó, Onsager Relations and Eulerian Hydrodynamic Limit for Systems with Several Conservation Laws. J. Stat. Phys. **112**, 497-521 (2003).

[87] B. Tóth and B. Valkó, Perturbation of singular equilibria of hyperbolic two-component systems: A universal hydrodynamic limit, Commun. Math. Phys. **256**, 111-157 (2005).

[88] C.A. Tracy and H. Widom, On the asymmetric simple exclusion process with multiple species, J. Stat. Phys. **150**, 457–470 (2013).

[89] C.N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. **19**, 1312–1315 (1967).