LIE ALGEBRAS AND DEGENERATE AFFINE HECKE ALGEBRAS OF TYPE $A$

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Abstract. We construct a family of exact functors from the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of $\mathfrak{sl}_n$-modules to the category of finite-dimensional representations of the degenerate affine Hecke algebra $H_\ell$ of $GL_\ell$. These functors transform Verma modules to standard modules or zero, and simple modules to simple modules or zero. Any simple $H_\ell$-module can thus be obtained.

Introduction

The classical Frobenius-Schur-Weil duality gives a remarkable correspondence between the category of finite-dimensional representations of the symmetric group $\mathfrak{S}_\ell$ and the category of finite-dimensional representations of the special (or general) linear group $SL_\ell$. Its generalizations have been studied in e.g. [5, 6, 12, 14, 20] where $\mathfrak{S}_\ell$ is replaced by other algebras, e.g. the Hecke algebras, the (degenerate) affine Hecke algebras or the double affine Hecke algebras, and $SL_\ell$ is replaced by the corresponding quantum groups.

In this paper, we present a new direction in generalizing the classical duality. Let $\mathcal{O}(\mathfrak{sl}_n)$ denote the BGG category of representations of the complex Lie algebra $\mathfrak{sl}_n$, and let $\mathcal{R}(H_\ell)$ denote the category of finite-dimensional representations of the degenerate (or graded) affine Hecke algebra $H_\ell$ of $GL_\ell$. To each weight $\lambda$ of $\mathfrak{sl}_n$ such that $\lambda + \rho$ is dominant integral (where $\rho$ is the half sum of the positive roots), we associate a functor $F_\lambda$ from $\mathcal{O}(\mathfrak{sl}_n)$ to $\mathcal{R}(H_\ell)$. When we take $\lambda = 0$ and restrict the functor $F_0$ to the category of finite-dimensional representations of $\mathfrak{sl}_n$, we obtain the classical duality.

To be more precise, let $V_n = \mathbb{C}^n$ be the vector representation of $\mathfrak{sl}_n$ and $M(\lambda)$ the highest weight Verma module with highest weight $\lambda$.

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For $X \in \text{obj } O(\mathfrak{sl}_n)$, we construct in §2.2 an action of $H_\ell$ on $X \otimes V_n^\otimes \ell$ commuting with the $\mathfrak{sl}_n$-action. This induces an $H_\ell$-action on a certain subquotient $F_\lambda(X)$ (see (2.1.1)) of $X \otimes V_n^\otimes \ell$. When $\lambda + \rho$ is dominant integral, $F_\lambda(X)$ is identified with $\text{Hom}_{\mathfrak{sl}_n}(M(\lambda), X \otimes V_n^\otimes \ell)$. (This space is an analogue of the space of the conformal blocks in the conformal field theory. See [1].)

Under the assumption that $\lambda + \rho$ is dominant integral, we prove that

1. $F_\lambda$ is exact,
2. $F_\lambda$ sends a Verma module to a “standard module” unless it is zero.

Here the standard module is an induced module from a certain one dimensional representation of a parabolic subalgebra of $H_\ell$, and it has a unique simple quotient.

Moreover, in the case $n = \ell$, we prove that

3. $F_\lambda$ sends a simple module to a simple module unless it is zero.

We should remark that our proof of (3) relies on the formula (A.3.2) in Appendix. This formula is a consequence of the fact that the irreducible decompositions of the standard modules of $H_\ell$ ([13]) and those of the Verma modules ([2, 4]) are both described by the Kazhdan-Lusztig polynomials.

We also determine when the image of the functor is non-zero (Theorem 3.3.1, Theorem 3.4.1). Furthermore, it follows from Zelevinsky’s results in [24] that any simple $H_\ell$-module with “integral weights” is of the form $F_\lambda(L)$ for some weight $\lambda$ such that $\lambda + \rho$ is dominant integral and some simple $\mathfrak{sl}_\ell$-module $L$ (see Corollary 3.5.1 for the precise statement).

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1. Basic Definitions

1.1. Root data. Let $t_n$ be an $n$-dimensional complex vector space with the basis $\{\epsilon^\vee_i \mid i = 1, \ldots, n\}$ and the inner product defined by $\langle \epsilon^\vee_i | \epsilon^\vee_j \rangle = \delta_{ij}$. Let $t_n^\ast$ be its dual and $\{\epsilon_i\}$ be the dual basis of $\{\epsilon^\vee_i\}$, which are orthonormal basis with respect to the induced inner product: $\langle \epsilon_i | \epsilon_j \rangle = \delta_{ij}$. The natural pairing between $t_n$ and $t_n^\ast$ will be denoted by $\langle \cdot, \cdot \rangle : t_n^\ast \times t_n \to \mathbb{C}$. Put $\mathfrak{h}_n^\ast = \{\sum_{i=1}^n \lambda_i \epsilon^\vee_i \in t_n^\ast \mid \sum_{i=1}^n \lambda_i = 0\}$ and $\mathfrak{h}_n = t_n / \mathbb{C} \epsilon^\vee$, where $\epsilon^\vee = \sum_{i=1}^n \epsilon^\vee_i$. so that $\mathfrak{h}_n$ and $\mathfrak{h}_n^\ast$ are dual to each
other. Define the roots and the simple roots by $\alpha_{ij} = \epsilon_i - \epsilon_j$ $(i \neq j)$, $\alpha_i = \alpha_{i,i+1}$ respectively and put

$$R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \quad (1.1.1)$$

$$R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \quad (1.1.2)$$

$$\Pi_n = \{\alpha_i \mid i = 1, \ldots, n-1\}, \quad (1.1.3)$$

then $R_n \subseteq h_n^*$ is a root system of type $A_{n-1}$. Define the coroots and the simple coroots by $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_j^\vee$ $(i \neq j)$ and $h_i = \alpha_i^\vee$, respectively.

Let $W_n \subseteq \text{GL}(t_n^*)$ be the Weyl group associated to the above data, which is by definition generated by the reflections $s_\alpha$ $(\alpha \in R_n)$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad (\lambda \in t_n^*). \quad (1.1.4)$$

We often write $s_{\alpha_{ij}} = s_{ij}$. Observe that $W_n$ preserves $h_n^* \subseteq t_n^*$ and $W_n$ is isomorphic to the symmetric group $\mathfrak{S}_n$. We often use another action of $W_n$ on $h_n^*$, which is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W_n, \ \lambda \in h_n^*), \quad (1.1.5)$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R_n^+} \alpha$. Put

$$Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z} \alpha_i \subseteq h_n^*, \quad Q_n^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i, \quad (1.1.6)$$

$$P_n = \{\lambda \in h_n^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z} \text{ for all } i = 1, \ldots, n-1\}, \quad (1.1.7)$$

$$P_n^+ = \{\lambda \in h_n^* \mid \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \ldots, n-1\}. \quad (1.1.8)$$

An element of $P_n$ (resp. $P_n^+$) is called an integral (resp. dominant integral) weight.

1.2. Lie algebras of type $A$. Let $\mathfrak{sl}_n$ be the complex Lie algebra of type $A_{n-1}$. We introduce an invariant inner product on $\mathfrak{sl}_n$ by $(x|y) = \text{tr}_{\mathbb{C}^n}(xy)$, where $\mathbb{C}^n$ denotes the vector representation of $\mathfrak{sl}_n$. We identify $h_n$ (introduced in the previous subsection) and the Cartan subalgebra of $\mathfrak{sl}_n$ as inner product spaces. Let $\mathfrak{sl}_n = n_+ \oplus h_n \oplus n_-$ be the triangular decomposition with $n_\pm = \oplus_{\alpha \in R_n^+} (\mathfrak{sl}_n)_\pm$, where $(\mathfrak{sl}_n)_\alpha$ denotes the root space corresponding to $\alpha \in R_n$. We choose a set of root vectors $\{e_\alpha \in (\mathfrak{sl}_n)_\alpha \mid \alpha \in R_n\}$ such that $(e_\alpha|e_{-\alpha}) = 1$ holds for all $\alpha \in R_n^+$. Let $\{h_i\}_{i=1,\ldots,n-1}$ be the dual basis of the coroots $\{h_i\}_{i=1,\ldots,n-1}$ in $h_n$, so
that \((h^i|h_j) = \delta_{ij}\). We define the special elements of \(\mathfrak{sl}_n \otimes \mathfrak{sl}_n\) by

\[
 r = \frac{1}{2} \sum_{i=1}^{n-1} h^i \otimes h_i + \sum_{\alpha \in R^+_n} e_\alpha \otimes e_{-\alpha},
\]

(1.2.1)

\[
 \Omega = \sum_{i=1}^{n-1} h^i \otimes h_i + \sum_{\alpha \in R^+_n} (e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha),
\]

(1.2.2)

which will be used later. For an \(\mathfrak{h}_n\)-module \(X\) and \(\lambda \in \mathfrak{h}^*_n\), put

\[
 X_\lambda = \{ v \in X \mid hv = \langle \lambda, h \rangle v \text{ for all } h \in \mathfrak{h}_n \}
\]

(1.2.3)

\[
P(X) = \{ \lambda \in \mathfrak{h}^*_n \mid X_\lambda \neq 0 \}
\]

(1.2.4)

The space \(X_\lambda\) is called the weight space of weight \(\lambda\) and an element of \(P(X)\) is called a weight of \(X\).

1.3. The BGG category \(\mathcal{O}\).

Let \(U(\mathfrak{sl}_n)\) denote the universal enveloping algebra of \(\mathfrak{sl}_n\). Let \(\mathcal{O}(\mathfrak{sl}_n)\) denote the category whose objects are those \(\mathfrak{sl}_n\)-modules \(X\) such that

(i) \(X\) is finitely generated over \(U(\mathfrak{sl}_n)\),

(ii) \(X\) is \(\mathfrak{n}_+\)-locally finite i.e. \(\dim \mathbb{C} U(\mathfrak{n}_+) v < \infty\) for each \(v \in X\),

(iii) \(X = \oplus_{\lambda \in \mathfrak{h}^*_n} X_\lambda\).

The morphisms of \(\mathcal{O}(\mathfrak{sl}_n)\) are, by definition, all the \(\mathfrak{sl}_n\)-homomorphisms. The category \(\mathcal{O}(\mathfrak{sl}_n)\) is closed under the operations such as taking sub-modules, forming quotient modules, finite direct sums, and tensor products with finite-dimensional modules. For \(\lambda \in \mathfrak{h}^*_n\) let \(M(\lambda)\) denote the highest weight Verma module with highest weight \(\lambda\). The unique simple quotient of \(M(\lambda)\) will be denoted by \(L(\lambda)\). The modules \(M(\lambda)\) and \(L(\lambda)\) are objects of \(\mathcal{O}(\mathfrak{sl}_n)\).

Let \(\chi_\lambda : Z(U(\mathfrak{sl}_n)) \to \mathbb{C}\) denote the infinitesimal character of \(M(\lambda)\) (i.e. \(zv = \chi_\lambda(z)v\) for all \(z \in Z(U(\mathfrak{sl}_n))\) and \(v \in M(\lambda)\)). It is known that \(\chi_\lambda = \chi_\mu\) if and only if \(\lambda = w \circ \mu\) for some \(w \in W_n\). Let \([\lambda]\) denote the orbit \(W_n \circ \lambda\) and put \(Z_\lambda = \text{Ker} \chi_\lambda \subset Z(U(\mathfrak{sl}_n))\). Define the full subcategory \(\mathcal{O}(\mathfrak{sl}_n)_{[\lambda]}\) of \(\mathcal{O}(\mathfrak{sl}_n)\) by

\[
 \text{obj} \mathcal{O}(\mathfrak{sl}_n)_{[\lambda]} = \{ X \in \text{obj} \mathcal{O}(\mathfrak{sl}_n) \mid (Z_\lambda)^k X = 0 \text{ for some } k \}.
\]

Then any \(X \in \text{obj} \mathcal{O}(\mathfrak{sl}_n)\) admits a decomposition

\[
 X = \bigoplus_{[\lambda]} X^{[\lambda]},
\]

(1.3.1)

such that \(X^{[\lambda]} \in \text{obj} \mathcal{O}(\mathfrak{sl}_n)_{[\lambda]}\). The correspondence \(X \mapsto X^{[\lambda]}\) gives an exact functor on \(\mathcal{O}(\mathfrak{sl}_n)\).
1.4. \textbf{n-homology of $\mathfrak{sl}_n$-modules.} We proof some facts on the zeroth $n$-homology space

$$H_0(n\_X) = X/n\_X$$  \hspace{1cm} (1.4.1)

of a $\mathfrak{sl}_n$-module $X$. The space $H_0(n\_X)$ has a natural $\mathfrak{h}_n$-module structure, and it is known that

$$P(H_0(n\_X)) \subseteq W_n \circ \mu,$$  \hspace{1cm} (1.4.2)

for $X \in O(\mathfrak{sl}_n)[\mu] \ (\mu \in \mathfrak{h}_n^*)$. Hence we have

\textbf{Lemma 1.4.1.} For any $X \in \text{obj } O(\mathfrak{sl}_n)$ and $\lambda \in \mathfrak{h}_n^*$, we have

$$H_0(n\_X) = H_0(n\_X^[\lambda])\_\lambda,$$

where $X^[\lambda]$ is defined by the decomposition (1.3.1).

Therefore we have the natural surjection

$$(X^[\lambda]) \lambda \rightarrow H_0(n\_X)\lambda.$$  \hspace{1cm} (1.4.3)

\textbf{Proposition 1.4.2.} Let $\lambda + \rho \in P_n^+$. Then the above map (1.4.3) is bijective.

\textbf{Proof.} By Lemma 1.4.1, it is enough to show that $(n\_X)\lambda = 0$ assuming $X \in \text{obj } O(\mathfrak{sl}_n)[\lambda]$. To show this, note that

$$P(X) \subseteq \{ \lambda - \beta | \beta \in Q_n^+ \},$$  \hspace{1cm} (1.4.4)

where we used the assumption $\lambda + \rho \in P_n^+$. Hence we have $P(n\_X) \subseteq \{ \lambda - \beta | \beta \in Q_n^+ \{0\} \}$, which implies $\lambda \notin P(n\_X)$ as required. \hfill $\square$

\textbf{Remark 1.4.3.} (i) There also exists a canonical bijection

$$\text{Hom}_{\mathfrak{sl}_n}(M(\lambda), X) \rightarrow (X^[\lambda])_{\lambda}$$

if $\lambda + \rho \in P_n^+$.

(ii) Similar arguments prove that Proposition 1.4.2 holds for any $\lambda \in \mathfrak{h}_n^*$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{>0}$ for all $\alpha \in R_n^+$. But we do not need this.

1.5. \textbf{Degenerate affine Hecke algebras.} Let $\ell \in \mathbb{Z}_{\geq 1}$. Let $S(t_{\ell})$ be the symmetric algebra of $t_{\ell}$, which is isomorphic to the polynomial ring over $t_{\ell}^*$.  

\textbf{Definition 1.5.1.} The \textit{degenerate (graded) affine Hecke algebra} $H_{\ell}$ of $GL_{\ell}$ is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_{\ell} \cong \mathbb{C}[W_{\ell}] \otimes S(t_{\ell})$, where $\mathbb{C}[W_{\ell}]$ denotes the group algebra of $W_{\ell}$.  

(ii) The subspaces $\mathbb{C}[W] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(t_\ell)$ are subalgebras of $H_\ell$ (their images will be identified with $\mathbb{C}[W_\ell]$ and $S(t_\ell)$ respectively).

(iii) The following relations hold in $H_\ell$:
\[ s_\alpha \xi - s_\alpha(\xi)s_\alpha = -\langle \alpha, \xi \rangle \quad (\alpha \in \Pi_\ell, \ \xi \in t_\ell). \tag{1.5.1} \]

The following two lemmas are well known.

**Lemma 1.5.2.** The center $Z(H_\ell)$ of $H_\ell$ is
\[ S(t_\ell)^{W_\ell} := \{ p \in S(t_\ell) \subset H_\ell \mid w(p) = p \text{ for all } w \in W_\ell \}. \]

**Lemma 1.5.3.** There exists a unique algebra homomorphism $ev : H_\ell \to \mathbb{C}[W_\ell]$ (called the evaluation homomorphism) such that
\[ ev(w) = w \ (w \in W_\ell), \ ev(\epsilon_\gamma) = \sum_{1 \leq j < i} s_{ji} \ (i = 1, \ldots, \ell). \]

2. **Exact Functor $F_\lambda$**

2.1. Let $V_n := \mathbb{C}^n$ be the vector representation of $\mathfrak{sl}_n$ and let $u_i \in V_n$ $(i = 1, \ldots, n)$ be the vector with only non-zero entry 1 in the $i$-th component. Let $\ell$ be another positive integer. For $X \in \text{obj} \mathcal{O}(\mathfrak{sl}_n)$, we regard $X \otimes V_n^{\otimes \ell}$ as a $\mathfrak{sl}_n$-module. For $\lambda \in \mathfrak{h}_n^*$, we put
\[ F_\lambda(X) = H_0(n_-, X \otimes V_n^{\otimes \ell})_\lambda. \tag{2.1.1} \]

**Proposition 2.1.1.** Let $\lambda + \rho \in P_n^+$. Then $F_\lambda$ is an exact functor from $\mathcal{O}(\mathfrak{sl}_n)$ to the category of finite-dimensional vector spaces.

**Proof.** Follows from Proposition 1.4.2 because $(\cdot) \otimes V_n^{\otimes \ell}, (\cdot)^{[\lambda]}$ and $(\cdot)_\lambda$ are all exact functors. $\square$

**Remark 2.1.2.** The space $F_\lambda(X)$ for $\lambda \in P_n^+$ has been studied by Zelevinsky [23]. He proved that $F_\lambda$ transforms the BGG resolution of a finite-dimensional simple $\mathfrak{sl}_n$-modules to an exact sequence.

2.2. **$H_\ell$-action.** We shall define an action of $H_\ell$ on $F_\lambda(X)$ as follows. For $i = 0, 1, \ldots, \ell$, define $\pi_i : \mathfrak{sl}_n \to \mathfrak{sl}_n^{\otimes \ell+1}$ by $\pi_i(g) = 1^{\otimes i} \otimes g \otimes 1^{\otimes \ell-i}$. We let $\mathbb{C}[W]$ act on $X \otimes V_n^{\otimes \ell}$ naturally via permutations of components of $V_n^{\otimes \ell}$. The image of $w \in W_n$ in $\text{End}_\mathbb{C}(X \otimes V_n^{\otimes \ell})$ will be denoted by the same symbol. Note that as operators on $X \otimes V_n^{\otimes \ell}$, the following equality holds:
\[ s_{ij} = \Omega_{ij} + \frac{1}{n} \quad (1 \leq i \neq j \leq \ell), \tag{2.2.1} \]
where \( \Omega_{ij} = (\pi_i \otimes \pi_j)(\Omega) \) and \( \Omega \) is as in (1.2.2). Consider the following operators on \( X \otimes V \otimes \ell \):

\[
y_i = \Omega_{0i} + \sum_{1 \leq j < i} \left( \Omega_{ji} + \frac{1}{n} \right) + \frac{n-1}{2} (i = 1, \ldots, \ell). \tag{2.2.2}
\]

**Lemma 2.2.1.** As operators on \( X \otimes V \otimes \ell \), the following equations holds:

\[
[y_i, y_j] = 0 \quad (i, j = 1, \ldots, \ell), \tag{2.2.3}
\]

\[
s_i y_j - y_{s_i(j)} s_i = -\langle \alpha_i, \epsilon_j \rangle \quad (i = 1, \ldots, \ell - 1, j = 1, \ldots, \ell). \tag{2.2.4}
\]

**Proof.** The first equality (2.2.3) follows easily from the following relations:

\[
[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0, \quad [\Omega_{ij}, \Omega_{km}] = 0,
\]

where \( i, j, k, m \in \{1, \ldots, \ell\} \) are all distinct. To show (2.2.4), recall the homomorphism \( \text{ev} \) in Lemma 1.5.3, with which the operator \( y_j \) can be written as

\[
y_j = \Omega_{0j} + \text{ev}(\epsilon_j^\vee) + \frac{n-1}{2}. \tag{2.2.5}
\]

Now the equality (2.2.4) follows by using \( s_i \Omega_{0j} = \Omega_{0s_i(j)} s_i \) \( (i = 1, \ldots, \ell - 1) \). \( \square \)

**Theorem 2.2.2.** (i) (c.f. [9]). There exists a unique homomorphism

\[
H_\ell \to \text{End}_{\mathfrak{sl}_n}(X \otimes V \otimes \ell)
\]

such that

\[
\epsilon_i^\vee \mapsto y_i \quad (i = 1, \ldots, \ell),
\]

\[
w \mapsto w \quad (w \in W_\ell).
\]

(ii) The above homomorphism induces an action of \( H_\ell \) on \( F_\lambda(X) \).

Evidently the correspondence \( X \mapsto F_\lambda(X) \) defines a functor from the category \( \mathcal{O}(\mathfrak{sl}_n) \) to the category \( \mathcal{R}(H_\ell) \) of finite-dimensional \( H_\ell \)-modules. Some remarks about this functor are in the sequel.

**Remark 2.2.3.** (i) The above construction of the functor \( F_\lambda \) arose from [1], in which representations of the degenerate double affine Hecke algebra are constructed from representations of the affine Lie algebra \( \hat{\mathfrak{sl}}_n \) using the Knizhnik-Zamolodchikov connections in the conformal field theory.

(ii) Let us consider the case \( \lambda = 0 \). Then for any \( X \in \text{obj} \mathcal{O}(\mathfrak{sl}_n) \), the action of \( H_\ell \) on \( F_0(X) \) factors the evaluation homomorphism in Lemma 1.5.3. Namely, \( F_0 \) is regarded as a functor from \( \mathcal{O}(\mathfrak{sl}_n) \) to the category
of finite-dimensional representations of $W_\ell$. Restricting the functor $F_0$ to the category of finite-dimensional representations of $\mathfrak{sl}_n$, we obtain the classical Frobenius-Schur-Weil duality (see [23]).

3. Images of Verma modules and simple modules

3.1. Standard modules and their simple quotients. We will determine explicitly how Verma modules and their simple quotients in $\mathcal{O}(\mathfrak{sl}_n)$ are transformed by the functor $F_\lambda$. We first introduce some $H_\ell$-modules. A pair $[a, b]$ of complex numbers such that $b - a + 1 \in \mathbb{Z}_{\geq 0}$ is called a segment. For a segment $[a, b]$ such that $b - a + 1 = \ell$, there exists a unique one-dimensional representation $C_{[a, b]} = C[1_{[a, b]}]$ of $H_\ell$ (we put $H_0 = \mathbb{C}$ for convenience) such that

$$w1_{[a, b]} = 1_{[a, b]} \quad (w \in W_\ell),$$

$$\varepsilon_i^j 1_{[a, b]} = (a + i - 1)1_{[a, b]} \quad (i = 1, \ldots, \ell).$$

Let $\Delta := ([a_1, b_1], \ldots, [a_k, b_k])$ be an ordered sequence of segments such that $b_i - a_i + 1 = \ell_i$ and $\ell = \sum_{i=1}^k \ell_i$. Regard $H_{\ell_1} \otimes H_{\ell_2} \otimes \cdots \otimes H_{\ell_k}$ as a subalgebra of $H_\ell$. Define an $H_\ell$-module $\mathcal{M}(\Delta)$ by

$$\mathcal{M}(\Delta) = H_\ell \otimes_{H_{\ell_1} \otimes \cdots \otimes H_{\ell_k}} (C_{[a_1, b_1]} \otimes \cdots \otimes C_{[a_k, b_k]}).$$

Evidently $\mathcal{M}(\Delta)$ is a cyclic module with a cyclic weight vector

$$1_\Delta := 1_{[a_1, b_1]} \otimes \cdots \otimes 1_{[a_k, b_k]},$$

whose weight $\zeta_\Delta$ is given by

$$\langle \zeta_\Delta, \varepsilon_j^i \rangle = a_i + j - \sum_{k=1}^{i-1} \ell_k - 1 \quad \text{for} \quad \sum_{k=1}^{i-1} \ell_k < j \leq \sum_{k=1}^i \ell_k.$$ 

It is also obvious that $\mathcal{M}(\Delta) \cong \mathbb{C}[W_\ell/(W_{\ell_1} \times \cdots \times W_{\ell_k})]$ as a $\mathbb{C}[W_\ell]$-module. In particular

$$\dim \mathcal{M}(\Delta) = \frac{\ell!}{\ell_1! \cdots \ell_k!}.$$ 

3.2. Take a pair of weights $\lambda, \mu \in \mathfrak{h}_n^* \subset \mathfrak{t}_n^*$ of $\mathfrak{sl}_n$ such that $\lambda - \mu \in P(\mathfrak{t}_n)$. Then there exist integers $(\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geq 0}^n$ such that $\ell = \sum_{i=1}^n \ell_i$ and

$$\lambda - \mu \equiv \sum_{i=1}^n \ell_i \varepsilon_i \mod \mathbb{C}.$$
where $\epsilon = \epsilon_1 + \cdots + \epsilon_n$. For $\lambda, \mu \in \mathfrak{h}^*_n$, we associate an ordered sequence of segments

$$\Delta_{\lambda, \mu} := ([\mu'_1, \mu'_1 + \ell_1 - 1], \ldots, [\mu'_n, \mu'_n + \ell_n - 1]),$$

(3.2.2)

where $\mu'_i = (\mu + \rho, \epsilon_i)$. We put

$$\mathcal{M}(\lambda, \mu) = \mathcal{M}(\Delta_{\lambda, \mu}), \quad 1_{\lambda, \mu} = 1_{\Delta_{\lambda, \mu}};$$

(3.2.3)

where $1_{\Delta_{\lambda, \mu}}$ is as in (3.1.4). We call $\mathcal{M}(\lambda, \mu)$ a standard module if $\lambda + \rho \in P_n^+$. It is known that the standard module $\mathcal{M}(\lambda, \mu)$ has a unique simple quotient, which is denoted by $\mathcal{L}(\lambda, \mu)$ (see Theorem A.2.1).

3.3. Images of Verma modules. Our goal in this subsection is the following.

**Theorem 3.3.1.** For $\lambda, \mu \in P_n$, there is an isomorphism of $H_t$-modules

$$F_\lambda(M(\mu)) \cong \begin{cases} \mathcal{M}(\lambda, \mu) & \text{if } \lambda - \mu \in P(V_n^\otimes \ell), \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{M}(\lambda, \mu)$ is given by (3.2.3) and (3.1.3). In particular, if $\lambda + \rho \in P_n^+$ and $\lambda - \mu \in P(V_n^\otimes \ell)$, then $F_\lambda(M(\mu))$ has a unique simple quotient.

To prove Theorem 3.3.1, we prepare some lemmas. For $\mu \in \mathfrak{h}^*_n$, let $v_\mu$ denote the highest-weight vector of $M(\mu)$.

**Lemma 3.3.2.** For $\lambda, \mu \in P_n$, the natural inclusion $(V_n^\otimes \ell)_{\lambda - \mu} \hookrightarrow (M(\mu) \otimes V_n^\otimes \ell)_{\lambda}$ given by $u \mapsto v_\mu \otimes u$ induces an isomorphism as $W_t$-modules

$$\sim ((V_n^\otimes \ell)_{\lambda - \mu} \sim F_\lambda(M(\mu))).$$

(3.3.1)

In particular $F_\lambda(M(\mu)) = 0$ unless $\lambda - \mu \in P(V_n^\otimes \ell)$.

**Proof.** The lemma follows from the following fact (known as the tensor product formula): For any $\mu \in \mathfrak{h}^*_n$ and any $\mathfrak{sl}_n$-module $Y$ there exists a unique $\mathfrak{sl}_n$-isomorphism

$$M(\mu) \otimes Y \cong \text{Ind}^{U(\mathfrak{g}_n)}_{U(n+ \oplus \mathfrak{h})} (Cv_\mu \otimes Y),$$

which sends $v_\mu \otimes u$ to $v_\mu \otimes u \ (u \in Y)$. \hfill \square

Recall that $\{u_i\}_{i=1,\ldots,n}$ is the standard basis of $V_n$. Fix $\lambda, \mu \in P_n$ such that $\lambda - \mu \in P(V_n^\otimes \ell)$ and let $u_{\lambda, \mu} \in F_\lambda(M(\mu))$ be the image of

$$\tilde{u}_{\lambda, \mu} := v_\mu \otimes u_1^{\otimes \ell_1} \otimes \cdots \otimes u_n^{\otimes \ell_n} \in (M(\mu) \otimes V_n^\otimes \ell)_{\lambda},$$

where $\ell_i$ are as in (3.2.1). Let $\zeta_{\lambda, \mu} \in t_n^*$ be the weight of $1_{\lambda, \mu}$ with respect to the action of $t_n$:

$$\xi 1_{\lambda, \mu} = \langle \zeta_{\lambda, \mu}, \xi \rangle 1_{\lambda, \mu} \quad (\xi \in t^*_n) \quad (\text{see } (3.1.4) \ (3.1.5)).$$
Lemma 3.3.3. Let $\bar{y}_i$ denote the image of the operator $y_i$ (see (2.2.2)) in $\text{End}_C(F_{\lambda}(M(\mu)))$. Then we have

$$\bar{y}_i u_{\lambda,\mu} = (\zeta_{\lambda,\mu}, \epsilon_i^\vee) u_{\lambda,\mu} \quad (i = 1, \ldots, \ell).$$

Proof. Let $p : (M(\mu) \otimes V_\mu^\otimes)_{\lambda} \to F_{\lambda}(M(\mu))$ be the natural surjection. Then it can be checked that $\bar{y}_i u_{\lambda,\mu} = p(v)$, where

$$v = \left( \sum_{1 \leq j < i} (r_{ji} + \frac{1}{2n}) - \sum_{i < j \leq \ell} (r_{ij} + \frac{1}{2n}) - \pi_i \left( \frac{1}{2} \sum_{k=1}^{n-1} h_k h^k + \sum_{\alpha \in R_\alpha^\vee} e_{-\alpha} e_{\alpha} + \frac{1}{2n} \right) \right)
+ \frac{1}{2} \sum_{k=1}^{n-1} (\lambda + \mu, h_k^k) \pi_i(h_k) - \frac{1}{2} \left( \frac{\ell}{2n} \right) \bar{u}_{\lambda,\mu}.$$

Here $r_{ij} = (\pi_i \otimes \pi_j)(r)$ (r is as in (1.2.1)), and we used $r_{ij} + r_{ji} = \Omega_{ij}$. Now the statement follows from the following formulas

$$\left( r + \frac{1}{2n} \right) (u_j \otimes u_k) = \frac{1}{2} \delta_{jk} (u_j \otimes u_k) \quad \text{for} \ j \leq k,$$

$$\left( \frac{1}{2} \sum_{k=1}^{n-1} h_k h^k + \sum_{\alpha \in R_\alpha^\vee} e_{-\alpha} e_{\alpha} + \frac{1}{2n} \right) u_j = -\frac{1}{2} (n - 2j + 1) u_j,$$

which are proved by direct calculations. $\square$

Proof of Theorem 3.3.4. By Lemma 3.3.2 we have that

(i) $u_{\lambda,\mu}$ is a cyclic vector of $F_{\lambda}(M(\mu))$, and obviously

(ii) $wu_{\lambda,\mu} = u_{\lambda,\mu}$ for all $w \in W_{\ell_1} \times \cdots \times W_{\ell_n}$.

By Lemma 3.3.3 and (i)(ii) above, we have a surjective $H_{\ell}$-homomorphism $\mathcal{M}(\lambda, \mu) \to F_{\lambda}(M(\mu))$ which sends $1_{\lambda,\mu}$ to $u_{\lambda,\mu}$, and it is a bijection by Lemma 3.3.2.

3.4. Images of simple modules. Next let us suppose that $n = \ell$ and determine the images of simple modules.

Theorem 3.4.1. Let $\lambda + \rho \in P_{\ell}^+$ and $w \in W_{\ell}$ be such that $\lambda - w \circ \lambda \in P(V_{\ell}^\otimes)$. Then we have the following:

(i) If $w$ satisfies

$$\langle w \circ \lambda + \rho, h_i \rangle \leq 0 \quad \text{for any} \ i \in \{1, \ldots, \ell\} \ \text{such that} \ \langle \lambda + \rho, h_i \rangle = 0,$$

then

$$F_{\lambda}(L(w \circ \lambda)) \cong \mathcal{L}(\lambda, w \circ \lambda),$$

where $\mathcal{L}(\lambda, \mu)$ is a unique simple quotient of $\mathcal{M}(\lambda, \mu)$ (see Theorem 3.2.1).
(ii) If $w$ does not satisfy the condition (3.4.1), then
\[ F_{\lambda}(L(w \circ \lambda)) = 0. \]  
(3.4.3)

Remark 3.4.2. For $\lambda \in P_{\ell}^+ - \rho$, let $W_{\lambda+\rho}$ denote the stabilizer
\[ W_{\lambda+\rho} = \{ w \in W_{\ell} \mid w(\lambda + \rho) = \lambda + \rho \}, \]  
(3.4.4)
which is a parabolic subgroup of $W_{\ell}$. Let $w_L$ and $w_{LR}$ denote the unique longest element in the coset $W_{\lambda+\rho}w$ and $W_{\lambda+\rho}wW_{\lambda+\rho}$, respectively. Then the condition (3.4.1) is equivalent to
\[ w \circ \lambda = w_L \circ \lambda, \quad \text{or equivalently} \quad w \circ \lambda = w_{LR} \circ \lambda. \]  
(3.4.5)

Note that $w_L \circ \lambda = w_{LR} \circ \lambda$.

3.5. For $\zeta \in t_{\ell}^*$, let
\[ \gamma_{\zeta} : Z(H_{\ell}) = S(t_\ell)W_{\ell} \to \mathbb{C} \]  
(3.5.1)
be the homomorphism given by the evaluation at $\zeta$. The following is a consequence of Theorem 3.4.1 and the classification theorem of simple affine Hecke algebra modules (see Corollary A.2.4).

Corollary 3.5.1. Let $\lambda + \rho \in P_{\ell}^+$. Then any finite-dimensional simple $H_{\ell}$-module with the action of $Z(H_{\ell})$ via $\gamma_{\lambda+\rho}$ is isomorphic to $F_{\lambda}(L(w_L \circ \lambda))$ for some $w \in W_{\ell}$.

Remark 3.5.2. For $c \in \mathbb{C}$, let $t_c$ be the automorphism of $H_{\ell}$ given by
\[ t_c(s_i) = s_i \quad (i = 1, \ldots, \ell - 1), \quad t_c(\epsilon_i^\vee) = \epsilon_i^\vee + c \quad (i = 1, \ldots, \ell). \]

For an $H_{\ell}$-module $Y$, let $Y^c$ denote the $H_{\ell}$-module given by the composition
\[ H_{\ell} \overset{t_c}{\to} H_{\ell} \to \text{End}_{\mathbb{C}}(Y). \]

It is known that any simple $H_{\ell}$-module is isomorphic to
\[ H_{\ell} \overset{t_{(\ell, \lambda(i), \mu(i), c_i)}}{\to} H_{\ell} \otimes \cdots \otimes H_{\ell} \left( \mathcal{L}(\lambda^{(1)}, \mu^{(1)})^{c_1} \otimes \cdots \otimes \mathcal{L}(\lambda^{(k)}, \mu^{(k)})^{c_k} \right), \]
for some $\{(\ell, \lambda(i), \mu(i), c_i)\}_{i=1, \ldots, k}$. Here $(\ell_1, \ldots, \ell_k) \in \mathbb{Z}_{>0}$ is a partition of $\ell$, $c_i$ is a complex number, and $(\lambda(i), \mu(i)) \in P_{\ell_i} \times P_{\ell_i}$ satisfying $\lambda(i) + \rho \in P_{\ell_i}^+$ and $\lambda(i) - \mu(i) \in P(V_{\ell_i}^{\otimes \ell_i})$ (see [10]).
3.6. **Proof of Theorem 3.4.1.** For \( w, y \in W_n \) such that \( w \leq y \) (where \( \leq \) denotes the Bruhat order in \( W_n \)), let \( P_{w,y}(q) \in \mathbb{Z}[q] \) denote the Kazhdan-Lusztig polynomial of the Hecke algebra associated to \( W_n \) (see [16, 17]). We put for convenience \( P_{w,y}(q) = 0 \) for \( w \not\leq y \).

The key formula in the following proof of Theorem 3.4.1 is the following formula (see Theorem A.1.1, Theorem A.3.1, and Corollary A.3.2):

\[
\mathcal{M}(\lambda, w \circ \lambda) : L(\lambda, y \circ \lambda) = P_{w_{LR}, y_{LR}}(1) = [M(w_L \circ \lambda) : L(y_L \circ \lambda)].
\]

(3.6.1)

Here \( \lambda \in \mathfrak{h}_t^* \) and \( w, y \in W_\ell \) are assumed to satisfy \( \lambda + \rho \in P_\ell^+ \) and \( \lambda - w \circ \lambda, \lambda - y \circ \lambda \in P(V_\ell^\vee) \), and \( [M : N] \) denotes the multiplicity of \( N \) in the composition series of \( M \), and \( y_{LR} \) (resp. \( y_L \)) denotes the longest element in the coset \( W_{\lambda+\rho}yW_{\lambda+\rho} \) (resp. \( yW_{\lambda+\rho} \)).

First we show the following lemma.

**Lemma 3.6.1.** Let \( \lambda + \rho \in P_\ell^+ \) and \( w \in W_\ell \) be such that \( \lambda - w \circ \lambda \in P(V_\ell^\vee) \).

- (i) If \( w \in W_\ell \) satisfies the condition (3.4.1), then \( F_\lambda(L(w \circ \lambda)) \neq 0 \).
- (ii) If \( w \in W_\ell \) does not satisfy the condition (3.4.1), then \( F_\lambda(L(w \circ \lambda)) = 0 \).

**Proof.** We first prove (ii). Suppose that \( w \) does not satisfy the condition (3.4.1). Then we can find \( s_i \in W_{\lambda+\rho} \) such that \( \langle w \circ \lambda + \rho, h_i \rangle > 0 \). This inequality means that \( M(w \circ \lambda) \) contains \( M(s_iw \circ \lambda) \) as a (proper) submodule, and hence \( F_\lambda(L(w \circ \lambda)) \) is a quotient of \( F_\lambda(M(w \circ \lambda)/M(s_iw \circ \lambda)) \). Since \( s_i \in W_{\lambda+\rho} \), we have \( \lambda - s_iw \circ \lambda \in P(V_\ell^\vee) \) and \( \dim \mathcal{M}(\lambda, w \circ \lambda) = \dim \mathcal{M}(\lambda, s_iw \circ \lambda) \) by (3.1.6), and thus we get \( F_\lambda(L(w \circ \lambda)) = 0 \).

Let us prove (i). Assume that \( w \) satisfies the condition (3.4.1). Then by Remark 3.4.2, we can assume that \( w \) is the longest element in \( W_{\lambda+\rho}wW_{\lambda+\rho} \). We can write in the Grothendieck group of \( \mathcal{O}(\mathfrak{sl}_n) \) as

\[
M(w \circ \lambda) = L(w \circ \lambda) + \sum_{y > w} P_{w,y}(1)L(y \circ \lambda)
\]

(3.6.2)

(see Theorem A.1.1), and the sum runs over those elements \( y \in W_\ell \) such that \( y \) is longest in \( yW_{\lambda+\rho} \) and \( y > w \). Applying \( F_\lambda \) to (3.6.2), we have

\[
\mathcal{M}(\lambda, w \circ \lambda) = F_\lambda(L(w \circ \lambda)) + \sum_{y > w} P_{w,y}(1)F_\lambda(L(y \circ \lambda))
\]

(3.6.3)

in the Grothendieck group of \( \mathcal{R}(H_\ell) \).

Now, let us assume that \( F_\lambda(L(w \circ \lambda)) = 0 \). Since \( \mathcal{M}(\lambda, w \circ \lambda) : L(\lambda, w \circ \lambda) \) is non-zero, there must be a summand \( F_\lambda(L(y \circ \lambda)) \) in the right
hand side of (3.6.3) such that
\[ F_\lambda (L(y \circ \lambda)) : \mathcal{L}(\lambda, w \circ \lambda) > 0. \]  
(3.6.4)

Since \( F_\lambda (L(y \circ \lambda)) \) is a non-zero quotient of \( F_\lambda (M(y \circ \lambda)) \), we have
\[ [\mathcal{M}(\lambda, y \circ \lambda) : \mathcal{L}(\lambda, w \circ \lambda)] \geq [F_\lambda (L(y \circ \mu)) : \mathcal{L}(\lambda, w \circ \lambda)] > 0. \]  
(3.6.5)

On the other hand, (3.6.1) implies
\[ [\mathcal{M}(\lambda, y \circ \lambda) : \mathcal{L}(\lambda, w \circ \lambda)] = P_{w,y}^w \mathcal{L}(\lambda, y \circ \lambda)] = 0 \]  
since \( l(y_{LR}) \geq l(y) > l(w) = l(w_{LR}) \). This contradicts (3.6.5) and shows that \( F_\lambda (L(y \circ \lambda)) \neq 0 \).
\[ \square \]

Let us complete the proof of Theorem 3.4.1. Assume that \( w \in W_\ell \) satisfies \( \lambda - w \circ \lambda \in P(V_\ell \otimes \ell) \) and \( w \) is the longest element in \( W_{\lambda + \rho}^+ \) (i.e. \( w = w_{LR} \)). We suppose that \( F_\lambda (L(w \circ \lambda)) \) has a constituent (a simple subquotient) other than \( \mathcal{L}(\lambda, w \circ \lambda) \), and will deduce a contradiction. Such a constituent is isomorphic to \( \mathcal{L}(\lambda, y \circ \lambda) \) for some \( y = y_{LR} \in W_\ell \) such that \( \lambda - y \circ \lambda \in P(V_\ell \otimes \ell) \) (see Corollary A.2.4):
\[ [F_\lambda (L(y \circ \lambda)) : \mathcal{L}(\lambda, y \circ \lambda)] \geq 1. \]  
(3.6.6)

Since \( F_\lambda (L(y \circ \lambda)) \) is a non-zero quotient of \( \mathcal{M}(\lambda, y \circ \lambda) \) by Lemma 3.6.1, we have
\[ [F_\lambda (L(y \circ \lambda)) : \mathcal{L}(\lambda, y \circ \lambda)] \geq 1. \]  
(3.6.7)

Combining (3.6.3) and inequalities (3.6.6) (3.6.7), we have
\[ [\mathcal{M}(\lambda, w \circ \lambda) : \mathcal{L}(\lambda, y \circ \lambda)] \geq [F_\lambda (L(w \circ \lambda)) : \mathcal{L}(\lambda, y \circ \lambda)] + P_{w,y}(1)[F_\lambda (L(y \circ \lambda)) : \mathcal{L}(\lambda, y \circ \lambda)] \geq 1 + P_{w,y}(1), \]
which contradicts (3.6.1) since \( y = y_{LR} \) and \( w = w_{LR} \).
\[ \square \]

Appendix A. Some facts from representation theory

We will review some facts used in this paper.

A.1. Composition series of Verma modules of \( \mathfrak{sl}_n \). For \( w, y \in W_n \) such that \( w \leq y \), let \( P_{w,y}(q) \in \mathbb{Z}[q] \) denote the Kazhdan-Lusztig polynomial of the Hecke algebra associated to \( W_n \). We put for convenience \( P_{w,y}(q) = 0 \) for \( w \not\leq y \). Let \( W_{\lambda + \rho}^+ := \{ w \in W_n \mid w(\lambda + \rho) = \lambda + \rho \} \) be the stabilizer.
Theorem A.1.1. ([2, 14]). Let \( \lambda + \rho \in P_n^+ \) and \( w \in W_n \). Then any composition factor of \( M(w \circ \lambda) \) is isomorphic to \( L(y \circ \lambda) \) for some \( y \in W_n \), and its multiplicity is given by

\[
[M(w \circ \lambda) : L(y \circ \lambda)] = P_{w,yR}(1)
\]  

(A.1.1)

where \( y_R \) is the longest element in the right coset \( yW_{\lambda+\rho} \).

A.2. Finite-dimensional representations of \( H_\ell \). We review a classification of finite-dimensional simple \( H_\ell \)-modules following [24] in terms of our parameterizations. (Note that the representation theory of \( H_\ell \) is related to that of the corresponding affine Hecke algebra by Lusztig [18].)

Theorem A.2.1. ([24] Theorem 6.1-(a)], see also [19, Theorem 5.2]) Suppose that \( \lambda, \mu \in h^n_* \) satisfy \( \lambda + \rho \in P_n^+ \) and \( \lambda - \mu \in P(V_n^*) \). Then \( M(\lambda, \mu) \) has a unique simple quotient, which is denoted by \( L(\lambda, \mu) \).

Lemma A.2.2. ([24] Theorem 6.1-(b)]) Suppose that \( \lambda + \rho \in P_n^+ \) and \( \mu, \eta \in P_n \) satisfy \( \lambda - \mu \in P(V_n^*) \), \( \lambda - \eta \in P(V_n^*) \). Then the following conditions are equivalent:

(i) \( M(\lambda, \mu) \cong M(\lambda, \eta) \).
(ii) \( L(\lambda, \mu) \cong L(\lambda, \eta) \).
(iii) There exists \( w \in W_{\lambda+\rho} \) such that \( \eta = w \circ \mu \).

Let us restrict ourselves to the case \( n = \ell \). In this case, the module

\[
M(\lambda, \lambda) = H_\ell \otimes_{S(t_\ell)} \mathbb{C}_{\lambda+\rho},
\]

is isomorphic to the regular representation as a \( \mathbb{C}[W_\ell] \)-module, where \( \mathbb{C}_{\lambda+\rho} \) is a one-dimensional \( S(t_\ell) \)-module determined by the weight \( \lambda + \rho \in h_\ell^* \subset t_\ell^* \). This module is called the principal series representation and studied e.g. in [3, 13, 19].

Theorem A.2.3. Let \( \lambda \in h_\ell^* \) be such that \( \lambda + \rho \in P_\ell^+ \subset t_\ell^* \).

(i) ([3, 19]) Any finite-dimensional simple module with action of \( Z(H_\ell) \) via \( \gamma_{\lambda+\rho} \) (where \( \gamma_{\lambda+\rho} \) is as in (3.5.1)) is a constituent of \( M(\lambda, \lambda) \).

(ii) ([24], Theorem 6.1-(c), Theorem 7.1]) For \( w \in W_\ell \) such that \( \lambda - w \circ \lambda \in P(V_\ell^*) \), the module \( L(\lambda, w \circ \lambda) \) is a constituent of \( M(\lambda, \lambda) \), and any constituent is isomorphic to \( L(\lambda, w \circ \lambda) \) for some \( w \in W_\ell \) such that \( \lambda - w \circ \lambda \in P(V_\ell^*) \).

Put

\[
S(\lambda) := \{ w \in W_\ell \mid \lambda - w \circ \lambda \in P(V_\ell^*) \} \subseteq W_\ell
\]

and let \( \overline{S(\lambda)} \) denote the image of \( S(\lambda) \) in the double coset \( W_{\lambda+\rho} \backslash W_\ell / W_{\lambda+\rho} \).
Corollary A.2.4. Let $\lambda + \rho \in P^+_\ell$. Then there exists a one to one correspondence between $S(\lambda)$ and the set of equivalent classes of finite-dimensional simple $H_\ell$-modules with the action of $Z(H_\ell)$ via $\gamma_{\lambda+\rho}$, which is given by $W_{\lambda+\rho}wW_{\lambda+\rho} \mapsto L(\lambda, w \circ \lambda)$.

A.3. Multiplicity formulas. Let us recall the multiplicity formula for (degenerate) affine Hecke algebras of $GL_\ell$. Zelevinsky conjectured in [21] that the multiplicity of simple modules in the composition series of standard modules is given in terms of the intersection cohomologies concerning the quiver variety. He proved in [22] that these intersection cohomologies are expressed by the Kazhdan-Lusztig polynomials of the symmetric group. Zelevinsky’s conjecture was proved by Ginzburg [13] (see also [11]) in more general situations. The result is rephrased as follows:

Theorem A.3.1 ([13]). Suppose that $\lambda+\rho \in P^+_\ell$ and $y, w \in W_\ell$ satisfy $\lambda - w \circ \lambda \in P(V_\ell^{\otimes \ell})$ and $\lambda - y \circ \lambda \in P(V_\ell^{\otimes \ell})$. Then

$$[\mathcal{M}(\lambda, w \circ \lambda) : \mathcal{L}(\lambda, y \circ \lambda)] = P_{w_{LR}y_{LR}}(1),$$
(A.3.1)

where $w_{LR}$ and $y_{LR}$ denote the longest elements in the double coset $W_{\lambda+\rho}wW_{\lambda+\rho}$ and $W_{\lambda+\rho}yW_{\lambda+\rho}$ respectively.

Combining Theorem A.1.1 and Theorem A.3.1 we have the following identity.

Corollary A.3.2. Assume that $\lambda+\rho \in P^+_\ell$ and that $w, y \in W_\ell$ satisfy $\lambda - w \circ \lambda \in P(V_\ell^{\otimes \ell})$ and $\lambda - y \circ \lambda \in P(V_\ell^{\otimes \ell})$. Then we have

$$[\mathcal{M}(\lambda, w \circ \lambda) : \mathcal{L}(\lambda, y \circ \lambda)] = [\mathcal{M}(w_{L} \circ \lambda) : L(y_{L} \circ \lambda)],$$
(A.3.2)

where $w_{L}$ and $y_{L}$ denote the longest elements in the left coset $W_{\lambda+\rho}w$ and $W_{\lambda+\rho}y$ respectively.

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