INVERSE PROBLEMS FOR JACOBI OPERATORS WITH FINITELY SUPPORTED PERTURBATIONS

EVGENY L. KOROTYAEV, EKATERINA LEONOVA

Abstract. We solve the inverse problem for Jacobi operators on the half lattice with finitely supported perturbations, in particular, in terms of resonances. Our proof is based on the results for the inverse eigenvalue problem for specific finite Jacobi matrices and theory of polynomials. We determine forbidden domains for resonances and maximal possible multiplicities of real and complex resonances.

1. Introduction and main results

1.1. Introduction. We consider Jacobi operators $J$ acting on $\mathbb{L}^2(\mathbb{N})$ and given by

$$ (Jf)_x = a_x f_{x+1} + b_x f_x + a_{x-1} f_{x-1}, \quad x \in \mathbb{N} = \{1, 2, 3, \ldots\}, $$

where $f = (f_x)_1^\infty \in \mathbb{L}^2(\mathbb{N})$ and formally $f_0 = 0$. For two real sequences $(a_x)_1^\infty$ and $b = (b_x)_1^\infty \in \mathbb{L}^\infty(\mathbb{N})$ define a perturbation of $J$ by

$$ q = q(a, b) = (b_1, a_1 - 1, b_2, a_2 - 1, \ldots). $$

We assume that $q$ is finitely supported and belongs to the class $X_k, k \in \mathbb{N}$ given by

$$ X_k = \left\{ h \in \mathbb{L}^\infty(\mathbb{N}) : h_{2x} > -1 \forall x \in \mathbb{N} \text{ and } h_k \neq 0, h_x = 0 \forall x > k \right\}. $$

(1.2)

Note that if $k \in \{2p - 1, 2p\}$ for some $p \in \mathbb{N}$, then we have

- if $x > p$, then $a_x - 1 = b_x = 0$,
- if $k = 2p - 1$, then $a_p = 1, b_p \neq 0$,
- if $k = 2p$, then $a_p \neq 1$.

For a set $Y \subset \mathbb{Z}$ (below $Y = \mathbb{N}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ or $\mathbb{Z}$) define the spaces $\mathbb{L}^2(Y)$ and $\mathbb{L}^\infty(Y)$ as the spaces of sequences $(h_n)_{n \in Y}$ equipped with the norms

$$ ||h||_2 = \left( \sum_{n \in Y} |h_n|^2 \right)^{\frac{1}{2}} \geq 0, \quad ||h||_\infty = \sup_{n \in Y} |h_n| $$

(1.3)

respectively. Recall the main properties of $J$, see, e.g., [40]. It has purely absolutely continuous spectrum $[-2, 2]$ plus a finite number of simple eigenvalues on the set $\mathbb{R} \setminus [-2, 2]$. Thus the spectrum of $J$ has the form

$$ \sigma(J) = \sigma_{ac}(J) \cup \sigma_d(J), \quad \sigma_{ac}(J) = [-2, 2], \quad \sigma_d(J) \subset \mathbb{R} \setminus [-2, 2], $$

(1.4)

see Fig. 1. The eigenvalues of $J$ are denoted by

$$ \lambda_{-n} < \cdots < \lambda_{-1} < -2 < 2 < \lambda_{n+} \cdots < \lambda_1, $$

(1.5)
for some \( n_+ \geq 0 \). The operator \( J \) with a perturbation \( q \in \mathcal{X}_k \), where \( k \in \{2p - 1, 2p\} \) has a representation as a semi-infinite matrix given by

\[
\begin{pmatrix}
\lambda_{-1} & \lambda_{-2} & \lambda_{-3} & \sigma_{ac}(J) \\
\times & \times & \times & \times \\
\end{pmatrix}
\]

\[2 \times \sqrt{2} \times \lambda_2 \times \lambda_1 \]

**Figure 1.** The spectrum of \( J \)

Define a new spectral variable \( z = z(\lambda) \) by

\[
\lambda = \lambda(z) = z + \frac{1}{z}, \quad z \in \mathbb{D}, \quad \lambda \in \Lambda := \mathbb{C} \setminus [-2, 2],
\]

where \( \mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}, r > 0 \) is the disc, and \( \mathbb{D} = \mathbb{D}_1 \). Here \( \lambda(z) \) is a conformal mapping from \( \mathbb{D} \) onto \( \Lambda \). Its inverse function \( z(\lambda) \) satisfies

\[
z(\lambda) = \frac{1}{2} - \sqrt{\frac{\lambda^2}{4} - 1}, \quad z(\lambda) = \frac{1 + \sigma(1)}{\lambda} \quad \text{as} \quad \lambda \to \infty.
\]

(1.8)

For the operator \( J \) we define a finite Jacobi operator \( J_p \) acting on \( \mathbb{C}^p \) and given by:

\[
\left( J_p f \right)_x = a_x f_{x+1} + b_x f_x + a_{x-1} f_{x-1}, \quad f = (f_x)_x \in \mathbb{C}^p, \quad x \in \mathbb{N}_p = \{1, 2, \ldots, p\},
\]

(1.9)

where formally \( f_0 = f_{p+1} = 0 \). We define a matrix \( J^1_p \) that is obtained from \( J_p \) by deleting the first row and the first column. Similarly, the matrix \( J^2_{p,1} \) is obtained from \( J_p \) by deleting the last row and the last column. If we delete both the first and the last row and column, we obtain a matrix \( J^1_{p,1} \). The operators corresponding to these matrices are self-adjoint.

The operator \( J_p \) has \( p \) simple eigenvalues labeled by \( \mu_0 < \mu_1 < \ldots < \mu_{p-1} \).

The operator \( J^1_{p,1} \) has \( p - 2 \) simple eigenvalues labeled by \( \nu_1 < \nu_2 < \ldots < \nu_{p-2} \).

The operator \( J^2_{p,1} \) has \( p - 1 \) simple eigenvalues labeled by \( \tau_1 < \tau_2 < \ldots < \tau_{p-1} \).

The operator \( J^3_p \) has \( p - 1 \) simple eigenvalues labeled by \( \varrho_1 < \varrho_2 < \ldots < \varrho_{p-1} \).

The numbers \( \tau_j \) are called Dirichlet eigenvalues, \( \varrho_j \) are called Neumann eigenvalues, \( \mu_j \) and \( \nu_j \) are called mixed eigenvalues. Recall a well-known relation (see, e.g., [41])

\[
\mu_0 < \varrho_1, \tau_1 < \mu_1 < \varrho_2, \tau_2 < \mu_2, \varrho_2 < \ldots < \mu_{p-1}, \nu_{p-1}.
\]

(1.10)

where \( \overline{uv} \) denotes \( \min\{u, v\} \leq \max\{u, v\} \). In order to describe eigenvalues of \( J \) we use the set \( n_* \) defined by

\[
n_* = (-n_-, \ldots, -1, 1, \ldots, n_+).
\]

(1.11)
For a Jacobi operator $J$ we introduce the Jost solution $(\psi_x(z))_0^\infty$ of the equation
\begin{equation}
    a_{x-1} \psi_{x-1} + a_x \psi_x + b_x \psi = (z + z^{-1}) \psi_x, \quad |z| > 0, \quad x \in \mathbb{Z}_+,
\end{equation}
with initial conditions
\begin{equation}
    \psi_x(z) = z^x \quad \forall x > p.
\end{equation}
For the matrix $J$ we define the Jost function by $\psi_0(z)$. The Jost function $\psi_0(z)$ is a real polynomial of order $k$. In particular, we have
\begin{equation}
    \psi_0(z) = \begin{cases}
        \frac{1}{A_0} + O(z) & \text{as } z \to 0 \\
        C z^k (1 + o(1)) & \text{as } z \to +\infty,
    \end{cases}
\end{equation}
where the constants $A_0 = a_p a_{p-1} \cdots a_1$ and $C \neq 0$ (see [40]). Note that the function $\psi_0$ has $\Re = n_+ + n_- \geq 0$ simple zeros in $\mathbb{D}$ given by
\begin{equation}
    s_j = z(\lambda_j), \quad \lambda_j = s_j + \frac{1}{s_j}, \quad j \in n_+,
\end{equation}
\begin{equation}
    -1 < s_{-n_-} < \ldots < s_{-1} < 0 < s_1 < \ldots < s_{n_+} < 1.
\end{equation}
The function $\psi_0$ can also have a finite number of zeros in $\mathbb{C} \setminus \mathbb{D}$ which are called resonances. The only possible zeros in $\{|z| = 1\}$ are $\pm 1$, and these zeros are always simple. They are called virtual states or resonances.

For two sequences $f = (f_x)_0^\infty$, $u = (u_x)_0^\infty$ we introduce the Wronskian by
\begin{equation}
    \{f, u\}_x = a_x (f_x u_{x+1} - u_x f_{x+1}), \quad x \geq 0.
\end{equation}
Note that if $f$ and $u$ are some solutions of the Jacobi equation (1.12), then $\{f, u\}_x$ does not depend on $x$. For each $J$ introduce finite Jacobi matrices $J^{\pm}$ on $\mathbb{C}^p$ by
\begin{equation}
    J^{\pm} = J_p \pm a_p^2 e_p^\top e_p,
\end{equation}
where the vector $e_p = (0, \ldots, 0, 1) \in \mathbb{C}^p$. Each matrix $J^{\pm}$ has $p$ simple eigenvalues $(\alpha_n^{\pm})^p$. Below we show that they satisfy the following relations, see Figure 2:
\begin{equation}
    \alpha_1^- < \alpha_1^+ < \alpha_2^- < \alpha_2^+ < \ldots < \alpha_p^- < \alpha_p^+,
\end{equation}
\begin{equation}
    \alpha_1^- < \mu_0, \quad \tau_j < \alpha_{j+1}^- < \mu_j, \quad j \in \mathbb{N}_{p-1},
\end{equation}
\begin{equation}
    \mu_{p-1}^- < \alpha_p^+, \quad \mu_{j-1}^- < \alpha_j^+, \quad j \in \mathbb{N}_{p-1}.
\end{equation}

1.2. Main results. Let $\#(A, (c, d))$ denote the number of eigenvalues of an operator $A$ in the interval $(c, d)$, counting multiplicity. Now we describe the location of the eigenvalues of the operators $J^{\pm}$ and $J, J_p$ and $J_{p,1}$.

**Theorem 1.1.** Let the operator $J$ be defined by (1.1) with a perturbation $q \in \mathbb{X}_k$, where $k \in \{2p-1, 2p\}$. Let $\lambda_{-1} < \cdots < \lambda_{-n_-} < -2$ and $2 < \lambda_{n_+} < \cdots < \lambda_1$ be its eigenvalues. Then they satisfy
\begin{equation}
    n_- = \#(J^-, (-\infty, -2)) \leq p, \quad n_+ = \#(J^+, (2, +\infty)) \leq p,
\end{equation}
\begin{equation}
    n_- + n_+ \leq \left\lfloor \frac{k}{2} \right\rfloor + 1,
\end{equation}
and
\begin{equation}
    \alpha_1^- < \lambda_{-1} < \mu_0 < \alpha_1^+ < \tau_1 < \alpha_2^- < \lambda_{-2} < \mu_1 < \ldots
\end{equation}
\begin{equation}
    \cdots < \tau_{n_-+1} < \alpha_{n_-}^- < \lambda_{n_-} < \min\{-2, \alpha_{n_-}^+\}.
\end{equation}
\[
\begin{array}{cccccccc}
\alpha_1^- & \times & \mu_0 & \alpha_1^+ & \times & \tau_1 & \alpha_2^- & \times & \mu_1 & \alpha_2^+ & \times & \tau_2 & \alpha_3^- & \times \\
\end{array}
\]

**Figure 2.** The eigenvalues of \( J_p \) and \( J^\pm \)

\[
\max\{2, \alpha_{p-n+1}^-\} < \lambda_{n+} < \alpha_{p-n+1}^+ < \tau_1 < \ldots \\
< \mu_{p-2} < \lambda_2 < \alpha_{p-1}^+ < \tau_{p-1} < \alpha_p^- < \mu_{p-1} < \lambda_1 < \alpha_p^+. \quad (1.23)
\]

Here \([x]\) is the floor function. In particular, if \( k > 1 \), then the operator \( J \) has at least \( p - 1 \) resonances.

**Remark.** In the case of Schrodinger operators on the half-line it is also possible to evaluate the number and location of eigenvalues of the operator on the half-line using the eigenvalues of the operator on the finite interval, see [32]. The relations from [32] are similar to (1.20-1.23). But in the continuous case the role of the eigenvalues \((\alpha_j^\pm)\) is played by the eigenvalues of the Sturm-Liouville problem on a finite interval with mixed boundary conditions, namely, Dirichlet condition on the one side and Neumann condition on the other side. Thus, there is a principal difference between the continuous and discrete cases.

In Section 5 (Examples 1 and 2) we describe a location of eigenvalues and resonances depending on the coefficients of the matrix \( J \) for \( k = 1, 2 \). The case \( k = 1 \) is trivial, while in the case \( k = 2 \) there are several options for the arrangement of eigenvalues and resonances. It is possible to give such description for larger \( k \), but the number of different arrangements grows very quickly.

For a real \( t > 0 \) we define a set \( \mathcal{E}_p(t) \) of strictly increasing sequences in \( \mathbb{R}^p \) by

\[
\mathcal{E}_p(t) = \left\{ s = (s_k)_1^p \in \mathbb{R}^p : -t < s_1 < s_2 < \cdots < s_p < t \right\}.
\]

Below we need a set \( \mathcal{E}_p = \mathcal{E}_p(\infty) \). For two interlacing sequences \((x_n)_0^p\) and \((y_n)_1^p\) we define an increasing sequence \( x \star y \) by

\[
x \star y = (x_0, y_1, x_1, y_1, \ldots, x_{p-1}, y_p, x_p) \in \mathcal{E}_{2p+1}. \quad (1.24)
\]

We introduce a set

\[
X_p = \mathcal{X}_{2p} \cup \mathcal{X}_{2p-1} = \left\{ h \in \ell^\infty(\mathbb{N}) : h_{2x} > -1 \ \forall x \in \mathbb{N}, \ |h_{2p-1}| + |h_{2p}| \neq 0, \ h_x = 0 \ \forall x > 2p \right\}
\]

We prove some analogue of the Hochstadt result [17] (see Theorem 3.1), which is used in the proof of main theorems.

**Proposition 1.2.** Let the operator \( J \) be defined by (1.1) with a perturbation \( q \in X_p, p > 0 \). Let \((\alpha_j^\pm)_1^p\) be the eigenvalues of the operators \( J^\pm \). Then the mapping

\[
\alpha^+ \star \alpha^- : X_p \to \mathcal{E}_{2p},
\]

given by \( q \mapsto \alpha^+ \star \alpha^- \) is a a real analytic isomorphism between \( X_p \) and \( \mathcal{E}_{2p} \). Moreover, there is an algorithm to recover \( q \) from the numbers \( \{\alpha_j^\pm\} \).
The resonances of $J$ can also be defined as the poles of its resolvent meromorphically continued in $\mathbb{C}$, see [9]. The S-matrix for $J$ is given by

$$S(z) = \frac{\psi_0(z^{-1})}{\psi_0(z)} = e^{-i2\xi(z)}, \quad \xi(z) = \arg \psi_0(z), \quad |z| = 1,$$  \quad (1.25)

where $\xi$ is called the phase shift. The function $\psi_0(z)$ has no zeros in $\mathbb{D} \setminus \left([-1, s_-] \cup [s_1, 1]\right)$. Therefore, we can choose a branch of $\log \psi_0(z)$ in this domain such that $\log \psi_0(0)$ is real, which uniquely defines the function $\xi(z)$. The function $S$ has a meromorphic continuation to the whole complex plane. Then the poles of $S(z)$ are exactly the resonances of $J$. Recall that the function $\xi(z)$ is continuous in the circle $\{|z| = 1\} \setminus \{\pm 1\}$. It satisfies

$$\xi(1+0i) = -\pi n_+ + \frac{m+}{2}, \quad \xi(1-0i) = \pi \left(2n_+ + n_+ + \frac{3m+}{2}\right), \quad (1.26)$$

$$\xi(-1 \pm 0i) = \pi (n_- + n_+) \pm \frac{m-}{2}\pi, \quad (1.27)

where $m_{\pm} \leq 1$ is the multiplicity of $\pm 1$ as a zero of $\psi_0$. Define a set $X^+_p$ by

$$X^+_p = X^+_{2p-1} \cup X^+_{2p}, \quad \text{where} \quad X^+_k = \left\{ q \in X_k : \psi_0(z, q) \neq 0 \quad \forall z \in \mathbb{D}\right\}. \quad (1.28)

Recall that we have $z(\lambda) = \frac{1}{2} - \sqrt{\frac{\lambda}{4}} - 1, \quad \lambda \in \Lambda$. Below we consider a function $\xi(z(\lambda+0i)), \quad \lambda \in [-2, 2]$. Note that for the conformal mapping $\lambda = z + \frac{1}{z}$ we have

$$z([-2, 2] \pm 0i) = \{z \in \mathbb{C}^\pm : |z| = 1\},$$

$$\xi(z(\lambda + 0i)) = -\xi(z(\lambda - 0i)) \quad \forall \quad \lambda \in [-2, 2].\nonumber$$

We formulate the first results about inverse problems for Jacobi operators with finitely supported perturbations on the half lattice.

**Theorem 1.3.** Let the operator $J$ be defined by (1.1) with a perturbation $q \in X^+_p$. Then the function $F(\lambda) = \frac{2\xi(z(\lambda+0i))}{2p+1} + \arg z(\lambda + 0i)$ is strongly increasing in $\lambda \in (-2, 2)$, and each equation $F(\lambda) = -\frac{\pi}{2p+1}, \quad n \in \mathbb{N}_{2p}$ has a unique solution $\omega_n \in (-2, 2)$ which satisfies

$$\omega_{2j-1} = \alpha_{j-1}^+, \quad \omega_{2j} = \alpha_{j}^-, \quad j \in \mathbb{N}_p. \quad (1.29)$$

The mapping $\omega : X^+_p \to \mathcal{E}_{2p}(2)$ given by

$$q \mapsto \omega = (\omega_n)^{2p}_{1} \quad (1.30)$$

is a bijection between $X^+_p$ and $\mathcal{E}_{2p}(2)$. Moreover, there is an algorithm to recover the coefficients $a, b$ from the numbers $(\omega_j)^{2p}_{1}$.

**Remark.** 1) If $q \in X^+_p$ is given, then we compute $(\omega_n)^{2p}_{1} \in \mathcal{E}_{2p}(2)$, where $\omega_{2j-1} = \alpha_{j}^+$ and $\omega_{2j} = \alpha_{j}^-$. Conversely, if $(\omega_n)^{2p}_{1} \in \mathcal{E}_{2p}(2)$ is given, then we can compute the unique coefficients $a, b$. In the proof of this theorem we only use well-known results on the inverse eigenvalue problem for specific finite Jacobi matrices and theory of polynomials. Marchenko equations are not used in the proof.

2) It is possible to use other eigenvalues (for example, $\mu_j$ and $\tau_j$) instead of $\alpha_j^+$ in this theorem. However, $\alpha_j^\pm$ are natural, since due to Theorem 1.4 the numbers $\alpha_j^\pm$ control the eigenvalues of $J$ and we just need to stay that they belong to $[-2, 2]$, which yields that $J$ has no eigenvalues. In the case with $\mu_j, \tau_j$, we need more complicated conditions to be hold.
3) The recovering procedure is described in the proof of Proposition 1.2. We also give an example of recovering the perturbation \( q \) in the case \( p = 2 \) in Section 5, Example 6.

Now we define the set characterising all possible locations of the eigenvalues and resonances.

**Definition R.** Let \( \mathcal{R}_k \subset \mathbb{C}^k, k \in \mathbb{N} \) be a set of vectors \( r = (r_n)^k_1 \) such that:

R1) \( 0 < |r_1| \leq |r_2| \leq \ldots \leq |r_k| \),

R2) a polynomial \( f(z) = \prod_{1}^{k}(z - r_n) \) is real on the real line,

R3) all zeros of \( f \) in \( \mathbb{D} \) are real and simple. Denote the zeros of \( f \) in \( \mathbb{D} \) as \( s_j, j \in n \) for some \( n_-, n_+ \geq 0 \) arranged by

\[
-1 < s_{-n} < \ldots < s_{-1} < 0 < s_1 < \ldots < s_{n} < 1,
\]

and let

\[
f(s_j^{-1}) \neq 0 \quad \forall j \in n_+, \tag{1.31}
\]

R4) \( f \) has an odd number \( \geq 1 \) of zeros on each interval \( (s_j^{-1}, s_{j-1}^{-1}) \), \( j = n_\setminus \{-1\} \) and an even number \( \geq 0 \) of zeros on each of intervals \( (1, s_{n_+}^{-1}) \) and \( (s_{n_-}^{-1}, -1) \).

The set of \( \mathcal{R}_k \subset \mathbb{C}^k, k \in \mathbb{N} \) is some analogue of the Jost class functions for Schrödinger operator with a compactly supported potential on the half-line from [24].

We sometimes write \( \psi_n(q), r_n(q), \ldots \) instead of \( \psi_n, r_n, \ldots \) when several potentials are being dealt with. Now we construct the mapping \( r : \mathcal{X}_k \to \mathcal{R}_k, k \in \mathbb{N} \) by

\[
r(q) = (r_n)^k_1 \in \mathcal{R}_k, \tag{1.32}
\]

where \( r = (r_n)^k_1 \) is the sequence of roots of the corresponding Jost function \( \psi_0(z) \).

**Theorem 1.4.** Each mapping \( r : \mathcal{X}_k \to \mathcal{R}_k, k \in \mathbb{N} \) is a bijection between \( \mathcal{X}_k \) and \( \mathcal{R}_k \). There is an algorithm to recover the perturbation \( q \) from \( r(q) \).

**Remark.** This theorem solves the inverse resonance problem for Jacobi operators on the half-lattice with finitely-supported perturbations. In our proof we do not use a Marchenko equations from [8] to show that the mapping \( r \) is a surjection, like it is usually done. Our proof consists of two parts. Firstly we consider the case when the Jost function of \( J \) does not have any zeros in unit circle and prove Theorem 1.3. Secondly we prove by induction that we can add roots to the unit circle. We also present the algorithm to recover the matrix \( J \) from its eigenvalues and resonances. We only use theory of polynomials and some simple lemmas proved beforehand in the second part.

We also prove a corollary of Theorem 1.4 which shows that the operator \( J \) is uniquely determined by the values of the phase shift function \( \xi \) at a finite number of points.

**Corollary 1.5.** Consider two Jacobi operators \( J_1 \) and \( J_2 \) with perturbations \( q_1, q_2 \in \mathcal{X}_k, k \geq 1 \). Let \( \xi_1 \) and \( \xi_2 \) be the corresponding phase shift functions. Let \( (w_j)^k_{1} \) be a sequence of distinct numbers on \( \mathbb{C}_{-} \cap \{|z| = 1\} \) such that \( \xi_1(w_j) = \xi_2(w_j) \) for all \( j \in \mathbb{N}_k \). Then we have \( J_1 = J_2 \).
Remark. 1) One can see from the proof that this corollary stays true if the numbers \((w_j)^k\) are chosen from the set \(|z| = 1\) \(\setminus \{\pm 1\}\) such that \(w_i \neq \overline{w}_j\) for all \(i, j \in \mathbb{N}_k\).

2) For the uniqueness we use a finite sequence \(\{\xi(w_j)\}_k\) for some \((w_j)^k\). Note that the number of parameters is equal to the number of perturbations in \(J\). We do not need eigenvalues and norming constants, like in general theory, see, e.g., [9].

Below we discuss the location of resonances and specify the forbidden domain.

**Theorem 1.6.** Let \(\psi_0\) be the Jost function for some operator \(J\) defined by (1.1) with a perturbation \(q \in \mathcal{X}_k\). Then all real resonances of multiplicity greater than 1 and non-real resonances belong to the disc \(|z| < R_o\), where \(R_o\) is given by:

- if \(k = 2p\) (i.e., \(a_p \neq 1\)), then
  \[
  R_o^2 = \frac{1}{\beta|1-a_p^2|}, \quad \beta = \begin{cases} 
  1, & \text{if } a_p < 1 \\
  |s_1s_{-1}|, & \text{if } a_p > 1
  \end{cases}
  \]  
  \[(1.33)\]

- if \(k = 2p - 1\) (i.e., \(a_p = 1, b_p \neq 0\)), then
  \[
  R_o^2 = \frac{1}{\beta|b_p|}, \quad \beta = \begin{cases} 
  |s_{-1}|, & \text{if } b_p < 0 \\
  s_1, & \text{if } b_p > 0
  \end{cases}
  \]  
  \[(1.34)\]

Here \(s_1 > 0\) and \(s_{-1} < 0\) are the roots of \(\psi_0\) in \(\mathbb{D}\) with the smallest absolute value, see (1.14). If there are none of those, they are replaced with 1.

Moreover, these estimates are sharp: if \(a_p\) (or \(b_p\)) and \(s_{\pm 1}\) are fixed, then one of the resonances can be arbitrarily close to the boundary of the disc for some Jacobi operator \(J\).

Now we determine the maximal value of positive simple resonances.

**Theorem 1.7.** Let \(\psi_0\) be the Jost function for some operator \(J\) defined by (1.1) with a perturbation \(q \in \mathcal{X}_k\). Then all its positive resonances of multiplicity 1 belong to the interval \((1, R_+\)], where \(R_+\) is defined by:

- if \(k = 2p\) (i.e., \(a_p \neq 1\)), then
  \[
  R_+ = \frac{1}{\beta|1-a_p^2|}, \quad \beta_+ = \begin{cases} 
  s_1, & \text{if } a_p < 1 \\
  \max\{|s_1s_{-1}|, |s_{-1}|\}, & \text{if } a_p > 1
  \end{cases}
  \]  
  \[(1.35)\]

- if \(k = 2p - 1\) (i.e., \(a_p = 1, b_p \neq 0\)), then
  \[
  R_+ = \frac{1}{\beta_+|b_p|}, \quad \beta_+ = \begin{cases} 
  |s_1s_{-1}|, & \text{if } b_p < 0 \\
  \max\{|s_1s_{-1}|, |s_{-1}|\}, & \text{if } b_p > 0
  \end{cases}
  \]  
  \[(1.36)\]

Here \(s_1 > 0\) and \(s_{-1} < 0\) are the roots of \(\psi_0\) in \(\mathbb{D}\) with the smallest absolute value, see (1.14). If there are none of those, they are replaced with 1.

Moreover, these estimates are sharp: if \(a_p\) (or \(b_p\)) and \(s_{\pm 1}\) are fixed, then one of the resonances can be arbitrarily close to \(R_+\) for some Jacobi operator \(J\).

Remark. 1) Similar result is proved for negative resonances in Corollary 4.2

2) For Schrodinger operators on the half line, the forbidden domain for resonances is determined by some logarithmic curve, see e.g., [21].

3) Thus, we specified the forbidden domain for the resonances of the operator \(J\). Namely, we obtained that if \(k\) and \(a_p\) (or \(b_p\)) and the smallest eigenvalue are fixed, that all non-real
resonances and resonances of multiplicity greater than 1 are located inside a ring of a fixed radius, while simple real resonances belong to two fixed intervals, see Fig. 4.

Now we discuss the maximal possible multiplicity of real-valued and complex resonances.

**Corollary 1.8.** Let $J$ be a Jacobi operator defined by (1.1) with a perturbation $q \in X_k$. Let its Jost function $\psi_0$ have $N = n_+ + n_-$ roots in $\mathbb{D}$ and let $r_o$ be a resonance of $J$. Then the maximal possible multiplicity of $r_o$ equals $M$, where $M$ is given by:

i) if $k = 2p$ and $a_p < 1$, then

$$M = \begin{cases} p - \mathfrak{N}, & \text{if } r_o \in \mathbb{C} \setminus \mathbb{R} \\ \min(2p - 2\mathfrak{N} + 1, 2p), & \text{if } r_o \in \mathbb{R} \end{cases},$$

(1.37)

ii) if $k = 2p$ and $a_p > 1$, then

$$M = \begin{cases} p + 1 - \max(n_+, 1) - \max(n_-, 1), & \text{if } r_o \in \mathbb{C} \setminus \mathbb{R} \\ 2p + 3 - 2\max(n_+, 1) - 2\max(n_-, 1), & \text{if } r_o \in \mathbb{R} \end{cases},$$

(1.38)

iii) if $k = 2p - 1$ and $b_p > 0$, then

$$M = \begin{cases} p - \max(n_+, 1) - n_-, & \text{if } r_o \in \mathbb{C} \setminus \mathbb{R} \\ 2p + 1 - 2\max(n_+, 1) - 2n_-, & \text{if } r_o \in \mathbb{R} \end{cases},$$

(1.39)

iv) if $k = 2p - 1$ and $b_p < 0$, then

$$M = \begin{cases} p - \max(n_-, 1) - n_+, & \text{if } r_o \in \mathbb{C} \setminus \mathbb{R} \\ 2p + 1 - 2\max(n_+, 1) - 2n_+, & \text{if } r_o \in \mathbb{R} \end{cases}.$$  

(1.40)

A complex resonance of the maximal multiplicity can be located in any point in $\mathbb{C} \setminus (\mathbb{R} \cup \mathbb{D})$. If $s_j$, $j \in n_*$ are the roots of $\psi_0$ in $\mathbb{D}$, then a real resonance of the maximal multiplicity can be located in any of the intervals $(s_j^{-1}, s_{j-1}^{-1})$, $j = -n_- + 1, \ldots, -1, 2, \ldots, n_+.$

**Remark.** Thus roughly speaking the theorem gives that the maximal multiplicity of resonances $M = p - n_- - n_+$. It follows from Theorems 1.1 and 1.4 that the numbers $n_+$ and $n_-$ can be arbitrary, under the condition $n_- + n_+ \leq \lfloor \frac{k}{2} \rfloor + 1.$
Finally we describe the behavior of the coefficients of the operator $J$ when one of resonances moves to infinity.

**Corollary 1.9.** Let the operator $J$ with a perturbation $q \in \mathfrak{X}_k$ be defined by (1.1). Let $\{r_n\}_1^k$ be the set of roots of its Jost function and let $\{s_j\}, j \in n_*$ be the set of its eigenvalues. We introduce the pair $(r, s)$ which satisfies one of the following conditions:

i) $r$ is some non-real resonances and $s = \bar{r}$.

ii) $r$ is some resonance from the interval $(s_{j+1}^{-1}, s_j^{-1})$ for some $j \in \{2, 3, \ldots, n_+ - 1\}$, or $r$ is some resonance from the interval $(1, s_{n_+ - 1}^{-1})$ and $j = n_+$. In both cases $s = s_j$.

Then the pair $(r, s)$ can be moved to any pair $(r_*, s_*)$ on $\mathbb{R} \setminus [-s_1^{-1}, s_1^{-1}]$ or any two conjugate points in $\mathbb{C} \setminus (\mathbb{R} \cup \mathbb{D})$, and there exists a matrix $\tilde{J}$ with $\tilde{q} \in \mathfrak{X}_k$, corresponding to the obtained set of roots. There also exists a matrix $J_o$ with a perturbation $q_o(a^*, b^*) \in \mathfrak{X}_{k-2}$ which Jost function has the roots $\{r_n\}_1^k \setminus \{r, s\}$.

In particular, if $k = 2p$ (i.e., $a_p \neq 1$), then we have

$$a_p = 1 - \frac{1 - (a_p^o)^2 + o(1)}{2|a|} \quad \text{as} \quad |a| \to +\infty,$$

where $a = r_\ast s_\ast$. If $k = 2p - 1$ (i.e., $a_p = 1, b_p \neq 0$), then we have

$$b_p = \frac{b_p^o}{|a|} \to 0 \quad \text{as} \quad |a| \to +\infty.$$

Moreover, in both cases $\tilde{q} \to q_o$ as $|a| \to \infty$.

**Remark.** 1) While in the first case the points $r$ and $s$ can be moved to infinity only together, preserving the conjugacy, in the second case we can move both points or fix one of them and move another one to infinity, and still have $|a| \to +\infty$.

2) Similar result holds true for negative eigenvalues. We discuss analogous results for real resonances and the eigenvalues of the smallest absolute value in Corollary 1.3.

3) Note that we cannot move the eigenvalues out of the circle without changing the location of resonances, as it will break the condition R4.

4) Similar problems were considered by Marletta and Weikard in [35]. They only studied the case when all roots of the Jost function have a large absolute value. Our results describe the change of the matrix $J$ when only one or two roots are moved to infinity. After applying Corollary 1.9 enough times, we obtain the result from [35].

1.3. **Short review.** A lot of papers are devoted to resonances of one-dimensional Schrödinger operators with compactly supported potentials, see Froese [12], Hitrik [16], Korotyaev [24], Simon [39], Zworski [43] and references therein. Inverse problems (uniqueness, reconstruction, characterization) in terms of resonances were solved by Korotyaev for a Schrödinger operator with a compactly supported potential on the real line [26] and the half-line [24]. See also Zworski [42], Brown-Knowles-Weikard [17] concerning the uniqueness. The resonances for periodic plus a compactly supported potential were considered by Fissova [11], Korotyaev [28], Korotyaev-Schmidt [34]. Christiansen [10] discussed resonances for steplike potentials. The ”local resonance” stability problems were considered in [25], [29], [37]. Resonances for three and fourth order differential operator with compactly supported coefficients on the line were studied in [31], [3]. Resonances for Stark operator with compactly supported potentials
were discussed in [13], [30]. Inverse resonance problems for Dirac operators with a compactly supported potential on the half line is solved in [33].

We briefly describe the results on the resonances for finitely supported perturbations of 1dim discrete Laplacian. Inverse problem on the real line was solved by Korotyaev [28], see also Brown-Naboko-Weikard [6] about the uniqueness. Bledsoe [4], Marletta and Weikard [38] studied stability for Jacobi operators. Uniqueness and stability are also discussed in [36]. The case of periodic Jacobi operators with finitely supported perturbations was studied by Iantchenko and Korotyaev [20, 21, 22] and Kozhan [35]. The scattering problem for Jacobi operators with matrix-valued coefficients was studied by Aptekarev and Nikishin [1].

The plan of this paper is as follows. In Sect. 2 we discuss the standard facts about Jost functions and fundamental solutions. In Sect. 3 we prove the main results about the inverse problem. In Sect. 4 we prove the main results about a location of resonances. In Sect. 5 we consider some examples.

2. Preliminaries

In order to prove the main theorems, we need the following asymptotics

**Lemma 2.1.** Let \( q(a, b) \in X_k, k \in \{2p, 2p - 1\} \). Then

i) Each function \( \psi_{p-n}(z), n = 0, ..., p \) is a real polynomial and satisfies

\[
\psi_p(z) = \frac{z^p}{a_p}, \quad \psi_{p-n}(z) = \frac{z^{p+n}}{A_{p-n}} \left( c_p - \frac{c_p B_{n-1} + b_p}{z} + O(z^{-2}) \right),
\]

\[
\psi_0(z) = \frac{z^{2p}}{A_0} \left( c_p - \frac{c_p B_{p-1} + b_p}{z} + O(z^{-2}) \right),
\]

as \( z \to \infty \), where \( B_j = \sum_{s=1}^j b_{p-s}, A_x = a_x a_{p-1} \cdot a_x \) and \( c_x = 1 - a_x^2 \). Moreover,

\[
\psi_{p-n}(z) = \frac{z^{p-n}}{A_{p-n}} \left( 1 - z \sum_{p-n+1}^p b_k + O(z^2) \right), \quad \psi_0(z) = \frac{1 + B_p z + O(z^2)}{A_0} \quad \text{as} \quad z \to 0.
\]

ii) The Jost function \( \psi_0(z) \) has the form:

\[
\psi_0(z) = \frac{1 - a_p^2}{A_0} \prod_{1}^{2p} (z - r_n), \quad \prod_{1}^{2p} r_n = \frac{1}{1 - a_p^2}, \quad \text{if} \quad a_p \neq 1,
\]

\[
\psi_0(z) = -\frac{b_p}{A_0} \prod_{1}^{2p-1} (z - r_n), \quad \prod_{1}^{2p-1} r_n = \frac{1}{b_p}, \quad \text{if} \quad a_p = 1, b_p \neq 0,
\]

\[
\sum_{1}^{k} r_n = \sum_{1}^{p} b_n.
\]

**Proof.** i) Using the equation \( a_{x-1} f_{x-1} = (\lambda - b_x) f_x - a_x f_{x+1} \) and \( \psi_x = z^x, \ x > p \), we obtain: if \( n = 0 \), then

\[
\psi_p = \frac{(\lambda - b_{p+1}) \psi_{p+1} - \psi_{p+2}}{a_p} = \frac{(z + z^{-1}) z^{p+1} - z^{p+2}}{a_p} = \frac{z^p}{a_p},
\]
if \( n = 1 \), then
\[
\psi_{p-1} = \frac{(\lambda - b_p)\psi_p - a_p\psi_{p+1}}{a_{p-1}} = \frac{(z + z^{-1} - b_p)zp - a_p^2zp}{a_pa_{p-1}} = \frac{zp+1c_p - b_pzp + zp-1}{a_pa_{p-1}},
\]
(2.8)

if \( n = 2 \), then
\[
\psi_{p-2} = \frac{(\lambda - b_{p-1})\psi_{p-1} - a_{p-1}\psi_p}{a_{p-2}} = \frac{(z + z^{-1} - b_{p-1})(zp+1c_p - b_pzp + zp-1) - a_{p-1}^2zp}{a_pa_{p-1}a_{p-2}}
\]
\[
= \frac{zp^{p+2}}{a_{p}a_{p-1}a_{p-2}} \left( c_p - \frac{b_p + c_pb_{p-1}}{z} + \frac{c_p + c_{p-1} + b_pb_{p-1}}{z^2} - \frac{b_p + b_{p-1}}{z^3} + \frac{1}{z^4} \right).
\]
(2.9)

Repeating this procedure we obtain (2.1), (2.2).

ii) Asymptotic (2.1) yield (2.4). The identities (2.4) and \( \psi_0(0) = \frac{1}{\lambda_0} > 0 \) (see (2.3)) gives (2.5), (2.6) follows from (2.3) and (2.5). □

For each eigenvalue \( \lambda_j \) of \( J \) we define a norming constant \( c_j \) by
\[
c_j = \sum_{x \geq 1} \psi_x(s_j)^2 > 0.
\]
(2.10)

There exists a following identity, see e.g., [Ca]:
\[
c_j = a_1\psi'_0(s_j)\psi_1(s_j)z'(\lambda_j), \quad s_j \in (-1, 1). \tag{2.11}
\]

Let \( #(f, I) \) denote the number of zeros of \( f \) on the set \( I \). We describe the location of eigenvalues and real resonances of the operator \( J \). We will use it to prove Theorems 1.1 and 1.5–1.6.

**Lemma 2.2.** Let \( q \in \mathbb{X}_k, k \in \{2p, 2p - 1\} \) and let \(-1 < s_{-n_-} < \ldots < s_{-1} < 0 < s_1 < \ldots < s_{n_+} < 1 \) be all zeros of \( \psi_0(z) \) on \((-1, 1)\). Let \( s_j^{-1} = \hat{s}_j \). Then we have
\[
c_j = a_1z'(\lambda_j)\psi'_0(s_j)\psi_1(s_j) = s_ja_1\psi'_0(s_j)\psi_0(\hat{s}_j) > 0, \quad \forall j \in n_.,
\]
(2.12)

\((-1)^j\psi'_0(s_j) > 0, \quad s_j(-1)^j\psi_0(\hat{s}_j) > 0, \quad \forall j \in n_.,
\]
(2.13)

\( #(\psi_0, (\hat{s}_{j+1}, \hat{s}_j)) = \text{odd} \geq 1, \quad \forall j \in n_{n_-+1},
\]
(2.14)

\( #(\psi_0, (\hat{s}_j, \hat{s}_{j-1})) = \text{odd} \geq 1, \quad \forall j \in [-n_-+1 : -1],
\]
(2.15)

\( #(\psi_0, (s_{n_+}, \hat{s}_{n_+})) = \text{even} \geq 0, \quad #(\psi_0, (\hat{s}_{-n_-}, s_{-n_-})) = \text{even}.
\]
(2.16)

Moreover, if \( k = 2p \) (i.e., \( a_p \neq 1 \)), then we have
\[
\begin{cases}
a_p < 1 \implies #(\hat{s}_{\pm}, \pm \infty) = \text{odd} \geq 1, \quad #(\hat{s}_{n_\pm}, \pm \infty) \geq n_\pm \\
a_p > 1 \implies #(\hat{s}_{\pm}, \pm \infty) = \text{even} \geq 0, \quad #(\hat{s}_{n_\pm}, \pm \infty) \geq n_\pm - 1
\end{cases}
\]
(2.17)

if \( k = 2p - 1 \) (i.e., \( a_p = 1, b_p \neq 0 \)), then we have
\[
\begin{cases}
b_p > 0 \implies #(\hat{s}_1, \infty) = \text{odd} \geq 1, \quad #(\hat{s}_{n_+}, + \infty) \geq n_+ \\
b_p < 0 \implies #(\hat{s}_1, \infty) = \text{even} \geq 0, \quad #(\hat{s}_{n+}, + \infty) \geq n_+ - 1
\end{cases}
\]
(2.18)
Proof. Denote \( f^\circ(z) = f\left(\frac{1}{z}\right) \). Computing the Wronskian for the functions \( \psi \) and \( \psi^\circ \), we obtain
\[
\{\psi, \psi^\circ\}_0 = \psi^\circ_0 \psi_1 - \psi^\circ_1 \psi_0,
\]
\[
\{\psi, \psi^\circ\}_{p+1} = \psi^\circ_{p+1} \psi_{p+2} - \psi^\circ_{p+2} \psi_{p+1} = z - \frac{1}{z} = z\lambda'(z).
\]
Then at \( z = s_j \) we have
\[
\psi_1(s_j) = \frac{s_j \lambda(s_j)}{\psi^\circ_0(s_j)},
\]
and, substituting this identity into \(\text{(2.11)}\), we obtain \(\text{(2.12)}\).

The function \( \psi_0(z), \ z \in (-1, 1) \) has the zeros \( \{s_j\} \), and \( \psi_0(0) > 0 \). Then we have \( \psi'_0(s_1) < 0 \) and \( \psi_0(s_2) > 0 \) etc. Hence due to \(\text{(2.12)}\) we obtain \(\text{(2.13)}\). Other identities follow directly from this.

Now let us study the relationship between the eigenvalues of the operator \( J \) and those of the finite Jacobi matrix \( J_p \). Introduce the fundamental solutions \( \varphi = (\varphi_x(\lambda))^\circ_0 \) and \( \vartheta = (\vartheta_x(\lambda))^\circ_0 \) of the equation
\[
a_{x-1}f_{x-1} + a_x f_{x+1} + b_x f_x = \lambda f_x, \quad (\lambda, x) \in \mathbb{C} \times \mathbb{Z}_+,
\]
with initial conditions
\[
\varphi_0 = \vartheta_1 = 0, \quad \varphi_1 = \vartheta_0 = 1.
\]
Note that for \( z \in \mathbb{D} \) the functions \( \varphi_x(\lambda(z)) \) and \( \vartheta_x(\lambda(z)) \) are polynomials in \( z + z^{-1} \). Therefore, they can be considered outside of \(\mathbb{D} \) too. Now we can write down the Wronskian for the Jost and fundamental solutions. For \( x = 0 \) we have
\[
\{\psi, \varphi\}|_{x=0} = \psi_0 \varphi_1 - \psi_1 \varphi_0 = \psi_0.
\]
Since the Wronskian of two fundamental solutions does not depend on \( x \), we have at \( x = p \):
\[
\psi_0 = \{\psi, \varphi\}|_{x=p} = a_p(\psi_p \varphi_{p+1} - \psi_{p+1} \varphi_p) = z^p \varphi_{p+1} - a_p z^{p+1} \varphi_p.
\]
Similar arguments for \( \psi \) and \( \vartheta \) yield
\[
\psi_1(z) = -z^p \vartheta_{p+1}(\lambda(z)) + a_p z^{p+1} \vartheta(\lambda(z)).
\]
Recall that \( \psi^\circ(z) = \psi\left(\frac{1}{z}\right) \). Computing the Wronskian \(\text{(1.15)}\) for \( \psi \) and \( \psi^\circ \) and using the fact that \( \varphi(\lambda(z)) = \varphi(\lambda(\frac{1}{z})) \), we obtain
\[
\varphi_{p+1}(\lambda(z))(z^2 - 1) = z^{p+2} \psi_0\left(\frac{1}{z}\right) - \frac{\psi_0(z)}{z^p},
\]
\[
a_p \varphi_p(\lambda(z))(z^2 - 1) = z^{p+1} \psi_0\left(\frac{1}{z}\right) - \frac{\psi_0(z)}{z^{p+1}}.
\]
Recall that
\[
(\mu_j)_0^{p-1} \text{ are the roots of } \varphi_{p+1}(\lambda) \text{ and the eigenvalues of } J_p,
\]
\[
(\tau_j)_0^{p-1} \text{ are the roots of } \varphi_p(\lambda) \text{ and the eigenvalues of } J_{p,1},
\]
\[
(\nu_j)_1^{p-1} \text{ are the roots of } \vartheta_{p+1}(\lambda) \text{ and the eigenvalues of } J_{p}^1,
\]
\[
(\nu_j)_2^{p-2} \text{ are the roots of } \vartheta_p(\lambda) \text{ and the eigenvalues of } J_{p,1}^1.
\]
We define two functions \( u^\pm \) by
\[
u_j^p = \varphi_{p+1} + a_p \varphi_p.
\]
Denote the characteristic polynomials of the matrices \( J_p, J_{p,1} \) and \( J^\pm \) by \( \mathcal{D}_p, \mathcal{D}_{p,1} \) and \( \mathcal{D}^\pm \) respectively. In order to prove Theorems \(\text{1.3} \) and \(\text{1.4} \) we recall a well-known result about fundamental solutions and study the connection between functions \( u^\pm(\lambda) \) and matrices \( J^\pm \).
Lemma 2.3. Let $J$ be a Jacobi operator acting on $\ell^2(\mathbb{N})$ defined by $(1.1)$ with a perturbation $q(a,b) \in \mathcal{X}_k$, $k \in \{2p-1, 2p\}$. Then

i) the fundamental solutions $\varphi(\lambda)$ satisfy

$$\varphi_x(\lambda) = \frac{\lambda^{x-1}}{a_1 \cdots a_{x-1}} \left( 1 - \frac{b_1 + \cdots + b_{x-1}}{\lambda} + O(\lambda^{-2}) \right) \quad \text{as} \quad \lambda \to +\infty, \ x \in \mathbb{N}_p. \quad (2.26)$$

ii) the functions $u^\pm(\lambda)$ defined by $(2.25)$ satisfy $D^\pm = (-1)^p A_1 a_p u^\pm$.

Proof. i) Using the equation $a_x f_{x+1} = (\lambda - b_x) f_x - a_{x-1} f_{x-1}$ and $\varphi_0 = 0$, $\varphi_1 = 1$, we obtain:

if $x = 1$, then we have

$$\varphi_2 = \frac{(\lambda - b_1)\varphi_1 - \varphi_0}{a_1} = \frac{\lambda - b_1}{a_1}, \quad (2.27)$$

if $x = 2$, then we have

$$\varphi_3 = \frac{(\lambda - b_2)\varphi_2 - a_1 \varphi_1}{a_2} = \frac{(\lambda - b_2)(\lambda - b_1) - a_1^2}{a_1 a_2} = \frac{\lambda^2 - (b_1 + b_2)\lambda + b_1 b_2 - a_1^2}{a_1 a_2}. \quad (2.28)$$

Repeating this procedure we obtain $(2.26)$. ii) Recall that the matrices $J^\pm$ are given by

$$J^\pm = \begin{pmatrix}
  b_1 & a_1 & 0 & 0 & \cdots & 0 \\
  a_1 & b_2 & a_2 & 0 & \cdots & 0 \\
  0 & a_2 & b_3 & a_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & a_{p-2} & b_{p-1} \\
  0 & 0 & 0 & 0 & a_{p-1} & b_p \pm a_p^2
\end{pmatrix}. \quad (2.29)$$

It follows from the asymptotics $(2.26)$ that

$$D_p = (-1)^p A_1 \varphi_{p+1}, \quad D_{p,1} = \frac{(-1)^{p-1} A_1}{a_p} \varphi_p. \quad (2.30)$$

Using the linearity of the determinant over rows, we divide the last row of $J^\pm$ into two parts and obtain

$$D^\pm = D_p \pm a_p^2 D_{p,1} = (-1)^p A_1 a_p (\varphi_{p+1} \mp a_p \varphi_p) = (-1)^p A_1 a_p u^\pm.$$

Recall a corollary from [15], see page 241.

Lemma 2.4. Let $n > 2$, let $A$ be Hermitian $n \times n$ matrix and let $v \in \mathbb{C}^n$ be nonzero. If $A$ and $A + vv^*$ do not have similar eigenvectors, then

$$\lambda_j(A) < \lambda_j(A + vv^*) < \lambda_{j+1}(A), \quad j \in \mathbb{N}_{n-1},$$

$$\lambda_n(A) < \lambda_n(A + vv^*).$$

Lemma 2.5. Let $J$ be a Jacobi operator acting on $\ell^2(\mathbb{N})$ defined by $(1.1)$ with a perturbation $q(a,b) \in \mathcal{X}_{2p}$, $p \in \mathbb{N}$. Let the operators $J^\pm$ be defined by $(1.16)$ and let $(\alpha_j^\pm)_{j=1}^p$ be their eigenvalues. Then the relations $(1.17)$ hold true and

$$\alpha_j^+ \to +\infty, \quad \alpha_j^- \to -\infty \quad \text{as} \quad a_p \to +\infty,$$

$$\alpha_n^+ \to \mu_n \quad \forall n \in \mathbb{N}_p \quad \text{as} \quad a_p \to 0. \quad (2.31)$$
Proof. The relations (1.17-1.19) follow from Lemma 2.4 if we take \( v = (0, \ldots, 0, a_p) \). Note that the fundamental solution \( \varphi_p(\lambda) \) does not depend on \( a_p \), while \( \varphi_{p+1} = \frac{\varphi_p}{a_p} \), where \( \varphi_{p+1} \) is the corresponding fundamental solution for \( a_p = 1 \). We will consider the eigenvalues \( \alpha_n^+ \), the proof for \( \alpha_n^- \) is similar.

i) Let \( a_p \to +\infty \). It follows from (2.26) that the second term of the polynomial \( \varphi_{p+1}(\lambda) \) is equal to \( -\frac{B_p}{A_p} \lambda^{p-1} \), and the first term of \( -a_p\varphi_p(\lambda) \) is equal to \( -\frac{a_p^2}{A_p} \lambda^{p-1} \). The sum of these terms is the second term of \( u^+(\lambda) \), and after dividing it by the leading term of \( u^+(\lambda) \), which is equal to the leading term of \( \varphi_{p+1}(\lambda) \), its absolute value tends to infinity as \( a_p \) tends to infinity. Then the absolute value of the sum \( \sum_m \alpha_j^+ \) of its roots also tends to infinity. Since all roots of \( u(\lambda) \), except for \( \alpha_p^+ \), are bounded, we have \( \alpha_p^+ \to +\infty \).

ii) Let \( a_p \to 0 \). We can see from (2.29) that the matrices \( J^\pm \) are obtained from the matrix \( J_p \) through the one-dimensional perturbation. When \( a_p \) tends to 0, this perturbation also tends to 0. Thus, the eigenvalues \( \alpha_n^\pm \) of these matrices tend to those of \( J \). ■

In order to prove the main theorems, we need the following property of the phase shift \( \xi(z) \).

Lemma 2.6. Let the operator \( J \) be defined by (1.1) with a perturbation \( q \in X_{\epsilon}^p \), where \( k \geq 1 \), and let the function \( \xi(z) \) be its phase shift. Then the relations (1.26-1.27) hold true.

Proof. We use arguments from [2], where similar relations are proved in for discrete Schrodinger operator. We know that \( \psi_0 \) has \( N \) simple zeros in \( \mathbb{D} \), all of them on the real line, and possibly a simple zero in \( \pm 1 \). Applying the argument principle to \( \psi_0(z) \) along the unit circle, we see that the total variation of \( \arg \psi_0 \) along the unit circle in the counterclockwise direction is given by

\[
\text{var} \arg \psi_0 = 2\pi \left( N + \frac{m_+ + m_-}{2} \right). \tag{2.32}
\]

Since for \( |z| = 1 \) we have \( \xi(\frac{1}{z}) = \xi(\bar{z}) = -\xi(z) \), half of this change is obtained as \( z \) moves from 1 to \(-1 \). Therefore, the total variation of \( \arg \psi_0 \) along the unit circle from 1 to \(-1 \) in the counterclockwise direction is given by

\[
\text{var}_+ \arg \psi_0 = \pi \left( N + \frac{m_+ + m_-}{2} \right). \tag{2.33}
\]

One can see that \( \psi_0(1) \) is real, and the sign of it depends on the amount of zeros in \((0,1] \). Therefore, we have \( \xi(1 + 0i) = -\pi n_+ + \frac{\pi m_+}{2} \). Adding the change \((2.32 - 2.33) \), we obtain the relations (1.26-1.27). ■

3. PROOF OF MAIN THEOREMS

Proof of Theorem 1.1. It follows from (1.23) that if \( \alpha_p^+ < 2 \), then \( \lambda_{n_+} < \alpha_p^+ \), and there are no eigenvalues in \([2, +\infty) \). If \( \alpha_p^+ = 2 \), then there is a virtual state \( z = 1 \), but still no positive eigenvalues. We fix a perturbation \( q_0(a_0, b_0) \) from \( X_{2p} \) or \( X_{2p-1} \), such that \( \alpha_p^+(q_0) = 2 \). Define the perturbation \( q_\varepsilon = q_\varepsilon(a_\varepsilon, b_\varepsilon) \), \( \varepsilon \in \mathbb{R} \) by

\[
(q_\varepsilon)_x = 0, \quad x > 2p, \quad (a_\varepsilon, b_\varepsilon)_x = (a_x, b_x + \varepsilon), \quad x \leq p. \tag{3.1}
\]

We sometimes write \( \psi(z, \varepsilon), \alpha_n^\varepsilon \) etc. instead of \( \psi(z), \alpha_n \), etc. when several perturbations \( q_\varepsilon \) are being dealt with. The operator corresponding to the perturbation \( q_\varepsilon \) we denote by \( J_\varepsilon \). Note that the eigenvalues \( \alpha_n^\varepsilon \) satisfy

\[
\alpha_n^+ (\varepsilon) = \alpha_n^+ (0) + \varepsilon, \quad \alpha_n^- (\varepsilon) = \alpha_n^- (0) + \varepsilon. \tag{3.2}
\]
Due to (1.15) the Jost function for $J(\varepsilon)$ for $\lambda = z + \frac{1}{z}$, $\lambda \in (0, +\infty)$ has the form

$$\psi(z, \varepsilon) = z^p \varphi_{p+1}(\lambda, \varepsilon) - a_p z^{p+1} \varphi_p(\lambda, \varepsilon) = z^p \varphi_{p+1}(\lambda - \varepsilon, 0) - a_p z^{p+1} \varphi_p(\lambda - \varepsilon, 0). \quad (3.3)$$

The function $\psi(z, \varepsilon)$ is entire in $z, \varepsilon$ and satisfies

$$\psi(1, 0) = 0, \quad \psi(1, 0) \neq 0. \quad (3.4)$$

Then due to Implicit Function Theorem there exists a function $z(\varepsilon)$, analytic in small disk $\{|\varepsilon| < \varepsilon\}$ such that $\psi(z(\varepsilon), \varepsilon) = 0$ in the disc $\{|\varepsilon| < \varepsilon\}$. Here $\lambda_{n+}(z) = z(\varepsilon) + \frac{1}{z(\varepsilon)}$ for $\varepsilon > 0$.

We have $z(\varepsilon) = 1 + z_1 \varepsilon + O(\varepsilon^2)$. Then from (3.3) we obtain for $z = z(\varepsilon)$ as $\varepsilon \to 0$:

$$\varphi_{p+1}(\lambda - \varepsilon) = a_p z(\varepsilon) \varphi_p(\lambda - \varepsilon) = a_p(\varphi_p(2) + \varepsilon(z_1 \varphi_p(2) - \dot{\varphi}_p(2))) + O(\varepsilon^2), \quad (3.5)$$

which yields $z_1 = -\frac{\dot{\varphi}_{p+1}(2) - a_p \dot{\varphi}_p(2)}{a_p \varphi_p(2)} < 0$, where $\dot{u} = \frac{du}{d\varepsilon} u$. If $\varepsilon > 0$ is small enough, then (3.2) yields $\alpha_+^p(\varepsilon) = \varepsilon$ and $\alpha_+^p(\varepsilon) = \alpha_-^p(q_0) + \varepsilon < 2$. Thus we obtain $\alpha_+^p(\varepsilon) < 2 < \lambda_{n+} < \alpha_+^p(\varepsilon)$. If $\varepsilon$ is increasing, then all eigenvalues $\alpha_-^p(\varepsilon) < 2 < \lambda_{n+} < \alpha_+^p(\varepsilon)$ move monotonically to the right.

We define a matrix $J^1 \in \mathfrak{X}_{k-1}$ which is be obtained from $J$ by deleting the first row and the first column. One can check that the Jost function for $J^1$ is equal to $\frac{\psi_1(z)}{z}$. It is expressed in terms of the fundamental solutions by

$$\frac{\psi_1(z)}{z} = a_p z^p \partial_p(\lambda(z)) - z^{p-1} \partial_{p+1}(\lambda(z)). \quad (3.7)$$

Denote the eigenvalues of the matrices $(J^1)^\pm$ as $(\gamma^\pm)_1^{[k]}$ and denote the eigenvalues of $J^1$ as $\omega_j$.

At $\varepsilon_1 = 2 - \gamma_{p-1}^+(0)$ we have $\gamma_{p-1}^+(\varepsilon) = 2$.

Note that if $\gamma_{p-1}^+ \leq 2$, then from (3.7) we deduce that the operator $J^1$ does not have eigenvalues in $(2, +\infty)$. Let $\varepsilon \in \varepsilon_1 + y$, where $\varepsilon_1 = 2 - \gamma_{p-1}^+(0)$. Then due to 3.7 we obtain for $\lambda = z + \frac{1}{z}$, $\lambda \in (0, +\infty)$:

$$F(z, \varepsilon) = \frac{\psi_1(z, \varepsilon)}{z} = a_p z^p \partial_p(\lambda - \varepsilon, \varepsilon_1) - z^{p-1} \partial_{p+1}(\lambda - \varepsilon, \varepsilon_1). \quad (3.8)$$

The function $F(z, \varepsilon)$ is entire in $z, \varepsilon$ and satisfies

$$F(1, 0) = 0, \quad F_0(0, 0) \neq 0. \quad (3.9)$$

Due to Implicit Function Theorem there is a function $z_0(y)$, analytic in small disc $\{|y| < \varepsilon_0\}$ such that $F(z_0(y), y) = 0$ in the disc $\{|y| < \varepsilon_0\}$. Here we have $\omega_{n+}(\varepsilon) = z_0(y) + \frac{1}{z_0(y)}$, $y = \varepsilon - \varepsilon_1$.

We have $z_0(y) = 1 + y_1 + O(y^3)$ as $y \to 0$. Then from (3.8) we obtain for $z = z_0(y)$ as $y \to 0$:

$$\partial_{p+1}(\lambda - y) = a_p z_0(y) \partial_p(\lambda - y, \varepsilon_1) = a_p(\partial_p(2, \varepsilon_1) + y(1 \partial_p(2, \varepsilon_1) - \dot{\varphi}_p(2, \varepsilon_1))) + O(y^2), \quad (3.10)$$

$$\partial_{p+1}(\lambda - y) = \partial_{p+1}(2, \varepsilon_1) - y \dot{\varphi}_p(2, \varepsilon_1) \partial_{p+1}(2, \varepsilon_1), \quad (3.11)$$

which yields $z_1 = -\frac{\dot{\varphi}_{p+1}(2, \varepsilon_1) + a_p \dot{\varphi}_p(2, \varepsilon_1)}{a_p \varphi_p(2, \varepsilon_1)} < 0$.

If $y > 0$ is small enough, then $\gamma_{p-1}^+(\varepsilon) = 2 + y$ and $\gamma_{p-1}^-(\varepsilon) = \gamma_{p-1}^-((0 + 0 < 2$, since there is the basic relation (1.17). Thus we obtain

$$\gamma_{p-1}^-(\varepsilon) < 2 < \omega_{n+}(\varepsilon) < \gamma_{p-1}^+(\varepsilon), \quad \omega_{n+}(\varepsilon) < \lambda_{n+}(\varepsilon).$$

It is important that $\omega_{n+}(\varepsilon) < \lambda_{n+}(\varepsilon)$ for any $\varepsilon$, since $\sigma_d(J) \cap \sigma_d(J^1) = \emptyset$. If $\varepsilon$ is increasing, the eigenvalues $\gamma_{p-1}^-(\varepsilon) < \omega_{n+}(\varepsilon) < \gamma_{p-1}^+(\varepsilon)$ and $\omega_{n+}(\varepsilon) < \lambda_{n+}(\varepsilon)$ move monotonically to the
right. New eigenvalues do not appear until \( \varepsilon = \varepsilon_2, \alpha_{p-1}^+ (\varepsilon_2) = 0 \). We can repeat the above arguments for any \( q_0 \) and \( \varepsilon \) to get the other inequalities. The proof for the interval \((-\infty, -2)\) is similar. Thus, we obtain (1.22). The estimate (1.23) follows from Lemma 2.2. \( \Box \)

In order to prove Proposition 1.2 we need the following Hochstadt result from [17].

**Theorem 3.1.** Let \( J_p \) be a finite Jacobi matrix on \( \mathbb{C}^p \) defined by (1.3). Let \((\mu_j)^{p-1} \) and \((\tau_j)^{p-1} \) be its mixed and Dirichlet eigenvalues. Then the mapping \( \mu \star \tau : \mathbb{R}^{p-1} \times \mathbb{R}^p \to \mathcal{E}_{2p-1} \) given by \( q = (a, b) \mapsto \mu \star \tau \) is a real analytic isomorphism between \( \mathbb{R}^{p-1} \times \mathbb{R}^p \) and \( \mathcal{E}_{2p-1} \).

Moreover, there exists an algorithm to recover the Jacobi matrix from its eigenvalues.

**Remark.** Analyticity of the mapping was proved by Simon and Geszttesy [15]. The recovery algorithm is given below, in the proof of Proposition 1.2.

**Proof of Proposition 1.2.** It follows from Lemma 2.3 that the numbers \((\alpha_j^\pm)_{p}^1\) are the roots of the polynomials \( u^-(\lambda) \) and that we have

\[
D_p = \frac{D^+ + D^-}{2}, \quad D_{p,1} = \frac{D^+ - D^-}{2a_p^2}.
\]

(3.12)

Also note that \( J_{p,1} = J_1^\pm \). Now we determine the coefficients of \( J \). Using the trace formula, we obtain

\[
\text{Tr} J^+ - \text{Tr} J^- = \Sigma_j^+ b_j - \Sigma_j^- a_j = (\Sigma_j^+ b_j + a_j^2) - (\Sigma_j^- b_j - a_j^2) = 2a_p^2.
\]

(3.13)

Since the coefficient \( a_p \) is positive, it is determined. It follows from the trace formulas for the matrices \( J_p \) and \( J_{p,1} \) that the second term of \( \chi(\lambda) \) equals \(-\Sigma_j^- b_j\), while the one of \( \chi_1(\lambda) \) equals \(-\Sigma_j^+ b_j\). Taking the difference, we obtain \( b_p \). All the other coefficients are the same for the matrices \( J_p \) and \( J^+ \). Thus, we can find the coefficients of the matrix \( J^+ \) and therefore find the matrix \( J_p \) too. We use a standart algorithm from [17]. Let the last \( k \) rows of the matrix \( J^+ \) be already found. Denote the standart basis in \( \mathbb{C} \) as \((\delta_j)_1^p\). From [17] we obtain that

\[
((J^+)^k \delta_p, \delta_p) = -\sum_{j=1}^{p} (a_j^+)_{\delta_j}^k D_{p,1}(a_j^+) / (D^+) (a_j^+), \quad k = 1, \ldots, p.
\]

(3.14)

On the other hand, direct calculations show that

\[
((J^+)^{2k+1} \delta_p, \delta_p) = d_1 b_{p-k} + d_2 a_{p-k} + \ldots + d_{2k} a_{p-1} + d_{2k-1} b_p,
\]

(3.15)

where the coefficients \( d_i \) depend only on \( a_{p-k}, b_{p-k+1}, \ldots, a_{p-1}, b_p \) and therefore are known, and \( d_1 \neq 0 \). Thus, the right parts of (3.14) and (3.15) are equal, and everything except \( b_{p-k} \) is known, which allows us to determine \( b_{p-k} \) from the obtained equation. After that we can consider \(((J^+)^{2k+2} \delta_p, \delta_p) \) and use a similar principle to find \( a_{p-k-1} \), so now the last \( k + 1 \) rows are known. Proceeding like this, we can find all the elements of the matrix \( J^+ \) and, since we already found \( a_p \) and \( b_p \), of the matrix \( J \). The continuity of the corresponding mapping follows from the results of [17]. \( \Box \)

In order to prove Theorem 1.3 we need to determine the derivative of the phase shift \( \xi(z) \).

**Lemma 3.2.** Let the operator \( J \) be defined by (1.1) with a perturbation \( q \in \mathfrak{X}_k^+ \), where \( k \in \{2p - 1, 2p\} \). Then for \( \lambda \in (-\infty, 2) \) its phase function \( \xi(z) \) satisfies

\[
\sqrt{1 - \frac{\lambda^2}{4} (\xi(z(\lambda)))^2} = \sum_{r_j \in \mathbb{R}} \frac{r_j \text{Re} z - 1}{|z - r_j|^p} + \sum_{r_j \in \mathbb{C}} \frac{\text{Re} r_j \text{Re} z - 1}{|z - r_j|^p},
\]

(3.16)

where \( z(\lambda) = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - 1} \) and \((r_j)_1^k\) are the roots of the Jost function \( \psi_0(z) \).
Proof. Consider the S-matrix

\[ S(z) = \frac{\psi_0(z^{-1})}{\psi_0(z)} = e^{-2i\xi(z)}. \]  \hspace{1cm} (3.17)

where \( |z| = 1, \ \text{Im} \ z < 0. \) Taking the derivative of the both parts of (3.17), we obtain:

\[ -\frac{\psi'_0(\frac{1}{z})z'(\lambda)}{z^2\psi_0(z)} - \frac{\psi_0'(\frac{1}{z})\psi_0(z)z'(\lambda)}{\psi_0(z)} = -2ie^{-2i\xi(z)}\xi'(z)z'(\lambda), \]  \hspace{1cm} (3.18)

which yields

\[ \frac{\psi'_0(\frac{1}{z})}{z^2\psi_0(\frac{1}{z})} + \frac{\psi_0'(z)}{\psi_0(z)} = 2i\xi'(z). \]  \hspace{1cm} (3.19)

From Lemma 2.1 we have \( \psi_0(z) = C\Pi_{j=1}^k (z - r_j). \) Then

\[ \frac{\psi'_0(z)}{\psi_0(z)} = \sum_{j=1}^k \frac{1}{z - r_j} = \sum_{j=1}^k \frac{\bar{z} - \bar{r}_j}{|z - r_j|^2}, \]  \hspace{1cm} (3.20)

and

\[ \frac{\psi'_0(\frac{1}{z})}{z^2\psi_0(\frac{1}{z})} + \frac{\psi_0'(z)}{\psi_0(z)} = \sum_{j=1}^k \frac{\bar{z} - \bar{r}_j + \bar{z}^2(z - r_j)}{|z - r_j|^2} = \sum_{j=1}^k \frac{2\bar{z} - \bar{r}_j(1 + \bar{z}^2)}{|z - r_j|^2}. \]  \hspace{1cm} (3.21)

Recall that \( z'(\lambda) = -\frac{\bar{z}}{2\sqrt{4z^2 - 1}} \) (here we use the positive branch of the square root). Then for the full derivative we have

\[ \sqrt{1 - \frac{\lambda^2}{4}} \xi'(z)z'(\lambda) = -\sum_{j=1}^k \frac{z(2\bar{z} - \bar{r}_j(1 + \bar{z}^2))}{4|z - r_j|^2} = \sum_{j=1}^k \frac{\bar{r}_j \text{Re} \ z - 1}{2|z - r_j|^2}. \]  \hspace{1cm} (3.22)

Since \( \psi(z) \) is a polynomial with real coefficients, for every complex root \( r_j \) there is also a root \( \bar{r}_j. \) Taking the sum of the corresponding parts of (3.22) we obtain (3.16). \[ \square \]

Proof of Theorem 1.3. Substitution of (2.23, 2.24) into (2.25) gives

\[ (z^2 - 1)u^+(z(\lambda)) = (z - 1)(z^{p+1}\psi_0(\frac{1}{z}) + \frac{1}{z^p}\psi_0(z)), \ \lambda \in (-2, 2). \]  \hspace{1cm} (3.23)

Recall that the roots of \( u^+(\lambda(z)) \) are \( \alpha_j^+, \ j \in \mathbb{N}_p. \) For \( |z| = 1, \ \text{Im} \ z < 0 \) we rewrite \( (z + 1)u^+(\lambda(z)) = 0 \) by

\[ \psi_0(z)(z^{2p+1}S(z) + \bar{z}) = 0. \]  \hspace{1cm} (3.24)

Since we consider \( |z| = 1, \) we can express (3.24) in terms of arguments. This, we obtain that \( z(\alpha_j^+) \) are the roots of

\[ \frac{2\xi(z)}{2p + 1} + \text{arg} \ z = -\frac{(2\pi - 1)n}{2p + 1}, \ n \in \mathbb{N}_p. \]  \hspace{1cm} (3.25)

The proof for \( \alpha_j^- \) is similar. Let \( F(\lambda) = \frac{2\xi(z(\lambda + 0i))}{2p + 1} + \text{arg} \ z(\lambda + 0i). \) Using the result of Lemma 3.2 we obtain that

\[ \sqrt{1 - \frac{\lambda^2}{4}} z'(\lambda + 0i)F'(\lambda) = \frac{1}{2} + \frac{1}{2p + 1} \sum_{j=1}^k \frac{\bar{r}_j \text{Re} \ z - 1}{2|z - r_j|^2} + \frac{1}{4p + 2} \sum_{j=1}^k \frac{\bar{r}_j \text{Re} \ z - 1}{|z - r_j|^2 + 1}. \]
For real roots $r_j$ and $|z| = 1$ we have

$$\frac{\dot{r}_j \text{Re} z - 1}{|z - r_j|^2} + 1 = \frac{|z|^2 + |r_j|^2 - r_j \text{Re} z - 1}{|z - r_j|^2} = \frac{|r_j|^2 - r_j \text{Re} z}{|z - r_j|^2} > 0,$$ \hspace{1cm} (3.26)

since $|r_j| \geq 1$. The proof for non-real roots is similar. Therefore, $F'(\lambda) > 0$ for $\lambda \in (-2, 2)$, so the function $F(\lambda)$ is strongly increasing. We also have $F(-2) = -\pi$ and $F(2) = 0$, so all of the roots of $F'(\lambda) + \frac{2(2+1)n}{2n+1}$ for $n \in \mathbb{N}$ lie in $[-2, 2]$. That means that all $p$ roots of $u^+(z)$ are the roots of $F(\lambda) + \frac{2(n-1)n}{2n+1}$ for such $n$. The proof for $u^-(z)$ is similar. It suffices to use Proposition 1.2 and obtain a bijection.

In order to prove Theorem 1.4 we need a simple corollary of Theorem 1.1.

**Corollary 3.3.** Let $J$ be a Jacobi matrix given by (1.1) with a perturbation $q \in X_k$ and let $(s_j)$, $j \in n^*$, be the roots of its Jost function in $\mathbb{D}$. Let $(\mu_n)_{0}^{p-1}$ be eigenvalues of the matrix $J_p$.

i) If $\mu_{p-1} > 2$, then we have $s_1 < z(\mu_{p-1})$.

ii) If $\mu_0 < -2$, then we have $s_{-1} > z(\mu_0)$.

**Proof.** If $\mu_{p-1} > 2$, then (1.23) gives that $\lambda_1 > \mu_{p-1}$. Since the function $z(\lambda)$ decreases when $\lambda > 2$, we have $s_1 = z(\lambda_1) < z(\mu_{p-1})$, which proves i). Proof of ii) is similar. \[\square\]

**Proof of Theorem 1.4.** First of all, we need to show that for every operator $J$ with perturbation $q \in X_k$ we have $(r_j)^k_i \in K_k$. It follows directly from Lemma 2.2.

After that we prove the injection. Let the set $(r_n)^k_i$ be given. Then the polynomial $\psi_0(z)$, which roots are exactly $(r_n)^k_i$, counting multiplicities, is determined up to multiplication by a constant. We can use Lemma 2.1 to find $a_p$. Since we have (2.23,2.24), the functions $\varphi_{p+1}(z)$ and $\varphi_p(z)$ are also determined up to multiplication by a constant, which means that their roots are uniquely determined. It suffices to use the fact that these roots define the matrix $J$, uniquely, which follows from Theorem 3.1.

Now we prove the surjection. We start with the case when $|\psi_0(z)| > 0$ in $\mathbb{D}$. It follows from Theorem 1.3 that there is a matrix $J$ which Jost function is exactly $\psi_0(z)$. We can recover it using the algorithm from Theorem 1.3.

Now let us use induction by the pair of numbers $k$ and the amount $m$ of roots of $\psi(z)$ in $\mathbb{D}$. The case $k = 0$ is already proved, the case $k = 1$ is obvious. Fix $k$ and $m$ and assume that the surjection is proved for any $(\tilde{m}, \tilde{k}) < (m, k)$. Let $k = 2p$ be even, the proof for the case $k = 2p - 1$ is similar. Consider a function $\psi(z)$ with $m$ zeros in $\mathbb{D}$ and let $s^0$ be one of its zeros with the smallest absolute value. Consider the function $\psi_0^0(z) = \frac{\psi_0(z)}{z-s^0}$. It is easy to see that $\psi_0^0(z)$ also suffices R1-R3 and that it has one less root in $\mathbb{D}$. It means that there is a finite Jacobi matrix $J^0$ such that $\psi_0^0(z)$ is the Jost function of the corresponding matrix $J^0$. Denote the fundamental solutions for this matrix as $\varphi_{p+1}^0(\lambda)$ and $\varphi_p^0(\lambda)$. Substituting $\psi_0^0$ into (2.23), we obtain

$$\varphi_{p+1}^0(\lambda) = -s^0 \varphi_{p+1}^0(\lambda) + a_p \varphi_p^0(\lambda).$$ \hspace{1cm} (3.27)

Denote the roots of $\varphi_{p+1}^0(\lambda)$ as $\mu_n^0$. They are all real and simple. Then we have

$$\varphi_{p+1}(\mu_n^0) = a_p \varphi_p^0(\mu_n^0).$$ \hspace{1cm} (3.28)

Since the roots of $\varphi_{p+1}(\lambda)$ and $\varphi_p^0(\lambda)$ alternate, we have

$$\varphi_p^0(\mu_{n-1}^0) \varphi_p^0(\mu_n^0) < 0, \hspace{1cm} n = 0, 1, \ldots, p - 1,$$ \hspace{1cm} (3.29)
and, consequently,
\[ \varphi_{p+1}(\mu_{n-1}^0)\varphi_{p+1}(\mu_n^0) < 0, \quad n = 0, 1, \ldots, p-1. \]  
(3.30)

This means that there is a root of \( \varphi_{p+1} \) between any two roots of \( \varphi_p^o \). Since the sign of \( \varphi_{p+1}(\lambda) \) at \( \lambda \to -\infty \) is similar to the one of \( \varphi_{p+1}^o(\lambda) \) and different from the sign of \( \varphi_p^o(\mu_n^0) \), the last zero of \( \varphi_{p+1} \) has to be smaller that \( \mu_n^0 \). Thus, all zeros of \( \varphi_{p+1} \) are real and simple. Similar arguments can be applied to obtain that
\[ a_p\varphi_p(\lambda) = a_p(\lambda - s^o)\varphi_p^o(\lambda) - \varphi_{p+1}(\lambda). \]  
(3.31)

and that all roots of \( \varphi_p(\lambda) \) are real and simple. It suffices to show that the roots of \( \varphi_{p+1}(\lambda) \) and \( \varphi_p(\lambda) \) alternate. Let \( \mu \) be a root of \( \varphi_{p+1}(\lambda) \). Then we have
\[ a_p\varphi_p^o(\mu) = s^o\varphi_{p+1}^o(\mu). \]  
(3.32)

Substituting (3.32) into (3.31), we obtain that
\[ \varphi_p(\mu) = \varphi_{p+1}^o(\mu)(s^o\mu - (s^o)^2 - 1). \]  
(3.33)

It follows from Lemma 3.3 that \( \mu_{p-1}^o < s^o + \frac{1}{s^o} \), and we have already proved that \( \mu_{p-1}^o > \mu_{p-1} \). Then \( \mu_{p-1} < s^o + \frac{1}{s^o} \), so the sign of the second factor in (3.33) is the same for all eigenvalues \( \mu \). Then we have
\[ \text{sign } \varphi_p(\mu) = \text{sign } \varphi_{p+1}^o(\mu). \]  
(3.34)

We already know that the roots of \( \varphi_{p+1} \) and \( \varphi_p^o \) alternate, so (3.34) implies that the roots of \( \varphi_p \) and \( \varphi_{p+1} \) also alternate. Thus, it follows from Theorem 3.1 that there is a finite Jacobi matrix \( J_p \) such that \( \varphi_p \) and \( \varphi_{p+1} \) are its fundamental solutions. We can use Lemma 2.1 to find \( a_p \). Adding the element \( a_p \) to the matrix \( J_p \), we obtain the matrix \( J \), such that \( (r_n^o)_1 \) is the sequence of roots of its Jost function. Therefore, the surjection is proved.

Therefore, the recovery algorithm looks like follows:
1. Determine \( a_p \) using Lemma 2.1.
2. Using the sequence \( (r_n^o)_1 \), find the function \( \psi_0(z) \) up to multiplication by a constant.
3. Determine the functions \( \varphi_p(z) \) and \( \varphi_{p+1}(z) \) up to multiplication by a constant using the formulas (2.23) (2.24).
4. Calculate the eigenvalues \( (\mu_j^o)_{0}^{p-1} \) and \( (\tau_j^o)_{0}^{p-1} \).
5. Recover the matrix \( J_p \) using Theorem 3.1.

Proof of Corollary 1.5

We have \[ \frac{\psi_0(q_1, w_j^{-1})}{\psi_0(q_1, w_j)} = \frac{\psi_0(q_2, w_j^{-1})}{\psi_0(q_2, w_j)}. \] Therefore, we can obtain that
\[ w_j^k \psi_0(q_1, w_j^{-1})\psi_0(q_2, w_j) = w_j^k \psi_0(q_2, w_j^{-1})\psi_0(q_1, w_j). \]  
(3.35)

Note that these equalities also hold true if we swap \( w_j \) and \( w_j^{-1} \). Consider the functions \( G_1(z) = z^k\psi_0(q_1, z^{-1})\psi_0(q_2, z) \) and \( G_2(z) = z^k\psi_0(q_2, z^{-1})\psi_0(q_1, z) \). These are polynomials of degree \( 2k \) which coincide at \( 2k + 1 \) points (including 1). Therefore, these polynomials are equal. Consider the identity
\[ z^k\psi_0(q_1, z^{-1})\psi_0(q_2, z) = z^k\psi_0(q_2, z^{-1})\psi_0(q_1, z). \]  
(3.36)

Let \( r \) be some zero of \( \psi_0(q_2, z) \). It is also a zero of the right part of (3.36). It follows from Lemma 2.2 that \( r \neq 0 \) and \( \psi_0(q_2, r^{-1}) \neq 0 \). Therefore, we have \( \psi_0(q_1, r) = 0 \). We can swap \( \psi_0(q_1, z) \) and \( \psi_0(q_2, z) \) and obtain that these polynomials have similar roots. It suffices to use Theorem 1.4 and obtain that \( J_1 = J_2 \).
4. Location of resonances

Here we state and proof the results about location of resonances. We give estimates on absolute value of resonances of $J$ depending on its coefficients and the eigenvalues of $J$ and $J^\pm$. We also consider the case when one of the resonances of $J$ is moved to infinity.

The first result of this section follows from Theorem 1.1 and Lemma 2.2 and describes the situation when all resonances of $J$ are real and are mostly of the same sign.

**Corollary 4.1.** i) Let $k = 2p$ and $a_p < 1$. Then

$$\{J \text{ has exactly } p \text{ resonances and all of them are positive}\} \iff \{\alpha_1^+ > 2\}.$$

ii) Let $k = 2p$ and $a_p > 1$, $\alpha_1^+ > 2$.

If $\alpha_1^- < -2$, then $J$ has exactly $p - 1$ resonances.

If $\alpha_1^- \geq -2$, then $J$ has exactly $p - 1$ positive resonances and $1$ negative one.

iii) Let $k = 2p - 1$. Then

$$\{J \text{ has exactly } p - 1 \text{ resonances and all of them are positive and } b_p < 0\} \iff \{\alpha_1^+ > 2\}.$$

**Remark.** Similar arguments can be used when $\alpha_p^− < -2$. Then the roots of the Jost function are negative.

**Proof.** i) Let $\alpha_1^+ > 2$. Then there are exactly $p$ roots $s_1, \ldots, s_p$ of $\psi_0$ in $(0, 1)$. If $a_p < 1$, then it follows from Lemma 2.1 that there are at least $p$ roots of $\psi_0$ in $(\frac{1}{a_p}, +\infty)$. Since degree of $\psi_0$ is equal to $2p$, there are no other roots.

Now, let $J$ have exactly $p$ resonances, all of them are positive. Then it follows from Theorem 1.1 that $J$ has exactly $p$ eigenvalues. If there is a negative eigenvalue, then it follows from Lemma 2.1 that there is at least one negative resonance. Thus, all $p$ eigenvalues are positive. Then we can use Theorem 1.1 to obtain that $\alpha_j^+ > 2$, $j \in \mathbb{N}_p$.

ii) Let $\alpha_1^+ > 2$ and $\alpha_1^- < -2$. Then it follows from Lemma 2.1 that there are at least $p - 1$ roots of $\psi_0$ in $(\frac{1}{a_p}, +\infty)$. Since $\psi_0(0) > 0$ and $\psi_0(z) \to -\infty$ as $z \to -\infty$, the last root has to be in $(-\infty, 0)$. If $\alpha_1^- < -2$, then it has to be an eigenvalue. Thus, there are $1$ negative eigenvalue, $p$ positive eigenvalues and $p - 1$ positive resonances. If $\alpha^- \geq -2$, then the last root is a negative resonance and there are $p$ positive eigenvalues, $p - 1$ positive resonances and $1$ negative resonance.

iii) If $\alpha_1^+ > 2$, then there are exactly $p$ roots $s_1, \ldots, s_p$ of $\psi_0$ in $(0, 1)$. If $b_p > 0$, then it follows from Lemma 2.1 that there are at least $p$ roots of $\psi_0$ in $(\frac{1}{s_p}, +\infty)$. Since degree of $\psi_0$ is equal to $2p - 1$, we get a contradiction. Thus, $b_p < 0$. Using the Lemma once again, we obtain the result.

Now, let $J$ have exactly $p - 1$ positive resonances and let $b_p < 0$. Then $J$ has exactly $p$ eigenvalues. If there is a negative eigenvalue, then it follows from Lemma 2.1 that there is at least one negative resonance. Thus, all eigenvalues are positive, so $\alpha_j^+ > 2$, $j \in \mathbb{N}_p$.

**Proof of Theorems 1.6 and 1.7.** We will consider the case $k = 2p$, the proof for the other case is similar. From Lemma 2.1 we know that the product of all roots of $\psi_0(z)$ is equal to $\frac{1}{1 - a_2}$. If $\psi_0(z)$ has no roots in the unit circle, then the theorem is proved, since the absolute value of all other roots is greater than $1$. Let there be some roots in the real circle. We already proved that all of them are real and simple.

Let $a_p < 1$. Consider the positive roots $0 < s_1 < \cdots < s_{n_p} < 1$ of $\psi_0(z)$. Lemma 2.2 shows us that there is at least one root of $\psi_0(z)$ in every interval $(\frac{1}{s_i}, \frac{1}{s_{i-1}})$ for $i \in \{2 : n_+\}$. In addition,
there is a root in the interval \((\frac{1}{s_1}, +\infty)\). The product of these roots with all \(s_i\) is greater than 1. Doing the same with negative roots, we obtain that the absolute value of the product of all roots of \(\psi_0(z)\) in the real line is greater than 1. From this we conclude that the absolute value of any resonance outside of the real line is smaller than \(\frac{1}{\sqrt{1-a_p^2}}\), as they always come in conjugate pairs. Similar arguments work for real resonances of multiplicity greater than 1. As for the simple real resonances, the greater of them is the one that is located in the interval \((\frac{1}{s_1}, +\infty)\). Therefore, the product of all other resonances is no less than \(\frac{1}{s_1\sqrt{1-a_p^2}}\), which gives us the desired estimate.

If \(a_p > 1\), then we can not guarantee that there is a real root of \(\psi_0(z)\) that is greater than \(\frac{1}{s_1}\), similar for negative roots. Therefore, we can only state that the product of the resonances outside of the real line is smaller than \(\frac{1}{s_{-1}s_1\sqrt{a_p^2-1}}\), which gives us the estimate for non-real roots and real roots of multiplicity greater than 1. Note that there may be no real roots, for example, in \((-1, 0)\). Then \(s_{-1}\) is replaced by 1. With simple real roots, there are two cases. Firstly, the greatest real root can lie in \((1, s_1^{-1})\). If it is greater than \(s_1^{-1}\), then there is another root like that, as it follows from lemma 2.2 that there is an even amount of resonances in \((s_1^{-1}, +\infty)\). Therefore, the greatest root has to be smaller than \(\frac{1}{s_{-1}\sqrt{a_p^2-1}}\).

**Theorem 4.2.** Let \(\psi_0\) be the Jost function for some operator \(J\) defined by \([1,1]\) with a perturbation \(q \in \mathfrak{X}_k\). Then all its negative resonances of multiplicity 1 belong to the interval \((-R_-, -1)\), where \(R_-\) is defined by:

- If \(k = 2p\) (i.e., \(a_p \neq 1\)), then
  \[
  R_- = \frac{1}{\beta_-|1-a_p^2|}, \quad b_- = \begin{cases} 
  \max\{\frac{|s_1|}{a_p^2-1}, s_1\}, & \text{if } a_p < 1 \\
  |s_1|, & \text{if } a_p > 1
  \end{cases} \tag{4.1}
  \]

- If \(k = 2p - 1\) (i.e., \(a_p = 1, b_p \neq 0\)), then
  \[
  R_- = \frac{1}{\beta_-|b_p|}, \quad \beta_- = \begin{cases} 
  \max\{\frac{|s_1|}{|b_p|}, 1\}, & \text{if } b_p < 0 \\
  |s_1s_{-1}|, & \text{if } b_p > 0
  \end{cases} \tag{4.2}
  \]

**Proof.** The proof is similar to the proof of Theorem 1.6.

**Proof of Corollary 1.8.** We start with considering the case \(k = 2p, a_p < 1\). If \(\psi_0(z)\) has exactly \(n_+\) positive eigenvalues \(s_1 < \cdots < s_{n_+}\), then there are at least \(n_+ - 1\) roots of \(\psi_0(z)\) in \((s_{n_+}, s_1^{-1})\), and there can be exactly \(n_+ - 1\) roots, one in each interval \((s_{j+1}^{-1}, s_j^{-1})\). There also has to be an odd number of roots in \((s_1^{-1}, +\infty)\), which gives us at least \(2n_+\) roots in \((0, +\infty)\). After considering the negative half-line we obtain at least \(2n_+\) roots on the real line, which means that we can have at most \(p - 2n_+\) pairs of conjugate complex roots left. If we add real roots instead, we can obtain the maximal multiplicity \(2p - 2n_+ + 1\) of the root in any one of the intervals \((s_{j+1}^{-1}, s_j^{-1})\).

Consider the case \(a_p > 1\) and let both \(n_+\) and \(n_-\) be greater than 0. Then we can have one less root on every half line (it follows from \((2.17)\)), which allows us to have \(p - 2n_+ + 1\) pairs of conjugate complex roots or a real root with multiplicity \(2p - 2n_+ + 3\). If \(n_+\) is equal to 0, then there is still at least one positive resonance, since the function \(\psi_0(z)\) has to change its sign between 0 and \(+\infty\). Therefore, the estimate is similar to the case when \(n_+ = 1\). Similar arguments are applied to \(n_-\), and therefore we obtain the estimate.
Now, consider the case \( k = 2p - 1 \). Let both \( n_+ \) be greater than 1. Similarly, it follows from Lemma 2.1 that if \( b_p > 0 \), then \( \psi_0(z) \) has at least \( 2n_+ \) roots in \((0, +\infty)\) and at least \( 2n_- - 1 \) roots in \((-\infty, 0)\), which leaves us with no more than \( p - 9t \) pairs of complex roots or with a real root with multiplicity \( 2p - 29t + 1 \). If \( n_- = 0 \), then there is a positive resonance, as in the case ii), so we can replace \( n_+ \) with 1. The case \( b_p < 0 \) gives similar result. ■

**Proof of Corollary 1.9**

i) Existence of the matrices \( \tilde{J} \) and \( J_0 \) follows from Theorem 1.4. Let \( k = 2p \) for some \( p > 0 \). Then it follows from (2.4) that

\[
\Pi^p_{2p} r_n = \frac{1}{1 - a^2_p}, \quad \Pi^p_{2p} r_n = \frac{1}{1 - (a^o_{p-1})^2}.
\]

(4.3)

Then for \(|a| \to +\infty\) we have

\[
\tilde{a}_p^2 = 1 - \frac{|r|^2}{\Pi^p_{2p} r_n \left| P^p_{2p} r \right|^2} = 1 - \frac{1 - (a^o_{p-1})^2}{|a|} \to 1.
\]

(4.4)

Therefore, we have

\[
\tilde{a}_p = \left(1 - \frac{1 - (a^o_{p-1})^2}{|a|}\right)^{1/2} = 1 - \frac{1 - (a^o_{p-1})^2}{2|a|} + o\left(\frac{1}{|a|}\right).
\]

(4.5)

Similarly, for \( k = 2p - 1 \) we obtain

\[
\tilde{b}_p = \frac{|r|^2}{\Pi^{p-1}_{2p} r_n \left| P^{p-1}_{2p} r \right|^2} = \frac{b^p_{p-1}}{|a|} \to 0.
\]

(4.6)

Let \( \tilde{\phi}(z) \) be the Jost function of the operator \( \tilde{J} \) and let \( \phi_0(z) \) be the Jost function for \( J_0 \). Then we have

\[
\tilde{\phi}(z) = \tilde{C} \psi_0(z) \frac{(z - r_*)(z - s_*)}{(z - r)(z - s)}, \quad \phi_0(z) = \frac{C_o \psi_0(z)}{(z - r_*)(z - s_*)}
\]

(4.7)

for some \( \tilde{C}, C_o \neq 0 \). Substituting (4.7) into (2.23), we obtain

\[
\varphi_{p+1}(\tilde{q}, z)(z^2 - 1) = \tilde{C} \left( z^{p+2} \psi_0 \frac{(z - r_*)(z - s_*)}{(z - r)(z - s)} - \psi_0(z) \frac{(z - r_*)(z - s_*)}{z^p(z - r)(z - s)} \right),
\]

\[
\varphi_{p+1}(q, z)(z^2 - 1) = C_o \left( z^{p+2} \psi_0 \frac{(z - r_*)(z - s_*)}{(z - r)(z - s)} - \psi_0(z) \frac{(z - r_*)(z - s_*)}{z^p(z - r)(z - s)} \right).
\]

(4.8)

(4.9)

Recall that the functions \( \varphi_{p+1}(q_0, \lambda(z)) \) and \( \varphi_{p+1}(q_0, \lambda(z)) \) are both polynomials of degree \( p \). After dividing (4.8) by \(|a|\) we obtain that all coefficients of the polynomial

\[
\frac{\varphi_{p+1}(\tilde{q}, \lambda(z))}{C_o |a|} - \frac{\varphi_{p+1}(q_0, \lambda(z))}{C_o}
\]

(4.10)

tend to zero as \(|a|\) tends to infinity. Therefore, the mixed eigenvalues \((\hat{\mu}_j)^{p-1}_0\) of the matrix \( \tilde{J}_p \) tend to those of the matrix \((J_0)^{p-1}_0\). The proof for the Dirichlet eigenvalues \((\tilde{\tau}_j)^{p-1}_0\) is similar. Now we can use the continuity of the mapping \( q \to \mu \ast \tau \) to obtain that \( \tilde{q} \to q_0 \) as \(|a| \to \infty\).

ii) We will prove the existence of the operator \( \tilde{J} \), the rest of the proof is similar to the proof of the previous case. Let \( 1 < j < n_+ \). If we move the eigenvalue \( s_j \) out of the unit circle, we need to have an odd number of resonances in the interval \((s_j^{-1}, s_j^{-1})\) to satisfy R4. Since there was an odd number of resonances both in \((s_j^{-1}, s_j^{-1})\) and \((s_j^{-1}, s_j^{-1})\), then we have an even number in total. If we move one of these resonances out of the interval, this condition
is satisfied. As we cannot break other conditions, we need to move both points out of the interval \([s_{-1}^{-1}, s_{1}^{-1}]\), or to the complex plane, preserving the conjugacy. The proof for the case \(j = n_+\) is similar.

We can also formulate an analogue of Corollary 1.9 for real resonances.

**Corollary 4.3.** Let the operator \(J\) with \(q \in \mathfrak{X}_k\) be defined by (1.1) and let \((r_n)_1^k\) be the set of roots of its Jost function.

i) Let \(r\) be some resonance lying in the interval \((s_{-1}^{-1}, +\infty)\), where \(s_1\) is a smallest positive eigenvalues of \(J\), or let \(r = s_1\). Then \(r\) can be moved to any point \(\tilde{r} \in \mathbb{R} \setminus [s_{-1}^{-1}, s_{1}^{-1}]\), and there exists a matrix \(\tilde{J}\) with \(\tilde{q} \in \mathfrak{X}_k\) corresponding to the obtained set of roots. There also exists a matrix \(J_o\) with \(q_0 = (a^o, b^o) \in \mathfrak{X}_{k-1}\), which Jost function has the roots \(\{r_n\}_1^k \setminus \{r^o\}\).

ii) If \(k = 2p\) \((a_p \neq 1)\), then for \(|\tilde{r}| \rightarrow +\infty\) we have

\[
a_p = 1 - \frac{1 - (a_{p-1}^o)^2}{2\tilde{r}} + o\left(\frac{1}{\tilde{r}}\right).
\]

If \(k = 2p - 1\) \((a_p = 1, b_p \neq 0)\), then for \(\tilde{r} \rightarrow +\infty\) we have

\[
b_p = \frac{b_{p-1}}{\tilde{r}} \rightarrow 0.
\]

Moreover, in both cases \(\tilde{q} \rightarrow q_o\) as \(\tilde{r} \rightarrow \infty\). Similar results hold true for negative \(r\).

**Proof.** The proof is similar to the proof of Corollary 1.9 \(\blacksquare\)

5. Examples

**Example 1.** Consider the case \(k = a_1 = 1, b_1 \neq 0\). Then we have

\[
\psi_0 = -b_1 z + 1, \quad r_1 = 1/b_1.
\]  
(5.1)

If \(|b_1| < 1\), then \(r_1\) is a resonance. If \(|b_1| > 1\), then \(r_1\) is an eigenvalue. If \(|b_1| = 1\), then \(r_1\) is a virtual state.

**Example 2.** Consider the case \(k = 2, b_1 \neq 0, a_1 \neq 0\). Then we have

\[
\psi_0 = \frac{z^2 c_1 - b_1 z + 1}{a_1} = \frac{c_1}{a_1} (z - r_1)(z - r_2), \quad r_1 r_2 = \frac{1}{c_1}, \quad r_2 + r_1 = \frac{b_1}{c_1},
\]  
(5.2)

where

\[
r_{1,2} = \frac{b_1 \pm \sqrt{D}}{2c_1}, \quad D = b_1^2 - 4c_1, \quad c_1 = 1 - a_1^2.
\]  
(5.3)

There are three cases:

1. If \(D < 0\), then \(r_{1,2}\) are complex resonances.
   If \(r_1 \in \mathbb{C} + \mathbb{D}_1\), then \(r_2 = \overline{r}_1\) and \(c_1 = \frac{1}{|r_1|^2} < 1, \quad b_1 = \frac{2\text{Re}r_1}{|r_1|^2}\).

2. \(D = 0\), then \(r_1 = r_2 = \frac{b_1}{2c_1}\) is a real resonance of multiplicity 2.

3. If \(D > 0\), then \(r_{1,2}\) are real.
   Moreover, in the last case we obtain:
   1) If \(r_1, r_2 > 1\), then \(c_1 = \frac{1}{r_1 r_2} > 0, b_1 = \frac{r_1 + r_2}{c_1} > 0\).
   2) If \(r_1, r_2 < -1\), then \(c_1 = \frac{1}{r_1 r_2} > 0, b_1 = \frac{r_1 + r_2}{c_1} < 0\).
   3) If \(r_1 < -1, r_2 > 1\), then \(c_1 = \frac{1}{r_1 r_2} < 0, b_1 = \frac{r_1 + r_2}{c_1}\).
   4) If \(r_1 < -1, r_2 \in (0, 1)\), then \(c_1 < 0, b_1 > 0\).
   5) If \(r_1 \in (-1, 0), r_2 > 1\), then \(c_1 < 0, b_1 > 0\).
6) The cases $r_1, r_2 \in (-1, 0)$ or $r_1, r_2 \in (0, 1)$ are impossible, since we have Theorem 1.1.

**Example 3.** Consider a discrete Schrödinger operator $J_h = J_0 + hV_p$, where $h \neq 0$, $J_0$ is a discrete Laplacian on $\mathbb{N}$ and $V$ is a step potential given by

$$(Vf)_x = \begin{cases} f_x, & x \leq p \\ 0, & x > p \end{cases}.$$  

(5.4)

The matrix $J$ has a perturbation $q \in \mathfrak{X}_{2p-1}$, $p \in \mathbb{N}$. Direct calculations show that for this matrix we have $\varphi_n(\lambda) = U_{n-1}(\frac{\lambda-\lambda_1}{2})$ and

$$\psi_0(z) = z^p U_p(\frac{\lambda_z - h}{2}) - z^{p+1} U_{p-1}(\frac{\lambda_z - h}{2}),$$  

(5.5)

where $U_n(\lambda)$ are Chebyshev polynomials of the second kind. Then we obtain

$$\mu_0 = h - 2 \cos\left(\frac{\pi}{p+1}\right), \quad \mu_j = h - 2 \cos\left(\frac{j+1}{2} \pi \right), \quad j \in \mathbb{N}_{p-1},$$  

(5.6)

$$\alpha_j^+ = h - 2 \cos\left(\frac{2\pi j}{2p+1}\right), \quad \alpha_j^- = h - 2 \cos\left(\frac{-2\pi j}{2p+1}\right), \quad j \in \mathbb{N}_p,$$  

(5.7)

and $|h - \mu_n| \leq 2$, $n \in [0 : p-1]$. The bound states and resonances are located as follows.

1. If $h > 2 + 2 \cos\left(\frac{2\pi}{2p+1}\right)$, then the matrix $J$ has $p$ eigenvalues $(\lambda_n)_1^p$ and

$$2 < \lambda_1 < \lambda_1^- < \lambda_2 < \cdots < \lambda_p < \alpha_p^-.$$  

(5.8)

There are $p$ eigenvalues $(s_n)_1^p$, $s_n = z(\lambda_n)$. All $p - 1$ resonances $(r_n)_1^{p-1}$ are real and simple, and $r_i \in (s_{i+1}^{-1}, s_i^{-1})$. If $h$ tends to infinity, then it follows from (5.7) and (5.8) that

$$s_j = \frac{1 + o(1)}{h}, \quad j \in \mathbb{N}_p, \quad r_j = h + o(h), \quad j \in \mathbb{N}_{p-1}.$$  

(5.9)

Note that if we take $(Vf)_x = \epsilon_x f_x$, $x \leq p$, for some $\epsilon_x \in \mathbb{R}$, then $\frac{1}{h} J_h = \frac{1}{h} J_0 + V_p$, where $\frac{1}{h} J_0 \to 0$ as $h \to +\infty$. Therefore, in this case we have

$$s_j = \frac{1 + o(1)}{h \epsilon_j}, \quad j \in \mathbb{N}_p, \quad r_j = h \epsilon_j + o(h), \quad j \in \mathbb{N}_{p-1}.$$  

(5.10)

2. If $h$ decreases to $h = 2 + 2 \cos\left(\frac{2\pi}{2p+1}\right)$, then $s_p$ tends to 1. After that, $s_p$ turns into a new resonance. Direct calculations show that it moves to the right, while $r_{p-1}$ moves to the left, until they collide at the point $r^*$ at $h = h^*_p$ and become a resonance of multiplicity 2.

3. While $h^*_p > h > 2 + 2 \cos\left(\frac{4\pi}{2p+1}\right)$, the operator $J$ has $p - 1$ eigenvalues $(\lambda_n)_{1}^{p-1}$, and

$$2 < \lambda_{p-1} < \lambda_2 < \cdots < \lambda_1 < \alpha_p^+.$$  

(5.11)

There are $p - 1$ bound states $(s_n)_{1}^{p-1}$, $s_n = z(\lambda_n)$. There are $p - 3$ resonances $(r_n)_{1}^{p-3}$ that are still real and simple. It follows from Theorem 1.1 that $r_i \in (s_{i+1}^{-1}, s_i^{-1})$. Two other resonances $r_{p-2}$ and $r_{p-3} = r_{p-2}$ are located near $r^*$.

4. While $h$ is decreasing further, the largest bound state $s_{p-1}$ grows until becomes a resonance and then grows some more before it colliding with $r_{p-3}$ at $h = h^*_p$, generating two complex resonances. After that, the procedure similar to the one described in 2-3 repeats again with the bound state $s_{p-2}$ and so on, until there are no bound states left.

5. If $h$ tends to zero, then all resonances tend to infinity. In particular, we have

$$|r_j| = \frac{1}{2^{p-1} \sqrt{|4h|}} + o(1) \text{ as } h \to 0, \quad j \in \mathbb{N}_{2p-1}.$$  

(5.12)
Take \( \frac{\lambda(z)-h}{2} = \cos \theta \). Then the points \( r^*, h^* \), are defined by the following equations.

\[
r^* = \sin\frac{(p + 1)\theta}{\sin \theta},
\]

\[
r^* \sin \theta = \frac{(r^*)^2 - 1}{2}((p + 1) \cos(p + 1)\theta - p \cos \theta).
\]

Note that \( \psi_0(z) = 0 \) is equivalent to

\[
z = \frac{\sin(p + 1)\theta}{\sin \theta}.
\]

If we take the inverse of \(5.15\) and sum it with \(5.15\), we obtain

\[
\lambda(z) = 2 \cos \theta + h = \frac{\sin(p + 1)\theta}{\sin \theta} + \frac{\sin(p)\theta}{\sin(p + 1)\theta}.
\]

Therefore,

\[
h = \frac{\sin^2(p + 1)\theta + \sin^2(p)\theta - 2 \sin(p)\theta \sin(p + 1)\theta \cos \theta}{\sin(p + 1)\theta \sin(p)\theta} = \frac{\sin^2 \theta}{\sin(p + 1)\theta \sin(p)\theta}.
\]

If we take \( \cos \theta = \eta \), we obtain

\[
h = \frac{1 - \eta^2}{\eta - T_{2p+1}(\eta)}.
\]

This gives us \( 2p - 1 \) simple roots, \( m \) roots in the unit circle (the number \( m \) depends on the location of the numbers \( \alpha_n \), see Theorem 1.1), \( m - 1 \) roots on the real line outside of it and \( p - m \) pairs of complex roots. If \( \eta \) tends to 1, then \( \eta - T_{p+1}(\eta) \) tends to zero, and the fraction in the right part of \(5.18\) tends to some \( C \neq 0 \). Since \( 2p - 1 \) is an odd number, the situation when \( \eta \to -1 \) is similar. Thus, if \( h \) tends to zero, then we have \( |\eta| \to +\infty \). In particular, \( |\eta| = \frac{1}{2^{p-1} \sqrt{|4h|} + o(1)} \), which implies

\[
|r_n| = \frac{1}{2^{p-1} \sqrt{|4h|} + o(1)} \quad \text{as} \quad h \to 0.
\]

**Example 4.** Let again \( J_h = \alpha J_0 + hV \), where \( h \neq 0, \alpha > 0, \alpha \neq 1 \). This matrix has a perturbation \( q \in X_{2p-1} \). Then we have \( \tilde{\varphi}_n(\lambda) = U_{n-1}(\frac{\lambda-h}{2\alpha}) \), where \( U_n(\lambda) \), and

\[
\mu_0 = h - 2\alpha \cos\left(\frac{\pi}{p+1}\right), \quad \mu_j = h - 2\alpha \cos\left(\frac{\pi j+1}{p+1}\right), \quad j \in \mathbb{N}_{p-1},
\]

\[
\alpha_j^+ = h - 2\alpha \cos\left(\frac{2\pi j}{2p+1}\right), \quad \alpha_j^- = h - 2\alpha \cos\left(\frac{-\pi + 2\pi j}{2p+1}\right), \quad j \in \mathbb{N}_{p}.
\]

The dynamics of the eigenvalues when \( h \) is changed is similar to the previous example. The change of \( \alpha \) leads to the change of distances between \( \alpha^+_j, \mu_j \) and, therefore, between the eigenvalues of the matrix \( J \):

\[
|\lambda_{j+1} - \lambda_j| \to \infty \quad \text{as} \quad |\alpha| \to \infty, \quad j \in n_\bullet \setminus n_+,
\]

\[
|\lambda_{j+1} - \lambda_j| \to 0 \quad \text{as} \quad |\alpha| \to 0, \quad j \in n_\bullet \setminus n_+.
\]

**Example 5.** Consider a matrix \( J \) with a perturbation \( q(a,b) \in X_{2p} \), \( p \in \mathbb{N} \), such that \( b_n = 0, n \in \mathbb{N}_p \). Below we prove that its Jost function \( \psi_0 \) satisfies \( \psi_0(-z) = \psi_0(z) \), \( z \in \mathbb{C} \), which means that both eigenvalues and resonances of \( J \) come in pairs, symmetric with respect to zero.
In fact, we prove a stronger statement, namely, that for such operators we have
\[
\psi_{2x}(-z) = \psi_{2x}(z), \quad \psi_{2x+1}(-z) = -\psi_{2x+1}(z), \quad \forall (x, z) \in \mathbb{N} \times \mathbb{C}.
\] (5.24)

We prove it by induction. We know that for \( x > p \) we have \( \psi_x(z) = z^x \), which gives is the base. Now, let (5.24) be proved for \( x \geq n \). Then we can use (1.12) to calculate
\[
\psi_{2n-1}(-z) = \frac{(-z - \frac{1}{z})\psi_{2n}(-z) - a_p \psi_{2n+1}(-z)}{a_p - 1}.
\] (5.25)

Since \( \psi_{2n}(-z) = \psi_{2n}(z) \) and \( \psi_{2n+1}(z) = -\psi_{2n+1}(z) \), we obtain that \( \psi_{2n-1}(z) = -\psi_{2n-1}(z) \). The next step, which is to prove that \( \psi_{2n-2}(-z) = \psi_{2n-2}(z) \), is similar. Hence, (5.24) is proved for all \( x \in \mathbb{Z} \).

Conversely, let there be some \( j \) for which \( b_j \neq 0 \). Take the maximal \( j \) out of these numbers. Then it follows from the first formula in (2.3) that \( \psi_{p-j}(z) = \frac{z^{p-j}}{\lambda_{p-j}} \left( 1 - zb_j + O(z^2) \right) \) as \( \zeta \to 0 \). Therefore, \( \psi_{p-j}(z) \neq \pm \psi_{p-j}(z) \), so the inverse of this statement is also true.

**Example 6. The algorithm for Theorem 1.3.** The following example demonstrates the algorithm for Theorem 1.3 that was described in the proof of Proposition 1.2. Let the operator \( J \) have a perturbation \( q \in X_p^+ \). Then it follows from Theorem 1.3 that there is a bijection between \( X_p^+ \) and \( \mathcal{E}_{2p}(2) \) defined by (1.30). Namely, that we can recover the perturbation \( q \) from the sequence of numbers \( \omega = (\omega_n) \) which solve the equation \( \frac{2\pi(z(\lambda+0i))}{2p+1} + \arg(z(\lambda+0i)) = -\frac{\pi n}{2p+1}, \quad n \in \mathbb{N}_{2p} \). Consider the example of the recovering process in the case \( p = 2 \). Let \( \omega = (\omega_j)^4 \) be \( (-1, -\frac{1}{2}, \frac{1}{2}, 1) \). Then it follows from the formulas (3.12,3.13) that we have
\[
\alpha_1^- = -1, \quad \alpha_2^- = \frac{1}{2}, \quad \alpha_1^+ = -\frac{1}{2}, \quad \alpha_2^+ = 1,
\]
\[
a_2 = \left( \frac{\alpha_1^+ + \alpha_2^+ - (\alpha_1^+ + \alpha_2^+)}{2} \right)^2 = \frac{\sqrt{2}}{2},
\]
\[
\mathcal{D}^-(\lambda) = (\lambda + 1)(\lambda - \frac{1}{2}), \quad \mathcal{D}^+(\lambda) = (\lambda + \frac{1}{2})(\lambda - 1),
\]
\[
\mathcal{D}_{2,1}(\lambda) = \frac{\mathcal{D}^+(\lambda) - \mathcal{D}^-(\lambda)}{2a_2^2} = -\lambda, \quad (\mathcal{D}^+)’ = 2\lambda - \frac{1}{2}.
\]
Consider the matrix \( J^+ \). It follows from (3.14) that
\[
(J^+\delta_2, \delta_2) = -\frac{\alpha_1^+ \mathcal{D}_{2,1}(\alpha_2^+)}{(\mathcal{D}^+)’(\alpha_1^+)} - \frac{\alpha_2^+ \mathcal{D}_{2,1}(\alpha_2^+)}{(\mathcal{D}^+)’(\alpha_2^+)} = \frac{1}{2}.
\]
On the other hand, direct calculations show that
\[
(J^+\delta_2, \delta_2) = b_2 - a_2^2 = b_2 - \frac{1}{2}.
\]
Therefore, we have \( b_2 - \frac{1}{2} = \frac{1}{2} \) and \( b_2 = 1 \). Proceeding like this, we obtain that
\[
((J^+)^2\delta_2, \delta_2) = -\frac{(\alpha_1^+)^2 \mathcal{D}_{2,1}(\alpha_1^+)}{(\mathcal{D}^+)’(\alpha_1^+)} - \frac{(\alpha_2^+)^2 \mathcal{D}_{2,1}(\alpha_2^+)}{(\mathcal{D}^+)’(\alpha_2^+)} = \frac{3}{4},
\]
while direct calculations show that
\[
((J^+)^2\delta_2, \delta_2) = a_1^2 + (b_2 - a_2)^2 = a_1^2 + \frac{1}{4}.
\]
Since $a_1 > 0$, it is determined, and we have $a_1 = \frac{\sqrt{2}}{2}$. Finally, we have

$$((J^+)^3 \delta_2, \delta_2) = -\frac{(\alpha_1^+)^3 D_{2,1}(\alpha_1^+)}{(D^+)'(\alpha_1^+)} - \frac{(\alpha_2^+)^3 D_{2,1}(\alpha_2^+)}{(D^+)'(\alpha_2^+)} = \frac{5}{8},$$

while direct calculations show that

$$((J^+)^3 \delta_2, \delta_2) = b_1 a_1^2 + 2b_2 a_1^2 - 2a_1^2 a_2^2 + (b_2 - a_2)^3 = \frac{b_1}{2} + \frac{5}{8},$$

from which we can conclude that $b_1 = 0$. Therefore, all coefficients of $J$ are found.

**Remark.** One can see that it takes some time to recover the perturbation $q$ from the numbers $(\omega_j)_j^{2p}$ even in the case $p = 2$. However, such operations are easily performed using a computer.

**References**

[1] A. Aptekarev, E. Nikishin, *The scattering problem for a discrete Sturm-Liouville operator*, Sb. Math., **49**:2 (1984), 325–355.

[2] T. Aktosun, V. Papanicolaou, and A. Rivero, *Darboux transformation for the discrete Schrödinger equation*, Electronic Journal of Differential Equations, **112** (2019), 1–34.

[3] A. Badanin and E. Korotyaev, *Resonances of 4-th order differential operators*, Asymp. Anal. **111** (2019) 137–177.

[4] M. Bledsoe, *Stability of the inverse resonance problem for Jacobi operators*, Integral Equations Operator Theory **74** (2012), no. 4, 481–496.

[5] C. de Boor and G. H. Golub, *The numerically stable reconstruction of a Jacobi matrix from spectral data*, Linear Algebra Appl. **21** (1978), no. 3, 245–260.

[6] B. Brown, S. Naboko and R. Weikard, *The inverse resonance problem for Jacobi operators*, Bull. London Math. Soc. **37** (2005), 727–37.

[7] B. Brown, I. Knowles and R. Weikard, *On the inverse resonance problem*, J. London Math. Soc. **68** (2003), no. 2, 383–401.

[8] K. M. Case and S. C. Chiu, *The discrete version of the Marchenko equations in the inverse scattering problem*, J. Math. Phys. **14** (1973), 1643–1647.

[9] K. M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14** (1973), 594–603.

[10] T. Christiansen, *Resonances for step-like potentials: forward and inverse results*. Trans. Amer. Math. Soc. **358** (2006), no. 5, 2071–2089.

[11] N. Firsova, *Resonances of the perturbed Hill operator with exponentially decreasing extrinsic potential*. Mathematical Notes, **36** (1984), no 5, 854-861.

[12] R. Froese, *Asymptotic distribution of resonances in one dimension*. J. Diff. Eq. **137** (1997), no. 2, 251-272.

[13] R. Froese and I. Herbst, *Resonances in the one dimensional Stark effect in the limit of small field*. Schrödinger operators, spectral analysis and number theory, 133167, Springer Proc. Math. Stat., 348, Springer, Cham, 2021.

[14] G. S. Guseinov, *Scattering problem for the infinite Jacobi matrix*, Izv. Akad. Nauk Arm. SSR, Mat. **12** (1977), 365–379.

[15] F. Gesztesy and B. Simon, *m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, J. Anal. Math. **73** (1997), 267–297.

[16] M. Hitrik, *Bounds on scattering poles in one dimension*. Comm. Math. Phys. **208** (1999), no. 2, 381–411.

[17] H. Hochstadt, *On the construction of a Jacobi matrix from spectral data*, Linear Algebra Appl. **8** (1974), 435–446.

[18] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 2012.

[19] A. Iantchenko and E. Korotyaev, *Schrödinger operator on the zigzag half-nanotube in magnetic field*, Math. Model. Nat. Phenom. **5** (2010), no. 4, 175–197.
[20] A. Iantchenko and E. Korotyaev, *Periodic Jacobi operator with finitely supported perturbation on the half-lattice*, Inverse Problems 27 (2011), no. 11, 115003, 26 pp.

[21] A. Iantchenko and E. Korotyaev, *Resonances for periodic Jacobi operators with finitely supported perturbations*, J. Math. Anal. Appl. 388 (2012), no. 2, 1239–1253.

[22] A. Iantchenko and E. Korotyaev, *Periodic Jacobi operator with finitely supported perturbations: the inverse resonance problem*, J. Differential Equations 252 (2012), no. 3, 2823–2844.

[23] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Annals of Mathematics. Second Series 158 (2003), no. 1, 253–321.

[24] E. Korotyaev, *Inverse resonance scattering on the half line*, Asymptot. Anal. 37 (2004), no. 3-4, 215–226.

[25] E. Korotyaev, *Stability for inverse resonance problem*, Int. Math. Res. Not. 73 (2004), 3927–3936.

[26] E. Korotyaev, *Inverse resonance scattering on the real line*, Inverse Problems 21 (2005), no. 1, 325–341.

[27] E. Korotyaev, *Inverse resonance scattering for Jacobi operators*, Russ. J. Math. Phys. 18 (2011), 427–439.

[28] E. Korotyaev, *Resonance theory for perturbed Hill operator*, Asympt. Anal. 74 (2011), no. 3-4, 199–227.

[29] E. Korotyaev, *Estimates of 1D resonances in terms of potentials*, Journal d’Analyse Math. 130 (2016), 151–166.

[30] E. Korotyaev, *Resonances for 1d Stark operators*, Journal of Spectral Theory, 7 (2017), No 3, 699–732.

[31] E. Korotyaev, *Resonances of third order differential operators*, Journal of Math. Analysis and Appl., 478 (2019), No 1, 82–107.

[32] E. Korotyaev, *Eigenvalues of Schrödinger operators on finite and infinite intervals*, Math. Nachrichten, 294 (2021), no. 11, 2188–2199.

[33] E. Korotyaev and D. Mokeev, *Inverse resonance scattering for Dirac operators on the half-line*, Analysis and Mathematical Physics 11 (2021), no. 1, Paper No. 32, 26 pp.

[34] E. Korotyaev and K. Schmidt, *On the resonances and eigenvalues for a 1D half-crystal with localized impurity*, J. Reine Angew. Math. 2012, Issue 670, 217–248.

[35] R. Kozhan, *Finite range perturbations of finite gap Jacobi and CMV operators*, Advances in Mathematics 301 (2016), 204–226.

[36] M. Marletta, S. Naboko, R. Shterenberg, and R. Weikard, *On the inverse resonance problem for Jacobi operators – uniqueness and stability*, J. Anal. Math. 117 (2012), 221–248.

[37] M. Marletta, R. Shterenberg, and R. Weikard, *On the Inverse Resonance Problem for Schrödinger Operators*, Commun. Math. Phys., 295 (2010), 465–484.

[38] M. Marletta, R. Weikard, *Stability for the inverse resonance problem for a Jacobi operator with complex potential*, Inverse Problems 23 (2007), 1677–1688.

[39] B. Simon, *Resonances in one dimension and Fredholm determinants*, J. Funct. Anal. 178 (2000), no. 2, 396–420.

[40] M. Toda, *Theory of nonlinear lattices, 2nd ed.*, Springer Series in Solid-State Sciences, vol. 20, Springer-Verlag, Berlin, 1989.

[41] P. van Moerbeke, *The spectrum of Jacobi matrices*, Invent. Math. 37 (1976), no. 1, 45-81.

[42] M. Zworski, *A remark on isopolar potentials*, SIAM J. Math. Anal. 32 (2001), 1324–1326.

[43] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. 73 (1987), 277–296.

SAINT-PETERSBURG STATE UNIVERSITY, UNIVERSITETSKAYA NAB. 7/9, ST. PETERSBURG, 199034, RUSSIA AND HSE UNIVERSITY, 3A KANTEMIROVSKAYA ULITSA, ST. PETERSBURG, 194100, RUSSIA, KOROTYAEV@GMAIL.COM, E.KOROTYAEV@SPBU.RU, ELEONOVA@HSE.RU