DETERMINING THE MAXIMUM DIFFERENCE BETWEEN THE NUMBER OF ATOMS AND NUMBER OF COATOMS OF A BRUHAT INTERVAL OF THE SYMMETRIC GROUP

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Abstract. We determine the largest difference between the number of atoms and number of coatoms of a Bruhat interval of $S_n$.

1. Introduction

Much work has been done on understanding the structure of Bruhat intervals of the symmetric group (see, e.g., [BW82], [Hul03] and [BB05] along with references therein). Recently, particular interest has arisen in understanding the number of atoms and coatoms of Bruhat intervals of the symmetric group [AR06, Kob11]. There, the maximum number of atoms and coatoms of an interval of a given length is determined. In this note, we determine the largest difference between the number of atoms and the number of coatoms of a Bruhat interval of $S_n$.

Our main results are

**Theorem A.** Let $\mathcal{I}$ be the set of intervals in $S_n$ and for $I \in \mathcal{I}$, let $a(I)$ and $c(I)$ denote the number of atoms and coatoms of $I$ respectively. Then

$$\max_{I \in \mathcal{I}} c(I) - a(I) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 1.$$ 

**Theorem B.** Let $n \geq 4$. An interval $I = [u, v] \subset S_n$ maximizes $c(I) - a(I)$ if and only if $c(I) = \left\lfloor \frac{n^2}{4} \right\rfloor$ and $a(I) = n - 1$.

2. Facts about Bruhat Intervals in $S_n$

We will be needing the following definition and two results.

**Definition 2.1.** [TW14 Definition 4.9] Let $u \leq v$ be permutations in $S_n$, and let $\mathcal{T}(u, v) := \{ t \in T : u < ut \leq v \}$ and $\overline{\mathcal{T}}(u, v) := \{ t \in T : v > vt \geq u \}$ be the transpositions labeling the cover relations corresponding to the atoms and coatoms in the interval. Define a labeled graph $G^a$ (resp. $G^{coat}$) on $[n]$ such that $G^a$ (resp. $G^{coat}$) has an edge between $a$ and $b$ if and only if the transposition $(ab) \in \mathcal{T}(u, v)$ (resp. $(ab) \in \overline{\mathcal{T}}(u, v)$). Let $B^a_{u,v}$ be the partition of $[n]$ whose blocks are the connected components of $G^a$. Similarly, define partition $B^{coat}_{u,v}$ whose blocks are the connected components of $G^{coat}$.

The next result allows us to relate the atoms and coatoms of a Bruhat interval.

**Proposition 2.2.** [TW14 Proposition 4.10] Let $[u, v] \subset S_n$. The labeled graphs $G^a$ and $G^{coat}$ have the same connected components.

The following result gives a sharp upper bound on the number of coatoms an interval of $S_n$ can have.

**Proposition 2.3.** [AR06 Proposition 2.1] For every positive integer $n$,

$$\max_{v \in S_n} \# \overline{\mathcal{T}}([1, v]) = \lfloor n^2/4 \rfloor.$$ 

The final result describes the permutations $v$ for which the number of coatoms is maximal.

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Proposition 2.4. \cite{AR06} Proposition 2.9] For every positive integer \( n \),
\[
\# \{ v \in S_n \mid \mathcal{I}(1,v) = \lfloor n^2/4 \rfloor \} = \begin{cases} 
  n, & \text{if } n \text{ is odd; } \\
  n/2, & \text{if } n \text{ is even. }
\end{cases}
\]

Each such permutation has the form
\[
v = [t+m+1, t+m+2, \ldots, n, t+1, t+2, \ldots, t+m, 1, 2, \ldots, t],
\]
where \( m \in \{ \lfloor n/2 \rfloor, \lceil n/2 \rceil \} \) and \( 1 \leq t \leq n-m \).

3. Largest gap between number of atoms and coatoms of an interval in the Symmetric Group

In this section, we consider the question of how large a gap can there be between the number of atoms and coatoms of a Bruhat interval of the Symmetric group \( S_n \). The first result is a simple inequality that will be used later in finding a maximum.

Lemma 3.1. For all \( k_1, k_2 \in \mathbb{N} \) with \( k_1 \geq 2 \),
\[
\lfloor k_1^2/4 \rfloor + \lfloor k_2^2/4 \rfloor + 1 < \lfloor (k_1 + k_2)^2/4 \rfloor.
\]

Proof. We have
\[
\lfloor k_1^2/4 \rfloor + \lfloor k_2^2/4 \rfloor \leq \lfloor k_1^2/4 + k_2^2/4 \rfloor.
\]
Therefore it suffices to observe that for \( k_1, k_2 \geq 2 \),
\[
k_1^2/4 + k_2^2/4 + 1 < (k_1 + k_2)^2/4 = k_1^2/4 + k_2^2/4 + \frac{k_1 k_2}{2}.
\]
\[\square\]

We now prove the main result of this note, which states that the largest difference between the number of coatoms and atoms of an interval of \( S_n \) is equal to \( \lceil n^2/4 \rceil - n + 1 \).

Theorem 3.2. Let \( \mathcal{I} \) be the set of intervals in \( S_n \) and for \( I \in \mathcal{I} \), let \( a(I) \) and \( c(I) \) denote the number of atoms and coatoms of \( I \) respectively. Then
\[
\max_{I \in \mathcal{I}} c(I) - a(I) = \lceil n^2/4 \rceil - n + 1.
\]

Proof. Let \( I \in \mathcal{I} \) and consider \( G^\text{at} \) and \( G^\text{coat} \) as in definition 2.1 By Proposition 2.2, \( G^\text{at} \) and \( G^\text{coat} \) have the same connected components. Let \( K_i, i = 1, 2, \ldots, m \), be the connected components of \( G^\text{at} \) and \( G^\text{coat} \), and let \( k_i \geq 1 \) denote their respective number of vertices. Let \( p \) be the number of active components and \( q = m - p \). By Proposition 2.3
\[
c(I) \leq \sum_{i=1}^{m} \lfloor k_i^2/4 \rfloor.
\]
Also,
\[
a(I) \geq \sum_{i=1}^{m} (k_i - 1).
\]
Therefore
\[
c(I) - a(I) \leq \sum_{i=1}^{m} \lfloor k_i^2/4 \rfloor - (k_i - 1).
\]
We maximize (2) over possible \( K_i \). Let \( f(x) = \lfloor x^2/4 \rfloor - x + 1 \), so that
\[
c(I) - a(I) \leq \sum_{i=1}^{m} f(k_i).
\]
By Lemma 3.1 if \( k_1, k_2 \geq 2 \), then
\[
f(k_1 + k_2) = f(k_1 + k_2) + f(1) > f(k_1) + f(k_2).
\]
Therefore
\[ \sum_{i=1}^{m} f(k_i) \leq qf(1) + (p-1)f(1) + f(n-q) = f(n-q). \]

Since
\[ \Delta[f](n) = f(n+1) - f(n) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} - 1 & \text{if } n \text{ is odd} \end{cases} \]
the function \( f : \mathbb{N} \to \mathbb{R} \) is monotonically increasing. It follows that for every \( I \in \mathcal{I} \),
\[ c(I) - a(I) \leq f(n). \]

Next we show that the value \( f(n) \) is attained for some interval \( I = [u, v] \). We consider any \( v \) of the form (1). By Proposition 2.4, the interval \([1, v]\) has \( \lfloor n^2/4 \rfloor \) coatoms. The identity permutation has exactly \( n-1 \) elements covering it. \( \square \)

**Theorem 3.3.** Let \( n \geq 4 \). An interval \( I = [u, v] \subset S_n \) maximizes \( c(I) - a(I) \) if and only if \( c(I) = \lfloor n^2/4 \rfloor \) and \( a(I) = n-1 \).

**Proof.** From the proof of Theorem 3.2,
\[ c(I) - a(I) \leq f(n-q). \]

The assumption that \( n \geq 4 \) implies that \( f(n) > f(n-q) \) for every \( q > 0 \). Moreover, we know that the maximum value of \( f(n) \) is attainable. Therefore \( c(I) - a(I) \) is maximized only if \( q = 0 \). So assume that \( q = 0 \). Let \( K \) be the single connected component of \( G^{at} \) and \( G^{coat} \) which contains \( n \) vertices. Then
\[ c(I) \leq \lfloor n^2/4 \rfloor \]
and
\[ a(I) \geq n-1. \]
\( \square \)

**Corollary 3.4.** Let \( n \geq 4 \). Suppose that \( I = [u, v] \subset S_n \) is an interval for which \( c(I) - a(I) \) is maximized. Then \( v \) is of the form (1).

**Proof.** By Proposition 2.4 the number of coatoms is \( \lfloor n^2/4 \rfloor \) only for \( v \) of the form (1). \( \square \)

A family of intervals for which the optimal value \( c(I) - a(I) = \lfloor n^2/4 \rfloor - n + 1 \) is attained is given by
\[ I = [1, v] \]
for \( v \) as in (1). There exist other intervals for which this maximum is attained. For example, in \( S_4 \), the intervals for which the maximum is attained are
\[ [1234, 3412], [1234, 4231], [1243, 4231], [2134, 4231]. \]

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