ON NON-LOCALLY CONNECTED BOUNDARIES OF CAT(0) SPACES

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Abstract. In this paper, we study CAT(0) spaces with non-locally connected boundary. We give some condition of a CAT(0) space whose boundary is not locally connected.

1. Introduction and preliminaries

In this paper, we study proper CAT(0) spaces with non-locally connected boundary. A metric space $X$ is said to be proper if every closed metric ball is compact. Definitions and basic properties of CAT(0) spaces and their boundaries are found in [1].

Let $X$ be a proper CAT(0) space and let $\gamma$ be an isometry of $X$. The translation length of $\gamma$ is the number $|\gamma| := \inf \{d(x, \gamma x) \mid x \in X\}$, and the minimal set of $\gamma$ is defined as $\text{Min}(\gamma) = \{x \in X \mid d(x, \gamma x) = |\gamma|\}$. An isometry $\gamma$ of $X$ is said to be hyperbolic, if $\text{Min}(\gamma) \neq \emptyset$ and $|\gamma| > 0$ (cf. [2]). For a hyperbolic isometry $\gamma$ of a proper CAT(0) space $X$, $\gamma^\infty$ is the limit point of the boundary $\partial X$ to which the sequence $\{\gamma_i x_0\}$ converges, where $x_0$ is a point of $X$.

In this paper, we define a reflection of a geodesic space as follows: An isometry $r$ of a geodesic space $X$ is called a reflection of $X$, if

1. $r^2$ is the identity of $X$,
2. $X \setminus F_r$ has exactly two convex connected components $X^+_r$ and $X^-_r$ and
3. $rX^+_r = X^-_r$,

where $F_r$ is the fixed-points set of $r$. We note that “reflections” in this paper need not satisfy the condition (4) $\text{Int} F_r = \emptyset$ in [2].

A CAT(0) space $X$ is said to be almost extendible, if there exists a constant $M > 0$ such that for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and $\zeta$ passes within $M$ of $y$. In [3], Ontaneda has proved that a CAT(0) space on which some

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In [3] and [2], Mihalik, Ruane and Tschantz have proved some nice results about CAT(0) groups with (non-)locally connected boundary.

The purpose of this paper is to prove the following theorem.

**Theorem 1.1.** Let $X$ be a proper and almost extendible CAT(0) space, let $\gamma$ be a hyperbolic isometry of $X$ and let $r$ be a reflection of $X$. If

1. $\gamma^\infty \not\in \partial F_r$,
2. $\gamma(\partial F_r) \subset \partial F_r$ and
3. $\text{Min}(\gamma) \cap F_r = \emptyset$,

then the boundary $\partial X$ of $X$ is not locally connected.

### 2. Topology of the boundary of a CAT(0) space

In this section, we recall topology of the boundary of a CAT(0) space.

Let $X$ be a proper CAT(0) space and $x_0 \in X$. The **boundary of $X$ with respect to $x_0$**, denoted by $\partial_{x_0} X$, is defined as the set of all geodesic rays issuing from $x_0$. Then the topology on $X \cup \partial_{x_0} X$ is defined by the following conditions:

1. $X$ is an open subspace of $X \cup \partial_{x_0} X$.
2. For $\alpha \in \partial_{x_0} X$ and $R, \epsilon > 0$, let
   
   $$U_{x_0}(\alpha; R, \epsilon) = \{ x \in X \cup \partial_{x_0} X \mid x \not\in B(x_0, R), \ d(\alpha(R), \xi_x(R)) < \epsilon \},$$

   where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from $x_0$ to $x$ ($\xi_x = x$ if $x \in \partial_{x_0} X$). Then for each $\epsilon_0 > 0$, the set
   
   $$\{ U_{x_0}(\alpha; R, \epsilon_0) \mid R > 0 \}$$

   is a neighborhood basis for $\alpha$ in $X \cup \partial_{x_0} X$.

This is called the cone topology on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of $X$ ([4], [2]).

Here the following lemma is known.

**Lemma 2.1.** Let $X$ be a proper CAT(0) space and let $x_0 \in X$. For $\alpha \in \partial_{x_0} X$ and $R, \epsilon > 0$, let

$$U'_{x_0}(\alpha; R, \epsilon) = \{ x \in X \cup \partial_{x_0} X \mid x \not\in B(x_0, R), \ d(\alpha(R), \text{Im} \xi_x) < \epsilon \},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from $x_0$ to $x$ ($\xi_x = x$ if $x \in \partial_{x_0} X$). Then for each $\epsilon_0 > 0$, the set

$$\{ U'_{x_0}(\alpha; R, \epsilon_0) \mid R > 0 \}$$

is also a neighborhood basis for $\alpha$ in $X \cup \partial_{x_0} X$. 

Let $X$ be a proper CAT(0) space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in $X$. The boundary of $X$, denoted by $\partial X$, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray $\xi$ is denoted by $\xi(\infty)$. For each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial x_0 X$ with $\xi(\infty) = \alpha$. Thus we may identify $\partial X$ with $\partial x_0 X$ for each $x_0 \in X$ ([1], [3]).

3. Proof of the theorem

We prove Theorem 1.1.

Proof of Theorem 1.1. Let $X$ be a proper and almost extendible CAT(0) space, let $\gamma$ be a hyperbolic isometry of $X$ and let $r$ be a reflection of $X$ such that

(1) $\gamma^\infty \not\in \partial F_r$,
(2) $\gamma(\partial F_r) \subset \partial F_r$ and
(3) $\Min(\gamma) \cap \partial F_r = \emptyset$.

Since $X$ is almost extendible, there exists a constant $M > 0$ such that for each pair of points $x, y \in X$, there is a geodesic ray $\zeta : [0, \infty) \to X$ such that $\zeta(0) = x$ and $\zeta$ passes within $M$ of $y$. By (1), $\gamma^\infty \not\in \partial F_r$. Since $\partial F_r$ is a closed set in $\partial X$, there exist $R > 0$ and $\epsilon > 0$ such that $U'_x(\gamma^\infty ; R, \epsilon) \subset \partial F_r = \emptyset$ by Lemma 2.1.

Let $x_0 \in \Min(\gamma)$ and let $\xi : [0, \infty) \to X$ be the geodesic ray in $X$ such that $\xi(0) = x_0$ and $\xi(\infty) = \gamma^\infty$. Then $\Im \xi \subset \Min(\gamma)$. Since $\Min(\gamma) \cap \partial F_r = \emptyset$ by (3) and $\xi(\infty) = \gamma^\infty \not\in \partial F_r$ by (1), there exists a number $K > 0$ such that $d(\xi(K), F_r) > M$. Let $N = d(x_0, rx_0)$. For an enough large number $i_0 \in \NN$,

$$U'_x(\gamma^\infty ; i_0 |\gamma|, N + K + M) \subset U'_x(\gamma^\infty ; R, \epsilon).$$

We prove that $U'_x(\gamma^\infty ; i_0 |\gamma|, N + K + M) \cap \partial X$ is not connected for any $i_0 \leq i \in \NN$. This implies that $\partial X$ is not locally connected, because $\{U'_x(\gamma^\infty ; i_0 |\gamma|, N + K + M) \cap \partial X \mid i \in \NN, i \geq i_0\}$ is a neighborhood basis of $\gamma^\infty$ in $\partial X$.

Let $i \in \NN$ such that $i \geq i_0$. Then $\gamma^i r \gamma^{-i}$ is a reflection of $X$ and $F_{\gamma^i r \gamma^{-i}} \subset F_{\gamma^i r}$. Here by (3),

$$F_{\gamma^i r \gamma^{-i}} \cap \Min(\gamma) = \gamma^i F_r \cap \gamma^i \Min(\gamma) = \gamma^i (F_r \cap \Min(\gamma)) = \emptyset.$$ 

Let $X \setminus F_{\gamma^i r \gamma^{-i}} = X_i^+ \cup X_i^-$, where $X_i^+$ and $X_i^-$ are convex connected components, and $x_0 \in X_i^+$. We consider the geodesic ray $\gamma^i r \xi$ such that $\gamma^i r \xi(0) = \gamma^i r x_0$ and $\gamma^i r \xi(\infty) = \gamma^i r \gamma^\infty$. Since $x_0 \in X_i^+$, $\Im \xi \subset X_i^+$.
and $\text{Im} \gamma^i r\xi \subset X_i^-$. Hence $\gamma^i r\xi(K) \in X_i^-$. Here
\[
d(\gamma^i r x_0, \gamma^i r\xi(K)) = d(x_0, \xi(K)) = K \text{ and } d(\gamma^i r\xi(K), \gamma^i r F_r) = d(\xi(K), F_r) > M.
\]
By the definition of the number $M$, there exists a geodesic ray $\zeta_i : [0, \infty) \to X$ such that $\zeta_i(0) = x_0$ and $\zeta_i$ passes within $M$ of $\gamma^i r\xi(K)$. Since $d(\gamma^i r\xi(K), \gamma^i r F_r) > M$, $\zeta_i(\infty) \in \partial X^-$. Because if $\zeta_i(\infty) \in \partial X \setminus \partial X_i^{-} = \partial(X_i^+ \cup \gamma^i F_r)$ then $\text{Im} \zeta_i \subset X_i^+ \cup \gamma^i F_r$, since $X_i^+ \cup \gamma^i F_r$ is convex and $\zeta_i(0) = x_0 \in X_i^+$. Then
\[
d(\gamma^i x_0, \text{Im} \zeta_i) \leq d(\gamma^i x_0, \gamma^i r x_0) + d(\gamma^i r x_0, \gamma^i r\xi(K)) + d(\gamma^i r\xi(K), \text{Im} \zeta_i)
\leq d(x_0, r x_0) + d(x_0, \xi(K)) + M = N + K + M.
\]
We note that $\xi(\infty) = \gamma^\infty$ and $\gamma^i x_0 = \xi(i|\gamma|)$, since $x_0 \in \text{Min}(\gamma)$. Hence $\zeta_i \in U'_{x_0}(\gamma^{\infty}; i|\gamma|, N + K + M)$.

Now we show that there does not exist a path from $\gamma^\infty$ to $\zeta_i(\infty)$ in $U'_{x_0}(\gamma^{\infty}; i|\gamma|, N + K + M) \cap \partial X$. Since $\gamma^\infty \in \partial X_i^+$ and $\zeta_i(\infty) \in \partial X_i^-$, such pass must intersect with $\partial F_{\gamma^i r \gamma^{-1}}$. Here
\[
\partial F_{\gamma^i r \gamma^{-1}} = \partial(\gamma^i F_r) = \gamma^i(\partial F_r) \subset \partial F_r,
\]
by (2). We note that $U'_{x_0}(\gamma^{\infty}; R, \epsilon) \cap \partial F_r = \emptyset$ and
\[
U'_{x_0}(\gamma^{\infty}; i|\gamma|, N + K + M) \subset U'_{x_0}(\gamma^{\infty}; i_0|\gamma|, N + K + M)
\subset U'_{x_0}(\gamma^{\infty}; R, \epsilon).
\]
Hence
\[
U'_{x_0}(\gamma^{\infty}; i|\gamma|, N + K + M) \cap \partial F_{\gamma^i r \gamma^{-1}} = \emptyset.
\]
Thus there does not exist a path between $\gamma^\infty$ and $\zeta_i(\infty)$ in $U'_{x_0}(\gamma^{\infty}; i|\gamma|, N + K + M) \cap \partial X$.

Therefore $\partial X$ is not locally connected. \hfill \square

4. Remark

Every CAT(0) space on which some group acts geometrically (i.e. properly and cocompactly by isometries) is proper (\cite{p.132}) and almost extendible (\cite{K}).

In \cite{K}, Ruane has proved that $\partial \text{Min}(\gamma)$ is the fixed-points set of $\gamma$ in $\partial X$, i.e.,
\[
\partial \text{Min}(\gamma) = \{ \alpha \in \partial X \mid \gamma \alpha = \alpha \}.
\]
Hence, for example, if $\partial F_r \subset \partial \text{Min}(\gamma)$ then $\gamma(\partial F_r) = \partial F_r$ and the condition (2) in Theorem \cite{L} holds.
A Coxeter system $(W, S)$ defines a Davis complex $\Sigma(W, S)$ which is a CAT(0) space ([2] and [7]). Then the Coxeter group $W$ acts geometrically on $\Sigma(W, S)$ and each $s \in S$ is a reflection of $\Sigma(W, S)$.

For example, as an application of Theorem 1.1, we can obtain the following corollary.

**Corollary 4.1.** Let $(W, S)$ be a right-angled Coxeter system and let $\Sigma(W, S)$ be the Davis complex of $(W, S)$. Suppose that there exist $s_0, s_1, u_0 \in S$ such that

1. $o(s_0s_1) = \infty$,
2. $o(s_0u_0) = \infty$ and
3. $s_0t = ts_0$ and $s_1t = ts_1$ for each $t \in T$,

where $T = \{ t \in S \mid tu_0 = u_0t \}$ and $\hat{T}$ is the subset of $S$ such that $W_{\hat{T}}$ is the minimum parabolic subgroup of finite index in $W_T$. Then the boundary $\partial \Sigma(W, S)$ is not locally connected.

**Proof.** Let $\gamma = s_0s_1$ and $r = u_0$. Then $\gamma$ is a hyperbolic isometry of $\Sigma(W, S)$ by (1), $r$ is a reflection of $\Sigma(W, S)$ and $\partial F_r = \partial \Sigma(W_{\hat{T}}, \hat{T})$. Here by (3),

$$\gamma(\partial F_r) = (s_0s_1)\partial \Sigma(W_{\hat{T}}, \hat{T}) = \partial \Sigma(W_{\hat{T}}, \hat{T}) = \partial F_r.$$ 

Also

$$\gamma^\infty = (s_0s_1)^\infty \notin \partial \Sigma(W_{\hat{T}}, \hat{T}) = \partial F_r,$$

and $\operatorname{Min}(\gamma) \cap F_r = \emptyset$ by (2). Thus the conditions in Theorem 1.1 hold, and $\partial \Sigma(W, S)$ is not locally connected.

Corollary 4.1 is a special case of Theorem 3.2 in [6]. We can also obtain Corollary 4.1 from Theorem 3.2 in [6].

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