ASYMPTOTIC ANALYSIS OF THE 2D CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS IN UNBOUNDED DOMAINS: GLOBAL ATTRACTORS AND UPPER SEMICONTINUITY

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Abstract. In this work, we carry out the asymptotic analysis of the two dimensional convective Brinkman-Forchheimer (CBF) equations, which characterize the motion of incompressible fluid flows in a saturated porous medium. We establish the existence of a global attractor in both bounded (using compact embedding) and Poincaré domains (using asymptotic compactness property). In Poincaré domains, the estimates for the Hausdorff as well as fractal dimensions of the global attractors are also obtained. We then show an upper semicontinuity of global attractors for the 2D CBF equations. We consider an expanding sequence of simply connected, bounded and smooth subdomains $\Omega_m$ of the Poincaré domain $\Omega$ such that $\Omega_m \to \Omega$ as $m \to \infty$. If $A_m$ and $A$ are the global attractors of the 2D CBF equations corresponding to $\Omega$ and $\Omega_m$, respectively, then we show that for large enough $m$, the global attractor $A_m$ enters into any neighborhood $U(A)$ of $A$. The presence of Darcy term in the CBF equations helps us to obtain the above mentioned results in general unbounded domains also. Finally, we discuss about the quasi-stability property of the semigroup associated with the 2D CBF equations in bounded domains and establish the existence of finite fractal dimensional global as well as exponential attractors.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open connected set (bounded or Poincaré domain). We take smooth boundary in $\mathbb{R}^2$ in the case of bounded domains. We say that $\Omega$ (can be unbounded) is a Poincaré domain (see page 306, [46]) if the Poincaré inequality is satisfied, that is, there exists constant $\lambda_1 > 0$ such that
\[
\int_{\Omega} |\phi(x)|^2 dx \leq \frac{1}{\lambda_1} \int_{\Omega} |\nabla \phi(x)|^2 dx, \quad \text{for all } \phi \in H_0^1(\Omega).
\] (1.1)

Note that if $\Omega$ is bounded in some direction, then the Poincaré inequality holds. For $x = (x_1, x_2) \in \mathbb{R}^2$, one can think $\Omega$ is included in a region of the form $0 < x_1 < L$. This work is concerned about the asymptotic analysis of solutions of the two dimensional convective Brinkman-Forchheimer (CBF) equations in bounded domains, Poincaré domains and general unbounded domains. Let us denote by $u(x, t): \Omega \times [0, T] \to \mathbb{R}^2$, the velocity of the fluid,
\( p(x, t) : \Omega \times [0, T] \rightarrow \mathbb{R} \), the pressure of the fluid and \( f(x, t) : \Omega \times [0, T] \rightarrow \mathbb{R}^2 \), an external time-dependent force. The CBF equations are given by

\[
\begin{aligned}
\frac{\partial u}{\partial t} &- \mu \Delta u + (u \cdot \nabla)u + \alpha |u|^{r-1}u + \nabla p = f, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \Omega \times (0, T), \\
u(0) &= u_0 \quad \text{in } \Omega, \\
\int_{\Omega} p(x, t) dx &= 0, \quad \text{in } (0, T).
\end{aligned}
\]  

(1.2)

The final condition in (1.2) is imposed for the uniqueness of the pressure \( p \). The constant \( \mu \) represents the positive Brinkman coefficient (effective viscosity), the positive constants \( \alpha \) and \( \beta \) represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. The absorption exponent \( r \in [1, \infty) \).

The CBF equations (1.2) describe the motion of incompressible fluid flows in a saturated porous medium (cf. [35]). The model given above is recognized to be more accurate when the flow velocity is too large for the Darcy’s law to be valid alone, in addition, the porosity is not too small. It can be easily seen that for \( \alpha = \beta = 0 \), we obtain the classical 2D Navier-Stokes equations. The nonlinearity of the form \( |u|^{r-1}u \) can be found in tidal dynamics as well as non-Newtonian fluid flows (see [3, 32, 34], etc and the references therein). The global solvability of the system (1.2) in two and three dimensional bounded domains is available in [4, 23, 31], etc.

The study of long time behavior of nonlinear dynamical system is an interesting branch of applied mathematics and it is essential in understanding many natural phenomena. For an extensive study on infinite dimensional dynamical systems in mathematical physics, the interested referred to see [9, 46], etc. The existence of global attractors as well as their finite dimensionality results for the two dimensional Navier-Stokes equations (NSE) in bounded domains have been obtained in [26, 46], etc. The author in [38] showed that the global attractor exists for the 2D NSE in Poincaré domains with forces \( f \in V' \). In this case, the finite dimensionality of the global attractor is also obtained in [38]. The upper semicontinuity of the global attractors for the 2D NSE is established in [19]. For an extensive literature on the existence of global attractors and related problems for the 2D NSE, the interested readers are referred to see [11, 12, 16, 26, 38, 39, 43, 46] etc, and the references therein.

The authors in [21] considered finite energy solutions for the 2D NSE with linear damping in the plane and showed that the corresponding dynamical system possesses a global attractor. Upper bounds for the number of asymptotic degrees (determining modes and nodes) of freedom for the 2D NSE with linear damping in periodic domains is obtained in [20]. The authors in [23] established the existence of regular dissipative solutions and global attractors for the 3D Brinkmann-Forchheimer equations with a nonlinearity of arbitrary polynomial growth rate. Using of the theory of evolutionary systems due to Cheskidov [8], the authors in [19] showed the existence of a strong global attractor for the critical 3D convective Brinkman-Forchheimer equations (\( r = 3 \)). The existence of a global attractor for the Brinkman-Forchheimer equations in bounded domains have been discussed in the works [17, 18], etc. The existence of global attractors for the dynamical systems generated by weak as well as strong solutions of the 3D NSE with damping in bounded domains has been
obtained in the works [28, 37, 41, 42], etc. As mentioned in the paper [23], the major difficulty in working with bounded domains $\Omega \subset \mathbb{R}^3$ is that $P_{\mathbb{H}}(|u|^{r-1}u)$ (here $P_{\mathbb{H}} : L^p(\Omega) \to \mathbb{H}$, $p \in [2, \infty)$, is the Helmholtz-Hodge projection) need not be zero on the boundary, and $P_{\mathbb{H}}$ and $-\Delta$ are not necessarily commuting (for a counter example, see Example 2.19, [40]). Moreover, $\Delta u |_{\partial \Omega} \neq 0$ in general and the term with pressure will not disappear (see [23]), while taking the inner product with $\Delta u$ to the first equation in (1.2). Therefore, the equality (1.3)

$$
\int_{\Omega} (-\Delta u(x)) \cdot |u(x)|^{r-1}u(x) dx
= \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{r-1} dx + 4 \left[ \frac{r-1}{(r+1)^2} \right] \int_{\Omega} |\nabla u(x)|^{\frac{2r}{r+1}}^2 dx
= \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{r-1} dx + \frac{r-1}{4} \int_{\Omega} |u(x)|^{r-3} |\nabla u(x)|^2 dx,
$$

may not be useful in the context of bounded domains. But in periodic domains or whole space the above equality holds true. Due to this technical difficulty, it appears to the author that some of the results obtained in the above works may not be true in bounded domains.

In this work, we study the asymptotic analysis of solutions of (1.2) under the following three cases:

(i) the domain $\Omega \subset \mathbb{R}^2$ is bounded with smooth boundary $\partial \Omega$,

(ii) an unbounded open set in $\mathbb{R}^2$ without any regularity assumption on its boundary $\partial \Omega$ and with the only assumption that the Poincaré inequality (1.1) holds on it (Poincaré domain),

(iii) a general unbounded domain.

We show the existence of global attractors in both bounded and Poincaré domains for the system (1.2). In Poincaré domains case, we are able to prove that the solution map $S(t) : \mathbb{H} \to \mathbb{H}$ is Fréchet differentiable with respect to the initial data for the absorption exponent $r \geq 1$ and hence we obtain estimates for the Hausdorff and Fractal dimensions of such attractors. Then, we establish an upper semicontinuity of global attractors for the 2D CBF equations. We take an expanding sequence of simply connected, bounded and smooth subdomains $\Omega_m$ of the Poincaré domain $\Omega$. If $\mathcal{A}_m$ and $\mathcal{A}$ are the global attractors of (1.2) corresponding to $\Omega$ and $\Omega_m$, respectively, then we show that for large enough $m$, the global attractor $\mathcal{A}_m$ enters into any neighborhood $U(\mathcal{A})$ of $\mathcal{A}$. We remark that the above mentioned results are true in general unbounded domains also (the presence of Darcy term in (1.2) helps us to get such results). Finally, we discuss about the quasi-stability of the semigroup $S(t) : \mathbb{H} \to \mathbb{H}$ and establish the existence of finite fractal dimensional global as well as exponential attractors for the 2D CBF equations in bounded domains.

The rest of the paper is organized as follows. In the next section, we provide an abstract formulation of the system (1.2) and give the necessary function spaces needed to prove the global solvability results of the system (1.2). We also discuss about the existence and uniqueness of the weak as well as strong solutions to the system (1.2) in the same section. In section 3 for $f \in V'$, we show that the system (1.2) in bounded domains possesses a global attractor, using the compact embedding of $V \subset \mathbb{H}$ (Remark 3.6 and Theorem 3.7). The existence of a global attractor for the system (1.2) in Poincaré domains is obtained in section 4 using asymptotic compactness property (Theorem 4.4). For the critical exponent $r \geq 1$, the estimates for the Hausdorff as well as Fractal dimensions of the global attractor...
for the system (1.2) in Poincaré domains is obtained in section 5 (Theorem 5.2). In section 6 we prove an upper semicontinuity of global attractors for the 2D CBF equations (Lemma 6.1 and Theorem 6.5). In the final section, we describe the quasi-stability property of the semigroup associated with the 2D SCBF equations in bounded domains and establish the existence of finite fractal dimensional global as well as exponential attractors (Theorem 7.3) using Ladyzhenskaya’s squeezing property (Proposition 7.1) and the results available in [9].

2. Mathematical Formulation

This section is devoted for providing the necessary function spaces needed to obtain the global solvability results of the system (1.2) as well as to discuss about the well-posedness of the system (1.2).

2.1. Functional setting. Let us define the space
\[ \mathcal{V} := \{ u \in C_0^\infty(\Omega; \mathbb{R}^2) : \nabla \cdot u = 0 \}, \]
where \( C_0^\infty(\Omega; \mathbb{R}^2) \) is the space of all infinitely differentiable functions with compact support in \( \Omega \). Let \( \mathbb{H} \) and \( \mathbb{V} \) denote the completion of \( \mathcal{V} \) in \( L^2(\Omega; \mathbb{R}^2) \) and \( H^1(\Omega; \mathbb{R}^2) \) norms respectively. For bounded domains, under some smoothness assumptions on the boundary, we characterize the spaces \( \mathbb{H} \) and \( \mathbb{V} \) as
\[ \mathbb{H} := \{ u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \}, \]
with the norm defined by \( \| u \|_H := \int_{\Omega} |u(x)|^2 dx \), where \( n \) is the outward normal to the boundary \( \partial \Omega \) and \( u \cdot n|_{\partial \Omega} \) should be understood in the sense of trace in \( H^{-1/2}(\partial \Omega) \) (cf. Theorem 1.2, Chapter 1, [44]),
\[ \tilde{H}^p := \{ u \in L^p(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \}, \]
with the norm denoted by \( \| u \|_{\tilde{H}^p} := \int_{\Omega} |u(x)|^p dx \), for \( p \in (2, \infty) \) and
\[ \mathbb{V} := \{ u \in H^1(\Omega) : \nabla \cdot u = 0 \}, \]
with the norm given by \( \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} |\nabla u(x)|^2 dx \). Using the Poincaré inequality (1.1), one can easily see that this norm is equivalent to the norm \( \| u \|_V^2 := \int_{\Omega} |\nabla u(x)|^2 dx \). The inner product in the Hilbert space \( \mathbb{H} \) is denoted by \( \langle \cdot, \cdot \rangle \), in \( \mathbb{V} \) is represented by \( [\cdot, \cdot] \) and the induced duality, for instance between the spaces \( \mathbb{V} \) and its dual \( \mathbb{V}' \), and \( \tilde{H}^p \) and its dual \( \tilde{H}^{p-1} \) is expressed by \( \langle \cdot, \cdot \rangle \). Note that \( \mathbb{V} \) is densely and continuously embedded into \( \mathbb{H} \) and \( \mathbb{H} \) can be identified with its dual \( \mathbb{H}' \) and we have the Gelfand triple: \( \mathbb{V} \subset \mathbb{H} \equiv \mathbb{H}' \subset \mathbb{V}' \). Remember that for a bounded domain \( \Omega \), the embedding of \( \mathbb{V} \subset \mathbb{H} \) is compact. In the rest of the paper, we also use the notation \( H^2(\Omega) := H^2(\Omega; \mathbb{R}^2) \) for the second order Sobolev spaces.

2.2. Linear operator. Let \( P_H : L^p(\Omega) \to \mathbb{H} \), for \( p \in [2, \infty) \) be the Helmholtz-Hodge projection (1.7). For \( p = 2 \), it is an orthogonal projection and for \( 2 < p < \infty \), \( P_H \) is a bounded linear operator. Let us define
\[ A u := -P_H \Delta u, \quad u \in D(A) := \mathbb{V} \cap H^2(\Omega). \]
Remember that the operator \( A \) is a non-negative, self-adjoint operator in \( \mathbb{H} \) and
\[ \langle A u, u \rangle = \| u \|_V^2, \quad \text{for all} \quad u \in \mathbb{V}, \quad \text{so that} \quad \| A u \|_{\mathbb{V}'} \leq \| u \|_V. \]
Remark 2.1. It is well known that for a bounded domain $\Omega$, the operator $A$ is invertible and its inverse $A^{-1}$ is bounded, self-adjoint and compact in $H$. Hence the spectrum of $A$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues, and there exists an orthogonal basis $\{e_k\}_{k=1}^\infty$ of $H$ consisting of eigenvectors of $A$ such that $Ae_k = \lambda_k e_k$, for all $k \in \mathbb{N}$. Since $u = \sum_{j=1}^\infty (u, e_j)e_j$, we have $Au = \sum_{j=1}^\infty \lambda_j(u, e_j)e_j$. Thus, it is immediate that

$$\|\nabla u\|_H^2 = \langle Au, u \rangle = \sum_{j=1}^\infty \lambda_j |(u, e_j)|^2 \geq \lambda_1 \sum_{j=1}^\infty |(u, e_j)|^2 = \lambda_1 \|u\|_H^2,$$

for all $u \in V$, which is the Poincaré inequality. Note that the constant appearing in (1.1) is different from the first eigenvalue $\lambda_1$ and for convenience, we are using the same notation. In the rest of the paper, these constants will be chosen according to the context. We use the following interpolation inequality in the sequel: If for some $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma = \theta \alpha + (1-\theta)\beta$ and $\theta \in [0, 1]$, then we have (see [9])

$$\|A^\theta u\|_H \leq \|A^{\alpha} u\|_H^{\theta} \|A^{\beta} u\|_H^{1-\theta}, \quad \text{for all } u \in D(A^{\alpha}) \cap D(A^{\beta}) \cap D(A^{\gamma}). \quad (2.2)$$

Remark 2.2. The following important estimates are needed in the rest of the paper:

(i) Using Ladyzhenskaya inequality (Lemma 1, Chapter 1, [25]), we have

$$\|u\|_{L^4} \leq 2^{1/4}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}, \quad u \in H_0^1(\Omega). \quad (2.3)$$

(ii) Applying Gagliardo-Nirenberg inequality (Theorem 1, [36]), we obtain

$$\|u\|_{L^p} \leq C\|u\|_{L^2}^{1-p} \|\nabla u\|_{L^2}^{p}, \quad u \in H^{1/2}_0(\Omega), \quad (2.4)$$

for all $p \in [2, \infty)$ and

$$\|\nabla u\|_{L^4} \leq C\|u\|_{L^2}^{3/4} \|u\|_{L^2}^{1/4}, \quad u \in H^2(\Omega). \quad (2.5)$$

From the inequality (2.4) and Poincaré’s inequality (1.1), it is clear that $V \subset L^p$, for all $p \in [2, \infty)$.

(iii) An application of Agmon’s inequality yields

$$\|u\|_{L^\infty} \leq C\|u\|_{H^{1/2}}^{1/2} \|u\|_{H^2}^{1/2}, \quad u \in H^2(\Omega). \quad (2.6)$$

2.3. Trilinear form and bilinear operator. We define the trilinear form $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ by

$$b(u, v, w) = \int_{\Omega} (u(x) \cdot \nabla)v(x) \cdot w(x)dx = \sum_{i,j=1}^2 \int_{\Omega} u_i(x) \frac{\partial v_j(x)}{\partial x_i}w_j(x)dx.$$ 

If $u, v$ are such that the linear map $b(u, v, \cdot)$ is continuous on $V$, the corresponding element of $V'$ is denoted by $B(u, v)$. We represent $B(u) = B(u, u) = P_{H}(u \cdot \nabla)u$. Using an integration by parts, it is immediate that

$$\begin{cases} b(u, v, v) = 0, \quad \text{for all } u, v \in V, \\ b(u, v, w) = -b(u, w, v), \quad \text{for all } u, v, w \in V. \end{cases} \quad (2.7)$$

It can be easily seen that $B$ maps $L^4$ (and so $V$) into $V'$ and

$$|\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \|u\|_{L^4}^2 \|\nabla v\|_H \leq \sqrt{2}\|u\|_H \|\nabla u\|_H \|v\|_V,$$
for all \( \mathbf{v} \in \mathbb{V} \), so that
\[
\|B(\mathbf{u})\|_V \leq \sqrt{2}\|\mathbf{u}\|_H\|\nabla \mathbf{u}\|_H \leq \frac{\sqrt{2}}{\lambda_1^{1/4}}\|\mathbf{u}\|_V^2,
\] for all \( \mathbf{u} \in \mathbb{V} \), (2.8)
using the Poincaré inequality.

2.4. Nonlinear operator. Next, we define the nonlinear operator
\[
C(\mathbf{u}) := P_H(|\mathbf{u}|^{r-1}\mathbf{u}).
\]
It can be easily seen that \( \langle C(\mathbf{u}), \mathbf{u} \rangle = \|\mathbf{u}\|_{L^{r+1}}^{r+1} \) and the operator \( C(\cdot) : \tilde{L}^{r+1} \to \tilde{L}^{r+1} \). Also, the Gateaux derivative of \( C(\cdot) \) is given by \( C'(\mathbf{u})\mathbf{v} = rP_H(|\mathbf{u}|^{r-1}\mathbf{v}) \), for all \( \mathbf{u}, \mathbf{v} \in \tilde{L}^{r+1} \). Furthermore, for any \( r \in [1, \infty) \) and \( \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V} \), we have (cf. [31]),
\[
\langle C(\mathbf{u}_1) - C(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \geq 0.
\] (2.9)

2.5. Abstract formulation, weak and strong solutions. Let us take the projection \( P_H \) onto the system (1.2) to write down the abstract formulation of the system (1.2) as:
\[
\begin{cases}
\frac{d\mathbf{u}(t)}{dt} + \mu \mathbf{u}(t) + B(\mathbf{u}(t)) + \alpha \mathbf{u}(t) + \beta C(\mathbf{u}(t)) = f(t), & t \in (0, T),
\mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{H},
\end{cases}
\] (2.10)
where \( f \in L^2(0, T; \mathbb{V}') \). For convenience, we used \( f \) instead of \( P_H f \). Let us now provide the definition of weak and strong solutions of the system (2.10).

Definition 2.3. A function \( \mathbf{u} \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \) with \( \partial_t \mathbf{u} \in L^2(0, T; \mathbb{V}') \), is called a weak solution to the system (2.10), if for \( f \in L^2(0, T; \mathbb{V}') \), \( \mathbf{u}_0 \in \mathbb{H} \) and \( \mathbf{v} \in \mathbb{V} \), \( \mathbf{u}(\cdot) \) satisfies:
\[
\begin{cases}
\langle \partial_t \mathbf{u}(t) + \mu \mathbf{u}(t) + B(\mathbf{u}(t)) + \alpha \mathbf{u}(t) + \beta C(\mathbf{u}(t)), \mathbf{v} \rangle = \langle f(t), \mathbf{v} \rangle,
\lim_{t \to 0} \int_\Omega \mathbf{u}(t)\mathbf{v}dx = \int_\Omega \mathbf{u}_0\mathbf{v}dx,
\end{cases}
\] (2.11)
and the energy equality
\[
\frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \mu \|\mathbf{u}(t)\|_{\mathbb{V}}^2 + \alpha \|\mathbf{u}(t)\|_{\mathbb{H}}^2 + \beta \|\mathbf{u}(t)\|_{L^{r+1}}^{r+1} = \langle f(t), \mathbf{u}(t) \rangle,
\] (2.12)
for a.e. \( t \in (0, T) \).

Definition 2.4. A function \( \mathbf{u} \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A)) \) with \( \partial_t \mathbf{u} \in L^2(0, T; \mathbb{V}') \), \( \mathbf{u}_0 \in \mathbb{V} \), \( \mathbf{u}(\cdot) \) satisfies (2.10) as an equality in \( \mathbb{V} \), for a.e. \( t \in (0, T) \).

Remark 2.5. 1. Using Gagliardo-Nirenberg’s interpolation inequality, we obtain
\[
\int_0^T \|C(\mathbf{u}(t))\|_H^2 dt \leq \int_0^T \|\mathbf{u}(t)\|_{L^{2r}}^{2r} dt \leq C \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{L^{2r}}^{2r-2} \int_0^T \|\mathbf{u}(t)\|_V^2 dt < \infty.
\] (2.13)
2. Since \( \mathbb{V} \subset \mathbb{H} \subset \mathbb{V}' \), \( \mathbf{u} \in L^2(0, T; \mathbb{V}) \) and \( \partial_t \mathbf{u} \in L^2(0, T; \mathbb{V}') \), an application of Theorem 3, section 5.9, [15] yields \( \mathbf{u} \in C([0, T]; \mathbb{V}) \) and the following energy equality is satisfied:
\[
\|\mathbf{u}(t)\|_H^2 + 2\mu \int_0^t \|\mathbf{u}(s)\|_V^2 ds + 2\alpha \int_0^t \|\mathbf{u}(s)\|_H^2 ds + 2\beta \int_0^t \|\mathbf{u}(s)\|_{L^{r+1}}^{r+1} ds
\]
\[
= \|\mathbf{u}_0\|_H^2 + 2 \int_0^t \langle f, \mathbf{u}(s) \rangle ds,
\] (2.14)
for all \( t \in [0, T] \).

3. Since \( \mathbb{H} \subset \mathbb{V} \subset D(A) \), the condition \( u \in L^2(0, T; D(A)) \) with \( \partial_t u \in L^2(0, T; \mathbb{H}) \) implies that \( u \in C([0, T]; \mathbb{V}) \).

Let us now state the existence and uniqueness theorem for the system (2.10). A proof of the following theorem can be obtained from Theorem 3.4, [31]. A local monotonicity property of the linear and nonlinear operators and a generalization of the Minty-Browder technique are exploited in the proof of Theorem 3.4, [31].

**Theorem 2.6.** For \( u_0 \in \mathbb{H} \) and \( f \in L^2(0, T; \mathbb{V}') \), there exists a unique weak solution \( u(\cdot) \) to the system (2.10) in the sense of Definition 2.3 satisfying the energy equality (2.14).

**Theorem 2.7.** For \( u_0 \in \mathbb{V} \) and \( f \in L^2(0, T; \mathbb{H}) \), there exists a unique strong solution \( u(\cdot) \) to the system (2.10) in the sense of Definition 2.4.

In two dimensional bounded and Poincaré domains, one can obtain the existence of a unique strong solution with the regularity given in Theorem 2.7 (cf. [31]) using the estimate

\[
\| (C(u), A u) \| \leq \| C(u) \|_{\mathbb{H}} \| A u \|_{\mathbb{H}} = \| u \|_{\mathbb{L}^2} \| A u \|_{\mathbb{H}} \leq C \| u \|_{\mathbb{H}}^{-1} \| u \|_{\mathbb{V}} \| A u \|_{\mathbb{H}} \\
\leq \frac{\mu}{4} \| A u \|_{\mathbb{H}}^2 + \frac{1}{\mu} \| u \|_{\mathbb{L}^{2r-2}}^2 \| u \|_{\mathbb{V}}^2.
\]

But in three dimensional bounded domains, due to the technical difficulty described in [23, 31] (see (1.3) also), getting a strong solution with the regularity given in Theorem 2.7 may not be possible.

### 3. Global Attractor (Bounded Domains)

In this section, we discuss the existence of a global attractor for the CBF equations in two dimensional bounded domains. We assume that \( f \in \mathbb{V}' \) is independent of \( t \) in (2.10).

**Lemma 3.1.** Let \( u(\cdot) \) be the unique weak solution of the system (2.10). Then, we have

\[
\frac{1}{t} \int_0^t \left[ \mu \| u(s) \|_{\mathbb{V}}^2 + \alpha \| u(s) \|_{\mathbb{H}}^2 + \beta \| u(s) \|_{\mathbb{L}^{r+1}}^{r+1} \right] \mathrm{d}s \leq \frac{\| u_0 \|_{\mathbb{H}}^2}{\mu} + \frac{1}{\mu} \| f \|_{\mathbb{V}'}^2. \tag{3.1}
\]

**Proof.** Let us take the inner product with \( u(\cdot) \) to the first equation in (2.10) to get

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{\mathbb{H}}^2 + \mu \| \nabla u(t) \|_{\mathbb{H}}^2 + \alpha \| u(t) \|_{\mathbb{H}}^2 + \beta \| u(t) \|_{\mathbb{L}^{r+1}}^{r+1} = \langle f, u(t) \rangle, \tag{3.2}
\]

where we used the fact that \( \langle B(u), u \rangle = 0 \). Using the Cauchy-Schwarz inequality and Young’s inequality, we estimate \( \| f, u \| \) as

\[
| \langle f, u \rangle | \leq \| f \|_{\mathbb{V}'} \| u \|_{\mathbb{V}} \leq \frac{\mu}{2} \| u \|_{\mathbb{V}}^2 + \frac{1}{2\mu} \| f \|_{\mathbb{V}'}^2. \tag{3.3}
\]

Applying (3.3) in (3.2) and then integrating from 0 to \( t \), we obtain

\[
\| u(t) \|_{\mathbb{H}}^2 + \mu \int_0^t \| u(s) \|_{\mathbb{V}}^2 \mathrm{d}s + 2\alpha \int_0^t \| u(s) \|_{\mathbb{H}}^2 \mathrm{d}s + 2\beta \int_0^t \| u(s) \|_{\mathbb{L}^{r+1}}^{r+1} \mathrm{d}s \leq \| u_0 \|_{\mathbb{H}}^2 + \frac{t}{\mu} \| f \|_{\mathbb{V}'}^2. \tag{3.4}
\]

From (3.4), we easily obtain (3.1). \( \square \)
Note that whenever a Cauchy problem on a Banach space $X$ is well posed in the sense of Hadamard (for proper notion of weak solution) for all initial data $x(0) = x_0 \in X$, then the corresponding (forward) solutions $x(t)$ can be written in the form $x(t) = S(t)x_0$, where the semigroup $S(t)$ is uniquely determined by the dynamical system. Let $u(t), t \geq 0$ be the unique weak solution of the system (2.10) with $u_0 \in \mathbb{H}$. Thanks to the existence and uniqueness of weak solution for the system (2.10) (Theorem 2.6), we can define a continuous semigroup \{S(t)\}_{t \geq 0} in $\mathbb{H}$ by

$$S(t)u_0 = u(t), \quad t \geq 0,$$

where $u(\cdot)$ is the unique weak solution of (2.10) with $u(0) = u_0 \in \mathbb{H}$.

Next, we prove the existence of a global attractor for the semigroup $S(t), t \geq 0$ defined on $\mathbb{H}$ for the 2D CBF (2.10) in bounded domains. Our first aim is to establish the existence of an absorbing ball in $\mathbb{H}$ for $S(t), t \geq 0$. We use the compact embedding of $V$ in $\mathbb{H}$ to get the existence of a global attractor for $S(t), t \geq 0$ in $\mathbb{H}$. In the next result, we show that the operator $S(t) : \mathbb{H} \to \mathbb{H}$, for $t \geq 0$, is Lipschitz continuous on bounded subsets of $\mathbb{H}$.

**Lemma 3.2.** The map $S(t) : \mathbb{H} \to \mathbb{H}$, for $t \geq 0$, is Lipschitz continuous on bounded subsets of $\mathbb{H}$.

**Proof.** Let us take $S(t)u_0 = u(t)$ and $S(t)v_0 = v(t)$, for all $t \geq 0$. Then $w(t) := u(t) - v(t)$ satisfies:

$$\frac{d}{dt}w(t) + \mu Aw(t) + B(w(t), u(t)) + B(v(t), w(t)) + \alpha w(t)
+ \beta[C(u(t)) - C(v(t))] = 0,$$  \label{3.6}

in $V'$ for a.e. $t \in (0,T)$. Let us take the inner product with $w(\cdot)$ to the first equation in (3.6) to find

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{\mathbb{H}}^2 + \mu \|\nabla w(t)\|_{\mathbb{H}}^2 + \alpha \|w(t)\|_{\mathbb{H}}^2 + \beta \langle C(u(t)) - C(v(t)), w(t) \rangle = -b(w(t), u(t), w(t)),$$  \label{3.7}

where we used the fact that $b(v, w, w) = 0$. Integrating from 0 to $t$, we obtain

$$\|w(t)\|_{\mathbb{H}}^2 + 2\mu \int_0^t \|w(s)\|_{\mathbb{H}}^2 ds + 2\alpha \int_0^t \|w(s)\|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \langle C(u(s)) - C(v(s)), w(s) \rangle ds$$

$$= \|w_0\|_{\mathbb{H}}^2 - 2 \int_0^t b(w(s), u(s), w(s)) ds.$$  \label{3.8}

Using Hölder’s and Ladyzhenskaya’s inequalities, we estimate $|b(w, u, w)|$ as

$$|b(w, u, w)| \leq \|w\|_{L^4}^2 \|\nabla u\|_{L^4} \leq \sqrt{2} \|w\|_{\mathbb{H}} \|\nabla w\|_{\mathbb{H}} \|\nabla u\|_{\mathbb{H}} \leq \frac{\mu}{2} \|w\|_{\mathbb{H}}^2 + \frac{1}{\mu} \|u\|_{\mathbb{H}}^2 \|w\|_{\mathbb{H}}^2.$$  \label{3.9}

Let us use (3.9) and (2.9) in (3.8) to deduce that

$$\|w(t)\|_{\mathbb{H}}^2 + \mu \int_0^t \|w(s)\|_{\mathbb{H}}^2 ds \leq \|w_0\|_{\mathbb{H}}^2 + \frac{2}{\mu} \int_0^t \|u(s)\|_{\mathbb{H}}^2 \|w(s)\|_{\mathbb{H}}^2 ds.$$  \label{3.10}

Applying Gronwall’s inequality in (3.10), we obtain

$$\|w(t)\|_{\mathbb{H}}^2 \leq \|w_0\|_{\mathbb{H}}^2 \exp \left( \frac{2}{\mu} \int_0^t \|u(s)\|_{\mathbb{H}}^2 ds \right),$$  \label{3.11}
and
\[ \|w(t)\|_H^2 + \mu \int_0^t \|w(s)\|_V^2 \, ds \leq \|w_0\|_H^2 \exp \left( \frac{4}{\mu} \int_0^t \|u(s)\|_V^2 \, ds \right), \]

for all \( t \in [0, T] \). Thus, it is immediate that
\[
\|S(t)u_0 - S(t)v_0\|_H \leq \|u_0 - v_0\|_H \exp \left( \frac{1}{\mu} \int_0^t \|u(s)\|_V^2 \, ds \right)
\]

\[
\leq \|u_0 - v_0\|_H \exp \left\{ \frac{1}{\mu^2} \left( \|u_0\|_H^2 + \frac{\mu \|f\|^2}{\mu^2} \right) \right\},
\]

where we used (3.4). Thus, the map \( S(t) : \mathbb{H} \to \mathbb{H} \), for \( t \geq 0 \), is Lipschitz continuous on bounded subsets of \( \mathbb{H} \). \( \square \)

### 3.1. Absorbing ball in \( \mathbb{H} \)

In this subsection, we show that the semigroup \( S(t) \) has a bounded absorbing set in \( \mathbb{H} \).

**Proposition 3.3.** The set
\[
B_1 := \left\{ v \in \mathbb{H} : \|v\|_H \leq M_1 \equiv \frac{1}{\mu} \sqrt{\frac{2}{\lambda_1}} \|f\|_V' \right\},
\]

is a bounded absorbing set in \( \mathbb{H} \) for the semigroup \( S(t) \). That is, given a bounded set \( B \subset \mathbb{H} \), there exists an entering time \( t_B > 0 \) such that \( S(t)B \subset B_1 \), for all \( t \geq t_B \).

**Proof.** From (3.2), we have
\[
\frac{d}{dt}\|u(t)\|_H^2 + \mu\lambda_1\|u(t)\|_H^2 + 2\alpha\|u(t)\|_H^2 + 2\beta\|u(t)\|_{L^{r+1}}^2 \leq \frac{1}{\mu}\|f\|_V'^2,
\]

and an application of Gronwall’s inequality yields
\[
\|u(t)\|_H^2 \leq \|u_0\|_H^2 e^{-\mu\lambda_1 t} + \frac{1}{\mu^2\lambda_1}\|f\|_V'^2,
\]

for all \( t \geq 0 \). Moreover, we have
\[
\limsup_{t \to \infty} \|u(t)\|_H^2 \leq \frac{1}{\mu^2\lambda_1}\|f\|_V'^2.
\]

That is, we obtain
\[
\limsup_{t \to \infty} \|u(t)\|_H \leq \frac{1}{\sqrt{\mu^2\lambda_1}} \|f\|_V'.
\]

Furthermore, it follows from (3.16) and (3.18) that the set (3.14) is absorbing in \( \mathbb{H} \) for the semigroup \( S(t) \). Hence, the following uniform estimate is valid:
\[
\|u(t)\|_H \leq M_1, \text{ where } M_1 = \sqrt{\frac{2}{\mu^2\lambda_1}} \|f\|_V',
\]

for \( t \) large enough (\( t \gg 1 \) or \( t \geq t_B \)) depending on the initial data. From (3.2), we infer that
\[
\mu \int_t^{t+\theta} \|u(s)\|_V^2 \, ds + 2\beta \int_t^{t+\theta} \|u(s)\|_{L^{r+1}}^2 \, ds \leq \|u(t)\|_H^2 + \frac{\theta}{\mu}\|f\|_V'^2,
\]

(3.20)
for all $\theta > 0$. Using (3.17) in (3.20), we find
\[
\limsup_{t \to \infty} \left[ \mu \int_t^{t+\theta} \|u(s)\|_V^2 ds + \beta \int_t^{t+\theta} \|u(s)\|_{E^{r+1}}^2 ds \right] \leq \frac{1}{\mu} \left( \frac{1}{\mu \lambda_1} + \theta \right) \|f\|_{V^2}^2. \tag{3.21}
\]

For $u_0 \in B_1$, we further have
\[
\sup_{t \geq 0} \sup_{u_0 \in B_1} \|u(t)\|_H^2 \leq \frac{3}{\mu^2 \lambda_1} \|f\|_{V^2}^2 =: M_1^2. \tag{3.22}
\]
and
\[
\int_t^{t+\theta} \left[ \mu \|u(s)\|_V^2 + \beta \|u(s)\|_{E^{r+1}}^2 \right] ds \leq \frac{1}{\mu} \left( \frac{3}{\mu \lambda_1} + \theta \right) \|f\|_{V^2}^2. \tag{3.23}
\]
for all $t \geq 0$ and $\theta > 0$.

3.2. **Absorbing ball in $V$.** Let us now show that the semigroup $S(t)$ has an absorbing ball in $V$.

**Proposition 3.4.** For $f \in H$ and $r \in [1, \infty)$, we have
\[
\|u(t)\|_V^2 \leq \frac{1}{\mu} \left\{ \|f\|_H^2 + M_1(M_1 + 2\|f\|_H) \right\} e^{\left\{ \frac{C M_2}{\mu^3}(M_1 + 2\|f\|_H) + \frac{C M_2 M_2^2 - 2}{\mu} \right\}} = M_2, \tag{3.24}
\]
for all $t \geq t_B + r$, where $r > 0$ and $t_B$ depends on the initial data given in Proposition 3.3.

**Proof.** Let us take the inner product with $Au(\cdot)$ the first equation in (2.10) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_H^2 + \mu \|Au(t)\|_H^2 + \alpha \|\nabla u(t)\|_H^2
= - \langle B(u(t)), Au(t) \rangle - \langle C(u(t)), Au(t) \rangle + \langle f, Au(t) \rangle. \tag{3.25}
\]
We estimate $|\langle f, Au(t) \rangle|$, using Cauchy-Schwarz and Young’s inequalities as
\[
|\langle f, Au(t) \rangle| \leq \|f\|_H \|Au(t)\|_H \leq \frac{\mu}{4} \|Au(t)\|_H^2 + \frac{1}{\mu} \|f\|_H^2. \tag{3.26}
\]
Using Hölder’s, Agmon’s and Young’s inequalities, we estimate $|\langle B(u), Au \rangle|$ as
\[
|\langle B(u), Au \rangle| \leq \|u\|_{L^\infty} \|u\|_V \|Au\|_H \leq C \|u\|_{H}^{1/2} \|u\|_V \|Au\|_H^{3/2}
\leq \frac{\mu}{4} \|Au\|_H^2 + \frac{C}{\mu^3} \|u\|_H^2 \|u\|_V^4. \tag{3.27}
\]
For $r \in [1, \infty)$, we estimate $\beta |\langle C(u), Au \rangle|$ using Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities as
\[
\beta |\langle C(u), Au \rangle| \leq \beta |\langle C(u)\|_H \|Au\|_H \leq \beta \|u\|_{E^{2r}} \|Au\|_H \leq C \beta \|u\|_{E^{r+1}} \|u\|_V \|Au\|_H
\leq \frac{\mu}{4} \|Au\|_H^2 + \frac{C \beta^2}{\mu} \|u\|_{E^{2r-2}}^2 \|u\|_V^2. \tag{3.28}
\]
Note that $|\langle f, u \rangle| \leq \|f\|_H \|u\|_H \leq M_1 \|f\|_H$, for all $t \geq t_B$. From (3.2), we obtain
\[
\mu \int_0^t \|u(s)\|_V^2 ds \leq \|u_0\|_H^2 + 2\|f\|_H \|u(t)\|_H \leq \|u_0\|_H^2 + 2M_1 \|f\|_H
\leq M_1(M_1 + 2\|f\|_H), \tag{3.29}
\]
for $u_0 \in B_1$ (otherwise on can keep $\|u_0\|_H$ as such in (3.29) and take $t \geq t_B$).
Moreover, multiplying (3.25) by $t$, using (3.26)–(3.28), we also infer that
\[
\frac{d}{dt} [t\|u(t)\|^2_V] + \frac{\mu t}{2} \|Au(t)\|^2_H + 2\alpha t\|u(t)\|^2_V \\
\leq \frac{2t}{\mu} \|f\|^2_H + \|u(t)\|^2_V + \frac{CM^2}{\mu^3} t\|u(t)\|^4_V + \frac{C\beta^2}{\mu} t\|u(t)\|^2r-2 \|u(t)\|^2_V, \tag{3.30}
\]
for $r \geq 1$. Integrating the above inequality from 0 to $t$, we find
\[
t\|u(t)\|^2_V + \frac{\mu}{\mu} \int_0^t s\|Au(s)\|^2_H ds + 2\alpha \int_0^t s\|u(s)\|^2_V ds \\
\leq \frac{t^2}{\mu} \|f\|^2_H + \int_0^t \|u(s)\|^2_V ds + \frac{CM^2}{\mu^3} \int_0^t s\|u(s)\|^4_V ds + \frac{C\beta^2 M^2r-2}{\mu} \int_0^t s\|u(s)\|^2_V ds. \tag{3.31}
\]
Applying Gronwall’s inequality in (3.31), we further have
\[
t\|u(t)\|^2_V \leq \left( \frac{t^2}{\mu} \|f\|^2_H + \int_0^t \|u(s)\|^2_V ds \right) \exp \left\{ \frac{CM^2}{\mu^3} \int_0^t \|u(s)\|^4_V ds + \frac{C\beta^2 M^2r-2}{\mu} \right\} \\
\leq \left( \frac{t^2}{\mu} \|f\|^2_H + \frac{M^2}{\mu} (M_1 + 2\|f\|_H) \right) \exp \left\{ \frac{CM^2}{\mu^3} (M_1 + 2\|f\|_H) + \frac{C\beta^2 M^2r-2}{\mu} \right\}, \tag{3.32}
\]
where we used (3.29). It can be further deduced that
\[
\sup_{0<\varepsilon<\tau \leq 1} \|u(t)\|^2_V \leq \left( \frac{1}{\mu} \|f\|^2_H + \frac{M^2}{\mu} (M_1 + 2\|f\|_H) \right) \exp \left\{ \frac{CM^2}{\mu^3} (M_1 + 2\|f\|_H) + \frac{C\beta^2 M^2r-2}{\mu} \right\}. \tag{3.33}
\]
Since our problem (2.10) is invariant under $t$-translations, from (3.33), we get a uniform estimate for $\|u(t)\|^2_V$, $t > \gamma \varepsilon > 0$, that is,
\[
\sup_{0<\varepsilon<\tau \leq \infty} \|u(t)\|^2_V \leq \left\{ \|f\|^2_H + M_1 (M_1 + 2\|f\|_H) \right\} \exp \left\{ \frac{CM^2}{\mu^3} (M_1 + 2\|f\|_H) + \frac{C\beta^2 M^2r-2}{\mu} \right\}, \tag{3.34}
\]
for $0 < \varepsilon \leq 1$, and hence the estimate (3.24) follows.

**Remark 3.5.** 1. From Proposition 3.4, it is clear that for any $u_0 \in H$,
\[
\|S(t)u_0\|_V \leq M_2, \text{ for all } t \geq t_B + r, \tag{3.35}
\]
for all $r > 0$. We denote $B_2$ as the absorbing ball of radius $M_2$ in $V$.

2. Using (3.26)–(3.28) in (3.25) and then integrating from $t$ to $t + \theta$, we find
\[
\|u(t + \theta)\|^2_V + \frac{\mu}{2} \int_t^{t+\theta} \|Au(s)\|^2_H ds \\
\leq \|u(t)\|^2_V + \frac{2\theta}{\mu} \|f\|^2_H + \frac{C}{\mu^3} \int_t^{t+\theta} \|u(s)\|^2_H \|u(s)\|^4_V ds + \frac{C\beta^2}{\mu} \int_t^{t+\theta} \|u(s)\|^2r-2 \|u(s)\|^2_V ds \\
\leq \left( M_2^2 + \frac{2\theta}{\mu} \|f\|^2_H + \frac{CM^2 M_1^2 \theta}{\mu^3} + \frac{C\beta^2 M^2r-2 M_2^2 \theta}{\mu} \right), \tag{3.36}
\]
for all $\theta \geq 0$ and $t \geq t_B + r$. 


Remark 3.6. For \( f \in \mathbb{V}' \), one can show the existence of an absorbing ball in \( \mathbb{V} \) in the following way. These type of estimates for the 2D Navier-Stokes equations are due to Ladyzhenskaya (see [27]). Taking the inner product with \( u_t(\cdot) \) to the first equation in (2.10), we find

\[
\|u_t(t)\|_{H^2}^2 + \frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{H^2}^2 + \frac{\alpha}{2} \frac{d}{dt} \|u(t)\|_{H^2}^2 \\
= -b(u(t), u(t), u_t(t)) - \beta (\mathcal{C}(u(t)), u_t(t)) + \langle f, u_t(t) \rangle.
\]

We estimate the three terms on the right hand side of the equality (3.37) as

\[
|\mathcal{C}(u_t)| \leq \|u\|_{L^4} \|\nabla u_t\|_{H^1}, \\
|\langle f, u_t \rangle| \leq \|f\|_{\mathbb{V}'} \|\nabla u_t\|_{H^1}.
\]

Using the above estimates in (3.37), we obtain

\[
\|u_t(t)\|_{H^2}^2 + \frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{H^2}^2 + \frac{\alpha}{2} \frac{d}{dt} \|u(t)\|_{H^2}^2 \\
\leq \|u(t)\|_{L^4}^2 \|\nabla u_t(t)\|_{H^1} + \|u(t)\|_{L^4}^2 \|\nabla u_t(t)\|_{H^1} + \|f\|_{\mathbb{V}'} \|\nabla u_t(t)\|_{H^1}.
\]

for \( r \in [1, \infty) \).

Let us now differentiate (2.10) with respect to \( t \) again to find

\[
u_{tt}(t) + \mu A u_t(t) + \alpha u_t(t) = -B(u(t), u_t(t)) - B(u_t(t), u(t)) - \beta \mathcal{C}'(u(t)) u_t(t),
\]

for all \( t \in (0, T) \). Taking the inner product with \( u_t(\cdot) \) to the first equation in (3.39), we get

\[
\frac{1}{2} \frac{d}{dt} \|u_t(t)\|_{H^2}^2 + \frac{\mu}{2} \|\nabla u_t(t)\|_{H^2}^2 + \frac{\alpha}{2} \|u_t(t)\|_{H^2}^2 + \frac{\beta}{2} \|u(t)\|_{L^4}^2 \|u_t(t)\|_{H^1}^2 = -b(u_t(t), u(t), u_t(t)).
\]

Using Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate \( b(u_t, u, u_t) \) as

\[
|b(u_t, u, u_t)| \leq \|\nabla u\|_{\mathbb{V}} \|u_t\|_{L^4}^2 \leq \sqrt{2} \|\nabla u\|_{\mathbb{V}} \|u_t\|_{H^1} \|u_t\|_{H^1} \leq \frac{\mu}{2} \|\nabla u_t\|_{H^1}^2 + \frac{1}{\mu} \|\nabla u\|_{\mathbb{V}}^2 \|u_t\|_{H^1}^2.
\]

Substituting (3.41) in (3.40), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|u_t(t)\|_{H^2}^2 + \frac{\mu}{2} \|\nabla u_t(t)\|_{H^2}^2 + \frac{\alpha}{2} \|u_t(t)\|_{H^2}^2 + \frac{\beta}{2} \|u(t)\|_{L^4}^2 \|u_t(t)\|_{H^1}^2 \leq \frac{1}{\mu} \|\nabla u(t)\|_{H^1}^2 \|u_t(t)\|_{H^2}^2.
\]

Using (3.38) and (3.42), we further have

\[
\frac{d}{dt} \left[ \|u_t(t)\|_{H^2}^2 + 2\mu t \|\nabla u(t)\|_{H^2}^2 + 2\alpha t^2 \|u_t(t)\|_{H^2}^2 + 2\beta t^2 \|u(t)\|_{L^4}^2 \|u_t(t)\|_{H^1}^2 \right] \\
\leq 2t \|u_t(t)\|_{H^2}^2 + 2\mu t \|\nabla u(t)\|_{H^2}^2 + 2\alpha t^2 \|u_t(t)\|_{H^2}^2 + 2\beta t^2 \|u(t)\|_{L^4}^2 \|u_t(t)\|_{H^1}^2 \\
+ 4t \|u_t(t)\|_{H^1} \|u(t)\|_{H^1} + 4t \|f\|_{\mathbb{V}'} \|\nabla u_t(t)\|_{H^1}.
\]
We estimate the terms \(4t\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}, 4t\|f\|_{V'}\|\nabla\mathbf{u}\|_{L^2}\) and \(4\beta t\|\mathbf{u}\|_{H}^2\|\nabla\mathbf{u}\|_{H}^2\) as follows:

\[
4t\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2} \leq 4\sqrt{2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} \leq \varepsilon \mu t^2\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{8}{\varepsilon \mu}\|\mathbf{u}\|_{H}^2\|\nabla\mathbf{u}\|_{H}^2,
\]

\[
4\gamma t\|f\|_{V'}\|\nabla\mathbf{u}\|_{L^2} \leq \varepsilon \mu t^2\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{4}{\varepsilon \mu}\|f\|_{V'}^2,
\]

\[
4\beta t\|\mathbf{u}\|_{H}^2\|\nabla\mathbf{u}\|_{H}^2 \leq \beta t^2\|\mathbf{u}\|_{H}^2\|\nabla\mathbf{u}\|_{H}^2 + 4\|\mathbf{u}\|_{H}^2^2,
\]

where we used Ladyzhenskaya’s and Young’s inequalities. Using the above estimates in (3.43) and choosing \(\varepsilon = \frac{1}{2}\), we deduce that

\[
\frac{d}{dt}\left[ t^2\|\mathbf{u}(t)\|_{H}^2 + 2\mu t\|\nabla\mathbf{u}(t)\|_{H}^2 \right] + \frac{\mu t^2}{2}\|\nabla\mathbf{u}(t)\|_{H}^2 + 2\alpha t^2\|\mathbf{u}(t)\|_{H}^2 + \beta t^2\|\mathbf{u}(t)\|_{H}^2^2 \leq 2\mu\|\nabla\mathbf{u}(t)\|_{H}^2^2 + \frac{t^2}{\mu}\|\nabla\mathbf{u}(t)\|_{H}^2\|\mathbf{u}(t)\|_{H}^2 + 32\|\mathbf{u}(t)\|_{H}^2\|\nabla\mathbf{u}(t)\|_{H}^2 + 16\|f\|_{V'}^2 + 4\|\mathbf{u}(t)\|_{H}^2^2.
\]

(3.44)

Integrating the inequality (3.44), we find

\[
t^2\|\mathbf{u}(t)\|_{H}^2 + 2\mu\|\nabla\mathbf{u}(t)\|_{H}^2 \leq 2\mu\int_0^t\|\nabla\mathbf{u}(s)\|_{H}^2 ds + \frac{1}{\mu}\int_0^t s^2\|\nabla\mathbf{u}(s)\|_{H}^2\|\mathbf{u}(s)\|_{H}^2 ds + 32\int_0^t\|\mathbf{u}(s)\|_{H}^2\|\nabla\mathbf{u}(s)\|_{H}^2 ds + \frac{16t}{\mu}\|f\|_{V'}^2 + 4\int_0^t\|\mathbf{u}(s)\|_{H}^2^2 ds.
\]

(3.45)

An application of Gronwall’s inequality in (3.45) yields

\[
t^2\|\mathbf{u}(t)\|_{H}^2 + 2\mu\|\nabla\mathbf{u}(t)\|_{H}^2 \leq \left\{ 2\mu\int_0^t\|\mathbf{u}(s)\|_{H}^2 ds + \frac{16t}{\mu}\|f\|_{V'}^2 + 4\int_0^t\|\mathbf{u}(s)\|_{H}^2^2 ds + 16\mu\int_0^t\|\mathbf{u}(s)\|_{H}^2\|\mathbf{u}(s)\|_{H}^2 ds \right\} \times \exp\left\{ \frac{1}{\mu}\int_0^t\|\mathbf{u}(s)\|_{H}^2^2 ds \right\}.
\]

(3.46)

Thus, from (3.46), it is immediate that

\[
t\|\mathbf{u}(t)\|_{V'}^2 \leq \frac{1}{\mu}\left\{ \left( M_1^2 + \frac{t}{\mu}\|f\|_{V'}^2 \right) + \frac{8t}{\mu}\|f\|_{V'}^2 + 2tM_1^2 + \frac{16M_1^2}{\mu^2}\left( M_1^2 + \frac{t}{\mu}\|f\|_{V'}^2 \right) \right\} \times \exp\left\{ \frac{1}{\mu}\left( M_1^2 + \frac{t}{\mu}\|f\|_{V'}^2 \right) \right\}.
\]

(3.47)

where we used (3.4) for \(\mathbf{u}_0 \in \mathcal{B}_1\) and \(t \geq t_B\). It can be further deduced that

\[
\sup_{0 < \varepsilon < t \leq 1}\|\mathbf{u}(t)\|_{V'}^2 \leq \frac{1}{\mu\varepsilon}\left\{ \left( M_1^2 + \frac{1}{\mu}\|f\|_{V'}^2 \right) + \frac{8}{\mu}\|f\|_{V'}^2 + 2M_1^2 + \frac{16M_1^2}{\mu^2}\left( M_1^2 + \frac{1}{\mu}\|f\|_{V'}^2 \right) \right\} \times \exp\left\{ \frac{1}{\mu}\left( M_1^2 + \frac{1}{\mu}\|f\|_{V'}^2 \right) \right\} =: M_3,
\]

(3.48)
Since our problem (2.10) is invariant under $t$-translations, from (3.48), we get a uniform estimate for $\|u(t)\|_V^2$, $t > \varepsilon > 0$, that is,

$$\sup_{0 < \varepsilon < t < \infty} \|u(t)\|_V^2 \leq M_3, \quad \varepsilon \leq 1,$$

(3.49)

for all $t \geq t_B + k$, for some $k > 0$.

**Theorem 3.7.** The dynamical system associated with the 2D CBF equations (2.10) possesses an attractor $\mathcal{A}_{\text{glob}}$ that is compact, connected, and global in $\mathbb{H}$. This attractor is a bounded set in $\mathbb{V}$. Moreover, $\mathcal{A}_{\text{glob}}$ attracts the bounded sets of $\mathbb{H}$ and is also maximal among the functional invariant sets bounded in $\mathbb{H}$.

**Proof.** Let us denote the right hand side of (3.24) as $M_2$. We then conclude that the ball $\mathcal{B}_2 = B(0, M_2)$ of $\mathbb{V}$ is an absorbing set in $\mathbb{V}$, which is compact in $\mathbb{H}$, for the semigroup $S(t)$. Moreover, if $B$ is any bounded set in $\mathbb{H}$, then $S(t)B \subset \mathcal{B}_2$, for $t \geq t_B + r$. This shows the existence of an absorbing set in $\mathbb{V}$ and also that the operators $S(t)$ are uniformly compact, that is, for every bounded set $B$, there exists $t_0$ which is dependent on $B$ such that $\bigcup_{t \geq t_0} S(t)B$ is relatively compact in $\mathbb{H}$. Thus, using the Theorem 1.1, Chapter I, [46], we obtain that the dynamical system associated with the 2D CBF equations (2.10) possesses an attractor $\mathcal{A}_{\text{glob}}$ that is compact, connected, and global in $\mathbb{H}$. Also, $\mathcal{A}_{\text{glob}}$ attracts the bounded sets of $\mathbb{H}$ and $\mathcal{A}_{\text{glob}}$ is maximal among the functional invariant sets bounded in $\mathbb{H}$. $\square$

4. **Global Attractor (Poincaré Domains)**

Let us now consider the case of Poincaré domains. We first show a result on the weak continuity of the semigroup $\{S(t)\}_{t \geq 0}$. Similar results for 2D Navier-Stokes equations have been obtained in [38] and we follow this work for our model also.

**Lemma 4.1.** Let $\{u^n_0\}_{n \in \mathbb{N}}$ be a weakly convergent sequence in $\mathbb{H}$ converging to $u_0 \in \mathbb{H}$. Then

$$S(t)u^n_0 \wrightarrow S(t)u_0 \quad \text{in} \quad \mathbb{H} \quad \text{and} \quad t \geq 0,$$

(4.1)

$$S(\cdot)u^n_0 \wrightarrow S(\cdot)u_0 \quad \text{in} \quad L^2(0,T;\mathbb{V}), \quad \text{for all} \quad T > 0, \quad \text{and}$$

(4.2)

$$S(\cdot)u^n_0 \wrightarrow S(\cdot)u_0 \quad \text{in} \quad L^{r+1}(0,T;L^r), \quad \text{for all} \quad T > 0.$$  

(4.3)

**Proof.** Let $u^n(t) = S(t)u^n_0$ and $u(t) = S(t)u_0$, for $t \geq 0$. From (3.16) and (3.4), we have

$$\{u_n\}_{n \in \mathbb{N}} \text{ is bounded in } L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}) \cap L^{r+1}(0,T;L^r), \quad \text{for all} \quad T > 0,$$

(4.4)

and the estimate

$$\sup_{0 \leq t \leq T} \|u^n(t)\|_H^2 + 2\mu \int_0^T \|u^n(t)\|_V^2 dt + 2\alpha \int_0^T \|u^n(t)\|_H^2 dt + 2\beta \int_0^T \|u^n(t)\|_{L^{r+1}}^2 dt$$

$$\leq C(\|u_0\|_{\mathbb{H}}, T, \|f\|_{\mathbb{V}}).$$

(4.5)

Remember that $\frac{du^n}{dt} = f - \mu Au^n - \alpha u^n - B(u^n) - \beta C(u^n)$ in $\mathbb{V}'$, for a.e. $t \in (0,T)$. For all $\psi \in L^2(0,T;\mathbb{V})$, it can be easily seen that

$$\int_0^T \left| \frac{du^n(t)}{dt}, \psi(t) \right| dt$$

It can be easily seen that
\[ \leq \mu \int_0^T |\langle Au^n(t), \psi(t) \rangle| dt + \alpha \int_0^T |\langle u^n(t), \psi(t) \rangle| dt \\
+ \beta \int_0^T |\langle \mathcal{C}(u^n(t)), \psi(t) \rangle| dt + \int_0^T |\langle f, \psi(t) \rangle| dt \\
\leq \mu \int_0^T \|u^n(t)\|_\mu \|\psi(t)\|_\nu dt + \alpha \int_0^T \|u^n(t)\|_H \|\psi(t)\|_H dt \\
+ \beta \int_0^T \|u^n(t)\|_{L^2} \|\psi(t)\|_H dt + \int_0^T \|f\|_\nu \|\psi(t)\|_\nu dt \\
\leq \left\{ \mu \left( \int_0^T \|u^n(t)\|_\nu^2 dt \right)^{1/2} + \sqrt{2} \sup_{t \in [0, T]} \|u^n(t)\|_\mu \left( \int_0^T \|u_n(t)\|_\mu^2 dt \right)^{1/2} + \sqrt{T} \|f\|_\nu, \\
+ C \left( \int_0^T \|u^n(t)\|_{H^1} dt \right)^{1/2} + C\beta \|u^n(t)\|_{H^{-1}} \left( \int_0^T \|u^n(t)\|_\nu^2 dt \right)^{1/2} \right\} \left( \int_0^T \|\psi(t)\|_\nu^2 dt \right)^{1/2} \\
\leq C(\|u_0\|_H, T, \mu, \beta, \|f\|_\nu). \quad (4.6) \]

Thus, we have
\[ \left\{ \frac{du^n}{dt} \right\}_{n \in \mathbb{N}} \text{ is bounded in } L^2(0, T; \nu'), \text{ for all } T > 0. \quad (4.7) \]

For all \( \psi \in \mathcal{V} \) and \( 0 \leq t \leq t + \tau \leq T \), with \( T > 0 \), we have
\[
(u^n(t + \tau) - u^n(t), \psi) = \int_t^{t+\tau} \left\langle \frac{du^n}{ds}(s), \psi \right\rangle ds \leq \|\psi\|_\nu \int_t^{t+\tau} \left\| \frac{du^n}{dt}(s) \right\|_\nu ds \\
\leq \left\| \frac{du^n}{dt} \right\|_{L^2(0, T; \nu')} \tau^{1/2} \|\psi\|_\nu \leq C(T)\tau^{1/2} \|\psi\|_\nu, \quad (4.8) \]

where \( C(\cdot) \) is a positive constant independent of \( n \). Since \( \mathbf{v} = u^n(t + \tau) - u^n(t) \in \mathcal{V} \), for a.e. \( t \in (0, T) \), choosing it in \( (4.8) \), we obtain
\[ \|u^n(t + \tau) - u^n(t)\|_H^2 \leq C(T)\tau^{1/2} \|u^n(t + \tau) - u^n(t)\|_\nu. \quad (4.9) \]

Integrating from 0 to \( T - \tau \), we further find
\[
\int_0^{T-\tau} \|u^n(t + \tau) - u^n(t)\|_H^2 dt \leq C(T)\tau^{1/2} \int_0^{T-\tau} \|u^n(t + \tau) - u^n(t)\|_\nu dt \\
\leq C(T)\tau^{1/2} \left( \int_0^{T-\tau} \|u^n(t + \tau) - u^n(t)\|_\nu^2 dt \right)^{1/2} \\
\leq \tilde{C}(T)\tau^{1/2}, \quad (4.10) \]

where we used the Cauchy-Schwarz inequality and \( (4.3) \). Also, \( \tilde{C}(T) \) is another positive constant independent of \( n \). Furthermore, we have
\[ \lim_{\tau \to 0} \sup_n \int_0^{T-\tau} \|u^n(t + \tau) - u^n(t)\|_{L^2(\Omega_R)}^2 dt = 0, \quad (4.11) \]

for all \( R > 0 \), where \( \Omega_R = \Omega \cap \{ x \in \mathbb{R}^2 : |x| \leq R \} \).
Let us now consider a truncation function $\chi \in C^1(\mathbb{R}^+)$ with $\chi(s) = 1$ for $s \in [0, 1]$ and $\chi(s) = 0$ for $s \in [2, \infty)$. For each $R > 0$, let us define

$$u^{n,R}(x) := \chi\left(\frac{|x|^2}{R^2}\right)u^n(x), \text{ for } x \in \Omega_{2R}.$$ 

It can easily be seen from (4.11) that for all $T, R > 0$,

$$\lim_{n \to \infty} \int_0^T \|u^{n,R}(t + \tau) - u^{n,R}(t)\|_{L^2(\Omega_{2R})}^2 d\tau = 0,$$

for all $T, R > 0$. Moreover, from (4.14), for all $T, R > 0$, we infer that

$$\{u^{n,R}\}_{n \in \mathbb{N}} \text{ is bounded in } L^\infty(0, T; L^2(\Omega_{2R})) \cap L^2(0, T; H_0^1(\Omega_{2R})).$$ (4.13)

Since the injection $H_0^1(\Omega_{2R}) \subset L^2(\Omega_{2R})$ is compact, we can apply Theorem 16.3, (30) (see Theorem 13.3, (15)) to obtain

$$\{u^{n,R}\}_{n \in \mathbb{N}} \text{ is relatively compact in } L^2(0, T; L^2(\Omega_{2R})), \quad \text{for all } T, R > 0. \quad (4.14)$$

From (4.11), we further infer that

$$\{u^n\}_{n \in \mathbb{N}} \text{ is relatively compact in } L^2(0, T; L^2(\Omega_R)), \quad \text{for all } T, R > 0. \quad (4.15)$$

Using the estimates (4.4) and (4.15), by a diagonal argument, we can extract a subsequence $\{u_{n_j}\}_{j \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ such that

$$\begin{cases}
u_{n_j} \overset{w^*}{\rightharpoonup} \tilde{\nu}, & \text{in } L^\infty(0, T; \mathbb{H}), \\
u_{n_j} \overset{w}{\rightharpoonup} \tilde{u}, & \text{in } L^2(0, T; \mathbb{V}), \\
u_{n_j} \overset{w}{\rightharpoonup} \tilde{u}, & \text{in } L^r+1(0, T; \overline{L}^{r+1}), \\
u_{n_j} \rightarrow \tilde{u}, & \text{in } L^2(0, T; L^2(\Omega_R)),
\end{cases} \quad (4.16)$$

as $j \to \infty$, for all $T, R > 0$, for some

$$\tilde{\nu} \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}).$$

Using the convergence given in (4.16), one can easily pass limit in the equation for $u^n$ to obtain that $\tilde{\nu}$ is a solution of (2.10) (in the weak sense) with $\tilde{\nu}(0) = u_0$ (cf. [24]). Since the solution of (2.10) is unique, we further have $\tilde{\nu} = \tilde{u}$. Uniqueness also gives us that the whole sequence $\{u^n\}_{n \in \mathbb{N}}$ converges to $u$. From the second and third convergences given in (4.16), we have

$$S(\cdot)u^n_0 = u_n \overset{w^*}{\rightharpoonup} u = S(\cdot)u_0 \text{ in } L^2(0, T; \mathbb{V}) \text{ and } L^{r+1}(0, T; \overline{L}^{r+1}),$$

which proves (4.12) and (4.3).

Let us now show the convergence given in (4.11). From the final convergence given (4.16), we immediately have $u^n(t)$ converges strongly to $u(t)$ in $L^2(\Omega_R)$, for a.e. $t \geq 0$ and all $R > 0$. Thus, for all $\psi \in \mathcal{V}$ and a.e. $t \in \mathbb{R}^+$, we have

$$\left(\psi, u^n(t) \right) \rightharpoonup \left(\psi, u(t) \right), \quad \text{a.e. } t \in \mathbb{R}^+.$$ \quad (4.17)

Furthermore, using (4.5) and (4.6), one can easily show that $\{(u^n(t), \psi)\}_{n \in \mathbb{N}}$ is equibounded and equicontinuous on $[0, T]$, for all $T > 0$. Therefore, for all $\psi \in \mathcal{V}$, we get

$$(u^n(t), \psi) \to (u(t), \psi), \quad \text{for all } t \in \mathbb{R}^+.$$ \quad (4.18)
The convergence in (4.1) follows easily from (4.18), making use of the fact that \( \mathcal{V} \) is dense in \( \mathbb{H} \). \[ \square \]

4.1. Asymptotic compactness. In this subsection, we prove the existence of a global attractor for the system (2.10) in Poincaré domains. This follows from the following general result.

**Theorem 4.2** (Theorem I.1.1, [46], [26], [38]). Let \( \mathcal{E} \) be a complete metric space and let \( \{ S(t) \}_{t \geq 0} \) be a semigroup of continuous (nonlinear) operators in \( \mathcal{E} \). If (and only if) \( \{ S(t) \}_{t \geq 0} \) possesses an absorbing set \( \mathcal{B} \) bounded in \( \mathcal{E} \) and is asymptotically compact in \( \mathcal{E} \), then \( \{ S(t) \}_{t \geq 0} \) possesses a (compact) global attractor \( \mathcal{A}_{\text{glob}} = \omega(\mathcal{B}) \). Furthermore, if \( t \mapsto S(t)u_0 \) is continuous from \( \mathbb{R}^+ \) into \( \mathcal{E} \) and \( \mathcal{B} \) is connected in \( \mathcal{E} \), then \( \mathcal{A}_{\text{glob}} \) is connected in \( \mathcal{E} \).

Note that \( \mathcal{B}_1 \) is a bounded absorbing set in \( \mathbb{H} \) for the semigroup \( S(t) \) (see Proposition 3.3). Let us now show the asymptotic compactness of the semigroup \( \{ S(t) \}_{t \geq 0} \) using the energy equation (3.2). We say that the semigroup \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact in a given metric space if

\[
\{ S(t_n)u_n \} \text{ is precompact,}
\]

whenever

\[
\{ u_n \}_n \text{ is bounded and } t_n \to \infty.
\]

We follow the work [38] for getting asymptotic compactness of the semigroup \( \{ S(t) \}_{t \geq 0} \).

Let us first define \( \langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R} \) by

\[
\langle u_1, u_2 \rangle = \mu [u_1, u_2] - \mu \frac{\lambda_1}{2} (u_1, u_2), \quad \text{for all } u_1, u_2 \in \mathbb{V},
\]

where \( (\cdot, \cdot) \) and \([\cdot, \cdot]\) denotes the inner products in \( \mathbb{H} \) and \( \mathbb{V} \), respectively. Let us take \( u_1 = u_2 = u \) in the above expression to find

\[
\langle u \rangle^2 = \langle u, u \rangle = \mu \| u \|^2_\mathbb{V} - \mu \frac{\lambda_1}{2} \| u \|^2_\mathbb{H} \geq \mu \| u \|^2_\mathbb{V} - \mu \frac{\lambda_1}{2} \| u \|^2_\mathbb{H} = \mu \frac{2}{2} \| u \|^2_\mathbb{V}.
\]

Moreover, we have

\[
\mu \frac{2}{2} \| u \|^2_\mathbb{V} \leq \langle u \rangle^2 \leq \mu \| u \|^2_\mathbb{V}, \quad \text{for all } u \in \mathbb{V},
\]

and hence \( \langle \cdot, \cdot \rangle \) defines an inner product in \( \mathbb{V} \) with norm \( \| \cdot \|_\mathbb{V} = \langle \cdot, \cdot \rangle \)^{1/2}, and is equivalent to \( \| \cdot \|_\mathbb{V} \).

**Proposition 4.3.** The semigroup \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact in \( \mathbb{H} \).

**Proof.** Let us add and subtract \( \mu \lambda_1 \| u \|^2_\mathbb{H} \) in (3.2) to find

\[
\frac{d}{dt} \| u(t) \|^2_\mathbb{H} + \mu \lambda_1 \| u(t) \|^2_\mathbb{H} + 2 \| u(t) \|^2 + 2 \alpha \| u(t) \|^2_\mathbb{H} + 2 \beta \| u(t) \|^{r+1}_{\mathbb{L}^{r+1}} = 2 \langle f, u(t) \rangle,
\]

for any solution \( u(t) = S(t)u_0 \), \( u_0 \in \mathbb{H} \). Then, using the variation of constant formula, we obtain

\[
\| u(t) \|^2_\mathbb{H} = \| u_0 \|^2_\mathbb{H} e^{-\mu \lambda_1 t} + 2 \int_0^t e^{-\mu \lambda_1 (t-s)} \left[ \langle f, u(s) \rangle - \left( \langle u(s) \rangle^2 + \alpha \| u(t) \|^2_\mathbb{H} + \beta \| u(s) \|^{r+1}_{\mathbb{L}^{r+1}} \right) \right] ds.
\]

Since \( u(t) = S(t)u_0 \), one can rewrite (4.20) as

\[
\| \mathcal{S}(t)u_0 \|^2_\mathbb{H}
\]
\[
\psi \in H \quad \text{for all} \quad T > 0,
\]

Since \( S(t) \) is absorbing, there exists a time \( t_B > 0 \) such that \( S(t)B \subset B_1 \), for all \( t \geq t_B \), so that for large enough \( t_n \), say \( t_n \geq t_B \), \( S(t)u_n \in B_1 \). Hence, the sequence \( \{S(t_n)u_n\}_{n_k} \) is weakly precompact in \( H \). Thus, since \( B_1 \) is closed and convex, we have

\[
\{S(t_n)u_n\}_{n_k} \overset{w}{\rightharpoonup} W \quad \text{in} \quad H,
\]

for some subsequence \( \{S(t_n)u_n\}_{n_k} \) of \( \{S(t_n)u_n\}_{n} \) and \( W \in B_1 \). Similarly, for each \( T > 0 \), one can show that \( S(t_n - T)u_n \in B_1 \), for all \( t_n \geq T + t_B \). Thus, we obtain \( S(t_n - T)u_n \) is precompact in \( H \), and by using a diagonal argument and passing to a further subsequence (if necessary), we can assume that

\[
\{S(t_n - T)u_n\}_{n_k} \overset{w}{\rightharpoonup} W_T, \quad \text{in} \quad H,
\]

for all \( T \in \mathbb{N} \) with \( W_T \in B_1 \). From (4.22), we know that \( (S(t_n)u_n, \psi) \rightarrow (W, \psi) \), for all \( \psi \in H \). Using the weak continuity of \( S(t) \) established in Lemma 4.1 (see (4.1)), we have

\[
W = \lim_{k \to \infty} (S(t_n)u_n, \psi) = \lim_{k \to \infty} (S(T)S(t_n - T)u_n, \psi) = (S(T) \lim_{k \to \infty} S(t_n - T)u_n, \psi) = S(T)W_T.
\]

Hence, we get \( W = S(T)W_T \), for all \( T \in \mathbb{N} \). Since \( \{S(t_n)u_n\}_{n_k} \overset{w}{\rightharpoonup} W \) weakly in \( H \) and using the weakly lower-semicontinuity property of \( H \), we also have

\[
\|W\| \leq \liminf_{k \to \infty} \|S(t_n)u_n\|_H.
\]

Our next aim is to show that

\[
\limsup_{k \to \infty} \|S(t_n)u_n\|_H \leq \|W\|_H.
\]

For \( T \in \mathbb{N} \) and \( t_n > T \), from (4.21), we have

\[
\|S(t_n)u_n\|_H^2 = \|S(T)S(t_n - T)u_n\|_H^2
\]

\[
= \|S(t_n - T)u_n\|_H^2 e^{-\mu_1 T} + 2 \int_0^T e^{-\mu_1 (T-s)} \left[ \langle f, S(s)S(t_n - T)u_n \rangle - \langle S(s)S(t_n - T)u_n \|_H^2 - \beta \|S(s)S(t_n - T)u_n\|_{L^r}^r \right] ds.
\]

Since \( S(t_n - T)u_n \in B_1 \), we can easily get

\[
\limsup_{k \to \infty} e^{-\mu_1 T}\|S(t_n - T)u_n\|_H^2 \leq M_1^2 e^{-\mu_1 T}.
\]

Using the weak continuity result given in (4.2) and the convergence \( \{S(t_n - T)u_n\}_{n_k} \overset{w}{\rightharpoonup} W_T \) in \( H \), we have

\[
S(\cdot)S(t_n - T)u_n \overset{w}{\rightharpoonup} S(\cdot)W_T \quad \text{weakly in} \quad L^2(0, T; V)
\]

\[
S(\cdot)S(t_n - T)u_n \overset{w}{\rightharpoonup} S(\cdot)W_T \quad \text{weakly in} \quad L^{r+1}(0, T; \mathbb{H}^{r+1}).
\]
Now, we consider
\[
\int_0^T \|e^{-\mu_1 (T-s)} f\|_{\mathcal{V}}^2 \, ds = \int_0^T e^{-2\mu_1 (T-s)} \|f\|_{\mathcal{V}}^2 \, ds = \|f\|_{\mathcal{V}}^2 \left( \frac{1 - e^{-2\mu_1 T}}{2\mu_1} \right) \leq \frac{\|f\|_{\mathcal{V}}^2}{2\mu_1} < +\infty,
\]
(4.29)
and hence the mapping \( s \mapsto e^{-\mu_1 (T-s)} f \in L^2(0, T; \mathcal{V}) \). Thus, we find
\[
\lim_{k \to \infty} \int_0^T e^{-\mu_1 (T-s)} \langle f, S(s)S(t_{n_k} - T)u_{n_k} \rangle \, ds = \int_0^T e^{-\mu_1 (T-s)} \langle f, S(s)W_T \rangle \, ds.
\]
(4.30)
Furthermore, since \( \langle \cdot \rangle \) defines a norm on \( \mathcal{V} \) equivalent to the norm \( \| \cdot \|_{\mathcal{V}} \) and \( 0 < e^{-\mu_1 T} \leq e^{-\mu_1 (T-s)} \leq 1 \), for all \( s \in [0, T] \), one can easily see that \( \left( \int_0^T e^{-\mu_1 (T-s)} \| \cdot \|_{\mathcal{V}}^2 \, ds \right)^{1/2} \) defines a norm on \( L^2(0, T; \mathcal{V}) \) equivalent to the norm \( \left( \int_0^T \langle \cdot \rangle^2 \, ds \right)^{1/2} \). Using (4.27) and the weakly lower-semicontinuity property of the norm, we get
\[
\int_0^T e^{-\mu_1 (T-s)} \|S(s)W_T\|_{\mathcal{V}}^2 \, ds \leq \liminf_{k \to \infty} \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{\mathcal{V}}^2 \, ds.
\]
Thus, we have
\[
\limsup_{k \to \infty} \left[ -2 \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{\mathcal{V}}^2 \, ds \right]
\leq -2 \liminf_{k \to \infty} \left[ \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{\mathcal{V}}^2 \, ds \right]
\leq -2 \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{\mathcal{V}}^2 \, ds.
\]
(4.31)
Note that \( 0 < e^{-\mu_1 T} \leq e^{-\mu_1 (T-s)} \leq 1 \), for all \( s \in [0, T] \), and thus it is immediate that \( \left( \int_0^T e^{-\mu_1 (T-s)} \| \cdot \|_{\mathcal{H}}^2 \, ds \right)^{1/2} \) defines a norm on \( L^2(0, T; \mathcal{H}) \), which is equivalent to the standard norm. Using (4.28) and the weakly lower-semicontinuity property of the norm, we get
\[
\int_0^T e^{-\mu_1 (T-s)} \|S(s)W_T\|_{\mathcal{H}}^2 \, ds \leq \liminf_{k \to \infty} \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{\mathcal{H}}^2 \, ds.
\]
(4.32)
Once again using the fact that \( 0 < e^{-\mu_1 T} \leq e^{-\mu_1 (T-s)} \leq 1 \), for all \( s \in [0, T] \), one can easily see that \( \left( \int_0^T e^{-\mu_1 (T-s)} \| \cdot \|_{L_{r+1}^{r+1}}^2 \, ds \right)^{1/(r+1)} \) defines a norm on \( L^{r+1}(0, T; \mathcal{L}^{r+1}) \) equivalent to the norm \( \left( \int_0^T \| \cdot \|_{L_{r+1}^{r+1}}^2 \, ds \right)^{1/(r+1)} \). Using (4.28) and the weakly lower-semicontinuity property of the norm, we get
\[
\int_0^T e^{-\mu_1 (T-s)} \|S(s)W_T\|_{L_{r+1}^{r+1}}^{r+1} \, ds \leq \liminf_{k \to \infty} \int_0^T e^{-\mu_1 (T-s)} \|S(s)S(t_{n_k} - T)u_{n_k}\|_{L_{r+1}^{r+1}}^{r+1} \, ds.
\]
(4.33)
Using (4.26)-(4.33) in (4.25) and then taking \( \limsup \) in (4.25), we obtain
\[
\limsup_{k \to \infty} \|S(t_{n_k})u_{n_k}\|_{\mathcal{H}}^2
\]
The linearized flow around $u$ at the attractor $A$

Remark 4.5. For functional invariant sets bounded in $H$, that is, compact invariant sets in $H$, implies $\{ \text{semigroup} \}\{S(t)u_n\}_{n \in \mathbb{N}} \to W$ strongly in $H$. Moreover, $A_{\text{glob}}$ is connected in $H$ and is maximal for the inclusion relation among all the functional invariant sets bounded in $H$.

Using the asymptotic compactness of the semigroup $\{S(t)\}_{t \geq 0}$ and Theorem 4.4 we have the following result:

**Theorem 4.4.** Let $\Omega$ be an open Poincaré domain. Assume that $\mu > 0$ and $f \in V'$. Then the semigroup $\{S(t)\}_{t \geq 0}$ associated with the 2D CBF system (2.10) possesses a global attractor $A_{\text{glob}}$ in $H$, that is, a compact invariant set in $H$, which attracts all bounded sets in $H$. Moreover, $A_{\text{glob}}$ is connected in $H$ and is maximal for the inclusion relation among all the functional invariant sets bounded in $H$.

**Remark 4.5.** For $f \in V'$, as we have proved in Remark 3.6, one can show that the global attractor $A_{\text{glob}}$ obtained in Theorem 4.4 is included and bounded in $V$.

5. **Dimension of the Attractor**

In this section, we analyze the dimension of the global attractor $A_{\text{glob}}$ obtained in section 4 (Poincaré domains). We estimate the bounds for the Hausdorff as well as fractal dimensions of the global attractor $A_{\text{glob}}$.

Let $u_0 \in H$ and $u(t) = S(t)u_0$, for $t \geq 0$ be the unique weak solution of the system (2.10). The linearized flow around $u(\cdot)$ is given by the following equation:

\[
\begin{aligned}
\frac{d}{dt}\xi(t) + \mu A\xi(t) + B(\xi(t), u(t)) + B(u(t), \xi(t)) + \alpha \xi(t) + \beta C'(u(t))\xi(t) &= 0, \\
\xi(0) &= \xi_0,
\end{aligned}
\]

for a.e. $t \in (0, T)$. As in the case of nonlinear problem, one can show that there exists a unique solution $\xi \in L^\infty(0, T; H) \cap L^2(0, T; V)$, for all $T > 0$. Furthermore, $\xi \in L^2(0, T; V')$ implies $\xi \in C([0, T]; H)$, for all $T > 0$. We define a map $A(t; u_0) : H \to H$ by setting
Let us define \( \Lambda(t; u_0) \), and its Fréchet differentiable with respect to the initial data, and its Fréchet derivative \( D_{u_0}(S(t)u_0)\xi_0 = \Lambda(t; u_0)\xi_0 \). Moreover, (5.2) is satisfied.

Proof. Let us define

\[ \eta(t) := u(t) - v(t) - \xi(t) = S(t)(u_0 - v_0) - \xi(t). \]

Then \( \eta(t) \) satisfies:

\[
\begin{aligned}
\frac{d}{dt} \eta(t) &+ \mu A\eta(t) + B(\eta(t), u(t)) + B(u(t), \eta(t)) - B(w(t), w(t)) + \alpha \eta(t) \\
+ \beta [C(u(t)) - C(v(t)) - C'(u(t))\xi(t)] &= 0, \\
\eta(0) &= 0,
\end{aligned}
\]

for a.e. \( t \in (0, T) \) in \( V' \), where \( w(t) = u(t) - v(t) \). Let us take the inner product with \( \eta(\cdot) \) to the first equation in (5.4) to obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\eta(t)\|^2_H + \mu \|\eta(t)\|^2_V + \alpha \|\eta(t)\|^2_H \\
= -b(\eta(t), u(t), \eta(t)) + b(w(t), w(t), \eta(t)) - \beta \langle C(u(t)) - C(v(t)) - C'(u(t))\xi(t), \eta(t) \rangle.
\end{aligned}
\]

We estimate \( |b(\eta, u, \eta)| \) using Hölder’s, Ladyzhenskaya’s and Young’s inequalities as

\[
|b(\eta, u, \eta)| \leq \|u\|_V \|\eta\|_H^2 \leq \sqrt{2} \|u\|_V \|\eta\|_H \|\eta\|_V \leq \frac{\mu}{4} \|\eta\|_V^2 + \frac{2\|u\|_V^2}{\mu} \|\eta\|_H^2,
\]

\[
|b(w, w, \eta)| = |b(w, \eta, w)| \leq \|\eta\|_V \|w\|_H^2 \leq \sqrt{2} \|\eta\|_V \|w\|_H \|w\|_V \leq \frac{\mu}{4} \|\eta\|_V^2 + \frac{2}{\mu} \|w\|_H^2 \|w\|_V^2.
\]

In order to estimate the term \( -\beta \langle C(u) - C(v) - C'(u)\xi, \eta \rangle \), we consider the cases \( r = 1, r = 2, r = 3 \) and \( r > 3 \) separately. For \( r = 1 \), its can be easily seen that

\[
-\beta \langle C(u) - C(v) - C'(u)\xi, \eta \rangle = -\beta \|\eta\|_H^2.
\]

For \( r = 2 \), using Taylor’s formula (see Theorem 7.9.1, [10]), Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we obtain

\[
-\beta \langle C(u) - C(v) - C'(u)\xi, \eta \rangle \\
= -\beta \left\langle \int_0^1 C'(\theta u + (1 - \theta)v)d\theta(u - v) - C'(u)\xi, \eta \right\rangle
\]
\[ -\beta\langle C'(u) - C'(u)\xi, \eta \rangle = -\beta\langle C'(u) - C'(u)\xi, \eta \rangle + \beta \int_0^1 [C'(u) + (1 - \theta)v]d\theta(u - v), \eta \rangle \]
\[ = -\beta\|u\|^2_{H^2} + 2\beta \int_0^1 \langle [u] - \|\theta u + (1 - \theta)v]\|u - v), \eta \rangle d\theta \]
\[ \leq -\beta\|u\|^2_{H^2} + 2\beta \int_0^1 (1 - \theta)\|u - v\|^2, |\eta|\rangle d\theta \]
\[ \leq -\beta\|u\|^2_{H^2} + \beta\|u - v\|^2_{L^2}\|\eta\|_H \]
\[ \leq -\beta\|u\|^2_{H^2} + \frac{\mu^2}{2}\|\eta\|^2_{H^2} + \frac{\beta^2}{\mu}\|w\|^2_{H^2}\|\eta\|^2_{L^2}. \]

For the critical case \( r = 3 \), once again using Taylor’s formula, Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we find
\[ -\beta\langle C(u) - C(u)\xi, \eta \rangle = -\beta\langle C''(u)(u - v) + \frac{1}{2} \int_0^1 C''(\theta u + (1 - \theta)v)d\theta(u - v)\|u - v), \eta \rangle \]
\[ = -\beta\|u\|_{H^2} + 3\beta \int_0^1 \langle P_H((u - v)\|u - v)\|\theta u + (1 - \theta)v)\|u - v), \eta \rangle d\theta \]
\[ \leq -\beta\|u\|_{H^2} + 3\beta \int_0^1 \|\theta u + (1 - \theta)v\|_{L^4}\|u - v\|^2_{L^4}\|\eta\|_{H^2} d\theta \]
\[ \leq -\beta\|u\|_{H^2} + 6\sqrt{2}\beta\left(\|u\|_{L^4} + \|v\|_{L^4}\right)\|w\|_{H^2}\|w\|_{V}\|\eta\|_{H^2}^{1/2}\|\eta\|_V^{1/2} \]
\[ \leq -\beta\|u\|_{H^2} + \frac{\mu}{4}\|\eta\|^2_{L^2} + \frac{3}{16\mu^{1/3}}(6\sqrt{2}\beta)^{4/3}\left(\|u\|_{L^4} + \|v\|_{L^4}\right)^{4/3}\|w\|_{H^2}\|w\|_{V}\|\eta\|_{H^2}^{2/3} \]
\[ \leq -\beta\|u\|_{H^2} + \frac{\mu}{4}\|\eta\|^2_{L^2} + \frac{72\beta^2}{\mu}\|w\|^2_{H^2}\|w\|^2_{V}. \]

For \( r > 3 \), using Taylor’s formula, Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities, we obtain
\[ -\beta\langle C(u) - C(u)\xi, \eta \rangle = -\beta\|u\|_{H^2} + \frac{\beta\|u\|_{H^2}^2}{2} \int_0^1 \langle P_H((u - v)(u - v)\|\theta u + (1 - \theta)v)\|u - v), \eta \rangle d\theta \]
\[ \leq -\beta\|u\|_{H^2} + \frac{\beta\|u\|_{H^2}^2}{2} \int_0^1 \|\theta u + (1 - \theta)v\|_{L^2}^2\|w\|_{H^2}\|\eta\|_{H^2} d\theta \]
\[ \leq -\beta\|u\|_{H^2} + \beta\|u\|_{L^4}^2 + \beta\|v\|_{L^4}^2\right)\|w\|_{L^2}^2\|\eta\|_{H^2} + \beta\|u - v\|_{L^2}^2 + \beta\|u - v\|_{L^2}^2 = 3, \] once again using Taylor’s formula, Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we find
\[ \frac{d}{dt}\|\eta(t)\|^2_{H^2} + \frac{\mu}{2}\|\eta(t)\|^2_{L^2} + 2\alpha\|\eta(t)\|^2_{L^2}. \]
An application of Gronwall’s inequality in (5.9) gives
\[
0 \leq \begin{cases}
\frac{4}{\mu} \|u(t)\|^2_{H^2} \|\eta(t)\|^2_{H^2} + \frac{4}{\mu} \|u(t)\|_{L^4}^2 \|w(t)\|_{L^4}^2 + \frac{4}{\mu} \|w(t)\|_{L^4}^2 + \|u(t)\|_{L^4}^2 + \mu \|\eta(t)\|^2_{H^2}, \\
\frac{4}{\mu} \|\eta(t)\|^2_{H^2} + \frac{4}{\mu} (1 + 36\beta^2) \|w(t)\|^2_{H^2} \|\eta(t)\|^2_{H^2} + \mu \left( \|u(t)\|_{L^4}^2 + \|v(t)\|_{L^4}^2 \right) \|\eta(t)\|^2_{H^2}, \\
\frac{4}{\mu} \|\eta(t)\|^2_{H^2} + \frac{4}{\mu} \|w(t)\|^2_{H^2} \|\eta(t)\|^2_{H^2} + 2\beta r(r-1) \left( \|u(t)\|_{L^4}^2 + \|v(t)\|_{L^4}^2 \right)^r - 2 \|\eta(t)\|^2_{H^2}
\end{cases}
\]
for \( r = 1, 2, 3 \) and \( r > 3 \), respectively. From the estimate (3.10) (see Lemma 3.2), we find
\[
\|w(t)\|^2_{H^2} + \mu \int_0^t \|w(s)\|^2_{V} ds \leq \|w_0\|^2_{H^2} \exp \left\{ \frac{4}{\mu^2} \left( \|u_0\|^2_{H^2} + \frac{t}{\mu} \|f\|_{V}^2 \right) \right\},
\]
for all \( t \in [0, T] \), where we used (3.4). For \( r = 3 \), using (5.8) in (5.7) and then integrating from 0 to \( t \), we find
\[
\|\eta(t)\|^2_{H^2} + \frac{\mu}{2} \int_0^t \|\eta(s)\|^2_{V} ds + 2\alpha \int_0^t \|\eta(s)\|^2_{H^2} ds
\leq \frac{4}{\mu} \int_0^t \|u(s)\|_{L^4}^2 \|\eta(s)\|^2_{H^2} ds + \mu \int_0^t \left( \|u(s)\|_{L^4}^2 + \|v(s)\|_{L^4}^2 \right) \|\eta(s)\|^2_{H^2} ds
+ \frac{4}{\mu} (1 + 36\beta^2) \int_0^t \|w(s)\|^2_{H^2} \|\eta(s)\|^2_{V} ds
\leq \frac{4}{\mu} \int_0^t \|u(s)\|_{L^4}^2 \|\eta(s)\|^2_{H^2} ds + \mu \int_0^t \left( \|u(s)\|_{L^4}^2 + \|v(s)\|_{L^4}^2 \right) \|\eta(s)\|^2_{H^2} ds
+ \frac{4}{\mu^2} (1 + 36\beta^2) \|w_0\|^4_{H^2} \exp \left\{ \frac{8}{\mu^2} \left( \|u_0\|^2_{H^2} + \frac{t}{\mu} \|f\|_{V}^2 \right) \right\},
\]
(5.9)
An application of Gronwall’s inequality in (5.7) gives
\[
\|\eta(t)\|^2_{H^2} \leq \frac{4}{\mu^2} (1 + 36\beta^2) \exp \left\{ \frac{4}{\mu^2} \left( \|u_0\|^2_{H^2} + \frac{t}{\mu} \|f\|_{V}^2 \right) \right\} \|w_0\|^4_{H^2}
\times \exp \left( \frac{8}{\mu} \int_0^t \|u(s)\|^2_{L^4} ds + \mu \int_0^t \left( \|u(s)\|_{L^4}^2 + \|v(s)\|_{L^4}^2 \right) ds \right)
\leq \frac{4}{\mu^2} (1 + 36\beta^2) \exp \left\{ \left( \frac{12}{\mu^2} + \frac{\mu}{\beta^2} \right) \left( \|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \frac{t}{\mu} \|f\|_{V}^2 \right) \right\} \|w_0\|^4_{H^2},
\]
de (5.10)
where we used (3.4). Thus, by the definition of \( \eta \), it is immediate that
\[
\frac{\|u(t) - v(t) - \xi(t)\|_{H^2}}{\|u_0 - v_0\|_{H^2}} \leq \vartheta(t) \|u_0 - v_0\|_{H^2},
\]
(5.11)
where \( \vartheta(t) = \frac{2}{\mu} \sqrt{(1 + 36\beta^2) \exp \left\{ \left( \frac{6}{\mu^2} + \frac{\mu}{\beta^2} \right) \left( \|u_0\|^2_{H^2} + \|v_0\|^2_{H^2} + \frac{\mu}{\beta} \|f\|_{V}^2 \right) \right\} } \), and hence the differentiability of the semigroup \( S(t) \) with respect to the initial data as well as (5.2) and (5.3) follows.

For \( r > 3 \), a calculation similar to (5.10) yields
\[
\|\eta(t)\|^2_{H^2} \leq \left\{ \frac{4}{\mu} \int_0^t \|w(s)\|^2_{H^2} \|w(s)\|^2_{V} ds + 2\beta r(r-1) \int_0^t \|w(s)\|^4_{L^4} ds \right\}
\]
Theorem 5.2. The global attractor obtained in Theorem 4.4 has finite Hausdorff and fractal dimensions, which can be estimated as

$$\dim_H(\mathcal{A}_{\text{glob}}) \leq 1 + \frac{\tilde{\kappa} \|f\|_{\mathcal{V}'}^2}{\mu^4 \lambda_1}$$

and

$$\dim_F(\mathcal{A}_{\text{glob}}) \leq 2 \left( 1 + \frac{2\tilde{\kappa} \|f\|_{\mathcal{V}'}^2}{\mu^4 \lambda_1^4} \right),$$

for some absolute constant $\tilde{\kappa}$.

Proof. We can rewrite (5.11) as

$$\frac{d\xi}{dt} = \mathcal{F}(u)\xi = -[\mu A\xi + B(\xi, u) + B(u, \xi) + \alpha \xi + \beta C'(u)]\xi.$$  (5.15)
We define the numbers $q_m$, $m \in \mathbb{N}$ by

$$q_m = \limsup_{t \to \infty} \sup_{\mathbf{u}_0 \in \mathcal{A}_{\text{glob}}^\prime} \sup_{t_i \in \mathcal{H}} \frac{1}{t} \int_0^t \operatorname{Tr}(\mathcal{F}'(\mathbf{S}(s)\mathbf{u}_0) \circ q_m(s))ds,$$

where $q_m(s) = q_m(s; \mathbf{u}_0, q_1, \ldots, q_m)$ is the orthogonal projector of $\mathcal{H}$ onto the space spanned by $\Lambda(t; \mathbf{u}_0)q_1, \ldots, \Lambda(t; \mathbf{u}_0)q_m$. From section V.3.4, (46) (see Proposition V.2.1 and Theorem V.3.3), we infer that if $q_m < 0$, for some $m \in \mathbb{N}$, then the global attractor $\mathcal{A}_{\text{glob}}$ has finite Hausdorff and fractal dimensions estimated respectively as

$$\dim_{\mathcal{H}}^\mathcal{H}(\mathcal{A}_{\text{glob}}) \leq m, \quad (5.16)$$

$$\dim_{\mathcal{F}}^\mathcal{F}(\mathcal{A}_{\text{glob}}) \leq m\left(1 + \max_{1 \leq j \leq m} \frac{(q_j)}{m} \right). \quad (5.17)$$

Our next aim is to estimate the number $q_m$. Let $\mathbf{u}_0 \in \mathcal{A}_{\text{glob}}^\prime$, $q_1, \ldots, q_m \in \mathcal{H}$ and set $\mathbf{u}(t) = \mathbf{S}(t)\mathbf{u}_0$ and $\xi_j(t) = \Lambda(t; \mathbf{u}_0)q_j$, $t \geq 0$ and $1 \leq j \leq m$. Let us consider $\{\phi_1(t), \ldots, \phi_m(t)\}$ as an orthonormal basis in $\mathcal{H}$ for span$\{\xi_1(t), \ldots, \xi_m(t)\}$. Note that $\|\phi_j\|_{\mathcal{H}} = 1$, for all $1 \leq j \leq m$. Since $\xi_j \in L^2(0,T;\mathcal{V})$, we know that $\xi_j(t) \in \mathcal{V}$, for a.e. $t \in (0,T)$. By the Gram-Schmidt orthogonalization process, we can assume that $\phi_j(t) \in \mathcal{V}$. Then, one can see that

$$\operatorname{Tr}(\mathcal{F}'(\mathbf{u}(s)) \circ q_m(s)) = \sum_{j=1}^m (\mathcal{F}'(\mathbf{u}(s))\phi_j, \phi_j)$$

$$= \sum_{j=1}^m \{-[\mu A\phi_j + B(\phi_j, \mathbf{u}) + B(\mathbf{u}, \phi_j) + \alpha\phi_j + \beta C'(\mathbf{u})\phi_j, \phi_j]\}$$

$$= \sum_{j=1}^m \left\{-\mu\|\phi_j\|_{\mathcal{V}}^2 - \alpha\|\phi_j\|_{\mathcal{H}}^2 - \beta\|\mathbf{u}\|_{\mathcal{H}}^2\phi_j\|_{\mathcal{H}}^2 - \langle B(\phi_j, \mathbf{u}), \phi_j \rangle \right\}. \quad (5.18)$$

Calculating similarly as in page 82, (38), we find

$$\left| \sum_{j=1}^m \langle B(\phi_j, \mathbf{u}), \phi_j \rangle \right| \leq \frac{\mu}{2} \sum_{j=1}^m \|\phi_j\|_{\mathcal{V}}^2 + \frac{\tilde{\kappa}}{2\mu} \|\mathbf{u}\|_{\mathcal{V}}^2, \quad (5.19)$$

for an absolute constant $\tilde{\kappa}$. Thus, from (5.18), we infer that

$$\operatorname{Tr}(\mathcal{F}'(\mathbf{u}(s)) \circ q_m(s)) \leq -\frac{\mu}{2} \sum_{j=1}^m \|\phi_j\|_{\mathcal{V}}^2 - \alpha \sum_{j=1}^m \|\phi_j\|_{\mathcal{H}}^2 - \beta \sum_{j=1}^m \|\mathbf{u}(s)\|_{\mathcal{H}}^2\phi_j(s)\|_{\mathcal{H}}^2 + \frac{\tilde{\kappa}}{2\mu} \|\mathbf{u}(s)\|_{\mathcal{V}}^2$$

$$\leq -\frac{\mu\lambda_1}{2} \sum_{j=1}^m \|\phi_j\|_{\mathcal{H}}^2 + \frac{\tilde{\kappa}}{2\mu} \|\mathbf{u}(s)\|_{\mathcal{V}}^2 \leq \frac{\mu\lambda_1}{2} m + \frac{\tilde{\kappa}}{2\mu} \|\mathbf{u}(s)\|_{\mathcal{V}}^2. \quad (5.20)$$

Let us define the energy dissipation flux as

$$\mathcal{E} = \mu\lambda_1 \limsup_{t \to \infty} \sup_{\mathbf{u}_0 \in \mathcal{A}_{\text{glob}}^\prime} \frac{1}{t} \int_0^t \|\mathbf{S}(s)\mathbf{u}_0\|_{\mathcal{V}}^2 ds. \quad (5.21)$$
Note that the energy dissipation flux $E$ is finite due to the estimate (3.1) and $E \leq \frac{\kappa}{\mu} \|f\|_{V'}^2$. Then, from (5.20), we have

$$q_m = \limsup_{t \to \infty} \sup_{u_0 \in \mathcal{A}_{\text{glob}}} \sup_{\|\theta_i\| \leq 1} \frac{1}{t} \int_0^t \text{Tr}(\mathcal{F}'(S(s)u_0) \circ q_m(s))ds \leq -\frac{\mu \lambda_1}{2} m + \frac{\kappa}{2\mu^2 \lambda_1} E,$$

(5.22)

for all $m \in \mathbb{N}$. Hence, if $m' \in \mathbb{N}$ is defined by $m' - 1 \leq \frac{\kappa}{\mu \lambda_1} E < m'$, then $q_{m'} < 0$ and thus from (5.16), we find

$$\dim_{\mathcal{H}}^{\mathcal{A}_{\text{glob}}} \leq m' \leq 1 + \frac{\kappa}{\mu^3 \lambda_1^2} E \leq 1 + \frac{\kappa \|f\|_{V'}^2}{\mu^4 \lambda_1^4}.$$

(5.23)

Furthermore, if $m'' \in \mathbb{N}$ is defined by $m'' - 1 < \frac{2\kappa}{m' \lambda_1^2} E \leq m''$, then using Lemma VI.2.2, 46, we have

$$q_{m''} < 0 \text{ and } \left|\frac{(q_j)_+}{|q_{m''}|}\right| \leq 1, \text{ for all } j = 1, \ldots, m''.$$

(5.24)

Hence from (5.17), we obtain

$$\dim_{\mathcal{H}}^{\mathcal{A}_{\text{glob}}} \leq 2m'' \leq 2 \left(1 + \frac{2\kappa}{\mu^3 \lambda_1^2} E\right) \leq 2 \left(1 + \frac{2\kappa \|f\|_{V'}^2}{\mu^4 \lambda_1^4}\right),$$

(5.25)

which completes the proof. \qed

**Remark 5.3.** For the force $f \in V'$, if we take $\lambda_1^{1/2}$ for a characteristic length for the problem, one can regard $\|f\|_{V'}^{1/2} \lambda_1^{1/4}$ as a characteristic velocity and define the Reynolds number $\text{Re} = \frac{\|f\|_{V'}^{1/2}}{\nu \lambda_1^{1/2}}$. We can also define the generalized Grashof number $G = \frac{\|f\|_{V'}}{\nu^2 \lambda_1^{1/2}} = \text{Re}^2$. Thus, from (5.13) and (5.14), we infer that

$$\dim_{\mathcal{H}}^{\mathcal{A}_{\text{glob}}} \leq 1 + \kappa G^2 = 1 + \kappa \text{Re}^4 \text{ and } \dim_{\mathcal{H}}^{\mathcal{A}_{\text{glob}}} \leq 2(1 + 2\kappa G^2) = 2(1 + 2\kappa \text{Re}^4),$$

for an absolute constant $\kappa$.

### 6. Upper Semicontinuity of Global Attractors

In this section, we establish an upper semicontinuity of global attractors for the 2D CBF equations. Let $\{\Omega_m\}_{m=1}^{\infty}$ be an expanding sequence of simply connected, bounded and smooth subdomains of $\Omega$ (for example, one can take $\Omega = \mathbb{R} \times (-L, L)$) such that $\Omega_m \to \Omega$ as $m \to \infty$. We consider

$$\begin{cases}
\frac{\partial u_m(x,t)}{\partial t} - \mu \Delta u_m(x,t) + (u_m(x,t) \cdot \nabla)u_m(x,t) + \alpha u_m(x,t) + \beta |u_m(x,t)|^{\gamma-1}u_m(x,t) \\
+ \nabla p_m(x,t) = f_m(x), \\
\nabla \cdot u_m(x,t) = 0, \\
u_m(x,0) = u_{0,m}(x),
\end{cases}$$

(6.1)

for $(x,t) \in \Omega \times (0, T)$. In (6.1), $f_m(\cdot)$ and $u_{0,m}(\cdot)$ are given by

$$f_m(x) = \begin{cases} f(x), & x \in \Omega_m, \\
0, & x \in \Omega \setminus \Omega_m,
\end{cases} \text{ and } u_{0,m}(x) = \begin{cases} u_0(x), & x \in \Omega_m, \\
0, & x \in \Omega \setminus \Omega_m.
\end{cases}$$

(6.2)
If we take orthogonal projection $P_H$ onto the system (6.1), we get
\[
\begin{cases}
\frac{du_m(t)}{dt} + \mu u_m(t) + B(u_m(t)) + \alpha u_m(t) + \beta C(u_m(t)) = f_m, \quad t \in (0, T), \\
u_m(0) = u_{0,m}, v_m(0) = v_{0,m} = 0,
\end{cases}
\]
where $f_m$ and $u_{m,0}$ are defined as in (6.2) (projected). Throughout this section, we differentiate the $\mathbb{H}$-spaces defined in $\Omega$ and $\Omega_m$ as $\mathbb{H}_\Omega$ and $\mathbb{H}_{\Omega_m}$, respectively and similar modifications are made for other spaces also. Furthermore, we assume that $f \in \mathbb{H}_\Omega$. Let $u_m \in L^\infty(0,T;\mathbb{H}_{\Omega_m}) \cap L^2(0,T;\mathbb{V}_{\Omega_m})$ be the unique weak solution of the system (6.3) with $\nabla \cdot u_m \in L^2(0,T;\mathbb{V}_{\Omega_m}^\prime)$ and hence $u_m \in C([0,T];\mathbb{H}_{\Omega_m})$. The 2D CBF model (6.3) possesses an attractor $\mathcal{A}_m$ that is compact, connected, and global in $\mathbb{H}_{\Omega_m}$ (see Theorem 3.7 and Remark 3.6). Our aim in this section is to check whether the global attractors $\mathcal{A}$ and $\mathcal{A}_m$ (for simplicity of notations) of the 2D CBF equations corresponding to $\Omega$ and $\Omega_m$, respectively, have the upper semicontinuity, when $m \to \infty$. The upper semicontinuity of the Klein-Gordon-Schrödinger equations on $\mathbb{R}^n$, $n \leq 3$ is established in [29] and 2D Navier-Stokes equations is obtained in [49]. We follow the works [29, 49], etc for getting such a result for 2D CBF equations. We are providing a complete proof of the upper semicontinuity results, as we are improving some of the estimates used in [49] and we are handling the nonlinear damping term separately.

**Lemma 6.1.** Assume that $u_0 \in B_1$, where $B_1$ is the absorbing set defined in (3.14). Then, for any $\varepsilon > 0$, there exist $T_1(\varepsilon)$ and $m_1(\varepsilon)$ such that the solution $u(\cdot)$ of the problem (1.2) with the initial condition $u_0$ satisfies:
\[
\int_{\Omega_m} |u(x,t)|^2 dx \leq \varepsilon,
\]
for $t \geq T_1(\varepsilon)$ and $m \geq m_1(\varepsilon)$, where $T_1(\varepsilon)$ and $m_1(\varepsilon)$ are constants depending only on $\varepsilon$.

**Proof.** Let us first define a cutoff function $\chi(\cdot) \in C^1(\mathbb{R}^2; \mathbb{R})$ as
\[
\chi(x) = \begin{cases}
0, & \text{if } |x| < 1, \\
(2 - |x|^2)(2 - |x|^2)^2, & \text{if } 1 \leq |x| < 2, \\
1, & \text{if } |x| \geq 2,
\end{cases}
\]
and note that $0 \leq \chi(x) \leq 1$. Let us set $\chi_R(x) = \chi(\frac{x}{R})$, for sufficiently large $R > 1$, then
\[
\chi_R(x) = \begin{cases}
0, & \text{if } |x| < R, \\
\left(\frac{|x|^2}{R^2} - 1\right)^2 (2 - \frac{|x|^2}{R^2}), & \text{if } R \leq |x| < 2R, \\
1, & \text{if } |x| \geq 2R,
\end{cases}
\]
and
\[
\nabla \chi_R(x) = \begin{cases}
0, & \text{if } |x| < R \text{ or } |x| \geq 2R, \\
\frac{4}{R^2} \left(\frac{|x|^2}{R^2} - 1\right) \left(2 - \frac{|x|^2}{R^2}\right) x, & \text{if } R \leq |x| < 2R.
\end{cases}
\]
It can be easily seen that
\[
\|\nabla \chi_R\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{R},
\]
where $C = 12$. Taking divergence in the first equation in (1.2), we find
\[
-\Delta p = \nabla \cdot ([u \cdot \nabla]u) + \beta \nabla \cdot [\|u\|^{-1} u] = \nabla \cdot (\nabla \cdot (u \otimes u)) + \beta \nabla \cdot [\|u\|^{-1} u]
\]
so that
\[ p = (-\Delta)^{-1} \left[ \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) + \beta \nabla \cdot [\|u\|^{-1} u], \right] \]

and for all \( \theta \geq 0 \), we have
\[
\int_{t}^{t+\theta} \|p(s)\|_{L^2(\Omega)}^2 ds \leq C \int_{t}^{t+\theta} \|u(s)\|_{H^1_0(\Omega)}^4 ds + C\beta \int_{t}^{t+\theta} \|u(s)\|_{H^2(\Omega)}^{2r} ds
\]
\[
\leq C \left( \sup_{s \in [t,t+\theta]} \|u(s)\|_{H^1_0}^2 + C\beta \sup_{s \in [t,t+\theta]} \|u(s)\|_{H^2(\Omega)}^{2r-2} \right) \int_{t}^{t+\theta} \|u(s)\|_{V_0}^2 ds
\]
\[
\leq \frac{C(M^2 + \beta M^2 r^{-2})}{\mu^2} \left( \frac{2}{\mu \lambda_1} + \theta \right) \|f\|_{V_0}^2,
\]

where we used (3.20). Taking the inner product with \( \chi_R^2(x) u(t, x) \) to the first equation in (1.2), we find
\[
\frac{1}{2} \frac{d}{dt} \|\chi_R u(t)\|_{L^2(\Omega)}^2 = -\mu(\nabla u(t), \nabla(\chi_R^2 u(t))) - ((u(t) \cdot \nabla) u(t), \chi_R^2 u(t)) - \alpha \|\chi_R^2 u(t)\|_{L^2(\Omega)}^2
\]
\[
- \beta (\|u(t)\|^{-1} u(t), \chi_R^2 u(t)) - (\nabla p(t), \chi_R^2 u(t)) - (f, \chi_R^2 u(t)).
\]

Using Hölder’s and Young’s inequalities, we estimate \(-\mu(\nabla u, \nabla(\chi_R^2 u))\) as
\[
-\mu(\nabla u, \nabla(\chi_R^2 u)) = -\mu(\nabla u, \chi_R \nabla(\chi_R u)) - \mu(\nabla u, \nabla(\chi_R) \chi_R u)
\]
\[
= -\mu(\nabla(\chi_R u) - \nabla(\chi_R) \chi_R u, \nabla(\chi_R u)) - \mu(\nabla u, \nabla(\chi_R) \chi_R u)
\]
\[
= -\mu \|\nabla(\chi_R u)\|_{L^2(\Omega)}^2 + \mu(\nabla(\chi_R) \chi_R u, \nabla(\chi_R) \chi_R u) - \mu(\nabla u, \nabla(\chi_R) \chi_R u)
\]
\[
\leq -\frac{\mu}{4} \|\nabla(\chi_R u)\|_{L^2(\Omega)}^2 + \frac{C\mu}{R} \|u\|_{H^1_0}^2 + \frac{C\mu}{R} \|\nabla u\|_{H^1_0}^2
\]
\[
\leq -\frac{\mu}{4} \|\nabla(\chi_R u)\|_{L^2(\Omega)}^2 + \frac{C\mu M_0^2}{R} + \frac{C\mu}{R} \|u\|_{V_0}^2.
\]

We consider the term \(-(u \cdot \nabla) u, \chi_R^2 u\) from (6.6) and using an integration by parts, divergence free condition, Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate it as
\[
-(u \cdot \nabla) u, \chi_R^2 u = -\sum_{i,j=1}^{2} \int_{\Omega} u_i(x) \frac{\partial u_j(x)}{\partial x_i} \chi_R(x) u_j(x) dx
\]
\[
= -\frac{1}{2} \sum_{i,j=1}^{2} \int_{\Omega} \chi_R^2(x) u_i(x) \frac{\partial u_j(x)}{\partial x_i} dx
\]
\[
= \sum_{i,j=1}^{2} \int_{\Omega} \chi_R(x) \frac{\partial \chi_R(x)}{\partial x_i} u_i(x) u_j^2(x) dx
\]
Once again an integration by parts and divergence free condition yields

\[ \begin{align*}
\leq & \left\| \nabla \chi_R \right\|_{\infty} \left( \left\| \chi_R u \right\|_{L^2(\Omega)} \right) \left\| u \right\|_{L^2(\Omega)}^2 \\
\leq & \frac{C}{R} \left( \left\| u \right\|_{\mathbb{V}_\Omega}^2 + \left\| u \right\|_{\mathbb{H}_\Omega}^2 \right) \leq \frac{C}{R} \left( \left\| u \right\|_{\mathbb{V}_\Omega}^2 + M_1^2 \right). 
\end{align*} \tag{6.8} \]

Next, we estimate the term \( \beta(\left\| u \right\|^{-1} u, \chi_R^2 u) \) as

\[ \beta(\left\| u \right\|^{-1} u, \chi_R^2 u) = \beta(\chi_R[\left\| u \right\|^{-1} u], \chi_R u) \leq \beta \left\| \chi_R[\left\| u \right\|^{-1} u] \right\|_{L^2(\Omega)} \left\| \chi_R u \right\|_{L^2(\Omega)} \]

\[ \leq \frac{\mu}{4} \left( \left\| \nabla (\chi_R u) \right\|_{L^2(\Omega)}^2 + \frac{\beta_2^2}{\mu \lambda_1} \left\| \chi_R[\left\| u \right\|^{-1} u] \right\|_{L^2(\Omega)}^2 \right). \tag{6.9} \]

Once again an integration by parts and divergence free condition yields

\[ \begin{align*}
-(\nabla p, \chi_R^2 u) &= (p, \nabla \cdot (\chi_R^2 u)) = 2(p, \chi_R \nabla (\chi_R) \cdot u) \leq 2 \left\| p \right\|_{L^2(\Omega)} \left\| \nabla \chi_R \right\|_{L^\infty(\mathbb{R}^2)} \left\| \chi_R u \right\|_{L^2(\Omega)} \\
\leq & \frac{C}{R} \left\| p \right\|_{L^2(\Omega)} \left\| u \right\|_{\mathbb{H}_\Omega} \leq \frac{C}{R} \left( \left\| p \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{\mathbb{H}_\Omega}^2 \right) \leq \frac{C}{R} \left( \left\| p \right\|_{L^2(\Omega)}^2 + M_1^2 \right). \tag{6.10} \end{align*} \]

Using the Cauchy-Schwarz inequality, we estimate \(- (f, \chi_R^2 u) \) as

\[ \begin{align*}
-(f, \chi_R^2 u) \leq & \left\| \chi_R f \right\|_{L^2(\Omega)} \left\| \chi_R u \right\|_{L^2(\Omega)} \leq M_1 \left\| \chi_R f \right\|_{L^2(\Omega)}. \tag{6.11} \end{align*} \]

Combining (6.8)-(6.11) and substituting it in (6.6), we find

\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| \chi_R u(t) \right\|_{L^2(\Omega)}^2 + \mu \left\| \nabla (\chi_R u(t)) \right\|_{L^2(\Omega)}^2 + \alpha \left\| \chi_R u(t) \right\|_{L^2(\Omega)}^2 \\
\leq & \frac{2M_1}{\mu \lambda_1} \left\| \chi_R f \right\|_{L^2(\Omega)} + \frac{C}{R} \left[ (\mu + 1) \left\| u(t) \right\|_{\mathbb{V}_\Omega}^2 + \left\| p(t) \right\|_{L^2(\Omega)}^2 + M_2^2 (1 + \mu + M_1^2) \right] \\
&+ \frac{\beta_2^2}{\mu \lambda_1} \left\| \chi_R[\left\| u \right\|^{-1} u(t)] \right\|_{L^2(\Omega)}^2. \tag{6.12} \end{align*} \]

Using variation of constants formula, we obtain

\[ \begin{align*}
\left\| \chi_R u(t) \right\|_{L^2(\Omega)}^2 &\leq \left\| \chi_R u_0 \right\|_{L^2(\Omega)}^2 e^{-\mu \lambda_1 t} + \frac{2M_1}{\mu \lambda_1} \left\| \chi_R f \right\|_{L^2(\Omega)} + \frac{C M_2^2}{R \mu \lambda_1} \left[ 1 + \mu + M_1^2 \right] \\
&+ \frac{C(\mu + 1)}{R} \int_0^t e^{-\mu \lambda_1 (t-s)} \left[ \left\| u(s) \right\|_{\mathbb{V}_\Omega}^2 + \left\| p(s) \right\|_{L^2(\Omega)}^2 \right] ds \\
&+ \frac{\beta_2^2}{\mu \lambda_1} \int_0^t e^{-\mu \lambda_1 (t-s)} \left\| \chi_R[\left\| u \right\|^{-1} u(s)] \right\|_{L^2(\Omega)}^2 ds. \tag{6.13} \end{align*} \]

Let \( k \) be an integer such that \( k \leq t \leq k+1 \) and consider

\[ \begin{align*}
e^{-\mu \lambda_1 t} \int_0^t e^{\mu \lambda_1 s} \left[ \left\| u(s) \right\|_{\mathbb{V}_\Omega}^2 + \left\| p(s) \right\|_{L^2(\Omega)}^2 \right] ds \\
\leq & e^{-\mu \lambda_1 t} \sum_{j=0}^k e^{\mu \lambda_1 (j+1)} \int_j^{j+1} \left[ \left\| u(s) \right\|_{\mathbb{V}_\Omega}^2 + \left\| p(s) \right\|_{L^2(\Omega)}^2 \right] ds \\
\leq & e^{-\mu \lambda_1 t} \sum_{j=0}^k e^{\mu \lambda_1 (j+1)} \int_j^{j+1} \left[ \left\| u(s) \right\|_{\mathbb{V}_\Omega}^2 + \left\| p(s) \right\|_{L^2(\Omega)}^2 \right] ds \\
\leq & e^{-\mu \lambda_1 t} \frac{C(1 + M_2^2 + M_1^2 \alpha - 2)}{\mu^2} \left( \frac{2}{\mu \lambda_1} + 1 \right) \left\| f \right\|_{\mathbb{V}_\Omega}^2, \\
e^{-\mu \lambda_1 t} C(1 + M_2^2 + M_1^2 \alpha - 2) e^{\mu \lambda_1 (k+1) \mu \lambda_1 - 1} \left\| f \right\|_{\mathbb{V}_\Omega}^2 e^{\mu \lambda_1 (k+1) \mu \lambda_1 - 1}. \end{align*} \]
\[ \leq \frac{C(1+M_t^2 + M_t^{2r-2})}{\mu^2} \left( \frac{2}{\mu\lambda_1} + 1 \right) \|f\|_{V_0^\prime}^2 e^{2\mu\lambda_1} - 1. \]  

(6.14)

Note that

\[ \int_0^t e^{-\mu\lambda_1(t-s)} \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds \]

\[ = \left( \int_{\{s:|u(s)|^r \leq R\}} + \int_{\{s:R \leq |u(s)|^r \leq 2R\}} + \int_{\{s:|u(s)|^r \geq 2R\}} \right) e^{-\mu\lambda_1(t-s)} \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds. \]

Using the definition of \( \chi_R \), the first integral in the above equality becomes zero. For \( R \leq |u(s)|^r \leq 2R \), we find

\[ \int_{\{s:R \leq |u(s)|^r \leq 2R\}} e^{-\mu\lambda_1(t-s)} \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds \]

\[ = \frac{1}{R^8} \int_{\{s:R \leq |u(s)|^r \leq 2R\}} e^{-\mu\lambda_1(t-s)} \left( (|u(s)|^{2r} - R^2)^2(2R^2 - |u(s)|^{2r})^2 \right) \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds \]

\[ \leq \frac{1}{R^8} \int_0^t e^{-\mu\lambda_1(t-s)} \left( (|u(s)|^{2r} + R^2)(2R^2 + |u(s)|^{2r})^2 \right) \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds \]

\[ \leq \frac{CM_{1}^{8r-2}}{\mu^2 R^8} \left( \frac{2}{\mu\lambda_1} + 1 \right) \|f\|_{V_0^\prime}^2 e^{2\mu\lambda_1} - 1, \]

where we used a calculation similar to (6.14). For \( |u(s)|^r \geq 2R \), it can be easily seen that

\[ \int_{\{s:|u(s)|^r \geq 2R\}} e^{-\mu\lambda_1(t-s)} \|\chi_R[|u(s)|^{r-1}u(s)]\|_{L^2(\Omega)}^2 \, ds \]

\[ = \int_{\{s:|u(s)|^r \geq 2R\}} e^{-\mu\lambda_1(t-s)} \|1\|_{L^2(\Omega)}^2 \, ds \leq \int_0^t e^{-\mu\lambda_1(t-s)} \left( |u(s)|^r \right) \frac{2R}{\|u(s)\|_{L^2(\Omega)}} \, ds \]

\[ \leq \frac{1}{4R^2} \int_0^t e^{-\mu\lambda_1(t-s)} \|u(s)\|_{L^2(\Omega)}^{2r} \, ds \leq \frac{C}{R^2} \int_0^t \|u(s)\|_{H^1_0}^{2r-2} \|u(s)\|_{V_0}^2 \, ds \]

\[ \leq \frac{CM_{2}^{7r-2}}{\mu^2 R^2} \left( \frac{2}{\mu\lambda_1} + 1 \right) \|f\|_{V_0^\prime}^2 e^{2\mu\lambda_1} - 1. \]

Thus, from (6.13), we immediately have

\[ \|\chi_R u(t)\|_{L^2(\Omega)}^2 \]

\[ \leq \|\chi_R u_0\|_{L^2(\Omega)}^2 e^{-\mu\lambda_1 t} + \frac{2M_t}{\mu\lambda_1} \chi_R f \|_{L^2(\Omega)}^2 + \frac{CM_{1}^2}{R\mu\lambda_1} \left( 1 + \mu + M_t^2 \right) \]

\[ + \frac{C}{R^2} \left[ (1 + M_t^2 + M_t^{2r-2})(1 + \mu) + \frac{M_t^{2r-2}}{R} + \frac{M_t^{2r-2}}{R^2} \right] \left( \frac{2}{\mu\lambda_1} + 1 \right) \|f\|_{V_0^\prime}^2 e^{2\mu\lambda_1} - 1. \]  

(6.15)

Since \( f \in H_0 \), it can be easily seen that \( \lim_{R \to \infty} \|\chi_R f\|_{L^2(\Omega)} = 0 \). Hence, for any \( \varepsilon > 0 \), there exist \( m_1(\varepsilon) \) and \( T_1(\varepsilon) \) such that

\[ \|\chi_{m_{1}(\varepsilon)} u(t)\|_{L^2(\Omega)}^2 < \varepsilon, \quad \text{for all } t > T_1(\varepsilon). \]
Moreover, for \( m > m_1(\varepsilon) \), we finally have
\[
\|u(t)\|^2_{L^2(\Omega \setminus \Omega_m)} \leq \|\chi_{m_1(\varepsilon)} u(t)\|^2_{L^2(\Omega)} \leq \varepsilon, \tag{6.16}
\]
for all \( t > T_1(\varepsilon) \).

One can prove the following Lemma in a similar fashion as in Lemma 6.1.

**Lemma 6.2.** Assume that \( u_{m,0} \in \mathcal{B}_{1,m} \), where \( \mathcal{B}_{1,m} \) is the absorbing set defined in (3.14). Then, for any \( \varepsilon > 0 \), there exist \( T_2(\varepsilon) \) and \( m_2(\varepsilon) \) such that the solution \( u_m(\cdot) \) of the problem (6.1) with the initial condition \( u_0 \) satisfies:
\[
\int_{\Omega_m \setminus \Omega_k} |u_m(x,t)|^2 dx \leq \varepsilon, \tag{6.17}
\]
for \( t \geq T_2(\varepsilon) \) and \( m \geq k \geq m_2(\varepsilon) \), where \( T_2(\varepsilon) \) and \( m_2(\varepsilon) \) are constants depending only on \( \varepsilon \).

### 6.1. Upper semicontinuity

Let us now prove the main results of this section, the upper semicontinuity of the global attractors \( \mathcal{A}_m \) to the global attractor \( \mathcal{A} \) as \( m \to \infty \).

**Lemma 6.3.** If \( u_{m,0} \in \mathcal{A}_m, m = 1, 2, \ldots \), then there exists a \( u_0 \in \mathcal{A} \), such that up to a subsequence
\[
S_m(\cdot)u_{m,0} \overset{w}{\rightharpoonup} S(\cdot)u_0 \quad \text{in} \quad L^2(-T, T; V_\Omega), \tag{6.18}
\]
and
\[
S_m(t)u_{m,0} \overset{w}{\rightharpoonup} S(t)u_0 \quad \text{in} \quad H_\Omega, \quad \text{for each} \quad t \in \mathbb{R}. \tag{6.19}
\]

**Proof.** Given that \( u_{m,0} \in \mathcal{A}_m, m = 1, 2, \ldots \) and hence \( S_m(t)u_{m,0} \in \mathcal{A}_m \), for each \( t \in \mathbb{R} \) and using (3.14), we find
\[
\|S_m(t)u_{m,0}\|_H \leq M_1, \quad \text{for all} \quad t \in \mathbb{R}. \tag{6.20}
\]
Estimates like (3.4) ensures that for each \( T > 0 \), the sequence \( \{S_m(t)u_{m,0}\}_{m \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(-T, T; H_\Omega) \cap L^2(-T, T; V_\Omega) \) and \( \{\frac{\partial}{\partial t}S_m(t)u_{m,0}\}_{m \in \mathbb{N}} \) is bounded in \( L^2(-T, T; V_{\Omega}') \). These uniform bounds guarantee us the existence of a subsequence \( \{u_{m_k,0}\}_{k \in \mathbb{N}} \) of \( \{u_{m,0}\}_{m \in \mathbb{N}} \) such that
\[
\begin{cases}
S_{m_k}(t)u_{m_k,0} \overset{w}{\rightharpoonup} u \quad \text{in} \quad L^\infty(-T, T; H_\Omega), \\
S_{m_k}(t)u_{m_k,0} \overset{w}{\rightharpoonup} u \quad \text{in} \quad L^2(-T, T; V_\Omega), \\
S_{m_k}(t)u_{m_k,0} \overset{w}{\rightharpoonup} u \quad \text{in} \quad L^{r+1}(-T, T; \tilde{L}_{\Omega}^{r+1}), \\
\frac{\partial}{\partial t}S_{m_k}(t)u_{m_k,0} \overset{w}{\rightharpoonup} \frac{\partial u}{\partial t} \quad \text{in} \quad L^2(-T, T; V_{\Omega}').
\end{cases} \tag{6.21}
\]
From the first convergence in (6.21), it is clear that
\[
S_{m_k}(t)u_{m_k,0} \overset{w}{\rightharpoonup} u \quad \text{in} \quad L^2(-T, T; H_\Omega). \tag{6.22}
\]
Arguing similarly as in Lemma 4.1, one can show that \( u(\cdot) \) is a weak solution to (2.10) defined on \( \mathbb{R} \) and \( u \in C(\mathbb{R}; H_\Omega) \). Thus, we can write \( u(t) = S(t)u(0) \) and obtain
\[
S_{m_k}(\cdot)u_{m_k,0} \overset{w}{\rightharpoonup} S(\cdot)u(0) \quad \text{in} \quad L^2(-T, T; H_\Omega), \tag{6.23}
\]
which proves (6.18).
Our next aim is to establish (6.19). Let us now fix $t^* \in [-T, T]$. From the estimate (6.20), we know that the sequence $\{S_m(t^*)u_{m,0}\}_{m \in \mathbb{N}}$ is bounded in $\mathbb{H}_\Omega$. Therefore there exists $\tilde{u} \in \mathbb{H}_\Omega$ and a subsequence $\{S_{m_k}(t^*)u_{m_k,0}\}_{k \in \mathbb{N}}$ of $\{S_m(t^*)u_{m,0}\}_{m \in \mathbb{N}}$ such that $S_{m_k}(t^*)u_{m_k,0} \rightharpoonup \tilde{u}$ as $k \to \infty$ in $\mathbb{H}_\Omega$. Thus $u(t)$ is a solution of (2.10) with $u(t^*) = \tilde{u}$. Since $t^* \in [-T, T]$ is arbitrary, we obtain
\[ S_{m_k}(t)u_{m_k,0} \rightharpoonup S(t)u(0) \quad \text{in} \quad \mathbb{H}, \quad \text{for each} \quad t \in \mathbb{R}, \quad (6.24) \]
as desired. Now, it is left to show that $u_0 \in \mathcal{A}$. Using the weakly lower-semicontinuity of norm, (6.24) and (6.26), we find
\[ \|S(t)u(0)\|_{\mathbb{H}} \leq \liminf_{k \to \infty} \|S_{m_k}(t)u_{m_k,0}\|_{\mathbb{H}} \leq M_1, \quad \text{for all} \quad t \in \mathbb{R}, \quad (6.25) \]
which implies that the solution $S(t)u(0)$ defined on $\mathbb{R}$ and bounded. Hence, $S(t)u(0) \in \mathcal{A}$, for all $t \in \mathbb{R}$ and in particular $u(0) = u_0 \in \mathcal{A}$, which completes the proof. \hfill \square

The following Lemma improves the result obtained in Lemma 6.3.

**Lemma 6.4.** If $u_{m,0} \in \mathcal{A}_m$, $m = 1, 2, \ldots$, then there exists $u_0 \in \mathcal{A}$ such that up to a subsequence
\[ u_{m,0} \to u_0 \quad \text{strongly in} \quad \mathbb{H}. \quad (6.26) \]

**Proof.** The proof is similar to that of Proposition 4.3. A calculation similar to (4.21) yields
\begin{align*}
\|u_m(t)\|_{\mathbb{H}}^2 &= \|u_m(s)\|_{\mathbb{H}}^2 e^{-\mu \lambda_1(t-s)} + 2 \int_s^t e^{-\mu \lambda_1(t-s)} \left[ \langle f, u_m(s) \rangle - \langle u_m(s) \rangle^2 - \alpha \|u_m(s)\|_{\mathbb{H}}^2 - \beta \|u_m(s)\|_{\mathbb{H}^{r+1}}^2 \right] ds.
\end{align*}
(6.27)
Note that $u_{m,0} = S_m(T)S_m(-T)u_{m,0}$, for each $T \in \mathbb{R}$. Using (6.27), we find
\begin{align*}
\|u_{m,0}\|_{\mathbb{H}}^2 &= \|S_m(T)S_m(-T)u_{m,0}\|_{\mathbb{H}}^2 \\
&= \|S_m(T_0)S_m(-T)u_{m,0}\|_{\mathbb{H}}^2 e^{\mu \lambda_1(T-T_0)} + 2 \int_{T_0}^T e^{-\mu \lambda(t-s)} \langle f, S_m(s)S_m(-T)u_{m,0} \rangle ds \\
&- 2 \int_{T_0}^T e^{-\mu \lambda(t-s)} \left[ \langle S_m(s)S_m(-T)u_{m,0} \rangle^2 + \alpha \|S_m(s)S_m(-T)u_{m,0}\|_{\mathbb{H}}^2 \right] ds \\
&+ \beta \|S_m(s)S_m(-T)u_{m,0}\|_{\mathbb{H}^{r+1}}^2 ds.
\end{align*}
(6.28)
Using Lemma 6.3 (see (6.19)), one can easily see that there exists a $u_0 \in \mathcal{A}$ such that up to a subsequence
\[ u_{m,0} \rightharpoonup u_0 \quad \text{weakly in} \quad \mathbb{H}_\Omega, \quad (6.29) \]
and the weakly lower-semicontinuity property of the $\mathbb{H}_\Omega$ norm gives
\[ \|u_0\|_{\mathbb{H}_\Omega} \leq \liminf_{m \to \infty} \|u_{m,0}\|_{\mathbb{H}_\Omega}. \quad (6.30) \]
Since $u_{m,0} \in \mathcal{A}_m$, the solution $S_m(t)u_{m,0}$ of the problem (6.3) is bounded on $\mathbb{R}$ and $S_m(t)u_{m,0} \in \mathcal{A}_m$, for all $t \in \mathbb{R}$. Hence, for each $T \geq 0$, one can easily get $S_m(-T)u_{m,0} \in \mathcal{A}_m$. Once again using Lemma 6.3, we can find a $u_T \in \mathcal{A}$ such that, up to a subsequence
\[ S_m(t)S_m(-T)u_{m,0} \rightharpoonup u_T \quad \text{weakly in} \quad \mathbb{H}_\Omega, \quad (6.31) \]
for all $t \in \mathbb{R}$. Proceeding similarly as in Proposition 4.3, we obtain
\[
\limsup_{m \to \infty} \|u_{m,0}\|_{H}^{2} \leq \|u_{0}\|_{H}^{2} + (M_{1}^{2} - \|S(T^{*})u_{T}\|_{H}^{2})e^{-\mu \lambda_{1}(T-T^{*})} \leq \|u_{0}\|_{H}^{2} + M_{1}^{2}e^{-\mu \lambda_{1}(T-T^{*})},
\]
(6.32)
for all $T > T^{*}$. Since $H_{\Omega}$ is a Hilbert space, passing $T \to \infty$ in (6.32) and using (6.29)-(6.30), we get (6.26), which completes the proof. □

Using Lemmas 6.3 and 6.4, we obtain the following main theorem:

**Theorem 6.5.** Assume that $f \in H$ and $\mu > 0$. Let $A$ and $A_{m}$ be the global attractors corresponding to the problems (2.10) and (6.3), respectively. Then
\[
\lim_{m \to +\infty} \text{dist}_{H_{\Omega}}(A_{m}, A) = 0,
\]
(6.33)
where $\text{dist}_{H_{\Omega}}(A_{m}, A) = \sup_{u \in A_{m}} \text{dist}_{H_{\Omega}}(u, A)$ is the Hausdorff semidistance of the space $H_{\Omega}$.

**Proof.** We prove the Theorem by using a contradiction. Let us assume that (6.33) does not hold true. Then there exists a fixed $\varepsilon_{0} > 0$ and a sequence $u_{m} \in A_{m}$ such that
\[
\text{dist}_{H_{\Omega}}(u_{m}, A) \geq \varepsilon_{0} > 0, \quad m = 1, 2, \ldots.
\]
(6.34)
Using Lemma 6.4, we can find a subsequence $\{u_{m_{k}}\}$ of $\{u_{m}\}$ such that $\lim_{k \to +\infty} \text{dist}_{H_{\Omega}}(u_{m_{k}}, A) = 0$, which is a contradiction to (6.34) and it completes the proof. □

**Remark 6.6.** We can consider the asymptotic behavior of solutions of the 2D CBF equations (1.2) in general unbounded domains also. In that case, one has to take the norm defined on $V$ space as $\|u\|_{V}^{2} := \|u\|_{L_{2}}^{2} + \|\nabla u\|_{L_{2}}^{2}$. The Darcy coefficient $\alpha > 0$ appearing in (1.2) helps us to get the existence of global attractors. For instance, from the inequality (3.15), one can deduce that
\[
\|u(t)\|_{H}^{2} \leq \|u_{0}\|_{H}^{2}e^{-\alpha t} + \frac{1}{\mu \alpha}\|f\|_{V}^{2},
\]
for all $t \geq 0$. Thus, we obtain
\[
B_{1} := \left\{ v \in H : \|v\|_{H} \leq M_{1} \equiv \sqrt{\frac{2}{\mu \alpha}\|f\|_{V}} \right\},
\]
is a bounded absorbing set in $H$ for the semigroup $S(t)$. For the asymptotic compactness property, we can define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ by
\[
\langle u_{1}, u_{2} \rangle = \mu(\nabla u_{1}, \nabla u_{2}) + \frac{\alpha}{2}(u_{1}, u_{2}), \quad \text{for all } u_{1}, u_{2} \in V.
\]
Similar changes can be made in Theorems 5.2 and 6.5 also.

7. Quasi-stability and existence of global as well as exponential attractors

In this section, we consider the 2D CBF equations (2.10) in bounded domains. We show that the semigroup associated with the system (2.10) is quasi-stable. Moreover, making use of Theorem 3.4.11, [9], we show the existence of global as well as exponential attractors having finite fractal dimension. Quasi-stability and the existence of global as well as exponential attractors for the 2D Navier-Stokes equations and related hydrodynamic type equations have been established in [9]. We point out that our model does not fall in the hydrodynamic type...
the first equation in (7.4) to obtain the following system:

\[ \|S(t)y_1 - S(t)y_2\|_X \leq a_s \|y_1 - y_2\|_X, \quad \text{for every } y_1, y_2 \in B \text{ and } t \in [0, t_s], \]  \tag{7.1}

and

\[ \|P_t y_1 - P_t y_2\|_X \leq q \|y_1 - y_2\|_X + n_1(y_1 - y_2) + n_2(P_t y_1 - P_t y_2), \]  \tag{7.2}

for every \( y_1, y_2 \in B \). Then, under the conditions (7.1) and (7.2), using Proposition 3.4.10, \[9\], we infer that the system \((X, S(t))\) is quasi-stable on \(B \subset X\). We follow Proposition 4.5.9, \[9\] to obtain the following Ladyzhenskaya squeezing property.

**Proposition 7.1** (Ladyzhenskaya’s squeezing property). Let \(Q_m = I - P_m\) and \(f \in H\). Then, for every \(0 < q < 1\), \(0 < T^* \leq T < \infty\), there exists \(m_* = m(T^*, T, r, q)\) such that

\[ \|Q_m[S(t)u - S(t)u_*]\|_H \leq q\|u - u_*\|_H, \quad \text{for all } t \in [T^*, T], \ m \geq m_*, \]  \tag{7.3}

for any \(u\) and \(u_*\) from the set \(\mathcal{D}\), where \(\mathcal{D} = \{u \in V : \|S(t)u\|_V \leq R, \ \text{for all } t \in [0, T]\}\).

**Proof.** Let us define \(w = u - u_*\), where \(u(t) = S(t)u\) and \(u_* = S(t)u_*\). Then \(w(\cdot)\) satisfies the following system:

\[
\begin{cases}
\frac{d}{dt} w(t) + \mu A w(t) + \alpha w(t) + B(w(t), u(t)) + B(u_*(t), w(t)) + C(u(t)) - C(u_*(t)) = 0, \\
w(0) = w_0,
\end{cases}
\tag{7.4}
\]

in \(H\) for all \(t \in (0, T)\), where \(w_0 = u - u_*\). Note that for \(\varphi \in V\), we have \(P_m \varphi = \sum_{j=1}^{m} \lambda_j^{1/2}(\varphi, e_j)e_j\), \(A^{1/2} P_m \varphi = \sum_{j=1}^{m} \lambda_j^{1/2}(\varphi, e_j)e_j\), \(Q_m \varphi = \sum_{j=m+1}^{\infty} (\varphi, e_j)e_j\), \(A^{1/2} Q_m \varphi = \sum_{j=m+1}^{\infty} \lambda_j^{1/2}(\varphi, e_j)e_j\),

\[
\|A^{1/2} Q_m \varphi\|_H^2 = \sum_{j=m+1}^{\infty} \lambda_j \| (\varphi, e_j)\|^2 \geq \lambda_{m+1} \sum_{j=m+1}^{\infty} \| (\varphi, e_j)\|^2 = \lambda_{m+1} \|Q_m \varphi\|_H^2,
\]

and

\[
\|A^{1/2} P_m \varphi\|_H^2 = \sum_{j=1}^{m} \lambda_j \| (\varphi, e_j)\|^2 \leq \lambda_m \sum_{j=1}^{m} \| (\varphi, e_j)\|^2 = \lambda_m \|P_m \varphi\|_H^2.
\]

That is, we get

\[
\|Q_m \varphi\|_V \geq \sqrt{\lambda_{m+1}} \|Q_m \varphi\|_H \quad \text{and} \quad \|P_m \varphi\|_V \leq \sqrt{\lambda_m} \|P_m \varphi\|_H.
\tag{7.5}
\]

We define \(p(t) = P_m w(t)\) and \(q(t) = Q_m w(t)\). Let us take the inner product with \(q(t)\) to the first equation in (7.4) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|q(t)\|_V^2 + \mu \|q(t)\|_V^2 + \alpha \|q(t)\|_H^2 = -b(w(t), u(t), q(t)) - b(u_*(t), w(t), q(t)) - \langle C(u(t)) - \beta C(u_*(t)), q(t) \rangle,
\tag{7.6}
\]

equations described in \[9\] due to the presence of the nonlinear damping term. Quasi-stability results for the three dimensional Kelvin-Voigt fluid flow equations with “fading memory” is obtained in \[33\].
Similarly, we estimate $b(w, u, q)$ as
\[
|b(w, u, q)| = |b(w, q, u)| \leq \|q\|_V\|w\|_{L^4}\|u\|_{L^4} \leq \frac{\mu}{8}\|q\|_V^2 + \frac{4}{\mu}\|w\|_V\|u\|_V\|u\|_V
\]
\[
\leq \frac{\mu}{8}\|q\|_V^2 + \frac{\varepsilon}{4}\|w\|_V^2 + \frac{16}{\mu^2\varepsilon}\|u\|_H^2\|u\|_V^2\|w\|_H^2.
\]
Similarly, we estimate $b(u_*, w, q)$ as
\[
|b(u_*, w, q)| \leq \frac{\mu}{8}\|q\|_V^2 + \frac{\varepsilon}{4}\|w\|_V^2 + \frac{16}{\mu^2\varepsilon}\|u_*\|_H^2\|u_*\|_V^2\|w\|_H^2.
\]
For $r = 1$, we know that $-\beta\langle C(u) - C(u_*), q \rangle = -\beta\|Q_m w\|_H^2$. For $r > 1$, using Taylor's formula, Hölder's Gagliardo-Nirenberg's and Young's inequalities, we estimate $-\beta\langle C(u) - C(u_*), q \rangle$ as
\[
\beta\langle C(u) - C(u_*), q \rangle = \beta\left|\int_0^1 C'(\theta u + (1 - \theta)u_*)wd\theta, q\right|
\]
\[
\leq \beta r \sup_{0 \leq \theta \leq 1}\|\theta u + (1 - \theta)u_*\|_{L^{r+1}}\|q\|_{L^{r+1}}
\]
\[
\leq \beta Cr\left(\|u\|_{L^{r+1}} + \|u_*\|_{L^{r+1}}\right)^r\langle w\rangle\|w\|_V
\]
\[
\leq \frac{\mu}{4}\|q\|_V^2 + \frac{C\beta^2}{\mu}\left(\|u\|_{L^{r+1}} + \|u_*\|_{L^{r+1}}\right)^2\|w\|_{L^{r+1}}^2
\]
\[
\leq \frac{\mu}{4}\|q\|_V^2 + \frac{C\beta^2}{\mu}\left(\|u\|_{H^{2}}\|u\|_{V} + \|u_*\|_{H^{2}}\|u_*\|_{V}\right)^2\|w\|_{V}^{2(r-1)}\|w\|_{V}^{4(r-1)}.
\]
Combining the above estimates and substituting it in (7.6), we deduce that
\[
\frac{d}{dt}\|q(t)\|_H^2 + \mu\|q(t)\|_V^2 + \alpha\|q(t)\|_V^2
\]
\[
\leq \varepsilon\|w(t)\|_V^2 + \frac{32}{\mu^2\varepsilon}\left(\|u(t)\|_H^2\|u(t)\|_V^2 + \|u_* (t)\|_H^2\|u_* (t)\|_V^2\right)\|w(t)\|_H^2
\]
\[
+ \frac{C\beta^2}{\mu}\left(\|u(t)\|_{H^{2}}\|u(t)\|_{V}^2 + \|u_* (t)\|_{H^{2}}\|u_* (t)\|_{V}^2\right)^2\|w(t)\|_{V}^{2(r-1)}\|w(t)\|_{V}^{4(r-1)}
\]
for a.e. $t \in [0, T]$. Since $u, u_* \in D$, we find $\|u(t)\|_V, \|u_* (t)\|_V \leq R$, for all $t \in [0, T]$, so that from the above inequality, we deduce that
\[
\frac{d}{dt}\|q(t)\|_H^2 + \mu\lambda_{m+1}\|q(t)\|_H^2 \leq \varepsilon\|w(t)\|_V^2 + \frac{CR^4}{\mu^2\varepsilon}\|w(t)\|_H^2 + \frac{C\beta^2 R^{2(r-1)}}{\mu}\|w(t)\|_{V}^{2(r-1)}\|w(t)\|_{V}^{4(r-1)}
\]
(7.7)
where we used (7.5) also. A variation of constants formula yields
\[
\|Q_m w(t)\|_H^2 \leq \|Q_m w(0)\|_H^2 e^{-\mu\lambda_{m+1} t} + \varepsilon\int_0^t e^{-\mu\lambda_{m+1} (t-s)}\|w(s)\|_V^2 ds
\]
\[
+ \frac{CR^4}{\mu^2\varepsilon}\int_0^t e^{-\mu\lambda_{m+1} (t-s)}\|w(s)\|_H^2 ds
\]
\begin{align}
&+ \frac{C\beta^2 R^{2(r-1)}}{\mu} \int_0^t e^{-\mu \lambda_{m+1}(t-s)} \|w(s)\|_{\mathbb{H}}^{2(r-1)+1} \|w(s)\|_V^{\frac{1}{r}} \, ds \\
&\leq \|w(0)\|_{\mathbb{H}}^2 e^{-\mu \lambda_{m+1} t} + \varepsilon \int_0^t \|w(s)\|^2_V \, ds + \frac{CR^4}{\mu^3 \varepsilon \lambda_{m+1}} \sup_{s \in [0,t]} \|w(s)\|_{\mathbb{H}}^2 \\
&+ \frac{C\beta^2 R^{2(r-1)}}{\mu^2 \lambda_{m+1}} \sup_{s \in [0,t]} \|w(s)\|_V^{2(r-1)+1} \left( \int_0^t \|w(s)\|_V^{\frac{1}{r}} \, ds \right),
\end{align}

for all \( t \in [0,T] \). Using (3.12), for \( r > 1 \), we deduce that
\[
\|Q_m w(t)\|_{\mathbb{H}}^2 \leq \left\{ e^{-\mu \lambda_{m+1} t} + \left( \varepsilon + \frac{CR^4}{\mu^3 \varepsilon \lambda_{m+1}} + \frac{C\beta^2 R^{2(r-1)}}{\mu^2 \lambda_{m+1}} \right) e^{\frac{4\beta^2}{r}} \right\} \|w(0)\|_{\mathbb{H}}^2,
\]
for every \( \varepsilon > 0 \). The above inequality easily implies (7.3). The case of \( r = 1 \) is easy. \( \square \)

**Proposition 7.2**: (Quasi-stability). For every \( 0 < q < 1 \), \( 0 < a \leq b < \infty \), and a forward invariant set \( \mathcal{B} \), which is a bounded set in \( \mathbb{H} \), there exists \( m = m(a,b,q,B) \) such that
\[
\|S(t)u - S(t)u_*\|_{\mathbb{H}} \leq q \|S(r)u - S(r)u_*\|_{\mathbb{H}} + \|P_m[S(t)u - S(t)u_*]\|_{\mathbb{H}},
\]
for every \( t \in [a+r,b+r] \) and for all \( u, u_* \in \mathcal{B} \) and \( r \geq 0 \). Furthermore, the system \((S(t), \mathbb{H})\) is quasi-stable on \( \mathcal{B} \subset \mathbb{H} \).

**Proof.** We know that
\[
\|S(t)u - S(t)u_*\|_{\mathbb{H}} \leq \|Q_m[S(t)u - S(t)u_*]\|_{\mathbb{H}} + \|P_m[S(t)u - S(t)u_*]\|_{\mathbb{H}},
\]
and hence from (7.3), we obtain
\[
\|S(t)u - S(t)u_*\|_{\mathbb{H}} \leq q \|u - u_*\|_{\mathbb{H}} + \|P_m[S(t)u - S(t)u_*]\|_{\mathbb{H}},
\]
for every \( t \in [a,b] \) and for all \( u, u_* \in \mathcal{B} \). It is clear that \( \|P_m[\cdot]\|_{\mathbb{H}} \) is a seminorm on \( \mathbb{H} \). Thus, the condition (7.2) holds with \( \Xi = \mathbb{H} \), \( n_1 \equiv 0 \) and \( n_2 = \|P_m[\cdot]\|_{\mathbb{H}} \). The Lipschitz property (7.1) follows from (3.13). Therefore, applying Proposition 3.4.10, [9], we know that the system \((S(t), \mathbb{H})\) is quasi-stable on \( \mathcal{B} \subset \mathbb{H} \). Using Exercise 4.3.12, [9], we infer that the relation in (7.11) can be written in the uniform form (7.10). \( \square \)

Then, based on Theorem 3.4.11, [9], we have the following result.

**Theorem 7.3.** Let \( f \in \mathbb{H} \). Then the global attractor \( A_{\text{glob}} \) of the \((\mathbb{H}, S(t))\) generated by the dynamical system (2.10) is a bounded set in \( \mathbb{V} \) and possesses the following properties:

1. \( A_{\text{glob}} \) has a finite fractal dimension \( \text{dim}_F^H(A_{\text{glob}}) \) in \( \mathbb{H} \).
2. For any full trajectory \( \{u(t) : t \in \mathbb{R}\} \) from the attractor, \( u(t) \) is an absolutely continuous function with values in \( \mathbb{H} \) and
\[
\sup_{t \in \mathbb{R}} \left\{ \|u_t(t)\|_{\mathbb{H}} + \|u(t)\|_{\mathbb{V}} + \|B(u(t))\|_{\mathbb{H}} + \|C(u(t))\|_{\mathbb{H}} \right\} \leq C.
\]

Moreover, the system \((\mathbb{H}, S(t))\) possesses an exponential attractor \( A_{\text{exp}} \), whose fractal dimension \( \text{dim}_F^H(A_{\text{exp}}) \) is finite in the phase space \( \mathbb{H} \).
Substituting (7.14) in (7.13), we deduce that

\[ \|u(t + \theta) - u(t)\|_{\mathbb{H}} \leq q\|u(t + \theta - 1) - u(t - 1)\|_{\mathbb{H}} + \|P_m[u(t + \theta) - u(t)]\|_{\mathbb{H}}, \]

for every \( t \in \mathbb{R} \) and for full trajectory \{u(t) : t \in \mathbb{R}\} from the attractor. It can be easily seen that

\[
\|P_m[u(t + \theta) - u(t)]\|_{\mathbb{H}} = \left\|P_m\left[\int_t^{t+\theta} \frac{du}{d\tau}(\tau)d\tau\right]\right\|_{\mathbb{H}} \\
\leq \int_t^{t+\theta} \|P_m[-\mu Au(\tau) + B(u(\tau)) + \alpha u(\tau) + \beta C(u(\tau))]\|_{\mathbb{H}}d\tau \\
\leq C(m, A_{\text{glob}})|\theta|. \tag{7.14}
\]

Substituting (7.14) in (7.13), we deduce that

\[
(1 - q) \sup_{t \in \mathbb{R}} \|u(t + \theta) - u(t)\|_{\mathbb{H}} \leq C(m, A_{\text{glob}})|\theta|, \tag{7.15}
\]

and on passing \( \theta \to 0 \), we finally obtain (7.12).

In order to prove the existence of the fractal exponential attractors \( A_{\exp} \), we use the second part of Theorem 3.4.11, [9]. For this, we need to check the Hölder continuity property

\[
\|S(t_1)u - S(t_2)u\|_{\mathbb{H}} \leq C_{B,T}|t_1 - t_2|^\gamma, \quad t_1, t_2 \in [0, T], \quad u \in \mathcal{B},
\]

for some \( 0 < \gamma \leq 1 \), on some forward invariant absorbing set \( \mathcal{B} \) for \( (\mathbb{H}, S(t)) \). On an absorbing set \( \mathcal{B} = \{u \in \mathbb{V} : \|u\|_{\mathbb{V}} \leq M\} \) in \( \mathbb{V} \), we have

\[
\|u_\varepsilon(t)\|_{\mathbb{V}} \leq \mu\|Au(t)\|_{\mathbb{V}} + \|B(u(t))\|_{\mathbb{V}} + \alpha\|u(t)\|_{\mathbb{V}} + \beta\|C(u(t))\|_{\mathbb{V}} + \|f\|_{\mathbb{V}} \\
\leq \mu\|u(t)\|_{\mathbb{V}} + \sqrt{2}\|u(t)\|_{\mathbb{H}}\|u(t)\|_{\mathbb{V}} + C\alpha\|u(t)\|_{\mathbb{H}} + C\beta\|u(t)\|_{\mathbb{H}}^{-1}\|u(t)\|_{\mathbb{V}} + \|f\|_{\mathbb{V}} \\
\leq C(\mu, \alpha, \beta, \|f\|_{\mathbb{V}}, M) < \infty.
\]

Thus, using interpolation inequality (2.2), we obtain

\[
\|S(t_1)u - S(t_2)u\|_{\mathbb{H}} \leq \|S(t_1)u - S(t_2)u\|_{\mathbb{V}}^{1/2}\|S(t_1)u - S(t_2)u\|_{\mathbb{H}}^{1/2} \\
\leq \sqrt{2M}\left[\int_{t_2}^{t_1} \frac{dS(\tau)u}{d\tau} \right]^{1/2} \|S(t_1)u - S(t_2)u\|_{\mathbb{H}}^{1/2} \\
\leq C(\mu, \alpha, \beta, \|f\|_{\mathbb{V}}, M)|t_1 - t_2|^{1/2},
\]

for all \( u \in \mathcal{B} \). From the relation (7.13), we also get

\[
\|S(t)u - S(t)u_*\|_{\mathbb{H}} \leq e^{\frac{\alpha T}{2}}\|u - u_*\|_{\mathbb{H}},
\]

for all \( u, u_* \in \mathcal{B} \) and \( t \in [0, T] \). Thus the existence of the exponential attractors \( A_{\exp} \) follows by using Theorem 3.4.11, [9] (see Theorem 3.4.7, [9] also). The bounds for the fractal dimensions of \( A_{\text{glob}} \) and \( A_{\exp} \) can be derived from Theorems 3.4.11 and 3.2.3, [9] (see Theorem 5.2 also).
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