Low energy wave packet tunneling from a parabolic potential well through a high potential barrier

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Abstract

The problem of wave packet tunneling in a potential $V(x) = \left(\frac{m\omega^2}{2}\right) (x^2 - \delta x^\nu)$ with $\nu > 2$ is considered in the case when the barrier height is much greater than $\hbar\omega$ and the difference between the average energy of the packet and the oscillator ground state energy $\hbar\omega/2$ is sufficiently small. The universal Poisson distribution of the partial tunneling rates from the oscillator energy levels is discovered. The explicit expressions for the tunneling rates of different types of packets (coherent, squeezed, even/odd, thermal, etc.) are given in terms of the exponential and modified Bessel functions. The tunneling rates turn out very sensitive to the energy distributions in the packets, and they may exceed significantly the tunneling rate from the energy state with the same average number of quanta.

Key words: Quantum tunneling; Coherent and squeezed states

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1 Introduction

Usually, the problem of quantum tunneling through potential barriers was considered under the assumption that the initial state possessed a well defined energy, i.e., for the quasistationary states. The propagation of the Gaussian wave packets through a rectangular barrier was studied, e.g., in [1]. Recently, the problem of decay of coherent and squeezed packets confined initially in a deep potential well of a finite depth was considered by several authors [2, 3, 4]. The contribution of dissipation was studied, e.g., in [5, 6, 7, 8, 9]. However, in all those papers the influence of the wave packet shape on the transition or escape rates was analyzed in the framework of numerical calculations only.

The aim of the present article is to present simple analytical formulas for the decay rate in the case when a wave packet is localized initially near to the bottom of a deep potential well

\[ V(x) = \frac{1}{2}m\omega^2(x^2 - \delta x^\nu), \quad \nu > 2. \]  

(1)

The special cases of this potential were considered in [4, 5, 8, 10] (\(\nu = 3\)) and [11] (\(\nu = 4\)). More precisely, we assume that the potential is close to the harmonic one for relatively small values of \(|x|\), while it goes to \(+\infty\) when \(x \to -\infty\), so that we have a single barrier at \(x > 0\). Besides, it is implied that the potential is given by Eq. (1) provided \(x^2 - \delta x^\nu > 0\), whereas at large values of \(x\) (to the right of the barrier) it tends to some constant value.

If the initial state were described by means of the diagonal density matrix \(\hat{\rho} = \sum \rho_n |n\rangle \langle n|\), then the total decay rate \(\gamma\) would be a sum of partial rates \(\gamma_n\) taken with proper weights [8, 10, 12, 13],

\[ \gamma = \sum_n \rho_n \gamma_n. \]

(2)

For pure superpositions of many wave functions with different energies \(E_n\) the situation, in principle, may be more complicated due to the possibility of quantum interference effects. However, under certain conditions Eq. (2) can be applied to the pure states, as well. Suppose that we have a single high (unidimensional) barrier from the right, so that the motion can be considered as free at \(x > L\). Let us designate the probability of discovering the particle in the well and under the barrier as \(w_L = \int_{-\infty}^L \psi(x)^2 \, dx\). Then an immediate consequence of the quantum continuity equation is the relation \(\dot{w}_L = -j(L)\), where \(j = (i\hbar/2m)(\psi \nabla \psi^* - \psi^* \nabla \psi)\)
is the usual current density. If the wave function of the packet has the form

$$
\psi = \sum_n c_n \psi_n e^{-iE_n t/\hbar}
$$

(it holds for decaying states, as well, provided that $t \ll \gamma^{-1}$), then

$$
\dot{j} = \sum_n |c_n|^2 \dot{j}_n + \sum_{m \neq n} j_{mn} e^{i(E_n - E_m) t/\hbar}.
$$

Since the second sum consists of a large number of rapidly oscillating terms with different phases and frequencies, it turns practically into zero for $|E_n - E_m| t/\hbar \gg 1$, and we get Eq. (2) with $\rho_n = |c_n|^2$, $j_n = \gamma_n$, and $\gamma = \bar{j}$, the overbar meaning the averaging over fast oscillations. This result holds under the condition $\hbar \gamma \ll |E_n - E_m|$, where suffices $n, m$ correspond to all the coefficients $c_n$ that yield significant contributions to the expansion (3). In the special case of potential (1) with $\nu = 3$, Eq. (3) was actually derived in [4] in the framework of the quasiclassical method proposed in [2]. In the present paper we pursue two main goals: i) to find an analytical expression for the partial decay rate $\gamma_n$ in the potential (1), ii) to calculate the sum (2) explicitly for different physically interesting initial wave packets.

## 2 Partial decay rates from a parabolic well

To calculate the partial tunneling rates we use the standard quasiclassical formula [8, 13]

$$
\gamma_n = \frac{\omega}{2\pi} \exp \left\{ -\frac{2}{\hbar} \sqrt{2m} \int_{x_1}^{x_2} \sqrt{V(x) - E_n} \, dx \right\}.
$$

(4)

It is convenient to rewrite it in the form

$$
\gamma_n^{(\nu)} = \frac{\omega}{2\pi} \exp \left\{ -\frac{2V_0}{\lambda_\nu \hbar \omega} F_\nu \left( 2\lambda_\nu \frac{E_n}{V_0} \right) \right\},
$$

(5)

where $V_0$ is the barrier height:

$$
V_0 = m\omega^2 \lambda_\nu \delta^{-2/(\nu-2)}, \quad \lambda_\nu = \frac{\nu - 2}{2\nu} \left( \frac{2}{\nu} \right)^{2/(\nu-2)}.
$$

Function $F_\nu(t)$ is given by the integral

$$
F_\nu(t) = \int_a^b dx \left[ x^2 - x^\nu - t \right]^{1/2},
$$

(6)
where \( a < b \) are positive solutions to the equation

\[
x^2 - x^\nu - t = 0.
\]  

(7)

It is known that for \( \nu = 3 \) and \( \nu = 4 \) the integral (6) can be expressed in terms of the complete elliptic integrals of the first and second kind [14]

\[
K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}}, \quad E(z) = \int_0^1 \frac{1-z^2x^2}{1-x^2} dx.
\]

At \( \nu = 4 \) the roots of Eq. (7) can be found explicitly, and formula (3.155.9) of [14] yields

\[
F_4(t) = \frac{1}{3} (1 + \xi^2)^{-3/2} \left[ (1 + \xi^2) E \left( \sqrt{1 - \xi^2} \right) - 2\xi^2 K \left( \sqrt{1 - \xi^2} \right) \right],
\]

(8)

\[
\xi^2 = \frac{1 - \sqrt{1 - 4t}}{1 + \sqrt{1 - 4t}}.
\]

At \( \nu = 3 \) we get

\[
F_3(t) = \frac{2}{15} (1 - \xi^2 + \xi^4)^{-5/4} \left\{ 2 (1 - \xi^2 + \xi^4) E \left( \sqrt{1 - \xi^2} \right) - 2\xi^2 (1 + \xi^2) K \left( \sqrt{1 - \xi^2} \right) \right\},
\]

(9)

where \( \xi^2 = (a - c)/(b - c) \), and \( c \) is the negative root of Eq. (7). Another expression for \( \gamma_n^{(3)} \) was given in [4].

Assuming that coefficient \( \delta \) in the potential energy (1) is sufficiently small, we have \( Q \equiv V_0/\hbar \omega \gg 1 \). Then \( m\omega x^2/\hbar = 2\nu Q/(\nu - 2) \gg 1 \), where \( x^* \) is the position of the maximum of the potential energy. This means that the energies of the low levels practically coincide with the harmonic oscillator energy \( E_n = \hbar \omega (n + \frac{1}{2}) \). For these levels \( E_n \ll V_0 \), so we need the expansions of the exact expressions (8) and (9) at \( t \ll 1 \). For \( \nu = 4 \), the known asymptotics of the complete elliptic integrals [14] results in the formula

\[
F_4(t) = \frac{1}{3} + \frac{t}{4} \ln \left( \frac{t}{16e} \right) + \frac{3}{32} t^2 \ln t + \mathcal{O} \left( t^2 \right).
\]

(10)

For \( \nu = 3 \) the roots of the cubic equation (7) read (to within an accuracy of the order of the order of \( t^{3/2} \))

\[
a = \sqrt{t} \left( 1 + \frac{\sqrt{t}}{2} \right), \quad c = -\sqrt{t} \left( 1 - \frac{\sqrt{t}}{2} \right), \quad b = 1 - t, \quad \xi^2 = 2\sqrt{t}(1 - \sqrt{t}).
\]

However, the expansion of \( F_3(t) \) does not contain odd powers of \( \sqrt{t} \):

\[
F_3(t) = \frac{4}{15} + \frac{t}{4} \ln \left( \frac{t}{64e} \right) + \mathcal{O} \left( t^2 \ln t \right).
\]

(11)
Both the expressions, (10) and (11), contain the same term \((t/4) \ln(t/e)\). This coincidence is not accidental: the leading term in the expansion of the integral (6) at \(t \to 0\) equals \((t/4) \ln(t/e)\) for any \(\nu > 2\). Indeed, the contribution to this integral of the domain near the left turning point can be represented as

\[
F_{\text{left}} = \int_{\sqrt{t}}^{A} \sqrt{x^2 - t} \, dx + \cdots ,
\]

where \(A\) is some finite number. Thus \(F_{\text{left}} = (t/4) \sinh(2z) - tz/2 + \cdots\), where \(A = \sqrt{t} \cosh(z)\).

In the limit of \(A^2/t \gg 1\) we get

\[
F_{\text{left}} = A^2/2 + (t/4)(\ln t - 1) - (t/2) \ln(2A) + \mathcal{O}(t^2/A^2).
\]

In the vicinity of the right turning point we put \(x = 1 - \varepsilon\) and write

\[
F_{\text{right}} = \int_{t/(\nu - 2)}^{B} (\nu - 2) \varepsilon - t \, d\varepsilon + \cdots = \frac{2}{3} B^{3/2} \sqrt{\nu - 2} - t \sqrt{\frac{B}{\nu - 2}} + \mathcal{O}(t^2).\]

As to the integral in the limits from \(A\) to \(B\), it has an obvious power expansion with respect to \(t\). Consequently, the following expansion holds:

\[
F_{\nu}(t) = f_{\nu}^{(0)} + (t/4) \ln(t/e) - f_{\nu}^{(1)} t + \mathcal{O}(t^2 \ln t),
\]

where coefficients \(f_{\nu}^{(0)}\) and \(f_{\nu}^{(1)}\) depend on the concrete value of the exponent \(\nu\). Then the partial tunneling rate reads

\[
\gamma_{n}^{(\nu)} = \frac{\omega}{2\pi} \exp \left\{ -\frac{2Q}{\lambda_{\nu}} f_{\nu}^{(0)} + (n + 1/2) \left[ \ln \left( \frac{eQ}{2(n + 1/2)\lambda_{\nu}} \right) + 4f_{\nu}^{(1)} \right] + \mathcal{O}\left( \frac{n^2 \ln Q}{Q} \right) \right\}. \tag{12}
\]

Now let us notice that the factor \((n + 1/2)[\ln(n + 1/2) - 1]\) is the leading term of Stirling’s asymptotical formula

\[
\ln(n!) = (n + 1/2)[\ln(n + 1/2) - 1] + (1/2) \ln(2\pi) + \mathcal{O}(1/n),
\]

which works quite well even at \(n \approx 1\). Consequently, under the restriction

\[
n^2 \ln(Q)/Q \ll 1 \tag{13}
\]

the partial decay rates are given by the Poisson distribution:

\[
\gamma_{n}^{(\nu)} = \gamma_{0}^{(\nu)} \frac{\lambda_{\nu}^n}{n!}, \tag{14}
\]
$$\chi_\nu = \mu_\nu Q, \quad \mu_\nu = (2\lambda_\nu)^{-1}\exp\left[4f_\nu^{(1)}\right], \quad \gamma_\nu^{(\nu)} = \omega \sqrt{\frac{\chi_\nu}{2\pi}} \exp\left[-\frac{2Q}{\chi_\nu^{(0)}}\right].$$

Specifically,

$$\mu_3 = 432, \quad \mu_4 = 64, \quad \gamma_0^{(3)} = \omega \sqrt{\frac{216}{\pi}} Q \exp\left(-\frac{36}{5} Q\right), \quad \gamma_0^{(4)} = \omega \sqrt{\frac{32}{\pi}} Q \exp\left(-\frac{16}{3} Q\right).$$

Strictly speaking, the right-hand side of Eq. (4) contains some additional preexponential factor $G(V_0, \omega, E_n)$. But this factor is a smooth function of energy. This means that $G \approx G_0 [1 + \mathcal{O}(n/Q)]$, while the leading exponential term was approximated with an accuracy of the order of $n^2 \ln Q/Q$. Consequently, the influence of the preexponential factor can be neglected under the restriction (13). Note that our expression for $\gamma_0^{(3)}$ coincides identically with the result of [8], where a special attention was paid to the correct calculation of the preexponential term.

Formula (14) seems to be a universal distribution of the partial decay rates from the low energy levels, which holds for any potential of the form $V(x) = (m\omega^2/2)[x^2 - u(x)]$ with $|u(x)| \ll x^2$ at $x \to 0$, provided that conditions $Q \gg 1$ and (13) are fulfilled. The concrete form of $u(x)$ is responsible for the precise value of the coefficient $\mu$. Since $\chi_\nu \gg 1$, the total decay rate turns out very sensitive to the details of the distribution $\rho_n$.

### 3 Decay of a slightly deformed ground state

The simplest example of the initial wave packet corresponds to a coherent state, i.e., an eigenstate of the operator $\hat{a} = (m\omega\hat{x} + i\hat{p})/\sqrt{2m\omega\hbar}$. Then

$$\rho_n(\alpha) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}, \quad |\alpha|^2 = \bar{n},$$

and Eqs. (2), (14) and (13) result in the formula (we drop the suffix $\nu$)

$$\gamma_{coh} = \gamma_0 \exp\left(-|\alpha|^2\right) I_0(2|\alpha| \sqrt{\chi}) = \gamma_0 \exp\left(-\bar{n}\right) I_0\left(2\sqrt{\chi\bar{n}}\right),$$

$I_0(z)$ being the modified Bessel function. To evaluate the domain of its validity, we notice that the maximal contribution to the sum (2) is given by the terms with $n_{max} \sim \sqrt{\chi\bar{n}}$, and the width of the effective distribution $\gamma_n \rho_n$ is obviously less than $\sqrt{n_{max}}$. Thus the requirement (13) results in the inequality $|\alpha|^2 = \bar{n} \ll 1/\ln Q$, so that the term $\exp\left(-\bar{n}\right)$ can be omitted.
However, the condition $\bar{n} \ll 1/\ln Q$ does not exclude the possibility of $\sqrt{\chi \bar{n}} \gg 1$. In this case we have

$$\gamma_{coh} = \gamma_0 \left( 4\pi \sqrt{\chi \bar{n}} \right)^{-1/2} \exp \left( 2\sqrt{\chi \bar{n}} \right) \gg \gamma_0.$$  \hspace{1cm} (17)

The most general pure Gaussian state (which is called frequently in the current literature as a “squeezed state”: see [15] and references therein) can be considered as an eigenstate of a linear combination of the operators $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{b}|\beta uv\rangle = \beta|\beta uv\rangle, \quad \hat{b} = u\hat{a} + v\hat{a}^\dagger, \quad |u|^2 - |v|^2 = 1$$  \hspace{1cm} (18)

(for simplicity, we confine ourselves to the case of linear uniform transformations). The corresponding level population distribution reads [15, 16]

$$\rho_n = |\langle n|\beta uv\rangle|^2 = \frac{1}{|u|n!} \left| \frac{v}{2u} \right|^n \exp \left[ -|\beta|^2 + \Re \left( \frac{\beta^2 v^*}{u} \right) \right] \left| H_n \left( \frac{\beta}{\sqrt{2uv}} \right) \right|^2,$$  \hspace{1cm} (19)

where $H_n(x)$ is the Hermite polynomial. To calculate the sum (2) we need a formula for $\sum g^n H_n(x)H_n(x^*)/(n!)^2$. It can be easily found, if one takes the known generating function of the Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{g^n}{(n!)^2} H_n(x) = \exp \left( 2xz - z^2 \right),$$

multiplies both sides by the complex conjugated functions, puts $z = \sqrt{g} \exp(i\varphi)$, and integrates the product over $\varphi$. The result is

$$\sum_{n=0}^{\infty} \frac{g^n}{(n!)^2} |H_n(x)|^2 = \frac{1}{2\pi} \int_0^{2\pi} \exp \left[ 4|x|\sqrt{g} \cos(\varphi + \psi) - 2g \cos(2\varphi) \right] d\varphi, \quad x = |x|e^{i\psi}.$$  \hspace{1cm} (20)

In this way we get the expression

$$\gamma_{sq} = \frac{\gamma_0}{2\pi |u|} \int_0^{2\pi} \exp \left[ 2 \left| \frac{\beta}{u} \right| \sqrt{\chi} \cos(\varphi + \psi) - \chi \left| \frac{v}{|u|} \right| \cos(2\varphi) \right] d\varphi, \quad \psi = \arg \left( \frac{\beta}{\sqrt{uv}} \right)$$  \hspace{1cm} (20)

(we neglect the contribution of the terms proportional to $|\beta|^2$ in the argument of the exponential function, since it is very small under the conditions $\chi \gg 1$ and $|\beta| \ll 1$, which ensure the validity of Eq. (20)). At $v = 0$ we arrive again at Eq. (14).

For a slightly squeezed state with $\chi|v| \ll |\beta|\sqrt{\chi}$ (this inequality implies $|v| \ll 1$, so that $|u|$ can be replaced by unity) the integral in (20) can be calculated with the aid of the steepest descent method, provided that $|\beta|\sqrt{\chi} \gg 1$. The integrand assumes its maximal value at $\varphi = -\psi$. Thus we get

$$\gamma_{sq}(\beta, v) = \gamma_{coh}(\beta) \exp \left[ -\chi |v| \cos(2\psi) \right].$$
Consequently, the decay rate of a squeezed packet may be both greater and less than the decay rate of the coherent packet with the same mean energy, depending on the value of the phase difference $\psi$. (This qualitative result was obtained in [15, 17] for $\nu = 3$, although the quantitative estimations were not quite correct, since the importance of logarithmic terms in the expansion of $F_3(t)$ was underestimated.) To elucidate the situation, we take into account the formulas for the average number of quanta and its variance in the squeezed state [15]

$$\bar{n} = |v|^2 \left(1 + |\beta|^2\right) + |u|^2 \left[|\beta|^2 - 2\text{Re}\left(\frac{\beta^2 v^*}{u}\right)\right],$$

$$\sigma_n = \bar{n}^2 - (\bar{n})^2 = 2\bar{n} \left[2|u|^2 - 1\right] - 2|v|^4 - |\beta|^2.$$ 

They can be simplified significantly if $|v| \ll 1, |u| \approx 1$:

$$\bar{n} \approx |\beta|^2 [1 - 2|v| \cos(2\psi)], \quad \sigma_n \approx 2\bar{n} - |\beta|^2.$$ 

Finally, we get

$$\gamma_{sq} = \gamma_0 \left[(4\pi)^2 \chi \bar{n}\right]^{-1/4} \exp \left[2\sqrt{\chi \bar{n}} + \frac{1}{2} \chi S\right], \quad S = (\sigma_n - \bar{n}) / \bar{n}, \quad (21)$$

where $S$ is the known Mandel’s parameter characterizing the type of photon statistics. In this case the super-Poissonian statistics enhances the tunneling rate, while the sub-Poissonian one suppresses it. A different dependence of the tunneling rate on the degree of squeezing was found in Ref. [4]. But its authors performed the numerical calculations in a quite different domain of parameters: $Q = 10$ and $|\beta| \geq 1$, where our approach cannot be applied.

In the case of a squeezed vacuum ($\beta = 0$) the right-hand side of Eq. (21) coincides with the known integral representation of the modified Bessel function. Then

$$\gamma_v = \gamma_0 I_0(|v|) = \gamma_0 I_0(\chi \sqrt{\bar{n}}), \quad (22)$$

provided that $|v|^2 = \bar{n} \ll (Q \ln Q)^{-1}$ (for this reason we put $|u| = 1$). If $\chi |v| \gg 1$ and $|\beta| \ll |v| \sqrt{\chi}$, then one can use again the steepest descent method. Now we have two extremal points: $\varphi = \pi/2$ and $\varphi = 3\pi/2$, so

$$\gamma_v(\beta) = \gamma_0 \left(2\pi \chi |v|\right)^{-1/2} \exp(\chi |v|) \cosh(2|\beta| \sqrt{\chi} \sin \psi). \quad (23)$$

In this case we have $\gamma_v(\beta) > \gamma_v$, although Mandel’s parameter

$$S = \left[|v|^2 - 2|\beta|^2 |v| \cos(2\psi)\right] / \left(|v|^2 + |\beta|^2\right)$$

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may be both positive and negative, in spite of the requirement $|\beta| \ll |v| \sqrt{\chi}$.

The level populations in a Gaussian mixed state with zero mean values of the quadratures are expressed in terms of the Legendre polynomials [18, 19, 20]:

$$\rho_n = 2(4d + 2T + 1)^{-1/2} \left( \frac{4d + 1 - 2T}{4d + 1 + 2T} \right)^{n/2} P_n \left( \frac{4d - 1}{(4d + 1)^2 - 4T^2} \right)^{1/2}.$$  

(24)

Parameters $d$ and $T$ are related to the “degree of mixing” of the quantum state and the mean quantum number:

$$d^{-1} = 4 \left[ \text{Tr} (\hat{\rho}^2) \right]^2, \quad T = 1 + 2\bar{n}, \quad T \geq 2\sqrt{d}.$$  

(25)

In this case the sum in the right-hand side of Eq. (2) is reduced to the known generating function of the Legendre polynomials (see, e.g., Eq. 10.10(40) from [21]). Since we are restricted with the inequality $\bar{n} \ll 1$, it is convenient to introduce a small parameter $\varepsilon \leq \bar{n} \ll 1$ according to the relations $\text{Tr} (\hat{\rho}^2) \approx 1 - 2\varepsilon$, $d \approx 1/4 + \varepsilon$. Then we get a simple formula

$$\gamma_{\text{gauss}}(\chi, \bar{n}, \varepsilon) = \gamma_0 \exp(\chi\varepsilon) I_0 \left( \chi \sqrt{\bar{n}} - \varepsilon \right).$$  

(26)

At $\varepsilon = 0$ it coincides with (22). In the thermal state we have $\varepsilon = \bar{n}$, $\rho_n = \bar{n}^n / (1 + \bar{n})^{n+1}$, and $\gamma_{\text{therm}} = \gamma_0 \exp(\chi \bar{n})$. The last expression holds provided that $\bar{n} \ll (Q \ln Q)^{-1/2}$. Note that this restriction does not forbid the inequality $\chi \bar{n} \gg 1$. With the same value of $\bar{n}$, at $\chi \bar{n} \gg 1$ the tunneling rate from the squeezed (pure) vacuum state turns out much greater than that from the thermal one. However, the thermal state decays faster than the coherent one under the same conditions. These examples show that the decay rates are very sensitive to the details of the energy distribution in the wave packet, so it is difficult to find a general law.

An example of a Gaussian packet with nonzero means of the quadratures is the mixture of the coherent and thermal states [18, 20, 22], when the “shifted Planck distribution function” is expressed in terms of the Laguerre polynomials:

$$\rho_n = \frac{n_{th}^n}{(1 + n_{th})^{n+1}} \exp \left[ - \frac{|\alpha|^2}{1 + n_{th}} \right] L_n \left( - \frac{|\alpha|^2}{n_{th}(1 + n_{th})} \right).$$  

(27)

In this case sum (2) can be calculated exactly with the aid of Eq. 10.12(18) from [21]. For $n_{th} \ll 1$ and $|\alpha|^2 \ll 1$ the total decay rate equals the product of the coherent and thermal decay rates:

$$\gamma_{\text{shift}}(|\alpha|, n_{th}) = \gamma_0 \exp(\chi n_{th}) I_0 (2|\alpha|\sqrt{\chi}) = \gamma_{\text{therm}}\gamma_{\text{coh}}.$$  

(28)
An example of a non-Gaussian wave packet is the even coherent state introduced in [23],

$$|\alpha;+\rangle = \left\{ 2 \left[ 1 + \exp \left( -2|\alpha|^2 \right) \right] \right\}^{-1/2} (|\alpha\rangle + | -\alpha\rangle),$$

with the quantum distribution function

$$\rho_{2n}^{(\pm)} = \frac{|\alpha|^{4n}}{(2n)! \cosh |\alpha|^2}; \quad \rho_{2n+1}^{(\pm)} = 0, \quad \bar{n}^{(+)} = |\alpha|^2 \tanh |\alpha|^2.$$

In this case, the total decay rate is proportional to the sum of the usual and the modified Bessel functions of the argument $2|\alpha|\sqrt{\chi}$, provided that $|\alpha|^2 \ll (\mu_\nu \ln Q)^{-1}$. Then $\bar{n} = |\alpha|^4$, and $\sigma_n = 2\bar{n}$ at $|\alpha| \ll 1$. Therefore

$$\gamma_+ = \frac{1}{2} \gamma_0 \left[ I_0 \left( 2\bar{n}^{1/4} \sqrt{\chi} \right) + J_0 \left( 2\bar{n}^{1/4} \sqrt{\chi} \right) \right].$$

We see that the even coherent state is less stable with respect to tunneling than the Glauber coherent state with the same value of $\bar{n} \ll 1$. For all distributions, $\gamma \approx \gamma_0$ if $\chi \bar{n} \ll 1$.

### 4 Decay of slightly deformed excited states

An odd coherent state [23]

$$|\alpha;-\rangle = \left\{ 2 \left[ 1 - \exp \left( -2|\alpha|^2 \right) \right] \right\}^{-1/2} (|\alpha\rangle - | -\alpha\rangle),$$

$$\rho_{2n+1}^{(-)} = \frac{|\alpha|^{4n+2}}{(2n+1)! \sinh |\alpha|^2}; \quad \rho_{2n}^{(-)} = 0, \quad \bar{n}^{(-)} = |\alpha|^2 \coth |\alpha|^2$$

is an example of the deformed first excited oscillator state at $|\alpha| \ll 1$. Its decay rate equals

$$\gamma_- = \frac{\gamma_1}{2\chi |\alpha|^2} \left[ I_0 \left( 2|\alpha|\sqrt{\chi} \right) - J_0 \left( 2|\alpha|\sqrt{\chi} \right) \right], \quad \gamma_1 = \gamma_0 \chi.$$

The limitations on $|\alpha|$ are the same as above, but now $|\alpha|^4 = 3 (\bar{n} - 1)$.

Another example is the odd squeezed state [23]:

$$|z;-\rangle = \exp \left[ \frac{z}{2} (\hat{a}^\dagger)^2 \right] |1\rangle, \quad \rho_{2n+1}^{(z)} = \frac{(2n+1)!}{(n!)^2} |z|^{2n} \frac{z}{2}, \quad \rho_{2n}^{(z)} = 0.$$

The expression for the total decay rate is similar to formula (22) for the squeezed vacuum state:

$$\gamma_z = \gamma_1 I_0 (\chi |z|),$$
provided that $|z|^2 = (\bar{n} - 1)/3 \ll (Q \ln Q)^{-1}$.

It is not difficult to perform the calculations also for two families of deformed $m$-quantum states. The first one corresponds to the photon-added coherent states (PACS) [24]:

$$|\alpha, m\rangle = (\hat{a}^\dagger)^m |\alpha\rangle, \quad \rho_n^{\text{pacs}} = \frac{n!|\alpha|^{2(n-m)}}{m![(n-m)!]^2}, \quad n \geq m$$

(we assume that $|\alpha| \ll 1$). Then the total decay rate is given by the formula similar to (16),

$$\gamma_{\text{pacs}} = \gamma_m I_0 (2|\alpha|\sqrt{\chi}), \quad |\alpha|^2 \ll 1 / \ln Q,$$

but with another meaning of parameter $|\alpha|$, since now $\bar{n} - m = (m + 1)|\alpha|^2 = \sigma_n$.

The second family consists of the displaced number states [25, 26, 27, 28, 29] $|m, \alpha\rangle = \exp (\alpha \hat{a}^\dagger - \alpha^* \hat{a}) |m\rangle$, whose quantum distribution function is expressed in terms of the associated Laguerre polynomials:

$$\rho_n^{\text{disp}} = \frac{n!}{m!} \left[|\alpha|^{(n-m)} L_n^{(m-n)} (|\alpha|^2) \right]^2 \exp (-|\alpha|^2).$$

In this case, using the identity 10.12(19) from [21]

$$\sum_{n=0}^{\infty} z^n L_n^{(\alpha-n)} (x) = e^{-zx} (1 + z)^\alpha$$

and applying the same approach that led to Eq. (20), one can obtain the formula

$$\sum_{n=0}^{\infty} y^n \left[L_n^{(m-n)} (x) \right]^2 = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \exp (-2x\sqrt{y} \cos \varphi) (1 + y + 2\sqrt{y} \cos \varphi)^m.$$

Consequently, the total decay rate can be expressed as some combination of the modified Bessel functions of $2|\alpha|\sqrt{\chi}$ with different integer indices. However, in the most interesting case, when $2|\alpha|\sqrt{\chi} \gg 1$, the steepest descent method leads to a simple formula

$$\gamma_{\text{disp}} = \gamma_m (4\pi|\alpha|\sqrt{\chi})^{-1/2} \exp (2|\alpha|\sqrt{\chi}), \quad |\alpha|^2 = \bar{n} - m \ll 1 / \ln Q, \quad \sigma_n = (2n + 1)|\alpha|^2.$$

## 5 Conclusion

Two new results seem to be the most important. Firstly, we have found the universal Poisson distribution of the partial decay rates from the energy eigenstates in the parabolic potential well for a wide class of potential barriers. Secondly, we have demonstrated that the tunneling
decay rates are very sensitive to the shape of the wave packet. In particular, if one has initially not an exact energy eigenstate, but a combination (pure or mixed) of the states with different energies, then the decay rates may be quite different, even when the average energy, coordinate and momentum variances, etc., are almost the same. This fact may be important for the analysis of various phenomena related to the tunnel effect, when the initial state is not known absolutely exactly.

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