Biharmonic hypersurfaces with three distinct principal curvatures in spheres

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We obtain a complete classification of proper biharmonic hypersurfaces with at most three distinct principal curvatures in sphere spaces with arbitrary dimension. Precisely, together with known results of Balmuş-Montaldo-Oniciuc, we prove that compact orientable proper biharmonic hypersurfaces with at most three distinct principal curvatures in sphere spaces $S^{n+1}$ are either the hypersphere $S^n(1/\sqrt{2})$ or the Clifford hypersurface $S^{n_1}(1/\sqrt{2}) \times S^{n_2}(1/\sqrt{2})$ with $n_1 + n_2 = n$ and $n_1 \neq n_2$. Moreover, we also show that there does not exist proper biharmonic hypersurface with at most three distinct principal curvatures in hyperbolic spaces $H^{n+1}$.

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1 Introduction

It is well known that the theory of harmonic maps plays a central role in various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical points of the energy functional

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$$

for smooth maps $\phi : (M^n, g) \to (\bar{M}^m, \langle \cdot, \cdot \rangle)$.

Biharmonic maps $\phi : (M^n, g) \to (\bar{M}^m, \langle \cdot, \cdot \rangle)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g,$$

where $\tau(\phi) = \text{trace} \nabla d\phi$ is the tension field of $\phi$ that vanishes for harmonic maps. For biharmonic map, the bitension field satisfies the associated Euler-Lagrange equation (see [16])

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace} R^\bar{M}(d\phi, \tau(\phi))d\phi = 0,$$

where $R^\bar{M}$ is the curvature tensor

$$R^\bar{M}(U, V) = \nabla_U \nabla^\bar{M} V - \nabla^\bar{M} \nabla_U V - \nabla^\bar{M} [U, V], \quad U, V \in \mathcal{X}(\bar{M}),$$

and $\Delta$ is the rough Laplacian given by

$$\Delta = -\sum_{k=1}^n \left( \nabla^\phi_{e_k} \nabla^\phi_{e_k} - \nabla^\phi_{e_k} \nabla^\phi_{e_k} \right)$$

for a local orthonormal frame field $\{e_k\}_{k=1}^n$ defined on $(M^n, g)$.

The above equation shows that $\phi$ is a biharmonic map if and only if its bi-tension field $\tau_2(\phi)$ vanishes. Equivalently, for an immersion $\phi : (M^n, g) \to (\bar{M}^m, \langle \cdot, \cdot \rangle)$ between Riemannian manifolds, the mean curvature

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vector field $\mathbf{H}$ satisfies the following fourth order elliptic semi-linear PDE
\begin{equation}
\Delta \mathbf{H} + \text{trace} R^M(d\phi, \mathbf{H})d\phi = 0.
\end{equation}
(1.1)
Obviously, any minimal immersion, i.e. immersion satisfying $\mathbf{H} = 0$, is biharmonic. The non-harmonic biharmonic immersions are called proper biharmonic.

In a different setting, B. Y. Chen in the middle of 1980s initiated the study of biharmonic submanifolds in a Euclidean space by the condition $\Delta \mathbf{H} = 0$, where $\Delta$ is the rough Laplacian of submanifolds with respect to the induced metric. It is easy to see that both notions of biharmonic submanifolds in Euclidean spaces coincide with each other.

The study of biharmonic submanifolds is nowadays a very active subject. There is a challenging biharmonic conjecture of B. Y. Chen made in 1991 [7]:

**Chen’s conjecture:** The only biharmonic submanifolds of Euclidean spaces are the minimal ones.

Ten years later, in 2001 Caddeo, Montaldo and Oniciuc [5] made the following generalized Chen’s conjecture:

**Generalized Chen’s conjecture:** Every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.

Recently, the Generalized Chen’s conjecture was proved to be wrong by Y. L. Ou and L. Tang in [19], who constructed examples of proper-biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature. However, the original Chen’s conjecture is still open so far. Also, the Generalized Chen’s conjecture is still open in its full generality for ambient spaces with constant non-positive sectional curvature. For more recent developments of Chen’s conjecture and Generalized Chen’s conjecture, please refer to Chen’s recent survey article [11] and reference therein.

In contrast, the class of proper biharmonic submanifolds in sphere spaces is rather rich and very interesting. The complete classifications of biharmonic hypersurfaces in $S^3$ and $S^4$ were obtained by Balmuş, Caddeo, Montaldo and Oniciuc in [5], [4]. Moreover, the authors in [3] classified biharmonic hypersurfaces with at most two distinct principal curvatures in $S^n$ with arbitrary dimension. There are also some results on biharmonic submanifolds in general ambient space, e.g. [18].

For what concerns biharmonic hypersurfaces with three distinct principal curvatures in spheres, Balmuş-Montaldo-Oniciuc in [4] proved the following non-existence result: there do not exist compact constant mean curvature (CMC) proper-biharmonic hypersurfaces with three distinct principal curvatures in $S^n$ everywhere.

In the present paper, we concentrate on biharmonic hypersurfaces with three distinct principal curvatures in space forms with arbitrary dimension. Firstly, we prove that biharmonic hypersurface $M^n$ with at most three distinct principal curvatures in space forms necessarily has constant mean curvature. Combining with Balmuş et al.’s nice work on this subject, we can achieve a complete classification of compact proper biharmonic hypersurfaces with at most three distinct principal curvatures in spheres with arbitrary dimension, and without any other assumptions. Hence, our results extend all the known results mentioned above for biharmonic hypersurfaces in spheres. At last, with a similar argument we also show that there does not exist proper biharmonic hypersurface with at most three distinct principal curvatures in hyperbolic spaces $H^{n+1}$.

### 2 Preliminaries

Let $M^n$ be an orientable hypersurface isometrically immersed into a space form $R^{n+1}(c)$ with constant sectional curvature $c$. Denote the Levi-Civita connections of $M^n$ and $R^{n+1}(c)$ by $\nabla$ and $\bar{\nabla}$, respectively. Let $X$ and $Y$ denote vector fields tangent to $M^n$ and let $\xi$ be a unite normal vector field. Then the Gauss and Weingarten formulas (cf. [8], [9]) are given, respectively, by
\begin{equation}
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),
\end{equation}
\begin{equation}
\bar{\nabla}_X \xi = -AX,
\end{equation}
where $h$ is the second fundamental form, and $A$ is the Weingarten operator. It is well known that the second fundamental form $h$ and the Weingarten operator $A$ are related by
\begin{equation}
\langle h(X, Y), \xi \rangle = \langle AX, Y \rangle.
\end{equation}
The mean curvature vector field $\overrightarrow{H}$ is given by
\[
\overrightarrow{H} = \frac{1}{n} \text{trace } h.
\]
(2.4)

Moreover, the Gauss and Codazzi equations are given respectively by
\[
R(X, Y)Z = c((Y, Z)X - (X, Z)Y) + (AY, Z)AX - (AX, Z)AY,
\]
\[
(\nabla_X A)Y = (\nabla_Y A)X,
\]
where $R$ is the curvature tensor of hypersurface $M^n$ and $(\nabla_X A)Y$ is defined by
\[
(\nabla_X A)Y = (\nabla_X)AY - A(\nabla_X Y)
\]
for all $X, Y, Z$ tangent to $M^n$.

Assume that $\overrightarrow{H} = H\xi$ and $H$ denotes the mean curvature.

By identifying the tangent and the normal parts of the biharmonic condition (1.1) for hypersurfaces in space forms $R^{n+1}(c)$, we obtain the following characterization result for $M^n$ to be biharmonic (see also [6], [4], [9]).

**Theorem 2.1** The immersion $x : M^n \rightarrow R^{n+1}(c)$ of a hypersurface $M^n$ in an $n + 1$-dimensional space form $R^{n+1}(c)$ is biharmonic if and only if
\[
\begin{aligned}
\Delta H + H \text{ trace } A^2 &= ncH, \\
2A \text{ grad } H + nH \text{ grad } H &= 0.
\end{aligned}
\]
(2.6)

Clearly, it follows from (2.6) that the only umbilical proper biharmonic hypersurface in $S^{n+1}$ is an open part of $S^n(1/\sqrt{2})$.

Recall those known results on biharmonic hypersurfaces with at most two distinct principal curvatures in $S^{n+1}$ developed by Balmaș et al. in the last ten years.

**Theorem 2.2** ([5]) Let $M^2$ be a proper biharmonic surface in spheres $S^3$. Then $M^2$ is an open part of $S^2(1/\sqrt{2}) \subset S^3$.

**Theorem 2.3** ([3]) Let $M^n$ be a proper biharmonic hypersurface with at most two distinct principal curvatures in $S^{n+1}$. Then $M^n$ is either an open part of hypersphere $S^n(1/\sqrt{2})$ or Clifford hypersurface $S^n(1/\sqrt{2}) \times S^{n_1}(1/\sqrt{2})$ with $n_1 + n_2 = n$ and $n_1 \neq n_2$. Moreover, if $M^n$ is complete, then either $M^n$ is the hypersphere $S^n(1/\sqrt{2})$ or the Clifford hypersurface $S^n(1/\sqrt{2}) \times S^{n_1}(1/\sqrt{2})$ with $n_1 + n_2 = n$ and $n_1 \neq n_2$.

For biharmonic hypersurfaces with at most three distinct principal curvatures, Balmaș et al. obtained in [4] the following results.

**Theorem 2.4** Let $M^3$ be a biharmonic hypersurface of the space form $E^3(c)$. Then $M^3$ has constant mean curvature.

**Theorem 2.5** The only compact proper biharmonic hypersurfaces of $S^1$ are the hypersphere $S^3(1/\sqrt{2})$ and the torus $S^1(1/\sqrt{2}) \times S^2(1/\sqrt{2})$.

**Theorem 2.6** There exist no compact proper biharmonic hypersurfaces of constant mean curvature and with three distinct principal curvatures in the unit Euclidean spheres.

### 3 Biharmonic hypersurfaces with three distinct principal curvatures in $R^{n+1}(c)$

We will concentrate on an orientable biharmonic hypersurface $M^n$ in a space form $R^{n+1}(c)$ with $n \geq 4$. With the techniques developed by B. Y. Chen in [10] (see also [4, 12–15]), we firstly prove the following result.

**Theorem 3.1** Let $M^n$ be an orientable proper biharmonic hypersurface with at most three distinct principal curvatures in $R^{n+1}(c)$. Then $M^n$ has constant mean curvature.
It is known that the set $M_A$ of all points of $M$, at which the number of distinct eigenvalues of the Weingarten operator $A$ (i.e., the principal curvatures) is locally constant, is open and dense in $M^n$. Therefore, as $M^n$ has at most three distinct principal curvatures everywhere, one can work only on the connected component of $M_A$ consisting by points where the number of principal curvatures is three (it is already known that on the connected components of $M_A$ where the number of distinct principal curvatures is one or two, $M^n$ is CMC, i.e., the mean curvature is constant; in the end, by passing to the limit, $H$ will be constant on the whole $M^n$). On that connected component, the principal curvature functions of $A$ are smooth.

We now suppose that, on the component, the mean curvature $H$ is not constant. Thus, there is a point $x_0$ where $(\text{grad } H)(x_0) \neq 0$. In the following, we will work on an neighborhood of $x_0$ where $(\text{grad } H)(x_0) \neq 0$ at any point.

In view of the second equation of (2.6), we have that grad $H$ is an eigenvector of the Weingarten operator $A$ with the corresponding principal curvature $-\frac{n}{2} H$. Without loss of generality, we choose $e_1$ such that $e_1$ is parallel to grad $H$, and therefore the Weingarten operator $A$ of $M^n$ takes the following form with respect to a suitable orthonormal frame $\{e_1, \ldots, e_n\}$.

$$ A = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} , $$

(3.1)

where $\lambda_i$ are the principal curvatures and $\lambda_1 = -\frac{n}{2} H$. Since $e_1$ is parallel to grad $H$, we compute

$$ \text{grad } H = \sum_{i=1}^{n} e_i(H)e_i $$

and hence

$$ e_1(H) \neq 0, \quad e_i(H) = 0, \quad i = 2, 3, \ldots, n. $$

(3.2)

We write

$$ \nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \quad i, j = 1, 2, \ldots, n. $$

(3.3)

We compute the compatibility conditions $\nabla_{e_i} \langle e_i, e_j \rangle = 0$ and $\nabla_{e_i} \langle e_j, e_j \rangle = 0$, which imply respectively that

$$ \omega_{ij}^i = 0, \quad \omega_{ij}^k + \omega_{kj}^i = 0, $$

(3.4)

for $i \neq j$ and $i, j, k = 1, 2, \ldots, n$. Furthermore, we deduce from (3.1) and (3.3) and the Codazzi equation that

$$ e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ij}^j, $$

(3.5)

$$ (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_i - \lambda_j)\omega_{kj}^i $$

(3.6)

for distinct $i, j, k = 1, 2, \ldots, n$.

It follows from (3.2) and (3.3) that

$$ [e_i, e_j](H) = 0, \quad i, j = 2, 3, \ldots, n, \quad i \neq j, $$

which yields

$$ \omega_{ij}^1 = \omega_{j1}^i, $$

(3.7)

for distinct $i, j = 2, 3, \ldots, n$.

We claim that $\lambda_j \neq \lambda_1$ for $j = 2, 3, \ldots, n$. In fact, if $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (3.5) we have that

$$ 0 = (\lambda_1 - \lambda_j)\omega_{j1}^1 = e_1(\lambda_j) = e_1(\lambda_1), $$

(3.8)

which contradicts the first expression of (3.2).
By the assumption, $M^n$ is a nondegenerate hypersurface with three distinct principal curvatures. Without loss of generality, we assume that

$$
\lambda_2 = \lambda_3 = \ldots = \lambda_p = \alpha, \\
\lambda_{p+1} = \lambda_{p+2} = \ldots = \lambda_n = \beta
$$

for $\frac{n+1}{2} \leq p < n$. The multiplicities of principal curvatures $\alpha$ and $\beta$ are $p - 1$ and $n - p$, respectively.

By the definition (2.4) of $\vec{H}$, we have $n H = \sum_{i=1}^n \lambda_i$. Hence

$$
\beta = \frac{2nH - (p-1)\alpha}{n-p}. 
$$

(3.9)

Since $\lambda_j \neq \lambda_1$ for $j = 2, \ldots, n$, we obtain

$$
\alpha \neq \frac{n}{2} H - \frac{3n}{2(n-1)} H - \frac{n^2 - (p-3)n}{2(p-1)} H. 
$$

(3.10)

We will derive some information from (3.5).

Since $n \geq 4$, it follows from (3.9) that $p - 1 \geq 2$. For $i, j = 2, 3, \ldots, p$ and $i \neq j$ in (3.5), one has

$$
e_i(\alpha) = 0, \quad i = 2, 3, \ldots, p. 
$$

(3.11)

Depending on the multiplicity $n - p$ of the principal curvature $\beta$, we consider two cases:

**Case A:** $n - p \geq 2$. In this case, for $i, j = p + 1, \ldots, n$ and $i \neq j$ in (3.5) we have

$$
e_i(\beta) = 0, \quad i = p + 1, \ldots, n. 
$$

(3.12)

Hence, it follows directly from (3.2), (3.9), (3.11) and (3.12) that

$$
e_i(\alpha) = 0, \quad i = 2, \ldots, n. 
$$

(3.13)

**Case B:** $n - p = 1$. Then (3.11) reduces to

$$
e_i(\alpha) = 0, \quad i = 2, \ldots, n - 1. 
$$

(3.14)

In this case, we will show that $e_\nu(\alpha) = 0$ in the following.

Let us compute $[e_1, e_i](H) = (\nabla_{e_1} e_i - \nabla_{e_i} e_1)(H)$ for $i = 2, \ldots, n$. From the first expression of (3.4), we have $\omega_{11}^i = 0$. For $j = 1$ and $i \neq 1$ in (3.5), by (3.2) we have $\omega_{ij}^1 = 0 (i \neq 1)$. Hence we have

$$
e_i(\beta) = 0, \quad i = 2, \ldots, n. 
$$

(3.15)

By (3.14), with a similar way we can show that

$$
e_i(\alpha) = 0, \quad i = 2, \ldots, n - 1. 
$$

(3.16)

For $j = 1, k, i \neq 1$ in (3.6) we have

$$
(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1,
$$

which together with (3.7) yields

$$
\omega_{ik}^1 = 0, \quad i \neq j, \quad i, j = 2, \ldots n. 
$$

(3.17)

Combining (3.17) with the second equation of (3.4) gives

$$
\omega_{11}^i = 0, \quad i \neq j, \quad i, j = 2, \ldots n. 
$$

(3.18)

It follows from (3.5) that

$$
\omega_{11}^i = \frac{e_i(\lambda_i)}{\lambda_1 - \lambda_i}, \quad i = 2, \ldots n. 
$$

(3.19)
For \( k = 2 \) and \( i = n \) in (3.6), we have
\[
(\lambda_n - \lambda_j)\omega_{2n}^i = (\lambda_2 - \lambda_j)\omega_{n2}^i,
\]
which yields
\[
\omega_{2n}^i = 0, \quad j = 3, \ldots n - 1.
\]
Hence, from the first expression of (3.4) and (3.17) we get
\[
\omega_{2n}^i = 0, \quad j = 1, 3, \ldots n.
\] (3.20)
Also, (3.5) yields
\[
\omega_{2n}^2 = \frac{e_n(\alpha)}{\lambda_n - \alpha}.
\] (3.21)

In the following we will derive a useful equation.

From the Gauss equation and (3.1) we have \( R(e_2, e_n)e_1 = 0 \). Recall the definition of Gauss curvature tensor
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]
It follows from (3.16), (3.18–21) and (3.4) that
\[
\nabla_{e_2} \nabla_{e_n} e_1 = \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} e_2,
\]
\[
\nabla_{e_n} \nabla_{e_2} e_1 = e_n \left( \frac{e_1(\alpha)}{\lambda_1 - \alpha} \right) e_2 + \frac{e_1(\alpha)}{\lambda_1 - \alpha} \sum_{k=3}^{n} \omega_{kn}^k e_k,
\]
\[
\nabla_{(e_2, e_n)} e_1 = \frac{e_n(\alpha)e_1(\alpha)}{(\lambda_n - \lambda)(\lambda_1 - \alpha)} e_2 - \frac{e_1(\alpha)}{\lambda_1 - \alpha} \sum_{k=3}^{n} \omega_{kn}^k e_k.
\]

Hence
\[
e_n \left( \frac{e_1(\alpha)}{\lambda_1 - \alpha} \right) = \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} - \frac{e_n(\alpha)e_1(\alpha)}{(\lambda_n - \alpha)(\lambda_1 - \alpha)}.
\] (3.22)
Note that \( \lambda_1 = -\frac{2}{n}H \) and \( \lambda_n = \beta = \frac{2}{n}H - (n - 2)\alpha \) in this case.

It follows from (3.5) that
\[
\omega_{1i}^i = -\omega_{1i}^i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i}.
\] (3.23)
Consider the first equation of biharmonic equations (2.6). It follows from (3.1) and (3.19) that
\[
-e_1 e_1(H) + \left( \frac{(n - 2)e_1(\alpha)}{\lambda_1 - \alpha} - \frac{e_1(\lambda_n)}{\beta_1 - \lambda_n} \right) e_1(H) + H \left( \lambda_1^2 + (n - 2)\alpha^2 + \lambda_n^2 \right) = ncH.
\] (3.24)
Differentiating (3.24) along \( e_n \), by (3.2), (3.15) and (3.22) we get
\[
\frac{2}{\lambda_1 - \lambda_n} \left( \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{\alpha}{\lambda_1 - \alpha} \right) e_1(H)e_n(\alpha) + H \left( -3nH + 2(n - 1)\alpha \right)e_n(\alpha) = 0.
\]
If \( e_n(\alpha) \neq 0 \), then the above equation becomes
\[
\frac{2}{\lambda_1 - \lambda_n} \left( \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{\alpha}{\lambda_1 - \alpha} \right) e_1(H) + H \left( -3nH + 2(n - 1)\alpha \right) = 0.
\] (3.25)
Differentiating (3.25) along \( e_n \), using (3.22) and (3.25) one has
\[
\frac{2n(4 - n)H + 2(n - 2)(n - 1)\alpha}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} \left( \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{\alpha}{\lambda_1 - \alpha} \right) e_1(H)
\]
\[+ H \left( -7n + 10 \right)nH + 4(n - 1)(n - 2)\alpha = 0.
\] (3.26)
Therefore, combining (3.26) with (3.25) gives
\[(n - 2)H[3nH - 2(n - 1)\alpha]^2 = 0,\]
which implies that
\[\alpha = \frac{3n}{2(n - 1)}H.\]
This contradicts (3.10). Hence, we have that \(e_n(\alpha) = 0\).

Now we are ready to express the connection coefficients of hypersurfaces.

**Lemma 3.2** Let \(M^n\) be a biharmonic hypersurface with non-constant mean curvature in spheres \(S^{n+1}\), whose shape operator given by (3.1) with respect to an orthonormal frame \(\{e_1, \ldots, e_n\}\). Then we have

\[
\nabla_{e_1} e_1 = 0; \quad \nabla_{e_i} e_1 = \frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_i, \quad i = 2, \ldots, n;
\]

\[
\nabla_{e_j} e_i = \sum_{k=2, k \neq j}^{p} \omega^k_{ij} e_k, \quad i = 1, \ldots, n, \quad j = 2, \ldots, p, \quad i \neq j;
\]

\[
\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=2, k \neq i}^{p} \omega^k_{ii} e_k, \quad i = 2, \ldots, p;
\]

\[
\nabla_{e_i} e_j = \sum_{k=p+1, k \neq j}^{n} \omega^k_{ij} e_k, \quad i = 1, \ldots, n, \quad j = p + 1, \ldots, n, \quad i \neq j;
\]

\[
\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=p+1, k \neq i}^{n} \omega^k_{ii} e_k, \quad i = p + 1, \ldots, n,
\]
where \(\omega^k_{ij} = -\omega^l_{ij}\) for \(i \neq j\) and \(i, j, k = 1, \ldots, n\).

**Proof.** For \(j = 1\) and \(i = 2, \ldots, n\) in (3.5), by (3.2) we get \(\omega^1_{ii} = 0\). Moreover, by the first and second expressions of (3.4) we have

\[
\omega^1_{ij} = 0, \quad i = 1, \ldots, n. \tag{3.27}
\]

For \(i = 1, j = 2, \ldots, n\) in (3.5), we obtain

\[
\omega^i_{1j} = -\omega^1_{ij} = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}, \quad j = 2, \ldots, n. \tag{3.28}
\]

For \(i = p + 1, \ldots, n, j = 2, \ldots, p\) in (3.5), by (3.2) we have

\[
\omega^i_{jj} = 0. \tag{3.29}
\]

Similarly, for \(i = 2, \ldots, p, j = p + 1, \ldots, n\) in (3.5), we also have

\[
\omega^i_{jj} = 0. \tag{3.30}
\]

For \(i = 1, \) by choosing \(j, k = 2, \ldots, p\) or \(k, j = p + 1, \ldots, n\) \(j \neq k\) in (3.6), we have

\[
\omega^1_{kk} = 0. \tag{3.31}
\]

For \(i = 2, \ldots, p\) and \(j, k = p + 1, \ldots, n, j \neq k\) in (3.6), we get

\[
\omega^i_{ki} = 0. \tag{3.32}
\]

For \(i = 2, \ldots, p, j = 1\) and \(k = p + 1, \ldots, n\) in (3.6), one has

\[
(\alpha - \lambda_1)\omega^1_{ki} = (\beta - \lambda_1)\omega^1_{ki},
\]
which together with (3.7) and the second expression of (3.4) gives
\[
\omega_{ki}^1 = \omega_{ik}^1 = \omega_{ki}^k = 0. \tag{3.33}
\]
For \(i = 2, \ldots, p, k = 1\) and \(j = p + 1, \ldots, n\) in (3.6), we obtain
\[
(\beta - \alpha)\omega_{ij}^1 = (\lambda_1 - \alpha)\omega_{ij}^1,
\]
which together with (3.33) yields
\[
\omega_{ij}^1 = \omega_{ji}^1 = 0. \tag{3.34}
\]
Combining (3.27–3.34) with (3.4) completes the proof of the lemma. \(\square\)

Define two smooth functions \(A\) and \(B\) as follows:
\[
A = \frac{e_1(\alpha)}{\lambda_1 - \alpha}, \quad B = \frac{e_1(\beta)}{\lambda_1 - \beta}. \tag{3.35}
\]
One can compute the curvature tensor \(R\) by Lemma 3.2, and apply the Gauss equation for different values of \(X, Y\) and \(Z\). After comparing the coefficients with respect to the orthonormal basis \(\{e_1, \ldots, e_n\}\) we get the following:

- \(X = e_1, Y = e_2, Z = e_1, \)
  \[
e_1(A) + A^2 = -\lambda_1\alpha - c; \tag{3.36}
\]
- \(X = e_1, Y = e_n, Z = e_1, \)
  \[
e_1(B) + B^2 = -\lambda_1\beta - c; \tag{3.37}
\]
- \(X = e_n, Y = e_2, Z = e_n, \)
  \[
  AB = -\alpha\beta - c. \tag{3.38}
\]

Note that (3.38) is obtained by comparing the coefficient of \(e_2\) in the equation.

Compute the first equation of biharmonic equations (2.6) again. It follows from (3.1) and Lemma 3.2 that
\[
-e_1e_1(H) - [(p - 1)A + (n - p)B]e_1(H) + H \left[ \lambda_1^2 + (p - 1)\alpha^2 + (n - p)\beta^2 \right] = ncH. \tag{3.39}
\]

Lemma 3.3 The functions \(A\) and \(B\) are related by
\[
[(4 - p)A + (3 + p - n)B]e_1(H) + \frac{3n^2(n + 6 - p)}{4(n - p)}H^3 - \frac{3n(n - 2 + 4p)}{2(n - p)}H^2\alpha + \frac{3n(p - 1)}{n - p}H\alpha^2 - 3c(n + 1)H = 0. \tag{3.40}
\]

Proof. From (3.35), (3.36) and (3.37) respectively reduce to
\[
e_1e_1(\alpha) + 2Ae_1(\alpha) - Ae_1(\lambda_1) + (\lambda_1\alpha + c)(\lambda_1 - \alpha) = 0, \tag{3.41}
\]
\[
e_1e_1(\beta) + 2Be_1(\beta) - Be_1(\lambda_1) + (\lambda_1\beta + c)(\lambda_1 - \beta) = 0. \tag{3.42}
\]
By (3.9), it follows from the second expression of (3.35) that
\[
e_1(\alpha) = \frac{n - p}{p - 1} \left( \frac{3n}{2(n - p)}e_1(H) - e_1(\beta) \right),
\]
\[
= \frac{3n}{2(p - 1)}e_1(H) - \frac{n - p}{p - 1} B(\lambda_1 - \beta). \tag{3.43}
\]
Similarly, we have
\[ e_1(\beta) = \frac{3n}{2(n-p)} e_1(H) - \frac{p-1}{n-p} A(\lambda_1 - \alpha). \] (3.44)

Substitute (3.9) into (3.42). Eliminating \( e_1e_1(H) \) and \( e_1e_1(\alpha) \), from (3.38), (3.39) and (3.41-44) we obtain the desired Equation (3.40).

By the second expression of (3.35) and (3.9), (3.44) reduces to
\[ e_1(H) = -\left[ \frac{p-1}{3} H + \frac{2(p-1)}{3n} \alpha \right] A + \left[ \frac{n+3-p}{3} H + \frac{2(p-1)}{3n} \alpha \right] B. \] (3.45)

Substituting (3.45) into (3.40), by (3.38) we have
\[
(4-p)(p-1)(nH + 2\alpha)A^2 + (3+p-n)[n(n+3-p)H - 2(p-1)\alpha]B^2
\]
\[
= \frac{9n^2(n+6-p)}{4(n-p)} H^3 - \frac{3n^2(p-1)(2p - 2n - 15)}{2(n-p)} H^2 \alpha
\]
\[
+ \frac{n(p-1)(-2p^2 + 2pn + 11p + n - 12)}{n-p} H \alpha^2 - \frac{2(p-1)^2(2p - n - 1)}{n-p} \alpha^3
\]
\[
+ c(2p^2 - 5p - 2n^2 - 4nH) + 2c(p-1)(2p - n - 1)\alpha.
\] (3.46)

Multiplying A and B successively on the Equation (3.40), using (3.38) one gets respectively
\[
(4-p)A^2 e_1(H) = (3+p-n)(\alpha \beta + c) e_1(H)
\] (3.47)
\[
+ \left[ \frac{9n^2(n+6-p)}{4(n-p)} H^3 - \frac{3n^2(p-1)(2p - 2n - 15)}{2(n-p)} H^2 \alpha
\]
\[
+ \frac{n(p-1)(-2p^2 + 2pn + 11p + n - 12)}{n-p} H \alpha^2 - \frac{2(p-1)^2(2p - n - 1)}{n-p} \alpha^3
\]
\[
+ c(2p^2 - 5p - 2n^2 - 4nH) + 2c(p-1)(2p - n - 1)\alpha \right] A = 0.
\]

(3 + p - n)B^2 e_1(H) = (4-p)(\alpha \beta + c) e_1(H)
\] (3.48)
\[
+ \left[ \frac{9n^2(n+6-p)}{4(n-p)} H^3 - \frac{3n^2(p-1)(2p - 2n - 15)}{2(n-p)} H^2 \alpha
\]
\[
+ \frac{n(p-1)(-2p^2 + 2pn + 11p + n - 12)}{n-p} H \alpha^2 - \frac{2(p-1)^2(2p - n - 1)}{n-p} \alpha^3
\]
\[
+ c(2p^2 - 5p - 2n^2 - 4nH) + 2c(p-1)(2p - n - 1)\alpha \right] B = 0.
\]

Differentiating (3.40) along \( e_1 \), and using (3.36–37) and (3.39) we get
\[
\left[ (4-p) \left( \frac{n}{2} H \alpha - A^2 - c \right) + (3+p-n) \left( \frac{n}{2} H \beta - B^2 - c \right) \right] e_1(H)
\]
\[
- [(4-p)A + (3+p-n)B] [(p-1)A + (n-p)B] e_1(H)
\]
\[
+ [(4-p)A + (3+p-n)B] \left[ \frac{n^2}{4} H^3 + (p-1)H \alpha^2 + (n-p)H \beta^2 - ncH \right]
\]
\[
+ \left[ \frac{9n^2(n+6-p)}{4(n-p)} H^2 - \frac{3n^2(n-2+4p)}{n-p} H \alpha + \frac{3n(p-1)}{n-p} \alpha^2 - 3c(n+1) \right] e_1(H)
\]
\[
- \frac{3n(n-2+4p)}{2(n-p)} H^2 e_1(\alpha) + \frac{6n(p-1)}{n-p} Hae_1(\alpha) = 0.
\] (3.49)

Substituting (3.47), (3.48), (3.40) into (3.49), and using the first expression of (3.35) we obtain
\[
\left[ \frac{3n^2(2n-2p+21)}{4(n-p)} H^2 - \frac{3n(5p+1)}{n-p} H \alpha + \frac{(p-1)(2n+7)}{n-p} \alpha^2 - 2c(n+5) \right] e_1(H)
\]
\[
+ \left[ \frac{n^2(2pn-2p^2+7n+17p+30)}{4(n-p)} H^3 - \frac{3n(3np+2p^2+4p-3n-6)}{2(n-p)} H^2 \alpha
\]
\[
+ \frac{(p-1)(2np-2n+p-4)}{n-p} H \alpha^2 - c(2pn+3p+4n)H \right] A
\]
Moreover, it follows from (3.45) that the above equation further reduces to

\[
\left[ \frac{9}{4} n^3 (3n - 2p + 17) H^3 - \frac{3}{2} n^2 (-6p^2 + 11np + 43p - 11n - 37) H^2 \alpha \\
+ n(p - 1)(4np - 4n + 26p + 1) H \alpha^2 - 2(p - 1)^2 (2n + 7) \alpha^3 \\
+ cn(n - p)(-4np - 14n + 7p - 10) H + 4c(n - p)(n + 5)(p - 1) \alpha \right] A \\
- \left[ \frac{9}{2} (2n - 2p + 3) H^3 + \frac{9}{2} n^2 (2p^2 + n^2 - 3np - 7n + n - 3) H^2 \alpha \\
- 2n(p - 1)(2n^2 - 2np + 4n - 13p - 18) H \alpha^2 - 2(p - 1)^2 (2n + 7) \alpha^3 \\
+ cn(n - p)(-4n^2 + 4np - 11n + 9p + 21) H - 4c(n - p)(n + 5)(p - 1) \alpha \right] B = 0.
\] (3.50)

Now all the desired Equations (3.38), (3.46) and (3.50) concerning A and B are obtained.

In order to write handily, we introduce several notions: L, M denoting the coefficients of A and B respectively in (3.50), and N denoting the right hand side of quality in Equation (3.46). Then (3.46) and (3.50) become

\[
(4 - p)(p - 1)(nH + 2\alpha) A^2 \\
+ (3 + p - n)[n(n + 3 - p) H - 2(p - 1) \alpha] B^2 = N,
\] (3.51)

\[
LA - MB = 0.
\] (3.52)

Multiplying LM on the Equation (3.51), using (3.52) and (3.38) we can eliminate both A and B. Hence, we have

\[
(4 - p)(p - 1)(nH + 2\alpha) M^2 \left( \frac{\frac{3}{2} nH \alpha - (p - 1) \alpha^2}{n - p} + c \right) \\
+ (3 + p - n)[n(n + 3 - p) H - 2(p - 1) \alpha] L^2 \left( \frac{\frac{3}{2} nH \alpha - (p - 1) \alpha^2}{n - p} + c \right) \\
+ LMN = 0.
\] (3.53)

In view of (3.53), we note that the equation should take the following form:

\[
a_{90} H^9 + a_{81} H^8 \alpha + a_{72} H^7 \alpha^2 + a_{63} H^6 \alpha^3 + a_{54} H^5 \alpha^4 + a_{45} H^4 \alpha^5 \\
+ a_{36} H^3 \alpha^6 + a_{27} H^2 \alpha^7 + a_{18} H \alpha^8 + a_{09} \alpha^9 + c(a_{70} \alpha + a_{61} H \alpha^2) \\
+ a_{52} H^5 \alpha^2 + a_{43} H^4 \alpha^3 + a_{34} H^3 \alpha^4 + a_{25} H^2 \alpha^5 + a_{16} H \alpha^6 + a_{07} \alpha^7 \\
+ a_{50} H^5 + a_{41} H^4 \alpha + a_{32} H^3 \alpha^2 + a_{23} H^2 \alpha^3 + a_{14} H \alpha^4 + a_{05} \alpha^5 \\
+ a_{30} H^3 + a_{21} H^2 \alpha + a_{12} H \alpha^2 + a_{03} \alpha^3) = 0,
\] (3.54)

where the coefficients \( a_{ij} \) (\( i, j = 0, \ldots, 9 \)) are constants concerning \( n \) and \( p \).

From (3.53), (3.50) and (3.46), we compute \( a_{90} \) and \( a_{99} \) as follows

\[
a_{90} = \frac{729n^6(n - p + 6)(3n - 2p + 17)(2n - 2p + 3)}{32(n - p)}, \quad a_{99} = 0.
\]

Since \( n > p \), it is easy to see that \( a_{90} \neq 0 \).
Note that $\alpha$ is not constant in general. In fact, if $\alpha$ is a constant, then (3.54) becomes an algebraic equation of $H$ with constant coefficients. Thus, the real function $H$ satisfies a polynomial equation $q(H) = 0$ with constant coefficients, therefore it must be a constant. The conclusion follows immediately.

Consider an integral curve of $e_i$ passing through $p = \gamma(t_0)$ as $\gamma(t), t \in I$. Since $e_i(H) = e_i(\alpha) = 0$ for $i = 2, \ldots, n$ and $e_1(H), e_1(\alpha) \neq 0$, we can assume $t = t(\alpha)$ and $H = H(\alpha)$ in some neighborhood of $\alpha_0 = \alpha(t_0)$.

From the first expression of (3.35), (3.45) and (3.52), we have

\[
\frac{dH}{d\alpha} = \frac{dH}{dt} \frac{dt}{d\alpha} = \frac{e_1(H)}{e_1(\alpha)} \Rightarrow
\]

\[
= -\left(\frac{p-1}{3} H + \frac{2(p-1)}{3n} \alpha\right) A + \left(\frac{n+3-p}{3} H + \frac{2(p-1)}{3n} \alpha\right) B
\]

\[
= \frac{2(p-1)}{3n} + \left(\frac{n+3-p}{3n} H + \frac{2(p-1)}{3n} \alpha\right) B
\]

\[
= \frac{2(p-1)}{3n} + \frac{2((n+3-p)H - 2(p-1)\alpha)L}{3n(nH + 2\alpha)M}.
\]

Differentiating (3.54) with respect to $\alpha$ and substituting $\frac{dH}{d\alpha}$ from (3.55), combining these with (3.50) we get another algebraic equation of twelfth degree concerning $H$ and $\alpha$

\[
b_{12,0}H^{12} + b_{11,1}H^{11}\alpha + b_{10,2}H^{10}\alpha^2 + b_{9,3}H^9\alpha^3 + b_{8,4}H^8\alpha^4 + b_{7,5}H^7\alpha^5
\]

\[
+ b_{6,6}H^6\alpha^6 + b_{5,7}H^5\alpha^7 + b_{4,8}H^4\alpha^8 + b_{3,9}H^3\alpha^9 + b_{2,10}H^2\alpha^{10} + b_{1,11}H\alpha^{11}
\]

\[
+ b_{0,12}\alpha^{12} + c(b_{10,0}H^{10} + b_{10,1}H^9\alpha + b_{10,2}H^8\alpha^2 + b_{10,3}H^7\alpha^3 + b_{10,4}H^6\alpha^4
\]

\[
+ b_{11,5}H^5\alpha^5 + b_{11,6}H^4\alpha^6 + b_{11,7}H^3\alpha^7 + b_{11,8}H^2\alpha^8 + b_{11,9}H\alpha^9 + b_{11,10}\alpha^{10} + b_{11,11}\alpha^{11}
\]

\[
+ b_{12,6}H^6\alpha^6 + b_{12,7}H^5\alpha^7 + b_{12,8}H^4\alpha^8 + b_{12,9}H^3\alpha^9 + b_{12,10}H^2\alpha^{10} + b_{12,11}H\alpha^{11}
\]

\[
+ b_{13,8}H^8\alpha^8 + b_{13,9}H^7\alpha^9 + b_{13,10}H^6\alpha^{10} + b_{13,11}H^5\alpha^{11}
\]

\[
+ b_{14,10}H^{10}\alpha^{10} + b_{14,11}H^{11}\alpha^{11}
\]

\[
= 0,
\]

where the coefficients $b_{ij}$ (i, j = 0, \ldots, 12) are constants concerning $n$ and $p$.

Note that Equation (3.56) is non-trivial and different from (3.54).

We rewrite (3.54) and (3.56) respectively in the following forms

\[
\sum_{i=0}^{8} q_i(H)\alpha^i = 0,
\]

\[
\sum_{j=0}^{12} \tilde{q}_j(H)\alpha^j = 0,
\]

where $q_i(H)$ and $\tilde{q}_j(H)$ are polynomials concerning function $H$.

We may eliminate $\alpha$ between the two equations of (3.57). Multiplying $\tilde{q}_{12}(H)\alpha^4$ and $q_8(H)$ respectively on the first and second equations of (3.57), we obtain a new polynomial equation of $\alpha$ with eleventh degree. Combining this equation with the first equation of (3.57), we successively obtain a polynomial equation of $\alpha$ with tenth degree. In a similar way, by using the first equation of (3.57) and its consequences we are able to gradually eliminate $\alpha$.

At last, we obtain a non-trivial algebraic polynomial equation of $H$ with constant coefficients. Therefore, we conclude that the real function $H$ must be a constant, which contradicts our original assumption.

In summary, we have proved Theorem 3.1 as stated in the beginning part in this section.

Now we present our main theorem in the following.

**Theorem 3.4** Let $M^n$ be an orientable compact proper biharmonic hypersurface with at most three distinct principal curvatures in $\mathbb{S}^{n+1}$. Then $M^n$ is either the hypersphere $\mathbb{S}^n(1/\sqrt{2})$ or the Clifford hypersurface $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2})$ with $n_1 + n_2 = n$ and $n_1 \neq n_2$. 

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Proof. We only need to deal with the case of proper biharmonic hypersurfaces $M^n$ with three distinct principal curvatures in $S^{n+1}$. According to Theorem 3.1, $M^n$ has constant mean curvature $H$. Hence, Theorem 2.6 implies that this case is impossible, which together with Theorem 2.3 leads to the conclusion. □

Remark 3.5 Theorem 3.4 extends Balmuş et al’s results of Theorems 2.3, 2.5 and 2.6 in [3] and [4].

A result due to Oniciuc [17] says that a CMC biharmonic immersion in a space form $R^n(c)$ for $c \leq 0$ is minimal. Hence, combining this with Theorem 3.1 implies immediately that

**Theorem 3.6** There exist no proper biharmonic hypersurfaces with at most three distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$ or hyperbolic spaces $\mathbb{H}^{n+1}$ with arbitrary dimension.

Remark 3.7 T. Hasanis and T. Vlachos [15] proved that there exists no proper biharmonic hypersurface in $\mathbb{E}^4$ (see also [12]). And, it was proved recently by the author in [14] that there exists no proper biharmonic hypersurface with at most three distinct principal curvatures in Euclidean spaces. Thus, Theorem 3.6 recovers all the results in [15], [12], [14] and [4] for hyperfaces in $\mathbb{H}^4$.

Remark 3.8 Note that Theorem 3.6 gives an affirmative partial answer to the Generalized Chen’s conjecture.

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