Abstract

As a preparation for the numerical study of the SU(2) gauge theory with gluinos, the spectral properties of the fermion matrix and the masses of pseudoscalar and scalar states are investigated in the quenched approximation. The behaviour of the disconnected fermion diagrams on small lattices is also studied.

1 Introduction

The most remarkable developments in non-perturbative Quantum Field Theory (QFT) in recent years have been triggered by the exact solution of certain supersymmetric gauge theories by Seiberg and Witten \cite{SeibergWitten}. These exact results are exploiting the electric-magnetic duality and imply rigorous proofs of confinement and chiral symmetry breaking. They open the door for a new non-perturbative procedure: after solving the theory in a highly symmetric point of the parameter space one can calculate the renormalized effective action by deformations or by expansions in symmetry breaking parameters. (For examples, see ref. \cite{SeibergWitten, MontvayKoutsoumbas}.)

The conventional approach of non-perturbative QFT based on lattice regularization is in some sense opposite. There, in the cut-off theory, the symmetries are broken, with the important exception of exact local gauge symmetry, and have to be restored in the continuum limit by
parameter tuning. In spite of the great successes of the Seiberg-Witten approach, it would be
important to establish a connection to lattice regularization by investigating supersymmetric
gauge theories on the lattice. A possible way to do this was put forward some time ago by
Curci and Veneziano [4]. (For the case of $N = 2$ supersymmetric gauge theories considered in
[1] see also [3].)

The simplest supersymmetric gauge theory is pure SU(2) Yang-Mills theory with massless
gluinos. In case of massive gluinos the non-vanishing gluino mass softly breaks supersymmetry.
Recently some possible fermion simulation algorithms for theories with gluinos have been devel-
oped and tested [6, 7]. A numerical simulation with dynamical gluinos is in progress [8, 9]. As
a first preparatory step, so called “quenched” studies are useful, where the gauge configurations
are prepared in pure gauge theory.

In the present paper we investigate some interesting features of the SU(2) gauge theory
with gluinos in the quenched approximation. The spectral properties of the fermion operator
are studied, which are important in the class of local bosonic fermion algorithms [10] we shall
apply. First information about the interesting region of the fermion hopping parameter, where
hadronic masses are small in lattice units, can be obtained from determining the masses of
pseudoscalar and scalar bound states. In our case these are pion-like, sigma-meson-like or
eta-meson-like, states which are held together by the colour forces among the gluinos in the
adjoint representation. (In case of SU(2) this means colour triplets.) Therefore, we shall call
them adjoint-pion (shortly a-pion), adjoint-sigma (shortly a-sigma) and adjoint-eta (shortly
a-eta), respectively. For the a-eta state, the computation of the disconnected fermion diagrams
is necessary.

In the next section we report on our quenched calculations carried out at the bare gauge
coupling parameter $\beta \equiv 4/g^2 = 2.3$, which roughly corresponds to the lower edge of the scaling
region in pure SU(2) lattice gauge theory with Wilson action.

2 Numerical results

2.1 Spectral boundaries of the fermion matrix

The fermion matrix $Q$ relevant for gluinos can be defined as

$$Q_{yv,xu} \equiv Q_{yv,xu}[U] \equiv \delta_{yx} \delta_{vu} - K \sum_{\mu=1}^{4} \left[ \delta_{y,x+\hat{\mu}} (1 + \gamma_\mu) V_{vu,\mu x} + \delta_{y+\hat{\mu},x} (1 - \gamma_\mu) V_{T vu,\mu y} \right]. \quad (1)$$

Here, as usual, $x, y, \ldots$ denote lattice sites, $u, v, \ldots$ colour triplet indices, $\hat{\mu}$ the unit vector
in direction $\mu = 1, 2, 3, 4$ and $K$ the hopping parameter determining the gluino mass. The
gauge link variable in the adjoint representation $V_{x\mu}$ is connected to the usual fundamental link
variable $U_{x\mu}$ by the relation

$$V_{rs,x\mu} \equiv V_{rs,x\mu}[U] \equiv 2 \text{Tr}(U_{x\mu}^T U_{x\mu}^s) = V_{rs,x\mu}^* = V_{rs,x\mu}^{-1T}, \quad (2)$$
if $T_r$ is a group generator (for $SU(2)$ $T_r = \frac{1}{2} \tau_r$ is proportional to the Pauli matrix $\tau_r$).

In local bosonic updating algorithms the spectral properties of $Q$ and $Q^\dagger Q$ are important, because the polynomial approximations defining the bosonic action are optimized on the eigenvalues. In the two-step local bosonic algorithm [6] the polynomials are optimized in an interval $[\epsilon, \lambda]$ containing the eigenvalues $\rho$ of $Q^\dagger Q$. For a hermitean matrix as $Q^\dagger Q$ the extremal eigenvalues $\rho_{\text{min}} \equiv \min \rho$ and $\rho_{\text{max}} \equiv \max \rho$ can be obtained by known iterative methods [11] (for recent work on the spectrum of the Wilson-Dirac operator in QCD see [12]). In the same way, the extremal eigenvalues of the hermitean matrices

$$Q_R \equiv \frac{1}{2}(Q + Q^\dagger), \quad Q_I \equiv \frac{1}{2i}(Q - Q^\dagger)$$

(3) can also be determined. Since for vectors $v$ normalized to unity we have

$$Q = Q_R + iQ_I, \quad \text{Re} \langle v|Q|v \rangle = \langle v|Q_R|v \rangle, \quad \text{Im} \langle v|Q|v \rangle = \langle v|Q_I|v \rangle,$$

(4) the spectral boundaries of $Q$ itself can also be estimated. For instance, if $\lambda$ denotes eigenvalues of $Q$ and $\lambda_R$ eigenvalues of $Q_R$, we have

$$\lambda_{R\text{min}} \equiv \min \lambda_R = \min \text{Re} \langle v|Q|v \rangle \leq \min \text{Re} \lambda,$$

(5) and similarly for the maximum $\lambda_{R\text{max}}$ of $\lambda_R$ and the extrema $\lambda_{I\text{min,max}}$ of $\lambda_I$. The numerical simulation results for the quenched averages of these extremal eigenvalues at $\beta = 2.3$ for $0.180 \leq K \leq 0.205$ on $8^3 \cdot 16$ lattices are shown in table 1. The distributions of $\rho_{\text{min}}$ and $\lambda_{R\text{min}}$ for $K = 0.200$ are also shown in figures 1 and 2, respectively. The statistics is based on at least 320 independent gauge configurations per hopping parameter value. The iteration for the extremal eigenvalues was stopped if the relative deviation of the new estimate from the previous one was less than $10^{-3}$.

| $K$  | $\rho_{\text{min}}$ | $\rho_{\text{max}}$ | $\lambda_{R\text{min}}$ | $\lambda_{R\text{max}}$ | $\lambda_{I\text{max}} = -\lambda_{I\text{min}}$ |
|------|---------------------|----------------------|--------------------------|--------------------------|----------------------------------|
| 0.180| 0.00879(2)          | 4.4873(6)            | -0.0801(1)               | 2.0690(2)                | 0.86357(5)                       |
| 0.190| 0.00434(2)          | 4.764(1)             | -0.1375(2)               | 2.1287(3)                | 0.91151(9)                       |
| 0.200| 0.00197(1)          | 5.050(1)             | -0.1958(2)               | 2.1885(3)                | 0.95948(9)                       |
| 0.205| 0.00138(2)          | 5.194(1)             | -0.2245(2)               | 2.2178(3)                | 0.98373(9)                       |

As table 1 and figure 1 show, the lower limit of the spectrum of $Q^\dagger Q$ is of the order of $10^{-3}$, and the ratio of the largest to smallest eigenvalues is $10^3$-$10^4$. This is the range one also encounters in “unquenched” numerical simulations with dynamical gluinos [1]. It is interesting to observe that the lower limit of the spectrum of $Q_R$ defined in (3) is negative in the whole
range of the hopping parameter considered. Although the minimal real part of the eigenvalues of $Q$ might be somewhat larger, it is probably also negative. This shows that in the relevant hopping parameter range the eigenvalues of $Q$ do not remain in the half plane with positive real part, but surround zero in the complex plane from all sides.

2.2 Masses of a-pion and a-sigma

The connected contributions to pseudoscalar and scalar bosons are

$$\text{Tr}\left\{\gamma_5 Q_{yx}^{-1} \gamma_5 Q_{xy}^{-1}\right\}, \quad \text{Tr}\left\{Q_{yx}^{-1} Q_{xy}^{-1}\right\},$$

respectively. These give the correlation functions of the corresponding flavour non-singlet mesons in models with at least two (adjoint) fermion flavours ($N_a \geq 2$). Such states, which we call a-pion and a-sigma, respectively, do not exist in the gluino model, which is equivalent to an (adjoint) flavour number of $N_a = \frac{1}{2}$. Nevertheless, the masses of the a-pion and a-sigma are interesting, because the hopping parameter region where they become small in lattice units is the interesting one, where also other physical colour singlet hadrons become light.

Similarly to the pion in quenched QCD, one expects that the a-pion mass-squared vanishes linearly as a function of $1/K$. This is the remnant of chiral symmetry for $N_a \geq 2$ in the quenched approximation, because the bare quark mass in lattice units is $m_q = \frac{1}{4}(1/K - 1/K_0)$. Here $K_0 = K_0(\beta)$ is the hopping parameter where the pion mass vanishes. Of course, since the gluino corresponds to $N_a = \frac{1}{2}$, in the gluino model there is no analogous chiral symmetry and the exact value of the “critical” hopping parameter ($K_{cr}$) where the a-pion has zero mass is not directly relevant. It is interesting just as an indicator of the scaling region where the continuum limit has to be performed.

The a-pion mass at $\beta = 2.3$ as a function of $1/K$ is shown in figure 3. The linear extrapolation works very well and gives $K_{cr} = 0.2151(3)$. The masses of a-pion and a-sigma are displayed together in figure 4. As expected, the a-sigma mass is higher and does not seem to vanish at $K_{cr}$. The results in figures 3 and 4 were obtained on an $8^3 \cdot 16$ lattice. The masses were always extracted from timeslice correlations by a fit

$$c_0 + c_1 \left\{e^{-mt} + e^{-m(T-t)}\right\},$$

with $T$ being the lattice time extension. For comparison, in figure 4 at $K = 0.180$ also the masses on $4^3 \cdot 8$ are included, which do not deviate too much from the corresponding masses on $8^3 \cdot 16$. This shows that, not too close to $K_{cr}$, even a $4^3 \cdot 8$ lattice can give first indications on the mass spectrum.
2.3 Disconnected contributions: the mass of a-eta

The correlation function of the pseudoscalar meson made out of Majorana gluinos can be obtained from the general formulas given in ref. [6]:

\[
\langle (\overline{\Psi}_y \gamma_5 \Psi_y)(\overline{\Psi}_x \gamma_5 \Psi_x) \rangle = \langle \text{Tr} \{\gamma_5 Q_{yy}^{-1}\text{Tr} \{\gamma_5 Q_{xx}^{-1}\} - 2 \text{Tr} \{\gamma_5 Q_{yx}^{-1}\gamma_5 Q_{xy}^{-1}\} \rangle_U .
\]  (8)

Here the index \(U\) denotes an expectation value in the path integral over gauge variables. A similar formula is also valid for scalar mesons, where \(\gamma_5\) is replaced by the unit Dirac matrix.

Since in QCD the correlation analogous to (8), apart from the factor 2 in front of the connected contribution, belongs to \(\eta\) and \(\eta'\) mesons, one can call the corresponding boson a-eta.

As discussed in ref. [6], the expectation value in eq. (8) can also be expressed by a “noisy estimator” in terms of the \(n\) bosonic pseudofermion fields \((\phi)\). In the present case we have

\[
\langle (\overline{\Psi}_y \gamma_5 \Psi_y)(\overline{\Psi}_x \gamma_5 \Psi_x) \rangle =
= 4 \sum_{j,k=1}^{n} \left[ (\overline{\phi}_{ky} \phi_{kj}) + (\phi_{kj}^* \overline{\phi}_{ky}) \right] \left[ (\overline{\phi}_{jx} \phi_{jx}) + (\phi_{jx}^* \overline{\phi}_{jx}) \right] - 2 \delta_{jk} \delta_{yx} (\phi_{jx}^* \phi_{jx}) \rangle_{\phi^U} ,
\]  (9)

with the notation

\[
\overline{\phi}_{jx} \equiv \sum_y (\gamma_5 Q + \mu_j)_{yx} \phi_{jy} .
\]  (10)

In the last formula \(\mu_j\) \((j = 1, \ldots, n)\) denote the real parts of the roots of the polynomial used in the local bosonic updating algorithm.

The advantage of the noisy estimator in eq. (9) is that in principle it can be obtained with very little work from the pseudoscalar fields during unquenched updating. Unfortunately, this does not work in practice: the result is too noisy and the signal cannot be obtained in this way [9]. What remains is the evaluation of the fermion propagators appearing in eq. (8).

The “connected” contribution in the second term has been discussed in the previous subsection. The behaviour of the first “disconnected” contribution on \(4^3 \cdot 8\) lattice at \(\beta = 2.3, K = 0.180\) is shown in figure 4. Compared to the connected term, the disconnected contribution is two to three orders of magnitude smaller. On the other hand, the disconnected contribution decreases considerably slower as a function of the distance. It shows an effective mass of the order of \(m_{\text{disc}} \simeq 0.8\), whereas the effective mass in the connected contribution (the \(a\)-pion mass) is about \(m_{\text{conn}} \simeq 1.3\). This behaviour is qualitatively similar to the one observed in QCD [13].

As argued in ref. [13], in the quenched approximation one can also extract a “quenched” \(a\)-eta mass from the ratio of the disconnected to connected contributions. Denoting the ratio of the two terms at timeslice distance \(t\) by \(R(t)\), one has

\[
R(t) = \text{const.} + t \frac{m_{\eta}^2}{2m_\pi} .
\]  (11)

Here \(m_\eta\) and \(m_\pi\) stand generically for the masses of the flavour singlet and flavour non-singlet pseudoscalar mesons, respectively. In our case we get from (11) a quenched \(a\)-eta mass of
about $m_{\text{a-eta}} \simeq 0.1$, although the linear fit on our small $4^3 \cdot 8$ lattice is not very good. Of course, we are finally interested in numerical results with dynamical gluinos, where the linear behaviour in (11) is irrelevant, and the a-eta mass can be obtained from a usual kind of fit of the combination in eq. (8) according to eq. (7). The results shown in figure 5 were obtained by direct evaluation of the necessary inverses on 100 independent configurations. In fact, for a more precise calculation of the disconnected contributions, especially on larger lattices, there are better methods known (see ref. [14]).

In summary: the quenched calculations give useful information on the spectral properties of fermion matrices and the range of hopping parameters relevant for the continuum limit, which help in setting up numerical simulations with dynamical gluinos.

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Figure 1: The quenched distribution of the lower spectrum boundary $\rho_{\text{min}}$ of $Q^\dagger Q$ on $8^3 \cdot 16$ lattice at $\beta = 2.30$, $K = 0.20$. 
Figure 2: The quenched distribution of the lower spectrum boundary $\lambda_{R_{\text{min}}}$ of $Q_R = (Q + Q^\dagger)/2$ on $8^3 \cdot 16$ lattice at $\beta = 2.30$, $K = 0.20$. 
Figure 3: The a-pion mass squared as a function of the inverse hopping parameter on $8^3 \times 16$ lattice at $\beta = 2.3$. The line is a linear fit used to determine $K_{cr}$.
Figure 4: The a-pion and a-sigma mass as a function of the inverse hopping parameter on $8^3 \times 16$ lattice at $\beta = 2.3$. The open symbols refer to $4^3 \times 8$. The curves show linear fits for the mass squares.
Figure 5: The disconnected a-eta correlation as a function of the timeslice distance on $4^3 \cdot 8$ lattice at $\beta = 2.3$, $K = 0.18$. The curve is a fit according to (7) for $0 \leq t \leq 4$, giving a mass $m_{\text{disc}} = 0.84(15)$. 

$\beta = 2.3$, $K = 0.18$

mass $= 0.84 \pm 0.15$

$\chi^2 = 0.72$ (t=0.4, $c_0 = 3.31 \times 10^{-6}$)