CONE-VOLUME MEASURE AND STABILITY

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Abstract. We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This implies, among others, a conjectured best possible inequality for the $U$-functional of a convex body. For both results we provide stronger versions in the sense of stability inequalities.

1. Introduction

Let $K^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in K^n$ is a convex compact subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $\text{int}(K) \neq \emptyset$. As usual, we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\| \cdot \|$. $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{ x \in \mathbb{R}^n : \| x \| = 1 \}$. The norm associated to a $o$-symmetric convex body $K \in K^n$ is denoted by $\| \cdot \|_K$, i.e., $\| x \|_K = \min \{ \lambda \geq 0 : x \in \lambda K \}$.

For $K \in K^n$, we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and $\nu_K$ to denote the Gauß map assigning the exterior unit normal $\nu_K(x)$ to an $x \in \partial K$, where $\partial K$ consists of all points in the boundary $\partial K$ of $K$ having an unique outer normal vector. If the origin $o$ lies in the interior of $K \in K^n$, the cone-volume measure of $K$ on $S^{n-1}$ is given by

$$V_K(\omega) = \int_{\omega} \frac{h_K(u)}{n} dS_K(u) = \int_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x) \rangle}{n} d\mathcal{H}_{n-1}(x),$$

where $\omega \subset S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the $k$-dimensional Hausdorff-measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $V(\cdot)$ for the $n$-dimensional volume.

The name cone-volume measure stems from the fact that if $K$ is a polytope with facets $F_1, \ldots, F_m$ and corresponding exterior unit normals $u_1, \ldots, u_m$, then

$$V_K(\omega) = \sum_{i=1}^{m} V([o, F_i]) \delta_{u_i}(\omega).$$

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Here $\delta_u$ is the Dirac delta measure on $S^{n-1}$ at $u \in S^{n-1}$, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_L \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_L\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_L]$. With this notation $[o, F_i]$ is the cone with apex $o$ and basis $F_i$.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guédon, S. Mendelson and A. Naor [6], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [8, 9], M. Gromov and V.D. Milman [17], M. Ludwig [28], M. Ludwig and M. Reitzner [29], E. Lutwak, D. Yang and G. Zhang [32], A. Naor [34], A. Naor and D. Romik [35], G. Paouris and E. Werner [36], A. Stancu [42], G. Zhu [45, 46].

In particular, cone-volume measures are the subject of the logarithmic Minkowski problem, which is the particular interesting limiting case $p = 0$ of the general $L_p$-Minkowski problem – one of the central problems in convex geometric analysis. It is the task:

Find necessary and sufficient conditions for a Borel measure $\mu$ on $S^{n-1}$ to be the cone-volume measure $V_K$ of $K \in \mathcal{K}^n$ (with $o$ in its interior).

In the recent paper [9], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang characterize the cone-volume measures of origin-symmetric convex bodies. In order to state their result we have to introduce the subspace concentration condition. We say that a Borel measure $\mu$ on $S^{n-1}$ satisfies the subspace concentration condition if for any linear subspace $L \subseteq \mathbb{R}^n$, we have

\begin{equation}
\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1}),
\end{equation}

and equality in (1.3) for some $L$ implies the existence of a complementary linear subspace $\tilde{L}$ such that

\begin{equation}
\mu(\tilde{L} \cap S^{n-1}) = \frac{\dim \tilde{L}}{n} \mu(S^{n-1}),
\end{equation}

and hence $\text{supp } \mu \subseteq L \cup \tilde{L}$, i.e., the support of the measure “lives” in $L \cup \tilde{L}$.

Via the subspace concentration condition, the logarithmic Minkowski problem was settled in [9] in the symmetric case.

**Theorem 1.1** ([9]). A non-zero finite even Borel measure on $S^{n-1}$ is the cone-volume measure of an origin-symmetric convex bodies if and only if it satisfies the subspace concentration condition.

This result was proved earlier for discrete measures on $S^1$, i.e., for polygons, by A. Stancu [40, 41]. For cone-volume measures of origin-symmetric polytopes (cf. (1.2)) the necessity of (1.3) was independently shown by M. Henk, A. Schürmann and J.M.Wills [23] and B. He, G. Leng and K. Li [22].

We recall that the centroid of a $k$-dimensional convex compact set $M \subset \mathbb{R}^n$ is defined as

\[ c(M) = \mathcal{H}_k(M)^{-1} \int_M x \, d\mathcal{H}_k(x). \]
The centroid seems also be the right and natural position of the origin in order to extend Theorem 1.1 to arbitrary convex bodies. In fact, in [24] it was shown by M. Henk and E. Linke that the necessity part of Theorem 1.1 also holds for polytopes with centroid at the origin, i.e.,

**Theorem 1.2** ([24]). Let \( K \in \mathcal{K}^n \) be a polytope with centroid at the origin. Then its cone-volume measure \( V_K \) satisfies the subspace concentration condition.

Our first result is an extension of Theorem 1.2 to convex bodies.

**Theorem 1.3.** Let \( K \in \mathcal{K}^n \) with centroid at the origin. Then its cone-volume measure satisfies the subspace concentration condition.

While the subspace concentration condition is also the sufficiency property to characterize cone-volume measures among even non-trivial Borel measures, the cone-volume measure of a convex body \( K \in \mathcal{K}^n \) whose centroid is the origin should satisfy some extra properties. For example, in Proposition 4.1 we prove that the measure of any open hemisphere is at least \( \frac{1}{2^n} \).

If the origin is not the centroid of the convex body, then the subspace concentration condition may not hold anymore. In fact, it was recently shown by G. Zhu [45] that for unit vectors \( u_1, \ldots, u_m \in S^{n-1} \) in general position, \( m \geq n + 1 \), and arbitrary positive numbers \( \gamma_1, \ldots, \gamma_m \) there exists a polytope \( P \) with outer unit normals \( u_i \) with \( V_P(\{u_i\}) = \gamma_i, 1 \leq i \leq m \). In other words, Zhu settled the logarithmic Minkowski problem for discrete measures whose support is in general position. In general, the centroid of such a polytope \( P \) is not the origin, and a full characterization of cone-volume measures of arbitrary polytopes/bodies is still a challenging and important problem.

We note that (1.4) is a kind of condition on the cone-volume measure which is independent of the choice of the origin.

**Lemma 1.4.** If \( K \in \mathcal{K}^n \) with \( o \in \text{int} K \), and \( \text{supp} V_K \subset L \cup \tilde{L} \) for the proper complementary linear subspaces \( L, \tilde{L} \subset \mathbb{R}^n \), then

\[
V_K(L \cap S^{n-1}) = \frac{\dim L}{n} \mu(S^{n-1}).
\]

Let us provide the simple argument leading to Lemma 1.4. It follows from Minkowski’s uniqueness theorem that \( K = M + \tilde{M} \) where \( M, \tilde{M} \) are contained in affine spaces orthogonal to \( L, \tilde{L} \), respectively. By Fubini’s theorem, we conclude (1.4) for \( V_K \) and the subspaces \( L, \tilde{L} \).

For a convex body \( K \) containing the origin in its interior, E. Lutwak, D. Yang and G. Zhang [30] defined the \( \text{SL}(n) \) invariant quantity \( U(K) \) as an integral over subsets \( (u_1, \ldots, u_n) \in S^{n-1} \times \cdots \times S^{n-1} \), by

\[
U(K) = \left( \int_{u_1 \wedge \ldots \wedge u_n \neq 0} dV_K(u_1) \cdots dV_K(u_n) \right)^{\frac{1}{n}}.
\]
where \( u_1 \wedge \ldots \wedge u_n \neq 0 \) means that the vectors \( u_1, \ldots, u_n \) are linearly independent. The \( U \)-functional has been proved useful in obtaining strong inequalities for the volume of projection bodies \([30]\). For information on projection bodies we refer to the books by Gardner \([15]\) and Schneider \([39]\), and for more information on the importance of centro-affine functionals we refer to C. Haberl and L. Parapatits \([21, 29]\) and the references within.

We readily have \( U(K) \leq V(K) \), and equality holds if and only if \( V_K(L \cap S^{n-1}) = 0 \) for any non-trivial subspace of \( \mathbb{R}^n \) according to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang \([10]\). As a consequence of Theorem 1.3 we prove here a lower bound on \( U(K) \) in terms of \( V(K) \) which was conjectured in \([10]\).

**Theorem 1.5.** Let \( K \in K^n \) with centroid at the origin. Then

\[
U(K) \geq \frac{(n!)^{1/n}}{n} V(K),
\]

with equality if and only if \( K \) is a parallelepiped.

In particular, \( U(K) > (1/e) V(K) \). For polytopes, Theorem 1.5 was shown in \([24]\), where the special cases if \( K \) is an origin-symmetric polytope, or if \( n = 2, 3 \) were verified by B. He, G. Leng and K. Li \([22]\), and G. Xiong \([44]\), respectively.

In order to state another consequence of Theorem 1.5 we need the notation of an isotropic measure, going back to K.M. Ball’s reformulation of the Brascamp-Lieb inequality in \([2]\). A Borel measure \( \mu \) on \( S^{n-1} \) is called isotropic if

\[
\text{Id}_n = \int_{S^{n-1}} u \otimes u \, d\mu(u),
\]

where \( \text{Id}_n \) is the \( n \times n \)-identity matrix and \( u \otimes u \) the standard tensor product, i.e., \( u \otimes u = uu^\top \).

Equating traces shows that for an isotropic measure \( \mu(S^{n-1}) = n \). The subspace concentration condition of a Borel measure \( \mu \) on \( S^{n-1} \) is equivalent to have an isotropic normalized linear image of \( \mu \), i.e., that is, there exists a \( \Phi \in \text{GL}(n) \) such that

\[
\text{Id}_n = \int_{S^{n-1}} \frac{\Phi u}{\|\Phi u\|} \otimes \frac{\Phi u}{\|\Phi u\|} \, d\mu(u).
\]

The equivalence in this general form is due to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang \([10]\), while the discrete case was earlier handled by E. A. Carlen, and D. Cordero-Erausquin \([11]\), and J. Bennett, A. Carbery, M. Christ and T. Tao \([7]\) in their study of the Brascamp-Lieb inequality. Moreover, the case of a measure \( \mu \) when strict inequality holds for all subspaces in \([13]\) is due to B. Klartag \([27]\). Isotropic measures on \( S^{n-1} \) are discussed also e.g. in F. Barthe \([3, 4]\), E. Lutwak, D. Yang and G. Zhang \([31, 33]\). We note that isotropic measures on \( \mathbb{R}^n \) play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits \([25]\), see, e.g., F. Barthe and
D. Cordero-Erausquin [5], O. Guedon and E. Milman [20] and B. Klartag [26].

Now from Theorem 1.5 and by the equivalence (1.5) we immediately conclude

**Corollary 1.6.** Every convex body $K \in \mathcal{K}^n$ has an affine image, whose cone-volume measure is isotropic.

This, in particular, answers a question posed by E. Lutwak, D. Yang and G. Zhang [32].

In order to present stronger stability versions of Theorem 1.3 and Theorem 1.5 we need two notions of distance between the “shapes” of two convex bodies. Let $K, M \in \mathcal{K}^n$, and let $K' = K - c(K), M' = M - c(M)$ be their translates whose centroids are the origin. Then we define

$$
\delta_{\text{hom}}(K, M) = \min \{ \lambda \geq 0 : \exists t > 0, M' \subset tK' \subset e^{\lambda}M' \},
$$

$$
\delta_{\text{vol}}(K, M) = \frac{V(M' \Delta tK')}{V(M)}, \quad t = \frac{V(M)^{1/n}}{V(K)^{1/n}},
$$

where $A \Delta B$ denotes the symmetric difference of two sets, i.e., $A \Delta B = A \setminus B \cup B \setminus A$.

Then both $\delta_{\text{hom}}$ and $\delta_{\text{vol}}$ are metrics on the space of convex bodies in $\mathbb{R}^n$ whose volumes are 1, and centroids are the origin.

**Theorem 1.7.** Let $K \in \mathcal{K}^n$ with centroid at the origin, and let

$$
V_K(L \cap S^{n-1}) > \frac{d - \varepsilon}{n} V(K)
$$

for a non-trivial linear subspace $L$ with $\dim L = d$ and $\varepsilon \in (0, \varepsilon_0)$. Then there exist an $(n - d)$-dimensional compact convex set $C \subset L^\perp$, and a complementary $d$-dimensional compact convex set $M$ such that

$$
\delta_{\text{hom}}(K, C + M) \leq \gamma_h \varepsilon^{1/(5n)} \quad \text{and} \quad \delta_{\text{vol}}(K, C + M) \leq \gamma_v \varepsilon^{1/5},
$$

where $\varepsilon_0, \gamma_h, \gamma_v > 0$ depend only on $n$.

Here $L^\perp$ denotes the orthogonal complement of $L$, and $M$ is called a complementary compact convex set of $C$, if the linear spaces generated by $M$ and $C$ are complementary.

Observe that the range of $\varepsilon$, i.e., $\varepsilon_0$, in Theorem 1.7 has to depend on the dimension. For if, let $K \in \mathcal{K}^n$ be a simplex whose centroid is the origin, and let $L$ be generated by $d$ outer normals of the simplex, $d \in \{1, \ldots, n-1\}$. Then we have $V_K(L \cap S^{n-1}) = \frac{d}{n+1} V(K)$.

Actually, if $L$ is 1-dimensional, then a more precise version of Theorem 1.7 holds.

**Theorem 1.8.** Let $K \in \mathcal{K}^n$ with centroid at the origin, and let

$$
V_K(L \cap S^{n-1}) > \frac{1 - \varepsilon}{n} V(K)
$$
for a linear subspace $L$ with $\dim L = 1$ and $\varepsilon \in (0, \tilde{\varepsilon}_0)$. Then there exist $(n - 1)$-dimensional compact convex set $C \subset L^\perp$ with $c(C) = 0$, and $x, y \in \partial K$ such that $y = -e^a x$ where $|s| < \tilde{\gamma}_v \varepsilon^{1/n}$, $[x, y] + C \subset K$, and

$$K \subset [x, y] + (1 + \gamma_\varepsilon \varepsilon^{1/n})C \quad \text{and} \quad V(K) \leq (1 + \gamma_\varepsilon \varepsilon^{1/n})V([x, y] + C),$$

where $\tilde{\varepsilon}_0, \tilde{\gamma}_h, \tilde{\gamma}_v > 0$ depend only on $n$.

We use this theorem in order to deduce the following stability version of Theorem 1.5.

**Theorem 1.9.** Let $K \in \mathcal{K}^n$ with centroid at the origin, and let

$$U(K) \leq (1 + \varepsilon)^{\frac{1}{n}} V(K)$$

for $\varepsilon \in (0, \varepsilon_*)$. Then there exists a $K$ containing parallelepiped $P$, such that for any facet $F$ of $P$, we have

$$H_{n-1}(F \cap K) \geq (1 - \gamma_* \varepsilon^{1/n})H_{n-1}(F),$$

where $\varepsilon_*, \gamma_* > 0$ depend only on $n$. In particular, we have

$$(1 - \gamma \varepsilon^{1/n})P \subset K \quad \text{and} \quad V(P \setminus K) \leq \gamma \varepsilon^{1/n} V(K).$$

The paper is organized as follows. In the next section we collect some basic facts and notations from convexity which will be used later on. The third section is devoted to the proof of Theorem 1.3. In Section 4 we show another characteristic property of cone-volume measures of convex bodies with centroid at the origin. The proofs of Theorem 1.7, 1.8 are given in Section 8 and are prepared in Sections 5–7. Finally, in Section 9 we prove Theorem 1.5.

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**2. Preliminaries**

Good general references for the theory of convex bodies are provided by the books of Gardner [15], Gruber [18], Schneider [39] and Thompson [43].

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of convex body $K \in \mathcal{K}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\}.$$

A boundary point $x \in \partial K$ is said to have a unit outer normal (vector) $u \in S^{n-1}$ provided $\langle x, u \rangle = h_K(u)$. $x \in \partial K$ is called singular if it has more than one unit outer normal, and $\partial_s K$ is the set of all non-singular boundary points. It is well known that the set of singular boundary points of a convex body has $H_{n-1}$-measure equal to 0. For each Borel set $\omega \subset S^{n-1}$, the inverse spherical image of $\omega$ is the set of all points of $\partial K$ which have an outer unit normal belonging to $\omega$. Since the inverse spherical image of $\omega$ differs from
\( \nu^{-1}_K(\omega) \subseteq \partial_* K \) by a set of \( \mathcal{H}_{n-1} \)-measure equal to 0, we will often make no distinction between the two sets.

For \( K \in K^n \) the Borel measure \( S_K \) on \( S^{n-1} \) given by
\[
S_K(\omega) = \mathcal{H}_{n-1}(\nu^{-1}_K(\omega))
\]
is called the (Aleksandrov-Fenchel-Jessen) surface area measure. Observe that
\[
V(K) = V_{K}(S^{n-1}) = \int_{S^{n-1}} \frac{h_K(u)}{n} dS_K(u).
\]

As usual, for two subsets \( C, D \subseteq \mathbb{R}^n \) and reals \( \nu, \mu \geq 0 \) the Minkowski combination is defined by
\[
\nu C + \mu D = \{ \nu c + \mu d : c \in C, d \in D \}.
\]

By the celebrated Brunn-Minkowski inequality we know that the \( n \)-th root of the volume of the Minkowski combination is a concave function. More precisely, for two convex compact sets \( K_0, K_1 \subset \mathbb{R}^n \) and for \( \lambda \in [0, 1] \) we have
\[
(2.1) \quad V((1 - \lambda) K_0 + \lambda K_1)^{1/n} \geq (1 - \lambda) V(K_0)^{1/n} + \lambda V(K_1)^{1/n}
\]
with equality for some \( 0 < \lambda < 1 \) if and only if \( K_0 \) and \( K_1 \) lie in parallel hyperplanes or are homothetic, i.e., there exist \( t \in \mathbb{R}^n \) and \( \mu \geq 0 \) such that \( K_1 = t + \mu K_0 \) (see also [16]).

Let \( f : C \to \mathbb{R}_{>0} \) be a positive function on an open convex subset \( C \subset \mathbb{R}^n \) with the property that there exists a \( k \in \mathbb{N} \) such that \( f^{1/k} \) is concave. Then by the (weighted) arithmetic-geometric mean inequality
\[
\begin{align*}
f((1 - \lambda) x + \lambda y) &= \left( f^{1/k}((1 - \lambda) x + \lambda y) \right)^k \\
&\geq \left( (1 - \lambda)f^{1/k}(x) + \lambda f^{1/k}(y) \right)^k \\
&\geq f^{1-\lambda}(x) \cdot f^\lambda(y).
\end{align*}
\]
This means that \( f \) belongs to the class of log-concave functions which by the positivity of \( f \) is equivalent to
\[
\ln f((1 - \lambda) x + \lambda y) \geq (1 - \lambda) \ln f(x) + \lambda \ln f(y)
\]
for \( \lambda \in [0, 1] \). Hence, for all \( x, y \in C \) there exists a subgradient \( g(y) \in \mathbb{R}^n \) such that (cf., e.g., [38, Sect. 23])
\[
(2.2) \quad \ln f(x) - \ln f(y) \leq \langle g(y), x - y \rangle.
\]
If \( f \) is differentiable at \( y \), the subgradient is the gradient of \( \ln f \) at \( y \), i.e., \( g(y) = \nabla \ln f = \frac{1}{f(y)} \nabla f(y) \).

For a subspace \( L \subseteq \mathbb{R}^n \), let \( L^\perp \) be its orthogonal complement, and for \( X \subseteq \mathbb{R}^n \) we denote by \( X|L \) its orthogonal projection onto \( L \), i.e., the image of \( X \) under the linear map forgetting the part of \( X \) belonging to \( L^\perp \).
Here, for a convex body \( K \in \mathcal{K}^n \) and a \( d \)-dimensional subspace \( L \), \( 1 \leq d \leq n - 1 \), we are interested in the function measuring the volume of \( K \) intersected with planes parallel to \( L^\perp \), i.e., in the function
\[
(2.3) \quad f_{K,L} : L \to \mathbb{R}_{\geq 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)),
\]
where \( k = n - d \) is the dimension of \( L^\perp \). By the Brunn-Minkowski inequality and the remark above, \( f_{K,L} \) is a log-concave function on \( K \) which is positive at least in the relative interior of \( K \) (cf. [1]). \( f_{K,L} \) is also called the \( k \)-dimensional X-ray of \( K \) parallel to \( L^\perp \) (cf. [15]). By well-known properties of concave functions we also know

**Proposition 2.1.**

i) \( \ln f_{K,L} \) – and thus \( f_{K,L} \) – is continuous on \( \text{int}(K) \cap L \). Moreover, \( \ln f_{K,L} \) – and thus \( f_{K,L} \) – are Lipschitzian on any compact subset of \( \text{int}(K) \cap L \).

ii) \( \ln f_{K,L} \) – and thus \( f_{K,L} \) – is on \( \text{int}(K) \cap L \) almost everywhere differentiable, i.e., there exists a dense subset \( D \subseteq \text{int}(K) \cap L \), where \( \nabla f_{K,L} \) exists.

*Proof.* For i) see, e.g., [39, Theorem 1.5.3], and for ii) see, e.g., [38, Theorem 25.5].

Now for \( K \in \mathcal{K}^n \) with centroid at 0, i.e., \( c(K) = 0 \), we have by Fubini’s theorem with respect to the decomposition \( L \oplus L^\perp \)
\[
0 = \int_K x \, d\mathcal{H}_n(x)
= \int_{K \cap L} \left( \int_{(\hat{x} + L^\perp) \cap K} \hat{x} \, d\mathcal{H}_k(\hat{x}) \right) \, d\mathcal{H}_d(\hat{x})
= \int_{K \cap L} f_{K,L}(\hat{x}) \, c((\hat{x} + L^\perp) \cap K) \, d\mathcal{H}_d(\hat{x}).
\]
Writing \( c((\hat{x} + L^\perp) \cap K) = \hat{x} + \hat{y} \) with \( \hat{y} \in L^\perp \) gives
\[
(2.4) \quad \int_{K \cap L} f_{K,L}(\hat{x}) \, \hat{x} \, d\mathcal{H}_d(\hat{x}) = 0.
\]

### 3. Proof of Theorem 1.3

For the proof of Theorem 1.3 we first establish some more properties of the function \( f_{K,L} \), where we always assume that \( L \subset \mathbb{R}^n \) is a \( d \)-dimensional linear subspace, \( 1 \leq d \leq n - 1 \), with \( k \)-dimensional orthogonal complement \( L^\perp \). We recall that a function \( f \) is said to be upper semicontinuous on \( K \) if whenever \( x, y_m \in K \) for \( m \in \mathbb{N} \) and \( y_m \) tends to \( x \), then
\[
f(x) \geq \limsup_{m \to \infty} f(y_m).
\]

**Lemma 3.1.** The function \( f_{K,L} \) is upper semicontinuous on \( K \cap L \).
Proof. Let \( x, y_m \in K|L \) for \( m \in \mathbb{N} \) be such that \( \lim_{m \to \infty} y_m = x \). According to the Blaschke selection principle (cf., e.g., [39]), we may assume that the sequence of compact convex sets
\[
C_m = [(y_m + L^\perp) \cap K] - y_m \subset L^\perp
\]
tends to a compact convex set \( C \subset L^\perp \) in the Hausdorff topology. Since the \( k \)-volume of a compact convex set in \( L^\perp \) is a continuous functional, we have \( H_k(C) = \lim_{m \to \infty} f_{K,L}(y_m) \). However, \( x + C \subset K \), and therefore \( f_{K,L}(x) \geq H_k(C) \). \( \square \)

An immediate consequence is that for sequences from the relative interior of \( K|L \), \( f_{K,L} \) behaves “continuously”, i.e.,

Corollary 3.2. Let \( o \in \text{int} \ K \) and \( x \in K|L \). Then \( \lim_{m \to \infty} f_{K,L}(e \frac{-1}{m} x) = f_{K,L}(x) \).

Proof. Since \( o \in \text{int} \ K \), we get by the concavity of \( f^{1/k}_{K,L} \) that
\[
f_{K,L}(e \frac{-1}{m} x) \geq e \frac{1}{m} f_{K,L}(x).
\]
Since \( f_{K,L} \) is also upper semicontinuous on \( K|L \) by Lemma 3.1, we conclude the corollary. \( \square \)

Although the gradient \( \nabla f_{K,L} \) might not be bounded, its norm belongs to the space \( L^1(K|L) \) of absolute integrable functions.

Lemma 3.3. \( \|\nabla f_{K,L}\| \in L^1(K|L) \), and thus the function \( x \mapsto \langle \nabla f_{K,L}(x), x \rangle \) is in \( L^1(K|L) \), as well.

Proof. Let \( f = f_{K,L} \). Since \( \nabla f(x) = \nabla (f^{1/k}(x)) = k f^{\frac{1}{1-k}}(x) \nabla f^{\frac{1}{k}}(x) \) for almost all \( x \in K|L \), it is sufficient to prove \( \|\nabla h\| \in L^1(K|L) \) for the concave function \( h = f^{\frac{1}{k}} \). However, by the Brunn-Minkowski theorem, the graph \( X \) of the function \( h \) over \( K|L \) is part of the boundary of a \( (d+1) \)-dimensional compact convex set. Thus
\[
\int_{K|L} \|\nabla h\| d\mathcal{H}_d(x) \leq \int_{K|L} \sqrt{1 + \|\nabla h\|^2} d\mathcal{H}_d(x) = \mathcal{H}_d(X) < \infty.
\]
\( \square \)

The next two statements, which are the core ingredients of the proof of Theorem 1.3 have been proved in the special case of polytopes in [24].

Proposition 3.4. If \( o \in \text{int} \ K \), then
\[
n V_K(L \cap S^{n-1}) = d V(K) + \int_{K|L} \langle \nabla f_{K,L}(x), x \rangle d\mathcal{H}_d(x).
\]

Proof. Let \( f = f_{K,L} \), and let \( F(x) = f(x)x \) for \( x \in K|L \), which is a Lipschitz vector field on any compact subset of \( \text{int} \ K|L \) (cf. (2.1) i)). To state
the Gauß-Green divergence theorem for Lipschitz vector fields on Lipschitz domains, we follow W.F. Pfeffer [37]. Naturally,

\[ E_m = e^\frac{1}{m} K \subset (\text{int } K) \]

is a compact Lipschitz domain for \( m \geq 1 \), and hence \( \partial_s E_m = \partial E_m \) according to Proposition 4.1.2 in [37], where \( \partial(E_m) \) denotes the (relative) boundary with respect to the linear space \( L \).

Therefore Theorem 6.5.4 in [37] (going back to H. Federer [13]) yields that

\[ \int_{\partial E_m} \langle F(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}_{d-1}(x) = \int_{E_m} \text{div} F(x) \, d\mathcal{H}_d(x). \]  

(3.1)

If \( y \in \partial(K|L) \) then \( \nu_{K|L}(y) = \nu_{E_m}(e^\frac{1}{m} y) \); thus the left hand side of (3.1) is

\[ \int_{\partial E_m} \langle F(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}_{d-1}(x) = e^{-(d-1)} \int_{\partial(K|L)} \langle F(e^\frac{1}{m} y), \nu_{K|L}(y) \rangle \, d\mathcal{H}_{d-1}(y) \]

\[ = e^\frac{1}{m} \int_{\partial(K|L)} f(e^\frac{1}{m} y) \langle y, \nu_{K|L}(y) \rangle \, d\mathcal{H}_{d-1}(y). \]

Therefore, Corollary 3.2 and the Lebesgue dominated convergence theorem yield

\[ \lim_{m \to \infty} \int_{\partial E_m} \langle F(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}_{d-1}(x) = \int_{\partial(K|L)} f(y) \langle y, \nu_{K|L}(y) \rangle \, d\mathcal{H}_{d-1}(y). \]

(3.2)

Now, in order to evaluate the right hand side let \( X = \partial K \cap (L^+ + \partial(K|L)) \). Then the set of smooth points of \( \partial K \) in \( X \), i.e., \( \partial X \cap X \) coincides with the set of points in \( \nu_K^{-1}(L \cap S^{n-1}) \). In addition, if \( z \in X \cap \partial X \), then \( \nu_{K|L}(y) = \nu_{K}(z) \) for \( y = z|L \), and thus (1.1) and (3.2) give

\[ \lim_{m \to \infty} \int_{\partial E_m} \langle F(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}_{d-1}(x) = \int_X \langle z, \nu_{K}(z) \rangle \, d\mathcal{H}_{n-1}(z) \]

\[ = n V_K(L \cap S^{n-1}). \]  

(3.3)

Next, if \( \nabla f(x) \) exists at \( x \in \text{int } (K|L) \), then

\[ \text{div} F(x) = df(x) + \langle x, \nabla f(x) \rangle. \]

Therefore the right hand side of (3.1) is (cf. Proposition 2.1 ii), Lemma 3.3

\[ \int_{E_m} \text{div} F(x) \, d\mathcal{H}_d(x) = d \int_{E_m} f(x) \, d\mathcal{H}_d(x) + \int_{E_m} \langle x, \nabla f(x) \rangle \, d\mathcal{H}_d(x). \]

Since \( \int_{K|L} f(x) \, d\mathcal{H}_d(x) = V(K) \), we deduce that

\[ \lim_{m \to \infty} \int_{E_m} \text{div} F(x) \, d\mathcal{H}_d(x) = dV(K) + \int_{K|L} \langle x, \nabla f(x) \rangle \, d\mathcal{H}_d(x). \]

(3.4)

Combining (3.1), (3.3) and (3.4) completes the proof of the proposition. \( \Box \)
If $K$ is an $o$-symmetric convex body, we know by the Brunn-Minkowski inequality (2.1) that $f_{K,L}(x)$ attains its maximum at the origin $0$. Hence, in view of (2.2) we know that $\langle \nabla f_{K,L}(x), x \rangle \leq 0$ for almost every $x \in K|L$. Although, this is no longer true for bodies with centroid at 0, the next proposition shows that it is true in the average.

**Proposition 3.5.** If $c(K) = o$, then
\[
\int_{K|L} \langle \nabla f_{K,L}(x), x \rangle \, d\mathcal{H}_d(x) \leq 0,
\]
with equality if and only if $f_{K,L}$ is constant on $K|L$.

**Proof.** Again, let $f = f_{K,L}$ and let $g : K|L \to L$ be a subgradient of $f$. For $z \in (\text{int } K)|L$, applying (2.2) to $y = o$ and $x = z$ first, and next to $y = z$ and $x = o$, we deduce that
\[
\langle g(z), z \rangle \leq \ln f(z) - \ln f(0) \leq \langle g(o), z \rangle,
\]
where $g$ is a subgradient of $f$. In particular, if $\nabla f$ exists at $z \in (\text{int } K)|L$, then $\langle \nabla f(z), z \rangle \leq \langle g(o), zf(z) \rangle$. Together with the property $c(K) = 0$ we get from (2.4) that
\[
\int_{K|L} \langle \nabla f(z), z \rangle \, d\mathcal{H}_d(z) \leq \int_{K|L} \langle g(o), zf(z) \rangle \, d\mathcal{H}_d(z) = 0.
\]

Let us assume that equality holds in (3.6), and hence for almost all $z \in (\text{int } K)|L$ in (3.5). In particular, we have $\ln f(x) - \ln f(0) = \langle g(o), x \rangle$, and in turn $f(x) = f(0)e^{\langle g(o), x \rangle}$ for almost all $x \in (\text{int } K)|L$. Since $f$ is continuous on $(\text{int } K)|L$, Corollary 3.2 yields that $f(x) = f(0)e^{\langle g(o), x \rangle}$ for all $x \in K|L$. However $f^\frac{1}{d}$ is concave, therefore $g(o) = o$, or in other words, $f$ is constant. \hfill \Box

Now, we are ready to give the proof of Theorem 1.3.

**Proof of Theorem 1.3** Combining Propositions 3.4 and 3.5 yields that
\[
V_K(L \cap S^{n-1}) = \frac{d}{n} V(K) + \frac{1}{n} \int_{K|L} \langle \nabla f_{K,L}(x), x \rangle \, d\mathcal{H}_d(x) \leq \frac{d}{n} V(K).
\]

Let us assume that equality holds, and hence $f_{K,L}(x) = f_{K,L}(o)$ for $x \in K|L$ according to Proposition 3.5. Let $C(x) = K \cap (x + L^\perp)$ for $x \in K|L$. For any $x \in K|L$, there exists $\eta > 0$ such that $-\eta x \in K|L$, and hence
\[
\frac{\eta}{1+\eta} C(x) + \frac{1}{1+\eta} C(-\eta x) \subset C(o).
\]
Therefore $f_{K,L}(x) = f_{K,L}(-\eta x) = f_{K,L}(o)$ and the equality characterization of the Brunn-Minkowski inequality (2.1) implies that $C(x)$ is a translate of $C(o)$.

Choose linearly independent $v_1, \ldots, v_d \in K|L$ such that $v_0 = -v_1 - \ldots - v_d \in K|L$, as well. By $\sum_{i=0}^d v_i = o$ we have $\sum_{i=0}^d \frac{1}{d+1} C(v_i) \subset C(o)$, and we
deduce that $\sum_{i=0}^{d} \frac{1}{\eta_i} c(C(v_i)) = c(C(o))$. In particular,
$c(C(o)) \in \Pi = \text{aff}\{c(C(v_0)), \ldots, c(C(v_d))\}$,
where $\text{aff}\{\}$ denotes the affine hull. Next let $x \in K \setminus L$. There exists $\eta > 0$ such that $-\eta x \in [v_0, \ldots, v_d]$, and so $\lambda x + \sum_{i=0}^{d} \lambda_i v_i = o$, where $\lambda + \sum_{i=0}^{d} \lambda_i = 1$ and $\lambda, \lambda_i \geq 0$ for $i = 0, \ldots, d$. It follows as above that $\lambda C(x) + \sum_{i=0}^{d} \lambda_i C(v_i) \in C(o)$, and hence $c(C(x)) \in \Pi$, as well. Therefore, writing $M = \Pi \cap (K + L^\perp)$ and $C = C(o) - c(C(o))$, we get $K = C + M$. In particular, $\text{supp} V_K \subset L \cup \Pi^\perp$ and $L \cap \Pi^\perp = \{0\}$.

4. Another property of the cone-volume measure if the centroid is the origin

Let us recall two basic notions about convex bodies. Firstly, a convex body in $\mathbb{R}^n$ is called a cylinder if it is of the form $[p, q] + C$ for $p, q \in \mathbb{R}^n$ and an $(n-1)$-dimensional convex compact set $C$; $p + C$ and $q + C$ are called bases of the cylinder.

Secondly, let $v \in S^{n-1}$, and let $M$ be a convex body in $\mathbb{R}^n$. For any $t$ with $-h_M(-v) < t < h_M(v)$, we replace the section $M \cap (tv + v^\perp)$ with the $(n-1)$-ball of the same $(n-1)$-measure, centered at $tv$ in $tv + v^\perp$. Here, $v^\perp$ is the abbreviation for the linear space orthogonal to $v$.

The closure $\tilde{M}$ of the union of these $(n-1)$-balls is called the Schwarz rounding of $M$ with respect to $\mathbb{R}v$. It is a convex body by the Brunn-Minkowski theorem, and readily satisfies $V(\tilde{M}) = V(M)$. If $\tilde{M}$ is a cylinder, then all sections of the form $\tilde{M} \cap (tv + v^\perp)$ are of the same $(n-1)$-measure, and hence the equality case of the Brunn-Minkowski theorem yields that $M$ is a cylinder, as well. For more on Schwarz rounding we refer to [13].

**Proposition 4.1.** Let $K \in \mathcal{K}^n$ with $c(K) = o$ and $V(K) = 1$. Then

$$V_K(\Omega) \geq \frac{1}{2n},$$

for any open hemisphere $\Omega \subset S^{n-1}$. Equality holds if and only if $K$ is a cylinder whose generating segment is orthogonal to the linear $(n-1)$-space bounding the hemisphere $S$.

**Proof.** Let $\Omega \subset S^{n-1}$ be an open hemisphere, and let $v \in S^{n-1}$ such that

$$\Omega = \{u \in S^{n-1} : \langle u, v \rangle > 0\}.$$

For any convex body $M \in \mathcal{K}^n$ with $o \in \text{int} M$ and $x \in M|v^\perp$, let

$$f_M(x) = \max\{t \in \mathbb{R} : x + tv \in M\},$$

and let $\varphi_M(x) = x + f_M(x)v$.

In particular the points of $\partial M$ where all exterior normals have acute angle with $v$ are of the form $\varphi_M(x)$ for $x \in \text{int} M|v^\perp$. Therefore

$$V_M(\Omega) = V(\Xi_M) \quad \text{for} \quad \Xi_M = \bigcup_{x \in M|v^\perp} [o, \varphi_M(x)].$$
For \( x \in (\text{int} M[v^\perp]) \setminus \{o\} \), let \( z = \theta^{-1} x \in \partial M[v^\perp] \) for some \( \theta \in (0,1) \). Since

\[ [\varphi_M(z), o, \varphi_M(o)] \subset \Xi_M, \]

we have

\[ (4.1) \quad x + \mathbb{R}v \text{ intersects } \Xi_M \text{ in a segment of length at least } (1 - \theta)\|\varphi_M(o)\|, \]

with equality if and only if \([\varphi_M(z), \varphi_M(o)] \subset \partial M\).

Now, let \( \lambda = f_K(o) \), and hence \( \lambda v \in \partial K \). After a linear transformation we may assume that the tangent hyperplane \( H \) at \( \lambda v \) is given by \( H = \lambda v + v^\perp \).

We shake \( K \) down to \( H \), i.e., for each \( x \in K[v^\perp] \), we translate the section \( (x + \mathbb{R}v) \cap H \) by \((\lambda - f_K(x))v \) and hence one endpoint lands in \( H \). We write \( K' \) to denote the resulting convex body, which satisfies

\[ K'[v^\perp] = K[v^\perp] = C - \lambda v \quad \text{for } C = K' \cap H. \]

In addition \( V(K') = V(K) = 1 \), and \( \Xi_{K'} \) is the cone \([o, C]\).

For \( x \in (\text{int} K[v^\perp]) \setminus \{o\} \), it follows by (4.1) that \( x + \mathbb{R}v \) intersects \( \Xi_K \) in a segment of length at least the length of \( \Xi_{K'} \cap (x + \mathbb{R}v) \). Therefore, Fubini’s theorem yields

\[ (4.2) \quad V(\Xi_K) \geq V(\Xi_{K'}). \]

Furthermore, Fubini’s theorem implies that

\[ \langle c(K'), u \rangle = \langle c(K), u \rangle = 0 \quad \text{for } u \in v^\perp; \]

\[ \langle c(K'), v \rangle \geq \langle c(K), v \rangle = 0 \quad \text{with equality if and only if } K' = K. \]

We deduce

\[ (4.3) \quad c(K') = \eta v \quad \text{for } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = K. \]

Next let \( \widetilde{K} \) be the Schwarz rounding of \( K' \) with respect to \( \mathbb{R}v \). It follows from the rotational symmetry of \( \widetilde{K} \) that \( \langle c(\widetilde{K}), u \rangle = 0 \) for \( u \in v^\perp \), and by Fubini’s theorem that \( \langle c(\widetilde{K}), v \rangle = \langle c(K'), v \rangle \), which in turn yield by (4.3) and \( V(\widetilde{K}) = V(K') = 1 \) that

\[ (4.4) \quad c(\widetilde{K}) = c(K') = \eta v \quad \text{for } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = K. \]

We conclude by (4.2) and (4.4) that

\[ (4.5) \quad V(\Xi_K) \geq V(\Xi_{\widetilde{K} - c(\widetilde{K})}) \quad \text{with equality if and only if } K' = K. \]

Finally we compare \( \widetilde{K} \) to the cylinder \( Z \) over the \((n-1)\)-ball \( H \cap \widetilde{K} \), where \( V(Z) = V(\widetilde{K}) = 1 \) and \( Z \) and \( K \) lie on the same side of \( H \). We deduce from the rotational symmetry of \( Z \) that \( \langle c(Z), u \rangle = 0 \) for \( u \in v^\perp \). On the other hand, the rotational symmetry of \( \widetilde{K} \) and \( K[v^\perp] = (H \cap \widetilde{K}) - \lambda v \) yield that

\[ \langle x, v \rangle > -h_Z(-v) > \langle y, v \rangle \quad \text{for all } x \in \text{int} Z \setminus \widetilde{K} \text{ and } y \in \widetilde{K} \setminus Z. \]

Therefore,

\[ c(Z) = \tau v \quad \text{for } \tau \geq \eta, \text{ with } \tau = \eta \text{ if and only if } Z = \widetilde{K}. \]

We conclude by (4.3) and (4.5) that

\[ V(\Xi_K) \geq V(\Xi_{\widetilde{Z} - c(\widetilde{Z})}) = 1/(2n) \quad \text{with equality if } K' = K \text{ and } Z = \widetilde{K}. \]
In turn, we get Proposition 4.1.

5. Some properties of the symmetric volume distance

First we show that the distance \( \delta_{\text{hom}} \) can be estimated in terms of \( \delta_{\text{vol}} \). These types of estimates have been around, only we were not able to locate them in the form we need.

Lemma 5.1. Let \( K \in K^n \) with \( c(K) = o \).

(i) If \( Q \subset K \) is a convex body with \( V(K\setminus Q) \leq tV(K) \) for \( t \in (0, \frac{1}{\mu}) \), then \( (1-(et)^{1/n})K \subset Q \).

(ii) If \( Q \) is a convex body with \( V(K\Delta Q) \leq tV(K) \) for \( t \in (0, \frac{1}{\mu}) \), then \( (1-(et)^{1/n})K \subset Q \subset (1+4(et)^{1/n})K \).

Proof. The main tool is the following result due to B. Grünbaum [19]. If \( M \in K^n \), and \( H^+ \) is a half space containing \( c(M) \), then

\[
V(M \cap H^+) \geq V(M)/e.
\]

To prove (i), let \( \lambda = 0 \) if \( o \notin \text{int } Q \), and let \( \lambda > 0 \) be maximal with the property that \( \lambda K \subset Q \) otherwise. In addition, let \( x = o \) if \( o \notin \text{int } Q \), and let \( x \) be a common boundary point of \( Q \) and \( \lambda K \) otherwise. Therefore, there exists a half space \( H^+_1 \) such that \( x \) lies on its boundary, and \( H^+_1 \cap \text{int } Q = \emptyset \). Now there exists a \( y \in K \) such that \( x = \lambda y \), and hence \( x \) is the centroid of \( x + (1-\lambda)K = \lambda y + (1-\lambda)K \subset K \). It follows from (5.1) that

\[
tV(K) \geq V(H^+_1 \cap K) \geq V(H^+_1 \cap (x + (1-\lambda)K)) \geq V((1-\lambda)K)/e,
\]

and thus \( t \geq \frac{(1-\lambda)^n}{e} \).

To prove (ii), we observe that \( \lambda K \subset Q \) for \( \lambda = 1-(et)^{1/n} \) by (i). We may assume that \( Q\setminus K \neq \emptyset \), and let \( \mu > 1 \) be minimal with the property that \( Q \subset \mu K \). For a common boundary point \( z \) of \( Q \) and \( \mu K \), let \( w \in \bar{K} \) such that \( z = \mu w \). In particular, \( w \) is the centroid of

\[
w + \frac{\lambda(\mu-1)}{\mu} K \subset \frac{1}{\mu} z + \frac{\mu-1}{\mu} Q \subset Q.
\]

In addition there exists a half space \( H^+_2 \) such that \( w \) lies on its boundary, and \( H^+_2 \cap \text{int } K = \emptyset \). We deduce again from (5.1) that

\[
tV(K) \geq V(H^+_2 \cap Q) \geq V \left( H^+_2 \cap \left( w + \frac{\lambda(\mu-1)}{\mu} K \right) \right) \geq \frac{\lambda^n(\mu-1)^n}{\mu^n e} V(K).
\]

Now \( t < \frac{1}{4n e} \) yields that \( \lambda > \frac{1}{2} \) and \( 2(et)^{1/n} < \frac{1}{2} \), which in turn implies that \( \mu \leq (1-2(et)^{1/n})^{-1} < 1 + 4(et)^{1/n} \).

\[\square\]

Corollary 5.2. Let \( K, Q \in K^n \). Then

\[
\delta_{\text{hom}}(K, Q) \leq 12 \delta_{\text{vol}}(K, Q)^{1/n} \quad \text{if } \delta_{\text{vol}}(K, Q) < \frac{1}{4n e},
\]

\[
\delta_{\text{vol}}(K, Q) \leq 3n \delta_{\text{hom}}(K, Q) \quad \text{if } \delta_{\text{hom}}(K, Q) < \frac{1}{2n e}.
\]
Proof. We use that $1 + s < e^{s} < 1 + 2s$ and $1 - s < e^{-s} < 1 - \frac{s}{2}$ if $s \in (0, 1)$.

We may assume that $c(K) = c(Q) = o$, and $V(K) = V(Q) = 1$. In particular, $V(K \Delta Q) = \delta_{\text{vol}}(K, Q)$, and hence the estimates for the exponential function and Lemma 5.1 yield with $s = \delta_{\text{vol}}(K, Q)$ that

$$e^{-2s^{1/n}s^{1/n}}K \subset (1 - (se)^{1/n})K \subset Q \cap K \subset Q.$$  

Using the analogous formula $e^{-2s^{1/n}s^{1/n}}Q \subset K$, we conclude the first estimate.

For the second estimate, let $t = \delta_{\text{hom}}(K, Q)$. It follows that $e^{-t}K \subset Q \subset e^{t}K$, thus $V(K \Delta Q) \leq e^{nt} - e^{-nt} < 3nt$. $\square$

Our next goal is Lemma 5.4 stating that one does not need to insist on the common centroid in the definition of $\delta_{\text{vol}}$. We prepare the argument by the following observation.

Lemma 5.3. Let $K \in K_{n}$ and $x \in \mathbb{R}^{n}$. Then

$$V(K \Delta (x + K)) \leq 2n\|x\|_{K-K}V(K).$$

Proof. We may assume that $x \neq o$. Let $y, z \in K$ such that $x = \|x\|_{K-K}(y - z)$, and hence

$$\|x\|_{K-K} = \|x\|/\|y - z\|.$$  

Applying Steiner symmetrization with respect to the hyperplane $x^\perp$ shows that

$$V(K) \geq \frac{\|y - z\|}{n} \mathcal{H}_{n-1}(K|x^\perp).$$

We deduce by Fubini’s theorem that

$$V(K \Delta (x + K)) \leq 2\|x\| \mathcal{H}_{n-1}(K|x^\perp) \leq 2n\|x\|_{K-K}V(K).$$  

Lemma 5.4. Let $K, Q \in K_{n}$ with $c(K) = o$ and $V(K \Delta Q) \leq tV(K)$ for $t \in (0, \frac{1}{4e})$. Then

$$\|c(Q)\|_{K-K} \leq 4nt \quad \text{and} \quad \delta_{\text{vol}}(K, Q) \leq 9n^{2}t.$$  

Proof. We may assume that $V(K) = 1$, and the minimal volume so called Löwner ellipsoid $E$ containing $K - K$ is a ball (see, e.g., [18]). In particular, $n^{-1/2}E \subset K - K \subset E$, and the Brunn-Minkowski and Rogers-Shephard theorems yield that $2^{n} \leq V(K - K) \leq \left(\frac{2n}{n}\right)$. Since the volume of a centrally convex body over the volume of its Löwner ellipsoid is at least $2^{n}/(nV(B^{n}))$ according to K. Ball [2], we have

$$2^{n} \leq V(E) \leq \left(\frac{2n}{n}\right) \frac{n!}{2^{n}}V(B^{n}) \leq \sqrt{3} \cdot \frac{2^{n}n^{n}}{e^{n}}V(B^{n}).$$

It follows that

$$2^{n} \leq V(E) \leq \left(\frac{2n}{n}\right) \frac{n!}{2^{n}}V(B^{n}) \leq \sqrt{3} \cdot \frac{2^{n}n^{n}}{e^{n}}V(B^{n}).$$

(5.2) \quad \frac{2}{\sqrt{\pi}} B^{n} \subset K - K \subset nB^{n} \quad \text{and} \quad \frac{1}{n} \|x\| \leq \|x\|_{K-K} \leq 2\|x\|. 
Therefore, to prove Lemma 5.4 it is sufficient to verify the corresponding estimate for \(\|c(Q)\|\).

If \(c(Q) = o\), then we are done, otherwise let \(u = c(Q)/\|c(Q)\|\). We have \(Q \subset 2K \subset 2nB^n\) by Lemma 5.1 and (5.2), and \(V(Q) \geq 1 - t\) implies \(V(Q)^{-1} < 2\). By (5.2) we also have

\[
\|c(Q)\|_{K - K} \leq 2\|c(Q)\| = 2V(Q)^{-1}\langle u, c(Q) \rangle = 2V(Q)^{-1}\left\| \int_Q \langle u, x \rangle \, dx \right\|,
\]

and since \(c(K) = o\) we get

\[
\|c(Q)\|_{K - K} \leq 2V(Q)^{-1}\left\| \int_Q \langle u, x \rangle \, dx \right\|
\]

(5.3)

\[
= 2V(Q)^{-1}\left\| \int_{Q \setminus K} \langle u, x \rangle \, dx - \int_{K \setminus Q} \langle u, x \rangle \, dx \right\|
\]

\[
4 \int_{K \setminus Q} |\langle u, x \rangle| \, dx \leq 4nt.
\]

Let \(K' = K + c(Q)\), thus Lemma 5.3 and (5.3) imply that \(V(K \Delta K') \leq 8n^2t\). We observe that \(Q' = c(Q) + \frac{V(Q)^{-1/n}(Q - c(Q))}{V(Q)}\) satisfies \(c(Q') = c(Q)\), \(V(Q') = 1\), and \(V(Q' \Delta Q) \leq t\) by \(1 - t \leq V(Q) \leq 1 + t\) (cf. Lemma 5.1). Therefore

\[
\delta_{vol}(K, Q) = V(K' \Delta Q') \leq V(K' \Delta K') + V(K \Delta Q) + V(Q \Delta Q') < 9n^2t.
\]

\[\square\]

6. Some consequences of the stability of the Brunn-Minkowski inequality

Concerning the Brunn-Minkowski theory, including the properties of mixed volumes, the main reference is R. Schneider [39]. We use the Brunn-Minkowski theory in \(L^1\) in the terminology of Theorem 1.7 whose dimension is \(k = n - d\). For \(k, m \geq 1\), let

\[
I_m^k = \{(i_1, \ldots, i_m) : i_j \in \mathbb{N}, j = 1, \ldots, m \text{ and } i_1 + \ldots + i_m = k\}.
\]

For compact convex sets \(C_1, \ldots, C_m\) in \(\mathbb{R}^k\) and \((i_1, \ldots, i_m) \in I_m^k\), the non-negative mixed volumes \(V(C_1, i_1; \ldots; C_m, i_m)\) were defined by H. Minkowski in a way such that if \(\alpha_1, \ldots, \alpha_m \geq 0\), then

\[
H_k \left( \sum_{j=1}^m \alpha_j C_j \right) = \sum_{(i_1, \ldots, i_m) \in I_m^k} V(C_1, i_1; \ldots; C_m, i_m) \alpha_1^{i_1} \cdots \alpha_m^{i_m}, \tag{6.1}
\]

The mixed volume \(V(C_1, i_1; \ldots; C_m, i_m)\) actually depends only on the \(C_j\) with \(i_j > 0\), does not depend on the order how the pairs \(C_j, i_j\) are indexed, and we frequently ignore the pairs \(C_j, i_j\) with \(i_j = 0\). We have \(V(C_1, k) = \)
\[ H_k(C_1), \text{ and } V(C_1, i_1; C_m, i_m) > 0 \text{ if each } C_j \text{ is } k\text{-dimensional.} \] It follows by the Alexandrov-Fenchel inequality that

\[ (6.2) \quad V(C_1, i_1; \ldots; C_m, i_m)^k \geq \prod_{j=1}^{m} H_k(C_j)^{i_j}. \]

An important special case of (6.2) is the classical Minkowski inequality, which says

\[ (6.3) \quad V(C_1, i; C_2, k - 1)^k \geq H_k(C_1) H_k(C_2)^{k-1}. \]

Equality holds for \( k \)-dimensional \( C_1 \) and \( C_2 \) in the Minkowski inequality (6.3) if and only if \( C_1 \) and \( C_2 \) are homothetic. We remark that the equality conditions in the Alexandrov-Fenchel inequality (6.2) are not yet clarified in general.

Now the Alexandrov-Fenchel inequality (6.2), and actually already the Minkowski inequality (6.3) yields the classical (general) Brunn-Minkowski theorem stating that if \( C_1, \ldots, C_m \) are compact convex sets in \( \mathbb{R}^k \), and \( \alpha_1, \ldots, \alpha_m \geq 0 \), then (cf. (2.1))

\[ (6.4) \quad H_k \left( \sum_{j=1}^{m} \alpha_j C_j \right)^{1/k} \geq \sum_{j=1}^{m} \alpha_j H_k(C_j)^{1/k}. \]

Equality holds for \( k \)-dimensional \( C_1, \ldots, C_m \) and positive \( \alpha_1, \ldots, \alpha_m \) in the Brunn-Minkowski inequality (6.4) if and only if \( C_1 \) and \( C_j \) are homothetic for \( j = 2, \ldots, m \).

We need the following stability version of the Minkowski inequality (6.3) due to A. Figalli, F. Maggi and A. Pratelli [14]. If \( C_1, C_2 \) are \( k \)-dimensional compact convex sets in \( \mathbb{R}^k \), and

\[ (6.5) \quad V(C_1, 1; C_2, k - 1)^k \leq (1 + \varepsilon) H_k(C_1) H_k(C_2)^{k-1} \]

for small \( \varepsilon \geq 0 \), then [14] proves that

\[ (6.6) \quad \delta_{\text{vol}}(C_1, C_2) \leq \tilde{\gamma}_v \varepsilon^{1/2} \]

where the explicit \( \tilde{\gamma}_v > 0 \) depends only on the dimension \( k \).

We remark that here we only work out the estimate with respect to the symmetric volume distance \( \delta_{\text{vol}} \), and then just use Corollary 5.2 for \( \delta_{\text{hom}} \). Actually, V.I. Diskant [12] proved that (6.5) implies

\[ (6.7) \quad \delta_{\text{hom}}(C_1, C_2) \leq \tilde{\gamma}_h \varepsilon^{1/k} \]

for an unknown \( \tilde{\gamma}_h > 0 \) depending only on \( k \). We note that (6.6) and Corollary 5.2 readily yields a version of (6.7) with exponent \( \frac{1}{k} \) instead of \( \frac{1}{k} \).

Combining the stability versions (6.6) and (6.7) with Lemma 5.3 and Lemma 5.4 leads to the following stability version of the Brunn-Minkowski inequality.
Lemma 6.1. For any $k \geq 1$, $m \geq 2$ and $\omega \in (0,1]$, there exist positive $\varepsilon_0(k,m,\omega)$ and $\gamma(k,m,\omega)$ depending on $k$, $m$ and $\omega$ such that if $k$-dimensional compact convex sets $C_0, C_1, \ldots, C_m$ in $\mathbb{R}^k$, and $\alpha_1, \ldots, \alpha_m > 0$ satisfy that $\alpha_i/\alpha_j \geq \omega$ and $\mathcal{H}_k(C_i) = V$ for $i = 1, \ldots, m,$ and

$$\alpha_1 C_1 + \ldots + \alpha_m C_m \subset C_0$$

for $\varepsilon \in (0, \varepsilon_0(k,m,\omega))$, then for $i = 1, \ldots, m$, we have

$$\delta_{\text{vol}}(C_i, C_0) \leq \gamma(k,m,\omega)\varepsilon^{1/2};$$

and deduce that

$$\|c(C_0) - \sum_{i=1}^{m} \alpha_i c(C_i)\|_{C_0-C_0} \leq (\alpha_1 + \ldots + \alpha_m)\gamma(k,m,\omega)\varepsilon^{1/2}.$$

Proof. Since $\mathcal{H}_k(\alpha_1 C_1 + \ldots + \alpha_m C_m) \geq (\alpha_1 + \ldots + \alpha_m)k^kV$ according to the Brunn-Minkowski inequality, we may assume that $\alpha_1 C_1 + \ldots + \alpha_m C_m = C_0$ by Lemma 5.4. For $1 \leq i < j \leq m$, we apply the Alexandrov-Fenchel inequality (6.2) to each term in (6.1) except for $k\alpha_j \alpha_j^{-1}V(C_i, 1; C_j, k-1)$ and deduce that

$$k\alpha_i \alpha_j^{-1}V(C_i, 1; C_j, k-1) \leq k\alpha_i \alpha_j^{-1}V + (\varepsilon - 1)(\alpha_1 + \ldots + \alpha_m)k^kV.$$

Here $(\alpha_1 + \ldots + \alpha_m)^k \leq (\frac{m}{\omega})^k \alpha_i \alpha_j^{-1}$, and hence

$$V(C_i, 1; C_j, k-1) \leq \left(1 + \frac{2}{k} \left(\frac{m}{\omega}\right)^k \varepsilon\right)V.$$

Thus (6.8) yield

$$\delta_{\text{vol}}(C_i, C_j) \leq \gamma(k,m,\omega)\varepsilon^{1/2}$$

for $\gamma(k,m,\omega)$ depending only on $k$, $m$ and $\omega$. To compare to $C_0$, we may assume that $V = 1$, $\alpha_1 + \ldots + \alpha_m = 1$ and $c(C_i) = 0$ for $i = 1, \ldots, m$. Let $M = C_1 \cap \ldots \cap C_m$.

It follows from (6.8) that

$$\mathcal{H}_k(C_i \setminus M) \leq m \cdot \gamma(k,m,\omega)\varepsilon^{1/2}, \quad i = 1, \ldots, m,$$

and hence $\mathcal{H}_k(M) \geq 1 - m \cdot \gamma(k,m,\omega)\varepsilon^{1/2}$. Since $M \subset C_i$ for $i = 1, \ldots, m$ yields $M \subset C_0 = \sum_{i=1}^{m} \alpha_i C_i$, and $\mathcal{H}_k(C_0) \leq \varepsilon$, we deduce

$$\mathcal{H}_k(C_0 \Delta C_i) \leq 2\gamma(k,m,\omega)\varepsilon^{1/2}, \quad i = 1, \ldots, m.$$

Therefore Lemma 5.3 and Lemma 5.4 imply the required estimates for $\delta_{\text{vol}}(C_i, C_0)$ and $c(C_0)$. \qed

To prove the next Proposition 6.3 we need the following observation.

**Lemma 6.2.** If $M$ is a convex body in $\mathbb{R}^d$ such that $-M \subset \eta M$ for $\eta \geq 1$, then there exists an $d$-simplex $T \subset M$ whose centroid is the origin such that $M \subset \eta d^{3/2}T$. 

Proof. We may assume that the John ellipsoid $E$ of maximal volume contained in $M \cap (-M)$ is Euclidean ball, and let $T \subset M \cap (-M)$ be an inscribed regular simplex. Then $\eta^{-1}M \subset M \cap (-M) \subset \sqrt{d}E \subset d^{3/2}T$. \hfill $\square$

For Proposition 6.3 we use the notation of the previous sections, i.e., $K \in K^n$ is a convex body with $c(K) = o, d, k \in \{1, \ldots, n-1\}$ with $d+k = n$, and $L$ is a $d$-dimensional linear subspace. For $x \in K \setminus L$, we set

$$f(x) = f_{K,L}(x) = \mathcal{H}_k(K \cap (x+L^\perp)).$$

**Proposition 6.3.** There exist $t_0, \gamma > 0$ depending on $n$ with the following properties. Let $t \in (0, t_0)$, let $M_s \subset K \setminus L$ be a $d$-dimensional convex compact set, and let $K_\ast = K \cap (M_s + L^\perp)$. If $e^{-t} \leq f(x)/f(o) \leq e^t$ holds for any $x \in M_s$, then there exist a $k$-dimensional compact convex set $C \subset L^\perp$, and a complementary $d$-dimensional compact convex set $M$ such that

$$\delta_{\text{vol}}(K, C + M) \leq \gamma \max \left\{ \frac{V(K \setminus K_\ast)}{V(K)}, t^{1/2} \right\}.$$

**Proof.** Since $c(K) = o$ we have $-K \subset nK$. Hence $-K \setminus L \subset nK \setminus L$ and we may choose, according to Lemma 6.2, $v_0, \ldots, v_d \in e^{-s}K \setminus L$, for some $s > 0$, such that $v_0 + \ldots + v_d = o$, and

$$e^{-s}K \setminus L \subset n^{5/2}[v_0, \ldots, v_d].$$

(6.9)

For $x \in e^{-s}K \setminus L$, let $K(x) = K \cap (x+L^\perp)$, and let

$$\tilde{K}(x) = \frac{f(o)^{1/k}}{f(x)^{1/k}} K(x), \quad \text{and hence } \mathcal{H}_k(\tilde{K}(x)) = f(o).$$

We define

$$A = \text{aff}\{c(K(v_0)), \ldots, c(K(v_d))\},$$

$$M = \{y \in A : (y + L^\perp) \cap e^{-s}K \neq \emptyset\},$$

$$C = K(o) - c(K(o)).$$

We compare $K_\ast$ to $M + C$. To this end we consider the affine bijection $\varphi : L \to A$ defined by the correspondance $\{\varphi(x)\} = A \cap (x + L^\perp)$ for $x \in L$. In particular,

$$\varphi(v_i) = c(K(v_i)), \; i = 0, \ldots, d \quad \text{and} \quad \varphi(o) = \frac{1}{d+1} \sum_{i=0}^d c(K(v_i)).$$

(6.11)

Let $x \in e^{-s}K \setminus L$. We have $\frac{-1}{2n^{5/2}} x \in \left[\frac{1}{2}[v_0, \ldots, v_d]\right]$ according to (6.9), thus

$$\frac{-1}{2n^{5/2}} x = \sum_{i=0}^d \alpha_i v_i \quad \text{where } \sum_{i=0}^d \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq \frac{1}{2(d+1)}, \; i = 0, \ldots, d.$$
We define
\[
\tilde{\beta} = \frac{\beta f(x)^{1/k}}{f(o)^{1/k}} \quad \text{where} \quad \beta = \frac{1}{1 + 2n^{5/2}};
\]
\[
\tilde{\beta}_i = \frac{\beta_i f(v_i)^{1/k}}{f(o)^{1/k}} \quad \text{where} \quad \beta_i = \frac{\alpha_i 2n^{5/2}}{1 + 2n^{5/2}}, \quad i = 0, \ldots, d,
\]
and hence \(\beta + \sum_{i=0}^d \beta_i = 1\) and \(\beta x + \sum_{i=0}^d \beta_i v_i = o\). The condition on the function \(f\) yields that
\[
e^{-t/k} \leq \tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d \leq e^{t/k},
\]
and the ratio of any two of \(\tilde{\beta}, \tilde{\beta}_0, \ldots, \tilde{\beta}_d\) is at least \(1/(4n^{5/2})\). In particular,
\[
e^t(\tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d) f(o) \geq \mathcal{H}_k(K(o)),
\]
and the convexity of \(K\) implies (cf. (6.10))
\[
\tilde{\beta} K(x) + \sum_{i=0}^d \beta_i \tilde{K}(v_i) = \beta K(x) + \sum_{i=0}^d \beta_i K(v_i) \subset K(o).
\]

We deduce from Lemma 6.1, the stability version of the Brunn-Minkowski inequality, that there exists \(\gamma^* > 0\) depending on \(n\) such that for \(i = 0, \ldots, d\), we have
\[
\delta_{vol}(K(v_i), K(o)), \quad \delta_{vol}(K(x), K(o)) \leq \gamma^* t^{1/2},
\]
\[
\left\| c(K(o)) - \beta c(K(x)) - \sum_{i=1}^d \beta_i c(K(v_i)) \right\|_{K(o) - K(o)} \leq \gamma^* t^{1/2}.
\]

Naturally, if \(k = 1\), then even \(t\) can be written instead of \(t^{1/2}\) on the right hand side of (6.12) and (6.13), but we ignore this possibility.

First we assume that \(x = o\). In this case (6.11) and (6.13) yield
\[
\|c(K(o)) - \varphi(o)\|_{K(o) - K(o)} \leq \gamma^* t^{1/2}.
\]

Next let \(x \in e^{-s}K|L\) be arbitrary. We have \(\beta \varphi(x) + \sum_{i=0}^d \beta_i \varphi(v_i) = \varphi(o)\) because \(\varphi\) is affine. We recall that \(C = K(o) - c(K(o))\). Let
\[
w = c(K(o)) - \beta c(K(x)) - \sum_{i=1}^d \beta_i c(K(v_i)).
\]
Since $\beta \varphi(x) = \varphi(o) - \sum_{i=0}^{d} \beta_i \varphi(v_i)$, we have
\[
\|c(K(x)) - \varphi(x)\|_{C^{-C}} = \frac{\|\beta c(K(x)) - \beta \varphi(x)\|_{C^{-C}}}{\beta} \\
\leq \frac{\|\beta c(K(x)) + w - \beta \varphi(x)\|_{C^{-C}}}{\beta} + \frac{\|w\|_{C^{-C}}}{\beta} \\
= \frac{\|c(K(o)) - \varphi(o) - \sum_{i=0}^{d} \beta_i (c(K(v_i)) - \varphi(v_i))\|_{C^{-C}}}{\beta} \\
+ \frac{\|c(K(o)) - \beta c(K(x)) - \sum_{i=1}^{d} \beta_i c(K(v_i))\|_{C^{-C}}}{\beta}.
\]
As $\varphi(v_i) = c(K(v_i))$ according to (6.11), it follows by (6.13) and (6.14) that
(6.15) $\|c(K(x)) - \varphi(x)\|_{C^{-C}} \leq \frac{2\gamma^*}{\beta} \cdot t^{1/2} < 6n^{5/2} \gamma^* t^{1/2}$.

For $x \in e^{-s} K|L$, we deduce in order from (6.13), (6.12) and (6.10) that
\[
\mathcal{H}_k((C + \varphi(x))\Delta K(x)) \leq \mathcal{H}_k((C + \varphi(x))\Delta (C + c(K(x)))) \\
+ \mathcal{H}_k((C + c(K(x)))\Delta (\tilde{K}(x) - c(K(x)) + c(K(x)))) \\
+ \mathcal{H}_k((\tilde{K}(x) - c(K(x)) + c(K(x)))\Delta K(x)) \\
< 9n^{5/2} \gamma^* t^{1/2} \mathcal{H}_k(C).
\]
Hence, by Fubini’s theorem we get
\[
V(K, \Delta (M + C)) < 9n^{5/2} \gamma^* t^{1/2} V(M + C)
\]
and Lemma 5.3 yields the required estimate for $\delta_{\text{vol}}$. □

7. Some more properties of $f_{K,L}(x)$

Here we establish some more properties of the log-concave function (cf. (2.3))

\[
f_{K,L} : L \rightarrow \mathbb{R}_{\geq 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)),
\]
and use the notation as introduced in Section 2, i.e., $K \in \mathcal{K}^n$ is an $n$-dimensional convex body with $c(K) = 0$, $L$ is a $d$-dimensional subspace $L$, $1 \leq d \leq n-1$, and we set $k = n-d$. Since we will keep $K$ and $L$ fixed, we just write $f(x)$ instead of $f_{K,L}(x)$. As in Section 2 let $g(x)$ be the subgradient of $f(x)$, and we recall that $g(x) = \nabla f(x)/f(x)$ almost everywhere on int $(K)|L$.

For $\eta \geq 0$, we set
\[
M_\eta = \{x \in K|L : \ln f(x) - \ln f(o) \geq \langle g(o), x \rangle - \eta\}, \\
K_\eta = K \cap (M_\eta + L^\perp).
\]
Since $\ln f$ is concave, both $M_\eta$ and $K_\eta$ are compact and convex.

Lemma 7.1. Let $\eta \geq 0$. Then
\[
\int_{K|L} \langle \nabla f(x), x \rangle d\mathcal{H}_d(x) \leq -\eta V(K \setminus K_\eta).
\]
Proof. Let \( x \in (\text{int } K)|L \) and \( \eta \geq 0 \), and let us assume \( \ln f(x) - \ln f(o) \leq \langle g(o), x \rangle - \eta \). Then by (2.2) we have \( \langle g(x), x \rangle \leq \langle g(o), x \rangle - \eta \). Hence if \( \nabla f \) exists at \( x \in (\text{int } K)|L \), then
\[
\langle \nabla f(x), x \rangle \leq 0 \quad \text{provided that } x \in M_\eta,
\]
\[
\langle \nabla f(x), x \rangle \leq \langle g(o), f(x)x \rangle - f(x)\eta \quad \text{provided that } x \notin M_\eta.
\]
We conclude the lemma by (2.4) and \( V(K \setminus K_\eta) = \int_{(K|L) \setminus M_\eta} f(x) \, dx \). \( \square \)

Lemma 7.2. Let \( \eta \in [0, 1] \). If \( V(K \setminus K_\eta) \leq V(K)/(2^n e) \), then
\[
e^{-\tau} \leq \frac{f(x)}{f(o)} \leq e^\tau \quad \text{for } \tau = 7n^{3/2}\eta^{1/2} \quad \text{and } x \in M_\eta.
\]

Proof. By Lemma 5.1 we have \( \frac{1}{2} K \subset K_\eta \), and \( f(x) \geq f(o)e^{\langle g(o), x \rangle - \eta} \) for \( x \in K_\eta \). We claim that for \( \pm y \in K_\eta \)
\[
(|g(o), y|) \leq 3\sqrt{k\eta}.
\]
The concavity of \( f^{1/k} \) yields that
\[
f(o)^{1/k} \geq \frac{f(y)^{1/k} + f(-y)^{1/k}}{2} \geq \frac{f(o)^{1/k} e^{-\eta/k} e^{\langle g(o), y \rangle/k} + e^{\langle g(o), -y \rangle/k}}{2} \geq f(o)^{1/k} e^{\eta/k} \left( 1 + \frac{(\langle g(o), y \rangle)}{2k} \right)^2.
\]
Since \( e^t < 1 + 2t \) for \( t \in [0, 1] \), we conclude (7.1).

It follows from \( \frac{1}{2} K \subset K_\eta \) and \( -K \subset nK \) that \( \frac{1}{2} (K|L) \subset M_\eta \) and \( -(K|L) \subset n(K|L) \). In particular, if \( x \in M_\eta \) is arbitrary, then \( \pm y \in M_\eta \) for \( y = \frac{1}{2n} x \). We deduce from (7.1) that \( |\langle g(o), x \rangle| = 2n|\langle g(o), y \rangle| \leq 6n\sqrt{k\eta} \).
Therefore, the lemma follows from \( f(o)e^{\langle g(o), x \rangle - \eta} \leq f(x) \leq f(o)e^{\langle g(o), x \rangle} \). \( \square \)

8. PROOFS OF THEOREM 1.7 AND THEOREM 1.8

For the proofs of the two stability theorems 1.7 and 1.8 let \( K \in K^n \) with \( c(K) = o \), and let
\[
V_K(L \cap S^{n-1}) > \frac{d - \varepsilon}{n} V(K)
\]
for a non-trivial linear subspace \( L \) with \( \dim L = d \) and \( \varepsilon \in (0, (2^n e)^{-5}) \). As before, for \( x \in K|L \) let
\[
f(x) = H_k(K \cap (x + L^\perp)).
\]
According to Proposition 3.3 the condition on \( V_K(L \cap S^{n-1}) \) is equivalent with
\[
\int_{K|L} \langle \nabla f(x), x \rangle dH_d(x) > -\varepsilon V(K).
\]
Proof of Theorem 1.7. We set $\eta = \varepsilon^{4/5}$, and use the notation of Lemma 7.1. It follows from (8.1) and Lemma 7.1 that

$$V(K \setminus K_\eta) < \varepsilon^{1/5} V(K) < V(K)/(2^n),$$

and from Lemma 7.2 that

$$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \quad \text{for} \quad t = 7n^{3/2} \varepsilon^{2/5} \quad \text{and} \quad x \in M_\eta.$$

We assume that $\varepsilon$ is small enough in order to apply Proposition 6.3 with $M_* = M_\eta$ and $t = 7n^{3/2} \varepsilon^{2/5}$. We deduce the existence of an $(n-d)$-dimensional compact convex set $C \subset L^\perp$, and complementary $d$-dimensional compact convex set $M$ such that

$$\delta_{\text{vol}}(K, C + M) \leq \gamma \varepsilon^{1/5}.$$

In turn Corollary 5.2 implies that

$$\delta_{\text{hom}}(K, C + M) \leq \gamma \varepsilon^{1/(5n)},$$

completing the proof of Theorem 1.7.

\[\square\]

Proof of Theorem 1.8. Here we have $d = 1$. We may assume that $L = \mathbb{R}$, and $K|L = [-a, b]$ where $0 < a \leq b$. Since $c(K) = o$ implies $-K \subset nK$ according to B. Grünbaum [19], we have $b \leq na$.

We set $\eta = \varepsilon^{2/3}$, and use again the notation of Lemma 7.1. We deduce from (8.1) and Lemma 7.1 that

$$V(K \setminus K_\eta) < \varepsilon^{1/3} V(K) < V(K)/(2^n),$$

and from Lemma 7.2 that

$$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \quad \text{for} \quad t = 7n^{3/2} \varepsilon^{1/3} \quad \text{and} \quad x \in M_\eta.$$

It follows from Lemma 5.1 and (8.2) that $\frac{1}{2}[-a, b] \subset M_\eta$, therefore the concavity of $\ln f$ and (8.3) yield that

$$f(x) \leq e^{2t} f(o) \quad \text{for} \quad x \in [-a, b].$$

Let $M_\eta = [-a_\eta, b_\eta]$ for $a_\eta, b_\eta > 0$. Since $K \setminus K_\eta$ contains two cones, one with base $K(-a_\eta)$ and height $a - a_\eta$, and one with base $K(b_\eta)$ and height $b - b_\eta$, we get by (8.3), (8.2) and (8.4) that

$$\frac{a - a_\eta + b - b_\eta}{n} e^{-t} f(o) \leq \frac{a - a_\eta + b - b_\eta}{n} (f(-a_\eta) + f(b_\eta))$$

$$\leq V(K \setminus K_\eta) < \varepsilon^{1/3} V(K) \leq \varepsilon^{1/3} e^{2t} f(o)(a + b).$$

In particular,

$$\mathcal{H}_1(M_\eta) = a_\eta + b_\eta > (1 - 2n^{1/2}) (a + b).$$
Here and below $\gamma_1, \gamma_2, \ldots$ denote positive constants depending on $n$. We deduce by (8.3) that if $\varepsilon$ is small enough, then
\[
af(-a) + bf(b) = \int_{V} (1 - \varepsilon) V(K) > (1 - \varepsilon) \mathcal{H}_1(M) e^{-t} f(o) > (1 - \gamma_1 \varepsilon^{\frac{1}{3}})(a + b)f(o).
\]
Since $b \geq a$ and $\frac{a}{a + b} \geq \frac{1}{n+1}$ by $b \leq na$, (8.4) implies that if $\varepsilon$ is small enough, then
\[
f(-a), f(b) \geq (1 - \gamma_2 \varepsilon^{\frac{1}{3}}) f(o).
\]
As $f$ is concave, we have
\[
f(x) \geq (1 - \gamma_2 \varepsilon^{\frac{1}{3}}) f(o) \quad \text{for} \quad x \in [-a, b].
\]
However $\frac{a}{a + b} C(b) + \frac{b}{a + b} C(-a) \subset C(o)$, where $C(x) = K \cap (x + L^\perp)$. Thus Lemma 6.1 yields that
\[(8.5) \quad \delta_{\text{val}}(C(o), C(-a)) \leq \gamma_3 \varepsilon^{\frac{1}{3}} \quad \text{and} \quad \delta_{\text{val}}(C(o), C(b)) \leq \gamma_3 \varepsilon^{\frac{1}{3}}.
\]
Hence, with
\[
\tilde{C} = (C(-a) - \tilde{x}) \cap (C(b) - \tilde{y}) \quad \text{for} \quad \tilde{x} = c(C(-a)) \quad \text{and} \quad \tilde{y} = c(C(b)).
\]
It follows from (8.4) and (8.5) that
\[
[x, y] + \tilde{C} \subset K \quad \text{and} \quad V(K) \leq (1 + \gamma_4 \varepsilon^{\frac{1}{3}}) V([x, y] + \tilde{C}).
\]
Using Lemma 5.1, we replace $\tilde{C}$ by a suitably smaller homothetic copy $C$ such that $c(C) = o$, and obtain that there exist $x \in \tilde{x} + \tilde{C}$ and $y \in \tilde{y} + \tilde{C}$ satisfying $o \in [x, y], e^{-s}||x|| \leq ||y|| \leq e^{s}||x||$ for $s = \gamma_5 \varepsilon^{\frac{1}{3}}$, and
\[
[x, y] + C \subset K \quad \text{and} \quad V(K) \leq (1 + \gamma_6 \varepsilon^{\frac{1}{3}}) V([x, y] + C).
\]
Finally, if $z \in [-a, b]$, then $-z/n \in [-a, b]$ and $\frac{1}{n+1} C(z) + \frac{n}{n+1} C(-z/n) \subset C(o)$. Therefore Lemma 5.1, Lemma 6.1 and the estimates above imply
\[
K \subset [x, y] + (1 + \gamma_5 \varepsilon^{\frac{1}{3}}) C,
\]
completing the proof of Theorem 1.8. \hfill \Box

9. Stability of the $U$-functional $U(K)$

Let $m \in \{1, \ldots, n\}$. In this section, a finite sequence $u_1, \ldots, u_m$ always denote points of $S^{n-1}$, and by $\text{lin} \{X\}$ we denote the linear hull of a set $X$. As in [24], we define $\sigma_m(K) > 0$ by
\[
\sigma_m(K)^m = \int_{u_1 \wedge \ldots \wedge u_m \neq 0} 1 \, dV_K(u_1) \cdots dV_K(u_m).
\]
In particular, $\sigma_1(K) = V(K), \sigma_n(K) = U(K)$, and for $m < n$, we have (9.1)
\[
\sigma_{m+1}(K)^{m+1} = \int_{u_1 \wedge \ldots \wedge u_m \neq 0} (V(K) - V_K(S^{n-1} \cap \text{lin}\{u_1, \ldots, u_m\})) \, dV_K(u_1) \cdots dV_K(u_m).
\]
As \( V_K(S^{n-1} \cap \text{lin}\{u_1, \ldots, u_m\}) \leq \frac{m}{n}V(K) \) for linearly independent \( u_1, \ldots, u_m \) according to Theorem 1.3, we deduce that

\[
\sigma_{m+1}(K)^{m+1} \geq \left(1 - \frac{m}{n}\right)V(K)\sigma_m(K)^m.
\]

Therefore the inequality of Theorem 1.5 follows from

\[
U(K)^n \geq \frac{1}{n}V(K)\sigma_{n-1}(K)^{n-1} \geq \ldots \geq \left(\frac{n-1}{n}\right)^{n-1}V(K)\sigma_1 = n!V(K)^n.
\]

Now we assume that

\[
U(K) \leq (1 + \varepsilon)\left(n!\right)^{1/n}V(K)
\]

where \( \varepsilon > 0 \) is small enough to satisfy all estimates below. In particular, \( \varepsilon < \frac{1}{n!}\tilde{\varepsilon}_0 \), where \( \tilde{\varepsilon}_0 \) comes from Theorem 1.8. Applying (9.1) for \( m = 1 \), (9.2) for \( m \geq 2 \), and using \( (1 + \varepsilon)^n \frac{n-1}{n} < \frac{n-1}{n} + 2n\varepsilon \) gives

\[
(9.3) \int_{S^{n-1}} (V(K) - V_K(S^{n-1} \cap \text{lin}\{u\})) dV_K(u) \leq \left(\frac{n-1}{n} + 2n\varepsilon\right)V(K)^2.
\]

For any \( X \subset S^{n-1} \), there exists \( u \in X \) maximizing \( V_K(S^{n-1} \cap \text{lin}\{u\}) \) because different 1-dimensional subspaces have disjoint intersections with \( S^{n-1} \). We consider linearly independent \( v_1, \ldots, v_n \in S^{n-1} \) such that \( \sigma_1 \) maximizes \( V_K(S^{n-1} \cap \text{lin}\{u\}) \) for \( u \in S^{n-1} \), and \( v_i \) maximizes \( V_K(S^{n-1} \cap \text{lin}\{u\}) \) for all \( u \in S^{n-1} \cap \text{lin}\{v_1, \ldots, v_{i-1}\} \) if \( i = 2, \ldots, n \). Let \( L = \text{lin}\{v_1, \ldots, v_{n-1}\} \), and let \( V_K(S^{n-1} \cap \text{lin}\{v_n\}) = \left(\frac{1}{n} - t\right)V(K) \), and hence \( t \in [0, \frac{1}{n}] \) (cf. (1.3)). Thus we have

\[
(9.4) \quad \sigma_i = V_K(S^{n-1} \cap \text{lin}\{v_i\}) \geq (\frac{1}{n} - t)V(K) \quad \text{for} \quad i = 1, \ldots, n,
\]

\[
(9.5) \quad \sigma_{n} = V_K(S^{n-1} \cap \text{lin}\{u\}) \leq (\frac{1}{n} - t)V(K) \quad \text{for} \quad u \in S^{n-1} \setminus L
\]

We deduce from (9.3), (9.5) and \( V_K(S^{n-1} \cap \text{lin}\{u\}) \leq \frac{1}{n}V(K) \) for \( u \in S^{n-1} \cap L \) that

\[
(n\varepsilon + t)V(K)V_K(S^{n-1} \setminus L) + (n\varepsilon) \left(V_K(S^{n-1} \cap L) \leq (\frac{n\varepsilon}{n} + 2n\varepsilon) V(K)^2
\]

Since \( V_K(S^{n-1} \setminus L) \geq \frac{1}{n}V(K) \) according to Theorem 1.3, we conclude that \( t \leq 2n^2\varepsilon \). In particular, \( V_K(S^{n-1} \cap \text{lin}\{v_i\}) \geq (\frac{1}{n} - 2n\varepsilon)V(K) \) for \( i = 1, \ldots, n \) by (9.4).

From Theorem 1.8 we find for \( i = 1, \ldots, n \), that there exist an \( (n - 1) \)-dimensional compact convex set \( C_i \subset v_i^\perp \) with \( c(C_i) = o \), and \( x_i, y_i \in \partial K \) such that \( y_i = -\varepsilon^i x_i \), where \( |s_i| < \tilde{\varepsilon}_0 \), and for \( i = 1, \ldots, n \), we have

\[
(9.6) \quad [x_i, y_i] + C_i \subset K
\]

\[
(9.7) \quad V(K \setminus ([x_i, y_i] + C_i)) \leq n\tilde{\varepsilon}_0^\frac{1}{n}V(K)
\]

\[
(9.8) \quad K \subset [x_i, y_i] + (1 + 2\tilde{\varepsilon}_0^\frac{1}{n})C_i.
\]

We may assume that \( v_i \) is an exterior normal at \( x_i, i = 1, \ldots, n \). After a linear transformation of \( K \), we may also assume that \( v_1, \ldots, v_n \) form and
orthonormal system, and \( \langle v_i, x_i - y_i \rangle = 2 \). In particular,
\[
(9.9) \quad e^{-\tau} < \langle v_i, x_i \rangle, \langle -v_i, y_i \rangle < e^{\tau}, \quad \tau = n\gamma_n\varepsilon^{\frac{1}{m}}.
\]
In what follows, we write \( \gamma_1, \gamma_2, \ldots \) for positive constants depending on \( n \) only. It follows from (9.6), (9.7) and (9.9) that
\[
(9.10) \quad 1 - \gamma_1\varepsilon \tau^\frac{1}{m} < \mathcal{H}_{n-1}(C_i)/\mathcal{H}_{n-1}(C_j) < 1 + \gamma_1\varepsilon \tau^\frac{1}{m} \quad \text{for } i, j \in \{1, \ldots, n\}.
\]
For any \( i \neq j \in \{1, \ldots, n\} \), we write
\[
\begin{align*}
\hat{w}_i(v_j) &= h_{C_i}(v_j) + h_{C_i}(-v_j), \\
\hat{a}_i(v_j) &= \max \left\{ \mathcal{H}_{n-2}(C_i \cap (tv_j + v_j^\perp)) : -h_{C_i}(-v_j) \leq t \leq h_{C_i}(v_j) \right\},
\end{align*}
\]
and recall that \( h_{C_i}(x) \) denotes the support function. Hence \( \hat{w}_i(v_j) \) is the width of \( C_i \) in the direction of \( v_j \). Calculating \( \mathcal{H}_{n-1}(C_i) \) by integrating along \( \mathbb{R}v_j \) leads to
\[
(9.11) \quad \frac{1}{n-1} \hat{w}_i(v_j)\hat{a}_i(v_j) \leq \mathcal{H}_{n-1}(C_i) \leq \hat{w}_i(v_j)\hat{a}_i(v_j) \quad \text{for } i \neq j \in \{1, \ldots, n\}.
\]
Let \( p \neq q \in \{1, \ldots, n\} \). We choose \( t_1 \geq t_2 \geq t_0 \) such that
\[
\begin{align*}
\langle v_p, t_1x_p \rangle &= h_{C_q}(v_p), \\
\langle -v_p, t_2x_p \rangle &= h_{C_q}(-v_p), \\
\mathcal{H}_{n-2}(C_q \cap (t_2x_p + v_p^\perp)) &= a_q(v_p).
\end{align*}
\]
It follows from (9.9) and (9.8) that
\[
(9.12) \quad t_1 - t_0 > w_q(v_p)/2,
\]
\[
C_q \cap (t_2x_p + v_p^\perp) \subset t_2x_p + (1 + 2\gamma_h\varepsilon \tau^\frac{1}{m})C_p.
\]
Therefore \( a_p(v_q) \geq (1 + 2\gamma_h\varepsilon \tau^\frac{1}{m})^{-(n-2)}a_q(v_p) \), and hence interchanging the role of \( p \) and \( q \) leads to
\[
1 - \gamma_2\varepsilon \tau^\frac{1}{m} < a_q(v_p)/a_p(v_q) < 1 + \gamma_2\varepsilon \tau^\frac{1}{m}.
\]
We deduce from (9.10) and (9.11) that
\[
(9.13) \quad \frac{1}{2n} < \frac{w_p(v_q)}{w_q(v_p)} < 2n.
\]
Now combining (9.6) and (9.8) shows that
\[
(9.14) \quad h_{C_p}(v_q) \leq \langle x_q - t_mx_p, v_q \rangle \leq (1 + 2\gamma_h\varepsilon \tau^\frac{1}{m})h_{C_p}(v_q) \quad \text{for } m = 0, 1,
\]
and hence
\[
\left| \langle (t_1 - t_0)x_p, v_q \rangle \right| \leq 2\gamma_h\varepsilon \tau^\frac{1}{m} h_{C_p}(v_q) < 2\gamma_h\varepsilon \tau^\frac{1}{m} w_p(v_q).
\]
Applying (9.12), (9.13), and the analogous argument to \( y_q \) implies that
\[
(9.15) \quad \left| \langle x_p, v_q \rangle \right|, \left| \langle y_p, v_q \rangle \right| \leq \gamma_3\varepsilon \tau^\frac{1}{m}.
\]
Let \( P \) be the parallelepiped
\[
P = \{ x \in \mathbb{R}^n : \langle x, v_i \rangle \leq \langle x, v_i \rangle, \langle x, -v_i \rangle \leq \langle y_i, -v_i \rangle, i = 1, \ldots, n \},
\]
and hence each facet of \( P \) contains one of \( x_i + C_i, y_i + C_i \), \( i = 1, \ldots, n \). We claim that

\[(9.16) \quad \frac{1}{4n} P \subset K.\]

We suppose that \( (9.16) \) does not hold, and seek a contradiction. Possibly reversing the orientation of some of the \( v_i \), we may assume that

\[(9.17) \quad z = \frac{1}{4n} \sum_{i=1}^{n} \langle x_i, v_i \rangle v_i \notin K.\]

In particular, \( \|z\| \leq \frac{1}{2\sqrt{n}} \) by \( (9.9) \), and there exists \( u \in S^{n-1} \) such that

\[(9.18) \quad \langle u, z \rangle > \langle u, x \rangle \text{ for } x \in K.\]

There exists \( v_p \) such that \( \|u, v_p\| \geq 1/\sqrt{n} \), and hence \( (9.9) \) and \( (9.15) \) yield that \( \langle u, x_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{16n}} \) if \( \langle u, v_p \rangle \geq 1/\sqrt{n} \), and \( \langle u, y_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{16n}} \) if \( \langle u, v_p \rangle \leq -1/\sqrt{n} \). However \( \langle u, z \rangle \leq \|z\| \leq \frac{1}{2\sqrt{n}} \), contradicting \( (9.17) \).

Therefore we conclude \( (9.16) \).

For \( i = 1, \ldots, n \), let

\[\Xi_{2i-1} = [o, x_i + C_i] \quad \text{and} \quad \Xi_{2i} = [o, y_i + C_i].\]

Since the basis of the cones \( \Xi_1, \ldots, \Xi_{2n} \) lie in different facets of \( P \), the interiors of \( \Xi_1, \ldots, \Xi_{2n} \) are pairwise disjoint. By \( (9.7) \) and \( (9.9) \) we know

\[V(\Xi_j) \geq \left( \frac{1}{2n} - \gamma_5 \varepsilon^{\frac{1}{16n}} \right) V(K),\]

and so we get

\[V(\Xi) > (1 - 2n\gamma_5 \varepsilon^{\frac{1}{16n}}) V(K) \quad \text{for} \quad \Xi = \bigcup_{j=1}^{2n} \Xi_j \subset K.\]

We conclude from \( (9.16) \) that

\[V(P\setminus K) \leq V(P\setminus \Xi) = (4n)^n V((\frac{1}{4n} P)\setminus \Xi) \leq (4n)^n V(K\setminus \Xi) \leq \gamma_6 \varepsilon^{\frac{1}{16n}} V(K).\]

Therefore \( (9.7) \) yields

\[\mathcal{H}_{n-1}((P \cap v_i^p)\setminus C_i) \leq \gamma_7 \varepsilon^{\frac{1}{16n}} \mathcal{H}_{n-1}(C_i), \quad i = 1, \ldots, n,\]

and Lemma \( 5.1 \) implies that \( (1 - \gamma_8 \varepsilon^{\frac{1}{16n}}) P \subset K \), completing the proof of Theorem \( 1.9 \).
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