ABC IMPLIES A ZSIGMONDY PRINCIPLE FOR RAMIFICATION

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Abstract. Let $K$ be a number field or a function field of characteristic 0. If $K$ is a number field, assume the abc-conjecture for $K$. We prove a variant of Zsigmondy’s theorem for ramified primes in preimage fields of rational functions in $K(x)$ that are not postcritically finite. For example, suppose $K$ is a number field and $f \in K[x]$ is not postcritically finite, and let $K_n$ be the field generated by the $n$th iterated preimages under $f$ of $\beta \in K$. We show that for all large $n$, there is a prime of $K$ that ramifies in $K_n$ and does not ramify in $K_m$ for any $m < n$.

1. Introduction

Let $K$ be either a number field or a function field of characteristic 0 of transcendence degree 1 over its field of constants. Let $\phi \in K(x)$ be a rational function. Recall that the morphism $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is postcritically finite if the forward orbit of the ramification locus of $\phi$ is a finite set. Let $\phi$ be a non-postcritically finite rational function of degree $d \geq 2$ and let $\beta \in \mathbb{P}^1(K)$. As is usual in dynamics, we use $\phi^n$ to denote the map $\phi$ composed with itself $n$ times. For each $n \geq 1$, let

$$K_n = K(\phi^{-n}(\beta)) = K(\gamma \in K : \phi^n(\gamma) = \beta).$$

It is a theorem of the first author and coauthors [BJJ+15] that for any $\beta \in \mathbb{P}^1(K)$, there are infinitely many primes in $K$ that ramify in $\bigcup_{n=1}^{\infty} K_n$. The main idea of the theorem is to produce prime divisors of $\phi^n(\alpha) - \beta$ for $\alpha$ a critical point of $\phi$ with canonical height $h_{\phi}(\alpha) > 0$. The fact that there are infinitely many such primes follows from [Sil93]. Various authors (see [Zsi92, Elk91, Ric07, Kri13, IS09, FG11, GNT13] for example) have sought to show that not only are there infinitely many primes that divide $\phi^n(\alpha) - \beta$ for some $n$, but the stronger statement that there exists an $N$ such that for all $n > N$, there is a prime that divides $\phi^n(\alpha) - \beta$ that does not divide $\phi^m(\alpha) - \beta$ for any $m < n$. If

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this is true, one might say that there are infinitely many primes dividing 

\( \phi^n(\alpha) - \beta \) for some \( n \) because after a certain point each “new iterate” 

\( \phi^n(\alpha) - \beta \) gives a “new prime” dividing \( \phi^n(\alpha) - \beta \). This is sometimes 

referred as the “Zsigmondy principle”, after Zsigmondy \cite{Zsi92} who 

studied these questions in the context of primitive divisors of \( a^n - b^n \).

In this paper, we prove a Zsigmondy principle for ramification for 
certain types of rational functions, including polynomials. Our results 
are conditional on the abc conjecture when \( K \) is a number field. For 
polynomials, our result is the following. Recall that if \( K \) is a function 
field with field of constants \( k \), \( f \) is said to be isotrivial if there is an 
element \( \sigma \in \overline{\kappa}(x) \) of degree one such that 

\[ \sigma \circ \phi \circ \sigma^{-1} \in \overline{k}(x). \]

**Theorem 1.1.** Let \( K \) be a number field or a function field of characteristic 0. Let \( f \in K[x] \) be a polynomial with \( \deg f \geq 2 \) that is not 
postcritically finite. If \( K \) is a number field, assume the abc conjecture 
for \( K \). If \( K \) is a function field, assume that \( f \) is not isotrivial. Then, 
for all sufficiently large \( n \), there exists a prime of \( K \) that ramifies in 
\( K(f^{-n}(\beta)) \) and does not ramify in \( K(f^{-m}(\beta)) \) for \( m < n \).

Our most general theorem is most easily stated in terms of grand 
orbits. The orbit or forward orbit of \( \beta \in \mathbb{P}^1(K) \) is 

\[ O_{\phi}(\beta) = \{ \phi^n(\beta) : n \geq 0 \} = \{ \beta, \phi(\beta), \phi^2(\beta), \ldots \}. \]

The backward orbit of \( \beta \) is 

\[ O_{\phi}(\beta) = \{ \alpha \in \mathbb{P}^1(K) : \phi^n(\alpha) = \beta \text{ for some } n \geq 0 \} = \bigcup_{n=0}^{\infty} \phi^{-n}(\beta). \]

The grand orbit of \( \beta \) is the backward orbit of the forward orbit, that is, 

\[ \mathcal{G}O_{\phi}(\beta) = \{ \alpha \in \mathbb{P}^1(K) : \phi^m(\alpha) = \phi^n(\beta) \text{ for some } m, n \in \mathbb{Z}_{\geq 0} \}. \]

Grand orbits under \( \phi \) partition \( \mathbb{P}^1(K) \) into equivalence classes. A point 
\( \beta \) is said to be exceptional for \( \phi \) if its grand orbit is a finite set. It is 
well known that if \( \beta \) is exceptional for \( \phi \), then (up to conjugacy by a 
fractional linear transformation) either \( \phi \) is a polynomial and \( \beta = \infty \), 
or \( \phi(x) = x^d \) for some \( d \in \mathbb{Z} \) and \( \beta \in \{0, \infty\} \).

A point \( \beta \in \mathbb{P}^1(K) \) is periodic if \( \phi^n(\beta) = \beta \) for some \( n > 0 \) and 
preperiodic if \( \phi^n(\beta) = \phi^m(\beta) \) for some \( n > m \geq 0 \). A point that is not 
preperiodic is wandering. We define a grand orbit to be preperiodic 
if one (equivalently any) of its points is preperiodic, and wandering 
otherwise.

We now state the main theorem. If \( K \) is a number field, we will 
assume that the abc conjecture holds for \( K \). If \( K \) is a function field
of characteristic 0, the abc conjecture is a theorem of Mason-Stothers \cite{Mas84, Sto81} (see also Silverman \cite{Sil84}). As we now consider rational maps from $\mathbb{P}^1(K)$ to itself, it is possible for $\infty$ to arise as a preimage of $K$, in which case we simply declare that $K(\infty) = K$.

**Theorem 1.2.** Let $\phi \in K(x)$ with $\deg \phi \geq 2$. Suppose that $\phi$ is not postcritically finite and that $\beta \in \mathbb{P}^1(K)$ is not exceptional for $\phi$. If $K$ is a number field, assume the abc conjecture for $K$. If $K$ is a function field, assume that $\phi$ is not isotrivial. For all sufficiently large $n$, there exists a prime of $K$ that ramifies in $K(\phi^{-n}(\beta))$ and does not ramify in $K(\phi^{-m}(\beta))$ for $m < n$.

The restriction that $\phi$ be non-isotrivial is not a serious one. Indeed, we can treat the case of isotrivial rational functions by a fairly elementary argument, provided that $\beta$ is not in the field of constants of $K$. See Theorem 5.1.

Theorem 1.2 immediately produces Theorem 1.1 as a special case, since a polynomial of degree $d$ has at most $d - 1$ critical points other than the point at infinity (which is of course a fixed point). For rational functions in general we have the following theorem, which shows that a new prime ramifies at every two levels in the tower of fields $K_n$.

**Theorem 1.3.** Let $\phi \in K(x)$ with $\deg \phi \geq 2$. Suppose that $\phi$ is not postcritically finite and that $\beta \in \mathbb{P}^1(K)$ is not exceptional for $\phi$. If $K$ is a number field, assume the abc conjecture for $K$. If $K$ is a function field, assume that $\phi$ is not isotrivial. For all sufficiently large $n$, there exists a prime of $K$ that ramifies in $K(\phi^{-n}(\beta))$ and does not ramify in $K(\phi^{-m}(\beta))$ for $m \leq n - 2$.

One of our motivations for proving Theorem 1.2 was an application to the growth rate of Galois groups of iterates of polynomials. The group $\text{Gal}(K_n/K)$ injects into $\text{Aut}(T_n)$, the automorphism group of the complete $d$-ary rooted tree of height $n$ where $d = \deg \phi$. The group $\text{Aut}(T_n)$ is isomorphic to an iterated wreath product of the symmetric group $S_d$, so $|\text{Aut}(T_n)|$ grows doubly exponentially in $n$. It is expected that in many cases the index $|\text{Aut}(T_n) : \text{Gal}(K_n/K)|$ remains bounded as $n \to \infty$, which implies that the degree of the splitting field of $\phi^n(x) - \beta$ over $K$ grows doubly exponentially for large $n$. Odoni proved that generic polynomials have this property, as well as the particular polynomial $x^2 - x + 1$ \cite{Odo85, Odo88}; Juul \cite{Jun15} proved that generic rational functions have this property. Stoll proved that an infinite family of quadratic polynomials \cite{Sto92} have this property. Boston and Jones \cite{BJ09} have proposed a dynamical analog of the Serre open
image theorem (see [Ser72]), and we hope to use the techniques of this paper to treat some special cases of this problem, in particular the case of cubic polynomials.

It follows from our main theorem that the growth rate for many non-postcritically finite rational maps is at least simply exponential (conditional on the abc conjecture when $K$ is a number field). For example, this includes all polynomial maps.

**Corollary 1.4.** Suppose that $K$, $\phi \in K(x)$, and $\beta \in \mathbb{P}^1(K)$ satisfy the assumptions of Theorem 1.2. Then there exists $C$ such that for all sufficiently large $n$, $[K(\phi^{-n}(\beta)) : K] \geq C^n$.

The strategy of our proof combines the approaches of [GNT13] and [BIJ15]. We begin with Lemma 3.1 which gives a necessary condition for $K_n$ to ramify over $p$; this is adapted from [BGH13]. We then prove Lemma 3.2 which gives a sufficient condition for a prime $p$ to ramify in $K_n$. Note that the condition in both Lemmas has do to with whether or not a suitable iterate of a critical point under $\phi$ meets $\beta$ at $p$. We then use a so-called “Roth-abc” result (see Proposition 2.2) to show that for each critical point $\alpha$ the quantities $\phi^n(\alpha) - \beta_j$ have very few repeated factors for large $n$ and suitable preimages $\beta_j$ of $\beta$. This is done in Lemma 4.2. We are also able to bound the contribution to $h(\phi^n(\alpha) - \beta_j)$ coming from primes that divide $\phi^m(\alpha') - \beta_j$ for some $m < n$ and $\alpha'$ some critical point of $\phi$. This is done in Lemmas 4.1 and 4.3 (note that in the application of Lemma 4.3 it is crucial that the number of wandering grand orbits of $\phi$ is small). Putting these together along with some other simple estimates gives a prime $p$ such that $v_p(\phi^n(\alpha) - \beta_j) = 1$ for some suitable $\beta_j$ of $\beta$ with the property that $p$ does not ramify in $K_m$ for any $m < n$. Applying Lemma 3.2 then gives our main result, Theorem 1.2.

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2. Preliminaries

Let $K$ be a number field or a function field of characteristic 0 with transcendence degree 1 over its field of constants $k$. Let $\phi \in K(x)$ be a rational function of degree $d \geq 2$. If $K$ is a number field, let $\mathfrak{o}_K$ be the ring of integers of $K$. If $K$ is a function field, choose a prime $q$ and let $\mathfrak{o}_K = \{ z \in K : v_p(z) \geq 0 \text{ for all primes } p \neq q \text{ of } K \}$. For any prime $p$, let $k_p$ be the residue field $\mathfrak{o}_K/p$.

We use the notion of good reduction as introduced by Morton and Silverman [MS94]. Let $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ be a morphism, written
in homogeneous coordinates as \( \phi([X : Y]) = [P(X, Y) : Q(X, Y)] \), where \( P, Q \in \mathfrak{o}_K[X, Y] \) are homogeneous polynomials of the same degree without any common factor in \( \overline{K}[X,Y] \). Letting \( P_0(X,Y)=P(X,Y) \) and \( Q_0(X,Y)=Q(X,Y) \), we recursively define \( P_{m+1}(X,Y)=P(P_m(X,Y),Q_m(X,Y)) \) and \( Q_{m+1}=Q(P_m(X,Y),Q_m(X,Y)) \). We let \( p_m(X)=P_m(X,1) \) and \( q_m(X)=Q_m(X,1) \).

Let \( \phi_p=[P_p:Q_p] \), where \( P_p, Q_p \in k_p[X,Y] \) are the reductions of \( P \) and \( Q \) modulo \( p \). We say that \( \phi \) has \textit{good reduction} at \( p \) when \( \max(\deg P_p, \deg Q_p) = \max(\deg P, \deg Q) \) and \( P_p \) and \( Q_p \) have no common factor in \( k_p[X,Y] \). When this is the case, \( \phi_p \) induces a non-constant morphism from \( \mathbb{P}^1_{k_p} \) to itself. When this map is separable, we say that \( \phi \) has \textit{good separable reduction} at \( p \).

2.1. Heights. For a rational prime \( p \) of \( K \), define

\[
N_p = \frac{1}{[K : Q]} \log \#k_p
\]

if \( K \) is a number field and

\[
N_p = [k_p : k]
\]

if \( K \) is a function field. As in [GNT13], normalizing by the degree of the number field will make it easier to state proofs in the same way for both number fields and function fields.

If \( K \) is a number field, the height of \( z \in K \) is defined as

\[
h(z) = -\sum_{\text{primes } p \text{ of } \mathfrak{o}_K} \min(v_p(z),0)N_p + \frac{1}{[K : Q]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \max(\log |\sigma(z)|,0)
\]

where the second sum is taken over all maps \( \sigma : K \to \mathbb{C} \) (in particular, complex conjugate embeddings are not identified). We extend \( h \) to \( \mathbb{P}^1(K) \) by setting \( h(\infty) = 0 \). If \( K \) is a function field, instead the height of \( z \in K \) is

\[
h(z) = -\sum_{\text{primes } p \text{ of } \mathfrak{o}_K} \min(v_p(z),0)N_p.
\]

In either case, for \( z \neq 0 \) the product formula gives the inequality

\[
\sum_{v_p(z)>0} v_p(z)N_p \leq h(z).
\]

We will use the Call-Silverman canonical height \( h_{\phi} \), which is defined by

\[
h_{\phi}(x) = \lim_{n \to \infty} \frac{h(\phi^n(x))}{d^n}.
\]

This limit exists by the same telescoping series argument that shows the existence of the Nerón-Tate height on an elliptic curve. See [CS93]
for details. The canonical height satisfies the following important properties for some absolute constant $C_\phi$ and for every $x \in K$:

$$h_\phi(\phi(x)) = dh_\phi(x),$$

and

$$|h(x) - h_\phi(x)| \leq C_\phi.$$

It follows immediately from these properties that $h_\phi(x) \neq 0$ if and only if $h(\phi^n(x)) \to \infty$ as $n \to \infty$.

If $K$ is a number field, then for $n \geq 2$ we define the height of the nonzero $n$-tuple $(z_1, z_2, \ldots, z_n) \in K^n$ by

$$h(z) = -\sum_{\text{primes } p \text{ of } o_K} \min(v_p(z_1), \ldots, v_p(z_n))N_p + \frac{1}{[K : Q]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \max(\log |\sigma(z_1)|, \ldots, \log |\sigma(z_n)|)$$

2.2. The $abc$-conjecture. For $z_1, \ldots, z_n \in K^\times$, we define

$$I(z_1, \ldots, z_n) = \{\text{primes } p \text{ of } o_K \mid v_p(z_i) \neq v_p(z_j) \text{ for some } i, j\}$$

and

$$\text{rad}(z_1, \ldots, z_n) = \sum_{p \in I(z_1, \ldots, z_n)} N_p.$$

With this notation, we assume the $abc$-conjecture as follows.

**Conjecture 2.1.** Let $K$ be a number field. For any $\epsilon > 0$, there exists a constant $C_{K, \epsilon}$ such that for all $a, b, c \in K^\times$ with $a + b = c$, we have

$$h(a, b, c) < (1 + \epsilon) \text{rad}(a, b, c) + C_{K, \epsilon}.$$

We will make use of the following estimate, sometimes called “Roth-abc” as in [GNT13], which holds for number fields conditionally on the $abc$-conjecture and is true unconditionally for function fields of characteristic 0. The following combines Propositions 3.4 and 4.2 from [GNT13].

**Proposition 2.2.** Let $K$ be a number field or function field of characteristic 0. If $K$ is a number field, suppose that the $abc$-conjecture holds for $K$. Let $F \in K[x]$ be a polynomial of degree at least 3 with no repeated factors and let $\epsilon > 0$. Then there exists $C_{F, \epsilon}$ such that for all $x \in K$,

$$\sum_{v_p(F(x)) > 0} N_p \geq (\deg F - 2 - \epsilon)h(x) + C_{F, \epsilon}.$$

Note that in the case where $K$ is a function field, the result does not follow from $abc$ but instead requires Yamanoi’s proof [Yam04] of the
Vojta conjecture for algebraic points on curves over function fields of characteristic 0.

2.3. Base extension. Certain arguments are made more easily after passing from our number field or function field $K$ to a finite extension $L$ of $K$. We will quickly show that our results are true over $K$ exactly when they are true over a finite extension.

Lemma 2.3. Let $K$ be a number field or function field of characteristic 0, let $L$ be a finite extension of $K$, let $p$ be a finite prime of $K$ that does not ramify in $L$, and let $q$ be a finite prime of $L$ such that $q | p$. Then, for any finite Galois extension $M$ of $K$, the prime $p$ ramifies in $M$ if and only if $q$ ramifies in the compositum $M \cdot L$.

Proof. Suppose that $p$ does not ramify in $M$. Then $p$ does not ramify in $M \cdot L$ since $p$ does not ramify in $L$. Thus, any prime $q$ of $L$ such that $q | p$ cannot ramify in $M \cdot L$.

Suppose that $p$ ramifies in $M$. Since $M$ is Galois over $K$, this means that $e(m/p) > 1$ for any $m | p$ in $M$. Thus, for any $r | p$ in $L \cdot M$, we have $e(r/p) > 1$. Since $e(q/p) = 1$, we must have $e(r/p) = e(r/q)$; hence, $e(r/q) > 1$ so $q$ ramifies in $M \cdot L$. □

Lemma 2.4. Let $K$ be a number field or function field of characteristic 0, let $\beta \in K$, and let $\phi$ be a rational function with coefficients in $K$. Let $L$ be a finite extension of $K$. Then the following statements are equivalent:

(a) For all sufficiently large $n$, there is a finite prime $p$ of $K$ such that $p$ ramifies in $K(\phi^{-n}(\beta))$ and $p$ does not ramify in $K(\phi^{-m}(\beta))$ for $m < n$.

(b) For all sufficiently large $n$, there is a finite prime $q$ of $L$ such that $q$ ramifies in $L(\phi^{-n}(\beta))$ and $q$ does not ramify in $L(\phi^{-m}(\beta))$ for $m < n$.

Proof. Let $S$ be the set of finite primes of $K$ that ramify in $L$ and let $T$ be the set of primes of $L$ that lie over primes in $S$.

Suppose that (a) holds. Then, since $S$ is finite, for all sufficiently large $n$, there is a finite prime $p \notin S$ of $K$ such that $p$ ramifies in $K(\phi^{-n}(\beta))$ and $p$ does not ramify in $K(\phi^{-m}(\beta))$ for $m < n$. If $q$ is a prime of $L$ such that $q | p$, then $q$ ramifies in $L(\phi^{-n}(\beta))$ and $q$ does not ramify in $L(\phi^{-m}(\beta))$ for $m < n$, by Lemma 2.3.

Likewise, if (b) holds, then, since $T$ is finite, for all sufficiently large $n$, there is a finite prime $q \notin T$ of $L$ such that $q$ ramifies in $L(\phi^{-n}(\beta))$ and $q$ does not ramify in $L(\phi^{-m}(\beta))$ for $m < n$. If $p$ is a prime of $K$ such that $q | p$, then $p$ ramifies in $K(\phi^{-n}(\beta))$ and $p$ does not ramify in $K(\phi^{-m}(\beta))$ for $m < n$, again by Lemma 2.3. □
By Lemma 2.4 it suffices to prove Theorem 1.2 over a finite extension \( L \) of \( K \). We argue here that it also suffices to prove the Theorem after replacing \( \phi \) with \( \phi^\sigma = \sigma \circ \phi \circ \sigma^{-1} \) for any Möbius transformation \( \sigma \in L(x) \), and replacing \( \beta \) with \( \sigma(\beta) \). Note that for any \( \phi \in K(x) \) and \( \beta \in \mathbb{P}^1(K) \), the hypotheses of Theorem 1.2 (\( \phi \) is postcritically finite, \( \beta \) is non-exceptional, and the condition on wandering grand orbits intersecting \( R_\phi \)) are invariant under this change of variables. This is because \( \alpha \) is a critical point of \( \phi \) if and only if \( \sigma(\alpha) \) is a critical point of \( \phi^\sigma \), and because the map \( \sigma \) induces a bijection from the grand orbits of \( \phi \) to the grand orbits of \( \phi^\sigma \) that preserves their structure as grand orbits. Thus, we may assume that \( \phi \) has a fixed point defined over \( K \), and, after changing variables, we may assume that \( \phi(\infty) = \infty \). Note that this means that \( \deg P_m \geq \deg Q_m \) for all \( m \) and that when \( \phi \) has good reduction at \( \mathfrak{p} \), the leading coefficient of \( P_m \) is not divisible by \( \mathfrak{p} \) for all \( m \).

### 3. Criteria for ramification

To prove Theorem 1.1, we need some conditions for ramification in preimage fields. The necessary condition is an adaptation of a standard result about ramification in \( p \)-adic fields, for example [BGH+13, Lemma 1]. Recall that \( K \) is either a number field or a function field of characteristic 0. From this point forward, for \( \phi \in K(x) \) and \( \beta \in \mathbb{P}^1(K) \), we use the notation \( K_n = K(\phi^{-n}(\beta)) \) as defined in the introduction.

**Proposition 3.1.** Let \( \phi \in K(x) \) and \( \beta \in K \). Let \( \mathfrak{p} \) be a prime of \( K \) such that \( \phi \) has good separable reduction and \( v_\mathfrak{p}(\beta) \geq 0 \). If \( \mathfrak{p} \) ramifies in \( K_n \), there exists \( \alpha \in R_\phi \) such that \( v_\mathfrak{p}(\phi^m(\alpha) - \beta) > 0 \) for some \( m \) with \( 1 \leq m \leq n \).

**Proof.** Let \( (p_n)_\mathfrak{p} \) and \( (q_n)_\mathfrak{p} \) denote the reductions of \( p_n \) and \( q_n \) at \( \mathfrak{p} \), and let \( \beta'_\mathfrak{p} \) denote the reduction of \( \beta \) at \( \mathfrak{p} \). Since \( K_n \) is the splitting field of \( p_n(X) - \beta q_n(X) \), it follows that if \( K_n \) ramifies at \( \mathfrak{p} \) then \( F(X) = (p_n)_\mathfrak{p}(X) - \beta'_\mathfrak{p}(q_n)_\mathfrak{p}(X) \) has a multiple root. Thus, there is a root of \( F(X) \) that is also a root of the derivative of \( F(X) \).

Note that if \( \gamma \) is a root of both \( F(X) \) and \( F'(X) \), then \( \gamma \) is also a root of \( (p_n)_\mathfrak{p}(X)(q_n)_\mathfrak{p}(X) - (p_n)_\mathfrak{p}(X)(q_n)_\mathfrak{p}'(X) \). Since \( (\phi^\mathfrak{p})^n \) is separable at \( \mathfrak{p} \), we see that \( (p_n)_\mathfrak{p}(X)(q_n)_\mathfrak{p}(X) - (p_n)_\mathfrak{p}(X)(q_n)_\mathfrak{p}'(X) \) is not identically zero. Hence, all of its roots are the reduction modulo \( \mathfrak{p} \) of a root of \( p_n(X)q_n(X) - p_n(X)q_n(X) \). Therefore, there is a critical point \( \alpha \) of \( \phi^n \) that reduces to a root of \( (p_n)_\mathfrak{p}(X) - \beta(q_n)_\mathfrak{p}(X) \) at \( \mathfrak{p} \). This means that \( v_\mathfrak{p}(\phi^m(\alpha) - \beta) > 0 \). \( \square \)
Proposition 3.2. Let $\phi \in K(x)$ and $\beta \in K$. For all primes $p$ of $K$ such that $\phi$ has good separable reduction at $p$ and $v_p(\beta) \geq 0$, if there exists a critical point $\alpha$ of $\phi$ such that $\phi^n(\alpha) \neq \infty$ and $v_p(\phi^n(\alpha) - \beta) = 1$, then $p$ ramifies in $K_n$.

Proof. This is the criterion that forms the main argument of [BIJ+15, Theorem 5]. We provide a brief proof here. First note that by Lemma 2.3, we may assume without loss of generality that $\alpha \in K$, as otherwise we can replace $K$ by $K(\alpha)$.

Since $K_n$ is the splitting field of $p_n(X) - q_n(X)\beta$ and $\alpha \in K$, it follows that $K_n$ is also the splitting field of the polynomial $p_n(X + \alpha) - q_n(X + \alpha)\beta$. We write

$$p_n(X + \alpha) - q_n(X + \alpha)\beta = a_kX^k + \cdots + a_0.$$ 

Note that $v_p(a_0) = v_p((\phi^n(\alpha) - \beta)q_n(\alpha)) = 1$, because $v_p(q_n(\alpha)) = 0$ since $v_p(\beta) \geq 0$ and $\phi^n$ has good reduction at $p$. Also note that $v_p(a_k) = 0$, again using the fact that $\phi^n$ has good reduction at $p$.

Now, $p_n(X + \alpha) - q_n(X + \alpha)\beta$ is congruent mod $p$ to $p_n(X + \alpha) - q_n(X + \alpha)\phi^n(\alpha)$, because $v_p(\phi^n(\alpha) - \beta) > 0$. We have that $X^e$ divides $p_n(X + \alpha) - q_n(X + \alpha)\phi^n(\beta)$, where $e > 1$ is the ramification index of $\alpha$, so there is an $\ell > 1$ such that $v_p(a_j) > 0$ for $k = 0, \ldots, \ell - 1$ and $v_p(a_\ell) = 0$. Thus, the first segment of the $p$-adic Newton polygon of $p_n(X + \alpha) - q_n(X + \alpha)\beta$ is the line from $(0, 1)$ to $(\ell, 0)$. Therefore, $p_n(X + \alpha) - q_n(X + \alpha)$ has a root $\gamma$ such that $v_p(\gamma) = 1/\ell$, which means that $K_n$ ramifies over $K$ at $p$. (See [Kob77, IV.3] for summary of the theory of Newton polygons.)

\[ \square \]

In the next section, we will use Propositions 3.1 and 3.2 in tandem to show the existence of primes that ramify in the $n$th preimage field but do not ramify earlier.

4. Proofs of Main Theorems

To prove Theorem 1.2, we want to reduce to the case where the base point $\beta$ is non-periodic and non-postcritical. This ensures that the preimage sets $\phi^n(\beta)$ are of size $d^n$, and in particular, that the numerator of $\phi^n(x) - \beta$ is a squarefree polynomial. This will allow us to easily use the Roth-abc estimate of Proposition 2.2. Of course, in general $\beta$ may be periodic or postcritical. Let $t$ be the smallest positive integer such that no element of $\phi^{-t}(\beta) \setminus \phi^{-(t-1)}(\beta)$ is periodic or postcritical. Let $\{\beta_1, \ldots, \beta_N\}$ denote $\phi^{-t}(\beta) \setminus \phi^{-(t-1)}(\beta)$. Note that if $x \in \phi^{-n}(\beta)$ for some $n > t$, and $x$ is not periodic, not critical, and not postcritical, then $x \in \bigcup_{j=1}^N O_{\phi}^{-}(\beta_j)$. By the discussion at
the end of Section [2], we may adjoin the critical points of $\phi$ and the points $\beta_1, \ldots, \beta_N$ to $K$, and also make a change of variables such that $\phi(\infty) = \infty$.

We construct a finite set of bad primes $S$ for which we may not be able to control ramification. Let $S$ contain the primes $p$ where $\phi$ does not have good separable reduction at $p$, $v_p(\beta_j) \neq 0$ for some $\beta_j$, $v_p(\beta_j - \beta_k) > 0$ for $j \neq k$, or $v_p(\phi^m(\gamma) - \beta_j) > 0$ for some $m \in \{0, \ldots, t - 1\}$ and some $\gamma \in R_\phi$. Theorem [1,2] will be a straightforward consequence of the following three lemmas.

**Lemma 4.1.** Let $\alpha \in \mathbb{P}^1(K)$ with $h_\phi(\alpha) > 0$ and let $\beta_1, \ldots, \beta_N$ be as above. If $K$ is a number field, assume the abc conjecture for $K$. Let $\delta > 0$. For $n > 0$, let $Z(n)$ denote the set of primes $p \notin S$ such that

$$\min(v_p(\phi^n(\alpha) - \beta_i), v_p(\phi^n(\alpha) - \beta_j)) > 0$$

for some $0 < m < n$ and some $i, j$ between 1 and $N$. Then there exists a constant $C_\delta$ such that

$$\sum_{p \in Z(n)} N_p \leq \delta d^n h_\phi(\alpha) + C_\delta$$

for all sufficiently large $n$.

**Proof.** Let $F(X) = \prod_{i=1}^N (X - \beta_i)$. Then $F$ divides the numerator of $\phi^t$ (because $\phi^t(\beta_i) = 0$ for all $i$), none of the $\beta_i$ are periodic, and $\phi^t(\beta_i) \neq 0$ for all $i$ and any $\ell = 0, \ldots, t - 1$. Then Proposition 5.1 of [CNT13] asserts that if $Z'(n)$ is the set of primes $p$ such that $\min(v_p(\phi^{m+t}(\alpha)), v_p(F(\phi^n(\alpha)))) > 0$, then for any $\delta > 0$, there is a constant $C_\delta$ such that

$$\sum_{p \in Z'(n)} N_p \leq \delta h(\phi^n(\alpha)) + C_\delta$$

for all $n$. If $p \notin S$ and

$$\min(v_p(\phi^n(\alpha) - \beta_i), v_p(\phi^n(\alpha) - \beta_j)) > 0,$$

then $v_p(\phi^{m+t}(\alpha)) > 0$, since $\phi^t(\beta_j) = 0$, and $v_p(F(\phi^n(\alpha))) > 0$ since $\beta_i$ is a root of $F$. Thus, we see that $Z(n) \subseteq Z'(n)$. Using the properties of $h_\phi$ established in Section [2], namely that $h_\phi(\phi(x)) = d h_\phi(x)$ and that $|h(x) - h_\phi(x)|$ is bounded independently of $x$, our proof is complete. \qed

**Lemma 4.2.** Let $\beta_i$ be as above. If $K$ is a number field, suppose that the abc-conjecture holds for $K$. For every $\epsilon > 0$, there is a constant $C_{\epsilon}$ such that

$$\sum_{v_p(\phi^n(\alpha) - \beta_j) = 1} N_p \geq (d - \epsilon) d^{n-1} h_\phi(\alpha) + C_{\epsilon}.$$
Proof. Choose \(m > 0\) such that \(3/d^m < \epsilon/d\). Since \(\beta_j\) is not in the post-critical set, for any \(m\), the set of solutions to \(\phi^m(x) = \beta_j\) consists of exactly \(d^m\) distinct points. Thus, \(p_m(X) - \beta_j q_m(X)\) has no repeated roots. Thus, using Proposition 2.2 and the fact that \(|h - h_\phi|\) is bounded, there is a constant \(C_1\) such that

\[
\sum_{v_p(p_m(x) - \beta_j q_m(x))=1} N_p \geq (d^m - 3)h_\phi(x) + C_1
\]

for all \(x \in K\). Letting \(x = \phi^{n-m}(\alpha)\), we see there is a constant \(C_2\) such that

\[
\sum_{v_p(\phi^n(x) - \beta_j q_m(x))=1} N_p \geq (1-\epsilon/d)d^m d^{n-m}h_\phi(\alpha) + C_1 \geq (d-\epsilon)d^{n-1}h_\phi(\alpha) + C_2.
\]

For all but at most finitely many \(p\) we have \(v_p(\phi^n(\alpha)) = v_p(F(\phi^n(\alpha)))\), so the Lemma follows immediately.

Lemma 4.3. Let \(G\) be a set of critical points of \(\phi\) that all have the same grand orbit. Let \(Y(i, j)\) be the set of primes \(p\) such that \(v_p(\phi^n(x) - \beta_j q_m(x)) > 0\) for some \(\gamma \in G\). Let \(M_G = \max_{\gamma \in G} h_\phi(\gamma)\). Then, for all \(n\), we have

\[
\sum_{i=1}^{n-1} \sum_{j=1}^{N} \sum_{p \in Y(i, j)} N_p \leq N \left( \frac{1}{d-1} \right) d^n M_G + O(n).
\]

Proof. Let \(\alpha \in G\) be the critical point of largest canonical height \(h_\phi(\alpha)\). For every \(\gamma \in G\), we have \(\phi^n(\alpha) = \phi^m(\gamma)\) for some \(n, m \geq 0\), so \(d^n h_\phi(\alpha) = d^m h_\phi(\gamma)\) and \(m \geq n\). In other words, \(\alpha\) is the “farthest forward” critical point in the grand orbit. So except for \(1 \leq i \leq m-n\), the primes that divide \(\phi^i(\gamma) - \beta_j\) also divide \(\phi^k(\alpha) - \beta_j\) for some \(k\).

The indicated initial values of \(i\) have a finite contribution to the sum that can be absorbed into the \(O(n)\) term.

By the product formula and properties of heights we have

\[
\sum_{v_p(\phi^i(\alpha) - \beta_j) > 0} N_p \leq h(\phi^i(\alpha) - \beta_j) \leq d^i h(\alpha) + h(\beta_j) + C_\phi.
\]

So we can use the estimation

\[
\sum_{i=1}^{n-1} \sum_{j=1}^{N} \sum_{p \in Y(i, j)} N_p \leq N M_G \frac{d^n - 1}{d-1} + n C_{\phi,\beta_1,\ldots,\beta_N} + O(n)
\]

and the lemma follows.

Now, we are ready to prove Theorem 1.2.
Proof of Theorem 1.2. Assume that φ ∈ K(x) is not postcritically finite and that β ∈ P^1(K) is not exceptional for φ. Let β_1, . . . , β_N be as above. If necessary, replace K with K(α, β_1, . . . , β_N) (by Lemma 2.3 this loses no generality). Let g be the number of wandering grand orbits that R_φ intersects (we have g ≤ d − 1) and let α ∈ R_φ be a critical point of maximum canonical height h_φ(α). Observe that h_φ(α) > 0, because if every critical point has canonical height 0, then φ is postcritically finite. This follows from the fact that if K is a number field, then any nonpreperiodic point must have positive canonical height by Northcott’s theorem, while if K is a function field, Baker [Bak09] and Benedetto [Ben05] have proved that any nonpreperiodic point has positive canonical height whenever φ is not isotrivial. Hence, we may apply Lemma 4.1 to the orbit of α.

Let X(n) be the set of primes p ∉ S such that
1. v_p(φ^n(α) − β_j) = 1 for some 1 ≤ j ≤ N.
2. v_p(φ^m(α) − β_j) ≤ 0 for all 1 ≤ m ≤ n − 1 and 1 ≤ j ≤ N, and
3. v_p(φ^m(γ) − β_j) ≤ 0 for every critical point γ not in the same grand orbit as α, and all 1 ≤ m ≤ n − 1 and 1 ≤ j ≤ N.

For the critical points γ in the same grand orbit as α, we have φ^n(γ) = φ^s(α) for some positive integers s, u with s ≤ u (α is the farthest forward critical point in its grand orbit as in the proof of Lemma 4.3).

So if v_p(φ^m(γ) − β_j) > 0 for some m < n, then either m < u and p ∈ S, or φ^(s−u+m)(α) ≡ β_j (mod p), and p ∉ X(n) because v_p(φ^n(α) − β_j) ≤ 0 for m < n if p ∈ S. Therefore if p ∈ X(n), then v_p(φ^m(γ) − β_j) ≤ 0 for every critical point γ of φ and every m < n. Thus by Propositions 3.1 and 3.2 p ramifies in K_α and does not ramify in K_m for m < n.

We show that X(n) is nonempty for all large n. By Lemma 4.2 for a given j and any ε > 0 we have
\[ \sum_{v_p(φ^n(α) − β_j) = 1} N_p \geq (d−ε)d^{n−1}h_φ(α) + C_ε. \]

It follows that
\[ \sum_{v_p(φ^n(α) − β_j) = 1 \text{ for some } j} N_p \geq N(d−ε)d^{n−1}h_φ(α) + C_ε \]
because the primes p such that v_p(φ^n(α) − β_j) > 0 for j = j_1 and j = j_2 are divisors of β_{j_1} − β_{j_2}, so these primes are contained in S, and their contribution to the sum can be absorbed into the constant C_ε.

Now we apply Lemma 4.1 to α and each β_j, and we apply Lemma 4.3 to the grand orbits not containing α that intersect R_φ. There are at most d − 2 such wandering grand orbits; any preperiodic grand orbits contribute at most an O(n) term to the sum because the term M_G
coming from Lemma 4.3 is zero. Now we subtract the conclusion of Lemma 4.1 \((N\text{ times})\) and Lemma 4.3 \((g - 1\text{ times})\) from the conclusion of Lemma 4.2. This gives the following: for every \(\epsilon > 0\) and \(\delta > 0\), there are constants \(C_\epsilon, C_\delta,\) and \(C\) such that, for all sufficiently large \(n\), we have

\[
\sum_{p \in \mathcal{X}(n)} N_p \geq N(d - \epsilon)d^{n-1}h_\phi(\alpha) + C_\epsilon - N\delta d^n h_\phi(\alpha) - C_\delta
\]

\[- (g - 1)N \frac{1}{d - 1}d^n h_\phi(\alpha) + Cn
\]

\[
\geq d^n h_\phi(\alpha)N \left(1 - \epsilon d^{-1} - \delta - \frac{d - 2}{d - 1}\right) + Cn.
\]

Choosing \(\epsilon\) and \(\delta\) small enough, this quantity is positive for all large \(n\), and we are done.

□

Proof of Theorem 1.3. By the chain rule, the critical points of \(\phi^2\) are either critical points of \(\phi\) or preimages of these points under \(\phi\), so the critical points of \(\phi^2\) lie in at most \(#R_\phi \leq 2d - 2\) distinct grand orbits. We have \(2d - 2 < d^2 - 1\) because \(d > 1\). Applying Theorem 1.2 to the map \(\phi^2\) and the point \(\beta\), and also to a distinct point in \(\phi^{-1}(\beta)\) (which exists because \(\beta\) is not exceptional) yields the result.

□

Proof of Corollary 1.4. By Theorem 1.1 for all sufficiently large \(n\) there is a prime of \(K\) that ramifies in \(K_{n+1}\) but not in \(K_n\). Therefore the kernel of the natural surjection \(\text{Gal}(K_{n+1}/K) \to \text{Gal}(K_n/K)\) is nontrivial, so it must be at least order 2. The result follows.

□

5. The isotrivial case

In this section we treat the case of isotrivial rational functions. The techniques here are much more elementary than in the rest of the paper.

Theorem 5.1. Let \(K\) be a function field of characteristic 0 with field of constants \(k\), and let \(\phi \in K(x)\) be a rational function of degree greater than one. Suppose that there is a finite extension \(K'\) of \(K\) and \(\sigma \in K'(x)\) such that \(\sigma \phi \sigma^{-1} \in k'(x)\), where \(k'\) is the algebraic closure of \(k\) in \(K'\). Then we have the following:

(a) If \(\sigma(\beta) \in k'\), then there are at most finitely many primes of \(K\) that ramify in \(\bigcup_{n=1}^\infty K_n\).

(b) If \(\sigma(\beta) \notin k'\) and \(\phi\) is not postcritically finite, then for all sufficiently large \(n\), there exists a prime of \(K\) that ramifies in \(K_n\) and does not ramify in \(K_m\) for \(m < n\).
Proof. Suppose that $\sigma(\beta) \in k'$. Then, if $\phi^n(\alpha) = \beta$, we have

$$\sigma\phi\sigma^{-1}(\sigma(\alpha)) = \sigma(\beta) \in k'.$$

Since $\sigma\phi\sigma^{-1} \in k'(x)$, where $k'$ is algebraic over $k$, it follows that $\sigma(\alpha) \in \bar{k}$. Thus, $\alpha$ is in the compositum $\bar{k} \cdot K'$. Since $K'$ ramifies over at most finitely many primes of $K$ and $\bar{k} \cdot K'$ is unramified everywhere over $K'$, we see that $\bar{k} \cdot K'$ ramifies over at most finitely many primes of $K$. Thus, there are only finitely many primes of $K$ that ramify in $\bigcup_{n=1}^{\infty} K_n$.

Now suppose that $\sigma(\beta) \notin k'$. After passing to a finite extension, we may assume that all the critical points of $\phi$ are defined over $K'$. Let $\phi^\sigma$ denote $\sigma\phi\sigma^{-1}$. Since every critical point of $\phi^\sigma$ is simply $\sigma(z)$ for a critical point $z$ of $\sigma$ and every critical point of $\phi^\sigma$ is algebraic over $k$, we see then that every critical point of $\phi^\sigma$ is in $k'$.

Now, note that $\sigma(\beta)$ is not algebraic over $k'$, and that $K'$ is therefore a finite extension of $k'(\sigma(\beta))$. For any critical point $\alpha'$ of $\phi^\sigma$ and any $m$, we see that $(\phi^\sigma)^m(\alpha') - \sigma(\beta)$ generates a prime in $k'(\sigma(\beta))$. Since $\phi^\sigma$ is not postcritically finite, there is a critical point $\alpha$ of $\phi^\sigma$ such that $(\phi^\sigma)^m(\alpha) \neq (\phi^\sigma)^n(\alpha')$ for any $n < m$ and any critical point $\alpha \neq \alpha'$. Thus, for every $n > 0$, there is a prime $m$ of $k'(\sigma(\beta))$ such that $v_m((\phi^\sigma)^n(\alpha) - \sigma(\beta)) = 1$ and $v_m((\phi^\sigma)^m(\alpha') - \sigma(\beta)) = 0$ for all $m < n$. Then, by Proposition 3.2 and 3.1 this prime $m$ ramifies in $k'(\sigma(\beta))((\phi^\sigma)^{-n}(\sigma(\beta)))$ and does not ramify in $k'(\sigma(\beta))((\phi^\sigma)^{-m}(\sigma(\beta)))$ for any $m < n$. Note that since $\sigma$ is defined over $K'$ and $(\phi^\sigma)^n = \sigma\phi^n\sigma^{-1}$, we see that for any $z$ we have $(\phi^\sigma)^n(z) = \sigma(\beta)$ if and only if $\phi^n(\sigma(z)) = \beta$. Thus, by Lemma 2.4 it follows that for all but finitely many $n$, there is a prime $q$ of $K'$ such that $q$ ramifies in $L(\phi^{-n}(\beta))$ but $q$ does not ramify in $L(\phi^{-m}(\beta))$ for any $m < n$. Applying Lemma 2.4 again, we see that for all but finitely many $n$, there is a prime $p$ of $K$ such that $p$ ramifies in $K(\phi^{-n}(\beta))$ but $p$ does not ramify in $K(\phi^{-m}(\beta))$ for any $m < n$, as desired.

\[\square\]
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