Area spectrum of the $d$-dimensional Reissner–Nordström black hole in the small charge limit

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Abstract
A conjecture by Hod states that for the black hole horizon the spacing of its area spectrum is determined by the asymptotic value of its quasinormal frequencies. Recently to overcome some difficulties, Maggiore proposes some changes to the original Hod’s conjecture. Taking into account the modifications proposed by Maggiore we calculate the area quantum of the $d$-dimensional Reissner–Nordström black hole in the small charge limit.

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1. Introduction
Taking into account semiclassical arguments and assuming that the horizon area of a non-extremal black hole behaves as an adiabatic invariant, Bekenstein proposes a discrete and evenly spaced spectrum for the horizon area [1–4]. Thus, at least in the semiclassical limit, he proposes that the mathematical form of the area spectrum is

$$A_n = \epsilon n \hbar,$$  \hspace{1cm} (1)

where $n = 0, 1, 2, \ldots, \hbar$ stands for the reduced Planck constant, and $\epsilon$ is a dimensionless parameter of order 1. It is believed that a quantum theory of gravity allows us to determine the value of the parameter $\epsilon$ (in the case that the quantum theory confirms that the area spectrum of the black hole horizon takes the form (1)).

At the present time we do not know a complete quantum theory of gravity. Nevertheless, supposing that the area spectrum is of the form (1) and using different semiclassical methods we can calculate the parameter $\epsilon$ (see [5–34] for some examples). In the previous references, for the parameter $\epsilon$, the values $\epsilon = 8\pi$ and $\epsilon = 4\ln(j)$, with $j = 2, 3, \ldots$ are often found.
We note that several methods used to calculate the area spectrum produce the value
\( \epsilon = 8\pi \) \[1, 2, 13, 15, 17–22\] and for black hole horizons Medved proposes that this value for \( \epsilon \) is universal \[16\]. Nevertheless, assuming a strict statistical interpretation of the black hole entropy, Bekenstein and Mukhanov find that the parameter \( \epsilon \) must be equal to \( \epsilon = 4 \ln(j) \) \[5, 6\]. In an interesting proposal Hod uses the Bohr correspondence principle and the real part of the asymptotic quasinormal frequencies (QNF) to determine the value of the integer \( j \) \[7\]. Briefly, Hod proposes that for the four-dimensional Schwarzschild black hole the emission of a quantum produces a change in the black hole mass given by

\[
\Delta M = \hbar \omega_R,
\]

where \( \omega_R \) stands for the real part of the asymptotic QNF\(^1\). From the known value of \( \omega_R \) for the four-dimensional Schwarzschild black hole \[35–37\] Hod finds \( j = 3 \) \[7\]. Based on these ideas, Kunstatter expounds a different method to get equivalent results for the area quantum to those by Hod \[8\], at least for some spacetimes. For the results on the area spectrum of the Reissner–Nordström black hole that produces the original Hod’s conjecture see \[10–12\].

Although Hod’s conjecture works for Schwarzschild black holes, some difficulties arise when it is used in other black holes. For example, for the four-dimensional Kerr black hole it is expected a discrete and equally spaced area spectrum \[1–4, 38\], however based on Hod’s conjecture Setare and Vagenas find a discrete but not equally spaced area spectrum \[9\]. Also it is known that the real part of the asymptotic QNF is not universal, that is, the real part of the asymptotic QNF may depend on the black hole type or even on the field type \[35–37, 39, 40\].

Supposing that the quasinormal modes of a black hole can be described as the oscillations of a damped oscillator and taking into account Hod’s conjecture, Maggiore \[19\] proposes that in the semiclassical limit the area spectrum of the event horizon is determined by the asymptotic value of the so-called physical frequency defined by

\[
\omega_{p,k} = \sqrt{\omega_R^2 + \omega_I^2},
\]

where \( \omega_I \) stands for the imaginary part of the asymptotic QNF. Making this change and using a similar method to that proposed by Hod \[7\], Maggiore finds that for the four-dimensional Schwarzschild black hole its area spectrum takes the form (1) with \( \epsilon = 8\pi \). This result for \( \epsilon \) coincides with the value calculated by other methods \[15–18\].

The consequences of Maggiore’s proposal have been studied in several spacetimes \[20–34\]. Based on the ideas by Hod, Kunstatter and Maggiore, we calculate the area quantum of the \( d \)-dimensional Reissner–Nordström black hole in the small charge limit \( d \geq 4 \). We follow the methods already used for the slowly rotating four-dimensional Kerr black hole \[21, 22\]. Thus, for the area spectrum of the Reissner–Nordström black hole our calculation is an update of the results obtained with the original Hod’s conjecture \[10–12\]. We also extend the result to non-extreme Reissner–Nordström black hole.

This paper is organized as follows. In section 2 we calculate the area quantum of the \( d \)-dimensional Schwarzschild black hole. Our main objective in section 2 is to explain the methods that we shall use in the remainder of the paper. Using Hod–Maggiore and Kunstatter–Maggiore methods in section 3 we calculate the area quantum of the \( d \)-dimensional Reissner–Nordström black hole in the small charge limit and then with other method we extend the results to the non-extremal Reissner–Nordström black hole. We also compare our results with those already published. Finally in section 4 we summarize the obtained results.

\(^1\) This proposal is usually known as Hod’s conjecture.

\(^2\)
2. Area spectrum of the Schwarzschild black hole

Using the ideas by Hod, Kunstatter and Maggiore [7, 8, 19] in the following section we shall calculate the quantum of area for the $d$-dimensional Reissner–Nordström black hole in the small charge limit. For illustrative purposes in this section we expound a similar calculation for the $d$-dimensional Schwarzschild black hole. As far as we know this computation does not appear in the literature, but it follows from the calculation for the four-dimensional Schwarzschild black hole in a straightforward way [19, 20].

Note that in this paper to calculate the area spectra we do not use the original Hod’s conjecture [7]. Our main motivation to use Maggiore’s proposal are the examples of spacetimes whose QNF are purely imaginary [28] and therefore for their horizons Hod’s conjecture predicts a continuous area spectrum. Thus, our aim is to investigate the predictions for the area spectra of the Schwarzschild and Reissner–Nordström black holes that produce the modifications proposed by Maggiore [19].

The $d$-dimensional Schwarzschild black hole whose metric takes the form

$$ds^2 = -\left(1 - \frac{2\mu}{r^{d-3}}\right)dt^2 + \left(1 - \frac{2\mu}{r^{d-3}}\right)^{-1}dr^2 + r^2d\Sigma_{d-2}^2,$$

(4)

where $d\Sigma_{d-2}^2$ stands for the line element of the $(d-2)$-dimensional unit sphere, the parameter $\mu$ is related to the mass $M$ of the Schwarzschild black hole by (see appendix A of [39])

$$M = \frac{(d-2)\Omega_{d-2}}{8\pi \mu},$$

(5)

with $\Omega_{d-2}$ denoting the area of a unit $(d-2)$-dimensional sphere:

$$\Omega_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}.$$

(6)

We also note that for the $d$-dimensional Schwarzschild black hole the area of its event horizon is given by

$$A = \Omega_{d-2}r_+^{d-2},$$

(7)

where $r_+$ is the radius of its event horizon, $r_+ = (2\mu)^{1/(d-3)}$.

In order to exploit the ideas by Hod, Kunstatter and Maggiore to calculate the area and entropy quanta of the black hole under study, we need to know exactly the QNF or at least its asymptotic limit [7, 8, 19]. For the $d$-dimensional Schwarzschild black hole the asymptotic QNF of the gravitational perturbations are equal to [35–37, 39, 41]

$$\omega = \frac{d - 3}{4\pi (2\mu)^{1/(d-3)}} \ln(3) + i\frac{d - 3}{2(2\mu)^{1/(d-3)}}\left(k + \frac{1}{2}\right), \quad k \in \mathbb{N}, \quad k \to \infty.$$

(8)

From the previous formula we get that for the $d$-dimensional Schwarzschild black hole the physical frequency (3) is

$$\omega_{p,k} = \frac{d - 3}{2(2\mu)^{1/(d-3)}}k.$$

(9)

We shall use two methods to calculate the area quantum from the asymptotic QNF. The first method is based on the ideas by Hod [7] and Maggiore [19]. This method works as follows. Based on Hod’s ideas, Maggiore proposes that the mass of the four-dimensional Schwarzschild black hole changes in discrete steps determined by [19]:

$$\Delta M = \hbar \Delta \omega,$$

(10)
where $\Delta \omega$ is equal to

$$
\Delta \omega = \omega_{p,k+1} - \omega_{p,k} = \frac{d - 3}{2(2\mu)^{1/(d-3)}}
$$

with $d = 4$. We propose that formula (10) is valid for the $d$-dimensional Schwarzschild black hole and taking into account formulas (5), (6) and (7) we find that a small change in the mass of the Schwarzschild black hole produces a change in its horizon area given by

$$
\Delta A = \frac{2(2\mu)^{1/(d-3)}}{d - 3} - 8\pi \Delta M.
$$

(12)

Using formulas (10) and (11) we find that the area quantum of the $d$-dimensional Schwarzschild black hole is

$$
\Delta A = 8\pi \hbar.
$$

(13)

Thus, $\epsilon = 8\pi$ for this black hole and from the Bekenstein–Hawking area-entropy relation [42] we obtain the entropy quantum:

$$
\Delta S = 2\pi.
$$

(14)

The second method that we shall use to calculate the area quantum of the event horizon is based on the ideas by Kunstatter [8] and Maggiore [19]. This method works as follows. First we note that for a system with energy $E$ and oscillation frequency $\omega$ the quantity

$$
I = \int \frac{dE}{\omega}
$$

(15)

is an adiabatic invariant [8]. Furthermore, we recall that in the semiclassical limit Bohr–Sommerfeld quantization rule states that the adiabatic invariant $I$ has an equally spaced spectrum; thus, $I = n\hbar$, $n \in \mathbb{N}$ [8]. From these facts and taking into account the ideas by Kunstatter [8] and Maggiore [19], it is proposed that for the calculation of the area spectrum the quantity $\Delta \omega$ of formula (11) is the appropriate oscillation frequency [21]. Thus, for the Schwarzschild black hole we need to calculate the adiabatic invariant [8, 20, 21]:

$$
I = \int \frac{dM}{\Delta \omega}.
$$

(16)

For the $d$-dimensional Schwarzschild black hole we get

$$
I = \frac{d - 2}{d - 3} \frac{2^{(d-2)/(d-3)} \Omega_{d-2}}{8\pi} \int \mu^{1/(d-3)} d\mu = \frac{A}{8\pi}.
$$

(17)

Hence, Bohr–Sommerfeld quantization rule and the previous expression for $I$ imply that in the semiclassical limit the area spectrum of the $d$-dimensional Schwarzschild black hole takes the form

$$
A_n = 8\pi \hbar n.
$$

(18)

From this expression for the area spectrum we find the area quantum $\Delta A$ of formula (13) (and therefore the entropy quantum $\Delta S$ of expression (14)). Thus, for the Schwarzschild black hole the two methods produce identical values for $\Delta A$ and $\Delta S$.

We note that expression (13) for the area quantum and formula (14) for the entropy quantum of the $d$-dimensional Schwarzschild black hole are equal to those previously obtained for the four-dimensional Schwarzschild black hole [19, 20]. Thus, for the Schwarzschild black hole the area and entropy quanta are independent of the spacetime dimension.

At this point we note that in the method based on Hod [7] and Maggiore [19] proposals the area spectrum of the event horizon has the mathematical form (1), that is, a discrete and equally spaced area spectrum is assumed and the method gives us the value of the area quantum. In contrast the method based on ideas by Kunstatter [8] and Maggiore [19] produces a mathematical expression for the area spectrum and the value for the area quantum is a by product.
3. Area spectrum of the Reissner–Nordström black hole

In this section we use Hod’s conjecture [7] with the changes suggested by Maggiore [19] to calculate the area quantum of the $d$-dimensional Reissner–Nordström black hole in the small charge limit and then with a different method we extend these results to non-extreme Reissner–Nordström black hole. Hence, for the Reissner–Nordström black hole we revise and update some results obtained with the original Hod’s proposal [10]–[12]. Our calculation for the $d$-dimensional Reissner–Nordström black hole in the small charge limit is similar to that for the slowly rotating four-dimensional Kerr black hole [21, 22].

The $d$-dimensional Reissner–Nordström black hole whose metric is

$$ds^2 = -(1 - \frac{2\mu}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}})dt^2 + (1 - \frac{2\mu}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}})^{-1}dr^2 + r^2 d\Sigma_{d-2}^2,$$

where we define $d\Sigma_{d-2}^2$ as in the previous section, the parameters $\mu$ and $q$ are related to the Arnowitt–Deser–Misner mass $M$ and electric charge $Q$ of the black hole by the expressions

$$\mu = \frac{8\pi}{\Omega_{d-2}(d-2)} M, \quad q^2 = \frac{2}{(d-2)(d-3)} Q^2,$$

with $\Omega_{d-2}$ already defined in formula (6).

For the $d$-dimensional Reissner–Nordström black hole the area of its event horizon $A$ and its electrostatic potential at the horizon $\Phi_+$ are

$$A = \Omega_{d-2} r_+^{d-2}, \quad \Phi_+ = \frac{\Omega_{d-2} Q}{4\pi (d-3)r_+^{d-3}},$$

where $r_+$ stands for the radius of the event horizon

$$r_+^{d-3} = \mu + \sqrt{\mu^2 - q^2}.$$

We note that the radius of the inner horizon is determined by

$$r_-^{d-3} = \mu - \sqrt{\mu^2 - q^2}.$$

First we focus on the area quantum of the $d$-dimensional Reissner–Nordström black hole in the small charge limit; thus, we assume that $q \ll \mu$. Our main reasons are as follows.

1. It is believed that extreme and near extreme black holes are highly quantum objects [14, 15, 38, 43]; therefore, we think that the semiclassical methods explained in the previous section are not useful for extreme or near extreme black holes.

2. In the small charge limit there is an explicit formula for the asymptotic QNF of the Reissner–Nordström black hole (see formula (3.20) of [44]). Generally the asymptotic QNF of the Reissner–Nordström black hole are given implicitly [36, 37, 39, 44].

For the $d$-dimensional Reissner–Nordström black hole in the small charge limit the asymptotic QNF of the coupled gravitational and electromagnetic perturbations reduce to [44]:

$$\omega = \frac{d-3}{4\pi (2\mu)^{1/(d-3)}} \ln \left( 3 + 4 \cos \left( \frac{d-3}{2d-5} \pi \right) \right) + \frac{i(d-3)}{2(2\mu)^{1/(d-3)}} \left( k + \frac{1}{2} \right),$$

with $k \in \mathbb{N}, k \to \infty$. It is convenient to comment that the asymptotic QNF (24) do not correspond to those for the Schwarzschild black hole (8) as the electric charge goes to zero. In the zero charge limit the difference between (8) and (24) is in the logarithmic term of the real part for the asymptotic QNF. For example, when $d = 4$ in the result for Schwarzschild appears the factor $\ln(3)$ while in the result for Reissner–Nordström the corresponding factor is $\ln(5)$. 

5
In [44] it is shown that this fact is due to a discrete change in the topology of Stokes lines used in the computation when the damping of the QNF for the Reissner–Nordström black hole increases.

Furthermore in [44] evidence is presented that for the asymptotic QNF of the Reissner–Nordström black hole when we increase the damping there is a range such that the real part varies from the Schwarzschild value to near zero and then the real part goes to the value for the Reissner–Nordström black hole given in (24). Since we only use the methods with the modifications proposed by Maggiore for our problem, this fact does not change the analysis because we must calculate the physical frequency in the limit \( k \to \infty \) and in this case the real part of the asymptotic QNF becomes negligible and hence its variation does not change in an appreciable way the value of the physical frequency does. Moreover note that the imaginary part of the asymptotic QNF is continuous when we go from the asymptotic QNF for the Reissner–Nordström black hole to those for the Schwarzschild black hole.

From the asymptotic QNF (24) we get the quantity \( \Delta \omega \) corresponding to the Reissner–Nordström black hole in the small charge limit is

\[
\Delta \omega = \frac{d - 3}{2(2\mu)^{1/(d-3)}},
\]

and as for the \( d \)-dimensional Schwarzschild black hole we propose that formula (10) is valid for the \( d \)-dimensional Reissner–Nordström black hole (for the four-dimensional Kerr black hole see [21, 22]). Hence, from formulas (10) and (20) we find

\[
\Delta M = \frac{\hbar(d - 3)}{2(2\mu)^{1/(d-3)}},
\]

(Taking into account the first formula of (21) we obtain that, in the small charge limit, a small variation in the mass produces a change in the horizon area of the Reissner–Nordström black hole equal to

\[
\Delta A \approx \Omega d^2 \left( \frac{d - 2}{d - 3} \right) 2\mu \approx \left( \frac{2(2\mu)^{1/(d-3)}}{d - 3} \right) \frac{8\pi}{\mu} \Delta M,
\]

and using formula (26) we get that the area quantum of the \( d \)-dimensional Reissner–Nordström black hole in the small charge limit is \( \Delta A = 8\pi \hbar \) (and hence \( \epsilon = 8\pi \)).

Also, to calculate the area quantum of the Reissner–Nordström black hole in the small charge limit we can use the second method explained in the previous section. To leading order, that is, keeping terms of order \( (q/2\mu)^2 \) in the calculation, we note that in the small charge limit the area of the Reissner–Nordström event horizon is approximately

\[
A \approx \Omega d^2 \left( \frac{d - 2}{d - 3} \right) 2\mu \approx \frac{d - 2}{d - 3} \frac{d^2}{(2\mu)^{1-(d-3)}}.
\]

Following Vagenas [21] and Medved [22] we propose that for the Reissner–Nordström black hole the adiabatic invariant \( I \) of expression (16) transforms into

\[
I = \int \frac{dM - \Phi_+ dQ}{\Delta \omega},
\]

(see also [9, 28, 34]). To leading order we find that for the Reissner–Nordström black hole in the small charge limit the adiabatic invariant (29) becomes

\[
I = \frac{(d - 2)\Omega d^2}{4\pi(d - 3)} \int (2\mu)^{1/(d-3)} \, d\mu - \frac{\Omega d^2 2^{(d-2)/(d-3)}}{8\pi (d - 3)^2 \mu^{(d-3)/(d-3)}} \int Q \, dQ
\]

\[
\approx \frac{\Omega d^2}{8\pi} \left( \frac{2\mu}{(2\mu)^{1/(d-3)}} \right) \frac{d^2}{d - 3} \int Q \, dQ
\]

\[
= \frac{\Omega d^2}{8\pi} \left( \frac{2\mu}{(2\mu)^{1/(d-3)}} \right) \frac{d^2}{d - 3} \int Q \, dQ
\]

\[
\approx \frac{\Omega d^2}{8\pi} \left( \frac{2\mu}{(2\mu)^{1/(d-3)}} \right) \frac{d^2}{d - 3} \int Q \, dQ
\]
Comparing formulas (28) and (30) we find that in the small charge limit

\[ I \approx \frac{A}{8\pi}, \]  

and from Bohr–Sommerfeld quantization rule we get that for the \( d \)-dimensional Reissner–Nordström black hole in the small charge limit its area spectrum to leading order is \( A_0 \approx 8\pi \hbar n \). Therefore, to leading order, the area quantum that we get with the second method is identical to that previously calculated.

As we comment at the end of section 2, the first method that we used previously gives the value for the area quantum, while the second method produces an expression for the area spectrum. Doubtless, in the framework of the Kunstatter method, it is convenient to find the expression of the area spectrum when we kept higher order terms in the expansion in \( (q/2\mu) \) already used. Keeping terms of order \( (q/2\mu)^4 \), we find that for the Reissner–Nordström black hole in the small charge limit the area of its event horizon is (compare with (28))

\[ A \approx \Omega_{d-2}(2\mu)^{(d-2)/(d-3)} -\frac{d-2}{d-3} (2\mu)^{(d-4)/(d-3)} \frac{q^2}{(2\mu)^{(3d-10)/(d-3)}}. \]  

(32)

In a similar way, to the same order in \( (q/2\mu) \), we get that the adiabatic invariant \( I \) (29) is

\[ I \approx \frac{1}{8\pi} \Omega_{d-2}(2\mu)^{(d-2)/(d-3)} -\frac{1}{8\pi} \frac{d-2}{d-3} (2\mu)^{(d-4)/(d-3)} \frac{q^2}{(2\mu)^{(3d-10)/(d-3)}}. \]  

(33)

From these expressions we obtain that to this order the area of the event horizon \( A \) and the adiabatic invariant \( I \) do not satisfy (31). Thus, to higher order and with the approximations that we use to find expression (33), we see that the Kunstatter–Maggiore method predicts that for the Reissner–Nordström black hole in the small charge limit its area spectrum does not grow linearly. Hence, to leading order it is true that both methods give the same area quantum, but to higher order, the Kunstatter method does not predict an equal space area spectrum and therefore the area quantum that it gives is different from that produced by the first method\(^2\). For the already studied slowly rotating four-dimensional Kerr black hole [21, 22, 33] we guess that only to leading order the two methods used to calculate the area quantum produce the same value.

We note that in the calculation of expressions (32) and (33) for the area and the quantity \( I \) we expand to order \( (q/2\mu)^4 \) the factor \( (1 - q^2/\mu^2)^{1/2} \), but we use the physical frequency given in formula (25). It is convenient to calculate the corrections of the asymptotic QNF for the Reissner–Nordström black hole, and use these to find a new expression for the adiabatic invariant \( I \), in order to determine whether it satisfies relation (31).

Thus, to leading order, for the \( d \)-dimensional Reissner–Nordström black hole in the small charge limit its area quantum is equal to that of the \( d \)-dimensional Schwarzschild black hole. As for the Schwarzschild black hole, to leading order the Hod–Maggiore and Kunstatter–Maggiore methods yield that the area quantum of the Reissner–Nordström black hole in the small charge limit does not depend on the spacetime dimension. Moreover, for these two black holes the two methods that we have used to calculate their area spectra to leading order produce identical results.

From our results we point out that for the Schwarzschild black hole and the Reissner–Nordström black hole in the small charge limit, the area and entropy spectra are equally spaced

\(^2\) I thank to Referee for suggesting this analysis.
to leading order, but we point out that there are black holes for which their entropy spectra are evenly spaced but not their area spectra [20, 26].

Note that when we use the first method (see formula (27)) we implicitly assume that the change in the horizon area is determined by the change in the mass of the black hole, and thus we ignore the changes in the other parameters (and the changes that these produce in the horizon area). In particular for the Reissner–Nordström black hole we assume that the electric charge does not change or its change is much smaller than that of the mass.

Recently Kwon and Nam proposed that the appropriate adiabatic invariant to be quantized is similar to that of the Schwarzschild black hole (see formula (16)), even for charged and rotating black holes, in contrast to the proposals by Vagenas [21], Medved [22], and of formula (29). For the three-dimensional spinning BTZ black hole they find that the addition and subtraction of the areas for the outer and inner horizons are quantized. Using the method described in [31], Kwon and Nam [45] analyze the case of the Reissner–Nordström black hole and they obtain a result consistent with that presented above for the area quantum in the small charge limit.

It is convenient to comment that in [45] the surface areas of the event and inner horizons are quantized for the Reissner–Nordström black hole, similar to the proposal by Makela and Repo [46] and the results obtained with the original Hod’s conjecture [11, 12]. Furthermore, we note that in the process of quantization, Kwon and Nam [45] propose that we must consider on equal basis the quasinormal modes, the total transmission modes, and the total reflection modes, but in their calculation of the area spectrum of the Reissner–Nordström black hole they only consider the total transmission modes and the total reflection modes; thus, in their computation they do not take into account the QNM, because for the coupled electromagnetic and gravitational perturbations an explicit expression for the asymptotic QNF is not known (except in the small charge limit).

Nevertheless for the Reissner–Nordström black hole the asymptotic QNF of the uncharged massless Dirac field can be determined in the explicit form. For the four-dimensional Reissner–Nordström black hole the asymptotic QNF of the Dirac field are calculated by Cho in [47] (see formula (55) of [47]). We note that in the $d$-dimensional Reissner–Nordström black hole the Dirac equation simplifies to a pair of Schrödinger-type equations with effective potentials [48]:

$$V_{\pm} = -\kappa^2 \left( \frac{1}{r^2} - \frac{2\mu}{r^{d-1}} + \frac{q^2}{r^{2d-4}} \right)$$

$$= \mp i\kappa \left( 1 - \frac{2\mu}{r^{d-3}} + \frac{q^2}{r^{2d-6}} \right)^{1/2} \left( - \frac{1}{r^2} + \frac{\mu(d-1)}{r^{d-1}} - \frac{q^2(d-2)}{r^{2d-4}} \right),$$

(34)

where $\kappa$ is related to the eigenvalues of the Dirac operator on the $(d-2)$-dimensional sphere. From expressions (34) for the effective potentials we get

$$\lim_{r \to \infty} V_{\pm} = 0, \quad \lim_{r \to 0} V_{\pm} \approx \pm \frac{\alpha}{\kappa^{(d-2)/(2d-5)}},$$

(35)

with $\alpha$ being a constant and $x$ being the tortoise coordinate of the Reissner–Nordström black hole.

Hence, we can use the results of [36, 39] to find that in the $d$-dimensional Reissner–Nordström black hole the asymptotic QNF of the Dirac field are determined by formula (3.20) of [39] with $j = 1$. Thus, for $d \geq 4$ the asymptotic QNF of the Dirac field are

$$\omega = i\kappa^{+}\kappa$$

(36)

This parameter $j$ is different from that used in section 1.
where
\[ \kappa^+ = \frac{d - 3}{2r_+} \left( 1 - \frac{r_+^{d-3}}{r_q^{d-3}} \right) \] (37)
is the surface gravity of the event horizon for the Reissner–Nordström black hole. Note that the asymptotic QNF (36) are purely imaginary and in contrast to QNF (24) these are not restricted to the small charge limit.

From (36) for the Dirac field we obtain that the step in the physical frequency is \( \Delta \omega = \kappa^+ \).

Following Kwon and Nam [31, 45], instead the adiabatic invariant (29) in what follows we consider the quantity:
\[ I_{KN} = \int \frac{dM}{\Delta \omega}. \] (38)

Making the change of variable \( u = \mu + \sqrt{\mu^2 - q^2} \) we find that the value of the previous integral is
\[ I_{KN} = \frac{\Omega_{d-2} r_+^{d-2}}{8\pi} = \frac{A}{8\pi}. \] (39)

According to the proposal by Kwon and Nam in the semiclassical limit \( I_{KN} \) has an equally spaced spectrum of the form \( I_{KN} = n \hbar \). Hence, we obtain that for the \( d \)-dimensional Reissner–Nordström black hole the area quantum of its event horizon is \( \Delta A = 8\pi \hbar \) which is consistent with that already obtained. It is convenient to remark that this result is valid for the non-extremal Reissner–Nordström black holes and it is not restricted to the small charge limit.

Note that Kwon and Nam state that \( I_{KN} \) (38) is an action variable and therefore in the proposal by Kunstatter and Maggiore it is the physical quantity that must be quantized. We believe that for the quantity \( I_{KN} \) another interpretation is possible. The quantity \( I_{KN} \) is quantized when we assume that the changes in the charge or angular momentum of the black hole are neglected and hence only the changes in the mass are relevant. From this viewpoint the previous result that we obtain for the Dirac field deserves additional study.

Based on his conjecture [7], Hod studies the area spectrum of the four-dimensional Reissner–Nordström black hole [11, 12]. Taking into account the existence of two distinct families for the asymptotic QNF of the charged Klein–Gordon field (see formulas (5) and (6) of [12]), he finds that a family of QNF leads to an area quantum equal to \( \Delta A = 4\hbar \ln(2) \) for the surface area of the event horizon, whereas the second family of QNF leads to an area quantum equal to \( \Delta A = 4\hbar \ln(3) \) for the total surface area (the area of the outer horizon plus the surface area of the inner horizon). The last result reminds us the proposal by Makela and Repo [46], who suggest that for a multihorizon black hole the object that must be quantized in equal steps is the total area of the horizons.

Note that, in the small charge limit, formulas (5) and (6) of [12] lead to the same physical frequency and using the previous methods we get that, in the small charge limit, both families of asymptotic QNF lead to the same result for the area quantum of the Reissner–Nordström event horizon in four dimensions.

It is convenient to comment that for the Reissner–Nordström black hole with \( d \geq 4 \), the additional dimensions are included in the term \( d\Sigma_{d-2} \) of metric (19) and we can expect that to leading order the result for the area quantum does not depend on the dimension of the spacetime because the additional dimensions are ‘hidden’. We believe that the problem is not straightforward, because the \((t, r)\) sector of the metric for the \( d \)-dimensional Reissner–Nordström black hole depends on the dimension of the metric (for example see the factor \( 2\mu/r_+^{d-3} \)). Similar comments are valid for the \( d \)-dimensional Schwarzschild black hole.
In [10, 14, 15, 17, 46] we find other results on the area spectrum of the Reissner–Nordström black hole (mainly in four dimensions). Setare [10] analyzes the four-dimensional extreme Reissner–Nordström black hole. Thus, [10] and this paper study different limits of the Reissner–Nordström solution and it is not possible to compare the obtained results for the area quantum. Also in [10] it is not shown that the area of the extreme Reissner–Nordström black hole behaves as an adiabatic invariant and it is expected that the extreme Reissner–Nordström black hole is a quantum object. Thus, it is not straightforward that the Bekenstein and Hod ideas can be used to calculate the area quantum of the extreme Reissner–Nordström black hole.

In the small charge limit the area of the inner horizon is small and the total area is approximately the area of the event horizon; thus, we may say that our results do not distinguish whether the total area or the event horizon surface is quantized; nevertheless we note that in the second method, to leading order we identify the area of the event horizon with the value of the integral for the adiabatic invariant $I$ (29) of the Reissner–Nordström black hole (see also (39)) and hence we implicitly quantize the area of the outer horizon, and not the total area.

Based on reduced phase-space quantization Barvinsky et al [14] calculate the area spectrum of the $d$-dimensional Reissner–Nordström black hole (and other charged black holes). In [14] it is quantized the quantity $A_B = A - A_0$, where $A$ is the area of the event horizon, $A_0$ is a function that depends on a positive power of the electric charge and is related to the horizon area of the extremal Reissner–Nordström black hole. In the small charge limit we can assume that $A_0$ is small and hence $A_B \approx A$. For the four-dimensional Reissner–Nordström black hole they find the two parameter area spectrum [14] (a similar area spectrum is found for the $d$-dimensional Reissner–Nordström black hole):

$$A_B = 4\pi \bar{h}(2n + p + 1),$$

where $n, p = 0, 1, 2, \ldots$, with $p$ determining the charge spectrum $Q = \pm \sqrt{hp}$. From formula (40), if $Q$ is a constant, then $\Delta A_B = 8\pi \bar{h}$, which is equal to our result. Thus, in the small charge limit and for constant $Q$ our results to leading order are equal to those by Barvinsky et al [14]. Moreover the area quantum that we get with the Kwon and Nam proposal is identical to $\Delta A_B$. In contrast to the method of Barvinsky et al [14] the methods that we use in this paper do not determine the charge spectrum of the Reissner–Nordström black hole.

A similar analysis to that of Barvinsky et al [14] is developed by Ropotenko [17] and Medved [15] for black holes whose metric can be written in the Schwarzschild-like form. In particular, for the area spectrum of the Reissner–Nordström black hole they find $A_n = 8\pi \bar{h}n$. To leading order our result for the area quantum in the small charge limit coincides with the value which is reported in [15, 17], and the area spectrum that we get with the Kwon and Nam proposal is identical to that given in [15, 17].

4. Summary

From the results of [19–22] and from those of the previous sections, (for the $d$-dimensional Schwarzschild black hole, the $d$-dimensional Reissner–Nordström black hole in the small charge limit (to leading order), and the slowly rotating four-dimensional Kerr black hole) we obtain that their area quanta are equal to $\Delta A = 8\pi \bar{h}$ and coincide with the results for $\Delta A$ already calculated [15–18]. At least to leading order, note that this value of the area quantum is independent of the black hole parameters, the spacetime dimension and the field parameters. We also note that for these three black holes, to leading order the two methods used in [19–22] and in the previous sections give the same value for the area quantum.
In contrast to Hod’s original conjecture [7], the previous results show that Hod’s conjecture with the changes suggested by Maggiore [19] is applicable to black holes with two horizons, at least far from the extremal limit (see the examples of the slowly rotating four-dimensional Kerr black hole [21, 22], the $d$-dimensional Reissner–Nordström black hole in the small charge limit to leading order and of the non-extremal Reissner–Nordström black hole). Thus, for the black holes of the Einstein–Maxwell theory, it is probable that for the dimensionless parameter $\epsilon$ of formula (1) the value $\epsilon = 8\pi$ is generic (at least in some limit) as is suggested by Medved [16].

Here for the non-extremal Reissner–Nordström black holes we expand the analysis proposed by Kwon and Nam. Furthermore, with the methods of Hod–Maggiore and Kunstatter–Maggiore we analyze the Reissner–Nordström black hole in the small charge limit. A question that deserves further research is to study whether the Hod–Maggiore and Kunstatter–Maggiore methods of the previous sections can be used when we eliminate this restriction on the mass and charge, but the black hole is still far from extremality. Note that for this case in the Kunstatter–Maggiore method the corrections of higher order must be considered.

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