STUDY OF QUANTUM SYMMETRIES FOR VERTEX-TRANSITIVE GRAPHS USING INTERTWINNER SPACES.

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Abstract. We study the quantum automorphism group of vertex-transitive graphs using intertwiner spaces of the magic unitary matrix associated to this quantum subgroups of $S_n^+$. We also give some applications to quantum symmetries of circulant graphs.

1. Introduction

A quantum permutation group on $n$ points is a compact quantum group acting faithfully on the classical space consisting of $n$ points. The following facts were discovered by Wang [18].

1. There exists a largest quantum permutation group on $n$ points, denoted $S_n^+$, and called the quantum permutation groups on $n$ points.

2. The quantum group $S_n^+$ is infinite-dimensional if $n \geq 4$, and hence in particular an infinite compact quantum group can act faithfully on a finite classical space.

Very soon after Wang’s paper [18], the representation theory of $S_n^+$ was described by Banica [2]: it is similar to the one of $SO(3)$ and can be described using tensor categories of non-crossing partitions. This description, further axiomatized and generalized by Banica-Speicher [8], led later to spectacular connections with free probability theory, see e.g. [11].

The next natural question was the following one: does $S_n^+$ have many non-classical quantum subgroups, or is it isolated as an infinite quantum group acting faithfully on a finite classical space?

In order to find quantum subgroups of $S_n^+$, the quantum automorphism group of a finite graph was defined in [12, 15]. This construction indeed produced many examples of non-classical quantum permutation groups, answering positively to the above question. The known results on the computation of quantum symmetry groups of graphs are summarized in [5] where the description of the quantum symmetry group of vertex-transitive graphs of small order (up to 11) is given with an exception for the Petersen graph. The quantum symetries of Petersen graph was study in [17] where the autor proves that Petersen graph has no quantum symmetry.

The present paper is a contribution to the study of quantum automorphism groups of finite graphs: we study some way to prove that a vertex-transitiv graph has no quantum symmetries, with the study of associated intertwinners spaces (as in [8]). With our results we can better understand the quantum symmetries of some circulant graph, as $C_{13}(3, 4)$.

The paper is organized as follows. Section 2 is preliminary section: we recall some basic facts about compact quantum groups, quantum permutation groups and quantum automorphism group of finite graphs. Section 3 is devoted to intertwiner spaces for classical subgroups of $S_n$. In Sections 4 we study one particular intertwinner space associated to vertex-transitive graphs. In section 5 we extend the study to the quantum case, and introduce the notion of $B$-clos graphs. To finish the sections 6,7,8 and 9 are devoted to applications of results from section 5.
2. COMPACT QUANTUM GROUPS AND QUANTUM AUTOMORPHISM GROUP OF FINITE GRAPHS

We first recall some basic facts concerning compact quantum groups. The books [10][15] are convenient references for this topic, and all the definitions we omit can be found there. All algebras in this paper will be unital as well as all algebra morphisms, and \( \otimes \) will denote the minimal tensor product of \( C^*-\)algebras as well as the algebraic tensor product, this should cause no confusion.

**Definition 2.1.** A Woronowicz algebra is a \( C^*-\)algebra \( A \) endowed with a \( *\)-morphism \( \Delta : A \rightarrow A \otimes A \) satisfying the coassociativity condition and the cancellation law

\[
\Delta(A)(A \otimes 1) = A \otimes A = \Delta(A)(1 \otimes A)
\]

The morphism \( \Delta \) is called the comultiplication of \( A \).

The category of Woronowicz algebras is defined in the obvious way. A commutative Woronowicz algebra is isomorphic with \( C(G) \), the algebra of continuous functions on a compact group \( G \), unique up to isomorphism, and the category of compact quantum groups is defined to be the category dual to the category of Woronowicz algebras. Hence to any Woronowicz algebra \( A \) corresponds a unique compact quantum group \( G \) according to the heuristic notation \( A = C(G) \).

Woronowicz’s original definition for matrix compact quantum groups [19] is still the most useful to produce concrete examples, and we have the following fundamental result [20].

**Theorem 2.2.** Let \( A \) be a \( C^*-\)algebra endowed with a \( *\)-morphism \( \Delta : A \rightarrow A \otimes A \). Then \( A \) is a Woronowicz algebra if and only if there exists a family of unitary matrices \( (u_\lambda)_{\lambda \in \Lambda} \in M_{d_\lambda}(A) \) satisfying the following three conditions:

1. The \( *\)-subalgebra \( A_0 \) generated by the entries \( u^\lambda_{ij} \) of the matrices \( (u^\lambda)_{\lambda \in \Lambda} \) is dense in \( A \).
2. For \( \lambda \in \Lambda \) and \( i, j \in \{1, \ldots, d_\lambda\} \), one has \( \Delta(u^\lambda_{ij}) = \sum_{k=1}^{d_\lambda} u^\lambda_{ik} \otimes u^\lambda_{kj} \).
3. For \( \lambda \in \Lambda \), the transpose matrix \( (u^\lambda)^t \) is invertible.

In fact, the \( *\)-algebra \( A_0 \) in the theorem is canonically defined, and is what is now called a compact Hopf algebra: a Hopf \( *\)-algebra having all its finite-dimensional comodules equivalent to unitary ones (see [10][15]). The counit and antipode of \( A_0 \), denoted, respectively, \( \epsilon \) and \( S \), are referred to as the counit and antipode of \( A \). The Hopf \( *\)-algebra \( A_0 \) is called the algebra of representation functions on the compact quantum group \( G \) dual to \( A \), with another heuristic notation \( A_0 = \mathcal{O}(G) \).

Conversely, starting from a compact Hopf algebra, the universal \( C^*-\)completion yields a Woronowicz algebra in the above sense: see [10][15]. In fact there are possibly several different \( C^*-\)norms on \( A_0 \), but we will not be concerned with this question.

As usual, a (compact) quantum subgroup \( H \subset G \) corresponds to a surjective Woronowicz algebra morphism \( C(G) \rightarrow C(H) \), or to a surjective Hopf \( *\)-algebra morphism \( \mathcal{O}(G) \rightarrow \mathcal{O}(H) \).

We refer the reader to [10][15] for large classes of examples, including \( q\)-deformations of classical compact Lie groups. In the present paper, we will be interested in the following fundamental example, due to Wang [18]. First we need some terminology. A matrix \( u \in M_n(A) \) is said to be orthogonal if \( u = \bar{u} \) and \( uu^t = I_n = u^tu \). A matrix \( u \) is said to be magic unitary if all its entries are projections, all distinct elements of a same row or same column are orthogonal, and sums of rows and columns are equal to 1. A magic unitary matrix is orthogonal.

**Definition 2.3.** The \( C^*-\)algebra \( A_s(n) \) is defined to be the universal \( C^*-\)algebra generated by variables \( (u_{ij})_{1 \leq i, j \leq n} \), with relations making \( u = (u_{ij}) \) a magic unitary matrix.

The \( C^*-\)algebra \( A_s(n) \) admits a Woronowicz algebra structure given by

\[
\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \epsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}
\]

The associated compact quantum group is denoted by \( S_n^+ \), i.e.

\[
A_s(n) = C(S_n^+),
\]
Definition 2.4. A quantum permutation algebra is a Woronowicz algebra quotient of $A_s(n)$ for some $n$. Equivalently, it is a Woronowicz algebra generated by the coefficients of a magic unitary matrix.

We now come to quantum group actions, studied e.g. in [16]. They correspond to Woronowicz algebra coactions. Recall that if $B$ is a $C^*$-algebra, a (right) coaction of Woronowicz algebra $A$ on $B$ is a $*$-homomorphism $\alpha : B \to B \otimes A$ satisfying the coassociativity condition and

$$\alpha(B)(1 \otimes A) = B \otimes A$$

Wang has studied quantum groups actions on finite-dimensional $C^*$-algebras in [18], where the following result is proved.

Theorem 2.5. The Woronowicz algebra $A_s(n)$ is the universal Woronowicz algebra coacting on $\mathbb{C}^n$, and is infinite-dimensional if $n \geq 4$.

The coaction is constructed in the following manner. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$. Then the coaction $\alpha : \mathbb{C}^n \to \mathbb{C}^n \otimes A_s(n)$ is defined by the formula

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji}$$

We refer the reader to [18] for the precise meaning of universality in the theorem, but roughly speaking this means that $S_n^+$ is the largest compact quantum group acting on $n$ points, and deserves to be called the quantum permutation group on $n$ points.

Equivalently, Wang’s theorem states that any Woronowicz algebra coacting faithfully on $\mathbb{C}^n$ is a quotient of the Woronowicz algebra $A_s(n)$, and shows that quantum groups acting on $n$ points correspond to Woronowicz algebra quotient of $A_s(n)$, and hence to quantum permutation algebras. In particular, there is a surjective Woronowicz algebra morphism $A_s(n) \to C(S_n)$, yielding a quantum group embedding $S_n \subset S_n^+$. More directly, the existence of the surjective morphism $A_s(n) \to C(S_n)$ follows from the fact that $C(S_n)$ is the universal commutative $C^*$-algebra generated by the entries of a magic unitary matrix. See [18] for details.

The complete study of $S_4^+$ is in [6]. We will study some subgroups of $A_s(n)$ using vertex-transitives graphs as in [5].

We now recall the definition of the quantum automorphism group of a finite graph $X$ using [3] [9].

For a finite graph $X$ with $n$ vertices, it is convenient to also call $X$ the set of vertices of $X$. The complement graph of $X$ will be denoted by $X^c$. If $i$ and $j$ are two vertices of $X$ we use the notation $i \sim_X j$ when they are connected and $i \not\sim_X j$ when they are not (or simply $i \sim j$ when no confusion can arise).

Definition 2.6. The adjacency matrix of $X$ is the matrix $d_X = (d_{ij})_{1 \leq i, j \leq n} \in M_n(0,1)$ given by $d_{ij} = 1$ if $i, j$ are connected by an edge, and $d_{ij} = 0$ if not.

The classical automorphism group of $X$ will be denoted by $\text{Aut}(X)$ (this is a subgroup of $S_n$) and we have the following way to characterize its elements.

Proposition 2.7. Identifying $\sigma \in S_n$ to the associated permutation matrix $P_\sigma \in M_n(\{0,1\})$, we have:

$$\sigma \in \text{Aut}(X) \iff d_X P_\sigma = P_\sigma d_X$$

This characterization in the classical case leads to the following natural definition of the quantum automorphism group of a finite graph, see [3].

Definition 2.8. Associated to a finite graph $X$ is the quantum permutation algebra

$$A(X) = A_s(n)/\langle d_X u = ud_X \rangle$$

where $n$ is the number of vertices of $X$. 

3
The quantum automorphism group corresponding to \( A(X) \) is the quantum automorphism group of \( X \), denoted \( \mathbb{G}_X \). In this way we have a commuting diagram of Woronowicz algebras:

\[
\begin{array}{ccc}
A_s(n) = C(S^+_n) & \rightarrow & A(X) = C(\mathbb{G}_X) \\
\downarrow & & \downarrow \\
C(S_n) & \rightarrow & C(\text{Aut}(X))
\end{array}
\]

with the kernel of the right arrow being the commutator ideal of \( A(X) \).

**Example 2.9.** For the graph with \( n \) vertices and no edges we have \( A(X) = A_s(n) \), so \( \mathbb{G}_X = S^+_n \). Moreover we have \( A(X^c) = A(X) \), because \( ud_X = d_X u \) and \( ud_{X^c} = d_{X^c} u \) are equivalent when \( u \) is magic unitary.

If \( X = C_n \) is the \( n \)-cycle graph one can show that for \( n \neq 4 \), \( A(C_n) \) is commutative, thus \( A(C_n) = C(\text{Aut}(C_n)) \) and therefore \( \text{Aut}(C_n) = \mathbb{G}_{C_n} = \mathbb{D}_n \), where \( \mathbb{D}_n \) is the \( n \)-dihedral group. For more examples see \([5]\).

### 3. Intertwiner spaces in the classical case

We use the notations from \([8]\). Let \( G \) be a subgroup of \( S_n \) (we identify the elements \( \sigma \) of \( G \) to their associated permutation matrix \( P_{\sigma} \in M_n(\{0,1\}) \)). The intertwiner spaces are the following \( C_G(k,l) \):

\[
C_G(k,l) = \{ T \in \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}) : \forall g \in G, T g^\otimes k = g^\otimes l T \}
\]

with \((k,l) \in \mathbb{N} \times \mathbb{N} \).

**Remark 3.1.** By Frobenius duality we have

\[
C_G(k,l) \simeq C_G(0,k+l)
\]

We denote by \( C_G \) the set of all \( C_G(k,l) \) and we define the elements \( U, M, C \) as follow:

\[
U \in C_G(0,1) : \quad U(1) = \sum_{k=0}^{n-1} e_k
\]

\[
M \in C_G(2,1) : \quad M(e_i \otimes e_j) = \delta_{i,j} e_i
\]

\[
S \in C_G(2,2) : \quad S(e_i \otimes e_j) = e_j \otimes e_i
\]

and their duals:

\[
U^* \in C_G(1,0) : \quad U^*(e_i) = 1
\]

\[
M^* \in C_G(1,2) : \quad M^*(e_i) = e_i \otimes e_i
\]

\[
S^* \in C_G(2,2) : \quad S^* = S
\]

From \([8]\) we get the following proposition on the structure of \( C_G \).

**Proposition 3.2.** \([8]\), Proposition 1.2] The collection of vector spaces \( C_G(k,l) \) is a tensor category with duals, in the sense that it has the following properties:

1. \( (T, T' \in C_G) \Rightarrow T + T' \in C_G \)
2. \( (T, T' \in C_G) \Rightarrow T \otimes T' \in C_G \)
3. If \( T, T' \in C_G \) are composable, then \( T \circ T' \in C_G \)
4. \( (T \in C_G) \Rightarrow T^* \in C_G \)
5. \( I_d \in C_G(1,1) \)

We use a version of a Tannaka-Krein duality theorem about Woronowicz algebra (in \([21]\)). The use we make is an analogous of theorem 1.4 in \([8]\) for the study of compact groups containing \( S_n \).
Theorem 3.3. The construction $G \to C_G$ induce a bijection between subgroups $G \subset S_n$ and the symmetric tensor categories with duals, $C$, satisfying

$$C_{S_n} \subset C \subset C_{\{1\}}$$

where $C_{\{1\}}(k,l) = C(k,l)$.

$C_G$ is spanned as symmetric tensor category by endomorphism that characterize the relations satisfying by the coefficients of element of $G$ (view as matrices). The matrix $u = (u_{ij})_{0 \leq i,j \leq n-1}$ is in $S_n$ if:

i. $\sum_{k=0}^{n-1} u_{ik} = \sum_{k=0}^{n-1} u_{kj} = 1$

ii. $u_{ij}u_{j2} = \delta_{j1,j2}u_{ij1}$ et $u_{i1j}u_{12j} = \delta_{i1,i2}u_{i1j}$

iii. The $u_{ij}$ commute

Then we can check that:

(i) $\Leftrightarrow U, U^* \in C_G(0,1)$

(ii) $\Leftrightarrow M, M^* \in C_G(2,1)$

(iii) $\Leftrightarrow S \in C_G(2,2)$

so we get:

$$C_{S_n} = \langle U, M, S \rangle_{+, \otimes, \ast}$$

There is a more explicit way to describe $C_{S_n}$ by using the set $P(k,l)$ of all partitions of $[1, \ldots, k+l]$, see [3].

Remark 3.4. (3) + (4) + (ii) $\Rightarrow$ (5) since $I_d = M \circ M^*$.

When $G = \text{Aut}(X)$ we use this following description

Proposition 3.5. If $X$ is a vertex-transitive finite graph, then

$$C_{\text{Aut}(X)} = \langle U, M, S, d_X \rangle_{+, \otimes, \ast}$$

Proof. Elements of $\text{Aut}(X)$ are those of $S_n$ which commute with $d_X$, and by definition

$$(d_X \in C_{\text{Aut}(X)}(1,1)) \Leftrightarrow (\forall \sigma \in \text{Aut}(X), P_\sigma d_X = d_X P_\sigma)$$

which gives us the result. \qed

4. Spaces $C_{\text{Aut}(X)}(1,1)$ and $T^X$

In this part $G$ is a transitive subgroup of $S_n$ and we will consider that $S_n$ is the set of permutations over $\{0,1, \ldots, n-1\}$.

Definition 4.1. For all $i \in [0,n-1]$, we denote by $G_i$ the stabiliser of $i$ in $G$. Then we denote by $O_i^0$ the $G_i$-orbit over $[0,n-1]$ with $O_i^0 = \{0\}$.

For $i \in [1,n-1]$ we choose $\sigma_i \in G$ such that $\sigma_i(0) = i$ and we define:

$$O_i^s := \sigma_i(O_i^0)$$

Proposition 4.2. The $O_i^s$ has the following properties:

i. The $O_i^s$, for $0 \leq s \leq r$, are the $G_i$-orbit over $[0,n-1]$.

ii. The definition of $O_i^s$ does not depend on the choice of $\sigma_i \in G$ such that $\sigma_i(0) = i$.

iii. $\forall \sigma \in G, \forall i,j \in [0,n-1], \forall s_1, s_2 \in [0,r]$ we have

$$\sigma(O_i^{s_1}) = O_{\sigma(i)}^{s_1} \text{ and } \sigma(O_i^{s_1} \cap O_j^{s_2}) = O_{\sigma(i)}^{s_1} \cap O_{\sigma(j)}^{s_2}$$
Proof. (i): Let \( j_0 \in O^*_i = \sigma_i(O^s_0) \), \( j_0 = \sigma_i(i_0) \) with \( i_0 \in O^s_0 \).

\[
j \in O^*_i = \sigma_i(O^s_0) \iff \exists j' \in O^s_0, \ j = \sigma_i(j') \]

\[
\iff \exists \sigma' \in G_0, \ j = \sigma'\sigma(i_0) \]

\[
\iff \exists \sigma' \in G_0, \ j = \sigma_i\sigma'\sigma_{i}^{-1}(j_0) \]

\[
\iff \exists \sigma' \in \sigma_i G_0 \sigma_i^{-1}, \ j = \sigma'(j_0) \]

\[
\iff \exists \sigma' \in G_{\sigma_i(1)}, \ j = \sigma'(j_0) \]

\[
\iff j \text{ is in the } G_i\text{-orbit of } j_0
\]

which show that the \( O^*_i \) are the annonced orbits.

(ii): Let \( \sigma \in G \) such that \( \sigma(0) = i \), then

\[
k \in \sigma(O^s_0) \iff \exists k_0 \in O^s_0, k = \sigma(k_0) \iff \exists k_0 \in O^s_0, k = \sigma_i(\sigma_i^{-1}\sigma(k_0))
\]

since \( \sigma_i^{-1}\sigma(0) = 0 \), so \( \sigma_i^{-1}\sigma \in G_0 \). By definition of \( O^s_0 \) we obtain \( \sigma_i^{-1}\sigma(k_0) \in O^s_0 \) and we get

\[
k \in \sigma(O^s_0) \implies k \in \sigma_i(O^s_0)
\]

which mean

\[
\sigma(O^s_0) \subset \sigma_i(O^s_0)
\]

then using the cardinal we find \( \sigma(O^s_0) = \sigma_i(O^s_0) \) and the definition of \( O^*_i \) does not depend of the choice on \( \sigma_i \).

(iii): The first equality comes from this following calculation:

\[
\sigma(O^s_1) = \sigma\sigma_i(O^s_0) = O^{s_1}_{\sigma_i(0)} = O^{s_1}_{\sigma(i)}
\]

and the second from:

\[
\sigma(O^s_1 \cap O^s_2) = \sigma(O^{s_1}_i) \cap \sigma(O^{s_2}_j) = O^{s_1}_{\sigma(i)} \cap O^{s_2}_{\sigma(j)}
\]

\( \square \)

**Definition 4.3.** For every \( s \in [0, r] \) we define \( T_s \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \) by

\[
T_s(e_j) = \sum_{i \in O^*_j} e_i
\]

and we denote

\[
T^G := \text{Vect}(T_s \mid 0 \leq s \leq r)
\]

By definition the family \( (T_0, \ldots, T_r) \) is free and it’s a basis of \( T^G \).

**Remark 4.4.** This definition of \( T_s \) is not canonical because there is a choice made in the order of orbit’s labels. But the definition of \( T^G \) is canonical.

We now present an other way to build this endomorphism \( T_s \) to get new properties about.

**Proposition 4.5.** For \( s \in [0, r] \), we denote \( O^s \) the orbit of the diagonal action of \( G \) on \( [0, n-1] \times [0, n-1] \) which contained \( O^s_0 \times \{0\} \). The \( (O^0, \ldots, O^r) \) are orbits of this action and for all \( s \in [0, r] \) we have

\[
[T_s]_{i,j} = \delta_{(i,j) \in O^s}
\]

**Proof.** We first prove that all the orbits are of the form: \( O^s \).

Indeed if \( (i, j) \in [0, n-1] \times [0, n-1] \) then there exists \( s \in [0, r] \) such that \( i \in O^*_j \).

Let \( \sigma \in G \) such that \( \sigma(0) = j \). We obtain \( i \in \sigma(O^s_0) \), so

\[
(i, j) = \sigma(O^s_0 \times \{0\})
\]

which mean \( (i, j) \in O^s \). It gives also

\[
(i, j) \in O^s \iff i \in O^*_j
\]
And by definition we have

\[ [T_s]_{i,j} = \delta_{i,\sigma(j)} \]

as required.

This description exactly correspond to a coherent configuration as in [13].

**Lemma 4.6.** If \( \sigma \in G \subset S_n \) then,

\[ P_s T_s = T_s P_\sigma \]

**Proof.** We recall that by definition \( P_\sigma(e_i) = e_{\sigma(i)} \).

Let \( s \in [0, r] \) and \( j \in [0, n-1] \):

\[ P_\sigma T_s(e_j) = \sum_{i \in O^*} e_{\sigma(i)} = \sum_{k \in \sigma(O^*_j)} e_k = \sum_{k \in \sigma(O^*_j)} T_s(e_{\sigma(j)}) = T_s P_\sigma(e_j) \]

so we get \( P_s T_s = T_s P_\sigma \). □

**Lemma 4.7.** For \( f \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \) the following propositions are equivalent:

1. \( \forall \sigma \in G, P_\sigma f = f P_\sigma \)
2. \( \forall \sigma \in G, \forall i, j \in [0, n-1], a_{i,\sigma(j)} = a_{\sigma(i),j} \) where the \( a_{i,j} \) are coefficients of the matrix of \( f \) in the canonical basis \( (e_0, \ldots, e_{n-1}) \).
3. \( f \in T^G \)

**Proof.** (1) \( \Leftrightarrow \) (2):

\[ \forall \sigma \in G, P_\sigma f = f P_\sigma \iff \forall \sigma \in G, \forall i, j \in [0, n-1], [P_\sigma f]_{i,j} = [f P_\sigma]_{i,j} \]

\[ \iff \forall \sigma \in G, \forall i, j \in [0, n-1], a_{\sigma^{-1}(i),j} = a_{i,j} \]

\[ \iff \forall \sigma \in G, \forall i, j \in [0, n-1], a_{i,j} = a_{\sigma(i),\sigma(j)} \]

(3) \( \Rightarrow \) (1): Come quickly from lemma 4.6.

(2) \( \Rightarrow \) (3): Let \( s \in [0, r] \) and \( j_1 \neq j_2 \in O^*_s \).

Then there exists \( \sigma \in G_0 \) such that \( \sigma(j_1) = j_2 \). And by assumptions we have

\[ a_{j_1,0} = a_{\sigma(j_1),\sigma(0)} = a_{j_2,0} := a_{s,0} \]

and then

\[ f(e_0) = \sum_{s=0}^r a_{s,0} T_s(e_0) \]

Let \( i \in [1, n-1] \). As \( G \) act transitively it exists \( \sigma \in G \) such that \( \sigma(0) = i \). Hence

\[ f P_\sigma(e_0) = f(e_i) \]

so

\[ P_\sigma f(e_0) = \sum_{s=0}^r a_{s,0} P_\sigma T_s(e_0) = \sum_{s=0}^r a_{s,0} T_s(e_i) \]

by lemma 4.6. Which mean

\[ f(e_i) = \sum_{s=0}^r a_{s,0} T_s(e_i) \]

hence

\[ f = \sum_{s=0}^r a_{s,0} T_s \in T^G \]

□

**Remark 4.8.** For (2) \( \Leftrightarrow \) (3) we also can see that (2) mean that the function \( (i, j) \mapsto a_{i,j} \) is constant over the orbits \( O^*_s \), which mean \( f \in T^G \) by using proposition 4.5.
From now $X$ will denote a finite vertex-transitive graph. The set of vertices of $X$ will be denoted by $[0, n-1]$ and its adjacency matrix by $d_X$ in the canonical basis $(e_0, \ldots, e_{n-1})$ of $\mathbb{C}^n$. Then, $G = \text{Aut}(X)$ satisfy all the assumptions to define

$$ \text{Aut}_i(X) = G_i \quad \text{and} \quad T^X := T^{\text{Aut}(X)} $$

**Remark 4.9.** Lemma 4.7 show that $T^X$ is stable by composition and that $d_X \in T^X$ so we have:

$$ P_\sigma d_X = d_X P_\sigma \iff \sigma \in \text{Aut}(X) \iff \forall s \in [0, r], \ P_\sigma T_s = T_s P_\sigma $$

Moreover the definition of $T^X$ only depend on $\text{Aut}(X)$ so

$$ T^X = T^{X^c} $$

since $\text{Aut}(X) = \text{Aut}(X^c)$.

To resume, for this automorphism group of a vertex-transitive graph, we have this proposition.

**Proposition 4.10.** If $X$ is a vertex-transitive finite graph, then

$$ C_{\text{Aut}(X)} = \langle U, M, S, d_X \rangle_{+, \circ, \otimes, *} $$

$$ C_{\text{Aut}(X)}(1, 1) = T^X $$

**Proof.** The first equality is a recall from proposition 4.10 and the second is a direct consequence of Lemma 4.7 which say that $C_G(1, 1) = T^G$ and for $G = \text{Aut}(X)$ we have $T^G = T^X$. $\square$

In the following pages we want to study an equivalent of this intertwiner spaces for the quantum case.

5. **The quantum case.**

**Definition 5.1.** A $n$-quantum permutation algebra is a quotients $A$ of

$$ A_s(n) = \langle u_{ij}, 1 \leq i, j \leq n \mid u \text{ unitary matrix} \rangle $$

They correspond to quantum subgroups of $S_{n^+}$.

Always using the same notation of [8], the intertwiner spaces are:

$$ C_A(k, l) = \{ T \in C(k, l) \mid T u^\otimes k = u^\otimes l T \} $$

where $u^\otimes k$ is the $n^k \times n^k$-matrix $(u_{i_1 j_1} \ldots u_{i_k j_k})_{i_1 \ldots i_k j_1 \ldots j_k}$.

**Remark 5.2.** By Frobenius duality we still have:

$$ C_A(k, l) \simeq C_A(0, k + l) $$

The collection of the $C_A(k, l)$ for $(k, l) \in \mathbb{N} \times \mathbb{N}$, denoted $C_A$, is still a tensor category with duals in the sense of proposition 3.2. The Tannaka-Krein duality is as follows.

**Theorem 5.3.** Construction of $A \rightarrow C_A$, induce a bijection between the $n$-permutations quantum algebra and the tensor categories with duals, $C$, satisfying

$$ C_{A_s(n)} \subset C \subset C_{\{1\}} $$

Similary to the classical case we have:

$$ C_{A_s(n)} = \langle U, M \rangle_{+, \circ, \otimes, *} $$

This result appears first in [2] and more recently in [7]. In [8], we can fine a another classical description of $C_{A_s(n)}(k, l)$, using non crossing partition of $[0, n-1]$.

First, we can check that:

$$ C_{A_s(n)}(0, 1) = \text{Vect}(e), \quad \text{where} \quad e = \sum_{i=0}^{n-1} e_i $$

$$ C_{A_s(n)}(1, 1) = \text{Vect}(\text{Id}, J) $$

where $J$ is the matrix whose all entries are equal to one.
**Lemma 5.4.** Let $\mathcal{A}$ be an $n$-permutation quantum algebra and $G^0 \subset S_n$ the permutation group such that:

$$\text{com}(\mathcal{A}) = C(G^0)$$

where $\text{com}(\mathcal{A})$ is the abelianised algebra of $\mathcal{A}$. Then

$$C_A \subset C_{G^0}$$

and in particular

$$C_A(1,1) \subset T^{G_0}$$

**Proof.** This result is a direct consequence of the definition. The abelianisation correspond to the add of $S$ in the generators of $C_A$. □

For $A(X)$ we have $\text{com}(A(X)) = C(\text{Aut}(X))$ so $G^0 = \text{Aut}(X)$ and this lemma tell us that

$$C_A(X)(1,1) \subset T^X$$

We obtain the natural generalisation of the proposition [4.10] for $A(X)$ in the quantum case.

**Proposition 5.5.** For every finite graph $X$ we have:

$$C_A(X) = \langle U, M, d_X \rangle_{+, \circ, \otimes, *}$$

**Proof.** $A(X)$ is defined by the magic unitary matrix characterized by the following relations :

i. $\sum_{k=0}^{n-1} u_{ik} = \sum_{k=0}^{n-1} u_{kj} = 1$

ii. $u_{ij} u_{ij} = \delta_{j_1,j_2} u_{ij_1}$ et $u_{ij_1} u_{ij_2} = \delta_{i_1,i_2} u_{ij}$

iii. $ud_X = d_X u$

and as

(i) $\iff U \in C_A(X)(0,1)$

(ii) $\iff M \in C_A(X)(2,1)$

(iii) $\iff d_X \in C_A(X)(1,1)$

it ends the proof. □

**Corollary 5.6.** $X$ has no quantum symmetry if and only if $S \in C_{A(X)}$.

**Proof.** We know that $X$ has no quantum symmetry if and only if $C_A(X) = C_{\text{Aut}(X)}$ but $C_A(X) = \langle U, M, d_X \rangle_{+, \circ, \otimes, *}$ and $C_{\text{Aut}(X)} = \langle U, M, S, d_X \rangle_{+, \circ, \otimes, *}$ so it works.

We can also check that $S \in C_{A(X)}$ mean that the $u_{i,j}$ (who spanned $A(X)$) commute hence by definition $X$ has no quantum symmetry if and only if $A(X)$ is commutative. □

Now we want to understand better $C_A(X)(1,1)$ using the space $T^X$ as in the classical case. For that, we use the following Hadamard product on $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$.

**Definition 5.7.** Let $f, g \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, we denote $f \times g \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ the Hadamard product of $f$ and $g$ define by

$$f \times g = M \circ (f \otimes g) \circ M^*$$

**Proposition 5.8.** This product satisfy the following properties:

1. If $f, g \in C_A(X)(1,1)$ then $f \times g \in C_A(X)(1,1)$.

2. Let $f, g \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$, written on the matrix-form $f = (a_{ij})$ and $g = (b_{ij})$. Then we have $f \times g = (a_{ij} b_{ij})$.

3. If $f, g \in T^X$ are written on the form

$$f = \sum_{i=0}^{r} a^f_i T_i, \quad g = \sum_{i=0}^{r} a^g_i T_i$$

then we have

$$f \times g = \sum_{i=0}^{r} a^f_i a^g_i T_i \in T^X$$

which mean $a^f g = a^f a^g$. 

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Proof.
(1): Trivial since \( M, M^* \in C_{A(X)}(1,1) \) and \( C_{A(X)} \) is stable by composition and tensor product.

(2):
\[
\begin{align*}
f \times g(e_j) &= M(f(e_j) \otimes g(e_j)) \\
&= M\left( \sum_{i_1=0}^{n-1} a_{i_1,j} e_{i_1} \otimes \sum_{i_2=0}^{n-1} b_{i_2,j} e_{i_2} \right) \\
&= M\left( \sum_{i_1,i_2=0}^{n-1} a_{i_1,j} b_{i_2,j} (e_{i_1} \otimes e_{i_2}) \right) \\
&= \sum_{i,j=0}^{n-1} a_{ij} b_{ij} e_i \\
&= \sum_{i=0}^{n-1} a_{ij} b_{jj} e_i
\end{align*}
\]
which gives us the result.

(3): When \( f, g \in T^X \) we have this following calculation:
\[
\begin{align*}
f \times g(e_j) &= M\left( \sum_{i_1=0}^{r} \alpha_{i_1}^f T_{i_1}(e_j) \otimes \sum_{i_2=0}^{r} \alpha_{i_2}^g T_{i_2}(e_j) \right) \\
&= \sum_{i_1,i_2=0}^{r} \alpha_{i_1}^f \alpha_{i_2}^g (M(T_{i_1}(e_j) \otimes T_{i_2}(e_j))) \\
&= \sum_{i=0}^{r} \alpha_{i}^f \alpha_{i}^g T_i(e_j) \quad \text{since } O^j \cap O^j = \delta_{i_1,i_2} O^j_{i_1}
\end{align*}
\]
So \( f \times g = \sum_{i=0}^{r} \alpha_{i}^f \alpha_{i}^g T_i \).

This theorem make summarize the link between \( \text{Aut}(X), A(X) \) and \( T^X \).

**Theorem 5.9.** Let \( X \) be a vertex-transitive graph, then
\[
C_{\text{Aut}(X)} = \langle U, M, S, d_X \rangle_{+, o, \otimes, *} = \langle U, M, S, T^X \rangle_{+, o, \otimes, *} := \langle U, M, S, T_1, \ldots, T_r \rangle_{+, o, \otimes, *}
\]
\[
C_{A(X)} = \langle U, M, d_X \rangle_{+, o, \otimes, *} = \langle U, M, C[d_X] \rangle_{+, o, \otimes, *}
\]

**Proof.** With proposition 4.10 we obtain
\[
\langle U, M, S, T^X \rangle_{+, o, \otimes, *} \subset C_{\text{Aut}(X)} = \langle U, M, S, d_X \rangle_{+, o, \otimes, *}
\]
using \( d_X \in T^X \) we obtain the inverse inclusion and the equality.

The second equality is the proposition 5.5.

**Definition 5.10.** A vertex-transitive graph \( X \) is called \( \mathcal{B} \)-clos if
\[
C_{A(X)}(1,1) = T^X
\]
which is equivalent to \( T^X \subset C_{A(X)} \).

The quantum symmetries of a graph \( \mathcal{B} \)-clos are easier to study because he satisfies:
\[
C_{A(X)} = \langle U, M, T^X \rangle_{+, o, \otimes, *}
\]
which mean \( A(X) \) only depend on its automorphism group, \( \text{Aut}(X) \), in the sense of this following theorem.
Theorem 5.11. Let $X$ and $Y$ be two graphs $B$-clos with the same number $n$ of vertices. Then the following three propositions are equivalent:

1. $\mathcal{T}^X = \mathcal{T}^Y$
2. $\text{Aut}(X) = \text{Aut}(Y)$
3. $A(X) = A(Y)$

Proof.

(1) $\Rightarrow$ (3):

$\mathcal{T}^X = \mathcal{T}^Y \Rightarrow \langle U, M, \mathcal{T}^X \rangle_{+,\otimes,*} = \langle U, M, \mathcal{T}^Y \rangle_{+,\otimes,*} \\
\Leftrightarrow C_{A(X)} = C_{A(Y)} \\
\Leftrightarrow A(X) = A(Y)$

(3) $\Rightarrow$ (2): By passing to the abelianised.

(2) $\Rightarrow$ (1): Since $\mathcal{T}^X$ is canonicaly define by $\text{Aut}(X)$. □

6. Applications to circulant $p$-graphs

In this section we study quantum symmetries of some circulant graphs, define as follow.

**Definition 6.1.** A graph with $n$ vertices is called circulant if its automorphism group contains a cycle of length $n$, and hence a copy of the cycle $\mathbb{Z}_n$. We also call $n$-circulant graph for a circulant graph with $n$ vertices.

If we denoted by $(i_1, i_2, \ldots, i_n)$ a cycle of length $n$ in a $n$-circulant graph then all pairs $(i_k, i_{k+1})$ for all $k \in [1, n]$ (with $i_{n+1} := i_1$) have the same nature (connected or not). So, modulo complementation, we just need to study quantum symmetries where this above pairs are connected.

As Banica and Bichon in [5] we will concentrate our study on vertex-transitive graphs. First we remark that for any prime $p$ all vertex-transitive graph with $p$ vertices are circulant. Then vertex-transitive circulant graphs are exactly those describe by this following definition.

**Definition 6.2.** We denoted by $C_n(k_1, \ldots, k_r)$, with $1 < k_1 < \cdots < k_r \leq \lfloor \frac{n}{2} \rfloor$, $k_i \in \mathbb{N}$, the graph obtained by drawing the $n$-cycle $C_n$, then connecting all pairs of vertices at distance $k_i$ on the circle, for all $i$.

Which mean that for all $i, j \in [0, n - 1]$:

$i \sim j \iff (i - j) \mod n \in \{\pm 1, \pm k_1, \ldots, \pm k_r\}$

**Example 6.3.** The complete graph with $n$ vertices, name $K_n$ is circulant and here are two other examples of circulant graphs:

i. $X = C_6(2) = (3K_2)^c$:
We also need more definition (from [4]) to study those circulant graphs.

For a \( n \)-circulant graph we suppose that their vertices are elements of \( \mathbb{Z}_n \). Then for every \( k, i, j \in \mathbb{Z}_n \): \( i \sim j \Rightarrow i + k \sim j + k \). We denote by \( \mathbb{Z}_n^* \) the group of invertible elements of the ring \( \mathbb{Z}_n \). The following definition comes from [4].

**Definition 6.4.** Let \( X \) be a circulant graph with \( n \) vertices:

- The set \( S \subset \mathbb{Z}_n \) is given by \( i \sim j \Leftrightarrow j - i \in S \).
- The group \( E \subset \mathbb{Z}_n^* \) consists of elements \( a \in \mathbb{Z}_n^* \) such that \( aS = S \).
- The order of \( E \) is denoted \( k \), and called type of \( X \).

We can check that the set \( S \) is \( \{\pm 1, \pm k_1, \ldots, \pm k_r\} \) from definition 6.2. In the general case the type of a circulant graph is always even because \(-1\) of order 2 is in \( E \).

**Remark 6.5.** By definition \( E \) is a subgroup of \( \mathbb{Z}_n^* \) so when \( n = p \) is prime \( p - 1 \) is divided by the order of \( E \) so there exists \( \lambda \) in \( \mathbb{N}^* \) such that:

\[
p = \lambda k + 1
\]

The study in [4] use the the notion of 2-maximal as follows.

**Definition 6.6.** [4, Definition 4.2] Let \((R, +, \cdot)\) a ring such that \( 2 \in R^* \) where \( R^* \) is the group of invertible element of \( R \). An even subgroup \( G \subset R^* \) is called 2-maximal if:

\[
a - b = 2(c - d)
\]

with \( a, b, c, d \in G \) implies \( a = \pm b \).

**Theorem 6.7.** [4, Theorem 4.4] If \( E \subset \mathbb{Z}_p \) is 2-maximal (with \( p \geq 5 \)) then \( X \) has no quantum symmetry.

**Theorem 6.8.** [4, Theorem 5.3] A type \( k \) circulant \( p \)-graphe with \( p > 6\varphi(k) \) has no quantum symmetry. Where \( \varphi \) is the Euler function.

It means that we just need to study circulant \( p \)-graph which satisfy \( p \leq 6\varphi(k) \).

For \( k \leq 12 \) we have:

| \( k \) | 4 | 6 | 8 | 10 | 12 |
|-------|---|---|---|----|----|
| \( 6\varphi(k) \) | 36 | 1296 | 1296 | 1296 | 1296 |
To study all this circulant graphs with $p \leq 6e(k)$ we look at the space $T^X$ for this graph. For the orbitals of this $p$-graphs we have those following results from [14].

**Proposition 6.9.** If $X$ is a non trivial circulant $p$-graph then for all $i \in [0, p - 1]$, we have the following isomorphism

$$\Phi_i : E \to \text{Aut}_i(X)$$

$$e \to (\sigma_e : i + k \mapsto i + ke)$$

It means that for all $s \in [1, r]$, the orbitals $O_s^0$ are of the form:

$$O_s^0 = y_s E$$

with a choice of $y_s$ such that:

$$1 = y_1 < y_2 < \cdots < y_r$$

Then we also obtain:

$$O_s^i = i + y_s E$$

**Proposition 6.10.** Every circulant $p$-graphs satisfy

$$C[d_X] = T^X$$

so every circulant $p$-graph is $B$-clos.

**Proof.** From proposition [4,10] we already know that $C[d_X] \subset T^X$. We prove the other inclusion by checking the dimensions of this two spaces. First, the dimension of $T^X$ is the number $"r + 1"$ of $\text{Auto}(X)$-orbit and we know from [6.4] that this orbital are $\{0\}$ and the $y_sE$ with $s \in [1, r]$. So, except the trivial one, this orbital are all of length $k = |E|$ so we have: $1 + r \times k = p$, which gives us:

$$\dim (T^X) = r + 1 = \frac{p - 1}{k} + 1$$

Moreover, the dimension of $C[d_X]$ is the number of eigen values of $d_X$ but, from [1], all the non trivial eigenspaces of $d_X$ are $k$-dimensional so we have:

$$\dim (C[d_X]) = \frac{p - 1}{k} + 1 = \dim (T^X)$$

which ends the proof. □

This result with theorem [5,11] gives us this following proposition.

**Proposition 6.11.** Let $X_1$ and $X_2$ be two circulant $p$-graphs, then

$$E_1 = E_2 \implies A(X_1) = A(X_2)$$

**Proof.** From [1] or [14] we know that $E$ characterizes $\text{Aut}(X)$ for $p$-graphs. So with theorem [5,11] we have:

$$E_1 = E_2 \implies T^{X_1} = T^{X_2} \implies A(X_1) = A(X_2)$$

This proposition is useful because it says that the study of quantum symmetries for $p$-graphs depend only on $E$. Then we can reduce our study to the case $S = E$ since circulant-graph satisfy $E \subset S$. For $p$ and $k$ fixed there exists at most one graph to study since $E$ is the unique subgroup of order $k$ in $(\mathbb{Z}_p^*, \times)$.

**Example 6.12.** There exist 15 graphs of type 6 for $p = 31$ which are all of the form $C_{31}(5, 6, U)$, where $U$ is any union of $(2, 10, 12)$, $(4, 7, 11)$, $(3, 13, 15)$ and $(8, 9, 14)$ (except union of the fourth). In all the cases, $E = \{\pm 1, \pm 5, \pm 6\}$ so we just need to study $C_{31}(5, 6)$.  

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We now give a new way to find if \( X \) has quantum symmetries, using the duality of the corollary \([5.9]\). We recall that the number of non trivial orbits of \([0, p - 1]\) under the action of \( \text{Aut}_0(X) \) is \( r \) with

\[
p = 1 + rk
\]

and \( O_s^i = i + ysE \) (with \( y_1 = 1 \)). We can see that \( r = \lambda \) from remark \([5.5]\).

**Proposition 6.13.** Let \( X \) be a circulant \( p \)-graph such that

for every \( s \) in \([1, r]\), there exist \( t_s^1, t_s^2 \) in \([1, r]\) such that \( |O_s^i \cap O_s^j| = 1 \)

Then \( X \) has no quantum symmetry.

To prove it we need the following lemma, characteristic of circulant \( p \)-graphs.

**Lemma 6.14.** Let \( i, j, k \in [0, p - 1] \) and \( s, s_i, s_j \in [1, r] \) such that

\[
j \in O_s^i \quad \text{and} \quad O_{s_i}^{s_i} \cap O_{s_j}^{s_j} = \{k\},
\]

then \( O_s^i \cap O_s^{s_j} = \{j\} \) and \( O_s^j \cap O_s^{s_i} = \{i\} \).

**Proof.** We know by symmetry of element in \( T^X \) that \( j \in O_s^i \Leftrightarrow i \in O_s^j \) hence, by symmetry, we just need to prove \( O_s^i \cap O_s^{s_j} = \{j\} \). 

\[
k \in O_s^{s_j} \Rightarrow j \in O_s^{s_i} \quad \text{so we have} \quad j \in O_s^{s_i} \cap O_s^{s_j}.
\]

We can now prove that \( O_s^{s_i} \cap O_s^{s_j} \) has no other element.

Assume \( j, j' \in O_s^{s_i} \cap O_s^{s_j} \). We have

\[
k \in O_s^{s_i} \cap O_s^{s_j} \cap O_s^{s_j}
\]

Moreover as \( j, j' \in O_s^{s_i} \) there exist \( e, e' \in E \) such that

\[
j = i + y_se \quad \text{and} \quad j' = i + y_se'
\]

We denote \( \sigma := \Phi_i(e' \sigma) \) where \( \Phi_i \) is the isomorphism of the proposition \([6.9]\). Hence we have \( \sigma(i) = i \) and \( \sigma(j') = i + y_se'e'^{-1} = j \). Using \((ii)\) of the proposition \([4.2]\) we obtain

\[
\{\sigma(k)\} \subset \sigma(O_s^{s_i} \cap O_s^{s_j} \cap O_s^{s_j}) \subset \sigma(O_s^{s_i} \cap O_s^{s_j}) = O_s^{s_i} \cap O_s^{s_j} = O_s^{s_i} \cap O_s^{s_j} = \{k\}
\]

which mean \( k = \sigma(k) \). \( k \in O_s^{s_i} \) hence there exists \( e'' \in E \) such that \( k = i + y_se'' \) and we obtain

\[
i + y_se''e'^{-1} = \sigma(k) = k = i + y_se''
\]

and as \( s_i \neq 0 \) it implies that \( e = e' \) hence \( j = j' \), which ends the proof. \( \square \)

Here is now the proof of the proposition \([6.13]\).

**Proof.** We put \( t_0^1 = t_0^2 = 0 \). We define the following element of \( C_A(X) \) for all \( s \in [0, r] \).

\[
h_s := (M \otimes M) \circ (id \otimes T_{t_0^1} \otimes T_{t_0^2} \otimes id) \in C_A(X)(4, 2)
\]

\[
f_s := h_s \circ (id \otimes (M^* \otimes M) \otimes id) \circ (T_s \otimes T_{t_0^1} \otimes T_{t_0^2} \otimes T_s) \circ (M^* \otimes M^*) \in C_A(X)(2, 2)
\]

\[
g_s := (id \otimes M) \circ (id \otimes T_{T_s} \otimes id) \circ (M^* \otimes id) \in C_A(X)(2, 2)
\]

\[
F := \sum_{s=0}^{\lambda} f_s \circ g_s \in C_A(X)(2, 2)
\]

First remark that \( f_0 = id_{(C^p)^{\otimes 2}} \).

Then for \( s \in [1, r] \), we denote \( k_s \) the element of \([0, p - 1]\) such that

\[
O_s^{t_0^1} \cap O_s^{t_0^2} = \{k_s\}
\]
Lemma 6.15. This two following propositions are equivalent:

- There exist \( s_1, s_2 \in [1, r] \) such that \( |\mathcal{O}^{s_1}_0 \cap \mathcal{O}^{s_2}_1| = 1 \)
- For all \( s \in [1, r] \), there exist \( t^1_s, t^2_s \in [1, \lambda] \) such that \( |\mathcal{O}^{t^1_s}_0 \cap \mathcal{O}^{t^2_s}_y| = 1 \)

**Proof.** It’s a direct consequence of proposition 4.2 and definition of orbitals \( \mathcal{O}^s \) because for all \( s \in [1, r] \), there exists \( \sigma_s \in \text{Aut}(X) \) such that \( \sigma_s(0) = 0 \) and \( \sigma_s(1) = y_s \). \( \square \)
Theorem 6.16. Let $X$ be a circulant $p$-graph such that

There exist $s_1, s_2 \in [1, r]$ such that $|O_0^{s_1} \cap O_1^{s_2}| = 1$

Then $X$ has no quantum symmetry.

Remark 6.17. Using explicit description of $O_i^x$ with the set $E$, the condition over $X$ in the theorem 6.16 is equivalent to

There exist $x, x' \in [1, p-1]$ such that $|(xE) \cap (x'E + 1)| = 1$

We denote $\Gamma_X = (\beta_{s_1,s_2})$ the matrix with entries defined by:

$$\beta_{s_1,s_2} = |O_0^{s_1} \cap O_1^{s_2}| = |(y_{s_1}E) \cap (y_{s_2}E + 1)|$$

First the matrix $\Gamma_X$ is symmetrical since there exists $\sigma \in Aut(X)$ such that $\sigma(0) = 1$ and $\sigma(1) = 0$. Then assumptions over $X$ in theorem 6.16 are equivalent to say that at least one entry of $\Gamma_X$ is equal to 1. Using this, we study quantum symmetries of $p$-graphs of types 4, 6, 8 and 10.

6.1. Graphs of type 4: By 6.8 we just need to study the case $p \leq 36$. There is 4 such graphs of type 4 (counting only the one satisfying $S = E$).

- $X = C_{29}(12)$ ($r = 7$):
  $$O_0^1 = E = \{1, 12, 17, 28\}, \quad O_1^2 = 2E + 1 = \{3, 6, 25, 28\}$$
  $$O_0^1 \cap O_1^2 = \{28\}$$
  so theorem 6.16 holds.

- $X = C_{17}(4)$ ($r = 4$):
  $$O_0^1 = E = \{1, 4, 13, 16\}, \quad O_1^2 = 2E + 1 = \{3, 9, 10, 16\}$$
  $$O_0^1 \cap O_1^2 = \{16\}$$
  so theorem 6.16 holds.

- $X = C_{13}(5)$ ($r = 3$):
  $$O_0^1 = E = \{1, 5, 8, 12\}, \quad O_1^2 = 2E + 1 = \{3, 4, 11, 12\}$$
  $$O_0^1 \cap O_1^2 = \{12\}$$
  so theorem 6.16 holds.

- $X = K_5$: the complete graph with 5 vertices has quantum symmetries.

6.2. Graphs of type 6: By 6.8 we just need to study the case $p \leq 36$. There is 3 such graphs of type 6 (counting only the one satisfying $S = E$).

- $X = C_{31}(1, 5, 6)$ ($r = 5$):
  $$O_0^1 = E = \{1, 5, 6, 25, 26, 30\}, \quad O_1^2 = 2E + 1 = \{3, 11, 13, 20, 22, 30\}$$
  $$O_0^1 \cap O_1^2 = \{30\}$$
  so theorem 6.16 holds.

- $X = C_{19}(1, 7, 8)$ ($r = 3$):
  $$O_0^1 = E = \{1, 7, 8, 11, 12, 18\}, \quad O_1^2 = 2E + 1 = \{3, 4, 6, 15, 17, 18\}$$
  $$O_0^1 \cap O_1^2 = \{18\}$$
  so theorem 6.16 holds.

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\[ X = C_{13}(3, 4) \ (r = 2): \] in this case we have
\[
\mathcal{O}_0^1 = E = \{1, 3, 4, 9, 10, 12\}, \quad \mathcal{O}_0^2 = 2E = \{2, 5, 6, 7, 8, 11\}
\]
\[
\mathcal{O}_1^1 = E + 1 = \{0, 2, 4, 5, 10, 11\}, \quad \mathcal{O}_1^2 = 2E + 1 = \{3, 6, 7, 8, 9, 12\}
\]
hence
\[
\Gamma_X = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}
\]
so theorem 6.16 does not hold.

- \( X = K_7 \): the complete graph with 7 vertices has quantum symmetries.

6.3. **Graphs of type 8**: By 6.8 we just need to study the case \( p \leq 1296 \). There is 48 such graphs of type 8 (counting only the one satisfying \( S = E \)). Here is the study of the five which have less than 100 vertices, with theorem 6.16.

- \( X = C_{97}(22, 33, 47) \ (r = 12): \)
\[
\mathcal{O}_0^1 = E = \{1, 22, 33, 47, 50, 64, 75, 96\}, \quad \mathcal{O}_0^2 = 2E = \{3, 4, 32, 45, 54, 67, 95, 96\}
\]
\[
\mathcal{O}_0^1 \cap \mathcal{O}_0^2 = \{96\}
\]
so theorem 6.16 holds.

- \( X = C_{89}(12, 34, 37) \ (r = 11): \)
\[
\mathcal{O}_0^1 = E = \{1, 12, 34, 37, 52, 55, 77, 88\}, \quad \mathcal{O}_0^2 = 2E = \{3, 16, 22, 25, 66, 69, 75, 88\}
\]
\[
\mathcal{O}_0^1 \cap \mathcal{O}_0^2 = \{88\}
\]
so theorem 6.16 holds.

- \( X = C_{73}(10, 22, 27) \ (r = 9): \)
\[
\mathcal{O}_0^1 = E = \{1, 10, 22, 27, 46, 51, 63, 72\}, \quad \mathcal{O}_0^2 = 2E = \{3, 20, 21, 30, 45, 54, 55, 72\}
\]
\[
\mathcal{O}_0^1 \cap \mathcal{O}_0^2 = \{72\}
\]
so theorem 6.16 holds.

- \( X = C_{41}(3, 9, 14) \ (r = 5): \)
\[
\mathcal{O}_0^2 = 2E = \{2, 6, 13, 18, 23, 28, 35, 39\}, \quad \mathcal{O}_0^3 = 8E = \{9, 11, 12, 18, 25, 31, 32, 34\}
\]
\[
\mathcal{O}_0^2 \cap \mathcal{O}_0^3 = \{18\}
\]
so theorem 6.16 holds.

- \( X = C_{17}(2, 4, 8) \ (r = 2): \) in this case we have
\[
\mathcal{O}_0^2 = E = \{1, 2, 4, 8, 9, 13, 15, 16\}, \quad \mathcal{O}_0^3 = 3E = \{3, 5, 6, 7, 10, 11, 12, 14\}
\]
\[
\mathcal{O}_1^1 = E + 1 = \{0, 2, 3, 5, 9, 10, 14, 16\}, \quad \mathcal{O}_1^2 = 3E + 1 = \{4, 6, 7, 8, 11, 12, 13, 15\}
\]
hence
\[
\Gamma_X = \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}
\]
so theorem 6.16 does not hold.

The other graphs of type 8 were compute the same way. For all of them theorem 6.16 holds.
6.4. **Graphs of type** 10: By theorem 6.16 we just need to study the case \( p \leq 1296 \). There is 51 such graphs of type 10 (counting only the one satisfying \( S = E \)). Here is the study of the five which have less than 100 vertices, with theorem 6.16.

- \( X = C_{71}(5, 14, 17, 25) \) (\( r = 7 \)):
  \[
  O_1^1 = E = \{1, 5, 14, 17, 25, 46, 54, 57, 66, 70\}
  O_1^2 = 2E + 1 = \{3, 11, 22, 29, 35, 38, 44, 51, 62, 70\}
  O_0^1 \cap O_1^2 = \{70\}
  \]
  so theorem 6.16 holds.

- \( X = C_{61}(3, 9, 20, 27, 34, 41, 52, 58, 60) \) (\( r = 6 \)):
  \[
  O_0^2 = 2E = \{2, 6, 7, 18, 21, 40, 43, 54, 55, 59\}
  O_1^4 = 4E + 1 = \{5, 13, 15, 20, 26, 37, 43, 48, 50, 58\}
  O_0^2 \cap O_1^4 = \{43\}
  \]
  so theorem 6.16 holds.

- \( X = C_{41}(4, 10, 16, 18) \) (\( r = 4 \)): we have
  \[
  \Gamma_X = \begin{pmatrix}
  0 & 3 & 2 & 4 \\
  3 & 3 & 2 & 2 \\
  2 & 2 & 4 & 2 \\
  4 & 2 & 2 & 2
  \end{pmatrix}
  \]

- \( X = C_{31}(2, 4, 8, 15) \) (\( r = 3 \)): we have
  \[
  \Gamma_X = \begin{pmatrix}
  3 & 4 & 2 \\
  4 & 2 & 4 \\
  2 & 4 & 4
  \end{pmatrix}
  \]

- \( X = K_{11} \): the complete graph with 11 vertices has quantum symmetries.

The other graphs of type 10 with \( p \geq 71 \) were compute the same way. For all of them, theorem 6.16 holds.

The two next section are devoted to the study of \( C_{13}(3, 4) \) and \( C_{17}(2, 4, 8) \). In section 7 and 8 we proves that \( C_{13}(3, 4) \) and \( C_{17}(2, 4, 8) \) have no quantum symmetry. Then all the \( p \)-graphs of type 4, 6 or 8 (except complete graphs) have no quantum symmetry. Is it the case for bigger type? It’s an open question.

7. **\( C_{13}(3, 4) \) has no quantum symmetry**

The graph \( C_{13}(3, 4) \) is \( B \)-clo by 6.10 but don’t satisfy assumptions of theorem 6.16. Indeed, here are its orbitals:

| \( i \) | \( O_1^1 \) | \( O_1^2 \) |
|---|---|---|
| 0 | \{1, 3, 4, 9, 10, 12\} | \{2, 5, 6, 7, 8, 11\} |
| 1 | \{2, 4, 5, 10, 11\} | \{3, 6, 7, 8, 9, 12\} |
| 2 | \{3, 5, 6, 11, 12\} | \{4, 7, 8, 9, 10\} |
| 3 | \{4, 6, 7, 12, 0, 2\} | \{5, 8, 9, 10, 11\} |
| 4 | \{5, 7, 8, 0, 1, 3\} | \{6, 9, 10, 11, 12\} |
| 5 | \{6, 8, 9, 1, 2, 4\} | \{7, 10, 11, 12, 0, 3\} |
| 6 | \{7, 9, 10, 2, 3, 5\} | \{8, 11, 12, 0, 1, 4\} |
| 7 | \{8, 10, 11, 3, 4, 6\} | \{9, 12, 0, 1, 2, 5\} |
| 8 | \{9, 11, 12, 4, 5, 7\} | \{10, 0, 1, 2, 3, 6\} |
| 9 | \{10, 12, 0, 5, 6, 8\} | \{11, 1, 2, 3, 4, 7\} |
| 10 | \{11, 0, 1, 6, 7, 9\} | \{12, 2, 3, 4, 5, 8\} |
| 11 | \{12, 1, 2, 7, 8, 10\} | \{0, 3, 4, 5, 6, 9\} |
| 12 | \{0, 2, 3, 8, 9, 11\} | \{1, 4, 5, 6, 7, 10\} |
We remark that elements of $O_i^1$ are the neighbors of $i$ and the elements of $O_i^2$ are the vertices non connected to $i$ in $C_{13}(3, 4)$. It means that for every two couples $(i, j)$ and $(k, l)$ of vertices of same nature (connected or not) there exists an automorphism $\sigma \in \text{Aut}(C_{13}(3, 4))$ such that:

$$\sigma(i) = k \quad \text{and} \quad \sigma(j) = l$$

**Definition 7.1.** We denote $D = M \circ (\text{id} \otimes M)$ the function of $C_{A(C_{13}(3, 4))}(3, 1)$.

Then for all $i, j, k \in [0, 12]$, we have:

$$D(e_i \otimes e_j \otimes e_k) = \delta_{i,j} \delta_{i,k} e_i$$

**Lemma 7.2.** There exists an intertwiner $G_1 \in C_{A(C_{13}(3, 4))}(2, 1)$ such that:

$$G_1(e_0 \otimes e_2) = e_1$$

**Proof.** We put $H_1 = M \circ (T_1 \otimes T_1)$ and $H_4 = M \circ (T_2 \otimes T_2)$, then:

$$H_1(e_0 \otimes e_2) = M \circ (T_1 \otimes T_1)(e_0 \otimes e_2)$$

$$= M ((e_1 + e_3 + e_4 + e_9 + e_{10} + e_{12}) \otimes (e_1 + e_3 + e_5 + e_6 + e_{11} + e_{12}))$$

$$= e_1 + e_3 + e_{12}$$

$$H_4(e_0 \otimes e_2) = M \circ (T_2 \otimes T_2)(e_0 \otimes e_2)$$

$$= M ((e_2 + e_5 + e_6 + e_7 + e_8 + e_{11}) \otimes (e_0 + e_4 + e_7 + e_8 + e_9 + e_{10}))$$

$$= e_7 + e_8$$

$$T_2(e_7 + e_8) = 2e_0 + 2e_1 + 2e_2 + e_3 + e_5 + e_6 + e_9 + e_{10} + e_{12}$$

We define the intertwiner $G \in C_{A(C_{13}(3, 4))}(2, 1)$ by:

$$G = D \circ (T_1 \otimes T_2 \otimes T_1) \circ (\text{id} \otimes H_4 \otimes \text{id}) \circ (M^* \otimes M^*)$$

So we get:

$$G(e_0 \otimes e_2) = D \circ (T_1 \otimes T_2 \otimes T_1) \circ (\text{id} \otimes H_4 \otimes \text{id}) \circ (M^* \otimes M^*)(e_0 \otimes e_2)$$

$$= D \circ (T_1 \otimes T_2 \otimes T_1)(e_0 \otimes (e_7 + e_8) \otimes e_2)$$

$$= D ((e_1 + e_3 + e_4 + e_9 + e_{10} + e_{12}) \otimes T_2(e_7 + e_8) \otimes (e_1 + e_3 + e_5 + e_6 + e_{11} + e_{12}))$$

$$= 2e_1 + e_3 + e_{12}$$

Finally with $G_1 = G - H_1 \in C_{A(C_{13}(3, 4))}(2, 1)$ we have:

$$G_1(e_0 \otimes e_2) = 2e_1 + e_3 + e_{12} - (e_1 + e_3 + e_{12}) = e_1$$

as required. \(\square\)

**Lemma 7.3.** For all $k \in [0, 12]$ there exists an intertwiner $F_k \in C_{A(C_{13}(3, 4))}(2, 1)$ such that:

$$F_k(e_0 \otimes e_1) = e_k$$

**Proof.**

First we define $F_0 = M \circ (\text{id} \otimes (U \circ U^*))$ and $F_1 = M \circ ((U \circ U^*) \otimes \text{id})$ to have:

$$F_0(e_0 \otimes e_1) = e_0, \quad \text{and} \quad F_1(e_0 \otimes e_1) = e_1$$

Then, we consider the following intertwiners of $C_{A(C_{13}(3, 4))}(2, 1)$:

$$H_1 = M \circ (T_1 \otimes T_1), \quad H_2 = M \circ (T_1 \otimes T_2), \quad H_3 = M \circ (T_2 \otimes T_1), \quad H_4 = M \circ (T_2 \otimes T_2)$$

and we obtain:

$$H_1(e_0 \otimes e_1) = e_4 + e_{10}, \quad H_2(e_0 \otimes e_1) = e_3 + e_9 + e_{12}$$

$$H_3(e_0 \otimes e_1) = e_2 + e_5 + e_{11}, \quad H_4(e_0 \otimes e_1) = e_6 + e_7 + e_8$$
We remark that: $T_1(e_4 + e_{10}) = 2e_0 + 2e_1 + 2e_7 + e_3 + e_5 + e_6 + e_8 + e_9 + e_{11}$, so we obtain:

$$
D \circ (T_2 \otimes T_1 \otimes T_2) \circ (\text{id} \otimes H_1 \otimes \text{id}) \circ (M^* \otimes M^*)(e_0 \otimes e_1)
$$

$$
= D \circ (T_2 \otimes T_1 \otimes T_2) (e_0 \otimes (e_4 + e_{10}) \otimes e_1)
$$

$$
= D ((e_2 + e_5 + e_6 + e_7 + e_8 + e_{11}) \otimes T_1(e_4 + e_{10}) \otimes (e_3 + e_6 + e_7 + e_8 + e_9 + e_{12}))
$$

$$
= 2e_7 + e_6 + e_8
$$

hence $\text{F}_7 = D \circ (T_2 \otimes T_1 \otimes T_2) \circ (\text{id} \otimes H_1 \otimes \text{id}) \circ (M^* \otimes M^*) - H_4$ works.

Then: $\text{F}_3 = D \circ (T_1 \otimes T_1 \otimes T_2) \circ (\text{id} \otimes F_7 \otimes \text{id}) \circ (M^* \otimes M^*)$ and $\text{F}_{11} = D \circ (T_2 \otimes T_1 \otimes T_1) \circ (\text{id} \otimes F_7 \otimes \text{id}) \circ (M^* \otimes M^*)$ work since:

$$
\text{F}_3(e_0 \otimes e_1) = D \circ (T_1 \otimes T_1 \otimes T_2) \circ (\text{id} \otimes F_7 \otimes \text{id})(e_0 \otimes e_0 \otimes e_0 \otimes e_1)
$$

$$
= D \circ (T_1 \otimes T_1 \otimes T_2)(e_0 \otimes e_7 \otimes e_1)
$$

$$
= D (T_1(e_0) \otimes (e_3 + e_4 + e_6 + e_7 + e_{10} + e_{11}) \otimes T_2(e_1)) = e_3
$$

and

$$
\text{F}_1(1,e_0 \otimes e_1) = D \circ (T_2 \otimes T_1 \otimes T_1) \circ (\text{id} \otimes F_7 \otimes \text{id})(e_0 \otimes e_0 \otimes e_0 \otimes e_1)
$$

$$
= D \circ (T_2 \otimes T_1 \otimes T_1)(e_0 \otimes e_7 \otimes e_1)
$$

$$
= D (T_2(e_0) \otimes (e_3 + e_4 + e_6 + e_7 + e_{10} + e_{11}) \otimes T_1(e_1)) = e_{11}
$$

Finally we use:

$$
T_2(e_4 + e_{10}) = 2e_2 + 2e_{12} + e_3 + e_4 + e_5 + e_6 + e_8 + e_9 + e_{10} + e_{11}
$$

$$
T_2(e_9) = e_1 + e_2 + e_3 + e_4 + e_7 + e_{11}
$$

$$
T_2(e_3) = e_1 + e_5 + e_8 + e_9 + e_{10} + e_{11}
$$

to find the other intertwinners we use:

$$
\text{F}_2 = D \circ (T_2 \otimes T_2 \otimes T_1) \circ (\text{id} \otimes H_1 \otimes \text{id}) \circ (M^* \otimes M^*) - H_3
$$

$$
\text{F}_{12} = D \circ (T_1 \otimes T_2 \otimes T_2) \circ (\text{id} \otimes H_1 \otimes \text{id}) \circ (M^* \otimes M^*) - H_2
$$

$$
\text{F}_5 = H_3 - F_2 - F_{11}
$$

$$
\text{F}_9 = H_2 - F_3 - F_{12}
$$

$$
\text{F}_4 = D \circ (T_1 \otimes T_2 \otimes T_1) \circ (\text{id} \otimes F_9 \otimes \text{id}) \circ (M^* \otimes M^*)
$$

$$
\text{F}_8 = D \circ (T_2 \otimes T_2 \otimes T_2) \circ (\text{id} \otimes F_3 \otimes \text{id}) \circ (M^* \otimes M^*)
$$

$$
\text{F}_{10} = H_1 - F_4
$$

$$
\text{F}_6 = H_4 - F_7 - F_8.
$$

These functions well satisfy $F_k(e_0 \otimes e_1) = e_k$ as required

\[ \square \]

**Theorem 7.4.** The graph $C_{13}(3,4)$ has no quantum symmetry.

**Proof.** We first remark a property about the functions $F_k$. For any connected vertices $i$ and $j$, as 0 and 1 are connected there exists $\sigma \in \text{Aut}(C_{13}(3,4))$ such that: $\sigma(0) = i$ and $\sigma(1) = j$. Hence, for all $k \in [0,12]$:

$$
F_k(e_i \otimes e_j) = F_k(e_{\sigma(0)} \otimes e_{\sigma(1)}) = P_\sigma \circ F_k(e_0 \otimes e_1) = e_{\sigma(k)} \quad (*)
$$

Then we need to built new intertwinners:

- The vertex 4 is a neighbor of 0 and 1 so we can consider $\mu, \mu' \in \text{Aut}(C_{13}(3,4))$ such that:

  $$
  \mu(0) = 0, \quad \mu(1) = 4, \quad \mu'(0) = 4, \quad \mu'(1) = 1
  $$

  and we denote by $x,y \in [0,12]$ the vertices such that $\mu(x) = 1$ and $\mu'(y) = 0$ to define:

$$
F = (F_x \otimes F_y) \circ (\text{id} \otimes (M^* \circ F_4 \otimes \text{id}) \circ (M^* \otimes M^*)
$$
So by (*) we have:

\[
F(e_0 \otimes e_1) = (F_x \otimes F_y)(\text{id} \otimes (M^* \circ F_4) \otimes \text{id})(e_0 \otimes e_0 \otimes e_1 \otimes e_1)
\]

\[
= (F_x \otimes F_y)(e_0 \otimes e_4 \otimes e_4 \otimes e_1) = F_x(e_0 \otimes e_4) \otimes F_y(e_4 \otimes e_1)
\]

\[
= e_{\mu(x)} \otimes e_{\mu'(y)} = e_1 \otimes e_0
\]

Then if \(i\) and \(j\) are connected, with \(\sigma\) such that \(\sigma(0) = i\) and \(\sigma(1) = j\) we have:

\[
F(e_i \otimes e_j) = F(e_{\sigma(0)} \otimes e_{\sigma(1)}) = P_2^{\otimes 2} \circ F(e_0 \otimes e_1) = e_{\sigma(1)} \otimes e_{\sigma(0)} = e_j \otimes e_i
\]

- The vertex 1 is a neighbor of 0 and 2 so we can consider \(\tau \in \text{Aut}(C_{13}(3, 4))\) such that:

\[
\tau(0) = 1 \quad \text{and} \quad \tau(1) = 2
\]

et we denote by \(z \in [0, 12]\) the integer such that \(\tau(z) = 0\) to define:

\[
G = (F_2 \otimes F_z) \circ (\text{id} \otimes (M^* \circ G_1) \otimes \text{id}) \circ (M^* \circ M^*)
\]

where \(G_1\) is the function of the lemma \(7.2\). Then by (*), we have:

\[
G(e_0 \otimes e_2) = (F_2 \otimes F_z)(\text{id} \otimes (M^* \circ G_1) \otimes \text{id})(e_0 \otimes e_0 \otimes e_2 \otimes e_2)
\]

\[
= (F_2 \otimes F_z)(e_0 \otimes e_1 \otimes e_1 \otimes e_2) = F_2(e_0 \otimes e_1) \otimes F_z(e_1 \otimes e_2)
\]

\[
= e_2 \otimes e_{\tau(z)} = e_2 \otimes e_0
\]

If \(i\) and \(j\) are not connected, with \(\sigma\) such that \(\sigma(0) = i\) and \(\sigma(2) = j\) we have:

\[
G(e_i \otimes e_j) = G(e_{\sigma(0)} \otimes e_{\sigma(2)}) = P_2^{\otimes 2} \circ G(e_0 \otimes e_2) = e_{\sigma(2)} \otimes e_{\sigma(0)} = e_j \otimes e_i
\]

- For \(s \in \{0, 1, 2\}\) we define:

\[
g_s := (\text{id} \otimes M) \circ (\text{id} \otimes T_s \otimes \text{id}) \circ (M^* \otimes \text{id}) \in C_{A(C_{13}(3, 4))(2, 2)}
\]

This intertwiner are easy because:

\[
g_0(e_i \otimes e_j) = 1_{i=j}(e_i \otimes e_i), \quad g_1(e_i \otimes e_j) = 1_{i\sim j}(e_i \otimes e_j), \quad g_0(e_i \otimes e_j) = 1_{i\not\sim j}(e_i \otimes e_j)
\]

- To finish the proof we use the last intertwiner:

\[
H := g_0 + F \circ g_1 + G \circ g_2 \in C_{A(C_{13}(3, 4))}(2, 2)
\]

We have:

\[
H(e_i \otimes e_j) = \begin{cases} 
  e_i \otimes e_i & \text{if } i = j \\
  F(e_i \otimes e_j) & \text{if } i \sim j \\
  G(e_i \otimes e_j) & \text{if } i \not\sim j 
\end{cases} = e_j \otimes e_i
\]

hence \(S = H \in C_{A(C_{13}(3, 4))}\) and the graph \(C_{13}(3, 4)\) has no quantum symmetry by corollary \(5.6\).

\[\square\]

This result gives us also the complete study of quantum symmetries of vertex-transitive graphs of order 13:

| \(X\)     | \(\text{Aut}(X)\) | \(G_X\) |
|----------|-------------------|--------|
| \(K_{13}\) | \(S_{13}\)       | \(S_{13}^3\) |
| \(C_{13}\) | \(D_{13}\)       | \(D_{13}\) |
| \(C_{13}(2)\) | \(D_{13}\)       | \(D_{13}\) |
| \(C_{13}(2, 5)\) | \(D_{13}\)       | \(D_{13}\) |
| \(C_{13}(2, 6)\) | \(D_{13}\)       | \(D_{13}\) |
| \(C_{13}(3)\) | \(D_{13}\)       | \(D_{13}\) |
| \(C_{13}(5)\) | \(G_1 := Z_{13} \times Z_4\) | \(G_1\) |
| \(C_{13}(3, 4)\) | \(G_2 := Z_{13} \times Z_6\) | \(G_2\) |

To complete the work of \(5\) we should also study the graphs of order 12.
8. $C_{17}(2, 4, 8)$ has no quantum symmetry

The graph $C_{17}(2, 4, 8)$ is $\mathcal{B}$-clos (by 6.10) but don’t satisfy assumptions of theorem 6.16

Indeed, here are its orbitals:

| $i$ | $\mathcal{O}_1^i$ | $\mathcal{O}_2^i$ |
|-----|------------------|------------------|
| 0   | $\{1, 2, 4, 8, 9, 13, 15, 16\}$ | $\{3, 5, 6, 7, 10, 11, 12, 14\}$ |
| 1   | $\{2, 3, 5, 9, 10, 14, 16, 0\}$ | $\{4, 6, 7, 8, 11, 12, 13, 15\}$ |
| 2   | $\{3, 4, 6, 10, 11, 15, 0, 1\}$ | $\{5, 7, 8, 9, 12, 13, 14, 16\}$ |
| 3   | $\{4, 6, 7, 12, 0, 2\}$ | $\{5, 8, 9, 10, 11, 1\}$ |
| 4   | $\{5, 7, 8, 0, 1, 3\}$ | $\{6, 9, 10, 11, 12, 2\}$ |
| 5   | $\{6, 8, 9, 1, 2, 4\}$ | $\{7, 10, 11, 12, 0, 3\}$ |
| 6   | $\{7, 9, 10, 2, 3, 5\}$ | $\{8, 11, 12, 0, 1, 4\}$ |
| 7   | $\{8, 10, 11, 3, 4, 6\}$ | $\{9, 12, 0, 1, 2, 5\}$ |
| 8   | $\{9, 11, 12, 4, 5, 7\}$ | $\{10, 0, 1, 2, 3, 6\}$ |
| 9   | $\{10, 12, 0, 5, 6, 8\}$ | $\{11, 1, 2, 3, 4, 7\}$ |
| 10  | $\{11, 0, 1, 6, 7, 9\}$ | $\{12, 2, 3, 4, 5, 8\}$ |
| 11  | $\{12, 1, 2, 7, 8, 10\}$ | $\{0, 3, 4, 5, 6, 9\}$ |
| 12  | $\{0, 2, 3, 8, 9, 11\}$ | $\{1, 4, 5, 6, 7, 10\}$ |

As in $C_{13}(3, 4)$, the elements of $\mathcal{O}_1^i$ are the neighbors of $i$ and the elements of $\mathcal{O}_2^i$ are the vertices non connected to $i$ in $C_{13}(3, 4)$. For every two couples $(i, j)$ and $(k, l)$ of vertices of same nature (connected or not) there exists an automorphism $\sigma \in \text{Aut}(C_{17}(2, 4, 8))$ such that:

$$\sigma(i) = k \quad \text{and} \quad \sigma(j) = l$$

**Definition 8.1.** We denote $D = M \circ (\text{id} \otimes M)$ the function of $C_{A(C_{17}(2, 4, 8))}(3, 1)$.

Then for all $i, j, k \in [0, 16]$, we have:

$$D(e_i \otimes e_j \otimes e_k) = \delta_{i,j}\delta_{k,i}e_i$$

**Lemma 8.2.** For all $k \in \mathcal{O}_0^1 \cap \mathcal{O}_1^1 = \{2, 9, 16\}$, there exists an intertwiner $F_k \in C_{A(C_{17}(2, 4, 8))}(2, 1)$ such that:

$$F_k(e_0 \otimes e_1) = e_k$$

**Proof.** For all $i, j \in \{0, 1\}$, we define: $H_{i,j} = M \circ (T_i \otimes T_j) \in C_{A(C_{17}(2, 4, 8))}(2, 1)$ then we have:

$$H_{1,1}(e_0 \otimes e_1) = e_2 + e_9 + e_{16}, \quad H_{1,2}(e_0 \otimes e_1) = e_4 + e_8 + e_{13} + e_{15}$$

$$H_{2,1}(e_0 \otimes e_1) = e_3 + e_5 + e_{10} + e_{14}, \quad H_{2,2}(e_0 \otimes e_1) = e_6 + e_7 + e_{11} + e_{12}$$

Then for $H = H_{1,1}$, we have:

$$H(e_0 \otimes e_1) = 3(e_0 + e_1) + 2(e_8 + e_{15} + e_3 + e_{10} + e_7 + e_{11}) + (e_4 + e_{13} + e_5 + e_{14} + e_6 + e_{12})$$

Now we can see that if $S \in C_{A(C_{17}(2, 4, 8))}(2, 1)$ then $((T_i \otimes S \otimes T_j) \circ (M^* \otimes M^*)) (e_0 \otimes e_1)$ gives us only the ”$e_k$” of $S(e_0 \otimes e_1)$ that are also in $H_{i,j}(e_0 \otimes e_1)$. For example:

$$G_{2} := D \circ (T_2 \otimes H \otimes T_1) \circ (M^* \circ M^*) : e_0 \otimes e_1 \mapsto 2(e_3 + e_{10}) + e_5 + e_{14}$$

$$G_{9} := D \circ (T_2 \otimes H \otimes T_2) \circ (M^* \circ M^*) : e_0 \otimes e_1 \mapsto 2(e_7 + e_{11}) + e_6 + e_{12}$$

$$G_{16} := D \circ (T_1 \otimes H \otimes T_2) \circ (M^* \circ M^*) : e_0 \otimes e_1 \mapsto 2(e_8 + e_{15}) + e_4 + e_{13}$$

Then we define:

$$S_2 = T_1 \circ (G_2 - H_{2,1}), \quad S_9 = T_1 \circ (G_9 - H_{2,2}), \quad S_{16} = T_1 \circ (G_{16} - H_{1,2})$$

and we have:

$$S_2(e_0 \otimes e_1) = T_1(e_3 + e_{10}) = 2e_2 + e_9 + e_{16} + \ldots \text{"other } e_k\ldots$$

$$S_9(e_0 \otimes e_1) = T_1(e_3 + e_{10}) = e_2 + 2e_9 + e_{16} + \ldots \text{"other } e_k\ldots$$

$$S_{16}(e_0 \otimes e_1) = T_1(e_3 + e_{10}) = e_2 + e_9 + 2e_{16} + \ldots \text{"other } e_k\ldots$$
To finish the proof we define this following intertwiners of $C_{A(C_4(2,4,8))}(2,1)$:

\[
F_2 = D \circ (T_1 \otimes S_2 \otimes T_1) \circ (M^* \circ M^*) - H_{1,1}
F_9 = D \circ (T_1 \otimes S_0 \otimes T_1) \circ (M^* \circ M^*) - H_{1,1}
F_2 = D \circ (T_1 \otimes S_{16} \otimes T_1) \circ (M^* \circ M^*) - H_{1,1}
\]

to obtain:

\[
F_2(e_0 \otimes e_1) = (2e_2 + e_9 + e_{16}) - (e_2 + e_9 + e_{16}) = e_2
F_9(e_0 \otimes e_1) = (e_2 + 2e_9 + e_{16}) - (e_2 + e_9 + e_{16}) = e_9
F_{16}(e_0 \otimes e_1) = (e_2 + e_9 + 2e_{16}) - (e_2 + e_9 + e_{16}) = e_{16}
\]

□

**Theorem 8.3.** The graph $C_{17}(2,4,8)$ has no quantum symmetry.

**Proof.** As for $C_{13}(3,4)$ we check that for any connected vertices $i$ and $j$, as 0 and 1 are connected there exists $\sigma \in \text{Aut}(C_{17}(2,4,8))$ such that: $\sigma(0) = i$ and $\sigma(1) = j$. Hence, for all $k \in \{2, 9, 16\}$:

\[
F_k(e_i \otimes e_j) = F_k(e_{\sigma(0)} \otimes e_{\sigma(1)}) = P_{\sigma} \circ F_k(e_0 \otimes e_1) = e_{\sigma(k)} \quad (\ast)
\]

We recall that $\{2,9,10\} = O_0^1 \cap O_1^1$. so

\[
\sigma(\{2,9,10\}) = O_{\sigma(0)}^1 \cap O_{\sigma(1)}^1 = O_0^1 \cap O_1^1
\]

by proposition 1.2.

The vertices 0 and 2 are connected and 1 $\in O_0^1 \cap O_1^1$ so there exists $x \in \{2,9,16\}$ such that $F_x(e_0 \otimes e_2) = e_1$. The same way, as 1 and 2 are connected and $0 \in O_1^1 \cap O_2^1$, there exists $y \in \{2,9,16\}$ such that $F_y(e_2 \otimes e_1) = e_0$

Then we define:

\[
L = (F_x \otimes F_y) \circ (id \otimes (M^* \circ F_2) \otimes id) \circ (M^* \circ M^*) \in C_{A(C_4(2,4,8))}(2,2)
\]

So we have:

\[
L(e_0 \otimes e_1) = (F_x \otimes F_y) \circ (id \otimes (M^* \circ F_2) \otimes id)(e_0 \otimes e_0 \otimes e_1 \otimes e_1) = (F_x \otimes F_y)(e_0 \otimes e_2 \otimes e_2 \otimes e_1) = F_x(e_0 \otimes e_2) \otimes F_y(e_2 \otimes e_1) = e_1 \otimes e_0
\]

Then if $i$ and $j$ are connected, with $\sigma$ such that $\sigma(0) = i$ and $\sigma(1) = j$ we have:

\[
L(e_i \otimes e_j) = L(e_{\sigma(0)} \otimes e_{\sigma(1)}) = P_{\sigma}^2 \circ L(e_0 \otimes e_1) = e_{\sigma(1)} \otimes e_{\sigma(0)} = e_j \otimes e_i
\]

As for $C_{13}(3,4)$, for $s \in \{0,1,2\}$ we define:

\[
g_s := (id \otimes M) \circ (id \otimes T_s \otimes id) \circ (M^* \otimes id) \in C_{A(C_4(2,4,8))}(2,2)
\]

This $g_s$ are as follows:

\[
g_0(e_i \otimes e_j) = 1_{i=j}(e_i \otimes e_i), \quad g_1(e_i \otimes e_j) = 1_{i=j-1}(e_i \otimes e_j), \quad g_0(e_i \otimes e_j) = 1_{i\neq j}(e_i \otimes e_j)
\]

Then for $L' = L \circ g_1$ we have:

\[
L'(e_i \otimes e_j) = \begin{cases}
L(e_i \otimes e_j) & \text{if } i \sim j \\
0 & \text{if } i \not\sim j
\end{cases}
= \begin{cases}
e_j \otimes e_i & \text{if } i \sim j \\
0 & \text{if } i \not\sim j
\end{cases}
\]

We can now use the fact that $C_{17}(2,4,8)$ is self-adjoint, so as $C_{A((C_4(2,4,8))^\circ)}(2,2) = C_{A(C_4(2,4,8))}(2,2)$ there exists $L'' \in C_{A(C_4(2,4,8))}(2,2)$ such that:

\[
L''(e_i \otimes e_j) = \begin{cases}
e_j \otimes e_i & \text{if } i \not\sim j \text{ and } i \neq j \\
0 & \text{if } i \sim j \text{ or } i = j
\end{cases}
\]

To finish the proof we use the last intertwiner:

\[
G := g_0 + L' + L'' \in C_{A(C_4(2,4,8))}(2,2)
\]
We have $G(e_i \otimes e_j) = e_j \otimes e_i$, hence $S = G \in C_{A(C_{17}(2,4,8))}$ and the graph $C_{17}(2,4,8)$ has no quantum symmetry by corollary \[5.6\]

It should be interested to study the same way $C_{41}(4,10,16,18)$ and $C_{31}(2,4,8,15)$ to check if all the circulant $p$-graphs of order 10 have no quantum symmetry.

9. Applications in the general case

Theorem 9.1. Let $X$ be a $B$-clos graph, such that

For every $s \in [1, r]$, there exist $j_s \in O^s_0$, $k_s \in [0, n - 1]$ and $t^1_s, t^2_s, t^3_s, t^4_s, t^5_s \in [0, r]$ such that

$$O^t_{0} \cap O^t_{j_s} = \{k_s\}, \quad O^t_{0} \cap O^t_{k_s} = \{j_s\}, \quad O^t_{k_s} \cap O^t_{j_s} = \{0\}$$

Then $X$ has no quantum symmetry.

Remark 9.2. It is equivalent to replace the assumption ”there exists $j_s \in O^s_0$” by ”for every $j_s \in O^s_0$” because if it holds for one $j_s$, then, applying elements of $Aut_0(X)$ to the third wishes equality we get the same property for every elements of the orbit of $j_s$ under action of $Aut_0(X)$, which is $O^s_0$.

Proof. It’s nearly the same demonstration of proposition \[6.13\] with some quick changes. We put $t^1_0 = t^2_0 = t^3_0 = t^4_0 = t^5_0 = 0$. Then we define the following intertwiners of $C_{A(X)}$ for every $s \in [0, r]$.

\[h_s := (M \otimes M) \circ (id \otimes T_{t^2_s} \otimes T_{t^3_s} \otimes id) \in C_{A(X)}(4, 2)\]

\[f_s := h_s \circ (id \otimes (M^* \circ M) \otimes id) \circ (T_{s} \otimes T_{t^4_s} \otimes T_{t^5_s}) \circ (M^* \otimes M^*) \in C_{A(X)}(2, 2)\]

\[g_s := (id \otimes M) \circ (id \otimes T_{s} \otimes id) \circ (M^* \otimes id) \in C_{A(X)}(2, 2)\]

\[F := \sum_{s=0}^{r} f_s \circ g_s \in C_{A(X)}(2, 2)\]

We check that $f_0 = id_{(\mathbb{C}^n)^{\otimes 2}}$, then for $s \in [1, r]$, we have

$$O^t_{0} \cap O^t_{j_s} = \{k_s\}$$

We now consider two vertices $i$ and $j$ such that $j \in O^s_i$. Let $\sigma \in Aut(X)$ such that $\sigma(0) = i$. We know that $j_s \in O^s_0$ hence $\sigma(j_s) \in O^s_i$. So there exists $\sigma' \in Aut_i(X)$ such that $\sigma'(\sigma(j_s)) = j$. Hence, using $(iii)$ of proposition \[4.2\] we have:

\[\{\sigma'(\sigma(k_s))\} = \sigma'(O^t_0 \cap O^t_{j_s}) = O^t_{\sigma'(\sigma(0))} \cap O^t_{\sigma'(\sigma(j_s))} = O^t_i \cap O^t_j \]  

(1.1)

Then we also obtain

\[\{i\} = \sigma'(0) = \sigma'(O^t_{k_s} \cap O^t_{j_s}) = O^t_{\sigma'(\sigma(k_s))} \cap O^t_{j_s} \]  

(1.2)

\[\{j\} = \sigma'(j_s) = \sigma'(O^t_0 \cap O^t_{k_s}) = O^t_i \cap O^t_{\sigma'(\sigma(k_s))} \]  

(1.2)
Which allow us to do this following computation for every \( i, j \in [0, p - 1] \) such that \( j \in \mathcal{O}_i^p 
less \\
\\
\begin{align*}
  f_s(e_i \otimes e_j) &= h_s \circ (\text{id} \otimes (M^* \circ M) \otimes \text{id}) \left( T_s(e_i) \otimes T_{t_1}^s(e_i) \otimes T_{t_2}^s(e_j) \otimes T_{t_3}^s(e_j) \right) \\
  &= h_s \left( T_s(e_i) \otimes \sum_{k \in \mathcal{O}_i^1 \cap \mathcal{O}_j^2} (e_k \otimes e_k) \otimes T_{t_3}^s(e_j) \right) \\
  &= h_s \left( T_s(e_i) \otimes e_{\sigma'(k_1)} \otimes e_{\sigma'(k_2)} \otimes T_{t_3}^s(e_j) \right) \\
  &= (M \otimes M) \left( T_s(e_i) \otimes T_{t_1}^s(e_{\sigma'(k_1)}) \otimes T_{t_2}^s(e_{\sigma'(k_2)}) \otimes T_{t_3}^s(e_j) \right) \\
  &= \left( \sum_{a \in \mathcal{O}_i^1 \cap \mathcal{O}_j^2} e_a \right) \otimes \left( \sum_{b \in \mathcal{O}_j^2 \cap \mathcal{O}_j^2} e_b \right) \\
  &= e_j \otimes e_i 
\end{align*}
\\

The end of the proof is exactly the same of proposition \([5.13]\) we have \( S = F \in C_A(X) \). \Box

Application to the graph \( \text{Pr}(C_6) \):

Here is a particular application to the theorem \([9.1]\) in a case non treated with method of Banica and Bichon in \([5]\): \( X = \text{Pr}(C_6) = K_2 \sqcup C_6 \).

![Graph Pr(C_6)](image)

First remark that \( \text{Pr}(C_6) \) is \( \mathcal{B} \)-clos. Orbitals are as follow:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
i & \mathcal{O}_i^1 & \mathcal{O}_i^2 & \mathcal{O}_i^3 & \mathcal{O}_i^4 & \mathcal{O}_i^5 & \mathcal{O}_i^6 & \mathcal{O}_i^7 \\
\hline
0 & \{1\} & \{2, 10\} & \{3, 11\} & \{5, 9\} & \{4, 8\} & \{6\} & \{7\} \\
1 & \{0\} & \{3, 11\} & \{2, 10\} & \{4, 8\} & \{5, 9\} & \{7\} & \{6\} \\
2 & \{3\} & \{0, 4\} & \{1, 5\} & \{7, 11\} & \{6, 10\} & \{8\} & \{9\} \\
3 & \{2\} & \{1, 5\} & \{0, 4\} & \{6, 10\} & \{7, 11\} & \{9\} & \{8\} \\
4 & \{5\} & \{2, 6\} & \{3, 7\} & \{1, 9\} & \{0, 8\} & \{10\} & \{11\} \\
5 & \{4\} & \{3, 7\} & \{2, 6\} & \{0, 8\} & \{1, 9\} & \{10\} & \{11\} \\
6 & \{7\} & \{4, 8\} & \{5, 9\} & \{3, 11\} & \{2, 10\} & \{0\} & \{1\} \\
7 & \{6\} & \{5, 9\} & \{4, 8\} & \{2, 10\} & \{3, 11\} & \{1\} & \{0\} \\
8 & \{9\} & \{6, 10\} & \{7, 11\} & \{1, 5\} & \{0, 4\} & \{2\} & \{3\} \\
9 & \{8\} & \{7, 11\} & \{6, 10\} & \{0, 4\} & \{1, 5\} & \{3\} & \{2\} \\
10 & \{11\} & \{0, 8\} & \{1, 9\} & \{3, 7\} & \{2, 6\} & \{4\} & \{5\} \\
11 & \{10\} & \{1, 9\} & \{0, 8\} & \{2, 6\} & \{3, 7\} & \{5\} & \{4\} \\
\hline
\end{array}
\]
The following board show, for every value of $s \in [1,7]$, the existence of the $t^s_i$ such that assumptions of theorem 9.1 be satisfied.

| $s$ | $j_s$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{k_s\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{j_s\}$ | $O^{\ell_2}_{k_s} \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_{k_s} \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
|-----|------|---------------------------------|----------------------------------|---------------------------------|----------------------------------|
| 1   | {1}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{1\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{1\}$ | $O^{\ell_2}_t \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_t \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 2   | {2}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{10\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{2\}$ | $O^{\ell_2}_5 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_5 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 3   | {3}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{11\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{3\}$ | $O^{\ell_2}_3 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_3 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 4   | {5}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{9\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{5\}$ | $O^{\ell_2}_5 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_5 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 5   | {4}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{8\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{4\}$ | $O^{\ell_2}_4 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_4 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 6   | {6}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{6\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{6\}$ | $O^{\ell_2}_6 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_6 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |
| 7   | {7}  | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{7\}$ | $O^{\ell_2}_0 \cap O^{\ell_2}_{t^s_i} = \{7\}$ | $O^{\ell_2}_7 \cap O^{\ell_2}_{t^s_i} = \{0\}$ | $O^{\ell_2}_7 \cap O^{\ell_2}_{t^s_i} = \{0\}$ |

It does proves that $Pr(C_6)$ has no quantum symmetry by theorem 9.1.

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