ON THE CAUCHY PROBLEM FOR A FOUR-COMPONENT CAMASSA-HOLM TYPE SYSTEM

ZENG ZHANG  
Department of Mathematics, Sun Yat-sen University  
Guangzhou, 510275, China  

ZHAOYANG YIN  
Department of Mathematics, Sun Yat-sen University  
Guangzhou, 510275, China  
and  
Faculty of Information Technology  
Macau University of Science and Technology, Macau, China  

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Abstract. This paper is concerned with a four-component Camassa-Holm type system proposed in [37], where its bi-Hamiltonian structure and infinitely many conserved quantities were constructed. In the paper, we first establish the local well-posedness for the system. Then we present several global existence and blow-up results for two integrable two-component subsystems.

1. Introduction. Recently, Li, Liu and Popowicz [37] proposed the following four-component Camassa-Holm system:

\[
\begin{align*}
    m_1_t + (\Gamma m_1)_x + n_2(g_1 g_2 - \Gamma) + m_1(f_2 g_2 + 2 f_1 g_1) &= 0, \\
    m_2_t + (\Gamma m_2)_x - n_1(g_1 g_2 - \Gamma) - m_2(f_1 g_1 + 2 f_2 g_2) &= 0, \\
    n_1_t + (\Gamma n_1)_x - m_2(f_1 f_2 - \Gamma) - n_1(f_2 g_2 + 2 f_1 g_1) &= 0, \\
    n_2_t + (\Gamma n_2)_x + m_1(f_1 f_2 - \Gamma) + n_2(f_1 g_1 + 2 f_2 g_2) &= 0, \\
    m_i = u_i - u_{ixx}, n_i = v_i - v_{ixx}, &i = 1, 2,
\end{align*}
\]

where \( \Gamma \) is an arbitrary function, and

\[
    f_1 = u_2 - v_1, f_2 = u_1 + v_2, g_1 = v_2 + u_{1x}, g_2 = v_1 - u_{2x}.
\]

In [37], the authors constructed the bi-Hamiltonian structure and the infinitely many conserved quantities of Eq.(1).

Since \( \Gamma \) is an arbitrary function, Eq.(1) recovers many known Camassa-Holm type equations from reductions. As \( m_1 = u_2, m_2 = n_1, u_2 = \frac{1}{4}, \Gamma = u_1 \), Eq.(1) is

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reduced to the standard Camassa-Holm (CH) equation

\[ m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \]  

(2)

which was derived by Camassa and Holm [4] in 1993 as a model for the unidirectional propagation of shallow water waves over a flat bottom. It has a bi-Hamiltonian structure [9, 31] and is completely integrable [4, 8]. One of the remarkable properties of the CH equation is the existence of peakons [1, 4, 5, 21, 22]. It is worth pointing out that the peakons, as solitary waves with a peak at the crest, model accurately the famous wave of greatest height pattern of the free-boundary incompressible Euler equations [10, 11, 14, 49]. The Cauchy problems of the CH equation has been studied extensively: It has been shown that the CH equation is locally well-posed [12, 16, 23, 27, 28, 38, 47]. It has global strong solutions [7, 12, 16, 27, 28] and also blow-up solutions in finite time [7, 12, 15, 19, 27, 28]. Besides, it has global weak solutions [3, 13, 20, 55].

As \( u_1 = u_2 = 0, v_1 = 1, \Gamma = v_2, \) Eq.(1) is reduced to the Degasperis-Procesi (DP) equation [25]

\[ m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}. \]  

(3)

The Degasperis-Procesi equation can be regarded as another model for nonlinear shallow water dynamics [17, 18]. It was proved in [24] that the DP equation has a bi-Hamiltonian structure and infinitely many conserved laws, and admits peakon solutions which are analogous to CH peakons. Besides, the DP equation also has shock peakons [26, 41]. The Cauchy problems of the DP equation has also been studied extensively. See [6, 26, 40, 57, 58] for the study of local well-posedness, blow-up phenomena, global strong solutions and global weak solutions.

As \( m_1 = m_2 = n_1, m_2 = n_1, \Gamma = 4(u_1 + u_{1x})(u_2 - u_{2x}), \) Eq.(1) is reduced to the cubic Camassa-Holm equation [29, 30, 43, 44]

\[ m_t + 4(m(u^2 - u_x^2))_{xx} = 0, \quad m = u - u_{xx}. \]  

(4)

Its Lax pair, peakon and cusped soliton solutions, local well-posedness and blow-up phenomena have been studied in [32, 44].

As \( u_1 = u_2 = 0, v_1 = v_2, \Gamma = v_1v_2, \) Eq.(1) is reduced to the Novikov equation [42]

\[ m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}. \]  

(5)

Its Lax pair, peakon and soliton solutions, local well-posedness, global existence and blow-up phenomena have been studied in [33, 35, 42, 50, 51].

As \( m_1 = m_2, m_2 = m_1, u_1 = k_1u_2 + k_2, \Gamma = 4(k_1(u_2^2 - u_{2x}^2) + k_2u_2), \) Eq.(1) is reduced to the generalized Camassa-Holm equation with both quadratic and cubic nonlinearly [29]

\[ m_t + 4k_1([u^2 - u_x^2]_x + 4k_2(2mu_x + um_x)) = 0, \quad m = u - u_{xx}, \quad k_1, k_2 \in \mathbb{R}. \]  

(6)

One can refer to [39, 45] for the study of its Lax pair, peakons, weak kinks, kink-peakon interaction, and local well-posedness.

As \( m_1 = m_2, m_2 = n_1, \) making the change of variable

\[ (m, n, H) = (-2\sqrt{2}m_2, 2\sqrt{2}m_1, -\Gamma), \]
Eq. (1) is reduced to the following two-component Camassa-Holm system proposed by Xia, Qiao and Zhou in [54]:
\[
\begin{aligned}
& m_t = (mH)_x + mH - \frac{1}{2} m(u - u_x)(v + v_x), \\
& n_t = (nH)_x - nH + \frac{1}{2} n(u - u_x)(v + v_x), \\
& m = u - u_{xx}, n = v - v_{xx},
\end{aligned}
\]
where $H$ is an arbitrary function. Xia, Qiao and Zhou [54] provided the Lax pair and the infinitely many conservations laws for the system (7), and studied the bi-Hamiltonian structures and multi-peakon solutions of the following four integrable two-component equations:

**Case 1.** $H = \frac{1}{2} (uv - u_x v_x)$,
\[
\begin{aligned}
& m_t = \frac{1}{2} [(uv - u_x v_x) m - \frac{1}{2} (uv_x - vu_x) m], \\
& n_t = \frac{1}{2} [(uv - u_x v_x) n + \frac{1}{2} (uv_x - vu_x) n], \\
& m = u - u_{xx}, n = v - v_{xx},
\end{aligned}
\]
which is exactly the dispersionless version of the system derived in [45, 52].

**Case 2.** $H = \frac{1}{2} (u - u_x)(v + v_x)$,
\[
\begin{aligned}
& m_t = \frac{1}{2} [(u - u_x)(v + v_x) m], \\
& n_t = \frac{1}{2} [(u - u_x)(v + v_x) n], \\
& m = u - u_{xx}, n = v - v_{xx},
\end{aligned}
\]
which is exactly the equation derived by Song, Qu and Qiao [48].

**Case 3.** $H = 0$,
\[
\begin{aligned}
& m_t = -\frac{1}{2} (u - u_x)(v + v_x) m, \\
& n_t = \frac{1}{2} (u - u_x)(v + v_x) n, \\
& m = u - u_{xx}, n = v - v_{xx},
\end{aligned}
\]

**Case 4.** $H = \frac{1}{2} (uv_x - vu_x)$,
\[
\begin{aligned}
& m_t = \frac{1}{2} [(uv_x - vu_x) m - \frac{1}{2} (uv - u_x v_x) m], \\
& n_t = \frac{1}{2} [(uv_x - vu_x) n + \frac{1}{2} (uv - u_x v_x) n], \\
& m = u - u_{xx}, n = v - v_{xx}.
\end{aligned}
\]

Recently, Yan, Qiao and Yin [56] studied the local well-posedness and derived a precise blow-up scenario and a blow-up result for the strong solutions to Eq. (8). Later, Zhang and Yin [59] improved the results stated in [56] for Eq. (8) and obtained several global existence or blow-up results for Eq. (9).

For more research on other multi-component Camassa-Holm type systems, one can refer to [34, 36, 46, 53].

The aim of this paper is to establish the local well-posedness for the Cauchy problem of Eq. (1) in Besov spaces, and present several global existence and blow-up results for Eq. (10)-Eq. (11).

The rest of our paper is then organized as follows. In Section 2, we recall the Littlewood-Paley decomposition and some basic properties of the Besov spaces. In Section 3, we establish the local well-posedness for Eq. (1). The last section is devoted to proving several global existence and blow-up results for Eq. (10)-Eq. (11).
Throughout the paper, $C > 0$ stands for a generic constant, $A \lesssim B$ denotes the relation $A \leq CB$. Since all function spaces in this paper are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in the notations of function spaces if there is no ambiguity.

2. Preliminaries. In this section, we recall several useful lemmas which will be used in the sequel.

**Definition 2.1.** [2] Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p, r}$ consists of all $u \in S' (\mathbb{R})$ such that

$$
\|u\|_{B^s_{p, r}} \overset{\text{def}}{=} \left\| \left(2^{js} |\triangle_j u|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{l^r (\mathbb{Z})} < \infty,
$$

there $\triangle_j$ is the Littlewood-Paley decomposition operator [2].

Let us give some classical properties of the Besov spaces.

**Lemma 2.2.** [2] The set $B^s_{p, r}$ is a Banach space, and satisfies the Fatou property, namely, if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B^s_{p, r}$, then an element $u$ of $B^s_{p, r}$ and a subsequence $u_{\psi(n)}$ exist such that

$$
\lim_{n \to \infty} u_{\psi(n)} = u \text{ in } S' \text{ and } \|u\|_{B^s_{p, r}} \leq \text{Climinf}_{n \to \infty} \|u_{\psi(n)}\|_{B^s_{p, r}}.
$$

**Lemma 2.3.** [2] Let $m \in \mathbb{R}$ and $f$ be an $S^m$-multiplier (i.e. $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies that for each multi-index $\alpha$, there exists a constant $C_{\alpha}$ such that $|\partial^\alpha f (\xi)| \leq C_{\alpha} (1 + |\xi|)^m-|\alpha|, \forall \xi \in \mathbb{R}$). Then the operator $F(D)$ is continuous from $B^s_{p, r}$ to $B^{s-m}_{p, r}$.

**Lemma 2.4.** [23] (i) For $s > 0$ and $1 \leq p, r \leq \infty$, there exists $C = C(d, s)$ such that

$$
\|uv\|_{B^s_{p, r}} \leq C (\|u\|_{L^\infty} \|v\|_{B^s_{p, r}} + \|v\|_{L^\infty} \|u\|_{B^s_{p, r}}).
$$

(ii) If $1 \leq p, r \leq \infty$, $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$, $(s_2 \geq \frac{1}{p}, \text{if } r = 1)$ and $s_1 + s_2 > \max \{0, \frac{2}{p} - 1\}$, there exists $C = C(s_1, s_2, p, r)$ such that

$$
\|uv\|_{B^{s_1 + s_2}_{p, r}} \leq C (\|u\|_{B^{s_1}_{p, r}} \|v\|_{B^{s_2}_{p, r}}).
$$

**Lemma 2.5.** [2, 23] Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, $s > -\min \{\frac{1}{p_1}, 1 - \frac{1}{p_1}\}$. Assume $f_0 \in B^s_{p_1, r}, F \in L^1 (0, T; B^s_{p_1, r}), v \in L^\rho (0, T; B^{-M}_{\infty, \infty})$ for some $\rho > 1$ and $M > 0$, and

$$
\partial_x v \in L^1 (0, T; B_{p_1, \infty}^{s-1} \cap \mathcal{L}^\infty), \quad \text{if } s < 1 + \frac{1}{p_1},
$$

$$
\partial_x v \in L^1 (0, T; B_{p_1, r}^{s-1}), \quad \text{if } s > 1 + \frac{1}{p_1}, \text{ or } \{s = 1 + \frac{1}{p_1} \text{ and } r = 1\}.
$$

Then the following transport equation

$$
\begin{cases}
7xt + \nu \cdot \nabla f = F \\
7t = 0 = f_0,
\end{cases}
$$

has a unique solution $f \in C([0, T]; B^s_{p_1, r})$, if $r < \infty$, or $f \in L^\infty (0, T; B^s_{p_1, r}) \cap \left( \bigcap_{s' < s} C([0, T]; B^{s'}_{p_1, r}) \right)$, if $r = \infty$.

Moreover, the following inequality holds true:

$$
\|f(t)\|_{B^s_{p, r}} \leq \|f_0\|_{B^s_{p, r}} + \int_0^t \|F(\tau)\|_{B^s_{p, r}} d\tau + C \int_0^t V_\rho (\tau)\|f(\tau)\|_{B^s_{p, r}} d\tau
$$

(15)
with
\[ V'_{p_1}(t) = \begin{cases} \| \partial_x v(t) \|_{B_{p_1}^{s,1}} \cap L^\infty, & \text{if } s < 1 + \frac{1}{p_1}, \\ \| \partial_x v(t) \|_{B_{p_1}^{s-1,1}}, & \text{if } s > 1 + \frac{1}{p_1} \text{ or } \{ s = 1 + \frac{1}{p_1} \text{ and } r = 1 \}. \end{cases} \] (16)

As a consequence of Lemma 2.3 and the Young inequality, we have the following inequalities which will be frequently used in the sequel.

**Proposition 1.** For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), we have
\[
\| u \|_{B_{p,r}^s} = \| (1 - \partial_x^2)^{-1} m \|_{B_{p,r}^{s-2}} \approx \| m \|_{B_{p,r}^{s-2}}, \\
\| u \|_{L^\infty} = \| \frac{1}{2} e^{-|x|} * m \|_{L^\infty} \leq \| m \|_{L^\infty}, \\
\| u_x \|_{L^\infty} = \| \frac{1}{2} (\text{sign}(x) e^{-|x|}) * m \|_{L^\infty} \leq \| m \|_{L^\infty},
\]
where \( m = u - u_{xx} \).

3. **Local well-posedness.** In this section, we study the local well-posedness for Eq. (1). We rewrite Eq. (1) as follows:
\[
\begin{cases}
M_t + \Gamma M_x + A(\Gamma, \Gamma_x)M + B(U, U_x)M = 0, \\
M|_{t=0} = M_0,
\end{cases}
\] (17)
where \( M = (m_1, m_2, n_1, n_2)^T \), \( M_0 = (m_{10}, m_{20}, n_{10}, n_{20})^T \), \( U = (u_1, u_2, v_1, v_2)^T \), and
\[
A(\Gamma, \Gamma_x) = \begin{pmatrix}
\Gamma_x & 0 & 0 & -\Gamma \\
0 & \Gamma_x & \Gamma & 0 \\
0 & \Gamma & \Gamma_x & 0 \\
-\Gamma & 0 & 0 & \Gamma_x
\end{pmatrix},
\]
\[
B(U, U_x) = \begin{pmatrix}
f_2 g_2 + 2 f_1 g_1 & 0 & 0 & g_1 g_2 \\
0 & -(f_1 g_1 + 2 f_2 g_2) & -g_1 g_2 & 0 \\
0 & -f_1 f_2 & -(f_2 g_2 + 2 f_1 g_1) & 0 \\
f_1 f_2 & 0 & 0 & f_1 g_1 + 2 f_2 g_2
\end{pmatrix},
\]
where
\( f_1 = u_2 - v_{1x} \), \( f_2 = u_1 + v_{2x} \), \( g_1 = v_2 + u_{1x} \), \( g_2 = v_1 - u_{2x} \).

**Theorem 3.1.** Let \( 1 \leq p, r \leq \infty \), \( s > \max\{1 - \frac{1}{p}, \frac{1}{r}\} \), and \( M_0 \in B_{p,r}^s \). Suppose that \( \Gamma = \Gamma(u_1, u_2, v_1, v_2, u_{1x}, u_{2x}, v_{1x}, v_{2x}) \) is a polynomial of degree \( l \). Then there exists a time \( T > 0 \) such that Eq. (17) has a unique solution \( M \in L^\infty(0, T; B_{p,r}^s) \) with
\[
E^s_{p,r}(T) \triangleq \begin{cases}
C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1}), & \text{if } r < \infty, \\
\bigcap_{s' < s} \left(C([0, T]; B_{p,r}^{s'}) \cap C^1([0, T]; B_{p,r}^{s'-1})\right), & \text{if } r = \infty.
\end{cases}
\]
Further, suppose that \( T^* > 0 \) is the maximal existence time of the corresponding solution \( M \) to Eq. (17). If \( T^* \) is finite, then we have
\[
\int_0^{T^*} \| M(\tau) \|_{L^\infty}^q d\tau = \infty,
\]
where \( q = \max\{l, 2\} \).
Proof of Theorem 3.1. We shall proceed as follows.

**Step 1.** Uniqueness and continuity with respect to the initial data. We can get (a) if \( s > \max\{1 - \frac{1}{p}, \frac{1}{p}\} \), but \( s \neq 2 + \frac{1}{p} \), then

\[
\| (M^1 - M^2)(t) \|_{B_{p,r}^{-1}} \leq \| M_{10} - M_{20} \|_{B_{p,r}^{-1}} e^{\int_0^t (\| M^1(\tau) \|_{B_{p,r}^0} + \| M^2(\tau) \|_{B_{p,r}^0}) + 1) d\tau}
\]

(b) if \( s = 2 + \frac{1}{p} \), then

\[
\| (M^1 - M^2)(t) \|_{B_{p,r}^{-1}} \leq \| M_{10} - M_{20} \|_{B_{p,r}^{-1}} e^{\int_0^t (\| M^1(\tau) \|_{B_{p,r}^0} + \| M^2(\tau) \|_{B_{p,r}^0}) + 1) d\tau}
\]

where \( \theta \in (0, 1) \).

**Step 2.** Existence. We can obtain

(i). constructing approximate solutions

\[
\begin{cases}
M^n + 1 + \Gamma_n M^n = -A^m M^n - B^n M^n, \\
M^n_{|t=0} = M_0,
\end{cases}
\]

where \( M^n = (m_1^n, m_2^n, n_1^n, n_2^n, \cdots, n_N^n)^T, U^n = (u_1^n, u_2^n, v_1^n, v_2^n, \cdots, v_N^n)^T, \Gamma^n = \Gamma(U^n, U^n), A^n = A(\Gamma^n, \Gamma^n), B^n = B(U^n, U^n) \).

(ii). \((M^n)_{n \in \mathbb{N}}\) is bounded in \( L^\infty(0, T; B_{p,r}^*) \),

(iii). \((M^n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \( L^\infty(0, T; B_{p,r}^{-1}) \).

(iv). passing to the limit and to concluding that \( M \) is a solution of Eq. (17).

**Step 3.** If \( T^* < \infty \), and \( \int_0^{T^*} \| M(\tau) \|_{L^\infty} d\tau < \infty \), then we have \( \limsup_{t \to T^*} M(t) \|_{L^\infty} < \infty \). Thus we can extent the solution \( M \) beyond \( T^* \), which is a contradiction with the assumption of \( T^* \).

For more details about the proof, one can refer to the proofs of Theorem 3.1 and Theorem 3.2 in [59], where the system we deal with is

\[
\begin{cases}
M_t = H(U, U_z)M + A(H, H_z)M + B(U, U_z)M, \\
M_{|t=0} = M_0,
\end{cases}
\]

where \( M = (m_1, \cdots, m_N, n_1, \cdots, n_N)^T, M_0 = (m_0, \cdots, m_{N_0}, n_1, \cdots, n_{N_0})^T, U = (u_1, \cdots, u_N, v_1, \cdots, v_N)^T, H = H(U, U_z) \) is a polynomial of degree \( l \), and

\[
A(H, H_z) = \begin{pmatrix}
0 & H_z I_{N \times N} + H I_{N \times N} \\
0 & H_z I_{N \times N} - H I_{N \times N}
\end{pmatrix},
\]

\[
B(U, U_z) = \begin{pmatrix}
B_{11} & 0 \\
0 & B_{22}
\end{pmatrix},
\]

with

\[
B_{11} = \frac{1}{(N + 1)^2} \begin{pmatrix}
(u_1 - u_{1x})(v_1 + v_{1x}) & \cdots & (u_1 - u_{1x})(v_N + v_{Nx}) \\
\vdots & \ddots & \vdots \\
(u_N - u_{Nx})(v_1 + v_{1x}) & \cdots & (u_N - u_{Nx})(v_N + v_{Nx})
\end{pmatrix}
\]

\[
+ \frac{1}{(N + 1)^2} \sum_{i=1}^{N} (u_i - u_{ix})(v_i + v_{ix}) I_{N \times N}.
\]
Theorem 3.2. More precisely, we have

We use the classical ODE theory to set up the local well-posedness of Eq. (22).

The terms \( A(\Gamma, \Gamma_x) \) and \( B(U, U_x) \) in this paper can be estimated exactly in the same way as we did for the terms \( A(H, H_x) \) and \( B(U, U_x) \) in [59]. Thus following along exactly the same lines of the proofs of Theorem 3.1 and Theorem 3.2 in [59], one can prove the theorem. For the sake of simplicity, we omit the details here. ☐

Note that, if \( \Gamma = 0 \), then Eq. (1) is reduced to the following ODE system:

\[
\begin{aligned}
M_t + A(M)M &= 0, \\
M_{t=0} &= M_0,
\end{aligned}
\]

where \( M = (m_1, m_2, n_1, n_2)^T \), \( M_0 = (m_{10}, m_{20}, n_{10}, n_{20})^T \), \( U = (u_1, u_2, v_1, v_2)^T = (1 - \partial^2_{x})^{-1}M, f_1 = u_2 - v_1, f_2 = u_1 + v_2, g_1 = v_2 + u_1, g_2 = v_1 - u_2, \) and

\[
A(M) = \begin{pmatrix}
0 & 0 & g_1 & g_2 \\
0 & 0 & -f_1 & f_2 \\
0 & -f_2 & 0 & 0 \\
f_2g_2 + 2f_1g_1 & 0 & 0 & 0
\end{pmatrix}.
\]

We can use the classical ODE theory to set up the local well-posedness of Eq. (22).

Theorem 3.2. Let \( X = L^\infty \), or \( X = B^s_{p,r} \cap L^\infty \) with \( s > 0 \). Assume \( M_0 \in X \). Then, there exists a time \( T > 0 \) depending only on \( \|M_0\|_X \) such that Eq. (22) has a unique solution \( M \in C^\infty([0, T]; X) \). Further, suppose that \( T^* > 0 \) is the maximal existence time of the corresponding solution \( M \) to Eq. (22). If \( T^* \) is finite, then we have

\[
\int_0^{T^*} \|M(\tau)\|^2_{L^\infty} d\tau = \infty.
\]

Proof of Theorem 3.2. If \( X = L^\infty \), using Proposition 1, one can readily have that

\[
\|A(M)M\|_{L^\infty} \leq \|A(M)\|_{L^\infty} \|M\|_{L^\infty} \lesssim \left( \|U\|_{L^\infty} + \|U_x\|_{L^\infty} \right)^2 \|M\|_{L^\infty} \lesssim \|M\|_{L^\infty}^2,
\]

and

\[
\|A(M_1)M_1 - A(M_2)M_2\|_{L^\infty} \\
\lesssim \|A(M_1) - A(M_2)\|_{L^\infty} \|M_1\|_{L^\infty} + \|A(M_2)\|_{L^\infty} \|M_1 - M_2\|_{L^\infty} \\
\lesssim (\|U_1 - U_2\|_{L^\infty} + \|U_{1x} - U_{2x}\|_{L^\infty}) (\|U_1\|_{L^\infty} + \|U_{1x}\|_{L^\infty}) \\
+ \|U_2\|_{L^\infty} + \|U_{2x}\|_{L^\infty}) \|M_1\|_{L^\infty} + \|A(M_2)\|_{L^\infty} \|M_1 - M_2\|_{L^\infty} \\
\lesssim (\|M_1\|_{L^\infty} + \|M_2\|_{L^\infty})^2 \|M_1 - M_2\|_{L^\infty}.
\]

On the other hand, if \( X = B^s_{p,r} \cap L^\infty \) with \( s > 0 \), using Lemma 2.4 and Proposition 1, we get

\[
\|A(M)M\|_{B^s_{p,r} \cap L^\infty} \lesssim \|A(M)\|_{B^s_{p,r} \cap L^\infty} \|M\|_{B^s_{p,r} \cap L^\infty} \\
\lesssim (\|U\|_{B^s_{p,r} \cap L^\infty} + \|U_x\|_{B^s_{p,r} \cap L^\infty})^2 \|M\|_{B^s_{p,r} \cap L^\infty} \\
\lesssim \|M\|_{B^s_{p,r} \cap L^\infty}^2.
\]
and
\[ \|A(M_1)M_1 - A(M_2)M_2\|_{B^s_{p,r} \cap L^\infty} \leq \|A(M_1) - A(M_2)\|_{B^s_{p,r} \cap L^\infty} \|M_1\|_{B^s_{p,r} \cap L^\infty} + \|A(M_2)\|_{B^s_{p,r} \cap L^\infty} \|M_1 - M_2\|_{B^s_{p,r} \cap L^\infty} \]
\[ \leq \left(\|U_1 - U_2\|_{B^s_{p,r} \cap L^\infty} + \|U_1 - U_2\|_{B^s_{p,r} \cap L^\infty}\right) \left(\|M_1\|_{B^s_{p,r} \cap L^\infty} + \|M_2\|_{B^s_{p,r} \cap L^\infty}\right) \|M_1 - M_2\|_{B^s_{p,r} \cap L^\infty} \]
\[ \leq \left(\|M_1\|_{B^s_{p,r} \cap L^\infty} + \|M_2\|_{B^s_{p,r} \cap L^\infty}\right)^2 \|M_1 - M_2\|_{B^s_{p,r} \cap L^\infty}. \]
Thus, for \( X = L^\infty \), or \( X = B^s_{p,r} \cap L^\infty \) with \( s > 0 \), \( A(M)M \) is locally Lipschitz with respect to the norm \( X \). Therefore, using the classical Picard scheme, or using the Banach fixed point theorem, or applying Theorem 3.2 in [2], one can readily prove the existence and uniqueness of this theorem.

As for the blow-up criterion, since the case \( X = L^\infty \) is much more simple, we only consider the case \( X = B^s_{p,r} \cap L^\infty \) with \( s > 0 \). Using Lemma 2.4, we get
\[ \|A(M)M\|_{B^s_{p,r} \cap L^\infty} \leq \|A(M)\|_{B^s_{p,r}} \|M\|_{L^\infty} + \|A(M)\|_{L^\infty} \|M\|_{L^\infty} \]
\[ \leq \left(\|f_1\|_{B^s_{p,r}} + \|f_2\|_{B^s_{p,r}} + \|f_1\|_{L^\infty} + \|f_2\|_{B^s_{p,r}}\right) \|M\|_{L^\infty} + \|f_1\|_{B^s_{p,r}} \|M\|_{L^\infty} \]
\[ \leq \left(\|f_1\|_{B^s_{p,r}} + \|f_2\|_{B^s_{p,r}}\right) \|M\|_{L^\infty} + \|f_1\|_{B^s_{p,r}} \|M\|_{L^\infty} \]
\[ \leq \|M\|_{L^\infty}^2 \|M\|_{B^s_{p,r} \cap L^\infty}. \]
The Gronwall lemma then implies that if \( T^* < \infty \), and \( \int_0^{T^*} \|M(\tau)\|_{B^s_{p,r} \cap L^\infty} d\tau < \infty \), then we have \( \limsup_{t \to T^*} \|M(t)\|_{B^s_{p,r} \cap L^\infty} < \infty \). We can extend the solution \( M \) beyond \( T^* \), which is a contradiction with the assumption of \( T^* \). This completes the proof of the theorem.

Utilizing Theorem 3.1-3.2, we readily obtain the following corollaries.

**Corollary 1.** Let \( M_0 \in B^s_{p,r} \) with \( 0 < s < \infty \) and \( s > 0 \). Assume that \( \limsup_{t \to T^*} \|M(t)\|_{B^s_{p,r} \cap L^\infty} = \infty \). Then the corresponding solution \( M \) to Eq.(17) blows up in finite time if and only if \( T^* = \infty \).

**Corollary 2.** Let \( X = L^\infty \), or \( X = B^s_{p,r} \cap L^\infty \) with \( s > 0 \). Assume that \( M_0 \in X \) and \( T^* > 0 \) is the maximal existence time of the corresponding solution \( M \) to Eq.(22). Then the solution \( M \) blows up in finite time if and only if \( \limsup_{t \to T^*} \|M(t)\|_{L^\infty} = \infty \).

**Remark 1.** Apparently, for every \( s \in \mathbb{R}, B^s_{2,2} = H^s \). Theorem 3.1 and Corollary 1 hold true in the corresponding Sobolev spaces \( H^s \) with \( s > \frac{1}{2} \). Theorem 3.2 and Corollary 2 hold true in the corresponding Sobolev spaces \( H^s \) with \( s > 0 \).

4. **Global existence and blow-up phenomena for the two-component subsystems.** As already mentioned in the introduction, if \( m_1 = n_2, m_2 = n_1 \) and \((m, n) = (\sqrt{2}m_2, \sqrt{2}m_1)\), then Eq.(22) is reduced to Eq.(10), and if \( m_1 = n_2, m_2 = n_1 \), \((m, n, H) = (\sqrt{2}m_2, \sqrt{2}m_1, -\Gamma)\) and \( H = \frac{1}{2}(uv_x - vu_x)\), then
Eq. (17) is reduced to Eq. (11). Let us now present several precise global existence and blow-up results for Eq. (10) and Eq. (11).

**Theorem 4.1.** Let \( m_0, n_0 \in L^\infty \) (or \( m_0, n_0 \in H^s \cap L^\infty \) with \( s > 0 \)). Assume that supp \( m_0 \in [b, \infty) \), supp \( n_0 \in (-\infty, a] \), with \( a \leq b \). Then the corresponding solution \((m, n)\) to Eq. (10) exists globally in time.

**Proof of Theorem 4.1.** Note that,

\[
m(t, x) = m_0(x) \exp\{-\int_0^t \frac{1}{2}(u - u_x)(v + v_x)(\tau, x)d\tau\},
\]

\[
n(t, x) = n_0(x) \exp\{\int_0^t \frac{1}{2}(u - u_x)(v + v_x)(\tau, x)d\tau\},
\]

we have

\[\text{supp } m(t) \in [b, +\infty), \text{ supp } n(t) \in (-\infty, a].\]

On the other hand, since

\[u(t, x) = (1 - \partial_x^2)^{-1} m(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} m(t, y)dy,\]

\[v(t, x) = (1 - \partial_x^2)^{-1} n(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} n(t, y)dy,\]

we have

\[u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y)dy = 0, \text{ if } x \leq b,\]

\[v(t, x) + v_x(t, x) = e^x \int_x^{\infty} e^{-y} n(t, y)dy = 0, \text{ if } x \geq a.\]

Thus, \((u - u_x)(v + v_x) = 0\). Therefore \((m, n)(t, x) = (m_0(x), n_0(x))\) exists globally in time.

\[\square\]

**Theorem 4.2.** Let \( m_0, n_0 \in H^s \cap L^\infty \) with \( s > 0 \), and \( m_0, n_0 \) do not change sign. Then the corresponding solution \((m, n)\) to Eq. (10) exists globally in time.

**Proof of Theorem 4.2.** Let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq. (10). One can assume without loss of generality that \( m_0 \geq 0, n_0 \geq 0 \) for all \( x \in \mathbb{R} \). Thus, (23) and (24) imply that

\[m(t, x) \geq 0, \ n(t, x) \geq 0, \ \forall (t, x) \in [0, T) \times \mathbb{R}. \]  

(25)

Noticing

\[u(t, x) = (1 - \partial_x^2)^{-1} m(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} m(t, y)dy,\]

\[u_x(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^y m(t, y)dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y)dy,\]

we obtain

\[u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y)dy \geq 0, \]

(26)

\[u(t, x) + u_x(t, x) = e^x \int_x^{\infty} e^{-y} m(t, y)dy \geq 0. \]

(27)
which leads to
\[
|u_x(t, x)| \leq u(t, x), \forall (t, x) \in [0, T) \times \mathbb{R}.
\] (28)

Similar arguments yield
\[
|v_x(t, x)| \leq v(t, x), \forall (t, x) \in [0, T) \times \mathbb{R}.
\] (29)

Using Eq.(10), we get
\[
\frac{d}{dt} \int_{\mathbb{R}} m(v + v_x) dx = \frac{d}{dt} \int_{\mathbb{R}} n(u - u_x) dx = \int_{\mathbb{R}} [m_t(v + v_x) + n_t(u - u_x)] dx
\]
\[
= \int_{\mathbb{R}} \frac{1}{2}(u - u_x)(v + v_x)(n(u - u_x) - m(v + v_x)) dx
\]
\[
= \int_{\mathbb{R}} -\frac{1}{2}(u - u_x)(v + v_x) \partial_x((u - u_x)(v + v_x)) dx = 0.
\]

Using Eq.(10) again, and combining the above equality with (25)-(28) give rise to
\[
\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2)
\]
\[
= \int_{\mathbb{R}} (m_t u + n_t v)(t, x) dx
\]
\[
= \int_{\mathbb{R}} [- (u - u_x)(v + v_x) mu + (u - u_x)(v + v_x) n v](t, x) dx
\]
\[
\leq \frac{1}{2} \left( \|((u - u_x)u)(t)\|_{L^\infty} + \|(v + v_x) m)(t)\|_{L^1} + \||(v + v_x)v)(t)\|_{L^\infty} \|((u - u_x) n)(t)\|_{L^1} \right)
\]
\[
\leq \|u(t)\|_{L^\infty} \|(v_0 + v_{0x}) m_0\|_{L^1} + \|v(t)\|_{L^\infty} \|(u_0 - u_{0x}) n_0\|_{L^1}
\]
\[
\leq \frac{1}{2} \left( \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \right) \left( \|(v_0 + v_{0x}) m_0\|_{L^1} + \|(u_0 - u_{0x}) n_0\|_{L^1} \right).
\]

The Gronwall inequality then yields that \( \forall t \in [0, T) \),
\[
\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leq \left( \|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2 \right) e^{\left( \|(v_0 + v_{0x}) m_0\|_{L^1} + \|(u_0 - u_{0x}) n_0\|_{L^1} \right) t},
\] (30)

which leads to
\[
\|m(t)\|_{L^\infty} + \|n(t)\|_{L^\infty} \leq \left( \|m_0\|_{L^\infty} + \|n_0\|_{L^\infty} \right) e^{\int_0^t \frac{1}{2}(u - u_x)(v + v_x)\|_{L^\infty} d\tau}
\]
\[
\leq \left( \|m_0\|_{L^\infty} + \|n_0\|_{L^\infty} \right) e^{\int_0^t \frac{1}{2}u(\tau)\|_{L^\infty} \|v(\tau)\|_{L^\infty} d\tau}
\]
\[
\leq \left( \|m_0\|_{L^\infty} + \|n_0\|_{L^\infty} \right) e^{\int_0^t \|u(\tau)\|_{H^1} \|v(\tau)\|_{H^1} d\tau}
\]
\[
\leq C \exp\{C^t - 1\}, \forall t \in [0, T),
\] (31)

Therefore, in view of Theorem 3.2 and Corollary 2, we conclude that the solution \((m, n)\) exists globally in time. \(\square\)

For Eq.(11), we first consider the following initial value problem
\[
\begin{cases}
q_t(t, x) = -\frac{1}{2} (uv_x - vu_x)(t, q), \ t \in [0, T),
\end{cases}
\]
\[
q(0, x) = x, \ x \in \mathbb{R}.
\] (32)
Lemma 4.3. Let \( m_0, n_0 \in H^s \) \( (s > \frac{1}{2}) \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq.(11). Then Eq.(32) has a unique solution \( q \in C^1([0, T] \times \mathbb{R}; \mathbb{R}) \). Moreover, the mapping \( q(t, \cdot) \) \( (t \in [0, T]) \) is an increasing diffeomorphism of \( \mathbb{R} \), with
\[
q_x(t, x) = \exp\left( \int_0^t -\frac{1}{2} (mv - nu) (\tau, q(\tau, x)) d\tau \right).
\] (33)

Proof. According to Theorem 3.1, we get that \( m, n \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1}) \) with \( s > \frac{1}{2} \), from which we deduce that \( \frac{1}{2} (uv_x - vu_x) \) is bounded and Lipschitz continuous in the space variable \( x \) and of class \( C^1 \) in time variable \( t \), then the classical ODE theory ensures that Eq.(32) has a unique solution \( q \in C^1([0, T] \times \mathbb{R}; \mathbb{R}) \). Differentiating Eq.(32) with respect to \( x \) gives
\[
\begin{align*}
q_{x1}(t, x) &= -\frac{1}{2} (mv - nu) (t, q) q_x(t, x), \quad t \in [0, T), \\
q_x(0, x) &= 1, \quad x \in \mathbb{R},
\end{align*}
\] (34)

which leads to (33). So, the mapping \( q(t, \cdot) \) \( (t \in [0, T]) \) is an increasing diffeomorphism of \( \mathbb{R} \).

Lemma 4.4. Let \( m_0, n_0 \in H^s \) \( (s > \frac{1}{2}) \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq.(11). Then, we have for all \( t \in [0, T) \),
\[
m(t, q(t, x)) q_x(t, x) = m_0(x) \exp\left( -\frac{1}{2} \int_0^t (uv - u_x v_x) (\tau, q(\tau, x)) d\tau \right),
\] (35)
\[
n(t, q(t, x)) q_x(t, x) = n_0(x) \exp\left( \frac{1}{2} \int_0^t (uv - u_x v_x) (\tau, q(\tau, x)) d\tau \right). \quad \text{(36)}
\]

Proof. Combining Eq.(32), Lemma 4.3 and Eq.(11), we have
\[
\frac{d}{dt} (m(t, q(t, x)) q_x(t, x)) = (m_t(t, q) + m_x(t, q) q_t(t, x)) q_x(t, x) + m(t, q) q_{x1}(t, x)
\]
\[
= (m_t(t, q) - \frac{1}{2} (uv_x - vu_x) m_x(t, q)) q_x(t, x)
\]
\[
= -\frac{1}{2} (uv - u_x v_x)(t, q) m(t, q) q_x(t, x).
\]

Therefore, the Gronwall inequality yields (35). Similar arguments lead to (36). This completes the proof of the lemma.

Lemma 4.5. Let \( m_0, n_0 \in H^s \) \( (s > \frac{1}{2}) \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \((m, n)\) to Eq.(11). Assume that \( m_0 \) and \( n_0 \) do not change sign. Then there exists a constant \( C \) depending only on \( \|v_0 m_0\|_{L^1}, \|u_0 n_0\|_{L^1}, \|u_0\|_{H^1} \) and \( \|v_0\|_{H^1} \) such that
\[
|u_x(t, x)| \leq |u(t, x)|, \quad |v_x(t, x)| \leq |v(t, x)|,
\] (37)
\[
\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq C e^{C t}, \quad \forall t \in [0, T]. \quad \text{(38)}
\]

Proof. Without loss of generality, we assume that \( m_0 \geq 0, n_0 \geq 0 \). Next, from (35) and (36), we have \( m, n \geq 0 \). Thus \( u(t, x) = (1 - \partial_x^2)^{-1} m(t, x) \geq 0, v(t, x) = (1 - \partial_x^2)^{-1} n(t, x) \geq 0 \). Thanks to (26)-(29), we have (37) holds true. By Eq.(11),
we have
\[
\frac{d}{dt} \int_{\mathbb{R}} (mv)(t,x)dx = \frac{d}{dt} \int_{\mathbb{R}} (nu)(t,x)dx = \int_{\mathbb{R}} (m_{t}v + n_{t}u)(t,x)dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} ((uv_{x} - vu_{x})(-mv_{x} - nu_{x}) + (uv - v_{x}u_{x})(-mv + nu))(t,x)dx
\]
\[
= - \frac{1}{2} \int_{\mathbb{R}} \partial_{x}((uv - u_{x}v_{x})(uv_{x} - vu_{x}))(t,x)dx = 0.
\]

Finally, using Eq.(11) again and using the above conservation laws, we have
\[
\frac{1}{2} \frac{d}{dt}(\|u(t)\|_{H^{1}}^{2} + \|v(t)\|_{H^{1}}^{2}) = \int_{\mathbb{R}} (m_{t}u + n_{t}v)(t,x)dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} ((uv_{x} - vu_{x})(-mv_{x} - nu_{x}) + (uv - v_{x}u_{x})(-mv + nu))(t,x)dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} ((u^{2} - u_{x}^{2})vm + (v^{2} - v_{x}^{2})un)(t,x)dx
\]
\[
\leq \frac{1}{2} \|((u^{2} - u_{x}^{2})v)(t)\|_{L^{\infty}} \|mv(t)\|_{L^{1}} + \|(v^{2} - v_{x}^{2})u(t)\|_{L^{\infty}} \|nu(t)\|_{L^{1}}
\]
\[
\leq C(\|u(t)\|_{L^{\infty}}^{2} + \|v(t)\|_{L^{\infty}}^{2}) \leq C(\|u(t)\|_{H^{1}}^{2} + \|v(t)\|_{H^{1}}^{2}).
\]

Then the Gronwall lemma yields the desired inequality (38). This completes the proof of the lemma.

**Theorem 4.6.** Let $m_{0}, n_{0} \in H^{s}$ ($s > \frac{1}{2}$), and let $T > 0$ be the maximal existence time of the corresponding solution $(m,n)$ to Eq.(11). Set $Q(t,x) = -\frac{1}{2}(uv_{x} - u_{x}v)(t,x)$. Assume that $m_{0}, n_{0}$ do not change sign, and that there exists some $x_{0} \in \mathbb{R}$ such that $N(0,x_{0}) = |m(0,x_{0})| + |n(0,x_{0})| > 0$ and $Q_{x}(0,x_{0}) = -\frac{1}{2}(m_{0}v_{0} - n_{0}u_{0})(x_{0}) \leq a_{0}$, where $a_{0}$ is the unique negative solution to the following equation
\[
1 + a_{0}(\frac{\alpha}{N(0,x_{0})}) + N(0,x_{0}) \int_{0}^{g(-\frac{\alpha}{N(0,x_{0})})} f(s)ds = 0,
\]
with $f(x) = \exp(e^{C_{x} - 1}) - 1$, $x \geq 0$, $g(x) = \frac{1}{2} \log \left( \log(x + 1) + 1 \right)$, $x \geq 0$.

Then the solution $(m,n)$ blows up at a time $T_{0} \leq g(-\frac{Q_{x}(0,x_{0})}{N(0,x_{0})})$.

**Prof of Theorem 4.6.** It follows from Eq.(11) that
\[
Q_{xt} + Q(Q_{x})_{x} + Q_{x}^{2}
\]
\[
= \frac{1}{2} \left( (1 - \partial_{x}^{2})^{-1} \left( (Q_{x}v) + \partial_{x}(Q_{x}v_{x}) - \frac{1}{2}(uv - u_{x}v_{x})n \right) \right)m
\]
\[
- \left( (1 - \partial_{x}^{2})^{-1} \left( (Q_{x}u) + \partial_{x}(Q_{x}u_{x}) + \frac{1}{2}(uv - u_{x}v_{x})m \right) \right)n
\]
\[
+ \frac{1}{2}(uv - u_{x}v_{x})(mv + nu).
\]
Applying Lemma 4.5 to the first term and the second term on the right hand side of the above inequality yields
\[
(1 - \partial_{x}^{2})^{-1} \left( (Q_{x}v) + \partial_{x}(Q_{x}v_{x}) \right)(t,x)m(t,x)
\]
Integrating from 0 to $t$

By Lemma 4.4, we get

$$\leq \|(1 - \partial_x^2)^{-1}((Q_xv) + \partial_x(Q_xv_x)(t)\|_{L^\infty} |m(t, x)|$$

$$= \left\| \frac{1}{2} e^{-|x|} \ast \left( -\frac{1}{2} (mv - nu)v \right) \right\|_{L^\infty}$$

$$+ \left\| \frac{1}{2} (\text{sign}(x)e^{-|x|}) \ast \left( -\frac{1}{2} (mv - nu)v \right) \right\|_{L^\infty} |m(t, x)|$$

$$\leq C(\|u\|_{L^\infty} + \|v\|_{L^\infty})(\|v\|_{L^\infty} + \|v_x\|_{L^\infty})(\|e^{-|x|} \ast m\|_{L^\infty})$$

$$+ \|e^{-|x|} \ast n\|_{L^\infty} |m(t, x)|$$

$$\leq C(\|u\|_{L^\infty} + \|v\|_{L^\infty})(\|v\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{L^\infty} + \|v\|_{L^\infty})|m(t, x)|$$

$$\leq C e^{Ct} |m|(t, x),$$

where we have used the fact that $m, n$ do not change sign. The left terms can be treated in the same way. We have

$$-(1 - \partial_x^2)^{-1} \left( \frac{1}{2} (uw - u_xv_x)n \right) m - (1 - \partial_x^2)^{-1} \left( (Q_xu) + \partial_x(Q_xu_x) \right)$$

$$+ \left( \frac{1}{2} (uv - u_xv_x)m \right) n + \frac{1}{2} (uv - u_xv_x)(mv + nu) \leq C e^{Ct} (|m| + |n|)(t, x).$$

Plunging the above two inequalities into (39) yields

$$Q_{xt}(t, x_0) + (Q_x(Q_x)_x)(t, x_0) + Q^2_x(t, x_0) \leq C e^{Ct} (|m| + |n|)(t, x_0).$$

By Lemma 4.4, we get

$$\frac{d}{dt} Q_x(t, q(t, x_0)) + Q^2_x(t, q(t, x_0)) \leq C e^{Ct} (|m| + |n|)(t, q(t, x_0))$$

$$\leq C e^{Ct} N(0, x_0) \exp \left( \int_0^t \frac{1}{2} (mv - nu)(\tau, q(\tau, x_0)) d\tau \right)$$

$$\times \exp \left( \frac{1}{2} \int_0^t \|(uv - v_xu_x)(\tau)\|_{L^\infty} d\tau \right)$$

$$= C e^{Ct} N(0, x_0) \exp(\int_0^t -Q_x(\tau, q(\tau, x_0)) d\tau) \exp \left( \frac{1}{2} \int_0^t \|(uv - u_xv_x)(\tau)\|_{L^\infty} d\tau \right).$$

Again using Lemma 4.5, we have

$$\exp \left( \frac{1}{2} \int_0^t \|(uv - v_xu_x)(\tau)\|_{L^\infty} d\tau \right) \leq \exp(C \int_0^t e^{C\tau} d\tau) = \exp(e^{Ct} - 1),$$

from which it follows that

$$\frac{d}{dt} (Q_x(t, q(t, x_0)) \exp(\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau)) \leq C e^{Ct} \exp(e^{Ct} - 1) N(0, x_0).$$

Integrating from 0 to $t$ yields

$$\frac{d}{dt} e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} = e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} Q_x(t, q(t, x_0))$$

$$\leq Q_x(0, x_0) + N(0, x_0) \int_0^t \exp(e^{C\tau} - 1) C e^{C\tau} d\tau$$

$$= Q_x(0, x_0) + N(0, x_0)(\exp(e^{Ct} - 1) - 1).$$
Integrating again from 0 to \( t \) yields
\[
\left( e^{\int_0^t \inf_{\tau \in R} Q_x(\tau, x) \, d\tau} \right) \leq e^{\int_0^t Q_x(\tau, 0, x) \, d\tau} \leq 1 + Q_x(0, x_0) t + N(0, x_0) \int_0^t (\exp(e^{C\tau} - 1) - 1) \, d\tau. \tag{40}
\]

Next, we consider the following function
\[
F(a, t) = 1 + a t + N(0, x_0) \int_0^t f(s) \, ds, a \leq 0,
\]
where \( f(x) = \exp(e^{Cx} - 1) - 1, x \geq 0. \) It is easy to see that
\[
\min_{t \geq 0} F(a, t) = F(a, g(-\frac{a}{N(0, x_0)})) = 1 + ag(-\frac{a}{N(0, x_0)}) + N(0, x_0) \int_0^{g(-\frac{a}{N(0, x_0)})} f(s) \, ds \triangleq G(a), \tag{41}
\]
where \( g(x) = g(x) = \frac{1}{C} \log(\log(x + 1) + 1), x \geq 0, \) is the inverse function of \( f. \)

Differentiating \( G(a) \) with respect to \( a \), we obtain
\[
\frac{d}{da} G(a) = g(-\frac{a}{N(0, x_0)}) - g'(-\frac{a}{N(0, x_0)}) \frac{a}{N(0, x_0)}
+ g'(-\frac{a}{N(0, x_0)}) \frac{a}{N(0, x_0)} \frac{a}{N(0, x_0)}
= g(-\frac{a}{N(0, x_0)}) > 0, \ a < 0.
\]

Notice that
\[
\lim_{a \to -\infty} g(-\frac{a}{N(0, x_0)}) = +\infty.
\]

Thus, we deduce that
\[
\lim_{a \to -\infty} G(a) = -\infty,
\]
which, together with that fact that \( G(0) = 1 \) and the continuity of \( G \), yields that there exists a unique \( a_0 < 0 \) satisfies \( G(a_0) = 0. \) Therefore, \( G(a) \leq 0 \) if \( a \leq a_0. \) Combining this with (40), if \( Q_x(0, x_0) \leq a_0, \) we may find a time \( 0 < T_0 \leq g(-\frac{Q_x(0, x_0)}{N(0, x_0)}) \) such that
\[
e^{\int_0^t \inf_{\tau \in R} Q_x(\tau, x) \, d\tau} \to 0, \text{ as } t \to T_0,
\]
which implies that
\[
\liminf_{t \to T_0} \inf_{x \in R} Q_x(t, x) \to -\infty, \text{ as } t \to T_0.
\]

Thus, we can get \( \limsup_{t \to T_0} \|m(t)\|_{L^\infty} = \infty \) or \( \limsup_{t \to T_0} \|n(t)\|_{L^\infty} = \infty. \) According to Corollary 1, we conclude that the solution \((m, n)\) blows up at the time \( T_0. \)

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E-mail address: zhangzeng534534@163.com
E-mail address: mcsyzy@mail.sysu.edu.cn