Degenerations, transitions and quantum cohomology

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Given a singular variety I discuss the relations between quantum cohomology of its resolution and smoothing. In particular, I explain how toric degenerations helps with computing Gromov–Witten invariants, and the role of this story in “Fanosearch” programme [7]. The challenge is to formulate enumerative symplectic geometry of complex 3-folds in a way suitable for extracting invariants under blowups, contractions, and transitions.

Topologically a conifold transition is a surgery on a real 6-dimensional manifold switching a 3-sphere with a 2-sphere. The boundary of the tubular neighbourhood of a 2-sphere or a 3-sphere is isomorphic to $S^2 \times S^3$, so one can replace a tubular neighbourhood of one of those spheres with a tubular neighbourhood of another one. Smith–Thomas–Yau [17] show that this procedure makes sense in the context of symplectic manifolds, replacing a Lagrangian $S^3$ with a symplectic $S^2$. There are two natural ways of choosing a $S^2$ related by a flop. Note that similar procedure in real dimension 4, namely replacing a Lagrangian $S^3$ with a symplectic $S^2$, does not change the topology of underlying manifold (as was observed already by Brieskorn), and can be compensated merely by changing the symplectic form on the same underlying manifold. For the conifold transition of 6-dimensional manifolds topology obviously changes, since the Euler number of two manifolds differs by $2 = e(S^2) - e(S^3)$.

A local algebro-geometric picture for the conifold transition looks as follows. Consider a function $f : \mathbb{C}^4 \to \mathbb{C}$ given in some coordinates $x, y, z, w$ by a non-degenerate quadratic form $f = (xy - zw)$

A 3-fold conifold singularity (or an ordinary double point in dimension 3) is locally given by equation ($f = 0$), the fiber of map $f$ over 0

$$X_0 := \{ u \in \mathbb{C}^4 : f(u) = 0 \} = \{ (x, y, z, w) \in \mathbb{C}^4 : xy = zw \}$$

It has a smoothing $X_t$ for $t \in \mathbb{C}$ given as a fiber of $f$ over $t$

$$X_t := \{ u \in \mathbb{C}^4 : f(u) = t \} = \{ (x, y, z, w) \in \mathbb{C}^4 : xy = zw + t \}$$

and two small resolutions $\hat{X}_0 = \Gamma_0, \hat{X}_0' = \Gamma_0'$, given as Zariski closures of the graphs of rational maps $\phi, \phi' : X_0 \to \mathbb{C}P^1$

$$\phi(x, y, z, w) := (x : z) = (w : y)$$

$$\phi'(x, y, z, w) := (x : w) = (z : y)$$

Here all $X_t$ are isomorphic as complex manifolds $^1$, and as a real 6-manifold $X_1$ is isomorphic to $T^* S^3$, the total space of the (co)tangent bundle on $S^3$ $^2$. The small resolutions $\hat{X}_0$ and $\hat{X}_0'$ are isomorphic as abstract complex manifolds to the so-called local $\mathbb{C}P^1$, that is the total space of the bundle $\mathcal{O}_{\mathbb{C}P^1}(-1)^{\oplus 2}$.

The global picture is similar: we look for a singular complex threefold $Y_{\text{sing}}$, such that $\text{Sing} Y_{\text{sing}}$ equals to $N$ ordinary double points. It always has $2^N$ small resolutions which we denote by $Y_{\text{res}}$, however some of them may fail to be quasi-projective. The question of existence of a smoothing $^3$ is more subtle: versal deformation space for the conifold singularity is smooth and described above, but in some situations there are local-to-global obstructions. It was observed that for projective manifolds the existence of a projective

$^1$Thanks to a $\mathbb{C}^*$-action on the total space $(x, y, z, w) \to (\lambda x, \lambda y, \lambda z, \lambda w)$, compatible with the natural action on the base $t \to \lambda^2 t$.

$^2$It is easier to see in coordinates $z_1, \ldots, z_4$, where function $f = xy - zw$ has a form $f = \sum_{k=1}^4 z_k^2$ and for real positive $t$. Then if $z_k = x_k + iy_k$, two vectors in $\mathbb{R}^4 x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$ are pairwise-orthogonal and square of norms differ by $|x|^2 - |y|^2 = t$.

$^3$A smoothing of $Y_{\text{sing}}$ is a flat projective morphism $f : \mathcal{Y} \to \Delta$ to a disc $\Delta$, such that $f^{-1}(0) = Y_{\text{sing}}$ and for some $t \in \Delta$ the fiber $f^{-1}(t)$ is a smooth complex threefold $Y_{\text{sm}}$. In this situation we say that $Y_{\text{sm}}$ degenerates to $Y_{\text{sing}}$. 

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resolution and of a smoothing are mirror dual to each other. Friedman [8] shows that a smoothing always exists and unique (that is the versal deformation space is smooth) for Fano threefolds $Y_{\text{sing}}$. For Calabi–Yau threefolds $Y_{\text{sing}}$ a smoothing exists iff there is a linear relation between exceptional symplectic 2-spheres in $Y_{\text{res}}$, and a projective resolution exists iff there is a linear relation between vanishing Lagrangian 3-spheres in $Y_{\text{sm}}$. For proper toric varieties $Y_{\text{sing}}$ Gelfand–Kapranov–Zelevinsky demonstrate the existence of a projective small resolution.

Since in complex dimension 2 conifold transitions do not affect the topology, they can be used to compute Gromov–Witten invariants and find mirror potentials for non-toric del Pezzo surfaces of degrees 5 and 4. These surfaces have toric degenerations with $A_1$ singularities, and the crepant resolutions of these degenerations are smooth toric weak del Pezzo surfaces. There are variety of methods to compute holomorphic curves and discs on toric surfaces (equivariant method of Givental, tropical method of Mikhalkin, Cho–Oh, Fukaya–Oh–Ohta–Ono, Chan–Lau–Leung, etc), and the results of these computations can be transferred back to non-toric del Pezzo. Also in dimension 2 it is quite easy to construct symplectic birational invariants, for example a surface $S$ is rational iff its quantum cohomology $QH(S)$ is generically semi-simple [3] 4.

For the threefolds situation is quite different. There are theorems relating quantitative aspects of Gromov–Witten theories of a manifold and its blowup [10, 3] 5 or conifold transition [13, 14, 4, 12]. However none of them is suitable yet to demonstrate numerous qualitative relations we expect to hold: birational/transition invariance of symplectic rational connectedness, non-triviality of GW invariants, sharp birational invariants. In the remainder I discuss what we know and what we would like to know about the blowups and transitions of threefolds in the context of “birational symplectic topology”.

In my thesis [9] I described all conifold transitions from Fano threefolds to toric threefolds, thus answering a question posed by Batyrev in [2]. The case when $Y_{\text{sing}}$ is a toric Fano is arguably the simplest: by Friedman a smoothing $Y_{\text{sm}}$ exists and its topology is unique, so one just have to identify it, by computing some invariants and using Fano–Iskovskikh–Mori–Mukai’s classification. There are 100 conifold transitions from smooth Fano to smooth toric weak Fano, and approximately one half (44) of all non-toric Fanos have at least one such transition, on average two. To a toric $Y_{\text{sing}}$ (or $Y_{\text{res}}$) one can associate a Laurent polynomial $W = \sum_v z^v$ where the summation runs over all vertices $v$ of the fan polytope of $Y_{\text{sing}}$ (equiv. $Y_{\text{res}}$, since the resolution is small). Batyrev put a conjecture that constant terms of powers $W^d$ are equal to Gromov–Witten invariants $\langle \psi^{d-2} [pt] \rangle_{0,1,d}$, that count rational curves on $Y_{\text{sm}}$ passing through a generic point with “tangency conditions” prescribed by a power of psi-class. Nishinou–Nohara–Ueda shown that in fact $W$ above is the Floer potential function, that counts pseudo-holomorphic discs of Maslov index two on $Y_{\text{sn}}$ with a boundary on a Lagrangian torus, obtained as a symplectic transport of a fiber of a moment map $Y_{\text{sing}} \to \Delta$. With Bondal in [4] we proved a conjecture of Batyrev by expressing the described GW invariant as a polynomial of the numbers of holomorphic discs, using the relation between holomorphic curves, passing through a point, and holomorphic discs bounded on a Lagrangian torus, in the tropical limit. Same invariants were also computed in [6] by alternative (and less uniform) methods for all Fano threefolds. Also Batyrev and Kreuzer described degenerations of Fano threefolds to nodal half-anticanonical hypersurfaces in toric fourfolds, and it turns out that almost every Fano has such a degeneration.

In [13] Li and Ruan give a relation between Gromov–Witten theories of two threefolds linked by a conifold transition, and very recently Iritani and Xiao in [12] reformulated it as a relation between two quantum connections 6: the quantum connection of $Y_{\text{sm}}$ (on cohomology of even degree) is obtained from the quantum connection of $Y_{\text{res}}$ as a residue of a sub-quadratic. Similar invariants could (and should) be also obtained for the quantum connections of the blowups.

I expect that some sharp birational invariants could be extracted from Gromov–Witten theory of threefolds 7, and that they should be related to the monodromy group of the quantum connection. Much work still has to be done to understand better the relation between these monodromies, and to extract an invariant out of it.

Finally, the relation between conifold transitions and its effect on classical topology should be clarified further. A hypothesis usually referred to as “Reid’s dream” [15] says that all Calabi–Yau threefolds may be connected by a network of conifold transitions. There is an extensive experimental evidence towards this — most

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4However I do not know any a priori proof that semi-simplicity persists under the birational contractions, even in dimension 2. As an illustration of the problem: non-algebraic $K3$ surface does not have any non-trivial Gromov–Witten invariant, but its blowup in a point does — an invariant counting curves in an exceptional class!

5On practice it is often preferable to use less direct ways of computation as in [6], thanks to their more compact collection of the enumerative data.

6This work prompted me to speak on this subject here.

7For more details I refer to my recent talk http://www.math.stonybrook.edu/Videos/Birational/video.php?f=20150411-1-Galkin in Stony Brook on “New techniques in birational geometry” conference.
of known Calabi–Yau threefolds were shown to be connected by a network. But there are some classical invariants of the transition, such as the fundamental group. There are a couple of families of non-simply-connected threefolds without holomorphic 1-forms and $K = 0$ (e.g. 16 families were found by Batyrev and Kreuzer as crepant resolutions of hypersurfaces in toric fourfolds), so these threefolds are clearly disconnected from the conjectural “simply-connected network”. Smith–Thomas–Yau using conifold degenerations observed that possibly there are lots of non-Kähler symplectic Calabi–Yau threefolds, and Fine–Panov–Petrunin shown that the fundamental group of symplectic Calabi–Yau threefolds can take infinitely many different values. Non-simply-connected manifold could not be rationally connected, non-rationally-connected manifold cannot be symplectically rationally connected, non-symplectically-rationally-connected manifolds cannot have semi-simple quantum cohomology, which means at least some element in $QH(Y)$ of $Y$ with $\pi_1(Y) \neq 0$ should be nilpotent.

This suggests an interesting question: given a non-contractible loop $\gamma$ on $Y$, associate to it a nilpotent in $QH(Y)$. One possible approach to this could be via Givental’s interpretation of the quantum $D$-module of $Y$ with $S^1$-equivariant Floer theory on a free loop space $LY$. We find amusing the contrast with Seidel’s representation $Sei : \pi_1(\text{Symp}Y) \to QH(Y)^*$ that associates an invertible element to a loop in symplectomorphisms.

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8 Both $S^3$ (resp. $S^2$) has codimension at least 3 in $Y_{sm}$ (resp. $Y_{res}$), hence $\pi_1(Y_{sm}) = \pi_1(Y_{sm} \setminus \cup S^3_k) = \pi_1(Y_{res} \setminus \cup S^2_k) = \pi_1(Y_{res})$. 

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