Mirror Symmetry and Projective Geometry of Fourier-Mukai Partners

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Abstract. This is a survey article on mirror symmetry and Fourier-Mukai partners of Calabi-Yau threefolds with Picard number one based on recent works \[HoTa1,2,3,4\]. For completeness, mirror symmetry and Fourier-Mukai partners of K3 surfaces are also discussed.

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References
1. Introduction

Derived categories of coherent sheaves on projective varieties are attracting attentions from many aspects of mathematics for the last decades. Among them, the derived categories of coherent sheaves on Calabi-Yau manifolds have been attracting special attentions since they are conjecturally related to symplectic geometry by the homological mirror symmetry due to Kontsevich [Ko] and also to the geometric mirror symmetry due to Strominger-Yau-Zaslow [SYZ]. In this article, we will survey on the derived categories of Calabi-Yau manifolds of dimension two and three focusing on the so-called Fourier-Mukai partners and their mirror symmetry.

As defined in the text, smooth projective projective varieties $X$ and $Y$ are called Fourier-Mukai partners to each other if their derived categories of bounded complexes of coherent sheaves are equivalent, $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. When $X$ and $Y$ are K3 surfaces, the study of the derived equivalence goes back to the works by Mukai in '80s [Mu1] and Orlov in '90s [Or]. For completeness, we start our survey with a brief summary of their results, and also the mirror symmetry interpretations made in [HLOY1]. About the Fourier-Mukai partners of Calabi-Yau threefolds, little is known except a general result that two Calabi-Yau threefolds are derived equivalent if they are birational [Br2]. In [BC][Ku2], it has been shown that an interesting example of a pair of Calabi-Yau threefolds $X$, $Y$ of Picard number one (Grassmannian-Pfaffian Calabi-Yau threefolds) due to Rødland [Ro] is the case of non-trivial Fourier-Mukai partners which are not birational. In particular, it has been recognized in [Ku2, Ku1] that the classical projective duality between the Grassmannian $G(2,7)$ and the Pfaffian variety $\text{Pf}(4,7)$ in the construction of $X$ and $Y$ plays a prominent role, and a notion called homological projective duality has been introduced in [Ku1]. Recently, it has been found by the present authors [HoTa1, 2, 3, 4] that the projective duality of $G(2,7)$ and $\text{Pf}(4,7)$ has a natural counterpart in the projective duality between the secant varieties of symmetric forms and these of the dual forms. In this setting, we naturally came to two Calabi-Yau threefolds $X$ and $Y$ of Picard numbers one which are derived equivalent but not birational to each other. Calabi-Yau manifold $X$ is the so-called three dimensional Reye congruence (whose two dimensional counterpart has been studied in [Co]), and $Y$ is given by a linear section of double quintic symmetroids (see Section 5).

In the construction of $Y$ and also in the proof of the derived equivalence to $X$, birational geometry of the double quintic symmetroids has been worked out in detail in [HoTa3]. It has been found that the birational geometry of symmetroids itself contains interesting projective geometry of quadrics [Ty].

This article is aimed to be a survey of the works [HoTa1, 2, 3, 4] on mirror symmetry and Fourier-Mukai partners of the new Calabi-Yau manifolds of Picard number one, and also interesting birational geometry of the double quintic symmetroids which arises in the constructions. In order to clarify the entire picture of the subjects, we have included previous works on K3 surfaces and also the Rødland’s example. Since the expository nature of this article, most of the proofs for the statements are omitted referring to the original papers.

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2. Fourier-Mukai partners of K3 surfaces

2.1. Counting formula of Fourier-Mukai partners. Let $X$ be a K3 surface, i.e., a smooth projective surface with $K_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. We have a symmetric bilinear form $(\ast, \ast)$ on $H^2(X, \mathbb{Z})$ by the cup product. Then $(H^2(X, \mathbb{Z}), (\ast, \ast))$ is an even unimodular lattice of signature $(3, 19)$, which is isomorphic to $L_{K3} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ where $U$ is the hyperbolic lattice $(\mathbb{Z},_{\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}})$. Denote by $NS_X = Pic(X)$ the Picard (Néron-Severi) lattice and set $\rho(X) = \text{rk} NS_X$. $NS_X$ is the primitive sub-lattice in $H^2(X, \mathbb{Z})$ and has signature $(1, \rho(X) - 1)$. The orthogonal complement $T_X = (NS_X)^\perp$ in $H^2(X, \mathbb{Z})$ is called transcendental lattice. $T_X$ has signature $(2, 20 - \rho(X))$. The extension $\tilde{H}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$ is called Mukai lattice.

Let us denote by $\omega_X$ the nowhere vanishing holomorphic two form of $X$ which is unique up to constant. Then the Global Torelli theorem says that K3 surfaces $X$ and $X'$ are isomorphic iff there exists a Hodge isometry, i.e., a lattice isomorphism $\varphi : H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ which satisfies $\varphi(\mathbb{C}\omega_X) = \mathbb{C}\omega_{X'}$. Extending earlier works by Mukai [Mu1] in 80’, Orlov [Or] has formulated a similar Global Torelli theorem for the derived categories of coherent sheaves on K3 surfaces:

**Theorem 2.1 ([Mu1] [Or]).** K3 surfaces $X$ and $X'$ are derived equivalent, $D^b(X) \simeq D^b(X')$, if and only if there exists a Hodge isometry of transcendental lattices $(T_X, \mathbb{C}\omega_X) \simeq (T_{X'}, \mathbb{C}\omega_{X'})$.

Due to the uniqueness theorem of primitive embeddings into indefinite lattices (see Theorem A.1 in Appendix), we note that the Hodge isometry $(T_X, \mathbb{C}\omega_X) \simeq (T_{X'}, \mathbb{C}\omega_{X'})$ above always extends to that of the Mukai lattice $(\tilde{H}(X, \mathbb{Z}), \mathbb{C}\omega_X) \simeq (\tilde{H}(X', \mathbb{Z}), \mathbb{C}\omega_{X'})$, and hence we can rephrase the above theorem in terms of the Hodge isometry of Mukai lattices.

Consider smooth projective varieties $X$ and $Y$. $Y$ is called Fourier-Mukai partner of $X$ if $D^b(Y) \simeq D^b(X)$. We denote the set of Fourier-Mukai partners (up to isomorphisms) of $X$ by

$$FM(X) = \{ Y \mid D^b(Y) \simeq D^b(X) \} / \text{isom}.$$ 

For a K3 surface $X$, the set $FM(X)$ consists of K3 surfaces (see [Hu] Cor.10.2 for example) and its cardinality is known to be finite, i.e. $|FM(X)| < \infty$ in [BM].

Studying all possible obstructions for extending a Hodge isometry $(T_X, \mathbb{C}\omega_X) \simeq (T_{X'}, \mathbb{C}\omega_{X'})$ between the transcendental lattices to the corresponding Hodge isometry $(H^2(X, \mathbb{Z}), \mathbb{C}\omega_X) \simeq (H^2(Y, \mathbb{Z}), \mathbb{C}\omega_Y)$, the following counting formula has been obtained:

**Theorem 2.2 ([HLOY2]).** For a K3 surface $X$, we have

$$|FM(X)| = \sum_{\mathcal{G}(NS_X) = \{S_1, \ldots, S_N\}} \left| \text{O}(S_i) \setminus \text{O}(A_{S_i}) \right| / \text{O}_{\text{Hodge}}(T_X, \mathbb{C}\omega_X),$$

where $\mathcal{G}(NS_X)$ is the isogeny classes of the lattice $NS_X$, $A_{S_i} = (S^*/S, q : S^*/S \rightarrow \mathbb{Q}/\mathbb{Z})$ is the discriminant of the lattice $S_i$, and $O(S_i)$ and $O(A_{S_i})$ are isometries of $S_i$ and $A_{S_i}$. $\text{O}_{\text{Hodge}}(T_X, \mathbb{C}\omega_X)$ is the Hodge isometries of $(T_X, \mathbb{C}\omega_X)$.

We refer to [HLOY2] for the details (see also [HP]). Since the isogeny classes of a lattice are finite, the counting formula contains the earlier result $|FM(X)| < \infty$. 

When $X$ is a K3 surface with $p(X) = 1$ and $\deg(X) = 2n$, the counting formula coincides with the result in \[ \text{[Og]} \] (obtained by counting the so-called over-lattices);

\[(2.1) \quad |FM(X)| = 2^{p(n) - 1} = \frac{1}{2} |O(A_{NSX})|,\]

where $p(n)$ is the number of prime factors of $n$ (we set $p(1) = 1$). In fact, much is known by \[ \text{[Ml3]} \] in this case that we have

\[ FM(X) = \{ M_X(r,h,s) \mid n = rs, (r, s) = 1 \}, \]

in terms of the moduli space of stable vector bundles $E$ on $X$ with Mukai vector $(r, h, s) = h(E)\sqrt{td_X}$ in $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ (see also \[ \text{[HLOY3]} \]). We will study in detail the first non-trivial example of $|FM(X)| \neq 1 (n = 6)$ in Subsection 2.3.

### 2.2. Marked $M$-polarized K3 surfaces

A K3 surface $X$ with a choice of isomorphism $\phi : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$ is called a marked K3 surface $(X, \phi)$. Marked K3 surfaces $(X, \phi)$ and $(X', \phi')$ are isomorphic if there exists an isomorphism $f : X \rightarrow X'$ satisfying $\phi' = \phi \circ f^*$. By the Global Torelli theorem, $(X, \phi)$ and $(X', \phi')$ are isomorphic iff there exists a Hodge isometry $\varphi : (H^2(X', \mathbb{Z}), \omega_{X'}) \rightarrow (H^2(X, \mathbb{Z}), \omega_{X})$ such that $\varphi' = \phi \circ \varphi$ (see \[ \text{[BlIP]} \] for more details of K3 surfaces).

Consider a lattice $M$ of signature $(1, t)$ and fix a primitive embedding $i : M \hookrightarrow L_{K3}$. A marked K3 surface $(X, \phi)$ is called marked $M$-polarized K3 surface if $\varphi^{-1}(M) \subset NS_X$ (where we write $\varphi^{-1}(M) = (\varphi^{-1} \circ i)(M)$ for short). Marked $M$-polarized K3 surfaces $(X, \phi)$ and $(X', \phi')$ are isomorphic if there exists a lattice isomorphism $\varphi : L_{K3} \rightarrow L_{K3}$ such that

\[(2.2) \quad H^2(X, \mathbb{Z}) \xrightarrow{\phi} L_{K3} \xrightarrow{i} M \]

\[ \xrightarrow{\varphi} \quad H^2(X', \mathbb{Z}) \xrightarrow{\phi'} L_{K3} \xrightarrow{i} M \]

and the composition $(\varphi')^{-1} \circ \varphi \circ \phi : (H^2(X, \mathbb{Z}), \omega_{X}) \rightarrow (H^2(X', \mathbb{Z}), \omega_{X'})$ is a Hodge isometry. The lattice isomorphism $\varphi$ in (2.2) is an element of the group

\[ \Gamma(M) = \{ g \in O(L_{K3}) \mid g(m) = m \ (\forall m \in M) \}. \]

Consider the orthogonal lattice $M^\perp = (i(M))^\perp$. Then there is a natural injective homomorphism $\Gamma(M) \rightarrow O(M^\perp)$. The image is known to be described by the kernel $O(M^\perp)^* := \text{Ker} \{ O(M^\perp) \rightarrow O(A_{M^\perp}) \}$ of the natural homomorphism to the isometries of the discriminant $A_{M^\perp}$ (see \[ \text{[Do]} \text{Prop.3.3]}.\]

A marked K3 surfaces $(X, \phi)$ determines the period points $\phi(\omega_X)$ in the period domain $D = \{ [\omega] \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \omega) > 0 \}$. By the surjectivity of the period map, $D$ gives a classifying space of the (not necessarily projective) marked K3 surfaces. Then, by the Global Torelli theorem, the quotient $D/O(L_{K3})$ classifies the isomorphism classes of (not necessarily projective) marked K3 surfaces.

From the definition, it is easy to deduce that marked $M$-polarized K3 surfaces are classified by the period points in the following domain

\[ D(M^\perp) := \{ [\omega] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) \mid (\omega, \omega) = 0, (\omega, \omega) > 0 \}, \]

which has two connected components $D(M^\perp) = D(M^\perp)^+ \cup D(M^\perp)^-$. Let us define $O^+(M^\perp) \subset O(M^\perp)$ to be the isometries of $M^\perp$ which preserve the orientations of
all positive two spaces in $M^\perp \otimes \mathbb{R}$. Then the isomorphisms classes of marked $M$-polarized K3 surfaces are classified by the following quotient,

$$(2.3) \quad \mathcal{D}(M^\perp)/O(M^\perp)^* \simeq \mathcal{D}(M^\perp)^+/O^+(M^\perp)^* = \mathcal{D}(M^\perp)^-/O^+(M^\perp)^*,$$

where $O^+(M^\perp)^* := O^+(M^\perp) \cap O(M^\perp)^*$ is the monodromy group which acts on the period points $\phi(\omega_X) \in \mathcal{D}(M^\perp)^\pm$ of marked $M$-polarized K3 surfaces $(X, \phi)$.

2.3. $M$-polarizable K3 surfaces. Let us fix a primitive lattice embedding $i : M \hookrightarrow L_{K3}$ as in the preceding subsection. Following [HLOY1], we call a K3 surface $X$ $M$-polarizable if there is a marking $\phi : H^2(X, \mathbb{Z}) \sim \sim L_{K3}$ such that $(\phi^{-1} \circ i)(M) \subset NS_X$. Two $M$-polarizable K3 surfaces $X$ and $X'$ are defined to be isomorphic if there exists lattice isomorphisms $\varphi : L_{K3} \sim \sim L_{K3}$ and $g : M \sim \sim M$ which make the following diagram commutative:

$$\begin{array}{ccc}
H^2(X, \mathbb{Z}) & \xrightarrow{\phi} & L_{K3} \\
\sim \sim & \sim \sim & \sim \sim \\
H^2(X', \mathbb{Z}) & \xrightarrow{\varphi} & L_{K3} \\
& \sim \sim & \sim \sim \\
& g & g \\
& \sim \sim & \sim \sim \\
& M & M \\
\end{array}$$

(2.4)

and the composition $(\phi')^{-1} \circ \varphi \circ \phi : (H^2(X, \mathbb{Z}), \mathbb{C}\omega_X) \rightarrow (H^2(X', \mathbb{Z}), \mathbb{C}\omega_{X'})$ is a Hodge isometry. Note that, as we see in the diagram, the definition of the isomorphism is slightly generalized for the $M$-polarizable K3 surfaces. Hence, although $M$-polarizable K3 surfaces $X$ are obtained by forgetting the marking $\phi$ from the marked $M$-polarized K3 surfaces $(X, \phi)$, their isomorphism classes are possibly different. We saw in the last subsection that the isomorphism classes of marked $M$-polarized K3 surfaces are classified by the quotient $\mathcal{D}(M^\perp)/O(M^\perp)^*$. On the other hand, the classifying space of the isomorphism classes of $M$-polarizable K3 surfaces is given by a similar quotient of $\mathcal{D}(M^\perp)$ but with a group which resides between $O(M^\perp)^*$ and $O(M^\perp)$.

2.4. Mirror symmetry of K3 surfaces. In [Do], Dolgachev defined mirror symmetry of marked $M$-polarized K3 surfaces. To summarize his construction/definition, let us fix a primitive embedding $i : M \hookrightarrow L_{K3}$ of a lattice $M$ of signature $(1, t)$ and assume that the orthogonal lattice $M^\perp$ has a decomposition $M^\perp = M \oplus U$, i.e.,

$$M \oplus M^\perp = M \oplus U \oplus \tilde{M} \subset L_{K3},$$

where $U$ is the hyperbolic lattice. Since the signature of $\tilde{M}$ is $(1, \tilde{t}) = (1, 19 - t)$, the primitive embedding $i : M \hookrightarrow L_{K3}$ naturally introduces marked $\tilde{M}$-polarized K3 surfaces. Marked $\tilde{M}$-polarized K3 surfaces are classified by $\mathcal{D}(M^\perp)$, while marked $M$-polarized K3 surfaces are classified by $\mathcal{D}(M^\perp)$.

For a general marked $M$-polarized K3 surface $(X, \phi)$ and a general marked $\tilde{M}$-polarized K3 surface $(\tilde{X}, \tilde{\phi})$, we have the following isomorphisms:

$$(2.5) \quad NS_X \simeq M, \quad T_X \simeq U \oplus \tilde{M}; \quad NS_{\tilde{X}} \simeq \tilde{M}, \quad T_{\tilde{X}} \simeq U \oplus M,$$

and observe the exchange of the algebraic and transcendental cycles (up to the factor $U$). This exchange is the hallmark of the mirror symmetry of K3 surfaces. Also we see the so-called “mirror map” [LY] for K3 surfaces in the following isomorphisms
M-polarizable K3 surfaces, the corresponding group becomes larger.  

D-nice interpretation from the monodromy group which acts on the period domain |V(M)|. Namely, we understand the isomorphisms (2.5) as (for the complexified Kähler moduli spaces of the other hand, for a general marked K3 surface, the number of |FM| is symmetric due to the Serre duality for K3 surfaces. Since |O_{X}| and −|I_{X}| explain the additional factor U in U ⊕ M. The above isomorphisms are consequences of the homological mirror symmetry due to Kontsevich [Ko].

2.5. Homological mirror symmetry. There is a slight asymmetry in the exchange of the Picard lattices and the transcendental lattices in (2.3). This can be remedied by considering the (numerical) Grothendieck group together with a (non-degenerate) pairing ([E], [F]) = −χ(E, F) where χ(E, F) = ∑(-1)^i dim Ext_{O_X}^i(E, F). Namely, we understand the isomorphisms (2.5) as  

\[ T_X ≃ U ⊕ M ≃ (K(X), (*, **)), T_{X̄} ≃ U ⊕ M̄ ≃ (K(X̄), (*, **)). \]

Note that the form (*, **) is symmetric due to the Serre duality for K3 surfaces. Also we note that K(X) contains [O_{x}] and −[I_{x}], in addition to [O_D] = [O_X] − [O_Y], for D ∈ Pic(X) (likewise for K(X̄)). By Riemann-Roch theorem, it is easy to see that [O_{x}] and −[I_{x}] explain the additional factor U in U ⊕ M. The above isomorphisms are consequences of the homological mirror symmetry due to Kontsevich [Ko]. But we refrain from going into the details about this in this article.

2.6. FM(X) and mirror symmetry. Let us consider the case M_n = (2n), i.e., (Zh, h^2 = 2n) in detail. We first note that we can embed the lattice M_n into the hyperpolaric lattice U by making a primitive embedding (2n) ⊕ (-2n) ⊂ U. Then, since primitive embedding i : M_n ↪ L_{K3} is unique up to isomorphism due to Theorem A.2 we may assume that the embedding i : M_n ↪ L_{K3} is given by  

\[ M_n ⊕ M_n^⊥ = (2n) ⊕ (U ⊕ M_n) ⊂ L_{K3} \]

where M_n^⊥ := (i(M_n))^⊥ = (-2n) ⊕ U ⊕ E_{8}(-1)^{⊗2}.  

Let (X, φ) be a marked M_n-polarized K3 surface, and h be its polarization (h^2 = 2n). Then we have |FM(X)| = 2^{p(n)-1} from the counting formula. On the other hand, for a general marked M_n-polarized K3 surface (X, φ), we have |FM(νX)| = 1 since ν = 19 and A_{M_n} ≃ Z/2nZ (see [HLOY1] Cor.2.6 and also [Mul] Proposition 6.2].

It has been argued in [HLOY1] that the number |FM(X)| = 2^{p(n)-1} has a nice interpretation from the monodromy group which acts on the period domain D(M_n^⊥)^{⊥} for the mirror marked polarized M_n-polarized K3 surfaces. Roughly speaking, the number |FM(X)| appears as the covering degree of the map from D(M_n^⊥)^{⊥}/O^+(M_n^⊥)^{⊥} to the corresponding quotient for the isomorphism classes of M_n-polarizable K3 surfaces.

We have determined, in Subsection 2.2, the monodromy group of the marked M_n-polarized K3 surfaces by O^+(M_n^⊥)^{⊥} = O^+(M_n^⊥) ∩ O(M_n^⊥)^{⊥}. As for the M_n-polarizable K3 surfaces, the corresponding group becomes larger.
Lemma 2.3 ([HLOY1] Lem.1.14, Def.1.15). The monodromy group of the $M_n$-polarizable K3 surfaces is given by $O^+(M_n^+)/\{\pm \text{id}\}$.

By definition, for $M_n$-polarizable K3 surfaces $\tilde{X}, \tilde{X}'$, we have markings $\phi, \phi'$ such that $(\tilde{X}, \tilde{\phi})$ and $(\tilde{X}', \tilde{\phi}')$ are marked $M_n$-polarized K3 surfaces. Then, the above lemma can be deduced from the following diagram which describes the isomorphism of $M_n$-polarizable K3 surfaces:

$$
\begin{array}{ccc}
H^2(\tilde{X}, \mathbb{Z}) & \xrightarrow{\phi} & L_{K3} \\
\downarrow & & \downarrow \phi \\
H^2(\tilde{X}', \mathbb{Z}) & \xrightarrow{\phi'} & L_{K3}
\end{array}
$$

(2.8)

Here we sketch the proof of the lemma: Suppose an element $h \in O(M_n^+)$ is given. Since primitive embedding $M_n^+ = U \oplus M_n \hookrightarrow L_{K3}$ is unique by Theorem [1.2], $h$ extends to an isomorphism $\varphi : L_{K3} \hookrightarrow L_{K3}$ and also determines an isomorphism $g : M_n \hookrightarrow M_n$ on the orthogonal complement of $M_n^+$. By the surjectivity of the period map, we see that $\varphi$ extends to an isomorphism of $M_n$-polarizable K3 surfaces. From the relation $\mathcal{D}(M_n^+)/O(M_n^+) \simeq \mathcal{D}(M_n^+)/O^+(M_n^+)$ and the fact that $\{\pm \text{id}\}$ has a trivial action on $\mathcal{D}(M_n^+)$, the group $O^+(M_n^+)/\{\pm \text{id}\}$ identifies the $M_n$-polarizable K3 surfaces which are isomorphic to each other. In this sense, we can call the quotient group $O^+(M_n^+)/\{\pm \text{id}\}$ the monodromy group of $M_n$-polarizable K3 surfaces.

Now we can see the FM number $|FM(X)| = 2^p(n) - 1$ as the covering degree of the map

$$
\mathcal{D}(M_n^+)/O^+(M_n^+) \rightarrow \mathcal{D}(M_n^+)/O^+(M_n^+),
$$

which we evaluate for $n \neq 1$ (see [HLOY1] Theorem 1.18 for details) as

$$
|O^+(M_n^+)/\{\pm \text{id}\}| = 2^p(n) - 1,
$$

where we recall the fact that $\{\pm \text{id}\}$ acts trivially on the domain. The covering degree can be explained by the nontrivial actions of $g$ in the diagram [2.8], which implies that $(\tilde{X}, \tilde{\phi})$ and $(\tilde{X}', \tilde{\phi}')$ are related by Hodge isometries that have nontrivial actions on the Picard lattice. The monodromy group $O^+(M_n^+)$ comes from the Dehn twists which preserve (the cohomology classes of) generic symplectic forms (Kähler forms) $\kappa_X$ ([HLOY1] Thm.1.9]). Then the covering group represents isomorphisms of K3 surfaces which do not preserve the (cohomology classes of) generic symplectic forms $\kappa_X$. This is the mirror symmetry interpretation of $FM(X)$ made in [ibid], where the relation of the Dehn twist to $Autoq D^h(X)$ has been discussed in more detail.

2.7. An example due to Mukai. Here we consider an explicit construction of the $M_6 = (12)$-polarized K3 surfaces due to Mukai [Mu4]. We see general properties discussed in the last subsections for this specific example, and make an observation that will be shared with the examples of Calabi-Yau threefolds in the subsequent sections. Note that $FM(X) = \{X, Y\}$ with $Y \simeq \mathcal{M}_X(2, h, 3)$ for general $M_6$-polarized K3 surfaces $X$. 

2.7.1. Linear sections of $\text{OG}(5,10)$. Let us consider orthogonal Grassmannian $\text{OG}(5,10)$ which parametrizes maximal isotropic subspaces of $\mathbb{C}^{10}$ with a fixed non-degenerate quadratic form. $\text{OG}(5,10)$ has two connected components $\text{OG}^{\pm}(5,10)$, which are isomorphic to each other. $\text{OG}^{+}(5,10) \cong \text{OG}^{-}(5,10)$ is called spinor variety $\mathbb{S}_5$ (of dimension 10), and can be embedded into the projective space $\mathbb{P}(S_{16})$ of the spin representation of $SO(10)$. $\text{OG}^{+}(5,10)$ is the Hermitian symmetric space $SO(10,\mathbb{R})/U(5)$, and its Picard group is generated by the ample class of the above spinor embedding. The projective dual variety (discriminant variety) $\mathbb{S}_5^*$ in the dual projective space $\mathbb{P}(S_{16}^*)$ is known to be isomorphic to $\mathbb{S}_5$. Mukai [Mu] constructed a smooth K3 surface of degree 12 (with Picard group $\mathbb{Z}h$) by considering a complete linear section $X = \mathbb{S}_5 \cap H_1 \cap \ldots \cap H_8$ and observed that the moduli space of stable vector bundles $M_X(2, h, 3)$ over $X$ is isomorphic to a K3 surface $Y$, which is defined in the dual variety $\mathbb{S}_5^*$ in the following way: Let $L_8$ be a general 8-dimensional linear subspace in $S_{16}$ and by $L^\perp_8$ its orthogonal space in $S_{16}$. Then the K3 surfaces $X$ and $Y$ above are given by the “orthogonal linear sections to each other”.

Due to the isomorphism $Y \simeq M_X(2, h, 3)$ (see [IM] for a proof), we can write the equivalence $\Phi_P : D^b(Y) \simeq D^b(X)$ using the universal bundle $P$ over $X \times Y$ as the kernel of the Fourier-Mukai transform $\Phi_P(-) = R\pi_Y(\mathcal{L}_{\pi_Y}(-) \otimes P)$.

2.7.2. Mirror family of $\mathcal{M}_6$-polarized K3 surfaces. Let us consider marked $\mathcal{M}_6$-polarized K3 surfaces, which are the mirror K3 surfaces of $X$ as defined in Subsection 2.4. Their isomorphism classes are classified by the points on the quotient of the period domain $D(\mathcal{M}_6^\perp)$ by the group $O(\mathcal{M}_6^\perp)^\ast$. Noting that $D(\mathcal{M}_6^\perp) \simeq V(\mathcal{M}_6)$ consists of two copies of the upper half plane $\mathbb{H}_+$ and an isomorphism $O^+(\mathcal{M}_6^\perp)^\ast \simeq \Gamma_0(6)_{6+}$ (see [DG] Thm.(7.1), Rem.(7.2)), we have

$$D(\mathcal{M}_6^\perp)^+ / O^+(\mathcal{M}_6^\perp)^\ast \simeq \mathbb{H}_+ / \Gamma_0(6)_{6+},$$

see Fig.1. On the other hand, we have an isomorphism $O^+(\mathcal{M}_6^\perp) / \{ \pm \text{id} \} \simeq \Gamma_0(6)_{+}$ for the monodromy group of the $\mathcal{M}_6^\perp$-polarizable K3 surfaces [iibid] (see also [HOly1] Thm.5.5)). For these two groups, we have the following presentations:

\begin{align}
\Gamma_0(6)_{+} & = \langle \left( \begin{array}{cc} 1 & 1 \\
0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & -\frac{\sqrt{6}}{\sqrt{3}} \\
\sqrt{3} & 0 \end{array} \right), \left( \begin{array}{cc} 2\sqrt{3} & 3 \sqrt{3} \\
2 \sqrt{3} & \sqrt{3} \end{array} \right) \rangle =: \langle T_0, S_1, S_2S_1 \rangle, \\
\Gamma_0(6)_{6+} & = \langle \left( \begin{array}{cc} 1 & 1 \\
0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & -\frac{\sqrt{6}}{\sqrt{3}} \\
\sqrt{3} & 0 \end{array} \right), \left( \begin{array}{cc} 5 & 2 \\
12 & 5 \end{array} \right) \rangle =: \langle T_0, S_1, (S_2S_1)^2 \rangle,
\end{align}

with $S_2 = \left( \begin{array}{cc} -\sqrt{2} & \frac{\sqrt{6}}{\sqrt{3}} \\
-3\sqrt{2} & \sqrt{3} \end{array} \right)$. Explicit relations of $\Gamma_0(6)_{+}$ and $\Gamma_0(6)_{6+}$ to $O^+(\mathcal{M}_6^\perp) / \{ \pm \text{id} \}$ and $O^+(\mathcal{M}_6^\perp)^\ast$, respectively, are given by fixing an isomorphism $\mathcal{M}_6^\perp \simeq (\mathbb{Z}^{63}, \Sigma_6)$ with $\Sigma_6 = \left( \begin{array}{ccc} 0 & 0 & 1 \\
0 & 12 & 0 \\
1 & 0 & 0 \end{array} \right)$ and an anti-homorphism $R : \text{PSL}(2, \mathbb{R}) \to SO(2,1,\mathbb{R})$,

$$R : \left( \begin{array}{cc} a & b \\
c & d \end{array} \right) \mapsto \left( \begin{array}{cc} a^2 & -2ac & -\frac{a^2}{d^2} \\
-ab & ad+bc & -\frac{a^2}{d^2} \\
-6c^2 & 12bd & d^2 \end{array} \right) \in SO(2,1,\mathbb{R}),$$

where $SO(2,1,\mathbb{R}) = \{ g \in \text{Mat}(3,\mathbb{R}) \mid \langle g \Sigma_6 g \rangle = \Sigma_6 \}$. Here, we naturally consider $O^+(\mathcal{M}_6^\perp), O^+(\mathcal{M}_6^\perp)^\ast$ in $SO(2,1,\mathbb{R})$ (and the image of $O^+(\mathcal{M}_6^\perp) \to SO(2,1,\mathbb{R})$, $g \mapsto (\det g)g$ for $O^+(\mathcal{M}_6^\perp) / \{ \pm \text{id} \}$). The group index $[\Gamma_0(6)_{+} : \Gamma_0(6)_{6+}] = 2$ is
2.7.3. Monodromy calculations. As we see in Fig.1, there are two cusps in \( \mathbb{H}_+/\Gamma_0(6)_{+6} \). By Proposition 2.4 below, we see that these two are identified by the action of an element \( \Gamma_0(6)_{+6} \backslash \Gamma_0(6)_{+6} \). In fact, these cusps correspond to the maximally unipotent monodromy (MUM) points at \( x = 0 \) and \( x = \infty \) of (2.10), which we read in the following Riemann’s \( \mathcal{P} \) scheme:

\[
\begin{array}{ccc}
0 & a_1 & a_2 \\
0 & 0 & 1 \\
0 & 1 & 1 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\infty & 0 & 0
\end{array}
\]

with \( a_1 := 17 - 12\sqrt{2} \), \( a_2 := 17 + 12\sqrt{2} \) (see [Lo] for a general definition of MUM points). The relation of these cusps becomes explicit by constructing an integral basis of the solutions of the Picard-Fuchs equation (2.10) which is compatible with the mirror isomorphism \( T_X \cong (K(X), -\chi(\ast, \ast)) \) in (2.7). Since the construction is general for other K3 surfaces [Ho] and also parallel to that for Calabi-Yau threefolds (see [Ho, Ta, Sect.2]), we briefly sketch it here. Firstly, we set up the local solutions about the MUM point \( x = 0 \) of the form \( w_0(x) = 1 + O(x) \) and

\[
\begin{align*}
w_1(x) &= w_0(x) \log(x) + w_1^{reg}(x), \\
w_2(x) &= -w_0(x)(\log x)^2 + 2w_1(x) \log x + w_2^{reg}(x)
\end{align*}
\]
The two local solutions are related under an analytic continuation along a path with \( w(z) = \frac{1}{z} \) around \( z = 0 \) requiring \( w_0(z) = z(1 + O(z)) \) and \( \tilde{w}_2(z) = z(\tilde{c}z + O(z^2)) \). Using these, we set the following ansatz for the integral basis:

\[
\Pi(x) = N_x \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{(z)} \end{pmatrix}, \quad \Pi(z) = N_z \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{(z)} \end{pmatrix},
\]

where \( N_x \) and \( N_z \) are unknown constants and \( n_k := \frac{1}{(z)} \). These forms are expected in general to give an integral basis which represents the mirror isomorphism \( T_X \cong (K(X), -\chi(\ast, \ast)) \) with the bilinear form \( \Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \). The constants \( N_x, N_z \) are determined by the Griffiths transversalities:

\[
\begin{align*}
\Pi \Sigma_n \Pi &= \Pi \Sigma_n \frac{d^2}{dz^2} \Pi = 0, \\
\tilde{\Pi} \Sigma_n \tilde{\Pi} &= \Pi \Sigma_n \frac{d^2}{dz^2} \tilde{\Pi} = 0.
\end{align*}
\]

The following results are parallel to those in [Ho [a]], Prop.2.10:

**Proposition 2.4.** (1) The ansatz (2.11) with \( N_x = N_z = -1 \) satisfies (2.12).

(2) The two local solutions are related under an analytic continuation along a path through the upper half plane by \( \Pi(x) = U_{xz} \Pi(z) \) with \( U_{xz} = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & -3 \\ -2 & -3 & -1 \end{pmatrix} \).

(3) Monodromy matrices \( M_x \) of \( \Pi(x) \) of the \( \Pi(z) \) around each singular point \( x = c \) of (2.10) are given by

| \( x = 0 \) | \( a_1 \) | \( a_2 \) | \( \infty \) |
|---|---|---|---|
| \( M_x \) | \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -6 & -12 & 1 \end{pmatrix} | \begin{pmatrix} -24 & 120 & 25 \\ -10 & 49 & 10 \end{pmatrix} | \begin{pmatrix} 49 & -168 & -24 \\ 21 & -71 & -10 \end{pmatrix} |

| \( M_x \) | \begin{pmatrix} 25 & 120 & -24 \\ -15 & 71 & 13 \end{pmatrix} | \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} | \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} |


and satisfy \( M_0 M_a M_{a_2} M_{\infty} = \text{id} \) and \( \tilde{M}_c = U_{xz}^{-1} M_x U_{xz} \) with \( U_{xz}^{-1} = \begin{pmatrix} -1 & 5 & 1 \\ -2 & 5 & 1 \\ 3 & 2 & 2 \end{pmatrix} \).

(4) \( M_x \)’s and \( U_{xz} \) are given in terms of generators of \( \Gamma_0(6)_+ \) in \( \mathbb{Z}_6 \) by

\[
M_0 = R(T_0^{-1}), \quad M_{a_1} = -R(S_1), \quad M_{a_2} = -R(S_2 S_1 S_2), \quad U_{xz} = R(S_1 S_2).
\]

In particular \( M_0 M_{a_1} M_{a_2} \in O(M_6^+) \) and \( U_{xz} \) \( \in O(M_6^+) \) \( \setminus O(M_6^+) \) with the symmetric form \( \Sigma_6 \).

In Fig. 2.1, we see that the modular action of the element \( S_1 S_2 \in \Gamma_0(6)_+ \setminus \Gamma_0(6)_+6 \) on \( \mathbb{H}_+ \) identifies the image of \( D_+ \) with that of \( D_- \) by exchanging the two cusps points.

### 2.7.4. FM functor \( \Phi_F \) and Auteq \( D^6(X) \)

We can read more from the mirror isomorphism \( T_X \cong (K(X), -\chi(\ast, \ast)) \) which comes from the monodromy calculations. Let us note that the integral basis \( \Pi(x) = \Pi_1, \Pi_2, \Pi_3 \) in Proposition 2.4 explicitly determines the corresponding basis \( (\gamma_1, \gamma_2, \gamma_3) \) of the transcendental lattice \( T_X \). As for the basis of the lattice \( (K(X), -\chi(\ast, \ast)) \), we may take

\[
([E_1], [E_2], [E_3]) = ([O_x], [O_h] + 6[O_x], -[I_x]),
\]

with \( 0 \to O_X(-h) \to O_X \to O_h \to 0 \), and \( O_x \) the skyscraper sheaf and \( I_x \) the ideal sheaf of a point \( x \in X \). Note that we choose \([E_2] \) so that \( ch([O_h] + 6[O_x]) = h \), and...
hence we can verify \((-\chi([\mathcal{E}_1]), [\mathcal{E}_2])\) = \(\Sigma_0\) by Riemann-Roch theorem. Identifying these two bases, we have an explicit isomorphism \(T_X \cong (K(X), -\chi(*, *))\) (this can be done in general [Ho Sect.2.4]).

Actually, the identification of the two basis above is somehow canonical from the viewpoint of homological mirror symmetry, since we can show that the topology of \(\gamma_1\) is isomorphic to the real two torus, i.e. \(\gamma_1 \cong T^2\). The identification of such torus cycle with \(\mathcal{O}_x\) is justified from many aspects of the homological mirror symmetry \(D^b\text{Fuk}(X) \cong D^b(X)\) (see [Ko SYZ]). Note also that \(\gamma_1\) is isotropic in \(T_X\) and choosing such a vector in \(T_X\) determines (almost uniquely, i.e., up to signs) other bases with the specified intersection numbers in the entries of \(\Sigma_0\). Similar construction of the basis of \(\Pi(z)\) (or the cycles \(\gamma_1, \gamma_2, \gamma_3\)) and the identification \(\gamma_1 \approx T^2\) with \(\mathcal{O}_y\) are valid for \((K(Y), -\chi(*, *))\). We denote by \(h'\) the polarization of \(Y\).

Now recall that the Fourier-Mukai functor \(\Phi_P : D^b(Y) \cong D^b(X)\) is defined by the kernel \(P\), the universal bundle over \(X \times Y = X \times M_X(2, h, 3)\), and hence we have \(\Phi_P(\mathcal{O}_y) = \mathcal{P}_y\) with the Mukai vector \(\text{ch}(\mathcal{P}_y)\sqrt{\text{Todd}X} = 2 + h + 3v\) (\(v := \text{ch}(\mathcal{O}_x)\)). From this, we have

\[
\text{ch}(\Phi_P(\mathcal{O}_y)) = \text{ch}(\mathcal{P}_y) = 2 + h + v = 3v + h + 2(1 - v) = 3\text{ch}([\mathcal{E}_1]) + \text{ch}([\mathcal{E}_2]) - 2\text{ch}([\mathcal{E}_3]),
\]

and identify this in the 1st column of the connection matrix \(U_{xz} = R(S_1S_2^{-1})\) (note that we identify \(\gamma_1\) with \(\mathcal{O}_y\)). This leads us to a conjecture that the continuation of the cycles \(\gamma_1, \gamma_2, \gamma_3\) to \(\gamma_1, \gamma_2, \gamma_3\) corresponds to the Fourier-Mukai functor \(\Phi_P : D^b(Y) \cong D^b(X)\). Note that the analytic continuation of \(\Pi(x)\) connects cycles in the fibers around \(x = 0\) and those around \(x = \infty\), but actually it comes from a Dehn twist of \(X\) because the local family around \(x = 0\) and \(x = \infty\) are isomorphic as the family of \(M_0\)-polarizable K3 surfaces. Dehn twists around \(x = 0, a_1, a_2, \infty\) are easy to be identified from the standard forms of the monodromy matrices \(M_0, M_{a_1}, M_{a_2}\) and \(M_\infty\). They can be identified, respectively, with the following Fourier-Mukai functors (see e.g. [ST]):

\[
(-) \otimes \mathcal{O}_X(h), \quad \Phi_{\mathcal{I}_{\Delta}(X)} \quad \text{and} \quad \Phi_P \circ \Phi_{\mathcal{I}_{\Delta}(Y)} \circ \Phi_P^{-1},
\]

where \(\mathcal{I}_{\Delta}(X)\) (resp. \(\mathcal{I}_{\Delta}(Y)\)) is the ideal sheaf of the diagonal \(\Delta \subset X \times X\) (resp. \(\Delta \subset Y \times Y\)) and \(h'\) is the polarization of \(Y\). From the above considerations, and taking the monodromy relation into account, we naturally come to a conjecture that the group \(\text{Auto}D^b(X)\) is generated by the shift functor and the following Fourier-Mukai functors:

\[
(-) \otimes \mathcal{O}_X(h), \quad \Phi_{\mathcal{I}_{\Delta}(X)} \quad \text{and} \quad \Phi_P \circ \Phi_{\mathcal{I}_{\Delta}(Y)} \circ \Phi_P^{-1}.
\]

2.8. Some other aspects. From the example in the previous subsection, one may expect some relation between the Fourier-Mukai numbers \(|FM(X)|\) and the numbers of MUM points in \(D(\tilde{M}^1)/O(\tilde{M}^1)^*\). In fact, S. Ma [Ma] (see also [Ha]) showed that the counting formula in Theorem 2.2 allows such interpretation if we identify MUM points with the standard cusps in the Baily-Satake compactification of \(D(\tilde{M}^1)/O(\tilde{M}^1)^*\). From this viewpoint, we can read the counting formula as the number of non-isomorphic decompositions of \(\tilde{M}^1\) into \(\tilde{M}^1 = U \oplus M\) modulo the

\[\text{1The correspondence between the Chern characters} \: \text{ch}(\mathcal{P}_y) = \text{ch}(\Phi_P(\mathcal{O}_y)) \: \text{for} \: \mathcal{P} = \gamma \rightarrow X \: (Y \in FM(X)) \: \text{and the elements in} \: \Gamma_0(n)_{\pm} \setminus \Gamma_0(n)_{\pm+n} \: \text{in general has been worked in} \: [\text{Kaw}].\]
actions of $O(\tilde{M}^\perp)^*$. Non-standard cusps are 0-dimensional boundary points which correspond to the decompositions $M^\perp = U(m) \oplus M \ (m > 1)$. In ref. [Ma], the counting formula has been generalized to incorporate non-standard cusps, and it has been shown that the generalized formula counts the number of twisted Fourier-Mukai partners, i.e., K3 surfaces $Y$ satisfying $D^b(X) \simeq D^b(Y, \alpha)$ where $\alpha$ is an element of the Brauer Group $Br(Y)$. See references [HS, Ca] for the derived categories of twisted sheaves on $Y$.

3. Fourier-Mukai partners of Calabi-Yau threefolds I

We define Calabi-Yau 3-folds by smooth, projective, three dimensional varieties $X$ over $\mathbb{C}$ which satisfy $K_X \simeq \mathcal{O}_X$, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. It is known, due to Bridgeland [Br2], that birational Calabi-Yau 3-folds $X, Y$ are derived equivalent, i.e., $D^b(X) \simeq D^b(Y)$. Except this general theorem, however, not much is known about the Fourier-Mukai partners of Calabi-Yau 3-folds. Here and in the next section, we focus on two examples of pairs of Calabi-Yau 3-folds with Picard number one which are Fourier-Mukai partners but not birational to each other. In both cases, some similarity to the example of Mukai in the last section will be observed in the fact that suitable projective dualities play important roles in their constructions and also their derived equivalences.

3.1. Grassmannian and Pfaffian Calabi-Yau threefolds. The first example is Calabi-Yau 3-folds due to Rødland. Let $G(2, 7)$ be the Grassmannian of two dimensional subspaces in $\mathbb{C}^7$. Consider the Plücker embedding of $G(2, 7)$ into $\mathbb{P}(\wedge^2 \mathbb{C}^7)$. Then the projective dual of $G(2, 7)$ is the Pfaffian variety $\text{Pf}(4, 7)$ in the dual projective space $\mathbb{P}(\wedge^2 (\mathbb{C}^*)^7)$, i.e., the locus $\{ [c_{ij}] \in \mathbb{P}(\wedge^2 (\mathbb{C}^*)^7) \mid \text{rank} (c_{ij}) \leq 4 \}$. Let us consider general 7 dimensional linear subspace $L_7 \subset \wedge^2 (\mathbb{C}^*)^7$ and its orthogonal subspace $L_7^\perp \subset \wedge^2 \mathbb{C}^7$. Then, similarly to the construction in Subsection 2.7.1 we define

$$X = G(2, 7) \cap \mathbb{P}(L_7^\perp) \subset \mathbb{P}(\wedge^2 \mathbb{C}^7), \ Y = \text{Pf}(4, 7) \cap \mathbb{P}(L_7) \subset \mathbb{P}(\wedge^2 (\mathbb{C}^*)^7).$$

$X$ and $Y$, respectively, are called Grassmannian and Pfaffian Calabi-Yau 3-folds.

Proposition 3.1 (Rødland [Ro]). When $L_7$ is general, both $X$ and $Y$ are smooth Calabi-Yau 3-folds with Picard number one and the following invariants:

- $H^3_X = 42$, $c_2(X).H_X = 84$, $h^{1,1}(X) = 1$, $h^{2,1}(X) = 50$
- $H^3_Y = 14$, $c_2(Y).H_Y = 56$, $h^{1,1}(Y) = 1$, $h^{2,1}(Y) = 50$

where $H_X$ and $H_Y$ are the ample generators of the Picard groups, respectively.

As for the smoothness, it is further known that $X$ is smooth if and only if $Y$ is smooth [BC]. The equal Hodge numbers might indicate a possibility that $X$ and $Y$ were birational to each other [Ba2]. However, looking the degrees $H^3_X = 42$ and $H^3_Y = 14$ together with $\rho(X) = \rho(Y) = 1$, we see that this is not the case.
In \[Ro\], Rødland studied mirror symmetry of Pfaffian Calabi-Yau threefold \(Y\) and constructed a mirror family \(\mathcal{Y} = \{\tilde{Y}_x\}_{x \in \mathbb{P}^2}\) by the so-called orbifold mirror construction. His construction starts with a special family of Pfaffian Calabi-Yau 3-folds which admits a Heisenberg group action \([GiPo]\). By finding a suitable subgroup of the Heisenberg group as the orbifold group, and making a crepant resolutions for the singularities in the orbifold mirror construction, the desired mirror Calabi-Yau 3-folds \(\tilde{Y}\) with Hodge numbers \(h^{1,1}(\tilde{Y}) = 50, h^{2,1}(\tilde{Y}) = 1\) was obtained. Independently, mirror symmetry of Grassmannian Calabi-Yau 3-folds \(X\) was studied in \[BCKvS\] by the method of toric degeneration of Grassmannians. It was recognized by these authors that the Picard-Fuchs differential equations for these two families have exactly the same form but they are distinguished by two different MUM points of the equation, as we have witnessed in the equation (2.11).

3.2. Derived equivalence \(D^b(X) \simeq D^b(Y)\). As described in the previous subsection, there are similarities in their constructions between the example of Fourier-Mukai partners in Subsection 2.7 and the Grassmannian and Pfaffian Calabi-Yau 3-folds \(X\) and \(Y\). It is natural to expect that \(X\) and \(Y\) are derived equivalent. In fact, the derived equivalence is supported from the analysis of Gauged Linear Sigma Model (GLSM) in physics \[HT\]. The derived equivalence has been proved mathematically in \[BC\] and \[Kn2\] (see also \[BDFI\], \[ADS\] for recent progresses).

Let \(\mathcal{Y}\) be the Pfaffian variety \(\text{Pf}(4,7)\). \(\mathcal{Y}\) is singular along \(\mathcal{Y}_{\text{sing}} = \{[c_{ij}] \mid \text{rk} c \leq 2\}\) and has a natural (Springer-type) resolution

\[
\tilde{\mathcal{Y}} = \{([c], [w]) \mid w \subset \ker c\} \subset \mathcal{Y} \times G(3,7).
\]

Since it is easy to see that all the fibers of the projection \(\rho : \tilde{\mathcal{Y}} \rightarrow G(3,7)\) are isomorphic to \(\mathbb{P}^5\), \(\tilde{\mathcal{Y}}\) is smooth. Let us denote \(G(2,7)\) by \(\mathcal{X}\). Then we have \(X = \mathcal{X} \cap \mathbb{P}(L^7)\) and also we can write \(Y = \tilde{\mathcal{Y}} \cap \mathbb{P}(L_7)\) since \(\mathcal{Y}_{\text{sing}}\) is away from \(\mathbb{P}(L_7)\) for general \(L_7\). Let us summarize our settings into the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{\rho} & \tilde{\mathcal{Y}} \\
\downarrow & & \downarrow \pi \\
G(2,7) & \xrightarrow{\pi} & G(3,7) & \text{Y}
\end{array}
\]

The proofs of the derived equivalence in \[BC\] and \[Kn2\] uses a natural incidence correspondence between the two Grassmannians in the diagram, which is given by

\[
\Delta_0 = \{([\xi], [w]) \mid \dim(\xi \cap w) \geq 1\} \subset G(2,7) \times G(3,7).
\]

To sketch the proofs, let us consider the ideal sheaf \(\mathcal{I}_{\Delta_0}\) of \(\Delta_0\) and define its pullback \(\mathcal{I} := (\text{id} \times \rho)^*\mathcal{I}_{\Delta_0}\) on \(\mathcal{X} \times \mathcal{Y}\). The restriction \(I := \mathcal{I} |_{\mathcal{X} \times \mathcal{Y}}\) is an ideal sheaf on \(\mathcal{X} \times \mathcal{Y}\). We regard \(I\) as an object in \(D^b(\mathcal{X} \times \mathcal{Y})\) and defines the Fourier-Mukai functor \(\Phi_I(-) := R\pi_X_*(L\pi_Y^*(-) \otimes I)\), where \(\pi_X\) and \(\pi_Y\) are projections to \(\mathcal{X}\) and \(\mathcal{Y}\). Then, Borisov and Cadararu proved the following.
Theorem 3.2 ([BC, Theorem 6.2]). \( \Phi_f(-) : D^b(Y) \to D^b(X) \) is an equivalence.

The proof of the above theorem is based on the following theorem for smooth projective varieties \( X, Y \) and a Fourier-Mukai functor \( \Phi_P(-) = R\pi_X_*(L\pi_Y^*(-) \otimes \mathcal{P}) \) with an object \( \mathcal{P} \in D^b(X \times Y) \) (see [BO, Thm.1.1], [Br2, Thm.1.1], [Hu, Cor. 7.5, Prop. 7.6]):

**Theorem 3.3.** If \( \mathcal{P} \) a coherent sheaf on \( X \times Y \) flat over \( Y \), then \( \Phi_P : D^b(Y) \to D^b(X) \) is fully faithful if and only if the following two conditions are satisfied:

(i) For any point \( x \in X \), it holds \( \text{Hom}(\mathcal{P}_x, \mathcal{P}_x) \simeq \mathbb{C} \), and

(ii) if \( x_1 \neq x_2 \), then \( \text{Ext}^i(\mathcal{P}_{x_1}, \mathcal{P}_{x_2}) = 0 \) for any \( i \).

Under these conditions, \( \Phi_P \) is an equivalence if and only of \( \dim X = \dim Y \) and \( \mathcal{P} \otimes \pi_Y^*\omega_Y \simeq \mathcal{P} \otimes \pi_Y^*\omega_Y \).

It has been proved that the ideal sheaf \( I \) is flat over \( Y \), and in fact, defines a flat family of curves parametrized by \( Y \) [BC, Prop. 4.4]. The condition \( \text{Hom}(I_y, I_y) \simeq \mathbb{C} \) follows from a general property of ideal sheaves of subschemes of dimension \( \leq 1 \) in smooth projective 3-folds [ibid,Prop. 4.5]. Hence, verifying the cohomology vanishings

\[
\text{Ext}^i(I_{y_1}, I_{y_2}) = 0 \quad (y_1 \neq y_2)
\]

is the main part of the proof given in [ibid].

Kuznetsov formulates the derived equivalence as a consequence of the homological projective duality (HPD) between \( G(2, 7) \) and \( Pf(4, 7) \) (precisely, the non-commutative resolution of \( Pf(4, 7) \)). In the proof given in [Ku2, Theorem 6.2] the following locally free resolution of the ideal sheaf \( \mathcal{I} \) on \( X \times Y \) plays an important role:

\[
(3.3) \quad 0 \to S^2\mathcal{U} \otimes \mathcal{O}_Y \to \mathcal{U} \otimes \mathcal{Q} \to \mathcal{O}_X \otimes \wedge^2 \mathcal{Q} \to \mathcal{I} \otimes \mathcal{O}_{X \times Y}(1, (1, 0)) \to 0,
\]

where \( \mathcal{U} \) is the universal bundle on \( G(2, 7) \), \( \mathcal{Q} \) is the universal quotient bundle on \( G(3, 7) \) and \( \mathcal{O}_{X \times Y}(1, (1, 0)) := (\mathcal{O}_X(1) \boxtimes \rho^* \mathcal{O}_{G(3, 7)}(1)) \) (see [ibid, Lemma 8.2]). The restriction of \( \mathcal{B}_{34} \) to \( X \times \{y\} \) is nothing but the Eagon-Northcott complex which was used for the proof of the vanishings \([3.3] \) in [BC, Prop. 3.6]. Although we do not go into the details of HPD, but for the comparison with the corresponding results in another example in the next section it is useful to summarize some of the main results in [Ku2]. For that, let us introduce the following notation for the sheaves that appear in \([3.3] \):

\[
E_3 = S^2\mathcal{U}, \quad E_2 = \mathcal{U}, \quad E_1 = \mathcal{O}_X; \quad F_3 = \mathcal{O}_Y, \quad F_2 = \mathcal{Q}, \quad F_1' = \wedge^2 \mathcal{Q},
\]

and define the following full subcategories \( \mathcal{A}_i \subset D^b(X) \) \((i = 0, \ldots, 6) \) and \( \mathcal{B}_k \subset D^b(Y) \) \((k = 0, \ldots, 13) \):

\[
(3.5) \quad \langle E_3, E_2, E_1 \rangle = \mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_6 \subset D^b(X),
\]

\[
\langle F_3^*, F_2^*, F_1^* \rangle = \mathcal{B}_0 = \mathcal{B}_1 = \cdots = \mathcal{B}_{13} \subset D^b(Y),
\]

where we set \( F_i := F_i^*/\mathcal{O}_Y(a, b) = \rho^* \mathcal{O}_{G(3, 7)}(a) \otimes \pi^* \mathcal{O}_Y(b) \).

**Theorem 3.4 ([Ku2, Theorem 4.1]).** Denote by \( \mathcal{A}_i(a), \mathcal{B}_i(a) \) the twists of \( \mathcal{A}_i, \mathcal{B}_i \) by \( \mathcal{O}_X(a) \) and \( \pi^* \mathcal{O}_Y(a) \), respectively. Then

(i) \( \langle \mathcal{A}_0(1), \cdots, \mathcal{A}_6(6) \rangle \) is a Lefschetz decomposition of \( D^b(X) \), and

(ii) \( \langle \mathcal{B}_{13}(-13), \cdots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle \) is a dual Lefschetz decomposition of \( D^b(Y) \),
where \(\hat{D}^b(Y) \subset D^b(\hat{Y})\) is a full subcategory which is equivalent to \(D^b(Y, R)\), the bounded derived category of coherent sheaves of right \(R\)-modules on \(Y\) with \(R = \pi_*\text{End}(O_{\hat{Y}} \oplus \rho^*\hat{U})\) and \(\hat{U}\) the universal bundle on \(G(3, 7)\).

A (dual) Lefschetz decomposition is a special form of a semi-orthogonal decomposition of a triangulated category [BO]. In our case, the vanishings

\[
\text{Hom}^•_{D^b(\hat{Y})}(B_i(-i), B_j(-j)) = 0 \quad (i < j),
\]

which are implied in (ii) of the above theorem, entail the desired vanishings [LM].

3.3. BPS numbers. As noted in the previous subsection, the ideal sheaf \(I_y(y \in Y)\) defines a family of curves on \(X\). It can be shown by explicit calculations with Macaulay2 that

**Proposition 3.5.** For a general point \(y \in Y\), the ideal sheaf \(I_y\) defines a smooth curve on \(X\) of genus 6 and degree 14.

Expecting some relations to the moduli problems of ideal sheaves on \(X\), such as Donaldson-Thomas invariants of \(X\) [PT] or BPS numbers [HST], it is interesting to seek a possibly related number in the table of the BPS numbers calculated in [HK]. The relevant part of the table to the curves of Proposition 3.5 reads as follows (with \(d = 14\)):

\[
\begin{array}{c|cccccccc}
  g & 0 & \cdots & 6 & 7 & 8 & 9 & 10 \\
  n_y^X(d) & 2.67\times10^{19} & \cdots & 123676 & 392 & 7 & 0 & 0 \\
\end{array}
\]

Unfortunately the BPS number \(n_y^X(14) = 123676\) is rather large to find a relation to the curve defined by \(I_y\). However, as noted in [HoTa] (4-1.6), we can observe that \(n_8(14) = 7\) counts a well-known family of curves studied by Mukai, i.e., curves that are linear sections of \(G(2, 6)\). Such curves appear in our setting as \(G(2, 6) \cap \mathbb{P}(L_7^+) \subset G(2, 7) \cap \mathbb{P}(L_7^+) = X\), and hence they are naturally parameterized by \(\mathbb{P}^6 \simeq \{G(2, 6) \subset G(2, 7)\}\). General members of this family are smooth and of genus 8 and degree 14. Then, following the counting “rule” of BPS numbers [AV], we explain the number \(n_8(14) = 7\) as

\[
n_8(14) = (-1)^{\dim \mathbb{P}^6} e(\mathbb{P}^6) = 7.
\]

The counting “rule” also tells us that such a generically smooth family of curves of genus \(g\) contributes to the numbers \(n_h(d) \ (h \leq g)\) in a specified way [ibid]. Thus our observation above indicates that there are contributions from at least two different families of (generically) smooth curves in the BPS numbers \(\{n_h(14)\}_{h \leq 8}\) in [LM].

3.4. Mirror symmetry. Consider the mirror family \(\hat{Y} = \{\hat{Y}_x\}_{x \in \mathbb{P}^1}\) obtained from the orbifold mirror construction [Ro]. The Picard-Fuchs differential equation satisfied by the period integrals \(w(x) = \int_{\gamma} \Omega(\hat{Y}_x) \ (\gamma \in H_3(\hat{Y}_0, \mathbb{Z}))\) has been determined
by Rødland as $D_x w(x) = 0$ with

$$D_x = -9 \theta_x^3 - 3x(15 + 102 \theta_x + 272 \theta_x^2 + 340 \theta_x^3 + 173 \theta_x^4) - 2x^2(1083 + 4773 \theta_x + 7597 \theta_x^2 + 5032 \theta_x^3 + 1129 \theta_x^4) + 2x^2(6 + 675 \theta_x + 2353 \theta_x^2 + 2628 \theta_x^3 + 843 \theta_x^4) - x^4(26 + 174 \theta_x + 478 \theta_x^2 + 608 \theta_x^3 + 295 \theta_x^4) + x^5(\theta_x + 1)^4,$$

and $\theta_x = x \frac{d}{dx}$. As described in Subsection 3.1 the operator $D_x$ is the same as that of $X$ in [BCKvS, ES] and Gromov-Witten invariants of $X$ and $Y$ are calculated, respectively, from the MUM points at $x = 0$ and $z = \frac{1}{x} = 0$. Although the geometry of the family is rather complicated (cf. Subsection 4.4), monodromy calculations proceed in a similar way to Subsection 2.7. The Riemann’s $\mathcal{P}$-scheme is

$$\left\{ \begin{array}{cccccc}
0 & \alpha_1 & \alpha_2 & \alpha_3 & 3 & \infty \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 3 & 1 \\
0 & 2 & 2 & 2 & 4 & 1 \\
\end{array} \right\},$$

where $\alpha_k$ are the (real) roots of the ‘discriminant’ $1 - 57x - 289x^2 + x^3 = 0$ and $x = 3$ is an apparent singularity with no monodromy (with order $\alpha_2 < 0 < \alpha_1 < \alpha_3 < \alpha_3$). The symplectic and integral basis of the solution can be obtained by making ansatz similar to those in Subsection 2.7 (see also [BM, EF]). In fact, its full details are completely parallel to [HoTa] (2.5.1)-(2.5.7) assuming two local solutions of the forms,

$$\Pi(x) = N_x \begin{pmatrix} 1 & 0 & 0 & 0 & \beta & \kappa \gamma & 0 & -\kappa/6 \\
0 & 0 & 0 & 0 & 1 & \beta & a \kappa/2 & 0 \\
\gamma & \beta & 0 & -\kappa/6 & 0 & 1 & 0 & 0 \\
\end{pmatrix} , \tilde{\Pi}(z) = N_z \begin{pmatrix} 1 & 0 & 0 & 0 & \beta & \kappa \gamma & 0 & -\kappa/6 \\
0 & 0 & 0 & 0 & 1 & \beta & a \kappa/2 & 0 \\
\gamma & \beta & 0 & -\kappa/6 & 0 & 1 & 0 & 0 \\
\end{pmatrix}. $$

Here we summarize only the results of the monodromy matrices.

**Proposition 3.6.** (1) When $N_x = N_z = 1, a = \bar{a} = 0$ and

$$(\kappa, \beta, \gamma) = \left(H_x^3, -\frac{\alpha_1 H_x}{2}, -\frac{\zeta(3)(\gamma, X)}{2 \pi i x^3} \right), \tilde{\kappa}(\tilde{x}, \tilde{\beta}, \tilde{\gamma}) = \left(H_y^3, -\frac{\alpha_1 H_y}{2}, -\frac{\zeta(3)(\gamma, Y)}{2 \pi i y^3} \right),$$

the solutions $\Pi(x)$ and $\tilde{\Pi}(z)$ are integral and symplectic with respect to the symplectic form $S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$ These are analytically continued along a path in the upper-half plane as $\Pi_x(x) = U_{xx} \tilde{\Pi}(z)$ by a symplectic matrix $U_{xx} = \begin{pmatrix} -3 & 7 & -1 & -4 \\
0 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -14 & 0 & -5 \\
\end{pmatrix}$ with its inverse $U_{xx}^{-1} = \begin{pmatrix} -5 & 7 & -1 & -4 \\
0 & 5 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -14 & 0 & -3 \\
\end{pmatrix}$.

(2) The monodromy matrices $M_c$ of $\Pi(x)$ ($\tilde{M}_c$ of $\tilde{\Pi}(z)$) around each singular point $c$ are symplectic with respect to $S$, and they are given by (with $\tilde{M}_c = U_{xx}^{-1} M_c U_{xx}$)

| $x = 0$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\infty$ |
|---|---|---|---|---|
| $M_1$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-14 & 21 & -1 & 1 \\
\end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$ | $\begin{pmatrix} 15 & -14 & 2 & 4 \\
-7 & 6 & -1 & 2 \\
-49 & -49 & 8 & 14 \\
-49 & -49 & -7 & -13 \\
\end{pmatrix}$ | $\begin{pmatrix} 1 & 42 & 0 & 9 \\
0 & 1 & 0 & 0 \\
0 & -196 & 0 & -12 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$ | $\begin{pmatrix} 85 & -14 & 16 & 42 \\
6 & -6 & 1 & 2 \\
-32 & 7 & -62 & -168 \\
-33 & 28 & -6 & -13 \\
\end{pmatrix}$ |
| $\tilde{M}_1$ | $\begin{pmatrix} -27 & 322 & -8 & 126 \\
13 & -125 & 4 & -50 \\
42 & 385 & -13 & 155 \\
\end{pmatrix}$ | $\begin{pmatrix} 1 & 70 & 0 & 25 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$ | $\begin{pmatrix} -27 & 0 & -8 & 16 \\
14 & 1 & 4 & -8 \\
49 & 0 & -14 & 29 \\
\end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
7 & 14 & 1 & 0 \\
-7 & -7 & -1 & 1 \\
\end{pmatrix}$ |
and satisfy $M_{a_2}M_0M_{a_1}M_{a_3}M_{\infty} = \text{id}$.

As before, the integral basis $\Pi(x) = (\Pi_1, \Pi_2, \Pi_3, \Pi_4)$ implicitly determines the corresponding integral cycles $\gamma_i$, likewise for $\tilde{\Pi}(z)$ with the corresponding integral cycles $\tilde{\gamma}_i (i = 1,..,4)$. From the geometry of the family, one can see that $\gamma_1 \approx \tilde{\gamma}_1 \approx T^3$ and also $\gamma_4 \approx \tilde{\gamma}_4 \approx S^3$ about the topologies of the cycles. Form the homological mirror symmetry, these cycles may be identified with the skyscraper sheaves $O_x, O_y (x \in X, y \in Y)$ and the structure sheaves $O_X, O_Y$ as was the case in Subsection 2.7.4. Unfortunately we do not see directly the relation $\text{ch}(\Phi_I(O_y)) = \text{ch}(I_y)$ in the 1st column of $U_{xz}$ as before. However, we believe that if we take suitable auto-equivalences into account, in other words, if we change the path of the analytic continuation, we can identify the Chern character in the connection matrix. Recently, precise analysis of the so-called hemisphere partition functions of GLSMs [HR] have been developed. The analysis provides a concrete recipe to connect the cycles to the objects in derived category (of matrix factorizations), and also reproduces the connection matrix of the analytic continuation [EHKR]. We expect that the new method provides us new insights into more details of the above problem. Also, the significant progresses made in refs [Ha, BDFIK, DS] in the mathematical aspects of GLSMs are expected to provide us powerful tools to look into the derived categories of Fourier-Mukai partners and also their mirror symmetry.

4. Fourier-Mukai partners of Calabi-Yau threefolds II

Here we continue our exposition by the second example which was found recently by the present authors [HoTa1,2,3,4].

4.1. Reye congruences Calabi-Yau 3-folds and double coverings. In [HoTa1], we have found that Rødland’s construction of a pair of Calabi-Yau 3-folds has a natural counterpart in the projective space of symmetric matrices $\mathbb{P}(S^2C^5)$. Hereafter, we will fix $V = C^5$ and denote by $V_k$ a $k$-dimensional subspace of $V$.

We have found in [ibid] that the tower of secant varieties of $v_2(\mathbb{P}(V))$ in $\mathbb{P}(S^2V)$ and the corresponding (reversed) tower in $\mathbb{P}(S^2V^*)$ entail a similar duality of Calabi-Yau 3-folds. For the construction, we start with $S^2\mathbb{P}(V)$, i.e., the symmetric product of $\mathbb{P}(V)$ as the counterpart of the Grassmannian $G(2, 7) \subset \mathbb{P}(\wedge^2C^7)$. $S^2\mathbb{P}(V)$ is the first secant variety of $v_2(\mathbb{P}(V))$ and can be considered as the rank 2 locus of symmetric matrices $[c_{ij}] \in \mathbb{P}(S^2V)$. It is singular along the $v_2(\mathbb{P}(V))$, i.e., the rank 1 locus. The precise definition of the Pfaffian counterpart will be introduced in the next section, but here we only describe the resulting Calabi-Yau 3-fold starting with the rank 4 locus in the dual projective space $\mathbb{P}(S^2V^*)$,

$$\mathcal{H} := \{[a_{ij}] \in \mathbb{P}(S^2V^*) \mid \det(a_{ij}) = 0\}.$$
$\mathcal{H}$ is singular along the locus $\mathcal{H}_3$ with $\mathcal{H}_k := \{ \text{rk}(a_{ij}) \leq k \}$. As before, we consider a general five dimensional linear subspace $L_5 \subset S^2 V^*$ and its orthogonal linear subspace $L_5^\perp \subset S^2 V$. Then we define

$$X = S^2 \mathbb{P}(V) \cap \mathbb{P}(L_5^\perp) \subset \mathbb{P}(S^2 V), \quad H = \mathcal{H} \cap \mathbb{P}(L_5) \subset \mathbb{P}(S^2 V^*) .$$

**Proposition 4.1** (Hosono-Takagi [HoTa1]). (1) When $L_5$ is general, $X$ is a smooth Calabi-Yau 3-fold with $\text{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ and the following invariants:

$$H^3_X = 35, c_2(H_X) = 50, h^{1,1}(X) = 1, h^{2,1}(X) = 51,$$

where $H_X$ is the generator of the free part of $\text{Pic}(X)$.

(2) When $L_5$ is general, $H$ is a determinantal quintic hypersurface in $\mathbb{P}(L_5) \simeq \mathbb{P}^5$, which is singular along a smooth curve $C_H$ of genus 26 and degree 20 with $A_1$ type singularities.

(3) There is a double covering $Y \to X$ branched along $C_H$. Furthermore, $Y$ is a smooth Calabi-Yau 3-fold with $\text{Pic}(Y) = \mathbb{Z}H_Y$ and

$$H^3_Y = 10, c_2(Y) = 40, h^{1,1}(Y) = 1, h^{2,1}(Y) = 51.$$

If we do parallel constructions with $V = \mathbb{C}^4$, we obtain an Enriques surface for $X$. From historical reasons, this Enriques surface $X$ is called Reye congruence, or more precisely, Cayley model of Reye congruence (see [Co]). In our case of $V = \mathbb{C}^5$, Reye congruence $X$ is a Calabi-Yau 3-fold and is paired with another Calabi-Yau 3-fold $Y$ as above. It is easy to see that $Y$ is not birational to $X$ by the same arguments as described below Proposition 3.1. In addition to this, we can show the derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, which will be sketched in the next subsection. Here it should be worth while noting the following interesting properties of $X$ and $Y$ (HoTa3 Prop. 3.5.3, 4.3.4, HoTa4 Prop. 3.2.1):

**Proposition 4.2.** (1) $\pi_1(X) \simeq \mathbb{Z}_2$. (2) $\pi_1(Y) \simeq 0$ and the Brauer group of $Y$ contains a non-trivial 2-torsion element.

As argued in [ibid., Sect.9.2], one can show an exact sequence,

$$0 \to \mathbb{Z}_2 \to \text{Br}(Y)_{\text{tors}} \to \text{Br}(X) \to 0.$$

If $\text{Br}(Y) \simeq \mathbb{Z}_2$, then $\text{Br}(X) \simeq 0$ and this indicates the invariance of the product of (abelianization of) $\pi_1$ and the Brauer group, but not each factor, under the derived equivalence (see [Ad S] for details).

### 4.2. Derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

Here we sketch our proof of the derived equivalence. As we saw in the preceding subsection, our construction of the pair $(X, Y)$ is parallel to Rødland’s construction of Grassmannian-Pfaffian Calabi-Yau manifolds. We can pursue this parallelism toward the proof of the derived equivalence, although the projective geometries become more involved, and we have only partial results about the HPD (corresponding to Theorem 3.4) in our case.

#### 4.2.1. Resolutions.

Let $\mathcal{X} := S^2 \mathbb{P}(V)$. $X$ is defined by a linear section of $\mathcal{X}$ as $X = \mathcal{X} \cap \mathbb{P}(L_5^\perp)$. We see that $\mathcal{X}$ plays a similar role of $G(2, 7)$ in Rødland’s example, however there is a difference in that $\mathcal{X}$ is singular along the Veronese embedding of $\mathbb{P}(V)$, $v_2(\mathbb{P}(V)) \subset \mathcal{X} \subset \mathbb{P}(S^2 V)$. For this singularity, we have the...
The morphism $Z$ (4.1) from $Hilb$ where $Hilb$ is the Hilbert scheme of two points on $P(V)$ and $f$ is the Hilbert-Chow morphism. The morphism $g$ sends points $x \in X$ to the points $g(x) \in G(2, V)$ representing the lines determined by $x$. The fiber over $[V_2] \in G(2, V)$ is $g^{-1}([V_2]) \simeq S^2 \mathbb{P}(V_2) \simeq \mathbb{P}^2$. By our genericity assumption of $L_5$, $X = X \cap \mathbb{P}(L_5^\perp)$ is smooth (see Proposition 4.1) and hence $\mathbb{P}(L_5^\perp)$ is away from the singularity of $X$, therefore we may consider our linear intersection in $X$; i.e., $X = X \cap \mathbb{P}(L_5^\perp)$. Again, by the same reasoning, we have $g(X) \simeq X$, i.e., we have isomorphic image $g(X)$ of $X$ in $G(2, V)$. Historically, the image $g(X) \subset G(2, V)$ is called a Reye congruence.

$\mathcal{H}$ is singular along the rank $\leq 3$ locus $\mathcal{H}_3$. Expecting a (partial) resolution of the singularity, we consider the following (Springer-type) pairing of singular quadrics and planes therein (cf. (3.1)):

$$Z := \{(Q, [II]) \mid \mathbb{P}(II) \subset Q\} \subset \mathcal{H} \times G(3, V),$$

where $[Q] \in \mathcal{H}$ represents the point corresponding to a singular quadric $Q$. It is easy to see that all the fibers of the projection $Z \to G(3, V)$ are isomorphic to $\mathbb{P}^8$ since they consist of quadrics that contain a fixed plane $\mathbb{P}(II) \subset \mathbb{P}(V)$. Hence, we see that $Z$ is smooth. However we have $\dim Z = 6 + 8 = 14$, while $\dim \mathcal{H} = \dim \mathbb{P}(S^2 V) - 1 = 13$, and hence $Z \to \mathcal{H}$ can not be a resolution of $\mathcal{H}$ that we expect. To remedy the situation, we consider the Stein factorization $Y$ of the morphism $Z \to \mathcal{H}$ as follows:

$$Z \xrightarrow{\pi_Z} \mathbb{P}(\mathbb{P}^8, \text{bundle}) \xrightarrow{\pi} G(3, V)$$

$$\mathcal{H} \subset \mathbb{P}(S^2 V^*),$$

where $\pi_Z : Z \to Y$ has connected fibers and $\rho_Y$ is a finite morphism by definition. From the above dimension counting, the connected fibers generically have dimension $\dim Z - \dim \mathcal{H} = 1$. As for the finite morphism $\rho_Y$, looking into the families of planes in a singular quadric, it is easy to see that $\rho_Y$ is generically $2 : 1$ and has its ramification along the singular locus $\text{Sing}(\mathcal{H}) = \mathcal{H}_3$. This corresponds to the covering we observed in (3) of Proposition 4.1. In fact, about the singular locus of $Y$, we can see $\text{Sing}(Y) = \mathcal{H}_3$ [HOT3, Prop.5.7.2] where we identify the inverse image $\rho_Y^{-1}(\mathcal{H}_3)$ in $Y$ with $\mathcal{H}_3$. Hence the covering $Y$ changes the singular locus of $\mathcal{H}$ to a smaller one. If the linear subspace $L_5$ is general, then since $\mathbb{P}(L_5) \cap \mathcal{H}_2 = \emptyset$, the singularities in the linear section $H = \mathcal{H} \cap \mathbb{P}(L_5)$ is removed by $\rho_Y$. This is exactly the smooth double covering $Y$ in (3) of Proposition 4.1. We write the double cover of $H$ simply by $Y = Y \cap \mathbb{P}(L_5)$ with understanding the pullback of $\mathbb{P}(L_5)$ to $Y$. A natural resolution $Z \to Y$ follows by studying geometries of singular quadrics $\mathcal{H}$ [HOT3], which is interesting by itself from the projective geometry of
quadrics \[ Y \]. Birational geometry of \( Y \) and \( \widetilde{Y} \) will be described in Section 3 by introducing other birational models of \( Y \).

It would be helpful now to write our \( X \) and \( Y \) in terms of the resolutions \( \widetilde{X} \) and \( \widetilde{Y} \) as

\[
X = \widetilde{X} \cap \mathbb{P}(L_5^1), \quad Y = \widetilde{Y} \cap \mathbb{P}(L_5).
\]

The derived equivalence follows from certain ideal sheaf on \( \widetilde{Y} \times \widetilde{X} \) constructed in a parallel way to the Grassmannian-Pfaffian Calabi-Yau 3-folds. The following proposition is a part of the birational geometry of \( Y \) (see Fig. 5.2):

**Proposition 4.3.** (1) There exists a resolution \( \rho_{\widetilde{Y}} : \widetilde{Y} \rightarrow Y \).

(2) There exists a blow-up \( \mathcal{Y}_2 \rightarrow \widetilde{Y} \), and over \( \mathcal{Y}_2 \) there is a generically conic bundle \( \pi_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \mathcal{Y}_2 \) that admits a morphism \( \mu_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow G(3, V) \).

We summarize the resolutions and morphisms as follows (cf. (4.3)):

\[
\begin{array}{cccc}
\mathcal{Y} & \xrightarrow{\rho_{\mathcal{Y}}} & \widetilde{Y} & \xrightarrow{\rho_{\mathcal{Y}}} & \mathcal{Y}_2 \\
\pi_{\mathcal{Y}_2} & \quad & \mu_{\mathcal{Y}_2} & \quad & g \\
\text{G}(3, V) & \quad & \text{G}(2, V) & \quad & \text{G}(3, V) \\
\end{array}
\]

4.2.2. **Incidence relation** \( \Delta_0 \). In the diagram (4.2), we introduce the following incidence relation \( \Delta_0 \):

\[
\Delta_0 = \{(V_3, [V_2]) \mid V_3 \cap V_2 \subset G(3, V) \times G(2, V)\}
\]

and consider its ideal sheaf \( \mathcal{I}_{\Delta_0} \). Pulling this back to \( \mathcal{Y}_2 \times \widetilde{X} \), we obtain \( \mathcal{I}_{\Delta_2} = (\mu_{\mathcal{Y}_2} \times g)^* \mathcal{I}_{\Delta_0} \). Since the variety \( \Delta_0 \) is nothing but the flag variety \( F(2, 3, V) \), we have locally free resolution,

\[
0 \rightarrow \wedge^4 (W^* \boxplus \mathcal{F}) \rightarrow \wedge^3 (W^* \boxplus \mathcal{F}) \rightarrow \wedge^2 (W^* \boxplus \mathcal{F}) \rightarrow W^* \boxplus \mathcal{F} \rightarrow \mathcal{I}_{\Delta_2} \rightarrow 0,
\]

where

\[
0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_{G(3, V)} \rightarrow W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{F} \rightarrow V \otimes \mathcal{O}_{G(2, V)} \rightarrow \mathcal{J} \rightarrow 0
\]

are the universal sequences on the Grassmannians \( G(3, V) \) and \( G(2, V) \) (\( \text{rk} \mathcal{U} = 3, \text{rk} \mathcal{F} = 2 \)), respectively. Roughly speaking, the direct image \((\bar{\rho}_{\mathcal{Y}_2} \times \text{id})_* \circ (\pi_{\mathcal{Y}_2} \times \text{id})_* \mathcal{I}_{\Delta_2}\) is the ideal sheaf \( \mathcal{I} \) on \( \mathcal{Y} \times \widetilde{X} \) which corresponds to the one used in the Grassmannian-Pfaffian case in [BC] and [Ko2]. In actual calculation of the direct image, however, we need to use the structure of the conic bundle. Hence we first restrict the generically conic bundle to a conic bundle \( \pi_{\mathcal{Y}_2}^0 : \mathcal{Y}_2^0 \rightarrow \mathcal{Y}^0 := \mathcal{Y}_2 \setminus \mathcal{P}_\sigma \), where \( \mathcal{P}_\sigma \) is a certain subvariety of dimension 7, and define \( \mathcal{I}^0 := (\bar{\rho}_{\mathcal{Y}_2} \times \text{id})_* \circ (\pi_{\mathcal{Y}_2}^0 \times \text{id})_* \mathcal{I}_{\Delta_2} \) with the corresponding restriction \( \bar{\rho}_{\mathcal{Y}_2}^0 : \mathcal{Y}^0 \rightarrow \widetilde{Y} \). Then \( \mathcal{I} = \iota_* \mathcal{I}^0 \) under the inclusion \( \iota : \mathcal{Y}^0 \hookrightarrow \widetilde{Y} \) is the precise definition of the ideal sheaf \( \mathcal{I} \).

4.2.3. **Derived equivalence.** The proof of derived equivalence in [HoTa1] proceeds by constructing the Fourier-Mukai functor with the kernel \( \mathcal{I} = \mathcal{I}_{\mathcal{Y} \times X} \) as in Subsection 3.2. In the paper [ibid], we have obtained a locally free resolution of the ideal sheaf \( \mathcal{I} \) starting with (4.3). To describe the results, we introduce locally free sheaves on \( \widetilde{Y} \).
Proposition 4.4. There exists locally free sheaves $\tilde{S}_L, \tilde{T}, \tilde{Q}$ on $\tilde{Y}$ which satisfy

$$
\pi_{Y*}\{\mu_2^*\mathcal{O}_{G(3,V)}(1)\} \cong \tilde{\rho}_Y^*\tilde{S}_L, \quad \pi_{Y*}\{\mu_2^*\mathcal{W}\} \cong \tilde{\rho}_Y^*\tilde{T},
$$

$$
\pi_{Y*}\{(\mu_2^*\mathcal{S}_2\mathcal{W}) \otimes \mu_2^*\mathcal{O}_{G(3,V)}(-1)\} \cong \tilde{\rho}_Y^*(\tilde{Q} \otimes \mathcal{O}_{\tilde{Y}}(-M_{\tilde{Y}})),
$$

where $M_{\tilde{Y}}$ is the divisor corresponding to $\rho_{\tilde{Y}}^* \circ \rho_{\tilde{Y}}^* \mathcal{O}_{\tilde{Y}}(1)$.

Proof. See [HoTa4] Prop.5.6.4 and [HoTa3] Prop.6.1.2,6.2.3.

We denote by $L_{\tilde{Y}}$ (resp. $H_{\tilde{Y}}$) the divisor on $\tilde{Y}$ corresponding to $g^*\mathcal{O}_{G(2,V)}(1)$ (resp. $g^*\mathcal{O}_{\tilde{X}}(1)$). Then, we have

Theorem 4.5 ([HoTa4] Theorem 5.1.3). We have the following locally free resolution:

$$
0 \to \tilde{S}_L \otimes \mathcal{O}_{\tilde{Y}} \to \tilde{T}^* \otimes g^*\mathcal{F}^* \to (\mathcal{O}_{\tilde{Y}} \otimes g^*\mathcal{S}_2^*\mathcal{F}^*) \oplus (\tilde{Q}^*(M_{\tilde{Y}}) \otimes \mathcal{O}_{\tilde{Y}}(L_{\tilde{Y}})) \to \mathcal{I} \otimes (\mathcal{O}_{\tilde{Y}}(M_{\tilde{Y}}) \otimes \mathcal{O}_{\tilde{Y}}(2L_{\tilde{Y}})) \to 0.
$$

Extracting each term of the above resolution of $\mathcal{I}$, we define the following notation:

$$(E_3, E_2, E_{1a}, E_{1b}) = (\tilde{S}_L, \tilde{T}, \mathcal{O}_{\tilde{Y}}, \tilde{Q}^*(M_{\tilde{Y}})),
$$

$$(F_3, F_2, F_{1a}, F_{1b}) = (\mathcal{O}_{\tilde{Y}}, g^*\mathcal{F}^*, g^*\mathcal{S}_2^*\mathcal{F}^*, \mathcal{O}_{\tilde{Y}}(L_{\tilde{Y}})),
$$

and set $F_{1a} = F_{1a}/\mathcal{O}_{\tilde{Y}}(-H_{\tilde{Y}} + 2L_{\tilde{Y}})$. Now corresponding to (3.5) in Subsection 3.2 we define the following full-subcategories

$$(E_{3}, E_{2}, E_{1a}, E_{1b}) = A_0 = A_1 = \cdots = A_9 \subset D^b(\tilde{Y}),$$

$$(F_{10}, F_{11}, F_{12}, F_{13}) = B_0 = B_1 = \cdots = B_4 \subset D^b(\tilde{Y}).$$

Theorem 4.6 ([HoTa4] Theorem 3.4.5, 8.1.1]). Denote by $A_i(a), B_i(b)$ the twists of $A_i, B_i$ by $aM_{\tilde{Y}}$ and $bH_{\tilde{Y}}$, respectively. Then

(i) $(A_0, A_1(1), \cdots, A_9(9))$ is a Lefschetz collection in $D^b(\tilde{Y})$, and

(ii) $(B_1(-4), \cdots, B_1(-1), B_0)$ is a dual Lefschetz collection in $D^b(\tilde{Y})$.

In particular the following vanishings hold:

$$\text{Hom}_{D^b(\tilde{Y})}^*(A_i(i), A_j(j)) = 0 \quad (i > j).$$

$$\text{Hom}_{D^b(\tilde{Y})}^*(B_i(-i), B_j(-j)) = 0 \quad (i < j).$$

Although it is implicit in the above theorem, the (dual) Lefschetz collections (i) and (ii) above indicate that there exist some non-commutative resolutions of $\tilde{Y}$ and $\tilde{X}$, respectively, and furthermore, they are expected to be HPD with each other. This should be contrasted to Theorem 3.3 where non-commutative resolution has appeared only for the Pfaffian variety $\mathcal{Y}$. Of course, this difference is due to the fact that both $\tilde{Y}$ and $\tilde{X}$ are singular varieties in our case. See [Ku3] for a recent survey about known examples of HPDs.

As in Subsection 3.2 the derived equivalence follows from the flatness of the ideal sheaf $I = \mathcal{I}|_{Y \times X}$ over $X$ and the vanishing properties in Theorem 3.3.

Theorem 4.7 ([HoTa4] Theorem 8.0.3]). The restriction $I = \mathcal{I}|_{Y \times X}$ defines a scheme $\mathcal{C}$ flat over $X$, and an equivalence $\Phi_I : D^b(Y) \to D^b(X)$ with $\Phi_I(-) = R\pi_{X*}(\pi_Y^*(-) \otimes I)$.

The proof given in [HoTa4] Sect.8 proceeds in a similar way to [BC] and only uses the vanishing properties in Theorem 1.6.
4.3. BPS numbers. The ideal sheaf $I$ describes a family of curves on $Y$ parametrized by $x \in X$. In particular, in [HoTa4], an interesting relation of them to some BPS number of $Y$ has been observed. Here we start with the following proposition:

**Proposition 4.48 ([HoTa4 Sect.3, Prop.7.2.2]).** The ideal sheaf $I = \mathcal{I}_Y \times_X \mathcal{I}_X$ defines a flat family $\{C_x\}_{x \in X}$ whose general members are smooth curves of genus 3 and degree 5 in $Y$.

The curve $C_x$ appears from the incidence relation $\Delta_0$ in $G(3, V) \times G(2, V)$. Recall $X = \tilde{X} \cap \mathbb{P}(L_5^5)$ and the morphism $g : \tilde{X} \to G(2, V)$. Then $g(x) (x \in X)$ determines a line $l_x = \mathbb{P}(V_{2,x})$. Then we have

$$\Delta_0|_{G(3,V) \times \{g(x)\}} = \{[\Pi] \in G(3, V) \mid l_x \subset \mathbb{P}(\Pi)\}.$$ 

Now let us recall the definition of $\mathcal{V}$ in (4.1) and $Y = \mathcal{V} \cap \mathbb{P}(L_5)$. We define

$$\mathcal{Z}_x := \{([Q], [\Pi]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q, [Q] \in \mathbb{P}(L_5)\},$$

and

$$\gamma_x := \mathcal{Z}_x \cap \pi^{\mathcal{V}}(Y) = \{([Q], [\Pi]) \mid l_x \subset \mathbb{P}(\Pi) \subset Q, [Q] \in \mathbb{P}(L_5)\}.$$ 

When $Y$ is smooth, then $Y = \tilde{Y} \cap \mathbb{P}(L_5) = \mathcal{V} \cap \mathbb{P}(L_5)$, i.e., $\rho^{-1}_\mathcal{V}(\mathbb{P}(L_5))$ is away from the singular locus $\text{Sing}(\mathcal{V})$. On the other hand, over $\mathcal{V} \setminus \text{Sing}(\mathcal{V})$ the Stein factorization $\mathcal{Z} \to \mathcal{V}$ has the structure of a conic bundle which is isomorphic to the generically conic bundle $\mathcal{Z}_2 \to \mathcal{V}_2$ over $\mathcal{V}_2 \setminus (\hat{\rho}_2 \circ \rho_\mathcal{V})^{-1}(\text{Sing}(\mathcal{V}))$ (see [ibid.Sect.2.3] and also the next section). Therefore we have $C_x = \pi_\mathcal{V}(\gamma_x)$ for the family of curves on $Y$. We can further study the following properties:

**Proposition 4.49.** (1) $\tilde{\gamma}_x = \rho_\mathcal{V} \circ \pi_\mathcal{Z}(\gamma_x) = \rho_\mathcal{V}(C_x)$ is a plane quintic curve in $H = \mathcal{H} \cap \mathbb{P}(L_5)$ with 3 nodes and arithmetic genus 6 for general $x \in X$.

(2) When $x \in X$ is general, $\tilde{\gamma}_x$ is away from the branch locus $C_H \subset H$ and $C_x \to \tilde{\gamma}_x$ is the normalization map.

(3) For general $x \in X$, there exists a 'shadow' curve $C'_x$ of genus 3 and degree 5 with the properties $\rho^{-1}_\mathcal{V}(\tilde{\gamma}_x) = C_x \cup C'_x$ and $C_x \cap C'_x = \rho^{-1}_\mathcal{V}(3\text{ nodes of } \tilde{\gamma}_x)$.

We refer to [ibid Sect. 3, Fig.1] for details, but only remark that the plane curve $\tilde{\gamma}_x$ can be written explicitly by $\tilde{\gamma}_x = \{[Q] \in H \mid l_x \subset Q\}$. Considering the condition $l_x \subset Q$ under $x \in \tilde{X} \cap \mathbb{P}(L_5^5)$, we see easily that $\tilde{\gamma}_x$ is a plane curve $H \cap P_x$ with

$$P_x = \{[a_{ij}] \in \mathbb{P}(L_5^5) \mid t^4_za_{ij}z = t^4wAw = 0 ([v], [w] \in l_x)\} \simeq \mathbb{P}^2,$$

where $A = ([a_{ij}])$ is the symmetric matrix corresponding to a point $[a_{ij}]$. Note that $x \in \tilde{X} \cap \mathbb{P}(L_5^5)$ implies $t^4_za_{ii}w = 0$, which is one of the three conditions for $l_x \subset Q$. We depict the claims in Proposition 4.49 in Fig. 4.1.

As claimed in Proposition 4.49, there are two (distinct) families of curves $\{C_x\}_{x \in X}$ and $\{C'_x\}_{x \in X}$ in $Y$ parametrized $X$. These two are smooth curves of genus 3 and degree 5 for general $x \in X$, and interestingly, can be identified in the BPS numbers calculated in [HoTa1]. The relevant part of the table of BPS numbers reads as follows:

\[
\begin{array}{c|ccccc}
  g & 0 & 1 & 2 & 3 & 4 & 5 \\
  n_Y^3(d) & 12279982850 & 571891188 & 3421300 & 100 & 0 & 0 \\
\end{array}
\]

with $d = 5$. As discovered in [ibid], we can exactly identify the two families in the BPS number $n_Y^3(5) = 100$ as

$$n_Y^3(5) = (-1)^\dim X c(X) \times 2 = -(-50) \times 2$$
Fig. 4.1. Shadow curve $C'_x$. Two intersecting curves $C_x$ and $C'_x$ in $Y$ cover the plane quintic curve $\gamma_x$ in $H$. $C_H$ is the curve of the branch locus.

following the counting “rule” described in Subsection 3.3. This indicates that the BPS numbers, which are preferred in physics interpretations [GV] to other mathematical invariants such as Donaldson-Thomas invariants, has a nice moduli interpretation in some cases although their mathematical definition (as invariants of manifolds) is difficult in general [HST].

4.4. Mirror symmetry. In Subsection 3.4, we have only described the monodromy properties of Picard-Fuchs differential equation for the mirror family of Rødland’s Pfaffian Calabi-Yau 3-fold. This is partially because the geometry of the mirror family is rather involved. Our second example of FM partners $\{X, Y\}$ of $\rho = 1$ has a nice feature from this perspective. We have a rather simple description for the mirror family of Reye congruence Calabi-Yau 3-folds $X$ in terms of special form of determinantal quintic hypersurfaces in $\mathbb{P}^4$.

Recall the definition $X = S^2\mathbb{P}(V) \cap \mathbb{P}(L_5^5) \subset \mathbb{P}(S^2V)$. Using the fact $S^2\mathbb{P}(V) = \mathbb{P}(V) \times \mathbb{P}(V)/\mathbb{Z}_2$, it is easy to see the isomorphism $X \simeq \tilde{X}/\mathbb{Z}_2$ with

$$
(4.5) \quad \tilde{X} = \left( \mathbb{P}^4 \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right. \right)_{2,52},
$$

where the superscripts 2, 52 represent the Hodge numbers $h^{1,1}$ and $h^{2,1}$, respectively. The r.h.s of (4.5) is a common notation in physics literatures to represent complete intersections of five (generic) $(1,1)$-divisors in $\mathbb{P}^4 \times \mathbb{P}^4$. In our case, we should read this as the complete intersection of five generic and symmetric $(1,1)$-divisors which correspond to five linear forms in $\mathbb{P}(S^2V)$ determined by $L_5 \subset S^2V^*$. Note that when $L_5$ is taken in general position, $X$ is smooth which means that the $\mathbb{Z}_2$ action on $X$ is free.

For concreteness, let us take a basis of $L_5$ by $A_k = (a_{ij}^{(k)})$ ($k = 1, ..., 5$). Then the defining equations of $\tilde{X}$ are given by $f_1 = f_2 = ... = f_5 = 0$ with $f_k = \sum_{i,j} z_i a_{ij}^{(k)} w_j$ and $([z],[w]) \in \mathbb{P}^4 \times \mathbb{P}^4$. If we introduce a notation $A(z) = \left( \sum_i z_i a_{ij}^{(k)} \right)_{1 \leq k,j \leq 5}$ for the $5 \times 5$ matrix defined by $A_k$, then we have

$$
\tilde{X} = \{ ([z],[w]) \in \mathbb{P}^4 \times \mathbb{P}^4 \mid A(z)w = 0 \}.
$$
There exists a crepant resolution $\tilde{X}$ and Proposition 4.10

It is easy to deduce that the projection of $\tilde{X}$ to the first factor of $\mathbb{P}^4 \times \mathbb{P}^4$ is a determinantal quintic hypersurface,

$$Z = \{ [z] \in \mathbb{P}^4 \mid \det A(z) = 0 \}.$$  

**Proposition 4.10 ([HtT2]).** (1) When the linear subspace $L_5 \subset S^2V^*$ is general, the quintic hypersurface $Z$ is singular at 50 ordinary double points (ODPs) where $\text{rk} A(z) = 3$. (2) The morphism $\pi_1 : \tilde{X} \to Z$ is a small resolution of the 50 ODPs.

Details can be found in [ibid, Prop.3.3]. Here we summarize properties of $X, \tilde{X}$ and $Z$ in the left of the following diagrams:

![Diagram](4.6)

For the construction of mirror family of $X$, we invoke the orbifold mirror construction, which schematically described in the right diagram of (4.6). Namely, we start with a certain special form $A_{sp}(z)$ of $A(z)$ (or the linear subspace $L_5$) to define $Z_{sp} = \{ \det A_{sp}(z) = 0 \}$. $Z_{sp}$ is singular in general, and so is $\tilde{X}_{sp} := \{ A_{sp}(z) w = 0 \} \subset \mathbb{P}^4 \times \mathbb{P}^4$. Finding a suitable crepant resolution $\tilde{X}^* \to \tilde{X}_{sp}$, which is compatible with the $\mathbb{Z}_2$ action of exchanging the two factors of $\mathbb{P}^4 \times \mathbb{P}^4$, we obtain a mirror family of $X$ by the quotient $X^* = \tilde{X}^*/\mathbb{Z}_2$. In the final process, we usually need to find a suitable finite group $G_{orb}$ (called orbifold group) to arrive at the desired properties $h^{1,1}(X) = h^{2,1}(X^*)$ and $h^{2,1}(X) = h^{1,1}(X^*)$, however interestingly it turns out that $G_{orb} = \{ \text{id} \}$ in our case.

The special form $A_{sp}(z)$ found in [HtT2] corresponds to a linear subspace $L_5 = \langle A_1, A_2, \ldots, A_5 \rangle$ with $A_1, A_2, \ldots, A_5$ in order given by

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Using these special form of $A_k$, we have $Z_{sp}(a) := \{ \det A_{sp}(z) = 0 \} \subset \mathbb{P}^4$ where

$$\text{det } A_{sp}(z) = \begin{vmatrix}
z_1 z_2 (z_1 + a z_2) & a z_1 & 0 & 0 & 0 \\
0 & z_2 & 0 & 0 & 0 \\
0 & 0 & z_1 & 0 & 0 \\
0 & 0 & 0 & z_1 & 0 \\
0 & 0 & 0 & 0 & z_1
\end{vmatrix} = a^5 (z_1 z_2 z_3 z_4 z_5 + (z_1 + a z_2) (z_2 + a z_3) (z_3 + a z_4) (z_4 + a z_5) (z_5 + a z_1)).$$

By coordinate change, it is easy to see that $Z_{sp}(a)_{a^5}$ defines a family of Calabi-Yau threefolds over $\mathbb{P}^1$ by $[-a^5, 1] \in \mathbb{P}^1$.

**Proposition 4.11.** (1) When $a^5$ is general $(a^5 \neq -\frac{1}{3}, 1 - 11a^5 + a^{10} \neq 0)$, $Z_{sp}(a)$ is singular along 5 lines of $A_2$ singularities and 10 lines of $A_1$ singularities.  

(2) $\tilde{X}_{sp}(a) := \{ ([z], [w]) \mid A_{sp}(z) w = 0 \}$ partially resolves the singularities in (1) to 20 lines of $A_1$ singularities.  

(3) There exists a crepant resolution $\tilde{X}^*(a) \to \tilde{X}_{sp}(a)$. And $\tilde{X}^*(a)$ for general $a^5$ is a smooth Calabi-Yau 3-fold with Hodge numbers $h^{1,1} = 52, h^{2,1} = 2$.

More details of the singularities and their resolutions can be found in [ibid]. For general $a^5$, we can see that $\tilde{X}^*(a)$ admits a free $\mathbb{Z}_2$ action, and hence $X^*(a) = \tilde{X}^*(a)/\mathbb{Z}_2$ is a Calabi-Yau 3-fold with Hodge numbers $h^{1,1} = 26, h^{2,1} = 1$. We have then a family $X^* := \{ X^*(a) \}_{[-a^5, 1] \in \mathbb{P}^1}$ of Calabi-Yau 3-folds over $\mathbb{P}^1$.  

Details can be found in [ibid, Prop.3.3]. Here we summarize properties of $X, \tilde{X}$ and $Z$ in the left of the following diagrams:

![Diagram](4.6)
Proposition 4.12 ([HoTa2 Prop.6.9]). $X^*$ is a mirror family of Reye congruence Calabi-Yau 3-fold $X$.

We omit the monodromy calculations which correspond to those in Subsection 3.4 since they are reported in [ibid, Prop.2.10].

Remark. (1) Set $x = -a^5$, then from the defining equation (4.7) we observe that both $x = 0$ and $x = \infty$ are MUM points. In [HoTa1], Gromov-Witten invariants ($g \leq 14$) of Reye congruence $X$ have been calculated from the MUM degeneration at $x = 0$ and the invariants of Fourier-Mukai partner $Y$ from $x = \infty$. We believe that our mirror family $X^*$ provides us a nice example to study the geometry of mirror symmetry [SYZ, GrS1, GrS2, RuS] when non-trivial Fourier-Mukai partners exist. It is interesting, although accidental, that in (4.7) we come across to the geometry of quintic from which the study of mirror symmetry started [Ge, GP, CdOGP].

(2) If we focus on the form of Picard-Fuchs differential operators in [AESZ, ES, DM], there are many other examples which exhibit two MUM points. Among them, a nice example has been identified in [Mi] with the mirror family of the Calabi-Yau 3-fold given by general linear sections of a Schubert cycle in the Cayley plane $E_6/P_1$. It is expected that this Calabi-Yau 3-fold has a non-trivial Fourier-Mukai partner [ibid] [Ga]. Also the mirror family of the Calabi-Yau 3-folds given by the intersection of two copies of Grassmannians $X = G(2, 5) \cap G(2, 5) \subset P^9$ [Kan, Kap] shows two MUM points whose interpretation seems slightly different from those we have seen in this article. The two MUM points seems to correspond Fourier-Mukai partners which are diffeomorphic but not bi-holomorphic. It would be interesting to investigate these new examples in more detail.

(3) In [Hor], the pair of Reye congruence Calabi-Yau 3-fold $X$ and its Fourier-Mukai partner $Y$ have been understood in the language of Gauged linear sigma modes along the arguments used for the Grassmannian-Pfaffian example. Extending these arguments, many other examples have been worked out in [HK] by calculating the so-called “two sphere partition” in physics [JKLMR].
5. Birational Geometry of the Double Symmetroid $\mathcal{Y}$

We describe the birational geometry of the double (quintic) symmetroid $\mathcal{Y}$ and its resolution $\widetilde{\mathcal{Y}}$. We will see intensive interplay of the projective geometry of quadrics and that of relevant Grassmannians. In this section, we fix $V = \mathbb{C}^5$ and retain all the notations introduced in the last section. This section is an exposition of the results whose details are contained in \cite{HoTa3, HoTa4}.

5.1. Generically conic bundle $\mathcal{Z} \to \mathcal{Y}$. We describe the (connected) fibers of $\mathcal{Z} \to \mathcal{Y}$ of the Stein factorization $\mathcal{Z} \to \mathcal{Y} \to \mathcal{H}$ in \eqref{eq:5.1}. Recall the definition $\mathcal{H} = \{ [a_{ij}] \in \mathbb{P}(\mathbb{S}^2V^*) \mid \det a = 0 \}$ and

\[
\mathcal{Z} := \{ ([Q], [\Pi]) \mid \mathbb{P}(\Pi) \subset Q \} \subset \mathcal{H} \times G(3, V),
\]

i.e., $\mathcal{Z}$ consists of pairs of singular quadric and (projective) plane therein. The notation $[Q] \in \mathcal{H}$ above indicates that we identify points $[a_{ij}] \in \mathcal{H}$ with the corresponding quadric $Q$ in $\mathbb{P}(V)$. Since $\dim \mathcal{Z} - \dim \mathcal{H} = 1$, we have generically one dimensional fibers for $\pi_Z : \mathcal{Z} \to \mathcal{Y}$. It is easy to deduce the fibers of $\pi_Z : \mathcal{Z} \to \mathcal{Y}$ from those of $\mathcal{Z} \to \mathcal{H}$:

The fibers of $\mathcal{Z} \to \mathcal{H}$ over a point $[Q]$ consists of planes contained in the quadric $Q$. In Fig. 5.1, depending on the rank of $[Q] = [a_{ij}]$, the corresponding quadric $Q$ is depicted schematically. Let us define reduced quadric $\overline{Q}$ to be the smooth quadric naturally defined in $\mathbb{P}(V/\text{Ker}(a_{ij}))$. Then, as is clear in Fig. 5.1, $\overline{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, a smooth conic, two points and one point depending on $\text{rk} Q = 4, 3, 2$ and $1$, respectively. Singular quadrics $Q$ are then described by the cones over the reduced quadric $\overline{Q}$ with the vertex $\text{Ker} Q := \mathbb{P}(\text{Ker}(a_{ij}))$. The fibers of $\pi_Z : \mathcal{Z} \to \mathcal{Y}$ over $y \in \mathcal{Y}$ are given by connected families of planes contained in the quadric $Q_y = \rho_y(y)$. We summarize the connected fibers:

(a) When $\text{rk} Q_y = 4$, the fiber is the $\mathbb{P}^1$-families of planes which corresponds to one of the two possible rulings of $\overline{Q}_y \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

(b) When $\text{rk} Q_y = 3$, the fiber is the $\mathbb{P}^1$-family of planes parametrized by the conic $\overline{Q}_y$.

(c) When $\text{rk} Q_y = 2$, the fiber is the planes parametrized by $(\mathbb{P}^3)^* \sqcup_{1:\text{pt}} (\mathbb{P}^3)^*$ where $(\mathbb{P}^3)^*$ parametrizes planes in $\mathbb{P}^3$ and $A \sqcup_{1:\text{pt}} B$ represents the union with $a \in A$ and $b \in B$ (one point from each) are identified.

(d) When $\text{rk} Q_y = 1$, the fiber is the planes parametrized by $(\mathbb{P}^3)^*$.

We remark that, in the case of (a), one of the two possible $\mathbb{P}^1$-families of planes is specified (by the definition of Stein factorization) when we take $y \in \mathcal{Y}$. This and the other cases explain the finite morphism $\rho_\mathcal{Y} : \mathcal{Y} \to \mathcal{H}$ which is $2 : 1$ over $\mathcal{H}_4 \setminus \mathcal{H}_3$ and branched over $\mathcal{H}_3$. We say that a point $y \in \mathcal{Y}$ has rank $i$ if rank $a_y = i$ for $\rho_\mathcal{Y}(y) = [a_y]$, and define $G_\mathcal{Y} := \{ y \in \mathcal{Y} \mid \text{rk} y \leq 2 \}$. Note that $\dim G_\mathcal{Y} = \dim \mathcal{H}_2 = 8$.

Proposition 5.1. (1) Sing $\mathcal{H} = \mathcal{H}_3$ and Sing $\mathcal{Y} = G_\mathcal{Y} (= \mathcal{H}_2)$.

(2) $\pi_Z : \mathcal{Z} \to \mathcal{Y}$ is a generically conic bundle with the conics in $G(3, V)$.

Proof. (1) Sing $\mathcal{H} = \mathcal{H}_3$ follows from the basic properties of secant varieties. For the latter claim Sing$\mathcal{Y} = G_\mathcal{Y}$, we refer to \cite{HoTa3 Prop.5.7.2}.
the plane $\tilde{P}$ which actually defines a conic in $\text{G}(3, V)$. Proposition 5.1. The planes $\tilde{P}$ specific forms, we define the following subset of planes in $\text{G}(3, V)$:

$$\left\{ \begin{array}{c}
\eta = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\end{array} \right\}. $$

$\mathcal{H}$

Fig.5.1. Quadrics and planes therein. Quadrics $Q$ are depicted for each rank, $\text{rk} Q = 4, 3, 2, 1$. When $\text{rk} Q = 4$, there are two connected fibers of $Z' \rightarrow H$.

(2) Over $\mathcal{V} \setminus \text{G}(\mathcal{V})$, the fibers of $\pi_{\mathcal{V}} : Z' \rightarrow \mathcal{V}$ consists of smooth $\mathbb{P}^1$-families of planes in $\text{G}(3, V)$. As we see in the next subsection, it is easy to see that these are smooth conics on $\text{G}(3, V)$.

5.2. Birational model $\overline{\mathcal{V}}$ of $\mathcal{V}$. Let us consider a quadric $Q$ of rank 4 and 3, in order, and a $\mathbb{P}^1$-family of planes in $Q$.

First for a quadric $Q$ of rank 4, let us denote the vertex of $Q$ (the kernel of $(a_{ij})$) by $(v)$. Then, one of the $\mathbb{P}^1$-family of plane described in (a) in Subsection 5.1 takes the following form:

$$\left\{ \begin{array}{c}
\{ [a_{s,t}] \} := \{ (c(s,t), d(s,t), v) \mid [s,t] \in \mathbb{P}^1 \},
\end{array} \right.$$ 

where $c(s,t), d(s,t) \in V$ are linear in $s, t$ and spans the $(c(s,t), d(s,t)) \simeq \mathbb{P}^1$ which gives the ruling $\tilde{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. One of the key observations is that for such a $\mathbb{P}^1$-family of plane we have a conic $q$ in $\mathbb{P}(\wedge^3 V)$ by

$$q := \left\{ [c \wedge d \wedge v] = [A_0 s^2 + A_1 s t + A_2 t^2] \mid [s,t] \in \mathbb{P}^1 \right\},$$

which actually defines a conic in $\text{G}(3, V)$ by the Plücker embedding $\text{G}(3, V) \subset \mathbb{P}(\wedge^3 V)$. We note that conic $q$ resides in the plane $\mathbb{P}_q$ which is uniquely determined by the $\mathbb{P}^1$-family,

$$\mathbb{P}_q := (A_0, A_1, A_2) \subset \mathbb{P}(\wedge^3 V).$$

When $\text{rk} Q = 3$, we start with $\{ [a_{s,t}] \} = \{ (d(s,t), v_1, v_2) \mid [s,t] \in \mathbb{P}^1 \}$ with $v_1, v_2$ being bases of Ker $(a_{ij})$ and $d(s,t) = s^2 v_3 + s t v_4 + t^2 v_5$ parametrizing the conic $\tilde{Q}$ in $\mathbb{P}(V/\text{Ker}(a_{ij}))$. Again, we have the corresponding conic $q$ in $\text{G}(3, V)$ and also the plane $\mathbb{P}_q \subset \mathbb{P}(\wedge^3 V)$ which contains the conic $q$.

The conics $q$ above explain the generically conic bundle $Z' \rightarrow \mathcal{V}$ claimed in Proposition 5.1. The planes $\mathbb{P}_q \subset \mathbb{P}(\wedge^3 V)$ and conics $q$ will play central roles in the description of the resolution $\mathcal{V} \rightarrow \mathcal{V}$. Here noting that the planes $\mathbb{P}_q$ above have a specific forms, we define the following subset of planes in $\mathbb{P}(\wedge^3 V)$:

$$\overline{\mathcal{V}} = \{ [U] \in \text{G}(3, \wedge^3 V) \mid U = \tilde{U} \wedge v \text{ for some } v \in \mathbb{P}(V) \},$$

where we regard $\tilde{U}$ as an element in $\mathbb{P}(\wedge^2 (V/V_1))$ with $V_1 = \mathbb{C}v$. To introduce a (reduced) scheme structure on the subset $\overline{\mathcal{V}}$, we consider a linear morphism
\[ \varphi : S^2(\wedge^3 V) \to V \]  
by the composition of the following natural linear morphisms:
\begin{equation}
\varphi : S^2(\wedge^3 V) \to S^2(\wedge^2 V^*) \to (\wedge^2 V^*) \wedge (\wedge^2 V^*) \to \wedge^4 V^* \simeq V.
\end{equation}

We define \( \varphi_U := \varphi|_{\overline{U}} \) to be the natural restriction of \( \varphi \) for a fixed subspace \( \{ U \} \in G(3, \wedge^3 V) \). Then, we have the following proposition:

**Proposition 5.2.**  (1) \( U \subseteq \wedge^3 V \) decomposes as \( U = \bar{U} \cap v \) if and only if \( \text{rk} \varphi_U \leq 1 \).

(2) The scheme \( \{ \{ U \} \in G(3, \wedge^3 V) \mid \text{rk} \varphi_U \leq 1 \} \) is nonreduced along the singular locus of its reduced structure.

The proof of the above proposition follows by writing the rank condition explicitly for the matrix representing \( \varphi_U \) under suitable bases (see [H61Ta3 Subsect.5.3, 5.4]). Hereafter, we consider \( \overline{\mathcal{Y}} \) as the scheme with the reduced structure on \( \{ \{ U \} \in G(3, \wedge^3 V) \mid \text{rk} \varphi_U \leq 1 \} \).

**Proposition 5.3.** \( \mathcal{Y} \) and \( \overline{\mathcal{Y}} \) are birational.

*Proof.* By definition of the Stein factorization, points \( y \in \mathcal{Y} \) are specified by the connected fibers of \( \mathcal{Y} \to \mathcal{Y} \), which are generically given by conics \( q \) in \( G(3, V) \). Hence we can write general points \( y \in \mathcal{Y} \) by \( y = ([Q_y], q_y) \) where \( [Q_y] = \rho(y) \) and the corresponding conic \( q_y \) which is a \( \mathbb{P}^1 \)-family of planes contained in \( Q_y \). Rational map \( \mathcal{Y} \to \overline{\mathcal{Y}} \) has been described already above by \( y = ([Q_y], q_y) \to \mathbb{P}_{q_y} \) for \( y \in \mathcal{Y} \setminus \mathcal{G}_{\mathcal{Y}} \). To describe the inverse rational map \( \overline{\mathcal{Y}} \to \mathcal{Y} \), we note that the following isomorphism for \( U = \bar{U} \cap v \in \wedge^3 V \):
\begin{equation}
\mathbb{P}(U) \cap G(3, V) \in \mathbb{P}(\wedge^3 V) \simeq \mathbb{P}(\bar{U}) \cap G(2, V/V_1) \text{ in } \mathbb{P}(\wedge^2(V/V_1)),
\end{equation}

where \( V_1 = Cv \). Since \( G(2, V/V_1) \simeq G(2, 4) \) is the Plücker quadric, when \( U \) is general, the r.h.s. determines a smooth conic on \( G(2, V/V_1) \) and in turn a smooth conic on \( G(3, V) \). We can see that this is the inverse rational map. \( \square \)

Obviously, the inverse rational map \( \overline{\mathcal{Y}} \to \mathcal{Y} \) is not defined when \( \mathbb{P}(U) \cap G(3, V) = \mathbb{P}(U) \), i.e. \( \mathbb{P}(U) \subset G(3, V) \). There are two cases where \( \mathbb{P}(U) \subset G(3, V) \) occurs for \( \{ U \} \in \overline{\mathcal{Y}} \): The first one is when \( \mathbb{P}(U) \) is given by the Plücker image of the plane
\[ \mathbb{P}_{V_2} := \{ \Pi \mid V_2 \subset \Pi \subset V \} \simeq \mathbb{P}^2 \]
in \( G(3, V) \) for some \( V_2 \). The second one is given by the Plücker image of the plane
\[ \mathbb{P}_{V_1V_4} := \{ \Pi \mid V_1 \subset \Pi \subset V_4 \} \simeq \mathbb{P}^2 \]
in \( G(3, V) \) for some \( V_1 \) and \( V_4 \). The plans of the form \( \mathbb{P}_{V_2} \) and \( \mathbb{P}_{V_1V_4} \), respectively, are called \( \rho \)-planes and \( \sigma \)-planes. These planes determine the following loci in \( \overline{\mathcal{Y}} \):
\begin{equation}
\mathbb{P}_\rho := \{ \{ U \} \mid \exists V_2 \text{ s.t. } U = V/V_2 \wedge (\wedge^2 V_2) \},
\end{equation}
\begin{equation}
\mathbb{P}_\sigma := \{ \{ U \} \mid \exists V_1, V_4 \text{ s.t. } V_1 \subset V_4, U = \wedge^2(V_4/V_1) \wedge V_1 \}.
\end{equation}

Note that \( \mathbb{P}_\rho \simeq G(2, V) \) and \( \mathbb{P}_\sigma \simeq F(1, 4, V) \).
5.3. Sing $\mathcal{F}$ and resolutions of $\mathcal{F}$. We consider the reduced structure on $\mathcal{F}$ as described in the preceding subsection. Then writing the condition $\text{rk} \varphi_U \leq 1$, we can study the singularities of $\mathcal{F}$ explicitly.

**Proposition 5.4.** (1) $\mathcal{F}$ is singular along $\mathcal{F}_p \simeq G(2,V)$.
(2) Define $\mathcal{Y}_3 := \{(U,[V]) \mid U = U \wedge V \} \subset \mathcal{F} \times \mathbb{P}(V)$, then the natural projection $\mathcal{Y}_3 \to \mathcal{F}$ is a resolution of the singularity.
(3) $\mathcal{Y}_3$ is isomorphic to the Grassmannian bundle $G(3,\wedge^2 T_{\mathbb{P}(V)}(-1))$ over $\mathbb{P}(V)$.
(4) The singularities of $\mathcal{F}$ are the affine cone over $\mathbb{P}^1 \times \mathbb{P}^5$ along $\mathcal{F}_p$, and there is an (anti-)flip to another resolution $\mathcal{F} \to \mathcal{F}$ which fits into the following diagram:

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{(\text{anti-})\text{flip}} & \mathcal{Y}_3 \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\rho_\mathcal{F}} & \mathcal{Y} \\
\end{array}
\]

**Proof.** (1) and (4) follows directly by writing the condition $\text{rk} \varphi_U \leq 1$, see [HoTa3 Prop.5.4.2, 5.4.3]. Global descriptions of the blow-up $\mathcal{Y}_2 \to \mathcal{Y}_3$ will be given in Proposition 5.4.10. (3) We consider $\tilde{U} \simeq \mathbb{C}^3$ as a subspace in $\wedge^2(V/V_1)$. Then claim is clear since $T_{\mathbb{P}(V)}(-1)|_{V_1} \simeq V/V_1$. (2) follows from (3).

We denote by $\mathcal{P}_p$ the exceptional set (which is contracted to $\mathcal{F}_p$) of the resolution $\mathcal{Y}_3 \to \mathcal{F}$ and by $\mathcal{P}_\sigma \simeq \mathcal{P}_p$ the proper transform of $\mathcal{F}_\sigma$. It is easy to observe the following isomorphisms:

\[
\mathcal{P}_p \simeq F(1,2,V) \simeq T_{\mathbb{P}(V)}(-1), \quad \mathcal{P}_\sigma \simeq F(1,4,V) \simeq T_{\mathbb{P}(V)}(-1)^*.
\]

These loci $\mathcal{P}_p$ and $\mathcal{P}_\sigma$ in $G(3,\wedge^3 T_{\mathbb{P}(V)}(-1))$ will be interpreted in the next section.

In the diagram (5.4), we have included the content of the following theorem:

**Theorem 5.5.** There is a morphism $\rho_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}$ which contracts an exceptional divisor $F_{\mathcal{Y}}$ to the singular locus $G_{\mathcal{Y}}$ of $\mathcal{Y}$.

The above theorem is one of the main results of [HoTa3]. We refer to [ibid, Subsect. 5.7, and Fig.2] for details. Also, for the proof of Theorem 1.6, we used a natural flattening of the fibers of $\bar{F}_\mathcal{Y} \to G_{\mathcal{Y}}$ constructed in [ibid,Section 7]. Below, we describe the construction of the morphism $\rho_\mathcal{Y}$ briefly.

5.4. The resolution $\rho_\mathcal{Y} : \mathcal{F} \to \mathcal{Y}$. We formulate a rational map $\varphi_{DS} : \mathcal{F} \dashrightarrow \mathcal{H}$ which extends to $\bar{\varphi}_{DS} : \mathcal{F} \to \mathcal{H}$. Then the Stein factorization of $\bar{\varphi}_{DS}$ gives the claimed morphism $\rho_\mathcal{Y} : \mathcal{Y} \to \mathcal{Y}$ [ibid,Prop.5.6.1].

The key relation for the construction is the following decomposition:

\[
\wedge^3 (\wedge^2(V/V_1)) \simeq S^2(V/V_1) \oplus S^2(V/V_1)^*,
\]
as irreducible $so(\wedge^2(V/V_1)) \simeq sl(V/V_1)$-modules. We called this double spin decomposition since the r.h.s. is $V_{2\lambda_s} \oplus V_{2\lambda_s}$ with the spinor and conjugate spinor weights $\lambda_s$ and $\lambda_s$, respectively. $G(3,\wedge^2(V/V_1))$ consists of 3-spaces in $\wedge^2(V/V_1)$. We have also $\text{OG}(3,\wedge^2(V/V_1))$ which consists of isotropic 3-spaces with respect to
the natural symmetric form $\wedge^2(V/V_1) \times \wedge^2(V/V_1) \to \wedge^4(V/V_1) \cong \mathbb{C}$. We denote by $\text{OG}^\pm(3,\wedge^2(V/V_1))$ the connected components of $\text{OG}(3,\wedge^2(V/V_1))$.

If we consider the above decomposition fiberwise for $\wedge^2 T_{P(V)}(-1)$, then we have the following embedding:

\[
(5.7) \quad i : \mathcal{F}_3 = G(3,\wedge^2 T_{P(V)}(-1)) \hookrightarrow \mathbb{P}(S^2 T_{P(V)}(-1) \oplus S^2 T_{P(V)}(-1)^\ast).
\]

**Proposition 5.6. The following properties hold for the loci $\mathcal{P}_\rho$ and $\mathcal{P}_\sigma$ in $\mathcal{F}_3$:**

1. $i(\mathcal{P}_\rho) = v_2(T_{P(V)}(-1))$, $i(\mathcal{P}_\sigma) = v_2(T(-1)^\ast)$.
2. $\mathcal{P}_\rho = \text{OG}^+(3,\wedge^2 T_{P(V)}(-1))$, $\mathcal{P}_\sigma = \text{OG}^-(3,\wedge^2 T_{P(V)}(-1)^\ast)$.

**Proof.** (1) The claimed relations follow from the isomorphisms (5.5) and the form of the embedding (5.7). We can also verify the claim explicitly by writing the decomposition (5.6) (see Appendix B). (2) The points $[V_1, V_2] \in F(1, 2, V) \simeq \mathcal{P}_\rho$ determine the corresponding points $([U], [V_1]) \in \mathcal{P}_\rho$ with $[U] = [(V/V_2 \wedge (V_2/V_1)] \in G(3,\wedge^2(V/V_1))$. Then we verify $U \wedge \tilde{U} = 0$. Similarly, points $([U], [V_1]) \in \mathcal{P}_\sigma$ have the forms $[U] = [\wedge^2(V_4/V_1)]$ for some $V_4$. Again, $\tilde{U} \wedge \tilde{U} = 0$ is clear. Claim follows since all maximally isotropic subspaces in $\wedge^2(V/V_1)$ take either of these two forms.

Now we consider the following sequence of (rational) morphisms:

\[
(5.8) \quad \mathcal{F}_3 \rightarrow \mathbb{P}(S^2 T(-1) \oplus S^2 T(-1)^\ast) \rightarrow \mathbb{P}(S^2 T(-1)^\ast) \rightarrow \mathbb{P}(S^2 V^\ast),
\]

where and hereafter we write $T(-1)$ for $T_{P(V)}(-1)$ to simplify formulas. In the middle, we consider the projection to the second factor. The injection in the right is defined by the dual of the canonical surjection $V \otimes O_{P(V)} \to T(-1) \to 0$. Since the image of the composition is in $\mathcal{H} \subset \mathbb{P}(S^2 V^\ast)$, we have a rational map, $\phi_{DS} : \mathcal{F}_3 \rightarrow \mathcal{H}$.

**Proposition 5.7.** (1) The rational map $\phi_{DS}$ defines a morphism $\phi_{DS} : \mathcal{F}_3 \setminus \mathcal{P}_\rho \simeq \overline{\mathcal{F}} \setminus \mathcal{P}_\rho \to \mathcal{H}$. In particular, it induces a rational map $\varphi_{DS} : \overline{\mathcal{F}} \dashrightarrow \mathcal{H}$ whose indeterminacy locus is $\mathcal{P}_\rho$.

2. $\phi_{DS}(\mathcal{P}_\sigma) = \varphi_{DS}(\mathcal{P}_\sigma) = \mathcal{H}_1$.

3. The rational map $\varphi_{DS} : \overline{\mathcal{F}} \dashrightarrow \mathcal{H}$ extends to a morphism $\hat{\varphi}_{DS} : \hat{\overline{\mathcal{F}}} \to \mathcal{H}$.

**Proof.** (1) and (2) follow from the claim (1) in Theorem 5.6 and the definition $\varphi_{DS}$ with the Plücker embedding (5.7). We can verify (3) explicitly by writing the rational map $\varphi_{DS}$ and extending it to the blow-up $\overline{\mathcal{F}} \to \mathcal{F}$ (see [HoTa3 Prop.5.5.3]).

**Theorem 5.8.** $\hat{\varphi}_{DS} : \hat{\overline{\mathcal{F}}} \to \mathcal{H}$ factors as $\hat{\overline{\mathcal{F}}} \to \hat{\mathcal{F}} \xrightarrow{\rho} \mathcal{H}$ with the morphism $\rho : \hat{\mathcal{F}} \to \mathcal{H}$ in (4.4). This defines the resolution $\rho_{\mathcal{H}} : \hat{\overline{\mathcal{F}}} \to \mathcal{F}$.

**Proof.** The claim basically follows from the Stein factorization. In [ibid, Section 5.6, Fig.2], the fibers of $\hat{\varphi}_{DS} : \hat{\overline{\mathcal{F}}} \to \mathcal{H}$ have been described completely, and the claim is clear from the results there.

**Remark.** We describe the inverse image of the rational map $\phi_{DS}$. Let us fix $[a] \in \mathcal{H}$. When we fix (a choice of) $V_1 \subset \text{Ker} a$, we have naturally “reduced matrix” $[a_{V_1}] \in \mathbb{P}(S(V/V_1)^\ast)$ (as the inverse image of $[a]$ under the embedding $\mathbb{P}(S^2 T(-1)^\ast) \hookrightarrow \mathbb{P}(S^2 V^\ast)$ in (5.8)). Consider the restriction $\phi_{V_1} := \phi_{DS}|_{\pi_3^{-1}([V_1])}$
to the fiber \(\pi_3^{-1}([V_1]) = G(3, \wedge^2(V/V_1))\) of \(\pi_3: \mathcal{Z}_3 \rightarrow \mathbb{P}(V)\), and similar restriction \(i_{V_1}: G(3, \wedge^2(V/V_1)) \hookrightarrow \mathbb{P}(S^2(V/V_1) \oplus S^2(V/V_1))\) of the Plücker embedding (5.7). Then, over the fiber \(\pi_3^{-1}([V_1])\), the rational map \(\phi_{DS}: \mathcal{Z}_1 -\rightarrow \mathcal{Y}\) is basically given by the projection \(\mathbb{P}(S^2(V/V_1) \oplus S^2(V/V_1)) \rightarrow \mathbb{P}(S^2(V/V_1))\) sending \([v_{ij}, w_{kl}]\) to \([v_{ij}]\). The ideal of the Plücker embedding in terms coordinate \([v_{ij}, w_{kl}]\) turns out to have rather nice form as shown in Appendix B. Using the results listed in Appendix [5.5.2] we can prove the following properties of the inverse image of \(\phi_{DS}\):

1. When \(\text{rk}\ a = 4\), \(V_1\) is unique and we have \(i_{V_1} \circ \phi_{V_1}^{-1}(a) = [\pm \sqrt{\det a_{V_1}} a_{V_1}^{-1}, a_{V_1}]\).
2. When \(\text{rk}\ a = 3\). For any \(V_1 \subset \text{Ker}\ a\), we have \(i_{V_1} \circ \phi_{V_1}^{-1}(a) = 0\).
3. When \(\text{rk}\ a = 2\). For each choice of \(V_1 \subset \text{Ker}\ a\), we have \(i_{V_1} \circ \phi_{V_1}^{-1}(a) \simeq \mathbb{P}^1 \times \mathbb{P}^1\).
4. When \(\text{rk}\ a = 1\). For each choice of \(V_1 \subset \text{Ker}\ a\), we have \(i_{V_1} \circ \phi_{V_1}^{-1}(a) \simeq \mathbb{P}(1^3, 2)\).

Let us denote by \(G_p\) the exceptional set of the resolution \(\tilde{\phi}_{DS}: \mathcal{Y} \rightarrow \mathcal{F}\). Then, since \(\mathcal{Y}_3 \backslash P_p \simeq \mathcal{F}_3 \backslash P_p \simeq \mathcal{F} \backslash G_p\), we can identify \(\phi_{DS}, \varphi_{DS}\) and \(\tilde{\phi}_{DS}\) with each other over these complement sets. Then the above results indicate that \(\tilde{\phi}_{DS}^{-1}(a) (\text{rk}\ a = 3)\) is contained in the exceptional set \(G_p\) (and this is indeed the case [ibid, Lemma 5.6.2]). Note also that from 3) and 4) and \(\dim G_{\varphi} = 8 (G_{\varphi} \simeq \mathcal{A}_2)\), we see that \(\tilde{\phi}_{DS}^{-1}(a)\) is a divisor in \(\mathcal{F}\), which is nothing but the divisor \(F_{\varphi}\) that appeared in Theorem [5.5.2]. Full details of 1)–4) can be found in [ibid, Section 5.6] (see also [ibid, Fig.2]).

5.5. Generically conic bundles. We describe the generically conic bundle \(\pi_{2'}: \mathcal{Z}_2 \rightarrow \mathcal{Y}_2\) which has appeared in [5.2]. The basic idea is the same as that we used in the proof of Proposition [5.2] i.e., to consider the intersection \(\mathbb{P}(U) \cap G(3, V) \simeq \mathbb{P}(U) \cap G(3, V/V_1)\) for \(U = U \wedge V_1\).

5.5.1. Generically conic bundle \(\mathcal{F} \rightarrow \mathcal{F}\). Let us fix the embedding \(G(3, V) \subset \mathbb{P}(\wedge^3 V)\). We recall the definition

\[
\mathcal{F} = \{[U] \in G(3, \wedge^3 V) \mid U = \tilde{U} \wedge V_1 \text{ for some } V_1 \subset V\}.
\]

Then from the isomorphism \([5.2]\), we have generically conic bundle by

\[
\mathcal{F} : = \{([c], [U]) \mid [c] \in \mathbb{P}(U) \cap G(3, V), [U] \in \mathcal{F} \} \subset G(3, V) \times \mathcal{F},
\]

with the natural projection \(\mathcal{F} \rightarrow \mathcal{F}\). As explained in Subsection [5.5.2] the fibers \(\mathbb{P}(U) \cap G(3, V)\) over point \([U]\) are conics for \([U] \in \mathcal{F} \backslash (\mathcal{F}_p \cup \mathcal{F}_\sigma)\) while they are \(p\)-planes and \(\sigma\)-planes (\(\simeq \mathbb{P}(U)\)) for \([U] \in \mathcal{F}_p\) and \([U] \in \mathcal{F}_\sigma\), respectively.

5.5.2. Generically conic bundle \(\mathcal{Z}_3 \rightarrow \mathcal{Y}_3\). The generically conic bundle \(\mathcal{F} \rightarrow \mathcal{F}\) naturally extends to \(\mathcal{Z}_3 \rightarrow \mathcal{Y}_3\) by the isomorphism \(\mathbb{P}(U) \cap G(3, V) \simeq \mathbb{P}(U) \cap G(2, V/V_1)\) for \(U = \tilde{U} \wedge V_1\). To describe it, let us introduce the universal bundles for the Grassmannian bundle \(\pi_{3'}: \mathcal{Z}_3 = G(3, \wedge^2 T(-1)) \rightarrow \mathbb{P}(V)

\[
0 \rightarrow S \rightarrow \pi_{3'}^\wedge \mathbb{P}(V) \rightarrow Q \rightarrow 0.
\]

Denote by \(\mathbb{P}(S)\) the universal planes over \(\mathcal{Z}_3\), whose fiber over \(([\tilde{U}], [V_1])\) is \(\mathbb{P}(U)\). Now, consider Grassmannian bundle \(\pi_{G'}: G(2, T(-1)) \rightarrow \mathbb{P}(V)\), and define

\[
\mathcal{Z}_3 := G(2, V/V_1) \times_{\mathbb{P}(V)} \mathcal{Y}_3,
\]

with the natural projections \(\pi_{G'}: \mathcal{Z}_3 \rightarrow G(2, T(-1))\) and \(\pi_{3'}: \mathcal{Z}_3 \rightarrow \mathcal{Y}_3\). By definition, the fiber of \(\pi_{3'}\) over the points \(([\tilde{U}], [V_1])\) is \(\mathcal{Z}_3 \backslash (\mathcal{F}_p \cup \mathcal{F}_\sigma)\) is

\[
G(2, V/V_1) \cap \mathbb{P}(U),
\]
which are conics isomorphic to $\mathbb{P}(U) \cap G(3, V)$ with $U = \bar{U} \wedge V_1$, i.e., the fibers of $\mathcal{K} \to \mathcal{T}$ over $[U]$. As before the fibers over $\mathcal{P}_\rho$ and $\mathcal{P}_\sigma$ are the $\rho$-planes and $\sigma$-planes, respectively.

Noting the isomorphism $G(2, T(-1)) \simeq F(1, 3, V)$, the following lemma is clear:

**Lemma 5.9.** There is a natural morphism $\rho_G : G(2, T(-1)) \to G(3, V)$.

**Fig. 5.2. Generically conic bundles.** Generically conic bundles in the text are schematically described. The proper transforms of $\mathcal{T}_\sigma$ are written by the same letter $\mathcal{P}_\sigma$ for simplicity.

### 5.5.3. **Generically conic bundle** $\mathcal{Z}_2 \to \mathcal{Z}_2$. As described in Proposition 4.7, $\mathcal{Z}_2$ is given as the blow-up of $\mathcal{Y}_3$ along $\mathcal{P}_\rho$. We denote the exceptional divisor of the blow-up by $F_\rho$ (note that $F_\rho$ is a divisor).

**Proposition 5.10.** (1) We have $N_{\mathcal{P}_\rho/\mathcal{Y}_3} = S^2 S^* \otimes \pi_\rho^* \mathcal{O}_{\mathbb{P}(V)}(1)|_{\mathcal{P}_\rho}$ for the normal bundle of $\mathcal{P}_\rho \subset \mathcal{Y}_3$, and hence $F_\rho = \mathbb{P}(S^2 S^*|_{\mathcal{P}_\rho})$.

(2) The fibers of $F_\rho \to \mathcal{P}_\rho$ can be identified with the conics in the $\rho$-planes parametrized by $\mathcal{P}_\rho$.

**Proof.** (1) We have seen in Proposition 4.7 that $\mathcal{P}_\rho = OG(3, \wedge^2 T(-1))$, i.e., one of the connected components of $OG(3, \wedge^2 T(-1)) \subset G(3, \wedge^2 T(-1))$. The orthogonal Grassmannian consists maximally isotropic subspaces with respect to the symmetric form on the universal bundle $S$ induced from

$$\wedge^2 T(-1) \times \wedge^2 T(-1) \to \wedge^4 T(-1) \simeq \mathcal{O}_{\mathbb{P}(V)}(1).$$
Hence it is given by the zero locus of the section of the bundle $S^2 S^* \otimes \pi^* \mathcal{O}_\mathbb{P}(V)(1)$ over $G(3, \wedge^2 T(-1))$.

(2) The points $([\bar{U}], [V_1]) \in \mathcal{P}_\rho$ determine the $\rho$-planes $\mathbb{P}(\bar{U}) \subset \mathbb{P}(\wedge^2 (V/V_1))$. We can evaluate the fiber over a point $([\bar{U}], [V_1]) \in \mathcal{P}_\rho$ as

$$\mathbb{P}(S^2 S^*|([\bar{U}], [V_1])) = \mathbb{P}(S^2 \bar{U}^*)$$

which we identify with the conics in the $\rho$-plane.

Proposition 5.11. Let $\rho_2' : \mathcal{Z}_2 \to \mathcal{Z}_3$ be the blow-up of $\mathcal{Z}_3$ along $\pi_3^{-1}(\mathcal{P}_\rho)$, and $E_\rho$ be its exceptional divisor. Then $E_\rho \to F_\rho$ is the universal family of $\rho$-conics parametrized by $F_\rho$.

Proof. This follows by considering the normal bundle of $\pi_3^{-1}(\mathcal{P}_\rho)$ in $\mathcal{Z}_3$ carefully. We refer to [HoTa4, Prop.4.3.4] for the proof.

Now we summarize the above results into

Proposition 5.12. The natural morphism $\pi_2' : \mathcal{Z}_2 \to \mathcal{Y}_2$ between the blow-ups $\mathcal{Z}_2$ and $\mathcal{Y}_2$ is a generically conic bundle. Precisely, the fibers over $\mathcal{Y}_2 \setminus \mathcal{P}_\sigma$ are conics and the fibers over $\mathcal{P}_\sigma$ are $\sigma$-planes (where we use the same notation $\mathcal{P}_\sigma$ for the proper transform of $\mathcal{P}_\sigma$ in $\mathcal{Y}_3$).

We may summarize generically conic bundles into the following diagram:

\[ \begin{array}{ccc}
\mathcal{Z}_3 & \xrightarrow{\rho_3'} & \mathcal{Z}_2 \\
\downarrow{\pi_3'} & & \downarrow{\pi_2'} \\
\mathcal{Y}_3 & \xrightarrow{\rho_2} & \mathcal{Y}_2 \\
\mathcal{Z}_2 & \xrightarrow{\rho_2'} & \mathcal{Y}_2 \\
\mathcal{Y}_2 & \xrightarrow{\rho_2} & \mathcal{Y}_2 \\
\pi_3 & & \pi_2 \\
\pi_2 & & \pi_2 \\
\mathbb{P}(V) & \xrightarrow{\pi} & \mathbb{P}(V) \\
G(3, V) & \xrightarrow{\rho_G} & \mathbb{P}(V) \\
G(2, T(-1)) & \xrightarrow{\rho_G} & \mathbb{P}(V) \\
\end{array} \]

In the above diagram, we have included all the morphisms claimed in Proposition 4.3. In Fig. 5.2, we schematically depict the generically conic bundles, $\mathcal{Z}_2 \to \mathcal{Y}_2$, $\mathcal{Z}_3 \to \mathcal{Y}_3$, $\mathcal{Z}_2 \to \mathcal{Y}_2$ and also $\mathcal{Z}_2 \to \mathcal{Y}$ which follows from $\mathcal{Z}_2 \to \mathcal{Y}_2$ and $\mathcal{Z}_2 \to \mathcal{Y}$. 
APPENDIX A. Two Theorems on Indefinite Lattices

We summarize two theorems on indefinite lattices which we use in Section 2.

**Theorem A.1** ([Ni, Theorem 1.14.2]). Let \( L \) be an indefinite lattice and \( \ell(L) \) be the minimal number of generators of \( L^\vee/L \). If \( \text{rk} L \geq 2 + \ell(L) \), then the isogeny classes of \( L \) consists of \( L \) itself, \( G(L) = \{L\} \) and the natural group homomorphism \( O(L) \rightarrow O(A_L) \) is surjective.

**Theorem A.2** ([Ni, Theorem 1.14.4]). Let \( L \) be an even unimodular lattice with signature \((l_+, l_-)\) and \( M \) be an even lattice with signature \((m_+, m_-)\). If (i) \( \text{sgn}(L) - \text{sgn}(M) > 0 \) \((l_+ - m_+ > 0, l_- - m_- > 0)\) and (ii) \( \text{rk} L - \text{rk} M \geq 2 + l(A_M) \) hold, then primitive embedding \( L \hookrightarrow M \) is unique up to automorphism of \( L \).

APPENDIX B. Plücker Ideal of \( G(3, 6) \)

Let us fix a 4-dimensional space \( V_4 \) and write the double spin decomposition \([5, 6]\) as

\[
\wedge^3(\wedge^2 V_4) \simeq S^2 V_4 \oplus S^2 V_4^*.
\]

We fix a basis of \( V_4 \) and write the corresponding bases of \( \wedge^4 V_4 \) in terms of the index set \( \mathcal{I} = \{\{i, j\} \mid 1 \leq i < j \leq 4\} \) (where we regard \( \{i, j\} \) as an ordered set). Then we introduce the standard Plücker coordinate by \([p_{IJK}] \in \mathbb{P}(\wedge^3(\wedge^2 V_4))\). On the other hand, we introduce the homogeneous coordinate (which may be called double spin coordinate) by \([v_{ijw}, w_{kl}] \in \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*)\) with \(4 \times 4\) symmetric matrices \( v = (v_{ij}), w = (w_{kl})\). Writing the isomorphism of the above decomposition, we have a linear relation between \([p_{IJK}]\) and \([v_{ijw}, w_{kl}]\). Then the Plücker ideal \( I_G \) of the embedding \( G(3, \wedge^2 V_4) \subset \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*) \) follows from that of the standard embedding \( G(3, \wedge^2 V_4) \subset \mathbb{P}(\wedge^3(\wedge^2 V_4))\).

Let us introduce some notations. We define the signature function \( \epsilon_{IJ} (I, J \in \mathcal{I}) \) by the signature of the permutation of the “ordered” union \( I \cup J \), e.g., \( \{2, 4\} \cup \{1, 3\} = \{2, 4, 1, 3\} \). We also define the dual index \( \bar{I} \in \mathcal{I} \) of \( I \in \mathcal{I} \) by the property \( I \cup \bar{I} = \{1, 2, 3, 4\} \) (here \( \cup \) is the standard union).

**Proposition B.1** ([Ho3, Appendix A]). The Plücker ideal \( I_G \) of the embedding \( G(3, \wedge^2 V_4) \subset \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*) \) is generated by

\[
[v_{I,j}] - \epsilon_{IJ} \epsilon_{J,j} [w_{I,j}] \quad (I, J \in \mathcal{I}),
\]

\[(v.w)_{ij}, (v.w)_{ii} - (v.w)_{jj} \quad (i \neq j, 1 \leq i, j \leq 4),\]

where \([v_{I,j}], [w_{I,j}]\) represent the \(2 \times 2\) minors of \( v, w \) with the rows and columns specified by \( I \) and \( J \). \((v.w)_{ij}\) is the \(ij\)-entry of the matrix multiplication \(v.w\).

For \([v, w] \in V(I_G) \simeq G(3, 6)\), we have

1) \( \det v = \det w \),
2) \( v.w = \pm \sqrt{\det w} \cdot w_4 \),
3) \( \text{rk} v \neq 3 \) and \( \text{rk} w \neq 3 \),
4) \( \text{rk} v = 2 \leftrightarrow \text{rk} w = 2 \), and
5) \( \text{rk} v \leq 1 \leftrightarrow \text{rk} w \leq 1 \).

These are easy consequences from (B.1).
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