On Fifth Order KdV-Type Equation

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Abstract

The dynamics of the highly nonlinear fifth order KdV-type equation is discussed in the framework of the Lagrangian and Hamiltonian formalisms. The symmetries of the Lagrangian produce three commuting conserved quantities that are found to be recursively related to one-another for a certain specific value of the power of nonlinearity. The above cited recursion relations are obeyed with a second Poisson bracket which sheds light on the integrability properties of the above nonlinear equation. It is shown that a Miura-type transformation can be made to obtain the fifth order mKdV-type equation from the fifth order KdV-type equation. The spatial dependence of the fields involved is, however, not physically interesting from the point of view of the solitonic solutions. As a consequence, it seems that the fifth order KdV- and mKdV-type equations are completely independent nonlinear evolution equations in their own right.

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1 Introduction

In the classical soliton theory, the ideas of integrability, elastic scattering of solitons, stability and preservation of their shape, etc., are fairly well established to be very intimately connected. In recent years, the stability of the soliton solutions to the highly nonlinear fifth order KdV-type equations has been studied extensively [1–5]. In particular, the effect of the dispersion due to higher order derivative terms on solitons has been studied for the fifth order KdV- and mKdV-type equations which are useful in the context of fluid mechanics and plasma physics [2-4]. It has been shown that the nonzero dispersion effects lead to the narrowness of solitonic solutions [5]. Furthermore, it has been established that the solitons are stable for \( p < 8 \) where \( p \) corresponds to the power of nonlinearity [6] (see, e.g., eqn. (1) below). The analytic expression for the soliton solution to the fifth order \( KdV(FOKdV) \)-type equation and two analytic solutions to the fifth order \( mKdV(FOmKdV) \)-type equations have also been found [7].

It is obvious that the above equations, their stability and solutions are of considerable theoretical interests for the understanding of some physical processes in the realm of fluid mechanics and plasma physics.

It is a well known fact that 1+1 dimensional (super)integrable equations (e.g., (super)KdV and (super)Boussinesq, etc.) play a prominent role in the understanding of (super)string theory, W-type (super)algebras, statistical mechanics at critical temperature, 2D gravity theories, fluid mechanics, etc. Their integrability properties stem from the presence of an infinite number of commuting conserved quantities. The existence of second Hamiltonian structure, zero-curvature representation, Lax-pair formulation, multi-soliton solutions, Miura-type transformations, etc., are also some of the key features of integrability (see, e.g., [8–10]). The purpose of the present work is to study the highly nonlinear \( FOKdV \) and \( FOMKdV \) equations in the light of the above integrability properties as it is always worthwhile to look for a nonlinear evolution equation that is integrable or partially integrable [11]. To this end in mind, we first develop a Lagrangian formulation for the above dynamical equations following the procedure adopted by Cooper et al. [12] in the context of “compactons” and derive conserved quantities from the symmetries of the Lagrangian due to Noether theorem. We discuss a consistent Hamiltonian formulation for the above systems and concentrate on the Poisson bracket structures to study the commuting properties of the conserved quantities. It turns out that for a specific value of the power of nonlinearity, the conserved quantities are recursively related to one-another. However, the second Poisson bracket structure is found to be not a consistent bracket for an integrable system [13]. The \( FOMKdV \)-type equation is
derived from the \textit{FOKdV}-type equation by a Miura-type transformation. It turns out that the spatial dependence of the fields involved is such that the meaning of solitonic solutions to these equations is lost. In contrast to the usual \textit{KdV} and \textit{mKdV} equations which are related by Miura maps, this result implies that \textit{FOKdV} and \textit{FOMKdV} equations are independent in their own right. A consistent Lagrangian and Hamiltonian formulations for the \textit{FOMKdV} equation is also developed.

The material of our work is organized as follows. In Sec. 2, we derive the \textit{FOKdV} equation from the scalar field Lagrangian and discuss its symmetry properties. Sec. 3 is devoted to the Hamiltonian structures and discussion of the integrability properties. We try to understand how far and how much these properties can be discussed vis-a-vis the usual \textit{KdV} and \textit{mKdV} equations. In Sec. 4, we derive \textit{FOMKdV} equation from the starting \textit{FOKdV} equation by a Miura-type transformation and demonstrate that such type of relation entails the spatial dependence of the fields involved to be some fractional power over spatial variable($x$). As a consequence, it is difficult to provide some physical interpretation to this result in the language of solitonic solution. Finally, in Sec. 5, we make some concluding remarks and provide an outlook of the whole discussion.

## 2 Lagrangian Formulation

We derive here the \textit{FOKdV}-type equation

\[
 u_t + \gamma u_{5x} + \beta u_{3x} + \alpha u^p u_x = 0, \tag{1}
\]

from the first-order Lagrangian density \(L\) that is expressed in terms of the derivatives \((\phi_t = \partial_t\phi, \phi_x = \partial_x\phi)\) on a scalar field \(\phi\). The continuous symmetries of the action \((S = \int dt \int dx \, L)\) lead to the derivation of three conserved quantities due to Noether theorem. We begin with the following first-order Lagrangian

\[
 L(p) = \int dx \mathcal{L} \equiv \int dx \left[ \frac{1}{2} \phi_x \phi_t + \frac{\alpha (\phi_x)^{p+2}}{(p+1)(p+2)} - \frac{\beta}{2} \phi_{2x}^2 + \frac{\gamma}{2} \phi_{3x}^2 \right], \tag{2}
\]

where \(p > 0\) and c-number parameters \(\alpha, \beta, \gamma\) of (1) and (2) obey certain restrictions for the physically meaningful solutions to the equation (1) in the context of fluid mechanics and plasma physics (see, e.g., [4]). The Euler-Lagrange equation

\[
 \frac{d}{dx} \left( \frac{\delta L}{\delta \phi_x} \right) + \frac{d}{dt} \left( \frac{\delta L}{\delta \phi_t} \right) = 0, \tag{3}
\]

with

\[
 \frac{\delta L}{\delta \phi_t} = \Pi_\phi = \frac{\phi_x}{2}, \\
 \frac{\delta L}{\delta \phi_x} = \frac{1}{2} \phi_t + \frac{\alpha (\phi_x)^{p+1}}{(p+1)} + \beta \phi_{3x} + \gamma \phi_{5x}, \tag{4}
\]
leads to the following dynamical equation of motion

\[ \phi_{xt} + \alpha (\phi_x)^p \phi_{2x} + \beta \phi_{4x} + \gamma \phi_{6x} = 0. \] (5)

The identification \( \phi_x = u(x, t) \) leads to the derivation of (1).

We would like to dwell a bit more on the symmetries of the above Lagrangian as they provide a physical insight into the system described by it. Since only derivatives are present in the above Lagrangian, it is straightforward to check that the following transformation with a global infinitesimal parameter \( \varepsilon \)

\[ \delta_1 \phi = 2\varepsilon. \] (6)

is a symmetry transformation because \( \delta_1 \mathcal{L} = 0 \). Furthermore, the space and time translations \( x \rightarrow x - \varepsilon, t \rightarrow t - \varepsilon \) lead to the following transformations on the scalar field \( \phi \) (since \( \phi(x', t') = \phi(x, t) \))

\[ \delta_2 \phi = \varepsilon \phi_x, \]
\[ \delta_3 \phi = \varepsilon \phi_t. \] (7)

These are also the symmetry transformations of the action \( S \) as the above Lagrangian undergoes the following change

\[ \delta_2 \mathcal{L} = \varepsilon \mathcal{L}_x \equiv \frac{\partial}{\partial x}(\varepsilon \mathcal{L}), \]
\[ \delta_3 \mathcal{L} = \varepsilon \mathcal{L}_t \equiv \frac{\partial}{\partial t}(\varepsilon \mathcal{L}), \] (8)

where \( \varepsilon \) is a global infinitesimal parameter (as was the case in (6)).

The continuous symmetry transformations (6) and (7) imply presence of conserved quantities (\( I's \)) due to well-known Noether theorem. These are juxtaposed, with the identification \( \phi_x = u \), as follows

\[ I_1 = \int dx \phi_x \quad \rightarrow \quad \int dx u, \]
\[ I_2 = \frac{1}{2} \int dx \phi_x^2 \quad \rightarrow \quad \frac{1}{2} \int dx u^2, \]
\[ I_3 = \int dx \left( \frac{\beta}{2} \phi_x^2 - \frac{\alpha \phi_x^{p+2}}{(p+1)(p+2)} - \frac{\gamma}{2} \phi_x^2 \right), \]
\[ \rightarrow \int dx \left( \frac{\beta}{2} u_x^2 - \frac{\alpha u^{p+2}}{(p+1)(p+2)} - \frac{\gamma}{2} u_x^2 \right). \] (9)

Note that in the derivation of \( I_3 \), we have used

\[ I_3 = \int dx J_0 \equiv \int dx \left( \frac{\delta \phi}{\varepsilon} \frac{\partial \mathcal{L}}{\partial \phi_t} - \mathcal{L} \right), \] (10)
as the Lagrangian transforms to the time derivative of itself under global time translation (7). It can be easily seen that the time translation generator “energy” \( I_3 \) is nothing but the Hamiltonian function \( H(p) \) corresponding to the first-order Lagrangian function (2). It becomes transparent in the following Legendre transformation

\[
H(p) = \int dx [\Pi \phi_t] - L(p) \equiv I_3.
\]

The conservation laws for the quantities \( I_{1,2} \), corresponding to the “area” under \( u \) and “momentum”, can be understood directly from the form of the dynamical equation. It is obvious from (1) that \( I_1 \) is a conserved quantity due to the fact that r.h.s. of this equation is a total space derivative. For the conservation law of the space translation generator \( I_2 \), the equation of motion (1), implies that

\[
u u_t = -\frac{\partial}{\partial x} \left[ \alpha \frac{u^{p+2}}{(p+2)} + \beta (uu_{2x} - \frac{1}{2}u_x^2) + \gamma (uu_{4x} - u_x u_{3x} + \frac{1}{2}u_{2x}^2) \right].
\]

In general, for an equation of the type (1) with any order of odd derivatives on \( u \) and a nonlinear term of the type \( u^p \ u_x \), it can be seen that \( u u_t \) is always a total space derivative. It is because of the fact that for \( n = 0, 1, 2, 3 \)..............

\[
u (2n+1)x u = \frac{\partial}{\partial x} \left[ \sum_{r=0, n>r}^{n-1} (-1)^r u_{(2n-r)x} u_{rx} + (-1)^n \frac{1}{2}(u_{nx})^2 \right]
\]

where \( u_{0x} = u, r = 0, 1, 2, 3, \)..............\( (n-1) \). This demonstrates that the “momentum” \( I_2 \) is always a conserved quantity for the kind of equation we are discussing.

From the above Hamiltonian function, one can derive the dynamical equation (5) (and (1) with proper identification) by exploiting the following canonical Poisson-bracket structure for the scalar field

\[
\{ \phi(x, t), \phi_y(y, t) \}_{PB}^{(1)} = \delta(x - y),
\]

in the Hamilton equation of motion

\[
\phi_t = \{ \phi, H(p) \}_{PB}^{(1)} = -\gamma \phi_{5x} - \beta \phi_{3x} - \alpha \frac{\phi_{p+1}^x}{p+1}.
\]

It is evident that the total space derivative of the above equation is nothing but the dynamical equation (5) which, in turn, takes the form of equation (1) with the identification \( \phi_x = u \). Similar operation on the Poisson bracket (14) leads to the Hamiltonian structure on \( u \) (the first PB) as

\[
\{ u(x), u(y) \}_{PB}^{(1)} = \frac{\partial}{\partial x} \delta(x - y).
\]

Thus, we establish the fact that the Lagrangian and Hamiltonian formulations for the scalar field lead to the derivation of the Poisson-bracket structure for the \( u \) fields.
as has been taken by Karpman in Ref. [4] for the Hamiltonian function $I_3$. It is interesting to point out that all the conserved quantities of equation (9) commute with one-another under the choice of the Poisson bracket (16).

### 3 Second-Hamiltonian Structure and Recursion Relations

Here we demonstrate that for $p = 1$, these conserved quantities are recursively related to one-another for any arbitrary values of $\alpha, \beta, \gamma$ as there exists a second Poisson bracket for the equation of motion (1) with the Hamiltonian as the “momentum” expression $I_2$. Thus, to some extent, the FOKdV-type equation does mimic the usual KdV equation.

We know that the KdV equation is an integrable equation because it supports an infinite number of conserved quantities that are recursively related to one-another due to the existence of a “consistent” second Hamiltonian structure [13]. The first- and the second brackets satisfy the famous Jacobi identities. Here, we try to see how much and how far, the FOKdV equation can mimic the analogue of the recursion relations of KdV equation. To this end, it can be seen that the following Hamilton equations

$$u_t = \{u, I_3\}^{(1)} = \{u, I_2\}^{(2)},$$

(17)

with the first bracket (cf., equation (16))

$$\{u(x, t), u(y, t)\}^{(1)} = -D^{(1)} \delta(x - y),$$

(18)

and the second bracket

$$\{u(x, t), u(y, t)\}^{(2)} = D^{(2)} \delta(x - y),$$

(19)

where,

$$D^{(1)} = \partial \equiv \frac{\partial}{\partial y},$$

$$D^{(2)} = \left[ \gamma \partial^5 + \beta \partial^3 + C u^p \partial + \{(p + 1) C - \alpha \} u^{p-1} u_y \right],$$

(20)

lead to the derivation of equation (1) for any arbitrary value of the constant C. For the FOKdV-type equation, the analogue of KdV type recursion relations, with Poisson brackets (18) and (19), are

$$\hat{D}^{(1)} \left( \frac{\delta I_k}{\delta u} \right) = \hat{D}^{(2)} \left( \frac{\delta I_{k-1}}{\delta u} \right),$$

(21)

where $k \geq 2$ and operators $\hat{D}^{(1,2)}$ can be computed from the brackets (18) and (19). These operators for our discussion are

$$\hat{D}^{(2)} = -[\gamma \partial^5 + \beta \partial^3 + C u^p \partial - (C - \alpha) u^{p-1} \partial u],$$

$$\hat{D}^{(1)} = \partial.$$  

(22)
For \( k = 2 \), the above recursion relation leads to the following expression for \( I_2 \)

\[
I_2 = \frac{(C - \alpha)}{p(p + 1)} \int dx \ u^{p+1}.
\]  

(23)

However, the comparison with the starting momentum expression (i.e. \( I_2 = \frac{1}{2} \int dx \ u^2 \) of equation (9)), puts certain restrictions on \( p \) and \( C \) as given below:

\[
p = 1 \quad \text{and} \quad C = 1 + \alpha.
\]  

(24)

As a result, the operator \( \hat{D}^{(1)} \) remains intact but the operator \( \hat{D}^{(2)} \) modifies as follows:

\[
\hat{D}^{(2)} = -[\gamma \partial^5 + \beta \partial^3 + (1 + \alpha) \ u \partial - \partial u].
\]  

(25)

It will be noticed that the form of the equation (1) is changed because there is a restriction on \( p \) (i.e. \( p = 1 \)) but c-number parameters \( \alpha, \beta \) and \( \gamma \) remain intact as there are no restriction on them due to the recursion relations.

We go a step further to compute \( I_3 \) from the above recursion relation with the modified version of second Poisson bracket (19). The corresponding recursion relation with the above \( \hat{D}^{(2)} \) (cf. (25))

\[
\frac{d}{dx} \left( \frac{\delta I_3}{\delta u} \right) = \hat{D}^{(2)} \left( \frac{\delta I_2}{\delta u} \right),
\]  

(26)

leads to

\[
I_3 = \int dx \left( \frac{\beta}{2} u_x^2 - \frac{\alpha}{6} u^3 - \frac{\gamma}{2} u_{2x}^2 \right).
\]  

(27)

The comparison with the “energy” expression \( I_3 \) (for \( p = 1 \) in equation (9)), puts no constraint on \( \alpha, \beta, \gamma \) and all the three conserved quantities are recursively related. It can be checked that, for \( p = 1 \), all of them commute with one-another under the Poisson bracket structure (19) as well.

The next order recursion relation should yield the next conserved quantity \( I_4 \) if the second Hamiltonian structure is a “consistent” bracket of an integrable equation. In fact, the following expression comes out from the next recursion relation for any arbitrary value of \( \alpha, \beta \) and \( \gamma \):

\[
\frac{\delta I_4}{\delta u} = \gamma(u_{2x}^2 + uu_{4x} - 2u_xu_{3x}) + \frac{\alpha}{6} u^3 + \beta(u u_{2x} - u_x^2) + \beta^2 u_{4x} + \frac{\alpha^2}{3} u^3 + \gamma^2 u_{8x} + 2\beta \gamma u_{6x} + \alpha \beta \left( \frac{1}{2} u_{2x}^2 + 2uu_{2x} \right) + \alpha \gamma \left( \frac{7}{2} u_{2x}^2 + 2uu_{4x} + 3u_xu_{3x} \right).
\]  

(28)

To obtain the expression for \( I_4 \), one should see that the r.h.s. is a total variation of some quantity. It turns out that for the choice \( \alpha = 3 \), the r.h.s. reduces to a simpler form as given below

\[
\delta I_4 = \left[ 7\gamma \left( \frac{23}{14} u_{2x}^2 + uu_{4x} + u_xu_{3x} \right) + 7\beta(u u_{2x} + \frac{1}{2} u_x^2) + \beta^2 u_{4x} + \gamma^2 u_{8x} + \frac{7}{2} u^3 + 2\beta \gamma u_{6x} \right] \delta u.
\]  

(29)
Except the first term, all the other terms are total variation on a certain local quantity modulo some total space derivatives. This can be seen as follows:

\[ \frac{1}{2}u_x^2 + u u_{2x}\delta u = \frac{1}{4}\delta(u^2 u_{2x}) \]
\[ u_{8x}\delta u = \frac{1}{2}\delta(u_{4x}^2) \]
\[ u_{6x}\delta u = -\frac{1}{2}\delta(u_{3x}^2), \quad u_{4x}\delta u = \frac{1}{2}\delta(u_{2x}^2), \quad \frac{7}{2}u^3\delta u = \frac{7}{8}\delta(u^4). \]  

(30)

It is interesting to note that the first term cannot be cast as a total variation of a definite local quantity. The argument runs as follows. There are only four candidates with three \(u\) fields and four derivatives on them

\[ u^2 u_{4x}, u_{2x}^2 u, u u_x u_{3x}, u_x^2 u_{2x}, \]  

(31)

whose variation and/or the variation of their linear combination might produce the combination of the first term. It turns out that the variation on the last term is zero modulo some total space derivative term. All the other three candidates produce the same result due to the variation on them. For instance, for one of them the variation is

\[ (2u u_{4x} + 4u u_{3x} + 3u_{2x}^2)\delta u = \frac{1}{2}\delta(u^2 u_{4x}). \]  

(32)

Thus, the FOKdV equation is not an integrable equation because the second Hamiltonian structure is not consistent [13]. It can be also seen that the second-Poisson bracket does not satisfy the Jacobi identities (see, e.g., for more detail, Ref. [12] for a similar kind of situation).

4 Towards a Miura-type transformation

In this Sec., we derive the FOMKdV-type equation (i.e., \(p = 2\)) from the starting FOKdV equation by Miura-type transformation (see, e.g., [13]). This equation is also of theoretical interests in the context of fluid mechanics and plasma physics. We develop here the Lagrangian formulation for the FOMKdV and discuss in a nutshell the Poisson bracket structures for this system on similar lines as that of the FOKdV equation (cf. Sec. 2). The solitonic solution to this equation has been found in Ref. [7] where two analytic solutions have been obtained.

To start with, we know that for the usual KdV equation, there exists a transformation that relates the original KdV equation to the mKdV. The question we address is: Can one derive the fifth-order mKdV equation from the starting equation (1)? To answer this question we make the following transformation

\[ u(x,t) = v_x(x,t) - \frac{\alpha}{6\beta} v^2(x,t), \]  

(33)

in equation (1). It will be noticed that the inclusion of higher order derivative terms on the r.h.s. of equation (33) leads to more complicated expressions which
are difficult to handle. The ensuing equation is realized on $v(x,t)$ as

$$v_t + \gamma v_{5x} + \beta v_{3x} - \frac{\alpha^2}{6\beta} v^2 v_x = 0,$$

(34)

which can be recognized as the $FOMKdV$ and can be re-expressed in the form of a conservation law as given below:

$$v_t + \frac{\partial}{\partial x} \left[ \gamma v_{4x} + \beta v_{2x} - \frac{\alpha^2}{18\beta} v^3 \right] = 0.$$

(35)

It should be added, however, that for the validity of (35), the field $v(x,t)$ has to satisfy an extra condition given by

$$v_x v_{4x} + 2 v_{2x} v_{3x} = 0.$$

(36)

We have to find a solution for the above restriction if Miura transformation (33) has to make some sense. The equation (36) can be re-written as a total space derivative of a certain quantity, given by:

$$\frac{d}{dx} \left[ v_x v_{3x} + \frac{1}{2} (v_{2x})^2 \right] = 0.$$

(37)

The quantity in the square bracket has to be a constant. Choosing the constant to be zero, we obtain the $x$ dependence of $v$ as

$$v(x) = A x^{5/3},$$

(38)

for an arbitrary constant $A$. It will be noticed that equation (38) is a nontrivial solution for the Miura transformation and there is no constraint on the $t$-dependence. It is also evident that for such a choice ($v(x) \sim x^{5/3}$), the spatial dependence of $u$ field will also change accordingly. Thus, there is a problem about solitonic solutions for such kind of restriction. It is well known that for the usual $KdV$ equation, no such restrictions as (36–38) arise. For instance, if $\gamma = 0$ and $p = 1$ in the starting equation (1), there will be no restrictions like (36–38) and for any arbitrary $\alpha, \beta$, we have

$$u_t + \beta u_{3x} + \alpha u u_x = 0,$$

$$u(x,t) = v_x(x,t) - \frac{\alpha}{6\beta} v^2,$$

$$v_t + \beta v_{3x} - \frac{\alpha^2}{6\beta} v^2 v_x = 0,$$

(39)

as $KdV$ equation, Miura transformation and the $mKdV$ equation.

We conclude from equation (38) that $FOKdV$ and $FOMKdV$ equations are independent in their own right. There is no Miura type of mapping which can relate
these equations preserving the nature of stationary solitonic solutions that have been obtained in Ref. [7].

The \textit{FOMKdV} equation of motion (34) can be derived from the following first-order Lagrangian

$$ L = \int dx \left( \frac{1}{2} \phi_x \phi_t - \frac{\alpha^2}{72 \beta} (\phi_x)^4 - \frac{\beta}{2} (\phi_{2x})^2 + \frac{\gamma}{2} (\phi_{3x})^2 \right). \quad (40) $$

The Euler-Lagrange equation, emerging from the invariance of the action ($\delta S = 0$), leads to the following dynamical equation:

$$ \phi_{xt} - \frac{\alpha^2}{6 \beta} (\phi_x)^2 \phi_{2x} + \beta \phi_{4x} + \gamma \phi_{6x} = 0. \quad (41) $$

The identification $\phi_x = v(x, t)$ leads to the rederivation of (34). The analogue of the symmetry transformations (6) and (7) for the above Lagrangian lead to the following conserved currents due to the Noether theorem

$$ J_1 = \int dx \, v, \quad J_2 = \frac{1}{2} \int dx \, v^2, \quad J_3 = \int dx \left( \frac{\alpha^2}{72 \beta} \, v^4 + \frac{\beta}{2} v_x^2 - \frac{\gamma}{2} v_{2x}^2 \right). \quad (42) $$

The equation of motion (34) can be derived from the Hamilton equation as well. The corresponding Hamiltonian functions are none other than $J_3$ and $J_2$ as illustrated below:

$$ v_t = \{v, J_3\}^{(1)} = \{v, J_2\}^{(2)}, \quad (43) $$

where the first bracket and the second brackets are

$$ \{v(x, t), v(y, t)\}^{(1)} = -\frac{\partial}{\partial y} \delta(x - y), $$

$$ \{v(x, t), v(y, t)\}^{(2)} = [\gamma \partial^5 + \beta \partial^3 + C v^2 \partial + (3 C + \frac{\alpha^2}{6 \beta}) v_y v] \delta(x - y), \quad (44) $$

for arbitrary constant $C$ and $\partial = \frac{\partial}{\partial y}$.

\section*{5 Discussions}

The integrability properties play a pivotal role in furnishing the wealth of informations on the rich mathematical structure encoded in a given nonlinear evolution equation. This, in turn, leads to gain some deep physical insight into the phenomena that are described by these equations. In the light of this argument, we have studied the \textit{FOKdV} and \textit{FOMKdV} equations. We have developed a Lagrangian formulation for the \textit{FOKdV} equation and obtained the conserved quantities from
the symmetry properties of this Lagrangian. We have shown that these conserved quantities are recursively related to one-another for \( p = 1 \) due to the existence of bi-Hamiltonian structures. The recursion relations do not lead to the determination of more than three conserved quantities. Thus, as it appears, the \( FOKdV \) equation is not a completely integrable system.

The \( FOKdV \) equation mimics some of the key features of the partially integrable system. For instance, the soliton solutions are stable and preserve their shape for considerably large values of the power of nonlinearity. There exists a Miura type transformation that relates \( FOKdV \) and \( FOMKdV \) equations for the specific space-dependence of the fields on which these equations are realized. However, from the point of view of the solitonic solutions, this dependence is not an interesting feature. This leads to the conclusion that \( FOKdV \) and \( FOMKdV \) equations are independent in their own right and there exists no Miura-type mapping which can relate these equations without spoiling the solitonic solutions to these equations. Furthermore, for \( p = 1 \), the conserved quantities are recursively related to one-another (due to the existence of two Poisson bracket structures). One can also see that the \( FOKdV \) and \( FOMKdV \) equations can be expressed as an Abelian-type zero-curvature representation (\( F_{tx} = 0 \))

\[
F_{tx} = \partial_t A_x - \partial_x A_t \equiv \left[ \frac{\partial}{\partial t} + A_t, \frac{\partial}{\partial x} + A_x \right] = 0, \tag{45}
\]

where gauge connection \( A_x \) and \( A_t \) can be read off from the \( FOKdV \) and \( FOMKdV \) equations respectively, as listed below

\[
\begin{align*}
A_x &= u(x, t) \quad \text{and} \quad v(x, t), \\
A_t &= -(\gamma u_4 x + \beta u_2 x + \alpha \frac{u^{p+1}}{p + 1}) \quad \text{and} \quad \frac{\alpha^2}{18\beta} v^3 - \beta v_2 x - \gamma v_4 x. \tag{46}
\end{align*}
\]

It will be an interesting venture to study the integrability properties of these equations where we can use the Painleve test [11] and consider the multi-soliton solution by exploiting the Hirota’s bilinear form and its generalization [14]. It is worthwhile to mention that in Ref. [14], it has been claimed that the existence of three solitonic solutions imply integrability. These are the issues for future investigations and further works have to be done in this direction.

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