\( f(R) \) scalar-tensor cosmology by Noether symmetry

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(Dated: May 10, 2019)

In the framework of \( f(R) \) scalar-tensor cosmology, we use the Noether symmetry approach to find the cosmological models consistent with the Noether symmetry. We obtain the functions \( f(R) \) and \( H(a) \), or the corresponding differential equations, according to specific choices for the scalar field potential \( V(\phi) \), the Brans-Dicke function \( \omega(\phi) \), some cosmological parameters, and the constants of motion.

PACS numbers: 04.50.Kd, 98.80.-k

I. INTRODUCTION

It is known that the expansion of universe is currently undergoing a period of acceleration which is directly measured by observations such as Type Ia Supernovae [1], large scale structure [2], cosmic microwave background (CMB) radiation [3, 4], weak lensing [5], and baryon acoustic oscillations [6]. There are two common approaches to explain the current acceleration of the universe: The first one is to introduce some new cosmological components of energy sources contributing to the so-called “dark energy” in the framework of general relativity (for a review on dark energy, see, e.g., [7, 8]). The second one is to generalize \( R \) (Ricci scalar) gravity to some modified gravities [9, 11]. One of the most common modified gravities is \( f(R) \) gravity [12–14]. This theory relaxes the hypothesis that gravitational Lagrangian has to be a linear function of \( R \), and as a minimal extension introduces an effective action containing a generic \( f(R) \) function.

On the other hand, generalized actions of a scalar field nonminimally coupled to \( R \) gravity, as a generalization of Brans-Dicke theory [17], have been extensively studied [18]. In the present paper, we intend to more generalize such theories to include \( f(R) \) gravity with a scalar field nonminimally coupled to \( f(R) \) gravity. Explicitly, we aim to obtain the forms of \( f(R) \), appearing in such modified action, by demanding that the Lagrangian admits the desired Noether symmetry [19, 20] (for a study of the Noether symmetry in various cosmological models see [21]). We shall see that by demanding the Noether symmetry, we can either obtain the explicit forms of the function \( f(R) \) or at least find the differential equations which can be solved to obtain \( f(R) \).

In Sec. (I) we introduce the action of a \( f(R) \) scalar-tensor theory and obtain the corresponding field equations. In Sec. (III), we introduce in general the Noether symmetry approach, and in Sec. (IV) we apply it to the \( f(R) \) scalar-tensor cosmology. In Sec. (V), we obtain the forms of \( f(R) \) or the differential equations for \( f(R) \). Conclusions are given in Sec. (VI).

II. COSMOLOGY FROM SCALAR-TENSOR THEORIES

Let us consider the general action

\[
A = \int d^4x \sqrt{-g} \left( \phi^2 f(R) + 4\omega(\phi)g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right),
\]

where the scalar field \( \phi \) is nonminimally coupled to \( f(R) \), and \( \omega(\phi) \) and \( V(\phi) \) are respectively the Brans-Dicke parameter and the potential as generic functions of \( \phi \). In order to derive the cosmological equations in a FRW metric [22], one can define a canonical Lagrangian

\[
\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R}, \phi, \dot{\phi}),
\]

where \( \mathcal{Q} = \{ a, R, \phi, \dot{\phi} \} \) is the configuration space and \( T\mathcal{Q} = \{ a, \dot{a}, R, \dot{R}, \phi, \dot{\phi} \} \) is the related tangent bundle on which \( \mathcal{L} \) is defined, where a dot denotes derivative with respect to the cosmic time \( t \). The variable \( a \) is the scale factor in FRW metric, and all dynamical variables \( a, R, \phi \) are assumed to depend just on \( t \) to restore homogeneity and isotropy. The presence of Ricci scalar in the Lagrangian needs explanation. In fact, it is assumed that \( R \), as well as \( a \) and \( \phi \), is a canonical variable because it is generally used in canonical quantization of higher order gravitational theories. However, such a position seems arbitrary, since \( R \) is not independent of \( a \) and \( \dot{a} \). Hence, one can use the method of Lagrange multipliers to set \( R \) as a constraint of the dynamics

\[
\mathcal{A} = \int dt a^3 \left\{ \phi^2 f(R) + 4\dot{\phi}^2 \omega(\phi) - V(\phi) \right. + \lambda \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + k_0^2 \frac{a_0^2}{a^2} \right] \right\},
\]

where \( \lambda \) here is a Lagrange multiplier. The variation of action with respect to \( R \) gives \( \lambda = -\phi^2 f_R \) where \( f_R := \frac{df}{dr} \). Therefore, the above action can be rewritten as

\[
\mathcal{A} = \int dt a^3 \left\{ \phi^2 f(R) + 4\dot{\phi}^2 \omega(\phi) - V(\phi) \right. - \phi^2 f_R \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + k_0^2 \frac{a_0^2}{a^2} \right] \right\}.
\]
By integrating by parts, and neglecting a pure divergence we obtain the point-like FRW Lagrangian
\[
\mathcal{L} = a^3 \dot{a}^2 (f(R) - R f_R) - 6a^2 \ddot{a}^2 f_R - 12a^2 \dot{a} \dddot{a} f_R - 6a^2 \dddot{a} f_{RR} + 6a^3 \kappa f_R + a^3 \left[ 4\dot{a}^2 \omega(\phi) - V(\phi) \right].
\]
(4)
The equations of motion for \(a\), \(R\) and \(\phi\) are obtained respectively
\[
2f_{RR} \ddot{a} \dot{a} R^2 + 2f_{RR} \left[ 2a \dddot{a} \dot{a}^2 + 2\dot{a} \dot{a} \dddot{a} \dot{R} \right] + 2 \dot{a}^2 \dddot{a} \dot{R}^2 + 2f_{R} \left[ 4a \dddot{a} \dot{a} + \dot{a}^2 \dddot{a} + 2a \dot{a} \dddot{a} + 2a^2 \dddot{a} \right]
\]
\[
+ k \ddot{a}^2 - (1/2) a^2 \dot{a}^2 R + a^2 \dddot{a} f(R)
\]
\[
+ a^2 \left[ 4\dot{a}^2 \omega(\phi) - V(\phi) \right] = 0,
\]
(5)
Finally the total energy \(E_\mathcal{L}\), corresponding to the \((00)\) Einstein equation is obtained as
\[
6a^2 \dot{a} \dddot{a} f_{RR} + f_R \left[ 6 \dddot{a} \dddot{a} + 12 \dot{a} \dddot{a} \phi \dot{a}^2 + a^2 \dddot{a} \dot{R}^2 \right]
\]
\[
+ 6k \dddot{a} \dot{R}^2 + a^2 \dddot{a} f(R) - 8a^2 \dddot{a} \omega(\phi)
\]
\[
+ a^2 \left[ 4\dot{a}^2 \omega(\phi) - V(\phi) \right] = 0.
\]
(6)

### III. NOETHER SYMMETRY APPROACH

Let \(\mathcal{L}(\dot{q}, \ddot{q})\) be a canonical, non degenerate point-like Lagrangian subject to
\[
\frac{\partial \mathcal{L}}{\partial \ddot{q}} = 0,
\]
\[
\text{det} H_{ij} = \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0,
\]
(9)
where \(H_{ij}\) is the Hessian matrix and a dot denotes derivative with respect to the cosmic time \(t\). The Lagrangian \(\mathcal{L}\) is generally of the form
\[
\mathcal{L} = T(\dot{q}, \ddot{q}) - V(q),
\]
(10)
where \(T\) and \(V\) are the ‘kinetic energy’ (with positive definite quadratic form) and ‘potential energy’ respectively. The energy function associated with \(\mathcal{L}\) is defined
\[
E_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L},
\]
(11)
which is the total energy \(T + V\) as a constant of motion. Since our cosmological problem has a finite number of degrees of freedom, we consider only point transformations.

Any invertible transformation of the generalized positions \(Q^i = Q^i(q)\) induces a transformation of the generalized velocities
\[
\dot{Q}^i = \frac{\partial Q^i}{\partial q^j} \dot{q}^j,
\]
(12)
where the matrix \(J = \left\| \frac{\partial Q^i}{\partial q^j} \right\|\) is the Jacobian of the transformation, and it is assumed to be non-zero. On the other hand, an infinitesimal point transformation is represented by a generic vector field on \(Q\)
\[
X = \alpha^i(q) \frac{\partial}{\partial q^i}.
\]
(13)
The induced transformation \((12)\) is then represented by
\[
X_c = \alpha^i(q) \frac{\partial}{\partial q^i} + \left( \frac{d}{dt} \alpha^i \right) \frac{\partial}{\partial \dot{q}^i}.
\]
(14)
The Lagrangian \(\mathcal{L}(q, \dot{q})\) is invariant under the transformation by \(X\) provided that
\[
L_X \mathcal{L} = \alpha^i \frac{\partial \mathcal{L}}{\partial q^i} + \left( \frac{d}{dt} \alpha^i \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0,
\]
(15)
where \(L_X \mathcal{L}\) is the Lie derivative of \(\mathcal{L}\). Let us now consider the Lagrangian \(\mathcal{L}\) and its Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0.
\]
(16)
Contracting \((10)\) with \(\alpha^i\) gives
\[
\alpha^i \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = \alpha^i \left( \frac{\partial \mathcal{L}}{\partial q^i} \right).
\]
(17)
On the other hand, we can write
\[
\frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = \alpha^i \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) + \left( \frac{d \alpha^i}{dt} \right) \frac{\partial \mathcal{L}}{\partial q^i},
\]
(18)
in which the first term in the RHS can be replaced by the RHS of \((17)\), hence \((15)\) results in
\[
\frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}.
\]
(19)
The immediate consequence of this result is the Noether theorem which states: if \(L_X \mathcal{L} = 0\), then the function
\[
\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial q^i},
\]
(20)
is a constant of motion.
IV. NOETHER SYMMETRIES IN SCALAR-TENSOR COSMOLOGY

Considering the $f(R)$ scalar-tensor cosmology, the vector field associated with the Noether symmetry is

$$X = A \frac{\partial}{\partial a} + B \frac{\partial}{\partial \phi} + C \frac{\partial}{\partial R} + \dot{A} \frac{\partial}{\partial a} + \dot{B} \frac{\partial}{\partial \phi} + \dot{C} \frac{\partial}{\partial R}. \quad (21)$$

Now, the Noether symmetry exists if at least one of the functions $A, B$, or $C$ in the equation (21) is different from zero. To investigate the existence of Noether symmetry, we should write down the equation $L_X \mathcal{L} = 0$ as the following system of differential equations

$$3A\omega(\phi) + B a \frac{d\omega}{d\phi} - 3f_R a^2 \phi A + 2\omega(\phi) a^2 \partial_\phi B = 0, \quad (22)$$

$$f_R \left( A\phi + 2Ba + 2a\phi \partial_a A + 2a^2 \partial_a B \right) + a f_{RR} (C + a\partial_a C) = 0, \quad (23)$$

$$f_R \left( (2\phi \partial_R A + 2a\partial_R B) + 2f_{RR} (aB + A\phi) \right) + a\phi \left( C f_{RR} + a\partial_c A f_{RR} + \partial_c C f_{RR} \right) = 0, \quad (25)$$

$$\phi^2 f_{RR} \partial_\phi B = \frac{4}{3} \omega(\phi) a R_B + 2\phi f_R \partial_R A = 0, \quad (26)$$

$$\partial_R A = 0, \quad (27)$$

which are obtained by setting to zero the coefficients of the terms $\partial^2, R^2, \phi^2, aR, \partial\phi, \partial\phi R$ in $L_X \mathcal{L} = 0$. Finally, we have to satisfy the constraint

$$6k\phi^2 A_{fR} + 3a^2 \phi^2 A (f - f_{RR}) - 3a^2 V(\phi) A + 2a^3 \phi B (f - f_{RR}) - R f_{RR} a^3 \phi^2 C \quad (28)$$

$$- B a^3 \frac{dV}{d\phi} + 12ka^2 B + 6ka^2 f_{RR} C = 0.$$

A solution of (22)–(27) exists if explicit forms of $A, B$ and $C$ are found. By using Eq. (27), the equation (25) becomes

$$f_R (2a\partial_R B) + f_{RR} (2A\phi + 2a B + a\phi \partial_a A) + a\phi \partial_R (C f_{RR}) = 0, \quad (29)$$

which can be rewritten as

$$\partial_R (2aBf_R + a\phi C f_{RR}) + f_{RR} (2A\phi + a\phi \partial_a A) = 0, \quad (30)$$

and solved with respect to $R$ as

$$2Bf_R + \phi C f_{RR} = - \left( 2\frac{A\phi}{a} + \phi \partial_a A \right) f_R + h(a, \phi), \quad (31)$$

where $h(a, \phi)$ is the integration constant. From Eq. (31) we get

$$C = \frac{1}{\phi} \left[ - \left( 2B + 2\frac{A\phi}{a} + \phi \partial_a A \right) \frac{f_R}{f_{RR}} + h(a, \phi) \right]. \quad (32)$$

Inserting $C$ into Eq. (28) results in

$$f_R \left( A\phi - a\phi \partial_a A - \phi a^2 \frac{\partial^2 A}{\partial a^2} \right) + a (h + a\partial_a h) = 0, \quad (33)$$

which is solved for $A$ and $h$ as

$$A = \left( c_1 a + \frac{c_2}{a} \right) g(\phi) \quad \text{and} \quad h = \frac{\bar{c}}{a} \lambda(\phi), \quad (34)$$

where $c_1, c_2$ and $\bar{c}$ are integration constants, and $g(\phi)$ and $\lambda(\phi)$ are some generic functions of $\phi$. Substituting $A$ and $h$ into $C$ we obtain

$$C = - \left[ \frac{2B}{\phi} + \left( \frac{3c_1}{a} + \frac{c_2}{a^2} \right) g(\phi) \right] \frac{f_R}{f_{RR}} + \frac{\bar{c} \lambda(\phi)}{a\phi f_{RR}}. \quad (35)$$

We leave the constraint (28) as an equation to choose suitable potential $V(\phi)$ and $f_R$. The remaining equations governing $B, g(\phi), \omega(\phi)$ and $\lambda(\phi)$ are

$$3 \left( c_1 a + \frac{c_2}{a} \right) g(\phi) \omega(\phi) + aB \frac{d\omega}{d\phi} + 2a\omega(\phi) \partial_\phi B - 3f_R \phi \left( c_1 a + \frac{c_2}{a} \right) \frac{dg}{d\phi} = 0, \quad (36)$$

$$- \frac{1}{2} f_R \left( -c_1 a + \frac{c_2}{a} \right) \phi^2 \frac{dg}{d\phi} + \frac{\bar{c}}{2} \left( \lambda + \phi \frac{d\lambda}{d\phi} \right) \quad (37)$$

$$\phi^2 f_{RR} \left( c_1 a + \frac{c_2}{a} \right) \frac{dg}{d\phi} - \frac{4}{3} \omega(\phi) a \partial_\phi B = 0. \quad (38)$$

By taking $\lambda(\phi) = \lambda_0 \phi^{-1}$ in Eq. (37) the term proportional to $\bar{c}$ vanishes. The resulting equations (36), (37) and (38) are just consistent for constants $B(a, R, \phi) = B_0$ and $g(\phi) = g_0$. Equations (37) and (38) are satisfied by these solutions and Eq. (36) becomes

$$3 \left( c_1 a + \frac{c_2}{a} \right) g_0 \omega(\phi) + aB_0 \frac{d\omega}{d\phi} = 0. \quad (39)$$

One may set $c_2 = 0$ to convert this equation into a simple differential equation

$$\frac{d\omega}{d\phi} + k^2 \omega = 0, \quad (40)$$

where $k^2 = 3c_1 g_0 / B_0$. For the choices of the constants $c_1, g_0$ and $B_0$ resulting in $k^2 > 0$ we have the oscillating solutions

$$\omega(\phi) = \omega_0 \exp(\pm ik\phi), \quad (41)$$
whereas for the choices leading to $\kappa^2 < 0$, we obtain exponential solutions

$$\omega(\phi) = \omega_0 \exp(\pm \kappa \phi).$$

(42)

Therefore, we find

$$A = c_1 g_0 a, \quad B = B_0, \quad C = \frac{2B_0}{\phi} + 3c_1 g_0 \frac{f_R}{f_{RR}} + \frac{c\lambda_0}{\phi^2 f_{RR}}.$$  

(43)

The existence of non zero quantities $A$, $B$ and $C$ accounts for the Noether symmetry provided that $A$, $B$, $C$, $f_R$ and $V(\phi)$ satisfy the constraint (28). This equation may be converted to a differential equation for $f(R)$ as follows

$$f_R = \frac{1}{12ka^2c_1g} [3a^3\phi^2 c_1g_0 + 2a^3\phi B] f + \left[3a^3V(\phi)c_1g_0 + c\lambda_0(Ra^2 - 6k) + Ba^3dV/\phi \right].$$

(44)

We may find the constant of motion, namely the Noether charge as

$$\Theta_0 = A \frac{\partial \mathcal{L}}{\partial \Phi} + B \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + C \frac{\partial \mathcal{L}}{\partial R} = -6\phi a^2 c_1 g_0[2f_R(\Phi a) + \phi a \dot{R} f_{RR}]$$

$$- 12B_0 a^2 \dot{a} f_R + 8Ba^3 \phi \omega(\phi)$$

$$+ 6\phi a^2 \dot{a} \left[ 2B_0 \phi + 3c_1 g_0 \right] f_R$$

$$- 6c\lambda a \dot{a}.$$  

(45)

This equation may be written as

$$f_{RR} \dot{R} = -\frac{\Theta}{6a^3\phi^2 c_1 g_0} - 2 \frac{d}{dt} \ln(a) f_R$$

$$+ \frac{4B_0 \phi \omega(\phi)}{3a^2 c_1 g_0} + 3f_R H - \frac{c\lambda_0 H}{c_1 g_0 a \phi^2}.$$  

(46)

The Friedmann equation is obtained by construction of the zero Hamiltonian constraint as

$$H = \dot{a} \frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{R} \frac{\partial \mathcal{L}}{\partial \dot{R}} + \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}.$$  

(47)

$$= f + 6f_R \dot{R} H + 6f_R H^2 + 12f_R H(\frac{\dot{\phi}}{\phi})$$

$$- 4(\frac{\dot{\phi}}{\phi})^2 \omega(\phi) - f_R (R - \frac{6k}{a^2} - \frac{V(\phi)}{\phi^2}) = 0,$$

or

$$H = f - \frac{\Theta H}{\phi^2 a^3 c_1 g_0} + 12f_R H^2$$

$$- 6c\lambda_0 H^2 \frac{c_1 a \phi^2 g_0}{c_1 g_0} - f_R (R - \frac{6k}{a^2})$$

$$- \frac{V(\phi)}{\phi^2} + \frac{\dot{\phi}}{\phi^2} \omega(\phi) \left( 8HB_0 c_1 g_0 - 4\phi \right) = 0,$$

(48)

where we have used of (17) and that

$$\frac{\dot{\phi}}{\phi} = \frac{d}{dt} \ln(\phi), \quad \frac{d}{dt} \ln(\frac{a}{\phi}) = \frac{\dot{a}}{a} = H.$$  

V. $f(R)$ COSMOLOGICAL MODELS

To find some $f(R)$ scalar-tensor cosmological models consistent with the Noether symmetry, we first rewrite the constraint equation (11) as follows

$$c_1 a^2 (3a^4 \phi^2 f - 3a^4 V(\phi) - 12ka^2 \phi^2 f_R) g_0 = (50)$$

$$+ Ba^6 (2\phi f - \frac{dV}{d\phi}) = \epsilon \lambda_0 a^3 (Ra^2 - 6k).$$

Then, we study the different cases according to some specific choices for $\omega(\phi), V(\phi), B_0, g_0, c_1$ and $\Theta$. In all cases, except one, we will assume $\epsilon = 0$.

A. $B_0 = V(\phi) = 0$

In this case, Eq.(57) is reduced to

$$a^2 \phi^2 f - a^2 V(\phi) - 4k \phi^2 f_R = 0,$$  

(51)

or

$$f_R = \frac{a^2}{4k} f.$$  

(52)

Using (52) in the Friedmann equation (50), we obtain

$$f \left( \frac{5}{2} + \frac{3a^2 H^2}{k} - \frac{R a^2}{4k} \right) = \Theta H - \frac{\Theta H}{c_1 g_0 a^3} + 4\omega(\phi) \left( \frac{\dot{\phi}}{\phi^2} \right)^2.$$  

(53)

I) Imposing $\Theta = \omega(\phi) = 0$ leads to ($f \neq 0$)

$$R = 12H^2 + \frac{10k}{a^2}.$$  

(54)

Considering the general expression for the Ricci scalar

$$R = 12H^2 + 6a HH' + \frac{6k}{a^2},$$  

(55)

one may find the following differential equation by equating the RHS of (54) and (55)

$$(H^2)' - \frac{4k}{3a^2} = 0,$$  

(56)

where $'$ denotes derivative with respect to $a$. The differential equation (56) is simply solved as

$$H^2 = \frac{2k}{3a^2} + d_1,$$  

(57)

where $d_1$ is the integration constant. This equation determines the cosmological dynamics. Now, we obtain $f(R)$. To this end, we insert (57) into (54) to obtain $a(R)$.
Then, we calculate \( f_R \) by using \( a(R) \) and \( f_R = \frac{df}{da} \frac{da}{dR} \) as follows

\[
f_R = -\frac{df}{da} \frac{a^3}{4k}.
\]  

(58)

Finally, using (58) in (62) results in

\[
f(a) = \frac{d_2}{a}.
\]

(59)

where \( d_2 \) is another integration constant. Putting \( H^2 \) from (67) in (54), we obtain

\[
R = \frac{2k}{a^2} + 12d_1.
\]

(60)

Using this equation, we may transform \( f(a) \) into \( f(R) \) as

\[
f(R) = d_2 \left( \frac{R - 12d_1}{2k} \right)^{1/2}.
\]

(61)

This is viable for closed and open universes, \( k = \pm 1 \). For the flat universe, we may take \( d_2 = \sqrt{2k} \) so that

\[
f(R) = (R - 12d_1)^{1/2}.
\]

(62)

**II** Imposing \( \Theta \neq 0, \omega(\phi) = 0 \) in (89), results in

\[
f = \frac{2k\Theta}{c_1g_0a^3(2k - 3a^3HH')}.
\]

(63)

where use has been made of (88). Note that \( f_R = \frac{\partial f}{\partial a} \frac{da}{dR} \), so we may calculate separately \( \frac{\partial f}{\partial a} \) and \( \frac{da}{dR} \) using (63) and (64), respectively. Then, (62) casts into a differential equation for \( H^2 \) as

\[
(H^2)'' = \frac{(2k}{3H^2a^3} - \frac{5}{a}) \left( \frac{16k^2}{9a^6} \frac{1}{(H^2)'} + \frac{8k}{3a^3} \right) + \frac{5}{a} \left( \frac{2k}{3H^2a^3} - \frac{5}{a} \right)
\]

(64)

By solving this differential equation we may find \( H(a) \) which determines the cosmological dynamics. Moreover, we may insert \( H(a) \) into (55) to find \( a(R) \). Then, we can insert both \( H(a) \) and \( a(R) \) into (63) to obtain \( f(R) \).

**B.** \( B_0 = 0, \omega(\phi) = \frac{1}{2}m^2\phi^2 \)

In this case, Eq. (61) leads to

\[
f_R = \frac{a^2}{4k}f - \frac{ma^2}{2k}.
\]

(65)

By inserting \( f_R \) from (65) into the Friedmann equation (48), we obtain

\[
(10k + 12a^2H^2 - a^2R)f = \frac{4k\Theta H}{c_1g_0a^3\phi^2}
\]

(66)

\[
+ 5m^2k + 2ma^2(3H^2 - R/4) + 16k\omega(\phi)(\frac{\phi}{\phi^2})^2.
\]

\[I\] Imposing \( \Theta = \omega(\phi) = 0 \) in (66) leads to

\[
f = \frac{5k + 6a^2H^2 - a^2R/2}{10k + 12a^2H^2 - a^2R}m^2.
\]

(67)

By inserting \( R \) from (55) into (67) we obtain

\[
f = \frac{1}{2}m^2.
\]

(68)

Obviously, this case is not physically viable because it leads to a constant \( f \) with no cosmological solutions.

**II** Imposing \( \Theta \neq 0, \omega(\phi) = 0 \) in (69), and inserting \( R \) from (55) results in

\[
f = \frac{1}{2}m^2 + \frac{2k\Theta}{c_1g_0a^3(2k - 3a^3HH')}.
\]

(69)

Using \( f_R = \frac{\partial f}{\partial a} \frac{da}{dR}, \omega(\phi) \) and (67) we obtain the same differential equation for \( H^2 \) as (64). By solving this equation for \( H(a) \), the cosmological dynamics is obtained. By inserting \( H(a) \) into (55) we find \( a(R) \), and by using \( H(a) \) and \( a(R) \) in (70) we obtain \( f(R, \phi) \). Now, in order to remove \( \phi \) in favor of \( R \) within \( f(R, \phi) \), we first rewrite Eq. (22) as follows

\[
3c_1g_0a\omega(\phi) + B_0\frac{d\omega}{d\phi} = 0,
\]

(70)

where we have used of (13). This equation allows us to obtain \( \phi \) in terms of \( a \) as a function \( \phi(a) \). On the other hand, we have \( a(R) \) from (55). Therefore, combining \( \phi(a) \) and \( a(R) \) we may obtain \( \phi(R) \) by which we can replace \( \phi \) in \( f(R, \phi) \) in terms of \( R \) and obtain the desired \( f(R) \).

**C.** \( B_0 \neq 0, V(\phi) = 0 \)

In this case, Eq. (100) is reduced to

\[
f_R = \frac{(3c_1g_0\phi + 2B_0)a^2}{12kc_1g_0\phi}f.
\]

(71)

Inserting \( f_R \) from (71) into the Friedmann equation (48), and using the following definitions

\[
\begin{align*}
\alpha &= 30kc_1g_0\phi + 12kB_0, \\
\beta &= 12kc_1g_0\phi, \\
\gamma &= 36c_1g_0\phi + 2B_0, \\
\Delta &= c_1g_0\phi^2, \\
\xi &= \alpha - 6k\gamma, \\
\mu &= 4(\frac{\phi}{a})^2, \\
\nu &= 8B_0\phi/c_1g_0\phi^2,
\end{align*}
\]

(72)

together with (55), we obtain

\[
f \left( \frac{\xi - 6\gamma a^3H'}{\beta} \right) = \frac{\Theta H}{\Delta a^3} + (\mu - \nu H)\omega(\phi).
\]

(73)
I) Imposing $\Theta = 0, \omega(\phi) \neq 0$ in (73) leads to

$$f = \frac{D - EH}{G - Ma^3HH'},$$

(74)

where

$$\begin{aligned}
D &= \beta \mu \omega(\phi), \\
E &= \beta \nu \omega(\phi), \\
G &= \xi = \beta, \\
M &= 6\gamma.'
\end{aligned}$$

(75)

Using $f_R = \frac{\partial f}{\partial a^3} a^2H'$, (55), (70) and (74), we obtain a differential equation for $H^2$. By solving this equation for $H(a)$ we obtain the cosmological dynamics. Inserting $H(a)$ into (55) results in $a(R)$. Then, we may use $H(a)$ and $a(R)$ in (73), use Eq. (70) and $a(R)$ to obtain $\phi(R)$, and finally obtain $f(R)$.

II) Imposing $\Theta \neq 0, \omega(\phi) \neq 0$ in (73) leads to

$$f = \frac{a^3(\bar{D} - \bar{E}H) + FH}{a^3(G - a^3MHH')}$$

(76)

where

$$\begin{aligned}
\bar{D} &= \Delta \beta \mu \omega(\phi), \\
\bar{E} &= \Delta \beta \nu \omega(\phi), \\
\bar{F} &= \beta \Theta, \\
\bar{G} &= \beta \Delta, \\
M &= 6\gamma.'
\end{aligned}$$

(77)

Using $f_R = \frac{\partial f}{\partial a^3} a^2H'$, (55), (70) and (76), we obtain a differential equation for $H^2$. By solving this equation we obtain the cosmological dynamics $H(a)$. Moreover, similar to the previous case we may obtain $f(R)$.

D. $\bar{c} \neq 0, V(\phi) = g_0 = 0$

In this case, Eq. (35) reads as

$$C = \frac{2B}{\phi} \frac{f_R}{f_{RR}} + \frac{\bar{c} \lambda_0}{a^2 f_{RR}}.$$  

(78)

Inserting $C$ from (78) into (28), results in

$$(2B_0 a^3) f = a^3 B_0 \frac{dV}{d\phi} + \bar{c} \lambda_0 (Ra^2 - 6k).$$  

(79)

Unlike the previous procedure, in this case the constraint equation is not a differential equation containing $f_R$. To find an equation containing $f_R$ we use (41). We evaluate $f_{RR}R$ from (79) and put it in (41) to obtain

$$f + \frac{3c\lambda(\phi)}{aB_0} H - \frac{3c\lambda(\phi)}{aB_0} H^2 + 6f_R H^2$$

(80)

$$+ 12f_R H\left(\frac{\dot{\phi}}{\phi} - 4\left(\frac{\dot{\phi}}{\phi}\right)^2 \omega(\phi) - f_R(R - \frac{6k}{a^2}) - \frac{V(\phi)}{\phi^2}\right) = 0.$$  

Since $g_0 = 0$, we have $\omega(\phi) = 1$. Now, we insert (79) and (55) into (80) and obtain the required differential equation containing $f_R$.

$$f_R \left[-6H^2 + 12H\left(\frac{\dot{\phi}}{\phi}\right) - 6aHH'\right]$$

(81)

$$= -\frac{3c\lambda(\phi)}{aB_0} [H^2 + aHH' + H\frac{\lambda}{\lambda'}]$$

$$+ 4\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{1}{2\phi} \frac{dV}{d\phi} + \frac{V(\phi)}{\phi^2}.$$  

For simplicity we consider $V(\phi) = 0$, and obtain

$$f_R = \frac{3c\lambda(\phi)}{aB_0} [H^2 + aHH' + H\frac{\lambda}{\lambda'} - 4\left(\frac{\dot{\phi}}{\phi}\right)^2]$$

(82)

$$\frac{6H^2 - 12H\frac{\lambda}{\lambda'} + 6aHH'}.$$

This equation may be solved for $f(R)$ by using $H(\alpha)$, $a(\alpha)$, and $\phi(\alpha)$. Using $f_R = \frac{\partial f}{\partial a^3} a^2H'$, (55) and (79), a differential equation for $H^2$ is obtained as

$$(H^2)'' = \frac{1}{Ha^3(2a^2 + \alpha\mu) - a^4\beta} \times$$

(83)

$$\left[a^2(H^2)' [3a(H^2)'] + \frac{a\alpha}{2} (H^2)' + (8\alpha \frac{\dot{\phi}}{\phi} + 5a\mu)H - 5a\beta\right] +$$

$$4\alpha a(H^2)' (H^2') - 2H\frac{\dot{\phi}}{\phi} + 4a \alpha k \left[ 2\frac{(H^2)'}{a} - H^2 - H^2 \mu - H \right] + 43k,$$

where

$$\begin{aligned}
\alpha &= \frac{3c\lambda(\phi)}{B_0}, \\
\beta &= 4\left(\frac{\dot{\phi}}{\phi}\right)^2, \\
\mu &= \frac{\lambda}{\lambda'}.
\end{aligned}$$

This equation may be solved for $H^2(\alpha)$ by using $\phi(\alpha)$.

VI. CONCLUSIONS

In this paper, we have investigated the conditions for the existence of Noether symmetry in a $f(R)$ scalar-tensor theory of gravity in which the Ricci function $f(R)$, the scalar field potential $V(\phi)$ and the coupling function $\omega(\phi)$ are generally unknown. We have shown that the Noether symmetry may exist and further obtained a constraint between $f(R)$, $V(\phi)$ and $\omega(\phi)$. For specific choices of the functions $\omega(\phi), V(\phi)$, the parameters $B_0, g_0, c_1$, and the constant of motion $\Theta$, we have obtained explicitly the functions $f(R)$ and $H(\alpha)$. For other cases, we have found the corresponding differential equations which can only be solved numerically.

Acknowledgment

This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RIAAM) under research project NO.1/2077.
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