UNIVERSALITY OF THE PINCH TECHNIQUE GAUGE
BOSON SELF-ENERGIES

N. Jay Watson

Abstract

It is shown how the S-matrix pinch technique may be extended to the cases of external
scalar and gauge boson fields. Using this extension, the universality of the pinch technique
gauge-independent one-loop gauge boson self-energy in a general unbroken SU(N) gauge
theory is demonstrated explicitly.

Key-Words: pinch technique, self-energy

Number of figures: 3

December 1994
CPT-94/P.3133

anonymous ftp or gopher: cpt.univ-mrs.fr

\^email: watson@cptsu4.univ-mrs.fr
1. The pinch technique (PT) is a well-defined algorithm for the rearrangement of contributions from the conventional gauge-dependent $n$-point functions occurring in one-loop processes to construct gauge-independent one-loop self-energy-like, vertex-like and box-like functions in non-abelian gauge theories. First introduced by Cornwall [1]–[3] and further developed by Cornwall and Papavassiliou [4]–[5] in the context of QCD, the original motivation for the PT was to enable gauge-independent truncation schemes for non-perturbative approaches to QCD involving Schwinger-Dyson equations. Since then, the PT has also been applied to spontaneously broken gauge theories. The first such application was made by Papavassiliou [6] in the context of a simplified Georgi-Glashow model. More recently, the PT has been used by Degrassi and Sirlin [7] in the electroweak sector of the Standard Model to derive explicitly $\xi$-independent electroweak gauge boson self-energies in the class of renormalizable $R_\xi$ gauges. Subsequently, there have been various applications in electroweak phenomenology [8]–[12].

The PT is based on the observation that in a non-abelian gauge theory, one-loop diagrams which appear to give only vertex or box corrections to tree level processes in fact implicitly contain propagator-like components. This property is a fundamental consequence of the underlying non-abelian gauge symmetry of the theory, be it unbroken or broken. For example, in the four-fermion process $\psi_i(p) + \psi_i'(p') \rightarrow \psi_j(p+q) + \psi_j'(p'-q)$, the conventional gauge boson one-loop two-point function contribution is as shown schematically in fig. 1(a). This conventional two-point contribution is gauge-dependent. However, at the one-loop level there also occur the diagrams shown in figs. 1(b), (d) and (f) which implicitly contain propagator-like contributions. These propagator-like contributions occur when, in the external fermion lines, the vector–fermion–fermion interaction vertices coincide. These are the “pinch parts” of the diagrams, shown in figs. 1(c), (e) and (g). In a conventional $R_\xi$ gauge, the $\xi$-dependence of these pinch parts is such as to cancel exactly that of the conventional two-point function. The PT gauge-independent one-loop gauge boson self-energy is therefore constructed by adding to the conventional self-energy in fig. 1(a) the propagator-like contributions figs. 1(c), (e) and (g) extracted from the conventional vertex and box functions. The resulting function is illustrated schematically in fig. 1(h). The PT gauge-independent one-loop vector–fermion–fermion vertex and four-fermion box functions are then the conventional functions with the pinch part contributions subtracted.

There are three principle formulations of the PT: the intrinsic and the S-matrix formulations of Cornwall and Papavassiliou, and the current algebra formulation of Degrassi and Sirlin. The S-matrix and current algebra versions are explicitly formulated in the context of one-loop processes involving on-shell external fermions, while the intrinsic PT, although it avoids the explicit embedding in S-matrix elements, starts implicitly from the
consideration of fermionic processes. In order however for the self-energies constructed in the PT to be intrinsic properties of the gauge bosons, and so to be physically meaningful, clearly a necessary condition is that they should be universal, i.e. independent of whether the external fields in the processes from which the self-energies are constructed are fermions, scalars or gauge bosons. This necessary condition is in addition to that of gauge independence. It has been shown by Degrassi and Sirlin using an indirect method, still ultimately involving external fermions, that the same PT electroweak self-energies are obtained from the process \(e^+e^- \rightarrow W^+W^-\) as from \(e^+e^- \rightarrow e^+e^-\). However, a general and direct demonstration of the PT self-energies’ universality is lacking.

In this letter, it is shown how the S-matrix PT may be extended directly to the cases of external scalar and gauge boson fields. Using this extension, the universality of the PT gauge-independent one-loop gauge boson self-energy in a general unbroken SU(\(N\)) gauge theory is demonstrated explicitly.

2. We consider an unbroken SU(\(N\)) gauge theory coupled both to fermions of mass \(m\) in a multiplet \(\Psi\) with representation matrices \(T^a\) and scalars of mass \(M\) \((M^2 > 0)\) in a complex multiplet \(\Phi\) with representation matrices \(R^a\):

\[
\mathcal{L}_{\text{cl}} = -\frac{1}{4}(\partial_{\mu}A_{\alpha}^a - \partial_{\nu}A^{a\mu}_\alpha + gf^{abc}A^b_{\mu}A^c_{\nu})(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}_\mu + gf^{ade}A^d_{\mu}A^{e\nu}) + \overline{\Psi}[\gamma^\mu(\partial_{\mu} - igA^a_{\mu}T^a) - m]\Psi + [\overline{(\partial_{\mu} - igA^a_{\mu}R^a)\Phi}]^\dagger[(\partial^\mu - igA^{b\mu}R^b)\Phi] - M^2\Phi^\dagger\Phi - \frac{1}{4!}\lambda(\Phi^\dagger\Phi)^2. \tag{1}
\]

The Feynman rules for the interaction of the gauge boson \(A^a_\alpha\) with the fermion fields \(\psi_i\), \(\psi'_j\), the scalar fields \(\phi_k\), \(\phi_l\), the pair of gauge bosons \(A^m_\mu\), \(A^n_\nu\), the triplet of gauge bosons \(A^1_\lambda\), \(A^m_\mu\), \(A^n_\nu\) and the scalar and gauge fields \(\phi_k\), \(\phi_l\), \(A^m_\mu\) are shown in fig. 2. We work in the class of conventional \(R_\xi\) gauges.

In the S-matrix formulation of the PT, one considers the S-matrix elements for the scattering of on-shell fermions. For the four-fermion process \(\psi_i(p) + \psi'_j(p') \rightarrow \psi'_j(p + q) + \psi_j(p' - q)\), the S-matrix element \(T\) may be decomposed as

\[
T(s, t, m) = T_1(t) + T_2(t, m) + T_3(s, t, m) \tag{2}
\]

where \(s = (p + p')^2\) and \(t = q^2\). The component \(T_1\), depending (up to trivial external wavefunctions) only on the momentum transfer \(q^2\), is the full propagator-like component of the process. Because of their different dependences on the kinematic variables \(s\) and \(t\) and the fermion mass \(m\), the components \(T_1\), \(T_2\) and \(T_3\) must be individually gauge-independent.

The gauge-dependent contribution to the S-matrix element of the diagram in fig. 1(a)
involving the conventional gauge boson two-point function $\Pi(\xi, q^2)$ is given by

$$\text{Fig. 1(a)} = \left( \gamma_{J'} \gamma^\mu T_{J'}^a u_{J'} \right) \frac{-i}{q^2} i\Pi(\xi, q^2) \frac{-i}{q^2} \left( \gamma_{J'} \gamma^\mu T_{J}^a u_{J} \right)$$

The particular indices $i, j, i', j'$ are not summed. The effect of the PT algorithm is to extract the contributions (the pinch parts) of the diagrams in figs. 1(b), (d) and (f) which have exactly the propagator-like form of eq. (3), i.e. a function of $q^2$ between two tree level vector–fermion–fermion vertices. Adding these pinch contributions to the contribution eq. (3) gives the component $T_1(q^2)$ of the S-matrix element and defines the PT “effective” two-point function $\hat{\Pi}(q^2)$:

$$T_1(q^2) = \left( \gamma_{J'} \gamma^\mu T_{J'}^a u_{J'} \right) \frac{-i}{q^2} i\hat{\Pi}(q^2) \frac{-i}{q^2} \left( \gamma_{J'} \gamma^\mu T_{J}^a u_{J} \right).$$

Because the component $T_1$ is gauge-independent, so must be the function $\hat{\Pi}(q^2)$. The PT gauge-independent one-loop “effective” gauge boson self-energy tensor is then given by

$$\hat{\Pi}_{\mu\nu}^{ab}(q) = \left( g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) \delta^{ab} \hat{\Pi}(q^2).$$

The gauge boson on, say, the r.h.s. of fig. 1(a), rather than coupling to the fermion pair $\psi_i, \psi_j$, may however couple instead at tree level to a pair of scalar fields $\phi_k, \phi_l$, or a pair of gauge bosons $A^m_{\mu}, A^n_{\nu}$, or a triplet of gauge bosons $A^l_{\lambda}, A^m_{\mu}, A^n_{\nu}$, or a triplet of scalar and gauge fields $\phi_k, \phi_l, A^m_{\mu}$. Thus, at the one-loop level, rather than considering the process $\psi_i \psi_i' \rightarrow \psi_j \psi_j'$, we can instead consider the process $\phi_k \psi_i' \rightarrow \phi_l \psi_j'$, or $A^m_{\mu} \psi_i' \rightarrow A^n_{\nu} \psi_j'$, or $A^l_{\lambda} \psi_i' \rightarrow A^m_{\mu} A^n_{\nu} \psi_j'$, or $\phi_k \psi_i' \rightarrow \phi_l A^m_{\mu} \psi_j'$. In each case, the contribution to the corresponding S-matrix element of the diagram analogous to fig. 1(a) involving the conventional gauge boson two-point function is given by eq. (3) with the vector–fermion–fermion vertex term $\gamma_{J'} \gamma^\mu T_{J'}^a u_{J'}$ replaced by the corresponding Feynman rule from fig. 2, together with any appropriate polarization vectors. Clearly, the gauge dependence of this diagram is independent of the particular on-shell external particles.

For the given set of external particles, there then remain the one-loop diagrams analogous to the vertex and box corrections figs. 1(b), (d) and (f) in the four-fermion case. Just as in the four-fermion case, these diagrams implicitly contain propagator-like components (pinch parts), defined as the components proportional to functions of $q^2$ between the appropriate tree level Feynman rules of fig. 2. These pinch parts may be added to the contribution analogous to eq. (3) from the conventional two-point function to specify the “effective” two-point function for the process, giving the full propagator-like contribution to the corresponding S-matrix element. The requirement of universality of the PT gauge boson self-energy $\hat{\Pi}_{\mu\nu}^{ab}(q)$ is that it be this “effective” two-point function, independent of the species of external particle. We will now show that this is indeed the case.
In the S-matrix PT with external fermions, the identification of the pinch parts is made using the elementary Ward identity

\[ \mathbf{k} = (\mathbf{p} - m) - (\mathbf{p} - \mathbf{k} - m) = S^{-1}(p; m) - S^{-1}(p - k; m) \]

in the numerator of the Feynman integral for a given diagram, where \( k_\mu \) is the four-momentum carried away by the gauge boson and \( p_\mu \) and \( (p - k)_\mu \) are the four-momenta of the adjacent fermions. In general, in a conventional \( R_\xi \) gauge, such factors of four-momentum \( k_\mu \) arise both from the longitudinal components of the gauge field propagators and from triple gauge vertices. For example, in fig. 1(b), with \( p_\mu \) the four-momentum of the incoming fermion \( \psi_i \), the fermion line inside the loop has propagator \( iS(p - k; m) \).

Using the Ward identity eq. (7) for a four-momentum factor \( k_\mu \) contracted with the Dirac matrix \( \gamma^\mu \) associated with the lower vector–fermion–fermion vertex, the first term from eq. (7) gives zero contribution since \((\mathbf{p} - m)u_i(p) = 0\) for the on-shell fermion, while the second term exactly cancels the fermion propagator, so giving a pinch part. A similar effect occurs in the upper vertex for a four-momentum factor \((q + k)_\nu \) contracted with \( \gamma^\nu \) and acting on \( \overline{\psi}_j(p + q) \). Thus, by using the Ward identity eq. (7), the total propagator-like contribution of the diagram may be found. The propagator-like contributions of the diagrams shown in figs. 1(d) and (f) are identified in a similar way, with in these cases four-momentum factors coming only from the gauge propagators.

Once it is established that the quantity being calculated is gauge-independent, one is at liberty to choose the gauge in which the calculation is simplest. Choosing the Feynman gauge \( \xi = 1 \), the gauge field propagators \( iD_{\mu\nu}^{ab} \) are proportional to \( g_{\mu\nu} \) and so the only possible source of four-momentum factors with which to generate pinch parts is the triple gauge vertex\(^3\). Thus, in the Feynman gauge, the diagrams in figs. 1(d) and (f) have vanishing pinch parts\(^3\) and the entire pinch contribution to the PT self-energy is given by the pinch part of the diagram in fig. 1(b).

\(^2\) In the class of background field \( R_\xi \) gauges, the effect of choosing the Feynman gauge \( \xi_q = 1 \) for the quantum gauge fields is to eliminate even the background–quantum–quantum field triple gauge vertex as a source of such four-momentum factors. This is the reason behind the recent observations \(^13\)–\(^16\) that at one loop the background gauge boson two-point function calculated in the Feynman quantum gauge is identical to the PT “effective” two-point function, both in QCD and the SM electroweak sector. For \( \xi_q \neq 1 \) however, this result no longer holds since the background gauge boson two-point function is \( \xi_q \)-dependent. Thus, only for \( \xi_q = 1 \) does the background field two-point function give the “effective” two-point function between two tree level background gauge boson vertices. Clearly, for \( \xi_q \neq 1 \), the PT algorithm can be applied in the background field formulation to construct again this “effective” two-point function \(^13\).

\(^3\) For the case of massless gauge bosons, the pinch part fig. 1(e) of the diagram fig. 1(d) in fact vanishes for all values of \( \xi \) in dimensional regularization.
For the cases of external scalars and gauge bosons, pinch terms are generated by similar factors of four-momentum. For the scalars, the pinch process is exactly similar to the fermion case. However, in the case of gauge bosons, there are also pinch terms which arise from the additional triple gauge vertices involved. These additional pinch terms are the novel feature of the application of the PT to the case of external gauge bosons, and must be taken into account.

The demonstration of the universality of the PT gauge boson self-energy consists in choosing the Feynman gauge and then showing that, for each of the possible sets of external fields \( \psi_i \psi_j, \phi_k \phi_l, A^m_{\mu} A^n_{\nu}, A_\lambda A^m_{\mu} A^n_{\nu} \) and \( \phi_k \phi_l A^m_{\mu} \) to which the gauge boson \( A^a_\alpha \) couples at tree level, the pinch part of the diagram analogous to fig. 1(b) is the same. The relevant half of these diagrams and their pinch parts\(^5\) are shown in fig. 3. It is easy to show that all other one-loop diagrams\(^5\) have zero pinch part in this gauge, so that the universality is then proved.

In each case, the gauge boson \( A^b_\beta (A^c_\gamma) \) in the lower (upper) part of the loop is taken to have four-momentum \( k (−k − q) \) flowing into the triple gauge vertex. This vertex may be decomposed as originally suggested by 't Hooft\(^8\):

\[
\Gamma^{abc}_{\alpha\beta\gamma} = f^{abc} \left( \Gamma^{F}_{\alpha\beta\gamma} + \Gamma^{P}_{\alpha\beta\gamma} \right)
\]

where

\[
\Gamma^{F}_{\alpha\beta\gamma}(q, k, −q − k) = (2k + q)\alpha g_{\beta\gamma} - 2q_{\beta} g_{\gamma\alpha} + 2q_{\gamma} g_{\alpha\beta}
\]

\[
\Gamma^{P}_{\alpha\beta\gamma}(q, k, −q − k) = -k_{\beta} g_{\gamma\alpha} - (k + q)_{\gamma} g_{\alpha\beta}.
\]

The part \( \Gamma^{F}_{\alpha\beta\gamma} \) gives no pinch contribution and obeys a simple QED-like Ward identity \( q^\alpha \Gamma^{F}_{\alpha\beta\gamma}(q, k, −q − k) = [k^2 − (q + k)^2]g_{\beta\gamma} \) involving the difference of two inverse gauge field propagators in the Feynman gauge. The part \( \Gamma^{P}_{\alpha\beta\gamma} \) gives two pinch contributions, one from \( k_{\beta} \), the other from \( (k + q)_{\gamma} \).

i) External fermion pair \( \psi_i, \psi_j \)

For the case of a pair of external fermion fields \( \psi_i, \psi_j \), the subamplitude for the r.h.s. of the diagram fig. 3(a) is

\[
i g_{\gamma\gamma} T^c_{jr} \frac{i}{\not p - \not k - m + ie} g_{\gamma\beta} T^b_{ri}
\]

(for brevity we omit the spinors). The pinch parts of this subamplitude are generated by factors of four-momentum \( k^2 \) and \((k + q)^2 \) multiplying this expression. Using \( \not k = \)

\(^4\) It is emphasized that the vertices involving four gauge bosons in figs. 3(d) and (f) are not the usual tree level vertices from the lagrangian eq. (1).

\(^5\)The one-loop diagrams involving the tree level \( \phi \phi \phi \phi, AAAAA \) or \( \phi \phi AA \) vertices with a pair of fields from the four-boson vertex appearing as external fields result in contributions which do not have the kinematic form of a function of \( q^2 \) between two of the appropriate tree level vertices of fig. 2.
\( \not{p} - m - (\not{p} - \not{k} - m) \) and \( (\not{p} - m) u_i(p) = 0 \) for the on-shell incoming fermion, the pinch part due to a factor \( k^\beta \) is

\[
g^2 \gamma_\gamma (T^c T^b)_{ji}. \tag{12}\]

Similarly, using \( \not{k} + \not{q} = \not{p} + \not{q} - m - (\not{p} - \not{k} - m) \) and \( \Pi_j (p + q)(\not{p} + \not{q} - m) = 0 \) for the on-shell outgoing fermion, the pinch part due to a factor \( (k + q)^\gamma \) is

\[
g^2 \gamma_\gamma (T^c T^b)_{ji}. \tag{13}\]

Adding these contributions due to the part \( \Gamma^P_{\alpha \beta \gamma} \) of the triple gauge vertex and using \( f^{abc} T^c T^b = -\frac{1}{2} i N T^a \) gives the pinch part fig. 3(b) of the diagram fig. 3(a):

\[
\text{Fig. 3(a)} \big|_{\text{pinch}} = -i g^2 N \mu^{2\epsilon} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 (k + q)^2} \gamma_j T_{ji} \tag{14}\]

(we use always dimensional regularization in \( n = 4 - 2\epsilon \) dimensions and with 't Hooft mass scale \( \mu \)).

ii) External scalar pair \( \phi_k, \phi_l \)

For the case of a pair of external scalar fields \( \phi_k, \phi_l \), the subamplitude for the r.h.s. of the diagram fig. 3(c) is

\[
g(2p - k + q) \gamma_R c_i R^c_R b \frac{i}{(p - k)^2 - M^2 + i\epsilon} g(2p - k) \beta R^b_R c_k. \tag{15}\]

As in the case of fermions, the pinch parts of this subamplitude are generated by factors of four-momentum \( k^\beta \) and \( (k + q)^\gamma \) multiplying this expression. Using \( k^\beta(2p - k)_\beta = p^2 - M^2 - [(p - k)^2 - M^2] \) and \( p^2 - M^2 = 0 \) for the on-shell incoming scalar particle, the pinch part due to a factor \( k^\beta \) is

\[
g^2 (2p - k + q) \gamma (R^c R^b)_{lk}. \tag{16}\]

Similarly, using \( (k + q)^\gamma(2p - k + q)_\gamma = (p + q)^2 - M^2 - [(p - k)^2 - M^2] \) and \( (p + q)^2 - M^2 = 0 \) for the on-shell outgoing scalar particle, the pinch part due to a factor \( (k + q)^\gamma \) is

\[
g^2 (2p - k) \beta (R^c R^b)_{lk}. \tag{17}\]

Adding these contributions due to the part \( \Gamma^P_{\alpha \beta \gamma} \) of the triple gauge vertex and using \( f^{abc} R^c R^b = -\frac{1}{2} i N R^a \) gives the pinch part fig. 3(d) of the diagram fig. 3(c):

\[
\text{Fig. 3(c)} \big|_{\text{pinch}} = -i g^2 N \mu^{2\epsilon} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 (k + q)^2} \gamma_j T_{ji} \tag{18}\]

6
iii) External gauge boson pair \( A_{\mu}^m, A_{\nu}^n \)
For the case of a pair of external gauge boson fields \( A_{\mu}^m, A_{\nu}^n \), the subamplitude for the r.h.s. of the diagram fig. 3(e) is
\[
g f^{\text{ncr}}[(2k - p + q)_{\nu}g_{\gamma}^\rho + (2p - k + q)_{\gamma}g_{\rho}^\nu + (-p - k - 2q)^{\rho}g_{\nu\gamma}] \\
\times \frac{-i}{(p - k)^2}g f^{\text{mr}}[(2k - p)_{\mu}g_{\beta\rho} + (-p - k)_{\rho}g_{\beta\mu} + (2p - k)_{\beta}g_{\mu\rho}],
\]
(19)

Exactly as in the fermion and scalar cases, pinch contributions are generated by factors of \( k^\beta \) and \( (q + k)^{\gamma} \) multiplying this expression. Using \( p^2 = 0 \) and \( (p + q)^2 = 0 \) for the on-shell incoming and outgoing external gauge bosons, these factors give respectively
\[
ig f^{\text{ncr}} f^{\text{mr}}[(2k - p + q)_{\nu}g_{\gamma}^\rho + (2p - k + q)_{\gamma}g_{\rho}^\nu + (-p - k - 2q)^{\rho}g_{\nu\gamma}]g_{\mu\rho}
\]
(20)
and
\[
ig f^{\text{ncr}} f^{\text{mr}}[(2k - p)_{\mu}g_{\beta\rho} + (-p - k)_{\rho}g_{\beta\mu} + (2p - k)_{\beta}g_{\mu\rho}]g_{\rho}^\nu.
\]
(21)

However, there is also a pinch term generated within the subamplitude itself from the contraction \((-p - k - 2q)^{\rho}(-p - k)_{\mu} = (p - k)^2 + 4kp + 2pq + 2qk\) in eq. (19). This generates a pinch term
\[
-ig f^{\text{ncr}} f^{\text{mr}} g_{\nu\gamma}g_{\beta\mu}
\]
(22)
independent of any other factors in the overall amplitude.

Adding the two contributions due to the part \( \Gamma_{\alpha\beta\gamma}^P \) of the triple gauge vertex and the contribution proportional to the full triple gauge vertex \( \Gamma_{\alpha\beta\gamma} \) and using the identity \( f^{abc} f^{ncr} f^{mr} = \frac{1}{2} N f^a_{\alpha\beta\gamma} \) gives the pinch part fig. 3(f) of the diagram fig. 3(e):
\[
\text{Fig. 3(e)} |_{\text{pinch}} = -ig f^{\text{ncr}} f^{\text{mr}} N_{\mu}^{2c} \int \frac{d^4 k}{(2\pi)^n} \frac{1}{k^2(k + q)^2} \\
\times g f^{a_{\alpha\beta\gamma}} [(2q + p)_{\alpha}g_{\mu\nu} + (-2q - p)_{\mu}g_{\nu\alpha} + (q - p)_{\nu}g_{\alpha\mu}].
\]
(23)

iv) External gauge boson triple \( A_{\lambda}^l, A_{\mu}^m, A_{\nu}^n \)
For the case of three external gauge boson fields \( A_{\lambda}^l, A_{\mu}^m, A_{\nu}^n \), there are three distinct diagrams figs. 3(g), (h) and (i) which must be taken into account. For the first of these, fig. 3(g), the subamplitude for the r.h.s. of the diagram is
\[
-ig f^{\text{rl}} f^{\text{rcd}}(g_{\lambda\gamma} g_{\delta\epsilon} - g_{\lambda\delta} g_{\gamma\epsilon}) + f^{\text{rld}} f^{\text{rcn}}(g_{\lambda\nu} g_{\gamma\delta} - g_{\lambda\delta} g_{\nu\gamma}) + f^{\text{rlc}} f^{\text{rmd}}(g_{\lambda\nu} g_{\gamma\delta} - g_{\lambda\delta} g_{\nu\gamma}) \\
\times \frac{-i}{(p - k)^2}g f^{\text{bmn}}[(2k - p)_{\mu}g_{\beta\delta} + (2p - k)_{\beta}g_{\mu\delta} + (-p - k)_{\delta}g_{\beta\mu}],
\]
(24)
The pinch part of this subamplitude is generated by a factor \( k^\beta \) from \( \Gamma_{\alpha\beta\gamma}^P \), which, using
\[ p^2 = 0 \text{ for the on-shell gauge boson } A_{\mu}^m, \text{ gives} \]
\[
g^4 f^{abc} f^{bmd} \left[ f^{\text{frm} \text{f}^{\text{rcl}}} (g_{\lambda \alpha} g_{\nu \mu} - g_{\lambda \mu} g_{\nu \alpha}) 
+ f^{\text{frd} \text{f}^{\text{rnc}}} (g_{\lambda \nu} g_{\alpha \mu} - g_{\lambda \alpha} g_{\nu \mu}) 
+ f^{\text{frd} \text{f}^{\text{rmd}}} (g_{\lambda \mu} g_{\nu \alpha} - g_{\lambda \nu} g_{\alpha \mu}) \right] . \tag{25} \]

For the second of the diagrams, fig. 3(h), the subamplitude for the r.h.s. of the diagram is
\[
-ig^2 [f^{\text{rm} \text{f}^{\text{rd}}} (g_{\mu \lambda} g_{\beta \delta} - g_{\mu \delta} g_{\beta \lambda}) + f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \beta} g_{\lambda \delta} - g_{\mu \delta} g_{\lambda \beta}) + f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \alpha} g_{\lambda \nu} - g_{\mu \nu} g_{\alpha \lambda})] 
\times \frac{-i}{(p-k+r)^2} g f^{\text{cdn}} [(-p-2q-r-k)^\delta g_{\nu \gamma} + (2p-k+q+2r) \gamma g_{\nu} + (2k-p+q-r) g_{\gamma}]. \tag{26} \]

The pinch part of this subamplitude is generated by a factor \((k + q)\gamma\) from \(\Gamma^P_{\alpha \beta \gamma}\), which, using \((p+q+r)^2 = 0\) for the on-shell gauge boson \(A_\nu^n\), gives
\[
g^4 f^{abc} f^{cdn} [f^{\text{rm} \text{f}^{\text{rd}}} (g_{\mu \lambda} g_{\nu \alpha} - g_{\mu \nu} g_{\alpha \lambda}) 
+ f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \alpha} g_{\lambda \nu} - g_{\mu \nu} g_{\alpha \lambda}) 
+ f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \nu} g_{\alpha \lambda} - g_{\mu \lambda} g_{\nu \alpha})]. \tag{27} \]

Lastly, for the diagram fig. 3(i), the subamplitude for the r.h.s. is
\[
-ig^2 [f^{\text{rm} \text{f}^{\text{rd}}} (g_{\mu \gamma} g_{\nu \delta} - g_{\mu \delta} g_{\nu \gamma}) + f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \beta} g_{\gamma \delta} - g_{\mu \delta} g_{\beta \gamma}) + f^{\text{rmc} \text{f}^{\text{rmd}}} (g_{\mu \alpha} g_{\lambda \nu} - g_{\mu \nu} g_{\alpha \lambda})] 
\times \frac{-i}{(r-k)^2} g f^{\text{bld}} [(-k-\nu) \delta g_{\beta \lambda} + (2r-k) \beta g_{\lambda}^\delta + (2k-r) \delta g_{\beta}]. \tag{28} \]

The pinch term for this subamplitude is generated by a factor \(k^\beta\) from \(\Gamma^P_{\alpha \beta \gamma}\), which, using \(r^2 = 0\) for the on-shell gauge boson \(A_\lambda^1\), gives
\[
g^4 f^{abc} f^{bld} [f^{\text{rm} \text{f}^{\text{rd}}} (g_{\mu \alpha} g_{\nu \lambda} - g_{\mu \lambda} g_{\nu \alpha}) 
+ f^{\text{rmd} \text{f}^{\text{rbd}}} (g_{\mu \nu} g_{\alpha \lambda} - g_{\mu \lambda} g_{\nu \alpha}) 
+ f^{\text{rmc} \text{f}^{\text{rmd}}} (g_{\mu \nu} g_{\alpha \lambda} - g_{\mu \lambda} g_{\nu \alpha})]. \tag{29} \]

Defining the group-theoretic quantity
\[
f(abcd) = f^{\text{f}^{\text{ab}} f^{\text{bml}} f^{\text{cmm}} f^{\text{dnk}}} \tag{30} \]
there then exist the following identities:
\[
f(abcd) = f(bcda) = f(badc) \tag{31} \]
\[
f(abcd) - f(abdc) = -\frac{1}{2} N f^{\text{abn}} f^{\text{cdn}}. \tag{32} \]
Using these identities, the three pinch terms proportional to eqs. (25), (27) and (29) may be combined to give the pinch part fig. 3(j) of the diagrams figs. 3(g), (h) and (i):

\[
\text{Figs. 3(g) + (h) + (i)}_{\text{pinch}} = -ig^2 N\mu^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+q)^2} \\
\times -ig^2 \{ \text{f}_\text{ral} \text{f}_\text{rmn} (g_{\mu\alpha} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\alpha}) \\
+ \text{f}_\text{ran} \text{f}_\text{rlm} (g_{\lambda\alpha} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\alpha}) \\
+ \text{f}_\text{ram} \text{f}_\text{rln} (g_{\lambda\alpha} g_{\mu\nu} - g_{\mu\lambda} g_{\alpha\nu}) \}.
\]  

(33)

v) External scalar and gauge boson triple \( \phi_k, \phi_l, A^m_{\mu} \)

Finally, for the case of three external scalar and gauge boson fields \( \phi_k, \phi_l, A^m_{\mu} \), there are again three diagrams, figs. 3(k), (l) and (m), which contribute. In an exactly similar way to the previous cases, pinch parts are generated by factors \( k^\beta \) and \( (k+q)^\gamma \) from \( \Gamma^P_{\alpha\beta\gamma} \).

Adding these contributions gives the pinch part fig. 3(n) of the diagrams figs. 3(k), (l) and (m):

\[
\text{Figs. 3(k) + (l) + (m)}_{\text{pinch}} = -ig^2 N\mu^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(k+q)^2} \\
\times -ig^2 \{ R^a, R^m \}_{lk} g_{\alpha\mu}.
\]  

(34)

We see that in each case, the pinch part eqs. (14), (18), (23), (33) and (34) is given by the tree level Feynman rule from fig. 2 for the gauge boson \( A^a_{\alpha} \) coupling to the given external particles, multiplied by the same function of \( q^2 \). Given that all other diagrams have zero pinch part, the universality is therefore proved.

The gauge-independent universal PT self-energy is now obtained by adding the pinch contribution, multiplied by an inverse propagator \( iq^2 \) and a factor two (since it occurs on each side of the diagram), to the conventional self-energy:

\[
i\hat{\Pi}(q^2) = i\Pi(\xi = 1, q^2) + 2Ng^2 \mu^2 \int \frac{d^n k}{(2\pi)^n} \frac{q^2}{k^2(k+q)^2}.
\]  

(35)

At asymptotic \( q^2 \), this PT self-energy has the behaviour expected from the RG \( \beta \) function.

4. It has been shown here how the S-matrix PT may be extended to include as external particles all five of the different combinations \( \psi_i \psi_j, \phi_k \phi_l, A^m_{\mu} A^n_{\nu}, A^i_{\lambda} A^m_{\nu} A^n_{\nu} \) and \( \phi_k \phi_l A^m_{\mu} \) to which the gauge boson \( A^a_{\alpha} \) couples at tree level. The universality of the PT one-loop “effective” gauge boson two-point function has then been demonstrated explicitly.

The interest of this result is that it indicates how the concept of a gauge-independent effective charge \( g(q^2) \) valid at all \( q^2 \), not just the asymptotic region governed by the RG \( \beta \) function, may be extended from abelian QED, where it arises naturally, to non-abelian
gauge theories. The crucial point is the recognition that the required quantity is the “effective” two-point function $\Pi_{\mu}^{ab}$ defined between any two tree level vertices involving $A_{\mu}^a$ and $A_{\nu}^b$. That this is precisely the quantity constructed in the PT is the PT’s distinguishing feature. In order, however, to promote this effective charge to a generally applicable method for accounting for a well-defined, infinite subset of gauge-independent diagrams, it is necessary to demonstrate that the PT “effective” two-point function remains simultaneously gauge-independent and universal in processes involving off-shell external fields. If, as seems likely, both these properties persist, then this effective charge would have immediate applications in QCD renormalon calculus and, when extended to broken theories, in electroweak phenomenology. Work is under way in this direction.

I wish to thank Eduardo de Rafael for many useful discussions. This work was supported by EC grant ERB4001GT933989.

References

[1] J.M. Cornwall, in Proceedings of the French-American Seminar on Theoretical Aspects of Quantum Chromodynamics, Marseille, France, 1981, ed. J.W. Dash (Centre de Physique Théorique report no. CPT-81/P-1345, 1982).

[2] J.M. Cornwall, Phys. Rev. D26 (1982) 1453.

[3] J.M. Cornwall, W.S. Hou and J.E. King, Phys. Lett. B153 (1988) 173.

[4] J.M. Cornwall and J. Papavassiliou, Phys. Rev. D40 (1989) 3474.

[5] J. Papavassiliou, Phys. Rev. D47 (1992) 4728.

[6] J. Papavassiliou, Phys. Rev. D41 (1990) 3179.

[7] G. Degrassi and A. Sirlin, Phys. Rev. D46 (1992) 3104.

[8] G. Degrassi, B. Kniehl and A. Sirlin, Phys. Rev. D48 (1993) 3963.

[9] J. Papavassiliou and K. Philippides, Phys. Rev. D48 (1993) 4255.

[10] J. Papavassiliou and C. Parrinello, Phys. Rev. D50 (1994) 3059.

[11] J. Papavassiliou and A. Sirlin, Phys. Rev. D50 (1994) 5951.

[12] J. Papavassiliou, Phys. Rev. D50 (1994) 5958.

[13] E. de Rafael and N.J. Watson, unpublished.
[14] A. Denner, G. Weiglein and S. Dittmar, Phys. Lett. B333 (1994) 420.

[15] S. Hashimoto et al., HUPD-9408, YNU-HEPTh-94-104, hep-ph 9406271.

[16] A. Denner, G. Weiglein and S. Dittmar, BI-TP 94/50, UWITP 94/03, hep-ph 9410338.

[17] J. Papavassiliou, hep-ph 9410385.

[18] G. ’t Hooft, Nucl. Phys. B33 (1971) 173.
**Figure Captions**

**Fig. 1.** (a) The conventional one-loop gauge boson two-point function contribution to the four-fermion process $\psi_i\psi_i' \rightarrow \psi_j\psi_j'$. (b)–(g) The remaining one-loop diagrams involving gauge bosons (the external leg corrections associated with (d) are not shown), together with their pinch parts. (h) The PT gauge-independent “effective” gauge boson two-point function.

**Fig. 2.** The Feynman rules for the five different sets of fields to which a single gauge boson $A^a_\alpha$ couples at tree level.

**Fig. 3.** The Feynman diagrams giving pinch parts in the Feynman gauge for the five different sets of external fields $\psi_i\psi_j$, $\phi_k\phi_l$, $A^m_\mu A^n_\nu$, $A^I_\lambda A^m_\mu A^n_\nu$ and $\phi_k\phi_l A^m_\mu$. 
\[
\psi_j(p'-q) \quad \psi_j(p+q)
\]
\[
\psi_Y(p') \quad \psi_I(p)
\]

(a)

(b) + reversed diagram

(c) + reversed diagram

(d) + reversed diagram

(e) + reversed diagram

(f) + crossed diagram

(g)

(h)

Fig. 1
\[ \psi_j(p+q) \]

\[ A^a_\alpha(q) \]

\[ \psi_i(p) \]

\[ ig\gamma_\alpha T^a_{ji} \]

\[ \phi_l(p+q) \]

\[ A^a_\alpha(q) \]

\[ ig(2p+q)\alpha R^a_{lk} \]

\[ (b) \]

\[ A^a_\alpha(q) \]

\[ A^m_\mu(p) \]

\[ \phi_k(p) \]

\[ g f^a_{mn}(2p+q)\alpha g_{\mu\nu} \]

\[ +(-2q-p)\mu g_{\nu\alpha} \]

\[ +(q-p)\nu g_{\alpha\mu} \]

\[ (c) \]

\[ A^a_\alpha(p+q+r) \]

\[ A^m_\mu(p) \]

\[ A^l_\lambda(r) \]

\[ -ig^2[f_{\alpha\mu\lambda\nu}(g_{\mu\alpha}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\alpha}) \]

\[ +fr_{\alpha}fr_{\mu}(g_{\lambda\alpha}g_{\mu\nu} - g_{\mu\nu}g_{\lambda\alpha}) \]

\[ +fr_{\mu}fr_{\lambda}(g_{\lambda\alpha}g_{\mu\nu} - g_{\mu\nu}g_{\lambda\alpha})] \]

\[ (d) \]

\[ \phi_l(p+q+r) \]

\[ A^a_\alpha(q) \]

\[ A^m_\mu(r) \]

\[ \phi_k(p) \]

\[ -ig^2\{R^a, R^m\}_{lk} g_{\alpha\mu} \]

\[ (e) \]

\[ 14 \]  

Fig. 2
\[
\begin{align*}
A_\alpha(q) &\rightarrow \phi_l(p+q+r) A_\mu^m(r) \\
&\downarrow p-k \\
&\phi_k(p) \\
&\phi_s(p+q+r) \\
A_\alpha(q) &\rightarrow p-k+r \\
&\downarrow A_\mu^m(r) \\
&\phi_k(p) \\
&\phi_s(p+q+r) \\
A_\alpha(q) &\rightarrow r-k \\
&\downarrow A_\mu^m(r) \\
(k) &\text{pinch} \\
(l) &\text{pinch} \\
(m) &\text{pinch}
\end{align*}
\]