CONFORMAL EMBEDDINGS VIA HEAT KERNEL

Zhitong Su

Abstract
For any n-dimensional compact Riemannian Manifold $M$ with smooth metric $g$, by employing the heat kernel embedding introduced by Bérard-Besson-Gallot’94, we intrinsically construct a canonical family of conformal embeddings $C_{t,k}: M \rightarrow \mathbb{R}^q(t)$, with $t > 0$ sufficiently small, $q(t) \gg t^{-\frac{n}{2}}$, and $k$ as a function of $O(t^l)$ in proper sense. Our approach involves finding all these canonical conformal embeddings, which shows the distinctions from the isometric embeddings introduced by Wang-Zhu’15.

0 Introduction

Let $(M, g)$ be an n-dimensional compact Riemannian manifold, the following classical problem, called the isometric embedding problem is studied in differential geometry. Does there exist an embedding $u: M \rightarrow \mathbb{R}^N$ for some $N$ such that the induced metric is $g$? In 1956, J.Nash famously proved in [6] that there exists a $C^s$-class isometric embedding for $g \in C^s$, with $s \geq 3$ or $s = \infty$. Furthermore, for any compact $n$-dimensional Riemannian manifold, the optimal value of $N$ he found was $N = \frac{3}{2}n(n+1)+4n$.

In [6], Nash developed an iteration nowadays known the Nash-Moser theorem, to address the problem of losing differentiability when taking the inverse of a differential operator. Decades later, M.Günther (1989, [2]) significantly simplified Nash’s proof by using a different iteration, which avoids the loss of differentiability. This allows one to simply use the usual Banach fixed point theorem to conclude the proof. His approach can also be found in the proceedings [3] of ICM 1990 Kyoto, where Günther gave an invited talk on his proof.

Nash and Günther’s construction of the isometric embedding is highly flexible. By employing this method, any embedding $u: M \rightarrow \mathbb{R}^N$ such that the induced metric is less than or equal to $g$ can be used as a start to produce the isometric embedding. This great flexibility, on the other hand, often result in the isometric embeddings being noncanonical.

Contrastingly, in 1994, Bérard, Besson, and Gallot [1] constructed an ‘asymptotically isometric’ embedding using the heat kernel of the manifold. This embedding, referred to as normalized heat kernel embedding, maps a compact Riemannian manifold $M$ into $l^2$, the space of square summable series, and is constructed as follows:

$$
\Psi_t: x \mapsto \sqrt{2(4\pi)}^n t^{n+2} \cdot \{e^{-\hat{\lambda}_j t/2} \phi_j(x)\}_{j \geq 1}, \text{ for } t > 0,
$$

where $\hat{\lambda}_j$ is the $j$th eigenvalue of the Laplacian $\Delta = \text{tr}_g \nabla^2$ of $(M, g)$, here $\nabla$ is the Levi-Civita connection, and $\{\phi_j\}_{j \geq 0}$ is the $L^2$ orthonormal eigenbasis of $\Delta$. It is worth noting that the embedding $\Psi_t$ is canonical due to the fact that it is constructed by the heat kernel and therefore the spectre geometry of $(M, g)$ uniquely determines it. A more precise formula that justifies the above statement is the following, indicating that $\Psi_t$ tends to an isometry in the following sense:

$$
\Psi_t^* g_{\text{can}} = g + \frac{t}{3} \left( \frac{1}{2} \text{Scal}_g \cdot g - \text{Ric}_g \right) + O(t^2),
$$

where the $g_{\text{can}}$ is the standard Euclidean metric in $l^2$, $\text{Scal}_g$ is the scalar curvature of $(M, g)$, $\text{Ric}_g$ is the Ricci curvature of $(M, g)$, and the convergence is in the $C^r$ sense for any $r > 0$. 
In light of the fact that Nash and Günther’s isometric methods ([6],[2]) being flexible but far from being canonical, and Bérard, Besson, and Gallot’s heat kernel embedding ([1]) being canonical but not yet exactly isometric, Wang and Zhu (2015, [8]) embarked on a study aimed at finding a canonical isometric embedding of a compact Riemannian manifold into $\mathbb{R}^q$ for $q \gg 1$ by using the heat kernel of the compact manifold. Their approach begins by first modifying the heat kernel embedding $\Psi_t$ in [1] to a better approximation with an error term of $O(t^l)$ for any $l \geq 2$, and continues by perturbing such an ‘almost isometric’ embedding to an isometric one. In other words, by using the $\Psi_t$ in [1], they find a canonical family of ‘almost’ isometric embeddings $\tilde{\Psi}_t : M \to \ell^2$ such that

$$\tilde{\Psi}_t^* g_{\text{can}} = g + O(t^l)$$

in the $C^r$ sense for $r > 0$, $l \geq 1$. Subsequently, they find a unique $C^{\infty, \alpha}$ isometric embedding $I_t : M \to \mathbb{R}^{q(t)}$ such that

$$\|I_t - \tilde{\Psi}_t\|_{C^{\infty, \alpha}(M)} = O(t^{l+\frac{1}{2}\frac{s+\alpha}{2}}),$$

where $q(t) \geq t^{-\frac{1}{2}r}$, $s + \alpha < l + \frac{1}{2}$, $s \geq 2$, $t \in (0, t_0)$ for some $t_0 > 0$ depending on $s, \alpha, l$, and $g$. Additionally, we note Bérard, Besson, and Gallot’s heat kernel embeddings can be used in many other ways, for further references, historical context, and other uses of this heat kernel embedding, see [7].

From the point of view of Kähler geometry and complex geometry, such a canonical isometric embedding is good, but one may seek more embeddings of this canonical type. In analogy to the Kodaira embedding (see e.g. [4]) in Kähler geometry that preserves the holomorphic structure, we have managed to find a family of canonical embeddings of compact Riemannian manifolds that preserve the conformal structure. Indeed, an isometric one is already a conformal one, but by starting with the heat kernel ‘almost’ isometric embedding and looking into Günther’s method, we have shown that by requesting the result map to be conformal and keeping each step done canonically, one can find a family of canonical conformal embeddings of $(M, g)$ into Euclidean space, with the isometric embedding as one special case among them. Throughout this paper, conformal embeddings are refer to the embeddings that are conformal, see Definition 1.3. In the following we present the main theorems, by fixing the constant $\rho > 0$, and $0 < \alpha < 1$, and using Einstein summation notation throughout this paper.

**Theorem 0.1.** Let $(M, g)$ be a smooth $n$-dimensional compact Riemannian manifold without boundary, $g$ is the smooth Riemannian metric of $M$. Then for any integer $l \geq 1$, there exists a family of canonical almost conformal embeddings $\Psi_{t,g(t),\eta_i} : M \to \ell^2$ about $t$ and dependent on $\eta_1 \leq i \leq l - 1 \in C^{\infty}(M, g)$ uniquely, such that

$$\Psi_{t,g(t),\eta_i}^* g_{\text{can}} - \frac{\text{tr}_g \Psi_{t,g(t),\eta_i}^* g_{\text{can}}}{n} g = O(t^l)$$

as $t \to 0_+$, where the convergence is in $C^r(M, g)$ sense for any $r \geq 0$.

Note here that the $\Psi_{t,g(t),\eta_i}$ is an almost conformal embeddings since the error term $O(t^l)$ is small when $t \to 0_+$. And it is a canonical embedding in the sense that it is determined by the geometry of $(M, g)$.

As we will see in Proposition 1.5, here $\Psi_{t,g(t),h_i}$ depends on $\eta_i$ in the following way: for given $\eta_i \in C^{\infty}(M, g)$, and for each $1 \leq i \leq l - 1$, there exists a $h_i \in \Gamma(\text{Sym}^{2}(T^* M))$ uniquely determined by $\eta_i$, such that $\frac{\text{tr}_{g(t)} h_i}{n} = \eta_i$. And $\Psi_{t,g(t),\eta_i}$ is given by the heat kernel embedding of $(M, g(t))$, where $g(t) := g + \sum_{i=1}^{l-1} h_i(t) t^i$, $g(t) := g + \sum_{i=1}^{l-1} h_i(t)$.

Given this $\Psi_{t,g(t),\eta_i}$, as quoted verbatim from [8], we have the following definition:

**Definition 0.2.** (Truncated embedding) Let

$$\Pi_q : \ell^2 \to \mathbb{R}^q$$

be the projection of $\ell^2$ to the first $q$ components. To get a finite-dimensional almost conformal embedding, we introduce the truncated embedding

$$\Psi_{q(t)} := \Pi_q \circ \Psi_{t,g(t),\eta_i} : (M, g) \to \ell^2 \to \Pi_q \mathbb{R}^q(t).$$
And the following is the second part of our main theorem:

**Theorem 0.3.** Under the proceeding assumption, we have:

For any integer $s \geq 2$ satisfying $s + \alpha < l + \frac{1}{2}$, there exists a constant $t_0 > 0$ depending on $s, \alpha, l, g$ and $\rho \in \mathbb{R}_{>0}$, such that for any $0 < t < t_0$, there exists a family $(C_t, K)$ of conformal embeddings $C_{t,k}$, $K := \{k| k \in C^{s,\alpha}(M, g), \|k\|_{C^{s,\alpha}(M)} = O(t^{l})\}$, such that for any $k \in K$, each truncated embedding $\Psi_{t,n}$ can be perturbed to a unique $C^{s,\alpha}$ conformal embedding

$$C_{t,k} : M \to \mathbb{R}^{q(t)},$$

where the dimension $q(t) \geq t^{-\frac{s}{2} - 1}$.

Moreover, the resulting conformal map satisfies the estimate:

$$\|C_{t,k} - \Psi_{t,g(t),n}\|_{C^{s,\alpha}} = O(t^{l + \frac{1}{2} - s - \alpha}).$$

In this context, we still start with the ‘almost’ isometric heat kernel embedding $\Psi_t$ in [1], which encapsulates only the intrinsic information of $(M, g)$. Consequently, our resulting $C_{t,k}$ differs from $\Psi_t$ by at most $O(t)$ in the $C^{s,\alpha}$ sense. Some might argue that this conformal result closely resembles an isometric one. To answer that, we want to emphasize that one can indeed multiply a nonzero constant to $C_{t,k}$ to obtain another conformal embedding. More generally, one can compose it with any Möbius transformation of $\mathbb{R}^N$, the one-point compactification of $\mathbb{R}^N$, to achieve another conformal embedding. However, it is important to note that such operations are by no means due to the intrinsic property of $(M, g)$; they lack canonicity. Hence, to maintain the canonical nature of our result, we shall work with the heat kernel embedding $\Psi_t$.

It is also worth to note that finding the optimal dimension $q(t)$ is not our goal here, as lower dimensions can result in less canonical embeddings.

The main techniques in this article can be described as a process of ‘recovering the trace’, and can be outlined as follows. Notice that a map $u : M \to \mathbb{R}^N$ being free (see Definition 2.1) is a strong condition, which will ensure the existence and uniqueness of the solution to the equation (see Lemma 2.2):

$$P(u) \cdot v = [f \ h]^T.$$ (0.5)

However, to address our conformal question, we need to look into the equation

$$P_c(u) \cdot v = [f \ h - \frac{\nabla h}{n} g]^T,$$ (0.6)

where $P_c(u)$ is obtained from $P(u)$ by subtracting its own trace of the second derivative part (see Definition 2.1). The challenge lies in the fact that $P_c(u)$ is not of full rank. To overcome this difficulty, we point out that the solution of (0.5) is one special solution of (0.6). And precisely describing the kernel of $P_c(u)$ allows us to obtain all the solutions in the following manner: ‘solutions of (0.6)’ = ‘one special solution’ + ‘kernel of $P_c(u)$’(see Remark 2.4). In this context the ‘one special solution’ corresponds to the isometric embeddings attained by Wang and Zhu in [8], hence we will closely follow their construction. Additionally the ‘kernel of $P_c(u)$’ corresponds to the trace that are recovered. This coincides with the general perspective that isometric embeddings are special cases of conformal ones.

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1 Heat kernel embeddings and modifications to almost conformal maps

Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold with smooth metric \(g\). Denote the eigenvalues of the Laplacian of \((M, g)\) as \(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\), and let \(\{\phi_j\}_{j \geq 0} \subset C^\infty(M)\) be a corresponding \(L^2\)-orthonormal basis of the real eigenfunctions. In other words, this means \(\Delta_g \phi_j = -\lambda_j \phi_j\), and \(\int_M \phi_i \phi_j \text{d} \text{vol}_g = \delta_{ij}\), for \(i, j \geq 0\). The heat kernel of \((M, g)\) is:

\[ H(x, y, t) = \sum_{s=1}^{\infty} e^{-\lambda_s t/2} \phi_s(x) \phi_s(y), \]

where \(x, y \in M\), and \(t > 0\). Recall the definition in \([1]\) regarding almost isometric heat kernel embeddings into \(\ell^2\):

**Definition 1.1.** We call the family of maps

\[ \Phi_t : M \rightarrow \ell^2 \quad x \mapsto \{e^{-\lambda_j t/2} \phi_j(x)\}_{j \geq 1} \quad \text{for } t > 0 \]

the heat kernel embeddings, and call \(\Psi_t = \sqrt{2(4\pi)^{n/4} t^{-n/2}} \cdot \Phi_t\) the normalized heat kernel embeddings.

And the main theorem in \([1]\) can be phrased as the following:

**Theorem 1.2.** For \(t \rightarrow 0_+\), there is an expansion

\[ \Psi_t^* g_{\text{can}} = g + \sum_{i=1}^{l} t^i A_i(g) + O(t^{i+1}), \]

with

\[ A_1 = \frac{1}{3} \left( \frac{1}{2} S_g \cdot g - \text{Ric}_g \right), \]  

(1.1)

where the \(g_{\text{can}}\) is the metric of \(\ell^2\), \(S_g\) is the Scalar curvature, the \(A_i\)'s are universal polynomials of the covariant differentiations of the metric \(g\) and its curvature tensors up to order \(2i\), and the convergence is in the sense of \(C^r\) for any \(r \geq 0\).

As in Proposition 5 of \([8]\), we can perturb the family of maps to the family that’s in the form of an almost conformal map. The idea, compared to the isometric case, is to require it to be isometric to some conformal metric \(\lambda^2 g\), instead of the metric \(g\) itself. Here is the definition of conformal map upon which we base our understanding:

**Definition 1.3.** Assume \(f\) is an embedding from \((M, g_M)\) to \((N, g_N)\), which both are Riemannian manifolds, and \(M\) is of dimension \(m\). \(f\) is a **conformal map** from \(M\) to \(f(M)\), iff

\[ f^* g_N - \frac{\text{tr}_{g_M} f^* g_N}{m} g_M = 0, \]  

(1.2)

where the integer \(m\) is the dimension of manifold \(M\). Note this is equivalent to define a conformal map as the \(f\) satisfying \(f^* g_N = \lambda^2 g_M\) for some \(\lambda \in \mathbb{R}\). An embedding that is a conformal map is called a conformal embedding.

With a slight abuse of language, we also refer to a smooth map or a smooth immersion \(f\) as a **conformal map** as long as it satisfies the equation (1.2). Much attention will be focused on seeking immersions that satisfy (1.2), and ultimately showing the immersions are embeddings. Therefore, such an abuse won’t affect our results.
Remark 1.4. For a 2-tensor $\alpha \in \Gamma(\text{Sym}^{\otimes 2}(T^*M))$, the commonly encountered equation in this paper is the following:

$$\alpha - \frac{\text{tr}_g \alpha}{n} g = 0,$$

where we define the left-hand side as $\text{tr}_g^+(\alpha) := \alpha - \frac{\text{tr}_g \alpha}{n} g$, referred to as the conformal linear operator or traceless linear operator. The use of the notation ‘perpendicular’ is justified, as it satisfies $\langle \alpha - \frac{\text{tr}_g \alpha}{n} g, \frac{\text{tr}_g \alpha}{n} g \rangle = 0$, where we employ the inner product of 2-tensors induced by $g$.

Proposition 1.5. For any $l > 1$, $\eta_i \in C^\infty(M, g)$, $1 \leq i \leq l - 1$, there are $h_i \in \Gamma(\text{Sym}^{\otimes 2}(T^*M))$ uniquely determined by $\eta_i$ satisfying $\frac{1}{n}\text{tr}_g h_i = \eta_i$, such that for the family of metrics

$$g(s) = g + \sum_{i=1}^{l-1} s^i h_i,$$

the induced metric from the heat kernel embeddings with $\Psi_{t, g(s)} : (M, g(s)) \to \ell^2$ satisfies the estimate

$$||\Psi_{t, g(s)}^* g_{\text{can}} - \frac{\text{tr}_g \Psi_{t, g(s)}^* t g_{\text{can}}}{n} g||_{C^r(M, g)} \leq C(g, l, r) t^l,$$

for any $r > 0$, where the constant $C(g, l, r)$ depends only on $l, r$ and the geometry of $(M, g)$.

This proposition is, as we mentioned, a conformal version of Proposition 5 in [8]. Hence, the proof also uses the same method, with more attention to the trace part.

Proof. Like in [8], we assume the family of metrics $g(s)$ can be expressed as:

$$g(s) = g + \sum_{i=1}^{l-1} h_i s^i$$

with $h_i \in \Gamma(\text{Sym}^{\otimes 2}(T^*M)).$

Our objective is to determine the proper $h_i$’s. Let $G(s, t) := \Psi_{t, g(s)}^* g_{\text{can}} = g(s) + t A_1(g(s)) + t^2 A_2(g(s)) + \cdots$, then after letting $s = t \to 0$, and define $A_{i,j}(h_1, \cdots, h_j) := \frac{\partial^j}{\partial s^j} A_i(g(s))$, we have

$$\Psi_{t, g(s)}^* g_{\text{can}} - \frac{\text{tr}_g \Psi_{t, g(s)}^* t g_{\text{can}}}{n} g = G(s, t) - \frac{\text{tr}_g G(s, t)}{n} g$$

$$= g - \frac{\text{tr}_g g}{n} \cdot g + \frac{\text{tr}_g h_1}{n} \cdot g + t^2 \left( \frac{\text{tr}_g h_2}{n} \cdot g + \frac{\text{tr}_g A_1(g)}{n} \cdot g + t \frac{\text{tr}_g A_{1,1}(h_1)}{n} \cdot g + \frac{\text{tr}_g A_{1,1}(h_1)}{n} \right)$$

$$+ t^2 \left( \frac{\text{tr}_g A_2(g)}{n} \cdot g + \frac{\text{tr}_g A_{2,1}(h_1)}{n} \cdot g + t \frac{\text{tr}_g A_{2,1}(h_1)}{n} \cdot g + \frac{\text{tr}_g A_{2,1}(h_1)}{n} \right)$$

$$+ \cdots + O(t^l).$$

Then we need to find proper $h_i$ such that for $1 \leq k \leq l - 1$, all the terms of $t^k$ vanish:

$$h_1 - \frac{\text{tr}_g h_1}{n} \cdot g = -A_1(g) + \frac{\text{tr}_g A_1(g)}{n} \cdot g,$$

$$h_2 - \frac{\text{tr}_g h_2}{n} \cdot g = -A_2(g) + \frac{\text{tr}_g A_2(g)}{n} \cdot g - A_{1,1}(h_1) + \frac{\text{tr}_g A_{1,1}(h_1)}{n} \cdot g,$$

$$\cdots = \cdots.$$  

\[1.4\]
Here, the k-th equation depends on \( h_1, \ldots, h_{k-1} \); thus, the \( h_i \) will be found inductively. We shall observe the first equation of (1.5), using the conformal operator \( \text{tr}_g^+ \) defined in 1.4:

\[
\text{tr}_g^+(h_1) = -\text{tr}_g^+(A_1(g)).
\]  

(1.6)

To solve this equation, we can explicitly express all the solutions of \( h_1 \). First, given the geometric meaning of \( \text{tr}_g^+ \) as taking the traceless part of a symmetric 2-tensor, we point out that at each point \( x \in M, \text{Ker}(\text{tr}_g^+) \subset \text{Sym}^2 T_x^* M \) is of 1 dimension, which corresponds to the trace part of a 2-tensor, and such 1 dimension is generated by \( g(x) \). The method we use to find the solution reflects the discussion we have had on recovering the trace in the Introduction.

Notice that \( \text{Ker}(\text{tr}_g^+) \) is generated by \( g \), and that \( h_1 = -A_1(g) \) is one of the solutions. Therefore, all the \( h_1 \) satisfying (1.6) have to be in the following form

\[
h_1 = -A_1(g) + \frac{\text{tr}_g A_1(g)}{n}g + \eta_1 \cdot g.
\]

(1.7)

Here \( \eta_1 \in C^\infty(M, \mathbb{R}) \) is a globally smooth function. The expression of \( h_1 \) in (1.7) as a solution implies that \( \eta_1 = \frac{\text{tr}_g h_1}{n} \).

Hence, for each \( h_1 \) in the form of \(-A_1(g) + \eta_1 \cdot g\), after fixing one \( \eta_1 \in C^\infty(M, g) \), the equation

\[
h_2 = \frac{\text{tr}_g h_1}{n} \cdot g = -A_2(g) + \frac{\text{tr}_g A_2(g)}{n}g - A_{1,1}(h_1) + \frac{\text{tr}_g A_{1,1}(h_1)}{n}g
\]

can be solved for \( h_2 \). As in the \( h_1 \) case, all the \( h_2 \) have to be in the following form:

\[
h_2 = -A_2(g) - A_{1,1}(h_1) + \frac{\text{tr}_g (A_2(g) + A_{1,1}(h_1))}{n}g + \eta_2 \cdot g, \quad \eta_2 \in C^\infty(M, \mathbb{R}),
\]

(1.8)

where the \( \eta_2 \) satisfies \( \eta_2 = \frac{\text{tr}_g h_2}{n} \).

Now we can have an explicit expression of \( h_i \), \( 1 \leq i \leq l - 1 \) inductively. After determining \( h_i \), the way of \( g(t) = g + \sum_{i=1}^{l-1} h_i t^i \) approaches to \( g \) is determined. Then, \( \Psi_{t,g(t)} \) will satisfy:

\[
(\Psi_{t,g(t)})^* g_{\text{can}} - \frac{\text{tr}_g (\Psi_{t,g(t)})^* g_{\text{can}}}{n} \cdot g = O(t^i)
\]

in the \( C^r \) sense for any \( r \geq 0 \). □

**Definition 1.6.** (Canonical almost conformal embedding). Given \( \eta_i \in C^\infty(M, g) \), we call the \( \Psi_{t,g(t),\eta_i} : M \to \ell^2 \) constructed above the (modified) canonical almost conformal embedding, and denote as

\[
\tilde{\Psi}_{t,\eta_i} := \Psi_{t,g(t),\eta_i}.
\]

Please note, here and throughout this paper, \( \eta_i \in C^\infty(M, g) \) always serve as a sequence of smooth functions to be given. Then, to obtain the embedding into \( \mathbb{R}^q \), we truncate off the terms beyond the first \( q \) ones, in the following sense:

**Definition 1.7.** (Truncated embedding) Let

\[
\Pi_q : \ell^2 \to \mathbb{R}^q
\]

be the projection of \( \ell^2 \) to the first \( q \) components. To get a finite-dimensional almost conformal embedding, we introduce the truncated embedding

\[
\Psi_{t,\eta_i}^q(t) := \Pi_q \circ \tilde{\Psi}_{t,\eta_i} : (M, g) \to \ell^2 \to \mathbb{R}^q(t).
\]

The following Proposition estimates the truncated tail approaches to 0 exponentially, which is due to [8, Proposition 9].
Proposition 1.8 ([8, Proposition 9]). Consider a compact family \( \{g_s\}_{s \in \text{Com}} \) of smooth metrics defined on a compact \( n \)-dimensional Riemannian manifold \( M \), where \( g_s \) smoothly depends on the parameter \( s \), and \( \text{Com} \) denotes this compact family of metrics. For any point \( x \) in \( M \), let \( \{x^k\}_{1 \leq k \leq n} \) represent the normal coordinates with respect to the metric \( g_s \). Then, for any multi-indices \( \alpha \) and \( \beta \), and a \( q(t) \) satisfying \( q(t) \geq t^{-\left(\frac{n}{2} + \rho\right)} \),

\[
\sum_{j \geq q(t)+1} e^{-\lambda_j t} D^\alpha \phi_j D^\beta \phi_j \leq C e^{\left( -t^{-\frac{n}{2}} \right)}
\]

for all \( l \geq 1 \). The convergence is uniform across all points \( x \in M \) and all metrics \( s \in \text{Com} \) in the \( C^r \)-norm for any \( r \geq 0 \).

Notice that as long as \( \rho > 0 \), the right-hand side of the inequality approaches to 0, and then the inequality holds. This explains the role of the constant \( \rho \) in the main theorem.

The following corollary applies the former discussion to our conformal case.

Corollary 1.9. Given any \( l \geq 1 \), for \( q = q(t) \geq C t^{-\left(\frac{n}{2} + \rho\right)} \), the truncated modified heat kernel embedding \( \Psi^{(t)}_{l,\eta} : (M, g) \to \mathbb{R}^{(t)} \) still satisfies the asymptotic formula

\[
(\Psi^{(t)}_{l,\eta})^* g_{\text{can}} = \frac{\text{tr}_g(\Psi^{(t)}_{l,\eta})^* g_{\text{can}}}{n} \cdot g + O(t^l)
\]

in the \( C^r \) sense for any \( r \geq 0 \).

Proof. One can easily use the estimate in Proposition 1.8 to prove this, noting that the truncated heat kernel embedding \( \Psi^{(t)}_{l,\eta} \) corresponds to some metric \( g_s \), and the fact \( e^{-t^{-\frac{n}{2}}} < t^l \) for any \( l > 1 \) as \( t \to 0^+ \).

\[\square\]

2 Günther’s iteration and the modification to conformal case

We may start by stating some conventions. Assume \( u \in C^\infty(M, \mathbb{R}^N) \) is a smooth embedding, and let the metric \( g \in C^{2,\alpha}(M, \text{Sym}^2 T^*M) \). Eventually, as we will see in the later sections, \( u \) is meant to represent \( \Psi_t \) and \( \Psi^{(t)}_{l,\eta} \).

Let’s begin with the assumption that we are given \( u \) that is already almost a conformal map, which satisfies

\[
\nabla u \cdot \nabla u - \frac{\text{tr}_g(\nabla u \cdot \nabla u)}{n} = -f + \frac{\text{tr}_g f}{n} g,
\]

where \( f \) is a ‘small’ symmetric 2-tensors, and \( \nabla \) is the Levi-Civita connection of \((M, g)\). To attain our goal of this paper, it will suffice to find a map \( u + v : M \to \mathbb{R}^N \) that solves the equation

\[
\nabla (u + v) \cdot \nabla (u + v) - \frac{\text{tr}_g(\nabla (u + v) \cdot \nabla (u + v))}{n} g = 0.
\]

Then, after comparing the former two equations, our goal becomes to find a \( v \in C^{s,\alpha}(M, \mathbb{R}^N), s \geq 2 \), satisfying the conformal embedding equation:

\[
\nabla u \cdot \nabla v - \frac{\text{tr}_g(\nabla u \cdot \nabla v)}{n} g + \nabla v \cdot \nabla u - \frac{\text{tr}_g(\nabla v \cdot \nabla u)}{n} g + \nabla v \cdot \nabla v - \frac{\text{tr}_g(\nabla v \cdot \nabla v)}{n} g = f - \frac{\text{tr}_g f}{n} g.
\]
2.1 Free mappings.

In the following paragraphs, we would like to state the facts about free mapping and apply it to our conformal case. Throughout this paper, when we mention $C^{\infty, \alpha}$, $s \geq 2$, we will fix $0 < \alpha < 1$. Also, since finding the optimistic $N$ is not the goal of this paper, we could take $N$ to always be greater than or equal to $n + \frac{1}{4}n(n + 1)$.

**Definition 2.1.** A $C^{\infty}$ embedding $u : M \to \mathbb{R}^N$ is free if, for every fixed $x \in M$, the $n + \frac{1}{4}n(n + 1)$ many vectors in $\mathbb{R}^N$:

$$\partial_i u(x), \partial_i \partial_j u(x), 1 \leq i, j \leq n$$

form a $\min(N, n + \frac{1}{2}n(n + 1))$-dimensional linear subspace of $\mathbb{R}^N$. Note that this definition is independent of the choice of coordinates. Denote such a subspace as $\text{Span}\{\partial_i u(x), \partial_i \partial_j u(x)\}$. In this paper, we would also always denote a global linear operator $P(u)$ as follows:

$$P(u) := \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix},$$

and another global operator $P_c(u)$ that will be useful for conformal case:

$$P_c(u) := \begin{bmatrix} \nabla u \\ \nabla \nabla u - \frac{u_a \nabla u}{n} g \end{bmatrix}.$$

To clarify the definition, we want to point out that it is true that for any point $x \in M$, $P$ could be thought of as a map $C^{\infty}(M) \to T_x^* M \oplus \text{Sym}^2 T_x^* M$. However, in our discussion, $P$ is always applied to a fixed free mapping $u$, which makes it a linear operator. The definition of $P(u)$ can be viewed as follows:

$$C^{\infty}(M, \mathbb{R}^N) \xrightarrow{P|_u} \mathbb{R}^N \otimes (T_x^* M \oplus \text{Sym}^2 T_x^* M) \xrightarrow{\text{in normal coordinates}} \mathcal{L}(\mathbb{R}^N, \mathbb{R}^{n + \frac{3}{2}(n + 1)})$$

$$u \quad \mapsto \quad \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix} \quad \xrightarrow{\text{in normal coordinates}} \quad \begin{bmatrix} \nabla_i u_l \\ \nabla_j \nabla_k u_l \end{bmatrix} \quad \text{for } 1 \leq i, j, k \leq n, \quad 1 \leq l \leq N.$$

For $u = (u_1, \ldots, u_N) \in C^{\infty}(M, \mathbb{R}^N)$, which is a free embedding, at each point $x \in M$ and with respect to the normal coordinates $\{x^i\}_{1 \leq i \leq n}$ centered at $x$, we denote $P(u)(x)$ as a $\frac{n(n + 3)}{2} \times N$ matrix in the following ordering of index, for $i \neq j$, $1 \leq i, j, k \leq n$:

$$P(u)(x) = \begin{bmatrix} \nabla_i u_1 & \nabla_i u_2 & \ldots & \nabla_i u_N \\ \nabla_j \nabla_j u_1 & \nabla_j \nabla_j u_2 & \ldots & \nabla_j \nabla_j u_N \\ \nabla_k \nabla_k u_1 & \nabla_k \nabla_k u_2 & \ldots & \nabla_k \nabla_k u_N \end{bmatrix}(x).$$

Also, notice that although the explicit expression depends on the choice of local coordinates, the rank of $P(u)(x)$ is independent of local coordinates given $u$ is a free mapping. When $u$ is a free mapping and $N \geq \frac{n(n + 3)}{2}$, $P(u)(x)$ has a rank of $\frac{n(n + 3)}{2}$.

Similarly, we have the expression of $P_c(u)(x)$, for $i \neq j, 1 \leq i, j, k, p \leq n$:

$$P_c(u)(x) = \begin{bmatrix} \nabla_i u_1 \\ \nabla_i \nabla_j u_1 \\ \nabla_k \nabla_k u_1 - \frac{1}{p} \sum_p \nabla_p \nabla_p u_1 \end{bmatrix}.$$  \hspace{1cm} (2.4)

Notice that here and in most discussions in this paper, it could be useful for us to make the argument pointwise, which allows us to pick the the normal coordinates such that the Christoffel symbol vanishes at the point $x$. 

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Lemma 2.2. Let $s \geq 2$, for a free embedding $u \in C^\infty(M, \mathbb{R}^N)$, and for $f \in C^{s,\alpha}(M, T^*M)$, $h \in C^{s,\alpha}(M, \text{Sym}^{\otimes 2}T^*M)$, there exists a unique $v \in C^{s,\alpha}(M, \mathbb{R}^N)$ such that

$$P(u) \cdot v = \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix} v = \begin{bmatrix} f \\ h \end{bmatrix}, \text{ and } v(x) \perp \text{Ker} P(u)(x).$$

(2.5)

Proof. By the definition of $u$ being a free mapping of $u$, for each point $x \in M$, the $P(u)(x)$ is of full rank, hence $P(u)(x) : \mathbb{R}^N \rightarrow T^*_x M \oplus \text{Sym}^{\otimes 2}T^*_xM$ is surjective, therefore the solution $v \in \mathbb{R}^N$ exists. After forcing $v(x) \perp \text{Ker} P(u)(x)$, such $v(x)$ is unique.

Having the pointwise solution for $v$, to solve the equation (2.5) globally, we notice that $P(u)$, $h$, $f$ are globally defined, and due to the differentiability of $h$ and $f$, we can obtain the globally defined $v \in C^{s,\alpha}(M, \mathbb{R}^N)$.

Proposition 2.3. Let $s \geq 2$, for a free embedding $u \in C^\infty(M, \mathbb{R}^N)$, and for $f \in C^{s,\alpha}(M, T^*M)$, $h \in C^{s,\alpha}(M, \text{Sym}^{\otimes 2}T^*M)$, then there exists a unique $v' \in C^{s,\alpha}(M, \mathbb{R}^N)$ satisfying the following equation:

$$P_c(u) \cdot v' = \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix} v' = \begin{bmatrix} f \\ h - \frac{tr_g h}{n} g \end{bmatrix}, \text{ and } v'(x) \perp \text{Ker} P_c(u)(x).$$

(2.6)

Moreover on each point $x \in M$, we have $\text{rank}(P_c(u)(x)) = \text{dim}(\text{Im} P(u)(x)) = n + \frac{n(n+1)}{2} - 1$, and $\text{dim}(\text{Ker} P_c(u)(x)) = \text{dim}(\text{Ker} P(u)(x)) + 1$.

Proof. We will first show the existence and uniqueness of the solution on a fixed point $x \in M$. We notice that, without restricting $v'(x) \perp \text{Ker} P_c(u)(x)$, given any $f$ and $h$, the unique solution $v_0$ such that

$$P(u) \cdot v_0 = \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix} v_0 = \begin{bmatrix} f \\ h \end{bmatrix}, \text{ and } v_0(x) \perp \text{Ker} P(u)(x)$$

(2.7)

also solves

$$P_c(u) \cdot v_0 = \begin{bmatrix} \nabla u \\ \nabla \nabla u \end{bmatrix} \begin{bmatrix} f \\ h - \frac{tr_g h}{n} g \end{bmatrix},$$

and this is due to the linearity of trace operator. More precisely, the equation $(\nabla_i \nabla_i u, v) = h_{ii}$ from (2.7) would imply

$$\langle \nabla_i \nabla_i u - \frac{tr_g(\nabla \nabla u)}{n} g_{ii}, v \rangle = h_{ii} - \frac{tr_g h}{n} g_{ii}.$$

Hence, by linear algebra, if a $v'$ solves $P_c(u) \cdot v' = \begin{bmatrix} f, h - \frac{tr_g h}{n} g \end{bmatrix}^T$, $v'(x) \perp \text{Ker} P(u)(x)$ for given $f$ and $h$, then such a $v'$ has to be in the form $v' = v_0 + w$, where $v_0$ is the unique vector attained by (2.7), and $w$ is an arbitrary vector in $\text{Ker} P_c(u)(x)$. Such a solution space will definitely have intersection with $\text{Ker} P_c(u)^T(x)$, hence we showed the existence of the solution. After forcing $v'(x) \perp \text{Ker} P(u)(x)$, we get the uniqueness of the solution. By the same reason as in former lemma, the global solution $v' \in C^{s,\alpha}(M, \mathbb{R}^N)$ also exists and is unique.

Notice that since $\text{dim}(\text{Im} P(u)(x)) = n + \frac{n(n+1)}{2}$, then to show $\text{dim}(\text{Ker} P_c(u)(x)) = \text{dim}(\text{Ker} P(u)(x)) + 1$ can be reduced to show $\text{dim}(\text{Im} P_c(u)(x)) = n + \frac{n(n+1)}{2} - 1$. Hence, it will suffice to show that $\text{Im} P(u)(x) = \text{Span}\{\nabla_i u(x), \nabla_i \nabla_j u(x) - \sum_{p=1}^n (\nabla_p \nabla_p u(x)) \frac{n}{n} \}$ is of dimension $\frac{n(n+3)}{2} - 1$ as a subspace of $\mathbb{R}^N$, where each $\nabla_i u(x), \nabla_i \nabla_j u(x)$ is viewed as a vector in $\mathbb{R}^N$, and the $i, j$ are with respect to a normal coordinates around the point $x$.

Recall the expression of $P_c(u)(x)$ we have in (2.4), we can see each of the elements is a linear combination of the others when $i = j$:

$$\nabla_i \nabla_i u(x) - \frac{1}{n} \sum_{p=1}^n (\nabla_p \nabla_p u(x)) = \sum_{k \neq i} \nabla_k \nabla_i u(x) - \frac{1}{n} \sum_{p=1}^n (\nabla_p \nabla_p u(x)),$$
hence we can see \( \dim(\text{Span}\{\nabla_i u(x), \nabla_j u(x) - \frac{\sum_{p=1}^{n} (\nabla_p \nabla_p u(x))}{n}\}) \leq \frac{n(n+3)}{2} - 1 \). Also notice that by direct summing with a one dimensional space, we have:

\[
\text{Span}\{\nabla_i u(x), \nabla_j u(x) - \frac{\sum_{p=1}^{n} (\nabla_p \nabla_p u(x))}{n}\} \oplus \{ \frac{\sum_{p=1}^{n} (\nabla_p \nabla_p u(x))}{n}\} \supseteq \text{Span}\{\nabla_i u(x), \nabla_j u(x)\},
\]

therefore the space \( \text{Im} P_c(u)(x) \) is of \( n \) in \( T^*_x M \oplus \text{Sym}^2 T^*_x M \).

**Remark 2.4.** The former proposition states the decomposition that

\[
\text{Ker} P_c(u)(x) = \text{Ker} P(u)(x) \oplus \{ w(x) \}
\]

for each point \( x \in M \) where \( w \in \mathbb{R}^n \). Moreover, we can describe the generator \( w(x) \) precisely here. Let \( w \in C^{s,\alpha}(M, \mathbb{R}^n) \) be the unique one such that

\[
P(u) \cdot w = \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad \text{and } w(x) \perp \text{Ker} P(u)(x).
\]

By Lemma 2.2, such a \( w \) exists and is unique, then we have that \( P_c(u) \cdot w = 0 \), hence \( w \in \text{Ker} P_c(u)(x) \). By the definition of \( w \), especially that \( w(x) \perp \text{Ker} P(u)(x) \), we see this \( w \) is exactly the one in the decomposition.

Hence for all the \( v' \in C^{s,\alpha}(M, \mathbb{R}^n) \) that satisfies \( v'(x) \perp \text{Ker} P(u)(x) \) and solves \( P_c(u) \cdot v' = [f - h - \frac{\nabla h}{n}g] \), it has to be in the form that

\[
v' = v_0 + k \cdot w,
\]

where \( v_0 \) is the unique solution of \( P(u) \cdot v_0 = [f - h] \), \( v'(x) \in \text{Ker} P(u)(x) \), \( w \) is the unique vector defined above, and \( k \in C^{s,\alpha}(M, \mathbb{R}) \).

Following this, even though it is not closely related to our later goal, if restrict \( v'(x) \perp \text{Ker} P_c(u)(x) \), we can precisely describe the coefficient \( k \) for that unique \( v' \). Since \( v' \perp \text{Ker} P_c(u)(x) \), \( w \in \text{Ker} P_c(u)(x) \), we have

\[
v' = v_0 - \frac{\langle v_0, w \rangle_{\mathbb{R}^n}}{\langle w, w \rangle_{\mathbb{R}^n}} w.
\]

Notice that \( \langle w, w \rangle_{\mathbb{R}^n} \neq 0 \) for any \( x \in M \), so the expression is a well defined global one. Also, notice that even though \( \langle h - \frac{\nabla h}{n}g, g \rangle = 0 \), the inner product \( \langle v_1, w \rangle_{\mathbb{R}^n} \) isn’t always equal to zero.

### 2.2 Günther’s lemma in conformal case.

In this subsection, we would follow Günther [2] to perform a detailed computation, summarizing the results as a lemma at the end.

Let \( \nabla \) be the Levi-Civita connection of \((M, g)\). The Laplacian we use in the following is the connection Laplacian, \( \Delta := \text{tr} \nabla^2 \), applicable to all the functions and tensors of at least \( C^2 \) smooth. The Ricci curvature is defined as \( R^i_k := R_{ikl}^l \) in my convention. Here, and throughout this paper, the well-known Einstein notation is applied. Given \( u \in C^\infty(M, \mathbb{R}^n) \) as a free embedding, and given the smooth metric \( g \). Let \( s \geq 2 \), our goal is to find \( v \in C^{s,\alpha}(M, \mathbb{R}^n) \) satisfying \((2.3)\). To achieve this, we must examine the left-hand side of \((2.3)\):

\[
\nabla u \cdot \nabla v - \frac{\text{tr}_g(\nabla u \cdot \nabla v)}{n} g + \nabla v \cdot \nabla u - \frac{\text{tr}_g(\nabla v \cdot \nabla u)}{n} g + \nabla v \cdot \nabla v - \frac{\text{tr}_g(\nabla v \cdot \nabla v)}{n} g.
\]

To begin with, we need to first look into the term with trace:

\[
\nabla u \cdot \nabla v + \nabla v \cdot \nabla u + \nabla v \cdot \nabla v.
\]

In this definition, the eigenvalues are the \( \lambda_i \)'s satisfying \( \Delta f + \lambda f = 0 \), hence \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \). Then, for some positive constant number \( e \), the \( \Delta - e \) is an isomorphism between \( C^{s,\alpha} \) and \( C^{s-2,\alpha} \) of functions
and tensors of various sizes on compact $M$, for $s \geq 2$. Additionally, it has a unique inverse denoted as $(\Delta - e)^{-1}$. Note that for $s \geq 2$ and $v \in C^{s,\alpha}$, the terms in $(2.10)$ are 2-tensors of $C^{s-1,\alpha}$. We can apply $\Delta - e$ to both sides of the 2-tensors and set $(2.10)$ equal to $f$:

\[
(\Delta - e)(\nabla u \cdot \nabla v) + (\Delta - e)(\nabla v \cdot \nabla v) + (\Delta - e)(\nabla u \cdot \nabla v) = (\Delta - e)f,
\]

where the third derivatives of $v$ are to be considered as distributions, and this won’t affect the following computation.

We carefully treat each term using a local coordinates, denoted as $\{x^i\}_{1 \leq i \leq n}$, which are not necessarily normal coordinates. The first term to compute is the quadratic term $(\Delta - e)(\nabla_i v \cdot \nabla_j v)$ involving $v$:

\[
(\Delta - e)(\nabla_i v \cdot \nabla_j v)dx^i \otimes dx^j \\
= \nabla_i(\nabla_i v \cdot \nabla_j v dx^j) - e(\nabla_i v \cdot \nabla_j v dx^i \otimes dx^j) \\
= \nabla_i \nabla_i v \cdot \nabla_j v dx^j + \nabla_i(\nabla_i v dx^j) \cdot \nabla_j v dx^j \\
+ \nabla_i \nabla_i v \cdot \nabla_i(\nabla_j v dx^j) + \nabla_i v \cdot \nabla_i(\nabla_j v dx^j) - e(\nabla_i v \cdot \nabla_j v dx^i \otimes dx^j) \\
= \nabla_i(\nabla_i v dx^j) \cdot (\nabla_j v dx^j) + (\nabla_i v dx^i) \cdot \nabla_j v dx^j \\
+ 2\nabla_i(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) - e(\nabla_i v \cdot \nabla_j v dx^i \otimes dx^j) \\
= \nabla_i(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) + R^k_{ij} \nabla_i v \cdot \nabla_j v dx^i \otimes dx^j \\
+ \nabla_i \nabla_i v \cdot \nabla_i(\nabla_j v dx^j) + 2\nabla_i(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) - e(\nabla_i v \cdot \nabla_j v dx^i \otimes dx^j) \\
= \nabla_i(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) + R^k_{ij} \nabla_i v \cdot \nabla_j v dx^i \otimes dx^j \\
- \nabla_j(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) - e(\nabla_i v \cdot \nabla_j v dx^i \otimes dx^j) \\
= 2L_{ij}(v, v)dx^i \otimes dx^j + \nabla_i(\nabla_j v dx^j) \cdot \nabla_j(\nabla_i v dx^i) dx^j + \nabla_j(\nabla_i v dx^i) \cdot \nabla_i(\nabla_j v dx^j) dx^j.
\]

For brevity, we denoted
\[
L_{ij}(v, v)dx^i \otimes dx^j := \nabla_i(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) - \nabla_j(\nabla_i v dx^j) \cdot \nabla_i(\nabla_j v dx^j) dx^i + \frac{1}{2} e(\nabla_i v \cdot \nabla_j v) + \frac{1}{2}(R^k_{ij} \nabla_i v + R^k_{ji} \nabla_j v) \cdot \nabla_k v dx^i \otimes dx^j.
\]

For the other terms involving $u$ and $v$, we get the following by switching the Laplacian and the covariant derivative:

\[
(\Delta - e)(\nabla_i u dx^j \cdot \nabla_j v dx^j) \\
= (\Delta - e)(\nabla_j(\nabla_i u dx^j \cdot v) dx^j) - (\Delta - e)(\nabla_j(\nabla_i u dx^j) dx^j \cdot v) \\
= \nabla_j((\Delta - e)(\nabla_i u dx^j \cdot v) dx^j) + \{2R^k_{ij} \nabla_k(\nabla_i u \cdot v) + R^k_{ij}(\nabla_i u \cdot v) (-\Gamma^m_{kn}) \} + \nabla^k(\nabla_i u \cdot v) \\
+ g^{kl} R^m_{ijm} \nabla_n(\nabla_i u \cdot v) (-\Gamma^m_{ik}) + R^m \nabla_n(\nabla_i u \cdot v) + R^m \nabla_n(\nabla_i u \cdot v) (-\Gamma^m_{ik}) dx^k \otimes dx^j \\
- (\Delta - e)(\nabla_i \nabla_j u \cdot v dx^i \otimes dx^j) - (\Delta - e)(\nabla_i(\nabla_j u \cdot v dx^i) dx^j).
\]

Similar computation for the other one,

\[
(\Delta - e)(\nabla_i u dx^j \cdot \nabla_i v dx^j) \\
= \nabla_i((\Delta - e)(\nabla_i u dx^j \cdot v) dx^j) + \{2R^k_{ij} \nabla_k(\nabla_i u \cdot v) + R^k_{ij}(\nabla_i u \cdot v) (-\Gamma^m_{kn}) \} + \nabla^k(\nabla_i u \cdot v) \\
+ g^{kl} R^m_{ijm} \nabla_n(\nabla_i u \cdot v) (-\Gamma^m_{ik}) + R^m \nabla_n(\nabla_i u \cdot v) + R^m \nabla_n(\nabla_i u \cdot v) (-\Gamma^m_{ik}) dx^k \otimes dx^j \\
- (\Delta - e)(\nabla_i \nabla_i u \cdot v dx^i \otimes dx^j) - (\Delta - e)(\nabla_i(\nabla_i u \cdot v dx^i) dx^j).
\]

We could denote the following notion of $r_{ij}^n$, for $w = w_n dx^n \in C^{s,\alpha}(M, T^*_x M)$:

\[
r_{ij}^n w_n dx^i \otimes dx^j := \{2R^k_{ij} \nabla_k w_n + R^k_{ij} m w_n (-\Gamma^m_{kn}) \} + \nabla^k R^k_{ij} w_n \\
+ g^{kl} R^m_{ijm} w_n (-\Gamma^m_{ik}) + R^m w_n (-\Gamma^m_{ik}) dx^k \otimes dx^j.
\]

(2.13)
Combining everything, we get:

\[(\Delta - \varepsilon)(\nabla_i u \cdot \nabla_j v \, dx^i \otimes dx^j + \nabla_j u \cdot \nabla_i v \, dx^i \otimes dx^j + \nabla_i v \cdot \nabla_j v \, dx^i \otimes dx^j)\]

\[= \nabla_j ((\Delta - \varepsilon)(\nabla_i u \, dx^i \cdot v + \nabla_i v \cdot \Delta v) \, dx^i + \nabla_i ((\Delta - \varepsilon)(\nabla_j u \, dx^j \cdot v) + \Delta v \cdot \nabla_j v \, dx^j) \, dx^i + \{2L_{ij}(v, v) + r^n_{ij}(\nabla_n v \cdot v + R^n_{ij} \nabla_n (\nabla_j u \cdot v) + r^n_{ij} (\nabla_n u \cdot v) + R^n_{ij} \nabla_n (\nabla_i u \cdot v)\} \, dx^i \otimes dx^j\]

\[= 2(\Delta - \varepsilon)(\nabla_i (\nabla_j u \, dx^i) \, dx^j \cdot v).\]

Noticed that the last few terms only involves \(\nabla_n u \cdot v\), Günther made the observation that after forcing the \(\nabla_i u \cdot v \, dx^i\) equal to \(-(\Delta - \varepsilon)^{-1}\{\Delta v \cdot \nabla_i v \, dx^i\}\), the equation

\[f = \nabla v \cdot \nabla u + \nabla u \cdot \nabla v + \nabla v \cdot \nabla v \tag{2.14}\]

can be reduced to the following:

\[\nabla_i (\nabla_j u \, dx^j) \, dx^i \cdot v = \frac{1}{2}(\Delta - \varepsilon)^{-1}((2L_{ij}(v, v) + (r^n_{ij} + r^n_{ji})) (\nabla_n u \cdot v) \]

\[+ R^n_{ij} \nabla_n (\nabla_i u \cdot v) + R^n_{ij} \nabla_n (\nabla_j u \cdot v) \} \, dx^i \otimes dx^j) - \frac{1}{2} f_{ij} \, dx^i \otimes dx^j\]

\[= \frac{1}{2}(\Delta - \varepsilon)^{-1}((2L_{ij}(v, v) + (r^n_{ij} + r^n_{ji})) (-\{\Delta v \cdot \nabla v\}) n\]

\[= R^n_{ij} \nabla_n (\nabla_i u \cdot v) + R^n_{ij} \nabla_n (\nabla - \varepsilon)^{-1} \{\Delta v \cdot \nabla v\}) j \, dx^i \otimes dx^j)\]

\[= - \frac{1}{2} f_{ij} \, dx^i \otimes dx^j.\]

Here, again, for any 1-form \(w, w_j\) is meant to be the coefficient of \(w\) with respect to \(dx^j\), i.e., \(w = w_j \, dx^j\).

Note that \(L_{ij}(v, v)\) is a quadratic form about \(v\). Then, for \(s \geq 2\), we define \(Q(v, v) \in C^{s, \alpha}(M, \mathbb{R}^N)\) as follows:

\[\nabla u \cdot Q(v, v) = (\Delta - \varepsilon)^{-1}\{\Delta v \cdot \nabla v\}\]

\[\nabla u \cdot Q(v, v) = \frac{1}{2}(\Delta - \varepsilon)^{-1}((2L_{ij}(v, v) + (r^n_{ij} + r^n_{ji})) (-\{\Delta v \cdot \nabla v\}) n\]

\[+ R^n_{ij} \nabla_n (\nabla_i u \cdot v) + R^n_{ij} \nabla_n (\nabla - \varepsilon)^{-1} \{\Delta v \cdot \nabla v\}) j \, dx^i \otimes dx^j).\]

By the freeness of \(u\), such a \(Q(v, v)\) exist and is unique if we require \(Q(v, v)(x) \perp \text{Ker} P(u)(x)\) for any \(x \in M\). Hence the conformal equation (2.3) is reduced to:

\[f - \frac{\operatorname{tr} f}{n} g = \nabla_j (\nabla_i u \, dx^i \cdot \{v - Q(v, v)\}) \, dx^j - \sum_{i=j=1}^{n} \nabla_j (\nabla_i (\nabla_j u \, dx^j \cdot \{v - Q(v, v)\}) \, dx^i)\]

\[+ \nabla_i (\nabla_j u \, dx^j \cdot \{v - Q(v, v)\}) \, dx^i - \sum_{i=j=1}^{n} (\nabla_i (\nabla_j u \, dx^j \cdot \{v - Q(v, v)\}) \, dx^i)\]

\[= 2\nabla_i (\nabla_j u \, dx^j \cdot \{v - Q(v, v)\}) \, dx^i - \sum_{i=j=1}^{n} (\nabla_i (\nabla_j u \, dx^j) \, dx^i \cdot \{v - Q(v, v)\})\]

As like in Remark 1.4, by denoting \(\operatorname{tr}^+_g u(v)\) as the following

\[\operatorname{tr}^+_g u(v) := \nabla (\nabla u \cdot v) - \frac{\operatorname{tr} g (\nabla u \cdot v)}{n} g + \nabla (v \cdot \nabla u) - \frac{\operatorname{tr} g (v \cdot \nabla u)}{n} g - 2\nabla \nabla u \cdot v + \frac{\operatorname{tr} g (\nabla_j u \cdot v)}{n} g \tag{2.16}\]

the equation can be simply written as:

\[\operatorname{tr}^+_g u(v - Q(v, v)) = f - \frac{\operatorname{tr} f}{n} g.\]

Now we are ready to state the conformal version of Günther’s lemma.
Lemma 2.5 ( Günther’s Lemma with conformal operator). Assume $u \in C^\infty(M, \mathbb{R}^N)$ a free embedding, then to solve the conformal embedding equation

$$\nabla u \cdot \nabla v - \frac{\nabla v}{n} g + \nabla v \cdot \nabla u = \frac{\nabla v}{n} g + \nabla v \cdot \nabla v - \frac{\nabla v}{n} g = f - \frac{\nabla f}{n} g$$

(2.17)

can be reduced to solve the following:

$$\nabla u \cdot v = (\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \},$$

$$\nabla \nabla u \cdot v = -\frac{1}{2}(\Delta - e)^{-1}([2L_{ij}(v,v) + (r_{ij} + r_{ji})((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_n

+ R_{ij}^n \nabla_n((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_i + R_{ji}^n \nabla_n((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_j]dx^i \otimes dx^j) + \frac{1}{2}f,$$

(2.18)

where $L_{ij}(v,v)$ is defined in (2.12), and $r_{ij}^n$ is defined in (2.13).

If we define a quadratic term $Q(v,v) \in C^{s,\alpha}_v(M, \mathbb{R}^N)$, $s \geq 2,$ and $Q(v,v)(x) \perp \text{Ker}P_c(u)(x)$ as follows:

$$\nabla u \cdot Q(v,v) = (\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \},$$

$$\nabla \nabla u \cdot Q(v,v) = -\frac{1}{2}(\Delta - e)^{-1}([2L_{ij}(v,v) + (r_{ij} + r_{ji})((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_n

+ R_{ij}^n \nabla_n((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_i + R_{ji}^n \nabla_n((\Delta - e)^{-1} \{ \nabla v \cdot \nabla v \})_j]dx^i \otimes dx^j).$$

(2.19)

Then the conformal embedding equation will be formulated simply as:

$$\text{tr}^+_u(v - Q(v,v)) = f - \frac{\nabla f}{n} g,$$

(2.20)

where $\text{tr}^+_u$ is defined in (2.16).

3 The properties of $P_c$, and its family of right inverses $E_c$

3.1 Singularity of $P_c(\Psi_t)P^T_c(\Psi_t)$.

In Proposition 3.1, we have seen that $P_c(\Psi_t)$ is not of full rank in normal coordinates at a point $x \in M$. Hence, naturally, $P_c(\Psi_t)P^T_c(\Psi_t)(x)$ is singular.

To provide another perspective, in this subsection, we will explicitly present the matrix expression of $P_c(\Psi_t)P^T_c(\Psi_t)$ and compute its rank. We shall begin by introducing the following linear algebra proposition.

**Proposition 3.1.** Let $\sigma \in (-\frac{1}{n-1}, 1)$. Then the $n \times n$ matrix

$$\Xi_n(\sigma) := [\theta_{ij}]_{1 \leq i,j \leq n}$$

(3.1)

with $\theta_{ii} = 1$ and $\theta_{ij} = \sigma$ $(i \neq j)$ is invertible. And the condition for $\sigma > -\frac{1}{n-1}$ is sharp, more precisely, $\Xi_n(-\frac{1}{n-1})$ is not invertible and of rank $n - 1$.

**Proof.** The invertibility of $\Xi_n(\sigma)$ when $-\frac{1}{n-1} < \sigma < 1$ is due to [8, Corollary 26]. We only need to verify the rank of $\Xi_n(-\frac{1}{n-1})$ is $n - 1$. Indeed,

$$(n - 1) \cdot \Xi_n(-\frac{1}{n-1}) = \begin{bmatrix}
(n - 1) & -1 & \cdots & -1 \\
-1 & (n - 1) & \cdots & -1 \\
\cdots & \cdots & \cdots & \cdots \\
-1 & -1 & \cdots & (n - 1)
\end{bmatrix} = nI_n - J_n,$$

we can easily see it is of rank $n - 1$. □
Proposition 3.2. Express $P_c(\Psi_t)$ with respect to normal coordinates in the neighbourhood of $x \in M$ in the following way:

$$P_c(\Psi_t)(x) = [\nabla_i \Psi_t(x) \nabla_j \Psi_t(x) \nabla_k \nabla_p \Psi_t(x) - \frac{\sum_p \nabla_p \nabla_p \Psi_t}{n}(x)]^T$$  \hspace{1cm} (3.2)

$i \neq j, 1 \leq i, j, k, p \leq n$, then the matrix can be expressed locally as the following when $t \to 0^+$:

$$P_c(\Psi_t)P_c^T(\Psi_t)(x) = \left[ I_n + O(t) \right] - \frac{1}{2t} \cdot \left( \left[ I_{2(n-1)} \frac{O(t)}{n} 0 \right] \left[ 3 (\frac{1}{3}) - \frac{n+2}{n} J_n \right] + O(t) \right).$$  \hspace{1cm} (3.3)

Moreover, the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix

$$\left[ I_{2(n-1)} \frac{O(t)}{n} 0 \right] \left[ 3 (\frac{1}{3}) - \frac{n+2}{n} J_n \right]$$  \hspace{1cm} (3.4)

is of rank $\frac{n(n+1)}{2} - 1$. Hence the matrix expression of $P_c(\Psi_t)P_c^T(\Psi_t)(x)$ is not invertible when $t \to 0^+$.

Proof. The expression (3.3) can be achieved by employing the formulas in [8, Proposition 21] and direct computation, which we omit here.

To see the rank of (3.4), notice that

$$3 (\frac{1}{3}) - \frac{n+2}{n} J_n = \frac{2n-2}{n} \cdot \frac{\Xi(-\frac{1}{n-1})}{n-1}. $$

By the sharpness of Proposition 3.1, we know $\Xi(-\frac{1}{n-1})$ is not invertible and hence the whole matrix is of rank $\frac{n(n+1)}{2} - 1$.

The following proposition is due to [8, Corollary 29], which is interesting to compare it with the case of $P_c(\Psi_t)P_c^T(\Psi_t)$.

Proposition 3.3. For each point $x \in M$, and with respect to normal coordinates around $x$, we have:

$$P(\Psi_t)P^T(\Psi_t)(x) = \left[ I_n + O(t) \right] - \frac{1}{2t} \cdot \left( \left[ I_{2(n-1)} \frac{O(t)}{n} 0 \right] \left[ 3 (\frac{1}{3}) - \frac{n+2}{n} J_n \right] + O(t) \right).$$

This matrix is invertible, and its inverse is

$$\left( P(\Psi_t)P^T(\Psi_t)(x) \right)^{-1} = \left[ I_n + O(t) \right] - 2t \cdot \left( \left[ I_{2(n-1)} \frac{O(t^2)}{n} 0 \right] \left( 3 (\frac{1}{3}) - \frac{n+2}{n} J_n \right)^{-1} + O(t) \right).$$  \hspace{1cm} (3.5)

3.2 Construction of $E_c$.

In the last subsection, we see that the $P_c(\Psi_t)P_c^T(\Psi_t)(x)$ is of rank $\frac{n(n+3)}{2} - 1$. This implies that we cannot expect to find a right inverse operator of $P_c$, which illustrates the difference between the local conformal embedding question and the local isometric one.
Recall in Proposition 1.5, the remainder term $O(t^i)$ is a symmetric 2-tensor subtracting its own trace, we denote it as $h$. Also, denote the bundle $G := \{s - \frac{tr{s}}{n} \cdot g | s \in \text{Sym}^{\otimes 2}T^*M\}$ of the Riemannian manifold $(M, g)$, we have $h \in G$, then the following theorem constructs the inverse of $P_c(Ψ_t)$ that we need.

**Theorem 3.4.** For $q \geq t^{-\frac{3}{2}-\rho}$, assume $Ψ_t \in C^\infty(M, \mathbb{R}^q)$ is defined as before, and define the traceless 2-tensor bundle $G := \{s - \frac{tr{s}}{n} \cdot g | s \in \text{Sym}^{\otimes 2}T^*M\}$.

Then $P_c(Ψ_t)$ has a family of right inverses $E_c(Ψ_t) : C^{s,α}(M, T^*M) \times C^{s,α}(M, G) \rightarrow C^{s,α}(M, \mathbb{R}^q)$, $s \geq 2$. More precisely, there exist $E_c(Ψ_t)$ such that for $h \in G$, $v' \in C^{s,α}(M, \mathbb{R}^q)$, and $v'(x) \perp \text{Ker}P(Ψ_t)(x)$,

$$E_c(Ψ_t)(0, h) = v' \iff \begin{pmatrix} 0 \\ h \end{pmatrix} = P_c(Ψ_t) \cdot v'.$$ (3.6)

Such $E_c(Ψ_t)$ can be expressed as $E_c(Ψ_t)(0, h) = E(0, h) + kE(0, g)$ for some $k \in C^{s,α}(M, g)$, where $E(Ψ_t)$ is the right inverse of $P(Ψ_t)$ defined as $E(Ψ_t)(x) := P^T(Ψ_t)(x)[P(Ψ_t)P^T(Ψ_t)(x)]^{-1}$ for each point $x \in M$ with respect to the normal coordinates.

Note that here, the $h \in G$ is meant to be the small difference term $f - \frac{tr{f}}{n} g$ in the conformal embedding equation. Before we provide the proof, a lemma of inverse of matrix about the section 2.1, we have

**Lemma 3.5 ([8, Lemma 28]).** Consider two symmetric invertible matrices $A_1(t)$ of size $m_1 \times m_1$ and $A_2(t)$ of size $m_2 \times m_2$, where $||(A_i(t))^{-1}|| \leq \rho_0$ for $i = 1, 2$ and $t \in (0, t_0]$. Additionally, let $b(t)$ be an $m_2 \times m_1$ matrix such that $\|b(t)\|$ approaches zero as $t$ tends to zero from the right. Then, for sufficiently small $t > 0$, the inverse matrix of the block matrix $\begin{bmatrix} A_1(t) & b^T(t) \\ b(t) & A_2(t) \end{bmatrix}$ can be expressed as:

$$\begin{bmatrix} A_1^{-1}(t) & c^T(t) \\ c(t) & A_2^{-1}(t) \end{bmatrix} \begin{bmatrix} I_{m_1} + b^T(t)c(t)^{-1} & 0 \\ 0 & (I_{m_2} + b(t)c^T(t))^{-1} \end{bmatrix}$$

where $c(t)$, a $m_2 \times m_1$ matrix, is defined as $c(t) = A_2^{-1}(t)b(t)A_1^{-1}(t)$. In particular,

$$\|c(t)\| \leq \|(A_2(t))^{-1}\| \|b(t)\| \|(A_1(t))^{-1}\|.$$

**Proof of Theorem 3.4.** The proof is presented in normal coordinates near the point $x$. Note that we are working with any $q \geq t^{-\frac{3}{2}-\rho}$, including the case $q = \infty$, which is the same as considering $\ell^2$. We first show that when $k = 0$, $E(Ψ_t) = P^T(Ψ_t)(x)[P(Ψ_t)(x)P^T(Ψ_t)(x)]^{-1}$ is a right inverse of $P_c(Ψ_t)$. Having the definitions and expressions of $P(Ψ_t)$ and $P_c(Ψ_t)$ as $n + \frac{n(n+1)}{2} \times q$ matrix as in section 2.1, we have the following expression:

$$P_c(Ψ_t) = P(Ψ_t) - \frac{1}{n} \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} P(Ψ_t),$$ (3.7)

and

$$\begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} = n \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix},$$

hence $P^T_c(Ψ_t) \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} = 0$. Due to Proposition 3.3, we know that the inverse of $\frac{n(n+3)}{2} \times \frac{n(n+3)}{2}$ matrix $P(Ψ_t)P^T(Ψ_t)$ exists, then we have the following matrix computation:

$$P_c(Ψ_t)P^T_c(Ψ_t)[P(Ψ_t)P^T(Ψ_t)]^{-1}$$

$$= \left( P(Ψ_t) - \frac{1}{n} \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} P(Ψ_t) \right) P^T(Ψ_t)[P(Ψ_t)P^T(Ψ_t)]^{-1}$$

$$= I \frac{n(n+3)}{2} - \frac{1}{n} \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix}.$$
Moreover, $h = f - \frac{nr_g}{n} g$, $f \in \text{Sym}^{\geq 2} T^* M$, we would write $(0, h)^T$ as the same way of $P_c(\Psi_t)$:

$$
\begin{bmatrix}
0 & 0 & \cdots & 0 & f_{12} & \cdots & f_{n-1,n} & f_{11} - \frac{1}{n} \sum_{k=1}^n f_{kk} & \cdots & f_{nn} - \frac{1}{n} \sum_{k=1}^n f_{kk}
\end{bmatrix}^T.
$$

Then in such expression we obtain

$$
\begin{bmatrix}
0 & 0 \\
0 & J_n
\end{bmatrix} \begin{bmatrix}
0 \\
h
\end{bmatrix} = 0.
$$

Hence if $v' = E(\Psi_t) \begin{bmatrix}
0 \\
h
\end{bmatrix} = P^T(\Psi_t) [P(\Psi_t) P^T(\Psi_t)]^{-1} \begin{bmatrix}
0 \\
h
\end{bmatrix}$, then multiplying $P_c(\Psi_t)$ on both sides we get:

$$
P_c(\Psi_t) v' = P_c(\Psi_t) P^T(\Psi_t) [P(\Psi_t) P^T(\Psi_t)]^{-1} \begin{bmatrix}
0 \\
h
\end{bmatrix} = (I_{\frac{n(n+3)}{2}} - \frac{1}{n} \begin{bmatrix}
0 & 0 & J_n
\end{bmatrix} \begin{bmatrix}
0 \\
h
\end{bmatrix} \begin{bmatrix}
0 \\
h
\end{bmatrix}) = \begin{bmatrix}
0 \\
h
\end{bmatrix}, \quad (3.8)
$$

which proves that $E(\Psi_t)(x) = P^T(\Psi_t)(x) [P(\Psi_t) P^T(\Psi_t)]^{-1}$ is an honest right inverse of $P_c(\Psi_t)$ pointwise.

Next, following our discussion in Remark 2.4, knowing $\text{Ker} P_c(\Psi_t)(x)/\text{Ker} P(\Psi_t)(x)$ is of dimension 1, we want to show $\text{Ker} P_c(\Psi_t)(x)/\text{Ker} P(\Psi_t)(x)$ is generated by $E(\Psi_t)(0, g)(x)$. By the definition of $E(\Psi_t) = P^T(\Psi_t) [P(\Psi_t) P^T(\Psi_t)]^{-1}$, we know $P(\Psi_t)E(\Psi_t)(0, g)(x)$ is not 0; hence, $E(\Psi_t)(0, g)(x)$ is automatically in $\text{Ker} P(\Psi_t)(x)^2$, hence only need to show $P_c(\Psi_t)E(\Psi_t)(0, g) = 0$. The direct computation $P_c(\Psi_t)E(\Psi_t)(0, g)$ goes as follows:

$$
P_c(\Psi_t)(x) E(\Psi_t)(0, g) = P_c(\Psi_t) P^T(\Psi_t) [P(\Psi_t) P^T(\Psi_t)]^{-1} \begin{bmatrix}
0 \\
g
\end{bmatrix} = (I_{\frac{n(n+3)}{2}} - \frac{1}{n} \begin{bmatrix}
0 & 0 & J_n
\end{bmatrix} \begin{bmatrix}
0 \\
g
\end{bmatrix} \begin{bmatrix}
0 \\
g
\end{bmatrix}) \begin{bmatrix}
0 \\
g
\end{bmatrix} = \begin{bmatrix}
0 \\
g
\end{bmatrix}.
$$

Recall that we defined the truncated embedding $\Psi^{q(t)}_{t, \eta}$ in Definition 0.2. Due to Proposition 1.8, we can obtain the exact same result for $\Psi^{q(t)}_{t, \eta}$, as the order of the difference term between $\Psi^{q(t)}_{t, \eta}$ and $\Psi_t$ is much larger than the order used in this theorem. We present the version of $\Psi^{q(t)}_{t, \eta}$ as a proposition here.

**Proposition 3.6.** For $q \geq t^{-\frac{3}{2} - \epsilon}$, $t \to 0^+$, assume $\Psi^{q(t)}_{t, \eta} \in C^{\infty}(M, \mathbb{R}^{q(t)})$ to be defined as before, where $q(t) = q$.

Then the $P^{q(t)}_{c}(\Psi_t)$ has a family of right inverses $E_c(\Psi^{q(t)}_{t, \eta}) : C^{s,\alpha}(M, T^* M) \times C^{s,\alpha}(M, G) \to C^{s,\alpha}(M, \mathbb{R}^{q(t)})$, $s \geq 2$. More precisely, there exist $E_c(\Psi^{q(t)}_{t, \eta})$ such that for $h \in G$, $v' \in C^{s,\alpha}(M, \mathbb{R}^{q(t)})$, and $v'(x) \perp \text{Ker} P(\Psi^{q(t)}_{t, \eta})(x)$,

$$
E_c(\Psi^{q(t)}_{t, \eta})(0, h) = v' \iff \begin{bmatrix}
0 \\
h
\end{bmatrix} = P_c(\Psi^{q(t)}_{t, \eta}) \cdot v'. \quad (3.9)
$$

Such $E_c(\Psi^{q(t)}_{t, \eta})$ can be expressed as $E_c(\Psi^{q(t)}_{t, \eta})(0, h) = E(0, h) + kE(0, g)$ for some $k \in C^{s,\alpha}(M, g)$, where $E(\Psi^{q(t)}_{t, \eta})$ is defined as $E(\Psi^{q(t)}_{t, \eta})(x) := P^T(\Psi^{q(t)}_{t, \eta})(x) [P(\Psi^{q(t)}_{t, \eta}) P^T(\Psi^{q(t)}_{t, \eta})(x)]^{-1}$ for each point $x \in M$ and with respect to the normal coordinates around $x$.

### 3.3 Estimates about $E$.

In order to apply the implicit function theorem, we need to prepare the estimates of the norm $\|E(\Psi_t)\|_{C^{s,\alpha}}$, and the norm $\|E(\Psi_t)(0, h)\|_{C^{s,\alpha}}$ which is the right inverse operator with $(0, h)$ as the input.

First, in order to do the computation, we state the following analytic preliminaries:

---
Remark 3.7.  1. If \( u \in C^{s+1}(M) \), and manifold \( M \) is compact, then for a fixed \( 0 < \alpha < 1 \), there is a \( C \) just about \( M, \alpha, k \), such that
\[
\|u\|_{C^{s,\alpha}} < C\|u\|_{C^{s+1}}. \tag{3.10}
\]

2. Assuming the norm on the right-hand side exists, after using the former observation we have for functions \( u, v \) on compact \( M \), and constant \( C \) about \( s \)
\[
\|uv\|_{C^{s,\alpha}} < C_s\|u\|_{C^{s,\alpha}}\|v\|_{C^{s,\alpha}}, \tag{3.11}
\]
and even a finer estimate, for \( 0 \leq r < s \), and constant \( C \) only about \( k \):
\[
\|uv\|_{C^{s,\alpha}} < K(\|u\|_{C^{s,\alpha}}\|v\|_{C^{r,\alpha}} + \|v\|_{C^{s,\alpha}}\|u\|_{C^{r,\alpha}}) + C_k(\|u\|_{C^{s-1,\alpha}}\|v\|_{C^{r-1,\alpha}}). \tag{3.12}
\]

Lemma 3.8. As \( t \to 0_+ \), the Hölder derivatives satisfy
\[
[D^\beta \Psi_t(x)]_{\alpha;M} \leq C t^{-\frac{|\beta|+1+\alpha}{2}},
\]
\[
\|\Psi_t(x)\|_{C^{s,\alpha}(M)} \leq C t^{-\frac{s+\alpha}{2}}
\]
for some constant \( C > 0 \).

Proof. The estimate about \( \Phi_t \) is due to [8, Proposition 24], which is
\[
[D^\beta \Phi_t(x)]_{\alpha;M} \leq C t^{-\frac{|\beta|+\alpha}{2}}; \|\Phi_t(x)\|_{C^{s,\alpha}} \leq C t^{-\frac{s}{2} + \frac{\alpha}{2}}.
\]
Then recall the normalized \( \Psi_t \) is defined by \( \Psi_t = \sqrt{2}(4\pi)^{n/4}t^{-\frac{n+2}{2}} \cdot \Phi_t \), we will have the inequalities of Hölder derivatives in this lemma.

Then we need the estimates of \( E(\Psi_t)(x) = P^T(\Psi_t)(x)(P(\Psi_t)P^T(\Psi_t)(x))^{-1} \) which is due to [8, Corollary 31] as follows.

Proposition 3.9 ([8, Corollary 31]). Recall
\[
E(\Psi_t) : C^{s,\alpha}(M, T^*M) \times C^{s,\alpha}(M, \text{Sym}^{\otimes 2}T^*M) \to C^{s,\alpha}(M, \mathbb{R}^q),
\]
then for \( q \geq C t^{-\frac{n}{2} - \rho} \), \( E(\Psi_t) \) has the \( C^{s,\alpha} \) estimate
\[
\|E(\Psi_t)\|_{C^{s,\alpha}(M)} \leq C t^{-\frac{s+\alpha}{2}}, \tag{3.13}
\]
as well as the operator norm
\[
\|E(\Psi_t)\|_{op} \leq C t^{-\frac{s+\alpha}{2}} \tag{3.14}
\]
for a constant \( C \).

4 Günther’s implicit function theorem

To solve the equation (2.20), which can have multiple solutions, we seek the solutions that can be expressed as follows:
\[
E_c(\Psi_t)(0, -\frac{1}{2} f) = E(\Psi_t)(0, -\frac{1}{2} f + k \cdot g) = v - Q(v, v). \tag{4.1}
\]
Since \( P_c(\Psi_t) \) and \( \text{tr}^\perp_g \) refer to the same concept but in different notations, we can easily see the solutions of (4.1) are also the solutions of (2.20), where the \( u \) in the former equation is meant to be \( \Psi_t \) here. Recall that the definition of \( Q(v, v) \) is given as in (2.19).
Lemma 4.1 ([8, Proposition 33]). For any \( v \in C^{s,\alpha}(M,\mathbb{R}^q) \), we have
\[
\|Q(\Psi_t)(v,v)\|_{C^{s,\alpha}(M,\mathbb{R}^q)} \leq C(e,k,\alpha,M,g) t^{-\frac{2\alpha}{\nu_1}} \|v\|_{C^{s,\alpha}(M,\mathbb{R}^q)}^2.
\]
(4.2)

Remark 4.2. Following the definition of \( Q(v,v) \) and the former lemma, we can easily notice that \( Q \) is a bilinear operator, which also has a norm estimate. In our article, we only need the following for \( u, v \in C^{s,\alpha}(M,\mathbb{R}^q) \):
\[
\|Q(\Psi_t)(v,v) - Q(u,u)\|_{C^{s,\alpha}(M,\mathbb{R}^q)}
\leq C(e,k,\alpha,M,g) t^{-\frac{\alpha}{\nu_1}} \|v-u\|_{C^{s,\alpha}(M,\mathbb{R}^q)} \left( \|v\|_{C^{s,\alpha}(M,\mathbb{R}^q)} + \|u\|_{C^{s,\alpha}(M,\mathbb{R}^q)} \right).
\]
(4.3)

Notice that we are still with the normalised embedding \( \Psi_t \), but not the modified conformal one \( \tilde{\Psi}_t \).

Our objective is to show that the sequence of \( v_0 = 0 \) and, for \( l = 0, 1, 2, \ldots \), iterating \( v_l \) by the equation:
\[
v_{l+1} := E(\Psi_t)(0, -\frac{1}{2} h) + Q(v_l, v_l).
\]
(4.6)

Our objective is to show that the sequence of \( \{v_l\} \) converges in \( C^{2,\alpha} \). Using the former lemma, we have
\[
\|v_{l+1}\|_{C^{2,\alpha}} \leq C(e,2,\alpha,M,g) t^{-\frac{2\alpha}{\nu_1}} \|v_l\|_{C^{2,\alpha}}^2 + \frac{1}{2} \|E(\Psi_t)(0, h)\|_{C^{2,\alpha}}.
\]

By imposing the condition
\[
C(e,2,\alpha,M,g) t^{-\frac{2\alpha}{\nu_1}} \|E(\Psi_t)(0, h)\|_{C^{2,\alpha}} < \frac{1}{2},
\]
we obtain
\[
2\|E(\Psi_t)(0, h)\|_{C^{2,\alpha}} \|v_{l+1}\|_{C^{2,\alpha}} < \|v_l\|_{C^{2,\alpha}}^2 + \|E(\Psi_t)(0, h)\|_{C^{2,\alpha}}^2.
\]

Therefore, by induction from \( l = 0 \), we have that for all \( l \)
\[
\|v_l\|_{C^{2,\alpha}} < \|E(\Psi_t)(0, h)\|_{C^{2,\alpha}}.
\]
(4.7)

Next, we need to show \( \{v_k\} \) is a Cauchy sequence:
\[
\|v_{l+1} - v_l\|_{C^{2,\alpha}} \leq C(e,2,\alpha,M,g) t^{-\frac{2\alpha}{\nu_1}} \|v_l - v_{l-1}\|_{C^{2,\alpha}} \left( \|v_l\|_{C^{2,\alpha}} + \|v_l\|_{C^{2,\alpha}} + \|v_{l-1}\|_{C^{2,\alpha}} \right)
\leq 2C(e,2,\alpha,M,g) t^{-\frac{2\alpha}{\nu_1}} \|E(\Psi_t)(0, h)\|_{C^{2,\alpha}} \|v_l - v_{l-1}\|_{C^{2,\alpha}}.
\]
By enforcing the condition $2C(e, 2, \alpha, M, g)t^{-\frac{2\alpha}{1+\alpha}}\|E(\Psi_t)(0, h)\|_{C^{2,\alpha}} < \frac{1}{2}$, we obtain $\|v_{t+1} - v_t\|_{C^{2,\alpha}} < \frac{1}{2}\|v_t - v_{t-1}\|_{C^{2,\alpha}}$, demonstrating that $\{v_t\}$ is indeed a Cauchy sequence. Hence, we identify a unique solution $v \in C^{2,\alpha}$, which is the limit of the bounded Cauchy sequence $\{v_t\}$.

Finally, we shall extend the regularity of the solution $v \in C^{2,\alpha}$ to $C^{s,\alpha}$ for $s \geq 3$. This extension is achieved by showing that $\|v_t\|_{C^{s,\alpha}}$ is bounded. Similar to the $C^{2,\alpha}$ case, we have

$$\|v_{t+1}\|_{C^{s,\alpha}} \leq C(e, k, \alpha, M, g)t^{-\frac{4\alpha}{1+\alpha}}\|v_t\|_{C^{s,\alpha}} + \frac{1}{2}\|E(\Psi_t)(0, h)\|_{C^{s,\alpha}},$$

(4.8)

if we again enforce:

$$C(e, k, \alpha, M, g)t^{-\frac{4\alpha}{1+\alpha}}\|E(\Psi_t)(0, h)\|_{C^{s,\alpha}} < \frac{1}{2},$$

then we have $\|v_t\|_{C^{s,\alpha}} < \|E(\Psi_t)(0, h)\|_{C^{s,\alpha}}$. Notice that here $t \to 0_+$, so requiring:

$$C(e, k, \alpha, M, g)\|E(\Psi_t)(0, h)\|_{C^{s,\alpha}} < \frac{1}{2}t^{-\frac{4\alpha}{1+\alpha}}$$

would also imply that $\|E(\Psi_t)(0, h)\|_{C^{s,\alpha}}$ is bounded. Hence, the theorem got proven. \qed

5 The main theorem: conformal embeddings

In this section, we would use the Propositions and Theorems prepared earlier to prove the main theorem of this paper. To recap Section 4, we worked on the heat kernel embedding $\Psi_t$ that maps to $\ell^2$. It is worth noting that the modified conformal embedding $\tilde{\Psi}_t$ still holds the same estimates as shown in Lemma 3.8 and Proposition 3.9, given the fixed choice of $\{h_i\}$ picked in Proposition 1.5. This holds because $\tilde{\Psi}_t$ is also the heat kernel of $g_t$ for each $t$, and the variation of $g_t$ is in a small compact interval $[0, t_0]$, which doesn’t affect the estimate compared to $t$. Furthermore, due to the estimate for the truncated tail in Proposition 1.8, when $q > t^{-\frac{5}{2} - \rho}$, the truncated conformal embedding $\Psi^{(t)}_{t, v_t} : (M, g) \to \mathbb{R}^{q(t)}$ also satisfies the estimates presented in Lemma 3.8 and Proposition 3.9.

The main theorem can be divided to two propositions: the first one claims that we could find a family of conformal immersion $C_t$ depending on a function $k \in C^{s,\alpha}(M)$ of $O(t^l)$, and the second one checks that this $C_t$ is one to one, hence an embedding.

**Proposition 5.1 (Conformal immersion).** Under the conditions of main theorem, there exists $t_0 > 0$ depending on $(g, \rho, \alpha)$, such that for the integer $q = q(t) \geq t^{-\frac{5}{2} - \rho}$, $0 < t < t_0$, the modified canonical heat kernel embedding $\Psi_t$ can be truncated to

$$\Psi^{(t)}_{i, v_t} = \Pi_q \circ \tilde{\Psi}_t : (M, g) \to \mathbb{R}^{q(t)} \subset \ell^2$$

and can be perturbed to a family of conformal immersion $C_{t,k}$, such that for $k \in C^{s,\alpha}(M)$ of $O(t^l)$, each $\Psi^{(t)}_{i, v_t}$ can be perturbed to a unique $C^{s,\alpha}(M)$ conformal embedding

$$C_{t,k} : M \to \mathbb{R}^{q(t)}.$$

Moreover, the resulting conformal map satisfies:

$$\|C_{t,k} - \Psi^{(t)}_{i, v_t} \|_{C^{s,\alpha}} = O(t^{l + \frac{1 - \epsilon - \alpha}{2}}).$$

**Proof.** The proof is to apply Theorem 4.3 to our case. We use the estimates in Proposition 3.9 and notice that the construction of $P(\Psi_t)P^T(\Psi_t)$ in Proposition 3.3, the estimates of $E(\Psi_t)$ in Proposition 3.9, and the Theorem 4.3 all work in the same way for the truncated embedding $\Psi^{(t)}_{i, v_t}$, provided that the estimate for the part that is truncated off approaches to 0 exponentially as presented in Proposition 1.8.
Indeed, under the condition of the main theorem $s + \alpha < l + \frac{1}{2}$, denote $h$ as the error term $O(t^l)$ in (1.10), i.e. $h := (\Psi_{t,\eta}^{q(t)})^* g_{can} - \frac{\text{tr}_g(\Psi_{t,\eta}^{q(t)})^* g_{can}}{n} g = O(t^l)$. In order to use the Theorem 4.3, the $k \in C^{s,\alpha}(M, g)$ need to be of $O(t^l)$, and use the construction in Proposition 3.4, we have
\[
 t^{-\frac{s + \alpha}{n}} \|E(\Psi_{t,\eta}^{q(t)})(0, h - 2k \cdot g)\|_{C^{s,\alpha}} < Ct^{-\frac{s + \alpha}{n}} \|\nabla_i \nabla_j \Psi_{t,\eta}^{q(t)}\|_{1 \leq i \leq j \leq n} \cdot O(t) \cdot (h - 2k \cdot g) \|_{C^{s,\alpha}} < Ct^{-\frac{s + \alpha}{n}} \cdot (t^{-\frac{k + 1 + \alpha}{n}}) \cdot t \cdot t^l < Ct^{-s - \alpha + \frac{1}{2} + l} \to 0, \quad \text{as } t \to 0_+.
\]
Here we used the fact that the first position of the input of $E(\Psi_{t,\eta}^{q(t)})$ is 0, hence we only need to consider the lower block of the matrix expression of $E(\Psi_{t,\eta}^{q(t)})$. Then by Theorem 4.3, for each fixed $k \in C^{s,\alpha}(M, g)$ of $O(t^l)$, we get the unique solution $v_k \in C^{s,\alpha}(M, \mathbb{R}^q)$ satisfying:
\[
 E_c(\Psi_{t,\eta}^{q(t)})(0, -\frac{1}{2} h) = E(\Psi_{t,\eta}^{q(t)})(0, -\frac{1}{2} h + k \cdot g) + Q(v_k, v_k) = v_k, \tag{5.1}
\]
notice we denote it as $v_k$ for it really depends on the fixed $k \in C^{s,\alpha}$. By the Theorem 3.4 and the discussion in Section 2.2, such a $v_k$ satisfies:
\[
 h = \nabla \Psi_{t,\eta}^{q(t)} \cdot \nabla v_k - \frac{\text{tr}_g(\nabla \Psi_{t,\eta}^{q(t)} \cdot \nabla v_k)}{n} g + \nabla v_k \cdot \nabla \Psi_{t,\eta}^{q(t)} = \frac{\text{tr}_g(\nabla v_k \cdot \nabla \Psi_{t,\eta}^{q(t)})}{n} g \tag{5.2}
\]
Hence we attain the conformal immersions that we seek:
\[
 C_{t,k} := \Psi_{t,\eta}^{q(t)} + v_k. \tag{5.3}
\]
Note here the $v_k$ depends on $k \in C^{s,\alpha}(M)$. For the difference term $v_k$, as the computation shown in (4.7), we know $\|v_k\|_{C^{s,\alpha}} < \|E_c(\Psi_{t,\eta}^{q(t)})(0, h)\|_{C^{s,\alpha}} < C t^{l+\frac{1}{2} - \frac{4n}{s+\alpha}}$, hence for $t > 0$ sufficiently small, for different $k'$ and $k''$, $C_{t,k'} - C_{t,k''} = v_{k'} - v_{k''}$ is controlled by $\|E(\Psi_{t,\eta}^{q(t)}(0, k \cdot g))\|_{C^{s,\alpha}} < C t^{l+\frac{1}{2} - \frac{4n}{s+\alpha}}$.

The only thing left to show is that the map we found is injective:

**Proposition 5.2 (Injectivity).** Let $(M, g)$ be a compact Riemannian manifold with smooth metric $g$. Then, there exists a positive constant $\delta_0$ such that for $0 < t \leq \delta_0$ and $q(t) \geq Ct^{-\frac{s+\alpha}{n}}$, the truncated heat kernel mapping $\tilde{\Psi}_{t,\eta}^{q(t)} : M \to \mathbb{R}^q$ possesses the property of point distinguishability. In other words, for any $x \neq y$ on $M$, one has $\tilde{\Psi}_{t,\eta}^{q(t)}(x) \neq \tilde{\Psi}_{t,\eta}^{q(t)}(y)$. The property of point distinguishability also holds true for the perturbed almost conformal immersion $\tilde{\Psi}_{t,\eta}^{q(t)}(x)$, which satisfying: $\tilde{\Psi}_{t,\eta}^{q(t)} g_{can} = \frac{\text{tr}_g(\tilde{\Psi}_{t,\eta}^{q(t)} g_{can})}{n} g + O(t^l)$, and so is the conformal mapping $C_{t,k}$ for any $k \in C^{s,\alpha}(M)$ of $O(t^l)$.

**Proof.** It can be easily obtained by the same argument in [8, Proposition 36], since the almost conformal mapping $\tilde{\Psi}_t$ is also the heat kernel of some metric $g_t$.

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Zhitong Su
School of Mathematics
Sun Yat-sen University
Guangzhou 510275, P.R. China
E-mail address: suzht@mail.sysu.edu.cn