Multiple scattering of plane compressional waves by two cylindrical core-shell inclusions with surface effect

Sihe Wang, Dongxia Lei, and Zhiying Ou
Multiple scattering of plane compressional waves by two cylindrical core-shell inclusions with surface effect

Sihe Wang, Dongxia Lei, and Zhiying Ou

AFFILIATIONS
School of Science, Lanzhou University of Technology, Lanzhou 730050, People’s Republic of China

Electronic mail: zhiyingou@l63.com

ABSTRACT
Elastic wave propagation at nanoscale exhibits some special properties due to surface/interface effect. Scattering of plane compressional waves (P-wave) by two nanocylindrical core-shell inclusions in an elastic solid medium is investigated in this study. The wave fields of the core-shell structure are given by the eigenfunction expansion method and Graf addition theorem. The effect of factors such as surface energy, center distance, and thickness of the liner under different incident wave frequencies has been discussed. The results show that as the radius of the core-shell inclusions shrinks to nanometers, surface energy becomes a dominant factor that affects the scattering of elastic waves. The interaction between two core-shell inclusions in multiple scattering phenomena is discussed at the same time.

I. INTRODUCTION

The scattering of elastic waves by heterogeneous bodies in elastic media plays a very important role in the industry, such as mineral exploration, and in nanostructured devices and piezoelectric materials. The applications of sonar and explosion technologies all rely on the scattering principle of elastic waves. Pao and Mow discussed the scattering of elastic waves by elliptic cylinders and the dynamic stress concentration factor (DSCF) around inhomogeneous media with the wave function expansion method. Jain and Kanwal applied the wave function expansion method to study scattering of elastic waves by cylindrical inclusions and sphere inclusions. Datta and Shah studied the scattering of elastic waves by shallow cylindrical holes in semi-infinite space by matching the progressive method. Liu and Wang introduced the conformal transformation method to derive the dynamic response of a two-layer closely spaced circular tunnel to a simple harmonic wave in a plane.

With the increasing demand for nanomaterials, the size of components is reaching nanometers. Nanostructures possess different physical, optical, and mechanical properties from macroscale structures. Surface effect can be ignored only when the size of the structure is much greater than the atomic scale. At the nanoscale, the ratio of the interface area to the volume of the material is relatively high, and the effect of interface stress on the size-dependent behavior of materials is much more significant. However, the classical theory of continuum mechanics merely assesses the influence of surface/interface energy on the response of solids under elastic wave loading. In order to solve this problem, Gurtin et al. came up with a continuum surface model (the Gurtin-Murdoch model), considering the interface stress to reflect the surface effect with a mathematical approach and laid a theoretical foundation for the research and development of surface elasticity. This theory proves that surface elasticity is effective for investigating the mechanical behavior of nanomaterials.

In recent years, the influence of the inclusion of surface energy on the mechanical behavior of inhomogeneous nanomaterials has been extensively studied with the Gurtin-Murdoch model. Miller and Shenoy, and Dingreville and Qu analyzed the significance of surface effect on the size-dependent behavior of nanostructured components. Wang et al. studied the multiple scattering of plane harmonic compressional waves by circular holes at the nanoscale using the wave function expansion method. On this basis, in 2009, Wang applied the Graf addition theorem to the model to obtain...
the expression of wave field, studied the multiple scattering of plane P-waves by two nanocylindrical holes in an elastic solid at different frequencies, and revealed the influence of the distance between two holes and the surface effect on the DSCF around the holes in the multiple scattering phenomenon. In 2012, Ou and Lee applied the eigenfunction expansion method to study the scattering of P-waves and SV-waves by a single nanosized coated fiber with surface effect lying in an infinite plane. The characteristics of the DSCF related to inhomogeneous material constants, interface stress, and thickness of the coating were obtained. Describing the model with the mapping relation of the multipolar coordinate system, Ou and Lee studied the scattering of plane compression waves by the interaction of two cylindrical inclusions with surface effect. The DSCF near the inclusion is related to both the size of the inclusion and the distance between inclusions. Only the separation between the two inclusions is relatively long, and the interaction of the two inclusions can be neglected, which is consistent with the scattering of waves by a single inclusion. Fang et al. and Yang et al. studied dynamic stress concentration caused by the scattering of cylindrical nano-inclusion waves in half-space using the method of simulating right-angled edges with equivalent large circular arcs. They found that the interface effect around inclusions depends not only on the properties of materials and the shape of voids/inclusions but also on the surface properties and radius size of voids/inclusions. Qiang et al. applied the plane wave expansion method, adopting the scattering waves from the separated cylindrical nanoholes. The effect of thickness of the layer and wavenumber on wave propagation was studied with Stroh formalism by Qian et al. in 2019. The results revealed that the thickness of the layer had a great impact on the scattering of SH and lambda waves in matrix composite materials.

Some achievements were possible in the field of scattering of elastic waves by macroscopic inhomogeneous media and defects (inclusions, cavities, and cracks) with surface effect. Elastic wave scattering has been extensively studied in single inclusion scattering, single hole scattering, and multi-inclusion scattering, but, as far as we know, the influence of multi-core-shell structure on elastic wave scattering has not been studied sufficiently.

This paper mainly investigates the multiple scattering of plane P-waves by two cylindrical core-shell inclusions with interface effect. Considering the interfacial stress at the two interfaces of the core-shell inclusions, the solution for the elastic field near the core-shell inclusions caused by incident P-waves is derived using the wave function expansion method and Graf addition theorem under the framework of surface elasticity theory. The DSCF at the inner and outer interfaces of the core-shell inclusions is obtained by numerical calculations. The description of the model and the statement of the problem are given in Sec. II, and boundary conditions are determined and are given in Sec. III. The results and discussion are given in Sec. IV, and a review of surface elasticity and detailed description of boundary conditions based on surface elasticity are given in the Appendix.

II. STATEMENT OF THE PROBLEM

Based on surface elasticity theory, considering the scattering of plane P-waves by two cylindrical core-shell inclusions with radius \( a_1 \) and \( b_1 (\eta = 1, 2) \) embedded in an infinite matrix, as shown in

\[ a_2 = a_1 + h \quad \text{and} \quad b_2 = b_1 + h, \]

where \( h \) is the thickness of the core-shell inclusions. The distance between the two core-shell inclusions is \( 2d \), and \( L_1 \) and \( L_2 \) indicate the inner and outer interfaces, respectively. The elastic constants and the densities of the inclusion, liner, and matrix are \( X_j, \mu_j \), and \( \rho_j (j = 1, 2, 3) \), where the superscripts 1, 2, and 3 represent the quantities associated with the inclusion, liner, and matrix, respectively. Referring to the global coordinate system \((x, y)\), the \( y \)-axis passes through the center of the two core-shell inclusions, and the origin \( O \) is located at the midpoint of the two centers. For convenience, two local polar coordinate systems \((r_\eta, \theta_\eta)\) are established, as shown in Fig. 1, and the plane strain condition in the \( x, y \) plane is assumed to be \( \sigma_{zz} = \psi (\sigma_{xx} + \sigma_{yy}) \).

Assuming that an incident P-wave is propagating in the positive \( x \) direction (Fig. 1), in terms of the Bessel function, in the local polar coordinate system \((r_\eta, \theta_\eta)\), the incident P-wave can be expanded as

\[ \psi^{(s)}(\theta_\eta) = \varphi_0 (\theta_\eta) \sum_{n=0}^{\infty} A_n^{(s)} f_\eta (a_1 (\theta_\eta)) \cos n\theta_\eta, \quad (1) \]

where \( \varphi_0 \) is the amplitude of the incident wave, \( \xi_n \) is the Neumann factor \((\xi_n = 1, \text{for } n = 0 \text{ and } \xi_n = 2, \text{for } n > 0)\), \( f_\eta (x) \) is the \( n \)th order Bessel function of the first kind, \( a_1^{(s)} = \omega / c_\eta^{(s)} \) is the wave number of the matrix, and \( c_\eta^{(s)} = \sqrt{(\lambda^{(s)} + 2\mu^{(s)})/\rho^{(s)}} \) is the velocity of the P-wave in the matrix. The time-dependent term \( e^{-\imath \omega t} \) is suppressed throughout.

When the incident wave encounters the two core-shell inclusions, the P-wave and SV-wave are reflected from each core-shell inclusion at \( L_2 \). For scattered waves, the displacement potentials are

\[ \psi^{(s)}(\theta_\eta) = \varphi_0 (\theta_\eta) \sum_{n=0}^{\infty} A_n^{(s)} H_\eta^{(1)} (a_1 (\theta_\eta)) \cos n\theta_\eta, \quad (2) \]

\[ \psi^{(s)}(\theta_\eta) = \varphi_0 (\theta_\eta) \sum_{n=0}^{\infty} B_n^{(s)} H_\eta^{(1)} (\beta_1 (\theta_\eta)) \sin n\theta_\eta, \quad (3) \]
where $A_n^{(i)}$ and $B_n^{(i)}$ can be derived from the equations under boundary conditions of $L_1$ and $L_2$, $H_n^{(i)}(\cdot)$ is the $n$th order Hankel function of the first kind, $\psi_n^{(i)} = \omega/c_{\psi}^{(i)}$ is the shear wave number of the matrix, and the velocity of the shear wave in the matrix is $c_{\psi}^{(i)} = \sqrt{\mu^{(i)}/\rho^{(i)}}$.

The P-wave and SV-wave transmitted to the inclusions are as follows:

$$\psi_1^{(i)} = \int_0^\infty \sum_{n=0}^\infty A_n^{(i)} J_n(\alpha^{(1)} r_n) \cos n\theta_n, \quad (4)$$

$$\psi_2^{(i)} = \int_0^\infty \sum_{n=0}^\infty B_n^{(i)} J_n(\beta^{(1)} r_n) \sin n\theta_n, \quad (5)$$

where $A_n^{(i)}$ and $B_n^{(i)}$ can be derived from the equations under boundary conditions of $L_1$ and $L_2$, $\alpha^{(1)} = \omega/c_{\alpha}^{(1)}$ and $\beta^{(1)} = \omega/c_{\beta}^{(1)}$, where $c_{\psi}^{(1)} = \sqrt{\lambda^{(1)} + 2\mu^{(1)}}/\rho^{(1)}$ and $c_{\alpha}^{(1)} = \sqrt{\mu^{(1)}/\rho^{(1)}}$.

The P-wave and SV-wave reverberating in the inclusions are given as follows:

$$\psi_3^{(i)} = \int_0^\infty \sum_{n=0}^\infty A_n^{(i)} J_n(\alpha^{(1)} r_n) + C_n^{(i)} H_n^{(1)}(\alpha^{(1)} r_n) \cos n\theta_n, \quad (6)$$

$$\psi_4^{(i)} = \int_0^\infty \sum_{n=0}^\infty B_n^{(i)} J_n(\beta^{(1)} r_n) + D_n^{(i)} H_n^{(1)}(\beta^{(1)} r_n) \sin n\theta_n, \quad (7)$$

where $A_n^{(i)}, B_n^{(i)}, C_n^{(i)},$ and $D_n^{(i)}$ can be derived from the equations under boundary conditions of $L_1$ and $L_2$, $H_n^{(1)}(\cdot)$ is the $n$th order Hankel function of the second kind, $c_{\alpha}^{(2)} = \omega/c_{\alpha}^{(2)}$ and $\beta^{(2)} = \omega/c_{\beta}^{(2)}$, where $c_{\psi}^{(2)} = \sqrt{\lambda^{(2)} + 2\mu^{(2)}}/\rho^{(2)}$ and $c_{\alpha}^{(2)} = \sqrt{\mu^{(2)}/\rho^{(2)}}$.

Using the Graf addition theorem, [19,22] the following functions give the mapping relationship between two local polar coordinate systems:

$$H_n^{(1)}(kr_1)e^{i\theta_1} = \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2kd)m(kr_1)e^{im\theta_1}, \quad (8)$$

$$H_n^{(1)}(kr_1)e^{i\theta_1} = \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2kd)m(kr_1)e^{im\theta_1}. \quad (9)$$

Therefore, in $(r_1, \theta_1)$, in terms of Eq. (8), the scattering waves of the core-shell inclusion 2 shown in Fig. 1 can be written as

$$\psi_1^{(2)} = \int_0^\infty \sum_{n=0}^\infty A_n^{(3)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\alpha^{(3)} d)m(\alpha^{(1)} r_2) \cos m\theta_2, \quad (10)$$

$$\psi_2^{(2)} = \int_0^\infty \sum_{n=0}^\infty B_n^{(3)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\beta^{(3)} d)m(\beta^{(1)} r_2) \sin m\theta_2. \quad (11)$$

In a coordinate system $(r_1, \theta_1)$, using Eqs. (2), (3), (10), and (11), the total wave field in the matrix near core-shell inclusion 1 can be written as

$$\psi_1^{(3)} = \psi_1^{(1)} + \psi_1^{(2)} + \psi_2^{(2)}$$

$$= \int_0^\infty \sum_{n=0}^\infty \xi_n^{(1)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\alpha^{(1)} d)m(\alpha^{(1)} r_2) \cos m\theta_2$$

$$+ \int_0^\infty \sum_{n=0}^\infty A_n^{(3)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\alpha^{(3)} d)m(\alpha^{(3)} r_2) \cos m\theta_2,$$

$$\psi_2^{(3)} = \psi_2^{(1)} + \psi_2^{(2)}$$

$$= \int_0^\infty \sum_{n=0}^\infty \xi_n^{(1)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\beta^{(1)} d)m(\beta^{(1)} r_2) \sin m\theta_2$$

$$+ \int_0^\infty \sum_{n=0}^\infty B_n^{(3)} \sum_{m=-\infty}^{\infty} i^{m-n}H_m^{(1)}(2\beta^{(3)} d)m(\beta^{(3)} r_2) \sin m\theta_2.$$
III. BOUNDARY CONDITIONS WITH SURFACE/INTERFACE STRESS

Substituting Eq. (A7) into Eq. (A8) and then into Eq. (A5), we can use total displacement potentials to express the stress component as follows (referring to the Appendix):

\[
\begin{align*}
\sigma_{\alpha\beta}(r) & = \lambda \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{r \partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \\
& + 2\mu \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \\
\sigma_{\alpha \beta \delta}(r) & = \mu \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right] \\
& + 2\mu \left[ \frac{1}{r^2} \frac{\partial \phi}{\partial r} + \frac{1}{r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\partial \theta} \right]
\end{align*}
\]

(26)

According to the requirement of continuity of displacement, in a coordinate system \((r_1, \theta_1)\), at interfaces \(L_1(r_1 = a_1)\) and \(L_2(r_1 = a_2)\),

\[
\begin{align*}
u^{(1)}_1 & = u^{(1)}_2, \quad \nu^{(1)}_1 = u^{(1)}_2, \quad r_1 = a_1, \\
u^{(2)}_2 & = u^{(2)}_1, \quad \nu^{(2)}_2 = u^{(2)}_1, \quad r_1 = a_2.
\end{align*}
\]

(27a)

(27b)

In a coordinate system \((r_2, \theta_2)\), at interfaces \(L_1(r_2 = b_1)\) and \(L_2(r_2 = b_2)\),

\[
\begin{align*}
u^{(1)}_1 & = u^{(1)}_2, \quad \nu^{(1)}_1 = u^{(1)}_2, \quad r_2 = b_1, \\
u^{(2)}_2 & = u^{(2)}_1, \quad \nu^{(2)}_2 = u^{(2)}_1, \quad r_2 = b_2.
\end{align*}
\]

(28a)

(28b)

With the interface effect, the equilibrium equations can be deduced (referring to the Appendix),

\[
\begin{align*}
\sigma^{(1)}_{r_1 r_1} - \sigma^{(2)}_{r_1 r_1} & = \frac{\sigma^{(1)}_{\theta_1 \theta_1}}{a_1} - \frac{\sigma^{(2)}_{\theta_1 \theta_1}}{a_1} - \frac{1}{a_1} \frac{\partial \sigma^{(1)}_{\theta_1 \theta_1}}{\partial r_1}, r_1 = a_1, \\
\sigma^{(3)}_{r_1 r_1} - \sigma^{(4)}_{r_1 r_1} & = \frac{\sigma^{(3)}_{\theta_1 \theta_1}}{a_2} - \frac{\sigma^{(4)}_{\theta_1 \theta_1}}{a_2} - \frac{1}{a_2} \frac{\partial \sigma^{(3)}_{\theta_1 \theta_1}}{\partial r_1}, r_1 = a_2, \\
\sigma^{(5)}_{r_1 r_1} - \sigma^{(6)}_{r_1 r_1} & = -\frac{1}{b_1} \frac{\partial \sigma^{(5)}_{\theta_1 \theta_1}}{\partial b_1}, r_2 = b_1, \\
\sigma^{(7)}_{r_1 r_1} - \sigma^{(8)}_{r_1 r_1} & = -\frac{1}{b_2} \frac{\partial \sigma^{(7)}_{\theta_1 \theta_1}}{\partial b_2}, r_2 = b_2.
\end{align*}
\]

(29a)

(29b)

(29c)

(29d)

Substituting Eq. (A9) into Eq. (A10) and then into Eqs. (29a)–(29d) (referring to the Appendix), the boundary conditions at \(L_1\) are

\[
\begin{align*}
\sigma^{(1)}_{r_1 r_1} - \sigma^{(2)}_{r_1 r_1} & = S_1(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_1} - S_1 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_1}, r_1 = a_1, \\
\sigma^{(3)}_{r_1 r_1} - \sigma^{(4)}_{r_1 r_1} & = -S_1(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_1} + S_1 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_1}, r_1 = a_1, \\
\sigma^{(5)}_{r_1 r_1} - \sigma^{(6)}_{r_1 r_1} & = S_2(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} - S_2 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_1 = a_2, \\
\sigma^{(7)}_{r_1 r_1} - \sigma^{(8)}_{r_1 r_1} & = -S_2(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} + S_2 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_1 = a_2.
\end{align*}
\]

(30a)

(30b)

(30c)

(30d)

and at \(L_2\), we have

\[
\begin{align*}
\sigma^{(1)}_{r_2 r_2} - \sigma^{(2)}_{r_2 r_2} & = S_1(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} - S_1 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_2 = b_1, \\
\sigma^{(3)}_{r_2 r_2} - \sigma^{(4)}_{r_2 r_2} & = -S_1(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} + S_1 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_2 = b_1, \\
\sigma^{(5)}_{r_2 r_2} - \sigma^{(6)}_{r_2 r_2} & = S_2(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} - S_2 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_2 = b_2, \\
\sigma^{(7)}_{r_2 r_2} - \sigma^{(8)}_{r_2 r_2} & = -S_2(1 - v^{(1)}) \frac{\partial \phi^{(1)}}{\partial r_2} + S_2 v^{(1)} \frac{\partial \phi^{(1)}}{\partial r_2}, r_2 = b_2.
\end{align*}
\]

(31a)

(31b)

(31c)

(31d)

where

\[
S_1 = \frac{2\mu + \lambda'}{2\mu' + \lambda}, \quad S_2 = \frac{2\mu + \lambda'}{2\mu' + \lambda}; \quad S_1' = \frac{2\mu + \lambda}{2\mu' + \lambda'}, \quad S_2' = \frac{2\mu + \lambda}{2\mu' + \lambda'}.}
\]

(32)

where \(s_f = 1, 2, 1', \text{ and } 2'\) are a dimensionless quantity that represents surface effect at the nanoscale. Molecular dynamic simulation shows that the absolute values of \(\mu'/\mu(2)\) and \(\lambda'/\mu(2)\) are on the order of smaller nanometers. From Eq. (32), we can see that for a macroscopic inclusion, if the values of \(r_1\) and \(r_2\) are very large, then \(s_f = 1, 2, 1', \text{ and } 2'\) \(\ll 1\), so the effect of the interfaces is rather small. As the radius of the core-shell inclusions shrinks to the nanoscale, \(s_f, s_2, s_{1'}, s_{1''}\) and \(s_{2'}\) become relatively great, and the effects of the interfaces should be considered. With the boundary conditions in Eqs. (27a), (27b), (28a), (28b), (30a)–(30d), and (31a)–(31d), we can get a linear algebraic equation system to determine the coefficients \(A^{(0)}_{\alpha}, B^{(0)}_{\alpha}, C^{(0)}_{\alpha}\), and \(D^{(0)}_{\alpha}\) \((j = 1, 2, 3; n = 0, 1, 2, 3, \ldots, N)\). From the results of the equation system, we got the value of \(A^{(0)}_{\alpha}, B^{(0)}_{\alpha}, C^{(0)}_{\alpha}\), and \(D^{(0)}_{\alpha}\) \((j = 1, 2, 3; n = 0, 1, 2, 3, \ldots, N)\), and the stress fields under the incident P-wave can be calculated. The truncation number \(N\) is governed by the following inequality:
As \( \alpha = 3 \) is the soft material contains a hard liner, and the other case \( \alpha = 0.3 \) is the soft liner in the hard material.

When the distance between the two core-shell inclusions is much longer than the radius of the core-shell inclusions, the surface effect between the two core-shell inclusions can be neglected, so the stress field around the two cylindrical core-shell inclusions is the same as that around a single core-shell inclusion. It has been verified that the results are consistent with those in Ref. 20, which proves the validity of this study.

A. Low frequency incident wave with \( \alpha = 0.1 \)

For an incident wave with low frequency with \( \alpha = 0.1 \), the DSCF is symmetrical to the \( y \)-axis. The DSCF along the core-shell inclusions for different values of \( s_z \) is shown in Figs. 2(a) and 2(b). For \( d/a = 1.25 \), the situation with a small distance, the significance of the interaction effect is relatively large. For a hard liner \( (\mu^* = 3) \), as shown in Fig. 2(a), as \( s_z \) increases, the DSCF increases considerably to around \( \theta_1 = \pm \frac{\pi}{2} \), while it slightly decreases to around \( \theta_1 = 0 \) and \( \theta_1 = \pm \pi / 2 \). For a soft liner \( (\mu^* = 0.3) \), as shown in Fig. 2(b), it can be seen that the change of the \( s_z = 0.5 \) curve on \( \theta_1 = -\pi / 2 \) to \( \theta_1 = \pi \) is quite different from the other three curves; the \( s_z = 0.5 \) curve has a peak at \( \theta_1 = 0 \) and has a larger DSCF at \( \theta_1 = \pm \pi / 2 \). In Figs. 2(a) and 2(b), it has been shown that the DSCF of a hard liner is smaller than that of a soft liner at interface \( L_2 \) without any interface stress \( (s_z = 0) \). Also, the effect of interface energy on the DSCF near two core-shell inclusions has a close tendency compared with the previous results for a single core-shell inclusion.

For separations with different parameters between the two core-shell inclusions, Figs. 3(a) and 3(b) show the DSCF around core-shell inclusion 1. In the case where the distance between the two core-shell inclusions is large enough (greater than 10\( a \)), the interaction effect can be neglected.

Figures 4(a) and 4(b) show the DSCF at interface \( L_2 \) for various values of liner thickness \( h \) under \( s_z = 0.5 \) for \( \mu^* = 3 \) and \( \mu^* = 0.3 \). For a hard liner \( (\mu^* = 3) \), as shown in Fig. 4(a), as the thickness of the liner increases, the DSCF goes down dramatically when near \( \theta_1 = \pm \frac{\pi}{2} \) and slightly increases when near \( \theta_1 = 0 \) and \( \theta_1 = \pm \pi / 2 \); the maximum value of the DSCF usually appears at \( \theta_1 = \pm \frac{\pi}{2} \) for

\[
\frac{\xi_{n+1}}{\sum_{n=0}^{N} \xi_{n+1} A_n^{(3)} A_{n+1}^{(3)}} < 10^{-6},
\]

IV. RESULTS AND DISCUSSION

The inhomogeneity of solid materials is a key factor that affects wave scattering, and dynamic stress concentration will occur at the deformation of defective materials. Based on these assumptions, choosing the DSCF around the core-shell inclusion interfaces as an indicator represents the effect of surface energy on multiple scattering.

We define the DSCF as

\[
\text{DSCF} = \left| \frac{\sigma_{0,0}^{(3)}}{\sigma_{0,0}^{(1)}} \right|,
\]

where \( \sigma_{0,0}^{(3)} \) is the bulk stress of the material in a matrix of core-shell inclusion 1. \( \sigma_{0,0}^{(3)} = -\rho^{(3)} [(\rho^{(3)} - \rho^{(2)}) \phi_0] \) represents the stress intensity distribution along the wave motion path.

Dynamic stress concentration caused by scattering of a non-uniform single core-shell inclusion has been discussed in detail by Ou and Lee. 20 This paper focuses on the interaction and surface energy between two core-shell inclusions for incident waves of various frequencies. The DSCF depends on the interface parameters \( s_1, s_2, s_1', \) and \( s_2', \) liner thickness \( h, \) shear modulus ratio \( \mu_1^* = \mu_2^* = \mu_1 \) and \( \mu_2^* = \mu_2 / \mu_3^* \), Poisson ratio \( \nu_1 = \nu_2 = \nu_3 \), and number ratio of compression waves \( a_1^* = a_2^* / a_1^* \) and \( a_2^* = a_2^* / a_1^* \).

If the two core-shell inclusions have the same radius, \( a_1 = b_1 \) and \( a_2 = b_2 \), then \( s_1 = s_1' \) and \( s_2 = s_2' \). Obviously, the problem is symmetry of the \( x \)-axis, so we only need to consider the scattering field around a core-shell inclusion; hence, we chose core-shell inclusion 1 here. To simplify the calculation, it is assumed that the properties of the inclusion and matrix are the same, but different from the liner. Thus, we let \( \mu_1^* = \mu_2^* = \mu^* \) and \( a_1^* = a_2^* = a^* \). The first case \( \mu^* = 3 \) is the soft material contains a hard liner, and the other case \( \mu^* = 0.3 \) is the soft liner in the hard material.
different thicknesses of the liner. However, for a soft liner ($\mu^* = 0.3$), Fig. 4(b) shows the irregular change of liner thickness on the stress concentration at interface $L_2$; the peak value of the DSCF does not always appear at $\theta_1 = \pm \pi/2$; however, when $h = 1.15$, it appears at $\theta_1 = 0$ and $\theta_1 = \pm \pi$.

B. High frequency incident wave with $\alpha^{(3)} R_2 = \pi$

For an incident wave with high frequency with $\alpha^{(3)} R_2 = \pi$, the scattering phenomenon is the leading element, and the DSCF is still symmetrical to the $y$-axis. Figures 5(a) and 5(b) show the DSCF along $L_2$ near core-shell inclusion 1, which is quite different from that of the case of an incident wave under low frequency. Multiple peaks of the DSCF appeared along $L_2$, and the difference is caused by the interference effect of the incident and reflected waves. For an incident wave with high frequency, the maximum DSCF appears at $\theta_1 = \pm \pi$, which is shown in Fig. 5(a). Figure 5(b) shows the case with a soft liner; as $s_2$ goes up, the DSCF decreases almost across the whole range compared with the case of $s_2 = 0$. 

**FIG. 3.** DSCF along $L_2$ under different values of $d/a$ with $\alpha^{(3)} R_2 = 0.1, s_2 = 0.5, h = 1.10$. (a) $\mu^* = 3, \alpha^* = 1.5$. (b) $\mu^* = 0.3, \alpha^* = 0.7$.

**FIG. 4.** DSCF along $L_2$ under different values of $h$ with $\alpha^{(3)} R_2 = 0.1, s_2 = 0.5, d/a = 1.25$. (a) $\mu^* = 3, \alpha^* = 1.5$. (b) $\mu^* = 0.3, \alpha^* = 0.7$.

**FIG. 5.** DSCF along $L_2$ under different values of $s_2$ with $\alpha^{(3)} R_2 = \pi, d/a = 1.25$, $h = 1.10$. (a) $\mu^* = 3, \alpha^* = 1.5$. (b) $\mu^* = 0.3, \alpha^* = 0.7$. 

\[ \text{AIP Advances} \ 9, 125011 (2019); doi: 10.1063/1.5112056 \]

© Author(s) 2019
Figures 6(a) and 6(b) show the DSCF near core-shell inclusion 1 for various distances between the two core-shell inclusions. For an incident wave with high frequency, the multiple scattering field was quite sensitive to the distance between the two core-shell inclusions. As the distance between the core-shell inclusions is larger than $50a$, stress distribution becomes similar to the case of a single core-shell inclusion.

The distributions of the DSCF along interface $L_2$ with different values of $h$ are shown in Figs. 7(a) and 7(b) under $s_2 = 0.5$ for $\mu^* = 3$ and $\mu^* = 0.3$. As shown in Fig. 7(a), the maximum DSCF appears at $\theta_1 = \pm \pi$ for a hard liner ($\mu^* = 3$). As shown in Fig. 7(b), the maximum value of the DSCF appears at $\theta_1 = \pm 0.7\pi$ for a soft liner ($\mu^* = 0.3$). Comparing Figs. 4(a) and 4(b) with Figs. 7(a) and 7(b), the results show that for an incident wave with low frequency, the DSCF changes regularly with the change in liner thickness; in other words, it shows a close connection with thickness because it depends largely on static loading. However, for an incident wave with high frequency, there are multiple peak values; with the change in thickness, the effect of multiple scattering on the DSCF varies dramatically and irregularly. DSCF is dependent irregularly on the thickness because the change of thickness on a small scale affects the propagating ability of the waves to reach the inclusion.

The results indicate that size effect and surface/interface energy have a great impact on the interaction of elastic waves and solid media, which affect the design criterion of nanosized parts inside the microelectrical mechanical system (MEMS). Although controlling the thickness of a nanobeam has proved to be an effective method in controlling the stiffness of a system, the response of the two core-shell inclusion systems is irregular under dynamic loading driven by harmonic waves, which would cause difficulties in calibration of the system. The response of two core-shell inclusion systems with various distances provides a better response pattern when considering the designing of sensors.

V. SUMMARY

In the frame of surface elasticity theory, the multiple scattering of the plane compressive wave by two nanocylindrical core-shell inclusions in an elastic solid and the dynamic stress concentration near the core-shell inclusion interface at different frequencies are studied. The displacement and stress fields of inhomogeneous materials are calculated using the eigenfunction expansion method and Graf addition theorem. According to the related boundary conditions and the orthogonality of triangular functions, the model can be described by linear equations. Intercepting the finite terms, the following conclusions are obtained through numerical analysis:

(1) When the radius of the core-shell inclusions decreases to the nanoscale, the interfacial effect becomes remarkable and should be considered. The interface elasticity tends to inhibit the DSCF near the core-shell inclusions, whether it is a low frequency incident wave or high frequency incident wave;
with the increase of $s_2$, the value of the DSCF almost decreased steadily in the full range.

(2) For an incident wave with high frequency, the distance of the two core-shell inclusions have a dramatic effect on the propagation of the wave, and the relatively close distance causes greater dynamic stress concentration around the outer interface. Along with the increase of the distance, the dynamic stress concentration declined. However, for an incident wave with lower frequency, the effect of distance on dynamic stress concentration is not as significant as that of the wave with high frequency.

(3) For waves at a low frequency and high frequency range, the influence of liner thickness has a different effect on the DSCF. For an incident wave with low frequency, the DSCF shows a very regular dependence on the thickness of the liner, but for a high frequency incident wave, the effect of thickness change on the DSCF is irregular.

This research helps in providing theoretical support for preparation and optimization of the designing of composites containing nanoinhomogeneity.

ACKNOWLEDGMENTS

The research work is supported by the National Science Foundation (Grant Nos. 11362009 and 11862014).

APPENDIX: BOUNDARY CONDITIONS BASED ON SURFACE ELASTICITY

The surface stress tensor $\sigma'_{\alpha\beta}$ is connected to surface energy density $\Gamma(\epsilon_{\alpha\beta})$ by\(^{(1)}\)

$$\sigma'_{\alpha\beta} = \Gamma \delta_{\alpha\beta} + \frac{\partial \Gamma}{\partial \epsilon_{\alpha\beta}}. \quad (A1)$$

The simplification of isotropic surfaces is often used to get an accurate enough result.\(^{(1,14)}\) The surface stress $\sigma'_{\alpha\beta}$ on the isotropic surface is given by\(^{(14)}\)

$$\sigma'_{\alpha\beta} = r^0 \delta_{\alpha\beta} + 2(\mu' - r^0) \delta_{\alpha 1} \epsilon_{1\beta} + (\lambda' + r^0) \epsilon_{1\beta} \delta_{\alpha\beta}. \quad (A2)$$

For an ideal interface, the equilibrium equations on the boundaries can be written as\(^{(14)}\)

$$t_a + \sigma'_{\alpha\beta} n_a = 0, \quad (\sigma_0) n_a n_j = \sigma'_{\alpha\beta} n_a n_j. \quad (A3)$$

The equilibrium equations and the isotropic constitutive relationships are given by

$$\sigma_{ij} = \frac{\partial^2 u_i}{\partial y^2}, \quad \epsilon_{ij} = \frac{1}{2} \left( u_{ij} + u_{ji} \right). \quad (A4)$$

$$\sigma = 2\mu \left( \epsilon_{ij} + \frac{\nu}{1 - 2\nu} \epsilon_k \delta_{ij} \right). \quad (A5)$$

The strain tensor is connected to the displacement vector $u$ by

For the plane strain problem, the displacement potentials can be determined by

$$u_{\epsilon_k} = \frac{1}{\eta_k} \frac{\partial \varphi}{\partial \eta_k} + 1 \frac{\partial \varphi}{\partial r_k}, \quad u_{\theta_k} = \frac{1}{\eta_k} \frac{\partial \varphi}{\partial \theta_k} - \frac{\partial \varphi}{\partial r_k}, \quad u_z = 0. \quad (A7)$$

The geometric relations are

$$\epsilon_{\epsilon_k \epsilon_k} = \frac{\partial u_{\epsilon_k}}{\partial \eta_k}, \quad \epsilon_{\theta_k \theta_k} = \frac{1}{\eta_k} \frac{\partial u_{\theta_k}}{\partial \eta_k} + \frac{u_{\epsilon_k}}{r_k}, \quad (A8)$$

$$\epsilon_{\epsilon_k \theta_k} = \frac{1}{2} \left( \frac{1}{\eta_k} \frac{\partial u_{\epsilon_k}}{\partial \eta_k} \right) + \frac{u_{\theta_k}}{r_k} + \frac{\partial u_{\theta_k}}{\partial \eta_k}. \quad (A9)$$

The boundary conditions on the core-shell inclusions to interpret surface effects have been formulated. The interface strain component $\epsilon_{\theta_k \theta_k}$ can be obtained from the bulk stresses as follows:

$$\epsilon_{\theta_k \theta_k} = \frac{1}{2\mu} \left( 1 - \nu \right) \sigma_{\theta_k \theta_k} - \nu \sigma_{\epsilon_k \epsilon_k}. \quad (A10)$$

REFERENCES

1. Z. Yang, G. Jiang, C. Sun, T. Tang, and Y. Yang, “The role of soil anisotropy on SH wave scattering by a lined circular elastic tunnel in an elastic half-space soil medium,” Soil Dyn. Earthquake Eng. 125, 105721 (2019).
2. X. Q. Fang, C. S. Zhu, J. X. Liu, and X. L. Liu, “Surface energy effect on free vibration of nano-sized piezoelectric double-shell structures,” Physica B 529, 41–56 (2018).
3. Y. X. Zhen, “Wave propagation in fluid-conveying viscoelastic single-walled carbon nanotubes with surface and nonlocal effects,” Physica E 86, 275–279 (2017).
4. M. Pang, Y. Q. Zhang, and W. Q. Chen, “Transverse wave propagation in viscoelastic single-walled carbon nanotubes with small scale and surface effects,” J. Appl. Phys. 117, 024305 (2015).
5. Y. H. Pao, C. C. Mow, and J. D. Achenbach, “Diffraction of elastic waves and dynamic stress concentration,” J. Appl. Mech. 40, 872 (1973).
6. D. L. Jain and R. P. Kanwal, “Scattering of elastic waves by linearly viscoelastic obstacles,” J. Appl. Mech. 50, 4087–4109 (1973).
7. D. L. Jain and R. P. Kanwal, “Scattering of elastic waves by an elastic sphere,” Int. J. Eng. Sci. 18, 829–839 (1980).
8. S. K. Datta and A. H. Shah, “Scattering of SH waves by embedded cavities,” Wave Motion 4, 265–283 (1982).
9. Q. J. Liu and R. Y. Wang, “Dynamic response of twin closely-spaced circular tunnels to harmonic plane waves in a full space,” Tunnelling Underground Space Technol. 32, 212–220 (2012).
10. M. E. Gurtin and A. L. Murdoch, “A continuum theory of elastic material surfaces,” Arch. Ration. Mech. Anal. 57, 291–323 (1975).
11. M. E. Gurtin and A. L. Murdoch, “Surface stress in solids,” Int. J. Solids Struct. 14, 431–440 (1978).
12. M. E. Gurtin, J. Weissmüller, and F. Larché, “A general theory of curved deformable interfaces in solids at equilibrium,” Philos. Mag. A 78, 1093–1109 (1998).
13 T. Chen, G. J. Dvorak, and C. C. Yu, "Size-dependent elastic properties of unidirectional nano-composites with interface stresses," Acta Mech. 188, 39–54 (2006).
14 P. Sharma, S. Ganti, and N. Bhave, "Effect of surfaces on the size-dependent elastic state of nano-inhomogeneities," Appl. Phys. Lett. 82, 535–537 (2003).
15 S. Sahmani, M. Bahrami, and M. M. Aghdam, "Surface stress effects on the postbuckling behavior of geometrically imperfect cylindrical nanoshells subjected to combined axial and radial compressions," Int. J. Mech. Sci. 100, 1–22 (2015).
16 R. E. Miller and V. B. Shenoy, "Size-dependent elastic properties of nanosized structural elements," Nanotechnology 11, 139–147 (2000).
17 R. Dingreville and J. Qu, "Surface free energy and its effect on the elastic behavior of nano-sized particles, wires and films," J. Mech. Phys. Solids 53, 1827–1854 (2005).
18 G. F. Wang, T. J. Wang, and X. Q. Feng, "Surface effects on the diffraction of plane compressional waves by a nanosized circular hole," Appl. Phys. Lett. 89, 231923 (2006).
19 G. F. Wang, "Multiple diffraction of plane compressional waves by two circular cylindrical holes with surface effects," J. Appl. Phys. 105, 013507 (2009).
20 Z. Y. Ou and D. W. Lee, "Effects of interface energy on scattering of plane elastic wave by a nano-sized coated fiber," J. Sound Vib. 331, 5623–5643 (2012).
21 Z. Y. Ou and D. W. Lee, "Effects of interface energy on multiple scattering of plane compressional waves by two cylindrical fibers," Int. J. Appl. Mech. 04, 1250040 (2012).
22 X. Q. Fang, L. L. Zhang, and J. X. Liu, "Dynamic stress around a cylindrical nano-inhomogeneity with an interface in a half-plane under anti-plane shear waves," Appl. Phys. A 106, 625–633 (2011).
23 Q. Yang, J. X. Liu, and X. Q. Fang, "Dynamic stress in a semi-infinite solid with a cylindrical nano-inhomogeneity considering nanoscale microstructure," Acta Mech. 223, 879–888 (2012).
24 W. Dingreville and J. Qu, "Surface free energy and its effect on the elastic behavior of nano-sized particles, wires and films," J. Mech. Phys. Solids 53, 1827–1854 (2005).
25 G. F. Wang, T. J. Wang, and X. Q. Feng, "Surface effects on the diffraction of plane compressional waves by a nanosized circular hole," Appl. Phys. Lett. 89, 231923 (2006).
26 Z. Y. Ou and S. D. Pang, "A screw dislocation interacting with a coated nano-inhomogeneity incorporating interface stress," Mater. Sci. Eng. A 528, 2762–2775 (2011).
27 J. Niiranen, V. Balobanov, J. Kiendl, and S. Hosseini, "Variational formulations, model comparisons and numerical methods for Euler–Bernoulli micro- and nano-beam models," Math. Mech. Solids 24, 312–335 (2019).