A Tulczyjew triple for classical fields

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Abstract

The geometrical structure known as the Tulczyjew triple has proved to be very useful in describing mechanical systems, even those with singular Lagrangians or subject to constraints. Starting from basic concepts of the variational calculus, we construct the Tulczyjew triple for first-order field theory. The important feature of our approach is that we do not postulate ad hoc the ingredients of the theory, but obtain them as unavoidable consequences of the variational calculus. This picture of field theory is covariant and complete, containing not only the Lagrangian formalism and Euler–Lagrange equations but also the phase space, the phase dynamics and the Hamiltonian formalism. Since the configuration space turns out to be an affine bundle, we have to use affine geometry, in particular the notion of the affine duality. In our formulation, the two maps \( \alpha \) and \( \beta \) which constitute the Tulczyjew triple are morphisms of double structures of affine-vector bundles. We also discuss the Legendre transformation, i.e. the transition between the Lagrangian and the Hamiltonian formulation of the first-order field theory.

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1. Introduction

Variational calculus is a natural language for describing statics of mechanical systems. All mathematical objects that are used in statics have direct physical interpretations. Moreover, similar mathematical tools are also widely used in other theories, like dynamics of particles or field theories. In classical mechanics, variational calculus was first used for deriving equations of motion of mechanical systems, i.e. the Euler–Lagrange equations.

In numerous works by Tulczyjew, for example in the book [45] and papers [39–44], one may find another philosophy of using variational calculus in mechanics and field theories. This philosophy, especially the one leading to the construction called the Tulczyjew triple, has been recently recognized by many theoretical physicists and mathematicians. The main advantage of the approach developed by Tulczyjew and his collaborators is its generality.
For example, using the Tulczyjew triple for autonomous mechanics, we can derive the phase equations for systems with singular Lagrangians and understand properly the Hamiltonian description of such systems. One can even discuss systems with more general generating objects than just a Lagrangian function, e.g., systems described by a family of Lagrangians or a Lagrangian function defined on a submanifold. For details, we refer to [47].

Another advantage of Tulczyjew’s approach is its flexibility. Being based on well-defined general principles, it can be easily adapted to different settings. No wonder that there were many attempts to generalize the Tulczyjew triple to more general contexts of mechanics on algebroids or different field theories (see e.g. [6, 33, 36]). In our earlier paper [16], we started with constructing a toy model of the triple for field theory in the simplest topological situation.

The purpose of this work is to construct the Tulczyjew triple for first-order field theory in a very general setting, i.e., in the case where fields are sections of some differential fibration with no additional structure assumed. The origins of the geometric structures we study lie in the rigorous formulation of the variational principle including boundary terms. We pay much attention to recognize physically important objects, like the phase space, phase dynamics, Legendre map, Hamiltonians, etc. These issues are usually not well elaborated in the literature, as the classical field theory models used to concentrate on the Euler–Lagrange equations. Of course, we also recover the commonly accepted Euler–Lagrange equations, this time without requiring any regularity of the Lagrangian.

Classical field theory is usually associated with the concept of multisymplectic structure. The literature on the subject is very rich, so we mention only a few main papers. The multisymplectic approach appeared first in the papers of the ‘Polish school’ [9, 25–27, 42]. Then, it was developed by Gotay, Isenberg, Marsden and others in [12–15]. The original idea of the multisymplectic structure has been thoroughly investigated and developed by many authors, see e.g., papers by Cariñena, Crampin, Ibort, Cantrijn, de León [2–4] and Echeverría-Enríquez, Muñoz-Lecanda [6] for general analysis of the multisymplectic structure and its application to the classical field theories, and by Forger, Paufler, Römer [7, 8], or Vankerschaver, Cantrijn, de León [49] for the discussion of more detailed problems associated with the structure. An interesting discussion of the problem can also be found in [24]. The Tulczyjew triple in the context of multisymplectic field theories appeared recently in [33]. One can also find a similar diagram, but with differences on the Hamiltonian side, in [10] (see also [11]). Another approach to field theory, based on differential forms on fiber bundles, is present in works by Krupková and collaborators, e.g., [30–32].

Our approach to the Lagrangian and Hamiltonian formalism developed in the paper is different. We do not use directly the multisymplectic formalism, building instead the triple out of natural morphisms of double structures of affine-vector bundles. Also we do not use the framework based on Klein’s ideology and do not concentrate on the Euler–Lagrange equations nor regular Lagrangians, since the phase dynamics is for us the principal object. Using the affine geometry as a tool and following guidelines of variational calculus, we arrive at spaces and maps on the Hamiltonian side of the triple. The variational problem we start with determines uniquely the phase space together with its canonical structure which is different from the one in [10]. Moreover, the canonical structure of the phase space is not a multisymplectic form, but a family of symplectic forms on fibers over the base manifold with values in the space of forms on the base. The two structures are related, but not identical. As far as we know, a similar research is being done independently, for instance, by Vitagliano [50, 51], and Guzmán. Just before submitting this paper, we spotted a preprint by Campos, Guzmán and Marrero [5] dealing with similar questions.

The starting point of our studies is a locally trivial fibration $\zeta : E \to M$ over a manifold $M$ of dimension $m$, whose sections represent fields, and the corresponding bundle $J^1E$ of first jets
of sections playing the role of kinematic configurations. A Lagrangian is a map \( L : J^1E \rightarrow \Omega^m \), where \( \Omega^k := \bigwedge^k T^*M \) is the bundle of \( k \)-forms on \( M \). The phase space of the theory turns out to be the bundle \( \mathcal{P} = V^*E \otimes_{\mathcal{E}} \xi^*(\Omega^{m-1}) \) denoted simply \( \mathcal{P} = V^*E \otimes_{\mathcal{E}} \Omega^{m-1} \). Here, \( V^*E \) is the dual of the vertical subbundle \( VE \) in \( TE \) and \( \xi^*(\Omega^{m-1}) \) is the pullback bundle of \( \Omega^{m-1} \) along the projection \( \xi : E \rightarrow M \).

The Lagrangian side of the Tulczyjew triple is constituted by a map
\[
\alpha : J^1\mathcal{P} \rightarrow V^*J^1E \otimes_{\mathcal{J}^1E} \Omega^m
\]
being a morphism of double structures of affine vector bundles associated with fibrations over \( \mathcal{P} \) and \( J^1E \). The vertical derivative \( dL \) of the Lagrangian is a section of the bundle \( V^*J^1E \otimes_{\mathcal{J}^1E} \Omega^m \rightarrow J^1E \) and the (implicit) phase dynamics is defined as a subset \( D \) of \( J^1\mathcal{P} \) being the inverse image by \( \alpha \) of the image of \( dL \), i.e. \( D = \alpha^{-1}(dL(J^1E)) \).

Similarly, the Hamiltonian side of the triple is built on another morphism of affine-vector bundles, fibered also over \( \mathcal{P} \) and \( J^1E \),
\[
\beta : J^1\mathcal{P} \rightarrow \mathcal{P}J^1E,
\]
where \( J^1E \) is the ‘affine dual’ of \( J^1E \), i.e. the bundle of affine maps from \( J^1E \) to \( \Omega^m \), and \( \mathcal{P}J^1E \) is the affine phase bundle of the affine line bundle \( \theta : J^1E \rightarrow \mathcal{P} \), an affine analog of the cotangent bundle. A Hamiltonian is a section of the bundle \( \theta \), i.e. \( H : \mathcal{P} \rightarrow J^1E \), so its affine differential \( dH \) can be viewed as a map \( dH : \mathcal{P} \rightarrow \mathcal{P}J^1E \) and defines a phase dynamics \( D = \beta^{-1}(dH(\mathcal{P})) \).

Like in the classical Tulczyjew triple, the double bundles \( V^*J^1E \otimes_{\mathcal{J}^1E} \Omega^m \) and \( \mathcal{P}J^1E \) are canonically isomorphic. On the other hand, the morphisms \( \alpha \) and \( \beta \) are no longer isomorphisms. It is nothing strange, since generalized objects usually do not posses all properties of their ancestors. However, one can obtain isomorphisms passing to some quotient bundle of \( J^1\mathcal{P} \) but for the price of loosing its natural interpretation as the first jet bundle, i.e. as representing differential operators. Note that in other generalizations of the Tulczyjew triple, e.g. related to a Dirac algebroid [17], the sides of the Tulczyjew triple are made out of relations, so not even maps.

It is obvious that being presented in a coordinate-free form, the whole theory is covariant. We would also like to stress that all the geometrical objects we construct are not just postulated \textit{ad hoc}, but discovered by starting from natural general principles and rigorous investigations of the geometry which arises in this way. Another nice feature of our approach is that it admits a straightforward generalization for field theories of higher orders. We postpone these issues, however, to a separate paper.

This work is organized as follows. In section 2, we present the conceptual background of variational theories which justifies mathematical constructions and physical interpretations in particular examples. After introducing some notation in section 3, we pass to the Lagrangian side of the triple in section 4. Section 5 is devoted to the Hamiltonian side of the triple.

2. Variational calculus in statics and mechanics

Let us start with the simplest case of statics of a mechanical system. We will assume that the set of all possible configurations of the system is a differential manifold \( Q \). The tangent and cotangent bundles
\[
\tau_Q : TQ \rightarrow Q, \quad \pi_Q : T^*Q \rightarrow Q
\]
will also be used. In statics, we are usually interested in equilibrium configurations of an isolated system, as well as a system with an interaction with other static systems. The system
alone or in interaction is examined by performing processes and calculating the cost of each process. We assume that all the processes are quasistatic, i.e. they are slow enough to produce negligible dynamical effects. Every process can be represented by a one-dimensional smooth oriented submanifold with boundary. It may happen that for some reasons, not all the processes are admissible, i.e. the system is constrained. All the information about the system is therefore encoded in three objects: the configuration manifold $Q$, the set of all admissible processes and the cost function that assigns a real number to every process. The cost function should fulfill some additional conditions, e.g. it should be additive in the sense that if we break a process into two subprocesses, then the cost of the whole process should be equal to the sum of the costs of the two subprocesses. Usually we assume that the cost function is local, i.e. for each process it is an integral of a certain positively homogeneous function $W$ on $TQ$.

There are distinguished systems, called regular, for which all the processes are admissible and the function $W$ is the differential of a certain function $U: Q \to \mathbb{R}$. In this case, $U$ is called the internal energy function.

An equilibrium point for the system is such a point $q \in Q$ that all the processes starting from $q$ have positive cost, at least initially, i.e. for some sufficiently small subprocess with the same initial point. Usually we formulate only a first-order necessary criterion for the equilibrium point. It states that a point $q$ is an equilibrium point of the system if

$$W(\delta q) \geq 0$$

for all vectors $\delta q \in T_qQ$ tangent to admissible processes. Vectors tangent to admissible processes are called admissible virtual displacements. The set of such vectors will be denoted with $\Delta$. It may happen that $\Delta \cap TQ$ does not project on the whole $Q$. We have then the set of admissible configurations $C = \tau_Q(\Delta \cap TQ)$. For regular systems, the equilibrium condition assumes the form

$$dU(q) = 0.$$ 

We examine the interaction between two systems by creating composed systems. We can compose systems that have the same configuration space $Q$. The composite system is described by the intersection of the sets of admissible processes and the sum of the cost functions. We describe our system by making a list of all systems that, composed with our system, have certain admissible configuration $q$ as an equilibrium point. We observe that at each admissible $q$, all the external systems interacting with our systems can be classified according to their influence on our system. Moreover, in every class, we can find a regular system; therefore, the whole class can be represented by the differential of the internal energy of that regular system. We call a force the class of external systems interacting with our system. The force is represented by a covector, i.e. an element of $T^*Q$. Instead of making a list of all external systems in equilibrium with our system at the point $q$, we can give a subset of $T^*_qQ$ representing those systems. We call the constitutive set the subset of $T^*_Q$ of all forces in equilibrium with our system at all admissible points. For a large class of static systems, the constitutive set contains the complete information about the system. The passage from the triple $(Q, \Delta \subset TQ, W)$ describing our system to the constitutive set $D \subset T^*Q$ is called the Fenchel–Legendre transformation.

Let us now use the above concepts to describe the autonomous dynamics of a non-relativistic particle. For simplicity, we will consider only the unconstrained case. Let us assume that the set of positions of the particle (possible configurations) is a smooth manifold $Q$. There are at least two approaches to the problem. The first deals with the finite time interval $[t_0, t_1]$, while the second with the infinitesimal time interval represented by the Dirac $\delta$-distribution at $t$. The finite case provides a useful representation of objects coming from statics, while an infinitesimal approach leads to differential equations for phase trajectories that are commonly
used in physics. We skip the details of the construction and provide here only a summary of the results of both approaches.

For the finite time interval, the configuration space is the space of all motions, i.e. pieces of smooth curves parameterized by the time \( \gamma : [t_0, t_1] \ni t \rightarrow \mathcal{Q} \). This space is no longer a standard, i.e. not a finite-dimensional nor a Banach, manifold; therefore, we have to precise the notions of a smooth function, a tangent vector and a covector. We will work with functions (usually called functionals) of the form

\[
S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t)) \, dt. \tag{2.1}
\]

In the above formula, \( L \) is a smooth function on \( T\mathcal{Q} \), called the Lagrangian, and \( \gamma(t) \) denotes the tangent prolongation of the curve \( \gamma \). A curve in the configuration space always comes from a homotopy, i.e. from a smooth map

\[
\chi : \mathbb{R}^2 \ni (s, t) \mapsto \chi(s, t) \in \mathcal{Q}.
\]

Restricting the domain of \( t \) to \([t_0, t_1]\) for every \( s \), we obtain a curve in the space of motions that is smooth by definition. The choice is justified by the fact that the composition of the curve with any function of the form (2.1) is a real function smooth in the usual sense.

A vector tangent to a manifold is usually defined as an equivalence class of curves. In our situation, we can adopt the same definition. Working with equivalence classes is difficult; therefore, we observe that each equivalence class at a configuration \( \gamma \) can be conveniently represented as a curve

\[
\delta \gamma : [t_0, t_1] \rightarrow T\mathcal{Q} \tag{2.2}
\]

such that \( t_0 \circ \delta \gamma = \gamma \). In differential geometry, we define covectors as equivalence classes of functions. Equivalence classes of functions on the space of motions are again too abstract objects; therefore, we need a convenient representation for them. The idea of such a representation is given by performing variations of the functional \( S \) and separating boundary terms like in the procedure of deriving the Euler–Lagrange equations:

\[
\langle \delta S, \delta \gamma \rangle = \int_{t_0}^{t_1} \langle \varepsilon L(t^2 \gamma(t)) \rangle \, dt + \langle PL(\gamma(t_1)), \delta \gamma(t_1) \rangle - \langle PL(\gamma(t_0)), \delta \gamma(t_0) \rangle, \tag{2.3}
\]

where \( \varepsilon L \) denotes the Euler–Lagrange variation of \( L \) that depends on the second prolongation \( t^2 \gamma \) of the motion \( \gamma \), and \( PL \) is a vertical differential of \( L \) with respect to the projection \( \pi \). We see that the convenient representation of a covector on the space of motions, i.e. an equivalence class of functions of the form (2.1) is a triple \((f, p_0, p_1)\), where

\[
f : [t_0, t_1] \rightarrow T^*\mathcal{Q}, \quad \pi \circ f = \gamma, \quad p_0 \in T_{\gamma(t_0)}^*\mathcal{Q}, \quad p_1 \in T_{\gamma(t_1)}^*\mathcal{Q}. \tag{2.4}
\]

The evaluation between \( \delta \gamma \) and \((f, p_0, p_1)\) is given by

\[
\langle (f, p_0, p_1), \delta \gamma \rangle = \int_{t_0}^{t_1} (f(t), \delta \gamma(t)) \, dt + \langle p_1, \delta \gamma(t_1) \rangle - \langle p_0, \delta \gamma(t_0) \rangle. \tag{2.5}
\]

The elements \( f, p_0 \) and \( p_1 \) have physical interpretation. The curve \( f \) is an external force acting on the particle during its motion, and \( p_0 \) and \( p_1 \) are the initial and the final momenta. The constitutive set consists of all triples \((f, p_0, p_1)\) such that the particle moves along the curve \( \gamma = \pi \circ f \), starting at \( \gamma(t_0) \) with the initial momentum \( p_0 \) and arriving at \( \gamma(t_1) \) with the final momentum \( p_1 \), while it is a subject to the force \( f \) along the motion. Having the constitutive set, we can discuss the isolated system, i.e. systems with the external force \( f = 0 \), as well as the system interacting with external forces of different kinds. We may interpret the momenta as a result of an interaction between the system and its past and its future. Therefore, it makes
no sense to keep the momenta equal to zero. The space of momenta is usually called the phase space of the system. We see that in our example the phase space is $T^*Q$.

The constitutive set, as defined above, is a complicated object. We would like to describe it in a more convenient way, e.g. using differential equations for curves in forces and momenta such that their solutions restricted to any interval $[t_0, t_1]$ lie in the constitutive set. We can obtain such equations using the infinitesimal approach to the dynamics.

Passing to the infinitesimal formulation, we replace the finite domain of the integration, $[t_0, t_1]$, with the Dirac $\delta$-distribution at the point $t$. We see that the configurations are now elements of $TQ$. Since the configuration space is again a manifold, we have natural notions of smooth functions, curves, tangent vectors and covectors. The ‘internal energy’ function is now just the Lagrangian, and its differential is an element of $T^*TQ$. Virtual displacements are vectors tangent to curves in $TQ$, i.e. elements of $TTQ$. We observe, however, that studying convenient representations of vectors and covectors for the finite formulation gives interesting results also in the infinitesimal limit. A virtual displacement of a configuration $\gamma(t)$ is a vector $\delta\gamma(t)$ in $TTQ$ such that

$$
\tau_{TQ}(\delta\gamma(t)) = \gamma'(t), \quad \tau_{TQ}(\delta\gamma(t)) = \delta\gamma(t).
$$

(2.6)

In turn, the infinitesimal limit of the virtual displacement $\delta\gamma$ that we had for the finite time interval is an element $b\delta\gamma(t) \in TTQ$ such that

$$
\tau_{TQ}(b\delta\gamma(t)) = \delta\gamma(t), \quad \tau_{TQ}(b\delta\gamma(t)) = \gamma'(t).
$$

(2.7)

The virtual displacement and its convenient representation are two elements of $TTQ$ related by the canonical flip

$$
\kappa_M : TTQ \longrightarrow TTQ.
$$

The constitutive set for our regular system in the infinitesimal setting is the graph of $dL$. Again, the convenient representation of elements of the cotangent bundle in the finite time interval formulation provides us with another useful interpretation of the constitutive set. For the finite time interval, the evaluation of $(f, p_0, p_1)$ on $\delta\gamma$ reads

$$
\int_{t_0}^{t_1} \langle f(t), \delta\gamma(t) \rangle \, dt + \langle p_1, \delta\gamma(t_1) \rangle - \langle p_0, \delta\gamma(t_0) \rangle.
$$

(2.8)

In the infinitesimal case, we get

$$
\langle f(t), \delta\gamma(t) \rangle + \frac{d}{dt} \langle p(t), \delta\gamma(t) \rangle.
$$

(2.9)

In the absence of external forces, another description of a constitutive set can be derived out of the equation

$$
\frac{d}{dt} \langle p(t), \delta\gamma(t) \rangle = \langle dL(\gamma'(t)), \delta\gamma(t) \rangle.
$$

(2.10)

The left-hand side is the so-called tangent evaluation between a vector tangent to $T^*Q$ and a vector tangent to $TQ$ such that they have common projections on $TQ$. More precisely, if $p : \mathbb{R} \to T^*Q$ and $\delta\gamma : \mathbb{R} \to TQ$ are two curves covering the same curve $\gamma : \mathbb{R} \to Q$, then

$$
\langle [tp(t), b\delta\gamma(t)] \rangle = \frac{d}{dt} \langle p(t), \delta\gamma(t) \rangle.
$$

(2.11)

We have used here the same kind of brackets $\langle \cdot, \cdot \rangle$ as in (2.5), because the above tangent evaluation is an infinitesimal version of the evaluation given in (2.5) in the case when $f = 0$. Since the vectors $b\delta\gamma(t)$ and $\delta\gamma(t)$ are related by the canonical flip $\kappa_Q,$

$$
\kappa_Q(\delta\gamma(t)) = b\delta\gamma(t),
$$

where
the differential of the Lagrangian \( dL(\gamma(t)) \) and the tangent vector \( tp(t) \) are related by the Tulczyjew \( \alpha_Q \) (which is dual to \( \kappa_Q \)),
\[
\alpha_Q: T^*TQ \longrightarrow T^*Q.
\]
In this way, we have obtained another description of the constitutive set, called the phase dynamics and given by the formula
\[
T^*M \supset D = \alpha_Q^{-1}(dL(TQ)).
\]

If the system is autonomous, then the constitutive set for any time \( t \) is the same. The condition for a curve \( p: \mathbb{R} \supset I \rightarrow T^*Q \) to be a phase trajectory of the system is that
\[
\forall t \in I \quad tp(t) \in D.
\]
The dynamics \( D \) can be understood as a differential equation for a curve, \( p: \mathbb{R} \rightarrow T^*Q \), covering the same curve in \( Q \). A curve \( \gamma: I \rightarrow Q \) satisfies, in turn, the corresponding Euler–Lagrange equation, if the curve \( I \ni t \mapsto \alpha_Q^{-1}(dL(\gamma(t))) \in TT^*Q \) is the tangent prolongation of its projection to \( T^*Q \).

External forces can be included in the picture as follows. Equation (2.10) completed with the force reads as
\[
(f(t), \delta \gamma(t)) + \frac{d}{dt}(p(t), \delta \gamma(t)) = (dL(\gamma(t)), \delta t \gamma(t)).
\]
(2.12)
The set of all elements of \( TT^*Q \) with fixed projections on \( T^*Q \) and \( TQ \) is an affine space modeled on the appropriate fiber of \( T^*Q \). The force \( f \) can be therefore added to a vector tangent to the phase space. The map \( \alpha_Q \) can now be extended to the map
\[
\tilde{\alpha}_Q: T^*Q \times Q TT^*Q \longrightarrow T^*TQ, \quad \tilde{\alpha}_Q(f, u) = \alpha_Q(u + f).
\]
(2.13)
The dynamics with external forces (see [34]) is a subset \( \tilde{D} \) of \( T^*Q \times Q TT^*Q \) given by
\[
\tilde{D} = \tilde{\alpha}_Q^{-1}(dL(TQ)).
\]
(2.14)
All the structures needed for generating the dynamics from a Lagrangian (without external forces) can be summarized in the following commutative diagram of vector bundle morphisms
\[
\begin{array}{ccc}
TT^*Q & \xrightarrow{\alpha_Q} & T^*TQ \\
\downarrow{\tau_{T^*Q}} & & \downarrow{\tau_{T^*TQ}} \\
T^*Q & \xrightarrow{id} & T^*Q \\
\downarrow{\tau_Q} & & \downarrow{\tau_Q} \\
TQ & \xrightarrow{id} & TQ \\
\downarrow{\pi_Q} & & \downarrow{\pi_Q} \\
Q & & Q
\end{array}
\]
(2.15)
The map \( \alpha_Q \) is an isomorphism of double vector bundles. Recall that double vector bundles are, roughly speaking, manifolds equipped with two compatible vector bundle structures. The compatibility condition can be expressed e.g. as the commutation of the two Euler vector fields associated with these vector bundle structures. A precise definition of a double vector bundle together with its basic properties can be found in [35, 29, 22].

The map \( \alpha_Q \) is also a symplectomorphism between the symplectic manifolds \( (TT^*Q, d\tau_Q \omega_Q) \) and \( (T^*TQ, \omega_{TQ}) \), where \( d\tau_Q \omega_Q \) is the complete lift of the canonical symplectic form \( \omega_Q \) on \( T^*Q \) and \( \omega_{TQ} \) is the canonical symplectic form on \( T^*TQ \).
It may happen that the phase dynamics is an implicit differential equation, i.e. it is not the image of a vector field. In some cases, however, the phase dynamics is the image of a Hamiltonian vector field for some function $H : T^*Q \to \mathbb{R}$. So that we can write
\[
D = \beta_Q^{-1}(dH(T^*Q)),
\]
where $\beta_Q$ is the canonical isomorphism between $TT^*Q$ and $T^*T^*Q$ given by the canonical symplectic form $\omega_Q$ on $T^*Q$:
\[
\beta_Q : T^*T^*Q \longrightarrow T^*T^*Q, \quad \langle \beta_Q(v), w \rangle = \omega_Q(v, w).
\]
Let us recall that the canonical symplectic form $\omega_Q$ is defined as the differential
\[
\omega_Q = d\vartheta_Q
\]
of the Liouville form $\vartheta_Q$ given by
\[
\vartheta_Q(v) = \langle \tau_{T^*Q}(v), T\pi_Q(v) \rangle.
\]
The structures needed for Hamiltonian mechanics can be presented in the following commutative diagram:

\[
\begin{array}{c}
T^*T^*Q \\
\downarrow \beta_Q \\
T^*Q \\
\downarrow \pi_{T^*Q} \\
TQ \\
\end{array}
\quad
\begin{array}{c}
TT^*Q \\
\downarrow \pi_{TT^*Q} \\
T^*Q \\
\downarrow \pi_{T^*Q} \\
TQ \\
\end{array}
\]

The map $\beta_Q$ is an isomorphism of double vector bundles.

The formulation of the autonomous mechanics described above has at least two important features when compared with the ones in textbooks: it is very simple and can be easily generalized to more complicated cases including constraints, nonautonomous mechanics and mechanics on algebroids [20, 19, 21]. And last but not least, we need no regularity conditions for the Lagrangian. The Lagrangian and Hamiltonian can be functions, but one of them (or both) can be replaced by families of functions generating Lagrangian submanifolds in $T^*TQ$ and $T^*T^*Q$, respectively. It happens e.g. in the case of a relativistic particle in the Minkowski space [47]. Moreover, the generating object for dynamics on the Lagrangian side can be replaced by a 1-form different from $dL$. It happens e.g. for systems with friction. The crucial role is played by two mappings: $\alpha_Q$ and $\beta_Q$.

The two diagrams (2.18) and (2.19) glued together are called the Tulczyjew triple for mechanics. We would like to emphasize that the Lagrangian side of the Tulczyjew triple is not postulated, but derived from the variational calculus. The Hamiltonian side, present only in infinitesimal formulation, comes from the fact that $TT^*Q$ is equipped with two Liouville structures, i.e. is isomorphic to two different cotangent bundles $T^*T^*Q$ and $T^*TQ$ (see [48]).

The paper is devoted to deriving the Tulczyjew triple for field theory, i.e. to the case where configurations are sections of a certain fibration. Like in mechanics, the Lagrangian side of the triple appears as a result of the existence of the so-called convenient representations of equivalence classes representing tangent vectors and covectors. The Hamiltonian side is again related to the canonical isomorphism between certain cotangent bundles.
3. Notation

The notation used in papers concerning the geometry of classical field theory is usually very complicated, because iterated tangent functors have to be used. In this section, we will present a system of notation that will be used in the following sections. We will try to introduce some rules which make the notation systematic for the cost of the length of some symbols.

For $M$ being a smooth manifold of dimension $m$, we denote by

$$\tau_M : TM \rightarrow M \quad \text{and} \quad \pi_M : T^*M \rightarrow M$$

the tangent and cotangent bundles. Let $U \subset M$ be a domain of the local coordinate system $(q^i)_{i=1}^m$ on $M$. We have the adapted systems of coordinates $(q^i, \dot{q}^i)$ and $(q^i, p_j)$ on $\pi^{-1}_M(U)$ and $\pi^{-1}_M(U)$, respectively. The manifold $M$ is the manifold on which the field is defined; in applications it can be e.g. the spacetime. We will assume in the following that the manifold $M$ is oriented. This assumption allows us to use even forms instead of densities (which are not commonly used). Instead of the usual notation $\Lambda^k T^*M$ for the space of $k$-covectors on $M$, we will use the shorter symbol $\Omega^k$. The space $\Omega^k$ is a vector bundle over $M$. The vector space over the point $x \in M$ will be denoted by $\Omega^k_x$. For a local coordinate system $(q^i)$, we associate the volume form $\eta = dq^1 \wedge dq^2 \wedge \cdots \wedge dq^m$. We will denote by $\eta$ the contraction $i(\frac{\partial}{\partial q^i})\eta$.

Let

$$\zeta : E \rightarrow M$$

be a smooth locally trivial fibration with the total space of dimension $m+n$. The total space is the space of values of the field, i.e. a field is a local section of the bundle $\zeta$. In applications, the bundle $\zeta$ can have additional structures. On an open subset $V \subset E$ such that $\zeta(V) = U$, we can introduce a local coordinate system $(q^i, y^a)$ adapted to the structure of the bundle.

The space of vectors tangent to $E$ and vertical with respect to the projection onto $M$ will be denoted by $VE$. For the restriction of $\tau_E$ to the space of vertical vectors we will use the symbol $v_E$. The bundle

$$v_E : VE \rightarrow E$$

is therefore a vector bundle. The adapted coordinate system on $v^{-1}_E(V)$, coming from $(q^i, y^a)$ on $V$, will be denoted by $(q^i, y^a, \delta y^b)$. We will also need the dual vector bundle

$$\rho_E : V^*E \rightarrow E$$

with local coordinates $(q^i, y^a, p_b)$ defined on $\rho^{-1}_E(V)$.

The space of first jets of sections of the bundle $\zeta$ will be denoted $j^1\zeta$. By definition, the first jet $j^1\sigma(x)$ of the section $\sigma$ at the point $x$ is an equivalence class of sections having the same value $e$ at the point $x = \zeta(x)$ and such that the spaces tangent to the graphs of the sections at the point $e$ coincide. Therefore, there is a natural projection $j^1\zeta$ from the space $j^1\zeta$ onto the manifold $E$,

$$j^1\zeta : \quad j^1\zeta \ni j^1\sigma(e) \mapsto e \in E.$$  

Moreover, every jet $j^1\sigma(e)$ can be identified with the linear map from $T_eM$ to $T_eE$ being the tangent map $T\sigma$ restricted to the space $T_eM$. It is easy to see that linear maps coming from jets at the point $e$ form an affine subspace of the space $Lin(T_eM, T_eE)$ of all linear maps from $T_eM$ to $T_eE$. A map belongs to this subspace if, composed with $T\zeta$, it gives the identity. Using the tensorial representation, we can therefore write the inclusion

$$j^1E \subset T_eM \otimes T_eE.$$  

The affine subspace $j^1E$ is modeled on a vector subspace of maps having values in vectors tangent to $E$ at the point $e$ and vertical with respect to the fibration $\zeta$. In the tensor
representation, the model vector space is $T^r J T^1 M \otimes V^r E$. Summarizing, the bundle $J^1 E \to E$ is an affine subbundle in the vector bundle 
\[ \xi^*(T^r M) \otimes E T E \to E \]
modeled on the vector bundle 
\[ \xi^*(T^r M) \otimes E V E \to E. \]

The symbol $\xi^*(T^r M)$ denotes the pullback of the cotangent bundle $T^r M$ along the projection $\xi$. In the following, we will omit the symbol of the pullback, writing simply $T^r M \otimes E T E$ and $T^r M \otimes E V E$.

Using the adapted coordinates $(q^i, y^a)$ in $V \subset E$, we can construct the induced coordinate system $(q^i, y^a, y^b_j)$ on $(J^1 \xi)^{-1}(V)$ such that, for any section $\sigma$ given by $n$ functions $\sigma^i(q^i)$, we have 
\[ y^b_j(q^i(\sigma(x))) = \frac{\partial \sigma^b}{\partial q^i}(q^i(x)). \]

In the tensorial representation, the jet $J^1 \sigma(e)$ can be written as 
\[ dq^i \otimes \frac{\partial}{\partial q^i} + \frac{\partial \sigma^a}{\partial q^i}(q^i(x)) \, dq^i \otimes \frac{\partial}{\partial y^a}, \]
where we have used local bases of sections of $T^r M$ and $T E$ coming from the chosen coordinates.

In the following, we will have to use iterated tangent functors $J^r$ and $V$. All jet spaces we will use are spaces of jets of sections of bundles over $M$. All vertical tangent vectors are vertical with respect to the projection onto $M$. It means that $J^1 V E$ is the space of jets of sections of the bundle $\xi \circ V E$; therefore, the projection onto $V E$ will be denoted by $J^1(\xi \circ V E)$. Similarly, $V J^1 E$ is the space of vectors tangent to $J^1 E$ and vertical with respect to the projection onto $M$. The projection to $J^1 E$ will be denoted by $V J^1 E$. Both spaces, $V J^1 E$ and $J^1 V E$, play a very important role in the Lagrangian formulation of the classical field theory. We will discuss the structure of these spaces in the next section. Now, we would like to point out that the coordinate system $(q^i, y^a)$ on $V \subset E$ allows us to construct coordinate systems 
\[ (q^i, y^a, y^b_j, \delta y^c, \delta y^d_k) \quad \text{on} \quad (J^1 \xi \circ V J^1 E)^{-1}(V) \subset V J^1 E \]  
\[ \text{(3.1)} \]
and 
\[ (q^i, y^a, \delta y^b, y^c_f, \delta y^d_k) \quad \text{on} \quad (V E \circ J^1(\xi \circ V E))^{-1}(V) \subset J^1 V E. \]  
\[ \text{(3.2)} \]

4. Lagrangian formulation

In the first-order field theory, a Lagrangian is a map from the space of first jets of sections of the bundle $\xi$ to scalar densities on $M$, covering the identity on $M$. Since we assumed that $M$ is oriented, we can identify densities with $m$-covectors. A Lagrangian $L$ is therefore a map 
\[ L : J^1 E \to \Omega^m \]  
\[ \text{(4.1)} \]
covering the identity on $M$. In coordinates $L(q^i, y^a, y^b_j) = \ell(q^i, y^a, y^b_j) \eta$ where $\eta$ is an appropriate function and $\eta$ denotes the volume form associated with the coordinate system (see section 3). The space $J^1 E$ is often called the space of infinitesimal configurations for the first-order field theory. Let us recall that in statics and other variational theories, all information about the system is contained in a constitutive set which is a subset of the cotangent bundle of the configuration space. In mechanics, we developed another description of a constitutive set, using the so-called convenient representations of covectors. The convenient representation approach led to differential equations for phase trajectories of a system. In field theory, we can
adopt the same scheme. In the infinitesimal approach, the space of infinitesimal configurations is a manifold and the role of the internal energy is played by the Lagrangian. Since variations are vectors vertical with respect to the projection onto \( M \), the constitutive set is given as an image of the vertical differential of the Lagrangian. Taking into account that the Lagrangian has values in \( \Omega^1 \), we get that the constitutive set is a subset of \( V^*J E \otimes J E \otimes \Omega^1 \). In mechanics, studying convenient representations led us to the concept of momentum and external force. Let us do the same for the field theory.

### 4.1. The phase space

The first step is recognizing the phase space for the first-order field theory on the fibration \( \zeta \). We can use the calculus of variations as the guideline for the problem. Let \( D \) be a compact region in \( M \) with the smooth boundary \( \partial D \) such that it is contained in a domain of coordinates \( U \). We will perform now the standard calculations in coordinates that lead to the Euler–Lagrange equations, but we will not assume that the variations vanish on the boundary. In the following, we will denote by \( (\sigma^a) \) the functions defining the local section \( \sigma \), and by \( (\sigma^a, \delta\sigma^a) \) the functions defining its variation, i.e. a vertical vector field on \( E \) along the section \( \sigma \). The action functional \( S \) evaluated on \( \sigma \) gives

\[
S[\sigma] = \int_D \ell(q^i, \sigma^a, \partial \sigma^a / \partial q^j) \eta,
\]

and the variation of \( S \) evaluated on the variation of \( \sigma \) gives

\[
\langle \delta S, \delta \sigma \rangle = \int_D \left( \frac{\partial \ell}{\partial \sigma^a} \delta \sigma^a + \frac{\partial \ell}{\partial \partial q^j} \frac{\partial \delta \sigma^a}{\partial q^j} \right) \eta.
\] (4.2)

Using the Stokes theorem, we obtain

\[
\langle \delta S, \delta \sigma \rangle = \int_D \left( \frac{\partial \ell}{\partial \sigma^a} \delta \sigma^a - \frac{\partial \ell}{\partial q^j} \frac{\partial \delta \sigma^a}{\partial q^j} \right) \delta \sigma^a \eta + \int_{\partial D} \frac{\partial \ell}{\partial \partial q^j} \delta \sigma^a \eta_i.
\] (4.3)

The term integrated over \( D \) gives the definition of the external forces that are now interpreted as sources of a field, while the term integrated over the boundary \( \partial D \) gives the definition of the momenta. An object that can be integrated over the boundary of the region \( D \) is an \((m-1)\)-form; therefore, the results of the evaluation of momenta over the variations should lie in the bundle of \((m-1)\)-covectors on the base manifold \( M \). We point out, however, that the momenta should be evaluated on variations rather than on infinitesimal configurations. It is a special case of autonomous mechanics when variations and infinitesimal configurations are represented by the same geometrical object (tangent vector); therefore, we can evaluate momenta on velocities. We cannot evaluate the momenta on first jets, at least not in the usual sense. The phase space for the first-order field theory on the bundle \( \zeta \) is therefore the space

\[
\mathcal{P} = V^*E \otimes \Omega^{m-1}.
\] (4.4)

It is a vector bundle over \( E \). We will denote the corresponding fibration by

\[
\pi : \mathcal{P} \rightarrow E.
\]

Using a base of sections of the bundle \( \rho_E \) and a base \( (\eta_i) \) in \( \Omega^{m-1} \), we can construct local linear coordinates on \( \pi^{-1}(V) \):

\[
(q^i, y^o, p^h),
\] (4.5)

such that a section of the bundle \( \pi \) is represented as

\[
p^h(q, y) dy^h \otimes \eta_j.
\]
The role of external forces is played by the source of a field. The source is represented by a section of the bundle $V^*E \otimes_{\omega} \Omega^m \rightarrow M$.

All the above calculations can be done in a coordinate-free form. We postpone it to section 4.4.

4.2. The structure of iterated bundles

Passing from (4.2) to (4.3), we have used implicitly a canonical mapping

$$\kappa : \bigvee_1 V^*E \rightarrow J^1 VE$$

which is analogous to the canonical flip $\kappa_M : TTM \rightarrow TTM$. The role of $\kappa$ in the field theory is similar to that of $\kappa_M$ in mechanics, where the map $\kappa_M$ is used to obtain a convenient representation of a variation of an infinitesimal configuration. The idea of using convenient representations comes from mechanics formulated for finite time intervals. In the field theory, instead of the finite time intervals, we have the bounded domains of integration $D \subset M$. A configuration is then a section of $\xi$ restricted to $D$ and its virtual displacement is represented by an equivalence class of curves in the space of sections. Curves in the space of sections come from vertical homotopies, i.e. maps

$$\chi : U \times I \rightarrow E$$

such that $D$ is contained in an open set $U$ and $I$ is a neighborhood of 0 in $\mathbb{R}$. The verticality means that for any $t$ we have $\xi(\chi(x, t)) = x$. Fixing $x$ we obtain a vertical curve in $E$, while fixing $t$ we obtain a local section of $\xi$. Restricting the domain of $\chi$ to $D \subset U$, we obtain a curve in configurations. The curves are classified, as usual, with the use of some functions on configurations of the type of an action functional. We observe that equivalence classes are conveniently represented by vertical vector fields $\delta\sigma$ along a sections $\sigma$ over $D$. In the infinitesimal approach, a configuration is the first jet $j^1 \sigma(x)$ of a section and its variation is represented by a vector $\delta j^1 \sigma$ tangent to the space of first jets and vertical with respect to the projection on $M$. From the convenient representation, we obtain the first jet of a vertical vector field along a section, i.e. an element of $J^1 VE$.

Let us be more precise and define the map $\kappa$ using representatives of elements of $J^1 VE$ and $\bigvee_1 V^*E$. For a vertical homotopy $\chi : U \times I \rightarrow E$ of local sections of $\xi$, and for a point $x_0 \in M$, we can create two objects. Taking the first jet of the section $x \mapsto \chi(x, t)$ at $x_0$, we obtain the curve

$$t \mapsto j^1 \chi(x_0, t)$$

in $J^1 E$ which is vertical with respect to the projection onto $M$. The vector

$$t j^1 \chi(x_0, 0),$$

tangent to this curve at $t = 0$, is an element of $\bigvee_1 V^*E$. On the other hand, we can first take vectors tangent to vertical curves $t \mapsto \chi(x, t)$ at $t = 0$, obtaining a vertical vector field along the section $x \mapsto \chi(x, 0)$. The vector field

$$x \mapsto t \chi(x, 0)$$

is a section of the bundle $\xi \circ \rho_E : VE \rightarrow M$. Taking the first jet of this section at the point $x_0$, we obtain the element $j^1 t \chi(x_0, 0)$ of $J^1 VE$.

**Definition 4.1.** The map $\kappa : \bigvee_1 V^*E \rightarrow J^1 VE$ is uniquely determined by

$$\kappa(t j^1 \chi(x_0, 0)) = j^1 t \chi(x_0, 0).$$
The definition is correct as both sides of (4.6) are independent of the choice of the representative \( \chi \) for an element of \( \mathcal{V}^1_1 E \). Any element \( \nu \) of \( \mathcal{V}^1_1 E \) over \( x \) can be represented as \( \nu = \kappa(j_1^2 \delta \sigma(x)) \) for some \( \delta \sigma \) of the bundle \( \zeta \circ \nu_E \). The same can be done for any element of \( \mathcal{V}^1 J E \) with the use of the isomorphism

\[
\kappa_{(l,1)} : J^1 \mathcal{V}^E \longrightarrow \mathcal{V}^1 J E
\]

which is defined analogously to \( \kappa \), using representatives of elements of \( \mathcal{V}^1 J E \) and \( J^1 \mathcal{V}^E \). It is convenient to use the notation \( \delta \sigma'(x) = \kappa_{(l,1)}(j_1^2 \delta \sigma(x)) \). In local coordinates (see (3.1), (3.2)), the map \( \kappa \) reads

\[
\kappa(q', y^i, y^b_j, \delta y^c, \delta y^d_k) = (q', y^i, \delta y^c, y^b_j, \delta y^d_k).
\]

(4.7)

It is easy to see that \( \kappa \) is an involution, a diffeomorphism of \( \mathcal{V}^1 J E \) and \( J^1 \mathcal{V}^E \), and moreover a morphism of double bundle structures that are described in the following.

Both spaces, \( J^1 \mathcal{V}^E \) and \( \mathcal{V}^1 J E \), are double bundles in the sense that they carry the structure of two compatible fibrations. The space \( J^1 \mathcal{V}^E \) is fibered over \( \mathcal{V}^E \) and the fibration is an affine bundle modeled on the vector bundle \( T^* M \otimes_{\nu_E} \mathcal{V}^E \to \mathcal{V}^E \). The projection from \( J^1 \mathcal{V}^E \) onto \( \mathcal{V}^E \) is \( j_1^1(\zeta \circ \nu_E) \), since the first jets are calculated with respect to the projection on \( M \). In adapted coordinates, we have

\[
(q', y^i, \delta y^b, y^c_j, \delta y^d_k) \mapsto (q', y^i, \delta y^b).
\]

The space \( J^1 \mathcal{V}^E \) is also fibered over \( J^1 E \) and the corresponding projection will be denoted by \( J^1 \nu_E \). In adapted coordinates, the second projection reads as

\[
(q', y^i, \delta y^b, y^c_j, \delta y^d_k) \mapsto (q', y^i, y^c_j).
\]

The compatibility of the two bundle structures means that the model vector bundle \( T^* M \otimes_{\nu_E} \mathcal{V}^E \) is in fact a double vector bundle. For the concept of a double affine bundle and its model double vector bundle, we refer to [23]. We have therefore two commutative diagrams of bundle projections:

\[
\begin{array}{ccc}
J^1 \mathcal{V}^E & \longrightarrow & j_1^1(\zeta \circ \nu_E) \\
\downarrow & & \downarrow \\
J^1 E & \longrightarrow & J^1 E
\end{array}
\]

\[
\begin{array}{ccc}
T^* M \otimes_{\nu_E} \mathcal{V}^E & \longrightarrow & \mathcal{V}^E \\
\downarrow & & \downarrow \\
T^* M \otimes_{\nu_E} \mathcal{V}^E & \longrightarrow & \mathcal{V}^E
\end{array}
\]

(4.8)

The space \( \mathcal{V}^1 J E \) also carries the structure of a double bundle. One of the bundles is an affine bundle and the other is a vector bundle. In adapted coordinates, the vector bundle fibration

\[
\nu_M : \mathcal{V}^1 J E \longrightarrow \mathcal{V}^1 J E
\]

reads as

\[
(q', y^i, y^b_j, \delta y^c, \delta y^d_k) \mapsto (q', y^i, y^b_j).
\]

The second fibration is the affine bundle fibration

\[
\mathcal{V}^1 J : \mathcal{V}^1 J E \longrightarrow \mathcal{V}^E
\]

which we obtain by applying the vertical tangent functor \( V \) to the projection \( J^1 \zeta : J^1 E \to E \). The model vector bundle for this affine bundle is \( V(\mathcal{V}^E \otimes_{\nu_E} T^* M) \to \mathcal{V}^E \).
In the adapted coordinates, the affine projection $\mathcal{V}^1\zeta$ reads as

$$ (q', y', y_j', \delta y_j', \delta y^j) \longmapsto (q', y', \delta y^j). $$

The compatibility condition of the two projections can be expressed as the assumption that the model space $\mathcal{V}(\mathbb{E} \otimes E \mathcal{T}^*M) \simeq \mathcal{V}(\mathbb{E} \otimes E \mathcal{T}^*M)$ is a double vector bundle. The structure of $\mathcal{V}^1E$ can be summarized in the following two commutative diagrams:

$$ \begin{align*}
\mathcal{V}^1E & \xrightarrow{\nu_E} \mathcal{V}E \\
J^1E & \xrightarrow{\nu_E} \mathcal{V}E
\end{align*} $$

$$ \begin{align*}
\mathcal{V}^1E \mathcal{T}^*M & \xrightarrow{\nu_E} \mathcal{V}E \mathcal{T}^*M \\
J^1E \mathcal{T}^*M & \xrightarrow{\nu_E} \mathcal{V}E \mathcal{T}^*M
\end{align*} $$

where the second diagram represents the model double vector bundle. Let us summarize our observations as follows.

**Theorem 4.1.** The bundles $\mathcal{V}^1E$ and $J^1\mathcal{V}E$ are canonically double affine-vector bundles. The map $\kappa : \mathcal{V}^1E \rightarrow J^1\mathcal{V}E$ is an isomorphism of double bundle structures covering the identities on side bundles.

Note that on the level of the model double vector bundles, our map $\kappa$ corresponds to the canonical flip $\kappa_E : T\mathbb{E} \rightarrow T\mathbb{E}$ restricted to vertical vectors.

Another example of a space equipped with the double structure of a vector-affine bundle is the space $J^1\mathcal{P}$ of first jets of the bundle $\zeta \circ \pi$. As the bundle of jets, it is fibered over $\mathcal{P}$ and the fibration is an affine fibration modeled on a vector fibration $\mathcal{V}\mathcal{P} \otimes \mathcal{T}^*M \rightarrow \mathcal{P}$. The vector bundle structure on $J^1\mathcal{P}$ comes from the jet prolongation of the vector bundle projection $\pi : \mathcal{P} \rightarrow E$. The double-bundle structure of $J^1E$ can be summarized in the following two diagrams:

$$ \begin{align*}
\mathcal{V}^1\mathcal{P} & \xrightarrow{\nu_{\mathcal{P}}} \mathcal{V}\mathcal{P} \\
J^1\mathcal{P} & \xrightarrow{\nu_{\mathcal{P}}} \mathcal{V}\mathcal{P}
\end{align*} $$

$$ \begin{align*}
\mathcal{V}^1\mathcal{P} \mathcal{T}^*M & \xrightarrow{\nu_{\mathcal{P}}} \mathcal{V}\mathcal{P} \mathcal{T}^*M \\
J^1\mathcal{P} \mathcal{T}^*M & \xrightarrow{\nu_{\mathcal{P}}} \mathcal{V}\mathcal{P} \mathcal{T}^*M
\end{align*} $$

where the second diagram represents the model double vector bundle. Elements of the model vector bundle $\mathcal{V}\mathcal{P} \otimes_{\mathcal{P}} \mathcal{T}^*M$ can be added to elements of $J^1\mathcal{P}$. In $\mathcal{V}\mathcal{P} \otimes_{\mathcal{P}} \mathcal{T}^*M$, there is a subbundle of vectors which are vertical with respect to the projection on $E$, i.e. vectors $v$ such that $\nabla\pi (v) = 0$. Vectors vertical with respect to the projection on $E$ have the first component tangent to the corresponding fiber of the bundle $\pi$. As usual, vectors tangent to a fiber of a vector bundle can be identified with elements of the fiber itself. Therefore, if $\nabla\pi (v) = 0$, then $v$ can be identified with an element of $\mathcal{P} \otimes_{\mathcal{P}} \mathcal{T}^* = \mathcal{V}\mathcal{E} \otimes_{\mathcal{E}} \Omega^{m-1} \otimes_{\mathcal{E}} \mathcal{T}^*M$. Note that adding vectors vertical with respect to the projection on $E$ does not change the right-hand side projection, i.e. if $p$ is a local section of $\zeta \circ \pi$, then

$$ J^1\pi (j^1 p(x) + v) = J^1\pi (j^1 p(x)). $$

In coordinates, if $j^1 p(x) = (x^i, y^a, p^j_b, y^j_k, p^j_{di})$ and $v = (x^i, y^a, v^j_b)$, i.e. $v = v^j_b \eta_j \otimes dx^b$, then

$$ j^1 p(x) + v = (x^i, y^a, p^j_b, y^j_k, p^j_{di} + v^j_{di}). $$
In section 4.1, we have observed that sources of fields are represented by sections of the bundle $\mathcal{V}^*E \otimes_E \Omega^m \to M$. Since

$$\mathcal{V}^*E \otimes_E \Omega^m \subset \mathcal{V}^*E \otimes_E \Omega^{m-1} \otimes_E \mathcal{T}^*M,$$

we see that elements of the total space of the bundle of sources can be added to first jets from $J^1\mathcal{P}$ without changing any of the projections. This operation will be needed in construction of the field phase equations with sources in section 4.3.

In the next section, we will construct the main map of the Lagrangian formulation of the field theory that maps covectors on the space of infinitesimal configurations to their convenient representations. For that we will need an evaluation between the space $J^1\mathcal{P}$ of first jets of sections of the bundle

$$\zeta \circ \pi : \mathcal{P} \longrightarrow M$$

and the space $J^1\mathcal{V}E$ of first jets of vertical virtual displacements. More precisely, we will construct an evaluation between the bundle

$$J^1\pi : J^1\mathcal{P} \longrightarrow J^1E$$

and the bundle

$$J^1\mathcal{V}E : J^1\mathcal{V}E \longrightarrow J^1E$$

with values in the pullback of the bundle $\Omega^m \to M$ by $\zeta \circ J^1\xi$. The pairing is an adaptation of the tangent pairing in the context of field theory.

Let $p : M \supset U \longrightarrow \mathcal{P}$ be a local section of the momentum bundle. We denote by $\sigma$ the underlying section of the bundle $\xi$, i.e. $\sigma : M \supset U \longrightarrow E$ is such that $p \circ \pi = \sigma$. Let also $\delta \sigma : M \supset U \longrightarrow \mathcal{V}E$ be a vertical vector field along the section $\sigma$. There is a natural evaluation between $\mathcal{V}E$ and $\mathcal{P}_\sigma = \mathcal{V}^*E \otimes \Omega^{m-1}_{\mathcal{X}_\sigma}$; therefore, $\langle p, \delta \sigma \rangle$ is an $(m-1)$-form defined on $U \subset M$. We can define the evaluation between $J^1p(x_0)$ and $J^1\delta \sigma(x_0)$ using the formula

$$\langle J^1p(x_0), J^1\delta \sigma(x_0) \rangle = d\langle p, \delta \sigma \rangle(x_0),$$

so that the evaluation is a map

$$\langle \cdot, \cdot \rangle : J^1\mathcal{P} \times J^1\mathcal{V}E \longrightarrow \Omega^m.$$

In coordinates, if $\sigma$ is given by local functions $(\sigma^a)$, if the momentum is represented as

$$p_\alpha \, dy^a \otimes \eta_i,$$

and $\delta \sigma$ as

$$\delta \sigma^b = \frac{\partial}{\partial y^b},$$

then

$$\langle p, \delta \sigma \rangle(x) = p_\alpha(x) \delta \sigma^a(x) \eta_i,$$

and

$$d\langle p, \delta \sigma \rangle(x_0) = \left( \frac{\partial p^b_\alpha}{\partial q^i}(x_0) \delta \sigma^b(x_0) + p_\alpha'(x_0) \frac{\partial \delta \sigma^b}{\partial q^i}(x_0) \right) \eta_i;$$

therefore, the evaluation $\langle \cdot, \cdot \rangle$ in coordinates reads as

$$\langle \langle q^i, y^a, p_\alpha^b, y_\ell, p_\alpha^\ell \rangle, \langle q^i, y^a, \delta y^b, y_\ell, \delta y_\ell \rangle \rangle = p^i_\alpha \delta y^a + p^i_\alpha \delta y^\ell.$$

In this way, we get the following.

**Theorem 4.2.** There is a canonical pairing between the vector bundles $J^1\pi : J^1\mathcal{P} \to J^1E$ and $J^1\mathcal{V}E : J^1\mathcal{V}E \to J^1E$, defined by (4.11), which in local coordinates reads as (4.13).
4.3. The map $\alpha$

Let us now define the main geometrical object of the Lagrangian formulation of the first-order field theory.

**Definition 4.2.** The relation
\[ \alpha : J^1 P \longrightarrow V^* J^1 E \otimes \Omega^m, \]
given by the condition
\[ \langle\langle u, \kappa(w) \rangle\rangle = \langle \alpha(u), w \rangle \] (4.14)
for all $w$ having the same projection on $J^1 E$ as $u$, will be called the Lagrangian relation.

In this case, the relation $\alpha$ is actually a mapping. In coordinates, we have
\[ \alpha : (q^i, y^a, p^b_j, y^i_{,k}, p^l_{,di}) \mapsto \left( q^i, y^a, y^i_{,k}, \sum_l p^l_{,di}, p^b_j \right). \] (4.15)

The map $\alpha$ is a field-theoretical analog of the Tulczyjew $\alpha_M$ in mechanics. It relates covectors on the space of infinitesimal configurations which are elements of $V^* J^1 E \otimes \Omega^m$ to their convenient representations. In the simplest case, when sources of the field are equal to 0, a convenient representation of a covector is the first jet of a section of the momentum bundle. If there are no constraints for infinitesimal configurations of the system described by the Lagrangian $L$, the constitutive set of the system is given as an image of $J^1 E$ by the vertical differential $dL$ (understood as a map from $J^1 E$ to $V^* J^1 E \otimes \Omega^m$). Using the map $\alpha$, we can obtain a convenient representation of the constitutive set as a differential inclusion which we understand as a condition for sections of the momentum bundle. This differential inclusion is called the phase dynamics of the field.

**Definition 4.3.** The phase dynamics of the field, when sources are equal to 0, is the subset $D$ of $J^1 P$ given by
\[ D = \alpha^{-1}(dL(J^1 E)). \] (4.16)

The phase dynamics is also called the Lagrangian field equations. Let us note that obtaining the Lagrangian field equations from a Lagrangian is in our theory very simple. We do not require any regularity of the Lagrangian.

**Definition 4.4.** We say that a section $p : M \rightarrow P$ is a solution of the Lagrange field equations if
\[ J^1 p(x) \in D. \]

In coordinates, it means that
\[ \sum_j \frac{\partial p^j_b}{\partial q^l} = \frac{\partial L}{\partial y^j}, \quad p^b_j = \frac{\partial L}{\partial y^j_{,b}}. \]

The equations, known as the Euler–Lagrange equations for field theory, are consequences of the Lagrange field equations.
The Lagrangian side of the Tulczyjew triple for the first-order classical field theory can be presented in the following diagram:

\[
\begin{array}{c}
\mathcal{P} \\
\downarrow \pi \\
E
\end{array}
\quad \xrightarrow{\alpha} \quad
\begin{array}{c}
\mathcal{P} \\
\downarrow \pi \\
E
\end{array}
\quad \xrightarrow{\rho \circ \varepsilon}
\]

There is one projection in the above diagram that needs explanation. It is the projection \(\xi: \mathcal{V}^* J^1 E \otimes J^1 E \mathcal{O}_m \rightarrow \mathcal{P}\).

Let us fix a point \(v \in J^1 E\) and denote \(e = j^1 \zeta(v)\) and \(x = \zeta(e)\). In the vector space \(\mathcal{V}_v J^1 E\), there is a subspace of those vectors which are tangent to fibers of the projection \(j^1 \zeta: J^1 E \rightarrow E\).

Since each such fiber is an affine subspace of \(T^*_x M \otimes T_x E\), the space of vectors tangent to the fiber is isomorphic to its model vector space which is \(T^*_x M \otimes \mathcal{V}_x E\). An element of \(\mathcal{V}_v J^1 E \otimes \mathcal{O}_m\), treated as a linear function on \(\mathcal{V}_v J^1 E\) with values in \(\mathcal{O}_m\), can be restricted to the subspace of vectors tangent to the fibers. The restriction is an element of \(T^x M \otimes \mathcal{V}_x E \otimes \mathcal{O}_m \approx \mathcal{V}_x E \otimes \mathcal{O}_m^{-1} = \mathcal{P}_x\).

Summarizing, the projection \(\xi\) is a restriction of a covector to the subspace of vectors tangent to fibers of a certain projection. It provides the Legendre map, defined by the Lagrangian, from the space of infinitesimal configurations to the space of momenta,

\[
\lambda: J^1 E \rightarrow \mathcal{P}, \quad \lambda(v) = \xi(dL(v)).
\]

Let us summarize our observations as follows.

**Theorem 4.3.** The bundle \(\mathcal{V}^* J^1 E \otimes J^1 E \mathcal{O}_m\) is canonically a double affine-vector bundle fibered over \(\mathcal{P}\) and \(J^1 E\). The map \(\alpha: J^1 \mathcal{P} \rightarrow \mathcal{V}^* J^1 E \otimes J^1 E \mathcal{O}_m\) defined by (4.14) is a morphism of double affine-vector bundles.

Sources of the field can also be included in the picture. Recall that sources are sections of the bundle \(\mathcal{V}^* E \otimes \mathcal{O}_m \rightarrow M\) and that an element of the total space of this bundle can be added to the elements of \(J^1 \mathcal{P}\). We define the extended map \(\tilde{\alpha}: \mathcal{V}^* E \otimes \mathcal{O}_m \times J^1 \mathcal{P} \rightarrow \mathcal{V}^* J^1 E \otimes J^1 E \mathcal{O}_m\) by the formula

\[
\tilde{\alpha}(\rho, j^1 p) = \alpha(\rho + j^1 p).
\]

**Definition 4.5.** The phase field dynamics with sources is the subset \(\tilde{D}\) of \(\mathcal{V}^* J^1 E \otimes J^1 E \mathcal{O}_m \times J^1 \mathcal{P}\) given by

\[
\tilde{D} = \tilde{\alpha}^{-1}(dL(J^1 E)).
\]

**Definition 4.6.** We say that a pair of sections \(p: M \rightarrow \mathcal{P}\) and \(\rho: M \rightarrow \mathcal{V}^* E \otimes \mathcal{O}_m\) is a solution of phase field dynamics with sources if

\[
\rho(x) + j^1 p(x) \in \tilde{D}.
\]

In coordinates, it means that

\[
\sum_j \frac{\partial p_j}{\partial q^j} + \rho_b = \frac{\partial L}{\partial y^b}, \quad p'_b = \frac{\partial L}{\partial y^b}.
\]
4.4. Phase space: geometrical version

In section 4.1, using a coordinate calculation, we have split the differential of a Lagrangian into two parts: the Euler–Lagrange part and the total differential part. We have used formula (4.3) to determine the phase space for the problem. Now we can do the same intrinsically, i.e. without using any specific choice of coordinates (see [46]).

Let us fix a point \( x_0 \) in \( M \). Given a covector \( \varphi \in T^{*}_{x_0}M \), we can choose a local function \( f \) on \( M \) such that \( f(x_0) = 0 \) and \( df(x_0) = \varphi \). Now we define

\[
F(\varphi, v) = \kappa(1^{j} \xi(x_0)), \quad \text{where} \quad v = \kappa(1^{j} \delta\sigma(x_0)) \text{ and } \xi(x) = f(x)\delta\sigma(x).
\]

(4.21)

It is clear that the value of \( F \) depends only on the covector \( \varphi \) and the vector \( v \), and not on the representatives. We have then defined the map

\[
F : T^{*}_{x_0}M \times \mathcal{V}_{\delta\sigma(x_0)}J^{1}E \longrightarrow \mathcal{V}_{\xi(x_0)}J^{1}E
\]

which is bilinear. In coordinates, if

\[
\varphi = \varphi_i \, dq^i, \quad v = \delta y^a \frac{\partial}{\partial y^a} + \delta y^j \frac{\partial}{\partial y^j}.
\]

then

\[
F(\varphi, v) = \varphi_j \delta y^a \frac{\partial}{\partial y^a}.
\]

We can see that the projection of \( F(\varphi, v) \) on \( \mathcal{V} E \) is zero, i.e. \( F(\varphi, v) \) is vertical with respect to \( j^{1} \xi \).

For a one-form \( \mu \) on \( J^{1}E \) with values in \( \Omega^{n} \), we define a one-form \( i_{E} \mu \) on \( J^{1}E \) with values in \( \Omega^{n-1} \) by the formula

\[
\langle \mu, F(\varphi, v) \rangle = \varphi \wedge \langle i_{E} \mu, v \rangle.
\]

(4.22)

In coordinates, if

\[
\mu = ((\mu^0)_a \, dy^a + (\mu^1)_a \, dy^a \wedge \eta) \otimes \eta,
\]

then

\[
i_{E} \mu = ((\mu^1)_a \, dy^a) \otimes \eta.
\]

The form \( i_{E} \mu \) is \( j^{1} \xi \)-horizontal, i.e. it vanishes on vectors vertical with respect to the projection \( j^{1} \xi \).

There is an operation of total differential \( d_{E} \) defined for forms on jet bundles with values in \( \Omega^{n} \). For example, if \( \mu \) is a one-form on \( J^{1}E \) with values in \( \Omega^{k} \), then \( d_{E} \mu \) is a one-form on \( J^{1+1}E \) with values in \( \Omega^{k+1} \) given by the formula

\[
\langle d_{E} \mu, (j^{1+1} \delta\sigma(x_0)) \rangle = d(\langle \mu, j^{1} \delta\sigma \rangle)(x_0).
\]

(4.23)

Recall that \( \delta\sigma(x_0) = \kappa_{1,1}(j^{1} \delta\sigma(x_0)) \) (see definition 4.1). For the coordinate expression of \( d_{E} \), we need the differential operator \( D_{j} \) of total derivative with respect to \( q^{j} \):

\[
D_{j} = \frac{\partial}{\partial q^{j}} + y^{a}_{,j} \frac{\partial}{\partial y^{a}} + y^{a}_{,m} \frac{\partial}{\partial y^{a}_{,m}} + \cdots + y^{a}_{,A} \frac{\partial}{\partial y^{a}_{,A}} + \cdots,
\]

acting on functions on jet spaces. If \( f \) is a function on \( J^{k}E \), then \( D_{j} f \) is a function on \( J^{k+1}E \). Here \( A \) is a multi-index \( A = j_1 j_2 \ldots j_i \). For

\[
\mu = f \, dy^{a}_{,A} \otimes dq^{j} \wedge dq^{j} \wedge \cdots \wedge dq^{h}
\]

we get

\[
d_{E} \mu = (D_{j}(f) \, dy^{a}_{,A} + f \, dy^{a}_{,A}) \otimes dq^{j} \wedge dq^{j} \wedge \cdots \wedge dq^{h}.
\]

(4.24)
Applying \( d_M \) to \( i_F \mu \) for \( \mu \) being a one-form on \( J^1E \) with values in \( \Omega^m \), we get that \( d_M i_F \mu \) is a one-form on \( J^2E \) with values in \( \Omega^m \). Let
\[
E(\mu) = (\tau_1^2)^* \mu - d_M i_F \mu \quad \text{and} \quad P(\mu) = i_F \mu,
\]
(4.25)
where \( \tau_1^2 \) is the canonical projection
\[
\tau_1^2: J^2E \rightarrow J^1E.
\]
We have
\[
(\tau_1^2)^* \mu = E(\mu) + d_M P(\mu)
\]
and both forms \( E(\mu) \) and \( P(\mu) \) are horizontal with respect to the projection on \( E \). In particular, if \( v \in \mathcal{V} J^2E \) and \( \tau_1^2 \tilde{v}(v) = 0 \), then
\[
\langle E(\mu), v \rangle = 0.
\]
Indeed, let us take a representative \( \delta \sigma \) such that \( v = \delta \sigma^2(x_0) \). Since \( v \) is vertical, \( \delta \sigma(x_0) = 0 \), so
\[
\langle E(\mu), v \rangle = \langle \mu, \tau_1^2(v) \rangle - \langle d_M i_F, v \rangle
\]
\[
= \langle \mu, \delta \sigma^1(x_0) \rangle - \langle d_M i_F, \delta \sigma^2(x_0) \rangle = \langle \mu, \delta \sigma^1(x_0) \rangle - d((i_F \mu) \circ j^1, \delta \sigma^1)(x_0).
\]
(4.26)
The vector \( \delta \sigma^1(x_0) \) is also vertical with respect to the projection on \( E \); therefore, we can find a function \( f \) vanishing at \( x_0 \) and a vector \( u = \delta \omega^1(x_0) \) such that
\[
\delta \sigma^1(x_0) = F(df(x_0), u),
\]
i.e. we can write (for the first part of formula (4.26))
\[
\langle \mu, \delta \sigma^1(x_0) \rangle = \langle \mu, F(df(x_0), \delta \omega^1(x_0)) = df(x_0) \wedge (i_F \mu (j^1, \delta \sigma^1(x_0), \delta \omega^1(x_0)).
\]
Using the fact that \( f(x_0) = 0 \), we can write that
\[
df(x_0) \wedge (i_F \mu (j^1, \delta \sigma^1(x_0), \delta \omega^1(x_0)) = d(f(i_F \mu \circ j^1, \delta \omega^1)(x_0).\)
Now, let us concentrate on the second part of formula (4.26). Since the form \( i_F \mu \) is horizontal, the value of
\[
\langle i_F \mu (j^1 \sigma(x)), \delta \sigma^1(x) \rangle
\]
depends only on the jet \( j^1 \sigma(x) \) of the base section and on \( \delta \sigma(x) \). The value of the differential
\[
d_M ((i_F \mu \circ j^1, \delta \sigma^1)) (x_0)
\]
(4.27)
depends therefore on the second jet \( j^2 \sigma(x_0) \) and the first jet \( \delta \sigma^1(x_0) \). This means that in formula (4.27), we can substitute the section \( \delta \sigma \) by the section \( x \mapsto \xi(x) = f(x) \delta \omega(x) \) that covers the same section \( \sigma \) and has the same first jet at \( x_0 \). We can now continue the calculation started in (4.26):
\[
\langle j^1 \sigma(x_0), \delta \sigma^1(x_0) \rangle - d((i_F \mu \circ j^1, \delta \sigma^1))(x_0)
\]
\[
= d(f(i_F \mu \circ j^1, \delta \omega^1) - (i_F \mu \circ j^1, \sigma, \kappa (j^1 \xi)))(x_0) = d(i_F \mu \circ j^1, \sigma, f \delta \omega^1 - \kappa (j^1 \xi))(x_0).
\]
(4.28)
Let us now observe that \( f(x) \delta \omega^1(x) = \kappa (j^1 \xi(x)) \) is a vertical vector for any \( x \), because \( f(x) \delta \omega^1(x) \) projects onto \( f(x) \delta \omega(x) \) and \( \kappa (j^1 \xi(x)) \) projects on \( \delta \omega(x) \). Using the horizontality of \( i_F \mu \), we see that
\[
\langle i_F \mu \circ j^1, \sigma, f \delta \omega^1 - \kappa (j^1 \xi) \rangle
\]
equals 0 on the whole neighborhood of \( x_0 \); its differential is therefore equal to zero.
We have shown that for any one-form \( \mu \) on \( J^1E \), the form \( E(\mu) \) is horizontal. The form \( P(\mu) \) is also horizontal by definition. For \( \mu = dL \), we can therefore define two maps:

\[
\mathcal{E}(dL) : J^2E \longrightarrow V^* \otimes_E \Omega^m, \quad \langle \mathcal{E}(L)(j^2\sigma(x)), \delta\sigma(x) \rangle = \langle E(dL)(j^1\sigma(x)), \delta\sigma^2(x) \rangle
\]

(4.29)

and

\[
\mathcal{P}(dL) : J^1E \longrightarrow V^* \otimes_E \Omega^{m-1}, \quad \langle \mathcal{P}(L)(j^1\sigma(x)), \delta\sigma(x) \rangle = \langle E(dL)(j^1\sigma(x)), \delta\sigma^1(x) \rangle,
\]

(4.30)

such that formula (4.3) assumes the form

\[
\langle \delta S, \delta\sigma \rangle = \int_D \langle \mathcal{E}(dL)(j^2\sigma(x)), \delta\sigma(x) \rangle + \int_{jD} \langle \mathcal{P}(dL)(j^1\sigma(x)), \delta\sigma(x) \rangle. \tag{4.31}
\]

4.5. Simple example: electrostatics

Let us write the Lagrangian side of the Tulczyjew triple for electrostatics. In this particular example, we have also included sources of the field. Since it is a nonrelativistic theory, we have decided to use a three-dimensional affine space \( A \) as our playground. For the affine space, tangent and cotangent bundles are trivial, so it is possible to present mathematical objects that appear in the theory in a simple way. The model vector space for the affine space \( A \) will be denoted with \( V \). It is a three-dimensional vector space equipped with a symmetric non-degenerate positive definite bilinear form \( g \) representing the metric. For the affine space \( A \), we have trivial tangent and cotangent bundles

\[
\tau_A : A \times V \longrightarrow A, \quad \pi_A : A \times V^* \longrightarrow A.
\]

Moreover, because of the presence of the metric, there is a canonical isomorphism

\[
\tilde{g} : V \longrightarrow V^*, \quad v \longmapsto g(v, \cdot),
\]

and a canonical scalar density \( g \) represented (because of orientation) by a 3-form.

The potential of the electrostatics is a scalar field; therefore, we take \( E = A \times \mathbb{R} \) and \( M = A \). The main bundle of the theory is the trivial bundle

\[
p_{\tau_A} : A \times \mathbb{R} \longrightarrow A.
\]

The space of the first jets of sections of the above bundle can be identified with

\[
J^1E = A \times \mathbb{R} \times V^*.
\]

The first jet of a section \( \phi \) at a point \( x \) is just \((x, \phi(x), d\phi(x))\). Since in our case \( \Omega^m(M) = A \times \bigwedge^3 V^* \), we get that the Lagrangian is a map

\[
L : A \times \mathbb{R} \times V^* \longrightarrow A \times \bigwedge^3 V^*
\]

covering the identity on \( A \) which reads as

\[
L(x, r, \mu) = \frac{1}{2} (\mu, \tilde{g}^{-1}(\mu)) g.
\]

If \( \phi \) is a section of \( p_{\tau_A} \) (i.e. a function on \( A \)), then

\[
L(x, \phi(x), d\phi(x)) = \frac{1}{2} (d\phi(x), \tilde{g}^{-1}(d\phi(x))) g.
\]

Let us look at the other spaces involved in the theory, namely

\[
V^* J^1E \otimes \Omega^m(M) = A \times \mathbb{R} \times V^* \times ((\mathbb{R} \times V) \otimes \bigwedge^3 V^*) \simeq A \times \mathbb{R} \times V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^*.
\]
An element of the above space will be denoted by \((x, r, \mu, \eta, \vartheta)\). There are two projections:

\[ \nu : A \times \mathbb{R} \times V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \rightarrow A \times \mathbb{R} \times V^* \]
and

\[ \xi : A \times \mathbb{R} \times V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \rightarrow A \times \mathbb{R} \times V^*. \]

The graph of the differential of the Lagrangian is a subset defined as

\[ dL(A \times \mathbb{R} \times V^*) = \{(x, r, \mu, 0, \tilde{g}^{-1}(\mu) \cdot g), \ x \in A, \ r \in \mathbb{R}, \ \mu \in V^* \}. \]

We see from the above that the phase space of the theory is

\[ \mathcal{P} = A \times \mathbb{R} \times \bigwedge^2 V^*. \]

We are looking for an equation for a section of the phase bundle over \(A\), i.e. for a map

\[ A \ni x \mapsto (x, \varphi(x), E(x)). \quad (4.32) \]

The Legendre map that associates the phase with a configuration is

\[ \lambda : A \times \mathbb{R} \times V^* \rightarrow A \times \mathbb{R} \times \bigwedge^2 V^*, \quad \lambda(x, r, \mu) = (x, r, \tilde{g}^{-1}(\mu) \cdot g). \]

Elements of \(\bigwedge^2 V^*\), i.e. two-forms, can also be interpreted as vector densities. In the case of electrostatics, the two-form or the vector density that we obtain here, integrated over a surface in \(A\), gives the flux of the electrostatic field through the surface.

The space \(J^1\mathcal{P}\) is in our case

\[ J^1\mathcal{P} = A \times \mathbb{R} \times \bigwedge^2 V^* \times (V^* \otimes \bigwedge^2 V^*). \]

An element of the above space will be denoted by \((x, r, p, \mu, \nu)\). Using the same symbols \((x, r, \mu)\) does not lead to any confusion, because the objects denoted by those symbols are conserved by every map we use. The analog of the space of external forces for the electrostatic field is

\[ V^* E \otimes \Omega^3(M) = A \times \mathbb{R} \times \bigwedge^3 V^*, \]

whose elements are \((x, r, \rho)\). The extended map \(\tilde{\alpha}\) reads as

\[ A \times \mathbb{R} \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times (V^* \otimes \bigwedge^2 V^*) \rightarrow A \times \mathbb{R} \times V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^*, \]

\[ (x, r, \rho, p, \mu, \nu) \mapsto (x, r, \mu, \text{Alt}(\nu) - \rho, p). \]

It means that the inverse image of \(dL(A \times \mathbb{R} \times V^*)\) by \(\alpha\) is the set \(D\) of all \((x, r, \rho, p, \mu, \nu)\) such that

\[ p = \tilde{g}^{-1}(\mu) \cdot g, \quad \text{Alt}(\nu) = \rho. \quad (4.33) \]

The set \(D\) represents an equation for part \((4.32)\) of the phase bundle: if \(r = \varphi(x)\) and \(p = E(x)\), we get

\[ \mu = d\varphi(x), \quad \nu = j^1E(x), \]
and the equations are
\[ E(x) = \tilde{g}^{-1}(d\phi(x)) \cdot g, \quad (4.34) \]
\[ dE(x) = \rho. \quad (4.35) \]

It is easy to see that substituting (4.34) into (4.35), we obtain
\[ d(\tilde{g}^{-1}(d\phi(x)) \cdot g) = \rho \]

which can be written as
\[ (\Delta \phi) g = \rho, \quad (4.36) \]
i.e. the Poisson equation for the potential of the electrostatic field produced by the charge density \( \rho \).

All the mathematical objects of the theory have clear physical meaning: the field itself is a potential for an electrostatic field. The electrostatic field is a covector field rather than a vector field; however, in the presence of a metric we can canonically translate one into the other. The phase is a vector density associated with the field and used to calculate the flux of the electrostatic field through a surface.

4.6. Scalar field

Let us end the section devoted to the Lagrangian side of the Tulczyjew triple for field theory with the example of a scalar field. The theory of a scalar field is based on the trivial fibration
\[ pr_M : E = M \times \mathbb{R} \to M, \]
where \( M \) is a manifold. Fields are therefore functions on \( M \). The first jet of a function \( \varphi : M \to \mathbb{R} \) at \( x \) is identified with a pair \( (\varphi(x), d\varphi(x)) \in \mathbb{R} \times T^*M \), i.e. \( J^1E \simeq \mathbb{R} \times T^*M \).

The following spaces are needed for the Lagrangian side of a Tulczyjew triple:
\[ \mathcal{P} \simeq V^*E \otimes \Omega^{m-1} \simeq \mathbb{R} \times \Omega^{m-1}, \]
\[ V^*J^1E \otimes \Omega^m \simeq \mathbb{R} \times T^*M \times_M \Omega^m \times_M J^1\Omega^{m-1}, \]
\[ J^1P \simeq \mathbb{R} \times T^*M \times_M J^1\Omega^{m-1}. \]

The map \( \alpha : J^1\mathcal{P} \to V^*J^1E \otimes \Omega^m \) reads as
\[ \alpha(\varphi, f, J^1p(x)) = (\varphi, f, dp(x), p(x)), \]
where \( (\varphi, f, J^1p(x)) \in \mathbb{R} \times T^*M \times_M J^1\Omega^{m-1} \) and \( x \mapsto p(x) \) is any representative of \( J^1p(x) \).

5. Hamiltonian formulation

In this section, we will construct the Hamiltonian side of the Tulczyjew triple. The name ‘Hamiltonian’ is usually associated with the time evolution of the system. In our approach, the Hamiltonian side of the triple gives just another way of generating phase dynamics. The interpretation of the Hamiltonian itself strongly depends on the particular theory. In some theories (electrostatics, electrodynamics), the Hamiltonian is related to the concept of energy of the field. In other theories, e.g., gravitation things get more complicated (see e.g. [28]).

The space of infinitesimal configurations \( J^1E \) is an affine bundle over \( E \); therefore, it is necessary to use the affine geometry and the notion of affine duality. In the following, we will recall this notion and construct an affine analog of the cotangent bundle. Since the affine bundle that appears in field theory, i.e. \( J^1\zeta : J^1E \to E \), has a reach internal structure, we decided to work first with simpler objects, and then apply the results to \( J^1\zeta \).
5.1. The affine-dual bundle

Let us first recall some facts from the geometry of affine spaces (for more details, see e.g \[19\]). Let \( \tau : A \to N \) be an affine bundle modeled on a vector bundle \( \nu : V \to N \). Let us also fix a one-dimensional vector space \( U \). We will use the symbol \( \nu^* \) for the projection \( V^* \otimes_N U \to N \).

The vector space of all affine maps from a fiber \( A_q, \ q \in N \), to \( U \) will be denoted by \( \text{Aff}(A_q, U) \). Every affine map has its linear part; therefore, \( \text{Aff}(A_q, U) \) is fibered over \( \text{Lin}(V_q, U) \cong V_q^* \otimes U \). Collecting the spaces \( \text{Aff}(A_q, U) \) point by point in \( N \), we obtain a vector bundle

\[
\tau^\uparrow : \text{Aff}(A, U) \to N
\]

and an affine bundle

\[
\theta : \text{Aff}(A, U) \to V^* \otimes_N U,
\]

The fibration \( \theta \) is an affine bundle modeled on the trivial vector bundle

\[
pr_1 : V^* \otimes_N U \times U \to V^* \otimes_N U.
\]

In the case \( U = \mathbb{R} \), the space \( \text{Aff}(A_q, \mathbb{R}) \) is called the affine dual of \( A_q \) and denoted \( A^*_q \).

It is always useful to write geometrical objects in coordinates. We will use a set of coordinates adapted to the structure. Let \( (x') \) denote a local system of coordinates in \( O \subset N \). Choosing a local basis \( e = (e_\alpha) \) of sections of \( V \), the dual basis \( \epsilon = (\epsilon^\alpha) \) of sections of \( V^* \), a reference section \( a_0 \) \( a_0 : O \to A \), and a non-zero vector \( u \in U \), we can construct the adapted system of coordinates

\[
(x', f^\alpha) \text{ in } A \quad \text{such that} \quad f^\alpha(a) = \epsilon^\alpha(a - a_0(q)), \quad q = \tau(a),
\]

and the adapted system of coordinates

\[
(x', \varphi_a, r) \text{ in } \text{Aff}(A, U)
\]

such that

\[
\varphi(a) = (\varphi_a \epsilon^\alpha(a - a_0(q)) + r)u, \quad q = \tau(a)
\]

for \( \varphi \in \text{Aff}(A, U) \). In coordinates, the projection \( \theta \) is expressed as the projection onto first two sets of coordinates \( (x', \varphi_a) \).

In the family of all smooth maps from \( \text{Aff}(A, U) \) to \( U \), we distinguish maps \( \Psi : \text{Aff}(A, U) \to U \) which are affine along fibers of \( \theta \) and satisfy

\[
\Psi(\varphi + u) = \Psi(\varphi) - u.
\]

Property (5.4) implies that in every fiber \( \theta^{-1}(p) \) of the fibration \( \theta \) there is exactly one point \( \varphi_p \) such that \( \Psi(\varphi_p) = 0 \). It means that the set \( \Psi^{-1}(0) \) is the graph of a section \( \Sigma_\varphi \) of the fibration \( \theta \). On the other hand, having a section \( \Sigma \) we can define

\[
\Psi_\Sigma(\varphi) = \Sigma(\theta(\varphi)) - \varphi.
\]

It is obvious that \( \Psi_\Sigma \) satisfies condition (5.4). Therefore, we have a one-to-one correspondence between smooth maps which are affine along fibers and satisfy condition (5.4) on one hand, and smooth sections of the bundle \( \theta \) on the other.

The differentials of maps satisfying property (5.4) are such covectors on \( \text{Aff}(A, U) \) with values in \( U \) that, restricted to vectors tangent to fibers of \( \theta \), they give \(-id\). The submanifold of such covectors will be denoted \( K_{-id} \). It is a coisotropic submanifold of \( T^* \text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U \) with respect to the canonical symplectic structure on \( T^* \text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U \) with values in \( U \). Using the local system of coordinates (5.3), we can construct the adapted system of coordinates on \( T^* \text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U \):

\[
(x', \varphi_a, r, \sigma_i, f^\alpha, \rho).
\]
In the above coordinates, the submanifold $K_{\text{id}}$ is given by the condition $\rho = -1$ and the canonical symplectic structure on $T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U$ reads as

$$\omega_{\text{Aff}(A, U)} = d\sigma_i \wedge d\xi^i + df^\alpha \wedge dq_\alpha + d\rho \wedge dr.$$  \hfill (5.6)

Having a coisotropic submanifold, we can perform a symplectic reduction. Leaves of characteristic foliation are orbits of the cotangent lift of the natural action of $U$ on $\text{Aff}(A, U)$. The reduced manifold will be denoted by $\text{PAff}(A, U)$. In coordinates, the reduction is the map

$$K_{\text{id}} \ni (x^i, \varphi_u, r, \sigma_i, f^\alpha, -1) \longmapsto (x^i, \varphi_u, \sigma_i, f^\alpha) \in \text{PAff}(A, U).$$

Elements of $\text{PAff}(A, U)$ can also be interpreted as equivalence classes of sections of the bundle $\theta$ with respect to the following equivalence relation. Since $\text{Aff}(A, U)$ is fibered over $V^* \otimes N U$ and the fibration is modeled on trivial fibration with the fiber being $U$, the difference $\Sigma_2 - \Sigma_1$ of two sections is a map from $V^* \otimes N U$ to $U$. It is therefore clear what means that $d(\Sigma_2 - \Sigma_1)(p) = 0$ for some $p \in V^* \otimes U$. We say that two pairs $(p_1, \Sigma_1)$ and $(p_2, \Sigma_2)$ are equivalent if and only if $p_1 = p_2$ and $d(\Sigma_2 - \Sigma_1)(p_1) = 0$. The equivalence class of $(p, \Sigma)$ is sometimes denoted by $d\Sigma(p)$ and called the differential of the section $\Sigma$ at the point $p$. The manifold $\text{PAff}(A, U)$ is obviously fibered over $V^* \otimes N U$. The fibration is an affine bundle modeled on $T^*(V^* \otimes N U) \otimes_{\text{Aff}(A, U)} U \to V^* \otimes N U$.

The above construction of $\text{PAff}(A, U)$ is analogous to the construction of $\text{PZ}$ for an affine bundle $Z \to M$, modeled on the trivial bundle $M \times \mathbb{R} \to M$ given in [19]. The only difference is that $\mathbb{R}$ is replaced by the one-dimensional vector space $U$.

The affine bundle $\text{PAff}(A, U)$ is actually a double bundle. The second bundle structure is inherited from the double bundle $T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U$. Let us first recall that the structure of the double vector bundle $T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U$

$$T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U \rightarrow \text{Aff}(A, U)$$

$$\xrightarrow{\iota} \tau^1$$

$$\xrightarrow{(\tau^1)^*} \text{Aff}(A, U)^* \otimes N U$$

is given by the two commuting Euler vector fields associated with the two vector bundle structures: the canonical one,

$$\nabla_1 = \sigma_i \frac{\partial}{\partial \sigma_i} + f^\alpha \frac{\partial}{\partial f^\alpha} + \rho \frac{\partial}{\partial \rho},$$

and the second one,

$$\nabla_2 = \varphi_u \frac{\partial}{\partial \varphi_u} + f^\alpha \frac{\partial}{\partial f^\alpha} + r \frac{\partial}{\partial r}.$$  \hfill (5.7)

The second projection $\pi_1$ can be understood as follows. The covector $\psi \in T^*\text{Aff}(A, U) \otimes U$ can be restricted to vectors tangent at $\psi$ to the fiber of the projection $\text{Aff}(A, U) \to N$, i.e. to the space $\text{Aff}(A, U)$. Any such fiber is a vector space; therefore, vectors tangent to the fiber can be identified with elements of the fiber itself. This leads to the identification of the restriction of $\psi$ with an element of the dual to the fiber, i.e. an element of $\text{Aff}(A, u)^* \otimes U$. We denote by $(\tau^1)^*$ the projection from $\text{Aff}(A, U)^* \otimes N U$ to $N$.

The coisotropic submanifold $K_{\text{id}}$ is an affine subbundle of the canonical bundle structure $\pi_{\text{Aff}(A, U)}$, and a vector subbundle (over a submanifold, see [22]) of the second bundle structure $\pi_1$. The image $\pi_1(K_{\text{id}})$ consists of all elements $h$ of $\text{Aff}(A, u)^* \otimes U$ that satisfy property (5.4):

$$h(\varphi + u) = h(\varphi) - u.$$
Elements of $\text{Aff}(A_q, U)^* \otimes U$ satisfying property (5.4) are in a one-to-one correspondence with elements of $A_q$ itself. A natural identification can be established as follows. Every $a \in A_q$ gives rise to a linear map on $\text{Aff}(A_q, U)$ with values in $U$ by evaluation, i.e. $h_a : \phi \mapsto \phi(a)$; however, $h_a$ does not satisfy (5.4). Property (5.4) is satisfied by $-h_a$, i.e. the map $\phi \mapsto -\phi(a)$.

For dimensional reasons, every element of $\text{Aff}(A_q, U)^* \otimes U$ that satisfies property (5.4) is of the form $-h_a$ for some $a \in A_q$. The Euler vector field $\nabla_2$ is tangent to $K_{-id}$ and projectable with respect to the symplectic reduction. It gives rise to a vector bundle structure $\mathcal{P}\text{Aff}(A, U) \to A$.

We have therefore the double bundle

$$
\begin{array}{ccc}
V^* \otimes N U & \overset{\nu^*}{\longrightarrow} & A \\
\mathcal{P}\text{Aff}(A, U) & \overset{P \theta}{\longrightarrow} & \mathcal{P} \\
\mathcal{P}\text{Aff}(A, U) & \overset{P \varsigma}{\longrightarrow} & A
\end{array}
$$

with the left projection being an affine bundle and the right projection being a vector bundle.

We can repeat the above constructions for $A = (j^1 \zeta \circ \zeta)^{-1}(x)$, i.e. the fiber of the bundle $J^1 E \to M$ that is fibered over $N = E$, $V = (\zeta \circ \rho_E)^{-1}(x)$ and $U = \Omega^m$. For simplicity, we will denote $(j^1 \zeta \circ \zeta)^{-1}(x)$ with $(J^1_e)$. The space $\text{Aff}(J^1_e, \Omega^m)$ will be called the affine dual of $J^1_e$ and denoted by $J^1_e$. Usually the affine dual of an affine space is the vector space off all affine functions on the affine space with real values. Here we replace real numbers with top forms on $M$, but we keep the name. As a result of the above general construction, we get the affine dual $J^1_e$ together with the affine fibration

$$
\theta_e : J^1_e \longrightarrow \mathcal{P}_e
$$

and the correspondence between sections of $\theta_e$ and maps which are affine along fibers of $\theta_e$ with values in $\Omega^m$. Collecting affine dual spaces $J^1_e$ point by point in $M$, we obtain a vector bundle

$$
{j^1 \zeta} : J^1 E \longrightarrow E
$$

and an affine bundle

$$
\theta : J^1 E \longrightarrow \mathcal{P}.
$$

Sections of the fibration $\theta$ are in a one-to-one correspondence with maps from $J^1 E$ to $\Omega^m$ covering the identity on $M$, affine along fibers of $\theta$, and satisfying property (5.4). We also have the manifold $\mathcal{P}J^1 E$ fibered over $\mathcal{P}$ and equipped with the canonical family of symplectic forms with values in $\Omega^m$, parameterized by points in $M$ and obtained by reduction from $V^* J^1 E \otimes_{J^1 E} \Omega^m$.

We then expect that the Hamiltonian description of the first-order field theory will be connected with the fibration $\theta : J^1 E \to \mathcal{P}$ and the space $\mathcal{P}J^1 E$. In particular, a Hamiltonian is a section of $\theta$ and the differential of this Hamiltonian at $p \in \mathcal{P}$ is an element of $\mathcal{P}J^1 E$. 

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therefore, the image of the differential of a Hamiltonian is a subset of \( P J^1 E \). Diagram (5.8) for \( J^1 E \) takes the form

\[
\begin{array}{c}
\text{\( \mathcal{P} \)} \\
\downarrow \pi \\
\text{\( E \)} \\
\end{array} \\
\begin{array}{c}
\text{\( P J^1 E \)} \\
\text{\( J^1 E \)} \\
\text{\( \mathcal{P} \)} \\
\end{array} \\
\begin{array}{c}
\text{\( \mathcal{P} \)} \\
\downarrow \pi \\
\text{\( E \)} \\
\end{array}
\]

(5.9)

**Theorem 5.1.** The bundle \( P J^1 E \) carries a canonical structure of a double affine-vector bundle structure (5.9).

Let us end this section by constructing coordinates in \( P J^1 E \) adapted to the structure of the double bundle. In section 3, we have introduced coordinates \((q^i, y^a, y^b_j)\) in \( J^1 E \) and in section 4.1 coordinates \((q^i, y^a, p^j_a)\) in \( \mathcal{P} \). Since \( J^1 E \) is fibered over \( \mathcal{P} \) and the fibration is an affine bundle modeled on the trivial bundle \( \mathcal{P} \times_M \Omega^m \to \mathcal{P} \), it will be convenient to use coordinates \((q^i, y^a, p^j_a, r)\) in \( J^1 E \) such that \( r \) is an affine coordinate along the fibers of \( \theta \). If \( \varphi = (q^i, y^a, p^j_a, r) \) and \( J^1(\sigma(x)) = (q^i, y^a, y^b_j) \), then

\[
\varphi(J^1(\sigma(x))) = p^j_b y^b_j + r.
\]

In \( V^* J^1 E \otimes_{J^1 E} \Omega^m \) we therefore have the adopted coordinate system \((q^i, y^a, p^j_a, r, \xi_a, y^b_k, \rho)\) and the coisotropic submanifold \( K_{-\rho} \) is given by the condition \( \rho = -1 \). Diagram (5.7) in the case \( A = J^1 E \) takes the form

\[
\begin{array}{c}
\text{\( \pi^* J^1 E \otimes_{J^1 E} \Omega^m \)} \\
\downarrow \pi \\
\text{\( J^1 E \)} \\
\end{array} \\
\begin{array}{c}
\text{\( \pi^* J^1 E \otimes_{J^1 E} \Omega^m \)} \\
\downarrow \pi \\
\text{\( E \)} \\
\end{array}
\]

(5.10)

that in coordinates reads as

\[
\begin{array}{c}
(q^i, y^a, p^j_a, r, \xi_a, y^b_k, \rho) \\
\downarrow \pi \\
(q^i, y^a, p^j_a, r) \\
\end{array} \\
\begin{array}{c}
(q^i, y^a, p^j_a, r) \\
\downarrow \pi \\
(q^i, y^a) \\
\end{array} \\
\begin{array}{c}
(q^i, y^a, y^b_j, \rho) \\
\downarrow \pi \\
(q^i, y^a) \\
\end{array}
\]

(5.11)
After the reduction, we obtain coordinates \((q^i, y^a, p^j_a, \xi_a, y^b_k)\) in \(P^1 E\). Diagram (5.9) in coordinates takes the form

\[
\begin{array}{ccc}
(q^i, y^a, p^j_a, \xi_a, y^b_k) & \xrightarrow{P_{\theta}} & (q^i, y^a, p^j_a) \\
& \xrightarrow{\pi} & (q^i, y^a) \\
& \xrightarrow{j^i \zeta} & (q^i, y^a, y^b_k)
\end{array}
\]

The family of symplectic forms with values in \(\Omega^m\) parameterized by points of \(M\) is given in coordinates as

\[
\omega_{P^1 E} = (d\xi_a \wedge dy^a + dy^b_k \wedge dp^j_b) \otimes \eta.
\]

5.2. The map \(\beta\)

In this section, we will construct the map \(\beta\) which will be used in deriving the phase dynamics of the field from a Hamiltonian. It will be a field-theoretical version of \(\rho_0\) (see (2.19)). In mechanics, there is a well-known formula relating Lagrangians to Hamiltonians, namely

\[
H(p) = \langle p, v \rangle - L(v).
\]

The origin of this formula lies in the procedure of composing symplectic relations [37, 38, 1]. Recall that a symplectic relation between symplectic manifolds \((P_1, \omega_1), (P_2, \omega_2)\) is a Lagrangian submanifold in \((P_1 \times P_2, \omega_1 - \omega_2)\). If we deal with cotangent bundles, we can think of generating objects for symplectic relations. For example, it is well known that there is a canonical symplectomorphism \(R_F : T^*F \rightarrow T^*F^*\) for any vector bundle \(\tau : F \rightarrow M\). The graph of \(R_F\) is the Lagrangian submanifold in \((T^*F \times T^*F^*, \omega_F - \omega_{F^*})\) generated by the evaluation of covectors and vectors \(F \times_M F^* \ni (f, \varphi) \mapsto \varphi(f) \in \mathbb{R}\) in the following sense. The evaluation is a function defined on the submanifold \(F \times_M F^* \subset F \times F^*\); therefore, it generates a Lagrangian submanifold in \(T^*(F \times F^*)\). The space \(T^*(F \times F^*)\) can be naturally identified with \((T^*F \times T^*F^*)\). To get a Lagrangian submanifold with respect to the form \(\omega_F - \omega_{F^*}\), we apply the transformation

\[
T^*F \times T^*F^* \ni (\zeta_1, \zeta_2) \mapsto (\zeta_1, -\zeta_2) \in T^*F \times T^*F
\]

to the generated submanifold. In the adapted coordinates \((x^i, f^a)\) in \(F\), \((x^i, f^a, \sigma_i, \psi_a)\) in \(T^*F\) and \((x^i, \psi_a, \sigma_i, h^a)\) in \(T^*F^*\), the isomorphism \(R_F\) reads

\[
R_F(x^i, f^a, \sigma_i, \psi_a) = (x^i, \psi_a, \sigma_i, -f^a).
\]
To clarify all sign problems, let us mention that the family (5.16) is a generating object for a symplectic relation between a single point and $T^*F^*$. For a generating object of the Lagrangian submanifold $R_F(dL(F))$, we have to take $-H$.

In mechanics, the whole procedure is applied to $F = TQ$. Then, $-H$ is the generating object of a Lagrangian submanifold $D_H$ in $T^*T^*Q$. The dynamics, i.e., a subset of $TT^*Q$, is obtained as the inverse image of $D_H$ by the map $\beta_Q : TT^*Q \rightarrow T^*T^*Q$:

$$\beta_Q = \alpha_Q \circ R_{TQ}. \tag{5.17}$$

In this case, $\beta_Q$ is also associated with $\omega_Q$:

$$\beta_Q(v) = \omega_Q(v, \cdot) \quad \text{for} \quad v \in TT^*Q. \tag{5.18}$$

In the classical field theory, we need an affine version of the above procedure, since in the space of infinitesimal configurations $J^1E$ we only have an affine structure on fibers over $E$. Moreover, we have to replace real-valued Lagrangians with Lagrangians taking values in the vector bundle $\Omega^m$.

For simplicity, let us first work with an affine bundle $A \rightarrow N$ and a vector space $U$ like in the previous section. The canonical evaluation between elements of $A$ and elements of $\text{Aff}(A, U)$ is now a map defined on a submanifold $A \times_N \text{Aff}(A, U) \subset A \times \text{Aff}(A, U)$ with values in $U$. The evaluation in coordinates (chosen as in the previous section) reads

$$A \times_N \text{Aff}(A, U) \ni (x^i, f^a, \varphi_u, r) \longmapsto (f^a \varphi_u + r)u \in U.$$ 

The canonical evaluation generates a Lagrangian submanifold of

$$(T^*A)^{(A \times \text{Aff}(A, U)) \otimes (A \times \text{Aff}(A, U))} U$$

with respect to the canonical $U$-valued symplectic form. There is an identification of the above cotangent bundle with

$$T^*A \otimes_A U \times T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U.$$ 

The canonical symplectic form is identified with $\omega_A + \omega_{\text{Aff}(A, U)}$. If we use the adapted coordinates $(x^i, f^a, \sigma_i, \psi_u)$ in $T^*A \otimes_A U$ and appropriate coordinates $(x^i, \varphi_u, r, \sigma_i', h^a, \rho)$ in $T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U$, we get the generated Lagrangian submanifold given by the conditions

$$x^i = x^i, \quad \sigma_i = -\sigma_i', \quad \psi_u = \varphi_u, \quad f^a = h^a, \quad \rho = 1.$$ 

To get a symplectic relation out of that submanifold, we have to change signs in the fiber of $T^*\text{Aff}(A, U) \otimes_{\text{Aff}(A, U)} U$. The graph of $\tilde{R}_A$ is therefore given by the conditions

$$x^i = x^i, \quad \sigma_i = \sigma_i', \quad \psi_u = \varphi_u, \quad f^a = -h^a, \quad \rho = -1.$$ 

We observe that the relation $\tilde{R}_A$ is a map defined on the submanifold $K_{-id}$ with values in $T^*A \otimes_A U$. In coordinates,

$$\tilde{R}_A(x^i, \varphi_u, r, \sigma_i, f^a, -1) = (x^i, -f^a, \sigma_i, \varphi_u). \tag{5.19}$$

We recall that $K_{-id}$ consists of the differentials of $U$-valued functions on $\text{Aff}(A, U)$ satisfying property (5.4). Since $\tilde{R}_A$ is constant on leaves of the characteristic foliation of $K_{-id}$, it reduces to a symplectomorphism

$$R_A : T^*A \otimes_A U \longrightarrow \text{PAff}(A, U). \tag{5.20}$$

For any function $L : A \rightarrow U$, from $dL(\tilde{A}) \subset T^*A \otimes_A U$ we obtain a Lagrangian submanifold of $\text{PAff}(A, U)$. As a generating object, we can choose a family of functions on $\text{Aff}(A, U)$ parameterized by elements of $A$:

$$H : A \times_N \text{Aff}(A, U) \longrightarrow U, \quad H(a, \varphi) = L(a) - \varphi(a). \tag{5.21}$$
Note that $H$ satisfies condition (5.4), i.e.

$$H(a, \varphi + u) = H(a, \varphi) - u.$$  

It generates, of course, a submanifold of $T^*\text{Aff}(A, U) \otimes U$ which, after reduction, equals $R_A(dL(A))$. If the family reduces to a single function, the latter corresponds to a certain section $\Sigma_H$ of the bundle $\theta : \text{Aff}(A, U) \to V^* \otimes U$.

Applying again the above constructions to $A = J^1_1E$ and recalling that we use the notation $\text{Aff}(J^1_1E, \Omega^m) = J^1_1E$, we get a diffeomorphism

$$R_j : V^*J^1_1E \otimes \Omega^m \to P^jE$$

which, restricted to every fiber over $M$, is a symplectomorphism with respect to appropriate symplectic $\Omega^m_\varphi$ valued forms. We will denote with $\beta$ the composition

$$\beta = \alpha \circ R_j,$$ 

$$\beta : J^1_1P \to P^jE. \quad (5.22)$$

In local coordinates, we get

$$\beta(q^i, y^a, p^j_b, y^b_k, p^j_{ab}) = \left(q^i, y^a, p^j_b, \sum_k p^j_{ik}, -y^b_k\right). \quad (5.23)$$

**Theorem 5.2.** There is a canonical isomorphism of double affine-vector bundles,

$$R_{j_1E} : V^*J^1_1E \otimes \Omega^m \to P^jE.$$ 

This isomorphism induces a morphism $\beta : J^1_1P \to P^jE$ of double affine-vector bundles via $\beta = \alpha \circ R_{j_1E}$.

The map $\beta$ constitutes the Hamiltonian side of the Tulczyjew triple for the classical field theory:

$$\begin{align*}
\beta & : J^1_1P \to P^jE, \\
\beta & : J^1_1P \to P^jE.
\end{align*} \quad (5.24)$$

The above construction shows that, in the first-order field theory, a Hamiltonian is not a density-valued function on the phase space, but a section of certain affine bundle over phase space with one-dimensional fibers. Differentials of such sections are elements of an affine analog of the cotangent bundle. From the construction, we obtain a family of Hamiltonian sections parameterized by elements of $J^1_1E$:

$$\Sigma_H : J^1_1E \times E P \to J^1_1E, \quad (5.25)$$

which corresponds to a family of density-valued maps

$$H : J^1_1E \times E J^1_1E \to \Omega^m, \quad H(j^1_1\sigma, \varphi) = L(j^1_1\sigma) - \varphi(j^1_1\sigma). \quad (5.26)$$

In some cases, the above family reduces to a single generating section. It happens e.g. in electrostatics (see section 5.4). In such cases, we obtain $\mathcal{D}$ from the image of the differential of the Hamiltonian section $H : P \to J^1_1E$ by means of the map $\beta$. More precisely,

$$\mathcal{D} = \beta^{-1}(dH(P)). \quad (5.27)$$

Like in the Lagrangian case, the process of generating phase dynamics from a Hamiltonian is very simple.
5.3. Structure of the phase space

The phase space $\mathcal{P}$ is fibered over $E$. The fibration is a vector bundle. The space of vertical vectors $\mathcal{V}\mathcal{P}$ is therefore a double vector bundle fibered over $\mathcal{V}E$ and $\mathcal{P}$,

$$
\begin{array}{c}
\mathcal{P} \\
\downarrow \theta \\
E \\
\uparrow \nu_E
\end{array}
\begin{array}{c}
\mathcal{V}\mathcal{P} \\
\downarrow \nu_P \\
\mathcal{V}E \\
\uparrow \nu
\end{array}
$$

We define a one-form $\vartheta_P$ on $\mathcal{P}$ with values in $\Omega^{m-1}$ by the formula

$$
\vartheta_P(\delta p) = \langle \nu_P(\delta p), V_\theta(\delta p) \rangle.
$$

(5.28)

In coordinates, for $\delta p = (x^i, y^a, p^j_a, \delta y^c, \delta p^k_d)$, we get

$$
\vartheta_P(\delta p) = p^j_a \delta y^a \eta^i + p^k_d \delta p^k_d \otimes \eta.
$$

(5.29)

The form $\vartheta_P$ is an analog of the canonical Liouville form on a cotangent bundle. Applying $d_M$ to $\vartheta_P$, we obtain a one-form on $J^1\mathcal{P}$ with values in $\Omega^m$ which in coordinates reads

$$
d_M \vartheta_P = p^j_a \delta y^a \otimes \eta + p^k_d \delta p^k_d \otimes \eta.
$$

(5.30)

and which is an analog of $d_M \vartheta_M$, the Liouville form on $T^*M$.

Applying vertical differential $d$ to $d_M \vartheta_P$, we obtain a two-form on $J^1\mathcal{P}$ which can be treated as a family of presymplectic forms with values in $\Omega^m$ parameterized by points of $M$,

$$
\omega_{J^1\mathcal{P}} = d_M d \vartheta_P = d p^j_a \wedge dy^a \otimes \eta + dp^k_d \wedge dp^k_d \otimes \eta.
$$

(5.31)

It is easy to see in coordinates that

$$
\beta^\ast \omega_{J^1\mathcal{P}} = \omega_{J^1\mathcal{P}}.
$$

(5.32)

There is an alternative construction of the map $\beta$ that uses the language of differential forms on fiber bundles. The crucial role in the construction is played by the two-form $d \vartheta_P$ with values in $\Omega^{m-1}$ [52].

5.4. Electrostatics

The Hamiltonian side of the Tulczyjew triple for electrostatics is simplified, because fields are sections of the trivial bundle $pr_A : \mathbb{A} \times \mathbb{R} \to \mathbb{R}$. The bundle $J^1pr_A$ is therefore a vector bundle:

$$
J^1pr_A : J^1\mathcal{P} = \mathcal{A} \times \mathbb{R} \times \mathbb{V}^* \to \mathcal{A} \times \mathbb{R}.
$$

The affine dual to this vector bundle is again the Cartesian product. Note that $\Omega^m \simeq \mathbb{A} \times \bigwedge^3 \mathbb{V}^*$.

We have the following identifications:

$$
J^1\mathcal{E} \simeq \mathcal{A} \times \mathbb{R} \times \mathbb{V}^*,
$$

$$
J^1\mathcal{E} \simeq \mathcal{A} \times \mathbb{R} \times \bigwedge^2 \mathbb{V}^* \times \bigwedge^3 \mathbb{V}^*,
$$

$$
\mathcal{P} \simeq \mathcal{A} \times \mathbb{R} \times \bigwedge^2 \mathbb{V}^*.
$$

The bundle $\theta : J^1\mathcal{E} \to \mathcal{P}$ is trivial, i.e.

$$
\theta : \mathcal{A} \times \mathbb{R} \times \bigwedge^2 \mathbb{V}^* \to \mathcal{A} \times \mathbb{R} \times \bigwedge^2 \mathbb{V}^*.
$$
There is a natural action of $J$ in the fibers of the above bundle by addition:

$$
\left(\bigwedge^3 V^*\right) \times \left(A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^*\right) \to A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^*.
$$

$$(u, (x, r, p, \lambda)) \mapsto (x, r, p, \lambda + u).$$

The canonical evaluation between $(x, r, \mu) \in J^1E$ and $(x, r, p, \lambda) \in J^1E$ with values in $\bigwedge^3 V^*$ reads

$$
\langle (x, r, \mu), (x, r, p, \lambda) \rangle = \mu \wedge p + \lambda.
$$

The above evaluation generates a relation $\mathcal{R}$ between the cotangent bundle of the space of infinitesimal configurations,

$$
V^*J^1E \otimes \Omega^m \simeq A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times \bigwedge^3 V^*.
$$

and the cotangent bundle of the affine dual,

$$
V^*J^1E \otimes \Omega^m \simeq A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \mathbb{R}.
$$

An element $(x, r, \mu, \sigma, \omega) \in V^*J^1E \otimes \Omega^m$ is in the relation $\mathcal{R}$ with an element $(x, r, p, \lambda, a, b, c) \in V^*J^1E \otimes \Omega^m$, if and only if

$$
p = \omega, \quad a = -\sigma, \quad b = \mu, \quad c = 1.
$$

There is an action of $\bigwedge^3 V^*$ in the cotangent bundle $V^*J^1E \otimes \Omega^m$ lifted from the action in $J^1E$. The image of $\mathcal{R}$ (defined by $c = 1$) is invariant with respect to the lifted action and the quotient space:

$$
P_J^1E \quad \text{can be identified with} \quad A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times \bigwedge^3 V^*.
$$

The graph of $\mathcal{R}$ is also invariant with respect to the lifted action; therefore, there exists the quotient relation between $V^*J^1E \otimes \Omega^m$ and $P_J^1E$. This quotient relation is actually a map,

$$
\tilde{\mathcal{R}} : V^*J^1E \otimes \Omega^m \to P_J^1E,
$$

$$(x, r, \mu, \sigma, \omega) \mapsto (x, r, \mu, -\sigma, \mu).$$

Composing the map $\tilde{\mathcal{R}}$ with $\alpha$ from the Lagrangian side, we get the map $\beta$

$$
\beta : V^* \otimes \Omega^m \times E J^1P \to P_J^1E,
$$

i.e.

$$
A \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \bigwedge^2 V^* \times \bigwedge^3 V^* \times \mathbb{R} \times \bigwedge^2 V^* \times \bigwedge^3 V^*, \quad (x, r, \rho, p, \mu, v) \mapsto (x, r, p, \rho - \lambda g(v), \mu).
$$

Equation (4.35) for a section of the phase bundle $P \to A$ can be generated from a Hamiltonian. The generating family

$$
h(x, r, p, \lambda, \mu) = L(x, r, \mu) - \mu \wedge p - \lambda g
$$

reduces to a section $H$ of the bundle $\theta$. Since the bundle is trivial, this section can be represented by the map

$$
H : P \to \bigwedge^3 V^*, \quad H(x, r, p) = \frac{1}{2}(sp) \wedge p.
$$
where ∗ is the Hodge-star associated with the metric g. The Hamiltonian \( H \) generates the subset

\[ dH(P) \subset PJ^1E, \quad dH(P) = \{ (x, r, p, a, b) : a = 0, b = *p \}. \]

The inverse image of \( dH(P) \) by \( \beta \) is

\[ \beta^{-1}(dH(P)) = \{ (x, r, p, ρ, μ) : ρ = \text{Alt}(ν), μ = *p \}. \]

Comparing (5.34) with (4.33), we see that

\[ D = \beta^{-1}(dH(P)). \]

5.5. Scalar field

Since \( J^1E = \mathbb{R} \times T^*M \rightarrow M \times \mathbb{R} = E \) is a vector bundle, the Hamiltonian side of the Tulczyjew triple is simplified. An affine map \( A : \mathbb{R} \times T^*M \rightarrow \Omega^m \) is determined by \( p \in \Omega^{m-1} \) and \( a \in \Omega^m \):

\[ A(φ, f) = f \wedge p + a. \]

We therefore have

\[ J^1E \simeq \mathbb{R} \times \Omega^{m-1} \times M \Omega^m, \]
\[ PJ^1E \simeq \mathbb{R} \times \Omega^{m-1} \times_M T^*M \times_M \Omega^m. \]

The map \( β : J^1P \rightarrow PJ^1E \) reads as

\[ β(φ, f, j^1p) = (φ, p, f, dp(x)), \]

where \( x \mapsto p(x) \) is any representative of \( j^1p \).

6. Conclusions

We have constructed the Tulczyjew triple for the first-order field theory, starting from fundamental concepts of calculus of variations. Our results can be summarized in the following diagram of affine and vector bundle morphisms:

\[ \begin{array}{ccccccc}
\pi & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
J^1E & \xrightarrow{β} & J^1P & \xrightarrow{α} & V^*J^1E \otimes \Omega^m \\
\beta & & \beta & & \beta & & \beta \\
\xi & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
J^1E & \xrightarrow{π} & J^1E & \xrightarrow{j^1π} & J^1E & \xrightarrow{j^1π} & J^1E \\
π & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E & \xrightarrow{j^1E} & E & \xrightarrow{j^1E} & E & \xrightarrow{j^1E} & E \\
\end{array} \]

The left-hand side of the triple is Lagrangian, the right-hand side is Hamiltonian and the phase dynamics lives in the middle. The phase dynamics \( D \) being a subset of \( J^1P \) is interpreted as a condition for first jets of sections of the momentum bundle. It can be obtained from a Lagrangian as

\[ D = \alpha^{-1}(dL(J^1E)) \]
or from a Hamiltonian as
\[ \mathcal{D} = \beta^{-1}(d\Sigma_H(P)). \]

The Hamiltonian is a section of an affine bundle \( \theta \) with fibers modeled on the vector spaces of volume elements on \( M \).

The Lagrangian side of the triple looks similarly in almost all papers devoted to the Tulczyjew triple for field theory. However, we would like to emphasize that all the spaces and maps that we use are constructed, not postulated, and have clear interpretation in the language of variational calculus. It is also interesting that the phase space \( \mathcal{P} \) is not dual to the space of infinitesimal configurations \( J^1E \).

To construct the Hamiltonian side of the triple, we have used the notion of affine duality. Geometrical language for the first-order theory is not the only place in classical mathematical physics where affine structures are needed. Constructions that were used in section 5 are similar to those needed in time-dependent mechanics [19] or in the intrinsic formulation of Newtonian mechanics [18]. We also expect that the notion of affine duality will play an important role in higher-order theories. The Hamiltonian formulation of the first-order field theory is based on the canonical isomorphism \( PJ^1E \cong V^*J^1E \otimes \Omega^m \) generated by the evaluation between \( J^1E \) and its affine-dual bundle.

In the Tulczyjew triple for mechanics, all three spaces, \( T^*T^*Q \), \( TT^*Q \) and \( T^*TQ \), are isomorphic. It is not the case in the field theory. The middle space is not isomorphic to \( PJ^1P \) and \( V^*J^1E \otimes \Omega^m \). To have an isomorphism between all three spaces, we could replace the space \( J^1P \) with the quotient space with respect to a certain equivalence relation. We decided not to do it, because passing to the quotient we would lose the obvious interpretation of the constitutive set as a first-order differential equation. However, it is clear that only very special differential inclusions \( \mathcal{D} \subset J^1P \) are generated by some Lagrangian or Hamiltonian.

The spaces \( PJ^1E \) and \( V^*J^1E \otimes \Omega^m \) are equipped with canonical two-forms that, restricted to every fiber, are symplectic forms with values in \( \Omega^m \). We also have a canonical presymplectic form on fibers of the bundle \( J^1P \to M \). The phase space \( \mathcal{P} \) possesses a canonical one-form with values in \( \Omega^{m-1} \) which is an analog of the canonical Liouville form on \( T^*M \).

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