Bôcher theorems and applications

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Abstract: We establish maximum principles and Bôcher type theorems for super-harmonic and fractional super-harmonic nonnegative functions on a punctured ball. Connecting maximum principles with Bôcher type theorems is a crucial observation.

Keywords: fractional Laplacian, singular solution, Bôcher theorem.

1 Introduction

The well-known Bôcher theorem [1 5 25] for the nonnegative harmonic function states:

Bôcher theorem: If $v(x)$ is nonnegative and harmonic on $B_1(0) \setminus \{0\}$ and $v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$, then there is a constant $a \geq 0$ such that for all $x \in B_1(0) \setminus \{0\} \subset \mathbb{R}^n$ with $n \geq 2$ that

\[
\begin{cases}
(i) & v(x) \in L^1_{\text{loc}}(B_1(0)), \\
(ii) & -\Delta v(x) = a\delta_0, \text{ on } B_1(0),
\end{cases}
\]

where $\delta_0$ is the Delta distribution concentrated at the origin.

The original proof is given by Bôcher [5] by using some non-obvious properties of the level surfaces of a harmonic function. Later, it was proved by Helms [20] by taking advantage of the potential theory and the theory of super-harmonic functions, Kellogg [22] using series expansions for spherical harmonics. Recently, Axler [1] gave a simpler method basing on the minimum principle, Harnack inequality and the solvability of the Dirichlet problem in a unit ball.

Brézis-Lions [6] obtained another Bôcher type theorem:

Let $v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$ and $v(x) \geq 0$, a.e. in $B_1(0)$ be such that

\[
\begin{align*}
& \Delta v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\}) \text{ in the sense of distribution on } B_1(0), \\
& -\Delta v(x) \geq -Dv(x) - f(x), \ D > 0, \ a.e. \ in \ B_1(0), \text{ with } f \in L^1_{\text{loc}}(B_1(0)).
\end{align*}
\]

Then $v(x) \in L^1_{\text{loc}}(B_1(0))$ and there exists $\phi(x) \in L^1_{\text{loc}}(B_1(0))$ and $a \geq 0$ such that

\[
-\Delta v(x) = \phi(x) + a\delta_0, \text{ in } \mathcal{D}'(B_1(0)).
\]

In [6], they rely heavily on the assumption $f(x) \in L^1_{\text{loc}}(B_1(0))$ and the sphere average method.

In the last decades, the problems on the equations of fractional Laplacian have attracted a lot of attention from scientists of both mathematical and physical science. However, due to lack of the sphere average for the fractional Laplacian at present, we need to find some new method to deal with Bôcher theorems for super-harmonic functions. In particular, we give a uniform approach to derive Bôcher theorems for both the Laplacian and fractional Laplacian cases.

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With these Bôcher type theorems, we derive several maximum principles for fractional super-harmonic functions on a punctured ball. The classical work on the maximum principles in Berestycki-Nirenberg-Varadhan [4] and Caffarelli-Nirenberg-Spruck [8] are inspirational. Also, our maximum principles assume very basic regularity of $v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$. These maximum principles are the useful tools to deal with singular solutions for both equations of Laplacian and fractional Laplacian. In particular, these are essential to study the equations of Laplacian or fractional Laplacian by Kelvin transform and method of moving planes. See the references [14, 15, 16, 12, 13, 21].

Recently, Caffarelli, Jin, Sire and Xiong in [11] have obtained the local asymptotic symmetry of singular solutions to the nonlinear equation of the fractional Laplacian for $n \geq 2$ by using extension formulations for fractional Laplacians established by Caffarelli and Silvestre [10]. This result can be also obtained for the fractional case in the framework of [24], where the authors considered the Laplacian case. At last, we also refer readers some important and related work on the equations of the Laplacian or fractional Laplacian, [2, 3, 18, 19] for the symmetry property, [4, 7] for maximum estimates. The paper is organized as follows. In Section 2 we give the proofs of the Bôcher type theorems, references therein.

Throughout this paper, $C$ and $C_i$ denote general positive constants which may vary in different estimates. The paper is organized as follows. In Section 2 we give the proofs of the Bôcher type theorems. In Section 3 we apply these theorems to deduce maximum principles in punctured balls.

## 2 Bôcher type theorems

In this section, we give a uniform proof of Bôcher theorems for both super-harmonic functions and fractional super-harmonic functions.

First, we prove the Bôcher theorem for super-harmonic functions.

**Theorem 2.1.** Let $v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$ be a nonnegative solution in $\mathbb{R}^n$ ($n \geq 2$) to

$$
- \Delta v(x) + c(x)v(x) = f(x) \geq 0 \text{ on } B_1(0) \setminus \{0\}
$$

for some $f(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$ and $c(x) \leq D$ with some constant $D$, then

(i) $v(x), f(x) \in L^1_{\text{loc}}B_1(0)$,

(ii) $- \Delta v(x) + c(x)v(x) = f(x) + a_0$ on $B_1(0)$,

for some constant $a_0 \geq 0$. Here and hereafter, $\delta_0$ is the Delta distribution concentrated at the origin, and all the inequalities and identities are in the sense of distribution. The assumption (2.1) is defined as $\int_{B_1(0) \setminus \{0\}} v(x)(-\Delta)\varphi(x) + c(x)v(x)\varphi(x)dx = \int_{B_1(0) \setminus \{0\}} f(x)\varphi(x)dx \geq 0$, $0 \leq \forall \varphi(x) \in C^\infty_0(B_1(0) \setminus \{0\})$, for which we say that $v(x)$ is super-harmonic on the punctured ball $B_1(0) \setminus \{0\}$. Our conclusion (2.2) means that $\int_{B_1(0)} v(x)(-\Delta)\phi(x) + c(x)v(x)\phi(x)dx = \int_{B_1(0)} (f(x)\phi(x) + a\phi(0))dx \geq 0$, $0 \leq \forall \phi(x) \in C^\infty_0(B_1(0))$.

**Proof.** First of all, we rewrite (2.1) as

$$
- \Delta v(x) + Dv(x) = f(x) + (D - c(x))v(x) \triangleq \tilde{f}(x) \geq f(x) \geq 0.
$$

If from (2.3), we have proved that $- \Delta v(x) + Dv(x) = \tilde{f}(x) + a_0$ with some $a \geq 0$, then we have also obtained $- \Delta v(x) + c(x)v(x) = - \Delta v(x) + Dv(x) + (c(x) - D)v(x) = \tilde{f}(x) + a_0 + (c(x) - D)v(x) = \tilde{f}(x) + a_\delta_0$, that is, we have proved (2.2). Thus, in the following, we just consider the special case $c(x) \equiv D$ and derive (2.2). Without loss of generality, we set $D \geq 2$.

From the assumption $v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})$ and the interior estimate, we have

$$
\int_{B_{\frac{1}{2}}(0)} |(\nabla v(x)) + v(x)|dx \leq C.
$$
Then there exists a constant $R \in (\frac{1}{4}, \frac{3}{4})$ such that
\[
\int_{\partial B_R(0)} (|\nabla v(x)| + v(x))dx \leq C. \quad (2.5)
\]
Without loss of generality, we set $R = 1/2$. Then define $\phi_\epsilon(|x|) = \phi_\epsilon(r)$ with $\epsilon \ll 1$ as
\[
\begin{align*}
\Delta \phi_\epsilon(|x|) &= \frac{(r\phi_\epsilon'(r))^\prime}{r} = 2D\phi_\epsilon(r), \\
\phi_\epsilon(|x|)|_{|x|=\epsilon} &= 0, \\
\phi_\epsilon'(|x|)|_{|x|=\epsilon} &= \frac{1}{c\ln(1/\epsilon)} > 0,
\end{align*}
\] (2.6)
which implies that
\[
\begin{align*}
\phi_\epsilon'(r) &= \frac{1}{r\ln(1/\epsilon)} + 2D \frac{1}{r} \int_\epsilon^r s\phi_\epsilon(s)ds, \\
\phi_\epsilon(r) &= \frac{\ln(r/\epsilon)}{\ln(1/\epsilon)} + 2D \int_\epsilon^r \frac{1}{\tau} \left( \int_\epsilon^\tau s\phi_\epsilon(s)ds \right)d\tau,
\end{align*}
\] (2.7)
After direct computation, we have the following estimates
\[
0 < \frac{1}{r\ln(1/\epsilon)} \leq \phi_\epsilon'(r) \leq \frac{1}{r} + 2D, \quad \text{when} \quad \frac{1}{\ln(1/\epsilon)} \leq 1, \\
\frac{\ln(r/\epsilon)}{\ln(1/\epsilon)} \leq \phi_\epsilon(r) \leq 2\frac{\ln(r/\epsilon)}{\ln(1/\epsilon)}, \quad \text{when} \quad r \leq \frac{1}{\sqrt{D}}. \quad (2.8)
\]
After a direct calculation, when $n \geq 2$ and $\epsilon \leq \frac{1}{\sqrt{D}}$ we have
\begin{align*}
-\Delta \phi_\epsilon(|x|) &\leq -2D\phi_\epsilon(|x|), \quad \phi_\epsilon(\frac{1}{\sqrt{D}}) \geq \frac{1}{2}, \quad \phi_\epsilon(\epsilon) = 0, \quad \frac{\partial \phi_\epsilon}{\partial r} \geq 0, \quad \lim_{\epsilon \to 0} \frac{\partial \phi_\epsilon}{\partial r} = 0, \quad \forall \ r > 0. \quad (2.9)
\end{align*}
Testing the equation (2.11) with $\phi_\epsilon(|x|)$, and using (2.5), we obtain
\[
\begin{align*}
\int_{B_{\epsilon}(0) \setminus B_{\epsilon}(0)} f(x)\phi_\epsilon(|x|)dx + &D \int_{B_{\epsilon}(0) \setminus B_{\epsilon}(0)} v(x)\phi_\epsilon(|x|)dx + \int_{\partial B_{\epsilon}(0)} v(x)\frac{\partial \phi_\epsilon}{\partial r}d\sigma \\
\leq & \int_{\partial B_{\epsilon}(0)} \left( v(x)\frac{\partial \phi_\epsilon}{\partial r} - \phi_\epsilon(r_0)\frac{\partial v}{\partial r} \right)d\sigma = \int_{\partial B_{\epsilon}(0)} \left( \frac{v(x)}{\ln^\epsilon} - \frac{\partial v}{\partial r} \right)d\sigma \\
\leq & C - \int_{\partial B_{\epsilon}(0)} \frac{v(x)}{\ln^\epsilon}d\sigma \leq C, \quad \text{where} \ r_0 = \frac{1}{\sqrt{D}}.
\end{align*}
\] (2.10)
Noticing $v(x) \geq 0$, $\frac{\partial \phi_\epsilon}{\partial r} \geq 0$ and $\lim_{\epsilon \to 0} \frac{\partial \phi_\epsilon}{\partial r} = 0$, by letting $\epsilon \to 0$ in (2.11), we have
\[
\int_{B_1(0)} f(x)dx = \lim_{\epsilon \to 0} \int_{B_{\epsilon}(0) \setminus B_{\epsilon}(0)} f(x)\phi_\epsilon(|x|)dx \leq -\int_{\partial B_{\epsilon}(0)} \frac{\partial v}{\partial r}d\sigma \leq C, \quad (2.11)
\]
which implies $f(x)$ is locally integrable in $B_1(0)$ and
\[
\int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial r}d\sigma \leq 0. \quad (2.12)
\]
(2.10) also implies
\[
C \geq \int_{\partial B_{r_0}(0)} v(x)\frac{\partial \phi_\epsilon}{\partial r}d\sigma = \int_{\partial B_{r_0}(0)} \frac{v(x)}{\ln^\epsilon}d\sigma, \quad \text{for} \ \epsilon \in (0, r_0),
\]
which is equivalent to
\[
\int_{\partial B_r(0)} v(x) d\sigma \leq C r \ln \frac{1}{r}, \quad \text{for } 0 < r < r_0. \tag{2.13}
\]

Obviously, (2.10) yields that
\[
\int_{B_{r_0}(0)} v(x) dx = \int_0^{r_0} dr \int_{S^{n-1}} v(r \omega) r^{n-1} d\omega \leq C \int_0^{r_0} r \ln \frac{1}{r} dr \leq C, \tag{2.14}
\]

which together with the assumption \( v(x) \in L^1_{\text{loc}}(B_1(0)) \) leads to \( v(x) \in L^1_{\text{loc}}(B_1(0)) \). Hence, we have proved that both \( v(x) \) and \( f(x) \in L^1_{\text{loc}}(B_1(0)) \).

Next, we prove \( v(x) \) satisfies (2.2) by three steps.

**Step 1.** we claim that there exists a constant \( a \) and a constant vector \( \vec{b} \) such that
\[
- \Delta v(x) + Dv(x) = f(x) + a \delta_0 + (\vec{b} \cdot \nabla) \delta_0, \quad \text{in } B_1(0). \tag{2.15}
\]

First of all, we rewrite a test function \( \phi(x) \in C^\infty_0(B_1(0)) \) as \( \phi(x) = (\phi(0) + \nabla \phi(0) \cdot x) \eta(x) + H(x) \), where \( H(x) \) is a smooth function with quadratic or higher order of \( x \) in \( B_1(0) \), and \( \eta(x) \) is a positive smooth function supported in \( B_1(0) \). Then
\[
\int_{B_1(0)} v(x)(-\Delta + D) \phi(x) dx = \int_{B_1(0)} v(x)(-\Delta + D)(\phi(0) + \nabla \phi(0) \cdot x) \eta(x) dx + \int_{B_1(0)} v(x)(-\Delta + D)H(x) dx
\]
\[
= \phi(0) \int_{B_1(0)} v(x)(-\Delta + D) \phi(x) dx + \phi(0) \int_{B_1(0)} v(x)(-\Delta + D)(\nabla \phi(0) \cdot x) \eta(x) dx + \int_{B_1(0)} v(x)(-\Delta + D)H(x) dx
\]
\[
+ \lim_{\epsilon \to 0} \int_{B_1(0) \setminus B_{\epsilon}(0)} f(x) \eta(x) dx + \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(0)} \frac{\partial H(x)}{\partial n} - v(x) \frac{\partial H(x)}{\partial n} d\sigma
\]
\[
= \phi(0) \int_{B_1(0)} v(x)(-\Delta + D) \eta(x) dx + \int_{B_1(0)} f(x)(\phi(x) - (\phi(0) + \nabla \phi(0) \cdot x) \eta(x)) dx + \int_{\partial B_{\epsilon}(0)} \frac{\partial H(x)}{\partial n} - v(x) \frac{\partial H(x)}{\partial n} d\sigma
\]
\[
\int_{B_1(0) \setminus B_{\epsilon}(0)} |\nabla v(x)| dx \leq C \int_{B_1(0) \setminus B_{\epsilon}(0)} |\Delta v(x)| dx + C \int_{B_1(0) \setminus B_{\epsilon}(0)} v(x) dx.
\]
}

Then we need to prove the last limit in (2.16) is zero. In fact, by using gradient estimate of \( v(x) \)
\[
\int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} |\nabla v(x)| dx \leq C \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} |\Delta v(x)| dx + C \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(x) dx,
\]
we know that there exists some \( R \in \left(\frac{1}{2}, 1\right) \) such that
\[
\int_{\partial B_R(0)} |\nabla v(x)| d\sigma \leq C \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} |\Delta v(x)| dx + C \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(x) dx.
\]
Then, by scaling, we have
\[
\int_{\partial B_{R_0}(0)} |\nabla v(x)| d\sigma \leq C \int_{B_1(0) \setminus B_{\frac{3}{2}}(0)} f(x) dx + C \frac{1}{\epsilon^2} \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(x) dx, \tag{2.17}
\]
which together with the fact that \( H(x) \sim |x|^2 \) implies that
\[
\left| \lim_{\epsilon \to 0} \int_{\partial B_{R_0}(0)} \left( H(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial H(x)}{\partial n} \right) d\sigma \right| = \left| \lim_{\epsilon \to 0} \int_{\partial B_{R_0}(0)} \left( H(x) \frac{\partial v(x)}{\partial n} - v(x) \frac{\partial H(x)}{\partial n} \right) d\sigma \right| \leq C \lim_{\epsilon \to 0} \left( \epsilon^2 |\nabla v(x)| + \epsilon v(x) d\sigma \right) \leq C \lim_{\epsilon \to 0} \left( \epsilon^2 \int_{B_1(0) \setminus B_{\frac{3}{2}}(0)} f(x) dx + C \lim_{\epsilon \to 0} \epsilon \int_{\partial B_{R_0}(0)} v(x) d\sigma \right) = C \lim_{\epsilon \to 0} \left( \epsilon^2 \int_{B_1(0) \setminus B_{\frac{3}{2}}(0)} f(x) dx \right) + C \lim_{\epsilon \to 0} \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(x) dx + C \lim_{\epsilon \to 0} \epsilon \int_{\partial B_{R_0}(0)} v(x) d\sigma) := I + II + III = 0,
\]
where \( I = 0 \) and \( II = 0 \) are from the integrality of \( f(x) \) and \( v(x) \) in \( B_1(0) \), and \( III = 0 \) from the estimate (2.13). Therefore, we know the last limit of (2.16) is zero, that is, we have proved that
\[
\int_{B_1(0)} v(x)(-\Delta + D)\phi(x) dx = \phi(0)a + \nabla \phi(0) \cdot \vec{b} + \int_{B_1(0)} f(x) \phi(x) dx, \tag{2.18}
\]
for any test function \( \phi(x) \in C_0^\infty(B_1(0)), \phi(x) \geq 0 \). This completes the proof of the claim (2.15).

**Step 2.** We prove the vector \( \vec{b} \) in (2.15) must be zero. First, write \( v(x) \) in the sense of distribution
\[
v(x) = \begin{cases} 
\omega(x) + \frac{ac_n}{|x|^{n-2}} + \frac{\vec{b} \cdot x}{|x|^n} + h(x), & \text{for } n \geq 3, \; x \in B_1(0), \\
\omega(x) - ac_n \ln |x| + \frac{\vec{b} \cdot x}{|x|^2} + h(x), & \text{for } n = 2, \; x \in B_1(0),
\end{cases} \tag{2.19}
\]
where \( c_n > 0 \) and \( h(x) \) is harmonic and hence bounded, \( \omega(x) \) is a solution of \(-\Delta \omega(x) = f(x) - Dv(x)\). We consider the negative part \( v^-(x) = -\min\{v(x), 0\} \). In particular, we have
\[
v^-(x) = \left( \frac{\vec{b} \cdot x}{|x|^n} + \omega(x) + \frac{ac_n}{|x|^{n-2}} + h(x) \right)^- \geq \frac{\vec{b} \cdot x}{|x|^n} - w(x) - h(x) - \frac{ac_n}{|x|^{n-2}}, \tag{2.20}
\]
For \( n \geq 3 \), we know \( \omega(x), h(x) \) and \( \frac{a}{|x|^{n-2}} \) are in \( L^{\frac{2n-4}{2n-4}}(\Omega) \) for any \( \Omega \subset B_1(0) \). Besides, we can choose a conic domain \( \Omega = \left\{ \left[ \frac{\vec{b}}{|\vec{b}|} \cdot x \right] > \frac{|x|^2}{2} \right\} \subset B_1(0) \) such that \( \| \vec{b} \|_{L^{\frac{2n-4}{2n-4}}(\Omega)} = \infty \). These estimates and (2.20) yield that \( \int_{\Omega} (v^-(x))^{\frac{2n-4}{2n-4}} dx > 0 \), that is, there are some point \( x_0 \in \Omega \subset B_1(0) \) such that \( v(x_0) < 0 \). This is a contradiction with the assumption \( v(x) \geq 0 \) in \( B_1(0) \). Hence, we have proved that \( \vec{b} = 0 \), that is,
\[
-\Delta v(x) + Dv(x) = f(x) + a\delta_0 \quad \text{on} \quad B_1(0), \tag{2.21}
\]
in the sense of distribution. The case of \( n = 2 \) can be proved similarly.
Step 3. We prove $a \geq 0$ in (2.21). First, we have from (2.21) that

$$\int_{B_1(0)} f(x) \phi(x) + a \phi(0) dx = \int_{B_1(0)} v(x)(-\Delta + D)\phi(x) dx$$

$$= \lim_{\epsilon \to 0} \int_{B_1(0) \setminus B_0(0)} v(x)(-\Delta + D)\phi(x) dx$$

$$= \int_{B_1(0)} f(x) \phi(x) dx - \lim_{\epsilon \to 0} \int_{\partial B_x(0)} \phi(x) \frac{\partial v}{\partial n} d\sigma + \lim_{\epsilon \to 0} \int_{\partial B_x(0)} v(x) \frac{\partial \phi}{\partial n} d\sigma$$

which shows that

$$a = -\lim_{\epsilon \to 0} \int_{\partial B_1(0)} \frac{\partial v}{\partial n} d\sigma \geq 0. \quad (2.23)$$

In fact, denoting $F(s) = \int_{\partial B_1(0)} \frac{\partial v}{\partial n} d\sigma$, from (2.12) we know $F(1) \leq 0$. In the same way, we also have $F(\epsilon) \leq 0$ for any $\epsilon \in (0, 1)$. Then letting $\epsilon \to 0$, we have $a \geq 0$.

Remark 1. Theorem 2.1 for $n \geq 2$ has been proved in [24]. To show that it is also true for $n = 1$, we give some new ideas in the proof. In addition, we have some counterexamples for $n = 1$:

1. $v(x) = |x|$, and $-v''(x) = -2\delta_0$, $a = -2 < 0$, where $a$ is in (2.13);
2. $v(x) = \begin{cases} 0, & x > 0, \\ 1, & x < 0, \end{cases}$ and $-v''(x) = \delta'(0)$, $b = 1 \neq 0$, where $b$ is in (2.15);
3. $v(x) = |x|^\theta$, $0 < \theta < 1$, then $-\Delta v(x) = \theta(1 - \theta)|x|^{\theta-2}$ is not integrable in $B_1(0)$.

Remark 2. It does not hold if the super-harmonicity in Theorem 2.1 is changed into sub-harmonicity. In fact, we have the counterexample: when $n \geq 2$, if $v(x) = |x|^{-\theta}$ with $\theta > n - 2$, then $-\Delta v(x) = \theta(n - 2 - \theta)|x|^{-\theta-2} < 0$, but $f(x) = \theta(n - 2 - \theta)|x|^{-\theta-2}$ is not integrable in $B_1(0)$.

It need to mention that under the additional assumption $v(x) \in C^2(B_2(0) \setminus \{0\})$, Ghergu and Taliaferro in their book [17] gave another proof for the above theorem, where they also rely on the method of sphere average.

Next, we consider the case of fractional Laplacian [9], which defined as

$$(-\Delta)^s v(x) = C_{n,s}\text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2s}} dy, \quad 0 < s < 1, \quad (2.24)$$

where $P.V.$ stands for the Cauchy principle value. Then, we define that

$$L_\alpha = \left\{ v : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|v(y)|}{1 + |y|^{n+\alpha}} dy < +\infty \right\}.$$

It is easy to see that for any $w \in L_{2s}$, $(-\Delta)^s w$ as a distribution is well-defined: $(-\Delta)^s w(\phi) = \int_{\mathbb{R}^n} w(x)(-\Delta)^s \phi(x) dx, 0 \leq \forall \phi \in C_0^\infty(\mathbb{R}^n)$.

Here, we need several interesting lemmas to get Bôcher type theorem for the fractional Laplacian. They are also interesting in their own.

The first one is a simple application of the work of [25].

**Lemma 2.2.** Assume that $u \in L_{2s}$ and $(-\Delta)^s u(x) = 0$, $\forall x \in B_1(0)$, then

$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{2s}} dy \leq C, \quad \forall x \in B_{\frac{1}{2}}(0). \quad (2.25)$$
Lemma 2.3. If \( f(x) \in L^1_{loc}(\Omega) \) for the domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 1 \), and \((-\Delta)^s w(x) \geq f(x) \) in \( \Omega \), then \((-\Delta)^s w(x) \geq f_s(x) \) in \( \Omega \), with \( \Omega_s = \{ x \mid B_s(x) \subset \Omega \} \). Here the mollification \( w_s(x) = \rho_s * w(x) \), with \( \rho_s(x) = e^{-\rho(\frac{x}{s})}, \quad 0 \leq \rho(x) \in C_0^\infty(B_1(0)) \), and \( \int_{B_1(0)} \rho(x)dx = 1 \).

In particular, if letting \( f(x) = 0 \), when \( w(x) \) is nonnegative and fractional super-harmonic (fractional sub-harmonic) in the domain \( \Omega \) in \( \mathbb{R}^n \), then the mollification \( w_s(x) = w * \rho_s(x) \) is also fractional super-harmonic (fractional sub-harmonic) in the domain \( \Omega_s \).

Proof. By the definition of the fractional Laplacian and the mollifier, it holds that

\[
(-\Delta)^s w_s(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{\int_{\mathbb{R}^n} \rho_s(x-z)w(z)dz - \int_{\mathbb{R}^n} \rho_s(y-z)w(z)dz}{|x-y|^{n+2s}} dy
\]

\[
= \int_{\mathbb{R}^n} w(z)(-\Delta)^s \rho_s(x-z)dz = \int_{\mathbb{R}^n} w(z)(-\Delta)^s \rho_s(x-z)dz
\]

\[
= \int_{\mathbb{R}^n}(-\Delta)^s w(z) \rho_s(x-z)dz \geq \int_{\mathbb{R}^n} f(z) \rho_s(x-z)dz = f_s(x),
\]

where we have used the fact \((-\Delta)^s g(x-z) = (-\Delta)^s g(x-z)\). This proves Lemma 2.3.

Lemma 2.4. If two functions \( u(x), v(x) \in L^2_{loc} \) and \( f(x), g(x) \in L^1_{loc}(\mathbb{R}^n) \) with \( n \geq 1 \), and satisfy

\[ (-\Delta)^s u(x) \leq f(x), \quad (-\Delta)^s v(x) \leq g(x), \]

then for the function \( w(x) = \max\{u(x), v(x)\} \), we have in the sense of distribution that

\[ (-\Delta)^s w(x) \leq f(x) \chi_{u(x) > v(x)} + g(x) \chi_{u(x) \leq v(x)}, \tag{2.26} \]

where \( \chi_w(x) = \begin{cases} 1, & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) \leq 0. \end{cases} \) On the other hand, if

\[ (-\Delta)^s u(x) \geq f(x), \quad (-\Delta)^s v(x) \geq g(x), \]

then for the function \( w(x) = \min\{u(x), v(x)\} \), we have in the sense of distribution that

\[ (-\Delta)^s w(x) \geq f(x) \chi_{u(x) \leq v(x)} + g(x) \chi_{u(x) > v(x)}. \tag{2.27} \]

Proof. In the first step, we consider the case that the functions \( u(x) \) and \( v(x) \) are smooth. Without loss of generality, we set \( w(x) = u(x) \) in \( \Omega \) and \( w(x) = v(x) \) in \( \Omega^c \). For simplicity, we denote \( h(x) = f(x) \chi_{u(x) > v(x)} + g(x) \chi_{u(x) \leq v(x)} \). For any nonnegative test function \( \phi(x) \) and fixed \( \delta > 0 \), we want to prove that

\[
\int_{\mathbb{R}^n} w(x)(-\Delta)^s \phi(x)dx \leq \int_{\mathbb{R}^n} h(x)\phi(x)dx
\]

\[
= \int_{\Omega} (-\Delta)^s u(x)\phi(x)dx + \int_{\Omega^c} (-\Delta)^s v(x)\phi(x)dx. \tag{2.28} \]

To this end, we first prove that for a fixed \( \delta > 0 \) it holds

\[
\int_{\Omega} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{w(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx
\]

\[
= \int_{\Omega} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{u(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{v(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx
\]

\[
= \int_{\Omega} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{u(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega^c} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{v(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx
\]

\[+ \int_{\Omega^c} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{v(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega^c} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{v(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx. \]
\[
\begin{align*}
&= \int_{\Omega} \int_{|x-y| \leq \delta} \phi(x)(u(x) - u(y)) \, dy \, dx + \int_{\Omega} \int_{|x-y| \leq \delta} u(x)(\phi(x) - \phi(y)) \, dy \, dx \\
&\quad + \int_{\Omega^c} \int_{|x-y| \leq \delta} |x-y|^{n+2s} \phi(x)(v(x) - v(y)) \, dy \, dx + \int_{\Omega^c} \int_{|x-y| \leq \delta} |x-y|^{n+2s} v(x)(\phi(x) - \phi(y)) \, dy \, dx \\
&\quad + \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)u(x) - u(y)) \, dy \, dx - \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)(u(x) - u(y)) \, dy \, dx \\
&\quad + \int_{\Omega^c} \int_{|x-y| \leq \delta} |x-y|^{n+2s} \phi(x)(v(x) - v(y)) \, dy \, dx - \int_{\Omega^c} \int_{|x-y| \leq \delta} |x-y|^{n+2s} v(x)(\phi(x) - \phi(y)) \, dy \, dx \\
&\quad + \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)(u(x) - u(y)) \, dy \, dx + \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)v(x) - v(y)) \, dy \, dx \\
&\quad + \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)(u(x) - u(y)) \, dy \, dx - \int_{\Omega^c} \int_{|x-y| \leq \delta} \phi(x)(v(x) - v(y)) \, dy \, dx
\end{align*}
\]

From the assumptions \( \phi(x) \in C^\infty_c(\mathbb{R}^n) \) and \( u(x), v(x) \) are smooth, we know the above integrals are bounded. Then letting \( \delta \to 0 \), we can immediately obtain

\[
\int_{\mathbb{R}^n} w(x)(-\Delta)^s \phi(x) \, dx \leq \int_{\Omega} (-\Delta)^s u(x) \phi(x) \, dx + \int_{\Omega^c} (-\Delta)^s v(x) \phi(x) \, dx,
\]

which implies that it holds in the sense of distribution

\[
(-\Delta)^s w(x) \leq h(x) = f(x)\chi_{u(x)>v(x)} + g(x)\chi_{u(x)\leq v(x)}.
\]

Second, we consider the case that the functions \( u(x) \) and \( v(x) \) are not smooth. From Lemma 2.3 and the assumptions \( (-\Delta)^s u(x) \leq f(x) \) and \( (-\Delta)^s v(x) \leq g(x) \), we have \( (-\Delta)^s u(x) \leq f_\epsilon(x) \) and \( (-\Delta)^s v_\epsilon(x) \leq g_\epsilon(x) \), where \( u_\epsilon(x), v_\epsilon(x) \) are the mollifications of \( u(x) \) and \( v(x) \) respectively. Define \( w_\epsilon = \max\{u_\epsilon(x), v_\epsilon(x)\} \). The first step implies that \( (-\Delta)^s w_\epsilon \leq f(x)\chi_{u_\epsilon(x)>v_\epsilon(x)} + g(x)\chi_{u_\epsilon(x)\leq v_\epsilon(x)} \). Then letting \( \epsilon \to 0 \), and from the facts: \( (-\Delta)^s w_\epsilon \to (-\Delta)^s w(x) \) in distribution, \( f_\epsilon(x) \to f(x) \) and \( g_\epsilon(x) \to g(x) \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \), we derive \( (-\Delta)^s w(x) \leq f(x)\chi_{u(x)>v(x)} + g(x)\chi_{u(x)\leq v(x)} \). This completes the proof of Lemma 2.3.

**Remark 3:** The special but essential case of Lemma 2.3 when \( f(x) = g(x) = 0 \) has been proved in [27] under the assumptions \( u(x) \) and \( v(x) \in L_2 \) and are lower semi-continuous. Here we need not lower semi-continuity. On the other hand, most of people use the form \( \max\{f(x), g(x)\} \). Here, we have a sharper upper bound \( f(x)\chi_{u(x)>v(x)} + g(x)\chi_{u(x)\leq v(x)} \) instead.

**Remark 4:** For the equation of Laplacian, the similar result to Lemma 2.3 can be proved by using integration by parts. The result for Laplacian operator is well-known and widely used.

The following is the Bôcher theorem for the fractional super-harmonic functions.
Theorem 2.5. Let $v(x) \in L_{2s}$ with $n > 2s$ be a nonnegative solution to

$$(-\Delta)^sv(x) + cv(x) = f(x) \geq 0 \text{ on } B_1(0)\setminus\{0\}$$  (2.32)

for some $f(x) \in L^1_{\text{loc}}(B_1(0)\setminus\{0\})$ and $c(x) \leq D$ with some constant $D$, then there exists a constant $a > 0$ such that

(i) $v(x), f(x) \in L^1_{\text{loc}}(B_1(0))$,
(ii) $(-\Delta)^sv(x) + cv(x) = f(x) + a\delta_0$ on $B_1(0)$.  (2.33)

Proof. As in Theorem 2.11 we only consider the special case $c(x) \equiv D$, since it follows that the general cases also hold. The integrability of $v(x)$ in $B_1(0)$ is from the fact $v(x) \in L_{2s}$. Thus, we first prove that $f(x)$ is locally integrable in $B_1(0)$.

Define two functions $w_\epsilon(x) := \max\{1 - \frac{\epsilon}{|x|^{n-2s}} - \frac{2}{|x|}, 0\}$ and $\eta(x)$ which is a nonnegative, non-increasing, smooth function supported in $|x| < \frac{3}{4}$ and $\eta(x) = 1$ when $|x| \leq 1/2$. The mollified function $w_\epsilon^\delta(x) = w_\epsilon * \rho_\delta(x)$ is also useful to our proof, where $\rho_\delta(x)$ is a mollifier. By using integration by parts for the fractional case, we have

$$\int_{B_{\frac{3}{4}}(0)} f(x)w_\epsilon^\delta(x)dx$$

$$= \int_{B_{\frac{3}{4}}(0)} f(x)(1 - \eta(x))w_\epsilon^\delta(x)dx + \int_{\mathbb{R}^n} f(x)\eta(x)w_\epsilon^\delta(x)dx$$

$$\leq C + \int_{\mathbb{R}^n} f(x)\eta(x)w_\epsilon^\delta(x)dx = C + D\int_{\mathbb{R}^n} v(x)\eta(x)w_\epsilon^\delta(x)dx + \int_{\mathbb{R}^n} v(x)(-\Delta)^s(\eta(x)w_\epsilon^\delta(x))dx$$

$$\leq C + \int_{\mathbb{R}^n} v(x)w_\epsilon^\delta(x)(-\Delta)^s\eta(x)dx + \int_{\mathbb{R}^n} v(x)\eta(x)(-\Delta)^s w_\epsilon^\delta(x)dx \leq 0,$$

$$(-\Delta)^s w_\epsilon^\delta(x) \leq 0$$

from Lemma 2.3-2.4

$$\leq C + \int_{\mathbb{R}^n} v(x)\int_{|x-y| \leq \delta} \frac{(\eta(x) - \eta(y))(w_\epsilon^\delta(x) - w_\epsilon^\delta(y))}{|x-y|^{n+2s}}dydx$$

$$\leq C + \int_{\mathbb{R}^n} v(x)\int_{|x-y| \leq \delta} \frac{(\eta(x) - \eta(y))(w_\epsilon^\delta(x) - w_\epsilon^\delta(y))}{|x-y|^{n+2s}}dy\bigg|dx.$$

Denoting

$$\left|\int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))(w_\epsilon^\delta(x) - w_\epsilon^\delta(y))}{|x-y|^{n+2s}}dy\right|\bigg|\int_{|x-y| \geq \frac{1}{4}} \frac{1}{|x-y|^{n+2s}}dy \leq C.$$  (2.36)

Then we show that $II$ is bounded. There are two cases.

Case 1: $|x| \leq \frac{1}{2}$, Then $|y| < 1$, which means $\eta(x) - \eta(y) = 0$. As a consequence, $II = 0$. 

Case 2: $|x| > \frac{1}{2}$, Then $|y| > 1$, which means $\eta(x) - \eta(y) = 1$. As a consequence, $II = 0$. 

We have shown that $I$ and $II$ are bounded. 

Since both $\eta(x)$ and $w_\epsilon^\delta(x)$ are smooth and bounded,

$$I \leq C \int_{|x-y| \geq \frac{1}{4}} \frac{1}{|x-y|^{n+2s}}dy \leq C.$$  (2.36)
Case 2: \(|x| > \frac{1}{2}\). It is easy to see that \(\eta(x) - \eta(y) \leq c_1 |x - y|\) and \(w^\delta_\epsilon(x) - w^\delta_\epsilon(y) \leq c_2 |x - y|\), where \(c_1\) and \(c_2\) are independent of \(\epsilon\) and \(\delta\). So

\[
II = \int_{|x - y| < \frac{1}{4}} \frac{(\eta(x) - \eta(y))(w^\delta_\epsilon(x) - w^\delta_\epsilon(y))}{|x - y|^{n+2s}} dy \leq C \int_{|x - y| < \frac{1}{4}} \frac{1}{|x - y|^{n+2s}} dy \leq C. \tag{2.37}
\]

Combining (2.34), (2.35), (2.36) and (2.37), we obtain

\[
\int_{B_{\frac{1}{4}}(0)} f(x)w^\delta_\epsilon(x) dx \leq C. \tag{2.38}
\]

Taking \(\delta \to 0\) and then \(\epsilon \to 0\), we obtain \(f(x)\) is integrable in \(B_{\frac{1}{4}}(0)\), which together with the assumption \(f \in L^1_{\text{loc}}(B_1(0) \setminus \{0\})\) implies that \(f(x)\) is integrable in \(B_1(0)\).

Next, we prove (2.33)(ii) in three steps.

**Step 1:** We claim that the following identity holds in the sense of distribution

\[
(-\Delta)^s v(x) + Dv(x) = f(x) + a\delta_0 + (\vec{b} \cdot \nabla)\delta_0 \text{ in } B_1(0). \tag{2.39}
\]

Given a test function \(\phi(x) \in C^\infty_0(B_1(0))\), we linearize it as \(\phi(x) = (\phi(0) + \nabla \phi(0) \cdot x)\eta(x) + H(x)\), where \(H(x)\) is a smooth function with quadratic or higher order of \(x\) in \(B_1(0)\), and \(\eta(x)\) is a positive smooth function supported in \(B_1(0)\). Define \(\rho(x)\) as a smooth nonnegative function compactly supported in \(B_2(0)\), \(\rho(x) = 1, x \in B_1(0)\), and \(\rho_\epsilon(x) = \rho(x)\). Then, we have

\[
\begin{align*}
\int_{\mathbb{R}^n} & v(x)((-\Delta)^s + D)\phi(x) dx \\
= & \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\phi(0) + \nabla \phi(0) \cdot x)\eta(x) dx + \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)H(x) dx \\
= & \phi(0) \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)\eta(x) dx + \nabla \phi(0) \cdot \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\eta(x)) dx \\
+ & \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(1 - \rho_\epsilon(x))H(x) dx + \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\rho_\epsilon(x)H(x)) dx \\
= & \phi(0) \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)\eta(x) dx + \nabla \phi(0) \cdot \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\eta(x)) dx \\
+ & \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(x)((1 - \rho_\epsilon(x))H(x)) dx + \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\rho_\epsilon(x)H(x)) dx \\
= & \phi(0) \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)\eta(x) dx + \nabla \phi(0) \cdot \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\eta(x)) dx \\
+ & \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} f(x)((1 - \rho_\epsilon(x))(\phi(x) - \phi(0) + \nabla \phi(0) \cdot x)\eta(x)) dx \\
+ & \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\rho_\epsilon(x)H(x)) dx \\
= & \phi(0)a_1 + \nabla \phi(0) \cdot \vec{b} + \int_{\mathbb{R}^n} \phi(x)f(x) dx + \phi(0) \int_{\mathbb{R}^n} f(x)\eta(x) dx \\
+ & \nabla \phi(0) \cdot \int_{\mathbb{R}^n} f(x)x\eta(x) dx + \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\rho_\epsilon(x)H(x)) dx \\
+ & \nabla \phi(0) \cdot \int_{\mathbb{R}^n} f(x)x\eta(x) dx + \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)((-\Delta)^s + D)(\rho_\epsilon(x)H(x)) dx
\end{align*}
\]
\[\phi(0) a + \nabla \phi(0) \cdot \mathbf{b} + \int_{\mathbb{R}^n} \phi(x) f(x) dx + D \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)(-\Delta)^s(\rho_\epsilon(x) H(x)) dx.\] (2.40)

Next, we estimate the last two limits above, and show they are zero.

Since \(\rho_\epsilon(x)\) is compactly supported in \(B_\delta(0)\) and \(H(x)\) is a function of \(O(|x|^2)\) near the origin, then

\[|\rho_\epsilon(x) H(x)| \leq \epsilon^2,\] (2.41)

which together with the integrability of \(v(x)\) in \(B_\delta(0)\) yields that the second limit in the last line of (2.40) is zero. Thus, in the following, we consider the first limit in the last line of (2.40).

By the definition of fractional Laplacian and setting \(0 < \delta < \epsilon \ll 1\), we have

\[(-\Delta)^s(\rho_\epsilon(x) H(x)) = C_{n,s} \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_\delta(0)} 2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y) dy \]
\[= C_{n,s} \lim_{\delta \to 0} \int_{B_\epsilon(0) \setminus B_\delta(0)} 2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y) dy + C_{n,s} \int_{B_{2\epsilon}(0) \setminus B_\delta(0)} 2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y) dy \]
\[:= I + II.\] (2.42)

Another observation is that when \(|y|\) is small enough, there exists a constant \(C > 0\) independent of \(x, y\) and \(\epsilon\), such that \(|2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y)| < C|y|^2\). In fact,

\[|2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y)| \leq C|\nabla^2(\rho_\epsilon(x) H(x))||y|^2 \leq C|\nabla^2 \rho_\epsilon(x) H(x)| + C|\nabla^2 H(x)\rho_\epsilon(x)||y|^2| + C|\nabla^2 H(x)\rho_\epsilon(x)||y|^2| \leq C|y|^2.\] (2.43)

Combining (2.41) and (2.43), we obtain

\[|2\rho_\epsilon(x) H(x) - \rho_\epsilon(x + y) H(x + y) - \rho_\epsilon(x - y) H(x - y)| \leq C\min\{\epsilon^2, |y|^2\}.\]

If \(|x| \leq 1\),

\[|I| \leq C \int_{B_{2\epsilon}(0)} |y|^2 |y|^{n+2s} dy \leq C\epsilon^{2-s},\]
\[|II| \leq C \int_{B_{2\epsilon}(0)} \epsilon^2 |y|^{n+2s} dy \leq C\epsilon^{2-s},\] (2.44)

which implies that

\[|(-\Delta)^s(\rho_\epsilon(x) H(x))| \leq C\epsilon^{2-s}.\] (2.45)

If \(|x| > 1\), since \(H(x)\) and \(\rho_\epsilon(x)\) are both compactly supported functions,

\[|(-\Delta)^s(\rho_\epsilon(x) H(x))| = \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus |x-y| \leq \delta} \frac{\rho_\epsilon(x) H(x) - \rho_\epsilon(y) H(y)}{|x-y|^{n+2s}} dy \]
\[\leq C \int_{\mathbb{R}^n} |\rho_\epsilon(y) H(y)| dy \leq C\epsilon^2 \frac{1 + |x|^{n+2s}}{1 + |x|^{n+2s}},\] (2.46)

\(\rho_\epsilon(y) \leq \epsilon^2\) and is compactly supported in \(|y| \leq \epsilon\)
From (2.45) and (2.46), it holds that \( \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)(-\Delta)^s(\rho_\epsilon H(x))dx = 0. \)

Therefore, we have completed the step 1.

**Step 2.** We derive that the vector \( \vec{b} \) in (2.39) is zero. Since \( v(x) \) satisfies (2.39), \( v(x) \) can be represented as

\[
v(x) = \omega(x) + \frac{ac_n}{|x|^{n-2s}} + \frac{\vec{b} \cdot x}{|x|^{n-2s+2}} + h(x), \quad x \in B_1(0),
\]

(2.47)
in the sense of distribution. Here \( c_n > 0, \omega(x) \) is the newtonian potential of the integrable function \( f(x) - Dv(x) \), and \( h(x) \) is \( s \)-harmonic function. From Lemma 2.2 we know that \( h(x) \) is bounded.

Next, we show that \( \vec{b} = 0 \). First, it is easy to see that \( \omega(x), h(x) \) and \( \frac{\vec{a}}{|x|^{n-2s}} \) are in \( L^{\frac{n}{n-2s+2}}(\Omega) \) for any \( \Omega \in B_1(0) \). Then choose a conic domain \( \Omega = \{ |\vec{b}| \cdot x| > \frac{|x|}{2}, \vec{b} \cdot x < 0 \} \in B_1(0) \). We can find that \( \| \frac{\vec{b}}{|x|^{n-2s+2}} \|_{L^{\frac{n}{n-2s+2}}(\Omega)} = \infty \). Similar to step 2 in Theorem 2.1 these estimates imply that the negative part \( v^{-}(x) \) of \( v(x) \) is positive, which is a contradiction with \( v(x) \geq 0 \) in \( \Omega \). As a result, we have that \( \vec{b} = 0 \).

**Step 3:** We prove that \( a \geq 0 \) in (2.47). First, a conclusion of (2.47) is

\[
v(x) = w(x) + \frac{a}{|x|^{n-2s}} + h(x),
\]

(2.48)
where \( h(x) \) is a bounded \( s \)-harmonic function.

Assume \( a < 0 \), Consider the average of \( w(x) \) in a \( \delta \)-ball and take \( \epsilon = \sqrt{\delta} \). It holds that

\[
\int_{B_\delta(0)} |w(x)|dx \leq \frac{C}{\delta^n} \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{|f(y) - Dv(y)|}{|x - y|^{n-2s}}dydx + \frac{C}{\delta^n} \int_{B_\delta(0)} \int_{B_{\delta}(0)} \frac{|f(y) - Dv(y)|}{|x - y|^{n-2s}}dydx
\]

\[
= \frac{C}{\delta^n} \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{|f(y) - Dv(y)|}{|x - y|^{n-2s}}dydx + \frac{C}{\delta^n} \int_{\epsilon = \sqrt{\delta}} \frac{|f(y) - Dv(y)|}{|x - y|^{n-2s}}dydx
\]

\[
\leq \frac{C}{\delta^n} (\int_{B_{2\epsilon}(0)} |f(y) - Dv(y)|dy + \frac{\delta^{n-2s}}{2})
\]

Then considering the average of \( \frac{a}{|x|^{n-2s}} \) in the \( \delta \) ball, we have

\[
\int_{B_\delta(0)} \frac{a}{|x|^{n-2s}}dx \leq \frac{a}{\delta^{n-2s}}.
\]

(2.50)
And choose \( \epsilon \) small enough such that

\[
\int_{B_{2\epsilon}(0)} |f(y) - Dv(y)|dy < \frac{-a}{2C}.
\]

(2.51)
Taking \( \delta \) small enough, and from (2.49) + (2.51), we have

\[
\int_{B_\delta(0)} v(x)dx \leq \int_{B_\delta(0)} (|w(x)| + \frac{a}{|x|^{n-2s}} + h(x))dx \leq \frac{a}{\delta^{n-2s}} - \frac{a}{2\delta^{n-2s}} < 0,
\]

(2.52)
which is a contradiction with the assumption \( v(x) \geq 0 \) in \( \Omega \). Hence, \( a \geq 0 \). This completes the proof of Theorem 2.5.
3 Maximum principles

In this section, we will derive several maximum principles for both super-harmonic functions and fractional super-harmonic functions on punctured balls. These maximum principles mainly rely on the Bôcher type theorems given in the last section.

**Theorem 3.1.** (Maximum Principle on a punctured ball)
Assume that \( v(x) \in L^1_{\text{loc}}(B_R(0) \setminus \{0\}) \), and satisfies in the sense of distribution that
\[
\begin{align*}
-\Delta v(x) + a(x)v(x) &\geq 0, \ x \in B_R(0) \setminus \{0\}, \\
v(x) &\geq m > 0, \ x \in \partial B_R(0), \\
v(x) &\geq 0, \ x \in \mathbb{R}^n, \ n \geq 2,
\end{align*}
\] (3.1)
Here \( a(x) \leq D \) for some constant \( D \), then there exists a positive constant \( c = c(n, s, D) \) depending on \( n, s \) and \( D \) only, such that \( v(x) \) satisfies in the sense of distribution
\[
v(x) \geq cm, \ x \in B_R(0) \setminus \{0\}.
\] (3.2)
In particular, when \( a(x) \equiv 0 \), we have the following estimate
\[
v(x) \geq m, \ x \in B_R(0) \setminus \{0\}.
\] (3.3)

**Proof.** Without loss of generality, we can assume that \( 0 \leq a(x) \leq D \), since it is easier to be treated when \( a(x) < 0 \). As in the proof of Theorem 2.1 we only need to consider the special case \( a(x) \equiv D \). First, we consider the case that \( w(x) \) is smooth. Thus, we introduce the function
\[
w(r) = me^{-\frac{r}{\sqrt{2(n-2)}}} \text{ with } r < R, \text{ which satisfies}
\]
\[
\begin{align*}
-\Delta w(|x|) + Dw(|x|) &\leq 0, \ x \in B_R(0), \\
w(|x|)|_{\partial B_R(0)} &\equiv m.
\end{align*}
\] (3.4)
Then the function \( u(x) = v(x) - w(x) \) satisfies that
\[
\begin{align*}
-\Delta u(x) + Du(x) &\geq 0, \ x \in B_R(0), \\
u(x)|_{\partial B_R(0)} &\geq 0.
\end{align*}
\] (3.5)
From the maximum principle for \( u(x) \) on \( B_R(0) \), we know that \( u(x) \geq 0 \) for \( x \in B_R(0) \), that is, 
\[
v(x) \geq w(|x|) = me^{-\frac{r}{\sqrt{2(n-2)}}} \geq me^{-\frac{R}{\sqrt{2}}}.\] This proves (3.2) and (3.3) when \( v(x) \) is smooth.

Second, when \( v(x) \) is not smooth, we consider the mollification \( v_\epsilon(x) = (\rho_\epsilon * v)(x) \). From Theorem 2.1 and the fact that if \(-\Delta v(x) \geq 0 \) then \(-\Delta v_\epsilon(x) \geq 0 \), it follows from the result in the first step, we know that 
\[
v_\epsilon \geq me^{-\frac{r}{\sqrt{2(n-2)}}} \text{ when } x \in B_{R-\epsilon}(0). \] Then letting \( \epsilon \to 0 \), we have 
\[
v(x) \geq e^{-\frac{R}{\sqrt{2}}} \text{ when } x \in B_{R}(0) \setminus \{0\}.\] This proves Theorem 3.1.

**REMARK 5.** Theorem 3.1 does not hold when \( n = 1 \). In fact, for the special case \( c(x) \equiv 0 \), the function \( v(x) = |x| \) satisfies 3.1 but obviously does not satisfy 3.3.

**Theorem 3.2.** (Fractional maximum principle on a punctured ball)
Assume that \( w(x) \in \mathcal{L}_{2s} \), and satisfies in the sense of distribution that
\[
\begin{align*}
(-\Delta)^s w(x) + a(x)w(x) &\geq 0, \ \text{on } B_r(x^0) \setminus \{x^0\}, \\
w(x) &\geq m > 0, \ \text{on } B_r(x^0) \setminus B_{\frac{r}{2}}(x^0), \\
w(x) &\geq 0, \ \text{in } \mathbb{R}^n, \ n > 2s,
\end{align*}
\] (3.6)
Here \( a(x) \leq D \) for some constant \( D \), then there exists a positive constant \( c = c(n, s, D) \) depending on \( n, s \) and \( D \) only, such that \( w(x) \) satisfies in the sense of distribution
\[
w(x) \geq cm, \ x \in B_r(x^0) \setminus \{x^0\}.
\] (3.7)
Then using the conclusion in the first step, we know there exist suitable positive constants the standard mollifier. From Lemma 2.3 and Theorem 2.5, we have (if we select $0 < c$ satisfying $0 < c < \bar{c}$ such that $\epsilon B(0, \rho)$ is independent of $\epsilon$, then we have proved that there exists a positive constant $\epsilon < 1$ such that $w(x) \geq \epsilon m \geq cm$ in $B_{1-\epsilon}(x^0)$, where $c > 0$ is independent of $\epsilon$. Then letting $\epsilon \to 0$, we can immediately derive $w(x) \geq cm$, $x \in B_1(x^0)\setminus \{x^0\}$ for some $c > 0$.

Finally, making scaling $\tilde{w}(x) = w(rx)$, we know from the first step that if
\begin{align*}
\tilde{w}(x) \geq 0, \quad x \in \mathbb{R}^n, \\
(-\Delta)^s \tilde{w}(x) + D\tilde{w}(x) \geq 0, \quad x \in B_1\left(\frac{x^0}{r}\right)\setminus \left\{\frac{x^0}{r}\right\}, \\
\tilde{w} \geq m > 0, \quad x \in B_1\left(\frac{x^0}{r}\right)\setminus \left\{\frac{x^0}{r}\right\},
\end{align*}
(3.8)
then there exists some positive constant $c$ depending on $n$ and $s$ only such that
\[ \tilde{w}(x) \geq cm, \quad \text{in} \quad B_1\left(\frac{x^0}{r}\right)\setminus \left\{\frac{x^0}{r}\right\}. \]
This proves Theorem 3.2.

We emphasize here the importance of the Böcher type Theorem 2.5: the nonnegative fractional super-harmonic function on the punctured ball $B(0)\setminus \{0\}$ is also actually a fractional super-harmonic function on the whole ball $B(0)$ in the sense of distribution.

In fact, connecting Theorem 2.5 and Theorem 3.2 is the crucial observation.

REMARK 6. There is an obvious difference between (3.3) in Theorem 3.1 and (3.7) in Theorem 3.2 for the special case $a(x) \equiv 0$ in (3.1) and (3.6), that is, there exists a positive constant $c < 1$ in
which is resulted by the non-locality of the fractional Laplacian. For example, give a function defined in $\mathbb{R}^n$ with $n > 2s$ that
\[
u(x) = \begin{cases} 1 - \epsilon \rho(x), & |x| \leq 1, \\ 1 - \eta(x), & |x| > 1, \end{cases}
\] (3.9)
where
\[
\begin{align*}
\rho(x) & \in C^\infty(\mathbb{R}^n), \quad \rho(x) = 0, \quad |x| \geq \frac{1}{2}, \quad \max_{|x| \leq 1} \rho(x) = \rho(0) = 1, \\
\eta(x) & \in C^\infty(\mathbb{R}^n), \quad \eta(x) = 0, \quad |x| \leq 1, \quad \eta(x) = 1, \quad |x| \geq 2.
\end{align*}
\] (3.10)
Obviously, $u(x) = 1$ in $B_1(0) \setminus B_{\frac{1}{2}}(0)$. After a direct calculation, if choosing $\epsilon$ sufficiently small we have for $x \in B_1(0)$ that
\[
(-\Delta)^s u(x) = C_{n,s} \lim_{\delta \to 0} \int_{\mathbb{R}^n \setminus \{|x-y| \leq \delta\}} \frac{-\epsilon \rho(x) + \epsilon \rho(y) + \eta(y)}{|x-y|^{n+2s}} dy \geq -\epsilon C_1 (-\Delta)^s \rho(x) + C_3 \geq -C_2 \epsilon + C_3 > 0.
\] (3.11)

However, $\min_{x \in \overline{B_1(0)} \setminus \{0\}} u(x) < 1$.

Next, we define the standard notations for the method of moving plane
\[
\begin{align*}
x = (x_1, x') \in \mathbb{R}^n, \quad \bar{x} = (-x_1, x'), \\
T = \{(x_1, x') \in \mathbb{R}^n | x_1 = 0\}, \\
H = \{(x_1, x') \in \mathbb{R}^n | x_1 < 0\}, \quad \bar{H} = \{(x_1, x') \in \mathbb{R}^n | x_1 > 0\}.
\end{align*}
\]

The following maximum principle is corresponding to the method of moving planes.

**Theorem 3.3.** (Fractional maximum principle for anti-symmetric functions)
Assume that $w(-x_1, x') = -w(x_1, x')$, $\forall x \in H$, and $B_r(x^0) \subset H$,
\[
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s w(x) + a(x) w(x) \geq 0, \text{ on } B_r(x^0) \setminus \{x^0\}, \\
w(x) \geq m > 0, \text{ on } B_r(x^0) \setminus B_{\frac{1}{2}}(x^0); \ w(x) \geq 0, \text{ on } H,
\end{array} \right.
\end{align*}
\] (3.12)
and $a(x) \leq D$ for some constant $D$. Then there exists a positive constant $c = c(n, s, D)$ depending on $n, s$ and $D$ only, such that $w(x)$ satisfies in the sense of distribution
\[
w(x) \geq cm, \quad x \in B_r(x^0) \setminus \{x^0\}.
\] (3.13)

**Proof.** As in the proof of Theorem 3.1 we only need to consider the special case $a(x) \equiv D$. First of all, we set $B_1(x^0) \subset H$, which yields that $d := \lambda - x_1^0 \geq 1$. In the first step, we consider the case that $w(x)$ is smooth and $r = 1$. Suppose (3.13) is not true, then there exists $c > 0$ such that
\[
0 \leq \min_{\overline{B_{\frac{1}{2}}(x^0)}} w(x) = w(\bar{x}) < cm.
\] (3.14)

On the other hand, we have
\[
\begin{align*}
0 & \leq (-\Delta)^s w(\bar{x}) + Dw(\bar{x}) = C_{n,s} \left( P.V. \int_{H} \frac{w(\bar{x}) - w(y)}{|\bar{x} - y|^{n+2s}} dy + \int_{H} \frac{w(\bar{x}) - w(y)}{|\bar{x} - y|^{n+2s}} dy \right) + a(\bar{x}) w(\bar{x}) \\
& = C_{n,s} \left( P.V. \int_{H} \frac{w(\bar{x}) - w(y)}{|\bar{x} - y|^{n+2s}} dy + \int_{H} \frac{w(\bar{x}) + w(y)}{|\bar{x} - y|^{n+2s}} dy \right) + Dw(\bar{x}) \\
& = C_{n,s} P.V. \int_{H} \frac{1}{|\bar{x} - y|^{n+2s}} dy + a(\bar{x}) w(\bar{x}) \\
& + C_{n,s} \int_{H} \frac{w(\bar{x}) - w(y)}{|\bar{x} - y|^{n+2s}} dy + a(\bar{x}) w(\bar{x}) \\
& := I + II + a(\bar{x}) w(\bar{x}) < I + II + Dcm.
\end{align*}
\] (3.15)
For $II$, we have
\[ II \leq C_3(\bar{x})w(\bar{x}) < cmC_3(n, s), \quad \text{where} \quad C_3(\bar{x}) := \int_H \frac{C_{n,s}}{|\bar{x} - \bar{y}|^{n+2s}} dy < \infty. \quad (3.16) \]

Then we can estimate $I$ as
\[
I = C_{n,s} \int_{H \setminus B_1(x^0)} (w(\bar{x}) - w(y)) \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy \\
+ C_{n,s} \int_{B_1(x^0) \setminus B_2(x^0)} (w(\bar{x}) - w(y)) \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy \\
+ C_{n,s} \text{P.V.} \int_{B_2(x^0)} (w(\bar{x}) - w(y)) \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy \leq C_{n,s} \int_{H \setminus B_1(x^0)} (w(\bar{x}) - w(y)) \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy \\
+ C_{n,s} \int_{B_1(x^0) \setminus B_2(x^0)} (w(\bar{x}) - w(y)) \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy \\
- (1-c)mC_{n,s} \int_{B_1(x^0) \setminus B_2(x^0)} \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy < cmC_1(\bar{x}) - (1-c)mC_2(\bar{x}),
\]

where
\[
C_1(\bar{x}) = C_{n,s} \int_{H \setminus B_1(x^0)} \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy, \\
C_2(\bar{x}) = C_{n,s} \int_{B_1(x^0) \setminus B_2(x^0)} \left( \frac{1}{|\bar{x} - y|^{n+2s}} - \frac{1}{|\bar{x} - \bar{y}|^{n+2s}} \right) dy,
\]

if we set $0 \leq c = \inf_{\bar{x} \in B_2(x^0)} C_1(\bar{x}) + C_2(\bar{x}) + C_3(\bar{x}) + B$, it yields from (3.15)–(3.17) that
\[
(-\Delta)^s w(\bar{x}) + Dw(\bar{x}) < 0,
\]
which leads to a contradiction. This proves $w(x) \geq cm$ in $B_1(x^0)$.

Second, we consider the case that $w(x)$ is not a smooth function. From Lemma 2.23 and Theorem 2.45 we have $(-\Delta)^s w_\epsilon(x) \geq 0$ in $B_1 \setminus \epsilon(0)$. Then using the conclusion in the first step, we know there exists a positive constant $\tilde{c} < 1$ such that $w_\epsilon(x) \geq \tilde{c} m \epsilon \geq cm$ in $B_1 \setminus \epsilon(0)$, where $c > 0$ is independent of $\epsilon$. Then letting $\epsilon \to 0$, we can immediately derive $w(x) \geq cm$, $x \in B_1(x^0) \setminus \{x^0\}$ for some $c > 0$.

Finally, making scaling $\bar{w}(x) = w(rx)$, we know from the first step that if
\[
\begin{cases}
\bar{w}(x) \geq 0, \ x \in \mathbb{R}^n, \\
(-\Delta)^s \bar{w}(x) + D\bar{w}(x) \geq 0, \ x \in B_1 \left( \frac{x^0}{r} \right) \setminus \left\{ \frac{x^0}{r} \right\}, \\
\bar{w}(x) \geq m > 0, \ x \in B_1 \left( \frac{x^0}{r} \right) \setminus B_{\frac{x^0}{r}},
\end{cases}
\]
then there exists some positive constant $c$ depending on $n$ and $s$ only such that
\[
\bar{w}(x) \geq cm, \text{ in } B_1 \left( \frac{x^0}{r} \right) \setminus \left\{ \frac{x^0}{r} \right\}.
\]

Therefore, we complete the proof. \(\square\)
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