Toroidal Lie superalgebras and free field representations

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Abstract. A loop-algebraic presentation is given for toroidal Lie superalgebras of classical types. Based on the loop superalgebra presentation free field realizations of toroidal Lie superalgebras are constructed for types $A(m, n)$, $B(m, n)$, $C(n)$ and $D(m, n)$.

1. Introduction

Lie superalgebras and Lie algebras are both important classes of algebraic structures with ample applications in mathematics and particularly mathematical physics. Since Kac’s classification of finite-diemsional simple Lie (super)algebra [K1], there have been various works on their representations and realizations.

Based on fermionic realizations [F, KP], A. Feingold and I. Frenkel [FF] realized classical affine Lie algebras using fermionic fields and bosonic fields respectively. Their constructions have been generalized to other algebras such as extended affine Lie algebras [G], affine Lie superalgebras [KW], Tits-Kantor-Köcher algebras [T], Lie algebras with central extensions [L], two-parameter quantum affine algebras [JZ] and others.

Toroidal Lie (super)algebras are generalizations of affine (super)Lie algebras and enjoyed many favorite properties similar to affine Lie (super)algebras. In the case of 2-toroidal Lie algebras, the Moody-Rao-Yakonuma presentation shows that the special toroidal Lie algebras have a similar algebraic

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structure like the affine Lie algebras [MRY], in particular, the double affine Lie algebras are one subclass. Using the MRY-presentation, the first named author and collators realized uniformly [JM, JMX] 2-toroidal Lie algebras of all classical types using bosonic fields or fermionic fields with help of a ghost field, which also included the newly discovered bosonic/fermionic realizations for orthogonal/symplectic types. In the case of super toroidal Lie algebras we recently constructed a loop-like toroidal Lie superalgebra of type $B(0, n)$ using bosonic fields and a ghost field in [JX]. That work suggests that the loop-algebra presentation may be given similarly.

The purpose of this paper is to give a Moody-Rao-Yokonuma presentation for the toroidal superalgebras of classical types, and then use the presentations to construct representations of toroidal Lie superalgebras of type $A, B, C, D$. Similar like the special case of [JX] we use mixed bosons and fermions as well as a ghost field.

The structure of this paper is as follows. In section 2, we collect the preliminaries needed. In section 3, we define a Lie superalgebra $T(X)$ to each type $X$ and give a loop-algebra presentation for all classical Lie toroidal superalgebras. In section 4, free field representation of $T(X)$ is constructed for each case. In the appendix we list all the extended distinguished Cartan matrices.

2. Preliminaries

A Lie superalgebra $g = g_0 \oplus g_1$ is a $\mathbb{Z}_2$-graded vector space equipped with a bilinear map $[\cdot, \cdot] : g \times g \to g$ such that

1) $[g_\alpha, g_\beta] \subseteq g_{\alpha+\beta}$ ($\mathbb{Z}_2$ - gradation),
2) $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ (graded antisymmetry),
3) $[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]$ (graded Jacobi identity).

where $\alpha, \beta \in \mathbb{Z}_2$ and $a, b, c$ are homogenous elements.

Let $M, N \in \mathbb{N}$ and $V = V_T \oplus V_\bar{T}$, where $\dim V_T = M, \dim V_\bar{T} = N$. Then the associative algebra $\text{End}V$ is equipped with $\mathbb{Z}_2$-grading $\text{End}V = \text{End}_0 V \oplus \text{End}_\bar{T} V$, where $\text{End}_0 V = \{ a \in \text{End}V \mid a(V_0) \subseteq V_{s+\alpha} \}$. For any two homogenous elements $a, b \in \text{End}V$, we define a superbracket

$$[a, b] = ab - (-1)^{p(a)p(b)}ba$$
and extended bilinearly, then \( \text{End} V \) becomes a Lie superalgebra called general linear superalgebra and denoted by \( \mathfrak{gl}(M|N) \). Let \( \text{str} \) be the subspace on \( \mathfrak{gl}(M|N) \), then \( \mathfrak{sl}(M, N) = \{ a \in \mathfrak{gl}(M|N) | \text{str} a = 0 \} \) is an ideal of \( \mathfrak{gl}(M|N) \) of codimensional 1 called special linear superalgebras. Note that \( \mathfrak{sl}(N, N) \) contains the 1-dimensional ideal consisting of \( \lambda I_{2N} \). We set

\[
A(m, n) = \mathfrak{sl}(m + 1, n + 1), \quad \text{for } m \neq n, m, n \geq 0
\]

\[
A(n, n) = \mathfrak{sl}(m + 1, n + 1)/ \langle I_{2n+2} \rangle \quad n > 0
\]

### 2.1. The orthosymplectic superalgebras.

Let \((\cdot | \cdot)\) be a non-degenerate bilinear form on \( V \) such that \((V_0^1 | V_0^1) = 0\), the restriction of \((\cdot | \cdot)\) to \( V_1 \) is symmetric and to \( V_0 \) is skewsymmetric, so that \( N = 2n \) is even. For \( \alpha = 0, 1 \), let

\[
\mathfrak{osp}(M|N)_{\alpha} = \{ a \in \mathfrak{gl}(M|N)_{\alpha} | (a(x) | y) + (-1)^{\alpha p}(x | a(y)) = 0, x, y \in V \}
\]

and \( \mathfrak{osp}(M|N) = \mathfrak{osp}(M|N)_0 \oplus \mathfrak{osp}(M|N)_1 \), then \( \mathfrak{osp}(M|N) \) becomes a simple subsuperalgebra of \( \mathfrak{gl}(M|N) \) which is called the orthosymplectic superalgebras. We denote by respectively

\[
B(m, n) = \mathfrak{osp}(2m + 1|2n), \quad m \geq 0, n \geq 1;
\]

\[
C(n) = \mathfrak{osp}(2|2n), \quad n \geq 1;
\]

\[
D(m, n) = \mathfrak{osp}(2m|2n), \quad m \geq 0, n \geq 1
\]

Let \((\cdot | \cdot)\) be an even symmetric invariant bilinear form on linear superalgebra \( \mathfrak{g} \), then the associated affine superalgebra \( \mathfrak{g}^{(1)} \) is

\[
\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d
\]

with the following commutation relations

\[
[a(m), b(n)] = [a, b](m + n) + m\delta_{m, -n}(a|b)K, \quad [d, a(m)] = -ma(m)
\]

where \( K \) is central and \( a(m) = a \otimes t^m; a, b \in \mathfrak{g}; m, n \in \mathbb{Z} \).

In the framework of Lie superalgebras a special theory is developed for what one calls the Kac-Moody superalgebra which is the corresponding theory to that of Kac-Moody algebra and we have the following well-known result.

**Proposition 2.1.** The linear superalgebras and their associated affine superalgebras are both Kac-Moody superalgebras.
The following proposition plays crucial role in defining the loop-like toroidal linear superalgebras and the equalities is usually called the Serre type relations of Kac-Moody superalgebras (cf. [Van]).

**Proposition 2.2.** Let $g(A, \tau)$ be a Kac-Moody superalgebra with Chevalley generators $e_i, f_i (1 \leq i \leq n)$, then the following relations hold for all $i, j$:

- If $a_{ii} = a_{ij} = 0$, then $\text{ad}_{e_i}(e_j) = \text{ad}_{f_i}(f_j) = 0$;
- If $a_{ii} = 0, a_{ij} \neq 0$, then $\left(\text{ad}_{e_i}\right)^2(e_j) = \left(\text{ad}_{f_i}\right)^2(f_j) = 0$;
- If $a_{ii} \neq 0$, then for $i \neq j$, $\left(\text{ad}_{e_i}\right)^{1-2\frac{a_{ii}}{a_{ij}}}(e_j) = \left(\text{ad}_{f_i}\right)^{1-2\frac{a_{ii}}{a_{ij}}}(f_j) = 0$.

Among the simple root systems of linear superalgebra $g$, there exists a simple root system which the number of odd roots is the smallest. Such a simple root system is called the distinguished simple root system and the associated Cartan matrix is called distinguished Cartan matrix. Furthermore from the distinguished simple (co)root system, Cartan matrix and Chevalley generators of $g$, one can obtain those of $g^{(1)}$ in an almost same way adopted in the Lie algebra setting. For convenience, we list the extended distinguished Cartan matrix of $g$ in the appendix.

Fix a vector space $V = V_0 \oplus V_1$. A formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, where $a(n) \in \text{End} V$, is called a field if for any $v \in V$ one has $a(n)v = 0$ for $n \gg 0$. Given a field $a(z)$, we always assume that all the coefficients have the same parity denoted by $p(a)$ and let

$$a(z)_- = \sum_{n \geq 0} a(n) z^{-n-1}, \quad a(z)_+ = \sum_{n < 0} a(n) z^{-n-1}$$

For any two fields $a(z), b(z)$, we define their normally ordered product as follows:

$$:a(z)b(w): = a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-$$

It follows from this definition that normally ordered product satisfies the supercommutativity, i.e. $:a(z)b(w): = (-1)^{p(a)p(b)} :b(w) a(z):$ and $:a(z)b(w):$ is also a field with the parity of $p(a) + p(b)$. The contraction of any two fields $a(z), b(w)$ is defined to be:

$$\overline{a(z)b(w)} = a(z)b(w)^- :a(z)b(w):$$

Furthermore, one can define the normally ordered product of more than two fields inductively from right to left.

The following well-known Wick’s theorem ([FLM]) is extremely useful for calculating the OPE of two normally ordered product of free fields (cf. [K2]).
THEOREM 2.3. Let $x^1, x^2, \ldots, x^M$ and $y^1, y^2, \ldots, y^N$ be two collections of fields with definite parity and the notation of normally ordering. Suppose these fields satisfy the following properties hold:

1) $\{ x^i y^j, z^k \} = 0$, for all $i, j, k$ and $z = x$ or $y$;
2) $\{ x^i \pm, y^j \pm \} = 0$, for all $i, j$.

hereafter $\{ \cdot, \cdot \}$ means the superbracket. then we have that

$$x^1 x^2 \cdots x^M :: y^1 y^2 \cdots y^N :$$

$$= \sum_{s=0}^{\min(M,N)} \sum_{i_1 < \cdots < i_s

where the subscript $(i_1, \ldots, i_s, j_1, \ldots, j_s)$ means the fields $x^{i_1}, \ldots, x^{i_s}, y^{j_1}, \ldots, y^{j_s}$ are removed and the sign $\pm$ is obtained by the rule: each permutation of the adjacent odd fields changes the sign.

In the final preparation of formal calculus, we recall the definition of the formal delta-function:

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n$$

which is formally defined to be the series expansions in two directions:

$$\partial^{(j)}_w \delta(z - w) = i_{z,w} \frac{1}{(z - w)^{j+1}} - i_{w,z} \frac{1}{(z - w)^{j+1}}$$

where $\partial^{(j)}_w = \partial^{(j)}/j!$ and the symbol $i_{z,w}$ (resp. $i_{w,z}$) means power series expansion in the domain $|z| > |w|$ (resp. $|z| > |w|$). In additional, the equality holds when both sides make sense: $f(z, w) \delta(z - w) = f(z, z) \delta(z - w)$. Usually we will drop $i_{z,w}$ if it is clear from the context.

3. Loop-like toroidal Lie superalgebras $\mathfrak{T}(X)$

Let $R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ be the complex commutative ring of Laurant polynomials in two variables $s, t$. Let $\mathfrak{g}$ be a complex Lie superalgebra. The loop Lie superalgebra $L(\mathfrak{g}) := \mathfrak{g} \otimes R$ is defined under the Lie superbracket $[x \otimes a, y \otimes b] = [x, y] \otimes ab$. Let $\Omega_R$ be the $R$-module of Kähler differentials of $R$ spanned by $da, a \in R$, and let $d\Omega_R$ be the space of exact forms. The quotient $\Omega_R/d\Omega_R$ has a basis consisting of $s^{m-1} d^n ds, s^n t^{n-1} dt, s^{-1} ds$, where $m, n \in \mathbb{Z}$. Here $\bar{a}$ denotes the coset $a + d\Omega_R$.

Let $T(\mathfrak{g})$ be the toroidal superalgebra Lie superalgebra:

$$T(\mathfrak{g}) = \mathfrak{g} \otimes R \oplus \Omega_R/d\Omega_R$$
with the Lie superbrackets \((x, y \in \mathfrak{g}, a, b \in R)\)

\[
[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x|y)(da)b, \quad [T(\mathfrak{g}), \Omega_R/d\Omega_R] = 0
\]

and the parities are specified by:

\[
p(x \otimes a) = p(x), \quad p(\Omega_R/d\Omega_R) = 0
\]

**Theorem 3.1.** [IK] The toroidal Lie superalgebra \(T(\mathfrak{g})\) is the universal central extension of the loop Lie superalgebra \(\mathfrak{g} \otimes R\).

Here a universal central extension is assumed to be perfect, thus there is a unique universal central extension as in the Lie algebra cases [Ga, W].

Next we would like to give a loop algebra presentation for the toroidal Lie superalgebras of classical types as the Moody-Rao-Yokonuma presentation for toroidal Lie algebras [MRY].

Let \(\mathfrak{g}\) be the Lie superalgebra associated with the extended distinguished Cartan matrix \(A = (a_{ij})\) of type \(X^{(1)}\). Let \(Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_r\) be its root lattice, where \(r = m + n + 1; m + n; n + 1\) respectively for \(X = A(m, n); B(m, n), D(m, n); X = C(n)\). The odd simple roots are:

\[
\Pi \cap \Delta_T = \begin{cases} 
\{\alpha_0, \alpha_n\} & \text{if } X = A(m, n); \\
\{\alpha_n\} & \text{if } X = B(m, n), D(m, n); \\
\{\alpha_0, \alpha_1\} & \text{if } X = C(n)
\end{cases}
\]

The standard invariant form is given by \((\alpha_i, \alpha_j) = d_i a_{ij}\), where

\[
(d_0, d_1, \cdots, d_r) = \begin{cases} 
(1,1,\cdots,1,-1,\cdots,-1), & \text{if } X = A(m, n); \\
(2,1,\cdots,1,1/2), & \text{if } X = B(0, n); \\
(2,1,\cdots,1,-1,\cdots,-1,-1/2), & \text{if } X = B(m, n), m \geq 1; \\
(1,1,-1,\cdots,-1,2), & \text{if } X = C(n); \\
(2,1,\cdots,1,-1,\cdots,-1), & \text{if } X = D(m, n).
\end{cases}
\]

Note that \(d_i = \frac{1}{2}(\alpha_i, \alpha_i)\) for non-isotropic roots.

**Theorem 3.2.** The toroidal Lie superalgebra \(T(\mathfrak{g})\) is isomorphic to the Lie superalgebra generated by

\[
\{\mathcal{K}, \alpha_i(k), x_i^\pm(k) | 0 \leq i \leq r, k \in \mathbb{Z}\}
\]
with parities given as: \(0 \leq i \leq r, k \in \mathbb{Z}\)

\[
p(K) = p(\alpha_i(k)) = 0, \quad p(x^\pm_i(k)) = p(\alpha_i)
\]

subject to the following relations:

1) \([K, \alpha_i(k)] = [K, x^\pm_i(k)] = 0;\)
2) \([\alpha_i(k), \alpha_j(l)] = k(\alpha_i|\alpha_j)\delta_{m-n}K;\)
3) \([\alpha_i(k), x^\pm_j(l)] = \pm(\alpha_i|\alpha_j)x^\pm_j(k + l);\)
4) \([x^+_i(k), x^-_j(l)] = 0, \text{ if } i \neq j;\)
5) \([x^+_i(k), x^+_j(l)] = 0;\)

\[
[x^+_i(k), x^-_j(l)] = 0, \text{ if } a_{ii} = a_{ij} = 0, i \neq j;
\]

\[
[x^+_i(k), [x^+_j(k), x^-_j(l)]] = 0, \text{ if } a_{ii} = 0, a_{ij} \neq 0;
\]

\[
[x^+_i(k), \cdots, [x^+_i(k), x^+_j(l)]] = 0, \text{ if } a_{ii} \neq 0, i \neq j.
\]

Let \(\mathfrak{T}(X)\) be the algebra generated by \(\{K, \alpha_i(k), x^\pm_i(k) | 0 \leq i \leq r, k \in \mathbb{Z}\}\) defined in Theorem 3.2. The algebra \(\mathfrak{T}(X)\) is a \(Q \times \mathbb{Z}\)-graded Lie superalgebra under the grading: \(\text{deg} K = (0, 0), \text{deg} \alpha_i(k) = (0, k), \text{deg} x^\pm_i(k) = (\pm \alpha_i, k)\). We denote the subspace of degree \((\alpha, k)\) by \(\mathfrak{T}_{\alpha, k}^0\), then \(\mathfrak{T}(X) = \bigoplus_{(\alpha, k) \in Q \times \mathbb{Z}} \mathfrak{T}_{\alpha, k}^0(X)\). We remark that the center of this algebra is contained in the subalgebra generated by \(\mathfrak{T}_{\alpha, k}^0(X), k, n \in \mathbb{Z}\).

**Proposition 3.3.** The following map defines a surjective homomorphism from loop-like toroidal superalgebra \(\mathfrak{T}(X)\) to the algebra \(T(X)\):

\[
K \mapsto s^{-1}ds
\]

\[
\alpha_i(k) \mapsto d_i(h_i \otimes s^k + \delta_{i0}s^{k-i}dt), \quad i = 0, \cdots, n
\]

\[
x^+_i(k) \mapsto e_i \otimes s^k, \quad i = 1, \cdots, n
\]

\[
x^-_i(k) \mapsto f_i \otimes s^k, \quad i = 1, \cdots, n
\]

\[
x^+_0(k) \mapsto e_0 \otimes s^k t^{-1},
\]

\[
x^-_0(k) \mapsto -f_0 \otimes s^k t
\]

**Proof.** It is straightforward to check the elements on the right hand satisfy the relation 1)-4). Note that \(a_{ii} = 2\) if \(a_{ii} \neq 0\), then the last relation is the direct result of Proposition 2.1. \(\square\)
Proof of Theorem 3.2. It follows from Proposition 3.3 that the loop-like toroidal Lie superalgebra $\mathfrak{T}(X)$ is a central extension of the 2-loop superalgebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}, s, s^{-1}]$, as the kernel is contained in the subspace $\oplus_{n,k} \mathfrak{g}_k$, which is clearly central by the commutation relations. On the other hand it is straightforward to check that the algebra $\mathfrak{T}(X)$ is a central extension of the loop algebra $L(\mathfrak{g})$ and therefore the central extension is isomorphic to $\mathfrak{T}(x)$ by the similar arguments as in [MRY].

It will be convenient to rewrite the relations in terms of generating series.

**Proposition 3.4.** The relations of $\mathfrak{T}(X)$ can be written as follows.

1') $[\mathcal{K}, \alpha_i(z)] = [\mathcal{K}, e_i^+(z)] = 0$;

2') $[\alpha_i(z), \alpha_j(w)] = (\alpha_i|\alpha_j)\partial_w \delta(z - w)\mathcal{K}$;

3') $[\alpha_i(z), e_j^-(w)] = \pm(\alpha_i|\alpha_j)e_j^-(w)\delta(z - w)$;

4') $[e_i^+(z), e_j^-(w)] = 0, \text{ if } i \neq j$;

where $\alpha_i(z) = \sum_{k \in \mathbb{Z}} \alpha_i(k)z^{-k-1}$, $e_i^+(z) = \sum_{k \in \mathbb{Z}} e_i^+(k)z^{-k-1}$.

4. Free field realizations of $\mathfrak{T}(X)$

In this section, we shall construct free field representations of toroidal superalgebras defined in last section.

For type $A(m,n)$, let $\varepsilon_i(0 \leq i \leq n + m + 3)$ be an orthonormal basis of the vector space $\mathbb{C}^{n+m+4}$ and $\delta_i = \sqrt{-1} \varepsilon_{m+1+i}$ $(1 \leq i \leq n + 1)$. For type $B(m,n)$, $D(m,n)$ let $\varepsilon_i(0 \leq i \leq n + m + 1)$ be an orthonormal basis of the vector space $\mathbb{C}^{n+m+2}$ and $\delta_i = \sqrt{-1} \varepsilon_{n+1+i}$ $(1 \leq i \leq m + 1)$. For type $C(n)$, let $\varepsilon_i(0 \leq i \leq n + m + 2)$ be an orthonormal basis of the vector space $\mathbb{C}^{n+3}$ and denote by $\delta_i = \sqrt{-1} \varepsilon_{1+i}$ $(1 \leq i \leq n + 1)$. Then the distinguished simple root systems, positive root systems and longest distinguished root of the orthosymplectic superalgebras can be expressed in terms of vectors $\varepsilon_i$'s
and $\delta_i$’s as follows:

i) Type $A(m, n)$ ($m, n \geq 1$):
   \[\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_m = \varepsilon_m - \varepsilon_{m+1}, \alpha_{m+1} = \varepsilon_m - \delta_1, \alpha_{n+1} = \delta_1 - \delta_2, \ldots, \alpha_{n+m+1} = \delta_n - \delta_{n+1}\};\]
   \[\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, \delta_k \pm \delta_{l} | 1 \leq i < j \leq n, 1 \leq k < l \leq m\}\]
   \[\cup \{\delta_k - \varepsilon_i | 1 \leq i \leq n, 1 \leq k \leq m \};\]
   \[\theta = \alpha_1 + \cdots + \alpha_{m+n+1} = \varepsilon_1 - \delta_{n+1};\]

ii) Type $B(0, n)$ ($n \geq 1$):
   \[\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n\};\]
   \[\Delta_+ = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\} \cup \{\varepsilon_i, 2\varepsilon_i | 1 \leq i \leq n\};\]
   \[\theta = 2\alpha_1 + \cdots + 2\alpha_n = 2\varepsilon_1;\]

iii) Type $B(m, n)$ ($m \geq 1, n \geq 1$):
   \[\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n - \delta_1, \alpha_{n+1} = \delta_1 - \delta_2, \ldots, \alpha_{n+m+1} = \delta_m - \delta_m, \alpha_{n+m} = \delta_m\}\]
   \[\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, \delta_k \pm \delta_l | 1 \leq i < j \leq n, 1 \leq k < l \leq m\}\]
   \[\cup \{2\varepsilon_i, \delta_k, \varepsilon_i \pm \delta_k | 1 \leq i \leq n, 1 \leq k \leq m \};\]
   \[\theta = 2\alpha_1 + \cdots + 2\alpha_{n+m} = 2\varepsilon_1;\]

iv) Type $D(m, n)$ ($m \geq 2, n \geq 1$):
   \[\Delta_+ = \{\varepsilon_i \pm \varepsilon_j, \delta_k \pm \delta_l | 1 \leq i < j \leq n, 1 \leq k < l \leq m\},\]
   \[\cup \{2\delta_k, \varepsilon_i \pm \delta_l | 1 \leq i \leq n, 1 \leq k \leq n\}\]
   \[\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n - \delta_1, \alpha_{n+1} = \delta_1 - \delta_2, \alpha_{n+m+1} = \delta_m - \delta_m, \alpha_{n+m} = \delta_m + \delta_m\}\]
   \[\theta = 2\alpha_1 + \cdots + 2\alpha_{n+m-2} + \alpha_{n+m-1} + \alpha_{n+m} = 2\varepsilon_1;\]

v) Type $C(n)$ ($n \geq 1$):
   \[\Pi = \{\alpha_1 = \varepsilon_1 - \delta_1, \alpha_2 = \delta_1 - \delta_2, \ldots, \alpha_n = \delta_{n-1} - \delta_n, \alpha_{n+1} = 2\delta_n\};\]
   \[\Delta_+ = \{\delta_k \pm \delta_l | 1 \leq k < l \leq n\} \cup \{2\delta_k, \varepsilon_1 \pm \delta_k | 1 \leq k \leq n\};\]
   \[\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n + \alpha_{n+1} = \varepsilon_1 + \delta_1;\]
We introduce $\bar{\sigma} = \varepsilon_0 + \delta_{n+2}$ for type $A(m, n)$, $\bar{\sigma} = \varepsilon_0 + \delta_{m+1}$ for type $B(m, n), D(m, n)$ and $\bar{\sigma} = \varepsilon_0 + \delta_{n+1}$ for type $C(n)$, then $\alpha_0 = \bar{\sigma} - \theta$. Furthermore we define

$$\beta = \begin{cases} 
\varepsilon_1 - \bar{\sigma}, & \text{for } A(m, n); \\
\varepsilon_1 - \frac{1}{2} \bar{\sigma}, & \text{for } B(m, n), D(m, n); \\
\delta_1 - \bar{\sigma}, & \text{for } C(n).
\end{cases}$$

Then we have $(\beta|\beta) = 1, (\beta|\varepsilon_i) = \delta_{1i}, \alpha_0 = -\beta + \delta_{n+1}$ for type $A(m, n)$ $(\beta|\beta) = 1, (\beta|\varepsilon_i) = \delta_{1i}, \alpha_0 = -2\beta$ for type $B(m, n), D(m, n)$ and $(\beta|\beta) = -1, (\beta|\delta_i) = -\delta_{1i}, \alpha_0 = -\beta - \varepsilon_1$ for type $C(n)$.

Let $\mathcal{P}_{\bar{\sigma}}$ and $\mathcal{P}_\tau$ be the vector spaces defined for each case as follows:

$$\mathcal{P}_{\bar{\sigma}} = \begin{cases} 
\text{span}_C\{\bar{\sigma}, \varepsilon_i|1 \leq i \leq m+1\}, & \text{for } A(m, n) \\
\text{span}_C\{\bar{\sigma}, \varepsilon_i|1 \leq i \leq n\}, & \text{for } B(m, n), D(m, n) \\
\text{span}_C\{\delta_i|1 \leq i \leq n\}, & \text{for } C(n).
\end{cases}$$

$$\mathcal{P}_\tau = \begin{cases} 
\text{span}_C\{\delta_i|1 \leq i \leq m+1\}, & \text{for } A(m, n) \\
\text{span}_C\{\delta_i|1 \leq i \leq n\}, & \text{for } B(m, n), D(m, n) \\
\text{span}_C\{\varepsilon_i\}, & \text{for } C(n).
\end{cases}$$

and set $\mathcal{C}_0 = \mathcal{P}_{\bar{\sigma}} \oplus \mathcal{P}_\tau^*, \mathcal{C}_\tau = \mathcal{P}_\tau \oplus \mathcal{P}_\tau^*$, then we form the superspace: $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_\tau$.

We define a bilinear form on $\mathcal{C}$ by: for $a, b \in \mathcal{P}_{\bar{\sigma}} \cup \mathcal{P}_\tau$

$$\langle b^*, a \rangle = -(-1)^p(a)p(b) \langle a, b^* \rangle = (a, b);$$

$$\langle b, a \rangle = -(-1)^p(a)p(b) \langle a^*, b^* \rangle = 0,$$

Let $\mathcal{A}(\mathbb{Z}^{2n+2|2m})$ be the associated superalgebra generated by

$$\{u(k)|u \in \mathcal{C}_\tau \cup \mathcal{C}_\tau, k \in \mathbb{Z}\}$$

with the parity $p(u(k)) = p(u)$ and the defining relations

$$u(k)v(l) - (-1)^{p(k)p(l)}u(k)v(l) = \langle u, v \rangle \delta_{k,-l}$$

for $u, v \in \mathcal{C}_\tau \cup \mathcal{C}_\tau$ and $k, l \in \mathbb{Z}$.

The representation space of the superalgebras $\mathcal{A}(\mathbb{Z}^{2n+1|2m})$ is defined to be the following vector space:

$$V(\mathbb{Z}^{n+1|m}) = \bigotimes_{a_i} \left( \bigotimes_{k \in \mathbb{Z}_+} \mathbb{C}[a_i(-k)] \bigotimes_{k \in \mathbb{Z}_+} \mathbb{C}[a_i^*(-k)] \right)$$

where $a_i$ runs though any basis in $\mathcal{P}_{\bar{\sigma}}$ and $\mathcal{P}_\tau$, consisting of, say $\bar{\sigma}, \varepsilon_i$'s and $\delta_k$'s. The superalgebra $\mathcal{A}(\mathbb{Z}^{2n+1|2m})$ acts on the space by the usual action: $a(-k)$ act as creation operators and $a(k)$ as annihilation operators.
To construct \( T(B(m, n)) \), we need to extend the superalgebra \( \mathcal{A}(\mathbb{Z}^{2n+1|2m}) \) by adding a new \( e \) to the basis of the odd space \( \mathcal{C}_T \) and extending the bilinear form define above by \( \langle e, e \rangle = -2 \), \( \langle e, \mathcal{C} \rangle = 0 \). The larger superalgebra is denoted by \( \mathcal{A}(\mathbb{Z}^{2n+1|2m+1}) \) and its representation space is defined to be

\[
V(\mathbb{Z}^{n+1|m+1}) = V(\mathbb{Z}^{n+1|m}) \bigotimes_{k \in \mathbb{Z}_+} e(-k)
\]

with \( e(-k) \) (resp. \( e(k) \)) act as creation (resp. annihilation) operators and \( e(0) \) as scalar \( \sqrt{-1} \).

For any \( u \in \mathcal{C}_0 \cup \mathcal{C}_1 \), we define the following formal power series with coefficients coming from the superalgebra \( \mathcal{A}(\mathbb{Z}^{2n+1|2m}) \):

\[
u(z) = \sum_{k \in \mathbb{Z}} u(k) z^{-k-1}
\]

It is easy to see that \( u(z) \) is a field on \( V \) with the parity \( \pi(u) \).

**Proposition 4.1.** The basic operator product expansions are: for \( u, v \in \mathcal{C}_0 \cup \mathcal{C}_1 \), we have

\[
u(z) v(w) = \frac{\langle u, v \rangle}{z - w}
\]

In particular we have for \( a, b \in \mathcal{T}_0 \cup \mathcal{T}_T \)

\[
\begin{align*}
\langle a(z) b(w) \rangle &= \frac{\langle a, b^* \rangle}{z - w}, \\
\langle a(z) b^*(w) \rangle &= \frac{\langle a^*, b \rangle}{z - w}
\end{align*}
\]

\[
e(z) a(w) = e(z) a^*(w) = 0; \quad e(z) e(w) = \frac{-2}{z - w}
\]

**Proof.** One can prove the proposition similarly to Proposition 3.1 in [JMX].

**Proposition 4.2.** For \( u, v \in \mathcal{C}_0 \cup \mathcal{C}_1 \), we have

\[
[u(z), v(w)] = \langle u, v \rangle \delta(z - w),
\]

\[
[e(z), u(w)] = 0, \quad [e(z), e(w)] = -2 \delta(z - w)
\]

**Proof.** In fact, we have

\[
[u(z), v(w)] = u(z) v(w) - (-1)^{p(a)p(b)} v(w) u(z)
\]

\[
= \frac{\langle u, v \rangle}{z - w} - (-1)^{p(a)p(b)} \frac{\langle v, u \rangle}{w - z}
\]

\[
= \langle u, v \rangle \delta(z - w)
\]

where we have used the fact : \( u(z) v(w) := (-1)^{p(a)p(b)} : v(w) u(z) : \)
The following proposition can be proved by the Wick’s theorem.

**Proposition 4.3.** The commutators among normally ordered products obey the following rules:

1) If \( r_i \in \mathcal{C}_\mathbb{T}, 1 \leq i \leq 4 \), then

\[
\begin{align*}
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] &= \langle r_1, r_4 \rangle : r_2(z) r_3(z) : \delta(z - w) + \langle r_1, r_3 \rangle : r_2(z) r_4(z) : \delta(z - w) \\
&+ \langle r_2, r_3 \rangle : r_1(z) r_4(z) : \delta(z - w) + \langle r_2, r_4 \rangle : r_1(z) r_3(z) : \delta(z - w) \\
&+ \left( \langle r_1, r_4 \rangle \langle r_2, r_3 \rangle + \langle r_1, r_3 \rangle \langle r_2, r_4 \rangle \right) \partial w \delta(z - w)
\end{align*}
\]

2) If \( r_1, r_2 \in \mathcal{C}_\mathbb{T}, r_3 \in \mathcal{C}_\mathbb{T} \) and \( r_4 \in \mathcal{C}_\mathbb{T} \cup \{ e \} \), then

\[
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] = 0
\]

3) If \( r_1, r_3 \in \mathcal{C}_\mathbb{T} \) and \( r_2, r_4 \in \mathcal{C}_\mathbb{T} \cup \{ e \} \), then

\[
\begin{align*}
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] &= \langle r_1, r_4 \rangle : r_2(z) r_3(z) : \delta(z - w) - \langle r_1, r_3 \rangle : r_2(z) r_4(z) : \delta(z - w) \\
&+ \langle r_2, r_3 \rangle : r_1(z) r_4(z) : \delta(z - w) - \langle r_2, r_4 \rangle : r_1(z) r_3(z) : \delta(z - w) \\
&+ \left( \langle r_1, r_4 \rangle \langle r_2, r_3 \rangle - \langle r_1, r_3 \rangle \langle r_2, r_4 \rangle \right) \partial w \delta(z - w)
\end{align*}
\]

4) If \( r_1, r_3 \in \mathcal{C}_\mathbb{T} \) and \( r_2, r_4 \in \mathcal{C}_\mathbb{T} \cup \{ e \} \), then

\[
\begin{align*}
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] &= \langle r_1, r_3 \rangle : r_2(z) r_4(z) : \delta(z - w) + \langle r_2, r_4 \rangle : r_1(z) r_3(z) : \delta(z - w) \\
&+ \langle r_1, r_3 \rangle \langle r_2, r_4 \rangle \partial w \delta(z - w)
\end{align*}
\]

5) If \( r_1, r_2, r_3 \in \mathcal{C}_\mathbb{T} \) and \( r_4 \in \mathcal{C}_\mathbb{T} \cup \{ e \} \), then

\[
\begin{align*}
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] &= \langle r_1, r_3 \rangle : r_2(z) r_4(z) : \delta(z - w) + \langle r_2, r_3 \rangle : r_1(z) r_4(z) : \delta(z - w)
\end{align*}
\]

6) If \( r_1, r_2 \in \mathcal{C}_\mathbb{T} \) and \( r_3 \in \mathcal{C}_0, r_4 \in \mathcal{C}_\mathbb{T} \cup \{ e \} \), then

\[
\begin{align*}
\left[ : r_1(z) r_2(z) ; r_3(w) r_4(w) : \right] &= -\langle r_1, r_4 \rangle : r_2(z) r_3(z) : \delta(z - w) + \langle r_1, r_3 \rangle : r_2(z) r_4(z) : \delta(z - w) \\
&- \langle r_2, r_3 \rangle : r_1(z) r_4(z) : \delta(z - w) + \langle r_2, r_4 \rangle : r_1(z) r_3(z) : \delta(z - w) \\
&+ \left( \langle r_1, r_3 \rangle \langle r_2, r_4 \rangle - \langle r_1, r_4 \rangle \langle r_2, r_3 \rangle \right) \partial w \delta(z - w)
\end{align*}
\]
In the following we will define the field operators to each generating series of \( \mathfrak{T}(X) \).

(i) For type \( A(m, n) \), we define

\[
x^+_i(z) = \begin{cases} 
\sqrt{-1} \delta_{n+1}(z) \beta^*(z) ; & \text{if } i = 0; \\
\sqrt{-1} \epsilon_i(z) \epsilon^*_{i+1}(z) ; & \text{if } 1 \leq i \leq m; \\
\epsilon_{m+1}(z) \delta^*_i(z) ; & \text{if } i = m + 1; \\
\delta_{i-m-1}(z) \delta^*_i(z) ; & \text{if } m + 2 \leq i \leq m + n + 1.
\end{cases}
\]

\[
x^-_i(z) = \begin{cases} 
\sqrt{-1} \beta(z) \delta^*_{n+1}(z) ; & \text{if } i = 0; \\
\sqrt{-1} \epsilon_i(z) \epsilon^*_{i+1}(z) ; & \text{if } 1 \leq i \leq m; \\
\epsilon_{m+1}(z) \delta^*_i(z); & \text{if } i = m + 1 \\
\delta_{i-m-1}(z) \delta^*_i(z); & \text{if } m + 2 \leq i \leq m + n + 1.
\end{cases}
\]

\[
\alpha_i(z) = \begin{cases} 
\epsilon_i(z) \epsilon^*_{i+1}(z) ; \quad & \text{if } i = 0; \\
\epsilon_{m+1}(z) \epsilon^*_{m+1}(z) ; \quad & \text{if } 1 \leq i \leq m; \\
\delta_1(z) \delta^*_1(z) ; \quad & \text{if } i = m + 1 \\
\delta_{i-m-1}(z) \delta^*_i(z) ; \quad & \text{if } m + 2 \leq i \leq m + n + 1.
\end{cases}
\]

(ii) For type \( B(0, n) \), we define

\[
x^+_i(z) = \begin{cases} 
\frac{1}{2} \beta^*(z) \beta^*(z) ; & \text{if } i = 0; \\
\epsilon_i(z) \epsilon^*_{i+1}(z) ; & \text{if } 1 \leq i \leq n - 1; \\
\epsilon_n(z) e(z) ; & \text{if } i = n.
\end{cases}
\]

\[
x^-_i(z) = \begin{cases} 
\frac{1}{2} \beta(z) \beta(z) ; & \text{if } i = 0; \\
- \epsilon_{i+1}(z) \epsilon^*_i(z) ; & \text{if } 1 \leq i \leq n - 1; \\
\epsilon^*_n(z) e(z) ; & \text{if } i = n.
\end{cases}
\]

\[
\alpha_i(z) = \begin{cases} 
-2 \beta(z) \beta^*(z) ; & \text{if } i = 0; \\
\epsilon_i(z) \epsilon^*_i(z) ; \quad & \text{if } 1 \leq i \leq n - 1; \\
\epsilon_n(z) e(z) ; & \text{if } i = n.
\end{cases}
\]

(iii) For type \( B(m, n) \) \((m \geq 1)\), the \( x^+_i(z), \alpha_i(z) \) \((0 \leq i \leq n - 1)\) is same as those of type \( B(0, n) \) and the others is defined as:

\[
x^+_i(z) = \begin{cases} 
\epsilon_n(z) \delta^*_i(z) ; & \text{if } i = n; \\
\delta_{i-n}(z) \delta^*_{i-n+1}(z) ; \quad & \text{if } n + 1 \leq i \leq n + m - 1; \\
\delta_m(z) e(z) ; & \text{if } i = n + m.
\end{cases}
\]

\[
x^-_i(z) = \begin{cases} 
\delta_1(z) \epsilon^*_n(z) ; & \text{if } i = n; \\
\delta^*_{i-n}(z) \delta_{i-n+1}(z) ; \quad & \text{if } n + 1 \leq i \leq n + m - 1; \\
\delta^*_m(z) e(z) ; & \text{if } i = n + m.
\end{cases}
\]
\[ \alpha_i(z) = \begin{cases} 
\varepsilon_{n}(z)\varepsilon_{n}^{\ast}(z) : & \text{if } i = n; \\
\delta_{i}(z)\delta_{i}^{\ast}(z) : & \text{if } i = 1; \\
\delta_{i-n}(z)\delta_{i-n}^{\ast}(z) : & \text{if } n + 1 \leq i \leq n + m - 1; \\
\delta_{m}(z)\delta_{m}^{\ast}(z) : & \text{if } i = n. 
\end{cases} \]

(iv) For type \( D(m, n) \) \( (m \geq 1) \), the \( x_{i}^{\pm}(z) \), \( \alpha_{i}(z) \) \( (0 \leq i \leq m + n - 1) \) is same as those of type \( B(m, n) \) and the only difference is the last one:

\[ x_{m+n}(z) = \delta_{m-1}(z)\delta_{m}(z); \quad x_{m+n}^{-}(z) = \delta_{m-1}^{\ast}(z)\delta_{m}^{\ast}(z); \]

\[ \alpha_{m+n}(z) = \delta_{m-1}(z)\delta_{m}^{\ast}(z) + \delta_{m}(z)\delta_{m}^{\ast}(z). \]

(v) For type \( C(n) \), we define

\[ x_{i}^{\pm}(z) = \begin{cases} 
\beta^{\ast}(z)\varepsilon_{1}^{\ast}(z) : & \text{if } i = 0; \\
\varepsilon_{1}(z)\delta_{1}^{\ast}(z) : & \text{if } i = 1 \\
\sqrt{-1}: \delta_{i-1}(z)\delta_{i}^{\ast}(z) : & \text{if } 2 \leq i \leq n; \\
\frac{1}{2}: \delta_{n}(z)\delta_{n}(z) : & \text{if } i = n + 1. 
\end{cases} \]

\[ x_{i}^{-}(z) = \begin{cases} 
\beta(z)\varepsilon_{1} : & \text{if } i = 0; \\
\delta_{i}(z)\varepsilon_{1}^{\ast}(z) : & \text{if } i = 1 \\
\sqrt{-1}: \delta_{i}(z)\delta_{i}^{\ast}(z) : & \text{if } 2 \leq i \leq n; \\
\frac{1}{2}: \delta_{n}(z)\delta_{n}^{\ast}(z) : & \text{if } i = n + 1. 
\end{cases} \]

\[ \alpha_{i}(z) = \begin{cases} 
\varepsilon(z_{1})\varepsilon_{1}^{\ast}(z) : - \beta(z)\beta^{\ast}(z) : & \text{if } i = 0; \\
\delta_{i}(z)\varepsilon_{1}^{\ast}(z) : - \delta_{1}(z)\delta_{1}^{\ast}(z) : & \text{if } i = 1; \\
\delta_{i-1}(z)\delta_{i}^{\ast}(z) : - \delta_{i}(z)\delta_{i}^{\ast}(z) : & \text{if } 2 \leq i \leq n; \\
2: \delta_{n}(z)\delta_{n}(z) : & \text{if } i = n + 1. 
\end{cases} \]

**Theorem 4.4.** The field operators defined above give rise to a representation of level \(-1\) of \( \Xi(X) \) for type \( A(m, n) \), \( B(m, n) \) and \( D(m, n) \), of level \( 1 \) for type \( C(n) \) respectively.

**Proof.** To prove the theorem, we need to check the field operators defined above satisfy all the relations listed in proposition 3.4 for each case.

i) Type \( A(m, n) \)

\[ [x_{0}^{\pm}(z), x_{0}^{-}(w)] = -[\delta_{n+1}(z)\beta^{\ast}(z); \beta(z)\delta_{n+1}^{\ast}(z) :] \]

\[ = \left( \delta_{n+1}(z)\delta_{n+1}^{\ast}(z) : - \beta(z)\beta^{\ast}(z) : \right)\delta(z - w) + \partial_{w}\delta(z - w) \]

\[ = -\left( \alpha_{0}(z)\delta(z - w) + (-1)\partial_{w}\delta(z - w) \right) \]

\[ [\alpha_{0}(z), x_{0}^{\pm}(w)] = 0 = \pm(\alpha_{0}, \alpha_{0})x_{0}^{\pm}(w)\delta(z - w). \]
For $1 \leq i \leq m$, we have

$$[x_i^+(z), x_i^-(w)] = [\varepsilon_i(z) \varepsilon_{i+1}^*(z) :: \varepsilon_i(z) \varepsilon_i^*(z) :]$$

$$= -\left( [\varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_i(z) \varepsilon_{i+1}^*(z) :] \delta(z - w) + \partial_w \delta(z - w) \right)$$

$$= -\frac{2}{(\alpha_i, \alpha_i)} (\alpha_i(z) \delta(z - w) + (-1) \partial_w \delta(z - w) \cdot)$$

$$[\alpha_i(z), x_i^+(w)] = \sqrt{-1} [\varepsilon_i(z) \varepsilon_i^*(z) : - : \varepsilon_i(z) \varepsilon_i^*(z) :]$$

$$= 2\sqrt{-1} [\varepsilon_i(z) \varepsilon_i^*(z) : \delta(z - w)$$

$$= (\alpha_i, \alpha_i) x_i^+(w) \delta(z - w)$$

$$[\alpha_i(z), x_i^-(w)] = - (\alpha_i, \alpha_i) x_i^-(w) \delta(z - w)$$

For $m + 2 \leq i \leq m + n + 1$, we have

$$[x_i^+(z), x_i^-(w)] = [\delta_{i-m}^*(z) \delta_{i-m}^*(z) :: \delta_{i-m}^*(z) \delta_{i-m}^*(z) :]$$

$$= -\left( [\delta_{i-m}^*(z) \delta_{i-m}^*(z) : - : \delta_{i-m}^*(z) \delta_{i-m}^*(z) :] \delta(z - w) + \partial_w \delta(z - w) \right)$$

$$= -\alpha_i(z) \delta(z - w) + (-1) \partial_w \delta(z - w) \cdot)$$

$$[\alpha_i(z), x_i^+(w)] = [\delta_{i-m}^*(z) \delta_{i-m}^*(z) : - : \delta_{i-m}^*(z) \delta_{i-m}^*(z) :]$$

$$= -2 [\delta_{i-m}^*(z) \delta_{i-m}^*(z) : \delta(z - w) = (\alpha_i, \alpha_i) x_i^+(w) \delta(z - w)$$

$$[\alpha_i(z), x_i^-(w)] = - (\alpha_i, \alpha_i) x_i^-(w) \delta(z - w)$$

It is straightforward to check for all $i \neq j$ that $[x_i^+(z), x_j^-(w)] = 0$ and

$$[\alpha_i(z), \alpha_j(w)] = (\alpha_i, \alpha_j) \partial_w \delta(z - w) \cdot (-1)$$

for example, we have

$$[\alpha_i(z), \alpha_i+1(w)] = -[\varepsilon_i(z) \varepsilon_i^*(z) :: \varepsilon_i(z) \varepsilon_i^*(z) :]$$

$$= \partial_w \delta(z - w) = (\alpha_i, \alpha_i) \partial_w \delta(z - w) \cdot (-1)$$
We proceed to check the Serre relations

$$[x^+_n(z_1), [x^+_n(z_2), x^+_n(w)]]$$

$$= -\sqrt{-1}[: \delta_{n+1}(z_1)\beta^*(z_1) ; : \delta_{n+1}(z_2)\beta^*(z_2) ; : \varepsilon_1(w)\varepsilon_2^*(w) :]$$

$$= -\sqrt{-1}[: \delta_{n+1}(z_1)\beta^*(z_1) ; : \delta_{n+1}(w)\varepsilon_2^*(w) :]\delta(z_2 - w)$$

$$= 0$$

all the others relations are proved similarly.

ii) The construction for type $B(0, n)$ is almost the same to that presented in [JX] and the only difference is that we let $(e, e) = -2$ while it is 1 in [JX].

iii) Since the field operators for type $B(m, n)$ ($m \geq 1$) is the same as type $B(0, n)$ for $0 \leq i \leq n - 1$, we only to check field operators for $n \leq i \leq n + m$ satify the relations. First, we have by Proposition 4.3 4) that

$$[x^+_n(z), x^-_n(w)] = [: \varepsilon_n(z)\delta^*_n(z) ; : \varepsilon_n(w)\delta^*_1(w) :]$$

$$= - (\varepsilon_n(z)\delta^*_n(z) - \delta_1(z)\delta^*_1(z)\delta(z - w) + \partial_w\delta(z - w))$$

and by Proposition 4.3 5) and 6)

$$[\alpha_n(z), x^+_n(w)] = [\varepsilon_n(z)\delta^*_n(z) - \delta_1(z)\delta^*_1(z) ; : \varepsilon_n(w)\delta^*_1(w) :] = 0 = (\alpha_n, \alpha_n), x^+_n(w)\delta(z - w)$$

and

For $n + 1 \leq i \leq m + n - 1$, we have by proposition 4.3 3) that

$$[x^+_i(z), x^+_i(w)] = [: \delta_{i-n}(z)\delta^*_n(z) ; : \delta_{i-n}(w)\delta^*_n(w) :]$$

$$= (\delta_{i-n}(z)\delta^*_n(z) - \delta_{i-n}(w)\delta^*_n(w) : \delta(z - w) + (-1) \cdot \partial_w\delta(z - w))$$

$$[\alpha_i(z), x^+_i(w)] = [\delta_{i-n}(z)\delta^*_n(z) - \delta_{i-n}(w)\delta^*_n(w) :] = 0 = (\alpha_i, \alpha_i), x^+_i(w)\delta(z - w)$$

$$[x^+_i(z), x^-_i(w)] = [\delta_{i-n}(z)\delta^*_n(z) - \delta_{i-n}(w)\delta^*_n(w) :] = 0 = (\alpha_i, \alpha_i), x^-_i(w)\delta(z - w)$$

$$[x^+_{n+m}(z), x^-_{n+m}(w)] = [: \delta_m(z)e(z) ; : \delta^*_m(w)e(w) :]$$

$$= 2 : \delta_m(z)\delta^*_m(z) \delta(z - w) - 2\partial_w\delta(z - w)$$

$$= -\frac{2}{(\alpha_{n+m}, \alpha_{n+m})}(\alpha_{n+m}(z)\delta(z - w) + (-1) \cdot \partial_w\delta(z - w))$$
Finally, we check that 
\[ u(z) = 0. \]

\[ [\alpha_{n+m}(z), x_{n+m}^+(w)] = \left[ \delta_m(z) \delta_n^*(z) ; : \delta_m(z)e(z) : \right] = -\partial_w \delta(z-w) = (\alpha_{n+m}, \alpha_{n+m})x_{n+m}^+(w)\delta(z-w) \]

\[ [\alpha_{n+m}(z), x_{n+m}^-(w)] = -(\alpha_{n+m}, \alpha_{n+m})x_{n+m}^-(w)\delta(z-w) \]

One can check that \( [x_i^+(z), x_j^-(w)] = 0 \) for \( i \neq j \) one by one using Proposition 4.3.

In what follows, we check those relations concerning on \( \alpha_i(z) \) \( n \leq i \leq n+m \). First, one has

\[ [\alpha_{i-1}(z), \alpha_i(w)] = -[:, \varepsilon_n(z) \varepsilon_{n+i}(z) ; : \varepsilon_n(w) \varepsilon_{n+i}(w) :] = \partial_w \delta(z-w) = (\alpha_{i-1}, \alpha_i) \partial_w \delta(z-w) \cdot (-1) \]

Then we have, for \( n \leq i \leq n+m-1 \), that

\[ [\alpha_i(z), \alpha_{i+1}(w)] = -[:, \delta_{i-n+1}(z) \delta_{i-n+1}^*(z) ; : \delta_{i-n+1}(w) \delta_{i-n+1}^*(w) :] = -\partial_w \delta(z-w) = (\alpha_i, \alpha_{i+1}) \partial_w \delta(z-w) \cdot (-1) \]

Next, we proceed to check the Serre type relations. First of all, we have \( [x_i^+(z), x_i^-(w)] = 0 \) for \( 0 \leq i \leq m+n \) and \( [x_i^-(z), x_i^+(w)] = 0 \) for \( 0 \leq i \leq m+n, i \neq n-1, n+1 \). Then we check that

\[ [x_i^+(z_1), [x_i^+(z_1), x_{i-1}^+(w)]] = -[:, \varepsilon_n(z_1) \delta_{i-n}^*(z_1) ; : \delta_{i-n}^*(w) \varepsilon_n(z_1) :] \delta(z_2-w) = 0 \]

\[ [x_{i-1}^+(z_1), [x_{i-1}^+(z_1), x_i^+(w)]] = -[:, \varepsilon_n-1(z_1) \delta_{i-n+1}^*(z_1) ; : \delta_{i-n+1}^*(w) \varepsilon_n-1(z_1) :] \delta(z_2-w) = 0 \]

Similarly, we have for \( n \leq i \leq n+m-2 \)

\[ [x_i^+(z_1), [x_i^+(z_1), x_{i+1}^+(w)]] = [x_i^+(z_1), [x_i^+(z_1), x_i^+(w)]] = 0 \]

Finally, we check that

\[ [x_{i+n}^+(z_1), [x_{i+n}^+(z_1), [x_{i+n}^+(z_1), x_{i+n}^+(w)]]] = -[:, \delta_m(z_1) \varepsilon(z_1) ; : \delta_m(z_2) \varepsilon(z_2) ; : \varepsilon(w) \delta_{m+n-1}(w) :] \delta(z_3-w) = 0 \]

\[ = -2[:, \delta_m(z_1) \varepsilon(z_1) ; : \delta_m(w) \delta_{m-1}(w) :] \delta(z_2-w) \delta(z_3-w) = 0 \]

and \( [x_{i+n}^+(z_1), [x_{i+n}^+(z_2), x_{i+n}^+(w)]] = 0. \)
iv) For type $D(m, n)$, note that the difference between $B(m, n)$, then it is sufficient to check that

\[
[x_{n+m}^+, x_{n+m}^-] = [\delta_{m-1}(z) \delta_m(z) ; ; \delta_m^*(w) \delta_m^*(w)]
\]

\[
= -\left( : \delta_{m-1}(z) \delta_m^*(z) ; ; \delta_m(z) \delta_m^*(z) : \right) \delta(z - w) - \partial_w \delta(z - w)
\]

\[
= -\frac{2}{(\alpha_{n+m}, \alpha_{n+m})} \left( \alpha_{n+m}(z) \delta(z - w) + (-1) \cdot \partial_w \delta(z - w) \right)
\]

where we have used the fact if $u(z)$ is an odd field, then $: u(z)u(z) := 0$.

\[
[\alpha_{n+m}(z), x_{n+m}^+(w)] = [\delta_{m-1}(z) \delta_m^*(z) ; ; \delta_m(z) \delta_m^*(z) ; ; \delta_{m-1}(z) \delta_m(z) ; ; \delta_m(z) \delta_m^*(z) ; ; \delta_{m-1}(z) \delta_m(z) :]
\]

\[
= -2 : \delta_{m-1}(z) \delta_m(z) : \delta(z - w) = (\alpha_{n+m}, \alpha_{n+m}) x_{n+m}^+(w) \delta(z - w)
\]

\[
[x_{n+m}^+(z), x_{n+m}^-(w)] = -(\alpha_{n+m}, \alpha_{n+m}) x_{n+m}^-(w) \delta(z - w)
\]

\[
[x_{n+m}^-(z), x_{n+m}^+(w)] = [\alpha_{n+m}(z), x_{n+m}^-(w)] = 0, 0 \leq i \leq n + m - 1
\]

\[
[\alpha_{n+m-2}(z), \alpha_{n+m}(w)] = -[\delta_{m-1}(z) \delta_m^*(z) ; ; \delta_m(z) \delta_m^*(z) ; ; \delta_{m-1}(z) \delta_m(z) ; ; \delta_m(z) \delta_m^*(z) ; ; \delta_{m-1}(z) \delta_m(z) :]
\]

\[
= \partial_w \delta(z - w) = (\alpha_{n+m-2}, \alpha_{n+m}) \partial_w \delta(z - w) \cdot (-1)
\]

\[
[\alpha_i(z), \alpha_{n+m}(w)] = 0, i \neq n + m - 2
\]

The Serre relations need checking here are those concerning $x_{m+n}^\pm(z)$ and $x_{m+n-2}^\pm(z)$, for example

\[
[x_{m+n}^+(z_1), x_{m+n}^+(z_2), x_{m+n-2}^+(w)]
\]

\[
= -[\delta_{m-1}(z_1) \delta_m(z_1) ; ; \delta_m(z_2) \delta_m(z_2) ; ; \delta_{m-2} \delta_{m-2} ; ; \delta_{m-2} \delta_{m-2} : \delta(z_2 - w) = 0
\]

all others can be proved similarly.

v) For type $C(n)$, we first check relations $3')$ and $4')$ in Proposition 4.3

\[
[x_0^+(z), x_0^-(w)] = [\beta^*(z) \varepsilon_1^*(z) ; ; \beta(z) \varepsilon_1(z) ;]
\]

\[
= -\left( : \varepsilon(z_1) \varepsilon_1^*(z) ; ; \beta(z) \beta^*(z) : \right) \delta(z - w) + 1 \cdot \partial_w \delta(z - w)
\]

\[
= -(\alpha_0(z) \delta(z - w) + 1 \cdot \partial_w \delta(z - w))
\]

\[
[x_0^+(z), x_0^-(w)] = 0 = (\alpha_0, \alpha_0) x_0^+(w) \delta(z - w)
\]

\[
[x_1^+(z), x_1^-(w)] = [\varepsilon_1(z) \delta_1^*(z) ; ; \varepsilon_1(z) \delta_1(z) ;]
\]

\[
= -\left( \varepsilon(z_1) \varepsilon_1^*(z) ; ; \delta_1(z) \delta_1^*(z) : \right) \delta(z - w) + 1 \cdot \partial_w \delta(z - w)
\]

\[
= -(\alpha_1(z) \delta(z - w) + 1 \cdot \partial_w \delta(z - w))
\]

\[
[x_1^+(z), x_1^-(w)] = 0 = (\alpha_1, \alpha_1) x_1^+(w) \delta(z - w)
\]
\[ [x_i^+(z), x_i^-(w)] = -\delta_\alpha (z) \delta_i^+ (z) : : \delta_i(z) \delta_i^+(z) : ]
\]
\[ = -\frac{2}{(\alpha_i, \alpha_i)} (\alpha_i(z) \delta(z - w) + 1 \cdot \partial_w \delta(z - w)) \]
\[ 2 \leq i \leq n. \]

\[ [\alpha_i(z), x_i^+(w)] = \sqrt{-1} \delta_i(z) \delta_i^+(z) : : \delta_i(z) \delta_i^+(z) : ]
\]
\[ = -2\sqrt{-1} \delta_i(z) \delta_i^+(z) : \delta(z - w) \]
\[ = (\alpha_i, \alpha_i) x_i^+(w) \delta(z - w) \]
\[ [\alpha_i(z), x_i^-(w)] = -(\alpha_i, \alpha_i) x_i^-(w) \delta(z - w) \]

\[ [x_{n+1}^+(z), x_{n+1}^-(w)] = \frac{1}{4} \delta(z) \delta_n(z) : : \delta_n(z) \delta_n(z) : ]
\]
\[ = \frac{2}{4} (2 \delta(z) \delta_n^+(z) : \delta(z - w) + 1 \cdot \partial_w \delta(z - w)) \]
\[ = -\frac{2}{(\alpha_{n+1}, \alpha_{n+1})} (\alpha_{n+1} z - w + 1 \cdot \partial_w \delta(z - w)) \]

\[ [\alpha_{n+1}(z), x_{n+1}^+(w)] = [\delta(z) \delta_{n+1}^+(z) : : \delta(z) \delta_n(z) : ]
\]
\[ = -2 \delta(z) \delta_{n+1}^+(z) : \delta(z - w) = (\alpha_{n+1}, \alpha_{n+1}) x_{n+1}^+(w) \delta(z - w) \]
\[ [\alpha_{n+1}(z), x_{n+1}^-(w)] = -(\alpha_{n+1}, \alpha_{n+1}) x_{n+1}^-(w) \delta(z - w) \]
\[ [x_i^+(z), x_j^-(w)] = 0, \quad \text{for } i \neq j \]

Next, we will check the relation 3\(\ddagger\) by straightforward calculus
\[ [\alpha_0(z), \alpha_0(w)] = [\alpha_1(z), \alpha_1(w)] = 0 \]

\[ [\alpha_i(z), \alpha_i(w)] = [\delta_\alpha z - 1(z) \delta_i^+(z) : : \delta_i(z) \delta_i^+(z) : ]
\]
\[ + [\delta_\alpha z - 1(w) \delta_i^+(w) : : \delta_i(z) \delta_i^+(z) : ] \quad 2 \leq i \leq n \]
\[ = -2 \partial_w \delta(z - w) = (\alpha_i, \alpha_i) \partial_w \delta(z - w) \]
\[ [\alpha_{n+1}(z), \alpha_{n+1}(w)] = 4 [\delta(z) \delta^+_n(z) : : \delta(z) \delta^+_n(z) : ]
\]
\[ = -4 \partial_w \delta(z - w) = (\alpha_{n+1}, \alpha_{n+1}) \partial_w \delta(z - w) \]

all others are proved similarly.

Finally, we proceed to check the Serre type relations as follows:
\[ [x_i^+(z), x_i^+(w)] = 0, \quad 0 \leq i \leq n + 1 \]
\[ [x_0^+(z), x_i^+(w)] = [x_i^+(z), x_i^+(w)] = 0, \quad 3 \leq i \leq n + 1 \]
\[ [x_0^+(z_1), [x_0^+(z_2), x_i^+(z_3)]] = [\beta^*(z_1) \epsilon_i^*(z_1) : : \beta^*(z) \epsilon_i^*(z) : ] \delta(z_2 - w) = 0 \]
\[
[x_n^+(z_1), [x_n^+(z_2), [x_n^+(z_3), x_{n+1}^+(z_w)]]]
= \sqrt{-1} : \delta_{n-1}(z_1) \delta_n^*(z_1) ; : \delta_{n-1}(z_2) \delta_n^*(z_2) ; : \delta_{n-1}(w) \delta_n(w) :: \delta(z_3 - w)
= -\sqrt{-1} : \delta_{n-1}(z_1) \delta_n^*(z_1) ; : \delta_{n-1}(w) \delta_{n-1}(w) :: \delta(z_2 - w) \delta(z_3 - w)
= 0
\]

all others are proved similarly and this completes the proof. \(\square\)

5. Appendix

In this appendix, we list the extended distinguished Cartan matrix of affine superalgebra of type \(A, B, C, D\) for convenience.

1) Type \(A(m, n)\)

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
-1 & 2 & -1 & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & \ddots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & \cdots & 0 & 0 & -1 & 0 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & -1 & 2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 2 & -1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-1 & \cdots & \cdots & \cdots & 0 & -1 & 2 \\
\end{pmatrix}
\]

2) Type \(B(0, n)\) \((n \geq 1)\)

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
. & . & . & \cdots & . & . & . \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2 \\
\end{pmatrix}
\]
3) type $B(m,n)^{(1)} (m \geq 1, n \geq 1)$

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-2 & 2 & -1 & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & \ddots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & -1 & 0 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & -1 & 2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & 2 & -1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 & -1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -2 & 2 \\
\end{pmatrix}$$

4) type $C(n)^{(1)} (n \geq 1)$

$$\begin{pmatrix}
0 & -2 & -1 & \cdots & 0 \\
-2 & 0 & -1 & \cdots & 0 \\
-1 & -1 & 2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & 2 & -1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & -1 & 2 & -2 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 \\
\end{pmatrix}$$

5) type $D(m,n)^{(1)} (n \geq 1)$

$$\begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-2 & 2 & -1 & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & \ddots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & -1 & 0 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & -1 & 2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & 2 & -1 & -1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & 2 \\
\end{pmatrix}$$
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