Multi-Dimensional Hermite Polynomials in Quantum Optics

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Abstract. We study a class of optical circuits with vacuum input states consisting of Gaussian sources without coherent displacements such as downconverters and squeezers, together with detectors and passive interferometry (beam-splitters, polarisation rotations, phase-shifters etc.). We show that the outgoing state leaving the optical circuit can be expressed in terms of so-called multi-dimensional Hermite polynomials and give their recursion and orthogonality relations. We show how quantum teleportation of photon polarisation can be modelled using this description.

Suppose we have an optical circuit, that is, a collection of various optical components. It is usually important to know what the outgoing state of this circuit is. In this paper, we give a description of the outgoing state for a special class of optical circuits with a special class of input states.

First, in section 1 we define this class of optical circuits and show that they can be described by so-called multi-dimensional Hermite polynomials. In section 2 we give an example of this description. Section 3 discusses the Hermite polynomials, and finally, in section 4 we briefly consider the effect of imperfect detectors on the outgoing state.

1. The Optical Circuit

What do we mean by an optical circuit? We can think of a black box with incoming and outgoing modes of the electro-magnetic field. The black box transforms a state of the incoming modes into a (different) state of the outgoing modes. The black box is what we call an optical circuit. We can now take a more detailed look inside the black box. We will consider three types of components.

First, the modes might be mixed by beam-splitters, or they may pick up a relative phase shift or polarisation rotation. These operations all belong to a class of optical components which preserve the photon number. We call them passive optical components.

Secondly, we may find optical components such as lasers, down-converters or (optical) parametric amplifiers in the black box. These components can be viewed as photon sources, since they do not leave the photon number invariant. We will call these components active optical components.

And finally, the box will generally include measurement devices, the outcomes of which may modify optical components on the remaining modes depending on the detection outcomes. This is called feed-forward detection. We can immediately simplify optical circuits using feed-forward detection, by considering the family of...
fixed circuits corresponding to the set of measurement outcomes (see also Ref. [1]). In addition, we can postpone the measurement to the end, where all the optical components have ‘acted’ on the modes.

These three component types have their own characteristic mathematical description. A passive component yields a unitary evolution $U_i$, which can be written as

$$U_i = \exp\left(-i\kappa \sum_{jk} c_{jk} \hat{a}_j \hat{a}^\dagger_k - \text{H.c.}\right),$$

where H.c. denotes the Hermitian conjugate. This unitary evolution commutes with the total number operator $\hat{n} = \sum_j \hat{a}^\dagger_j \hat{a}_j$.

Active components also correspond to unitary transformations, which can be written as $\exp(-itH_I^{(j)})$. Here $H_I^{(j)}$ is the interaction Hamiltonian associated with the $j$th active component in a sequence. This Hamiltonian does not necessarily commute with the total number operator. To make a typographical distinction between passive and active components, we denote the $i$th passive component by $U_i$, and the $j$th active component by its evolution in terms of the interaction Hamiltonian.

The mathematical description of the (ideal) measurement will correspond to taking the inner product of the outgoing state prior to the measurement with the eigenstate corresponding to the measurement.

Now that we have the components of an optical circuit of $N$ modes, we have to combine them into an actual circuit. Mathematically, this corresponds to applying the unitary evolutions of the successive components to the input states. Let $|\psi_{\text{in}}\rangle$ be the input state and $|\psi_{\text{prior}}\rangle$ the output state prior to the measurement. We then have (with $K > 0$ some integer)

$$|\psi_{\text{prior}}\rangle = U_K e^{-itH_I^{(K)}} \ldots U_1 e^{-itH_I^{(1)}} U_0 |\psi_{\text{in}}\rangle,$$

where it should be noted that $U_i$ might be the identity operator $\mathbb{1}$ or a product of unitary transformations corresponding to passive components:

$$U_i = \prod_k U_{i,k}.$$  

When the (multi-mode) eigenstate corresponding to the measurement outcome for a limited set of modes labelled $1,\ldots,M$ with $M < N$ is given by $|\gamma\rangle = |n_1, n_2, \ldots, n_M\rangle$ with $M$ the number of detected modes out of a total of $N$ modes, and $n_i$ the number of photons found in mode $i$, the state leaving the optical circuit in the undetected modes is given by

$$|\psi_{\text{out}}\rangle_{M+1,\ldots,N} = \langle \gamma | \psi_{\text{prior}} \rangle_{1,\ldots,N}.$$  

In this paper, we study the outgoing states $|\psi_{\text{out}}\rangle$ for a special class of optical circuits. First, we assume that the input state is the vacuum on all modes. Thus, we effectively study optical circuits as state preparation devices. Secondly, our class of optical circuits include all possible passive components, but only active components with quadratic interaction Hamiltonians:

$$H_I^{(j)} = \sum_{kl} \hat{a}^\dagger_k R^{(j)}_{kl} \hat{a}^\dagger_l + \sum_{kl} \hat{a}_k R^{(j)*}_{kl} \hat{a}_l,$$

where $R^{(j)}$ is some complex symmetric matrix. This matrix determines the behaviour of the $j$th active component, which can be any combination of down-converters.
and squeezers. Finally, we consider ideal photo-detection, where the eigenstate corresponding to the measurement outcome can be written as $|\gamma\rangle = |n_1, \ldots, n_M\rangle$.

The class of optical circuits we consider here is not the most general class, but it still includes important experiments like quantum teleportation [2], entanglement swapping [3] and the demonstration of GHZ correlations [4]. In section 2 we show how teleportation can be modelled using the methods presented here.

The state $|\psi\rangle$ prior to the photo-detection can be written in terms of the components of the optical circuit as

$$|\psi\rangle = U_K e^{-iH_{I(K)}^t} \cdots U_1 e^{-iH_{I(1)}^t} |0\rangle .$$

(6)

The creation and annihilation operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ for mode $i$ satisfy the standard canonical commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \quad \text{and} \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 ,$$

(7)

with $i, j = 1 \ldots N$.

For any unitary evolution $U$, we have the relation

$$U e^H U^\dagger = \sum_{l=0}^{\infty} \frac{U R^l U^\dagger}{l!} = \sum_{l=0}^{\infty} \frac{(U R U^\dagger)^l}{l!} = e^{U R U^\dagger} .$$

(8)

Furthermore, if $U$ is due to a collection of only passive components, such an evolution leaves the vacuum invariant: $U |0\rangle = |0\rangle$. Using these two properties it can be shown that Eq. (6) can be written as

$$|\psi\rangle = \exp \left[ -\frac{1}{2} \sum_{i,j=1}^N \hat{a}_i^\dagger A_{ij}^{(1)} \hat{a}_j^\dagger + \frac{1}{2} \sum_{i,j=1}^N \hat{a}_i A_{ij} \hat{a}_j \right] |0\rangle ,$$

(9)

where $A$ is some complex symmetric matrix. We will now simplify this expression by normal-ordering this evolution.

Define $(\hat{a}, A\hat{a}) \equiv \sum_{i,j} \hat{a}_i A_{ij} \hat{a}_j$. As shown by Braunstein [8], we can rewrite Eq. (8) using two passive unitary transformations $U$ and $V$ as:

$$|\psi\rangle = U e^{-\frac{1}{2} (\hat{a}^\dagger A^{(1)} \hat{a})} + \frac{1}{2} (\hat{a}, A \hat{a}) V^T |0\rangle ,$$

(10)

where $A$ is a diagonal matrix with real non-negative eigenvalues $\lambda_i$. This means that, starting from vacuum, the class of optical circuits we consider here is equivalent to a set of single-mode squeezers, followed by a passive unitary transformation $U$ and photo-detection. Since $A$ is diagonal, we can write Eq. (10) as

$$|\psi\rangle = U \left( \prod_{i=1}^N \exp \left[ -\frac{\lambda_i^*}{2} (\hat{a}_i^\dagger)^2 + \frac{\lambda_i}{2} \hat{a}_i^2 \right] \right) |0\rangle .$$

(11)

We can now determine the normal ordering of every factor $\exp[-\frac{\lambda_i^*}{2} (\hat{a}_i^\dagger)^2 + \frac{\lambda_i}{2} \hat{a}_i^2]$ separately. Note that the operators $\hat{a}_i^2$, $(\hat{a}_i^\dagger)^2$ and $2\hat{a}_i^\dagger \hat{a}_i + 1$ generate an $su(1,1)$ algebra. According to Refs. [4,5,8], this may be normal-ordered as

$$e^{-\frac{\lambda_i^*}{2} (\hat{a}_i^\dagger)^2 + \frac{\lambda_i}{2} \hat{a}_i^2} = e^{-\frac{\lambda_i^*}{2} \hat{a}_i^2} e^{-2 \ln(\cosh |\frac{\lambda_i}{2}|) \hat{a}_i^\dagger \hat{a}_i} e^{\frac{\lambda_i}{2} \hat{a}_i^2} ,$$

(12)

where $\tilde{\lambda}_i = \lambda_i/|\lambda_i|$. In general, when $L_\pm$ and $L_0$ are generators of an $su(1,1)$ algebra (i.e., when $A$ is unitary) we find [8]

$$e^{-\frac{1}{2}(\tau L_+ + \tau^* L_-)} = e^{-\tau \cosh |\tau| L_0} e^{-2 \ln(\cosh |\tau|) L_0} e^{\tau \cosh |\tau| L_-} ,$$

(13)
with \( \tau \) a complex coupling constant and \( \hat{\tau} \) its orientation in the complex plane. When we now apply this operator to the vacuum, the annihilation operators will vanish, leaving only the exponential function of the creation operators. We thus have

\[
|\psi\rangle = U e^{-\frac{1}{2}(\hat{\sigma}^1, \hat{\Lambda}^* \hat{\sigma}^1)} V^T |0\rangle = e^{-\frac{1}{2}(\hat{\sigma}^1, B \hat{\sigma}^1)} |0\rangle ,
\]

with \( B \equiv U \Lambda^* U^\dagger \), again by virtue of the invariance property of the vacuum. This is the state of the interferometer prior to photo-detection. It corresponds to multi-mode squeezed vacuum.

The photo-detection itself can be modelled by successive application of annihilation operators. Every annihilation operator \( \hat{a}_i \) removes a photon in mode \( i \) from the state \( |\psi\rangle \). Suppose the optical circuit employs \( N \) distinct modes. We will now detect \( M \) modes, finding \( n_1 + \ldots + n_M = N_{\text{tot}} \) photons (with \( M < N \)).

These modes can be relabelled 1 to \( M \). The vector \( \vec{n} \) denotes the particular detector ‘signature’: \( \vec{n} = (n_1, \ldots, n_M) \) means that \( n_1 \) photons are detected in mode 1, \( n_2 \) in mode 2, and so on. The freely propagating outgoing state \( |\psi_{\vec{n}}\rangle \) can then be described as

\[
|\psi_{\vec{n}}\rangle = 1_{1..M} \langle n_1, \ldots, n_M | \psi \rangle_{1..N} = c_{\vec{n}} \langle 0 | \hat{a}^{n_1}_1 \cdots \hat{a}^{n_M}_M | \psi \rangle .
\]

At this point we find it convenient to introduce the \( N \)-mode Bargmann representation [9]. The creation and annihilation operators obey the commutation relations given in Eq. (7). We can replace these operators with c-numbers and their derivatives according to

\[
\hat{a}_i^\dagger \to \alpha_i \quad \text{and} \quad \hat{a}_i \to \partial_i \equiv \frac{\partial}{\partial \alpha_i}.
\]

The commutation relations then read

\[
[\partial_i, \alpha_j] = \delta_{ij} \quad \text{and} \quad [\partial_i, \partial_j] = [\alpha_i, \alpha_j] = 0 .
\]

The state created by the optical circuit in this representation (prior to the detections, analogous to Eq. (14)) in the Bargmann representation is

\[
\psi(\vec{\alpha}) = \exp \left[ -\frac{1}{2}(\vec{\alpha}^T B \vec{\alpha}) \right] = \exp \left[ -\frac{1}{2} \sum_{ij} \alpha_i B_{ij} \alpha_j \right] .
\]

Returning to Eq. (15), we can write the freely propagating state after detection of the auxiliary modes in the Bargmann representation as

\[
\psi_{\vec{n}}(\vec{\alpha}) \propto c_{\vec{n}} \partial^{n_1}_1 \cdots \partial^{n_M}_M e^{-\frac{1}{2}(\vec{\alpha}'^T B \vec{\alpha}')} \bigg|_{\vec{\alpha}'=0} ,
\]

up to some normalisation factor, where \( \vec{\alpha}' = (\alpha_1, \ldots, \alpha_M) \). By setting \( \vec{\alpha}' = 0 \) we ensure that no more than \( n_i \) photons are present in mode \( i \). It plays the role of the vacuum bra in Eq. (14).

Now that we have an expression for the freely propagating state emerging from our optical setup after detection, we seek to simplify it. We can multiply \( \psi_{\vec{n}}(\vec{\alpha}) \) by the identity operator \( I \), written as

\[
I = (-1)^{2N_{\text{tot}}} \exp \left[ -\frac{1}{2}(\vec{\alpha}, B \vec{\alpha}) \right] \exp \left[ \frac{1}{2}(\vec{\alpha}, B \vec{\alpha}) \right] ,
\]
where $N_{\text{tot}}$ is the total number of detected photons. We then find the following expression for the unnormalised freely propagating state created by our optical circuit:

$$\psi_{\vec{n}}(\vec{\alpha}) \propto c_{\vec{n}}(-1)^{N_{\text{tot}}} H_{\vec{n}}^B(\vec{\alpha}) e^{-\frac{1}{2}(\vec{\alpha},B\vec{\alpha})} \bigg|_{\vec{\sigma} = 0}. \quad (21)$$

Now we introduce the so-called multi-dimensional Hermite polynomial, or MDHP for short:

$$H_{\vec{n}}^B(\vec{\alpha}) = (-1)^{N_{\text{tot}}} e^{\frac{1}{2}(\vec{\alpha},B\vec{\alpha})} \frac{\partial^{n_1}}{\partial \alpha_1^{n_1}} \cdots \frac{\partial^{n_M}}{\partial \alpha_M^{n_M}} e^{-\frac{1}{2}(\vec{\alpha},B\vec{\alpha})}. \quad (22)$$

The use of multi-dimensional Hermite polynomials and Hermite polynomials of two variables have previously been used to describe $N$-dimensional first-order systems [10, 11] and photon statistics [12, 13, 14]. Here, we have shown that the lowest order of the outgoing state of optical circuits with quadratic components (as described by Eq. (9)) and conditional photo-detection can be expressed directly in terms of an MDHP.

In physical systems, the coupling constants (the $\lambda_i$’s) are usually very small (i.e., $\lambda_i \ll 1$ or possibly $\lambda_i \lesssim 1$). This means that for all practical purposes only the first order term in Eq. (21) is important (i.e., for small $\lambda_i$’s we can approximate the exponential by 1). Consequently, studying the multi-dimensional Hermite polynomials yield knowledge about the typical states we can produce using Gaussian sources without coherent displacements. In section 3 we take a closer look at these polynomials, but first we consider the description of quantum teleportation in this representation.

2. Example: Quantum Teleportation

As an example of how to determine the outgoing state of an optical circuit, consider the teleportation experiment by Bouwmeester et al. [2]. The optical circuit corresponding to this experiment consists of eight incoming modes, all in the vacuum state. Physically, there are four spatial modes $a$, $b$, $c$ and $d$, all with two polarisation components $x$ and $y$. Two down-converters create entangled polarisation states; they belong to the class of active Gaussian components without coherent displacements. Mode $a$ undergoes a polarisation rotation over an angle $\theta$ and modes $b$ and $c$ are mixed in a 50:50 beam-splitter. Finally, modes $b$ and $c$ emerging from the beam-splitter are detected with polarisation insensitive detectors and mode $a$ is detected using a polarisation sensitive detector. The state which is to be teleported is therefore given by

$$|\Psi\rangle = \cos \theta |x\rangle - \sin \theta |y\rangle. \quad (23)$$

The state prior to the detection and normal ordering (corresponding to Eq. (22)) is given by ($\tau$ is a coupling constant)

$$|\psi_{\text{prior}}\rangle = U_{BSU} e^{\tau(\vec{\sigma}^t, L\vec{\sigma})/2 + \tau^*(\vec{\sigma}^t, L\vec{\sigma})/2 + \tau(\vec{\sigma}^t, L\vec{\sigma})/2 + \tau^*(\vec{\sigma}^t, L\vec{\sigma})/2} |0\rangle, \quad (24)$$

with

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (25)$$
and $\vec{u} = (\hat{a}_x^+, \hat{a}_y^+, \hat{b}_x^+, \hat{b}_y^+)$, $\vec{v} = (\hat{c}_x^+, \hat{c}_y^+, \hat{d}_x^+, \hat{d}_y^+)$. This can be written as

$$|\psi_{\text{prior}}\rangle = \exp \left[ \frac{\tau}{2} (\vec{a}^+, A\vec{a}^+) + \frac{\tau^*}{2} (\vec{a}, A\vec{a}) \right] |0\rangle ,$$

(26)

with $\vec{a} = (\hat{a}_x, \ldots, \hat{d}_y)$ and $A$ the (symmetric) matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -\sin \theta & \cos \theta & -\sin \theta & \cos \theta & 0 & 0 \\ 0 & \cos \theta & \sin \theta & \cos \theta & \sin \theta & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (27)$$

We now have to find the normal ordering of Eq. (26). Since $A$ is unitary, the polynomial $(\vec{a}^+, A\vec{a}^+)$ is a generator of an $su(1, 1)$ algebra. According to Truax [8], the normal ordering of the exponential thus yields a state

$$|\psi_{\text{prior}}\rangle = \exp \left[ \frac{\xi}{2} (\vec{a}^+, A\vec{a}^+) \right] |0\rangle ,$$

(28)

with $\xi = (\tau \tanh |\tau|)/|\tau|$. The lowest order contribution after three detected photons is due to the term $\xi^2 (\vec{a}^+, A\vec{a}^+)^2/8$. However, first we write Eq. (28) in the Bargmann representation:

$$\psi_{\text{prior}}(\vec{\alpha}) = \exp \left[ \frac{\xi}{2} (\vec{\alpha}, A\vec{\alpha}) \right] ,$$

(29)

where $\vec{\alpha} = (\alpha_{a_x}, \ldots, \alpha_{d_y})$ and $\vec{\alpha}' = (\alpha_{a_x}, \ldots, \alpha_{c_y})$. The polarisation independent photo-detection is then modelled by the differentiation $\partial_{b_x} \partial_{c_y} - \partial_{b_y} \partial_{c_x}$. Given a detector hit in mode $a_x$, the polarisation sensitive detection of mode $a$ is modelled by $\partial_{a_x}$:

$$\psi_{\text{out}}(\vec{\alpha}) = \partial_{a_x} \left( \partial_{b_x} \partial_{c_y} - \partial_{b_y} \partial_{c_x} \right) \exp \left[ \frac{\xi}{2} (\vec{\alpha}, A\vec{\alpha}) \right] \bigg|_{\vec{\alpha}'=0} .$$

(30)

The outgoing state in the Bargmann representation is thus given by

$$\psi_{\text{out}}(\vec{\alpha}) = (\cos \theta \alpha_{d_x} + \sin \theta \alpha_{d_y}) e^{\frac{\xi}{2} (\vec{\alpha}, A\vec{\alpha})} ,$$

(31)

which is the state teleported from mode $a$ to mode $d$ in the Bargmann representation. This procedure essentially amounts to evaluating the multi-dimensional Hermite polynomial $H^A_{\vec{n}}(\vec{\alpha})$. Note that the polarisation independent detection of modes $b$ and $c$ yield a superposition of the MDHP’s.

3. The Hermite Polynomials

The one-dimensional Hermite polynomials are of course well known from the description of the linear harmonic oscillator in quantum mechanics. These polynomials may be obtained from a generating function $G$. Furthermore, there exist two recursion relations and an orthogonality relation between them. The theory of multi-dimensional Hermite polynomials with real variables has been developed by Appell and Kempé de Fériet [15] and in the Bateman project [16]. Mizrahi derived an expression for real MDHP’s from an $n$-dimensional generalisation of the Rodriguez formula [17]. We will
now give the generating function for the complex MDHP’s given by Eq. (22) and consecutively derive the recursion relations and the orthogonality relation (see also Ref. [11]).

Define the generating function $G_B(\vec{\alpha}, \vec{\beta})$ to be

$$G_B(\vec{\alpha}, \vec{\beta}) = e^{(\vec{\alpha}, B \vec{\beta}) - \frac{1}{2} (\vec{\beta}, B \vec{\beta})} = \sum_{\vec{n}} \frac{\beta^{n_1}_{1!} \cdots \beta^{n_M}_{M!}}{\vec{n}} H_B^{\vec{\beta}}(\vec{\alpha}) .$$  (32)

$G_B(\vec{\alpha}, \vec{\beta})$ gives rise to the MDHP in Eq. (22), which determines this particular choice. Note that the inner product $(\vec{\alpha}, B \vec{\beta})$ does not involve any complex conjugation. If complex conjugation was involved, we would have obtained different polynomials (which we could also have called multi-dimensional Hermite polynomials, but they would not bear the same relationship to optical circuits).

In the rest of the paper we use the following notation: by $\vec{n} - e_j$ we mean that the $j$-th entry of the vector $\vec{n} = (n_1, \ldots, n_M)$ is lowered by one, thus becoming $n_j - 1$. By differentiation of both sides of the generating function in Eq. (32) we can thus show that the first recursion relation becomes

$$\frac{\partial}{\partial \alpha_i} H_B^{\vec{\beta}}(\vec{\alpha}) = \sum_{j=1}^{M} B_{ij} n_j H_B^{\vec{\beta}}(\vec{n}_{-e_j}) (\vec{\alpha}) .$$  (33)

The second recursion relation is given by

$$H_B^{\vec{\beta}}(\vec{n} + e_i) (\vec{\alpha}) - \sum_{j=1}^{M} B_{ij} \alpha_j H_B^{\vec{\beta}}(\vec{n}) (\vec{\alpha}) + \sum_{j=1}^{M} B_{ij} n_j H_B^{\vec{\beta}}(\vec{n}_{-e_j}) (\vec{\alpha}) = 0 ,$$  (34)

which can be proved by mathematical induction using

$$\sum_{k=1}^{M} B_{ik} n_k H_B^{\vec{n}_{-e_k + e_i}} (\vec{\alpha}) - B_{ii} H_B^{\vec{n}_{-e_i}} (\vec{\alpha}) = \sum_{k=1}^{M} B_{ik} m_k H_B^{\vec{m}_{+e_i}} (\vec{\alpha}) .$$  (35)

Here, we have chosen $\vec{m} = \vec{n} - e_k$.

The orthogonality relation is somewhat more involved. Ultimately, we want to use this relation to determine the normalisation constant of the states given by Eq. (21). To find this normalisation we have to evaluate the integral

$$\int_{\mathbb{C}^N} d\vec{\alpha} \psi_{\vec{n}}^* (\vec{\alpha}) \psi_{\vec{m}} (\vec{\alpha}) .$$  (36)

The state $\psi_{\vec{n}}$ includes $|\vec{\alpha}'=0\rangle$, which translates into a delta-function $\delta(\vec{\alpha}')$ in the integrand. The relevant integral thus becomes

$$\int_{\mathbb{C}^N} d\vec{\alpha} e^{-Re(\vec{\alpha}, B \vec{\beta})} [H_B^{\vec{n}} (\vec{\alpha})]^* H_B^{\vec{m}} (\vec{\alpha}) \delta(\vec{\alpha}') .$$  (37)

From the orthonormality of different quantum states we know that this integral must be proportional to $\delta_{\vec{n}, \vec{m}}$.

Since in the Bargmann representation we are only concerned with the functional relationship between $\alpha_i$ and $\partial_{\alpha_i}$ and not the actual values, we can choose $\alpha_i$ to be real. To stress this, we write $\alpha_i \to x_i$. The orthogonality relation is thus derived from

$$\int_{\mathbb{R}^N} d\vec{x} \psi_{\vec{n}}^* (\vec{x}) \psi_{\vec{m}} (\vec{x}) = \int_{\mathbb{R}^N} d\vec{x} e^{-(\vec{x}, Re(B)\vec{x})} H_B^{\vec{n}} (\vec{x}) H_B^{\vec{m}} (\vec{x}) \delta(\vec{x}') .$$  (38)
where \(\delta(\vec{x}')\) is the real version of \(\delta(\vec{a}')\). Following Klauder [1] we find that

\[
\int d\vec{x} e^{-\left(\vec{x},\text{Re}(B)\vec{x}\right)} H^B_{m}(\vec{x}) H^B_{\bar{m}}(\vec{x}) = \left(-1\right)^{N_{\text{tot}}} \int d\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} \partial_{\vec{x}}^\dagger e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} H^B_{m}(\vec{x}) ,
\]

(39)

where \(\partial_{\vec{x}}\) is the differential operator \(\partial_{\vec{x}_1} \cdots \partial_{\vec{x}_M}\) acting solely on the exponential function. We now integrate the right-hand side by parts, yielding

\[
\left(-1\right)^{N_{\text{tot}}} \int d\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} \partial_{\vec{x}}^\dagger e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} H^B_{m}(\vec{x}) = \left(-1\right)^{N_{\text{tot}}} \int d^i\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} \partial_{\vec{x}}^\dagger e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} H^B_{m}(\vec{x}) \bigg|_{x_i=-\infty}^{+\infty}
\]

(40)

and

\[
\left(-1\right)^{N_{\text{tot}}} \int d\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} \partial_{\vec{x}}^\dagger e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} \partial_{\vec{x}} e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} H^B_{m}(\vec{x}) ,
\]

(41)

where \(d^i\vec{x} = dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_N\). The left-hand term is equal to zero when \(\text{Re}(B)\) is positive definite, i.e., when \(\langle \vec{x},\text{Re}(B)\vec{x} \rangle > 0\) for all non-zero \(\vec{x}\). Repeating this procedure \(n_i\) times yields

\[
\int d\vec{x} e^{-\left(\vec{x},\text{Re}(B)\vec{x}\right)} H^B_{m}(\vec{x}) H^B_{\bar{m}}(\vec{x}) = \left(-1\right)^{N_{\text{tot}}+n_i} \int d\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} \partial_{\vec{x}}^\dagger e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} \partial_{\vec{x}} e^{-\frac{1}{2}(\vec{x},B^*\vec{x})} H^B_{\bar{m}}(\vec{x}) .
\]

(41)

When there is at least one \(n_i > m_i\), differentiating the MDHP \(n_i\) times to \(x_i\) will yield zero. Thus we have

\[
\int d\vec{x} e^{-\left(\vec{x},\text{Re}(B)\vec{x}\right)} H^B_{m}(\vec{x}) H^B_{\bar{m}}(\vec{x}) = 0 \quad \text{for} \quad \vec{n} \neq \vec{m}
\]

(42)

when \(\text{Re}(B)\) is positive definite and \(n_i \neq m_i\) for any \(i\). The case where \(\vec{n}\) equals \(\vec{m}\) is given by

\[
\int d\vec{x} e^{-\frac{1}{2}(\vec{x},B\vec{x})} H^B_{\vec{m}}(\vec{x}) H^B_{\vec{m}}(\vec{x}) = \delta_{\vec{m}} \mathcal{N} ,
\]

(43)

where \(\delta_{\vec{n},\vec{m}}\) denotes the product of \(\delta_{n_m}\) with \(1 \leq i \leq N\). Here, \(\mathcal{N}\) is equal to

\[
\mathcal{N} \equiv 2^{N_{\text{tot}}} B^n_{n_1} \cdots B^n_{n_N} n_1! \cdots n_N! \left|\pi^{-1}B\right|^{-\frac{1}{4}} .
\]

(44)

For the proof of this identity we refer to Ref. [1].

4. Imperfect Detectors

So far, we only considered the use of ideal photo-detection. That is, we assumed that the detectors tell us exactly and with unit efficiency how many photons were present in the detected mode. However, in reality such detectors do not exist. In particular we have to incorporate losses (non-perfect efficiency) and dark counts. Furthermore, we have to take into account the fact that most detectors do not have a single-photon resolution (i.e., they cannot distinguish a single photon from two photons) [4].

This model is not suitable when we want to include dark counts. These unwanted light sources provide thermal light, which is not of the form of Eq. (3). In single-shot experiments, however, dark counts can be neglected when the detectors operate only within a narrow time interval.
We can model the efficiency of a detector by placing a beam splitter with transmission amplitude $\eta$ in front of a perfect detector [14]. The part of the signal which is reflected by the beam-splitter (and which will therefore never reach the detector) is the loss due to the imperfect detector. Since beam splitters are part of the set of optical devices we allow, we can make this generalisation without any problem. We now trace out all the reflected modes (they are truly ‘lost’), and end up with a mixture in the remaining undetected modes.

Next, we can model the lack of single-photon resolution by using the relative probabilities $p(n|k)$ and $p(m|k)$ of the actual number $n$ or $m$ of detected photons conditioned on the indication of $k$ photons in the detector (as described in Ref. [14]). We can determine the pure states according to $n$ and $m$ detected photons, and add them with relative weights $p(n|k)$ and $p(m|k)$. This method is trivially generalised for more than two possible detected photon numbers.

Finally, we should note that our description of this class of optical circuits (in terms of multi-dimensional Hermite polynomials) is essentially a one-way function. Given a certain setup, it is relatively straightforward to determine the outgoing state of the circuit. The other way around, however, is very difficult. As exemplified by our efforts in Ref. [1], it is almost impossible to obtain the matrix $B$ associated with an optical circuit which produces a particular predetermined state from a Gaussian source.

5. Conclusions

In this paper, we have derived the general form of squeezed multi-mode vacuum states conditioned on photo-detection of some of the modes. To lowest order, the outgoing states in the Bargmann representation are proportional to multi-dimensional Hermite polynomials. As an example, we showed how teleportation can be described this way.

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