CONCENTRATION PHENOMENA FOR THE SCHRÖDINGER-POISSON SYSTEM IN $\mathbb{R}^2$

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Abstract. We perform a semiclassical analysis for the planar Schrödinger-Poisson system
\[
\begin{aligned}
-\varepsilon^2 \Delta \psi + V(x)\psi &= E(x)\psi & \text{in } \mathbb{R}^2, \\
-\Delta E &= |\psi|^2 & \text{in } \mathbb{R}^2,
\end{aligned}
\]
where $\varepsilon$ is a positive parameter corresponding to the Planck constant and $V$ is a bounded external potential. We detect solution pairs $(u_\varepsilon, E_\varepsilon)$ of the system $(SP_\varepsilon)$ as $\varepsilon \to 0$, leaning on a nongeneracy result in [3].

1. Introduction

We are concerned with the planar Schrödinger-Poisson system
\[
\begin{aligned}
-\varepsilon^2 \Delta \psi + V(x)\psi &= E(x)\psi & \text{in } \mathbb{R}^2, \\
-\Delta E &= |\psi|^2 & \text{in } \mathbb{R}^2,
\end{aligned}
\]
which presents some special features, because of the different nature of the Newtonian potential in two-dimensional space. This system has been derived in $\mathbb{R}^3$ by R. Penrose in [21] in his description of the self-gravitational collapse of a quantum mechanical system (see also [20, 22, 19, 18]). The rigorous mathematical study of the nonlinear Schrödinger equation with nonlocal nonlinearity, involving a Coulomb type convolution potential, dates back to the seminal papers by Lieb [14] and Lions [15]. Successively in [24] Wei and Winter studied the semiclassical limit for the Schrödinger-Poisson system, after showing the nondegeneracy of the least energy solutions of a related limiting system (see also [13]). We also mention the papers [6, 8, 9, 17] where variational and topological methods have been employed to derive concentration phenomena for generalized NLS equations with more general nonlocal nonlinearity in dimensional $d \geq 3$, where the nondegeneracy properties of the linearized operators do not hold.

The rigorous study of the Schrödinger-Poisson system in $\mathbb{R}^2$ remained open for long time, since it appears more delicate. Differently from the Coulomb potential, the Newton potential in $\mathbb{R}^2$ is sign-changing and it presents singularities at zero and infinity. Moreover we recall that the Poisson equation $-\Delta E = |\psi|^2$ determines the solution $E : \mathbb{R}^2 \to \mathbb{R}$ only up to harmonic functions, and every semibounded harmonic function is constant in $\mathbb{R}^2$. Therefore if $\psi \in L^\infty(\mathbb{R}^2)$ and $E$ solves the Poisson equation under suitable additional assumption at infinity, such as $E(x) \to -\infty$ as $|x| \to +\infty$, then we have $E(x) = \Phi_\psi(x) + c$, where $c$ is a constant and $\Phi_\psi$ is the convolution of fundamental solution of $-\Delta$ in $\mathbb{R}^2$ with $|\psi|^2$, namely
\[
\Phi_\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} |\psi(y)|^2 dy.
\]

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In literature, apart from some numerical results in [12], existence and uniqueness results of spherically symmetric solutions of (1.1) were proved by Stubbe and Vuffray [5], for $V \equiv 1$, using shooting methods for the associated ODE system (see also [4] for the one-dimensional case).

In [16] Masaki proved a global well-posedness of the Cauchy problem for (1.1) in a subspace of $H^1(\mathbb{R}^2)$, where $E(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|y|}{|x-y|} |\psi(y)|^2 dy$, which means $E(0) = 0$.

In the more natural case, $E$ coincides with the Newtonian potential $\Phi$ of $|\psi|^2$, the Schrödinger-Poisson system with a constant potential can be written as the following Schrödinger equation with a nonlocal nonlinearity:

\[
- \Delta u + u = \frac{1}{2\pi} \left[ \log \frac{1}{|\cdot|} \ast |u|^2 \right] u, \quad x \in \mathbb{R}^2.
\]

For such an integro-differential equation, unlike the 3D case, the applicability of variational tools is not straightforward, because the usual Sobolev spaces do not provide a good environment to work in. In [23] Stubbe tackled this problem by setting a suitable variational framework for (1.2) within the space

\[
X = \left\{ u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \log(1 + |x|)|u(x)|^2 dx < \infty \right\},
\]

endowed with the norm

\[
\|u\|_X^2 = \int_{\mathbb{R}^2} \left( |\nabla u|^2 + |u|^2 \right) dx + \int_{\mathbb{R}^2} \log(1 + |x|)|u(x)|^2 dx.
\]

The space $X$ provides a reasonable variational framework, but its norm does not detect the invariance of the problem under translations; furthermore the quadratic part of the energy functional associated to (1.2) is not coercive on $X$. These difficulties enforced the implementation of new variational ideas and estimates to treat nonlinear Schrödinger equation with nonlocal nonlinearities involving logarithmic type convolution potential [7, 10, 11]. In particular in [10], the authors proved the existence result of an unique positive ground state solution $U$ to (1.2). Sharp asymptotics and the nondegeneracy of the ground state solution $U$ has been proved in [3].

In the present paper we study the existence of solution pairs of the Schrödinger-Poisson system as the parameter $\varepsilon \to 0^+$. This study presents some new aspects with respect to the 3D case, since the Newtonian potential in $\mathbb{R}^2$ does not scale algebraically.

The semiclassical analysis remained in the background until very recent years and, to the best of our knowledge, it has only been treated by Masaki in [16] via WKB approximation.

Here we adapt some perturbation method developed in [1, 2] in the variational framework $X$ where the norm depends on the weight $x \mapsto \log(1 + |x|)$. This makes it more involved to apply a finite dimensional reduction.

In the rest of the paper we will consider a potential function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following condition:

\[ (V) \ V \in C^2(\mathbb{R}^2), \ \inf_{x \in \mathbb{R}^2} V(x) > 0 \text{ and } \sup_{x \in \mathbb{R}^2} \left[ |V(x)| + \sum_{j=1}^{2} |\partial_j V(x)| + \sum_{i,j=1}^{2} |\partial^2_{ij} V(x)| \right] < +\infty. \]
Setting \( v(x) = \varepsilon \psi(x) \), the system (1.1) can be written
\[
\begin{cases}
-\varepsilon^2 \Delta v + V(x)v = Ev & \text{in } \mathbb{R}^2, \\
-\varepsilon^2 \Delta E = |v|^2 & \text{in } \mathbb{R}^2.
\end{cases}
\]

Our main existence result can be summarized as follows.

**Theorem 1.1.** Suppose that \( V \) satisfies (V) and has a non-degenerate critical point \( x_0 \), i.e. \( \nabla V(x_0) = 0 \) and \( D^2 V(x_0) \) is either positive- or negative-definite. Then, for every \( \varepsilon > 0 \) sufficiently small, the system (1.3) possesses a solution \((v_\varepsilon, E_\varepsilon)\) such that
\[
v_\varepsilon(x) \simeq U \left( \frac{x - x_0}{\varepsilon} \right), \quad E_\varepsilon(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \log \frac{\varepsilon}{|x - z|} |v_\varepsilon(z)|^2 \, dz
\]
where \( U \) is the unique (up to translations) positive ground state solution of the limiting equation
\[
-\Delta u + V(x_0)u = \frac{1}{2\pi} \left[ \log \frac{1}{| \cdot |} \ast |u|^2 \right] u, \quad x \in \mathbb{R}^2.
\]

Remark 1.2. In Theorem (1.1) we have \( E_\varepsilon(x) = \varepsilon^{-2} \Phi_{v_\varepsilon}(x) + c_\varepsilon \) where \( \Phi_{v_\varepsilon}(x) = \log \frac{1}{| \cdot |} \ast v_\varepsilon^2 \) and \( c_\varepsilon = \varepsilon^{-2} \log \varepsilon \| v_\varepsilon \|_2^2 \). Coming back to the system (1.1), we derive the existence of the solution pair \((\varepsilon^{-1}v_\varepsilon, E_\varepsilon)\) for \( \varepsilon > 0 \) small.

### 2. Functional setting

Without loss of generality, we will assume that \( x_0 = 0 \) and \( V(0) = 1 \). Setting \( u(x) = v(\varepsilon x) \) and \( \omega(x) = E(\varepsilon x) \), the system (1.3) becomes
\[
\begin{cases}
-\Delta u + V(\varepsilon x)u = \omega(x)u & \text{in } \mathbb{R}^2, \\
-\Delta \omega = |u|^2 & \text{in } \mathbb{R}^2.
\end{cases}
\]

The second equation in (2.1) can be explicitly solved with respect to \( \omega \). Choosing \( \omega \) as the convolution of the fundamental solution of \(-\Delta\) in \( \mathbb{R}^2 \) with \( |u|_\varepsilon^2 \), this system can be written as the single nonlocal equation
\[
-\Delta u + V(\varepsilon x)u = \frac{1}{2\pi} \left[ \log \frac{1}{| \cdot |} \ast |u|^2 \right] u, \quad x \in \mathbb{R}^2.
\]

We consider the functional space
\[
X = \left\{ u \in H^1(\mathbb{R}^2) \mid |u|_\ast < +\infty \right\},
\]
where
\[
|u|_\ast^2 = \int_{\mathbb{R}^2} \log (1 + |x|) |u(x)|^2 \, dx.
\]

We endow \( X \) with the norm
\[
\|u\|_X^2 = \|u\|_{H^1}^2 + |u|_\ast^2
\]
and the associated scalar product
\[
\langle u \mid v \rangle_X = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla v + uv] \, dx + \int_{\mathbb{R}^2} \log(1 + |x|)u(x)v(x) \, dx.
\]

The norms in \( H^1(\mathbb{R}^2) \) and \( L^q(\mathbb{R}^2) \) will be denoted by \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_q \), respectively.
It is known that the space \( X \) is compactly embedded in \( L^p(\mathbb{R}^2) \) for any \( p \in [2, +\infty) \) (cf. Lemma 2.2 in [10]).

Solutions to (2.2) correspond to critical points of the energy functional \( I_\varepsilon : X \to \mathbb{R} \) defined by

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + V_\varepsilon|u|^2 \, dx - \frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left( \frac{1}{|x-y|} \right) |u(x)|^2 |u(y)|^2 \, dx \, dy,
\]

where we set \( V_\varepsilon(x) = V(\varepsilon x) \).

We observe that

\[
\|u\|_2^2 = \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V_\varepsilon|u|^2 \right] \, dx + |u|^2_{a_0}.
\]

(2.3)

This can be considered as an equivalent norm on \( X \) by virtue of assumption (V). The functional \( I_\varepsilon \) fails to be continuous on the Sobolev space \( H^1(\mathbb{R}^2) \). On the contrary, arguing as in [10, Lemma 2.2], we can infer the following regularity result on \( X \).

**Proposition 2.1.** If \( V \) satisfies (V), then \( I_\varepsilon \) is a functional of class \( C^2 \) on \( X \).

3. **Limiting Equation**

We consider the planar integro-differential equation

\[
-\Delta u + u = \frac{1}{2\pi} \left[ \log \left( \frac{1}{|x|} \right) \ast |u|^2 \right] u, \quad \text{in } \mathbb{R}^2,
\]

which has the rôle of a limiting problem for (2.2). We define the energy functional \( I : X \to \mathbb{R} \) associated to (3.1):

\[
I(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{8\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(|x-y|)|u(x)|^2 |u(y)|^2 \, dx \, dy.
\]

For future reference, we introduce some shorthand: let us set

\[
B(f, g) = -\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x-y| f(x)g(y) \, dx \, dy,
\]

so that

\[
I(u) = \|u\|_{H^1}^2 - \frac{1}{4} B(u^2, u^2).
\]

It follows from [10, Lemma 2.2] that \( I \) is of class \( C^2 \) and that

\[
I'(u)[\varphi] = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla \varphi + u \varphi] - B(u^2, u\varphi)
\]

\[
I''(u)[\varphi, \psi] = \int_{\mathbb{R}^2} [\nabla \varphi \cdot \nabla \psi + \varphi \psi] - B(u^2, \varphi \psi) - 2B(u \varphi, u \psi).
\]

It has been proved in [10, Theorem 1.1] that the restriction of \( I \) to the associated Nehari manifold

\[
\mathcal{N} = \{ u \in X \setminus \{0\} \mid I'(u)[u] = 0 \}
\]

attains a global minimum. Moreover, every minimizer \( u \in \mathcal{N} \) of \( I|_{\mathcal{N}} \) is a solution of (3.1) which does not change sign and obeys the variational characterization

\[
I(u) = \inf_{u \in \mathcal{N}} \sup_{t \in \mathbb{R}} I(tu).
\]

From [10, Theorem 1.3] we have the following result.
Theorem 3.1. Every positive solution $u \in X$ of (3.1) is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover $u$ is unique, up to translation in $\mathbb{R}^2$.

Moreover, from [3, Theorem 1], the sharp asymptotics of the radially symmetric positive solution of (3.1) are known.

Theorem 3.2. If $u \in X$ is a radially symmetric positive solution of (3.1), there exists $\mu > 0$ such that, as $|x| \to +\infty$,

$$u(x) = \frac{\mu + o(1)}{\sqrt{|x|}(|\log |x|)|^{1/4}} \exp \left( -\sqrt{M}e^{-1/M} \int_1^{||x||^{1/M}} \sqrt{\log s} \, ds \right),$$

where $M = (2\pi)^{-1} \int_{\mathbb{R}^2} |u|^2 \, dx$.

We consider the linearization on a positive solution $u$ of (3.1). Let $L(u) : \tilde{X} \to L^2(\mathbb{R}^2)$ be the linear operator defined by

$$L(u) : \varphi \mapsto -\Delta \varphi + (1 - w)\varphi + 2u \left( \log \frac{2\pi}{2\pi} \ast (u\varphi) \right),$$

where

$$w : \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x - y|} |u(y)|^2 \, dy$$

and

$$(3.2) \quad \tilde{X} = \left\{ \varphi \in X \mid \text{for every } \psi \in C^\infty_c(\mathbb{R}^2) : \int_{\mathbb{R}^2} \varphi L(u) \psi = \int_{\mathbb{R}^2} f \psi \right\}$$

By standard arguments, one easily shows that $L(u)$ is a self-adjoint operator acting on $L^2(\mathbb{R}^2)$ with domain $\tilde{X}$. Also, differentiating the equation (3.1), it is clear that $\alpha_1 \partial_{x_1} u + \alpha_2 \partial_{x_2} u \in \ker L(u)$ for every $\alpha_1, \alpha_2 \in \mathbb{R}$.

The following result has been proved in [3, Theorem 3].

Theorem 3.3. Let $u \in X$ be a positive solution of (3.1). Then

$$\ker L(u) = \left\{ \gamma \cdot \nabla u \mid \gamma \in \mathbb{R}^2 \right\}.$$ 

The functional-analytic properties of the second derivative of $I$ will play a crucial rôle in our analysis.

Lemma 3.4. Let $u \in X$ be a positive solution of (3.1). The operator $I''(u)$ is a Fredholm operator of index zero from $X$ to its dual space $X^*$.

Proof. We will actually prove that $I''(u) = A + K$, where $A$ is a bounded invertible operator and $K$ is a compact operator on $X$. 

Set $c^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} u^2(y) dy$. For any $\varphi \in X$ and $\psi \in X$, we have

$$I''(u)[\varphi, \psi] = \int_{\mathbb{R}^2} \left[ \nabla \varphi(x) \nabla \psi(x) + \varphi(x) \psi(x) \right] dx$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| u^2(y) \varphi(x) \psi(x) dx dy$$

$$+ \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| u(y) \varphi(y) u(x) \psi(x) dx dy$$

$$= \int_{\mathbb{R}^2} \left( \nabla \varphi(x) \nabla \psi(x) + \varphi(x) \psi(x) + c^2 \log(1 + |x|) \varphi(x) \psi(x) \right) dx$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| - \log(1 + |x|) u^2(y) \varphi(x) \psi(x) dx dy$$

$$+ \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| u(y) \varphi(y) u(x) \psi(x) dx dy.$$ 

We have deduced the decomposition $I''(u) = A + K$, where the operators $A$ and $K$ act as follows:

(3.3) \[ \langle A\varphi, \psi \rangle = \int_{\mathbb{R}^2} \left( \nabla \varphi \cdot \nabla \psi + \varphi \psi + c^2 \log(1 + |x|) \varphi(x) \psi(x) \right) dx \]

and

(3.4) \[ \langle K\varphi, \psi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| - \log(1 + |x|) u^2(y) \varphi(x) \psi(x) dx dy \]

$$+ \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| u(y) \varphi(y) u(x) \psi(x) dx dy.$$ 

Equation (3.3) implies that the correspondence

$$u \in X \mapsto \langle Au, u \rangle$$

is an equivalent norm on $X$. It follows that the operator $A$ is invertible from $X$ to $X^\ast$.

We claim that $K$ is compact from $X$ to $X^\ast$. Indeed, let $\{\varphi_n\}_n \subset X$ be a sequence such that $\varphi_n \rightharpoonup 0$ as $n \to +\infty$. It follows that $\|\varphi_n\|_X \leq D$ for any $n \in \mathbb{N}$.

We prove that

(3.5) \[ \lim_{n \to +\infty} \sup_{\|\psi\|_X = 1} |\langle K\varphi_n, \psi \rangle| = 0. \]

Fix $\varepsilon > 0$ and $\psi \in X$ such that $\|\psi\|_X = 1$. Since $u \in X$, there exists $M > 0$ such that

$$\frac{D}{2\pi} \int_{|y| > M} \log(1 + |y|) u^2(y) dy < \frac{\varepsilon}{4}$$

and

$$\frac{D}{\pi} \int_{|y| > M} u^2(y) dy < \frac{\varepsilon}{4}.$$
We evaluate
\[\langle K\varphi_n, \psi \rangle = \frac{1}{2\pi} \int_{|y|>M} \int_{\mathbb{R}^2} \left[ \log(1 + |y|) - \log(1 + |x|) \right] u^2(y) \varphi_n(x) \psi(x) \, dx \, dy + \frac{1}{2\pi} \int_{|y|\leq M} \int_{\mathbb{R}^2} \left[ \log(1 + |x|) - \log(1 + |x|) \right] u^2(y) \varphi_n(x) \psi(x) \, dx \, dy - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x|} \right) u^2(y) \varphi_n(x) \psi(x) \, dx \, dy + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |y|) u(y) \varphi_n(y) u(x) \psi(x) \, dx \, dy - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x|} \right) u(y) \varphi_n(y) u(x) \psi(x) \, dx \, dy.\]

Recalling the elementary inequality \(\log(1 + |x|) \leq \log(1 + |x|) + \log(1 + |y|)\) for \(x \in \mathbb{R}^2, y \in \mathbb{R}^2\), we have that
\[|\langle K\varphi_n, \psi \rangle| \leq \frac{1}{2\pi} \int_{|y|>M} u^2(y) \, dy \int_{\mathbb{R}^2} \left[ 2 \log(1 + |x|) + \log(1 + |y|) \right] |\varphi_n(x)||\psi(x)| \, dx + \frac{1}{2\pi} \int_{|y|\leq M} u^2(y) \, dy \int_{\mathbb{R}^2} \log \left( \frac{1 + |x|}{1 + |x|} \right) ||\varphi_n(x)||\psi(x)|| \, dx
\]
\[+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x|} \right) u^2(y) ||\varphi_n(x)||\psi(x)|| \, dx \, dy + \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x|) u(y) ||\varphi_n(y)||u(x)||\psi(x)|| \, dy \]
\[+ \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left( 1 + \frac{1}{|x|} \right) u(y) ||\varphi_n(y)||u(x)||\psi(x)|| \, dx \, dy.\]

Firstly, we estimate
\[\frac{1}{2\pi} \int_{|y|>M} u^2(y) \, dy \int_{\mathbb{R}^2} \left[ 2 \log(1 + |x|) + \log(1 + |y|) \right] ||\varphi_n(x)||\psi(x)|| \, dx \leq \left( \frac{1}{\pi} \int_{|y|>M} u^2(y) \, dy \right) ||\varphi_n||_X ||\psi||_X + \frac{1}{2\pi} \int_{|y|>M} \log(1 + |y|) u^2(y) \, dy \right) ||\varphi_n||_2 ||\psi||_2
\]
\[\leq \frac{D}{\pi} \left( \int_{|y|>M} u^2(y) \, dy \right) + \frac{D}{2\pi} \left( \int_{|y|>M} \log(1 + |y|) u^2(y) \, dy \right) \leq \frac{\varepsilon}{2}.
\]

We claim that for every \(M > 0\), there exists \(L > 0\) such that for any \(y \in \mathbb{R}^2\) with \(|y| \leq M\) and for any \(x \in \mathbb{R}^2\) we have
\[|\log \frac{1 + |x|}{1 + |x|} < L.\]

Indeed for any \(x \in \mathbb{R}^2\) and \(y \in \mathbb{R}^2, |y| \leq M\) we have
\[\frac{1 + |x|}{1 + |x|} \leq 1 + M.\]

Now take \(R = 2M - 1 > 0\), we have that \(\frac{M}{1 + |x|} < 1/2\) for any \(x \in \mathbb{R}^2\), and \(|x| \geq R\).
It follows that for any \( x, y \in \mathbb{R}^2 \) with \(|x| \geq |y|, |x| \geq R\) and \(|y| \leq M\):

\[
\frac{1 + |x - y|}{1 + |x|} \geq \frac{1 + |x| - |y|}{1 + |x|} \geq 1 - \frac{|y|}{1 + |x|} \geq 1 - \frac{M}{1 + |x|} > \frac{1}{2}
\]

On the other hand, if \(|x| \leq R\):

\[
\frac{1 + |x - y|}{1 + |x|} \geq \frac{1}{1 + R} = \frac{1}{2M}.
\]

Conversely if \(|x| \leq |y|\), we infer that \(|x| \leq M\) and

\[
\frac{1 + |x - y|}{1 + |x|} \geq \frac{1}{1 + M}.
\]

We conclude that there exists \( L > 0 \) such that (3.6) holds. It follows that

\[
\frac{1}{2\pi} \int_{|y| \leq M} \log \left( \frac{1 + |x - y|}{1 + |x|} \right) |u^2(y)||\varphi_n(x)||\psi(x)| \, dx \, dy
\]

\[
\leq \frac{L}{2\pi} \int_{|y| \leq M} u^2(y) \, dy \int_{\mathbb{R}^2} |\varphi_n(x)||\psi(x)| \, dx \leq \frac{L}{2\pi} \left( \int_{|y| \leq M} u^2(y) \, dy \right) \|\varphi_n\|_2 \|\psi\|_2
\]

\[
\leq \frac{\Gamma}{2\pi} \|\varphi_n\|_2 \|\psi\|_X = \frac{\Gamma L}{2\pi} \|\varphi_n\|_2,
\]

where \( \Gamma = \int_{|y| \leq M} u^2(y) \, dy \).

By Hardy-Sobolev-Littlewood inequality we have

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left( 1 + \frac{1}{|x - y|} \right) u^2(y)||\varphi_n(x)||\psi(x)| \, dx \, dy
\]

\[
\leq \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x - y|} u^2(y)||\varphi_n(x)||\psi(x)| \, dx \, dy \leq c_1 \|u\|_{8/3}^2 \|\varphi_n\|_{8/3} \|\psi\|_{8/3}
\]

\[
\leq c_2 \|u\|_{8/3}^2 \|\varphi_n\|_{8/3} \|\psi\|_X = c_2 \|u\|_{8/3}^2 \|\varphi_n\|_{8/3}
\]

where \( c_1, c_2 > 0 \) are suitable constants. Moreover we can take \( R > 0 \) such that

\[
\frac{D}{\pi} \left( \int_{|y| > R} \log(1 + |y|) u^2(y) \, dy \right)^{1/2} \|u\|_2 < \frac{\varepsilon}{4}.
\]
We have
\[
\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x - y|) u(y) |\varphi_n(y) u(x)| |\psi(x)| \, dx \, dy
\]
\[
\leq \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x|) u(y) |\varphi_n(y) u(x)| |\psi(x)| \, dx \, dy
\]
\[
+ \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |y|) u(y) |\varphi_n(y) u(x)| |\psi(x)| \, dx \, dy
\]
\[
\leq \frac{1}{\pi} \|u\|_2 \|u\|_X \|\varphi_n\|_2 \|\psi\|_X
\]
\[
+ \frac{1}{\pi} \int_{|y| \leq R} \log(1 + |y|) u(y) |\varphi_n(y)| \, dy \int_{\mathbb{R}^2} u(x) |\psi(x)| \, dx
\]
\[
+ \frac{1}{\pi} \int_{|y| > R} \log(1 + |y|) u(y) |\varphi_n(y)| \, dy \int_{\mathbb{R}^2} u(x) |\psi(x)| \, dx
\]
\[
\leq \frac{1}{\pi} \|u\|_X^2 \|\varphi_n\|_2 \frac{1}{\pi} \log(1 + R) \|u\|_2^2 \|\varphi_n\|_2 \|\psi\|_2
\]
\[
+ \frac{D}{\pi} \left( \int_{|y| > R} \log(1 + |y|) u^2(y) \, dy \right)^{1/2} \|u\|_2 \|\psi\|_X
\]
\[
\leq \frac{1}{\pi} \|u\|_X^2 \|\varphi_n\|_2 \frac{1}{\pi} \log(1 + R) \|u\|_2^2 \|\varphi_n\|_2
\]
\[
+ \frac{D}{\pi} \left( \int_{|y| > R} \log(1 + |y|) u^2(y) \, dy \right)^{1/2} \|u\|_2
\]
\[
\leq \frac{1}{\pi} \left( 1 + \log(1 + R) \right) \|u\|_X^2 \|\varphi_n\|_2 + \frac{\varepsilon}{4}.
\]

By the Hardy-Sobolev-Littlewood inequality we have
\[
\frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left( 1 + \frac{1}{|x - y|} \right) |\varphi_n(y) u(y) u(x)| |\psi(x)| \, dx \, dy
\]
\[
\leq \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x - y|} u(y) |\varphi_n(y) u(x)| |\psi(x)| \, dx \, dy
\]
\[
\leq \frac{1}{\pi} \|u\|_{S/3}^2 \|\varphi_n\|_{S/3} \|\psi\|_X = \frac{1}{\pi} \|u\|_{S/3}^2 \|\varphi_n\|_{S/3}.
\]

Finally we conclude that
\[
(3.7) \sup_{\psi \in X} \langle (K\varphi_n, \psi) \rangle \leq \frac{3\varepsilon}{4} + c_3 \|u\|_{S/3}^2 \|\varphi_n\|_{S/3} + \frac{\Gamma L}{2\pi} \|\varphi_n\|_2 + \frac{1}{\pi} (1 + \log(1 + R)) \|u\|_X^2 \|\varphi_n\|_2
\]

for some \( c_3 \) positive constant. Taking into account that \( X \) is compactly embedded into \( L^s(\mathbb{R}^2) \) for any \( s \in [2, +\infty) \) \([10]\), we derive that \( \|\varphi_n\|_2 \to 0 \) and \( \|\varphi_n\|_{S/3} \to 0 \) as \( n \to +\infty \). Therefore there exists \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \)
\[
c_3 \|u\|_{S/3}^2 \|\varphi_n\|_{S/3} + \frac{\Gamma L}{2\pi} \|\varphi_n\|_2 + \frac{1}{\pi} (1 + \log(1 + R)) \|u\|_X^2 \|\varphi_n\|_2 < \frac{\varepsilon}{4}.
\]

We derive that \( \lim_{n \to +\infty} \langle (K\varphi_n, \psi) \rangle = 0 \), uniformly with respect to \( \psi \). Therefore \( K \) is compact and the proof is complete. \( \square \)
**Definition 3.5.** In the sequel, we will denote by $U$ the unique positive solution of (3.1) such that

$$U(0) = \max_{x \in \mathbb{R}^2} U(x).$$

From the non-degeneracy result, we can infer the following convexity property of $I''(U)$.

**Proposition 3.6.** The operator $I''(U)$ has only one negative eigenvalue, and therefore there exists $\delta > 0$ such that

$$I''(U)[v, v] \geq \delta \|v\|_X^2$$

for every $v \perp_X \text{span}\{U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\}$, where $\perp_X$ means orthogonality with respect to the inner product $\langle \cdot | \cdot \rangle_X$.

**Proof.** Since

$$-\Delta U + U + \frac{1}{2\pi} \left[ \log*|U|^2 \right] U = 0,$$

we find that

$$I''(U)[U, U] = \langle \mathcal{L}(U)U, U \rangle = -2 \left( \int_{\mathbb{R}^2} |\nabla U|^2 + \int_{\mathbb{R}^2} |U|^2 \right) < 0.$$

Let now $\varphi \in \ker I''(U)$, namely $\varphi \in X$ and $I''(U)\varphi = 0$ in $X^*$. It follows that $I''(U)\varphi = 0$ also in $\tilde{X}^*$, but $\varphi \in \tilde{X}$, so that $\mathcal{L}(U)\varphi = 0$. Hence $\varphi \in \text{span}\{\partial_1 U, \partial_2 U\}$.

On the other hand, if $\varphi \in \text{span}\{\partial_1 U, \partial_2 U\}$, then $\mathcal{L}(U)\varphi = 0$ in $\tilde{X}^*$. Let $\psi \in X$. By density, $\psi$ is the limit in $X$ of a sequence $g_n \in C^\infty_0(\mathbb{R}^2)$. It follows that

$$I''(U)[\varphi, \psi] = \lim_{n \to +\infty} I''(U)[\varphi, g_n] = \lim_{n \to +\infty} \langle \mathcal{L}(U)\varphi, g_n \rangle = 0$$

and thus $\varphi \in \ker I''(U)$. This shows that $\ker I''(U) = \text{span}\{\partial_1 U, \partial_2 U\}$.

Taking into account that $U$ is a Mountain Pass solution, by Proposition 3.1, we deduce that there exists $\delta > 0$ such that (3.8) holds. $\square$

4. The perturbation technique

We will look for solutions to (2.2) near the embedded submanifold $Z = \{z_\xi \mid \xi \in \mathbb{R}^2\}$, where we set $z_\xi(x) = U(x - \xi)$. Although the norm of $X$ is not invariant under the group of translations defined on $X$ by

$$\tau_\xi u: x \in \mathbb{R}^2 \mapsto u(x - \xi),$$

the elementary inequality

$$\log(1 + |x - y|) \leq \log(1 + |x| + |y|) \leq \log(1 + |x|) + \log(1 + |y|)$$

yields that $u \in X$ and $\xi \in \mathbb{R}^2$ implies $\tau_\xi u \in X$. It follows that $U(\cdot - \xi) = \tau_\xi U \in X$ for every $\xi \in \mathbb{R}^2$. The invariance under translation of $I$ then implies that $Z$ is a manifold of critical points of $I$.

We will show that each point of $Z$ is an approximate critical point of $I_\varepsilon$, and that there exists a true critical point of $I_\varepsilon$ located in a tubular neighborhood of $Z$, provided $\varepsilon$ is small enough.
Let assumption (V) be satisfied. Then there exists a constant $C > 0$ such that, for every $\xi \in \mathbb{R}^2$ and every $\varepsilon > 0$ sufficiently small, we have
\[
\|I'_\xi(z_\xi)\| \leq C \left( \varepsilon \|\nabla V(0)\| + \varepsilon^2 \right).
\]

Proof. Since $z_\xi$ is a critical point of $I$, it follows easily that
\[
|I'_\xi(z_\xi)[v]|^2 \leq \|v\|^2 \int_{\mathbb{R}^2} |V(\varepsilon x) - 1|^2 |z_\xi|^2 \, dx
\]
for any $v \in X$. Using the boundedness of $D^2V$ and the exponential decay of $z_\xi$ at infinity, we can prove easily that
\[
\int_{\mathbb{R}^2} |V(\varepsilon x) - 1|^2 |z_\xi|^2 \, dx \leq C\varepsilon^2 \|\nabla V(0)\|^2 + C\varepsilon^4.
\]

Proposition 4.2. There exists a constant $\tilde{C} > 0$ and a constant $M > 0$ such that for every $\xi \in \mathbb{R}^2$, $|\xi| \leq M$, we have
\[
(4.1) \quad I''(z_\xi)[\varphi, \varphi] \geq \tilde{C} \|\varphi\|^2_X
\]
for every $\varphi \perp_X \left( \text{span} \left\{ z_\xi, \frac{\partial z_\xi}{\partial x}, \frac{\partial z_\xi}{\partial y} \right\} \right)$, where $\perp_X$ means orthogonality with respect to the inner product $\langle \cdot, \cdot \rangle_X$.

Proof. For the sake of simplicity we denote here $\perp_X$ by $\perp$. In order to get a contradiction, we suppose that there exists a sequence $\{\xi_n\}_n$ in $\mathbb{R}^2$ such that $\xi_n \rightarrow 0$ and there exists a sequence $\{\varphi_n\}_n \subset X$ such that $\varphi_n \in \left( \text{span} \left\{ z_{\xi_n}, \frac{\partial z_{\xi_n}}{\partial x}, \frac{\partial z_{\xi_n}}{\partial y} \right\} \right) \perp$, $\varphi_n \rightarrow \tilde{\varphi}$ in $X$ and in $H^1(\mathbb{R}^2)$, $\varphi_n \rightarrow \tilde{\varphi}$ in $L^2(\mathbb{R}^2)$,
\[
\|\varphi_n\|_X = 1 \quad \text{for every } n \in \mathbb{N},
\]
and
\[
I''(z_{\xi_n})[\varphi_n, \varphi_n] \leq \frac{1}{n}.
\]
Assume that $\tilde{\varphi} \neq 0$. Then,
\[
\frac{1}{n} \geq I''(z_{\xi_n})[\varphi_n, \varphi_n] = I''(U)[\varphi_n, \varphi_n] + I''(z_{\xi_n})[\varphi_n, \varphi_n] - I''(U)[\varphi_n, \varphi_n]
\]
\[
\geq I''(U)[\varphi_n, \varphi_n] - \|I''(z_{\xi_n}) - I''(U)\| \|\varphi_n\|_X^2 = I''(U)[\varphi_n, \varphi_n] - o(1)
\]
as $n \rightarrow +\infty$. Indeed, the functional $I''$ is continuous at the point $U$, and the exponential decay of $U$ at infinity (see Theorem 3.2) immediately yields that $z_{\xi_n} \rightarrow U$ strongly in $X$.

We claim that $\tilde{\varphi} \perp U$, $\tilde{\varphi} \perp \frac{\partial U}{\partial x}$ and $\tilde{\varphi} \perp \frac{\partial U}{\partial y}$ in $X$. We only prove the first orthogonality property, the other two being similar. By assumption, we have that $\varphi_n \perp z_{\xi_n}$, $\varphi_n \perp \frac{\partial z_{\xi_n}}{\partial x}$, $\varphi_n \perp \frac{\partial z_{\xi_n}}{\partial y}$ for every $n \in \mathbb{N}$. Now,
\[
\langle \varphi_n \mid U \rangle_X = -\langle \varphi_n \mid z_{\xi_n} - U \rangle_X.
\]
We will show that the linear operator $R$ exists and is continuous, so that the equation $\bar{\varphi} = (\varepsilon, \xi)$ shows that $\bar{\varphi} \perp \frac{\partial R}{\partial y}$. As a consequence, $0 \geq \liminf_{n \to +\infty} I''(\varphi_n, \varphi_n) \geq \liminf_{n \to +\infty} I''(U)[\varphi_n, \varphi_n] \geq I''(U)[\bar{\varphi}, \bar{\varphi}] \geq \delta \|\bar{\varphi}\|^2_X$.

Here we have used Theorem 3.6 and the fact that the linear operator $I''(U)$ is the sum of a lower semicontinuous operator $A$ and of a compact operator $K$ introduced in (3.3) and (3.4). This shows that $\varphi = 0$.

But now, exactly as before,

$$\frac{1}{n} \geq I''(U)[\varphi_n, \varphi_n] - o(1) = (A\varphi_n, \varphi_n) + \langle K\varphi_n, \varphi_n \rangle - o(1) \geq C\|\varphi_n\|_X - o(1) \geq C - o(1),$$

a contradiction. \qed

In what follows, for each $z_\xi \in Z$, we denote by $P^\varepsilon_\xi$ the orthogonal projection of $X$ onto $\left(T_{z_\xi} Z\right)^\perp$, where $X$ is endowed with the norm (2.3) (depending on $\varepsilon$) and $\perp$ is the orthogonality with respect the associated inner product. We aim to construct, for every $z_\xi \in Z$, an element $w = w(\varepsilon, \xi) \in \left(T_{z_\xi} Z\right)^\perp$ such that

(4.2) \[ P^\varepsilon_\xi I'_\xi(z_\xi + w) = 0 \]

and

$$\left(\text{Id} - P^\varepsilon_\xi\right)I'_\xi(z_\xi + w) = 0.$$  

Clearly, the point $u_\varepsilon = z_\xi + w(\varepsilon, z_\xi)$ will be a critical point of $I_\varepsilon$, i.e. a solution to (2.2).

To solve the auxiliary equation (4.2) we first write

$$P^\varepsilon_\xi I'_\xi(z_\xi + w) = P^\varepsilon_\xi I'_\xi(z_\xi) + P^\varepsilon_\xi I''_\xi(z_\xi)[w] + R(z_\xi, w).$$

We will show that $R(z_\xi, w) = o(\|w\|)$ uniformly with respect to $z_\xi \in Z$ for $|\xi|$ bounded. Then we will show that the linear operator

$$B_{\varepsilon, \xi} = - \left(P^\varepsilon_\xi I''_\xi(z_\xi)\right)^{-1}$$

exists and is continuous, so that the equation $P^\varepsilon_\xi I'_\xi(z_\xi + w) = 0$ is equivalent to

$$w = B_{\varepsilon, \xi} \left(P^\varepsilon_\xi I'_\xi(z_\xi) + R(z_\xi, w)\right),$$

a fixed-point problem in the unknown $w \in \left(T_{z_\xi} Z\right)^\perp$.

**Lemma 4.3.** Let $M$ be the constant introduced in Proposition 4.2. For $\varepsilon$ sufficiently small, the operator $L_\xi = P^\varepsilon_\xi \circ I'_\xi(z_\xi) \circ P^\varepsilon_\xi$ is invertible, and there exists a constant $C > 0$ such that

$$\left\| L_\xi^{-1} \right\| \leq C,$$

for every $\xi \in \mathbb{R}^2$ with $|\xi| \leq M$. 

Proof. Let \( \xi \in \mathbb{R}^2 \), \(|\xi| \leq M\). For simplicity we denote here \( P^\varepsilon_\xi \) by \( P_\xi \). We write \( \left(T_{\xi Z} \right)^\perp = V_1 \oplus V_2 \), where

\[
V_1 = \text{span}\{P_\xi z_\xi\}
\]
\[
V_2 = \left(\text{span}\{z_\xi\} \oplus T_{\xi Z} \right)^\perp,
\]
so that \( V_1 \perp V_2 \). We claim that for \( \varepsilon \to 0^+ \)

\[
(4.3) \quad \|z_\xi - P_\xi z_\xi\| = o(1), \quad I''_\varepsilon(z_\xi)[z_\xi, \cdot] = \left(\frac{1}{\pi} \log \|z_\xi\|^2\right)z_\xi + o(1).
\]

It follows from (4.3) that

\[
L_\xi(z_\xi) = P_\xi \circ I''_\varepsilon(z_\xi)[P_\xi z_\xi] = P_\xi \left(I''_\varepsilon(z_\xi)[z_\xi, \cdot] + o(1)\right)
\]
\[
= P_\xi \left(-\frac{1}{\pi} \log \|z_\xi\|^2 \right)z_\xi + o(1)
\]
\[
= \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x-y|\|z_\xi(x)\|^2\|z_\xi(y)\|^2 \, dx \, dy\right) z_\xi + o(1).
\]

As a consequence, the operator \( L_\xi \), in matrix form with respect to the decomposition \( \left(T_{\xi Z} \right)^\perp = V_1 \oplus V_2 \), can be written as

\[
L_\xi = \begin{pmatrix}
\left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x-y|\|z_\xi(x)\|^2\|z_\xi(y)\|^2 \, dx \, dy\right) \text{Id} + o(1) & o(1) \\
o(1) & A_\xi
\end{pmatrix}
\]

where the operator \( A_\xi \) satisfies \( A_\xi \geq C^{-1} \text{Id} \) according to (4.1) in Proposition 4.2.

It now follows from (3.5) that \( L_\xi \) is negative definite on \( V_1 \) and thus globally invertible on \( \left(T_{\xi Z} \right)^\perp \). It remains to prove the previous claim.

Recalling the definition of \( z_\xi(x) = U(x - \xi) \) and the exponential decay of \( U \) at infinity, we see that

\[
\langle z_\xi \mid \partial_{ij} z_\xi \rangle = -\langle z_\xi \mid \partial_{x_i} z_\xi \rangle = -\langle z_\xi \mid \partial_{x_i} z_\xi \rangle x + \int_{\mathbb{R}^2} (V(\varepsilon x) - 1) z_\xi \partial_{x_i} z_\xi \, dx
\]
\[
= o(1) \quad \text{as} \quad \varepsilon \to 0
\]

for every \( i \in \{1, \ldots, n\} \). Therefore, \( \|z_\xi - P_\xi z_\xi\| = o(1) \) as \( \varepsilon \to 0 \). This proves the first part of (4.3). The second identity is proved as follows: we compute

\[
I''_\varepsilon(z_\xi)[z_\xi, v] = I''_\varepsilon(z_\xi)[z_\xi, v] + \int_{\mathbb{R}^2} (V(\varepsilon x) - 1) z_\xi v \, dx
\]

and recall that \( z_\xi \) solves

\[
-\Delta z_\xi + z_\xi = \frac{1}{2\pi} \left[ \log \frac{1}{\|z_\xi\|^2} \right] z_\xi.
\]

Since \( \int_{\mathbb{R}^2} (V(\varepsilon x) - 1) z_\xi v \, dx \) for \( \varepsilon \) small, we conclude that, for any \( v \in X \), we have

\[
I''_\varepsilon(z_\xi)[z_\xi, v] = I''_\varepsilon(z_\xi)[z_\xi, v] + \int_{\mathbb{R}^2} (V(\varepsilon x) - 1) z_\xi v \, dx = \left\langle \left(\frac{1}{\pi} \log \|\cdot\| \, \|z_\xi\|^2\right) z_\xi \mid v \right\rangle + o(1)\|v\|.
\]

\( \square \)
Proposition 4.4. Let assumption (V) be satisfied. Then for every \( \varepsilon \) small, there exists a unique \( w = w(\varepsilon, \xi) \in (T_{z\xi}Z)^d \) with \( |\xi| \leq M \) such that \( I_\varepsilon'(z\xi + w(\varepsilon, \xi)) \in T_{z\xi}Z \). The function \( \varepsilon, \xi \mapsto w(\varepsilon, \xi) \) is of class \( C^1 \) with respect to \( \xi \), and there holds

\[
\|w(\varepsilon, \xi)\| \leq C \left( \varepsilon|\nabla V(0)| + \varepsilon^2 \right)
\]

Moreover, the function \( \Theta_\varepsilon(\xi) = I_\varepsilon(z\xi + w(\varepsilon, \xi)) \) is of class \( C^1 \) and the condition \( \Theta_\varepsilon'(\xi_0) = 0 \) implies \( I_\varepsilon'(z_{\xi_0} + w(\varepsilon, \xi_0)) = 0 \).

Proof. Let us recall that our aim is to construct a solution \( w \in (T_{z\xi}Z)^d \) to (4.2). We write

\[
I_\varepsilon'(z\xi + w) = I_\varepsilon'(z\xi) + I_\varepsilon''(z\xi)[w] + R(z\xi, w),
\]

where

\[
R(z\xi, w) = I_\varepsilon'(z\xi + w) - I_\varepsilon'(z\xi) - I_\varepsilon''(z\xi)[w].
\]

By the invertibility of \( L_\xi = P_\xi^* \circ I_\varepsilon'(z\xi) \circ P_\xi^* \) (see Lemma 4.3), the function \( w \) solves (4.2) if and only if

\[
(4.6) \quad w = N_{\varepsilon, \xi}(w),
\]

where

\[
N_{\varepsilon, \xi}(w) = -L_\xi^{-1} \left( P_\xi^* \circ I_\varepsilon'(z\xi) + P_\xi^* R(z\xi, w) \right).
\]

We can now show that, for \( \varepsilon \) sufficiently small, equation (4.6) can be solved by means of the Contraction Mapping Theorem.

First of all, understanding the \( L^2 \)-duality, we have

\[
I_\varepsilon'(z\xi + w) = -\Delta z\xi + V_\varepsilon z\xi - \Delta w + V_\varepsilon w + \frac{1}{2\pi} \left[ \log * (z\xi + w)^2 \right] (z\xi + w),
\]

\[
I_\varepsilon'(z\xi) = -\Delta z\xi + V_\varepsilon z\xi + \frac{1}{2\pi} \left[ \log * |z\xi|^2 \right] z\xi
\]

and

\[
I_\varepsilon''(z\xi)[w] = -\Delta w + V_\varepsilon w + \frac{1}{2\pi} \left[ \log * |w|^2 \right] w + \frac{1}{\pi} \left[ \log * (z\xi w) \right] z\xi.
\]

Therefore, again with respect to the \( L^2 \)-duality,

\[
R(z\xi, w) = I_\varepsilon'(z\xi + w) - I_\varepsilon'(z\xi) - I_\varepsilon''(z\xi)[w]
\]

\[
= \frac{1}{\pi} \left[ \log * (z\xi w) \right] w + \frac{1}{2\pi} \left[ \log * |w|^2 \right] w + \frac{1}{2\pi} \left[ \log * |z\xi|^2 \right] z\xi.
\]

We have

\[
(4.7) \quad \|R(z\xi, w)\| \leq C \left( \|w\|^2 + o(\|w\|^2) \right)
\]

as \( \|w\| \to 0 \).
Indeed we have for any $\phi \in X$

\[
\pi |R(z_\xi, w), \phi| \leq \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x - y| z_\xi(x) w(x) w(y) \phi(y) \, dx \, dy \right|
\]

\[
+ \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x - y| w(x)^2 z_\xi(y) \phi(y) \, dx \, dy \right| + \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x - y| w(x)^2 \phi(y) \, dx \, dy \right|
\]

\[
\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x|) + \log(1 + |y|) z_\xi(x) w(x)^2 w(y)^2 \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x|) + \log(1 + |y|) w(x)^2 |z_\xi(y)| \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + |x|) + \log(1 + |y|) |w(x)|^2 |z_\xi(y)| \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + \frac{1}{x - y}) |z_\xi(x)| |w(x)|^2 |w(y)| \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + \frac{1}{x - y}) |w(x)|^2 |w(y)| \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log(1 + \frac{1}{x - y}) |w(x)|^2 |z_\xi(y)| \phi(y) \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |z_\xi(x)||w(x)||w(y)||\phi(y)| \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |w(x)|^2 |z_\xi(y)||\phi(y)| \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |w(x)|^2 |w(y)||\phi(y)| \, dx \, dy
\]

\[
\leq \|w\|_2 \|\phi\|_2 \|z_\xi\|_2 \|w\|_x + \|z_\xi\|_2 \|w\|_2 \|w\|_x \|\phi\|_x + \|z_\xi\|_2 \|w\|_2 \|w\|_2 \|w\|_x \|\phi\|_x
\]

\[
+ \|w\|_2 \|\xi\|_2 \|\phi\|_2 + \|w\|_2 \|w\|_3 \|\phi\|_2 + \|w\|_2 \|w\|_3 \|w\|_x \|\phi\|_x
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |z_\xi(x)||w(x)||w(y)||\phi(y)| \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |w(x)|^2 |z_\xi(y)||\phi(y)| \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{x - y} |w(x)|^2 |w(y)||\phi(y)| \, dx \, dy
\]

\[
\leq \|w\|_2 \|\phi\|_2 \|z_\xi\|_2 \|w\|_x + \|z_\xi\|_2 \|w\|_2 \|w\|_x \|\phi\|_x + \|z_\xi\|_2 \|w\|_2 \|w\|_2 \|w\|_x \|\phi\|_x
\]

\[
+ \|w\|_2 \|\phi\|_2 + \|w\|_2 \|w\|_3 \|\phi\|_2 + \|w\|_2 \|w\|_3 \|w\|_x \|\phi\|_x
\]

\[
+ C \|w\|_{S/3} \|\phi\|_{S/3} + C \|w\|_{S/3} \|z_\xi\|_{S/3} \|\phi\|_{S/3} + C \|w\|_{S/3} \|\phi\|_{S/3}
\]

for some suitable positive constant $C$. Since the norm in (3.3) is equivalent to $\| \cdot \|_X$ and $\phi \in X$ is arbitrary, we have

\[(4.8) \quad \| R(z_\xi, w) \| \leq C_1 \| z_\xi \|_2 \| w \|_2^2 + C_2 \| w \|_3^3 \]

for some $C_1, C_2$ positive constants and thus we infer (4.7). In a similar way we can deduce that

\[(4.9) \quad \| R(z_\xi, w_1) - R(z_\xi, w_2) \| \leq C (\| w_1 \| + \| w_2 \| + o(\| w_1 - w_2 \|)) \| w_1 - w_2 \|
\]

Using Lemma 4.1, (4.7) and (4.9), we find that

\[
\| N_{z_\xi}(w) \| \leq C \left( \varepsilon \| \nabla V(0) \| + \varepsilon^2 + \| w \|_2^2 + o(\| w \|_2^2) \right)
\]

\[
\| N_{z_\xi}(w_1) - N_{z_\xi}(w_2) \| \leq C (\| w_1 \| + \| w_2 \| + o(\| w_1 - w_2 \|)) \| w_1 - w_2 \|. 
\]
As a consequence, the operator \( N_{\varepsilon, \xi} \) is a contraction on the closed subset
\[
W_C = \left\{ w \in (T_z\xi \mathcal{Z})^\perp \mid \|w\| \leq C \left( \varepsilon |\nabla V(0)| + \varepsilon^2 \right) \right\},
\]
provided that \( C > 0 \) is sufficiently large, and \( \varepsilon > 0 \) is sufficiently small. The Contraction Mapping Theorem yields a unique fixed point \( w = w(\varepsilon, \xi) \) of \( N_{\varepsilon, \xi} \) in \( W_C \) such that (4.4) holds.

The last statements of the Proposition are proved by a straightforward modification of the arguments contained in [2, pp. 129–130], so we present only a sketch of the ideas.

Let us define the map \( H : \mathbb{R}^2 \times X \times \mathbb{R}^2 \times \mathbb{R}^2 \to X \times \mathbb{R}^2 \),
\[
H(\xi, w, \alpha, \varepsilon) = \left( I_\varepsilon'(z_\xi + w) - \sum_{i=1}^{2} \alpha_i \partial_{x_i} z_\xi \right), \quad \left( \langle w \mid \partial_{x_1} z_\xi \rangle, \langle w \mid \partial_{x_2} z_\xi \rangle \right).
\]
In particular, \( w \in (T_z\xi \mathcal{Z})^\perp \) solves the equation \( P_{\xi} I_\varepsilon'(z_\xi + w) = 0 \) if and only if \( H(\xi, w, \alpha, \varepsilon) = 0 \).

With estimates similar to those we have shown above, we can prove that \( \frac{\partial H}{\partial (w, \alpha)}(\xi, 0, 0, \varepsilon) \) is uniformly invertible in \( \xi \) for \( \varepsilon \) small enough. By the Implicit Function Theorem, the map \( \xi \mapsto (w_\xi, \alpha_\xi) \) is of class \( C^1 \).

Differentiating the identity \( H(\xi, w_\xi, \alpha_\xi, \varepsilon) = 0 \) with respect to \( \xi \), we obtain
\[
\frac{\partial H}{\partial \xi}(\xi, w, \alpha, \varepsilon) + \frac{\partial H}{\partial (w, \alpha)}(\xi, w, \alpha, \varepsilon) \frac{\partial (w_\xi, \alpha_\xi)}{\partial \xi} = 0,
\]
hence
\[
\|\partial_\xi w\| \leq C \left\| \frac{\partial H}{\partial (w, \alpha)}(\xi, w, \alpha, \varepsilon)[\partial_\xi z_\xi, \alpha] \right\|
\]
\[
\leq C \left( \|I_\varepsilon''(z_\xi + w)[\partial_\xi z_\xi]\| + |\alpha| + \|w\| \right).
\]
It now follows easily that (4.4) holds.

5. The reduced functional

Following [2], the manifold
\[
Z_\varepsilon = \left\{ z_\xi + w(\varepsilon, \xi) \mid \xi \in \mathbb{R}^2, \ |\xi| \leq M, \ \varepsilon \ll 1 \right\}
\]
is a natural constraint for \( I_\varepsilon \), in the sense that any critical point of \( I_\varepsilon \) constrained to \( Z_\varepsilon \) is a free critical point of \( I_\varepsilon \). To prove the existence of a critical point of the functional \( I_\varepsilon \), it is therefore sufficient to show that the constrained functional \( \Theta_\varepsilon : \overline{B(0, M)} \subset \mathbb{R}^2 \to \mathbb{R} \) defined by
\[
\Theta_\varepsilon(\xi) = I_\varepsilon(z_\xi + w)
\]
possesses a critical point. To this aim, we evaluate
\[ \Theta_\varepsilon(\xi) = I(z_\xi + w) + \frac{1}{2} \int_{\mathbb{R}^2} (V_\varepsilon - 1) |z_\xi + w|^2 \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla (z_\xi + w)|^2 + |z_\xi + w|^2 \, dx \]
\[ + \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| |z_\xi(x) + w(x)|^2 |z_\xi(y) + w(y)|^2 \, dx \, dy \]
\[ + \frac{1}{2} \int_{\mathbb{R}^2} (V_\varepsilon - 1) |z_\xi + w|^2 \, dx \]
\[ = I(z_\xi) + \frac{1}{2} \int_{\mathbb{R}^2} (V_\varepsilon - 1) |z_\xi + w|^2 + R_\varepsilon(w), \]
where
\[ R_\varepsilon(w) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla w|^2 + w^2) \, dx + \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| |w(x)|^2 |w(y)|^2 \, dx \, dy \]
\[ + \int_{\mathbb{R}^2} (\nabla z_\xi \cdot \nabla w + z_\xi w) \, dx \]
\[ + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| |z_\xi(x)w(x) - z_\xi(y)w(y)|^2 \, dx \, dy \]
\[ + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| |z_\xi(x)w(x) - z_\xi(y)w(y)| \, dx \, dy \]
\[ + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x - y| |z_\xi(x)w(x) - z_\xi(y)w(y)| \, dx \, dy. \]

According to Proposition 4.4, the function \( \Theta_\varepsilon \) can be expanded as
\[ \Theta_\varepsilon(\xi) = b_0 + \frac{1}{2} \int_{\mathbb{R}^2} (V(\varepsilon x) - 1) |z_\xi + w|^2 \, dx + o(\varepsilon^2), \]
where \( b_0 = I(z_\xi) = I(U) \). Let us define \( Q_2 = D^2 V(0) \) and the function \( \Gamma: \mathbb{R}^2 \to \mathbb{R} \),
\[ \Gamma(\xi) = \int_{\mathbb{R}^2} Q_2(x) |z_\xi(x)|^2 \, dx. \]

From now on, we will suppose for the sake of definiteness that \( x_0 = 0 \) is a proper local minimum of \( V \), so that \( D^2 V(0) \) is a positive-definite quadratic form. The case of a proper local maximum can be treated analogously.

**Lemma 5.1.** The point \( \xi = 0 \) is a strict local minimum for \( \Gamma \).

**Proof.** By oddness, \( \partial_1 \partial_2 \Gamma(0) = 0 \). Since \( \nabla Q_2(x) \cdot x = 2Q_2(x) > 0 \), we conclude that \( D^2 \Gamma(0) \) is positive-definite. \( \square \)

We fix a number \( \bar{\xi} > 0 \) in such a way that \( \bar{\xi} < M \) and
\[ \Gamma(\xi) > \Gamma(0) \]
for every \( \xi \in \mathbb{R} \setminus \{0\} \), where \( B = B(0, \bar{\xi}) \).

**Lemma 5.2.** For \( \varepsilon > 0 \) sufficiently small, there results \( \Theta_\varepsilon(0) < \inf_{|\xi| = \bar{\xi}} \Theta_\varepsilon(\xi) \).
Proof. We recall the asymptotic expansion \((5.1)\) and observe that
\[
\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (V_{\varepsilon} - 1) |z_{\varepsilon} + w|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^2} Q_2 |z_{\varepsilon}|^2 \, dx = \frac{1}{2} \Gamma(\xi).
\]
Hence
\[
\Theta_{\varepsilon}(\xi) - \Theta_{\varepsilon}(0) = \frac{1}{2} \varepsilon^2 (\Gamma(\xi) - \Gamma(0)) + o(\varepsilon^2).
\]
It now follows from the choice of \(\tilde{\xi}\) that \(\Theta_{\varepsilon}(\xi) - \Theta_{\varepsilon}(0) > 0\) if \(|\xi| = \tilde{\xi}\) and \(\varepsilon > 0\) is small enough. The proof is complete. \(\square\)

Proof of Theorem 1.1. We have just shown that the function \(\Theta_{\varepsilon}\) must have a minimum at some \(\xi = \xi(\varepsilon)\) in the ball \(B \subset B(0, M)\). This gives rise to a critical point \(u_{\varepsilon} = z_{\varepsilon} + w(\varepsilon, \xi) \in Z^\varepsilon\) of the functional \(I_{\varepsilon}\) with \(\varepsilon \sim 0\). Now, for every \(\xi \in \overline{B}\),
\[
0 \leq \Theta_{\varepsilon}(\xi) - \Theta_{\varepsilon}(\xi(\varepsilon)) = \frac{1}{2} \varepsilon^2 (\Gamma(\xi) - \Gamma(\xi(\varepsilon))) + o(\varepsilon^2);
\]
as \(\varepsilon \to 0\), we may assume that \(\xi(\varepsilon) \to \xi_0\) and we obtain \(\Gamma(\xi) - \Gamma(\xi_0) \geq 0\) for every \(\xi \in \overline{B}\). Our choice of \(\xi\) forces \(\xi_0 = 0\), so that \(\xi(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Hence \(u_{\varepsilon} = z_{\varepsilon(\varepsilon)} + w(\varepsilon, \xi(\varepsilon)) \to U\).

Coming back to the system \((1.3)\) we obtain the existence of pairs of solution \((v_{\varepsilon}, E_{\varepsilon})\) where
\[
v_{\varepsilon}(x) = u_{\varepsilon}\left(\frac{x}{\varepsilon}\right) \simeq U\left(\frac{x}{\varepsilon}\right)
\]
and
\[
E_{\varepsilon}(x) = \omega\left(\frac{x}{\varepsilon}\right) = -\int_{\mathbb{R}^2} \log \left|\frac{x}{\varepsilon} - y\right| |u_{\varepsilon}(y)|^2 \, dy
= -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \log \left|\frac{x - z}{\varepsilon}\right| |u_{\varepsilon}\left(\frac{z}{\varepsilon}\right)|^2 \, dz
= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \log \left|\frac{\varepsilon}{x - z}\right| |v_{\varepsilon}(z)|^2 \, dz.
\]
Therefore we have \(E_{\varepsilon}(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \log \frac{1}{|x - z|} |v_{\varepsilon}(z)|^2 \, dz + c_{\varepsilon}\), with \(c_{\varepsilon} = \frac{\log \varepsilon}{\varepsilon^2} \|v_{\varepsilon}\|^2_2\). \(\square\)

Remark 5.3. Our Theorem 1.1 can be slightly generalized. Indeed, we can assume that the potential \(V\) has a non-degenerate critical point at some \(x_0\), in the sense \(\nabla V(x_0) = 0\) and there exists an integer \(m \geq 1\) such that \(D^{2m}V(x_0)\) is either positive- or negative-definite. The proof then requires only a higher-order expansion of \(I_{\varepsilon}(z + w)\) in \(\varepsilon\). We omit the details for brevity.

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