THE CATEGORY $[T^c]^{\text{op}}$ AS FUNCTORS ON $T^b_c$

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Abstract. We revisit an old assertion due to Rouquier, characterizing the perfect complexes as bounded homological functors on the bounded complexes of coherent sheaves. The new results vastly generalize the old statement—first of all the ground ring is not restricted to be a field, any commutative, noetherian ring will do. But the generalization goes further, to the abstract world of approximable triangulated categories.

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0. Introduction

In [7, Corollary 7.51(ii)] there is the claim that, if $X$ is any scheme projective over a field $R$, then any finite, $R$–linear homological functor $H : D^b_{\text{coh}}(X) \to R$–mod is of the form $\text{Hom}(A, -)$ with $A \in D^{\text{perf}}(X)$. The terminology is explained in

Definition 0.1. Let $R$ be a noetherian, commutative ring, and $S$ an $R$–linear triangulated category. In this article an $R$–linear homological functor $H : S \to R$–Mod will be called locally finite if $H^i(s)$ is a finite $R$–module for all $s \in S$ and $i \in \mathbb{Z}$, and if, for any fixed object $s \in S$, we have $H^i(s) = 0$ for $i \ll 0$.

The functor $H$ is called finite if it is locally finite, and in addition $H^i(s) = 0$ for $i \gg 0$.

From the main results of current article one can easily deduce the following improvement on [7, Corollary 7.51(ii)].

2000 Mathematics Subject Classification. Primary 18E30, secondary 14F05.

Key words and phrases. Derived categories, $t$–structures, homotopy limits.

The research was partly supported by the Australian Research Council.

1There is a brief outline following [7, Corollary 7.51(ii)], hinting at how a proof might proceed. But it is too sketchy for the author of the current article to follow.
Theorem 0.2. Let $R$ be a noetherian, commutative ring, let $X$ be a scheme proper over $R$, and assume that every closed subvariety of $X$ admits a regular alteration in the sense of de Jong [3, 4]. Consider the following functors

\[ \tilde{\iota} : [D_{\text{perf}}(X)]^{\text{op}} \to [D_{\text{coh}}^{-}(X)]^{\text{op}} \to \text{Hom}_{R}[D_{\text{coh}}^{b}(X), R\text{-Mod}] \]

That is: $\tilde{\iota}$ is the inclusion, and $\tilde{\mathcal{Y}}$ is the functor taking an object $A \in D_{\text{coh}}^{-}(X)$ to the map $\text{Hom}(A, -)$, viewed as an $R$–linear functor $D_{\text{coh}}^{b}(X) \to R\text{-Mod}$.

Then the functor $\tilde{\mathcal{Y}}$ is full and its essential image consists of the locally finite homological functors. The functor $\tilde{\mathcal{Y}} \circ \tilde{\iota}$ is fully faithful, and its essential image consists of the finite homological functors.

One can generalize this to the world of approximable triangulated categories, see [6] for the theory. With the notation as in [3], Theorem 0.2 is an immediate consequence of [5, Theorem 2.3], [6, Example 3.6] and the following general assertion.

Theorem 0.3. Let $R$ be a noetherian, commutative ring. Let $T$ be an $R$–linear, approximable triangulated category, and assume there is a compact generator $H \in T$ such that $\text{Hom}(H, \Sigma^{i}H)$ is a finite $R$–module for all $i \in \mathbb{Z}$. Let $T_{c}^{-}$ and $T_{c}^{b}$ be the ones corresponding to the preferred equivalence class of $t$–structures, and assume there is an object $G \in T_{c}^{b}$ and an integer $N > 0$ with $T = \langle G \rangle_{N}$.

Consider the following functors

\[ [T_{c}]^{\text{op}} \xrightarrow{\tilde{\iota}} [T_{c}^{-}]^{\text{op}} \xrightarrow{\tilde{\mathcal{Y}}} \text{Hom}_{R}[T_{c}^{b}, R\text{-Mod}] \]

That is: $\tilde{\iota}$ is the inclusion, and $\tilde{\mathcal{Y}}$ is the functor taking an object $A \in T_{c}^{-}$ to the map $\text{Hom}(A, -)$, viewed as an $R$–linear functor $T_{c}^{b} \to R\text{-Mod}$.

Then the functor $\tilde{\mathcal{Y}}$ is full and its essential image consists of the locally finite homological functors. The functor $\tilde{\mathcal{Y}} \circ \tilde{\iota}$ is fully faithful, and its essential image consists of the finite homological functors.

Acknowledgements. The author would like to thank Jesse Burke and Bregje Pauwels for corrections and comments on an earlier draft.

1. Conventions

Let $T$ be a triangulated category, let $G \in T$ be an object, and let $a \leq b$ and $n > 0$ be integers, possibly infinite. We will be using the notation of [5, Theorem 0.16 and Corollary 1.11], we briefly remind the reader. The subcategory $\langle G \rangle_{[a, b]}^{[n]} \subset T$ is the full subcategory of $T$ containing $\Sigma^{-i}G$ for all $a \leq i \leq b$, closed under direct sums and direct summands, and whose objects are obtainable using no more than $n$ extensions. If $T$ has coproducts we will also consider the big versions: the subcategory $\text{Copro}_{n}(G[a, b])$ contains $\Sigma^{-i}G$ for all $a \leq i \leq b$, is closed under all coproducts, and its objects are
obtainable using no more than \( n \) extensions. Finally the subcategory \( \langle G \rangle_n \) is the closure in \( \mathcal{T} \) of the subcategory \( \text{Coprod}_n(G[a,b]) \) under direct summands.

We adopt the convention that, if any of the integers \( a, b, n \) is left out, it should be taken to be infinite. For example \( \langle G \rangle_{\infty} = \langle G \rangle_{[\infty, \infty]} \) and \( \langle G \rangle^{[a,b]}_{\infty} = \langle G \rangle^{[a,b]}_{\infty} \).

We also remind the reader of some constructions from [6]. Let \( \mathcal{T} \) be a triangulated category. In [6 Definition 0.10] we declared two \( t \)-structures \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) and \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) to be equivalent if there exists an integer \( A > 0 \) with \( \mathcal{T}^{\leq -A} \subset \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq A} \). Now assume \( \mathcal{T} \) has coproducts and a compact generator \( G \). Then [6 Definition 0.14] defines a preferred equivalence class of \( t \)-structures, namely the one containing the \( t \)-structure generated by \( G \) in the sense of Alonso, Jeremías and Souto [1]—in the notation of the paragraph above the \( t \)-structure generated by \( G \) has \( \mathcal{T}^{\leq 0} = \langle G \rangle^{[\infty, \infty]} \). And [6 Definition 0.16] allows one to construct two subcategories \( \mathcal{T}^b_{\cdot} \subset \mathcal{T}_{\cdot} \subset \mathcal{T} \). If the compact generator \( G \in \mathcal{T} \) is such that \( \text{Hom}(G, \Sigma^i G) = 0 \) for \( i > 0 \), then \( \mathcal{T}^b_{\cdot} \subset \mathcal{T}^c_{\cdot} \) are both thick subcategories of \( \mathcal{T} \), see [6 Proposition 2.10].

2. LEMMAS THAT DON’T REQUIRE APPROXIMABILITY

We begin with the easy part of Theorem [0.3] showing that the images of the functors \( \mathcal{Y} \) and \( \mathcal{Y} \circ \mathcal{I} \) are contained where the theorem asserts they should be.

**Lemma 2.1.** Let \( R \) be a commutative, noetherian ring. Suppose \( \mathcal{T} \) is an \( R \)-linear triangulated category. Suppose \( \mathcal{T} \) has a single compact generator \( H \), such that \( \text{Hom}(H, \Sigma^i H) \) is a finite \( R \)-module for every \( i \in \mathbb{Z} \), and vanishes when \( i \gg 0 \).

Then \( \text{Hom}(A, \Sigma^i B) \) is a finite \( R \)-module whenever \( A \in \mathcal{T}^- \) and \( B \in \mathcal{T}^b_{\cdot} \). For fixed \( A \) and \( B \) it vanishes when \( i \ll 0 \).

If \( A \) belongs to \( \mathcal{T}^c \subset \mathcal{T}_- \), then \( \text{Hom}(A, \Sigma^i B) \) also vanishes for \( i > 0 \).

**Proof.** Fix \( A \) and \( B \). Since \( A \in \mathcal{T}^- \) and \( B \in \mathcal{T}^+ \) there will be some integer \( m > 0 \) such that \( \text{Hom}(A, \Sigma^i B) = 0 \) for \( i < -m \).

We need to prove the finiteness for fixed \( i \), and without loss we may assume \( i = 0 \). Shifting if necessary we may assume \( B \in \mathcal{T}^{\geq 0} \). But \( A \in \mathcal{T}^- \) means that there must exist a triangle \( E \rightarrow A \rightarrow D \) with \( E \in \mathcal{T}^c \) and \( D \in \mathcal{T}^{\leq -2} \). In the exact sequence

\[
\text{Hom}(D, B) \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(E, B) \rightarrow \text{Hom}(\Sigma^{-1} D, B)
\]

we have \( \text{Hom}(D, B) = 0 = \text{Hom}(\Sigma^{-1} D, B) \), hence \( \text{Hom}(A, B) \cong \text{Hom}(E, B) \). The finiteness of \( \text{Hom}(E, B) \) is contained in [6 Lemma 7.2].

Now assume \( A \in \mathcal{T}^c \). The vanishing of \( \text{Hom}(A, \Sigma^i B) \) for \( i > 0 \) follows from [6 Lemma 2.8].

**Lemma 2.2.** Let \( \mathcal{T} \) be a triangulated category with coproducts, and let \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) be a \( t \)-structure on \( \mathcal{T} \). Let \( G \in \mathcal{T}^b_{\cdot} \) be an object, and let \( N > 0 \) be an integer.

Then there exists an integer \( M > 0 \) so that any map \( A \rightarrow C \), with \( A \in \mathcal{T}^{\leq 0} \) and \( C \in \langle G \rangle^{-m, \infty}_N \), must factor through some object \( B \in \langle G \rangle^{-m, M}_N \).
If we further assume that $\mathcal{T}^{\geq 0}$ is closed under coproducts, then the integer $M > 0$ may be chosen so that any map $A \rightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in (G)_N^{[-m, \infty]}$, will factor through some object $B \in (G)_N^{[m, M]}$.

In both statements we allow $m > 0$ to be (possibly) infinite.

Proof. Because $G \in \mathcal{T}^b$ we may choose an integer $K > 0$ so that $G \in \mathcal{T}^{\leq -K} \cap \mathcal{T}^{\leq K}$. Fix such a $K$; for any integer $n > 0$ we clearly have $(G)_n^{[K, \infty]} \subset \mathcal{T}^{\geq 0}$, and if $\mathcal{T}^{\geq 0}$ is closed under coproducts then we also have $(G)_n^{[m, \infty]} \subset \mathcal{T}^{\geq 0}$.

Next we proceed to prove the assertion of the lemma beginning with “if we further assume”. We leave to the reader the case with $C \in (G)_N^{[-m, \infty]}$. We are given that $C \in (G)_N^{[-m, \infty]}$, and by [3] Corollary 1.11 we have $$\frac{(G)_n^{[-m, \infty]}}{(G)_n^{[-m, \infty]}} = \text{smd} \left( \text{Coprod}_N \left( G[-m, \infty] \right) \right).$$

Therefore we may choose an object $C'$ with $C \oplus C' \in \text{Coprod}_N \left( G[-m, \infty] \right)$, and any map $f : A \rightarrow C$ obviously factors as the composite $A \xrightarrow{f} C \xrightarrow{i} C \oplus C' \xrightarrow{\pi} C$, meaning $f$ factors through $C \oplus C'$. It therefore suffices to show that, with $K > 0$ as in the paragraph above, we have

(i) Any map $f : A \rightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in \text{Coprod}_N \left( G[-m, \infty] \right)$, must factor through some object $B \in \text{Coprod}_N \left( G[-m, (2K + 1)N] \right)$.

Now we proceed by induction on $N$. If $N = 1$ we are given a map $A \rightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in \text{Coprod}_1 \left( G[-m, \infty] \right)$. But

$$\text{Coprod}_1 \left( G[-m, \infty] \right) = \text{Coprod}_1 \left( G[-m, K] \right) \bigoplus \text{Coprod}_1 \left( G[K + 1, \infty] \right)$$

As $\text{Coprod}_1 \left( G[K + 1, \infty] \right)$ is contained in $\mathcal{T}^{\geq 1}$ and $A \in \mathcal{T}^{\leq 0}$, the map $A \rightarrow C$ must factor through $B \in \text{Coprod}_1 \left( G[-m, K] \right)$. We have proved an improvement on (i) in the case $N = 1$.

Next assume we know (i) for all integers $\leq N$, and keep in mind that, for $N = 1$, we proved an improvement on (i) in the last paragraph. Now let $S = \mathcal{T}^{\leq 0}$ and put

$$X = \text{Coprod}_N \left( G[-m, \infty] \right) \quad A = \text{Coprod}_N \left( G[-m, (2K + 1)(N + 1)] \right)$$

$$Z = \text{Coprod}_1 \left( G[-m, \infty] \right) \quad C = \text{Coprod}_1 \left( G[-m, K] \right)$$

By the induction any pair of maps $s \rightarrow x$ and $s \rightarrow z$, with $s \in S$, $x \in X$ and $z \in Z$, factor (respectively) as $s \rightarrow a \rightarrow x$ and $s \rightarrow c \rightarrow z$, with $a \in A$ and $c \in C$.

Now let $d$ be an object of $(\Sigma^{-1}C) \ast S$. As $\Sigma^{-1}C \subset \mathcal{T}^{\leq 2K + 1}$ and $S = \mathcal{T}^{\leq 0}$ we deduce that $d \in \mathcal{T}^{\leq 2K + 1} = \Sigma^{-2K - 1}S$, and induction tells us that any map $d \rightarrow x$, with $x \in X$, must factor as $d \rightarrow a \rightarrow x$ with $a \in A$. The hypotheses of [5] Lemma 1.5] are satisfied, hence any morphism $s \rightarrow X \ast Z = \text{Coprod}_{N+1} \left( G[-m, \infty] \right)$ factors through $A \ast C \subset \text{Coprod}_{N+1} \left( G[-m, (2K + 1)(N + 1)] \right)$.$\Box$
Lemma 2.3. Let \( \mathcal{T} \) be a triangulated category with coproducts, as well as a \( t \)-structure \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\). Let \( G \in \mathcal{T}^b \) be an object, and let \( N > 0 \) be an integer. Then there exists an integer \( M > 0 \) so that

(i) Any map \( C \rightarrow A \), with \( A \in \mathcal{T}^{\geq 0} \) and \( C \in \langle G \rangle_{N}^{[-M,m]} \), must factor through some object \( B \in \langle G \rangle_{N}^{[-M,m]} \).

(ii) Suppose \( \mathcal{T}^{\geq 0} \) is closed under coproducts. Then any map \( C \rightarrow A \), with \( A \in \mathcal{T}^{\geq 0} \) and \( C \in \langle G \rangle_{N}^{[-\infty,m]} \), must factor through some object \( B \in \langle G \rangle_{N}^{[-M,m]} \).

In both statements the integer \( m > 0 \) is possibly infinite.

Proof. Because \( G \in \mathcal{T}^b \) we may choose an integer \( K > 0 \) so that \( G \in \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K} \). Fix such a \( K \). The fact that \( \mathcal{T}^{\leq 0} = \mathcal{T}^{\geq -1} \) tells us that \( \mathcal{T}^{\leq 0} \) is (automatically) closed under coproducts, and hence for any integer \( n > 0 \) we have \( \langle G \rangle_{n}^{[-\infty,-K]} \subset \langle G \rangle_{n}^{[-\infty,-K]} \subset \mathcal{T}^{\leq 0} \).

The rest of the proof is just the dual of the proof of Lemma 2.2, we sketch the proof of assertion (ii) and leave (i) to the reader. Any map \( C \rightarrow A \), with \( A \in \mathcal{T}^{\geq 0} \) and \( C \in \langle G \rangle_{N}^{[-\infty,m]} \), must factor through some \( B \in \text{Coproduct}_{1}(G[-K,m]) \).

More precisely: let \( S = \mathcal{T}^{\geq 0} \) and

\[
\begin{align*}
\mathcal{X} &= \text{Coproduct}_{N}(G^{[-\infty,m]}) \\
\mathcal{Z} &= \text{Coproduct}_{1}(G^{[-\infty,m]}) \\
\mathcal{A} &= \text{Coproduct}_{N}(G^{[-(2K+1)(N+1),m]}) \\
\mathcal{E} &= \text{Coproduct}_{1}(G^{[-K,m]})
\end{align*}
\]

The hypothesis that \( \mathcal{T}^{\geq 0} \) is closed under coproducts gives that \( \Sigma \mathcal{E} \subset \mathcal{T}^{\geq 2K-1} = \Sigma^{2K+1} \mathcal{S} \), and hence \( \mathcal{S} \cdot \Sigma \mathcal{E} \subset \mathcal{S} \cdot \Sigma^{2K+1} \mathcal{S} \subset \Sigma^{2K+1} \mathcal{S} \). Now induction coupled with [5, Lemma 1.5] guarantee that any map \( y \rightarrow s \), with \( y \in \mathcal{Z} \star \mathcal{X} \) and \( s \in \mathcal{S} = \mathcal{T}^{\geq 0} \), must factor through \( B \in \mathcal{E} \star \mathcal{A} \subset \text{Coproduct}_{N+1}(G^{[-(2K+1)(N+1),m]}) \). \( \square \)

Lemma 2.4. Let \( \mathcal{T} \) be a triangulated category with coproducts, as well as a \( t \)-structure \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) with \( \mathcal{T}^{\geq 0} \) is closed under coproducts. Let \( G \in \mathcal{T}^b \) be an object, let \( N > 0 \) be an integer, and assume \( \mathcal{T} = \langle G \rangle_{N} \).

Then there exists an integer \( M > 0 \) such that, for any integers \( a \leq b \),

\[
\begin{align*}
\mathcal{T}^{\geq 0} \subset \langle G \rangle_{N}^{[-M,\infty]} & , & \mathcal{T}^{\leq 0} \subset \langle G \rangle_{N}^{[-\infty,M]} & & , & \mathcal{T}^{\geq a} \cap \mathcal{T}^{\leq b} \subset \langle G \rangle_{N}^{[a-M,b+M]}.
\end{align*}
\]

Proof. For the given integer \( N \) and object \( G \), pick an integer \( M \) so that Lemmas 2.2 and 2.3 both hold.

Let \( a \) be an integer, and suppose \( A \) is an object in \( \mathcal{T}^{\geq a} \). The identity map \( A \rightarrow A \) is a morphism from \( A \in \langle G \rangle_{N} \) to \( A \in \mathcal{T}^{\geq a} \), and Lemma 2.3 (with \( m = \infty \)) guarantees that the map factors through an object \( B \in \langle G \rangle_{N}^{[-a-M,\infty]} \). As \( A \) is a direct summand of \( B \) it must also lie in \( \langle G \rangle_{N}^{[-a-M,\infty]} \). The case \( a = 0 \) gives the first assertion of the Lemma.
Now assume \( A \in \mathcal{T}^{\leq a} \cap \mathcal{T}^{\leq b} \) with \( a \) possibly equal to \(-\infty\). By the first half of the Lemma, already proved, \( A \) must lie in \((G)^{(a-M,\infty)}_N\). The identity map \( A \to A \) is a morphism from \( A \in \mathcal{T}^{\leq b} \) to \( A \in \mathcal{T}^{(a-M,\infty)}_N \), and Lemma 2.2 guarantees that it factors through some \( B \in \mathcal{T}^{(a-M,b+M)}_N \). Therefore, \( A \), being a direct summand of \( B \), must also belong to \((G)^{(a-M,b+M)}_N\).

\[ \square \]

**Lemma 2.5.** Let \( \mathcal{T} \) be a triangulated category with coproducts and a single compact generator \( H \). Assume \( \text{Hom}(H, \Sigma^n H) = 0 \) for \( n \gg 0 \).

Let \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) be a \( t \)-structure on \( \mathcal{T} \), in the preferred equivalence class. Given an integer \( K > 0 \) and a collection of objects \( \{X_i \in \mathcal{T}^{\leq K} \cap \mathcal{T}^{\geq -K} \mid i \in \mathbb{Z}\} \), then the map

\[
\prod_{i=-\infty}^{\infty} \Sigma^i X_i \xrightarrow{\varphi} \prod_{i=-\infty}^{\infty} \Sigma^i X_i
\]

is an isomorphism.

**Proof.** Because the \( t \)-structure is in the preferred equivalence class and \( H \) is compact, \([6, \text{Observation 0.12}]\) tells us that there is some integer \( A > 0 \) with \( H \in \mathcal{T}^{\geq A-1} \), and therefore \( \text{Hom}(H, -) \) vanishes on \( \mathcal{T}^{\geq A} = \left(\mathcal{T}^{\geq A-1}\right)^{\perp} \). And \([6, \text{Lemma 2.8}]\) allows us to assume, possibly after increasing \( A \), that \( \text{Hom}(H, \mathcal{T}^{\leq -A}) = 0 \). Therefore the functor \( \text{Hom}(H, -) \) vanishes on the union \( \mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A} \). As \( X_i \) is assumed to belong to \( \mathcal{T}^{\leq K} \cap \mathcal{T}^{\geq -K} \), we have that \( \text{Hom}(H, \Sigma^n X_i) = 0 \) whenever \( |n| > A + K \).

Any morphism \( \Sigma^n H \to \prod_{i=-\infty}^{\infty} \Sigma^i X_i \) will factor through a finite subcoproduct by the compactness of \( H \), and any map \( \Sigma^n H \to \prod_{i=-\infty}^{\infty} \Sigma^i X_i \) will factor through a finite subproduct by the vanishing of \( \text{Hom}(H, \Sigma^{i-n} X_i) \) for all but finitely many \( i \). Therefore the functor \( \text{Hom}(\Sigma^n H, -) \) takes the map \( \varphi \) to an isomorphism, for every \( n \in \mathbb{Z} \), and as \( H \) is a generator the map \( \varphi \) must be an isomorphism.

\[ \square \]

**Proposition 2.6.** Let \( \mathcal{T} \) be a triangulated category with coproducts, and assume it has a compact generator \( H \) with \( \text{Hom}(H, \Sigma^n H) = 0 \) for \( n \gg 0 \). Suppose \( G \) is an object in \( \mathcal{T}^b \), where we mean the \( \mathcal{T}^b \) that comes from the preferred equivalence class of \( t \)-structures. Assume that there exists an integer \( N > 0 \) with \((G)^{\leq a}_N = \mathcal{T} \).

Suppose \( F \in \mathcal{T}^c \) is an object such that \( \text{Hom}(F, \Sigma^n G) = 0 \) for \( n \gg 0 \). Then \( F \) is compact.

**Proof.** In the preferred equivalence class choose a \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) so that \( \mathcal{T}^{\geq 0} \) is closed under coproducts. We are given that \( G \in \mathcal{T}^b \), hence we may choose an integer \( K > 0 \) with \( G \in \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K} \). Because \( F \in \mathcal{T}^c \), for any integer \( i > 0 \) we may form a triangle \( E_i \xrightarrow{\alpha_i} F \to D_i \to \) with \( E \in \mathcal{T}^c \) and \( D_i \in \mathcal{T}^{\leq -i-2} \). For any \( X \in \mathcal{T} \) we have the exact sequence

\[
\text{Hom}(D_i, X) \longrightarrow \text{Hom}(F, X) \xrightarrow{\text{Hom}(\alpha_i, X)} \text{Hom}(E_i, X) \longrightarrow \text{Hom}(\Sigma^{-1} D_i, X)
\]
and if $X \in \mathcal{T}^{\geq-i-K}$ then $\text{Hom}(D_i, X) = 0 = \text{Hom}(\Sigma^{-1}D_i, X)$. Thus $\text{Hom}(\alpha_i, X)$ is an isomorphism for all $X \in \mathcal{T}^{\geq-i-K}$.

Now choose an integer $A > 0$ so that $\text{Hom}(F, \Sigma^iG) = 0$ for all $i \geq A$. Pick an integer $i \geq A$; the vanishing of $\text{Hom}(F, \Sigma^iG)$ implies the vanishing of $\text{Hom}(E_i, \Sigma^iG)$, and the compactness of $E_i$ implies the vanishing of of $\text{Hom}(E_i, \Sigma^i\text{Coproд}_1(G))$. But $\mathcal{T}^{\geq-i-K}$ is closed under coproducts and contains $\Sigma^iG$, hence $\Sigma^i\text{Coproд}_1(G)$ is contained in $\mathcal{T}^{\geq-i-K}$, and the vanishing of $\text{Hom}(E_i, \Sigma^i\text{Coproд}_1(G))$ implies the vanishing of $\text{Hom}(F, \Sigma^i\text{Coproд}_1(G))$.

Thus $\text{Hom}(F, -)$ vanishes on $\Sigma^i\text{Coproд}_1(G)$ for any $i \geq A$. Now every object $X \in \text{Coproд}_1(G(-\infty,-A])$ can be written as $\prod_{i=A}^{\infty} \Sigma^iX_i$ with $X_i \in \text{Coproд}_1(G) \subset \mathcal{T}^{\geq-K} \cap \mathcal{T}^{\leq K}$. By Lemma 2.5 there is an isomorphism $\prod_{i=A}^{\infty} \Sigma^iX_i \cong \prod_{i=A}^{\infty} \Sigma^iX_i$, and hence $\text{Hom}(F, -)$ vanishes on the category $\text{Coproд}_1(G(-\infty,-A])$.

It immediately follows that $\text{Hom}(F, -)$ also vanishes on the category $\mathcal{T}^\infty(-\infty,-A]$. But now Lemma 2.4 establishes the existence of some integer $m > 0$ with $\mathcal{T}^{\leq-m} \subset \mathcal{T}^\infty(-\infty,-A]$, and hence $\text{Hom}(F, -)$ vanishes on $\mathcal{T}^{\leq-m}$. On the other hand $F$ belongs to $\mathcal{T}_c^-$, hence we may choose a triangle $E \to F \to D \to$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq-m}$, and as the map $F \to D$ must vanish we have that $F$ is a direct summand of $E \in \mathcal{T}^c$. \hfill \Box

**Lemma 2.7.** Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator $H$. Assume $\text{Hom}(H, \Sigma^nH) = 0$ for $n \gg 0$. Suppose $G$ is an object in $\mathcal{T}^b_c$, where we mean the $\mathcal{T}^b_c$ that comes from the preferred equivalence class of $t$-structures.

For any integer $N > 0$ there exists an integer $M > 0$ so that any map $F \to Y$, with $F \in \mathcal{T}^c_c$ and $Y \in \langle G \rangle_N^{[a,b]}$, factors through an object in $\langle G \rangle_N^{[a,b]}$. We allow $b$ to be infinite.

**Proof.** Choose a $t$-structure in the preferred equivalence class, and pick it so that $\mathcal{T}^{\geq0}$ is closed under coproducts. By hypothesis $G$ is contained in $\mathcal{T}^b_c$, hence we may choose and fix an integer $K > 0$ with $G \in \mathcal{T}^{\geq-K}$. Then, for any integer $N > 0$, we have

$$\langle G \rangle_N^{[a,b]} \subset \langle G \rangle_N^{[a,b]} \subset \mathcal{T}^{\geq a-K}.$$

Now we prove the lemma by induction on $N$. Suppose first that $N = 1$; then any object $Y \in \langle G \rangle_1^{[a,b]}$ is a direct summand of an object in $\text{Coproд}_1(G[a,b])$, hence we may assume $Y$ belongs to $\text{Coproд}_1(G[a,b]) \subset \langle G \rangle_1^{[a,b]}$. Because $F \in \mathcal{T}^c_c$ we may choose triangle $\Sigma^{-1}D \to E \to F \to D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\geq a-K-2}$, and hence

$$\text{Hom}(\Sigma^{-1}D, \mathcal{T}^{\geq a-K}) = 0 = \text{Hom}(\Sigma^{-1}D, \mathcal{T}^{\geq a-K}).$$

Thus, for any object $Y \in \text{Coproд}_1(G[a,b])$, the natural map $\text{Hom}(F, Y) \to \text{Hom}(E, Y)$ is an isomorphism. But $E$ is compact and $Y$ is a coproduct, hence any map $E \to Y$ factors through a finite subcoproduct. This completes the proof in the case $N = 1$.

Now assume we know the Lemma for all integers $i$ with $0 < i \leq N$. Put

$$\mathcal{S} = \mathcal{T}^c_c, \quad \mathcal{A} = \langle G \rangle_1^{[a,b]}, \quad \mathcal{C} = \langle G \rangle_N^{[a,b]}, \quad \mathcal{X} = \langle G \rangle_1^{[a,b]}, \quad \mathcal{Z} = \langle G \rangle_N^{[a,b]}.$$
By the hypotheses of the lemma \( S = \mathcal{T}_c \) is triangulated and contains \( \mathcal{C} \). The inductive hypothesis, coupled with [6] Lemma 1.5 and Remark 1.6], give that any map \( s \to y \), with \( s \in S \) and \( y \in X \ast \mathbb{Z} \), must factor through an object \( b \in A \ast \mathcal{C} \). The lemma follows immediately.

**Proposition 2.8.** Suppose \( \mathcal{T} \) is a triangulated category, and assume it has a compact generator \( H \) with \( \text{Hom}(H, \Sigma^i H) = 0 \) for \( i > 0 \). Let \( \mathcal{T}_c^b \) be the one corresponding to a preferred \( t \)-structure.

Suppose there is an object \( G \in \mathcal{T}_c^b \) and an integer \( N > 0 \) with \( \mathcal{T} = \langle \overline{G} \rangle_N \). Then \( \mathcal{T}_c^b = \langle \overline{G} \rangle_N \).

**Proof.** Let \( \langle \mathcal{T}^0, \mathcal{T}^0 \rangle \) be a \( t \)-structure in the preferred equivalence class, and choose it so that \( \mathcal{T}^0 \) is closed under coproducts.

Take any object \( A \in \mathcal{T}_c^b \). Because \( A \) lies in \( \mathcal{T}_c^b \) is must belong to \( \mathcal{T}_c^b \) for some integers \( a \leq b \), and Lemma 2.4 guarantees that it must belong to \( \langle G \rangle_N^{[a-M,b+M]} \) for some \( M > 0 \). But then the identity \( A \to A \) is a morphism from the object \( A \in \mathcal{T}_c^b \) to the object \( A \in \langle G \rangle_N^{[a-M,b+M]} \), and Lemma 2.7 gives that it must factor through an object of \( B \in \langle G \rangle_N^{[a-M,b+M]} \subset \langle G \rangle_N \). As \( \langle G \rangle_N \) is closed under direct summands it must contain \( A \).

**Lemma 2.9.** Suppose \( \mathcal{T} \) is a triangulated category with coproducts, and assume that \( \mathcal{T} \) has a compact generator \( H \) with \( \text{Hom}(H, \Sigma^i H) = 0 \) for \( i > 0 \). Suppose further that there is an object \( G \in \mathcal{T}_c^b \) and an integer \( N > 0 \) with \( \mathcal{T} = \langle \overline{G} \rangle_N \).

Assume \( \langle \mathcal{T}^0, \mathcal{T}^0 \rangle \) is a \( t \)-structure in the preferred equivalence class. Then there exists an integer \( M > 0 \) such that, for any object \( Y \in \mathcal{T}_c^b \) and any integer \( i \in \mathbb{Z} \), the \( t \)-structure truncation morphism \( Y \to Y^{\geq -i} \) may be factored as \( Y \to Y^{\geq -i-2M} \to C \to Y^{\geq -i} \) with \( C \in \langle G \rangle_N^{-i-M,\infty} \).

**Proof.** Replacing the \( t \)-structure by an equivalent one if necessary, we may assume \( \mathcal{T} \) is closed under coproducts. Choose an integer \( M > 0 \) as in Lemma 2.4, in particular \( \mathcal{T}^{\geq -i} \subset \langle G \rangle_N^{-i-M,\infty} \). Assume further that \( M \) is large enough so that \( G \in \mathcal{T}_c^b \) belongs to \( \mathcal{T}^{\geq -M} \).

We are given the map \( Y \to Y^{\geq -i} \), with \( Y \in \mathcal{T}_c^b \) and \( Y^{\geq -i} \) in \( \mathcal{T}^{\geq -i} \), and by the choice of \( M \) we have that \( \mathcal{T}^{\geq -i} \) is contained in \( \langle G \rangle_N^{-i-M,\infty} \). Now Lemma 2.7 permits us to factor \( Y \to Y^{\geq -i} \) through an object \( C \in \langle G \rangle_N^{-i-M,\infty} \). This far we have composites \( Y \to C \to Y^{\geq -i} \).

But \( G \) is an object of \( \mathcal{T}^{\geq -M} \), hence \( \langle G \rangle_N^{-i-M,\infty} \) must be contained in \( \mathcal{T}^{\geq -i-2M} \). The map \( Y \to C \) must therefore factor canonically through the \( t \)-structure truncation, and we have our factorization \( Y \to Y^{\geq -i-2M} \to C \to Y^{\geq -i} \).

**Corollary 2.10.** Suppose \( \mathcal{T} \) is a triangulated category with coproducts, and assume there is an object \( G \in \mathcal{T}_c^b \) and an integer \( N \) with \( \mathcal{T} = \langle \overline{G} \rangle_N \). Assume further that \( \mathcal{T} \) has a compact generator \( H \) with \( \text{Hom}(H, \Sigma^i H) = 0 \) for \( i > 0 \). For any object \( Y \in \mathcal{T}_c^b \) we may
choose an inverse sequence \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \) so that the subsequence
\( \cdots \to E_7 \to E_5 \to E_3 \to E_1 \to E_0 \) lies in \( \mathcal{T}^b_c \) while the subsequence
\( \cdots \to E_8 \to E_6 \to E_2 \to E_0 \) is a subsequence of \( \cdots \to Y_{\geq -3} \to Y_{\geq -2} \to Y_{\geq -1} \to Y_{\geq 0} \).

**Proof.** The construction of the sequence \( E_i \) is just by iterating Lemma 2.9. \( \square \)

In view of Corollary 2.10, the next lemma becomes interesting.

**Lemma 2.11.** Suppose \( Y \in \mathcal{T}^{-}_c \) is an object, mapping to an inverse system \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \) in \( \mathcal{T}^b_c \). Assume this inverse system is pro-isomorphic to
\( \cdots \to F_3 \to F_2 \to F_1 \to F_0 \), and what we know about \( F_i \) is that for any \( n > 0 \) there
exists an \( m > 0 \) so that the map \( Y_{\geq -n} \to F_{i-n}^c \) is an isomorphism for all \( i \geq m \).

When we view \( \text{Hom}(Y, -) \) as a functor on \( \mathcal{T}^b_c \), it is equal to the colimit of \( \text{Hom}(E_i, -) \).

**Proof.** Any object \( Z \in \mathcal{T}^b_c \) belongs to \( \mathcal{T}^{\geq -n} \) for some \( n > 0 \), and the sequence \( \text{Hom}(F_i, Z) \)
becomes stable and isomorphic to \( \text{Hom}(Y, Z) \) for \( i \gg 0 \). Hence \( \text{Hom}(Y, -) \) is the colimit
of the sequence \( \text{Hom}(F_i, -) \). Therefore it must also be the colimit of the ind-isomorphic sequence \( \text{Hom}(E_i, -) \). \( \square \)

### 3. Lemmas that require approximability

**Lemma 3.1.** Suppose \( \mathcal{T} \) is a weakly approximable triangulated category, and let \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) be a \( t \)-structure in the preferred equivalence class. Suppose \( \cdots \to Z_3 \to Z_2 \to Z_1 \to Z_0 \) is an inverse sequence of objects in \( \mathcal{T} \) so that, for any integer \( n \), the functor \((-)^{\geq n}\)
takes it to a sequence that is eventually stable.

Let \( Z = \text{Holim} \, Z_i \) and consider the natural map \( f_i : Z \to Z_i \). Then for \( i \gg 0 \) the
functor \((-)^{\geq n}\) takes the map \( f_i \) to an isomorphism.

More precisely: there exists an integer \( L > 0 \) so that, whenever the sequence \( \cdots \to Z_3^{\geq -L} \to Z_2^{\geq -L} \to Z_1^{\geq -L} \to Z_0^{\geq -L} \) is constant, the map \( f_i^{\geq 0} : Z^{\geq 0} \to Z_i^{\geq 0} \) is an isomorphism.

**Proof.** Let \( G \) be a compact generator for \( \mathcal{T} \) and suppose \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) is a \( t \)-structure in the preferred equivalence class. By [2] Lemma 2.9 there exists an integer \( A > 0 \) so that, if \( \text{Hom}(\Sigma^i G, X) = 0 \) for all \( i \leq A \), then \( X \) must belong to \( \mathcal{T}^{\leq 0} \). Choose and fix such an integer \( A \). Then choose an integer \( B > 0 \) with \( \text{Hom}(G, \mathcal{T}^{\leq -B}) = 0 \).

By shifting and passing to a subsequence it suffices to prove the "moreover" assertion, and we assert that \( L = A + B + 1 \) works. Suppose therefore that the sequence \( \cdots \to Z_3^{\geq A-B-1} \to Z_2^{\geq A-B-1} \to Z_1^{\geq A-B-1} \to Z_0^{\geq A-B-1} \) is constant, and consider the
commutative diagram in which the rows are triangles

\[
\begin{array}{ccc}
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i \\
\downarrow & & \downarrow \\
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i
\end{array}
\]

1-shift

\[
\begin{array}{ccc}
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i \\
\downarrow & & \downarrow \\
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i
\end{array}
\]

1-shift

We may complete it to a 3 × 3 diagram of triangles

\[
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow & & \downarrow \\
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i \\
\downarrow & & \downarrow \\
\prod_{i=0}^{\infty} Z_i^{\leq -A-B-2} & \rightarrow & \prod_{i=0}^{\infty} Z_i \\
\end{array}
\begin{array}{cc}
\downarrow & \downarrow \\
X & \rightarrow \\
\prod_{i=0}^{\infty} (\Sigma^{-1} Z_i)^{\leq -A-B-1} & \rightarrow & \prod_{i=0}^{\infty} Z_i^{\leq -A-B-2}
\end{array}
\]

where the middle column is the definition of \( Z = \text{Holim} Z_i \). The inverse sequence \( Z_i^{\geq -A-B-1} \) is constant, hence the object \( Y \) is canonically isomorphic to \( Z_i^{\geq -A-B-1} \).

What we know about the object \( X \) is that it sits in the triangle given by the left column

\[
\prod_{i=0}^{\infty} (\Sigma^{-1} Z_i)^{\leq -A-B-1} \rightarrow X \rightarrow \prod_{i=0}^{\infty} Z_i^{\leq -A-B-2}
\]

By the choice of \( B \) we have that \( \text{Hom}(\Sigma^i G, -) \) kills the two outside terms whenever \( i \leq A+1 \), hence the choice of \( A \) gives that the two outside terms lie in \( \mathcal{T}^{\leq -1} \). Therefore \( X \in \mathcal{T}^{\leq -1} \), and it follows that the truncation \((-)^{\geq 0}\) takes the map \( Z \rightarrow Y = Z_i^{\geq -A-B-1} \) to an isomorphism. \( \square \)

**Proposition 3.2.** Suppose \( \mathcal{T} \) is a weakly approximable triangulated category, and let \((\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})\) be a t-structure in the preferred equivalence class. Let \( Y \) be any object in \( \mathcal{T} \), assume \( \cdots \rightarrow Z_3 \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0 \) is an inverse sequence of objects in \( \mathcal{T} \), and let \( f_* : Y \rightarrow Z_* \) be a map from \( Y \) to the inverse system. Suppose that, for any integer \( i > 0 \), there exists an integer \( N > 0 \) so that

\[
n \geq N \quad \Rightarrow \quad f^{\geq -i}_n : Y^{\geq -i} \rightarrow Z_n^{\geq -i} \text{ is an isomorphism.}
\]

If \( Z = \text{Holim} Z_n \) is the homotopy inverse limit, then the (non-canonical) map \( f : Y \rightarrow Z \) is an isomorphism.
Applying to the inverse system $Z_n = Y^{≥−n}$, we learn that $T$ is left-complete with respect to any $t$-structure in the preferred equivalence class.

**Proof.** By [2, Lemma 2.7] the $t$-structure is nondegenerate, hence it suffices to prove that the morphism $ϕ^{≥−i} : Y^{≥−i} → Z^{≥−i}$ is an isomorphism for every $i ∈ Z$. By shifting it suffices to prove this for $i = 0$. Let $L > 0$ be the integer in the “moreover” part of Lemma 3.1, choose an integer $N$ so large that $f_{N}^{≥−L} : Y^{≥−L} → Z_{N}^{≥−L}$ is an isomorphism whenever $n ≥ N$, and apply the functor $(-)^{≥0}$ to the commutative triangle below

$$
\begin{array}{ccc}
Y & \xrightarrow{f_N} & Z_N^{≥−L} \\
\downarrow f & & \downarrow \varphi_N \\
Z & & 
\end{array}
$$

Lemma 3.1 teaches us that $ϕ_{N}^{≥0}$ is an isomorphism. Since $f_{N}^{≥0}$ is also an isomorphism, the commutativity forces $f^{≥0}$ to be an isomorphism. □

**Lemma 3.3.** Let $T$ be a weakly approximable triangulated category, and choose a $t$-structure $(T^{≤0}, T^{≥0})$ in the preferred equivalence class. Suppose $⋯ → Z_3 → Z_2 → Z_1 → Z_0$ is an inverse sequence of objects in $T_c$ so that, for any integer $n$, the functor $(−)^{≥n}$ takes it to a sequence that is eventually stable.

Then $Z = \mathrm{Holim} Z_i$ belongs to $T_c$.

**Proof.** Since $T$ is weakly approximable, the “moreover” part of Lemma 3.1 provides an integer $L > 0$ such that, given an inverse sequence $⋯ → Z_3 → Z_2 → Z_1 → Z_0$ in $T$ with $⋯ → Z_{3}^{≥−L} → Z_{2}^{≥−L} → Z_{1}^{≥−L} → Z_{0}^{≥−L}$ all isomorphisms, we have that the map $[\mathrm{Holim} Z_i]^{≥0} → Z_0^{≥0}$ is an isomorphism. Also: since $T$ is weakly approximable it has a compact generator $H$, and [6, Corollary 2.14] permits us to choose an integer $B > 0$ such that

(i) $\hom(H, T^{≤−B}) = 0$.

(ii) Every object $X ∈ T_c^−$ admits a triangle $W → X → D$, with $W ∈ \langle H \rangle^{[−B, ∞)}$ and $D ∈ T^{≤0}$.

OK: we have chosen the integers $L$ and $B$ and it’s time to get to work. Suppose we are given in $T_c^−$ a sequence $⋯ → Z_3 → Z_2 → Z_1 → Z_0$ satisfying the hypotheses, put $Z = \mathrm{Holim} Z_i$, we need to show that $Z ∈ T_c^−$. Choose any integer $m > 0$. We choose an integer $i > 0$ so that the maps in the subsequence $⋯ → Z_{i+2}^{≥−m−2B+1} → Z_{i+1}^{≥−m−2B+1} → Z_i^{≥−m}$ are all isomorphisms. By (ii) we may choose a triangle $W → Z_i → D$ with $W ∈ \langle H \rangle^{[−m−B, ∞)}$ and $D ∈ T^{≤−m}$. By the choice of $L$ the map $Z_{i}^{≥−m−2B+1} → Z_{i}^{≥−m−2B+1}$ is an isomorphism, and in the triangle $Z → Z_i → D$ we have $D ∈ T^{≤−m−2B}$. Therefore the composite $W → Z_i → D$ is a map from $W ∈ \langle H \rangle^{[−m−B, ∞)}$ to $D ∈ T^{≤−m−2B}$ and vanishes by (i). We may therefore factor
\( W \rightarrow Z_i \) as \( W \rightarrow Z \rightarrow Z_i \). Completing to an octahedron

\[
\begin{array}{ccc}
W & \rightarrow & Z \\
\downarrow & & \downarrow \\
W & \rightarrow & Z_i \\
\downarrow & & \downarrow \\
\tilde{D} & \rightarrow & \tilde{D}
\end{array}
\]

produces a triangle \( W \rightarrow Z \rightarrow \tilde{D} \), with \( W \in \mathcal{T} \). The triangle \( \Sigma^{-1} \tilde{D} \rightarrow D' \rightarrow D \), coupled with the facts that \( \Sigma^{-1} \tilde{D} \in \mathcal{T}_{\leq -m-2B+1} \) and \( D \in \mathcal{T}_{\leq -m} \), guarantee that \( D' \in \mathcal{T}_{\leq -m} \). □

**Notation 3.4.** In the next lemmas \( R \) will be a commutative ring and \( \mathcal{T} \) will be an \( R \)-linear, weakly approximable triangulated category. Let \( \mathcal{T}_c^b \) and \( \mathcal{T}_c^- \) be understood with respect to the preferred equivalence class of \( \ell \)-structures. We will be considering two functors

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\mathcal{Y}} & \text{Hom}_R\left(\mathcal{T}_c^{\text{op}}, R\text{-Mod}\right) \\
\mathcal{T}^{\text{op}} & \xrightarrow{\tilde{\mathcal{Y}}} & \text{Hom}_R\left(\mathcal{T}_c^b, R\text{-Mod}\right)
\end{array}
\]

The object \( A \in \mathcal{T} \) goes under the functors, respectively, to

\[
\mathcal{Y}(A) = \text{Hom}(\cdot, A), \quad \tilde{\mathcal{Y}}(A) = \text{Hom}(A, \cdot).
\]

For \( \mathcal{Y}(A) \) the variable \( (\cdot) \) takes its values in \( \mathcal{T}_c \), while in the case of \( \tilde{\mathcal{Y}}(A) \) the variable \( (\cdot) \) lies in \( \mathcal{T}_c^b \).

We will mostly be concerned with the restrictions of \( \mathcal{Y} \) and \( \tilde{\mathcal{Y}} \) to the subcategory \( \mathcal{T}_c^- \).

**Proposition 3.5.** Let the conventions be as in Notation 3.4, and assume there is an object \( G \in \mathcal{T}_c^b \) and an integer \( N \) with \( \mathcal{T} = \langle G \rangle_N \).

Then restriction to \( \mathcal{T}_c^- \) of the map \( \tilde{\mathcal{Y}} \) is full. More generally: any morphism \( \varphi : \tilde{\mathcal{Y}}(b) \rightarrow \tilde{\mathcal{Y}}(a) \), with \( b \in \mathcal{T}_c^- \) and \( a \in \mathcal{T} \), is equal to \( \tilde{\mathcal{Y}}(f) \) for some \( f : a \rightarrow b \).

**Proof.** By Corollary 2.10 we may construct for \( b \) an inverse sequence \( \cdots \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \) pro-isomorphic to the sequence \( \cdots \rightarrow b^{2^{-3}} \rightarrow b^{2^{-2}} \rightarrow b^{2^{-1}} \rightarrow b^{2^0} \), and do it in such a way that \( C_i \) all belong to \( \mathcal{T}_c^b \). Lemma 2.11 tells us, moreover, that \( \tilde{\mathcal{Y}}(b) \) is the colimit of \( \tilde{\mathcal{Y}}(C_i) \).

But \( \tilde{\mathcal{Y}}(C_i) \) is representable, hence the composite \( \tilde{\mathcal{Y}}(C_i) \rightarrow \tilde{\mathcal{Y}}(b) \xrightarrow{\varphi} \tilde{\mathcal{Y}}(a) \) is a morphism from a representable functor. Yoneda tells us that it must be \( \tilde{\mathcal{Y}} \) of a unique map \( f_i : a \rightarrow C_i \). These maps are compatible, and hence all factor through a morphism \( f : a \rightarrow z \) with \( z = \text{Holim}\ C_i \). Because the sequence \( C_i \) is pro-isomorphic to the sequence \( b^{2^{-i}} \) we have that \( z = \text{Holim}\ b^{2^{-1}} \), and Proposition 3.2 gives an isomorphism \( b \cong z \). We have produced a morphism \( f : a \rightarrow b \), and it's now obvious that \( \tilde{\mathcal{Y}}(f) = \varphi \). □
Lemma 3.6. Let the conventions be as in Notation 3.4, and assume there is an object \( G \in \mathcal{T}_b^c \) and an integer \( N \) with \( \mathcal{T} = \langle G \rangle_N \).

Given two morphisms \( f, g : X \rightarrow Y \) in the category \( \mathcal{T}_c^\geq \), we have \( \overline{\gamma}(f) = \overline{\gamma}(g) \) if and only if \( \overline{\gamma}(f) = \overline{\gamma}(g) \).

Proof. Choose and fix a \( t \)-structure \( (\mathcal{T}^\leq_0, \mathcal{T}^\geq_0) \) in the preferred equivalence class.

Suppose \( \overline{\gamma}(f) = \overline{\gamma}(g) \), let \( S \in \mathcal{T}_b^c \) be an object, and let \( h : Y \rightarrow S \) be an element of \( \overline{\gamma}(Y)(S) = \text{Hom}(Y, S) \). Because \( S \in \mathcal{T}_b^c \) we may choose an integer \( m > 0 \) with \( S \in \mathcal{T}^\leq_{-m+1} \). Now the object \( X \) belongs to \( \mathcal{T}_c^\leq \), hence there is a triangle \( E \xrightarrow{\alpha} X \rightarrow D \) with \( E \in \mathcal{T}_c^c \) and \( D \in \mathcal{T}^\leq_{-m} \). Because \( \overline{\gamma}(f) = \overline{\gamma}(g) \) the two composites

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & X \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{h} & S
\end{array}
\]

must be equal. Hence so are the longer composites

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & X \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{h} & S
\end{array}
\]

Therefore the map \( h(f - g) : X \rightarrow S \) must factor as \( X \rightarrow D \rightarrow S \), but as \( D \in \mathcal{T}^\leq_{-m} \) and \( S \in \mathcal{T}^\leq_{-m+1} \) the map \( D \rightarrow S \) must vanish. Therefore \( hf = hg \). Since this is true for every \( h \) we have \( \overline{\gamma}(f) = \overline{\gamma}(g) \).

Next suppose \( \overline{\gamma}(f) = \overline{\gamma}(g) \), let \( E \in \mathcal{T}_c^c \) be an object, and let \( \alpha : E \rightarrow X \) be an element in \( \overline{\gamma}(X)(E) = \text{Hom}(E, X) \). By [3, Lemma 2.8] we may choose an integer \( m > 0 \) with \( \text{Hom}(E, \mathcal{T}^\leq_{-m}) = 0 \). By Lemma 2.9 the map \( Y \rightarrow Y^\geq_{-m+1} \) factors as \( Y \xrightarrow{h} S \rightarrow Y^\geq_{-m+1} \) with \( S \in \mathcal{T}_b^c \). Now the two composites

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Y & \xrightarrow{h} & S
\end{array}
\]

are equal because \( \overline{\gamma}(f) = \overline{\gamma}(g) \), hence the longer composites

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & X \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{h} & S
\end{array}
\]

must also be equal. In other words: the map \( Y \rightarrow Y^\geq_{-m+1} \) annihilates the map \( (f - g)\alpha : E \rightarrow Y \), and hence \( (f - g)\alpha \) must factor as \( E \rightarrow Y^\leq_{-m} \rightarrow Y \). But \( \text{Hom}(E, \mathcal{T}^\leq_{-m}) = 0 \), and we deduce that \( f\alpha = g\alpha \). As this is true for every \( \alpha \) we have \( \overline{\gamma}(f) = \overline{\gamma}(g) \).

Lemma 3.7. Let \( R \) be a noetherian ring. Let the conventions be as in Notation 3.4, except that weak approximability is no longer enough—for this Lemma we assume \( \mathcal{T} \) approximable. Suppose also that there is an object \( G \in \mathcal{T}_b^c \) and an integer \( N \) with \( \mathcal{T} = \langle G \rangle_N \). Assume further that there is a compact generator \( G' \in \mathcal{T} \) with \( \text{Hom}(G', \Sigma^i G') \) a finite \( R \)-module for every \( i \in \mathbb{Z} \).
Suppose we are given an $R$–linear cohomological functor $H : \mathcal{T}_c^b \rightarrow R\text{-Mod}$, as well as an object $Y \in \mathcal{T}_c^c$ such that $H$ is a direct summand of $\overline{\gamma}(Y)$. Then there exists an object $Y' \in \mathcal{T}_c^c$ and an isomorphism $H \cong \overline{\gamma}(Y')$.

Proof. We are given that $H$ is a direct summand of $\overline{\gamma}(Y)$, and the composite $\overline{\gamma}(Y) \rightarrow H \rightarrow \overline{\gamma}(Y)$ is an idempotent endomorphism $\varphi : \overline{\gamma}(Y) \rightarrow \overline{\gamma}(Y)$. By Proposition 3.3 there is a morphism $f : Y \rightarrow Y$ in the category $\mathcal{T}_c^c$ with $\overline{\gamma}(f) = \varphi$. Because $\overline{\gamma}(f)$ is idempotent we have that $\overline{\gamma}(f) = \overline{\gamma}(f^2)$, and Lemma 3.5 informs us that $\overline{\gamma}(f) = \overline{\gamma}(f^2)$. Therefore $\overline{\gamma}(f)$ is idempotent, and corresponds to the projection to a direct summand of $\overline{\gamma}(Y)$. The finiteness hypotheses on $\text{Hom}(G', \Sigma^i G')$ guarantee that $\overline{\gamma}(Y)$ is a locally finite cohomological functor, hence so is any direct summand, and by [6, Theorem 7.20] there exists an object $Y' \in \mathcal{T}_c^c$ and morphisms $Y \xrightarrow{\alpha} Y' \xrightarrow{\beta} Y$ with $\overline{\gamma}(f) = \overline{\gamma}(\beta \alpha)$ and $\overline{\gamma}(\alpha \beta) = \text{id}_{\overline{\gamma}(Y')} = \overline{\gamma}(\text{id}_{Y'})$.

Lemma 3.6 informs us that $\overline{\gamma}(f) = \overline{\gamma}(\beta \alpha)$ and $\overline{\gamma}(\alpha \beta) = \overline{\gamma}(\text{id}_{Y'})$. Thus $\overline{\gamma}(f)$ factors as

$$
\overline{\gamma}(Y) \xrightarrow{\overline{\gamma}(\alpha)} \overline{\gamma}(Y') \xrightarrow{\overline{\gamma}(\beta)} \overline{\gamma}(Y)
$$

while

$$
\overline{\gamma}(Y') \xrightarrow{\overline{\gamma}(\beta)} \overline{\gamma}(Y) \xrightarrow{\overline{\gamma}(\alpha)} \overline{\gamma}(Y')
$$

composes to the identity. The current lemma follows. \hfill \square

4. Proof of the main theorems

The proof of the main theorems will be by applying to the current situation the lemmas of [6, Section 6]. More precisely

Notation 4.1. With the conventions of Notation 3.4, assume given an object $G \in \mathcal{T}_c^b$ and an integer $N > 0$ such that $\mathcal{T} = (G)_N^c$. Assume also that we have fixed a $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, with $\mathcal{T}^{\geq 0}$ closed under coproducts. Let $\mathcal{A}$ be the heart of the $t$–structure, and $\mathcal{H} : \mathcal{T} \rightarrow \mathcal{A}$ the standard homological functor.

The lemmas of [6, Section 6] will be applied to the category $\mathcal{T}^{\text{op}}$. The subcategory $\mathcal{S}$ of [6, Notation 6.1] will be $\mathcal{S} = [\mathcal{T}_c^b]^{\text{op}}$.

The next definition and lemma are similar to what works in [6, Section 7]. The reader might wish to compare the definition below, and the lemma that follows, with [6, Definition 7.3 and Lemma 7.5].

Definition 4.2. A powerful $(G)_n$–approximating sequence is an inverse system $\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E_1 \rightarrow E_0$ in $\mathcal{T}$, so that

(i) Each $E_m$ belongs to $(G)_n$.

(ii) The map $\mathcal{H}^i(E_{m+1}) \rightarrow \mathcal{H}^i(E_m)$ is an isomorphism whenever $i \geq -m$.

Suppose we are also given an object $F \in \mathcal{T}$, together with

(iii) A map from $F$ to the approximating system $E_*$.
(iv) The map in (iii) is such that \( \mathcal{H}^i(F) \to \mathcal{H}^i(E_m) \) is an isomorphism whenever \( i \geq -m \).

Then we declare \( E_* \) to be a powerful \( \langle G \rangle_n \)-approximating system for \( F \).

**Lemma 4.3.** With the conventions of Definition 4.2 we have

(i) Given an object \( F \in \mathcal{T} \) and a powerful \( \langle G \rangle_n \)-approximating system \( E_* \) for \( F \), then the (non-canonical) map \( F \to \text{Holim} E_i \) is an isomorphism.

(ii) Any \( \langle G \rangle_n \)-powerful approximating system \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \) has a subsequence which is a powerful \( \langle G \rangle_n \)-approximating system of the homotopy limit \( F = \text{Holim} E_i \). Moreover, \( F \) belongs to \( \mathcal{T}_c^- \).

**Proof.** Assertion (i) is contained in Proposition 3.2 and the “moreover” part of (ii) is contained in Lemma 3.3.

Let \( L > 0 \) be the integer of Lemma 3.3. Then Lemma 3.1 says that, for any \( \langle G \rangle_n \)-powerful approximating system \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \) and with \( F = \text{Holim} E_m \), the maps \( \mathcal{H}^i(F) \to \mathcal{H}^i(E_{m+L}) \) are isomorphisms whenever \( i \geq -m \). In other words, the subsequence \( \cdots \to E_{3+L} \to E_{2+L} \to E_{1+L} \to E_L \) is a powerful approximating sequence for \( F \). This proves the first half of (ii). \( \square \)

**Remark 4.4.** Let us now explain how to specialize [6 Lemma 6.5] to the framework of this section. Suppose we are given an object \( \hat{B} \in \mathcal{T}_c^- \) and a powerful \( \langle G \rangle_n \)-approximating system \( \mathcal{B}_* \) for \( \hat{B} \). Lemma 2.11 informs us that the natural map \( \text{colim} \tilde{y}(\mathcal{B}_i) \to \tilde{y}(\hat{B}) \) is an isomorphism. Thus

(i) A powerful \( \langle G \rangle_n \)-approximating system \( \mathcal{B}_* \) for \( \hat{B} \) is an approximating system for \( \tilde{y}(\hat{B}) \) in the sense of [6 Definition 4.1]. Moreover: Lemma 4.3(ii) tells us that the map \( \hat{B} \to \text{Holim} \mathcal{B}_m \) is an isomorphism, hence a powerful \( \langle G \rangle_n \)-approximating system for \( \hat{B} \), as in Definition 4.2, is also an approximating system for \( \hat{B} \) as in [6 Remark 4.6].

Note also that in Lemma 4.3(ii) we learned that any powerful \( \langle G \rangle_n \)-approximating system \( \mathcal{B}_* \) has a subsequence which is a powerful \( \langle G \rangle_n \)-approximating system of \( \text{Holim} \mathcal{B}_i \).

Next assume we are given

(ii) A morphism \( \hat{\beta} : \hat{B} \to \hat{C} \) in the category \( \mathcal{T}_c^- \).

(iii) Two integers \( n \) and \( n' \), as well as a powerful \( \langle G \rangle_n \)-approximating system \( \mathcal{B}_* \) for \( \hat{B} \) and a powerful \( \langle G \rangle_{n'} \)-approximating system \( \mathcal{C}_* \) for \( \hat{C} \).

The dual of [6 Lemma 4.5] allows us to choose a subsequence of \( \mathcal{B}'_* \subset \mathcal{B}_* \) and a map of sequences \( \hat{\beta}_* : \mathcal{B}'_* \to \mathcal{C}_* \) compatible with \( \hat{\beta} : \hat{B} \to \hat{C} \). A subsequence of a powerful \( \langle G \rangle_n \)-approximating sequence is clearly a powerful \( \langle G \rangle_{n'} \)-approximating sequence, hence \( \mathcal{B}'_* \) is a powerful \( \langle G \rangle_n \)-approximating sequence for \( \hat{B} \). Now as in [6 Lemma 6.5] we extend \( \hat{\beta}_* : \mathcal{B}'_* \to \mathcal{C}_* \) to a sequence of triangles, in particular for each \( m > 0 \) this gives
a morphism of triangles

\[
\begin{array}{c}
\Sigma^{-1} \mathcal{B}'_{m+1} \xrightarrow{\Sigma^{-1} \beta_{m+1}} \Sigma^{-1} \mathcal{C}_{m+1} \\
\Sigma^{-1} \mathcal{B}'_{m} \xrightarrow{\Sigma^{-1} \beta_{m}} \Sigma^{-1} \mathcal{C}_{m} \\
\Sigma^{-1} \mathcal{B}'_{m} \xrightarrow{\Sigma^{-1} \gamma_{m}} \Sigma^{-1} \mathcal{C}_{m} \\
\mathcal{A}_{m+1} \xrightarrow{\alpha_{m+1}} \mathcal{B}'_{m+1} \\
\mathcal{A}_{m} \xrightarrow{\alpha_{m}} \mathcal{B}'_{m} \\
\mathcal{E}_{m+1} \xrightarrow{\beta_{m+1}} \mathcal{E}_{m+1} \\
\mathcal{E}_{m} \xrightarrow{\beta_{m}} \mathcal{E}_{m}
\end{array}
\]

Applying the functor \( \mathcal{H}^i \) with \( i \geq -m + 1 \) yields a commutative diagram in the heart of \( \mathcal{T} \) where the rows are exact, and where vertical maps away from the middle are isomorphisms. By the 5-lemma the middle vertical map, i.e. the map \( \mathcal{H}^i(\mathfrak{A}_{m+1}) \to \mathcal{H}^i(\mathfrak{A}_m) \), must also be an isomorphism when \( i \geq -m + 1 \). We conclude that a subsequence of \( \mathfrak{A}_* \) is a powerful \( \langle G \rangle_{n'+n} \)-approximating system. Put \( \hat{A} = \text{Holim} \mathfrak{A}_* \). By Lemma 2.11(ii) the object \( \hat{A} \) belongs to \( \mathcal{T}_c \), and Lemma 2.11 coupled with Proposition 3.5 guarantee that the weak triangle \( A \xrightarrow{u} B \xrightarrow{v} C \to \Sigma A \) of [6, Lemma 6.5] is isomorphic to the image under \( \hat{y} \) of a weak triangle \( \hat{A} \xrightarrow{\hat{u}} \hat{B} \xrightarrow{\hat{v}} \hat{C} \to \Sigma \hat{A} \) in the category \( \mathcal{T}_c \).

**Lemma 4.5.** Suppose \( H \) is a locally finite \( \langle G \rangle_1 \)-homological functor. Then there is a surjection \( \hat{y}(F)|_{\langle G \rangle_1} \to H \), where \( F \in \mathcal{T}_c^+ \) has a powerful \( \langle G \rangle_1 \)-approximating system \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \). Moreover: the system may be chosen so that the maps \( E_{m+1} \to E_m \) are split epimorphisms.

**Proof.** We have that \( H(\Sigma^i G) \) is a finite \( R \)-module for every \( i \in \mathbb{Z} \), and vanishes when \( i \ll 0 \). For each \( i \) with \( H(\Sigma^i G) \neq 0 \) choose a finite number of generators \( \{ f_{ij}, j \in J_i \} \) for the \( R \)-module \( H(\Sigma^i G) \). By Yoneda every \( f_{ij} \in H(\Sigma^i G) \) corresponds to a morphism \( \varphi_{ij} : \hat{y}(\Sigma^i G) \to H \). Let \( F \) be defined by

\[
F = \prod_{i \in \mathbb{Z}, j \in J_i} \Sigma^i G \cong \prod_{i \in \mathbb{Z}, j \in J_i} \Sigma^i G
\]

where the isomorphism of the coproduct and the product is by Lemma 2.5. Let the morphism \( \varphi : \hat{y}(F) \to H \) be given by

\[
\hat{y}(F)|_{\langle G \rangle_1} \xrightarrow{\bigoplus_{i \in \mathbb{Z}, j \in J_i} \hat{y}(\Sigma^i G)} \hat{y}(F) \xrightarrow{(\varphi_{ij})} H
\]

where \( (\varphi_{ij}) \) stands for the row matrix with entries \( \varphi_{ij} \); on the \( i, j \) summand the map is \( \varphi_{ij} \). Finally: because \( G \in \mathcal{T}_c^b \) there is an integer \( B > 0 \) with \( \Sigma^B G \in \mathcal{T}^{\leq -1} \). For \( m > 0 \) we define

\[
E_m = \bigoplus_{i \leq m+B} \bigoplus_{j \in J_i} \Sigma^i G
\]

The sum is finite by hypothesis, making \( E_m \) an object of \( \langle G \rangle_1 \). The obvious map \( E_{m+1} \to E_m \) is a split epimorphism, and in the decomposition \( F \cong E_m \oplus \hat{F} \) we have that \( \hat{F} \), being the coproduct of \( \Sigma^i G \) for \( i \geq m + B \), belongs to \( \mathcal{T}^{\leq -m-1} \). Therefore
the map \( \mathcal{H}^i(F) \to \mathcal{H}^i(E_m) \) is an isomorphism if \( i \geq -m \), making the \( E_* \) a powerful \( (G)_1 \)-approximating system for \( F \).

And now the time has come to prove the main results.

**Theorem 4.6.** Let \( R \) be a noetherian, commutative ring. Let \( \mathcal{T} \) be an \( R \)-linear, approximable triangulated category, and assume there is a compact generator \( G' \in \mathcal{T} \) such that \( \text{Hom}(G', \Sigma^i G') \) is a finite \( R \)-module for all \( i \in \mathbb{Z} \). Let \( \mathcal{T}^- \) and \( \mathcal{T}_c \) be the ones corresponding to the preferred equivalence class of \( t \)-structures, and assume there is an object \( G \in \mathcal{T}_c \) and an integer \( N > 0 \) with \( \mathcal{T} = \langle G \rangle_N \).

Then the functor \( \tilde{y}: \mathcal{T}^- \to \text{Hom}[\mathcal{T}_c, R\text{-Mod}] \) if full, and the essential image consists of the locally finite homological functors.

**Proof.** The fact that the functor \( \tilde{y} \) is full was proved in Proposition 8.3 and the fact that its image is contained in the locally finite homological functors was shown in Lemma 2.1. What needs proof is that every locally finite homological functor can be realized as \( \tilde{y}(F) \) for some \( F \in \mathcal{T}^- \).

Suppose therefore that \( H \) is a locally finite \( \mathcal{T}_c \)-homological functor. Therefore \( H|_{(G)_1} \) is a locally finite \( (G)_1 \)-homological functor, and Lemma 4.5 produces an object \( F_1 \in \mathcal{T}_c \), with a powerful \( (G)_1 \)-approximating system, and an epimorphism \( \tilde{y}(F_1)|_{(G)_1} \to H|_{(G)_1} \). From [6, Corollary 4.4] it follows that we may lift the natural transformation to all of \( \mathcal{T}_c \); there is a natural transformation \( \varphi_1 : \tilde{y}(F_1) \to H \) so that \( \varphi_1|_{(G)_1} : \tilde{y}(F_1)|_{(G)_1} \to H|_{(G)_1} \) is surjective.

Next we proceed inductively. Suppose we have constructed \( F_n \in \mathcal{T}_c \), with a powerful \( (G)_n \)-approximating system, and a natural transformation \( \varphi_n : \tilde{y}(F_n) \to H \), and assume that \( \varphi_n|_{(G)_1} : \tilde{y}(F_n)|_{(G)_1} \to H|_{(G)_1} \) is surjective. Since both \( H|_{(G)_1} \) and \( \tilde{y}(F_n)|_{(G)_1} \) are locally finite and the ring \( R \) is noetherian, the kernel of \( \varphi_n|_{(G)_1} \) is also locally finite. Lemma 2.1 permits us to find a surjection to the kernel: there is an object \( F' \in \mathcal{T}_c \), with a powerful \( (G)_1 \)-approximating system \( \cdots \to E_3 \to E_2 \to E_1 \to E_0 \) in which all the connecting maps \( E_{m+1} \to E_m \) are split epimorphisms, and an exact sequence \( \tilde{y}(F')|_{(G)_1} \to \tilde{y}(F_n)|_{(G)_1} \to H|_{(G)_1} \). Now [6, Corollary 4.4] allows us to lift the map to \( \mathcal{T}_c \). We deduce:

(i) There is a morphism \( \alpha : F_n \to F' \) so that the sequence below is exact

\[
\begin{array}{ccc}
\tilde{y}(F')|_{(G)_1} & \xrightarrow{\tilde{y}(\alpha)|_{(G)_1}} & \tilde{y}(F_n)|_{(G)_1} \\
\alpha|_{(G)_1} & \xrightarrow{\varphi_n|_{(G)_1}} & H|_{(G)_1}
\end{array}
\]

Forget for a second the exactness; the vanishing of the composite in (i), coupled with [6, Lemma 6.5], allows us to construct

(ii) With the notation of [6, Definition 6.2], and working in the category \( \mathcal{T}^\text{op} \) and with \( S = \langle \mathcal{T}_c \rangle^\text{op} \), there is an object \( F_{n+1} \in \mathcal{T}_c \) with a powerful \( (G)'_{n+1} \)-approximating system, a weak triangle in \( \mathcal{T}_c \) of the form \( F_{n+1} \xrightarrow{\beta_n} F_n \xrightarrow{\alpha} F' \to \Sigma F_{n+1} \), and a
morphism $\varphi_{n+1} : \tilde{y}(F_{n+1}) \to H$ so that the following triangle commutes

$$\begin{array}{ccc}
\tilde{y}(F_n) & \xrightarrow{\tilde{y}(\alpha)} & \tilde{y}(F_{n+1}) \\
\downarrow{\varphi_n} & & \downarrow{\varphi_{n+1}} \\
H & & H
\end{array}$$

This inductively constructs an inverse sequence in $\mathcal{T}_c$ of the form $\cdots \to F_4 \overset{\beta_3}{\to} F_3 \overset{\beta_2}{\to} F_2 \overset{\beta_1}{\to} F_1$, as well as compatible maps $\varphi_n : \tilde{y}(F_n) \to H$.

Now the map $\varphi_1 : \tilde{y}(F_1) \to H$ restricts to an epimorphism on $\langle G \rangle_1$ by construction, and the exactness of the sequence in $\langle G \rangle_1$ coupled with [6, Lemma 6.6] informs us, by induction, that $\varphi_n : \tilde{y}(F_n) \to H$ restricts to an epimorphism on $\langle G \rangle_n$. By Proposition 2.8 we have $\mathcal{T}_c^b = \langle G \rangle_N$, hence $\varphi_N$ is an epimorphism.

Now apply [6, Lemma 6.7] to the diagram

$$\begin{array}{cccc}
\tilde{y}(F_N) & \xrightarrow{\tilde{y}(\alpha)} & \tilde{y}(F_{N+n}) & \xrightarrow{\tilde{y}(\beta_{N+n})} \\
\downarrow{\varphi_{n+N+1}} & & \downarrow{\varphi_{n+N+1}} & & \downarrow{\varphi_{n+N+1}} \\
H & & H & & H
\end{array}$$

Induction on $n \geq 0$ teaches us that the map $\tilde{y}(F_N)|_{\langle G \rangle_n} \to \tilde{y}(F_{N+n})|_{\langle G \rangle_n}$ annihilates the kernel of $\tilde{y}(F_N)|_{\langle G \rangle_n} \to H|_{\langle G \rangle_n}$. If we put $n = N$ and remember that $\langle G \rangle_N = \mathcal{T}_c^b$, we have that the map $\tilde{y}(F_N) \to \tilde{y}(F_{2N})$ and the epimorphism $\tilde{y}(F_N) \to H$ have the same kernel. Thus $\tilde{y}(F_N) \to \tilde{y}(F_{2N}) \to H$ factors as $\tilde{y}(F_N) \to H \to \tilde{y}(F_{2N}) \to H$, making $H$ a direct summand of $\tilde{y}(F_{2N})$. Lemma 3.7 produces an object $Y \in \mathcal{T}_c^b$ with $H = \tilde{y}(Y)$. \hfill $\square$

**Theorem 4.7.** Let the notation be as in Theorem 4.6. The essential image under $\tilde{y}$ of the subcategory $\mathcal{T}_c^b \subset \mathcal{T}_c$ is precisely the finite homological functors. Moreover the restriction of $\tilde{y}$ to $\mathcal{T}_c^b$ is fully faithful: it induces an equivalence of $[\mathcal{T}_c^{\text{op}}]$ with the category of finite homological functors $\mathcal{T}_c^b \to R\text{-mod}$.

The “moreover” part can even be strengthened as follows: for any pair of objects $a \in \mathcal{T}_c^b$ and $b \in \mathcal{T}_c^b$ the natural map is an isomorphism

$$\text{Hom}(a, b) \cong \text{Hom}[\tilde{y}(b), \tilde{y}(a)]$$

**Proof.** The fact that $\tilde{y}(A)$ is finite when $A \in \mathcal{T}_c^b$ follows from Lemma 2.7—the essential image under $\tilde{y}$ of the subcategory $\mathcal{T}_c^b \subset \mathcal{T}_c$ is contained in the finite functors.

Now suppose $H : \mathcal{T}_c^b \to R\text{-mod}$ is a finite homological functor. Since finite homological functors are locally finite Theorem 4.6 tells us that there exists an object $A \in \mathcal{T}_c^b$...
and an isomorphism $\tilde{\gamma}(A) \cong H$. It suffices to prove that $A \in T_c$. But the finiteness tells us that, for the object $G \in T_c^b$ with $T = \langle G \rangle_N$ of the hypotheses of Theorem 4.6, we must have that $H^i(G) \cong \text{Hom}(A, \Sigma^i G) = 0$ for $i \gg 0$. By Proposition 2.6 $A$ must be compact.

It remains to prove the full faithfulness, or rather the strengthened version. We already know the surjectivity of the map

$$\text{Hom}(a, b) \longrightarrow \text{Hom}\left[\tilde{\gamma}(b), \tilde{\gamma}(a)\right],$$

that was part of Theorem 4.6. Suppose therefore that we have two morphisms $f, g : a \longrightarrow b$ with $\tilde{\gamma}(f) = \tilde{\gamma}(g)$. Then Lemma 3.6 informs us that $\gamma(f) = \gamma(g)$, and as $\gamma(a) = \text{Hom}(a, -)$ is representable we deduce from Yoneda that $f = g$. □

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