The role of a form of vector potential — normalization of the antisymmetric gauge

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Results obtained for the antisymmetric gauge \( A = [Hy, −Hx]/2 \) by Brown and Zak are compared with those based on pure group-theoretical considerations and corresponding to the Landau gauge \( A = [0, Hx] \). Imposing the periodic boundary conditions one has to be very careful since the first gauge leads to a factor system which is not normalized. A period \( N \) introduced in Brown’s and Zak’s papers should be considered as a magnetic one, whereas the crystal period is in fact \( 2N \). The ‘normalization’ procedure proposed here shows the equivalence of Brown’s, Zak’s, and other approaches. It also indicates the importance of the concept of magnetic cells. Moreover, it is shown that factor systems (of projective representations and central extensions) are gauge-dependent, whereas a commutator of two magnetic translations is gauge-independent. This result indicates that a form of the vector potential (a gauge) is also important in physical investigations.

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I. INTRODUCTION

The discovery of the quantum Hall effect \( \square \) led to remarkable interest in two-dimensional electron systems subjected to a magnetic field \( \square \). Since 1980 authors working in different fields — from applied to mathematical physics — have considered related problems and many new features have been observed and discussed \( \square \). One of the most interesting questions is the dynamic of two-dimensional electrons in a periodic potential and an external magnetic field \( \square \). The first results, in the tight binding approximation, were presented by Peierls \( \square \) shortly after Landau’s discovery of the quantization of electron states in a magnetic field. A new impact was due to Brown and Zak \( \square \), who independently introduced magnetic translation operators in two different, but equivalent, ways. Both approaches were based on group-theoretical considerations and led to the broadening of the Landau levels and quantization of a magnetic field \( \square \). Although more than thirty years have passed, their papers are still considered as fundamental ones \( \square \). Brown and Zak proved that the problem considered is in fact two-dimensional and their investigations confirmed the importance of projective representations and central extensions in quantum physics \( \square \). On the other hand, Zak’s and Brown’s results were not gauge-independent — only a completely antisymmetric vector potential was considered by both authors. An attempt to consider gauge-equivalent vector potentials leads to some ambiguities and misconceptions if it is not done carefully. A bit simpler and more clear results can be obtained from pure group-theoretical considerations. For example, Divakaran and Rajagopal did not consider gauges at all and they worked with central extensions and projective representations only \( \square \). However, pure mathematical description may not provide us with an intuitive image of the physical phenomena. Moreover, many experiments and theories indicate the importance of vector potential \( \square \), so it is necessary to include gauges and potentials in considerations.

The aim of this paper is to show sources of misconceptions, ambiguities, and unexpected gauge-dependence of the problem. In particular, factor systems of projective representations and central extensions introduced by Brown and Zak have been carefully checked and compared with those obtained from pure group-theoretical considerations \( \square \). It occurs that they can be considered as standard but they are not normalized \( \square \). This last fact is the main source of differences between Brown’s and Zak’s approaches. Moreover, it indicates points at which a form of the vector potential is important, i.e., the points at which the problem is not gauge-independent.

In this paper we propose a procedure of ‘normalization’ of those factor systems, which enables us to identify irreps introduced by Brown and Zak. A comparison of these irreps with those obtained for central extensions of finite translation groups leads to a concept of the so-called magnetic cells \( \square \) and shows that Brown and Zak considered in fact finite lattices with a period \( 2N \) not \( N \).

For the sake of clarity, the following simplifications arising from the quoted papers are assumed. Position \( (r, R) \), momentum \( (p) \), and vector potential \( (A) \) are considered to be two-dimensional vectors. Note that \( r = (x, y) \) is any vector of \( \mathbb{R}^2 \), whereas \( R = (X, Y) \in \mathbb{Z}^2 \) denotes a vector of a square lattice with \( a_1 = \hat{x} \) and \( a_2 = \hat{y} \), so the area of the elementary cell is equal to \( 1 \). The magnetic field is perpendicular to the \( x-y \) plane and \( H = \hat{z} \). The periodic boundary conditions are imposed on representations of \( \mathbb{Z}^2 \) and the periods are equal, i.e., \( N_1 = N_2 = N \); the finite translation group and its representations can be considered equivalently.

The paper is organized as follows. In Sec. \( \square \) the most fundamental formulas of Brown’s and Zak’s papers are recalled and equivalence of their approaches are indicated. Basic properties of projective representations are briefly
II. DIFFERENT DESCRIPTIONS OF MAGNETIC TRANSLATION GROUPS

From the algebraic point of view there are two equivalent descriptions of the magnetic translation operators. Brown investigated a projective representation of the translation group $T$ and then imposed the magnetically periodic boundary conditions on it. On the other hand, Zak introduced a closed set of noncommuting operators which, in fact, form a covering group $T'$ of $T$ so its standard (vector) representations are projective representations of $T'$. The finiteness of these representations was again achieved by imposing the periodic boundary conditions. These two approaches are related by a formula which follows from the induction procedure if one constructs representations of the covering group. Since $T'$ is a central extension of $T$ by the group $U(1)$ (or its subgroup referred to hereafter as a group of factors and denoted $F'$) then its (vector) representations can be written as

$$ \Xi[\alpha, \mathbf{R}] = \Gamma(\alpha) D(\mathbf{R}), $$

(1)

where $\alpha \in F \subset U(1)$, $\mathbf{R} \in T$, $\Gamma$ is a vector representation of $F$ and $D$ is a projective representation of $T$. A factor system $m(\mathbf{R}, \mathbf{R}')$ of this representations is determined by the relation

$$ D(\mathbf{R}) D(\mathbf{R}') = m(\mathbf{R}, \mathbf{R}') D(\mathbf{R} + \mathbf{R}'), $$

(2)

whereas the multiplication rule for $T'$ reads

$$ [\alpha, \mathbf{R}][\alpha', \mathbf{R}'] = [\alpha \alpha' \mu(\mathbf{R}, \mathbf{R}'), \mathbf{R} + \mathbf{R}'] $$

(3)

with $\mu(\mathbf{R}, \mathbf{R}')$ being a factor system of a central extension. These factor systems are related to each other by the formula

$$ m(\mathbf{R}, \mathbf{R}') = \Gamma[\mu(\mathbf{R}, \mathbf{R}')]. $$

(4)

This relation establishes the equivalence of both approaches. Moreover, both authors assumed the antisymmetric vector potential (gauge) $\mathbf{A} = (\mathbf{H} \times \mathbf{r})/2 = [H_y, -H_x]/2$ and were not able to generalize their considerations to other gauges, in particular their approaches did not include the Landau gauge. On the other hand, their results and some conclusions are different in some points which will be discussed here and compared with the results obtained for the Landau gauge.

All considerations and formulas given above are also valid for a finite group $T_N$ and its (finite-dimensional) representations. In addition, we can apply to this case a version of the Burnside theorem which reads that nonequivalent irreducible projective representations of $T_N$ with the same factor system $m(\mathbf{R}, \mathbf{R}')$ satisfy the following condition

$$ \sum_{j} |jD|^2 = |T_N| = N^2, $$

(5)

where $j$ labels nonequivalent representations (there is no expression for a number of these representations) and $|jD|$ denotes the dimension of $jD$. Since $F$ is an Abelian group then it has $|F|$ irreducible nonequivalent representations and each of them determines different (nonequivalent) factor system $m(\mathbf{R}, \mathbf{R}')$ according to (3). It follows from (5) that irreducible representations of $T_N'$ determined by (3) satisfy the Burnside theorem.

Brown defined a magnetic translation operator as

$$ \hat{T}(\mathbf{R}) = \exp[-i\mathbf{R} \cdot (\mathbf{p} - e\mathbf{A}/c)/\hbar], $$

(6)

where $\mathbf{p}$ is the kinetic momentum and $\mathbf{A}$ is the vector potential such that $\nabla \times \mathbf{A} = \mathbf{H}$. These operators form a projective representation of $T$ with a factor system

$$ m(\mathbf{R}, \mathbf{R}') = \exp[-\pi i (\mathbf{R} \times \mathbf{R}') \cdot \mathbf{H}/\varphi_0] $$

(7)

where $\varphi_0 = ch/e$. Brown showed that one can impose the periodic boundary conditions $\hat{T}(N_a)\psi = \psi$ if (notice simplifications assumed in this paper — in fact, $H$ denotes hereafter the magnetic flux through one primitive cell)
\[ H = \frac{l}{N}\varphi_0 , \]  

where \( l \) is mutually prime with \( N \), i.e. \( \text{gcd}(l,N) = 1 \). Hence a factor system of a finite projective representation \( lD \) for \( H \) satisfying (8) is given as

\[ m_l([X,Y], [X',Y']) = \exp[-\pi i l(XY' - YX')/N] . \]

Brown showed that there is the unique (up to equivalence) irreducible projective representation with a dimension \( N \) and matrix elements

\[ lD_{jk}[X,Y] = \exp\left[\pi i \frac{ LX}{N} (Y + 2j)\right] \delta_{j,k-Y} , \]

where \( j,k = 0,1, \ldots, N-1 \) and \( \delta_{j,k} \) is calculated modulo \( N \) (Brown labeled rows and columns by \( j,k = 1,2, \ldots, N \)). Zak considered a covering group of the translation group consisting of operators

\[ \tau(R|R_1, \ldots, R_j) = \hat{T}(R)\exp[2\pi i\phi(R_1, \ldots, R_n)/\varphi_0] , \]

where \( \sum_i R_i = R \) and \( \phi(R_1, R_2, \ldots, R_j) \) is the flux of the magnetic field through a polygon enclosed by a loop consisting of the vectors \( R_1, R_2, \ldots, R_j, -R \). The periodicity condition was the same as (3) for even \( N \) but for odd \( N \) Zak proved that the condition

\[ H = 2\frac{l}{N}\varphi_0 \]

should be satisfied. This condition implies that for even \( N \) the number of different factors is \( 2N \), whereas it equals \( N \) for odd \( N \). The result obtained by Zak agreed with Azbel’s considerations \( 10 \) who showed that wave functions had to be periodic functions of \( H \) with the period \( 2\varphi_0 \). It is worthwhile noting that Zak also worked with the antisymmetric gauge. Zak did not introduce a factor system in an explicit way (it was not necessary in his constructions for odd \( N \)) with the period \( 2\varphi_0 \). However, Zak used different representations. However, for even \( N \) Brown and Zak used different representations. However, for even \( N \) (\( b = 2 \)) we have

\[ \frac{l}{2}D_{jk}[\tau([X,Y][X,Y])] = \exp\left[2\pi i \frac{ LX}{6N} (Y + 2k)\right] \delta_{j,k-Y} , \]

where \( b = 1,2 \) for \( N \) odd and even, respectively. It is obvious that this representation corresponds to the irreducible representation \( \Gamma(\alpha) = \alpha \) of the factor group \( F \), so \( m_l(R,R') = \mu_l(R,R') \). According to (8) and (12) changes of \( H \) are related to changes of \( l \) but they were interpreted in different ways. In Brown’s considerations \( H \) determines a factor system of projective representations in a direct way — different values of \( H \) satisfying (8) lead to nonequivalent projective representations. On the contrary, Zak considered different (nonequivalent) central extensions of \( T \) with factor systems \( \mu_l \). However, Zak assumed that only the representations \( \Gamma(\alpha) = \alpha \) were physical whereas the others were rejected as nonphysical in further considerations \( 10 \). It means, according to Zak, that for a central extension with a factor system \( \mu_l \) one has to find projective representations \( lD \) with a factor system \( m_l = \Gamma(\mu_l) = \mu_l \). The same result can be obtained while considering only the factor system \( \mu_1 \) and next all irreducible representations \( \Gamma_l \) of \( F \) such that \( \Gamma_l(\mu_1) = m_l \). Thus all representations necessary in physical applications, considered by Zak as representations of different although isomorphic groups, can be obtained by use of ‘nonphysical’ representations \( \Gamma_l \) with \( l > 1 \). Nevertheless, it seems that the representations introduced by Brown \( 10 \) could be used in Zak’s approach to construct (vector) representations of \( T' \) (finite or not) according to (3). A comparison of (10) and (13) shows that for odd \( N \) Brown and Zak used different representations. However, for even \( N \) \( (b = 2) \) we have

\[ \frac{l}{2}D_{jk}[X,-Y] = \frac{l}{2}D_{jk}[\tau([X,Y][X,Y])] , \]

where the sign ‘\(-\)’ originates from a different choice of the sign of \( e \) assumed by Zak \( 10 \) (in Zak’s approach eigenvectors of \( D[1,0] \) are permuted by \( D[0,1] \) in the opposite direction than that assumed in Brown’s definition).

The third approach is based on pure group-theoretical considerations and consists in determination of all possible central extensions of a finite group \( \mathbb{Z}_N^2 \) (in general \( \mathbb{Z}_{N_1} \otimes \mathbb{Z}_{N_2} \)) by an infinite \( \langle U(1) \rangle \) or finite \( \langle C_N \rangle \) \( \{\alpha \in \mathbb{C} \mid \alpha^N = 1\} \)
group of factors $F$. It was shown, by means of the Mac Lane method, that all nonequivalent factor systems corresponding to finite magnetic translation groups can be written as

$$\mu_k([X, Y], [X', Y']) = \exp(2\pi i YX'/N)$$

with $k = 0, 1, \ldots, N - 1$. Some important facts have to be mentioned:

- This formula resembles the Landau gauge $A = [0, Hx]$; recently it has been shown that this convergence is not accidental.
- The fraction $k/N$ can be interpreted as $H/\varphi_0$ so the resulting numbers constitute a periodic function of $H$ with the period $\varphi_0$, in agreement with Brown’s result but contrary to Zak’s and Azbel’s results.
- There are no additional conditions imposed on $k$ and on $N$ (i.e., the results are valid for both odd and even $N$ and for $\gcd(k, N) > 1$).

Taking into account only parameters $k = l$ mutually prime with $N$ it can be shown that $N$-dimensional irreducible projective representations of $T_N$ (or ‘physical’ vector representations of the extension of $T_N$ by $C_N$) have the following matrix elements

$$\frac{1}{l}D_{jk}[X, Y] = \exp\left(2\pi i \frac{l}{N} X_j \right) \delta_{j,k-Y}.$$  \hspace{1cm} \text{(17)}

Representations with $\gcd(k, N) > 1$ were briefly discussed elsewhere but the difference between odd and even $N$ was not considered there.

### III. PROJECTIVE REPRESENTATIONS — STANDARD AND NORMALIZED FACTOR SYSTEMS

To compare different descriptions of the magnetic translation groups we have to discuss not only projective representations themselves but also their factor systems. To begin with we recall now some definitions related to factor systems and their properties.\cite{13} As one can see factor systems appear in the definition of a projective representation\cite{14} and in the multiplication rule for a central extension of groups\cite{18}. Factor systems $m_l$ are determined directly (as in Brown’s approach) or via factor systems $\mu_l$ for central extensions by means of the ‘physical’ representations $\Gamma(\alpha) = \alpha$ and the formula (4).

A factor system $m: T \times T \to \mathbb{C}$ has to satisfy the following condition:\cite{13, 17}

$$m(R, R')m(R + R', R'') = m(R', R'')m(R, R + R')$$

for all $R, R', R''$. A trivial factor system $t(R, R')$ is determined by any mapping $f: T \to \mathbb{C}$ according to

$$t(R, R') = f(R)f(R')/f(R + R').$$

Since $T$ is Abelian then each trivial factor system is symmetric, i.e., $t(R, R') = t(R', R)$. If

$$m'(R, R') = t(R, R')m(R, R')$$

then factor systems $m$ and $m'$ are called equivalent. Notice, however, that the projective representations determined by equivalent factor systems are nonequivalent.\cite{14} Since all factor systems for a given $T$ form an Abelian group $\Phi$ and a set of trivial factor systems $\Theta$ is its normal subgroup then elements of the factor group $M = \Phi/\Theta$ (known as the Schur multiplicator) correspond to representatives of classes of equivalent factor systems. A factor system is called standard if it satisfies

$$m(R, 0) = m(0, R) = 1, \quad \forall R.$$  \hspace{1cm} \text{(21)}

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A factor system of an \(N\)-dimensional projective representation is normalized if
\[
m(R, R') \in C_N, \quad \forall R, R'
\] (i.e., each factor is the \(N\)th root of 1).

It is well known that the Schur multiplicator of \(\mathbb{Z}_N^2\) is \(C_N \mathfrak{Z}^2\) so the factor system (13) is normalized and standard since \(m_k([0,0],[X,Y]) = 1\). Moreover, it is periodic with respect to \(Y\) and \(X'\) — the period is equal to \(N\). In particular we have \(m_k([N,0],[X,Y]) = m_k([X,Y],[N,N]) = 1\), etc. On the other hand, the factor system (14) of \(N\)-dimensional representations (11) or (13) is not normalized because some of factors do not belong to \(C_N\) but to \(C_{2N}\) instead. It also means that this system is standard, because \(m_k([0,0],[X,Y]) = 1\), but it appears that \(N\) does not serve as a period because, for example, \(m_l([0,N],[1,0]) = \exp(\pi i l) = (-1)^l\). This fact stirs up a conflict between the conditions obtained by Brown and those obtained by Zak and, moreover, leads to difficulties in studying magnetic translations for the antisymmetric gauge. Of course, one may work with factor systems (and, hence, representations) which are neither standard nor normalized, but such considerations have to be carried very carefully and results obtained have to be carefully interpreted, too. Brown and Zak did not check normalization of their factor systems and this led to ambiguity of their results [cf. (8) and (12)].

At first let us notice that Brown took into account one requirement only, namely
\[
\hat{T}(Na_j)\hat{T}(R) = \hat{T}(R)\hat{T}(Na_j),
\] (23)
i.e., that \(\hat{T}(Na_j)\) would commute with any other operator. On the other hand, Zak demanded in addition that
\[
m(Na_j, Na_k) = 1,
\] (24)
i.e., that \(\hat{T}(Na_j)\) should behave as a constant factor. As follows from (8)
\[
m_l([N,0],[0,N]) = (-1)^{ln},
\] so for odd \(N\) the magnetic field \(H\) has to be twice as high as in (8). Note that representations (17), corresponding to the Landau gauge, satisfy, for both odd and even \(N\), the following stronger condition
\[
\frac{1}{2}D(Na_j)\frac{1}{2}D(R) = \frac{1}{4}D(R + Na_j) = \frac{1}{4}D(R),
\] (25)
i.e., both \(Na_j\) and \(\hat{T}(Na_j)\) are equal to the unit element in the translation group \(T_N\) and in the group of magnetic translation operators, respectively. The condition (12) (for odd \(N\)) removes these problems but, however, leads to another question why odd and even \(N\) should be considered separately while both cases can be evidently treated as one with the use of the Landau gauge.

While considering restrictions imposed on \(H\) by the periodic boundary conditions with the Landau gauge, i.e., the standard factor system (13), one can see that the condition \(H = k\pi /N\) is sufficient. So, it seems that Brown’s approach is well-supported. Moreover, it should be noted that Zak weakened his requirements and later on he considered only Brown’s condition (14). To enlighten the problem we have to check whether the factor systems (14) and (15) are equivalent or not. At first note that the group-theoretical commutator of operators of any projective representations (of an Abelian group \(T\)) is equal to
\[
c(R, R') = D(R)D(R')D(R')^{-1} = \frac{m(R, R')}{m(R, R')},
\] (26)
then it is the same for all equivalent factor systems. Since the equivalence of factor systems means the equivalence of vector potentials (gauges) (14) then the above commutator does not depend on \(A\) but rather on \(H\) and in this sense this commutator only (not a factor system) has the physical meaning — if \(D(R)\) represents a lattice translation in the presence of a magnetic field then the commutator corresponds to a loop determined by vectors \(R, R', -R, -R'\) and its value depends on the flux through a nonprimitive, in general, cell determined by these vectors. So, Brown’s requirement (23) leading to the condition (8) was based on a reasonable assumption. However, the factor system considered was not normalized which led to a disagreement with Zak’s results.

IV. EQUIVALENCE OF FACTOR SYSTEMS AND REPRESENTATIONS

Let us consider a mapping \(f_w[X,Y] = \exp(2\pi i wXY), w \in \mathbb{R}\), which determines the following trivial factor system
\[ t_w([X, Y], [X', Y']) = \exp[-2\pi i w(XY' + YX')]. \] (27)

The factors obtained belong to \( C_N \), i.e., \( t \) is standard and normalized, if \( w = j/N \). For example for \( j = k \) the factor system (13) is transformed to

\[ \mathbf{\Pi}_k([X, Y], [X', Y']) = (\mu_k \circ t_{k/N})([X, Y], [X', Y']) = \exp(-2\pi i k XY'/N), \] (28)

which corresponds to another form of the Landau gauge \( \mathbf{\bar{A}} = [-Hy, 0] \). It is important that if \( t \) is not normalized then a new factor system \( m' = tm \) is not normalized, too. This is, however, the case which leads to the factor system (14) determined by Brown and Zak — one has to put \( w = k/2N \). This, and the previous discussion on the commutator, proves that the stronger condition introduced by Zak following from (24) is superfluous. It can be easily shown for odd \( N \), since for \( l \) mutually prime with odd \( N \) also gcd(2\( l \), \( N \)) = 1 (the mapping \( l \to 2l \) is an automorphism of \( \mathbb{Z}_N \) which changes the order of elements only). Therefore in the formulas obtained by Brown, (8)–(10), one can replace \( l < N \) in the following way

\[
\begin{cases}
  0, & \text{for } l = 0 \\
  2k = 2l', & \text{for even } l \neq 0 \\
  2k - 1 = 2(k + N') = 2l', & \text{for odd } l,
\end{cases}
\] (29)

where \( N' = (N - 1)/2 \), \( k = 1, 2, \ldots, N' \), and \( l' = 1, 2, \ldots, 2N' = N - 1 \). In this way a relation similar to (14) is obtained

\[ \{D_{jk}[X, -Y] = \frac{l}{2}D_{jk} \left[ \tau([X, Y])([X, Y]) \right], \] (30)

where \( l \) and \( l' \) are interrelated by (23). In the same way one can transform the factor system (15) into (14). If gcd\( (l, N) \) = 1 then \( l \) is replaced by \( l' \), so

\[ m_l([X, Y], [X', Y']) = \exp(2\pi i (2l')YX'/N) \] (31)

and next \( w \) is taken to be \( l'/N \). The factor system obtained

\[ t_{w_m}[X, Y], [X', Y'] = \exp[\pi i l(YX' - XY')/N] \] (32)

is exactly the same as (14). In a sense, we have performed a ‘normalization’ of the factor system used by Brown and Zak. In other words, the projective representations (10) do not satisfy the condition (24) for odd \( N \) since they are not given in normalized form. Such a form can be obtained by substitution \( l \to 2l' \) which leads to the condition (12) determined by Zak.

Anyway, this way of normalization is not possible in the case of even \( N \), since in general gcd\( (l, N) \) \( \neq \) gcd\( (2l, N) \). However, we can use a hint given by Zak, who did not exploit it in full. At the end of his paper Zak noticed that a finite magnetic translation group contains \( N^3 \) elements for odd \( N \) whereas for even \( N \) the number of elements is two times bigger. It suggests that a group considered by him was, in fact, an extension of \( T_N \) by \( C_{2N} \) — the factors obtained were not normalized since they did not belong to the multipicator of \( T_N \).

In a previous paper [13] it was shown that central extensions of \( T_N \) by \( C_{2N} \) which correspond to magnetic translation groups have factor systems

\[ 2m_k([X, Y], [X', Y']) = \exp \left( \frac{\pi i}{N} 2k XY' \right) \] (33)

with \( k = 0, 1, \ldots, N - 1 \). A mapping \( f_w \) assigns to each \([X, Y]\) an element of \( C_{2N} \) so it is well defined for \( w = k/2N \), i.e.,

\[ f_w[X, Y] = \exp \left( \frac{\pi i}{N} k XY \right). \]

Note that the product \( kXY \) is calculated modulo \( 2N \) and, therefore, \( f_w \) is a multivalued function: in \( \mathbb{Z}_N \) numbers \( X \) and \( X + N \) represent the same element, whereas

\[ f_w[X + N, Y] = (-1)^{kY} \exp \left( \frac{\pi i}{N} k XY \right) \]

is not equal to \( f_w[X, Y] \), in a general case. To calculate a trivial factor system \( t_w \) according to (19) one has to determine \( f_w(R + R') \). Let us assume, at this moment only formally, that a sum of vectors in this formula will not be calculated modulo \( N \). Then
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lattices does not depend on the parity of $N$). It means, in particular, that the antisymmetric gauge for $N = 2$ can be introduced only if one considers the $4 \times 4$ lattice with $H = 0$ (a trivial case) or $H = \varphi_0/2$.

In this work the descriptions of the magnetic translation group proposed by Brown and Zak were compared with the results obtained by means of the Mac Lane method. The first authors assumed the antisymmetric gauge, whereas the Mac Lane method led to the Landau gauge. Due to a factor $\frac{1}{2}$ in the antisymmetric gauge some problems arise when one introduces the magnetically periodic boundary conditions. More careful considerations put forward by Zak have the additional condition (12) for an odd period $N$ when one introduces the magnetically periodic boundary conditions. More careful considerations put forward by Zak have the additional condition (12) for an odd period $N$ when one introduces the magnetically periodic boundary conditions. More careful considerations put forward by Zak have the additional condition (12) for an odd period $N$ when one introduces the magnetically periodic boundary conditions.

Let us also remind that in this work $H$ in fact denotes the magnetic flux through one primitive cell. Therefore, according to (8) or (34), the total flux through the $N \times N$ lattice is equal to

$$\Phi = lN\varphi_0,$$

i.e., to an integer multiplicity of the fluxon. To introduce the antisymmetric gauge one has to consider a $(2N) \times (2N)$ lattice and even $l = 2l'$. Hence the total flux equals $\Phi = 4l'N\varphi_0$, so the flux through one $N \times N$ magnetic cell is equal to $l'N\varphi_0$, which is consistent with the previous value (13), and the flux trough one primitive cell is equal to $H_0 = l'\varphi_0/N$. On the other hand, the flux through one primitive cell of the $(2N) \times (2N)$ lattice (assuming the Landau gauge) is $H_L = l\varphi_0/(2N)$, so in general it is a half of $H_0$. Therefore we can set up the certain procedure: For a given magnetic field $H_0$ the antisymmetric gauge can be introduced if the magnetically periodic boundary conditions admit two times smaller $H_L$. For the sake of illustration let us consider the $(2N) \times (2N)$ lattice and $H = \varphi_0/N$. Then, from (13), one obtains a commutator corresponding to the primitive vectors $[1, 0]$ and $[0, 1]$ as

$$c([1, 0], [0, 1]) = \exp(-2\pi i/N).$$

The formula (9) gives the following values of the corresponding factors [see (23)]

$$1m([1, 0], [0, 1]) = \exp(-\pi i/N)$$

and

$$1m([0, 1], [1, 0]) = \exp(\pi i/N),$$

whereas (13) leads to

$$2m([1, 0], [0, 1]) = 1$$

and

$$2m([0, 1], [1, 0]) = \exp(2\pi i/N).$$

So, the flux through the primitive cell, corresponding to the commutator and independent of the gauge, was decomposed in two different ways into fluxes through ‘primitive’ triangles. However, the first decomposition (related to the antisymmetric gauge) is not possible for the minimal flux $H = \varphi_0/(2N)$. If considering any other trivial factor system (23) determined by the parameter $w \in \mathbb{R}$ one can obtain many other decompositions of the commutator into factors. It can be viewed as the decomposition of the flux through the primitive cell into fluxes through the ‘lower’ and ‘upper’ triangle. In particular, the other Landau gauge, corresponding to the factor system (34), changes roles of these triangles since one obtains

$$\overline{m}([1, 0], [0, 1]) = \exp(2\pi i/N)$$

and

$$\overline{m}([0, 1], [1, 0]) = 1.$$

Despite the fact that the physical properties are gauge-independent we have noticed that the form of the vector potential $A$ has a certain importance in the mathematical description of a system. One has to be especially very careful considering projective representations or extensions of groups, since some equivalent factor systems are neither standard nor normalized. However, it may occur that in certain applications or in other descriptions of the same problem it is more convenient to use such a form of $A$. 


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