Quench dynamics of a strongly interacting resonant Bose gas

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We explore the dynamics of a Bose gas following its quench to a strongly interacting regime near a Feshbach resonance. Within a self-consistent Bogoliubov analysis we find that after the initial condensate-quasiparticle Rabi oscillations, at long time scales the gas is characterized by a nonequilibrium steady-state momentum distribution function, with depletion, condensate density and contact that deviate strongly from their corresponding equilibrium values. These are in a qualitative agreement with recent experiments on $^{85}\text{Rb}$ by Makotyn, et al.\cite{1}. Our analysis also suggests that for sufficiently deep quenches close to the resonance the nonequilibrium state undergoes a phase transition to a fully depleted state, characterized by a vanishing condensate density.

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Experimental realizations of trapped degenerate atomic gases coupled with field-tuned Feshbach resonances (FR)\cite{2} have lead to studies of quantum states of matter in previously unexplored, extremely coherent, strongly interacting regimes. Some of the notable early successes include a realization of paired s-wave superfluidity and the corresponding BCS-to-Bose-Einstein Condensate (BEC) crossover\cite{3,4,5}, phase transitions driven by species imbalance\cite{6,7}, and the superfluid-to-Mott insulator transition in optical lattices\cite{8}. More recently, much of the attention has turned to nonequilibrium analogs of these quantum states, made possible by unmatched high tunability (adiabatic or sudden quench) of system parameters, such as FR interactions and single-particle (e.g., trap and lattice) potentials in atomic gases. Quenched dynamics of FR fermionic gases have been extensively explored theoretically, predicting coherent post-quench oscillations\cite{9,10} and topological nonequilibrium steady states and phase transitions\cite{11}.

In bosonic gases such quench studies date back to seminal work on $^{85}\text{Rb}$\cite{12}, illustrating coherent Rabi-like oscillations between atomic and molecular condensates\cite{13}. More recently, oscillations have also been observed in quasi-2D bosonic $^{133}\text{Cs}$, following shallow quenches between weakly-repulsive interactions\cite{14}. Given that resonant bosonic gases are predicted to exhibit atomic-to-molecular superfluid phase transition (rather than just a fermionic smooth BCS-BEC crossover) and other interesting phenomenology\cite{15}, we expect their quenched dynamics to be even richer, providing further motivation for our study.

Fundamentally, such Bose gases become unstable upon approach to a FR (where two-particle scattering length $a_s$ diverges) due to a growth of the three-body loss rate, $\gamma_3 \propto n^2 a^4_s$ relative to the two-body scattering rate $\gamma_2 \propto n^{1/3} a^2_s$. On general grounds, in the limit of $n a_s \gg 1$, these rates are expected and found\cite{1} to saturate at an order of Fermi energy $\epsilon_F = \hbar^2 k_F^2 / 2m$ ($k_F \equiv n^{1/3}$), exhibiting universality akin to unitary Fermi gases\cite{16,18}. However, as was recently discovered in $^{85}\text{Rb}$\cite{1}, quenches on the molecular ($a_s > 0$) side of the resonance, even near the unitarity, the three-body rate appears to be more than an order of magnitude slower than the two-body rate (both proportional to Fermi energy, as expected), thereby opening up a window of time scales for metastable strongly-interacting nonequilibrium dynamics.

Stimulated by these experiments\cite{1} and taking the aforementioned slowness of $\gamma_3 \ll \gamma_2$ as an empirical fact, in this Letter we study the upper-branch effectively repulsive dynamics of a three-dimensional gas of strongly interacting bosonic atoms following a deep detuning quench close to the unitary point on the molecular side ($a_s > 0$) of the FR. Before turning to the analysis, we present highlights of our results that show qualitative agreement with JILA experiments\cite{1} and discuss limit of their validity.

Following a sudden shift in interaction from $g_0 = 4\pi a_0 / m$ to $g_f = 4\pi a_f / m$ (we take $\hbar = 1$ throughout) leaves the system in an excited state of the shifted Hamiltonian, $H_f$ that leads to a nontrivial dynamics associated with Rabi-like oscillations between an atomic condensate and Bogoliubov quasi-particles. Although such oscillations have indeed been seen in shallow quenches\cite{12,14}, they do not appear to have been observed in deep quenches to unitarity\cite{1}. We find that oscillations at frequencies corresponding to different momenta $k$ decohere on longer time scales, set by inverse of the Bogoliubov spectrum. We thus observe that momentum distribution function, $n_k(t) = \langle 0^- | a_k(t)^\dagger a_k(t) | 0^- \rangle$ approaches a steady state form at a time scale growing as $\sim 1/k^2$ ($\sim 1/k$) for momenta larger (smaller) than coherence momentum, $k_c = \sqrt{\pi a f n}$. The full distribution function reaches a nonequilibrium steady state at the longest time scale $R \sqrt{m/(g_f n)}$ set by the cloud size, $R$, beyond which it is characterized by a time-independent momentum distribution function illustrated in Fig.2. Within our self-consistent Bogoliubov theory $n_k(t \rightarrow \infty) \equiv n_k^\infty$ never fully thermalizes, although is expected to on physical grounds if Bogoliubov quasi-particle collisions are
taken into account. However, if this latter rate is significantly slower than the interaction energy (set by the chemical potential), we would expect our prediction for $n_k(t)$ (Fig. 1) and its long-time steady-state form (illustrated in Fig. 2).

![Graph of $\hat{n}_k(t)$](image)

**FIG. 1:** Time evolution of the (column-density) momentum distribution function, $\hat{n}_k(t) \equiv n^{-1/3}V^{-1} \int dk_z n_k(t)$ following a scattering length quench $k_Fa_0 = 0.01 \rightarrow k_Fa_f = 0.6$ in a resonant Bose gas.

$\hat{n}_k(t)$

$$n^\infty_k = \frac{k^4 + 8\pi k^2 (a_0 n + \tilde{a}_f n + \tilde{a}_f n^\infty) + 64\pi^2 \tilde{a}_f n^\infty n(\tilde{a}_f + a_0)}{2\sqrt{k^2 (k^2 + 16\pi \tilde{a}_f n^\infty) (k^2 + 16\pi a_0 n) (k^2 + 16\pi \tilde{a}_f n^\infty)}}$$

$$-\frac{1}{2}$$

(1)

to capture the nonequilibrium momentum distribution observed in JILA experiments. In Eq. (1) $\tilde{a}_f = a_f/\sqrt{1 + k_F^2 a_f^2}$ is the effective finite-density scattering length and $n^\infty_c \equiv n_c(t \rightarrow \infty)$ is the asymptotic steady-state value of the condensate density, $n_c(t) = n - V^{-1} \sum_{k \neq 0} n_k(t)$, that is self-consistently determined by the depletion $n_d(t) = V^{-1} \sum_{k \neq 0} n_k(t)$, illustrated in Fig. 3.

After time set by $m/(4\pi a_f n)$ the depletion saturates at a nonequilibrium value, $n^\infty_d(k_Fa_f)$ (calculated below), that deviates significantly from the adiabatic depletion, i.e., the equilibrium value corresponding to the scattering length $a_f$.

Finally, we find that for $k_Fa_f > k_Fa_{fc} = 1.35$, the asymptotic condensate density is driven to zero; this contrasts with the equilibrium state, where in 3d at $T = 0$ the gas is a BEC at arbitrary strong interactions, $k_Fa_s$.

Our analysis thus suggests that for a sufficiently deep quench, the system undergoes a nonequilibrium phase transition to a non-BEC steady state. We conjecture that the nonequilibrium transition is set by the critical depth of the quench for which the associated excitation energy, $E_{exc} = \langle 0^+ | H_f | 0^+ \rangle - \langle 0_f | H_f | 0_f \rangle$ ($\langle 0_f \rangle$ is the ground state of $H_f$) significantly exceeds $\epsilon_F$.

We obtain these results using a self-consistent Bogoliubov theory, implemented in two steps. Firstly, to ensure that condensate fraction $n_c(t)$ remains positive (equivalently, the depletion $n_d(t)$ does not exceed total atom number $n$, as it can at strong coupling within straight Bogoliubov theory), we solve the Heisenberg equations of motion for the finite momentum quasiparticles using time-dependent Bogoliubov Hamiltonian, where $n_c(t)$ appears as a self-consistently determined function. This approximation is the bosonic analog of the self-consistently determined BCS gap function in fermionic systems and for a uniform state is equivalent to solving the Gross-Pitaevskii equation for the condensate $\Phi_0$ in conjunction with Heisenberg equations for the finite momentum excitations $a_k$. Our second “beyond-Bogoliubov” approximation is the replacement of the scattering length $a_f$ by the density dependent scattering amplitude $|f(k_F, a_f)| = a_f/\sqrt{1 + k_F^2 a_f^2} \equiv \tilde{a}_f$.

![Graph of $n_d(t)$](image)

**FIG. 3:** Time evolution of the condensate depletion, $n_d(t)$ as a function of pulse duration $t$, following a scattering length quench $a_0 \rightarrow a_f$ in a resonant Bose gas. The inset shows an approximate collapse of the scaled depletion, with only a weak dependence on other parameters.
This qualitatively captures the crossover from the two-atom regime, \( a_f \ll n^{-1/3} \) to a finite density limit, when \( a_f \) reaches inter-particle spacing and the scattering amplitude saturates at \( \sim k_F^{-1} \). While the details of the crossover function are ad-hoc, our qualitative predictions are insensitive to these details and depend on the limiting values of the two regimes.

After the \( a_0 \rightarrow a_f \) quench the atomic resonant gas with a free dispersion \( \epsilon_k = k^2/2m \) is governed by a single-channel bosonic Hamiltonian \( H_f = \sum_k \epsilon_k a_k^\dagger a_k + g_f^{(0)} \sum_{k_1,k_2,q} a_{k_1}^\dagger a_{k_1+q} a_{k_2} a_{k_2+q} \), with the final interaction parameter, \( g_f \) and corresponding scattering length, \( a_f \), tunable via a magnetic field near a Feshbach resonance. Motivated by the experiments\([1]\), we focus on the initial states and their subsequent evolution, that, although possibly strongly depleted and time-dependent, are confined to a well-established condensate (focusing for simplicity on periodic boundary conditions without a trap). This allows us to make progress in treating the resonant interactions, by expanding in finite-momentum quasi-particle fluctuations about a macroscopically occupied \( k = 0 \) state, and thereby to reduce the Hamiltonian to the quadratic form, \( H_f(t) \approx \frac{v}{2} N^2 - \sum_k \sum_i \epsilon_k + g_f n_c(t) \), where

\[
H_f^B(t) = \frac{1}{2} \sum_{k \neq 0} \left( a_k^\dagger a_{-k} \right) \left( \epsilon_k + g_f n_c(t) \right) \left( \frac{\epsilon_k + g_f n_c(t)}{c} \right) \left( a_k^\dagger a_{-k} \right)
\]

(2)

with a new ingredient that the time-dependent condensate density is self-consistently determined by \( n_c(t) = \frac{1}{V} \sum_k n_f^{(0)} |a_k(t)|^2 \), where \( n_f^{(0)} = \frac{1}{V} \sum_k E_f^{(0)} |\psi_k(t)|^2 \). We solve numerically the initial, pre-quench Hamiltonian, \( H_f^B = \sum_k E_f^{(0)} \alpha_k^\dagger \alpha_k + \text{const.} \), characterized by a pre-quench scattering length, \( a_0 \).

The corresponding Heisenberg equation of motion \( i \sigma_z \partial_t \Phi_k(t) = H^B(t) \cdot \Phi_k(t) \) for \( \Phi_k(t) = (a_k(t), a_k^\dagger(t)) \) is conveniently encoded in terms of a time-dependent Bogoliubov transformation \( U_k(t), \Phi_k(t) = U_k(t) \Psi_k \), where \( \Psi_k = (\beta_k, \beta_k^\dagger) \) are time-independent bosonic reference operators (ensured by \( |u_k(t)|^2 - |v_k(t)|^2 = 1 \), that diagonalize the Hamiltonian at the initial time \( t = 0^+ \) after the quench, \( H_f^B(0^+) = \sum_k E_f^{(0')} |\beta_k^\dagger \beta_k + \text{const.} \). Equivalently, \( U_k(0^+) |\psi_k(0^+) = E_f^{(0')} |\psi_k(0^+) = \sqrt{E_f^{(0')} + 2q_f n_c^{(0')}}, \) fixing the initial condition \( u_k(0^+) = \sqrt{1/2 + g_f n_c^{(0')}}, \) for spinor \( \psi_k(t) \equiv (u_k(t), v_k(t)), \) that evolves according to

\[
i \sigma_z \partial_t \psi_k(t) = \hat{H}_k(t) \cdot \psi_k(t).
\]

(3)

For a given condensate density \( n_c(t) \) the solution for the evolution operator \( U(t) \) can be found exactly\([20]\) and for a slowly evolving \( n_c(t) \) is well approximated by an instantaneous Bogoliubov transformation for \( H_f^B(t), \)

\[
U(t) = \left( u_k(t) e^{-i \int_0^t E_f(\tau')} \right) v_k(t) e^{-i \int_0^t E_f(\tau')} \right) u_k(t) e^{-i \int_0^t E_f(\tau')} \right),
\]

(4)

where

\[
\begin{align*}
 u_k(t) &= \sqrt{\frac{1}{2} \left( \frac{\epsilon_k + g_f n_c(t)}{E_f(t)} + 1 \right)}, \\
v_k(t) &= -\sqrt{\frac{1}{2} \left( \frac{\epsilon_k + g_f n_c(t)}{E_f(t)} - 1 \right)},
\end{align*}
\]

(5, 6)

and \( E_f(t) = \sqrt{\epsilon_k^2 + 2g_f n_c(t) \epsilon_k} \). Using this and the relation of the atomic operators at the initial time to pre- and post-quench Bogoliubov operators

\[
U(0^-) \left( \frac{\alpha_k}{\alpha_{-k}} \right) = U(0^+) \left( \frac{\beta_k}{\beta_{-k}} \right),
\]

(7)

we have

\[
\left( \frac{\alpha_k}{\alpha_{-k}} \right) = U(t) U^{-1}(0^+) U(0^-) \left( \frac{\alpha_k}{\alpha_{-k}} \right).
\]

(8)

We can now compute arbitrary dynamic atomic correlators, such as the structure function, the RF spectroscopy signal, and the momentum distribution function in terms of these time-dependent matrices\([20]\). Focusing on the momentum distribution function (measured in the JILA experiments\([1]\), we find

\[
n_c(t) = \langle 0^- | a_k^\dagger(t) a_k(t) | 0^- \rangle, \]

\[
= \frac{k^4 + k^2 (\sigma + 1 + \hat{n}_c) + 2 \sigma \hat{n}_c + 2 \hat{n}_c (1 - \sigma) \sin^2 \phi}{2k \sqrt{(k^2 + 2\hat{n}_c)(k^2 + 2\sigma)(k^2 + 2)}}
\]

(9)

with normalized \( \hat{\phi} = k/\sqrt{8\pi a_f}, \hat{n}_c(t) \equiv n_c(t)/\hat{\phi}, n, \sigma = a_0/\hat{a}_f \), and \( \hat{t} = t/\hat{\phi} \).

We close this equation by using it to self-consistently compute the time-dependent condensate density \( n_c(t) = n - 1/V \sum_k n_c(t) \equiv n - n_d(t) \). We solve numerically the corresponding equation for \( \hat{n}_c(t) \)

\[
1 - \hat{n}_c(t) = \hat{n}_d(\hat{t}) = \frac{2}{\pi} (\frac{a_f}{\hat{\phi}_0})^{1/2} \int_0^\infty dk \hat{\phi}^2 n_k(\hat{t}, \sigma, \hat{n}_c(\hat{t})),
\]

(10)

with the solution for \( n_d(\hat{t}) \) illustrated in Fig.3.
long-time limit (averaging away the oscillatory component) Eq.\ref{11} reduces to an equation for $n_c^\infty(k_Fa_f)$, which then determines the steady-state momentum distribution function, $n_k^\infty$ in Eq\ref{1}. The associated long-time depletion $n_d^\infty = \frac{8}{3\sqrt{\pi}} \sqrt{na_f^2}$ at the corresponding $a_f$. As is clear from the inset of Fig\ref{2}, the long-time momentum distribution function $n_k^\infty$ exhibits a large momentum $1/k^4$ tail, $n_k^\infty \rightarrow \frac{C}{k^4}$. We find that the corresponding nonequilibrium contact, $C$\ref{21,22} is given by

\[ C = 16\pi^2 k_F^4 \left( (k_F a_f - k_F a_0)^2 + (k_F a_f n_c^\infty)^2 \right) \]  

and is illustrated in Fig\ref{3}.

Finally, we observe that our long-time solution $n_c^\infty$ of Eq\ref{11} monotonically decreases with $k_F a_f$, vanishing at the critical quench value of $k_F a_f = 1.35$. This suggests\cite{20} that such deep quenches excite the zero-temperature Bose gas to high enough energies, $E_{exc}$, so as to fully deplete the condensate and to drive a nonequilibrium transition to a non-BEC state. This is not an unreasonable nonequilibrium counterpart of a thermal BEC-to-normal gas transition.

To summarize, we studied the dynamics of a resonant Bose gas, following a deep quench to a large positive scattering length. Utilizing a self-consistent extension of a Bogoliubov theory, that allows us to approximately account for a large depletion and a time-dependent condensate density, we computed the nonequilibrium momentum distribution function, $n_k(t)$ and a variety of properties derived from it. These show reasonable qualitative agreement with recent experiments\cite{21}, but also leave many interesting open questions for future studies\cite{20}.

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