THE ELECTROMAGNETIC WAVES GENERATED BY DIELECTRIC NANOPARTICLES

XINLIN CAO * AHCENE GHANDRICHE ** AND MOURAD SINI ‡

ABSTRACT. We estimate the electromagnetic fields generated by a cluster of dielectric nanoparticles embedded into a background made of a vacuum. The dielectric nanoparticles are small scaled but enjoy high contrast of their relative permittivity. Such scales/contrasts can be ensured using the Lorentz model with incident frequencies chosen appropriately close to the undamped resonance (appearing in the Lorentz model). Under certain ratio between their size and contrast, these nanoparticles generate resonances, called dielectric resonances. These resonances are characterized and computed via the spectrum of the electric Newtonian operator, stated on the support of nanoparticles, projected on the space of divergence-free fields with vanishing boundary normal components. We characterize the dominant field generated by a cluster of such dielectric-resonating nanoparticles. In this point-interaction approximation, the nanoparticles can be distributed to occupy volume-like domains or low dimensional hypersurfaces where periodicity is not required. The form of these approximations suggests that the effective electromagnetic medium, equivalent to the cluster of such nanoparticles, is a perturbation of the magnetic permeability and not the electric permittivity. The cluster can be tuned such that the equivalent permeability has positive or negative values (while the permittivity stays unchanged).

1. Introduction

1.1. Background. In the recent years, in the engineering community, there was a high gain of interests in studying electromagnetic wave propagating in resonating dielectric nanostructures [9, 20, 27, 30]. These media are composed of the homogeneous background in which we inject nano-scaled particles. In addition to be small scaled, these particles enjoy high values of their relative index of refraction with a relatively small $Q$-factor (i.e. the ratio between the absorption and diffusion coefficients is very small). These properties allow them to enjoy interesting and useful optical properties. The smallness of the $Q$-factor allow them to be less lossy as compared with other types of nanoparticles as the plasmonic ones. As such, they are more suited to be used in imaging techniques that require remote measurements, for instance. The high contrasts of the refraction index allow them to resonate at certain scales size/contrast (or at certain ranges of incident frequencies). Such resonance’s effects are very attractive in both imaging and material sciences. They are also potentially applicable for the design of highly nonlinear material (i.e. the Kerker effect).

Our interest in this topic is to understand the interaction between the light and the nanostructure. Precisely, we want to estimate the perturbations of the used incident field due to the presence of these nanostructures. Looking at the problem under this angle, we wish to derive the dominating terms in the expansion of the scattered field taking into account the whole structure of the composite, namely the (potentially high) number of the nanoparticles, their sizes and the high contrasts of the related indices of refraction. The derived close form of the dominating term will allow us to tune the structure, at will, so that the equivalent material will enjoy needed properties as sign changing and nonlinearity of the effective electric or magnetic susceptibility.

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* RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. Email: xinlin.cao@ricam.oeaw.ac.at. This author is supported by the Austrian Science Fund (FWF): P 32660.

** RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. Email: ahcene.ghandriche@ricam.oeaw.ac.at. This author is supported by the Austrian Science Fund (FWF): P 30756-NBL.

‡ RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. Email: mourad.sini@oeaw.ac.at. This author is partially supported by the Austrian Science Fund (FWF): P 30756-NBL and P 32660.
In this work, we focus on deriving these approximations for general shapes. This follows the lines of our previous works on point-interaction approximations (called also the Foldy-Lax approximation) for the electromagnetic fields. In [13, 14], we derived the close form of the dominating term in the case that the contrasts of the permittivity and the permeability of the particles are moderate. This dominating term is given in a form of a superposition of electric poles and magnetic dipoles with attached weights. These attached weights (that encode the structure of the composite) are vectors that are computable by solving an invertible algebraic system. The analysis is based on the related Lippmann-Schwinger system of equations coupling the electric and magnetic fields. This Lippmann-Schwinger system is defined through an operator that couples the electric vector Newtonian operator and the Magnetization operator. As the contrasts of the particles are moderate, both the two operators play equivalent roles. That is why the derived approximation in [13, 14] involves contributions from the electric and magnetic polarization tensor with equivalent roles. The equivalent media generated by such moderately contrasting nanoparticles have been derived in [15] that confirms that both the electric permittivity and the magnetic permeability are perturbed. However, in the case of dielectric nanoparticles, as in our current work, the permittivity has high contrast while the permeability stays moderate (or unchanged). In this case, the effect of the vector Newtonian operator dominates the one of the Magnetization operator. In addition, under critical scales between the sizes and the contrasts, of the particles, one can excite a subfamily of the eigenvalues of the Newtonian operator. These are related to the projection of this Newtonian operator on the subspace of divergence free fields with zero normal components. This makes the analysis more involved and subtle than that in [13, 14]. However, this is worth the efforts. Indeed, we show that even though the contrasts of the nanoparticles are due to the permittivity, the dominating field is of magnetic type. In other words, the point-interaction approximation, due to the presence of the whole cluster of dielectric nanoparticles, suggests that the effective medium will be a perturbation of the magnetic permeability and not the electric permittivity. The justification of such effective medium, in different scenarios including the Moiré metamaterials, see [34], will be done in a forthcoming work.

Wave propagation in resonating media is highly attractive and, in the very recent years, we witness a rapid growth of the number of published works in the mathematical community (in addition to the relatively large engineering literature). These resonances occur if the particles are small-sized and enjoy high or negative contrasts of their materials. In effect, they are related to the spectrum of the volumetric operator (Newtonian potential operator) or surface operators (Neumann-Poincaré operator or related Magnetization operator) appearing in the resolvent operator modeling the wave propagation. With such contrasts of the materials, one can excite these resonances and enhance the values of scattering coefficients (or generally the polarization tensors). Such enhancements are useful in many applications as in the effective medium theory. In this direction, let us cite the following works [2–6] for the acoustic propagation. As far as the effective medium theory is concerned, few results are known for the electromagnetism with highly contrasting or negative permittivity or permeability. We can cite [1, 10–12, 16–18, 28, 32, 33] who assume periodicity and derive the equivalent coefficients, via the homogenization theory, for dielectric nanoparticles. The results provided in the current work is a contribution to fill in this gap.

1.2. Main results. Let $D$ be a bounded and Lipschitz-regular domain in $\mathbb{R}^3$. We assume that $D$ is a non-magnetic material meaning that the permeability $\mu$ is constant everywhere and equals to the one of the vacuum $\mu_0$. The electric permittivity is equal to the one of the vacuum outside $D$, $\epsilon_0$, but inside, we assume it to have different values $\epsilon_p$. We denote by the relative permittivity and permeability as $\epsilon_r := \frac{\epsilon}{\epsilon_0}$ and $\mu_r := \frac{\mu}{\mu_0}$. With such notations, we have $\epsilon_r = 1$ outside $D$ while $\mu_r = 1$ in the whole space $\mathbb{R}^3$.

The electromagnetic scattering of time-harmonic plane waves from the body $D$ reads as follows

\begin{equation}
\begin{cases}
\text{Curl}(E^T) - ik H^T = 0 & \text{in } \mathbb{R}^3 \\
\text{Curl}(H^T) + ik \epsilon_r E^T = 0 & \text{in } \mathbb{R}^3,
\end{cases}
\end{equation}

where $E$ and $H$ are the electric and magnetic fields, respectively.
where the total field \((E^T, H^T)\) is of the form \((E^T := E^{inc} + E^s, H^T := H^{inc} + H^s)\) with and incident plane wave \((E^{inc}, H^{inc})\) of the form
\[
E^{inc}(x, \theta) = \theta \cdot (i k \theta \cdot x) \quad \text{and} \quad H^{inc}(x, \theta) = (\theta \times \theta) \exp(i k \theta \cdot x),
\]
and the scattered field \((E^s, H^s)\) satisfies the Silver-Müller radiation conditions (SMRC) at infinity:
\[
\sqrt{|\mu|} e^{k|\theta|} H^s(x) \times \frac{x}{|x|} - E^s(x) = O\left(\frac{1}{|x|^2}\right).
\]

This problem is well posed in appropriate Sobolev spaces, see \([19, 29]\), and we have the behaviors
\[
(1.2) \quad E^s(x) = \frac{e^{ik|x|}}{|x|} \left(E^\infty(\hat{x}) + O(|x|^{-1})\right), \quad \text{as} \quad |x| \to \infty,
\]
and
\[
H^s(x) = \frac{e^{ik|x|}}{|x|} \left(H^\infty(\hat{x}) + O(|x|^{-1})\right), \quad \text{as} \quad |x| \to \infty,
\]
where \((E^\infty(\hat{x}), H^\infty(\hat{x}))\) is the corresponding electromagnetic far field pattern of \((1.1)\) in the propagation direction \(\hat{x} := \frac{x}{|x|} \).

Next, we present the needed assumptions on the model \((1.1)\) to derive our results.

I. **Assumptions on the cluster of particles.** Suppose that each component \(D_m\) of \(D\) is of the form \(D_m = a_b m + z_m, \ m = 1, \cdots, N\), which is characterized by the parameter \(a > 0\) and the location \(z_m\). We denote \(a := \max_{1 \leq m \leq M} \text{diam}(D_m), \ d := \min_{1 \leq m, j \leq M} d_{m,j} := \min_{1 \leq m, j \leq M} \text{dist}(D_m, D_j)\). We take
\[
(1.3) \quad \kappa = O(d^{-3}) \quad \text{and} \quad a \sim a^t
\]
with the nonnegative parameter \(t \leq 1\). For simplicity of the exposition, we assume that the shapes of \(B_m\)'s are the same and we denote \(B := B_m\). The domain \(B\) is a bounded Lipschitz domain that contains the origin.

II. **Assumption on the shape of \(B\).** Define the vector Newtonian operator \(N^0\). Denote \(e_n^{(1)}\) as the corresponding eigenfunctions over the subspace \(H_0(\text{div} = 0)\). Since \(H_0(\text{div} = 0) \equiv \text{Curl}(H_0(\text{Curl}) \cap H(\text{div} = 0))\), see for instance \([8]\), then we have
\[
(1.4) \quad e_n^{(1)} = \text{Curl}(\phi_n) \quad \text{with} \quad \nu \times \phi_n = 0 \quad \text{and} \quad \text{div}(\phi_n) = 0.
\]
We assume that, for a certain \(n_0\),
\[
\int_B \phi_{n_0}(y) \ dy \neq 0.
\]

III. **Assumptions on the permittivity and permeability of each particle.** In order to investigate the electromagnetic scattering of dielectric nanoparticles with high contrast electric permittivity parameter, we assume that for a constant \(\varsigma\),
\[
(1.5) \quad \eta := \epsilon_r - 1 = \varsigma a^{-2}, \quad \text{with} \quad \varsigma < 1, \quad \mathcal{A} \ll 1,
\]
and the magnetic permeability \(\mu_r\) to be moderate, namely \(\mu_r = 1\).

IV. **Assumption on the used incident frequency \(k\).** There exists a positive constant \(c_0\) such that
\[
(1.6) \quad 1 - k^2 \eta a^2 \lambda^{(1)}_{n_0} = \pm \epsilon_0 a^h, \quad a \ll 1,
\]
where \(\lambda^{(1)}_{n_0}\) is the eigenvalue corresponding to \(e^{(1)}_{n_0}\).

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1 The number of particles that can be distributed in a given bounded (3D) domain is \(\kappa = O(d^{-3})\). The condition on \(\varsigma\), i.e. \(\varsigma < 1\), is needed only if we distribute the maximum number of particles, i.e. \(\kappa \sim d^{-3}\). In addition, \(\varsigma\) can be complex and in this case we need \(\text{Re}(\varsigma) < 1\) with \(\text{Im}(\varsigma) \ll 1\), see Remark \([1, 2]\) for an example.
Remark 1.1. The conditions (1.5) and (1.6) can be derived from the Lorentz model by choosing appropriate incident frequency \( k \). Indeed, we recall the Lorentz model for the relative permittivity \( \epsilon_r = 1 - \frac{k_p^2}{k_0^2 - k^2 - ik\gamma} \) where \( k_p \) is the plasmonic resonance, \( k_0 \) is the undamped resonance and \( \gamma \) is the damping frequency. We also recall the eigenvalues \( \lambda_{n_0}^{(1)} \) of the Newtonian operator stated on \( B \), and the contrast parameter \( \eta = \epsilon_r - 1 \). If the frequency of the incident wave \( k \) is chosen to be real and \( k^2 \) close to the undamped resonance frequency \( k_0^2 \) with
\[
k^2 - k_0^2 = \frac{k_p^2 a^2 \lambda_{n_0}^{(1)} k_0^2}{(1 + \text{Re}(c_0) a^2) + \frac{\text{Im}^2(c_0) a^2}{(1 + \text{Re}(c_0) a^2)} - k_p^2 a^2 \lambda_{n_0}^{(1)} k_0^2} = k_p^2 a^2 \lambda_{n_0}^{(1)} k_0^2 [1 + O(a^2)]
\]
and
\[
\gamma k = \frac{\text{Im}(c_0) a^2 (k^2 - k_0^2)}{(1 + \text{Re}(c_0) a^2)} = \pm \text{Im}(c_0) a^4 \lambda_{n_0}^{(1)} k_0^2 k_p^2 [1 + O(a^2)],
\]
then we obtain \( \text{Re}(\eta) = a^{-2} (\lambda_{n_0}^{(1)} k_0^2)^{-1} + O(a^2) \) and \( \text{Im}(\eta) = \mp \text{Im}(c_0) (\lambda_{n_0}^{(1)} k_0^2)^{-1} + O(a^4) \). With these choices, we see that \( \frac{k^2}{k_0^2} \sim 1 \). In addition, we derive the needed relation \( 1 - k^2 \eta a^2 \lambda_{n_0}^{(1)} = \pm c_0 a^2 \), \( \alpha \ll 1 \) with a given value of \( c_0 \in \mathbb{C} \). In practice, an additional condition on the sign of \( \text{Im}(c_0) \) might be needed to ensure the non-negativity of the damping, i.e. \( \text{Im}(\epsilon_r) \geq 0 \). In particular, if \( \text{Im}(c_0) = 0 \) then \( \gamma = 0 \) and then we get the Drude’s model.

Remark 1.2. From the previous remark we deduce that \( \varsigma \), given in (1.6), can be approximated by
\[
\varsigma = \frac{1}{\lambda_{n_0}^{(1)} k_0^2} \left( 1 + \frac{\text{Im}(c_0) a^2}{(1 + O(a^2))} \right) = \frac{1}{\lambda_{n_0}^{(1)} k_0^2} + O(a^2),
\]
and the condition \( \text{Re}(\varsigma) < 1 \) will be fulfilled as long as \( 1 < \lambda_{n_0}^{(1)} k_0^2 \) with \( a \ll 1 \). If \( \text{Im}(c_0) = 0 \) then \( \varsigma \) is real valued.

Based on the above conditions, we are now in a position to state our main result.

**Theorem 1.3.** Let the conditions (I, II, III, IV) on the electromagnetic scattering problem (1.1), by the multiple particles, \( D_1, \ldots, D_N \), be satisfied. For \( t \) and \( h \) in \( [0, 1] \) such that
\[
3 - 3t - h \geq 0, \quad \text{and} \quad \frac{9}{11} < h < 1, \tag{1.7}
\]
the far field of the scattered wave admits the following expansion
\[
E^\infty(\hat{x}) = -ik\eta \sum_{m=1}^{N} e^{ik\hat{z} \cdot \hat{x}} Q_m + O(a^h), \tag{1.8}
\]
where \( (Q_m)_{m=1, \ldots, N} \) is the vector solution to the algebraic system
\[
Q_m - \eta k^2 a^{5-h} \sum_{j \neq m}^{N} \mathbf{P}_0 \cdot \Upsilon_k(z_m, z_j) \cdot Q_j = ik a^{5-h} \mathbf{P}_0 \cdot H^{\text{inc}}(z_m), \tag{1.9}
\]
where \( \Upsilon_k \) is the dyadic Green’s function given by (2.3) and \( \mathbf{P}_0 \) is the polarization matrix defined by
\[
\mathbf{P}_0 := \sum_{m} \int_{B} \phi_{n_0, m}(y) dy \otimes \int_{B} \phi_{n_0, m}(y) dy,
\]
where \( \phi_{n_0, m} \) fulfills
\[
\epsilon_{n_0, m}^{(1)} = \text{Curl}(\phi_{n_0, m}), \quad \text{div}(\phi_{n_0, m}) = 0, \quad \nu \times \phi_{n_0, m} = 0 \quad \text{and} \quad N\left(\epsilon_{n_0, m}^{(1)}\right) = \lambda_{n_0}^{(1)} \epsilon_{n_0, m}^{(1)}. \tag{1.10}
\]
\(^2\)The limit value \( h = 1 \) can be handled at the expense of assuming the constant \( c_0 \) appearing in (1.6) to be large enough. The lower bound is only sufficient and not optimal.
In particular, (1.9) is invertible under the condition that
\[
\frac{k^2 |\eta| a^5}{d^3 \left| 1 - k^2 \eta a^2 \lambda_{10}^{(1)} \right|} \|P_0\| < 1.
\]

We observe from (1.8)-(1.9) that if the number of the particles behaves like \( \aleph \sim d^{-3} (\sim a^{-3t}) \), with \( 3 - 3t - h = 0 \), and the particles are distributed in a bounded domain \( \Omega_{\text{eff}} \), then, at least formally, the electric field is equivalent to the one generated by an effective medium given by \( (\epsilon_{\text{eff}} := \epsilon_0, \mu_{\text{eff}} := \mu_0 + \mu_{\text{pert}} \chi_{\Omega}) \) where \( \mu_{\text{pert}} \) is related to the cluster of particles and it can be positive or negative (depending on the chosen sign in (1.6)). In particular, we see that even though the dielectric nanoparticles are merely generated by the contrasts of their permittivity (and not their permeability), the effective medium is a perturbation of the permeability and not the permittivity.

Before closing this introduction, let us mention that quite recently, after closing the content of the current work, we got aware of the work [7], where the authors derived similar approximation formulas in the case of a single dielectric nanoparticle. We note that the error term in (1.8), of the order \( O(a^{-3t}) \), takes into account the number of particles of the order \( \aleph = O(a^{-3t}) \) with \( t \) satisfying (1.7). For the case of a single particle, this error term is smaller and it is dominated by the first term in (1.8), with \( \aleph = 1 \). This can be seen at the end of Section 4 with more details.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries including the Lippmann-Schwinger system of equations for the solution to (1.1) and the needed decomposition of the vector \( L^2 \) space via the subspaces \( H_0(\text{div} = 0) \), \( H_0(\text{Curl} = 0) \) and \( \nabla \text{Harmonic} \). Then we present some needed a-priori estimates of the projections of the solution on these three subspaces, respectively, which will be proved later in Section 5. In Section 3, we first investigate the general expressions of the linear algebraic system and the corresponding invertibility conditions. Then we formulate the precise form of the linear algebraic systems associated to the projection of the total wave onto the two subspaces \( H_0(\text{div} = 0) \) and \( \nabla \text{Harmonic} \) of \( L^2 \), respectively. Detailed proofs will be showed later in Section 6. Section 4 is devoted to show the rigorous proof of our main theorem, in particular the Foldy-Lax approximation of the far-field. Finally, in Section 5 and Section 6, we give the detailed proofs of the a-priori estimates introduced in Section 2 and the construction of the linear algebraic systems in Section 3, respectively.

2. Some preliminary knowledge and a-prior estimates.

2.1. Decomposition of \( L^2 \). The following direct sum provide a useful decomposition of \( L^2 \) (see [20], page 314)
\[
L^2 = H_0(\text{div} = 0) \oplus H_0(\text{Curl} = 0) \oplus \nabla \text{Harmonic}
\]
(2.1)
where
\[
H_0(\text{div} = 0) := \left\{ E \in (L^2(D))^3, \text{div} E = 0, \nu \cdot E = 0 \text{ on } \partial D \right\},
\]
\[
H_0(\text{Curl} = 0) := \left\{ E \in (L^2(D))^3, \text{Curl} E = 0, \nu \times E = 0 \text{ on } \partial D \right\},
\]
and
\[
\nabla \text{Harmonic} := \left\{ E : E = \nabla \psi, \psi \in H^1(D), \Delta \psi = 0 \right\}.
\]
From the decomposition (2.1), we define \( \frac{1}{P}, \frac{2}{P} \) and \( \frac{3}{P} \) to be the natural projectors as follows
\[
\frac{1}{P} := L^2 \rightarrow H_0(\text{div} = 0), \quad \frac{2}{P} := L^2 \rightarrow H_0(\text{Curl} = 0) \quad \text{and} \quad \frac{3}{P} := L^2 \rightarrow \nabla \text{Harmonic}.
\]
2.2. Lippmann-Schwinger integral formulation of the solutions. Recall the Green’s function for the Helmholtz operator
\[
\Phi_k(x, y) = \frac{1}{4\pi} e^{ik|x-y|}, \quad x \neq y,
\]
and the corresponding dyadic Green’s function
\[
(2.3) \quad \Upsilon_k(x, y) := \text{Hess} \Phi_k(x, y) + k^2 \Phi_k(x, y) I, \quad x \neq y.
\]
For any vector function \( F \), define the Newtonian potential operator \( N^k \) and the Magnetization operator \( \nabla M^k \) as follows:
\[
(2.4) \quad N^k(F)(x) := \int_D \Phi_k(x, y)F(y)\,dy \quad \text{and} \quad \nabla M^k(F)(x) := \nabla \int_D \Phi_k(x, y) \cdot F(y)\,dy.
\]
The solution to \((1.1)\) of the integral form can be formulated as the following proposition.

**Proposition 2.1.** The solution to the electromagnetic scattering problem \((1.1)\) satisfies
\[
(2.5) \quad E^T(x) - \eta \int_D \Upsilon_k(x, y) \cdot E^T(y)\,dy = E^{inc}(x), \quad x \in D,
\]
where \( \eta := \varepsilon_r - 1 \) is the contrast of the electric permittivity, and \( \Upsilon_k(x, y) \) is the dyadic Green’s function defined by \((2.3)\). Equivalently, we have
\[
(2.6) \quad E^T(x) + \eta \nabla M^k(E^T)(x) - k^2 \eta N^k(E^T)(x) = E^{inc}(x), \quad x \in D,
\]
by virtue of the Newtonian operator \( N^k \) and the Magnetization operator \( \nabla M^k \) given by \((2.4)\).

The proposition can be proved by utilizing the Stratton-Chu formula directly, see [19, Theorem 6.1] for more detailed discussions.

In addition, in \((2.3)\), we have
\[
H_{\text{ess}} \Phi_k(x, y) = H_{\text{ess}} \Phi_0(x, y) + H_{\text{ess}}(\Phi_k - \Phi_0)(x, y)
\]
\[
= H_{\text{ess}} \Phi_0(x, y) + \frac{k^2}{4\pi} \sum_{n \geq 1} \frac{(ik)^{n+1}}{(n+1)!} Hess(||x-y||^n)
\]
\[
= H_{\text{ess}} \Phi_0(x, y) - \frac{k^2}{2} \Phi_0(x, y) I_3 - \frac{i k^3}{24 \pi} I_3 + \frac{k^2}{2} \Phi_0(x, y) \frac{A(x, y)}{||x-y||^2}
\]
\[
+ \frac{1}{4\pi} \sum_{n \geq 3} \frac{(ik)^{n+1}}{(n+1)!} Hess(||x-y||^n),
\]
where \( A \) is the matrix given by \( A(x, y) := (x-y) \otimes (x-y) \). Then, we can write the Magnetization operator \( \nabla M^k \) with the help of \((2.4)\) as
\[
\nabla M^k(F)(x) = \nabla M^0(F)(x) + \frac{k^2}{2} N^0(F)(x) + \frac{i k^3}{12 \pi} \int_D F(y)\,dy - \frac{k^2}{2} \int_D \Phi_0(x, y) \frac{A(x, y) \cdot F(y)}{||x-y||^2}\,dy
\]
\[
- \frac{1}{4\pi} \sum_{n \geq 3} \frac{(ik)^{n+1}}{(n+1)!} \int_D Hess(||x-y||^n) \cdot F(y)\,dy.
\]
(2.8)

Similarly, we write an analogous formula for the Newtonian potential operator \( N^k \) as
\[
(2.9) \quad N^k(F)(x) = N^0(F)(x) + \frac{i k}{4\pi} \int_D F(y)\,dy + \frac{1}{4\pi} \sum_{n \geq 1} \frac{(ik)^{n+1}}{(n+1)!} \int_D ||x-y||^n F(y)\,dy.
\]
For shortness reason, we use the notation \( \nabla M \) instead of \( \nabla M^0 \) and \( N \) instead of \( N^0 \) in the subsequent analyses.

Thanks to Green’s formulas, and recalling the \( L^2 \)-space decomposition \((2.1)\), we have
\[
(2.10) \quad \forall E \in H_0(\text{div} = 0), \quad \nabla M(E) = 0 \quad \text{and} \quad \forall E \in H_0(\text{Curl} = 0), \quad \nabla M(E) = E,
\]
and other nice properties for the Magnetization operator, such as self-adjointness, positivity, spectrum, boundedness \((\|\nabla M\| = 1)\), invariance of \(\nabla \text{Harmonic}\), etc., can be found in [31] and [22].

2.3. A-prior Estimates. Based on the decomposition (2.1), we present here some necessary a-prior estimates derived from the Lippmann-Schwinger equation (2.6), which play an important role in the proof of our main results. The proof of the propositions and lemmas in this subsection shall be given later in Section 5.

Following the notations in (2.2), we denote
* \(\left(\lambda_n^{(1)}; e_n^{(1)}\right)\) as the eigensystem of the operator \(N\) projected on the subspace \(H_0(\text{div} = 0)\),
* \(\left(\lambda_n^{(2)}; e_n^{(2)}\right)\) as the eigensystem of the operator \(N\) projected on the subspace \(H_0(\text{Curl} = 0)\),
* \(\left(\lambda_n^{(3)}; e_n^{(3)}\right)\) as the eigensystem of the operator \(\nabla M\) on the subspace \(\nabla \text{Harmonic}\).

For the existence and the construction of \(\left(\lambda_n^{(j)}; e_n^{(j)}\right)\), \(n \in \mathbb{N}, j = 1, 2, 3\), we refer to Section 5 of [23].

Now, suppose \(E_T\) solves (1.1) with the integral formulation (2.6). Then the projection of \(E_T\) onto three subspaces \(H_0(\text{div} = 0)\), \(H_0(\text{Curl} = 0)\) and \(\nabla \text{Harmonic}\) can be respectively represented as \(1^*P(E_T)\), \(2^*P(E_T)\) and \(3^*P(E_T)\). Then we have the following estimates.

2.3.1. Estimation for one particle.

**Proposition 2.2.** Under Assumptions (I, II, III, IV), consider the electromagnetic scattering problem (1.1) with only one distributed particle. Let \(k\) fulfill

\[
(2.11) \quad k^2 := \frac{1 \mp c_0 a h}{\eta a^2 \lambda_n^{(1)}(0)} \sim 1,
\]

where \(\lambda_n^{(1)}(0)\) is an eigenvalue of the Newtonian potential operator in the subspace \(H_0(\text{div} = 0)\). Then for \(h < 2\), there holds

\[
(2.12) \quad \|\tilde{E}_T\|_{L^2(B)} = O(a^{1-h}),
\]

and in particular,

\[
(2.13) \quad 2^*P(\tilde{E}_T) = 0,
\]

where \(\tilde{E}_T\) is the total wave of \(E_T\) after scaling from \(D\) to \(B\).

The detailed proof of Proposition 2.2 can be seen in Subsection 5.1. Based on the above a-priori estimate for one particle, we also investigate the estimation of the solution in the presence of multiple particles.

2.3.2. Estimation for several particles.

**Proposition 2.3.** Under Assumptions (I, II, III, IV), consider the electromagnetic scattering problem (1.1) for multiple particles \(D_j, j = 1, 2, \cdots, \aleph\). Then for \(t\) and \(h\) satisfying

\[
(2.14) \quad 3 - 3t - h \geq 0, \quad \text{and} \quad \frac{9}{11} < h < 1,
\]

there holds the estimation

\[
(2.15) \quad \max_j \left\|\frac{1}{P}(\tilde{E}_T)\right\|_{L^2(B)}^2 \lesssim a^{2-2h} \quad \text{and} \quad \max_j \left\|\frac{3}{P}(\tilde{E}_T)\right\|_{L^2(B)}^2 \lesssim a^{3+h}.
\]

From (2.15), we expect that for \(j = 1, 2, \cdots, \aleph, \frac{1}{P}(\tilde{E}_T)\) dominates. To clear the structure of the paper, we also postpone the proof of Proposition 2.2 to Subsection 5.2.
2.3.3. Estimation for the scattering coefficient. Recall the solution to the electromagnetic scattering problem with the form (2.6). Let \( W \) be the solution to
\[
(I + \eta \nabla M^{-k} - k^2 \eta N^{-k}) (W) (x) = \mathcal{P}(x, z), \quad x \in D,
\]
where \( \nabla M^{-k} \) and \( N^{-k} \) are the adjoint operators to \( \nabla M^k \) and \( N^k \) introduced in (2.4), respectively, and \( \mathcal{P}(x, z) \) is the matrix form of the vector \( (x - z) \), which can be expressed by
\[
\mathcal{P}(x, z) = \begin{pmatrix}
(x - z)_1 I_3 \\
(x - z)_2 I_3 \\
(x - z)_3 I_3
\end{pmatrix},
\]
with \( (x - z)_i, i = 1, 2, 3 \), being the corresponding components. Define the scattering coefficient as
\[
\mathcal{C} := \int_D W(x) \, dx = \int_D (I + \eta \nabla M^{-k} - k^2 \eta N^{-k})^{-1} (\mathcal{P}(\cdot, z))(x) \, dx.
\]
Then there holds the following estimates of \( \mathcal{C} \).

Proposition 2.4. The scattering coefficient \( \mathcal{C} \) defined by (2.17) satisfies
\[
\mathcal{C} \sim a^{4-h} \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \mathcal{P} \left( N \left( e_n^{(3)} \right) \right) ; e_n^{(1)} \langle \mathcal{P}(\cdot, 0) ; e_n^{(1)} \rangle \otimes \int_B e_n^{(3)}(x) \, dx = \mathcal{O} \left( a^{6-h} \right),
\]
where \( \lambda_n^{(3)} \), \( e_n^{(1)} \) and \( e_n^{(3)} \) are of the eigensystem introduced in Subsection 2.3. Moreover, after scaling from \( D \) to \( B \), the solution \( W \) to (2.16) satisfies
\[
\left\| \frac{1}{2} \mathcal{P} \left( \tilde{W} \right) \right\|_{L^2(B)} = \mathcal{O}(a^{-h}), \quad \left\| \frac{1}{2} \mathcal{P} \left( \tilde{W} \right) \right\|_{L^2(B)} = \mathcal{O}(a^3), \quad \left\| \frac{3}{2} \mathcal{P} \left( \tilde{W} \right) \right\|_{L^2(B)} = \mathcal{O}(a^3).
\]

We refer to Subsection 5.3 for the proof of Proposition 2.4.

3. Linear algebraic system and corresponding invertibility discussions.

In this section, we first present a linear algebraic system of general form and study the corresponding invertibility condition. Then, by projecting the solution \( E \) onto the two subspaces \( \mathbb{H}_0(\text{div} = 0) \) and \( \mathcal{H}_{\text{Harmonic}} \), we give the precise formulation of the linear algebraic systems with respect to \( \tilde{\mathcal{P}} (E^T) \) and \( \tilde{\mathcal{P}} (E^T) \), respectively. The estimates for the significant polarization tensors appearing in the linear algebraic systems related to those two subspaces are also investigated. These expressions and estimations are essentially related to our main result of the Foldy-Lax approximation to the far-field.

3.1. General linear algebraic system and corresponding invertibility condition. Recall that \( D_j, j = 1, 2, \cdots, N \), are the cluster of particles introduced in Assumption (I, II, III, IV). Denote by \([P_{D_j}^3], j = 1, 2, \cdots, N\), the polarization tensors, which carry the information of the geometry of the scatterer and the material parameters for the electromagnetic scattering problem (1.1).

Proposition 3.1. Let \( J_j, j = 1, 2, \cdots, N \), be a sequence of vectors. The linear algebraic system
\[
Q_{j_0} - \eta \sum_{j \neq j_0}^N [P_{D_{j_0}}^3] \cdot \mathcal{T}_k(z_{j_0}, z_j) \cdot Q_j = J_{j_0}, \quad j_0 = 1, 2, \cdots, N,
\]
is invertible under the condition that
\[
|\eta| \max_{j=1, \cdots, N} \left\| [P_{D_j}^3] \right\|_{L^\infty(\Omega)} d^{-3} < 1.
\]

We refer to Section 5 for the detailed proof of this proposition. On the basis of Proposition 3.1, next, we present two concrete forms of the linear algebraic systems with respect to \( \tilde{\mathcal{P}}(E_j^T) \) and \( \tilde{\mathcal{P}}(E_j^T) \), \( j = 1, 2, \cdots, N \), which shall play an essential role in the study of our main approximation result for the far-field.
3.2. Construction of the linear algebraic system for $\mathbb{P}(E^T)$. Recall that

\begin{equation}
\mathbb{H}_0(\text{div} = 0) = \text{Curl}(\mathbb{H}_0(\text{Curl}) \cap \mathbb{H}(\text{div} = 0)).
\end{equation}

For $j = 1, 2, \cdots, N$, $\mathbb{P}(E^T_j)$ denotes the projection of the total wave $E^T$ onto the subspace $\mathbb{H}_0(\text{div} = 0)$ with respect to the particle $D_j$. Then from (3.3), we can write

\begin{equation}
\mathbb{P}(E^T_j) = \text{Curl}(F_j) \quad \text{with} \quad \nu \times F_j = 0, \ \text{div}(F_j) = 0.
\end{equation}

Setting

\begin{equation}
Q_j := \int_{D_j} F_j(y) \, dy,
\end{equation}

we can construct the following precise form of the linear algebraic system with respect to $\mathbb{P}(E^T_j)$, with the help of the Lippmann-Schwinger equation (2.6).

**Proposition 3.2.** Under Assumption (I, II, III, IV), there holds the following linear algebraic system associated with the electromagnetic scattering problem (1.1) related to $\mathbb{P}(E^T_j)$, $j_0 = 1, 2, \cdots, N$, as

\begin{equation}
Q_{j_0} - \eta k^2 a^{5-h} \sum_{j \neq j_0}^N P_0 \cdot \nu_k(z_{j_0}, z_j) \cdot Q_j = i k a^{5-h} P_0 \cdot H^{inc}(z_{j_0}) + \text{Error},
\end{equation}

where $P_0$ is defined by (1.10) and

\begin{equation}
\text{Error} := O(a^{9-2h} \, d^{-4}) + O(a^{6-h}).
\end{equation}

Moreover, the linear algebraic system (3.6) is invertible if

\begin{equation}
\frac{k^2}{d^3} \frac{|\eta| a^5}{\left| 1 - k^2 \eta a^2 \lambda_{j_0}^{(1)} \right|} \|P_0\| < 1.
\end{equation}

It is easy to see from (3.1) and (3.6) that in Proposition 3.2, we take

\begin{equation}
[P_{D_{j_0}}] := k^2 a^{5-h} P_0 \quad \text{and} \quad J_{j_0} := i k a^{5-h} P_0 \cdot H^{inc}(z_{j_0}) + \text{Error}.
\end{equation}

The detailed discussions on the construction of Proposition 3.2 can be found in Section 6. In a similar manner, in $\nabla^N\text{Harmonic}$, we also have the following form of the linear algebraic system with respect to $\mathbb{P}(E^T_j)$, $j = 1, 2, \cdots, N$.

3.3. Construction of the linear algebraic system for $\mathbb{P}(E^T)$. We set here

\begin{equation}
Q_j := \int_{D_j} \mathbb{P}(E^T_j)(y) \, dy.
\end{equation}

**Proposition 3.3.** Under Assumption (I, II, III, IV), there holds the following linear algebraic system associated with the electromagnetic problem (1.1) with respect to $\mathbb{P}(E^T_{j_0})$, $j_0 = 1, 2, \cdots, N$, as

\begin{equation}
Q_{j_0} - \eta \sum_{j \neq j_0}^N P_1 \cdot \nu_k(z_{j_0}, z_j) \cdot Q_j = P_1 \cdot E_{j_0}^{inc}(z_{j_0}) + O\left(a^{\min(6.8-3t)}\right),
\end{equation}

where $P_1$ is the polarization matrix as defined by

\begin{equation}
P_1 := a^3 \sum_n \frac{1}{1 + \eta \lambda_{j_0}^{(3)}} \int_B e_n^{(3)}(y) \, dy \otimes \int_B e_n^{(3)}(y) \, dy = a^3 \sum_n \frac{1}{1 + \eta \lambda_{j_0}^{(3)}} \sum_m \int_B e_{n,m}^{(3)}(y) \, dy \otimes \int_B e_{n,m}^{(3)}(y) \, dy,
\end{equation}

\text{with} \quad \nu \times F_j = 0, \ \text{div}(F_j) = 0.
where $e^{(3)}_{n,m}$ is the eigenvalue such that $\nabla M(e^{(3)}_{n,m}) = \lambda^{(3)}_{n,m} e^{(3)}_{n,m}$. Moreover, the linear algebraic system \eqref{eq:3.3} is invertible if

\begin{equation}
\frac{|\eta| \alpha^3}{d^3} \left\| \sum_n \frac{1}{1 + \eta \lambda^{(3)}_n} \int_B e^{(3)}_n(y) dy \otimes \int_B e^{(3)}_n(y) dy \right\| < 1.
\end{equation}

Combining with \eqref{eq:3.1} and \eqref{eq:3.9}, we notice that in Proposition \ref{prop:3.3} we take $[P_{D_{j_0}}] = P_1$ and

\[ J_{j_0} := P_1 \cdot E^{inc}_{j_0}(z_{j_0}) + \mathcal{O}\left(a^{\min(6, h - 3)}\right). \]

The proof of Proposition \ref{prop:3.3} can be seen in Section \ref{sec:proof}

**Remark 3.4.** Indeed, the linear algebraic systems \eqref{eq:3.6} and \eqref{eq:3.9} can be seen as two particular cases of the following general form

\begin{equation}
\begin{bmatrix}
I + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}
\end{bmatrix}
\begin{bmatrix}
\int_{D_1} F_1(y) dy \\
\vdots \\
\int_{D_3} F_3(y) dy \\
\int_{D_1} \mathcal{P} E_1^T(y) dy \\
\vdots \\
\int_{D_3} \mathcal{P} E_3^T(y) dy
\end{bmatrix}
\begin{bmatrix}
\int_{D_1} F_1(y) dy \\
\vdots \\
\int_{D_3} F_3(y) dy \\
\int_{D_1} \mathcal{P} E_1^T(y) dy \\
\vdots \\
\int_{D_3} \mathcal{P} E_3^T(y) dy
\end{bmatrix}
\begin{bmatrix}
\int_{D_1} F_1(y) dy \\
\vdots \\
\int_{D_3} F_3(y) dy \\
\int_{D_1} \mathcal{P} E_1^T(y) dy \\
\vdots \\
\int_{D_3} \mathcal{P} E_3^T(y) dy
\end{bmatrix}
\begin{bmatrix}
\int_{D_1} F_1(y) dy \\
\vdots \\
\int_{D_3} F_3(y) dy \\
\int_{D_1} \mathcal{P} E_1^T(y) dy \\
\vdots \\
\int_{D_3} \mathcal{P} E_3^T(y) dy
\end{bmatrix}
\begin{bmatrix}
i k a^{5-h} P_0 \cdot H^{inc}(z_1) \\
\vdots \\
i k a^{5-h} P_0 \cdot H^{inc}(z_8) \\
P_1 \cdot E^{inc}(z_1) \\
\vdots \\
P_1 \cdot E^{inc}(z_8)
\end{bmatrix}
+ \text{Error},
\end{equation}

where

\[
B_{11} := -\eta k^2 a^{5-h} \begin{pmatrix}
0 & P_0 \cdot \Upsilon_k(z_1, z_2) & \cdots & P_0 \cdot \Upsilon_k(z_1, z_8) \\
P_0 \cdot \Upsilon_k(z_2, z_1) & 0 & \cdots & P_0 \cdot \Upsilon_k(z_2, z_8) \\
\vdots & \vdots & \ddots & \vdots \\
P_0 \cdot \Upsilon_k(z_8, z_1) & P_0 \cdot \Upsilon_k(z_8, z_2) & \cdots & 0
\end{pmatrix},
\]

\[
B_{22} := -\eta \begin{pmatrix}
0 & P_1 \cdot \Upsilon_k(z_1, z_2) & \cdots & P_1 \cdot \Upsilon_k(z_1, z_8) \\
P_1 \cdot \Upsilon_k(z_2, z_1) & 0 & \cdots & P_1 \cdot \Upsilon_k(z_2, z_8) \\
\vdots & \vdots & \ddots & \vdots \\
P_1 \cdot \Upsilon_k(z_8, z_1) & P_1 \cdot \Upsilon_k(z_8, z_2) & \cdots & 0
\end{pmatrix}
\]

and $I$ is the identity matrix. The linear algebraic system \eqref{eq:3.12} is the one corresponding to the Lippmann-Schwinger equation given by:

\begin{equation}
E^T(x) + \eta \nabla M^k(E^T)(x) - k^2 \eta N^k(E^T)(x) = E^{inc}(x), \quad x \in D,
\end{equation}

seeing \eqref{eq:2.1}. Since there are two operators presented in \eqref{eq:3.13}, namely the Magnetization operator and the Newtonian operator, we distinguish the following two cases in terms of the choice of the incident frequencies to enhance the scattered electromagnetic fields.

(1) **On the case when the incident frequencies are chosen to generate plasmonic resonances.** In this case, we can prove that the field corresponding to the Magnetization operator $\nabla M(\cdot)$ is the dominating one in equation \eqref{eq:3.13}, and consequently, the matrix block $B_{22}$ dominates the other blocks. Hence, \eqref{eq:3.12} will be reduced to \eqref{eq:3.9}. The analysis of this case for a single plasmonic nano-particle was the object of \cite{23}. 

(2) **On the case when the incident frequencies are chosen to generate dielectric resonances.** In this case, we can prove that the field generated by the Newtonian operator $N(\cdot)$ is the dominating one in equation (3.6), and consequently, the matrix block $B_{11}$ dominates the other blocks. Hence, $\mathcal{O}(3.6)$ will be reduced to $\mathcal{O}(3.6)$. The goal of the current work is to analyze this case in details.

Furthermore, regardless of the chosen resonances, we can prove the smallness of the two matrices $B_{12}$ and $B_{21}$, seeing the proof of Proposition 3.2 and Proposition 3.3.

### 4. Proof of Theorem 1.3

With all the necessary propositions presented in the previous sections, we give the proof of our main result in this section as follows.

**Proof of Theorem 1.3** We have

$$E_{\infty}(\hat{x}) = \eta (I - \hat{x} \otimes \hat{x}) \int_D e^{i k \hat{x} \cdot y} E_T(y) \, dy, \quad D = \bigcup_{m=1}^N D_m$$

where $E_T^m(\cdot) := E_T(\cdot)|_{D_m}$ and $D_m = z_m + a \mathbb{B}$. By developing near the center, we obtain

$$E_{\infty}(\hat{x}) = \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N \int_{D_m} \left[ e^{i k \hat{x} \cdot z_m} + i k e^{i k \hat{x} \cdot z_m} (y - z_m) \cdot \hat{x} \right] E_T^m(y) \, dy$$

$$- \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N \int_{D_m} \frac{k^2}{2} (y - z_m) \cdot \hat{x}^2 \int_0^1 (1-t) e^{i k \hat{x} \cdot (z_m + t(y-z_m))} dt \, E_T^m(y) \, dy$$

$$= \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N e^{i k \hat{x} \cdot z_m} \left[ \int_{D_m} E_T^m(y) \, dy + i k \int_{D_m} \hat{x} \cdot (y - z_m) E_T^m(y) \, dy \right]$$

$$+ \mathcal{O} \left( a^2 \max_m \| E_T^m \|_{L^2(D_m)} \right).$$

Based on Proposition 2.2, we split $E_T^m$ as $E_T^m := \hat{P}(E_T^m) + \mathbb{P}(E_T^m)$ and we plug it into the previous equation to obtain

$$E_{\infty}(\hat{x}) = \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N e^{i k \hat{x} \cdot z_m} \left[ \int_{D_m} \frac{3}{2} \hat{P}(E_T^m)(y) \, dy + i k \int_{D_m} \hat{x} \cdot (y - z_m) \frac{1}{2} \mathbb{P}(E_T^m)(y) \, dy \right]$$

$$+ \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N e^{i k \hat{x} \cdot z_m} \left[ i k \int_{D_m} \hat{x} \cdot (y - z_m) \frac{3}{2} \mathbb{P}(E_T^m)(y) \, dy \right] + \mathcal{O} \left( a^2 \max_m \| E_T^m \|_{L^2(D_m)} \right).$$

Since from (2.15) of Proposition 2.3, we have that

$$\max_m \| E_T^m \|_{L^2(D_m)} = a^2 \max_m \left\| \hat{E}_m \right\|_{L^2(B)} \lesssim a^2 \cdot a^{1-h} = \mathcal{O} \left( a^{2-h} \right),$$

in which $\| E_T^m \|_{L^2(D_m)}$ is dominated by $\| \hat{P}(E_T^m) \|_{L^2(D_m)}$. Combining with the Cauchy-Schwarz inequality and the scaling property, we can derive

$$E_{\infty}(\hat{x}) = \eta (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^N e^{i k \hat{x} \cdot z_m} \left[ \int_{D_m} \frac{3}{2} \hat{P}(E_T^m)(y) \, dy + i k \int_{D_m} \hat{x} \cdot (y - z_m) \frac{1}{2} \mathbb{P}(E_T^m)(y) \, dy \right]$$

$$+ \mathcal{O} \left( a^2 \max_m \| \hat{E}_m \|_{L^2(B)} \right) + \mathcal{O} \left( a \right).$$
where we use the fact that \( \mathbb{N} = O(d^{-3}) \sim a^{-3t} \) with \( t \leq 1 - \frac{\alpha}{3} \), in \( \mathbb{R}^3 \), which is imposed by the invertibility condition of the linear algebraic systems. In particular, using Proposition 3.3 again, for \( \frac{1}{11} < h < 1 \), we can further simplify (4.1) as

\[
E_\infty(\hat{x}) = \eta \left( I - \hat{x} \otimes \hat{x} \right) \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \left[ \int_{D_m} \frac{3}{\mathbb{P}} (E_m^T) (y) \, dy + i k \int_{D_m} \hat{x} \cdot (y - z_m) \frac{1}{\mathbb{P}} (E_m^T) (y) \, dy \right] + O(a).
\]

By (3.4), we know that there holds

\[
E_\infty(\hat{x}) = \eta \left( I - \hat{x} \otimes \hat{x} \right) \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \int_{D_m} \frac{3}{\mathbb{P}} (E_m^T) (y) \, dy - \eta i k \left( I - \hat{x} \otimes \hat{x} \right) \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \hat{x} \times \int_{D_m} F_m(y) \, dy + O(a),
\]

(4.2)

where the vector \( \int_{D_m} \frac{3}{\mathbb{P}} (E_m^T) (y) \, dy \) is the solution to (3.9). Indeed, in (3.9), by denoting

\[
S_{\text{source},j_0} := P_1 \cdot E_{\text{Inc},j_0}(z_{j_0}),
\]

it yields,

\[
\int_{D_{j_0}} \frac{3}{\mathbb{P}} (E_{j_0}^T) (x) \, dx - \eta \sum_{j=1}^\infty \eta \sum_{j \neq j_0}^{\infty} P_1 \cdot T_k(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathbb{P}} (E_j^T) (y) \, dy = S_{\text{source},j_0} + O(a^{\min(6,8-3t)}).
\]

Under the invertibility condition (3.11), in Proposition 3.3, now we can give the estimation for the first term in (4.2) on the basis of (4.4) as follows.

\[
L_1 := \eta \left( I - \hat{x} \otimes \hat{x} \right) \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \int_{D_m} \frac{3}{\mathbb{P}} (E_m^T) (y) \, dy
\]

\[
|L_1| \lesssim \eta \left( I - \hat{x} \otimes \hat{x} \right) \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} |S_{\text{source},m}| \lesssim |\eta| \sum_{m=1}^\infty |S_{\text{source},m}|,
\]

using the definition of \( S_{\text{source},m} \) in (4.3), we obtain that

\[
|L_1| \lesssim |\eta| |P_1| \mathbb{N} (|\eta| a^5 \mathbb{N}) = O(a^h),
\]

where we use the fact that \( d \sim a^t \), \( t \leq 1 - \frac{\alpha}{3} \) and \( \mathbb{N} = O(d^{-3}) \), seeing (1.3).

Therefore, equation (4.2) becomes,

\[
E_\infty(\hat{x}) = -i k \eta \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \left[ \hat{x} \times \int_{D_m} F_m(y) \, dy \right] + O(a^h) = -i k \eta \sum_{m=1}^\infty e^{i k \hat{x} \cdot z_m} \hat{x} \times Q_m + O(a^h),
\]

following the notation (3.3) presented in Proposition 3.2 where the vector \( (Q_m)_{m=1, \ldots, \mathbb{N}} \) is the solution to the linear algebraic system (3.2).

Consider the linear algebraic system

\[
\dot{Q}_m - \eta k^2 a^{5-h} \sum_{j=1}^{\infty} P_0 \cdot T_k(z_m, z_j) \cdot \dot{Q}_j = i k a^{5-h} P_0 \cdot \dot{H}_{\text{Inc}}(z_m),
\]

(4.6)
Substracting (4.6) from (3.6), it is clear that there holds

\[(4.7) (Q_m - \hat{Q}_m) - \eta k^2 a^{5-h} \sum_{j=1}^{N} P_0 \cdot \mathcal{Y}_j (z_m, z_j) \cdot (Q_j - \hat{Q}_j) = i k a^{5-h} P_0 \cdot (H^{Inc}(z_m) - \hat{H}^{Inc}(z_m)) + \text{Error},\]

where \(\hat{H}^{Inc}(z_m) = -i k^{-1} a^{h-5} P_0^{-1} \cdot \text{Error}\) and \(\text{Error}\) possesses the expression (3.7).

Combining with the right hand side of (3.6) as well as the invertibility condition (3.8), we can know by direct verification that for any \(m = 1, 2, \cdots, N\), \(\hat{Q}_m\) fulfills

\[
\left( \sum_{m=1}^{N} |\hat{Q}_m|^2 \right)^{\frac{1}{2}} \leq \frac{k a^{5-h} \|P_0\|}{1 - \frac{k^2 \eta a^5}{d^3 \left(1 - k^2 \eta a^2 \lambda_{n_0}^{(1)}\right)} \|P_0\|} \left( \sum_{j=1}^{N} |\hat{H}^{Inc}(z_m)|^2 \right)^{\frac{1}{2}},
\]

which indicates that in (4.7), we have

\[
\left( \sum_{m=1}^{N} |Q_m - \hat{Q}_m|^2 \right)^{\frac{1}{2}} \leq \frac{k a^{5-h} \|P_0\|}{1 - \frac{k^2 \eta a^5}{d^3 \left(1 - k^2 \eta a^2 \lambda_{n_0}^{(1)}\right)} \|P_0\|} \left( a^{2h-10} k^{-2} \|P_0^{-1}\|^2 \sum_{m=1}^{N} |\text{Error}|^2 \right)^{\frac{1}{2}} \leq \frac{\|P_0\| \|P_0^{-1}\|}{k^2 \eta a^5} \frac{\|P_0\|}{d^3 \left(1 - k^2 \eta a^2 \lambda_{n_0}^{(1)}\right)} \sum_{m=1}^{N} |\text{Error}|.\]

Combining with the a-priori estimates (2.15) given in Proposition 2.3 by direct calculations, we can deduce that

\[E_{\infty}(\vec{x}) = -i k \eta \sum_{m=1}^{N} e^{ik \cdot z_m} \hat{x} \cdot Q_m + \mathcal{O}(a^h)\]

\[= -i k \eta \sum_{m=1}^{N} e^{ik \cdot z_m} \hat{x} \cdot \hat{Q}_m - i k \eta \sum_{m=1}^{N} e^{ik \cdot z_m} \hat{x} \cdot (Q_m - \hat{Q}_m) + \mathcal{O}(a^h)\]

\[= -i k \eta \sum_{m=1}^{N} e^{ik \cdot z_m} \hat{x} \cdot \hat{Q}_m + \mathcal{O}\left( |\eta| \left( \sum_{m=1}^{N} |e^{ik \cdot z_m}|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{m=1}^{N} |Q_m - \hat{Q}_m|^2 \right)^{\frac{1}{2}} \right) + \mathcal{O}(a^h)\]

\[= -i k \eta \sum_{m=1}^{N} e^{ik \cdot z_m} \hat{x} \cdot \hat{Q}_m + \mathcal{O}(a^h)\]

by taking \(t \leq 1 - \frac{h}{\eta}\) for \(0 < h < 1\), which completes the proof. \(\square\)

5. Proof of the a-priori estimates of Section 2

5.1. Proof of Proposition 2.2. Recall the Lippmann-Schwinger equation in \(L^2(D)\) presented in Proposition 2.1 that

\[E^T + \eta \nabla M^k (E^T) - k^2 \eta N^k (E^T) = E^{Inc}, \quad \text{in } D.\]

After scaling to \(B\), we obtain

\[(5.1) \tilde{E}^T + \eta \nabla M^k \tilde{E}^T - k^2 \eta a^2 N^k \tilde{E}^T = \tilde{E}^{Inc}, \quad \text{in } B.\]

To study (5.1) in \(L^2\), we project it onto each subspace by writing \(\tilde{E}^T\) over \(e^{(i)}_n\) for \(i = 1, 2, 3\).
1. Taking the $L^2(B)$-inner product with respect to $e_{n}^{(1)}$, then

$$\langle \tilde{E}^T, e_{n}^{(1)} \rangle + \eta \langle \nabla M^{ka}(\tilde{E}^T), e_{n}^{(1)} \rangle - k^2 \eta a^2 \langle N^{ka}(\tilde{E}^T), e_{n}^{(1)} \rangle = \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle.$$ 

By using the fact that $\nabla M^{ka}$ is vanishing on $H_0(\text{div} = 0)$, we obtain

$$\langle \tilde{E}^T, e_{n}^{(1)} \rangle = \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle + k^2 \eta a^2 \left( \frac{i k a}{4 \pi} \int_B \tilde{E}^T(y) \, dy, e_{n}^{(1)} \right) + k^2 \eta a^2 \frac{1}{4 \pi} \sum_{j \geq 1} (ika)^{j+1} \left( \int_B \frac{|y|^j}{(j+1)!} \tilde{E}^T(y) \, dy, e_{n}^{(1)} \right).$$

From the mutual orthogonality of the decomposed three subspaces introduced in (2.1), since constant vectors are in $\nabla \text{Harmonic}$, we deduce that $\int_B e_{n}^{(1)}(y) \, dy = 0$, $\forall n \in \mathbb{N}$. Then,

$$\langle \tilde{E}^T, e_{n}^{(1)} \rangle = \frac{1}{(1 - k^2 \eta a^2 \lambda_{n}^{(1)})} \left[ \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle + k^2 \eta a^2 \left( \frac{i k a}{4 \pi} \int_B \tilde{E}^T(y) \, dy, e_{n}^{(1)} \right) \right].$$

Taking the modulus, there holds

$$\left| \langle \tilde{E}^T, e_{n}^{(1)} \rangle \right| \lesssim \frac{1}{1 - k^2 \eta a^2 \lambda_{n}^{(1)}} \left[ \left| \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle \right| + a^2 \left( \sum_{j \geq 1} \left| \int_B \frac{1}{(j+1)!} \tilde{E}^T(y) \, dy, e_{n}^{(1)} \right|^2 \right)^{\frac{1}{2}} \right].$$

This implies,

$$\left\| \frac{1}{\tilde{P}} (\tilde{E}^T) \right\|^2_{L^2(B)} := \sum_n \left| \langle \tilde{E}^T, e_{n}^{(1)} \rangle \right|^2 \lesssim \sum_n \left| \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle \right|^2 + a^{4-2h} \sum_n \sum_{j \geq 1} \left| \int_B \frac{1}{(j+1)!} \tilde{E}^T(y) \, dy, e_{n}^{(1)} \right|^2 \lesssim \sum_n \left| \langle \tilde{E}^{nc}, e_{n}^{(1)} \rangle \right|^2 + a^{4-2h} \left\| \tilde{E}^T \right\|^2_{L^2(B)}.$$ (5.2)

2. Taking the $L^2(B)$-inner product with respect to $e_{n}^{(2)}$, then

$$\langle \tilde{E}^T, e_{n}^{(2)} \rangle + \eta \langle \nabla M^{ka}(\tilde{E}^T), e_{n}^{(2)} \rangle - k^2 \eta a^2 \langle N^{ka}(\tilde{E}^T), e_{n}^{(2)} \rangle = \langle \tilde{E}^{nc}, e_{n}^{(2)} \rangle.$$ 

Passing to the adjoint operators for both $N^{ka}$ and $\nabla M^{ka}$, then

$$\langle \tilde{E}^T, e_{n}^{(2)} \rangle + \eta \langle \tilde{E}^T, \nabla M^{-ka}(e_{n}^{(2)}) \rangle - k^2 \eta a^2 \langle \tilde{E}^T, N^{-ka}(e_{n}^{(2)}) \rangle = \langle \tilde{E}^{nc}, e_{n}^{(2)} \rangle.$$ (5.3)

Since, for $x \in B$, we have

$$\nabla M^{-ka}(e_{n}^{(2)})(x) := \nabla \int_B \Phi_{-ka}(x,y) \cdot e_{n}^{(2)}(y) \, dy = -\nabla \text{div} \int_B \Phi_{-ka}(x,y) e_{n}^{(2)}(y) \, dy = -\nabla \text{div} N^{-ka}(e_{n}^{(2)})(x).$$

Using the identity that $\Delta + \text{Curl} \circ \text{Curl} = \nabla \text{div}$, we get

$$\nabla M^{-ka}(e_{n}^{(2)}) = (-ka)^2 N^{-ka}(e_{n}^{(2)}) + e_{n}^{(2)} - \text{Curl} \left( N^{-ka}(\text{Curl } e_{n}^{(2)}) - SL^{-ka}(\nu \times e_{n}^{(2)}) \right),$$
where $SL^{-ka}$ is the Single Layer operator
\[
SL^{-ka}(\nu \times e_n^{(2)})(\cdot) := \int_{\partial B} \Phi_{-ka}(\cdot, y) \cdot (\nu \times e_n^{(2)}(y)) \, d\sigma(y).
\]
Since $\text{Curl} \left( e_n^{(2)} \right) = 0$ in the domain and $\nu \times e_n^{(2)} = 0$ on the boundary, the previous equation will be reduced to
\[
(5.4) \quad \nabla M^{-ka} \left( e_n^{(2)} \right) = (-ka)^2 N^{-ka} \left( e_n^{(2)} \right) + e_n^{(2)}.
\]
By plugging this into equation (5.3), we obtain
\[
\langle \tilde{E}^T, e_n^{(2)} \rangle (1 + \eta) = \langle \tilde{E}^{\text{Inc}}, e_n^{(2)} \rangle.
\]
Since $\tilde{E}^{\text{Inc}} \in \mathbb{H} (\text{div} = 0) = (\mathbb{H}_0 (\text{div} = 0) \oplus \mathcal{H} (\text{harmonic}) \downarrow \mathbb{H}_0 (\text{Curl} = 0)$, then
\[
(5.5) \quad \left\| \frac{2}{P} \left( \tilde{E}^T \right) \right\|^2_{L^2(B)} := \sum_n \left| \langle \tilde{E}^T, e_n^{(2)} \rangle \right|^2 = 0,
\]
which proves (2.13).

(3) Taking the $L^2(B)$-inner product with respect to $e_n^{(3)}$, then
\[
\langle \tilde{E}^T, e_n^{(3)} \rangle + \eta \langle \nabla M^{ka}(\tilde{E}^T), e_n^{(3)} \rangle - k^2 \eta a^2 \langle N^{ka}(\tilde{E}^T), e_n^{(3)} \rangle = \langle \tilde{E}^{\text{Inc}}, e_n^{(3)} \rangle.
\]
Since on this subspace, we have $\nabla M(e_n^{(3)}) = \lambda_n^{(3)} e_n^{(3)}$, then we obtain that
\[
\langle \tilde{E}^T, e_n^{(3)} \rangle \left( 1 + \eta \lambda_n^{(3)} \right) = \langle \tilde{E}^{\text{Inc}}, e_n^{(3)} \rangle + k^2 \eta a^2 \langle N^{ka}(\tilde{E}^T), e_n^{(3)} \rangle - \eta \langle (\nabla M^{ka} - \nabla M)(\tilde{E}^T), e_n^{(3)} \rangle
\]
\[
(5.6) \quad \langle \tilde{E}^T, e_n^{(3)} \rangle = \frac{1}{(1 + \eta \lambda_n^{(3)})} \left[ \langle \tilde{E}^{\text{Inc}}, e_n^{(3)} \rangle + \eta \langle (k^2 a^2 N^{ka} - \nabla M^{ka} + \nabla M)(\tilde{E}^T), e_n^{(3)} \rangle \right].
\]
Using (2.18) and (2.19), we rewrite the second term on the right hand side of the previous equation as
\[
(5.7) \quad \mathfrak{S} \left( \tilde{E}^T \right) := \frac{(ka)^2}{2} N \left( \tilde{E}^T \right) + \frac{(ka)^2}{2} \int_{B} \Phi_{0}(\cdot, y) \frac{A(\cdot, y) \cdot \tilde{E}^T(\cdot, y)}{||x - y||^2} \, dy + \frac{i (ka)^3}{6 \pi} \int_{B} \tilde{E}^T(\cdot, y) \, dy
\]
\[
+ \frac{1}{4 \pi} \sum_{j \geq 1} (ika)^{j+1} \int_{B} Hess_{\frac{x}{j+1}} \left( ||x - y||^2 \right) \cdot \tilde{E}^T(\cdot, y) \, dy + \frac{k^2 a^2}{4 \pi} \sum_{j \geq 1} (ika)^{j+1} \int_{B} \frac{||-y||^j}{(j+1)!} \tilde{E}^T(\cdot, y) \, dy.
\]
Then equation (5.6) becomes,
\[
\langle \tilde{E}^T, e_n^{(3)} \rangle = \frac{\langle \tilde{E}^{\text{Inc}}, e_n^{(3)} \rangle}{\left( 1 + \eta \lambda_n^{(3)} \right)} \left[ \frac{(ka)^2}{2} \langle N \left( \tilde{E}^T \right), e_n^{(3)} \rangle \right.
\]
\[
+ \frac{(ka)^2}{2} \langle \int_{B} \Phi_{0}(\cdot, y) \frac{A(\cdot, y) \cdot \tilde{E}^T(\cdot, y)}{||x - y||^2} \, dy, e_n^{(3)} \rangle + \frac{i (ka)^3}{6 \pi} \langle \int_{B} \tilde{E}^T(\cdot, y) \, dy, e_n^{(3)} \rangle
\]
\[
+ \frac{1}{4 \pi} \sum_{j \geq 1} (ika)^{j+1} \langle \int_{B} Hess_{\frac{x}{j+1}} \left( ||x - y||^2 \right) \cdot \tilde{E}^T(\cdot, y) \, dy, e_n^{(3)} \rangle
\]
\[
+ \frac{k^2 a^2}{4 \pi} \sum_{j \geq 1} (ika)^{j+1} \langle \int_{B} \frac{||-y||^j}{(j+1)!} \tilde{E}^T(\cdot, y) \, dy, e_n^{(3)} \rangle \right],
\]
(5.8)
and
\[
\left| (\tilde{E}^T, e_n^{(3)}) \right|^2 \lesssim \left| \left\langle \tilde{E}^{Tnc}, e_n^{(3)} \right\rangle \right|^2 \left\| \frac{\hat{E}^{Tnc}}{1 + \eta \lambda_n^{(3)}} \right\|^2 + a^4 \left| \left\langle N \left( \tilde{E}^T \right), e_n^{(3)} \right\rangle \right|^2 + a^4 \left| \left\langle \Phi_0(\cdot, y) \cdot \frac{A \cdot \tilde{E}^T(y)}{\|x-y\|^2} \right\rangle \right|^2 \\
+ a^6 \left\| \int_B \tilde{E}^T(y) dy, e_n^{(3)} \right\|^2 + a^8 \sum_{j=3}^{\infty} \left( \left| \left\langle \frac{Hess \left( \|\cdot - y\|^j \right)}{(j+1)!} \cdot \tilde{E}^T(y) \right\rangle \right|^2 \right) \\
+ a^8 \sum_{j=1}^{\infty} \left( \left| \left\langle \frac{\tilde{E}^T(y) dy, e_n^{(3)}}{(j+1)!} \right\rangle \right|^2 \right)
\]

Taking the series with respect to \( n \) on the both sides and using the continuity of the Newtonian potential operator, we deduce that
\[
\left\| \frac{3}{B} \left( \tilde{E}^T \right) \right\|^2_{L^2(B)} \lesssim a^4 \left\| \frac{3}{B} \left( \tilde{E}^{Tnc} \right) \right\|^2_{L^2(B)} + a^4 \left\| \tilde{E}^T \right\|^2_{L^2(B)} + a^6 \left\| \tilde{E}^T \right\|^2_{L^2(B)} \\
+ a^8 \left\| \tilde{E}^T \right\|^2_{L^2(B)} \sum_{j=3}^{\infty} \int_B \int_B \frac{Hess \left( \|\cdot - y\|^j \right)}{(j+1)!^2} dydx \\
+ a^8 \left\| \tilde{E}^T \right\|^2_{L^2(B)} \sum_{j=1}^{\infty} \int_B \int_B \frac{\|\cdot - y\|^j}{(j+1)!^2} dydx.
\]

We prove that the two series, on \( j \) index, are converging. For the first one, we have\(^3\)
\[
S_1 := \sum_{j=3}^{\infty} \int_B \int_B \frac{Hess \left( \|\cdot - y\|^j \right)}{(j+1)!^2} dydx \leq |B|^2 \sum_{j=3}^{\infty} \frac{j \left( diam(B)^2 \right)^{(j-2)}}{(j+1)!^2} \ < +\infty,
\]
and for the second one, we have
\[
S_2 := \sum_{j=1}^{\infty} \int_B \int_B \frac{\|\cdot - y\|^j}{(j+1)!^2} dydx \leq |B|^2 \sum_{j=1}^{\infty} \frac{\left( diam(B) \right)^j}{(j+1)!^2} \ < +\infty,
\]
where \( diam(B) \) is the diameter of the domain \( B \).\(^4\) Equation (5.9) becomes,
\[
\left\| \frac{3}{B} \left( \tilde{E}^T \right) \right\|^2_{L^2(B)} \lesssim a^4 \left\| \frac{3}{B} \left( \tilde{E}^{Tnc} \right) \right\|^2_{L^2(B)} + a^4 \left\| \tilde{E}^T \right\|^2_{L^2(B)}.
\]
Now, by adding (5.10), (5.5) and (5.12), we can obtain that for
\[
h < 2,
\]
there holds the following formula
\[
\left\| \tilde{E}^T \right\|^2_{L^2(B)} := \sum_{j=1}^{3} \left\| \frac{3}{B} \left( \tilde{E}^T \right) \right\|^2_{L^2(B)}.
\]

\(^3\)Straightforward computations allow us to deduce that
\[
Hess \left( \|\cdot - y\|^j \right) = j \left( \|\cdot - y\|^{j-2} I + j \left( \|\cdot - y\|^{j-4} (x - y) \otimes (x - y) \right) \right).
\]

\(^4\)Recall that \( \sum a_n \) converge if \( \lim_{n} (a_{n+1}/a_n) \to n \to +\infty 0.\)
Proof.

(5.16)

where

Lemma 5.1. \( \nabla \) Since the operators \( T \)

\( (5.15) \)

Therefore we deduce that

\( \langle E^{inc}, e_n^{(1)} \rangle \), we deduce:

To compute \( \langle \tilde{E}^{inc}, e_n^{(1)} \rangle \), we write \( e_n^{(1)} = Curl (\phi_n) \), then with integration by parts, we have

Then,

Therefore we deduce that

This ends the proof of Proposition 2.2.

5.2. Proof of Proposition 2.3.

Before presenting the proof of Proposition 2.3, to write short formulas, we set the following volume integral operator:

Since the operators \( \nabla M^{-k} \) and \( N^{-k} \) are the adjoint operators of \( \nabla M^k \) and \( N^k \), respectively, we deduce that \( T_{-k} \) is the adjoint operator of \( T_k \). In addition, we state the following lemma.

Lemma 5.1. Recall that \( e_n^{(1)} \) is the eigenfunction given by Subsection 2.3. Then there holds

\( (5.15) \)

\( (5.16) \)

Proof.

\( (5.16) \)

The remainder of the proof consists in taking the inverse of the operator appearing on the left hand side.

Now we prove Proposition 2.3 as follows. Recall the Lippmann-Schwinger equation \( (5.6) \)

\( E^T(x) + \eta \sum_{j=1}^{N} \nabla M^k(\eta J^T_j(x) - k^2 \eta \sum_{j=1}^{N} N^k(\eta J^T_j(x) = E^{inc}(x), \ x \in D = \bigcup_{j=1}^{N} D_j. \)
By Proposition 2.2 from [5.5], we can write $E^T_{j} = \frac{1}{P} (E^T_{j}) + \frac{3}{P} (E^T_{j}), j = 1, \cdots, N$. We split the study into two parts.

i) **Estimation of**

For $x \in D_{j_0}$, we have

$$\textbf{E}_k \left(E_{j_0}^T \right)(x) + \eta \sum_{j \neq j_0}^N \left(\nabla_x M^k - k^2 N^k \right)(E_j^T)(x) = E_{j_0}^{inc}(x).$$

Taking the inverse operator of $\textbf{E}_k$ on the both sides of (5.17), we get

$$E_{j_0}^T (x) + \eta \sum_{j \neq j_0}^N \left(\nabla_x M^k - k^2 N^k \right)(E_j^T)(x) = \textbf{E}_k^{-1} \left(E_{j_0}^{inc} \right)(x),$$

$$\frac{1}{P} (E_{j_0}) (x) - \eta \sum_{j \neq j_0}^N \textbf{E}_k^{-1} \int_{D_j} \Upsilon (x, y) \cdot \frac{1}{P} (E_j^T)(y) dy = \textbf{E}_k^{-1} \left(E_{j_0}^{inc} \right)(x) - \frac{3}{P} (E_{j_0}) (x) + \eta \sum_{j \neq j_0}^N \textbf{E}_k^{-1} \int_{D_j} \Upsilon (x, y) \cdot \frac{3}{P} (E_j^T)(y) dy.$$

Now, we have

$$\int_{D_j} \Upsilon (x, y) \cdot \frac{1}{P} (E_j^T)(y) dy := \left(- \nabla_x M^k + k^2 N^k \right) \left(\frac{1}{P} (E_j^T) \right)(x) = k^2 N^k \left(\frac{1}{P} (E_j^T) \right)(x)$$

$$= k^2 \int_{D_j} \Phi_k(x, y) \frac{1}{P} (E_j^T)(y) dy.$$

Expanding $\frac{1}{P}$ near the center $z_j$ and using the fact that $\int_{D_j} \frac{1}{P} (E_j^T)(y) dy = 0$, we obtain

$$\int_{D_j} \Upsilon (x, y) \cdot \frac{1}{P} (E_j^T)(y) dy = k^2 \nabla_x \left(\Phi_k \right)(z_{j_0}, z_j) \cdot \int_{D_j} P(y; z_j) \cdot \frac{1}{P} (E_j^T)(y) dy$$

$$+ k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{1} \cdot H_{x} \left(\Phi_k \right)(z_{j_0}, z_j + t(y - z_j)) \cdot \left(\Phi_k \right)(y - z_j) dt \cdot \frac{1}{P} (E_j^T)(y) dy$$

$$+ \text{Err}(x, j_0, j, \frac{1}{P} (E_j^T)),$$

where

$$\text{Err}(x, j_0, j, \frac{1}{P} (E_j^T)) :=$$

$$- k^2 \int_{D_j} \int_0^1 H_{x} \left(\Phi_k \right)(z_{j_0} + t(x - z_{j_0}), z_j) \cdot \left(\Phi_k \right)(x - z_{j_0}) dt \cdot \frac{1}{P} (E_j^T)(y) dy$$

$$+ k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{1} \cdot \nabla_x \left(\Phi_k \right)(z_{j_0} + t(y - z_j)) \cdot \left(\Phi_k \right)(z_j) dt \cdot \frac{1}{P} (E_j^T)(y) dy$$

$$+ k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{1} \cdot \nabla_x \left(\Phi_k \right)(z_{j_0} + t(y - z_j)) \cdot \left(\Phi_k \right)(z_j) dt \cdot \frac{1}{P} (E_j^T)(y) dy$$

$$+ k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{1} \cdot \nabla_x \left(\Phi_k \right)(z_{j_0} + t(y - z_j)) \cdot \left(\Phi_k \right)(z_j) dt \cdot \frac{1}{P} (E_j^T)(y) dy$$

$5$Recall the Taylor expansion for a function of several variables:

$$f(x + h) = f(x) + \sum_{k=1}^n \frac{1}{k!} d^k f_x (h^k) + \int_0^1 (1 - t)^n \cdot d^{n+1} f_x (h^{n+1}) dt.$$
\[
\int_0^1 \int_0^1 \frac{d}{dx} \left( \nabla_y \left( H \mathcal{E}_k \Phi_k \right) \right) \left( z_{j_0} + \rho s(x - z_{j_0}), z_j + t(y - z_j) \right) \cdot (x - z_{j_0})^{[1]} \, dp \cdot \mathcal{P}(x, z_{j_0}) \, ds \\
(5.18) \quad \cdot (y - z_j) \, dt \cdot \frac{1}{\mathcal{P}} \left( E_j^T \right)(y) \, dy,
\]
and
\[
\int_{D_j} \mathcal{Y}(x, y) \cdot \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy = \mathcal{Y}(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
+ \int_{D_j} \int_0^1 \nabla_y \left( \mathcal{Y} \right) (z_{j_0}, z_j + t(y - z_j)) \cdot \mathcal{P}(y, z_j) \, dt \cdot \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
+ Err(x, j_0, j, \frac{3}{\mathcal{P}} \left( E_j^T \right)),
\]
where
\[
Err(x, j_0, j, \frac{3}{\mathcal{P}} \left( E_j^T \right)) = \int_0^1 \nabla_y \left( \mathcal{Y} \right) (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) \, dt \cdot \int_{D_j} \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
+ \int_{D_j} \int_0^1 \left[ \int_0^1 \nabla_y \left( \mathcal{Y} \right) (z_{j_0} + s(x - z_{j_0}), z_j + t(y - z_j)) \cdot \mathcal{P}(x, z_{j_0}) \, ds \right] \, dy.
\]
(5.19) \quad \cdot \mathcal{P}(y, z_j) \, dt \cdot \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy.

Then,
\[
\frac{1}{\mathcal{P}} \left( E_{j_0}^T \right)(x) - \eta k^2 \sum_{j \neq j_0}^N T_k^{-1} \nabla_y \left( \Phi_k \right)(z_{j_0}, z_j) \cdot \int_{D_j} \mathcal{P}(y, z_j) \cdot \frac{1}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
- \eta k^2 \sum_{j \neq j_0}^N T_k^{-1} \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{\perp} \cdot H \mathcal{E}_k \left( \Phi_k \right)(z_{j_0}, z_j + t(y - z_j)) \cdot (y - z_j) \, dt \cdot \frac{1}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
= T_k^{-1} \left( E_{j_0}^{Inc} \right)(x) - \frac{3}{\mathcal{P}} \left( E_{j_0}^T \right)(x) + \eta \sum_{j \neq j_0}^N T_k^{-1} \mathcal{Y}(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
+ \eta \sum_{j \neq j_0}^N T_k^{-1} \int_{D_j} \int_0^1 \nabla_y \left( \mathcal{Y} \right)(z_{j_0}, z_j + t(y - z_j)) \cdot \mathcal{P}(y, z_j) \, dt \cdot \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy \\
+ \eta \sum_{j \neq j_0}^N T_k^{-1} \left[ Err(x, j_0, j, \frac{3}{\mathcal{P}} \left( E_j^T \right)) + Err(x, j_0, j, \frac{3}{\mathcal{P}} \left( E_j^T \right)) \right].
\]

Scaling to \( B \), taking the inner product with respect to \( e_n^{(1)} \cdot \cdot \cdot \) and using the adjoint operator of \( T_{ka} \), i.e. \( T_{ka} = T_{-ka} \), we obtain
\[
\langle \frac{1}{\mathcal{P}} \left( E_{j_0}^T \right), e_n^{(1)} \rangle \quad - \quad \eta k^2 a^4 \sum_{j \neq j_0}^N \int_B \mathcal{P}(y; 0) \cdot \frac{1}{\mathcal{P}} \left( E_j^T \right)(y) \, dy, T_{-ka}^{-1} \left( e_n^{(1)} \right) \\
- \eta k^2 a^5 \sum_{j \neq j_0}^N \int_B \int_0^1 (1-t)y^{\perp} \cdot H \mathcal{E}_k \left( \Phi_k \right)(z_{j_0}, z_j + ta \cdot y) \cdot ydt \cdot \frac{1}{\mathcal{P}} \left( E_j^T \right)(y) \, dy, T_{-ka}^{-1} \left( e_n^{(1)} \right) \\
= \langle E_{j_0}^{Inc} \rangle, T_{-ka}^{-1} \left( e_n^{(1)} \right) + \eta a^3 \sum_{j \neq j_0}^N \int_B \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy, T_{-ka}^{-1} \left( e_n^{(1)} \right) \\
+ \eta a^4 \sum_{j \neq j_0}^N \int_B \int_0^1 \nabla_y \left( \mathcal{Y} \right)(z_{j_0}, z_j + ta \cdot y) \cdot \mathcal{P}(y; 0) \, dy, T_{-ka}^{-1} \left( e_n^{(1)} \right) \\
+ \eta a^5 \sum_{j \neq j_0}^N \int_B \int_0^1 \nabla_y \left( \mathcal{Y} \right)(z_{j_0}, z_j + ta \cdot y) \cdot \mathcal{P}(y; 0) \, dt \cdot \frac{3}{\mathcal{P}} \left( E_j^T \right)(y) \, dy, T_{-ka}^{-1} \left( e_n^{(1)} \right).
(5.20) \[ + \eta \sum_{j \neq j_0}^{N} \langle \tilde{E}^r(x, j_0, j, \mathbb{P} (E^T_j)) + \tilde{E}^r(x, j_0, j, \mathbb{P} (E^T_j)), T_{-k}^{-1} (e^{(1)}_n) \rangle. \]

Using (5.14) of Lemma 5.1, we rewrite (5.20) as

\[
\begin{align*}
\langle \mathbb{P} (\tilde{E}^T_{j_0}), e^{(1)}_n \rangle &= \frac{\eta k^2 a^4}{(1 - k^2 \eta a^2 \lambda_n)} \sum_{j \neq j_0}^{N} \langle \nabla_x (\Phi_k I) (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle \\
&\quad + \frac{\eta k^2 a^5}{(1 - k^2 \eta a^2 \lambda_n)} \sum_{j \neq j_0}^{N} \left\langle \int_B \int_0^1 (1 - t) y^\perp \cdot \text{Hess} (\Phi_k) (z_{j_0}, z_j + t a y) \cdot ydt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \right\rangle \\
&\quad + \eta k^2 a^5 \sum_{j \neq j_0}^{N} \left\langle \int_B \int_0^1 (1 - t) y^\perp \cdot \text{Hess} (\Phi_k) (z_{j_0}, z_j + t a y) \cdot ydt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, R_n \right\rangle \\
&\quad + \frac{\langle \tilde{E}^{\text{inc}}_{j_0}, e^{(1)}_n \rangle}{\left(1 - k^2 \eta a^2 \lambda_n \right)} + \frac{\eta a^3}{\left(1 - k^2 \eta a^2 \lambda_n \right)} \sum_{j \neq j_0}^{N} \langle \Upsilon (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle \\
&\quad + \eta a^3 \sum_{j \neq j_0}^{N} \langle \Upsilon (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, R_n \rangle \\
&\quad + \eta a^4 \sum_{j \neq j_0}^{N} \langle \nabla_x (\Phi_k I) (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, R_n \rangle \\
&\quad + \frac{\eta a^4}{\left(1 - k^2 \eta a^2 \lambda_n \right)} \sum_{j \neq j_0}^{N} \langle \int_B \int_0^1 y \cdot \Upsilon (z_{j_0}, z_j + t a y) \cdot \mathbb{P}(y; 0)dt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle \\
&\quad + \eta a^4 \sum_{j \neq j_0}^{N} \langle \int_B \int_0^1 y \cdot \Upsilon (z_{j_0}, z_j + t a y) \cdot \mathbb{P}(y; 0)dt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, R_n \rangle \\
&\quad + \eta \sum_{j \neq j_0}^{N} \langle \tilde{E}^r(x, j_0, j, \mathbb{P} (E^T_j)) + \tilde{E}^r(x, j_0, j, \mathbb{P} (E^T_j)), e^{(1)}_n \rangle + \langle \tilde{E}^{\text{inc}}_{j_0}, R_n \rangle.
\end{align*}
\]

As the eigenfunctions $e^{(1)}_n$, $n = 1, 2, \ldots$, which are in $\mathbb{H}_0 (\text{div} = 0)$, are orthogonal to the constant vectors, which are in $\nabla\text{Harm} \text{onic}$, then we deduce that

\[
\begin{align*}
\langle \Upsilon (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle &= 0, \\
\langle \nabla_x (\Phi_k I) (z_{j_0}, z_j) \cdot \int_B \mathbb{P}(y; 0) \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle &= 0, \\
\langle \int_B \int_0^1 y \cdot \Upsilon (z_{j_0}, z_j + t a y) \cdot \mathbb{P}(y; 0)dt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle &= 0, \\
\langle \int_B \int_0^1 y^\perp \cdot \text{Hess} (\Phi_k) (z_{j_0}, z_j + t a y) \cdot ydt \cdot \mathbb{P} (\tilde{E}_{j}^T) (y) dy, e^{(1)}_n \rangle &= 0.
\end{align*}
\]
Then, after taking the squared modulus, we get

\[
\| \mathbb{P} \left( \hat{E}_{T_{j_0}} \right) \|^2 \approx \frac{\langle \hat{E}_{T_{j_0}}^{\text{nc}}, e_n^{(1)} \rangle^2}{1 - k^2 \eta a^2 \lambda_n^{(1)}}^2 + a^2 \mathbb{N} \sum_{j \neq j_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ a^4 \mathbb{N} \sum_{j \neq j_0}^N \left| \nabla \left( \Phi_k I \right) (z_{j_0}, z_j) \cdot \int_B \mathcal{P}(y; 0) \cdot \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ a^4 \mathbb{N} \sum_{j \neq j_0}^N \left| \nabla \left( \Phi_k I \right) (z_{j_0}, z_j) \cdot \int_B \mathcal{P}(y; 0) \cdot \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ a^4 \mathbb{N} \sum_{j \neq j_0}^N \left| \nabla \left( \Phi_k I \right) (z_{j_0}, z_j) \cdot \int_B \mathcal{P}(y; 0) \cdot \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ |\gamma|^2 \sum_{j \neq j_0}^N \left| \nabla \left( \Phi_k I \right) (z_{j_0}, z_j) \cdot \int_B \mathcal{P}(y; 0) \cdot \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2.
\]

Taking the series with respect to the index $n$, we obtain

\[
\| \mathbb{P} \left( \hat{E}_{T_{j_0}} \right) \|^2 \approx \frac{\langle \hat{E}_{T_{j_0}}^{\text{nc}}, e_n^{(1)} \rangle^2}{1 - k^2 \eta a^2 \lambda_n^{(1)}}^2 + \sum_{n \neq n_0}^N \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ a^2 \mathbb{N} \sum_{j \neq j_0}^N \sum_n \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ a^4 \mathbb{N} \sum_{j \neq j_0}^N \sum_n \left| \nabla \left( \Phi_k I \right) (z_{j_0}, z_j) \cdot \int_B \mathcal{P}(y; 0) \cdot \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \sum_n \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \sum_n \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2 \\
+ \left| \hat{\gamma} \right|^2 \sum_{j \neq j_0}^N \sum_n \left| \mathcal{Y}(z_{j_0}, z_j) \cdot \int_B \mathbb{P} \left( \hat{E}_T \right) (y) \, dy, R_n \right|^2.
\]
\[
+ |\eta|^2 \sum_{j \neq j_0}^{N} \sum_{n} \left| \langle \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)) + \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)), R_n \rangle \right|^2 + \sum_{n} |\tilde{E}_{\text{inc}}(x, j_0, R_n)|^2
\]
\[
+ a^6 \sum_{j \neq j_0}^{N} \sum_{n} \left| \int_0^1 \int_B (1-t)y^T \cdot H_{\text{ess}}(\Phi_k)(z_j, z_j + aty) \cdot y \ dt \cdot \mathbb{P} \left( E_j^T \right) (y) dy, R_n \right|^2
\]
(5.21)
\[
+ a^4 \sum_{j \neq j_0}^{N} \sum_{n} \left| \int_0^1 \int_B \nabla \cdot (\mathbb{Y}(z_j, z_j + tay)) \cdot \mathbb{P}(y, 0) dt \cdot \mathbb{P} \left( E_j^T \right) (y) dy, R_n \right|^2
\]

We set
\[
I := \sum_{j \neq j_0}^{N} \sum_{n} \frac{1}{1 - k^2 \eta a^2 \lambda_n^{(1)} \lambda_n^{(2)}} \left| \langle \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)) + \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)), e_n^{(1)} \rangle \right|^2,
\]
then, obviously, we have
\[
|I| \lesssim a^{-2h} \sum_{j \neq j_0}^{N} \sum_{n} \left| \langle \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)), e_n^{(1)} \rangle \right|^2 + a^{-2h} \sum_{j \neq j_0}^{N} \sum_{n} \left| \langle \tilde{E}_{\text{err}}(x, j_0, j, \mathbb{P}(E_j^T)), e_n^{(1)} \rangle \right|^2
\]
(5.22)
\[
\lesssim a^{-2h} \sum_{j \neq j_0}^{N} \left\| \tilde{E}_{\text{err}}(\cdot, j_0, j, \mathbb{P}(E_j^T)) \right\|_{L^2(B)}^2 + a^{-2h} \sum_{j \neq j_0}^{N} \left\| \tilde{E}_{\text{err}}(\cdot, j_0, j, \mathbb{P}(E_j^T)) \right\|_{L^2(B)}^2.
\]

Next, to compute an upper bound for \(I\), we split the computations into two parts.

(a) Computing \(\left\| \tilde{E}_{\text{err}}(\cdot, j_0, j, \mathbb{P}(E_j^T)) \right\|_{L^2(B)}^2\).

Firstly, we rewrite \(\text{Err}(x, j_0, j, \mathbb{P}(E_j^T))\), given by (5.18), as

\[
\text{Err}(x, j_0, j, \mathbb{P}(E_j^T)) :=
\]
\[
- k^2 \int_{D_j} \int_0^1 \nabla \left( H_{\text{ess}}(\Phi_k) \right) (z_j, z_j + t(x - z_j), y) \cdot (x - z_j) dt \cdot \mathbb{P} \left( E_j^T \right) (y) dy
\]
\[
- k^2 \int_{D_j} \int_0^1 H_{\text{ess}}(\Phi_k - \Phi_0) (z_j, z_j + t(x - z_j), y) \cdot (x - z_j) dt \cdot \mathbb{P} \left( E_j^T \right) (y) dy
\]
\[
+ k^2 \int_{D_j} \int_0^1 \int_0^1 \nabla \left( H_{\text{ess}}(\Phi_k) \right) (z_j, z_j + t(x - z_j), z_j + s(y - z_j)) \cdot \mathbb{P}(y, z_j) ds \cdot (x - z_j) dt
\]
\[
\cdot (y - z_j) \mathbb{P} \left( E_j^T \right) (y) dy
\]
\[
+ k^2 \int_{D_j} \int_0^1 (1-t)(y - z_j) \cdot \nabla \left( H_{\text{ess}}(\Phi_k) \right) (z_j, z_j + t(y - z_j)) \cdot \mathbb{P}(x, z_j) \cdot (y - z_j) dt \mathbb{P} \left( E_j^T \right) (y) dy
\]
\[
+ k^2 \int_{D_j} \int_0^1 (1-t)(y - z_j) \cdot \nabla \left( H_{\text{ess}}(\Phi_k) \right) (z_j + \rho s(x - z_j), z_j + t(y - z_j)) \cdot (x - z_j) dt \mathbb{P} \left( E_j^T \right) (y) dy
\]
(5.23)
\[
\cdot (y - z_j) dt \mathbb{P} \left( E_j^T \right) (y) dy.
\]

---

\textsuperscript{6}The difference between (5.18) and (5.23) is merely technical. We keep the first term on the right hand side of (5.18) with the help of the Taylor expansion of \(\text{Hess} N^k(\cdot \cdot)\), then we split \(\text{Hess} N^k\) into \(\text{Hess} N\) and \(\text{Hess} (N^k - N)\).
Secondly, for the shortness reason, we denote

\[ Err(\cdot, j_0, j, \frac{1}{P}(E_j^T)) = \sum_{\ell=1}^{5} J_\ell(\cdot) \]

and we compute an estimation of the \(L^2\)-norm of each \(J_\ell(\cdot)\), \(\ell = 1, \cdots, 5\). For this, we have:

(i) Estimation of

\[ J_1(x) := -k^2 \int_{D_1} \int_{0}^{1} \text{Hess}(\Phi_0)(z_{j_0} + t(x - z_{j_0}), y) \cdot (x - z_{j_0}) dt \cdot (y - z_j) \frac{1}{P}(E_j^T)(y) \ dy. \]

For simplicity, we set \(\beta_j\), a scalar function, to be an arbitrary component of the vector field \(\frac{1}{P}(E_j^T)\). We also set \(J_{1,j}\) the corresponding \(j\)th component of \(J_1\). Then,

\[ J_{1,j}(x) = -k^2 \int_{D_1} \int_{0}^{1} \text{Hess}(\Phi_0)(z_{j_0} + t(x - z_{j_0}), y) \cdot (x - z_{j_0}) dt \cdot (y - z_j) \beta_j(y) \ dy. \]

Scaling to \(B\), we obtain

\[ J_{1,j}(x) = -k^2 a^5 \int_{0}^{1} \text{Hess} N_B \left( \frac{\beta_j(\cdot) \chi_B(\cdot)}{a} \right) \left( z_{j_0} + t a x \right) \cdot x \ dt. \]

Now, taking the norm on the both sides, we get

\[ \left\| J_{1,j} \right\|_{L^2(B)} \lesssim a^5 \left\| \int_{0}^{1} \text{Hess} N_B \left( \frac{\beta_j(\cdot) \chi_B(\cdot)}{a} \right) \left( z_{j_0} + t a \cdot \right) dt \right\|_{L^2(B)} \]

\[ \lesssim a^5 \int_{0}^{1} \left\| \text{Hess} N_B \left( \frac{\beta_j(\cdot) \chi_B(\cdot)}{a} \right) \left( z_{j_0} + t a \cdot \right) \right\|_{L^2(B)} dt \]

\[ \lesssim a^5 \left\| \text{Hess} N_B \left( \frac{\beta_j(\cdot) \chi_B(\cdot)}{a} \right) \right\|_{L^2(\mathbb{R}^3)}. \]

Due to Theorem 9.9, formula 9.28, page 230 of [24], we have

\[ \left\| \text{Hess} N_B \left( \frac{\beta_j(\cdot) \chi_B(\cdot)}{a} \right) \right\|_{L^2(\mathbb{R}^3)} = \left\| \frac{\beta_j(\cdot)}{a} \right\|_{L^2(B)}. \]

This implies,

\[ \left\| J_{1,j} \right\|_{L^2(B)} \lesssim a^5 \left\| \frac{\beta_j}{a} \right\|_{L^2(B)}, \]

Since \(\beta_j\) is chosen as an arbitrarily component, we deduce that

\[ \left\| J_1 \right\|_{L^2(B)} = o \left( a^5 \left\| \frac{1}{P}(E_j^T) \right\|_{L^2(B)} \right). \]

(ii) Estimation of

\[ J_2(x) := -k^2 \int_{D_1} \int_{0}^{1} \text{Hess}(\Phi_k - \Phi_0)(z_{j_0} + t(x - z_{j_0}), y) \cdot (x - z_{j_0}) dt \cdot (y - z_j) \frac{1}{P}(E_j^T)(y) \ dy. \]
Recall the expansion \( [2.7] \). Then we have

\[
Hess (\Phi_k - \Phi_0)(x, y) = -\frac{k^2}{2} \Phi_0(x, y) \frac{I - i}{24\pi} L_3 + \frac{k^2}{2} \Phi_0(x, y) \frac{A(x, y)}{\|x - y\|^2} + \frac{1}{4\pi} \sum_{n \geq 3} \frac{(ik)^n + 1}{(n + 1)!} Hess(\|x - y\|^n).
\]

Using this we deduce that the dominant term of \( J_2(x) \) is

\[
J_2(x) \approx \frac{k^4}{2} \int_{D_j} \int_0^1 \Phi_0(z_{j_0} + t(x - z_{j_0}), y) dt \cdot (y - z_j)^{\frac{1}{2}} \mathbb{P}(E_j^T)(y) dy.
\]

By taking the norm, we get

\[
\|J_2\|_{L^2(D_{j_0})} \lesssim \left\| \int_{D_j} \int_0^1 \Phi_0(z_{j_0} + t(-z_{j_0}), y) dt \cdot (y - z_j)^{\frac{1}{2}} \mathbb{P}(E_j^T)(y) dy \right\|_{L^2(D_{j_0})}
\]

\[
(5.26) \quad \lesssim a^5 |z_j - z_{j_0}|^{-1} \left\| \frac{1}{2} \mathbb{P}(E_j^T) \right\|_{L^2(D_j)}.
\]

(iii) Estimation of

\[
J_3 := k^2 \int_{D_j} \int_0^1 \int_0^1 \left[ \nabla_y \left( \frac{Hess \left( \Phi_k \right)}{y} \right) z_{j_0} + t(x - z_{j_0}), y \right] \cdot \mathcal{P}(y, z_j) ds \cdot (x - z_{j_0}) dt \cdot (y - z_j)^{\frac{1}{2}} \mathbb{P}(E_j^T)(y) dy
\]

\[
\|J_3\|_{L^2(D_{j_0})} \lesssim \left\| \int_{D_j} \int_0^1 \int_0^1 \left[ \nabla_y \left( \frac{Hess \left( \Phi_k \right)}{y} \right) z_{j_0} + t(-z_{j_0}), y \right] \cdot \mathcal{P}(y, z_j) ds \cdot (-z_{j_0}) dt \cdot (y - z_j)^{\frac{1}{2}} \mathbb{P}(E_j^T)(y) dy \right\|_{L^2(D_{j_0})}
\]

\[
(5.27) \quad \lesssim a^6 |z_j - z_{j_0}|^{-4} \left\| \frac{1}{2} \mathbb{P}(E_j^T) \right\|_{L^2(D_j)}.
\]

(iv) Estimation of

\[
J_4 := k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{\frac{1}{2}} \cdot \nabla_x \left( \frac{Hess \left( \Phi_k \right)}{y} \right) z_{j_0}, z_j + t(y - z_j) \cdot \mathcal{P}(x, z_{j_0}) \cdot (y - z_j) dt \frac{1}{2} \mathbb{P}(E_j^T)(y) dy.
\]

Taking the \( L^2(D_{j_0}) \)–norm on the both sides, we obtain that

\[
(5.28) \quad \|J_4\|_{L^2(D_{j_0})} \lesssim a^6 |z_j - z_{j_0}|^{-4} \left\| \frac{1}{2} \mathbb{P}(E_j^T) \right\|_{L^2(D_j)}.
\]

(v) Estimation of

\[
J_5 := k^2 \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{\frac{1}{2}} \cdot \left[ \int_0^1 d \nabla_x \left( \frac{Hess \left( \Phi_k \right)}{y} \right) z_{j_0}, z_j + t(y - z_j) \cdot (x - z_{j_0})^{[1]} dp \cdot \mathcal{P}(x, z_{j_0}) ds \right] \cdot (y - z_j) dt \frac{1}{2} \mathbb{P}(E_j^T)(y) dy,
\]

\[
\|J_5\|_{L^2(D_j)} \lesssim \left\| \int_{D_j} \int_0^1 (1 - t)(y - z_j)^{\frac{1}{2}} \right\|_{L^2(D_j)}.
\]
and after scaling, we obtain that

\[
\mathbb{E}rr(\cdot, j_0, j, \frac{1}{\mathbb{P}} (E_j^T)) = J_6 + J_7.
\]

(i) Estimation of

\[
J_6 := \int_0^1 \int_0^1 \left[ \int_0^1 \nabla_x (\nabla_y (Y)) (z_{j_0} + s(x - z_{j_0}), z_j) \cdot \mathbb{P}(x, z_{j_0}) ds \right] dy,
\]

\[
\|J_6\|_{L^2(D_{j_0})} \leq a^4 |z_j - z_{j_0}|^{-4} \left\| \mathbb{P} \left( E_j^T \right) \right\|_{L^2(D_j)}.
\]

(ii) Estimation of

\[
J_7 := \int_D \int_0^1 \left[ \int_0^1 \nabla_x (\nabla_y (Y)) (z_{j_0} + s(x - z_{j_0}), z_j) \cdot \mathbb{P}(x, z_{j_0}) ds \right] dy,
\]

\[
\|J_7\|_{L^2(D_{j_0})} \leq a^5 |z_j - z_{j_0}|^{-5} \left\| \mathbb{P} \left( E_j^T \right) \right\|_{L^2(D_j)}.
\]

Clearly,

\[
\mathbb{E}rr(\cdot, j_0, j, \frac{1}{\mathbb{P}} (E_j^T)) = \mathcal{O} \left( a^8 |z_j - z_{j_0}|^{-8} \left\| \mathbb{P} \left( E_j^T \right) \right\|_{L^2(D_j)}^2 \right),
\]

and after scaling, we obtain that

\[
\mathbb{E}rr(\cdot, j_0, j, \frac{1}{\mathbb{P}} (E_j^T)) = \mathcal{O} \left( a^8 |z_j - z_{j_0}|^{-8} \left\| \mathbb{P} \left( E_j^T \right) \right\|_{L^2(B_j)}^2 \right).
\]
Going back to (5.22) and using the estimations (5.30) and (5.31), we can derive that

\[
|I| \lesssim a^{-2h} \sum_{j \neq j_0}^{N} \left\| \tilde{E}_{\text{err}}(\cdot, j_0, j, \bar{P} (E_j^T)) \right\|^2_{L^2(B)} + a^{-2h} \sum_{j \neq j_0}^{N} \left\| \tilde{E}_{\text{err}}(\cdot, j_0, j, \bar{P} (E_j^T)) \right\|^2_{L^2(B)} \\
\lesssim a^{10^{-2h}} \sum_{j \neq j_0}^{N} \frac{1}{d_j^2} \left\| \bar{P} (\tilde{E}_j^T) \right\|^2 + a^{12^{-2h}} \sum_{j \neq j_0}^{N} \frac{1}{d_j^2} \left\| \bar{P} (\tilde{E}_j^T) \right\|^2 + a^{8^{-2h}} \sum_{j \neq j_0}^{N} \frac{1}{d_j^2} \left\| \bar{P} (\tilde{E}_j^T) \right\|^2 \\
\lesssim (a^{10^{-2h}} d^{-3} + a^{12^{-2h}} d^{-8}) \max_j \left\| \bar{P} (\tilde{E}_j^T) \right\|^2 + a^{8^{-2h}} d^{-8} \max_j \left\| \bar{P} (\tilde{E}_j^T) \right\|^2.
\]  

(5.32)

We also need to estimate

\[
I^* := \sum_n \frac{1}{1 - k^2 \eta a^2 \lambda_n^{(1)}} \left[ \sum_{k=1}^{N} \left( \left\| \tilde{E}_{\text{err}}(x, j_0, k, \bar{P} (E_k^T)) \right\| + \left\| \tilde{E}_{\text{err}}(x, j_0, k, \bar{P} (E_k^T)) \right\| \right) \\
\cdot \sum_{i > k}^{N} \left( \left\| \tilde{E}_{\text{err}}(x, j_0, i, \bar{P} (E_i^T)) \right\| + \left\| \tilde{E}_{\text{err}}(x, j_0, i, \bar{P} (E_i^T)) \right\| \right). \]

By using (5.30) and (5.31), we obtain

\[
|I^*| \lesssim a^{-2h} \sum_{k=1}^{N} \left( a^5 d_{k,j_0}^{-1} \left\| \bar{P} (\tilde{E}_k^T) \right\| + a^6 d_{k,j_0}^{-4} \left\| \bar{P} (\tilde{E}_k^T) \right\| + a^4 d_{k,j_0}^{-4} \left\| \bar{P} (\tilde{E}_k^T) \right\| \right) \\
\cdot \sum_{i > k}^{N} \left( a^5 d_{i,j_0}^{-1} \left\| \bar{P} (\tilde{E}_i^T) \right\| + a^6 d_{i,j_0}^{-4} \left\| \bar{P} (\tilde{E}_i^T) \right\| + a^4 d_{i,j_0}^{-4} \left\| \bar{P} (\tilde{E}_i^T) \right\| \right)
\]

\[
\lesssim a^{-2h} \left( a^5 d^{-3} + a^6 d^{-4} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\| \right) + a^4 d^{-4} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\|^2 \\
\lesssim a^{-2h} \left( a^{10} d^{-6} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\|^2 \right) + a^8 d^{-8} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\|^2.
\]

Finally,

\[
|I^*| = \Theta \left( a^{10^{-2h}} d^{-6} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\|^2 + a^{8^{-2h}} d^{-8} \max_i \left\| \bar{P} (\tilde{E}_i^T) \right\|^2 \right).
\]  

(5.33)

The analysis of the terms \( \sum_n |\langle \cdots, R_n \rangle|^2 \) is more technical. For an arbitrary vector field \( F \), we have

\[
\langle F, R_n \rangle = \frac{1 + c_0 a^h}{4 \pi \lambda_n^{(1)}(1 - k^2 \eta a^2 \lambda_n^{(1)})} \sum_{\ell \geq 1} (-ika)^{\ell+1} \langle F, T_{-k,a}^{-1} \rangle \int_B \frac{|y|^\ell}{(\ell + 1)!} c_n^{(1)}(y) dy \\
\approx \frac{1}{1 - k^2 \eta a^2 \lambda_n^{(1)}} \sum_{\ell \geq 1} (-ika)^{\ell+1} \langle T_{k,a}^{-1} F \rangle, \int_B \frac{|y|^\ell}{(\ell + 1)!} c_n^{(1)}(y) dy.
\]  

(5.16)
Taking the modulus, with the help of the Cauchy-Schwartz inequality, we obtain that
\[
|\langle F, R_n \rangle|^2 \lesssim \frac{a^4}{1 - k^2 \eta a^2 \lambda_n^{(1)}} \sum_{\ell \geq 1} (-ika)^{\ell+1} \left| \int_B T_{k_a}^{-1}(F)(x) \frac{|x|^{\ell}}{(\ell + 1)!} dx, e_n^{(1)} \right|^2.
\]
Then,
\[
\sum_n |\langle F, R_n \rangle|^2 \lesssim a^{4-2h} \sum_{\ell \geq 1} \left| \int_B T_{k_a}^{-1}(F)(x) \frac{|x|^{\ell}}{(\ell + 1)!} dx \right|^2_{L^2(B)}
\]
\[
\lesssim a^{4-2h} \left\| T_{k_a}^{-1}(F) \right\|^2_{L^2(B)} \sum_{\ell \geq 1} \int_B \int_B \left( \frac{|y-x|^2}{(\ell + 1)!} \right)^{\ell} dx dy,
\]
where we have the convergence of the series \( \sum_{\ell \geq 1} \int_B \int_B \left( \frac{|y-x|^2}{(\ell + 1)!} \right)^{\ell} dx dy \). Since we are approaching an eigenvalue of the first family, the dominant part of \( \left\| T_{k_a}^{-1}(F) \right\|^2_{L^2(B)} \) is \( \left\| T_{k_a}^{-1}(\mathbb{1} P(F)) \right\|^2_{L^2(B)} \) and then
\[
\sum_n |\langle F, R_n \rangle|^2 \lesssim a^{4-2h} \left\| T_{k_a}^{-1}(\mathbb{1} P(F)) \right\|^2_{L^2(B)} \lesssim a^{4-2h} \sum_{\ell} \frac{1}{1 - k^2 \eta \lambda^{(1)}_\ell a^2} \lesssim a^{4-4h} \left\| \mathbb{1} P(F), e_n^{(1)} \right\|^2.
\]
Finally,
\[
(5.34) \quad \sum_n |\langle F, R_n \rangle|^2 = O \left( a^{4-4h} \left\| \mathbb{1} P(F), e_n^{(1)} \right\|^2 \right).
\]
Remark that when \( F \) or \( \mathbb{1} P(F) \) is a constant vector field, using the representation \( e_\ell^{(1)} = \text{Curl}(\phi_\ell) \), \( \text{div}(\phi_\ell) = 0, \nu \times \phi_\ell = 0 \) and integration by parts, the series \( \sum_{\ell} \frac{1}{1 - k^2 \eta \lambda^{(1)}_\ell a^2} \left\| \mathbb{1} P(F), e_n^{(1)} \right\|^2 \) is vanishing. In this case, consequently, \( \sum_n |\langle F, R_n \rangle|^2 \) will be zero.

By utilizing (5.32), (5.33) and (5.34), the equation (5.21) takes the following form
\[
\left\| \mathbb{1} P \left( \hat{E}_{j0}^T \right) \right\|^2 \lesssim \left\| \langle \hat{E}_{j0}^{inc}, e_n^{(1)} \rangle \right\|^2 + \sum_{n \neq n_0} \left\| \langle \hat{E}_{j0}^{inc}, e_n^{(1)} \rangle \right\|^2 + a^{4-4h} \left\| \mathbb{1} P \left( \hat{E}_{j0}^{inc}, e_n^{(1)} \right) \right\|^2
\]
\[
+ a^{6-2h} d^{-6} \max_j \left\| \mathbb{1} P \left( \hat{E}_{j}^T \right) \right\|^2 + a^{4-2h} d^{-8} \max_j \left\| \mathbb{1} P \left( \hat{E}_{j}^T \right) \right\|^2
\]
\[
+ a^{-4h} \sum_{j \neq j_0} \left\| \mathbb{1} P \left( \hat{E}_{j0}, \hat{E}_{j}^T, \mathbb{1} P \left( \hat{E}_{j}^T \right) \right), e_n^{(1)} \right\|^2.
\]
(5.35)

For the last term, we know that
\[
\left\| \mathbb{1} P \left( \hat{E}_{j0}, \hat{E}_{j}^T, \mathbb{1} P \left( \hat{E}_{j}^T \right) \right), e_n^{(1)} \right\|^2
\]
where we can see from (5.18) that

\[
\left| \tilde{\text{Err}}(x, j_0, j, \mathbb{P}(E_j^T)) \right| \lesssim a^5 \int_B \int_0^1 Hess(\Phi_{ka})(z_{j_0} + tax, z_j) \cdot x \, dt \cdot y \cdot \mathbb{P}\left(\tilde{E}_j^T \right)(y) \, dy.
\]

Then there holds

\[
\left| \langle \text{Err}(x, j_0, j, \mathbb{P}(E_j^T)), e_{n_0}^{(1)} \rangle \right|^2 \lesssim a^{10} \int_B \int_0^1 Hess(\Phi_{ka})(z_{j_0} + tax, z_j) \cdot x \, dt \cdot y \cdot \mathbb{P}\left(\tilde{E}_j^T \right)(y) \, dy \, dx
\]

\[
\lesssim a^{10} \int_B \int_0^1 Hess(\Phi_{ka})(z_{j_0} + tax, z_j) \cdot x \, dt \cdot y \cdot \mathbb{P}\left(\tilde{E}_j^T \right)(y) \, dy \, dx
\]

\[
\lesssim a^{10} \frac{1}{|z_{j_0} - z_j|^8} \left\| \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2.
\]

Similarly, from (5.19), we deduce that

\[
\left| \langle \text{Err}(x, j_0, j, \mathbb{P}(E_j^T)), e_{n_0}^{(1)} \rangle \right|^2 \lesssim a^8 \int_B \int_0^1 \nabla(\Upsilon_{ka})(z_{j_0} + tax, z_j) \cdot \mathbb{P}(x, 0) \, dt \int_B \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right)(y) \, dy \cdot e_{n_0}^{(1)}(x) \, dx
\]

\[
\lesssim a^8 \frac{1}{|z_{j_0} - z_j|^8} \left\| \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2.
\]

Combining with (5.36) and (5.37), it is easy to see that in (5.35)

\[
a^{-4h} N \sum_{j \neq j_0}^N \left| \left\langle \mathbb{P}\left(\tilde{\text{Err}}(x, j_0, j, \mathbb{P}(E_j^T)) \right) + \text{Err}(x, j_0, j, \mathbb{P}(E_j^T)) \right), e_{n_0}^{(1)} \right\|^2
\]

\[
\lesssim a^{-4h} \sum_{j \neq j_0}^N \left( a^{10} \frac{1}{|z_{j_0} - z_j|^8} \left\| \frac{1}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2 + a^8 \frac{1}{|z_{j_0} - z_j|^8} \left\| \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2 \right)
\]

\[
\lesssim a^{10 - 4h} d^{-6} \max_j \left\| \frac{1}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2 + a^{8 - 4h} d^{-8} \max_j \left\| \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2.
\]

Now, we compare the orders of \(a\) in (5.38) with those in the following terms

\[
a^{6 - 2h} d^{-6} \max_j \left\| \frac{1}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2 + a^{4 - 2h} d^{-8} \max_j \left\| \frac{3}{\mathbb{P}} \left(\tilde{E}_j^T \right) \right\|_2^2,
\]

and by direct computations, we derive that the condition

\[
4 - s - 2h \geq 0,
\]

is sufficient to ensure that (5.39) is dominated.
For the other terms, we have
\[
\langle \tilde{E}^{inc}_{j_0}, \epsilon^{(1)}_n \rangle = \langle \tilde{E}^{inc}_{j_0}, \text{Curl} (\phi_n) \rangle = \langle \text{Curl} (\tilde{E}^{inc}_{j_0}), \phi_n \rangle = i k a \langle \tilde{H}^{inc}_{j_0}, \phi_n \rangle,
\]
and this implies
\[
\sum_{n \neq n_0} \left| \langle \tilde{E}^{inc}_{j_0}, \epsilon^{(1)}_n \rangle \right|^2 + a^{4-4h} \left| \frac{1}{\mathbb{P}} \langle \tilde{E}^{inc}_{j_0}, \epsilon^{(1)}_n \rangle \right|^2 \sim a^2 + a^{6-4h} = o(a^{(2.6-4h)}) = o(a^2),
\]
for \( h \leq 1 \) by (5.40). We end up with the following equation
\[
\left| \frac{1}{\mathbb{P}} \langle \tilde{E}^T_{j_0} \rangle \right|^2 \lesssim a^{2-2h} \left| \langle \tilde{H}^{inc}_{j_0}, \phi_{n_0} \rangle \right|^2 + a^{6-2h} d^{-6} \max \left| \frac{1}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2
\]
\[
+ a^{4-2h} d^{-8} \max \left| \frac{3}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2 + o(a^2).
\]
Taking the max \( (\cdot) \) with respect to \( j_0 \) on the both sides, we obtain that
\[
\max_{j_0} \left| \frac{1}{\mathbb{P}} \langle \tilde{E}^T_{j_0} \rangle \right|^2 \lesssim a^{2-2h} \max_{j_0} \left| \langle \tilde{H}^{inc}_{j_0}, \phi_{n_0} \rangle \right|^2 + a^{6-2h} d^{-6} \max_{j_0} \left| \frac{1}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2
\]
\[
+ a^{4-2h} d^{-8} \max_{j_0} \left| \frac{3}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2 + o(a^2).
\]
Assuming that \( a^{6-2h} d^{-6} \sim a^{6-2h-6t} \) is small enough or, the following sufficient condition
\[
(5.41) \quad 3 - h - 3t > 0
\]
is satisfied, then,
\[
(5.42) \quad \max_{j_0} \left| \frac{1}{\mathbb{P}} \langle \tilde{E}^T_{j_0} \rangle \right|^2 \lesssim a^{2-2h} \max_{j_0} \left| \langle \tilde{H}^{inc}_{j_0}, \phi_{n_0} \rangle \right|^2 + a^{4-2h} d^{-8} \max_{j_0} \left| \frac{3}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2.
\]
The goal of the next part is to derive an analogous formula to (5.42).

ii) Estimation of \( \max_j \left| \frac{3}{\mathbb{P}} \langle \tilde{E}^T_j \rangle \right|^2 \)

For \( x \in D_{j_0} \), we have
\[
\left( I + \eta \nabla M^k \right) \left( E_{j_0}^T \right) (x) + \eta \sum_{j \neq j_0} \nabla M^k \left( E_{j_0}^T \right)(x) - k^2 \eta \sum_{j=1}^N N^k \left( E_{j_0}^T \right)(x) = E_{j_0}^{inc}(x).
\]
Using the integration by parts, we deduce that \( \nabla M^k \left( \frac{1}{\mathbb{P}} \langle E_{j_0}^T \rangle \right) = 0, \ j = 1, \cdots , N \). Then
\[
\left( I + \eta \nabla M^k \right) \left( \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle \right) (x) = E_{j_0}^{inc}(x) + k^2 \eta N^k \left( \frac{1}{\mathbb{P}} \langle E_{j_0}^T \rangle \right)(x) - \frac{1}{\mathbb{P}} \langle E_{j_0}^T \rangle (x) + k^2 \eta N^k \left( \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle \right)(x)
\]
\[
(5.43) \quad + k^2 \eta \sum_{j \neq j_0} \int_{D_j} \Phi_k(x, y) I \cdot \frac{1}{\mathbb{P}} \langle E_{j_0}^T \rangle (y) dy + \eta \sum_{j \neq j_0} \int_{D_j} \Psi_k(x, y) \cdot \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle (y) dy,
\]
where \( \Psi_k (\cdot, \cdot) \) is given by (2.3). By Taylor expansion near the center \( z_j \), we obtain that
\[
\int_{D_j} \Psi_k(x, y) \cdot \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle (y) dy = \Psi_k(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle (y) dy
\]
\[
+ \int_0^1 \nabla \Psi_k (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathbb{P}(x, z_{j_0}) dt \cdot \int_{D_j} \frac{3}{\mathbb{P}} \langle E_{j_0}^T \rangle (y) dy.
\]

\footnote{The limit condition \( 3 - h - 3t = 0 \) can also be considered, see Remark 5.3}
\[ + \int_{D_j} \int_0^1 \nabla_y (Y_k) (x, z_j + t(y-z_j)) \cdot P(y, z_j) \, dt \cdot \frac{3}{\eta} (E_j^T) (y) \, dy, \]

and

\[ (5.44) \quad \int_{D_j} \Phi_k(x, y) I \cdot \hat{P} (E_j^T) (y) \, dy = \Phi_k(x, z_j) I \cdot \int_{D_j} \hat{P} (E_j^T) (y) \, dy \]

\[ + \int_{D_j} \int_0^1 \nabla_y (\Phi_k I) (x, z_j + t(y-z_j)) \cdot P(y, z_j) \, dt \cdot \hat{P} (E_j^T) (y) \, dy. \]

The first integral on the right hand side of (5.44) is vanishing and the second term by Taylor expansion give us that

\[ \int_{D_j} \Phi_k(x, y) I \cdot \hat{P} (E_j^T) (y) \, dy \]

\[ = - \int_{D_j} \int_0^1 \int_0^1 H_{ess} (\Phi_k) (z_{jo} + h(x-z_{jo}), z_j + t(y-z_j)) \cdot (x-z_{jo}) \, dh \cdot (y-z_j) dt \cdot \frac{3}{\eta} (E_j^T) (y) \, dy. \]

Plugging this expansion into equation (5.44), we can deduce that

\[ \left( I + \eta \nabla_x M^k \right) \left( \frac{3}{\eta} P (E_j^T) \right) (x) = E_{j0}^{Inc} (x) + k^2 \eta N^k \left( \frac{1}{\eta} P (E_j^T) \right) (x) + k^2 \eta N^k \left( \frac{3}{\eta} P (E_j^T) \right) (x) \]

\[ - k^2 \eta \sum_{j \neq jo}^N \int_{D_j} \int_0^1 \int_0^1 H_{ess} (\Phi_k) (z_{jo} + h(x-z_{jo}), z_j + t(y-z_j)) \cdot (x-z_{jo}) \, dh \cdot (y-z_j) dt \cdot \frac{3}{\eta} (E_j^T) (y) \, dy \]

\[ + \eta \sum_{j \neq jo}^N \int_{D_j} \int_0^1 \nabla_x (Y_k) (x, z_j + t(y-z_j)) \cdot P(x, z_j) \, dt \cdot \frac{3}{\eta} (E_j^T) (y) \, dy \]

\[ + \eta \sum_{j \neq jo}^N \int_{D_j} \int_0^1 \nabla_y (Y_k) (x, z_j + t(y-z_j)) \cdot P(y, z_j) \, dt \cdot \frac{3}{\eta} (E_j^T) (y) \, dy. \]

Scaling to the domain \( B \), and by taking the inner product with respect to \( e_n^{(3)} \), we obtain that

\[ (5.45) \quad \langle \frac{3}{\eta} P (E_j^T), e_n^{(3)} \rangle = \frac{1}{1 + \eta \lambda^3_n} \left[ \langle E_{j0}^{Inc}, e_n^{(3)} \rangle + \eta a^3 \sum_{j \neq jo}^N (Y_k(z_{jo}, z_j) \cdot \int_B \frac{3}{\eta} P (E_j^T) (y) \, dy, e_n^{(3)} + Error(n, jo) \right], \]

where

\[ Error(n, jo) := \]

\[ k^2 \eta a^2 \langle N^k a (\frac{1}{\eta} P (E_j^T)), e_n^{(3)} \rangle - \eta \left( \langle \nabla_x M^k - \nabla_x M \right) \left( \frac{3}{\eta} P (E_j^T) \right), e_n^{(3)} \rangle \]

\[ + k^2 \eta a^2 \langle N^k a (\frac{3}{\eta} P (E_j^T)), e_n^{(3)} \rangle \]

\[ + \eta a^4 \sum_{j \neq jo}^N \int_0^1 \nabla_x (Y_k) (z_{jo} + t a \cdot z_j) \cdot P(x, 0) \, dt \cdot \int_B \frac{3}{\eta} P (E_j^T) (y) \, dy, e_n^{(3)} \rangle \]

\[ - k^2 \eta a^5 \sum_{j \neq jo}^N \int_B \int_0^1 \int_0^1 H_{ess} (\Phi_k a) (z_{jo} + h a \cdot z_j + t a y) \cdot x \, dh \cdot y \, dt \cdot \frac{3}{\eta} P (E_j^T) (y) \, dy, e_n^{(3)} \rangle \]

\[ + \eta a^4 \sum_{j \neq jo}^N \int_B \int_0^1 \nabla_y (Y_k) (z_{jo} + a \cdot z_j + t a y) \cdot P(y, 0) \, dt \cdot \frac{3}{\eta} P (E_j^T) (y) \, dy, e_n^{(3)} \rangle. \]
We have

\[
\begin{align*}
k^2 \eta a^2 \left< N^ka \left( \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> &= k^2 \eta a^2 \left< N \left( \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> + k^2 \eta a^2 \left< (N^ka - N) \left( \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> \\
&= k^2 \eta a^2 \sum_{\ell \geq 1} \left< \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right), e^{(1)}_\ell \lambda^{(1)}_\ell \right> \left< e^{(1)}_\ell, e_n^{(3)} \right> + k^2 \eta a^2 \left< (N^ka - N) \left( \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right>
\end{align*}
\]

Similarly, using (2.11) and (2.8), we rewrite

\[
\begin{align*}
\eta \left< \left( \nabla M^ka - \nabla M \right) \left( \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> &= \frac{1 + c_0 a h^2}{2\lambda^{(1)}_n} \left< N \left( \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> + \eta \frac{i (ka)^3}{12 \pi} \int_B \left< \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right)(y) \right> dy, e_n^{(3)} \right> \]

Then, Error\((n, j_0)\) takes the form

\[
\begin{align*}
\text{Error}(n, j_0) := \frac{1 + c_0 a h^2}{4 \pi \lambda^{(1)}_n} \sum_{\ell \geq 1} \frac{(ika)^{\ell+1}}{\ell+1} \left< \int_B \left| - y \right|^\ell \left< \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right)(y) \right> dy, e_n^{(3)} \right> - \frac{1 + c_0 a h^2}{2 \lambda^{(1)}_n} \left< N \left( \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right> \\
- \eta \frac{i (ka)^3}{12 \pi} \int_B \left< \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right)(y) \right> dy, e_n^{(3)} \right> + \frac{1 + c_0 a h^2}{2 \lambda^{(1)}_n} \left< \int_B \Phi_0 \left( \cdot \right), y \right> \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right)(y) dy, e_n^{(3)} \right> \\
+ \eta \frac{i (ka)^3}{12 \pi} \int_B \left< \frac{3}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right)(y) \right> dy, e_n^{(3)} \right> + k^2 \eta a^2 \left< (N^ka \left( \frac{1}{\mathbb{P}} \left( \mathbb{E}^T_{j0} \right) \right), e_n^{(3)} \right>
\end{align*}
\]
Taking the squared modulus on the both sides of (5.45), it yields
\[
\left| \left( \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T, \epsilon_n^{(3)} \right) \right)^2 \right| \lesssim |\eta|^{-2} \left( \left| \mathbb{E}_{j_0}^{inc} + \epsilon_n^{(3)} \right|^2 + a^2 \sum_{j \neq j_0} |\gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy, \epsilon_n^{(3)} |^2 \right) + a^2 \sum_{j = 1}^N |\gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy, \epsilon_n^{(3)} |^2.
\]

Taking the series with respect to \( n \), we can further deduce that
\[
\left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T \right) \right\|^2 \lesssim |\eta|^{-2} \left( \left\| \mathbb{E}_{j_0}^{inc} \right\|^2 + a^2 \sum_{j \neq j_0} |\gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy \right)^2 + \sum_n |\text{Error}(n, j_0)|^2
\]
\[+ a^2 \sum_{j = 1}^N \left( \gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy \right) \sum_{j > j_0} \gamma_k(z_{j_0}, z_i) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{i}^T \right) (y) dy \right].
\]

Using the fact that \( \gamma_k(z_{j_0}, z_j) \approx d_{j_0}^{-3} \), it is easy to get the following upper bound
\[
\sum_{j \neq j_0} |\gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy |^2 \lesssim \sum_{j \neq j_0} \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2 \lesssim d_6^{-6} \max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2,
\]
and thus
\[
I^{**} := \sum_{j = 1}^N |\gamma_k(z_{j_0}, z_j) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) (y) dy | \sum_{j > j_0} \gamma_k(z_{j_0}, z_i) \cdot \int_B \frac{3}{\mathbb{P}} \left( \mathbb{E}_{i}^T \right) (y) dy | \lesssim \sum_{j = 1}^N d_6^{-3} \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2 \sum_{j > j_0} d_6^{-3} \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{i}^T \right) \right\|^2 \approx d_6^{-6} |\log(d)|^2 \max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2.
\]
Therefore,
\[
(5.47) \quad \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T \right) \right\|^2 \lesssim |\eta|^{-2} \left\| \mathbb{E}_{j_0}^{inc} \right\|^2 + a^6 |\log(d)|^2 d_6^{-6} \max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2 + |\eta|^{-2} \sum_n |\text{Error}(n, j_0)|^2.
\]

Taking the max \( \cdot \) on both sides of (5.47), we have
\[
\max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T \right) \right\|^2 \lesssim |\eta|^{-2} \max_j \left\| \mathbb{E}_{j_0}^{inc} \right\|^2 + a^6 |\log(d)|^2 d_6^{-6} \max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j}^T \right) \right\|^2 + |\eta|^{-2} \sum_n |\text{Error}(n, j_0)|^2.
\]

Remark that the coefficient \( a^6 |\log(d)|^2 d_6^{-6} \sim a^6 |\log(d)|^2 \) is small, then we obtain
\[
(5.48) \quad \max_j \left\| \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T \right) \right\|^2 \lesssim |\eta|^{-2} \max_j \left\| \mathbb{E}_{j_0}^{inc} \right\|^2 + |\eta|^{-2} \max_j \sum_n |\text{Error}(n, j_0)|^2.
\]

To finish the estimation of (5.48), we need to estimate \( \max_j \sum_n |\text{Error}(n, j_0)|^2 \). For this, taking the squared modulus on the both sides of (5.46), we get
\[
|\text{Error}(n, j_0)|^2 \lesssim a^4 \sum_{\ell \geq 1} \left| \int_B \left( \frac{\mathbb{E}_{j_0}^T}{\ell + 1} \right) dy, \epsilon_n^{(3)} \right|^2 + \left| \mathcal{N} \left( \frac{3}{\mathbb{P}} \left( \mathbb{E}_{j_0}^T \right) \right), \epsilon_n^{(3)} \right|^2.
\]
\[ + a^2 \left| \left\langle \int_B \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) (y) dy, e_n^{(3)} \right\rangle \right|^2 + \left| \left\langle \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot \mathbb{P}(\tilde{E}^T_{j_0}(y))}{\| \cdot - y \|^2} dy, e_n^{(3)} \right\rangle \right|^2 \]

\[ + a^4 \sum_{\ell \geq 1} \left| \left\langle \int_B \frac{\text{Hess}_y \left( \cdot, - y \right)^\ell}{(\ell + 1)!} \cdot \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) (y) dy, e_n^{(3)} \right\rangle \right|^2 + \left| \left\langle N^{ka} \left( \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right), e_n^{(3)} \right\rangle \right|^2 \]

\[ + a^4 \mathbb{R} \sum_{j \neq j_0} \left| \left\langle \int_0^1 \nabla \left( \Psi_{ja} \right) (z_{j_0} + t a \cdot, z_j) \cdot \mathcal{P}(x, 0) dt \cdot \int_B \frac{3}{P} \left( \tilde{E}^T_j \right) (y) dy, e_n^{(3)} \right\rangle \right|^2 \]

\[ + a^0 \mathbb{R} \sum_{j \neq j_0} \left| \left\langle \int_B \int_0^1 \int_0^1 \text{Hess}_y \left( \Phi_{ka} \right) (z_{j_0} + h a \cdot, z_j + t a y) \cdot x dy \cdot y dt \frac{3}{P} \left( \tilde{E}^T_j \right) (y) dy, e_n^{(3)} \right\rangle \right|^2 \]

\[ + a^4 \mathbb{R} \sum_{j \neq j_0} \left| \left\langle \int_B \int_0^1 \nabla \left( \Psi_{ja} \right) (z_{j_0} + a \cdot, z_j + t a y) \cdot \mathcal{P}(y, 0) dt \cdot \frac{3}{P} \left( \tilde{E}^T_j \right) (y) dy, e_n^{(3)} \right\rangle \right|^2 \].

From \[2.9\], we can know that

\[ (5.49) \quad N^{ka} \left( \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right) = N \left( \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right) + ika \frac{\pi}{4\pi} \int_D \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) (y) dy + \frac{1}{4\pi} \sum_{n \geq 1} (ika)^{n+1} (n+1)! \int_D \| x - y \|^n \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) (y) dy. \]

It is obvious that in \[5.19\], \( N \left( \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right) \) dominates. Thus by taking the series with respect to \( n \) index, using the continuity of the Newtonian potential operator, we get

\[ \sum_n |\text{Error}(n, j_0)|^2 \lesssim a^4 \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 + \sum_{\ell \geq 1} \int_B \int_B \frac{\| x - y \|^{2\ell}}{((\ell + 1)!)^2} dy dx + \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 \]

\[ + a^4 \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 + \sum_{\ell \geq 1} \int_B \int_B \frac{\| \text{Hess}_y \left( x - y \right)^\ell \|}{((\ell + 1)!)^2} dy dx + a^4 \mathbb{R} d^{-8} \max_j \left\| \frac{3}{P} \left( \tilde{E}^T_j \right) \right\|^2 \]

\[ + a^6 \mathbb{R} d^{-6} \max_j \left\| \frac{1}{P} \left( \tilde{E}^T_j \right) \right\|^2. \]

For the convergence of the two series appearing on the right hand side, we refer to \[5.10\] and \[5.11\]. Then,

\[ \sum_n |\text{Error}(n, j_0)|^2 \lesssim a^4 \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 + \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 + a^4 \mathbb{R} d^{-8} \max_j \left\| \frac{3}{P} \left( \tilde{E}^T_j \right) \right\|^2 + a^6 \mathbb{R} d^{-6} \max_j \left\| \frac{1}{P} \left( \tilde{E}^T_j \right) \right\|^2. \]

Taking the \( \max (\cdot) \) with respect to \( j_0 \) index, we obtain that

\[ \max_{j_0} \sum_n |\text{Error}(n, j_0)|^2 \lesssim \max (1; a^4 \mathbb{R} d^{-8}) \max_j \left\| \frac{3}{P} \left( \tilde{E}^T_j \right) \right\|^2 + \max (a^4; a^6 \mathbb{R} d^{-6}) \max_j \left\| \frac{1}{P} \left( \tilde{E}^T_j \right) \right\|^2. \]

Going back to \[5.38\] and plugging the previous result, we can derive that

\[ \max_{j_0} \left\| \frac{3}{P} \left( \tilde{E}^T_{j_0} \right) \right\|^2 \lesssim |\eta|^{-2} \max_{j_0} \| \tilde{E}^{tac}_{j_0} \|^2. \]

\[ ^8 \text{We neglect the estimation of } \sum_n \left| \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot \mathbb{P}(\tilde{E}^T_{j_0}(y))}{\| \cdot - y \|^2} dy, e_n^{(3)} \right|^2 \text{ since the singularity of the corresponding kernel is like the one of the Newtonian operator.} \]
We assume that $|\eta|^{-2} a^4 \sim a^{g-8} - s$ is small enough, which is equivalent to the following condition

$$4 - 4t - \frac{s}{2} > 0.$$  

(5.50)

In particular, when the distribution of the cluster of the nanoparticles is made over a volume, we have $S \sim d^{-3}$. Therefore, we have $s \leq 3t$. Coupling this with (5.51), i.e. $t \leq 1 - \frac{h}{3}$, then (5.51) is satisfied only if

$$h > \frac{9}{11}.$$  

(5.51)

Then

$$\max_{j_0} \left\| \frac{2}{3} \left( \tilde{E}_{j_0}^T \right) \right\|^2 \lesssim |\eta|^{-2} \max_{j_0} \left\| \tilde{E}_{j_0}^{inc} \right\|^2 + |\eta|^{-2} \max_{j_0} \left( a^4; a^6 \right) \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2. $$

For fixed $j_0$, we have

$$\left\| \tilde{E}_{j_0}^{inc} \right\|^2 = \sum_n \left\| \langle \tilde{E}_{j_0}^{inc}, e_n^{(1)} \rangle \right\|^2 + \sum_n \left\| \langle \tilde{E}_{j_0}^{inc}, e_n^{(3)} \rangle \right\|^2$$

$$= \sum_n \left\| \langle \tilde{E}_{j_0}^{inc}, \text{Curl} (\phi_n) \rangle \right\|^2 + O(1)$$

$$= k^2 a^2 \sum_n \left\| \tilde{H}_{j_0}^{inc}, \phi_n \right\|^2 + O(1) = O a^2 + O(1) = O(1).$$

Then,

$$\max_{j_0} \left\| \frac{2}{3} \left( \tilde{E}_{j_0}^T \right) \right\|^2 \lesssim a^4 + |\eta|^{-2} \max_{j_0} \left( a^4; a^6 \right) \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2. $$

(5.52)

Now, from (5.50) and (5.52), we deduce that

$$\max_j \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h} + a^{4-2h} d^{-8} \left[ a^4 + a^4 \max_{j} \left( a^4; a^6 \right) \right] \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2$$

$$\lesssim a^{2-2h} + a^{8-2h} d^{-8} + a^{8-2h} d^{-8} \max_{j} \left( a^4; a^6 \right) \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2.$$  

If there holds

$$2 - s - 6t < 0,$$

then $\max_{j} \left( a^4; a^6 \right) = a^6 d^{-6}$, which indicates that

$$\max_j \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h} + a^{8-2h} d^{-8} + a^{14-2h} d^{-14} \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2. $$

From (5.41) and (5.50), it is direct to verify that $7 - 7t - h - \frac{s}{2} > 0$. Hence, we can derive

$$\max_j \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h} + a^{8-2h} d^{-8}. $$

(5.53)

Otherwise, if there holds

$$2 - s - 6t \geq 0,$$

then $(a^4; a^6) = a^4$, which indicates that

$$\max_j \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h} + a^{8-2h} d^{-8} + a^{12-2h} d^{-8} \max_{j} \left\| \frac{1}{P} \left( \tilde{E}_j^T \right) \right\|^2.$$
By using (5.41) and the fact that \( t < 1 \), it is automatically fulfilled that \( 12 - 2h - 8t > 0 \) and therefore (5.53) still holds.

Following a similar argument above, for \( \mathbb{P} \left( \tilde{E}_j^T \right) \), we can also know from (5.42) and (5.52) that

\[
\max_j \left\| \frac{3}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^4 + a^4 \cdot \max(a^4; a^6 \cdot d^{-6}) \max_j \left\| \frac{1}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \\
\lesssim a^4 + a^4 \cdot \max(a^4; a^6 \cdot d^{-6}) \left[ a^{2-2h} + a^{1-2h} d^{-8} \max_j \left\| \frac{3}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \right] \\
\lesssim a^4 + a^{12-2h} d^{-6}.
\]

Since the sufficient conditions (5.30), (5.41) and (5.50) give us \( t < \frac{5}{7} \), we see that

\[
\max_j \left\| \frac{1}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h}.
\]

Moreover,

\[
(5.54) \quad \max_j \left\| \frac{3}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim \begin{cases} a^4 & \text{if } t < \frac{2}{3}, \\
a^{12-2h} d^{-6} & \text{if } \frac{2}{3} \leq t < \frac{5}{7}. \end{cases}
\]

Since \( N = O(d^{-3}) \sim a^{-3t} \) as \( d \sim a^t \), by recalling that \( t \leq 1 - \frac{h}{3} \), then for \( \frac{2}{3} < h < 1 \), we have

\[
(5.55) \quad \max_j \left\| \frac{1}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{2-2h} \quad \text{and} \quad \max_j \left\| \frac{3}{\mathbb{P}} \left( \tilde{E}_j^T \right) \right\|^2 \lesssim a^{3+h},
\]

which completes the proof.

**Remark 5.2.** Recall the sufficient condition (5.50). Particularly, in the regime that

\[
t = 1 - \frac{h}{3}, \quad \text{and} \quad s = 3t,
\]

the expression \( 3 - h - 3t \), given on the left hand side of (5.41), will be vanishing and then the condition will not be accomplished. To be able to handle this situation, we assume that \( \eta \) satisfies \( |\eta| = \zeta a^{-2} \), with \( \zeta < 1 \).

### 5.3. Proof of Proposition 2.4

Here, we present the estimation for the scattering coefficient. Similar to Lemma 5.1 we first give the following formulation before proving Proposition 2.4.

**Lemma 5.3.** Recall that \( e_n^{(3)}, n = 1, 2, \ldots \), are the eigenfunctions of \( \nabla M \) in the subspace \( \nabla \text{Harmonic} \). Then we have

\[
T_{ka}^{-1} \left( e_n^{(3)} \right) = \frac{1}{1 + \eta \lambda_n^{(3)}} e_n^{(3)} \\
+ \frac{1}{2 \lambda_n^{(1)}} \left[ T_{ka}^{-1} N \left( e_n^{(3)} \right) + T_{ka}^{-1} \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{||\cdot - y||} dy \right]^{(3)} + R_n,
\]

where

\[
R_n^{(3)} := \frac{1}{1 + \eta \lambda_n^{(3)}} T_{ka}^{-1} \left[ \frac{\pm c_0 a^h}{2 \lambda_n^{(1)}} N \left( e_n^{(3)} \right) + \frac{\pm c_0 a^h}{2 \lambda_n^{(1)}} \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{||\cdot - y||} dy + \frac{i \eta k^3 a^3}{6\pi} \int_B e_n^{(3)}(y) dy \right]^{(3)} \\
+ \frac{\eta}{4\pi} \sum_{\ell \geq 3} \frac{(ika)^{\ell+1}}{(\ell + 1)!} \int_B \text{Hess} \left( ||\cdot - y||^\ell \right) \cdot e_n^{(3)}(y) dy - \frac{(1 \pm c_0 a^h)}{4\pi \lambda_n^{(1)}} \sum_{\ell \geq 2} \frac{(ika)^{\ell+1}}{(\ell + 1)!} \int_B ||\cdot - y||^\ell e_n^{(3)}(y) dy.
\]
Proof. Let us compute $T_{ka}(e_n^{(3)})$ as follows:

$$T_{ka}(e_n^{(3)}) = (I + \eta \nabla M)(e_n^{(3)}) + \eta (\nabla M \times - \nabla M) \left( e_n^{(3)} \right) - k^2 \eta a^2 N \left( e_n^{(3)} \right)$$

(5.57)

By combining with (2.8) and (2.9), we deduce that

$$[(\nabla M \times - \nabla M) - k^2 a^2 N \times ka] \left( e_n^{(3)} \right) = -\frac{k^2 a^2}{2} N \left( e_n^{(3)} \right) - \frac{k^2 a^2}{2} \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{\|\cdot - y\|^2} dy$$

$$- \frac{i k^3 a^3}{6\pi} \int_B e_n^{(3)}(y) dy - \frac{1}{4\pi} \sum_{n \geq 3} (ika)^{n+1} \int_B \text{Hess} (\|\cdot - y\|^n) \cdot e_n^{(3)}(y) dy$$

$$- \frac{k^2 a^2}{4\pi} \sum_{n \geq 1} (ika)^{n+1} (n+1)! \int_B \|\cdot - y\|^n e_n^{(3)}(y) dy,$$

and therefore we see that (5.57) can be further written as

$$T_{ka}(e_n^{(3)}) = \left(1 + \eta \lambda_n^{(3)}\right) e_n^{(3)} - \frac{k^2 a^2}{2} N \left( e_n^{(3)} \right) - \frac{k^2 a^2}{2} \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{\|\cdot - y\|^2} dy$$

$$- \frac{i k^3 a^3}{6\pi} \int_B e_n^{(3)}(y) dy - \frac{1}{4\pi} \sum_{n \geq 3} (ika)^{n+1} \int_B \text{Hess} (\|\cdot - y\|^n) \cdot e_n^{(3)}(y) dy$$

(5.58)

Taking the inverse of $T_{ka}$ on the both sides of (5.58), we obtain that

$$e_n^{(3)} = \left(1 + \eta \lambda_n^{(3)}\right) T_{ka}^{-1} e_n^{(3)} - \frac{k^2 a^2}{2} T_{ka}^{-1} N \left( e_n^{(3)} \right)$$

$$- \eta k^2 a^2 \int_B \Phi_0(\cdot, y) A(\cdot, y) \cdot e_n^{(3)}(y) dy - \frac{i k^3 a^3}{6\pi} \int_B e_n^{(3)}(y) dy T_{ka}^{-1} (1)$$

$$- \frac{\eta}{4\pi} T_{ka}^{-1} \sum_{n \geq 3} (ika)^{n+1} (n+1)! \int_B \text{Hess} (\|\cdot - y\|^n) \cdot e_n^{(3)}(y) dy$$

(5.59)

Dividing by $\left(1 + \eta \lambda_n^{(3)}\right)$ on the both sides of (5.59) and after rearranging terms, we derive that

$$T_{ka}^{-1} e_n^{(3)} = \frac{1}{\left(1 + \eta \lambda_n^{(3)}\right)} e_n^{(3)} - \frac{1}{\left(1 + \eta \lambda_n^{(3)}\right)^2} k^2 a^2 \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{\|\cdot - y\|^2} dy$$

$$+ \frac{1}{\left(1 + \eta \lambda_n^{(3)}\right)} \int_B e_n^{(3)}(y) dy T_{ka}^{-1} (1)$$

$$+ \frac{i k^3 a^3}{6\pi} \int_B e_n^{(3)}(y) dy T_{ka}^{-1} (1)$$

$$+ \frac{\eta}{4\pi} \sum_{n \geq 3} (ika)^{n+1} (n+1)! \int_B \text{Hess} (\|\cdot - y\|^n) \cdot e_n^{(3)}(y) dy.$$
Now we use the result given in Lemma 5.1 and the fact that the term \( C \int_B \cdot \cdot \cdot \) given by (2.17). After scaling to the domain \( \Omega \), we can deduce (5.56) from (5.60), which completes the proof.

We first give the estimation for the scattering coefficient \( \mathcal{C} \).

Proof of Proposition 2.4. We first give the estimation for the scattering coefficient \( \mathcal{C} \). Recall the definition of \( \mathcal{C} \) given by (2.11). After scaling to the domain \( B \), we obtain

\[
\mathcal{C} = a^4 \int_B T_{\alpha_0}^{-1}(\mathcal{P}(\cdot, 0)) \cdot x \, dx.
\]

By expanding over the basis and using the fact that \( \int_B e^{(1,2)}_n(x) dx = 0 \), we get

\[
\mathcal{C} = a^4 \sum_n \langle T_{\alpha_0}^{-1}(\mathcal{P}(\cdot, 0)), e^{(3)}_n \rangle \int_B e^{(3)}_n(x) dx = a^4 \sum_n \langle \mathcal{P}(\cdot, 0), T_{\alpha_0}^{-1}(e^{(3)}_n) \rangle \int_B e^{(3)}_n(x) dx.
\]

From the expression (5.56) in Lemma 5.3, in order to extract the dominant term of (5.56), we need to analyse the term \( S := T_{\alpha_0}^{-1}N(e^{(3)}_n) \). For this we have

\[
S := T_{\alpha_0}^{-1} \left[ \frac{1}{\mathbb{P}} \left( N(e^{(3)}_n) \right) + \frac{2}{\mathbb{P}} \left( N(e^{(3)}_n) \right) + \frac{3}{\mathbb{P}} \left( N(e^{(3)}_n) \right) \right],
\]

\[
S = \sum_{\ell} \frac{1}{\mathbb{P}} \left( N(e^{(3)}_n) \right), e^{(1)}_\ell T_{\alpha_0}^{-1} \left[ e^{(1)}_\ell \right] + \sum_{\ell} \frac{2}{\mathbb{P}} \left( N(e^{(3)}_n) \right), e^{(2)}_\ell T_{\alpha_0}^{-1} \left[ e^{(2)}_\ell \right]
\]

\[
+ \sum_{\ell} \frac{3}{\mathbb{P}} \left( N(e^{(3)}_n) \right), e^{(3)}_\ell T_{\alpha_0}^{-1} \left[ e^{(3)}_\ell \right].
\]

Now we use the result given in Lemma 5.1 and the fact that \( T_{\alpha_0} \), when restricted to \( \mathbb{H}_0(Curl = 0) \), is equal to \( (1 + \eta) I \) to deduce that

\[
S = \sum_{\ell} \frac{1}{\mathbb{P}} \left( N(e^{(3)}_n) \right), e^{(1)}_\ell \left( \frac{1}{1 - k^2 \eta a^2 \lambda^{(3)}_\ell} e^{(1)}_\ell + R_{\ell} \right) + (1 + \eta)^{-1} \frac{2}{\mathbb{P}} \left( N(e^{(3)}_n) \right)
\]

\[
+ \sum_{\ell} \frac{3}{\mathbb{P}} \left( N(e^{(3)}_n) \right), e^{(3)}_\ell T_{\alpha_0}^{-1} \left[ e^{(3)}_\ell \right].
\]
For the last term, we repeat the same computations as those done in Lemma 5.3 to deduce that
\[ \sim \sum_{\ell} \langle \mathcal{P} \left( N \left( e_n^{(3)} \right) \right), e_n^{(3)} \rangle \frac{1}{1 + \eta \lambda_n^{(3)}} e_n^{(3)}. \]

Finally, the dominant term of \( S \) is given by
\[ S \sim \sum_{\ell} \frac{1}{\mathcal{P}} \left( N \left( e_n^{(3)} \right) \right), e_n^{(1)} \frac{1}{1 - k^2 \eta a^2 \lambda_n^{(1)}} e_n^{(1)}. \]

Since
\[ T_{-k \alpha}^{-1} \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot e_n^{(3)}(y)}{||\cdot - y||} dy \]
has the same behaviour as \( S \), the corresponding dominant term is also given by (5.62). Using (5.56) and (5.52), the equation (5.61) takes the following form
\[ \mathcal{E} \sim a^4 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \int_B e_n^{(3)}(x) dx \]
\[ + \frac{a^4}{2 \lambda_n^{(1)}} \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \sum_{\ell} \frac{1}{\mathcal{P}} \left( N \left( e_n^{(3)} \right) \right), e_n^{(1)} \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \int_B e_n^{(3)}(x) dx. \]

Remark that the second term, when taking the index \( \ell = n_0 \), will dominate the first one. Consequently,
\[ \mathcal{E} \sim a^{4-h} \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \int_B e_n^{(3)}(x) dx = O \left( a^{6-h} \right). \]

Next, we prove (2.19). Recall that \( W \) solves (2.10). By scaling (2.16) from \( D \) to \( B \), we obtain that
\[ T_{-k \alpha} \left( \tilde{W} \right)(x) = a^{\mathcal{P}(x, 0)}, \quad x \in B. \]

We project \( \tilde{W} \) onto the subspaces introduced in (2.2) as \( \tilde{W} = \frac{1}{\mathcal{P}} \left( \tilde{W} \right) + \frac{2}{\mathcal{P}} \left( \tilde{W} \right) + \frac{3}{\mathcal{P}} \left( \tilde{W} \right) \). Then (5.64), using the fact that \( \nabla M^k \equiv 0 \) on \( \mathcal{H}_0(\text{div} = 0) \) and (5.4), becomes
\[ (I - k^2 \eta a^2 N) \left( \frac{1}{\mathcal{P}} \left( \tilde{W} \right) \right)(x) + (1 + \eta) \frac{2}{\mathcal{P}} \left( \tilde{W} \right)(x) + (I + \eta \nabla M) \left( \frac{3}{\mathcal{P}} \left( \tilde{W} \right) \right)(x) = a^{\mathcal{P}(x, 0)} - T(x), \]
where
\[ T(x) := \eta \left( \nabla M^{-k \alpha} - \nabla M \right) \left( \frac{3}{\mathcal{P}} \left( \tilde{W} \right) \right)(x) - k^2 \eta a^2 \left( N^{-k \alpha} - N \right) \left( \frac{3}{\mathcal{P}} \left( \tilde{W} \right) \right)(x), \]
which can be further rewritten as
\[ T(x) = -\eta \frac{(k \alpha)^2}{2} N \left( \frac{3}{\mathcal{P}} \left( \tilde{W} \right) \right)(x) + i \eta \frac{(k \alpha)^3}{6 \pi} \int_B \frac{3}{\mathcal{P}} \left( \tilde{W} \right)(y) dy - \eta \frac{(k \alpha)^2}{2} \int_B \Phi_0(x, y) \frac{A(x, y) \cdot \frac{3}{\mathcal{P}} \left( \tilde{W} \right)(y)}{||x - y||^2} dy \]
\[ - \frac{1}{4 \pi} \eta \int_B H e s s \left( \frac{|x - y|^2}{(\ell + 1)!} \right) \cdot \frac{3}{\mathcal{P}} \left( \tilde{W} \right)(y) dy \]
\[ - \frac{k^2 \eta a^2}{4 \pi} \int_B \frac{|x - y|^2}{(\ell + 1)!} \cdot \frac{3}{\mathcal{P}} \left( \tilde{W} \right)(y) dy \]
\[ \frac{k^2 \eta a^2}{4 \pi} \int_B \frac{|x - y|^2}{(\ell + 1)!} \cdot \frac{1}{\mathcal{P}} \left( \tilde{W} \right)(y) dy, \]
By using (2.8) and (2.9). Taking the inner product of (5.65) with respect to \(e^{(1)}_n\), we obtain
\[
\langle \mathcal{P}(\mathcal{W}), e^{(1)}_n \rangle = \frac{1}{1 - k^2 \eta a^2 \lambda^{(1)}_n} \left[ a \langle \mathcal{P}(\cdot, 0), e^{(1)}_n \rangle - \langle T, e^{(1)}_n \rangle \right].
\]
By taking the squared modulus and summing with respect to the index \(n\), we deduce that
\[
\|\mathcal{P}(\mathcal{W})\|^2 \lesssim \sum_n \frac{1}{1 - k^2 \eta a^2 \lambda^{(1)}_n} \left[ \langle \mathcal{P}(\cdot, 0), e^{(1)}_n \rangle \right]^2 + a^{-2h} \sum_n \|T, e^{(1)}_n\|^2.
\]
We need to compute \(\sum_n \|\langle T, e^{(1)}_n \rangle\|^2\). Indeed,
\[
\|\langle T, e^{(1)}_n \rangle\|^2 \lesssim \left| \int B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot \mathcal{P}(\mathcal{W})}{\|x - y\|^2} dy, e^{(1)}_n \right|^2 + a^4 \sum_{\ell \geq 3} \left| \int B \frac{Hess \left[ \mathcal{P}(\cdot, 0) \right]}{(\ell + 1)!} \cdot \mathcal{P}(\mathcal{W}) \right| (y) dy, e^{(1)}_n \right|^2
= a^6 \sum_{\ell \geq 2} \left| \int B \left[ \frac{|\cdot - y|^{\ell}}{(\ell + 1)!} \right] \mathcal{P}(\mathcal{W}) \right| (y) dy, e^{(1)}_n \right|^2.
\]
Then,
\[
\sum_n \|\langle T, e^{(1)}_n \rangle\|^2 \lesssim \left| \int B \Phi_0(\cdot, y) \frac{A(\cdot, y) \cdot \mathcal{P}(\mathcal{W})}{\|x - y\|^2} dy \right|^2 + a^4 \sum_{\ell \geq 3} \left| \int B \frac{Hess \left[ \mathcal{P}(\cdot, 0) \right]}{(\ell + 1)!} \cdot \mathcal{P}(\mathcal{W}) \right| (y) dy \right|^2
= a^6 \sum_{\ell \geq 2} \left| \int B \left[ \frac{|\cdot - y|^{\ell}}{(\ell + 1)!} \right] \mathcal{P}(\mathcal{W}) \right| (y) dy \right|^2
\]
\[
\lesssim \|\mathcal{P}(\mathcal{W})\|^2 + a^6 \|\mathcal{P}(\mathcal{W})\|^2.
\]
Equation (5.68) takes the following form
\[
\|\mathcal{P}(\mathcal{W})\|^2 \lesssim \sum_n \frac{1}{1 - k^2 \eta a^2 \lambda^{(1)}_n} \left[ \langle \mathcal{P}(\cdot, 0), e^{(1)}_n \rangle \right]^2 + a^{-2h} \|\mathcal{P}(\mathcal{W})\|^2.
\]
We compute \(\|\mathcal{P}(\mathcal{W})\|^2\) in a similar manner. First, we take the inner product on the both sides of (5.65) with respect to \(e^{(2)}_n\), to obtain
\[
(1 + \eta) \langle \mathcal{P}(\mathcal{W}), e^{(2)}_n \rangle = \left[ a \langle \mathcal{P}(\cdot, 0), e^{(2)}_n \rangle - \langle T, e^{(2)}_n \rangle \right].
\]
After taking the squared modulus and the series with respect to \(n\), we obtain:
\[
\|\mathcal{P}(\mathcal{W})\|^2 \lesssim \frac{1}{(1 + \eta)^2} a^2 \sum_n \|\langle \mathcal{P}(\cdot, 0), e^{(2)}_n \rangle\|^2 + a^4 \sum_n \|\langle T, e^{(2)}_n \rangle\|^2.
\]
The computation of \(\sum_n \|\langle T, e^{(2)}_n \rangle\|^2\) can be made similar to the computation of \(\sum_n \|\langle T, e^{(1)}_n \rangle\|^2\). Indeed, from (5.69), we deduce
\[
\sum_n \|\langle T, e^{(2)}_n \rangle\|^2 \lesssim \|\mathcal{P}(\mathcal{W})\|^2 + a^6 \|\mathcal{P}(\mathcal{W})\|^2.
\]
Thus (5.71) takes the form

\begin{equation}
\left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\|^2 \lesssim \frac{1}{1 + \eta^2} a^2 \sum_n \left| \langle \mathcal{P}(\cdot, 0), e_n^{(2)} \rangle \right|^2 + a^4 \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 + a^{10} \left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\|^2.
\end{equation}

Next, we estimate \( \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 \). From (5.65), we have

\begin{equation}
\left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 \lesssim \frac{1}{1 + \eta^2} a^2 \left[ \left| \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \right|^2 + \left| \langle \mathcal{T}, e_n^{(3)} \rangle \right|^2 \right],
\end{equation}

(5.73)

\begin{align*}
\left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 & \lesssim \left| \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \right|^2 + a^4 \sum_n \left| \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \right|^2 + a^{10} \sum_n \left| \langle \mathcal{T}, e_n^{(3)} \rangle \right|^2.
\end{align*}

For \( \left| \langle \mathcal{T}, e_n^{(3)} \rangle \right|^2 \), we have

\begin{align*}
\left| \langle \mathcal{T}, e_n^{(3)} \rangle \right|^2 & \lesssim \left| \langle \mathcal{N} (\frac{3}{\mathcal{P}} (\tilde{W})), e_n^{(3)} \rangle \right|^2 + a^2 \left| \int_B \frac{Hess \left( \cdot, -y \right)^{\ell}}{\ell + 1} \frac{3}{\mathcal{P}} (\tilde{W}) (y) dy, e_n^{(3)} \right|^2 + a^4 \sum_{\ell \geq 2} \left| \int_B \frac{Hess \left( \cdot, -y \right)^{\ell}}{\ell + 1} \frac{3}{\mathcal{P}} (\tilde{W}) (y) dy, e_n^{(3)} \right|^2.
\end{align*}

Taking the series with respect to \( n \) index and utilizing the continuity of the Newtonian potential operator, we obtain

\begin{align*}
\sum_n \left| \langle \mathcal{T}, e_n^{(3)} \rangle \right|^2 & \lesssim \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 + a^2 \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 + a^4 \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 + a^{10} \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 + a^6 \left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\|^2.
\end{align*}

and therefore (5.73) becomes

\begin{equation}
\left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2 \lesssim \left| \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \right|^2 + a^{10} \left\| \frac{3}{\mathcal{P}} (\tilde{W}) \right\|^2.
\end{equation}

Plugging (5.74) into (5.70), we can deduce

\begin{align*}
\left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\|^2 & \lesssim a^2 \sum_n \left| \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \right|^2 + a^{6-2h} \sum_n \left| \langle \mathcal{P}(\cdot, 0), e_n^{(3)} \rangle \right|^2 + a^{6} \left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\|^2.
\end{align*}

This implies,

\begin{equation}
\left\| \frac{1}{\mathcal{P}} (\tilde{W}) \right\| = \mathcal{O} (a^{1-h}).
\end{equation}
Combining with (5.75) and (5.74), we deduce that

\[ \left\| \frac{2}{\mathcal{P}} \left( \bar{W} \right) \right\| = O \left( a^3 \right). \]

Similarly, from (5.75), (5.76) and (5.77), we derive that

\[ \left\| \frac{3}{\mathcal{P}} \left( \bar{W} \right) \right\| = O \left( a^3 \right). \]

\[ \square \]

6. Derivation of the linear algebraic systems stated in Section 3

6.1. Proof of Proposition 3.1 We first give the proof of the invertibility condition for the general form of the linear algebraic system as follows.

**Proof of Proposition 3.1** We rewrite the linear algebraic system (3.1) in the following matrix form

\[
\begin{pmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_N
\end{pmatrix} - \eta \begin{pmatrix}
0 & [P_{D_1}] \cdot \Upsilon_{k}^1 & \cdots & [P_{D_1}] \cdot \Upsilon_{k}^N \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} \begin{pmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_N
\end{pmatrix} = \begin{pmatrix}
J_1 \\
J_2 \\
\vdots \\
J_N
\end{pmatrix},
\]

where \( \Upsilon_{k}^j := \Upsilon_k(z_j, z_j) \) and we set \( \| \bar{P}_0 \| := \max_{j=1,\ldots,N} \| [P_{D_j}] \|_{L^\infty(\Omega)}. \)

Taking the inner product on both sides of (6.1) with the following vector

\[
(P_{D_1} \cdot Q_1, [P_{D_2}] \cdot Q_2, \ldots, [P_{D_N}] \cdot Q_N)^T,
\]

by direct computations, we obtain from (6.1) that

\[
\sum_{j=1}^{N} \langle Q_j, [P_{D_j}] \cdot Q_j \rangle - \eta \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} \Upsilon_{j}^0 \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle = \sum_{j=1}^{N} \langle J_j, [P_{D_j}] \cdot Q_j \rangle.
\]

We rewrite (6.2) as follows

\[
\sum_{j=1}^{N} \langle Q_j, [P_{D_j}] \cdot Q_j \rangle - \eta \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} \Upsilon_{j}^0 \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle \\
\quad - \eta \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} (\Upsilon_{j} - \Upsilon_{j}^0) \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle = \sum_{j=1}^{N} \langle J_j, [P_{D_j}] \cdot Q_j \rangle.
\]

Denote

\[
\Gamma^0 := \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} \Upsilon_{j}^0 \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle.
\]

Since \( \Upsilon_{j}^0 \) is harmonic, by the Mean Value Theorem we know that

\[
\Upsilon_{j}^0(z_\ell, z_j) = \frac{1}{|B_\ell|} \frac{1}{|B_j|} \int_{B_\ell} \int_{B_j} \Upsilon^0(x, z) \, dx \, dz,
\]

where \( B_\ell \) and \( B_j \) are two balls of radius \( \frac{d}{2} \), centered at \( z_\ell \) and \( z_j \), respectively. Then

\[
\Gamma^0 = \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} \frac{1}{|B_\ell|} \frac{1}{|B_j|} \int_{B_\ell} \int_{B_j} \Upsilon^0(x, z) \, dx \, dz \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle \\
= \frac{36}{\pi^2 d^6} \sum_{j=1}^{N} \left\langle [P_{D_j}] \cdot \sum_{j \neq \ell} \int_{B_\ell} \int_{B_j} \Upsilon^0(x, z) \, dx \, dz \cdot Q_j, [P_{D_j}] \cdot Q_\ell \right\rangle.
\]
We denote
\( V_m := [P_{D_m}] \cdot Q_m \) for \( m = 1, 2, \ldots, N \),
and for \( m = 1, \ldots, N \),
\[
\Pi(x) := \begin{cases} V_m & \text{in } B_m, \\
0 & \text{otherwise.}
\end{cases}
\Lambda(x) := \begin{cases} Q_m & \text{in } B_m, \\
0 & \text{otherwise.}
\end{cases}
\text{[\text{\[P\]}]} := \begin{cases} [P_{D_m}] & \text{in } B_m, \\
0 & \text{otherwise.}
\end{cases}
\]

Set \( \Omega := \bigcup_{m=1}^{N} B_m \), then \( \Gamma^0 \) becomes
\[
\Gamma^0 = \frac{36}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \langle [\text{\[P\]}] \cdot \mathbf{T}^0(x, z) \cdot \Lambda(x), \Pi(z) \rangle \, dx \, dz - \frac{36}{\pi^2 d^6} \sum_{m=1}^{N} \int_{B_m} \int_{B_m} \langle [P_{D_m}] \cdot \mathbf{T}^0(x, z) \cdot Q_m, V_m \rangle \, dx \, dz.
\]

It is direct to see that
\[
\int_{B_m} \int_{B_m} \mathbf{T}^0(x, z) \, dx \, dz = \int_{B_m} \int_{B_m} \text{Hess} \Phi_0(x, z) \, dx \, dz = - \int_{B_m} \nabla \int_{B_m} \nabla \Phi_0(x, z) \, dx \, dz = \int_{B_m} \text{Hess} \Phi_0(x, z) \, dx \, dz.
\]

From [24, eq. (1.12)], we know that for \( B_m \), there holds:
\[
\mathbf{N}_{B_m}(1)(x) = \frac{d^2}{8} - \frac{|x|^2}{6} \text{ henceforth } \text{Hess } \mathbf{N}_{B_m}(1)(x) = -\frac{1}{3} I, \quad x \in B_m.
\]

Then,
\[
\int_{B_m} \int_{B_m} \mathbf{T}^0(x, z) \, dx \, dz = \int_{B_m} \text{Hess} \Phi_0(1)(x) \, dx = -\frac{\pi d^3}{18} I.
\]

Therefore, we obtain
\[
\Gamma^0 = \frac{36}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \langle [\text{\[P\]}] \cdot \mathbf{T}^0(x, z) \cdot \Lambda(x), \Pi(z) \rangle \, dx \, dz - \frac{36}{\pi^2 d^6} \sum_{m=1}^{N} \left( [P_{D_m}] \cdot \left( -\frac{\pi d^3}{18} I \right) \cdot Q_m, V_m \right),
\]
\[
= \frac{36}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \langle [\text{\[P\]}] \cdot \mathbf{T}^0(x, z) \cdot \Lambda(x), \Pi(z) \rangle \, dx \, dz + \frac{2}{\pi d^3} \sum_{m=1}^{N} |V_m|^2.
\]

Now we proceed to estimate the difference term by denoting
\[
\mathcal{R}_k := \Gamma^k - \Gamma^0 = \sum_{\ell=1}^{N} \left( [P_{D_{\ell}}] \cdot \sum_{j=1}^{N} (\mathbf{T}^0_{j \ell} - \mathbf{T}^0_{\ell \ell}) \cdot Q_j, V_\ell \right).
\]

Combining with [2.7], we derive by direct computations that
\[
\mathbf{T}^0_{j \ell} - \mathbf{T}^0_{\ell j} = \text{Hess} \Phi_k(z_\ell, z_j) - \text{Hess} \Phi_0(z_\ell, z_j) + k^2 \Phi_k(z_\ell, z_j) I \quad (x \neq y)
\]
\[
= \frac{k^2}{2} \Phi_0(z_\ell, z_j) I - \frac{ik^3}{12 \pi} I + \frac{k^2}{2} \Phi_0(z_\ell, z_j) (\ell - z_j) \otimes (z_\ell - z_j) \frac{1}{\|z_\ell - z_j\|^2}
\]
\[
+ \frac{1}{4\pi} \sum_{n \geq 3} \frac{(ik)^n}{(n+1)!} \text{Hess} \left( \|z_\ell - z_j\|^n \right) + k^2 \frac{1}{4\pi} \sum_{n \geq 0} \frac{(ik)^n}{(n+1)!} |z_\ell - z_j|^n.
\]
Then $\mathcal{R}_k$ becomes

$$
\mathcal{R}_k = \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \langle [P_{D_\ell}] \cdot \left( \frac{ik^3}{12\pi} \right) , Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \langle [P_{D_\ell}] \cdot \left( \frac{k^2}{2} \frac{1}{4\pi|z_\ell - z_j|} \right) \cdot Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \langle [P_{D_\ell}] \cdot \left( \frac{k^2}{4\pi} \sum_{n \geq 0} (ik)^{n+1} Hess(\|z_\ell - z_j\|^n) \right) \cdot Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \langle [P_{D_\ell}] \cdot \left( \frac{k^2}{4\pi} \sum_{n \geq 0} (ik)^{n+1} |z_\ell - z_j|^n \right) \cdot Q_j , V_\ell \rangle.
$$

With the notation (6.4), we further derive

$$
\mathcal{R}_k = -\sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{ik^3}{12\pi} \langle [P_{D_\ell}]Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{k^2}{2} \frac{1}{4\pi|z_\ell - z_j|} \langle [P_{D_\ell}]Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{k^2}{4\pi} \sum_{n \geq 3} (ik)^{n+1} Hess(\|z_\ell - z_j\|^n) \langle [P_{D_\ell}]Q_j , V_\ell \rangle 
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{k^2}{4\pi} \sum_{n \geq 0} (ik)^{n+1} |z_\ell - z_j|^n \langle [P_{D_\ell}]Q_j , V_\ell \rangle.
$$

Since by the Mean Value Theorem, there holds that

$$
\frac{1}{4\pi|z_\ell - z_j|} = \frac{1}{|B_\ell|} \int_{B_\ell} \Phi_0(x, y) \, dx \, dy = \frac{36}{\pi^2 d^6} \int_{B_\ell} \Phi_0(x, y) \, dx \, dy,
$$

then

$$
\mathcal{R}_k = -\frac{ik^3}{12\pi} \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \langle [P_{D_\ell}]Q_j , V_\ell \rangle 
+ \frac{18k^2}{\pi^2 d^6} \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \int_{B_\ell} \Phi_0(x, y) \langle [P_{D_\ell}]Q_j , V_\ell \rangle \, dx \, dy
+ \frac{18k^2}{\pi^2 d^6} \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \int_{B_\ell} \Phi_0(x, y) \langle \frac{(z_\ell - z_j)}{|z_\ell - z_j|}, Q_j \rangle \langle [P_{D_\ell}] \cdot \frac{(z_\ell - z_j)}{|z_\ell - z_j|}, V_\ell \rangle \, dx \, dy
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{1}{4\pi} \sum_{n \geq 3} (ik)^{n+1} Hess(\|z_\ell - z_j\|^n) \langle [P_{D_\ell}]Q_j , V_\ell \rangle
+ \sum_{\ell=1}^{R} \sum_{j \neq \ell}^{R} \frac{k^2}{4\pi} \sum_{n \geq 0} (ik)^{n+1} |z_\ell - z_j|^n \langle [P_{D_\ell}]Q_j , V_\ell \rangle.
$$

(6.9)
Considering the second term in (6.9), we know that

\[
L_6 := \sum_{\ell=1}^{N} \sum_{j \neq \ell} \int_{B_j} \int_{B_j} \Phi_0(x,y) \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle \, dx \, dy
\]

\[
= \sum_{\ell,j=1}^{N} \int_{B_j} \int_{B_j} \Phi_0(x,y) \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle \, dx \, dy - \sum_{m=1}^{N} \int_{B_m} \int_{B_m} \Phi_0(x,y) \langle V_m, V_m \rangle \, dx \, dy
\]

(6.10)

\[
= \int_{\Omega} \int_{\Omega} \Phi_0(x,y) |P| \Lambda(x) \, dx \, dy - \sum_{m=1}^{N} \int_{B_m} \int_{B_m} \Phi_0(x,y) \langle V_m, V_m \rangle \, dx \, dy.
\]

By utilizing (6.6) again, there holds

\[
\int_{B_m} \int_{B_m} \Phi_0(x,y) \, dx \, dy = \int_{B_m} N(1)(y) \, dy = \int_{B_m} \left( \frac{d^2}{8} - \frac{|y|^2}{6} \right) \, dy = \frac{\pi d^6}{60}.
\]

Thus, (6.10) can be written as

\[
L_6 = \int_{\Omega} \int_{\Omega} \Phi_0(x,y) |P| \Lambda(x) \Pi(y) \, dx \, dy - \sum_{m=1}^{N} \frac{\pi d^6}{60} |V_m|^2,
\]

which implies that in (6.9), we have

\[
\mathcal{R}^k = \frac{ik^3}{12\pi} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle + \frac{18k^2}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \Phi_0(x,y) |P| \Lambda(x) \Pi(y) \, dx \, dy - \frac{3k^2}{10\pi d} \sum_{m=1}^{N} |V_m|^2
\]

\[
+ \frac{18k^2}{\pi^2 d^6} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \int_{B_j} \int_{B_j} \Phi_0(x,y) \left\langle \frac{(z_\ell - z_j)}{|z_\ell - z_j|}, Q_j \right\rangle \left\langle [P_{D_{\ell}}] \cdot \frac{(z_\ell - z_j)}{|z_\ell - z_j|}, V_\ell \right\rangle \, dx \, dy
\]

\[
+ \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{1}{4\pi} \sum_{n \geq 3}^{N} \frac{(ik)^{n+1}}{(n+1)!} \text{Hess}(|z_\ell - z_j|^n) \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle
\]

(6.11)

Now, we give the estimation of \( \mathcal{R}^k \) term by term.

- Estimation of

\[
L_{7,1} := -\frac{ik^3}{12\pi} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle,
\]

\[
|L_{7,1}| \leq \frac{ik^3}{12\pi} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle \lesssim k^3 \sum_{\ell=1}^{N} \sum_{j \neq \ell} \langle [P_{D_{\ell}}]Q_j, V_\ell \rangle \lesssim k^3 \sum_{\ell=1}^{N} \sum_{j \neq \ell} \langle [P_{D_{\ell}}]Q_j, [P_{D_{\ell}}]Q_\ell \rangle
\]

(6.12)

\[
\lesssim k^3 \|[P_0]\|^2 \left( \sum_{j=1}^{N} |Q_j| \right)^2 \lesssim k^3 \|[P_0]\|^2 \sum_{j=1}^{N} |Q_j|^2.
\]

- Estimation of

\[
L_{7,2} := \frac{18k^2}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \Phi_0(x,y) |P| \Lambda(x) \, dx \, dy.
\]

Following the definition of the Newtonian potential operator, we have

\[
|L_{7,2}| \leq \frac{18k^2}{\pi^2 d^6} \int_{\Omega} \int_{\Omega} \Phi_0(x,y) |P| \Lambda(x) \, dx \, dy \lesssim k^2 d^{-6} \int_{\Omega} N(\Pi)(x) \cdot [P] \Lambda(x) \, dx.
\]
\[ \lesssim k^2 d^{-6} \|N(\Pi)\|_{L^2(\Omega)} \|P\|_{L^2(\Omega)} \lesssim k^2 d^{-6} \|N\|_{L(\mathbb{R};L^2(\Omega))} \|\Pi\|_{L^2(\Omega)} \|P_0\| \|\Lambda\|_{L^2(\Omega)} \]

\[ \lesssim k^2 d^{-3} \|N\|_{L(\mathbb{R};L^2(\Omega))} \left( \sum_{m=1}^{N} |V_m|^2 \right)^{\frac{1}{2}} \|P_0\| \left( \sum_{m=1}^{N} |Q_m|^2 \right)^{\frac{1}{2}} \]

(6.13)

\[ \lesssim k^2 d^{-3} \|N\|_{L(\mathbb{R};L^2(\Omega))} \|P_0\|^2 \sum_{m=1}^{N} |Q_m|^2. \]

- Estimation of

\[ L_{7,3} := -\frac{3k^2}{10\pi d} \sum_{m=1}^{N} |V_m|^2, \]

(6.14)

\[ |L_{7,3}| \lesssim k^2 d^{-1} \sum_{m=1}^{N} |V_m|^2 = k^2 d^{-1} \sum_{m=1}^{N} |P_{D_m}| Q_m|^2 = k^2 d^{-1} \|P_0\|^2 \sum_{m=1}^{N} |Q_m|^2. \]

- Estimation of

\[ L_{7,4} := \frac{18k^2}{\pi^2 d^6} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \int_{B_\ell} \int_{B_j} \Phi_0(x, y) \left( \frac{z_{\ell} - z_j}{|z_{\ell} - z_j|} \right) Q_j \left( P_{D_{\ell}} \right) \frac{(z_{\ell} - z_j)}{|z_{\ell} - z_j|} \, dx \, dy, \]

\[ |L_{7,4}| \lesssim k^2 \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{1}{|z_{\ell} - z_j|} \|Q_j\| \lesssim k^2 \|P_0\| \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{1}{|z_{\ell} - z_j|} |Q_j|. \]

Then, by Cauchy-Schwartz inequality, for both of the indices \( \ell \) and \( j \), we deduce that

\[ |L_{7,4}| \lesssim k^2 \|P_0\| \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{1}{|z_{\ell} - z_j|} |Q_j|. \]

(6.15)

\[ d^{-2} \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{(ik)^{n+1}}{4\pi (n+1)!} H_{ess}(\|z_{\ell} - z_j\|^n) (|P_{D_{\ell}}| Q_j, V_{\ell}). \]

Similar to (5.10), by direct verification, we can prove that

\[ \left| \sum_{n \geq 3} \frac{(ik)^{n+1}}{(n+1)!} H_{ess}(\|z_{\ell} - z_j\|^n) \right| \lesssim \sum_{n \geq 3} \frac{n k^{n+1} \|z_{\ell} - z_j\|^{n-2}}{(n+1)!} < +\infty, \]

which, after taking the modulus, implies

\[ |L_{7,5}| \lesssim k^4 \sum_{\ell=1}^{N} \sum_{j \neq \ell} |P_{D_{\ell}}| Q_j |V_{\ell}| \lesssim k^4 \|P_0\|^2 \sum_{j=1}^{N} |Q_j|^2. \]

Similarly, we also obtain that

\[ L_{7,6} := \sum_{\ell=1}^{N} \sum_{j \neq \ell} \frac{k^2}{4\pi} \sum_{n \geq 0} \frac{(ik)^{n+1}}{(n+1)!} \|z_{\ell} - z_j\|^n (|P_{D_{\ell}}| Q_j, V_{\ell}), \]
(6.17) \[ |L_{7,6}| = \left| \sum_{j=1}^{N} \left( \sum_{k 
eq l} \frac{k^2}{4|\eta|}\sum_{n \geq 0} \frac{(kn+1)}{(n+1)}|z_j - \tilde{z}_j|^{n} \langle (P_{D_k})Q_j, V_j \rangle \right) \right| = O \left( k^3 \left\| P_0 \right\|^2 \sum_{j=1}^{N} \langle Q_j \rangle^2 \right). \]

Combining with (6.12), (6.13), (6.14), (6.15), (6.16) and (6.17), we derive the following estimate:

(6.18) \[ |R_k| \lesssim \sum_{j=1}^{6} L_{7,j} = O \left( k^2 \left\| P_0 \right\|^2 \sum_{j=1}^{N} \langle Q_j \rangle^2 \left( k^3 N \left\| L_{L^2(\Omega);L^2(\Omega)} \right\| \right) \right). \]

Rewriting the linear algebraic system (6.3) by virtue of the expressions of \( \Gamma^0 \) and \( R_k \), we get

(6.19) \[ \sum_{j=1}^{N} \langle Q_j, V_j \rangle - \eta \frac{36}{\pi^{3/2}} \int_{\Omega} \int_{\Omega} \langle [P] \cdot \nabla \eta(x, z) \cdot \Lambda(x), \Pi(z) \rangle \ dx \ dz - \frac{2\eta}{\pi d^3} \sum_{m=1}^{N} |V_m|^2 - \eta R_k = \sum_{j=1}^{N} \langle J_j, V_j \rangle. \]

For the term \( \frac{2\eta}{\pi d^3} \sum_{m=1}^{N} |V_m|^2 \), it is easy to see that

(6.20) \[ \left\| \frac{2\eta}{\pi d^3} \sum_{m=1}^{N} |V_m|^2 \right\| \lesssim a^{-2} d^{-3} \sum_{m=1}^{N} |Q_m|^2 = a^{-2} d^{-3} \sum_{m=1}^{N} \left\| \left[P_{D_m}\right] \cdot Q_m \right\|^2 = a^{-2} d^{-3} \left\| P_0 \right\|^2 \sum_{m=1}^{N} |Q_m|^2. \]

By substituting (6.18) and (6.20) into (6.19), we can further deduce that

(6.21) \[ \sum_{j=1}^{N} \langle J_j, V_j \rangle + O \left( a^{-2} \left\| P_0 \right\|^2 \sum_{j=1}^{N} |Q_j|^2 \left( k^3 N + d^{-3} k^2 \left\| L_{L^2(\Omega);L^2(\Omega)} \right\| \right) \right), \]

since \( d^{-3} \) and \( N \) are of the same order with respect to \( a \).

Recall the notations (6.5), where we have

(6.22) \[ \sum_{j=1}^{N} \langle V_j, Q_j \rangle = \frac{6}{\pi d^3} \langle [\Pi], \Lambda \rangle_{L^2(\Omega)}. \]

Then we can derive from (6.21) the linear algebraic system in terms of \( \Pi \) and \( \Lambda \) as follows

(6.23) \[ \langle [\Pi], \Lambda \rangle_{L^2(\Omega)} - \frac{6\eta}{\pi d^3} \int_{\Omega} \int_{\Omega} \langle [P] \cdot \nabla \eta(x, z) \cdot \Lambda(x), \Pi(z) \rangle \ dx \ dz = -\frac{6\eta}{\pi d^3} \int_{\Omega} \left[ P \right] \cdot \nabla M(\Lambda)(z) \cdot \Pi(z) \ dz, \]

which fulfills the estimate

(6.24) \[ -\frac{6\eta}{\pi d^3} \int_{\Omega} \left[ P \right] \cdot \nabla M(\Lambda)(z) \cdot \Pi(z) \ dz \lesssim \frac{1}{\eta} \left\| \nabla \right\|_{L^2(\Omega);L^2(\Omega)} \left\| \Pi \right\|_{L^2(\Omega)} \]

Knowing that \( \left\| \nabla \right\|_{L^2(\Omega);L^2(\Omega)} = 1 \), see Theorem 2.1 of [21], we deduce that

(6.25) \[ -\frac{6\eta}{\pi d^3} \int_{\Omega} \left[ P \right] \cdot \nabla M(\Lambda)(z) \cdot \Pi(z) \ dz \lesssim \frac{1}{\eta} \left\| \left[ P_0 \right] \right\| d^{-3} \left\| \Lambda \right\|_{L^2(\Omega)} \left\| \Pi \right\|_{L^2(\Omega)}. \]
Thus, a sufficient condition for the linear algebraic system to be invertible is that
\[ |\eta| \| [P_0] \| d^{-3} < 1, \]
which completes the proof.

Now, we construct the linear algebraic system related to \( \frac{1}{L}(E_j^T) \), \( j = 1, 2, \cdots, N \), in the following subsection.

6.2. **Construction of the linear algebraic system in Proposition 3.2.**

**Proof of Proposition 3.2.** Recall the Lippmann-Schwinger equation \( (2.10) \)

\[ E^T(x) + \eta \nabla M_k (E^T)(x) - k^2 \eta N_k (E^T)(x) = E^{Inc}(x), \quad x \in D = \bigcup_{j=1}^{N} D_j, \]

which can be rewritten, for \( x \in D_{j_0} \), as

\[ T_k (E_{j_0}^T)(x) - \eta \sum_{j=1}^{N} \int_{D_j} Y_k(x, y) \cdot E_j^T(y) dy = E_{j_0}^{Inc}(x), \quad \text{for} \quad x \in D_{j_0}, \]

where \( Y_k(\cdot, \cdot) \) is given by \( (2.3) \). Considering the equation \( (6.24) \), and successively, inverting the operator \( T_k \), multiplying on the both sides by \( \mathcal{P}(x, z_{j_0}) \) and integrating over \( D_{j_0} \), then with the help of the adjoint operator of \( T_k \) and the definition of \( W \), seeing \( (2.10) \), we obtain that

\[ \int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot E_{j_0}^T(x) \, dx = \eta \int_{D_{j_0}} W(x) \cdot \sum_{j=1}^{N} \int_{D_j} Y_k(x, y) \cdot E_j^T(y) \, dy \, dx.
\]

For \( j = 1, \cdots, N \), since from Proposition \( (2.2) \) we know that \( \frac{2}{L}(E_j^T) = 0 \), we write \( E_j^T \) as \( E_j^T = \frac{3}{L}(E_j^T) + \frac{3}{L}(\tilde{E}_j^T) \) and we plug it into the previous equation to get

\[ \int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot \frac{3}{L}(E_{j_0}^T)(x) \, dx = \eta \int_{D_{j_0}} W(x) \cdot \sum_{j=1}^{N} \int_{D_j} Y_k(x, y) \cdot \frac{3}{L}(E_j^T)(y) \, dy \, dx
\]

\[ = \int_{D_{j_0}} W(x) \cdot E_{j_0}^{Inc}(x) \, dx - \int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot \frac{3}{L}(E_{j_0}^T)(x) \, dx
\]

\[ + \eta \int_{D_{j_0}} W(x) \cdot \sum_{j=1}^{N} \int_{D_j} Y_k(x, y) \cdot \frac{3}{L}(E_j^T)(y) \, dy \, dx.
\]

Next, we estimate the last two terms on the right hand side of \( (6.25) \). To do this, we have

\[ J_1 := \int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot \frac{3}{L}(E_{j_0}^T)(x) \, dx
\]

\[ J_1 = a^4 \int_{B} \mathcal{P}(x, 0) \cdot \frac{3}{L}(\tilde{E}_{j_0}^T)(x) \, dx
\]

\[ |J_1| \leq a^4 \| \mathcal{P}(\cdot, 0) \|_{L^2(B)} \left\| \frac{3}{L}(\tilde{E}_{j_0}^T) \right\|_{L^2(B)} \lesssim a^4 \left\| \frac{3}{L}(\tilde{E}_{j_0}^T) \right\|_{L^2(B)}.
\]

Similarly, we set

\[ J_2 := \eta \sum_{j=1}^{N} \int_{D_{j_0}} W(x) \cdot \int_{D_j} Y_k(x, y) \cdot \frac{3}{L}(E_j^T)(y) \, dy \, dx.
\]
By expanding the fundamental solution \( \Upsilon_k(\cdot, \cdot) \) near the center \( z_j \), we obtain

\[
\Upsilon_k(x, y) = \Upsilon_k(z_{j_0}, z_j) + \int_0^1 \nabla_x (\Upsilon_k) (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) \, dt \\
+ \int_0^1 \nabla_y (\Upsilon_k) (x, z_j + t(y - z_j)) \cdot \mathcal{P}(y, z_j) \, dt.
\]

(6.26)

Then \( J_2 \) becomes

\[
J_2 := \eta \sum_{j=1 \atop j \neq j_0}^N \int_{D_{j_0}} W(x) \, dx \cdot \Upsilon_k(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathcal{P}(E_j^T)} (y) \, dy \\
+ \eta \sum_{j=1 \atop j \neq j_0}^N \int_{D_{j_0}} W(x) \cdot \int_0^1 \nabla_x (\Upsilon_k) (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) \, dt \, dx \cdot \int_{D_j} \frac{3}{\mathcal{P}(E_j^T)} (y) \, dy \\
+ \eta \sum_{j=1 \atop j \neq j_0}^N \int_{D_{j_0}} W(x) \cdot \int_{D_j} \int_0^1 \nabla_y (\Upsilon_k) (x, z_j + t(y - z_j)) \cdot \mathcal{P}(y, z_j) \, dt \cdot \int_{D_j} \frac{3}{\mathcal{P}(E_j^T)} (y) \, dy \, dx.
\]

Keeping only the first dominant term, by using the estimate for the scattering coefficient in Proposition we deduce that

\[
J_2 \sim a^{4-h} \sum_{j=1 \atop j \neq j_0}^N \Upsilon_k(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{\mathcal{P}(E_j^T)} (y) \, dy \\
|J_2| \lesssim a^{4-h} \sum_{j=1 \atop j \neq j_0}^N \frac{1}{|z_{j_0} - z_j|^3} \left\| \frac{3}{\mathcal{P}(E_j^T)} \right\|_{L^2(D_j)} \lesssim a^{7-h} \, d^{-3} \, |\log(d)| \, \max_j \left\| \frac{3}{\mathcal{P}(E_j^T)} \right\|_{L^2(B)}.
\]

Thus, combining with the estimations of \( J_1 \) and \( J_2 \), we rewrite (6.25) as

\[
\int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot \frac{1}{\mathcal{P}(E_{j_0}^T)} (x) \, dx \\
- \eta \int_{D_{j_0}} W(x) \cdot \sum_{j=1 \atop j \neq j_0}^N \int_{D_j} \Upsilon_k(x, y) \cdot \frac{1}{\mathcal{P}(E_j^T)} (y) \, dy \, dx \\
= \int_{D_{j_0}} W(x) \cdot E_{j_0}^{inc} (x) \, dx + \mathcal{O} \left( a^4 \left\| \frac{3}{\mathcal{P}(E_{j_0}^T)} \right\|_{L^2(B)} \right) \\
+ \mathcal{O} \left( a^{7-h} \, d^{-3} \, |\log(d)| \, \max_j \left\| \frac{3}{\mathcal{P}(E_j^T)} \right\|_{L^2(B)} \right).
\]

Next, we write \( W \) as \( W = \frac{1}{\mathcal{P}(W)} + \frac{2}{\mathcal{P}(W)} + \frac{3}{\mathcal{P}(W)} \) and we plug it into the above equation to get

\[
\int_{D_{j_0}} \mathcal{P}(x, z_{j_0}) \cdot \frac{1}{\mathcal{P}(E_{j_0}^T)} (x) \, dx \\
- \eta \int_{D_{j_0}} \frac{1}{\mathcal{P}(W)} (x) \cdot \sum_{j=1 \atop j \neq j_0}^N \int_{D_j} \Upsilon_k(x, y) \cdot \frac{1}{\mathcal{P}(E_j^T)} (y) \, dy \, dx \\
- \eta \int_{D_{j_0}} \frac{2}{\mathcal{P}(W)} (x) \cdot \sum_{j=1 \atop j \neq j_0}^N \int_{D_j} \Upsilon_k(x, y) \cdot \frac{1}{\mathcal{P}(E_j^T)} (y) \, dy \, dx \\
- \eta \int_{D_{j_0}} \frac{3}{\mathcal{P}(W)} (x) \cdot \sum_{j=1 \atop j \neq j_0}^N \int_{D_j} \Upsilon_k(x, y) \cdot \frac{1}{\mathcal{P}(E_j^T)} (y) \, dy \, dx.
\]
\[
\begin{align*}
&= \int_{D_{j_0}} \frac{1}{P}(W)(x) \cdot E^{Inc}_{j_0}(x) \, dx \\
&\quad + \int_{D_{j_0}} \frac{2}{P}(W)(x) \cdot E^{Inc}_{j_0}(x) \, dx + \int_{D_{j_0}} \frac{3}{P}(W)(x) \cdot E^{Inc}_{j_0}(x) \, dx \\
&\quad + \mathcal{O} \left( a^4 \left\| \frac{1}{P}(E^T_{j_0}) \right\|_{L^2(B)} \right) + \mathcal{O} \left( a^{7-h} \, d^{-3} \, \max_j \left\| \frac{3}{P}(E^T_j) \right\|_{L^2(B)} \right) + \mathcal{O}(a^6),
\end{align*}
\]

(6.27)

Since \( E^{Inc}_{j_0} \in \mathbb{H}(\text{div} = 0) \perp \mathbb{H}_0(\text{Curl} = 0) \) and, by construction, \( \frac{2}{P}(W) \in \mathbb{H}_0(\text{Curl} = 0) \), we deduce that

\[
J_3 := \int_{D_{j_0}} \frac{2}{P}(W)(x) \cdot E^{Inc}_{j_0}(x) \, dx = 0.
\]

Denote

\[
J_4 := \int_{D_{j_0}} \frac{3}{P}(W)(x) \cdot E^{Inc}_{j_0}(x) \, dx
\]

then

\[
|J_4| \leq \left\| \frac{3}{P}(W) \right\|_{L^2(D_{j_0})} \left\| E^{Inc}_{j_0} \right\|_{L^2(D_{j_0})} \lesssim a^3 \left\| \frac{3}{P}(W) \right\|_{L^2(B)} = \mathcal{O}(a^6),
\]

by utilizing (5.76). On the left hand side of (6.27), we need to estimate

\[
J_{5,\ell} := \eta \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \nabla \Phi_k(x, y) \cdot \frac{1}{P}(E^T_j)(y) \, dy \, dx, \quad \ell = 2, 3
\]

\[
= -\eta \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \nabla \Phi_k(x, y) \cdot \frac{1}{P}(E^T_j)(y) \, dy \, dx
\]

\[
+ \eta k^2 \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \Phi_k(x, y) \frac{1}{P}(E^T_j)(y) \, dy \, dx.
\]

By using integration by parts and the fact that \( \frac{1}{P}(E^T_j) \in \mathbb{H}_0(\text{div} = 0) \), we can prove that

\[
\int_{D_j} \nabla \Phi_k(\cdot, y) \cdot \frac{1}{P}(E^T_j)(y) \, dy = 0, \quad j = 1, \ldots, N.
\]

Then,

\[
J_{5,\ell} = \eta k^2 \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \Phi_k(x, y) \frac{1}{P}(E^T_j)(y) \, dy \, dx.
\]

We rewrite \( J_{5,\ell} \), by Taylor expansion and the fact that \( \int_{D_j} \frac{1}{P}(E^T_j)(y) \, dy = 0 \), as

\[
J_{5,\ell} = \eta k^2 \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \nabla \Phi_k(x, z_j) \cdot (y - z_j) \frac{1}{P}(E^T_j)(y) \, dy \, dx
\]

\[
+ \eta k^2 \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \int_0^1 (1-t)(y-z_j)^\perp \cdot \nabla \Phi_k(y) \, dt \frac{1}{P}(E^T_j)(y) \, dy \, dx
\]

\[
+ \eta k^2 \int_{D_{j_0}} \frac{\ell}{P}(W)(x) \cdot \sum_{j=1, j \neq j_0}^N \int_{D_j} \int_0^1 (1-t)(y-z_j)^\perp \cdot Hess(\Phi_k)(y) \, dt \frac{1}{P}(E^T_j)(y) \, dy \, dx.
\]
Technically, the analyses for the case \( \ell = 2 \) and the case \( \ell = 3 \) are a bit different. So, we split the analyses into two parts.

* Case \( \ell = 2 \),

Recall that we have \( \int_{D_{j0}} \frac{2}{p} (W) (x) dx = 0 \) and remark that the second term of \( J_{5,2} \) is of the form

\[
\int_{D_{j0}} \frac{2}{p} (W) (x) \cdot V dx,
\]

where \( V \) is a constant vector. Consequently, the second term is vanishing. Then,

\[
J_{5,2} = \eta k^2 \int_{D_{j0}} \frac{2}{p} (W) (x) \cdot \sum_{j=1}^{N} \int_{D_j} \int_{0}^{1} (1 - t) (y - z_j) \frac{1}{p} (E_j^T) (y) dy dx
\]

\[
+ \eta k^2 \int_{D_{j0}} \frac{2}{p} (W) (x) \cdot \sum_{j=1}^{N} (1 - t) (y - z_j^\perp) \cdot \left( \int_{0}^{1} \frac{1}{p} (E_j^T) (y) dy dx \right)
\]

\[
\left[ \int_{0}^{1} \nabla_x \left( \text{Hess}(\Phi_k) \right) (z_j + s(x - z_{j0}), z_j + t(y - z_j)) \cdot \mathcal{P}(x, z_{j0}) ds \right] (y - z_j) \frac{1}{p} (E_j^T) (y) dy dx.
\]

Clearly, for the above equation, the first term is more dominant compared with the second one. Using Taylor expansion near \( z_{j0} \), we obtain

\[
J_{5,2} = -\eta k^2 \int_{D_{j0}} \frac{2}{p} (W) (x) \cdot \sum_{j=1}^{N} \int_{D_j} \left[ \int_{0}^{1} \text{Hess} (\Phi_k) (z_{j0} + s(x - z_{j0}), z_j) \cdot (x - z_{j0}) dt \right] (y - z_j)
\]

\[
\frac{1}{p} (E_j^T) (y) dy dx
\]

\[
+ \eta k^2 \int_{D_{j0}} \frac{2}{p} (W) (x) \cdot \sum_{j=1}^{N} \int_{D_j} \int_{0}^{1} (1 - t) (y - z_j^\perp) \cdot \left[ \int_{0}^{1} \text{Hess} (\Phi_k) (z_j + s(x - z_{j0}), z_j + t(y - z_j)) \cdot \mathcal{P}(x, z_{j0}) ds \right] (y - z_j) \frac{1}{p} (E_j^T) (y) dy dx
\]

\[
|J_{5,2}| \lesssim a^3 \left\| \frac{2}{p} (W) \right\|_{L^2(D_{j0})} \sum_{j \neq j_0}^{N} \left| z_j - z_{j0} \right|^{-3} \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(D_j)} + a^4 \left\| \frac{2}{p} (W) \right\|_{L^2(D_{j0})} \sum_{j \neq j_0}^{N} \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(D_j)} \left| z_j - z_{j0} \right|^4
\]

\[
\lesssim a^6 \left\| \frac{2}{p} (\tilde{W}) \right\|_{L^2(B)} \max_j \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(B)} \left( d^{-3} \log(d) \right) + a^7 \left\| \frac{2}{p} (\tilde{W}) \right\|_{L^2(B)} \max_j \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(B)} d^{-4}
\]

\[
\lesssim a^6 \left\| \frac{2}{p} (\tilde{W}) \right\|_{L^2(B)} \max_j \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(B)} d^{-4}.
\]

Knowing the estimation of \( \left\| \frac{2}{p} (\tilde{W}) \right\|_{L^2} \), given by (5.77), we deduce that

\[
(6.31) \quad J_{5,2} = \Theta \left( a^9 d^{-4} \max_j \left\| \frac{1}{p} (E_j^T) \right\|_{L^2(B)} \right).
\]
\[ J_{5,3} = \eta k^2 \int_{D_{j_0}}^3 \mathcal{P}(W)(x) \cdot \sum_{j \neq j_0}^N \int_{D_j} \nabla \Phi_k(x, z_j) \cdot (y - z_j) \frac{1}{\mathcal{P}}(E_j^T)(y) dy \, dx \]
\[ + \eta k^2 \int_{D_{j_0}}^3 \mathcal{P}(W)(x) \cdot \sum_{j \neq j_0}^N \int_{D_j}^1 (1 - t)(y - z_j)^{1/2} \cdot \text{Hess} \Phi_k(z_{j_0}, z_j + t(y - z_j)) \cdot (y - z_j) \, dt \]
\[ \int_{D_{j_0}}^1 \frac{1}{\mathcal{P}}(E_j^T)(y) dy \, dx \]
\[ + \eta k^2 \int_{D_{j_0}}^3 \mathcal{P}(W)(x) \cdot \sum_{j \neq j_0}^N \int_{D_j}^1 (1 - t)(y - z_j)^{1/2} \]
\[ \left[ \int_0^1 \nabla_x \left( \text{Hess} \Phi_k \right)(z_{j_0} + s(x - z_{j_0}), z_j + t(y - z_j)) \cdot \mathcal{P}(x, z_{j_0}) ds \right] \cdot (y - z_j) \, dt \frac{1}{\mathcal{P}}(E_j^T)(y) dy \, dx. \]

Since \( \int_{D_{j_0}}^3 \mathcal{P}(W)(x) \, dx \neq 0 \), after expanding the first term near \( z_{j_0} \) and keeping the dominant term of \( J_{5,3} \), we deduce that
\[ J_{5,3} \simeq \eta k^2 \int_{D_{j_0}}^3 \mathcal{P}(W)(x) \, dx \cdot \sum_{j = 1}^N \nabla_y (\Phi_k L)(z_{j_0}, z_j) \cdot \int_{D_j} \mathcal{P}(y, z_j) \cdot \frac{1}{\mathcal{P}}(E_j^T)(y) dy \]
\[ |J_{5,3}| \lesssim |\eta| \|1\|_{L^2(D_{j_0})} \left\| \frac{3}{\mathcal{P}}(W) \right\|_{L^2(D_{j_0})} \sum_{j \neq j_0}^N |z_{j_0} - z_j|^{-2} \|\mathcal{P}(0, z_j)\|_{L^2(D_j)} \left\| \frac{\mathcal{P}}{\mathcal{P}}(E_j^T) \right\|_{L^2(D_j)} \]
\[ \lesssim a^2 \cdot a^3 \cdot d^{-3} \left\| \frac{\mathcal{P}}{\mathcal{P}}(\tilde{W}) \right\|_{L^2(B)} \max_j \left\| \frac{\mathcal{P}}{\mathcal{P}}(\tilde{E}_j^T) \right\|_{L^2(B)}. \]

With the estimation of \( \left\| \frac{\mathcal{P}}{\mathcal{P}}(\tilde{W}) \right\|_{L^2} \), given by (67), we can deduce that
\[ (6.32) \quad J_{5,3} = \mathcal{O} \left( a^8 d^{-3} \max_j \left\| \frac{1}{\mathcal{P}}(E_j^T) \right\|_{L^2(B)} \right). \]

Finally, combining with (6.31) and (6.32), we obtain that
\[ (6.33) \quad J_{5,\ell} = \mathcal{O} \left( a^8 d^{-3} \max_j \left\| \frac{1}{\mathcal{P}}(\tilde{E}_j^T) \right\|_{L^2(B)} \right). \]

The equation (6.27), using (6.28), (6.29) and (6.33), will be reduced to
\[ \int_{D_{j_0}}^3 \mathcal{P}(x, z_{j_0}) \cdot \frac{1}{\mathcal{P}}(E_{j_0}^T)(x) \, dx \quad - \quad \eta \int_{D_{j_0}}^3 \frac{1}{\mathcal{P}}(W)(x) \cdot \sum_{j \neq j_0}^N \int_{D_j} Y_k(x, y) \cdot \frac{1}{\mathcal{P}}(E_j^T)(y) \, dy \, dx \]
\[ = \int_{D_{j_0}}^3 \frac{1}{\mathcal{P}}(W)(x) \cdot E_{j_0}^\text{inc}(x) \, dx + \mathcal{O}(a^6) + \mathcal{O} \left( a^4 \left\| \frac{3}{\mathcal{P}}(\tilde{E}_{j_0}^T) \right\|_{L^2(B)} \right) \]
\[ + \mathcal{O} \left( a^{7-h} d^{-3} \log(d) \max_j \left\| \frac{3}{\mathcal{P}}(\tilde{E}_j^T) \right\|_{L^2(B)} \right). \]
Since $\mathbb{P} (E_j^T), \mathbb{P} (W) \in H_0(\text{div} = 0)$, we denote

$$ \mathbb{P} (E_j^T) = \text{Curl} (F_j) \quad \text{and} \quad \mathbb{P} (W) = \text{Curl} (A), $$

with

$$ \nu \times A = 0, \text{ div} (A) = 0, \nu \times F_j = 0, \quad \text{and} \quad \text{div} (F_j) = 0, \quad j = 1, \cdots, N. $$

Plugging all these formulas into (6.34), using (6.30) and integration by parts, we can obtain that

$$ \begin{align*}
\mathcal{M} \cdot & \int_{D_{j_0}} F_{j_0} (x) \, dx - \eta k^2 \sum_{j = 1}^{N} \int_{D_{j_0}} A (x) \cdot \text{Curl}_x \Phi_k (x, y) \, C_{j} (F_j) (y) \, dy \, dx \\
= & \int_{D_{j_0}} A (x) \cdot \text{Curl} \left( E_{j_0}^{inc} \right) (x) \, dx + \mathcal{O} (a^6) + \mathcal{O} \left( a^4 \left\| \mathbb{P} \left( E_{j_0}^{T} \right) \right\|_{L^2 (B)} \right) \\
& + \mathcal{O} \left( a^7 \, d^{-3} \left| \log (d) \right| \max_j \left\| \mathcal{P} \left( E_{j_0}^{T} \right) \right\|_{L^2 (B)} \right) + \mathcal{O} \left( a^8 \, d^{-3} \max_j \left\| \mathbb{P} \left( E_{j_0}^{T} \right) \right\|_{L^2 (B)} \right),
\end{align*} $$

where $\mathcal{M}$ is the constant matrix given by

$$ \mathcal{M} := \text{Curl}_x (\mathcal{P} (x, z_{j_0})) = \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. $$

For the second term on the left hand side of (6.37), we have:

$$ \begin{align*}
\text{Curl}_x & \int_{D_{j}} \Phi_k (x, y) \, C_{j} (F_j) (y) \, dy \\ &= \text{Curl}_x \int_{D_{j}} \text{Curl}_y \left( \Phi_k (x, y) \, F_j (y) \right) \, dy \\
&+ \text{Curl}_x \int_{\partial D_{j}} \Phi_k (x, y) \nu (y) \times F_j (y) \, dy \\
&= \int_{D_{j}} \text{Curl}_x \circ \text{Curl}_y \left( \Phi_k (x, y) \, F_j (y) \right) \, dy \\
&= \int_{D_{j}} \left( -\Delta \Phi_k (x, y) + \nabla \text{div} \Phi_k (x, y) \right) \, F_j (y) \, dy \\
&= \int_{D_{j}} \left( k^2 \Phi_k (x, y) \, F_j (y) + H_{\text{ess}} \Phi_k (x, y) \cdot F_j (y) \right) \, dy \\
&= \int_{D_{j}} \Phi_k (x, y) \cdot F_j (y) \, dy.
\end{align*} $$

$^9$ We use, for 2 arbitrary vectors, the following relation $\text{Curl} (a) \cdot b = \text{Curl} (b) \cdot a + \text{div} (a \times b)$
$^10$ For any vector field $a (\cdot)$, we have the following formula

$$ \text{Curl}_x (\Phi_k (x, y) \, a (y)) + \text{Curl}_y (\Phi_k (x, y) \, a (y)) = \Phi_k (x, y) \, \text{Curl}_x (a (y)). $$. 
For the right hand side of (6.37), we have

\[ (6.41) \int_{D_{\rho_0}} A(x) \cdot \text{Curl} \left( E_{\rho_0}^{\text{inc}}(x) \right) dx = i k \int_{D_{\rho_0}} A(x) \cdot H_{\rho_0}^{\text{inc}}(x) dx. \]

Then, from (6.40) and (6.41), the equation (6.37) becomes

\[
\mathbf{M} \cdot \int_{D_{\rho_0}} F_{\rho_0}(x) \, dx - \eta k^2 \sum_{j=1}^{N} \int_{D_{\rho_0}} A(x) \cdot \int_{D_{j}} \Upsilon_{k}(x, y) \cdot F_{j}(y) \, dy \, dx
\]

\[ = i k \int_{D_{\rho_0}} A(x) \cdot H_{\rho_0}^{\text{inc}}(x) dx + \mathcal{O} \left( a^6 \right) + \mathcal{O} \left( a^4 \left\| \frac{3}{2} \left( \nabla \mathcal{E}_{j}^{T} \right) \right\|_{L^2(B)} \right)
\]

\[ + \mathcal{O} \left( a^{7-h} d^{-3} \max_{j} \left\| \frac{1}{2} \left( \nabla \mathcal{E}_{j}^{T} \right) \right\|_{L^2(B)} \right) + \mathcal{O} \left( a^{8} d^{-3} \max_{j} \left\| \frac{1}{2} \left( \mathcal{E}_{j}^{T} \right) \right\|_{L^2(B)} \right). \]

Next, expanding \( H_{\rho_0}^{\text{inc}}(\cdot) \) near \( z_{\rho_0} \) and \( \Upsilon_{k}(\cdot, \cdot) \) near the center \((z_{\rho_0}, z_{j})\), similar to (6.20), we obtain

\[
\mathbf{M} \cdot \int_{D_{\rho_0}} F_{\rho_0}(x) \, dx - \eta k^2 \sum_{j=1}^{N} \int_{D_{\rho_0}} A(x) \, dx \cdot \Upsilon_{k}(z_{\rho_0}, z_{j}) \cdot \int_{D_{j}} F_{j}(y) \, dy
\]

\[ = i k \int_{D_{\rho_0}} A(x) \cdot H_{\rho_0}^{\text{inc}}(z_{\rho_0})
\]

\[ + \eta k^2 \sum_{j \neq \rho_0} \int_{D_{\rho_0}} A(x) \cdot \int_{0}^{1} \nabla \left( H_{\rho_0}^{\text{inc}} \right) \left( z_{\rho_0} + t(x - z_{\rho_0}) \right) \cdot (x - z_{\rho_0}) \, dt \, dx
\]

\[ + \eta k^2 \sum_{j \neq \rho_0} \int_{D_{\rho_0}} A(x) \cdot \int_{0}^{1} \nabla \left( \Upsilon_{k} \right) \left( z_{\rho_0} + t(x - z_{\rho_0}), z_{j} \right) \cdot \mathcal{P}(x, z_{\rho_0}) \, dt \, dx \cdot \int_{D_{j}} F_{j}(y) \, dy
\]

\[ + \mathcal{O} \left( a^6 \right) + \mathcal{O} \left( a^4 \left\| \frac{3}{2} \left( \mathcal{E}_{\rho_0}^{T} \right) \right\|_{L^2(B)} \right)
\]

\[ + \mathcal{O} \left( a^{7-h} d^{-3} \max_{j} \left\| \frac{3}{2} \left( \mathcal{E}_{j}^{T} \right) \right\|_{L^2(B)} \right) + \mathcal{O} \left( a^{8} d^{-3} \max_{j} \left\| \frac{1}{2} \left( \mathcal{E}_{j}^{T} \right) \right\|_{L^2(B)} \right). \]

Now we estimate the error terms appearing on the right hand side of (6.42) as follows. Firstly, set

\[
J_6 := i k \int_{D_{\rho_0}} A(x) \cdot \int_{0}^{1} \nabla \left( H_{\rho_0}^{\text{inc}} \right) \left( z_{\rho_0} + t(x - z_{\rho_0}) \right) \cdot (x - z_{\rho_0}) \, dt \, dx,
\]

\[
J_6 = -k^2 \int_{D_{\rho_0}} A(x) \cdot \int_{0}^{1} e^{i k \theta \left( x - z_{\rho_0} \right)} \theta \cdot (x - z_{\rho_0}) \, dt \, dx,
\]

\[
\left| J_6 \right| \leq \left\| A \right\|_{L^2(D_{\rho_0})} \left\| \nabla \right\|_{L^2(D_{\rho_0})} \leq a^4 \left\| A \right\|_{L^2(B)}.
\]

Using the variant of Friedrich inequality on the subspace \( H_0(\text{Curl}) \cap H(\text{div}) = 0 \), we have

\[ |J_6| \leq a^4 \left\| \text{Curl} \left( \hat{A} \right) \right\|_{L^2(B)} = a^5 \left\| \text{Curl} \left( \hat{A} \right) \right\|_{L^2(B)} = a^5 \left\| \hat{W} \right\|_{L^2(B)} = a^5 \left\| \hat{W} \right\|_{L^2(B)} \leq \mathcal{O} \left( a^{6-h} \right). \]

Then,

\[ (6.43) \quad J_6 = \mathcal{O} \left( a^{6-h} \right). \]
Secondly, set
\[
J_7 := \eta k^2 \sum_{j \neq j_0}^{N} \int_{D_{yj_0}} A(x) \cdot \int_{D_j} \int_{0}^{1} \nabla (\Upsilon_k) (z_j, t(x - z_j)) \cdot P(x, z_j) \, dt \, dx \cdot \int_{D_{yj_0}} F_j(y) \, dy
\]
and
\[
|J_7| \lesssim a^{-\frac{1}{2}} \sum_{j \neq j_0}^{N} \|A\|_{L^2(D_{yj_0})} \left\| \int_{0}^{1} \nabla (\Upsilon_k) (z_j, t(x - z_j)) \cdot P(x, z_j) \, dt \right\|_{L^2(D_{yj_0})} \|F_j\|_{L^2(D_{yj_0})}
\]

(6.44)
\[
\lesssim a^2 \sum_{j \neq j_0}^{N} \|A\|_{L^2(D_{yj_0})} \|z_j - z_{j_0}\|^{-4} \|F_j\|_{L^2(D_{yj_0})} \lesssim a^2 \cdot a^3 \sum_{j \neq j_0}^{N} \|A\|_{L^2(D_{yj_0})} \|z_j - z_{j_0}\|^{-4} \|\tilde{F}_j\|_{L^2(D_{yj_0})}
\]
\[
\leq a^5 d^{-4} \|A\|_{L^2(B)} \max_j \|\tilde{F}_j\|_{L^2(B)} \lesssim a^7 d^{-4} \max_j \|\tilde{F}_j\|_{L^2(B)}.
\]

By applying the Friedrich type inequality for \(\tilde{F}_j\), \(j = 1, \ldots, N\), we have:
\[
\|\tilde{F}_j\|_{L^2(B)} \lesssim \|\text{Curl} (\tilde{F}_j)\|_{L^2(B)} \lesssim a \|\text{Curl}(F_j)\|_{L^2(B)} = O \left( a \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right).
\]
Then we can further derive that:

(6.45)
\[
J_7 = O \left( a^{8-h} d^{-4} \max_j \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right).
\]

Finally, denote
\[
J_8 := \eta k^2 \sum_{j \neq j_0}^{N} \int_{D_{yj_0}} A(x) \cdot \int_{D_j} \int_{0}^{1} \nabla (\Upsilon_k) (x, z_j + t(y - z_j)) \cdot P(y, z_j) \, dt \cdot F_j(y) \, dy \, dx,
\]
and
\[
|J_8| \leq |\eta| \sum_{j \neq j_0}^{N} \|A\|_{L^2(D_{yj_0})} \left\| \int_{D_j} \int_{0}^{1} \nabla (\Upsilon_k) (\cdot, z_j + t(y - z_j)) \cdot P(y, z_j) \, dt \cdot F_j(y) \, dy \right\|_{L^2(D_{yj_0})}
\]
\[
\leq a^2 \|A\|_{L^2(D_{yj_0})} \sum_{j \neq j_0}^{N} \|z_j - z_{j_0}\|^{-4} \|F_j\|_{L^2(D_{yj_0})}.
\]

We remark that the previous formula follows the same as (6.44), and then, in a straightforward manner, we deduce that

(6.46)
\[
J_8 = O \left( a^{8-h} d^{-4} \max_j \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right).
\]

Taking into account the estimations (6.42, 6.43) and (6.44), the equation (6.42) becomes,
\[
M \cdot \int_{D_{yj_0}} F_j(x) \, dx \quad - \quad \eta k^2 \sum_{j \neq j_0}^{N} \int_{D_{yj_0}} A(x) \, dx \cdot \Upsilon_k(z_{j_0}, z_j) \cdot \int_{D_j} F_j(y) \, dy
\]
\[
= i k \int_{D_{yj_0}} A(x) \, dx \cdot H^T_{yj_0}(z_{j_0}) + O \left( a^{8-h} d^{-4} \max_j \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right)
\]
\[
\quad + O \left( a^{6-h} \right) + O \left( a^4 \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right) + O \left( a^{7-h} d^{-3} |\log(d)| \max_j \| \mathbb{P} \left( \tilde{E}_j^T \right) \|_{L^2(B)} \right).
\]

(6.47)
Remark that $\frac{1}{4} \left( \mathbf{M}^T \cdot \mathbf{M} \cdot \mathbf{M}^T \right) \cdot \mathbf{M} = \mathbf{I}$ and denote $\mathcal{A}_{j_0}$ to be

\[ (6.48) \quad \mathcal{A}_{j_0} := \frac{1}{4} \left( \mathbf{M}^T \cdot \mathbf{M} \cdot \mathbf{M}^T \right) \cdot \int_{D_{j_0}} A(x) \, dx \in \mathbb{C}^{3 \times 3}. \]

Then, the equation (6.47), after multiplying each side by $\frac{1}{4} \left( \mathbf{M}^T \cdot \mathbf{M} \cdot \mathbf{M}^T \right)$, becomes

\[ (6.49) \quad \int_{D_{j_0}} F_{j_0}(y) \, dy - \eta k^2 \sum_{j \neq j_0}^N \mathcal{A}_{j_0} \cdot \mathcal{Y}_j(z_{j_0}, z_j) \cdot \int_{D_j} F_j(y) \, dy = i k \mathcal{A}_{j_0} \cdot H_0^{inc}(z_{j_0}) + \text{Error}, \]

where

\[ \text{Error} := \mathcal{O} \left( \alpha^8 - h \right) \max_j \frac{1}{2} \left\| \mathbf{P} \left( \tilde{E}_j \right) \right\|_{L^2(B)} + \mathcal{O} \left( \alpha^6 - h \right) \]

\[ (6.50) \quad + \mathcal{O} \left( \alpha^4 \right) \left\| \mathbf{P} \left( \tilde{E}_j \right) \right\|_{L^2(B)} + \mathcal{O} \left( \alpha^7 - h \right) \left( \log(d) \right) \max_j \left\| \mathbf{P} \left( \tilde{E}_j \right) \right\|_{L^2(B)} \]

Next, we estimate $\mathcal{A}_{j_0}$. Since $\frac{1}{4} \left( \mathbf{M}^T \cdot \mathbf{M} \cdot \mathbf{M}^T \right)$ is a constant matrix, the estimation of $\mathcal{A}_{j_0}$ is the same as that of $\int_{D_{j_0}} A(x) \, dx$. We have

\[ \int_{D_{j_0}} A(x) \, dx = a^3 \int_B \tilde{A}(x) \, dx = a^3 \int_B \tilde{A}(x) \cdot I \, dx = -a^3 \int_B \tilde{A}(x) \cdot \text{Curl} \left( \Omega(x) \right) \, dx, \]

where $\Omega(x)$ is the matrix given by

\[ \Omega(x) := \begin{pmatrix} 0 & x_3 & 0 \\ 0 & 0 & x_1 \\ x_2 & 0 & 0 \end{pmatrix}, \]

and, with integration by parts, we get

\[ \int_{D_{j_0}} A(x) \, dx = -a^3 \int_B \text{Curl} \left( \tilde{A} \right)(x) \cdot \Omega(x) \, dx = -a^4 \int_B \text{Curl} \left( \tilde{A} \right)(x) \cdot \Omega(x) \, dx = \int_B \mathbf{P} \left( \tilde{W} \right) (x) \cdot \Omega(x) \, dx. \]

By expanding $\mathbf{P} \left( \tilde{W} \right)$ over the basis $\left( e_n^{(1)} \right)_n$, we obtain

\[ \int_{D_{j_0}} A(x) \, dx = -a^4 \sum_n \frac{1}{2} \left( \mathbf{P} \left( \tilde{W} \right) ; e_n^{(1)} \right) \cdot \left( \int_B e_n^{(1)}(x) \cdot \Omega(x) \, dx \right)^{Tr}. \]

We write $e_n^{(1)}$ as $\text{Curl} \left( \phi_n \right)$ and using integration by parts to get

\[ \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle = \langle \mathcal{P}(\cdot, 0), \text{Curl} \left( \phi_n \right) \rangle = \langle \text{Curl} \left( \mathcal{P}(\cdot, 0) \right), \phi_n \rangle = \langle \mathcal{M}, \phi_n \rangle \]

and

\[ \int_B e_n^{(1)}(x) \cdot \Omega(x) \, dx = \int_B \text{Curl} \left( \phi_n \right)(x) \cdot \Omega(x) \, dx = \int_B \phi_n(x) \cdot \text{Curl} \left( \Omega \right)(x) \, dx = -\int_B \phi_n(x) \, dx. \]

Then

\[ \int_{D_{j_0}} A(x) \, dx = a^{5 - h} \langle \mathcal{M}, \phi_{n_0} \rangle \cdot \left( \int_B \phi_{n_0}(x) \, dx \right)^{Tr}. \]
By substituting (6.52) into (6.49), after rearranging the terms, we obtain the linear algebraic with the form

\[
\begin{align*}
- a^5 \sum_{n \neq n_0} \frac{1}{(1 - k^2 \eta a^2 \lambda_n^{(1)})} & \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \cdot \left( \int_B e_n^{(1)}(x) \cdot \mathcal{Q}(x) \, dx \right)^T \\
+ a^4 \sum_{n} \frac{1}{(1 - k^2 \eta a^2 \lambda_n^{(1)})} & \langle T, e_n^{(1)} \rangle \cdot \left( \int_B e_n^{(1)}(x) \cdot \mathcal{Q}(x) \, dx \right)^T .
\end{align*}
\]

Set

\[
J_{9,1} := a^5 \sum_{n \neq n_0} \frac{1}{(1 - k^2 \eta a^2 \lambda_n^{(1)})} \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \cdot \left( \int_B e_n^{(1)}(x) \cdot \mathcal{Q}(x) \, dx \right)^T ,
\]

then

\[
|J_{9,1}| \leq a^5 \sum_{n \neq n_0} \left| \langle \mathcal{P}(\cdot, 0), e_n^{(1)} \rangle \right| \left| \langle \mathcal{Q}, e_n^{(1)} \rangle \right| \sim a^5.
\]

And for the term \( J_{9,2} \) given by

\[
J_{9,2} := a^4 \sum_{n} \frac{1}{(1 - k^2 \eta a^2 \lambda_n^{(1)})} \langle T, e_n^{(1)} \rangle \cdot \left( \int_B e_n^{(1)}(x) \cdot \mathcal{Q}(x) \, dx \right)^T ,
\]

we can check, from the definition of the term \( T(\cdot) \), see for instance (5.66), that

\[
J_{9,2} \sim a^4 \sum_{n} \frac{1}{(1 - k^2 \eta a^2 \lambda_n^{(1)})} \left( \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y)}{||-y||^2} \cdot \frac{3}{2} \mathcal{P}(\mathcal{W})(y)dy, e_n^{(1)} \right) \cdot \left( \int_B e_n^{(1)}(x) \cdot \mathcal{Q}(x) \, dx \right)^T 
\]

\[
|J_{9,2}| \lesssim a^{4-h} \sum_{n} \left| \left( \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y)}{||-y||^2} \cdot \frac{3}{2} \mathcal{P}(\mathcal{W})(y)dy, e_n^{(1)} \right) \right| \left| \langle \mathcal{Q}, e_n^{(1)} \rangle \right| 
\]

\[
\lesssim a^{4-h} \left\| \int_B \Phi_0(\cdot, y) \frac{A(\cdot, y)}{||-y||^2} \cdot \frac{3}{2} \mathcal{P}(\mathcal{W})(y)dy \right\| \lesssim a^{4-h} \left\| \frac{3}{2} \mathcal{P}(\mathcal{W}) \right\| \lesssim \lesssim \lesssim a^{7-h} .
\]

Using the estimations \( J_{9,1} \) and \( J_{9,2} \), we obtain that

\[
\int_{D_{\phi_0}} \mathcal{A}(x) \, dx = a^{5-h} \left\langle \mathcal{M}, \phi_{n_0} \right\rangle \cdot \left( \int_B \phi_{n_0}(x) \, dx \right)^T + \mathcal{O}(a^5).
\]

Seeing the definition of \( \mathcal{A}_{J_0} \) in (6.48), by a straightforward computation, we deduce that

\[
(6.52) \quad \mathcal{A}_{J_0} = a^{5-h} \left( \left( \int_B \phi_{n_0}(y) \, dy \right) \otimes \left( \int_B \phi_{n_0}(y) \, dy \right) \right) + \mathcal{O}(a^5).
\]

By substituting (6.52) into (6.49), after rearranging the terms, we obtain the linear algebraic with the form (6.50). Moreover, from the a-priori estimates for \( \mathcal{P}(E_j^T) \) and \( \mathcal{P}(E_j^T) \) given by (2.15b) in Proposition 2.3, the error term in (6.50) can be simplified as (3.7).

As for the invertibility condition (3.2), indeed, utilizing the precise representation (6.51) of the tensor \( \mathcal{A}_{J_0} \) and combining with (6.52), it is direct to verify that \( \left\| \mathcal{P}_0 \right\| \) of the invertibility condition (3.2) in Proposition 3.1 can be replaced by

\[
\frac{k^2 a^5}{1 - k^2 \eta a^2 \lambda_{n_0}^{(1)}} \left\| \int_B \phi_{n_0}(y) \, dy \otimes \int_B \phi_{n_0}(y) \, dy \right\| ,
\]

which yields (3.3).
6.3. Construction of the linear algebraic system in Proposition 3.3. Similar to Proposition 3.2, we construct the precise form of the linear algebraic system related to \( F_j(T_j) \), \( j = 1, 2, \ldots, N \), in Proposition 3.3 as follows.

**Proof of Proposition 3.3.** Recall again the Lippmann-Schwinger system of the equation

\[ T_k(E_{j_0}^T)(x) - \eta \sum_{j \neq j_0}^N \int_{D_j} \mathcal{Y}_k(x, y) \cdot E_{j_0}^T(y) dy = E_{j_0}^{inc}(x), \quad \text{in } D_{j_0}, \ j_0 = 1, \ldots, N. \]

Define \( V \) as

\[ V := T_k^{-1}(I), \]

Inverting the operator \( T_k \), integrating over \( D_{j_0} \), and using the definition of \( V \), see for instance (6.53), we obtain that

\[ \int_{D_{j_0}} E_{j_0}^T(x) dx - \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \mathcal{Y}_k(x, y) \cdot E_{j_0}^T(y) dy dx = \int_{D_{j_0}} V(x) \cdot E_{j_0}^{inc}(x) dx, \]

or in the following form

\[ \int_{D_{j_0}} \frac{3}{P} (E_{j_0}^T)(x) dx - \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \mathcal{Y}_k(x, y) \cdot \frac{3}{P} (E_{j}^T)(y) dy dx \]

\[ - \eta k^2 \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \Phi_k(x, y) P (E_{j}^T)(y) dy dx = \int_{D_{j_0}} V(x) \cdot E_{j_0}^{inc}(x) dx. \]

We rewrite the previous equation, with the help of (6.26), as

\[
\int_{D_{j_0}} \frac{3}{P} (E_{j_0}^T)(x) dx - \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) dx \cdot \mathcal{Y}_k(z_{j_0}, z_j) \cdot \int_{D_j} \frac{3}{P} (E_{j}^T)(y) dy dx = \int_{D_{j_0}} V(x) \cdot E_{j_0}^{inc}(x) dx
\]

\[
+ \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \int_0^1 \nabla_y (\mathcal{Y}_k)(x, z_j + t(y - z_j)) \cdot P(y, z_j) dt \cdot \frac{3}{P} (E_{j}^T)(y) dy dx
\]

\[
+ \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_0^1 \nabla_x (\mathcal{Y}_k)(x, z_j + t(x - z_{j_0}), z_j) \cdot P(x, z_{j_0}) dt \cdot \int_{D_j} \frac{3}{P} (E_{j}^T)(y) dy dx
\]

\[ (6.54) \]

Then, we estimate the last three terms on the right hand side of (6.54). Set,

\[ J_{10} := \eta \sum_{j \neq j_0}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \int_0^1 \nabla_y (\mathcal{Y}_k)(x, z_j + t(y - z_j)) \cdot P(y, z_j) dt \cdot \frac{3}{P} (E_{j}^T)(y) dy dx \]

\[ |J_{10}| \leq |\eta| \|V\|_{L^2(D_{j_0})} \sum_{j \neq j_0}^N \left[ \int_{D_{j_0}} \int_{D_j} \int_0^1 \nabla_y (\mathcal{Y}_k)(x, z_j + t(y - z_j)) \cdot P(y, z_j) dt \right]^{\frac{1}{2}} \| \frac{3}{P} (E_{j}^T) \|_{L^2(D_j)}
\]

\[ \lesssim a^2 \|V\|_{L^2(D_{j_0})} \sum_{j \neq j_0}^N |z_j - z_{j_0}|^{-4} \| \frac{3}{P} (E_{j}^T) \|_{L^2(D_j)}. \]
Then,

\[ J_{10} = \mathcal{O}\left( a^2 \| V \|_{L^2(D_{j_0})} d^{-4} \max_j \frac{3}{\| E_j^T \|_{L^2(D_j)}} \right) \]

Denote

\[ J_{11} := \eta \sum_{j=1}^N \int_{D_{j_0}} V(x) \cdot \int_0^1 \nabla (\mathcal{Y}_k (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) dt \cdot \int_{D_j} \frac{3}{\| E_j^T \|} (y) dy dx \]

\[ J_{11} = \eta \sum_{j=1}^N \int_{D_{j_0}} V(x) \cdot \int_0^1 \nabla (\mathcal{Y}_k (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) dt \cdot \int_{D_j} \frac{3}{\| E_j^T \|} (y) dy \]

\[ |J_{11}| \leq a^{-\frac{3}{2}} \| V \|_{L^2(D_{j_0})} \sum_{j=1}^N \left| \int_0^1 \nabla (\mathcal{Y}_k (z_{j_0} + t(x - z_{j_0}), z_j) \cdot \mathcal{P}(x, z_{j_0}) dt \right| \| E_j^T \|_{L^2(D_j)} \]

\[ \lesssim a^2 \| V \|_{L^2(D_{j_0})} \sum_{j=1}^N |z_j - z_{j_0}|^{-4} \frac{3}{\| E_j^T \|_{L^2(D_j)}}. \]

Then,

\[ J_{11} = \mathcal{O}\left( a^2 \| V \|_{L^2(D_{j_0})} d^{-4} \max_j \frac{3}{\| E_j^T \|_{L^2(D_j)}} \right) \]

Also, we denote \( J_{12} \) to be

\[ J_{12} := \eta k^2 \sum_{j=1}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \Phi_k(x, y) \frac{1}{\| E_j^T \|} (y) dy dx \]

\[ = \eta k^2 \sum_{j=1}^N \int_{D_{j_0}} V(x) \cdot \int_{D_j} \left[ \Phi_k(x, z_j) + \nabla \Phi_k(x, z_j) \cdot (y - z_j) \right. \]

\[ + \frac{1}{2} \int_0^1 (y - z_j)^{\frac{1}{2}} \cdot H_{ess} (\Phi_k)(x, z_j + t(y - z_j)) \cdot (y - z_j) dt \left]. \frac{1}{\| E_j^T \|} (y) dy dx. \]

For the first term on the right hand side of the above equation, it is clear that:

\[ \int_{D_{j_0}} V(x) \cdot \int_{D_j} \Phi_k(x, z_j) \frac{1}{\| E_j^T \|} (y) dy dx = \int_{D_{j_0}} V(x) \cdot \Phi_k(x, z_j) \int_{D_j} \frac{1}{\| E_j^T \|} (y) dy dx = 0. \]

And for the last two terms, there holds

\[ L_{4,1} := \sum_{j=1}^N \int_{D_{j_0}} V(x) \int_{D_j} \nabla \Phi_k(x, z_j) \cdot (y - z_j) \cdot \frac{1}{\| E_j^T \|} (y) dy dx, \]

\[ |L_{4,1}| \leq \sum_{j=1}^N \int_{D_{j_0}} V(x) \int_{D_j} \nabla \Phi_k(x, z_j) \cdot (y - z_j) \cdot \frac{1}{\| E_j^T \|} (y) dy dx \]

\[ \lesssim a^4 \| V \|_{L^2(D_{j_0})} \sum_{j=1}^N \frac{1}{|z_{j_0} - z_j|^2} \frac{1}{\| E_j^T \|_{L^2(D_j)}}. \]
and we estimate the last term as

\[
\frac{1}{3} \| \mathbf{F} \|_{L^2(D_{j0})} \left( \begin{array}{c} \delta \end{array} \right) \| \mathbf{E}^j \|_{L^2(D_j)} \leq \mathcal{O} \left( a^{\frac{5}{2} - h d^{-3}} \| V \|_{L^2(D_{j0})} \right),
\]

and

\[
L_{4.2} := \sum_{j \neq j_0}^{N} \int_{D_{j0}} V(x) \int_{D_j} \int_{0}^{1} (y - z_j)^{-\frac{1}{2}} \cdot H_{\text{ess}} (\Phi_k)(x, z_j + t(y - z_j)) \cdot (y - z_j) dt \cdot \mathbf{F}^j \mathbf{E}^j (y) dy dx.
\]

Hence, \( J_{12} \) can be estimated as

\[
J_{12} = \mathcal{O} \left( a^{\frac{5}{2} - h d^{-3}} \| V \|_{L^2(D_{j0})} \right).
\]

Recall \( (6.54) \). Using \( (6.55), (6.56) \) and \( (6.57) \), we deduce that

\[
\int_{D_{j0}}^{3} \mathbf{P}^T \left( \mathbf{E}^T_{j0} \right) (x) dx - \eta \sum_{j \neq j_0}^{N} \int_{D_{j0}} V(x) dx \cdot \mathbf{Y}_k (z_{j0}, z_j) \cdot \mathbf{P}^T \mathbf{E}^j (y) dy = \int_{D_{j0}} V(x) \cdot \mathbf{E}^{nc}_{j0} (x) dx
\]

which can be reduced, using the conditions on \( t \) and \( h \) given by \( (1.27) \), to:

\[
\int_{D_{j0}}^{3} \mathbf{P}^T \left( \mathbf{E}^T_{j0} \right) (x) dx - \eta \sum_{j \neq j_0}^{N} \mathbf{c}_{j0} \cdot \mathbf{Y}_k (z_{j0}, z_j) \cdot \mathbf{P}^T \mathbf{E}^j (y) dy = \int_{D_{j0}} V(x) \cdot \mathbf{E}^{nc}_{j0} (x) dx
\]

(6.58)

where \( \mathbf{c}_{j0} := \int_{D_{j0}} V(x) dx \). Moreover,

\[
\int_{D_{j0}} V(x) \cdot \mathbf{E}^{nc}_{j0} (x) dx = \mathbf{c}_{j0} \cdot \mathbf{E}^{nc}_{j0} (z_{j0}) + \int_{D_{j0}} V(x) \cdot \left( \nabla \mathbf{E}^{nc}_{j0} (z_{j0} + t(x - z_{j0})) \cdot (x - z_{j0}) dt dx,
\]

and we estimate the last term as

\[
\left\| \int_{D_{j0}} V(x) \cdot \int_{0}^{1} \nabla \mathbf{E}^{nc}_{j0} (z_{j0} + t(x - z_{j0})) \cdot (x - z_{j0}) dt dx \right\| \leq a^{\frac{5}{2}} \| V \|_{L^2(D_{j0})}.
\]

Therefore, equation \( (6.58) \) takes the following form

\[
\int_{D_{j0}}^{3} \mathbf{P}^T \left( \mathbf{E}^T_{j0} \right) (x) dx - \eta \sum_{j \neq j_0}^{N} \mathbf{c}_{j0} \cdot \mathbf{Y}_k (z_{j0}, z_j) \cdot \mathbf{P}^T \mathbf{E}^j (y) dy = \mathbf{c}_{j0} \cdot \mathbf{E}^{nc}_{j0} (z_{j0}) + \mathcal{O} \left( a^{\frac{5}{2}} \| V \|_{L^2(D_{j0})} \right) + \mathcal{O} \left( a^{\frac{5}{2} - h d^{-3}} \| V \|_{L^2(D_{j0})} \right).
\]

(6.59)

Next, we compute the estimation of \( \mathcal{E}_{j0} \). Recalling the definition of the matrix \( V \) in \( (6.58), \) i.e. \( V := T^{-1}_{-k} (I) \), and scaling to the domain \( B \), we obtain

\[
\tilde{V} := T^{-1}_{-k \alpha} (I).
\]

(6.60)
Now, to give the estimation of \( e_{j_0} \), we compute \( \langle \tilde{V}, e^{(3)}_{n} \rangle, j = 1, 2, 3 \). Indeed, we have
\[
0 = \int_B e^{(1)}_n(x) \, dx = \int_B I \cdot e^{(1)}_n(x) \, dx \tag{6.64} = \int_B T_{-ka} (\tilde{V}(x)) \cdot e^{(1)}_n(x) \, dx = \int_B \tilde{V}(x) \cdot T_{ka} (e^{(1)}_n)(x) \, dx.
\]
Knowing that \( \nabla M^{ka} (e^{(1)}_n) = 0 \), combining with \( \tag{5.4} \), we deduce from the above equation that
\[
\langle \tilde{V}, e^{(1)}_n \rangle = \frac{k^2 \eta a^2}{4\pi} \sum_{\ell \geq 1} \frac{(ika)^{\ell+1}}{(\ell + 1)!} \int_B \tilde{V}(x) \cdot \int_B \|x-y\|^\ell e^{(1)}_n(y) \, dy \, dx.
\]
Similarly, we have
\[
0 = \int_B e^{(2)}_n(x) \, dx = \int_B I \cdot e^{(2)}_n(x) \, dx \tag{6.60} = \int_B T_{-ka} (\tilde{V}(x)) \cdot e^{(2)}_n(x) \, dx = \int_B \tilde{V}(x) \cdot T_{ka} (e^{(2)}_n)(x) \, dx \tag{5.4} = (1+\eta) \langle \tilde{V}, e^{(2)}_n \rangle.
\]
In the subspace \( \nabla \text{Harmonic} \), we have
\[
\int_B e^{(3)}_n(x) \, dx = \int_B I \cdot e^{(3)}_n(x) \, dx \tag{6.60} = \int_B T_{-ka} (\tilde{V}(x)) \cdot e^{(3)}_n(x) \, dx = \int_B \tilde{V}(x) \cdot T_{ka} (e^{(3)}_n)(x) \, dx = \left(1 + \eta \lambda_n \right) \langle \tilde{V}, e^{(3)}_n \rangle.
\]
Then,
\[
\langle \tilde{V}, e^{(3)}_n \rangle = \left(1 + \eta \lambda_n \right)^{-1} \int_B e^{(3)}_n(x) \, dx + \eta \int \langle \tilde{V}, e^{(3)}_n \rangle \left(-k^2 a^2 N^{ka} + (\nabla M^{ka} - \nabla M) (e^{(3)}_n)\right) \, dx
\]
\[
= \frac{1}{1 + \eta \lambda_n} \int_B e^{(3)}_n(x) \, dx + \eta \int \langle \tilde{V}, S (e^{(3)}_n) \rangle \, dx,
\]
where \( S \) is given by \( \tag{5.7} \). Now,
\[
e^{(3)}_{j_0} = a^3 \int_B \tilde{V}(x) \, dx = a^3 \sum_n \langle \tilde{V}, e^{(3)}_n \rangle \otimes \int_B e^{(3)}_n(x) \, dx \tag{5.63}
\]
\[
= a^3 \sum_n \left[ \frac{1}{1 + \eta \lambda_n} \int_B e^{(3)}_n(x) \, dx + \eta \int \langle \tilde{V}, S (e^{(3)}_n) \rangle \, dx \right] \otimes \int_B e^{(3)}_n(x) \, dx
\]
\[
= a^3 \sum_n \left[ \frac{1}{1 + \eta \lambda_n} \int_B e^{(3)}_n(x) \, dx + \eta \int \langle \tilde{V}, S_d (e^{(3)}_n) \rangle \, dx \right] \otimes \int_B e^{(3)}_n(x) \, dx
\]
\[
+ a^3 \eta \sum_n \left[ \frac{1}{1 + \eta \lambda_n} \langle \tilde{V}, S_r (e^{(3)}_n) \rangle \otimes \int_B e^{(3)}_n(x) \, dx \right],
\]
where, from \( \tag{5.7} \), we have
\[
S_d (e^{(3)}_n)(x) = \frac{a^2 k^2}{2} N(e^{(3)}_n)(x) + \frac{a^2 k^2}{2} \int_B \Phi_0(x, y) A(x, y) \cdot e^{(3)}_n(y) \, dy
\]
and
\[
S_r (e^{(3)}_n)(x) := \frac{i(ak)^3}{6\pi} \int_B e^{(3)}_n(y) \, dy + \frac{1}{4\pi} \sum_{\ell \geq 3} \frac{(ika)^{\ell+1}}{(\ell + 1)!} \int_B Hess_x \|x-y\|^\ell e^{(3)}_n(y) \, dy
\]
\[
+ \frac{k^2 a^2}{4\pi} \sum_{\ell \geq 1} \frac{(ika)^{\ell+1}}{(\ell + 1)!} \int_B \|x-y\|^\ell e^{(3)}_n(y) \, dy.
\]
First, we estimate
\[
J_{13} := a^3 \eta \sum_n \left[ \frac{1}{1 + \eta \lambda_n} \langle \tilde{V}, S_r (e^{(3)}_n) \rangle \otimes \int_B e^{(3)}_n(x) \, dx \right] \simeq a^3 \sum_n \langle \tilde{V}, S_r (e^{(3)}_n) \rangle \otimes \int_B e^{(3)}_n(x) \, dx.
\]
By keeping the dominant term of (6.60), we see that $J_{13}$ fulfills

$$J_{13} \simeq a^6 \sum_n \langle \tilde{V}, \int_B e_n^{(3)}(x) \rangle \otimes \int_B e_n^{(3)}(x) dx$$

$$|J_{13}| \lesssim a^6 \left( \sum_n \left( \left| \langle \tilde{V}, \int_B e_n^{(3)}(x) \rangle \right|^2 \right)^{\frac{1}{2}} \left( \sum_n \left| \int_B e_n^{(3)}(x) dx \right|^2 \right)^{\frac{1}{2}} \right) = O \left( a^6 \| \tilde{V} \|_{L^2(B)} \right).$$

Then equation (6.64) becomes

$$C_{64) \becomes a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \left[ \int_B e_n^{(3)}(x) dx + \eta \langle \tilde{V}, S_d(e_n^{(3)}) \rangle \right] \otimes \int_B e_n^{(3)}(x) dx + O \left( a^6 \| \tilde{V} \|_{L^2(B)} \right).$$

With the help of the expression of $S_d(e_n^{(3)})$ in (6.64), we have

$$C_{j_o} = a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \left[ \int_B e_n^{(3)}(x) dx + \eta \langle \tilde{V}, N(e_n^{(3)}) \rangle \right] \otimes \int_B e_n^{(3)}(x) dx + O \left( a^6 \| \tilde{V} \|_{L^2(B)} \right).$$

Then, using the fact that $k^2 \eta a^2 = \frac{1}{\lambda^{(1)}_{n_0}}$ given by (2.11), there holds

$$C_{j_o} = a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \left[ \int_B e_n^{(3)}(x) dx + \frac{1}{2 \lambda^{(1)}_{n_0}} \langle \tilde{V}, N(e_n^{(3)}) \rangle \right] \otimes \int_B e_n^{(3)}(x) dx + O \left( a^6 \| \tilde{V} \|_{L^2(B)} \right).$$

It is easy to verify that for

$$L_5 := a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \frac{c_0 a^h}{2 \lambda^{(1)}_{n_0}} \langle \tilde{V}, N(e_n^{(3)}) \rangle \otimes \int_B e_n^{(3)}(x) dx,$$

$$|L_5| \lesssim a^{5+h} \sum_n |\langle \tilde{N}(\tilde{V}), e_n^{(3)} \rangle| \otimes \int_B e_n^{(3)}(x) dx$$

$$\lesssim a^{5+h} \| \tilde{N}(\tilde{V}) \|_{L^2(B)} = O \left( a^{5+h} \| \tilde{V} \|_{L^2(B)} \right).$$

Hence,

$$C_{j_o} = a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \left[ \int_B e_n^{(3)}(x) dx + \frac{1}{2 \lambda^{(1)}_{n_0}} \langle \tilde{V}, N(e_n^{(3)}) \rangle \right] \otimes \int_B e_n^{(3)}(x) dx + O \left( a^{5+h} \| \tilde{V} \|_{L^2(B)} \right),$$

where we deduce that

$$C_{67} \sim a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \int_B e_n^{(3)}(x) dx \otimes \int_B e_n^{(3)}(x) dx \sim a^5.$$  

Next, we estimate the $L^2(B)$-norm of $\tilde{V}$. For this, using (6.61), (6.62) and (6.63), we obtain that

$$\| \tilde{V} \|_{L^2(B)}^2 = \sum_n |\langle \tilde{V}, e_n^{(3)} \rangle|^2 + \sum_n |\langle \tilde{V}, e_n^{(3)} \rangle|^2.$$

11To write short, in (6.63), the estimation of the second term of the right side is neglected because the singularity of the corresponding kernel behaves like the one of the Newtonian operator.

12The smallness of the term $a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \frac{c_0 a^h}{2 \lambda^{(1)}_{n_0}} \langle \tilde{V}, N(e_n^{(3)}) \rangle \otimes \int_B e_n^{(3)}(x) dx$, can be proved using the estimation of $\| \tilde{V} \|_{L^2(B)}$, given by (6.63).
By injecting the expression (6.69), we deduce the invertibility condition (3.11) for the linear algebraic system (3.9).

We use all these estimates to deduce that

$$\sum_n \langle S_d^* (\tilde{V}) , e_n^{(3)} \rangle^2 \lesssim a^4 \sum_n \langle N (\tilde{V}) , e_n^{(3)} \rangle^2 \lesssim a^4 \| \tilde{V} \|^2.$$ 

In addition, we have

$$\sum_n \sum_{\ell \geq 1} \left| \int_B \tilde{V}(x) \cdot \int_B \frac{\| x - y \|^{\ell+1}}{(\ell+1)!} e_n^{(1)}(y) dy dx \right|^2 = \sum_n \sum_{\ell \geq 1} \left| \int_B \int_B \tilde{V}(x) \frac{\| x - y \|^{\ell+1}}{(\ell+1)!} dx \cdot e_n^{(1)}(y) dy \right|^2 \lesssim \sum_n \left\| \int_B \tilde{V}(x) \frac{\| x - y \|^{\ell+1}}{(\ell+1)!} dx \right\|^2 \lesssim \| \tilde{V} \|^2.$$

We use all these estimates to deduce that

$$\| \tilde{V} \|^2_{L^2(B)} \lesssim a^4 \sum_n \langle I , e_n^{(3)} \rangle^2 \sim a^4.$$

Going back to (6.59), by using (6.68), we get

$$\int_{\Gamma} \frac{3}{\mathcal{P}} (E^T_j) (x) dx - \sum_{j=1}^N \mathcal{C}_{j0} \cdot \kappa (z_{j0}, z_j) \cdot \int_D \frac{3}{\mathcal{P}} (E^T_j) (y) dy = \mathcal{C}_{j0} \cdot E^{inc}_{j0} (z_{j0}) + o \left( a^{\min(6.8-h-3)} \right).$$

By injecting the expression (6.67), of $\mathcal{C}_{j0}$, into (6.69), after rearranging terms, we derive the linear algebraic system of the form (3.9). Finally, replacing $\| [P] \|$ in (3.22) by

$$a^3 \sum_n \frac{1}{1 + \eta \lambda_n^{(3)}} \int_B e_n^{(3)}(y) dy \otimes \int_B e_n^{(3)}(y) dy,$$

we deduce the invertibility condition (3.11) for the linear algebraic system (3.9). 

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