Entanglement measure for the universal classes of fractons

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Abstract

We introduce the notion of entanglement measure for the universal classes of fractons as an entanglement between occupation-numbers of fractons in the lowest Landau levels and the rest of the many-body system of particles. This definition came as an entropy of the probability distribution à la Shannon. Fractons are charge-flux systems classified in universal classes of particles or quasiparticles labelled by a fractal or Hausdorff dimension defined within the interval $1 < h < 2$ and associated with the fractal quantum curves of such objects. They carry rational or irrational values of spin and the spin-statistics connection takes place in this fractal approach to the fractional spin particles. We take into account the fractal von Neumann entropy associated with the fractal distribution function which each universal class of fractons satisfies. We consider the fractional quantum Hall effect-FQHE given that fractons can model Hall states. According to our formulation entanglement between occupation-numbers in this context increases with the universality classes of the quantum Hall transitions considered as fractal sets of dual topological quantum numbers filling factors. We verify that the Hall states have stronger entanglement between occupation-numbers and so we can consider this resource for fracton quantum computing.

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I. INTRODUCTION

Entanglement is a foundational characteristic of quantum systems and, more recently, has been considered as a physical resource for quantum information processing. There is a great effort to find right answers to questions like how to measure and classify quantum correlations for implement quantum computation [1]. An entangled state is a state which cannot be written as a direct product of states of the parts of a composite quantum system. Several entanglement measures are known, for bipartite system but no well defined measure exist for mixed multipartite systems. Hence a general definition of entanglement measure is an open problem.

In strongly correlated systems such as fractional quantum Hall effect-FQHE, these ideas can give us some insight into the understanding of this macroscopic complex system. FQHE presents subtle properties of entanglement for different Hall states and in the literature is suggested that the theory of entanglement developed in the context of quantum computing is a suitable tool for this investigation. The relevance of this perspective, for example, to a deeper understanding of quantum phase transitions, has been emphasized by leading researches. In [2] entanglement properties of the Laughlin wave functions for filling factors $(f = 1/m, \text{with } m \text{ an odd number})$ and for wave functions generated by the K-matrix of quantum Hall liquid $(f = 2/(2m + 1))$ have been studied considering an entanglement measure of indistinguishable fermions. Another one, in terms of occupation-numbers for fermions, was discussed in [3].

In this Letter, we introduce the notion of entanglement measure for the universal classes of fractons\(^1\). These objects are charge-flux systems which carry rational or irrational values of spin and are classified in universal classes of particles or quasiparticles labelled by a fractal or Hausdorff dimension\(^2\) defined in the interval $1 < h < 2$. The spin of the particles are related to the Hausdorff dimension by $h = 2 - 2s$, $0 < s < \frac{1}{2}$. This expression is a physical analogous to the fractal dimension formula of the graph of the functions, $\Delta(\Gamma) = 2 - H$, in the context of the fractal geometry, where $H$ is known as Hölder exponent, with $0 < H < 1$ [4].

The bounds of the fractal dimension $1 < \Delta(\Gamma) < 2$ need to be obeyed in order for a function to be a fractal (a continuous function but not differentiable [4]). The bounds

\(^1\)By universal class of fractons we mean a set of particles with rational or irrational values of spin which satisfy a specific fractal distribution function in the same way that fermions constitute a universal class of particles with semi-integer values of spin satisfying the Fermi-Dirac distribution function.

\(^2\)The fractal dimension $h$ can be defined by

$$h - 1 = \lim_{R \to 0} \frac{\ln (L/l)}{\ln (R)},$$

where $L$ is the perimeter of a closed curve, $\Gamma$, and $l$ is the usual length for the resolution $R$. The curve is covering with $\frac{l}{R}$ spheres of diameter $R$ and so a fractal curve is scale invariant, self-similar and has a non-integer dimension [4].
of our parameter \( h \) are defined such that, for \( h = 1 \) we have fermions, for \( h = 2 \) we have bosons, and for \( 1 < h < 2 \) we have fractons. The H"older exponent characterizes irregular functions which appear in diverse physical systems [5]. The fractal character of the quantum paths has been observed in the path integral approach of the quantum mechanics [6]. Thus, the fractal dimension is a geometrical parameter associated with the quantum curves of fractons. Alternatively, the fractal properties of the quantum paths can be extracted from the propagators of the particles in the momentum space [7,9] and so our expression relating \( h \) and \( s \) can once more be justified (for another view, see below). The physical formula introduced by us, when we consider the spin-statistics relation \( \nu = 2s \), is written as

\[
\nu = 2 - h, \quad 0 < \nu < 1.
\]

(1)

The statistical weight for the universal classes of fractons is given by [7]

\[
W[h, n] = \frac{[G + (nG - 1)(h - 1)]!}{[nG]! [G + (nG - 1)(h - 1) - nG]!}
\]

(2)

and from the condition of the entropy be a maximum, we obtain the fractal distribution function

\[
n[h] = \frac{1}{\mathcal{Y}[\xi] - h}.
\]

(3)

The function \( \mathcal{Y}[\xi] \) satisfies the equation

\[
\xi = \left\{ \mathcal{Y}[\xi] - 1 \right\}^{h-1} \left\{ \mathcal{Y}[\xi] - 2 \right\}^{2-h},
\]

(4)

with \( \xi = \exp \{(\epsilon - \mu)/KT\} \). The statistical weight can be written in terms of gamma function

\[
W[x, y] = \frac{\Gamma(x + y + 1)}{\Gamma(x + 1)\Gamma(y + 1)},
\]

(5)

where \( x[h] = N = nG \) and \( y[h] = G + (N-1)h+1-2N, \) such that for bosons \( y[h = 2] = G - 1 \) and for fermions \( y[h = 1] = G - N \). For particles with spin defined in the interval \( 0 \leq s \leq \frac{1}{2} \), we obtain \( y[s] = G - (N-1)2s - 1 \), so in the large \( N \) limit we have \( h = 2 - 2s \) and for the statistical parameter within the interval \( 0 \leq \nu \leq 1 \), we obtain \( y[\nu] = G - (N-1)\nu - 1 \), and again in the large \( N \) limit \( h = 2 - \nu \), so \( \nu = 2s \), i.e. the spin-statistics connection is established. We can check that all the interpolating expressions have the same bounds. Each expression reduces to the other one.

We understand the fractal distribution function as a quantum-geometrical description of the statistical laws of nature, since the quantum path is a fractal curve (a point noted by Feynman) and this reflects the Heisenberg uncertainty principle. The Eq.(3) embodies nicely this subtle information about the quantum paths associated with the particles.
We can obtain for any class its distribution function considering the Eqs.(3,4). For example, the universal class \( h = \frac{3}{2} \) with distinct values of spin \( \{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \} \), has a specific fractal distribution

\[
  n \left[ \frac{3}{2} \right] = \frac{1}{\sqrt{\frac{3}{4} + \xi^2}}. \tag{6}
\]

This result coincides with another one of the literature of fractional spin particles for the statistical parameter \( \nu = \frac{1}{2} \) [10], however our interpretation is completely distinct. This particular example, shows us that the fractal distribution is the same for all the particles into the universal class labelled by \( h \) and with different values of spin (consider the fractal spectrum, for a simple check). Thus, we emphasize that in our formulation the spin-statistics connection is valid for such fractons. The authors in [10] ignored this possibility. Therefore, our results give another perspective for the fractional spin particles or anyons [11]. By the way, we can obtain straightforward the Hausdorff dimension associated to the quantum paths of the particles with any value of spin. This constitutes a fine result of our approach.

We also have

\[
  \xi^{-1} = \left\{ \Theta[Y] \right\}^{h-2} - \left\{ \Theta[Y] \right\}^{h-1}, \tag{7}
\]

where

\[
  \Theta[Y] = \frac{Y[\xi] - 2}{Y[\xi] - 1} \tag{8}
\]

is the single-particle partition function. We verify that the classes \( h \) satisfy a duality symmetry defined by \( \tilde{h} = 3 - h \). So, fermions and bosons come as dual particles. As a consequence, we extract a fractal supersymmetry which defines pairs of particles \((s, s + \frac{1}{2})\). This way, the fractal distribution function appears as a natural generalization of the fermionic and bosonic distributions for particles with braiding properties. Therefore, our approach is a unified formulation in terms of the statistics which each universal class of particles satisfies, from a unique expression we can take out any distribution function.

The fractal von Neumann entropy per state in terms of the average occupation number is given as [7,8]

\[
  S_G[h, n] = K \left[ 1 + (h - 1)n \right] \ln \left\{ \frac{1 + (h - 1)n}{n} \right\} - \left[ 1 + (h - 2)n \right] \ln \left\{ \frac{1 + (h - 2)n}{n} \right\} \tag{9}
\]

and it is associated with the fractal distribution function Eq.(3).

In [8] we have considered, a microstate probability as

\[
  \mathcal{P}(n) = pn^Gq^{[n(h-2)+1]G-(h-1)}, \tag{10}
\]

where \( nG = N \) is the number of particles, \( G \) is the number of states and \( p + q = 1 \). Also we have that the total probability is unity.
\[
\sum_n W(n)P(n) = 1, \quad (11)
\]
and differentiating this expression with respect to \( p \), we find
\[
n \left[ \frac{q}{p} + 2 - h \right] G = G - (h - 1), \quad (12)
\]
and as \( G \gg (h - 1) \) (1 \( < h \) \( < 2 \)), in the large \( G \) limit, we recover the Eq.(3), with the definition of the function \( \left[ \frac{q}{p} + 2 \right] \equiv \mathcal{Y}[\xi] \), which satisfies the Eq.(4).

The microstate probabilities are written as \( p = \frac{1}{\mathcal{Y}[\xi]-1} \) and \( q = \frac{\mathcal{Y}[\xi]-2}{\mathcal{Y}[\xi]-1} \), and the Eq.(3) as \( n[h] = \frac{p}{q+(2-h)p} \). On the other hand, the von Neumann entropy in terms of the matrix density is found to be [12]
\[
\frac{S}{K} = -\text{Tr} \rho \ln \rho \quad (13)
\]
\[
= - \sum_n W(n)P(n) \ln P(n), \quad (14)
\]
and considering our previous definitions we reobtain the fractal von Neumann entropy as
\[
\mathcal{S}_G[h] = n K \left[ (\mathcal{Y}[\xi] - 1) \ln (\mathcal{Y}[\xi] - 1) - (\mathcal{Y}[\xi] - 2) \ln (\mathcal{Y}[\xi] - 2) \right]. \quad (15)
\]
Finally, in terms of \( p \) and \( q \), we get
\[
\frac{\mathcal{S}_G[h]}{K} = \frac{1}{q + (2 - h)p} \left\{ -p \ln p - q \ln q \right\}. \quad (16)
\]

Now, we define an entanglement measure\(^3\) for the universal classes of fractons in terms of the probability distribution \( p \) as:
\[
\mathcal{E}[h,p] = \frac{1}{1 - (h - 1)p} \left\{ -p \log_2 p - (1 - p) \log_2 (1 - p) \right\}, \quad (17)
\]
where \( 0 \leq p \leq 1 \), is the probability of the system to be in a microstate with entanglement between occupation-numbers of the modes considered empty, partially or completely filled. Thus, an entangled state of fermions, for example, with one particle and three modes is written as
\[
|\Phi> = c_1|110> + c_2|101> + c_3|011>,
\]
with \( p = |c_i|^2 \) and \( \sum_i |c_i|^2 = 1 \). For fractons, for example of the class \( h = \frac{3}{2} \), with 3 modes and 4 particles\(^4\) we have the configuration

\(^3\) Or entanglement fractal von Neumann entropy.

\(^4\) In particular, we have here the maximum of two particles for each mode, so the entanglement between the 3 modes presents this configuration.
\[ |\Psi| = c_1|121 > + c_2|022 > + c_3|211 > + c_4|202 > + c_5|112 > + c_6|220 > , \]

and the amount of entanglement is given by:

\[
\mathcal{E}[h = \frac{3}{2}, p] = \frac{2}{2 - p} \left\{ -p \log_2 p - (1 - p) \log_2 (1 - p) \right\} 
= \sum_{i=1}^{6} \left\{ \frac{2}{2 - |c_i|^2} \left[ -|c_i|^2 \log_2 |c_i|^2 - (1 - |c_i|^2) \log_2 (1 - |c_i|^2) \right] \right\}. \tag{18}
\]

The \( \mathcal{E}(17) \) for fermions reduces to

\[
\mathcal{E}[h = 1, p] = -p \log_2 p - (1 - p) \log_2 (1 - p), \tag{20}
\]

and the amount of entanglement for the entangled state considered above is given by

\[
\mathcal{E}[h = 1, p] = \sum_{i=1}^{3} \left\{ -|c_i|^2 \log_2 |c_i|^2 - (1 - |c_i|^2) \log_2 (1 - |c_i|^2) \right\}. \tag{21}
\]

The expression \( \mathcal{E}(20) \) coincides with another one in Ref. \[3\] for the entanglement measure between occupation-numbers of different single particle basis states in the context of the FQHE. On the other hand, in Ref. \[2\] an entanglement measure of indistinguishable fermions was considered taking into account the Laughlin wave functions and those ones generated by the K-matrix of the quantum Hall liquid. Therefore, our approach consider different systems of particles ( fractons ), i.e., charge-flux systems which carry rational or irrational values of spin, so in this way we obtain results in agreement with those ones reported in Refs. \[2,3\]. These points we will discuss in the next section.

II. ENTANGLEMENT IN THE FQHE

In \[7\] we have considered a fractal approach to the FQHE \[13\] with Hall states modeled by fractons. According to our formulation the quantum Hall state associated with a specific filling factor is a fracton state with value of spin \( s = \nu/2 \). The filling factor which characterizes the quantization of the Hall resistance, has the same value of the statistical parameter, i.e. \( \nu = f(\text{numerically}) \), where \( f \) is defined by \( f = N \phi_0 / \phi \), and \( N \) is the electron number, \( \phi_0 \) is the quantum unit of flux and \( \phi \) is the flux of the external magnetic field throughout the sample. The spin-statistics relation is given by \( \nu = 2s = 2\phi / \phi_0 \), where \( \phi \) is the flux associated with the charge-flux system which defines the fracton \( (h, \nu) \). In this way the universality classes of the quantum Hall transitions satisfy some properties of a subgroup of the modular group \( SL(2, \mathbb{Z}) \) related with the Farey sequences of rational numbers. The transitions allowed are those generated by the condition \( |p_2q_1 - p_1q_2| = 1 \), with \( \nu_1 = \frac{p_1}{q_1} \) and \( \nu_2 = \frac{p_2}{q_2} \) \[14,7\]. This way, we define the universality classes of the quantum Hall transitions in terms of fractal sets labelled by the Hausdorff dimension. We verify that the filling factors experimentally observed appear into the classes \( h \) and from the definition of duality between the fractal sets, we note that the FQHE occurs in pairs of dual filling factors. These quantum numbers get their topological character from the fractal dimension associated with the quantum paths. Our results show clearly which the FQHE has a fractal-like structure.
and this deeper feature is revealed by robust mathematical concepts. Another fractal formulation to the FQHE was discussed in [15] and considered as an approach to be explored for understanding the subtle properties of the FQHE [16]. However, we observe that our program anticipated this suggestion just considering properly ideas of the fractal geometry [7,8].

The entanglement properties of fractons ( Hall states ) can be now analyzed considering the Eq.(17). For distinct classes of fractons we have verified that \( \mathcal{E}[h, p] = \mathcal{E}[h, 1 - p] \). We can check for the classes of particles \( h = 1, \frac{1}{3}, \frac{2}{3}, \frac{5}{3} \), via the graphic \( \mathcal{E} \times p \), \( 0 \leq p \leq 1 \), which the entanglement fractal von Neumann entropy is a concave function and increases in the interval \( 1 < h < 2 \), for instance, \( \mathcal{E}[h = 4/3] < \mathcal{E}[h = 3/2] < \mathcal{E}[h = 5/3] \). Consider now, the sequence

\[
\cdots \rightarrow \left\{ \begin{array}{c} 2 \ 4 \ 8 \ \cdots \\ \frac{3}{2} \ \frac{3}{2} \ \frac{3}{2} \ \cdots \\ \end{array} \right\}_{h=4} \rightarrow \left\{ \begin{array}{c} 1 \ 3 \ 5 \ \cdots \\ \frac{2}{2} \ \frac{2}{2} \ \frac{2}{2} \ \cdots \\ \end{array} \right\}_{h=2} \rightarrow \left\{ \begin{array}{c} 1 \ 5 \ 7 \ \cdots \\ \frac{3}{3} \ \frac{3}{3} \ \frac{3}{3} \ \cdots \\ \end{array} \right\}_{h=4} \rightarrow \cdots ,
\]

and for the entanglement measure written in terms of the filling factors

\[
\mathcal{E}[2 - \nu, p] = \frac{1}{1 - (1 - \nu)p} \left\{ -p \log_2 p - (1 - p) \log_2 (1 - p) \right\},
\]

with \( 0 < \nu < 1 \), and we confirm that \( \mathcal{E}[\nu = 2/3] < \mathcal{E}[\nu = 1/2] < \mathcal{E}[\nu = 1/3] \). For the other members of the classes we need to consider the fractal spectrum Eq.(1). This way, we verify that the Eq.(17) for the class \( h \), is the same for all the members of the class and so, in terms of their entanglement content, different Hall states are equivalent. The understanding that something in this sense can be provided by a quantitative theory of entanglement for complex quantum systems was envisaged by Osborne-Nielsen in [17]. Therefore, we have obtained a result, in the context of the FQHE, which just realizes this perception. Observe that our approach gives information about the entanglement for any possible wave function associated with a specific value of the filling factor. In another route we can consider the LLL for fractons, i.e. if the temperature is sufficiently low and \( \epsilon < \mu \), we can check that the mean occupation number Eq.(3) is given by \( n = \frac{1}{2 - h} \), and so the fractal parameter \( h \) regulates the number of particles in each quantum state. For \( h = 1, n = 1; h = 2, n = \infty \); etc. At \( T = 0 \) and \( \epsilon > \mu \), \( n = 0 \) if \( \epsilon > \epsilon_F \) and \( n = \frac{1}{2 - h} \) if \( \epsilon < \epsilon_F \), hence we get a step distribution, taking into account the Fermi energy \( \epsilon_F \) and \( h \neq 2 \). We can check that for \( h = \frac{4}{3}, \frac{5}{3}, \frac{2}{3} \) we obtain \( n = \frac{3}{2}, \frac{2}{1}, \frac{3}{2} \), respectively. In the first case we have three particles for two states, in the second case two particles for one state and in the last case three particles for one state. So when we run in the interval \( 1 < h < 2 \) we gain more particles for each possible state. In some sense fractons can be understood as quasifermions when near the universal class \( h = 1 \) and as quasibosons when near the universal class \( h = 2 \). The entanglement of the FQHE increases because we have more particles ( fractons ) and less states. On the other hand, in terms of the filling factors, the average occupation number can be written as \( n = \frac{1}{\nu} \), \( 0 < \nu < 1 \); \( n = \frac{1}{2 - \nu} \), \( 1 < \nu < 2 \); \( n = \frac{1}{\nu - 2} \), \( 2 < \nu < 3 \); etc. We obtain the pairs \( (\nu = \frac{2}{3}, n = \frac{3}{2}); (\nu = \frac{1}{2}, n = \frac{2}{3}); (\nu = \frac{1}{3}, n = \frac{3}{1}); (\nu = \frac{4}{3}, n = \frac{3}{2}); (\nu = \frac{3}{2}, n = \frac{2}{3}); (\nu = \frac{5}{3}, n = \frac{3}{2}); (\nu = \frac{5}{3}, n = \frac{2}{3}); (\nu = \frac{7}{3}, n = \frac{3}{2}); (\nu = \frac{7}{3}, n = \frac{2}{3}); \) etc. The behaviour of the step distribution confirms our former analysis: the ground state of the FQHE is a stronger entangled state and the entanglement between occupation-numbers of fractons in the LLL and the rest of the system shows us quantum correlations which can be quantified.
All these results agree with the entanglement properties of the Laughlin wave functions and those generated by the K-matrix [2]. On the other hand, the FQHE understood in terms of the composite fermions or composite bosons are non-entangled as observed in [3], so in contrast, fractons appear as a suitable system for study the quantum correlations of the FQHE. Thus the suggestion that ideas of the quantum information science can give insights for understanding some complex quantum systems [17] is manifested in our definition of entanglement measure for the universal classes of fractons. The universality classes of the quantum Hall transitions as fractal sets of dual topological quantum numbers filling factors, according to our formulation, have increasing entanglement in the interval $1 < h < 2$ and this suggests fracton qubits as a physical resource for quantum computing. In the literature, FQHE qubits associated with the geometrical characteristic of the fractional spin particles have been exploited [18]. The quantum Hall phase transitions discussed by us were obtained considering global properties as the modular symmetry and the Hausdorff dimension associated to the quantum paths of the particles, so some peculiarities of the FQHE, in particular, do not depend on the dynamical aspects or other details of this strongly interacting system [7,8].

III. CONCLUSIONS

The discussion of the FQHE in terms of fractons shows us the potential application of this physical support for the implementation of quantum computing. The topological character of these objects is crucial against problems of the decoherence. Entangled fracton states can be considered as a stable resource for a topological quantum computation, i.e., a fault-tolerant quantum computation. We observe again that fractons carry rational or irrational values of spin and they obey the spin-statistics connection. In the literature geometric phases and the concept of anyons have been explored for implement quantum gates because they are robust against random noise of the environment [19].

Finally, we have introduced the notion of entanglement measure for the universal classes of fractons, where concepts of the fractal geometry appear naturally and gives us the opportunity to extract information about quantum correlations of the ground state of the FQHE. We can obtain the entanglement properties of any possible wave function associated with a specific value of the filling factor such as: Laughlin wave functions [13], wave functions generated by the K-matrix of quantum Hall liquid [20], the Pfaffian trial wave function [21] associated with nonabelian quantum statistics, Jain functions [22], Halperin functions [23], etc. This way, in some sense, we have elaborated a unifying framework for understanding subtle properties of complex quantum systems. Our approach reveals us the fractal nature of this FQHE-phenomenon [7,8]. The possibility of to establish a bridge between quantum information theory and the quantum Hall transitions goes to the direction of some ideas of research in the literature and considered of extreme importance [17]. As we saw, our results agree with other ones [2,3] and open an avenue for we speculate on a fracton quantum computing.
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