Closed superstring moduli tree-level two-point scattering amplitudes in Type IIB orientifold on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

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Abstract

We reconsider the two-point string scattering amplitudes of massless Neveu-Schwarz–Neveu-Schwarz states of Type IIB orientifold superstring theory on the disk and projective plane in ten dimensions and analyse the $\alpha'$ expansion. Then we consider the unoriented Type IIB theory on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ where two-point string scattering amplitudes of complex Kähler moduli and complex structures of the untwisted sector are computed on the disk and projective plane. New results are obtained together with known ones. Finally, the comparison between string scattering amplitudes results at $\alpha'^2$-order and the (curvature)$^2$ terms in the low energy effective action of D-branes and $\Omega$-planes is performed in both cases.

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1 Introduction

Perturbative string theories are characterised by a genus expansion controlled by $g_s = e^{\langle \Phi \rangle}$, i.e. the string coupling. This expansion in world-sheet topologies where vertex operators are inserted amounts to n-points string scattering amplitudes. When strings propagate in a non-trivial background, the world-sheet action describes an interacting 2d field theory that usually is not exactly solvable, but can be explored perturbatively. This second expansion is known as the $\alpha'$-expansion or derivative expansion. For each fixed world-sheet topology, a string scattering amplitude contains an infinite power series expansion in $\alpha'$, i.e terms of higher order in momenta that lead to an infinite tower of higher derivative terms in the Low Energy Effective Action (LEEA). Through this perturbative double expansion one can investigate the possible corrections to the LEEA. On the other hand, corrections to the LEEA can be induced not only by the inclusion of the "physical" D-branes and $\Omega$-planes, but also in the non-perturbative regime by world-sheet instantons and D-branes instantons. D-branes are extended objects [1] to which R-R p-forms can couple and also where open strings can end, i.e the loci where gauge groups in the sense of quantum field theories appear. Instead, $\Omega$-planes are typical objects of unoriented string theory, i.e. when orientifold projection $\Omega$ (world-sheet parity operator) is gauged, part of the initial spectrum is truncated since only states that are invariant under the exchange of left moving with right moving survive. Type-I string theory is the result of $\Omega$-gauging of Type IIB, but when compactification schemes and T-duality are taken into account, more general orientifold projections can be performed (as discussed below), producing Type II orientifold (or open descendant) models [2, 3, 4, 5, 6, 7, 8]. World-sheet instantons are non-perturbative in $\alpha'$ expansion and perturbative in $g_s$ since they can appear at tree level in the $g_s$ expansion, especially in four

\footnote{For completeness NS-NS 2-form, common to all the string theories, is in general coupled to the fundamental string $F_1$.}
dimensional string vacuum configurations when closed string world-sheets wraps non-trivial 2-cycles of the internal compactification manifold [9]. While objects that are non-perturbative in $g_s$ expansion such a D-branes instantons (or ED-branes) and NS5-branes instantons can wrap different non-trivial cycles which characterised the geometry of the internal compactified space [30, 38].

The role of the compactification process and the shape of the internal geometry clearly goes beyond the study of non-perturbative contributions to the LEEA, in the sense that it is a necessary step that one has to consider in order to make contact with four dimensional physics starting from theories with extra dimensions. With LEEA one means that only the massless fields of a given string model are considered as the starting point for the construction of the field configuration that solves the e.o.m. of the effective theory. The assumption on the starting space-time geometry is that it can be factorised into a four dimensional Minkowskian geometry $\mathcal{M}_4$ times a six dimensional compact manifold $X_6$, where the specific characteristics of the latter induce a non-trivial influence on the four dimensional physics. The simplest internal manifold that one can consider is a $T^6$-torus, which is a generalisation of the Kaluza-Klein compactification on a circle [10], and falls in the class of toroidal compactification. The main problem with this kind of compactification is the amount of supersymmetry in four dimensions that is maximal, i.e. $\mathcal{N} = 8$ (Type II) or $\mathcal{N} = 4$ (Heterotic and Type-I), because all the initial supercharges are conserved. Proceeding with this purely geometrical point of view, i.e. where only the internal metric is non-trivial yet constant and all the other fields vanish, promising compactified models with less supersymmetry in four dimensions are Calabi-Yau compactifications [11], with the internal manifold a CY compactified models with less supersymmetry in four dimensions are Calabi-Yau compactifications [11], with the internal manifold a CY 3 which admits a Ricci-flat Kähler metric [12, 13] thus it satisfies Einstein equations in vacua. CY compactifications preserve one quarter of supersymmetries of the initial ten dimensional string models, i.e. the field content can be organised in $\mathcal{N} = 2$ supermultiplets (Type II) or $\mathcal{N} = 1$ supermultiplets (Heterotic). Compactifications produce a rather large number of light neutral scalar fields (experimentally unobserved up to now) called moduli fields. Their role is to parametrise the size and shape of the compactification manifold $X_6$ or the position of the D-branes, and moreover their vev directly influences several parameters like gauge couplings and masses of other fields in the four dimensional EA. The main difference between Heterotic compactifications and Type II compactifications is the origin of these moduli, i.e. if they come from NS-NS states only (Heterotic) or if there are a mixing with the RR part (Type II). In the context of CY compactifications the origin of the moduli fields give important constraints on the moduli space geometry, which from the supergravity LEEA point of view, characterise the kinetic terms for the moduli (at tree level). The moduli space for Type II theories on $CY_3$ is a direct product space [15, 30]

$$\mathcal{M} = Q \times S_K$$

where $S_K$ a special Kähler manifold and $Q$ a quaternionic manifold. Special Kähler manifold $S_K$ describes only geometric moduli fields (moduli coming from NS-NS states) which fall into vector multiplets of the $\mathcal{N} = 2$. While, the quaternionic manifold $Q$ contains both geometric and non-geometric moduli (coming from RR states) and the universal dilaton, all collected in hypermultiples. The number of these $\mathcal{N} = 2$ supermultiplets dictates the complex or real dimension of the respective manifold: $h^{2,1}$ ($h^{1,1}$) complex dimension for $S_K$ manifold in Type IIB(A), $4(h^{1,1} + 1)$ ($4(h^{2,1} + 1)$) real dimension for $Q$ Type IIB(A). Where $h^{1,1}$ and $h^{2,1}$ are the non-trivial Hodge numbers for a Calabi-Yau threefolds, related to the Kähler moduli and the complex structure moduli respectively. In both Type II theories $S_K$ vectormultiplet moduli space doesn’t receive any kinds of $g_s$ corrections, it is exact at string tree level. In Type IIB this moduli space is also exact, at tree level, in the $\alpha'$ expansion. On the other hand hypermultiplet moduli space $Q$, receives perturbative and non-perturbative string corrections. Perturbative $g_s$ corrections, like world-sheet instantons wrapping two-cycles in the CY manifold, enter in Type IIA. While non-perturbative D-brane and NS-brane instantons wrapping supersymmetric cycles inside the CY, enter both Type II theories: ED(-1), ED1, ED3, ED5 and ENS5 brane instantons in Type IIB while ENS5 and ED2 brane instantons in Type IIA. Something special happens for fixed values of non-geometric moduli, i.e. when they become non dynamical fields, because the quaternionic

\footnote{In Type IIA some $\alpha'$ string effects can shape the $S_K$ moduli space dictated by classical geometry.}

\footnote{All the even-cycles are supersymmetric while, among odd-cycles, only 3-cycles.}
manifold $\mathcal{Q}$ can be written as a direct product
\[ \mathcal{Q} \sim \frac{SU(1,1)}{U(1)} \times \hat{\mathcal{K}} \]  
with $\hat{\mathcal{K}}$ special Kähler manifold describing only the $h_{1,1}(h^{2,1})$ hypermultiplets which contain the geometric moduli and $SU(1,1)/U(1)$ the moduli space of the universal complex axion-dilaton hypermultiplet of Type IIB (A) [15, 16]. In the case where geometric moduli are fixed, the resulting moduli space for the axion-dilaton and the non-geometrical moduli would be $SU(1,h_{1,1} + 2)/U(1) \times SU(h^{1,1} + 2)$ with $h_{1,1}$ the number of these hypermultiplets in Type IIB ($h^{2,1}$ for Type IIA) [15, 16].

From the string point of view this kind of compactifications can be used as a reliable 4-dimensional approximation of the 10-dimensional supergravity theory, when only the tree-level approximation in the $\alpha'$-expansion is considered, i.e. in the large volume regime where the $R$ length scale which characterises the size of CY manifold is large. This because, as we said, when non trivial curved background is considered, the 2d-field theory becomes an interacting field theory, thus one loses powerful tolls of the initially free 2d-CFT as vertex operators formulation, two-point functions definition that are fundamental for the construction of a four dimensional effective field theories by string S-matrix approach.

Compactifications that make able the S-matrix approach are Toroidal orbifolds. Essentially they are the result of a toroidal background $T^d$ under the action of discrete group of isometries (orbifold group)
\[ X_6 = \frac{T^6}{\mathbb{Z}_N}, \quad X_6 = \frac{T^6}{\mathbb{Z}_N \times \mathbb{Z}_M} \]  
which acts identifying some points and leaving fixed others on the six-torus $T^6$ lattice. At the fixed points, orbifolds fail to be manifolds, because they are a singular points on $X^6$. But outside the singularities, the resulting geometry is locally flat, thus one can use CFT methods. Sometimes orbifolds are referred as CY limits because the singularities can be removed using a blowing up process. The fields content of orbifold compactifications, is a truncation of the six-torus $T^6$ compactification to invariant states under the orbifold action. They fall into $\mathcal{N} = 2$ supermultiples (Type II) or $\mathcal{N} = 1$ supermultiplets (Heterotic) in four dimensions, as compactification on CY3. The fields content of Type-I superstring both on toroidal orbifolds and on CY3 can also be organised in $\mathcal{N} = 1$ supermultiplets in four dimensions, but in the literature sometimes appears as a special case of more general class of Type II orientifold compactifications, where in general the final amount of supersymmetries in four dimensions is $\mathcal{N} = 1$ due to the non-trivial orientifold projection, and the resulting spectrum is a truncation to states that are invariant under the orientifold action. In Type II orientifold compactifications extended objects as D-branes and $\Omega$-planes have to be included for consistency of the theories. The moduli are collected into chiral multiplets, and orientifold projections at tree level doesn’t destroy the shape of closed string moduli space $\mathcal{L}_{(1,1)}$ until open strings moduli (wilson lines) coming from D-branes that wrap internal directions, enter the spectrum. Even if there is yet no evidence of supersymmetry, the main attention is focused on theories with $\mathcal{N} = 1$ supersymmetry in four dimensions because the presence of chiral matter is need in order to make contact between string theories and the Standard Model. The informations on the structure of any $\mathcal{N} = 1$ supergravity Lagrangian in four dimensions are encoded by Kähler potential $\mathcal{K}(\phi, \bar{\phi})$, superpotential $W(\phi)$ and the (matrix) gauge kinetic function $f(\phi)$. In the context of string compactification theories these functions depend non-trivially from the complex scalar fields $\phi$ which arise from compactification process. The resulting four dimensional LEEA for the bosonic fields can be summarised as
\[ \mathcal{L}_{\mathcal{N}=1}^{SG} = \frac{R}{2k_4^2} \mathcal{K}_{IJ}(\phi, \bar{\phi}) \nabla_\mu \phi^I \nabla^\mu \phi^J - V(\phi, \bar{\phi}) - \frac{1}{8} \text{Re}(f_{ab}(\phi)) F^a_{\mu\nu} F^{b \mu\nu} - \frac{1}{8} \text{Im}(f_{ab}(\phi)) F^a_{\mu\nu} F^{b \mu\nu} + \ldots \]  
where $a, b$ label the gauge group representation and $\mathcal{K}_{IJ}$ the Kähler metric
\[ \mathcal{K}_{IJ}(\phi, \bar{\phi}) = \frac{\partial^2 \mathcal{K}(\phi, \bar{\phi})}{\partial \phi^I \partial \bar{\phi}^J} \]  

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6 For completeness, there are also several abstract CFT constructions that bring solutions to this problem. For instance Gepner models, not discussed in this paper.

7 We don’t consider this general case, but for completeness, when D-branes moduli are dynamical fields the geometry of moduli space is described by both closed and open string moduli, i.e in principle there is no factorisation, and the shape is more complicated.
with $I, J$ running both on geometrical and non geometrical moduli, while the scalar potential $V(\phi, \bar{\phi})$ with its $F$-term and $D$-term respectively is given by

$$V(\phi, \bar{\phi}) = e^{k^2 K} \left( K^{I \bar{J}} \partial_I \partial_{\bar{J}} \mathcal{W} - 3k_2^2 |W|^2 \right) + \frac{1}{2} (\Re e(f^{-1}))_{ab} \partial^a D^b$$

with $\nabla_I \equiv \partial_I + k_2^2 (\partial_I K)$ the Kähler covariant derivative. By the computation of tree-level string scattering amplitudes the explicit form of the terms in the Lagrangian can be derived and compared with the terms that coming from the dimensional reduction of the chosen higher dimensional string model. Until the $N = 1$ supersymmetry is unbroken it is also known which kinds of corrections the three relevant functions can receive. Kähler potential $K$ can in principle have corrections both perturbative and non-perturbative in $\alpha'$ and $g_s$. The superpotential $V$, by general non-renormalization theorems, can receives only non-perturbative $g_s$ corrections while, to the gauge kinetic function $f$, can appear corrections both perturbative (up to one-loop) and non-perturbative in $g_s$. These kinds of corrections are needed in order to arrive at some string solutions which can describe a positive vacuum energy (de Sitter vacua) and stabilise the moduli to positive mass-squared values. If they remain massless they can mediate long range forces and from the cosmological point of view overclose the universe.

The latter problem comes from the fact that at tree-level in $\alpha'$ and $g_s$ expansions, the scalar potential $V(\phi, \bar{\phi})$ (1.6) is non-scale type $^9$, i.e. is identically zero both for the Kähler moduli (geometrical and non-geometrical) and open moduli (when present), leaving them unfixxed. This happens in general because the superpotential $V$ at tree-level can depends only to the complex axion-dilaton and the complex structure moduli (geometric moduli), thus it is not able to generate all the $F$-terms needed.

The KKLT [18] and the LVS [19] are the two principal scenarios that took into account the Kähler moduli stabilisation adding non-perturbative contributions (in both) and perturbative ones (in LVS only). On these topics are based, for instance, the challenges on building cosmological string models which can link the inflation phase to the cosmological standard model (CSM) beyond the solutions of the classical theory. Our work is based on the computation of tree-level string scattering amplitudes useful to extract informations on the structure of LEEA at specific order in $g_s$ expansion, i.e from disk and projective plane worldsheet.

The paper is organized as follows. In Section 1 an introduction with motivations and general aspects, that we recall in the paper, are given. In Section 2 closed string scattering amplitudes, in $D \leq 10$ dimensions, on the disk $D_2$ and real projective plane $RP_2$ is computed: historical disk amplitudes computations and results are detailed reviewed [21, 22, 23, 25] and a preamble on the role of real projective plane is furnished, due to the presence of this surface and extended objects as $\Omega_p$-planes in unoriented theories. In Section 3 we perform an overview on Type IIB orientifold model and focus on Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with construction of: vertex operators for the closed untwisted Kähler moduli ($T$) and complex structure moduli ($U$) of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ in presence of $D$-brane, $\Omega$-plane and tool blocks of the two-point functions on $D_2$ and $RP_2$. In Section 4 two-point string scattering amplitudes with untwisted Kähler moduli ($T$) and complex structure moduli ($U$) of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ on disk $D_2$ and projective plane $RP_2$ are computed and discussed. It was also checked that there are no corrections: to the tree-level Kähler potential $K$ of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ when scattering amplitudes on $RP_2$ are considered [27], order $\alpha'$-corrections to the Einstein-Hilbert term $R$ from the amplitudes considered [27]. In Section 5 is performed a comparison at $\alpha'^2$-order between string scattering amplitudes results and (curvature)$^5$ terms in LEEA of $D$-branes and $\Omega$-planes, starting from a generic Type IIB orientifold model in high dimensions, going to the specific case of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. Finally, in Section 6, discussions and new perspectives follows.

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8 The fermionic terms can be obtained by supersymmetry.
9 Without D-term.
10 The axion-dilaton and complex structure moduli can in general stabilised turning on fluxes [17].
11 An exhaustive overview on inflation models by string theories approaches can be find in [20].
2 Scattering Amplitudes from $D_P$-brane and $\Omega_P$-plane

The purpose of this Section is to review ten (or less) dimensional two-point closed string scattering amplitudes, involving massless states, on the disk and real projective plane and their leading $\alpha'$ contributions (in particular $\alpha'^2$), in the context of unoriented string theory where objects like $D_P$-branes and $\Omega_P$-planes are necessary for consistency reasons. There are many works that approach these topics [21, 22, 23, 25], but for our purposes it is instructive to reproduce these results and point out precisely the $\alpha'^2$ contributions that will be present in low energy effective action as already mentioned in the works [25, 28, 29] for the disk and [26] for the projective plane. In the following one can find the general setup of scattering amplitudes including the specific choice of fixing gauge for the invariance under the specific conformal killing group together with results and implications. All the computational technicalities are relegated to the appendices.

2.1 $A_{D_2}^{(NS-NS,NS-NS)}$

The starting point is the computation of scattering amplitudes of two massless NS-NS states from $D_P$-branes, which at tree-level involves the disk $D_2$ as worldsheet. To this aim one is forced to use vertex operators different pictures in order to cancel the vacuum extra charge of the superghosts. In addition, the presence of the $D_P$-brane is taken into account by introducing the reflection matrix $\mathcal{R}$, a diagonal matrix with entries $+1$ in the Neumann (Poincaré preserving) directions and $-1$ in the Dirichlet (Poincaré non-preserving) directions. This information is encoded in the vertex operators as follows:

$$\mathcal{W}_{NS-NS(\pm 1)}^{(E,k,z,\bar{z})} = E_{MN} : \mathcal{V}_{\mathcal{R}}^{M}(k,z) \mathcal{V}_{\mathcal{R}}^{N}(k,\bar{z}) := E_{MN} \mathcal{R}_{\mathcal{Q}}^{N} : \mathcal{V}_{\mathcal{R}}^{M}(k,z) :: \mathcal{V}_{\mathcal{R}}^{Q}(k,\bar{z}) : \quad (2.1)$$

In particular, the form of the vertices in the two standard choices $(0,0)$ and $(-1,-1)$ of the superghost picture is

$$\mathcal{W}_{NS-NS(-1,-1)}^{(E,k,z,\bar{z})} = E_{MN} \mathcal{R}_{\mathcal{R}}^{N} : e^{-\phi} \psi^{M} e^{ikX}(z) :: e^{-\phi} \psi^{P} e^{ik\mathcal{R}X}(\bar{z}) :$$

$$\mathcal{W}_{NS-NS(0,0)}^{(E,k,z,\bar{z})} = E_{MN} \mathcal{R}_{\mathcal{R}}^{N} : \left( i\partial X^{M} + \frac{\alpha'}{2}(k\bar{\psi}) \psi^{M} \right) e^{ikX}(z) :: \left( i\bar{\partial} X^{P} + \frac{\alpha'}{2}(k\bar{\psi}) \psi^{P} \right) e^{ik\mathcal{R}X}(\bar{z}) : \quad (2.2)$$

The scattering amplitude is then given by

$$A_{D_2}^{(NS-NS,NS-NS)} = g_s^2 C_{D_2} \int_{\mathcal{H}^+} \frac{d^2 k \, d^2 \bar{z}}{V_{\mathcal{R}}} \langle \mathcal{W}_{NS-NS(-1,-1)}^{(E_1,k_1,z_1,\bar{z}_1)} \mathcal{W}_{NS-NS(0,0)}^{(E_2,k_2,z_2,\bar{z}_2)} \rangle_{\mathcal{H}^+}$$

$$= \frac{g_s^2 C_{D_2}}{\alpha'} \int_{\mathcal{H}^+} \frac{d^2 k \, d^2 \bar{z}}{V_{\mathcal{R}}} \left( E_{M_1 N_1} E_{M_2 N_2} \mathcal{R}_{\mathcal{R}}^{N_1} \mathcal{R}_{\mathcal{R}}^{N_2} : e^{-\phi} \psi^{M_1} e^{ik_1 \mathcal{X}}(z_1) :: e^{-\phi} \psi^{P_1} e^{ik_1 \mathcal{R}X}(\bar{z}_1) : \times \right. \left. \left( \frac{\alpha'}{2} (k_2 \bar{\psi}) \psi^{M_2} + \frac{\alpha'}{2} (k_2 \bar{\psi}) \psi^{P_2} \right) e^{ik_2 X}(z_2) \right)$$

$$= \frac{g_s^2 C_{D_2}}{\alpha'} \int_{\mathcal{H}^+} \frac{d^2 k \, d^2 \bar{z}}{V_{\mathcal{R}}} \langle e^{-\phi}(z_1) :: e^{-\phi}(\bar{z}_1) : \left( M^{(1)} + M^{(2)} + M^{(3)} + M^{(4)} \right) \rangle \quad (2.3)$$

where $M^{(i)}$'s indicate the different sub-amplitudes that one has to calculate, as for instance

$$M^{(1)} = E_{M_1 N_1} E_{M_2 N_2} \mathcal{R}_{\mathcal{R}}^{N_1} \mathcal{R}_{\mathcal{R}}^{N_2} : \psi^{M_1} e^{ik_1 \mathcal{X}}(z_1) :: \psi^{P_1} e^{ik_1 \mathcal{R}X}(\bar{z}_1) :: i\partial X^{M_2} e^{ik_2 \mathcal{X}}(z_2) :: i\bar{\partial} X^{P_2} e^{ik_2 \mathcal{R}X}(\bar{z}_2) : \quad (2.4)$$

In order to compute all the contractions the relevant two-point functions are

$$\langle e^{-\phi}(z) e^{-\phi}(\bar{z}) \rangle_{D_2} = \frac{1}{(z-\bar{z})}, \quad \langle X^{M}(z) X^{N}(\bar{z}) \rangle_{D_2} = \frac{\alpha'}{2} \eta^{MN} \log|z - \bar{z}|$$

$$\langle X^{M}(z) X^{N}(\bar{w}) \rangle_{D_2} = -\frac{\alpha'}{2} \eta^{MN} \log(z - \bar{w}), \quad \langle X^{M}(z) X^{N}(w) \rangle_{D_2} = -\frac{\alpha'}{2} \eta^{MN} \log(z - w)$$

$$\langle \psi^{M}(z) \psi^{N}(\bar{\psi}) \rangle_{D_2} = \frac{\eta^{MN}}{(z - \bar{\psi})}, \quad \langle \psi^{M}(z) \psi^{N}(\bar{w}) \rangle_{D_2} = \frac{\eta^{MN}}{(z - \bar{w})}, \quad \langle \psi^{M}(z) \psi^{N}(w) \rangle_{D_2} = \frac{\eta^{MN}}{(z - w)} \quad (2.5)$$

In order to compute all the contractions the relevant two-point functions are
The explicit result of the $\mathcal{M}^{(i)}$’s calculation, being quite cumbersome, is reported in [54] (appendix A.2). Moreover, one has to remove the redundancy related to the CKG of $D_2$. Identifying the disk with $\mathcal{H}^+ = S_2/\mathcal{P}_{D_2}$, the residual CKG is $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2$, i.e. the subgroup of $SL(2, \mathbb{C})$ that preserves the involution $\mathcal{P}_{D_2}(z) = \bar{z}$. The finite transformations are [30, 31, 32]

$$z \mapsto z' = \frac{az + b}{cz + d}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \quad \text{i.e.} \quad \begin{cases} a, b, c, d \in \mathbb{R} \\ ad - cb = 1 \to \det M = 1 \end{cases}$$

(2.6)

The corresponding Lie algebra is $\mathfrak{sl}_2(\mathbb{R})$, which generators are $2 \times 2$ traceless real matrices [33]. The infinitesimal transformation can be obtained from the finite transformation expanding it around $a = d = 1; c = b = 0$ as follows

$$\delta(z') = \delta(\frac{az + b}{cz + d}) \bigg|_{a=d=1; c=b=0}$$

$$= \frac{z}{cz + d} \delta a + \frac{1}{cz + d} \delta b - \frac{az + b}{(cz + d)^2} \delta d - \frac{a z^2 + b z}{(cz + d)^2} \delta c \bigg|_{a=d=1; c=b=0}$$

$$= \delta a + \delta d = 0 \to \begin{cases} \delta b + 2 \delta a z - \delta c z^2 \\ p := \delta b, q := 2 \delta a, m := -\delta c \end{cases}$$

$$= p + q z + m z^2 \quad p, q, m \in \mathbb{R}.$$  

Writing $z = x + iy$ we have

$$\delta(x' + iy') = p + q(x + iy) + m(x + iy)^2 \to \begin{cases} \delta x' = p + qx + m(x^2 - y^2) \\ \delta y' = qy + 2mxy \end{cases}.$$  

(2.8)

The $PSL(2, \mathbb{R})$ symmetry allows to fix the positions of three chiral vertex operators\footnote{In the case with two closed string one can fix two chiral(anticrhal) and one antichiral(chiral).}. In the specific case, there are two closed strings and, using the doubling trick, the amplitude can be written

$$\int \frac{dz_1 dz_2 d\bar{z}_2}{V_{CKG}} \langle : V_L^1(z_1) :: V_L^1(\bar{z}_1) :: V_L^2(z_2) :: V_L^2(\bar{z}_2) : \rangle; \quad V_{CKG} = \int dp dq dm$$

(2.9)

where the integral over the parameters is clearly divergent without fixing the symmetry. Writing the surface element $dz d\bar{z}$ in terms of its real components [30]

$$dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 = 4dx_1 dy_1 dx_2 dy_2$$

(2.10)

and using (2.6) it is possible to fix, for instance, $x_1, x_2$ and $y_1$:

$$4dx_1 dy_1 dx_2 dy_2 = 4 |J| dp dq dm dy_2; \quad J = \text{det} \begin{vmatrix} x_1 & 0 & 1 \\ (x_1^2 - y_1^2) & 2x_1y_1 & (x_2^2 - y_2^2) \end{vmatrix} = y_1(y_1^2 - y_2^2) + y_1(x_1 - x_2)^2$$

(2.11)

The same result can be obtained by inserting three unintegrated vertices, i.e putting $cV$ instead of $\int V$, where $c$ is the reparametrization ghost. This means that the $\langle |ccc| \rangle$ ghost correlator has to reproduce exactly the Jacobian of the $PSL(2, \mathbb{R})$ transformation. Fixing the symmetry using directly the $PSL(2, \mathbb{R})$ or the correct $\langle |ccc| \rangle$ correlator must give the same result. In [30] one can verify that in all most common examples such as 3 unintegrated closed vertices on the sphere or three unintegrated open vertices on the disk, the Jacobian of the transformation can be written as a $\langle |ccc| \rangle \langle |\bar{c}\bar{c}\bar{c}| \rangle$ and $\langle |ccc| \rangle$ ghost correlator respectively. The cases of purely closed or open/closed amplitudes on the disk
are less direct. The reason is that \( PSL(2, \mathbb{R}) \) acts on the complex variable \( z \) in non trivial way (2.6). In the purely-closed amplitudes, the Jacobian (2.11) can not be reproduced by a single \(|cc\rangle\) insertion, but can be reproduced by inserting the linear combination

\[
|\langle c(z_1)c(\bar{z}_1)c(z_2)\rangle + \langle c(z_1)c(\bar{z}_1)c(z_2)\rangle| = |(z_1 - \bar{z}_1)(\bar{z}_1 - z_2)(z_1 - z_2) + (z_1 - \bar{z}_1)(\bar{z}_1 - \bar{z}_2)(z_1 - \bar{z}_2)|
\]

\[
= |4y_1(y_1^2 - y_2^2) + 4y_1(x_1 - x_2)^2|
\]

that takes into account both the position of the vertices in \( \mathcal{H}^+ \) as well as those on the images in \( \mathcal{H}^- \).

This way one gets

\[
dz_1dz_2d\bar{z}_1d\bar{z}_2 = 4|J|dpdqdm\ dqy_2 \equiv |\langle c(z_1)c(\bar{z}_1)c(z_2)\rangle + \langle c(z_1)c(\bar{z}_1)c(z_2)\rangle|dpdqdm\ dqy_2
\]

(2.13)

One can see that specializing conveniently the points as

\[
z_1 \mapsto z_1' = i; \quad z_2 \mapsto z_2' = iy; \quad \Leftrightarrow \quad x_1 = 0; \quad y_1 = 1; \quad x_2 = 0, \quad y_2 = y
\]

\[
z_1' = -i; \quad z_2' = -iy \quad \Leftrightarrow \quad x_1 = 0, \quad y_1 = 1; \quad x_2 = 0, \quad y_2 = y
\]

(2.14)

inserting these in (2.13), with the help of (2.11) and (2.12), the final integration measure takes the following form

\[
dz_1dz_2d\bar{z}_1d\bar{z}_2 = 4(1-y^2)dpdqdm\ dqy.
\]

(2.15)

After this procedure the integral becomes:

\[
\mathcal{A}^{(NS-NS,NS-NS)}_{D_2} = \frac{4g^2C_{D_2}}{\alpha'} \int_0^1 dy \left\{ \frac{4y}{(1+y)^2} \right\}^{-\alpha'} \left\{ \frac{1-y}{1+y} \right\}^{-\alpha'} \frac{4(1-y^2)}{16y^2}
\]

(2.16)

where

\[
a_1 = -\frac{\alpha'^2}{4} \left\{ \frac{\text{Tr}(E_1^T E_2)k_1 R k_1}{2} + \text{Tr}(R E_1)k_1 E_2 k_1 - k_2 E_1 R E_2 k_1 + k_2 R E_1^T E_2 k_1 + \frac{k_2 E_1 E_2 k_1}{2} \right\}
\]

\[
a_2 = -\frac{\alpha'^2}{4} \left\{ \frac{\text{Tr}(E_1^T E_2)k_1 R k_1}{2} - \frac{\text{Tr}(E_1 R E_2 R)k_2 R k_2}{2} - \text{Tr}(R E_1)k_1 E_2 R k_2 + \text{Tr}(R E_1)k_2 R E_2 R k_2 \right\}
\]

\[
a_3 = \frac{\alpha'}{2} \text{Tr}(R E_1) \text{Tr}(R E_2).
\]

(2.17)

The complete derivation of the \( a_i \) coefficients can be found in [54] (appendix A.3). According to [21, 22, 23], performing in (2.16) the change of variable

\[
y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}
\]

(2.18)

the result reads

\[
\frac{4g^2C_{D_2}}{\alpha'} \left\{ a_1 B[(-\alpha's+1);(-\alpha't/4)] + a_2 B[(-\alpha's);(-\alpha't/4+1)] + a_3 (1+\alpha's) B[(-\alpha's-1);(-\alpha't/4+1)] \right\}.
\]

(2.19)

The latter result, using the \( \Gamma \) function properties, can be written as

\[
\mathcal{A}^{(NS-NS,NS-NS)}_{D_2} = \frac{4g^2C_{D_2}}{\alpha'} \left( -\alpha' s a_1 - \alpha' t \frac{\alpha'}{4} \hat{a}_2 \right) \frac{\Gamma(-\alpha's)\Gamma(-\alpha't/4)}{\Gamma(-\alpha's - \alpha't/4 + 1)}
\]

(2.20)
that is symmetric under the \((1 \leftrightarrow 2)\) exchange as in \([26, 21, 22, 23]\) with

\[
\hat{a}_2 = a_2 + \left(\alpha' s + \alpha' \frac{t}{4}\right) a_3.
\] (2.21)

In order to identify the leading \(\alpha'\) corrections one expands the combination of gamma functions in the limit \(\alpha' \to 0\):

\[
\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t/4)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} = 4 \frac{\alpha' s \alpha' t}{\alpha' s \alpha' t} - \zeta(2) - \left(\alpha' s + \alpha' \frac{t}{4}\right) \zeta(3) + O(\alpha'^2)
\] (2.22)

combining this with the terms above, one can find that the amplitude \((2.20)\) exhibits open string poles in the s-channel and closed string poles in the t-channel as expected. There are also terms proportional to \(Tr(RE_1)Tr(RE_2)\) constant in \(\alpha'\). The \(\alpha'^2\) corrections coming from \((2.20)\) reads

\[
g_c^2 C_{d_2} \zeta(2) \left\{ 4s a_1 + t a_2 + \left(\alpha' s + \alpha' \frac{t}{4}\right) t a_3 \right\}.
\] (2.23)

### 2.2 \(A_{RP_2}(NS−NS, NS−NS)\)

Scattering amplitudes of two massless NS-NS states from \(O_p\)-planes involve at tree level a world-sheet with the topology of the real projective plane. They can be dealt with in analogy to the disk amplitudes. \(O_p\)-planes are the fixed loci of the space-time involution whose combined action with the world-sheet parity operator \(\Omega\) (possibly dressed with suitable action on fermions) realizes the unoriented projection together with \((9-p)\) T-dualities. The real projective plane is a quotient of the Riemann sphere via the anti-conformal involution \(J_{RP_2}(z) = -1/\bar{z}\). Possible choices for the fundamental region are thus the upper-half-plane or the unit disk \([24]\). Vertex operators on the real projective plane must be defined consistently with the involution. The combination that takes into account all of these characteristics is

\[
W_{NS−NS(q,\bar{q})}(E, k, z, \bar{z}) \to_{RP_2} W_{\Omega NS−NS(q,\bar{q})}(E, k, z, \bar{z})
\] (2.24)

where

\[
W_{\Omega NS−NS(q,\bar{q})}(E, k, z, \bar{z}) = \frac{1}{2} E_{MN} \left\{ h^M_{(q)}(k, z) W^N_{(\bar{q})}(\bar{k}, \bar{z}) : + : W^M_{(\bar{q})}(\bar{k}, \bar{z}) W^N_{(q)}(k, z) : \right\}
\] (2.25)

Using the doubling trick it is possible to arrive at the final form of the vertex operator

\[
W_{\Omega NS−NS(q,\bar{q})}(E, k, z, \bar{z}) = \frac{1}{2} E_{MN} \left\{ R_{PQ} \nu^M_{(q)}(k, z) \nu^P_{(\bar{q})}(\bar{k}, \bar{z}) : + R_{Q} \nu^M_{(q)}(k, z) \nu^P_{(\bar{q})}(\bar{k}, \bar{z}) : \right\}
\] (2.26)

In the standard pictures, one has

\[
W_{\Omega NS−NS(−1,−1)}(E, k, z, \bar{z}) = \frac{E_{MN} R^P_{N}}{2} \left\{ e^{-\phi} \psi \psi^P e^{ikX(z)} e^{-\phi} \psi \psi^P e^{ikX(\bar{z})} : + (z \leftrightarrow \bar{z}) \right\}
\] (2.27)

\[
W_{\Omega NS−NS(0,0)}(E, k, z, \bar{z}) = \frac{E_{MN} R^P_{N}}{2} \left\{ i \partial^X^M + \frac{\alpha'}{2} (k \psi) \psi^M e^{ikX(z)} : + (z \leftrightarrow \bar{z}) \right\}
\]

Now one can starts with the computation of the two-point scattering amplitude from \(O_p\)-plane off massless NS-NS states. Due to the involution of the real projective plane there are several sub-amplitudes \(\Lambda_i\) that one has to consider

\[
A_{\Omega(RP_2)}(NS−NS, NS−NS) = \frac{g_c^2 C_{d_2}}{V_{CGK}} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{|z_1 - z_2|} \langle W_{\Omega NS−NS(−1,−1)}(E_1, k_1, z_1, \bar{z}_1) W_{\Omega NS−NS(0,0)}(E_2, k_2, z_2, \bar{z}_2) \rangle_{RP_2} = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4
\] (2.28)
where a representative $\Lambda_i$ has the following form

$$\Lambda_1 = g_c^2 C_{RP_2} \alpha' \int \frac{d^2 z_1 d^2 z_2}{V_{CG}} E_{M_1 N_1} E_{M_2 N_2} R_{P_1}^M R_{P_2}^N \langle e^{-\phi} \psi_1^M e^{i k_1 X(z_1)} : e^{-\phi} \psi_1^N e^{i k_1 RX(z_1)} : \rangle$$

\[
= g_c^2 C_{RP_2} \alpha' \int \frac{d^2 z_1 d^2 z_2}{V_{CG}} \langle e^{-\phi}(z_1) : e^{-\phi}(z_2) \rangle \left( M^{(1)}_{\Lambda_1} + M^{(2)}_{\Lambda_1} + M^{(3)}_{\Lambda_1} + M^{(4)}_{\Lambda_1} \right)
\]

and once again can be separated like the disk amplitude in $M_{\Lambda_i}^{(t)}$'s terms. The complete set of sub-amplitudes can be found in [54] (appendix A.4), and the basic two-point functions used to evaluate them are

\[
\begin{align*}
\langle e^{-\phi}(z) e^{-\phi}(\bar{z}) \rangle_{RP_2} &= \frac{1}{1 + z \bar{z}}, \quad \langle X^M(z) X^N(\bar{z}) \rangle_{RP_2} = -\frac{\alpha'}{2} \eta^{MN} \log |1 + z \bar{z}| \\
\langle X^M(z) X^N(\bar{w}) \rangle_{RP_2} &= -\frac{\alpha'}{2} \eta^{MN} \log (1 + z \bar{w}), \quad \langle X^M(z) X^N(w) \rangle_{RP_2} = -\frac{\alpha'}{2} \eta^{MN} \log (z - w) \\
\langle \psi^M(z) \psi^N(\bar{z}) \rangle_{RP_2} &= \eta^{MN}_{\bar{z}}(1 + z \bar{z}), \quad \langle \psi^M(z) \psi^N(w) \rangle_{RP_2} = \eta^{MN}_{z - w}
\end{align*}
\]

Using the definitions of the usual kinematical invariants (see appendix (A.1)) for the Koba-Nielsen factors, it is convenient to combine the $\Lambda$-subamplitudes in pairs as follows\textsuperscript{13}

\[A = (\Lambda_1 + \Lambda_1) + (\Lambda_2 + \Lambda_3).\]

As one knows, in order to compute the integral in (2.28), one has to fix the redundancy of the conformal killing group of $RP_2$. The CKG of $RP_2$ is the subgroup of $SL(2,\mathbb{C})$ that is invariant under the anti-conformal involution $J_{RP_2}(z)$, and it comes out to be $SU(2)$. In this case the finite transformation reads

\[z \mapsto z' = \frac{u z + v}{\bar{v} z + \bar{u}}, \quad L = \left( \begin{array}{c} u \\ \bar{v} \end{array} \right) \in SU(2) \quad i.e. \quad \left\{ \begin{array}{l} u = 1 + i \beta \\ v = \alpha \equiv \gamma + i \lambda \\ \beta, \gamma, \lambda \in \mathbb{R} \\ |u|^2 + |v|^2 = 1 \end{array} \right.\]

and the $su(2)$ Lie algebra described by $2 \times 2$ traceless complex matrices. In agreement with [34] the infinitesimal transformation is

\[\delta(z') = \delta \left( \frac{u z + v}{\bar{v} z + \bar{u}} \right) \bigg|_{\beta = \gamma = \lambda = 0} = \frac{i z (\gamma + i \lambda) z + (1 - i \beta)}{[(\gamma + i \lambda) z + (1 - i \beta)]^2} \bigg|_{\beta = \gamma = \lambda = 0} \delta \beta + \frac{i z [(\gamma + i \lambda) z + (1 - i \beta)]^2}{[(\gamma + i \lambda) z + (1 - i \beta)]^2} \bigg|_{\beta = \gamma = \lambda = 0} \delta \gamma + \frac{i z [(\gamma + i \lambda) z + (1 - i \beta)]^2}{[(\gamma + i \lambda) z + (1 - i \beta)]^2} \bigg|_{\beta = \gamma = \lambda = 0} \delta \lambda\]

that produces

\[\left\{ \begin{array}{l} 2iz \delta \beta + (1 + z^2) \delta \gamma + i (1 - z^2) \delta \lambda \\ e := 2 \delta \beta, \quad f := \delta \gamma, \quad g := \delta \lambda \\
\end{array} \right. = (f + ig) + i e z + (f - ig) z^2, \quad e, f, g \in \mathbb{R}\]

Taking $z = q + it$,

\[\delta(q' + it') = (f + ig) + ie(q + it) + (f - ig)(q + it)^2 \rightarrow \left\{ \begin{array}{l} \delta q' = [1 + (q^2 - t^2)] f - e t + 2 t q g \\
\delta t' = 2 t f + e q + [1 - (q^2 - t^2)] g
\end{array} \right.\]

\textsuperscript{13} No picture changing is needed to combine together the $\Lambda$-subamplitudes. We checked that the amplitude is picture changing invariant, i.e. there is no dependence on the picture distribution for the vertex operators, as argued in [26].
one can fix three positions choosing the value of, for instance, \( q_1, t_1 \) and \( q_2 \) using the \( SU(2) \) symmetry, thus
\[
dz_1 dz_1 dz_2 dz_2 = 4 dq_1 dt_1 dq_2 dt_2 = 4 |J| df dg de dt_2
\] (2.36)

where the Jacobian \( J \) is
\[
J = \det \begin{vmatrix} 1 + (q_1^2 - t_1^2) & 2q_1 t_1 & 1 + (q_2^2 - t_2^2) \\ 2q_1 t_1 & 1 - (q_1^2 - t_1^2) & 2q_2 t_2 \\ -t_1 & -t_2 & 0 \end{vmatrix}
\] (2.37)
\[
= t_1 \{[1 + (q_1^2 + t_1^2)][1 + (q_2^2 - t_2^2)]\} - t_2[1 + (q_1^2 + t_1^2)] \{2q_2 t_1 + [1 - (q_2^2 + t_2^2)]\}
\]
The combination of \(|\langle ccc \rangle| \) ghost correlators is again
\[
|\langle c(z_1) c(\bar{z}_1) c(z_2) \rangle - \langle c(z_1) c(\bar{z}_1) c(z_2) \rangle| = |\langle 1 + z_1 \bar{z}_1 \rangle(z_1 - z_2) - (1 + z_1 \bar{z}_1)(\bar{z}_1 - \bar{z}_2)(1 + z_1 \bar{z}_2)\|
\]
\[
= 2t_1 \{[1 + (q_1^2 + t_1^2)][1 + (q_2^2 - t_2^2)]\} - 2t_2[1 + (q_1^2 + t_1^2)] \{2q_2 t_1 + [1 - (q_2^2 + t_2^2)]\}
\] (2.38)

then
\[
dz_1 dz_1 dz_2 dz_2 = 4 |J| df dg de dt_2 = 2 \langle c(s_1) c(\bar{s}_1) c(s_2) \rangle - \langle c(s_1) c(\bar{s}_1) c(s_2) \rangle |df dg de dt_2 \] (2.39)

In particular, for the specific case of interest a convenient choice is to fix the vertices in
\[
\begin{align*}
z_1 &= 0; \quad z_2 = iy \to q_1 = 0, \quad t_1 = 0; \quad q_2 = 0, \quad t_2 = y \\
\bar{z}_1 &= 0; \quad \bar{z}_2 = -iy \to q_1 = 0, \quad t_1 = 0; \quad q_2 = 0, \quad t_2 = y
\end{align*}
\] (2.40)

with this choice many terms vanish (see [54] appendix A.4). From (2.39) using (2.37) and (2.38) one gets
\[
ds_1 ds_1 ds_2 ds_2 = 4 y df dg de dy
\] (2.41)

In this setup, there only remain the following two contributions
\[
\begin{align*}
\Lambda_1 + \Lambda_4 &= \frac{2g^2 C_{R\alpha}}{\alpha'} \int_0^1 dy^2 (1 + y^2)^{-\alpha'} y^{-2a' \frac{1}{2}} \left( \frac{a_1}{y^2} + \frac{a_2}{(1 + y^2)^2} \right) + \frac{a_3 (1 + \alpha')}{(1 + y^2)^2} \\
\Lambda_2 + \Lambda_3 &= \frac{2g^2 C_{R\alpha}}{\alpha'} \int_0^1 dy^2 (1 + y^2)^{-\alpha'} y^{-2a' \frac{1}{2}} \left( \frac{a_1}{y^2} + \frac{a_2}{y^2(1 + y^2)} + \frac{a_3 (1 + \alpha')}{(1 + y^2)^2} \right)
\end{align*}
\] (2.42)

with
\[
\begin{align*}
a_1 &= -\frac{\alpha'^2}{4} \left\{ -\frac{1}{2} \text{Tr}(E_1^T E_2)k_1 R k_1 - \text{Tr}(RE_1)k_1 E_2 k_1 + k_2 E_1 R E_2 k_1 - k_2 R E_1 E_2^T k_1 - \frac{1}{2} k_2 E_1 E_2^T k_1 \\
&\quad - k_2 R E_1^T E_2 k_1 - \frac{1}{2} k_2 E_1 E_2^T k_1 + (1 \leftrightarrow 2) \right\} \\
\frac{1}{2} \{ \text{Tr}(E_1^T E_2)k_1 R k_1 - \frac{1}{2} \text{Tr}(E_1 R E_2)k_2 R k_2 + \text{Tr}(RE_1)k_2 R E_2 k_1 + \frac{1}{2} k_1 R E_1 E_2^T R k_2 + \\
\frac{1}{2} k_1 R E_1^T E_2 R k_2 - \text{Tr}(RE_1)k_1 E_2 R k_2 + (1 \leftrightarrow 2) \left\} \\
\frac{1}{2} \{ \text{Tr}(RE_1) \} \text{Tr}(RE_2)
\end{align*}
\] (2.43)

and the details on the terms inside the \( a_4 \) are treated in [54] (appendix A.5). Using the integral representation of the hypergeometric function
\[
2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 du u^{\beta - 1}(1 - u)^{\gamma - \beta - 1}(1 - uz)^{-\alpha}
\] (2.44)
with $\gamma = \beta + 1$ and $z = -1$, one obtains

\[
\Lambda_1 + \Lambda_4 = \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ a_1 2F_1(\alpha's, -\alpha't/4; -\alpha't/4 + 1; -1) \right. + a_2 2F_1(\alpha's + 1, -\alpha't/4 + 1; -\alpha't/4 + 2; -1)
+ a_3 (1 + \alpha's) 2F_1(\alpha's + 2, -\alpha't/4 + 1; -\alpha't/4 + 2; -1) \left. \right\}
\]

\[
\Lambda_2 + \Lambda_3 = \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ a_1 2F_1(\alpha's, -\alpha'u/4; -\alpha'u/4 + 1; -1) \right. + a_2 2F_1(\alpha's + 1, -\alpha'u/4; -\alpha'u/4 + 1; -1)
+ a_3 (1 + \alpha's) 2F_1(\alpha's + 2, -\alpha'u/4 + 1; -\alpha'u/4 + 2; -1) \left. \right\}
\]

that with the help of the identity

\[
b 2F_1(a, a + b; a + 1; -1) + a 2F_1(b, a + b; b + 1; -1) = \frac{\Gamma(a + 1)\Gamma(b + 1)}{\Gamma(a + b)}
\]

it is straightforward to see that all the subamplitudes $\Lambda_i$ combine together, because for instance

\[
2F_1(\alpha's, -\alpha'u/4; -\alpha'u/4 + 1; -1) = \frac{\Gamma(\alpha't/4)\Gamma(\alpha'u/4) - (-\alpha'u/4)\Gamma(\alpha't/4)}{\Gamma(\alpha's + 1)2F_1(\alpha's, -\alpha't/4; -\alpha't/4 + 1; -1)}
\]

and exploiting the $\Gamma$ function properties, one arrives at the same conclusion as [26]

\[
\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 = \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ a_1(\alpha's) \Gamma(\alpha't/4)\Gamma(\alpha'u/4) - (-\alpha'u/4)\Gamma(\alpha't/4) \right. + a_2(-\alpha't/4)\Gamma(-\alpha'u/4)
+ a_3(1 + \alpha's)(-\alpha't/4)(-\alpha'u/4)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
+ a_3(1 + \alpha's)(-\alpha'u/4 + 1)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4 + 1)
\]

\[
= \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ \alpha's a_1 - \alpha't/4 a_2 \right\} \Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
\]

\[
= \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ \alpha's a_1 - \alpha't/4 a_2 \right\} \Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
\]

\[
= \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ \alpha's a_1 - \alpha't/4 a_2 \right\} \Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
\]

\[
= \frac{2g_c^2 C_{RP2}}{\alpha'} \left\{ \alpha's a_1 - \alpha't/4 a_2 \right\} \Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
\]

where $\hat{a}_2 = a_2 - \frac{\alpha' a_3}{4}$. Taking the limit $\alpha' \to 0$ as in (2.22) the amplitude (2.48) exhibits only closed string poles in the $t$-channel and $u$-channel due to the presence of $\Omega_P$-planes where no open strings can be added. As for the disk case we extract the terms proportional to $\alpha'^2$:

\[
g_c^2 C_{RP2} \zeta(2) \left\{ -2s a_1 + \frac{t}{2} a_2 + \alpha' ut \frac{8}{a_3} \right\}
\]

in agreement with [26, 35]. At this point it is also interesting to compare the pole structures of both disk and real projective plane amplitudes

\[
A_{D2} = \frac{4g_c^2 C_{D2}}{\alpha'} (-\alpha's a_1^{D2} - \alpha't/4 a_2^{D2}) \Gamma(-\alpha's)\Gamma(-\alpha't/4)
\]

\[
\frac{\Gamma(-\alpha's - \alpha't/4 + 1)}{\Gamma(-\alpha's - \alpha't/4 + 1)}
\]

\[
A_{RP2} = \frac{2g_c^2 C_{RP2}}{\alpha'} (-\alpha's a_1^{RP2} - \alpha't/4 a_2^{RP2}) \Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)
\]

\[
\frac{\Gamma(-\alpha't/4 - \alpha'u/4 + 1)}{\Gamma(-\alpha't/4 - \alpha'u/4 + 1)}
\]

where it is straightforward to verify that the factor inside the round bracket is the same in both the amplitudes since $a_1^{RP2} = -a_1^{D2}$ and $a_2^{RP2} = a_2^{D2}$. We match the pole expansion in the $t$-channel, using
Our analysis is done on the full amplitudes (2.50), not on partial subamplitudes as in [26].

The following representations

\[ A_{D_2} \sim \frac{\Gamma(\alpha' t + \alpha' u) \Gamma(-\alpha' t)}{\Gamma(\alpha' t + \alpha' u + 1)} = -\frac{\sin\left(\frac{\pi(\alpha' t)}{\alpha'}\right)}{\pi} \Gamma\left(-\alpha' t + \alpha' u\right) \Gamma\left(-\alpha' t + \alpha' u + 1\right) \]

\[ A_{RP_2} \sim \frac{\Gamma(-\alpha' t) \Gamma(\alpha' t)}{\Gamma(-\alpha' t - \alpha' u + 1)} = \frac{\sin\left(\frac{\pi(\alpha' t + \alpha' u)}{\pi}\right)}{\pi} \Gamma\left(-\alpha' t - \alpha' u\right) \Gamma\left(-\alpha' t + \alpha' u + 1\right) \]

where has been used \( \alpha' = -\alpha' t - \alpha' u / 4 \) and the Gamma function property \( \Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z) \). In (2.51) the \( t \)-channel poles are due to \( \Gamma(-\alpha' t / 4) \) and the matching of the Residue at the pole \( \alpha' t / 4 = n \) is equal to

\[ \text{Res}(A_{D_2})|_{\alpha' t = n} \sim \left(-\frac{1}{n}\right)^{n+1} \frac{\sin\left(\frac{\pi(\alpha' t)}{\alpha'}\right)}{\pi n!} \Gamma\left(-\alpha' t + \alpha' u\right) \Gamma\left(n + \alpha' u\right) \]

\[ \text{Res}(A_{RP_2})|_{\alpha' t = n} \sim \frac{\sin\left(\frac{\pi(\alpha' t + \alpha' u)}{\pi}\right)}{\pi n!} \Gamma\left(-\alpha' t - \alpha' u\right) \Gamma\left(n + \alpha' u\right) \]

with \( n \) integer. At fixed \( n \), one can verify that, for \( n \)-odd the poles have the same sign while for \( n \)-even the sign of the poles are opposite in agreement with [26].

Moreover the mass-square \( M_n^2 \), in the \( t \)-channel, for the closed string can be expressed as

\[ M_n^2 = \frac{4n}{\alpha'} := \frac{2}{n} (2n) = \frac{2}{\alpha'} \left((N_a + \tilde{N}_a) + (N_b + \tilde{N}_b) - \delta^{(NS-R)}\right) \]

where it is straightforward to verify that only the GSO-projected states are permitted.

### 3 Type IIB orientifold on \( T^6 / \mathbb{Z}_2 \times \mathbb{Z}_2 \)

Our aim is to compute string two-point scattering amplitude involving closed string moduli, at tree-level in \( g_s \) expansion, in particular on the first worldsheet spanned by open string and closed unoriented string, i.e the disk \( D_2 \) and the real projective plane \( RP_2 \) respectively, postponing to Section 4 the amplitude calculations. In this section our attention is focused on the formulation of the needed tools, as the explicit construction of the relevant vertex operators and two-point functions on these specific worldsheet surfaces, in order to make contact between the string approach and the supergravity construction of the four dimensional LEEA. We firstly summarise the basic aspects of Type IIB orientifold on \( T^6 / \mathbb{Z}_2 \times \mathbb{Z}_2 \), giving an overview on: Type IIB toroidal orbifold \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) spectrum, the extended objects that the orientifold projection \( \Omega \sigma \) induces and the resulting parametrisation of the moduli space at tree-level in a specific situations from the supergravity point of view.

From the general Toroidal orbifold (1.3) the Type IIB on \( T^6 / \mathbb{Z}_2 \times \mathbb{Z}_2 \) which we consider is the one with Abelian orbifold group \( \Gamma \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \) without discrete torsion [36, 2, 3, 4] and satisfying the constraints needed to obtain \( N = 2 \) supersymmetry [37, 30, 38], "before" the orientifold projection. The root lattice of the underlying \( T^6 \) torus is chosen in a way that it factorises as

\[ T^6 = \bigotimes_{i=1}^{3} T^2_i \]

where the two-tori \( T^2_i \) are the blocks-diagonal part of \( T^6 \), and the elements of the group orbifold are \( G_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \{1, \theta_1, \theta_2, \theta_3\} \) with

\[ \theta_1 = (+, -, +) \quad \theta_2 = (-, +, -) \quad \theta_3 = (-, -, +) \]

\[ ^{14} \text{Our analysis is done on the full amplitudes (2.50), not on partial subamplitudes as in [26].} \]
where each element $\theta_I$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts as it were a single $\mathbb{Z}_2$ thus leaves invariant the $\mathbb{T}_j^4$ lattice base and flips with a minus sign the lattice of the corresponding two-torus $\mathbb{T}_{j\neq I}^2$. The torus partition function of this model is the truncation onto invariant states of the torus partition function of Type IIB on $\mathbb{T}^6$ torus, obtained by the action of a defined orbifold projection operator on the latter [4] (as the GSO projection operator in superstring). The resulting states of the spectrum are organised in sectors: the untwisted sector and the twisted sectors. The twisted sectors are characterised by strings that satisfy periodicity condition imposed by the orbifold group elements and their presence ensure the modular invariance of the torus partition function. As for CY compactification, the spectrum of toroidal orbifold is expressed using the Hodge classes $H^{p,q}$ and their dimensions $h^{p,q}$ are collected into the Hodge diamond that for the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ is [36]

$$
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 3 & 51 & 0 \\
0 & 51 & 3 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
$$

(3.3)

where $p$ and $q$ span the columns and the rows respectively. The massless spectrum of the Type IIB orbifold is contained in the matrix (3.3), since its entries count the number of super-multiplets in the low-energy effective field theory and $h^{1,1}, h^{2,1}$

$$
\begin{align*}
h^{1,1} &\equiv h^{1,1}_{\text{utw}} + h^{1,1}_{\text{tw}} = 3 + 0 \\
h^{2,1} &\equiv h^{2,1}_{\text{utw}} + h^{2,1}_{\text{tw}} = 3 + 48 \\
\end{align*}
$$

(3.4)

that are splitted into the untwisted (utw) and twisted (tw) sectors mutually. In terms of $N = 2$ supermultiplets in 4-dimensions the moduli are organised in hypermultiplets and vectormultiplets as follows

$$
\begin{align*}
\text{Utw} : (h^{1,1}_{\text{utw}} + 1) &\equiv (3 + 1) \quad \text{Hyper}, \quad h^{2,1}_{\text{utw}} \equiv 3 \quad \text{Vector} \\
\text{Tw} : h^{1,1}_{\text{tw}} &\equiv 0 \quad \text{Hyper}, \quad h^{2,1}_{\text{tw}} \equiv 48 \quad \text{Vector} \\
\end{align*}
$$

(3.5)

where we have take into account also the axion-dilaton hypermultiplet (or universal hypermultiplet). Our attention is on the untwisted moduli and on the moduli space that parameterise. Starting form the Type IIB on $\mathbb{T}^6$ in 4-dimensions with $N = 8$ supersymmetries, one knows that the 70 scalar fields are collected in the gravitational multiplet (unique multiplet) and parametrise a moduli space that is $E_{7(7)}/SU(8)$ which can be reduced to a factorised moduli space parameterised only by the geometrical moduli (as in Heterotic case [14])

$$
\frac{SU(1,1)}{U(1)} \times \frac{SO(6,6)}{SO(6) \times SO(6)}
$$

(3.6)

when the non-geometrical moduli are frozen [15] with the first coset refers to the axion-dilaton. The reduction of this coset (3.6) due to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action is the moduli space spanned by the untwisted moduli (3.5)[16]

$$
\frac{SU(1,1)}{U(1)} \times \left( \frac{SU(1,1)}{U(1)} \times \frac{SO(2,h^{1,1}_{\phi})}{SO(2) \times SO(h^{1,1}_{\phi} - 1)} \right) \times \left( \frac{SU(1,1)}{U(1)} \times \frac{SO(2,h^{2,1}_{\phi})}{SO(2) \times SO(h^{2,1}_{\phi} - 1)} \right)
$$

(3.7)

in agreement (when only geometrical moduli are dynamical fields) with the moduli space factorisation

$$
\frac{SU(1,1)}{U(1)} \times \tilde{S}_{K}^{h^{1,1}} \times S_{K}^{h^{2,1}}
$$

(3.8)

for Type IIB compactification on $CY_3$ with $N = 2$ supersymmetries (1.1) with both $\tilde{S}_{K}$ and $S_{K}$ special Kähler manifold parametrised by the hypermultiplets and vectormultiplets respectively . In the supergravity language the scalar fields in the hypermultiplets and vectormultiplets are respectively the complex Kähler moduli $t^I$ and the complex structure $u^I$ while the universal complex axion-dilaton is $s$. The Kähler potential (1.5) associated to the space (3.7) can be written as

$$
\kappa_4^2 K = - \ln(s + \bar{s}) - \ln \prod_{I=1}^{3}(t^I + \bar{t}^I) - \ln \prod_{I=1}^{3}(u^I + \bar{u}^I)
$$

(3.9)
with $\kappa_4$ the physical gravitational coupling constant in 4-dimensions introduced in (1.4). $\mathcal{N} = 1$ models in 4-dimensions can be obtained taking the orientifold projection $\Omega \sigma$ of Type II theories. To make clearer how to obtain these models we explain the key steps needed. The main feature that orientifold projection $\Omega \sigma$ introduces is the presence of non-dynamical $\Omega_p$-planes, with $P$ the dimensions of the worldvolume $[1, 4]$. Moreover different kinds of orientifold projections $\Omega \sigma$ on Type IIB induces specific $\Omega_p$-planes summarised in Table 1, where the discrete involution $I_n(n = 0, 2, 4, 6)$ acts both on the internal space and on the directions perpendicular to the $\Omega_p$-planes as a reflection fixing also the positions of the $\Omega_p$-planes, while $(-1)^F_L$ ensures that the $\Omega \sigma$ operator square to unity, as required for an involution. Additional $\Omega$-planes can appear when the orbifold group $\mathcal{G}_t \cong \mathbb{Z}_2$ contains $\mathbb{Z}_2$ elements because they mix with $\Omega I_n$ giving : $\Omega_{9-(4-n)}$-planes for $I_n = I_0, I_4$ or $\Omega_{5+(n-2)}$-planes for $I_n = I_2, I_6$. To each kind of $\Omega_p$-plane a stack of $D_F$-branes have to be added for consistency. The truncation of the spectrum of the Type IIB theories onto $\Omega \sigma$ invariant states takes into account that: worldsheet parity projection $\Omega$ exchanges the left moving part with right moving part of the string, the $(-1)^F_L$ assigns a $(+)$-eigenvalue and $(-)$-eigenvalue to the NS-NS and R-R states respectively, while the pullback of $\Omega$ acts on the $H^{p,q}$ Hodge classes of the toroidal orbifold (or CY manifold) dividing each classes into two subclasses characterised by $(+)$-eigenvalue $H^{p,q}_{+}$ and $(-)$-eigenvalue $H^{p,q}_{-}$ and on the unique 3-form $\Omega_3$ with a $(+)$-eigenvalue or $(-)$-eigenvalue mutually for $I_n = I_0, I_4$ or $I_n = I_2, I_6$ [17]. The states in the spectrum that survives the $\Omega \sigma$ projection are those with overall $(+)$-eigenvalue. Now all the moduli fields fall into chiral multiples of $\mathcal{N} = 1$ in 4-dimensions, but due to the orientifold projection $\Omega \sigma$, when one has to complexify the Kähler moduli $t$, the net distinction between geometrical and non-geometrical moduli for the $\mathcal{N} = 2$ case is lost because in principle they mix, thus the linear combinations allowed in general are those with the real Kähler moduli coming from the same Hodge subclasses, i.e both in $H^{p,q}_{+}$ or $H^{p,q}_{-}$ [17]. Complex structure moduli $u^I$ are complex by itself thus there is no confusion.

Now, we continue to explore the Type IIB orientifold on $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with a focus on the closed untwisted sector, because the moduli in the twisted sectors can be consider frized. We choose as orientifold $\Omega \sigma$ operator the worldsheet parity operator $\Omega$ ($\sigma = I_0$) which by itself introduces $\Omega_9$-plane (see Table 1). The mixing of worldsheet parity operator $\Omega$ with the three $\theta_1 = \mathbb{Z}_2$ elements (3.2) of $\mathcal{G}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ orbifold group add three different $\Omega_{9}^I$-planes fixed at the $\mathbb{Z}_2$ invariant loci respectively. Both $\Omega_0$ and $\Omega_{9}^I$ are balance by stacks of $D_9$- and $D_7^I$-branes respectively, where the specific number of branes into each stack and the gauge group localised on the $D$-branes worldvolume, can be pointed out performing the partition function of the orientifold model where one has to include all the one-loop oriented and unoriented worldsheet surfaces torus ($\mathcal{F}$) and Klein bottle ($\mathcal{K}$) from closed string, annulus ($\mathcal{A}$) and Mōbius strip ($\mathcal{M}$) from open string. We do not discuss this point in a detailed way, see for instance [4], but for completeness the model admits coincident 32 $D_9$-branes on top of the $\Omega_9$-plane that wrap the full internal space, three sets of coincident 32 $D_7^I$ -branes on top of the $\Omega_{9}^I$-planes wrapped along the $\mathbb{T}^2$-torus with $(USp(16))^4$ gauge group [27, 41]. The orientifold $\Omega$ action splits the Hodge classes $H^{1,1}$ and $H^{2,1}$, as said before, but does not reduce the number of untwisted moduli

| $\Omega \sigma$ | $\Omega I_0$ | $\Omega I_2 (-1)^F_L$ | $\Omega I_4$ | $\Omega I_6 (-1)^F_L$ |
|--------------|-------------|----------------|-------------|----------------|
| $\Omega_p$   | $\Omega_9$  | $\Omega_7$   | $\Omega_5$ | $\Omega_3$    |

**Table 1**

The main feature that orientifold projection and the orbifold projection, thus one can take both the projections at the same time for instance considering the full orientifold group, i.e. $\mathcal{G} = \mathcal{G}_r \cup \Omega \sigma \mathcal{G}_r$.

15. There is no dictated order between the orientifold projection and the orbifold projection, thus one can take both the projections at the same time for instance considering the full orientifold group, i.e. $\mathcal{G} = \mathcal{G}_r \cup \Omega \sigma \mathcal{G}_r$.

16. We want to stress that in that case Type IIB orientifold $\equiv$ Type I.
in the same Hodge subclass $H^{1,1}_+$. The complex axion-dilaton $s$ contains the scalar dual to the $C_2$-RR field as imaginary part and the four-dimensional dilaton $\phi$ as real part. The complex structures $u^t$ remain purely geometrical moduli. The orientifold projection doesn’t modify the shape of the moduli space parametrised by untwisted moduli (3.7) at tree level, it remains a direct product of moduli space for the $s$ axion-dilaton $SU(1,1)/U(1)$, the complex Kähler $t^I \ (SU(1,1)/U(1))^3$ and complex structure $u^{i} \ (SU(1,1)/U(1))^3$ [15, 38, 17, 37], thus also the Kähler potential $K$ (3.9) is the same. Some modifications can enter when open string moduli (Wilson lines) coming from $D$-branes wrapping the internal space, are taken into account: the space of moduli space is no longer total factorisable (see for instance [17]), the definitions of complex Kahler moduli $t$ (by open string moduli from $D_9$-branes) and of axion-dilaton $s$ (by open string moduli from $D_9^I$-branes) become more involved [42, 17, 27] and the gauge group $(Usp(16))^4$ can be broken\footnote{When some open string moduli take non-trivial vev but we do not discuss these cases.} [43, 27].

### 3.1 Derivation of the compactified vertex operators and their properties

In this section the main steps needed to build the vertex operators and the two-point functions for the closed untwisted moduli of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ (and \textit{mutatis mutandis} for each models that have similar behaviour), are described. The compactifications of Type II string theories with $\mathcal{N} = 2$ supersymmetries in four-dimensions allow a two-dimensional superconformal field theory (SCFT) description on string worldsheet that is: local $N = (1, 1)$ in the external space and global $N = (2, 2)$ in the internal space with central charges $(c^{\text{ext}}, c^{\text{int}}) = (-9, -9)$ and $(c^{\text{int}}, c^{\text{int}}) = (9, 9)$ respectively [17, 30]. Before the orientifold projection $\Omega\sigma$, the string spectrum originating from the global $N = (2, 2)$ SCFT can be put in correspondence with the fields content of the Type II string theories with $\mathcal{N} = 2$ supersymmetries in four-dimensions [17]. For instance looking at the massless moduli, this happens because at this stage there is no mixing between geometrical and non geometrical moduli. Thus one can start from the vertex operators for the geometrical moduli coming from the NSNS fields (graviton $g$ and Kalb-Ramond $B_2$) because their explicit form can be deduced by the two-dimensional non-linear sigma-model. Then using the supersymmetry transformations the vertex operators for the non-geometrical moduli originating from the RR fields (2-form $C_2$ and 4-form $C_4$) can be obtained.\footnote{Also the fermionic vertex operators can be obtained by supersymmetry transformations.} The closed string complex moduli coming from the CFT point of view, can be considered as the supergravity scalar fields (or Kähler coordinates sometimes) of the $\mathcal{N} = 2$ space-time supermultiplets when a flip of imaginary and real part is made because, the supergravity construction, imposes certain constraints on the structure of the LEEA. When the orientifold projection $\Omega\sigma$ is considered, the correspondence between vertex operators and supergravity moduli fields ends because the $N = (2, 2)$ string states do not represent scalars of $\mathcal{N} = 1$ chiral supermultiplets in four-dimensions. A mixing, as said, between NSNS moduli and RR moduli due to the action of $\Omega\sigma$ takes place. As a consequence, when one calculates string scattering amplitudes, one can extract the specific contribution of the real or imaginary part of a given complex modulus, taking for instance a linear combination of the $N = (2, 2)$ SCFT vertex operators in order to mimic the scattering between the NSNS or RR moduli that survive the $\Omega\sigma$ projection.

The two-dimensional non-linear sigma model (closed bosonic part) that describes the string propagation in the background of Type IIB orientifold on $\mathbf{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, with orientifold projection performed by the worldsheet parity operator $\Omega$ only, is

$$
S = \frac{1}{4\pi} \int d^2z \sum_{l=1}^{3} \partial X^m(z) \bar{\partial} \bar{X}^n(z) (g^I)_{mn}(X) \quad (3.11)
$$

where due to the $\Omega$ projection, the metric $g$ of the factorisable $\mathbf{T}^6 = \bigotimes_{I=1}^{3} \mathbf{T}_I^2$ torus survives the projection while the Kalb-Ramond $B_2$ is projected out.\footnote{As known also the dilaton $\phi$ survives.} The metric and its inverse for each $\mathbf{T}_I^2$ torus...
are respectively

$$(g^I)_{mn} = \frac{T^I_1}{U^I_2} \left( \frac{1}{U^I_1} \left| U^I_1 \right|^2 \right) \quad (g^I)_{mn} = \frac{1}{T^I_2 U^I_1} \left( \left| U^I_1 \right|^2 - U^I_1 \right) \left( -U^I_1 \right)$$

(3.12)

with $I \in \{1, 2, 3\}$ and $[m, n] \in \{[4, 5, 6, 7, 8, 9]\}$. From the CFT point of view, the geometric untwisted moduli fields (3.10) describe the deformations of the underlying sigma-model, i.e. they are the real parameters of the complex $\mathbf{T}_1^I$ tori: $U^I_1$ and $U^I_2$ are the real and imaginary part of the complex structure moduli $U^I(U^I)$ that parametrise the shape of the $\mathbf{T}_1^I$ torus, while the size of the $\mathbf{T}_1^I$ torus is parametrised by $T^I_2$ that is the imaginary part of the complex Kähler moduli $T^I(T^I)$ (the real part comes form the RR $C_2$-form no longer from the NSNS $B_2$-form). These are the complex moduli in the CFT description thus their definition is flip respect to (3.10) supergravity description. Now expanding (3.11), for instance, along the first torus $\mathbf{T}_1^I$ (the same happens for the other two $\mathbf{T}^2_1$ tori) new functions of internal bosonic field $X^m$ can be defined

$$
\frac{1}{4\pi} \int d^2 z \left[ \partial X^4 \partial \bar{X}^4 \frac{T^I_1}{U^I_2} + \partial X^4 \partial \bar{X}^5 \frac{T^I_1}{U^I_2} U^I_1 + \partial X^5 \partial \bar{X}^4 \frac{T^I_1}{U^I_2} U^I_1 + \partial X^5 \partial \bar{X}^5 \frac{T^I_1}{U^I_2} \left| U^I_1 \right|^2 \right]
$$

$$
= \frac{1}{4\pi} \int d^2 z 2 \left[ \sqrt{\frac{T^I_1}{2U^I_2}} \partial (X^4 + \bar{U}^I X^5) \sqrt{\frac{T^I_1}{2U^I_2}} \overline{\partial (X^4 + U^I \bar{X}^5)} \right]
$$

$$
= \frac{1}{2\pi} \int d^2 z \partial Z^I(z) \overline{\partial \bar{Z}^I(z)}
$$

(3.13)

where the new internal bosonic fields $Z^I, \bar{Z}^I$ are identified as \cite{40}\cite{21}

$$
Z^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (X^{2I+2} + \bar{U}^I X^{2I+3})(z), \quad \bar{Z}^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (X^{2I+2} + U^I \bar{X}^{2I+3})(z)
$$

(3.14)

while the supersymmetric partners, i.e the internal fermionic fields $\Psi^I, \tilde{\Psi}^I$ are \cite{40}

$$
\Psi^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (\psi^{2I+2} + \bar{U}^I \psi^{2I+3})(z), \quad \tilde{\Psi}^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (\psi^{2I+2} + U^I \psi^{2I+3})(z)
$$

$$
\tilde{\Psi}^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (\psi^{2I+2} + \bar{U}^I \psi^{2I+3})(z), \quad \bar{\Psi}^I(z) = \sqrt{\frac{T^I_1}{2U^I_2}} (\psi^{2I+2} + U^I \psi^{2I+3})(z)
$$

(3.15)

3.1.1 Compactified vertex operators on $S_2$ and $D_2$

Using the definitions of internal bosonic (3.14) and fermionic fields (3.15) one can construct the NS-NS vertex operators for the holomorphic (anti holomorphic) untwisted complex Kähler moduli $T^I(T^I)$ and the untwisted complex structure moduli $U^I(U^I)$. The known holomorphic building block for the NS sector for uncompactified states are

$$
\mathcal{V}_{(-1)}^\mu(k, z) = e^{-\phi \bar{j}^\mu(z)} e^{ikX(z)}
$$

$$
\mathcal{V}_{(0)}^\mu(k, z) = \sqrt{2 \alpha^\prime} \left( i \partial X^\mu + \frac{\alpha^\prime}{2} (k \bar{\psi}) \psi^\mu \right) e^{ikX(z)}
$$

(3.16)

where (-1) and (0) are the ghost picture and, the tensor product with the anti holomorphic part, provides the closed vertex operators in the NS-NS sector \cite{30}

$$
\mathcal{W}(k, E)_{(-1, -1)} = E_{\mu \nu} : e^{-\phi \bar{j}^\mu(z)} e^{-\bar{j} \bar{\psi}^\nu(z)} e^{ikX(z)} :
$$

$$
\mathcal{W}(k, E)_{(0, 0)} = E_{\mu \nu} \sqrt{2 \alpha^\prime} \left( i \partial X^\mu + \frac{\alpha^\prime}{2} (k \bar{\psi}) \psi^\mu \right) e^{ikX(z)} \left( i \partial \bar{X}^\nu + \frac{\alpha^\prime}{2} (k \bar{\psi} \psi) \right) e^{ik\bar{X}(z)} :
$$

(3.17)
with the polarisation tensor $E_{\mu
u}$ encoding the properties of the state that the vertex has to represent. Taking the compactification of the vertex operators (3.17) on the underlying background $T^6 = \bigotimes_{I=1}^{3} T_I$ torus, the NS-NS vertex operator for the untwisted complex Kähler modulus $T^I$ in the canonical ghost picture ($-1,1$) reads

$$\mathcal{V}_{T^I(-1,1)}(E, k, z, \bar{z}) = E_{mn}[T^I] \cdot \mathcal{V}_{(-1)}(k, z) \mathcal{V}_{(-1)}(k, \bar{z}) := \left( \frac{\partial (g')}{\partial T^I} \right) : \psi \phi \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \left\{ \left( \frac{\partial (g')}{\partial T^I} \right)_{[2I+2][2I+2]} + \left( \frac{\partial (g')}{\partial T^I} \right)_{[2I+3][2I+3]} \right\} e^{-\phi} \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \frac{1}{(T^I - T^J)} \left\{ T_2^I \left( T_2^J - T_2^I \right) \right\} e^{-\phi} \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \frac{2}{(T^I - T^J)} : \bar{\Psi} I e^{-\phi} e^{ikX(z)} \Psi I e^{-\bar{\phi}} e^{ik\bar{X}(\bar{z})} :$$

(3.18)

where the polarisation tensor $E_{mn}[T^I]$ can be determined taking the variation of the sigma-model (3.11) with respect to the complex Kähler modulus $T^I$. The NS-NS vertex operator for the untwisted complex structure modulus $U^I$ in the same ghost picture ($-1,1$) reads

$$\mathcal{V}_{U^I(-1,1)}(E, k, z, \bar{z}) = E_{mn}[U^I] \cdot \mathcal{V}_{(-1)}(k, z) \mathcal{V}_{(-1)}(k, \bar{z}) := \left( \frac{\partial (g')}{\partial U^I} \right) : \psi \phi \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \left\{ \left( \frac{\partial (g')}{\partial U^I} \right)_{[2I+2][2I+2]} - \left( \frac{\partial (g')}{\partial U^I} \right)_{[2I+3][2I+3]} \right\} e^{-\phi} \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \frac{T_2^I}{(U^I - U^I)^2} \left( T_2^J - T_2^I \right) \left( T_2^J - T_2^I \right) e^{-\phi} \psi^m e^{ikX(z)} e^{-\bar{\phi}} \bar{\psi}^n e^{ik\bar{X}(\bar{z})} :$$

$$= \frac{2}{(U^I - U^I)^2} : \bar{\Psi} I e^{-\phi} e^{ikX(z)} \Psi I e^{-\bar{\phi}} e^{ik\bar{X}(\bar{z})} :$$

(3.19)

From the $N = (2,2)$ SCFT point of view (3.18) and (3.19) can be written in a more general form [17]

$$\mathcal{V}_{T^I(-1,1)}(E, k, z, \bar{z}) = \mathcal{E}_{I} [T^I] : \Delta^I(z, \bar{z}) e^{-\phi} \phi^{ikX(z)} e^{-\bar{\phi}} \bar{\phi}^{ik\bar{X}(\bar{z})} :, \quad I = 1, \ldots, h^{1,1}_{\Delta^I}$$

$$\mathcal{V}_{U^I(-1,1)}(E, k, z, \bar{z}) = \mathcal{E}_{I} [U^I] : \Sigma^I(z, \bar{z}) e^{-\phi} \phi^{ikX(z)} e^{-\bar{\phi}} \bar{\phi}^{ik\bar{X}(\bar{z})} :, \quad I = 1, \ldots, h^{2,1}_{\Sigma^I}$$

where $\Delta^I(z, \bar{z})$ and $\Sigma^I(z, \bar{z})$ are conformal field with conformal dimensions $h, \bar{h} = (1/2, 1/2)$ respect to the internal $N = (2,2)$ SCFT and charged under the couple of U(1) currents $(J, \bar{J})$: $(1,1)$ for the field $\Delta^I(z, \bar{z})$, $(-1,1)$ for the field $\Sigma^I(z, \bar{z})$. The complex conjugate conformal field $\bar{\Delta}^I(z, \bar{z})$ and $\bar{\Sigma}^I(z, \bar{z})$ have the same conformal dimensions and opposite charges respectively. The vertex operators for the antiholomorphic complex Kähler modulus $T^I$ and the complex structure modulus $U^I$ in the ($-1,1$) ghost pictures, together the vertex operators in the $(0,0)$ ghost pictures follow

$$\mathcal{W}_{T^I(-1,1)}(E, k, z, \bar{z}) = E_{mn}[\bar{T}^I] : e^{-\bar{\phi}} \phi^{m} e^{ik\bar{X}(\bar{z})} e^{-\bar{\phi}} \bar{\phi}^{n} e^{ik\bar{X}(\bar{z})} :$$

$$= -\frac{2}{(T^I - T^I)} : \bar{\Psi} I e^{-\bar{\phi}} e^{ik\bar{X}(\bar{z})} \Psi I e^{-\phi} e^{ikX(z)} :$$

(3.20)
\[ W_{U(1,-1)}(E, k, z, \bar{z}) = E_{mn} [U^I] : e^{-\phi \psi^m} e^{ikX} (z) e^{-\phi \bar{\psi}^m} e^{ik\bar{X}} (\bar{z}) : \\
= \frac{2}{(U^I - U^{-I})} \bar{\psi}^I e^{-\phi \psi^m} e^{ikX} (z) \bar{\psi}^I e^{-\phi \bar{\psi}^m} e^{ik\bar{X}} (\bar{z}) : \]

\[ W_{T(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'} E_{mn} [T^I] : \left( i \partial X^m + \frac{\alpha'}{2} (k \psi) \psi^m \right) e^{ikX} (z) \left( i \partial \bar{X}^m + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^m \right) e^{ik\bar{X}} (\bar{z}) : \\
= \frac{4}{\alpha' (T^I - \bar{T}^I)} \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ikX} (z) \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ik\bar{X}} (\bar{z}) : \]

\[ W_{T(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'} E_{mn} [T^I] : \left( i \partial X^m + \frac{\alpha'}{2} (k \psi) \psi^m \right) e^{ikX} (z) \left( i \partial \bar{X}^m + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^m \right) e^{ik\bar{X}} (\bar{z}) : \\
= -\frac{4}{\alpha' (T^I - \bar{T}^I)} \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ikX} (z) \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ik\bar{X}} (\bar{z}) : \]

\[ W_{\tilde{T}(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'} E_{mn} [U^I] : \left( i \partial X^m + \frac{\alpha'}{2} (k \psi) \psi^m \right) e^{ikX} (z) \left( i \partial \bar{X}^m + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^m \right) e^{ik\bar{X}} (\bar{z}) : \\
= -\frac{4}{\alpha' (U^I - U^{-I})} \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ikX} (z) \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ik\bar{X}} (\bar{z}) : \]

\[ W_{\tilde{T}(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'} E_{mn} [U^I] : \left( i \partial X^m + \frac{\alpha'}{2} (k \psi) \psi^m \right) e^{ikX} (z) \left( i \partial \bar{X}^m + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^m \right) e^{ik\bar{X}} (\bar{z}) : \\
= \frac{4}{\alpha' (U^I - U^{-I})} \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ikX} (z) \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k \bar{\psi}) \bar{\psi}^I \right) e^{ik\bar{X}} (\bar{z}) : \]

(3.21)

When calculation of string scattering amplitudes involve oriented surfaces as the sphere $S_2$ and the disk $D_2$, these vertex operators can be used.

### 3.1.2 Compactified vertex operators on $RP_2$

When unoriented surface as the real projective plane $RP_2$ is considered, the usual string vertex operators cannot be used due to the presence of $\Omega P$-planes induced by the action of $\Omega \sigma$ as summarized in Table 1 [26, 30]. Closed vertex operators in a generic picture $(q, \bar{q})$ for a generic $\Omega P$-plane, represent string states that are invariant under the $\Omega \sigma$ action and manifestly invariant under the involution $\Theta_{RP_2}(z) = -1/z$

\[ W^\otimes_{(q, \bar{q})}(k, E) = \frac{1}{2} \left( E_{\mu \nu} \Psi^{\nu}_{(q)}(k, z) \Psi^{\nu}_{(\bar{q})}(k, \bar{z}) + E_{\mu \nu} R^\mu_{\rho} R^\nu_{\gamma} \Psi^{\rho}_{(q)}(k R, \bar{z}) \Psi^{\gamma}_{(\bar{q})}(k R, z) \right). \quad (3.22) \]

The vertex is a symmetric combination of holomorphic and antiholomorphic vertices, since $\Omega$ operator exchanges the left-movers with the right-movers of the string while the reflection matrix $R$, defined in the next section, manifests the action of $\sigma = I_n$, that acts as a reflections in the $n$ directions perpendicular to the $\Omega P$-plane. Moreover the operator $I_n$ can be viewed as the result of $n \equiv (9 - p)$ $T$-dualities of Type I theory.\(^\text{22}\) For the untwisted moduli $T^I$ and $U^I$ the vertex operators $W^\otimes_{(q, \bar{q})}$ in the $(-1, -1)$, $(0, 0)$ picture read

\[ W^\otimes_{T(-1, -1)}(E, k, z, \bar{z}) = \frac{1}{(T^I - T^{-I})} \left( \Psi^I e^{-\phi} e^{ikX}(z) \Psi^I e^{-\phi} e^{ik\bar{X}}(\bar{z}) + \{ L, k, z \} \leftrightarrow \{ R, k R, \bar{z} \} \right) \]

\[ W^\otimes_{T(-1, -1)}(E, k, z, \bar{z}) = \frac{1}{(T^I - T^{-I})} \left( \Psi^I e^{-\phi} e^{ikX}(z) \Psi^I e^{-\phi} e^{ik\bar{X}}(\bar{z}) + \{ L, k, z \} \leftrightarrow \{ R, k R, \bar{z} \} \right) \]

\(^{22}\) For instance a single $T$-duality acts on the worldsheet parity operator $\Omega$ as $T^{-1} \Omega T = \Omega I_1$. 

19
\[ W^\otimes_{T^I(1,-1)}(E, k, z, \bar{z}) = -\frac{1}{(U^I-U^J)} \left\{ :\Psi^I e^{-\phi} e^{ikX(z)} \bar{\Psi}^I e^{-\bar{\phi}} e^{ik\bar{X}(\bar{z})} : + : \{L, k, z\} \leftrightarrow \{R, kR, \bar{z}\} \right\} \]

\[ W^\otimes_{T^I(1,-1)}(E, k, z, \bar{z}) = \frac{1}{(U^I-U^J)} \left\{ :\bar{\Psi}^I e^{-\bar{\phi}} e^{ik\bar{X}(\bar{z})} \Psi^I e^{-\phi} e^{ikX(z)} : + : \{L, k, z\} \leftrightarrow \{R, kR, \bar{z}\} \right\} \]

\[ W^\otimes_{T^I(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'(T^I-T^J)} \left\{ : i\partial \bar{Z}^I + \frac{\alpha'}{2}(k\psi)\bar{\Psi}^I \right\} e^{ikX(z)} \left( i\partial \bar{Z}^I + \frac{\alpha'}{2}(k\bar{\psi})\Psi^I \right) e^{ik\bar{X}(\bar{z})} : + : \{L, k, z\} \leftrightarrow \{R, kR, \bar{z}\} \right\} \]

\[ W^\otimes_{T^I(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'(U^I-U^J)} \left\{ : i\partial \tilde{Z}^I + \frac{\alpha'}{2}(k\psi)\tilde{\Psi}^I \right\} e^{ikX(z)} \left( i\partial \tilde{Z}^I + \frac{\alpha'}{2}(k\bar{\psi})\tilde{\Psi}^I \right) e^{ik\tilde{X}(\tilde{z})} : + : \{L, k, z\} \leftrightarrow \{R, kR, \bar{z}\} \right\} \]

\[ W^\otimes_{T^I(0,0)}(E, k, z, \bar{z}) = \frac{2}{\alpha'(T^I-T^J)} \left\{ : i\partial \bar{Z}^I + \frac{\alpha'}{2}(k\psi)\bar{\Psi}^I \right\} e^{ikX(z)} \left( i\partial \bar{Z}^I + \frac{\alpha'}{2}(k\bar{\psi})\Psi^I \right) e^{ik\bar{X}(\bar{z})} : + : \{L, k, z\} \leftrightarrow \{R, kR, \bar{z}\} \right\} \]

where the building block (3.16) and the polarisation tensor \( E_{mn} \) coming from the variation of the sigma model (3.11) respect to \( T^I(U^I) \) have been used.

### 3.1.3 Two-point functions for the \( Z, \Psi \) system

When string scattering amplitudes on surfaces with boundaries as the disk \( D_2 \) is considered, the two-point functions as is know have to be modified due to the \( \mathbb{Z}_2 \) involution \( \mathcal{J}_{D_2}(z) = \bar{z} \) which takes the complex plane \( \mathbb{C}(S_2) \) and gives the upper half planes \( \mathcal{H}_+(D_2) \). This implies an interaction between the left-moving and right-moving parts of closed string fields [30, 21]

\[ \langle \partial X^M(z_1) \bar{\partial} \bar{X}^N(\bar{z}_2) \rangle_{D_2} = -\frac{\alpha'}{2} \frac{R^{MN}}{(z_1 - \bar{z}_2)^2} \]

\[ \langle \psi^M(z_1) \bar{\psi}^N(\bar{z}_2) \rangle_{D_2} = \frac{R^{MN}}{(z_1 - \bar{z}_2)} \]

\[ \langle \phi(z_1) \bar{\phi}(\bar{z}_2) \rangle_{D_2} = -\ln(z_1 - \bar{z}_2) \]

(3.24)

where the non trivial interaction between holomorphic and antiholomorphic part can be obtained using the doubling trick.\(^{23}\) In this way each right-moving field of the closed string vertex operator is replaced by

\[ \tilde{X}^M(z) \to R^{M_N} X^N(z), \quad \bar{\psi}^M(\bar{z}) \to R^{M_N} \psi^N(\bar{z}), \quad \bar{\phi}(\bar{z}) \to \phi(z) \]

(3.25)

and with the help of the standard two-point functions on the sphere [30, 21], obtain (3.24) on the disk is straightforward. The presence of both closed and open strings, especially for the latter, involves the presence of D-brane on which one can impose Neumann or Dirichlet (or mixed ones) boundary

\(^{23}\) Extend the fields to the entire complex plane.
conditions on the directions parallel and transverse to the brane respectively. The reflection matrix $\mathcal{R}$ allows to impose that conditions and it reads

$$\mathcal{R}^{MN} = \begin{cases} g_{ab}, & a, b = 0, \ldots, p \quad (NN) \\ -g_{ij}, & i, j = p + 1, \ldots, 9 \quad (DD) \end{cases} \quad M = 0, \ldots, 9 \quad (3.26)$$

where Neumann boundary conditions are imposed on coordinates $X^a(\psi^i)$ for $0 \leq a \leq p$ and Dirichlet boundary conditions on coordinates $X^i(\psi^i)$ for $p + 1 \leq i \leq 9$. In this paper we don’t consider the mixed boundary conditions case, i.e ND (DN).24

When orbifolds compactifications are taken into account, one has to specify either vertex operators (sections 3.1.1 and 3.1.2) and two-point functions relative to the internal part that locally, aside singular points, looks like a $T^6$ torus. In our case the internal six-torus factorize $T^6 = \otimes_{I=1}^3 T_I^2$ and the matrix $R$ (3.26) in the internal directions takes the form

$$\mathcal{R}^{mn} = \begin{cases} (g^I)^{mn}, & (NN) \\ -(g^I)^{mn}, & (DD) \end{cases} \quad (3.27)$$

where the metric and the associated boundary conditions refers to which $T^2$-torus the specific D-brane wraps (as discussed at the beginning of Section 3), with the index $\{I, m, n\}$ the same of (3.12). For the D9-branes (3.27) reads

$$\mathcal{R}^{mn}_{D9} = \otimes_{I=1}^3 (g^I)^{mn} \quad (NN), \quad [m, n] \in \{[4, 5]; [6, 7]; [8, 9]\} \quad (3.28)$$

that are characterised only by Neumann boundary conditions because they wrap the full $T^6$. Taking for instance the set of D5$_1$ from the three sets of D5-branes, one has

$$\mathcal{R}^{mn}_{D5_1} = \begin{cases} (g^I)^{mn}, & (NN) \\ -(g^I)^{mn}, & (DD) \end{cases}, \quad [m, n] \in [4, 5] \quad (3.29)$$

where the Neumann boundary conditions refers to the $T^2_I$ torus that D5$_1$ branes wrap while the Dirichlet boundary conditions refers to the $T^2_T \otimes T^2_3$ torus transverse to the D5$_1$ branes.25 Concerning the correlators, on the sphere $S_2$ one has for the internal bosonic fields $Z$ (3.14) [40]

$$\langle \partial Z^I(z_1) \partial \tilde{Z}^J(z_2) \rangle_{S_2} = \sqrt{\frac{T_I}{2U_I}} \sqrt{\frac{T_J}{2U_J}} \left[ \langle \partial X^{2I+2}(z_1) \partial X^{2J+2}(z_2) \rangle + \tilde{U}^I \langle \partial X^{2I+3}(z_1) \partial X^{2J+2}(z_2) \rangle ight. \right.$

$$\left. + \tilde{U}^J \langle \partial X^{2I+2}(z_1) \partial X^{2J+3}(z_2) \rangle + \tilde{U}^I \tilde{U}^J \langle \partial X^{2I+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]$$

$$= -\sqrt{\frac{T_I}{2U_I}} \sqrt{\frac{T_J}{2U_J}} \frac{\alpha'}{2(z_1 - z_2)^2} \left[ g^{[2I+2][2J+2]} + \tilde{U}^I g^{[2I+3][2J+2]} + \tilde{U}^J g^{[2I+2][2J+3]} + \tilde{U}^I \tilde{U}^J g^{[2I+3][2J+3]} \right]$$

$$= -\frac{\alpha' \delta_{I,J}}{2(z_1 - z_2)^2} \quad (3.30)$$

$$\langle \partial Z^I(z_1) \partial Z^J(z_2) \rangle_{S_2} = \sqrt{\frac{T_I}{2U_I}} \sqrt{\frac{T_J}{2U_J}} \left[ \langle \partial X^{2I+2}(z_1) \partial X^{2J+2}(z_2) \rangle + \tilde{U}^I \langle \partial X^{2I+3}(z_1) \partial X^{2J+2}(z_2) \rangle ight. \right.$

$$\left. + \tilde{U}^I \langle \partial X^{2I+2}(z_1) \partial X^{2J+3}(z_2) \rangle + \tilde{U}^I \tilde{U}^J \langle \partial X^{2I+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]$$

$$= -\sqrt{\frac{T_I}{2U_I}} \sqrt{\frac{T_J}{2U_J}} \frac{\alpha'}{2(z_1 - z_2)^2} \left[ g^{[2I+2][2J+2]} + \tilde{U}^I g^{[2I+3][2J+2]} + \tilde{U}^J g^{[2I+2][2J+3]} + \tilde{U}^I \tilde{U}^J g^{[2I+3][2J+3]} \right]$$

$$= 0 \quad (3.31)$$

24 That enters when twisted fields are considered.
25 The internal directions where the D5$_1$ branes are fixed.
where in both (3.30) and (3.31) the $\delta_{I,J}$ is due to the vanishing of the off diagonal block matrix $g^{mn}$ when $I \neq J$, while only (3.31) vanishes also in the case $I = J$. The two-point functions for the compactified bosons $Z$ on the sphere are

$$
\langle \partial Z^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{s_2} = \langle \partial \bar{Z}^I(z_1) \partial Z^J(z_2) \rangle_{s_2} = \langle \partial \bar{Z}^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{s_2} = \langle \partial \bar{Z}^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{s_2} = 0
$$

(3.32)

while for the compactified fermions $\Psi$ (3.15) one has

$$
\langle \Psi^I(z_1) \Psi^J(z_2) \rangle_{s_2} = \langle \bar{\Psi}^I(z_1) \bar{\Psi}^J(z_2) \rangle_{s_2} = \langle \bar{\Psi}^I(z_1) \bar{\Psi}^J(z_2) \rangle_{s_2} = 0
$$

(3.33)

where in the fermionic analog of (3.30), (3.31) the standard fermionic correlator on the sphere is used [30, 21]. To generalize the result to the disk $D_2$, one has to calculate the correlators on the upper-half plane $\mathbb{H}^+$. First of all using the doubling trick one is able to replace the right part according to

$$
\bar{Z}^I(z) \rightarrow R^I \bar{Z}^I(z), \quad \bar{\Psi}^I(z) \rightarrow R^I \bar{\Psi}^I(z), \quad \bar{\Psi}^I(z) \rightarrow R^I \bar{\Psi}^I(z)
$$

(3.34)

where the reflection matrix $R$ expressed in a non compact form is $R^I = (R^I)^m_n$ with $\{I, m, n \}$ as (3.12). More specifically, one can use $(R^I)^{mn} = \delta^m_a \delta^m_n$ where $\delta^m_a$ that is +1(-1) for $NN(DD)$-directions, $(g^I)^{mn}$ the internal metric (3.12) and $a$ labels the type of $D_P$-brane. Examples of correlators for the boson fields $Z$ on $\mathbb{H}^+$ are

$$
\langle \partial Z^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{D_2} = \langle \partial Z^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{s_2} R^I
$$

$$
= \sqrt{\frac{T^I}{2 U^J}} \sqrt{\frac{T^J}{2 U^I}} \left[ (R^I)^{[2J+2][2J+2]} \langle \partial X^{2J+2}(z_1) \partial X^{2J+2}(z_2) \rangle + U^I (R^I)^{[2J+2][2J+2]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]
$$

$$
+ U^I (R^I)^{[2J+3][2J+3]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle + \bar{U}^I U^J (R^I)^{[2J+3][2J+3]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]
$$

$$
= - \frac{\alpha'}{2(\bar{z}_1 - \bar{z}_2)^2} \left[ \langle \partial X^{2J+2}[2J+2] \rangle + U^I (R^I)^{[2J+2][2J+3]} + \bar{U}^I U^J (R^I)^{[2J+3][2J+3]} \right]
$$

$$
= \frac{\alpha'}{2(\bar{z}_1 - \bar{z}_2)^2} \left[ \langle \partial X^{2J+2}[2J+2] \rangle + U^I (R^I)^{[2J+2][2J+3]} + \bar{U}^I U^J (R^I)^{[2J+3][2J+3]} \right]
$$

(3.35)

$$
\langle \partial Z^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{D_2} = \langle \partial Z^I(z_1) \partial \bar{Z}^J(z_2) \rangle_{s_2} R^I
$$

$$
= \sqrt{\frac{T^I}{2 U^J}} \sqrt{\frac{T^J}{2 U^I}} \left[ (R^I)^{[2J+2][2J+2]} \langle \partial X^{2J+2}(z_1) \partial X^{2J+2}(z_2) \rangle + U^I (R^I)^{[2J+2][2J+2]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]
$$

$$
+ \bar{U}^I (R^I)^{[2J+3][2J+3]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle + \bar{U}^I U^J (R^I)^{[2J+3][2J+3]} \langle \partial X^{2J+3}(z_1) \partial X^{2J+3}(z_2) \rangle \right]
$$

$$
= \frac{\alpha'}{2(\bar{z}_1 - \bar{z}_2)^2} \left[ \langle \partial X^{2J+2}[2J+2] \rangle + U^I (R^I)^{[2J+2][2J+3]} + \bar{U}^I U^J (R^I)^{[2J+3][2J+3]} \right]
$$

(3.36)
where \((R^J)^n_m = (\delta^J)^n_m\) and the \(\delta_{IJ}\) are used as before. Thus besides (3.32) and (3.33) one has

\[
\langle \partial Z^I(z_1)\bar{\partial} \bar{Z}^J(z_2) \rangle_{D_2} = 0, \quad \langle \bar{\partial} \bar{Z}^I(z_1) \partial Z^J(z_2) \rangle_{D_2} = 0,
\]

\[
\langle \partial Z^I(z_1) \bar{\partial} \bar{Z}^J(z_2) \rangle_{D_2} = -\frac{\alpha' R^I_a \delta_{IJ}}{2(z_1 - z_2)^2}, \quad \langle \bar{\partial} \bar{Z}^I(z_1) \partial Z^J(z_2) \rangle_{D_2} = -\frac{\alpha' R^I_a \delta_{IJ}}{2(z_1 - z_2)^2}
\]

(3.37)

for the compactified bosons and

\[
\langle \Psi^I(z_1) \bar{\Psi}^J(z_2) \rangle_{D_2} = 0, \quad \langle \bar{\Psi}^I(z_1) \Psi^J(z_2) \rangle_{D_2} = 0,
\]

\[
\langle \Psi^I(z_1) \bar{\Psi}^J(z_2) \rangle_{D_2} = \frac{R^I_a \delta_{IJ}}{(z_1 - z_2)^2}, \quad \langle \bar{\Psi}^I(z_1) \Psi^J(z_2) \rangle_{D_2} = \frac{R^I_a \delta_{IJ}}{(z_1 - z_2)^2}
\]

(3.38)

for the compactified fermions \(\Psi\).\(^{26}\) At tree level the presence of \(\Omega_P\)-planes in orientifold models suggest that one has to consider the internal bosonic and fermionic correlators on real projective plane \(RP_2\). Taking the action of \(Z_2\) involution \(\jmath_{RP_2}(z) = -\frac{1}{z}\) on \(S_2\) one obtains the \(RP_2\), that is a disk \(D_2\) with antipodal points on the boundary identified \([47, 26, 30]\). The basic two-point functions on the \(RP_2\), employing the method of images and the doubling trick, are \([30]\)

\[
\langle X^M(z)X^N(\bar{w}) \rangle_{RP_2} = -\frac{\alpha'}{2} R^{MN} \ln(1 + z\bar{w})
\]

\[
\langle \psi(z) \bar{\psi}^N(\bar{w}) \rangle_{RP_2} = \frac{R^{MN}}{(1 + z\bar{w})}
\]

\[
\langle \phi(z) \bar{\phi}(\bar{w}) \rangle_{RP_2} = -\ln(1 + z\bar{w})
\]

(3.39)

with \(R^{MN}\) the reflection matrix (3.26). The two-point functions for the compactified bosons \(Z\) and fermions \(\Psi\) can be performed using the building block (3.39), the correlators on the sphere \(S_2\) (3.32) and (3.33) respectively. They differ from those on \(H^+\) only in the \(z\bar{w}\) dependence, thus

\[
\langle \partial Z^I(z_1) \bar{\partial} \bar{Z}^J(z_2) \rangle_{RP_2} = 0, \quad \langle \bar{\partial} \bar{Z}^I(z_1) \partial Z^J(z_2) \rangle_{RP_2} = 0,
\]

\[
\langle \partial Z^I(z_1) \bar{\partial} \bar{Z}^J(z_2) \rangle_{RP_2} = -\frac{\alpha' R^I_a \delta_{IJ}}{2(1 + z_1 \bar{z}_2)^2}, \quad \langle \bar{\partial} \bar{Z}^I(z_1) \partial Z^J(z_2) \rangle_{RP_2} = -\frac{\alpha' R^I_a \delta_{IJ}}{2(1 + z_1 \bar{z}_2)^2}
\]

(3.40)

for the compactified bosons and for the compactified fermions

\[
\langle \Psi^I(z_1) \bar{\Psi}^J(z_2) \rangle_{RP_2} = 0, \quad \langle \bar{\Psi}^I(z_1) \Psi^J(z_2) \rangle_{RP_2} = 0,
\]

\[
\langle \Psi^I(z_1) \bar{\Psi}^J(z_2) \rangle_{RP_2} = \frac{R^I_a \delta_{IJ}}{(1 + z_1 \bar{z}_2)}, \quad \langle \bar{\Psi}^I(z_1) \Psi^J(z_2) \rangle_{RP_2} = \frac{R^I_a \delta_{IJ}}{(1 + z_1 \bar{z}_2)}
\]

(3.41)

with \(R^I_a\) that is \(+1(-1)\) for \(NN(DD)\)-directions and \(\alpha\) labels the type of \(\Omega_P\)-planes.

### 4 Scattering Amplitudes of the closed untwisted moduli in Type IIB \(T^6/\mathbb{Z}_2 \times \mathbb{Z}_2\) orientifold

In this section string scattering amplitudes with two untwisted closed string moduli on the disk \(D_2\) \([40]\) are reviewed and extended to the real projective plane \(RP_2\) world-sheet surface, as aspected at tree-level when unoriented string models are considered. String scattering amplitudes on the disk \(D_2\) with fundamental region the upper-half plane \(H^+\) that we will compute are

\(^{26}\) Since we are not taking fluxes the matrix \(R\) has only diagonal components and one is in the simplified case of \([40]\), in which one has to send the fluxes \(f^J\) to zero.
\( A_0 (T^I, T^J) + A_0 (T^I, T^J) = g_c^2 C_{D_2} \int_{\mathbb{H}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \langle W_{T^I(z_1, z_1)} W_{T^J(z_2, z_2)} \rangle \right) + \left( T \leftrightarrow U \right) \)

where the different kinds of \( D \)-branes are labelled by \( a \in \{9, 5\} \). On the real projective plane \( \mathbb{R}P_2 \) the string scattering amplitudes involving the same states taking the unit disk \(|z| \leq 1\) as fundamental region, reads

\[ A_0 (T^I, T^J) + A_0 (T^I, T^J) = g_c^2 C_{\mathbb{R}P_2} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \langle W_{T^I(z_1, z_1)} W_{T^J(z_2, z_2)} \rangle \right) + \left( T \leftrightarrow U \right) \]

\[ A_0 (U^I, U^J) + A_0 (U^I, U^J) = g_c^2 C_{\mathbb{R}P_2} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \langle W_{T^I(z_1, z_1)} W_{T^J(z_2, z_2)} \rangle \right) + \left( T \leftrightarrow U \right) \]

with \( \alpha \in \{9, 5\} \) that label the different kinds of \( \Omega \)-planes. In Section 3.1 the derivation of the compactified vertex operators and the structure that enters in the specific scattering amplitudes were provided. Moreover the doubling trick needed to make the correlation among the left and the right field, for convenience, will not be manifest in the definition of the vertex operators, as in Section 2.

### 4.1 \( \mathcal{A}^{D_2}_a (T^I, T^J) \) and \( \mathcal{A}^{D_2}_a (T^I, T^J) \)

Let start with the first set of string scattering amplitudes in (4.1) that involves two untwisted Kähler moduli \( T^I \). The amplitude which mixes \( T \) and \( T \) Kähler moduli

\[ A_0 (T^I, T^J) = g_c^2 C_{D_2} \int_{\mathbb{H}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \langle W_{T^I(-1, -1)}(E_1, k_1, z_1, \bar{z}_1) W_{T^J(0, 0)}(E_2, k_2, z_2, \bar{z}_2) \rangle \right) \]

\[ \overset{-}{=} \frac{8g_c^2 C_{D_2}}{\alpha'(T^I - T^J)(T^J - T^J)} \int_{\mathbb{H}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle e^{-\psi} \bar{\Psi}^J e^{ik_1 X(z_1)} e^{-\bar{\phi}} \bar{\Psi}^I e^{ik_1 \bar{X}(\bar{z}_1)} \rangle \]

\[ \overset{-}{=} \frac{8g_c^2 C_{D_2}}{\alpha'(T^I - T^J)(T^J - T^J)} \int_{\mathbb{H}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle e^{-\phi} (z_1) : e^{-\bar{\phi}} (\bar{z}_1) \langle \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)} + \mathcal{M}^{(4)} \rangle \]  

where the \( \mathcal{M}^{(i)} \)'s \( (i = 1, 2, 3, 4) \) are all the possible different contractions with for instance

\[ \mathcal{M}^{(1)} = \langle \bar{\Psi}^J e^{ik_1 X(z_1)} \bar{\Psi}^I e^{ik_1 \bar{X}(\bar{z}_1)} : i \partial Z^J e^{ik_2 \bar{X}(\bar{z}_2)} \partial \bar{Z}^J e^{ik_2 X(z_2)} \rangle \]  

For the uncompactified fields the relevant two-point functions are\(^{27} (2.5) \), while for the compactified field the two-point functions are \((3.32), (3.33), (3.37) \) and \((3.38) \).

The explicit derivation of \( \mathcal{M}^{(i)} \)'s and the explanation as why some combinations vanish are in [54]

\(^{27} \) Where one has to reinsert the reflection matrix \( R \) on the two points correlation function which involves left and right field.
(appendix A.6). Putting all together one gets

\[ A_\alpha(T^I, \bar{T}^J) = - \frac{8g_s^2 C_{D_2}}{\alpha'(T^I - \bar{T}^I)(T^J - \bar{T}^J)} \int_{\mathcal{V}_{CKG}} d^2 z_1 d^2 z_2 \left( \frac{|z_1 - \bar{z}_1||z_2 - \bar{z}_2|}{|z_1 - \bar{z}_2|^2} \right)^{-\alpha'} \left( \frac{|z_1 - \bar{z}_2|^2}{|z_1 - \bar{z}_2|^2} \right)^{-\alpha' \frac{4}{5}} \frac{1}{(z_1 - \bar{z}_2)} \]

Using the $PSL(2, R)$ symmetry (see from (2.6)) in order to fix vertex operators at [21, 22, 23]

\[ z_1 = i, \quad \bar{z}_1 = -i, \quad z_2 = iy, \quad \bar{z}_2 = -iy \]

and inserting the c-ghost determinant (2.15), one obtains

\[ A_\alpha(T^I, \bar{T}^J) = \frac{8g_s^2 C_{D_2}}{(T^I - \bar{T}^I)(T^J - \bar{T}^J)} \left\{ \alpha' t \frac{4}{4} \xi_{a} \bar{\xi}_{a} \left( \alpha' s + \alpha' \frac{t}{4} \right) + \alpha' s \delta_{I,J} \left( \alpha' s + \alpha' \frac{t}{4} \right) \right\} \frac{1}{\Gamma(-\alpha') \Gamma(-\alpha' t/4)} \frac{1}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} \]

that exploiting the substitution (2.18) [21, 22, 23] and the $\Gamma$ function properties becomes

\[ A_\alpha(T^I, \bar{T}^I) = \frac{8g_s^2 C_{D_2}}{(T^I - \bar{T}^I)^2} \left\{ \alpha' t \frac{4}{4} \xi_{a} \bar{\xi}_{a} \left( \alpha' s + \alpha' \frac{t}{4} \right) + \alpha' s \delta_{I,J} \left( \alpha' s + \alpha' \frac{t}{4} \right) \right\} \frac{1}{\Gamma(-\alpha') \Gamma(-\alpha' t/4 + 1)} \]

Of course two different kind of contributions are present, the diagonal one:28

- $I = J$

\[ \frac{8g_s^2 C_{D_2}}{(T^I - \bar{T}^I)^2} \left( \alpha' s + \alpha' \frac{t}{4} \right) \frac{2}{\Gamma(-\alpha' s + \alpha' \frac{t}{4})} \frac{\Gamma(-\alpha') \Gamma(-\alpha' t/4)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} = \]

\[ A_\alpha(T^I, \bar{T}^I) = \frac{8g_s^2 C_{D_2}}{(T^I - \bar{T}^I)^2} \left\{ \frac{4s}{t} + \frac{t}{4s} + 2 + \alpha' u^2 \right\} \frac{1}{16} (-\zeta(2) + O(\alpha')) \]

and the off-diagonal one:

- $I \neq J$

\[ \frac{8g_s^2 C_{D_2} \xi_{a} \bar{\xi}_{a} \alpha' \frac{t}{4} \xi_{a} \bar{\xi}_{a} \left( \alpha' s + \alpha' \frac{t}{4} \right) \Gamma(-\alpha') \Gamma(-\alpha' t/4)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} = \]

\[ A_\alpha(T^I, \bar{T}^J) = \frac{8g_s^2 C_{D_2} \xi_{a} \bar{\xi}_{a} \alpha' \frac{t}{4} \xi_{a} \bar{\xi}_{a} \left( \alpha' s + \alpha' \frac{t}{4} \right) \Gamma(-\alpha') \Gamma(-\alpha' t/4 + 1)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} \]

where has been used, as in [26, 40], the gamma function expansion

\[ \frac{\Gamma(-\alpha') \Gamma(-\alpha' t/4)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)} = \frac{1}{\alpha'^2} \frac{4}{st} - \zeta(2) + O(\alpha') \]

Taking the pair $(T, T)$, i.e two $T$ Kähler moduli (the same holds with a pair of $\bar{T}$) instead of the Kähler moduli pair $(T, \bar{T})$, the resulting scattering, summed with (4.8), gives us informations on the geometrical modulus $T_2^I$ (imaginary part of $T$) as we will see at the end of this section. As before all

\[ (\xi_{a})^2 = 1 \]
the salient features of the computation are shown, thus

$$\mathcal{A}_a(T^I, T^J) = g_s^2 C_{a_2} \int_{\mathbb{R}^4} \frac{d^2 z_I d^2 z_J}{V_{CKG}} \langle \mathcal{W}_{T^I(-1,-1)}(E_1, k_1, \bar{z}_1, \bar{z}_2) \rangle$$

$$= \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \int_{\mathbb{R}^4} \frac{d^2 z_I d^2 z_J}{V_{CKG}} \langle \tilde{\Psi} I e^{-\phi} e^{ik_1 X(z_1)} \tilde{\Psi} I e^{-\phi} e^{ik_1 X(z_1)} : (i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \psi) \bar{\psi}) e^{ik_2 X(z_2)} : (i \partial \bar{Z}^J + \frac{\alpha'}{2} (k_2 \psi) \bar{\psi}) e^{ik_2 X(z_2)} : \rangle$$

$$= \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \int_{\mathbb{R}^4} \frac{d^2 z_I d^2 z_J}{V_{CKG}} \langle e^{-\phi}(z_1) \cdot e^{-\phi}(z_1) : (\mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)} + \mathcal{M}^{(4)}) \rangle$$

(4.12)

where we take the opportune vertex operators from Section (3.1). One of the several $\mathcal{M}^{(i)}$’s terms reads

$$\mathcal{M}_1 = \langle \tilde{\Psi} I e^{ik_1 X(z_1)} \tilde{\Psi} I e^{ik_1 X(z_1)} : (i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \psi) \bar{\psi}) e^{ik_2 X(z_2)} : (i \partial \bar{Z}^J + \frac{\alpha'}{2} (k_2 \psi) \bar{\psi}) e^{ik_2 X(z_2)} : \rangle$$

(4.13)

while the other $\mathcal{M}^{(i)}$’s and their results, using (2.5) and (3.32), (3.33), (3.37) and (3.38), are relegated to [54] (appendix A.6). Exploiting the $PSL(2, R)$ symmetry to fix the vertex operators at (4.6), gives

$$\mathcal{A}_a(T^I, T^J) = \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \left( \frac{4 y}{(1+y)^2} \right)^{-\alpha' t} \left( \frac{1-y}{1+y} \right)^{-\alpha' t} \frac{4(1+y)^2}{16y^2} \left( \frac{(1+\alpha') \mathcal{R}_{\mathfrak{a}} \mathcal{R}_{\mathfrak{a}}}{4} - \frac{\alpha' \delta_{l,J} (\mathcal{R}_{\mathfrak{a}})^2}{4y(1+y)^2} \right)$$

that after the change of variable (2.18) leads to

$$\mathcal{A}_a(T^I, T^J) = \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \left( \frac{4 y}{(1+y)^2} \right)^{-\alpha' t} \left( \frac{1-y}{1+y} \right)^{-\alpha' t} \frac{4(1+y)^2}{16y^2} \left( \frac{(1+\alpha') \mathcal{R}_{\mathfrak{a}} \mathcal{R}_{\mathfrak{a}}}{4} - \frac{\alpha' \delta_{l,J} (\mathcal{R}_{\mathfrak{a}})^2}{4y(1+y)^2} \right) \Gamma(-\alpha' s) \Gamma(-\alpha' t/4)$$

(4.14)

The two different cases are

- $I = J$

$$\mathcal{A}_a(T^I, T^I) = \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)^2} \left( -\alpha^2 t^2 + \frac{\alpha' t}{4} \right) \Gamma(-\alpha' s) \Gamma(-\alpha' t/4)$$

(4.16)

$$= \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)^2} \left( -\alpha^2 t^2 + \frac{\alpha' t}{4} \right) \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' t/4)}{\Gamma(-\alpha' s - \alpha' t/4 + 1)}$$

- $I \neq J$

$$\mathcal{A}_a(T^I, T^J) = \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \left( \frac{\alpha' t}{4} \right) \left( -\alpha^2 s - \alpha^2 t^2 - \frac{\alpha' t}{4} \right) \Gamma(-\alpha' s) \Gamma(-\alpha' t/4)$$

(4.17)

$$= \frac{8 g_s^2 C_{a_2}}{(T^I - T^I)(T^J - T^J)} \left( 1 + \frac{1}{4s} \frac{\alpha^2 t}{4} \right) \left( -\alpha' s - \alpha' t/4 + 1 \right)$$

As said in Section 3.1 the vertex operator associated to the NS-NS untwisted modulus field $T_2^I$ (the imaginary part of the complex Kähler moduli $T^I$), is given by the linear combination

$$W_{T_2^I}(E, z, \bar{z}, k) = -\frac{i}{2} \left( W_{T^I(\theta)}(E, z, \bar{z}, k) - W_{\bar{T}^I(\bar{\theta})}(E, z, \bar{z}, k) \right)$$

(4.18)

thus the true string scattering amplitude that involves two NS-NS untwisted moduli $T_2^I$ is

$$A_a(T_2^I, T_2^J) = \frac{1}{4} \left( A_a(T^I, T^J) - A_a(T^J, T^J) - A_a(T^J, T^J) + A_a(T^I, T^I) \right).$$

(4.19)

The results for the two distinct cases using (4.8), (4.15) are

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29 The same results hold for $(T, T)$ and $(T, T)$ amplitudes.
\[ T \moduli = \left( \frac{4g_s^2 C_{D_2}}{(T^I - T^J)^2} \right) \left\{ \frac{4s}{t} + \frac{t}{2s} + 2 + \alpha^2 \frac{(u^2 + t^2)}{16} (-\zeta(2) + O(\alpha')) \right\} \] (4.20)

\[ \begin{align*}
\mathcal{A}_a(T^I_2, T^J_2) &= \frac{8g_s^2 C_{D_2} \mathcal{R}_a^I \mathcal{R}_a^J}{(T^I - T^J)(T^J - T^I)} \left\{ 1 + \frac{t}{4s} - \frac{\alpha^2 t u}{16} (-\zeta(2) + O(\alpha')) \right\} \\
\mathcal{A}_a(T^I_2, T^J_2) &= \frac{8g_s^2 C_{D_2} \mathcal{R}_a^I \mathcal{R}_a^J}{(T^I - T^J)(T^J - T^I)} \left\{ 1 + \frac{t}{4s} - \frac{\alpha^2 t u}{16} (-\zeta(2) + O(\alpha')) \right\}
\end{align*} \] (4.21)

where on D9-branes \( a = 9 \) and \( \mathcal{R}_9^I = +1 \) while on D5f-branes \( a = 5_f \) one has \( \mathcal{R}_9^I = +1 \) and \( \mathcal{R}_9^J = -1 \). As expected there are no off diagonal mixing at tree level for the kinetic terms between different Kähler moduli \( T^I_2 \) because the closed t-pole channel is absent in (4.21), while (4.20) suggests that the Kähler potential has the expected shape (3.9) [40, 17, 38, 27].

### 4.2 A\(_{D_2}^J (U^I, \bar{U}^J)\) and A\(_{D_2}^{U^I, \bar{U}^J}\)

The next set of amplitudes in (4.1) involves two complex structure moduli \( U^I \). Aside the different vertex operator definition for the complex structure \( U^I \), the main steps are clearly the same as for the Kähler modulus \( T^I_2 \), thus

\[ \mathcal{A}_a(U^I, \bar{U}^J) = g^2 C_{D_2} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \mathcal{W}_{c^I(-1,-1)}(E_1, k_1, z_1, \bar{z}_1) \mathcal{W}_{c^J(0,0)}(E_2, k_2, z_2, \bar{z}_2) \right) \]

\[ = -\frac{8g_s C_{D_2}}{\alpha'(U^I - U^J)(U^J - U^I)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \Psi^I e^{-\phi} e^{ik_1 X(z_1)} \bar{\Psi}^I e^{-\bar{\phi}} e^{ik_1 \bar{X}(\bar{z}_1)} : \right) \]

\[ \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \bar{\psi}) \bar{\psi} \right) e^{ik_2 X(z_2)} \left( i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \bar{\psi}) \bar{\psi} \right) e^{ik_2 \bar{X}(\bar{z}_2)} : \]

\[ = -\frac{8g_s C_{D_2}}{\alpha'(U^I - U^J)(U^J - U^I)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( e^{-\phi(z_1)} : e^{-\bar{\phi}(\bar{z}_1)} : \right) \left( \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)} + \mathcal{M}^{(4)} \right) \] (4.22)

that with the help of (2.5), (3.32), (3.33), (3.37) and (3.38), lead to \( \mathcal{M} \)'s terms as

\[ \mathcal{M}^{(1)} = \left\{ \Psi^I e^{ik_1 X(z_1)} \bar{\Psi}^I e^{ik_1 \bar{X}(\bar{z}_1)} : i \partial \bar{Z}^I e^{ik_2 X(z_2)} i \partial \bar{Z}^I e^{ik_2 \bar{X}(\bar{z}_2)} : \right\} . \] (4.23)

with details on all \( \mathcal{M}^{(i)} \)'s collected in [54] (appendix A.6). At this point the amplitude for the pair \( (U^I, \bar{U}^J) \) reads

\[ \mathcal{A}_a(U^I, \bar{U}^J) = -\frac{8g_s C_{D_2}}{\alpha'(U^I - U^J)(U^J - U^I)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \left( \left| z_1 - \bar{z}_1 \right| \left| z_2 - \bar{z}_2 \right| \right)^{-\alpha'} \left( \left| z_1 - z_2 \right|^2 \right)^{-\alpha''} \frac{1}{(z_1 - \bar{z}_1)} \]

\[ \left\{ -\frac{\alpha'^2 (\mathcal{R}_a^I)^2 \delta_{I,J}}{2(z_2 - \bar{z}_2)(z_1 - \bar{z}_1) - (z_2 - \bar{z}_2)(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} \right\} \] (4.24)

that exploiting (4.6) to fix the vertex operators and inserting c-ghost determinant (2.15) becomes

\[ -\delta_{I,J} \frac{8g_s C_{D_2}}{(U^I - U^J)(U^J - U^I)} (\alpha' s) \int_0^1 dy \left( \frac{4y}{(1+y)^2} \right)^{-\alpha'} \left( \frac{(1-y)^2}{(1+y)^2} \right)^{-\alpha''} \left( \frac{1+y}{y(1-y)} \right) \left( \frac{(\mathcal{R}_a^I)^2(1-y)}{y(1+y)} \right) \] (4.25)

and finally after the change of variable (2.18) one gets

\[ \mathcal{A}_a(U^I, \bar{U}^J) = \delta_{I,J} \frac{8g_s C_{D_2}}{(U^I - U^J)(U^J - U^I)} (\alpha' s) \frac{\Gamma(-\alpha') \Gamma(-\alpha'/4)}{\Gamma(-\alpha' - \alpha'/4+1)} \] (4.26)

Separating the two cases and using the gamma function expansion (4.11), one has

\[ (\mathcal{R}_a^I)^2 = 1 \]
\[ I = J \]

\[
A_a(U^I, U^I) = \frac{8g_a^2C_{D_3}}{(U^I - U^I)^2} \left\{ \frac{4s}{t} + \alpha'^2 s^2 (-\zeta(2) + O(\alpha')) \right\}
\]

(4.27)

\[ I \neq J \]

\[
A_a(U^I, U^J) = 0
\]

(4.28)

which give the same results in all cases \( a \in \{9, 51\} \). computing the amplitude which involves the complex structure moduli pair \((U^I, U^J)\) we anticipate that one arrives at a vanishing result. Considering the vertex operators in Section (3.1) one has

\[
A_a(U^I, U^J) = g_a^2 C_{D_3} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle \Psi^{I} e^{-\phi} e^{ik_1 X}(z_1) \bar{\Psi}^{I} e^{-\phi} e^{ik_1 \bar{X}}(\bar{z}_1) : \rangle
\]

\[
= \frac{8g_a^2 C_{D_3}}{\alpha'(U^I - U^I)(U^J - U^J)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle \Psi^{J} e^{-\phi} e^{ik_1 X}(z_2) \bar{\Psi}^{J} e^{-\phi} e^{ik_1 \bar{X}}(\bar{z}_2) : \rangle
\]

(4.29)

where the specific combination of two-point functions (3.32), (3.33), (3.37) and (3.38), for the compactified field in the \( M^{(i)}'s \) computation as for instance

\[
\mathcal{M}^{(1)} = \langle \Psi^{J} e^{ik_1 X}(z_1) \bar{\Psi}^{I} e^{ik_1 \bar{X}}(\bar{z}_1) : i\partial \bar{Z} \rangle
\]

(4.30)

are responsible for the vanishing of all the \( \mathcal{M}^{(i)}'s \) terms as one can see in [54] (appendix A.6), so the final result is

\[
A_a(U^I, U^J) = 0
\]

(4.31)

independently if \( I = J \) or \( I \neq J \). The complex structure \( U^I \) moduli is a purely geometrical moduli thence no vertex redefinition is needed in contrast to the Kähler modulus \( T^I \). Thus (4.31) and (4.28) show that there is no mixing between different complex structure \( U \) moduli, while (4.27) is in agreement with the shape of the tree-level Kähler potential for this model (3.9) [40, 17, 38, 27].

4.3 \( A_a^{D_3}(T^I, \bar{T}^J) \) and \( A_a^{D_3}(T^J, U^I) \)

The last set of amplitudes in (4.1) involves one Kähler modulus \( T^I \) and one complex structure modulus \( U^I \). We anticipate that this kind of amplitudes are zero and, for these reasons, we give less details here in contrast to the previous cases. The amplitude which involves the pair \((T, \bar{T})\), with the explicit form of vertex operator taken from the Section (3.1), is equal to

\[
A_a(T^I, \bar{T}^J) = g_a^2 C_{D_3} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle W_{T^I(1, -1)}(E_1, k_1, z_1, \bar{z}_1) \rangle \langle W_{U^J(0, 0)}(E_2, k_2, z_2, \bar{z}_2) \rangle
\]

\[
= \frac{8g_a^2 C_{D_3}}{\alpha'(T^I - T^I)(U^J - U^J)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle \bar{\Psi}^{I} e^{ik_1 X}(z_1) \Psi^{I} e^{-\phi} e^{ik_1 \bar{X}}(\bar{z}_1) : \rangle
\]

\[
= \frac{8g_a^2 C_{D_3}}{\alpha'(T^I - T^I)(U^J - U^J)} \int_{\mathcal{N}^+} \frac{d^2 z_1 d^2 z_2}{V_{\text{CKG}}} \langle e^{-\phi}(z_1) \cdot e^{-\bar{\phi}}(\bar{z}_1) : \rangle \left( \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)} + \mathcal{M}^{(4)} \right)
\]

(4.32)
where as one can see in [54] (appendix A.6) that the $\mathcal{M}^{(i)}$’s terms are all zero due to the vanishing of particular two-point functions that enter in the definition of $\mathcal{M}^{(i)}$’s. So the amplitude is zero when $I = J$, $I \neq J$ and for $a \in \{9, 5_l\}$

$$ A_a(T^I \bar{U}^J) = 0 \quad (T \leftrightarrow U) . \quad (4.33) $$

The same happens for to the second amplitude in this specific set

$$ A_a(T^I, U^J) = -g_c^2 C_{d_2} \int_{n^+} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (W_{T^I, T^I}(-1,-1)}(E_1, k_1, z_1, \bar{z}_1)W_{U^J, U^J}(E_2, k_2, z_2, \bar{z}_2)) $$

$$ = -\frac{8g_c^2 C_{d_2}}{\alpha'(T^I - T^I)(J^J - U^J)} \int_{n^+} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (\bar{\Psi}^I e^{-\phi} e^{ik_1 X}(z_1)\bar{\Psi}^I e^{-\phi} e^{ik_1 X}(\bar{z}_1) : (i\partial Z^I + \frac{\alpha'}{2}(k_2 z)\bar{\Psi}^I) (i\partial \bar{Z}^I + \frac{\alpha'}{2}(k_2 \bar{z})\bar{\Psi}^I) e^{ik_2 \bar{X}}(\bar{z}_2) : ) $$

$$ = -\frac{8g_c^2 C_{d_2}}{\alpha'(T^I - T^I)(J^J - U^J)} \int_{n^+} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (e^{-\phi}(z_1) :: e^{-\phi}(\bar{z}_1) :: ) \left( \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{M}^{(3)} + \mathcal{M}^{(4)} \right) \quad (4.34) $$

which vanishes for the same reasons (no difference between $I = J$ and $I \neq J$) as one can see in [54] (appendix A.6)

$$ A_a(T^I, U^J) = 0 \quad (4.35) $$

In a rigorous way also in this case one should take the vertex for the NS-NS $T^I$ Kähler modulus (4.18), but the result will be same. So equations (4.35) and (4.33) confirm again the shape of the Kähler potential (3.9).

### 4.4 $A_{\alpha}^{RP_2} (T^I, \bar{T}^J)$ and $A_{\alpha}^{\bar{R}P_2} (T^I, T^J)$

The technology applied to the scattering amplitude on $RP_2$ world-sheet for the uncompactified model is useful to understand how one has to treat the scattering of two moduli on $RP_2$ when compactified model is considered. The set of amplitudes involving untwisted moduli calculated, as said, are

$$ A_a(T^I, \bar{T}^J) + A_a(T^I, T^J) = g_c^2 C_{\alpha} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (W_{T^I, T^I}(z_1, \bar{z}_1)W_{\bar{T}^J, \bar{T}^J}(z_2, \bar{z}_2)) $$

$$ A_a(U^I, U^J) + A_a(U^I, T^J) = g_c C_{\alpha} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (W_{U^I, U^I}(z_1, \bar{z}_1)W_{U^J, \bar{T}^J}(z_2, \bar{z}_2)) $$

$$ A_a(U^I, \bar{T}^J) + A_a(T^I, U^I) = g_c C_{\alpha} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} (W_{T^I, U^I}(z_1, \bar{z}_1)W_{\bar{T}^J, \bar{T}^J}(z_2, \bar{z}_2)) $$

$$ A_a(T^I, \bar{T}^J) + A_a(T^I, T^J) = g_c^2 C_{\alpha} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} \left( W_{T^I, T^I}(-1,-1)}(E_1, k_1, z_1, \bar{z}_1)W_{T^I, T^I}(-1,-1)}(E_2, k_2, z_2, \bar{z}_2)) = \sum_{i=1}^{4} \Lambda_i \quad (4.36) $$

Details on the vertex operators construction on $RP_2$ are reported in the Section (3.1). The first set of scattering amplitudes, for the pair $(T, \bar{T})$ massless Kähler moduli in the picture $(-1,-1); 0,0$ is

$$ A_a(T^I, \bar{T}^J) = g_c^2 C_{\alpha} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} \left( W_{T^I, T^I}(-1,-1)}(E_1, k_1, z_1, \bar{z}_1)W_{T^I, T^I}(-1,-1)}(E_2, k_2, z_2, \bar{z}_2)) = \sum_{i=1}^{4} \Lambda_i \quad (4.37) $$

where the $\Lambda_i$’s are different sub-amplitudes due to the shape of vertex operators, as one can see in Section (3.1). Each $\Lambda$’s sub-amplitude behaves as a single disk scattering amplitude therefore, for
instance, \( \Lambda_1 \) is equal to

\[
\Lambda_1 = - \frac{2g^2 C_{RP_2}}{\alpha'(T^I - T^I)(T^J - T^J)} \int_{|z| \leq 1} d^2 z_1 d^2 z_2 \frac{\langle \hat{\Psi}^I e^{-\phi} e^{ik_1 X(z_1)} \hat{\Phi}^J e^{-\tilde{\phi}} e^{ik_1 \tilde{X}(\tilde{z}_1)} \rangle}{V_{CKG}} \cdot (i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \psi)) e^{ik_2 X(z_2)} (i \partial \bar{Z}^J + \frac{\alpha'}{2} (k_2 \bar{\psi})) e^{ik_2 \tilde{X}(\tilde{z}_2)}/(4.38)
\]

where again \( \mathcal{M}^{(j)} \)'s with \((i, j = 1, 2, 3, 4)\) are the different contraction terms that one meets in each \( \Lambda_i \)'s sub-amplitude, as for instance

\[
\mathcal{M}^{(1)}_{\Lambda_1} = \langle \hat{\Psi}^I e^{ik_1 X(z_1)} \hat{\Phi}^J e^{ik_1 \tilde{X}(\tilde{z}_1)} : i \partial \bar{Z}^I e^{ik_2 X(z_2)} i \partial \bar{Z}^J e^{ik_2 \tilde{X}(\tilde{z}_2)} : \rangle. (4.39)
\]

The \( \mathcal{M}^{(j)} \)'s can be calculated using the two-point functions for the uncompactified fields and the two-point functions for the compactified fields (3.32), (3.33), (3.40) and (3.41). The derivation is equal to the disk case, aside the difference due to the involution \( \mathcal{I}(z) = -1/z \) that characterizes the projective plane. As for the disk cases the list of all \( \Lambda \) sub-amplitudes and the details on their \( \mathcal{M} \) terms are in [54] (appendix A.7). Summing the \( \Lambda \)'s sub-amplitudes with the same Koba-Nielsen factor one gets

\[
\Lambda_1 + \Lambda_4 = - \frac{2g^2 C_{RP_2}}{\alpha'(T^I - T^I)(T^J - T^J)} \int_{|z| \leq 1} d^2 z_1 d^2 z_2 \frac{\langle \hat{\Psi}^I e^{-\phi} e^{ik_1 X(z_1)} \hat{\Phi}^J e^{-\tilde{\phi}} e^{ik_1 \tilde{X}(\tilde{z}_1)} \rangle}{V_{CKG}} \cdot (\frac{(1 + |z_1|^2)(1 + |z_2|^2)}{|1 + z_1 \bar{z}_2|^2}) \cdot \alpha_s \bar{\delta}_{I,J} (\frac{|z_1 - z_2|^2}{|1 + z_1 \bar{z}_2|^2}) \cdot \frac{(\alpha_s \bar{\delta}_{I,J})^2}{(1 + |z_1|^2)(1 + |z_2|^2)\bar{z}_2 - z_1 |^2}). (4.40)
\]

The vertices, using the \( SU(2) \) symmetry can be fixed at

\[z_1 = 0; \quad \bar{z}_1 = 0; \quad z_2 = iy; \quad \bar{z}_2 = -iy (4.41)\]

and inserting in (4.40) the c-ghost contribution (from (2.32)) becomes

\[
\Lambda_1 + \Lambda_4 = - \frac{2g^2 C_{RP_2}}{(T^I - T^I)(T^J - T^J)} \int_0^1 dy^2 (1 + y^2)^{-\alpha_s(y^2)} - \alpha_s \bar{\delta}_{I,J} \left\{ (1 + \alpha_s \bar{\delta}_{I,J}) \frac{(1 + \alpha_s \bar{\delta}_{I,J})}{(1 + y^2)^2} \right\}.
\]

\[
\Lambda_2 + \Lambda_3 = - \frac{2g^2 C_{RP_2}}{(T^I - T^I)(T^J - T^J)} \int_0^1 dy^2 (1 + y^2)^{-\alpha_s(y^2)} - \alpha_s \bar{\delta}_{I,J} \left\{ (1 + \alpha_s \bar{\delta}_{I,J}) \frac{(1 + \alpha_s \bar{\delta}_{I,J})}{(1 + y^2)^2} \right\}.
\]

\[31 \text{ Where one has to reinsert the reflection matrix } \mathcal{R} \text{ on the two points correlation function which involves left and right field.}
\]

\[32 (\bar{\mathcal{R}}_2^J)^2 = 1\]
Using the definition of the $2F_1$ hypergeometric function (2.44) in (4.51) one obtains

$$
\begin{align*}
\Lambda_1 + \Lambda_4 &= -\frac{2g_c^2 C_{\text{RP}_2}}{(T^I - T^I)(T^J - T^J)} \left\{ \mathcal{R}_\alpha^I \mathcal{R}_\alpha^J \left( (1+\alpha')s_2 F_1(\alpha's+2, -\alpha't/4+1; -\alpha't/4+2; -1) \right) \right. \\
&\quad \quad - \delta_{I,J} \alpha' s_2 F_1(\alpha's+1, -\alpha't/4; -\alpha't/4+1; -1) \\
\end{align*}
$$

(4.43)

$$
\begin{align*}
\Lambda_2 + \Lambda_3 &= -\frac{2g_c^2 C_{\text{RP}_2}}{(T^I - T^I)(T^J - T^J)} \left\{ \mathcal{R}_\alpha^I \mathcal{R}_\alpha^J \left( (1+\alpha')s_2 F_1(\alpha's+2, -\alpha'u/4+1; -\alpha'u/4+2; -1) \right) \right. \\
&\quad \quad - \delta_{I,J} \alpha' s_2 F_1(\alpha's+1, -\alpha'u/4+1; -\alpha'u/4+2; -1) \\
\end{align*}
$$

(4.44)

where using the identity (2.46), one is able to combine the apparently different $2F_1$ hypergeometric functions, as for instance

$$
\begin{align*}
\alpha' s_2 F_1(\alpha's+1, -\alpha't/4; -\alpha't/4+1; -1) = -\frac{(-\alpha'u/4+1)}{(-\alpha't/4+1)} s_2 F_1(\alpha's+2, -\alpha't/4+1; -\alpha't/4+2; -1) \\
+ \frac{\Gamma(-\alpha't/4+1)\Gamma(-\alpha'u/4+2)}{\Gamma(\alpha's+2)}
\end{align*}
$$

that together with the $\Gamma$ function properties allows one to sum all the $\Lambda_i$’s in a final compact result

$$
\mathcal{A}_\alpha(T^I, T^J) = -\frac{2g_c^2 C_{\text{RP}_2}}{(T^I - T^I)(T^J - T^J)} \left\{ \alpha's_2 F_1(\alpha's+2, -\alpha'u/4+1; -\alpha'u/4+2; -1) \right. \\
- \delta_{I,J} \alpha' s_2 F_1(\alpha's+1, -\alpha'u/4+1; -\alpha'u/4+2; -1) \\
\right\} \frac{\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)}{\Gamma(-\alpha't/4 - \alpha'u/4+1)}
$$

(4.45)

The second amplitude in this set involving the pair $(T^I, T^J)$ massless Kähler moduli in the picture (-1-1;00) reads

$$
\mathcal{A}_\alpha(T^I, T^J) = g_c^2 C_{\text{RP}_2} \int_{|z| \leq 1} \frac{dz_1 dz_2}{V_{\text{CKG}}} (W_{\tau^{(-1,-1)}}(E_1, k_1, z_1, z_1) W_{\tau^{(0,0)}}(E_2, k_2, z_2, z_2)) = \sum_{i=1}^{4} \Lambda_i
$$

(4.48)

where each $\Lambda_i$’s sub-amplitude admit the same representation as before, i.e behaves like a single disk amplitude, for instance

$$
\begin{align*}
\Lambda_1 &= \frac{2g_c^2 C_{\text{RP}_2}}{\alpha'(T^I - T^I)(T^J - T^J)} \int_{|z| \leq 1} \frac{dz_1 dz_2}{V_{\text{CKG}}} (e^{\phi(z_1)} e^{i\phi z_1}(z_1)): \tilde{\bar{\Psi}}^I e^{-\phi} e^{ik_1 X(z_1)} \bar{\tilde{\Psi}}^I e^{-\phi} e^{ik_1 X(z_1)} :
\end{align*}
$$

(4.49)
with for instance $\mathcal{M}^{(1)}_\Lambda$ equals to
\begin{equation}
\langle \bar{\Psi} I e^{ik_1X(z_1)} \Psi I e^{ik_2X(z_2)} : i\partial Z J e^{ik_2X(z_2)} i\partial Z J e^{ik_1X(z_1)} : \rangle .
\end{equation}

We collect all the $\Lambda$'s sub-amplitudes and $\mathcal{M}$'s terms in [54](appendix A.7). $SU(2)$ invariance allows one to fix the vertex operators at (4.41) and with the specific c-ghost correlator (2.41) the (4.49) reads
\begin{equation}
\begin{aligned}
\Lambda_1 + \Lambda_4 &= \frac{2g_c^2 C_{R_{\alpha}}}{(T^J - T^J)} \int_0^1 dy^2 (1 + y^2)^{-\alpha' s(y^2) - \alpha' s(y^2)} \left\{ \frac{1 + \alpha' s}{(1 + y^2)^2} - \frac{\alpha' s}{(1 + y^2)^2} \right\} \\
\Lambda_2 + \Lambda_3 &= \frac{2g_c^2 C_{R_{\alpha}}}{(T^J - T^J)} \int_0^1 dy^2 (1 + y^2)^{-\alpha' s(y^2) - \alpha' s(y^2)} \left\{ \frac{1 + \alpha' s}{(1 + y^2)^2} - \frac{\alpha' s}{(1 + y^2)^2} \right\} .
\end{aligned}
\end{equation}

In terms of the integral definition of Hypergeometric functions $2F_1$ (2.44) they are represented by
\begin{equation}
\begin{aligned}
\Lambda_1 + \Lambda_4 &= \frac{2g_c^2 C_{R_{\alpha}}}{(T^J - T^J)} \left\{ \frac{2F_1(1 + \alpha' s, 2, -\alpha' t/4 + 1; -\alpha' t/4 + 2; -1)}{(-\alpha' t/4 + 1)} - \frac{\alpha' s}{-\alpha' t/4 + 1} \right\} \\
\Lambda_2 + \Lambda_3 &= \frac{2g_c^2 C_{R_{\alpha}}}{(T^J - T^J)} \left\{ \frac{2F_1(1 + \alpha' s, 2, -\alpha' u/4 + 1; -\alpha' u/4 + 2; -1)}{(-\alpha' u/4)} - \frac{\alpha' s}{-\alpha' u/4} \right\} .
\end{aligned}
\end{equation}

that together with the identity for the Hypergeometric functions such as (4.44), the $\Lambda$'s sub-amplitudes can be combined as
\begin{equation}
\begin{aligned}
\frac{2g_c^2 C_{R_{\alpha}}}{(T^J - T^J)} \left\{ \frac{2F_1(1 + \alpha' s + 1, 1; -\alpha' t/4 + 1; -\alpha' u/4 + 1; -\alpha' u/4 + 1)}{(-\alpha' u/4 + 1)\Gamma(\alpha' s + 1)} - \frac{\alpha' s}{-\alpha' u/4 + 1} \right\}
\end{aligned}
\end{equation}

giving the final result
\begin{equation}
\begin{aligned}
\mathcal{A}_{\alpha} (T^I, T^J) &= \frac{2g_c^2 C_{R_{\alpha}}}{(T^I - T^J)^2 \Gamma(\alpha' s + 1)} \left\{ \frac{2F_1(1 + \alpha' s, 2, -\alpha' t/4 + 1; -\alpha' u/4 + 1)}{(-\alpha' u/4 + 1)\Gamma(\alpha' s + 1)} - \frac{\alpha' s}{-\alpha' u/4 + 1} \right\}
\end{aligned}
\end{equation}

\begin{itemize}
\item $I = J$
\begin{equation}
\mathcal{A}_{\alpha} (T^I, T^J) = \frac{2g_c^2 C_{R_{\alpha}}}{(T^I - T^J)^2} \left\{ \frac{\Gamma(\alpha' t/4)\Gamma(\alpha' u/4)}{\Gamma(-\alpha' t/4)} \right\}
\end{equation}
\end{itemize}

\begin{itemize}
\item $I \neq J$
\begin{equation}
\mathcal{A}_{\alpha} (T^I, T^J) = \frac{2g_c^2 C_{R_{\alpha}}}{(T^I - T^J)(T^J - T^J)} \left\{ \frac{\Gamma(\alpha' t/4)\Gamma(\alpha' u/4)}{\Gamma(\alpha' t/4 - \alpha' u/4 + 1)} \right\}
\end{equation}
\end{itemize}

In order to take the string scattering amplitude involving two NS-NS Kähler moduli $T^I_2$, as in the disk $D_2$, a redefinition of the vertex is needed
\begin{equation}
\mathcal{W}^{\oplus}_{(q, \bar{q})} (E, z, \bar{z}, k) = \frac{i}{2} \left( \mathcal{W}^{\oplus}_{(q, \bar{q})} (E, z, \bar{z}, k) - \mathcal{W}^{\oplus}_{(q, \bar{q})} (E, z, \bar{z}, k) \right)
\end{equation}
and the corresponding string scattering amplitude results for both the cases $I = J$ and $I \neq J$, come from the following linear combination

$$
\mathcal{A}_\alpha(T_2^I, T_2^J) = \frac{1}{4} \left( \mathcal{A}_\alpha(T^I, T^J) - \mathcal{A}_\alpha(T^I, \bar{T}^J) + \mathcal{A}_\alpha(\bar{T}^I, T^J) + \mathcal{A}_\alpha(T^I, \bar{T}^J) \right)
$$

(4.58)

- $I = J$

$$
\mathcal{A}_\alpha(T_2^I, T_2^J) = \frac{g_s^2 C_{RP_2}}{(T^I - T^J)^2} \left\{ \frac{u + t}{u} \right\} + \frac{\alpha'^2}{16} \left( u^2 + t^2 \right) \left( -\zeta(2) + O(\alpha') \right)
$$

(4.59)

- $I \neq J$

$$
\mathcal{A}_\alpha(T_2^I, T_2^J) = -\frac{g_s^2 C_{RP_2}}{(T^I - T^J)(T^I - \bar{T}^J)} \left\{ 1 + \alpha'^2 \frac{t u}{16} \left( -\zeta(2) + O(\alpha') \right) \right\}
$$

(4.60)

where $\Omega_9$-planes $\alpha = 9$, $\mathcal{R}^I_9 = +1$ while on $\Omega_5T$-planes $\alpha = 5T$ one has $\mathcal{R}^I_5 = +1$ and $\mathcal{R}^I_5 = -1$. As expected, only closed string pole channels are allowed. Moreover the results (4.59) and (4.59) tell us that the contribution of $RP_2$ does not spoil the shape of the Kähler potential (3.9) but confirms it.

4.5 $\mathcal{A}^{RP_2}_\alpha(U^I, \bar{U}^J)$ and $\mathcal{A}^{RP_2}_\alpha(U^I, U^J)$

The second set of amplitudes in (4.36) involves two massless complex structures $U$ where again the scattering amplitudes is taken in the picture $(-1, -1; 0, 0)$ and the specific vertex operators can be find in Section (3.1). The scattering amplitude for the pair $(U^I, \bar{U}^J)$ is

$$
\mathcal{A}_\alpha(U^I, \bar{U}^J) = g_s^2 C_{RP_2} \int_{\{z\leq 1\}} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} \langle \Psi^I e^{-\phi} e^{ik_1 X(z_1)} \bar{\Psi}^I e^{-\bar{\phi}} e^{ik_1 \bar{X}(\bar{z}_2)} : i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \bar{\psi}) e^{ik_2 X(z_2)} \bar{\partial} \bar{Z}^J + \frac{\alpha'}{2} (k_2 \bar{\psi}) e^{ik_2 X(z_2)} e^{ik_2 \bar{X}(\bar{z}_2)} : \rangle
$$

(4.61)

where, for instance, the $\Lambda_1$ sub-amplitude is equal to

$$
\Lambda_1 = -\frac{2g_s^2 C_{RP_2}}{\alpha'(U^I - U^J)(U^J - \bar{U}^J)} \int_{\{z\leq 1\}} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} \langle \Psi^I e^{-\phi} e^{ik_1 X(z_1)} \bar{\Psi}^I e^{-\bar{\phi}} e^{ik_1 \bar{X}(\bar{z}_2)} : i \partial \bar{Z}^I + \frac{\alpha'}{2} (k_2 \bar{\psi}) e^{ik_2 X(z_2)} \bar{\partial} \bar{Z}^J + \frac{\alpha'}{2} (k_2 \bar{\psi}) e^{ik_2 X(z_2)} \rangle e^{ik_2 \bar{X}(\bar{z}_2)}
$$

(4.62)

and correlators as $M_1^{(1)}$

$$
\langle \Psi^I e^{ik_1 X(z_1)} \bar{\Psi}^I e^{ik_1 \bar{X}(\bar{z}_2)} : i \partial \bar{Z}^I e^{ik_2 X(z_2)} i \bar{\partial} \bar{Z}^J e^{ik_2 \bar{X}(\bar{z}_2)} : \rangle
$$

(4.63)

can be calculated using (2.30), (3.32), (3.33), (3.40) and (3.41). The details, together with the others $\Lambda$’s and the relative $M_{\Lambda}$’s are reported in [54] (appendix A.7). As for the $TT$ case, one takes the sum of $\Lambda_i$’s sub-amplitudes characterized by the same Koba-Nielsen factor

$$
\Lambda_1 + \Lambda_4 = -\frac{2g_s^2 C_{RP_2}}{(U^I - U^J)(U^J - \bar{U}^J)} \int_{\{z\leq 1\}} \frac{d^2 z_1 d^2 \bar{z}_2}{V_{CKG}} \langle \frac{1 + |z_1|^2}{1 + |z_1|^2} \frac{1 + |z_2|^2}{1 + |z_2|^2} \rangle - \alpha' s \frac{|z_1 - z_2|^2}{|1 + z_1 \bar{z}_2|^2} - \alpha' \bar{s} \frac{|z_1 - z_2|^2}{|1 + z_1 \bar{z}_2|^2}
$$

(4.64)
The positions of vertices can be fixed at (4.41) using the SU(2) symmetry. In this way the \( \Lambda_i \)’s sub-amplitudes with the right insertion of c-ghost correlator (2.41) read

\[
\begin{align*}
\Lambda_1 + \Lambda_4 &= - \frac{2g_c^2C_{R\Phi_2} \delta_{I,J}}{(U^I - U^J)(U^J - U^I)}(\alpha') \left[ \frac{1}{(1 + y^2)^{\alpha' s}} \right] \left\{ - \frac{1}{(1 + y^2)} - \frac{1}{(1 + y^2)y^2} \right\} \\
\Lambda_2 + \Lambda_3 &= - \frac{2g_c^2C_{R\Phi_2} \delta_{I,J}}{(U^I - U^J)(U^J - U^I)}(\alpha') \left[ \frac{1}{(1 + y^2)^{\alpha' s}} \right] \left\{ - \frac{1}{(1 + y^2)y^2} - \frac{1}{(1 + y^2)^2} \right\}
\end{align*}
\]

and using (2.44) for the integral definition of \( \mathcal{F}_1 \) function one can write

\[
\begin{align*}
\Lambda_1 + \Lambda_4 &= - \frac{2g_c^2C_{R\Phi_2} \delta_{I,J}}{(U^I - U^J)(U^J - U^I)}(\alpha') \left[ \frac{2F_1(\alpha' s + 1, -\alpha' t/4 + 1; -\alpha' t/4 + 2; -1) - 2F_1(\alpha' s, -\alpha' t/4 + 1; -\alpha' t/4 + 1; -1)}{(-\alpha' t/4)} \right] \\
\Lambda_2 + \Lambda_3 &= - \frac{2g_c^2C_{R\Phi_2} \delta_{I,J}}{(U^I - U^J)(U^J - U^I)}(\alpha') \left[ \frac{2F_1(\alpha' s + 1, -\alpha' u/4; -\alpha' u/4 + 1; -1) - 2F_1(\alpha' s + 1, -\alpha' u/4 + 1; -\alpha' u/4 + 2; -1)}{(-\alpha' u/4)} \right].
\end{align*}
\]

At this point summing together all the \( \Lambda_i \) using the \( \mathcal{F}_1 \)-identity (2.46), the final result is

\[
\mathcal{A}_\alpha(U^I, U^J) = \frac{2g_c^2C_{R\Phi_2} \delta_{I,J}}{(U^I - U^J)(U^J - U^I)}(\alpha') \left[ \frac{\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(-\alpha' t/4 - \alpha' u/4 + 1)} \right]
\]

with \( \alpha \in \{9, 9/4\} \) while the two different cases are

- \( I = J \)

\[
\mathcal{A}_\alpha(U^I, U^I) = \frac{2g_c^2C_{R\Phi_2}}{(U^I - U^I)^2} \left\{ \frac{t}{u} + \frac{u}{t} + 2 + \alpha'^2s^2 (-\zeta(2) + O(\alpha')) \right\}
\]

- \( I \neq J \)

\[
\mathcal{A}_\alpha(U^I, U^J) = 0
\]

In this second set of amplitudes that involves the complex structures moduli \( U \), the second amplitude, characterized by the pair \( (U^I, U^J) \) is

\[
\mathcal{A}_\alpha(U^I, U^J) = g_c^2C_{R\Phi_2} \int_{|z| \leq 1} \frac{d^2z_1 d^2z_2}{V_{CKG}} \mathcal{W}^{\otimes(-1,-1)}(E_1, k_1, z_1, \bar{z}_1)W^{\otimes(0,0)}(E_2, k_2, z_2, \bar{z}_2) = \sum_{i=1}^{4} \Lambda_i
\]

Since the intermediates steps are equal to the \( (U^I, \bar{U}^I) \) case, here we give only an example of one type of \( \Lambda \)’s sub-amplitudes and of \( \mathcal{M} \)’s contraction terms

\[
\Lambda_1 = \frac{2g_c^2C_{R\Phi_2}}{\alpha'(U^I - U^I)(U^J - U^J)} \int_{|z| \leq 1} \frac{d^2z_1 d^2z_2}{V_{CKG}} \left( \Psi^I e^{-\phi} e^{ik_1X}(z_1) \bar{\Psi}^I e^{-\bar{\phi}} e^{ik_1X}(\bar{z}_1) : i\partial Z^I + \frac{\alpha'}{2} (k_2 \bar{\psi}^I) e^{ik_2X}(z_2) : e^{ik_2X}(\bar{z}_2) \right)
\]

(4.71)

and

\[
\mathcal{M}^{(1)}_{\Lambda_1} = \left( \Psi^I e^{ik_1X}(z_1) \bar{\Psi}^I e^{ik_1X}(\bar{z}_1) : i\partial Z^I e^{ik_2X}(z_2) i\partial \bar{Z}^I e^{ik_2X}(\bar{z}_2) : \right)
\]

(4.72)

but in contrast to the previous case, all the \( \mathcal{M} \)’s terms inside the \( \Lambda \)’s sub-amplitudes, taking the two-point functions \( (2.30) \) and \( (3.32) \), \( (3.33) \), \( (3.40) \), \( (3.41) \), vanish being equal to zero some correlators
Therefore the result of this amplitudes in both cases, \( I = J \) and \( I \neq J \), with \( \alpha \in \{9,5I\} \) is zero
\[
A_{\alpha}(U^I,U^J) = 0.
\]

The results (4.68), (4.69) and (4.73) confirm the tree-level Kähler potential (3.9).

### 4.6 \( A_{\alpha}^{RP_2}(T^I,\bar{U}^J) \) and \( A_{\alpha}^{RP_2}(T^I,U^J) \)

The last set of amplitudes in (4.36) involves one Kähler modulus \( T^I \) and one complex structure modulus \( U^J \). As before, the specific vertex operators are taken in the picture \((-1,-1;00)\) and are explicitly written in Section (3.1). We anticipate that this amplitude, as in the disk case, vanishes due to the particular combination of two-point functions for the compactified fields that are involved. But for completeness the intermediate steps are sketched only, while for the details the reader can consult [54]. Therefore one starts with
\[
A_{\alpha}(T^I,\bar{U}^J) = g_c^2 C_{RP_2} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \langle W_{T^I}^{\bar{z}_1}(E_1,k_1,z_1,\bar{z}_1) W_{U^J}^{\bar{z}_2}(E_2,k_2,z_2,\bar{z}_2) \rangle = \sum_{i=1}^4 \Lambda_i
\]
where for instance, the sub-amplitude \( \Lambda_1 \) is equal to
\[
\Lambda_1 = \frac{2g_c^2 C_{RP_2}}{\alpha'(T^I - T^J)(U^J - \bar{U}^J)} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \left( \tilde{\Psi}^I e^{-\phi} e^{ik_1 X} (z_1) \tilde{\Psi}^I e^{-\phi} e^{ik_1 \bar{X}} (\bar{z}_1) : \right)
\]
\[
= \frac{2g_c^2 C_{RP_2}}{\alpha'(T^I - T^J)(U^J - \bar{U}^J)} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \left( e^{-\phi} (z_1) :: e^{-\phi} (\bar{z}_1) : \right) \left( M^{(1)}_{\Lambda_1} + M^{(2)}_{\Lambda_1} + M^{(3)}_{\Lambda_1} + M^{(4)}_{\Lambda_1} \right)
\]
while the relevant \( M^{(1)}_{\Lambda_1} \) is
\[
\langle \tilde{\Psi}^I e^{ik_1 X} (z_1) \tilde{\Psi}^I e^{ik_1 \bar{X}} (\bar{z}_1) : i\partial \bar{Z}^J e^{ik_2 X} (z_2) i\bar{\partial} \bar{Z}^{\bar{J}} e^{ik_2 \bar{X}} (\bar{z}_2) \rangle.
\]

Considering the two sets of correlators (2.30) and (3.32), (3.33), (3.40), (3.41) for the uncompactified and compactified fields respectively, one is able to calculate all the \( M^{(1)}_{\Lambda_1} \)’s terms inside the \( \Lambda_i \)’s that are collected in [54] (appendix A.7). The final result, as anticipated, is zero
\[
A_{\alpha}(T^I,\bar{U}^J) = 0 \quad (T \leftrightarrow \bar{U}).
\]

for \( I = J \) and \( I \neq J \) with \( \alpha \in \{9,5I\} \), as in (4.33). It is not strange to suspect that also the second amplitude, in this given set, vanishes. To this aim one can easily verify that the scattering amplitude for the pair \((T,U)\)
\[
A_{\alpha}(T^I,U^J) = g_c^2 C_{RP_2} \int_{|z| \leq 1} \frac{d^2 z_1 d^2 z_2}{V_{CKG}} \langle W_{T^I}^{\bar{z}_1}(E_1,k_1,z_1,\bar{z}_1) W_{U^J}^{\bar{z}_2}(E_2,k_2,z_2,\bar{z}_2) \rangle = \sum_{i=1}^4 \Lambda_i
\]
gives, for the same reason said before, a vanishing result (see [54] appendix A.7) with \( \alpha \in \{9,5I\} \) as for the case \( I = J \) as for \( I \neq J \)
\[
A_{\alpha}(T^I,U^J) = 0 \quad (T \leftrightarrow U).
\]

To conclude this section, the tree-level Kähler potential (3.9) together with the consequently tree-level Kähler metric components needed to write the kinetic terms for the closed untwisted moduli in the LEFA for the Type IIB orientifold on \( T^6/Z_2 \times Z_2 \) (and \textit{mutatis mutandis} for all the models that have a similar moduli space), is confirmed by the directly string scattering amplitudes calculations.
5 Adding $\alpha'^2 R^2$ to LEEA

In this section we want to discuss which kind of terms can be produced if, at the $\alpha'^2$-order in the high derivative expansion of the LEEA, $R^2$ terms like the contraction of two Ricci tensors $R_{\lambda\sigma} R^{\lambda\sigma}$, Riemann tensors $R_{\lambda\sigma\beta\gamma} R^{\lambda\sigma\beta\gamma}$ and the square of scalar curvature $R^2$ are considered and in particular, if this kinds of terms are reproduced by the string scattering amplitudes considered in this paper.\textsuperscript{33} In $g_\ast$-expansion at sphere level, this kind of terms are absent in the action of Type II theories, Heterotic theories and in Type I theory due to the S-duality relation between the last two [50, 51, 24].\textsuperscript{34} To the tree-level actions for $D$-branes and $\Omega$-planes, terms as $R^2$ can be added because they are supported by S-duality relation [51]. More precisely, terms that have a behavior as $C e^{-\Phi} R^2$ in the $g_\ast$-expansion correspond to a tree-level string scattering amplitude on the disk or projective plane that, under S-duality, are mapped to a sphere tree-level terms $C' e^{-2\Phi} R^2$ that are allowed in Heterotic $g_\ast$-expansion [51, 25, 28, 29].

Therefore the Dirac-Born-Infeld (DBI) action for the $D$-branes and $\Omega$-planes at $\alpha'^2$-order reads [51, 25, 28, 29, 26]

$$\alpha'^2 \tau \int d^{p+1} x \ e^{-\Phi} \sqrt{-g} \left\{ a R_{\lambda\sigma\beta\gamma} R^{\lambda\sigma\beta\gamma} + b R_{\sigma\beta} R^{\sigma\beta} + c R^2 \cdots \right\}$$

(5.1)

where $\xi^\beta$ are the (intrinsic) $D$-brane and $\Omega$-plane world-volume coordinates, $g_{\alpha\beta}$ is the pull-back of the ten-dimensional metric $g_{MN}$ to the world-volume, $g_{\alpha\beta} = \partial_\lambda X^M (\xi) \partial_\beta X^N (\xi) g_{MN}$, indices $\alpha,\beta$ labelling the directions parallel to the $D$-brane and $\Omega$-plane and finally $\tau$ is a constant which includes the tension of $D$-brane and $\Omega$-plane respectively plus other constants. The DBI actions considered, are all in static gauge, i.e. the world-volume coordinates $\xi^\beta$ coincide with the string coordinate $X^M$ in the $p+1$-directions. The terms in (5.1) are not all the pull-back to the world-volume of the respective bulk terms but only the (Riemann)$^2$-term is. As explained in [25], at linearised level around flat space, the vanishing (in the vacuum) of bulk Ricci tensor gives on $D$-brane and $\Omega$-plane tree independent equations

$$R^L_{\lambda MN} = 0 \quad \text{with} \quad L \in \{ \lambda = 0, \ldots, p; \ l = p+1, \ldots, 9 \}$$

$$R^l_{\mu\lambda} = - R^l_{\mu\lambda} \equiv R_{\mu\lambda} ; \quad R^l_{mn} = - R^l_{m\lambda n} \equiv R_{mn}$$

(5.2)

and in order to build (Riemann)$^2$ terms for $D$-brane and $\Omega$-plane action, one can use only the tree linear independent Ricci tensors $R_{\mu\nu}, R_{\mu\lambda}, R_{\mu\lambda n}$. The scalar curvature $R$ can be obtained from $R_{\mu\nu}$ and $R_{\mu\lambda}$ contracting opportunely the indices. The dots inside (5.1) means that other terms with tensor components along the orthogonal directions to the world-volume (for the $D$-brane case only) can enter in general the action [51, 25, 28, 29, 26], but we concentrate our discussion only on the tangent part (5.1). In order to find the right combination in (5.1) between $R^l_{\mu\lambda n}, R^l_{\mu\lambda}$ and $R^l_{mn}$ one can take firstly the expansion of the terms in (5.1) using the linearized approximation for which, the spacetime metric, is expanded as $g_{\mu\nu} = h_{\mu\nu} + \hat{h}_{\mu\nu}$ with $h_{\mu\nu}$ the Minkowsky metric and $\hat{h}_{\mu\nu}$ the fluctuation around the Minkowsky metric, i.e. the graviton field. After that, match the terms up to two graviton fields $\hat{h}$ with the results of the string scattering amplitudes (2.23) and (2.49) specialized to the case of two gravitons (i.e. $E_{\mu\nu} = h_{\mu\nu}$) emitted and absorbed form $D$-brane and $\Omega$-plane.\textsuperscript{35} Recalling that $g^{\mu\nu} = \eta^{\mu\nu} - \hat{h}_{\mu\nu} + o(h^2)$ [52], the expansion up to terms with four derivatives and two gravitons $\hat{h}$ gives for (Riemann)$^2$ the following

$$a R_{\lambda\sigma\beta\gamma} R^{\lambda\sigma\beta\gamma} = a g_{\mu\nu} R^\mu_{\sigma\alpha\beta} g^{\sigma\alpha} g^{\beta\gamma} R^\lambda_{\epsilon\rho\sigma} \bigg|_{\hat{h}} \ = \ \frac{a}{4} \left\{ \partial_\alpha \partial_\beta \hat{h}_{\lambda\gamma} - \partial_\alpha \partial_\lambda \hat{h}_{\beta\gamma} - \partial_\beta \partial_\gamma \hat{h}_{\lambda\sigma} + \partial_\beta \partial_\sigma \hat{h}_{\lambda\gamma} \right\}$$

$$= \ \left\{ \partial_\alpha \partial_\beta \hat{h}_{\lambda\gamma} + \partial_\alpha \partial_\lambda \hat{h}_{\beta\gamma} - \partial_\beta \partial_\gamma \hat{h}_{\lambda\sigma} - \partial_\beta \partial_\sigma \hat{h}_{\lambda\gamma} - \partial_\alpha \partial_\beta \hat{h}_{\lambda\gamma} + \partial_\beta \partial_\sigma \hat{h}_{\lambda\gamma} \right\}$$

(5.3)

\textsuperscript{33} $R^2$ means general combination of Riemann, Ricci and scalar curvature that one can takes at this order.

\textsuperscript{34} In Heterotic case, not all but certain specific dilaton couplings are absent.

\textsuperscript{35} $\hat{h}_{\mu\nu} = k_d h_{\mu\nu}$ with $k_d$ the d-dimensional physical gravitational coupling.
for (Ricci)²

\[ bR_{\alpha\beta}R^{\alpha\beta} = b R^\lambda_{\alpha\lambda\beta} \sigma^\rho \sigma^\mu g_{\rho\mu} R^\gamma_{\gamma} \bigg|_h = b \left( \partial^\lambda \partial_\lambda \hat{h}_\beta + \partial^\lambda \partial_\beta \hat{h}_\lambda - \partial^\lambda \partial_\lambda \hat{h}_\beta \right) \left\{ \partial_\rho \partial^\rho \hat{h}^{\beta\gamma} + \partial_\rho \partial^\beta \hat{h}^{\gamma\rho} \right\} \]

and for (R)²

\[ cR^2 = c (g^{\mu\nu} R_{\mu\nu}) \bigg|_h = c \partial^\mu \partial^\rho \hat{h}_{\mu\nu} \partial_\alpha \partial^\beta \hat{h}_{\alpha\beta} \]

(5.4)

where the properties of symmetry and tracelessness of \( \hat{h} \) have been used. Performing in (5.1) also the expansion of \( \sqrt{-\hat{g}} = \sqrt{-q} (1 + O(\text{Tr}(\hat{h})) \) and integrating by parts the terms (5.3) and (5.4), apart from total derivative terms, one obtains

\[
aR^\lambda_{\lambda\alpha\beta} R^\lambda_{\alpha\beta} + bR_{\alpha\beta} R^{\alpha\beta} + cR^2 \bigg|_h = \left( a + \frac{b}{4} \right) \partial^\alpha \partial_\alpha \partial^\beta \partial_\beta \hat{h}_{\alpha\beta\lambda} + \left( 2a + \frac{b}{2} \right) \partial^\beta \partial_\beta \hat{h}_{\alpha\beta\lambda} \partial_\alpha \partial^\gamma \hat{h}^{\lambda\gamma},
\]

\[
+ \left( a + \frac{b}{2} + c \right) \partial^\beta \partial^\gamma \hat{h}_{\alpha\beta\lambda} \partial_\alpha \partial^\gamma \hat{h}^{\lambda\gamma}
\]

(5.6)

which symmetrising and transforming to momentum space becomes

\[
aR^\lambda_{\lambda\alpha\beta} R^\lambda_{\alpha\beta} + bR_{\alpha\beta} R^{\alpha\beta} + cR^2 \bigg|_h = \left( a + \frac{b}{4} \right) \left( k_1^\parallel \right)^2 \text{Tr}(\hat{h}_1 \hat{h}_2) + \left( a + \frac{b}{4} \right) \left( k_2^\parallel \right)^2 \text{Tr}(\hat{h}_1 \hat{h}_2)
\]

\[
+ \left( a + \frac{b}{2} + c \right) (k_j^\parallel h_j k_j^\parallel h_j) + (1 \leftrightarrow 2)
\]

(5.7)

On the other hand the results of scattering amplitudes (2.23) and (2.49) specialised to the two gravitons (i.e. \( E_{\mu\nu} = h_{\mu\nu} \)) case, using the transversality \( (k_i \cdot h_i)_\nu = 0 \), the condition on the trace \( \text{Tr}(\hat{h}) = 0 \) and the reflection maritrix equal to \( R_{\mu\nu} = \eta_{\mu\nu} \) (i.e. only NN directions of the D-brane and \( \Omega \)-plane) are

\[ \mp a^2 \tau \zeta(2) \left\{ \frac{1}{2} \text{Tr}(h_1 h_2) \left( k_1^\parallel \right)^2 + \left( k_1^\parallel \right)^2 \right\} + (1 \leftrightarrow 2)
\]

(5.8)

with \( \mp \tau \) equal to \( -g_C^2 C_{D_2} \) or \( g_C^2 C_{RP_2} / 2 \) for disk and projective plane respectively and where \( k_1^\parallel = -s \) is used (see appendix (A.1)). The cases with \( D_9 \)-brane and \( \Omega \)-plane bring to amplitudes that on-shell are zero, thus in order to match the terms one should use, for instance, the helicity formalism in 10-dimensions [53]. The cases with \( D_P \)-brane and \( \Omega \)-plane where \( P < 9 \) have no problems since in general \( k^\parallel \neq 0 \). The first two terms in (5.7) are those that the scattering of two gravitons (5.8) reproduce, while the third term \( k_j^\parallel h_j k_j^\parallel h_j \) as explained in [52] could produces negative probabilities in the theory unless new ghost fields were considered, thus one has to find a solution for the coefficients in (5.7) that cancel this \( \text{ill} \) term. Matching (5.7) with (5.8) one is able to fix only two coefficients, we chose to fix

\[ a = 1 - \frac{b}{4}; \quad c = -1 - \frac{b}{4}.
\]

(5.9)

Inserting these in (5.1), the combination between \( R^\lambda_{\mu\nu}, R_{\mu\nu} \) and \( R \) is equal to

\[ a^2 \tau \int d^{P+1} X e^{-\Phi} \sqrt{-g} \left\{ R^\lambda_{\lambda\alpha\beta} R^{\lambda\alpha\beta} - R^2 - \frac{b}{4} \left( R^\lambda_{\lambda\alpha\beta} R^{\lambda\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) \cdots \right\}
\]

(5.10)

the term multiplied by \( b/4 \) is Gauss-Bonnet-type and the coefficient \( b \) is unfixed using scattering amplitude of two gravitons, but it could be fixed by the calculation of string scattering amplitude of three gravitons.\(^{36}\) Setting for instance \( b = -4 \) one can find again the result in [25, 28, 29, 26]\(^{37}\). This

---

\(^{36}\) The Gauss-Bonnet term in four dimensions, as the world-volume of a \( D_3 \)-brane and \( \Omega_3 \)-plane, is a purely topological term.

\(^{37}\) In [25, 28, 29, 26] we think that they chose to fix \( c = 0 \) from the beginning, thus in this way \( a \) and \( b \) are uniquely fixed.

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analysis is valid in a general $d$-dimensional spacetime, but if one consider the compactification to four-dimensional spacetime of an higher-dimensional theory, more terms will appear from the dimensional reduction of the action (5.10) for instance. The Type IIB orientifold on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ considered, see Section 3 for details, is characterised by the presence of one set of $D_9$-branes on top of $\Omega_9$-plane and three sets of $D_5$-branes on top of $\Omega_5$-planes. In order to make the compactification to four-dimensions of the starting model in ten-dimensions, the set of $D_9$-branes on top of $\Omega_9$-plane have to wrap the full internal $T^6$-torus, while each set of $D_{5_I}$-branes on top of $\Omega_{5_I}$-plane have to wrap one $T^2_I$-torus respectively, with $I = 1, 2, 3$. In Type IIB orientifold on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, the action of each element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ works as it was a single $\mathbb{Z}_2$ that leaves invariant one of $T^2_I$ torus while flips the others $T^2_{\neq I}$ tori. Since we are focused on the moduli of the untwisted sector, and in particular on the geometric moduli which parametrised the $T^2_I$ that the $\theta_I$ orbifold element leaves invariant respectively, one can approximate locally the internal manifold as

$$\theta_I T^6 \to T^2_I \times \frac{(T^2_I \times T^2_K)}{\mathbb{Z}_2} \sim T^2_I \times K_3$$

(5.11)

where the resulting low energy effective action at the $\alpha'$-order can be found in [27, 36] setting to zero the open string moduli (Wilson lines)\(^{39}\), can be interpreted as a superposition of three copies of the Type I on $T^2 \times K_3$ model\(^{40}\) [42]. For simplicity we consider the set of $D_9$-branes, $\Omega_9$-plane and one set of set of $D_5$-branes, $\Omega_5$-plane. We add to the $\alpha'$-order action in [27, 42] the $\alpha'^2$-order terms that arising from the compactification of (5.10) on $T^2 \times K_3$ and match these with the results arising from the scattering amplitudes involving untwisted moduli in Section 4 at the $\alpha'^2$-order. The compactification process can be deal in two steps, first going from ten-dimensions to six-dimensions using the $K_3$ compactification of (5.11) for instance. The Type IIB orientifold on $\mathbb{R}^5/\mathbb{Z}_2 \times \mathbb{Z}_2$, while the $\tau$ matrix is specified the metric dependences of the (5.10) for instance. The Type IIB orientifold on $\mathbb{R}^5/\mathbb{Z}_2 \times \mathbb{Z}_2$, the action of each element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ works as it was a single $\mathbb{Z}_2$ that leaves invariant one of $T^2_I$ torus while flips the others $T^2_{\neq I}$ tori. Since we are focused on the moduli of the untwisted sector, and in particular on the geometric moduli which parametrised the $T^2_I$ that the $\theta_I$ orbifold element leaves invariant respectively, one can approximate locally the internal manifold as

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$$\alpha'^2 \tau_{(D_9, \Omega_9)} \int d^6 X e^{-\Phi_6} \sqrt{-g^{(6)} e^{2}} \left\{ R_{\lambda\sigma\alpha\beta} R^{\lambda\sigma\alpha\beta} - R^2 - \frac{b}{4} \left( R_{\lambda\sigma\alpha\beta} R^{\lambda\sigma\alpha\beta} - 4 R_{\sigma\beta} R^{\sigma\beta} + R^2 \right) \right\} \left( g^{(6)} \right)$$

$$+ R^2 \left( g^{(6)}, G_{K_3}^{(4)} \right) + R^2 \left( C_{K_3}^{(4)} \right) \cdots$$

(5.12)

where in the round brackets is specified the metric dependences of the $R^2$ terms, $\nu$ is the $K_3$ volume modulus and $\Phi_6$ the dilaton in six-dimensions which is related to the ten-dimensional dilaton as $\exp(-2\Phi_6) = \exp(-2\Phi_{10}) \nu^4$ [42]. From the (5.12) we consider only the $R^2 \left( g^{(6)} \right)$ since we are not focus neither on the scattering of $K_3$ moduli (which can be thought as the blowing-up of the untwisted moduli of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$) nor on the scattering of mixing $K_3$ moduli with the $T^2$ moduli. Then, the compactification of $T^2$ of the $R^2 \left( g^{(6)} \right)$ in (5.12) and of (5.10) specialised to $D_5$-branes and $\Omega_5$-plane, with the splitting of the world-volume metric $g^{(6)} = g_{\mu \nu}^{(4)} \times G_{mn}^{(2)}$ where $g_{\mu \nu}^{(4)}$ is the four-dimensional world-volume metric and $G_{mn}^{(2)}$ is the $T^2$ torus metric (3.12), are

$$\alpha'^2 \left\{ \tau_{(D_9, \Omega_9)} \int d^4 X e^{-\Phi_4} \nu^2 + \tau_{(D_9, \Omega_9)} \int d^4 X e^{-\Phi_4} \nu^{-2} \right\} \left( \sqrt{G^{(2)}} \right)^2 \sqrt{g^{(4)}} \left\{ R^2 \left( g^{(4)} \right) + R^2 \left( G^{(2)} \right) \right\}$$

$$+ \left\{ R_{LQAB} R_{LQAB} - R^2 - \frac{b}{4} \left( R_{LQAB} R_{LQAB} - 4 R_{QB} R^{QB} + R^2 \right) \right\} \left( g^{(4)}, G^{(2)} \right) \cdots$$

(5.13)

\(^{38}\) Adding the DBI-action for the $\Omega_I$-planes also.

\(^{39}\) Up to now the modifications of axion dilaton and kähler moduli definitions due to the presence of open string moduli are not considered [42, 27].

\(^{40}\) The Type I on $T^2 \times K_3$ model has one set of $D_9$-branes on top of $\Omega_9$-plane and one set of $D_5$-branes on top of $\Omega_5$-plane, and has $N = 2$ supersymmetry in four dimensions.

\(^{41}\) aside the six-dimensional dilaton redefinition for the $D_9$-branes, $\Omega_9$-plane
where $\Phi_t$ is the four-dimensional dilaton linked to the six-dimensional dilaton by $\exp(-2\Phi_t) = \exp(-2\Phi_6) \sqrt{G^{(2)}}$. $R^2 \left( g^{(4)} \right) + R^2 \left( G^{(2)} \right)$ are terms which contains only contractions between world-volume indices $(\{\mu, \nu\})$ or $T^2$-torus indices $(\{m, n\})$ respectively, while the terms inside the square bracket contains all the possible contractions with mixed indices, i.e $L = \{\lambda, l\}$. The terms that we want to compare with the string scattering amplitudes of Section 4 can be derived from the square bracket terms of (5.13). The possible mixed contractions of indices are for the (Riemann)$^2$

\[
R_{LQAB}R^{LQAB} = G_{\mu \nu \rho \sigma} \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + g_{\mu \nu} \partial^\rho \partial^\sigma + g_{\mu \rho} \partial^\nu \partial^\sigma + g_{\mu \sigma} \partial^\nu \partial^\rho + \frac{1}{2} \left( \partial^\mu \partial^\nu - \partial^\nu \partial^\mu \right) \left( \partial^\rho \partial^\sigma + \partial^\sigma \partial^\rho \right) - \frac{1}{2} \left( \partial^\rho \partial^\sigma \partial^\mu \partial^\nu + \partial^\sigma \partial^\rho \partial^\nu \partial^\mu \right) - \frac{1}{2} \left( \partial^\mu \partial^\sigma \partial^\rho \partial^\nu + \partial^\rho \partial^\mu \partial^\sigma \partial^\nu \right) - \frac{1}{2} \left( \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + \partial^\rho \partial^\sigma \partial^\mu \partial^\nu \right) - \frac{1}{2} \left( \partial^\mu \partial^\nu \partial^\sigma \partial^\rho + \partial^\rho \partial^\sigma \partial^\mu \partial^\nu \right) + \frac{1}{2} \left( \partial^\mu \partial^\nu \partial^\rho \partial^\sigma - \partial^\rho \partial^\sigma \partial^\mu \partial^\nu \right)
\]

for the (Ricci)$^2$

\[
R_{\mu \nu \rho \sigma} = \partial^\rho \partial^\sigma - \partial^\sigma \partial^\rho + \frac{1}{4} \left( \partial^\rho \partial^\sigma \partial^\mu \partial^\nu + \partial^\rho \partial^\sigma \partial^\nu \partial^\mu + \partial^\rho \partial^\nu \partial^\sigma \partial^\mu + \partial^\sigma \partial^\rho \partial^\nu \partial^\mu \right) - \frac{1}{4} \left( \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + \partial^\rho \partial^\sigma \partial^\mu \partial^\nu + \partial^\sigma \partial^\rho \partial^\nu \partial^\mu \right)
\]

and for (R)$^2$

\[
R = G_{\mu \nu} \partial^\mu \partial^\nu + \frac{1}{4} \left( \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + \partial^\rho \partial^\sigma \partial^\mu \partial^\nu + \partial^\rho \partial^\nu \partial^\sigma \partial^\mu + \partial^\sigma \partial^\rho \partial^\nu \partial^\mu \right) - \frac{1}{4} \left( \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + \partial^\rho \partial^\sigma \partial^\mu \partial^\nu + \partial^\rho \partial^\nu \partial^\sigma \partial^\mu + \partial^\sigma \partial^\rho \partial^\nu \partial^\mu \right)
\]

(5.14)

(5.15)

(5.16)

Since in Section 4 two-point string scattering amplitudes for the untwisted moduli has been performed, we want to extract from (5.14), (5.15) and (5.16) terms which involve four derivatives and two untwisted moduli using the linearised approximation for the spacetime (world-volume) metric $\eta_{\mu \nu} = \eta_{\mu \nu} + O(\tilde{h}_{\mu \nu})$, recalling that no mixed components of the metric $\tilde{g}_{\mu \nu}$ are present. With those approximations, several terms in (5.14), (5.15) and (5.16) are zero due to the vanishing of the Christoffel symbols, as for instance

\[
\Gamma^l_{\mu \nu} = \frac{1}{2} G^{l \alpha} \left( \partial_\mu \tilde{g}_{\nu \lambda} + \partial_\nu \tilde{g}_{\lambda \mu} - \partial_\lambda \tilde{g}_{\mu \nu} \right) = 0.
\]

(5.17)

The non zero terms in (5.14), (5.15) and (5.16) that can contain terms starting with four derivatives and two untwisted moduli (after integration by parts), are

\[
R_{LQAB}R^{LQAB} \rightarrow \left( \partial_\sigma \partial_\alpha G_{\lambda \beta} \right) G^{\beta \delta} G^{\lambda \epsilon} \left( \partial^\sigma \partial^\alpha \partial^\delta \partial^\epsilon \right)
\]

\[
R_{QBR^2} \rightarrow \frac{1}{4} \left( \partial_\sigma \partial_\alpha G_{\lambda \beta} \right) G^{\beta \delta} G^{\lambda \epsilon} \left( \partial^\sigma \partial^\alpha \partial^\delta \partial^\epsilon \right)
\]

\[
R^2 \rightarrow G^{\beta \delta} \left( \partial_\sigma \partial_\alpha G_{\lambda \beta} \right) G^{\lambda \epsilon} \left( \partial^\sigma \partial^\alpha \partial^\delta \partial^\epsilon \right)
\]

(5.18)

from which it is straightforward to verify that the round brackets term in the second line of (5.13), cancels by itself at this order, leaving the $b$ coefficient undetermined yet. The $\alpha^2$-order terms that can be read from the scattering amplitudes of untwisted moduli (geometric moduli) in Section 4 are: for two Kähler moduli $T$ (or better, the imaginary part of $T$, i.e. $T_2$) taking (4.20), (4.21), (4.59) and (4.60)

\[
A_{\alpha / \alpha}(T_2, T_2') = -\frac{\alpha^2 C(2)}{\left( T - T' \right)^2} \left( u^2 + t^2 \right) \; ; \; \; A_{\alpha / \alpha}(T_2, T_2') = \frac{\alpha^2 C(2) \partial_{\alpha / \alpha} \partial_{T_2} \partial_{T_2'}}{\left( T - T' \right) \left( T - T' \right)} tu
\]

(5.19)

This components are related to the graviphoton that we are set to zero.
for two complex structure $U$ from (4.27), (4.28), (4.31), (4.68), (4.69) and (4.73)

$$A_{a/\alpha}(U^I, \bar{U}^I) = -\frac{\alpha' C C(2)}{(U^I - \bar{U}^I)^2} s^2; \quad A_{a/\alpha}(U^I, \bar{U}^I) = 0; \quad A_{a/\alpha}(U^I, U^I) = A_{a/\alpha}(U^I, \bar{U}^I) = 0 \quad (5.20)$$

and finally, for one Kähler modulus $T$ and one complex structure $U$ from (4.33), (4.35), (4.77), (4.79)

$$A_{a/\alpha}(T^I, \bar{U}^I) = A_{a/\alpha}(T^I, \bar{U}^I) = 0; \quad A_{a/\alpha}(T^I, U^I) = A_{a/\alpha}(T^I, U^I) = 0 \quad (5.21)$$

with the label $a/\alpha$ for the D-brane and $\Omega$-plane respectively, and $C$ a constant. At this point, since even for the real ($U_1$) and the imaginary ($U_2$) part of the complex structure $U$, the relative vertex operators, can be written

$$W_{U_1^I(q,\bar{q})}(E, z, \bar{z}, k) = \frac{1}{2} \left( W_{U_1^I(q,\bar{q})}(E, z, \bar{z}, k) + W_{U_1^I(q,\bar{q})}(E, z, \bar{z}, k) \right)$$

$$W_{U_2^I(q,\bar{q})}(E, z, \bar{z}, k) = -\frac{i}{2} \left( W_{U_2^I(q,\bar{q})}(E, z, \bar{z}, k) - W_{U_2^I(q,\bar{q})}(E, z, \bar{z}, k) \right) \quad (5.22)$$

recalling what has been done for the imaginary ($T_2$) part of the Kähler modulus $T$ (4.18), the equations (5.20) and (5.21) tell us that even at $\alpha'^2$-order there are no mixing between: the real ($U_1$) and the imaginary ($U_2$) part of the complex structure $U$, the Kähler modulus $T$ (i.e. $T_2$) and the complex structure $U$ respectively. The string scattering amplitude results (5.19), (5.20) and (5.21) are in agreement with the non vanishing $(5.13)$, written exploiting (5.18)

$$R_{LQAB}R^{LQAB} - R^2 \rightarrow (\partial_0 \partial_0 G_{bl}) \left( G^{bI} G^{Ie} (\partial^e \partial^a G_{he}) - G^{bI} (\partial_0 \partial_0 G_{qe}) G^{qh} (\partial^q \partial^a G_{ph}) \right)$$

$$= (\partial_0 \partial_0 (G^I)_{bl}) (G^{I}^{bI}) (\partial^e \partial^a (G^I)_{he}) - \frac{1}{2} (G^{I})^{qh} (\partial_0 \partial_0 (G^I)_{qh}) (G^I)^{qh} (\partial^q \partial^a (G^I)_{ph}) + (I \leftrightarrow J) \quad (5.23)$$

with $G^I$ the metric of the $T^I_2$-torus (3.12). Concerning the term in (5.19) with $I \neq J$, this can be reproduced only by the scalar curvature squared $R^2$ term because the (Riemann) $g_{ij}$ term, involves metric tensors mutually contracted, but the metric of the $T^I_2$-torus factorises ($T^I_2 = \bigotimes_{i=1}^3 T^I_2$), thus there is no mixing between the latter.

At this point, considering only the terms with two derivatives acting on the same modulus $\varphi \in \{T_2, U_1, U_2\}$, the derivatives of the metric components $G^I$ gives

$$\partial_\varphi \partial_\varphi \left( G^{[2I+1|2I+2]} \right) \mid_{\varphi} = \left( \frac{\partial_\varphi \partial_\varphi T^I_2}{U_2^I} - \frac{T^I_2 \partial_\varphi \partial_\varphi U^I_2}{(U_2^I)^2} \right)$$

$$\partial_\varphi \partial_\varphi \left( G^{[2I+1|2I+2]} \right) \mid_{\varphi} = \left( \frac{\partial_\varphi \partial_\varphi T^I_2}{U_2^I} + \frac{T^I_2 \partial_\varphi \partial_\varphi U^I_2}{(U_2^I)^2} - \frac{T^I_2 U^I_2 \partial_\varphi \partial_\varphi U^I_2}{(U_2^I)^2} \right) \quad (5.24)$$

$$\partial_\varphi \partial_\varphi \left( G^{[2I+2|2I+2]} \right) \mid_{\varphi} = \left( \frac{U^I_2 \partial_\varphi \partial_\varphi U^I_2}{U_2^I} + \frac{T^I_2 U^I_2 \partial_\varphi \partial_\varphi U^I_2}{(U_2^I)^2} + \frac{T^I_2 ((U_2^I)^2 - (U_2^I)^2) \partial_\varphi \partial_\varphi U^I_2}{(U_2^I)^2} \right)$$

which inserted in (5.23), is straightforward to verify that the non vanishing terms are

$$\left( T^I_2, T^I_2 \right) : \frac{(\partial_\varphi \partial_\varphi T^I_2)(\partial^e \partial^a T^I_2)}{(T^I_2 - T^I_2)^2}; \quad \left( T^I_2, T^I_2 \right) : \frac{(\partial_\varphi \partial_\varphi T^I_2)(\partial^e \partial^a T^I_2)}{(T^I_2 - T^I_2)^2}$$

$$\left( U^I_2, U^I_2 \right) : \frac{(\partial_\varphi \partial_\varphi U^I_2)(\partial^e \partial^a U^I_2)}{(U^I_2 - U^I_2)^2}; \quad \left( U^I_2, U^I_2 \right) : \frac{(\partial_\varphi \partial_\varphi U^I_2)(\partial^e \partial^a U^I_2)}{(U^I_2 - U^I_2)^2} \quad (5.25)$$

while all the other combinations are zero and, aside constant factors, (5.25) exhibits the same moduli dependencies as the scattering amplitudes (5.19), (5.20) and (5.21).

6 Conclusion

We have explored perturbatively the LEFA for Type IIB orientifold models using string scattering approach, focusing our attention on tree-level string scattering amplitudes involving only closed string
as external states, on the disk $D_2$ and projective-plane $RP_2$ worldsheet. Indeed, when an unoriented model is considered, one have to put on the same footing the presence of extended objects as $D_p$-branes and as $\Omega_P$-planes which, under specific conditions, make together the given theory well define. Two-point closed string scattering amplitude has been chosen because this is the first non vanishing contribution on both worldsheet, since only on the projective-plane the one-point function is non zero due to the properties of its conformal Killing Group $SU(2)$. In this paper we resumed historical two-point disk [21, 22, 23, 26] and projective-plane [26] calculations in a pedagogical way, where only NS-NS external states are considered because, we are interested on higher derivative curvature corrections to LEEA of $D_p$-branes and $\Omega_P$-planes, more specific on $\alpha'$-order terms, i.e $(\text{curvature})^2$ terms. This was achieved matching the string scattering amplitude involving two-gravitons with the most general linear combination of $(\text{curvature})^2$ terms which the Dirac-Born-Infeld action of $D_p$-branes and $\Omega_P$-planes can contain at this order

$$\alpha'^2 C_T \int d^{p+1}x e^{-\Phi} \sqrt{-g} \left\{ a R_{\lambda \sigma \alpha \beta} R^{\lambda \sigma \alpha \beta} + b R_{\sigma \beta} R^{\sigma \beta} + c R^2 \ldots \right\}$$

(6.1)

arriving at the result

$$\alpha'^2 \int d^{p+1}x e^{-\Phi} \sqrt{-g} \left\{ R_{\lambda \sigma \alpha \beta} R^{\lambda \sigma \alpha \beta} - R^2 - \frac{b}{4} \left( R_{\lambda \sigma \alpha \beta} R^{\lambda \sigma \alpha \beta} - 4 R_{\sigma \beta} R^{\sigma \beta} + R^2 \right) \ldots \right\}$$

(6.2)

where the $b$ parameter remain unfixed at this order, but it will be fixed taking the two-point string scattering amplitudes with two-gravitinos as external states together the supersymmetric action to (6.1) or the three-point string scattering amplitude with three-gravitons which, in principle contains many terms than the two-point amplitude.

After this, our analysis focused on to a specific Type IIB orientifold model, i.e. Type IIB orientifold on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ since we wanted to extend the two-point disk calculation with untwisted moduli made in [40] to the case where also the projective-plane contribution is considered. In order to prepare the reader to new results for the projective-plane case, for consistence we reviewed the disk case [40], giving in Section 3.1 more space to the building of vertex operators for the untwisted moduli and to the two-point correlators for both the worldsheet. String scattering amplitudes with two-untwisted moduli are derived in Section 4. The $\alpha'$-expansion is also performed, and it was found that there are no $\alpha'$-order corrections at LEEA by projective-plane calculations confirming the tree-level shape of Kähler potential for the untwisted moduli

$$k^2 \mathcal{K} = - \ln \prod_{l=1}^{3} (T^I - \bar{T}^I) - \ln \prod_{l=1}^{3} (U^I - \bar{U}^I).$$

(6.3)

Finally in Section 5 we compared the $\alpha'^2$-order terms arising from the string scattering amplitudes in Section 4 with the terms produced by the compactification on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ of (6.1) checking that the only non-vanishing untwisted moduli contributions are

$$(T^I_2, T^I_2) : - \left( \frac{\partial_0 \partial_\sigma T^I_2}{(T^I - \bar{T}^I)^2} \right) \left( \frac{\partial^\sigma \partial^\alpha T^I_2}{(T^I - \bar{T}^I)^2} \right); \quad (T^I_2, T^J_2) : \left( \frac{\partial_0 \partial_\sigma T^I_2}{(T^I - \bar{T}^I) (T^J - \bar{T}^J)} \right) \left( \frac{\partial^\sigma \partial^\alpha T^J_2}{(T^J - \bar{T}^J)} \right)$$

$$(U^I_1, U^I_1) : - \left( \frac{\partial_0 \partial_\sigma U^I_1}{(U^I - \bar{U}^I)^2} \right) \left( \frac{\partial^\sigma \partial^\alpha U^I_1}{(U^I - \bar{U}^I)^2} \right); \quad (U^I_2, U^I_2) : \left( \frac{\partial_0 \partial_\sigma U^I_2}{(U^I - \bar{U}^I)^2} \right) \left( \frac{\partial^\sigma \partial^\alpha U^I_2}{(U^I - \bar{U}^I)^2} \right)$$

(6.4)

in agreement with string scattering amplitude results (4.20), (4.21), (4.59) and (4.60). A similar analysis will be made for two-point string scattering amplitudes for twisted moduli at the orbifold point were the CFT is well define, but a careful evaluation of vertex operators and two-point correlators is required in order to understand if some new results and behaviour could appear. Finally, we recall that all the details on the calculations can be find in [54].

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A

A.1 Kinematics: Closed strings as open strings

Firstly we would like show that a closed string can be described using open strings [21, 22, 23, 26, 40, 39]. This is due to the presence of extended objects like D-branes (oriented models which contain open string sector) and/or \( \Omega \)-planes (unoriented models), that project the total momentum \( k \) of the closed string in its parallel \( k^\parallel \) and orthogonal \( k^\perp \) components to the extended objects respectively, and allow the conservation of the momentum \( k \) only in the parallel direction [30, 40, 21].

Thus one has

\[
 k^M = (k^\parallel)^\mu + (k^\perp)^m = \left( \frac{k^M}{2} + \frac{(\mathcal{R} \cdot k)^M}{2} \right)^\parallel + \left( \frac{k^M}{2} - \frac{(\mathcal{R} \cdot k)^M}{2} \right)^\perp \quad \left[ M \in \{0, \ldots, d\}; \mu \in \{0, \ldots, p\} \right] \tag{A.1}
\]

\( \mathcal{R} := \mathcal{R}^{MN} = \begin{pmatrix} [\eta^{\mu\nu}]^\parallel & 0 \\ 0 & [-\delta^{mn}]^\perp \end{pmatrix} \) \quad \begin{cases} \text{Neumann direct. : } [\eta^{\mu\nu}]^\parallel \\ \text{Dirichlet direct. : } [-\delta^{mn}]^\perp \end{cases}

From this point of view it seems that the two projection of \( k \) can be described by two open-like-string independent momenta \( \{k/2; Rk/2\} \), thus the question is in which way the closed string mass-squared is split between these two open-like-string? Taking the mass-shell condition for \( k \) we can answer this question

\[
 -m^2 = k^2 = (k^\parallel + k^\perp)^2 = \left( \frac{k}{2} + \frac{\mathcal{R} \cdot k}{2} \right)^2 + \left( \frac{k}{2} - \frac{\mathcal{R} \cdot k}{2} \right)^2 \perp + 2 \left( \frac{k}{2} + \frac{\mathcal{R} \cdot k}{2} \right)^\parallel \cdot \left( \frac{k}{2} - \frac{\mathcal{R} \cdot k}{2} \right)^\perp \tag{A.2}
\]

and thinking the mass as a two-component vector, one can split the mass-squared closed string as follow

\[
 \vec{m} := \vec{m}_a + \vec{m}_b = \begin{pmatrix} m \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \quad ; \quad -\vec{m}_a + \vec{m}_b)^2 = 2 \left( \frac{k}{2} \right)^2 + 2 \left( \frac{\mathcal{R} \cdot k}{2} \right)^2 \tag{A.3}
\]

that gives rise to the following components \(-m_a^2 \equiv \frac{m^2}{2} = 2 \left( \frac{k}{2} \right)^2 \) and \(-m_b^2 \equiv \frac{m^2}{2} = 2 \left( \frac{\mathcal{R} \cdot k}{2} \right)^2 \), thus each open string-like momentum \( \{k/2; Rk/2\} \) brings one half of the original closed string mass \( \frac{m^2}{2} \). In the case of two massless closed string states, described by \( \{k_1, k_2\} \) momenta, one has

\[
 k_1 = k_1^\parallel + k_1^\perp = \left( \frac{k_1}{2} + \frac{\mathcal{R} \cdot k_1}{2} \right)^\parallel + \left( \frac{k_1}{2} - \frac{\mathcal{R} \cdot k_1}{2} \right)^\perp \tag{A.4}
\]

\[
 k_2 = k_2^\parallel + k_2^\perp = \left( \frac{k_2}{2} + \frac{\mathcal{R} \cdot k_2}{2} \right)^\parallel + \left( \frac{k_2}{2} - \frac{\mathcal{R} \cdot k_2}{2} \right)^\perp
\]

but only the parallel component of each \( k_i \) is constrained by momentum conservation, thus only this

\[43 \quad \text{Because only in the world-volume of the extended object the Poincaré group is unbroken.}\]
part enters directly in the construction of the kinematical invariants:

\[ \sum_{i=1}^{2} k_i^\| = k_1 + Rk_1 + k_2 + Rk_2 = 0; \quad \# s_{ij} = \left( \frac{4}{2} \right) = 6 \quad [i,j \in \{1,1,2,2\}; i \neq j; i < j] \]

\[

t\equiv (A.5)
\]

\[
s_{11} = -\left( \frac{k_1}{2} + \frac{Rk_1}{2} \right)^2 = -\left( \frac{k_1}{2} \right)^2 - \left( \frac{Rk_1}{2} \right)^2 - k_1 \cdot Rk_1 \frac{1}{2} = \frac{m_1^2}{4} + \frac{m_1^2}{4} - k_1 \cdot Rk_1 \frac{1}{2} = -k_1 \cdot Rk_1 \frac{1}{2}
\]

\[
s_{12} = -(k_1 + k_2)^2 = -k_1^2 - k_2^2 - 2k_1 \cdot k_2 = m_1^2 + m_2^2 - 2k_1 \cdot k_2 = -2k_1 \cdot k_2
\]

\[
s_{12} = -(Rk_1 + k_2)^2 = -(Rk_1)^2 - k_2^2 - 2Rk_1 \cdot k_2 = m_1^2 + m_2^2 - 2Rk_1 \cdot k_2 = -2Rk_1 \cdot k_2
\]

\[
s_{12} = -(Rk_1 + Rk_2)^2 = -(Rk_1)^2 - (Rk_2)^2 - 2Rk_1 \cdot Rk_2 = m_1^2 + m_2^2 - 2Rk_1 \cdot Rk_2 = -2Rk_1 \cdot Rk_2
\]

\[
s_{22} = -\left( \frac{k_2}{2} + \frac{Rk_2}{2} \right)^2 = -\left( \frac{k_2}{2} \right)^2 - \left( \frac{Rk_2}{2} \right)^2 - k_2 \cdot Rk_2 \frac{1}{2} = \frac{m_2^2}{4} + \frac{m_2^2}{4} - k_2 \cdot Rk_2 \frac{1}{2} = -k_2 \cdot Rk_2 \frac{1}{2}
\]

where all the \( Rk_i/2 \) momentum are now label with \( \tilde{i} \). Using the momentum conservation one can eliminate \( \tilde{2} \) and then

\[
\# s_{i\tilde{i}} = 3 \quad \text{linear dependent k.i.} \quad [i \in \{1,1,2\}]
\]

\[
\# s_{i\tilde{h}} = \left( \frac{3}{2} \right) = 3 \quad \text{linear independent k.i.} \quad [i, h \in \{1,1,2\}; i \neq h; l < h]
\]

\[
\text{(A.6)}
\]

\[
\begin{align}
    s_{12} & = -(k_1 + Rk_2)^2 = -(Rk_1 + k_2)^2 \equiv s_{12} := u; \quad s_{12} = -(Rk_1 + Rk_2)^2 = -(k_1 + k_2)^2 \equiv s_{12} := t \\
    s_{22} & = -\left( \frac{k_2}{2} + \frac{Rk_2}{2} \right)^2 = -\left( \frac{k_1}{2} + \frac{Rk_1}{2} \right)^2 \equiv s_{11} := s
\end{align}
\]

\[
\text{(A.7)}
\]

and one can also writes the on-shellness condition for \( \tilde{2} \) that produces

\[
\text{(A.8)}
\]

\[
\begin{align}
    s_{11} + s_{12} + s_{12} & = s + \frac{u}{4} + \frac{t}{4} = 0
\end{align}
\]

where \( s \) is the open string channel while both \( t \) and \( u \) are closed string channels.
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