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Georgios Fellouris & Alexander G. Tartakovsky

Department of Mathematics, University of Southern California, Los Angeles, California, USA

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Nearly Minimax One-Sided Mixture-Based Sequential Tests

Georgios Fellouris and Alexander G. Tartakovsky
Department of Mathematics, University of Southern California,
Los Angeles, California, USA

Abstract: We focus on one-sided, mixture-based stopping rules for the problem of sequential testing a simple null hypothesis against a composite alternative. For the latter, we consider two cases—either a discrete alternative or a continuous alternative that can be embedded into an exponential family. For each case, we find a mixture-based stopping rule that is nearly minimax in the sense of minimizing the maximal Kullback–Leibler information. The proof of this result is based on finding an almost Bayes rule for an appropriate sequential decision problem and on high-order asymptotic approximations for the performance characteristics of arbitrary mixture-based stopping times. We also evaluate the asymptotic performance loss of certain intuitive mixture rules and verify the accuracy of our asymptotic approximations with simulation experiments.

Keywords: Asymptotic optimality; Minimax tests; Mixtures rules; One-sided sequential tests; Open-ended tests; Power one tests.

Subject Classifications: 62L10; 62L15; 60G40.

1. INTRODUCTION

1.1. Problem Formulation and Literature Review

Let \( \{X_n\}_{n\in\mathbb{N}} \) be a sequence of independent and identically distributed (i.i.d.) observations (generally vectors, \( X_n \in \mathbb{R}^d \)) whose common distribution under the probability measure \( P_0 \) (the null hypothesis \( H_0 : P = P_0 \)) is \( F_0 \). There is no cost for sampling under \( P_0 \). However, sampling should be terminated as soon as possible if there is sufficient evidence against \( P_0 \) and in favor of a class of probability measures \( \mathcal{P} \) (an alternative hypothesis \( H : P \in \mathcal{P} \)). The problem is to find an \( \{\mathcal{T}_n\} \)-stopping
time that takes large values under $P_0$ and small values under every probability measure in $\mathcal{P}$, where $\mathcal{T}_n = \sigma(X_1, \ldots, X_n)$ is the sigma-algebra generated by the first $n$ observations $X_1, \ldots, X_n$, $n \geq 1$.

When $\mathcal{P}$ consists of a single probability measure, say $\mathcal{P} = \{P_1\}$, and the $P_1$-distribution of $X_1, F_1$, is absolutely continuous with respect to $F_0$, a definitive solution to this sequential hypothesis testing problem is the one-sided sequential probability ratio test (SPRT)

$$T_A^1 = \inf\{n \geq 1 : \Lambda_n^1 \geq A\}, \quad \inf\{\emptyset\} = \infty,$$

where $A > 1$ is a fixed level (threshold) and $\{\Lambda_n^1\}$ is the corresponding likelihood-ratio process; that is,

$$\Lambda_n^1 = \prod_{m=1}^{n} \frac{dF_1(X_m)}{dF_0(X_m)}, \quad n \in \mathbb{N}.$$

The stopping time $T_A^1$ is often called an open-ended test or a test of power one, because it does not terminate almost surely under $P_0$ ($P_0(T_A^1 < \infty) < 1/A$), whereas it terminates almost surely under $\mathcal{P}$; that is, $P_1(T_A^1 < \infty) = 1$. Furthermore, it follows from Chow et al. (1971, pp. 107–108) that if the threshold $A = A_2$ is selected so that $P_0(T_A^1 < \infty) = z$, then

$$E_{i}[T_A^1] = \inf_{T \in \mathcal{C}_z} E_{i}[T], \quad (1.1)$$

where $E_i$ denotes expectation with respect to $P_i$ and $\mathcal{C}_z = \{T : P_0(T < \infty) \leq z\}$ is the class of stopping times whose “error probability” is bounded by $z$, $0 < z < 1$.

When the alternative hypothesis is not simple, there have been extensions of the one-sided SPRT, but none of them exhibits such an exact optimality property as (1.1) under every probability measure associated with the alternative hypothesis $\mathcal{P}$. More specifically, suppose that $\mathcal{P} = \{P_{\theta}\}_{\theta \in \Theta_{[0]}}$ and that the $P_{\theta}$-distribution of $X_1$ belongs to the exponential family

$$\frac{dF_\theta(x)}{dF_0(x)} = e^{\theta x - \psi_\theta}, \quad \theta \in \Theta = \{\theta \in \mathbb{R} : E_0[e^{\theta X_1}] < \infty\}, \quad (1.2)$$

where $\psi_\theta = \log E_0[e^{\theta X_1}]$. Moreover, let $\Lambda_n^\theta$ be the likelihood ratio of $P_{\theta}$ versus $P_0$ based on the first $n$ observations; that is,

$$\Lambda_n^\theta = \prod_{k=1}^{n} \frac{dF_\theta(X_k)}{dF_0(X_k)} = \exp \left\{ \theta \sum_{k=1}^{n} X_k - n\psi(\theta) \right\}, \quad n \in \mathbb{N} \quad (1.3)$$

and let $I_\theta = E_0[\log \Lambda_n^\theta]$ denote the Kullback–Leibler divergence of $F_\theta$ versus $F_0$, where here and in what follows $E_\theta$ stands for expectation with respect to $P_\theta$.

A natural generalization of the one-sided SPRT is the threshold stopping time $\inf\{n \geq 1 : \Lambda_n^\theta \geq A\}$, where $\theta_n$ is an estimate of the unknown parameter $\theta$ at time $n$. Lorden (1973) followed a generalized likelihood ratio approach, where $\theta_n$ is taken to be the maximum likelihood estimator (MLE) of $\theta$ based on the first $n$ observations (see also Lai, 2001 for two composite hypotheses and two-sided tests). Robbins and Siegmund (1970, 1974) followed a non-anticipating estimation approach and
considered a one-step delayed estimator. For the latter approach, we also refer to Pollak and Yakir (1999), Pavlov (1990), Dragalin and Novikov (1999), and Lorden and Pollak (2005).

An alternative, mixture-based approach was used by Darling and Robbins (1968; see also Robbins, 1970), where the stopping rule has the form

\[ T_A = \inf\{n \geq 1 : \Lambda_n \geq A\} \quad (1.4) \]

with \( \{\Lambda_n\} \) being a weighted (mixed) likelihood-ratio statistic given by

\[ \Lambda_n = \int_{\Theta} \Lambda_n^\theta G(d\theta), \quad n \in \mathbb{N} \quad (1.5) \]

and \( G \) being an arbitrary distribution function on \( \Theta \). Assuming that \( G \) has a positive and continuous density with respect to the Lebesgue measure, Pollak and Siegmund (1975) obtained an asymptotic approximation for \( E_{\Lambda_n}[T] \) as \( A \to \infty \). Based on this approximation, Pollak (1978) proved that if \( \alpha = 1/A \) and \( \Theta \subset \Theta \) is an arbitrary, closed, finite interval, bounded away from 0, then

\[ \inf_{T \in \mathbb{C}_\alpha} \sup_{\Theta \subset \Theta} I_{\Lambda_n}[T] \geq |\log z| + \log \sqrt{|\log z|} + O(1) \quad \text{as } z \to 0, \quad (1.6) \]

where \( O(1) \) is bounded as \( z \to 0 \), and that this asymptotic lower bound is attained by any mixture rule whose mixing distribution has a positive and continuous density with support that includes \( \Theta \). Note that if \( T \) has finite expectation, then \( I_{\Lambda_n}[T] = E_{\Lambda_n}[\log \Lambda_n^\theta] \) is the total Kullback–Leibler information in the trajectory \( X_{1,T} = (X_1, \ldots, X_T) \) in favor of the hypothesis \( H_\theta : P = P_\theta \) versus \( H_0 : P = P_0 \), so that minimizing the maximal value of \( I_{\Lambda_n}[T] \) can be interpreted as minimizing the Kullback–Leibler information in the least favorable situation.

Lerche (1986) considered the problem of sequential testing for the drift of a Brownian motion in a Bayesian setup.

1.2. Main Contributions

One of the goals of this work is to extend the above work on mixture rules. In the framework of exponential families, we show that a particular choice of the mixing density leads to a mixture rule \( T_A \) that attains \( \inf_{T \in \mathbb{C}_\alpha} \sup_{\Theta \subset \Theta} I_{\Lambda_n}[T] \), not only up to an \( O(1) \) term as in Pollak (1978) but up to an \( o(1) \) term (see Theorem 3.1).

However, the main emphasis is on the case that the alternative hypothesis \( \mathcal{P} \) is a finite set, \( \mathcal{P} = \{P_1, \ldots, P_K\} \). In this setup, the weighted likelihood ratio statistic becomes

\[ \Lambda_n = \sum_{i=1}^K p_i \Lambda_n^i, \quad n \in \mathbb{N}, \quad (1.7) \]

where \( \Lambda_n^i = \prod_{m=1}^n \frac{dF_i(X_m)}{dF_0(X_m)} \), \( F_i \) is the \( P_i \)-distribution of \( X_1 \), which is assumed to be absolutely continuous with respect to \( F_0 \), and \( \{p_i\} \) is a probability mass function; that is, \( p_i \geq 0 \) for every \( i \) and \( \sum_{i=1}^K p_i = 1 \). This is a more general framework than that of an exponential family, in that the distributions \( F_i \) and \( F_0 \)
are not required to belong to the same (exponential) parametric family. Moreover, it can be seen as a discrete approximation to the continuous setup (1.2). Such an approximation is necessary in practice, since the continuously weighted likelihood ratio (1.5) is not usually implementable without such a discretization.

However, the main motivation for the discrete setup is that it arises naturally in many applications. Consider, for example, the so-called $L$-sample slippage problem, where there are $L$ sources of observations (“channels” or “populations”) and there are two possibilities for the distribution of each source (in and out of control). This problem has a variety of important applications, in particular in cybersecurity (see Tartakovsky et al., 2006a, b) and in target detection (see Tartakovsky and Veeravalli, 2004, Tartakovsky et al., 2003).

Our main contribution in the discrete setup is that we find a mixing distribution $\{p_i^0\}$ which makes the corresponding mixture test nearly minimax in the sense that it attains $\inf_{r \in e_1^c} \max_{1 \leq l \leq L} (I_lE[l])$ up to an $o(1)$ term as $\alpha \to 0$, where $I_l$ is the Kullback–Leibler distance between $F_l$ and $F_0$ (see Theorem 2.2). The main components of the proof are a nearly Bayes rule for a decision problem with non homogeneous sampling costs in $\mathcal{P}$, a high-order asymptotic expansion for $E[T_A]$ up to an $o(1)$ term as well as an asymptotic approximation for the “error probability” $P_0(T_A < \infty)$ as $A \to \infty$.

### 1.3. Misspecification and the Appropriate Minimax Criterion

As we will see, the expansion for $E[T_A]$ remains valid even when $p_i = 0$, as long as certain additional conditions are satisfied. That is, we allow the number of active components, $K = \#\{p_i : p_i \neq 0\}$, of an arbitrary mixture rule to be smaller than $K$. It is useful to incorporate this case in our analysis, since the “true” distribution may not be included in $\mathcal{P}$. For example, in the slippage problem, the actual number of out-of-control channels is typically not known in advance. Thus, the cardinality of $\mathcal{P}$ is $K = \sum_{l=1}^{L} \binom{l}{1} = 2^K - 1$. However, if a designer assumes that only one channel can be out of control, which is the hardest case to detect, the resulting mixture rule will assign a positive weight to only $L$ of the $K$ probability measures in $\mathcal{P}$, so that $K = L < K$. Another case where such a misspecification arises naturally is when approximating a continuous alternative hypothesis with a discrete set of points. Then, it is useful to evaluate the performance of the discrete mixture rule also between the points that were used for its design.

Finally, allowing some components of the mixing distribution to be 0 helps to explain why we chose to design a sequential test that attains asymptotically $\inf_{r \in e_1^c} \max_{1 \leq l \leq L} (I_lE[l])$ instead of $\inf_{r \in e_1^c} \max_{1 \leq l \leq L} E[l]$, which would be the straightforward minimax criterion. Indeed, in Subsection 2.6 we will see that when the Kullback–Leibler numbers $\{I_l\}$ are not identical, the latter criterion cannot be attained asymptotically, not even up to a first order, by a mixture rule that gives positive weights to all of its components. Thus, minimizing the maximal expected sample size is an inappropriate criterion, since it dictates the use of a sequential test, $T^*$, that will not even be uniformly first-order asymptotically optimal; that is, the ratio $E[T^*]/\inf_{r \in e_1^c} E[l]$ will not converge to 1 as $\alpha \to 0$ for every $1 \leq i \leq K$.

On the other hand, the criterion $\inf_{r \in e_1^c} \max_{1 \leq l \leq L} (I_lE[l])$ leads to a nontrivial mixture test with $p_i > 0$ for every $1 \leq i \leq K$, which (just like any other fully supported mixture rule) attains $\inf_{r \in e_1^c} E[l]$ as $\alpha \to 0$ up to a constant for every
1 \leq i \leq K. Moreover, it is a natural minimax criterion since, as we already mentioned above, \( \max_i (I E_i(T)) = \max_i E_i[\log \Lambda_i^T] \) is the maximum Kullback–Leibler distance between \( \mathcal{P} \) and \( \mathcal{P}_0 \) based on the observations up to time \( T \). Thus, this criterion provides a natural and meaningful way to express the minimax property and select a particular mixture rule for our problem.

### 1.4. Anscombe’s Condition and Nonlinear Renewal Theory

We would like at this point to highlight the connection of our work with the celebrated article of Anscombe (1952), where he insightfully introduced the notion of uniform continuity in probability and showed that it constitutes a sufficient condition for preserving convergence in distribution when using random times. More specifically, Anscombe called a sequence \( \{\xi_n\} \) uniformly continuous in probability (u.c.i.p) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
P\left( \max_{0 \leq k \leq n} |\xi_{n+k} - \xi_n| \geq \epsilon \right) < \epsilon \quad \text{for every } n \in \mathbb{N}. \tag{1.8}
\]

Moreover, he proved that if an u.c.i.p. sequence \( \{\xi_n\} \) converges in distribution to a random variable \( \xi \) as \( n \to \infty \), then \( \{\xi_n\} \) also converges to \( \xi \) as \( n \to \infty \). This theorem has had a profound impact on the field of sequential analysis, since it provided the basis for developing central limit theorems (CLTs) for stopped random walks and families of stopping times. However, the notion of uniform continuity in probability plays an important role in a much wider range of sequential problems, including the one we consider in this article. The reason is its deep connection with nonlinear renewal theory, which is the main tool that we use in order to describe the asymptotic performance of mixture rules. The corresponding analysis for continuous mixture rules was done by Pollak and Siegmund (1975), who first used such ideas before a general theory was presented by Lai and Siegmund (1977, 1979).

More specifically, assuming that \( p_i > 0 \), we can decompose the logarithm of the mixture statistic (1.7) as \( \log \Lambda_n = \log \Lambda_n^i + Y_n^i \), where \( Y_n^i \) is defined in (2.12) below. The idea then is that the asymptotic distribution of the overshoot \( \log(\Lambda_{T_A}/A) \) as \( A \to \infty \) will be the same as if \( Y_n^i \) was 0, as long as \( Y_n^i \), \( n = 1, 2, \ldots \) are “slowly changing” compared to \( \bar{P}_A (T_A < \infty) \), and it is also the basis for the high-order asymptotic expansion of \( E_i[T_A] \) (for which additional integrability and convergence conditions on \( Y_n^i \) are required).

Nonlinear renewal theory makes the above argument rigorous by formalizing the notion of a “slowly changing” sequence. Specifically, \( \{\xi_n\} \) is said to be slowly changing if it is uniformly continuous in probability and satisfies the probabilistic growth condition

\[
\max_{0 \leq k \leq n} |\xi_k| = o_p(n) \quad \text{as } n \to \infty; \tag{1.9}
\]

that is, \( n^{-1} \max_{0 \leq k \leq n} |\xi_k| \to 0 \) in probability. Therefore, uniform continuity in probability is at the core of nonlinear renewal theory and is the key condition that
allows us to understand the behavior of overshoots of perturbed random walks and, consequently, a variety of “sequential objects”, such as the mixture-based sequential tests that we consider in this article.

Finally, we should note that using Anscombe’s theorem we can establish the asymptotic normality of the (standardized) mixture stopping rules \( T_A \) as \( A \to \infty \). Whereas we do not need this property for our purposes, it is useful since it justifies using the expectation of \( T_A \) in order to quantify its performance.

1.5. Organization of the Article

The rest of the article is organized as follows. In Section 2, we focus on discrete mixture rules and study their asymptotic performance and optimality properties. In Section 3, we consider the case of an exponential family with continuous parameter. Section 4 illustrates our findings with simulation experiments in the normal case. In Section 5, we discuss ramifications of our work in testing of two hypotheses and in sequential change detection, and we conclude in Section 6.

2. DISCRETE MIXTURE RULES

In this section we assume that \( \mathcal{P} = \{P_i\}_{i=1}^{K} \) and we let \( \{p_i\} \) be an arbitrary probability mass function; that is, \( p_i \geq 0 \) for every \( i = 1, \ldots, K \) and \( \sum_{i=1}^{K} p_i = 1 \).

2.1. Notation and Assumptions

Let \( \Lambda_n \) be as defined in (1.7) and let \( Z_n = \log \Lambda_n \). Then the mixture rule (1.4) calls for stopping and accepting the hypothesis \( H : P \in \mathcal{P} \) (rejecting the null hypothesis \( H_0 : P = P_0 \)) at

\[
T_A = \inf \{ n \geq 1 : Z_n \geq \log A \},
\]

where \( T_A = \infty \) if there is no such \( n \). For every \( i = 1, \ldots, K \), we set

\[
\Lambda'_n = \prod_{m=1}^{n} \frac{dF_i(X_m)}{dF_0(X_m)} \quad \text{and} \quad Z'_n = \log \Lambda'_n = \sum_{m=1}^{n} \log \frac{dF_i(X_m)}{dF_0(X_m)}, \quad n \in \mathbb{N},
\]

and we define the one-sided SPRTs

\[
T'_A = \inf \{ n \geq 1 : \Lambda'_n \geq A \} = \inf \{ n \geq 1 : Z'_n \geq \log A \},
\]

where \( A > 1 \) is a fixed threshold.

For every \( i, j = 1, \ldots, K \), we assume that \( 0 < E_j[Z'_i] < \infty \), where \( E_j[\cdot] \) refers to expectation with respect to \( P_j \), and we set

\[
I_j = E_j[Z_i] \quad \text{and} \quad I_{ji} = E_j[Z'_i - Z_i] = I_j - E_j[Z_i];
\]

that is, \( I_j \) (\( I_{ji} \)) is the Kullback–Leibler divergence of \( F_j \) versus \( F_0 \) (\( F_i \)). Therefore, \( \{Z'_n\}_{n \geq 1} \) is a random walk under \( P_j \) whose increments have mean \( E_j[Z'_i] = I_j -
Mixture-Based Stopping Rules

If \( I_j > I_{j'} \), then, by renewal theory, the asymptotic distribution of the overshoot \( \eta_A^j = Z_{T_A} - \log A \) under \( P_j \) is well defined and we denote it as

\[
\mathcal{H}_{ij}(x) = \lim_{A \to \infty} P_j(\eta_A^j \leq x).
\]

More specifically, \( \mathcal{H}_{j|i} \) can be defined in terms of the ladder variables of the \( P_j \)-random walk \( \{Z_n^j\} \). For the sake of brevity, we write \( \mathcal{H}_i = \mathcal{H}_{j|i} \) for the asymptotic distribution of \( \eta_A^j \) under \( P_i \), which is always well defined since \( E_i[Z_0^j] = I_i > 0 \).

With a change of measure \( P_0 \mapsto P_i \), it can be easily shown that

\[
AP_0(T_A < \infty) = AE_0\left[1/\lambda_A 1\{T_A < \infty\}\right] = E_j\left[\exp(-\eta_A^j)1\{T_A < \infty\}\right] \to \delta_i \quad \text{as} \ A \to \infty,
\]

where \( \delta_i \) is the Laplace transform of \( \mathcal{H}_i \); that is,

\[
\delta_i = \int_0^\infty e^{-x} \mathcal{H}_i(dx) = \lim_{A \to \infty} E_j[e^{-\eta_A^j}].
\]

Note that the quantity \( \delta_i \) is also very important when designing the one-sided test \( T_A \). More specifically, Lorden (1977) showed that if \( c \) is the cost of every observation, then the one-sided SPRT \( T_A \) with \( A = \delta_i I_i/c \) attains \( \inf_T [P_0(T < \infty) + cE_i[T]] \), where the infimum is taken over all stopping times.

If \( E_i[\max(0, Z_1^2)] < \infty \), then from Wald’s identity, (2.4), and renewal theory (Woodroofe, 1982, Corollary 2.2), we have

\[
[I_j - I_{j'}]E_j[T_A] = \log A + \kappa_{j|i} + o(1) \quad \text{as} \ A \to \infty,
\]

where \( \kappa_{j|i} \) is the average of \( \mathcal{H}_{j|i} \); that is,

\[
\kappa_{j|i} = \int_0^\infty x \mathcal{H}_{j|i}(dx) = \lim_{A \to \infty} E_j[\eta_A^j].
\]

It is a direct consequence of (2.7) that

\[
I_jE_j[T_A] = \log A + \kappa_i + o(1) \quad \text{as} \ A \to \infty,
\]

where \( \kappa_i = \kappa_{j|i} \). In the next section, we show that the limiting average overshoots \( \kappa_1, \ldots, \kappa_K \) completely determine the (optimal) mixing distribution of the nearly minimax mixture rule.

If \( P_0(T_A < \infty) = \alpha \), where \( \alpha \) is a predefined number \( 0 < \alpha < 1 \), then (2.5) and (2.9) imply that

\[
I_jE_j[T_A] = |\log \alpha| + \log(\delta_i e^{\kappa_i}) + o(1) \quad \text{as} \ z \to 0.
\]

Due to (1.1), this is the optimal asymptotic performance under \( P_j \) up to an \( o(1) \) term. Therefore, asymptotic approximation (2.10) provides a benchmark for the performance of any stopping time under \( P_j \).
In order to study the performance of $T_A$ under $P_i$ even if $p_i = 0$, for every $i = 1, \ldots, K$ we define the index

$$i^* = \arg\max_{j \neq i} \mathbb{E}_i[Z_i^1] = \arg\min_{j \neq i} I_{ij}$$

(2.11)

and we assume that it is unique. When $p_i > 0$, this is obviously the case since $i^* = i$. On the other hand, when $p_i = 0$, $i^*$ represents the “active” index that is closest to $i$, in the sense of the Kullback–Leibler distance for the corresponding distributions. Thus, assuming that $i^*$ is unique, we exclude the case that there are two or more active indexes that are “equidistant” from $i$ when $p_i = 0$. Then, for every $i = 1, \ldots, K$, we have the decomposition $Z_n = Z_n^c + Y_n^o$, where

$$Y_n^o = \log p_i + \log \left(1 + \sum_{j \neq i} \frac{p_j \Lambda_j}{P_i \Lambda_i} \right), \quad n \in \mathbb{N}.$$  

(2.12)

Based on this decomposition and the fact that when $i^*$ is unique the sequence $\{Y_n^o\}$ is slowly changing, we are able to use nonlinear renewal theory and understand the asymptotic behavior of the mixture rule $T_A$. When $p_i = 0$ and $i^*$ is not unique, this decomposition is not valid and this case has to be considered separately. We do not consider this case here, since this would break the flow of the presentation without adding any insight to our main points. Methods similar to those developed in Dragalin et al. (2000) and Tartakovsky et al. (2003) can be used for this purpose.

Finally, in the case $p_i = 0$, we will also need the following Cramér-type condition:

**Condition 1.** For every $j \neq i^*$ with $p_j > 0$ there exists $\gamma_j > 0$ such that $g_j(\gamma_j) = 1$ and $g_j'(\gamma_j) < \infty$, where $g_j(t) = \mathbb{E}_j[e^{t(Z_i^1 - Z_i^2)}]$.

### 2.2. Modes of Asymptotic Optimality

Ideally, we would like to find an optimal test $T_{opt} \in \mathcal{C}_z$ that minimizes the expected sample size $\inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T]$ for all $i = 1, \ldots, K$, where $\mathcal{C}_z = \{T : P_i(T < \infty) \leq \alpha\}$. Since this is an extremely difficult task (if at all possible), we would like to find a test $T_o \in \mathcal{C}_z$ that attains $\inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T]$ at least asymptotically for all $i = 1, \ldots, K$. We distinguish the following three notions of asymptotic optimality. We say that $T_o$ minimizes $\inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T]$ to first-order if $\mathbb{E}_i[T_o] = \inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T](1 + o(1))$; to second-order if $\mathbb{E}_i[T_o] = \inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T] + O(1)$; and to third-order if $\mathbb{E}_i[T_o] = \inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T] + o(1)$, where $O(1)$ is asymptotically bounded and $o(1)$ an asymptotically vanishing term as $\alpha \to 0$.

Since the one-sided SPRT $T_A^i$ is exactly optimal under $P_i$, it follows from (2.10) that

$$\inf_{T \in \mathcal{C}_z} \mathbb{E}_i[T] = \frac{1}{I_i} \left[ |\log \alpha| + \log(\delta e^{\kappa}) \right] + o(1) \quad \text{as} \quad \alpha \to 0.$$  

Using this fact along with Theorem 2.1, we will see that a mixture rule is second-order asymptotically optimal under every $P_i \in \mathcal{D}$ if and only if it assigns positive
weights to all probability measures in the alternative hypothesis; that is, \( p_i > 0 \) for every \( i = 1, \ldots, K \). In other words, for every fully supported mixture test \( T_A \) with \( P_0(T_A < \infty) = \alpha \), the expectation \( E[T_A] \) has a bounded distance from \( \inf_{T \in \mathcal{A}} E[T] \) as \( A \to \infty \) for every \( i = 1, \ldots, K \).

### 2.3. Asymptotic Performance

The main result of this subsection is Theorem 2.1, which provides a high-order asymptotic approximation for \( E[T_A] \) as \( A \to \infty \). Its proof is based on Lemmas 2.1–2.4. In Lemma 2.1 we present the main properties of the sequence \( \{Y_n^r\} \), in Lemma 2.2 we obtain sufficient conditions for \( T_A \) to have power 1 under \( P_r \), and in Lemmas 2.3 and 2.4 we obtain asymptotic approximations for \( \log P_0(T_A < \infty) \) and \( E[T_A] \) in terms of the threshold \( A \).

**Lemma 2.1.** For every \( i \), \( P_i(Y_n^r \downarrow \log p_r) = 1 \), and hence the sequence \( \{Y_n^r\} \) is slowly changing under \( P_r \). Moreover, if either \( p_i > 0 \) or if \( p_i = 0 \) and Condition 1 is satisfied, then there exists \( \gamma_r > 0 \) such that the following asymptotic equality holds

\[
P_i \left( \max_{0 \leq k \leq n} |Y_k^r - \log p_r| > x \right) = O(e^{-\gamma_r x}) \quad \text{as} \quad x \to \infty.
\]  

(2.13)

**Proof.** From (2.12) it follows directly that \( Y_n^r \geq \log p_r \). Moreover, by the strong law of large numbers,

\[
\frac{1}{n} \log \frac{\Lambda_n^i}{\Lambda_n^r} = \frac{Z_n^i - Z_n^r}{n} \xrightarrow{\text{P}_i} E[Z_i^r - Z_i^r] = I_i^r - I_j^r \quad \text{for every} \quad j \neq i^*.
\]

Since \( I_{i^*} \leq I_{i^*} \) (by the definition of \( i^* \)), it follows that \( P_i(\Lambda_n^i/\Lambda_n^r \to 0) = 1 \) for every \( j \neq i^* \) with \( p_j > 0 \) and, consequently, \( P_i(Y_n^r \to \log p_r) = 1 \). As a result, \( \{Y_n^r\} \) satisfies (1.8) and (1.9). Thus, it is a slowly changing sequence under \( P_r \).

To prove (2.13), suppose first that \( p_i > 0 \). Then, \( i^* = i \) and \( \sum_{j \neq i^*} \Lambda_n^j/\Lambda_n^i \) is a \( P_r \)-martingale with mean \( K - 1 \). Thus, from (2.12) and Doob’s submartingale inequality we obtain

\[
P_i \left( \max_{0 \leq k \leq n} |Y_k^r - \log p_r| > x \right) = P_i \left( 1 + \max_{0 \leq k \leq n} \sum_{j \neq i^*} \frac{p_j \Lambda_n^j}{p_i \Lambda_n^i} > e^x \right)
\]

\[
\leq P_i \left( \sum_{j \neq i^*} \frac{\Lambda_n^j}{\Lambda_n^i} > p_i(e^x - 1) \right)
\]

\[
\leq \left( K - 1 \right) \frac{p_i(e^x - 1)}{p_i(e^x - 1)}
\]

which implies that (2.13) holds with \( \gamma_r = 1 \).

Suppose now that \( p_i = 0 \), in which case \( i^* \neq i \). Then, working as in (2.14) and using the following inclusion

\[
\left\{ \max_{0 \leq k \leq n} \sum_{j \neq i^*, p_j > 0} \frac{\Lambda_n^j}{\Lambda_n^i} > y \right\} \subset \bigcup_{j \neq i^*, p_j > 0} \left\{ \max_{0 \leq k \leq n} \frac{\Lambda_n^j}{\Lambda_n^i} > \frac{y}{K - 1} \right\}
\]
which holds for every positive constant \( y \), we obtain
\[
\Pr_i \left( \max_{0 \leq k \leq n} |Y''_k| - \log p_{r^*} > x \right) \leq \Pr_i \left( \max_{0 \leq k \leq n} \sum_{j \neq r^*, p_j > 0} \frac{\Lambda'_j}{\Lambda'_k} > p_{r^*}(e^x - 1) \right) \\
\leq \sum_{j \neq r^*} \Pr_i \left( \max_{0 \leq k \leq n} \frac{\Lambda'_j}{\Lambda'_k} > \frac{p_{r^*}(e^x - 1)}{K} \right) \\
= \sum_{j \neq r^*} \Pr_i \left( \max_{0 \leq k \leq n} [Z'_k - Z''_k] > x + \Theta(1) \right),
\]
where \( \Theta(1) \) is a term that is asymptotically bounded from above and from below as \( x \to \infty \). For every \( j \neq i^* \), the process \( \{Z''_n - Z'_n\}_{n \geq 1} \) is a \( \Pr\)-random walk whose increments have mean \( \mathbb{E}[Z''_n - Z'_n] \) < 0, which is negative due to the definition of \( i^* \). Thus, by Condition 1, for every \( j \neq i^* \) with \( p_j > 0 \) there exists a positive constant \( \gamma'_j > 0 \) such that
\[
\Pr_i \left( \max_{0 \leq k \leq n} [Z'_k - Z''_k] > x + \Theta(1) \right) = O(e^{-\gamma'_j x}),
\]
which implies that (2.13) is satisfied with \( \gamma'_j = \min \{\gamma_j : j \neq i^*, p_j > 0\} \). \( \square \)

**Lemma 2.2.** If either \( p_i > 0 \) or \( p_i = 0 \) but \( I_i > I_i^* \), then \( \Pr_i(T_A < \infty) = 1 \) \( \forall A > 1 \) and \( \Pr_i(T_A \to \infty) = 1 \) as \( A \to \infty \).

**Proof.** First of all, we observe that \( \{Z''_n\} \) is a \( \Pr\)-random walk whose increments have mean \( \mathbb{E}[Z''_n] = I_i - I'_i \). Due to the assumption of the lemma, the latter is positive, and therefore \( \Pr_i(Z''_n \to \infty) = 1 \) as \( n \to \infty \). Since
\[
T_A = \inf \{n \geq 1 : Z''_n + Y''_n \geq \log A\},
\]
and, by Lemma 2.1, \( \Pr_i(Y''_n \downarrow \log p_{r^*}) = 1 \) we conclude that \( T_A \) terminates \( \Pr\)-a.s. and that \( \Pr_i(T_A \to \infty) = 1 \) as \( A \to \infty \). \( \square \)

**Lemma 2.3.** For every \( A > 1 \), \( T_A \) is a test of level \( 1/A \); that is, \( \Pr_0(T_A < \infty) \leq 1/A \). Moreover, if for every \( i \) such that \( p_i > 0 \) the \( \Pr\)-distribution of \( Z'_i \) is non-arithmetic, then
\[
\Pr_0(T_A < \infty) \to \sum_{i:p_i>0} p_i \delta_i \quad \text{as} \quad A \to \infty.
\]

**Proof.** Define the probability measure \( P = \sum_{i:p_i>0} p_i P_i \). If \( p_i > 0 \), then by Lemma 2.2, \( \Pr_i(T_A < \infty) = 1 \), and therefore \( \Pr(T_A < \infty) = 1 \). Moreover,
\[
\frac{dP}{d\Pr_0} \bigg|_{\delta_i} = \Lambda_n = \sum_{i=1}^K p_i \Lambda'_n
\]
Therefore, if \( \mathbb{E}[\cdot] \) denotes expectation with respect to \( P \), change of measure \( \Pr_0 \mapsto P \) yields
\[
\Pr_0(T_A < \infty) = \mathbb{E}[e^{-Z_i^*}] = \mathbb{E} \left[ e^{-(Z_i^* - \log A)} \right] \leq 1,
\]
which proves the first assertion. Furthermore, from (2.17) and the definition of $P$ we have

$$AP_0(T_A < \infty) = \sum_{i : \rho_i > 0} p_i E_i \left[ e^{-(Z_{\rho_i} - \log A)} \right].$$

(2.18)

If $p_i > 0$, then $\rho = i$ and we have the decomposition $Z_n = T_i + Y_i$, where $\{Z^i_n\}$ is a $P_i$-random walk with positive mean $I_i$ and $\{Y^i_n\}$ is a slowly changing sequence under $P_i$. Therefore, if also the distribution of $Z^i_0$ is non-arithmetic, then $T_i - \log A$ converges weakly as $A \to \infty$ to $\mathcal{H}_i(\cdot)$ under $P_i$ (see Woodroofe, 1982, Theorem 4.1). Thus, recalling the definition of $\delta_i$ in (2.6) and applying the bounded convergence theorem, from (2.18) we obtain (2.15). This completes the proof.

Lemma 2.4. Suppose that $Z^i_0$ has a non-arithmetic distribution with a finite second moment under $P_i$. If either $p_i > 0$ or $I_i > I_{\mu}$ and Condition 1 holds, then

$$(I_i - I_{\mu}) E_i[T_A] = \log A + x_i \rho_i - \log p_i + o(1) \quad \text{as} \quad A \to \infty.$$  

(2.19)

Proof. Write $D_{\rho_i} = I_i - I_{\rho_i}$. Since $\{Z^i_n\}_{n \geq 1}$ is a $P_i$-random walk whose increments have non-arithmetic distribution and positive mean $E[Z^i_0] = D_{\rho_i}$, asymptotic approximation (2.19) follows from Woodroofe’s nonlinear renewal theorem (see Theorem 4.5 in Woodroofe, 1982), as long as the following conditions are satisfied:

(A1) $\{\max_{0 \leq k \leq n} |Y^\rho_k - \log p_i|\}_{n \geq 1}$ is a uniformly integrable sequence;

(A2) $\sum_{n = 0}^{\infty} P_i(|Y^\rho_n - \log p_i| \leq -n\varepsilon) < \infty$ for some $\varepsilon \in (0, D_{\rho_i})$;

(A3) $\{Y^\rho_n - \log p_i\}_{n \geq 1}$ converges in distribution;

(A4) $P_i(T_A \leq N_A) = o(1/N_A)$ as $A \to \infty$ for some $\varepsilon > 0$, where $N_A = \lfloor (\varepsilon \log A) / D_{\rho_i} \rfloor$.

Condition (A1) is satisfied because $\sup_n |Y^\rho_n - \log p_i|$ is $P_i$-integrable. Indeed, from (2.13), which holds if either $p_i > 0$ or Condition 1 holds (see Lemma 2.1), we have

$$E_i \left[ \sup_n |Y^\rho_n - \log p_i| \right] = \lim_{n \to \infty} \sum_{k = 0}^{n} p_i \left( \max_{0 \leq k \leq n} (Y^\rho_k - \log p_i) > x \right) dx < \infty.$$  

Condition (A2) is clearly satisfied, since $Y^\rho_n \geq \log p_i$ for every $n$, whereas condition (A3) is also satisfied, since $\{Y^\rho_n - \log p_i\}$ converges to $0$ $P_i$-a.s.

In order to verify (A4), we start with the following inclusion, which holds for every $n \in \mathbb{N}$ and $x > 0$,

$$\{ \max_{0 \leq k \leq n} Z_k > x \} \subseteq \{ \max_{0 \leq k \leq n} Z^\rho_k > x/2 \} \cup \{ \max_{0 \leq k \leq n} Y^\rho_k > x/2 \}$$

and which implies that

$$P_i(T_A \leq N_A) \leq P_i \left( \max_{0 \leq k \leq N_A} Z_k \geq \log A \right)$$

$$\leq P_i \left( \max_{0 \leq k \leq N_A} Z^\rho_k \geq \log \sqrt{A} \right) + P_i \left( \max_{0 \leq k \leq N_A} Y^\rho_k \geq \log \sqrt{A} \right).$$
Therefore, it suffices to show that both terms on the right-hand side are of order \( o(1/\log A) \) as \( A \to \infty \).

Consider the second term. If \( p_i > 0 \), then \( i^* = i \) and, by (2.14),

\[
P_i \left( \max_{0 \leq k \leq N_A} |Y_k - \log p_i| \geq \log A \right) \leq \frac{K - 1}{p_i(\sqrt{A} - 1)} = O(A^{-1/2}).
\]

Now, if \( p_i = 0 \) and Condition 1 is satisfied, then by (2.13)

\[
P_i \left( \max_{0 \leq k \leq N_A} |Y_k - \log p_i| \geq \log A \right) = O(A^{-1/2}) \quad \text{as} \quad A \to \infty.
\]

Finally, consider the first term. We have

\[
P_i \left( \max_{0 \leq k \leq N_A} |Z_k^* \geq \log A \right) = P_i \left\{ \max_{0 \leq k \leq N_A} (Z_k^* - D_{i^*}N_A) \geq \frac{1}{2} \log A - D_{i^*}N_A \right\}
\leq P_i \left\{ \max_{0 \leq k \leq N_A} (Z_k^* - D_{i^*}N_A) \geq \frac{1 - 2\varepsilon}{2\varepsilon} D_{i^*}N_A \right\}
\leq P_i \left\{ \max_{0 \leq k \leq N_A} (Z_k^* - D_{i^*}k) \geq \gamma N_A \right\}
\]

for some \( \gamma > 0 \). Write \( S_k = Z_k^* - D_{i^*}k \) and \( \sigma^2 = E_i S_1^2 \) (which is finite by the conditions of lemma). Note that \( \{S_k\}_{k \geq 1} \) is a zero-mean \( P_i \)-martingale, so that \( \{S_k^2\}_{k \geq 1} \) is a submartingale with respect to \( P_i \). Applying Doob's maximal submartingale inequality, we obtain

\[
P_i \left( \max_{0 \leq k \leq N_A} |S_k| \geq \gamma N_A \right) \leq \frac{1}{(\gamma N_A)^2} E_i \left[ S_{N_A}^2 \mathbb{1}_{\{\max_{0 \leq k \leq N_A} S_k \geq \gamma N_A\}} \right]
\leq \frac{1}{\gamma^2 N_A} E_i \left[ \frac{S_{N_A}^2}{N_A^2} \mathbb{1}_{\{\max_{0 \leq k \leq N_A} S_k \geq \gamma N_A\}} \right].
\]

First, it follows that

\[
P_i \left( \max_{0 \leq k \leq N_A} |S_k| \geq \gamma N_A \right) \leq \frac{\sigma^2}{\gamma^2 N_A} \to 0.
\]

Now, we show that

\[
E_i \left[ \left( \frac{S_{N_A}^2}{N_A} \right) \mathbb{1}_{\{\max_{0 \leq k \leq N_A} S_k \geq \gamma N_A\}} \right] \to 0, \quad N_A \to \infty
\]

as long as \( E_i S_1^2 = \sigma^2 < \infty \), which implies that

\[
P_i \left( \max_{0 \leq k \leq N_A} |S_k| > \gamma N_A \right) = o(1/N_A) \quad \text{as} \quad A \to \infty;
\]
that is, the desired result. By the CLT, 
\[ S^2_{N_A} / (N_A \sigma^2) \] converges as \( A \to \infty \) in distribution to a standard chi-squared random variable with one degree of freedom, \( \chi^2 \). Hence, for any \( L < \infty \) we have

\[
E_i \left( \frac{S^2_{N_A}}{N_A} I_{\{\max_{0 \leq k \leq N_A} S_k \geq \chi^2 \gamma N_A} \right) = E_i \left[ L \wedge \frac{S^2_{N_A}}{N_A} I_{\max_{0 \leq k \leq N_A} S_k \geq \chi^2 \gamma N_A} \right] \\
+ E_i \left[ \left( \frac{S^2_{N_A}}{N_A} - L \wedge \frac{S^2_{N_A}}{N_A} \right) I_{\max_{0 \leq k \leq N_A} S_k \geq \chi^2 \gamma N_A} \right] \\
\leq LP_i \left( \max_{0 \leq k \leq N_A} S_k > \gamma N_A \right) + E_i \left( \frac{S^2_{N_A}}{N_A} - L \wedge \frac{S^2_{N_A}}{N_A} \right) \\
\leq \frac{L \sigma^2}{\sigma^2 N_A} + \sigma^2 - E_i \left( L \wedge \chi^2 \sigma^2 / \sigma^2 \right) \\
\xrightarrow{N_A \to \infty} \sigma^2 - E_i \left( L \wedge \chi^2 \sigma^2 / \sigma^2 \right) \xrightarrow{L \to \infty} \sigma^2 (1 - 1) = 0.
\]

The proof is complete. \( \square \)

Now everything is prepared to obtain an asymptotic approximation for the expected sample size up to the negligible term \( o(1) \).

**Theorem 2.1.** Suppose that \( Z_1^\prime \) has a non-arithmetic distribution with a finite second moment under \( P_i \) and that either \( p_i > 0 \) or \( p_i = 0 \) and \( I_i > I_{i^*} \) and Condition 1 holds. Then

\[
(I_i - I_{i^*}) E_i[T_A] = |\log P_0(T_A < \infty)| + \log \left( \sum_{i:p_i > 0} p_i \delta_i \right) \\
+ \kappa_{i^*} - \log p_i + o(1) \quad \text{as } A \to \infty. \tag{2.20}
\]

**Proof.** Using (2.15), we obtain

\[
\log A = |\log P_0(T_A < \infty)| + \log \left( \sum_{i:p_i > 0} p_i \delta_i \right) + o(1). \tag{2.21}
\]

We can then obtain (2.20) combining (2.19) and (2.21). \( \square \)

**Remark 2.1.** If the desired error probability \( P_0(T_A < \infty) = \alpha \) is fixed in advance, usually it is not possible to choose the threshold \( A = A_\alpha \) so that \( T_A \) is a test of size \( \alpha \); that is, so that \( P_0(T_A < \infty) \) is exactly equal to \( \alpha \). Nevertheless, if \( A = \alpha^{-1} \sum_{i=1}^K p_i \delta_i \), then from (2.15) and (2.20) we have

\[
P_0(T_A < \infty) = \alpha (1 + o(1)), \\
(I_i - I_{i^*}) E_i[T_A] = |\log \alpha| + \log \left( \sum_{i:p_i > 0} p_i \delta_i \right) \\
+ \kappa_{i^*} - \log p_i + o(1) \quad \text{as } \alpha \to 0. \tag{2.22}
\]
The following corollary specializes Theorem 2.1 in the case that \( p_i > 0 \).

**Corollary 2.1.** Suppose that \( p_i > 0 \) and that \( Z_1 \) has a non-arithmetic distribution with a finite second moment under \( P_i \). If \( P_0(T_A < \infty) = \alpha \), then

\[
I_i E_i[T_A] = |\log \alpha| + \log \left( \sum_{i: p_i > 0} p_i \delta_i \right) + \alpha - \log p_i + o(1) \quad \text{as} \quad \alpha \to 0 \tag{2.23}
\]

and \( T_A \) is second-order asymptotically optimal under \( P_i \); that is,

\[
E_i[T_A] = \inf_{T \in \mathcal{C}_\alpha} E_i[T] + O(1) \quad \text{as} \quad \alpha \to 0. \tag{2.24}
\]

This corollary implies that the performance loss of a mixture rule is bounded as \( A \to \infty \) under every \( P_i \in \mathcal{P} \), as long as \( p_i > 0 \) for every \( i = 1, \ldots, K \). However, when the number of “active” components in the mixing distribution, \( \tilde{K} = \# \{ i : p_i > 0 \} \), is very large, only first-order asymptotic optimality can be attained. This is the content of the following corollary of Theorem 2.1.

**Corollary 2.2.** Suppose that \( p_i > 0 \) and that \( Z_1 \) has a non-arithmetic distribution with a finite second moment under \( P_i \). If \( P_0(T_A < \infty) = \alpha \) and \( \tilde{K} \to \infty \) so that \( \log \tilde{K} = o(|\log \alpha|) \), then \( T_A \) is first-order asymptotically optimal under \( P_i \); that is,

\[
E_i[T_A] = \inf_{T \in \mathcal{C}_\alpha} E_i[T] (1 + o(1)) \quad \text{as} \quad \alpha \to 0.
\]

### 2.4. A Nearly Minimax Discrete Mixture Rule

The proof of minimaxity is constructed based on an auxiliary Bayesian approach. The method is ideologically similar to that used by Lorden (1977) and goes back to the proof of optimality of Wald’s SPRT given by Wald and Wolfowitz (1948).

More specifically, consider the following Bayesian problem denoted by \( \mathcal{B}(\pi, \{ p_i \}, c) \). Let \( \pi \in (0, 1) \) be the prior probability of the null hypothesis \( H_0 : P = P_0 \) and assume that the losses associated with stopping at time \( T \) are 1 if \( T < \infty \) and the hypothesis \( H_0 \) is true and \( (c \cdot I_i) \times T \) if \( P_i \) is the true probability measure, where \( c > 0 \) is a fixed constant. Therefore, the cost of every observation under \( P_i \) is proportional to the difficulty of discriminating between \( F_i \) and \( F_0 \) measured by the Kullback–Leibler divergence \( I_i \). Since the prior probability of the alternative hypothesis \( H_1 : P \in \mathcal{P} = \bigcup_{i=1}^K P_i \) is \( (1 - \pi) \sum_{i=1}^K p_i = (1 - \pi) \), the Bayes (integrated) risk associated with an arbitrary stopping time \( T \) is

\[
\mathcal{R}_i(T) = \pi P_0(T < \infty) + c (1 - \pi) \sum_{i=1}^K p_i I_i E_i[T]. \tag{2.25}
\]

Moreover, for any positive constant \( Q \) such that \( Qc < \pi \), we consider the mixture rule \( T_{A_{Qc}} \), where

\[
A_{Qc} = \left( \frac{1 - Qc}{Qc} \right) \left( \frac{1 - \pi}{\pi} \right). \tag{2.26}
\]
These stopping times have a natural Bayesian interpretation. Indeed, write \( P^\pi = \sum_{i=1}^K p_i P_i \) and \( P^\pi = \pi P_0 + (1 - \pi) P^\pi \). Then
\[
P^\pi(\cdot | H_0) = \pi P_0(\cdot), \quad P^\pi(\cdot | H) = (1 - \pi) P^\pi(\cdot),
\]
and the posterior probability of the hypothesis \( H_0 \) takes the form
\[
\Pi_n = P^\pi(H_0 | F_n) = \frac{1}{1 + \frac{1 - \pi}{\pi} \Lambda_n}, \quad n \in \mathbb{N}.
\]
Thus, \( T_{A_{Qc}} \) is the first time that the posterior probability of the null hypothesis becomes smaller than \( Q_c \); that is,
\[
T_{A_{Qc}} = \inf \{ n \geq 1 : \Lambda_n \geq A_{Qc} \} = \inf \{ n \geq 1 : \Pi_n \leq Q_c \}.
\]

Solution of \( \mathcal{B}(\pi, \{ p_i \}, c) \) requires minimization of the expected loss (2.25). In the following lemma we establish Bayesian optimality of the mixture test \( T_{A_{Qc}} \) in the problem \( \mathcal{B}(\pi, \{ p_i \}, c) \) for sufficiently small \( c \).

**Lemma 2.5.** For any given \( \pi \in (0, 1) \) and \( Q > 1/e \), there exists \( c^* \) such that
\[
\mathcal{R}_c(T_{A_{Qc}}) = \inf_i \mathcal{R}_c(T) \quad \text{for every } c < \pi c^*,
\]
where infimum is taken over all stopping times.

The proof of Lemma 2.5 is methodologically similar to the proof of Lemma 2 in Pollak (1978; see also Lorden, 1967) and is presented in the Appendix. This lemma provides the basis for the following important theorem, which shows that a particular mixing distribution leads to a mixture rule that is almost minimax in the sense of minimizing the Kullback–Leibler information in the worst-case scenario up to an \( o(1) \) term.

**Theorem 2.2.** Let \( \mathcal{C}_2 = \{ T : P_0(T < \infty) \leq \alpha \} \) be the class of stopping times whose “error probabilities” are at most \( \alpha \), \( 0 < \alpha < 1 \). Suppose that \( E_1|Z_1|^2 < \infty \) and that \( Z_1 \) is \( P_1 \)-non-arithmetic. Then
\[
\inf_{T \in \mathcal{C}_2} \max_{i=1, \ldots, K} E_i[T] \geq |\log \alpha| + \log \left( \sum_{i=1}^K \delta_i e^{\kappa_i} \right) + o(1) \quad \text{as } \alpha \to 0,
\]
and this asymptotic lower bound is attained by the mixture rule \( T_\lambda = T_\lambda(p^0) \) defined in (2.1) whose mixing distribution is
\[
p_i^0 = \frac{e^{\kappa_i}}{\sum_{j=1}^K e^{\kappa_j}}, \quad i = 1, \ldots, K
\]
and whose error probability is exactly equal to \( \alpha \), that is, the threshold \( \lambda = A_\alpha \) is selected in such a way that \( P_0(T_\lambda(p^0) < \infty) = \alpha \).
Proof. Let \( \{p_i\} \) be an arbitrary mixing distribution, \( \pi = 1/2 \), \( Q > 1/e \) and choose \( c < 1/2Q \) so that \( P_0(T_{A_0} < \infty) = \pi \) (recall the definition of \( A_{Q_0} \) in 2.26). Then from (A.2) in the Appendix it follows that \( \pi \leq 2Qc \) and from the definition of \( R_c \) we obtain the following inequality:

\[
\frac{\pi}{2} + \frac{c}{2} \inf_{T \in \mathcal{C}} \max_{1 \leq i \leq K} I_i E_i[T] \geq \inf_{T \in \mathcal{C}} R_c(T). \tag{2.30}
\]

By Lemma 2.5, there exists \( c^* < 1/Q \) such that for every \( c < c^*/2 \) (and consequently for every \( \pi < Qc^* \)):

\[
\inf_{T \in \mathcal{C}} R_c(T) = R_c(T_{A_0}) = \frac{\pi}{2} + \frac{c}{2} \sum_{i=1}^{K} p_i E_i[T_{A_0}] . \tag{2.31}
\]

Consequently, from (2.30) and (2.31) it follows that

\[
\inf_{T \in \mathcal{C}} \max_{1 \leq i \leq K} I_i E_i[T] \geq \sum_{i=1}^{K} p_i E_i[T_{A_0}]. \tag{2.32}
\]

It remains to show that if \( \{p_i\} \) is chosen according to (2.29), then

\[
\sum_{i=1}^{K} p_i E_i[T_{A_0}] = | \log \pi | + \log \left( \sum_{i=1}^{K} \delta_i e^{\xi_i} \right) + o(1) \quad \text{as } \pi \to 0. \tag{2.33}
\]

Substituting the mixing distribution (2.29) in (2.23), we obtain that, as \( \pi \to 0 \),

\[
I_i E_i[T_{A_0}] = | \log \pi | + \log \left( \sum_{i=1}^{K} \delta_i e^{\xi_i} \right) + o(1), \quad i = 1, \ldots, K, \tag{2.34}
\]

which implies (2.28). Since by construction \( P_0(T_{A_0} < \infty) = \pi \), it also follows from (2.34) that

\[
\max_{1 \leq i \leq K} I_i E_i[T_A] = | \log \pi | + \log \left( \sum_{i=1}^{K} \delta_i e^{\xi_i} \right) + o(1) \quad \text{as } \pi \to 0
\]

whenever \( A = A_\pi \) is chosen so that \( P_0(T_A < \infty) = \pi \). The proof is complete. \( \square \)

Therefore, Theorem 2.2 implies that if the threshold \( A = A_\pi \) is selected so that \( P_0(T_A < \infty) = \pi \) and the mixing distribution \( p = p^0 \) is given by (2.29), then the test \( T_A(p^0) \) is third-order asymptotically minimax; that is, as \( \pi \to 0 \),

\[
\inf_{T \in \mathcal{C}} \max_{1 \leq i \leq K} (I_i E_i[T]) = \max_{1 \leq i \leq K} (I_i E_i[T_A(p^0)]) + o(1)
\]

and

\[
\max_{1 \leq i \leq K} (I_i E_i[T_A(p^0)]) = | \log \pi | + \log \left( \sum_{i=1}^{K} \delta_i e^{\xi_i} \right) + o(1).
\]
2.5. Asymptotic Minimax Performance of Mixture Rules

The minimax performance loss of an arbitrary mixture rule \( T_A = T_A(p) \) with mixing distribution \( p = \{ p_i \} \) and error probability \( P_0(T_A < \infty) = \alpha \) can be naturally defined as follows:

\[
\mathcal{L}_\alpha(T_A(p)) = \max_{1 \leq i \leq K} (I[E[T_A]]) - \inf_{T \in \Theta} \max_{1 \leq i \leq K} (I[E[T]]). \tag{2.35}
\]

Corollary 2.1 implies that if \( T_A(p) \) gives positive weights to all of its components, that is, \( p_i > 0 \) for every \( 1 \leq i \leq K \), then:

\[
I[E[T_A]] = |\log \alpha| + \log \left( \sum_{j=1}^{K} p_j \delta_j \right) + \kappa - \log p_i + o(1), \quad 1 \leq i \leq K,
\]

and, consequently,

\[
\max_{1 \leq i \leq K} (I[E[T_A]]) = |\log \alpha| + \log \left( \sum_{j=1}^{K} p_j \delta_j \right) + \kappa - \log \max_{1 \leq i \leq K} (e^{\kappa_i}/p_i) + o(1). \tag{2.36}
\]

Therefore, based on (2.28) and (2.36), for relatively small \( \alpha \) we can approximate the performance loss (2.35) of an arbitrary mixture rule \( T_A(p) \) with mixing distribution \( p = \{ p_i \} \) as follows:

\[
\mathcal{L}(p) = \log \left( \sum_{j=1}^{K} p_j \delta_j \right) \left( \max_{1 \leq i \leq K} (e^{\kappa_i}/p_i) \right) - \log \left( \sum_{j=1}^{K} e^{\kappa_i} \delta_j \right),
\]

where \( \mathcal{L}(p) = \lim_{\alpha \to 0} \mathcal{L}_\alpha(T_A(p)) \) is the limiting (asymptotic) loss.

Clearly, \( \mathcal{L}(p) > \mathcal{L}(p^0) = 0 \) for any \( p = \{ p_i \} \), where \( p^0 = \{ p_i^0 \} \) is the “optimal” mixing distribution defined in (2.29). Along with the uniform mixing distribution \( p^u = \{ p_i^u \} \), \( p_i^u = 1/K \) for every \( 1 \leq i \leq K \), which would be perhaps the first choice for practical implementation, consider the following mixing distributions:

\[
p_i^{KL} = \frac{I_i}{\sum_{j=1}^{K} I_j}, \quad p_i^{1/\delta} = \frac{1/\delta_i}{\sum_{j=1}^{K} (1/\delta_j)}, \quad p_i^{e^{\kappa_i}/\delta_i} = \frac{e^{\kappa_i}/\delta_i}{\sum_{j=1}^{K} (e^{\kappa_j}/\delta_j)}, \quad 1 \leq i \leq K, \tag{2.38}
\]

which resemble \( p^0 \) in that they all give more weight to those members of \( \Theta \) that are further from \( P_0 \). Notice also that in the completely symmetric case that the \( P_i \)-distribution of \( A_i \) does not depend on \( i \), these mixing distributions reduce to uniform mixing \( p^u \). Using (2.37), we obtain

\[
\mathcal{L}(p^{KL}) = \log \left( \sum_{j=1}^{K} \delta_j I_j \right) \left( \max_{1 \leq i \leq K} (e^{\kappa_i}/I_i) \right) / \sum_{j=1}^{K} \delta_j e^{\kappa_j},
\]

\[
\mathcal{L}(p^{1/\delta}) = \log \left( \sum_{j=1}^{K} \delta_j e^{\kappa_j} \right) / \sum_{j=1}^{K} \delta_j e^{\kappa_j},
\]

\[
\mathcal{L}(p^{e^{\kappa_i}/\delta_i}) = \log \left( \sum_{j=1}^{K} \delta_j e^{\kappa_j} \right) / \sum_{j=1}^{K} \delta_j e^{\kappa_j}.
\]
\[ \mathcal{L}(p) = \log \left( \frac{\sum_{j=1}^{K} \delta_i \left( \max_{1 \leq i \leq K} e^{\epsilon_j} \right)}{\sum_{j=1}^{K} \delta_i e^{\epsilon_j}} \right), \]

\[ \mathcal{L}(p^{c/\delta}) = \log \left( \frac{\sum_{j=1}^{K} e^{\epsilon_j} \left( \max_{1 \leq i \leq K} \delta_i \right)}{\sum_{j=1}^{K} \delta_i e^{\epsilon_j}} \right). \]

2.6. An Inefficient Minimax Mixture Rule

We close this section by explaining why we chose to work with a “modified” minimax criterion, instead of \( \inf_{T \in \mathcal{C}} \max_i E[T] \), which at first glance would be a more natural choice. The reason is that if we wanted to design a mixture rule \( T_A \) that would optimize the latter criterion (at least asymptotically), \( T_A \) should be an equalizer at least up to a first order; that is, \( E_i[T_A] / E_j[T_A] \) should be approaching 1 as \( A \to \infty \) for any \( 1 \leq i \neq j \leq K \). However, assuming that \( I_i > I_{i^*} \) and that Condition 1 holds for every \( 1 \leq i = 1, \ldots, K \), Theorem 2.1 implies that

\[ (I_i - I_{i^*})E[T_A] = |\log z(1 + o(1)), \quad i = 1, \ldots, K, \]

(2.39)

where \( z = P_{0}(T_A < \infty) \). Thus, a necessary condition for a mixture rule to attain \( \inf_{T \in \mathcal{C}} \max_i E[T] \) asymptotically is that

\[ I_i - I_{i^*} = I_j - I_{j^*}, \quad 1 \leq i \neq j \leq K. \]

(2.40)

But this condition is not satisfied in general by a nontrivial mixture stopping rule that gives a positive weight to all of its components. Indeed, if \( p_i > 0 \) for every \( i \), (2.40) holds only in the completely symmetric case that \( I_1 = \cdots = I_K \). In general, this condition is satisfied by any mixture rule for which

\[ p_i > 0 \Leftrightarrow I_i = \min_{j \neq i} [I_j - I_{j^*}]. \]

However, such a minimax mixture rule can be very inefficient—it is not even uniformly first-order asymptotically optimal unless we are dealing with the symmetric case. Consider, for example, the slippage problem with \( K \) populations and suppose that only one population can be out of control and that \( I_1 < I_2 = \cdots = I_K \). Then, if we wanted to attain \( \inf_{T \in \mathcal{C}} \max_i E[T] \), even asymptotically, we should use the one-sided SPRT \( T^1 \), which is optimal under \( P_1 \), but ignores all other states of the alternative hypothesis. This is clearly not a meaningful answer and shows that the seemingly natural minimax criterion \( \inf_{T \in \mathcal{C}} \max_i E[T] \) is not appropriate.

3. CONTINUOUS MIXTURE RULES FOR AN EXPONENTIAL FAMILY

3.1. Notation and Assumptions

In this section we assume that \( \Theta = \{P_\theta\}_{\theta \in \Theta} \), where \( \Theta \subset \Theta \) is a finite interval bounded away from 0 and that the \( P_\theta \)-distribution of \( X_1, F_\theta \), is defined by (1.2). Recall the definition of the likelihood ratio \( \Lambda_0^\theta \) in (1.3) and write

\[ S_n^0 = \log \Lambda_0^\theta = \theta \sum_{k=1}^{n} X_k - n \psi_\theta. \]
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Observe that $E_\theta[S_n^0] = E_\theta[\theta X_1 - \psi_\theta] = \theta \psi_\theta' - \psi_\theta = I_\theta$, where $I_\theta$ is the Kullback–Leibler divergence of $F_\theta$ and $F_0$. For every $\theta \in \Theta$, we define the corresponding one-sided SPRT and overshoot

$$T_\alpha^0 = \inf \{ n \geq 1 : S_n^0 \geq \log A \}, \quad \eta_\alpha^0 = S_n^0 - \log A \quad \text{on } \{ T_\alpha^0 < \infty \}.$$ 

For every $\theta, \bar{\theta} \in \Theta$ such that $E_{\bar{\theta}}[\theta X_1 - \psi_\theta] = \theta \psi_\bar{\theta}' - \psi_\bar{\theta} > 0$, we set

$$\delta_{\bar{\theta}} = \int_0^\infty e^{-x} \mathcal{H}_{\bar{\theta}}(dx), \quad \kappa_{\bar{\theta}} = \int_0^\infty x \mathcal{H}_{\bar{\theta}}(dx),$$

(3.1)

where $\mathcal{H}_{\bar{\theta}}$ is the asymptotic distribution of $\eta_\alpha^0$ under $P_{\bar{\theta}}$; that is, $\mathcal{H}_{\bar{\theta}}(x) = \lim_{n \to \infty} P_{\bar{\theta}}(\eta_\alpha^0 \leq x)$. For brevity’s sake, we write $\mathcal{H}_{\bar{\theta}} = \mathcal{H}_{\bar{\theta}}, \kappa_{\bar{\theta}} = \kappa_{\bar{\theta}, \theta}$, and $\delta_{\bar{\theta}} = \delta_{\bar{\theta}, \theta}$.

From (2.10) it follows that if $\alpha = P_{\bar{\theta}}(T_\alpha^0 < \infty)$, then the optimal asymptotic performance under $P_{\bar{\theta}}$ is

$$I_{\bar{\theta}} \inf_{T \in \mathcal{C}} E_\theta[T] = I_\theta E_\theta[T_\alpha^0] = | \log \alpha | + \log(\delta_{\bar{\theta}} e^{\kappa_{\bar{\theta}}}) + o(1) \quad \text{as } \alpha \to 0. \quad (3.2)$$

Recall that in the continuous parameter case the mixture test $T_\alpha$ is defined by (1.4) with the average likelihood ratio process $\Lambda_n$ given by (1.5). Below we assume that mixing distribution $G(\theta)$ has continuous density $g(\theta)$ with respect to the Lebesgue measure, in which case

$$\Lambda_n = \int_\Theta \exp(S_n^0 g(\theta)) d\theta, \quad n \in \mathbb{N}.$$  

3.2. Asymptotic Performance of Continuous Mixture Rules

The following lemma provides a higher-order asymptotic approximation for the expected sample size $E_\theta[T_\alpha]$ for large threshold values.

Lemma 3.1. If $g$ is a positive and continuous mixing density on $\overline{\Theta}$ and $P_\theta(T_\alpha < \infty) = \alpha$, then for every $\theta \in \Theta$

$$I_\theta E_\theta[T_\alpha] = | \log \alpha | + \log \left( \sqrt{\log \alpha} \right) - \frac{1 + \log(2\pi)}{2} + \log \left( \frac{e^{\kappa_{\bar{\theta}} \sqrt{\psi_{\bar{\theta}}/I_\theta}}}{g(\theta)} \right) + o(1) \quad \text{as } \alpha \to 0. \quad (3.3)$$

Proof. From Pollak and Siegmund (1975) and Woodroofe (1982, p. 68), it follows that for every $\theta \in \Theta$

$$I_\theta E_\theta[T_\alpha] = \log A + \log \sqrt{\log A} - \frac{1 + \log(2\pi)}{2} + \log \left( e^{\kappa_{\bar{\theta}} \sqrt{\psi_{\bar{\theta}}/I_\theta}} \right) + o(1) \quad \text{as } A \to \infty. \quad (3.4)$$
Moreover, from Corollary 1 in Woodroofe (1982, p. 67; see also Pollak, 1986) it follows that

\[ AP_0(T_A < \infty) \to \int_{-\infty}^{\infty} \delta_0 g(\theta) d\theta, \]

and, consequently,

\[ \log A = |\log x| + \log \left( \int_{-\infty}^{\infty} \delta_0 g(\theta) d\theta \right) + o(1). \tag{3.5} \]

We can now complete the proof by substituting (3.5) into (3.4). \hfill \Box

Asymptotic approximations (3.2) and (3.3) imply that any continuous mixture rule with positive and continuous density on \( \bar{\Theta} \) minimizes the expected sample size to first-order for every \( \theta \in \bar{\Theta} \); that is,

\[ E_\theta[T_A] = \inf_{T \in C_{\theta}} E_\theta[T], \quad \text{as } \alpha \to 0 \text{ for all } \theta \in \bar{\Theta}. \]

However, such a continuous mixture rule is not second-order asymptotically optimal for any \( \theta \in \bar{\Theta} \). More specifically, the following asymptotic equality holds

\[ E_\theta[T_A] - \inf_{T \in C_{\theta}} E_\theta[T] = O \left( \log(\sqrt{|\log x|}) \right) \quad \text{for all } \theta \in \bar{\Theta}. \]

In other words, the distance between \( E_\theta[T_A] \) and the optimal asymptotic performance (3.2) under \( P_\theta \) does not remain bounded as \( x \to 0 \) for any \( \theta \in \bar{\Theta} \).

### 3.3. A Nearly Minimax Continuous Mixture Rule

In the following theorem we show that a particular continuous mixture rule is third-order asymptotically minimax in the sense of minimizing the maximal Kullback–Leibler information \( \sup_{\theta} I_{\theta} E_\theta[T] \) in the class \( C_{\alpha} \) as \( \alpha \to 0 \).

**Theorem 3.1.** If the limiting average overshoot \( \kappa_0 \) is a continuous function on \( \bar{\Theta} \), then

\[
\inf_{T \in C_{\alpha}} \sup_{\theta \in \Theta} I_{\theta} E_\theta[T] \geq |\log x| + \log \sqrt{|\log x|} - \frac{1 + \log(2\pi)}{2} + \log \left( \int_{-\infty}^{\infty} \delta_0 e^{\kappa_0 \sqrt{\psi_0} / I_0} d\theta \right) + o(1) \quad \text{as } x \to 0, \tag{3.6}
\]

and this asymptotic lower bound is attained by the continuous mixture rule \( T_A(g^0) \) whose mixing density is

\[ g^0(\theta) = \frac{e^{\kappa_0 \sqrt{\psi_0} / I_0}}{\int_{-\infty}^{\infty} e^{\kappa_0 \sqrt{\psi_0} / I_0} d\theta}, \quad \theta \in \bar{\Theta} \tag{3.7} \]

and for which \( P_0(T_A(g^0) < \infty) = x. \)
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Proof. Lower bound (3.6) can be established following the same steps as in the proof of Theorem 2.2. The details are omitted.

In order to show that the mixture rule $T_\lambda(g^\theta)$ with mixing density (3.7) attains the asymptotic lower bound in (3.6), it suffices to substitute (3.7) into (3.3) to obtain that for every $\theta \in \Theta$

$$ I_\theta E_\theta[T_\lambda] = |\log x| + \log(\sqrt{|\log x|}) - \frac{1 + \log(2\pi)}{2} $$

$$ + \log \left( \int_0^1 \delta \exp \left( \int_0^\theta \psi_0 d\theta' / T_\theta \right) \right) + o(1) \quad \text{as} \quad x \to 0. \quad (3.8) $$

This completes the proof. \qed

Remark 3.1. Note that for (3.4) and (3.8) to hold, the mixing density (3.7) must be continuous, which requires that $\pi_0$ and $I_\theta = \theta \psi_0 - \psi_0$ are continuous. This is true at least when the distribution of $S_1^\theta$ is continuous.

Typically, the computation of the optimal mixing density (3.7) requires discretization. An example where such a discretization is not necessary is that of an exponential distribution. More specifically, suppose that $dF_0(x) = e^{-x} dx$ and $dF_\theta(x) = e^{-(1-\theta)x} dx$ for every $0 < \theta < 1$. Then $\psi_0 = -\log(1-\theta)$, $I_\theta = \theta / (1-\theta) + \log(1-\theta)$ and the exact distribution of the overshoot $\eta_1^\theta$ is exponential with rate $(1-\theta)/\theta$ for every $\lambda > 1$. Therefore, $\pi_0$ is an exponential distribution with rate $(1-\theta)/\theta$, which implies that $\pi_0 = \theta / (1-\theta)$ and $\delta = 1 - \theta$. As a result, mixing density (3.7) is completely specified up to the normalizing constant

$$ \int_0^1 \exp \left( \int_0^\theta \psi_0 d\theta' / T_\theta \right) d\theta = \int_0^\theta \frac{\exp \left( \theta / (1-\theta) \right)}{\sqrt{(1-\theta)(\theta + (1-\theta)\log(1-\theta))}} d\theta, \quad (3.9) $$

which can be computed numerically.

Unfortunately, $\pi_0$ and $\delta$ do not have analogous closed-form expressions in terms of $\theta$ in general. Therefore, it is typically difficult to compute optimal mixing density $g^\theta$. Thus, in practice it may be more convenient to choose mixing density $g$ from the class of probability density functions on the whole parameter space $\Theta$ that are conjugate to $f_\theta$, so that the resulting mixture rule is easily computable. However, such a mixture rule will only be second-order asymptotically minimax over $\Theta$, as shown by Pollak (1978).

In the following subsection, we consider another alternative to the nearly minimax continuous mixture rule; we approximate $\Theta$ with a discrete set of points and we use the corresponding nearly minimax discrete mixture test.

3.4. A Discrete Approximation

A practical alternative to the optimal continuous mixture rule is to approximate the interval $\Theta$ by a genuinely discrete set, $\Theta_k = \{\theta_1, \ldots, \theta_K\} \subset \Theta$. In this case, the discrete mixture likelihood ratio statistic takes the form

$$ \Lambda_n = \sum_{i=1}^K \rho_i e^{\theta_i} = \sum_{i=1}^K \rho_i \exp \left\{ \sum_{m=1}^n \left[ \theta_m X_m - \psi_{\theta_i} \right] \right\}, \quad n \in \mathbb{N}, $$
and, according to Theorem 2.2, the optimal mixing distribution \{p_i\} is given by (2.29). By Corollary 2.1, such a discrete mixture rule is second-order asymptotically optimal under \( P_\theta \) for every \( i = 1, \ldots, K \); that is, \( E_0[T_A] = \inf_{P \in \Theta_k} E_0[T] + O(1) \) for every \( i = 1, \ldots, K \). Moreover, it is asymptotically third-order minimax with respect to the Kullback–Leibler information; that is,

\[
\max_{1 \leq i \leq K} (I_0 E_0[T_A]) = \inf_{P \in \Theta_k} \max_{1 \leq i \leq K} (I_0 E_0[T]) + o(1).
\]

However, it is not even first-order asymptotically optimal under \( P_\theta \) when \( \theta \notin \Theta_k \). More specifically, we have the following corollary of Theorem 2.1, for which we write \( I_{0\theta} \) for the Kullback–Leibler divergence of the distributions \( F_\theta \) and \( F_{0\theta} \); that is,

\[
I_{0\theta} = E_0[S_1^\theta - S_1^{0\theta}] = E_0[(\theta - \theta^*)X_1 - (\psi_\theta - \psi_{0\theta})] = (\theta - \theta^*)\psi_\theta - (\psi_\theta - \psi_{0\theta}).
\]

**Corollary 3.1.** Suppose that \( \theta \in \Theta \setminus \Theta_k \) and that there exists a unique \( \theta^* = \arg \min_{\theta \in \Theta_k} I_{0\theta} \). If \( \psi_{0\theta} < 0^* \psi_{0\theta} \), then \( P_\theta(T_A < \infty) = 1 \). If also \( P_\theta(T_A < \infty) = \infty \), then

\[
[I_0 - I_{0\theta}] E_0[T_A] = |\log \omega| + \log \left( \sum_{i=1}^K p_i \delta_{\theta_i} \right) + \kappa_{0\theta} - \log p_{0\theta} + o(1) \quad \text{as} \quad \omega \to 0.
\]

**Proof.** From Lemma 2.2 it follows that \( P_\theta(T_A < \infty) = 1 \) as long as \( I_0 > I_{0\theta} \), or, equivalently,

\[
0^* \psi_{0\theta} > (\theta - \theta^*)\psi_\theta - (\psi_\theta - \psi_{0\theta}) \Leftrightarrow \psi_{0\theta} < 0^* \psi_{0\theta}.
\]

Moreover, since the random variable \( \theta^* X_1 - \psi_{0\theta} \) has non-arithmetic distribution with exponential moments under \( P_\theta \) for almost every \( \theta \) (see Lemma 6.4 in Woodroofe, 1982), the conditions of Theorem 2.1 are satisfied, and, consequently, we obtain (3.11). \( \square \)

4. **MONTE CARLO SIMULATIONS**

In this section, we illustrate the asymptotic formulas obtained in Section 2 and check their validity with simulation experiments in the Gaussian example where \( F_\theta(x) = \Phi(x) \) and \( F_i(x) = \Phi(x - i) \) for \( i = 1, 2, 3 \) (\( \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-r^2/2} dr \) is the standard normal distribution function). Thus, the observations are normally distributed with unit variance and mean that is equal to 0 under \( H_0 \) and is either 1 or 2 or 3 under \( H_i \) (\( K = 3 \)). In this example, the quantities \( \kappa_i \) and \( \delta_i \) can be computed with any precision using the following expressions:

\[
\kappa_i = 1 + \frac{i^2}{4} - i \sum_{n=1}^{\infty} \left[ \frac{1}{\sqrt{n}} \phi \left( \frac{i}{\sqrt{n}} \right) - \frac{i}{2} \Phi \left( -\frac{i}{2} \sqrt{n} \right) \right], \quad (4.1)
\]

\[
\delta_i = \frac{1}{i} \exp \left\{ -2 \sum_{n=1}^{\infty} \frac{1}{n} \Phi \left( -\frac{i}{2} \sqrt{n} \right) \right\}, \quad (4.2)
\]

(see, e.g., Woodroofe, 1982, p. 32).
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Table 1. Mixing distributions and quantities \( \kappa_i \) and \( \delta_i \)

| \( i \) | \( l_i \) | \( \kappa_i \) | \( \delta_i \) | \( p^{e/\delta} \) | \( p^0 \) | \( p^{KL} \) | \( p^{1/\delta} \) | \( p^* \) |
|------|-------|--------|--------|-------------|-------|---------|---------|-------|
| 1    | 0.5   | 0.718  | 0.560  | 0.25        | 0.066 | 0.071   | 0.176   | 0.33  |
| 2    | 2     | 1.747  | 0.320  | 0.125       | 0.185 | 0.286   | 0.307   | 0.33  |
| 3    | 4.5   | 3.146  | 0.190  | 0.85        | 0.749 | 0.643   | 0.517   | 0.33  |

In Table 1, we compute these quantities, the optimal mixing distribution (2.29), as well as the mixing distributions that we defined in (2.38). Using Table 1, we can compute the asymptotic performance loss (2.37) for each of the corresponding mixture rules:

\[
\mathcal{L}(p^{KL}) = 0.21, \quad \mathcal{L}(p^{1/\delta}) = 0.58, \quad \mathcal{L}(p^{e/\delta}) = 0.85, \quad \mathcal{L}(p^0) = 1.21.
\]

In Remark 2.1 we discussed that if we set \( A \) as

\[
A = \frac{\sum_{i=1}^{K} p_i \delta_i}{x}, \quad (4.3)
\]

where \( p = \{p_i\} \) is the mixing distribution that defines \( T_A(p) \), the probability \( P_0(T_A < \infty) \) is expected to be approximately equal to \( x \) for sufficiently small values of \( x \). In Table 2, we present the actual probabilities computed using Monte Carlo simulations. An importance sampling technique was used in these experiments, taking advantage of the representation \( P_0(T_A < \infty) = \sum_i p_i E_i[e^{-Z_T}] \) (see 2.18). This allowed us to evaluate a very low error probability with a reasonable number of Monte Carlo runs. It is seen that the formula (4.3) ensures extremely high accuracy of the approximation of the desired error probability for all mixing distributions.

Table 3 allows us to verify the accuracy of the asymptotic approximation (2.36) for the Kullback–Leibler information \( \max_i (I[E_i[T_A]]) \) for all studied probabilities of error \( x \leq 0.01 \). However, for uniform mixing distribution, the approximation (2.36) is considerably less accurate but improves significantly as the error probability goes to 0.

Table 2. Probability \( P(T_A < \infty) \) for different mixing distributions: the first column represents the desired error probabilities; the other columns represent the actual error probabilities obtained by Monte Carlo simulations when the threshold is chosen according to (4.3)

| \( x \)   | \( p^{e/\delta} \) | \( p^0 \) | \( p^{KL} \) | \( p^{1/\delta} \) | \( p^* \) |
|----------|-----------------|-------|---------|---------|-------|
| 10^{-1}  | 5.9979 10^{-2}  | 6.7037 10^{-2} | 8.0337 10^{-2} | 8.0029 10^{-2} | 8.9314 10^{-2} |
| 10^{-2}  | 9.1127 10^{-3}  | 9.4317 10^{-3} | 9.8754 10^{-3} | 9.8885 10^{-3} | 1.0049 10^{-2} |
| 10^{-4}  | 1.0104 10^{-4}  | 1.0107 10^{-4} | 1.0027 10^{-4} | 1.0038 10^{-4} | 1.0011 10^{-4} |
| 10^{-6}  | 1.0017 10^{-6}  | 1.0006 10^{-6} | 1.0009 10^{-6} | 1.0004 10^{-6} | 1.0008 10^{-6} |
| 10^{-8}  | 1.0008 10^{-8}  | 1.0033 10^{-8} | 1.0002 10^{-8} | 1.0017 10^{-8} | 1.0006 10^{-8} |
Table 3. The maximal expected Kullback–Leibler information $\max_i (I_i(E[T_i(p)])$ for optimal and uniform mixing distributions $p^0$ and $p^\alpha$. The threshold $A$ is selected according to (4.3)

| $\alpha$ | Monte Carlo | Approximation (2.36) | $\alpha$ | Monte Carlo | Approximation (2.36) |
|----------|-------------|---------------------|----------|-------------|---------------------|
| $10^{-1}$ | 4.99        | 4.31                | $10^{-1}$ | 5.04        | 5.52                |
| $10^{-2}$ | 6.36        | 6.61                | $10^{-2}$ | 6.88        | 7.82                |
| $10^{-4}$ | 10.99       | 11.21               | $10^{-4}$ | 11.87       | 12.42               |
| $10^{-6}$ | 15.65       | 15.82               | $10^{-6}$ | 16.59       | 17.03               |
| $10^{-8}$ | 20.33       | 20.42               | $10^{-8}$ | 21.29       | 21.63               |

5. EXTENSIONS

Despite the fact that one-sided tests have limited practical applications themselves, they can be used effectively in the more realistic problems of testing two (or more) hypotheses and in change point detection problems. Indeed, multi-hypothesis sequential tests and changepoint detection procedures are typically built based on combinations of one-sided tests; see, for example Lorden (1971, 1977), Tartakovsky et al. (2003), and Tartakovsky (1998). Therefore, the results of the present article may have certain implications for these more practical problems, some of which we now briefly discuss.

5.1. Two-Sided Mixture Sequential Tests

Suppose that we want to stop as soon as possible not only under $\mathcal{P}$ but also under $P_0$ and either reject $H_0$ or accept it. Then, a sequential test is a pair $(T, d_T)$ that consists of an $\mathcal{F}_T$-stopping time $T$ and an $\mathcal{F}_T$-measurable random variable $d_T$ that takes values in $\{0, 1\}$, depending on whether the null or the alternative hypothesis is accepted. When $\mathcal{P}$ consists of a single probability measure, say $\mathcal{P} = \{P_i\}$, the optimal test is Wald’s two-sided SPRT

$$T_{A,B}^i = \inf \left\{ n \geq 1 : \Lambda_n^i \geq A \text{ or } \Lambda_n^i \leq B \right\};$$

$$d_{T_{A,B}}^i = \begin{cases} 1 & \text{if } \Lambda_{T_{A,B}}^i \geq A \\ 0 & \text{if } \Lambda_{T_{A,B}}^i \leq B \end{cases},$$

where $0 < B < 1 < A$ are fixed thresholds. Indeed, as shown by Wald and Wolfowitz (1948), the SPRT attains both

$$\inf_{(T,d_T) \in \mathcal{C}_{z,B}^i} E_0[T] \text{ and } \inf_{(T,d_T) \in \mathcal{C}_{z,B}^i} E_i[T],$$

where $P_0(d_T = 1) = z$, $P_1(d_T = 0) = \beta$ and

$$\mathcal{C}_{z,B}^i = \{(T, d) : P_0(d_T = 1) \leq z \text{ and } P_i(d_T = 0) \leq \beta\}.$$
When the alternative hypothesis consists of a discrete set of probability measures, \( \mathcal{P} = \{ P_1, \ldots, P_K \} \), a natural generalization of the SPRT is the two-sided mixture rule

\[
T_{A,B} = \min \{ T_0(B), T_1(A) \}, \quad d_{x_n} = \mathbb{1}_{[T_1(A) < T_0(B)]},
\]

where

\[
T_0(B) = \inf \left\{ n \geq 1 : \sum_{i=1}^{K} q_i A^i_n \leq B \right\}, \quad T_1(A) = \inf \left\{ n \geq 1 : \sum_{i=1}^{K} p_i A^i_n \geq A \right\}
\]

and \( \{ q_i \}, \{ p_i \} \) are mixing distributions. We conjecture that if \( \{ p_i \} \) is chosen according to (2.29), then \( (T_{A,B}, d_{x_n}) \) is almost minimax, in the sense that it attains

\[
\inf_{(T,d_T) \in \mathcal{C}_{x,\beta}} \max_{i=1,\ldots,K} (I[E[T]])
\]

up to an \( o(1) \) term as \( x|\log \beta| + \beta|\log x| \to 0 \), where \( P_0(d_{x_n} = 1) = x \) and \( P_1(d_{x_n} = 0) = \beta \) and

\[
\mathcal{C}_{x,\beta} = \left\{ (T,d_T) : P_0(d_T = 1) \leq x \text{ and } \max_{i=1,\ldots,K} P_i(d_T = 0) \leq \beta \right\}.
\]

However, this statement does not follow directly from our results in this article. Moreover, it is not clear whether \( \inf_{(T,d_T) \in \mathcal{C}_{x,\beta}} E_0[T] \) is attained up to an \( o(1) \) term for some particular choice of \( \{ q_i \} \). This open problem will be addressed in the future.

### 5.2. Sequential Changepoint Detection

Suppose that a change occurs at an unknown time \( \nu \) so that the pre-change distribution of the sequence \( \{ X_n \} \) is \( F_0 \) and the post-change distribution belongs to the set \( \{ F_1, \ldots, F_K \} \). We denote by \( P_i \) the probability measure under which the change occurs at time \( \nu \) and the post-change distribution is \( F_i \). If \( \nu = \infty \) (there is never a change), then \( X_n \sim F_0 \) for every \( n \in \mathbb{N} \); that is, \( P^\infty_0 \equiv P_0 \). If \( \nu = 1 \) (the change occurs at the very beginning), then \( X_n \sim F_i \) for all \( n \in \mathbb{N} \); that is, \( P^1 \equiv P_i \). The goal is to detect the change as soon as possible after it occurs, avoiding false alarms. Thus, a detection rule is a stopping time \( T \), and one attempts to find such \( T \) that \( (T - \nu)^+ \) takes small values under every \( P^\nu_i \) but large values under \( P_0 \).

Lorden (1971) showed that there is a close link between change detection rules and one-sided sequential tests. Based on this connection, he proved that applying repeatedly the one-sided SPRT, \( T_i \), leads to a detection rule (the so-called cumulative sum [CUSUM] procedure) that is asymptotically optimal in the sense that it attains to first-order

\[
\inf_{T \in \mathcal{C}[T] \geq A} \bar{J}_i[T], \quad (5.1)
\]

where \( \bar{J}_i[T] \) is a minimax performance measure that quantifies the delay of the detection rule \( T \) when the post-change distribution is \( F_i \). Using Lorden’s method, it can be easily established that repeatedly applying a mixture-based sequential test \( T_A \) with \( p_i > 0 \) for all \( i = 1, \ldots, K \) leads to a detection procedure that attains to first-order (5.1) for every \( i = 1, \ldots, K \). However, the optimal choice of the mixing distribution remains an open problem that we plan to consider in the future.
6. CONCLUSIONS AND FINAL REMARKS

The main focus of this article is on discrete, mixture-based stopping rules for testing a simple null hypothesis against a composite alternative hypothesis. These rules arise naturally in important practical problems, such as the multi-sample slippage problem, where the statistician has to decide whether one of the populations has "slipped to the right of the rest," without specifying which one. Discrete mixture rules are also useful when the alternative hypothesis is continuous, since they have certain important advantages over their continuous counterparts. More specifically, they asymptotically minimize the expected sample size within a constant (not only to first-order) at all parameter values used for their design (but they are asymptotically suboptimal outside of these points). However, the most important advantage of discrete mixtures is that they are easily implementable, which is not usually the case with continuous mixture rules.

The main contribution of this article consists of finding an optimal mixing distribution both for discrete and continuous mixture rules. That is, for both cases, we find mixing distributions so that the resulting sequential tests are nearly minimax, in the sense that they minimize the maximal Kullback–Leibler information within a negligible term $o(1)$. We believe that the methods of the present article can be effectively used in the more practical problems of sequential testing two or more composite hypotheses and constructing nearly optimal mixture-based change-point detection procedures.

APPENDIX: PROOF OF LEMMA 2.5

We need to find a $c^*$ such that $R_c(T) \geq R_{c^*}(T_{A_{c^*}})$ for every stopping time $T$ and for every $c$ smaller than $\pi c^*$ or equivalently, for every $c$ that satisfies the inequality $Q_c < \pi Q_{c^*}$. Since $A_{Q_c}$ is defined so that $Q_c < \pi$, it is clear that $c^*$ must be chosen so that $Q_c < 1$.

Recalling that $\pi = P^*(H_0)$ is the prior probability of the null hypothesis $H_0$ as well as the definitions of the probability measure $P^*$ and the posterior process $\{\Pi_n\}_{n \geq 1}$, for any stopping time $T$, we have

$$
\pi P_0(T < \infty) = \sum_{n=1}^{\infty} P^*(T = n, \theta = 0)
$$

$$
= \sum_{n=1}^{\infty} E^*[\Pi_{|T=n}, \Pi_n] = E^*[\Pi_T 1_{[T<\infty]}]
$$

(A.1)

and

$$
c(1 - \pi) \sum_{i=1}^{K} p_i E_i[T] \geq c \left( \min_{1 \leq i \leq K} I_i \right) (1 - \pi) \sum_{i=1}^{K} p_i E_i[T]$$

$$
= c \left( \min_{1 \leq i \leq K} I_i \right) E^*[T].
$$

Therefore,

$$
R_c(T) \geq E^*[\Pi_T 1_{[T<\infty]}] + c \left( \min_{1 \leq i \leq K} I_i \right) T.
$$
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From this inequality it is clear that without any loss of generality we can restrict ourselves to \(P\)-a.s. finite stopping times. Since the process \(\{\Pi_n\}_{n \geq 0}\) is a bounded martingale with \(\Pi_0 = \pi\), we conclude that \(\mathbb{R}_-(T) \geq \mathbb{E}^\pi[\Pi_T] = \pi\) for every \(P\)-a.s. finite stopping time \(T\). Hence, it suffices to find \(c^*\) with \(Qc^* < 1\) such that for every \(c \leq \pi c^*\)

\[
\pi \geq \mathbb{R}_-(T_{A_{Qc^*}}) = \pi P_0(T_{A_{Qc^*}} < \infty) + c(1 - \pi) \sum_{i=1}^K p_i(I_iE_i[T_{A_{Qc^*}}]).
\]

From (2.27) and (A.1) it follows that

\[
\pi P_0(T_{A_{Qc^*}} < \infty) = \mathbb{E}^\pi[\Pi_{T_{A_{Qc^*}}}] \leq Qc. \tag{A.2}
\]

Therefore, we must find \(c^*\) with \(Qc^* \leq 1\) such that for every \(c \leq \pi c^*\)

\[
Qc + c(1 - \pi) \sum_{i=1}^K p_i(I_iE_i[T_{A_{Qc^*}}]) \leq \pi \iff (1 - \pi) \sum_{i=1}^K p_i(I_iE_i[T_{A_{Qc^*}}]) \leq \frac{\pi}{c} - Q. \tag{A.3}
\]

However, from (2.19) it follows that there exists a constant \(C > 0\), which does not depend on \(i\) and \(A\), such that \(I_iE_i[T_A] \leq \log A + C\) for any mixture rule \(T_A\). Therefore,

\[
(1 - \pi) \sum_{i=1}^K p_i(I_iE_i[T_{A_{Qc^*}}]) \leq (1 - \pi)[\log A_{Qc} + C]
= (1 - \pi) \left[ \log \left( \frac{1 - Qc}{Qc} \cdot \frac{\pi}{1 - \pi} \right) + C \right]
\leq (1 - \pi) \left[ \log \left( \frac{\pi}{Qc} \right) + \log \left( \frac{1}{1 - \pi} \right) + C \right]
\leq \frac{\pi}{Qc} \log \left( \frac{\pi}{Qc} \right) + (1 - \pi) \log \left( \frac{1}{1 - \pi} \right) + C
= \frac{\pi}{Qc} \log \left( \frac{\pi}{Qc} \right) + \left[ (1 - \pi) \log \left( \frac{1}{1 - \pi} \right) \right] + C.
\]

Since also \(Qc \leq \pi\), from the inequality \(\sup_{0 < x < 1} (x \log x) \leq e^{-1}\) we have

\[
(1 - \pi) \sum_{i=1}^K p_i(I_iE_i[T_{A_{Qc^*}}]) \leq \frac{\pi}{Qc} \log \left( \frac{\pi}{Qc} \right) + \left[ (1 - \pi) \log \left( \frac{1}{1 - \pi} \right) \right] + C. \tag{A.4}
\]

Hence, from (A.3) and (A.4) it follows that it suffices to find \(c^*\) with \(Qc^* < 1\) such that for \(c \leq \pi c^*\)

\[
\frac{\pi}{c} - \frac{\pi}{c} \cdot \frac{Qe - 1}{Qe} + e^{-1} + C \leq \frac{\pi}{c} - Q
\iff \frac{\pi}{c} \cdot \frac{Qe - 1}{Qe} \geq e^{-1} + Q + C
\iff \frac{c}{\pi} \leq \frac{Qe - 1}{Qe} \cdot \frac{1}{e^{-1} + Q + C}.
\]
Thus, it suffices to set
\[ e^* = \frac{Qe - 1}{Qe} \frac{1}{Q^{-1} + Q + C}, \]
and this is a valid choice since
\[ Qe^* \leq \frac{Qe - 1}{Qe} \frac{Q}{Q^{-1} + Q + C} < 1. \]

The proof is complete.

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