Smoothed Particle Hydrodynamics Simulations of Ultra-Relativistic Shocks with Artificial Viscosity

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ABSTRACT

We present a fully Lagrangian conservation form of the general relativistic hydrodynamic equations for perfect fluids with artificial viscosity in a given arbitrary background spacetime. This conservation formulation is achieved by choosing suitable Lagrangian time evolution variables, from which the generic fluid variables rest-mass density, 3-velocity, and thermodynamic pressure have to be determined. We present the corresponding equations for an ideal gas and show the existence and uniqueness of the solution. On the basis of the Lagrangian formulation we have developed a three-dimensional general relativistic Smoothed Particle Hydrodynamics (SPH) code using the standard SPH formalism as known from non-relativistic fluid dynamics. One-dimensional simulations of a shock tube and a wall shock are presented. With our method we can model ultra-relativistic fluid flows including shocks with relativistic $\gamma$-factors of even 1000.

Subject headings: hydrodynamics — methods: numerical — relativity — shock waves

1. INTRODUCTION

Modeling ultra-relativistic fluid flows is a great challenge for any relativistic hydro code. Numerical difficulties arise from strong relativistic shocks and from narrow physical structures (Norman & Winkler 1986). Typical examples in astrophysics for such extreme conditions are proto-stellar jets and blast waves of supernovae explosions. In recent years, the development of numerical algorithms for relativistic fluid dynamics went mainly along two different lines. First, there are the so called High Resolution Shock Capturing (HRSC) methods, which allow to obtain numerically discontinuous solutions of the relativistic hydrodynamic equations by solving local Riemann problems between adjacent numerical cells. Some recently developed HRSC techniques are those of Font et al. (1994), Falle & Komissarov (1996), Romero et al. (1996), Banyuls et al. (1997), Wen, Panaitescu, & Laguna (1997), Pons et al. (1998), and Komissarov (1999). However, employing analytic solutions of the Riemann shock tube problem, it is not surprising that these HRSC codes produce almost exact numerical solutions of flow structures including discontinuities. In the second type of algorithms, shocks are
treated numerically not as fluid discontinuities, but are rather spread over some small length with the help of an artificial viscosity. These algorithms either solve the dynamical equations of relativistic hydrodynamics on an Eulerian grid such as the finite difference schemes of Hawley, Smarr, & Wilson (1984b) and Norman & Winkler (1986) or by using computational nodes following the fluid motion such as the Lagrangian Smoothed Particle Hydrodynamics (SPH) methods of Kheyfets, Miller, & Zurek (1990) and Laguna, Miller, & Zurek (1993). All the latter methods seem to be limited even for mildly relativistic flows containing shocks. Norman & Winkler (1986) suggest that the appearance of numerical inaccuracies and instabilities is due to the time derivative of the relativistic $\gamma$-factor in the energy equation of these numerical schemes. This additional time derivative of a hydrodynamic variable renders the system of evolution equations non-conservative. For non-relativistic hydrodynamics, it is well-known that a numerical method based on non-conservative equations can produce a solution which looks reasonable but is entirely wrong if shocks or other discontinuities are involved (see LeVeque 1997 for a corresponding analysis of Burger's equation).

The main intention of this paper is to present a conservation formulation of the relativistic hydrodynamic equations that is designed for numerical methods. This particular formulation allows hydro codes to resolve ultra-relativistic shocks numerically with relativistic $\gamma$-factors of even 1000 by means of an artificial viscosity rather than using a Riemann solver. The conservation form of the relativistic equations is obtained by choosing suitable Lagrangian variables. Unfortunately, these Lagrangian dynamical variables can be expressed in terms of the generic fluid variables rest-mass density, 3-velocity, and thermodynamic pressure only through a set of nonlinear algebraic equations. However, we show that these equations can be solved analytically in a unique way.

For numerical work we discretize our set of partial differential equations by the Smoothed Particle Hydrodynamics (SPH) method introduced by Lucy (1977) and Gingold & Monaghan (1977). In recent years, SPH became a popular computational tool for numerically modeling complex three-dimensional fluid flows in astrophysics. This is primarily due to its computational simplicity and the absence of a computational grid. Furthermore, SPH is adaptive in the sense that its computational nodes follow the fluid.

SPH has been already applied to relativistic fluid flows. Kheyfets et al. (1990) developed a relativistically covariant version of the SPH technique by modeling the contact interactions with spatial smoothing functions in the local comoving frame of the fluid. Laguna et al. (1993) applied the SPH method to the so called ADM formalism of general relativity due to Arnowitt, Deser, & Misner (1962). In their relativistic SPH formulation they modified the flat space kernels of the Newtonian SPH method which can become anisotropic and are then no longer invariant under translations which leads to additional terms in the SPH equations. In contrast to these methods, we have developed a fully three-dimensional general relativistic SPH code on the basis of our set of Lagrangian conservation equations. This code employs the standard SPH approach as used in Newtonian theory with spherically symmetric kernels for all particles. Our code is restricted to ideal fluids with artificial viscosity using the ideal-gas equation of state. The influence of the fluid on the spacetime metric is neglected,
therefore, we consider only a background spacetime with a given but otherwise arbitrary metric. In addition, we neglect the fluid’s self-gravity and do not account for radiation or electromagnetic effects.

The layout of this paper is as follows. In section 2 we derive the formulation of the relativistic hydrodynamic equations in Lagrangian conservative form and present the equations for calculating the generic fluid properties rest-mass density, 3-velocity, and thermodynamic pressure from our suitably chosen Lagrangian variables. Section 3 gives a review of the standard SPH method that we use, including a prescription of its application to relativistic fluid flows in curved spacetimes and the implementation of an artificial viscosity. Numerical test calculations of one-dimensional ultra-relativistic flows with discontinuities are presented in section 4. Throughout this paper we set the speed of light $c$ and the Boltzmann’s constant $k$ to unity, i.e., $c = k = 1$. Latin indices $\{i, j, \ldots \}$ run from 1 to 3, Greek indices $\{\mu, \nu, \ldots \}$ from 0 to 3; the signature of the metric is $(-, +, +, +)$.

2. THE LAGRANGIAN CONSERVATION FORMULATION OF THE GENERAL RELATIVISTIC IDEAL FLUID EQUATIONS

Our starting point is the covariant formulation of the equations of relativistic hydrodynamics for a perfect fluid with artificial viscosity: the local conservation of baryon number

$$(pu^\mu)_{;\mu} = 0$$

and the local conservation of energy-momentum

$$T^{\mu\nu}_{\ ;\nu} = 0$$

with the stress-energy tensor of a perfect fluid

$$T^{\mu\nu} = (\rho w + q)u^\mu u^\nu + (p + q)g^{\mu\nu}.$$  

The semicolon denotes the covariant derivative and the Einstein summation convention is used. Here, $\rho$ is the rest-mass density, $u^\mu$ the 4-velocity of the fluid, $p$ the isotropic thermodynamic pressure, $w = 1 + \varepsilon + p/\rho$ the relativistic specific enthalpy with specific internal energy $\varepsilon$, $q$ the artificial viscous pressure, and $g^{\mu\nu}$ the spacetime metric. All thermodynamic quantities in the stress-energy tensor are measured in the local rest frame of the fluid. The artificial viscous pressure $q$, which appears in equation (3), can be introduced into the stress-energy tensor either by the substitution $p \rightarrow p + q$ or, equivalently, from the stress-energy tensor of a viscous fluid (Misner, Thorne, & Wheeler 1973; Landau & Lifshitz 1991) by ignoring the shear viscosity and heat conduction and replacing the bulk viscous pressure by $q$. An explicit expression for $q$ in terms of velocity gradients will be given in the subsequent section.

For a Lagrangian formulation of relativistic hydrodynamics suitable for SPH one has to break the unity of time and space inherent in the covariant formulation. This can be
accomplished by applying the ADM formalism of Arnowitt et al. (1962), where the spacetime is decomposed into an infinite foliation of spatial hypersurfaces \( \Sigma_t \) of constant coordinate time \( t \) by writing the line element as

\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = -(\alpha^2 - \beta^i \beta_i) \, dt^2 + 2\beta_i dx^i + \eta_{ij} dx^i dx^j .
\]

Here, \( \alpha \) is the lapse function, \( \beta^i \) the shift vector, and \( \eta_{ij} \) the spatial metric induced on \( \Sigma_t \) with \( \beta_i = \eta_{ij} \beta^j \) and \( \eta_{il} \eta^{lj} = \delta^j_i \). In this paper we consider only given background spacetimes, i.e., we do not solve the Einstein equations. Thus, \( g_{\mu\nu}, \alpha, \beta^i, \) and \( \eta_{ij} \) are given analytic functions of both space and time. From the definitions of \( \alpha \) and \( \beta^i \) the basis vector field \( \partial_t \) of the coordinate basis \( \{ \partial_t, \partial_i \} \) can be decomposed into normal and parallel components with respect to the hypersurfaces \( \Sigma_t \)

\[
\partial_t = \alpha n + \beta^i \partial_i , \tag{4}
\]

where \( n \) is the unit time-like vector field normal to the slices \( \Sigma_t \), i.e., \( n \cdot \partial_t = 0 \). Observers having \( n \) as 4-velocity are at rest in the slices \( \Sigma_t \) — they are called Eulerian observers. In the basis \( \{ n, \partial_i \} \) the 4-velocity of a fluid has the representation

\[
u = \gamma \left( n + \bar{v}^i \partial_i \right) ,
\]

whereas in the coordinate basis \( \{ \partial_t, \partial_i \} \) it follows from equation (4)

\[
u = u^\mu \partial_\mu = \frac{\gamma}{\alpha} \left( \partial_t + v^i \partial_i \right)
\]

with

\[
 v^i = \alpha \bar{v}^i - \beta^i .
\]

From the normalization condition of the 4-velocity \( u^\mu u_\mu = -1 \) the relativistic \( \gamma \)-factor is given by

\[
\gamma = \frac{1}{\sqrt{1 - \eta_{ij} \bar{v}^i \bar{v}^j}} .
\]

In the following, we use both, the 3-velocity of the fluid \( \bar{v}^i \) measured by Eulerian observers and the 3-velocity \( v^i \) in the coordinate basis.

We now derive the Lagrangian equations of relativistic hydrodynamics where the Lagrangian or total time derivative \( d/dt \) is defined as

\[
\frac{d}{dt} = \frac{\alpha}{\gamma} u^\mu \partial_\mu = \partial_t + v^i \partial_i .
\]

From the law of baryon-number conservation \( (1) \) we obtain

\[
0 = \partial_\mu \left( \sqrt{-g} \rho u^\mu \right) = \partial_t \left( \sqrt{-g} \rho \frac{\bar{v}^i}{\alpha} \right) + \partial_i \left( \sqrt{-g} \rho \frac{\bar{v}^i}{\alpha} v^i \right)
= \frac{dD^*}{dt} + D^* \partial_i v^i , \tag{5}
\]
where $g$ is the determinant of the spacetime metric $g^{\mu\nu}$, and the relativistic rest-mass density $D^*$ is defined by

$$D^* = \sqrt{-g} \gamma^{\alpha \rho} = \sqrt{\eta} \gamma^\rho .$$

(6)

Here, $\eta$ is the determinant of the spatial metric $\eta_{ij}$ obeying $\sqrt{\eta} = \sqrt{-g}/\alpha$. With the above definition of $D^*$ the relativistic continuity equation (5) has the same form as in the non-relativistic case. Note, however, that in equation (5) the expression for $\partial_i v^i$ is not the divergence of a 3-vector on $\Sigma_t$ but rather a sum of partial derivatives. Eulerian observers measure the relativistic rest-mass density $D = -\rho u \cdot n = \rho \gamma$. By rewriting equation (5) for $D^*$, we obtain the relativistic continuity equation

$$0 = \frac{dD^*}{dt} + D^* \partial_i v^i + D^* \frac{d}{dt} \ln \sqrt{\eta}$$

which, in contrast to equation (5), contains an additional source term. In this paper we will use both definitions of a relativistic rest-mass density, i.e., $D$ and $D^* = \sqrt{\eta} D$.

In order to derive our relativistic expressions for the energy and momentum equations, we first re-express the conservation law of energy-momentum (2) by using the continuity equation (1)

$$0 = T_{\nu}^{\mu} = \rho u^{\nu} \left[ (w + q) u^{\mu} \right] - \rho \gamma \frac{d}{dt} \left[ (w + q) u^{\mu} \right] - \frac{1}{2} \left[ (w + q) \right] g^{\alpha \beta,\mu} u^{\alpha} u^{\beta} .$$

(7)

One can easily show that the covariant derivative in the first term of equation (7) can be written as

$$u^{\nu} \left[ (w + q) u^{\mu} \right] = \gamma \frac{d}{dt} \left[ (w + q) u^{\mu} \right] - \frac{1}{2} \left( w + q \right) g^{\alpha \beta,\mu} u^{\alpha} u^{\beta} .$$

Thus, equation (7) becomes

$$\frac{d}{dt} \left[ (w + q) u^{\mu} \right] = -\frac{\alpha}{\rho \gamma} \left[ \partial_{\mu} (p + q) - \frac{1}{2} (\rho w + q) u^{\alpha} u^{\beta} g_{\alpha \beta,\mu} \right]$$

$$= -\frac{1}{D^*} \left[ \partial_{\mu} \left[ \sqrt{-g} (p + q) \right] - \frac{\sqrt{-g}}{2} T^{\alpha \beta} g_{\alpha \beta,\mu} \right] ,$$

(8)

where we have used the identity

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} \right) = \frac{1}{2} g^{\alpha \beta} g_{\alpha \beta,\mu} .$$

Taking the spatial components of equation (8), i.e., $\mu = i$, and using

$$u_i = g_{ij} u^j = \beta_i \gamma^\alpha + \eta_{ij} \gamma^\alpha v^j = \gamma \eta_{ij} v^j ,$$

we get the momentum equation

$$\frac{d}{dt} S_i = -\frac{1}{D^*} \left[ \partial_i \left[ \sqrt{-g} (p + q) \right] - \frac{\sqrt{-g}}{2} T^{\alpha \beta} \partial_{ij} g_{\alpha \beta} \right] ,$$

(9)
with the relativistic specific momentum $S_i$ defined by

$$S_i = \left( w + \frac{q}{\rho} \right) \gamma \eta_{ij} \bar{v}^j .$$

(10)

The component $DS_i = -T(n, \partial_i)$ of the stress-energy tensor field $T$ is the relativistic momentum density in the $i$-direction measured by Eulerian observers. The momentum equation (9), which, in a similar form, was also used by Laguna et al. (1993), can be applied immediately to the SPH method because it contains only spatial derivatives on its right hand side. In the Eulerian formulation an expression similar to equation (9) without artificial viscosity is given by Hawley, Smarr, & Wilson (1984a) for the momentum variable $DS_i$, which is more convenient for a Eulerian description. The Newtonian limit yielding the non-relativistic Euler equation is obvious from equation (9).

Taking the $\mu = 0$-component of equation (8), we obtain the relativistic energy equation

$$\frac{d}{dt} \left[ \left( w + \frac{q}{\rho} \right) u_0 \right] = -\frac{1}{D^*} \left[ \partial_i \left[ \sqrt{-g} (p + q) \right] - \frac{\sqrt{-g}}{2} T^{\alpha\beta} \partial_i g_{\alpha\beta} \right] ,$$

(11)

which, unfortunately, has time derivatives of hydrodynamic variables on both sides. Re-expressing the left hand side of equation (11) as

$$\left( w + \frac{q}{\rho} \right) u_0 = -\alpha \left( w + \frac{q}{\rho} \right) \gamma + \beta^i S_i ,$$

with

$$u_0 = g_{0\mu} u^\mu = \left( \beta^i \beta_i - \alpha^2 \right) \frac{\gamma}{\alpha} + \beta_i \frac{v^i}{\alpha} = \gamma \left( \eta_{ij} \beta^i \bar{v}^j - \alpha \right) ,$$

and rewriting the first term on the right hand side of equation (11) using equation (3),

$$\frac{1}{D^*} \partial_t \left[ \sqrt{-g} (p + q) \right] = \frac{1}{D^*} \frac{d}{dt} \left[ \sqrt{-g} (p + q) \right] - \frac{v^i}{D^*} \partial_i \left[ \sqrt{-g} (p + q) \right]$$

$$= \frac{d}{dt} \left( \sqrt{-g} (p + q) \right) - \frac{1}{D^*} \partial_i \left[ \sqrt{-g} (p + q) v^i \right] ,$$

we obtain the relativistic energy equation

$$\frac{d}{dt} \left[ \alpha E - \beta^i S_i \right] = -\frac{1}{D^*} \left[ \partial_i \left[ \sqrt{-g} (p + q) v^i \right] + \frac{\sqrt{-g}}{2} T^{\alpha\beta} \partial_i g_{\alpha\beta} \right] ,$$

(12)

with the total relativistic specific energy $E$ defined as

$$E = \left( w + \frac{q}{\rho} \right) \gamma - \frac{p + q}{D} .$$

(13)

The component $DE = T(n, n)$ of the stress-energy tensor field $T$ is the total relativistic energy density measured by Eulerian observers. As in the case of the relativistic momentum equation (9) the right hand side of equation (12) contains no time derivatives of hydrodynamic variables. It is, therefore, well suited for the SPH method in contrast to the energy equation used by Hawley et al. (1984a, 1984b) and Laguna et al. (1993) which has a non-conservative form containing two total time derivatives of hydrodynamical variables separately on both
sides of the equation. Norman & Winkler (1986) suggest that the additional time derivative of the relativistic \( \gamma \)-factor in the energy equation gives rise to numerical inaccuracies and instabilities. Note that for the special relativistic case equation (12) without an artificial viscosity can also be found in Monaghan (1992). In the non-relativistic case, equation (12) yields the energy equation for the total non-relativistic specific energy \( |v|^2/2 + \varepsilon_N \) written in a form similar to our expression (12)

\[
\frac{d}{dt} \left[ \frac{1}{2} |v|^2 + \left( w_N + \frac{q}{\rho_N} \right) - \frac{p + q}{\rho_N} \right] = -\frac{1}{\rho_N} \partial_i \left[ (p + q)v^i \right],
\]

where the index \( N \) denotes Newtonian quantities and \( w_N = \varepsilon_N + p/\rho_N \) is the non-relativistic specific enthalpy.

To close our system of hydrodynamical equations (5), (9), and (12), we have to add an equation of state of the form \( p = p(\rho, \varepsilon) \), which relates the thermodynamic pressure \( p \) to the rest-mass density \( \rho \) and the specific internal energy \( \varepsilon \). We restrict ourselves to the ideal-gas equation of state given by

\[
p = (\Gamma - 1)\rho\varepsilon
\]

with the ideal-gas adiabatic constant \( \Gamma \).

Our system of ideal fluid equations is now complete. We have derived the relativistic hydrodynamic equations (5), (9), and (12) in Lagrangian conservative form similar to their Newtonian counterparts. This was achieved by choosing suitable hydrodynamic variables \( D^*, S_i, \) and \( E \) defined in equations (6), (10), and (13). Because of the equation of state (14) and the use of the 3-velocity for moving particles in the Lagrangian numerical methods, we now need to calculate the generic hydrodynamic quantities \( \rho, \bar{v}^i, \) and \( p \) from these variables by solving a highly nonlinear system of equations. If an artificial viscous pressure \( q \) is included, a severe difficulty arises from \( q \) being usually expressed in terms of velocity gradients. Since there is no time evolution equation for \( q \), the character of the dynamic equations is thus changed which is the usual situation in the hydrodynamics of viscous flows. Due to the coupling of \( \bar{v}^i \) and \( q \), the suitably chosen dynamic variables actually have to be solved iteratively for the generic hydrodynamic variables. However, artificial viscosity operates only in the vicinity of shock transitions and is zero everywhere else. For simplicity one can calculate the generic variables in a single iteration taking \( q \) from the previous time step. Using the expression \( w = 1 + p/(G\rho) \) with \( G = 1 - 1/\Gamma \) for the total relativistic specific enthalpy, the specific energy \( E \) from equation (13) can be written as

\[
E = w\gamma - (w - 1)\frac{G}{\gamma} + \left( \gamma - \frac{1}{\gamma} \right) \frac{q}{\rho}
\]

\[
= \left( \gamma - \frac{G}{\gamma} \right) w + \frac{G}{\gamma} + \left( \gamma - \frac{1}{\gamma} \right) \frac{q}{\rho}.
\]

Solving equation (15) for the relativistic specific enthalpy \( w \) and adding the term \( q/\rho \), we obtain

\[
w + \frac{q}{\rho} = \frac{\tilde{E}\gamma - G}{\gamma^2 - G},
\]

where \( \tilde{E} \) is the total relativistic energy.
where the variable $\tilde{E}$ is given by

$$\tilde{E} = E + \frac{q}{\Gamma D}.$$  

Using

$$S^2 = \eta^{ij} S_i S_j = \left( w + \frac{q}{\rho} \right)^2 \left( \gamma^2 - 1 \right) \quad (17)$$

and inserting expression (16), the relativistic $\gamma$-factor can be determined explicitly from

$$0 = \left( S^2 - \tilde{E}^2 \right) \gamma^4 + 2G\tilde{E}\gamma^3 + \left( \tilde{E}^2 - 2GS^2 - G^2 \right) \gamma^2 - 2G\tilde{E}\gamma + G^2 \left( 1 + S^2 \right) \quad (18)$$

as the root of a polynomial of degree four. In the appendix we show that a solution of equation (18) exists and that it is unique for all allowed values of $G$, $\tilde{E}$, and $S^2$. With the value of $\gamma$ known, one can calculate first the rest-mass density $\rho$ from equation (6), then the thermodynamic pressure $p$ from equation (16) and the equation of state (14), and finally the velocity $\tilde{v}^i$ from equation (10) using $\eta^{ij} S_j = (w + q/\rho)\gamma\tilde{v}^i$.

### 3. THE SPH EQUATIONS

In this section we derive the SPH formalism for our set of relativistic hydrodynamic equations (5), (9), (12), and (14). Since in the previous section we have obtained the dynamic equations in an appropriate Lagrangian form, we can proceed in a way that is completely analogous to the Newtonian case. Before we derive our relativistic SPH equations, we give a brief introduction into the standard SPH formalism, which was invented independently by Lucy (1977) and Gingold & Monaghan (1977). For a review of the SPH method see Monaghan (1992).

SPH is a numerical method which discretizes the dynamic equations on a set of nodes, called particles, moving with the fluid. The final discrete equations are obtained in two separate steps. First, all hydrodynamic functions on $\Sigma_t$ are smoothed over a certain volume with the help of a smoothing kernel $W(|r - r'|, h)$, i.e., for a continuous function $f(r)$ one has

$$f(r) = \int f(r') \left. W(|r - r'|, h) \right| dr' + O(h^2) \quad (19)$$

where $r = \{x^i\}$ is the set of spatial coordinates $x^i$. The kernel $W$ is a smooth (differentiable) function with compact support of size $h$, the so called smoothing length, it is normalized to unity

$$\int W(|r - r'|, h) \left| dr' = 1 \right.,$$

and consequently, the smoothing error in equation (19) is $O(h^2)$. The above integrals extend over a region on $\Sigma_t$ around $r$ that contains the support of $W$. An example for $W$ is the cubic-spline kernel of Monaghan & Lattanzio (1985), and in our simulations we use this kernel throughout. The second step consists of an approximate evaluation of the above integral (19) at the particle positions $r_a$

$$\int f(r') \left. W(|r_a - r'|, h) \right| dr' \approx \sum_b \frac{f_b}{n_b} W_{ab} =: \langle f \rangle_a \quad (20)$$
Here, \( a \) and \( b \) label the particles which are distributed in space with number density \( n(r) \). We define \( f_a = f(r_a) \) for any function \( f \) and \( W_{ab} = W(|r_a - r_b|, h) \). The important point is that, applying the smoothing and discretization operations (19), (20), it is possible to derive approximate expressions for derivatives

\[
\langle \nabla f \rangle_a = \sum_b \frac{f_b}{n_b} \nabla_a W_{ab}, \quad \langle \partial_i f^i \rangle_a = \sum_b \frac{1}{n_b} f_b \cdot \nabla_a W_{ab},
\]

where \( f = \{ f^i \} \), \( \nabla = \{ \partial_i \} \), and \( \nabla_a W_{ab} \) is the gradient of the kernel \( W(|r_a - r_b|, h) \) taken with respect to the coordinates of particle \( a \).

With the discretization method described above we are now in the position to derive the SPH equations for our system of relativistic hydrodynamic equations (5), (9), (12), and (14), and nothing has to be changed compared to the standard (non-relativistic) SPH method. In the equations (20) and (21) we replace the number density \( n_a \) by the mass per particle \( m_a \), i.e., \( n_a = D^*_a/m_a \). Dropping the angle brackets \( \langle \rangle \) from now on, equation (20) leads to the SPH representation for the relativistic rest-mass density

\[
D^*_a = \sum_b m_b W_{ab}.
\]

From equation (5) and using

\[
D^* \nabla \cdot v = \nabla \cdot (D^* v) - v \cdot \nabla D^*,
\]

we obtain the SPH form of the relativistic continuity equation

\[
\frac{d}{dt} D_a^* = - \sum_b m_b (v_b - v_a) \cdot \nabla_a W_{ab}.
\]

Here, \( v = \{ v^i \} \) is just a shorthand notation for the set of components \( v^i \), whereas \( \nabla \cdot v \) stands for \( \partial_i v^i \) and is not the divergence of a vector field \( v \). As a consequence, the total mass is conserved exactly. Applying the Lagrangian time derivative to equation (22) with the smoothing length being constant in space and time, one can show that the SPH expression of the relativistic rest-mass density \( D^* \) in equation (22) automatically satisfies the relativistic continuity equation. Thus, in SPH the relativistic rest-mass density \( D^* \) can be computed either from equation (22) or equation (23). If, however, the density \( D = \gamma \rho \) is used instead of \( D^* \), a modification of the standard SPH method is required. Therefore, Laguna et al. (1993) multiply the flat-space kernel \( W(|r - r'|, h) \) of Newtonian SPH with \( 1/\sqrt{\eta(r')} \). In a curved spacetime this factor makes the kernel anisotropic, violates its translation invariance, and leads to additional terms in the SPH approximation of derivatives. Applying the relation \( D^* = \sqrt{\eta} D \), the relativistic SPH formulation of Laguna et al. (1993) can be cast into the standard SPH scheme which is considerably simpler. We therefore suggest that there is no need to modify SPH for given background spacetimes if the continuity equation is used in the conservative form (5) without source terms.
In order to derive the SPH form of the relativistic momentum equation we start from equation (9). Rewriting the pressure gradient as

\[ \frac{1}{D^*} \nabla \left[ \sqrt{-g} (p + q) \right] = \sqrt{-g} \left[ \nabla \left( \frac{p + q}{D^*} \right) + \frac{p + q}{D^*} \nabla D^* \right] + \frac{p + q}{D^*} \nabla \sqrt{-g} , \]

we obtain

\[ \frac{d}{dt} S_a = -\sqrt{-g} a \sum_b m_b \left( \frac{p_a + q_a}{D_a^*} + \frac{p_b + q_b}{D_b^*} \right) \nabla_a W_{ab} \]

\[ - \sqrt{-g} D_a^* \left[ (p_a + q_a) \nabla_a (\ln \sqrt{-g})_a - \frac{1}{2} T^\alpha_\beta a (g_{\alpha\beta})_a \right] , \quad (24) \]

where \( S_a = \{ S_i \}_a \), and the metric gradients \( \nabla \ln \sqrt{-g} \) and \( \nabla g_{\alpha\beta,t} \) can be calculated analytically. Thus, only the pressure gradient term in equation (24) has to be smoothed, and it has been symmetrized to conserve linear and angular momentum in SPH. Next, we proceed with the relativistic energy equation (12). In the expression

\[ \frac{1}{D^*} \nabla \cdot \left[ \sqrt{-g} (p + q) v \right] = \frac{\sqrt{-g}}{D^*} \nabla \cdot [(p + q) v] + \frac{p + q}{D^*} v \cdot \nabla \sqrt{-g} \]

we replace the first term on the right hand side by

\[ \frac{1}{D^*} \nabla \cdot [(p + q) v] = \frac{1}{D^*} v \cdot \nabla (p + q) + \frac{p + q}{D^*} \nabla \cdot v \]

\[ = v \cdot \left[ \nabla \left( \frac{p + q}{D^*} \right) + \frac{p + q}{D^*} \nabla D^* \right] \]

\[ + \frac{1}{2} \left[ \nabla \cdot \left( \frac{p + q}{D^*} v \right) - v \cdot \nabla \left( \frac{p + q}{D^*} \right) \right] \]

\[ + \frac{p + q}{D^*} \left[ \nabla \cdot (D^* v) - v \cdot \nabla D^* \right] . \]

With this combination of terms we obtain the SPH representation of the relativistic energy equation

\[ \frac{d}{dt} [\alpha E - \beta^i S_i]_a = -\sqrt{-g} a \sum_b m_b \left( \frac{p_a + q_a}{D_a^*} + \frac{p_b + q_b}{D_b^*} \right) (v_a + v_b) \cdot \nabla_a W_{ab} \]

\[ - \sqrt{-g} D_a^* \left[ (p_a + q_a) v_a \cdot \nabla_a (\ln \sqrt{-g})_a + \frac{1}{2} T^\alpha_\beta a (g_{\alpha\beta,t})_a \right] . \quad (25) \]

Note that equation (25) and the momentum equation (24) contain identical symmetric factors in front of the kernel gradients. As outlined in section 2, this form of the relativistic energy equation is well suited for SPH because it contains no time derivatives of hydrodynamic variables on its right hand side. Again, there is no need to smooth the derivatives of the metric \( \nabla \ln \sqrt{-g} \) and \( g_{\alpha\beta,t} \). The last equation that needs to be considered is the equation of state (14), and its SPH formulation reads

\[ p_a = (\Gamma - 1) \rho_a \varepsilon_a . \quad (26) \]
This completes our set of relativistic SPH equations (22), (24), (25), and (26) for computing general relativistic fluid flows in given arbitrary background spacetimes.

We now focus our attention to the implementation of an artificial viscosity term which is necessary to handle shock fronts. For the use of an artificial viscosity in the standard SPH method see Monaghan & Gingold (1983). In our simulations, we have used the following artificial viscous pressure

\[ q_a = \begin{cases} 
\rho_a w_a \left[ -\tilde{\alpha} c_a h_a (\nabla \cdot v)_a + \tilde{\beta} h_a^2 (\nabla \cdot v)_{a}^2 \right] & \text{if } (\nabla \cdot v)_a < 0 \\
0 & \text{otherwise}
\end{cases} \tag{27} \]

with

\[ (\nabla \cdot v)_a \approx \frac{\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{r_{ab}^2 + \tilde{\varepsilon} h_{ab}^2}, \tag{28} \]

where \( c_a = \sqrt{\Gamma p_a / (\rho_a w_a)} \) is the relativistic sound velocity measured in the rest frame of the fluid, \( \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{\varepsilon} \) are numerical parameters, \( \mathbf{v}_{ab}, \mathbf{r}_{ab} \) stand for the differences \( \mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b, \mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b \), and \( \bar{h}_{ab} = (h_a + h_b)/2 \) is the mean value of the smoothing lengths of particles \( a \) and \( b \). Without the enthalpy \( w_a \), the expression (27) for \( q \) is almost equivalent to the standard SPH form of the artificial viscosity invented by Monaghan & Gingold (1983). Including \( w_a \) into equation (27), the parameters \( \tilde{\alpha} \) and \( \tilde{\beta} \) can be chosen to be of order unity even for shocks with ultra-relativistic values of \( \gamma \). The \( \tilde{\alpha}\)-term, which is linear in the velocity differences, is similar to a physical shear and bulk viscosity, and the quadratic \( \tilde{\beta}\)-term is the standard von Neumann-Richtmyer (1950) artificial viscosity used in finite difference methods for handling high Mach-number shocks. According to Monaghan & Gingold (1983), the un-smoothed representation of the velocity divergence in equation (28) acts more directly on the relative motion of particle pairs and leads to a damping of irregular oscillations in shock transitions. In equation (28), the parameter \( \tilde{\varepsilon} \) has been introduced to avoid singularities, and a typical value is \( \tilde{\varepsilon} = 0.1 \).

Since the expression (27) for the artificial viscosity is not Lorentz invariant, we also performed calculations with a relativistically covariant formulation of the artificial viscous pressure

\[ q_a = \begin{cases} 
\rho_a \left[ -\tilde{\alpha} c_a h_a \theta_a + \tilde{\beta} h_a^2 \theta_a^2 \right] & \text{if } \theta_a < 0 \\
0 & \text{otherwise}
\end{cases} \tag{29} \]

where

\[ \theta_a = (w^{\mu};\mu)_a = \frac{1}{\alpha_a} \left[ \frac{d \gamma_a}{dt} + \gamma_a \left( (\nabla \cdot v)_a + \frac{d}{dt} (\ln \sqrt{\eta})_a \right) \right]. \]

However, this expression contains a time derivative of the \( \gamma \)-factor which destroys the explicit nature of the Lagrangian form of the hydrodynamic equations. One possibility to circumvent this problem is to take the backward time difference approximation \( \gamma_a(t) - \gamma_a(t - \Delta t)/\Delta t \) with time step \( \Delta t \) for the time derivative \( d\gamma_a/dt \). Since the non-covariant form (27) turned out to be quite appropriate for the resolution of shock structures, we used this approach in all our simulations and did not pursue the covariant relation (29).
To improve the local resolution of SPH, we allow the smoothing length \( h \) to vary in space according to the relativistic rest-mass density \( D^* \) via

\[
h_a = (h_0)_a \left[ \frac{(D_0^*)_a}{D_a} \right]^{1/d}, \tag{30}
\]

where \( d \) denotes the dimension of the spatial slices \( \Sigma_t \). For particles of equal mass the scaling law (30) indicates that the number of particles within the support of the kernel \( W \) is approximately kept constant in time. Thus, in regions where the gas is compressed, the smoothing length is increased while it is decreased in rarefaction zones. Since the smoothing length is now a function of the density \( D^* \), equation (22) (or eq. (23)) is now a nonlinear implicit relation for the density. We solve this approximately by inserting \( D^*_a \) from the previous time step into equation (30) to obtain an estimate for \( h_a \). Since the smoothing length is now a function of position and time, the SPH form of the continuity equation contains additional terms (Monaghan 1992). However, no such terms appear if the relativistic rest-mass density \( D^*_a \) is calculated from the computationally simpler equation (22).

4. NUMERICAL TESTS

Although we have developed a fully three-dimensional general relativistic SPH code, we restrict ourselves in this paper to the standard analytic test bed of one-dimensional special relativistic shock problems: the shock tube and the wall shock. For each simulation we show four diagrams of the numerical results together with the analytic solution for the fluid variables 3-velocity \( v := \bar{v}^1 = v^1 \), rest-mass density \( \rho \), thermodynamic pressure \( p \), and specific internal energy \( \varepsilon \). These variables are functions of the coordinate \( x := x^1 \in [0,100] \). The quality of the simulations is measured in terms of the relative error with respect to the analytic solution, i.e., for each hydrodynamic function \( f \) an error \( \Delta f \) is calculated from

\[
\Delta f = \frac{1}{Nf_{\text{max}}} \sum_{b=1}^{N} |f_b^0 - f_b^0| ,
\]

where the sum is over all \( N \) particles and \( f_b^0 \) stands for the analytic solution which has the maximum value \( f_{\text{max}}^0 \).

a) Shock Tube Tests

First we consider the shock tube problem, where initially a fluid at rest is divided by a diaphragm into two regions of different densities and internal energies. When the diaphragm is removed, a rarefaction wave travels into the warm and dense medium and a compression wave into the cold and lower density fluid. Between the two media a so called contact discontinuity is present. For the analytic solution of the relativistic shock tube problem we refer to Taub (1948), McKee & Colgate (1973), Hawley et al. (1984a), and Marti & Müller (1994).
Fig. 1.— Numerical result of an SPH calculation with 1000 particles (open circles) and analytic solution (solid line) for the non-relativistic shock tube problem (note that the units are arbitrary except for the velocity which is measured in units of $c$). The intermediate pressure, velocity, and relativistic $\gamma$-factor are $p_m = 0.303$, $v_m = 2.93 \times 10^{-3}$, and $\gamma_m = 1.000004$, the positions of the shock, the contact discontinuity, and the head and tail of the rarefaction wave are $x_s = 83.2$, $x_c = 67.6$, $x_h = 27.6$, and $x_t = 48.7$, respectively, and the velocity and relativistic $\gamma$-factor of the shock are $v_s = 5.54 \times 10^{-3}$ and $\gamma_s = 1.000015$.

As any relativistic hydro code has to be tested for the Newtonian limit, we start our series of shock tube simulations with a non-relativistic test case. Figure 2 shows the corresponding numerical and analytic solution for a gas with $\Gamma = 1.4$ at time $t = 6000$ with the initial conditions $\rho = 10^5$, $\varepsilon = 2.5 \times 10^{-5}$ for $x < 50$, and $\rho = 0.125 \times 10^5$, $\varepsilon = 2 \times 10^{-5}$ for $x > 50$ (note that the units are arbitrary except for the velocity which is measured in units of $c$). The numerical calculation was performed with 1000 particles, initial smoothing length $h = 0.6$ ($\approx 10$ interactions per particle), and artificial viscosity parameters $\bar{\alpha} = 1$ and $\bar{\beta} = 2$. In the initial distribution, particles were placed on a uniform grid, and the particles at $x > 50$ have a mass ten times smaller than the particles at $x < 50$. In the calculation of Figure 2 the largest relative error occurs in the fluid velocity where $\Delta v = 1.1\%$. 
Fig. 2.— Numerical result of an SPH calculation with 1000 particles (open circles) and analytic solution (solid line) for the relativistic shock tube problem. The intermediate pressure, velocity, and relativistic $\gamma$-factor are $p_m = 1.45$, $v_m = 0.714$, and $\gamma_m = 1.4$, the positions of the shock, the contact discontinuity, and the head and tail of the rarefaction wave are $x_s = 87.3$, $x_c = 82.1$, $x_h = 17.8$, and $x_t = 57.5$, respectively, and the velocity and relativistic $\gamma$-factor of the shock are $v_s = 0.828$ and $\gamma_s = 1.8$.

Next, a mildly relativistic shock tube is investigated with initial conditions $\rho = 10$, $\varepsilon = 2$ ($x < 50$), and $\rho = 1$, $\varepsilon = 10^{-6}$ ($x > 50$). Figure 2 shows the numerical result and the analytic solution for a gas with $\Gamma = 5/3$ at time $t = 45$. We used the same numerical parameters as in the non-relativistic shock tube problem, i.e., 1000 particles, $h = 0.6$, $\tilde{\alpha} = 1$, and $\tilde{\beta} = 2$. The largest relative error is $\Delta v = 1.0\%$. As can be seen in Figure 2, the velocity profile of a relativistic rarefaction wave is no longer linear as in the Newtonian case because of the relativistic velocity addition formula. Comparing our results with the simulations of Hawley et al. (1984b) and Laguna et al. (1993) for the same $\gamma = 1.4$ shock tube, we note that numerical inaccuracies due to the non-conservative formulation of their energy equation clearly show up in their results for the relativistic rest-mass density and specific internal energy. To investigate the convergence properties of our numerical method, we performed a calculation of the $\gamma = 1.4$ shock tube with 10000 particles and initial smoothing length $h = 0.1$ ($\approx 20$ interactions per particle). As can be seen in Figure 2, the numerical calculation covers
Fig. 3.— Numerical result of an SPH calculation with 10000 particles (points) and analytic solution (solid line) for the relativistic shock tube problem. The intermediate pressure, velocity, and relativistic $\gamma$-factor are $p_m = 1.45$, $v_m = 0.714$, and $\gamma_m = 1.4$, the positions of the shock, the contact discontinuity, and the head and tail of the rarefaction wave are $x_s = 87.3$, $x_c = 82.1$, $x_h = 17.8$, and $x_t = 57.5$, respectively, and the velocity and relativistic $\gamma$-factor of the shock are $v_s = 0.828$ and $\gamma_s = 1.8$.

the analytic solution almost exactly, and the largest relative error is reduced to $\Delta v = 0.2\%$.

When the relativistic $\gamma$-factor of the shock tube is increased by increasing the initial specific internal energy ratio, the region between the leading shock front and the trailing contact discontinuity becomes extremely thin and dense. Thus, without specially designed adaptive methods, it will be impossible to resolve these Lorentz contracted shells of matter, which are typical for relativistic fluid flows. In addition, the specific internal energy at the contact discontinuity may become negative in the SPH simulations. In order to avoid these difficulties, we consider now a simplified problem of a single shock front without the presence of a contact discontinuity and a nonlinear rarefaction wave.
Fig. 4.— Numerical result of an SPH calculation with about 250 particles (open circles) and analytic solution (solid line) for the relativistic $\gamma_i = 1.8$ wall shock problem. The post shock properties are $\rho_p = 5.66$, $p_p = 1.52$, and $\varepsilon_p = 0.803$, and the position and velocity of the shock front are $x_s = 64.3$ and $v_s = -0.178$ with $\gamma_s = 1.02$.

b) Wall Shock Tests

As a second test of our numerical method we modeled the wall shock problem of a cold relativistically moving fluid flowing towards a solid wall. As the fluid hits the wall, a shock front forms, which then travels upstream against the incoming fluid producing a hot and dense post shock region of zero velocity.

Figure 4 shows the numerical result and the analytic solution of a mildly relativistic wall shock for a gas with $\Gamma = 4/3$ at time $t = 200$ moving to the right with the reflecting wall at $x = 100$. The uniform initial fluid properties are $D_i = 1$, $v_i = 0.832$ with $\gamma_i = 1.8$, and $\varepsilon_i = 10^{-5}$. Initially, particles of equal mass are uniformly distributed in the simulation domain. At the location of the solid wall we use reflecting boundary conditions. The simulation of Figure 4 showing about 250 particles was performed with the initial smoothing length $h = 3$ ($\approx 10$ interactions per particle) and artificial viscosity parameters $\tilde{\alpha} = 0.25$ and $\tilde{\beta} = 0.5$. The rest-mass density $\rho$, the thermodynamic pressure $p$, and the specific internal energy $\varepsilon$ show
Fig. 5.— Numerical result of an SPH calculation with about 250 particles (open circles) and analytic solution (solid line) for the ultra-relativistic $\gamma_i = 1000$ wall shock problem. The post shock properties are $\rho_p \approx 4$, $p_p \approx 4/3 \times 10^3$, and $\varepsilon_p \approx 1000$, and the position and velocity of the shock front are $x_s \approx 33.3$ and $v_s \approx -1/3$ with $\gamma_s \approx 1.06$.

A spike-like feature (see Fig. 4), which is known in the literature as “wall heating” (Norman & Winkler 1986). The largest relative error appears in the specific internal energy where $\Delta \varepsilon = 1.1\%$.

The results of an ultra-relativistic shock simulation are shown in Figure 5. The initial velocity $v_i$ is increased to $v_i = 0.9999995$ which corresponds to a relativistic $\gamma$-factor of $\gamma_i = 1000$. All other parameters are identical with those of the previous wall shock calculation. Neglecting the pre-shock specific internal energy $\varepsilon_i$, one can show that in the ultra-relativistic limit $v_i \to 1$ the post shock properties $\rho_p$, $p_p$, $\varepsilon_p$, and the shock velocity $v_s$ are given by $\rho_p = D_i \Gamma / (\Gamma - 1)$, $p_p = \Gamma D_i \gamma_i$, $\varepsilon_p = \gamma_i$, and $v_s = -(\Gamma - 1)$. For the wall shock problem of Figure 5 we thus obtain $\rho_p \approx 4$, $p_p \approx 4/3 \times 10^3$, $\varepsilon_p \approx 1000$, and $v_s \approx -1/3$. At the time $t = 200$ the shock front has moved the distance $|v_s| t \approx 66.7$ to the left reaching the spatial position $x_s \approx 33.3$. The largest relative error in this case is $\Delta \rho = 1.1\%$.

A calculation of a mildly relativistic wall shock problem (with $\gamma_i = 2.24$) using artificial viscosity was also performed by Hawley et al. (1984b) with their time-explicit Eulerian finite
difference code. They tried several formulations of their non-conservative energy equation by omitting and including various artificial viscous pressure terms. They found that the \( q \partial_t \gamma \) term in their energy equation is unstable even for mildly relativistic wall shocks. This is verified by the calculations of Norman & Winkler (1986) using an implicit adaptive-mesh finite difference method, which shows the requirement of an implicit technique to handle the \( q \partial_t \gamma \) term. The fact that our method is able to treat even ultra-relativistic shocks without problems is entirely based on our conservation formulation of the relativistic hydrodynamic equations where no additional time derivatives of hydrodynamic variables appear such as the \( q \partial_t \gamma \) term. In order to verify that this numerical stability is not restricted to the SPH method, we also performed numerical simulations of the wall shock problem with a simple explicit finite difference upwind scheme. Figure 6 shows the numerical result and the analytic solution for the \( \gamma_i = 1000 \) wall shock at time \( t = 200 \) using a grid of 250 zones, and the maximum relative error occurs for the specific internal energy where \( \Delta \varepsilon = 2.2\% \).

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**Fig. 6.** Numerical result of a finite difference calculation with a grid of 250 zones (open circles) and analytic solution (solid line) for the ultra-relativistic \( \gamma_i = 1000 \) wall shock problem. The post shock properties are \( \rho_p \approx 4, \ p_p \approx 4/3 \times 10^3 \), and \( \varepsilon_p \approx 1000 \), and the position and velocity of the shock front are \( x_s \approx 33.3 \) and \( v_s \approx -1/3 \) with \( \gamma_s \approx 1.06 \).
5. SUMMARY

We have derived a fully Lagrangian conservative form of the general relativistic equations of hydrodynamics for a perfect fluid with artificial viscosity in a given arbitrary background spacetime. This has been achieved by choosing suitable Lagrangian time evolution variables. These variables are connected to the generic fluid variables rest-mass density $\rho$, 3-velocity $\vec{v}$, and thermodynamic pressure $p$ through a set of nonlinear algebraic equations. For an ideal gas, we have shown that these equations can be reduced to a single fourth-order equation with a unique solution which can be explicitly calculated in terms of various roots. For more complex equations of state the solution of the above algebraic equations has to be performed numerically, and the question of uniqueness is more complicated. Using our Lagrangian formulation, we have developed a three-dimensional general relativistic SPH code based on the standard SPH approach of non-relativistic fluid dynamics. The important point is that all metric factors from covariant derivatives have been absorbed in the definition of the Lagrangian variables and thus no longer appear in the fluid equations. As a result, the SPH kernels remain spherically symmetric and are the same for all particles. This is an essential difference to the covariant SPH approach of Kheyfets et al. (1990) using kernels defined in the local comoving frame of the fluid, and to the relativistic SPH method of Laguna et al. (1993) where, in curved spacetime, the kernels are anisotropic and have no translational symmetry which leads to additional terms in the SPH equations. The relativistic continuity equation (3), for example, contains no source term, hence we can identify the relativistic rest-mass density $D^\ast = \sqrt{\eta\gamma}\rho$ as the appropriate SPH density for smoothing the equations.

In this paper we have restricted ourselves to the numerical simulation of two different one-dimensional examples, i.e., the special relativistic shock tube and the wall shock. An empirical error estimate is obtained from a comparison of the numerical results with the corresponding analytic solutions. The SPH calculations with 1000 particles show a typical maximum relative error of about 1%, and an increase of the particle number with a decreasing smoothing length reduces this error in a uniform way. The wall-shock problem can be solved without any numerical difficulties for very large $\gamma$-factors (at least $\gamma = 1000$). The shock-tube case suffers from a resolution problem if $\gamma$ is large because a thin shell of matter builds up between the shock front and the contact discontinuity. This can only be resolved with the help of additional adaptive methods which increase the number of particles in this zone.

An important ingredient in our SPH formulation is the treatment of shock structures by an artificial viscosity rather than using a Riemann solver. This is very easy to implement even for the two- or three-dimensional case. Introducing an artificial viscous pressure $q$ is considered as a purely numerical tool that acts as a filter to smear out steep gradients in the hydrodynamic functions and suppresses unphysical oscillations. Thus, there is no need to use a covariant expression for $q$ in terms of velocity derivatives as long as the time evolution and the jump conditions of shocks are represented correctly. Since the jump conditions follow from the conservation of mass, momentum, and energy across the shock, it is obvious that a conservative form of the relativistic hydrodynamic equations is important for handling discontinuities in relativistic fluid dynamics numerically. The simulation of relativistic flows
with large \( \gamma \)-factors including shocks is the traditional domain of HRSC methods. However, with our formulation of the general relativistic hydrodynamic equations it is possible to model such flows using an artificial viscosity independent of the underlying numerical method, i.e., SPH or finite difference schemes.

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A. APPENDIX

In section 2 we have derived equation (18) from which the relativistic $\gamma$-factor can be calculated analytically as the root of a polynomial of degree four for given artificial viscous pressures $q$. Equation (18) is restricted to an ideal-gas equation of state (14). We will now show the existence and uniqueness of the solution of equation (18) for all allowed values of $G$, $S^2$, and $\tilde{E}$.

First, we rewrite equation (18) by substituting $\gamma = 1 + \delta$, $\delta \in [0, \infty)$ and obtain

$$0 = \frac{S^2 - \tilde{E}^2}{a_4} \delta^4 + 2 \frac{2S^2 - 2\tilde{E}^2 + G\tilde{E}}{a_3} \delta^3$$

$$+ \frac{6S^2 - 5\tilde{E}^2 + 6G\tilde{E} - 2GS^2 - G^2}{a_2} \delta^2$$

$$+ 2 \frac{2S^2 - \tilde{E}^2 + 2G\tilde{E} - 2GS^2 - G^2}{a_1} \delta + \frac{S^2(1 - G)^2}{a_0}.$$  \hspace{1cm} (A1)

With the degree of freedom $f \geq 3$ the ideal-gas adiabatic constant $\Gamma = 1 + 2/f$ and the variable $G = 1 - 1/\Gamma$ lie in the range

$$1 < \Gamma \leq \frac{5}{3} \ , \quad 0 < G \leq \frac{2}{5}. \hspace{1cm} (A2)$$

From equations (16) and (17) the variables $\tilde{E}$ and $S^2$ are given by

$$\tilde{E} = \left(\gamma - \frac{G}{\gamma}\right) \tilde{w} + \frac{G}{\gamma} \ , \quad S^2 = \tilde{w}^2 \left(\gamma^2 - 1\right) \geq 0,$$  \hspace{1cm} (A3)

where we have defined $\tilde{w} = w + q/\rho \geq 1$. Using the relations (A2) and the expressions (A3), we obtain for the coefficient $a_4$ in equation (A1)

$$a_4 = S^2 - \tilde{E}^2$$

$$= - \left(1 - 2G + \frac{G^2}{\gamma^2}\right) \tilde{w}^2 - 2G \left(1 - \frac{G}{\gamma^2}\right) \tilde{w} - \frac{G^2}{\gamma^2} < 0.$$ \hspace{1cm} (A4)

Thus, $S^2$ is limited to the range

$$0 \leq S^2 < \tilde{E}^2.$$  \hspace{1cm}

a) Existence

With the coefficients $a_4 < 0$ and $a_0 = S^2/\Gamma^2 \geq 0$ equation (A1) has at minimum one positive root of $\delta$. 



b) Uniqueness

To show the uniqueness of the solutions of equation (A1) for \( \delta \), we investigate the changes of signs of the coefficients \( a_3, a_2, \) and \( a_1 \) depending on the variable \( S^2 \):

\[
\begin{align*}
a_3 < 0 & \iff S^2 < \tilde{E}^2 - \frac{1}{2}G\tilde{E} =: S_3^2 , \\
a_2 < 0 & \iff S^2 < \frac{1}{2(3 - G)} \left( 5\tilde{E}^2 - 6G\tilde{E} + G^2 \right) =: S_2^2 , \quad \text{and} \\
a_1 < 0 & \iff S^2 < \frac{1}{2(1 - G)} \left( \tilde{E} - G \right)^2 =: S_1^2 .
\end{align*}
\]

Using

\[
\tilde{E} - G = (\tilde{w} - 1) \gamma \left( 1 - \frac{G}{\gamma^2} \right) + \gamma - G > 0
\]

and the relations (A2), one can show that

\[
S_2^2 > 0 , \quad \text{(A5)}
\]

\[
S_2^2 - S_1^2 = \frac{\Gamma}{1 + 2\Gamma} \left[ (2 - \Gamma)\tilde{E}^2 + 2(\Gamma - 1)G\tilde{E} - \Gamma G^2 \right] > \frac{\Gamma G^2}{1 + 2\Gamma} \left[ (2 - \Gamma) + 2(\Gamma - 1) - \Gamma \right] = 0 , \quad \text{and} \quad \text{(A6)}
\]

\[
S_2^2 = \frac{1}{2(3-G)} \left( 5\tilde{E}^2 - 6G\tilde{E} + G^2 \right) < \frac{1}{2(3-G)} \left( 5\tilde{E}^2 - 6G\tilde{E} + G\tilde{E} \right) < \frac{5}{2(3-G)} \left( \tilde{E}^2 - G\tilde{E} \right) < \tilde{E}^2 - G\tilde{E} < S_3^2 . \quad \text{(A7)}
\]

The relations (A5), (A6), and (A7) lead to

\[
0 < S_1^2 < S_2^2 < S_3^2 . \quad \text{(A8)}
\]

i) For \( a_0 = S_2/\Gamma^2 > 0 \) or \( S_2 > 0 \), respectively, we obtain from the relations (A4) and (A8) the following table of signs for the coefficients \( a_4, a_3, a_2, a_1, \) and \( a_0 \) depending on \( S^2 \):

| \( S^2 \)                           | \( a_4 \) | \( a_3 \) | \( a_2 \) | \( a_1 \) | \( a_0 \) |
|------------------------------------|-----------|-----------|-----------|-----------|-----------|
| \( S_3^2 < S^2 < \tilde{E}^2 \)    | -         | +         | +         | +         | +         |
| \( S^2 = S_3^2 \)                  | -         | 0         | +         | +         | +         |
| \( S_1^2 < S^2 < S_3^2 \)          | -         | -         | +         | +         | +         |
| \( S^2 = S_1^2 \)                  | -         | -         | 0         | +         | +         |
| \( 0 < S^2 < S_1^2 \)              | -         | -         | 0         | +         | +         |

For all values of \( S^2 \in (0, \tilde{E}^2) \) the series of coefficients \( a_4, a_3, a_2, a_1, \) and \( a_0 \) has exactly one change of sign. Therefore, in \( \delta \in (0, \infty) \) there exists only one root of equation (A1) for \( \delta \).
ii) For $a_0 = S^2/\Gamma^2 = 0$ or $S^2 = 0$, respectively, equation (A1) has the root $\delta = 0$. With the relations (A4) and (A8) the coefficients $a_4$, $a_3$, $a_2$, and $a_1$ are all less than zero. Thus, $\delta = 0$ is the only positive root of equation (A1) for $\delta$.

To summarize, with the restriction of the ideal-gas equation of state (14) a solution of equation (A1) for $\delta$ or of equation (18) for $\gamma$, respectively, exists and is unique. Thus, the fluid variables rest-mass density $\rho$, 3-velocity $\bar{v}^i$, and thermodynamic pressure $p$ can be calculated analytically in a unique way from the variables $D^*$, $S_i$, $E$, and $q$ from equations (18), (14), (14), (14), (14), and (14).
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