Multiple positivity and the Riemann zeta-function.

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Abstract

In the paper it is discussed the relations of the Riemann $\zeta$–function to classes of generating functions of multiply positive sequences according to Schoenberg (also called Pólya frequency sequences).

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1 Introduction.

This paper was inspired by works of G.Csordas, T.S.Norfolk, R.S.Varga ([4]) and D.K.Dimitrov ([9]). To formulate their results we need the following definition.

Definition 1. A real entire function $f$ is said to be in the Laguerre-Pólya class, written $f \in L - P$, if $f$ can be represented in the form

$$f(z) = C z^n e^{-\beta z^2 + \gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z) e^{-\alpha_k z},$$

where $C, \gamma, \alpha_k$ are real, $\beta \geq 0$, $n \in \mathbb{N} \cup \{0\}$, $\sum \alpha_k^2 < \infty$. 
Pólya and Schur ([25]) termed a real entire function $f$ as a function of type I in the Laguerre-Pólya class, written $f \in L - PI$, if one of the functions $f(z), f(-z), -f(z)$ or $-f(-z)$ can be represented in the form:

$$f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z),$$

(1)

where $C \geq 0, n \in \mathbb{N} \cup \{0\}, \gamma \geq 0, \alpha_k \geq 0, \sum \alpha_k < \infty$.

These classes were introduced by Laguerre (see [19], pp.168-178). Laguerre proved that $f \in L - PI$ iff $f$ is uniform limit, on compact subsets of $C$, of polynomials with only real nonpositive (or real nonnegative) zeros. Laguerre supposed that $f \in L - P$ iff $f$ is uniform limit, on compact subsets of $C$, of polynomials with only real zeros. This theorem was proved by Pólya later (see [24], p.54). We will write $f(z) = \sum_{k=0}^{\infty} a_k z^k \in L - P^+$, if $f \in L - PI$ and $a_k \geq 0$ for all $k = 0, 1, 2, \ldots$

In works ([4]), ([5]), ([6]), ([7]), ([9]), ([8]) it is discussed the different properties of the Laguerre-Pólya classes and the relations of the Riemann $\zeta-$function to these classes.

We proceed to briefly review here some of the nomenclature pertaining to the Riemann Hypothesis. To begin with, the Riemann $\xi$-function can be defined (see [30], p. 16, cf. [24], p.285) by

$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma(s/2) \zeta(s),$$

(2)

where $\zeta$ is the Riemann $\zeta$-function. It is known that $\xi$ is an entire function of order one and of maximal type (see, for example, [30], p.29). The function $\xi$ satisfies the following functional equation.

$$\xi(1 - s) = \xi(s).$$

Hence $\xi(s + 1/2)$ is an even real function and

$$\xi(s + 1/2) = \frac{1}{2} (s^2 - \frac{1}{4}) \pi^{-s/2 - 1/4} \Gamma(s/2 + 1/4) \zeta(s + 1/2) = \sum_{k=0}^{\infty} b_k s^{2k}.$$  

(3)

The coefficients $b_k$ can be found by the formula ([30])

$$b_k = 8 \frac{2^{2k}}{(2k)!} \int_{0}^{\infty} t^{2k} \Phi(t) dt, \quad k = 0, 1, \ldots, \quad (4)$$

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and

$$\Phi(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) \exp(-\pi n^2 e^{4t}).$$  \hspace{1cm} (5)$$

Note that Titchmarsh uses the symbol $\Xi$ for the function $\Xi(s) = \xi(is + \frac{1}{2})$ ([30], chapter II). The famous Riemann Hypothesis is the statement that all the zeros of $\Xi$ are real, that is $\Xi \in L - P$. For the various properties and characterizations of the Riemann $\zeta-$function see, for example, ([30]), ([10]), ([27]). The change of variables $z = s^2$ in (3) yields

$$\xi_1(z) = \xi(\sqrt{z} + 1/2) = \sum_{k=0}^{\infty} b_k z^k.$$  \hspace{1cm} (6)$$

Thus, $\xi_1$ is an entire function of order $\frac{1}{2}$ and the Riemann Hypothesis is equivalent to the statement that $\xi_1$ has only real negative zeros, so that $\xi_1 \in L - P^+$. The idea sketched above belongs to Pólya.

In works ([4]), ([5]), ([6]), ([7]), ([8]), ([9]) it is considered the following problem. Does the function $\Xi$ belong to $L - P$? In the same works authors investigate the equivalent problem. Does $\xi_1$ belong to $L - P^+$? In particular, they study the following necessary condition for $\xi_1 \in L - P^+$ (cf.([2]), ([5]), ([25])):

$$T_1(k) = b_k^2 - \frac{k+1}{k} b_{k+1} b_{k-1} \geq 0, \quad k \in N,$$  \hspace{1cm} (7)$$

where $b_k$ are Maclaurin coefficients of $\xi_1$. (In todays terminology, the inequalities of (7) are called Turan inequalities). In 1927 Pólya raised the question of whether or not the Turan inequalities (7) for the function $\xi_1$ are all valid. In 1995 G.Csordas, T.S.Norfolk and R.S.Varga ([4]) proved inequalities (7) for the function $\xi_1$. For various extensions and inequalities related to (7), we refer to ([2]) and ([9]). In ([2]) T.Craven and G.Csordas investigated certain polynomial invariants and used them to prove that a necessary condition that some function $f \in L - P^+$ is that its coefficients satisfy the following double Turan inequalities

$$T_2(k) = T_1(k)^2 - T_1(k+1)T_1(k-1) \geq 0, \quad k \in N,$$  \hspace{1cm} (7)$$

where $T_1(k)$ is defined by (7). But in ([7]) G.Csordas points out that the question whether or not the higher iterated Turan inequalities

$$T_n(k) = (T_{n-1}(k))^2 - T_{n-1}(k+1)T_{n-1}(k-1) \geq 0, \quad (k \geq n \geq 2)$$
hold for functions in the Laguerre-Pólya class remains open. In the same work the author pays attention to one more open problem: whether or not the double Turan inequalities are valid for the function $\xi_1$.

In this paper we also study the problem of whether or not the function $\xi_1 \in L^-P^+$. But our method is slightly different from the one described above. It is based on a characteristic property of $L^-P^+$, which was obtained by Aissen, Schoenberg, Whitney and Edrei in [1]. We need some definitions and notations.

**Definition 2.** Let $m$ be any positive integer. The sequence $\{a_k\}_{k=0}^\infty$ is called $m$-times positive (totally positive) if all minors of order $\leq m$ (of any order) of the infinite matrix

$$
\begin{vmatrix}
  a_0 & a_1 & a_2 & a_3 & \ldots \\
  0 & a_0 & a_1 & a_2 & \ldots \\
  0 & 0 & a_0 & a_1 & \ldots \\
  0 & 0 & 0 & a_0 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
$$

(8)

are non-negative. The multiply positive sequences (also called Pólya frequency sequences) were introduced by Fekete in 1912 (see [11]) in connection with the problem on the exact calculation of the number of positive zeros of the real polynomial. We will denote by $PF_m(PF_{\infty})$ the class of all $m$-times positive (totally positive) sequences. We will denote by $SPF_m$ the class of all sequences from $PF_m$ such that all minors of order $\leq m$ of matrix (8) without vanishing rows (columns) are positive.

We need also such notion.

**Definition 3.** Let $m$ be any positive integer. The sequence $\{a_k\}_{k=0}^\infty$ is called asymptotically $m$-times positive if there exists a positive integer $N$ such that all minors of matrix $A_N := (a_{N+j-l})$, $l = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, (a_k = 0$ for $k < 0)$ are nonnegative.

We will denote by $APF_m$ the class of all asymptotically $m$-times positive sequences. Obviously, $SPF_m \subset PF_m \subset APF_m$.

The classes of corresponding generating functions

$$
f(z) = \sum_{k=0}^{\infty} a_k z^k
$$
are also denoted by $PF_m$, $SPF_m$ and $APF_m$.

The class $PF_\infty$ was completely described by Aissen, Schoenberg, Whitney and Edrei in [1] (see also [16], p. 412):

**Theorem ASWE.** A function $f \in PF_\infty$ iff

$$f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} \frac{1 + \alpha_k z}{1 - \beta_k z},$$

where $C \geq 0$, $n \in \mathbb{Z}$, $\gamma \geq 0$, $\alpha_k \geq 0$, $\beta_k \geq 0$, $\sum(\alpha_k + \beta_k) < \infty$.

In connection with this result Schoenberg [28] formulated the problem of characterizing functions $f \in PF_m$, $m < \infty$, and investigated zero sets of polynomials $f \in PF_m$, $m < \infty$. He proved the following two theorems.

**Theorem A ([28]).** Let $f$ be a polynomial of degree $n$ and let $f \in PF_m$, $m < \infty$. Then $f(z) \neq 0$ for $z \in \{z : |\arg z| < \frac{\pi m}{n + m - 1}\}$.

**Theorem B ([29]).** Let $f$ be a polynomial, $f(0) > 0$. Then

$$f(z) \neq 0 \quad \text{for} \quad z \in \{z : |\arg z| < \frac{\pi m}{m + 1}\} \implies f \in PF_m.$$

Both estimates in Theorems A and B are sharp.

The following fact is a simple corollary of Theorem ASWE and (1).

**Theorem C.** Let $f$ be an entire function. Then $f \in PF_\infty$ iff $f \in L - P^+$.

So, the Riemann Hypothesis is equivalent to statement that $\xi_1 \in PF_\infty$. Since $PF_\infty = \bigcap_{m=1}^{\infty} PF_m$, this fact means that $\xi_1 \in PF_m$ for every $m \in \mathbb{N}$. In this paper we investigate the relations of the function $\xi_1$ to classes $APF_m$ and $PF_m$. With the help of Theorem B it is easy to prove the following fact.

**Theorem 1.** $\xi_1 \in PF_{44}$.

**Proof of Theorem 1.** It is well-known (see, for example [30], chapter XV) that

$$\xi(s) \neq 0 \quad \text{for} \quad s \in \{s : \frac{1}{2} < Re s \leq 1, \quad 0 \leq Im s \leq 14\}.$$  \hfill (9)

By (6) this implies that the function $\xi_1$ has no zeros in the angle $\{z : |\arg z| < \frac{43\pi}{44}\}$. It follows from (4), (5) and (6) that $\xi_1(0) > 0$. Since $\xi_1$ is an entire function of order $\frac{1}{2}$ by the Hadamard theorem (see, for example, [20], p. 24) we have

$$\xi_1(z) = C \prod_{k=1}^{\infty} (1 - \frac{z}{z_k}),$$
where \( C > 0, \sum_{k=1}^{\infty} \frac{1}{|z_k|} < \infty \) and \(|\pi - \arg z_k| \leq \frac{\pi}{44}\). Consider the polynomials \( P_N(z) = C \prod_{k=1}^{N} (1 - \frac{z}{z_k})\). For every \( N \in \mathbb{N} \) the polynomial \( P_N \) has no zeros in the angle \( \{ z : |\arg z| < \frac{44\pi}{44} \} \). So by Theorem B we have \( P_N \in PF_{44}, N \in \mathbb{N} \).

The function \( \xi_1 \) is the uniform limit, on compact subsets of \( \mathbb{C} \), of polynomials \( P_N \). Hence \( \xi_1 \in PF_{44} \). This completes the proof of Theorem 1.

Numerous calculations (see, for example, [21], [31]) show that the height of rectangle in (9) can be increased essentially. So, the constant \( 44 \) in Theorem 1 can be also increased essentially. But it is impossible to prove the Riemann hypothesis with the help of this method, because the above arguments use the fact that the Riemann \( \zeta \)-function does not vanish in the critical strip \( \{ s : \frac{1}{2} < \Re s < 1 \} \).

We make some comments. It follows from \( \xi_1 \in PF_2 \) that
\[
b_k^2 - b_{k+1}b_{k-1} \geq 0, \quad k \in \mathbb{N}.
\]

This inequalities are similar to Turan inequalities (7), but they are weaker. The double Turan inequalities mean also nonnegativity of some determinants.

More precisely,
\[
T_2(k) = b_kk! \begin{vmatrix} b_kk! & b_{k+1}(k+1)! & b_{k+2}(k+2)! \\ b_{k-1}(k-1)! & b_kk! & b_{k+1}(k+1)! \\ b_{k-2}(k-2)! & b_{k-1}(k-1)! & b_kk! \end{vmatrix}.
\]

The main results of this paper are the following theorems.

**Theorem 2.** \( \xi_1 \in APF_m \) for all \( m \in \mathbb{N} \).

Note, that the Riemann Hypothesis is equivalent to the following statement: \( \xi_1 \in PF_m \) for all \( m \in \mathbb{N} \).

**Theorem 3.** For every \( m \in \mathbb{N} \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) the following inclusions hold:

(i) \( e^{nz}\xi_1(z) \in PF_m \);

(ii) \( \cosh(n\sqrt{z})\xi_1(z) \in PF_m \).

Note, that multiplication by the function \( e^{nz} \) does not change zero set of \( \xi_1(z) \) and multiplication by the function \( \cosh(n\sqrt{z}) \) adds only negative zeros (but growth of the product \( \cosh(n\sqrt{z})\xi_1(z) \) is the same as the growth of \( \xi_1 \)). Hence the Riemann Hypothesis is equivalent to each of the statements:
\[
\exists n \in \mathbb{N} \forall m \in \mathbb{N} : e^{nz}\xi_1(z) \in PF_m
\]
or
\[ \exists n \in \mathbb{N} \, \forall m \in \mathbb{N} : \cosh(n\sqrt{z})\xi_1(z) \in PF_m. \]

The methods of proofs of Theorems 2, 3 are similar to those used in ([12]), ([13]), ([17]), ([18]).

2 The reduction of the proofs of Theorems 2 and 3 to the statements on the asymptotic behavior of some multiple integrals.

Let \( f(z) = \sum_{k=0}^{\infty} a_k(f)z^k, \quad a_k(f) \geq 0, \) be an entire function and let
\[
A_k(f) := \det \|a_{k+j-l}(f)\|_{l=0,\ldots,\nu-1; j=0,\ldots,\nu-1} \quad (a_k(f) = 0, \text{ for } k < 0). \tag{10}
\]
Further these determinants will play a principal role. We will denote by \( A_N(f) \) the matrix
\[
A_N(f) := (a_{N+j-l}(f)), \quad l = 0, 1, \ldots, m-1, \quad j = 0, 1, \ldots, N \in \mathbb{N} \tag{11}
\]
(\( a_k(f) = 0 \) for \( k < 0 \)).

In ([17]) the following fact was proved

**Lemma 1.** \( \forall k = 0, 1, 2, \ldots \forall \nu = 1, 2, \ldots \forall r > 0 \)
\[
A_k^\nu(f) = \frac{r^{-k}}{(2\pi)^\nu} \int_{-\pi}^{\pi} e^{-ik\theta} f(re^{i\theta})d\theta, \quad k \in \mathbb{Z},
\]
we have
\[
A_k^\nu(f) = \frac{r^{-k\nu}}{(2\pi)^\nu} \det \| \int_{-\pi}^{\pi} f(re^{i\theta})e^{-i(k+j)\theta}e^{il\theta}\|_{l,j=0}^{\nu-1}.
\]
By virtue of the Pólya composition formula ([26], p.48, Problem 68) for any 2 sets of functions \( \psi_0, \ldots, \psi_{\nu-1} \) and \( \varphi_0, \ldots, \varphi_{\nu-1} \) we have
\[
\det \| \int_{a}^{b} \psi_\alpha(x)\varphi_\beta(x)dx\|_{\alpha,\beta=0}^{\nu-1} =
\]

\[
\frac{1}{\nu!} \int_a^b \cdots \int_a^b \det \psi_\alpha(x_\beta)\|^{\nu-1}_{\alpha,\beta=0} \det \varphi_\alpha(x_\beta)\|^{\nu-1}_{\alpha,\beta=0} dx_0 \cdots dx_{\nu-1}.
\]

Applying this formula to \(\psi_\alpha(\theta) = f(re^{i\theta})e^{-i(k+\alpha)\theta}\) and \(\varphi_\alpha(\theta) = e^{i\alpha\theta}\) and the formula for the Vandermonde determinant we obtain the proof of Lemma 1.

Consider the multiple integrals

\[
I_\nu^k(\eta, f) = \int_{-\pi}^\pi \cdots \int_{-\pi}^\pi \Re \left\{ \prod_{j=1}^\nu \left( e^{-ik\theta_j} \frac{f(e^{\eta+i\theta_j})}{f(e^\nu)} \right) \right\}
\]

\[
\prod_{1 \leq \alpha < \beta \leq \nu} 4 \sin^2 \frac{\theta_\alpha - \theta_\beta}{2} d\theta_1 \cdots d\theta_\nu.
\]  

(13)

Since \(f(r) > 0\) for \(r \geq 0\) and the determinants \(A_\nu^k(f)\) are real such fact follows immediately from Lemma 1.

**Lemma 2.** \(\forall k = 0, 1, 2, \ldots \) \(\forall \nu = 1, 2, \ldots \) \(\forall \eta \in \mathbb{R}\)

\[
sign I_\nu^k(\eta, f) = sign A_\nu^k(f).
\]

Put

\[
f^1(z) = A\xi_1(z), \quad \text{where} \quad A > 0 \quad \text{is such that} \quad f^1(0) > 1. \quad (14)
\]

Let

\[
f^2(z) = e^{nz}f^1(z), \quad f^3(z) = \cosh (n\sqrt{z})f^1(z). \quad (15)
\]

Obviously Theorem 2 and Theorem 3 are equivalent to the following statements respectively

**Theorem 2'**. The function \(f^1 \in APF_m\) for all \(m \in \mathbb{N}\).

**Theorem 3'**. For every \(m \in \mathbb{N}\) there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) the following inclusions hold: \(f^p \in PF_m \quad (p = 2, 3)\).

Now we show that Theorem 2' is a corollary of the following fact.

**Proposition 1.** \(\forall \nu = 1, 2, \ldots, m \quad \exists N(\nu) \in \mathbb{N} \quad \forall k \geq N(\nu) \quad \exists \eta = \eta(k, \nu) > 0 \implies I_\nu^k(\eta, f^1) > 0\).

Let us deduce Theorem 2' from Proposition 1. By Proposition 1 and Lemma 2

\[
\forall m \in \mathbb{N} \exists N(m) \in \mathbb{N} \forall k \geq N(m) \forall \nu = 1, 2, \ldots, m \implies A_\nu^k(f^1) > 0,
\]

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that is all minors of order \( \nu = 1, 2, \ldots, m \) composed of consecutive rows and consecutive columns of matrix \( A_N(f^1) \) (defined by (10)) are positive. The statement on \( f^1 \in APF_m \) means nonnegativity of all minors of matrix \( A_N(f^1) \). As a matter of fact these minors are positive. It follows from such result of Schoenberg ([28]).

**Theorem D.** If all minors of order \( \nu = 1, 2, \ldots, m \) composed of consecutive rows and consecutive columns of matrix \( A \) are positive then all minors of \( A \) of order \( \nu = 1, 2, \ldots, m \) are positive.

Let

\[
f_\varepsilon^2(z) = e^z f^1(\varepsilon z), \quad \varepsilon = \frac{1}{n},
\]

\[
f_\varepsilon^3(z) = \cosh \sqrt{z} f^1(\varepsilon z), \quad \varepsilon = \frac{1}{n^2}.
\]

It is clear that

\[
\forall \varepsilon > 0 : (f(z) \in PF_m \iff f(\varepsilon z) \in PF_m).
\]

Thus, Theorem 3' follows from such fact.

**Theorem 3".** \( \forall m \in \mathbb{N} \ \exists \varepsilon_0(m) > 0 \ \forall \varepsilon, \ 0 < \varepsilon < \varepsilon_0(m) \ \forall p = 2, 3 \implies f_\varepsilon^p(z) \in PF_m. \)

We obtain Theorem 3" as a corollary of such proposition.

**Proposition 2.** \( \forall \nu \in \mathbb{N} \ \exists N(\nu) \in \mathbb{N} \ \forall \varepsilon, \ 0 < \varepsilon < 1 \ \forall k \geq N(\nu) \ \forall p = 2, 3 \implies I_k^\nu(\eta, f_\varepsilon^p) > 0. \)

By Proposition 2 and Lemma 2 we have

\[
\forall \nu \in \mathbb{N} \ \exists N(\nu) \in \mathbb{N} \ \forall \varepsilon, \ 0 < \varepsilon < 1 \ \forall k \geq N(\nu) \forall p = 2, 3 \implies A_{k}^{\nu}(f_\varepsilon^p) > 0.
\]

(18)

Now we need the following lemma from ([17]) which is similar to Theorem D.

**Lemma 3.** ([17]) Let \( \{a_k\}_{k=0}^\infty \) be a sequence of positive numbers such that \( \sum_{0}^{\infty} a_k < \infty \). Consider the matrix with \( \nu \) rows and infinitely many columns

\[
A_\nu = \|a_{j-i}\|_{i=0,\ldots,\nu-1; j=0,1,\ldots} \quad (a_k = 0, \text{ for } k < 0).
\]

Suppose that for every \( \nu = 1, \ldots, m \) the matrix \( A_\nu \) satisfies the following condition: all minors of order \( \nu \), composed of consecutive columns, are positive. Then \( \{a_k\} \in PF_m. \)
Note that
\[ f_1^1(z) \to e^z f_1^1(0), \quad f_2^2(z) \to \cosh(\sqrt{z}) f_1^1(0) \]
uniformly on any compact set of C-plane. It is well-known that \( e^z \in SPF_\infty \).

Since the matrix (8) of coefficients of \( \cosh(\sqrt{z}) \) is the submatrix of the matrix (8) of coefficients of \( e^z \) the function \( \cosh(\sqrt{z}) \in SPF_\infty \).

Therefore
\[ \forall \nu \in \mathbb{N} \quad \exists \varepsilon(\nu) > 0 \quad \forall \varepsilon, 0 < \varepsilon < \varepsilon_\nu \quad \forall k = 0, 1, \ldots, N(\nu) - 1 \]
\[ \forall p = 2, 3 \implies A_k^\nu(f_\varepsilon^p) > 0. \]

So, by (18)
\[ \forall \nu \in \mathbb{N} \quad \exists \varepsilon(\nu) > 0 \quad \forall \varepsilon, 0 < \varepsilon < \varepsilon_\nu \quad \forall k = 0, 1, 2 \ldots \quad \forall p = 2, 3 \implies A_k^\nu(f_\varepsilon^p) > 0. \]

Putting \( \tilde{\varepsilon} = \min\{\varepsilon(1), \ldots, \varepsilon(m)\} \) we obtain that the sequence of coefficients of \( f_\varepsilon^p, p = 2, 3 \) satisfies the assumptions of Lemma 3 and thus \( f_\varepsilon^p \in PF_m, p = 2, 3 \). We have proved that Theorem 3$''$ follows from Proposition 2.

Thus the proofs of Theorems 1 and 2 are reduced to the statements on positivity of corresponding integrals of the form (13). Let \( \vec{\theta} = (\theta_1, \ldots, \theta_\nu) \in \mathbb{R}^\nu \). We will use the \( l_\infty \)-norm. To prove Propositions 1 and 2 for \( f = f_1^1, f_2^2, f_3^3 \) we write the integral (13) as
\[
I_k^\nu(\eta, f) = \left( \int \cdots \int_{\|\vec{\theta}\| \leq \sigma} + \int \cdots \int_{\sigma < \|\vec{\theta}\| \leq \pi} \right) \mathbb{R} \left\{ \prod_{j=1}^{\nu} \left( e^{-ik\theta_j} \frac{f(e^{\eta+i\theta_j})}{f(e^{\eta})} \right) \right\} \times
\]
\[
\prod_{1 \leq \alpha < \beta \leq \nu} 4 \sin^2 \frac{\theta_\alpha - \theta_\beta}{2} d\theta_1 \cdots d\theta_\nu = J_1(f) + J_2(f),
\]
(19)
where \( \sigma = \sigma(f) > 0 \) and \( \eta = \eta(f) > 0 \) will be chosen with the help of reasoning usually applied in the saddle-point method. We will estimate \( J_1(f) \) from below and \( J_2(f) \) from above.

3 The estimates from below of the integrals \( J_1(f) \).

We need some properties of the function \( \xi_1 \).
Lemma 4. The function $\xi_1$ satisfies the following conditions:

(a) all coefficients of $\xi_1$ are positive;

(b) $\xi_1$ has no zeros in the angle $|\arg z| < \pi/2$;

(c) for $\theta$, $|\theta| \leq \pi/2$, we have

$$\log |\xi_1(re^{i\theta})| = \frac{1}{4} \sqrt{r} \log r \cos \frac{\theta}{2} - \frac{1}{4} \sqrt{r} \theta \sin \frac{\theta}{2} - \frac{1}{2} \sqrt{r} (\log 2 + 1 + \log \pi) \cos \frac{\theta}{2} + O(\log r), \quad r \to \infty;$$

(d) for $\theta$, $|\theta| < \pi$, we have

$$\log |\xi_1(re^{i\theta})| \leq \frac{1}{4} \sqrt{r} \log r \cos \frac{\theta}{2} + O(\sqrt{r} \log r);$$

(e) $\xi_1'(r)/\xi_1(r) = \frac{1}{8} \log r + O\left(\frac{1}{\sqrt{r}}\right), \quad r \to \infty.$

Proof. In virtue of (4), (5), (6) the statement (a) is obvious. The proposition (b) follows immediately from (6) and (9). To prove (c) we note that it follows from such well-known properties of the $\zeta-$function (see, for example [30], chapter I)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{Re } z > 1$$

and

$$\zeta(z) = \prod_p \left(1 - \frac{1}{p^z}\right)^{-1}, \quad \text{Re } z > 1$$

(here and further we denote prime numbers by $p$) that

$$|\zeta(z)| \leq C, \quad \text{Re } z \geq 2$$

and

$$|\zeta(z)|^{-1} \leq C, \quad \text{Re } z \geq 2.$$
we obtain (c).

Since all zeros of $\xi$–function lie in the strip $\{z : 0 < \text{Re } z < 1\}$ (see, for example [30], chapter III), than by (6) the number of zeros of the function $\xi_1$ in every angle $\{z : |\text{arg } z| < \psi\}$, $0 < \psi \leq \pi$, is finite. Hence, the indicator of the function $\xi_1$ is trigonometric. So, the inequality (d) is a well-known property of the indicator of an entire function (see, for example, [20], p.71).

Let us prove (e). It follows from the formula (see, for example, [30], chapter I)

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{m=1}^{\infty} \sum_{p} \frac{\log p}{p^{mz}}, \quad \text{Re } z > 1,$$

that

$$\left| \frac{\zeta'(r)}{\zeta(r)} \right| = O(1), \quad r \to \infty.$$

By (3) and the formula (see, for example, [14], chapter III)

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O\left(\frac{1}{|z|^2}\right), \quad |z| \to \infty, \quad |\text{arg } z| \leq \pi - \delta,$$

we have (e). Lemma 4 is proved.

Since the methods of proofs of Propositions 1, 2 are similar, we will prove these statements at the same time. We will denote by $b(\eta, f)$ the function

$$b(\eta, f) = \log f(e^{\eta}).$$

(20)

So,

$$b(\eta, f^1) = \log f^1(e^{\eta}) = \log(A\xi_1(e^{\eta}));$$

(21)

$$b(\eta, f^2_\varepsilon) = e^{\eta} + \log f^1(\varepsilon e^{\eta});$$

(22)

$$b(\eta, f^3_\varepsilon) = \log \cosh(e^{\eta/2}) + \log f^1(\varepsilon e^{\eta}).$$

(23)

Further we need such lemma.

**Lemma 5.** For $\eta \geq \eta_0$ the following inequalities are valid

$$b(\eta, f^1) \leq C\eta e^{\eta/2} \leq b'(\eta, f^1);$$

(24)

$$b'(\eta, f^1) \leq C\eta e^{\eta/2} \leq b(\eta, f^1).$$

(25)
For any $\eta \geq 0$ and every $\varepsilon, 0 < \varepsilon < 1$, the following inequalities hold

\[ b(\eta, f^2_\varepsilon) \leq C(e^\eta + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) \leq b'(\eta, f^2_\varepsilon); \quad (26) \]

\[ b'(\eta, f^2_\varepsilon) \leq C(e^\eta + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) \leq b(\eta, f^2_\varepsilon); \quad (27) \]

\[ b(\eta, f^3_\varepsilon) \leq C(e^{\eta/2} + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) \leq b'(\eta, f^3_\varepsilon); \quad (28) \]

\[ b'(\eta, f^3_\varepsilon) \leq C(e^{\eta/2} + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) \leq b(\eta, f^3_\varepsilon). \quad (29) \]

**Proof.** In virtue of (21) it is clear that inequalities (24) and (25) are the corollaries of the properties (c), (e) from Lemma 4.

To prove (26)-(29) we note that by (c) the following estimates are valid

\[ C\sqrt{r}\log r - C \leq \log \xi_1(r) \leq C\sqrt{r}\log r + C, \quad r \geq 0. \quad (30) \]

It follows from (e) that

\[ C\sqrt{r}\log r - C \leq \frac{\xi'_1(r)}{\xi_1(r)} \leq C\sqrt{r}\log r + C, \quad r \geq 0. \quad (31) \]

Hence, by (14) and (22) we have

\[ b(\eta, f^2_\varepsilon) \leq e^n + C\sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon) + C \leq C(e^n + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) \]
\[ \leq C(e^n + C\varepsilon e^{\eta}\left(\frac{\xi'_1(r)}{\xi_1(r)}\right)|_{r=\varepsilon e^n} + C) \leq Cb'(\eta, f^2_\varepsilon). \]

Inequality (26) is proved. The estimates (27), (28) and (29) can be obtained in the same way as (26). Lemma 5 is proved.

It follows from property (b) of Lemma 4, that for every $f = f^1, f^2, f^3$ the function $f(e^{\eta+i\theta}) \neq 0$ in the circle $\{\theta \in \mathbb{C} : |\theta| \leq \pi/4\}$. Thus, for every $f = f^1, f^2, f^3$ and $|\theta| \leq \pi/4$ we have the decomposition

\[ \log \left\{ e^{-ik\theta} \frac{f(e^{\eta+i\theta})}{f(e^n)} \right\} = -ik\theta + \sum_{j=1}^{\infty} b^{(j)}(\eta, f) \frac{i^j \theta^j}{j!}. \quad (32) \]
By virtue of property (a) of Lemma 4 for every \( f = f^1, f^2, f^3 \) the function \( b'(\eta, f) > 0 \), \( b'(\eta, f^1) \) is an increasing function of \( \eta \), \( b'(\eta, f^p), p = 2, 3 \), are the increasing functions of \( \eta \), and of \( \varepsilon \). Therefore for every \( f = f^1, f^2, f^3 \) there exists a positive integer \( k_0 = k_0(f) \) such that for every \( k \geq k_0(f) \) the equation

\[
b'(\eta, f) = k
\]

has the unique solution \( \eta = \eta(k, f) \). In so doing, we will have

\[
k_0(f^1) = \max\{ [b'(0, f^1)] + 1, 1 \}; \quad k_0(f^p) = \max\{ \max_{0 \leq \varepsilon \leq 1} [b'(0, f^p)] + 1, 1 \}, p = 2, 3.
\]

Note, that \( k_0(f^p), p = 2, 3 \), does not depend on \( \varepsilon, 0 < \varepsilon < 1 \).

We need the following Lemma.

**Lemma 6.** \( \exists C > 0 \) \( \forall f = f^1, f^2, f^3 \exists k_0(f) \) \( \forall k \geq k_0, k \in \mathbb{N}, \forall j \in \mathbb{N} : \)

\[
|b^{(j)}(\eta, f)| \leq \frac{4^j j!}{\pi^j} Ck.
\]

Moreover, \( k_0(f^2) \) and \( k_0(f^3) \) do not depend on \( \varepsilon, 0 < \varepsilon < 1 \).

**Proof.** For every \( f = f^1, f^2, f^3 \) we have \( f(e^{\eta + \varepsilon}) \neq 0 \) in the circle \( \{ z \in \mathbb{C} : |z| \leq \pi/4 \} \). Applying the Schwarz formula (see, for example, [26], Problem 231) to the function \( \log f(e^{\eta + z}) \) in the circle \( |z| \leq \pi/4 \), we obtain

\[
\log f(e^{\eta + z}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{\eta + \frac{\pi}{4}e^{i\tau}})| \frac{\pi e^{i\tau} + z}{\frac{\pi}{4} e^{i\tau} - z} d\tau + iC,
\]

where \( f = f^1, f^2, f^3 \). Differentiating with respect to \( z \), setting \( z = 0 \), we have

\[
|b^{(j)}(\eta, f)| \leq \frac{4^j j!}{\pi^{j+1}} \int_{-\pi}^{\pi} \log |f(e^{\eta + \frac{\pi}{4}e^{i\tau}})||d\tau = \]


\[
\frac{2^{2j+1}j!}{\pi^j} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} \log^+ |f(e^{\eta + \pi \tau}e^{i\tau})| d\tau - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{\eta + \pi \tau}e^{i\tau})| d\tau \right) = \frac{2^{2j+1}j!}{\pi^j} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} \log^+ |f(e^{\eta + \pi \tau}e^{i\tau})| d\tau - \log f(e^{\eta}) \right),
\]

where \( f = f^1, f^2, f^3 \). Taking into account (14) and (20), (21), (22), we obtain

\[
|b^{(j)}(\eta, f)| \leq \frac{2^{2j+1}j!}{\pi^j} \int_{-\pi}^{\pi} \log^+ |f(e^{\eta + \pi \tau}e^{i\tau})| d\tau,
\]

(37)

where \( f = f^1, f^2, f^3 \). By property (a) of Lemma 4 for \( f = f^1, f^2, f^3 \) we have

\[
\log^+ |f(e^{\eta + \pi \tau}e^{i\tau})| \leq \log f(e^{\eta + \pi \tau}) = b((\eta + \frac{\pi}{4}), f).
\]

(38)

In virtue of (24) and (33) we obtain

\[
b((\eta + \frac{\pi}{4}), f^1) \leq C\eta e^{\eta/2} \leq b'(\eta, f^1) = Ck.
\]

(39)

By (26) and (33) for all \( \varepsilon, 0 < \varepsilon < 1 \), we have

\[
b((\eta + \frac{\pi}{4}), f^2_{\varepsilon}) \leq C(e^{\eta} + \sqrt{\varepsilon} e^{\eta/2} (\eta + \log \varepsilon)) \leq b'(\eta, f^2_{\varepsilon}) = Ck.
\]

(40)

where \( f = f^1, f^2_{\varepsilon}, f^3_{\varepsilon} \). By property (a) of Lemma 4 for \( f = f^1, f^2_{\varepsilon}, f^3_{\varepsilon} \) we have

\[
\log^+ |f(e^{\eta + \pi \tau}e^{i\tau})| \leq \log f(e^{\eta + \pi \tau}) = b((\eta + \frac{\pi}{4}), f).
\]

(41)

In virtue of (24) and (33) we obtain

\[
b((\eta + \frac{\pi}{4}), f^1) \leq C\eta e^{\eta/2} \leq b'(\eta, f^1) = Ck.
\]

(42)

By (26) and (33) for all \( \varepsilon, 0 < \varepsilon < 1 \), we have

\[
b((\eta + \frac{\pi}{4}), f^2_{\varepsilon}) \leq C(e^{\eta} + \sqrt{\varepsilon} e^{\eta/2} (\eta + \log \varepsilon)) \leq b'(\eta, f^2_{\varepsilon}) = Ck.
\]

(43)

By (28) and (33) for all \( \varepsilon, 0 < \varepsilon < 1 \), we have

\[
b((\eta + \frac{\pi}{4}), f^3_{\varepsilon}) \leq C(e^{\eta/2} + \sqrt{\varepsilon} e^{\eta/2} (\eta + \log \varepsilon)) \leq b'(\eta, f^3_{\varepsilon}) = Ck.
\]

(44)
Substituting (42)-(44) into (41) and then into (37) we obtain statement of Lemma 6. Lemma 6 is proved.

It follows from (38) that for $\theta, |\theta| \leq \frac{\pi}{8}$ and for $f = f^1, f^2, f^3$ we have

$$|\tau(\theta, \eta, f)| \leq Ck \sum_{p=3}^{\infty} \frac{4^p|\theta|^p}{\pi^p} \leq B|\theta|^3k,$$

where $B > 0$. Let us choose

$$\sigma = \sigma(k) = \left(\frac{\pi}{3B_\nu}\right)^{1/3} k^{-1/3},$$

then for $\theta, |\theta| \leq \sigma$, we have

$$|\tau(\theta, \eta, f)| \leq \frac{\pi}{3\nu}.$$  \hspace{1cm} (46)

Applying (34), (36) for $\tilde{\theta}, ||\tilde{\theta}|| \leq \sigma$, we obtain

$$\text{Re} \prod_{j=1}^{\nu} \left\{ e^{-ik\theta_j} \frac{f(e^{\eta+i\theta_j})}{f(e^{\eta})} \right\} \geq \frac{1}{2} e^{-\frac{\pi}{3\nu}} \exp \left( -Ck \sum_{j=1}^{\nu} \theta_j^2 \right).$$

So, with the help of (45) we obtain

$$J_1(f) \geq \left(\frac{2}{\pi}\right)^{\nu-1} C f \cdots f \text{||} \theta \text{||} \leq \sigma \exp \left( -Ck \sum_{j=1}^{\nu} \theta_j^2 \right)$$

$$\prod_{1 \leq \alpha < \beta \leq \nu} (\theta_\alpha - \theta_\beta)^2 d\theta_1 \cdots d\theta_\nu =$$

$$\left(\frac{2}{\pi}\right)^{\nu-1} Ck^{-\frac{\nu}{2}} f \cdots f \text{||} \theta \text{||} \leq Ck^4 \exp \left( -C \sum_{j=1}^{\nu} u_j^2 \right)$$

$$\prod_{1 \leq \alpha < \beta \leq \nu} (u_\alpha - u_\beta)^2 du_1 \cdots du_\nu \geq C(\nu)k^{-\frac{\nu}{2}},$$

where $C(\nu) > 0$ depends only on $\nu$.

4 The estimate from above of the integrals $J_2(f)$.

The integration domain $\{\tilde{\theta} : \sigma < ||\tilde{\theta}|| \leq \pi\}$ in $J_2(f)$ is contained in the union of the domains

$$\{\tilde{\theta} : |\theta_1| \leq \pi, \cdots, |\theta_{j-1}| \leq \pi, \sigma < |\theta_j| \leq \pi, |\theta_{j+1}| \leq \pi, \cdots, |\theta_\nu| \leq \pi, \; j = 1, \cdots, \nu\}.$$
and the integrand has a majorant
\[ 2^{(\nu-1)\nu} \prod_{j=1}^{\nu} \left| \frac{f(e^{\eta+i\theta_j})}{f(e^{\eta})} \right|, \]
which is symmetric with respect to \( \theta_1, \ldots, \theta_\nu \). Therefore,
\[ |J_2(f)| \leq \nu 2^{(\nu-1)\nu} \int_{\sigma<|\theta|\leq\pi} \left| \frac{f(e^{\eta+i\theta})}{f(e^{\eta})} \right| d\theta \left( \int_{-\pi}^{\pi} \left| \frac{f(e^{\eta+i\theta})}{f(e^{\eta})} \right| d\theta \right)^{\nu-1}. \]

By property (a) of Lemma 4 and (14), (15) we have
\[ \left| \frac{f(e^{\eta+i\theta})}{f(e^{\eta})} \right| < 1. \]

Hence
\[ |J_2(f)| \leq \nu 2^{(\nu-1)(\nu+1)\pi(\nu-1)} \left( \int_{\sigma<|\theta|\leq\pi/2} f(e^{\eta+i\theta}) \, d\theta + \int_{\pi/2<|\theta|\leq\pi} f(e^{\eta+i\theta}) \, d\theta \right) \left( \int_{-\pi}^{\pi} \left| \frac{f(e^{\eta+i\theta})}{f(e^{\eta})} \right| d\theta \right)^{\nu-1} = C(\nu)(I_1(f) + I_2(f)). \quad (49) \]

To estimate the integral \( I_1(f) \) we note that by property (c) of Lemma 4 for \( \theta, \, |\theta| \leq \pi/2 \), we have
\[ \log |\xi_1(re^{i\theta})| - \log \xi_1 \leq \frac{1}{4} \sqrt{r} \log r \left( \cos \frac{\theta}{2} - 1 \right) - \frac{1}{4} \sqrt{r} \theta \sin \frac{\theta}{2} - \frac{1}{2} \sqrt{r} \left( \log 2 + 1 + \log \pi \right) \left( \cos \frac{\theta}{2} - 1 \right) + C \log r, \quad r \geq r_0. \]

Hence for \( \theta, \sigma \leq |\theta| \leq \pi/2 \), and \( r \geq 0 \) the following inequality is valid
\[ \log |\xi_1(re^{i\theta})| - \log \xi_1(r) \leq -C \sqrt{r} \log r \sigma^2 + C \log r + C. \quad (50) \]

By (14) and (25) for \( \theta, \sigma \leq |\theta| \leq \pi/2 \), we have
\[ \log |f^1(e^{\eta+i\theta})| - \log f^1(e^{\eta}) \leq -C \eta e^{-\eta/2} \sigma^2 + C \eta \]
\[ \leq -C b'(\eta, f^1) \sigma^2 + C \eta, \quad \eta \geq \eta_0. \quad (51) \]
Note that
\[ e^{\eta+i\theta} - e^{\eta} \leq -2/\pi^2 e^{\eta} \theta^2, \quad |\theta| \leq \pi/2; \quad (52) \]
\[ \log |\cosh(e^{(\eta+i\theta)/2})| - \log \cosh(e^{\eta/2}) \leq \]
\[ e^{\eta/2} (\cos \theta^2 - 1) + \log 2 \leq -\frac{1}{2\pi} e^{\eta/2} \theta^2 + \log 2, \quad |\theta| \leq \pi. \]

Then by (14), (16), (50) and (27) for all \( \varepsilon, 0 < \varepsilon < 1 \), and \( \eta > 0 \) we have
\[ \log |f_2^{\varepsilon}(e^{\eta+i\theta})| - \log f_2^{\varepsilon}(e^{\eta}) \leq \quad (54) \]
\[ -C(e^{\eta/2} + \sqrt{\varepsilon} e^{\eta/2}(\eta + \log \varepsilon)) \sigma^2 + C\eta + C \leq \]
\[ -Cb'(\eta, f_2^{\varepsilon}) \sigma^2 + C\eta + C, \quad \sigma \leq |\theta| \leq \pi/2, \]
and by (14), (17), (50) and (29) for all \( \varepsilon, 0 < \varepsilon < 1 \), and \( \eta > 0 \) we have
\[ \log |f_3^{\varepsilon}(e^{\eta+i\theta})| - \log f_3^{\varepsilon}(e^{\eta}) \leq \quad (55) \]
\[ -C(e^{\eta/2} + \sqrt{\varepsilon} e^{\eta/2}(\eta + \log \varepsilon)) \sigma^2 + C\eta + C \leq \]
\[ -Cb'(\eta, f_3^{\varepsilon}) \sigma^2 + C\eta + C, \quad \sigma \leq |\theta| \leq \pi/2. \]

From (51), (54), (55) with the help of (33) and (45) for \( \theta, \sigma < |\theta| \leq \pi/4, \)
and \( f = f_1^{\varepsilon}, f_2^{\varepsilon}, f_3^{\varepsilon} \) we obtain
\[ \log |f(e^{\eta+i\theta})| - \log f(e^{\eta}) \leq -Ck^{1/4} + C \log k + C \leq -Ck^{1/3}, \quad k \geq k_0(f), \quad (56) \]
where \( k_0(f_2^{\varepsilon}), k_0(f_3^{\varepsilon}) \) do not depend on \( \varepsilon \).

Hence
\[ I_1 \leq C \exp(-C(\nu)k^{1/4}). \quad (57) \]

Now we will estimate \( I_2 \). By the conditions (c) and (d) of Lemma 4 for \( \theta, \frac{\pi}{2} < |\theta| < \pi \) we have
\[ \log |\xi_1(r e^{i\theta})| - \log \xi_1(r) \leq \quad (58) \]
\[ -C \sqrt{r} \log r \theta^2 + a(\sqrt{r} \log r) \leq -C \sqrt{r} \log r + C, \quad r \geq 0. \]

By (14) and (25) for \( \theta, \frac{\pi}{2} < |\theta| < \pi \), we have
\[ \log |f_1^{\varepsilon}(e^{\eta+i\theta})| - \log f_1^{\varepsilon}(e^{\eta}) \leq -C\eta e^{\eta/2} \leq -Cb'(\eta, f_1^{\varepsilon}) + C, \quad \eta \geq \eta_0. \quad (59) \]
Then by (14), (16), (58) and (27) for all $\varepsilon, 0 < \varepsilon < 1$, and $\eta > 0$ we have

$$\log |f_\varepsilon^2(e^{\eta+i\theta})| - \log f_\varepsilon^2(e^{\eta}) \leq -C(e^{\eta} + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) + C \leq -C\theta'(\eta, f_\varepsilon^2) + C, \quad \frac{\pi}{2} < |\theta| < \pi,$$

and by (14), (17), (58) and (28) for all $\varepsilon, 0 < \varepsilon < 1$, and $\eta > 0$ we have

$$\log |f_\varepsilon^3(e^{\eta+i\theta})| - \log f_\varepsilon^3(e^{\eta}) \leq -C(e^{\eta/2} + \sqrt{\varepsilon}e^{\eta/2}(\eta + \log \varepsilon)) + C \leq -C\theta'(\eta, f_\varepsilon^3) + C, \quad \frac{\pi}{2} < |\theta| < \pi.$$

Thus by (33) for all $f = f_1, f_\varepsilon^2, f_\varepsilon^3$

$$I_2(f) \leq C \exp(-Ck), \quad k \geq k_0(f),$$

where $k_0(f_\varepsilon^2), k_0(f_\varepsilon^3)$ do not depend on $\varepsilon, 0 < \varepsilon < 1$.

From (48), (49), (57) and (62) we obtain Propositions 1 and 2.

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