INDUCED REPRESENTATIONS OF INFINITE-DIMENSIONAL GROUPS

A. V. KOSYAK

Abstract. The induced representation $\text{Ind}_H^G S$ of a locally compact group $G$ is the unitary representation of the group $G$ associated with unitary representation $S : H \to U(V)$ of a subgroup $H$ of the group $G$. Our aim is to develop the concept of induced representations for infinite-dimensional groups. The induced representations for infinite-dimensional groups in not unique, as in the case of a locally compact groups. It depends on two completions $\tilde{H}$ and $\tilde{G}$ of the subgroup $H$ and the group $G$, on an extension $\tilde{S} : \tilde{H} \to U(V)$ of the representation $S : H \to U(V)$ and on a choice of the $G$-quasi-invariant measure $\mu$ on an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $H\backslash G$. As the illustration we consider the “nilpotent” group $B_0^\mathbb{N}$ of infinite in both directions upper triangular matrices and the induced representation corresponding to the so-called generic orbits.

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1. Introduction

The induced representations were introduced and studied for a finite groups by F.G. Frobenius. Our aim is to develop the concept of induced representations for infinite-dimensional groups.

The content of the article is as follows. Section 2 is devoted to the notion of induced representations elaborated for a locally compact groups by G.W.Mackey [14, 15] and to the Kirillov orbit methods [4] for the nilpotent Lie groups $B(n, \mathbb{R})$.

In Section 3 we extend the notion of the induced representations for infinite-dimensional groups. We start the orbit method for infinite-dimensional “nilpotent” group $B_0^Z$, construct the induced representations corresponding to the generic orbits and study its irreducibility.

In Section 4 we remind the Gauss decomposition of $n \times n$ matrices (Subsection 4.1), and Gauss decomposition of infinite order matrices (Subsection 4.2).

More precisely, we give the well-known definition of the induced representations for a locally compact groups in Subsection 2.1. In Subsection 2.2 we remind the Kirillov orbit method for finite-dimensional nilpotent group $G_n = B(n, \mathbb{R})$. The induced representations, corresponding to a generic orbits of the group $G_n$ are discussed in Subsection 2.3. In the Subsection 2.4 we give a new proof of the irreducibility of the induced representations corresponding to a generic orbits in order to extend the proof of the irreducibility for infinite-dimensional “nilpotent” group $B_0^Z$.

In Subsection 3.1 we remind the definition of the regular and quasiregular representations of infinite-dimensional groups. As in the case of a locally compact group these representations are the particular cases of the induced representations. This gives us the hint how to define the induced representations for infinite-dimensional groups. The definition is done in Subsection 3.2. The questions concerning the development of the orbit method for infinite-dimensional “nilpotent” group $B_0^N$ and $B_0^Z$ are discussed in Subsection 3.3.

The completions of the initial groups $G$ are necessary to the definition of the induced representations for the initial infinite-dimensional group. The completions of the inductive limit $G = \lim_{\to} G_n$ of matrix groups $G_n$ are studied in Subsection 3.4 and 3.5. We show that the Hilbert-Lie groups appear naturally in the representation theory of the infinite-dimensional matrix group. We define a family of the Hilbert-Lie group $GL_2(a)$ (resp. $B_2(a)$), a Hilbert completions of the group $GL_0(2\infty, \mathbb{R}) = \lim_{\to} GL(2n - 1, \mathbb{R})$ (resp. $B_0^Z = \lim_{\to} B(2n - 1, \mathbb{R})$). We show that any continuous representation of the group $GL_0(2\infty, \mathbb{R})$ (resp. $B_0^Z$) is in fact continuous in some stronger topology, namely in a topology of a suitable Hilbert-Lie group $GL_2(a)$ (resp. $B_2(a)$) depending on the representation.

In Subsection 3.7 we construct the induced representations of the group $B_0^Z$ corresponding to a generic orbits. The irreducibility of these representations is studied in Subsection 3.8. The very first steps to describe some part of the dual for the group $B_0^N$ and $B_0^Z$ are mentioned in Subsection 3.9.

2. Induced representations, finite-dimensional case

2.1. Induced representations. The induced representation $\text{Ind}_{H}^{G} S$ is the unitary representation of a group $G$ associated with a unitary representation $S : H \to U(V)$ of a closed subgroup $H$ of the group $G$. For details, see [7], Section 2.1. Suppose that $X = H \setminus G$ is a right $G$–space and that $s : X \to G$ is a Borel section of the projection
Let us consider the representation $T$ given by the formula

$$[T(g)f](x) = A(x, g)f(xg) = S(h) \left( \frac{d\mu_s(xg)}{d\mu_s(x)} \right)^{1/2} f(xg),$$

where $\mu_s$ is appropriately chosen, then the following equalities are valid

$$d_\tau(g) = \frac{\Delta_G(h)}{\Delta_H(h)} d\mu_s(x)d_\tau(h),$$

$$\frac{d\mu_s(xg)}{d\mu_s(x)} = \frac{\Delta_H(h(x, g))}{\Delta_G(h(x, g))},$$

where $\Delta_G$ is a modular function on the group $G$ and $h(x, g) \in H$ is defined by the relation

$$s(x)g = h(x, g)s(xg).$$

Recall that a modular function on a group $G$ is a homomorphism $G \ni t \mapsto \Delta_G(t) \in \mathbb{R}_+$ defined by the equality $h^{rt} = \Delta_G(t)h$, where $h$ is the right Haar measure on $G$, $L$ is the left action of the group $G$ on itself and $h^{rt}(C) = h(tC)$.

**Remark 2.1.** If the group $G$ is unimodular, i.e. $\Delta_G \equiv 1$, and it is possible to select a subgroup $K$ that is complementary to $H$ in the sense that almost every element of $G$ can be uniquely written in the form

$$g = hk, \ h \in H, \ k \in K,$$

then it is natural to identify $X = H \setminus G$ with $K$ and to choose $s$ as the embedding of $K$ in $G$

$$s : K \mapsto G.$$ 

In such a case, the formula (2.3) assume the form

$$dg = \Delta_H(h)^{-1} d_\tau(h)d_\tau(k).$$

If both $G$ and $H$ are unimodular (or, more generally, if $\Delta_G(h)$ and $\Delta_H(h)$ coincide for $h \in H$), then there exist a $G$-invariant measure on $X = H \setminus G$. If it is possible to extend $\Delta_H$ to a multiplicative function on the group $G$, then there exist a quasi-invariant measure on $X$ which is multiplied by the factor $\Delta_H(g)^{-1}$ under translation by $g$.

Now we can define $\text{Ind}_H^G S$ (see [7], section 2.3.). Let $S : H \to U(V)$ be a unitary representation of a subgroup $H$ of the group $G$ in a Hilbert space $V$ and let $\mu$ be a measure on $X$ satisfying condition (2.3). Let $H$ denote the space of all vector-valued functions $f$ on $X$ with values in $V$ such that

$$\|f\|^2 := \int_X \|f(x)\|^2 d\mu(x) < \infty.$$
where

\[(2.9)\]

\[A(x, g) = \left[ \frac{\Delta_H(h)}{\Delta_G(h)} \right]^{1/2} S(h),\]

and where the element \(h = h(x, g)\) is defined by formula (2.4).

**Definition 2.2.** The representation \(T\) is called the unitary induced representation and is denoted by \(\text{Ind}^G_H S\).

**Remark 2.3.** The right (or the left) regular representation \(\rho, \lambda : G \to U(L^2(G, h))\) of a locally compact group \(G\) is a particular case of the induced representation \(\text{Ind}^G_H S\) with \(H = \{e\}\) and \(S = \text{Id}\). The quasiregular representation is a particular case of the induced representation with some closed subgroup \(H \subset G\) and \(S = \text{Id}\).

2.2. **Orbit method for finite-dimensional nilpotent group** \(B(n, \mathbb{R})\). See Kirillov [6] and [7], Chapter 7, §2, p.129-130, for details. "Fix the group \(G_n = B(n, \mathbb{R})\) of all upper triangular real matrices of order \(n\) with ones on the main diagonal. (The Kirillov notation for the group \(B(n, \mathbb{R})\) is \(N_+(n, \mathbb{R})\)).

The basic result of the method of orbits, applied to nilpotent Lie groups, is the description of a one-to-one correspondence between two sets:

a) the set \(\hat{G}\) of all equivalence classes of irreducible unitary representations of a connected and simply connected nilpotent Lie group \(G\),

b) the set \(\mathcal{O}(G)\) of all orbits of the group \(G\) in the space \(g^*\) dual to the Lie algebra \(g\) with respect to the coadjoint representation.

To construct this correspondence, we introduce the following definition. A subalgebra \(h \subset g\) is subordinate to a functional \(f \in g^*\) if

\[\langle f, [x, y] \rangle = 0 \quad \text{for all} \quad x, y \in h,\]

i.e. if \(h\) is an isotropic subspace with respect to the bilinear form defined by \(B_f(x, y) = \langle f, [x, y] \rangle\) on \(g\).

**Lemma 2.4** (Lemma 7.7, [7]). The following conditions are equivalent:

(a) a subalgebra \(h\) is subordinate to the functional \(f\),

(b) the image of \(h\) in the tangent space \(T_f\Omega\) to the orbit \(\Omega\) in the point \(f\) is an isotropic subspace,

(c) the map

\[x \mapsto \langle f, x \rangle\]

is a one-dimensional real representation of the Lie algebra \(h\).

If the conditions of Lemma 2.4 are satisfied, we define the one-dimensional unitary representation \(U_{f, H}\) of the group \(H = \exp h\) by the formula

\[U_{f, H}(\exp x) = \exp 2\pi i \langle f, x \rangle.\]

**Theorem 2.5** (Theorem 7.2, [7]). (a) Every irreducible unitary representation \(T\) of a connected and simply connected nilpotent Lie group \(G\) has the form

\[T = \text{Ind}^G_H U_{f, H},\]

where \(H \subset G\) is a connected subgroup and \(f \in g^*\);

(b) the representation \(T = \text{Ind}^G_H U_{f, H}\) is irreducible if and only if the Lie algebra \(h\) of the group \(H\) is a subalgebra of \(g\) subordinate to the functional \(f\) with maximal possible dimension;
(c) irreducible representations $T_{f_1,H_1}$ and $T_{f_2,H_2}$ are equivalent if and only if the functionals $f_1$ and $f_2$ belong to the same orbit of $\mathfrak{g}^*$.

Example 2.6. Let us consider the Heisenberg group $G_3 = B(3, \mathbb{R})$, its Lie algebra $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$. Fix the notations

$$G = B(3, \mathbb{R}) = \left\{ \left( \begin{array}{ccc} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{array} \right) \right\},$$

$$\mathfrak{g} = n_+(3, \mathbb{R}) = \left\{ \left( \begin{array}{ccc} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{array} \right) \right\}, \quad \mathfrak{g}^* = n_-(3, \mathbb{R}) = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{11} & y_{12} & 0 \end{array} \right) \right\}.$$

The adjoint action $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ of the group $G$ on its Lie algebra $\mathfrak{g}$ is:

$$\mathfrak{g} \ni x \mapsto \text{Ad}_x(t) := tx^{-1} \in \mathfrak{g}, \quad t \in G,$$

the pairing between the $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$\langle y, x \rangle \mapsto \langle y, x \rangle := \text{tr}(xy) = \sum_{1 \leq k < n \leq 3} x_{kn}y_{nk} \in \mathbb{R}.$$

Since $\text{tr}(tx^{-1}y) = \text{tr}(xt^{-1}yt)$ the coadjoint action of $G$ on the dual $\mathfrak{g}^*$ to $\mathfrak{g}$ is

$$\mathfrak{g}^* \ni y \mapsto \text{Ad}^*_y(t) := (t^{-1}yt)_- \in \mathfrak{g}^*, \quad t \in G,$$

where $(z)_-$ means that we take lower triangular part of the matrix $z$.

To calculate $\text{Ad}^*_y(t)$ explicitly for $n = 3$, we have

$$t^{-1}yt = \left( \begin{array}{ccc} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{array} \right)^{-1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{11} & y_{12} & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{array} \right),$$

hence

$$\text{Ad}^*_y(t) := (t^{-1}yt)_- = \left( \begin{array}{ccc} y_{21} - t_{23}y_{31} & 0 & 0 \\ y_{31} & y_{31}t_{12} + y_{32} & y_{31}t_{13} + y_{32}t_{23} \end{array} \right).$$

We have two type of the orbits $\mathcal{O}$:

1) if $y_{31} = 0$, then $(y_{21} \ 0 \ 0) \simeq (y_{21}, y_{32})$ for fixed $y_{21}$, $y_{32}$ is 0-dimensional orbit;

2) if $y_{31} \neq 0$, then $(y_{21} \ y_{32})$ is 2-dimensional orbits.

In the case 1) the point $f = (y_{21}, y_{32})$, the subordinate subalgebra $\mathfrak{h}$ coincide with all $\mathfrak{g}$, since $[\mathfrak{g}, \mathfrak{g}] = \langle E_{13} \rangle := \{ tE_{13} \mid t \in \mathbb{R} \}$. Corresponding one-dimensional representation of the algebra $\mathfrak{h} = \mathfrak{g}$ is

$$\mathfrak{g} \ni x \mapsto \langle f, x \rangle = \text{tr}(xf) = \text{tr} \left( \begin{array}{ccc} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{11} & y_{12} & 0 \end{array} \right) = x_{12}y_{21} + x_{23}y_{32} \in \mathbb{R}.$$
The corresponding one-dimensional representations of the subalgebras $\mathfrak{h}_i$, $i = 1, 2$ are

$$\mathfrak{h}_1 \ni x \mapsto \langle f, x \rangle = x_{13}y_{31} + x_{23}y_{32} \in \mathbb{R},$$
$$\mathfrak{h}_2 \ni x \mapsto \langle f, x \rangle = x_{12}y_{21} + x_{13}y_{31} \in \mathbb{R}.$$  

The corresponding representations $S$ of the subgroups $H_1$ and $H_2$ respectively are:

$$H_1 \ni \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(x) \mapsto \exp(2\pi i (x_{13}y_{31} + x_{23}y_{32})) \in S^1,$$

$$H_2 \ni \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(x) \mapsto \exp(2\pi i (x_{12}y_{21} + x_{13}y_{31})) \in S^1.$$  

In the case $H_1$ we have the decomposition $G_3 = \mathbb{R}^2 \ltimes B(2, \mathbb{R}) \simeq H_1 \ltimes \mathbb{R}$, indeed we have

$$G_3 \ni \begin{pmatrix} x_{12} & x_{13} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{12} & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^2 \ltimes B(2, \mathbb{R}),$$

hence the space $X = H_1 \backslash G_3$ is isomorphic to $B(2, \mathbb{R}) \simeq \mathbb{R}$ and $s$ can be choosing as the embedding $s : B(2, \mathbb{R}) \mapsto B(3, \mathbb{R})$.

$$B(2, \mathbb{R}) \ni \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: x \mapsto s(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in B(3, \mathbb{R}).$$

For general $n$ we have

(2.14)  

$$B(n+1, \mathbb{R}) = \mathbb{R}^n \ltimes B(n, \mathbb{R}).$$

To calculate the right action of $G$ on $X$ i.e. to find $h(x, t)$ such that

$$s(x)t = h(x, t)s(xt),$$

we have for $x \in B(2, \mathbb{R})$ and $t \in B(3, \mathbb{R})$

$$s(x)t = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + t_{12} & t_{13} + xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x + t_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}

= h(x, t)s(xt),$$

hence $h(x, t) = \begin{pmatrix} 1 & t_{12} & t_{13} + xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}.$

Finally, the induced unitary representation $\text{Ind}_{H_1}^G S$ have the following form in the Hilbert space $L^2(\mathbb{R}, dx)$ (case $H_1$ and $f = y_{31}E_{31}$):

(2.15)  

$$f(x) \mapsto S(h(x, t))f(xt) = \exp(2\pi i (t_{13} + t_{23}x)y_{31})f(x + t_{12}).$$

In the Kirillov [7] notations we have:

$$f(x) \mapsto \exp(2\pi i (c + bx)\lambda)f(x + a), \quad y_{31} = \lambda, \quad \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \cdot c \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$  

2.3. The induced representations, corresponding to a generic orbits, finite-dimensional case. We show following A. Kirillov [7] how the orbit method works for the nilpotent group $B(n, \mathbb{R})$ and small $n$.

For general $n \in \mathbb{N}$ the coadjoint action of the group $G_n$ on $\mathfrak{g}$ is as follows

$$t = I + \sum_{1 \leq k < m \leq n} t_{km}E_{km}, \quad y = \sum_{1 \leq m < k \leq n} y_{km}E_{km}, \quad t^{-1} := I + \sum_{1 \leq k < m \leq n} t^{-1}_{km}E_{km},$$

hence

$$(tyt^{-1})_{pq} = \sum_{m=1}^{q} (ty)_{pm}t^{-1}_{mq} = \sum_{m=1}^{q} \sum_{r=p}^{n} t_{pr}y_{rm}t^{-1}_{mq}, \quad 1 \leq p, q \leq n,$$

and

(2.16)  

$$\text{Ad}_t^*(y) = (t^{-1}yt)_{-} = I + \sum_{1 \leq q < p \leq n} (t^{-1}yt)_{pq}E_{pq}.$$
Example 2.7. Generic orbits for the group $G = B(n, \mathbb{R})$ (see [7], Example 7.9).

"The form of the action $\text{Ad}_t^*(y) = (t^{-1}yt)_-$ implies that $\text{Ad}_t^*$, $t \in G$ acts as follows: to a given column of $y \in \mathfrak{g}^*$, a linear combination of the previous columns is added and to a given row of $y$, a linear combination of the following rows is added. More generally, the minors $\Delta_k$, $k = 1, 2, ..., \left[\frac{n}{2}\right]$, consisting of the last $k$ rows and first $k$ columns of $y$ are invariant of the action. It is possible to show that if all the numbers $c_k$ are different from zeros, then the manifold given by the equation

$$\Delta_k = c_k, \ 1 \leq k \leq \left[\frac{n}{2}\right]$$

is a $G$-orbit in $\mathfrak{g}^*$. Hence generic orbits have codimension equal to $\left[\frac{n}{2}\right]$ and dimension equal to $\frac{n(n-1)}{2} - \left[\frac{n}{2}\right]$. To obtain a representation for such an orbit, we can take a matrix $y$ of the form

$$y = \begin{pmatrix} 0 & 0 \\ \Lambda & 0 \end{pmatrix},$$

where $\Lambda$ is the matrix of order $\left[\frac{n}{2}\right]$ such that all nonzero elements are contained in the anti-diagonal. It is easy to find a subalgebra of dimension $\left[\frac{n}{2}\right] \times \left[\frac{n+1}{2}\right]$ subordinate to the functional $y$. It consist of all matrices of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A$ is an $\left[\frac{n}{2}\right] \times \left[\frac{n+1}{2}\right]$ or $\left[\frac{n+1}{2}\right] \times \left[\frac{n}{2}\right]$ matrix."

Example 2.8. Let $G = B(5, \mathbb{R})$, $\mathfrak{g} = n_+(5, \mathbb{R})$, $\mathfrak{g}^* = n_-(5, \mathbb{R})$. We write the representations for generic orbit corresponding to the point $y = y_{51}E_{51} + y_{42}E_{42} \in \mathfrak{g}^*$. Set $\mathfrak{h}_3 = \{t - I \mid t \in H_3\}$ where

$$G = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad H_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & t_{14} & t_{15} \\ 0 & 1 & 0 & t_{24} & t_{25} \\ 0 & 0 & 1 & t_{34} & t_{35} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad \mathfrak{g}^* = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

The corresponding representation $S$ of the subgroup $H_3$ of the maximal dimension is:

$$H_3 \ni t \mapsto \exp(2\pi i(y(t - I))) = \exp(2\pi i[t_{15}y_{51} + t_{24}y_{42}]) \in S^1.$$

For the group $B(5, \mathbb{R})$ holds the following decomposition

$$B(5, \mathbb{R}) = B_3B(3)B(3) \text{ i.e. } x = x_3x(3)x(3),$$

where

$$B(3) = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad B(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} \\ 0 & 1 & 0 & x_{24} & x_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad B_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

We calculate $h(x, t)$ in the relation $s(x)t = h(x, t)s(xt)$, but first we fix the section $s : X = H \setminus G \mapsto G$ of the projection $p : G \mapsto X$. To define the section $s : X \mapsto G$ we show that in addition to the decomposition (2.18) the following decomposition $B(5, \mathbb{R}) = B(3)B_3B(3)$ also holds. Indeed, to find $h \in H_3 = B(3)$ such that $x = hx_3x(3)$, we get $x_3x(3)x(3) = hx_3x(3)$, hence

$$h = x_3x(3)x_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_{45} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} - x_{14}x_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x_{24} & x_{25} - x_{24}x_{45} & x_{25}x_{45} \\ 0 & 0 & 1 & x_{34} & x_{35} - x_{34}x_{45} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in B(3).$$

We have two different decompositions

$$B_3B(3)B(3) \ni x_3x(3)x(3) = hx_3x(3) \in B(3)B_3B(3), \quad \text{with } h = x_3x(3)x_3^{-1}. $$
Remark 2.9. For an arbitrary \( n, m \in \mathbb{N}, 1 < m < n \), we have for the group \( G_n = B(n, \mathbb{R}) \) two decompositions:

\[
G_n = B_m B(m) B(m) \ni x_m x(m) x^{(m)} = h x_m x^{(m)} \in B(m) B_m B(m), \quad h = x_m x(m) x_m^{-1},
\]

where

\[
B_m = \{ I + \sum_{m < k < r \leq n} x_{kr} E_{kr} \}, \quad B(m) = \{ I + \sum_{1 \leq k \leq m < r \leq n} x_{kr} E_{kr} \}, \quad B^{(m)} = \{ I + \sum_{1 \leq k < r \leq m} x_{kr} E_{kr} \}.
\]

Since \( X = B(m) \setminus G_n \) is isomorphic to \( B_m B^{(m)} \) by decomposition (2.19), the section \( s \) can be choosing, by Remark 2.1, as the embedding

\[
B_m B^{(m)} \ni x_m x^{(m)} \mapsto s(x_m x^{(m)}) = x_m x^{(m)} \in B_m B(m) B(m).
\]

Since \( s(x)t = h(x,t)s(x) \), we have \( h(x,t) = s(x)t(s(x))^{-1} \). It remains to calculate \( s(x)t \) and \( s(x) \).

Remark 2.10. We have

\[
h(x,t) - I = \begin{cases} 0, & \text{for } t \in B_m B^{(m)} \\ x^{(m)}(t - I) x_m^{-1}, & \text{for } t \in B(m) \\ \end{cases}
\]

Indeed, let \( t = t_m t^{(m)} \in B_m B^{(m)} \) then \( s(x)t = x_m x^{(m)} t_m t^{(m)} = x_m t_m x^{(m)} t^{(m)} \). We get also \( xt = x_m x^{(m)} t_m t^{(m)} = x_m t_m x^{(m)} t^{(m)} \), so \( s(x)t = x_m t_m x^{(m)} t^{(m)} \), hence \( s(x)t = s(x) \) and we get \( h(x,t) = e \). For \( t := t(m) \in B(m) \) and \( x = x_m x^{(m)} \in B_m B(m) \) we get

\[
s(x)t = x_m x^{(m)} t = x_m x^{(m)} t x^{(m)} x^{-1} x_m = x_m x^{(m)} \pi \bar{x}(m) x^{(m)} = h x_m x^{(m)} = h(x,t) s(x),
\]

where \( \bar{x}(m) = x^{(m)} t(x^{(m)})^{-1} \). Then we get by (2.19)

\[
h(x,t) = h = x_m \bar{x}(m) x^{-1} = x_m x^{(m)} t x^{(m)} x^{-1} = x_m x^{(m)} t x_m x^{(m)} x^{-1},
\]

(2.21)

\[
h(x,t) = \left( \begin{array}{cc} x^{(m)} & 0 \\ 0 & x_m \end{array} \right) \left( \begin{array}{cc} 1 & t-I \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} x^{(m)} & 0 \\ 0 & x_m \end{array} \right) = \left( \begin{array}{cc} 1 & x^{(m)}(t-I)x_m \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) H(x,t),
\]

where

\[
H(x,t) := x^{(m)}(t - I)x_m^{-1}.
\]

Denote by \( E_{kr}(t) := I + t E_{kr}, t \in \mathbb{R} \) the one-parameter subgroups of the groups \( B(n, \mathbb{R}) \). We would like to find the generators \( A_{kn} = \frac{d}{dt} T_{I+tE_{kn}} |_{t=0} \) of the induced representation \( T_t \) (2.28).

Set for \( G_n = B_m B(m) B(m) \) and \( 1 \leq k \leq m < r \leq n \)

\[
S_{kr}(t) := \langle y, (h(x,E_{kr}(tkr)) - I) \rangle, \quad A_{kr} = \frac{d}{dt} \exp(2\pi i S_{kr}(t)) |_{t=0} = 2\pi i S_{kr}(1).
\]

Let us denote by \( S \) the following matrix:

\[
S = (S_{kr})_{1 \leq k \leq m < r \leq n}, \quad \text{where} \quad S_{kr} = S_{kr}(1), \quad \text{then} \quad S = (2\pi i)^{-1}(A_{kr})_{k,r}.
\]

Lemma 2.11. Let \( B = (b_{kr})_{k=1}^{n} \in \text{Mat}(n, \mathbb{C}) \). Define the matrix \( C = (c_{kr})_{k=1}^{n} \in \text{Mat}(n, \mathbb{C}) \) by

\[
c_{kr} = \text{tr}(E_{kr} B), \quad 1 \leq k, r \leq n, \quad \text{then we have} \quad C = B^T,
\]

where \( E_{kr} \) are matrix units and \( B^T \) means transposed matrix to the matrix \( B \). The equality \( C = B^T \) holds also in the case when \( B \) is an arbitrary \( m \times n \) rectangular matrix. The statement is true also for matrices \( B \in \text{Mat}(\infty, \mathbb{C}) \).
Proof. Indeed, we have $\text{tr}(E_{kr}B) = b_{kr}$.\hfill\qed

We calculate now the matrix $S(t) = (S_{kr}(t_{kr}))_{k,r}$ and the matrix $S = (S_{kr}(1))_{k,r}$ using Lemma 2.11. Using (2.22) we have
\[
\langle y, h(x,t) - I \rangle = \text{tr}(H(x,t)y) = \text{tr}\left(x^{(m)}t_0x_m^{-1}y\right) = \text{tr}\left(t_0x_m^{-1}yx^{(m)}\right) = \text{tr}\left(t_0B(x,y)\right),
\]
where $t_0 = t - I$ and
\[
B(x,y) = x_m^{-1}yx^{(m)} \simeq \left( \begin{array}{cc} 1 & 0 \\ 0 & x_m^{-1} \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ y & 0 \end{array} \right) \left( \begin{array}{cc} x^{(m)} & 0 \\ 0 & x_m^{-1} \end{array} \right) = \left( \begin{array}{cc} x_m^{-1} & 0 \\ 0 & yx^{(m)} \end{array} \right).
\]
By definition we have
\[
S_{kr}(t_{kr}) = \langle y, (h(x,E_{kr}(t_{kr}))-I) \rangle = \text{tr}(t_{kr}E_{kr}B(x,y)),
\]
hence by Lemma 2.11 and (2.26) we conclude that
\[
S = (S_{kr}(1))_{k,r} = (\text{tr}(E_{kr}B(x,t)))_{k,r} = B^T(x,y) = (x^{(m)})^Ty^T(x_m^{-1})^T = \left( \begin{array}{cc} 0 & (x^{(m)})^Ty^T(x_m^{-1})^T \end{array} \right).
\]
So the induced representation $\text{Ind}_G^h(S) : G \to U(L^2(X,\mu))$ corresponding to the point $y \in g^*$ has the following form
\[
(2.28) \quad (T_{i,f})(x) = S(h(x,t)) \left( \frac{d\mu(x)}{d\mu(t)} \right)^{1/2} f(x), \quad f \in L^2(X,\mu), \quad x \in X = H\backslash G, \quad t \in G,
\]
where
\[
(2.29) \quad S(h(x,t)) = \exp(2\pi i \langle y, (h(x,t) - I) \rangle) = \exp\left(2\pi i \text{tr}((t - I)B(x,y))\right).
\]
We calculate $B(x,y)$ and $S$ for different groups $G_n$. For $G_5$ we get by (2.26):
\[
G_5 = \left\{ \left( \begin{array}{cccc} x_{12} & x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & 0 & 0 \\ x_{34} & x_{35} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) : \left\{ \begin{array}{c} x_{12} = \ldots = x_{35} \end{array} \right\} \right\}, \quad y = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad x^{(3)} = \left( \begin{array}{cccc} 1 & x_{12} & x_{13} \\ x_{12} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad x_3 = \left( \begin{array}{c} x_{45} \end{array} \right),
\]
\[
B(x,y) = \left( \begin{array}{cccc} x_{12}^{-1} & 0 & 0 & 0 \\ 0 & y_{12} & 0 & 0 \\ 0 & 0 & y_{35} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad x_3 = \left( \begin{array}{c} x_{45} \end{array} \right),
\]
hence by (2.27) we have
\[
(2.30) \quad S := B(x,y)^T = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & x_{12} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{35} \end{array} \right) \left( \begin{array}{c} x_{45} \\ y_{12} \\ y_{35} \\ y_{34} \end{array} \right) = \left( \begin{array}{c} x_{45} y_{12} x_{13} \\ y_{12} x_{13} y_{14} \\ y_{35} x_{14} \\ y_{35} x_{13} \end{array} \right).
\]

Remark 2.12. For the matrix $x = I + \sum_{1 \leq k < n \leq m} x_{kn}E_{kn} \in B(m,\mathbb{R})$ we denote by $x_{kn}^{-1}$ the matrix elements of the matrix $x^{-1}$, i.e. $x^{-1} = : I + \sum_{1 \leq k < n \leq m} x_{kn}^{-1}E_{kn} \in B(m,\mathbb{R})$. The explicit expressions for $x_{kn}^{-1}$ are as follows (see [3], formula (4.4))
\[
(2.31) \quad x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r-1} \sum_{k < i_1 < i_2 < \ldots < i_r < n} x_{ki_1,i_1i_2,\ldots,i_{r-1},i_r,n}, \quad k < n - 1.
\]

The generators $A_{kn} = \frac{d}{dt}T_{i,tE_{kn}}|_{t=0}$ of the one-parameter subgroups $E_{kn}(t) := I + tE_{kn}$, $t \in \mathbb{R}$ generated by the representation $T_t$ (2.28) are as follows (see (2.24) and (2.30))
\[
(2.32) \quad A_{12} = D_{12}, \quad A_{13} = D_{13}, \quad A_{23} = x_{12}D_{13} + D_{23}, \quad A_{45} = D_{45},
\]
\[
(2.33) \quad S = \frac{1}{2\pi i} \left( \begin{array}{cccc} A_{14} & A_{15} \\ A_{24} & A_{25} \\ A_{34} & A_{35} \end{array} \right) = \left( \begin{array}{cccc} x_{45} y_{12} x_{13} \\ y_{12} x_{13} y_{14} \\ y_{35} x_{14} \\ y_{35} x_{13} \end{array} \right).
\]
where $D_{kn} = \frac{\partial}{\partial x_{kn}}$. For example, to obtain the expression $A_{23} = x_{12}D_{13} + D_{23}$ we note that

$$B(3, \mathbb{R}) \ni x(I + tE_{23}) = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_{12} + tx_{12} & x_{13} + tx_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here we denote by $D_{kn} = D_{kn}(h)$ the operator of the partial derivative corresponding to the shift $x \mapsto x + tE_{kn}$ on the group $B_m \times B^{(m)} \ni x = (x_{kn})_{k,n}$ and the Haar measure $h$:

$$(2.34) \quad (D_{kn}(h)f)(x) = \frac{d}{dt} \left( \frac{df(x + tE_{kn})}{dt} \right)^{1/2} f(x + tE_{kn}) \big|_{t=0}, \quad D_{kn}(h) := \frac{\partial}{\partial x_{kn}}.$$

**Example 2.13.** Let $G = B(4, \mathbb{R}) = \{ \begin{pmatrix} 1 & x_{23} & x_{24} & x_{25} \\ 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 1 \end{pmatrix} \}$. The representations for generic orbit corresponding to the point $y = y_{43}E_{43} + y_{52}E_{52} \in g^*$. We calculate $S$ in two different ways. First using (2.26) we get

$$B(x, y) = x^{-1}_m y x^{(m)} = \begin{pmatrix} 1 & x_{23}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{43} & 0 \\ 0 & y_{52} \end{pmatrix} \begin{pmatrix} 1 & x_{23} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{43}x_{23}^{-1}y_{52} \\ y_{52} & x_{23}\end{pmatrix},$$

$$\frac{1}{2\pi i} \left( \begin{array}{c} A_{24} \\ A_{34} \\ A_{35} \end{array} \right) = S = B^T(x, y) = \begin{pmatrix} 1 & x_{23} \\ 0 & y_{43} \end{pmatrix} \begin{pmatrix} 0 & y_{52} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{52} \\ y_{52} & y_{52} \end{pmatrix},$$

$$A_{23} = D_{23}, \quad A_{45} = D_{45}.$$

From the other hand, by (2.21) we get $h(x, t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, where

$$(2.35) \quad H(x, t) = x^{(3)}(t-I)x^{-3} = \begin{pmatrix} 1 & x_{23} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_{24} & t_{25} \\ t_{34} & t_{35} \end{pmatrix} \begin{pmatrix} 1 & x_{23}^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_{24} + t_{23}t_{34}x_{23}^{-1} + t_{25} + t_{23}t_{35} \\ t_{34}x_{23}^{-1} + t_{25} + t_{23}t_{35} \end{pmatrix}.$$

Therefore,

$$\langle y, (h(x, t) - I) \rangle = h(x, t)^{34}y_{43} + h(x, t)^{25}y_{52} = t_{34}y_{43} + [(t_{24} + t_{23}t_{34})x_{23}^{-1} + t_{25} + t_{23}t_{35}]y_{52},$$

hence

$$S_2(t) := \begin{pmatrix} s_{24}(t_{24}) & s_{25}(t_{25}) \\ s_{34}(t_{34}) & s_{35}(t_{35}) \end{pmatrix} = \begin{pmatrix} t_{24}x_{23}^{-1} & t_{25}y_{52} \\ t_{34}y_{43} + t_{23}t_{34}x_{23}^{-1} & t_{23}t_{35}y_{52} \end{pmatrix}.$$

$$(2.36) \quad S_2 := S_2(1) = \begin{pmatrix} x_{23}^{-1}y_{52} \\ y_{52} \end{pmatrix} \begin{pmatrix} 1 & x_{23}^{-1}y_{52} \\ 0 & y_{52} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & y_{52} \end{pmatrix} \begin{pmatrix} 0 & y_{52} \\ 1 & 0 \end{pmatrix}. $$

**Example 2.14.** Let $G = B(6, \mathbb{R}), \quad g = \mathfrak{n}_+ (6, \mathbb{R}), \quad g^* = \mathfrak{n}_- (6, \mathbb{R})$. We write the representations for generic orbit corresponding to the point $y = y_{43}E_{43} + y_{52}E_{52} + y_{61}E_{61} \in g^*$. Set

$$G_6 = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad H_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & t_{14} & t_{15} & t_{16} \\ 0 & 1 & 0 & t_{24} & t_{25} & t_{26} \\ 0 & 0 & 1 & t_{34} & t_{35} & t_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{n}_3 = \{ t - I \mid t \in H_3 \}$. The corresponding representations $S$ of the subgroup $H_3$ is:

$$H_3 \ni \exp(t - I) = t \mapsto \exp(2\pi i \langle y, (t - I) \rangle) = \exp(2\pi i [t_{34}y_{43} + t_{25}y_{52} + t_{61}y_{61}]) \in S^1.$$

For the group $B(6, \mathbb{R})$ holds the following decomposition (see Remark 2.9)

$$(2.37) \quad B(6, \mathbb{R}) = B_3B(3)B(3) \quad \text{i.e.} \quad x = x_3x(3)x^{(3)},$$

where

$$x^{(3)} = \begin{pmatrix} 1 & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 1 & x_{23} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} & x_{16} \\ 0 & 1 & 0 & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. $$
We get by (2.26) and (2.27)

\[ B(x, y) = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{56}^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{43} \\ y_{52} & 0 & 0 \\ y_{61} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} \\ x_{56}^{-1} y_{61} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} \end{pmatrix} \]

hence

\[ S = B^T(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ 0 & y_{43} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{61} \\ y_{52} & 0 & 0 \\ y_{45}^{-1} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_{46}^{-1} & 1 & 0 \\ 0 & x_{46}^{-1} & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} \\ x_{56}^{-1} y_{61} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} \end{pmatrix} \]

Using again (2.24), (2.28) and Remark 2.10 we get the following expressions for the generators \( A_{kn} = \frac{4}{d t} T_{t + E_{kn}} \mid t = 0 \) of one-parameter subgroups \( I + t E_{kn}, \ t \in \mathbb{R} \):

\[ (2.38) \quad A_{12} = D_{12}, \ A_{13} = D_{13}, \ A_{23} = x_{12} D_{13} + D_{23}, \]

\[ (2.39) \quad A_{45} = D_{45}, \ A_{46} = D_{46}, \ A_{56} = x_{45} D_{46} + D_{56}, \]

\[ (2.40) \quad S = \frac{1}{2 \pi i} \begin{pmatrix} A_{14} & A_{15} & A_{16} \\ A_{24} & A_{25} & A_{26} \\ A_{34} & A_{35} & A_{36} \end{pmatrix} = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} \\ y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} & y_{56}^{-1} y_{61} x_{13} \end{pmatrix} \]

We recall the expressions for \( B(x, y) \) and hence for \( S = B(x, y)^T \) for small \( n \). For \( n = 4 \) we have

\[ B(x, y) = x_m^{-1} y_m^{(m)} = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{56}^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{43} \\ y_{52} & 0 & 0 \\ y_{61} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1} y_{52} & y_{43} + x_{45}^{-1} y_{52} x_{23} \\ y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} \end{pmatrix} \]

\[ S = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ 0 & y_{43} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{61} \\ y_{52} & 0 & 0 \\ y_{45}^{-1} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_{46}^{-1} & 1 & 0 \\ 0 & x_{46}^{-1} & 1 \end{pmatrix} \]

For \( G_2^3 \simeq B(6, \mathbb{R}) \) (see (2.41) for the notation \( G_n^m \)) holds:

\[ B(x, y) = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{56}^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{43} \\ y_{52} & 0 & 0 \\ y_{61} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{46}^{-1} y_{61} & x_{45}^{-1} y_{52} + x_{46}^{-1} y_{61} x_{12} & y_{43} + x_{45}^{-1} y_{52} x_{23} + x_{46}^{-1} y_{61} x_{13} \\ x_{56}^{-1} y_{61} & y_{52} + x_{56}^{-1} y_{61} x_{12} & y_{52} x_{23} + x_{56}^{-1} y_{61} x_{13} \end{pmatrix} \]

hence

\[ S = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ 0 & y_{43} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{61} \\ y_{52} & 0 & 0 \\ y_{45}^{-1} & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_{46}^{-1} & 1 & 0 \\ 0 & x_{46}^{-1} & 1 \end{pmatrix} \]

For \( G_3^3 \simeq B(8, \mathbb{R}) \) holds:

\[ y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
As before we have
\[
B(x, y) = \begin{pmatrix}
1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\
0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\
0 & 0 & 1 & x_{67}^{-1} \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
S = (x^{(m)})^T y^T (x^{-1})^T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
x_{01} & 1 & 0 & 0 \\
x_{02} & x_{12} & 1 & 0 \\
x_{03} & x_{13} & x_{23} & 1
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & y_{43} \\
0 & 0 & y_{52} & 0 \\
y_{70} & 0 & 0 & 0
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
x_{12} & x_{13} & 0 & 0 \\
x_{23} & 0 & 0 & 0 \\
x_{47} & x_{57} & 0 & 0
\end{pmatrix}.
\]

2.4. New proof of the irreducibility of the induced representations corresponding to a generic orbits.

Remark 2.15. By Kirillov’s Theorem [2.3] the induced representation \( T_{f, H} = \text{Ind}^G_H U_{f, H} \) is irreducible if and only if the Lie algebra \( \mathfrak{h} \) of the group \( H \) is a subalgebra of \( \mathfrak{g} \) subordinate to the functional \( f \) with maximal possible dimension.

The condition of “maximal possible dimension” is difficult to extend for the infinite-dimensional case. That is why in this section we give another proof of the irreducibility of the induced representation of a nilpotent group \( B(n, \mathbb{R}) \) that will be extended in Section 7.8 for the infinite-dimensional analog \( B_0^\mathbb{Z} \) of the group \( B(n, \mathbb{R}) \).

Let us consider a sequence of a Lie groups \( G_n^m \) and its Lie algebras \( \mathfrak{g}_n^m \), \( m \in \mathbb{Z}, n \in \mathbb{N} \) defined as follows
\[
G_n^m = \{ I + \sum_{m-n \leq k < n \leq m+n+1} x_{kn} E_{kn} \}, \quad \mathfrak{g}_n^m = \{ \sum_{m-n \leq k < n \leq m+n+1} x_{kn} E_{kn} \}.
\]

We note that for any \( m \in \mathbb{N} \) holds \( B_0^\mathbb{Z} = \lim_{n \to \infty} G_n^m \). We have the decomposition (see (2.9))
\[
G_n^m = B_{m,n} B(m, n) B^{(m,n)},
\]
where
\[
B_{m,n} = \{ I + \sum_{(k,r) \in \Delta_{m,n}} x_{kr} E_{kr} \}, \quad B(m,n) = \{ I + \sum_{(k,r) \in \Delta_{m,n}} x_{kr} E_{kr} \},
\]
\[
B^{(m,n)} = \{ I + \sum_{(k,r) \in \Delta_{m,n}} x_{kr} E_{kr} \},
\]
and
\[
\Delta_{m,n} = \{(k,r) \in \mathbb{Z}^2 \mid m-n \leq k \leq m < r \leq m+n+1 \},
\]
\[
\Delta_{m,n} = \{(k,r) \in \mathbb{Z}^2 \mid m+1 \leq k \leq r \leq m+n+1 \},
\]
\[
\Delta^{(m,n)} = \{(k,r) \in \mathbb{Z}^2 \mid m-n \leq k < r \leq m \}.
\]

The corresponding elements of the group \( G_n^m \) are as follows
\[
\begin{pmatrix}
1 & x_{m-n,m-n+1} & x_{m-n,m-1} & x_{m-n,m} & t_{m-n,m+1} & t_{m-n,m+2} & \ldots & t_{m-n,m+n+1} \\
0 & 1 & 0 & \ldots & x_{m-n,1} & x_{m-n,0} & \ldots & 1 \\
0 & 0 & \ldots & 1 & x_{m-1,1} & x_{m-1,0} & \ldots & 1 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]
The induced representation of the group $G^m_n$ is defined in the space $L^2(X,d\mu)$ by the following formula

$$(T^{m,y_n}_t f)(x) = S(h(x,t)) \left( \frac{d\mu(xt)}{d\mu(x)} \right)^{1/2} f(x), \quad f \in L^2(X,\mu), \ x \in X = H\backslash G, \ t \in G$$

where $X = B(m,n) \backslash G^m_n \cong B_{m,n} \times B^{(m,n)}$ (see (2.11)),

$$(2.43) \quad d\mu(x_m, x^{(m)}) = dx_m \otimes dx^{(m)} = \otimes_{(k,n) \in \Delta(m,n)} dx_{kn} \otimes \otimes_{(k,n) \in \Delta(m,n)} dx_{kn}$$

be the Haar measure on the group $B_{m,n} \times B^{(m,n)}$. Denote by $H_{m,n} = L^2(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)})$.

Theorem 2.16. The induced representation $T^{m,y_n}_n$ of the group $G^m_n$ defined by formula (2.42), corresponding to generic orbit $O_{y_n}$, generated by the point $y_n \in (\mathbb{C}^*)^n$,

$y_n = \sum_{r=0}^{n-1} y_{m+r+1,m-r} E_{m+r+1,m-r}$

is irreducible. Moreover the generators of one-parameter groups $A_{kr} = \frac{\partial}{\partial t} T^{m,y_n}_t |_{t=0}$ are as follows

$$A_{kr} = \sum_{s=m-n}^{k-1} x_{ks} D_{rs} + D_{kr}, \quad (k,r) \in \Delta(m,n), \quad A_{kr} = \sum_{s=m+1}^{k-1} x_{ks} D_{rs} + D_{kr}, \quad (k,r) \in \Delta(m,n),$$

$$(2\pi i)^{-1} (A_{kr})_{(k,r) \in \Delta(m,n)} = S^{(m)}(x) = (S_{kr})_{(k,r) \in \Delta(m,n)} = (x_m^{-1} y x^{(m)})^T.$$

The irreducibility of the induced representation of the group $G^m_n$ is based on the following lemma.

Lemma 2.17. Two von Neumann algebra $\mathfrak{A}^S$ and $\mathfrak{A}^x$ in the space $H_{m,n}$ generated respectively by the sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$ coincides, where

$$(2.44) \quad (U_{kr}(t)f)(x) = \exp(2\pi i S_{kr}(t))f(x), \quad (V_{kr}(t)f)(x) := \exp(2\pi i t x_{kr})f(x),$$

$\mathfrak{A}^S = (U_{kr}(t) = T^{m,y_n}_{t+tE_{kr}} = \exp(2\pi i S_{kr}(t)) | t \in \mathbb{R}, (k,r) \in \Delta(m,n))''$,

$\mathfrak{A}^x = (V_{kr}(t) := \exp(2\pi i t x_{kr}) | t \in \mathbb{R}, (k,r) \in \Delta(m,n) \cup \Delta(m,n))''$.

Proof. Using the decomposition (see (2.26) and (2.27))

$$(2.45) \quad S^{(m)}(x) = (x^{-1}_m y x^{(m)})^T = (x^{(m)})^T y T(x^{-1}_m)^T$$

we conclude that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$. Indeed, we get $V_{kr}(t) := \exp(2\pi i t x_{kr}) \in \mathfrak{A}^x$ hence the operators $x_{kr}$ of multiplication by the independent variable $f(x) \mapsto x_{kr} f(x)$ in the space $H_{m,n}$ are affiliated with the von Neumann algebra $\mathfrak{A}^x$, i.e. $x_{kr} \in \mathfrak{A}^x$ for $(k,r) \in \Delta(m,n) \cup \Delta(m,n)$.

Definition 2.18. Recall (c.f. e.g. [3]) that a non necessarily bounded self-adjoint operator $A$ in a Hilbert space $H$ is said to be affiliated with a von Neumann algebra $M$ of operators in this Hilbert space $H$, if $\exp(itA) \in M$ for all $t \in \mathbb{R}$. One then writes $A \in M$.

By (2.31) the matrix elements $x^{-1}_{kr}$ of the matrix $x^{-1}_{m} \in B_{m,n}$ are also affiliated $\eta \mathfrak{A}^x$. Using (2.45) we conclude that the matrix elements $S_{kr}, \in \Delta(m,n)$ of the matrix $S^{(m)}(x)$ are affiliated: $S_{kr} \in \mathfrak{A}^x$, $(k,r) \in \Delta(m,n)$, so $\mathfrak{A}^S \subseteq \mathfrak{A}^x$.

To prove that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$ we find the expressions of the matrix element of the matrix $x^{(m)} \in B^{(m,n)}$ and $x^{-1}_m \in B_{m,n}$ in terms of the matrix elements of the matrix $S^{(m)}(x) = (S_{kr})_{(k,r) \in \Delta(m,n)}$. To do that we connect the above decomposition $S^{(m)}(x) = $
Using the Theorem 4.1 we can find the matrix elements of the matrix \( x \) for \( t \) and \( L \). The latter decomposition (2.46) is in fact the Gauss decomposition of the matrix \( S J \) i.e. we get

\[
SJ = LDU, \quad \text{where} \quad L = (x^{(m)})^T, \quad D = y^T J, \quad U = J(x_m^{-1})^T J.
\]

Using the Theorem 4.1 we can find the matrix elements of the matrix \( x^{(m)} \in B^{(m,n)} \) and \( x_m^{-1} \in B_{m,n} \) in terms of the matrix elements of the matrix \( S_n^{(m)} \), hence we can also find the matrix elements of the matrix \( x_m \in B_{m,n} \). This finish the proof of the lemma. \( \square \)

We give below the expressions for \( S_n J \). For \( m = 3 \) and \( n = 1 \) i.e. for \( G_1^3 \) we have (remind that \( J^2 = I \))

\[
S_2 = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{12} & 0 \\ y_{43} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{12} & 0 \\ y_{43} & 0 \end{pmatrix} \begin{pmatrix} x_{23}^{-1} & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
S_2 J = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{12} & 0 \\ y_{43} & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{23}^{-1} \\ 0 & 1 \end{pmatrix}.
\]

For \( G_2^3 \) we get

\[
S_3 = \begin{pmatrix} x_{12} & 0 \\ x_{13} & x_{23} \end{pmatrix} \begin{pmatrix} y_{41} & 0 \\ 0 & y_{43} \end{pmatrix} \begin{pmatrix} x_{46} x_{56}^{-1} & x_{46}^{-1} \\ x_{45} & 1 \end{pmatrix},
\]

\[
S_3 J = \begin{pmatrix} x_{12} & 0 \\ x_{13} & x_{23} \end{pmatrix} \begin{pmatrix} y_{41} & 0 \\ 0 & y_{43} \end{pmatrix} \begin{pmatrix} 1 & x_{46}^{-1} \\ 0 & 1 \end{pmatrix}.
\]

For \( G_3^3 \) we have

\[
S_4 = \begin{pmatrix} x_{10} & 0 \\ x_{02} & x_{12} \\ x_{03} & x_{13} & x_{23} \end{pmatrix} \begin{pmatrix} y_{70} & 0 \\ 0 & y_{60} \\ 0 & y_{63} & 0 \end{pmatrix} \begin{pmatrix} x_{47} x_{67}^{-1} & x_{47}^{-1} \\ x_{46} x_{56}^{-1} & 1 \\ x_{45} & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
S_4 J = \begin{pmatrix} x_{10} & 0 \\ x_{02} & x_{12} \\ x_{03} & x_{13} & x_{23} \end{pmatrix} \begin{pmatrix} y_{70} & 0 \\ 0 & y_{60} \\ 0 & y_{63} & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{46}^{-1} \\ 0 & 1 \end{pmatrix}.
\]

Proof of the Theorem 2.16 The irreducibility follows from the Kirillov results (see Remark 2.15). To give another proof of the irreducibility of the induced representation consider the restriction \( T^{m,y_n} |_{B(m,n)} \) of this representation to the commutative subgroup \( B(m,n) \) of the group \( G_{m,n}^n \). Note that

\[
\mathfrak{A}^x = \{ \exp(2\pi i tx_{kr}) \mid t \in \mathbb{R}, \quad (k, r) \in \Delta_{m,n} \bigcup \Delta^{(m,n)} \bigcup \} = L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)}).
\]

By Lemma 2.17 the von Neumann algebra \( \mathfrak{A}^S \) generated by this restriction coincides with \( L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)}) \). Let now a bounded operator \( A \) in a Hilbert space \( \mathcal{H}^{m,n} \) commute with the representation \( T^{m,y_n} \). Then \( A \) commute by the above arguments with \( L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)}) \), therefore the operator \( A \) itself is an operator of multiplication by some essentially bounded function \( a \in L^\infty \) i.e. \( (Af)(x) = a(x)f(x) \) for \( f \in \mathcal{H}^{m,n} \). Since \( A \) commute with the representation \( T^{m,y_n} \) i.e. \( [A, T^{m,y_n}] = 0 \) for all \( t \in B_{m,n} \times B^{(m,n)} \) we conclude that

\[
a(x) = a(xt) \mod dx_m \otimes dx^{(m)} \quad \text{for all} \quad t \in B_{m,n} \times B^{(m,n)}.
\]
Since the measure $dh = dx_m \otimes dx^{(m)}$ is the Haar measure on $G = B_{m,n} \times B^{(m,n)}$, this measure is $G$-right ergodic. We conclude that $a(x) = \text{const} \mod dx_m \otimes dx^{(m)}$. □

3. Induced representations, infinite-dimensional case

3.1. Regular and quasiregular representations of infinite-dimensional groups. To define the induced representation we explain first how to define the regular representation of infinite-dimensional group $G$. Since the initial group in not locally compact there is neither Haar (invariant) measure on $G$ (Weil, [18]), nor a $G$-quasi-invariant measure (Xia Dao-Xing, [19]). We can try to find some bigger topological group $\tilde{G}$ and the $G$-quasi-invariant measure $\mu$ on $\tilde{G}$ such that $G$ is the dense subgroup in $\tilde{G}$. In this case we define the right or left regular representation of the group $G$ in the space $L^2(\tilde{G}, \mu)$ if $\mu^R_t \sim \mu$ (resp. $\mu^L_t \sim \mu$) for all $t \in G$ as follows:

$$T^R_{t, \mu}(f)(x) = (d\mu_xt/d\mu(x))^{1/2} f(x), \quad f \in L^2(\tilde{G}, \mu), \quad t \in G,$$

$$T^L_{t, \mu}(f)(x) = (d\mu^{-1}_t x/d\mu(x))^{1/2} f(t^{-1}x), \quad f \in L^2(\tilde{G}, \mu), \quad t \in G.$$

**Conjecture 3.1** (Ismagilov, 1985). The right regular representation $T^R_{t, \mu} : G \to U(L^2(\tilde{G}, \mu))$ is irreducible if and only if

1) $\mu^L_t \perp \mu \forall t \in G \setminus \{e\}$,

2) the measure $\mu$ is $G$-ergodic.

Analogously we can define the quasiregular representation. Namely, if $H$ is a closed subgroup of the group $G$, then on the space $X = H \tilde{G} = \tilde{H} \tilde{G}$ the right action of the group $G$ is well defined, where $\tilde{G}$ (resp. $\tilde{H}$) is some completion of the group $G$ (resp. $H$). If we have some $G$-right-quasi-invariant measure $\mu$ on $X$ one may define the “quasiregular representation” of the group $G$ in the space $L^2(X, \mu)$ as in a locally compact case:

$$(\pi^R_{t, \mu,X}(f))(x) = (d\mu_xt/d\mu(x))^{1/2} f(x), \quad t \in G.$$  

The regular and quasiregular representations for general infinite-dimensional groups were introduced and investigated in e.g. [1, 9, 10, 11, 13].

3.2. Induced representations for infinite-dimensional groups. The induced representation $\text{Ind}^G_H S$ of a locally-compact group is the unitary representation of the group $G$ associated with a unitary representation $S$ of a subgroup $H$ of the group $G$ (see Section 2).

As it was mentioned in section 2 (see [4] [7]) all unitary irreducible representations up to equivalence $\tilde{G}_n$ of the nilpotent group $G_n = B(n, \mathbb{R})$, are obtained as induced representations $\text{Ind}^G_{\tilde{G}_n} U_{f,H}$ associated with a point $f \in \mathfrak{g}_n$ and the corresponding subordinate subgroup $H \subset G_n$. The induced representation $\text{Ind}^G_{\tilde{G}_n} U_{f,H}$ is defined canonically in the Hilbert space $L^2(H \tilde{G}_n, \mu)$.

A. Kirillov [7], Chapter I, §4, p.10 says: ”The method of induced representations is not directly applicable to infinite-dimensional groups (or more precisely to a pair $G \supset H$) with an infinite-dimensional factor $H \tilde{G}$”.

Our aim is to develop the concept of induced representations for infinite-dimensional groups. Let we have the infinite-dimensional group $G$ and a unitary representation $S : H \to U(V)$ in a Hilbert space $V$ of a subgroup $H$ of the group $G$ such that the factor space $H \tilde{G}$ is infinite-dimensional.
In general, it is difficult to construct $G$-quasi-invariant measure on an infinite-dimensional homogeneous space $H \backslash G$. As is the case of the regular and quasiregular representations of infinite-dimensional groups $G$ (see Subsection 3.1), it is reasonable to construct some $G$-quasi-invariant measure on a suitable completion $\hat{H} \backslash \hat{G}$ of the initial space $H \backslash G$ in a certain topology, where $\hat{H}$ (resp. $\hat{G}$) is some completion of the group $H$ (resp. $G$). To go further we should be able to extend the representation $S : H \rightarrow U(V)$ of the group $H$ to the representation $\tilde{S} : \hat{H} \rightarrow U(V)$ of the completion $\hat{H}$ of the group $H$.

Finally, the induced representation of the group $G$ associated with a unitary representation $S$ of a subgroup $H$ will depend on two completions $\hat{H}$ and $\hat{G}$ of the subgroup $H$ and the group $G$, on an extension $\tilde{S} : \hat{H} \rightarrow U(V)$ of the representation $S : H \rightarrow U(V)$ and on a choice of the $G$-quasi-invariant measure $\mu$ on an appropriate completion $\hat{X} = \hat{H} \backslash \hat{G}$ of the space $H \backslash G$.

Hence the procedure of induction will not be unique but nevertheless well-defined (if a $G$-quasi-invariant measure on $H \backslash G$ exists). So the uniquely defined induced representation $\text{Ind}_{H}^{G} S$ in the Hilbert space $L^{2}(H \backslash G, V, \mu)$ (in the case of a locally-compact group $G$) should be replaced by the family of induced representations $\text{Ind}_{H,H}^{G,G,\mu}(\tilde{S}, S)$ in the Hilbert spaces $L^{2}(\hat{H} \backslash \hat{G}, V, \mu)$ depending on different completions $\hat{G}$ of the group $G$, completions $\hat{H}$ of the group $H$ and different $G$-quasi-invariant measures $\mu$ on $H \backslash G$.

**Example 3.2** ([11]). Regular representations $T^{R,\mu}$ of the infinite-dimensional group $G$ in the space $L^{2}(\hat{G}, \mu)$, associated with the completion $\hat{G}$ of the group $G$ and a $G$-right -quasi-invariant measure $\mu$ on $\hat{G}$, is a particular case of the induced representation (see Remark 2.3)

$$T^{R,\mu} = \text{Ind}_{e}^{G,G,\mu}(Id),$$
generated by the trivial representation $S = Id$ of the trivial subgroup $H = \{e\}$ (as in the case of a locally compact groups).

**Example 3.3** ([13]). Quasi-regular representations $\pi^{R,\mu,X}$ of the infinite-dimensional group $G$ in the space $L^{2}(X, \mu)$ where $X = \hat{H} \backslash \hat{G}$ and $H$ is some subgroup of the group $G$ is a particular case of the induced representation (see Remark 2.3)

$$\pi^{R,\mu,X} = \text{Ind}_{H,H}^{G,G,\mu}(Id),$$
generated by the trivial representation $S = Id$ of the completion $\hat{H}$ in the group $\hat{G}$ of the subgroup $H$ in the group $G$.

Let $G$ be an infinite-dimensional group and $S : H \rightarrow U(V)$ be a unitary representation in a Hilbert space $V$ of the subgroup $H \subset G$, such that the space $H \backslash G$ is infinite-dimensional. We give the following definition.

**Definition 3.4.** The induced representation

$$\text{Ind}_{H,H}^{G,G,\mu}(\tilde{S}, S),$$
generated by the unitary representations $S : H \rightarrow U(V)$ of the subgroup $H$ in the group $G$ is defined (similarly to (2.3) and (3.3)) as follows:

1) we should first find some completion $\hat{H}$ of the group $H$ such that

$$\tilde{S} : \hat{H} \rightarrow U(V)$$

is the continuous unitary representation of the group $\hat{H}$, such that $\tilde{S}|_{H} = S$, .
2) take any $G$-right-quasi-invariant measure $\mu$ on the an appropriate completion $\tilde{X} = \tilde{H} \backslash \tilde{G}$ of the space $X = H \backslash G$, on which the group $G$ acts from the right, where $\tilde{H}$ (resp. $\tilde{G}$) is a suitable completion of the group $H$ (resp. $G$),

3) in the space $L^2(\tilde{X}, V, \mu)$ of all vector-valued functions $f$ on $\tilde{X}$ with values in $V$ such that

$$\|f\|^2 := \int_{\tilde{X}} \|f(x)\|^2 d\mu(x) < \infty,$$

define the representation of the group $G$ by the following formula

$$(3.3) \quad (T_t f)(x) = S(\tilde{h}(x, t)) \left( \frac{d\mu(x t)}{d\mu(x)} \right)^{1/2} f(x t), \quad x \in \tilde{X}, \ t \in G,$$

where $\tilde{h}$ is defined by

$$\tilde{s}(x t) = \tilde{h}(x, t) \tilde{s}(x t).$$

The section $s : H \to G$ of the projection $p : G \to H$ should be extended to the appropriate section $\tilde{s} : \tilde{H} \to \tilde{G}$ of the extended projection $\tilde{p} : \tilde{G} \to \tilde{H}$.

The comparison of the induced representation for locally compact group and the above definition for infinite-dimensional groups may be given in the following table:

| 1 | $G$ | $G$ loc.comp. | $\dim G = \infty$ |
|---|---|---|---|
| 2 | $H$ | $H \subseteq G$ | $H \subseteq G$ |
| 3 | $S$ | $S : H \to U(V)$ | $S : H \to U(V) \Rightarrow S : H \to U(V)$ |
| 4 | $X$ | $X = H \backslash G$ | $\tilde{X} = \tilde{H} \backslash \tilde{G}$ |
| 5 | $\mathcal{H}$ | $L^2(X = H \backslash G, V, \mu)$ | $L^2(\tilde{X} = \tilde{H} \backslash \tilde{G}, V, \mu)$ |
| 6 | Ind | $\text{Ind}^G_H S$ | $\text{Ind}^{G,G,\mu}_{\tilde{H},\tilde{H}} (\tilde{s}, S)$ |
| 7 | $T_t$ | $(T_t f)(x) = S(\tilde{h}(x, t)) \left( \frac{d\mu(x t)}{d\mu(x)} \right)^{1/2} f(x t)$ | $(T_t f)(x) = \tilde{s}(\tilde{h}(x, t)) \left( \frac{d\mu(x t)}{d\mu(x)} \right)^{1/2} f(x t)$ |
| 8 | $p$ | $p : G \to X$ | $\tilde{p} : \tilde{G} \to \tilde{X}$ |
| 9 | $s$ | $s : X \to G$ | $s : H \backslash G \to \tilde{G}$ |
| 10 | $h(x, t)$ | $s(x t) = h(x, t) s(x t)$ | $\tilde{s}(x t) = h(x, t) \tilde{s}(x t)$ |

### 3.3. How to develop the orbit method for infinite-dimensional “nilpotent” group $B_0^N$ and $B_0^Z$?

We would like to develop the orbit method for infinite-dimensional “nilpotent” group $G = \lim_n G_n$ with $G_n = B(n, \mathbb{R})$. The corresponding Lie algebra $\mathfrak{g}$ is the inductive limit $\mathfrak{g} = \lim_n \mathfrak{b}_n$ of upper triangular matrices, so as the linear space it is isomorphic to the space $\mathbb{R}_0^\infty$ of finite sequences $(x_k)_{k \in \mathbb{N}}$ hence the dual space $\mathfrak{g}^*$ is isomorphic to the space $\mathbb{R}^\infty$ of all sequences $(x_k)_{k \in \mathbb{N}}$, but the latter space $\mathbb{R}^\infty$ is too large to manage with it, for example to equip with a Hilbert structure or to describe all orbits.

To make it less it is reasonable to increase the initial group $G$ or to make completion $\tilde{G}$ of this group in some stronger topology.

To develop the orbit method for groups $B_0^N$ and $B_0^Z$ we should answer some questions:

1. How to define the appropriate completion $\tilde{G}$ of the group $G$, corresponding Lie algebras $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) and corresponding dual spaces $\mathfrak{g}^*$ (resp. $\tilde{\mathfrak{g}}^*$)?
2. Which pairing should we use between $\mathfrak{g}$ and $\mathfrak{g}^*$?
(3) Let the dual space $\mathfrak{g}^*$, some element $f \in \mathfrak{g}^*$ and corresponding algebra $\mathfrak{h}$, subordinate to the element $f$, are chosen. How to define the corresponding induced representation $\text{Ind}_{H}^{G}U_{f,H}$ and study its irreducibility?

(4) Shall we get all irreducible representations of the corresponding groups, using induced representations?

(5) Find the criteria of irreducibility and equivalence of induced representations.

The problem of completion of the inductive limit group $G = \lim_{\cdots} G_{n}$, where $G_{n}$ are finite-dimensional classical groups were studied by A. Kirillov (\[5\], 1972) for the induced representations?

Ind_{H}^{G}U_{f,H}$ and study its irreducibility? for the inductive limit of classical groups. They described all unitary irreducible representations of the corresponding groups $G = \lim_{\cdots} G_{n}$, continuous in stronger topology, namely in the strong operator topology. The description of the dual $G$ of the initial group $G = \lim_{\cdots} G_{n}$ is much more complicated.

In \[8\] (see details in section 3.4) we have constructed for the group $\text{GL}_{0}(2\infty, \mathbb{R}) = \lim_{\cdots} \text{GL}(2n - 1, \mathbb{R})$ a family of the Hilbert-Lie groups $\text{GL}_{2}(a)$, $a \in \mathbb{A}$ such that

a) $\text{GL}_{0}(2\infty, \mathbb{R}) \subset \text{GL}_{2}(a)$ and $\text{GL}_{0}(2\infty, \mathbb{R})$ is dense in $\text{GL}_{2}(a)$ for all $a \in \mathbb{A}$,

b) $\text{GL}_{0}(2\infty, \mathbb{R}) = \cap_{a \in \mathbb{A}} \text{GL}_{2}(a)$,

c) any continuous representation of the group $\text{GL}_{0}(2\infty, \mathbb{R})$ is in fact continuous in some stronger topology, namely in a topology of a suitable Hilbert-Lie group $\text{GL}_{2}(a)$.

(1) Therefore, as we show in Sections 3.3, 3.4 it is sufficient to consider a Hilbert-Lie completions $B_{2}(a)$ of the initial group $B_{0}^{\infty}$.

(2) In this case the pairing between the corresponding Hilbert-Lie algebra $\mathfrak{b}_{2}(a)$ and its dual $\mathfrak{b}_{2}(a)^*$ is correctly defined by the trace (as in the finite-dimensional case).

3.1 We define in Section 3.7 the induced representations of the group $B_{0}^{\infty}$ corresponding to some orbits, generic orbits, using schema given in Section 3.2. We consider only the simplest example of $G$-quasi-invariant measures on $\bar{X} = \bar{H} \setminus G$, namely the infinite product of one-dimensional Gaussian measures.

3.2 How to construct the induced representation corresponding to an arbitrary orbit?

Conjecture 3.5. Two induced representations $\text{Ind}_{H_{1}}^{G_{1}}U_{f_{1,H_{1}}}$ and $\text{Ind}_{H_{2}}^{G_{2}}U_{f_{2,H_{2}}}$ are equivalent if and only if the corresponding measures $\mu_{1}$ and $\mu_{2}$ are equivalent and the functionals $f_{1}$ and $f_{2}$ belong to the same orbit of $(\hat{\mathfrak{g}})^*$.

3.4 Hilbert-Lie groups $\text{GL}_{2}(a)$. We show that the Hilbert-Lie groups appear naturally in the representation theory of infinite-dimensional matrix group. The remarkable fact is that for the inductive limit $G = \lim_{\cdots} G_{n}$ of matrix groups $G_{n} \subset \text{GL}(2n - 1, \mathbb{R})$ it is sufficient to consider only the Hilbert completions of the initial group $G$ and of the spaces $H \setminus G$.

Let us consider the group $\text{GL}_{0}(2\infty, \mathbb{R}) = \lim_{\cdots} \text{GL}(2n - 1, \mathbb{R})$ with respect to the symmetric embedding $i_{n}^{\infty} : G_{n} \to G_{n+1}, G_{n} \ni x \mapsto x + E_{-n, -n} + E_{nn} \in G_{n+1}$, where $G_{n} = \text{GL}(2n - 1, \mathbb{R})$. We consider here only the real matrices.

The Hilbert-Lie group $\text{GL}_{2}(a)$ we define (see \[8\]) by its Hilbert-Lie algebra $\mathfrak{g}l_{2}(a)$ with composition $[x, y] = xy - yx$

$$\mathfrak{g}l_{2}(a) = \{ x = \sum_{k,n \in \mathbb{Z}} x_{kn} E_{kn} | ||x||_{\mathfrak{g}l_{2}(a)}^{2} = \sum_{k,n \in \mathbb{Z}} | x_{kn} |^{2} a_{kn} < \infty \}, a \in \mathfrak{A}_{\text{GL}},$$

$$\text{GL}_{2}(a) = \{ I + x | (I + x)^{-1} = 1 + y \} , x, y \in \mathfrak{g}l_{2}(a) \}.$$
To be more precise, let us consider an analogue $\sigma_2(a)$ of the algebra of the Hilbert-Schmidt operators $\sigma_2(H)$ in a Hilbert space $H$:

$$\sigma_2(a) = \{ x = \sum_{k,n \in \mathbb{Z}} x_{kn} E_{kn} \mid \|x\|_{\sigma_2(a)}^2 = \sum_{k,n \in \mathbb{Z}} |x_{kn}|^2 a_{kn} < \infty \}. $$

**Lemma 3.6** ([8]). The Hilbert space $\sigma_2(a)$ is an (associative) Hilbert algebra (i.e. $\|xy\| \leq C\|x\|\|y\|$), $x, y \in \sigma_2(a)$ if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{GL}$ defined as follows:

$$\mathfrak{A}_{GL} = \{ a = (a_{kn})_{(k,n)\in\mathbb{Z}^2} \mid 0 < a_{kn} \leq Ca_{km}a_{mn}, \ k, n, m \in \mathbb{Z}, C > 0 \}. $$

We define the Hilbert-Lie algebra $\mathfrak{gl}_2(a)$ as the Hilbert space $\sigma_2(a)$ with an operation $[x, y] = xy - yx$. 

**Corollary 3.7.** The Hilbert space $\mathfrak{gl}_2(a)$ is a Hilbert-Lie algebra if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{GL}$. 

We remark also [8] that $\text{GL}_0(2\mathbb{C}, \mathbb{R}) = \cap_{a \in \mathfrak{A}_{GL}} \text{GL}_2(a)$. 

**Theorem 3.8** (Theorem 6.1 [8]). Every continuous unitary representation $U$ of the group $\text{GL}_0(2\mathbb{C}, \mathbb{R})$ in a Hilbert space $H$ can be extended by continuity to a unitary representation $U_2(a) : \text{GL}_2(a) \rightarrow U(H)$ of some Hilbert-Lie group $\text{GL}_2(a)$ depending on the representation.

3.5. **Hilbert-Lie groups** $B_2(a)$. Let us consider the following Hilbert-Lie group $B_2(a) := B_2^2(a)$

$$B_2(a) = \{ I + x \mid x \in b_2(a) \}, $$

where the corresponding Hilbert-Lie algebra $b_2(a) := b_2^2(a)$ is defined as

$$b_2(a) = \{ x = \sum_{(k,n) \in \mathbb{Z}^2, k<n} x_{kn} E_{kn} \mid \|x\|_{b_2(a)}^2 = \sum_{(k,n) \in \mathbb{Z}^2, k<n} |x_{kn}|^2 a_{kn} < \infty \}. $$

**Lemma 3.9** ([8]). The Hilbert space $b_2(a)$ (with an operation $(x, y) \mapsto xy$) is a Banach algebra if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2, k<n}$ satisfies the conditions

$$a = (a_{kn})_{k<n}, \ a_{kn} \leq Ca_{km}a_{mn}, \ k < m < n, \ k, m, n \in \mathbb{Z}. $$

Denote by $\mathfrak{A}$ the set of all weight $a$ satisfying the mentioned condition.

3.6. **Orbit method for infinite-dimensional “nilpotent” group** $B_2^a$, first steps. Take the group $B_2^a$, fix some its Hilbert completion i.e. a Hilbert-Lie group $B_2(a)$, $a \in \mathfrak{A}$ and the corresponding Hilbert-Lie algebra $\mathfrak{g} = b_2(a)$. The corresponding dual space $\mathfrak{g}^* = b_2^a(a)$ has the form

$$b_2^a(a) = \{ y = \sum_{(k,n) \in \mathbb{Z}^2, k>n} y_{kn} E_{kn} \mid \|y\|_{b_2^a(a)}^2 = \sum_{(k,n) \in \mathbb{Z}^2, k>n} |y_{kn}|^2 a_{kn}^{-1} < \infty \}. $$

The adjoint action $B_2(a) \rightarrow \text{Aut}(b_2(a))$ of the group $B_2(a)$ on its Lie algebra $b_2(a)$ is:

$$b_2(a) \ni x \mapsto \text{Ad}_t(x) := t x t^{-1} \in b_2(a), \ t \in B_2(a). $$

The pairing between $\mathfrak{g} = b_2(a)$ and $\mathfrak{g}^* = b_2^a(a)$ is correctly defined by the trace:

$$\mathfrak{g}^* \times \mathfrak{g} \ni (y, x) \mapsto \langle y, x \rangle := \text{tr}(xy) = \sum_{(k,n) \in \mathbb{Z}^2, k<n} x_{kn} y_{nk} \in \mathbb{R}. $$
The coadjoint action of the group $B_2(a)$ on the dual $\mathfrak{g}^* = \mathfrak{b}_2^*(a)$ to $\mathfrak{g} = \mathfrak{b}_2(a)$ is as follows: for $t \in B_2(x)$ and $y \in \mathfrak{b}_2^*(a)$
\[ t = I + \sum_{(k,n) \in \mathbb{Z}^2, k < n} t_{kn} E_{kn}, \quad y = \sum_{(k,n) \in \mathbb{Z}^2, k > n} y_{kn} E_{kn}, \quad t^{-1} := I + \sum_{(k,n) \in \mathbb{Z}^2, k < n} t_{kn}^{-1} E_{kn} \]
we have
\[ (t^{-1}yt)_{pq} = \sum_{m=-\infty}^{q} (t^{-1}y)_{pm} t_{mq} = \sum_{m=-\infty}^{q} \sum_{r=p}^{\infty} t_{pr}^{-1} y_{rm} t_{mq}, \quad (p, q) \in \mathbb{Z}^2, p > q, \]
hence
\[ (3.11) \quad \text{Ad}_t^*(y) = (t^{-1}yt)_{-} := I + \sum_{(p,q) \in \mathbb{Z}^2, p > q} (t^{-1}yt)_{pq} E_{pq}. \]
We consider four different type of orbits with respect to the coadjoint action of the group $B_2(a)$ in the dual space $\mathfrak{b}_2^*(a)$.

**Case 1)** The finite-dimensional orbits corresponding to a finite points $y = \sum_{(k,n) \in \mathbb{Z}, k > n} y_{kn} E_{kn} \in \mathfrak{b}_2^*(a)$ (finiteness of $y$ means that only finite number of $y_{kn}$ are nonzero). This orbits leads to the induced representations of an appropriate finite-dimensional groups $G_n^m$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ defined by (2.41). All irreducible unitary representations of the groups $G_n^m$ are completely described by the Kirillov orbit method hence the finite-dimensional orbits gives us the set $\bigcup_{n \in \mathbb{N}} \hat{G}_n^m \subset \hat{B}_0$ (see subsection 3.9).

**Remark 3.17** for embedding $\hat{G}_n^m \subset \hat{G}_{n+1}^m$.

**Case 2)** 0-dimensional orbits are of the form:
\[ \mathcal{O}_0 = y, \quad y \in \mathfrak{b}_2^*(a), \quad y = \sum_{k \in \mathbb{Z}} y_{k+1,k} E_{k+1,k}. \]
The Lie algebra $\mathfrak{b}_2(a)$ is subordinate to the functional $y$, $\langle y, [\mathfrak{b}_2(a), \mathfrak{b}_2(a)] \rangle = 0$ since
\[ [\mathfrak{b}_2(a), \mathfrak{b}_2(a)] = \{ x \in \mathfrak{b}_2(a) \mid x = \sum_{(k,n) \in \mathbb{Z}^2, k+1 < n} x_{kn} E_{kn} \}. \]
The one-dimensional representation of the Lie algebra $\mathfrak{b}_2(a)$ are
\[ \mathfrak{b}_2(a) \ni x \mapsto \langle y, x \rangle = \sum_{k \in \mathbb{Z}} x_{k,k+1} y_{k+1,k} \in \mathbb{R}. \]
Corresponding one-dimensional representations of the group $B_2(a)$ are as follows:
\[ (3.12) \quad B_2(a) \ni \exp(x) \mapsto \exp(2\pi i(\langle y, x \rangle)) = \exp(2\pi i \sum_{k \in \mathbb{Z}} x_{k,k+1} y_{k+1,k}) \subset S^1. \]
They are all irreducible and nonequivalent for different $y = \sum_{k \in \mathbb{Z}} y_{k+1,k} E_{k+1,k} \in \mathfrak{b}_2^*(a)$.

**Case 3)** Generic orbit is generated for an arbitrary $m \in \mathbb{Z}$ by a point $y \in \mathfrak{b}_2^*(a)$
\[ y = \sum_{p=0}^{\infty} y_{m+p+1,m-p} E_{m+p+1,m-p} \in \mathfrak{b}_2^*(a), \quad \text{with} \quad y_{m+p+1,m-p} \neq 0, \quad p + 1 \in \mathbb{N}. \]
Sections 3.7 and 3.8 are devoted to the study of this case.

**Case 4)** General orbits generated by an arbitrary non finite points
\[ y = \sum_{(k,n) \in \mathbb{Z}, k > n} y_{kn} E_{kn} \in \mathfrak{b}_2^*(a). \]
Problem. How to construct the induced representations for general orbits and study their irreducibility?

3.7. Construction of the induced representations of the group $B_0^\mathbb{Z}$ corresponding to a generic orbits. Consider more carefully the case 3). The irreducibility we shall study in the following subsection. Take as before the group $B_0^\mathbb{Z}$, fix some its Hilbert completion i.e. a Hilbert-Lie group $B_3(a)$, $a \in \mathfrak{a}$, the corresponding Hilbert-Lie algebra $\mathfrak{g} = \mathfrak{b}_2(a)$ and its dual $\mathfrak{g}^* = \mathfrak{b}_2^*(a)$ as in the previous subsection.

We shall write the analog of the induced representation of the group $B_0^\mathbb{Z}$ for generic orbits (see Examples 2.4, 2.8 and 2.14) corresponding to the point $y \in \mathfrak{b}_2^*(a)$ defined by (3.13) following steps 1)–3) of Definition 3.4.

Step 1) Extension of the representation $S : H \to U(V)$. For fixed $m \in \mathbb{Z}$, consider the decomposition

$$B^\mathbb{Z} = B_mB(m)B^{(m)}$$

similar to the decomposition (2.19), where $B^\mathbb{Z} = \{I + \sum_{k,n \in \mathbb{Z}, k < n} x_{kn}E_{kn}\}$,

$$B_m = \{I + \sum_{(k,r) \in \Delta_m} x_{kr}E_{kr}\}, \quad B(m) = \{I + \sum_{(k,r) \in \Delta(m)} x_{kr}E_{kr}\}, \quad B^{(m)} = \{I + \sum_{(k,r) \in \Delta^{(m)}} x_{kr}E_{kr}\},$$

$$\Delta_m = \{(k,r) \in \mathbb{Z}^2 \mid m + 1 \leq k < r\}, \quad \Delta(m) = \{(k,r) \in \mathbb{Z}^2 \mid k \leq m < r\},$$

and $\Delta^{(m)} = \{(k,r) \in \mathbb{Z}^2 \mid k < r \leq m\}$.

Since the algebras $\mathfrak{h}_0(m)$, $m \in \mathbb{Z}$ defined as follows $\mathfrak{h}_0(m) = \{t - I \mid t \in B_0(m)\}$, where $B_0(m) = B(m) \cap B^\mathbb{Z}$, are commutative, so $\langle y, [\mathfrak{h}_0(m), \mathfrak{h}_0(m)] \rangle = 0$, hence they are subordinate to the functional $y \in \mathfrak{g}^* = \mathfrak{b}_2^*(a)$. The corresponding one-dimensional representation of the algebra $\mathfrak{h}_0(m) = \mathfrak{h}(m) \cap \mathfrak{g}_0^\mathbb{Z}$ is

$$\mathfrak{h}_0(m) \ni x \mapsto \langle y, x \rangle = \sum_{p=0}^{\infty} x_{m-p,m+p+1}y_{m+p+1,m-p} \in \mathbb{R}.$$ 

The unitary representation of the corresponding group $H_0(m)$ is

$$H_0(m) \ni \exp(x) \mapsto S(\exp(x)) = \exp(2\pi i \langle y, x \rangle) \in S^1.$$ 

This representation can be extended to representation of the corresponding Hilbert-Lie group $H = H_2(m,a) = B(m) \cap B_2(a)$ (we note that $t = \exp(t - 1)$):

$$H_2(m,a) \ni \exp(x) \mapsto S(\exp(x)) = \exp(2\pi i \langle y, x \rangle) \in S^1.$$ 

In what follows we shall use a notation $B_2(m,a)$ for the group $H_2(m,a)$.

Step 2 a) Construction of the completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $X = H\backslash G$. It is difficult to construct an appropriate measure on the space $X_{m,0} = B_0(m)\backslash B^\mathbb{Z}$ since it is isomorphic to the space $\mathbb{R}_0^\infty \subset \mathbb{R}_0^\infty$. That is why we consider two homogeneous spaces, an appropriate completions of the space $X_{m,0}$:

$$X_{m,2}(a) = B_{m,2}(a)\backslash B_2(a), \quad X_m = B(m)\backslash B^\mathbb{Z}.$$ 

Since the decompositions holds

$$B_0^\mathbb{Z} = B_{m,0}B_0(m)B^{(m)}_0A, \quad B_2(a) = B_{m,2}(a)B_2(m,a)B^{(m)}_2(a), \quad B^\mathbb{Z} = B_mB(m)B^{(m)} = \mathfrak{g} \mathfrak{b}_2^*(a),$$

(see Remark 2.9), we have the following inclusions: $X_{m,0} \subset X_{m,2}(a) \subset X_m$, where $X_{m,0} \simeq B_{m,0} \times B^{(m)}_0$, $X_{m,2}(a) \simeq B_{m,2}(a) \times B^{(m)}_2(a)$, $X_m = B(m)\backslash B^\mathbb{Z} \simeq B \times B^{(m)}$.

Step 2 b) We construct a measure $\mu_b$ on the space $X_m$ with support $X_{m,2}(a)$ i.e. such that $\mu_b(X_{m,2}(a)) = 1$. That is we take $\tilde{X} = \tilde{H}\backslash\tilde{G} = B_2(m,a)\backslash B_2(a)$. 

Remark 3.10. On the space \( X_m \) we can take any \( B^Z_0 \)-quasi-invariant ergodic measure, construct the induced representation and study the irreducibility. We consider the simplest case of the Gaussian measure, the infinite product of one-dimensional Gaussian measure.

We construct the measure \( \mu_b \) on the space \( X_m \simeq B_m \times B^{(m)} \) as a product-measure \( \mu_b = \mu_{b,m} \otimes \mu_b^{(m)} \), where \( \mu_{b,m} \) (resp. \( \mu_b^{(m)} \)) is Gaussian product measure on the group \( B_m \) (resp. \( B^{(m)} \)) defined as follows:

\[
\begin{align*}
\mu_{b,m}(x_m) &= \otimes_{(k,n) \in \Delta_m} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n) \in \Delta_m} \sqrt{\frac{b_{kn}}{\pi}} \exp(-b_{kn}x_{kn}^2) dx_{kn}, \\
\mu_b^{(m)}(x^{(m)}) &= \otimes_{(k,n) \in \Delta^{(m)}} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n) \in \Delta^{(m)}} \sqrt{\frac{b_{kn}}{\pi}} \exp(-b_{kn}x_{kn}^2) dx_{kn}.
\end{align*}
\]

The corresponding Hilbert space is \( \mathcal{H}^m = L^2(X_m, \mu_b) = L^2(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}) \).

Lemma 3.11 (Kolmogorov’s zero-one law, [17]). We have \( \mu_{b,m} \otimes \mu_b^{(m)}(B_m 2(a) \times B_2^{(m)}(a)) = 1 \) if and only if

\[
\sum_{(k,n) \in \Delta(m) \cup \Delta^{(m)}} \frac{a_{kn}}{b_{kn}} < \infty.
\]

Lemma 3.12 ([9, 10]). The measure \( \mu_b = \mu_{b,m} \otimes \mu_b^{(m)} \) is \( B_{m,0} \times B_0^{(m)} \)-right-quasi-invariant i.e. \( (\mu_b)^{Rt} \sim \mu_b \) for all \( t \in B_{m,0} \times B_0^{(m)} \) if and only if

\[
S_{kn}^R(\mu_b) = \sum_{r = -\infty}^{k-1} \frac{b_{rn}}{b_{rk}} < \infty, \quad \text{for all}, \quad k < n \leq m.
\]

Step 3) The corresponding induced representation of the group \( B^Z_0 \) we defined as follows:

\[
(T^{m,y}_t f)(x) = S(h(x,t)) \left( \frac{d\mu_b^{(m)}(x^{(m)})}{d\mu_b(x)} \right)^{1/2} f(x), \quad x \in X_m, \ t \in G,
\]

where (see (3.21))

\[
S(h(x,t)) = \exp(2\pi i(y, h(x,t) - 1)) = \exp \left( 2\pi i \text{tr} \left( (t - I)B(x,y) \right) \right).
\]

3.8. Irreducibility of the induced representations of the group \( B^Z_0 \) corresponding to a generic orbits. Consider the induced representation \( T^{m,y}_t \) of the group \( B^Z_0 \) corresponding to a generic orbit \( O_y \), generated by the point \( y = \sum_{r=0}^{\infty} y_{m+r+1, m-r} E_{m+r+1, m-r} \in b^Z_2(a) \) defined by (3.16). Set for \( (k,r) \in \Delta(m) \)

\[
S_{kr}(t_{kr}) := \langle y, (h(x, E_{kr}(t_{kr})) - I) \rangle, \quad \text{then} \quad A_{kr} = \frac{d}{dt} \exp(2\pi i S_{kr}(t))|_{t=0} = 2\pi i S_{kr}(1).
\]

Let us denote by \( S^{(m)} = S \) the following matrix (compare with (2.23) and (2.24)):

\[
S = (S_{kr})_{(k,r) \in \Delta(m)}, \quad \text{where} \quad S_{kr} = S_{kr}(1).
\]

We calculate now the matrix \( S(t) = (S_{kr}(t_{kr}))_{(k,r) \in \Delta(m)} \) and the matrix \( S = (S_{kr}(1))_{(k,r) \in \Delta(m)} \) using analog of the Lemma 2.11. As in (2.22) we have

\[
\langle y, h(x, t) - I \rangle = \text{tr} \left( H(x, t)y \right) = \text{tr} \left( (x^{(m)}) t_0 x_m^{-1} y \right) = \text{tr} \left( t_0 x_m^{-1} y x^{(m)} \right) = \text{tr} \left( t_0 B(x, y) \right).
\]
where \( t_0 = t - I \) and for \( x_m \in B_m \), \( x^{(m)} \in B^{(m)} \) we denote
\[
B(x, y) = x_m^{-1} y x^{(m)} \cong \left( \begin{array}{cc} \frac{1}{x_m} & 0 \\ 0 & \frac{1}{y} \end{array} \right) \left( \begin{array}{cc} x^{(m)} & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & x_m^{-1} y x^{(m)} \end{array} \right).
\]

By definition we have (recall that \( E_{kn}(t_{kn}) = I + t_{kn} E_{kn} \))
\[
S_{kn}(t_{kn}) = \langle y, (h(x, E_{kn}(t_{kn})), - I) \rangle = \text{tr}(t_{kn} E_{kn} B(x, y)),
\]
hence by analog of the Lemma \ref{11} we conclude that
\[
S = (S_{kn}(1))_{k,r} = (\text{tr}(E_{kr} B(x, y)))_{k,r} = B^T(x, y) = (x^{(m)})^T y^T (x_m^{-1})^T = \left( \begin{array}{cc} 0 & \langle (x^{(m)})^T y^T (x_m^{-1})^T \rangle \end{array} \right).
\]
So, we have
\[
S(h(x, t)) = \exp(2\pi i \langle y, h(x, t) - I \rangle) = \exp \left( 2\pi i \text{tr} \left((t - I) B(x, y) \right) \right).
\]

Using results of \ref{12} we conclude that the following lemma holds.

**Lemma 3.13.** The measure \( \mu_b = \mu_{b,m} \otimes \mu_b^{(m)} \) is \( B_{m,0} \times B_0^{(m)} \)-right-ergodic if
\[
E(\mu_b) = \sum_{k<n \leq m} \frac{S_{kn}^R(\mu_b)}{b_{kn}} < \infty.
\]

**Theorem 3.14.** The induced representation \( T^{m,y} \) of the group \( B_0^Z \) defined by formula \ref{10}, corresponding to generic orbit \( O_y \), generated by the point
\[
y = \sum_{r=0}^{\infty} y_{m+r-1,m-r} E_{m+r-1,m-r} \in b_2^*(a) \]
is irreducible if the measure \( \mu_{b,m} \otimes \mu_b^{(m)} \) on the group \( B_m \times B^{(m)} \) is right \( B_{m,0} \times B_0^{(m)} \)-ergodic. Moreover the generators of one-parameter groups \( A_{kr} = \frac{d}{dt} T^{m,y}_{I + t E_{kr}} \big|_{t=0} \) are as follows
\[
A_{kr} = \sum_{s=-\infty}^{s_m} x_{ks} D_{rs} + D_{kr}, \ (k, r) \in \Delta^{(m)}, \quad A_{kr} = \sum_{s=m+1}^{s_m} x_{ks} D_{rs} + D_{kr}, \ (k, r) \in \Delta_m,
\]
\[
(2\pi i)^{-1}(A_{kr})_{(k,r) \in \Delta(m)} = S^{(m)} = (S_{kr})_{(k,r) \in \Delta(m)} = (x^{-1}_m y x^{(m)})^T.
\]

Here we denote by \( D_{kn} = D_{kn}(\mu_b) \) the operator of the partial derivative corresponding to the shift \( x \mapsto x + t E_{kn} \) and the measure \( \mu_b \) on the group \( B_m \times B^{(m)} \) \( x = I + \sum x_{kr} E_{kr} \):
\[
(D_{kn}(\mu_b)f)(x) = \frac{d}{dt} \left( \frac{d\mu_b(x + t E_{kn})}{d\mu_b(x)} \right)^{1/2} f(x + t E_{kn}) \big|_{t=0}, \quad D_{kn}(\mu_b) = \frac{\partial}{\partial x_{kn}} - b_{kn} x_{kn}.
\]

The irreducibility of the induced representation of the group \( B_0^Z \) follows from the following lemma.

**Lemma 3.15.** Two von Neumann algebra \( \mathfrak{A}^S \) and \( \mathfrak{A}^x \) in the space \( \mathcal{H}^m = L^2(X_m, \mu_b) \) generated respectively by the sets of unitary operators \( U_{kr}(t) \) and \( V_{kr}(t) \) coincides, where
\[
(U_{kr}(t)f)(x) = \exp(2\pi i S_{kr}(t)) f(x), \quad (V_{kr}(t)f)(x) := \exp(2\pi i t x_{kr}) f(x),
\]
\[
\mathfrak{A}^S = (U_{kr}(t) = T^{m,y}_{I + t E_{kr}} = \exp(2\pi i S_{kr}(t)) \mid t \in \mathbb{R}, \ (k, r) \in \Delta(m)),
\]
\[
\mathfrak{A}^x = (V_{kr}(t) = \exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, \ (k, r) \in \Delta_m \bigcup \Delta^{(m)}).\]
Proof. Using the decomposition (3.20)
\[ S^{(m)} = B(x, y)^T = (x^{-1}_myx^{(m)})^T = (x^{(m)})^Ty^T(x^{-1}_m)^T \]
we conclude that \( S^x \subseteq S^x \) (see the proof of Lemma 2.17).

To prove that \( S^x \supseteq S^x \) it is sufficient to find the expressions of the matrix element of the matrix \( x^{(m)} \in B^{(m)} \) and \( x^{-1}_m \in B_m \) in terms of the matrix elements of the matrix \( S^{(m)} = (S_{kr})_{(k,r) \in \Delta(m)} \). To do this we connect the above decomposition \( S^{(m)} = B(x, y)^T \) (see (3.19)) and the Gauss decomposition \( C = LDU \) for infinite matrices (see Theorem 3.2). By (3.19) we get \( B(x, y) = x^{-1}_myx^{(m)} \).

To find a matrix connected with the matrix \( S^{(m)} \), for which an appropriate decomposition \( LDU \) holds we recall the expressions for \( B(x, y) \) for small \( n \) and finite-dimensional groups \( G_n \) (see Example (2.14)). We note that \( J_2 \subseteq I \), where
\[ J_m \in \text{Mat}(\infty, \mathbb{R}), \quad J_m = \sum_{r \in \mathbb{Z}} E_{m+r+1,m-r}. \]
For \( G_3 \) we get
\[
B(x, y) = x^{-1}_myx^{(m)} = \left( \begin{array}{ccc}
1 & x_{45} & x_{46} \\
0 & 1 & x_{56} \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
0 & 0 & y_{43} \\
0 & 0 & y_{52} \\
y_{70} & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
1 & x_{01} & x_{02} \\
0 & 1 & x_{12} \\
0 & 0 & 0
\end{array} \right),
\]
(3.24) \[ B(x, y)J = \left( \begin{array}{ccc}
1 & x_{45} & x_{46} \\
0 & 1 & x_{56} \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
y_{43} & 0 & 0 \\
y_{52} & 0 & 0 \\
y_{70} & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
x_{23} & 1 & 0 \\
x_{03} & x_{02} & 1
\end{array} \right). \]

We use the infinite-dimensional analog of the latter presentation, i.e. instead of the group \( G_n = B(n, \mathbb{R}) \) consider the infinite-dimensional group \( B_0^\mathbb{Z} \) and do the same. Let
\[ x_m \in B_m, \quad x^{(m)} \in B^{(m)}, \quad y = \sum_{r=0}^{\infty} y_{m+r+1,m-r}E_{m+r+1,m-r} \in S^x(a) \]
and \( J = J_m = \sum_{r \in \mathbb{Z}} E_{m+r+1,m-r} \). Then we get \( S^T = B(x, y) = x^{-1}_myx^{(m)}. \)

Set \( C = C(x, y) = B(x, y)J \) then \( C = UDL \), more precisely we have:
(3.25) \[ B(x, y)J = x^{-1}_myJ_mJ_mx^{(m)}J_m = UDL, \quad \text{where} \quad U = x^{-1}_m, \quad D = yJ_m, \quad L = J_mx^{(m)}J_m, \]
(3.26) \[ C = B(x, y)J = \left( \begin{array}{ccc}
1 & x_{45} & x_{46} \\
0 & 1 & x_{56} \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
y_{43} & 0 & 0 \\
y_{52} & 0 & 0 \\
y_{70} & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
x_{23} & 1 & 0 \\
x_{03} & x_{02} & 1
\end{array} \right), \]
\[ C = \left( \begin{array}{cccc}
c_{11} & c_{12} & \ldots & c_{1n} \\
c_{21} & c_{22} & \ldots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nn}
\end{array} \right) = \left( \begin{array}{ccc}
u_{12} & \ldots & u_{1n} \\
v_{21} & \ldots & u_{2n} \\
0 & \ldots & 1
\end{array} \right) \left( \begin{array}{ccc}
d_1 & \ldots & d_n \\
0 & \ldots & d_n \\
0 & \ldots & d_n
\end{array} \right) \left( \begin{array}{ccc}
l_{11} & l_{12} & \ldots \\
0 & \ldots & 1
\end{array} \right). \]

To finish the proof of the Lemma it is sufficient to find the decomposition \( C = UDL \).

Let us suppose that we can find the inverse matrix \( C^{-1} \). Then by (3.25) holds \( C^{-1} = L^{-1}D^{-1}U^{-1} \) and we can use Theorem 3.2 to find
\[ L^{-1} = J_m(x^{(m)})^{-1}J_m, \quad D^{-1} = y^{-1}J_m, \quad U^{-1} = x_m. \]

Hence, we can find the matrix elements of the matrix \( (x^{(m)})^{-1} \in B^{(m)} \) and \( x_m \in B_m \) in terms of the matrix elements of the matrix \( C^{-1} = (S^TJ)^{-1} = (B(x, y)J)^{-1} \). Finally, we can also find the matrix elements of the matrix \( x^{(m)} \in B^{(m)} \) using formulas (2.31). This
Therefore we have to finish the proof of the lemma since in this case we have $x_{kr} \in \Delta_m \cup \Delta^{(m)}$. Hence $\mathfrak{A}^x \subseteq \mathfrak{A}^s$.

1) To find the inverse matrix $C^{-1}$ we write two decompositions:

\begin{equation}
C = L_1 D_1 U_1 = UDL, \quad C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1} = L^{-1} D^{-1} U^{-1}.
\end{equation}

2) Using (3.27) we can find $L_1, D_1$ and $U_1$ by Theorem 1.2. More precisely, for all $x \in \Gamma_G$, where

$$
\Gamma_C = \{x \in B_m \times B^{(m)} | M^{12\ldots,k}_{12\ldots,k}(C(x)) \neq 0, k \in \mathbb{N}\}
$$

holds the decomposition $C(x) = L_1 D_1 U_1$ and the matrix elements of the matrix $L_1, D_1$ and $U_1$ are rational functions in $c_{kn}(x)$.

3) We can find $(L_1)^{-1}$ and $(U_1)^{-1}$ using formulas (2.31). Note that $J_m L J_m, U$, and $J_m L J_m, U^{-1} \in B_2(a)$.

4) Using identity (3.27) we can calculate $C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1}$, since $L^{-1}, D^{-1}$ and $U^{-1}$ are well defined.

5) Using equality (3.27) we can find the decomposition $C^{-1} = L^{-1} D^{-1} U^{-1}$ of the matrix $C^{-1}$ by Theorem 1.2. In other words, the decompositions holds $C^{-1} = L^{-1} D^{-1} U^{-1}$ for all $x \in \Gamma_{G^{-1}}$, where

$$
\Gamma_{C^{-1}} = \{x \in B_m \times B^{(m)} | M^{12\ldots,k}_{12\ldots,k}(C^{-1}(x)) \neq 0, k \in \mathbb{N}\}
$$

and the matrix elements of the matrix $L^{-1}, D^{-1}$ and $U^{-1}$ are rational functions in matrix elements $c^{-1}_{kn}(x)$ of the matrix $C^{-1}$.

We make the last remark. Let us denote $(L_1)^{-1} = (L_{1;kmn})_{kn}, (D_1)^{-1} = \text{diag}(d_{1;kn})_k$ and $(U_1)^{-1} = (U_{1;kmn})_{kn}$. The decompositions $C = L_1 D_1 U_1$ and $C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1}$ hold for $x \in \Gamma_C \cap \Gamma_{C^{-1}}$, i.e. almost for all $x \in B_m \times B^{(m)}$ with respect to the measure $\mu_b$ since $\mu_b(\Gamma_C \cap \Gamma_{C^{-1}}) = 1$. We conclude that the convergence

\[ c^{-1}_{kn}(x) = \sum_{m \in \mathbb{N}} U^{-1}_{1;km} d^{-1}_{1;mn} L^{-1}_{1;mn}, \quad k, n \in \mathbb{N} \]

holds pointwise almost everywhere $x \in B_m \times B^{(m)} \mod \mu_b$. Since $U^{-1}_{1;km}, d^{-1}_{1;mn}$ and $L^{-1}_{1;mn} \eta \mathfrak{A}^s$ by 2) and 3), we conclude by Lemma 5.1 that $c^{-1}_{kn}(x) \eta \mathfrak{A}^s$. This finish the proof of the lemma.

**Proof of the Theorem 3.14** To prove the irreducibility of the induced representation consider the restriction $T^{m,y}_{|B_0(m)}$ of this representation to the commutative subgroup $B_0(m)$ of the group $B_0$. Note that

$$
\mathfrak{A}^x = \{ \exp(2\pi it x_{kr}) | t \in \mathbb{R}, (k,r) \in \Delta_m \cup \Delta^{(m)} \} = L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}).
$$

By Lemma 3.13, the von Neumann algebra $\mathfrak{A}^s$ generated by this restriction coincides with $\mathfrak{A}^s = L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)})$. Let now a bounded operator $A$ in the Hilbert space $\mathcal{H}^m$ commute with the representation $T^{m,y}$. Then $A$ commute by the above arguments with $L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)})$, therefore the operator $A$ itself is an operator of multiplication by some essentially bounded function $a \in L^\infty$ i.e. $(Af)(x) = a(x)f(x)$ for $f \in \mathcal{H}^m$. Since $A$ commute with the representation $T^{m,y}$ i.e. $[A,T^{m,y}_m] = 0$ for all $t \in B_{m,0} \times B_0^{(m)}$, where $B_{m,0} = B_m \cap B_0^Z$ and $B_0^{(m)} = B^{(m)} \cap B_0^Z$, we conclude that

$$
a(x) = a(xt) \mod \mu_{b,m} \otimes \mu_b^{(m)} \quad \text{for all} \quad t \in B_{m,0} \times B_0^{(m)}.
$$

Since the measure $\mu_{b,m} \otimes \mu_b^{(m)}$ on the group $B_m \times B^{(m)}$ is right $B_{m,0} \times B_0^{(m)}$-ergodic we conclude that $a(x) = \text{const} \mod dx_m \otimes dx^{(m)}$.

\[ \square \]
Remark 3.16. We would like to show that $T^{m,y} = \lim_n T^{m,y_n}$. To be more precise consider the projection $B_0^\mathbb{Z} \mapsto G^m_n$ of the group $B_0^\mathbb{Z}$ on the subgroup $G^m_n$ and all other projections: homogeneous spaces, measures, Hilbert spaces and representations:

$$X_m = B_m \times B^{(m)} \mapsto X_{m,n} = B_{m,n} \times B^{(m,n)}, \quad \mu_{b,m} \otimes \mu_{b}^{(m)} \mapsto \mu_{b,m,n} \otimes \mu_{b}^{(m,n)}$$

$$\mathcal{H}^m = L^2(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_{b}^{(m)}) \mapsto L^2(B_{m,n} \times B^{(m,n)}, \mu_{b,m,n} \otimes \mu_{b}^{(m,n)})$$

$$\cong L^2(B_{m,n} \times B^{(m,n)}, dx_{m,n} \otimes dx^{(m,n)}) = \mathcal{H}^{m,n}$$

$$T^{m,y} \mapsto T^{m,y_n}, \quad n \in \mathbb{N}.$$ 

Since the measure $\mu_{b,m,n} \otimes \mu_{b}^{(m,n)}$ is equivalent with the Haar measure (compare (2.43) and (3.14)) we conclude that the corresponding representations $T^{n,m,y_n}$ in the spaces $L^2(B_{m,n} \times B^{(m,n)}, \mu_{b,m,n} \otimes \mu_{b}^{(m,n)})$ and $T^{m,y_n}$ in the space $L^2(B_{m,n} \times B^{(m,n)}, dx_{m,n} \otimes dx^{(m,n)})$ are equivalent. This implies $T^{m,y} = \lim_n T^{m,y_n}$.

3.9. Dual description of the groups $B_0^\mathbb{N}$ and $B_0^\mathbb{Z}$. First steps. Let $\hat{G}$ be the dual of the group $G$. Our aim is to describe $\hat{G}$ for $G = \lim_n G_n$ where $G_n = B(n, \mathbb{R})$ is the group of all $n \times n$ upper triangular real matrices with units on the principal diagonal, i.e. we would like to describe the dual of the group $B_0^\mathbb{N}$ of infinite in one direction and $B_0^\mathbb{Z}$ infinite in both directions matrices. Consider the inductive limit $G = \lim_n G_n$ of nilpotent groups $G_n = B(n, \mathbb{R})$. The symmetric (resp. nonsymmetric) imbedding gives us two infinite-dimensional analog of “nilpotent” groups $B_0^\mathbb{Z}$ (resp. $B_0^\mathbb{N}$).

We do not know the description of all $\hat{G}$. We only know that the set $\hat{G}$ contains the following three classes of representations.

1) The set $\hat{G}$ contains $\bigcup_n \hat{G}_n$ i.e. $\hat{G} \supset \bigcup_n \hat{G}_n$. One may use Kirillov’s orbit method [4, 7] to describe $\hat{G}_n$. The embedding $\hat{G}_n \subset G_{n+1}$ is described in Remark 3.17.

2) We have $\hat{G} \backslash \bigcup_n \hat{G}_n \neq \emptyset$. Namely $\hat{G} \backslash \bigcup_n \hat{G}_n$ contains ”regular” $TR^{v,u}$ and ”quasiregular” $\pi^{R,v,u,x}$ representations of the group $G$ (see subsection 3.1).

3) Induced representations (see subsection 3.6).

It is natural together with the group $B_0^\mathbb{N}$ (resp. $B_0^\mathbb{Z}$) consider all Hilbert-Lie completion $B_2^\mathbb{N}(a)$ (resp. $B_2^\mathbb{Z}(a)$) and the group of all upper-triangular matrices $B^\mathbb{N}$ (resp. $B^\mathbb{Z}$) (see subsections 3.3, 3.4)

$$G_n \rightarrow B_0^\mathbb{N} \rightarrow B_2^\mathbb{N}(a) \rightarrow B^\mathbb{N} \rightarrow G_n.$$ $G_n \rightarrow B_0^\mathbb{Z} \rightarrow B_2^\mathbb{Z}(a) \rightarrow B^\mathbb{Z} \rightarrow G_n.$

Together with all imbedding and projections of all mentioned groups $G_n = B(n, \mathbb{R})$ we have:

$$B(n, \mathbb{R}) \xrightarrow{i_{n+1}^n} B(n+1, \mathbb{R}) \xrightarrow{i_{n}^\infty} B_0^\mathbb{N} \rightarrow B_2(a) \rightarrow B^\mathbb{N} \rightarrow B(n+1, \mathbb{R}) \xrightarrow{p_{n+1}^n} B(n, \mathbb{R}),$$

where the imbedding $i_{n+1}^n$ and the projections $p_{n+1}^n$ are defined as follows:

$$B(n, \mathbb{R}) \ni x \mapsto i_{n+1}^n(x) = x + E_{n+1,n+1} \in B(n+1, \mathbb{R}),$$

$$B(n+1, \mathbb{R}) \ni x = x^{n+1}x_n \mapsto p_{n+1}^n(x) = x_n \in B(n, \mathbb{R}),$$

where $x^{n+1} = I + \sum_{k=1}^{n} x_{kn+1}E_{kn+1}$, $x_n = I + \sum_{1 \leq m < n \leq} x_{km}E_{km}$. For groups $G^m_n \simeq B(2n, \mathbb{R})$ defined by (2.41) consider the homomorphism $p_{n+1}^{m,n} : G_{n+1}^m \rightarrow G^m_n$ defined as follows (for simplicity we define $p_{n+1}^{m,n}$ for $m = 0$)

$$G_{n+1}^0 \ni x = x^{n+1}x_n \mapsto p_{n+1}^{0,n}(x) = x_n \in G_0^n.$$
we conclude that
the orbit method for the Hilbert-Lie group
It leaves to describe
\( \hat{B} \) (resp. \( B \))

\[ (3.9) \]

Remark 3.17. The embedding \( \hat{B}(n, \mathbb{R}) \mapsto B(n + 1, \mathbb{R}) \) (resp. \( \hat{G}^m \mapsto \hat{G}^m_{n+1} \)) is induced by the homomorphism \( [3.9] \)

\[ p_{n+1}^m : B(n + 1, \mathbb{R}) \mapsto B(n, \mathbb{R}) \) (resp. by the homomorphism \( [3.9] \)

\[ p^m_{s,m,n} : G^m_{n+1} \mapsto G^m_n \). So for \( m \in \mathbb{Z} \) we get \( \bigcup_{n \in \mathbb{N}} \hat{G}^m_n \subset \hat{B}^\mathbb{Z}_0 \). Similarly, we have
\( \bigcup_{n \in \mathbb{N}} B(n, \mathbb{N}) \subset B^\mathbb{N}_0 \)

Let us denote by \( B^N_2(a) \) (resp. \( B^Z_2(a) \)) the completion of the subgroup \( B^N_0 \subset GL_0(2\mathbb{\infty}, \mathbb{R}) \) (resp. \( B^Z_0 \subset GL_0(2\mathbb{\infty}, \mathbb{R}) \)) in the Hilbert-Lie group \( GL_2(a) \). Since (see [8])

\[ B^N_0 = \bigcap_{a \in \mathbb{A}} B^N_2(a) \quad \text{(resp. } B^Z_0 = \bigcap_{a \in \mathbb{A}} B^Z_2(a) \text{)} \]

we conclude that
\( \hat{B}^N_0 = \bigcup_{a \in \mathbb{A}} \hat{B}^N_2(a) \) (resp. \( \hat{B}^Z_0 = \bigcup_{a \in \mathbb{A}} \hat{B}^Z_2(a) \)).

It leaves to describe \( \hat{B}^N_2(a) \) (resp. \( \hat{B}^Z_2(a) \)) for all \( a \in \mathbb{A} \). The problem of developing
the orbit method for the Hilbert-Lie group \( B^N_2(a) \) (resp. \( B^Z_2(a) \)) could be easier, since the corresponding Lie algebra \( b^N_2(a) \) (resp. \( b^Z_2(a) \)) is a Hilbert-Lie algebra, the dual \( (b^N_2(a))^* \) (resp. \( (b^Z_2(a))^* \)) and the pairing between \( b^N_2(a) \) (resp. \( b^Z_2(a) \)) and \( (b^N_2(a))^* \) (resp. \( (b^Z_2(a))^* \)) are well defined (see subsection 3.6).

Using \( [3.9] \) we conclude
\[ (3.28) \]

\[ B^N_0 = \lim_{n,t} B(n, \mathbb{R}), \quad B^N_0 = \lim_{t \to 0, a} B^N_2(a), \quad B^N = \lim_{t \to 0, p} B(n, \mathbb{R}), \]

\[ \hat{B}^N_0 \supset \hat{B}^N_2(a) \supset \hat{B}_0^N, \]

finally we conclude that
\[ (3.29) \]

\[ \hat{B}_0^N = \bigcup_{a \in \mathbb{A}} \hat{B}_2^N(a), \quad \hat{B}_0^N = \bigcup_{n \in \mathbb{N}} \hat{G}_n = \bigcup_{n \in \mathbb{N}} B(n, \mathbb{R}). \]

The similar relations holds also for groups \( B^Z_0 \subset B^Z_2(a) \subset B^Z \).

Definition 3.18. We call the representation of the group \( G = \lim_{n \to 0} G_n \) local if it depends
only on the elements of the subgroup \( G_n \) for some fixed \( n \in \mathbb{N} \).

The last relation in \( (3.28) \) and \( (3.29) \) we can reformulated as follows:

Theorem 3.19. (V.L. Ostrovsky, PhD dissertation, 1986). The class of all irreducible unitary local representations of the group \( B^N_0 = \lim_{t \to 0} B(n, \mathbb{R}) \) coincides with the class
\( \bigcup_{n} \hat{G}_n \).

4. Appendix 1. Gauss decompositions

4.1. Gauss decomposition of \( n \times n \) matrices. We need some decomposition of the matrix \( C \in \text{Mat}(n, \mathbb{C}) \). Let us denote by
\[ M_{i_1i_2...i_r}(C), \quad 1 \leq i_1 < ... < i_r \leq n, \quad 1 \leq j_1 < ... < j_r \leq n \]

the minors of the matrix \( C \) with \( i_1, i_2, ..., i_r \) rows and \( j_1, j_2, ..., j_r \) columns.
Theorem 4.1 (Gauss decomposition, [2]). A matrix $C \in \text{Mat}(n, \mathbb{C})$ admits the following decomposition $C = LDU$ (Gauss decomposition),

\[
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1n} \\
c_{21} & c_{22} & \ldots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \ldots & c_{nn}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
l_{21} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
l_{n1} & l_{n2} & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
d_1 & 0 & \ldots & 0 \\
d_2 & d_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
d_n & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
1 & u_{12} & \ldots & u_{1n} \\
0 & 1 & \ldots & u_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

where $L$ (resp. $U$) is lower (resp. upper) triangular matrix and $D$ a diagonal matrix if and only if all principal minors of the matrix $C$ are given by the formulas (see [2] Ch.II, §4, (44), (45))

\[
l_{mk} = \frac{M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, \quad u_{km} = \frac{M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, 1 \leq k < m \leq n, \]

\[
d_1 = M^1_1(C), \quad d_k = \frac{M^{1,2,\ldots,k}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, \quad 2 \leq k \leq n. \]

Proof. If we write $L^{-1}C = DU$, we get

\[
M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C) = M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(L^{-1}C) = M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(DU) = d_1 \ldots d_k,
\]

this implies (4.3). Moreover, we get also

\[
M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,m}(L^{-1}C) = M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,m}(DU) = d_1 \ldots d_k u_{km}, \quad k < m,
\]

this implies the second formula in (4.2). Similarly if we write $CU^{-1} = LD$ we get

\[
M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(CU^{-1}) = M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(C) = M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(LD) = d_1 \ldots d_k l_{mk}, \quad k < m,
\]

this implies the first formula in (4.2).

4.2. Gauss decomposition of infinite order matrices. Let us consider the infinite matrix $C, L, D, U \in \text{Mat}(\infty, \mathbb{C})$.

Theorem 4.2 (Gauss decomposition $C = LDU$). A matrix $C \in \text{Mat}(\infty, \mathbb{C})$ admits the following decomposition $C = LDU$ (Gauss decomposition),

\[
\begin{pmatrix}
c_{11} & c_{12} & \ldots & c_{1n} & \ldots \\
c_{21} & c_{22} & \ldots & c_{2n} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
c_{n1} & c_{n2} & \ldots & c_{nn} & \ldots
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \ldots & 0 & \ldots \\
l_{21} & 1 & \ldots & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
l_{n1} & l_{n2} & \ldots & 1 & \ldots
\end{pmatrix}
\begin{pmatrix}
d_1 & 0 & \ldots & 0 & \ldots \\
d_2 & d_2 & \ldots & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
d_n & 0 & \ldots & d_n & \ldots
\end{pmatrix}
\begin{pmatrix}
1 & u_{12} & \ldots & u_{1n} & \ldots \\
0 & 1 & \ldots & u_{2n} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & \ldots & 1 & \ldots
\end{pmatrix}
\]

where $L$ (resp. $U$) is lower (resp. upper) triangular matrix and $D$ a diagonal matrix of infinite order if and only if all principal minors of the matrix $C$ are different from zeros i.e. $M^{1,2,\ldots,k}_{1,2,\ldots,k}(C) \neq 0, \quad k \in \mathbb{N}$. Moreover the matrix elements of the matrices $L, U$ and $D$ are given by the same formulas as in the Theorem 4.1

\[
l_{mk} = \frac{M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, \quad u_{km} = \frac{M^{1,2,\ldots,k-1,m}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, \quad k, m \in \mathbb{N}, \quad k < m,
\]

\[
d_1 = M^1_1(C), \quad d_k = \frac{M^{1,2,\ldots,k}_{1,2,\ldots,k-1,k}(C)}{M^{1,2,\ldots,k-1,k}_{1,2,\ldots,k-1,k}(C)}, \quad k \in \mathbb{N}, \quad k > 1.
\]

Proof. The proof repeat word by word the proof of the Theorem 4.1. \qed
5. Appendix 2. One elementary fact concerning abelian von Neumann algebras

Let \( (X, \mathcal{F}, \mu) \) be a measurable space, with a finite measure \( \mu(X) < \infty \), where \( \mathcal{F} \) is a sigma-algebra. Consider the set \( (f_n) = (f_n)_{n \in \mathbb{N}} \) of measurable real valued functions on \( X \) i.e. \( f_n : X \to \mathbb{R}. \) Denote by \( B(H) \) the von Neumann algebra of all bounded operators in the Hilbert space \( H = L^2(X, \mu) \) and let \( \mathfrak{A}(f_n)(\in B(H)) \) be a von Neumann algebra generated by operators \( U_n(t) \) of multiplication by functions \( \exp(itf_n(x)), n \in \mathbb{N} \)

\[
\mathfrak{A}(f_n) = \left( U_n(t) = e^{itf_n} \mid n \in \mathbb{N}, t \in \mathbb{R} \right)^{\prime \prime}.
\]

We are interesting in the following question. Let \( f_n \to f \) as \( n \to \infty \) in some sense. When \( U(t) = e^{itf} \in \mathfrak{A}(f_n) \) for all \( t \in \mathbb{R}? \)

Since \( \mathfrak{A}(f_n) \) is a von Neumann algebra it is sufficient to find when the strong convergence of the unitary operators in the space \( H \) holds i.e. \( s.\lim_n U_n(t) = U(t) \), where the operators \( U_n(t), n \in \mathbb{N} \) and \( U(t) \) are defined as follows

\[
(U_n(t)g)(x) = e^{itf_n(x)}g(x), \quad (U(t)g)(x) = e^{itf(x)}g(x), \quad g \in L^2(X, \mu), \quad t \in \mathbb{R}.
\]

**Lemma 5.1.** Let \( f_n \to f \) as \( n \to \infty \) pointwise almost everywhere, then \( s.\lim_n U_n(t) = U(t) \) hence \( U(t) = e^{itf} \in \mathfrak{A}(f_n) \).

**Proof.** For \( g \in H \) we get

\[
\| (U_n(t) - U(t))g \|^2 = \int_X \left| (e^{itf_n(x)} - e^{itf(x)})g(x) \right|^2 d\mu(x) =
\]

\[
\int_X \left| e^{itf_n(x) - itf(x)} - 1 \right|^2 |g(x)|^2 d\mu(x) = \int_X \left| e^{it\alpha_n(x)} - 1 \right|^2 |g(x)|^2 d\mu(x) \to 0
\]

as \( n \to \infty \), if \( \alpha_n(x) := f_n(x) - f(x) \to 0 \) pointwise almost everywhere by Lebesgue’s dominated convergence theorem.

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Institute of Mathematics, Ukrainian National Academy of Sciences, 3 Tereshchenki’ska, Kyiv, 01601, Ukraine.

E-mail address: kosyak02@gmail.com