On the universality of minimum uncertainty states as approximate classical states

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Explaining the perceived classicality of the macroscopic world from quantum principles has been one of the promising approaches to the quantum-to-classical transition. Coherent states have been regarded as the closest to the classical, representing the best compromise allowed by the Heisenberg uncertainty relations. On the other hand, decoherence theory recognizes the crucial role of the environment in the quantum-to-classical transition and defines, in idealistic scenarios, its own preferred, robust states, so-called pointer states. Analyzing realistic open dynamics, where both interaction and free terms contribute, we show that these two notions are in general different. Connecting group theory and open dynamics, we derive general equations describing states most robust to thermal decoherence and study spin systems as an example. Although we concentrate on compact groups here, our method is more general and opens a new set of problems in open dynamics–finding physical pointer states.

I. INTRODUCTION

One of the possible approaches to understand the quantum-to-classical transition is to recognize that what we perceive as classicality is, in fact, a property of quantum states [1]. Traditionally, such “classical” states have been associated with the states minimizing the Heisenberg uncertainty relations [2, 3] as the best compromise [4–7]. However, decoherence theory [1, 8, 9] presents a competing view, where classical properties emerge dynamically as a result of interaction with the environment (for more advanced decoherence mechanisms see e.g. [10–16]). A question arises if these notions coincide in anything close to the real life? The seminal example of Ref. [17] suggests it may be so. However, a general analysis presented here shows that a broad class of realistic open dynamics defines its own notion of “closest to the classical”, different from the traditional notion of minimal uncertainty.

A major step in establishing decoherence as a promising mechanism of the quantum-to-classical transition, has been made with the introduction of the so called pointer states [18, 19]. Mathematically defined as pure states of the central system preserved by the combined system-environment dynamics, they were promoted to represent the actual physical state of affairs by the stability criterion [1, 18]: Decohered system is found in one of the dynamically stable states. It is thus the interaction with the environment that dynamically selects, the process being called einselection [20], the perceived classical properties as the properties possessed or momenta. There is also an important connection to the measurement problem via, what one may call, the decoherence-measurement duality. Substituting (central system) ↔ (measured system) and (environment) ↔ (measuring apparatus), einselection fills the gap left by the von Neumann measurement scheme by singling out states that are dynamically stable w.r.t. the measurement interaction as the preferred basis, in which the system is found after the measurement has been completed.

However, strictly dynamically stable states exist only for a narrow; idealized class of dynamics and in the most realistic situations no such states are available. The paradigmatic example is the intermediary regime, where interaction and free Hamiltonians are of a roughly comparable strength and not commuting. One way to relax the stability requirement is via, so called, predictability sieve idea [21] (see also [19]): Approximate pointer states are given by the states most immune to decoherence. Such states couple least to the environment and hence are the most stable and the most predictable. Their immunity can be quantified via various measures [8, 19], with purity being the simplest [22]: $s(\rho_S) = 1 - \text{Tr}[\rho_S^2]$, where $g_S = \text{Tr}_E[\rho_{SE}]$. The problem is, however, how to find these states? A prominent example of approximate pointer states has been found in the Quantum Brownian Motion (QBM) model [1, 9, 23, 24], where they happen to be the standard Glauber-Sudarshan coherent states [17]. Minimum uncertainty states have also been proven to be universal pointer states for free evolution [25] (cf. [19]).

In this work we build a general framework for finding approximately stable states of open dynamics in the intermediary regime. We do so under a broad assumption of an existence of some group structure behind the dynamics, which is insensitive to the type of the environment, i.e. spin or boson, as long as it is thermal. This assumption is broad enough to cover, among others, the canonical models of decoherence [9]. We analyze in detail the case of a compact, semi-simple group, but the method applies to other group structures. In fact, we show that the approximate pointer states, obtained in Ref. [8], can be derived readily from our construction. We explicitly derive general conditions defining approximate pointer states and although they are formulated

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II. PREDICTABILITY SIEVE IN THE PRESENCE OF A GROUP STRUCTURE

We consider a general thermal model, where a quantum system $S$ is linearly coupled to a thermal environment $E$. The evolution law for the reduced state $\rho$ of the system is then given by the well known expression (under the usually assumed Born-Markov approximation) [9]:

$$\dot{\rho}(t) = -i[H_0, \rho(t)] - \int_0^\infty d\nu(\tau) [A, [A(\tau), \rho(t)]] + i \int_0^\infty d\eta(\tau) [A, \{A(\tau), \rho(t)\}],$$

where $\{A, B\} = AB + BA$ is the anti-commutator, $A$ is the coupling observable and $H_0$ is the free Hamiltonian of the system. Further,

$$\nu(\tau) = \int_0^\infty d\omega J(\omega) \coth\left(\frac{\omega \beta}{2}\right) \cos(\omega \tau),$$

$$\eta(\tau) = \int_0^\infty d\omega J(\omega) \sin(\omega \tau)$$

are the noise and dissipation kernels with spectral density $J(\omega)$ and inverse temperature $\beta$, and

$$A(\tau) = e^{-iH_0\tau} A e^{iH_0\tau}$$

is the backwards time-evolved observable $A$ under the free evolution. Remarkably, the law (1) covers both oscillator and spin environments, the latter being described by the change $J(\omega) \rightarrow J(\omega) \tanh(\beta \omega/2)$.

The predictability sieve looks for those states of the system, which are least affected by the dynamics, given by Eq. (1). Following Ref. [17], we assume the initial state to be pure

$$\rho(0) = |\psi\rangle \langle \psi|$$

and ask how much entropy is produced during the evolution. The latter is conveniently measured by the linear entropy, which is defined as

$$s(\rho) = 1 - \text{Tr}[\rho^2].$$

Our goal is to estimate the entropy production due to Eq. (1). In general it is a difficult task and we are aware of only one model-specific estimate [17], obtained in the model of quantum Brownian motion. Here we make far less restrictive assumptions, namely that $H_0$ and $A$ can be identified with generators of some Lie group $G$ of dimension $N$. In particular, we take

$$H_0 \equiv X_N,$$

$$A \equiv \sum_{j=1}^N a_j X_j,$$

where $\{X_i\}_{i=1}^N$ are the generators of the Lie group $G$. We choose $H_0$ to be $X_N$ only for the sake of definiteness. Both $H_0$ and $A$ must be Hermitian for obvious physical reasons. Thus, we can assume that the Lie algebra in question is real and spanned by Hermitian operators, $X_j^\dagger = X_j$. Further assuming they can be represented in finite dimension, $G$ is a subgroup of sufficiently large unitary group and thus compact. In light of these identifications, the free evolution, given by Eq. (4), becomes the adjoint action of $G$ in its Lie algebra. In particular, we have

$$X_j(-\tau) = e^{-iX_N\tau} X_j e^{iX_N\tau} = \sum_k R_{jk}^N(-\tau) X_k,$$

where $R_{jk}^N(0) = \delta_{jk}$. The indices $i, j, k, \ldots \in 1 \ldots N$ are the Lie algebra indices of $G$. Matrix $R^N$ is nothing but the exponent of the structure constants $f_{ijk}$ of $G$, arranged into $N \times N$ matrix (the matrix of the ad$_{X_N}$ action) and is given by

$$R_{jk}^N(t) = [e^{it \text{ad}_{X_N}}]_{jk}, \quad [\text{ad}_{X_N}]_{jk} = if_{ijk} X_k,$$

where $f_{ijk}$ are defined via $[X_i, X_j] = \sum_k f_{ijk} X_k$. In what follows we will omit the index $N$ for simplicity, writing $R_{jk}(\tau)$. Substituting Eq. (9) into Eq. (1), we obtain

$$\dot{\rho}(t) = -i[X_N, \rho(t)] - \sum_{jkl} a_ia_j D_{jk} X_i, [X_k, \rho(t)]$$

$$+ i \sum_{jkl} a_ia_j \gamma_{jk} X_i, [X_k, \rho(t)],$$

where $\gamma_{jk}$ are the noise and dissipation kernels with spectral density $J(\omega)$ and inverse temperature $\beta$, and

$$D_{jk}(\tau) = e^{-iH_0\tau} D_{jk} e^{iH_0\tau}$$

is the backwards time-evolved observable $D_{jk}$ under the free evolution.
where we have introduced constants
\[ D_{jk} = \int_0^\infty d\tau \nu(\tau) R_{jk}(-\tau), \quad \gamma_{jk} = \int_0^\infty d\tau \eta(\tau) R_{jk}(-\tau). \]

We can now express the change of the entropy, given by Eq. (6), as
\[ \frac{1}{2} \dot{s} = \sum_{jkl} a_j a_l D_{jk} \left( \text{Tr} [\rho^2 \{ X_l, X_k \}] - 2 \text{Tr} [\rho X_l \rho X_k] \right) + \sum_{jklm} a_j a_l \gamma_{jk} f_{lkm} \text{Tr} [\rho^2 X_m], \]

To make the further analysis feasible, we may assume as a first approximation [17] that the state \( \rho(t) \) at the right hand side above is approximately pure and evolves according to the free evolution, i.e.,
\[ \rho(t) \approx e^{-ix_n t} \langle \psi | e^{ix_n t}. \]

Using Eq. (9) we obtain
\[ \frac{1}{2} \dot{s} \approx \sum_{jklm} a_j a_l D_{jk} R_{lm}(t) R_{kn}(t) C_{mn} + \]
\[ \sum_{jklm} a_j a_l \gamma_{jk} f_{lkm} R_{mn}(t) \langle X_n \rangle, \]

where
\[ C_{mn} = \langle \{ X_m, X_n \} \rangle - 2 \langle X_m \rangle \langle X_n \rangle \]
is the covariance matrix calculated in the initial state, i.e. \( \langle X_m \rangle = \langle \psi | X_m | \psi \rangle \) the average w.r.t. the initial state. To analyze Eqs. (15) and (16) further, we look closer at the structure of matrices \( R_{jk}(t) \). For compact group \( G \), there is a Killing-Cartan form, \( h_{jk} = \text{Tr}[\text{ad} X_j \text{ad} X_k] \) (see e.g. [26]) on the Lie algebra of \( G \) with a definite signature and which serves as a metric. Moreover, \( h_{jk} \) is preserved by the adjoint action (9). For simplicity, we will also assume that this metric is non-degenerate (so the group is semi-simple), but this is not crucial. We can then diagonalize the metric and assume \( h_{jk} \propto \delta_{jk} \) so that the matrices \( R_{jk}(t) \) of the adjoint action become orthogonal matrices. Since they are in the connected component of the unity by taking \( t \rightarrow 0 \) in Eq. (9), they are also special orthogonal, i.e. \( R(t) \in SO(N) \). Furthermore, from Eq. (9) we obviously have \( e^{ix_n t} X_N e^{-ix_n t} = X_N \), i.e. the \( N \)-th row has just one element \( \delta_{N,k} \). From the orthogonality \( R(t) \) is of the form:
\[ R(t) = \begin{bmatrix} SO(N-1) & 0 \\ 0 & 1 \end{bmatrix}, \]
i.e. it is a rotation around the axis of \( X_N \) as one could easily guess from Eq. (9). In most physical models the non-dynamical part \( a_N X_N \) of \( A \) is usually neglected but we keep it here for completeness. The \( SO(N-1) \) part can be further decomposed into a direct sum of 2D rotations by a change of basis in the Lie algebra. For odd \( N-1 \), i.e. even \( N \), one of the blocks will again be 1. Summarizing, we can write the following decomposition of the full matrix \( R(t) \):
\[ R(t) = O^T [\Omega_\alpha R_\alpha(t)] O, \quad R_\alpha(t) = \begin{bmatrix} \cos t \Omega_\alpha & \sin t \Omega_\alpha \\ -\sin t \Omega_\alpha & \cos t \Omega_\alpha \end{bmatrix}, \]
where \( O \in SO(N) \) (we prefer to use the full dimensional matrices extended by 1 on the diagonal for the ease of the index notation later) and the last block \( R_\alpha \) dominates. Additionally, for even \( N \) the one before the last block is also trivial, i.e., we have
\[ OR(t)O^T = \begin{bmatrix} R_1(t) & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \]

making \( f_{ijk} \) totally anti-symmetric as by definition \( f_{ijk} = -f_{ikj} \).

Summarizing, for each particular model, the l.h.s. of Eq. (15) is a finite linear combination of trigonometric functions and their squares, which, in principle, can be integrated (cf. [17]). However, the momentary value of the entropy production \( s(t) \) may not be the most indicative quantity due to its time fluctuations. We thus look at the long-time average of the entropy production, which is defined as
\[ \bar{s} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T [s(t) - s(0)] dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \dot{s}(t). \]

Here and in what follows \( T \) indicates averaging time, not the temperature. We apply the above time-averaging to Eq. (15) using Eq. (19). The first term gives
\[ \bar{R}_{lm}(t) R_{kn}(t) = \]
\[ = \sum_{k' m' n'} O_{l' k'} O_{k'' m} \sum_\alpha R_{m'n'}^{(\alpha)}(t) R_{k'n'}^{(\bar{\alpha})}(t) O_{m'n} O_{n'n} \]
\[ = \sum_{k' m' n'} O_{l' k'} \sum_\alpha \frac{1}{2} \left[ \delta_{m'n'}^{(\alpha)} s_{k'n'}^{(\bar{\alpha})} + \epsilon_{m'n'}^{(\alpha)} \epsilon_{k'n'}^{(\bar{\alpha})} \right] O_{m'n} O_{n'n}, \]
where \( R_{jm}^{(\alpha)} \) are the matrices of \( R_\alpha \) embedded naturally into the whole space by adding rows and columns of zeros,
i.e. $R_{lm}^{(a)} = 0$ when $jm$ are outside of the $\alpha$-subspace. We have also used the fact that
\begin{equation}
R_{lm}^{(a)}(t)R_{kn}^{(a)}(t) = \delta_{\alpha\beta} \frac{1}{2} \left[ \delta_{lm}^{(a)} \delta_{kn}^{(a)} + \epsilon_{lm}^{(a)} \epsilon_{kn}^{(a)} \right],
\end{equation}
for non-trivial blocks. Trivial blocks give simply 1. Here $\epsilon_{lm}$ is the totally anti-symmetric Levi-Civita symbol in dimension two. The Eq. (24) follows from a direct entry-by-entry averaging of $R_{\alpha}(t) \otimes R_{\beta}(t)$, using the explicit form, Eq. (19), where only the angular terms with the same angular velocity, i.e., $\sin^2 t \Omega_\alpha$ and $\cos^2 t \Omega_\alpha$ give non-zero contributions (We assume a generic situation with no special relations between eigenfrequencies $\Omega_\alpha$ for different $\alpha$). Let us define rotated generators as
\begin{equation}
\tilde{X}_m = \sum_m O_{mn} X_n
\end{equation}
and the corresponding covariance matrix $\tilde{C}_{nn}$. We recall that $X_N = X_N$ by the construction of $O$, cf. Eq. (18). We then have
\begin{equation}
\sum_{mm'nn'} \left[ \delta_{lm}^{(a)} \delta_{kn}^{(a)} + \epsilon_{lm}^{(a)} \epsilon_{kn}^{(a)} \right] O_{mn} O_{nn'} C_{m'n'} = \left[ \tilde{C}_\alpha + \epsilon \tilde{C}_\alpha e^T \right]_{kk} = \text{Tr}(\tilde{C}_\alpha) \delta_{kk},
\end{equation}
where $\tilde{C}_\alpha$ is the projection of $\tilde{C}$ on the $\alpha$-subspace and in the last step we used the symmetry of $\tilde{C}_\alpha$ (inherited from $\tilde{C}$). Using this, we can write
\begin{equation}
\sum_{jkl} a_j a_l D_{jk} O_{ij} O_{kl} = \sum_{\alpha} \sum_{ijkl} \tilde{a}_i \tilde{a}_l D_{ij}^{(\alpha)} O_{kl} O_{ik} = \sum_{\alpha} \sum_i \tilde{a}_i \tilde{a}_l D_{ij}^{(\alpha)}
\end{equation}
where we introduced a rotated vector $\tilde{a}_i \equiv \sum_j O_{ij} a_j$ and decomposed the matrix $D_{jk}$ by inserting decomposition, given by Eq. (19) into Eq. (12) and defined
\begin{equation}
D_{jk}^{(\alpha)} = \int_0^\infty d\tau \nu(\tau) R_{jk}^{(\alpha)}(-\tau) = \left[ D_\alpha \ f_\alpha \Omega_\alpha \ f_\alpha \Omega_\alpha \right].
\end{equation}
The coefficients $D_\alpha$ and $f_\alpha$ are the normal and anomalous diffusion coefficients [9], corresponding to the eigenfrequency $\Omega_\alpha$ and the inverse temperature $\beta$ and are given by
\begin{equation}
D_\alpha = \int_0^\infty d\tau \nu(\tau) \cos \tau \Omega_\alpha = \frac{\pi}{2}(\Omega_\alpha) \coth \left( \frac{\beta \Omega_\alpha}{2} \right)
\end{equation}
\begin{equation}
f_\alpha = -\frac{1}{\Omega_\alpha} \int_0^\infty d\tau \nu(\tau) \sin \tau \Omega_\alpha
\end{equation}
We now substitute Eqs. (23), (26), and (27) into Eq. (15) to finally obtain
\begin{equation}
\sum_{jklmn} a_j a_l D_{jk} R_{lm}^{(a)}(t) R_{kn}^{(a)}(t) C_{mn} = \frac{1}{2} \sum_{\alpha} \sum_{ij'kl'} \tilde{a}_i \tilde{a}_l D_{ij}^{(\alpha)} \text{Tr}[\tilde{C}_\alpha] \delta_{kl}'.
\end{equation}
and for even $N$ there is an additional term $\tilde{a}_k^{N-1} D_0 (\Delta \tilde{X}_{k-1}^2)$. Here $||\tilde{a}_\alpha||^2$ is the norm squared of the vector $\tilde{a}$ on the $\alpha$-subspace. We used the fact that matrices $D_\beta$ and $L_\alpha$ are supported in different subspaces for $\alpha \neq \beta$, and the explicit form, Eq. (28), to calculate the quadratic form. Further, in the last step we used the definition of $\tilde{C}_\alpha$, given by Eq. (17), but with the rotated generators (Eq. (25)). Moreover, $X_{\alpha N}, X_{\alpha 1}$ denote the two generators in the $\alpha$-subspace and $\Delta \tilde{X}_{\alpha 0}^2 = (\tilde{X}_\alpha)^2 - (X_{\alpha})^2$ are their variances in the initial state $|\psi\rangle$. Finally, we introduced $D_0 = \int d\tau \nu(\tau)$ equal to $\frac{\pi}{2} \lim_{\tau \to 0} \frac{f(\omega)}{\tau}$ for oscillator environments and $\frac{\pi}{2} \lim_{\tau \to 0} J(\omega)$ for spin environments; cf. Eq. (29). For oscillator models, $D_0$ is equal to $\pi/\beta$ for Ohmic $J(\omega)$, zero for super-Ohmic, and is divergent for sub-Ohmic case. Thus our method is not applicable for sub-Ohmic $J(\omega)$ unless $\alpha_N = 0$ (and $\alpha_{N-1} = 0$ for even $N$). We also note that all $D_\alpha \geq 0$ and hence Eq. (31) is non-negative too.
In the similar fashion we now analyze the term, Eq. (16), linear in $R_{jk}(t)$. Let us first assume odd group dimension $N$. It is then clear from Eqs. (18) and (19) that the only term that survives the time averaging is $R_{Nk}(t)R_{Nk}(t) = R_{Nk}(t)$, the rest being zero, so that
\begin{equation}
\sum_{jklmn} a_j a_l \gamma_{jk} f_{jkm} R_{mn}(t) \langle X_N \rangle = -\sum_{jkl} a_j a_l f_{Nkl} \langle X_N \rangle,
\end{equation}
where we have used the total antisymmetry of $f_{jkm}$, cf. Eq. (21). Let us now decompose matrices $\gamma_{jk}$ following the decomposition, Eq. (19). Substituting Eq. (19) into Eq. (12) we obtain
\begin{equation}
\gamma = \sum_\alpha O^T \gamma_\alpha O, \ \gamma_\alpha = \left[ -\Omega_\alpha^2 \ -\gamma_\alpha \Omega_\alpha \ -\gamma_\alpha \Omega_\alpha \right],
\end{equation}
where [9]
\begin{equation}
\Omega_\alpha^2 = -\int_0^\infty d\tau \gamma(\tau) \cos \tau \Omega_\alpha,
\end{equation}
\begin{equation}
\gamma_\alpha = \frac{1}{\Omega_\alpha} \int_0^\infty d\tau \gamma(\tau) \sin \tau \Omega_\alpha = \frac{\pi}{2} J(\Omega_\alpha)
\end{equation}
are the frequency shift and the momentum damping coefficients, corresponding to the eigenfrequency $\Omega_\alpha$. In the similar way, Eq. (19) implies via the differentiation of Eq. (10), the block-diagonal form of the $\text{ad}_{X_N}$ matrix as

$$ \text{ad}_{X_N} = \sum_{\alpha < \alpha_N} O^T c_\alpha O, \quad c_\alpha = \begin{bmatrix} 0 & -i\Omega_\alpha \\ \Omega_\alpha & 0 \end{bmatrix}, \quad (36) $$

where the last block is zero due to Eq. (18). Recalling that $[\text{ad}_{X_N}]_{jk} = i f_{Njk}$; using matrix/vector notation and the natural embeddings of $\gamma_\alpha$ and $c_\alpha$ into the whole space, we obtain from Eq. (32) that

$$ \sum_{jkl} a_j a_l \gamma_{jk} f_{Nkl} = -i \langle a|\gamma \cdot \text{ad}_{X_N} a \rangle \\
= -i \sum_{\alpha \beta} (a|O^T \gamma_\alpha O^T c_\beta O a) \\
= -i \sum_{\alpha < \alpha_N} \langle \tilde{a}|\gamma_\alpha \cdot c_\alpha \tilde{a} \rangle \\
= - \sum_{\alpha < \alpha_N} \langle \tilde{a}|\gamma_\alpha \rangle^2 \Omega_\alpha^2 \gamma_\alpha, \quad (37) $$

where we used the explicit forms of $\gamma_\alpha$ and $c_\beta$ and the fact that they are supported in different subspaces for $\alpha \neq \beta$. Thus, the linear term, Eq. (32), is equal to

$$ \sum_{\alpha < \alpha_N} \langle \tilde{a}|\gamma_\alpha \rangle^2 \Omega_\alpha^2 \gamma_\alpha \langle X_N \rangle, \quad (38) $$

For even dimension $N$, the situation is more complicated. There is one more non-zero element in $\tilde{R}_{mn}(t)$, corresponding to the $(N-1,N-1)$ element of the canonical form of $R(t)$: $[OR(t)O^T]_{N-1,N-1} = 1$, cf. Eq. (20). It leads to an additional term in Eq. (32):

$$ \sum_{jkmn} a_j a_l \gamma_{jk} f_{klm} O^{-1}_{N-1,m} O^{-1}_{N-1,n} \langle X_N \rangle \\
= - \sum_{jkmn} a_j a_l \gamma_{jk} \left( \sum_m O^{-1}_{N-1,m} f_{mkl} \right) \langle \tilde{X}_{N-1} \rangle \\
= i \langle a|\gamma \cdot \text{ad}_{\tilde{X}_{N-1}} a \rangle \langle \tilde{X}_{N-1} \rangle \\
= i \sum_\alpha \langle \tilde{a}|\gamma_\alpha \cdot \text{ad}_{\tilde{X}_{N-1}} \tilde{a} \rangle \langle \tilde{X}_{N-1} \rangle, \quad (39) $$

where $\text{ad}_{\tilde{X}_m}$ is the ad operator of the transformed generator $\tilde{X}_m = \sum_{mn} O_{mn} X_m'$ and $\text{ad}_{\tilde{X}_{N-1}} = O \text{ad}_{\tilde{X}_{N-1}} O^T$ is the matrix of $\text{ad}_{\tilde{X}_{N-1}}$ transformed to the new basis so that $[\tilde{X}_1, \tilde{X}_2] = i \sum_j [\text{ad}_{\tilde{X}_1}]_{jk} \tilde{X}_k$. This is as much as can be said in a general case.

Summarizing, for an odd dimension $N$, the asymptotic entropy production reads

$$ \frac{1}{2} \pi \approx \sum_{\alpha < \alpha_N} \langle \tilde{a}|\gamma_\alpha \rangle^2 \Omega_\alpha^2 \gamma_\alpha \left[ \Delta \tilde{X}_{N0}^2 + \Delta \tilde{X}_{N1}^2 + \frac{\Omega_\alpha^2 \gamma_\alpha \langle X_N \rangle}{D_\alpha} \right] $$

+ $a_N^2 D_0 \Delta \tilde{X}_{N}^2 \quad (40) $

For even $N$ it reads

$$ \frac{1}{2} \pi \approx \sum_{\alpha < \alpha_{N-1}} \langle \tilde{a}|\gamma_\alpha \rangle^2 \Omega_\alpha^2 \gamma_\alpha \left[ \Delta \tilde{X}_{N0}^2 + \Delta \tilde{X}_{N1}^2 + \frac{\Omega_\alpha^2 \gamma_\alpha \langle X_N \rangle}{D_\alpha} \right] $$

+ $a_{N-1}^2 D_0 \Delta \tilde{X}_{N-1}^2 + a_N^2 D_0 \Delta \tilde{X}_{N}^2 \quad (41) $

In the above equations $\gamma_\alpha / D_\alpha = (1/\Omega_\alpha) \tanh(\beta \Omega_\alpha / 2)$. To simplify the above expressions further, let us consider the high-temperature limit $\beta \approx 0$ when $\gamma_{jk} / D_\alpha \approx 0$ irrespectively of the environment model. Furthermore, we drop the non-dynamical term $\alpha N X_N$ from Eq. (8) as it is usually the case. This gives

$$ \frac{1}{2} \pi \approx \sum_{jk} g_{jk} \left[ \langle \tilde{X}_j \tilde{X}_k \rangle - \langle \tilde{X}_j \rangle \langle \tilde{X}_k \rangle \right], \quad (43) $$

where we introduced an environment-dependent metric

$$ g_{jk} = \begin{bmatrix} \langle \tilde{a}|\gamma_\alpha \rangle^2 \Omega_\alpha^2 \gamma_\alpha \langle X_N \rangle & 0 \\ 0 & 0 \end{bmatrix}, \quad (44) $$

where for even $N$ the last non-zero block is equal to $a_N^2 D_0$. Now comes our main argument: The metric $g_{jk}$ is in general different from the Killing-Cartan form $h_{jk}$, which in this case is $\approx 1$. For example, $g_{jk}$ can non-trivially depend on the temperature via the damping coefficients $D_\alpha$. Since $h_{jk}$ defines generalized coherent states as the states that minimize the $G$-invariant dispersion $\langle \Delta C \rangle = \sum_{jk} (\langle \tilde{X}_j \tilde{X}_k \rangle - (\langle \tilde{X}_j \rangle \langle \tilde{X}_k \rangle) \langle X_N \rangle$ [7], the states minimizing Eq. (43) are in general different from the coherent states for $G$. Thus, the most dynamically stable states are not the generalized coherent states, as one might have suspected based the singular evidence of Ref. [17]. Open dynamics, given by Eq. (1) with Eqs. (7), (8) and the predictability sieve single out different states than the generalized coherent states, depending on the physics of the problem, in particular on the interaction with the environment. In general, Eqs. (40)-(42) are difficult to solve, even in the high temperature regime (43), as we illustrate in the example of spin systems. This opens a whole new set of research problems: finding physical pointer states.

Although we have assumed a compact, semi-simple group $G$ as broad example, our method is not restricted to such groups only. For example, the entropy production...
equations derived above relied on \( \text{ad}_{X_N} \) being an orthogonal matrix. This can happen for non-compact groups too as we show below.

III. QUANTUM BROWNIAN MOTION CASE

A question arises how did coherent states appear in the seminal work [17] under the same approximations used above? The system analyzed there was the quantum Brownian motion model with \( H_0 = P^2/2M + M\Omega^2Q^2/2 \) and \( A = Q \). The group \( G \) that can be associated with the dynamics is generated by the operators \( X = \{ Q, P, 1, H_0 \} \) and is known as the oscillator group [27]. The group is non-compact (it is a projective representation of \( Sp(2, R) \)) but after the rescaling \( q = \sqrt{M}\Omega Q, \ p = (1/\sqrt{M})P \), the matrix of the adjoint action (see Eq. (9)) generated by \( h_0 = (q^2 + p^2)/2 \) becomes orthogonal (with the Lie algebra basis \( \{ q, p, 1, h_0 \} \) and \( [q, p] = i\Omega \)). In particular,

\[
R(t) = \begin{bmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{bmatrix}, \quad (45)
\]

The matrix \( R(t) \) is already in the canonical form, Eq. (20). We can thus use our procedure and obtain the high temperature entropy production equation (Eq. (43) with \( a_1 = 1 \) and the rest \( a_i \)'s zero):

\[
\bar{\sigma} \approx 2D \left[ \Delta q^2 + \Delta p^2 \right] = 2DM\Omega^2 \left[ \Delta Q^2 + \frac{\Delta P^2}{M^2\Omega^2} \right], \quad (46)
\]

Modulo an unimportant prefactor, the above equation is the same as obtained in Ref. [17] via a direct calculation. \( D \) is given by Eq. (29) with \( \Omega_0 = \Omega \). It is now so happens that the r.h.s. of Eq. (46) corresponds to an invariant dispersion for a subgroup \( H \) of \( G \)–the Heisenberg-Weyl group generated by \( \{ Q, P, 1 \} \). The minimization of Eq. (46) leads to the coherent states of \( H \), which are the celebrated Glauber-Sudarshan coherent states \( |\alpha\rangle \) [5, 6], and which are also coherent states for \( G \) [7]. This situation, however, is rather exceptional as e.g. it is well known that taking higher order than quadratic polynomials in \( Q, P \) will not lead to any group structures, which in turn is connected with the deep problem of the canonical quantization (see e.g. Groenewold's Theorem [28] and Ref. [29]).

IV. EXAMPLE OF SPIN SYSTEMS

As a further illustration of our findings, we consider the spin-boson model, described by the Hamiltonian:

\[
H = \Omega J_z + \sum_i \omega_i^2 a_i^\dagger a_i - J_z \sum_i (g_i a_i^\dagger + g_i^* a_i), \quad (47)
\]

where \( J_i \) are the spin operators and \( a_i, a_i^\dagger \) are the annihilation and creation operators, respectively, of the bath. This model is of central importance in modern physics, describing, among others, atom-photon interactions in the dipole approximation [30]. It falls within our framework with \( G = SU(2), H_0 = \Omega J_z, A = -J_z \). The adjoint matrix, Eq. (9), is then already in the canonical form and reads (after rescaling \( X_N \)):

\[
R(t) = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix}. \quad (48)
\]

From Eqs. (40) and (41), we immediately obtain the general asymptotic entropy production (with \( a_x = -1 \), the rest zero) as

\[
\bar{\sigma} \approx 2D \left[ \Delta J_z^2 + \Delta J_x^2 + \frac{\gamma}{D} \langle J_z \rangle \right], \quad (49)
\]

with \( \gamma/D = \tanh(\beta\Omega/2) \). Further, in the high temperature limit, we obtain

\[
\bar{\sigma} \approx 2D \left[ \Delta J_z^2 + \Delta J_x^2 \right]. \quad (50)
\]

The r.h.s. looks almost like the \( G \)-invariant dispersion \( \langle \Delta C \rangle = \Delta J_z^2 + \Delta J_x^2 + \Delta J_y^2 \), which defines the spin coherent states [7], but without the last term. This absence of the \( G \)-invariance, changes the minimization problem significantly (Eq. (50) is invariant only w.r.t. \( e^{i\phi J_z} \)). To this end, let us assume that \( J_z \)'s describe a spin-\( j \) system. It is then easy to calculate Eq. (49) for spin coherent states \( |\mathbf{n}\rangle = e^{i\mathbf{m} \cdot J} |j, -j\rangle \), where \( \mathbf{n} \) is a unit vector on a sphere.
\[ m = (\sin \phi, -\cos \phi, 0), \text{ and } J_z |j, -j\rangle = -j |j, -j\rangle. \]

More precisely, we obtain
\[ \frac{\bar{s}}{2D} \approx j \left( 1 - \frac{1}{2} \sin^2 \theta - \frac{\gamma}{D} \cos \theta \right), \tag{51} \]
with the minimum at \( \cos \theta = \gamma/D \) leading to
\[ \frac{\bar{s}_{\text{min}}}{2D} \approx \frac{j}{2} \left( 1 - \frac{\gamma^2}{D^2} \right). \tag{52} \]

In the special case of \( \beta \to 0 \), the above equation implies \( \bar{s}_{\text{min}}/2D = j/2 \). This is satisfied for spin-1/2 systems (e.g. a two-level atom), since then all pure states are coherent states by definition. However, this is not so already for spin-1. As we show in Appendix A, the minimum for high temperature is achieved for the following \( U(1) \) family of states:
\[ |\psi\rangle = \sqrt{\frac{5}{16}} (e^{i\psi} |1, 1\rangle + e^{-i\psi} |1, -1\rangle) + \sqrt{\frac{3}{8}} |1, 0\rangle, \tag{53} \]
and the minimum value is given by
\[ \frac{\bar{s}_{\text{min}}}{2D} = \frac{7}{16} \approx 0.4375 < 0.5, \tag{54} \]
which is strictly lower than the minimum for spin coherent states. Although the overlap \( \langle \psi|n \rangle \) can be as high as \( 1/2 + \sqrt{15}/8 \approx 0.98 \) for certain coherent states, the states \( |\psi\rangle \) are superpositions of two orthogonal spin coherent states (see Appendix A), showing their “non-classicality” w.r.t. coherent states. This illustrates our point that coherent states are in general not the most dynamically stable states and dynamics of open systems selects a whole new family of closest to classical states.

V. CONCLUSIONS

We analyzed the problem of finding states that are the most robust to decoherence for realistic open dynamics, where both interaction and free evolution contribute to the dynamics significantly and one term cannot be neglected compared to the other (the so called intermediary regime [9]). This problem is important from the fundamental point of view of the quantum-to-classical transition, where such robust states, called pointer states, play a pivotal role in attempts to explain the classical character of macroscopic world directly from quantum principles. This approach has been attracting considerable popularity recently due to the advances in the decoherence theory, but the intermediary regime has remained relatively uncharted due to its notorious difficulty. Here, we introduce a method which under relatively mild assumptions, including existence of a group structure behind the Hamiltonian and a form of a weak coupling, allows one to study how purity changes for an arbitrary initial state as a result of the interaction with the environment. Analyzing the case of compact, semi-simple Lie groups as an example, we derive general equations (40)-(42) for the asymptotic entropy production. Although these equations contain dispersions of certain observables, they are generically different from equations defining generalized coherent states, contrary to the well-known example of [17]. This defines in a systematic way a new class of “most classical states”, singled out by open dynamics. We further consider, as an example, the case of spin-1 systems and explicitly construct the most classical states based on our recipe and show that indeed the spin coherent states are not the most classical states singled out by the quantum dynamics. We, however, emphasize that finding the explicit form of the most classical state in a general dynamics is a challenging problem due to the lack of tools such as e.g. those used in finding the minimum uncertainty states [7].

An immediate and closely related question is to describe how the decoherence leads arbitrary states to these “most classical states”, e.g. along the lines of Refs. [19, 25]. Further, it would be very interesting to understand if advanced forms of decoherence like quantum Darwinism or Spectrum Broadcast Structures can be defined using these pointer states. We leave these questions for future research. As a final remark, we emphasize that our method is applicable to more general systems than the ones corresponding to the compact, semi-simple Lie groups, as the QBM example shows. Indeed, the method is based on the analysis of the group structure behind the free evolution, Eq. (9), and one can study other groups than the orthogonal group appearing here. This opens a possibility of systematic search for physical pointer states in yet more general situations.

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to zero. We have following:

\[
\begin{align*}
\frac{\partial L}{\partial u} &= 2 \left[ -2r^2(u + \bar{u}) + \left( \frac{\gamma}{D} + \mu \right) u \right] = 0 \\
\frac{\partial L}{\partial \bar{u}} &= 2 \left[ -2r^2(u + \bar{u}) - \left( \frac{\gamma}{D} - \mu \right) \bar{u} \right] = 0 \\
\frac{\partial L}{\partial v} &= 2 \left[ -2r^2(v - \bar{v}) + \left( \frac{\gamma}{D} + \mu \right) v \right] = 0 \\
\frac{\partial L}{\partial \bar{v}} &= 2 \left[ 2r^2(v - \bar{v}) - \left( \frac{\gamma}{D} - \mu \right) \bar{v} \right] = 0 \\
\frac{\partial L}{\partial r} &= 2r \left[ (\mu + 1) - 2((u + \bar{u})^2 + (v - \bar{v})^2) \right] = 0 \\
\frac{\partial L}{\partial \mu} &= u^2 + v^2 + \bar{u}^2 + \bar{v}^2 + r^2 - 1 = 0.
\end{align*}
\]

For \( r \neq 0 \), we have following set of equations equivalent to above equations.

\[
\begin{align*}
2r^2 &= \left( \frac{\gamma}{D} + \mu \right) u + \bar{u} = -\left( \frac{\gamma}{D} - \mu \right) \bar{u} + \bar{\bar{u}} = \left( \frac{\gamma}{D} + \mu \right) v + \bar{v} = \left( \frac{\gamma}{D} - \mu \right) \bar{v}
\end{align*}
\]

(A10a)

\[
\langle \mu + 1 \rangle = 2((u + \bar{u})^2 + (v - \bar{v})^2)
\]

(A10b)

\[
\begin{align*}
u^2 + \bar{v}^2 + \bar{u}^2 + v^2 + r^2 &= 1.
\end{align*}
\]

(A10c)

Taking \( k = \frac{\xi^2 + \mu}{\xi - \mu} \), we have \( \bar{u} = -ku \) and \( \bar{v} = kv \), then the above equations become

\[
\begin{align*}
-2r^2 &= \left( \frac{\gamma}{D} \right)^2 - \mu^2 \left( \frac{2}{2\mu} \right) \\
\langle \mu + 1 \rangle &= \left( \frac{\gamma}{D} \right)^2 - \mu^2 \left( 1 - r^2 \right) \\
u^2 + \bar{v}^2 &= \frac{1}{2} \left( \frac{\gamma}{D} - \mu \right)^2 - \mu^2 \left( 1 - r^2 \right)
\end{align*}
\]

(A11a)

(A11b)

(A11c)

In the case when \( \gamma/D = \mu \), we can define \( k' = 1/k \), 
\( u = -k'\bar{u} \) and \( v = k'\bar{v} \). Combining above equations, we get the following cubic polynomial in terms of \( \mu \):

\[
2\mu^3 - 3\mu^2 + \left( \frac{\gamma}{D} \right)^2 = 0.
\]

(A12)

Let \( \mu = \mu_0 \) be a real solution to the above equation giving rise to the minima. Then the pure state achieving the minima in Eq. (A7) is given by following parameters: \( u, v \) are arbitrary real numbers and

\[
r = \sqrt{\frac{\mu_0^2 - \left( \frac{\gamma}{D} \right)^2 \mu_0}{4\mu_0} \cdot u^2 + v^2 = \frac{1}{2} \left( \frac{\gamma}{D} \right)^2 + \mu_0^2 \left( 1 - r^2 \right)}.
\]

(A13)

The minimum value for the Eq. (A7) is given by

\[
\frac{\bar{\gamma}_{\min}}{2D} = \frac{1}{4\mu_0} \left[ \mu_0^2 - 3\mu_0^2 + \left( 4 - \frac{\gamma^2}{D^2} \right) \mu_0 - \frac{\gamma^2}{D^2} \right].
\]

(A14)

1. **High temperature regime** \( \bar{\gamma} \approx 0 \)

In this case, \( \mu_0 \) is a solution of the equation \( 2\mu^3 - 3\mu^2 = 0 \). Then the minimum of Eq. (A7) is obtained for \( \mu_0 = 3/2 \) and the pure state achieving the minimum is given by

\[
|\psi\rangle = q |1, 1\rangle + \sqrt{\frac{3}{8}} |1, 0\rangle + q^* |1, -1\rangle,
\]

(A15)

where \( |q|^2 = 5/16 \). The minimum value of Eq. (A7) is given by

\[
\frac{\bar{\gamma}_{\min}}{2D} = \frac{7}{16} = 0.4375.
\]

(A16)

Figure 1 shows that indeed 0.4375 is the minimum value of \( \bar{\gamma}/(2D) \). The random pure states on the plot are generated by first sampling two random complex numbers and one random real number using Mathematica functions `RandomComplex[]` and `RandomReal[]`, respectively and then normalizing these three random numbers (cf. Eq. (A3)). Further, for the state achieving the minima, we have

\[
\langle J_z \rangle = 0, \quad \Delta J_z = \frac{3}{8} - \text{Re}|q|^2, \quad \Delta J_z = \frac{3}{8} - \text{Im}|q|^2.
\]

(A17)

Their overlap with spin-1 coherent states is given by

\[
\langle \psi|n \rangle = \sqrt{\frac{5}{16}} e^{i\varphi} \left[ \cos^2 \frac{\theta}{2} + e^{-2i(\varphi + \phi)} \sin^2 \frac{\theta}{2} \right]
\]

\[
- \sqrt{\frac{5}{4}} e^{-i\phi} \sin \theta,
\]

(A18)
where the maximum overlap is with the coherent state, given by

\[ |\psi\rangle = \frac{1}{2} |1,-1\rangle + \frac{e^{i\psi}}{\sqrt{2}} |1,0\rangle + \frac{e^{2i\psi}}{2} |1,1\rangle \tag{A19} \]

and reads \((\sqrt{5} + \sqrt{3})^2/16 \approx 0.98\). In the above, we have identified the coherent state \(|n(\theta, \phi)\rangle\) with \(|\theta, \phi\rangle\). Further, decomposing \(|\psi\rangle\) into spin-coherent states shows that it can be represented as a sum of two orthogonal coherent states:

\[ |\psi\rangle = \frac{\sqrt{5} - \sqrt{3}}{4} |\frac{\pi}{2}, -\psi\rangle + \frac{\sqrt{5} + \sqrt{3}}{4} |\frac{\pi}{2}, \pi - \psi\rangle. \tag{A20} \]

\[ \mu_0 = \frac{1 + \sqrt{3}}{2}. \]

Then, the pure state achieving the minimum (that is the pointer state) is given by

\[ |\psi_0\rangle = q |1,1\rangle + \frac{1}{2} |1,0\rangle - kq^* |1,-1\rangle, \tag{A21} \]

where \(|q|^2(1 + k^2) = \frac{3}{4}\) and

\[ k = \frac{1 + \sqrt{3} + \sqrt{2}}{1 + \sqrt{3} - \sqrt{2}}. \tag{A22} \]

Then using Eq. (A14), the minimum value of Eq. (A7) is given by

\[ \frac{\bar{\pi}_{\text{min}}}{2D} = \frac{7 - 3\sqrt{3}}{8} \approx 0.225481. \tag{A23} \]

Further, for the state achieving the minima, we have

\[ \langle J_z \rangle = -\sqrt{\frac{2}{3}}; \]

\[ \Delta J_x^2 = \frac{1}{2} \left[ 1 - (1-k)^2 (\text{Re}|q|^2) + (1+k)^2 (\text{Im}|q|^2) \right]; \]

\[ \Delta J_y^2 = \frac{1}{2} \left[ 1 + (1+k)^2 (\text{Re}|q|^2) - (1-k)^2 (\text{Im}|q|^2) \right]. \tag{A26} \]

\[ \langle J_z \rangle = -1; \quad \Delta J_x^2 = \frac{1}{2} = \Delta J_y^2. \tag{A26} \]

Figure 2 shows that the time averaged mixedness decreases as we decrease temperature.