Quantum Mechanics from Focusing and Symmetry.

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Abstract

A foundation of quantum mechanics based on the concepts of focusing and symmetry is proposed. Focusing is connected to c-variables - inaccessible conceptually derived variables; several examples of such variables are given. The focus is then on a maximal accessible parameter, a function of the common c-variable. Symmetry is introduced via a group acting on the c-variable. From this, the Hilbert space is constructed and state vectors and operators are given a clear interpretation. The Born formula is proved from weak assumptions, and from this the usual rules of quantum mechanics are derived. Several paradoxes and other issues of quantum theory are discussed.

1 Introduction

Nobody doubts today that quantum mechanics is a true theory. But even though the calculations devised by the standard theory give accurate predictions that all researchers agree upon, the same theory is felt by many as being too abstract and too formal. From the very beginning [1] leading physicists and mathematicians have tried to find a new and better logical basis for the theory, and this search has continued until now [2,3,4,5,6]. The discussions around the foundation of quantum mechanics are not just theoretical; obscurities connected to foundational issues also has as a consequence that one may be uncertain whether certain applied statements should be classified as facts or myths [7].

Zeilinger [2] compares the situation with that of the special and general relativity theory, which both are based on firm foundational principles, and requests such principles for quantum theory. I agree that what one needs is not a new formal axiomatic formulation of the mathematical foundations of quantum mechanics, but completely new foundational conceptually defined principles. In this paper I propose two: Maximal focusing in a situation with inaccessible conceptually defined variables, and symmetry as given by a group acting on these variables. These principles will be made more precise shortly, and they
will be demonstrated in the paper to lead to the essential parts of quantum
theory. What may be lacking in the present development, is a description of
more complicated quantummechanical situations without symmetry. For these
situations we can refer to the extensive new foundational recent work by Doering
and Isham [8, 9, 10, 11] based on category theory. Note that group theory may
be looked upon as a specialization of category theory, a specialization which
illustrates many basic features of that theory.

In contrast to [8, 9, 10, 11], I will here want to keep the foundation simple.
In fact, part of my aim will be to connect the discussion to daily life concepts.
Also, a connection to statistical inference will be made clear, a connection which
is crucial, Quantum theory and statistical theory are both concerned with pre-
dictions based upon observations, yet all indications of any connection between
these two worlds seem up to now to be completely absent from the literature.
Finally, the explicit use of group theory in the foundation may have interest in
itself. The physics literature is full of examples where group theory has turned
out to be useful in a quantummechanical setting.

In agreement with the statistical tradition I will regard all measurement
apparatus as macroscopic, and I will use the ordinary statistical inference theory
on all measurements made. In particular, this implies that the measurement
problem in its ordinary quantum mechanical formulation where a quantum state
also is given to the measurement apparatus, is barely touched upon here. For
a recent survey paper where the measurement problem is related to various
interpretations of quantum theory, see Wallace [12].

Also, by following up the statistical way of thinking, the interpretation of
the quantum states advocated here is epistemic. In accordance with Niels Bohr’s
words 'It is wrong to think that the task of Physics is to find out what Nature
is. Physics concerns what we can say about nature.' [13], this can be said to be
in agreement with the classical Copenhagen school.

2 Briefly on statistical theory.

A basic concept is that of a statistical parameter $\lambda$. The fundamental questions
that we ask nature, are in terms of such a parameter: What is the value of $\lambda$?
Later in the present paper, all parameters are discrete, and such a question can
be answered literally in a simple way. For continuous parameters, statistical
theory offers the tools of point estimation, interval estimation and hypothesis
testing.

In any case, what lies behind these inference procedures, is a statistical
model: A probability distribution $P^\lambda(\cdot)$ of the observed data, given the value of
the parameter $\lambda$. This leads to a very rich theory, where inference statements
about $\lambda$ can be formulated in a precise way, and where one also may formulate
in various ways just how precise of these statements are. The simplest possible
statistical model is a perfect one, where each $P^\lambda$ is just 0 or 1 as functions of
the data, most often such that the sets where $P^\lambda$ is 1, are disjoint for different
$\lambda$. In this case the given observation leads to a unique estimated value $\hat{\lambda}$ This
simplification may be relevant in some physical applications.

In general, an estimator of a parameter $\lambda$ is a function $\hat{\lambda}(y)$ of the observations used to estimate $\lambda$. The problem of finding an estimator will in general not have a unique solution. One aim of statistics is to find good estimators.

There are two schools of statistical inference, the frequentist and the Bayesian. The Bayesians allow themselves to make use of, in addition to the statistical model, prior probabilities on $\lambda$. In the present paper I will also make extensive use of probabilities on the parameter space, but now introduced by Born’s formula, and thus being transition probabilities between different states. One way to look upon these probabilities are as priors for the next experiment, given the result of an earlier perfect experiment.

The main idea from statistical theory that I will use in this paper, is that of using parameters as intermediate quantities when asking questions about nature. For those who want to read more about statistical theory, I can recommend on the intermediate level books [14, 15] or more advanced books like [18, 19, 20].

3 Conceptually defined variables, and focusing.

There has been a long debate on hidden variables in quantum mechanics, see, e.g. [21, 22]. Today these words are met with much scepticism, but a related concept, hidden measurements, has been introduced successfully by Aerts [23, 24].

Hidden variables are assumed to take values. As a contrast, in this paper I will base much of the discussion upon conceptually defined variables, abbreviated $c$-variables, and denoted by $\phi$, which as a rule do not take values at all. To some, this may seem like a strange construction, but in fact both our daily language and the scientific language is full of such conceptual variables.

A design of experiment example. Consider a single patient who can be given one out of two possible treatments 1 and 2. Let $\lambda_1^1$ be his expected life time if he is given treatment 1; let $\lambda_2^1$ be his expected life time if he is given treatment 2. Consider the vector $\phi = (\lambda_1^1, \lambda_2^1)$. This variable can perhaps be given a hypothetical value, but it can never be given a real value which can be compared to observations. More precisely, $\phi$ can never be a parameter of a statistical model. But we can focus upon $\lambda_1^1$, and use this to make a model for the lifetime of the patient.

Counterfactual parameters. As a continuation of the previous example: In very many cases at the experimental design phase it is natural to define a conceptual variable or $c$-variable

$$\phi = (\lambda_1^1, \lambda_2^1, ..., \lambda_r^r)$$

at the outset. Only one parameter $\lambda_j^1$ is realized by the experiment; the rest are called counterfactuals. Counterfactual variables/parameters are also important in other cases than statistical experiments. They have turned out to be essential in causal reasoning; see the book by Pearl [25].

3
A special apparatus. An apparatus for a very special length measurement is so sensitive that it is destroyed after one single measurement. Let $\mu$ be the length which is to be measured. Assume furthermore that the measurement uncertainty $\sigma$, defined at the outset in a Bayesian way, only can be estimated by destroying the whole apparatus. Let $\phi = (\mu, \sigma)$. Then it is impossible to estimate the whole c-variable $\phi$, only $\mu$ or $\sigma$ can be estimated. In general, the choice of experimental question is essential.

Too many parameters. In linear prediction problems one often has a large set of potential explanatory variables, resulting in a formal regression model

$$y = \beta^1 x^1 + \beta^2 x^2 + \ldots + \beta^p x^p + e,$$

assuming the variables centered for simplicity. Here $e$ is an independent error term. When $p$ is large compared to the number $n$ of experimental units, in particular if $p$ is much larger than $n$, this can hardly be called a data model. And $\phi = (\beta^1, \beta^2, \ldots, \beta^p)$ is hardly a parameter vector, but more a c-variable. Again one must focus from this, but now the focusing is not necessarily related to a choice of experiment. What one can do in this prediction situation - and also does in practice - is to focus on a certain parameter

$$\theta = (\beta^{i_1}, \ldots, \beta^{i_q})$$

and the corresponding model, using both the data and the conceptual setting in the selection.

In most of the examples above, the c-variable $\phi$ is such that it is impossible to estimate it from the available data. It is then called inaccessible.

The focusing takes place by a question and an answer: To find out something about nature, one must not only look in all directions for facts, but often focus upon a parameter $\lambda$, an accessible part of $\phi$, and then look for answer to the question: What is the value of $\lambda$?

In my interpretation of quantum mechanics, two questions are called complementary if they are given by mutually exclusive functions of a common inaccessible c-variable $\phi$. More precisely, this means: Let the parameters of the two questions be $\lambda$ and $\mu$. Then the vector variable $\theta = (\mu, \lambda)$ is in itself an inaccessible c-variable. An example might be where $\mu$ is the theoretical position and $\lambda$ is the theoretical momentum of a single particle. The word ‘theoretical’ here just indicates that a measurement apparatus because of measurement uncertainties may give different values. This latter phenomenon is a rather trivial one, but provides the link from statistics to physics, in particular the link from statistical parameters to theoretical physical variables.
4 Symmetry in the parameter space.

Assume in this Section that a symmetry group $G$ acting on a parameter space $\Theta$ and simultaneously on the space of observations by

$$P^g(A) = P^g(Ag^{-1}).$$

Some examples may be scale change, translation or rotations. It will turn out later to be advantageous to place the group symbol to the right of the quantity to transform.

Note that $P^g(yg \in B)$ is equal to $P^g(y \in Bg^{-1})$, thus $P^g(y \in B)$; so transformation of observations is related to transformation of parameters in simple situations. In this paper we will concentrate on the group action on the parameters of potential statistical models or on a single model, thus on the parameter space $\Theta$.

4.1 Orbits. Transitive group.

Fix $\theta_0 \in \Theta$, a point in the parameter space. Consider the set $\{\theta_0 g : g \in G\}$, the set of all parameter values that are transforms of $\theta_0$. This is called the orbit of $G$ containing $\theta_0$.

Here is an example which may be illustrative: Let $\Theta$ be like the surface of the earth, and let $G$ act as does the rotation of the earth. Then the orbits will be the circles of latitude.

The space $\Theta$ is always partitioned into disjoint orbits. A useful way to look upon an orbit is that it is a minimal invariant set under the group.

If there is only one orbit, this will consist of the whole space $\Theta$. Then we say that the group is transitive, or more precisely: $G$ is acting transitively upon $\Theta$.

4.2 Invariant measure.

Under weak conditions there exists a right-invariant measure $\rho$ on the parameter space:

$$\rho(\Gamma g) = \rho(\Gamma) \text{ for } g \in G \text{ and all } \Gamma \subseteq \Theta.$$ 

The measure $\rho$ can be taken as a probability measure if $\Theta$ is compact. This measure $\rho$ is unique if and only if the parameter group is transitive. (In the noncompact case this uniqueness is up to a multiplicative constant.)

Whenever there is a natural symmetry group acting upon $\Theta$, in particular if it is transitive, there are many arguments for using the right invariant measure $\rho$ as a noninformative prior in Bayesian data analysis.

4.3 Subparameters and estimation of parameters.

A subparameter $\lambda = \lambda(\theta)$ is called permissible if

$$\lambda(\theta_1) = \lambda(\theta_2) \text{ implies } \lambda(\theta_1g) = \lambda(\theta_2g) \text{ for all } g.$$
Then $\lambda$ transforms under $G$ by

$$\lambda(\theta) \rightarrow (\lambda g)(\theta) = \lambda(\theta g).$$

For any subparameter $\lambda$ there is a maximal subgroup $G^a$ of $G$ under which $\lambda$ is permissible.

A group defined on the parameter space and conformably on the space of observations, may be useful in finding a good estimator for $\lambda$. An estimator which transforms under the group in the same way as the parameter, is called equivariant. More precisely, if $\lambda(\theta)$ is a permissible parameter, so that $(\lambda g)(\theta) = \lambda(\theta g)$, then an estimator $\hat{\lambda}$ is called equivariant if $\hat{\lambda}(y g) = (\hat{\lambda} g)(y)$ for all $g$ and $y$.

In the transitive case we have the following important result (see, e.g. [21]).

**Theorem 1.** The best equivariant estimator under quadratic loss is equal to the Bayes estimator with prior equal to the invariant measure $\rho$ (the Pitman estimator).

The Bayes estimator is computed as follows: First find the posterior parameter distribution using Bayes’ formula, a formula which is also recently advocated strongly for in a quantummechanical setting [16]. Then find the expected parameter under this distribution. Even though this in principle sounds quite straightforward, it often involves quite heavy calculations, calculations which in the recent statistical literature are solved by Markov Chain Monte Carlo techniques [17].

### 4.4 Model reduction.

Model reduction was introduced under focusing. It is important when you have few data. When there is a symmetry group $G$ acting upon the parameter space, one has the following requirement to impose:

The original parameter space is invariant under the group $G$. Therefore it is natural that the reduced parameter space also should be invariant under $G$. This implies that

*The reduced parameter space should be an orbit/ a set of orbits for $G$.*

Then within each orbit the Pitman estimator gives an optimal solution. This solution is unique when the reduced space is transitive, i.e., when the model reduction leads to a single orbit.

As another application of focusing, the group $G$ may first be defined on a larger $c$-variable space, and then from this induced on the parameter space. This is the procedure I will use when linking all this to quantum mechanics.

Then as a summary: It is useful to have a group defined, first it is useful for selecting a suitable prior, then in the analysis of subparameters and finally in connection to model reduction. Group theory in statistics may introduce some
abstract notions, but the concept of a symmetry group in itself is not abstract at all. More about the groups in statistics can be found in [18, 26].

5 A large scale EPR example.

Let $Z$ be a matrix of data values; let $t$ and $u$ be latent variables (unknown variables), and let $a$ and $b$ be parameters. Consider the multivariate latent variable statistical model

$$Z = ta' + ub'(+E)$$

with negligible error $E$. We assume that $Z$ is measured, but the rest is unknown to begin with. Call $\phi = (t, a, u, b)$ a c-variable.

Let us now have two distant stations. At station 1 it is possible to measure $t$ or $a$, but not both. At station 2 it is possible to measure $u$ or $b$, but not both.

Now assume that we measure $t$. Let $P = t(t't)^{-1}t'$, and let $v = (I - P)u$. Then we know the product

$$vb' = (I - P)Z.$$

This implies that we can find $bb'v = (v'v)bb'$, and hence in effect all of the unknown parameter $b$. (Note that in the model $b$ is only defined modulus a scale factor.) On the other hand, an essential part of the parameter $u$ remains unknown.

By the complementary experiment at station 1, namely measuring $a$, we obtain all info on $u$, while parts of $b$ remain unknown.

There is no direct action at a distance here, but by taking a decision on what to measure at station 1, we determine what parameter to get information on at station 2.

In my view this simple thought experiment, first formulated to me by Harald Martens, bears some relationship with the EPR experiment, an experiment which throughout the years has caused much discussion in the quantummechanical literature.

The focus is not upon what the values of $u$ and $b$ are, but upon what information we can get about $u$ and $b$. The corresponding general view in quantum theory is the epistemic view, explicitly introduced recently; see, e.g. Fuchs [27], but related to the classical Copenhagen interpretation of quantum mechanics.

6 An approach to electron spin.

To illustrate the general approach of this paper, I will describe the quantum-mechanical spin in a new way using a conceptually defined but inaccessible c-variable, then focusing, symmetry and model reduction.
As a start, model the angular momentum of a particle such as the electron by a vector $\phi$, a c-variable. Let the symmetry group $G$ be the group of rotations of the vector $\phi$, that is, the group that fixes the norm $\|\phi\|$.

Next, choose a direction $a$ in space, and focus upon the angular momentum component in this direction:

$$\theta^a = \|\phi\| \cos(\phi, a).$$

The largest subgroup $G^a$ with respect to which $\theta^a$ is permissible, is given by rotations around the axis $a$ together with a reflection in a plane perpendicular to $a$. However, the action on $\theta^a$ is just reflection.

Finally, introduce a model reduction: The orbits of $G^a$ as acting on $\theta^a$ are given by two-point sets $\{\pm \kappa\}$ together with the single point 0. A maximal model reduction is to one such orbit. In this case it does not matter which non-trival orbit we take, but to be definite, choose the single orbit $\{\pm 1\}$. Let $\lambda^a$ be the reduced parameter.

The parameter $\lambda^a$, taking one of the values $\pm 1$, is our parameter for experiments. Measuring apparatus for electron spin, Stern-Gerlach apparatus, are often described as perfect in textbooks, but from a statistical point of view they must usually be assumed to have errors of measurements. This is of some importance, since the state specification in our approach is connected to the statistical parameter $\lambda^a$.

More specifically, the important elements in this whole description are the c-variable $\phi$, the basic rotation group $G$, the direction of focusing $a$, the reduced group $G^a$ and the reduced parameter $\lambda^a$. The state of an electron spin will in our system consist of two elements:

A focused question: Given the direction $a$; what is the value of $\lambda^a$?

An answer: $\lambda^a = +1$ or $\lambda^a = -1$.

This rather concrete state concept should be contrasted to the conventional foundation of quantum theory. Shortly we will construct the ordinary Hilbert space from such a framework, and we will show that unit vectors in this Hilbert space can be put into correspondence with the states just defined.

## 7 Conventional quantum mechanics and the statistical approach.

Quantum mechanics has had an enormous empirical success, but its foundation is very formal. Briefly, the whole theory can be derived from 4 axioms; here taken from Isham [28]:

**Rule 1** The state of a quantum system is given by a unit vector $v$ in a Hilbert space $\mathbf{H}$.

**Rule 2** The observables of the system are represented by selfadjoint operators.
Rule 3 If the observable quantity \( \lambda \) is represented by the selfadjoint operator \( T \), and the state by \( v \), then the expected result of a perfect measurement is

\[
E_v(\lambda) = v^\dagger T v.
\]

**Note:** This implies that a state vector \( v \) is an eigenvector for \( T \) with eigenvalue \( \lambda_0 \) if and only if a perfect measurement of \( \lambda \) in this state gives a certain value \( \lambda_0 \). Eigenvectors for operators are important states; in fact all states can in some way be seen in this light.

Rule 4 Time development is given by the Schrödinger equation:

\[
i\hbar \frac{dv_t}{dt} = Hv_t,
\]

where \( \hbar \) is Planck’s constant, and where \( H \) is the special selfadjoint operator known as the Hamiltonian.

More extensive principles of quantum mechanics are given for instance by Volovich [29]. Such principles can also be derived from our approach by doing some more work. It should be pointed out that the state transformer or collapse of the wave packet here requires a different, in fact a statistical and non-formal, discussion; see later.

Our aim now is to derive the 4 rules above from the statistical approach: The background of this approach consists of a c-variable \( \phi \), a group \( G \) acting upon the range \( \Phi \) of \( \phi \), then a focused parameter \( \lambda^a \) with its reduced group \( G^a \).

**State:** The state consists of two elements:

a) A focused question: Given the focus \( a \), what is the value of \( \lambda^a \)?

b) An answer: \( \lambda^a = \lambda_k \).

Under suitable conditions now we want to find a Hilbert space \( H \), where the unit vectors \( v \) represent the states above, and where the operators \( T \) correspond to the parameters \( \lambda \). This will give a starting point for the relation to quantum theory.

8 Constructing the quantum space.

8.1 Parametric quantum space for a single choice of focusing.

Consider \( \phi \), the group \( G \) and the invariant measure \( \rho \). Focus on a choice \( a \) and a reduced parameter \( \lambda^a = \lambda^a(\phi) \) taking a discrete set of values \( \{\lambda_k\} \). Recall that \( \phi \) in general takes no values. Nevertheless it is possible to define a group
$G$ as acting upon the space $\Phi$ of $c$-variables. (Think of the example with two possible treatments and two potential expected life times for a single patient. It is possible to define a joint scale transformation acting on these expected life times.) Also, one can in a meaningful way define the $L^2$-space $L^2(\Phi, \rho)$.

Let the space $L^a$ consist of all functions in $L^2(\Phi, \rho)$ which are of the form

$$f(\phi) = \tilde{f}(\lambda^a(\phi)).$$

It is easily verified that this is a closed subspace of $L^2(\Phi, \rho)$, hence a Hilbert space.

A natural choice of operator $S^a$ consists of multiplying:

$$S^a f(\phi) = \lambda^a(\phi) f(\phi).$$

Then the eigenfunctions of $S^a$ are indicator functions of the sets $\{ \phi : \lambda^a(\phi) = \lambda_k \}$, constituting a complete orthogonal basis for $L^a$.

We will call $L^a$ a parametric quantum space. The parametric quantum space is simple, and it depends only upon the $c$-variable $\phi$ and the focused statistical parameter $\lambda^a$, plus the group $G$.

I intend to use this as the basic building stone, together with a group theoretic method of joining the parametric quantum spaces for different choices of focusing. There will be some necessary mathematical theory in this development, but mainly to bind the whole thing together with the formal quantum theoretical axioms. During the development, other links to statistical theory will turn out.

### 8.2 Maximally accessible parameter.

In the parametric quantum space the parameter $\lambda^a$ should be accessible, and maximally so. Recall that a statistical parameter is call accessible if, given the relevant context, it can be made estimable by doing a suitable experiment. As a background for this definition, we may look upon the set of possible parameters as connected to more than one single model. We are free to make decisions both in terms of experiment to perform and model to select. In this respect the distinction between parameter and $c$-variable may be not quite clear always. The important distinction is between what is accessible or not.

**Definition.** Make a partial ordering on the parameters so that $\lambda \ll \theta$ if there is a function $h$ such that $\lambda = h(\theta)$. We say that $\lambda$ is maximally accessible if it is maximal among the accessible parameters under this ordering.

Note that this is consistent with the fact that the $c$-variable $\phi$ usually is inaccessible, while each parameter $\lambda^a$ usually is accessible. The definition itself is also consistent: When $\theta$ is accessible, then so is $\lambda = h(\theta)$. And if $\lambda$ should be inaccessible with $\lambda = h(\theta)$, then $\theta$ is also inaccessible.
8.3 A modified space.

It is always possible to transform the abstract vector space \( L^a \) by a unitary transformation. If a quantum system is defined on a space \( L \), and \( W \) is a unitary operator, then a completely equivalent quantum system can be defined on \( H = WL \) by the correspondence \( v \rightarrow Wv, T \rightarrow WTW^\dagger \).

Let \( L^a \) be the parametric quantum space corresponding to the choice of focusing \( a \), and let \( W \) be any unitary operator acting on \( L^a \). Then \( H^a = WL^a \) is called a simple quantum space corresponding to the choice of focusing \( a \).

The simple quantum spaces will shortly be joined together to form ordinary quantum spaces. The choice of \( W \) will be made later.

8.4 Briefly on group representation theory.

Let a group \( G \) be given. A group representation \( V \) is a function to the set of operators on some Hilbert space such that

\[
V(gh) = V(g)V(h) \quad \text{for all } g, h \in G.
\]

A subspace \( H \) is said to be invariant under the representation \( V \) if \( V(g)v \in H \) whenever \( g \in G \) and \( v \in H \). An invariant space gives a subrepresentation of \( V \).

The right regular representation on \( L^2(\Phi, \rho) \) is defined by \( U(g)f(\phi) = f(\phi g) \).

It can be shown \cite{30} that this always is a unitary representation.

8.5 Coupling different focusings together.

As a basis for coupling together the different simple quantum spaces, we make the following assumption.

**Assumption 1.** For each pair of focused experiments \( a \) and \( b \) there is a group element \( g_{ab} \in G \) such that

\[
\lambda^b(\phi) = \lambda^a(\phi g_{ab}).
\]

This holds for the electron spin case by a straightforward verification. In general it is an assumption to the effect that the fundamental group \( G \) is large enough to contain transformations between the differently focused experiments. Recall that \( G^a \) is the maximal subgroup under which the parameter \( \lambda^a \) is permissible. We will look at two consequences of Assumption 1.

**Lemma 1** The groups \( G^a \) and \( G^b \) can be connected by the group element \( g_{ab} \):

\[
g^b = g_{ab}g^a g_{ab}^{-1}.
\]

**Proof:** We define \((\lambda^a g_{ab})(\phi) = \lambda^a(\phi g_{ab})\), which is consistent with earlier definitions. By permissibility each element \( g^a \) of the group \( G^a \) is determined by
its effect on $\lambda^a$. By Assumption 1 we must have that $\lambda^b g^b = \lambda^a g^a g_{ab}$ for some $g^b \in G^b$. Hence it follows that $\lambda^a g_{ab} g^b = \lambda^a g^a g_{ab}$, and the Lemma follows.

The next result has also been anticipated in notation used earlier.

**Lemma 2.** Assuming that each $\lambda^a$ takes discrete values $\lambda^a_k$, these values can be arranged such that $\lambda^a_k = \lambda_k$ is the same for each $a$ ($k = 1, 2, ...$).

**Proof:** By Assumption 1

\[
\{ \phi : \lambda^b(\phi) = \lambda^b_k \} = \{ \phi : \lambda^a(\phi g_{ab}) = \lambda^b_k \} = \{ \phi : \lambda^a(\phi) = \lambda^b_k \} g_{ba}.
\]

The sets in brackets on the lefthand side here are disjoint with union $\Phi$. But then the sets on the righthand side are disjoint with union $\Phi g_{ab} = \Phi$, and this implies that $\{ \lambda^b_k \}$ gives all possible values of $\lambda^b$.

Now make the following observation: If $f^a_k$ is the indicator that $\lambda^a$ equals $\lambda_k$, and $f^b_k$ is the indicator that $\lambda^b$ equals $\lambda_k$, then $f^b_k = U(g_{ab}) f^a_k$, where $U$ is the right regular representation of $G$. A consequence of this is that $L^b = U(g_{ab})L^a$.

And a consequence of this again is that $H^b = V(g_{ab})H^a$ for an element of the unitary representation $V(g) = UW(g)W^\dagger$.

From now on we restrict ourselves to the case with a finite number of parameter values $\lambda_k$. And also, without much loss of generalizations, we assume that the underlying group $G$ is compact. From standard mathematics we then have the following result:

**Theorem 2.** For a compact group, every irreducible unitary representation $V(g)$ can be written in as $V(g) = UW(g)W^\dagger$ for some $W$, with $U(g)$ being a subrepresentation of the right regular representation.

We now introduce another assumption, to the effect that there is a sufficient amount of focused questions to ask:

**Assumption 2.** The reduced groups $G^a, G^b, ...$ generate the whole group $G$.

**Theorem 3.** Fixing some preliminary $W_0$ and hence some set of unitary relations $V_0(g_{bc}) = W_0 U(g_{bc}) W_0^\dagger$ between simple Hilbert spaces $H^b$ and $H^c$, say, the fixed space $H^a$ is an invariant space for some abstract representation $V$ of the whole group $G$.

**Proof:** First we observe that $H^a$ is an invariant space for the subgroup $G^a$ under the right regular representation. This follows directly from the definitions. To extend this, we look at a product $g_1 g_2 g_3$, where $g_1 \in G^a$, $g_2 \in G^b$ and $g_3 \in G^c$. We can define a map from such elements to operators on $H^a$ by

\[
V(g_1 g_2 g_3) = U(g_1)(V_0(g_{ba}) U(g_2) V_0(g_{ab}))(V_0(g_{ca}) U(g_3) V_0(g_{ac})).
\]
With a similar definition for all products of elements from the subgroup, we verify for instance that 
\[ V(g_1g'_2g_2) = V(g_1g_2g_3) \]
when \( g_2 = g'_2g''_2 \in G^c \). Also, 
\[ V(g_1g_2g_3) = V(g_1)V(g_2)V(g_3) \]
It follows that 
\[ V(gh) = V(g)V(h) \]
on these products, since the last factor of \( g \) and the first factor of \( h \) either belong 
to the same subgroup or to different subgroups. In this way we see that \( V \) is a 
representation on the set of finite product of elements from the subgroups, and 
since by Assumption 2 these products generate \( G \), it is a representation of \( G \). 
In particular, one is able to take \( H^a \) as an invariant space for a representation 
\( V \) of this group.

**Choice of \( W \):** Now keep \( H^a \) fixed, but be free to change the matrix \( W_0 \) which 
fixes the relations to the other Hilbert spaces. Concretely, there is a group 
representation having \( H^a \) as an invariant space. By Theorem 2, we can choose 
\( W \) such that 
\[ V(g) = Wu(g)W^\dagger \]
is this representation.

**Theorem 4.** Then from 
\[ H^b = V(g_{ab})H^a \]
we have 
\[ H^a = H^b = H^c = ..., \]
and this can be taken as the quantum mechanical space \( H \).

*Example:* \( SU(2) \) gives a twodimensional invariant space for electron spin, 
coupled to the rotation group.

In the space \( H \) there are state vectors \( v^a_k \) which are transforms of indicators 
that \( \lambda^a(\phi) = \lambda_k \). Specifically we take 
\[ v^a_k = WF^a_k \]
with 
\[ f^a_k(\phi) = I(\lambda^a(\phi) = \lambda_k) \].

In the electron spin case and in other cases all unit vectors in \( H \) are of this 
form. In the electron spin case this can be verified directly by a Boch sphere 
argument. In general this amounts to an assumption to the effect that the set 
of focused questions is rich enough. On the mathematical side, one can argue 
from the fact that every unit vector can be considered as eigenvector of some 
operator.

**Definition** The vectors \( v^a_k \) are the state vectors with the statistical interpreta-
tion: Focused question: What is \( \lambda^a \)? Answer: \( \lambda^a = \lambda_k \).

In general, the vectors \( v^a_k \) are eigenvectors of the selfadjoint operator 
\[ T^a = WS^aW^\dagger \]
with eigenvalues \( \lambda^a_k \). Hence we have the result:

**Theorem 5.** For each choice of focused question \( a \) there is an operator \( T^a \) on 
the Hilbert space \( H \) which corresponds to the perfect experiment with parameter 
\( \lambda^a \). These operators have eigenvectors \( v^a_k \) with eigenvalues \( \lambda^a_k \).

Note that by the unitary transformation, the vectors \( v^a_k \) are unit vectors 
when the \( f^a_k \) are. In fact the unit vectors in \( H \) correspond in a unique way to a

13
question-and-answer pair if there is a correspondance at all:

Theorem 6. Assume that two vectors in \( H \) satisfy \( v^a_i = v^b_j \). Then the question: 'What is \( \lambda^a \)?' is answered by \( \lambda^a = \lambda_i \) if and only if the question: 'What is \( \lambda^b \)?' is answered by \( \lambda^b = \lambda_j \).

Proof: The assumption is that \( W f^a_i = W f^b_j \), hence \( f^a_i(\phi) = I(\lambda^a(\phi) = \lambda_i) = I(\lambda^b(\phi) = \lambda_j) \). Thus the level sets coincide, and the Theorem follows.

8.6 Operators and maximality.

The operators \( T^a \) will have the form

\[
T^a = \sum_k \lambda_k v^a_k v^{a\dagger}_k.
\]

Up to now we have assumed that all parameters \( \lambda^a \) are maximally accessible. This implies that \( T^a \) has nondegenerate eigenvalues.

In applied quantum mechanics one also allows operators with degenerate eigenvalues. In our setting these can be introduced through non-maximal parameters \( \mu = h(\lambda) \), and

\[
T = \sum_k \mu_k v^a_k v^{a\dagger}_k.
\]

Thus these parameters are also associated with operators in the obvious way. The quantum mechanical Rule 1 and Rule 2 hold for these operators and for the states \( v^a_k \).

9 Auxiliary quantities for one choice of focusing.

The state vector \( v^a_k \) has an arbitrary phase factor in our setting, since by definition \( W \) has an arbitrary phase factor. Therefore it is sometimes replaced by the one-dimensional projector \( v^a_k v^{a\dagger}_k \).

The information contained in this projector is again: A focused question: 'What is \( \lambda^a(\phi) \)?' has been asked, and the answer is: \( \lambda^a(\phi) = \lambda_k \).

Taking this as a point of departure, one can introduce other operators which play a crucial role in many applications of quantum mechanics, and which also can be connected to statistics.

9.1 Auxiliary quantity I: Density operator.

A \textit{density operator} is defined by:

\[
\sigma = \sum_k \pi_k v^a_k v^{a\dagger}_k.
\]
when a prior or posterior probability $\pi_k$ over $\lambda^a$ is given.

Conversely, given $\sigma$, one can reproduce the focused experiment itself, all levels of answers $k$ and the (prior or posterior) probabilities $\pi_k$ for all these $k$.

In the ordinary quantum mechanical tradition, a density operator is defined as any selfadjoint operator with trace 1.

9.2 Auxiliary quantity II: Effect.

Given some data $y$, an effect (cf. [31]) is defined by:

$$\mathcal{E} = \sum_j p^b_j(y) v_j^a v_j^b\dagger.$$  

An experiment with likelihood $p^b_j(y)$ and parameter $\lambda^b$ is the basis for $\mathcal{E}$.

Conversely, given $\mathcal{E}$, one can reproduce the focused experiment itself, all levels of answers $k$ and the likelihood $p^a_k(y)$ for the parameter values $\lambda_j$.

In the ordinary quantum mechanical tradition, an effect is defined as any selfadjoint operator with eigenvalues between 0 and 1.

Using these quantities, we now turn to the probability link between different focusings.

10 Born’s formula.

Born’s formula is fundamental in linking differently focused experiment:

$$P(\lambda^b = \lambda_j | \lambda^a = \lambda_k) = |v_j^a v_k^b\dagger|^2.$$  

It has been proved in a standard quantum mechanical setting under various assumptions by Deutsch [32], Wallace [33], Saunders [34], Aerts [35] and Zurek [36].

The crucial assumption by Zurek in his proof of Born’s formula is what he called envariance.

Couple the quantum system $S$ to an environment $E$, and then make the following assumption:

Every unitary transformation $U = u_S \otimes 1_E$ acting solely on $S$:

$$Uv_{SE} = (u_S \otimes 1_E)v_{SE} = v'_{SE}$$

can be undone by a transformation $V$ acting solely on the environment $E$:

$$Vv'_{SE} = (1_S \otimes u_E)v'_{SE} = v_{SE}$$

It was argued in various ways in [36] that this is a relatively weak assumption. We want to prove the formula in our setting from a somewhat different, but also weak, assumption.
10.1 Proof of Born’s formula via the Busch-Gleason theorem.

The proof will depend upon a recent variant (Busch [31]; Caves et al. [37]) of a well known mathematical result by Gleason [38]. One advantage of this result is that it is also valid for dimension 2, when the ordinary Gleason Theorem fails.

**Theorem 7.** Assume that there is a generalized probability measure \( \pi \) on the effects in a Hilbert space \( \mathcal{H} \); i.e., a set function satisfying

\[
\pi(\mathcal{E}) \geq 0 \text{ for all } \mathcal{E},
\]

\[
\pi(I) = 1,
\]

\[
\sum_i \pi(\mathcal{E}_i) = \pi(\mathcal{E}) \text{ for effects } \mathcal{E}_i \text{ whose sum is an effect } \mathcal{E}.
\]

Then \( \pi \) is necessarily of the form \( \pi(\mathcal{E}) = \text{tr}(\sigma \mathcal{E}) \) for some density operator \( \sigma \).

Now define \( \pi(v^b_j v^b_j|) = P(\lambda^b = \lambda_j|\lambda^a = \lambda_k) \), and extend by linearity to all effects. Such an effect will then correspond to a parameter \( \lambda^b \), an orthonormal set of Hilbert space vectors \( v^b_j \), each giving one value \( \lambda_j \) for \( \lambda^b \). Furthermore, the effect will correspond to a single data point \( y \) and for each \( j \) the probability \( p^b_j(y) \) of observing \( y \), given that \( \lambda^b = \lambda_j \). We then have

\[
\pi(\mathcal{E}^b) = \sum_j p^b_j(y)P(\lambda^b = \lambda_j|\lambda^a = \lambda_k) = P(\text{obs. } y \text{ through experim. } b|\lambda^a = \lambda_k).
\]

Imagine now \( n \) potential such experiments \( b_1, \ldots, b_n \), corresponding to \( n \) different questions and \( n \) different sets of likelihood \( p^b_j(y) \). An experimenter selects one of these randomly, each with probability \( 1/n \). He has available a measurement apparatus through which each of the single experiments above can be performed, that is, experiments with parameters \( \lambda^b \) and each with the single observation \( y \) as one possibility. The randomized experiment above can be imagined to correspond to a parameter \( \lambda^0 \) with this measurement apparatus. It is important that the randomization is performed blindly, i.e., the experimenter does not know which experiment is carried out. His likelihood in the randomized experiment for observing \( y \) is \( p^0_j(y) \) when \( \lambda^0 = \lambda_j \). And, by definition, \( \lambda^0 \) equals \( \lambda^b \) with probability \( 1/n \). Then this experiment also has an effect \( \mathcal{E}^0 \) and a generalized probability measure

\[
\pi(\mathcal{E}^0) = \sum_j p^0_j(y)P(\lambda^0 = \lambda_j|\lambda^a = \lambda_k).
\]

This gives

\[
\pi(\mathcal{E}^0) = \frac{1}{n} \sum_r P(\text{obs. } y \text{ through experim. } b_r|\lambda^a = \lambda_k) = \frac{1}{n} \sum_r \pi(\mathcal{E}^r)
\]

with \( \mathcal{E}^r = \mathcal{E}^{br} \).
On the other hand, $E^0$ is a randomization at the measurement apparatus of the experiments with effects $E^r$. This implies that any hypothetical observer knowing the result of the randomization will have the corresponding result:

$$E^{0'} = \sum_r I_r E^r,$$

where $I_r = 1$ if experiment $r$ was selected, otherwise $I_r = 0$. A similar formula should hold for partial knowledge. For instance, if the observer does not know whether experiment 1 or 2 was selected, the indicators $I_1$ and $I_2$ should both be replaced by $\frac{1}{2} I$, where $I$ is the indicator of experiment 1 or 2.

Since it should be a high degree of consistency between the observers knowing the randomization and a single observer not knowing the randomization at all, we expect that $E^0$ should be close to, if not equal to, $\sum_r E(I_r)E^r$. A weak condition is that these two effects should have the same generalized probability measure.

**Assumption 3.**

$$\pi(E^0) = \pi(\sum_r E(I_r)E^r).$$

With the simple randomization used here, this means

$$\pi(E^0) = \pi(\frac{1}{n} \sum_r E^r).$$

This implies that $\pi(\frac{1}{n} \sum_r E^r) = \frac{1}{n} \sum_r \pi(E^r)$, or, by changing the likelihood, if $\sum_r E^r$ is an effect:

$$\pi(\sum_r E^r) = \sum_r \pi(E^r).$$

Thus follows the most important premise of the Busch-Gleason theorem from our assumption. The other premises are easily proved, and it follows that there exists a density operator $\sigma$ such that $\pi(vv^\dagger) = v^\dagger \sigma v$ for all $v \in H$. For this density operator we have that $\sigma = \sum_j c_j u_j u_j^\dagger$, where $c_j$ are nonnegative constants adding to 1.

Inserting this gives $\pi(vv^\dagger) = \sum_j c_j |v^\dagger u_j|^2$. Specialize now to the particular case given by $v = v_k^a$ for some $k$. For this case one must have $\sum_j c_j |v_k^a u_j|^2 = 1$ and thus $\sum_j c_j (1 - |v_k^a u_j|^2) = 0$. This implies for each $j$ that either $c_j = 0$ or $|v_k^a u_j| = 1$. Since the last condition implies $u_j = v_k^a$ (modulus an irrelevant phase factor), and this is a condition which can only be true for one $j$, it follows that $c_j = 0$ for all other $j$ than this one, and that $c_j = 1$ for this particular $j$.

Summarizing this, we get $\sigma = v_k^a v_k^a$, and Born’s formula follows.

### 10.2 Consequence of Born’s formula I.

Let the state of a system be given by $v = v_k^a$, and let $\lambda = \lambda^b$ be an arbitrary parameter with corresponding operator $T = \sum_j \lambda_j v_j^a v_j^b$. 

17
Then the expected result of a perfect measurement will be:

\[ E_v(\lambda) = \sum_j \lambda_j P(\lambda^b = \lambda_j | \lambda^a = \lambda_k) \]

\[ = \sum_j \lambda_j (v_k^a v_j^b) (v_j^b v_k^a) = v_k^a (\sum_j \lambda_j v_j^b v_j^b) v_k^a = v^\dagger T v. \]

This is Rule 3 in the basis for conventional quantum mechanics.

### 10.3 Consequence of Born's formula II.

Let an experiment have focus on a parameter \( \lambda^b \) and likelihood \( p_j(y) \) for \( \lambda^b = \lambda_j \). Define an operator valued measure \( M \) by

\[ M(dy) = \sum_j p_j(y) v_j^b v_j^b \dagger \, dy. \]

These operators satisfy \( M(S) = I \) for the whole sample space \( S \) and are countably additive.

Assume now that the initial state is given by \( \lambda^a = \lambda_k \). Then the probability distribution for the result of experiment \( b \) is given by

\[ P(dy | \lambda^a = \lambda_k) = v_k^a \dagger M(dy) v_k^a. \]

A more general state assumption is a Bayesian one corresponding to this setting: Let the current state be given by the question: ‘What is the value of \( \lambda^a \)?’, and then the probabilities \( \pi(\lambda_k) \) for the different values \( \lambda_k \). Then, defining \( \sigma = \sum \pi(\lambda_k) v_k^a v_k^a \dagger \), we get

\[ P(dy) = \text{tr}(\sigma M(dy)). \]

Operator valued measures are increasingly being used in quantum statistical inference. Here they are traced back to a likelihood based concept.

### 10.4 Consequence III. Collapse of the wave packet.

Note that the density matrix \( v_k^a v_k^a \dagger \) is equivalent to a pure state \( v_k^a \). Similarly, a density matrix \( v_j^b v_j^b \dagger \) is equivalent to the statement that a perfect measurement giving \( \lambda^b = \lambda_j \) has just been performed. From this and from a straightforward application of Born’s formula one gets

**Theorem 8.** a) Assume an initial state \( v_k^a \), and assume that a perfect measurement of \( \lambda^b \) has been done without knowing that value. Then the state is described by a density matrix \( \sum_j |v_k^a v_j^b|^2 v_j^b v_j^b \dagger \).

b) After measurement \( \lambda^b = \lambda_j \) the state then changes to the vector \( v_j^b \). From a) this happens with probability \( |v_k^a v_j^b|^2 \).
This is the well known and much discussed collapse of the wave packet, or really of the density matrix. In our statistical interpretation this represents no problem. A similar ‘collapse’ occurs in Bayesian statistics each time an observation is made. Note that the reason why this collapse is non-problematic is that we do not assume a simple ontological interpretation of the state vector. The state represents just a question together with an ideal answer.

11 The state interpretation and ‘action at a distance’.

Consider two electrons with their spins modeled in our terminology by c-variables \( \phi^1 \) and \( \phi^2 \). Assume that these in some distant past have been in interaction so that \( \phi^1 = -\phi^2 \), what in conventional quantum mechanics is called a singlet state. But now the two electrons are far apart. This leads to the Einstein, Podolsky, Rosen (EPR) [39] situation as modified by Bohm.

Assume that we can make measurements on electron 1. We are then free to focus on some chosen parameter \( \lambda^a \). If we make a perfect measurement in that direction and get the result \( \lambda^a = \lambda_k \), then the spin state of this particle is fixed. But the spin state of the electron 2 is also determined, namely as \( \lambda^a = -\lambda_k \) by the known coupling. And the probability distribution of perfect measurements of spin components in any other direction for electron 2 is given by Born’s formula. All this assumes that \( \phi_1 + \phi_2 \) is an accessible c-variable, while each of \( \phi_1 \) and \( \phi_2 \) are inaccessible.

Like in the earlier macroscopic example: There is no action at a distance; we only ask questions and get answers. As we do in statistical investigations.

11.1 On Bell’s inequality.

Assume again the EPR situation, where spin components \( \lambda^a \) and \( \mu^b \) are measured in the directions given by unit vectors \( a \) and \( b \) on the two particles at distant sites \( A \) and \( B \). The measured values \( \hat{\lambda}^a \) and \( \hat{\mu}^b \) are each assumed to take values \( \pm 1 \). Let this be repeated 4 times: Two settings \( a, a' \) at site \( A \) combined with two settings \( b, b' \) at site \( B \). The CHSH version of Bell’s inequality then reads:

\[
E(\hat{\lambda}^a \hat{\mu}^b) \leq E(\hat{\lambda}^a \hat{\mu}^{b'}) + E(\hat{\lambda}^{a'} \hat{\mu}^b) + E(\hat{\lambda}^{a'} \hat{\mu}^{b'}) + 2 \quad (1)
\]

In fact, by a combinatorial argument we can easily show the seemingly stronger statement:

\[
\hat{\lambda}^a \hat{\mu}^b \leq \hat{\lambda}^a \hat{\mu}^{b'} + \hat{\lambda}^{a'} \hat{\mu}^b + \hat{\lambda}^{a'} \hat{\mu}^{b'} + 2 \quad (2)
\]

As is well known, the inequality \( (1) \) can be violated in the quantum mechanical case, and this is also well documented experimentally. There is a large literature on Bell’s inequality, and I will not try to summarize it here. The
derivation of (2) above is quite obvious, and the usual statement in the quantum-mechanical literature is that (1) follows under what is called local realism.

My own view is that quantum theory is a statistical theory, and should be interpreted as such. In this connection the comparison to a classical mechanical world picture, and the term ‘local realism’ inherited from this comparison, is not necessarily of interest. I am more interested in the comparison of ordinary statistical theory and quantum theory.

Now take a general statistical inference point of view on any situation that may lead to statements like (2) and (1). Then one must be prepared to take into account that there are really 4 experiments involved in these inequalities. Going from the inaccessible c-variable ±φ to the observations there are really three steps involved at each node: The components θ(φ) is selected, there is a model reduction λ = λ(θ), and finally the observation λ. Briefly: A model is selected, and there is an estimation within that model.

Turn now to general statistical theory: According to the conditionality principle, a principle on which there seems to be a fair amount of consensus among statisticians, inference in each experiment should always be conditional upon the experiment actually performed. Taking this into account, it may be argued that at least under some circumstances also in the microscopic case, different expectations should be used in complicated enough situations leading to (1), and then the transition from (2) to (1) is not necessarily valid.

This is dependent upon one crucial point, as seen from the conditionality principle as formulated above: When one has the choice between two experiments, the same parameter should be used in both. Here is a way to achieve this: Focus on the Stern-Gerlach apparatus which measures the spin. Make a fixed convention on how the measurement apparatus is moved from one location to another. Then define a parameter λ which is −1 at one end of the apparatus and +1 at the other end. By using λ as a common parameter for all experiments under choice, the conditionality principle can be applied, and (1) does not follow from (2).

The crucial point here is that the violation of the Bell inequality is not by necessity a phenomenon that makes the quantum world completely different from the rest of the world as we know it. It has recently been pointed out that variants of Bell’s inequality may be broken in macroscopic settings.

12 Briefly on ’paradoxes’ in quantum mechanics.

Schrödinger’s cat: A cat is contained in a box together with some radioactive substance and a bottle of poison. Decay of the substance leads to release of the poison.

The c-variable φ gives a complete description of the system, including the death status of the cat. The focused parameter λ may or may not include the latter.
**Wigner’s friend:** A person (Wigner) observes the world, and a friend also observes everything, including Wigner.

In principle a statistical model can be formulated excluding an observer or including an observer. There is no contradiction.

**Hidden variables:** There are no hidden variables in our treatment; there are only hidden conceptual c-variables \( \phi \), in general taking no value. This does not prevent us from defining a group action on the c-variable. Also, a parametric Hilbert space, a subspace of \( L^2(\Phi, \rho) \), can be defined in a meaningful way.

### 13 Continuous parameters.

So far, the focused parameter \( \lambda \) has been discrete. This has had the advantage that the Hilbert space could be constructed in a rigorous way. Some theory of continuous parameters can be simply developed, however, using c-variables and the group theory approach. A straightforward example is when the c-variable consists of the position and the momentum of a single particle. We consider the ideal values of these quantities, i.e., the corresponding parameters.

#### 13.1 Symmetry in space and time.

To begin with, I indicate a relativistic treatment in this subsection. As is known from special relativity, the four-vector of space-time positions \( \xi = (\xi_1, \xi_2, \xi_3, \xi_0 = c\tau) \) and the four-vector of momentum-energy \( \pi = (\pi_1, \pi_2, \pi_3, \pi_0 = c^{-1}\epsilon) \) transform according to the extended Lorentz transformation, the Poincaré transformation. This is the group which fixes \( c^2d\tau^2 = c^2d\tau^2 - \sum_{i=1}^{3} d\xi_i^2 \) and \( c^2m_0^2 = c^{-2}\epsilon^2 - \sum_{i=1}^{3} \pi_i^2 \), where \( m_0 \) is the rest mass. This may be a natural transformation group to link to the eightdimensional c-variable \( \phi = (\xi_1, \xi_2, \xi_3, \tau, \pi_1, \pi_2, \pi_3, \epsilon) \), associated with a particle at time \( \tau \). A very thorough treatment of the Poincaré group and its representations is given in [42]. For simplicity we will limit ourselves here to the subgroup \( G \) of translations. Several further subgroups may be of interest, for instance the group \( B_1 \) of translations in the \( \xi_1 \)-direction.

**Theorem 9.** \( V^F = \{ f : f(\phi) = q(\xi_1(\phi)) \} \) for some \( q \) is a subspace of \( L^2(\Phi, \rho) \) which is invariant under the group \( B_1 \). The right regular representations have the form \( U_1(g)q(\xi_1) = q(\xi_1g) = q(\xi_1 + b) \).

But by a Taylor expansion

\[
q(\xi_1 + b) = \sum_{k=0}^{\infty} \frac{b^k}{k!} \frac{\partial^k}{\partial \xi_1^k} q(\xi_1) = \exp(b \frac{\partial}{\partial \xi_1}) q(\xi_1) = \exp\left( \frac{ibP_1}{\hbar} \right) q(\xi_1),
\]

where \( P_1 \) is the familiar momentum operator

\[
P_1 = \frac{\hbar}{i} \frac{\partial}{\partial \xi_1}.
\]
Thus the particular group formulated above has a Lie group representation on an invariant space with a generator equal to the corresponding momentum operator of quantum mechanics. The proportionality constant $\hbar$ can be argued to be the same for all momentum components (and energy) by the conservation of the 4-vector. By similarly considering systems of particles one can argue that $\hbar$ is a universal constant.

In a similar way we can show

**Theorem 10.** Time translation $\tau \to \tau + t$ has a right regular representation as a Lie group with generator

$$\exp\left(-\frac{iHt}{\hbar}\right),$$

(3)

where $H$ is the Hamiltonian operator.

13.2 The Schrödinger equation.

I have just shown that for a single particle, the time translation $\tau \to \tau + t$ has the right regular representation found by (3). This can be generalized to systems of several particles, assuming an additive Hamiltonian, and assuming that the particles at some point of time were pairwise in contact, or at least so close with respect to space and velocity that relativistic time scale differences can be neglected. The operator (3) acts on the Hilbert space $\mathbf{H}$. In this subsection I work under the non-relativistic approximation.

Assume further that at time 0 a maximal measurement was done, so that the system is in some state $v_0 \in \mathbf{H}$. This means, according to my interpretation that some measurement with focused parameter $\lambda^a$ has been done, resulting in some value $\lambda_1$. The parameter $\lambda^a$ is associated with an operator $T^a$. After time $t$ the parameter $\lambda^a$ will have developed into $\lambda^a(t)$ with an operator $T^a(t) = \exp\left(-\frac{iHt}{\hbar}\right)T^a \exp\left(\frac{iHt}{\hbar}\right)$. Then, since after time $t$ the state vector $v_t$ should still be an eigenvector of $T^a(t)$, then with eigenvalue $\lambda^a(t) = \exp(-\frac{iHt}{\hbar})\lambda^a$, the state vector must be

$$v_t = \exp\left(-\frac{iHt}{\hbar}\right)v_0.$$

As is well known, the latter equation is just a formulation of the familiar Schrödinger equation

**Theorem 11.** The time development of the state vector can be found from

$$i\hbar \frac{\partial}{\partial t} v_t = Hv_t.$$
14 Concluding remarks.

The starting point of this paper is partly related to another discipline than quantum physics, namely statistics. From one point of view this may be an advantage, giving an opportunity to see the foundational problems with new eyes. On the other hand, the discussion here goes much further than what is common in the literature in mathematical statistics. The language of this paper may be seen as a synthesis of the existing languages in quantum theory on the one hand and that of statistical inference on the other hand.

Understanding across cultural barriers is crucial. If nothing else, this paper may point at the trivial fact - a fact that still may be difficult to accept - that different cultures exist in what we perceive as a very objective science. Sometimes it is useful to know that we can translate languages and learn from each other.

Statistics and quantum theory have lived side by side for 75 years and more without any appreciable interaction, that is, at least since the publication of von Neumann’s monograph [43]. Still one might infer from the present paper that deep conceptual links seem to exist.

Non-concrete concepts are crucial in almost any human discussion. Different focusing from these concepts is also common both in the verbal exchanges of daily life and in the literature of social sciences, say. The link from this to statistics/quantum theory should be developed further.

Then finally: The formal quantum theory has proved to be very powerful in calculations. But this should not imply that the logical foundation of the theory by necessity should be formal. In fact, the essential elements of quantum theory - as an epistemic discipline under focusing and symmetry - may seem to follow logically from rather simple assumptions.

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