OPTIMAL REGULARITY
OF FOURIER INTEGRAL OPERATORS
WITH ONE-SIDED FOLDS

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Abstract. We obtain optimal continuity in Sobolev spaces for the Fourier integral operators associated to singular canonical relations, when one of the two projections is a Whitney fold. The regularity depends on the type, $k$, of the other projection from the canonical relation ($k = 1$ for a Whitney fold). We prove that one loses $(4 + \frac{2}{k})^{-1}$ of a derivative in the regularity properties.

The proof is based on the $L^2$ estimates for oscillatory integral operators.

1. Introduction and results

The Fourier integral operators associated to singular canonical relations (i.e., which are not local graphs) fall out of scope of the classical theory of Fourier integral operators. Their regularity properties are still to be studied. The first step in this direction was due to the paper of R.B. Melrose and M.E. Taylor [MeTa85], who showed that the canonical relation with Whitney folds on both sides can be transformed to the canonical form. (This idea originates from the paper of Melrose [Me76].) Melrose and Taylor then derived the loss of $1/6$ of a derivative in the regularity properties of Fourier integral operators with folding canonical relations, vs. operators associated to local canonical graphs.

A corresponding result for oscillatory integral operators (with not necessarily homogeneous phase functions) was obtained by Y. Pan and C.D. Sogge [PaSo90], who also relied on the reduction of the folding canonical relation to the normal form. An independent analytical approach to such operators in $\mathbb{R}^1$ was used by D.H. Phong and E.M. Stein [PhSt91]. This approach was generalized for operators in $\mathbb{R}^n$ by S. Cuccagna [Cu97], who used fine almost orthogonal decompositions of the integral kernels. Let us also mention thorough investigations of oscillatory integral operators in $\mathbb{R}^1$ and related generalized Radon transforms in the plane by Phong and Stein [PhSt97] and by A. Seeger [Se93], [Se98]. We also refer to the survey of D.H. Phong [Ph94].
The relation of the regularity properties of Fourier integral operators to the rate of high-frequency decay of norms of oscillatory integral operators was used by A. Greenleaf and A. Seeger [GrSe94] for deriving the a priori continuity of Fourier integral operators associated to one-sided Whitney folds: in general, one loses up to $1/4$ of a derivative in the regularity properties. Let us mention a recent result [GrSe98] that for operators associated to canonical relations with cusp singularities on one side there is a loss of at most $1/3$ of a derivative.

In this paper, we are going to derive the optimal regularity properties of the Fourier integral operators associated to one-sided Whitney folds. Let us recall the standard framework. Let $X$ and $Y$ be $C^\infty$ manifolds of the same dimension $n$, and let $C$ be a homogeneous canonical relation $C \subset T^*X\setminus 0 \times T^*Y\setminus 0$, that is, $C$ is lagrangian with respect to the difference of the canonical symplectic forms lifted from $T^*X$ and $T^*Y$ onto $C$. If $C$ is locally a canonical graph, that is, both projections

$$\pi_L : C \to T^*X\setminus 0, \quad \pi_R : C \to T^*Y\setminus 0$$

are local diffeomorphisms, then the regularity properties of Fourier integral operators associated to $C$ are well-known, cf. L. Hörmander [Hö85]: Given a Fourier integral operator $\mathfrak{F} \in I^m(X, Y, C)$, then for any real $s$ there is a continuous map $\mathfrak{F} : H^s_{\text{comp}}(Y) \to H^{s-m}_{\text{loc}}(X)$. Here $H^s(X)$ is the standard Sobolev space of order $s$.

Now let us state the continuity of the Fourier integral operators associated to canonical relations with a Whitney fold on one side. The continuity turns out to depend on the type of the projection from the canonical relation on the other side (see the definition after the theorem).

**Theorem 1.1.** Let $\mathfrak{F} \in I^m(X, Y, C)$ be a Fourier integral operator associated to the homogeneous canonical relation $C \subset T^*X\setminus 0 \times T^*Y\setminus 0$, such that one of the projections $C \to T^*X\setminus 0$, $C \to T^*Y\setminus 0$ is a Whitney fold and the other is of type at most $k$ ($k = 1$ for a Whitney fold). Then, for any real $s$, $\mathfrak{F}$ defines a continuous map

$$\mathfrak{F} : H^s_{\text{comp}}(Y) \to H^{s-m-(4+\frac{2}{k})^{-1}}_{\text{loc}}(X).$$

This result is optimal, in the sense that there are operators associated to the singular canonical relations with one of the projections being a Whitney fold and the other of type $k$, when the smoothing stated above can not be improved.

The above-mentioned case of two-sided Whitney folds corresponds to $k = 1$.

The type of a map of corank at most 1 is to be defined as the highest order of vanishing of the determinant of its Jacobian in the directions of the kernel.
of its differential. Let $M$ and $N$ be two $C^\infty$ manifolds of the same dimension and let $\pi : M \rightarrow N$ be a smooth map. We assume that the rank of $\pi$ drops simply by 1: the corank of $d\pi$ is at most 1 and the differential $d(\det d\pi)$ does not vanish in some neighborhood of the critical variety

$$\Sigma(\pi) = \{ p \in M \mid \det d\pi|_p = 0 \}.$$ 

Let $V \in C^\infty(\Gamma(TM|_U))$ be a smooth vector field defined in some open neighborhood $U \subset M$ of a point $p_0 \in \Sigma(\pi)$, which generates the kernel of $d\pi$:

$$V|_U \neq 0, \quad V|_{U \cap \Sigma(\pi)} \in \text{Ker } d\pi.$$ 

**Definition.** The type of $\pi$ at a point $p_0 \in \Sigma(\pi)$ is the smallest $k \in \mathbb{N}$ such that

$$V^k(\det d\pi)|_{p_0} \neq 0.$$ 

The type of $\pi$ at $p \in M \setminus \Sigma(\pi)$ is defined to be 0.

Since we assume that the rank of $\pi$ drops simply and hence $d(\det d\pi) \neq 0$ on $\Sigma(\pi)$, any other smooth vector field $\tilde{V}$ which satisfies the conditions (1.1) can be represented in the neighborhood $U$ as $\tilde{V} = \varphi \cdot V + (\det d\pi) \cdot W$, where $\varphi$ is a smooth function on $M$ which does not vanish on $\Sigma(\pi)$ and $W$ is a smooth vector field. (Let us note that $\det d\pi$ is only defined up to a non-zero factor, depending on the choice of local coordinates.) As a consequence, the above Definition does not depend on the choice of $V$.

**Remark.** In the context of singular integral operators, the type conditions can be defined without the assumption that the rank drops simply, see for example [PhSt91], [Se93], [PhSt94], [Co97], and [Se98]. We use the above Definition since the assumption of Theorem 1.1 that one of the projections from the canonical relation is a Whitney fold guarantees that the other projection drops its rank simply by 1: This is because the corank of both projections is the same (and hence not greater than 1) and $\det d\pi_L$, $\det d\pi_R$ are equal up to a non-zero factor (and both vanish simply on the critical variety).

An example of a map of type at most $k$ is a map with a Morin $S_{1k}$-singularity, with $k$ units (see R. Thom [Th63] and B. Morin [Mo65]). For example, type $k = 1$ unambiguously corresponds to the Whitney fold. $k = 2$ for the Whitney Pleat, or the Simple Cusp, at the cusp point (and $k = 1$ at neighboring critical points); $k = 3$ for the Swallow Tail at the “swallow tail point”, et cetera.
Relation with oscillatory integral operators. We will use the results of Greenleaf and Seeger on the relation of Fourier integral operators and oscillatory integral operators. As they showed in [GrSe94], the regularity in Sobolev spaces of singular Fourier integral operators associated to singular canonical relations is determined by the rate of decay of the $L^2$-operator norm of oscillatory integral operators associated to similar canonical relations.

The oscillatory integral operators are of the form

$$T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)} \psi(x,\vartheta) u(\vartheta) \, d\vartheta, \quad \psi \in C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n), \quad \lambda \gg 1.$$  

We will write subscripts for two copies of $\mathbb{R}^n$: $x \in \mathbb{R}^n_L$, $\vartheta \in \mathbb{R}^n_R$. The canonical relation $C$ associated to the oscillatory integral operator (1.3) is given by

$$C = \{(x,S_x) \times (\vartheta,S_\vartheta) \mid x \in \mathbb{R}^n_L, \vartheta \in \mathbb{R}^n_R\} \subset T^*\mathbb{R}^n_L \times T^*\mathbb{R}^n_R.$$  

Here $S_x$ stands for the components of the 1-form $d_xS$, etc. Using the isomorphism $\mathbb{R}^n_L \times \mathbb{R}^n_R \cong C$, we write the projections from the canonical relation onto the first and second factors of $T^*\mathbb{R}^n_L \times T^*\mathbb{R}^n_R$ in the following form:

$$\pi_L : (x, \vartheta) \mapsto (x, S_x), \quad \pi_R : (x, \vartheta) \mapsto (\vartheta, S_\vartheta).$$  

The projections (1.4) are degenerate on the variety where the determinant of the mixed Hessian of $S$ vanishes. We will use the notation

$$h(x, \vartheta) = \det S_{x\vartheta},$$  

so that the common critical variety of the projections $\pi_L$, $\pi_R$ is given by

$$\Sigma = \{(x, \vartheta) \mid h(x, \vartheta) = 0\}.$$  

**Theorem 1.2.** Let $T_\lambda$ be an oscillatory integral operator (1.3) with a smooth compactly supported density $\psi(x,\vartheta)$ and a smooth (not necessarily homogeneous) phase function $S(x,\vartheta)$, $x, \vartheta \in \mathbb{R}^n$, such that one of the projections $\pi_L : (x, \vartheta) \mapsto (x, S_x)$ and $\pi_R : (x, \vartheta) \mapsto (\vartheta, S_\vartheta)$ from the associated canonical relation is a Whitney fold and the other is of type at most $k$.

Then there is the following estimate on $T_\lambda$:

$$\|T_\lambda\|_{L^2 \to L^2} \leq \text{const} \lambda^{-\frac{n}{2} + \frac{4}{2} + \frac{2}{k} - 1}.$$  

This result is optimal (see Section 4).  

Let us mention here that the most general estimates for oscillatory integral operators with real analytic phase functions in $\mathbb{R}^1$ are derived in [PhSt97].

According to Greenleaf and Seeger [GrSe94], Theorem 1.1 follows from Theorem 1.2. The proof of Theorem 1.2 is in Sections 2 and 3. The sharpness of the results is discussed in Section 4.
2. Dyadic creeping to the critical variety

We use the dyadic decomposition $\sum_{N \in \mathbb{Z}} \beta(2^N t) = 1$ (for $t > 0$), where $\beta(t)$ is a smooth function supported in $[1/2, 2]$, $0 \leq \beta(t) \leq 1$, $\beta(t) \equiv 1$ in a neighborhood of $t = 1$. We put $\beta_+(t) \equiv \beta(t)$, $\beta_-(t) \equiv \beta(-t)$, to take care of positive and negative values separately (usually we will not write $\pm$-subscripts). We define $\bar{\beta} \in C^\infty_{\text{comp}}([-2, 2])$ by $\bar{\beta}(t) = \beta(|t|)$ for $|t| \geq 1$, $\bar{\beta}(t) \equiv 1$ for $|t| \leq 1$. There is the following partition of 1 which depends on $N_0 \in \mathbb{Z}$:

\begin{equation}
1 = \sum_{\pm} \sum_{N \in \mathbb{Z}, N < N_0} \beta_\pm(2^N h(x, \vartheta)) + \bar{\beta}(2^{N_0} h(x, \vartheta)).
\end{equation}

We define the operators $T_{\lambda}^{\pm h}$ and $\overline{T}_{\lambda}^{h}$ by

\begin{equation}
T_{\lambda}^{\pm h} u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \beta_\pm(h^{-1} h(x, \vartheta)) u(\vartheta) \, d\vartheta,
\end{equation}

\begin{equation}
\overline{T}_{\lambda}^{h} u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) \bar{\beta}(h^{-1} h(x, \vartheta)) u(\vartheta) \, d\vartheta,
\end{equation}

(here $h = 2^{-N}$ refers to the magnitude of $h(x, \vartheta)$), and decompose (1.3) into a sum

\begin{equation}
T_{\lambda} = \sum_{\pm} \sum_{h > h_0(\lambda)}^{h \leq 2D} T_{\lambda}^{\pm h} + \overline{T}_{\lambda}^{h_0(\lambda)}, \quad h = 2^{-N}, \; N \in \mathbb{Z},
\end{equation}

where $D$ is the uniform bound on $|h(x, \vartheta)|$, and $h_0(\lambda)$ will be chosen so that the optimal estimates on $T_{\lambda}^{\pm h}$ and $\overline{T}_{\lambda}^{h}$ coincide when $h = h_0(\lambda)$. It suffices to consider the estimates on $T_{\lambda}^{\pm h}$, $\overline{T}_{\lambda}^{h}$ for $h < 1$.

We will prove the estimates $\|T_{\lambda}^{\pm h}\|_{L^2 \to L^2} \leq \text{const} \lambda^{-\frac{n}{2}} h^{-\frac{1}{2}}$ (Theorem 2.1) and $\|\overline{T}_{\lambda}^{h}\|_{L^2 \to L^2} \leq \text{const} \lambda^{-\frac{n}{2}} h^{\frac{1}{2}} + \frac{k}{2}$ (Theorem 3.1); these estimates meet at

\begin{equation}
h_0(\lambda) = \lambda^{-\frac{k}{2n+k}}.
\end{equation}

Using the corresponding estimates for the operators in the right-hand side of (2.4), we arrive at the statement of Theorem 1.2.
Theorem 2.1. Let the projection $\pi_L : (x, \vartheta) \mapsto (x, S_x)$ be a Whitney fold and let the projection $\pi_R : (x, \vartheta) \mapsto (\vartheta, S_{\vartheta})$ be of type at most $k$, for some $k \in \mathbb{N}$. Then, as long as $\hbar \geq \lambda^{-\frac{k}{2}}$, there is the following estimate:

$$\| T_{\lambda}^{\pm h} \|_{L^2 \to L^2} \leq \text{const} \lambda^{-\frac{n}{2}} h^{-\frac{1}{2}}. \tag{2.6}$$

Since both $T_{\lambda}^{h}$ and $T_{\lambda}^{-h}$ require the same argument, we will always restrict the consideration to $T_{\lambda}^{h}$. Also, unless otherwise stated, the norm $\| \|$ will refer to the $L^2$ operator norm.

Remark. According to (2.5), we are only interested in $\hbar \geq \lambda^{-\frac{k}{2}} + \epsilon$; we will only give the proof for this case. This proof has already appeared in the author’s paper [Co97], but we reproduce it for the sake of completeness.

The proof for $\hbar \geq \lambda^{-\frac{1}{2}}$ can be obtained by some elaboration of almost orthogonal decompositions. The estimate (2.6) becomes useless for $\hbar < \lambda^{-\frac{1}{2}}$.

The proof involves the spatial decomposition with respect to $\vartheta$, with the step $\hbar$. We use the notation $(T_{\lambda}^{h})_\Theta$ for $T_{\lambda}^{h}$ localized near the point $h\Theta \in \mathbb{R}^n$; here $\Theta$ is a point on the integer lattice $\mathbb{Z}^n$.

As long as $\hbar \geq \lambda^{-\frac{1}{2}} + \epsilon$, $\epsilon > 0$, the argument similar to the one used by S. Cuccagna [Cu97] shows that $\pi_L$ being a Whitney fold is a sufficient condition for the pieces $(T_{\lambda}^{h})_\Theta$ to be almost orthogonal with respect to different values of $\Theta \in \mathbb{Z}^n$:

$$\| (T_{\lambda}^{h})_\Theta (T_{\lambda}^{h})_W^* \|, \| (T_{\lambda}^{h})_\Theta^* (T_{\lambda}^{h})_W \| \leq \text{const} \lambda^{-n} h^{-1} |\Theta - W|^{-N},$$

for any $N \in \mathbb{N}$. Here $|\Theta - W|$ is the distance between the points $\Theta, W$ in $\mathbb{Z}^n$. Then, the Cotlar-Stein almost orthogonality lemma [St93] applies.

Let us derive the individual estimates on $(T_{\lambda}^{h})_\Theta$, which are similar to Hörmander’s estimates for non-degenerate oscillatory integral operators in $\mathbb{R}^n$. We consider the integral kernel of the composition $(T_{\lambda}^{h})_\Theta (T_{\lambda}^{h})_\Theta^*$:

$$K \left( (T_{\lambda}^{h})_\Theta (T_{\lambda}^{h})_\Theta^* \right)(x, y) = \int d^m \vartheta e^{i\lambda (S(x, \vartheta) - S(y, \vartheta))} \chi(h^{-1} \vartheta - \Theta) \times \ldots. \tag{2.7}$$

We integrate by parts in (2.7), using the operator

$$L_\vartheta = \frac{1}{i\lambda} \cdot \frac{(S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)) \cdot \nabla_{\vartheta}}{|S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)|^2}.$$

When acting on cut-offs, $\nabla_{\vartheta}$ contributes $h^{-1}$ (we will discuss this below in more details). This is the pay for decomposing $T_{\lambda}$ with respect to the values
of \( h(x, \vartheta) \); one takes over when integrating with respect to \( \vartheta \) in (2.7), since \((T^h_\lambda)_{\vartheta} \) has the support of size \( h \) in \( \vartheta \)-variables.

We then apply the Schur lemma, i.e., integrate with respect to \( x \) (or \( y \)) the absolute value of (2.7) with the extra factor \((1 + \lambda h |S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)|)^{-N} \). It is convenient to change the variables of integration: \( x \mapsto \eta = S_{\vartheta}(x, \vartheta) \),

\[
\int dx \, d\vartheta \rightarrow \int \frac{d\eta \, d\vartheta}{|\det S_{xx\vartheta}|}.
\]

The integration with respect to \( \eta \) contributes \((\lambda h)^{-n} \), the integration with respect to \( \vartheta \) contributes \( h^n \), and \(|\det S_{xx\vartheta}| \approx h \). This yields the estimate

\[
\| (T^h_\lambda)_{\vartheta} (T^h_\lambda)_{\vartheta}^* \| \leq \text{const} \lambda^{-n} h^{-1},
\]

and hence proves the bound (2.6) on \((T^h_\lambda)_{\vartheta} \).

We need to control that, during the integrations by parts, the derivative \( \nabla_{\vartheta} \) contributes at most \( \text{const} h^{-1} \) even when it acts on the denominator of \( L_{\vartheta} \) itself. For this, the map \( \pi_R |_{\vartheta} : x \mapsto S_{\vartheta}(x, \vartheta) \) needs to satisfy certain convexity condition on the support of \((T^h_\lambda)_{\vartheta} \):

\[
|S_{\vartheta}(x, \vartheta) - S_{\vartheta}(y, \vartheta)| \geq \text{const} |x - y|.
\]

As the matter of fact, \( \pi_R \) does not generally satisfy the condition (2.8) on the entire support of \((T^h_\lambda)_{\vartheta} \), and we are going to introduce one more localization.

In the rest of this section, we discuss the construction of this new localization (which will also be used in Section 3) and prove that (2.8) is valid on the support of each of the pieces of \((T^h_\lambda)_{\vartheta} \).

Since one of the projections from the associated canonical relation is a Whitney fold (we assume it is \( \pi_r \)), the rank of the mixed Hessian \( S_{x\vartheta} \) on the critical variety is equal to \( n - 1 \). We choose local coordinates \( x = (x', x_n) \) and \( \vartheta = (\vartheta', \vartheta_n) \) so that \( S_{x'\vartheta'} \) is non-degenerate. We introduce the vector field \( K_R \),

\[
K_R = \partial_{x_n} - S_{x_n\vartheta'}(x, \vartheta) S_{\vartheta'x'}(x, \vartheta) \partial_{x'};
\]

we wrote \( S_{\vartheta'x'}(x, \vartheta) \) for the inverse to the matrix \( S_{x'\vartheta'} \) at a point \((x, \vartheta)\). It can be checked immediately that this vector field satisfies

\[
K_R |_S \in \text{Ker} \, d\pi_R.
\]

Since the type of \( \pi_R \) is not greater than \( k \), we can assume, in the agreement with the definition of type of the map (see (1.2)), that on the support of the integral kernel of \( T_\lambda \)

\[
|K_R^{k'} h(x, \vartheta)| \geq \kappa > 0,
\]

for some positive constant \( \kappa \) and for some integer \( k' \leq k \). For definiteness, we assume that \( k' = k \) (this is the “worst” case).
We fix two smooth functions $\rho_-$ and $\rho_+$, supported in $(-\infty, 1]$ and $[-1, \infty)$, respectively, such that $\rho_-(t) + \rho_+(t) = 1$, $t \in \mathbb{R}$, and define the following partition of 1:

$$1 = \sum_\sigma \rho_\sigma^h(x, \theta), \quad \sigma = (\sigma_1, \ldots, \sigma_{k-1}), \quad \sigma_j = \pm 1,$$

where

$$(2.11) \quad \rho_\sigma^h(x, \theta) \equiv \prod_{j=1}^{k-1} \rho_{\sigma_j}(h^{-1} K_R^j h(x, \theta)).$$

We then split $T^h_\lambda$ into $\sum_\sigma T^h_{\lambda, \sigma}$, multiplying the integral kernel of $T^h_\lambda$ by the functions $\rho_\sigma^h(x, \theta)$:

$$(2.12) \quad T^h_{\lambda, \sigma} u(x) = \int e^{i \lambda S(x, \theta)} \beta(h^{-1} h) \rho_\sigma^h(x, \theta) \psi(x, \theta) u(\theta) d\theta.$$

**Proposition 2.2.** The map $\pi_R |_\theta : x \mapsto S_\theta(x, \theta)$ satisfies the convexity condition (2.8) on the support of the integral kernel of each $T^h_{\lambda, \sigma}$.

This proposition allows the integration by parts in (2.7), and thus finishes the proof of Theorem 2.1. The proof of the proposition itself is based on two lemmas below.

We consider the map $\pi_R |_\theta : x \mapsto \eta = S_\theta(x, \theta)$ as the composition

$$(2.13) \quad \pi_R |_\theta : x \mapsto \pi' \mapsto (\eta' = S_{\theta'}, x_n) \mapsto \pi'' \mapsto (\eta', \eta_n = S_{\theta_n}).$$

Here $\eta_n$ is considered as a function of $\eta'$ and $x_n$: $\eta_n(S_{\theta'}(x, \theta), x_n) = S_{\theta_n}(x, \theta)$.

**Remark.** The kernel of the differential $d\pi^s$ is certainly generated by the vector $(\partial_{x_n})_{\eta'}$ (the subscript refers to choosing $\eta'$ and $x_n$ as the independent variables), and there is a convenient relation

$$K_R = (\partial_{x_n})_{\eta'},$$

which motivated the definition (2.9) of $K_R$.

Since $\det S_{\theta', \theta'} \neq 0$, the map $\pi'$ in (2.13) is a diffeomorphism (at least locally), and hence we may assume that it satisfies

$$(2.14) \quad |\pi'(x) - \pi'(y)| \geq \text{const} |x - y|.$$
Now we work in the \((\eta', x_n)\)-space; we need to show that \(\pi^*\) in (2.13) satisfies

\[
|\pi^*(\eta', x_n) - \pi^*(\zeta', y_n)| \geq \text{const} \cdot \text{dist}[(\eta', x_n), (\zeta', y_n)],
\]

for appropriate ranges of the values of \(\eta', \zeta', x_n, \) and \(y_n\).

We denote by \(\mathcal{L}\) the line segment from the point \((\eta', x_n)\) to \((\zeta', y_n)\). Since the first \(n - 1\) components of \(\pi^*\) are identities, the inequality (2.15) is trivially satisfied if \(\mathcal{L}\) is outside the conic neighborhood of magnitude \(c\hbar\) (where \(c > 0\) is to be chosen later) of the directions \(\pm (\partial_{x_n})_{\eta'}\) in the \((\eta', x_n)\)-space.

Now let \(\mathcal{L}\) be inside the \(\hbar\)-cone around \(\pm (\partial_{x_n})_{\eta'}\); then the value of \(|\eta' - \zeta'|\) is bounded by \(c\hbar|x_n - y_n|\). According to the Mean Value theorem applied to \(\eta_n(\eta', x_n)\), there is the following bound from below for the left-hand side of (2.15):

\[
\begin{align*}
|\eta_n(\eta', x_n) - \eta_n(\zeta', y_n)| \\
\geq |x_n - y_n| \cdot \inf_{\mathcal{L}} |(\partial_{x_n})_{\eta'} \eta_n| - |\eta' - \zeta'| \cdot \sup_{\mathcal{L}} |\nabla_{\eta'} \eta_n| \\
\geq |x_n - y_n| \cdot \left(\inf_{\mathcal{L}} |(\partial_{x_n})_{\eta'} \eta_n| - c\hbar \sup_{\mathcal{L}} |\nabla_{\eta'} \eta_n|\right).
\end{align*}
\]

If we show that \(\inf_{\mathcal{L}} |(\partial_{x_n})_{\eta'} \eta_n|\) is of magnitude \(\hbar\), then we may choose \(c\) sufficiently small so that the inequality (2.8) follows.

The value of the derivative \((\partial_{x_n})_{\eta'} \eta_n\) can be determined from the decomposition \(\pi_R|_{\varphi} = \pi^* \circ \pi'\). Considering the determinants of the Jacobi matrices, \(J(\pi_R|_{\varphi}) = J(\pi^*) \cdot J(\pi')\), we obtain \(h(x, \vartheta) = (\partial_{x_n})_{\eta'} \eta_n \cdot \det S_{x' \vartheta'}\). Hence,

**Lemma 1.** There is the relation \((\partial_{x_n})_{\eta'} \eta_n = \frac{h(x, \vartheta)}{\det S_{x' \vartheta'}}\).

Now we only need to check that \(h \geq \text{const} \hbar\) everywhere on \(\mathcal{L}\), if the length of \(\mathcal{L}\), \(|\mathcal{L}|\equiv \text{dist}[(\eta', x_n), (\zeta', y_n)]\), is sufficiently small.

**Lemma 2.** If \(|\mathcal{L}| \leq \frac{1}{17}\), then \(h \geq \frac{\hbar}{4}\) everywhere on \(\mathcal{L}\).

We thus admit that the line segment \(\mathcal{L}\) could be not entirely on the support of the integral kernel of \(T^h_{\lambda, \sigma}\), where \(h \geq \hbar/2\).

**Proof.** Since both \((\eta', x_n)\) and \((\eta', y_n)\) are on the support of the integral kernel of the operator \(T^h_{\lambda, \sigma}\) defined by (2.12), we have

\[
(2.17) \quad h \geq \frac{\hbar}{2} \quad \text{at the points (}\eta', x_n\text{) and (}\zeta', y_n\text{),}
\]

\[
(2.18) \quad \sigma_j \cdot K^j_{\lambda} h \geq -\hbar, \quad j < k, \quad \text{at the points (}\eta', x_n\text{) and (}\zeta', y_n\text{),}
\]
for all \( j < k \). Also, according to (2.10),

\[
|K^k h| \geq \kappa \quad \text{everywhere.}
\]

Let \( t \) be a parameter on the line segment \( L \), changing from \( t = 0 \) at the point \((\eta', x_n)\) to \( t = |L| \) at the point \((\zeta', y_n)\); \( \partial_t = (\partial_{x_n})_{\eta'} \). We consider \( h|_L \) as a function of \( t \). As long as \( L \) is in the \( ch \)-cone around \( \pm (\partial_{x_n})_{\eta'} \),

\[
\partial_t h = K^j h(x, \vartheta)|_L \quad \text{modulo terms of magnitude } \ ch,
\]

for any \( j \leq k \).

If \( c \) is sufficiently small, then due to the inequalities (2.17)-(2.20) we have

\[
h(0) \geq \frac{h}{2}, \quad h(|L|) \geq \frac{h}{2},
\]

\[
\sigma_j h^{(j)}(0) \geq -2h, \quad \sigma_j h^{(j)}(|L|) \geq -2h, \quad \text{for all } j < k,
\]

and also

\[
\sigma_k h^{(k)}(t) > -3h, \quad \text{for all } 0 \leq t \leq |L|,
\]

where \( \sigma_k \) is equal to 1 or \(-1\). For our convenience, we have weakened the bound in the right-hand side of (2.22). We will base the rest of the argument on the following elementary inequality:

**Lemma.** Let \( f(t) \in C^1([0, l]) \). If there is a uniform bound \( \sigma f'(t) \geq -\epsilon \), where \( \sigma \) is a constant equal to \( \pm 1 \) and \( \epsilon > 0 \), then

\[
\min[f(0), f(l)] - \epsilon l \leq f(t) \leq \max[f(0), f(l)] + \epsilon l, \quad \text{for any } 0 \leq t \leq l.
\]

From (2.22) and from the above Lemma (where we take \( \epsilon = 3h \)) we conclude that \( \sigma_{k-1} h^{(k-1)}(t) \geq -2h - 2h|L| \geq -3h \), for any \( t \) between 0 and \( |L| \). Continuing by induction, we conclude that \( \sigma_1 h'(t) \geq -3h \). Therefore, again from the above Lemma, we deduce that everywhere between 0 and \( |L| \)

\[
h(t) \geq \frac{h}{2} - 3h|L|,
\]

and this is not less than \( \frac{h}{4} \) as long as \(|L| \leq \frac{1}{12} \).  \( \square \)
3. Almost orthogonal decompositions near the critical variety

Now we consider the operator $T^{\hbar}_{\lambda}$ defined by (2.3):

$$T^{\hbar}_{\lambda} u(x) = \int e^{i\lambda S(x,\vartheta)} \psi(x,\vartheta) \bar{\beta}(\hbar^{-1} h(x,\vartheta)) u(\vartheta) d\vartheta, \quad \psi \in C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\beta \in C^\infty_{\text{comp}}(\mathbb{R})$, supp $\beta \subset [-2, 2]$. The support of this operator contains the critical variety $\Sigma = \{ \det S_{x,\vartheta}(x,\vartheta) = 0 \}$.

**Theorem 3.1.** If the projection $\pi_L$ is a Whitney fold and the projection $\pi_R$ is of type at most $k$, then, as long as $\hbar \geq \lambda^{-\frac{1}{2}}$, there is the following estimate:

$$\|T^{\hbar}_{\lambda}\|_{L^2 \rightarrow L^2} \leq \text{const} \lambda^{-\frac{n}{2}+\frac{1}{2}}. \tag{3.1}$$

**Remark.** According to (2.5), we only need the estimate (3.1) for $\hbar = \lambda^{-\frac{k}{2k+1}}$. We will prove Theorem 3.1 assuming that

$$\hbar \geq \lambda^{-\frac{k}{2k+1}}, \tag{3.2}$$

to avoid unnecessary details.

We start with the decomposition of $T^{\hbar}_{\lambda}$ into

$$T^{\hbar}_{\lambda} = \sum_{\sigma} T^{\hbar}_{\lambda,\sigma}, \quad \sigma = (\sigma_1, \ldots, \sigma_{k-1}), \quad \sigma_j = \pm 1,$$

with respect to the signs of the derivatives $K^j_{\hbar} h$, $1 \leq j \leq k - 1$: We use introduced earlier functions $\rho^{\hbar}_{\sigma}(x,\vartheta)$ (see (2.11)) and define

$$T^{\hbar}_{\lambda,\sigma} u(x) = \int e^{i\lambda S(x,\vartheta)} \bar{\beta}(\hbar^{-1} h) \rho^{\hbar}_{\sigma}(x,\vartheta) \psi(x,\vartheta) u(\vartheta) d\vartheta. \tag{3.3}$$

On the support of the integral kernel of $T^{\hbar}_{\lambda,\sigma}$ the following inequalities are satisfied:

$$\sigma_j K^j_{\hbar} h(x,\vartheta) \geq -\hbar, \tag{3.4}$$

for all $j$ between 1 and $k - 1$. Let us mention that for $k = 1$ no decomposition is needed.

We will consider the operators $T^{\hbar}_{\lambda,\sigma}$ with different sets $\sigma$ separately. Given $\sigma$, we decompose the corresponding $T^{\hbar}_{\lambda,\sigma}$ into

$$T^{\hbar}_{\lambda,\sigma} = \sum_{X \in \mathbb{Z}^n} \sum_{\Theta \in \mathbb{Z}^n} \left( T^{\hbar}_{\lambda,\sigma} \right)_{X\Theta}, \tag{3.5}$$
where \((\overline{T}^{\hbar}_{\lambda,\sigma})_{X \Theta}\) is an operator with the integral kernel
\[
(3.6) \quad \chi(h^{-\frac{\lambda}{2}} x - X) \cdot K \left(\overline{T}^{\hbar}_{\lambda,\sigma}\right)(x, \vartheta) \cdot \chi(h^{-1} \vartheta - \Theta).
\]

Here \(K \left(\overline{T}^{\hbar}_{\lambda,\sigma}\right)(x, \vartheta)\) stands for the integral kernel of \(T^{\hbar}_{\lambda,\sigma}\). The estimate on each \((\overline{T}^{\hbar}_{\lambda,\sigma})_{X \Theta}\) is straightforward: The mixed Hessian \(S_{x \vartheta}\) is of rank at least \(n - 1\), while the \(x\)-support of the integral kernel is of size \(h^k\), and \(\vartheta\)-support is of size \(h^{\frac{1}{2}}\). Therefore, according to Hörmander’s estimate for non-degenerate oscillatory integrals in \(\mathbb{R}^{n-1}\) [Hö71] (in \(x', \vartheta'\)-variables) and to the Schur lemma (in \(x_n, \vartheta_n\)-variables), we conclude that
\[
(3.7) \quad \| (\overline{T}^{\hbar}_{\lambda,\sigma})_{X \Theta} \| \leq \text{const} \lambda^{-\frac{n-1}{2}} (h^k h)^{\frac{1}{2}}.
\]

This agrees with (3.1).

The almost orthogonality of the pieces localized near different points in the \(\vartheta\)-space is easy to establish. This orthogonality is proved identically to the almost orthogonality of pieces \((\overline{T}^{\hbar}_{\lambda,\sigma})_{X \Theta}\) from Section 2; again, we refer to [Cu97]. Therefore, we can assume that \(\Theta\) is the same for all \((\overline{T}^{\hbar}_{\lambda,\sigma})_{X \Theta}\), and we only need to prove the almost orthogonality with respect to different values of \(X\). We put for brevity
\[
\overline{\tau}_X \equiv \left(\overline{T}^{\hbar}_{\lambda,\sigma}\right)_{X \Theta}, \quad \overline{\tau}_Y \equiv \left(\overline{T}^{\hbar}_{\lambda,\sigma}\right)_{Y \Theta};
\]
the values of \(\sigma\) and \(\Theta\) are assumed to be the same for the rest of the section.

We claim that these operators are almost orthogonal:

**Proposition 3.2.** The operators \(\overline{\tau}_X = \left(\overline{T}^{\hbar}_{\lambda,\sigma}\right)_{X \Theta}\) are almost orthogonal with respect to different values of \(X \in \mathbb{Z}^n\):
\[
\| \overline{\tau}_X^* \overline{\tau}_Y \|_{L^2 \to L^2}, \quad \| \overline{\tau}_X^* \overline{\tau}_Y \|_{L^2 \to L^2} \leq \text{const} \tau^2 |X - Y|^{-N}, \quad \text{for any } N > 0.
\]

Here \(\tau\) is the estimate (3.7) which is valid for each operator \(\overline{\tau}_X\).

Now the statement of Theorem 3.1 would follow from the Cotlar-Stein lemma.

**Proof.** It suffices to consider the almost orthogonality for \(|X - Y| \geq 2\sqrt{n} + 1\), when the integral kernels of \(\overline{\tau}_X\) and \(\overline{\tau}_Y^*\) have no common support in \(x\). The almost orthogonality is straightforward for the compositions \(\overline{\tau}_X^* \overline{\tau}_Y\); this leaves us with \(\overline{\tau}_X^* \overline{\tau}_Y^*\).
The integral kernel of \( \bar{\pi}_X \bar{\pi}_Y^* \) is given by

\[
K(\bar{\pi}_X \bar{\pi}_Y^*)(x, y) = \int d\vartheta e^{i\lambda(S(x, \vartheta) - S(y, \vartheta))} \times \ldots .
\]

It is convenient to fix \( \vartheta \) and to work in the space \((\eta', x_n)\), which is the image of the diffeomorphism \( \pi' : x \mapsto (\eta'(x) \equiv S_{\vartheta}(x, \vartheta), x_n) \), which already appeared in (2.13). We denote by \( \mathcal{L} \) the line segment from \((\eta'(x), x_n)\) to \((\eta'(y), y_n)\); \( |\mathcal{L}| \) stays for the length of \( \mathcal{L} \). Without the loss of generality we assume \( |\mathcal{L}| \leq 1 \).

We will consider two cases:

- The \textit{vertical case}, when the line segment \( \mathcal{L} \) is within the conic neighborhood of magnitude

\[
\alpha = c h^{1 - \frac{1}{k}}, \quad \text{for some small } c > 0,
\]

of the directions \( \pm (\partial_{x_n})_{\eta'} \).

Let \( t \) be a parameter on the line segment \( \mathcal{L} \), which changes from \( t = 0 \) at \( \pi'_R|_{\vartheta}(x) \) to \( t = |\mathcal{L}| \) at \( \pi'_R|_{\vartheta}(y) \). Since \( \pi' \) is a diffeomorphism, we may assume that \( c_1 |x - y| \leq |\mathcal{L}| \leq c_2 |x - y| \), for some constants \( c_2 > c_1 > 0 \), and since \( |x - y| \approx h^{\frac{1}{k}} |Y - X| \) (with the error of magnitude \( h^{\frac{1}{k}} \)), we have

\[
C_1 h^{\frac{1}{k}} |X - Y| \leq |\mathcal{L}| \leq C_2 h^{\frac{1}{k}} |X - Y|, \quad C_2 > C_1 > 0.
\]

We may consider \( h|_\mathcal{L} \) as a function of \( t \). Since \( \mathcal{L} \) is in the \( \alpha \)-cone around \( \pm (\partial_{x_n})_{\eta'} \),

\[
\partial_t^j h(t) = K^j(h(x, \vartheta)|_\mathcal{L}) \quad \text{modulo terms of magnitude } \alpha = c h^{1 - \frac{1}{k}}.
\]

Therefore, the values of the derivatives \( h^{(j)}(t) \) are close to the values of \( K^j(h(x, \vartheta)|_\mathcal{L}) \). Since \( \sigma_j K^j h \geq -h \) at the points \((x, \vartheta)\) and \((y, \vartheta)\) (which are on the support of the integral kernel of \( T^h_{\lambda, \sigma} \)), we can take \( c \) small enough so that for any \( j < k \)

\[
\sigma_j h^{(j)}(0) \geq -h^{1 - \frac{1}{k}}, \quad \sigma_j h^{(j)}(|\mathcal{L}|) \geq -h^{1 - \frac{1}{k}}.
\]

According to (2.10), \( |K^k h| \geq \kappa > 0 \); hence (if \( c \) is sufficiently small) we also know that

\[
|h^{(k)}(t)| \geq \frac{\kappa}{2} > 0, \quad \text{for any } t \text{ between } 0 \text{ and } |\mathcal{L}|.
\]

Let us show what restriction this imposes on \(|\mathcal{L}|\).
Lemma. Let \( f(t) \in C^k(\mathbb{R}) \). Assume that for some \( l \), \( 0 \leq l \leq 1 \), for some set of \( k-1 \) numbers \( \sigma_j = \pm 1 \), \( 1 \leq j \leq k-1 \), and for some \( \epsilon > 0 \) the following conditions are satisfied:

\[
\sigma_j f^{(j)}(0) \geq -\epsilon \quad \text{and} \quad \sigma_j f^{(j)}(l) \geq -\epsilon, \quad 1 \leq j < k,
\]

\[
|f^{(k)}(t)| \geq \kappa > 0 \quad \text{for} \ 0 \leq t \leq l.
\]

Then

\[
|f(l) - f(0)| \geq \frac{\kappa^l}{k!} - (k-1)\epsilon.
\]

Similar inequalities appeared in [Ch85] and [PhSt97].

Proof. Due to (3.14), the function \( f^{(k-1)}(t) \) is monotone. From (3.13) we know that \( \sigma_{k-1} f^{(k-1)} \geq -\epsilon \) at \( t = 0 \) and \( t = l \), and, since \( |f^{(k)}(t)| \geq \kappa \), we derive that

\[
\text{either} \quad \sigma_{k-1} f^{(k-1)}(t) \geq \kappa t - \epsilon \quad \text{or} \quad \sigma_{k-1} f^{(k-1)}(t) \geq \kappa (l-t) - \epsilon,
\]

for any \( t \) between \( 0 \) and \( l \), depending on the relation between the signs of \( f^{(k)}(t) \) and \( \sigma_{k-1} \).

Assume that \( \sigma_{k-1} = 1 \) and that the first inequality in (3.16) is satisfied. If \( \sigma_{k-2} = 1 \), then \( f^{(k-2)}(0) \geq -\epsilon \), and we have:

\[
f^{(k-2)}(t) \geq \frac{\kappa^2}{2} t^2 - \epsilon t + f^{(k-2)}(0) \geq \frac{\kappa^2}{2} - 2\epsilon, \quad 0 \leq t \leq l.
\]

If instead \( \sigma_{k-2} = -1 \), then from \( f^{(k-2)}(l) \leq \epsilon \) and \( f^{(k-1)}(t) \geq \kappa t - \epsilon \) we derive

\[
f^{(k-2)}(t) \leq -\frac{\kappa^2}{2} t^2 + \epsilon (l-t) + f^{(k-2)}(l) \leq -\frac{\kappa (l-t)^2}{2} + 2\epsilon, \quad 0 \leq t \leq l.
\]

All other cases are treated similarly; each time we end up with one of the following bounds on \( f^{(k-2)}(t) \):

\[
\text{either} \quad \sigma_{k-2} f^{(k-2)}(t) \geq \frac{\kappa^2}{2} - 2\epsilon \quad \text{or} \quad \sigma_{k-2} f^{(k-2)}(t) \geq \frac{\kappa(l-t)^2}{2} - 2\epsilon,
\]

depending on the relation between signs of \( \sigma_{k-1} \) and \( \sigma_{k-2} \), and which of the inequalities in (3.16) is valid. We continue by induction and conclude that

\[
\text{either} \quad \sigma_1 f'(t) \geq \frac{\kappa^{k-1}}{(k-1)!} - (k-1)\epsilon \quad \text{or} \quad \sigma_1 f'(t) \geq \frac{\kappa(l-t)^{k-1}}{(k-1)!} - (k-1)\epsilon.
\]
In either case, $|f(l) - f(0)| \geq \frac{\kappa l^k}{k!} - (k - 1)\epsilon l$. □

According to (3.11), (3.12), and to the above lemma (with $\epsilon = \hbar^{1-\frac{1}{k}}$),

$$|h(l) - h(0)| \geq \frac{\kappa |L|^k}{2k!} - (k - 1)\hbar^{1-\frac{1}{k}}|L|.$$  

Since the left-hand side could not be greater than $4\hbar$, there is the following restriction on the length of $L$:

(3.17) \hspace{1cm} |L| \leq \text{const } \hbar^{\frac{1}{k}}.

Therefore, according to (3.9), $|Y - X| \leq \text{const}$. We conclude that for sufficiently large values of $|X - Y|$ the line segment $L$ is only allowed to be outside the conic neighborhood of magnitude $c\hbar^{1-\frac{1}{k}}$ (for certain small constant $c$) of the directions $\pm (\partial_{x_n})_{\eta'}$ in the $(\eta', x_n)$-space.

- We are thus left to consider the *horizontal case*, when the line segment $L$ is outside the $\alpha$-cone around $\pm (\partial_{x_n})_{\eta'}$, where $\alpha = ch^{1-\frac{1}{k}}$ and $c > 0$ is some small constant:

(3.18) \hspace{1cm} |\eta'(y) - \eta'(x)| \geq \sin \alpha \cdot (|\eta'(y) - \eta'(x)|^2 + |y_n - x_n|^2)^{1/2}.

We use (3.9) and obtain

(3.19) \hspace{1cm} |\eta'(y) - \eta'(x)| \geq \text{const } \alpha \hbar^{\frac{1}{k}} |Y - X| \geq \text{const } \hbar |Y - X|.

We integrate in (3.8) by parts, with the aid of the operator

$$L_{\theta} = \frac{1}{i\lambda} \frac{(S_{\theta}(x, \vartheta) - S_{\theta}(y, \vartheta)) \cdot \nabla_{\theta}}{|S_{\theta}(x, \vartheta) - S_{\theta}(y, \vartheta)|^2}.$$  

Each derivative $\nabla_{\theta}$ contributes at most $\hbar^{-1}$. According to (3.19), this also includes the case when the derivative falls on the denominator of $L_{\theta}$ itself. Therefore, each integration by parts yields the factor

(3.20) \hspace{1cm} \frac{\text{const}}{\lambda \hbar \cdot \hbar |Y - X|}.

According to (3.2), $\hbar \geq \lambda^{-\frac{1}{m+1}}$, and therefore $\lambda \hbar^2 \geq \lambda^{1-\frac{2m+2}{m+1}} = \lambda^{\frac{1}{m+1}}$. Repeated integration by parts in (3.8) yields powers of (3.20), and we gain arbitrarily large negative powers of $\lambda$ and $|Y - X|$. This proves the required almost orthogonality relations for the operators $\bar{\tau}_X$ with different indices $X \in \mathbb{Z}^n$, and concludes the proof of Proposition 3.2. □
4. Sharpness of the results

Let us consider a particular oscillatory integral operator $T^{(1,2)}_\lambda$,

$$T^{(1,2)}_\lambda u(x) = \int e^{i\lambda S(x,\vartheta)} \psi(x,\vartheta) u(\vartheta) \, d\vartheta, \quad x, \vartheta \in \mathbb{R},$$

with the phase function given by

$$S(x,\vartheta) = x^3\vartheta - x\vartheta^2.$$ 

The function $\psi \in C^\infty_{\text{comp}}(\mathbb{R} \times \mathbb{R})$ is supported in the unit ball centered in the origin in $\mathbb{R} \times \mathbb{R}$. We assume that near the origin $\psi \equiv 1$.

The projections from the associated canonical relation are represented by the maps

$$\pi_L : (x,\vartheta) \mapsto (x, S_x = 3x^2\vartheta - \vartheta^2), \quad \pi_R : (x,\vartheta) \mapsto (\vartheta, S_\vartheta = x^3 - 2x\vartheta),$$

which have the singularities of the Whitney fold ($k = 1$) and the simple cusp ($k = 2$), respectively. This is represented by the superscript $(1,2)$. (Note that the determinants of the Jacobi matrices of both projections are equal to $h(x,\vartheta) = 3x^2 - 2\vartheta$, so that $\partial_\vartheta h \neq 0$, $\partial_x^2 h \neq 0$.) According to Theorem 1.2, $\|T^{(1,2)}_\lambda\| \leq \text{const} \lambda^{-\frac{3}{10}}$. We are going to prove that this estimate is optimal.

**Proposition 4.1.** *The optimal rate of decay of $\|T^{(1,2)}_\lambda\|_{L^2 \to L^2}$ equals $3/10$.***

Let us assume that the operator $T^{(1,2)}_\lambda$ is bounded from $L^2$ to $L^2$ by

$$\|T^{(1,2)}_\lambda\| \leq \text{const} \lambda^{-d},$$

where $d$ is some positive real number (which *a priori* could be greater than $3/10$). We consider the family of operators,

$$T^{(1,2)}_{\lambda,R} u(x) = \int e^{i\lambda S(x,\vartheta)} \psi \left( \frac{x}{R}, \frac{\vartheta}{R^2} \right) u(\vartheta) \, d\vartheta, \quad x, \vartheta \in \mathbb{R},$$

where $R \geq 1$. All these operators are bounded from $L^2$ to $L^2$ (as long as $R < \infty$). We would like to know the behavior of their norms as $\lambda$ and $R$ become large.

We rescale $x$ and $\vartheta$ with the aid of some $\mu > 0$:

$$T^{(1,2)}_{\lambda,R} u(\mu x) = \int e^{i\lambda S(\mu x,\mu^2 \vartheta)} \psi \left( \frac{\mu x}{R}, \frac{\mu^2 \vartheta}{R^2} \right) u(\mu^2 \vartheta) \, d(\mu^2 \vartheta) = \mu^2 \int e^{i\lambda \mu^5 S(x,\vartheta)} \psi \left( \frac{\mu x}{R}, \frac{\mu^2 \vartheta}{R^2} \right) u(\mu^2 \vartheta) \, d\vartheta.$$
We put $\mu = R$ and use the assumption (4.3), getting
\[(4.5) \quad \|T^{(1,2)}_{\lambda,R} u(Rx)\|_{L^2} \leq \text{const} R^2 (\lambda R^5)^{-d} \|u(R^2 \vartheta)\|_{L^2}.
\]

Now we rescale $x$ and $\vartheta$ “back” and keep track of the powers of $R$, obtaining
\[R^{-\frac{1}{2}} \|T^{(1,2)}_{\lambda,R} u(x)\|_{L^2} \leq \text{const} R^2 (\lambda R^5)^{-d} R^{-1} \|u(\vartheta)\|_{L^2},
\]
which gives the following bound on $T^{(1,2)}_{\lambda,R}$:
\[(4.6) \quad \|T^{(1,2)}_{\lambda,R}\| \leq R^{\frac{3}{2} - 5d} \text{const} \lambda^{-d}.
\]

Now let us argue that the exponent $d = 3/10$ is optimal. Assuming $d > 3/10$, we could conclude from (4.6) that $\|T^{(1,2)}_{\lambda,R}\| \to 0$ as $R$ becomes large (and $\lambda$ is fixed). At the same time, if we take a function $u(\vartheta)$ supported in a small neighborhood of $\vartheta = 0$, then the image $T^{(1,2)}_{\lambda,R} u(x)$ would not change the values, in some small neighborhood of $x = 0$, when $R$ grows up. Therefore, $\|T^{(1,2)}_{\lambda,R} u(x)\|_{L^2}$ could not decrease, and we are facing the contradiction.

**Remark 1.** A slight modification of the proof shows that the decay $\sim \lambda^{-0.3}$ is the sharp result for the decrease of the norm of $T^{(1,2)}_{\lambda}$, in the sense that for each $\lambda$ one can choose a function $u^{(\lambda)} \in C^\infty_{\text{comp}}(\mathbb{R})$ supported in a small neighborhood of the origin such that
\[\|T^{(1,2)}_{\lambda} u^{(\lambda)}(x)\|_{L^2} \geq c \lambda^{-0.3} \|u^{(\lambda)}(x)\|_{L^2},
\]
with the constant $c > 0$ independent on $\lambda$.

**Remark 2.** Since (4.6) is valid with $d = 3/10$, the norm of the operator $T^{(1,2)}_{\lambda,R}$ defined by (4.4) does not increase as $R$ becomes large: $\|T^{(1,2)}_{\lambda,R}\| \leq \text{const} \lambda^{-3/10}$ independent on a particular value of $R$. Hence, the operators $T^{(1,2)}_{\lambda,R}$ converge (in the weak $L^2 \to L^2$ operator topology) to the non-compactly supported oscillatory integral operator $\tilde{T}^{(1,2)}_{\lambda}$, defined by
\[(4.7) \quad \tilde{T}^{(1,2)}_{\lambda} u(x) = \int_{\mathbb{R}} e^{i\lambda(x^3 \vartheta - xx^2)} u(\vartheta) \, d\vartheta, \quad x, \vartheta \in \mathbb{R}.
\]

$\tilde{T}^{(1,2)}_{\lambda}$ extends to a continuous operator on $L^2$ with the norm $\text{const} \lambda^{-3/10}$. 
Models of operators with higher order singularities.

We generalize the previous example and construct the canonical relation with the projection $\pi_\varkappa$ being a fold and $\pi_\mu$ being a map with a Morin $S_{\varkappa_k}$-singularity.

We fix $n \geq k - 1$ and introduce the phase function $S(x, \vartheta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ given by the polynomial

\begin{equation}
S(x, \vartheta) = (x_n^{k+1} + x_n^{k-1}x_{n-1} + \cdots + x_n^2x_{n-k+2})\vartheta_n + x_n \frac{\vartheta_n^2}{2} + x' \cdot \vartheta', \quad x = (x', x_n) \in \mathbb{R}^n, \quad \vartheta = (\vartheta', \vartheta_n) \in \mathbb{R}^n.
\end{equation}

The map $\pi_{\varkappa_R}$,

\[
\begin{bmatrix}
  x' \\
  x_n \\
  \vartheta' \\
  \vartheta_n
\end{bmatrix} \mapsto 
\begin{bmatrix}
  \vartheta' \\
  \vartheta_n
\end{bmatrix} = \begin{bmatrix}
  \vartheta' \\
  \vartheta_n
\end{bmatrix},
\]

has the canonical form [Mo65] of a map with a Morin $S_{\varkappa_k}$-singularity at the origin. Then, since $\det d\pi_\varkappa = \det d\pi_\mu = (k+1)x_n^{k+1} + \cdots + \vartheta_n$ vanishes of the first order in the direction of the kernel of $\pi_\varkappa$ (which is generated at $x = \vartheta = 0$ by $\vartheta_n$), $\pi_\varkappa$ is a Whitney fold.

We consider the oscillatory integral operator with the phase function (4.8),

\begin{equation}
T^{(1,k)}_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(x, \vartheta) u(\vartheta) d\vartheta, \quad x, \vartheta \in \mathbb{R}^n,
\end{equation}

where $\psi$ is a smooth function supported near the origin. According to Theorem 1.2,

\[\|T^{(1,k)}_\lambda\| \leq \text{const } \lambda^{-\frac{4}{2} + (4 + \frac{2}{k})^{-1}}.\]

**Proposition 4.2.** The optimal rate of decay of $\|T^{(1,k)}_\lambda\|$ equals $\frac{4}{2} - (4 + \frac{2}{k})^{-1}$.

**Proof.** The proof is similar to the proof of Proposition 4.1. If we rescale $x_n \mapsto \mu x_n$, then for $S$ to be homogeneous in $\mu$ we need to rescale $x$ and $\vartheta$ as follows:

\[x \to X_\mu(x) = (\mu^n x_1, \ldots, \mu^{n-j+1} x_j, \ldots, \mu x_n), \quad \vartheta \to \Theta_\mu(\vartheta) = (\mu^{2k+1-n} \vartheta_1, \ldots, \mu^{2k-n+j} \vartheta_j, \ldots, \mu^{k-1} \vartheta_{n-1}, \mu^k \vartheta_n).\]

We then have $S(X_\mu(x), \Theta_\mu(\vartheta)) = \mu^{2k+1} S(x, \vartheta)$. We define

\[T^{(1,k)}_{\lambda, R} u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, \vartheta)} \psi(X_{R^{-1}}(x), \Theta_{R^{-1}}(\vartheta)) u(\vartheta) d\vartheta.\]
We proceed similarly to the proof of Proposition 4.1 and obtain
\[ \| T^{(1,k)}_{\lambda,R} \| \leq \text{const}(\lambda R^{2k+1})^{-d} \left( \left| \frac{\partial X_R}{\partial x} \right| \right)^{\frac{1}{2}} \left( \left| \frac{\partial \Theta_R}{\partial \vartheta} \right| \right)^{\frac{1}{2}}. \]

Here \( \left| \frac{\partial X_R}{\partial x} \right|, \left| \frac{\partial \Theta_R}{\partial \vartheta} \right| \) are the determinants of the Jacobi matrices of the maps \( X_\mu(x) \) and \( \Theta_R(\vartheta) \) (which only depend on \( R \)). To simplify the rest, we notice that
\[ \frac{\partial X_R}{\partial x} \cdot \frac{\partial \Theta_R}{\partial \vartheta} = \text{diag}(R^{2k+1}, \ldots, R^{2k+1}, R^{k+1}), \]
and hence
\[ \| T^{(1,k)}_{\lambda,R} \| \leq \text{const}(\lambda R^{2k+1})^{-d} R^{n(2k+1) - k}. \]

Since the norm of \( T^{(1,k)}_{\lambda,R} \) can not decrease when \( R \) becomes large (according to the same arguments as in the proof of Proposition 4.1), we conclude that the rate of decay \( d \) can not be larger than \( \frac{n}{2} - \frac{k}{2(2k+1)}. \) \( \square \)

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REFERENCES

[Ch85] M. Christ, *Hilbert transforms along curves*, Ann. of Math. **122** (1985), 575–596.

[Co97] A. Comech, *Integral operators with singular canonical relations*, Spectral theory, microlocal analysis, singular manifolds (M. Demuth, E. Schrohe, B.-W. Schulze, and J. Sjöstrand, eds.), Akademie Verlag, Berlin, 1997, pp. 200–248.

[Cu97] S. Cuccagna, *\( L^2 \) estimates for averaging operators along curves with two sided \( k \) fold singularities*, Duke Journal **89** (1997), 203–216.

[GrSe94] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities*, J. Reine Angew. Math. **455** (1994), 35–56.

[GrSe98] ______, *Fourier integral operators with cusp singularities*, Amer. J. Math. **120** (1998), 1077–1119.

[Hö71] L. Hörmander, *Fourier integral operators*, Acta Math. **127** (1971), 79–183.

[Hö85] ______, *The analysis of linear partial differential operators*, Springer-Verlag, 1985.

[Me76] R.B. Melrose, *Equivalence of glancing hypersurfaces*, Invent. Math. **37** (1976), 165–191.

[MeTa85] R.B. Melrose and M.E. Taylor, *Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle*, Adv. in Math. **55** (1985), 242–315.

[Mo65] B. Morin, *Canonical forms of the singularities of a differentiable mapping*, C. R. Acad. Sci. Paris **260** (1965), 6503–6506.
Y.B. Pan and C.D. Sogge, Oscillatory integrals associated to folding canonical relations, Colloq. Math. 61 (1990), 413–419.

D.H. Phong, Singular integrals and Fourier integral operators, Essays on Fourier Analysis in honor of Elias M. Stein (C. Fefferman, R. Fefferman and S. Wainger, eds.), Princeton Univ. Press, 1994, pp. 287–320.

D.H. Phong and E.M. Stein, Radon transform and torsion, Internat. Math. Res. Notices 4 (1991), 49–60.

D.H. Phong and E.M. Stein, Models of degenerate Fourier integral operators and Radon transforms, Ann. of Math. 140 (1994), 703–722.

D.H. Phong and E.M. Stein, Newton polyhedron and oscillatory integral operators, Acta Math 179 (1997), 105–152.

A. Seeger, Degenerate Fourier integral operators in the plane, Duke Math. J. 71 (1993), 685–745.

A. Seeger, Radon transforms and finite type conditions, Journal Amer. Math. Soc. 11 (1998), 869-897.

E.M. Stein, Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, 1993.

R. Thom, Les singularités des applications differentiables, Ann. Inst. Fourier 6 (1963), 43–87.