On the realization problem of plane real algebraic curves as Hessian curves

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Abstract

The Hessian Topology is a subject having interesting relations with several areas, for instance, differential geometry, implicit differential equations, analysis and singularity theory. In this article we study the problem of realization of a real plane curve as the Hessian curve of a smooth function. The plane curves we consider are constituted either by only outer ovals or inner ovals. We prove that some of such curves are realizable as Hessian curves.

Keywords: Hessian curves, Hessian Topology, Real algebraic curves

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Introduction

It is well known that the points of a generic smooth surface in $\mathbb{R}^3$ ($\mathbb{RP}^3$) are classified as elliptic, parabolic and hyperbolic. The geometric structure of such a surface is described by the behavior of the sets conformed by these points, that is, the set of parabolic points is a smooth curve called parabolic curve whose complement is a disjoint union of elliptic and hyperbolic domains. A problem that has been studied for a long time is the description of this geometric structure, in particular the topology of the parabolic curve [13], [9], [5], [2], [3], among others. When the surface is expressed locally as the graph $z = f(x, y)$ of a smooth function $f$, the parabolic curve is the image under $f$, of the Hessian curve of $f$, which is the set of zeros of the Hessian polynomial of $f$, $\text{Hess } f := f_{xx}f_{yy} - f_{xy}^2$. Moreover, if the surface is the graph of a smooth function defined on the whole plane, then this local description is global, and so, to understand the behavior of the parabolic curve in this case, it is enough to study the Hessian curve.

Some topics concerning the Hessian Topology [1], [4] (problems 2000-1, 2000-2, 2001-1, 2002-1) are based in the study of the following problems. How many compact connected components can have the (smooth) parabolic curve of the graph of a real polynomial of degree $n$ in two variables? and, how are they distributed? [10], [11], [6], [7], [8]. Another such a problem is to determine the number of connected components of the parabolic curve of a smooth algebraic surface of fixed degree $n$ in the real projective space [14], [10], [6]. The realization problem of a smooth plane curve as the Hessian curve of a smooth function is another such a problem of this subject, which is treated here. The meaning of all these problems is to find out if the property of being Hessian curve imposes conditions on the topology and geometry of curves and surfaces.

A connected component of a compact smooth plane curve is called outer oval if it is contained in the outside component of the complement of each of the connected components of the curve. In this paper we give a family of real polynomials $\{f_k\}_{k \geq 2}$ such that the Hessian curve of $f_k$
consists exactly of \( k - 2 \) outer ovals. Jointly, in Theorem 2 we give a detailed description of the geometric structure of the graph of each one of these polynomials.

In Theorem 10 for each even natural number \( k \), we present real polynomials of degree \( k \) such that its Hessian curve is formed exactly by \( k - 2 \) concentric circles. We also describe the geometric structure of the graph of these polynomials. In an analogue form, we contribute to the odd case with a family of real functions \( \{ f_r \}_{r \geq 1} \) such that the Hessian curve of \( f_r \) are constituted by \( 2r - 1 \) concentric circles, Theorem 11.

To conclude, we show in Proposition 12, that if a smooth plane curve \( C \) is equivalent by an affine transformation on the plane to any of the previous Hessian curves, then \( C \) is also a Hessian curve.

**Preliminaries**

Points on a smooth surface (not necessarily generic) in \( \mathbb{R}^3 \) (\( \mathbb{R}P^3 \)) can be classified in terms of the maximal order of contact of the tangent lines to the surface at each point. A point \( p \) of a surface is **elliptic** if all lines tangent to the surface at \( p \) have a contact of order 2 with the surface at that point. The tangent lines having an order of contact at least three with the surface are known as **asymptotic lines**. A point \( p \) is **hyperbolic** if it has exactly two asymptotic lines. We say that an hyperbolic point is an **inflexion point** if it has at least one asymptotic line with order of contact greater than 3.

The points on the surface that are not elliptic or hyperbolic will be called **parabolic points** and the set constituted by them will be referred to as **parabolic curve**. This curve is formed by the following types of points (some of them may not appear). A **generic parabolic point** is a point which has exactly one asymptotic line and the order of contact of this line with the surface is 3. A **special parabolic point** (other authors call it Gaussian cusp or godron) is a point at which the parabolic curve is smooth locally, the 4-jet (without constant neither linear terms) of the function defining the surface at this point is not a perfect square and moreover, it has exactly one asymptotic line whose order of contact is at least 4. Finally, a **degenerate parabolic point** is a parabolic point that is not generic or special. It is important to mention that all these type of points are invariant under the action of the affine group on 3-space.

The directions determined on the \( xy \)-plane of the asymptotic lines under the projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^2, (x,y,z) \mapsto (x,y) \), are the solutions of the quadratic differential equation:

\[
 f_{xx}(x,y)dx^2 + 2f_{xy}(x,y)dxdy + f_{yy}(x,y)dy^2 = 0,
\]

where the quadratic differential form on the left will be referred to as the **second fundamental form of** \( f \). Its discriminant defined as

\[
 \Delta_{II_f} = f_{xy}^2 - f_{xx}f_{yy},
\]

allows us to characterize the type of point in the graph of \( f \). That is, \( (p,f(p)) \) is elliptic, parabolic or hyperbolic if \( \Delta_{II_f}(p) \) is negative, zero or positive, respectively.

**Results and Proofs**

1 **Outer ovals**

We recall that a connected component of a compact smooth plane curve is called **outer oval** if it is contained in the outside component of the complement of each of the connected components of the curve.
**Definition 1** A set of polynomials $l_1, \ldots, l_n \in \mathbb{R}[x, y]$ of degree one are in good position if for each $i = 1, \ldots, n$, the straight line $l_i(x, y) = 0$ contains no critical point of the function $\prod_{j \neq i} l_j$.

**Theorem 2** Let $m, n$ two natural numbers. Consider the polynomial

$$f(x, y) = \prod_{i=0}^{m} l_i(x, y) \prod_{j=0}^{n} g_j(x, y)$$

of degree $m+n+2$ given as the product of polynomials of degree one, $l_0(x, y) = y - ax$, $g_0(x, y) = y - bx$, $l_i(x, y) = x - a_i$, $g_j(x, y) = x - b_j$, with $a, b, a_i, b_j \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $a \neq b$ and $a_m < \cdots a_1 < 0 < b_1 < \cdots < b_n$. If the set of polynomials $\{l_i(x, y), g_j(x, y) : i = 0, \ldots, m, j = 0, \ldots, n\}$ are in good position, then the Hessian curve of $f$ consists exactly of $m+n$ outer ovals. Moreover, the graph of $f$ has $3(m+n)$ special parabolic points and its inflexion curve is the set $\{f(x, y) = 0\}$.

**Proof.** In order to prove that the inflexion curve is the set $\{f(x, y) = 0\}$, we state the following Remark which is a consequence of the definition of $f(x, y)$.

**Remark 3** The restriction of $f$ to any straight line $m \in \mathbb{R}^2$ is a one variable real polynomial such that all its inflexion points are non degenerated.

**Lemma 4** The inflexion curve of the graph of $f$ is the set $\{f(x, y) = 0\}$.

**Proof.** Because the order of contact of each straight line $l_i(x, y) = 0$, $i = 1, \ldots, n$, with the graph of $f$ is infinite then these lines are contained in the inflexion curve. Inversely, suppose that $p$ is a point of the inflexion curve and does not belong to the set $L := \{f(x, y) = 0\}$. So, at least one asymptotic direction of $p$ has order of contact equal to or greater than 4 with the graph of $f$. This implies that the restriction of $f$ to the line, which is the projection to the $xy$-plane of this asymptotic line, has a degenerated inflexion point. This contradicts Remark 3.

**Lemma 5** A point is in the intersection of the parabolic curve with the inflexion curve if and only if this point is a special parabolic point or a degenerated parabolic point.

**Proof.** If a point belongs to both curves, parabolic and inflexion, it has an asymptotic line with order of contact at least 4 with the surface. Then, the point is special parabolic or degenerated. Inversely, because a special parabolic point satisfies that its asymptotic line has order of contact at least 4 with the surface, it lies in the inflexion curve. It only remains to prove that any degenerated point lies in the inflexion curve. Note that a degenerated point has one or an infinite number of asymptotic lines. For the first case, if the Hessian curve is singular at this point, the asymptotic line has order of contact at least 4 with the surface, so it lies in the inflexion curve. Remaining in the first case, suppose that the Hessian curve is smooth at the point and the 4-jet (without constant neither linear terms) of the function is a perfect square. This implies that the point lies in the inflexion curve. For the case in which every tangent line at the point is an asymptotic line, we have that the second fundamental form of $f$ at the point is identically zero, that is, each tangent line has order of contact at least 4 with the surface.

**Lemma 6** The intersection of the parabolic curve with the inflexion curve consists of $3(m+n)$ points.
Proof. After some straightforward calculus we have that the restriction of the Hessian of $f$ to the straight line $x = a_i$, for $i = 1, \ldots, m$, is

$$
\text{Hess}_{x=a_i}(x,y) = -(2y - (a + b)a_i)^2 \left( \prod_{j=1}^{m} (a_i - b_j) \sum_{k=1}^{m} \prod_{s \neq k} (a_i - a_s) \right)^2
$$

where $C_i = \prod_{j=1}^{n} (a_i - b_j) \sum_{k=1}^{m} \prod_{s \neq k} (a_i - a_s)$ is a nonzero real number for each $i = 1, \ldots, m$.

Then $\text{Hess}_{x=a_i}(x,y) = 0$ if and only if $y = \frac{(a + b)a_i}{2}$, $i = 1, \ldots, m$. The points $\left( a_i, \frac{(a + b)a_i}{2} \right)$, $i = 1, \ldots, m$, are between the line $l_0(x,y) = 0$ and the line $g_0(x,y) = 0$ because $ba_i < \frac{(a + b)a_i}{2} < aa_i$. Moreover, the other points on the lines $l_i = 0$, $i = 1, \ldots, m$, are hyperbolic points.

By a similar analysis for the lines $x = b_j$ for $j = 1, \ldots, n$ we have that the points $\left( b_j, \frac{(a + b)b_j}{2} \right)$, $j = 1, \ldots, n$, are parabolic and they are in the interior of the compact segment $[ab_j, bb_j]$ of the line $x = b_j$.

Now, let us analyze the intersection of the parabolic curve with the straight lines $l_0(x,y) = 0$ and $g_0(x,y) = 0$. Replacing $y = ax$ in the Hessian polynomial of $f$ we obtain

$$
\text{Hess}(x,y)|_{y=ax} = -(a - b)^2 \left[ \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + x \left( \sum_{l=1}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + \sum_{k=1}^{n} \prod_{j \neq k}^{m} (x - b_j) \prod_{i=1}^{m} (x - a_i) \right) \right]^2.
$$

Note that the expression being inside of the square brackets is the first derivative of the one variable polynomial $g(x) = x \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j)$ of degree $m + n + 1$ which has $m + n + 1$ simple zeros and $m + n$ simple critical points. This implies that $\text{Hess}(x,y)|_{y=ax}$ has $m + n$ simple real roots.

By an analogous analysis for the straight line $y = bx$ we obtain that $\text{Hess}(x,y)|_{y=bx}$ has $m + n$ simple real roots. 

\begin{lemma}
If the set of polynomials $\{l_i(x,y), g_j(x,y) : i = 0, \ldots, m, j = 0, \ldots, n\}$ are in good position, then $\{f(x,y) = 0\}$ contains no degenerated parabolic point.
\end{lemma}

Before starting with the proof of this Lemma we state the following Remark which is obtained by a suitable grouping of terms of the Hessian polynomial of $f$.

\begin{remark}
The Hessian polynomial of $f$ can be written as

$$
\text{Hess}(x,y) = \beta(x) \left( y - \frac{(a + b)x}{2} \right)^2 + \alpha(x),
$$

where

$$
\beta(x) = \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j).
$$

\end{remark}
arguments are valid for the case that are all special points. (Lemma 7). We shall prove that the points lying in both curves, parabolic and inflexion, 
Proof. \[\alpha(x) = -(a - b)^2 \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) \]
\[+ 2x \left( \sum_{l=1}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + \sum_{k=1}^{n} \prod_{j \neq k}^{n} (x - b_j) \prod_{i=1}^{m} (x - a_i) \right) \]
\[+ \frac{x^2}{2} \left( \sum_{l=1}^{m} \sum_{s \neq l}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + \sum_{k=1}^{n} \sum_{t \neq k}^{n} \prod_{j \neq k}^{n} (x - b_j) \prod_{i=1}^{m} (x - a_i) \right) \]
\[+ 2 \sum_{k=1}^{n} \prod_{j \neq k}^{n} (x - b_j) \sum_{l=1}^{m} \prod_{i \neq l}^{m} (x - a_i) \right] \]
\[\beta(x) = 2 \left( y - \frac{(a + b)x}{2} \right)^2 \left[ \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) \left( \sum_{l=1}^{m} \sum_{s \neq l}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) \right) \right. \]
\[+ \sum_{l=1}^{m} \sum_{s \neq l}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + 2 \sum_{k=1}^{n} \prod_{j \neq k}^{n} (x - b_j) \prod_{i=1}^{m} (x - a_i) \right) \]
\[\left. - 2 \left( \sum_{l=1}^{m} \prod_{i \neq l}^{m} (x - a_i) \prod_{j=1}^{n} (x - b_j) + \sum_{k=1}^{n} \prod_{j \neq k}^{n} (x - b_j) \prod_{i=1}^{m} (x - a_i) \right) \right]. \]

Proof. (Lemma 7). We shall prove that the points lying in both curves, parabolic and inflexion, are all special points.

The next calculus shows that the second fundamental form of \( f \) at each point \( q \in \{ f(x, y) = 0 \} \) is nonzero. Let suppose that \( q \in \{ l_i(x, y) = 0 \} \) for some \( i \in \{0, \ldots, m\} \) (the following arguments are valid for the case that \( q \) belongs to some \( l_i(x, y) = 0 \)). So, \( f(x, y) = l(x, y)l_i(x, y) \) where \( l(x, y) = \prod_{k=0, k \neq i}^{m} l_k(x, y) \prod_{j=0}^{m} g_j(x, y) \). Then,
\[ f_{xx}(q) = 2 \frac{\partial l_i}{\partial x}(q) l_x(q), \quad f_{yy}(q) = 2 \frac{\partial l_i}{\partial y}(q) l_y(q), \]
\[ f_{xy}(q) = \frac{\partial l_i}{\partial y}(q) l_y(q) + \frac{\partial l_i}{\partial x} l_x(q). \]

The hypothesis of good position implies that \( q \) can not be critical point of \( l \). So, \( l_x(q) \neq 0 \) or \( l_y(q) \neq 0 \) and the second fundamental form of \( f \) at \( q \) is non degenerated. That is, each point has exactly one asymptotic line.

Now, let \( p \) be a point in the intersection of the parabolic curve with the inflexion curve. We shall prove that the Hessian curve at \( p \) is non singular. The proof will be in two cases.

- The point \( p \) lies in any of the lines \( l_0(x, y) = 0 \) or \( g_0(x, y) = 0 \). After an appropriate affine transformation of the \( xy \)-plane, we can suppose that such a line is the line \( y = 0 \). Denote by \( x_1, \ldots, x_{m+n+1} \) the intersection points of the line \( y = 0 \) with all other lines. Since \( f(x, y) = y \prod_{k=1}^{m+n+1} r_k(x, y) \), where \( r_k(x, y) \) are real polynomials of degree one, then,
\[ f_y(x, y)|_{y=0} = \prod_{k=1}^{m+n+1} r_k(x, y)|_{y=0}. \]
Define \( f_y(x,y) |_{y=0} = F(x) \). The polynomial \( F(x) \) is of degree \( m + n + 1 \) and its roots are real and simple because they are the points \( x_1, \ldots, x_{m+n+1} \). Moreover, it has \( m + n \) non degenerated critical points, that is, the function \( F'(x) = f_{xy}(x,y) |_{y=0} \) has \( m + n \) simple real zeroes: exactly one in the interior of each segment \([x_i, x_{i+1}]\), \( i = 1, \ldots, m + n \) and \( f_{xxy}(x,y) |_{y=0} \neq 0 \) at these roots. Note that

\[
(Hess f(x,y)) |_{y=0} = -(f_{xy}(x,y))^2 |_{y=0} = -(F'(x))^2.
\]

So, the point \( p \), after applying the affine transformation, has the form \( \tilde{p} = (\tilde{x}, 0) \), where \( \tilde{x} \) is one of the critical points of \( F \). Moreover, \( f_{xy}(x,y), f_{xx}(x,y) \) are zero at this point. Finally, since the second fundamental form of \( f \) at \( \tilde{p} \) is non zero and \( f_{xxy}(\tilde{p}) \neq 0 \), then

\[
\left( \frac{\partial}{\partial y} Hess f \right)(\tilde{p}) = (f_{xx} f_{yyy} + f_{xxy} f_{yy} - 2f_{xy} f_{xyy}) |_{\tilde{p}} = f_{xxy}(\tilde{p}) f_{yy}(\tilde{p}) \neq 0.
\]

- The point \( p \) lies in one of the lines \( l_1(x,y) = 0, \ldots, l_m(x,y) = 0, g_1(x,y) = 0, \ldots, g_n(x,y) = 0 \). In the proof of Lemma 6 it is shown that \( p \) has the form \( (x_0, \frac{(a+b)x_0}{2}) \) where \( x_0 \in \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \). So, by Remark 8 we have \( \frac{\partial}{\partial y} Hess f(p) = \alpha'(x_0) \neq 0 \).

Note that the asymptotic line at each one of these points is a line of the set \( \{l_i(x,y) = 0, g_j(x,y) = 0 : i = 0, \ldots, m, j = 0, \ldots, n\} \) because the parabolic curve is tangent to this set.

It only remains to prove that the 4-jet (without constant neither linear terms) of the function at these points is free square. For the points \( \left( a_i, \frac{(a+b)a_i}{2} \right) \), this assertion follows by noting that the coefficient of the \( xy^2 \) term is nonzero while the coefficient of \( y^4 \) is zero.

In order to prove the compacity of the parabolic curve consider the bounded connected components of the complement of \( \{f(x,y) = 0\} \) in the \( xy \)-plane. Take its closure of each one and the union of them. Let us denote by \( W \) the complement of this union.

**Lemma 9** The points belonging to the set \( W \) are hyperbolic.

**Proof.** We partition the set \( W \) in 4 regions which are represented in Figure 1.

Consider Region I. It was shown in the proof of Lemma 6 that the points lying in some of the straight lines \( l_i(x,y) = 0, g_j(x,y) = 0, \) with \( i = 0, \ldots, m, j = 0, \ldots, n \) of Region I (or II) are hyperbolic. So, it only remains to show that the points of Region I and belonging to the
positive or negative. That is \( f \) to this line is a one variable polynomial of degree 2 with two real roots (the intersections with the straight lines \( l_0(x, y) = 0, g_0(x, y) = 0 \)). So, \( f(q) \) and the derivative of \( f|_m \) at \( q \) are positive or negative. That is \( f|_m \) is convex or concave at \( q \), respectively.

Suppose that \( f(q) \) is positive. Denote by \( t \) the straight line tangent to the level curve \( f(x, y) = f(q) \) at \( q \). Because the derivative of \( f|_t \) is zero, the line \( t \) is different from the line \( m \). The function \( f|_t \) has a non degenerated critical point at \( q \) which is a maximum. So, the function \( f|_t \) is concave at \( q \). The two previous conclusions imply that \( (q, f(q)) \) is a hyperbolic point. The case \( f(q) < 0 \) is analogous.

The analysis for Region II is similar.

Now, we consider Region III which is the unbounded connected component determined by the lines \( x = b_n, y = ax \) and \( y = bx \). Let \( q \) be a point in such a set. Consider the straight lines \( m, l \) parallels to the lines \( g_0(x, y) = 0, l_0(x, y) = 0 \), respectively, and passing through \( q \). The lines \( m, l \) divide the \( xy \)-plane in four sectors, one of which, denoted by \( A \), is totally contained in Region III. Take a straight line \( t \) such that it passes through \( q \), it divides the sector \( A \) in two unbounded sectors and it is not parallel to any line of \( L \). So, the roots of \( f|_t \) are in the left side of \( q \).

If \( f(q) > 0 \), then the derivative of \( f|_t \) at \( q \) is positive. So \( f|_t \) is convex at \( q \). On the other hand, consider the straight line passing through \( q \) which is parallel to the \( y \)-axis. Denote it by \( r \). So, \( f|_r \) is concave at \( q \) because it is a one variable polynomial of degree two and \( f(q) > 0 \). Then, the point \( (q, f(q)) \) is hyperbolic. The case for Region IV is similar.

Since there is one special parabolic point at each compact segment of the straight lines \( g_0(x, y) = 0, l_0(x, y) = 0 \), there is at least one connected component of the Hessian curve on each compact polygon delimited by the lines \( g_i(x, y) = 0, l_i(x, y) = 0, i = 0, \ldots, m, j = 0, \ldots, n \). Because there are \( m + n \) such as compact polygons, then there are at least \( m + n \) connected components of the Hessian curve. Now, the goal is to prove that the Hessian curve has exactly \( m + n \) connected components. In fact, we shall show that the Hessian curve has at most \( 2(m + n) \) vertical tangent lines since this implies that the Hessian curve has at most \( m + n \) connected components.

Let us prove it. Consider the expression of the Hessian polynomial of \( f \) given in Remark \( 8 \). The parabolic points having a vertical tangent line to the parabolic curve are given by

\[
\left\{ \frac{\partial}{\partial y} \text{Hess} f(x, y) = 0 \right\} \cap \left\{ \text{Hess} f(x, y) = 0 \right\}.
\]

Note that if \( x_0 \) is a real root of the polynomial \( \beta(x) \) then \( \alpha(x_0) \neq 0 \) because in another case the points of the line \( x = x_0 \) would be parabolic and it contradicts Lemma \( 9 \). So, the parabolic points having a vertical tangent line are the points \( \left( x_0, \frac{(a+b)x_0}{2} \right) \) such that \( x_0 \) is a real root of \( \alpha(x) \). The proof concludes by noting that \( \alpha(x) \) is a one variable real polynomial of degree \( 2(m + n) \).

### 2 Concentric circles. Even case

**Theorem 10** Consider the real polynomial of degree \( 2n \)

\[
f(x, y) = \prod_{i=1}^{n} (x^2 + y^2 - m_i^2)
\]
with $0 < m_1 < m_2 < \cdots < m_n$, $m_i \in \mathbb{R}, i = 1, \ldots, n$. Then, its Hessian curve is non singular and it is formed by $2(n - 1)$ concentric circles. Moreover, the unbounded connected component of the complement of the Hessian curve on the $xy$-plane is elliptic.

**Proof.** After a straightforward calculus the Hessian polynomial of $f$ is

$$\text{Hess}(f) = 4s(x, y) t(x, y),$$

where

$$s(x, y) = \sum_{j=1}^{n} \prod_{i \neq j} (x^2 + y^2 - m_i^2) \quad \text{and} \quad t(x, y) = \sum_{j=1}^{n} \prod_{i \neq j} (x^2 + y^2 - m_i^2) + 2(x^2 + y^2) \sum_{j=1}^{n} \sum_{i \neq j} \prod_{i \neq j, l} (x^2 + y^2 - m_i^2).$$

Note that the graph of the Hessian of $f$, $z - \text{Hess}f(x, y) = 0$, is the rotation about the $z$-axis of the graph of the function $4\tilde{s}(x)\tilde{t}(x)$, where

$$\tilde{s}(x) = \sum_{j=1}^{n} \prod_{i \neq j} (x^2 - m_i^2) \quad \text{and} \quad \tilde{t}(x) = \sum_{j=1}^{n} \prod_{i \neq j} (x^2 - m_i^2) + 2(x^2) \sum_{j=1}^{n} \sum_{i \neq j} \prod_{i \neq j, l} (x^2 - m_i^2).$$

So, the solution curves of $s(x, y) = 0$ ($t(x, y) = 0$) correspond to the non negative real roots of $\tilde{s}(x)$ ($\tilde{t}(x)$). We shall prove now that $\tilde{s}(x)$ and $\tilde{t}(x)$ have $n - 1$ simple positive real roots, each one. Define the one variable real polynomial

$$\tilde{f}(x) := \prod_{i=1}^{n} (x^2 - m_i^2).$$

The degree of this polynomial is $2n$, it has $2n$ non zero simple roots, $2n - 1$ non degenerated critical points and $2n - 2$ simple inflexion points. Because

$$\tilde{f}_x(x) = 2x \prod_{j=1}^{n} (x^2 - m_i^2) = 2x \tilde{s}(x),$$

then $\tilde{s}(x)$ has $2n - 2$ non zero simple real roots from which, $n - 1$ are positive by symmetry. Moreover, since

$$\tilde{f}_{xx}(x) = 2 \sum_{j=1}^{n} \prod_{i \neq j} (x^2 - m_i^2) + 4x^2 \sum_{j=1}^{n} \sum_{i \neq j} \prod_{i \neq j, l} (x^2 - m_i^2) = \tilde{t}(x),$$

then $\tilde{t}(x)$ has $2n - 2$ simple real roots (different from the roots of $\tilde{s}(x)$). Because $\tilde{f}$ is symmetric with respect to the origin, then $\tilde{t}(x)$ has $n - 1$ positive roots. Moreover, the unbounded connected component of the complement of the Hessian curve on the $xy$-plane is elliptic because the functions $\tilde{s}$ and $\tilde{t}$ are positive if $x > m_n$. ■
3 Concentric circles. Odd case

Theorem 11 Consider the real valued differentiable function

\[ f(x, y) = \prod_{k=1}^{n} \frac{x^2 + y^2 - k^2}{x^2 + y^2 + 1}. \]

Its Hessian curve is non singular and it is formed by \(2n - 1\) concentric circles. Moreover, the unbounded connected component of the complement of the Hessian curve on the \(xy\)-plane is hyperbolic.

Proof. After a straightforward computation the Hessian polynomial of \(f\) is

\[ \text{Hess} f(x, y) = \frac{4s(x, y) t(x, y)}{(x^2 + y^2 + 1)^{2n+3}}, \]

where

\[ s(x, y) := \sum_{j=1}^{n} \left( (j^2 + 1) \prod_{i \neq j} (x^2 + y^2 - i^2) \right), \]

\[ t(x, y) := (1 - 3x^2 - 3y^2) \sum_{j=1}^{n} \left( (j^2 + 1) \prod_{i \neq j} (x^2 + y^2 - i^2) \right) + \]

\[ + 2(x^2 + y^2) \sum_{j=1}^{n} \left( (j^2 + 1) \sum_{l \neq j} \left( (l^2 + 1) \prod_{k \neq j, l} (x^2 + y^2 - k^2) \right) \right). \]

Note that the graph of the Hessian polynomial of \(f\), is the rotation about the \(z\)-axis of the graph of the function \(\frac{4s(x) t(x)}{(x^2 + 1)^{2n+3}}\), where

\[ \tilde{s}(x) = \sum_{j=1}^{n} \left( (j^2 + 1) \prod_{k \neq j} (x^2 - k^2) \right), \quad \tilde{t}(x) = (1 - 3x^2) \sum_{j=1}^{n} \left( (j^2 + \]

\[ + 1) \prod_{k \neq j} (x^2 - k^2) \right) + 2x^2 \sum_{j=1}^{n} \left( (j^2 + 1) \sum_{l \neq j} \left( (l^2 + 1) \prod_{k \neq j, l} (x^2 - k^2) \right) \right). \]

We analyze now the zeros of \(\tilde{s}(x)\) and \(\tilde{t}(x)\). Consider the differentiable function \(\tilde{f} : \mathbb{R} \to \mathbb{R}\)

\[ \tilde{f}(x) = \prod_{k=1}^{n} \frac{x^2 - k^2}{x^2 + 1}. \]

This function has exactly \(2n\) simple zeros from which \(n\) are positive. Moreover,

\[ \tilde{f}_x(x) = \frac{2x}{(x^2 + 1)^{n+1}} \sum_{j=1}^{n} \left( (j^2 + 1) \prod_{i \neq j} (x^2 - i^2) \right) = \frac{2x \tilde{s}(x)}{(x^2 + 1)^{n+1}}, \]

\[ \tilde{f}_{xx}(x) = \frac{2}{(x^2 + 1)^{n+2}} \left[ (1 - 3x^2) \sum_{j=1}^{n} \left( (j^2 + 1) \prod_{i \neq j} (x^2 - i^2) \right) + \]

\[ + 2x^2 \sum_{j=1}^{n} \left( (j^2 + 1) \sum_{l \neq j} \left( (l^2 + 1) \prod_{k \neq j, l} (x^2 - k^2) \right) \right) \right] = \frac{2 \tilde{t}(x)}{(x^2 + 1)^{n+2}}. \]
Since $\tilde{f}(x)$ has at least $2n - 1$ critical points and the degree of $\tilde{s}(x)$ is $2n - 2$ then, the polynomial $\tilde{s}(x)$ has exactly $2n - 2$ simple roots different from zero ($n - 1$ are positive). So, the set $s(x, y) = 0$ is a disjoint union of $n - 1$ concentric circles.

Because the critical points of $\tilde{f}(x)$ are non degenerated and it is symmetric respect to the origin, then $\tilde{f}(x)$ has at least $n - 1$ positive inflexion points in the interval $(-n, n)$, exactly one point between two consecutive critical points. That is, $\tilde{t}(x)$ has $n - 1$ positive real roots in the interval $[0, n)$.

Denote by $x_0$ the critical point of $\tilde{f}(x)$ lying in the interval $(n - 1, n)$. Note that the sign of $\tilde{s}(n - 1) = ((n - 1)^2 + 1) \prod_{k \neq n - 1}^{n} ((n - 1)^2 - k^2)$ is negative while the sign of $\tilde{s}(n) = (n^2 + 1) \prod_{i=1}^{n-1} (n^2 - i^2)$ is positive. From these remarks and because the function $\tilde{f}$ has exactly one critical point at each interval $(i, i+1)$, $i = 1, \ldots, n - 1$, we have that $x_0$ is a minimum. Moreover, $\tilde{t}(x_0)$ is positive. After a straightforward calculation, it can be seen that $\tilde{t}(n^2)$ is negative. So, by the main value Theorem, there exists a real number $c \in (x_0, n^2)$ such that $\tilde{t}(c) = 0$. It implies that $\tilde{t}(x)$ has $n$ positive real roots. That is, the set $t(x, y) = 0$ is a disjoint union of $n$ concentric circles different from the circles described by the equation $s(x, y) = 0$. We conclude that the Hessian curve of $f$ consists of $2n - 1$ concentric circles.

**Proposition 12** If a smooth plane curve $C$ is equivalent by an affine transformation of the plane, to a Hessian curve, then $C$ is also a Hessian curve.

**Proof.** Suppose that the curve $C$ is defined by $g(x, y) = 0$, where $g$ is a smooth real valued function defined on the plane. By hypothesis, the curve $C$ is equivalent by an affine transformation to a Hessian curve, that is, $g(x, y) = (\text{Hess } f) \circ T(x, y)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation of the plane whose determinant (of the linear part) is denoted by $J$. After a straightforward calculation, we have that

$$\text{Hess } \left( \frac{f \circ T(x, y)}{J} \right) = (\text{Hess } f) \circ T(x, y).$$

So, the curve $C$ is also a Hessian curve. ■

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