Estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations of the classes of \((\psi, \beta)\)–differentiable functions

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Abstract

We obtain the exact-order estimates for approximations by Fourier sums, best approximations and best orthogonal trigonometric approximations in metrics of spaces \(L_s\), \(1 \leq s < \infty\), of classes of \(2\pi\)–periodic functions, whose \((\psi, \beta)\)–derivatives belong to unit ball of the space \(L_\infty\).

We denote by \(L_p\), \(1 \leq p < \infty\), the space of \(2\pi\)–periodic functions \(f : \mathbb{R} \to \mathbb{C}\), summable to the power \(p\) on \([0, 2\pi]\), in which the norm is given by the formula \(\|f\|_p = \left(\frac{2\pi}{0} |f(t)|^p dt \right)^{\frac{1}{p}}\); and we denote by \(L_\infty\) the space of \(2\pi\)–periodic measurable and essentially bounded functions \(f : \mathbb{R} \to \mathbb{C}\) with the norm \(\|f\|_\infty = \operatorname{ess sup}_t |f(t)|\).

Let \(f : \mathbb{R} \to \mathbb{R}\) be the function from \(L_1\), whose Fourier series has the form

\[\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx},\]

where \(\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt}dt\) are Fourier coefficients of the function \(f\), \(\psi(k)\) is an arbitrary fixed sequence of real numbers and \(\beta\) is a fixed real number. Then, if the series

\[\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{\psi(|k|)} e^{i(kx + \frac{\beta}{2}\pi \text{sign}k)}\]

is the Fourier series of some function \(\varphi\) from \(L_1\), then this function is called the \((\psi, \beta)\)–derivative of the function \(f\) and denoted by \(f^\psi_\beta\). A set of functions \(f\), whose \((\psi, \beta)\)–derivatives exist is denoted by \(L^\psi_\beta\) (see \([1]\)).

If \(f \in L^\psi_\beta\) and, at the same time, \(f^\psi_\beta \in \mathfrak{N}\), where \(\mathfrak{N} \subseteq L_1\), then we say that the function \(f\) belongs to the class \(L^\psi_\beta \mathfrak{N}\). By \(B_{R,p}\) we denote the balls of the radius \(R\) of real–valued functions from \(L_p\), i.e., the sets

\[B_{R,p} := \{\varphi : \mathbb{R} \to \mathbb{R}, \|\varphi\|_p \leq R\}, \quad R > 0, \quad 1 \leq p \leq \infty.\]
In present paper as \( \mathfrak{N} \) we take the unit balls \( B_{1,p} \). Herewith, the functional classes \( L^\psi_{\beta,p} B_{1,p} \) are denoted by \( L^\psi_{\beta,p} \).

In the case \( \psi(k) = k^{-r}, r > 0 \), the classes \( L^\psi_{\beta,p} \) are well–known Weyl–Nagy classes \( W^r_{\beta,p} \).

For functions \( f \) from classes \( L^\psi_{\beta,p} \) we consider: \( L_s \)–norms of deviations of the functions \( f \) from their partial Fourier sums of order \( n - 1 \), i.e., the quantities

\[
\|\rho_n(f; \cdot)\|_s = \|f(\cdot) - S_{n-1}(f; \cdot)\|_s, \quad 1 \leq s \leq \infty,
\]

where

\[
S_{n-1}(f; x) = \sum_{k=-n+1}^{n-1} \hat{f}(k) e^{ikx};
\]

best orthogonal trigonometric approximations of the functions \( f \) in metric of space \( L_s \), i.e., the quantities of the form

\[
e^\perp_m(f)_s = \inf_{\gamma_m} \|f(\cdot) - S_{\gamma_m}(f; \cdot)\|_s, \quad 1 \leq s \leq \infty,
\]

where \( \gamma_m, m \in \mathbb{N} \), is an arbitrary collection of \( m \) integer numbers, and

\[
S_{\gamma_m}(f; x) = \sum_{k \in \gamma_m} \hat{f}(k) e^{ikx};
\]

and best approximations of the functions \( f \) in space \( L_s \), i.e., the quantities of the form

\[
E_n(f)_s = \inf_{t_{n-1} \in T_{2n-1}} \|f - t_{n-1}\|_s, \quad 1 \leq s \leq \infty,
\]

where \( T_{2n-1} \) is the subspace of all trigonometric polynomials \( t_{n-1} \) with real coefficients of degrees not greater than \( n - 1 \).

We set

\[
\mathcal{E}_n(L^\psi_{\beta,p})_s = \sup_{f \in L^\psi_{\beta,p}} \|\rho_n(f; \cdot)\|_s, \quad 1 \leq p, s \leq \infty,
\]

\[
e^\perp_n(L^\psi_{\beta,p})_s = \sup_{f \in L^\psi_{\beta,p}} e^\perp_n(f)_s, \quad 1 \leq p, s \leq \infty,
\]

\[
E_n(L^\psi_{\beta,p})_s = \sup_{f \in L^\psi_{\beta,p}} E_n(f)_s, \quad 1 \leq p, s \leq \infty.
\]

The following inequalities follow from given above definitions (4)–(6)

\[
E_n(L^\psi_{\beta,p})_s \leq \mathcal{E}_n(L^\psi_{\beta,p})_s, \quad e^\perp_{2n-1}(L^\psi_{\beta,p})_s \leq \mathcal{E}_n(L^\psi_{\beta,p})_s, \quad 1 \leq p, s \leq \infty.
\]

In present paper we solve the problem about finding the exact order estimates of the quantities \( \mathcal{E}_n(L^\psi_{\beta,\infty})_s, E_n(L^\psi_{\beta,\infty})_s \) and \( e^\perp_n(L^\psi_{\beta,\infty})_s \) for \( 1 \leq s < \infty, \beta \in \mathbb{R} \).

For the Weyl–Nagy classes the exact order estimates of the quantities \( \mathcal{E}_n(W^r_{\beta,p})_s \) and \( E_n(W^r_{\beta,p})_s \) are known for all admissible values of parameters \( r, p, s \) and \( \beta \), i.e., for \( r > \max \{ \frac{1}{p} - \frac{1}{s}, 0 \}, \beta \in \mathbb{R} \) and \( 1 \leq p, s \leq \infty \) (see, e.g., [2, p. 47–49]). What concerning the
best orthogonal trigonometric approximations \( e_n^\psi(W_{\beta,p})_s \), so order estimates are known for them (see [3]–[9]) for various (but not for all possible) values of the parameters \( r, p, s \) and \( \beta \).

Order estimates of the quantities (4)–(6) under certain restrictions for the parameters \( r, p, s \) and \( \beta \) were established in the works [1], [10]–[20]. However, the case \( p = \infty \), \( 1 \leq s \leq \infty \) for some or another reasons hasn’t been investigated yet.

We denote by \( P \) the set of positive, almost decreasing sequences \( \psi(k), k \geq 1 \), (we remind, that sequence \( \psi(k) \) almost decreases, if there exists a positive constant \( M \) such that for arbitrary \( k_1 \leq k_2 \) the following inequality is satisfied \( \psi(k_2) \leq M \psi(k_1) \)) such that

\[
\sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\psi_n(k+1) - \psi_n(k)| \leq K \psi(n),
\]

where

\[
\psi_n(k) = \begin{cases} 
0, & k < n, \\
\psi(k), & k \geq n,
\end{cases}
\]

and \( K \) is the quantity uniformly bounded with respect to \( n \).

**Theorem 1.** Let \( \psi \in P \), \( 1 \leq s < \infty \) and \( \beta \in \mathbb{R} \). Then

\[
E_n(L_{\beta,\infty}^\psi)_s \asymp \mathcal{E}_n(L_{\beta,\infty}^\psi)_s \asymp \psi(n). \tag{8}
\]

Here and in what follows, we write \( A(n) \asymp B(n) \) for positive sequences \( A(n) \) and \( B(n) \) to denote that there are positive constants \( K_1 \) and \( K_2 \) such that \( K_1 B(n) \leq A(n) \leq K_2 B(n) \), \( n \in \mathbb{N} \).

**Proof.** At first let’s prove that the following inequality is true

\[
\mathcal{E}_n(L_{\beta,\infty}^\psi)_s \leq K^{(1)} \psi(n), \quad 1 \leq s < \infty. \tag{9}
\]

In inequality (9) and henceforth by \( K^{(i)}, i = 1, 2, ... \) we denote quantities uniformly bounded with respect to \( n \).

If \( f \in L_{\beta,\infty}^\psi \), then

\[
\|f_{\beta}^\psi\|_s \leq (2\pi)^{\frac{1}{2}} \|f_{\beta}^\psi\|_\infty \leq (2\pi)^{\frac{1}{2}}, \tag{10}
\]

and so, it is obviously that

\[
L_{\beta,\infty}^\psi \subset L_{\beta}^\psi B(2\pi)^{\frac{1}{2}}, \subset L_{\beta}^\psi L_s, \quad 1 \leq s < \infty. \tag{11}
\]

The following proposition follows from the theorem 6.7.1 in [1].

**Proposition 1.** Let \( 1 < s < \infty \), \( \psi \in P \), \( f \in L_{\beta}^\psi L_s \) and \( \beta \in \mathbb{R} \). Then for arbitrary \( n \in \mathbb{N} \) there exists a positive constant \( K \), which is uniformly bounded with respect to \( n \) and \( f \) and such that

\[
\|\rho_n(f; x)\|_s \leq K \psi(n) E_n(f_{\beta}^\psi)_s. \tag{12}
\]

Taking into account (10), (11) and in view of proposition 1, we obtain the following estimates

\[
\mathcal{E}_n(L_{\beta,\infty}^\psi)_s \leq \mathcal{E}_n(L_{\beta}^\psi B(2\pi)^{\frac{1}{2}},_s) \leq (2\pi)^{\frac{1}{2}} K \psi(n), \quad 1 < s < \infty. \tag{13}
\]

Thus, the inequalities (9) are proved for \( 1 < s < \infty \).
Let’s show the rightness of correlation (9) for \( s = 1 \). We use the following statement (see, e.g., [2, p. 8]).

**Proposition 2.** Let \( 1 \leq q \leq p \leq \infty \). On this if \( f \in L_p \), then \( f \in L_q \) and

\[
\|f\|_q \leq (2\pi)^{\frac{1}{2} - \frac{1}{q}} \|f\|_p. \tag{14}
\]

By using (14) for \( q = 1, p = 2 \) and inequality (13) for \( s = 2 \), we obtain

\[
\mathcal{E}_n(L^\psi_{\beta,\infty})_1 = \sup_{f \in L^\psi_{\beta,\infty}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_1 \leq (2\pi)^{\frac{1}{2}} \sup_{f \in L^\psi_{\beta,\infty}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_2 = (2\pi)^{\frac{1}{2}} \mathcal{E}_n(L^\psi_{\beta,\infty})_2 \leq K^{(1)} \psi(n). \tag{15}
\]

The rightness of the inequality (9) follows from (13) and (15).

To obtain the lower bound of the quantity \( E_n(L^\psi_{\beta,\infty})_s \), we consider the following function

\[
f_1(t) = f_1(\psi; n; t) = \psi(n) \cos nt.
\]

It is obviously, that \( f_1 \in L^\psi_{\beta,\infty} \) and \( f_1 \perp \iota_{n-1} \) for arbitrary \( \iota_{n-1} \in \mathcal{T}_{2n-1} \). Therefore

\[
\int_{-\pi}^{\pi} (f_1(t) - \iota_{n-1}(t)) \cos nt \, dt = \int_{-\pi}^{\pi} f_1(t) \cos nt \, dt = \pi \psi(n) \quad \forall \iota_{n-1} \in \mathcal{T}_{2n-1}. \tag{16}
\]

On the other hand, taking into account the proposition 2 for \( q = 1, p = s \), we get

\[
\int_{-\pi}^{\pi} (f_1(t) - \iota_{n-1}(t)) \cos nt \, dt \leq \|f_1 - \iota_{n-1}\|_1 \leq (2\pi)^{1 - \frac{1}{s}} \|f_1 - \iota_{n-1}\|_s, \quad 1 \leq s \leq \infty, \quad \forall \iota_{n-1} \in \mathcal{T}_{2n-1}. \tag{17}
\]

In view of (16)–(17) we arrive at the inequalities

\[
E_n(L^\psi_{\beta,\infty})_s \geq E_n(f_1)_s = \inf_{\iota_{n-1} \in \mathcal{T}_{2n-1}} \|f_1 - \iota_{n-1}\|_s \geq \frac{1}{2} \psi(n), \quad 1 \leq s \leq \infty. \tag{18}
\]

Theorem 1 is proved.

We denote by \( B \) the set of positive sequences \( \psi(k), k \in \mathbb{N} \), for each of which there exists a positive constant \( K \) such that \( \psi(k) \leq K, k \in \mathbb{N} \). The sequences \( \psi(k) = k^{-r}, r > 0, \psi(k) = \ln^{-\varepsilon}(k + 1), \varepsilon > 0 \), etc. are representatives of the set \( B \).

**Theorem 2.** Let \( \psi \in P \cap B, 1 \leq s < \infty \) and \( \beta \in \mathbb{R} \). Then

\[
e_n(L^\psi_{\beta,\infty})_s \propto e_n(L^\psi_{\beta,\infty})_s \propto \psi(n). \tag{19}
\]

**Proof.** It is follows from the formulas (7) and (9), that under the conditions of the theorem 1, next inequalities are true

\[
e_n(L^\psi_{\beta,\infty})_s \leq \psi_{n-1}(L^\psi_{\beta,\infty})_s \leq e_n(L^\psi_{\beta,\infty})_s \leq K^{(1)} \psi(n). \tag{20}
\]
Now we determine a lower bound of the quantity $e_{2n}^\psi(L_{\beta}^{\psi}s)$. For this we use the well-known result of Rudin–Shapiro (see, e.g., lemma 6.32.1 in [21]).

**Proposition 3.** There exists sequence of numbers $\{\varepsilon_k\}_{k=0}^{\infty}$, such that $\varepsilon_k = \pm 1$ and

$$\left\| \sum_{k=0}^{m} \varepsilon_k e^{ikx} \right\|_{\infty} \leq 5\sqrt{m+1}, \quad m = 0, 1, \ldots$$  \hspace{1cm} (21)

Taking into account proposition 3 for $m = 2n - 1$, we choose the sequence of numbers $\{\xi_k\}_{k=0}^{\infty}$, $\xi_k = \pm 1$ such that

$$\left\| \sum_{k=0}^{2n-1} \xi_k e^{ikx} \right\|_{\infty} \leq 5\sqrt{2n}.$$  \hspace{1cm} (22)

We set

$$\psi(0) := \psi(1)$$

and consider the function

$$f_2(t) = f_2(\psi; n; t) := \frac{1}{10\sqrt{2n} + 2} \sum_{k=-2n+1}^{2n-1} \xi_k |\psi(k)| e^{ikt}.$$  \hspace{1cm} (23)

Since, according to definition of $(\psi, \beta)$–derivative and the inequality (22),

$$\left\| (f_2)^{\psi}_{\beta} \right\|_{\infty} = \frac{1}{10\sqrt{2n} + 2} \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(kt + \frac{\beta \pi}{2})} + \sum_{k=1}^{2n-1} \xi_k e^{i(-kt - \frac{\beta \pi}{2})} \right\|_{\infty} \leq \frac{1}{10\sqrt{2n} + 2} \left( \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(kt + \frac{\beta \pi}{2})} \right\|_{\infty} + \left\| \sum_{k=1}^{2n-1} \xi_k e^{i(-kt - \frac{\beta \pi}{2})} \right\|_{\infty} \right) = \frac{1}{5\sqrt{2n} + 1} \left\| \sum_{k=1}^{2n-1} \xi_k e^{ikt} \right\|_{\infty} \leq 1,$$

so $f_2 \in L_{\beta}^{\psi}_{\infty}$.

We consider the quantity

$$I = \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} (f_2(t) - S_{\gamma_{2n}}(f_2; t)) \sum_{k=-2n+1}^{2n-1} \xi_k e^{ikt} dt \right|.$$

By virtue of Holder’s inequality, proposition 2 and correlation (22) for $1 \leq s < \infty$, $rac{1}{s} + \frac{1}{s'} = 1$

$$I \leq \inf_{\gamma_{2n}} \left\| f_2(t) - S_{\gamma_{2n}}(f_2; t) \right\|_{s} \left\| \sum_{k=-2n+1}^{2n-1} \xi_k |\psi(k)| e^{ikt} \right\|_{s'} = \left( \int_{-\pi}^{\pi} \left| (f_2)^{\psi}_{\beta} (f_2) \right|^{s'} dt \right)^{\frac{1}{s'}} \left( \sum_{k=-2n+1}^{2n-1} \xi_k |\psi(k)| e^{ikt} \right)_{s'} \leq$$

$$\leq \left( \int_{-\pi}^{\pi} \left| (f_2)^{\psi}_{\beta} (f_2) \right|^{s'} dt \right)^{\frac{1}{s'}} \left( \sum_{k=-2n+1}^{2n-1} \xi_k |\psi(k)| e^{ikt} \right)_{s'} \leq 2(2\pi)^{\frac{1}{s'}} e_{2n}^\psi(f_2)_s \left\| \sum_{k=-2n+1}^{2n-1} \xi_k |\psi(k)| e^{ikt} \right\|_{\infty} \leq$$

$$\leq$$
\[
\leq 2\pi e^{\frac{1}{2n}(f_2)_s} \left( \left\| \sum_{k=0}^{2n-1} \xi_ke^{ikt} \right\|_{\infty} + \left\| \sum_{k=1}^{2n-1} \xi_ke^{-ikt} \right\|_{\infty} \right) \leq \\
\leq 2\pi e^{\frac{1}{2n}(f_2)_s} \left( 2 \left\| \sum_{k=0}^{2n-1} \xi_ke^{ikt} \right\|_{\infty} + 1 \right) \leq 2\pi(10\sqrt{2n} + 1) e^{\frac{1}{2n}(f_2)_s}. \quad (24)
\]

On the other hand, taking into account the orthogonality of trigonometric system \(\{e^{ikt}\}\) and the fact that \(\xi_k^2 = 1\), we obtain

\[
I = \frac{1}{10\sqrt{2n} + 2} \inf_{\gamma \geq 2n} \left( \int_{-\pi}^{\pi} \sum_{|k| \leq 2n-1, k \notin \gamma} \xi_k|\psi(|k|)|e^{ikt} \sum_{k=-2n+1}^{2n-1} \xi_k e^{ikt} dt \right) = \\
= \frac{\pi}{5\sqrt{2n} + 1} \inf_{\gamma \geq 2n} \sum_{|k| \leq 2n-1, k \notin \gamma} \psi(|k|). \quad (25)
\]

Since the sequence \(\psi(k)\) almost decreases, so

\[
\inf_{\gamma \geq 2n} \sum_{|k| \leq 2n-1, k \notin \gamma} \psi(|k|) \geq K(2) \inf_{\gamma \geq 2n} \sum_{|k| \leq 2n-1, k \notin \gamma} \psi(2n-1) = K(2)\psi(2n-1)(2n-1). \quad (26)
\]

In view of (24)–(26) we get

\[
e^{\frac{1}{2n}(f_2)_s} \geq K(2)\psi(2n-1)(2n-1) \geq (10\sqrt{2n} + 2)(10\sqrt{2n} + 1) \geq K(3)\psi(2n). \quad (27)
\]

Since, if \(\psi \in B\), so \(\psi(2n) \geq K(4)\psi(n)\), and, hence, taking into account (27), we find

\[
e^{\frac{1}{2n}(L^{\psi}_{\beta,\infty})_s} \geq e^{\frac{1}{2n}(f_2)_s} \geq K(5)\psi(n). \quad (28)
\]

Estimates (19) follow from (20) and (28). Theorem 2 is proved.

Corollary 1. Let \(r > 0, 1 \leq s < \infty\) and \(\beta \in \mathbb{R}\). Then

\[
e^{\frac{1}{2n}(W^r_{\beta,\infty})_s} \preceq e^{\frac{1}{2n-1}(W^r_{\beta,\infty})_s} \preceq n^{-r}. \quad (29)
\]
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