LOOP CORRECTIONS IN NONLINEAR COSMOLOGICAL PERTURBATION THEORY. II. TWO-POINT STATISTICS AND SELF-SIMILARITY

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ABSTRACT

We calculate the lowest order nonlinear contributions to the power spectrum, two-point correlation function, and smoothed variance of the density field, for Gaussian initial conditions and scale-free initial power spectra, \( P(k) \sim k^n \). These results extend and, in some cases, correct previous work in the literature on cosmological perturbation theory. Comparing with the scaling behavior observed in N-body simulations, we find that the validity of nonlinear perturbation theory depends strongly on the spectral index \( n \). For \( n < -1 \), we find excellent agreement over scales where the variance \( \sigma^2(R) \lesssim 10 \); however, for \( n \gtrsim -1 \), perturbation theory predicts deviations from self-similar scaling (which increase with \( n \)) not seen in numerical simulations. This anomalous scaling suggests that the principal assumption underlying cosmological perturbation theory, namely, that large-scale fields can be described perturbatively even when fluctuations are highly nonlinear on small scales, breaks down beyond leading order for spectral indices \( n \gtrsim -1 \). For \( n < -1 \), the power spectrum, variance, and correlation function in the scaling regime can be calculated using dimensional regularization.

Subject heading: galaxies: clusters: general — large-scale structure of universe — methods: numerical

1. INTRODUCTION

Several independent arguments suggest that the growth of cosmological density perturbations on large scales can be described by perturbation theory, even when the density and velocity fields are highly nonlinear on small scales. For example, analytic and numerical work showed that linear perturbation theory describes the evolution of the large-scale density power spectrum \( P(k) \), provided the initial spectrum falls off less steeply than \( k^n \) for small \( k \) (Zeldovich 1965; Peebles 1974, 1980; Peebles & Groth 1976). In addition, for Gaussian initial conditions, leading-order nonlinear perturbative calculations of higher order moments of the density field, e.g., the skewness and kurtosis, agree well with \( N \)-body simulations in the weakly nonlinear regime, where the variance of the smoothed density field \( \sigma^2(R) \equiv \langle \delta^2(R) \rangle \lesssim 0.5 - 1 \) (Juszkiewicz, Bouchet, & Colombi 1993; Bernardardeau 1994; Gaztañaga & Baugh 1995; Baugh, Gaztañaga, & Efstathiou 1995). Thus, leading-order perturbation theory has been shown to work surprisingly well in the cases where comparisons with numerical simulations have been made.

The success of leading-order cosmological perturbation theory raises questions: since the perturbation series is most likely asymptotic, what happens beyond leading order: does the agreement with simulations improve or deteriorate? More generally, what sets the limits of perturbation theory, beyond which it breaks down? These questions have become more urgent since it has been shown that leading-order perturbation theory appears to provide an adequate description even on scales where next-to-leading order and higher order perturbative contributions would be expected to become important.

To address these questions, one must calculate loop corrections, i.e., corrections beyond leading order, in nonlinear cosmological perturbation theory (NLCPT). This is the second paper of a series devoted to this topic. In the first paper (Scoccimarro & Frieman 1996), we applied diagrammatic techniques to calculate loop corrections to one-point cumulants of unsmoothed fields, such as the variance and skewness, for Gaussian initial conditions. Here we calculate the one-loop (first nonlinear) corrections to the power spectrum, the volume-averaged two-point correlation function, and the variance of the smoothed density field for scale-free initial spectra, \( P(k) \sim k^n \). While the linear power spectrum for the universe is not scale-free (on both observational and theoretical grounds), scale-free spectra are useful approximations over limited ranges of wave-number \( k \); they also have the advantage of yielding analytic closed form results. In a forthcoming paper, we will present one-loop corrections to the bispectrum (the three-point cumulant in Fourier space) and the skewness of the smoothed density field for scale-free and cold dark matter spectra.

One-loop corrections to the two-point correlation function and power spectrum have been previously studied in the literature (Juszkiewicz 1981; Vishniac 1983; Juszkiewicz, Sonoda, & Barrow 1984; Coles 1990; Suto & Sasaki 1991; Makino, Sasaki, & Suto 1992; Jain & Bertschinger 1994; Baugh & Efstathiou 1994). Multiloop corrections to the power spectrum were considered by Fry (1994), including the full contributions up to two loops and the most important terms at large \( k \) in three- and four-loop order. Some of our results overlap in particular with the analytic results for the one-loop power spectrum reported by Suto & Sasaki (1991) and in Makino et al. (1992). However, since we found that some of their expressions contain errors, we present complete corrected expressions for the one-loop power spectrum. One-loop corrections to the variance were studied numerically for Gaussian smoothing by Lokas et al. (1995). We correct some numerical errors in their results, extend them to include top-hat smoothing and the average two-point correlation function, and provide analytic derivation of some of the numerical results.

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The limiting behavior of the one-loop corrections on large scales leads us to reconsider the issue of self-similarity in perturbation theory (Davis & Peebles 1977; Peebles 1980). It is commonly accepted that the evolution of density perturbations from scale-free initial conditions in an Einstein–de Sitter universe is statistically self-similar. N-body simulations have generally found self-similar scaling for $-3 < n < 1$, although the results for $n < -1$ have been somewhat ambiguous due to problems of dynamic range in the simulations (Efstathiou et al. 1988; Bertshinger & Gelb 1991; Ryden & Gramann 1991; Gramann 1992; Colombi, Bouchet, & Hernquist 1995; Jain 1995; Padmanabhan et al. 1995). The issue of self-similarity in NLCPT has recently been investigated by Jain & Bertshinger (1995), who present arguments that the perturbative evolution is also self-similar. While we do not disagree with their calculations, we start from a different premise regarding cutoffs in the initial spectrum, which leads us to conclude that loop corrections break self-similar scaling if the spectral index $n \geq -1$. In the regime where we do find scaling, $n < -1$, we use dimensional regularization to calculate analytically the asymptotic behavior of the one-loop power spectrum, variance, and average correlation function; these results agree quite well with the “universal scaling” extracted from numerical simulations (Hamilton et al. 1991; Peacock & Dodds 1994; Jain, Mo, & White 1995).

The paper is organized as follows. In § 2 we discuss nonlinear perturbation theory and review the diagrammatic approach developed in Fry (1984), Goroff et al. (1986), and Scoccimarro & Frieman (1996) to the calculation of statistical quantities. Section 3 presents the results of one-loop calculations for the power spectrum, the smoothed variance, and average two-point correlation function. Self-similarity in perturbation theory is the subject of § 4. We discuss the conditions under which one-loop corrections exhibit self-similarity and compare our results with the universal scaling hypothesis, based on numerical simulations. Section 5 contains our conclusions.

2. DYNAMICS AND STATISTICS

2.1. The Equations of Motion

Assuming the universe is dominated by pressureless dust (e.g., cold dark matter), in the single-stream approximation (prior to orbit crossing) one can adopt a fluid description of the cosmological N-body problem. In this limit, the relevant equations of motion correspond to conservation of mass and momentum and the Poisson equation (e.g., Peebles 1980; Scoccimarro & Frieman 1996),

\[
\frac{\partial \delta(x, \tau)}{\partial \tau} + \nabla \cdot \left([1 + \delta(x, \tau)] \mathbf{v}(x, \tau)\right) = 0,
\]

(1a)

\[
\frac{\partial \mathbf{v}(x, \tau)}{\partial \tau} + \nabla \Phi(x, \tau) + \left[\mathbf{v}(x, \tau) \cdot \nabla\right] \mathbf{v}(x, \tau) = -\nabla \Phi(x, \tau),
\]

(1b)

\[
\nabla^2 \Phi(x, \tau) = \frac{2}{3} \Omega \mathcal{H}^2(\tau) \delta(x, \tau),
\]

(1c)

where the density contrast $\delta(x, \tau) \equiv \rho(x, \tau)/\bar{\rho} - 1$, with $\bar{\rho}(\tau)$ the mean density of matter, $v \equiv dx/d\tau$ represents the velocity field fluctuations about the Hubble flow, $\mathcal{H} \equiv d \ln a/d \tau$ is the conformal expansion rate, $a(\tau)$ is the cosmic scale factor, $x$ denotes comoving coordinates, $\tau = \int d\tau/a$ is the conformal time, $\Phi$ is the gravitational potential due to the density fluctuations, and $\Omega = \rho/\rho_c = 8\pi G \bar{\rho} a^2/3 \mathcal{H}^2$ is the density parameter. Note that we have implicitly assumed the Newtonian approximation to general relativity, valid on scales less than the Hubble length $a \mathcal{H}^{-1}$. We take the velocity field to be irrotational, so it can be completely described by its divergence $\theta \equiv \nabla \cdot \mathbf{v}$. Equations (1a)–(1c) hold in an arbitrary homogeneous and isotropic background universe which evolves according to the Friedmann equations; henceforth, for simplicity we assume an Einstein–de Sitter background, $\Omega = 1$ (with vanishing cosmological constant), for which $a \propto t^2$ and $3 \Omega a^2 H^2/2 = 6\pi^2$.

We take the divergence of equation (1b) and Fourier transform the resulting equations of motion according to the convention

\[
\tilde{A}(k, \tau) = \int \frac{d^3 x}{(2\pi)^3} \exp(-ik \cdot x) A(x, \tau),
\]

(2)

for the Fourier transform of any field $A(x, \tau)$, where, here and throughout, $k$ is a comoving wavenumber. This yields

\[
\frac{\partial \tilde{\delta}(k, \tau)}{\partial \tau} + \tilde{\delta}(k, \tau) = -\int d^3 k_1 d^3 k_2 \delta_D(k - k_1 - k_2) \alpha(k, k_1) \tilde{\delta}(k_1, \tau) \tilde{\delta}(k_2, \tau),
\]

(3a)

and

\[
\frac{\partial \tilde{\delta}(k, \tau)}{\partial \tau} + \mathcal{H}(\tau) \tilde{\delta}(k, \tau) + \frac{3}{2} \mathcal{H}^2(\tau) \tilde{\delta}(k, \tau) = -\int d^3 k_1 d^3 k_2 \delta_D(k - k_1 - k_2) \beta(k, k_1, k_2) \tilde{\delta}(k_1, \tau) \tilde{\delta}(k_2, \tau),
\]

(3b)

($\delta_D$ denotes the three-dimensional Dirac delta distribution), where the functions

\[
\alpha(k, k_1) \equiv \frac{k \cdot k_1}{k_1^2}, \quad \beta(k, k_1, k_2) \equiv \frac{k^2 (k_1 \cdot k_2)}{2 k_1^2 k_2^2},
\]

(4)
encode the nonlinearity of the evolution (mode coupling) and come from the nonlinear terms in the continuity equation (1a) and the Euler equation (1b), respectively.

2.2. Perturbation Theory Solutions

Equations (3a)–(3b) can be formally solved via a perturbative expansion,

\[
\tilde{\delta}(k, \tau) = \sum_{n=1}^{\infty} a^n(\tau)\delta_n(k), \quad \tilde{\theta}(k, \tau) = \mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau)\theta_n(k),
\]

where only the fastest growing mode is taken into account. At small \(a\), the series are dominated by their first terms, and since \(\theta_0(k) = -\delta_1(k)\) from the continuity equation, \(\delta_1(k)\) completely characterizes the linear fluctuations. The equations of motion (3a) and (3b) determine \(\delta_0(n)\) and \(\theta_0(n)\) in terms of the linear fluctuations,

\[
\delta_0(n) = \int d^3q_1 \cdots \int d^3q_n \epsilon_{\delta(k - q_1 - \cdots - q_n)F_n(q_1, \ldots, q_n)} \delta_1(q_1) \cdots \delta_1(q_n),
\]

and

\[
\theta_0(n) = -\int d^3q_1 \cdots \int d^3q_n \epsilon_{\delta(k - q_1 - \cdots - q_n)G_n(q_1, \ldots, q_n)} \delta_1(q_1) \cdots \delta_1(q_n),
\]

where \(F_n\) and \(G_n\) are symmetric homogeneous functions of the wave vectors \(\{q_1, \ldots, q_n\}\) with degree zero, which are constructed from the fundamental mode coupling functions \(a(k, k_1)\) and \(b(k, k_1, k_2)\) by a recursive procedure (see Goroff et al. 1986; Jain & Bertschinger 1994).

2.3. Diagrammatic Expansion of Statistical Quantities

In this work we focus on the nonlinear evolution of two-point cumulants of the density field, such as the power spectrum and the volume-average two-point correlation function, and their one-point counterpart, the variance. These are defined, respectively, by

\[
\langle \Delta(k, \tau)\Delta(k', \tau) \rangle = \delta_{0}(k + k')P(k, \tau),
\]

\[
\zeta(R, \tau) \equiv \int \zeta(x, \tau)\tilde{W}(x)d^3x = \int P(k, \tau)W(kR)d^3k,
\]

and

\[
\sigma^2(R, \tau) = \int P(k, \tau)\tilde{W}(kR)d^3k.
\]

Here \(\zeta(x, \tau) = \{ P(k, \tau)\exp(ik \cdot x)d^3k \) is the two-point correlation function and \(\tilde{W}(x)\) is a window function, with Fourier transform \(\tilde{W}(kR)\) which we take to be either a top-hat (TH) or a Gaussian (G),

\[
W_{\text{TH}}(u) = \frac{3}{u^2} [\sin(u) - u \cos(u)],
\]

\[
W_{G}(u) = \exp(-u^2/2).
\]

We are interested in calculating the nonlinear evolution of these statistical quantities from Gaussian initial conditions in the weakly nonlinear regime, \(\sigma(R) \ll 1\). A systematic framework for calculating correlations of cosmological fields in perturbation theory has been formulated using diagrammatic techniques (Goroff et al. 1986; Wise 1988; Scoccimarro & Frieman 1996). In this approach, contributions to \(p\)-point cumulants of the density field come from connected diagrams with \(p\) external (solid) lines and \(r = p - 1, p, \ldots\) internal (dashed) lines. The perturbation expansion leads to a collection of diagrams at each order, the leading order being tree diagrams, the next-to-leading order one-loop diagrams and so on. In each diagram, external lines represent the spectral components of the fields in which we are interested \([e.g., \delta(k, \tau)\]). Each internal line is labeled by a wavevector that is integrated over, and represents a linear power spectrum \(P_{11}(q, \tau)\). Vertices of order \(n\) (i.e., where \(n\) internal lines join) represent an \(n\)th-order perturbative solution \(\delta_n\), and momentum conservation is imposed at each vertex. Figure 1 shows the factors associated with vertices and internal lines (for more details including combinatoric factors, see Scoccimarro & Frieman 1996).

According to the diagrammatic rules, we can write the loop expansion for the power spectrum up to one-loop corrections as

\[
P(k, \tau) = P^{(0)}(k, \tau) + P^{(1)}(k, \tau) + \cdots,
\]

where the superscript \((n)\) denotes an \(n\)th-order contribution, the tree-level (zero loop) contribution is just the linear spectrum,

\[
P^{(0)}(k, \tau) = P_{11}(k, \tau),
\]
with \( a^2 < \delta_1(k)\delta_1(k') >_c = \delta_0(k + k')P_{11}(k, \tau) \), and the one-loop contribution consists of two terms,

\[
P^{(1)}(k, \tau) = P_{22}(k, \tau) + P_{13}(k, \tau),
\]

(14)

with (see Fig. 2)

\[
P_{22}(k, \tau) = 2 \int [F_0^G(k - q, q)]^2 P_{11}(|k - q|, \tau)P_{11}(q, \tau)d^3q,
\]

(15a)

\[
P_{13}(k, \tau) = 6 \int F_0^G(k, q, -q)P_{11}(k, \tau)P_{11}(q, \tau)d^3q.
\]

(15b)

Here \( P_{ij} \) denotes the amplitude given by the above rules for a connected diagram representing the contribution from \( \delta_0 \) to the power spectrum. We have assumed Gaussian initial conditions, for which \( P_{ij} \) vanishes if \( i + j \) is odd.

For the smoothed variance and average two-point correlation function we write

\[
\sigma^2(R) = \sigma^2_{11}(R)[1 + s^{(1)}(R) + \cdots]
\]

(16a)

and

\[
\tilde{\xi}(R) = \tilde{\xi}_{11}(R)[1 + \chi^{(1)}(R) + \cdots],
\]

(16b)

where \( \sigma^2_{ij}(R) \) and \( \tilde{\xi}_{ij}(R) \) denote the variance and average two-point correlation function in linear theory (given by eqs. [9] and [8] with \( P = P_{11} \)). The dimensionless one-loop amplitudes are

\[
s^{(1)}(R) \equiv \frac{1}{\sigma^2_{11}(R)} \int P^{(1)}(k, \tau)W^2(kR)d^3k,
\]

(17a)

\[
\chi^{(1)}(R) \equiv \frac{1}{\tilde{\xi}_{11}(R)} \int P^{(1)}(k, \tau)W(kR)d^3k.
\]

(17b)

A convenient property of these dimensionless amplitudes is the following: if one defines the correlation length \( R_{0}^{(1)} \) in linear theory as the scale where the smoothed linear variance is unity, \( \sigma^2_{11}[R_{0}^{(1)}] = 1 \), then \( s^{(1)}[R_{0}^{(1)}] = \sigma^2[R_{0}^{(1)}] - 1 \) is just the nonlinear correction to the variance at this scale.

3. RESULTS

3.1. One-Loop Power Spectrum

We consider a linear (tree-level) power spectrum \( P_{11}(k, \tau) \) given by a truncated power law

\[
Aa^2(\tau)k^n \quad \text{if } \epsilon \leq k \leq k_s,
\]

\[
0 \quad \text{otherwise},
\]

(18)

where \( A \) is a normalization constant, and the infrared and ultraviolet cutoffs \( \epsilon \) and \( k_s \) are imposed in order to regularize the required radial integrations (Scoccimarro & Frieman 1996). In a cosmological N-body simulation, they would correspond roughly to the inverse comoving box size and lattice spacing (or interparticle separation), respectively. In the absence of the

\[
(P_{11}) + \begin{array}{c}
\bullet
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\end{array}
\]

\[
(P_{22}) + \begin{array}{c}
\bullet
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\end{array} + \begin{array}{c}
\bullet
\end{array}
\]

\[
(P_{13})
\]

(19)

Fig. 2.—Diagrams for the power spectrum up to one loop. See eqs. (15a), (15b) for diagram amplitudes.
cut-off dependence of which gives

\[ p^{(1)} = p_{22} + p_{13}, \]

where the dimensionless one-loop spectrum is (18) and we have introduced dimensionless wavenumber variables

\[ x \equiv k/k_c, \quad \Lambda \equiv k_c/\epsilon. \]

From equation (15a) we obtain (defining \( t \equiv q/k \))

\[ p_{22}(n; x, \Lambda) \equiv x^4 1/49 \int_{1}^{\infty} dt t^2 \lambda_{\min}(x, t, \Lambda) J_{1}(2\pi x\Lambda) \lambda(x^2 + t^2 - 2\pi x\Lambda)\lambda_{\min}(x, t, \Lambda) \lambda_{\max}(x, t), \]

when \( x \leq 2 \); otherwise \( p_{22}(n; x, \Lambda) = 0 \). The constraint on the angular integration variable \( \lambda \equiv (k \cdot q)/(kq) \) comes from the cut-off dependence of \( P_{11}(|k - q|, \tau) \), which gives

\[ \lambda_{\min}(x, t) \equiv \max \left\{ 1, \frac{x^2 + t^2 - 1}{2xt} \right\}, \]

\[ \lambda_{\max}(x, t, \Lambda) \equiv \min \left\{ 1, \frac{x^2 + t^2 - \Lambda^{-2}}{2xt} \right\}. \]

Care must be taken when dealing with the limits imposed by equations (22a) and (22b), especially in the cases \( n = -1, -2 \). On the other hand, integration over angular variables in equation (15b) is straightforward, and we obtain:

\[ p_{13}(n; x, \Lambda) \equiv x^4 \int_{1/\Lambda}^{1} dt t^{n+2} \left[ \frac{6x^6 - 79x^4t^2 + 50x^2t^4 - 21t^6}{63x^4t^2} + \frac{(t^2 - x^2)^3(7t^2 + 2x^2)}{42x^4t^5} \ln \frac{|x + t|}{|x - t|} \right], \]

for \( \Lambda^{-1} \leq x \leq 1 \); otherwise \( p_{13}(n; x, \Lambda) = 0 \).

In the following subsections we give results for \( p_{13} \) and \( p_{22} \) in the limit \( \Lambda = k_c/\epsilon \rightarrow \infty \) up to terms of order \( \Lambda^0 \) for spectral indices \( n = 1, 0, -1, -2 \). We also give their asymptotic behavior at large scales. From the properties of the perturbation theory kernels \( F_n^{(0)} \) and equations (15a) and (15b), one would naively expect that as \( x \rightarrow 0 \),

\[ p_{22}(n; x, \Lambda) \sim (k/k_c)^4, \quad p_{13}(n; x, \Lambda) \sim x^{n+2} \sim (k/k_c)^n. \]

Although this is certainly correct for \( p_{22} \) when \( x \ll \Lambda^{-1} \) (i.e., \( k \ll \epsilon \)), the expressions below correspond to \( p_{22} \) and \( p_{13} \) after the limit \( \Lambda \rightarrow \infty \) has been taken and therefore exhibit a different kind of asymptotic behavior (corresponding to \( 0 < k < \epsilon \)). This deviation from the expected scaling, becomes more pronounced as \( n \) decreases since infrared effects (\( \epsilon \rightarrow 0 \)) become more important with the increase of large-scale power. As we will see, equation (24) is obeyed by \( p_{22} \) only when \( n \geq 1 \) and by \( p_{13} \) when \( n \geq 0 \).

Our results below are equivalent to those of Makino et al. (1992) in the case of \( p_{13} \), but they differ for \( p_{22} \) in two respects. First, their expressions for \( p_{22} \) are only valid for \( x < 1 \); they did not seem to consider that \( p_{22} \) is nonvanishing up to \( x = 2 \) and that the region \( 1 \leq x \leq 2 \) requires a separate integration when \( n \) is odd. In addition, for \( n \) even, their series expansions for the dilogarithms do not converge in this region. The second difference comes from the fact that for \( n = -1, -2 \), \( p_{22} \) develops a divergence at \( x = 1 \), and the expressions for \( p_{22} \) must be modified in a region of radius \( \Lambda^{-1} \) about \( x = 1 \). This singularity is integrable, giving a finite contribution to the one-loop correction to the variance. Finally, some of their resulting expressions for \( p^{(1)} \) contain typographic errors. We have checked our expressions by analytically integrating them and verifying that they correctly reproduce the unsmoothed one-loop coefficients \( s^{(1)} \) given by Scoccimarro & Frieman (1996). For comparison, we also plot (but do not give explicit expressions for) the one-loop power spectrum in the Zeldovich approximation.

### 3.1.1. \( n = 1 \)

For \( n = 1 \), the dimensionless one-loop power spectrum contributions are

\[ p_{13}(1; x) = -\frac{1}{21x} + \frac{482}{315} + \frac{181x^3}{315} - \frac{2x^5}{21} + \ln \left( \frac{1 + x}{1 - x} \right) \left( \frac{5 - 19x^2 + 25x^4 - 5x^6 + 10x^8}{210x^2} \right) + \frac{8x^5}{105} \ln \left( \frac{1 - x^2}{x^2} \right), \]

\[ p_{22}(1; x) = \frac{18x^4}{49} - \frac{13x^3}{98} + \frac{20x^6 + x^7}{1029 + 49}, \quad \text{if } x \leq 1, \]

\[ p_{22}(1; x) = -\frac{320}{1029x} - \frac{16x^3}{49} + \frac{2x^3}{3} - \frac{18x^4}{49} + \frac{5x^5}{42} + \frac{20x^6 + x^7}{1029 + 49}, \quad \text{if } 1 \leq x \leq 2, \]

and the one-loop reduced power spectrum is then (see Fig. 3):

\[ p^{(1)}(n = 1; x = k/k_c \leq 1) = -\frac{1}{21x} + \frac{482}{315} + \frac{181x^3}{315} + \frac{18x^4}{49} - \frac{6x^5}{294} + \frac{20x^6 + x^7}{1029 + 49} + \ln \left( \frac{1 + x}{1 - x} \right) \left( \frac{5 - 19x^2 + 25x^4 - 5x^6 + 10x^8}{210x^2} \right) + \frac{8x^5}{105} \ln \left( \frac{1 - x^2}{x^2} \right), \]
with \( p^{(1)}(x) = p_{22}(x) \) when \( 1 \leq x \leq 2 \). For \( x \ll 1 \) we obtain
\[
p_{13}(1; x) \approx p^{(1)}(1; x) \approx -\frac{12}{17} x^3 + \mathcal{O}(x^4),
\]
\[
p_{22}(1; x) \approx \frac{15}{68} x^4 + \mathcal{O}(x^5).
\]

3.1.2. \( n = 0 \)

For \( n = 0 \), the one-loop power spectrum contributions are
\[
p_{13}(0; x) = -\frac{1}{18x^2} + \frac{157}{756} - \frac{269x^2}{252} - \frac{15x^3}{84} - \frac{x^4}{21}
+ \ln \left(1 + \frac{x}{1 - x}\right) \left(\frac{x^2 - 1}{504x^3}\right)
- \left(14 + 43x^2 - 47x^4 + 12x^6\right) + \frac{x^3}{42} \left[\text{Li}_2(x) - \text{Li}_2(-x)\right]
\]
and
\[
p_{22}(0; x \leq 2) = \frac{25}{98} + \frac{25x}{196} - \frac{10x^2}{147} - \frac{5x^3}{392} + \frac{29\pi^2x^3}{294} + \frac{3x^4}{49} - \frac{65x^5}{1176} + \frac{x^7}{196}
+ \ln|x - 1| \left(\frac{25 - 15x^2 - 16x^4 + 6x^6 + 29x^4\ln x}{98x}\right)
- \frac{29x^3}{98} \left\{\frac{1}{2} \ln^2(x) + \text{Re}[\text{Li}_2(x)] + \text{Li}_2\left(\frac{x - 1}{x}\right)\right\},
\]
where \( \text{Li}_2(x) \) denotes the dilogarithm, defined by (Lewin 1981) as
\[
\text{Li}_2(x) = -\int_0^x \frac{\ln(1 - z)}{z} \, dz,
\]
which has the series expansion for small argument \( x \ll 1 \)
\[
\text{Li}_2(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^2}.
\]
The resulting one-loop power spectrum is (Fig. 4)
\[
p^{(1)}(0; x \leq 1) = -\frac{1}{18x^2} + \frac{2499}{5292} + \frac{25x}{196} - \frac{2003x^2}{1764} - \frac{5x^3}{392} + \frac{17\pi^2x^3}{196} + \frac{2x^4}{147} + \frac{65x^5}{1176} + \frac{x^7}{196}
+ \ln(1 + x) \left(\frac{x^2 - 1}{504x^3}\right)
- \left(14 + 43x^2 - 47x^4 + 12x^6\right) - \frac{29x^3}{196} \ln^2(x)
+ \left(-\frac{98 + 1299x^2 - 1170x^4 - 163x^6 + 132x^8 + 1044x^6\ln x}{3528x^3}\right) \ln(1 - x)
+ \frac{40x^3}{147} \text{Li}_2(x) - \frac{x^3}{42} \text{Li}_2(-x) - \frac{29x^3}{98} \text{Li}_2\left(\frac{x - 1}{x}\right),
\]
and $p^{(1)}(x) = p_{22}(x)$ when $1 \leq x \leq 2$.

For $x \ll 1$ we obtain

$$p_{13}(0; x) \approx p^{(1)}(0; x) \approx -\frac{344}{775} x^2 + O(x^3),$$

$$p_{23}(0; x) \approx \frac{29\pi^2}{196} x^3 + O(x^4).$$

3.1.3. $n = -1$

For $n \leq -1$, the one-loop contributions are infrared-divergent and therefore depend on the infrared cutoff through $\Lambda = k_*/\epsilon$,

$$p_{13}(-1; x, \Lambda) = -\frac{4x}{3} \ln \Lambda - \frac{1}{15x^3} + \frac{88}{315x} - \frac{11x}{189} - \frac{2x^3}{63}$$

$$+ \ln \left( \frac{1 + x}{1 - x} \right) \left( \frac{21 - 95x^2 + 225x^4 + 15x^6 + 10x^8}{630x^4} \right) + \frac{88x}{315} \ln \left( \frac{1 - x^2}{x^2} \right), \quad (38)$$

$$p_{23}(-1; x, \Lambda) = -\frac{2x}{3} \ln \Lambda + \frac{160}{147x} + \frac{x^3}{3} + \frac{2x^5}{49} + \frac{44x^4}{441} + \frac{x^7}{441} + \frac{2x^5}{49} + \frac{2x^7}{441} \ln(x) - \frac{(x - 1)^2}{147x} \left( 50 + 100x + 105x^2 + 12x^3 + 6x^4 \right) \Lambda$$

$$- \frac{(x - 1)^3}{882x} \left( 15 + 45x + 60x^2 + 60x^3 + 12x^4 + 4x^5 \right) \Lambda^3, \quad 1 - \Lambda^{-1} \leq x \leq 1 + \Lambda^{-1}, \quad (39)$$

$$p_{22}(-1; x) = \frac{320}{147x} - \frac{80x}{147} - \frac{2x^2}{3} + \frac{x^3}{3} + \frac{2x^5}{49} + \frac{44x^4}{441} + \frac{2x^7}{441} + \frac{2x^3}{3} \ln(x - 1), \quad 1 + \Lambda^{-1} \leq x \leq 2. \quad (40)$$

The one-loop correction is then (Fig. 5)

$$p^{(1)}(-1; x) = -\frac{1}{15x^3} + \frac{88}{315x} + \frac{643x}{1323} + \frac{2x^2}{3} + \frac{19x^3}{63} + \frac{44x^4}{441} + \frac{2x^5}{49} + \frac{x^7}{441} + \left( 21 - 95x^2 + 225x^4 + 15x^6 + 10x^8 \right)$$

$$\times \ln \left( \frac{1 + x}{1 - x} \right) + \frac{298x}{315} \ln(1 - x) + \frac{244x}{315} \ln(x) + \frac{88x}{315} \ln(1 + x), \quad x \leq 1 - \Lambda^{-1}, \quad (41)$$
while when $1 \leq x \leq 2$ we have $p^{(1)}(-1; x) = p_{22}(-1; x)$. For $x \ll 1$ we obtain

$$ p_{13}(-1; x, \Lambda) \approx -\frac{4x}{3} \ln \Lambda + \frac{107}{252x^2} + \frac{82}{x^2} + \frac{225}{315} x - \frac{176x}{315} \ln (x) + O(x^3), \quad (43) $$

$$ p_{22}(-1; x, \Lambda) \approx \frac{4x}{3} \ln \Lambda + \frac{80}{147} + \frac{4x}{3} \ln (x) + O(x^4), \quad (44) $$

$$ p^{(1)}(-1; x) \approx \frac{12272}{11025} x + \frac{244x}{315} \ln (x) + O(x^3). \quad (45) $$

In this limit, the divergences cancel. Note, however, that the $\Lambda \to \infty$ divergence in $p_{22}$ in the region $1 - \Lambda^{-1} \leq x \leq 1 + \Lambda^{-1}$ is uncancellation, although the “size” of the divergent region shrinks to zero.

3.1.4. $n = -2$

For $n = -2$, the one-loop terms are

$$ p_{13}(-2; x, \Lambda) = -\frac{4}{3} \Lambda - \frac{1}{12x^4} + \frac{107}{252x^2} + \frac{5}{2x} + \frac{82}{x^2} + \frac{225}{315} x - \frac{176x}{315} \ln (x) + O(x^3), \quad (46) $$

$$ p_{22}(-2; x, \Lambda) = \frac{2}{3} \Lambda + \frac{75}{1568x} + \frac{25x^2}{196x} + \frac{15x}{392} + \frac{29x^3}{392} + \frac{3x^5}{196} + \frac{x^7}{784} + \frac{\Lambda^2}{196x} (-50 + 82x^2 - 29x^4 - 3x^6 - 30x^2 \ln x) $$

$$ + \frac{\Lambda^4}{1568x} (25 - 60x^2 + 63x^4 - 24x^6 - 4x^8 + 44x^4 \ln x), \quad 1 - \Lambda^{-1} \leq x \leq 1 + \Lambda^{-1}. $$

Whereas when $x \leq 1 - \Lambda^{-1}$ or $x \geq 1 + \Lambda^{-1}$ we obtain

$$ p_{22}(-2; x, \Lambda) = -\frac{4}{3} \theta(1 - x) \Lambda + \frac{1}{2(x - 1)} + \frac{25x^2}{98x} + \frac{205}{392} + \frac{399}{392} + \frac{11x^2}{392} + \frac{53x^3}{1568} + \frac{3x^5}{196} + \frac{x^7}{784} $$

$$ - \frac{75}{96x^2} \ln^2 (x) + \ln |x - 1| \left( \frac{-1001 + 60x^2 - 11x^4 + 300 \ln x}{392x} \right) - \frac{75}{96x} \left\{ \Re \left[ Li_2(x) \right] + Li_2 \left( \frac{x - 1}{x} \right) \right\}, \quad (47) $$

where $\theta(x)$ is the step function. This gives (Fig. 6)

$$ p^{(1)}(-2; x) = -\frac{1}{12x^2} + \frac{107}{252x^2} + \frac{1}{2(x - 1)} + \frac{85x^2}{196x} + \frac{274}{352} - \frac{309x}{392} - \frac{1176}{1568} + \frac{3x^3}{196} + \frac{3x^5}{196} + \frac{x^7}{784} - \frac{75}{196x} \ln^2 (x) $$

$$ + \ln (1 - x) \left( \frac{-1001 + 60x^2 - 11x^4 + 300 \ln x}{392x} \right) + \ln \left( \frac{1 + x}{1 - x} \right) \frac{(x^2 - 1)}{168x^2} (-7 + 31x^2 + 4x^4 + 2x^6) $$

$$ + \frac{5}{14x} Li_2 (-x) - \frac{55}{49x} Li_2 (x) - \frac{75}{98x} Li_2 \left( \frac{x - 1}{x} \right), \quad x \leq 1 - \Lambda^{-1}, \quad (48) $$
otherwise \( p^{(1)}(-2; x) = p_{22}(-2; x) \) when \( 1 + \Lambda^{-1} \leq x \leq 2 \).

For \( x \ll 1 \) we obtain

\[
\begin{align*}
P_1 s(-2; x, \Lambda) & \approx -\frac{4}{3} \Lambda + \frac{5\pi^2}{28x} + O(x^0), \\
P_{22}(-2; x, \Lambda) & \approx \frac{4}{3} \Lambda + \frac{75\pi^2}{196x} + O(x^0), \\
p^{(1)}(-2; x) & \approx \frac{55\pi^2}{98x} + O(x^0).
\end{align*}
\]

### 3.2. One-Loop Smoothed Variance and Average Two-point Correlation Function

Using the results of the previous section for the power spectrum, we can calculate the one-loop corrections to the variance and average two-point function, equations \(9\) and \(8\). We focus on the corresponding one-loop amplitudes \( s^{(1)} \) and \( x^{(1)} \) defined in equations \(16\) and \(17\). From equation \((17), (18), (19)\), we can write

\[
s^{(1)}(n; k_c, R) \equiv \frac{(k_c R)^{2n+3}}{4[\Delta I_2(n, k_c, R, \Lambda)]^2} \int_0^{2k_c R} p^{(1)}[n; u(k_c, R)^{-1}, \Lambda]W^2(u)u^2 du \tag{52a}
\]

and

\[
x^{(1)}(n; k_c, R) \equiv \frac{(k_c R)^{2n+3}}{4[\Delta I_2(n, k_c, R, \Lambda)]^2} \int_0^{2k_c R} p^{(1)}[n; u(k_c, R)^{-1}, \Lambda]W(u)u^2 du , \tag{52b}
\]

where \( \Delta I_2(n, k_c, R, \Lambda) \equiv \Delta I_2(n, k_c, R) - \Delta I_2(n, \epsilon R) \), and similarly for \( \Delta I_3 \), with

\[
I_s(n, z) \equiv \int_0^z u^{n+2} W^2(u) du , \tag{53a}
\]

\[
I_z(n, z) \equiv \int_0^z u^{n+2} W(u) du . \tag{53b}
\]

In Appendix A we present analytic results for \( I_s \) and \( I_z \) with top-hat and Gaussian smoothing.

In Figures 7–10, we show results for \( s^{(1)} \) and \( x^{(1)} \) for top-hat and Gaussian smoothing, based on numerical integration of equations \((52a)\) and \((52b)\). Similar results for \( s^{(1)} \) for the Gaussian window function were presented by Lokas et al. (1995); comparison with our results shows very good agreement for scales such that \( k_c R \gtrsim 2 \). However, in the limit \( k_c R \to 0 \), they apparently find \( s^{(1)} \approx 0 \), instead of recovering the unsmoothed values reported in Scoccimarro & Frieman (1996) and in...
Fig. 7.—One-loop corrections to the average two-point correlation function $x^{(1)}$ and variance $s^{(1)}$ for $n = -2$ as a function of $k_c R$, for top-hat (TH) and Gaussian (G) smoothing. Solid curve shows $s_{TH}^{(1)}$, dotted curve corresponds to $s_G^{(1)}$, dashed curve shows $x_G^{(1)}$, and the long-dashed curve $x_{TH}^{(1)}$. The dot-dashed line shows the large-scale approximation given in eqs. (55) and (57). For $k_c R = 0$ we recover the unsmoothed results $s^{(1)} = x^{(1)} \approx 1.82$ (Scoccimarro & Frieman 1996).

conflict with our results shown here. The source of their error may have been to replace $2k_c R$ by $k_c R$ in the upper limit of integration in equation (52a); we found we could approximately reproduce their results with this change.

From equations (52a) and (52b), one can analytically calculate $s^{(1)}$ and $x^{(1)}$ in the limit $k_c R \gg 1$ by asymptotic expansion in $1/k_c R$: for large $k_c R$, all we need is the behavior of $p^{(1)}(n; x, \Lambda)$ for small $x$ given in the previous section. For $n = -2$ we obtain

$$s^{(1)}(-2; k_c R \gg 1) \approx s^{(1)}(-2; \infty) = \frac{55\pi^2}{392} \frac{I_a(-1, \infty)}{[I_a(-2, \infty)]^2},$$

(54)

Fig. 8.—Same as Fig. 7, but for $n = -1$. Dot-dashed curves correspond to the large-scale approximation to the one-loop coefficients, eqs. (59) and (60). We do not show $x_{TH}^{(1)}$, which undergoes large oscillations.
where we used equations (51) and (53a). An identical expression holds for $x^{(1)}$ upon replacing $I_x$'s by $I_z$'s. Using the results in Appendix A, we find the following for top-hat smoothing:

$$s_{TH}^{(1)}(-2; \infty) = \frac{1375}{1378} \approx 0.877 ,$$

$$x_{TH}^{(1)}(-2; \infty) = \frac{1139}{1440} \approx 0.748 .$$

For Gaussian smoothing we find

$$s_G^{(1)}(-2; \infty) = x_G^{(1)}(-2; \infty) = \frac{55\pi}{196} \approx 0.882$$

**Fig. 9.**—Same as Fig. 7, but for $n = 0$. Dot-dashed curves correspond to the large-scale approximation to the one-loop coefficients, eqs. (61) and (62).

**Fig. 10.**—Same as Fig. 7, but for $n = 1$. Dot-dashed curves correspond to the large-scale approximation to one-loop coefficients, eqs. (64) and (65).
in good agreement with the numerical results of Lokas et al. (1995), who found \( s^4(1; -2; \infty) \approx 0.86. \) Comparing these analytic results to Figure 7, we see excellent agreement except for \( x^4 \), which oscillates around the value given by equation (56). The reason for these oscillations can be partially understood from equation (54), using the results in Appendix A for finite \( k \), \( R \).

The behavior of \( x^4 \) is also due in part to the fact that smoothing with one top-hat window allows small-scale power in \( p^{1/2}(n; \mathbf{x}, \Lambda) \) to “leak in” to the average correlation function, while equation (54) assumes that only the large-scale power contributes. For this reason, in the following we will give analytical results only for Gaussian smoothing.

For \( n = -1 \), using equations (45) and (53a) we obtain

\[
 s^{(1)} (-1; k, R \gg 1) = \frac{3068}{11025} \frac{I_s(1, \infty)}{[I_s(-1, \infty)]^2} + \frac{244}{315} \int_0^\infty W^2(u) u^3 \ln u \, du - \frac{61}{315} \frac{I_s(1, \infty)}{[I_s(-1, \infty)]^2} \ln (k, R),
\]

and similarly for \( x^{(1)} \). Using Gaussian smoothing we find

\[
 s^{(1)} (-1; k, R \gg 1) = \frac{61}{11025} + \frac{61}{11025} (1 - \gamma_c) - \frac{122}{315} \ln (k, R) \approx 0.638 - 0.387 \ln (k, R),
\]

\[
 x^{(1)} (-1; k, R \gg 1) = \frac{61}{11025} + \frac{61}{11025} (1 - \gamma_c + \ln 2) - \frac{122}{315} \ln (k, R) \approx 0.773 - 0.387 \ln (k, R),
\]

where \( \gamma_c \approx 0.577216... \) is the Euler-Mascheroni constant. Equations (59) and (60) are plotted as the dot-dashed curves in Figure 8, which shows the excellent agreement with the results from the full numerical integration at large \( k, R \).

For \( n = 0 \), one must include the next to leading order term at small \( x \) in \( p^{1/2}(0; \mathbf{x}, \Lambda) \) since this gives rise to a constant term in the one-loop coefficients. We obtain

\[
 p^{1/2}(0; \mathbf{x}) \approx -\frac{244}{315} x^2 + \frac{20}{147} x^3 + \mathcal{O}(x^4),
\]

which gives (see dot-dashed curves in Fig. 9)

\[
 s^{(1)} (0; k, R \gg 1) = \frac{80\pi}{147} - \frac{122}{105} \sqrt{\frac{3}{\pi}} k, R \approx 1.710 - 0.655 k, R,
\]

\[
 x^{(1)} (0; k, R \gg 1) = \frac{80\pi}{147} - \frac{61}{105} \sqrt{\frac{3}{\pi}} k, R \approx 1.710 - 0.464 k, R.
\]

Similarly, for \( n = 1 \), we need the expansion

\[
 p^{1/2}(1; \mathbf{x}) \approx -\frac{122}{315} x^3 + \frac{18}{35} x^4 - \left( \frac{4973}{7350} + \frac{61}{105} \ln x \right) x^5 + \mathcal{O}(x^6),
\]

which leads to (see Fig. 10)

\[
 s^{(1)} (1; k, R \gg 1) = -\frac{4973}{7350} - \frac{4}{105} (11 - 6\gamma_c) + \frac{16}{35} \ln (k, R) + \frac{135}{392} - \frac{122}{315} (k, R)^2,
\]

\[
 \approx -0.964 + 0.457 \ln (k, R) + 0.610 k, R - 0.387 (k, R)^2,
\]

\[
 x^{(1)} (1; k, R \gg 1) = -\frac{4973}{7350} - \frac{4}{105} (11 - 6\gamma_c + 6 \ln 2) + \frac{16}{35} \ln (k, R) + \frac{135}{392} \sqrt{\frac{3}{\pi}} k, R - \frac{61}{315} (k, R)^2
\]

\[
 \approx -1.122 + 0.457 \ln (k, R) + 0.432 k, R - 0.194 (k, R)^2.
\]

Note that the one-loop coefficients are scale-dependent for \( n \geq -1 \); this is an indication of the breaking of self-similarity, as we discuss next. (A similar analysis to that above can be carried out for the one-loop corrections in the Zeldovich approximation, but details will not be given here.)

4. SELF-SIMILARITY AND PERTURBATION THEORY

4.1. Self-similar Solutions

Since there is no preferred scale in the dynamics of a self-gravitating pressureless perfect fluid in an Einstein–de Sitter universe, equations (1a)–(1c) admit self-similar solutions (see Peebles 1980). This means that the cosmological fields should scale with a self-similarity variable, given appropriate initial conditions: knowing the fields at a given time completely specifies their evolution. We can search for the appropriate self-similarity transformation by rewriting the fields as

\[
 \delta(x, \tau) \equiv \chi(y),
\]

\[
 \psi(x, \tau) \equiv \tau^\mu \psi(y),
\]

\[
 \Phi(x, \tau) \equiv \tau^{\alpha} \Phi(y),
\]

where \( y \equiv x/\tau^\nu \) is the similarity variable. Self-similarity is obeyed if we can rewrite equations (1a)–(1c) in terms of \( y \) only; this condition determines the indices \( \mu, \nu, \) and \( \nu \). Note that the term \( 1 + \delta(x, \tau) \) in the continuity equation (1a) precludes a power of
functions; & Peebles Using these results, the equations of motion can be rewritten in self-similar form as (1)

\[ \mu = v - 1 \]  

(67)

while the Poisson equation (1c) gives

\[ \gamma = 2(v - 1) \]  

(68)

These two conditions in turn guarantee that the Euler equation obeys self-similar evolution. The index \( v \) can be determined in terms of the spectral index \( n \) of the linear power spectrum: equation (66a) implies that the two-point correlation function obeys the self-similar scaling

\[ \xi(x, \tau) \equiv \Xi(x\tau^{-\gamma}) \]  

(69)

which upon Fourier transformation gives for the power spectrum

\[ P(k, \tau) \equiv \tau^{3\gamma}\mathcal{H}(k\tau) \]  

(70)

On the other hand, in the linear regime, for a power-law initial power spectrum with spectral index \( n \) in an Einstein–de Sitter universe, \( P_{\text{11}}(k, \tau) \sim \tau^4k^2 \); equating this with equation (70) fixes the remaining parameter in the similarity transformation,

\[ \nu = \frac{4}{n + 3} \]  

(71)

(A more general derivation of eq. [71] starts from the BBGKY hierarchy of equations for the phase space particle distribution functions; Davis & Peebles 1977.) Using these results, the equations of motion (1) can be rewritten in self-similar form as

\[ v\dot{\varphi}(y) - (1 + \varphi(y)) \dot{\varphi}(y) = 0 \]  

(72a)

\[ v\dot{\varphi}(y) - (1 + \nu)\dot{\varphi}(y) = \dot{\varphi}(y) \]  

(72b)

\[ \dot{\varphi}^2\dot{\varphi}(y) = 6\varphi(y) \]  

(72c)

where \( \dot{\varphi} \equiv \partial\varphi/\partial y \). The solutions of these equations in the linear regime are \( \varphi(y) \propto y^\gamma \) with \( \gamma = -2/v, 3/v \), which correspond to the well-known linear growing and decaying modes, \( \delta_i \propto a, a^{-3/2} \). The condition of self-similar evolution also fixes the spatial dependence of the fields, e.g., \( \delta_i(x) \propto x^{-2/3}\nu \) for the growing mode. However, we generally consider the density and velocity fields to be random fields, on which we impose initial conditions only statistically, e.g., by specifying the linear power spectrum for Gaussian initial conditions. A particular realization of the ensemble of initial conditions will not obey this spatial scaling, so the cosmological fields themselves will not be self-similar (Jain & Bertschinger 1995). Since NLCPT is built from linear solutions, the same conclusion applies to the expansions given by equation (5), which are not self-similar (except when the spatial dependence is fixed as above). Nevertheless, in some cases the statistical quantities of interest, such as the power spectrum, variance, and two-point correlation function, may exhibit self-similar scaling even if the fields in a given realization do not. We now consider the conditions under which this happens in perturbation theory.

### 4.2. Self-Similarity and Linear Perturbation Theory

The introduction of fixed (time-independent) cutoff scales \( \epsilon \) and \( k_c \) in the linear power spectrum (eq. [18]) breaks self-similarity because they do not scale with the self-similarity variable \( a(t)^{-1/2} \) (the Fourier space analog of \( \nu \)). The extent to which one can take the limits \( \epsilon \rightarrow 0 \) and \( k_c \rightarrow \infty \) will determine whether the statistical properties of the density field obey self-similar scaling. In the absence of cutoffs, the only physical scale that can be defined from the power spectrum is the correlation length, \( R_o \), defined by

\[ \sigma^2(R_o) = \int P(k, \tau)W^2(kR_0)dk \equiv 1 \]  

(73)

In linear perturbation theory, for scale-free initial conditions, one finds \( R_0 \propto a^{2/(n + 3)} \), which has the right time dependence to build the self-similarity variable. Consequently, in the absence of cutoffs, statistical quantities in linear theory evolve self-similarly with \( R_0 \), and we can write, e.g. (see eqs. [69] and [70]),

\[ \xi(r, \tau) \equiv \Xi(r/R_0) \]  

(74)

\[ R_0^{-3}P_{11}(k, \tau) \equiv \mathcal{P}_{11}(kR_0) \]  

(75)

The presence of cutoffs in the initial spectrum changes this situation. From equation (18) we have

\[ R_0^{-3} = 4\pi Aa^3 \int_{0}^{k_R_0} \kappa^{n+2}W^2(\kappa)dk \equiv 4\pi Aa^3[I(n, k, R_0) - I(n, \epsilon R_0)] \equiv 4\pi Aa^2 \Delta I(n, k, R_0, \Lambda) \]  

(76)

and \( R_0 \) will not scale as \( a^{2/(n + 3)} \) unless \( \Delta I(n, k, R_0, \Lambda) \) is time independent. Before examining under which conditions this is
true, it is useful to introduce another measure $\ell_o$ of the correlation length, defined in terms of the volume-averaged correlation function,

$$
\zeta(\ell_o) = \int P(k, \tau)W(\ell_o) d^3 k \equiv 1 ,
$$

and, therefore,

$$
\ell_o^{n+3} = 4\pi Aa^2 \int_{\ell_o}^{kR_o} k^{\alpha+2}W(\kappa) d\kappa \equiv 4\pi Aa^2 \int [I_3(n, k, \ell_o) - I_3(n, \ell_o)] \equiv 4\pi Aa^2 \Delta I_3(n, k, \ell_o, \Lambda) .
$$

Note that $R_o$ and $\ell_o$ both differ slightly from the conventional definition of the correlation length, $r_o$, in terms of the two-point correlation function, $\zeta(r_o) \equiv 1$.

A necessary condition for self-similarity in the variable $R_o(\ell_o)$ is the convergence of the integrals $\Delta I_\alpha(\Delta I_j)$ for small and large $\kappa$ [where $\kappa$ denotes $kR_o(\ell_o)$]; otherwise these quantities would be sensitive to the cutoffs and consequently time dependent. From equations (76) and (78), in order for the correlation length to scale self-similarly in linear theory, we must have at least the following conditions:

1. For a Gaussian filter, since $W(\kappa) \approx 1$ when $\kappa \to 0$, convergence in the infrared (small $\kappa$) requires $n > -3$. Convergence in the ultraviolet (large $\kappa$) is achieved for any $n$.
2. For a top-hat filter, convergence in the infrared requires $n > -3$. Since $W_{\text{TH}}(\kappa) \approx \kappa^{-2}$ as $\kappa \to \infty$, convergence in the ultraviolet requires $n < 1$ in equation (76) and $n < -1$ in equation (78).

Note the dependence of these conditions on the local averaging scheme (top-hat vs. Gaussian) and the type of statistics (variance vs. average correlation function).

While these conditions guarantee convergence of $\Delta I_{\sigma, c, \xi}$, they do not test the constancy of these quantities for different choices of parameters. In Figures 11, 12, 13, and 14, we show $\Delta I_{\sigma, c, \xi}(n, k, R_o, \Lambda)$ for $\Lambda \equiv k/T^2 = 10^3, 10^4, \infty$, as a function of $k$, $R_o$ for spectral indices $n = -2, -1, 0, 1$. These finite values of $\Lambda$ are comparable to the dynamic range currently achievable in $N$-body simulations. By definition, $k/R_o$ measures the ratio of the correlation length to the small-scale cutoff of the linear power spectrum. Therefore, in the region $k/R_o \approx 1$, the evolution of the correlation length is strongly affected by the cutoff $k_c$, and the $\Delta I_{\sigma, c, \xi}$ drop precipitously. By the time the correlation length has evolved to $k/R_o \gtrsim 3$, however, $\Delta I_{\sigma, c, \xi}$ reach their no-cutoff limits (in the cases where they converge). On the other hand, when $k/R_o \approx \Lambda$, i.e., $R_o \approx \kappa^{-1}$, the correlation length reaches the large-scale cutoff, and $\Delta I_{\sigma, c, \xi}$ again drop. This behavior is more dramatic as $n$ decreases (and therefore strongest for $n = -2$) because of the increase in large-scale power. As expected from the discussion above, the definition of the correlation length using Gaussian smoothing displays self-similar evolution over a longer time interval than the other possibilities, with top-hat smoothing becoming notoriously worse as $n$ increases.

The behavior of statistical quantities in linear perturbation theory with respect to self-similarity is determined directly from these considerations. We can write the linear power spectrum as (see eq. [75])

$$
\mathcal{P}_{11}(k R_o, k, R_o, \Lambda) \equiv \frac{(k R_o)^n}{4\pi \Delta I(n, k, R_o, \Lambda)} ,
$$

where $\Delta I$ depends on the local averaging scheme and statistical quantity used to define the correlation length, and $kR_o \ll 1$ in the linear regime. Unless otherwise noted, for power spectrum calculations we will use the choice in equation (76), i.e., $R_o$, with

![Graph](image-url)
Fig. 12.—Same as Fig. 11, but for $n = -1$

Fig. 13.—Same as Fig. 11, but for $n = 0$

Fig. 14.—Same as Fig. 11, but for $n = 1$
Gaussian smoothing. For the variance and average correlation function we obtain

\[ \sigma^2 \left( \frac{\mathbf{R}}{R_0}, k_c, R_0, \Lambda \right) = \frac{\Delta I_d(n, k_c, R_0, \Lambda)}{\Delta I_d(n, k_c, R_0, \Lambda)} \left( \frac{R}{R_0} \right)^{-(n+3)} \]  

(80)

\[ \bar{\xi} \left( \frac{\mathbf{R}}{R_0}, k_c, R_0, \Lambda \right) = \frac{\Delta I_d(n, k_c, R_0, \Lambda)}{\Delta I_d(n, k_c, R_0, \Lambda)} \left( \frac{R}{R_0} \right)^{-(n+3)} \]  

(81)

Now the question is under what conditions are the statistics self-similar in the variables \( kR_0 \) (for the power spectrum) and \( R/R_0 \) (variance and average correlation function). In the regime \( R \gg R_0 \) (where linear theory is valid) and

\[ 3 \lesssim k_c R_0 \leq k_c R \ll \Lambda \]  

(82)

Figures 11–14 show that \( \Delta I \) is approximately constant. Provided the conditions on \( n \) stated above hold, in this regime \( kR_0 \) and \( R/R_0 \) are self-similar variables; in linear perturbation theory, the power spectrum scales as

\[ \mathcal{P}_{11}(kR_0, k_c, R_0, \Lambda) \approx \frac{(kR_0)^n}{2\pi \Gamma(n+3)/2} \]  

(83a)

(see eq. [A13]), and

\[ \sigma^2(R/R_0, k_c, R_0, \Lambda) \approx \bar{\xi}(R/R_0, k_c, R_0, \Lambda) \approx (R/R_0)^{-(n+3)} \]  

(83b)

Note that, for top-hat smoothing, the requirement of self-similarity of the statistical quantities is, in practice, less restrictive than the requirement that \( R_0 \) or \( \ell_0 \) scale self-similarly. For example, although \( \ell_0 \), defined from top-hat smoothing, requires \( n \leq -1 \) to scale self-similarly, the average correlation function for \( n = 1 \) does scale self-similarly, i.e., it depends on \( R/R_0 \) only, in the regime given by equation (82) (see eq. [A8d])

\[ \bar{\xi} \left( \frac{\mathbf{R}}{R_0}, k_c, R_0, \Lambda \right) \approx \left( \frac{R}{R_0} \right)^{-3} \]  

(84)

where we have averaged over oscillatory behavior. However, it does not obey the “expected” \( \bar{\xi}(R/R_0, k_c, R_0, \Lambda) \approx (R/R_0)^{-(n+4)} \) scaling. This trend appears to be seen in the results from high-resolution scale-free \( N \)-body simulations (Colombi et al. 1995).

### 4.3. Self-Similarity and Nonlinear Perturbation Theory

We have examined the conditions under which linear perturbation theory obeys self-similar scaling. We now turn to the question of whether loop corrections in NLCP factor self-similar scaling.

Using the results of the previous section and equation (19), we can write the power spectrum up to one-loop corrections as

\[ \mathcal{P}(kR_0, k_c, R_0, \Lambda) \equiv \frac{(kR_0)^n}{4\pi \Delta I_d(n, k_c, R_0, \Lambda)} + \frac{(k_c R_0)^{2n+3}}{16\pi \Delta I_d(n, k_c, R_0, \Lambda)^2} p^{11} \left( n; \frac{k}{k_c}, \Lambda \right) \]  

(85)

where \( kR_0 > 1 \) in the weakly nonlinear regime. Self-similarity is maintained at the one-loop level if, in the scaling regime given by

\[ kR_0 \leq 1 \ll k_c, R_0 \ll \Lambda \]  

(86)

the dependence of \( \mathcal{P}(kR_0, k_c, R_0, \Lambda) \) on \( k_c, R_0 \) and \( \Lambda \) is negligible, that is, if

\[ p^{11} \left( n; \frac{k}{k_c} \to 0, \Lambda \to \infty \right) \approx a_n \left( \frac{k}{k_c} \right)^{2n+3} \]  

(87)

with \( a_n \) some constant which depends only on the spectral index \( n \). From the results of § 3.1, we see that this condition is only satisfied when \( n = -2 \). In this case we have (see eq. [51])

\[ \mathcal{P}(kR_0; n = -2) \approx \frac{1}{2\pi^{1/2}} (kR_0)^{-2} + \frac{55}{392} (kR_0)^{-1} \]  

(88)

which takes the self-similar form. For \( n = -1 \), one-loop diagrams yield logarithmic \( k_c \)-dependent corrections to self-similar scaling (see eq. [45]),

\[ \mathcal{P}(kR_0, k_c, R_0) \approx \frac{1}{2\pi} (kR_0)^{-1} + \frac{16}{11025\pi} (kR_0) \left[ 3068 + 2135 \ln \left( \frac{k}{k_c} \right) \right] \]  

(89)
The results for $n = 0, 1$ show a stronger power-law breaking of self-similarity (see eqs. [36] and [29]):

$$\mathcal{P}(k R_0, k, R_0) \approx \frac{(k R_0)^n}{2 \pi \Gamma([n + 3]/2)} \frac{61(k R_0)^{2n+3}}{315 \pi (n + 1) \Gamma^2([n + 3]/2)} \left(\frac{k}{k_c}\right)^n,$$

(90)

where $\eta = -(n + 1)$ is an exponent which measures the deviation from self-similar scaling, generally known as the anomalous dimension in the theory of critical phenomena (Goldenfeld 1992; Barenblatt 1979). A visual summary of these results is given in the next subsection, where we compare one-loop NLCPT to the universal scaling hypothesis first proposed by Hamilton et al. (1991). For the variance and average two-point correlation function, self-similarity breaking means that, in the expected scaling region given by equation (82), the one-loop coefficients $x^{(1)}$ and $x^{(1)}$ (see eq. [16]) are scale-dependent; in § 3.2, we found this to be the case when $n \geq -1$. Thus, for $n \geq -1$, we find generally that self-similarity is broken in perturbation theory by the first nonlinear (one-loop) corrections to the power spectrum. Our conclusions about self-similarity in NLCPT differ substantially from those of Jain & Bertschinger (1995). The primary difference stems from the fact that they take the small-scale cutoff in the linear power spectrum, $k_c$, to be time dependent in comoving coordinates. They choose $k_c$ to scale as the inverse correlation length, i.e., they fix $k_c R_0 \approx 1$ throughout the evolution. Their rationale for this is plausible: since NLCPT is expected to break down at scales below the correlation length within the framework of perturbation theory, one should perhaps restrict contributions to the power spectrum to scales larger than $R_0$, since depends on time, the cutoff to be the case when $NLCPT$ is expected to break down at scales below the correlation length within the framework of perturbation theory.

The results for $\mathcal{P}(k R_0, k, R_0)$, $\mathcal{P}(k R_0, k, R_0)$ show a stronger power-law breaking of self-similarity (see eqs. [36] and [29]):

$$\mathcal{P}(k R_0, k, R_0) \approx \frac{(k R_0)^n}{2 \pi \Gamma([n + 3]/2)} \frac{61(k R_0)^{2n+3}}{315 \pi (n + 1) \Gamma^2([n + 3]/2)} \left(\frac{k}{k_c}\right)^n,$$

(90)

The universal scaling hypothesis (Hamilton et al. 1991) asserts that the nonlinear evolution of the average two-point correlation function can be obtained from its linear counterpart by a universal scaling relation. This relation has been derived empirically from the study of numerical simulations for scale-free and CDM power spectra, but its main features can be understood on physical grounds (Nityananda & Padmanabhan 1994). These results have been extended by Peacock & Dodds (1994) to a universal relation between linear and nonlinear power spectra and by Jain et al. (1995) to include the dependence of the scaling relation on the spectral index $n$. In its current version, the universal scaling hypothesis for the dimensionless power spectrum,

$$\Delta(k) \equiv 4\pi k^3 P(k),$$

(91)

relates the linear ($\Delta_n$) to the nonlinear evolved ($\Delta_n$) spectrum via

$$\Delta_n(k) = B(n) \Phi \left[ \frac{\Delta_n(k)}{B(n)} \right], \quad k^3 \equiv \frac{k_c^3}{1 + \Delta_n(k)},$$

(92)

4.4. Comparison with Simulations: The Universal Scaling Hypothesis

The universal scaling hypothesis (Hamilton et al. 1991) asserts that the nonlinear evolution of the average two-point correlation function can be obtained from its linear counterpart by a universal scaling relation. This relation has been derived empirically from the study of numerical simulations for scale-free and CDM power spectra, but its main features can be understood on physical grounds (Nityananda & Padmanabhan 1994). These results have been extended by Peacock & Dodds (1994) to a universal relation between linear and nonlinear power spectra and by Jain et al. (1995) to include the dependence of the scaling relation on the spectral index $n$. In its current version, the universal scaling hypothesis for the dimensionless power spectrum,

$$\Delta(k) \equiv 4\pi k^3 P(k),$$

(91)

relates the linear ($\Delta_n$) to the nonlinear evolved ($\Delta_n$) spectrum via

$$\Delta_n(k) = B(n) \Phi \left[ \frac{\Delta_n(k)}{B(n)} \right], \quad k^3 \equiv \frac{k_c^3}{1 + \Delta_n(k)},$$

(92)
where the simulations are empirically fitted by setting $B(n) = (1 + n/3)^{1/3}$ and (Jain et al. 1995)

$$\Phi(z) = \frac{1 + 0.6z + z^2 - 0.2z^3 - 1.5z^{7/2} + z^4}{1 + 0.0037z^2}.$$  \hfill (93)

The Ansatz in equation (92) is equivalent to the hypothesis that the mean dimensionless pairwise velocity is a universal function of the average two-point correlation function (Hamilton et al. 1991; Nityananda & Padmanabhan 1994). Note that the linear spectrum is mapped to the nonlinear spectrum at smaller scale ($k_\ell > k$), reflecting the fact that nonlinearly evolving perturbations shrink in comoving coordinates. The specific relation between $k$ and $k_\ell$ in equation (92) can be obtained from the evolution equation for the average correlation function (conservation of pairs), according to which the mean number of neighbors within distance $r_e$ of a particle is time independent if decreases in time according to here $r = r_e(1 + \xi_e(r_e))$; here $r$ is a fixed scale, the radius of a sphere which encloses the same number of neighbors in the linear regime.

We now compare these expressions with the perturbative calculations of the previous sections. We can write the dimensionless power spectrum up to one-loop corrections as (see eq. [85])

$$\Delta(kR_0, k_\ell R_0, \Lambda) \equiv \frac{(kR_0)^{\alpha + 3}}{\Delta I(n, k, R_0)} + \frac{(k_\ell R_0)^{3\alpha}}{4[\Delta I(n, k, R_0)]^2} n^{(\alpha + 3) n - 1}.$$  \hfill (94)

In Figure 15 we compare the one-loop NLCPT relation between linear and nonlinear dimensionless power spectra with the universal scaling hypothesis (USH) for spectral index $n = -2$. The different dotted curves correspond to the predictions of NLCPT for $k_\ell R_0 = 10^3$, with $\alpha = 0.5 - 2$; since $5 \leq k_\ell R_0 \ll \Lambda = 10^3$, cutoff effects in the linear evolution of the correlation length are negligible (see eq. [82] and Fig. 11). Each curve extends down to scales such that $(kR_0)_{\text{max}} = 10^{\alpha - 1/2}$. The agreement between NLCPT and the USH (solid curve) is impressive all the way down to scales where $\Delta_\ell \approx 10$, far beyond the domain of linear perturbation theory (see also Lokas et al. 1995) for a comparison between NLCPT and $N$-body results for $n = -2$. For smaller scales than this, the one-loop power spectrum overestimates the nonlinear power spectrum. Note, however, that the turnover at $\Delta_\ell \approx 10$ in the USH curve marks the onset of the transition to the regime described by the stable clustering hypothesis, and therefore disagreement with NLCPT beyond this point is not surprising. The NLCPT curve for $\alpha = 2$ shows a slight disagreement with self-similarity at the large-scale end, which corresponds to the fact that, at this stage of evolution, these scales are not negligible compared to the size of the system. Otherwise, the agreement with self-similarity is excellent. For comparison, we also present the perturbative results to one-loop for the Zeldovich approximation (ZA) for $k_\ell R_0 = 100$ (dot-dashed curve). The agreement between ZA and the USH is not very good, although the results of the $N$-body simulations of Jain et al. (1995) display a scatter comparable to the difference. In fact, their earlier (higher redshift) outputs

![Figure 15](image_url)
seem to fit the ZA predictions shown here, suggesting that some of their scatter for \( n = -2 \) may be due to transients from the ZA initial conditions of the simulations.

In Figures 16, 17, and 18, we show the same comparison for spectral indices \( n = -1, 0, 1 \). As expected from the discussion in the previous sections and the fact that the USH is self-similar by construction, for \( n \geq -1 \) there is obvious disagreement between NCLPT and USH. This discord becomes more evident as time evolution proceeds (a change in the scale factor by \( \Delta a \) corresponds to \( \Delta a \equiv 10^{(n+3)\Delta a} \) as the scales considered decrease or as the spectral index increases. The dotted curves that drop after reaching a maximum end close to the scale where the one-loop corrections dominate over the tree-level contribution, driving \( \Delta \), negative (this happens at \( kR_0 < 1 \)). A similar conclusion regarding the disagreement between NLCPT and the USH has been recently noted by Bharadwaj (1995), who examined the hypothesis that the mean dimensionless
pairwise velocity is a universal function of the average correlation function. From the analysis of the BBGKY hierarchy in the weakly nonlinear regime, he showed that this hypothesis is not obeyed at the one-loop level for linear power spectra \( P_{\ell}(k) \propto k \exp(-k) \), consistent with our results for \( n = 1 \). In fact, we find that nonlinear perturbation theory fares worse than linear theory for \( n \geq -1 \).

A comparison of the USH with NLCPT to one-loop has also been carried out by Lokas et al. (1995), who concluded, based on numerical calculations of the \( s^{(1)} \) coefficients for Gaussian smoothing, that the agreement was good. Their main focus, however, was to understand whether nonlinear evolution decreases or increases the growth rate of structure with respect to that predicted by linear perturbation theory. In this regard, NLCPT qualitatively agrees with the USH in the sense that both predict that the growth rates are increased over linear theory when \( n = -2 \) and decreased when \( n = 0 \) (with \( n = -1 \) showing marginal behavior). However, as we have seen, the quantitative agreement is poor for \( n > 0 \); the scale dependence of the \( s^{(1)} \) coefficients in NLCPT disagrees with that of the USH, which by self-similarity requires \( s^{(1)} \approx \text{const.} \) in the scaling regime given by equation (86).

In fact, one can extract the \( s^{(1)} \) coefficients predicted by the USH from equations (92) and (93) by using a small-\( \Delta \) expansion,

\[
\Delta^{(1)} \approx \left[ 0.3 \left( \frac{3}{n+3} \right)^{1.3} - \left( \frac{n+3}{3} \right) \right] \Delta^2
\]

(compare to equation [94]). Using Gaussian smoothing, this gives

\[
s^{(1)}_{g}(\text{USH}) \approx \frac{2\Gamma(n+3)}{\Gamma^2((n+3)/2)} \left[ 0.3 \left( \frac{3}{n+3} \right)^{1.3} - \left( \frac{n+3}{3} \right) \right],
\]

which yields \( s^{(1)}_{g} \approx 0.58, -0.32, -3.56, -13.52 \) for \( n = -2, -1, 0, 1 \), respectively. These results should be taken with caution, however, because the range over which the next-to-leading order term in the fitting formula (93) dominates is quite narrow (Jain et al. 1995) and therefore sensitive to uncertainties in the fitting formulae. To quantify this uncertainty, one can, for example, calculate \( x^{(1)}_{1/2}(n) \) for \( n = -2 \) from the result in equation (95), which gives \( x^{(1)}_{1/2}(-2) \approx 0.50 \); on the other hand, a more direct calculation from the fitting formula for the average correlation function (analogous to eq. [93]) given in Jain et al. (1995) yields \( x^{(1)}_{1/2}(-2) \approx 0.75 \), in remarkable agreement with the perturbative result, equation (56). For \( n = -1, 0, 1 \), the USH fitting formula for the average correlation function yields \( x^{(1)}_{1/2} \approx -0.04, -0.55, -0.98 \). We also note that these results and equation (96) predict a similar behavior regarding the change of sign of the one-loop corrections as \( n \) increases, as pointed out by Lokas et al. (1995).

There has been some recent discussion in the literature about the slope of the \( \Delta_{c}(k) \) versus \( \Delta_{c}(k) \) relation in the quasi-linear regime (i.e., the transition from linear theory to stable clustering). In Figure 19 we show the slope \( \beta' \Delta_{c}(k) \equiv \{ \Delta_{c}(k) \}' \) as a function of \( \Delta_{c} \) for initial power spectrum \( n = -2 \). We compare one-loop NLCPT results with the USH fitting formula proposed by Jain et al. (1995) and the recently proposed fitting by Peacock & Dodds (1996), also based on USH. We see that
there is very good agreement with the latter in the expected regime ($\Lambda_x \lesssim 10$), while the agreement with the Jain et al. (1995) fit is not as good, although still reasonable. The slope $\beta$ peaks at about $\beta_{\text{max}} \approx 3 - 3.5$ at $\Lambda_x \approx 10$, and then it goes over to the stable clustering value $\beta = 1.5$ at large $\Lambda_x$. On the other hand, Padmanabhan (1996) has argued that generally the slope $\beta \approx 3$ in the quasi-linear regime ($1 \lesssim \Lambda_x \lesssim 100$), which partially captures the correct behavior, but not very accurately.

4.4.1. Self-Similarity via Dimensional Regularization

In the results presented so far, we imposed infrared and ultraviolet cutoffs on the initial power spectrum to obtain finite integrals. For $n = -2$, however, we have seen that the resulting one-loop power spectrum in the scaling regime is independent of the cutoffs and thus self-similar; this suggests that in this case the loop corrections can be alternatively calculated without the need to employ cutoffs. In fact, if the spectral index is in the range $-3 \leq n < -1$, one can show that one-loop NLCPT
presents self-similarity by considering $s^{(1)}$ in the no-cutoff limit and using dimensional regularization to regularize the required integrations (see Appendix B). Dimensional regularization is much simpler than the cutoff approach because it obviates the need for complicated constraints on the angular integration variable (see eq. [22]). It also makes possible the analytic calculation for noninteger values of $n$.

In Figure 20, we show $x_{0.1}^{(1)}$, derived via dimensional regularization for $-3 \leq n < -1$, and compare it with the predictions of the USH (Jain et al. 1995) taken directly from the fitting formula for the average correlation function (labeled USH-1) and from the dimensionless power spectrum Ansatz (USH-2). We see that the agreement is very good over most of this range. The NLCPT results diverge as $n \rightarrow -1$, as expected from the logarithmic divergence found as $k_{c} \rightarrow \infty$ in the cutoff calculation. As $n \rightarrow -3$, the NLCPT result goes to the unsmeared value $x^{(1)} = s^{(1)} = 4063/2205 \approx 1.843$ (Scoccimarro & Frieman 1996); this can be understood from the fact that large-scale fluctuations become dominant and averaging over small scales has a negligible effect. (By contrast, the USH has a singularity at $n = -3$ because of the way that the $n$ dependence is parametrized through $B(n)$ in the USH fitting formula; since the USH was extracted from simulations with $n \geq -2$, it should probably only be trusted in that range.) The linear part of the NLCPT curve is well described by an expansion about $n = -3$

$$x_{0.1}^{(1)}(n; \infty) = \frac{4063}{2205} - \frac{3679}{2205} (n + 3) \approx 1.843 - 1.168(n + 3).$$

(97)

Similar computations can be carried out in NLCPT for Gaussian smoothing and for the smoothed $s^{(1)}$ coefficient; the results show essentially the same features as displayed in Figure 20.

We see that for $n \approx -1.4$ the one-loop corrections vanish; this value of $n$ is a "fixed point" of nonlinear evolution in the sense that if $n \neq -1.4$ initially, one-loop corrections make the effective spectral index closer to $n = -1.4$. It is interesting to note that Colombi et al. (1995) find $n_{eff} \approx -1.4$ in the quasi-linear regime for $n = -2$ initial conditions. This value of $n_{eff}$ is also consistent with the observed power spectrum of galaxies at intermediate scales.

5. SUMMARY AND CONCLUSIONS

We have calculated the one-loop (first nonlinear) corrections to the power spectrum, the average two-point correlation function, and the variance of the density field, including smoothing effects for top-hat and Gaussian filters, for scale-free Gaussian initial conditions. For the power spectrum, these results should replace the expressions given in Makino et al. (1992) for the $p_{22}$ contribution on length scales smaller than the inverse ultraviolet cutoff of the linear spectrum ($k_{c} < k < 2k_{c}$). Our results for the one-loop corrections to the variance extend those of Lokas et al. (1995) to top-hat smoothing and correctly go over to the unsmeared values found in Scoccimarro & Frieman (1996) when the smoothing radius approaches zero.

We found that, when formulated with "fixed" comoving cutoffs, nonlinear perturbation theory beyond tree-level does not obey self-similar scaling when the spectral index $n \geq -1$. As a consequence, in this spectral range, NLCPT disagrees with the results of $N$-body simulations embodied in the self-similar universal scaling Ansatz (USH). For $-3 < n < -1$, however, we found that one-loop corrections do obey self-similar scaling and are in excellent agreement with the USH down to length scales where the dimensionless power spectrum $\Delta(k) \approx \sigma^2(R \sim 1/k) \approx 10$; below this scale, the $N$-body results make the transition to the highly nonperturbative regime described approximately by stable clustering.

We interpret the breaking of self-similarity for $n \geq -1$ as a signature of the breakdown of the fundamental assumption that the large-scale evolution of the density field can be calculated perturbatively when there are highly nonlinear fluctuations on small scales. In this instance, self-similarity breaking can be thought of as arising from an ultraviolet divergence in the one-loop corrections as the small-scale cutoff $k_{c}$ of the linear power spectrum goes to infinity. This problem is more severe as the spectral index $n$ increases, which is the expected behavior due to the increase in small-scale power.

Since for $n \geq -1$, the relative power on small scales is large, the question arises of whether the disagreement between the perturbative results and numerical simulations is due to the breakdown of perturbation theory (as we argued above) or to the use of the single-stream approximation, which neglects the effects of pressure gradients due to velocity dispersion. When these effects are included, one expects that the anisotropic velocity dispersion (nonradial motions) associated with small-scale effects are included, one expects that the anisotropic velocity dispersion (nonradial motions) associated with small-scale

okas (1995), Peebles (1990).

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APPENDIX A

EXPRESSIONS FOR $I_p$ AND $I_m$

Recall the definitions given in the text (see eqs. [53a], [53b]):

$$I_p(n, z) = \int_0^z u^{n+2} W^2(u) du$$  \hspace{1cm} (A1)

and

$$I_m(n, z) = \int_0^z u^{n+2} W(u) du.$$  \hspace{1cm} (A2)

Here we tabulate these integrals as a function of filter and spectral index.

A1. TOP-HAT SMOOTHING

$$I_p(-2, z) = \frac{3}{10z^3} \left[ -3 - 5z^2 + (3 - z^2 + 2z^4) \cos (2z) + (6z + z^3) \sin (2z) + 4z^5 \sin (2z) \right],$$  \hspace{1cm} (A3a)

$$I_p(-1, z) = \frac{9}{4} + \frac{9}{8z^4} \left[ -1 - 2z^2 + \cos (2z) + 2z \sin (2z) \right],$$  \hspace{1cm} (A3b)

$$I_p(0, z) = \frac{3}{2z^3} \left[ -1 - 3z^2 + (1 + z^2) \cos (2z) + 2z \sin (2z) + 2z^3 \sin (2z) \right],$$  \hspace{1cm} (A3c)

$$I_p(1, z) = \frac{9}{2} (1 - e^{-\ln 2}) + \frac{9}{4z^2} \left[ -1 + \cos (2z) - 2z^2 \sin (2z) + 2z^2 \ln (z) + 2z \sin (2z) \right],$$  \hspace{1cm} (A3d)

where $\text{Si} (z)$ and $\text{Ci} (z)$ denote the sine and cosine integrals, defined by

$$\text{Si} (z) \equiv \int_0^z \frac{\sin (u)}{u} \ du,$$  \hspace{1cm} (A4)

$$\text{Ci} (z) \equiv - \int_z^\infty \frac{\cos (u)}{u} \ du,$$  \hspace{1cm} (A5)

and $\gamma_e \approx 0.577216...$ is the Euler-Mascheroni constant. As $z \to \infty$, we have $\text{Ci} (z) \to 0$ and $\text{Si} (z) \to \pi/2$. Using the following expression of a top-hat filter in terms of Bessel functions

$$W_{TH}(u) = 3 \sqrt{\frac{\pi}{2}} u^{-3/2} J_{3/2}(u),$$  \hspace{1cm} (A6)

we obtain (for any $n$ in the range $-3 < n < 1$):

$$I_p(n, \infty) = 9\pi 2^{n-2} \frac{\Gamma(1 - n)\Gamma[(n + 3)/2]}{\Gamma^2(1 - n/2)\Gamma[(5 - n)/2]},$$  \hspace{1cm} (A7)

where $\Gamma(x)$ denotes the gamma function. For $I_q(n, z)$ we obtain

$$I_q(-2, z) = \frac{3}{2} \left[ \text{Si} (z) + \frac{\cos (z)}{z} - \frac{\sin (z)}{z^2} \right],$$  \hspace{1cm} (A8a)

$$I_q(-1, z) = \frac{3}{2} \left[ 1 - \frac{\sin (z)}{z} \right],$$  \hspace{1cm} (A8b)

$$I_q(0, z) = 3 [\text{Si} (z) - \sin (z)],$$  \hspace{1cm} (A8c)

$$I_q(1, z) = 3 [2 - 2 \cos (z) - z \sin (z)].$$  \hspace{1cm} (A8d)

Using equation (A6), we obtain ($-3 < n < 0$)

$$I_q(n, \infty) = 3\sqrt{\pi 2^n} \frac{\Gamma[(n + 3)/2]}{\Gamma(1 - n/2)}.$$  \hspace{1cm} (A9)
A2. GAUSSIAN SMOOTHING

\[ I_p(-2, z) = \frac{\sqrt{\pi}}{2} \text{erf}(z), \]  
(A10a)

\[ I_p(-1, z) = \frac{1}{2} [1 - \exp(-z^2)], \]  
(A10b)

\[ I_p(0, z) = \frac{\sqrt{\pi}}{4} \text{erf}(z) - \frac{1}{2} z \exp(-z^2), \]  
(A10c)

\[ I_p(1, z) = \frac{1}{2} - \frac{(1 + z^2)}{2} \exp(-z^2), \]  
(A10d)

where \( \text{erf}(z) \) denotes the error function,

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-u^2)du, \]  
(A11)

and for \( I_q(n, z) \) we have

\[ I_q(n, z) = 2^{(n+3)/2} I_p(n, z/\sqrt{2}). \]  
(A12)

When \( z \to \infty \) we obtain

\[ I_p(n, \infty) = \frac{1}{2} \Gamma\left(\frac{n+3}{2}\right). \]  
(A13)

APPENDIX B
DIMENSIONAL REGULARIZATION

To obtain the low-\( k \) behavior of the power spectrum for \( n < -1 \), one can use dimensional regularization to simplify the calculations considerably and get one-loop coefficients such as \( x^{(1)}(n; \infty) \) for \( n \) in the range \(-3 < n < -1\). Since we are interested in the limit \( k_c \to \infty \), all the integrals run from 0 to \( \infty \), and divergences are regulated by changing the dimensionality \( d \) of space; we take \( d = 3 + \epsilon \) and expand in \( \epsilon \ll 1 \). For power spectrum calculations, we need the following standard formula for dimensional-regularized integrals (Collins 1984):

\[
\int \frac{d^d q}{(q^2)^{n_1}[(k - q)^2]^{n_2}} = \frac{\Gamma(d/2 - v_1)\Gamma(d/2 - v_2)\Gamma(v_1 + v_2 - d/2)}{\Gamma(v_1)\Gamma(v_2)\Gamma(d - v_1 - v_2)} \pi^{d/2} k^{d-2v_1-2v_2}.
\]  
(B1)

When using this equation, divergences appear as poles in the gamma functions, which can be handled by the following expansion \( (n = 0, 1, 2, \ldots \text{ and } \epsilon \to 0) \):

\[
\Gamma(-n + \epsilon) = \left(\frac{-1}{n!}\right)^n \left\{ \frac{1}{\epsilon} + \psi(n + 1) + \frac{\epsilon}{2} \left[ \frac{\pi^2}{3} + \psi^2(n + 1) - \psi(n + 1) \right] + O(\epsilon^2) \right\},
\]  
(B2)

where \( \psi(x) \equiv d \ln \Gamma(x)/dx \) and

\[
\psi(n + 1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma, \]  
(B3a)

\[
\psi'(n + 1) = \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}, \]  
(B3b)

with \( \psi(1) = -\gamma \) and \( \psi'(1) = \pi^2/6 \). We can write the one-loop power spectrum as (see eq. [15]):

\[
P^{(1)}(k, \tau; n) = A^2 a^4(\tau) \int d^4 q d^4 q' [q - \bar{q}]^2 [F_2^Q(k - q, q) + 6k^\nu F_3^Q(k, q, -q)]. \]  
(B4)
In the numerators of the integrands, we can use the relation $2q \cdot k = -(k - q)^2 + k^2 + q^2$ to rewrite equation (B4) exclusively in terms of integrals of the form (B1). The resulting one-loop power spectrum contributions for $-3 < n < -1$ are

$$P_{22}(k, \tau; n) = \left\{ \frac{\Gamma((5/2 - n)/2)(n + 1/2)}{2\Gamma(2 - n/2)\Gamma(n - 1)} + \frac{3\Gamma((3/2 - n)/2)(n + 1/2)}{\Gamma(1 - n/2)\Gamma(2 - n/2)\Gamma(n)} + \frac{29\Gamma((1/2 - n)/2)(n + 1/2)}{4\Gamma(1 - n/2)\Gamma(n + 1)} \right. \]

$$+ 11\Gamma((1/2 - n)/2)(n + 1/2)\Gamma((n + 3)/2) + 5(1/2 - n)\Gamma((n + 1/2)\Gamma((n + 3)/2) - 2\Gamma(-1 - n/2)\Gamma(2 - n/2)\Gamma(n + 1) + 15\Gamma(-1 - n/2)\Gamma((n + 1)/2)\Gamma(n + 5/2) + 2\Gamma(1 - n/2)\Gamma(1/2 - n)\Gamma(n + 2) - 25\Gamma(-3 - 2n)\Gamma((n + 1)/2)\Gamma(n + 5/2) / 4\Gamma(2 - n/2)\Gamma(n + 3) + 75\Gamma(-3 - 2n)^2\Gamma((n + 3)/2) / 4\Gamma(2 - n/2)\Gamma(n + 3) \right\} k^{2n + 3} \),

$$P_{13}(k, \tau; n) = \left\{ -\Gamma((n + 1)/2)\Gamma((1 - n)/2) - 19\Gamma((n + 5)/2) \right\} A^2 a^4(\tau) k^{2n + 3} \)

Note that this implies that $P^{(1)} \propto k^{2n + 3}$, as required by self-similarity.

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