A generalization of carries processes and Eulerian numbers

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Abstract

We study a generalization of Holte’s amazing matrix, the transition probability matrix of the Markov chains of the 'carries' in a non-standard numeration system. The stationary distributions are explicitly described by the numbers which can be regarded as a generalization of the Eulerian numbers and the MacMahon numbers. We also show that similar properties hold even for the numeration systems with the negative bases.

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1 Introduction and statements of results

The transition probability matrix so-called 'amazing matrix' of the Markov chain of the 'carries' has very nice properties [5], and has unexpected connection to the Markov chains of riffle shuffles [2, 3]. Diaconis and Fulman [3] studies a variant of the carries process, type \( B \) carries process. Novelli and Thibon studies the carries process in terms of noncommutative symmetric functions [7]. This paper studies a generalization of the carries process which includes Diaconis and Fulman’s type \( B \) carries process as a special case. We study the transition probability matrices of the Markov chains of the carries in the numeration systems with non-standard digit sets. We show that the matrices have the eigenvectors which can be perfectly described by a generalization of Eulerian numbers and the MacMahon numbers [8, 6, 1, 3]. We also show that similar properties hold even for the numeration systems with negative bases.

1.1 Numeration system

Throughout the paper, \( b \) denotes a positive integer and \( \mathcal{D} = \{ d, d+1, \ldots, d+b-1 \} \) denotes a set of integers containing 0. Therefore, \(-b < d < b\). Then, we have a numeration system \((b, \mathcal{D})\): Suppose that an integer \( x \) has a representation of the form,

\[
x = (x_kx_{k-1}\cdots x_0)_b \overset{\text{def}}{=} x_0 + x_1b + x_2b^2 + \cdots + x_kb^k, \quad x_0, x_1, \ldots, x_k \in \mathcal{D}, x_k \neq 0.
\]

Then, it can be easily shown that this representation is uniquely determined for \( x \) and

\[
\{(x_kx_{k-1}\cdots x_0)_b \mid k \geq 0, x_0, x_1, \ldots, x_k \in \mathcal{D}\} = \begin{cases} \mathbb{Z} & d \neq 0, -b + 1, \\ \mathbb{N} & d = 0, \\ -\mathbb{N} & d = -b + 1. \end{cases}
\]

is closed under the addition, where \( \mathbb{N} \) denotes the set of non-negative integers.

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1.2 Carries process

Let \( \{X_{i,j}\}_{1 \leq i \leq n, j \geq 0} \) be the set of independent random variables each of which is distributed uniformly over \( \mathcal{D} \). Define the two stochastic processes \((A_0, A_1, A_2, \ldots)\) and \((C_0, C_1, C_2, \ldots)\) in the following way: \( C_0 = 0 \) with probability one. \((A_i)_{i \geq 0}\) is a sequence of \( \mathcal{D} \)-valued random variables satisfying

\[
A_i \equiv C_i + X_{1,i} + \cdots + X_{n,i} \pmod{b}, \quad i = 0, 1, 2, \ldots
\]

and

\[
C_i = \frac{C_{i-1} + X_{1,i-1} + \cdots + X_{n,i-1} - A_{i-1}}{b}, \quad i = 1, 2, 3, \ldots
\]

(See Figure 1.) It is obvious that \((C_0, C_1, C_2, \ldots)\) is a Markov process, which we call the carries process with \( n \) summands or simply \( n \)-carry process over \((b, \mathcal{D})\).

1.3 A generalization of Eulerian numbers

Let \( p \geq 1 \) be a real number and \( n \) a positive integer. Then we define an array of numbers \( v^{(p)}_{i,j}(n) \) for \( i = 0, 1, \ldots, n \) and \( j = 0, 1, \ldots, n+1 \) by

\[
v^{(p)}_{i,j}(n) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} [p(j-r)+1]^{n-i},
\]

(2)

and define \( v^{(p)}_{i,i-1}(n) = 0 \). We denote

\[
\binom{n}{j}_p = v^{(p)}_{0,j}(n),
\]

which can be regarded as a generalization of the Eulerian numbers. In fact, \( \{\binom{n}{j}_p\}_p \) forms the array of the ordinary Eulerian numbers when \( p = 1 \), and MacMahon numbers \([8, 6, 1, 3]\) when \( p = 2 \).

1.4 Statement of the result

Throughout the paper, \( \Omega = \Omega_n(b, \mathcal{D}) \) denotes the state space of the \( n \)-carry process over \((b, \mathcal{D})\), that is, the set of possible values of carries, and \( p_{i,j} \) denotes the transition probability \( \Pr(C_{i+1} = j \mid C_i = i) \) for \( i, j \in \Omega_n(b, \mathcal{D}) \). For computational convenience, it is desirable for the transition probability matrix to have indices starting from 0. We define \( \tilde{p}_{i,j} = p_{i+s,j+s} \), where \( s \) is the minimal element of \( \Omega_n(b, \mathcal{D}) \). Then, we define the matrix \( P \) by

\[
P = (\tilde{p}_{i,j})_{0 \leq i,j \leq \#\Omega-1},
\]

(3)
where \( \#\Omega \) denotes the size of the state space, which is explicitly computed in Lemma 1. \( P \) is the central object of this paper.

**Remark 1.** As we will show in the later sections, this matrix \( P \) which we regard as a generalization of Holte’s amazing matrix is determined only by \( b, n \) and \( p \), and therefore these amazing matrices with same \( n \) and \( p \) form a commutative family. Holte’s amazing matrix corresponds to the case when \( p = 1 \) and Diaconis and Fulman’s type B carries process corresponds to the case when \( p = 2 \).

**Theorem 1.** Let \( \Omega = \Omega_n(b,D) \) be the state space of the \( n \)-carry process over \((b,D)\), and \( m = \#\Omega_n(b,D) \). let \( p \) be defined by

\[
p = \begin{cases}
  \frac{1}{(n-1)(-l)} & (n-1)l \notin \mathbb{Z}, \\
  1 & (n-1)l \in \mathbb{Z},
\end{cases}
\]

where \( l = d/(b-1) \) and \( \{x\} = x - \lfloor x \rfloor \), and let \( V = \left(v_{i,j}^p(n)\right)_{0 \leq i,j \leq m-1} \). Then, the transition probability matrix \( P \) is diagonalized by \( V \):

\[VPV^{-1} = \text{diag}(1, b^{-1}, \ldots, b^{-(m-1)}) .\]

In particular, by Lemma 5, the probability vector \( \pi \) of the stationary distribution of the carries process is

\[
\pi = (\pi(s), \pi(s+1), \ldots, \pi(s+m-1)) = \frac{1}{p^nn!} \left\langle \begin{array}{c}
n \\
0 \\
\vdots \\
m-1
\end{array} \right\rangle^p .
\]

**Remark 2.** It is remarkable that our amazing matrix has the eigenvalues of the same form \( 1, 1/b, 1/b^2, \ldots \) as those of Holte’s amazing matrix.

**Corollary 1.** Let \( S_n \) be the sum of \( n \) independent random variables each of which is distributed uniformly over the unit interval \([0,1]\). Then, for all positive real numbers \( p \geq 1 \) and integers \( k \), the probability of \( S_n \) being in the interval \( \frac{1}{p} + [k-1,k] \) is

\[
\Pr \left( S_n \in \frac{1}{p} + [k-1,k] \right) = \frac{1}{p^nn!} \left\langle \begin{array}{c}
n \\
k-1
\end{array} \right\rangle^p .
\]

**Remark 3.** This corollary can be derived directly from the formula of the distribution of sums of independent uniform random variables in [4], and it is shown for the case \( p = 2 \) in [3].

**Example 1.** Let \( p \geq 1 \) be a real number. As will be shown in the later section, the array of generalized Eulerian numbers satisfies the following recursive relations

\[
\left\langle \begin{array}{c}
n+1 \\
k
\end{array} \right\rangle^p = (pk+1)\left\langle \begin{array}{c}
n \\
k
\end{array} \right\rangle^p + (p(n+1-k)-1)\left\langle \begin{array}{c}
n \\
k-1
\end{array} \right\rangle^p ,
\]

and the boundary conditions

\[
\left\langle \begin{array}{c}
n \\
0
\end{array} \right\rangle^p = 1, \quad \text{and} \quad \left\langle \begin{array}{c}
n \\
k
\end{array} \right\rangle^p = 0 \text{ for } k > n .
\]

(See Figure 2.)
The probability density function of $S_3$ described in Theorem 1 is

$$f(x) = \begin{cases} 
  x^2/2 & \text{if } 0 \leq x < 1, \\
  -(x - \frac{3}{2})^2 + \frac{3}{4} & \text{if } 1 \leq x < 2, \\
  (x - 3)^2/2 & \text{if } 2 \leq x \leq 3, \\
  0 & \text{otherwise.}
\end{cases}$$
The probability vectors appear in the third rows of the triangles of the generalized Eulerian numbers (Figure 2).

2 Proof

2.1 State space and transition probability

Lemma 1. Let \( \Omega = \Omega_n(b, \mathcal{D}) \) be the state space of the \( n \)-carry process over the numeration system \((b, \mathcal{D})\). Then, \( \Omega = \{s, s+1, \ldots, t\} \) with

\[
s = -[(n-1)(-l)] = [(n-1)l], \quad t = [(n-1)(l+1)],
\]

where \( l = d/(b-1) \). Therefore, the size of the state space \( \Omega \) is

\[
\#\Omega = \begin{cases} n+1 & (n-1)l \not\in \mathbb{Z}, \\ n & (n-1)l \in \mathbb{Z}. \end{cases}
\]

Proof. Suppose that we add \( n \) numbers,

\[
(x_1, x_1, x_1, \ldots, x_1, x_{1,0})_b, (x_2, x_2, x_2, x_2, \ldots, x_2, x_{2,0})_b, \ldots, (x_n, x_n, x_n, x_n, \ldots, x_n, x_{n,0})_b,
\]

and get the sum \((a_{i+1}, a_0, a_1, a_0)_b\) and the carries \( c_0, c_1, c_2, \ldots \). That is, \( c_0 = 0 \) and

\[
c_{i+1} = \frac{c_i + x_{1,i} + \cdots + x_{n,i} - a_i}{b},
\]

where \( a_i \in \mathcal{D} \) and \( a_i \equiv c_i + x_{1,i} + \cdots + x_{n,i} \pmod{b} \). Let \( F \) be defined by

\[
F = \{x_1 b^{-1} + x_2 b^{-2} + \cdots + x_m b^{-m} \mid x_i \in \mathcal{D}_b, m > 0 \in \mathbb{Z}\}.
\]
Then, $F$ is a dense subset of the interval $(l, l + 1)$, where $l = d/(b - 1)$.

$$c_i = c \iff \frac{(x_{1,i-1} \cdots x_{1,0})_b + \cdots + (x_{n,i-1} \cdots x_{n,0})_b}{b^i} = \frac{c b^i + (a_{i-1} \cdots a_0)_b}{b^i}$$

Since $(x_{1,i-1} \cdots x_{1,0})_b, \ldots, (x_{n,i-1} \cdots x_{n,0})_b \in F$, we have $c \in nF - F \subset ((n-1)l - 1, (n-1)(l+1) + 1)$. Conversely, if $c \in ((n-1)l - 1, (n-1)(l+1) + 1) \cap \mathbb{Z}$, then $c+F \subset nF$. Therefore $s$ is the smallest integer strictly greater than $(n-1)l - 1$ and $t$ is the greatest integer strictly smaller than $(n-1)(l+1) + 1$, that is,

$$s = -[(n-1)(-l)], \quad t = [(n-1)(l+1)].$$

\[\square\]

**Theorem 2.**

$$p_{i,j} = \frac{1}{b^n} \sum_{k=0}^{\left\lfloor \frac{d(n-1)+1}{b} \right\rfloor} (-1)^k \binom{n+1}{k} \left( n + b(j + 1 - k) - d(n-1) - i - 1 \right), \quad i, j \in \Omega_n(b, \mathcal{D}).$$

**Proof.** This proof is essentially the same as that of Holte [5] for the case when $\mathcal{D} = \{0, 1, \ldots, b-1\}$. $p_{i,j}$ is the probability of $C_{k+1} = j$ under $C_k = i$ for some $k > 0$. $C_{k+1} = j$ and $C_k = i$ implies there exists a number $a \in \mathcal{D}$, such that,

$$j = \frac{i + X_{1,k} + X_{2,k} + \cdots + X_{n,k} - a}{b}.$$ 

We count the number $N$ of the solutions $(x_1, x_2, \ldots, x_n, a) \in \mathcal{D}^{n+1}$ of the equation

$$bj + a = i + x_1 + x_2 + \cdots + x_n.$$ 

This is equal to the number of solutions $(x_1, x_2, \ldots, x_n, y) \in \mathcal{D}^n \times \{0, 1, \ldots, b-1\}$ for the equation

$$bj + d + b - 1 - i = x_1 + x_2 + \cdots + x_n + y.$$ 

By adding $d$ to the both sides, $N$ is equal to the number of solutions $(x_1, \ldots, x_n, z) \in \mathcal{D}^{n+1}$ for

$$b(j + 1) + 2d - 1 - i = x_1 + x_2 + \cdots + x_n + z.$$ 

Thus, $N$ is the coefficient of $x^{b(j+1)+2d-1-i}$ in $(x^d + x^{d+1} + \cdots + x^{d+b-1})^{n+1}$. Since

$$\sum_{k=0}^{\left\lfloor \frac{d(n-1)+1}{b} \right\rfloor} (-1)^k \binom{n+1}{k} x^{b(k)k} \left( \sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right),$$

$$= x^{d(n+1)} \left( \frac{1 - x^b}{1 - x} \right)^{n+1}$$

$$= x^{d(n+1)} \left( \frac{n+1}{k} \right) x^{bk} \left( \sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right),$$

$$= x^{d(n+1)} \left( \frac{n+1}{k} \right) x^{bk} \left( \sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right),$$

$$= x^{d(n+1)} \left( \frac{n+1}{k} \right) x^{bk} \left( \sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right),$$

$$= x^{d(n+1)} \left( \frac{n+1}{k} \right) x^{bk} \left( \sum_{r=0}^{\infty} \binom{n+r}{n} x^r \right),$$
we have

\[ N = \sum_{d(n + 1) + bk + r = (j + 1) + 2d - 1 - i} (-1)^k \binom{n + 1}{k} \binom{n + r}{n} x^r \]

\[ = \sum_{b(j + 1) + 2d - 1 - i - d(n + 1) - bk \geq 0} (-1)^k \binom{n + 1}{k} \binom{n + b(j + 1) + 2d - 1 - i - d(n + 1) - bk}{n} x^r \]

\[ j + 1 + \left\lfloor \frac{-1 + i + d(n-1)}{b} \right\rfloor = j - \left\lfloor \frac{i + d(n-1)}{b} \right\rfloor, p_{i,j} = \frac{N}{b^r} \text{ and the theorem follows.} \]

**Lemma 2.** Let \( P = (\tilde{p}_{i,j}) \) be the matrix defined by (3) and let \( p \) be defined by (4). Then,

\[ \tilde{p}_{i,j} = \frac{1}{b^p} \sum_{r=0}^{j} (-1)^r \binom{n + 1}{r} \binom{n + b(j + 1 - r) + \frac{b-1}{p} - i}{n} \]

**Proof.** Let \( s \) be the minimal element of the state space \( \Omega_n(b,D) \) of the \( n \)-carry process. Then, recall that

\[ s = - \left\lfloor (n - 1) \frac{-d}{b-1} \right\rfloor = \left\lfloor (n - 1) \frac{d}{b-1} \right\rfloor. \]

Therefore,

\[ \tilde{p}_{i,j} = p_{i+s,j+s} = \frac{1}{b^p} \sum_{r=0}^{j} (-1)^r \binom{n + 1}{r} \binom{n + b(j + s + 1 - r) - d(n - 1) - i - s - 1}{n} \]

\[ = n + (b - 1) s + b(j + 1 - r) - d(n - 1) - i - 1 \]

\[ = n + (b - 1) \left\lfloor (n - 1) \frac{-d}{b-1} \right\rfloor + b(j + 1 - r) - d(n - 1) - i - 1 \]

\[ = n + (b - 1) \left( (n - 1) \frac{-d}{b-1} \right) + b(j + 1 - r) - d(n - 1) - i - 1 \]

\[ = n + (b - 1) \frac{(n-1)d - (n-1)d \pmod{b-1} + b(j + 1 - r) - d(n - 1) - i - 1}{b-1} \]

\[ = n + (b - 1) \frac{(b-1) - (n-1)d \pmod{b-1} + b(j + 1 - r) - d(n - 1) - i - 1}{b-1} \]

\[ = n + (b - 1) \frac{(n-1)(-d)}{b-1} + b(j + 1 - r) - i \]

\[ = \begin{cases} n + (b - 1) \frac{(n-1)(-d)}{b-1} + b(j + 1 - r) - i & \frac{(n-1)(-d)}{b-1} \in \mathbb{Z}, \\ n + (b - 1) + b(j + 1 - r) - i & \frac{(n-1)(-d)}{b-1} \notin \mathbb{Z}. \end{cases} \]

Here, \( x \pmod{N} \) denotes the integer \( y \in \{0, 1, \ldots, N - 1\} \) such that \( x - y \in N \mathbb{Z} \). Then, we calculate a common upper bound of the range of the summation in (5):

\[ n + b(j + 1 - r) + \frac{b-1}{p} - i \geq n \iff r \leq j + \frac{b-1}{pb} - \frac{i}{p}. \]

Since \( p \geq 1 \) and \( b > 1 \), \( j + \frac{b-1}{pb} - \frac{i}{p} \leq j \). \qed
2.2 Generalized Eulerian numbers

Lemma 3. \[ v^{(p)}_{i,n+1} = 0. \]

Proof. \[ v^{(p)}_{i,n+1}(n) = \sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} [p(n+1-r)+1]^{n-i} \]

This is a linear combination of \[ \sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} r^k = 0, \quad k = 0, 1, \ldots, n. \]

Lemma 4. \[ v^{(p)}_{i,j}(n) = [p(n+1-j)-1]v^{(p)}_{i,j-1}(n-1) + (pj+1)v^{(p)}_{i,j}(n-1). \] (7)

Proof. The first term \( T_1 \) of right hand side of (7) can be rewritten as

\[ T_1 = \sum_{k=1}^{j} (-1)^{k-1} \binom{n}{k-1} [p(n+1-j)-1][p(j-k)+1]^{n-1-i}, \]

and the second term \( T_2 \)

\[ T_2 = \sum_{k=1}^{j} (-1)^k \binom{n}{k} (pj+1)[p(j-k)+1]^{n-1-i} + (pj+1)(pj+1)^{n-1-i}. \]

Thus the right hand side of (7) is

\[ T_1 + T_2 = \sum_{k=1}^{j} (-1)^k \left\{ -\binom{n}{k-1} [p(n+1-j)-1] + \binom{n}{k} (pj+1) \right\} [p(j-k)+1]^{n-1-i} + (pj+1)^{n-i} \]

\[ \sum_{k=1}^{j} (-1)^k \binom{n+1}{k} [p(j-k)+1]^{n-i} + (pj+1)^{n-i} \]

\[ = v^{(p)}_{i,j}(n). \]

Lemma 5. \[ \sum_{j=0}^{n} v^{(p)}_{i,j}(n) = \begin{cases} p^n n! & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \]
Proof. By Lemma 4, we have

\[
\sum_{j=0}^{n} v_{i,j}^{(p)}(n) = \sum_{j=0}^{n} \left\{ [p(n+1-j) - 1]v_{i,j-1}^{(p)}(n-1) + (pj+1)v_{i,j}^{(p)}(n-1) \right\}
\]

\[
= \sum_{j=0}^{n-1} [p(n-j) - 1]v_{i,j}^{(p)}(n-1) + \sum_{j=0}^{n-1} (pj+1)v_{i,j}^{(p)}(n-1)
\]

\[
= \sum_{j=0}^{n-1} pnv_{i,j}^{(p)}(n-1)
\]

\[
= pn \sum_{j=0}^{n-1} v_{i,j}^{(p)}(n-1)
\]

\[
= \left\{ \begin{array}{ll}
p^n n! & \text{if } i = 0, \\
0 & \text{if } i > 0. 
\end{array} \right.
\]

The following Proposition 1 shows a symmetry of the generalized Eulerian number.

Proposition 1. Let \( n \) be a positive integer. Then

\[
v_{i,n-1-j}^{(1)} = (-1)^i v_{i,j}^{(1)}(n) \quad \text{for } 0 \leq j \leq n-1.
\]

Let \( p > 1 \) and \( p^* \) be the real number satisfying

\[
\frac{1}{p} + \frac{1}{p^*} = 1.
\]

Then,

\[
v_{i,n-j}^{(p^*)}(n) = (-1)^i \left( \frac{p^*}{p} \right)^{n-i} v_{i,j}^{(p)}(n) \quad \text{for } 0 \leq j \leq n.
\]

Proof. We show the proof only for the second part. The first part can be proved in the same manner.
If \( p > 1 \) then \( p^* = p/(p - 1) \).

\[
v_{i,n-j}^{(p^*)}(n) = \sum_{k=0}^{n-j} (-1)^k \left( \begin{array}{c} n+1 \\ k \end{array} \right) (p^*(n-j-k)+1)^{n-i}
= \sum_{k=n+1-j}^{n+1} (-1)^k \left( \begin{array}{c} n+1 \\ k \end{array} \right) (p^*(n-j-k)+1)^{n-i}
= \sum_{k=0}^{j} (-1)^{n+1-k'} \left( \begin{array}{c} n+1 \\ n+1-k' \end{array} \right) (p^*(n-j-(n+1-k'))+1)^{n-i}
= \sum_{k=0}^{j} (-1)^{n+1-k'} \left( \begin{array}{c} n+1 \\ k' \end{array} \right) \left( \frac{p}{p-1} (k' - j - 1) + 1 \right)^{n-i}
= (-1)^n \left( \frac{1}{p-1} \right)^{n-i} \sum_{k=0}^{j} (-1)^k \left( \begin{array}{c} n+1 \\ k \end{array} \right) (p(j-k+1) - p + 1)^{n-i}
= \frac{(-1)^i}{(p-1)^{n-i}} \sum_{k=0}^{j} (-1)^k \left( \begin{array}{c} n+1 \\ k \end{array} \right) (p(j-k) + 1)^{n-i}
\]

\[\Box\]

2.3 Left eigenvectors

Proof of Theorem 1. The proof is essentially the same as that of Holte [5]. It suffices to show that

\[
\sum_{k=0}^{m-1} v_{i,k}^{(p)}(n) \tilde{p}_{k,j} = \frac{1}{b^0} v_{i,j}^{(p)}(n).
\]

We prove the theorem for the case in which \( p \neq 1 \), i.e., \( m = n + 1 \), and the other case can be proved in the same manner. By Lemma 2 we have

\[
\tilde{p}_{k,j} = \frac{1}{b^0} \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n+1 \\ r \end{array} \right) \left( n + K(j,r) - k \right),
\]

where we put \( K(j,r) = b(j - r) + \frac{b-1}{p} \) for the simplicity of the notation.

\[
\sum_{k=0}^{n} v_{i,k}^{(p)}(n) \tilde{p}_{k,j} = \sum_{k=0}^{n} \frac{1}{b^0} \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n+1 \\ r \end{array} \right) \left( n + K(j,r) - k \right) v_{i,k}^{(p)}(n)
= \frac{1}{b^0} \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n+1 \\ r \end{array} \right) \sum_{k=0}^{n} \left( n + K(j,r) - k \right) v_{i,k}^{(p)}(n)
= \frac{1}{b^0} \sum_{r=0}^{j} (-1)^r \left( \begin{array}{c} n+1 \\ r \end{array} \right) \{pK(j,r) + 1\}^{n-i}.
\]
The third equality in the above transformation is derived as follows: First recall that \( v_{i,k}(n) \) is the coefficient of \( x^k \) in
\[
\left( \sum_{\nu=0}^{n+1} (-1)^\nu \binom{n+1}{\nu} x^\nu \right) \left( \sum_{\mu=0}^\infty (p\mu + 1)^{n-i} x^\mu \right) = (1 - x)^{n+1} \left( \sum_{\mu=0}^\infty (p\mu + 1)^{n-i} x^\mu \right),
\]
and
\[
\frac{1}{(1 - x)^{n+1}} = \sum_{k=0}^\infty \binom{n+k}{n} x^k.
\]
Therefore, \( \sum_{k=0}^{K(j,r)} \binom{n+K(j,r)-k}{n} v_{i,k}(n) \) is the coefficient of \( x^{K(j,r)} \) in \( \sum_{\mu=0}^\infty (p\mu + 1)^{n-i} x^\mu \).

It can be easily confirmed that \( pK(j, r) + 1 = b(p(j - r) + 1) \) which completes the proof. \( \square \)

Theorem \( \square \) gives a way of finding a numeration system \((b, D)\) whose \(n\)-carry process has the stationary distribution of the form
\[
\pi = (\pi(s), \pi(s+1), \ldots, \pi(s+m-1)) = \frac{1}{p^n n!} \left( \begin{array}{c} n \\ 0 \\ \vdots \\ m-1 \end{array} \right)_p
\]
for any given \( n \) and rational \( p \geq 1 \). For instance, if \( p = \frac{K}{L} \) where \( K \) and \( L \) are coprime positive integers such that \( K \geq L \), then we can choose \( b \) and \( d \) as
\[
b = (n-1)K + 1, \quad d = -L.
\]

**Example 2.** We construct numeration systems for \( p = 2 \) and \( 5/3 \).

| \( n \) | \( b \) | \( D \) | \( P \) | \( V \) |
|---|---|---|---|---|
| 2 | 3 | \{-1, 0, 1\} | \( \frac{1}{3} \begin{pmatrix} 3 & 6 & 0 \\ 1 & 7 & 1 \\ 0 & 6 & 3 \end{pmatrix} \) | \( \frac{1}{3} \begin{pmatrix} 1 & 6 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \) |
| 3 | 5 | \{-1, 0, 1, 2, 3\} | \( \frac{1}{5} \begin{pmatrix} 10 & 80 & 35 & 0 \\ 4 & 68 & 52 & 1 \\ 1 & 52 & 68 & 4 \\ 0 & 35 & 80 & 10 \end{pmatrix} \) | \( \frac{1}{5} \begin{pmatrix} 1 & 23 & 23 & 1 \\ 1 & 5 & -5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix} \) |
| 4 | 7 | \{-1, 0, 1, 2, 3, 4, 5\} | \( \frac{1}{7} \begin{pmatrix} 35 & 826 & 1330 & 210 & 0 \\ 15 & 640 & 1420 & 325 & 1 \\ 5 & 470 & 1451 & 470 & 5 \\ 1 & 325 & 1420 & 640 & 15 \\ 0 & 210 & 1330 & 826 & 35 \end{pmatrix} \) | \( \frac{1}{7} \begin{pmatrix} 1 & 76 & 230 & 76 & 1 \\ 1 & 22 & 0 & -22 & -1 \\ 1 & 4 & -10 & 4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \) |

Table 1: Amazing matrices with \( p = 2 \)
\[
\begin{array}{c|c|c|c|c}
 n & b & D & P & V \\
\hline
2 & \{-3, -2, \ldots, 2\} & \frac{1}{p^2} \begin{pmatrix} 10 & 25 & 1 \\ 6 & 27 & 3 \\ 3 & 27 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 37/9 & 4/9 \\ 1 & -1/3 & -2/3 \\ 1 & -2 & 1 \end{pmatrix} \\
3 & \{-3, -2, \ldots, 7\} & \frac{1}{p^3} \begin{pmatrix} 84 & 804 & 439 & 4 \\ 56 & 745 & 520 & 10 \\ 35 & 676 & 600 & 20 \\ 20 & 600 & 676 & 35 \end{pmatrix} & \begin{pmatrix} 1 & 404/27 & 311/27 & 8/27 \\ 1 & 28/9 & -11/3 & -4/9 \\ 1 & -4/3 & -1/3 & 2/3 \\ 1 & -3 & 3 & -1 \end{pmatrix} \\
4 & \{-3, -2, \ldots, 12\} & \frac{1}{p^4} \begin{pmatrix} 715 & 20176 & 37390 & 7240 & 15 \\ 495 & 18000 & 38326 & 8680 & 35 \\ 330 & 15900 & 38960 & 10276 & 70 \\ 210 & 13900 & 39280 & 12020 & 126 \\ 126 & 12020 & 39280 & 13900 & 210 \end{pmatrix} & \begin{pmatrix} 1 & 3691/81 & 8891/81 & 2321/81 & 16/81 \\ 1 & 377/27 & -31/9 & -101/9 & -8/27 \\ 1 & 19/9 & -61/9 & 29/9 & 4/9 \\ 1 & -7/3 & 1 & 1 & -2/3 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix} \\
\end{array}
\]

Table 2: Amazing matrices with \( p = \frac{5}{3} \)

2.4 Sum of independent uniform random variables

Proof of Corollary 1 Let \( X_{i,j} (i = 1, 2, \ldots, n, j = 1, 2, \ldots) \) be independent random variables each distributed uniformly on \( D \). Then, for each integer \( k \geq 1 \), the random variables

\[
X_i^{(k)} = \frac{X_{i,1}}{b} + \frac{X_{i,2}}{b^2} + \cdots + \frac{X_{i,k}}{b^k}, \quad i = 1, 2, \ldots, n
\]

are independent random variables uniformly distributed over the set

\[
R_k = \left\{ \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots + \frac{x_k}{b^k} \mid x_i \in D \right\}.
\]

Therefore

\[
\lim_{k \to \infty} \Pr(X_i^{(k)} \in [a, b]) = b - a, \quad \text{for } a \geq b \in [l, l+1] \text{ and } i = 1, 2, \ldots, n.
\]

Let \( X_1, X_2, \ldots, X_n \) be independent random variables each of which is distributed uniformly over \([l, l+1]\). Then, for any integer \( c \in \Omega \),

\[
\lim_{k \to \infty} \Pr(X_1^{(k)} + X_2^{(k)} + \cdots + X_n^{(k)} \in c + [l, l+1]) = \Pr(X_1 + X_2 + \cdots X_n \in c + [l, l+1]).
\]

Since

\[
\lim_{k \to \infty} \Pr(X_1^{(k)} + X_2^{(k)} + \cdots + X_n^{(k)} \in c + [l, l+1]) = \pi(c),
\]

we have

\[
\pi(c) = \Pr(X_1 + X_2 + \cdots X_n \in c + [l, l+1]).
\]

Let \( p > 1 \) be a non-integral rational number, and suppose that we choose \( b \) and \( d \) so that

\[
p = \frac{1}{(n-1)(-l)}
\]
holds, which is always possible by (8). Then, we have
\[
\frac{1}{p} = (n-1)(-l) - \lfloor (n-1)(-l) \rfloor \iff nl + \frac{1}{p} = l - \lfloor (n-1)(-l) \rfloor = l + 1 - \lfloor (n-1)(-l) \rfloor \\
\iff nl + \frac{1}{p} + [k-1,k] = (s+k) + [l,l+1].
\]

Let \( Y_1, Y_2, \ldots, Y_n \) be independent random variables each uniformly distributed on the unit interval \([0,1] \).
\[
\Pr(Y_1 + Y_2 + \ldots + Y_n \in \frac{1}{p} + [k-1,k]) = \Pr((Y_1 + l) + \ldots + (Y_n + l) \in nl + \frac{1}{p} + [k-1,k])
\]
\[
= \Pr(X_1 + X_2 + \ldots + X_l \in nl + \frac{1}{p} + [k-1,k])
\]
\[
= \Pr(X_1 + X_2 + \ldots + X_l \in (s+k) + [l,l+1])
\]
\[
= \pi(s+k)
\]
\[
= \frac{1}{p^n n!} \left\langle \frac{n}{k} \right\rangle^p.
\]

When \( p \) is an integer, the statement is proved by a similar argument. Since both sides of the equation (5) are continuous function of \( p \), the statement of the theorem holds when \( p \) is irrational. \(\square\)

**Remark 4.** Corollary 1 gives a different proof and a new interpretation for Proposition 1 for the case \( i = 0 \).

### 3 Negative base

In this section, we consider carries processes over the numeration systems with the negative bases. Let \( b > 1 \) be an integer and \( \mathcal{D} = \{d, d + 1, \ldots, d + b - 1\} \) a set of integers containing 0. Suppose an integer \( x \) can be represented in the form:
\[
x = (x_l x_{l-1} \cdots x_0)_{-b} = x_l(-b)^l + x_{l-1}(-b)^{l-1} + \cdots + x_1(-b) + x_0,
\]
where \( l \) is a non-negative integer and \( x_l \neq 0 \). Then this representation is unique and the set
\[
\{(x_l x_{l-1} \cdots x_0)_{-b} | l \geq 0, x_k \in \mathcal{D}\}.
\]
is closed under the addition. We can define the \( n \)-carry process over the numeration system \((-b, \mathcal{D})\) in the same manner as the positive base case. Let \( \{X_{i,j}\}_{1 \leq i \leq n, j \geq 0} \) be a set of i.i.d. random variables each distributed uniformly on \( \mathcal{D} \). Then the carries process \((C_0, C_1, C_2, \ldots)\) is defined as follows:
\[
\Pr(C_0 = 0) = 1 \text{ and } \quad C_i = \frac{C_{i-1} + X_{1,j-1} + \cdots + X_{n,j-1} - A_{i-1}}{-b} \text{ for } i > 0,
\]
where \( A_j \) is \( \mathcal{D} \)-valued. These carries processes have the properties similar to those of the positive base cases. The proofs of the following Lemma 6, Theorem 3, and Lemma 7 are similar to those of Lemma 1, Theorem 2, and Lemma 2. The proof of Theorem 4 needs an additional combinatorial argument.
Lemma 6. Let $\Omega = \Omega_n(-b, D)$ be the state space of the $n$-carry process over the numeration system $(-b, D)$. Then, $\Omega = \{s, s + 1, \ldots, t\}$ with

$$s = -[(n - 1)(n)] = [(n - 1)|l], \ t = [(n - 1)(l + 1)],$$

where $l = -(d - b)/(b + 1)$. Therefore, the size of the state space $\Omega$ is

$$\#\Omega = \begin{cases} n + 1 & (n - 1)|l \notin \mathbb{Z}, \\ n & (n - 1)|l \in \mathbb{Z}. \end{cases}$$

Theorem 3. Let $i, j \in \Omega(-b, D)$. Then the transition probability $p_{i,j} = \Pr(C_{t+1} = j | C_{t} = i)$ for $t > 0$ is

$$p_{i,j} = \frac{1}{b^n} \left[ \frac{-1 + (1 - n)d}{b} \right]_{0}^{n} (-1)^r \binom{n + 1}{r} \binom{n - b(j - 1 + r) - i - 1 + (1 - n)d}{n}.$$ 

We denote

$$\tilde{p}_{i,j} = p_{i+s,j+s},$$

where $s$ is the minimal element of the state space $\Omega(-b, D)$ of the $n$-carry process over $(-b, D)$.

Lemma 7. Let $p$ be defined by

$$p = \begin{cases} \frac{1}{\{1\}} (n - 1)|l \notin \mathbb{Z} & (\Leftrightarrow m = n + 1), \\ 1 & (n - 1)|l \in \mathbb{Z} & (\Leftrightarrow m = n). \end{cases}$$

where $l = -(d - b)/(b + 1)$ and $m = \#\Omega_n(-b, D)$. Then,

$$\tilde{p}_{i,j} = \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n + 1}{r} \binom{n + b(n + 1 - j - r) - \frac{b+1}{p} - i}{n}. $$

Theorem 4. Let $P = (\tilde{p}_{i,j})_{0 \leq i, j \leq m-1}$ be the transition probability matrix of the $n$-carry process over the numeration system $(-b, D)$, and $m = \#\Omega(-b, D)$ be the size of the state space. Let $p$ be defined by

$$p = \begin{cases} \frac{1}{\{1\}} (n - 1)|l \notin \mathbb{Z} & (\Leftrightarrow m = n + 1), \\ 1 & (n - 1)|l \in \mathbb{Z} & (\Leftrightarrow m = n). \end{cases}$$

where $l = -(d - b)/(b + 1)$ and $\{x\} = x - \lfloor x \rfloor$. Let $V = (V_{i,j}^{(p)})_{0 \leq i, j \leq m-1}$. Then, we have

$$VPV^{-1} = \text{diag}(1, (-b)^{-1}, \ldots, (-b)^{m-1}).$$

In particular, the $n$-carry process over $(-b, D)$ has the stationary distribution

$$\pi = (\pi(s), \pi(s + 1), \ldots, \pi(s + m - 1)) = \frac{1}{p^m n!} \left( \begin{array}{c} n \\ 0 \\ 0 \\ \vdots \\ n \\ m - 1 \end{array} \right).$$

Proof. It suffices to show that

$$\sum_{k=0}^{m-1} V_{i,k}^{(p)}(n) \tilde{p}_{k,j} = \frac{1}{(-b)^{n}} V_{i,j}^{(p)}(n).$$

(10)
Recall that
\[ \tilde{p}_{k,j} = \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n+1}{r} \binom{n+K(j,r)-k}{n}, \]
where, we put \( K(j,r) = b(n+1-j-r) - \frac{b+1}{b} \) for the simplicity of the notation. Therefore,

\[
\text{L.H.S. of (10)} = \sum_{k=0}^{m} \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n+1}{r} \binom{n+K(j,r)-k}{n} v_{i,k}(p)(n) \\
= \frac{1}{b^n} \sum_{r=0}^{n-j} \sum_{k=0}^{K(j,r)} (-1)^r \binom{n+1}{r} \binom{n+K(j,r)-k}{n} v_{i,k}(p)(n) \\
= \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n+1}{r} K(j,r) \sum_{k=0}^{K(j,r)} \binom{n+K(j,r)-k}{n} v_{i,k}(p)(n) \\
= \frac{1}{b^n} \sum_{r=0}^{n-j} (-1)^r \binom{n+1}{r} [pK(j,r) + 1]^n - i \\
\hspace{2cm} \text{(We use the same argument as in the proof of Theorem 2)} \\
= \frac{1}{b^n} \sum_{r'=j+1}^{n+1} (-1)^{n+1-r'} \binom{n+1}{r+1-r'} [pK(j,n+1-r') + 1]^n - i \\
\hspace{2cm} \text{(We use the transformation } r' = n+1-r.) \\
= \frac{1}{b^n} \sum_{r=j+1}^{n+1} (-1)^{n+1-r} \binom{n+1}{r} [-b(p(j-r') + 1)]^n - i.
\]

\( \square \)

4 Concluding remarks

Many natural questions arise.

In the forthcoming paper, we will show a formula for the right eigenvectors, which involves Stirling numbers.

Our theorems hold only for the numeration systems \((b,D)\), where \(D\) consists of consecutive integers containing 0. For example, the 2-carry process over \((3,\{-1,0,4\})\) has rather large state space

\[ \Omega_2(3,\{-1,0,4\}) = \{-5,-4,\ldots,4\}, \]
and the transition probability matrix

\[
P = \frac{1}{9} \begin{pmatrix}
1 & 2 & 0 & 3 & 0 & 2 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 2 & 1 & 4 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 3 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 3 & 0 & 2 & 1 & 0 & 0 \\
0 & 2 & 0 & 0 & 2 & 1 & 4 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 3 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 0 & 2 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & 2 & 1 & 4 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 3 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 0 & 3 & 0 & 2 & 1
\end{pmatrix},
\]
whose characteristic polynomial \( \det(xI - P) \) is

\[
(x - 1)(3x - 1)(9x - 1)(531441x^7 - 19683x^5 + 5103x^4 - 1944x^3 - 297x^2 + 24x + 2).
\]

Although there are eigenvalues of the form \( 1, 1/3, 1/3^2 \), we have no knowledge on the rest of the eigenvalues. The difficulty comes from the geometric structure of the fundamental domain \( \{(x_l | x_{l-1} \ldots x_0)_b | l \geq 0, x_k \in \mathbb{D}\} \).

Diaconis and Fulman \([2,3]\) shows the relation between carries processes and shufflings for the case when \( p = 1 \) and \( 2 \). We do not know whether there exist some shufflings corresponding to the cases with \( p \neq 1, 2 \).

References

[1] C. Chow, I. M. Gessel, On the descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math., 38 (2007) 275–301.

[2] P. Diaconis, J. Fulman, Carries, shuffling, and an amazing matrix, Amer. Math. Monthly, 116 (2009) 780–803.

[3] P. Diaconis, J. Fulman, Carries, shuffling, and symmetric functions, Adv. App. Math., 43 (2009) 176–196.

[4] W. Feller, An Introduction to Probability Theory and its Applications, volume II, 2nd edition, John Wiley & Sons, Inc., New York-London-Sydney 1971.

[5] J. Holte, Carries, combinatorics, and an amazing matrix, Amer. Math. Monthly, 104 (1997) 138–149.

[6] P. A. MacMahon, The divisors of numbers, Proc. London Math. Soc., 19 (1921) 305–340.

[7] J.-C. Novelli, J.-Y. Thibon, Noncommutative symmetric functions and an amazing matrix, Adv. App. Math., 48 (2012) 528–534.

[8] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. http://oeis.org/A060187.