On the Exact Eigenstates and the Ground States Based on the Boson Realization for Many-Quark Model with $su(4)$ Algebraic Structure

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(Received April 24, 2009)

Based on the boson realization, the ground state and phase structure are investigated in a many-quark model with algebraic structure developed in our previous paper, in which the two-body pairing and particle-hole type interactions are active. The physical interpretation of the exact eigenstates obtained in the boson realization in our previous paper is given in a many-quark model with $su(4)$ algebraic structure.

§1. Introduction

Recently, the exact eigenstates and energy eigenvalues were obtained in a many-quark model, in which the two-body pairing and the particle-hole type interactions are active, by present authors in Ref.1), which is hereafter referred to as (I) in this paper. This model was called the modified Bonn model because, if the only pairing interaction is included, this model is reduced to the Bonn model developed by Petry et. al.2) It is pointed out in (I) that the original Bonn model has the $su(4)$ symmetry. In the original Bonn model, the colored state is energetically favorable while the color neutral triplet formation of quarks is realized as a nice feature of the model.3) Thus, this model leads to the pairing instability and the ground state is color superconducting state at any time.

However, the dynamics of quarks are governed by the quantum chromodynamics (QCD) which possesses a color $su(3)$ symmetry. Therefore, there is a room that the original Bonn model can be extended to the $su(3)$ symmetric model, not $su(4)$, by introducing the $su(4)$ symmetry breaking term.

In (I), we have formulated the modified Bonn model with color $su(3)$ symmetry introducing the particle-hole type interaction which breaks the $su(4)$ symmetry. In that paper, the exact eigenstates and the exact energy eigenvalues have been obtained in the boson space by using the Schwinger boson representation. In the triplet formation of quarks, the energy of the colored state is smaller than that of the color neutral triplet states if the particle-hole type interaction is switched off, namely, in the original Bonn model. However, the energy of the colored triplet states shifts up as the coupling strength of the particle-hole type interaction is increasing, while the energy of the color neutral triplet state is not changed. Thus, beyond a certain
critical value of the coupling strength, the color neutral triplet state is favorable energetically compared with the colored triplet state. As is similar to the case of the triplet formation, in the pairing states, the energy of colored pairing states is energetically favorable. But, the energy of these states increases as the coupling strength of the particle-hole type interaction increases, as is seen in the case of the colored triplet states in the triplet formation. Further, in Ref. 4), which is referred to as (II) in this paper, the exact eigenstates are described in a unified manner. However, in (I) and (II), it is not investigated which state, namely, the triplet state or the pairing state, is energetically favorable or not.

In this paper, as one of important purposes rested in our previous papers, we investigate the ground state in the modified Bonn model. Following the previous papers (I) and (II), we describe the exact eigenstates and energy eigenvalues of this many-quark model in the boson space by using the boson realization. We will give the phase diagram in this many-quark model under a certain parameterization.

This paper is organized as follows: In the next section, we recapitulate the exact eigenstates and the energy eigenvalues for this many-quark model in the boson realization. In §3, the physical interpretation of the exact eigenstates obtained in the boson representation of many-quark model is discussed in some cases by using the correspondences with the original fermion states. In §4, the ground state and the phase structure in this modified Bonn model are investigated. The last section is devoted to the discussions and concluding remarks.

§2. Recapitulation of the exact eigenstates and the energy eigenvalues for many-quark model with \( su(4) \) algebraic structure

In this section, we recapitulate the basic ingredients of the many quark model with \( su(4) \) algebraic structure, especially, of the energy eigenvalues and eigenstates for the model Hamiltonian following to (I).

2.1. The model

The model is formulated in terms of the generators of the \( su(4) \) algebra constructed by the bilinear forms of the quark creation and annihilation operators. The color quantum numbers are specified as \( i = 1, 2, 3 \). Each color state has the degeneracy \( 2\Omega \). Here, \( 2\Omega = 2j_s + 1 \) and \( j_s \) is a half integer. Thus, the maximum total quark number is \( 6\Omega \). An arbitrary single-particle state is specified as \( (i, m) \) with \( m = -j_s, -j_s + 1, \cdots, j_s - 1, j_s \), and is created and annihilated by the fermion operators \( c_{im}^* \) and \( c_{im} \). For simplicity, we neglect the degrees of freedom related to the isospin. We define the following bilinear forms for the fermion creation and annihilation operators:

\[
\begin{align*}
\tilde{S}_1 &= \sum_m c_{2m}^* c_{3m}^*, \\
\tilde{S}_2 &= \sum_m c_{3m}^* c_{1m}^*, \\
\tilde{S}_3 &= \sum_m c_{1m}^* c_{2m}^*, \\
\tilde{S}_{11} &= \sum_m c_{1m}^* c_{1m}, \\
\tilde{S}_{22} &= \sum_m c_{2m}^* c_{2m}, \\
\tilde{S}_{33} &= \sum_m c_{3m}^* c_{3m},
\end{align*}
\]
The fermion number operators with $i$

Here, $c_{i,m} = (-1)^{j_s - m} c_{i,-m}$. The operators in the definition (2.1) are generators of the $su(4)$ algebra:

\[ \tilde{S}^i_1 = \sum_m (c_{2m}^* c_{2m} + c_{3m}^* c_{3m}) - 2\Omega, \quad \tilde{S}^i_2 = \sum_m (c_{3m}^* c_{3m} + c_{1m}^* c_{1m}) - 2\Omega, \]
\[ \tilde{S}^3_3 = \sum_m (c_{1m}^* c_{1m} + c_{2m}^* c_{2m}) - 2\Omega, \quad \tilde{S}_1 = (\tilde{S}^1_1)^*, \quad \tilde{S}_2 = (\tilde{S}^2_2)^*, \quad \tilde{S}_3 = (\tilde{S}^3_3)^*. \]

The operators in the definition (2.1) are generators of the $su(4)$ algebra:

\[ \tilde{S}^i_1 = \tilde{S}^i_1, \quad (\tilde{S}^i_j)^* = \tilde{S}^j_i, \quad [\tilde{S}^i_1, \tilde{S}^i_j] = \delta_{ij}, \]
\[ [\tilde{S}^i_j, \tilde{S}^j_k] = \delta_{jk} \tilde{S}^i_1 + \delta_{ik} \tilde{S}^j_1 - \delta_{jl} \tilde{S}^j_1. \]

The fermion number operators with $i = 1, 2, 3$ read, respectively,

\[ \tilde{N}_1 = \Omega - \frac{1}{2} (\tilde{S}^1_1 - \tilde{S}^2_2 - \tilde{S}^3_3), \quad \tilde{N}_2 = \Omega - \frac{1}{2} (\tilde{S}^2_2 - \tilde{S}^3_3 - \tilde{S}^1_1), \]
\[ \tilde{N}_3 = \Omega - \frac{1}{2} (\tilde{S}^3_3 - \tilde{S}^1_1 - \tilde{S}^2_2) \]
and the total quark number is

\[ \tilde{N} = \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3 = 3\Omega + \frac{1}{2} (\tilde{S}^1_1 + \tilde{S}^2_2 + \tilde{S}^3_3). \]

As a sub-algebra, the $su(4)$ algebra contains the $su(3)$ algebra which is generated by

\[ \tilde{S}^2_1, \tilde{S}^1_2, \tilde{S}^3_2, \tilde{S}^1_3, \tilde{S}^3_1, \frac{1}{2} (\tilde{S}^2_2 - \tilde{S}^3_3), \tilde{S}^1_1 - \frac{1}{2} (\tilde{S}^2_2 + \tilde{S}^3_3). \]

The Casimir operator $\tilde{Q}^2$ for the $su(3)$ algebra reads

\[ \tilde{Q}^2 = \sum_{i \neq j} \tilde{S}^i_j \tilde{S}^j_i + 2 \left( \frac{1}{2} (\tilde{S}^2_2 - \tilde{S}^3_3) \right)^2 + \frac{2}{3} \left( \tilde{S}^1_1 - \frac{1}{2} (\tilde{S}^2_2 + \tilde{S}^3_3) \right)^2. \]

The many-quark model which we investigate in this paper has the pairing interaction and particle-hole type interaction in terms of usual many-fermion system. The Hamiltonian is written as

\[ \tilde{H}_m = \tilde{H} + \chi \tilde{Q}^2, \quad (\chi : \text{a real parameter}) \]

where $\tilde{H}$ and $\tilde{Q}^2$ are defined by

\[ \tilde{H} = -G \left( \tilde{S}^1_1 \tilde{S}^2_1 + \tilde{S}^2_2 \tilde{S}^3_2 + \tilde{S}^3_3 \right) \]
and Eq.(2.7). We omit a kinetic term for quarks. Hereafter, the coupling constant $G$ for the pairing interaction is set to 1 without loss of generality. The coupling $\chi$ represents the force strength of the particle-hole type interaction. If we only have $\tilde{H}$, namely, we take $\chi = 0$, this model is known as the Bonn model and obeys the
su(4) algebra. Thus, we call our model \((2.7)\) the modified Bonn model. Here, since \(\tilde{Q}^2\) is a Casimir operator for the \(su(3)\) algebra, then, the modified Bonn model still has a the color \(su(3)\) symmetry, that is,

\[
[ \tilde{H}_m, \tilde{S}_j^i ] = 0 , \quad i , j = 1 , 2 , 3 .
\] (2.8)

The Schwinger-type boson realization for the \(su(4)\) generators are given as

\[
\tilde{S}_i = \hat{a}_i^\dagger \hat{b} - \hat{a}^\dagger \hat{b}_i , \quad \tilde{S}_i = \hat{b}^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{a} , \quad \tilde{S}_i^j = (\hat{a}_i^\dagger \hat{a}_j - \hat{b}_j^\dagger \hat{b}_i) + \delta_{ij}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) ,
\] (2.9)

where \(\hat{a}_i, \hat{a}_i^\dagger, \hat{b}_i, \hat{b}_i^\dagger, \hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger\) \((i = 1, 2, 3)\) denote boson operators. Associated with the form \((2.9)\) in the boson representation, we define the \(su(1,1)\) algebra:

\[
\begin{align*}
\hat{T}_+ &= \hat{t}_+ + \hat{\tau}_+ , \quad \hat{T}_- = \hat{t}_- + \hat{\tau}_- , \quad \hat{T}_0 = \hat{t}_0 + \hat{\tau}_0 , \\
\hat{t}_+ &= \hat{b}^\dagger \hat{a}^\dagger , \quad \hat{t}_- = \hat{a} \hat{b} , \quad \hat{t}_0 = \frac{1}{2}(\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a}) + \frac{1}{2} , \\
\hat{\tau}_+ &= \sum_{i=1}^{3} \hat{b}_i^\dagger \hat{a}_i^\dagger , \quad \hat{\tau}_- = \sum_{i=1}^{3} \hat{a}_i \hat{b}_i , \quad \hat{\tau}_0 = \frac{1}{2} \sum_{i=1}^{3} (\hat{b}_i^\dagger \hat{b}_i + \hat{a}_i^\dagger \hat{a}_i) + \frac{3}{2} ,
\end{align*}
\] (2.10)

which satisfy

\[
\begin{align*}
[\hat{T}_+, \hat{T}_-] &= -2\hat{T}_0 , \quad [\hat{T}_0, \hat{T}_\pm] = \hat{T}_\pm , \\
[\hat{T}_\mu, \hat{S}_i] &= [\hat{T}_\mu, \hat{S}_i^j] = [\hat{T}_\mu, \hat{S}_i^j] = 0 , \quad \mu = \pm , 0 .
\end{align*}
\] (2.11)

Thus, we can deal with the modified Bonn model in the boson space. The Hamiltonian \((2.6)\) is replaced to

\[
\begin{align*}
\hat{H}_m &= \hat{H} + \chi \hat{Q}^2 , \\
\hat{H} &= - \left( \hat{S}_1^1 \hat{S}_1 + \hat{S}_2^2 \hat{S}_2 + \hat{S}_3^3 \hat{S}_3 \right) , \\
\hat{Q}^2 &= \sum_{i \neq j} \hat{S}_i^j \hat{S}_i^j + 2 \left( \frac{1}{2} \left( \hat{S}_2^2 - \hat{S}_3^3 \right) \right)^2 + \frac{2}{3} \left( \hat{S}_1^1 - \frac{1}{2} \left( \hat{S}_2^2 + \hat{S}_3^3 \right) \right)^2.
\end{align*}
\] (2.13)

with \(G = 1\) for the pairing interaction. The relative strength of the pairing interaction to the particle-hole type interaction is given as \(G/\chi = 1/\chi\) for \(G = 1\).

2.2. The exact eigenstates paying an attention to the pairing correlation

The orthogonal set specified by eight quantum numbers is constructed in (I), namely, \((I-4.1)\) as

\[
| \lambda \mu \nu \sigma_0 ; T T_0 \rangle = (\hat{T}_+)^{T_0-T} \hat{Q}^+ (\lambda \mu \nu \sigma_0) | \lambda \rho \sigma_0 T \rangle .
\] (2.15)

Here, the state \(| \lambda \rho \sigma_0 T \rangle\) is written as

\[
| \lambda \rho \sigma_0 T \rangle = (\hat{S}_3^3)^{2\lambda} (\hat{S}_1^1)^{2\rho} | \sigma_0, T \rangle ,
\]

\[
| \sigma_0, T \rangle = | m_1 \rangle = (\hat{b}_1^\dagger)^{2(\sigma_1 - \sigma_0)} (\hat{b}^\dagger)^{2\sigma_0} | 0 \rangle ,
\]

\[
\sigma_1 = T - 2
\] (2.16)
and the operators $\hat{S}^4$ in (2.10) and $\hat{Q}_+ (\lambda \mu \nu \rho)$ in (2.15) are represented as

$$\hat{S}^4 = \hat{S}^1 \left( \hat{S}^1 \frac{1}{2} (\hat{S}^2 + \hat{S}^3) \right) + (\hat{S}^2 \hat{S}^2 + \hat{S}^3 \hat{S}^3), \tag{2.17}$$

$$\hat{Q}_+ (\lambda \mu \nu \rho) = \sum_{\lambda \mu \nu \rho} \langle \lambda \lambda_0 \mu \nu_0 | \nu \nu_0 \rangle \sqrt{\frac{(\lambda - \lambda_0)!}{(2 \lambda)!(\lambda + \lambda_0)!}} \sqrt{\frac{(2 \mu)!}{(\mu + \mu_0)!(\mu - \mu_0)!}} \times (\hat{S}^3)^{\mu + \mu_0} (-\hat{S}^2)^{\mu - \mu_0} (\hat{S}^3)^{\lambda + \lambda_0}. \tag{2.18}$$

Further, we introduced $\Omega$-operator $\hat{\Omega}$, the number operators of color $i$ quarks, $\hat{N}_i$, and the total quark number operator $\hat{N}$ in this boson space, which have the eigenvalues $\Omega, n_1$ and $N$ acting on $|m_1\rangle$:

$$\hat{\Omega} = n_0 + \frac{1}{2} \left( \sum_{i=1}^{3} (\hat{a}_i^* \hat{a}_i + \hat{b}_i^* \hat{b}_i) + \hat{a}^* \hat{a} + \hat{b}^* \hat{b} \right), \tag{2.19}$$

$$\hat{N}_i = n_0 + \hat{a}^* \hat{a} + (\hat{b}_i^* \hat{b}_i - \hat{a}_i^* \hat{a}_i) + \sum_{j=1}^{3} \hat{a}_j^* \hat{a}_j, \tag{2.20}$$

$$\hat{N} = 3n_0 + 3\hat{a}^* \hat{a} + 2 \sum_{j=1}^{3} \hat{a}_j^* \hat{a}_j + \sum_{j=1}^{3} \hat{b}_j^* \hat{b}_j, \tag{2.21}$$

where $n_0$ implies $n_2 = n_3 = n_0$ in the state $|m_1\rangle$. We have

$$n_0 = \Omega - \sigma_1, \quad n_1 = \Omega + \sigma_1 - 2\sigma_0, \quad \text{i.e.}, \quad n_1 - n_0 = 2(\sigma_1 - \sigma_0). \tag{2.22}$$

Hereafter, we take the particle picture developed in (I). The results with respect to the hole picture are the similar to those of the particle picture. The eigenvalue equation for $\hat{H}_m$ and the eigenvalue are obtained as

$$\hat{H}_m |\lambda \mu \nu \rho; \sigma_0 TT_0\rangle = E_{\sigma_1 \sigma_0 \rho \lambda}^{(m)} |\lambda \mu \nu \rho; \sigma_0 TT_0\rangle, \quad E_{\sigma_1 \sigma_0 \rho \lambda}^{(m)} = E_{\sigma_1 \sigma_0 \rho \lambda} + \chi F_{\sigma_1 \sigma_0 \rho \lambda}, \tag{2.23}$$

$$E_{\sigma_1 \sigma_0 \rho \lambda} = -(2\lambda(2\sigma_0 + 1 - 2\rho - 2\lambda) + 2\rho(2\sigma_1 + 3 - 2\rho))$$

$$- 2\rho(2\Omega + 3 - 2n_0 - 2\rho) = E_{N_{n_0 n_1 \rho}},$$

$$F_{\sigma_1 \sigma_0 \rho \lambda} = 2\lambda(1 + \frac{2}{3}) \left( 2(\sigma_1 - \sigma_0) - 2\rho + \lambda \right) \left( 2(\sigma_1 - \sigma_0) - 2\rho + \lambda + 3 \right)$$

$$= G_{N_{n_0 n_1}} + 2E_{\sigma_1 \sigma_0 \rho \lambda},$$

$$G_{N_{n_0 n_1}} = 2(\Omega - n_0)(\Omega - n_0 + 3) + (\Omega - n_1)^2 - \frac{1}{3} (3\Omega - N)(3\Omega - N + 6). \tag{2.24}$$
Then, the eigenvalue equation and energy eigenvalue are derived in Eq. (2.16): 
\[ \Omega = n_0 - 2, \quad T = n_0 + \sigma_1, \]
\[ N = 3n_0 + 4(\lambda + \rho) + 2\sigma_1 - 2\sigma_0 = 2n_0 + n_1 + 4(\lambda + \rho). \] (2.25)

Here, we further used the relation (2.16), that is, \( T = \sigma_1 + 2 \).

2.3. Triplet formation

Next, we introduce another form for the orthogonal set. This state corresponds to the triplet formation, in which the fermion number changes in unit 3 as was mentioned in Eq. (1.3-38).

The energy eigenstates are constructed by
\[ \langle \lambda \mu \nu \sigma; t\tau TT \rangle = \langle \hat{T}_+ \rangle \hat{T}_0 - T \langle \lambda \mu \nu \sigma; t\tau T \rangle = \langle \hat{T}_+ \rangle \hat{T}_0 - T \langle \lambda \mu \nu \sigma; t\tau T \rangle, \]
\[ ||\lambda \mu \nu \sigma; t\tau T \rangle = \langle \hat{O}_+ (t\tau) \rangle \hat{T}_0^{-\tau} \langle \lambda \mu \nu \sigma; t\tau T \rangle \]
\[ = \sum_{t_0 t_0; t_0 + \tau_0 = T_0} \frac{(-1)^{t_0 - T} \Gamma(T - (t + \tau) + 1) \Gamma(2t) \Gamma(2\tau)}{\Gamma(t_0 - t + 1) \Gamma(t_0 + t) \Gamma(\tau_0 + \tau)} \times ||\lambda \mu \nu \sigma; \tau \tau_0 \rangle \otimes |t_0 \rangle, \]
\[ ||\lambda \tau T \rangle \]
\[ = \langle \hat{O}_+ (t\tau) \rangle \hat{T}_0^{-\tau} \langle \lambda \tau \rangle \otimes |t \rangle, \] (2.26)
where the operator \( \hat{Q}_+ (\lambda \mu \nu \sigma) \) is defined in Eq. (2.18) and \( \hat{O}_+ \) is introduced as
\[ \hat{O}_+ (t\tau) = \hat{\tau}_+ (t_0 + t + \epsilon)^{-1} - \hat{\tau}_+ (\tau_0 + \tau + \epsilon)^{-1}. \] (2.27)

The states appearing in (2.26) are defined as
\[ |\lambda \tau \rangle = (\hat{a}_0^\dagger)^{2\lambda} (\hat{b}_1^\dagger)^{2\tau - 3 - 2\lambda} |0 \rangle, \]
\[ |e \rangle = (\hat{b}_1^\dagger)^{2t - 1} |0 \rangle, \]
\[ ||\lambda \mu \nu \sigma; \tau \tau_0 \rangle = \hat{Q}_+ (\lambda \mu \nu \sigma) (\hat{\tau}_+)^{t_0 - T} |\lambda \tau \rangle, \]
\[ |tt_0 \rangle = (\hat{\tau}_+)^{t_0 - T} |t \rangle. \] (2.28)

Then, the eigenvalue equation and energy eigenvalue are derived as
\[ \hat{H}_m ||\lambda \mu \nu \sigma; t\tau TT \rangle = E_{T\tau \lambda}^{(m)} ||\lambda \mu \nu \sigma; t\tau TT \rangle, \]
\[ E_{T\tau \lambda}^{(m)} = E_{T\tau \lambda} + \chi F_{\tau \lambda}^{(t)}, \]
\[ E_{T\tau \lambda} = -(T - t - \tau)(T + t + \tau - 1) - 4t\lambda, \]
\[ F_{\tau \lambda}^{(t)} = 2\lambda(\lambda + 1) + \frac{2}{3}((2\tau - 3) - \lambda)((2\tau - 3) - \lambda + 3). \] (2.29)

Operation of \( \hat{\Omega} \) and \( \hat{N} \) on ||\lambda \tau TT \rangle leads to
\[ T = \Omega + 2 - n_0, \]
\[ t = \Omega + 1 - \frac{1}{3}(\tau - 2\lambda) - \frac{1}{3}N. \] (2.31)
Thus, $E_{Tt\tau\lambda}$ is rewritten as

$$\begin{align*}
E_{Tt\tau\lambda} &= -\left(\frac{1}{3}N - n_0 - \frac{1}{3}(2\tau - 3) - \frac{2}{3}\lambda\right) \\
&\quad \times \left(2\Omega + 3 - \frac{1}{3}N - n_0 + \frac{1}{3}(2\tau - 3) + \frac{2}{3}\lambda\right) \\
&\quad - 4\lambda\left(\Omega + \frac{1}{2} - \frac{1}{3}N - \frac{1}{6}(2\tau - 3) + \frac{2}{3}\lambda\right) .
\end{align*}$$

(2.32)

§3. Physical interpretation of the eigenstates

In this section, the physical interpretation of the various states is investigated for the Hamiltonian $\hat{H}$, namely, we put $\chi = 0$ in Eq. (2.13).

3.1. Pairing correlation

3.1.1. Correspondence to the original fermion states

From (2.25) for the quark number in the pairing state (2.16), we can obtain the following relation:

$$2\lambda = (N - (2n_0 + n_1))/2 - 2\rho .$$

(3.1)

From the construction of the state, $2\lambda$ and $2\rho$ must be positive integers. Therefore, $N - (2n_0 + n_1)$ have to be a positive even integer. This means that the change in the fermion number relatively to that of the minimum weight state is restricted to even number, i.e., it is of the pairing-type. From (2.22), the number of quarks in the state $|m_1\rangle$ is given as

$$N = n_1 + n_2 + n_3 = n_1 + 2n_0 , \quad (n_2 = n_3 = n_0)$$

(3.2)

and each quark number with color $i$ is also written as

$$|m_1\rangle : \begin{cases} 
\text{numbers of color 1} = n_1 \\
\text{numbers of color 2} = n_0 \\
\text{numbers of color 3} = n_0
\end{cases}$$

(3.3)

In order to investigate the physical interpretation of the exact eigenstates in the boson representation, we return to the original fermion states which correspond to the boson states in the boson representation. In the original fermion space, the operator $\hat{S}^4$ has the following correspondence:

$$\hat{S}^4 \longleftrightarrow \tilde{S}^4 = \frac{1}{2} \sum_m c^\dagger_{2m}c^\dagger_{3\bar{m}} \sum_m (c^\dagger_{2m}c_{2m} + c^\dagger_{3m}c_{3m} - 2c^\dagger_{1m}c_{1m})$$

$$+ \sum_{mm'} (c^\dagger_{3m}c^\dagger_{1m'}c_{2m'}c_{1m'} + c^\dagger_{1m}c^\dagger_{2m}c^\dagger_{3m}c_{1m}) .$$

(3.4)

Namely, the operator $\hat{S}^4$ creates two quarks whose colors are 2 and 3. Further, the operators $\hat{S}^3, \hat{S}^2, \hat{S}_1^3$ and $\hat{S}_1^2$ have the correspondences in the following manner:

$$\hat{S}^3 \leftrightarrow \tilde{S}^3 = \sum_m c^\dagger_{1m}c^\dagger_{2\bar{m}} ,$$
\[
\hat{S}_2^3 \leftrightarrow \hat{S}_2^3 = \sum_m c_{3m}^* c_{2m}, \\
\hat{S}_1^2 \leftrightarrow \hat{S}_1^2 = \sum_m c_{2m}^* c_{1m}, \\
\hat{S}_3^1 \leftrightarrow \hat{S}_3^1 = \sum_m c_{3m}^* c_{1m}.
\]

Thus, \(\hat{S}_3^3\) creates two quarks with color 1 and 2, \(\hat{S}_2^3\), \(\hat{S}_1^2\) and \(\hat{S}_3^1\) create one quark with color 3, 2 and 3 and annihilate one quark with color 2, 1 and 1, respectively, in terms of the original fermion space. Thus, the state (2.15) under \(T = T_0\) includes the following quarks:

- numbers of color 1: \(n_1 + 2\lambda - 2\mu\),
- numbers of color 2: \(n_0 + 2\rho + \lambda + \lambda_0 + \mu - \mu_0\),
- numbers of color 3: \(n_0 + 2\rho + \lambda + \lambda_0 + \mu + \mu_0\).

From (2.8), the Hamiltonian commutes with \(\hat{S}_2^1\), \(\hat{S}_1^1\) and \(\hat{S}_3^2\). Thus, if \(T = T_0\), the energy eigenvalue is determined by the state \(|\lambda\rho\sigma_0 T\rangle\) in Eq. (2.16). Hereafter, we use the notation \(E_{N\rho\sigma\rho_0}^\pm\) given in Eq. (2.24).

### 3.1.2. Color superconducting state where colors 2 and 3 are coupled

For the state \(|\lambda\rho\sigma_0 T\rangle\) in Eq. (2.16), let us adopt the following parameterization:

- \(n_0 = 0\), \(n_1 = 2\Omega\), \(2\rho = q'\), \(\mu = \mu_0 = 0\), \(\lambda = \lambda_0 = 0\).

Then, as is seen from Eq. (3.6), the state (2.16) under the above conditions includes the quarks as

- numbers of color 1: \(n_1 = 2\Omega\),
- numbers of color 2: \(2\rho = q'\),
- numbers of color 3: \(2\rho = q'\).

Here, the total quark number \(N\) is written as

\[
N = n_1 + 2\rho + 2\rho = 2\Omega + 2q'.
\]

Thus, in the case \(\chi = 0\), the energy eigenvalue (2.24) with \(\chi = 0\) is recast into

\[
E_{N\rho\sigma\rho_0}^\pm = -2\rho(2\Omega + 3 - 2\rho)
\]

In the original fermion system, the following state can be constructed:

\[
|\Psi(q', \Omega)\rangle = (S^1)^q' \prod_{m=1}^{\Omega} c_{1m}^* c_{1m}^* |0\rangle.
\]
This state is nothing but the color superconducting state in which the quarks with color 2 and 3 are coupled. The number of quarks with each color in this state is as follows:

\[
\begin{align*}
\text{numbers of color } 1 &= 2\Omega, \\
\text{numbers of color } 2 &= q', \\
\text{numbers of color } 3 &= q'.
\end{align*}
\] (3.12)

and the total quark number \(N\) is obtained as

\[
N = 2\Omega + 2q'.
\] (3.13)

For this state, the energy eigenvalue is derived easily in the fermion space and the result is as follows:

\[
E_N = -q'(2\Omega + 3 - q').
\] (3.14)

Thus, we conclude that the state described in the boson space, Eq.(2.16), under the conditions (3.7) corresponds to the color superconducting state in the original fermion space, in which the quarks with color 2 and 3 are coupled.

3.2. triplet formation

As was discussed in (I), the operators \(\hat{b}^*, \hat{a}^*, \{\hat{t}_\mu\} \) and \(\{\hat{\tau}_\mu\}\) commute with the \(su(3)\) generators. Namely, they are invariant under the group \(SU(3)\). Further, from Eq.(1-5-17), we derived the following relation in the state (2.26) with (2.28):

\[
\lambda = \mu = \nu = \nu_0 = 0 \quad \text{for} \quad \tau = \frac{3}{2}.
\] (3.15)

Thus, it is seen that under \(T = T_0\), the state (2.26) with \(\tau = 3/2\) is color neutral. Of course, if \(\tau > 3/2\), the state is colored one.

As is similar to the case of the pairing correlation, the energy eigenvalue in the case of triplet formation is determined by the state \(|\lambda\tau tT_0\rangle\rangle\) when \(T = T_0\). Hereafter, we restricted ourselves to the case \(T = T_0\).

3.3. Relation of the states with pairing correlation to those with triplet formation

In our previous paper (II), the state \(|lsrw\rangle\) is introduced and is defined as

\[
|lsrw\rangle = (\hat{S}^3)^{2l}(\hat{q}^1)^{2s}(\hat{B}^*)^{2\tau}(\hat{b}^*)^{2\nu}|0\rangle,
\] (3.16)

where the newly introduced operators \(\hat{q}^i\) and \(\hat{B}^*\) are defined as

\[
\hat{q}^i = \hat{b}^*\hat{b} - \hat{a}^*\hat{a}^i, \quad \hat{B}^* = \sum_i \hat{S}^i\hat{q}^i.
\] (3.17)

Then, the state with pairing correlation, \(|\lambda\rho\sigma_0T\rangle\) in Eq.(2.16), can be recast into

\[
|\lambda\rho\sigma_0\sigma_1\rangle = |\lambda, \sigma_1 - \sigma_0 - \rho, \rho, \sigma_1\rangle
\] (3.18)

with \(\sigma_1 = T - 2\). (See, (II-5-11).)
In the state (2.15) with $T = T_0$ and

$$\mu = \mu_0 = 0 \, , \quad \lambda = \lambda_0 = 0 \, , \quad n_0 = 0 \, , \quad n_1 = 2\rho \, ,$$

$$(\sigma_1 - \sigma_0 = \frac{1}{2}(n_1 - n_0)) \, , \quad (3.19)$$

the numbers of quarks with color $i$ are written from (3.6) as

- numbers of color 1 = $n_1$,
- numbers of color 2 = $2\rho = n_1$,
- numbers of color 3 = $2\rho = n_1$,

i.e., $2\rho = n_1 = \frac{N}{3}$.

Therefore, this state is color neutral. Here, the state (3.18) is written as

$$|00\rangle_{N/6} T - 2 \rangle \, . \quad (3.21)$$

For this state, the energy eigenvalue (2.23) with $\chi = 0$ is obtained as

$$E_{N \rho = N/6 \, n_0 = 0 \, n_1 = N/3} = -\frac{N}{3} \left(2\Omega + 3 - \frac{N}{3}\right) \, . \quad (3.22)$$

It will be shown in the following that this energy eigenvalue is identical with the one derived by the state which forms the color neutral quark triplet, namely $|\lambda\tau T\rangle$ in Eq.(2.26). From (2.32) with $n_0 = 0$ and $\tau = 3/2$, which means $\lambda = 0$ from Eq.(3.15), the energy eigenvalue in the case of the triplet formation is written as

$$E_{T=\Omega+2 \, t=\Omega+1/2+N/3 \, \tau=3/2 \, \lambda=0} = -\frac{N}{3} \left(2\Omega + 3 - \frac{N}{3}\right) \, . \quad (3.23)$$

Of course, $\tau = 3/2$ reveals that this state is the color neutral state. As was shown in (II), the state with triplet formation $|\lambda\tau T\rangle$ in Eq.(2.26) is nothing but the state $|lsrw\rangle$ in Eq.(3.16) with the following relations:

$$l = \lambda \, , \quad s = \frac{1}{2}(2\tau - 3 - 2\lambda) \, , \quad r = \frac{1}{2}(T - t - \tau) \, , \quad w = T - 2 \, , \quad (3.24)$$

which was given in (II-5·25). Thus, the above state with $n_0 = 0$ and $\tau = 3/2$ has the following parameters as

$$l = 0 \, , \quad s = 0 \, , \quad r = \frac{1}{2} \left(T - t - \frac{3}{2}\right) = \frac{N}{6} \, , \quad w = T - 2 \, , \quad (3.25)$$

where we used the relation in Eq.(2.31). Thus, this state is written as

$$|00N/6 \, T - 2 \rangle \, . \quad (3.26)$$

It should be noted that this triplet state is identical with the state with pairing correlation in Eq.(3.21).
§4. Ground state for \( \chi \neq 0 \)

In this section, the main aim of this paper is treated, namely, we investigate the ground state in our modified Bonn model whose Hamiltonian is given in Eq. (2.13). In the previous paper (I), we investigated the energy minimum state and its energy eigenvalue in the pairing correlation and triplet formation, respectively.

In the case of triplet formation, the energy of the colored state with \( \tau \neq 3/2 \) increases together with the particle-hole type coupling strength \( \chi \) increasing. Thus, if the coupling strength \( \chi \) is greater than a certain value, which was given in Eq. (I-5.25) or (I-5.26) for the particle or hole picture, respectively, the energy of the colored state is larger than that of the color neutral triplet state with \( \tau = 3/2 \) because the energy of the color neutral triplet has no change with respect to \( \chi \). Thus, the color neutrality is retained in our model.

As is similar to the case of the triplet formation, in the case of the pairing correlation in which the eigenstate is given in Eq. (2.15) or (2.16), the energy of the colored states is pushed up when the coupling strength \( \chi \) increases.

Here, let us search the ground state in the various coupling strength \( \chi \) numerically. Hereafter, we set \( n_0 = 0 \) whose parameter was used in (I). In (I), the energy minimum states were obtained in both the case of pairing correlation and triplet formation, respectively. As for the triplet formation, the energy is given in (3.23) for \( \chi = 0 \). As for the pairing correlation, the minimum energy states and minimum energy \( \mathcal{E} \) are determined analytically for various \( \chi \). We summarize them:

\[
(1) \quad -\frac{1}{6} \leq \chi \leq -\frac{1}{6} \\
\text{for } 0 \leq N \leq \frac{6(2\Omega - 3 - 6\chi)}{5 + 6\chi} \\
\mathcal{E} = E(N) + F(N) , \\
E(N) = -\frac{1}{4} N(4\Omega + 2 - N) , \\
F(N) = \frac{\chi}{6} N(N + 6) , \\
n_1 = 0 , \\
\text{for } \frac{6(2\Omega - 3 - 6\chi)}{5 + 6\chi} < N \leq 3\Omega \\
\mathcal{E} = E(N) + F(N) - \frac{N}{3\Omega} \left[(5N - 12\Omega + 18) + 6\chi(N + 6)\right] , \\
n_1 = \frac{N}{3} , \\
(2) \quad -\frac{1}{6} < \chi \leq \frac{1}{6}(2\Omega - 3) \\
\text{for } 0 \leq N \leq \frac{2\Omega}{1 + 2\chi} - 3 \\
\mathcal{E} = E(N) + F(N) , \\
E(N) = -\frac{1}{4} N(4\Omega + 2 - N) , \\
F(N) = \frac{\chi}{6} N(N + 6) , \\
n_1 = 0 , \\
\text{for } \frac{2\Omega}{1 + 2\chi} - 3 < N < 3\Omega - \frac{9}{2} - 9\chi
\]
\[ \mathcal{E} = E(N) + F(N) - \frac{1}{4(1 + 6\chi)} [(N - 2\Omega + 3) + 2\chi(N + 3)]^2 , \]
\[ n_1 = \frac{(N - 2\Omega + 3) + 2\chi(N + 3)}{1 + 6\chi} , \]
for \(3\Omega - \frac{9}{2} - 9\chi \leq N \leq 3\Omega \)
\[ \mathcal{E} = E(N) + F(N) - \frac{N}{36}[(5N - 12\Omega + 18) + 6\chi(N + 6)] , \]
\[ n_1 = \frac{N}{3} , \]
(3) \(\chi \geq \frac{1}{6}(2\Omega - 3)\)
for \(0 \leq N \leq 3\Omega \)
\[ \mathcal{E} = E(N) + F(N) - \frac{N}{36}[(5N - 12\Omega + 18) + 6\chi(N + 6)] , \]
\[ n_1 = \frac{N}{3} . \] 

In the case of the range \(3\Omega \leq N \leq 6\Omega\), \(N\) appearing in all the terms except \(E(N)\) should be replaced by \(6\Omega - N\) and as for \(E(N)\), the following should be used:
\[ E(N) = -\frac{1}{4}(N - 2\Omega)(6\Omega + 6 - N) . \] 

Here, we omit the case \(\chi < -(\Omega + 6)/(6(\Omega + 2))\) because we treat the model based on the original Bonn model which has the attractive pairing type interaction with \(\chi = 0\). If \(\chi\) is negative with large absolute value, the model mainly reveals the nature with the attractive particle-hole type interaction. Namely, this model gives a system with the attractive particle-hole type interaction plus relatively small attractive pairing interaction. Our purpose in this paper is to investigate the Bonn

![Fig. 1. The total energies are shown as a function of the particle number \(N\) with \(\Omega = 6\) in the case of \(\chi = 0\). The solid curve represents the minimum energy calculated in the case of the triplet formation. The dashed curve represents the minimum energy calculated in the case of the pairing correlation.](image)
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Fig. 2. The total energies are shown as a function of the particle number $N$ with $\Omega = 6$ in the case of $\chi = 1/4$. The solid curve represents the minimum energy calculated in the case of the triplet formation. The dashed curve represents the minimum energy calculated in the case of the pairing correlation.

Fig. 3. The total energies are shown as a function of the particle number $N$ with $\Omega = 6$ in the case of $\chi = 3/2$. The solid curve represents the minimum energy calculated in the case of the triplet formation. This curve is identical with the one which represents the minimum energy calculated in the case of the pairing correlation.

Our next task is to compare the energy of the state with pairing correlation in Eq. (2.16) with that with triplet formation in Eq. (2.26). In Fig. 1, the total energies are shown as a function of the particle number $N$ in the case $\chi = 0$ with $\Omega = 6$. The solid curve represents the minimum energy obtained in the case of the color neutral triplet formation. The dashed curve represents the minimum energy obtained in the case of the pairing correlation. It is shown that, partially, the energy of the colored pairing state is lower than that of the color neutral triplet state. If we introduce the $su(4)$ symmetry broken but color $su(3)$ symmetric term $\chi \hat{Q}^2$ in the original Bonn model, the situation is modified. In Fig. 2, the same results are depicted except for the coupling strength of the particle-hole type $\chi$ which we take $\chi = 1/4$. The energy of the colored state with pairing correlation is pushed up and this energy is close...
to that of the color neutral triplet formation. Finally, when the coupling strength is equal to $\chi = 3/2$ or greater than $3/2$ in the case $\Omega = 6$, the energy with pairing correlation is equivalent to that of the color neutral triplet, which is described in §3.3. In this region of the coupling strength, namely, $\chi \geq 3/2$ with $\Omega = 6$, the variable $n_1$ is equal to $N/3$ in the state with pairing correlation. Thus, the state with pairing correlation is reduced to the state with color neutral triplet as is mentioned in Eqs. (3.21) and (3.26). The calculated energies are shown in Fig. 3.

As was investigated in detail in (I), in the case of the state with pairing correlation in Eq. (2.16), the variable $n_1$ plays a role of the order parameter for the phase transition from pairing state to color neutral triplet state. If $n_1 = N/3$, the state is equivalent to the state with triplet formation. Thus, according to the value of $n_1$, we can give the phase diagram based on Eqs. (4-1) and (4-2) as is seen in Fig. 4 in which the phase diagram is shown with respect to the particle number $N$ (vertical axis) and the coupling strength $\chi$ (horizontal axis). The variables are fixed as $\Omega = 6$ and $n_0 = 0$, so the physical region is determined by the relation $0 \leq N \leq 6\Omega = 36$. The region (A) represents the one with $n_1 = 0$ and the states in this area are color-pairing states in general. The region (C) represents the one with $n_1 = N/3$. Thus, the states in this area are color neutral with triplet formation. The region (B) represents the transition region in which $0 < n_1 < N/3$. In the region (B), the order parameter $n_1$ is changed continuously with respect to the particle number $N$ under fixed $\chi$. Thus, the structure of the ground state is changed and it may be allowed to say that the phase change occurs.
§5. Discussions and concluding remarks

In the original Bonn model, that is, $\chi = 0$ in Eq. (2.13), the only pairing interaction between quarks with different colors is included, which may be regarded as an effective model of QCD. However, it is known that, in this model, the colored state such as the color superconducting state is energetically favorable. This fact leads to the pairing instability any time and to breaking of the color neutrality in the ground state, while this model reveals a nice feature that this model leads to the quark-triplet formation as a nucleon without the three-body correlation. As was pointed out in (I), the original Bonn model has the $su(4)$ symmetry. Thus, the room in which the $su(4)$ symmetry is broken but the color $su(3)$ symmetry is still retained is rest. Therefore, we introduced a particle-hole type interaction, $\chi \hat{Q}^2$, in the previous paper (I), where $\hat{Q}^2$ is the Casimir operator of the $su(3)$ sub-algebra.

In this effective model of QCD, we can control the strength of pairing interaction and the strength of particle-hole type interaction which are represented by $G$ and $\chi$, respectively. Here, we take the ratio $R = G/\chi$, in which we fix $G = 1$ in this paper. If the ratio $R$ is small, for example, $R \leq 2/3$ under the restriction $\Omega = 6$, the ground state is the color neutral triplet state in general. However, when $R$ increases from $2/3$, the ground state is no longer the color neutral quark triplet state under a specific value of the particle number $N$. Namely, the color pairing state becomes energetically stable. This fact indicates that the color pairing instability is realized in a certain region of $N$ when the strength of the pairing interaction, $G$, is relatively large for the strength of the particle-hole type interaction, $\chi$, that is, $R$ goes beyond a certain critical value. If it is allowed to use the terminology of the infinite nuclear and/or quark matter, this situation developed in our papers is resemble to the phase transition between the color superconducting and the normal nuclear phases. Of course, since we cannot deduce the interaction terms of the pairing and the particle-hole types from QCD directly, the parameter $R$ is unknown in this stage. However, it may be a possible scenario that $R$ depends on the baryon density in which $R$ may increases as the baryon density increases. As a result, the pairing instability may occur at high baryon density.

Acknowledgement

One of the authors (Y.T.) is partially supported by the Grants-in-Aid of the Scientific Research No. 18540278 from the Ministry of Education, Culture, Sports, Science and Technology in Japan.

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