Quantum liquid crystals in imbalanced Fermi gas: fluctuations and fractional vortices in Larkin-Ovchinnikov states

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(Dated: June 27, 2009)

We develop a low-energy model of a unidirectional Larkin-Ovchinnikov (LO) state. Because the underlying rotational and translational symmetries are broken spontaneously, this gapless superfluid is a smectic liquid crystal, that exhibits fluctuations that are qualitatively stronger than in a conventional superfluid, thus requiring a fully nonlinear description of its Goldstone modes. Consequently, at nonzero temperature the LO superfluid is an algebraic phase even in 3d. It exhibits half-integer vortex–dislocation defects, whose unbinding leads to transitions to a superfluid nematic and other phases. In 2d at nonzero temperature, the LO state is always unstable to a nematic superfluid. We expect this superfluid liquid-crystal phenomenology to be realizable in imbalanced resonant Fermi gases trapped isotropically.

The tunability of interactions through Feshbach resonances has led to a realization of an s-wave paired superfluidity and BCS-BEC crossover[1, 2, 3], as well as promises of more exotic states such as gapless p-wave[4] and periodic Fulde-Ferrell-Larkin-Ovchinnikov (FFLO)[5, 6] superfluidity in strongly correlated degenerate alkali gases. The latter enigmatic state has been thoroughly explored within a BCS mean-field studies[7, 8, 9] and is expected to be realizable in a population-imbalanced (polarized) Feshbach resonant Fermi gas[10, 11]. While recent experiments[12, 13], have confirmed much of the predicted phenomenology of phase separation[11, 14, 15] in such systems, the FFLO states have so far eluded definitive observation.

The simplest mean-field treatments[5, 6, 11] find that the FFLO type states are quite fragile, confined to a narrow range of polarization on the BCS side. However, motivated by earlier studies[7, 8] and based on the finding of a negative domain-wall energy in an otherwise uniform singlet BCS superfluid[9, 14], it has recently been argued that a more general periodic superfluid state that includes a larger set of collinear wavevectors may be significantly more stable. Much like a type-II superconductor undergoes a continuous transition into a vortex state at a lower-critical field \( H_{c1} \), which is significantly below the thermodynamic field, here too, a Zeeman-field driven domain-wall nucleation (with the density increasing above the lower critical \( h_{c1} \) field) allows a continuous mechanism for a transition from a singlet paired superfluid to a LO-like periodic state[7, 8, 17].

In this scenario the SF-LO transition is of a commensurate-incommensurate type as can be explicitly shown in one dimension (1d)[8, 17]. The imposed species imbalance (excess of the majority fermionic atoms) can be continuously accommodated by the subgap states localized on the self-consistently induced domain-walls, with this picture resembling the doping of polyacetylene[15]. Such LO state can also be thought of as a periodically ordered micro-phase separation between the normal and BCS states, that thus naturally replaces the macro-phase separation ubiquitously found in the BEC-BCS detuning-polarization phase diagram[11] (Fig.1).

With this motivation in mind, here we report on our study that is complementary to these microscopic mean-field investigations. Namely, assuming that the LO state is indeed energetically favorable over a region of a phase diagram, we explore its stability to low-energy fluctuations and the resulting phenomenology.

We demonstrate that the low-energy model of the LO state is that of two coupled smectics, whose moduli we derive from the BCS theory. Thus we show that a resonant imbalanced Fermi gas, confined to an isotropic trap is a natural realization of a quantum (superfluid) liquid
crystal, that unlike the solid state analogs\textsuperscript{4, 22} is not plagued by the underlying lattice potential that explicitly breaks continuous spatial symmetries.

We find that while it is stable to quantum fluctuations, in 3d a long-ranged LO order is marginally unstable at any nonzero $T$. The resulting superfluid state is an \textit{algebraic} phase, characterized by universal quasi-Bragg peaks and correlations that admit an asymptotically exact description\textsuperscript{22}. In contrast, crystalline LO phases with multiple noncollinear ordering wavevectors are stable against thermal fluctuations.

As with treatments of the LO state where long range order is assumed\textsuperscript{21, 24, 25}, we also find an unusual topological excitation – a half vortex bound to a half dislocation. For integer and half-integer vortex-superfluid nematic, depending on the relative energetics, controlled by a nontrivial exactly calculable\textsuperscript{26} phase, exhibiting universal power-law phonon correlations in microscopic details.

In 2d and nonzero $T$, the state is also an \textit{algebraic} phase, exhibiting universal power-law phonon correlations, controlled by a nontrivial exactly calculable\textsuperscript{20} fixed point. It displays short-range positional order with Lorentzian structure function peaks, and is thus unstable to proliferation of dislocations. The resulting state is either a “charge”-4 (four-fermion) superfluid or a non-superfluid nematic, depending on the relative energetics of aforementioned integer and half-integer vortex-dislocation defects. The latter normal nematic state is a (complementarily described\textsuperscript{22}) deformed Fermi surface state\textsuperscript{27, 28}.

\textit{MODEL:} We now present the highlights of our calculations. We begin with a Ginzburg-Landau theory that captures the system’s tendency to order into a finite wavevector paired state, with a preferred magnitude $Q_0$ and a spontaneously chosen direction. The corresponding free-energy density

$$
\mathcal{H} = J \left[ |\nabla^2 \Delta|^2 - 2Q_0^2 |\nabla \Delta|^2 + r |\Delta|^2 + \frac{\nu_1}{2} |\Delta|^4 + \frac{\nu_2}{2} J^2 + \ldots \right] \tag{1}
$$

can be derived from a microscopic BCS model\textsuperscript{6, 11, 20} near the upper-critical chemical potential difference, $h_{c2}$, with

$$
J &\approx \frac{0.61n}{\epsilon_F Q_0^2}, \quad Q_0 \approx \frac{1.81 \Delta_{BCS}}{2 \hbar v_F}, \quad r \approx \frac{3n}{4 \epsilon_F} \ln \left[ \frac{9h}{4h_{c2}} \right], \\
h_{c2} &\approx \frac{3}{4} \Delta_{BCS}, \quad \nu_1 \approx \frac{3n}{4 \epsilon_F \frac{\Delta_{BCS}^2}{Q_0^2}}, \quad \nu_2 \approx \frac{1.83nm^2}{\epsilon_F \Delta_{BCS}^2 Q_0^2} \tag{2}
$$

that can be more generally taken as phenomenological parameters to be determined experimentally. $n$, $\epsilon_F$, $v_F$, $\Delta_{BCS}$, and $m$ are the atomic density, Fermi energy and velocity, BCS ($h = 0$) gap and atomic mass, respectively. Near the lower-critical Zeeman field, $h_{c1}$, $Q_0(h)$ is expected to vanish with the species imbalance, as the system continuously transitions into a uniform singlet superfluid\textsuperscript{7, 8, 9}, with this and other moduli’s dependences derivable via fluctuating domain-walls methods\textsuperscript{23}. Above, $j$ is the supercurrent and the last term is crucial for getting a nonzero transverse (to $Q$) superfluid stiffness in the LO state. From the first two terms it is clear that the dominant instability and fluctuations are near a finite wavevector of magnitude $Q_0$. Thus, for $h < h_{c2}$, $r < JQ_0^4 = 0.61n/\epsilon_F$ and the system develops a pairing order parameter $\Delta(x) = \sum_Q \Delta_Q e^{iQx}$. \textsuperscript{22}

As with other crystallization problems, the choice of the set of $Q_n$’s is determined by the details of interactions and will not be addressed here. Motivated by LO findings\textsuperscript{2}, we focus on the unidirectional order characterized by a collinear set of $Q_n$’s. These fall into two, LO and FF universality classes. The LO (FF) states are characterized by breaking (preserving) translational and preserving (breaking) time-reversal symmetries. Low-energy properties of such states can be well captured with a single $\pm Q$ pair (LO) and a single $Q$ wavevector (FF) approximations.

Because it is the periodic LO state that is expected to be most stable\textsuperscript{6, 5, 8, 9, 16}, we focus on this more interesting case and only comment in passing on the homogeneous FF state. Within the LO approximation the pairing function is given by $\Delta_{LO}(x) = \Delta_+ (x) e^{iQx} + \Delta_- (x) e^{-iQx}$, where $\Delta_\pm = \Delta_Q e^{i\theta_\pm(x)}$ are the leading complex order parameters, whose amplitudes deep in the ordered LO state can be taken to be equal and constant, $\Delta_Q^2 \approx c \Delta_{BCS}^2 \ln(h_{c2}/h)$, thereby focusing on the two Goldstone modes $\theta_\pm(x)$. A slightly rearranged form of the LO order parameter $\Delta_{LO}(x)$ clarifies its physical interpretation

$$
\Delta_{LO}(x) = 2\Delta_Q e^{i\theta_+(x)} \cos(\mathbf{Q} \cdot \mathbf{x} + \theta_{sm}(x)), \tag{3}
$$

showing that it is a product of a superfluid order parameter and a unidirectional, spontaneously oriented (along $Q$) Cooper-pair density wave, i.e., simultaneously exhibiting the ODLRO and smectic order. The low-energy properties are respectively characterized by two Goldstone modes, the superconducting phase $\theta_{sc} \equiv \frac{1}{2}(\theta_+ + \theta_-)$ and the phonon displacement $u = \theta_{sm}/Q \equiv \frac{1}{2}(\theta_+ - \theta_-)/Q$. In contrast, the uniform FF state is characterized by a single $\Delta_Q$ amplitude and a Goldstone mode $\theta_Q$.

Substituting $\Delta_{LO}(x)$ into $\mathcal{H}$ we obtain a Hamiltonian density for the bosonic Goldstone modes of a generic LO state:

$$
\mathcal{H}_{LO} = \sum_{\alpha = \pm} \left[ \frac{K}{4} (\nabla^2 u_\alpha)^2 + \frac{B}{4} (\partial u_\alpha + \frac{1}{2} (\nabla u_\alpha)^2)^2 \right] + \gamma (\nabla u_+ - \nabla u_-)^2, \quad \alpha = \pm \tag{4}
$$

$$
\approx \frac{K}{2} (\nabla^2 u)^2 + \frac{B}{2} \left( \partial u + \frac{1}{2} (\nabla u)^2 \right)^2 + \frac{\gamma}{2} (\nabla \theta_{sc})^2,
$$

where we dropped constant and fast oscillating parts, chose $Q = Q_0 \hat{z}$, and defined phonon fields $u_\pm = \pm \theta_\pm/Q_0$. 

\textsuperscript{2}
and the bend \((K = 4JQ^3_0\Delta^3_0 \approx 2.4n\Delta^3_0/(\epsilon_FQ^3_0))\) and compressional \((B = 16JQ^4_0\Delta^2_0 \approx 9.8n\Delta^2_0/(\epsilon_F))\) elastic moduli.

This form (valid beyond above weak-coupling microscopic scopic derivation) is familiar from studies of conventional smectic liquid crystals [34], with rotational invariance encoded in two ways. Firstly, for a vanishing \(g\) the gradient elasticity in \(u_+\) (and \(u\)) only appears along \(Q\), namely \(\partial \equiv \nabla\) (compression), with elasticity transverse to \(Q\) of a “softer” Laplacian (curvature) type. Secondly, the elastic energy is an expansion in a rotationally-invariant strain tensor combination \(u^\pm_{QQ} = \partial u_+ + \frac{1}{2}(\nabla u_+)^2\), whose nonlinearities in \(u_+\) ensure that it is fully rotationally invariant even for large reorientations \(Q_0\rightarrow Q\) of the LO ground state.

A nonzero \(\varphi \equiv v_2Q^3_0\Delta^3_0/m^2 \approx 1.8n\Delta^3_0/(\epsilon_F\Delta^3_{BCS})\) coupling (minimized by a vanishing supercurrent \(\nabla \theta_+ + \nabla \theta_-\) removes the two independent rotational symmetries, orientationally locking the two smectics. This leads to the superconducting phase combination, \(\varphi_{sc} = \frac{1}{2}(\theta_+ + \theta_-)\) to be of a conventional XY (as opposed to “soft” smectic) gradient type. It is characterized by parallel \((\rho^\parallel \equiv B/Q^3_0)\) and transverse \((\rho^\perp \equiv 4\gamma/Q^3_0)\) superfluid stiffnesses, appearing in the second form of \(\mathcal{H}_{LO}\), equivalent to the first form at low energies of interest to us.

Physically, \(\rho^\parallel\) and \(\rho^\perp\) are superfluid stiffnesses for the supercurrent \(j = (j_+ + j_-)/2\) produced by the imbalance in the left \((j_-)\) and right \((j_+)\) supercurrent magnitudes and directions, respectively. We thus find that the LO state is a highly anisotropic superfluid, with

\[
\rho^\perp/\rho^\parallel = \frac{3}{4}(\Delta_Q/\Delta_{BCS})^2 \approx \ln(hc_2/h),
\]

a ratio that vanishes for \(h \rightarrow h^c\). We find the FF state to be even more exotic, characterized by an identically vanishing transverse superfluid stiffness, a reflection of the rotational invariance of the spontaneous current to an energy-equivalent ground state.

**FLUCTUATIONS:** The thermodynamics can be obtained in a standard way through a coherent path-integral. Although there are nontrivial issues of the interplay between the fermionic quasi-particles and the Goldstone modes, we can unambiguously show that at \(T = 0\) the superfluid and smectic orders (and thus the LO state) are stable to quantum fluctuations in \(d > 1\) [22].

In contrast, for \(T > 0\), \(\varphi_{sc} (\varphi_{sm})\) fluctuations diverge and ODLRO (smectic order) is destroyed for \(d \leq 2\) \((d \leq 3)\). Consequently, we find that the LO state is unstable to thermal fluctuations, displaying quasi-Bragg (Lorentzian) peaks in 3d \((2d)\) in its structure function. Thus in both cases the LO order parameter, \(\varphi\), vanishes and the state is qualitatively distinct from its mean-field form, at low \(T\) characterized by a “charge”-4 superfluid order parameter \(\Delta_{sc}^{(4)} \sim \Delta^2 \approx 1/2\epsilon_0^2(\epsilon_FQ^3_0)\).

Furthermore, in the presence of these divergent thermal fluctuations phonon nonlinearities in \(\mathcal{H}_{LO}\), Eq. 1 become important. They qualitatively modify correlations on scales larger than \(\xi_{NL} \sim [K^{3/2}/(B_1^2T)^{1/(3-d)}] \sim \kappa_1^2[\Delta^2_0Q/(\Delta^2_{BCS}T)]^{1/(3-d)}\) (on shorter scales the harmonic description above applies), giving universal power-laws, e.g., \(\langle(u(z,x)u(0,0))^{1/2}\sim \text{Max} [\theta^a, z^b], \) controlled by a nontrivial low \(T\) (order \(3 - d\)) fixed point [22], that has an exact description in 2d with \(\alpha = 1/2, \beta = 1/3[24]\).

In 3d, \(\xi_{NL} \sim e^{K^{3/2}/(B_1^2T)}\) and phonon correlations grow as a universal power of a logarithm, a result that is asymptotically exact. These elastic results of course only hold as long as dislocations remain bound or on scales shorter than the dislocation unbinding scale.

**DEFECTS:** We now turn to the discussion of topological defects and corresponding phases accessible by their unbinding. With two compact Goldstone modes \(\theta_{sc}, u\) (equivalently, \(\theta = \pm 2\pi n_u/a\)), defects are labeled by vortex and dislocation charges \((2\pi n_v, an_d)\). Ordinary vortex, \((2\pi, 0)\) and dislocation \((0, a)\) are clearly allowed, and in terms of the two smectic displacements these respectively correspond to the opposite and same signs of integer dislocations in \(u_\pm\). When proliferated they destroy the superfluid phase coherence and smectic periodic order, respectively, and either one is clearly sufficient to suppress the conventional LO order, \(\Delta_{LO}\).

However, because a sign change in \(\Delta_{LO}\) due to a \(d/2\)-dislocation in \(u\) can be compensated by a \(\pi\)-vortex in \(\theta_{sc}\) (thereby preserving a single-valuehood of \(\Delta_{LO}\) \(1/2\)-charge defects in \(\theta_{sc}\) and \(u\) are also allowed, but are confined into \((\pm \pi, \pm a/2)\) pairs [21, 24, 22].) In terms of the two coupled smectic displacement fields, \(u_+, u_-\) these correspond to an integer dislocation in one and no dislocation in the other.

**TRANSITIONS:** There are thus many paths of continuous transitions out of the LO (SF\(_2\)–Sm\(_Q\)) state. One is through an unbinding of ordinary integer \((0, a)\) dislocations in \(a\). This melts the smectic order in favor of a nematic, but retains a superfluid order, thereby transforming the LO state to a nematic “charge”-4 superfluid (SF\(_4\)–Nm). Another path, is by unbinding \(2\pi\) vortices in \(\theta_{sc}\). This destroys the superfluid order and converts the smectic positional order \(Q\) to 2Q (N-Sm\(_{2Q}\)). Finally, a third route out of the LO superfluid is through a direct proliferation of \((\pi, \pm a/2)\) fractional vortex-dislocation pairs, that destroy both smectic and superfluid orders, inducing a transition to a normal (non-superfluid) nematic (N-Nm). For 3d these possibilities, determined by the relative energetics of these different types of defects are illustrated in Fig 1. In 2d, the dislocation energy is finite and LO state is necessarily destabilized by thermal fluctuations to a “charge”-4 superfluid nematic, SF\(_4\)–Nm. Upon rotation the resulting nematic superfluid will display \(\pi\) vortices \((j \cdot v \cdot \hat{d} = h/4m)\), that (because of the nematic order) we expect to form a uniax-
ially distorted lattice. We note that this rich fluctuations-driven phase behavior contrasts sharply with a direct LO-N transition (described by $U(1) \times U(1)$ Landau theory $H_{mf} = r\left|\Delta_+^2 + \Delta_-^2\right| + \lambda_1\left|\Delta_+^4 + |\Delta_-|^4\right| + \lambda_2\left|\Delta_+^2|\Delta_-|^2\right|$) found in mean-field theory.

**FERMIONS**: We now turn to a discussion of the fermionic sector that we have so far ignored. Near $h_{c2}$ a single harmonic ($Q$ for FF and $\pm Q$ for LO states) approximation is sufficient. Unlike the simpler FF case (that can be diagonalized exactly with a two-component Nambu spinor $\tilde{\Psi}$), the LO state involves a three-component vacuum (see for example V. Gurarie, L. Radzihovsky, Annals of Physics 322 2 (2007), and references therein).

The interactions between the Goldstone modes and unpaired fermionic excitations. These are nothing but the unpaired fraction of the majority atoms. From the spectrum above it is clear that two distinct LO states are possible. One, LO1 exhibits a single (majority) species, LO2 characterized by both majority and minority fermion flavor. The Fermi surface volume difference is proportional to the species imbalance and the stiffness we derived near $h_{c2}$. It is extremely sensitive to thermal fluctuations that destroy its long-range positional order even in 3d, replacing it by an algebraic phase, that exhibits vortex fractionalization, where the basic superfluid vortex is half the strength of a vortex in a regular paired condensate. This should be observable via a doubling of a vortex density in a rotated state. Also under rotation, the high superfluid anisotropy leads to an imbalance-tunable strongly anisotropic vortex core and a lattice highly stretched along Q. Bragg peaks in the time-of-flight images can distinguish the periodic SF$_2$–Sm$_Q$ (superfluid smectic) state from the homogeneous SF$_3$–Nn (superfluid nematic), which are in turn distinguished from the N-Sm$_2$Q and N-N (normal smectic and nematic) by their superfluid properties, periodicity, collective modes, quantized vortices, and condensate peaks. Thermodynamic signatures will identify corresponding phase transitions.

We acknowledge support from the NSF DMR-0321848 (LR), DMR-0645691 (AV), the Miller and CU Faculty Fellowships (LR), and thank S. Kivelson for helpful comments on the manuscript. LR thanks Berkeley Physics Department for its hospitality.

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