Waveguide modal analysis of the ultrasonic medical instrument

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Abstract. Numerical procedure for analysis of eigenfrequencies and eigenfunctions of ultrasonic medical instrument waveguide is presented. A curved beam of variable cross-section is considered as a design scheme for the waveguide with respect to different boundary conditions. Test analysis for clamped-free curved waveguide is developed with MathCAD complex.

1. Introduction
In modern surgery, non-invasive technologies based on point penetration to the focus of the disease with the help of special tools play an increasing role [1, 2, 3]. Ultrasonic waveguides are used in a variety of medical applications. In surgical operations, it is efficient tools to perform various procedures. Ultrasonic surgical instruments have gained widespread acceptance and use in surgical procedures for tissue dissection, fragmentation and ablation applications and offer promise in interventional cardiology procedures. All such ultrasonic surgical systems use an ultrasonic frequency vibrating metal probe to achieve a desired effect. These device is capable of stopping bleeding because of the physical and chemical effects of ultrasound.

Surgical operations are performed with the help of various tools-nozzles at the ends of manipulators. When using ultrasonic medical instruments (UMI) an important role is played by the design of the waveguide, which provides the calculated mode of movement of the working edge (figure 1).

Figure 1. UMI curved waveguide.
The waveguide shape depends on the purpose of a particular UMI. This requires at the design stage to calculate the dynamic characteristics of waveguides of different geometries [10, 11, 12]. This work is devoted to the numerical determination of natural frequencies and oscillation forms of a curved waveguide with a variable cross-section. The obtained eigenforms will be used further to analyze the forced oscillations of the working edge of the UMI [4, 8].

2. Equations of small oscillations of a plane UMI curved waveguide in its own plane

To determine the eigenfrequencies of a UMI curved waveguide, let’s consider the small oscillations of a curved rod element (figure 2) with respect to its equilibrium state [13, 14].

Figure 2. Curved rod element.

Vector equations of curved rod small oscillations with respect to its equilibrium state are [5, 6]:

\[
\begin{align*}
\frac{m_i}{\partial t} &= \frac{\partial \Delta \mathbf{Q}}{\partial S} + \left( \Delta \Omega \ast \Delta \mathbf{Q} \right) + \left( \Delta \mathbf{M} \ast \Delta \mathbf{Q} \right) + \Delta \mathbf{q}; \\
\frac{I_0}{\partial t} \frac{\partial \Delta \omega}{\partial t} &= \frac{\partial \Delta \mathbf{M}}{\partial S} + \left( \Delta \Omega \ast \Delta \mathbf{M} \right) + \left( \Delta \mathbf{M} \ast \Delta \mathbf{M} \right) + \left( \Delta \mathbf{Q} \ast \Delta \mathbf{M} \right) + \Delta \mathbf{\mu}; \\
\frac{\partial \Delta \mathbf{V}}{\partial S} &= \left[ \Delta \Omega \ast \Delta \mathbf{V} \right] = \left[ \Delta \mathbf{\omega} \ast \mathbf{e}_1 \right]; \\
\frac{\partial \Delta \mathbf{\omega}}{\partial S} &= \frac{\partial \Delta \mathbf{M}}{\partial t} = \left[ \Delta \mathbf{\omega} \ast \Delta \mathbf{Q} \right]; \\
\Delta \mathbf{M} &= A \Delta \mathbf{\Omega};
\end{align*}
\]

Where

\( \Delta \mathbf{V} = \{V_1, V_2, V_3\}^T \) - Linear velocity of the waveguide element;

\( \Delta \mathbf{\omega} = \{\omega_1, \omega_2, \omega_3\}^T \) - Angular velocity of the waveguide element;

\( \Delta \mathbf{Q} = \{Q_1, Q_2, Q_3\}^T \) - Internal forces in the cross section of the waveguide;
\( \Delta \bar{M} = \{M_1, M_2, M_3\}^T \) - Internal bending and torsional moments in the cross section of the waveguide;

\( \bar{\Omega} = \frac{e_1}{\rho_1} + \frac{e_3}{\rho_3} = \chi_1 \bar{v}_1 + \chi_3 \bar{v}_3 \) - Darboux vector;

\( \bar{e}_1, \bar{e}_2, \bar{e}_3 \) - Natural coordinates unit vectors.

In projections on natural axes, the system of vector equation (1), in which the transition from linear velocity \( \Delta \bar{V} \) to linear displacements \( \bar{u} \), as well as from angular velocities \( \Delta \bar{\omega} \) to rotation angles \( \bar{\vartheta} \) is performed, becomes as follows [4, 5, 7]:

\[
\begin{align*}
\frac{\partial^2 U_1}{\partial t^2} &= \frac{\partial \Delta Q_1}{\partial S} + \chi_{20} \Delta Q_2 - \chi_{30} \Delta Q_3 + \Delta \chi_2 Q_{30} - \Delta \chi_3 Q_{20} + \Delta q_1; \\
\frac{\partial^2 U_2}{\partial t^2} &= \frac{\partial \Delta Q_2}{\partial S} + \chi_{30} \Delta Q_1 - \chi_{10} \Delta Q_3 + \Delta \chi_3 Q_{10} - \Delta \chi_1 Q_{30} + \Delta q_2; \\
\frac{\partial^2 U_3}{\partial t^2} &= \frac{\partial \Delta Q_3}{\partial S} + \chi_{10} \Delta Q_1 - \chi_{20} \Delta Q_2 + \Delta \chi_1 Q_{20} - \Delta \chi_2 Q_{10} + \Delta q_3; \\
\frac{\partial^2 \vartheta_1}{\partial t^2} &= \frac{\partial \Delta M_1}{\partial S} + \chi_{20} \Delta M_2 + \Delta \chi_2 M_{30} + \Delta \mu_1; \\
\frac{\partial^2 \vartheta_2}{\partial t^2} &= \frac{\partial \Delta M_2}{\partial S} + \chi_{30} \Delta M_1 + \Delta \chi_3 M_{10} + \Delta \mu_2; \\
\frac{\partial^2 \vartheta_3}{\partial t^2} &= \frac{\partial \Delta M_3}{\partial S} + \chi_{10} \Delta M_2 + \Delta \chi_1 M_{20} + \Delta \mu_3; \\
\frac{\partial U_1}{\partial S} + \chi_{20} U_3 - \chi_{30} U_2 &= 0; \\
\frac{\partial U_2}{\partial S} + \chi_{30} U_1 - \chi_{10} U_3 &= \vartheta_1; \\
\frac{\partial U_3}{\partial S} + \chi_{10} U_2 - \chi_{20} U_1 &= \vartheta_3; \\
\frac{\partial \vartheta_1}{\partial S} - \Delta \chi_1 &= \chi_{30} \vartheta_2 - \chi_{20} \vartheta_3; \\
\frac{\partial \vartheta_2}{\partial S} - \Delta \chi_2 &= \chi_{10} \vartheta_3 - \chi_{30} \vartheta_1; \\
\frac{\partial \vartheta_3}{\partial S} - \Delta \chi_3 &= \chi_{20} \vartheta_1 - \chi_{10} \vartheta_2; \\
\Delta M_1 &= A_1 \Delta \chi_1; \\
\Delta M_2 &= A_2 \Delta \chi_2; \\
\Delta M_3 &= A_3 \Delta \chi_3;
\end{align*}
\]

Where

\( A_i = GI_{s_ip}(z) \) - Waveguide torsional stiffness;
\[ A_2 = EI_x(z) \quad A_3 = EI_y(z) \] - Waveguide flexural stiffness.

Let’s consider small oscillations with respect to the natural unloaded state of the waveguide, i.e.
\[ Q_{10} = M_{i0} = 0; \Delta q_i = \Delta \mu_i = 0 \quad (3) \]

As an example we consider a flat circular rod of constant cross section of radius \( R \), for which \( \chi_{10} = 0; \chi_{20} = 0; \chi_{30} = \frac{1}{R} \) (figure 3).

![Figure 3. Flat circular rod of radius R.](image)

The oscillation equations for a plane ring are [5]:
\[
\begin{align*}
    m_0 \frac{\partial^2 U_1}{\partial t^2} &= \frac{\partial \Delta Q_1}{\partial s} - \frac{1}{R} \Delta Q_2; \\
    m_0 \frac{\partial^2 U_2}{\partial t^2} &= \frac{\partial \Delta Q_2}{\partial s} + \frac{1}{R} \Delta Q_1; \\
    I_3 \frac{\partial^2 \varphi_3}{\partial t^2} &= A_3 \frac{\partial \Delta \chi_3}{\partial s} + \Delta Q_3; \\
    \frac{\partial U_1}{\partial s} - \frac{1}{R} U_2 &= 0; \\
    \frac{\partial U_2}{\partial s} + \frac{1}{R} U_1 &= \varphi_3; \\
    \frac{\partial \varphi_3}{\partial s} - \Delta \chi_3 &= 0;
\end{align*}
\] (4)

Where
\[
\begin{align*}
    m_0 &= \rho A(s); \\
    I_3 &= \rho I_p(s);
\end{align*}
\]
3. Initial parameters numerical method for determination of natural frequencies and forms of oscillations

To determine the natural frequencies and waveforms of a UMI circular waveguide in its own plane by the Fourier method we separate the variables:

\[ U_i(t,s) = U_{i0}(s) \cos pt \]
\[ \vartheta_i(t,s) = \vartheta_{i0}(s) \cos pt \]
\[ \Delta Q_i(t,s) = Q_i(s) \cos pt \]
\[ \Delta \chi_i(t,s) = \chi_i(s) \cos pt \]  

(5)

After substitution equation (5) in equation (4), we get

\[ \frac{dQ_{i0}}{ds} - \frac{1}{R} Q_{20} + m_0 p^2 U_{i0} = 0; \]
\[ \frac{dQ_{20}}{ds} + \frac{1}{R} Q_{10} + m_0 p^2 U_{20} = 0; \]
\[ \frac{d\chi_{20}}{ds} + \frac{1}{A_3} Q_{20} + \frac{I_3 p^2}{A_3} \vartheta_{30} = 0; \]
\[ \frac{dU_{i0}}{ds} - \frac{1}{R} U_{20} = 0; \]
\[ \frac{dU_{20}}{ds} + \frac{1}{R} U_{10} - \vartheta_{30} = 0; \]
\[ \frac{d\vartheta_{30}}{ds} - \chi_{30} = 0; \]  

(6)

Or in matrix form

\[ Y' + A(P)Y = 0; \]

Where

\[ Y = [Q_1, Q_2, \chi_3, U_1, U_2, \vartheta_3]^T \]

(7)

\[ A(P) = \begin{bmatrix}
0 & -\frac{1}{R} & 0 & m_0 p^2 & 0 & 0 \\
\frac{1}{R} & 0 & 0 & 0 & m_0 p^2 & 0 \\
0 & \frac{1}{A_3} & 0 & 0 & \frac{I_3 p^2}{A_3} & 0 \\
0 & 0 & 0 & -\frac{1}{R} & 0 & 0 \\
0 & 0 & \frac{1}{R} & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 
\end{bmatrix} \]  

(8)

The solution of equation (7) gives
\[ \bar{Y}(z) = K(z)\bar{Y}(0); \]  

(9)

Where \( K(z) \) - the fundamental matrix of solutions

For numerical determination of \( K(L) \), we integrate the equation (7) from 0 to \( L \) six times, respectively, with the initial vectors

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix}
, \quad
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

Getting the columns of the fundamental matrix \( K(L) \):

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\]

Now we can get \( \bar{Y}(L) \) depending on \( \bar{Y}(0) \), using boundary conditions

\[ \bar{Y}(L) = K(L) \ast \bar{Y}(0) \]  

(10)

Or in expanded form

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\
K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6
\end{bmatrix}
= \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
0 \\
0 \\
0
\end{bmatrix}
\]

We obtain the following system of equations:

\[
\begin{align*}
K_{11}C_1 + K_{12}C_2 + K_{13}C_3 &= 0 \\
K_{21}C_1 + K_{22}C_2 + K_{23}C_3 &= 0 \\
K_{31}C_1 + K_{32}C_2 + K_{33}C_3 &= 0
\end{align*}
\]

(11)

The system of Eq. (11) has a nontrivial solution if

\[
\det(P) = \begin{vmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{vmatrix} = 0
\]

(12)
Equation (12) is a frequency equation for our problem and allows us to determine the eigenfrequencies of the curvilinear waveguide in its plane. After calculating the test version of the waveguide, the first three eigenfrequencies are obtained.

\[ P_1 = 17 \text{ rad/s}; \]
\[ P_2 = 86 \text{ rad/s}; \]
\[ P_3 = 273 \text{ rad/s}. \]

To determine the \( i \)-th eigenform of oscillations \( \psi_i(z) \), it is necessary to substitute the corresponding frequency \( P_i \) into the equation (11). In this case, one of the equations of the system is a linear combination of the other two [7]. Let’s consider the first two equations:

\[
\begin{align*}
K_{11}^{(i)} C_{1i} + K_{12}^{(i)} C_{2i} + K_{13}^{(i)} C_{3i} &= 0 \\
K_{21}^{(i)} C_{1i} + K_{22}^{(i)} C_{2i} + K_{23}^{(i)} C_{3i} &= 0
\end{align*}
\]

Assuming \( C_{1i} = 1 \), we get

\[
\begin{align*}
K_{12}^{(i)} C_{2i} + K_{13}^{(i)} C_{3i} &= -K_{11}^{(i)} \\
K_{22}^{(i)} C_{2i} + K_{23}^{(i)} C_{3i} &= -K_{21}^{(i)}
\end{align*}
\]

Let’s solve this equation with the Given-Find function of MathCad for \( P_1 = 17 \text{ rad/s} \). And then get the following values:

\[
\begin{align*}
C_{11} &= 1 \\
C_{21} &= -0.73 \\
C_{31} &= 1.46 \times 10^{-3}
\end{align*}
\]

Now the initial vector looks like

\[
\vec{Y}_0 = \begin{bmatrix} 1 \\ -0.73 \\ 1.46 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

We integrate it from 0 to \( L = 0.5\pi R \) and get the first form of transverse displacements \( \psi_1(z) \) (figure 4). Similarly, substituting into the system of equation (11) \( P_2 = 86 \text{ rad/s} \) and \( P_3 = 273 \text{ rad/s} \), we obtain \( \psi_2(z) \) and \( \psi_3(z) \) (figure 4).

Along with the transverse displacements for each natural frequency \( P_i \), the corresponding longitudinal displacements are obtained (figure 5) and the angles of rotation of the sections (figure 6), as well as the relevant internal forces and moments [9].
Figure 4. Natural forms of transverse displacements.

Figure 5. Natural forms of longitudinal oscillations.
4. Conclusions

In the present work the technique for exact numerical determination of natural frequencies and forms of oscillations of a UMI curved waveguide is presented. With the example of a plane circular waveguide of constant cross-section, this technique is implemented in the Mathcad package. The obtained eigenforms of oscillations can serve as a basis for analysis of UMI forced oscillations.

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