Iwasawa Effects in Multi-layer Optics

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Abstract

There are many two-by-two matrices in layer optics. It is shown that they can be formulated in terms of a three-parameter group whose algebraic property is the same as the group of Lorentz transformations in a space with two space-like and one time-like dimensions, or the \(Sp(2)\) group which is a standard theoretical tool in optics. Among the interesting mathematical properties of this group, the Iwasawa decomposition drastically simplifies the matrix algebra under certain conditions, and leads to a concise expression for the S-matrix for transmitted and reflected rays. It is shown that the Iwasawa effect can be observed in multi-layer optics, and a sample calculation of the S-matrix is given.

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I. INTRODUCTION

In a series of recent papers [1,2], Han, Kim and Noz have formulated polarization optics in terms of the two-by-two and four-by-four representations of the six-parameter Lorentz group. They noted that the Lorentz group properties can be found in optical materials. Indeed, there are many two-by-two matrices in layer optics [3–5]. In this paper, we reorganize them within the framework of the Lorentz group. We then derive a mathematical relation which can be tested experimentally. If a light wave hits a flat surface, a part of this beam becomes reflected and the remaining part becomes transmitted.

If there are multi-layers, this process repeats itself at each boundary. There has been a systematic approach to this problem based on the two-by-two S-matrix formalism [3–5]. This S-matrix consists of boundary and phase-shift matrices. The phase-shift matrices are complex and the S-matrix is in general complex.

However, in this paper, we show that these complex matrices can be systematically transformed into a set of real traceless matrices with three independent parameters. Then we can use the well-established mathematical procedure for them. This procedure is called the $Sp(2)$ group whose algebraic property is the same as that of the $SU(1,1)$ group which occupies a prominent place in optics from squeezed states of light [6]. However, the most pleasant aspect of the $Sp(2)$ group is that its algebras consist only of two-by-two matrices with real elements. When applied to a two-dimensional plane, they produce rotations and squeeze transformations [7].

It is known that these simple matrices produce some non-trivial mathematical results, namely Wigner rotations and Iwasawa decompositions [8]. The Wigner rotation means a rotation resulting from a multiplication of three squeeze matrices, and the Iwasawa decomposition means that a product of squeeze and rotation matrices, under certain conditions, leads to a matrix with one vanishing off-diagonal element. This leads to a substantial simplification in mathematics and eventually leads to a more transparent comparison of theory with experiments. This decomposition has been discussed in the literature in connection with polarization optics [9,10] and Wigner rotations [8]. In this paper, we study applications of this mathematical device in layer optics.

There are papers in the literature on applications of the Lorentz group in layer optics [2,11], but these papers are concerned with polarization optics. In this paper, we are dealing with reflections and transmissions of optical rays. We show that layers with alternate indexes of refraction can exhibit an Iwasawa effect and provide a calculation of the transmission and reflection coefficients. It is remarkable that the Lorentz group can play as the fundamental scientific language even in the physics of reflections and transmissions.

In Sec. II, we formulate the problem in terms of the S-matrix method widely used in optics [3]. In Sec. III, this S-matrix formalism is translated into the mathematical framework of the $Sp(2)$ group consisting of two-by-two traceless matrices with real elements. We demonstrate that there is a subset of these matrices with one vanishing non-diagonal element. It is shown possible to produce this set of matrices from multiplications of the matrices in the original set. This is called the Iwasawa decomposition. In Sec. IV, we transform the mathematical formalism of the Iwasawa decomposition into the real world, and calculate the reflection and transmission coefficients which can be measured in optics laboratories.
II. FORMULATION OF THE PROBLEM

Let us start with the S-matrix formalism of the layer optics. If a beam is incident on a plane boundary of a medium with a different index of refraction, the problem can be formulated in terms of two-by-two matrices \[3,5\]. If we write the column vectors

\[
\begin{pmatrix}
E^{(+)}_1 \\
E^{(-)}_1
\end{pmatrix}, \quad 
\begin{pmatrix}
E^{(+)}_2 \\
E^{(-)}_2
\end{pmatrix},
\]

(1)

for the incident, with superscript (+), and reflected, with superscript (-), for the beams in the first and second media respectively, then they are connected by the two-by-two S-matrix:

\[
\begin{pmatrix}
E^{(+)}_1 \\
E^{(-)}_1
\end{pmatrix} = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix} \begin{pmatrix}
E^{(+)}_2 \\
E^{(-)}_2
\end{pmatrix}.
\]

(2)

Of course the elements of the above S-matrix depend on reflection and transmission coefficients \[3\].

![FIG. 1. Multi-layer system. A light beam is incident on the first boundary, with transmitted and reflected rays. The transmitted ray goes through the first medium and hits the second medium again with reflected and transmitted rays. The transmitted ray goes through the second medium and hits the first medium. This cycle continues \(N\) times.](image)

Let us consider a light wave incident on a flat surface, then it is decomposed into transmitted and reflected rays. If \(E^{(+)}_1\) is the incident ray, the transmitted ray is \(E^{(+)}_2\), with

\[
E^{(+)}_2 = t_{12}E^{(+)}_1, \quad E^{(-)}_1 = r_{12}E^{(+)}_1.
\]

(3)
Thus, the S-matrix takes the form
\[
\begin{pmatrix}
E_1^{(+)} \\
E_1^{(-)}
\end{pmatrix} = \begin{pmatrix} 1/t_{12} & r_{12}/t_{12} \\ r_{12}/t_{12} & 1/t_{12} \end{pmatrix} \begin{pmatrix} E_2^{(+)} \\
0 \end{pmatrix}.
\] (4)

If the ray comes from the second medium in the opposite direction, the same matrix can be used for
\[
\begin{pmatrix}
0 \\
E_1^{(-)}
\end{pmatrix} = \begin{pmatrix} 1/t_{12} & r_{12}/t_{12} \\ r_{12}/t_{12} & 1/t_{12} \end{pmatrix} \begin{pmatrix} E_2^{(+)} \\
E_2^{(-)} \end{pmatrix}.
\] (5)

Since the magnitude of the reflection coefficient is smaller than one, and since \( t_{12}^2 + r_{12}^2 = 1 \), we can write the above matrix as
\[
\begin{pmatrix}
\cosh \eta & \sinh \eta \\
\sinh \eta & \cosh \eta
\end{pmatrix},
\] (6)
with
\[
r_{12} = \tanh \eta, \quad t_{12} = 1/\cosh \eta.
\] (7)

Since this describes both the reflection and transmission at the boundary, shall call this matrix “boundary matrix” \([12]\). It is a uni-modular matrix (determinant = 1). The mathematics of this form is well known. It can perform Lorentz boosts when applied to the longitudinal and time-like coordinates. Recently, it has been observed that it performs squeeze transformations when applied to the two-dimensional space of \( x \) and \( y \) \([7]\).

Next, if the ray travels within a given medium from one inner-surface to the other surface
\[
\begin{pmatrix}
E_a^{(+)} \\
E_a^{(-)}
\end{pmatrix} = \begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} E_b^{(+)} \\
E_b^{(-)} \end{pmatrix},
\] (8)
where the subscripts \( a \) and \( b \) are for the initial and final surfaces respectively. The above expression tells there is a phase difference of \( 2\delta \) between the rays. This phase difference depends on the index of refraction, wavelength and the angle of incidence \([3]\).

In this paper, we consider a multi-layer system consisting of two media with different indexes of refraction as is illustrated in Fig. 1. Then, the system consists of many boundaries and phase-shift matrices. After multiplication of all those matrices, the result will be one two-by-two matrix which we introduced as the S-matrix in Eq.(2). We are interested in this paper when this matrix takes special forms which can be readily tested experimentally.

If the ray hits the first medium from the air, as is illustrated in Fig 1, we write the matrix as
\[
\begin{pmatrix}
\cosh \lambda & \sinh \lambda \\
\sinh \lambda & \cosh \lambda
\end{pmatrix}.
\] (9)

Within the first medium, the phase shift matrix becomes
\[
\begin{pmatrix}
e^{-i\phi} & 0 \\ 0 & e^{i\phi}\end{pmatrix}.
\] (10)
When the beam hits the surface of the second medium, the corresponding matrix is

\[
\begin{pmatrix}
\cosh \eta & \sinh \eta \\
\sinh \eta & \cosh \eta \\
\end{pmatrix}.
\]  
(11)

Within the second medium, we write the phase-shift matrix as

\[
\begin{pmatrix}
e^{-i\xi} & 0 \\
0 & e^{i\xi} \\
\end{pmatrix}.
\]  
(12)

Then, when the beam hits the first medium from the second

\[
\begin{pmatrix}
\cosh \eta & -\sinh \eta \\
-\sinh \eta & \cosh \eta \\
\end{pmatrix}.
\]  
(13)

But if the thickness of the first medium is zero, and the beam exists to the air, then the system goes through the boundary matrix

\[
\begin{pmatrix}
\cosh \lambda & -\sinh \lambda \\
-\sinh \lambda & \cosh \lambda \\
\end{pmatrix}.
\]  
(14)

The net result is

\[
\begin{pmatrix}
\cosh \lambda & \sinh \lambda \\
\sinh \lambda & \cosh \lambda \\
\end{pmatrix} \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix} \begin{pmatrix}
\cosh \lambda & -\sinh \lambda \\
-\sinh \lambda & \cosh \lambda \\
\end{pmatrix},
\]  
(15)

with

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix} = \begin{pmatrix}
e^{-i\phi} & 0 \\
0 & e^{i\phi} \\
\end{pmatrix} \begin{pmatrix}
\cosh \eta & \sinh \eta \\
\sinh \eta & \cosh \eta \\
\end{pmatrix}
\times \begin{pmatrix}
e^{-i\xi} & 0 \\
0 & e^{i\xi} \\
\end{pmatrix} \begin{pmatrix}
\cosh \eta & -\sinh \eta \\
-\sinh \eta & \cosh \eta \\
\end{pmatrix}.
\]  
(16)

If the ray goes through \(N\) cycles of this pair of layers, the S-matrix becomes

\[
\begin{pmatrix}
\cosh \lambda & \sinh \lambda \\
\sinh \lambda & \cosh \lambda \\
\end{pmatrix} \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix}^N \begin{pmatrix}
\cosh \lambda & -\sinh \lambda \\
-\sinh \lambda & \cosh \lambda \\
\end{pmatrix}.
\]  
(17)

Thus, the problem reduces to looking into unusual properties of the core matrix

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix}^N.
\]  
(18)

We realize that the numerical computation of this expression is rather trivial these days, but we are still interested in the mathematical form which takes exceptionally simple form. It is still an interesting problem to produce mathematics which enables us to perform calculations without using computers. In Sec. \[II\], we shall consider mathematical simplification coming from one vanishing off-diagonal element.
III. MATHEMATICAL INSTRUMENT

The core matrix of Eq. (18) contains the chain of the matrices

$$ W = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix}. $$

(19)

The Lorentz group allows us to simplify this expression under certain conditions.

For this purpose, we transform the above expression into a more convenient form, by taking the conjugate of each of the matrices with

$$ C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. $$

(20)

Then $C_1W^{-1}$ leads to

$$ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}. $$

(21)

In this way, we have converted $W$ of Eq. (19) into a real matrix, but it is not simple enough.

Let us take another conjugate with

$$ C_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. $$

(22)

Then the conjugate $C_2C_1W^{-1}C_2^{-1}$ becomes

$$ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}. $$

(23)

The combined effect of $C_2C_1$ is

$$ C = C_2C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}, $$

(24)

with

$$ C^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} & -e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix}. $$

(25)

After multiplication, the matrix of Eq. (23) will take the form

$$ V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, $$

(26)

where $A, B, C,$ and $D$ are real numbers. If $B$ and $C$ vanish, this matrix will become diagonal, and the problem will become too simple. If, on the other hand, only one of these two elements become zero, we will achieve a substantial mathematical simplification and will be encouraged to look for physical circumstances which will lead to this simplification.

Let us summarize. we started in this section with the matrix representation $W$ given in Eq. (19). This form can be transformed into the $V$ matrix of Eq. (23) through the conjugate transformation
\[ V = CW C^{-1}, \]  
where \( C \) is given in Eq.(24). Conversely, we can recover the \( W \) representation by
\[ W = C^{-1} V C. \]  
For calculational purposes, the \( V \) representation is much easier because we are dealing with real numbers. On the other hand, the \( W \) representation is of the form for the S-matrix we intend to compute. It is gratifying to see that they are equivalent.

Let us go back to Eq.(23) and consider the case where the angles \( \phi \) and \( \xi \) satisfy the following constraints.
\[ \phi + \xi = 2 \theta, \quad \phi - \xi = \pi/2, \]  
thus
\[ \phi = \theta + \pi/4, \quad \xi = \theta - \pi/4. \]  
Then in terms of \( \theta \), we can reduce the matrix of Eq.(23) to the form
\[ \begin{pmatrix} (\cosh \eta) \cos(2\theta) & \sinh \eta - (\cosh \eta) \sin(2\theta) \\ \sinh \eta + (\cosh \eta) \sin(2\theta) & (\cosh \eta) \cos(2\theta) \end{pmatrix}. \]  
Thus the matrix takes a surprisingly simple form if the parameters \( \theta \) and \( \eta \) satisfy the constraint
\[ \sinh \eta = (\cosh \eta) \sin(2\theta). \]  
Then the matrix becomes
\[ \begin{pmatrix} 1 & 0 \\ 2 \sinh \eta & 1 \end{pmatrix}. \]  
This aspect of the Lorentz group is known as the Iwasawa decomposition [8], and has been discussed in the optics literature [9,10].

The matrices of the form is not so strange in optics. In para-axial lens optics, the translation and lens matrices are written as
\[ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \]  
respectively. These matrices have the following interesting mathematical property [2].
\[ \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}, \]  
and
\[ \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ u_1 & 1 \end{pmatrix}. \]  
We note that the multiplication is commutative, and the parameter becomes additive. These matrices convert multiplication into addition, as logarithmic functions do.
IV. POSSIBLE EXPERIMENTS

The question then is whether it is possible to construct optical layers which will perform this or similar calculation. In order to make contacts with the real world, let us extend the algebra to the form

\[
\begin{pmatrix}
1 & 0 \\
2 \sinh \eta & 1
\end{pmatrix}
\begin{pmatrix}
e^{-\eta} & 0 \\
0 & e^{\eta}
\end{pmatrix},
\]

which becomes

\[
\begin{pmatrix}
e^{-\eta} & 0 \\
2e^{-\eta} \sinh \eta & e^{\eta}
\end{pmatrix}.
\]

The square of this matrix is

\[
\left(\begin{array}{cc}
e^{-\eta} & 0 \\
2e^{-\eta} \sinh \eta & e^{\eta}
\end{array}\right)^2 = \left(\begin{array}{cc}
e^{-2\eta} & 0 \\
2(e^{-2\eta} + 1) \sinh \eta & e^{2\eta}
\end{array}\right).
\]

If we repeat this process,

\[
\left(\begin{array}{cc}
e^{-\eta} & 0 \\
2e^{-\eta} \sinh \eta & e^{\eta}
\end{array}\right)^N = \left(\begin{array}{cc}
e^{N\eta} & 0 \\
2b(\sinh \eta) & e^{-N\eta}
\end{array}\right),
\]

with

\[
b = e^{-N\eta} \sum_{k=1}^{N-1} e^{-2(k-1)\eta},
\]

which can be simplified to

\[
b = \frac{e^{-\eta} \sinh(N\eta)}{\sinh \eta}.
\]

Then we can write Eq.\,(44) as

\[
\left(\begin{array}{cc}
e^{-\eta} & 0 \\
2e^{-\eta} \sinh \eta & e^{\eta}
\end{array}\right)^N = \left(\begin{array}{cc}
e^{-N\eta} & 0 \\
2e^{-\eta} \sinh(N\eta) & e^{N\eta}
\end{array}\right).
\]

If we take into account the boundary between the air and the first medium,

\[
\left(\begin{array}{cc}
e^\lambda & 0 \\
0 & e^{-\lambda}
\end{array}\right) \left(\begin{array}{cc}
e^{-N\eta} & 0 \\
2e^{-\eta} \sinh(N\eta) & e^{N\eta}
\end{array}\right) \left(\begin{array}{cc}
e^{-\lambda} & 0 \\
0 & e^\lambda
\end{array}\right) = \left(\begin{array}{cc}
e^{-N\eta} & 0 \\
2e^{-(2\lambda+\eta)} \sinh(N\eta) & e^{N\eta}
\end{array}\right).
\]

Thus, the original matrix of Eq.(2) becomes

\[
\begin{pmatrix}
\cosh(N\eta) + ie^{-(\eta+2\lambda)} \sinh(N\eta) & -(1 + ie^{-(\eta+2\lambda)}) \sinh(N\eta) \\
-(1 - ie^{-(\eta+2\lambda)}) \sinh(N\eta) & \cosh(N\eta) - ie^{-(\eta-2\lambda)} \sinh(N\eta)
\end{pmatrix}.
\]

From the S-matrix formalism, the reflection and transmission coefficients are
\[ R = \frac{E_{a}^{-}}{E_{a}^{+}} = \frac{S_{21}}{S_{11}}, \]
\[ T = \frac{E_{a}^{+}}{E_{a}^{+}} = \frac{1}{S_{21}}. \] (46)

Thus, they become
\[ R = \frac{(1 - i e^{-(\eta+2\lambda)}) \sinh(N\eta)}{\cosh(N\eta) + i e^{-(\eta+2\lambda)} \sinh(N\eta)}, \]
\[ T = \frac{-1}{(1 - i e^{-(\eta+2\lambda)}) \sinh(N\eta)}. \] (47)

The above expression depends only the number of layer cycles \( N \) and the parameter \( \eta \), which was defined in terms of the reflection and transmission coefficients in Eq.(4). It is important also that the above simple form is possible only if the phase-shift parameters \( \phi \) and \( \xi \) should satisfy the relations given in Eq.(30) and Eq.(32). In summary, they should satisfy
\[ \cos(2\xi) = -\cos(2\phi), \quad \text{and} \quad \tanh \eta = \cos(2\xi). \] (48)

In setting up the experiment, we note that all three parameters \( \eta, \xi \) and \( \eta \) depend on the incident angle and the frequency of the light wave. The parameter \( \eta \) is derivable from the reflection and transmission coefficients which depend on both the angle and frequency. The angular parameters \( \xi \) and \( \phi \) depend on the optical path and the index of refraction which depend on the incident angle and the frequency respectively.

Now all three quantities in Eq.(48) are functions of the incident angle and the frequency. If we consider a three-dimensional space with the incident angle and frequency as the \( x \) and \( y \) axes respectively. All three quantities, \( \cos(2\xi) \), \( \cos(2\phi) \), and \( \tanh \eta \), will be represented by two-dimensional surfaces. If we choose \( \cos(2\xi) \) and \( \cos(2\phi) \), the intersection will be a line. This line will pass through the third surface for \( \tanh \eta \). The point at which the line passes through the surface corresponds to the values of the incident angle and frequency which will satisfy the two conditions given in Eq.(48).

**CONCLUDING REMARKS**

In this paper, we borrowed the concept of Iwasawa decomposition from well-known theorems in group theory. On the other hand, group theory appears in this paper in the form of two-by-two matrices with three independent parameters. The Iwasawa decomposition makes the algebra of two-by-two matrices even simpler. It is interesting to note that there still is a room for mathematical simplifications in the algebra of two-by-two matrices and that this procedure can be tested in optics laboratories.
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