Continuos particles in the Canonical Ensemble as an abstract polymer gas

Thiago Morais\textsuperscript{1,2} and Aldo Procacci\textsuperscript{1}

\textsuperscript{1} Departamento de Matemática UFMG 30161-970 - Belo Horizonte - MG Brazil
\textsuperscript{2} Departamento de Matemática UFOP 35400-000 - Ouro Preto - MG Brazil

December 11, 2013

Abstract

We revisit the expansion recently proposed by Pulvirenti and Tsagkarogiannis for a system of \( N \) continuous particles in the canonical ensemble. Under the sole assumption that the particles interact via a tempered and stable pair potential and are subjected to the usual free boundary conditions, we show the analyticity of the Helmholtz free energy at low densities and, using the Penrose tree graph identity, we establish a lower bound for the convergence radius which happens to be identical to the lower bound of the convergence radius of the virial series in the grand canonical ensemble established by Lebowitz and Penrose in 1964. We also show that the (Helmholtz) free energy can be written as a series in power of the density whose \( k \) order coefficient coincides, modulo terms \( o(N)/N \), with the \( k \)-order virial coefficient divided by \( k + 1 \), according to its expression in terms of the \( m \)-order (with \( m \leq k + 1 \)) simply connected cluster integrals first given by Mayer in 1942. We finally give an upper bound for the \( k \)-order virial coefficient which slightly improves, at high temperatures, the bound obtained by Lebowitz and Penrose.

1 Introduction

The rigorous approach to systems constituted by a large number of continuous classical interacting particles has been a deeply investigated subject during the last decades. In particular, the study of the low density phase, where the behavior of the system should be near to that of an ideal gas, has yielded some of the most impressive results in mathematical physics. It is remarkable to note that most of these results have been obtained between 1962 and 1968.

As far as a system (gas) of continuous particles is concerned, the Mayer series of the pressure (the pressure in powers of the fugacity) and the virial series of the pressure (the pressure in powers of the density) were known since the years around 1940. In particular, J. E. Mayer first gave the explicit expressions of the \( n \)-order Mayer series coefficient in term of a sum over connected graphs between \( n \) vertices of cluster integrals, and of the \( n \)-order virial series coefficient in term a sum over two-connected graphs between \( n + 1 \) vertices of irreducible cluster integrals (see. e.g. \cite{22} and references therein). However the question regarding the convergence of these series remained unanswered during the following two decades, since a “direct” upper bound of the type (Const.)\(^n\) on these \( n \) order coefficients, which would have guaranteed analyticity of these series, was generally considered rather prohibitive, due to the fact that the number of connected (or even two-connected) graph between \( n \) vertices is too large (i.e. order \( C^{n^2} \) with \( C > 1 \)).
The rigorous analysis of the Mayer series and the virial series of the pressure of the system of particles started to produce results with the work by Groeneweld [18] in 1962, who first gave a bound of the type (Const.)^n for the n-order Mayer coefficient, but under the assumption that the pair potential is non-negative. One year later, Penrose [28, 29] and independently Ruelle [38, 39] proved that the Mayer series of a system of continuous particles interacting via a stable and tempered pair potential (see ahead for the definition) is an analytic function for small values of the density, as well providing a lower bound for the convergence radius. One year later Lebowitz and Penrose [19] obtained a lower bound for the convergence radius of the virial series (the pressure in function of the density). These results were all obtained “indirectly”, (i.e. not by trying to bound directly the expressions of the coefficients as sum over connected graphs exploiting some cancelations) via the so called Kirkwood-Salszberg Equations (KSE), iterative relations between correlations functions of the system, and their possible use towards the control of the convergence of the Mayer series and virial series was glimpsed since the forties (see, e.g. [23, 15] and reference therein).

An alternative method to KSE was proposed in the same years by Penrose [30] who proved the convergence of the Mayer series of a system of particles interacting via a pair potential with a repulsive hard-core at short distance (but possibly attractive at large distance). To obtain this result he rewrote the sum over connected graphs of the n-order Mayer coefficient in terms of trees, by grouping together some terms, obtaining in this way the first example, as far as we know, of tree graph identity (TGI). It was only a decade later that Brydges and Federbush [5] were able to provide, for the second time, a proof of the analyticity of the Mayer series for the pressure of a continuous gas by directly bounding its n-term coefficients via a new type of TGI. Later, this direct approach based on TGI has been further developed and systematized by several authors (see e.g. [6], [20], [5], [1], [35], [34], [17], [42]). Method based on KSE and TGI are part of the so called Cluster Expansion (CE) method. TGI are nowadays much more popular than the old KSE tools, mainly because of their flexibility and adaptability, but it is worths to remark that, as far as continuous particle systems interacting via a stable and tempered pair potential are concerned, the bounds given in [38, 39, 28, 29, 19] obtained via KSE have never been beaten.

An other very popular tool in the framework of CE methods, which we need to mention here, is the so called abstract polymer gas (APG). The abstract polymer gas is basically a gas of subsets of some large set called polymers which posses a fugacity and interact via a hard core (non overlapping) potential. Such model is in fact an extremely general tool for investigating analyticity of thermodynamic functions of virtually any kind of lattice system and its study has also a very long history which remounts to the sixties. Indeed the Polymer gas, as a gas of subsets of the cubic lattice \( Z^d \), was originally proposed in the seminal papers by Gallavotti and Miracle Solé [12] (as a tool for the study of the Ising model at low temperature) and Gruber and Kunz [13] (as a model in its own right), where the analyticity of the pressure of such gas at low densities is proved via KSE methods. The same model was then studied also using TGI methods e.g. by Seiler [41] and Cammarota [7]. In 1986 Kotecky and Preiss [16] proposed a downright “abstract” polymer gas (polymers needed not to be subsets of an underlying set) and gave a proof of the convergence of the pressure not based on the usual KSE or TGI cluster expansion methods. They also provided criterion to estimate the convergence radius which improved those previously obtained via CE. The proof of the Kotecky-Preiss criterion was further simplified (and also slightly improved) by Dobrushin [8, 9] who reduced it to a simple inductive argument. The beautiful inductive approach for the polymer gas formulated by Dobrushin (which he called “no cluster expansion approach”) was then generalized and popularized by the work of Sokal [40] who also gave an extension of the APG convergence criterion for non-hard core repulsive pair interactions. Finally, the robustness of
Cluster Expansion has been recently revalued in [10] where the Dobrushin Kotecky-Preiss criterion has been improved via TGI arguments [10]. Moreover, always using TGI tools, further generalizations of the APG convergence criterion for non-hard core, non-repulsive pair interactions has been recently given in [32, 33], and [31].

Turning back to continuous particle systems, as mentioned above, all rigorous results about analyticity in the low density phase were obtained in the Grand Canonical Ensemble. Very recently Pulvirenti and Tsagkarogiannis [36] have obtained a proof of analyticity of the free energy of a system of continuous particles in the Canonical Ensemble. To get such a proof, they combined, within the cluster expansion methods, a standard Mayer expansion with the abstract polymer gas theory. In this work authors used the convergence criterion for the abstract polymer gas given in [2] and [25]. However, this is not the best criterion in the literature. In fact it is inferior not only to the recent Fernandez-Procacci criterion, but even to the Kotecky-Preiss criterion. Moreover, for technical reasons, in [36] the authors have used periodic boundary conditions (instead of the usual free boundary conditions), and consequently they had to assume some further condition on the pair potential beyond usual stability and temperness (see (2.3) in [36] and comments below).

In this note we revisit the calculations in the canonical ensemble proposed by Pulvirenti and Tsagkarogiannis, but under the sole assumption that particles interact via a tempered and stable pair potential and are subjected to the usual free boundary conditions. Using the Fernandez-Procacci criterion to check the convergence of the polymer expansion, we show the analyticity of the free energy at low densities in the canonical ensemble and establish a lower bound for the convergence radius which improves the bound given in [36] and is identical to the lower bound of the convergence radius of the virial series in the grand canonical ensemble established by Lebowitz and Penrose [19] in 1964. We also show that the (Helmholtz) free energy can be written as a series in power of the density, whose \( k \) order coefficient coincides, up to terms \( o(N)/N \), with the \( k \) order virial coefficient divided by \( k+1 \), according to its expression in terms of (simply) connected cluster integrals originally given by Mayer in 1942 (formula (49) in [21]). We finally give an upper bound for the \( k \)-order virial coefficient which slightly improves, at least at high temperatures, the bound obtained by Lebowitz and Penrose in [19].

## 2 Notations and results

Throughout the paper, if \( S \) is a set, then \( |S| \) denotes its cardinality. If \( n \) is an integer then we will denote shortly \([n] = \{1, 2, \ldots, n\}\).

### 2.1 Continuous particle system: Notations

We consider a system of \( N \) of classical, identical particles enclosed in a cubic box \( \Lambda \subset \mathbb{R}^d \) with volume \( V \) (and hence at fixed density \( \rho = N/V \)). We suppose \( N \) large (typically \( N \approx 10^{23} \)). We denote by \( x_i \in \mathbb{R}^d \) the position vector of the \( i^{th} \) particle and by \( |x_i| \) is modulus. We assume that there are no particles outside \( \Lambda \) (free boundary conditions) and that these \( N \) particles interact via a pair potential \( V(x_i - x_j) \), so that the configurational energy of the \( N \) particles in the positions \( (x_1, \ldots, x_N) \in \Lambda^N \) is given by

\[
U(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} V(x_i - x_j)
\]

We make the following assumptions on the pair potential \( V(x) \).
A. Stability: there exists $B \geq 0$ such that, for all $N \in \mathbb{N}$ and for all $(x_1, \ldots, x_N) \in \mathbb{R}^{dN}$,

$$
\sum_{1 \leq i < j \leq N} V(x_i - x_j) \geq -BN \quad (2.1)
$$

B. Temperness:

$$
C(\beta) = \int_{\mathbb{R}^d} |e^{-\beta V(x)} - 1| \, dx < +\infty \quad (2.2)
$$

The (configurational) partition function of this system in the canonical ensemble at fixed density $\rho = N/V$ and fixed inverse temperature $\beta \in \mathbb{R}^+$ is given by the following function

$$
Z_{\Lambda}(\beta, \rho) = \frac{1}{N!} \int_{\Lambda} dx_1 \ldots \int_{\Lambda} dx_N e^{-\beta \sum_{1 \leq i < j \leq N} V(x_i - x_j)}
$$

The thermodynamics of the system can be derived from the partition function $Z_{\Lambda}(\beta, \rho)$. In particular, the Helmholtz free energy per unit volume of the system is given by

$$
f(\beta, \rho) = \lim_{\Lambda, N \to \infty} \frac{1}{N/V} f_{\Lambda}(\beta, \rho) \quad (2.3)
$$

where $\Lambda \to \infty$ means that the size of the cubic box $\Lambda$ goes to infinity and

$$
f_{\Lambda}(\beta, \rho) = -\frac{1}{\beta V} \ln Z_{\Lambda}(\beta, \rho) \quad (2.4)
$$

We recall that the limit (2.3) is known to exists if the pair potential $V(x)$ satisfies stability and temperness (see e.g. [37]).

We will also make use in what follows of some notations concerning the system in the Grand Canonical Ensemble. We first recall the expression of the Grand Canonical Partition function of the system enclosed in the volume $\Lambda$, at fixed inverse temperature $\beta$ and fixed fugacity $\lambda$.

$$
\Xi_{\Lambda}(\beta, \lambda) = \sum_{N \geq 0} \frac{\lambda^N}{N!} \int_{\Lambda} dx_1 \ldots \int_{\Lambda} dx_N e^{-\beta \sum_{1 \leq i < j \leq N} V(x_i - x_j)}
$$

with the $N = 0$ term being equal to 1. The finite volume pressure of the system $P_{\Lambda}(\beta, \lambda)$ in the Grand Canonical ensemble is given by (see e.g. [19] or [37])

$$
\beta P_{\Lambda}(\beta, \lambda) = \frac{1}{V} \log \Xi_{\Lambda}(\beta, \lambda) = \sum_{n \geq 1} b_n(\beta, \lambda) \lambda^n \quad (2.5)
$$

where $b_1(\beta, \Lambda) = 1$ and, for $n \geq 2$,

$$
b_n(\beta, \Lambda) = \frac{1}{V n!} \int_{\Lambda} dx_1 \ldots \int_{\Lambda} dx_n \prod_{g \in G_n} \sum_{\{i,j\} \in E_g} e^{-\beta V(x_i - x_j)} - 1 \quad (2.6)
$$

where $G_n$ is the set of all connected graphs with vertex set $[n]$ and if $g \in G_n$ then its edge set is denoted by $E_g$. The r.h.s. of (2.5) is known as Mayer series and the term $b_n(\beta, \Lambda)$ is the $n$-order
Mayer coefficient (a.k.a. \(n\)-order connected cluster integral). We will need in the next sections an upper bound for \(|b_n(\beta, \Lambda)|\). This term has been bounded several times in the literature using both KSE or TGI methods. However, as far as we know, the bound obtained by Penrose nearly fifty years ago (see formula (6.12) in [28]) has never been beaten. The bound for \(|b_n(\beta, \Lambda)|\) in [28] is, uniformly in \(\Lambda\) and for all \(n \geq 2\), as follows

\[
|b_n(\beta, \Lambda)| \leq e^{2\beta B(n-2)} n^{n-2} \frac{[C(\beta)]^{n-1}}{n!}
\]

where

\[
C(\beta) = \int_{\mathbb{R}^d} |e^{-\beta V(x)} - 1| \, dx
\]

An easy computation (see again [19], [37]) shows that the finite volume density \(\rho = \rho_\Lambda(\beta, \lambda)\) of the system in the Grand Canonical Ensemble is given by

\[
\rho = \sum_{n \geq 1} n b_n(\beta, \Lambda) \lambda^n
\]

So that one can eliminate \(\lambda\) in (2.5) and (2.9) to obtain the so-called Virial expansion of the Pressure, i.e. the pressure in power of the density \(\rho = \rho_\Lambda(\beta, \lambda)\), in the Grand canonical Ensemble

\[
\beta P_\Lambda(\beta, \lambda) = \rho - \sum_{k \geq 1} \frac{k}{k+1} \beta_k(\beta, \Lambda) \rho^{k+1}
\]

where, as shown more than fifty years ago by Mayer (see e.g. [22] and reference therein)

\[
\beta_k(\beta, \Lambda) = \frac{1}{V \cdot k!} \int_{\Lambda} dx_1 \cdots \int_{\Lambda} dx_{k+1} \sum_{g \in G^*_k} \prod_{\{i,j\} \in E_g} [e^{-\beta V(x_i-x_j)} - 1]
\]

with \(G^*_k\) being the set of two-connected graphs with vertex set \([k+1]\). The term \(\beta_k(\beta, \Lambda)\) is also known in the literature as the irreducible cluster integral of order \(k\).

As remarked in the introduction, the Mayer series in the r. h. s. of (2.5) has been proved to converge absolutely, uniformly in \(\Lambda\) [28, 29, 38, 39], for any complex \(\lambda\) inside the disk

\[
|\lambda| < \frac{1}{e^{2\beta B + 1} C(\beta)}
\]

where \(B\) is the stability constant defined in (2.1) and \(C(\beta)\) is the function defined in (2.8). Moreover Lebowitz and Penrose [19] showed that the virial series in the r.h.s. of (2.10) converges for all complex \(\rho = \rho_\Lambda(\beta, \lambda)\), uniformly in \(\Lambda\), inside the disk

\[
|\rho| < g(e^{2\beta B}) \frac{1}{e^{2\beta B} C(\beta)}
\]

with

\[
g(u) = \max_{0 < w < 1} \left[ (1 + u)e^{-w} - 1 \right] w
\]

It is important to stress one again that the r.h.s. of (2.12), obtained in 1963, and the r.h.s. of (2.13), obtained one year later, still remain, as far as we know, the best lower bounds for the convergence radius of the Mayer series of the pressure and the convergence radius of the virial series of the pressure, respectively, of a system of continuous particles.
2.2 Results: a Theorem in the Canonical Ensemble

To state the results obtained in this paper (resumed in Theorem 1 below), let us introduce the following notations. Following [36], we put

\[
Z_\Lambda(\beta, \rho) = \frac{V^N}{N!} \tilde{Z}_\Lambda(\beta, \rho)
\]

where \(\tilde{Z}_\Lambda(\beta, \rho) = \int_\Lambda \cdots \int_\Lambda e^{-\beta \sum_{1 \leq i \leq j \leq N} V(x_i-x_j)}\) (2.15)

Let us define

\[
Q_\Lambda(\beta, \rho) = \frac{1}{V} \log \tilde{Z}_\Lambda(\beta, \rho)
\]

(2.16)

Then the finite-volume Helmholtz free energy \(f_\Lambda(\beta, \rho)\) defined in (2.4) can be written as

\[
f_\Lambda(\beta, \rho) = -\frac{1}{\beta} \left[ \frac{1}{V} \ln \left( \frac{V^N}{N!} \right) + Q_\Lambda(\beta, \rho) \right]
\]

(2.17)

where \(-\frac{1}{\betaV} \ln \frac{V^N}{N!}\) is the Helmholtz free energy of an ideal gas, and \(-\frac{1}{\beta}Q_\Lambda(\beta, \rho)\) is the part of the Helmholtz free energy due to the presence of the interaction \(V(x)\).

**Theorem 1** Let \(Q_\Lambda(\beta, \rho)\) be defined as in (2.16), then the following statements are true.

i) It holds that

\[
Q_\Lambda(\beta, \rho) = \sum_{k \geq 1} \frac{C_k(\beta, \Lambda)}{k+1} \rho^{k+1}
\]

(2.18)

where, for any fixed \(k\),

\[
\lim_{N \to \infty} C_k(\beta, \Lambda) = \sum_{n=1}^k (-1)^{n-1} \frac{(k-1+n)!}{k!} \sum_{\{m_2, \ldots, m_{k+1}\} \subset \mathbb{N}_0, \sum_{m_i=1}^{k+1} m_i=n, \sum_{i=2}^{k+1} (i-1)m_i=k} \prod_{i=2}^{k+1} \frac{[b_i(\beta, \Lambda)]^{m_i}}{m_i!}
\]

(2.19)

with the \(b_i(\beta, \Lambda)'s\) are the (simply connected) cluster integrals defined in (2.6).

ii) Let

\[
\rho_\beta^* = \frac{F(e^{2\beta B})}{e^{2\beta B}C(\beta)}
\]

(2.20)

where \(C(\beta)\) is the function defined in (2.8) and

\[
F(u) = \max_{a>0} \frac{\ln[1+u(1-e^{-a})]}{e^a[1+u(1-e^{-a})]}
\]

(2.21)

Then the series in the r.h.s. of (2.18) converges absolutely, uniformly in \(\Lambda\), in the (complex) disk \(|\rho| \leq \rho_\beta^*\).
iii) As soon as the density $\rho$ is such that $\rho \leq \rho_\beta^*$, the factors $c_k(\beta, \Lambda)$ in r.h.s. of (2.18) admit the bound, uniformly in $\Lambda$

$$|c_k(\beta, \Lambda)| \leq \left[\frac{1}{k+1} + (e^{a_\beta} - 1)e^{a_\beta k}\right] e^{2\beta B(k-1)(k+1)k} k! \left[C(\beta)\right]^k (2.22)$$

with $a_\beta^*$ being the unique value of $a \in (0, \infty)$ such that the function in the r.h.s. of (2.21) reaches its minimum value (i.e. reaches the value $\rho_\beta^*$).

**Remark 1.** The theorem above immediately implies that the infinite volume free energy $f(\beta, \rho)$ defined in (2.3) is also analytic in $\rho$ in the same disk $|\rho| \leq \rho_\beta^*$.

**Remark 2.** It is not difficult to check that $F(e^{2B\beta})$ is an increasing function of $\beta$ with $F(1) \approx 0.1448$ and $\lim_{\beta \to \infty} F(e^{2B\beta}) = e^{-1}$. Moreover $a_\beta^*$ is a decreasing function of $\beta$ with $a_{\beta=0}^* = 0.426...$ and $\lim_{\beta \to \infty} a_\beta^* = 0$. We can thus compare this result with the best lower bound for the convergence radius of the virial series in the grand canonical ensemble (the pressure as a function of $\rho$). Such bound was obtained by Lebowitz and Penrose in 1964. They found that the virial series is analytic in the open disk $|\rho| < R$ where (see formula (3.9) of [19])

$$R = g(e^{2B\beta}) e^{2B\beta C(\beta)} (2.23)$$

with

$$g(u) = \max_{0 < w < 1} \frac{[(1 + u)e^{-w} - 1]w}{u} (2.24)$$

It has been for us quite surprising to realize that $g(e^{2B\beta}) = F(e^{2B\beta})$, so that the lower bound (2.20) of the convergence radius of the Helmholtz free energy as a function of the density in the canonical ensemble and the lower bound (2.23) of the convergence radius of the virial series in the grand canonical ensemble given by Lebowitz and Penrose in [19] are in fact identical.

Indeed, first note that the function $G(w) = \frac{[(1 + u)e^{-w} - 1]w}{u}$ inside the max in r.h.s. of (2.24) is positive only in the interval $w \in (0, \ln(1 + u))$ and $G(0) = G(\ln(1 + u)) = 0$. So we can rewrite

$$g(u) = \max_{0 < w < \ln(1+u)} \frac{[(1 + u)e^{-w} - 1]w}{u} (2.25)$$

On the other hand, via the change of variables $w = \ln[1 + u(1 - e^{-a})]$ so that $e^{a} = u/(u + 1 - e^{u})$, the r.h.s. of (2.21) can be written

$$F(u) = \max_{a > 0} \frac{\ln[1 + u(1 - e^{-a})]}{e^{a} [1 + u(1 - e^{-a})]} = \max_{0 \leq w < \ln(1+u)} \frac{w}{e^{w}} \frac{(u + 1 - e^{u})}{u} = g(u) (2.26)$$

**Remark 3.** Formula (2.19) is also quite remarkable since it immediately implies that $c_k(\beta, \Lambda)$ coincides, modulo terms of order $o(N)$, with the $k$ order virial coefficient $\beta_k(\beta, \Lambda)$. Indeed, r.h.s. of (2.19) is, as was first shown by Mayer in 1942 [21], the representation of the two-connected cluster integral $\beta_k(\beta, \Lambda)$, defined in (2.11), in terms of the simply connected cluster integrals $b_i(\beta, \Lambda)$
\[ (i = 1, \ldots, k + 1), \text{ defined in (2.6), (see [21] formula 49, see also formula (29) p. 319 of [26]).} \]

Therefore we have, accordingly with [36], that

\[ C_k(\beta, \Lambda) = \left[ 1 + \frac{o(N)}{N} \right] \beta_k(\beta, \Lambda) \]  

(2.27)

**Remark 4.** We finally compare our bound for the virial coefficients (2.22) with that obtained by Lebowitz and Penrose. Their bound, as stated in formula (3.13) of [19] is

\[ k\beta_k(\beta, \Lambda) \leq \left[ \left( \frac{e^{2\beta B} + 1}{0.28952} \right)^k \right] \]

(2.28)

On the other hand bound (2.22) behaves asymptotically, taking for \( a^*_\beta \) its largest (and hence worst) value \( a^*_\beta = 0, 426 \), as

\[ \text{Const.} \left[ \left( \frac{e^{2\beta B} C(\beta)}{0.24026} \right)^k \right] \]

So our bound is asymptotically better than (2.28) for \( \beta \) small (i.e. high temperatures) and worst for \( \beta \) large (i.e. low temperatures).

### 3 Proof of Theorem 1

Following [36], we can easily rewrite \( \tilde{Z}_\Lambda(\beta, \rho) \) defined in (2.15) as a partition function of a hard core polymer gas. Indeed, using the Mayer trick we can rewrite the factor \( e^{-\beta \sum_{1 \leq i < j \leq N} V(x_i - x_j)} \) as

\[ e^{-\beta \sum_{1 \leq i < j \leq N} V(x_i - x_j)} = \prod_{1 \leq i < j \leq N} \left[ e^{-\beta V(x_i - x_j)} - 1 + 1 \right] = \]

\[ = \sum_{\{R_1, \ldots, R_r\} \in \pi_N} \xi(R_1) \cdots \xi(R_r) \]

where \( \pi_N = \text{set of all partitions of } [N] \equiv \{1, 2, \ldots, N\}, \) and

\[ \xi(R) = \begin{cases} 1 & \text{if } |R| = 1 \\ \sum_{g \in G_R, \{i,j\} \in E_g} \prod_{\{i,j\} \in E_g} \left[ e^{-\beta V(x_i - x_j)} - 1 \right] & \text{if } |R| \geq 2 \end{cases} \]

with \( G_R \) being the set of connected graphs with vertex set \( R \) and, given \( g \in G_R, \) \( E_g \) denotes the set of edges of \( g. \) Now define, for any \( R \subset [N] \) such that \( |R| \geq 2, \)

\[ \zeta_{|R|} = \int \Lambda \cdots \int \Lambda \prod_{i \in R} \frac{dx_i}{V} \xi(R) \]  

(3.1)

Note that \( \zeta_{|R|} \) depends only of the cardinality of the polymer \( R \) (the variables \( \{x_i\}_{i \in R} \) are mute variables). Observe also that

\[ \zeta_n = \frac{b_n(\beta, \Lambda)n!}{V^{n-1}} \]  

(3.2)
where, for any $n \geq 2$, $b_n(\beta, \Lambda)$ is the $n$ order coefficient of the Mayer series of the pressure defined in (2.6). It is now easy to check that

$$
\tilde{Z}_\Lambda(\beta, \rho) = \Xi_{[N]} = \sum_{n \geq 0} \sum_{\{R_1, \ldots, R_n\} \subset [N], |R_i| \geq 2, R_i \cap R_j = \emptyset} \zeta_{|R_1|} \cdots \zeta_{|R_n|}
$$

with the $n = 0$ term giving the factor 1. So the partition function of a continuous gas in the canonical ensemble is equal to the hard core polymer gas partition function $\Xi_{[N]}$ of a polymer gas in which the polymers are non overlapping subsets of the set $[N]$ with cardinality greater than one, and a polymer $R$ has activity $\zeta_{|R|}$ given by (3.1). Note that the activity $\zeta_{|R|}$ can be negative since, by (3.2), its signal is the same of the Mayer coefficient $b_{|R|}(\beta, \Lambda)$.

It is well known (see e.g. [7], [35]) that the logarithm of the hard core polymer gas partition function $\Xi_{[N]}$ can be written as

$$
\log \Xi_{[N]} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(R_1, \ldots, R_n) \in [N]^n} \phi^T(R_1, \ldots, R_n) \zeta_{|R_1|} \cdots \zeta_{|R_n|}
$$

with

$$
\phi^T(R_1, \ldots, R_n) = \begin{cases} 
1 & \text{if } n = 1 \\
\sum_{g \in G_n} (-1)^{|E_g|} & \text{if } n \geq 2
\end{cases}
$$

where $G(R_1, \ldots, R_n)$ is the graph with vertex set $[n]$ and edge set $E_G(R_1, \ldots, R_n) = \{\{i, j\} \subset [n] : R_i \cap R_j \neq \emptyset\}$, so $\sum_{g \in G_n}$ is the sum over all connected graphs with vertex set $[n]$ and $U(R, R')$ which are also subgraphs of $G(R_1, \ldots, R_n)$. In conclusion we can write the function $Q_\Lambda(\beta, \rho)$ defined (2.16) as

$$
Q_\Lambda(\beta, \rho) = \frac{1}{V} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(R_1, \ldots, R_n) \in [N]^n} \phi^T(R_1, \ldots, R_n) \zeta_{|R_1|} \cdots \zeta_{|R_n|}
$$

### 3.1 Convergence criterion: proof of Theorem 1, part ii

Once identity (3.6) has been established, or, in other words, once recognized that $Q_\Lambda(\beta, \rho)$ is the logarithm, divided by $V$ of the partition function of a subset polymer gas (according with the terminology used in [3]), we are in the position to prove part ii of Theorem 1. By the Fernandez-Procacci criterion [10], we have that $\log \Xi_{[N]}$ defined in (3.4) (and hence $Q_\Lambda(\beta, \rho)$) can be written as an absolutely convergent series for all complex activities $\zeta_{|R|}$ as soon as

$$
\sup_{i \in [N]} \sum_{R \subset [N], i \in R, |R| \geq 2} |\zeta_{|R|}| e^{a|R|} \leq e^a - 1
$$

A few line proof of this statement can also be found in [3] (see there Theorem 2.4). Now, since the sum the l.h.s. of (3.7) does not depend on $i \in [N]$ and using also the fact that the activity $\zeta_{|R|}$
depends only on the cardinality of the polymer $R$, we can rewrite the condition (3.7) as

$$\sum_{m=2}^{N} e^{am} C_m^\rho \leq e^a - 1 \quad (3.8)$$

where

$$C_m^\rho = |\zeta_m| \left( \frac{N - 1}{m - 1} \right) \quad (3.9)$$

An upper bound for the activity $\zeta_{|R|}$ of a polymer $R$ is now the key ingredient to implement the convergence criterion (3.7). Such a bound, recalling (3.2), follows immediately from the Penrose upper bound on the $n$-order Mayer coefficient given in (2.7). So we have

$$|\zeta_m| \leq V^{-m+1} e^{2\beta B(m-2)} m^{m-2} [C(\beta)]^{m-1} \quad (3.10)$$

and, recalling that $\rho = \frac{N}{V}$, we get that $C_m^\rho$ admits the following estimate

$$C_m^\rho \leq \rho^{m-1} e^{2\beta B(m-2)} m^{m-2} \frac{m^{m-2}}{(m-1)!} [C(\beta)]^{m-1} \quad (3.11)$$

where we have bound $(\frac{N - 1}{m - 1}) N^{m+1} \leq 1/(m - 1)!$.

Hence, the convergence condition (3.8) is true if

$$\sum_{m=2}^{N} [\rho e^a C(\beta) e^{2\beta B}]^{m-1} \frac{m^{m-2}}{(m-1)!} \leq e^{2\beta B} (1 - e^{-a}) \quad (3.12)$$

i.e. if

$$\sum_{n=1}^{\infty} n^{n-1} \left[ e^a \kappa \right]^{n-1} \leq 1 + e^{2\beta B} (1 - e^{-a}) \quad (3.13)$$

where $\kappa = 1/(\rho e^{2\beta B} C(\beta))$. Let now

$$K^* = \min_{a \geq 0} \inf \left\{ \kappa : \sum_{n=1}^{\infty} n^{n-1} \left[ e^a \kappa \right]^{n-1} \leq 1 + e^{2\beta B} (1 - e^{-a}) \right\}$$

As shown in [4] (see also [11] and [14]) $K^*$ can be written explicitly as

$$K^* = \min_{a > 0} \frac{e^a [1 + e^{2\beta B} (1 - e^{-a})]}{\ln[1 + e^{2\beta B} (1 - e^{-a})]}$$

So (3.8), and hence (3.7), hold for all complex $\rho$ as soon as

$$|\rho| \leq \rho_{\beta}^*$$

where

$$\rho_{\beta}^* = F(e^{2\beta B}) \frac{1}{e^{2\beta B} C(\beta)} \quad (3.14)$$

and

$$F(u) = \max_{a > 0} \frac{\ln[1 + u(1 - e^{-a})]}{e^a [1 + u(1 - e^{-a})]} \quad (3.15)$$

□
3.2 Free energy in powers of the density: proof of Theorem 1 part i

To prove formula (2.18), we reorganize the expansion (3.4), i.e.

\[ Q_{\Lambda}(\beta, \rho) = \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(R_1, \ldots, R_n) \in [N]^n} \phi^T(R_1, \ldots, R_n) \zeta_{|R_1|} \cdots \zeta_{|R_n|} \]

as a power series in the density \( \rho \). We use first of all the Penrose identity [30] [40] [10] [33] [14] which states (we use the notation of [10]) that

\[ \phi^T(R_1, \ldots, R_n) = (-1)^{n-1} \sum_{\tau \in T_n} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} \]

where \( P_{G(R_1, \ldots, R_n)} \) are the set of Penrose trees of the graph \( G(R_1, \ldots, R_n) \) with vertex set \([n]\) and rooted in a fixed vertex of \([n]\), e.g., with root in the vertex 1. Thus

\[ Q_{\Lambda}(\beta, \rho) = \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} \zeta_{|R_1|} \cdots \zeta_{|R_n|} \]  

(3.16)

We put

\[ \zeta_s = \rho^{s-1} \mu_s \]  

(3.17)

where, recalling (3.2),

\[ \mu_s = \frac{b_s(\beta, \Lambda)s!}{Ns-1} \]  

(3.18)

Then

\[
Q_{\Lambda}(\beta, \rho) = \frac{\rho}{N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \sum_{|R_i|=s_i} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} \zeta_{s_1} \cdots \zeta_{s_n} = \\
= \frac{\rho}{N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{s_1, \ldots, s_n \geq 2} \rho^{s_1-1} \mu_{s_1} \cdots \rho^{s_n-1} \mu_{s_n} \sum_{|R_i|=s_i} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} = \\
= \frac{\rho}{N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{k_1, \ldots, k_n \geq 1} \rho^{k_1} \mu_{k_1+1} \cdots \rho^{k_n} \mu_{k_n+1} \sum_{|R_i|=k_i+1} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} = \\
= \frac{\rho}{N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{k \geq n} \rho^k \sum_{k_1, \ldots, k_n \geq 1} \mu_{k_1+1} \cdots \mu_{k_n+1} \sum_{|R_i|=k_i+1} \mathbb{1}_{\tau \in P_{G(R_1, \ldots, R_n)}} .
\]
In conclusion we have proved formula (2.18) in Theorem 1, i.e. we have that

\[ Q_N(\beta, \rho) = \sum_{k \geq 1} \frac{\mathcal{C}_k(\beta, \Lambda)}{k + 1} \rho^{k + 1} \]

where

\[ \mathcal{C}_k(\beta, \Lambda) = \sum_{n=1}^{k} (-1)^{n-1} W_n(k) \]

and

\[ W_n(k) = \frac{k + 1}{n!} \sum_{\substack{s_1, \ldots, s_n \geq 2 \\ s_1 + \cdots + s_n = k + n}} \prod_{i=1}^{n} [b_{s_i}(\beta, \Lambda) s_i!] \mathcal{P}(s_1, \ldots, s_n) \]

with

\[ \mathcal{P}(s_1, \ldots, s_n) = \frac{1}{N^{k+1}} \sum_{\tau \in T_n} \sum_{\tau_1, \ldots, \tau_n \in [N]^{k+1}} \mathbb{I}_{\tau \in \mathcal{P}_G(R_1, \ldots, R_n)} \]

We now have to show formula (2.19). We start making the following observation. For fixed \( k \) and \( n \), the integers \( s_1, \ldots, s_n \) such that \( s_1 + \cdots + s_n = k + n \) define uniquely a \( k \)-tuple of integers \( \{m_2, \ldots, m_{k+1}\} \) such that \( m_i \in \{0, 1, 2, \ldots\} \) and \( \{\# j \in [n] : s_j = i\} = m_i \) with the property that \( \sum_{i=2}^{k+1} m_i = n \) and \( \sum_{i=2}^{k+1} (i - 1) m_i = k \) and so \( \prod_{j=1}^{n} [b_{s_j}(\beta, \Lambda) s_j!] = \prod_{i=2}^{k+1} [b_i(\beta, \Lambda) i!]^{m_i} \). Hence we can write

\[ W_n(k) = \frac{k + 1}{n!} \sum_{\substack{(m_2, \ldots, m_{k+1}) \geq 0 \\ \sum_{i=2}^{k+1} m_i = n \sum_{i=2}^{k+1} (i - 1) m_i = k \sum_{i=1}^{k+1} m_i = n}} \prod_{i=2}^{k+1} [b_i(\beta, \Lambda) i!]^{m_i} \mathcal{P}(s_1, \ldots, s_n) \]

Let us now calculate \( \mathcal{P}(s_1, \ldots, s_n) \) under the conditions that \( s_1, \ldots, s_n \) is an \( n \)-tuple of integers such that \( s_1 \geq 2 \) and \( s_1 + \cdots + s_n = k + n \) and such that it defines the \( k \)-tuple of integers \( \{m_2, \ldots, m_{k+1}\} \). Recalling thus (3.22), fix a tree \( \tau \in T_n \) and consider the factor

\[ w_{\tau} = \sum_{\tau \in \mathcal{P}_G(R_1, \ldots, R_n)} \mathbb{I}_{\tau \in \mathcal{P}_G(R_1, \ldots, R_n)} (3.24) \]
This factor is clearly a polynomial in $N$ and, in view of the limit $N \to \infty$, we will retain only the term of maximal degree in $N$. It is not difficult to see that once the tree $\tau \in T_n$ has been fixed, the contribution of the higher order in $N$ comes from the sum over polymers $R_i$ submitted to the following prescriptions.

- For any fixed vertex $i$ of the tree $\tau$ different from the root with degree $d_i$ and $i_1, \ldots, i_{d_i-1}$ children sum over $R_{i_1}, \ldots, R_{i_{d_i-1}}$ in such way that each one of the polymers $R_{i_1}, \ldots, R_{i_{d_i-1}}$ shares exactly one vertex with $R_i$ and all these $d_i-1$ vertices are distinct (so in particular we must have $d_i - 1 \leq |R_i|$), otherwise the contribution of this tree vanishes. This is because in such way the factors

$$\left( \frac{N}{|R_{i_1}| - 1} \right) \cdots \left( \frac{N}{|R_{i_{d_i-1}}| - 1} \right)$$

have the maximal power in $N$. For the root 1 of $\tau$, with degree $d_1$ and $i_1, \ldots, i_{d_1}$ children do analogously (but this time it must hold $d_1 \leq |R_1|$).

- In any fixed vertex $i$ of $\tau$ we can choose the remaining $|R_{ij}| - 1$ vertices in each polymer $R_{ij}$ associated to the $i_j$ child of $i$ ($j = 1, \ldots, s_i$) among $N$ vertices; actually the $|R_{ij}| - 1$ vertices in each $R_{ij}$ should be chosen among $N_{ij} < N$ vertices, where $N - k - n \leq N_{ij} \leq N - 1$, according to the constraints imposed by the Penrose condition $\tau \in P_G(R_1, \ldots, R_n)$, but, for $k$ fixed, we have that $k + n \leq 2k$, so that $N(1 - \frac{2k}{N}) \leq N_{ij} \leq N(1 - \frac{1}{N})$.

These prescription are exactly the same conditions used to calculate the r.h.s. of equation 3.20 in Lemma 3.4 of [11]. Proceeding thus analogously to the computation explained in Lemma 3.4 of [11] we have, for any $\tau \in T_n$

$$w_{\tau} = \left( \frac{N}{s_1} \right) \left( \frac{s_1}{d_1} \right) d_1! \prod_{i=2}^{n} \left( \frac{N}{s_i - 1} \right) \left( \frac{s_i}{d_i - 1} \right) (d_i - 1)! \left[ 1 + \frac{o(N)}{N} \right] \mathbb{I}_{d_1 \leq s_1, d_i \leq s_i + 1} =$$

$$= \frac{N^{s_1}}{s_1!} \left( \frac{s_1}{d_1} \right) d_1! \prod_{i=2}^{n} \frac{N^{s_i-1}}{(s_i - 1)!} \left( \frac{s_i}{d_i - 1} \right) (d_i - 1)! \left[ 1 + \frac{o(N)}{N} \right] \mathbb{I}_{d_1 \leq s_1, d_i \leq s_i + 1} =$$

$$= N^{k+1} \frac{s_1!}{s_1! d_1! (s_1 - d_1)!} d_1! \prod_{i=2}^{n} \frac{1}{(s_i - 1)! (d_i - 1)! (s_i - d_i + 1)!} \left( \frac{s_i}{d_i - 1} \right) (d_i - 1)! \left[ 1 + \frac{o(N)}{N} \right] \mathbb{I}_{d_1 \leq s_1, d_i \leq s_i + 1} =$$

$$= N^{k+1} \frac{1}{(s_1 - d_1)! (s_1 - d_1)!} \prod_{i=2}^{n} \frac{s_i}{s_i - d_i + 1)!} \left[ 1 + \frac{o(N)}{N} \right] \mathbb{I}_{d_1 \leq s_1, d_i \leq s_i + 1}$$

where $d_i$ is the degree of vertices $i$ in $\tau$ and we recall that for any tree $\tau \in T_n$ we have that $d_1 + \ldots + d_n = 2n - 2$. So, summing over all trees $\tau \in T_n$ and recalling Cayley formula, we have that

$$P(s_1, \ldots, s_n) = \sum_{\substack{d_1, \ldots, d_n: d_i \geq 1 \\
\sum_{i=1}^{n} d_i = 2n - 2 \\
1 \leq d_1 \leq s_1, 1 \leq d_i \leq s_i + 1}} \frac{(n - 2)!}{\prod_{i=1}^{n} (d_i - 1)!} \frac{1}{(s_1 - d_1)!} \prod_{i=2}^{n} \frac{s_i}{(s_i - d_i + 1)!} \left[ 1 + \frac{o(N)}{N} \right]$$
\[\begin{align*}
&= \frac{(n-2)!}{\prod_{i=1}^{n}(s_i-1)!} \sum_{l_1+\ldots+l_n \leq n-2} \frac{(s_1-1)!}{l_1!(s_1-l_1-1)!} \prod_{i=2}^{n} \frac{s_i!}{l_i!(s_i-l_i)!} \left[1 + \frac{o(N)}{N}\right] \\
&= \frac{(n-2)!}{\prod_{i=1}^{n}(s_i-1)!} \sum_{l_1+\ldots+l_n \leq n-2} \frac{(s_1-1)!}{l_1!(s_1-l_1-1)!} \prod_{i=2}^{n} \frac{s_i!}{l_i!(s_i-l_i)!} \left[1 + \frac{o(N)}{N}\right]
\end{align*}\]

In conclusion we have obtained

\[P(s_1,\ldots,s_n) = \frac{(n-2)!}{\prod_{i=2}^{n+1}((i-1)!)^{m_i}} \sum_{l_1+\ldots+l_n \leq n-2} \frac{(s_1-1)!}{l_1!(s_1-l_1-1)!} \prod_{i=2}^{n} \frac{s_i!}{l_i!(s_i-l_i)!} \left[1 + \frac{o(N)}{N}\right] \tag{3.25}\]

We now show the identity

\[\sum_{l_1+\ldots+l_n \leq n-2} \frac{(s_1-1)!}{l_1!(s_1-l_1-1)!} \prod_{i=2}^{n} \frac{s_i!}{l_i!(s_i-l_i)!} = \binom{k-1+n}{n-2} \tag{3.26}\]

Indeed, put \(t_1 = s_1 - 1\) and \(t_i = s_i\) for all \(i = 2,\ldots,n\). So we need to prove that for any \(n\)-tuple \(t_1,\ldots,t_n\) such that \(\sum_{i=1}^{n} t_i = n + k - 1\), \(t_1 \geq 1\) and \(t_i \geq 2\) for \(i \geq 2\).

\[\sum_{l_1+\ldots+l_n \leq n-2} \prod_{i=1}^{n} \frac{t_i}{l_i} = \binom{k-1+n}{n-2} \tag{3.27}\]

To prove this identity, suppose to have a set \(V\) with \(n + k - 1\) objects and let \(V = V_1 \cup V_2 \cup \ldots \cup V_n\) with \(V_1,\ldots,V_n\) being \(n\) disjoint sets with cardinality \(|V_1| = t_1,\ldots,|V_n| = t_n\). Let \(K_{n,k}\) be the number of ways to pick up \(n - 2\) objects from the \(n + k - 1\) objects of \(V\). Of course \(K_{n,k} = \binom{k-1+n}{n-2}\) and this is the r.h.s. of (3.27). On the other hand, since the sets \(V_1,\ldots,V_n\) form a partition of the set \(V\) we can also compute \(K_{n,k}\) by choosing, for each \(i \in [n]\), \(l_i\) objects from \(V_i\) which is done in \(\binom{t_i}{l_i}\) ways, and then summing over all possible \(n\)-tuple \(l_1,\ldots,l_n\) under the constraint that \(l_1 + \ldots + l_n = n - 2\), getting in such way that \(K_{n,k}\) is also equal to the l.h.s. of (3.27). This prove (3.27) and hence (3.26). Putting this into (3.25) we have

\[P(s_1,\ldots,s_n) = \frac{(n-2)!}{\prod_{i=2}^{n+1}((i-1)!)^{m_i}} \binom{k-1+n}{n-2} \left(1 + \frac{o(N)}{N}\right) \tag{3.28}\]

Note that (3.28) implies that the factor \(P(s_1,\ldots,s_n)\) is, at least modulo terms of order \(\frac{o(N)}{N}\), a symmetric function of \(s_1,\ldots,s_n\), i.e. it depends only on \(n, k\) and numbers \(\{m_2,\ldots,m_{k+1}\}\). Actually
one can easily realize that $\mathcal{P}(s_1, \ldots, s_n)$ is globally symmetric. Indeed, from definition (3.22) it immediately follows that $\mathcal{P}(s_1, \ldots, s_n)$ is invariant under permutation of $s_2, \ldots, s_n$ if the root of the Penrose trees is chosen to be 1. Moreover, by the construction of the Penrose identity, the r.h.s. of (3.22) do not depend on the choice of the root, so that in the end $\mathcal{P}(s_1, \ldots, s_n)$ is actually invariant under permutation of $s_1, \ldots, s_n$.

We can now plug (3.28) into (3.23) and we obtain

$$W_n(k) = \frac{k + 1}{n!} \left[ 1 + \frac{o(N)}{N} \right] \sum_{(m_i) = (m_2, \ldots, m_{k+1})}^{k+1} \prod_{i=2}^{k+1} \frac{b_i(\beta, \Lambda) i!^{m_i}}{m_i!} \frac{(n-2)!}{(i-1)!^{m_i}} \left( \frac{k - 1 + n}{n - 2} \right) \sum_{\substack{s_1, \ldots, s_n \geq 2 \atop \#j \in [n]: s_j = 1 = m_j}} 1$$

So, since

$$\sum_{\substack{s_1, \ldots, s_n \geq 2 \atop \#j \in [n]: s_j = 1 = m_j}} 1 = \frac{n!}{\prod_{i=2}^{k+1} m_i!}$$

we obtain

$$W_n(k) = (k + 1)(n - 2)! \left[ 1 + \frac{o(N)}{N} \right] \sum_{(m_i) = (m_2, \ldots, m_{k+1})}^{k+1} \prod_{i=2}^{k+1} \frac{b_i(\beta, \Lambda) i!^{m_i}}{m_i!} \left( \frac{k - 1 + n}{n - 2} \right)$$

and so

$$\mathcal{C}_k(\beta, \Lambda) = \left[ 1 + \frac{o(N)}{N} \right] \sum_{n=1}^{k} (-1)^{n-1} \frac{(k - 1 + n)!}{k!} \sum_{(m_i) = (m_2, \ldots, m_{k+1})}^{k+1} \prod_{i=2}^{k+1} \frac{b_i(\beta, \Lambda) i!^{m_i}}{m_i!}$$

which concludes the proof of part i) of Theorem 1. $\square$

As previously remarked, r.h.s. of (3.29) is, up to terms of order $\frac{o(N)}{N}$, exactly the expression given in formula (49) of [21] (see also (29) p. 319 of [26]). We can thus conclude that $\mathcal{C}_k(\beta, \Lambda)$ is, up to terms of order $\frac{o(N)}{N}$, the very same $k$ order virial coefficient as it is defined in formula (13.25) pag. 287 of [22]. So $\mathcal{C}_k$ can also be written in terms of a sum over two-connected graphs between $k + 1$ vertices (see e.g. (13.25) in [22]). Namely, $\mathcal{C}_k(\beta, \Lambda) = [1 + \frac{o(N)}{N}] \beta_k(\beta, \Lambda)$ where $\beta_k(b, \Lambda)$ is the virial coefficient defined in (2.11).
3.3 Bound for the free energy: proof of Theorem 1, part \textit{iii}

Let us define the positive term series

\[ |Q|_{\Lambda}(\beta, \rho) = \frac{1}{V} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in P_G(R_1, \ldots, R_n)} |\zeta_{R_1}| \ldots |\zeta_{R_n}| \]  \hfill (3.30)

Then, clearly

\[ |Q|_{\Lambda}(\beta, \Lambda) \leq |Q|_{\Lambda}(\beta, \rho) \]

Now

\[ |Q|_{\Lambda}(\beta, \rho) = \frac{\rho}{N} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k \geq 1} \sum_{s_1, \ldots, s_n = k+n} |s_1 + s_2 + \ldots + s_n| \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in P_G(R_1, \ldots, R_n)} |\mu_{s_1}| \ldots |\mu_{s_n}| = \]

\[ = \frac{\rho}{N} \sum_{k \geq 1} \sum_{n=1}^{k} \frac{1}{n!} \sum_{s_1, \ldots, s_n = k+n} \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in P_G(R_1, \ldots, R_n)} |\mu_{s_1}| \ldots |\mu_{s_n}| = \]

\[ = \sum_{k \geq 1} \frac{\rho^{k+1}}{k+1} \sum_{n=1}^{k} \frac{1}{n!} \sum_{s_1, \ldots, s_n = k+n} \prod_{i=1}^{n} |b_{s_i}(\beta, \Lambda)|_{s_i} \frac{k+1}{N^{k+1}} \sum_{\tau \in T_n} \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in P_G(R_1, \ldots, R_n)} \]

In conclusion we get

\[ |Q|_{\Lambda}(\beta, \rho) = \sum_{k \geq 1} \frac{|C|_{k}(\beta, \Lambda)}{k+1} \rho^{k+1} \]  \hfill (3.31)

where

\[ |C|_{k}(\beta, \Lambda) = \sum_{n=1}^{k} |W|_{n}(k) \]  \hfill (3.32)

with

\[ |W|_{n}(k) = \frac{k+1}{n!} \sum_{s_1, \ldots, s_n = 2, \ldots, k+n} \prod_{i=1}^{n} |b_{s_i}(\beta, \Lambda)|_{s_i} \mathcal{P}(s_1, \ldots, s_n) \]  \hfill (3.33)

Of course we have that

\[ |C|_{k}(\beta, \Lambda) \leq |C|_{k}(\beta, \Lambda) \]  \hfill (3.34)

We now obtain an upper bound for \( |Q|_{\Lambda}(\beta, \rho) \) as soon as \( \rho \leq \rho_{\beta}^{*} \). We start by writing, recalling (3.30)

\[ |Q|_{\Lambda}(\beta, \rho) = \frac{1}{V} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\tau \in T_n} u_{\tau} \]  \hfill (3.35)
where

\[ u_\tau = \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in \mathcal{P}_G(R_1, \ldots, R_n)} |\zeta_{R_1}| \cdots |\zeta_{R_n}| \]

We first observe that the \( n = 1 \) term in the sum in the r.h.s. of (3.35) can be written as

\[ \frac{1}{V} \sum_{\tau \in T_1} u_\tau = \frac{1}{V} \sum_{R_1 \in [N]} |\zeta_{R_1}| = \frac{1}{V} \sum_{s \geq 2} |\zeta_s| \sum_{R_1 \in [N]} 1 = \frac{1}{V} \sum_{s \geq 2} \binom{N}{s} |\zeta_s| = \frac{1}{V} \sum_{s \geq 2} \frac{N}{s} \left( \frac{N-1}{s-1} \right) |\zeta_s| = \rho \sum_{s \geq 2} \frac{1}{s} C_s^\rho \]

(3.36)

Now following \([11]\), for a fixed \( n \geq 2 \) and \( \tau \in T_n \), we can bound

\[ |u_\tau| \leq \sum_{(R_1, \ldots, R_n) \in [N]^n} \mathbb{1}_{\tau \in \mathcal{P}_G^s(R_1, \ldots, R_n)} |\zeta_{R_1}| \cdots |\zeta_{R_n}| \]

where \( \mathcal{P}_G^s(R_1, \ldots, R_n) \) are the weakly Penrose trees which are subsets of \( G(R_1, \ldots, R_n) \). Using once again lemma 3.4 in \([11]\) we have, for any \( \tau \in \mathcal{P}_G^s(R_1, \ldots, R_n) \) and any \( n \geq 2 \),

\[ |u_\tau| \leq \sum_{R_1 \subset [N]} |\zeta_{R_1}| \left( \begin{array}{c} R_1 \\ d_1 \end{array} \right) d_1! \prod_{i=2}^{n} \left[ \sup_{j \in [N]} \sum_{R_i \subset [N], j \in R_i \ | R_i | \geq (d_i - 1) \& 2} \left( \frac{|R_i|}{d_i} \right)^{d_i - 1} \right] |\zeta_{R_i}| \]

\[ \doteq w(d_1, \ldots, d_n) \]

where \( d_i \) is the degree of vertex \( i \) of \( \tau \). Hence, by Cayley formula and definition (3.9)

\[ \frac{1}{n!} \sum_{\tau \in T_n} |u_\tau| \leq \frac{1}{n!} \sum_{d_1, \ldots, d_n = 2n-2} d_1! \cdots d_n! \prod_{i=1}^{n} (d_i - 1)! w(d_1, \ldots, d_n) = \]

\[ = \frac{1}{n(n-1)} \sum_{d_1, \ldots, d_n = 2n-2} \sum_{R_1 \subset [N]} |\zeta_{R_1}| \left( \begin{array}{c} R_1 \\ d_1 \end{array} \right) d_1 \prod_{i=2}^{n} \left[ \sup_{j \in [N]} \sum_{R_i \subset [N], j \in R_i \ | R_i | \geq (d_i - 1) \& 2} \left( \frac{|R_i|}{d_i} \right)^{d_i - 1} \right] |\zeta_{R_i}| \]

\[ = \frac{N}{n(n-1)} \sum_{d_1, \ldots, d_n = 2n-2} \sum_{s_1 \geq 1} \frac{C_1^{s_1}}{s_1} \left( \begin{array}{c} s_1 \\ d_1 \end{array} \right) d_1 \prod_{i=2}^{n} \left[ \sum_{s_i \geq (d_i - 1) \& 2} \left( \frac{s_i}{d_i} \right) C_s^{s_i} \right] \]

we now use the trick first used in \([35]\) (see there section 3), so that, multiplying and dividing by \( \alpha^{n-1} \) (with \( \alpha > 0 \)), we get, for any \( n \geq 2 \),

\[ \frac{1}{n!} \sum_{\tau \in T_n} |u_\tau| \leq \frac{N \alpha^{-n+1}}{n(n-1)} \sum_{d_1, \ldots, d_n = 2n-2} \sum_{s_1 \geq 1} \frac{C_1^{s_1}}{s_1} \left( \begin{array}{c} s_1 \\ d_1 \end{array} \alpha^{d_1} \right) d_1 \prod_{i=2}^{n} \left[ \sum_{s_i \geq (d_i - 1) \& 2} \left( \frac{s_i}{d_i} \right) C_s^{s_i} \alpha^{d_i} \right] \]
\[
\leq \frac{N}{\alpha^{n-1}n(n-1)} \sum_{s_1 \geq 2} ^{n} \frac{C_{s_1}^{\rho}}{s_1} C_{s_2}^{\rho} \cdots C_{s_n}^{\rho} \sum_{d_1 = 1}^{s_1} \left( \frac{s_1}{d_1} \right) d_1 \alpha d_1 \prod_{i=2}^{n} \left[ \sum_{d_i-1=0}^{s_i} \left( \frac{s_i}{d_i-1} \right) \alpha d_i^{-1} \right]
\]

\[
= \frac{N}{\alpha^{n-1}n(n-1)} \sum_{s_1 \geq 2} ^{n} \frac{C_{s_1}^{\rho}}{s_1} C_{s_2}^{\rho} \cdots C_{s_n}^{\rho} \sum_{i=2}^{n} \left[ 1 + \alpha \right]^{s_i}
\]

\[
= \frac{N}{\alpha^{n-1}n(n-1)} \alpha \sum_{s_1 \geq 2} ^{n} \frac{C_{s_1}^{\rho}}{s_1} (1 + \alpha)^{s_1-1} \left[ \frac{1}{\alpha} \sum_{s \geq 2} (1 + \alpha)^{s} C_{s}^{\rho} \right]^{n-1}
\]

Now, choosing \( \alpha = e^{a_2^\beta} - 1 \), we get, by the convergence criterion (3.8), that for \( \rho \leq \rho_\beta^* \)

\[
\alpha \sum_{s \geq 2} (1 + \alpha)^{s} C_{s}^{\rho} \leq 1
\]

So we get, for \( \rho \leq \rho_\beta^* \)

\[
\frac{1}{V} \sum_{n \geq 2} \frac{1}{n!} \sum_{\tau \in T_n} |u_\tau| \leq \rho (e^{a_2^\beta} - 1) \sum_{s \geq 2} C_{s_1}^{\rho} e^{a_2^\beta(s_1-1)}
\]

(3.37)

and hence, (3.37) together with (3.36) yield the bound

\[
|Q|_{\Lambda}(\beta, \rho) \leq \rho \sum_{s \geq 2} \left( \frac{1}{s} + (e^{a_2^\beta} - 1)e^{a_2^\beta(s-1)} \right) C_{s}^{\rho}
\]

Therefore, recalling bound (3.11) for \( C_{s}^{\rho} \), we get

\[
|Q|_{\Lambda}(\beta, \rho) \leq \sum_{k \geq 1} \frac{\rho^{k+1}}{k+1} \left[ 1 + (k+1)(e^{a_2^\beta} - 1)e^{a_2^\beta k} \right] e^{2\beta B(k-1)\frac{(k+1)^{k-1}}{k!}[C(\beta)]^k}
\]

(3.38)

By comparing (3.38) with (3.31) we immediately get

\[
|C_k|_{\Lambda}(\beta, \Lambda) \leq \left[ \frac{1}{k+1} + (e^{a_2^\beta} - 1)e^{a_2^\beta k} \right] e^{2\beta B(k-1)\frac{(k+1)^{k}}{k!}[C(\beta)]^k}
\]

and, by (3.34), the bound (2.22) follows. \( \square \)

**Acknowledgments**

Aldo Procacci has been partially supported by the Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo à Pesquisa do estado de Minas Gerais (FAPEMIG - Programa de Pesquisador Mineiro).
References

[1] A. Adbesselam and V. Rivasseau (1995): Tree forests and jungles: a botanical garden for cluster expansions in Constructive physics, Proceedings, Palaiseau, France 1994, Lecture notes in physics n. 446.

[2] A. Bovier and M. Zahradník (2000): A simple inductive approach to the problem of convergence of cluster expansions of polymer models. J. Statist. Phys. 100, 765–78.

[3] R. Bissacot, R. Fernández and A. Procacci (2010): On the Convergence of Cluster Expansions for Polymer Gases, J. Statist. Phys., 139, 598–617.

[4] C. Borgs (2006): Absence of zeros for the chromatic polynomial on bounded degree graphs. Combin. Probab. Comput. 15, 63–74.

[5] D. Brydges and P. Federbush (1978): A new form of the Mayer expansion in classical statistical mechanics. J. Math Phys. 19, 2064 (4 pages).

[6] D. C. Brydges (1984): A short cluster in cluster expansions. In Critical Phenomena, Random Systems, Gauge Theories, Osterwalder, K. and Stora, R. (eds.), Elsevier, 129–83.

[7] C. Cammarota (1982): Decay of correlations for infinite range interactions in unbounded spin systems. Comm. Math. Phys., 85, 517–28.

[8] R. L. Dobrushin (1996): Estimates of semiinvariants for the Ising model at low temperatures. Topics in Statistics and Theoretical Physics, Amer. Math. Soc. Transl., 177, 59–81.

[9] R. L. Dobrushin (1996): Perturbation methods of the theory of Gibbsian fields. In Ecole d'Eté de Probabilités de Saint-Flour XXIV – 1994, Springer-Verlag (Lecture Notes in Mathematics 1648), Berlin–Heidelberg–New York, 1–66.

[10] R. Fernández and A. Procacci (2007): Cluster expansion for abstract polymer models. New bounds from an old approach, Commu. Math. Phys., 274, 123–140.

[11] R. Fernández and A. Procacci (2007): Regions Without Complex Zeros for Chromatic Polynomials on Graphs with Bounded Degree, Combin. Prob. Comp. 17, 225–238.

[12] G. Gallavotti; S. Miracle-Solé (1968): Correlation functions for lattice systems, Commun. Math Phys. 7, 274-288.

[13] C. Gruber and H. Kunz (1971): General properties of polymer systems. Comm. Math. Phys., 22, 133–61.

[14] B. Jackson, A. Procacci and A. D. Sokal (2013): Complex zero-free regions at large $|q|$ for multivariate Tutte polynomials (alias Potts-model partition functions) with general complex edge weights, J. Combin. Theory, Series B, 103, 21–45.

[15] J. G. Kirkwood (1946): The statistical mechanical theory of transport processes, J. Chem. Phys., 14, 180-201

[16] R. Kotecký and D. Preiss (1986): Cluster expansion for abstract polymer models. Comm. Math. Phys., 103, 491–498.
[17] T. Kuna, Yu. G. Kondratiev, and J. L. Da Silva (1998): *Marked Gibbs Measures via Cluster Expansion*, Methods Funct. Anal. Topology 4, 50–81.

[18] J. Groeneveld (1962): *Two theorems on classical many-particle systems*. Phys. Lett., 3, 50–51.

[19] J. L. Lebowitz and O. Penrose (1964): *Convergence of Virial Expansions*, J. Math. Phys. 7, 841-847.

[20] V. A. Malyshev (1980): *Cluster expansions in lattice models of statistical physics and quantum theory of fields*. Russian Mathematical Surveys, 35, 1–62.

[21] J. E. Mayer (1942): *Contribution to Statistical Mechanics*, J. Chem. Phys., 10, 629–643.

[22] J. E. Mayer and M. G. Mayer (1940): *Statistical Mechanics*, John Wiley & Sons, Inc. London: Chapman & Hall, Limited.

[23] J. E. Mayer (1947): Integral equations between distribution functions of molecules, J. Chem. Phys., 15, 187–201.

[24] S. Miracle-Solé (2000): *On the convergence of cluster expansions*. Physica A, 279, 244–249.

[25] F. R. Nardi, E. Olivieri and M. Zahradněk (1999): *On the Ising model with strongly anisotropic external field*, J. Statist. Phys. 97, 87–144.

[26] R. K. Pathria and P. D. Beale (2011): *Statistical mechanics, Third edition*, Elsevier, Amsterdam.

[27] Ch.-E. Pfister (1991): *Large deviation and phase separation in the two-dimensional Ising model*, Helv. Phys. Acta, 64, 953–1054.

[28] O. Penrose (1963): *Convergence of Fugacity Expansions for Fluids and Lattice Gases*, Journal of Mathematical Physics 4, 1312 (9 pages).

[29] O. Penrose (1963): *The Remainder in Mayer's Fugacity Series*, J. Math. Phys. 4, 1488 (7 pages).

[30] O. Penrose (1967): *Convergence of fugacity expansions for classical systems*. In Statistical mechanics: foundations and applications, A. Bak (ed.), Benjamin, New York.

[31] S. Poghosyan and D. Ueltschi (2009): *Abstract cluster expansion with applications to statistical mechanical systems*, J. Math. Phys. 50, no. 5, 053509, (17 pp).

[32] A. Procacci (2007): *Abstract Polymer Models with General Pair Interactions*, J. Stat Phys., 129, 171–188.

[33] A. Procacci (2009): *Erratum and Addendum: “Abstract Polymer Models with General Pair Interactions”*, J. Stat. Phys., 135, 779–786.

[34] A. Procacci, B. N. B. de Lima and B. Scoppola (1998): *A Remark on High Temperature Polymer Expansion for Lattice Systems with Infinite Range Pair Interactions*, Lett. Math. Phys., 45, 303–322.
[35] A. Procacci and B. Scoppola (1999): *Polymer gas approach to N-body lattice systems*. J. Statist. Phys. **96**, 49–68.

[36] E. Pulvirenti and D. Tsagkarogiannis (2012): *Cluster Expansion in the Canonical Ensemble*, Comm. math Phys. **316**, Issue 2, pp 289–306.

[37] D. Ruelle (1969): *Statistical mechanics: Rigorous results*. W. A. Benjamin, Inc., New York-Amsterdam.

[38] D. Ruelle (1963): *Correlation functions of classical gases*, Ann. Phys., **5**, 109–120.

[39] D. Ruelle (1963): *Cluster Property of the Correlation Functions of Classical Gases*, Rev. Mod. Phys., **36**, 580–584.

[40] A. D. Sokal (2001): *Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions*, Combin. Probab. Comput. **10**, 41–77.

[41] E. Seiler (1982): *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics **159**, Springer-Verlag, Berlin–Heidelberg–New York.

[42] D. Ueltschi (2004): *Cluster expansions and correlation functions*. Mosc. Math. J. **4**, 511–522.