STABILITY PROPERTIES OF STRONGLY MAGNETIZED SPINE-SHEATH RELATIVISTIC JETS

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ABSTRACT

We derive linearized relativistic magnetohydrodynamic (RMHD) equations describing a uniform axially magnetized cylindrical relativistic jet spine embedded in a uniform axially magnetized relativistically moving sheath. The displacement current is retained in the equations, so that effects associated with Alfvén wave propagation near light speed can be studied. A dispersion relation for the normal modes is obtained. Analytical solutions for the normal modes in the low- and high-frequency limits are found, and a general stability condition is determined. A trans-Alfvénic and even a super-Alfvénic relativistic jet spine can be stable to velocity shear–driven Kelvin-Helmholtz modes. The resonance condition for maximum growth of the normal modes is obtained in the kinetically and magnetically dominated regimes. Numerical solution of the dispersion relation verifies the analytical solutions and is used to study the regime of high sound and Alfvén speeds.

Subject headings: galaxies: jets — gamma rays: bursts — instabilities — ISM: jets and outflows — methods: analytical — MHD — relativity

1. INTRODUCTION

Relativistic jets are associated with active galactic nuclei (AGNs) and quasars (QSOs), with black hole binary star systems (microquasars), and are thought to be responsible for gamma-ray bursts (GRBs). In microquasar and AGN jets, proper motions of intensity enhancements show mildly superluminal for the microquasar jets 1.2 c (Mirabel & Rodriguez 1999), range from subluminal (c) to superluminal (6 c) along the M87 jet (Biretta et al. 1995, 1999), are up to 25 c along the 3C 345 jet (Zensus et al. 1995; Steffen et al. 1995), and have inferred Lorentz factors 100 in the GRBs (e.g., Piran 2005). The observed proper motions along microquasar and AGN jets imply speeds from 0.9 c up to 0.9999 c, and the speeds inferred for the GRBs are 0.99999 c.

Jets may be kinetically dominated at large scales and contain relatively weak magnetic fields, e.g., have equipartition or less between magnetic and gas pressure, but much stronger magnetic fields are possible close to the acceleration and collimation region. Here general relativistic magnetohydrodynamic (GRMHD) simulations of jet formation (e.g., Koide et al. 2000; Nishikawa et al. 2005; De Villiers et al. 2003, 2005; Hawley & Krolik 2006; McKinney 2006; Mizuno et al. 2006) and earlier theoretical work (e.g., Lovelace 1976; Blandford 1976; Blandford & Znajek 1977; Blandford & Payne 1982) invoke strong magnetic fields. In addition to strong magnetic fields, GRMHD simulation studies of jet formation indicate that highly collimated high-speed jets driven by the magnetic fields threading the ergosphere may themselves reside within a broad accretion disk wind (e.g., Nishikawa et al. 2005) or a narrower sheath outflow driven by the magnetic fields anchored in the inner accretion disk (e.g., McKinney 2006; Hawley & Krolik 2006; Mizuno et al. 2006).

Jet-wind structure is indicated by recent observations of winds in several QSOs with speeds 0.1–0.4 c (Chartas et al. 2002, 2003; Pounds et al. 2003a, 2003b; Reeves et al. 2003). Other observational evidence, such as limb brightening, has been interpreted as evidence for a slower external sheath flow surrounding a faster jet spine, e.g., in Mrk 501 (Giroletti et al. 2004), M87 (Perlman et al. 2001), and a few other radio galaxy jets (e.g., Swain et al. 1998; Giovannini et al. 2001). Additional circumstantial evidence, such as the requirement for large Lorentz factors suggested by the TeV BL Lac objects when contrasted with much slower observed motions, suggest the presence of a spine-sheath morpholo (Ghisellini et al. 2005). At scales of hundreds of kiloparsecs, Siemiginowska et al. (2007) have proposed a two-component (spine-sheath) model to explain the broadband emission from the PKS 1127–145 jet. A spine-sheath jet structure has been proposed based on theoretical arguments (e.g., Sol et al. 1989; Henri & Pelletier 1991; Laing 1996; Meier 2003). Similar structure has been investigated in the context of GRB jets (e.g., Rossi et al. 2002; Lazzati & Begelman 2005; Zhang et al. 2003, 2004; Morsony et al. 2007).

In order to study the effect of strong magnetic fields and the effect of a broad wind or narrower moving sheath around a jet or jet spine, I begin by adopting a simple system with no radial dependence of quantities inside the jet spine and no radial dependence of quantities outside the jet in the sheath. This “top-hat” configuration with magnetic fields parallel to the flow can be described exactly by the linearized relativistic magnetohydrodynamic (RMHD) equations. This system with no magnetic and flow helicity is stable to current-driven (CD) modes of instability (Istomin & Pariev 1994, 1996; Lyubarskii 1999). However, this system can be unstable to Kelvin-Helmholtz (KH) modes of instability (Hardee 2004). This approach allows us to look at the potential KH modes without complications arising from coexisting CD modes (see Baty et al. 2004), and predictions can be verified by numerical simulations (Mizuno et al. 2007).

This paper is organized as follows. In § 2, I present the dispersion relation arising from a normal mode analysis of the linearized RMHD equations. Analytical approximate solutions to the dispersion relation for various limiting cases are presented in § 3. In § 4, I verify the analytical solution through numerical solution of the dispersion relation. I summarize the stability results in § 5 and discuss the applicability of the present results in § 6. Derivation of the linearized RMHD equations and the normal mode dispersion relation is presented in Appendix A, and derivation of the analytical solutions is outlined in Appendix B.

2. THE RMHD NORMAL MODE DISPERSION RELATION

Let us analyze the stability of a spine-sheath system by modeling the jet spine as a cylinder of radius R, having a uniform proper density , a uniform axial magnetic field , and a uniform velocity . The external sheath is assumed to have...
a uniform proper density $\rho_e$, a uniform axial magnetic field $B_z = B_{e,z}$, a uniform velocity $u_e = u_{e,z}$, and extends to infinity in the radial direction. The sheath velocity corresponds to an outflow around the central spine if $u_{e,z} > 0$, or represents backflow when $u_{e,z} < 0$. The jet spine is established in static total pressure balance with the external sheath, where the total static uniform pressure is $P_s^* = P_e + B_z^2/8\pi = P_j^* = P_j + B_j^2/8\pi$, and the initial equilibrium satisfies the zeroth-order equations. Formally, the assumption of an infinite sheath means that a dispersion relation could be derived in the reference frame of the sheath, with results transformed to the source/observer reference frame. However, it is not much more difficult to derive a dispersion relation in the source/observer frame, in which analytical solutions to the dispersion relation take on simple revealing forms. In addition, this approach lends itself to modeling the propagation and appearance of jet structures viewed in the source/observer frame, e.g., helical structures in the 3C 120 jet (Hardee et al. 2005).

The general approach to analyzing the time-dependent properties of this system is to linearize the ideal RMHD and Maxwell equations, where the density, velocity, pressure, and magnetic field are written as $\rho = \rho_0 + \rho_t$, $v = u + v_t$, and $B = B_0 + B_t$, respectively, where subscript 1 refers to a perturbation to the equilibrium quantity with subscript 0. In addition, the Lorentz factor $\gamma^2 = (\gamma^2_0 + \gamma^2_1)$ and $\gamma^2_0 = 1 - \gamma^2_1$ (Vlahakis & Königl 2003) is defined by

$$\gamma_0^2 = 1 - \frac{\rho_0}{\rho_1}, \quad \gamma_1^2 = \frac{\rho_1}{\rho_0},$$

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and

$$\beta_0^2 = \frac{\eta_0^2 + \eta_1^2}{\eta_0^2 + \eta_1^2}, \quad \beta_1^2 = \frac{\eta_0^2 + \eta_1^2}{\eta_0^2 + \eta_1^2},$$

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where $\eta_0^2 = \eta_0^2 + \eta_1^2$, and $\eta_1^2 = \eta_0^2 + \eta_1^2$. In cylindrical geometry a random perturbation $\rho_t, v_t, B_t$, and $P_t$ can be considered to consist of Fourier components of the form

$$f_1(r, \phi, z, t) = f_1(r) \exp\left[i(kz + n\phi - \omega t)\right],$$

where flow is along the z-axis and r is in the radial direction, with the flow bounded by $r = R$.

In cylindrical geometry $n$ is an integer azimuthal wavenumber; for $n > 0$ waves propagate at an angle to the flow direction, and $+n$ and $-n$ give wave propagation in the clockwise and counterclockwise sense, respectively, when viewed in the flow direction. In equation (1) $n = 0, 1, 2, 3, 4$, etc., correspond to pinching, helical, elliptical, triangular, rectangular, etc., normal mode distortions of the jet, respectively.

Propagation and growth or damping of the Fourier components can be described by a dispersion relation of the form

$$\beta_j^2 = \frac{\chi_j}{\chi_e} H_n^{(1)}(\beta R) H_n^{(1)}(\beta R),$$

Linearization of the RMHD equations and derivation of this dispersion relation is outlined in Appendix A. In the dispersion relation $J_n$ and $H_n^{(1)}$ are Bessel and Hankel functions, respectively; the primes denote derivatives of the Bessel and Hankel functions with respect to their arguments. In equation (2)

$$\chi_j = \gamma_0^2 \gamma_1^2 W_j(\omega^2 - \eta_0^2 v_{e,0}^2),$$

$$\chi_e = \gamma_0^2 \gamma_1^2 W_e(\omega^2 - \eta_0^2 v_{e,0}^2),$$

and

$$\beta_j^2 = \frac{\gamma_0^2 (\omega^2 - \eta_0^2 v_{e,0}^2) (\omega^2 - \eta_0^2 v_{e,0}^2)}{\eta_0^2 \omega^2 - \eta_0^2 v_{e,0}^2 \eta_1^2},$$

$$\beta_0^2 = \frac{\gamma_0^2 (\omega^2 - \eta_0^2 v_{e,0}^2) (\omega^2 - \eta_0^2 v_{e,0}^2)}{\eta_0^2 \omega^2 - \eta_0^2 v_{e,0}^2 \eta_1^2}.$$
where \( \eta \equiv \gamma_j^2 W_j / \gamma_e^2 W_e \) and a “surface” Alfvén speed is defined by

\[
V_{Ae}^2 \equiv \left( \gamma_{Ae}^2 W_j + \gamma_{Ae}^2 W_e \right) \frac{B_j^2 + B_e^2}{4\pi W_j W_e}.
\]

In equation (9) note that \( \gamma_{Ae,j}^2 \equiv (1 - v_{j,e}^2/c^2)^{-1/2} = 1 + V_{Ae,j}^2/c^2 \). The jet is stable to low-frequency \( n > 0 \) surface wave modes perturbations when

\[
\gamma_j^2 \gamma_e^2 (u_j - u_e)^2 < \gamma_j^2 \gamma_e^2 \left( \frac{W_j}{\gamma_{Ae}} + \frac{W_e}{\gamma_{Ae}} \right) \frac{B_j^2 + B_e^2}{4\pi W_j W_e}.
\]

For example, with \( u_j \approx c \gg u_e \), \( \gamma_j^2 \approx 1 \), \( \gamma_{Ae}^2 \approx 1 \), \( B_j^2 \gg B_e^2 \), and using \( \gamma_{Ae}^2 = 1 + B_j^2 / 4\pi W_j c^2 \), the jet is stable when

\[
\gamma_j^2 < \left( 1 + \frac{B_j^2}{4\pi W_j c^2} \right) \gamma_{Ae}^2.
\]

(11a)

or with \( B_e = B_j \), \( W_e = W_j \), so that \( v_{A,j} = v_{A,e} \), and with \( \gamma_e \equiv \gamma_{A,e} = \gamma_{A,j} \), the jet is stable when

\[
\gamma_j^2 \gamma_e^2 (u_j - u_e)^2 < 4\gamma_j^2 (\gamma_e^2 - 1) c^2.
\]

Thus, the jet can remain stable to the surface wave modes even when the jet Lorentz factor exceeds the Alfvénic Lorentz factor.

In the low-frequency limit, the real part of the body wave solutions is given by

\[
kR \approx k_{nm} R \equiv \left[ \frac{v_{m0}^2 u_j^2 - v_{e0}^2 a_j^2}{\gamma_j^2 (u_j^2 - a_j^2)(u_e^2 - v_{e0}^2)} \right]^{1/2} \times \left[ (n + 2m - 1/2)\pi/2 + (-1)^m \epsilon_n \right],
\]

(12)

where \( n \) specifies the normal mode, \( m = 1, 2, 3, \ldots \) specify the first, second, third, etc., body wave solutions, and

\[
\epsilon_n \equiv \frac{\chi_e \beta_e}{\chi_j \beta_j} \left( \frac{\pi \beta_j R}{2} \right)^{1/2} J_0^2(\beta_j R) / H_n^2(\beta_j R).
\]

With no significant external magnetic field and no significant external flow, \( \epsilon_n = 0 \) as \( \chi_e = \gamma_e^2 \gamma_{Ae}^2 W_e (u_e^2 - v_{e0}^2) k^2 = 0 \). In this low-frequency limit the body wave solutions are either purely real or damped, exist only when \( k_{nm} R \) has a positive real part, and with \( |\epsilon_n| \ll 1 \) require that

\[
\left[ \frac{v_{m0}^2 u_j^2 - v_{e0}^2 a_j^2}{\gamma_j^2 (u_j^2 - a_j^2)(u_e^2 - v_{e0}^2)} \right] > 0.
\]

Thus, the body modes can exist when the jet is supersonic and super-Alfvénic, i.e., \( u_j^2 - a_j^2 > 0 \) and \( u_e^2 - v_{e0}^2 > 0 \), or in a limited velocity range given approximately by \( a_j^2 > u_j^2 > \gamma_j^2 / (1 + \gamma_j^2) a_j^2 \) when \( v_{A,j} \approx a_j \), where \( \gamma_{ij} \equiv (1 - a_j^2 / c^2)^{-1/2} \) is a sonic Lorentz factor.

### 3.2. Resonance

With the exception of the pinch fundamental mode, which can have a relatively broad plateau in the growth rate, all body modes and all surface modes can have a distinct maximum in the growth rate at some resonant frequency. The resonance condition can be evaluated analytically either in the fluid limit where \( a \gg V_{A} \) or in the magnetic limit where \( V_{A} \gg a \). Note that in the magnetic limit, magnetic pressure balance implies that \( B_j = B_e \). In these cases a necessary condition for resonance is that

\[
\frac{u_j - u_e}{1 - u_j u_e/c^2} > \frac{v_{ij} + v_{we}}{1 + v_{ij} v_{we}/c^2},
\]

where \( v_{ij} \equiv (a_j, v_{A,j}) \) and \( v_{we} \equiv (a_e, v_{A,e}) \) in the fluid or magnetic limits, respectively. When this condition is satisfied, it can be shown that the wave speed at resonance is

\[
v_w \approx v_w^* \equiv \frac{\gamma_{ij}(v_{we} v_{ij} u_j + \gamma_{ij} v_{ij} v_{ij})}{\gamma_{ij}(v_{we} v_{ij} + \gamma_{ij} v_{ij} v_{ij})},
\]

(15)

where \( \gamma_{ij} \equiv (1 - v_{ij}^2 / c^2)^{-1/2} \) is the sonic or Alfvénic Lorentz factor accompanying \( v_{ij} \equiv (a_j, v_{A,j}) \) and \( v_{we} \equiv (a_e, v_{A,e}) \) in the fluid or magnetic limits, respectively.

The resonant wave speed and maximum growth rate occur at a frequency given by

\[
\frac{\omega_R/v_w}{\omega_{nm} R/v_{we}} = \frac{(2n + 1)/4 + m\pi}{\left( \left(1 - u_e/v_w^2 \right)^2 - \left( v_{we}^2 - u_e^2 / c^2 \right)^2 \right)^{1/2}}.
\]

(16)

In equation (16), \( n \) specifies the normal mode, \( m = 0 \) specifies the surface wave, and \( m \geq 1 \) specifies the body waves. In the limit of insignificant shear flow, \( u_e = 0 \) and using equation (15) for \( v_w^* \) in equation (16) allows the resonant frequency to be written as

\[
\omega_{nm} R_j/v_{we} \approx \frac{(2n + 1)/4 + m\pi}{\gamma_e \left[ 1 - 2(u_e/u_j)(1 - v_{we}^2/c^2) - (v_{we}^2 - u_e^2 / u_j^2) \right]^{1/2}}.
\]

and this predicts a resonant frequency that is primarily a function of the sound and Alfvén wave speeds in the sheath. The effect of shear flow is best illustrated by assuming comparable conditions in the spine and sheath, \( \gamma_{ij} v_{ij} \sim v_{we} v_{ij} \), and assuming that \( \gamma_{ij} u_j > \gamma_{ij} u_e \) in which case

\[
\omega_{nm} R_j/v_{we} \approx \frac{(2n + 1)/4 + m\pi}{\gamma_e \left[ 2(u_e/u_j)(1 - v_{we}^2/c^2) - (v_{we}^2 - u_e^2 / u_j^2) \right]^{1/2}}.
\]

The term \( u_e/u_j \) in the denominator indicates that the resonant frequency increases as the shear speed, \( u_j - u_e \), declines. In the limit

\[
\frac{u_j - u_e}{1 - u_j u_e/c^2} \rightarrow \frac{v_{ij} + v_{we}}{1 + v_{ij} v_{we}/c^2},
\]

the resonant frequency \( \omega_{nm} R/v_{we} \rightarrow \infty \).

The resonant wavelength is given by \( \lambda \approx \lambda_{nm} = 2\pi v_w^*/\omega_{nm} \) and can be calculated from

\[
\lambda_{nm} = \frac{2\pi}{(2n + 1)/4 + m\pi} \left( \frac{\gamma_e}{v_{we}} \right) \left\{ \left( v_{w}^* - u_e \right)^2 - \left[ v_{we} - (v_{we} u_e/c^2) v_{w}^* \right]^2 \right\}^{1/2} R.
\]

(17)
Equations (15)–(17) provide the proper functional dependence of the resonant wave speed, frequency, and wavelength provided \((u_j/u_e)^2 \ll 1\) and \((v_{we}/u_j)^2 \ll 1\).

With the exception of the \(n = 0, m = 0\), fundamental pinch mode, a maximum spatial growth rate, \(k_{j,max}^m\), is approximated by

\[
k_{j}^{\text{max}} R \approx k_j^{\text{max}} R \equiv \frac{1}{2} \frac{v_{we}}{\gamma_j u_j} \ln|R|, \tag{18}
\]

where

\[
|R| \approx \left[\frac{4(\omega_{mn}^* R/v_{we})^2(1 - 2u_e/u_j)}{(\ln|R|/2)^2} + (\ln|R|/2)^2\right]^{1/2}. \tag{19}
\]

Equations (18) and (19) show that the maximum growth rate is primarily a function of the jet sound, Alfven, and flow speed through \(v_{we}/\gamma_j u_j\), and secondarily a function of the shear sound, Alfven, and flow speed through \((\omega_{mn}^* R/v_{we})^2(1 - 2u_e/u_j)\).

I can illustrate the dependencies of the maximum growth rate on sound, Alfven, and flow speeds by using

\[
\left(\frac{\omega_{mn}^* R}{v_{we}}\right)^2 (1 - 2u_e/u_j) \approx \frac{(1 - 2u_e/u_j)(2n + 1)\pi/4 + m\pi}{1 - 2(u_e/u_j)(1 - v_{we}^2/c^2) - (v_{we}^2 - u_e^2)/u_j^2},
\]

and if, say, \(u_e = 0\), then

\[
\left(\frac{\omega_{mn}^* R}{v_{we}}\right)^2 \ln|R| \approx 4\left(\frac{1 - v_{we}^2/u_j^2}{1 - v_{we}^2/u_e^2}\right)^{1/2} \frac{(2n + 1)\pi}{4 + m\pi}. \tag{20}
\]

Thus, \(|R|\) increases as \(\omega_{mn}^*\) increases for higher order modes with larger \(n\) and larger \(m\), and this result indicates an increase in the maximum growth rate for larger \(n\) and larger \(m\). When the sound or Alfven wave speed, \(v_{we}\), increases, \(|R|\) increases. This result indicates an increase in the maximum growth rate at the higher resonant frequency accompanying an increase in the sound or Alfven wave speed in the sheath.

The behavior of the maximum growth rate as the shear speed, \(u_j - u_e\), declines is best illustrated by considering the effect of an increasing wind speed where \((v_{we}^2 - u_j^2)/u_j^2 \ll 1\) is ignored. In this case,

\[
\left(\frac{\omega_{mn}^* R}{v_{we}}\right)^2 \ln|R| \approx 4\left(\frac{2n + 1)\pi}{4 + m\pi}, \tag{21}
\]

and \(|R|\) will remain relatively independent of \(\omega_{mn}^*\) even as \(\omega_{mn}^* \rightarrow \infty\) as the shear speed decreases. This result indicates a relatively constant maximum growth rate as the shear speed decreases.

In the fluid limit, decline in the shear speed ultimately results in a decrease in the growth rate and an increase in the spatial growth length. This decline in the growth rate is also indicated by equation (8), which, in the fluid limit, becomes

\[
\frac{\omega}{k} = \frac{\eta jy_j + u_e}{1 + \eta} \pm i \frac{\eta}{1 + \eta} (u_j - u_e). \tag{22}
\]

Equation (22) applies to frequencies below the resonant frequency \(\omega_{mn}^*\) and directly reveals the decline in growth rates as \((u_j - u_e) \rightarrow 0\).

In the magnetic limit, the resonant frequency \(\omega_{mn}^* R/v_{Ac} \rightarrow \infty\) as

\[
\frac{u_j - u_e}{1 - u_e^2/c^2} \rightarrow \frac{v_{Aj} + v_{Ac}}{1 + v_{Aj} v_{Ac}/c^2}. \tag{23}
\]

Here equation (8) indicates that the jet is stable when

\[
\gamma_j^2 \gamma_e^2 (u_j - u_e)^2 < V_A^2/s, \tag{24}
\]

and the jet will be stable as \(\omega_{mn}^* \rightarrow \infty\) when

\[
\gamma_j^2 \gamma_e^2 \left(1 - \frac{u_j u_e}{c^2} \right) < 2 \gamma_j^2 \gamma_e^2 \frac{v_{Ac}^2 + v_{Aj}^2}{(v_{Aj} + v_{Ac})^2} \left(1 + \frac{v_{Aj} v_{Ac}/c^2}{c^2}\right), \tag{25}
\]

where I have used an equality in equation (23) in equation (8) to obtain equation (24). Equation (24) indicates that a high jet speed relative to the Alfven wave speed is necessary for instability. For example, if \(v_p = v_{Aj} = v_{Ac}\) and \(u_e = 0\), the jet is stable at high frequencies provided

\[
\gamma_j^2 < (1 + v_p^2/c^2)^2 \gamma_e^4. \tag{25a}
\]

This high-frequency condition is slightly different from the low-frequency stabilization condition found when \(v_p = v_{Aj} = v_{Ac}\) and \(u_e = 0\) from equation (11b),

\[
\gamma_j^2 (u_j/c)^2 < 4 \gamma_e^2 (\gamma_e^2 - 1). \tag{25b}
\]

Note that equations (25a) and (25b) are identical in the large Lorentz factor limit. Equations (25a) and (25b) predict that stabilization at high frequencies occurs at somewhat higher jet speeds than stabilization at lower frequencies. Determination of stabilization at intermediate frequencies requires numerical solution of the dispersion relation. A nonnegligible positive external flow requires even higher jet speeds for the jet to be unstable. Thus, a strongly magnetized relativistic trans-Alfvenic jet is predicted to be KH stable, and a super-Alfvenic jet can be KH stable.

3.3. High-Frequency Limit

Provided the condition for resonance (eq. [14]) is met, the real part of the solutions to the dispersion relation in the high-frequency limit for fundamental, surface, and body modes is given by

\[
\frac{\omega}{k} \approx \frac{u_j \pm v_{we}}{1 \pm v_{we} u_j/c^2} \tag{26}
\]

and describes sound waves \(v_{we} = \eta_j\) or Alfven waves \(v_{we} = \eta_{Aj}\) propagating with and against the jet flow inside the jet. Unstable growing solutions are associated with the backward-moving (in the jet fluid reference frame) wave, but the growth rate is vanishingly small in the high-frequency limit.

4. NUMERICAL SOLUTION

OF THE DISPERSION RELATION

The behavior of solutions within an order of magnitude of the resonant frequency and for comparable sound and Alfven wave speeds can be investigated by numerical solution of the dispersion
In this section the basic behavior of the pinch fundamental (F), helical and elliptical surface (S) modes is investigated (1) as a function of varying sound speed in the external sheath or jet spine for a fixed sound speed in the jet spine or external sheath and no sheath flow, (2) as a function of equal sound speeds in the jet spine and external sheath for no sheath flow, and (3) as a function of sheath flow for a relatively high sound speed equal in the jet spine and external sheath. In general, only growing solutions are shown, and complexities associated with multiple crossing solutions are not shown. For all solutions shown the jet spine Lorentz factor and speed are set to $\gamma = 2.5$ and $u_j = 0.9165 c$, respectively. Sound speeds are input directly, with the only constant being the sheath number density. Total pressure and spine density are quantities computed for the specified sound speeds. The adiabatic index is chosen to be $\Gamma = 13/9$ when $0.1 \leq \gamma_{s}\,c \leq 0.5$, which is consistent with relativistically hot electrons and cold protons (Synge 1957). For sound speeds $\gamma_{s}\,c \sim c/\sqrt{3}$, the adiabatic index is set to $\Gamma = 4/3$. Solutions shown assume zero magnetic field. Test calculations with magnetic fields giving magnetic pressures of a few percent of the gas pressure and Alfvén wave speeds an order of magnitude less than the sound speeds give almost identical results.

In Figure 1 solutions in the left column are for a fixed jet spine sound speed $a_j = 0.3 c$, and in the right column are for a fixed external sheath sound speed $a_e = 0.3 c$. The solutions shown in Figure 1 confirm the accuracy of the low-frequency solutions to the pinch fundamental mode (eqs. [5] and [6]), and the helical and elliptical surface modes (eq. [8]). Note that fast or slow wave speeds are possible at low frequencies, depending on whether $\eta \simeq (\gamma_{s}\,a_e/\gamma_{e}\,a_j)^2$ in equation (8) is much greater or much less than 1, respectively. The numerical solutions to the dispersion relation show that the maximum growth rate is primarily a function of the jet spine sound speed and only secondarily a function of the external sheath sound speed, as indicated by equations (18)–(20). Where a distinct supersonic resonance exists, the resonant frequency is primarily a function of the external sheath sound speed, as predicted from equation (16). The analytical expression for the resonant frequency for the helical and elliptical surface modes provides the correct functional variation to within a constant multiplier provided $a_e \leq c/\sqrt{3}$ and $a_j < c/3$. A dramatic increase in the resonant frequency and modest increase in the growth rate for larger jet spine sound speeds indicates the transition to transonic behavior. Equation (15) for the resonant wave speed and equation (17) for the resonant wavelength also provide a reasonable approximation to the functional variations provided $a_e \leq c/\sqrt{3}$ and $a_j < c/3$. These results confirm the resonant solutions found in $\S$ 3.2. At frequencies more than an order of magnitude above resonance, the growth rate is greatly reduced, and solutions approach the high-frequency limiting form given by equation (26). Note that equation (26) allows only relatively high wave speeds at high frequencies because $a_j \leq c/\sqrt{3}$.

In Figure 2 the behavior of solutions to the fundamental/surface (left) and associated first body mode (right) shows how solutions change as the sound speed increases in both the jet spine and external sheath. Here I illustrate the transition from supersonic to transonic behavior for no flow in the sheath. At low frequencies the modes behave as predicted by the analytic solutions given in...
The solutions show the expected shift to a higher resonant frequency, which is primarily a function of the increased external sheath sound speed, and an accompanying increase in the resonant growth rate, which is primarily a function of the increased jet spine sound speed. The resonance disappears as sound speeds approach $c/\sqrt{3}$ and the jet becomes transonic, as predicted by the resonance condition in § 3.2.

In the transonic regime high-frequency fundamental/surface mode growth rates and wave speeds are identical, with wave speeds given by equation (26). Provided the jet is sufficiently supersonic, i.e., $a_{j,e} < 0.5 \, c$, the maximum growth rate of the first body mode is greater than that of the pinch fundamental mode, is comparable to that of the helical surface mode, and is less than that of the elliptical surface mode. A narrow damping peak shown for the helical first body (B1) solution when $a_{j,e} = 0.4 \, c$ is indicative of complexities in the body-mode solution structure. In the transonic regime growth of the first body mode is less than that of the pinch fundamental, helical surface, and elliptical surface modes.

Figure 3 illustrates the behavior of fundamental/surface and first body modes as a function of the sheath speed for equal sound speeds in spine and sheath of $a_{j,e} = 0.4 \, c$. For this value of the sound speeds, a sheath speed $u_s = 0$ provides a supersonic solution structure baseline. At low frequencies the surface modes behave as predicted by equation (8), and the wave speed rises as $u_s$ increases. As $u_s$ increases, the resonant frequency increases in accordance with equation (16). On the other hand, the growth rate at resonance does not vary significantly, in accordance with equations (18) and (19). When the sheath speed exceeds the sound speed, solutions make a transition from a supersonic to a transonic structure. Note that the transition point between supersonic and transonic behavior is similar but not identical for the helical and elliptical surface modes, i.e., occurs at a slightly lower sheath speed for the elliptical mode. The first body modes also show an increase in resonant frequency with little change in the maximum growth rate provided the sheath speed remains below the sound speed. A significant damping feature in the helical
first body (B1) panel is found. While a similar damping feature was not found for the pinch and elliptical first body mode, this does not indicate a significant difference, since the root finding technique does not find all structure associated with the body modes. The body mode solution structure is complex, with multiple solutions not shown here, and modest damping or growth can occur where solutions cross (e.g., Mizuno et al. 2007). When the sheath speed exceeds the sound speed, the maximum body mode growth rate declines significantly. This result is quite different from the transonic solution behavior illustrated in Figure 2 when \( a_j / c = 0.577 \) for no sheath flow. Thus, sheath flow affects stability in the relativistic jet, beyond that accompanying an increase to the maximum sound speed in the absence of sheath flow. The reduction in growth of the body modes in the presence of sheath flow provides the relativistic jet equivalent of nonrelativistic transonic/subsonic jet–solution behavior. At higher frequencies, wave speeds are identical and are given by equation (26). Note that high-frequency wave speeds are nearly independent of \( u_e \).

### 4.2. Magnetic Limit

In this subsection the basic behavior of pinch, helical, and elliptical modes is investigated (1) as a function of varying Alfven speed in the external sheath or jet spine for a fixed Alfven speed in the jet spine or external sheath and no sheath flow, (2) as a function of equal Alfven speeds in the jet spine and external sheath for no sheath flow, and (3) as a function of sheath speed for a relatively high Alfven speed equal in the jet spine and external sheath. In general, only growing solutions are shown, and complexities associated with multiple crossing solutions are not shown. For all solutions shown, the jet spine Lorentz factor and speed are set to \( \gamma = 2.5 \) and \( u_j = 0.9165 \) respectively. Alfven speeds are 2 orders of magnitudes larger than the sound speed and are determined by varying the sound speeds, but with a gas pressure fraction of order 0.01% of the total pressure. Only the sheath number density is held constant. The adiabatic index is set to \( \Gamma = 5/3 \) when \( a_j / c \ll 0.1 \), which is consistent with low gas pressures and temperatures.

The solutions shown in Figure 4 confirm the theoretical predictions in the magnetic limit, with behavior depending on the Alfven speed, like the behavior found for the sound speed (see Fig. 1). The pinch fundamental mode (not shown) has a growth rate almost entirely dependent on sound speeds, and is negligible in the magnetic limit, as predicted by equation (6). In Figure 4 solutions in the left column are for a fixed jet spine Alfven speed \( v_{Aj} = 0.3 \) c, and in the right column are for a fixed external sheath Alfven speed \( v_{Ae} = 0.3 \) c. The solutions shown confirm the accuracy of the low-frequency solutions for helical and elliptical surface modes given by equation (8). Note that low-frequency wave speeds can be high or low depending on the values of \( \eta = \gamma W_j / \gamma W_e, V_{Ae} / \gamma \), and \( V_{Aj} / \gamma \). The numerical solutions to the dispersion relation show that the maximum growth rate is primarily a function of the jet spine Alfven speed and only secondarily a function of the external sheath Alfven speed, as predicted by equations (18)–(20). The resonant frequency is primarily a function of the external sheath Alfven speed, as predicted by equation (16). The analytical expression for the resonant frequency of the helical and elliptical surface modes provides the correct functional variation to within a constant multiplier, provided \( v_{ Aj,e } < 0.5 \) c. The decrease in the growth rate for jet shear Alfven speeds \( v_{Ae} > 0.5 \) c indicates the transition toward trans-Alfvenic behavior. Equation (15) for the resonant wave speed and equation (17) for the resonant wavelength also provide a reasonable approximation to the functional variations for \( v_{ Aj,e } < 0.5 \) c. At frequencies more than an order of magnitude above resonance the growth rate is greatly reduced, and solutions approach the high-frequency limiting form given by equation (26). The surface modes have relatively slow wave speeds, \( \gamma_{ht} v_{ht} / c < 1 \) at high frequencies when the Alfven wave speed \( v_{Aj} > 0.5 \) c. Unlike the fluid case, the helical and elliptical surface modes are stabilized for Alfven speeds somewhat in excess of \( v_{Aj} \) \( \approx 0.8 \) c, in accordance with equations (8) and (24).

In Figure 5 the behavior of solutions to the pinch fundamental mode is shown, together with the helical and elliptical surface (left) and associated first body modes (right); the figure illustrates how solutions change as the Alfven speed increases in both the jet spine and external sheath. The sound speed is \( a_j / c = 0.2 \) c for the pinch fundamental mode, in order to illustrate the mode behavior with increasing Alfven speed. Sound speeds for all body modes and for helical and elliptical surface modes are \( a_j / c \approx 0.01 v_{Aj} \). Here the transition from super-Alfvenic to trans-Alfvenic behavior for no flow in the sheath is illustrated. At low frequencies the modes behave as predicted by the analytic solutions given in § 3.1. The growth rate of the pinch fundamental mode is reduced as the Alfven speed increases, as predicted by equation (6). The surface and body mode solutions show the expected shift to a higher resonant frequency, which is primarily a function of the increased sheath Alfven speed, and an accompanying increase in the resonant growth rate, which is primarily a function of the increased spine Alfven speed. The resonance moves to higher frequency, but the maximum growth rate is reduced for Alfven speeds \( v_{Aj,e} > 0.6 \) c, and all modes become stable at higher Alfven speeds, in accordance with equations (8) and (24). At high frequencies, wave speeds are given by equation (26).
Provided the jet is sufficiently super-Alfvenic, i.e., $v_{\text{Aj},e} < 0.6 \, c$, the maximum growth rate of the first body mode is much greater than that of the pinch fundamental mode, is comparable to that of the helical surface mode, and is less than that of the elliptical surface mode. A narrow damping peak shown for the elliptical body mode ($B_1$) solution when $v_{\text{Aj},e} = 0.6 \, c$, indicated by the arrow, reveals complexities in the body mode solution structure.

Figure 6 illustrates the behavior of fundamental/surface and first body modes as a function of the sheath speed for equal sound speeds in jet and sheath for different sheath flow speeds. The pinch fundamental mode sound speed is $a_{j,e} = 0.2 \, c$. Sound speeds for all other cases are $a_{j,e} \sim 0.01 v_{\text{Aj},e}$. Real and imaginary parts of the wavenumber as a function of angular frequency are shown as in Fig. 4. Locations of the maximum growth rate are indicated by the vertical solid lines. A vertical arrow (elliptical $B_1$) indicates a narrow damping feature. The lower panel shows the relativistic wave speed, $c_1 c_w/v_w$. Line colors indicate the Alfven speed in units of $c$: 0.10 (black), 0.20 (blue), 0.40 (green), and 0.60 (red).

Fig. 5.—Solutions to the dispersion relation for pinch fundamental, helical surface, elliptical surface (left), and the first body (right) modes are shown for equal sound speeds in jet and sheath and no sheath flow. The pinch fundamental mode sound speed is $a_{j,e} = 0.2 \, c$. Sound speeds for all other cases are $a_{j,e} \sim 0.01 v_{\text{Aj},e}$. Real and imaginary parts of the wavenumber as a function of angular frequency are shown as in Fig. 4. Locations of the maximum growth rate are indicated by the vertical solid lines. A vertical arrow (elliptical $B_1$) indicates a narrow damping feature. The lower panel shows the relativistic wave speed, $c_1 c_w/v_w$. Line colors indicate the Alfven speed in units of $c$: 0.10 (black), 0.20 (blue), 0.40 (green), and 0.60 (red).

Fig. 6.—Solutions to the dispersion relation for pinch fundamental, helical surface, elliptical surface (left) and the first body (right) modes are shown for equal sound speeds in jet and sheath for different sheath flow speeds. The pinch fundamental mode sound speed is $a_{j,e} = 0.2 \, c$. Sound speeds for all other cases are $a_{j,e} \sim 0.01 v_{\text{Aj},e} = 0.005 \, c$. Real and imaginary parts of the wavenumber as a function of angular frequency are shown as in Fig. 4. Locations of the maximum growth rate are indicated by the vertical solid lines. A vertical arrow indicates low-frequency damping of the pinch $B_1$ solutions. The lower panel shows the relativistic wave speed, $c_1 c_w/v_w$. Line colors indicate the sheath speed in units of $c$: 0.0 (black), 0.20 (blue), 0.30 (cyan), 0.40 (green), and 0.60 (red).

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rate is insensitive to the sheath speed and remains unstable at $u_{te} = 0.6 \ c$ even when all other modes are stabilized.

4.3. A High Sound and Alfvén Speed Magnetosonic Case

In this subsection the basic behavior of the pinch fundamental, helical surface, elliptical surface, and associated first body modes is illustrated for different sheath speeds. The pinch speeds span a solution structure from supersonic to transonic but still super-Alfvénic flow. Here the sound speeds in the jet spine and external sheath are set equal with $a_{je} = 0.577 \ c$, and Alfvén speeds are set equal with $v_{Ae} = 0.5 \ c$. The solutions for this case are shown in Figure 7. With no sheath flow, the fundamental/surface and first body modes show a typical supersonic and super-Alfvénic structure, albeit the pinch fundamental mode now has a maximum growth rate comparable to the helical and elliptical surface modes as a consequence of the high sound speed. The associated first body modes also have maximum growth rates comparable to the fundamental/surface modes. An increase in the sheath speed results in a decrease in the growth rate of the helical and elliptical surface modes at low frequencies, as predicted by equation (8). The low-frequency growth rate of the pinch fundamental also declines with increasing sheath speed. The resonant frequency increases with increasing sheath speed, as expected from the analytical and numerical studies performed in the fluid and magnetic limits, and the fundamental/surface modes take on a transonic structure for sheath speeds $0.4 \ c \geq u_{te} \geq 0.1 \ c$.

At high frequencies the fundamental/surface modes exhibit very high growth rates, provided the sheath flow remains below the Alfvén speed. On the other hand, the maximum growth rate of the first body modes declines as the sheath speed increases and is reduced severely when $u_{te} > 0.1 \ c$. This behavior is similar to what is found for nonrelativistic jets as flow enters the transonic and super-Alfvénic regime (Hardee & Rosen 1999). Additional increase in the sheath flow speed to $u_{te} > 0.4 \ c$ results in a decrease in the growth rate of the fundamental/surface modes. Solutions for the helical and elliptical surface modes shown in Figure 7 for a sheath speed $u_{te} = 0.5 \ c$ equal to the Alfvén speed illustrate some of the complexity associated with barely super-Alfvénic flow. Here limited growth is associated with both the slow and fast helical and elliptical surface solution pair. At slower sheath speeds in the super-Alfvénic regime, growth is associated with the slow surface solution, i.e., backward moving in the jet fluid reference frame. The yellow dash-dotted line extension at higher frequencies in the helical and elliptical surface panels indicates a damped solution. Solutions were very difficult to follow in this parameter regime, and it is possible that some solutions were not found. When the sheath speed $u_{te} > 0.5 \ c$ all modes are stabilized.

A choice of Alfvén speeds greater than sound speeds results in a more magnetic-like solution structure, such as that shown in § 4.2. A choice of Alfvén speeds more than a factor of 2 less than the sound speeds produces a more fluid-like solution structure, such as that shown in § 4.1. The more complicated solution structure illustrated in Figure 7 only occurs for a relatively narrow range of high sound speeds with similar or slightly lower Alfvén speeds. In general, the detailed solution structure for situations in which sound and Alfvén speeds are comparable must be examined individually (e.g., Mizuno et al. 2007); further investigation of these cases is beyond the scope of the present paper.

5. SUMMARY

The analytical and numerical work performed here provides for the first time a detailed analysis of the KH stability properties of a RMHD jet spine-sheath configuration that allows for relativistic motions of the sheath, sound speeds up to $c/\sqrt{3}$, and, by keeping the displacement current in the analysis, Alfvén wave speeds approaching light speed and large Alfvén Lorentz factors. In the fluid limit, the present results confirm an earlier, more restricted low-frequency analytical and numerical simulation study performed by Hardee & Hughes (2003). Provided the jet spine is supersonic and super-Alfvénic internally and also relative to the sheath, the helical, elliptical, and higher order surface modes and the pinch, helical, elliptical and higher order first body modes have a maximum growth rate at a resonant frequency. The pinch fundamental growth rate is significant only when the sound speeds $a_{je} \sim c/\sqrt{3}$. In general, the first body mode maximum growth rate is greater than the pinch fundamental mode, slightly greater than the helical surface mode, slightly less than the elliptical surface mode, and occurs at a higher frequency than the maximum growth rate for the fundamental/surface mode.
The basic KH stability behavior as a function of spine-sheath parameters is indicated by the analytic low-frequency surface mode solution and by the behavior of the resonant frequency. The analytic surface mode solution valid at frequencies below resonance is given by

$$\frac{\omega}{k} = \frac{\omega_c}{k} \pm i \frac{\omega_i}{k}$$

$$= \left( \frac{\eta u_j + u_e}{u_j - u_e} \pm i \sqrt{ \frac{1}{2} \left( u_j - u_e \right)^2 - \frac{V_A^2}{\gamma_e^2 \gamma_i^2} } \right),$$  

(27)

where

$$V_A^2 \equiv \left( \frac{\gamma_e^2 W_j + \gamma_i^2 W_e}{B_j^2 + B_e^2} \frac{B_j^2 + B_e^2}{4\pi W_j W_e} \right),$$  

(28)

and \( \eta \equiv \gamma_j^2 W_j/\gamma_i^2 W_e, V \equiv B^2/4\pi W, \) \( W \equiv \rho + |\Gamma/(\Gamma - 1)| P/c^2, \) \( \) and \( \gamma_A \equiv (1 - \gamma_i^2/c^2)^{1/2}. \) Equation (27) provides a temporal growth rate, \( \omega(k) \), and a wave speed, \( v_w = \omega_i/k. \) The reciprocal provides a spatial growth rate \( k(\omega) \) and growth length \( l = k_j^{-1}. \) Increase or decrease of the growth rate, dependence on physical parameters, and stabilization at frequencies/wavenumbers below resonance is directly revealed by \( \omega_i \) in equation (27). Note that higher jet Lorentz factors reduce \( \omega_i \) through the dependence on \( \eta. \)

The resonant frequency is

$$\omega^* \propto \frac{v_w}{\left[ 1 - u_c^2/v_w^2 \right]^{1/2} - \left( v_w/v_e - u_e/v_e \right)^2/c^2}^{1/2},$$  

(29)

where \( v_w \) is the wave speed at resonance, equation (15). The resonant frequency increases as the sheath sound or Alfvén wave speed, \( v_w \equiv (a_s, v_{we}) \), increases, and \( \omega^* \to \infty \) when the denominator decreases to zero as

$$\frac{u_j - u_e}{1 - u_j u_e/c^2} = \frac{v_{ij}^* + v_w}{1 + v_{ij} v_w/c^2},$$

where \( v_{ij} \equiv (a_s, v_{A,i}, v_{A,j}) \) in the fluid and magnetic limits, respectively. Since equation (27) applies below resonance, the overall behavior of the growth rate is indicated by \( \omega_i. \) Thus, growth rates decline to zero as \( (u_j - u_e)^2 - V_A^2/\gamma_i^2 \gamma_j^2 \to 0. \) The numerical analysis of the dispersion relation shows that the pinch fundamental and all first body modes are comparably or more readily stabilized, and thus the jet is KH stable when

$$\left( u_j - u_e \right)^2 - V_A^2/\gamma_i^2 \gamma_j^2 < 0.$$  

(30)

This stability condition takes on a particularly simple form when conditions in the spine and sheath are equal, i.e., \( B_e = B_j, \) \( W_e = W_j, \) so that \( v_{A,j} = v_{A,e}, \) and with \( \gamma_A \equiv \gamma_{A,e} = \gamma_{A,j} \)

$$\gamma_j^2 \gamma_i^2 \left( u_j - u_e \right)^2 < 4 \gamma_A^2 \left( \gamma_j^2 - 1 \right)^2$$

(31)

indicates stability. This result implies that a trans-Alfvénic relativistic jet with \( \gamma_j u_j \approx \gamma_A v_A \) will be KH stable and that even a super-Alfvénic jet with \( \gamma_j \gg \gamma_A \) can be KH stable.

6. DISCUSSION

Formally, the present results and expressions apply only to magnetic fields parallel to an axial spine-sheath flow in which conditions within the spine and the sheath are independent of radius and the sheath extends to infinity. A rapid decline in perturbation amplitudes in the sheath as a function of radius, governed by the Hankel function in the dispersion relation, suggests that the present results will apply to sheaths more than about 3 times the spine radius in thickness.

The relativistic jet is transonic in the absence of sheath flow only for spine and sheath sound speeds \( \sim c/\sqrt{3}. \) Only in this regime does the pinch fundamental have a significant growth rate, and in general, we do not expect the pinch fundamental to grow significantly on relativistic jets. On the other hand, the pinch first body mode can have a significant maximum growth rate and would dominate any axisymmetric structure. The elliptical and higher order surface modes have increasingly larger maximum growth rates at resonant frequencies higher than the helical surface mode, and the maximum first body mode growth rates for helical and elliptical modes are comparable to that of the surface modes. Nevertheless, we expect the helical surface mode to achieve the largest amplitudes in the nonlinear limit as a result of the reduced saturation amplitudes that accompany the higher resonant frequency and shorter resonant wavelengths associated with the higher order surface modes and all body modes.

In astrophysical jets we expect a toroidal magnetic field component, and possibly an ordered helical structure and accompanying flow helicity. Jet rotation (e.g., Bodo et al. 1996) or a radial velocity profile (e.g., Birkinshaw 1991) will modify the present results but will not stabilize the helical mode. Two-dimensional nonrelativistic slab jet theoretical results indicate that KH stabilization occurs when the velocity shear projected on the wavevector is less than the projected Alfvén speed (Hardnee et al. 1992). In the work presented here magnetic and flow field are parallel and project equally on the wavevector, which for the helical \( (n = 1) \) and elliptical \( (n = 2) \) modes lies at an angle \( \theta = \tan^{-1} (n/kR) \) relative to the jet axis. Provided magnetic and flow helicity and radial gradients in jet spine/sheath properties are not too large, we expect the present results to remain valid where \( u_{ij} \) and \( B_{ij} \) refer to the poloidal velocity and field components.

KH-driven normal mode structures move at less than the jet speed. The fundamental pinch mode moves backward in the jet frame at about the sound speed nearly independent of the sheath properties and thus moves at nearly the jet speed in the source/observer frame. Low-frequency and long-wavelength helical and higher order surface modes are advected with wave speed indicated by equation (27) and move slowly in the source/observer frame for light, i.e., \( \eta \equiv \gamma_j^2 W_j/\gamma_i^2 W_e < 1, \) and/or for magnetically dominated flows. Higher frequency (above resonance) and shorter wavelength normal mode structures move backward in the jet frame at the sound/Alfvén wave speed, have a wave speed nearly independent of the sheath properties, and can move slowly in the source/observer frame only for magnetically dominated flows.

Where flow and magnetic fields are parallel, current-driven (CD) modes are stable (Isotmin & Pariev 1994, 1996). Where magnetic and flow fields are helical, CD modes can be unstable (Lyubarskii 1999) in addition to the KH modes. CD and KH instability are expected to produce helically twisted structure. However, the conditions for instability, the radial structure, the growth rate, and the pattern motions are different. For example, KH modes grow more rapidly when the magnetic field is force-free (e.g., Appl 1996), and nonrelativistic simulation work (e.g., Lery et al. 2000; Baty & Keppens 2002; Nakamura & Meier 2004) indicates that CD-driven structure is internal to any spine-sheath interface and moves at the jet speed.

The differences between KH and CD instability can serve to identify the source of helical structure on relativistic jets and allow determination of jet properties near to the central engine. Perhaps
the observation of relatively low proper motions in the TeV BL Lac objects when intensity modeling requires high-flow Lorentz factors (Ghisellini et al. 2005) is an indication of a magnetically dominated KH-unstable spine-sheath configuration. The author acknowledges partial support through National Space Science and Technology Center (NSSTC/NASA) cooperative agreement NCC8-256 and by National Science Foundation award AST 05-06666 to the University of Alabama.

APPENDIX A

LINEARIZED RMHD EQUATIONS AND NORMAL MODE DISPERSION RELATION

In vector notation the relativistic MHD continuity equation, energy equation, and momentum equation can be written as

$$\frac{\partial}{\partial t} (\gamma \rho) + \nabla \cdot (\gamma \rho \mathbf{v}) = 0, \quad (A1)$$

$$\frac{\partial}{\partial t} \left[ \gamma^2 W - \frac{P}{c^2} + \frac{B^2}{8\pi c^2} \left(1 + \frac{\mathbf{v} \cdot \mathbf{B}}{c^2}\right)^2 \right] + \nabla \cdot \left[ \gamma^2 W \mathbf{v} + \frac{B^2}{4\pi c^2} \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \frac{\mathbf{B}}{4\pi c^2} \right] = 0, \quad (A2)$$

$$\gamma^2 W \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) = -\nabla P - \frac{\mathbf{v} \cdot \mathbf{E}}{c^2} \rho + \rho_0 \mathbf{E} + \frac{\mathbf{j} \times \mathbf{B}}{4\pi}. \quad (A3)$$

These equations along with Maxwell’s equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 4\pi \rho_q, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi}{c} \mathbf{j}, \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B},$$

and assuming ideal MHD with comoving electric field equal to zero

$$\mathbf{E} = -\frac{\mathbf{v} \times \mathbf{B}}{c},$$

provide the complete set of ideal RMHD equations. In the above $W \equiv \rho + [\gamma/(\gamma - 1)]P/c^2$ is the enthalpy, the Lorentz factor $\gamma = (1 - \mathbf{v} \cdot \mathbf{v}/c^2)^{-1/2}$, and $\rho$ is the proper density. The condition for isentropic flow is given by

$$\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \left( \frac{P}{\rho^2} \right) = 0.$$

The general approach to analyzing the time-dependent properties of this system is to linearize the ideal RMHD equations, where the density, velocity, pressure, and magnetic field are written as $\rho = \rho_0 + \rho_1$, $\mathbf{v} = \mathbf{u} + \mathbf{v}_1$ (we use $\rho_0 \equiv \mathbf{u}$ for notational reasons), $P = P_0 + P_1$, $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$, and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, where subscript 1 refers to a perturbation to the equilibrium quantity with subscript 0. In addition, $W = W_0 + W_1$, $\gamma = (\gamma_0 + \gamma_1)^2 \approx \gamma_0 + 2\gamma_0^3 \mathbf{u} \cdot \mathbf{v}_1/c^2$, and $\gamma_1 \approx \gamma_0^2 \mathbf{u} \cdot \mathbf{v}_1/c^2$. It is assumed that the initial equilibrium system satisfies the zeroth-order equations. The linearized continuity, energy, and momentum equations become

$$\frac{\partial}{\partial t} (\gamma_0 \rho_1 + \gamma_1 \rho_0) + \nabla \cdot [(\gamma_0 \rho_1 \mathbf{u} + \gamma_0 \rho_0 \mathbf{v}_1 + \gamma_1 \rho_0 \mathbf{u})] = 0, \quad (A4)$$

$$\frac{\partial}{\partial t} \left[ \gamma_0^2 W_1 - P_1/c^2 + 2\gamma_0^4 (\mathbf{u} \cdot \mathbf{v}_1/c^2) W_0 \right] + \nabla \cdot \left[ \gamma_0^2 W_1 \mathbf{u} + 2\gamma_0^4 (\mathbf{u} \cdot \mathbf{v}_1/c^2) W_0 \mathbf{u} + \gamma_0^2 W_0 \mathbf{v}_1 \right]
+ \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left[ B_0^2 (\mathbf{u} \cdot \mathbf{v}_1/c^2) + (1 + \mathbf{v}_1^2/c^2) \mathbf{B}_0 \cdot \mathbf{B}_1 - (\mathbf{u} \cdot \mathbf{B}_1/c + \mathbf{v}_1 \cdot \mathbf{B}_0/c) \mathbf{u} \cdot \mathbf{B}_1/c \right]
+ \frac{1}{4\pi c^2} \nabla \cdot \left( 2(B_0 \cdot \mathbf{B}_1) \mathbf{u} + B_0^2 \mathbf{v}_1 - (\mathbf{u} \cdot \mathbf{B}_0) \mathbf{B}_1 - (\mathbf{u} \cdot \mathbf{B}_1) \mathbf{B}_0 - (\mathbf{v}_1 \cdot \mathbf{B}_0) \mathbf{B}_0 \right) = 0, \quad (A5)$$

$$\gamma_0^2 W_0 \left( \frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v}_1 \right) = -\nabla P_1 - \frac{\mathbf{u} \cdot \mathbf{P}_1}{c^2} \frac{\partial P_1}{\partial t} + \mathbf{B}_1 \times \mathbf{P}_1/c. \quad (A6)$$

The linearized Maxwell equations become

$$\nabla \cdot \mathbf{B}_1 = 0, \quad \nabla \cdot \mathbf{E}_1 = 4\pi \rho_{q1}, \quad \nabla \times \mathbf{B}_1 = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E}_1 + \frac{4\pi}{c} \mathbf{j}_1, \quad \nabla \times \mathbf{E}_1 = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_1,$$

where I keep the displacement current in order to allow for strong magnetic fields and Alfvén wave speeds comparable to light speed. Under the assumption of ideal MHD, the equilibrium charge density $\rho_{e0} = 0$, the electric field

$$\mathbf{E}_1 = -\frac{\mathbf{u} \times \mathbf{B}_1 + \mathbf{v}_1 \times \mathbf{B}_0}{c}$$
is first order, the charge density \( \rho_{q1} = (\nabla \cdot E_1)/4\pi \) is also first order, and the electrostatic force term, \( \rho_{q1}E_1 \), is second order and dropped from the linearized momentum equation. The condition for isentropic perturbations becomes

\[
P_1 = \partial^2 \rho_1 = \left( \Gamma \frac{P_0}{\rho_0} \right) \rho_1.
\]

This basic set of linearized RMHD equations is similar to those found in Begelman (1998), but allows a relativistic zeroth-order velocity, i.e., \( \mathbf{v} = \mathbf{u} + \mathbf{v}_1 \) and \( \mathbf{u} \ll c \), whereas Begelman allowed only for relativistic first-order motions, \( \mathbf{v}_1 \).

In what follows let us model a jet as a cylinder of radius \( R \), having a uniform proper density, \( \rho_p \), a uniform axial magnetic field, \( B_j = B_{z,j} \), and a uniform velocity, \( u_j = u_{z,j} \). The external medium is assumed to have a uniform proper density, \( \rho_e \), a uniform axial magnetic field, \( B_e = B_{z,e} \), and a uniform velocity, \( u_e = u_{z,e} \). An external velocity could be the result of a wind or sheath outflow around a central jet, \( u_e > 0 \), or could represent backflow, \( u_e < 0 \), in a cocoon surrounding the jet. The jet is established in static total pressure balance with the external medium where the total static uniform pressure is \( P^*_e = P_e + B_z^2/8\pi = P^0_e \equiv P_j + B_j^2/8\pi \). Under these assumptions the linearized continuity and energy equations and the components of the momentum equation become

\[
\frac{\partial}{\partial t} \left( \gamma_0 \rho_1 + \gamma_1 \rho_0 \right) + u \frac{\partial}{\partial z} \left( \gamma_0 \rho_1 + \gamma_1 \rho_0 \right) + \gamma_0 \rho_0 \nabla \cdot \mathbf{v}_1 = 0,
\]

\[
\frac{\partial}{\partial t} \left( \gamma_0^2 W_1 - \frac{P_1}{c^2} + 2 \gamma_0^4 \frac{u z v_1}{c^2} W_0 \right) + u \frac{\partial}{\partial z} \left( \gamma_0^2 W_1 + 2 \gamma_0^4 \frac{u z v_1}{c^2} W_0 \right) + \gamma_0^2 W_0 \nabla \cdot \mathbf{v}_1 = 0,
\]

and

\[
\frac{\partial^2}{\partial z^2} \mathbf{v}_1 + \frac{u}{c^2} \mathbf{v}_1 = \left( \frac{\partial}{\partial t} P_1 + \frac{B_0}{4\pi} \left( \frac{\partial}{\partial z} B_{1z} + \frac{u}{c^2} \frac{\partial}{\partial t} B_{1z} \right) \right),
\]

\[
\frac{\partial^2}{\partial z^2} \mathbf{v}_1 + \frac{u}{c^2} \mathbf{v}_1 = \left( \frac{\partial}{\partial t} P_1 + \frac{B_0}{4\pi} \left( \frac{\partial}{\partial z} B_{1z} + \frac{u}{c^2} \frac{\partial}{\partial t} B_{1z} \right) \right),
\]

\[
\frac{\partial^2}{\partial z^2} \mathbf{v}_1 + \frac{u}{c^2} \mathbf{v}_1 = \left( \frac{\partial}{\partial t} P_1 + \frac{B_0}{4\pi} \left( \frac{\partial}{\partial z} B_{1z} + \frac{u}{c^2} \frac{\partial}{\partial t} B_{1z} \right) \right),
\]

where \( V_A^2 \equiv B_0^2/(4\pi W_0) \).

In cylindrical geometry perturbations \( \rho_1, \mathbf{v}_1, P_1, \) and \( B_1 \) can be considered to consist of Fourier components of the form \( f_i(r, \phi, z, t) = f_i(r) \exp \{ i(kz \pm n \phi - \omega t) \} \), where the flow is in the \( z \)-direction and \( r \) is in the radial direction, with the jet bounded by \( r = R \). In cylindrical geometry \( n \), an integer, is the azimuthal wavenumber, and for \( n > 0 \) waves propagate at an angle to the flow direction, where \( +n \) and \( -n \) refer to wave propagation in the clockwise and counterclockwise sense, respectively, when viewed outward along the flow direction. In general, the goal is to write a differential equation for the radial dependence of the total pressure perturbation \( P^*_1 \equiv P_1 + (\mathbf{B}_1 \cdot \mathbf{B}_0)/4\pi = P^0_1(r) \exp \{ i(kz \pm n \phi - \omega t) \} \). The differential equation can be obtained from the energy equation by using the momentum equation and writing the velocity components \( v_{1r}, v_{1\phi}, v_{1z} \) in terms of \( P^*_1, u, B_0 \).

The components of the linearized momentum equation (eqs. \([A9a]-[A9c]\)), along with \( \partial B_1/\partial t = -c(\nabla \times E_1) = \nabla \times (u \times B_1) + \nabla \times (v_1 \times B_0) \), are used to provide relations between \( B_{1r} \) and \( v_{1r} \), and \( B_{1\phi} \) and \( v_{1\phi} \),

\[
\frac{\partial}{\partial t} B_{1r} + \frac{u}{c^2} \frac{\partial}{\partial z} B_{1z} = B_0 \frac{\partial}{\partial z} v_{1z} + B_0 \frac{\partial}{\partial \phi} v_{1\phi},
\]

\[
\frac{\partial}{\partial t} B_{1\phi} + \frac{u}{c^2} \frac{\partial}{\partial z} B_{1z} = B_0 \frac{\partial}{\partial z} v_{1z} + B_0 \frac{\partial}{\partial \phi} v_{1\phi},
\]

and to provide a relation between \( B_{1z}, v_{1z}, \) and \( P^*_1 \),

\[
\left[ 1 + \frac{V_A^2}{c^2} \left( \frac{1}{a^2} + \frac{\Gamma}{1 - c^2} \right) \right] \left( \frac{\partial}{\partial t} B_{1z} + \frac{u}{c^2} \frac{\partial}{\partial z} B_{1z} \right) - \frac{V_A^2}{\gamma_0^2 c^2} \frac{\partial}{\partial t} B_{1z}
\]

\[
= B_0 \frac{\partial}{\partial z} v_{1z} + 2B_0 \gamma_0^2 \frac{u}{c^2} \left( \frac{\partial}{\partial z} v_{1z} + \frac{\partial}{\partial \phi} v_{1\phi} \right) + B_0 \frac{\partial}{\partial W_0} \left( \frac{1}{a^2} + \frac{\Gamma}{1 - c^2} \right) \left( \frac{\partial}{\partial t} P^*_1 + \frac{u}{c^2} \frac{\partial}{\partial t} P^*_1 \right) - B_0 \frac{\partial}{\partial z} \frac{\partial}{\partial t} P^*_1.
\]

Equations \([A9a]-[A9c]\), along with \( (\partial/\partial t, \partial/\partial r, \partial/\partial \phi, \partial/\partial z) f_i = (-i\omega, \partial/\partial r, \pm in, +ik) f_i(r) \exp \{ i(kz \pm n \phi - \omega t) \} \), allow the velocity components to be written as

\[
\frac{\partial}{\partial t} v_{1r} = -\frac{\partial}{\partial r} P^*_1 + i \frac{B_0}{4\pi} \left( k - \omega \frac{u}{c^2} \right) B_{1r},
\]

\[
\frac{\partial}{\partial t} v_{1\phi} = -\frac{\partial}{\partial \phi} P^*_1 + i \frac{B_0}{4\pi} \left( k - \omega \frac{u}{c^2} \right) B_{1\phi},
\]

\[
\frac{\partial}{\partial t} v_{1z} = -\left( k - \omega \frac{u}{c^2} \right) P^*_1 + i \frac{B_0}{4\pi} \left( k - \omega \frac{u}{c^2} \right) B_{1z}.
\]
Thus, the perturbed magnetic field components from equations (A10a)–(A10c) become

\[ B_{r1} = \frac{kn}{ku - \omega}B_0, \]  
\[ B_{\theta 1} = \frac{kn}{ku - \omega}B_0, \]  
\[ B_{z1} = \frac{1}{W_0}(ku - \omega)\frac{a^2 - (k - \omega u/c^2)(u/c^2)(u/c^2)}{(ku - \omega) + V^2_\Lambda (ku - \omega)/a^2 - (k - \omega u/c^2)(u/c^2)}v_1 B_0, \]

where

\[ a^2 = \left( \frac{1}{\alpha^2} + \frac{1}{\Gamma - 1} \right) \frac{1}{c^2} - \frac{1}{c^2} = \frac{\Gamma P}{\rho + \left[\frac{\Gamma}{(\Gamma - 1)}\right](P/c^2)}. \]

Using equations (A12a)–(A12c) for the perturbed magnetic field components, I obtain the following relations between the perturbed velocity components \( v_1 \) and the total pressure perturbation \( P^*_1 \):

\[ v_{r1} = C_r \frac{\partial}{\partial r} P^*_1 = i \frac{1}{r} \frac{\partial}{\partial r} P_1 = i \frac{1}{X} \frac{\partial}{\partial r} P_1 = i \frac{1}{X} \frac{\partial}{\partial r} \left( \frac{ku - \omega}{\gamma_t W_0 \gamma_\Lambda^2 (ku - \omega)^2 - (k - \omega u/c^2)^2 v_\Lambda^2} \right) \frac{\partial}{\partial r} P^*_1, \]  
\[ v_{\theta 1} = C_\theta P^*_1 = \mp \frac{n}{r} \frac{P_1}{P^*} = \mp \frac{n}{r} \frac{P_1}{r} \gamma_t W_0 \gamma_\Lambda^2 (ku - \omega)^2 - (k - \omega u/c^2)^2 v_\Lambda^2 \frac{P^*_1}{P_1}, \]

\[ v_{z1} = C_z P^*_1 = - \frac{1}{\gamma_t W_0} \left( (ku - \omega)(k - \omega u/c^2) \gamma_x - v_\Lambda \right) \left[ (ku - \omega)^2 + \gamma_x v_\Lambda^2 (ku - \omega)^2 - (k - \omega u/c^2)^2 \right] P^*_1, \]

where

\[ v_\Lambda^2 = \frac{V_\Lambda^2}{1 + V_\Lambda^2/c^2} \]

is the Alfvén wave speed, \( \gamma_\Lambda^2 = (1 - v_\Lambda^2/c^2)^{-1} = 1 + V_\Lambda^2/c^2 \) is an Alfvén Lorentz factor, and note that \( V_\Lambda^2 = \gamma_\Lambda^2 v_\Lambda^2 \).

Equations (A13a)–(A13c), relating the velocity components and total pressure perturbation, can be used to write

\[ \nabla \cdot v_1 = C_r \frac{\partial^2}{\partial r^2} P^*_1 + C_r \frac{\partial}{\partial r} P^*_1 + C_\theta \frac{\partial}{\partial \theta} P^*_1 + C_Z \frac{\partial}{\partial z} P^*_1 = i \frac{1}{X} \frac{\partial}{\partial r} P^*_1 + i \frac{1}{r} \frac{\partial}{\partial r} P^*_1 - \frac{n^2}{r} \frac{\partial}{\partial r} P^*_1 + \frac{1}{X} \frac{\partial}{\partial r} P^*_1 + \frac{1}{r} \frac{\partial}{\partial r} P^*_1 + \frac{1}{r} \frac{\partial}{\partial r} P^*_1 + i k C_z P^*_1. \]

Using the energy equation (eq. [A8]) written in the form

\[ -\gamma_t W_0 \nabla \cdot v_1 = \gamma_t \left( \frac{1}{\alpha^2} + \frac{1}{\Gamma - 1} \frac{1}{c^2} \right) \left( \frac{\partial}{\partial t} P_1 + u \frac{\partial}{\partial z} P_1 \right) - \frac{1}{c^2} \frac{\partial}{\partial t} P^*_1 - \gamma_t \left( \frac{1}{\alpha^2} + \frac{1}{\Gamma - 1} \frac{1}{c^2} \right) \left( \frac{\partial}{\partial t} B_{z1} + u \frac{\partial}{\partial z} B_{z1} \right) + \frac{1}{c^2} \frac{\partial}{\partial t} B_{z1} + 2 \gamma_t \frac{u}{c^2} \frac{\partial}{\partial z} W_0 \left( \frac{\partial}{\partial z} v_{z1} + u \frac{\partial}{\partial z} v_{z1} \right), \]

inserting \( \partial/\partial t + u \partial/\partial z \) \( B_{z1} = -B_0 (\nabla \cdot v_1 - \partial v_{z3}/\partial z) \), and using \( v_{z1} = C_z P^*_1 \) gives

\[ \nabla \cdot v_1 = -i \frac{Y}{\gamma_t W_0} \left[ 1 + \frac{V_\Lambda^2}{\gamma_0^2} \frac{Y}{(ku - \omega)} \right]^{-1} P^*_1 - \left[ 1 + \frac{V_\Lambda^2}{\gamma_0^2} \frac{Y}{(ku - \omega)} \right]^{-1} \left[ 2 \gamma_t^2 (ku - \omega) \frac{u}{c^2} - \frac{V_\Lambda^2}{\gamma_0^2} \frac{k}{(ku - \omega)} \right] C_z P^*_1, \]

where \( Y = \gamma_t^2 (ku - \omega)/a^2 - (k - \omega u/c^2)^2 u/c^2 \). Setting equations (A14) and (A15) equal gives us a differential equation for \( P^*_1 \) in the form of Bessel’s equation

\[ r^2 \frac{\partial^2}{\partial r^2} P^*_1 + r \frac{\partial}{\partial r} P^*_1 + (\beta^2 r^2 - n^2) P^*_1 = 0, \]

where

\[ \beta^2 = \frac{Y X}{\gamma_0^2 W_0} \left[ 1 + \frac{V_\Lambda^2}{\gamma_0^2} \frac{Y}{(ku - \omega)} \right]^{-1} + k X C_z + \left[ 1 + \frac{V_\Lambda^2}{\gamma_0^2} \frac{Y}{(ku - \omega)} \right]^{-1} \left[ 2 \gamma_t^2 (ku - \omega) \frac{u}{c^2} - \frac{V_\Lambda^2}{\gamma_0^2} \frac{k}{(ku - \omega)} \right] X C_z. \]
Considerable algebraic manipulation and regrouping provides the following form

\[
\beta^2 = \left\{ \frac{\gamma_0^2 \gamma_\lambda^2 (k u - \omega)^2 - (k - \omega u/c^2)^2 v_\lambda^2}{(a^2 + \gamma_\lambda^2 v_\lambda^2)(k u - \omega)^2 + \gamma_\lambda^2 v_\lambda^2 a^2 (k u - \omega)(k - \omega u/c^2)u/c^2} \right\}
\]

\[
\times \left\{ \left[ \left( a^2 + \gamma_\lambda^2 v_\lambda^2 \right)(k u - \omega)^2 + \gamma_\lambda^2 v_\lambda^2 a^2 (k u - \omega)(k - \omega u/c^2)u/c^2 \right] \left[ (k u - \omega)^2 - (k - \omega u/c^2)^2 a^2 \right] \right\}.
\]

which can be written in the compact form

\[
\beta^2 = \left[ \frac{\gamma_0^2 (\omega^2 - \kappa_\lambda^2 a^2)(\omega^2 - \kappa_\lambda^2 v_\lambda^2)}{v_{\text{ms}}^2 \omega^2 - \kappa_\lambda^2 v_\lambda^2 a^2} \right].
\]  

(A17)

where \( \omega^2 \equiv (\omega - ku)^2 \), \( \kappa_\lambda^2 \equiv (k - \omega u/c^2)^2 \), and where the fast magnetosonic speed perpendicular to the magnetic field is given by (e.g., Vlahakis & Königl 2003)

\[
v_{\text{ms}} \equiv (a^2 + v_\lambda^2 - a^2 v_\lambda^2/c^2)^{1/2} = (a^2/\gamma_\lambda^2 + v_\lambda^2)^{1/2}.
\]

This expression for \( \beta^2 \) reduces to the relativistic pure fluid form given in Hardee (2000) and to the nonrelativistic MHD form given in Hardee et al. (1997).

The solutions that are well behaved at jet center and at infinity are \( P_{ij}^*(r \leq R) = C_j J_{\pm n}(\beta_j r) \) and \( P_c^*(r \geq R) = C_e H_n^{(1)}(\beta_e r) \), respectively, where \( J_{\pm n} \) and \( H_{\pm n} \) are the Bessel and Hankel functions with arguments defined by

\[
\beta_j^2 \equiv \left[ \frac{\gamma_j^2 \left( \omega_j^2 - \kappa_j^2 a_j^2 \right) \left( \omega_j^2 - \kappa_j^2 v_{\lambda j}^2 \right)}{v_{\text{ms},j}^2 \omega_j^2 - \kappa_j^2 v_{\lambda j}^2 a_j^2} \right]
\]

and \( \beta_e^2 \equiv \left[ \frac{\gamma_e^2 \left( \omega_e^2 - \kappa_e^2 a_e^2 \right) \left( \omega_e^2 - \kappa_e^2 v_{\lambda e}^2 \right)}{v_{\text{ms},e}^2 \omega_e^2 - \kappa_e^2 v_{\lambda e}^2 a_e^2} \right] \).

In the above \( \omega_j^2 \equiv (\omega - ku_j)^2 \), \( \kappa_j^2 \equiv (k - \omega u_j/c^2)^2 \), \( \gamma_j^2 \equiv (1 - u_j^2/c^2)^{-1} \), and \( \gamma_e^2 \equiv (1 - v_{\lambda e}^2/c^2)^{-1} \). The jet flow speed and external flow speed are positive if flow is in the +z-direction. The condition that the total pressure be continuous across the jet boundary requires that

\[
C_j J_{\pm n}(\beta_j R) = C_e H_n^{(1)}(\beta_e R).
\]  

(A18)

The first derivative of the total pressure is given by \( \partial P_{ij}/\partial r = -iX_j v_{ij} \equiv -(\omega - ku)X_j \), where \( v_{ij} \equiv (\partial/\partial t + u \cdot \nabla) \xi_j = -i(\omega - ku)\xi_j \), and \( \xi_j \) is the fluid displacement in the radial direction. The radial displacement of the jet and external medium must be equal at the jet boundary, i.e., \( \xi_j(R) = \xi_e(R) \), from which it follows that

\[
\left. \frac{\beta_j}{-(\omega - ku_j)} C_j \frac{\partial J_n(\beta_j r)}{\partial (\beta_j r)} \right|_{r=R} = \left. \frac{\beta_e}{-(\omega - ku_e)} C_e \frac{\partial H_n^{(1)}(\beta_e r)}{\partial (\beta_e r)} \right|_{r=R}.
\]  

(A19)

Inserting \( C_j \) and \( C_e \) from equation (A18) leads to a dispersion relation describing the propagation of Fourier components, which can be written

\[
\left. \frac{\beta_j}{\chi_j} J_n(\beta_j R) \right|_{\beta_j} = \left. \frac{\beta_e}{\chi_e} H_n^{(1)}(\beta_e R) \right|_{\beta_e}, \]

(A20)

where the primes denote derivatives of the Bessel and Hankel functions with respect to their arguments, and with

\[
\chi_j \equiv \gamma_j^2 \gamma_{\lambda j}^2 W_j(\omega_j^2 - \kappa_j^2 v_{\lambda j}^2) \quad \text{and} \quad \chi_e \equiv \gamma_e^2 \gamma_{\lambda e}^2 W_e(\omega_e^2 - \kappa_e^2 v_{\lambda e}^2).
\]

This dispersion relation readily reduces to the nonrelativistic form given in Hardee et al. (1997), and describes the normal modes of a cylindrical jet where \( n = 0, 1, 2, 3, 4, \) etc., involve pinching, helical, elliptical, triangular, rectangular, etc., normal mode distortions of the jet, respectively.
APPENDIX B

ANALYTIC SOLUTIONS AND APPROXIMATIONS

Each normal mode $n$ contains a fundamental/surface wave and multiple body wave solutions to the dispersion relation. The low-frequency limiting form for the fundamental/surface modes are obtained in the limit where $\omega \to 0$ and $k \to 0$, but with $\omega/k \neq 0$. In this limit the dispersion relation for the fundamental ($n = 0$) and surface ($n > 0$) modes is given by

$$\chi_n \approx \begin{cases} -\frac{1}{2} \chi_e (\beta e R)^2 \left[ \ln \left( \frac{\beta e R}{2} \right) + \frac{\pi}{2} \epsilon - i \frac{\pi}{2} \right], & n = 0, \\ -\chi_e, & n > 0, \end{cases}$$

where $\beta e R \to 0$ and $\beta j R \to 0$, and I have used

$$\frac{J_n'(\beta e R)}{J_n(\beta e R)} \frac{H_n^{(1)}(\beta j R)}{L_n^{(1)}(\beta j R)} \approx \begin{cases} -\frac{1}{2} (\beta e R) (\beta j R) \left[ \ln \left( \frac{\beta e R}{2} \right) + \frac{\pi}{2} \epsilon - i \frac{\pi}{2} \right], & n = 0, \\ -\beta e / \beta j, & n > 0, \end{cases}$$

where $\epsilon$ is Euler's constant.

B1. FUNDAMENTAL PINCH MODE ($N = 0; M = 0$) IN THE LOW-FREQUENCY LIMIT

In the low-frequency limit, equation (B1) for the fundamental pinch mode becomes

$$\gamma_j^2 \gamma_{j,\ell} W_j \left( \omega_j^2 - \kappa_j^2 v_{\ell, j}^2 \right) \approx -\frac{1}{2} \gamma_j^2 \gamma_{\ell, j} \omega_j \left( \omega_j^2 - \kappa_j^2 v_{\ell, j}^2 \right) \left( \frac{\gamma_j^2 \left( \omega_j^2 - \kappa_j^2 a_j^2 \right) \left( \omega_j^2 - \kappa_j^2 v_{\ell, j}^2 \right)}{v_{\ell, j}^2 \omega_j - \kappa_j^2 v_{\ell, j}^2 a_j^2} \right) R^2 \left[ \ln \left( \frac{\beta e R}{2} \right) + \frac{\pi}{2} \epsilon - i \frac{\pi}{2} \right].$$

Here we have solutions

$$\omega_j^2 \approx \begin{cases} \kappa_j^2 v_{\ell, j}^2, & \kappa_j^2 a_j^2 \left( \frac{v_{\ell, j}^2 / v_{\ell, j}^2 - \delta}{1 - \delta} \right), \end{cases}$$

where

$$\delta \equiv -\frac{1}{2} \gamma_j^2 \gamma_{\ell, j} \frac{\gamma_{\ell, j}}{\gamma_j^2} \omega_j \left( \omega_j^2 - \kappa_j^2 v_{\ell, j}^2 \right) \frac{v_{\ell, j}^2}{v_{\ell, j}^2} \left( \frac{\gamma_j^2 \left( \omega_j^2 - \kappa_j^2 a_j^2 \right) \left( \omega_j^2 - \kappa_j^2 v_{\ell, j}^2 \right)}{v_{\ell, j}^2 \omega_j - \kappa_j^2 v_{\ell, j}^2 a_j^2} \right) R^2 \left[ \ln \left( \frac{\beta e R}{2} \right) + \frac{\pi}{2} \epsilon - i \frac{\pi}{2} \right]$$

and $\delta$ is complex. Thus, in the low-frequency limit, the fundamental pinch mode admits an Alfvén wave solution $\omega_j^2 = \kappa_j^2 v_{\ell, j}^2$, with $v_{\ell, j}^2 = v_{\ell, j}$ in the jet frame and a magnetosonic wave solution consisting of a growing and damped wave pair with complex phase velocity in the jet frame,

$$v_{\ell, j}^2 \simeq a_j^2 \left[ \frac{v_{\ell, j}^2}{v_{\ell, j}^2} + \delta \left( \frac{v_{\ell, j}^2}{v_{\ell, j}^2} - 1 \right) \right].$$

Previous work has shown the unstable growing solution associated with the backward-moving (in the jet fluid reference frame) wave.

B2. SURFACE MODES ($N > 0; M = 0$) IN THE LOW-FREQUENCY LIMIT

In the low-frequency limit the solution for all surface modes is most easily obtained from equation (B1) written in the form

$$\gamma_j^2 W_j \left( \omega - ku \right)^2 - \frac{V_j^2}{\gamma_j^2} \left( k^2 - \omega^2 / c^2 \right) = -\gamma_j^2 W_0 \left( \omega - ku \right)^2 - \frac{V_j^2}{\gamma_j^2} \left( k^2 - \omega^2 / c^2 \right),$$

where I have used $\chi \equiv \gamma_0^2 \gamma_j^2 W_0 (\omega^2 - \kappa_j^2) = \gamma_0^2 W_0 \left( \omega - ku \right)^2 - \left( k^2 - \omega^2 / c^2 \right) V_j^2 / \gamma_j^2$. The solution can be put in the form

$$\frac{\omega}{k} = \frac{(\eta u_j + u_e) \pm i \eta^{1/2} \left( u_j - u_e \right)^2 - V_0^2 / \gamma_j^2 \gamma_j^2 \gamma_j^2}{(1 + V_0^2 / \gamma_j^2 c^2) + \eta (1 + V_0^2 / \gamma_j^2 c^2)}.$$
where

\[ \eta \equiv \frac{\gamma^2 W_j}{\gamma_e^2 W_e} \quad \text{and} \quad V_{A,i}^2 \equiv \left( \frac{\gamma_{\Lambda,j} W_j + \gamma_{\Lambda,e} W_e}{\gamma_{\Lambda,j}^2 + \gamma_{\Lambda,e}^2} \right) \frac{B_j^2 + B_e^2}{4\pi W_j W_e}. \]

The jet is stable to surface mode perturbations when

\[ \gamma^2 c_s^2 (u_j - u_e)^2 < \gamma_{\Lambda,j}^2 \frac{W_j}{\gamma_{\Lambda,e}} + \frac{W_e}{\gamma_{\Lambda,e}} \left( \frac{B_j^2 + B_e^2}{4\pi W_j W_e} \right). \]  \( \text{(B8)} \)

Equation (B7) reduces to the relativistic fluid form found in Hardee & Hughes (2003) and reduces to the nonrelativistic MHD form found in Hardee & Rosen (2002).

**B3. BODY MODES \( N \geq 0; M \geq 1 \) IN THE LOW-FREQUENCY LIMIT**

In the low-frequency limit the real part of the body wave solutions can be obtained in the limit \( \omega = 0, k \neq 0, \) where the dispersion relation can be written in the form

\[ \cos \left[ \beta_R R - \frac{(2n + 1)\pi}{4} \right] \approx \epsilon_n = \frac{\chi_e}{\chi_j} \beta_e \left( \frac{\pi m}{2} \right)^{1/2} J'_m(\beta_R) H_n^{(1)}(\beta_R). \]

Here I have assumed that \( J_m(\beta_R) \approx (2/\pi \beta_R)^{1/2} \cos(\beta_R R - (2n + 1)\pi/4) \) applies. In the absence of a magnetic field and a flow surrounding the jet, \( \chi_e = 0, \epsilon_n = 0, \) and solutions are found from \( \beta_R R - (2n + 1)\pi/4 = \pm m \pi \pm \pi/2, \) where \( m \) is an integer. Provided \( \epsilon_n \approx \pi/2 \) and \( \theta \approx \cos^{-1} \epsilon_n \approx \pm (\pi/2 - \epsilon_n), \) solutions can be found from \( \beta_R R - (2n + 1)\pi/4 = \pm (m \pi + (\pi/2 - \epsilon_n)), \) where for \( \pm \epsilon_n \) the plus or minus sign is for \( m \) odd or even, respectively. In the limit \( \omega = 0 \) the solutions are given by

\[ kR \approx k_{nn} R = \frac{\sqrt{\gamma_e^2 c_s^2 (u_j^2 - v_{s,0}^2) \beta_e^2 - \gamma^2 (u_j^2 - a_j^2)(u_j^2 - v_{s,0}^2)}}{\sqrt{\gamma^2 (u_j^2 - a_j^2)(u_j^2 - v_{s,0}^2)}} \left( \frac{n + 2m - 1/2}{2} \right)^{1/2} \left( \frac{n + 2m - 1/2}{2} \right) (\pm 1\epsilon_n), \]

where I have set \( m \rightarrow m + 1 \) to be consistent with previous notation so that \( m = 1 \) corresponds to the first body mode. Equation (B10) reduces to the relativistic purely fluid form found in Hardee & Hughes (2003) and reduces to the nonrelativistic MHD form found in Hardee & Rosen (2002). I note here that there is an error in equations (5) in these previous articles in the treatment of the sign on \( \epsilon_n \) for even values of \( m. \)

**B4. THE RESONANCE CONDITION**

The resonance conditions are found by evaluating the transmittance, \( T, \) and reflectance, \( R, \) of waves at the jet boundary where \( T = 1 + R. \) With the dispersion relation written as

\[ \frac{1}{Z_j} \frac{J'_m(\beta_R)}{J_m(\beta_R)} = \frac{1}{Z_e} \frac{H_n^{(1)}(\beta_R)}{H_n^{(1)}(\beta_R)}, \]

\[ Z = \frac{\chi}{\beta} = \gamma^2 \gamma_{\Lambda}^2 W (\omega^2 - \kappa^2 v_{s,0}^2) \left[ \frac{(a_j^2 - \gamma_{\Lambda}^2 v_{s,0}^2) \omega^2 - \gamma^2 \kappa^2 \gamma_{\Lambda}^2 v_{s,0}^2}{\gamma^2 \gamma_{\Lambda}^2 (\omega^2 - \kappa^2 a_j^2)(\omega^2 - \kappa^2 v_{s,0}^2)} \right]^{1/2}, \]

and the reflectance is given by

\[ R = (Z_e - Z_j)/(Z_e + Z_j). \]

For a fluid containing no magnetic field, \( Z \) is a quantity related to the acoustic normal impedance (Gill 1965). When \( Z_e + Z_j \approx 0, \) \( R \) and \( T \) are large, and the reflected and transmitted waves have an amplitude much larger than the incident wave.

**B4.1. The Fluid Limit (Alfvén Speed \( \ll \) Sound Speed)**

For the case of a pure fluid,

\[ Z_{e,j} = \frac{W_{e,j} \left( \zeta_{e,j}^2 + \gamma_{e,j}^2 \kappa_{e,j}^2/k^2 \right)}{\zeta_{e,j}}, \]

\( \text{(B14)} \)
where $\chi / k^2 = W(\zeta^2 + \gamma^2 \kappa^2 / k^2) a^2$ and $\zeta \equiv \beta / k$. For nonrelativistic flows where $(a^2/c^2)(\omega/k) \ll 1, \gamma \approx 1$, and with adiabatic indices $\Gamma_j = \Gamma_e$, the reflectance $R = [(\zeta - \zeta_j)/(\zeta + \zeta_j)](1 + \zeta_j)$, and a supersonic resonance (Miles 1957) occurs when $\beta_e + \beta_j = k(\zeta + \zeta_j) = 0$. This resonance corresponds to the maximum growth rate.

I now generalize the results in Hardee (2000) to include flow in the external medium relative to the source/observer frame. Here $Z_e + Z_j = 0$ becomes

$$\Gamma_j \zeta_j \chi_e + \Gamma_e \zeta_e \chi_j = \Gamma_j \zeta_j \left( \frac{\gamma_j^2 \omega_j^2}{k^2} a_j^2 \right) + \Gamma_e \zeta_e \left( \frac{\gamma_e^2 \omega_e^2}{k^2} a_e^2 \right) = 0. \quad (B15)$$

A necessary condition for resonance is $\zeta_j < 0$ and $\zeta_e > 0$, and on the real axis

$$\frac{u_j - u_e}{1 - u_j a_j / c^2} > \frac{\omega}{k} > \frac{u_e + a_e}{1 + u_e a_e / c^2}. \quad (B16)$$

Equivalently, the resonance only exists when

$$\frac{u_j - u_e}{1 - u_j a_j / c^2} > \frac{a_j + a_e}{1 + a_j a_e / c^2}. \quad (B17)$$

To find the resonant real part of the phase velocity, I solve $\zeta_j^2 = \varepsilon^2 \zeta_j^2$, where here I set $\varepsilon \equiv (\Gamma_j \gamma_j^2 \omega_j^2 / k^2 a_j^2) / (\Gamma_e \gamma_e^2 \omega_e^2 / k^2 a_e^2) = 1$. The resulting quadratic equation can be written in the form

$$\left( \frac{a_j^2}{\gamma_j^2 \gamma_j} - \frac{a_j^2}{\gamma_j^2 \gamma_{ee}} \right) \left( \frac{\omega}{k} \right)^2 - 2 \left( \frac{\gamma_j^2 a_j^2 u_j}{\gamma_{ee}} - \frac{\gamma_j^2 a_j^2 u_e}{\gamma_{ee}} \right) \left( \frac{\omega}{k} \right) + \left[ \gamma_j^2 a_j^2 (u_j^2 - a_j^2) - \gamma_e^2 a_e^2 (u_e^2 - a_e^2) \right] = 0, \quad (B18)$$

where I have used $[\gamma_j^2 a_j^2 (1 - a_j^2 u_j^2 / c^4) - \gamma_e^2 a_e^2 (1 - a_e^2 u_e^2 / c^4)] = [a_j^2 (\gamma_j^2 \gamma_j^2) - a_j^2 (\gamma_j^2 \gamma_{ee}^2)]$ and $\gamma_j^2 \equiv (1 - a_j^2 / c^2)^{-1}$. The resonant solution to equation (B18) is

$$v^* = \frac{\omega}{k} = \frac{\gamma_j a_j u_j + \gamma_e a_e u_e}{\gamma_j \gamma_{ee} + \gamma_e (\gamma_{ee} a_j / a_e)} \quad \text{(B19)}$$

Inserting the resonant solution (eq. [B19]) into the expression for $\varepsilon$ gives $0.695 \leq \varepsilon^2 \leq (\Gamma_j^2 \gamma_j^4) / (\Gamma_e^2 \gamma_e^4) \leq 1.44$, where $2.78 \geq \Gamma^2 \gamma^4 < 1$. When $a_j = a_e$ and $\Gamma_j = \Gamma_e$, $\varepsilon^2 = 1$, and the resonant solution is exact. The small range on $\varepsilon$ $(0.83 \leq \varepsilon \leq 1.2)$ suggests that this solution remains relatively robust for unequal values of the sound speed and adiabatic index in the jet and external medium. When $u_e = 0$ this resonant solution is equivalent to the form given in Hardee (2000).

The resonant frequencies can be estimated using the large-argument forms for the Bessel and Hankel functions. In this limit the dispersion relation becomes

$$\frac{J_n(\beta) R H_n^{(1)}(\beta) R}{J_n(\beta) H_n^{(1)}(\beta) R} \approx i \tan \left( \frac{\beta R - 2n + 1}{4} \pi \right) = \frac{\chi_j}{\chi_e} \frac{\beta_j}{\beta_j} \quad \text{(B20)}$$

From the dispersion relation with $Z_e + Z_j \approx 0$, and $(\chi_j / \beta_j) / \chi_e \approx Z_j / Z_e \approx \beta_j / \beta_j \approx -1$, $\tan \left( \beta R - (2n + 1) \pi / 4 \right) \approx 0$ on the real axis. It follows that $|\beta R| \approx |\beta_j R| \approx (2n + 1) \pi / 4 + m \pi$ can be used to obtain an estimate for the resonant frequencies from $|\beta R| \approx (2n + 1) \pi / 4 + m \pi$, with the result that the resonant frequencies are given by

$$\frac{\omega_m \Re}{a_e} \approx \frac{\omega_m \Re}{a_e} \equiv \frac{(2n + 1) \pi / 4 + m \pi}{\gamma_e \left[ (1 - u_e/v_w^*)^2 - (a_j/a_e)^2 (c^2) \right]^{1/2}} \quad \text{(B21)}$$

In the absence of external flow, $u_e = 0$, and for $a_j \gg a_e$, and $1 \gg (ka_e / \omega)^2$, this expression reduces to the form given in Hardee (2000). The resonant frequency $\omega^* \rightarrow \infty$ as $(1 - u_e/v_w^*)^2 - (a_j/a_e)^2 (c^2) \rightarrow 0$, and an equivalent condition is

$$u_j - u_e = \frac{a_j + a_e}{1 + a_j a_e / c^2} \quad \text{(B22)}$$

Thus, the resonance moves to higher frequencies and $\omega^* \rightarrow \infty$ when the “shear” speed drops below a “surface” sound speed.
The behavior of the growth rate at resonance also can be found using the large-argument forms for the Bessel and Hankel functions. In this limit the reflectance can be written as

\[
\mathcal{R} = \frac{J_n(\beta_R)H_n^{(1)}(\beta_R) - J'_n(\beta_R)H_n^{(1)}(\beta_R)}{J_n(\beta_R)H_n^{(1)}(\beta_R) + J'_n(\beta_R)H_n^{(1)}(\beta_R)} \approx \exp \left[ -2i \left( \beta_R - \frac{2n + 1}{4\pi} \right) \right],
\]

(B23)

and

\[
\beta_R - \frac{2n + 1}{4\pi} \approx \frac{i}{2} \ln |\mathcal{R}| - \frac{\phi}{2},
\]

(B24)

where \( \mathcal{R} \equiv |\mathcal{R}|e^{i\phi} \). It follows that \( (\beta_R)_f \approx \frac{1}{2} \ln |\mathcal{R}| \), and since typically at resonance, \( |\omega - k_R u_j|/c_j > |k_R - \omega u_j/c| \), I can approximate \( \beta_j \) by

\[
\beta_j \equiv \beta_j^R + i\beta_j^F \approx \gamma_j \left[ \frac{(\omega - k_R u_j)}{a_j} - ik_l \left( \frac{u_j}{a_j} \right) \right].
\]

(B25)

It follows that \( (\beta_R)_f \approx -\gamma_j(u_j/a_j)k_f R \), and

\[
k_f R \approx -\frac{a_j}{2\gamma_j u_j} |\ln |\mathcal{R}||.
\]

(B26)

At resonance

\[
|\mathcal{R}| \approx \frac{\beta_j - \beta_c(\chi_j/\chi_c)}{\beta_j + \beta_c(\chi_j/\chi_c)} \approx \frac{\beta_j - \beta_c}{\beta_j + \beta_c} \approx \left[ \frac{(-2\beta_c^R)^2 + (\beta_j^R - \beta_c^R)^2}{(\beta_j^R + \beta_c^R)^2} \right]^{1/2},
\]

(B27)

where I have used \( (\beta_j^R - \beta_c^R) \approx -2\beta_c^R \) from the resonance condition on the real axis. It follows that

\[
|\mathcal{R}| \approx \left\{ \frac{4\gamma_j^2(\omega - k_R u_c)^2/a_j^2 + k_l^2 \left[ \gamma_j(u_j/a_j) - \gamma_c(u_c/a_c) \right] - [(\gamma_j a_j)/(\omega - k_R u_c)k_j]^2}{k_l^2 \left[ \gamma_j(u_j/a_j) + \gamma_c(u_c/a_c) \right] + [(\gamma_j a_j)/(\omega - k_R u_c)k_j]^2} \right\}^{1/2},
\]

(B28)

where I have used \( \beta_c \equiv \beta_c^R \equiv \gamma_c \left\{ \frac{\omega - k_R u_c}{a_c} - ik_l u_c/a_c + k_l a_c(\omega - k_R u_c) \right\} \).

If I assume that \( \gamma_j(\gamma_c a_c) \gg \gamma_c(\gamma_j a_j) \), with resonant wave speed \( \omega/k_R \approx u_j \) and \( u_c/a_j \ll 1 \), then

\[
|\mathcal{R}| \approx \left\{ \frac{4(\omega_m R/a_c)^2 (1 - 2u_c/u_j) + k_l^2 R^2 \left[ \gamma_j(u_j/a_j) - (a_j/u_j) (1 + u_c/u_j) \right]^2}{k_l^2 R^2 \left[ \gamma_j(u_j/a_j) + (a_j/u_j) (1 + u_c/u_j) \right]^2} \right\}^{1/2},
\]

(B29)

and since \( k_f R \approx -(a_j/2\gamma_j u_j) |\ln |\mathcal{R}|| \),

\[
|\mathcal{R}| \approx \left[ \frac{4(\omega_m R/a_c)^2 (1 - 2u_c/u_j) + (|\ln |\mathcal{R}||/2)^2}{(\ln |\mathcal{R}|/2)^2} \right]^{1/2}.
\]

(B30)

In this limit where \( v_m^r \approx u_j \) and \( u_c/v_m^r \approx u_c/u_j \), equation (B21) gives

\[
\left( \frac{\omega_m R}{a_c} \right)^2 (1 - 2u_c/u_j) \approx \frac{(1 - 2u_c/u_j)}{1 - 2(u_c/u_j)(1 - a_j^2/c^2) - a_j^2/u_j^2} \left[ (2n + 1)\pi/4 + m\pi \right]^2,
\]

and if say \( u_c = 0 \), then

\[
\left( |\mathcal{R}|^2 - 1 \right)^{1/2} \ln |\mathcal{R}| \approx 4 \left( 1 - a_j^2/u_j^2 \right)^{-1/2} \left[ (2n + 1)\pi/4 + m\pi \right].
\]

(B31)
Formally $|R| \to \infty$ as $\omega_{nm}^* \to \infty$ when the jet speed drops below the “surface” sound speed. This result applies only to the surface modes and not to the body modes as, in the fluid limit, the body modes do not exist when the jet speed drops below the jet sound speed. On the other hand, if say, $a^2_s/u_j^2 \ll 1$, then

$$\left(|R|^2 - 1\right)^{1/2} \ln|R| \approx 4(2n + 1)\pi/4 + m\pi.$$  \hfill (B32)

Formally $|R| \approx \text{constant}$, and the growth rate does not increase as $\omega_{nm}^* \to \infty$ when the sheath speed becomes comparable to the jet speed.

B4.2. The Magnetic Limit (Alfvén Speed $\gg$ Sound Speed)

For the magnetic limit in which magnetic pressure dominates gas pressure

$$Z_{e,j} = \gamma_{e,j}\gamma_{Ae,j}W_{e,j}v_{Ae,j}\left(\omega_{e,j}^2 - \kappa_{e,j}^2v_{Ae,j}^2\right)^{1/2}. \hfill (B33)$$

A necessary condition for resonance is $(\omega_e^2 - \kappa_e^2v_A^2) > 0$ and $(\omega_e^2 - \kappa_e^2v_A^2) < 0$ on the real axis with the result that $Z_e + Z_j = 0$ when

$$\frac{u_j - u_e}{1 - u_ju_e/c^2} > \frac{v_{Aj} + v_{Ae}}{1 + v_{Aj}v_{Ae}/c^2}. \hfill (B34)$$

This result is identical in form to the sonic case with sound speeds replaced by Alfvén wave speeds. The resonant solution for the real part of the phase velocity is obtained from

$$Z_j^2 = \gamma_j^2W_j^2V_{Aj}^2\left[\omega_j^2 - (k^2 - \omega^2/c^2)V_{Aj}^2/\gamma_j^2\right] = Z_e^2 = \gamma_e^2W_e^2V_{Ae}^2\left[\omega_e^2 - (k^2 - \omega^2/c^2)V_{Ae}^2/\gamma_e^2\right], \hfill (B35)$$

where I have used $\gamma_j^2\gamma_e^2(\omega_e^2 - \kappa_e^2v_A^2) = \gamma_j^2(\omega_e^2 - (k^2 - \omega^2/c^2)V_{Ae}^2/\gamma_e^2)$, and recall that $v_A^2 = V_{Ae}^2/\gamma_e^2$. The resulting quadratic equation can be written in the form

$$\left(\gamma_j^2W_j^2V_{Aj}^2 - \gamma_e^2W_e^2V_{Ae}^2\right)\left(\frac{\omega}{k}\right)^2 - 2\left(\gamma_j^2W_j^2V_{Aj}^2\mu_j - \gamma_e^2W_e^2V_{Ae}^2\mu_e\right)\left(\frac{\omega}{k}\right) + \left(\gamma_j^2W_j^2V_{Aj}^2\mu_j^2 - \gamma_e^2W_e^2V_{Ae}^2\mu_e^2\right) = 0, \hfill (B36)$$

where I have used $\gamma_j^2W_j^2V_{Aj}^2(k^2 - \omega^2/c^2)V_{Aj}^2/\gamma_j^2 = \gamma_j^2W_j^2V_{Aj}^2(k^2 - \omega^2/c^2)V_{Ae}^2/\gamma_e^2$ because pressure balance in the magnetically dominated case requires $W_jV_{Aj} = W_eV_{Ae}$. The resonant solution becomes

$$v_{e,j} = \frac{\omega}{k} = \frac{\gamma_jW_jV_{Aj} + \gamma_eW_eV_{Ae}u_e}{\gamma_jW_jV_{Aj} + \gamma_eW_eV_{Ae}} = \frac{\gamma_eV_{Aj}V_{Ae}v_{Aj} + \gamma_eV_{Ae}v_{Ae}}{\gamma_jV_{Aj}V_{Ae} + \gamma_eV_{Ae}} = \frac{\gamma_jV_{Aj}v_{Aj} + \gamma_eV_{Ae}}{\gamma_jV_{Aj} + \gamma_eV_{Ae}}, \hfill (B37)$$

where I have used $\gamma_jW_jV_{Aj} + \gamma_eW_eV_{Ae} = \gamma_jW_jV_{Aj} + \gamma_eW_eV_{Ae}$. This resonant solution has the same form as the sonic case with sound speeds and sonic Lorentz factors replaced by Alfvén wave speeds and Alfvén Lorentz factors.

As in the sonic case, the resonant frequencies are found from $|\beta_{R}| \approx (2n + 1)\pi/4 + m\pi$, with the result that the resonant frequencies are given by

$$\frac{\omega_{nm}R}{v_{Ae}} \approx \frac{\omega_{nm}^*}{v_{Ae}} = \frac{(2n + 1)\pi/4 + m\pi}{\gamma_e\left[\left(1 - u_e/v_{w}^2\right)^2 - (v_{Ae}/v_{e}^* - u_ev_{Ae}/c^2)^2\right]^{1/2}}. \hfill (B38)$$

Here the resonant frequency $\omega_{nm}^* \to \infty$ as $(1 - u_e/v_{w}^2)^2 - (v_{Ae}/v_{e}^* - u_ev_{Ae}/c^2)^2 \to 0$, and an equivalent condition is

$$\frac{u_j - u_e}{1 - u_ju_e/c^2} = \frac{v_{Aj} + v_{Ae}}{1 + v_{Aj}v_{Ae}/c^2}. \hfill (B39)$$

Thus, the resonance moves to higher frequencies as the “shear” speed approaches a “surface” Alfvén speed.

The behavior of the growth rate at resonance proceeds in the same manner as for the fluid limit but with sound speeds replaced by Alfvén wave speeds. The resonant growth rate is now given by

$$k_{f}R \approx -\frac{1}{2} \frac{v_{Aj}}{\gamma_j\mu_j} \ln|R|.$$  \hfill (B40)
If I assume that \( \gamma(\gamma_{\parallel}u_{\parallel}) \gg \gamma_{\parallel}(\gamma_{\parallel}v_{\parallel}) \), with resonant wave speed \( \omega/k \approx u_{\parallel} \) and \( u_{\parallel}/u_{\parallel} \ll 1 \), then

\[
|\mathcal{R}| \approx \left\{ 4\left(\frac{\omega_{\min}^2}{v_{\parallel}}\right)^2 \left[1 - 2u_{\parallel}/u_{\parallel}\right] + k^2 R^2 \frac{\left[\gamma_{\parallel}(u_{\parallel}/v_{\parallel}) - (v_{\parallel}/u_{\parallel})(1 + u_{\parallel}/u_{\parallel})\right]^2}{k^2 R^2 \left[\gamma_{\parallel}(u_{\parallel}/v_{\parallel}) + (v_{\parallel}/u_{\parallel})(1 + u_{\parallel}/u_{\parallel})\right]^2} \right\}^{1/2}. \tag{B41}
\]

Now from equation (B38),

\[
\left(\frac{\omega_{\min}^2}{v_{\parallel}}\right)^2 \left[1 - 2u_{\parallel}/u_{\parallel}\right] \approx \frac{1}{1 - 2(u_{\parallel}/u_{\parallel})(1 - v_{\parallel}^2/c^2)^2 - (v_{\parallel}^2 - u_{\parallel}^2)/u_{\parallel}^2} \left[\frac{(2n + 1)\pi}{2} + m\pi\right]^2,
\]

and if say \( u_{\parallel} = 0 \), then

\[
\left(\frac{|\mathcal{R}|^2 - 1}{\mathcal{R}}\right)^{1/2} \ln|\mathcal{R}| \approx 4\left[1 - v_{\parallel}^2/u_{\parallel}^2\right]^{-1/2} \left[\frac{(2n + 1)\pi}{4} + m\pi\right], \tag{B42}
\]

and \( |\mathcal{R}| \) increases as \( \omega_{\min}^2 \) increases when the jet speed decreases. However, when the shear speed drops below the “surface” Alfvén speed, the jet is stable. This result is quite different from the fluid limit, where the jet remains unstable when the shear speed drops below the “surface” sound speed. If I use \( u_{\parallel} - u_{\parallel} \) given by equation (B39) in equation (B8), it follows that the jet will be unstable when resonance disappears if

\[
\gamma_{\parallel}^2 \gamma_{\parallel}^2 \left[1 - u_{\parallel}^2/c^2\right]^2 > 2\gamma_{\parallel}^2 \gamma_{\parallel}^2 \frac{v_{\parallel}^2 + v_{\parallel}^2}{(v_{\parallel} + v_{\parallel})} \left[1 + v_{\parallel}^2 v_{\parallel}^2\right]^2, \tag{B43}
\]

where I have used \( (v_{\parallel}^2 + v_{\parallel}^2) = (W_{\parallel}/\gamma_{\parallel}^2 + W_{\parallel}/\gamma_{\parallel}^2)(B_{\parallel}^2 + B_{\parallel}^2)/(4\pi W_{\parallel}W_{\parallel}) \) as \( B_{\parallel} = B_{\parallel} \) from magnetic pressure balance. Formally, \( |\mathcal{R}| \to \infty \) as \( \omega_{\min}^2 \to \infty \) only for jet Lorentz factors greatly in excess of the Alfvénic Lorentz factor.

#### B5. Wave Modes at High Frequency

To obtain the behavior of wave modes at high frequency, I begin with the dispersion relation written in the form

\[
\beta_{\parallel}R = \frac{\chi_{\parallel}}{\chi_{\parallel}} \beta_{\parallel} \left[ \frac{J_n(\beta_{\parallel}R)}{\pm J_{n+1}(\beta_{\parallel}R)} + \frac{n/\beta_{\parallel}R}{J_n(\beta_{\parallel}R)} \right] \frac{H_{n+1}(\beta_{\parallel}R) - (n/\beta_{\parallel}R)H_{n}^{(1)}(\beta_{\parallel}R)}{H_{n+1}^{(1)}(\beta_{\parallel}R)}. \tag{B44}
\]

If I assume a large argument in the Hankel function with \( H_{n+1}^{(1)}(\beta_{\parallel}R) \approx \exp i[\beta_{\parallel}R - (2n + 1)\pi/4] \) and a small argument \( \beta_{\parallel}R \ll 1 \) in the Bessel function, then the dispersion relation becomes

\[
\beta_{\parallel}R \approx \left\{ \begin{array}{ll}
- \left( \frac{\chi_{\parallel}}{\chi_{\parallel}} \beta_{\parallel} \right) e^{-in/2}, & n = 0, \\
\beta_{\parallel} \left( \frac{\chi_{\parallel}}{\chi_{\parallel}} \beta_{\parallel} \right) e^{-in/2}, & n > 0,
\end{array} \right. \tag{B45}
\]

where I have used \( J_0(\beta_{\parallel}R)J_1(\beta_{\parallel}R) \approx 2/\beta_{\parallel}R \). At high frequency and large wavenumber, \( \chi_{\parallel} \) and \( \chi_{\parallel} \) are proportional to \( k^2 \), \( \beta_{\parallel} \) and \( \beta_{\parallel} \) are proportional to \( k \), and \( \beta_{\parallel}R = \gamma_{\parallel}k R \propto (kR)^{1/2} \) for \( n = 0 \). Thus, the internal solutions in the high-frequency and large-wavenumber limit are given by \( \beta_{\parallel}R \approx 0 \) and are found from

\[
\left( k u_{\parallel} - \omega \right)^2 - \left( k - \omega u_{\parallel}/c^2 \right)^2 a_{\parallel}^2 \left( k u_{\parallel} - \omega \right)^2 - \left( k - \omega u_{\parallel}/c^2 \right)^2 v_{\parallel}^2 \approx 0, \tag{B46}
\]

with solutions

\[
\frac{\omega}{k} \approx \frac{u_{\parallel} \pm v_{\parallel}}{1 \pm u_{\parallel} v_{\parallel}/c^2}, \tag{B47}
\]

where \( v_{\parallel} \equiv (a_{\parallel}, v_{\parallel}). \)
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