ABSTRACT

The Kerr solution is defined by a null congruence which is geodesic and shear free and has a singular line contained in a bounded region of space. A generalization of the Kerr congruence for a nonstationary case is obtained. We find a nonstationary shear free geodesic null congruence which is generated by a given analytical complex world line. Solutions of the Einstein equations are analyzed. It is shown that there exists complex radiative solution which is generalization of the Kerr solution and the Kinnersley accelerating solution for "photon rocket".

1. INTRODUCTION

The algebraically special solutions of the Einstein and Einstein-Maxwell equations are characterized by the existence of geodesic shear free principal null congruences. Setting up the congruence is a first step to solving the corresponding field equations. The Kerr theorem [1,2,3] gives a rule how to construct such congruences: An arbitrary geodesic shear-free null congruence in Minkowski space is defined by a function $Y$ which is a solution of the equation

$$F = 0,$$

where $F(\lambda_1, \lambda_2, Y)$ is an arbitrary analytic function of the projective twistor coordinates

$$\lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}, \quad Y.$$  

In this paper we consider Kerr-Schild space-times containing geodesic and shear free congruences. First we analyze one of the two principal null congruences of the Kerr metric. It belongs to a class of congruences for which the singularities are
contained in a bounded region of space \([4,5]\). In this case the equation \(F = 0\) may be solved in explicit form. We find a representation for the function \(F\) in which the congruence is defined by a source moving in complex Minkowski space \(CM^4\) along a straight complex world line. This representation reproduces, in the Kerr-Schild formalism, a retarded time construction which was considered before by Lind and Newman \([6,7]\).

Second, we obtain a nonstationary generalization of the principal null congruence of the Kerr metric, in which the congruence is generated by a given complex world line \(x_0(\tau)\) depending on a complex time \(\tau\). We find also the necessary conditions under which the congruence satisfies the Kerr theorem:

i) the world-line \(x_0(\tau)\) must have an analytical dependence on the complex time \(\tau\),

ii) \(\tau\) must be a "left" solution of the retarded time equation, corresponding to the "left" null plane of the complex light cone.

In the last section we analyze the field equations and find the complex solutions in which the metric is not vacuum but has a Ricci tensor corresponding the stress tensor of null radiation. This solutions contain as a particular case the Kerr solution and the Kinnersley accelerating solution for a "photon rocket" \([3,8]\).

In the notation we follow the work of Debney, Kerr and Schild \([1]\). In the four-dimensional space-time with signature \((+ + + -)\), the null tetrad \(e_1, e_2, e_3, e_4\) is given by

\[
g_{ab} = e^a_\mu e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{ab}. \tag{1.3} \]

\(e^3, e^4\) are real null vectors, \(e^1, e^2\) are complex conjugates. The Ricci rotation coefficients are

\[
\Gamma^a_{bc} = - e^a_{\mu;\nu} e^\mu_b e^\nu_c. \tag{1.4} \]

Let the principal null congruence which is being considered have the \(e^3\) direction as tangent. It will be geodesic if and only if \(\Gamma_{42} = 0\) and shear free if and only if \(\Gamma_{422} = 0\). The corresponding complex conjugate terms are \(\Gamma_{414} = 0\) and \(\Gamma_{411} = 0\).
The Kerr-Schild metric

\[ g_{\mu\nu} = \eta_{\mu\nu} + 2h e^3_\mu e^3_\nu \]  

(1.5)

has the property that the principal null direction \( e^3 \) is also a null direction with respect to the auxiliary Minkowski space with metric

\[ \eta = dx^2 + dy^2 + dz^2 - dt^2 = 2dudv + 2d\bar{\zeta}d\zeta \]  

(1.6)

where the null coordinates are related to Cartesian coordinates by

\[ 2^{1/4} \zeta = x + iy, \quad 2^{1/4} \bar{\zeta} = x - iy, \]
\[ 2^{1/4} u = z + t, \quad 2^{1/4} v = z - t \]  

(1.7)

The field of the null directions \( e^3 \) is defined by the complex function \( Y \)

\[ e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - YY dv. \]  

(1.8)

Complete the null tetrad \( e^\mu_a \) as follows:

\[ e^1 = d\zeta - Y dv; \]
\[ e^2 = d\bar{\zeta} - \bar{Y} dv; \]  

(1.9)

\[ e^4 = dv - he^3. \]

The inverse tetrad has the form

\[ \partial_1 = \partial_\zeta - \bar{Y} \partial_u; \]
\[ \partial_2 = \partial_\bar{\zeta} - Y \partial_u; \]
\[ \partial_3 = \partial_u - h \partial_4; \]
\[ \partial_4 = \partial_v + Y \partial_\zeta + \bar{Y} \partial_\bar{\zeta} - YY \partial_u. \]  

(1.10)

It was shown in [1] that

\[ \Gamma_{42} = \Gamma_{42a} e^a = -dY - hY, \]  

(1.11)

The congruence \( e^3 \) is geodesic if \( \Gamma_{424} = -Y, \) and is shear free if \( \Gamma_{422} = -Y, \) \( 2 = 0. \) Thus the function \( Y \) with conditions

\[ Y,2 = Y,4 = 0, \]  

(1.12)
defines a shear free geodesic congruence.

2. THE KERR THEOREM FOR THE GENERAL CONGRUENCES

Given the geodesic and shear free nature (1.12) of the null congruence, the Kerr theorem [1,2,3], (1.1) and (1.2) holds. Following [1], we consider the differential of the function $Y$ in the case of $Y_{2} = Y_{4} = 0$:

$$dY = Y_{a} e^{a} = Y_{1} e^{1} + Y_{3} e^{3}.$$  \hspace{1cm} (2.1)

$Z = Y_{1}$ is the complex expansion of the null congruence (expansion + $i$ rotation).

As the first step we work out the form of $Y_{3}$. By using relations (1.10) and their commutators we find

$$Z_{,2} = (Z - \bar{Z})Y_{;3}. \hspace{1cm} (2.2)$$

Straightforward differentiation of $Y_{;3}$ gives the equation

$$Y_{;32} = (Y_{;3})^{2}. \hspace{1cm} (2.3)$$

And by using (2.2) and (2.3) we obtain the equation

$$(Z^{-1}Y_{;3})_{,2} = \bar{Z}(Z^{-1}Y_{;3})^{2}. \hspace{1cm} (2.4)$$

This is a first-order differential equation for the function $Z^{-1}Y_{;3}$. The general solution has the form

$$Y_{;3} = Z(\phi - \bar{Y})^{-1}, \hspace{1cm} (2.5)$$

where $\phi$ is an arbitrary solution of the equation

$$\phi_{,2} = 0. \hspace{1cm} (2.6)$$

Substitution $Y_{;3}$ in (2.1) implies

$$Z^{-1}(\bar{Y} - \phi)dY = \phi(d\zeta - Y dv) + (du + Y d\bar{\zeta}). \hspace{1cm} (2.7)$$

Differentiating the function $F(\lambda_{1}, \lambda_{2}, Y)$ and comparing the result with (2.7) we find that

$$PZ^{-1} = - \frac{dF}{dY}, \hspace{1cm} P = \partial_{\lambda_{1}}F - \bar{Y}\partial_{\lambda_{2}}F, \hspace{1cm} \mu = \partial_{\lambda_{2}}F, \hspace{1cm} (2.8)$$
where the function $P$ is defined

$$P = \mu (\phi - \bar{Y}). \quad (2.9)$$

The singular region of the congruence where the complex divergence $Z$ blows up is defined by the system of equations

$$F = 0, \quad dF/dY = 0. \quad (2.10)$$

3. STATIONARY CONGRUENCES HAVING SINGULARITIES CONTAINED IN A BOUNDED REGION

The null congruence with tangent $e^3$ is said to be stationary if there exists a real timelike vectorfield $K$ such that $KY = 0$. The stationary congruences having singularities contained in a bounded region have been considered in papers [4, 5]. In this case function $F$ must be at most quadratic in $Y$,

$$F \equiv a_0 + a_1 Y + a_2 Y^2 + (qY + c)\lambda_1 - (pY + \bar{q})\lambda_2, \quad (3.1)$$

where coefficients $c$ and $p$ are real constants and $a_0, a_1, a_2, q, \bar{q}$, are complex constants. The solutions of the equation $F = 0$ and the equations for the singularities may be found in explicit form. An analysis shows that this case gives the Kerr solution up to the Poincare group of motions.

The form (3.1) of $F$ implies that

$$KY = 0. \quad (3.2)$$

where

$$K = c\partial_u + \bar{q}\partial_\zeta + q\partial_{\zeta} - p\partial_v. \quad (3.2a)$$

There is another equivalent form of $F$ which allows one to introduce a retarded time parameter. One can represent (3.1) in the form

$$F \equiv (\lambda_1 - \lambda_1^0)K\lambda_2 - (\lambda_2 - \lambda_2^0)K\lambda_1. \quad (3.3)$$
Here we introduce a complex world line \( x_0^\mu(\tau) \) parametrized by the complex time parameter \( \tau \). The coordinates of this world line are complex, \( x_0(\tau) = (\zeta_0, \bar{\zeta}_0, u_0, v_0) \in CM^4 \). Thus \( \bar{\zeta}_0 \) and \( \zeta_0 \) are not necessarily complex conjugates. The vector \( K \) may be expressed now in the form

\[
K(\tau) = \dot{x}_0^\mu(\tau) \partial_\mu,
\]

where the dot denotes \( \partial_\tau \).

In this section we consider a straight world line with 3-velocity \( \bar{v} \) in \( CM^4 \), for which the Kerr-Schild map (1.5) gives the Kerr solution:

\[
x_0^\mu(\tau) = x_0^\mu(0) + \xi^\mu \tau; \quad \xi^\mu = (1, \bar{v}),
\]

\( K = \xi^\mu \partial_\mu \) is independent of \( \tau \) and \( \xi^\mu \) is a Killing vector of the Kerr solution. The parameters in (3.3) are then defined

\[
\lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta},
\]

\[
\lambda_1^0(\tau) = \zeta_0(\tau) - Yv_0(\tau), \quad \lambda_2^0(\tau) = u_0(\tau) + Y\bar{\zeta}_0(\tau),
\]

where the twistor components with zero indices denote the values on the points of complex world-line \( x_0(\tau) \).

To obtain the geometrical picture of this representation we consider a complex light cone with the vertex at some point \( x_0 \) of the complex world line \( x_0^\mu(\tau) \). The solutions of the complex light cone equation

\[
(x_\mu - x_{0\mu})(x^\mu - x_0^\mu) = 0,
\]

split into two families of null planes: ”left” planes

\[
x_L = x_0(\tau) + \alpha e^1 + \beta e^3,
\]

(\( L \))

and ”right” planes

\[
x_R = x_0(\tau) + \alpha e^2 + \beta e^3,
\]

(\( R \))
where $\alpha$ and $\beta$ are arbitrary parameters on these planes. Obviously (L) and (R) are solutions of (3.8) with the metric (1.2). The twistor parameters $\lambda_1$ and $\lambda_2$ may be represented in the form

$$
\lambda_1 = x^\mu e^1_\mu, \quad \lambda_2 = x^\mu (e^3_\mu - \bar{Y} e^1_\mu).
$$

(3.9)

Substitution of (L) in (3.9) shows that for an arbitrary $Y$ the twistor parameters $\lambda_1$ and $\lambda_2$ are constants on the "left" planes

$$
\lambda_1 = \lambda^0_1(\tau); \quad \lambda_2 = \lambda^0_2(\tau),
$$

(3.10)

and the equation $F = 0$ is fulfilled. One obtains the relations

$$
K\lambda_1 = \dot{\zeta}_0 - Y \dot{v}_0 = \partial_\tau \lambda^0_1, \quad K\lambda_2 = \dot{u}_0 + Y \dot{\zeta}_0 = \partial_\tau \lambda^0_2.
$$

(3.11)

In spite of an explicit dependence of the parameters of the $F$ in (3.3) on $\tau$ via (3.7) this dependence is absent really in consequence of the relations

$$
\lambda^0_1(x_0(\tau)) = \lambda^0_1(x_0(0)) + \tau K\lambda_1, \quad \lambda^0_2(x_0(\tau)) = \lambda^0_2(x_0(0)) + \tau K\lambda_2,
$$

(3.12)

and cancellation of the terms proportional to $\tau$.

Thus (3.1) and (3.3) are equivalent forms of $F$. A real cut of a "left" null plane gives a real null line which is defined by the twistor coordinates $(\lambda^0_1(\tau), \lambda^0_2(\tau), Y)$. Thus the Kerr principal null congruence arises as the real cut of the family of the "left" null planes of the complex light cones the vertices of which lie on the straight complex world line $x_0(\tau)$. By writing the function $F$ in the form

$$
F = A Y^2 + B Y + C,
$$

(3.13)

where

$$
A = (\bar{\zeta} - \bar{\zeta}_0) \dot{v}_0 - (v - v_0) \dot{\zeta}_0; \\
B = (u - u_0) \dot{v}_0 + (\zeta - \zeta_0) \dot{\zeta}_0 - (\bar{\zeta} - \bar{\zeta}_0) \dot{\zeta}_0 - (v - v_0) \dot{u}_0; \\
C = (\zeta - \zeta_0) \dot{u}_0 - (u - u_0) \dot{\zeta}_0,
$$

(3.14)

one can find two explicit solutions for the function $Y(x)$

$$
Y_{1,2} = (-B \pm \Delta)/2A,
$$

(3.15)
where $\Delta = (B^2 - 4AC)^{1/2}$. On the other hand (2.8), (2.9) and (3.10) imply

$$Y = -(B + PZ^{-1})/2A,$$

(3.16)

and consequently

$$PZ^{-1} = \mp \Delta.$$  

(3.17)

This relation reflects a twofoldedness of the Kerr geometry. The complex radial coordinate $PZ^{-1}$ is related to the Kerr coordinates [1] by $PZ^{-1} = r + i \cos \theta$. A change of the sign corresponds to a transition from the positive $r$ sheet of the metric to the negative one where $r \leq 0$. By using (2.8), (2.9) and (3.4) we find

$$P = \dot{Y} K\lambda_1 + K\lambda_2 = \dot{x}_0^\mu(\tau)e_\mu^3.$$  

(3.18)

Like the function $F$ and vector $K$, the coefficients $A, B, C$, as well as $Y, P, Z^{-1}$ for the Kerr solution do not depend on $\tau$, in spite of the presence of $\tau$ in the formulas containing $x_0(\tau)$. Nevertheless this parameter $\tau$ may be defined for each point $x$ of the Kerr space-time and plays really the role of a retarded time parameter. Its value for a given point $x$ may be defined by using the solution $Y(x)$ and by forming the twistor parameters (2.8) which fix the ”left” null plane (3.10). A point of intersection of this plane with the complex world-line $x_0(\tau)$ gives a value of the ”left” retarded time $\tau_L$. Thus $\tau_L$ is in fact a complex scalar function on the space-time $\tau_L(x)$. Since $Kx^\mu = \dot{x}_0^\mu$, action $K$ on the light cone equation $(x^\mu - x_0^\mu(\tau_L))(x^\nu - x_0^\nu(\tau_L)) = 0$ yields

$$K(\tau_L)(x) = 1,$$

(3.19)

and then by using (3.10) one finds that

$$K\lambda_1^0 = K\lambda_1; \quad K\lambda_2^0 = K\lambda_2,$$

(3.20)

1The projective twistor parameters $\lambda_1, \lambda_2, Y$ can be written in twistor notation $Z = (\mu^A, \psi_A)$, $\mu^A = x^\mu\sigma_\mu^A\psi_A$, as follows $(\lambda_1, \lambda_2, Y, 1) = (\mu^0, \mu^1, \psi_0, \psi_1)/\psi_1$. The eq.(3.2) takes the form $K\psi_A = 0$ and eq.(3.20) can be written in twistor terms as $K\mu^{0A} = K\mu^A$. This equation can be obtained also from $K\tau = 1$ and from the relations

$$K\mu^A = \dot{x}_0^\mu\partial_\mu x^\nu\sigma_\nu^{A\dot{A}}\psi_A = \dot{x}_0^\mu\sigma_\nu^{A\dot{A}}\psi_A,$$

$$K\mu^{0A} = \dot{x}_0^\mu\partial_\mu x^\nu\sigma_\nu^{A\dot{A}}\psi_A = \dot{x}_0^\mu\sigma_\nu^{A\dot{A}}\psi_A K\tau.$$
On the ”left” null plane we can use (3.10) and express \( \Delta \) in the form

\[
\Delta = (u - u_0)\dot{v}_0 + (\zeta - \zeta_0)\dot{\xi} + (\bar{\zeta} - \bar{\zeta}_0)\dot{\bar{\xi}}_0 + (v - v_0)\dot{u}_0 = -\partial_\tau (x - x_0)^2 / 2 = \tau - t + \bar{V}\bar{R},
\]

which gives a retarded-advanced time equation

\[
\tau = t \mp PZ^{-1} + \bar{V}\bar{R}, \tag{3.21}
\]

and a simple expression for the solutions

\[
Y_1 = [(u - u_0)\dot{v}_0 + (\zeta - \zeta_0)\dot{\xi}]/[(v - v_0)\dot{\zeta}_0 - (\bar{\zeta} - \bar{\zeta}_0)\dot{\bar{\zeta}}_0],
\]

\[
Y_2 = [(v - v_0)\dot{u}_0 + (\bar{\zeta} - \bar{\zeta}_0)\dot{\bar{\zeta}}_0]/[(\bar{\zeta} - \bar{\zeta}_0)\dot{\bar{\xi}}_0 - (v - v_0)\dot{\zeta}_0].
\]

Only the first root \( Y_1 \) is compatible with the constraints (3.10) and satisfies the condition \( Y_1, Y_2 = 0 \).

4. NONSTATIONARY CASE. CONGRUENCES GENERATED BY A COMPLEX WORLD LINE

Now we would like to extend the above representation of the function \( F \) in the form (3.3), and introduce the ”left” retarded time parameter \( \tau_L \) for the nonstationary generalization of the Kerr congruence when \( x_0(\tau) \) is an arbitrary complex world-line parametrized by complex time parameter \( \tau \) (not only straight and so far not only analytical). For every point \( x \) of the space-time one can consider the complex light cone and the point \( x_0(\tau) \) of the intersection of this light cone with the complex world line. One may look for a solution for the parameter \( \tau \) of the corresponding light cone equation \((x^\mu - x_0^\mu)\eta_{\mu\nu}(x^\nu - x_0^\nu) = 0\) where the metric \( \eta_{\mu\nu} \) is (1.6), and the complex continuation of the transformation (1.7) between null co-ordinates \( x_0^\mu \) and Cartesian coordinates holds. Using the complex Euclidean space distance \( r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \) between the real point \( x \) and the complex point \( x_0(\tau) \), and parametrisation \( \tau = t_0 \), we split this as

\[
\tau = t - r(x, x_0(\tau)) \tag{4.1}.
\]
This is an implicit nonlinear equation for the retarded time coordinate $\tau$. Its solution $\tau(x)$ is a complex scalar function on the space-time. One can introduce $K(\tau) = \dot{x}_0^\mu(\tau) \partial_\mu$ and, by the same method as in stationary case, one finds that

$$K(\tau)\tau(x) = 1; \quad K\lambda_1^0 = K\lambda_1, \quad K\lambda_2^0 = K\lambda_2. \tag{4.2}$$

We can find now the conditions on the function $F$ defined by (3.4), which will guarantee that its solution $Y$ satisfies the differential equation (2.7) for the shear free and geodesic congruence. Differentiation (4.1) and comparison the result with (2.7) yields the following equations

$$\tau,2 = 0; \tag{4.3}$$

$$(\lambda_1 - \lambda_1^0)\partial_\tau K\lambda_2 - (\lambda_2 - \lambda_2^0)\partial_\tau K\lambda_1 = 0. \tag{4.4}$$

To satisfy the equation (4.3) we attempt to use a retarded time $\tau$, which is subject to the light cone constraint

$$(x_\mu - x_0_\mu(\tau))(x^\mu - x_0^\mu(\tau)) = 0. \tag{4.5}$$

Differentiation of (4.5) gives

$$\tau,2 = [2(x - x_0)e^1 + \bar{\tau},2 \partial_\tau(x - x_0)^2]/\partial_\tau[(x - x_0)^2], \tag{4.6}$$

and condition (4.3) takes the form

$$(x - x_0)e^1 = 0; \tag{4.7}$$

$$\partial_\tau x_0(\tau) = 0. \tag{4.8}$$

Equation (4.4), in consequence of (4.7) and the representation (3.9) for the twistor parameters, gives

$$(x - x_0)e^3 = 0; \tag{4.9}$$

But (4.7) and (4.9) are in fact the equations (3.10) of the "left" null plane. Thus the necessary conditions, under which the congruence is shear free and geodesic, are:
i) the world-line $x_0(\tau)$ must have an analytical dependence on the complex time $\tau$, or $\partial_\tau x_0(\tau) = 0$,

ii) $\tau$ must be a "left" solution of the retarded time equation - $\tau_L$, corresponding to an intersection of the "left" null plane with the world line $x_0(\tau)$.

From (3.4) and (3.9) one can find

$$K\lambda_1 = \dot{x}_0^\mu e_\mu^1, \quad K\lambda_2 = \dot{x}_0^\mu (e_\mu^3 - \bar{Y} e_\mu^1), \quad (4.10)$$

then by using (2.8), (2.9) and (4.1) we obtain

$$P = \dot{x}_0^\mu e_\mu^3, \quad \bar{P}_Y = \partial_Y P = \dot{x}_0^\mu e_\mu^1, \quad Y, 3 = -ZP / P. \quad (4.11)$$

5. ANALYSIS OF THE FIELD EQUATIONS

We have not been able to obtain non-trivial real vacuum or electrovacuum solutions for the above nonstationary generalization of the Kerr metric. However, we will present some results on the features of such solutions and show the existence of the complex radiating solutions, having a stress-tensor proportional to $e_\mu^3 e_\nu^3$, which included the accelerating Kinnersley solution for the photon rocket [3,8] as particular case. We shall assume that there is no electromagnetic field and the null radiation may be produced by an incoherent flow of the light-like particles like the case of the Kinnersley solution.

First note that all equations for the Kerr-Schild form of metric of the ref.[1] up to Eq. (5.50) remain valid for a nonstationary case if all variables are expressed in terms of the real function $P$. However, for the nonstationary generalization of the Kerr congruence considered above the function $P$ takes complex values, that leads to the necessity of introducing corrections for three cases, which we are going to consider.

The function $P$ may be defined via tangent directions of the complex world line (4.11) and has to be essentially complex for any non-trivial complex analytical world line. Correspondingly, the function $P(Y, \bar{Y}, \tau) = \dot{x}_0^\mu(\tau)e_\mu^3$ has to be complex in the general case too. Integration of the field equations gives then the complex
meanings of $M$ which are not compatible with the standard propositions of the real Kerr-Schild formalism [1]. However, we can obtain that the Kerr-Schild formalism admits the complex solutions of the Einstein equations.

Independently of the real or complex meaning of the metric we have

$$R_{24} = R_{22} = R_{44} = 0. \quad (5.1)$$

If the electromagnetic field is zero we have also

$$R_{12} = R_{34} = 0. \quad (5.2)$$

We mention that the equation

$$h_{,44} + 2(Z + \bar{Z})h_{,4} + 2ZZh = 0, \quad (5.4)$$

which follows from (5.2), admits not only real but also the complex solutions

$$h = (MZ + \bar{M}\bar{Z})/2, \quad (5.5)$$

where $M$ and $\bar{M}$ are complex functions, not necessarily complex conjugates, obeying the conditions $\bar{M}_{,4} = M_{,4} = 0$.

Next, the equation

$$R_{23} = 0, \quad (5.6)$$

for the complex $M$ acquires the form

$$M_{,2} - 3Z^{-1}\bar{Z}Y_{,3}M - (Z^{-2}/2)(M - \bar{M})Z\bar{Z},_{2} = 0, \quad (5.7)$$

containing the extra term

$$(Z^{-2}/2)(M - \bar{M})Z\bar{Z},_{2}. \quad (5.8)$$

The last gravitational field equation $R_{33} = -P_{33}$ takes the form

$$M_{,3} - Z^{-1}Y_{,3}M_{,1} - \bar{Z}^{-1}\bar{Y}_{,3}M_{,2} = Z^{-1}\bar{Z}^{-1}P_{33}/2, \quad (5.9)$$

where $-P_{33}$ is proportional to $e_{\mu}^{3}e_{\nu}^{3}$ that corresponds to the presence of null radiation.
The exact solutions we are looking for satisfy the restriction

\[ M = \bar{M}, \]  

leading to the canceling of the extra term (5.8). As result the equation (5.7) acquires the simple form and we get its complex solution

\[ M = m/P^3, \]  

where

\[ m_{,4} = m_{,2} = 0. \]  

From (5.10) we have

\[ P = \bar{P}. \]  

This is possible for the complex functions \( P \) and \( M \) only if \( \bar{P} \) and \( \bar{M} \) are considered as independent functions. Thus, the complex world lines \( x_0(\tau) \) and \( \bar{x}_0(\bar{\tau}) \) are not to be taken as complex conjugated. They may be considered rather to be parallel, since their tangent vectors coincides, \( \partial_{\tau}x_0 = \partial_{\bar{\tau}}\bar{x}_0 \) to provide the condition

\[ P = \partial_{\tau}x_0^\mu e_\mu^3 = \bar{P} = \partial_{\bar{\tau}}\bar{x}_0^\mu e_\mu^3. \]  

We have to speak now about the “right” and “left” world lines \( x_0 \) and \( \bar{x}_0 \), about the corresponding “right” and “left” time parameters \( \tau_L, \bar{\tau}_R \) and parameters \( Y, \bar{Y} \).

We introduce the differential operator

\[ D = \partial_3 - Z^{-1}Y_3 \partial_1 - \bar{Z}^{-1}\bar{Y}_3 \partial_2. \]  

Action of the operator \( D \) on the variables \( Y, \bar{Y} \) and \( \tau \) is following

\[ DY = D\bar{Y} = 0, \quad D\tau = P^{-1}, \]  

where the last relation follows from (4.2) and (4.11). If \( M \) is function of \( Y, \bar{Y} \) and \( \tau \) then the equation (5.9) takes the form

\[ \partial_{\tau}M = PZ^{-1}\bar{Z}^{-1}P_{33}/2, \]  

(5.17)
It is not really a field equation but a definition of $P_{33}$ of the null radiation. Substitution of (5.11) gives

$$P_{33} = 2PZ\bar{Z}\partial_{\tau}m/P^3 = Z\bar{Z}[-6m(\partial_\tau P) + 2P(\partial_\tau m)]/P^3, \tag{5.18}$$

The resulting metric is complex and has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + (m/P^3)(Z + \bar{Z})e^3_\mu e^3_\nu. \tag{5.19}$$

One can normalize $e^3$ by introducing $l = e^3/P$ so that

$$\dot{x}_0^\mu e^3_\mu = 1, \tag{5.20}$$

and the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} - m(\tilde{r}^{-1} + \bar{\tilde{r}}^{-1})l_\mu l_\nu, \tag{5.21}$$

where $\tilde{r}$ is a complex radial coordinate

$$\tilde{r} = PZ^{-1} = -dF/dY. \tag{5.22}$$

We can select also two particular cases. The first case corresponds to the known nonstationary solutions of Kinnersley [8] where congruence is twist free. The world line is real in this case, $Imx_0 = 0$ and $\tau$ is real too. The complex conjugate world lines coincide and so do the ”right” and ”left” retarded times.

Equation (5.11) has the solution

$$M(Y, \bar{Y}, u) = m/P^3, \tag{5.19}$$

where function $P$ depends now on real $u = \tau = \tilde{r}$ and $m_4 = m_2 = 0$. From the equation (5.18) one can see that metric is not vacuum. We get the real retarded-time construction considered by Kinnersley [8] leading to a generalization of the Vaidya shining-star metric that permits arbitrary acceleration of the source. The Kinnersley metric has the Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + Hl_\mu l_\nu = \eta_{\mu\nu} + 2(m/r)(\sigma_\mu/r)(\sigma_\nu/r). \tag{5.23}$$
The relation with our notations is following

\[ l^\mu = \sigma^\mu / r, \]  
(5.24)

where

\[ \sigma^\mu = x^\mu - x_0^\mu, \quad r = P Z^{-1}. \]  
(5.25)

The Kinnersley retarded time parameter \( u = \tau / \sqrt{(\dot{x}_0)^2} \), and \( \lambda^\mu(u) = \dot{x}_0^\mu(\tau) / \sqrt{(\dot{x}_0)^2} \). The metric has a Ricci tensor proportional to \( e^\mu_\nu e^\nu_\nu \) that corresponds to the presence of null radiation.

The second degenerate case corresponds to a straight complex world line having a constant complex tangent direction \( \xi = \partial_\tau x_0 = \text{const} \). The function \( P = \xi^\mu e^3_\mu \) is now independent of \( \tau \). The equation (5.17) is fulfilled with \( m = \text{const}, P_{33} = 0 \) and we get the exact complex vacuum solution which is generalization of the Kerr solution to the case of a complex Killing direction \( \xi^\mu \). The physical meaning of this solution is still unknown and may be obtained only after finding the real slice.

**CONCLUSION**

We describe nonstationary geodesic shear free principal null congruences in Kerr-Schild spaces which are generated by a complex world line. Similar ideas were considered also in papers [4-7] and in the case of a real world line in work [8] where an exact solution was pointed out. The present consideration in the Kerr-Schild formalism gives a convenient form of the function \( F \), of the Kerr theorem used in the retarded time construction and allows us to obtain an explicit representation of the congruence and singularities. We make precise also the necessary conditions on the complex world line and retarded time parameter. The analysis of field equations shows that there are complex and radiating metrics among the nonstationary generalizations of the Kerr solution.

The second feature is an appearance of the broken symmetry of complex conjugation and a possibility of the equivalent description of this situation in the class of double Kerr-Schild metrics [9,10].
Complex world lines may be considered in string theory as a particular class of relativistic strings [11,12]. A further application of the complex world line representation of shear free geodesic congruences has been pointed out in [12].

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