Time-Domain to Delay-Doppler Domain Conversion of OTFS Signals in Very High Mobility Scenarios

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Abstract—In Orthogonal Time Frequency Space (OTFS) modulation, information symbols are embedded in the delay-Doppler (DD) domain instead of the time-frequency (TF) domain. In order to ensure compatibility with existing OFDM systems (e.g., 4G LTE), most prior work on OTFS receivers consider a two-step conversion, where the received time-domain (TD) signal is firstly converted to a time-frequency (TF) signal (using an OFDM demodulator) followed by post-processing of this TF signal into a DD domain signal. In this paper, we show that the spectral efficiency performance of a two-step conversion based receiver degrades in very high mobility scenarios where the Doppler shift is a significant fraction of the communication bandwidth (e.g., control and non-payload communication (CNPC) in Unmanned Aircraft Systems (UAS)). We therefore consider an alternate conversion, where the received TD signal is directly converted to the DD domain. The resulting received DD domain signal is shown to be not the same as that obtained in the two-step conversion considered in prior works. The alternate conversion does not require an OFDM demodulator and is shown to have lower complexity than the two-step conversion. Analysis and simulations reveal that even in very high mobility scenarios, the SE achieved with the alternate conversion is invariant of Doppler shift and is significantly higher than the SE achieved with two-step conversion (which degrades with increasing Doppler shift).

Index Terms—OTFS, Doppler Spread, Very High Mobility.

I. INTRODUCTION

Recently, Orthogonal Time Frequency Space (OTFS) modulation has been introduced, which has been shown to be more robust towards channel induced Doppler shift when compared to multi-carrier systems (e.g., Orthogonal Frequency Division Multiplexing (OFDM)) [1]–[3]. This is because, in OTFS modulation, information is embedded in the delay-Doppler (DD) domain instead of the time-frequency (TF) domain.

Due to the sparseness of the effective channel in the DD domain [1]–[3], most OTFS receivers first convert the received time-domain (TD) OTFS signal to the DD domain and then perform detection in the DD domain [4]–[6]. In order to ensure compatibility with existing OFDM based receivers, most prior work on OTFS consider a two-step TD to DD domain conversion, where, i) the received TD signal is converted to a TF signal using the OFDM demodulator, and ii) this TF signal is then converted to the DD domain [1], [4]–[6]. Although the two-step conversion based OTFS receiver is significantly more robust to Doppler shift when compared to OFDM [1]–[3], in this paper we show that for very high mobility scenarios where the Doppler shift is a significant fraction of the communication bandwidth, the performance of the two-step conversion based OTFS receiver degrades with increasing Doppler shift.

In this paper, we therefore consider an alternate conversion, where the received TD signal is directly converted to the DD domain using its ZAK representation. There are possibly several variants of the ZAK representation [2], but in this paper we use the one considered by A. J. E. M. Janssen in [10]. Subsequently, we refer to an OTFS receiver based on this alternate conversion as the “ZAK receiver” and that based on the two-step conversion as the “two-step receiver”. The novel contributions of this paper are:

- In Section III we derive an expression for the ZAK representation of the transmitted TD OTFS signal in terms of the continuous delay and Doppler variables, $\tau$ and $\nu$ respectively.
- In Section IV we consider an alternate TD to DD domain conversion where the ZAK representation of the received TD signal is sampled at discrete points in the DD domain. The resulting expression for the sampled DD domain signal is not the same as that obtained in the two-step conversion considered in prior works.
- We show that the ZAK representation based alternate conversion has a smaller complexity when compared to the two-step conversion.
- Through analysis and simulations we show that in very high mobility scenarios, the spectral efficiency achieved by the ZAK receiver is almost invariant of the Doppler shift and is greater than the SE achieved by the two-step receiver (which degrades with increasing mobile speed). One such scenario is the control and non-payload communication (CNPC) between an Unmanned Aircraft System (UAS) and a Ground Station (GS). This involves low bandwidth communication (30 KHz) over a wireless channel where the Doppler shift can be very high due to high UAS speed [11], [12].

Notations: For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer smaller than or equal to $x$. $\mathbb{E}[\cdot]$ denotes the expectation operator. $\mathcal{CN}(0, \sigma^2)$ denotes the circular symmetric complex Gaussian distribution with variance $\sigma^2$. For any matrix $A$, $|A|$ denotes

1In [7], a time-domain based Linear Minimum Mean Squared Estimation (LMMSE) detector has been proposed. This detector considers the channel path delay and Doppler shifts to be integer multiples of the delay and Doppler domain resolution which results in a sparse banded structure of the inverse LMMSE equalizer matrix, due to which the equalizer matrix can be computed at low complexity. In the very high Doppler spread scenarios considered here, the sparse banded structure is not valid and therefore this LMMSE detector will have very high complexity in such scenarios.

2In [8], a low-complexity TD based OTFS equalizer has been proposed. The effective channel matrix for which equalization is performed, is exactly same as the effective DD domain channel in a two-step receiver, and therefore just like the two-step receiver, the performance of this equalizer will also degrade in very high mobility scenarios.
its determinant and $A[p, q]$ denotes the element in its $p$-th row and $q$-th column.

II. OTFS Modulation

In OTFS modulation, the delay-Doppler (DD) domain information symbols $x[k, l], k = 0, 1, \ldots, N - 1, l = 0, 1, \ldots, M - 1$ are firstly converted to Time-Frequency (TF) symbols $X[m, n], m = 0, 1, \ldots, M - 1, n = 0, 1, \ldots, N - 1$ through the inverse Symplectic Finite Fourier Transform (ISFFT) \(^{[1]}\), i.e.,

$$X[n, m] = \frac{1}{MN} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} x[k, l] e^{-j2\pi(m \cdot n - k \cdot l)}.$$  \hspace{1cm} (1)

For a given $T > 0$ and $\Delta f = 1/T$, these TF symbols are then used to generate the time-domain (TD) transmit signal which is given by the Heisenberg transform \(^{[1]}\), i.e.,

$$x(t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} X[n, m] g(t - nT) e^{j2\pi m \Delta f (t - nT)}$$ \hspace{1cm} (2)

where $g(\cdot)$ is the transmit pulse. When $g(t)$ is approximately time-limited to $[0, T]$, the Heisenberg transform in (2) is similar to OFDM where $X[n, m]$ is the symbol transmitted on the $m$-th sub-carrier ($m = 0, 1, \ldots, M - 1$) of the $n$-th OFDM symbol ($n = 0, 1, \ldots, N - 1$). Each OFDM symbol is of duration $T$ and the sub-carrier spacing is $\Delta f$, i.e., each OTFS frame has duration $NT$ and occupies bandwidth $M \Delta f$.

Let us consider a channel consisting of $L$ paths, where the $i$-th path has gain $h_i$, induces a delay $\tau_i$ ($0 < \tau_i < T$) and a Doppler shift $\nu_i$, $i = 1, 2, \ldots, L$. Then, with $x(t)$ as the transmitted signal, the received signal is given by \(^{[13]}\)

$$y(t) = \sum_{i=1}^{L} h_i x(t - \tau_i) e^{j2\pi \nu_i (t - \tau_i)} + n(t) \hspace{1cm} (3)$$

where $n(t)$ is the additive white Gaussian noise (AWGN) with power spectral density $N_0$. Most prior work consider a two-step OTFS receiver which is compatible with OFDM receivers and where the received TD signal $y(t)$ is first converted to a discrete TF signal through the Wigner transform i.e., $\hat{Y}[n, m] \triangleq \int_{-\infty}^{\infty} g(t - nT)y(t)e^{-j2\pi m \Delta f t} dt, \quad m = 0, 1, \ldots, M - 1, n = 0, 1, \ldots, N - 1$. This discrete TF signal is then converted to a DD domain signal through the Symplectic Finite Fourier Transform (SFFT) i.e., $\hat{x}[k', l'] \triangleq \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{Y}[n, m] e^{j2\pi \left( \frac{m}{M} - \frac{n}{N} \right) \Delta f}. \quad k' = 0, 1, \ldots, N - 1, l' = 0, 1, \ldots, M - 1$. With $g(t) = 1/\sqrt{T}, 0 \leq t < T$ and zero otherwise, from \(^{[1]}\), \(^{[2]}\), \(^{[3]}\) and the two-step receiver operations (i.e., Wigner transform and SFFT), it follows that

$$\hat{x}[k', l'] = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} x[k, l] \hat{h}[k, l', k, l] + \hat{n}[k', l'] \hspace{1cm} (4)$$

where $\hat{n}[k', l']$ are the DD domain noise samples and $\hat{h}[k', l', k, l]$ is given by \(^{[5]}\) (see top of next page).

III. The ZAK Representation of OTFS Signal

The ZAK transform of $x(t)$ is given by \(^{[10]}\)

$$Z_x(\tau, \nu) \triangleq \sqrt{T} \sum_{k=-\infty}^{\infty} x(t + kT) e^{-j2\pi k \nu T}, \quad -\infty < \tau < \infty, \quad -\infty < \nu < \infty. \hspace{1cm} (6)$$

Due to the following result, the $\tau-$ and $\nu-$ domains are called the “delay” and “Doppler” domain respectively, i.e., collectively we refer to them as the delay-Doppler (DD) domain.

Theorem 1: Let $x(t)$ be the transmitted signal and let there be only one channel path with a delay of $\tau_0$ and a Doppler shift of $\nu_0$. The ZAK transform of the noise-free received signal $y(t) = x(t - \tau_0)e^{j2\pi \nu_0 (t - \tau_0)}$ is then given by

$$Z_y(\tau, \nu) = e^{j2\pi \nu_0 (\tau - \tau_0)} Z_x(\tau - \tau_0, \nu - \nu_0) \hspace{1cm} (7)$$

i.e., delay and Doppler shift in time-domain results in a shift along the $\tau-$ and $\nu-$ domains by $\tau_0$ and $\nu_0$ respectively.

Proof: From (6), the ZAK transform of $y(t)$ is given by

$$Z_y(\tau, \nu) = \sqrt{T} \sum_{k=-\infty}^{\infty} y(t + kT) e^{-j2\pi \nu k T} \hspace{1cm} (8)$$

where step (a) follows from (6).

Let us consider $g(t)$ to be

$$g(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \leq t < T \\ 0, & \text{otherwise}. \end{cases} \hspace{1cm} (9)$$

From (6) it follows that in the interval $S \triangleq \{(\tau, \nu) \mid 0 \leq \tau < T, \quad 0 \leq \nu < \Delta f\}$, the ZAK transform of $g(t)$ in (9) is

$$Z_g(\tau, \nu) = 1, \quad 0 \leq \tau < T, \quad 0 \leq \nu < \Delta f. \hspace{1cm} (10)$$

From (6) it follows that for any $n \in \mathbb{Z}$

$$Z_g(\tau - nT, \nu) = \sqrt{T} \sum_{k=-\infty}^{\infty} g(t - nT + kT) e^{-j2\pi k \nu T} \hspace{1cm} (11)$$

i.e., $Z_g(\tau, \nu)$ is quasi-periodic along the delay domain \(^{[10]}\). From the above derivation it is clear that quasi-periodicity holds not just for $Z_g(\tau, \nu)$ but is valid for the ZAK transform of any finite energy time-domain signal. Also, for any $m \in \mathbb{Z}$, $Z_g(\tau, \nu + m \Delta f) = Z_g(\tau, \nu)$ (see (2.20) and (2.21) in \(^{[10]}\)). Using these facts along with (10) it follows that

$$Z_g(\tau, \nu) = e^{j2\pi \nu T}, \quad -\infty < \tau < \infty, \quad -\infty < \nu < \infty. \hspace{1cm} (12)$$

The following theorem gives the expression for the ZAK transform of the transmit OTFS signal $x(t)$.

Theorem 2: The ZAK transform of $x(t)$ in (2) is
where $Z_{y}(\tau, \nu)$ is given by (12).

Proof: See Appendix A.

From Theorem 2, it is clear that $\Phi_{k,l}(\tau, \nu)$ are the basis signals in the DD domain. Taking the expression of $Z_{y}(\tau, \nu)$ from (12) into (13), and using the fact that $T\Delta f = 1$, we get

\[
\Phi_{k,l}(\tau, \nu) = e^{j2\pi \frac{f}{M} \left( \frac{\nu - k\Delta f}{N} \right)} w_{1} \left( \nu - \frac{k\Delta f}{N} \right) w_{2} \left( \tau - \frac{fT}{M} \right),
\]

\[
w_{1}(\nu) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi n\nu T} \sin(\pi N \frac{\nu}{\Delta f}) N \sin(\pi \frac{\nu}{\Delta f})
\]

and

\[
w_{2}(\tau) = \frac{1}{M} \sum_{m=0}^{M-1} e^{j2\pi m\Delta f T} = e^{j\pi (M-1) \frac{\nu}{\Delta f}} \frac{M}{M \sin(\pi \frac{\nu}{\Delta f})}
\]

where step (a) follows from (6) and the fact that the received signal $y(t)$ is time-limited to the interval $0 < t < (N+1)T$.

This is because, the channel path delays are less than $T (0 < \tau_{i} < T, i = 1, 2, \cdots, L)$ and the transmit signal $x(t)$ is limited to the time interval $0 \leq t < NT$ as $y(t)$ is time-limited to $0 \leq t < T$ (see the expression for $x(t)$ in (7)). We also note that the discrete time-domain samples $y(nT + \frac{lT}{M})$, $n = 0, 1, \cdots, N$, $l = 0, 1, \cdots, M-1$ are obtained by sampling $y(t)$ at integer multiples of $1/(M\Delta f) = T/M$. The ZAK transform of $y(t)$ in (3) is given by

\[
Z_{y}(\tau, \nu) \triangleq \sum_{l=1}^{L} \sum_{i=1}^{N} h_{i} e^{j2\pi \nu (\tau - \tau_{i})} Z_{u}(\tau - \tau_{i}, \nu - \nu_{i}) + Z_{n}(\tau, \nu)
\]

where step (a) follows from Theorem 1 and step (b) follows from Theorem 2 and (14). In (16), $Z_{n}(\tau, \nu)$ is the ZAK transform of the AWGN $n(t)$. Using (16) in (15), we get

\[
Y[k', l'] \triangleq \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} x[k,l] \tilde{h}[k', l', k, l] + Z[k', l']
\]

\[
\tilde{h}[k', l', k, l] \triangleq \sum_{i=1}^{I} \sum_{i' \neq i} h_{i} \left( \frac{(k' - k)\Delta f - \nu_{i}}{N} \right) w_{1} \left( \tau_{i} - \frac{l' - l}{M} \right)
\]

\[
Z[k', l'] \triangleq Z_{n} \left( \frac{l'T}{M}, k'\Delta f/N \right)
\]

\[
h_{i} \left( k', l' \right) \triangleq h_{i} e^{j2\pi \frac{l'T}{M} \left( \frac{k'}{N} - \frac{l'}{M} \right)} e^{j2\pi \left( \frac{k'}{N} - \frac{l'}{M} \right) \left( \frac{k'}{N} + \frac{l'}{M} \right)}
\]

\[
k' = 0, 1, \cdots, N - 1, l' = 0, 1, \cdots, M - 1
\]

The noise samples in the DD domain are given by

\[
Z[k', l'] = Z_{n} \left( \frac{l'T}{M}, k'\Delta f/N \right) \approx \sqrt{T} \sum_{k=0}^{N} \left( \frac{l'T}{M} + kT \right) e^{-j2\pi k \frac{l'}{M} k' \Delta f/N}
\]

\[
= \sqrt{T} \sum_{k=0}^{N-1} n \left( \frac{l'T}{M} + kT \right) e^{-j2\pi k \frac{l'}{M} k' \Delta f/N} + \sqrt{T} n \left( \frac{l'T}{M} + NT \right)
\]

\[
k' = 0, 1, \cdots, N - 1, l' = 0, 1, \cdots, M - 1
\]

where step (a) follows from (6) and the fact that the received signal $y(t)$ is time-limited to $0 < t < NT$. By comparing the
expression for $\tilde{h}[k', l', k, l]$ in (5) and $\tilde{h}[k', l', k, l]$ in (17) it is clear that the received discrete DD domain signal for the two-step receiver (i.e., $\hat{x}[k', l']$ in (4)) is not the same as that for the ZAK receiver (see $Y[k', l']$ in (17)).

From (19) it is clear that for a given $l'$, $Y[k', l']$ can be computed for all $k' = 0, 1, \ldots, N - 1$ using a $N$-point Discrete Fourier Transform (DFT) whose complexity is $O(N \log N)$. Therefore the complexity of the alternate conversion, i.e., computing $Y[k', l']$ from $y(t)$ for all $k' = 0, 1, \ldots, N - 1$, $l' = 0, 1, \ldots, M - 1$ is $O(MN \log N)$. For the two-step conversion, the complexity of computing FFT is $O(M \log(MN))$ which is greater than $O(MN \log N)$. Further, the alternate conversion does not require an OFDM demodulator.

With information symbols $x[k, l] \sim$ i.i.d. $\mathcal{CN}(0, E)$, $E\int_{-\infty}^{\infty} x(t)^2 \, dt = E$ (follows from (27) in Appendix A). Since $x(t)$ is time-limited to $NT$ seconds, the average transmit power is $E/NT$. With a communication bandwidth of $M \Delta f$, the AWGN power at the receiver is $MNf_0$ and therefore the ratio of the transmit power to the receiver noise power is

$$\rho \triangleq \frac{E/NT}{MNf_0} = \frac{E}{MNf_0}. \tag{19}$$

Let the received DD domain samples $Y[k', l']$ and the transmitted information symbols $x[k, l]$ be arranged into vectors $y$ and $x$ respectively as given by (20) (see top of next page).

From (17) it then follows that

$$y = Hx + z \tag{21}$$

where $z$ is the vector of noise samples in the DD domain (see (20)) and $H \in \mathbb{C}^{MN \times MN}$ is the equivalent DD domain channel matrix. The element of $H$ in its $(l'+k'M+1)$-th row and $(l+M+1)$-th column is $\tilde{h}[k', l', k, l]$ (see (17)). Let $K_x \triangleq \mathbb{E}[zz^T]$ denote the covariance matrix of $z$. In (13), the AWGN samples $u \left( \frac{l' T + k T}{M} \right), l' = 0, 1, \ldots, M - 1, k = 0, 1, \ldots, N$ are i.i.d. $\mathcal{CN}(0, M \Delta f N_0) \tag{19}$ and therefore the element of $K_x$ in its $(k_1 M + l_1 + 1)$-th row and $(k_2 M + l_2 + 1)$-th column is given by

$$K_x[k_1 M + l_1 + 1, k_2 M + l_2 + 1] = \mathbb{E}[z[k_1, l_1]z^*[k_2, l_2]] = MNf_0 \left( \frac{1}{M} \right), k_1 = k_2, l_1 = l_2 \quad MNf_0 \left( \frac{1}{M} \right), k_1 \neq k_2, l_1 = l_2. \tag{22}$$

From (22) it is also clear that

$$\lim_{N \to \infty} \frac{K_x}{MNf_0} = I \tag{23}$$

where $I$ denotes the $MN \times MN$ identity matrix. With perfect knowledge of $H$ at the receiver and joint equalization of all $MN$ information symbols in the DD domain, the achievable spectral efficiency (SE) of the ZAK receiver is given by (14)

$$C_{zk} = \frac{1}{MN} \log_2 \left( 1 + \rho H^H \left( \frac{K_x}{MNf_0} \right)^{-1} H \right) \approx \frac{1}{MN} \log_2 \left( 1 + \rho H^H H \right) \left( \text{Large } N \right) \tag{24}$$

where step (a) follows from (19) and (21), and the approximation follows from (23).

Example: In the following we compare the ZAK receiver and the two-step receiver for a single-path channel (i.e., $L = 1$) with zero delay and a Doppler shift equal to an integer multiple of $\Delta f$, i.e., $\nu_k = q \Delta f, q \in \{0, 1, \ldots, M - 1\}$. From (5) and (17) we then have

$$\tilde{h}[k', l', k, l] = h_1 e^{j 2 \pi k' q} \delta[k - k'] \left[ \frac{1}{M} \sum_{m=0}^{M-1-q} e^{j 2 \pi m \frac{q}{M}} \right]$$

$$\tilde{h}[k', l', k', l'] = h_1 e^{j 2 \pi k' q} \delta[k - k'] \delta[l - l'] \tag{25}$$

where $\delta[k], k \in \mathbb{Z}$ is one when $k = 0$ and is zero for all other $k \neq 0$. Just as in (21) for the ZAK receiver, for the two-step receiver also, let $H \in \mathbb{C}^{MN \times MN}$ denote the effective DD domain channel matrix whose element in its $(k'M + l' + 1)$-th row and $(k'M + l + 1)$-th column is $\tilde{h}[k', l', k, l]$. The spectral efficiency achieved by the two-step receiver is $C = \log_2 \left( 1 + \rho H^H H \right) / (MN)$. Let $\tilde{I}[k_1, k_2, l_1, l_2]$ denote the element of $H^H H$ in its $(k_1 M + l_1 + 1)$-th row and $(k_2 M + l_2 + 1)$-th column. From (25) it then follows that

$$\tilde{I}[k_1, k_2, l_1, l_2] = \delta[k_1 - k_2] |h_1|^2 \frac{1}{M} \sum_{m=0}^{M-1-q} e^{j 2 \pi m \frac{(l_1 - l_2)}{M}}. \tag{26}$$

For the ZAK receiver, from (24) it follows that $H^H H = H H^H = |h_1|^2 I$, and therefore from (24) it follows that for large $N$, $C_{zk} \approx \log_2 (1 + \rho |h_1|^2)$ which is same as the capacity of an ideal single-path channel without any Doppler shift. Further, from (26) it follows that $H^H H$ is a block diagonal matrix where each block along the diagonal is a $M \times M$ matrix $A$ whose element in the $(l_1 + 1)$-th row and $(l_2 + 1)$-th column is $|h_1|^2 \frac{1}{M} \sum_{m=0}^{M-1-q} e^{j 2 \pi m \frac{(l_1 - l_2)}{M}}$, $l_1 = 0, 1, \ldots, M - 1, l_2 = 0, 1, \ldots, M - 1$. The spectral efficiency achieved by the two-step receiver is therefore $C = \frac{1}{MN} \log_2 |I + \rho A|$. One can check that $A$ is a positive semi-definite matrix and the DFT vectors $u_p = \frac{1}{\sqrt{M}} (1, e^{j 2 \pi p}, \ldots, e^{j 2 \pi (M-1)p})^T$, $p = 0, 1, \ldots, M - 1$ are a set of $M$ orthonormal eigenvectors of $A$ with corresponding eigen-values $\lambda_p = |h_1|^2$, $p = 0, 1, \ldots, M - q - 1$ and $\lambda_0 = 0$, $p = (M - q), \ldots, M - 1$. From this it follows that $C = \frac{1}{MN} \log_2 |I + \rho A| = \left( 1 - \frac{\rho}{|h_1|^2} \right) \log_2 (1 + \rho |h_1|^2) = \left( 1 - \frac{\rho}{|h_1|^2} \right) \log_2 (1 + \rho |h_1|^2)$ which is clearly less than $C_{zk}$. Therefore, if the ratio of the Doppler shift to the communication bandwidth (i.e., $\nu_1 / (M \Delta f)$) is high, the spectral efficiency of the two-step receiver will be significantly less than $C_{zk}$.

V. NUMERICAL SIMULATIONS

In this section we compare the average spectral efficiency (SE) achieved by the two-step OTFS receiver and the ZAK...
The averaging of the SE is w.r.t. usually not below a downlink CNPC frame is frequency, $\rho$ between an Unmanned Aircraft System (UAS) and a Ground receiver for control and non-payload communication (CNPC).

In Fig. 1 we plot the average SE achieved by the two-step (i.e., $\text{Average Spectral Efficiency} \; (\text{bits/s/Hz})$ $\rho$) $K_m$ [11]. For the en-route scenario a widely accepted $2.75$ is the speed of light, $\nu_h$ $15$ and $\nu'_h = \nu_h - \Delta \nu_h$ is the carrier substituting the expression for $\Phi_{k,l}(\tau,\nu)$.

\[ \Phi_{k,l}(\tau,\nu) \triangleq \sum_{m=0}^{N} \sum_{n=0}^{N} e^{2\pi n \Delta f (t - \frac{\nu}{f_T})}. \]

\[ g(\tau + k'T - nT)e^{-j2\pi
u'k'T} \]

\[ \sqrt{T} \sum_{\nu'} \Theta(\tau + k'T - nT) e^{-j2\pi \nu'k'T} \]

where step (a) follows from (6) and step (b) follows from substituting the expression for $\Phi_{k,l}(t)$ from (27) into the R.H.S. of step (a). The ZAK transform of $g(t - nT)$ is

\[ \sqrt{T} \sum_{\nu'} \Theta(\tau + k'T - nT) e^{-j2\pi \nu'k'T} \]

where $Z_g(\tau,\nu)$ is the ZAK transform of $g(t)$ and step (a) follows from the first and third equality in (11). Using (29) in (28) along with the fact that $e^{j2\pi \nu'k'T}(\tau + k'T) = e^{j2\pi \nu'f_T}\tau$ (since $T\Delta f = 1$) completes the proof.

**REFERENCES**

[1] R. Hadani, S. Rakib, M. Tsatsanis, A. Monk, A. J. Goldsmith, A. F. Molisch, “Orthogonal Time Frequency Space Modulation,” IEEE Wireless Comm. and Networking Conference (WCNC’17), March 2017.

[2] A. Monk, R. Hadani, M. Tsatsanis and S. Rakib, “OTFS - Orthogonal Time Frequency Space: A Novel Modulation Meeting 5G High Mobility and Massive MIMO Challenges,” arXiv:1608.02993 [cs.IT] 9 Aug. 2016.

[3] R. Hadani and A. Monk, “OTFS: A New Generation of Modulation Addressing the Challenges of 5G,” arXiv:1802.02623, 2018.

[4] P. Raviteja, K. T. Phan, Y. Hong and E. Viterbo, “Interference Cancellation and Iterative Detection for Orthogonal Time Frequency Space Modulation,” IEEE Trans. on Wireless Comm., vol. 17, no. 10, Oct. 2018.
[5] P. Raviteja, K. T. Phan, Q. Jin, Y. Hong and E. Viterbo, “Low-Complexity Iterative Detection for Orthogonal Time Frequency Space Modulation,” in Proc. IEEE Wireless Comm. and Net. Conf. (WCNC’18), Apr. 2018.

[6] M. K. Ramachandran and A. Chockalingam, MIMO-OTFS in high-doppler fading channels: Signal detection and channel estimation, in Proc. IEEE Global Commun. Conf. (GLOBECOM), Kansas City, MO, USA, Dec. 2018, pp. 206212.

[7] S. Tiwari, S. S. Das and V. Rangamgari, “Low Complexity MMSE Receiver for OTFS,” IEEE Communication Letters, vol. 23, no. 12, Dec. 2019.

[8] G. D. Surabhi and A. Chockalingam, “Low-Complexity Linear Equalization for OTFS Modulation,” IEEE Communications Letters, vol. 24, no. 2, Feb. 2020.

[9] J. Zak, “Finite Translations in Solid State Physics,” Phy. Rev. Lett., 19, pp. 1385 - 1387,1967.

[10] A. J. E. M. Janssen, “The Zak Transform: A Signal Transform for Sampled Time-Continuous Signals,” Philips J. Res., 43, pp. 23-69, 1988.

[11] H. W. Kim, K. Kang, K. Lim and J. Y. Ahn, “Performance Analysis for Terrestrial Radio waveform for Control and Non-Payload Communication for Unmanned Aircraft Systems,” IEEE International Conf. on Information and Comm. Tech. Convergence (ICTC’ 2016), pp. 759-761, Oct. 2016.

[12] E. Haas, “Aeronautical Channel Modeling,” IEEE Transactions on Vehicular Technology, vol. 51, no. 2, pp. 254-264, March 2002.

[13] P. A. Bello, “Characterization of Randomly Time-Variant Linear Channels,” IEEE Trans. Comm. Syst., vol. 11, pp. 360-393, 1963.

[14] D. N. C. Tse and P. Vishwanath, Fundamentals of Wireless Communication, Cambridge, U.K.: Cambridge Univ. Press, 2005.