ON A QUASILINEAR MEAN FIELD EQUATION WITH AN EXPONENTIAL NONLINEARITY

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Abstract. The mean field equation involving the $N$–Laplace operator and an exponential nonlinearity is considered in dimension $N \geq 2$ on bounded domains with homogeneous Dirichlet boundary condition. By a detailed asymptotic analysis we derive a quantization property in the non-compact case, yielding to the compactness of the solutions set in the so-called non-resonant regime. In such a regime, an existence result is then provided by a variational approach.

1. Introduction
We are concerned with the following quasilinear mean field equation
\begin{equation}
\begin{cases}
-\Delta_N u = \lambda \frac{V e^{u}}{\int_{\Omega} V e^{u} \, dx} & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\end{equation}
on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $\Delta_N u = \text{div}( |\nabla u|^{N-2} \nabla u)$ denotes the $N$–Laplace operator, $V$ is a smooth nonnegative function and $\lambda \in \mathbb{R}$. In the sequel, (1.1) will be referred to as the $N$–mean field equation.
In terms of $\lambda$ or $\rho = \lambda \int_{\Omega} V e^{u}$, the planar case $N=2$ on Euclidean domains or on closed Riemannian surfaces has strongly attracted the mathematical interest, as it arises in conformal geometry [18, 19, 44], in statistical mechanics [16, 17, 20, 46], in the study of turbulent Euler flows [29, 64] and in connection with self-dual condensates for some Chern-Simons-Higgs model [25, 28, 32, 37, 51, 52, 58].

For $N=2$ Brézis and Merle [15] initiated the study of the asymptotic behavior for solutions of (1.1) by providing a concentration-compactness result in $\Omega$ without requiring any boundary condition. A quantization property for concentration masses has been later given in [48], and a very refined asymptotic description has been achieved in [23, 47].

A first natural question concerns the validity of a similar asymptotic behavior in the quasilinear case $N > 2$, where the nonlinearity of the differential operator creates an additional difficulty. The only available result is a concentration-compactness result [2, 61], which provides a too weak compactness property towards existence issues for (1.1). Since a complete classification for the limiting problem
\begin{equation}
\begin{cases}
-\Delta_N U = e^{U} & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} e^{U} < \infty
\end{cases}
\end{equation}
is not available for $N > 2$ (except for extremals of the corresponding Moser-Trudinger’s inequality [43, 50] as opposite to the case $N = 2$ [21], the starting point of Li-Shafrir’s analysis [48] fails and a general quantization property is completely missing. Under a “mild” control on the boundary values of $u$, Y.Y.Li and independently Wolanski have proposed for $N=2$ an alternative approach based on Pohozaev identities, successfully applied also in other contexts [6, 7, 66]. The typical assumption on $V$ is the following:
\begin{equation}
\frac{1}{C_0} \leq V(x) \leq C_0 \quad \text{and} \quad |\nabla V(x)| \leq C_0 \quad \forall x \in \Omega
\end{equation}
for some $C_0 > 0$.

Pushing the analysis of [2, 61] up to the boundary and making use of the above approach, our first main result is the following:

**Theorem 1.1.** Let $u_k \in C^{1,\alpha}(\Omega)$, $\alpha \in (0,1)$, be a sequence of weak solutions to
\begin{equation}
-\Delta_N u_k = V_k e^{u_k} \quad \text{in } \Omega,
\end{equation}
where $V_k$ satisfies (1.3) for all $k \in \mathbb{N}$. Assume that
\begin{equation}
\sup_{k \in \mathbb{N}} \int_{\Omega} e^{u_k} < +\infty
\end{equation}
and
\begin{equation}
\text{osc}_{\partial\Omega} u_k = \sup_{\partial\Omega} u_k - \inf_{\partial\Omega} u_k \leq M
\end{equation}
for some $M \in \mathbb{R}$. Then, up to a subsequence, $u_k$ verifies one of the following alternatives: either

(i) $u_k$ is uniformly bounded in $L^\infty_{loc}(\Omega)$

or

(ii) $u_k \to -\infty$ as $k \to +\infty$ uniformly in $L^\infty_{loc}(\Omega)$

or

(iii) there exists a finite, non-empty set $S = \{p_1, \ldots, p_m\} \subset \Omega$ such that $u_k \to -\infty$ uniformly in $L^\infty_{loc}(\Omega \setminus S)$ and

$$V_k e^{\alpha_k} \to c_N \sum_{i=1}^m \delta_{p_i}$$

weakly in the sense of measures in $\Omega$ as $k \to +\infty$, where $c_N = N \left( \frac{2N}{N-1} \right)^{N-1} \omega_N$ with $\omega_N = |B_1(0)|$. In addition, if $\text{osc}_{\partial\Omega} u_k = 0$ for all $k$, alternatives (i)-(iii) do hold in $\Omega$, with $S \subset \Omega$ in case (iii).

Without an uniform control on the oscillation of $u_k$ on $\partial\Omega$, in general the concentration mass $\alpha_i$ in (1.6) at each $p_i$, $i = 1, \ldots, m$, just satisfies $\alpha_i \geq N^\epsilon \omega_N$, see [2] for details. Moreover, the assumption $\text{osc}_{\partial\Omega} u_k = 0$ is used here to rule out boundary blow-up. For strictly convex domains, one could simply use the moving-plane method to exclude maximum points of $u_k$ near $\partial\Omega$ as in [53]. For $N = 2$ this extra assumption can be removed by using the Kelvin transform to take care of non-convex domains, see [54, 60]. Although $N$–harmonic functions in $\mathbb{R}^N$ are invariant under Kelvin transform, such a property does not carry over to (1.4) due to the nonlinearity of $-\Delta_N$. To overcome such a difficulty, we still make use of the Pohozaev identity near boundary points, to exclude the boundary blow-up as in [56, 62].

Problem 1.2 has a $(N+1)$–dimensional family of explicit solutions $U_{\epsilon,p}(x) = U\left(\frac{|x|}{\epsilon}\right) - N \log \epsilon$, $\epsilon > 0$ and $p \in \mathbb{R}^N$, where

$$U(x) = \log \frac{F_N}{(1 + |x|^{-N})^N}, \quad x \in \mathbb{R}^N,$$

with $F_N = N \left( \frac{2N}{N-1} \right)^{N-1}$. As $\epsilon \to 0^+$, a description of the blow-up behavior at $p$ is well illustrated by $U_{\epsilon,p}$. Since

$$\int_{\mathbb{R}^N} e^{U_{\epsilon,p}} = c_N,$$

in analogy with Li-Shafrir’s result it is expected that the concentration mass $\alpha_i$ in (1.6) at each $p_i$, $i = 1, \ldots, m$, should be an integer multiple of $c_N$. The additional assumption $\text{osc}_{\partial\Omega} u_k < +\infty$ allows us to prove that all the blow-up points $p_i$, $i = 1, \ldots, m$, are “simple” in the sense $\alpha_i = c_N$

Concerning the $N$-mean field equation (1.1), as a simple consequence of Theorem 1.1 we deduce the following crucial compactness property:

**Corollary 1.2.** Let $\Lambda \subset [0, +\infty) \setminus c_N \mathbb{N}$ be a compact set. Then, there exists a constant $C > 0$ such that $\|u\|_\infty \leq C$ does hold for all $\lambda \in \Lambda$, all weak solution $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, of (1.1) and all $V$ satisfying (1.6).

In the sequel, we will refer to the case $\lambda \neq c_N \mathbb{N}$ as the non-resonant regime. Existence issues can be attacked by variational methods: solutions of (1.1) can be found as critical points of

$$J_\lambda(u) = \frac{1}{N} \int_{\Omega} |\nabla u|^N - \lambda \log \left( \int_{\Omega} V e^u \right), \quad u \in W^{1,N}_0(\Omega).$$

The Moser-Trudinger inequality [33] guarantees that the functional $J_\lambda$ is well-defined and $C^1$–Fréchet differentiable on $W^{1,N}_0(\Omega)$ for any $\lambda \in \mathbb{R}$. Moreover, if $\lambda < c_N$ the functional $J_\lambda$ is coercive and then attains the global minimum. For $\lambda = c_N$, $J_\lambda$ still has a lower bound but is not coercive anymore: in general, in the resonant regime $\lambda \in c_N \mathbb{N}$ existence issues are very delicate. When $\lambda > c_N$ the functional $J_\lambda$ is unbounded both from below and from above, and critical points have to be found among saddle points. Moreover, the Palais-Smale condition for $J_\lambda$ is not globally available, see [33], but holds only for bounded sequences in $W^{1,N}_0(\Omega)$.

The second main result is the following:

**Theorem 1.3.** Assume that the space of formal barycenters $\mathcal{M}_m(\overline{\Omega})$ of $\overline{\Omega}$ with order $m \geq 1$ is non-contractible. Then equation (1.1) has a solution in $C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, for all $\lambda \in (c_N m, c_N (m + 1))$.

For mean-field equations, such a variational approach has been introduced in [33] and fully exploited later by Djadli and Malchiodi [35] in their study of constant Q-curvature metrics on four manifolds. It has revealed to be very powerful in many contexts, see for example [1] [8] [34] [35] and references therein. Alternative approaches are available: the computation of the corresponding Leray-Schauder degree [23] [24], based on a very refined asymptotic analysis of blow-up solutions; perturbative constructions of Lyapunov-Schmidt in the almost resonant regime [5] [21] [22] [11] [17] [21] [22] [50]. For our problem a refined asymptotic analysis for blow-up solutions is still missing, and perturbation arguments are very difficult due to the nonlinearity of $\Delta_N$. A variational approach is the only reasonable way to attack existence issues, and in this way the analytic problem is reduced to a topological one concerning the non-contractibility of a model space, the so-called space of formal barycenters, characterizing the very low sublevels of $J_\lambda$. We refer to Section 4 for a definition
of $\mathcal{B}_m(\Omega)$. To have non-contractibility of $\mathcal{B}_m(\Omega)$ for domains $\Omega$ homotopically equivalent to a finite simplicial complex, a sufficient condition is the non-triviality of the $Z$-homology, see [11]. Let us emphasize that the variational approach produces solutions a.e. $\lambda \in (c_N m, c_N (m + 1))$, $m \geq 1$, and Corollary [12] is crucial to get the validity of Theorem [13] for all $\lambda$ in such a range.

The paper is organized as follows. In Section 2 we show how to push the concentration-compactness analysis [2, 61] up to the boundary, by discussing boundary blow-up and mass quantization. Section 3 is devoted to Theorem 1.3 and some comments concerning $\mathcal{B}_m(\Omega)$. In the appendix, we collect some basic results that will be used frequently throughout the paper.

2. CONCENTRATION-COMPACTNESS ANALYSIS

Even though representation formulas are not available for $\Delta_N$, the Brézis-Merle’s inequality [14] can be extended to $N > 2$ by different means:

**Lemma 2.1.** [2, 61] Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution of

$$-\Delta_N u = f \quad \text{in } \Omega$$

with $f \in L^1(\Omega)$. Let $\varphi$ be a $N$-harmonic function in $\Omega$ with $\varphi = u$ on $\partial \Omega$. Then, for every $\alpha \in (0, \alpha_N)$ there exists a constant $C = C(\alpha, |\Omega|)$ such that

$$\int_\Omega \exp \left[ \frac{|u(x) - \varphi(x)|}{\|f\|_{L^{\frac{N}{N-\alpha}}}} \right] \leq C,$$

where $\alpha_N = \frac{N}{N d_N \omega_N}$ and

$$d_N = \inf_{X \neq Y \in \mathbb{R}^N} \frac{|X|^{N-2}X - |Y|^{N-2}Y}{|X - Y|^N} > 0.$$

In addition, if $u = 0$ on $\partial \Omega$, (2.1) holds with $\alpha_N = \frac{N}{N \omega_N}$.

Under some smallness uniform condition on the nonlinear term, a-priori estimates hold true as follows:

**Lemma 2.2.** Let $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, be a sequence of weak solutions to (1.3), where $V_k$ satisfies (1.3) for all $k \in \mathbb{N}$. Assume that

$$\sup_k \int_{\Omega \cap B_{4R}} V_k e^{u_k} < N^2 d_N \omega_N$$

(2.2)

does hold for some $R > 0$, and $u_k$ satisfies $u_k = c_k$ in $\Omega \cap B_{4R}$, for $c_k \in \mathbb{R}$ if $\partial \Omega \cap B_{4R} \neq \emptyset$. Then

$$\sup_k \|u_k\|_{L^\infty(\Omega \cap B_{4R})} < +\infty.$$

(2.3)

**Proof.** Let $\varphi_k$ be the $N$-harmonic function in $\Omega \cap B_{4R}$ so that $\varphi_k = u_k$ on $\partial(\Omega \cap B_{4R})$. Choosing

$$\alpha \in \left( \frac{1}{\sup_{k} \int_{\Omega \cap B_{4R}} V_k e^{u_k}} \right) \frac{1}{N \omega_N}$$

(2.4)

in view of (2.2), by Lemma 2.1 we get that $e^{(u_k - \varphi_k)}$ is uniformly bounded in $L^q(\Omega \cap B_{4R})$, for some $q > 1$. Since $V_k \geq 0$, by the weak comparison principle we get that $c_k \leq \varphi_k \leq u_k$ in $\Omega \cap B_{4R}$. Since $\varphi_k = c_k$ on $\partial(\Omega \cap B_{4R})$ and

$$\sup_k \|\varphi_k\|_{L^\infty(\Omega \cap B_{4R})} \leq \sup_k \|u_k\|_{L^\infty(\Omega \cap B_{4R})} < +\infty$$

(2.5)

in view of (1.3) and (2.2), by Theorem 1.1 we get that $\varphi_k \leq C_0$ in $\Omega \cap B_{2R}$ uniformly in $k$, for some $C_0$. Since $e^{\varphi_k} \leq e^{C_0} e^{u_k - c_k}$, we get that $e^{\varphi_k}$ is uniformly bounded in $L^q(\Omega \cap B_{2R})$. Since $q > 1$, by Theorem 1.1 we deduce the validity of (2.3) in view of (2.4).

We can now prove our first main result:

**Proof (of Theorem 1.1).**

First of all, by (1.3) for $V_k$ and (1.2) we deduce that $V_ke^{\varphi_k}$ is uniformly bounded in $L^1(\Omega)$. Up to a subsequence, by the Prokhorov Theorem we can assume that $V_ke^{\varphi_k} \rightharpoonup \mu \in \mathcal{M}^+(\overline{\Omega})$ as $k \to +\infty$ in the sense of measures in $\overline{\Omega}$, i.e.

$$\int_{\Omega} V_k e^{\varphi_k} \varphi \to \int_{\Omega} \varphi d\mu \quad \text{as } k \to +\infty \quad \forall \varphi \in C(\overline{\Omega}).$$

A point $p \in \overline{\Omega}$ is said a regular point for $\mu$ if $\mu(\{p\}) < N^N \omega_N$, and let us denote the set of non-regular points as:

$$\Sigma = \{p \in \overline{\Omega}: \mu(\{p\}) \geq N^N \omega_N\}.$$

Since $\mu$ is a bounded measure, it follows that $\Sigma$ is a finite set. We complete the argument through the following five steps.

**Step 1** Letting

$$S = \{p \in \overline{\Omega}: \limsup_{k \to +\infty} \sup_{\partial \Omega \cap B_R(p)} u_k = +\infty \forall R > 0\},$$


there holds $S \cap \Omega = \Sigma \cap \Omega \ (S = \Sigma \text{ if } \text{osc}_{\Omega} u_k = 0 \text{ for all } k)$.

Letting $p_0 \in S$, assume that $p_0 \in \Omega$ or $u_k = c_k$ on $\partial \Omega$ for some $c_k \in \mathbb{R}$. In the latter case, notice that $u_k \geq c_k$ in $\Omega$ in view of the weak comparison principle. Setting

$$
\Sigma' = \left\{ p \in \overline{\Omega} : \mu(p) \geq N d_N \omega_N \right\},
$$

by Lemma 2.2 we know that $p_0 \in \Sigma'$. Indeed, if $p_0 \notin \Sigma'$, then (2.2) would hold for some $R > 0$ small, and then by Lemma 2.2 it would follow that $u_k$ is uniformly bounded from above in $\Omega \cap B_R(p_0)$, contradicting $p_0 \in S$. To show that $p_0 \in \Sigma$, the key point is to recover a good control of $u_k$ on $\partial(\Omega \cap B_R(p_0))$, for some $R > 0$, in order to drop $d_N$. Assume that $p_0 \notin \Sigma$, in such a way that

$$
\sup_k \int_{\Omega \cap B_R(p_0)} V_k e^{u_k} < N^r \omega_N
$$

for some $R > 0$ small. Since $\Sigma'$ is a finite set, up to take $R$ smaller, let us assume that $\partial(\Omega \cap B_R(p_0)) \cap \Sigma' \subset \{ p_0 \}$, and then by compactness we have that

$$
u_k \leq M \quad \text{in } \partial(\Omega \cap B_R(p_0)) \setminus B_R(p_0)
$$

in view of $S \cap \Omega \subset \Sigma' \cap \Omega$ and $S \subset \Sigma'$ if $\text{osc}_{\Omega} u_k = 0$ for all $k$. If $p_0 \in \Omega$, we can also assume that $\overline{B_R(p_0)} \subset \Omega$. If $p_0 \notin \Sigma$, $u_k = c_k$ on $\partial \Omega$ yields to $c_k \leq M$ in view of (2.6). In both cases, we have shown that (2.6) does hold in the stronger way:

$$
u_k \leq M \quad \text{in } \partial(\Omega \cap B_R(p_0)).
$$

Letting $w_k \in W_{0}^{1,N}(\Omega \cap B_R(p_0))$ be the weak solution of

$$
\begin{cases}
-\Delta_N w_k = V_k e^{u_k} & \text{in } \Omega \cap B_R(p_0) \\
w_k = 0 & \text{on } \partial(\Omega \cap B_R(p_0))
\end{cases}
$$

by (2.7) and the weak comparison principle we get that

$$
u_k \leq w_k + M \quad \text{in } \Omega \cap B_R(p_0).
$$

Applying Lemma 2.1 to $w_k$ in view of (2.6), it follows that

$$
\int_{\Omega \cap B_R(p_0)} e^{w_k} \leq e^{qM} \int_{\Omega \cap B_R(p_0)} e^{\xi w_k} \leq C
$$

for all $k$, for some $q > 1$ and $C > 0$. In particular, $u_k^q$ is uniformly bounded in $L^q(\Omega \cap B_R(p_0))$ and $V_k e^{u_k}$ is uniformly bounded in $L^q(\Omega \cap B_R(p_0))$. By Theorem A.1 it follows that $u_k$ is uniformly bounded from above in $\Omega \cap B_R(p_0)$, in contradiction with $p_0 \notin \Sigma$. So, we have shown that $p_0 \in \Sigma$, which yields to $S \cap \Omega \subset \Sigma \cap \Omega$ and $S \subset \Sigma$ if $\text{osc}_{\Omega} u_k = 0$ for all $k$.

Conversely, let $p_0 \in \Sigma$. If $p_0 \notin S$, one could find $R_0 > 0$ so that $u_k \leq M$ in $\Omega \cap B_{R_0}(p_0)$, for some $M \in \mathbb{R}$, yielding to

$$
\int_{\Omega \setminus B_{R_0}(p_0)} V_k e^{u_k} \leq C_0 e^{M} R_0^r, \ R \leq R_0,
$$

in view of (19). In particular, $\mu(\{p_0\}) = 0$, contradicting $p_0 \in \Sigma$. Hence $\Sigma \subset S$, and the proof of Step 1 is complete.

**Step 2** $S \cap \Omega = \emptyset$ ($S = \emptyset$) implies the validity of alternative (i) or (ii) in $\Omega$ (in $\overline{\Omega}$ if $\text{osc}_{\Omega} u_k = 0$ for all $k$).

Since $u_k$ is uniformly bounded from above in $L^\infty_\text{loc}(\Omega)$, then either $u_k$ is uniformly bounded in $L^\infty_\text{loc}(\Omega)$ or there exists, up to a subsequence, a compact set $K \subset \Omega$ so that $\min_{\Omega} u_k \to -\infty$ as $k \to +\infty$. The set $\Omega_k = \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta \right\}$ is a compact connected set so that $K \subset \Omega_k$, for $\delta > 0$ small. Since $u_k \leq M$ in $\Omega$ for some $M > 0$, the function $s_k = M - u_k$ is a nonnegative weak solution of $-\Delta_N s_k = -V_k e^{u_k}$ in $\Omega$. By the Harnack inequality in Theorem A.1 we have that

$$
\max_{\overline{\Omega}} s_k \leq C \left( \min_{\overline{\Omega}} s_k + 1 \right)
$$

in view of

$$
\| V_k e^{u_k} \|_{L^\infty(\Omega)} \leq C_0 e^{M}.
$$

In terms of $u_k$, it reads as

$$
\max_{\overline{\Omega}} u_k \leq M \left( 1 - \frac{1}{C} \right) + 1 + \frac{1}{C} \min_{\overline{\Omega}} u_k \to -\infty
$$

as $k \to +\infty$ for all $\delta > 0$ small, yielding to the validity of alternative (ii) in $\Omega$. Assume in addition that $u_k = c_k$ on $\partial \Omega$ for some $c_k \in \mathbb{R}$. Notice that $c_k \leq u_k \leq M$ in $\Omega$ for all $k$. If alternative (i) does not hold in $\overline{\Omega}$, up to a subsequence, we get that $c_k \to -\infty$. Since $u_k$ is uniformly bounded in $\Omega$, we apply Corollary A.3 to $s_k = u_k - c_k$, a nonnegative solution of $-\Delta_N s_k = V_k e^{u_k}$ with $s_k = 0$ on $\partial \Omega$, to get $s_k \leq M'$ in $\Omega$ for some $M' \in \mathbb{R}$. Hence, $u_k \leq M' + c_k \to -\infty$ in $\Omega$ as $k \to +\infty$, yielding to the validity of alternative (ii) in $\overline{\Omega}$. The proof of Step 2 is complete.
Step 3 \( S \cap \Omega \neq \emptyset \) implies the validity of alternative (iii) in \( \Omega \) (in \( \overline{\Omega} \) if \( \text{osc}_{\text{osc}} u_k = 0 \) for all \( k \)) with \( (1.6) \) replaced by the property:

\[
V_k e^{u_k} \rightharpoonup \sum_{i=1}^{m} \alpha_i \delta_{p_i} \tag{2.8}
\]

weakly in the sense of measures in \( \Omega \) (in \( \overline{\Omega} \)) as \( k \to +\infty \), with \( \alpha_i \geq N^N \omega_N \) and \( S \cap \Omega = \{ p_1, \ldots, p_m \} \) (\( S = \{ p_1, \ldots, p_m \} \)). Let us first consider the case that \( u_k \) is uniformly bounded in \( L^\infty(\Omega \setminus S) \). Fix \( p_0 \in S \) and \( R > 0 \) small so that \( B_R(p_0) \setminus S = \{ p_0 \} \). Arguing as in (2.4)-(2.5), we have that \( u_k \geq m \) on \( \partial(\Omega \cap B_R(p_0)) \) for some \( m \in \mathbb{R} \). Since \( u_k \) is uniformly bounded in \( L^\infty(\Omega \setminus S) \), by Theorem A.1 it follows that \( u_k \) is uniformly bounded in \( C^{\frac{1}{2}}_{\text{loc}}(\Omega \setminus B_R(p_0) \setminus \{ p_0 \}) \), for some \( \alpha \in (0, 1) \), and, up to a subsequence and a diagonal process, we can assume that \( u_k \to u \) in \( C^{\frac{1}{2}}_{\text{loc}}(\Omega \setminus B_R(p_0) \setminus \{ p_0 \}) \) as \( k \to +\infty \). By (1.6) on each \( V_k \), we can also assume that \( V_k \to V \) uniformly in \( \Omega \) as \( k \to +\infty \). Hence, there holds

\[
V_k e^{u_k} \rightharpoonup \mu = V e^\varphi \, dx + \alpha_0 \delta_{p_0} \tag{2.9}
\]

weakly in the sense of measures in \( \Omega \cap B_R(p_0) \) as \( k \to +\infty \), where \( \alpha_0 \geq N^N \omega_N \). Since

\[
\lim_{k \to +\infty} \int_{\Omega \cap B_R(p_0)} V_k e^{u_k} = \int_{\Omega \cap B_R(p_0)} V e^\varphi + \alpha_0 > \alpha_0
\]

in view of (2.10), for \( k \) large we can find a unique \( 0 < r_k < R \) so that

\[
\int_{\Omega \cap B_{r_k}(p_0)} V_k e^{u_k} = \alpha_0. \tag{2.10}
\]

Notice that \( r_k \to 0 \) as \( k \to +\infty \). Indeed, if \( r_k \geq \delta > 0 \) were true along a subsequence, one would reach the contradiction

\[
\alpha_0 \geq \int_{\Omega \cap B_{r_k}(p_0)} V_k e^{u_k} \to \int_{\Omega \cap B_{r_k}(p_0)} V e^\varphi + \alpha_0 > \alpha_0
\]

as \( k \to +\infty \) in view of (2.9)-(2.10). Denoting by \( \chi_A \) the characteristic function of a set \( A \), we have the following crucial property:

\[
\chi_{B_{r_k}(p_0)} V e^{u_k} \rightharpoonup \alpha_0 \delta_{p_0}
\]

weakly in the sense of measures in \( \Omega \cap B_R(p_0) \) as \( k \to +\infty \), as it easily follows by (2.10) and \( \lim_{k \to +\infty} \frac{r_k}{r} = 0 \).

We can now specialize the argument to deal with the case \( p_0 \in S \setminus \Omega \). Assume that \( R \) is small so that \( B_R(p_0) \subset \Omega \). Letting \( w_k \in W_0^{1,N}(B_R(p_0)) \) be the weak solution of

\[
\begin{cases}
-\Delta_N w_k = \chi_{B_{r_k}(p_0)} V_k e^{u_k} & \text{in } B_R(p_0) \\
w_k = 0 & \text{on } \partial B_{r_k}(p_0),
\end{cases}
\]

by the weak comparison principle there holds \( 0 \leq w_k \leq u_k - m \leq 0 \) in \( B_R(p_0) \) in view of \( 0 \leq \chi_{B_{r_k}(p_0)} V_k e^{u_k} \leq V e^{u_k} \). Arguing as before, up to a subsequence, by Theorem A.3 we can assume that \( w_k \rightharpoonup w \) in \( C^{\frac{1}{2}}_{\text{loc}}(B_R(p_0) \setminus \{ p_0 \}) \) as \( k \to +\infty \), where \( w \geq 0 \) is a \( N \)-harmonic and continuous function in \( B_R(p_0) \setminus \{ p_0 \} \) which solves

\[
-\Delta_N w = \alpha_0 \delta_{p_0} \quad \text{in } B_R(p_0)
\]

in a distributional sense. By Theorem A.3 we deduce that

\[
w \geq (N \omega_N)^{-1} \alpha_0 \frac{1}{\log \frac{1}{|x - p_0|} + C} \frac{1}{\log \frac{1}{|x - p_0|} + C} \quad \text{in } B_r(p_0)
\]

in view of \( \alpha_0 \geq N^N \omega_N \), for some \( C \in \mathbb{R} \) and \( 0 < r \leq \min \{ 1, R \} \). Since

\[
\int_{B_r(p_0)} e^{u_k} \leq e^{-m} \sup_k \int_{\Omega} e^{u_k} < +\infty
\]

in view of (2.11), as \( k \to +\infty \) we get that \( \int_{B_r(p_0)} e^w < +\infty \), in contradiction with (2.11):

\[
\int_{B_r(p_0)} e^w \geq e^C \int_{B_r(p_0)} \frac{1}{|x-p_0|^N} = +\infty.
\]

Since \( u_k \) is uniformly bounded from above and not from below in \( L^\infty(\Omega \setminus S) \), there exists, up to a subsequence, a compact set \( K \subset \Omega \setminus S \) so that \( \inf_K u_k \to -\infty \) as \( k \to +\infty \). Arguing as in Step 2 by simply replacing \( \text{dist}(\cdot, \Omega) \) with \( \text{dist}(\cdot, \partial \Omega \setminus S) \), we can show that \( u_k \to -\infty \) in \( L^\infty(\Omega \setminus S) \) as \( k \to +\infty \), and (2.8) does hold in \( \Omega \) with \( \{ p_1, \ldots, p_m \} = S \cap \Omega \).

If in addition \( u_k \in C^{1,\alpha} \) on \( \partial \Omega \) for some \( \alpha \in (0, 1) \), we can argue as in the end of Step 2 (by using Theorem A.2 instead of Corollary A.2) to get that \( u_k \to -\infty \) in \( L^\infty(\Omega \setminus S) \) as \( k \to +\infty \), yielding to the validity of (2.8) in \( \Omega \) with \( \{ p_1, \ldots, p_m \} = S \). The proof of Step 3 is complete.

To proceed further we make use of Pohozaev identities. Let us emphasize that \( u_k \in C^{1,\alpha}(\overline{\Omega}) \), \( \alpha \in (0, 1) \), and the classical Pohozaev identities usually require more regularity. In (2.7) a self-contained proof is provided in the quasilinear case, which reads in our case as:
Lemma 2.3. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain, $f$ be a locally Lipschitz continuous function and $0 \leq V \in C^1(\overline{\Omega})$. Then, there holds
\[
\int_{\Omega} \left[ N \, V + \langle x - y, \nabla V \rangle \right] F(u) = \int_{\partial \Omega} V \, F(u)(x - y, \nu) + |\nabla u|^{N-2} \langle x - y, \nabla u \rangle \, \partial_s u - \frac{|\nabla u|^N}{N} (x - y, \nu)
\]
for all weak solution $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, of $-\Delta_N u = V f(u)$ in $\Omega$ and all $y \in \mathbb{R}^N$, where $F(t) = \int_{-\infty}^t f(s) \, ds$ and $\nu$ is the unit outward normal vector at $\partial \Omega$.

Thanks to Lemma 2.3 in the next two Steps we can now describe the interior blow-up phenomenon and exclude the occurrence of boundary blow-up:

**Step 4** If $\text{osc}_{\Omega} u_k \leq M$ for some $M \in \mathbb{R}$, then $\alpha_i = c_N$ for all $p_i \in S \cap \Omega$.

Since $0 \leq u_k - \text{inf}_{\Omega} u_k \leq M$ on $\partial \Omega$, we have that $s_k = u_k - \text{inf}_{\Omega} u_k \geq 0$ satisfies
\[
\begin{cases}
-\Delta_N s_k = W_k e^{s_k} & \text{in } \Omega \\
0 \leq s_k \leq M & \text{on } \partial \Omega,
\end{cases}
\]
where $W_k = V_k e^{\text{inf}_{\Omega} u_k}$. Letting now $\varphi_k$ be the $N$-harmonic function in $\Omega$ with $\varphi_k = s_k$ on $\partial \Omega$, by the weak comparison principle we have that $0 \leq \varphi_k \leq M$ in $\Omega$. Since $\text{sup}_{\Omega} \int_{\Omega} W_k e^{s_k} < +\infty$ and $e^{s_k} \geq \delta s^k$ for all $s \geq 0$, for some $\delta > 0$, by Lemma 2.3 we deduce that $s_k \geq \varphi_k$ and then $s_k$ are uniformly bounded in $L^\infty(\Omega)$. Since $W_k e^{s_k} = V_k e^{u_k}$ is uniformly bounded in $L^\infty(\Omega \setminus S)$, by Theorem A.2 it follows as in Step 3 that, up to a subsequence, $s_k \to s$ in $C^{1,\alpha}_\text{loc}(\Omega \setminus S)$. Fix $p_0 \in S \cap \Omega$ and take $R_0 > 0$ small so that $B = B_{R_0}(p_0) \subset \subset \Omega$ and $\overline{B} \cap S = \{ p_0 \}$. The limiting function $s \geq 0$ is a $N$-harmonic and continuous function in $B \setminus \{ p_0 \}$ which solves
\[-\Delta_N s = \alpha_0 \, \delta \, \partial_{\nu} \partial_s s \quad \text{in } B,
\]
where $\alpha_0 \geq N^N \omega_N$. By Theorem A.2 we have that $s = \frac{\alpha_0}{N \omega_N} \Gamma(|x - p_0|) + H$, where $H \in L^\infty(\overline{B})$ does satisfy
\[
\lim_{x \to p_0} \frac{|x - p_0|}{|\nabla H(x)|} = 0.
\]

Applying the Pohozaev identity to $s_k$ on $B_{R}(p_0)$, $0 < R \leq R_0$, with $y = p_0$, we get that
\[
\int_{B_{R}(p_0)} [NW_k + \langle x - p_0, \nabla W_k \rangle] e^{s_k} = R \int_{\partial B_{R}(p_0)} \left[ W_k e^{s_k} + |\nabla s_k|^{N-2} (\partial_s s_k)^2 \right] (\partial_{\nu} \partial_s s_k)^2 - \frac{|\nabla s_k|^N}{N}.
\]
Since $S \cap \Omega \neq \emptyset$ and $V_k e^{s_k} = W_k e^{s_k}$, by Step 3 we get that $\int_{\partial B_{R}(p_0)} W_k e^{s_k} \to 0$ and
\[
\int_{B_{R}(p_0)} [NW_k + \langle x - p_0, \nabla W_k \rangle] e^{s_k} = N \int_{B_{R}(p_0)} V_k e^{u_k} + O \left( \int_{B_{R}(p_0)} |x - p_0| V_k e^{u_k} \right) \to N \alpha_0
\]
as $k \to +\infty$. Letting $k \to \infty$ we get that
\[
N \alpha_0 = R \int_{\partial B_{R}(p_0)} \frac{1}{|\nabla H|} - \left( \frac{\alpha_0}{N \omega_N} \right)^N \frac{|x - p_0|^2}{|x - p_0|^2} \left[ \frac{\alpha_0}{N \omega_N} \right] \frac{1}{|x - p_0|^2} + O \left( \frac{|x - p_0| |\nabla H| + |\nabla H|^2}{|x - p_0|^2} \right)^2
\]
in view of $s_k \to s = \alpha_0 \frac{1}{N \omega_N} \Gamma(|x - p_0|) + H$ in $C^{1,\alpha}_\text{loc}(B \setminus \{ p_0 \})$ as $k \to +\infty$. Letting $R \to 0$ we get that
\[
N \alpha_0 = \frac{N - 1}{N} \frac{\alpha_0}{N \omega_N} \frac{1}{N} \omega_N = c_N
\]
in view of A.2. Therefore, there holds
\[
\alpha_0 = N \left( \frac{N^2}{N - 1} \right)^{N-1} \omega_N = c_N
\]
for all $p_0 \in S \cap \Omega$, and the proof of Step 4 is complete.

**Step 5** If $\text{osc}_{\partial \Omega} u_k = 0$ for all $k$, then $S \subseteq \Omega$.

Assume now that $u_k = c_k$ on $\partial \Omega$. Since by the weak comparison principle $c_k \leq u_k$ in $\Omega$ for all $k$, the function $s_k = u_k - c_k$ is a nonnegative weak solution of
\[
\begin{cases}
-\Delta_N s_k = W_k e^{s_k} & \text{in } \Omega \\
s_k = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( W_k = V_k e^{s_k} \). Since \( W_k e^{s_k} = V_k e^{s_k} \) is uniformly bounded in \( L^1(\Omega) \), by Lemma 2.1 we have that \( s_k \) is uniformly bounded in \( L^\infty(\Omega) \). Since \( W_k e^{s_k} = V_k e^{s_k} \) is uniformly bounded in \( L^\infty_m(\Omega \setminus S) \), arguing as in Step 3, by Theorem \( \ref{thm:uniformly_bounded} \) it follows that \( s_k \) is uniformly bounded in \( C^{1,\alpha}_{loc}(\Omega \setminus S) \), \( \alpha \in (0,1) \), and, up to a subsequence, \( s_k \to s \) in \( C^{1,\alpha}_{loc}(\Omega \setminus S) \). We claim that \( s \in C^1(\overline{\Omega}) \).

If \( c_k \to -\infty \), we have that \( s \in C^{1,\alpha}_{loc}(\Omega \setminus S) \) is a nonnegative \( N \)-harmonic function in \( \Omega \setminus S \) with \( s = 0 \) on \( \partial \Omega \setminus S \). By Theorem \( \ref{thm:existence} \) we deduce that \( s = 0 \) in \( \Omega \), and then \( s \in C^1(\overline{\Omega}) \). Up to a subsequence, we can now assume that \( c_k \to c \in \mathbb{R} \) as \( k \to +\infty \) and \( S = \{ p_1, \ldots, p_m \} \subseteq \partial \Omega \) in view of Step 3. By \( \ref{thm:existence} \) \( s \in W_0^{1,q}(\Omega) \) for all \( q < N \) and is a distributional solution of

\[
\begin{cases}
-\Delta_N s = W e^s & \text{in } \Omega \\
0 & \text{on } \partial \Omega
\end{cases}
\]

(referred to as SOLA, Solution Obtained as Limit of Approximations), where \( W = V e^c \) and \( W e^s \in L^1(\Omega) \). By considering different \( L^1 \)-approximations or even \( L^1 \)-weak approximations of \( W e^s \in L^1(\Omega) \) one always get the same limiting SOLA \( \varphi \), which is then unique in the sense explained right now. Unfortunately, the sequence \( W_k e^{s_k} \) does not converge \( L^1 \)-weakly to \( \varphi \) as \( k \to +\infty \) because it keeps track that some mass is concentrating near the boundary points \( p_1, \ldots, p_m \). Given \( p = p_i \in S \) and \( \alpha = \alpha_i \), arguing as in \( \ref{eq:contradiction} \) we can find a radius \( r_k \to 0 \) as \( k \to +\infty \) so that

\[
\int_{B_r(p)} W_k e^{s_k} = \alpha.
\]

Let \( w_k \in W^{1,N}_0(\Omega \cap B_R(p)) \) be the weak solution of

\[
\begin{cases}
-\Delta_N w_k = \chi_{\Omega \cap B_R(p)} W_k e^{s_k} & \text{in } \Omega \cap B_R(p) \\
0 & \text{on } \partial(\Omega \cap B_R(p)),
\end{cases}
\]

where \( R < \frac{1}{2} \text{dist}(p, S \setminus \{p\}) \). Arguing as in Step 3, up to a subsequence, we have that \( w_k \to w \in C^1_{loc}(\Omega \cap B_R(p)) \) as \( k \to +\infty \), where \( w \) is \( N \)-harmonic and continuous in \( \Omega \cap B_R(p) \). If \( w > 0 \) in \( \Omega \cap B_R(p) \), by \( \ref{thm:maximum} \) \( \ref{thm:non_degeneracy} \) we have that

\[
\lim_{r \to 0^+} w(\sigma r + p) = -(\sigma, \nu(p))<0
\]

uniformly for \( \sigma \) with \( (\sigma, \nu(p)) \leq -\delta < 0 \). Thanks to \( \ref{eq:contradiction} \), as in Step 3 we still end up with the contradiction \( \int_{\Omega \cap B_R(p)} e^w = +\infty \). Therefore, by the strong maximum principle we necessarily have that \( w = 0 \) in \( \Omega \cap B_R(p) \). Since \( w_k \) is the part of \( s_k \) which carries the information on the concentration phenomenon at \( p \) and tends to disappear as \( k \to +\infty \), we can expect that \( s_k \) in the limit does not develop any singularities. We aim to show that \( e^s \in L^1(\Omega \cap B_R(p)) \) for all \( q \geq 1 \), by mimicking some arguments in \( \ref{thm:existence} \). Letting \( \varphi_k \) be the \( N \)-harmonic extension in \( \Omega \cap B_R(p) \) of \( s_k |_{\partial(\Omega \cap B_R(p))} \), for \( M, a > 0 \) we have that

\[
\int_{\Omega \cap B_R(p)} \left( |\nabla s_k|^N - 2 |\nabla w_k|^N - 2 |\nabla \varphi_k|^N - 2 |\nabla \varphi_k| \nabla [T_{M+a}(s_k - w_k - \varphi_k) - T_M(s_k - w_k - \varphi_k)] \right) = \int_{\Omega \cap B_R(p)} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}) [T_{M+a}(s_k - w_k - \varphi_k) - T_M(s_k - w_k - \varphi_k)] \leq a \int_{|s_k - w_k - \varphi_k| > M} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}),
\]

where the truncation operator \( T_M, M > 0 \), is defined as

\[
T_M(u) = \begin{cases} -M & \text{if } u < -M \\ u & \text{if } |u| \leq M \\ M & \text{if } u > M. \end{cases}
\]

The crucial property we will take advantage of is the following:

\[
\sup_k \int_{|s_k - w_k - \varphi_k| > M} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}) \to 0 \quad \text{as } M \to +\infty.
\]

Indeed, by \( \ref{thm:existence} \) notice that, up to a subsequence, we can assume that \( \varphi_k \to \varphi \) in \( C^1(\Omega \cap B_R(p)) \) as \( k \to +\infty \), where \( \varphi \) is the \( N \)-harmonic function in \( \Omega \cap B_R(p) \) with \( \varphi = s \) on \( \partial(\Omega \cap B_R(p)) \). Since \( s_k - w_k - \varphi_k \to s - \varphi \) uniformly in \( \Omega \cap (B_R(p) \setminus B_r(p)) \) as \( k \to +\infty \) for any given \( r \in (0, R) \), we can find \( M_r > 0 \) large so that

\[
\cup_k \{ |s_k - w_k - \varphi_k| > M \} \subset \Omega \cap B_r(p) \quad \forall M \geq M_r,
\]

and then

\[
\sup_k \int_{|s_k - w_k - \varphi_k| > M} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}) \leq \sup_k \int_{\Omega \cap B_r(p)} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}) \to 0
\]

for all \( M \geq M_r \). Since by \( \ref{thm:existence} \) and \( \ref{thm:non_degeneracy} \)

\[
\int_{\Omega \cap B_r(p)} (1 - \chi_{\Omega \cap B_R(p)} W_k e^{s_k}) \to \int_{\Omega \cap B_r(p)} W e^s
\]
as \( k \to +\infty \) and \( We^+ \in L^1(\Omega) \), for all \( \epsilon > 0 \) we can find \( r_\epsilon > 0 \) small so that

\[
\sup_k \int_{\Omega \cap B_{r_\epsilon}(p)} (1 - \chi_{\Omega \cap B_{r_\epsilon}(p)}) W_k e^{\epsilon k} \leq \epsilon,
\]
yielding to the validity of (2.17). Inserting (2.11) into (2.10) we get that, for all \( \epsilon > 0 \), there exists \( M_\epsilon \) so that

\[
\int_{\{ M < |s_k - w_k| \leq M + \epsilon \}} (|\nabla s_k|^{-2} |\nabla s_k - |\nabla w_k|^{-2} |\nabla w_k| - |\nabla \varphi_k|^{-2} |\nabla \varphi_k|) (s_k - w_k - \varphi_k) \leq \alpha \epsilon
\]
for all \( M \geq M_\epsilon \) and \( \alpha > 0 \). Recall that \( w_k \to 0 \), \( s_k \to s \) in \( C^1_{\text{loc}}(\Omega \cap B_R(p) \setminus \{p\}) \) and in \( W^{1,q}(\Omega \cap B_R(p)) \) for all \( q < N \) as \( k \to +\infty \) in view of [12, 13]. Since

\[
(\nabla s_k|^{-2} |\nabla s_k - |\nabla w_k|^{-2} |\nabla w_k|) (s_k - w_k - \varphi_k) \geq 0
\]
and \( \nabla \varphi_k \) behaves well, we can let \( k \to +\infty \) in (2.18) and by the Fatou Lemma get

\[
\frac{dN}{a} \int_{\{ M < |s - \varphi| \leq M + \epsilon \}} |\nabla (s - \varphi)|^N \leq \frac{1}{\alpha} \int_{\{ M < |s - \varphi| \leq M + \epsilon \}} (|\nabla s|^{-2} |\nabla s - |\nabla \varphi|^{-2} |\nabla \varphi|) (s - \varphi) \leq \epsilon
\]
for some \( dN > 0 \) and all \( M \geq M_\epsilon \). Introducing \( H_{M,a}(s) = T_{M,a}^{-1}(s - \varphi) - T_{M,a}^{-1}(s - \varphi) \) and the distribution \( \Phi_{s,\varphi}(M) = |x \in \Omega \cap B_R(p) : |s - \varphi(x)| > M \} \) of \( |s - \varphi(0) \), we have that

\[
\Phi_{s,\varphi}(M + a)^{\frac{1}{N-1}} \leq \left( \int_{\Omega \cap B_R(p)} |H_{M,a}(s)|^N \right)^{\frac{1}{N}} \leq (N^N \omega_N)^{-\frac{1}{N-1}} \int_{\Omega \cap B_R(p)} \nabla H_{M,a}(s)
\]
in view of the Sobolev embedding \( W^{1,1}_0(\Omega \cap B_R(p)) \to L^{\frac{N}{N-1}}(\Omega \cap B_R(p)) \) with sharp constant \( (N^N \omega_N)^{-\frac{1}{N-1}} \), see [39]. By the Hölder inequality and (2.19) we then deduce that

\[
\Phi_{s,\varphi}(M + a) \leq \frac{(N^N dN \omega_N)^{-\frac{1}{N-1}}}{a} \Phi_{s,\varphi}(M) - \Phi_{s,\varphi}(M + a)
\]
for all \( M \geq M_\epsilon \). By letting \( a \to 0^+ \) it follows that

\[
\Phi_{s,\varphi}(M) \leq -\frac{(N^N dN \omega_N)^{-\frac{1}{N-1}}}{a} \Phi_{s,\varphi}(M)
\]
for a.e. \( M \geq M_\epsilon \), and by integration in \( \{ M \geq M_\epsilon \} \)

\[
\Phi_{s,\varphi}(M) \leq \| \Omega \cap B_R(p) \| \exp \left( -\frac{(N^N dN \omega_N)^{-\frac{1}{N-1}}}{a} M \right)
\]
for all \( M \geq M_\epsilon \), in view of \( \Phi_{s,\varphi}(M) \leq \| \Omega \cap B_R(p) \| \). Given \( q \geq 1 \) we can argue as follows:

\[
\int_{\Omega \cap B_R(p)} e^{q|s - \varphi|} - \| \Omega \cap B_R(p) \| = q \int_{\Omega \cap B_R(p)} d\| x \| \frac{|s(x) - \varphi(x)|}{q} e^{qM} dM = q \int_0^\infty e^{qM} \Phi_{s,\varphi}(M) dM \leq \| \Omega \cap B_R(p) \| \left[ e^{qM} + q \int_{M_\epsilon}^\infty \exp \left( q - \frac{(N^N dN \omega_N)^{-\frac{1}{N-1}} M \right) \right] dM < +\infty
\]
by taking \( \epsilon \) sufficiently small. Since \( \varphi \in C^1(\Omega \cap B_R(p)) \), we get that \( e^s \) is a \( L^q \)-function near any \( p \in \Omega \), and then \( e^s \in L^q(\Omega) \) for all \( q \geq 1 \). By the uniqueness result in [39] and by Theorems A.1 A.4 we get that \( s \in C^{1,\alpha}(\Omega) \), for some \( \alpha \in (0, 1) \).

**Remark 2.4.** The proof of \( s \in C^{1,\alpha}(\Omega) \), \( \alpha \in (0, 1) \), might be carried over in a shorter way. Indeed, the function \( We^+ \in L^1(\Omega) \) can be approximated either in a strong \( L^1 \)-sense or in a weak measure-sense. In the former case, the limiting function \( z \) is an entropy solution of

\[
\begin{cases}
-\Delta_N z = We^+ & \text{in } \Omega \\
z = 0 & \text{on } \partial \Omega,
\end{cases}
\]

while in the latter we end up with \( s \) by choosing \( We^+ \) as the approximation in measure-sense. As consequence of the impressive uniqueness result in [39], \( s = z \) and then \( s \) is an entropy solution of (2.13) (see [12, 13] for the definition of entropy solution). Lemma 2.12 is proved in [2] for entropy solutions, and has been used there, among other things, to show that a solution \( s \) of (2.13) is necessarily in \( C^{1,\alpha}(\Omega) \), for some \( \alpha \in (0, 1) \). We have preferred a longer proof to give a self-contained argument which does not require to introduce special notions of distributional solutions (like SOLA, entropy and renormalized solutions, just to quote some of them).
Fix any \( p_0 \in \partial \Omega \) and take \( R_0 > 0 \) small so that \( B_{R_0}(p_0) \cap S = \{ p_0 \} \). Setting \( y_k = p_0 + \rho_k \nu(p_0) \) with \( 0 < R \leq R_0 \) and

\[
\rho_k = \int_{\partial \Omega \setminus B_R(p_0)} \langle x - p_0, \nu \rangle |\nabla u_k|^N, \\
\int_{\partial \Omega \setminus B_R(p_0)} \langle \nu(p_0), \nu \rangle |\nabla u_k|^N,
\]

we have that

\[
\int_{\partial \Omega \setminus B_R(p_0)} \langle x - y_k, \nu \rangle |\nabla u_k|^N = 0. \tag{2.20}
\]

Up to take \( R_0 \) smaller, we can assume that \( |\rho_k| \leq 2R \). Applying Lemma 2.3 to \( s_k \) on \( \Omega \cap B_R(p_0) \) with \( y = y_k \), we obtain that

\[
\int_{\Omega \setminus B_R(p_0)} [NW_k + \langle x - y_k, \nabla W_k \rangle]e^{\alpha k} = \int_{\partial \Omega \setminus B_R(p_0)} W_k e^{\alpha k} \langle x - y_k, \nu \rangle + \int_{\partial \Omega \setminus B_R(p_0)} \left[ |\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k \rangle \partial_x s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu \rangle \right]. \tag{2.21}
\]

We would like to let \( k \to +\infty \), but \( \partial(\Omega \cap B_R(p_0)) \) contains the portion \( \partial \Omega \cap B_R(p_0) \) where the convergence \( s_k \to s \) might fail. The clever choice of \( \rho_k, R \), as illustrated by \( \tag{2.22} \), leads to

\[
\int_{\partial \Omega \setminus B_R(p_0)} \left[ |\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k \rangle \partial_x s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu \rangle \right] = (1 - \frac{1}{N}) \int_{\partial \Omega \setminus B_R(p_0)} |\nabla u_k|^N \langle x - y_k, \nu \rangle = 0
\]

in view of \( \nabla s_k = \nabla u_k \) and \( \nabla s_k = -|\nabla s_k| \nu \) on \( \partial \Omega \) by means of \( s_k \to 0 \) on \( \partial \Omega \). Hence, \( \tag{2.22} \) reduces to

\[
N \int_{\Omega \setminus B_R(p_0)} V_k e^{\alpha k} = - \int_{\Omega \setminus B_R(p_0)} \langle x - y_k, \nabla V_k \rangle V_k e^{\alpha k} + \int_{\partial \Omega \setminus B_R(p_0)} V_k e^{\alpha k} \langle x - y_k, \nu \rangle + \int_{\partial \Omega \setminus B_R(p_0)} \left[ |\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k \rangle \partial_x s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu \rangle \right]. \tag{2.22}
\]

Since \( |x - y_k| \leq 3R \) and \( |\nabla V_k| \leq C_0^2 \) in \( \Omega \cap B_R(p_0) \) in view of \( \tag{1.3} \), by letting \( k \to +\infty \) in \( \tag{2.22} \) we get that

\[
N \mu(\Omega \cap B_R(p_0)) \leq 3R C_0^2 \mu(\Omega \cap B_R(p_0)) + 3C_0 \mu \frac{M}{\mu(\partial \Omega \cap B_R(p_0))} + 3R(1 + \frac{1}{N}) \int_{\partial \Omega \setminus B_R(p_0)} |\nabla s_k|^N
\]

in view of \( s_k \to s \) in \( C^{1,0}(\overline{\Omega} \setminus S) \). Since \( s \in C^1(\overline{\Omega}) \), by letting \( R \to 0 \) we deduce that \( \mu(\{ p_0 \}) = 0 \), and then \( p_0 \notin \Sigma = S \). Since this is true for all \( p_0 \in \partial \Omega \), we have shown that \( S \subset \Omega \), and the proof of Step 5 is complete.

The combination of the previous 5 Steps provides us with a complete proof of Theorem 1.1.

\[
\square
\]
3. A general existence result

The Moser-Trudinger inequality \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] states that, for some \( C_0 > 0 \), there holds

\[ \int_\Omega \exp(\alpha |u|^{N/n}) \, dx \leq C_0 \]

(3.1)

for all \( u \in W^{1,N}_0(\Omega) \) with \( \|u\|_{W^{1,N}_0(\Omega)} \leq 1 \) and all \( \alpha \leq \alpha_N = (N^N \omega_N)^{1/N} \), whereas \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \]

is false when \( \alpha > \alpha_N \). A simple consequence of \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \]

always referred to as the Moser-Trudinger inequality, is the following:

\[ \log \left( \int_\Omega e^{\alpha u} \, dx \right) \leq \frac{1}{Nc_N} \|u\|^N_{W^{1,N}_0(\Omega)} + \log C_0 \]

(3.2)

for all \( u \in W^{1,N}_0(\Omega) \), where \( c_N \) is defined in \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \]. Indeed, \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] follows by \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] by noticing

\[ u \leq \frac{N \alpha_N}{N-1} \left( \frac{N}{N-1} \right) \left( \frac{|u|^{N-1}}{N-1} \right) \leq \frac{1}{Nc_N} \|u\|^N_{W^{1,N}_0(\Omega)} + \alpha_N \left( \frac{1}{\|u\|_{W^{1,N}_0(\Omega)}} \right)^{N-1} \]

in view of the Young’s inequality. By \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] it follows that:

\[ J_\lambda(u) \geq \frac{1}{N} \left( 1 - \frac{\lambda}{c_N} \right) \|u\|^N_{W^{1,N}_0(\Omega)} - \lambda \log(C_0 C_\Omega) \]

for all \( u \in W^{1,N}_0(\Omega) \) in view of \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \], where \( J_\lambda \) is given in \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \]. Hence, \( J_\lambda \) is bounded from below for \( \lambda \leq c_N \) and coercive for \( \lambda \leq c_N \). Since the map \( u \in W^{1,N}_0(\Omega) \to V e^{\alpha u} \in L^2(\Omega) \) is compact in view of \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] and the embedding \( W^{1,N}_0(\Omega) \hookrightarrow L^2(\Omega) \) is compact, for \( \lambda < c_N \) we have that \( J_\lambda \) attains the global minimum in \( W^{1,N}_0(\Omega) \), and then \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] is solvable. In Theorem \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] we just consider the difficult case \( \lambda > c_N \). Notice that a solution \( u \in W^{1,N}_0(\Omega) \) of \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] belongs to \( C^{1,\alpha}(\Omega) \) for some \( \alpha \in (0,1) \), in view of \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] and Theorems \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \].

The constant \( \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \) is optimal as it follows by evaluating the inequality along

\[ U \left( \frac{x-p}{\epsilon} \right) - \frac{N^2}{N-1} \log \epsilon, \quad p \in \Omega, \]

as \( \epsilon \to 0 \), up to make a cut-off away from \( p \) so to have a function in \( W^{1,N}_0(\Omega) \). The function \( U \) is given in \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] and, as already mentioned in the Introduction, satisfies

\[ \int_{\mathbb{R}^N} e^U = c_N. \]

Indeed, the equation \(-\Delta U = e^U \) does hold pointwise in \( \mathbb{R}^N \setminus \{0\} \), and then can be integrated in \( B_R(0) \setminus B_r(0) \), \( 0 < r < R \), to get

\[ \int_{B_R(0) \setminus B_r(0)} e^U = \int_{\partial B_R(0)} |\nabla U|^N (\nabla U, \nu) + \int_{\partial B_r(0)} |\nabla U|^{N-2} (\nabla U, \nu), \]

where \( \nu(x) = \frac{x}{|x|} \). Letting \( \epsilon \to 0 \) and \( R \to +\infty \), we get that

\[ \int_{\mathbb{R}^N} e^U = N \left( \frac{N^2}{N-1} \right)^{N-1} \omega_N = c_N \]

in view of

\[ \nabla U = -\frac{N^2}{N-1} \frac{|x|^{N-2} x}{|x|^{N-2}}. \]

Since \( \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \) is optimal, the functional \( J_\lambda \) is unbounded from below for \( \lambda > c_N \), and our goal is to develop a global variational strategy to find a critical point of saddle type. The classical Morse theory states that a sublevel is a deformation retract of an higher sublevel unless there are critical points in between, and the crucial assumption on the functional is the validity of the so-called Palais-Smale condition. Unfortunately, in our context such assumption fails since \( J_\lambda \) admits unbounded Palais-Smale sequences for \( \lambda \geq c_N \); see \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \]. This technical difficulty can be overcome by using a method introduced by Struwe that exploits the monotonicity of the functional \( \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \) in \( \lambda \). An alternative approach has been found in \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \], which provides a deformation between two sublevels unless \( J_\lambda \) has critical points in the energy strip for some sequence \( \lambda_k \to \lambda \). Thanks to the compactness result in Corollary \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] and the a-priori estimates in Theorem \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] we have at hands the following crucial tool:

**Lemma 3.1.** Let \( \lambda \in (c_N, +\infty) \setminus c_NN \). If \( J_\lambda \) has no critical levels \( u \) with \( a \leq J_\lambda(u) \leq b \), then \( J_\lambda^t \) is a deformation retract of \( J_\lambda \), where

\[ J_\lambda^t = \{ u \in W^{1,N}_0(\Omega) : J_\lambda(u) \leq t \}. \]

To attack existence issues for \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] when \( \lambda \in (c_N, +\infty) \setminus c_NN \), it is enough to find any two sublevels \( J_\lambda^a \) and \( J_\lambda^b \) which are not homotopically equivalent.

Hereafter, the parameter \( \lambda \) is fixed in \( (c_N, +\infty) \setminus c_NN \). By Corollary \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] and Theorem \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] we have that \( J_\lambda \) does not have critical points with large energy. Exactly as in \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \], Lemma \[ \frac{1}{\alpha \sqrt{\pi N}} \leq C_0 \] can be used to construct a deformation retract of \( W^{1,N}_0(\Omega) \) unto very high sublevels of \( J_\lambda \). More precisely, we have the following
Lemma 3.2. There exists $L > 0$ large so that $J^L_{\lambda}$ is a deformation retract of $W^{1,N}_0(\Omega)$. In particular, $J^L_{\lambda}$ is contractible.

For the sake of completeness, we give some details of the proof.

Proof. Take $L \in \mathbb{N}$ large so that $J_{\lambda}$ has no critical points $u$ with $J_{\lambda}(u) \geq L$. By Lemma 3.1, $J^L_{\lambda}$ is a deformation retract of $J^{L+1}_{\lambda}$ for all $n \geq L$, and $\eta_n$ will denote the corresponding retraction map. Given $u \in W^{1,N}_0(\Omega)$ with $J_{\lambda}(u) > L$, by setting recursively

$$
\eta^{1,n}(s, u) = \eta_n(s, u) \\
\eta^{2,n}(s, u) = \eta_{n-1}(s-1, \eta_n(1, u)) \\
\vdots \\
\eta^{k+1,n}(s, u) = \eta_{n-k}(s-k, \eta^{k}(k, u)),
$$

for $s \geq 0$ we consider the following map

$$
\hat{\eta}(s, u) = \begin{cases} \\
\eta^{k+1,n}(s, u) & \text{if } n < J_{\lambda}(u) \leq n+1 \text{ for } n \in [k, k+1] \\
\eta_{k+1,n}(u) & \text{if } J_{\lambda}(u) \leq L.
\end{cases}
$$

Next, define $s_n$ as the first $s > 0$ such that $J_{\lambda}(\hat{\eta}(s, u)) = L$ if $J_{\lambda}(u) > L$ and as 0 if $J_{\lambda}(u) \leq L$. The map $\eta(t, u) = \eta(t s_n, u) : [0, 1] \times W^{1,N}_0(\Omega) \to W^{1,N}_0(\Omega)$ satisfies $\eta(1, u) \in J^L_{\lambda}$ for $u \in W^{1,N}_0(\Omega)$ and $\eta(t, u) = u$ for $(t, u) \in [0, 1] \times J^{L}_{\lambda}$.

Since $s_n$ depends continuously in $u$, the map $\eta$ is continuous in both variables, providing us with the required deformation retract.

Thanks to Lemmas 3.1 and 3.2 we are led to study the topology of sublevels for $J_{\lambda}$ with very low energy. The real core of such a global variational approach is an improved form 22 of the Moser-Trudinger inequality for functions $u \in W^{1,N}_0(\Omega)$ with a measure $\frac{\lambda^N}{|\Omega|}$ concentrated on several subdomains in $\Omega$. As a consequence, when $\lambda \in (c_N m, c_N (m + 1))$ and $J_{\lambda}(u)$ is very negative, the measure $\frac{\lambda^N}{|\Omega|}$ can be concentrated near at most $m$ points of $\Omega$, and can be naturally associated to an element $\sigma \in B_m(\Omega)$, where

$$
\mathcal{B}_m(\Omega) := \{ \sum_{i=1}^m \delta_{x_i} : t_i \geq 0, \sum_{i=1}^m t_i = 1, p_i \in \Omega \},
$$

has been first introduced by Bahri and Coron in [3] and is known in literature as the space of formal barycenters of $\Omega$ with order $m$. The topological structure of $\mathcal{B}_m(\Omega)$, $L > 0$ large, is completely characterized in terms of $B_m(\Omega)$. The non-contractibility of $\mathcal{B}_m(\Omega)$ let us see a change in topology between $J^L_{\lambda}$ and $J^{L+1}_{\lambda}$ for $L > 0$ large, and by Lemma 3.1 we obtain the existence result claimed in Theorem 1.3. Notice that our approach is simpler than the one in [33, 34, 35] (see also [4]), by using $\mathcal{B}_m(\Omega)$ instead of the Struwe’s monotonicity trick to bypass the general failure of PS-condition for $J_{\lambda}$.

As already explained, the key point of the proof is the following improvement of the Moser-Trudinger inequality:

**Lemma 3.3.** Let $\Omega_i$, $i = 1, \ldots, l + 1$, be subsets of $\Omega$ so that $\text{dist}(\Omega_i, \Omega_j) \geq \delta_0 > 0$, for $i \neq j$, and $\gamma_0 \in (0, \frac{1}{2})$. Then, for any $\epsilon > 0$ there exists a constant $C = C(\epsilon, \delta_0, \gamma_0)$ such that there holds

$$
\log(\int_\Omega \nu e^u \, dx) \leq \frac{1}{Nc_N(l+1-\epsilon)} ||u||_{W^{1,N}_0(\Omega)}^N + C
$$

for all $u \in W^{1,N}_0(\Omega)$ with

$$
\int_\Omega \nu e^u \geq \gamma_0 \quad i = 1, \ldots, l + 1.
$$

**Proof.** Let $g_1, \ldots, g_{l+1}$ be cut-off functions so that $0 \leq g_i \leq 1$, $g_i = 1$ in $\Omega_i$, $g_i = 0$ in $\{ \text{dist}(x, \Omega_i) \geq \frac{\delta_0}{2} \}$ and $||g_i||_{C^2(\Omega)} \leq C_0$. Since $g_i$, $i = 1, \ldots, l$, have disjoint supports, for all $u \in W^{1,N}_0(\Omega)$ there exists $i = 1, \ldots, l + 1$ such that

$$
\int_\Omega (g_i |\nabla u|)^N \leq \frac{1}{l+1} \int_{\Omega_i \cup \text{supp} g_i} |\nabla u|^N \leq \frac{1}{l+1} ||u||_{W^{1,N}_0(\Omega)}^N.
$$

(3.4)

Since by the Young’s inequality

$$
||g_i u||_{W^{1,N}_0(\Omega)}^N \leq (g_i |\nabla u| + |\nabla g_i| |u||)^N \leq (g_i |\nabla u|)^N + C_1 (|\nabla g_i| |u||)^{N-1} |\nabla g_i| |u|| + (|\nabla g_i| |u||)^N \leq [1 + \frac{\epsilon}{(l+1)(3\delta + 3 - \epsilon)}] (g_i |\nabla u|)^N + C_2 (|\nabla g_i| |u||)^N
$$

for all $\epsilon > 0$ and some $C_1 > 0$, $C_2 = C_2(\epsilon) > 0$, we have that

$$
||g_i u||_{W^{1,N}_0(\Omega)}^N \leq \int_\Omega (g_i |\nabla u|)^N + \frac{\epsilon}{(l+1)(3\delta + 3 - \epsilon)} ||u||_{W^{1,N}_0(\Omega)}^N + Nc_N C_3 ||u||_{L^N(\Omega)}^N,
$$

for all $u \in W^{1,N}_0(\Omega)$. Thus, we have for $u \in W^{1,N}_0(\Omega)$ with

$$
\int_\Omega \nu e^u \geq \gamma_0
$$

and $\epsilon > 0$, we have that

$$
\log(\int_\Omega \nu e^u \, dx) \leq \frac{1}{Nc_N(l+1-\epsilon)} ||u||_{W^{1,N}_0(\Omega)}^N + C
$$

(3.3)
where \( C_3 = \frac{C_2 \beta N}{N \epsilon N} \). Since \( g, u \in W^{1, N}_0(\Omega) \), by (3.2) and (3.4) it follows that
\[
\int_{\Omega} e^{g u} \leq C_\delta \exp \left( \frac{3}{N \gamma \epsilon (2l + 3 - \epsilon)} \|u\|^N_{W^{1, N}_0(\Omega)} + C_\delta \|u\|_{L^\infty(\Omega)}^N \right)
\] (3.5)
does hold for all \( u \in W^{1, N}_0(\Omega) \) and some \( i = 1, \ldots, l + 1 \).

Let \( \eta \in (0, |\Omega|) \) be given. Since \( \{ |u| \geq \alpha \} = \Omega \) and \( \lim \alpha \to \infty \| \{ |u| \geq \alpha \} \| = 0 \), the set
\[
A_\eta = \{ a \geq 0 : \| \{ |u| \geq \alpha \} \| \geq \eta \}
\]
is non-empty and bounded from above. Letting \( a_\eta = \sup A_\eta \), we have that \( a_\eta \geq 0 \) is a finite number so that
\[
\| \{ |u| \geq \alpha \| \| \geq \eta \}\ 
\]
in view of the left-continuity of the map \( a \to \| \{ |u| \geq \alpha \} \| \). Given \( \eta > 0 \) and \( u \in W^{1, N}_0(\Omega) \) satisfying (3.3), we can fix \( a = a_\eta \) and \( i = 1, \ldots, l + 1 \) so that (3.3) applies to \( \| |u| - 2a \|_+ \) yielding to
\[
\int_{\Omega} V e^u \leq \frac{1}{\gamma_0} \int_{\Omega} \frac{V e^u}{\|u\|_{L^2(\Omega)}} \leq \frac{C_\delta C_\gamma}{\gamma_0} \exp \left( \frac{3}{N \gamma \epsilon (2l + 3 - \epsilon)} \|u\|^N_{W^{1, N}_0(\Omega)} + 2a + C_\delta \|u| - 2a\|_L^\infty(\Omega) \right)
\]
in view of (3.3). By the Poincaré and Young inequalities and the first property in (3.3) it follows that
\[
2\varepsilon \leq \frac{2}{\eta} \int_{\{ |u| \geq \alpha \}} |u| \leq \frac{C\varepsilon}{\eta} \|u\|_{W^{1, N}_0(\Omega)} \leq \frac{3\varepsilon}{N \gamma \epsilon (2l + 3 - \epsilon)} \|u\|^N_{W^{1, N}_0(\Omega)} + C_\delta
\]
for some \( C_\delta > 0 \) and \( C_\delta = C(\varepsilon, \eta) > 0 \), and there holds
\[
\|u| - 2a\|_L^\infty(\Omega) \leq \eta \\|u\|^N_{W^{1, N}_0(\Omega)} \leq C\varepsilon \|u\|^N_{W^{1, N}_0(\Omega)}
\]
for some \( C_\varepsilon > 0 \) in view of the Hölder and Sobolev inequalities and the second property in (3.3). Choosing \( \eta \) small as
\[
\eta = \left( \frac{C\varepsilon}{C_\delta \gamma \epsilon (2l + 3 - \epsilon)} \right)^2,
\]
we finally get that
\[
\int_{\Omega} V e^u \leq \frac{C\delta C_\varepsilon \gamma \epsilon \gamma}{\gamma_0} \exp \left( \frac{1}{N \gamma \epsilon (2l + 3 - \epsilon)} \|u\|^N_{W^{1, N}_0(\Omega)} + C_\delta \right)
\]
for some \( C = C(\varepsilon, \delta_0, \gamma_0) \). \( \square \)

A criterion for the occurrence of (3.3) is the following:

**Lemma 3.4.** Let \( l \in \mathbb{N} \) and \( 0 < \epsilon, r < 1 \). There exist \( \bar{\epsilon} > 0 \) and \( \bar{r} > 0 \) such that, for every \( 0 \leq f \in L^l(\Omega) \) with
\[
\|f\|_{L^1(\Omega)} = 1 \ , \ \int_{\Omega \setminus \{j = p_i\}} f < 1 - \epsilon \ \ \forall \ p_1, \ldots, p_l \in \Omega,
\] (3.7)
there exist \( l + 1 \) points \( \bar{p}_1, \ldots, \bar{p}_{l+1} \in \Omega \) so that
\[
\int_{\Omega \setminus \{j = \bar{p}_i\}} f \geq \bar{\epsilon}, \quad B_{\bar{r}}(\bar{p}_i) \cap B_{\bar{r}}(\bar{p}_j) = \emptyset \ \ \forall \ i \neq j.
\]

**Proof.** By contradiction, for all \( \bar{\epsilon}, \bar{r} > 0 \) we can find \( 0 \leq f \in L^l(\Omega) \) satisfying (3.7) such that, for every \( (l + 1) \)-tuple of points \( p_1, \ldots, p_{l+1} \in \Omega \) the statement
\[
\int_{\Omega \setminus \{j = \bar{p}_i\}} f \geq \bar{\epsilon}, \quad B_{\bar{r}}(\bar{p}_i) \cap B_{\bar{r}}(\bar{p}_j) = \emptyset \ \ \forall \ i \neq j
\] (3.8)
is false. Setting \( \bar{r} = \frac{1}{\bar{r}} \), by compactness we can find \( h \) points \( x_i \in \Omega \), \( i = 1, \ldots, h \), such that \( \Omega \subset \bigcup_{i=1}^h B_{\bar{r}}(x_i) \). Setting \( \bar{r} = \frac{1}{\bar{r}} \), there exists \( i = 1, \ldots, h \) such that \( f \geq \bar{\epsilon} \). Let \( \{x_1, \ldots, x_h \} \) be the maximal set with respect to the property \( \int_{\Omega \setminus \{j = \bar{p}_i\}} f \geq \bar{\epsilon} \). Set \( j_1 = 1 \) and let \( X_1 \) denote the set
\[
X_1 = \Omega \cap \bigcup_{i \in \Lambda_1} B_{\bar{r}}(x_i) \subset \Omega \cap B_{\bar{r}}(x_{1j_1}), \quad \Lambda_1 = \{ i = 1, \ldots, j : B_{\bar{r}}(x_i) \cap B_{\bar{r}}(x_{1j_1}) = \emptyset \}.
\]
If non empty, choose \( j_2 \in \{1, \ldots, j \} \setminus \Lambda_1, \) i.e. \( B_{\bar{r}}(x_{1j_2}) \cap B_{\bar{r}}(x_{1j_1}) = \emptyset \). Let \( X_2 \) denote the set
\[
X_2 = \Omega \cap \bigcup_{i \in \Lambda_2} B_{\bar{r}}(x_i) \subset \Omega \cap B_{\bar{r}}(x_{1j_2}), \quad \Lambda_2 = \{ i = 1, \ldots, j : B_{\bar{r}}(x_i) \cap B_{\bar{r}}(x_{1j_2}) = \emptyset \}.
\]
Iterating this process, if non empty, at the \( l \)-th step we choose \( j_l \in \{1, \ldots, j \} \setminus \bigcup_{i=1}^{l-1} \Lambda_j \), i.e. \( B_{\bar{r}}(x_{1j_l}) \cap B_{\bar{r}}(x_{1j_l}) = \emptyset \) for all \( i = 1, \ldots, l - 1 \), and we define
\[
X_l = \Omega \cap \bigcup_{i \in \Lambda_l} B_{\bar{r}}(x_i) \subset \Omega \cap B_{\bar{r}}(x_{1j_l}), \quad \Lambda_l = \{ i = 1, \ldots, j : B_{\bar{r}}(x_i) \cap B_{\bar{r}}(x_{1j_l}) = \emptyset \}.
\]
By (3.8) the process has to stop at the $s$-th step with $s \leq l$. By the definition of $\tilde{r}$ we obtain
\[
\Omega \cap \bigcup_{i=1}^{j} B_r(\tilde{x}_i) \subset \bigcup_{i=1}^{\ell} X_i \subset \Omega \cap \bigcup_{i=1}^{j} B_{\tilde{r}}(\tilde{x}_j) \subset \Omega \cap \bigcup_{i=1}^{j} B_r(\tilde{x}_j)
\]
in view of $\{1, ..., j\} = \bigcup_{i=1}^{\ell} A_i$. Therefore, we have that
\[
\int_{\Omega \cap \bigcup_{i=1}^{j} B_r(\tilde{x}_i)} f \leq \int_{\Omega \cap \bigcup_{i=1}^{j} B_{\tilde{r}}(\tilde{x}_j)} f = \int_{(\Omega \cap \bigcup_{i=1}^{j} B_r(\tilde{x}_i)) \setminus (\bigcup_{i=1}^{j} B_{\tilde{r}}(\tilde{x}_j))} f < (h - j)\epsilon < \frac{\epsilon}{2}
\]
in view of the definition of $\tilde{x}_1, ..., \tilde{x}_j$.

Define $p_i$ as $\tilde{x}_i$ for $i = 1, ..., s$ and as $\tilde{x}_j$ for $i = s + 1, ..., l$. Since $\int_{\Omega \cap \bigcup_{i=1}^{j} B_r(p_i)} f < \frac{\epsilon}{2}$, we deduce that
\[
\int_{\Omega \cap \bigcup_{i=1}^{j} B_r(p_i)} f = \int_{\Omega} f - \int_{\Omega \cap \bigcup_{i=1}^{j} B_r(p_i)} f > 1 - \frac{\epsilon}{2} > 1 - \epsilon,
\]
contradicting the second property in (3.7). The proof is complete. 

As a consequence, we get that

**Lemma 3.5.** Let $\lambda \in (c_Nm, c_N(m+1))$, $m \in \mathbb{N}$. For any $0 < \epsilon, r < 1$ there exists a large $L = L(\epsilon, r) > 0$ such that, for every $u \in W^{1,N}_0(\Omega)$ with $J_\lambda(u) \leq -L$, we can find $m$ points $p_{i,u} \in \overline{\Omega}$, $i = 1, ..., m$, satisfying
\[
\int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u})} V e^{u} \leq \epsilon \int_{\Omega} V e^{u}.
\]

**Proof.** By contradiction there exist $\epsilon, r \in (0, 1)$ and functions $u_k \in W^{1,N}_0(\Omega)$ so that $J_\lambda(u_k) \to -\infty$ as $k \to +\infty$ and
\[
\int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{u_k} > \epsilon
\]
for all $p_1, ..., p_m \in \overline{\Omega}$, where $\hat{u}_k = u_k - \log \int_{\Omega} V e^{u_k}$. Since
\[
\int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{\hat{u}_k} = \int_{\Omega} V e^{\hat{u}_k} - \int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{\hat{u}_k} = 1 - \int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{\hat{u}_k},
\]
by (3.9) we get that
\[
\int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{\hat{u}_k} < 1 - \epsilon
\]
for all $m$-tuple $p_1, ..., p_m \in \overline{\Omega}$. Applying Lemma 3.3 with $l = m$ and $f = V e^{\hat{u}_k}$, we find $\tilde{\epsilon}, \tilde{r} > 0$ and $\tilde{p}_1, ..., \tilde{p}_{m+1} \in \overline{\Omega}$ so that
\[
\int_{\Omega \cap \bigcup_{i=1}^{m} B_r(p_{i,u_k})} V e^{\hat{u}_k} \geq \tilde{\epsilon} \int_{\Omega} V e^{\hat{u}_k}, \quad B_{\tilde{r}}(\tilde{p}_i) \cap B_{2\tilde{r}}(\tilde{p}_j) = \emptyset \forall i \neq j.
\]
Applying Lemma 3.3 with $\Omega_i = \Omega \cap B_r(\tilde{p}_i)$ for $i = 1, ..., m + 1$, $\delta_0 = 2\tilde{r}$ and $\gamma_0 = \epsilon$, it now follows that
\[
\log \left( \int_{\Omega} V e^{u_k} \right) \leq \frac{1}{NC_1(m+1-\eta)} \|u_k\|_{W^{1,N}_0(\Omega)}^N + C
\]
for all $\eta > 0$, for some $C = C(\eta, \delta_0, \gamma_0, a, b)$. Since $\lambda < c_N(m+1)$, we get that
\[
J_\lambda(u_k) = \frac{1}{N} \|u_k\|_{W^{1,N}_0(\Omega)}^N - \lambda \log \left( \int_{\Omega} V e^{u_k} dx \right) \geq \frac{1}{N} \left( 1 - \frac{\lambda}{C(\eta, m+1-\eta)} \right) \|u_k\|_{W^{1,N}_0(\Omega)}^N - C\lambda \geq -C\lambda
\]
for $\eta > 0$ small, in contradiction with $J_\lambda(u_k) \to -\infty$ as $k \to +\infty$. 

The set $\mathcal{M}(\overline{\Omega})$ of all Radon measures on $\overline{\Omega}$ is a metric space with the Kantorovich-Rubinstein distance, which is induced by the norm
\[
\|\mu\|_* = \sup_{\|\phi\|_{L^p(\Omega)} \leq 1} \int_{\Omega} \phi d\mu, \quad \mu \in \mathcal{M}(\overline{\Omega}).
\]

Lemma 3.3 can be re-phrased as

**Lemma 3.6.** Let $\lambda \in (c_Nm, c_N(m+1))$, $m \in \mathbb{N}$. For any $\epsilon > 0$ small there exists a large $L = L(\epsilon) > 0$ such that, for every $u \in W^{1,N}_0(\Omega)$ with $J_\lambda(u) \leq -L$, we have
\[
\text{dist}\left( \frac{V e^{u}}{\int_{\Omega} V e^{u}}, \mathcal{B}_m(\overline{\Omega}) \right) \leq \epsilon.
\] (3.10)
Proof. Given $\epsilon \in (0, 2)$ and $r = 1$, let $L = L(\frac{3}{2}, r) > 0$ be as given in Lemma 3.5. For all $u \in W^{1,N}_0(\Omega)$ with $J_\lambda(u) \leq -L$, let us denote for simplicity as $p_1, \ldots, p_m \in \Omega$ the corresponding points $p_{1,u}, \ldots, p_{n,u}$ such that

$$\int_{\Omega \cup \bigcup_{i=1}^m B_r(p_i)} V \phi u \leq \frac{\epsilon}{4} \int_{\Omega} V \phi u. \quad (3.11)$$

Define $\sigma \in \mathcal{B}_m(\Omega)$ as

$$\sigma = \sum_{i=1}^m t_i \delta_{p_i}, \quad t_i = \frac{\int_{A_{r,i}} V \phi u}{\int_{\Omega \cup \bigcup_{i=1}^m A_{r,i} B_r(p_i)} V \phi u},$$

where $A_{r,i} = (\Omega \cap B_r(p_i)) \setminus \bigcup_{i=1}^{m-1} B_r(p_i)$. Since $A_{r,i}$, $i = 1, \ldots, m$, are disjoint sets with $\bigcup_{i=1}^m A_{r,i} = \Omega \cup \bigcup_{i=1}^m B_r(p_i)$, we have that $\sum_{i=1}^m t_i = 1$ and

$$\left| \int_{\Omega} \phi \left[ V \phi u dx - \int_{\Omega} V \phi u d\sigma \right] \right| \leq \int_{\Omega \cup \bigcup_{i=1}^m B_r(p_i)} V \phi u \phi u - \int_{\Omega} V \phi u \phi u - \left( \int_{\Omega} V \phi u \right) \sum_{i=1}^m t_i \phi(p_i) \leq \frac{\epsilon}{4} \int_{\Omega} V \phi u + \sum_{i=1}^m \int_{A_{r,i}} V \phi u \phi u - \left( \int_{\Omega} V \phi u \right) t_i \phi(p_i) \leq \frac{\epsilon}{4} \int_{\Omega} V \phi u + \sum_{i=1}^m \int_{A_{r,i}} V \phi u \phi u - \left( \int_{\Omega} V \phi u \right) t_i \phi(p_i) \leq \frac{\epsilon}{4} \int_{\Omega} V \phi u + \sum_{i=1}^m \int_{A_{r,i}} V \phi u \phi u - \left( \int_{\Omega} V \phi u \right) \frac{\int_{\Omega \cup \bigcup_{i=1}^m B_r(p_i)} V \phi u - 1}{\sum_{i=1}^m \int_{A_{r,i}} V \phi u - 1} \int_{\Omega} V \phi u \leq \frac{\epsilon}{4} + r + \frac{\epsilon}{4 - \epsilon} \int_{\Omega} V \phi u.$$
where \( \sigma = \sum_{i=1}^{m} t_{i} \delta_{p_{i}} \in \mathfrak{M}_{m}(K) \) and \( \epsilon > 0 \). Since \( \varphi_{\epsilon, \sigma} \in W_{0}^{1,N}(\Omega) \), the map \( \Phi \) can be constructed as \( \Phi_{\epsilon_0} \), \( \epsilon_0 > 0 \) small, where

\[
\Phi_{\epsilon} : \mathfrak{M}_{m}(K) \to J_{\lambda}^{L}
\]

\[
\sigma \to \varphi_{\epsilon, \sigma}.
\]

To map \( \mathfrak{M}_{m}(K) \) into the very low sublevel \( J_{\lambda}^{L} \), the difficult point is to produce uniform estimates in \( \sigma \) as \( \epsilon \to 0 \). We have

**Lemma 3.7.** There hold

1. there exist \( C_0 > 0 \) and \( \epsilon_0 > 0 \) so that

\[
\left\| \frac{V_{\epsilon}^{\varphi_{\epsilon, \sigma}}}{\int_{\Omega} V_{\epsilon}^{\varphi_{\epsilon, \sigma}}} - \sigma \right\|_{*} \leq C_0 \epsilon
\]

for all \( 0 < \epsilon \leq \epsilon_0 \) and \( \sigma \in \mathfrak{M}_{m}(K) \);

2. \( J_{\lambda}(\varphi_{\epsilon, \sigma}) \to -\infty \) as \( \epsilon \to 0 \) uniformly in \( \sigma \in \mathfrak{M}_{m}(K) \).

**Proof.** Recall that

\[
U_{\epsilon, p}(x) = \log \left( \frac{F_{N} e^{\frac{x}{\epsilon}}} {\frac{N}{e^{\frac{-N}{\epsilon}}} + |x - p|^{\frac{N}{e^{\frac{-N}{\epsilon}}}}} \right).
\]

Fix \( \phi \in \text{Lip}(\Omega) \) with \( ||\phi||_{\text{Lip}(\Omega)} \leq 1 \). Since \( \varphi_{\epsilon, \sigma} \) is bounded from above in \( \Omega \setminus \Omega_{\frac{1}{2}} \) uniformly in \( \sigma \), we have that

\[
\int_{\Omega} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} \phi = \epsilon^{-\frac{N}{e^{\frac{-N}{\epsilon}}}} \sum_{i=1}^{m} \int_{\Omega_{\frac{1}{2}}(p_{i})} t_{i} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} \phi_{i} + O(1) = \epsilon^{-\frac{N}{e^{\frac{-N}{\epsilon}}}} \sum_{i=1}^{m} \int_{B_{\frac{1}{2}}(p_{i})} t_{i} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} \phi_{i} + O(1)
\]

(3.12)

as \( \epsilon \to 0 \) uniformly in \( \phi \) and \( \sigma \). We have used that

\[
\int_{B_{\frac{1}{2}}(p_{i})} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} \phi_{i} = \int_{B_{\frac{1}{2}}(0)} (\phi(p_{i}) + O(\epsilon)) e^{U} = c_{N} \phi(p_{i}) + O(\epsilon)
\]

doing as \( \epsilon \to 0 \), uniformly in \( \phi \) and \( \sigma \), in view of (3.11). Therefore, there holds

\[
\left| \int_{\Omega} \phi \left( \frac{V_{\epsilon}^{\varphi_{\epsilon, \sigma}}}{\int_{\Omega} V_{\epsilon}^{\varphi_{\epsilon, \sigma}}} \right) dx - d\sigma \right| \leq C_0 \epsilon
\]

for all \( \phi \in \text{Lip}(\Omega) \) with \( ||\phi||_{\text{Lip}(\Omega)} \leq 1 \), and then

\[
\left\| \frac{V_{\epsilon}^{\varphi_{\epsilon, \sigma}}}{\int_{\Omega} V_{\epsilon}^{\varphi_{\epsilon, \sigma}}} - \sigma \right\|_{*} \leq C_0 \epsilon
\]

for all \( \sigma \in \mathfrak{M}_{m}(K) \). Part (1) is proved.

For part (2), it is enough to show that

\[
\log \int_{\Omega} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} = \frac{N}{N - 1} \log \frac{1}{\epsilon} + O(1)
\]

(3.13)

\[
\frac{1}{N} \int_{\Omega} |\nabla \varphi_{\epsilon, \sigma}|^{N} \leq \frac{N}{N - 1} c_{N}^{m} \log \frac{1}{\epsilon} + O(1)
\]

(3.14)

as \( \epsilon \to 0 \) uniformly in \( \sigma \in \mathfrak{M}_{m}(K) \), in view of \( \lambda > m_{\epsilon,N} \). Estimate (3.13) follows by (3.14) with \( \phi = 1 \). As far as (3.14) is concerned, let us set \( \varphi_{\epsilon, \sigma} = \chi_{\Omega} \varphi_{\epsilon, \sigma} \). All the estimates below are uniform in \( \sigma \). Since

\[
\nabla \varphi_{\epsilon, \sigma} = -\frac{N^{2}}{N - 1} \sum_{i=1}^{m} t_{i} V_{\epsilon}^{\varphi_{\epsilon, \sigma}} \frac{1}{(e^{-\frac{N}{e^{\frac{-N}{\epsilon}}}} + |x - p|^{\frac{N}{e^{\frac{-N}{\epsilon}}}})} \frac{1}{(N + 1)|x - p_{i}|^{\frac{N}{e^{\frac{-N}{\epsilon}}}}} (x - p_{i})
\]

we have that \( ||\varphi_{\epsilon, \sigma}||_{C^{1}\left(\Omega \setminus \Omega_{\frac{1}{2}}\right)} = O(1) \) and then

\[
|\nabla \varphi_{\epsilon, \sigma}| = O(1)
\]

in \( \Omega \setminus \Omega_{\frac{1}{2}} \). Therefore we can write that

\[
\frac{1}{N} \int_{\Omega} |\nabla \varphi_{\epsilon, \sigma}|^{N} = \frac{1}{N} \int_{\Omega_{\frac{1}{2}}} |\nabla \varphi_{\epsilon, \sigma}|^{N} + O(1).
\]

(3.15)
We estimate $|\nabla \tilde{\varphi}_{\epsilon, \sigma}|$ in two different ways:

(i) $|\nabla \tilde{\varphi}_{\epsilon, \sigma}|(x) \leq \frac{N^2 \epsilon^2}{|x-p_i|}$, where $d(x) = \min \{|x-p_i| : i = 1, \ldots, m\}$;

(ii) $|\nabla \tilde{\varphi}_{\epsilon, \sigma}| \leq \frac{N^2 \epsilon^2 C_0 e^{-1}}{|x-p_i|}$ in view of

$$
\frac{\epsilon|x-p_i|}{\epsilon^{N-1} + |x-p_i|^{N-1}} \leq C_0
$$

by the Young’s inequality. By estimate (ii) we have that

$$
\int_{\Omega} |\nabla \tilde{\varphi}_{\epsilon, \sigma}|^N = \int_{\Omega} \left( \sum_{j=1}^m \int_{A_j \setminus B_r(p_j)} |\nabla \tilde{\varphi}_{\epsilon, \sigma}|^N \right) + O(1) \leq \sum_{j=1}^m \int_{A_j \setminus B_r(p_j)} |\nabla \tilde{\varphi}_{\epsilon, \sigma}|^N + O(1)
$$

(3.16)

in view of $\Omega = \bigcup_{j=1}^m B_r(p_j) \cup \bigcup_{j=1}^m \left(A_j \setminus B_r(p_j)\right)$, where $A_j = \{x \in \Omega : |x-p_j| = d(x)\}$. Since by estimate (i) we have that

$$
\int_{A_j \setminus B_r(p_j)} |\nabla \tilde{\varphi}_{\epsilon, \sigma}|^N \leq \frac{N^2}{N-1} \int_{A_j \setminus B_r(p_j)} |x-p_j|^N \leq \frac{N^2}{N-1} \int_{B_r(0)} |x|^N + O(1) = \frac{N^2}{N-1} e \log \frac{1}{\epsilon} + O(1)
$$

in terms of $R = \text{diam} \Omega$, by (3.15)-(3.16) we deduce the validity of (3.14). The proof is complete.

In order to prove that $\Psi \circ \Phi_\epsilon$ is homotopically equivalent to $\text{Id}_{\mathcal{B}_m(K)}$, we construct an explicit homotopy $H$ as follows:

$$
H : (0, 1] \to C(\mathcal{B}_m(K), \| \cdot \|); \mathcal{B}_m(K), \| \cdot \|_*), \ t \mapsto H(t) = \Psi \circ \Phi_{\epsilon t}.
$$

The map $H$ is continuous in $(0, 1]$ with respect to the norm $\| \cdot \|_{\mathcal{B}_m(K)}$. In order to conclude, we need to prove that for all $\epsilon = \epsilon_0$.

$$
\lim_{\epsilon \to \epsilon_0} \|H(t) - \text{Id}_{\mathcal{B}_m(K)}\|_{\mathcal{B}_m(K)} = \lim_{\epsilon \to \epsilon_0} \sup_{(\sigma, \epsilon) \in \mathcal{B}_m(K)} \|\Psi \circ \Phi_\epsilon - \sigma\|_* = 0,
$$

where $\epsilon = \epsilon_0$. Since $\Pi_m(\sigma) = \sigma$ and $\mathcal{B}_m(K)$ is a compact set in $(M, \| \cdot \|_*)$, by the continuity of $\Pi_m$ in $\| \cdot \|_*$ and Lemma 3.3(1) we deduce that

$$
\|\Psi \circ \Phi_\epsilon - \sigma\|_* = \|\Pi_m \left( \frac{V e^{r_\epsilon, \sigma}}{\Omega} \right) - \Pi_m(\sigma)\|_* \to 0
$$

as $\epsilon \to 0$, uniformly in $\sigma \in \mathcal{B}_m(K)$. Finally, we extend $H(t)$ at $t = 0$ in a continuous way by setting $H(0) = \text{Id}_{\mathcal{B}_m(K)}$.

Let us now discuss the main assumption in Theorem 3.3. In 3.1 it is claimed that $\mathcal{B}_m(\Omega)$ is non contractible for all $m \geq 1$ if $\Omega$ is non contractible too, as it arises for closed manifolds. However, by the techniques in 3.2 it is shown in 3.1 that $\mathcal{B}_m(X)$ is contractible for all $m \geq 1$, for a non contractible topological and acyclic (i.e. with trivial $Z$-homology) space $X$. A concrete example is represented by the punctured Poincaré sphere, and it is enough to take a tubular neighborhood $\Omega$ of it to find a counterexample to the claim in 3.1. A sufficient condition for the main assumption in Theorem 3.3 is the following:

**Theorem 3.8.** [3.1] Assume that $X$ is homotopically equivalent to a finite simplicial complex. Then $\mathcal{B}_m(X)$ is non contractible for all $m \geq 2$ if and only if $X$ is not acyclic (i.e. with non trivial $Z$-homology).

**Appendix**

Let us collect here some useful regularity estimates which have been frequently used throughout the paper. Concerning $L^\infty$ estimates, the general interior estimates in 3.3 are used here to derive also boundary estimates for solutions $u \in W^{1,N}(\Omega) = \{u \in W^{1,N}(\Omega) : |\partial \Omega = c\}, c \in \mathbb{R}$, through the *Schwarz reflection principle*.

Given $x_0 \in \partial \Omega$, we can find a smooth diffeomorphism $\psi$ from a small ball $B \subset \mathbb{R}^N$, $0 \in B$, into a neighborhood $V$ of $x_0$ in $\mathbb{R}^N$ so that $\psi(B \cap \{y_N = 0\}) = V \cap \partial \Omega$ and $\psi(B^+) = V \cap \Omega$, where $B^+ = B \cap \{y_N > 0\}$. Letting $u_0 \in W^{1,N}_c(\Omega)$ be a critical point of

$$
\frac{1}{p} \int_{\Omega} |\nabla u|^N - \int_{\Omega} f u, \ u \in W^{1,N}_c(\Omega),
$$

then $v_0 = u_0 \circ \psi$ is a critical point of

$$
I(v) = \int_{B^+} \left[ \frac{1}{N} |A(y) \nabla v|^N - f v \right] |\det \nabla \psi|, \ v \in V,
$$

in view of $|\nabla u|^N \circ \psi = |A \nabla v|^N$ in $B^+$ for $v = u \circ \psi$, where $A(y) = (D\psi^{-1})^t(\psi(y))$ is an invertible $N \times N$ matrix for all $y \in B^+$ and

$$
V = \{v \in W^{1,N}(B^+) : v = c \text{ on } y_N = 0 \text{ and } v = u_0 \circ \psi \text{ on } \partial B \cap \{y_N > 0\} \}.
$$

In the sequel, $g_e$ and $g^e$ denote the odd and even extension in $B$ of a function $g$ defined on $B^+$, respectively. Decomposing the matrix $A$ as

$$
A = \begin{pmatrix}
A' & a_1 \\
a_2 & a_{NN}
\end{pmatrix}
$$
with $a_1, a_2 : B^+ \to \mathbb{R}^{N-1}$, for $y \in B$ let us introduce

$$A^0 = \left( \frac{(A')^2}{(a_1)^2} + \frac{(a_2)^1}{(a_2)^N} \right).$$

The odd reflection $(v_0 - c_2 + c \in W^{1,N}(B)$ is a weak solution in $B$ of

$$-\text{div} A(y, \nabla v) = (f) \text{det} \nabla v|_1,$$

where $A : (y, p) \in B \times \mathbb{R}^N \to | \det \nabla \psi|^2 |A(y)p|^{N-2}(A(y))A(y)p \in \mathbb{R}^N$. In view of the invertibility of $A(y)$ for all $y \in B^+$, the map $A$ satisfies

$$|A(y, p)| \leq a|p|^{N-1}, \quad \langle p, A(y, p) \rangle \geq a^{-1}|p|^N$$

for all $y \in B$ and $p \in \mathbb{R}^N$, for some $a > 0$. Since $2c - u \leq u$ when $u \geq c$, thanks to $A.1$ we can now apply the general local interior estimates of J. Serrin in [63] to get:

**Theorem A.1.** Let $u \in W^{1,N}_0(\Omega)$ be a weak solution of

$$-\Delta_N u = f \quad \text{in } \Omega.$$ 

Assume that $f \in L^\infty(\Omega \cap B_{2R})$, $0 < \epsilon \leq 1$, and $u \in W^{1,N}(\Omega \cap B_{2R})$ satisfies $u = c$ on $\partial \Omega \cap B_{2R}$, $u \geq c$ in $\Omega \cap B_{2R}$ for some $c \in \mathbb{R}$ if $\partial \Omega \cap B_{2R} \neq \emptyset$. Then, the following estimates do hold:

$$\|u\|_{L^\infty(\Omega \cap B_{2R})} \leq C\|u^+\|_{L^N(\Omega \cap B_{2R})} + 1$$

$$\|u\|_{L^\infty(\Omega \cap B_{2R})} \leq C\|u\|_{L^N(\Omega \cap B_{2R})} + 1 \quad (\text{if } c = 0)$$

for some $C = C(N, a, \epsilon, R, \|f\|_{L^\infty(\Omega \cap B_{2R})})$.

Since the Harnack inequality in [63] is very general, it can be applied in particular when $A$ satisfies $A.1$, by allowing us to treat also boundary points through the Schwarz reflection principle. The following statement is borrowed from [59].

**Theorem A.2.** Let $u \in W^{1,N}_0(\Omega)$ be a nonnegative weak solution of $A.2$, where $f \in L^{\infty}(\Omega)$, $0 < \epsilon \leq 1$. Let $\Omega' \subset \Omega$ be a sub-domain of $\Omega$. Assume that $u \in W^{1,N}(\Omega' \cap \Omega')$ satisfies $u = 0$ on $\partial \Omega' \cap \Omega'$. Then, there exists $C = C(N, \epsilon, \Omega')$ so that

$$\sup_{\Omega'} u \leq C \left( \inf_{\Omega'} u + \|f\|_{L^{\infty}(\Omega \cap B_{2R})} \right).$$

By choosing $\Omega' = \Omega$ we deduce that

**Corollary A.3.** Let $u \in W^{1,N}_0(\Omega)$ be a weak solution of $-\Delta_N u = f$ in $\Omega$, where $f \in L^{\infty}(\Omega)$, $0 < \epsilon \leq 1$. Then, there exists a constant $C = C(N, \epsilon, \Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{\infty}(\Omega \cap B_{2R})}.$$ 

Thanks to Theorem A.1 by the estimates in [31] [39] [65] we now have that

**Theorem A.4.** Let $u \in W^{1,N}_0(\Omega)$ be a weak solution of $A.2$. Assume that $f \in L^\infty(\Omega \cap B_{2R})$, and $u \in W^{1,N}(\Omega \cap B_{2R})$ satisfies $u = 0$ on $\partial \Omega \cap B_{2R}$. Then, there holds $\|u\|_{C^\alpha(\Omega \cap B_{2R})} \leq C = C(N, a, R, \|f\|_{L^\infty(\Omega \cap B_{2R})})$, for some $\alpha \in (0, 1)$.

We will now consider $A.2$ with a Dirac measure $\delta_0$, as R.H.S. In our situation, the fundamental solution $\Gamma$ takes the form

$$\Gamma(|x|) = (N\omega_N)^{-\frac{1}{N-1}} \log \frac{1}{|x|}.$$ 

In a very general framework, Serrin has described in [63] the behavior of solutions near a singularity. In particular, every $N$-harmonic and continuous function $u$ in $\Omega \setminus \{0\}$, which is bounded from below in $\Omega$, has either a removable singularity at 0 or there holds

$$\frac{1}{C} \Gamma \leq u \leq CT$$

in a neighborhood of 0, for some $C \geq 1$. For the $p-$Laplace operator Kichenassamy and Veron [15] have later improved $A.3$ by expressing $u$ in terms of $\Gamma$. A combination of [63] [65] leads in our situation to:

**Theorem A.5.** Let $u$ be a $N$-harmonic continuous function in $\Omega \setminus \{0\}$, which is bounded from below in $\Omega$. Then there exists $\gamma \in \mathbb{R}$ such that

$$u - \gamma \Gamma \in L^\infty(\Omega)$$

and $u$ is a distributional solution in $\Omega$ of

$$-\Delta_N u = \gamma |\gamma|^{N-2}\delta_0.$$
with $|\nabla u|^{-1} \in L^1_{loc}(\Omega)$. Moreover, for $\gamma \neq 0$ there holds
\[
\lim_{x \to \infty} |x|^{\alpha} D^{\alpha}(u - \gamma \Gamma)(x) = 0
\]
for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ with length $|\alpha| = \alpha_1 + \ldots + \alpha_N \geq 1$.

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