Six-vertex model on a finite lattice: integral representations for nonlocal correlation functions

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Abstract. We consider the problem of calculation of correlation functions in the six-vertex model with domain wall boundary conditions. To this aim, we formulate the model as a scalar product of off-shell Bethe states, and, by applying the quantum inverse scattering method, we derive three different integral representations for these states. By suitably combining such representations, and using certain antisymmetrization relation in two sets of variables, it is possible to derive integral representations for various correlation functions. In particular, focusing on the emptiness formation probability, besides reproducing the known result, obtained by other means elsewhere, we provide a new one. By construction, the two representations differ in the number of integrations and their equivalence is related to a hierarchy of highly nontrivial identities.

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1. Introduction

The theory of quantum integrable models can be seen as developing mainly in two directions (see, e.g., [1]). One is related to various (algebraic, analytic and geometric) aspects of integrability (such as Bethe ansatz, Yang-Baxter relation, Baxter T-Q equation, Quantum Groups, etc) of quantum models and mostly deals with the problem of diagonalization of quantum integrals of motion (transfer matrices). The other is related to applications (in gauge and string theories, condensed matters, algebraic combinatorics, probability, etc.) and deals mostly with the problem of evaluation of correlation functions. The latter uses heavily results from the former, but despite significant progress of the theory in general, it still provides numerous challenging problems. One of them concerns the calculation of correlation functions of models with broken translational invariance (e.g., due to the boundary conditions), possibly in cases where keeping finite the size of the system is of importance for extracting interesting (physical or mathematical) information.

A prototypical example is provided by the six-vertex model with domain wall boundary conditions. Current interest in the model is mostly motivated by the occurrence of phase separation [2–5], which recently triggered a number of numerical studies [6–8] and analytical results [9–12]. The model is also of relevance for quantum quenches in the closely related Heisenberg XXZ quantum spin chain [13–16], and for $\mathcal{N} = 4$ super-Yang-Mills theory [17–19]. As for correlation functions of the six-vertex model in the bulk (or with periodic boundary conditions), many notable important results were obtained for the XXZ chain (see, among others, papers [20–28] and references therein). Although these results cannot be directly applied to the case of domain wall boundary conditions, some aspects, such as multiple integral representations, indeed do prove useful.

Historically, correlation functions of the six-vertex model with domain wall boundary conditions were first studied close to the boundary, where the problem notably simplifies [29–33]. Some results and techniques developed in these studies allowed to evaluate a bulk correlation function, the so-called emptiness formation probability, as a multiple integral [34,35]. Subsequent study of such integral representation made it possible to derive an exact analytic expression for the spatial curve separating ferroelectric order from disorder (the so-called ‘arctic’ curve) [36–39].

In [40], in order to extend previous progress with the emptiness formation probability to other correlation functions, a method for their systematic calculation was proposed. A specific nonlocal correlation function, named row configuration probability, was introduced, in terms of which various other (nonlocal and local) correlation functions can in principle be obtained by suitable summations over position parameters. The row configuration probability can be viewed as a product of two
factors, which are in fact components of off-shell Bethe states. These states are complementary to each other in the sense that they are build over different pseudo-vacuum states (all-spins-up and all-spins-down); their scalar product is exactly the partition function. The main observation in [40] is that these two factors can be represented as multiple integrals involving the same number of integrations.

The main difficulty of the proposed method concerns the possibility of performing suitable summations and integrations to simplify the resulting expressions. As already noted in [40], to reproduce the previously obtained integral representation for the emptiness formation probability, one has to deal with a nontrivial problem of antisymmetrization over two sets of integration variables. In this respect, the existence of some useful antisymmetrization identity allows to show that indeed the proposed approach can be useful in practice [41].

In the present paper, we overview the method of [40,41] and provide some improvements which allow us to obtain further results. We start by formulating the model as a scalar product of off-shell Bethe states; applying the Quantum Inverse Scattering Method (QISM) [42] (see also [1] and references therein), we derive three constructively different integral representations for the components of the off-shell Bethe states.

Next, we show how such representations can be combined to build various local or nonlocal correlation functions. In particular, focusing on the emptiness formation probability, we combine two out of the three available representations and show how the problem of evaluating some intricate multiple sums and integrals can be tackled so that one can recover the multiple integral representation first derived in [34]. The antisymmetrization relation proven in [41] plays a crucial role in this alternative derivation.

Finally, by combining a different pair of representations for the components of the off-shell Bethe states, we derive an alternative multiple integral representation for the emptiness formation probability. The existence of two essentially different integral representations for the same object leads to a hierarchy of identities involving the (generating function of the) one-point boundary correlation function. Further study of such identities is required for a full understanding of their implications.

The paper is organized as follows. In the next section, after giving some definitions and notations, we sketch the strategy of our derivation, and set up QISM in the context of the considered problem. The core calculation of the components of the off-shell Bethe states for the inhomogeneous model is contained in section 3. In section 4 we perform the homogeneous limit, and obtain representations in terms of multiple integrals. In section 5, starting from a suitable combination of two such representations, we reproduce the previously obtained integral representation for the emptiness probability. In section 6 we show that the same procedure, when applied to a different combination of components of off-shell Bethe states, leads to an alternative and essentially different integral representation for the emptiness formation probability.

2. The model

In this section we define the model, introduce some nonlocal correlation functions of interest, and formulate the problem in the framework the QISM.
2.1. The partition function. The six-vertex model with domain wall boundary conditions is a model of arrows lying on the edges of a square lattice with $N$ horizontal and $N$ vertical lines. Arrow configurations are constrained by the ‘ice-rule’, requiring each vertex to have the same number of incoming and outgoing arrows [43]. Boltzmann weights are assigned to the six possible vertex configurations of arrows allowed by the ice-rule. With no loss of generality, we require the model to be invariant under reversal of all arrows. We thus have three distinct Boltzmann weights, denoted $a$, $b$, $c$. The domain wall boundary conditions are defined as follows: all arrows on the left and right boundaries are outgoing while all arrows on the top and bottom boundaries are incoming, see figure [1].

To use QISM for calculations we will consider the inhomogeneous version of the model, in which the weights of the vertex being at the intersection of $k$th horizontal line and $\alpha$th vertical line are [44]

\[
\begin{align*}
    a_{\alpha k} &= a(\lambda_\alpha, \nu_k), \\
    b_{\alpha k} &= b(\lambda_\alpha, \nu_k), \\
    c_{\alpha k} &= c,
\end{align*}
\]  

(2.1)

where

\[
\begin{align*}
    a(\lambda, \nu) &= \sin(\lambda - \nu + \eta), \\
    b(\lambda, \nu) &= \sin(\lambda - \nu - \eta), \\
    c &= \sin 2\eta,
\end{align*}
\]  

(2.2)

and we enumerate vertical lines (labelled by Greek indices) from right to left, and horizontal lines (labelled by Latin indices) from top to bottom. The parameters $\lambda_1, \ldots, \lambda_N$ are assumed to be all different; the same is assumed about $\nu_1, \ldots, \nu_N$. To obtain the homogeneous model, after applying QISM, we set these parameters equal within each set, $\lambda_\alpha = \lambda$ and $\nu_k = \nu$ where, with no loss of generality, we can choose $\nu = 0$.

The partition function of the inhomogeneous model is defined as the sum over all possible configurations, each configuration being assigned its Boltzmann weight,
which is the product of all vertex weights over the lattice,

$$Z_N = \sum_C \prod_{\alpha=1}^N \prod_{k=1}^N w_{\alpha k}(C).$$

Here $w_{\alpha k}(C)$ takes values $w_{\alpha k}(C) = a_{\alpha k}, b_{\alpha k}, c_{\alpha k}$, depending on the configuration $C$. Because of (2.1), $Z_N = Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N)$ where the $\lambda$'s and $\nu$'s are regarded as ‘variables’; $\eta$ is regarded as a parameter (having the meaning of a ‘coupling constant’) and it is often omitted in notations. In QISM the dependence on $\lambda$’s and $\nu$’s play an important role (in particular, $Z_N$ is invariant under permutations within each set of variables).

The partition function may be written in determinantal form, as established by Izergin and Korepin in (2.3):

$$Z_N = \frac{\prod_{\alpha=1}^N \prod_{k=1}^N a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)}{\prod_{1 \leq \alpha < \beta \leq N} \prod_{1 \leq j < k \leq N} d(\nu_j, \nu_k)} \det M. \tag{2.3}$$

Here $M$ is an $N$-by-$N$ matrix with entries

$$M_{\alpha k} = \varphi(\lambda_\alpha, \nu_k), \quad \varphi(\lambda, \nu) = \frac{c}{a(\lambda, \nu)b(\lambda, \nu)}, \tag{2.4}$$

while $a(\lambda, \nu)$, $b(\lambda, \nu)$ and $c$ are defined in (2.2), and function $d(\lambda, \lambda')$, standing in the prefactor of (2.3), is

$$d(\lambda, \lambda') := \sin(\lambda - \lambda'). \tag{2.5}$$

In the homogeneous limit, i.e., when $\lambda_\alpha = \lambda$ and $\nu_k = 0$, expression (2.3) becomes

$$Z_N = \frac{[\sin(\lambda - \eta)\sin(\lambda + \eta)]^{N^2}}{\prod_{n=1}^{N-1} (n!)^2} \det N, \tag{2.6}$$

where the $N$-by-$N$ matrix $N$ has entries

$$N_{\alpha k} = \partial_{\lambda}^{+k-2} \varphi(\lambda), \quad \varphi(\lambda) := \varphi(\lambda, 0) = \frac{\sin 2\eta}{\sin(\lambda - \eta)\sin(\lambda + \eta)}. \tag{2.7}$$

For a detailed proof of (2.3) and (2.6), see [47]. For an alternative derivation, see [30, 34].

2.2. Off-shell Bethe states and correlation functions. An interesting property of the domain-wall six-vertex model, directly following from its peculiar boundary conditions and from the ice-rule, is that on the $s$th row (i.e., on the $N$ vertical edges between the $s$th and the $(s+1)$th horizontal lines, counted from the top, in our conventions) there are exactly $s$ arrows pointing up. It is thus natural to describe configurations on the $s$th row in terms of the positions $r_1, \ldots, r_s$, of such $s$ up arrows (counted from the right, and with $1 \leq r_1 < \cdots < r_s \leq N$, see figure 2). Note that these configurations can equivalently be described in terms of the complementary set of integers $\bar{r}_1, \ldots, \bar{r}_{N-s}$, denoting the position of the $N - s$ down arrows.

Let us now suppose we are assigned a given $s$th-row configuration on the $N \times N$, described by the positions $r_1, \ldots, r_s$ of the up arrows, and let us imagine cutting all the vertical edges between the $s$th and the $(s+1)$th horizontal lines of the $N \times N$ lattice, thus separating it into two smaller lattices, see figure 2 as a result, we obtain an upper lattice with $s$ horizontal and $N$ vertical lines, and a lower lattice with $N - s$ horizontal and $N$ vertical lines. The boundary conditions on these two
lattices are naturally inherited from the depicted procedure, being of domain wall type on three sides, and with $s$ up arrows at positions $r_1, \ldots, r_s$ (here, $N = 7$, $s = 2$, $r_1 = 2$, $r_2 = 4$); (b) The corresponding top and bottom portions resulting from splitting the original lattice in correspondence of the $s$th row.

We will denote by $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$ the partition functions of the six-vertex model on the upper and lower sublattices, respectively. The partition functions $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$ can be viewed as components of off-shell Bethe states in the context of the algebraic Bethe ansatz. Besides being of relevance on their own right, they find application in further investigations of the limit shape of the model [48], and in combinatorics [49]. But our main interest here is in the fact that $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$ can be used, if suitably combined, as building blocks to compute some useful correlation functions.

Our goal in the present paper is therefore twofold. First, to derive some convenient representations for the partition functions $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$. Second, to devise how they should be combined to evaluate more sophisticated correlation functions.

As for the first goal, we will rely on the QISM [42], and on some additional ingredients, developed in [32,34], which allows to obtain multiple integral representations for these quantities. Note that some representation for, say, $Z_{r_1, \ldots, r_s}^{\text{top}}$, has been known for quite a long time: we are referring to the ‘coordinate wavefunction’ representation, that follows from the equivalence of the algebraic and coordinate versions of the Bethe ansatz [50], see appendix A for details. However, for reasons that will become clear below, in relation to our second goal, such ‘coordinate wavefunction’ representation is not sufficient for our purposes. In the following we will work out two additional representations for the components of the off-shell Bethe states.

The partition functions $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$, when considered within QISM, depends on the spectral parameters $\lambda_1, \ldots, \lambda_N$, and $\nu_1, \ldots, \nu_s$, and on $\lambda_1, \ldots, \lambda_N$, $\nu_{s+1}, \ldots, \nu_N$, respectively. As functions of the spectral parameters, these two partition functions can be related to each other, due to the crossing symmetry of the six-vertex model.
We recall that the crossing symmetry is the symmetry of the Boltzmann weights under reflection of the vertex (together with the orientation of the arrows on its edges) with respect to the vertical (or horizontal) line, with the simultaneous interchange of $a_{\alpha k}$ and $b_{\alpha k}$. This interchange can be implemented by the substitution
\[ \lambda_{\alpha} \mapsto \pi - \lambda_{\alpha}, \quad \nu_{\alpha} \mapsto -\nu_{\alpha}, \]
in terms of the spectral parameters. It is easy to see that the crossing symmetry of the model implies the relation
\[ Z_{r_1, \ldots, r_s}^{\text{top}}(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_s) = Z_{r_1, \ldots, r_N-r_s}^{\text{bot}}(\pi - \lambda_1, \ldots, \pi - \lambda_N; -\nu_1, \ldots, -\nu_s). \tag{2.8} \]
By means of this relation, a given representation for, say, $Z_{r_1, \ldots, r_s}^{\text{top}}$ immediately leads to a corresponding representation for $Z_{r_1, \ldots, r_s}^{\text{bot}}$. However, the dependence of these two representations on the row configuration is in terms of two complementary set of integers $(r's$ and $\bar{r}'s)$, while to combine them conveniently to work out representations of correlation functions, we need the two representations for $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$ to be expressed both in terms of the same set of integers.

In view of these considerations, in the following we will work out two distinct and essentially different representations for the two partition functions, which cannot be obtained one from the other simply by applying the crossing symmetry relation (2.8). As a matter of fact, these two distinct representations comes out essentially from two slightly different ways of applying the QISM machinery. They both appear to differ significantly from the longly known ‘coordinate wavefunction’ representation.

We turn now to the second main goal of the present paper. As mentioned above, $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$, if suitably combined, can be used as building block to construct other correlation functions. As a first, simple, example, let us mention the row configuration probability, that is the probability of observing a given configuration of arrows on a given row of the lattice. More specifically, we denote by $H_{N,s}^{(r_1, \ldots, r_s)}$, the probability of observing on the $s$th row of the $N \times N$ lattice a configuration with $s$ up arrows at positions $r_1, \ldots, r_s$. This nonlocal correlation function was introduced in \[30\], and, independently, in \[51\], in the context of combinatorics.

It is clear that the row configuration probability can be written as
\[ H_{N}^{(r_1, \ldots, r_s)} = \frac{1}{Z_{N}} Z_{r_1, \ldots, r_s}^{\text{top}} Z_{r_1, \ldots, r_s}^{\text{bot}}, \tag{2.9} \]
thus reconducting its evaluation to that of $Z_{r_1, \ldots, r_s}^{\text{top}}$ and $Z_{r_1, \ldots, r_s}^{\text{bot}}$.

If we specialize the row configuration probability to the first line, by setting $s = 1$, we obtain the probability of observing the sole reversed arrow of the first row exactly on the $r$th vertical edge. This quantity, called one-point boundary correlation function, was introduced and calculated using QISM in \[30\].

Besides being interesting on its own right, the row configuration probability can be employed to build useful correlation functions, such as, for instance, the one-point correlation function, or polarization. Let us denote by $G_{N}^{(r,s)}$ the polarization at point $(r,s)$, that is the probability of observing an up arrow on the $r$th vertical edge of the $s$th row of the lattice. Such up arrow may be any of the $s$ up arrows occurring in the $s$th row. Let us suppose it is the $l$th one, $l = 1, \ldots, s$. We thus have to sum over the positions of the $l-1$ up arrows to its right, $1 \leq r_1 < \cdots < r_{l-1} < r$, and over the positions of the $s-l$ up arrows to its left, $r < r_{l+1} < \cdots < r_s \leq N$. 

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Figure 3. Emptiness formation probability: (a) Basic definition, as the probability of having on all edges within a rectangular region of size \((N - r) \times s\) in the top-left corner of the lattice, arrows pointing down or left (here is shown the case \(s = 2, r = 4\) and \(N = 7\)); (b) Equivalent definition, as the probability of having, between the \(s\)th and \((s + 1)\)th horizontal lines, \(N - r\) down arrows at positions \(N - r + 1, \ldots, N\); (c) In terms of the row configuration probability, as a sum over \(1 \leq r_1 < \ldots < r_s \leq r\), where the dashed line shows the border which cannot be passed by the positions \(r_1, \ldots, r_s\) of the up arrows in the summation.

The result must then be summed over all possible values of \(l\). We can thus write:

\[
G^{(r,s)}_N = \sum_{l=1}^{s} \sum_{1 \leq r_1 < \ldots < r_{l-1} < r < r_{l+1} < \ldots < r_s \leq N} H^{(r_1, \ldots, r_{l-1}, r, r_{l+1}, \ldots, r_s)}_N. \tag{2.10}
\]

Clearly, one has still to work out some convenient procedure to perform the multiple sums. As a training ground to tackle such problem, we will turn our attention towards some slightly simpler, although nonlocal correlation function, namely, the emptiness formation probability.

The emptiness formation probability, denoted by \(F^{(r,s)}_N\), describes the probability of having on all edges within a top-left rectangular region of size \((N - r) \times s\) arrows pointing down or left, see figure 3a. It was first introduced in [34], where it was shown to satisfy a recurrence relation in \(r, s\) and \(N\). Such recurrence relation can be solved, allowing to build an \(s\)-fold multiple integral representation.

The domain wall boundary conditions, together with the ice rule, imply that the emptiness formation probability may be equivalently defined as the probability of observing the last \(N - r\) arrows between the \(s\)th and \((s + 1)\)th horizontal lines to be all pointing down, see figure 3b. It can thus be expressed in terms of the row configuration probability, as a sum over \(1 \leq r_1 < \ldots < r_s \leq r\), see figure 3c:

\[
F^{(r,s)}_N = \sum_{r_s = s}^{r} \sum_{r_2 = 2}^{r-1} \sum_{r_1 = 1}^{r_2-1} H^{(r_1, \ldots, r_s)}_N. \tag{2.11}
\]

The problem of performing the multiple sum appears here to be simpler with respect to the above mentioned case of polarization, see (2.10). It will be addressed in...
section 5, where, basing on the results of [40] and [41], we will recover by different methods the s-fold multiple integral representation worked out in [34].

For later use, let us discuss here an alternative way to express the emptiness formation probability in terms of the row configuration probability, and of the partition functions $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$. Recalling that the emptiness formation probability $F_N^{(r,s)}$ vanishes for $s > r$, it is convenient to introduce the lattice coordinate $n := r - s$, with $n = 0, 1, \ldots, N - s$, giving the distance from the antidiagonal of the lattice. Then, by definition, $F_N^{(s+n,s)}$ is a weighted sum over all configurations of the domain-wall $N \times N$ lattice, conditioned to have an $(N - s - n) \times s$ frozen rectangle in the top-left corner. Due to diagonal symmetry, and symmetry under reversal of all arrows, we can equivalently sum over all configurations conditioned to have a frozen rectangular region of size $s \times (N - s - n)$ in the bottom right corner, see figure [4].

Let us now focus on the configurations of arrows on the $N$ vertical edges of the $(s+n)$th row, and denote by $r_1, \ldots, r_{s+n}$ the position of the $s+n$ up arrows. Due to the condition on the considered configurations, the first $s$ up arrows (counting from the right) will certainly occur at position 1, $\ldots, s$, that is $r_j = j, j = 1, \ldots, s$, see figure [4]. It can thus be expressed in terms of the row configuration probability as a sum over the position of the remaining $n = r - s$ arrows, $s + 1 \leq r_{s+1} < \ldots < r_{s+n} \leq N$, see figure [4]:

\[
F_N^{(s+n,s)} = \sum_{s+1 \leq r_{s+1} < \ldots < r_{s+n} \leq N} H_N^{(1,\ldots,s,r_{s+1},\ldots,r_{s+n})}.
\] (2.12)
This expression will be our starting point in section 6, where, resorting once more to the technique developed in [40, 41], we will obtain an \( n \)-fold (rather than \( s \)-fold) multiple integral representation for the emptiness formation probability \( F^{(r,s)}_N \).

2.3. Quantum Inverse Scattering Method formulation. We now define the main objects of QISM in relation to the model. First, let us consider vector space \( \mathbb{C}^2 \) and denote its basis vectors as the spin-up and spin-down states

\[
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

To each horizontal and vertical line of the lattice we associate a copy of the vector space \( \mathbb{C}^2 \). We also use the convention that an upward or right arrow corresponds to a spin-up state while a downward or left arrow corresponds to a spin-down state.

Next, we introduce the quantum \( L \)-operator, which can be defined as a matrix of the Boltzmann weights. Namely, to each vertex being intersection of the \( o \)th vertical line and the \( k \)th horizontal line we associate the operator \( L_{\alpha,k}(\lambda, \nu_k) \) which acts in the direct product of two vector spaces \( \mathbb{C}^2 \) (associated with the \( k \)th horizontal line) and in the ‘vertical’ space \( \mathcal{V}_\alpha = \mathbb{C}^2 \) (associated with the \( o \)th vertical line). Referring to the scattering matrix picture, we regard arrow states on the top and right edges of the vertex as ‘in’ indices of the \( L \)-operator while those on the bottom and left edges as ‘out’ ones. Explicitly, the \( L \)-operator reads

\[
L_{\alpha,k}(\lambda, \nu_k) = \sin(\lambda - \nu_k + \eta \tau^z_k \sigma^z_k) + \sin 2\eta(\tau^-_k \sigma^+_k + \tau^+_k \sigma^-_k).
\]

Here \( \tau \)'s (\( \sigma \)'s) are Pauli matrices of the corresponding vertical (horizontal) vector spaces.

Further, we introduce the monodromy matrix, which is an ordered product of \( L \)-operators. We define the monodromy matrix here as a product of \( L \)-operators along a vertical line, regarding the corresponding vertical space \( \mathcal{V}_\alpha \) as an ‘auxiliary’ space, and the tensor product of the \( N \) horizontal spaces, \( \mathcal{H} = \otimes_{k=1}^N \mathcal{H}_k \), as the quantum space. In defining the monodromy matrix it is convenient to think of \( L \)-operator as acting in \( \mathcal{V}_\alpha \otimes \mathcal{H} \) and, moreover, writing it as \( 2 \)-by-\( 2 \) matrix in \( \mathcal{V}_\alpha \), with the entries being quantum operators (acting in \( \mathcal{H} \)),

\[
L_{\alpha,k}(\lambda, \nu_k) = \begin{pmatrix} \sin(\lambda - \nu_k + \eta \tau^z_k \sigma^z_k) & \sin(2\eta) \sigma^+_k \\ \sin(2\eta) \sigma^-_k & \sin(\lambda - \nu_k - \eta \sigma^z_k) \end{pmatrix}_{[\mathcal{V}_\alpha]}.
\]

Here the subscript indicates that this is a matrix in \( \mathcal{V}_\alpha \) and \( \sigma^l_k \) (\( l = +, -, z \)) denote quantum operators in \( \mathcal{H} \) acting as Pauli matrices in \( \mathcal{H}_k \) and identically elsewhere.

The monodromy matrix is defined as

\[
T_{\alpha}(\lambda) = L_{\alpha,N}(\lambda, \nu_N) \cdots L_{\alpha,2}(\lambda, \nu_2)L_{\alpha,1}(\lambda, \nu_1)
\]

\[
= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[\mathcal{V}_\alpha]}.
\]

The operators \( A(\lambda) = A(\lambda; \nu_1, \ldots, \nu_N) \), etc, act in \( \mathcal{H} \). Operators \( A(\lambda), B(\lambda), C(\lambda), \) and \( D(\lambda) \), admit simple graphical interpretation as vertical lines of the lattice, with top and bottom arrows fixed. Let us introduce ‘all spins down’ and ‘all spins up’ states

\[
|\uparrow\rangle_k = \bigotimes_{k=1}^N |\uparrow\rangle_k, \quad |\downarrow\rangle_k = \bigotimes_{k=1}^N |\downarrow\rangle_k, \quad \tag{2.13}
\]
where \(|\uparrow\rangle_k\) and \(|\downarrow\rangle_k\) are basis vectors of \(\mathcal{H}_k\). In the case of domain wall boundary conditions each vertical line corresponds to an operator \(B(\lambda_\alpha)\) (where \(\alpha\) is the number of the horizontal line) while vectors \([2.13]\) describe states on the right and left boundaries; the partition function reads:

\[
Z_N = \langle \uparrow | B(\lambda_N) \cdots B(\lambda_2) B(\lambda_1) | \uparrow \rangle. \tag{2.14}
\]

To fit the row configuration probability \(H_{N,s}^{r_1,\ldots,r_s}\) into the framework of QISM, we consider the following decomposition of the monodromy matrix,

\[
T(\lambda) = T_{\text{bot}}(\lambda) T_{\text{top}}(\lambda), \tag{2.15}
\]

where \(T_{\text{top}}(\lambda)\) is defined as a product of the \(s\) first \(L\)-operators

\[
T_{\text{top}}(\lambda) = L_{\alpha,s}(\lambda, \nu_s) \cdots L_{\alpha,1}(\lambda, \nu_1), \tag{2.16}
\]

and \(T_{\text{bot}}(\lambda)\) as the product of the remaining \(N - s\) ones:

\[
T_{\text{bot}}(\lambda) = L_{\alpha,\nu}(\lambda, \nu_N) \cdots L_{\alpha,\nu_{s+1}}(\lambda, \nu_{s+1}).
\]

We correspondingly decompose the quantum space \(\mathcal{H}\) into a ‘top’ quantum space \(\mathcal{H}_{\text{top}} = \otimes_{k=1}^s \mathcal{H}_k\) and a ‘bottom’ quantum space \(\mathcal{H}_{\text{bot}} = \otimes_{k=s+1}^N \mathcal{H}_k\), with \(\mathcal{H} = \mathcal{H}_{\text{top}} \otimes \mathcal{H}_{\text{bot}}\). Correspondingly, we introduce the operators \(A_{\text{top}}(\lambda), \ldots, D_{\text{top}}(\lambda)\), and \(A_{\text{bot}}(\lambda), \ldots, D_{\text{bot}}(\lambda)\), as operator valued entries of the corresponding monodromy matrices \(T_{\text{top}}(\lambda)\) and \(T_{\text{bot}}(\lambda)\), respectively. Such a decomposition was originally introduced in the context of the so-called ‘two-site model’ \([1, 2]\).

It is useful to consider the corresponding decomposition of the ‘all spins up’ and ‘all spin down’ vectors. For example, we have \(|\uparrow\rangle = |\uparrow_{\text{top}}\rangle \otimes |\uparrow_{\text{bot}}\rangle\), where, to fit \([2.14]\), we set

\[
|\uparrow_{\text{top}}\rangle = \otimes_{k=1}^s |\uparrow\rangle_k, \quad |\uparrow_{\text{bot}}\rangle = \otimes_{k=s+1}^N |\uparrow\rangle_k,
\]

and an analogous decomposition for the ‘all spins down’ vector. It is easy to verify that the above defined vectors are eigenvectors of \(A_{\text{top}}(\lambda), D_{\text{top}}(\lambda),\) and \(A_{\text{bot}}(\lambda), D_{\text{bot}}(\lambda)\), respectively. In particular, we have:

\[
A_{\text{bot}}(\lambda)|\uparrow_{\text{bot}}\rangle = \prod_{k=s+1}^N a(\lambda, \nu_k) |\uparrow_{\text{bot}}\rangle, \tag{2.17}
\]

\[
\langle \psi_{\text{top}} | D_{\text{top}}(\lambda) = \langle \psi_{\text{top}} | \prod_{k=1}^s a(\lambda, \nu_k). \tag{2.18}
\]

Using the notation introduced above, the partition functions on the upper, \(N \times s\), lattice can be written, in the spirit of representation \([2.14]\),

\[
Z_{r_1,\ldots,r_s}^{\uparrow} = \langle \psi_{\text{top}} | D_{\text{top}}(\lambda_N) \cdots D_{\text{top}}(\lambda_{r_s+1}) B_{\text{top}}(\lambda_{r_s}) D_{\text{top}}(\lambda_{r_s-1}) \cdots D_{\text{top}}(\lambda_{r_1+1}) B_{\text{top}}(\lambda_{r_1}) D_{\text{top}}(\lambda_{r_1-1}) \cdots D_{\text{top}}(\lambda_1) |\psi_{\text{top}}\rangle. \tag{2.19}
\]

Similarly, for the partition function on the lower, \(N \times (N - s)\), lattice we have

\[
Z_{r_1,\ldots,r_s}^{\downarrow} = \langle \psi_{\text{bot}} | B_{\text{bot}}(\lambda_N) \cdots B_{\text{bot}}(\lambda_{r_s+1}) A_{\text{bot}}(\lambda_{r_s}) B_{\text{bot}}(\lambda_{r_s-1}) \cdots B_{\text{bot}}(\lambda_{r_1+1}) A_{\text{bot}}(\lambda_{r_1}) B_{\text{bot}}(\lambda_{r_1-1}) \cdots B_{\text{bot}}(\lambda_1) |\psi_{\text{bot}}\rangle. \tag{2.20}
\]

Formulas \([2.19]\) and \([2.20]\) are our starting point in computing \(Z_{r_1,\ldots,r_s}^{\downarrow}\) and \(Z_{r_1,\ldots,r_s}^{\uparrow}\).
3. The ‘top’ and ‘bottom’ partition functions

In this section we compute the components of the off-shell Bethe states (or partition functions) $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$ using the technique of the commutation relations for the entries of the quantum monodromy matrix (the RTT relation).

3.1. Fundamental commutation relations. One of the most basic relations of QISM is the so-called ‘RLL’ relation [1, 42], which reads

$$R_{\alpha\alpha'}(\lambda, \lambda') [L_{\alpha k}(\lambda, \nu) \otimes L_{\alpha' k}(\lambda', \nu)] = [L_{\alpha k}(\lambda', \nu) \otimes L_{\alpha' k}(\lambda, \nu)] R_{\alpha\alpha'}(\lambda, \lambda').$$

Here $R_{\alpha\alpha'}(\lambda, \lambda')$, called the $R$-matrix, is a matrix acting in the direct product of two auxiliary vector spaces, $V_\alpha \otimes V_{\alpha'}$, and it can be conveniently represented as a 4-by-4 matrix (we assume that the first space refers to the 2-by-2 blocks, while the second one to the entries in the blocks):

$$R_{\alpha\alpha'}(\lambda, \lambda') = \begin{pmatrix} f(\lambda', \lambda) & 0 & 0 & 0 \\ 0 & g(\lambda', \lambda) & 1 & 0 \\ 0 & 1 & g(\lambda', \lambda) & 0 \\ 0 & 0 & 0 & f(\lambda', \lambda) \end{pmatrix}.$$

Here the functions $f(\lambda', \lambda)$ and $g(\lambda', \lambda)$ are

$$f(\lambda', \lambda) = \frac{\sin(\lambda - \lambda' + 2\eta)}{\sin(\lambda - \lambda')}, \quad g(\lambda', \lambda) = \frac{\sin(2\eta)}{\sin(\lambda - \lambda')}.$$

It is to be mentioned that here and below we are mainly following notations and conventions of book [1].

The importance of the RLL relation above resides in that it implies the following relation, which, in turn, can be called RTT relation,

$$R_{\alpha\alpha'}(\lambda, \lambda') [T_{\alpha}(\lambda) \otimes T_{\alpha'}(\lambda')] = [T_{\alpha}(\lambda') \otimes T_{\alpha'}(\lambda)] R_{\alpha\alpha'}(\lambda, \lambda'). \quad (3.1)$$

This relation contains in total 16 commutation relations, between the operators $A(\lambda), B(\lambda), C(\lambda),$ and $D(\lambda)$. In the following we need only some of these commutation relations, namely

$$A(\lambda) A(\lambda') = A(\lambda') A(\lambda), \quad (3.2)$$

$$B(\lambda) B(\lambda') = B(\lambda') B(\lambda), \quad (3.3)$$

$$D(\lambda) D(\lambda') = D(\lambda') D(\lambda), \quad (3.4)$$

$$A(\lambda) B(\lambda') = f(\lambda, \lambda') B(\lambda') A(\lambda) + g(\lambda', \lambda) B(\lambda) A(\lambda'), \quad (3.5)$$

$$B(\lambda) A(\lambda') = f(\lambda', \lambda) A(\lambda') B(\lambda) + g(\lambda, \lambda') A(\lambda) B(\lambda'), \quad (3.6)$$

$$D(\lambda) B(\lambda') = f(\lambda', \lambda) B(\lambda') D(\lambda) + g(\lambda, \lambda') B(\lambda) D(\lambda'), \quad (3.7)$$

$$B(\lambda) D(\lambda') = f(\lambda', \lambda) D(\lambda') B(\lambda) + g(\lambda, \lambda') D(\lambda) B(\lambda'). \quad (3.8)$$

Taking into account relation (3.3) and using relation (3.5), one can obtain, in the usual spirit of the algebraic Bethe ansatz calculation (see [1, 42]), the relation:

$$A(\lambda_\alpha) \prod_{\beta=1}^{r-1} B(\lambda_\beta) = \sum_{\alpha=1}^{r} g(\lambda_\alpha, \lambda_\beta) \prod_{\beta=1}^{r} f(\lambda_\alpha, \lambda_\beta) \prod_{\beta=1}^{r} B(\lambda_\beta) A(\lambda_\alpha). \quad (3.9)$$
Similarly, taking into account (3.2) and using (3.6), one obtains
\[
B(\lambda_r) \prod_{\beta=1}^{r-1} A(\lambda_\beta) = \sum_{\alpha=1}^{r} g(\lambda_\alpha, \lambda_r) \prod_{\beta=1, \beta \neq \alpha}^{r} f(\lambda_\alpha, \lambda_\beta) \prod_{\beta=1, \beta \neq \alpha}^{r} A(\lambda_\beta) B(\lambda_\alpha). 
\]
Equation (3.10)

Analogously, relation (3.17) together with (3.3) give
\[
D(\lambda_r) \prod_{\beta=1}^{r-1} B(\lambda_\beta) = \sum_{\alpha=1}^{r} g(\lambda_r, \lambda_\alpha) \prod_{\beta=1, \beta \neq \alpha}^{r} f(\lambda_\beta, \lambda_\alpha) \prod_{\beta=1, \beta \neq \alpha}^{r} B(\lambda_\beta) D(\lambda_\alpha). 
\]
Equation (3.11)

Finally, due to (3.3) and (3.8), we have
\[
B(\lambda_r) \prod_{\beta=1}^{r-1} D(\lambda_\beta) = \sum_{\alpha=1}^{r} g(\lambda_r, \lambda_\alpha) \prod_{\beta=1, \beta \neq \alpha}^{r} f(\lambda_\beta, \lambda_\alpha) \prod_{\beta=1, \beta \neq \alpha}^{r} D(\lambda_\beta) B(\lambda_\alpha). 
\]
Equation (3.12)

Evidently, decomposition (2.15) for the monodromy matrices implies the existence of RTT relations, analogous to relations (3.1), for the ‘top’ and ‘bottom’ quantum spaces. These, in turn, contains all commutation relations between operators \(A_{\text{top}}(\lambda), \ldots, D_{\text{top}}(\lambda)\), and between operators \(A_{\text{bot}}(\lambda), \ldots, D_{\text{bot}}(\lambda)\), respectively; below we use commutation relations (3.9)–(3.12) for the ‘top’ and ‘bottom’ quantum spaces.

Looking at formulae (2.19) and (2.20), we anticipate that the four relations (3.9)–(3.12) lead to four different representations, two for \(Z_{r_1,\ldots,r_s}^{\text{top}}\), and two for \(Z_{r_1,\ldots,r_s}^{\text{bot}}\). In particular, the two resulting representation for, say \(Z_{r_1,\ldots,r_s}^{\text{top}}\), are essentially different, each one being in turn related through crossing symmetry to one of the two representations obtained for \(Z_{r_1,\ldots,r_s}^{\text{bot}}\).

\[\textbf{3.2. Application to the ‘bottom’ partition function.}\]

We first consider the computation of the lower sublattice partition function. Starting from representation (2.20), we can use the fundamental RTT relations, and, in particular, generalized commutation relation (3.9) to move all operators \(A_{\text{bot}}(\lambda)\) to the right, and make them act on \(|\uparrow_{\text{bot}}\rangle\), exploiting relation (2.17). Indeed, using \(s\) times commutation relation (3.9), acting on the right on the vector \(|\uparrow_{\text{bot}}\rangle\), and multiplying from the left with the vector \(\langle\psi_{\text{bot}}|B(\lambda_N)\cdots B(\lambda_{r_s+1})\rangle\), we obtain
\[
Z_{r_1,\ldots,r_s}^{\text{bot}} = \sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_s=1}^{r_s} \prod_{j=1}^{s} \prod_{k=s+1}^{N} a(\lambda_{\alpha_j}, \nu_k) \prod_{j=1}^{s} g(\lambda_{\alpha_j}, \lambda_{r_j}) \\
\times \prod_{\beta_1=1}^{r_1} f(\lambda_{\alpha_1}, \lambda_{\beta_1}) \prod_{\beta_2=1}^{r_2} f(\lambda_{\alpha_2}, \lambda_{\beta_2}) \cdots \prod_{\beta_s=1}^{r_s} f(\lambda_{\alpha_s}, \lambda_{\beta_s}) \\
\times Z_{N-s}[\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_s}; \nu_1, \ldots, \nu_s] 
\]
Equation (3.13)

Here \(Z_{N-s}[\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_s}; \nu_1, \ldots, \nu_s]\) denotes the partition function of the domain-wall six-vertex model on the \((N-s) \times (N-s)\) lattice, with spectral parameters \(\lambda_\alpha, \alpha \in \{1, 2, \ldots, N\} \setminus \{\alpha_1, \ldots, \alpha_s\}\) and \(\nu_k, k \in \{s+1, \ldots, N\}\); in other words, the square brackets indicate independence from the enclosed variables, in comparison with the ‘original’ sets \(\alpha_1, \ldots, \alpha_N\) and \(\nu_1, \ldots, \nu_N\).
To proceed further, it is convenient to introduce the function
\[ v_r(\lambda) := \prod_{\alpha=1}^{N} d(\lambda_\alpha, \lambda) \prod_{\alpha=1}^{r-1} e(\lambda_\alpha, \lambda), \]
where function \( d(\lambda, \lambda') \) has been defined in (2.20), and
\[ e(\lambda, \lambda') = \sin(\lambda - \lambda' + 2\eta). \]
We now reexpress functions \( f(\lambda, \lambda') \) and \( g(\lambda, \lambda') \) appearing in (3.14) in terms of functions \( d(\lambda, \lambda') \) and \( e(\lambda, \lambda') \), defined in (2.20) and (3.15). We also substitute the Izergin-Korepin expression (2.3) for the partition function appearing in (3.13). In this lengthy but standard computation we arrive at the formula:
\[
Z_{r_1, \ldots, r_s}^{\text{bot}} = \prod_{1 \leq \alpha < \beta \leq N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k) \prod_{s+1 \leq j < k \leq N} d(\nu_j, \nu_k) \prod_{1 \leq j < k \leq s} e(\lambda_\alpha, \lambda_\beta) \prod_{1 \leq j \leq s} v_{r_j}(\lambda_\alpha) \prod_{1 \leq j < k \leq s} \det M_{\alpha_1, \ldots, \alpha_s \alpha_1, \ldots, \alpha_s},
\]
Here the function \( \chi(\alpha, \beta) \) is defined as
\[
\chi(\alpha, \beta) = \begin{cases} 0, & \text{if } \alpha < \beta, \\ 1, & \text{otherwise,} \end{cases}
\]
while \( M_{\alpha_1, \ldots, \alpha_s \alpha_1, \ldots, \alpha_s} \) denotes the \((N-s) \times (N-s)\) matrix obtained from the matrix \( M \), see (2.3), by removing rows \( \alpha_1, \ldots, \alpha_s \), and the first \( s \) columns. Note that, since function \( v_r(\lambda) \) vanishes for \( \lambda = \lambda_\alpha \) (\( \alpha = r+1, \ldots, N \)), all the sums appearing in (3.14) can be extended up to the value \( N \). Finally, note that the functions \( e(\lambda, \mu) \) appearing in the denominator in last line of (3.16) are exactly compensated by corresponding functions in the numerators, that are hidden in the definition of functions \( v_r(\lambda) \). As a consequence, each term of the sum remain regular even in the limit where two \( \lambda \)'s differ exactly by \( 2\eta \).

It is worth to comment that in computing \( Z_{r_1, \ldots, r_s}^{\text{bot}} \) here we could have proceeded differently. Namely, starting from representation (2.20), we could have chosen to use commutation relation (3.10) to move all operators \( A_{\text{bot}}(\lambda) \) to the left, and make them act on \(|\psi_{\text{bot}}\rangle \). As a result, \( Z_{r_1, \ldots, r_s}^{\text{bot}} \) would have been expressed as an \((N-s)\)-fold sum of minors of order \( N-s \) of the matrix \( M \), see (2.4). In this way we would have arrived at an essentially different representation, in comparison with (3.16). As it becomes clear below, it is the combination of these two complementary representations, one for \( Z_{r_1, \ldots, r_s}^{\text{bot}} \), and another for \( Z_{r_1, \ldots, r_s}^{\text{top}} \), that may lead to useful representations for the row configuration probability and other correlation functions therefrom.

3.3. Application to the ‘top’ partition function. Let us turn to the partition function on the upper sublattice. Having in mind representation (2.19) for \( Z_{r_1, \ldots, r_s}^{\text{top}} \), we can use the fundamental RTT relations to move all \( B_{\text{top}}(\lambda) \)'s on one side, and all \( D_{\text{top}}(\lambda) \)'s on the other. As already outlined on the example of \( Z_{r_1, \ldots, r_s}^{\text{bot}} \), one can implement this procedure in two different ways. The first possibility is to
use commutation relation (3.11) to commute each of the $D_{\text{top}}(\lambda)$’s to the right, through $B_{\text{top}}(\lambda)$’s. The resulting expression appears to be dual (under crossing symmetry) to the one computed in section 3.2. We report it in appendix B for the sake of completeness.

Here we exploit the second possibility, namely, we commute the $B_{\text{top}}(\lambda)$’s to the right through the $D_{\text{top}}(\lambda)$’s. More specifically, we can use commutation relation (3.3) to move all $D_{\text{top}}(\lambda)$’s to the left, and make them act on $\langle \psi_{\text{top}} \rangle$, exploiting relation (2.13). Using $s$ times commutation relation (3.12), acting on the right on the vector $|\uparrow_{\text{top}} \rangle$, we obtain (3.18) as a multiple sum involving a sum of possible configuration, one can find that

$$Z_{r_1, \ldots, r_s}^{\text{top}} = \sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_s=1}^{r_s} \prod_{\alpha_s \neq \alpha_1, \ldots, \alpha_{s-1}, \beta \neq a_1, \ldots, a_s}^{N} \prod_{k=1}^{s} a(\lambda_\beta, \nu_k) \prod_{j=1}^{s} g(\lambda_\beta, \lambda_{\alpha_j}) f(\lambda_{r_j}, \lambda_{\alpha_j})$$

$$= \prod_{\beta_1=1}^{r_1} f(\lambda_{\beta_1}, \lambda_{\alpha_1}) \prod_{\beta_2=1}^{r_2} f(\lambda_{\beta_2}, \lambda_{\alpha_2}) \cdots \prod_{\beta_s=1}^{r_s} f(\lambda_{\beta_s}, \lambda_{\alpha_s}) \times Z_s (\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_s}; \nu_1, \ldots, \nu_s).$$

Here $Z_s (\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_s}; \nu_1, \ldots, \nu_s)$ denotes the partition function of the inhomogeneous model on the $s \times s$ lattice, with the indicated spectral parameters. Substituting the Izergin-Korepin partition function, one can further rewrite (3.18) as a determinant; for our purposes below the above representation appears to be sufficient.

As a side comment, it is worth to mention that for $s = 1$ the formula (3.18) reduces to a sum of $r_1$ terms. On the other hand, evaluating the weight of the sole possible configuration, one can find that

$$Z_{r_1}^{\text{top}} = \prod_{\alpha=r_1+1}^{N} a(\lambda_\alpha, \nu_1) e \prod_{\alpha=1}^{r_1-1} b(\lambda_\alpha, \nu_1).$$

(3.19)

Apparently, the equivalence of the two expressions is due to certain identity; a direct proof of the relevant identity can be found in [30] (see also [50]). A natural question is whether such identity generalizes to values of $s > 1$. The answer appears to be positive. We report such remarkable identity and discuss some particular cases in appendix C.

To conclude this section, we wish to emphasize that representations (3.13), or equivalently (3.16), for $Z_{\text{bot}}^{r_1, \ldots, r_s}$, and (3.18) for $Z_{\text{top}}^{r_1, \ldots, r_s}$, depend on the row configuration through $r_1, \ldots, r_s$ denoting the positions of the up arrows. We recall that these representations have been obtained by repeated use of (3.9) and (3.12), respectively. If instead we had worked out analogous derivations starting from relations (3.11) and (3.10), we would have obtained two different representations, depending on the row configuration through the complementary set of integers $\bar{r}_1, \ldots, \bar{r}_{N-s}$ denoting the position of the down arrows (see appendix B formula (3.2) for $Z_{\text{bot}}^{\bar{r}_1, \ldots, \bar{r}_{N-s}}$). It is clear that the two additional representations are simply related to (3.18) and (3.16) by the crossing symmetry transformation, see (2.8).

Finally, we stress once more that such representations are all essentially different from the so-called ‘coordinate wavefunction’ representation, that follows from the equivalence of the algebraic and coordinate Bethe ansatz [50]. In particular,
referring to $Z^{\text{top}}_{r_1, \ldots, r_s}$ for definiteness, representations (3.18), and (B.2) are both different from (A.1).

4. Integral representations for the ‘top’ and ‘bottom’ partition functions

In this section we derive representations for $Z^{\text{bot}}_{r_1, \ldots, r_s}$ and $Z^{\text{top}}_{r_1, \ldots, r_s}$ in terms of $s$-fold contour integrals.

4.1. Orthogonal polynomial representation for the ‘bottom’ partition function. Let us first consider $Z^{\text{bot}}_{r_1, \ldots, r_s}$. To start with, we evaluate the homogeneous limit for expression (3.16). We resort to the procedure successfully used in (3.16). It is based on the observation that the multiple sum in (3.16) reminds the Laplace expansion of some determinant, since the minor appearing in the last line depends on the summed indices only through their absence. Thus the first step is to rewrite representation (3.16) in a determinant form. For this purpose we set

$$\lambda_\alpha = \lambda + \xi_\alpha, \quad \alpha = 1, \ldots, N$$

where the $\xi$’s will be sent to zero in the limit (as well as the $\nu$’s). Keeping the $\xi$’s nonzero (and different from each other), and using the fact that for a function $f(x)$, regular near $x = \lambda$, the relation $\exp(\xi x) f(\lambda + \varepsilon) f(\lambda + x) = f(\lambda + \xi)$ is valid, we can bring (3.16) to the form

$$Z^{\text{bot}}_{r_1, \ldots, r_s} = \frac{\prod_{\alpha=1}^{N} \prod_{k=s+1}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)}{\prod_{1 \leq \alpha < \beta \leq N} a(\lambda_\beta, \lambda_\alpha) \prod_{s+1 \leq k \leq N} d(\nu_j, \nu_k)} \left| \begin{array}{cccc} \exp(\xi_1 \partial_{\varepsilon_1}) & \ldots & \exp(\xi_1 \partial_{\varepsilon_s}) & \varphi(\lambda_1, \nu_{s+1}) & \ldots & \varphi(\lambda_1, \nu_N) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \exp(\xi_N \partial_{\varepsilon_1}) & \ldots & \exp(\xi_N \partial_{\varepsilon_s}) & \varphi(\lambda_N, \nu_{s+1}) & \ldots & \varphi(\lambda_N, \nu_N) \end{array} \right| \prod_{j=1}^{s} \nu_j (\lambda + \varepsilon_j) \prod_{1 \leq j < k \leq s} \frac{1}{e(\lambda + \varepsilon_j, \lambda + \varepsilon_k)} \bigg|_{\varepsilon_1, \ldots, \varepsilon_s = 0}. \tag{4.1}$$

It is to be emphasized that this expression is still for the inhomogeneous model; it represents an equivalent way of writing the multiple sum in (3.16).

We can now perform the homogeneous limit along the lines of (4.1). Specifically, we send $\xi_1, \ldots, \xi_N$ and $\nu_1, \ldots, \nu_N$ to zero. The procedure is explained in full detail in (4.4). Factoring out the partition function of the entire lattice, see (2.6), we obtain

$$Z^{\text{bot}}_{r_1, \ldots, r_s} = \frac{Z_N \prod_{j=1}^{N} (N - j)!}{(ab)^N \det N} \left| \begin{array}{cccc} \varphi(\lambda) & \ldots & \partial^{N-s-1}_\lambda \varphi(\lambda) & 1 & \ldots & 1 \\ \partial_\lambda \varphi(\lambda) & \ldots & \partial^{N-s}_\lambda \varphi(\lambda) & \partial_{\varepsilon_1} & \ldots & \partial_{\varepsilon_s} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial^{N-1}_\lambda \varphi(\lambda) & \ldots & \partial^{N-s-2}_\lambda \varphi(\lambda) & \partial^{N-1}_{\varepsilon_1} & \ldots & \partial^{N-1}_{\varepsilon_s} \end{array} \right| \prod_{j=1}^{s} \frac{(\sin \varepsilon_j)^{N-r_j} [\sin(\varepsilon_j - 2\eta)]^{r_j-1}}{[\sin(\varepsilon_j + \eta)]^{N-s}} \prod_{1 \leq j < k \leq s} \frac{1}{\sin(\varepsilon_j - \varepsilon_k + 2\eta)} \bigg|_{\varepsilon_1, \ldots, \varepsilon_s = 0}. \tag{4.2}$$
where, in writing the determinant, we have changed the order of columns with respect to (4.1). Here and below, when considering the homogeneous model we use the short notation for the weights, \( a = a(\lambda, 0) \), \( b = b(\lambda, 0) \), and for the partition function, \( Z_N = Z_N(\lambda, \ldots, \lambda; 0, \ldots, 0) \).

In order to rewrite (4.2) in integral form, we first transform the \( N \times N \) determinant into some more convenient and smaller \( s \times s \) determinant, \( s \) given in terms of a set of orthogonal polynomials. These polynomials naturally emerge when the determinant of matrix \( \mathcal{N} \) entering the homogeneous partition function (2.6) is interpreted as a Gram determinant associated to certain integral measure.

The derivation of the \( s \times s \) determinant representation from (4.2) is based on the following facts. Let \( \{ P_n(x) \}_{n=0}^\infty \) be a set of orthogonal polynomials,

\[
\int P_n(x) P_m(x) \mu(x) \, dx = h_n \delta_{nm}, \tag{4.3}
\]

where the integration domain is assumed over the real axis. The weight \( \mu(x) \) is real nonnegative and we choose \( h_n \)’s such that \( P_n(x) = x^n + \ldots \), i.e., that the leading coefficient of \( P_n(x) \) is equal to one. Let \( c_n \) denote the \( n \)th moment of the weight \( \mu(x) \),

\[
c_n = \int x^n \mu(x) \, dx \quad n = 0, 1, \ldots.
\]

The orthogonality condition (4.3) and standard properties of determinants allow us to prove that

\[
\begin{vmatrix}
  c_0 & c_1 & \ldots & c_{n-1} \\
  c_1 & c_2 & \ldots & c_n \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n-1} & c_n & \ldots & c_{2n-2}
\end{vmatrix}
= h_0 h_1 \cdots h_{n-1}.
\]

More generally (see, e.g., [54]), for \( s = 1, \ldots, N \), the following formula is valid

\[
\begin{vmatrix}
  c_0 & c_1 & \ldots & c_{N-s-1} & 1 & 1 & \ldots & 1 \\
  c_1 & c_2 & \ldots & c_{N-s} & x_1 & x_2 & \ldots & x_s \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{N-1} & c_N & \ldots & c_{2N-s-2} & x_1^{N-1} & x_2^{N-1} & \ldots & x_s^{N-1}
\end{vmatrix}
= h_0 \cdots h_{N-s-1} \begin{vmatrix}
  P_{N-s}(x_1) & \ldots & P_{N-s}(x_s) \\
  P_{N-1}(x_1) & \ldots & P_{N-1}(x_s)
\end{vmatrix} \tag{4.4}
\]

In our case, \( c_n = \partial_\eta^n \varphi(\lambda) \), and the integration measure \( \mu(x) \, dx \) is given by the Laplace transform of the function \( \varphi(\lambda) \); for explicit expressions, see [3].

Following [3][34][55], we denote

\[
K_n(x) = \frac{n! \varphi^{n+1}}{h_n} P_n(x), \tag{4.5}
\]

where \( \varphi := \varphi(\lambda) \), and \( h_n \) is given in (4.3). We also introduce the functions

\[
\omega(\varepsilon) := \frac{\sin(\lambda + \eta)}{\sin(\lambda - \eta)} \frac{\sin \varepsilon}{\sin(\varepsilon - 2\eta)}, \quad \tilde{\omega}(\varepsilon) := \frac{\sin(\lambda - \eta)}{\sin(\lambda + \eta)} \frac{\sin \varepsilon}{\sin(\varepsilon + 2\eta)}. \tag{4.6}
\]

These are functions of \( \varepsilon \), with \( \lambda \) and \( \eta \) regarded as parameters.

Noting that

\[
\frac{\sin(\varepsilon_1 + \lambda + \eta) \sin(\varepsilon_2 + \lambda - \eta)}{\sin(\varepsilon_1 - \varepsilon_2 + 2\eta)} = \frac{1}{\varphi} \frac{[1 - \tilde{\omega}(\varepsilon_1)] [\omega(\varepsilon_2) - 1]}{\tilde{\omega}(\varepsilon_1) \omega(\varepsilon_2) - 1},
\]

\[
\phi(\varepsilon_1, \varepsilon_2) = \frac{1}{4} \frac{\sin(\lambda + \eta) \sin(\lambda - \eta)}{\sin(\lambda + 2\eta)} \frac{\sin \varepsilon_1}{\sin \varepsilon_2} \frac{\sin(\varepsilon_1 - 2\eta)}{\sin(\varepsilon_2 - 2\eta)}.
\]

\[
\phi(\varepsilon_1, \varepsilon_2) = \frac{1}{4} \frac{\sin(\lambda + \eta) \sin(\lambda - \eta)}{\sin(\lambda + 2\eta)} \frac{\sin \varepsilon_1}{\sin \varepsilon_2} \frac{\sin(\varepsilon_1 - 2\eta)}{\sin(\varepsilon_2 - 2\eta)}.
\]
we have, in virtue of (4.3), the following orthogonal polynomials representation:

$$Z^\text{bot}_{r_1,\ldots,r_s} = \frac{Z_N}{a^{\frac{2r-2(N-1)}{2}} c^r_e} \prod_{j=1}^{s} \left( \frac{a}{b} \right)^{r_j}$$

$$\times \left| \begin{array}{cccc}
K_{N-s}(\hat{\epsilon}_1) & \ldots & K_{N-s}(\hat{\epsilon}_s) \\
\vdots & \ddots & \vdots \\
K_{N-1}(\hat{\epsilon}_1) & \ldots & K_{N-1}(\hat{\epsilon}_s)
\end{array} \right|^{s} \prod_{j=1}^{s} \left\{ \frac{[\omega(\hat{\epsilon}_j)]^{N-r-j} [\tilde{\omega}(\hat{\epsilon}_j)]^{s-j}}{[\omega(\hat{\epsilon}_j) - 1]^{N-s}} \right\}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{1}{[\omega(\hat{\epsilon}_j) \omega(\hat{\epsilon}_k) - 1]} \bigg|_{\hat{\epsilon}_1,\ldots,\hat{\epsilon}_s = 0}. \quad (4.7)$$

This representation is valid for arbitrary values of the parameters of the model, independently of the regime.

### 4.2. Integral representation for the ‘bottom’ partition function.

Our aim now is to rewrite representation (4.7) as a multiple integral. This can be done using the procedure provided in [34], where it was worked out on the example of the emptiness formation probability.

A special role below is played by the one-point boundary correlation function, denoted $H_N^{(r)}$, which is exactly the $s$th-row configuration probability $H_{N,s}^{(r_1,\ldots,r_s)}$ in the special case of $s = 1$. In this case the partition function $Z^\text{bot}_r$ can be easily computed with the result $Z^\text{bot}_r = a^{r-1} b^{r-1} c$, while $Z^\text{bot}_r$ can be found from (4.7), that yields

$$H_N^{(r)} = K_{N-1}(\hat{\epsilon}_c) \left. \frac{[\omega(\hat{\epsilon})]^{N-r} \omega(\hat{\epsilon}) - 1}{}^{N-1} \right|_{\hat{\epsilon}_c = 0}. \quad (4.8)$$

The whole procedure of transforming representation (4.7) into a multiple integral representation is based on the following key identity (see [34] for a proof):

$$K_{N-1}(\hat{\epsilon}_c) f(\omega(\hat{\epsilon})) \bigg|_{\hat{\epsilon}_c = 0} = \frac{1}{2\pi i} \oint_{C_0} \frac{(z-1)^{N-1}}{z^N} h_N(z) f(z) \, dz. \quad (4.9)$$

Here, $f(z)$ is an arbitrary function regular at the origin, $C_0$ is a small simple closed counterclockwise contour around the point $z = 0$, and $h_N(z)$ (not to be confused with $h_n$ in (4.3)) is the generating function of the one-point boundary correlation function $H_N^{(r)}$,

$$h_N(z) = \sum_{r=1}^{N} H_N^{(r)} z^{r-1}. \quad (4.8)$$

Clearly, $h_N(0) = H_N^{(1)} = a^{2(N-1)} c Z_{N-1}$, and $h_N(1) = 1$.

Further, we introduce functions $h_{N,s}(z_1,\ldots,z_s)$, where the second subscript, $s = 1,\ldots,N$, refers to the number of arguments. These functions are defined as

$$h_{N,s}(z_1,\ldots,z_s) = \prod_{1 \leq j < k \leq s} (z_k - z_j)^{-1}$$

$$\times \left| \begin{array}{cccc}
n_1^{s-1} h_{N-s+1}(z_1) & \ldots & n_1^{s-1} h_{N-s+1}(z_s) \\
\vdots & \ddots & \vdots \\
n_s^{s-1} h_{N-s+1}(z_1) & \ldots & n_s^{s-1} h_{N-s+1}(z_s)
\end{array} \right| \left( z_1 - 1 \right)^{s-1} h_N(z_1) \ldots \left( z_s - 1 \right)^{s-1} h_N(z_s). \quad (4.10)$$

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The functions $h_{N,s}(z_1, \ldots, z_s)$ are symmetric polynomials of degree $N - 1$ in each of their variables. Due to the structure of (4.10), it is easy to check the relations

$$h_{N,s}(z_1, \ldots, z_s, 1) = h_{N,s-1}(z_1, \ldots, z_{s-1})$$  (4.11)

and

$$h_{N,s}(z_1, \ldots, z_s, 0) = h_N(0) h_{N-1,s-1}(z_1, \ldots, z_{s-1})$$  (4.12)

which will play a crucial role below. The functions $h_{N,s}(z_1, \ldots, z_s)$ can be viewed as the multi-variable generalizations of $h_N(z)$ and turn out to be alternative representations (with respect to the Izergin-Korepin partition function) for the partially inhomogeneous partition functions [34].

To proceed, in representation (4.7) it is convenient to express $\tilde{\omega}(\epsilon)$ in terms of $\omega(\epsilon)$, by means of the identity

$$\tilde{\omega}(\epsilon) = \frac{t^2 \omega(\epsilon)}{2\Delta t\omega(\epsilon) - 1},$$

where we have used the parametrization

$$\Delta := \frac{a^2 + b^2 - c^2}{2ab}, \quad t := \frac{b}{a}.$$

Next, resorting to identity (4.9), we obtain the multiple integral representation

$$Z_{bot}^{r_1, \ldots, r_s} = Z_N \prod_{j=1}^s \frac{t^{r_j}}{a(N-1)C^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s \frac{1}{z_j} \prod_{1 \leq j < k \leq s} \left[ (w_j - w_k)(t^2 w_j w_k - 2\Delta t w_j + 1) \right] \frac{dz}{(2\pi i)^s}.$$  (4.13)

This representation is the final formula for $Z_{bot}^{r_1, \ldots, r_s}$. It is valid for arbitrary values of the parameters of the model, independently of the regime.

Note that the expression (4.13) also implies, through the crossing symmetry transformation, an analogous $(N-s)$-fold integral representation for $Z_{top}^{r_1, \ldots, r_s}$, which depends on the row configuration through the positions $\bar{r}_1, \ldots, \bar{r}_{N-s}$ of the $N-s$ down arrows, for details, see appendix B, formula (B.4).

### 4.3. Integral representation for the ‘top’ partition function

Let us now turn to the ‘top’ partition function, $Z_{top}^{r_1, \ldots, r_s}$. Before proceeding, it is worth to mention that a multiple integral representation for such quantity has already been worked out in [40], basing on the well-known ‘coordinate wavefunction’ representation (A.1), that follows from the equivalence of the algebraic and coordinate Bethe ansatz [50]. It reads

$$Z_{top}^{r_1, \ldots, r_s} = c^s a^{s(N-1)} \prod_{j=1}^s t^{r_j - j} \oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^s \left[ (w_j - w_k)(t^2 w_j w_k - 2\Delta t w_j + 1) \right] \frac{dw}{(2\pi i)^s},$$  (4.14)

see appendix A for a derivation.

We will now derive another, significantly different multiple integral representation, which will play a crucial role in the following. Let us turn back to representation (3.18) for $Z_{top}^{r_1, \ldots, r_s}$, and note that the multiple sum therein can be interpreted
as the sum of residues of some function in the s-fold complex plane. Specifically, we can use the following identity

\[
\sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_s=1}^{r_s} \frac{1}{\prod_{\beta_1=1}^{r_1}(\lambda_{\alpha_1} - \lambda_{\beta_1})} \frac{1}{\prod_{\beta_2=1}^{r_2}(\lambda_{\alpha_2} - \lambda_{\beta_2})} \cdots \frac{1}{\prod_{\beta_s=1}^{r_s}(\lambda_{\alpha_s} - \lambda_{\beta_s})} 
\]

\[
= \oint_{C_{(\lambda)}} \cdots \oint_{C_{(\lambda)}} \prod_{1 \leq j < k \leq s} d(\zeta_j, \zeta_k) \prod_{1 \leq j \leq s} d(\zeta_j) \quad \times \quad F(\zeta_1, \ldots, \zeta_s) \frac{d^s \zeta}{(2\pi i)^s}, \quad (4.15)
\]

where \( C_{(\lambda)} := C_{\lambda_1} \cup \cdots \cup C_{\lambda_r} \) is a simple closed counterclockwise contour in the complex plane of the integration variable, enclosing points \( \lambda_1, \ldots, \lambda_r \) and no other singularity of the integrand. The function \( F(\zeta_1, \ldots, \zeta_s) \) is a generic analytic function of its variables, regular in each variable within the region delimited by \( C_{(\lambda)} \).

We now reexpress functions \( f(\lambda, \lambda) \) and \( g(\lambda, \lambda) \) appearing in representation (3.18) in terms of functions \( d(\lambda, \lambda) \) and \( e(\lambda, \lambda) \), see (2.5) and (3.15). Comparing the resulting expression with the left hand side of identity (4.15), we set

\[
F(\lambda_1, \ldots, \lambda_s) = \prod_{\beta=1}^{N} \prod_{k=1}^{s} a(\lambda_\beta, \nu_k) \prod_{j, k=1}^{s} \frac{1}{a(\lambda_j, \nu_k)} 
\]

\[
\times \prod_{\beta_1=1}^{r_1} e(\lambda_{\alpha_1}, \lambda_{\beta_1}) \cdots \prod_{\beta_s=1}^{r_s} e(\lambda_{\alpha_s}, \lambda_{\beta_s}) \prod_{1 \leq j < k \leq s} \frac{1}{e(\lambda_{\alpha_j}, \lambda_{\alpha_k})} \times Z_s(\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_s}; \nu_1, \ldots, \nu_s). \quad (4.16)
\]

As a result, we obtain the following multiple integral representation:

\[
Z_{r_1, \ldots, r_s}^{top} = \prod_{\beta=1}^{N} \prod_{k=1}^{s} a(\lambda_\beta, \nu_k) 
\]

\[
\times \oint_{C_{(\lambda)}} \cdots \oint_{C_{(\lambda)}} \prod_{\beta_1=1}^{r_1} e(\zeta_1, \lambda_{\beta_1}) \cdots \prod_{\beta_s=1}^{r_s} e(\zeta_s, \lambda_{\beta_s}) \prod_{1 \leq j < k \leq s} \frac{1}{e(\zeta_j, \zeta_k)} Z_s(\zeta_1, \ldots, \zeta_s; \nu_1, \ldots, \nu_s) \frac{d^s \zeta}{(2\pi i)^s}. \quad (4.17)
\]

It is worth to recall that, besides the already mentioned poles at \( \lambda_1, \ldots, \lambda_r \), the only other poles of the integrand within the strip of width \( \pi \) of the integrand, the poles they give rise to are only apparent. Indeed, focussing (4.16), a careful comparison with (3.18) shows that the functions \( e(\lambda_{\alpha_k}, \lambda_{\alpha_j}) \) appearing in the denominator have been introduced to conveniently rewrite the products of function \( e(\lambda_{\alpha_j}, \lambda_{\beta_k}) \) in the numerator. In other words, the pole induced by the functions
Using now the relation (4.19) and changing the integration variables \( \zeta \) separately in this limit. We get

\[
Z_{r_1, \ldots, r_s}^{\text{top}} = \sum_{k=1}^{s} \left[ a(\lambda, \nu_\ell) \right]^N \int_{C_\lambda} \cdots \int_{C_\lambda} \prod_{j=1}^{s} \frac{[e(\zeta_j, \lambda)]^{r_j-1}}{[d(\zeta_j, \lambda)]^{r_j}} \prod_{1 \leq j < k \leq s} \frac{d(\zeta_k, \zeta_j)}{e(\zeta_k, \zeta_j)} \times Z_s (\zeta_1, \ldots, \zeta_s; \nu_1, \ldots, \nu_s) \frac{d^s \zeta}{\prod_{j=1}^{s} [a(\zeta_j)]^s} \frac{1}{(2\pi)^s},
\]

(4.18)

where \( C_\lambda \) is a small simple closed counterclockwise contour around point \( \lambda \). This representation can be related to the so-called 'coordinate wavefunction' representation (A.1) by means of a suitable deformation of the integration contours; we refer to last part of appendix [A] for details.

Next, we perform the homogeneous limit in spectral parameters \( \nu_1, \ldots, \nu_s \). This is straightforward as well, since all expressions appearing in (4.18) are regular separately in this limit. We get

\[
Z_{r_1, \ldots, r_s}^{\text{top}} = a^{Ns} \int_{C_\lambda} \cdots \int_{C_\lambda} \prod_{j=1}^{s} \frac{[e(\zeta_j, \lambda)]^{r_j-1}}{[d(\zeta_j, \lambda)]^{r_j}} \prod_{1 \leq j < k \leq s} \frac{d(\zeta_k, \zeta_j)}{e(\zeta_k, \zeta_j)} \times Z_s (\zeta_1, \ldots, \zeta_s; 0, \ldots, 0) \frac{d^s \zeta}{\prod_{j=1}^{s} [a(\zeta_j)]^s} \frac{1}{(2\pi)^s}.
\]

To proceed further we need to use the following identity expressing the partially inhomogeneous partition function \( Z_N(\lambda_1, \ldots, \lambda_N) \equiv Z_N(\lambda_1, \ldots, \lambda_N; 0, \ldots, 0) \) in terms of the generating function for the one-point boundary correlation function (see [34] for further details and proof):

\[
Z_N(\lambda_1, \ldots, \lambda_N) = Z_N(\lambda, \ldots, \lambda) \prod_{j=1}^{N} \left( \frac{a(\lambda_j, 0)}{a(\lambda, 0)} \right)^{N-1} \times h_{N,N} (\gamma(\lambda_1 - \lambda), \ldots, \gamma(\lambda_N - \lambda)).
\]

(4.19)

Here, the function \( \gamma(\xi) \) also depends on \( \lambda \) (and \( \eta \)) as a parameter and reads:

\[
\gamma(\xi) \equiv \gamma(\xi; \lambda) = \frac{a(\lambda, 0) b(\lambda + \xi, 0)}{b(\lambda, 0) a(\lambda + \xi, 0)}.
\]

(4.20)

Using now the relation (4.19) and changing the integration variables \( \zeta_j \mapsto w_j = \gamma(\zeta_j - \lambda) \), we get

\[
Z_{r_1, \ldots, r_s}^{\text{top}} = Z_s a^{s(N-s)} \prod_{j=1}^{s} \int_{C_1} \cdots \int_{C_1} \prod_{j=1}^{s} \frac{(t^2 w_j - 2\Delta t + 1)^{r_j-1}}{(w_j - 1)^{r_j}} \times \prod_{1 \leq j < k \leq s} \frac{w_k - w_j}{t^2 w_j w_k - 2\Delta t w_j + 1} h_{s,s} (w_1, \ldots, w_s) \frac{d^s w}{(2\pi)^s}.
\]

(4.21)

Formula (4.21) is one of our main results here.

Note that our last integral representation (4.21) differs significantly from (4.14), which is based on the 'coordinate wavefunction' representation [50]. It appears that
these two representations can be related by a suitable deformation of integration contours. Such relation is most easily seen at the level of (4.18), that is for the inhomegeneous version of the model, see appendix A for details.

It is worth mentioning that the expression (4.21) also implies, through crossing symmetry, an analogous \((N-s)\)-fold integral representation for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{bot}}\), depending on the row configuration through the position \(\tilde{r}_1, \ldots, \tilde{r}_{N-s}\) of the \(N-s\) down arrows. This representation is given in appendix B, see formula (B.5).

Finally, we emphasize that the two procedures leading from the inhomegeneous representations (3.16) and (3.18) to the multiple integral representations (4.13) and (4.21) for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{bot}}\) and \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{top}}\) are quite different and cannot be interchanged.

To recapitulate, we have thus in total three different representations for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{top}}\) and three for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{bot}}\). Concerning \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{top}}\), we have the two representations (4.21) and (B.4), besides the well-known ‘coordinate wavefunction’ representation, see (4.14) for its multiple integral form. Corresponding representations for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{bot}}\), related to (4.21) and (B.4) by crossing symmetry, are given by (B.5) and (4.13), respectively. Each of these representations appears to depend on the row configuration either through the \(r\)'s or through the \(\bar{r}\)'s.

5. Emptiness formation probability

In this section, we show how the integral representation for the emptiness formation probability derived in [34] can be recovered from the integral representations for \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{bot}}\) and \(Z_{\tilde{r}_1, \ldots, \tilde{r}_s}^{\text{top}}\). The alternative derivation presented here relies on certain relation involving antisymmetrization with respect to two set of variables, recently proved in [41].

5.1. Antisymmetrization relations. Given a multivariate function, we introduce the antisymmetrizer

\[
\text{Asym}_{z_1, \ldots, z_s} f(z_1, \ldots, z_s) = \sum_{\sigma} (-1)^{[\sigma]} f(z_{\sigma_1}, \ldots, z_{\sigma_s}),
\]

where the sum is taken over the permutations \(\sigma : 1, \ldots, s \mapsto \sigma_1, \ldots, \sigma_s\), with \([\sigma]\) denoting the parity of \(\sigma\).

We discuss here two antisymmetrization relations playing a relevant role in the calculation of integral representations for the emptiness formation probability.

The first antisymmetrization relation originates from the following relation, established and proven by Kitanine et al., see [20], Prop. C1:

\[
\text{Asym}_{\lambda_1, \ldots, \lambda_s} \left[ \prod_{j, k=1}^{s} \frac{a(\lambda_j, \nu_k) \prod_{k=j+1}^{s} b(\lambda_j, \nu_k)}{\prod_{1 \leq j < k \leq s} e(\lambda_k, \lambda_j)} \right] = \frac{\prod_{1 \leq j < k \leq s} d(\lambda_k, \lambda_j)}{\prod_{j, k=1}^{s} \prod_{1 \leq j < k \leq s} e(\lambda_k, \lambda_j)} Z_s(\lambda_1, \ldots, \lambda_s; \nu_1, \ldots, \nu_s). \tag{5.1}
\]

Here, the functions involved are defined in (2.2), (2.5), and (3.15), and \(Z_s\) denotes the Izergin-Korepin partition function \(2^{23}\) for an \(s \times s\) lattice.

We are interested in the particular case of (5.1) where \(\nu_j = 0, j = 1, \ldots, s\). We set

\[
z_j = \gamma(-\lambda_j + \eta), \quad j = 1, \ldots, s, \tag{5.2}
\]
where the function $\gamma(\xi)$ is defined in (4.20). We intend to use (4.19), so it is also convenient to introduce the notation:

$$u_j = \gamma(\lambda_j - \lambda), \quad j = 1, \ldots, s.$$  

One has

$$u_j = -z_j - \frac{z_j - 1}{(t^2 - 2\Delta t)z_j + 1}, \quad j = 1, \ldots, s,$$  

where, as above, $\Delta = \cos 2\eta$ and $t \equiv b(\lambda, 0)/a(\lambda, 0)$. Resorting now to (4.19), relation (5.1) at $\nu_j = 0, \ j = 1, \ldots, s$ can be rewritten as the following antisymmetrization relation:

$$\text{Asym}_{z_1, \ldots, z_s} \left[ \prod_{j=1}^s \frac{1}{u_j} \prod_{1 \leq j < k \leq s} (t^2 z_j z_k - 2\Delta t z_k + 1) \right]$$

$$= (-1)^{\frac{s(s-1)}{2}} a_s^{(s-1)} \prod_{1 \leq j < k \leq s} (z_k - z_j) \prod_{j=1}^s \frac{1}{u_j} h_s(u_1, \ldots, u_s).$$  

Here, and everywhere below, we assume that $u_j \equiv u(z_j)$, with the function $u(z_j)$ defined by the right-hand side of (5.3). We refer for more details to [34].

The second antisymmetrization relation we wish to discuss reads [41]:

$$\text{Asym}_{x_1, \ldots, x_s \ y_1, \ldots, y_s} \left[ \prod_{j=1}^s \frac{(x_j y_j)^{s-j}}{1 - \prod_{i=1}^s x_i y_i} \prod_{1 \leq j < k \leq s} (x_j x_k - 2\Delta x_k + 1)(y_j y_k - 2\Delta y_k + 1) \right]$$

$$= \prod_{j,k=1}^s (x_j + y_k - 2\Delta x_j y_k) \det_{1 \leq j,k \leq s} [\psi(x_j, y_k)],$$  

where

$$\psi(x, y) = \frac{1}{(1 - xy)(x + y - 2\Delta xy)}.$$  

The relation (5.5) can be proven by induction in $s$, using the symmetries in the involved variables and comparing singularities of both sides, along the lines of the proof of the relation (5.1) given in [20], see appendix C therein.

It is to be mentioned that similar relations appear in connection with the theory of symmetric polynomials [56–60]. Relation (5.5) does not seem to be a particular case of any of them, even if sharing the property that its right-hand side is expressible in terms of the Izergin-Korepin partition function (2.3). Instead, it appears to extend to the trigonometric case some antisymmetrization relation originally derived in the rational case by Gaudin, see [53], Appendix B. Also, (5.5) generalizes some antisymmetrization relation given in [61], in the context of the asymmetric simple exclusion process.

It is convenient to introduce the notation

$$W_s(x_1, \ldots, x_s; y_1, \ldots, y_s) = \frac{\prod_{j,k=1}^s (x_j + y_k - 2\Delta x_j y_k)}{\prod_{1 \leq j < k \leq s} (x_k - x_j) \prod_{1 \leq j < k \leq s} (y_k - y_j)} \times \det_{1 \leq j,k \leq s} [\psi(x_j, y_k)].$$  

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Observe that \( W_s(x_1, \ldots, x_s; y_1, \ldots, y_s) \) is a rational function of the form
\[
W_s(x_1, \ldots, x_s; y_1, \ldots, y_s) = \frac{P_s(x_1, \ldots, x_s; y_1, \ldots, y_s)}{\prod_{j,k=1}^s (1 - x_j y_k)},
\] (5.7)
where \( P_s(x_1, \ldots, x_s; y_1, \ldots, y_s) \) is a polynomial of degree \( s - 1 \) in each variable, separately symmetric under permutations of the variables within each set.

The function (5.6) is closely related to the Izergin-Korepin partition function (2.3). Indeed, let us set
\[
x_j = \frac{a(\lambda_j, \xi + \eta)}{b(\lambda_j, \xi + \eta)}, \quad y_j = \frac{a(\xi, \nu_j)}{b(\xi, \nu_j)}, \quad j = 1, \ldots, s,
\] (5.8)
where \( \xi \) is an arbitrary parameter, to be fixed later on. Then, we have
\[
\det_{1 \leq j,k \leq s} [\psi(x_j, y_k)] = \frac{1}{e^{3s}} \prod_{j=1}^s [b(\lambda_j, \xi + \eta) b(\xi, \nu_j)]^2 \det_{1 \leq j,k \leq s} [\varphi(\lambda_j, \nu_k)].
\]
Plugging this into (5.6) yields
\[
W_s(x_1, \ldots, x_s; y_1, \ldots, y_s) = (-1)^s \frac{\prod_{j=1}^s [b(\lambda_j, \xi + \eta) b(\xi, \nu_j)]}{e^{3s} \prod_{j,k=1}^s b(\lambda_j, \nu_k)} \times Z_s(\lambda_1, \ldots, \lambda_s; \nu_1, \ldots, \nu_s). \quad (5.9)
\]
Note that here the parameter \( \xi \) enters only the prefactor and not the Izergin-Korepin partition function.

Consider now (5.9) in the partially homogeneous limit where \( \nu_j \to 0, \quad j = 1, \ldots, s \). To make contact with our previous discussion let us also identify \( \xi = \lambda \). By comparison of (5.8) with (5.2), in the limit we get
\[
x_j = t z_j, \quad y_j = t^{-1}, \quad j = 1, \ldots, s.
\]
Recalling (4.19), we thus obtain
\[
W_s(tz_1, \ldots, tz_s; t^{-1}, \ldots, t^{-1}) = \frac{(-1)^s Z_s}{e^{3s} b^{s(s-1)}} \prod_{j=1}^s \frac{1}{(z_j - 1) u_j^{s-1}} h_{s,s}(u_1, \ldots, u_s),
\] (5.10)
where, as usual, \( u_j \)'s and \( z_j \)'s are related by (5.3).

Another result of interest concerns the evaluation of the quantity \( P_s(x_1, \ldots, x_s; x_1^{-1}, \ldots, x_s^{-1}) \), see (5.7). From (5.7), (5.8), and (5.9), it follows that:
\[
P_s(x_1, \ldots, x_s; y_1, \ldots, y_s) = e^{s^2 - 2s} \prod_{j=1}^s \frac{1}{[b(\lambda_j, \xi + \eta) b(\xi, \nu_j)]^{s-1}} \times Z_s(\lambda_1, \ldots, \lambda_s; \nu_1, \ldots, \nu_s).
\]
Setting now \( \nu_j = \lambda_j - \eta, \quad j = 1, \ldots, s \), we get
\[
P_s(x_1, \ldots, x_s; x_1^{-1}, \ldots, x_s^{-1}) = e^{s^2 - 2s} \prod_{j=1}^s \frac{1}{[b(\lambda_j, \xi + \eta) b(\xi, \lambda_j - \eta)]^{s-1}} \times Z_s(\lambda_1, \ldots, \lambda_s; \lambda_1 - \eta, \ldots, \lambda_s - \eta). \quad (5.11)
\]
The last line is easily evaluated thanks to the recursion relation for the inhomogeneous partition function \[45\). We have
\[
Z_s(\lambda_1, \ldots, \lambda_s; \lambda_1 - \eta, \ldots, \lambda_s - \eta) = e^s \prod_{j,k=1 \atop j \neq k}^s \sin(\lambda_j - \lambda_k + 2\eta).
\]

Reexpressing now the right-hand side of \((5.11)\) in terms of variables \(x_1, \ldots, x_j\), we finally obtain
\[
P_s(x_1, \ldots, x_s; x_1^{-1}, \ldots, x_s^{-1}) = \prod_{j=1}^s \frac{1}{x_j^2 - 1} \prod_{j,k=1 \atop j \neq k}^s (x_j x_k - 2\Delta x_j + 1),
\]
which will turn out useful below.

### 5.2. Known integral representations.

In \[34\] various representations have been worked out for the emptiness formation probability \(F_N^{(r,s)}\). In particular, using the Yang-Baxter commutation relations, and next performing the homogeneous limit, the following representation in terms of the orthogonal polynomials was obtained:
\[
F_N^{(r,s)} = (-1)^s \frac{K_{N-s}(\partial_{\varepsilon_1}) \cdots K_{N-s}(\partial_{\varepsilon_s})}{K_{N-1}(\partial_{\varepsilon_1}) \cdots K_{N-1}(\partial_{\varepsilon_s})} \prod_{j=1}^s \left\{ \frac{[\omega(\varepsilon_j)]^{N-r}}{[\omega(\varepsilon_j) - 1]^N} \right\}
\]
\[
\times \prod_{1 \leq j < k \leq s} \frac{1 - \tilde{\omega}(\varepsilon_j)[\omega(\varepsilon_k) - 1]}{[\omega(\varepsilon_j)\omega(\varepsilon_k) - 1]} \bigg|_{\varepsilon_1, \ldots, \varepsilon_s = 0}. \tag{5.13}
\]

Here, we use the same notations as in section \[4.1\]. In particular, \(K_n(x)\) denote the orthogonal polynomials \[45\], associated to the Hankel matrix \[2.7\], and the functions \(\omega(\varepsilon)\) and \(\tilde{\omega}(\varepsilon)\) are defined in \[4.6\].

The identity \[4.9\] when applied to \(5.13\), yields for the emptiness formation probability the following multiple integral representation:
\[
F_N^{(r,s)} = (-1)^s \frac{Z_s}{s! a^{s(s-1)/2}} \int_{C_0} \cdots \int_{C_0} \frac{(t^2 - 2\Delta t)z_j + 1} {z_j(z_j - 1)^{s-j+1}} h_{N,s}(z_1, \ldots, z_s) \frac{d^sz}{(2\pi i)^s}. \tag{5.14}
\]

Here, \(C_0\) denotes, as before, a small simple anticlockwise oriented contour around the point \(z = 0\), and the function \(h_{N,s}(z_1, \ldots, z_s)\) is defined in \(4.10\).

Note that the integrand in \(5.14\) is not symmetric with respect to the permutation of the integration variables. However, the antisymmetrization relation \(5.4\) allows to write down the essentially equivalent representation
\[
F_N^{(r,s)} = \left\{ -1 \right\}^s Z_s a^{s(s-1)/2} \int_{C_0} \cdots \int_{C_0} \frac{(t^2 - 2\Delta t)z_j + 1} {z_j(z_j - 1)^s} \prod_{j,k=1 \atop j \neq k}^s \frac{z_k - z_j}{t^2 z_j z_k - 2\Delta t z_j + 1} h_{N,s}(z_1, \ldots, z_s)h_{s,s}(u_1, \ldots, u_s) \frac{d^sz}{(2\pi i)^s}. \tag{5.15}
\]
Here, $u_j$'s are given in terms of $z_j$'s by (5.3). The representation (5.15) with the symmetric integrand has been proved of importance, for example, in the evaluation of the phase separation curves of the model [37, 38].

Given integral representation (5.15) for the emptiness formation probability, a natural question concerns the possibility of deriving it by suitably combining the integral expressions obtained above for $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$. As we will show below, the answer is affirmative.

### 5.3. An alternative and simpler derivation.

We propose here an alternative derivation of (5.15), with respect to the one originally proposed in [34]. Here we start from the relation (2.11), and substitute the integral representations (4.14) and (4.13) for $Z_{r_1,\ldots,r_s}^{\text{bot}}$ and $Z_{r_1,\ldots,r_s}^{\text{top}}$ in the expression for the row configuration probability (2.9). An essential role in the derivation is played by relation (5.5).

For convenience, we change the integration variables $z_j \mapsto x_j/t$, $j = 1, \ldots, s$ in (4.13), that yields

$$Z_{r_1,\ldots,r_s}^{\text{bot}} = \frac{Z_N}{a^{s(N-1)}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s \frac{1}{x_j} \prod_{1 \leq j < k \leq s} \frac{x_k - x_j}{x_j x_k - 2\Delta x_j + 1}$$

$$\times h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right) \prod_{1 \leq j < k \leq s} [(y_k - y_j)(y_j y_k - 2\Delta y_k + 1)] \frac{d^s x}{(2\pi i)^s}, (5.16)$$

and also we change $w_j \mapsto 1/(ty_j)$ in (4.14), that yields

$$Z_{r_1,\ldots,r_s}^{\text{top}} = e^s a^{s(N-1)} \oint_{C_{1/s}} \cdots \oint_{C_{1/s}} \prod_{j=1}^s \frac{1}{(ty_j - 1)^{y_j}}$$

$$\times \prod_{1 \leq j < k \leq s} [(y_k - y_j)(y_j y_k - 2\Delta y_k + 1)] \frac{d^s y}{(2\pi i)^s}. (5.17)$$

Then, inserting into (2.10) and (2.11), we have

$$F_N^{(r,s)} = \oint_{C_{1/s}} \cdots \oint_{C_{1/s}} \frac{d^s y}{(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s y_j^{-1} (ty_j - 1)^s$$

$$\times \prod_{1 \leq j < k \leq s} [(y_k - y_j)(y_j y_k - 2\Delta y_k + 1)] \frac{1}{x_j x_k - 2\Delta x_j + 1} h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right)$$

$$\times \prod_{1 \leq r_1 < r_2 < \cdots < r_s \leq r} \prod_{j=1}^s \frac{1}{(x_j y_j)^{r_j}} \frac{d^s x}{(2\pi i)^s}. (5.17)$$

To prove that this representation indeed reduces to (5.15), one has to perform the multiple sum and evaluate $s$ integrations.

First, let us focus on the multiple sum in (5.17). It is clear that, due to the integrations around the points $x_j = 0$, $j = 1, \ldots, s$, one can extend the sum over the values $1 \leq r_1 < r_2 < \cdots < r_s \leq r$ to the values $-\infty < r_1 < r_2 < \cdots < r_s \leq r$, obtaining:

$$\sum_{-\infty < r_1 < r_2 < \cdots < r_s \leq r} \prod_{j=1}^s \frac{1}{(x_j y_j)^{r_j}} = \prod_{j=1}^s \frac{1}{(x_j y_j)^{r_j}} \frac{1}{\prod_{l=1}^s \frac{1}{(1 - t_j x_l y_l)}}. (5.18)$$
Hence, (5.17) simplifies to

\[
F_N^{(r,s)} = \oint_{C_{1/t}} \cdots \oint_{C_{1/t}} \frac{d^s y}{(2\pi)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s \frac{1}{(ty_j - 1)^s y_j^{r+s+j}} \prod_{1 \leq j < k \leq s} \frac{(y_k - y_j)(y_j y_k - 2\Delta y_k + 1)(x_k - x_j)}{x_j x_k - 2\Delta x_j + 1} \\
\times h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right) \frac{d^s x}{(2\pi)^s}.
\]

To perform the integrations, we first observe that the form of the integrand in (5.19) allows us to apply the antisymmetrization relation (5.5). We obtain

\[
F_N^{(r,s)} = \frac{1}{(s!)^2} \oint_{C_{1/t}} \cdots \oint_{C_{1/t}} \frac{d^s y}{(2\pi)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s x_j^j (ty_j - 1)^s y_j^{r+s+j} \prod_{1 \leq j < k \leq s} (x_k - x_j)^2 (y_k - y_j)^2 \\
\times \prod_{1 \leq j < k \leq s} \frac{(x_j - x_k)^2 (y_j - y_k)^2}{x_j x_k - 2\Delta x_j + 1} \prod_{j,k=1}^s (y_j - y_k) \Phi(y_1, \ldots, y_s) d^s y, \]

where we have used the notation (5.10).

To evaluate the integrals over the variables \(y_1, \ldots, y_s\), we resort to the following identity:

\[
\oint_{C_{w_1, \ldots, w_s}} \cdots \oint_{C_{w_1, \ldots, w_s}} \prod_{j \neq k}^s (y_j - y_k) \Phi(y_1, \ldots, y_s) \frac{d^s y}{(2\pi)^s} = s! \Phi(w_1, \ldots, w_s),
\]

where \(C_{w_1, \ldots, w_s}\) is a simple closed counterclockwise contour in the complex plane, enclosing points \(w_1, \ldots, w_s\, and \, \Phi(y_1, \ldots, y_s)\) is a generic symmetric function, analytic in each of its variables within a domain containing \(C_{w_1, \ldots, w_s}\). In our case \(w_j = t^{-1}, j = 1, \ldots, s\, and \, we\, obtain

\[
F_N^{(r,s)} = \frac{ts(r-1)}{s!} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^s \frac{1}{x_j^j} \prod_{j \neq k} x_j - x_k \frac{x_j - x_k}{x_j x_k - 2\Delta x_j + 1} \\
\times W_s(x_1, \ldots, x_s; t^{-1}, \ldots, t^{-1}) h_{N,s} \left( \frac{x_1}{t}, \ldots, \frac{x_s}{t} \right) \frac{d^s x}{(2\pi)^s}.
\]

Finally, changing back the integration variables \(x_j \mapsto tz_j, j = 1, \ldots, s\, and \, using (5.10), we recover (5.15).

6. Another representation for the emptiness formation probability

In this section we derive an alternative representation for the emptiness formation probability, which supplements the already known representation (5.15). Crucial ingredients in such derivation are: i) the use of relation (2.12) rather than (2.11) to express the emptiness formation probability in terms of \(Z_{r_1, \ldots, r_s}^{\text{top}}\) and \(Z_{r_1, \ldots, r_s}^{\text{bot}}\); and ii) the use of representation (4.21), rather than (4.14), for \(Z_{r_1, \ldots, r_s}^{\text{top}}\). Although
the starting point is quite different, the derivation has some similarities with that of section 5.3 in particular, we will again use the antisymmetrization relation (5.5).

6.1. Performing summations. Our starting point is relation (2.12), with $H_{(N-1)c^{s+n}}^{(r_1,...,r_{s+n})}$ given by (2.9). As for $Z_{(N-1)cs+n}^{(r_1,...,r_{s+n})}$, we resort to representation (4.21), which, in the present setup, reads

$$Z_{r_1,...,r_{s+n}}^{top} = Z_{s+n}a(s+n)(N-s-n) \times \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s+n} \frac{1}{w_j} \prod_{1 \leq j < k \leq s+n} \frac{w_k - w_j}{w_j w_k - 2\Delta w_j + 1} \times h_{s+n,s+n} \left( \frac{(2\Delta t - 1)w_1 - t}{t(tw_1 - 1)}, \ldots, \frac{(2\Delta t - 1)w_{s+n} - t}{t(tw_{s+n} - 1)} \right) \frac{d^{s+n}w}{(2\pi)^{s+n}}.$$

Having $r_j = j, j = 1, \ldots, s$, integration over variables $w_1, \ldots, w_s$ in this order, is easily done, since each time the pole at origin is of order one. We thus have

$$Z_{s+n,t_1,\ldots,t_n}^{top} = Z_{s+n}a(s+n)(N-s-n) \times \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{n} \frac{1}{w_j} \prod_{1 \leq j < k \leq n} \frac{w_k - w_j}{w_j w_k - 2\Delta w_j + 1} \times h_{s+n,n} \left( \frac{(2\Delta t - 1)w_1 - t}{t(tw_1 - 1)}, \ldots, \frac{(2\Delta t - 1)w_n - t}{t(tw_n - 1)} \right) \frac{d^n w}{(2\pi)^n}, \quad (6.1)$$

where we have used relation (4.11), and we have relabelled $r_{s+j} \rightarrow l_j := r_{s+j} - s, j = 1, \ldots, n$. Note that we would have not been able to evaluate these first s integration if we had started from expression (4.14), rather than (4.21), for $Z_{s+n,t_1,\ldots,t_n}^{top}$. We turn now to $Z_{s+n,t_1,\ldots,t_n}^{bot}$, and resort to representation (4.13), or equivalently, up to change of the integration variables, to (5.10), which, in the present setup, reads:

$$Z_{r_1,...,r_{s+n}}^{bot} = \frac{Z_{N}}{a(s+n)(N-1)c^{s+n}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s+n} \frac{1}{z_j} \times \prod_{1 \leq j < k \leq s+n} \frac{z_k - z_j}{z_j z_k - 2\Delta z_j + 1} h_{N,s+n} \left( \frac{z_1}{t}, \ldots, \frac{z_{s+n}}{t} \right) \frac{d^{s+n}z}{(2\pi)^{s+n}}.$$ 

Again, having $r_j = j, j = 1, \ldots, s$, the first s integrations are easily done, with the result

$$Z_{s+n,t_1,\ldots,t_n}^{bot} = \frac{Z_{N-s}}{c^n a^{s+N-1}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{n} \frac{1}{z_j} \times \prod_{1 \leq j < k \leq n} \frac{z_k - z_j}{z_j z_k - 2\Delta z_j + 1} h_{N-s,n} \left( \frac{z_1}{t}, \ldots, \frac{z_n}{t} \right) \frac{d^n z}{(2\pi)^n}, \quad (6.2)$$

where we have used the relation (4.12).
Substituting now (6.1) and (6.2) into (2.12), see (2.9), we have

\[
F_N^{(s+n,s)} = \frac{Z_{s+n}Z_{N-s}a^{2s(N-s-n)}}{Z_Nc^n a^n n^{-1}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{n} \frac{1}{1-tw_j} \times \sum_{1 \leq j_1 < \cdots < j_n \leq N-s} \prod_{1 \leq j < k \leq n} \frac{1}{(w_jz_j)^j} \prod_{1 \leq j < k \leq n} \frac{(w_k-w_j)(z_k-z_j)}{(w_jw_k-2\Delta w_j+1)(z_jz_k-2\Delta z_j+1)} \times h_{s+n,n} \left( \frac{(2\Delta t-1)w_1-t}{t(tw_1-1)}, \ldots, \frac{(2\Delta t-1)w_n-t}{t(tw_n-1)} \right) \times \left( \frac{z_1}{t}, \ldots, \frac{z_n}{t} \right) \frac{d^n w}{(2\pi i)^n} \frac{d^n z}{(2\pi i)^n} .
\]

Observing that, similarly to section 5.3, the integral over \( z_j \) vanishes for \( l_j < 0 \), we can extend the sum over all negative values. Using then formula (5.18), we obtain

\[
F_N^{(s+n,s)} = \frac{Z_{s+n}Z_{N-s}a^{2s(N-s-n)}}{Z_Nc^n a^n n^{-1}} \times \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{n} \frac{1}{1-tw_j} \times \prod_{1 \leq j < k \leq n} \frac{(w_j-w_k)(z_j-z_k)}{(w_jw_k-2\Delta w_j+1)(z_jz_k-2\Delta z_j+1)} \times h_{s+n,n} \left( \frac{(2\Delta t-1)w_1-t}{t(tw_1-1)}, \ldots, \frac{(2\Delta t-1)w_n-t}{t(tw_n-1)} \right) \times h_{N-s,n} \left( \frac{z_1}{t}, \ldots, \frac{z_n}{t} \right) \frac{d^n w}{(2\pi i)^n} \frac{d^n z}{(2\pi i)^n} .
\]

We have thus performed the summations in (2.12).

6.2. Deforming integration contours. We want now to address the question of performing integration with respect to variables \( z_1, \ldots, z_n \). First of all, along the lines of section 5.3 let us symmetrize the integrand in (6.3) by resorting to relation (5.5). We obtain

\[
F_N^{(s+n,s)} = \frac{Z_{s+n}Z_{N-s}a^{2s(N-s-n)}}{(n!)^2 Z_Nc^n a^n n^{-1}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{n} \frac{1}{1-tw_j} \times \prod_{j=1}^{n} \frac{(w_jz_j)^j}{(w_jw_k-2\Delta w_j+1)(z_jz_k-2\Delta z_j+1)} \times h_{s+n,n} \left( \frac{(2\Delta t-1)w_1-t}{t(tw_1-1)}, \ldots, \frac{(2\Delta t-1)w_n-t}{t(tw_n-1)} \right) \times P_{\delta}(w_1, \ldots, w_n; z_1, \ldots, z_n) h_{N-s,n} \left( \frac{z_1}{t}, \ldots, \frac{z_n}{t} \right) \frac{d^n w}{(2\pi i)^n} \frac{d^n z}{(2\pi i)^n} ,
\]

where we have used the notation (5.7).

Turning now to the integration with respect to variables \( z_1, \ldots, z_n \), we observe that the corresponding integration contours \( C_0 \) can be deformed into new contours \( C_{1/w} := C_{1/w_1} \cup \cdots \cup C_{1/w_n} \), enclosing the poles at \( 1/w_j, j = 1, \ldots, n \), induced by
the factor $\prod_{1 \leq j, k \leq n} (1 - w_j z_k)$ in the denominator of (6.3), without changing the result.

The crucial ingredient permitting such deformation of contours is that the poles induced by the double product in the second line of (6.4) give a vanishing contribution to the integral. This should not come as a surprise, since such double product is the remnant of analogous double products appearing in (3.10) and (4.17). As already commented thereafter, the poles associated to such double products are exactly compensated by corresponding zeroes. However, once the homogeneous limit is performed, one loose track of this fact, and the mechanism of cancellation becomes quite subtle. A direct and detailed description is therefore appropriate.

To start with, let us focus on variable $z_n$. The poles induced by the double product in the second line of (6.4) can be divided into a first set of $n-1$ poles at positions $(2\Delta z_j - 1)/z_j$, $j = 1, \ldots, n-1$, and a second set of $n-1$ poles at positions $1/(2\Delta - z_j)$, $j = 1, \ldots, n-1$. Concerning the first set of poles, recalling the symmetry of the integrand under interchange of the $z_j$'s, let us focus for definiteness on the pole corresponding to $j = 1$, and show that the residue of the integrand in (6.4) at $z_n = (2\Delta z_1 - 1)/z_1$ vanishes upon integration with respect to variable $z_1$ over contour $C_0$.

Indeed, let us inspect the small $z_1$ behaviour of the residue of the integrand at $z_n = (2\Delta z_1 - 1)/z_1$. Such behaviour results from the contribution of four terms. Let us analyze them in turn. For the first relevant term, it is easily checked that

$$\lim_{z_n \to 2\Delta z_1} \frac{1}{z_j} \left| \frac{1}{z_n - \frac{2\Delta z_1 - 1}{z_1}} \sim 1, \quad z_1 \to 0.\right.$$

The next term, which contains the considered pole, requires a careful but nevertheless straightforward calculation, that gives

$$\lim_{z_n \to 2\Delta z_1} \left( z_n - \frac{2\Delta z_1 - 1}{z_1} \right) \prod_{j=1}^{n} \frac{z_k - z_j}{z_j z_k - 2\Delta z_j + 1} \sim \frac{1}{z_1^2}, \quad z_1 \to 0.$$

As for the third relevant term, recalling that $P_n(w_1, \ldots, w_n; z_1, \ldots, z_n)$ is a polynomial of degree $n-1$ in each of its variables, it follows that

$$\frac{P_n(w_1, \ldots, w_n; z_1, \ldots, z_n)}{\prod_{1 \leq j, k \leq n} (1 - w_j z_k)} \bigg|_{z_n = \frac{2\Delta z_1 - 1}{z_1}} \sim z_1, \quad z_1 \to 0.$$

Finally, concerning the contribution of $h_{N,n}(z_1/t, \ldots, z_n/t)$, its calculation is non-trivial. However we can resort to the following property

$$h_{N,n} \left( \frac{z_1}{t}, \ldots, \frac{z_n}{t} \right) \bigg|_{z_n = \frac{2\Delta z_1 - 1}{z_1}} \sim z_1, \quad z_1 \to 0, \quad (6.5)$$

which follows from the determinantal structure of the Izergin-Korepin partition function, and has been proven in [35], see appendix therein.

In all, it follows that the residue of the integrand in (6.4) at $z_n = (2\Delta z_1 - 1)/z_1$ is $O(1)$ as $z_1 \to 0$, and thus vanishes upon integration with respect to variable $z_1$ over contour $C_0$. Due to the symmetry of the integrand under interchange of the $z_j$'s, the same holds for variables $z_2, \ldots, z_{n-1}$. The integration contour for $z_n$ can thus be deformed from $C_0$ to a new contour $C'$ enclosing the origin and the poles at $z_n = (2\Delta z_j - 1)/z_j$, $j = 1, \ldots, n-1$.  

30
Inspecting now the large $|z_n|$ behaviour of the integrand, we observe that the double product in the second line is $O(1)$. We also have

$$P_n(w_1, \ldots, w_n; z_1, \ldots, z_n) \sim |z_n|^{n-1}, \quad |z_n| \to \infty,$$

and

$$h_{N-s,n}(\frac{z_1}{t}, \ldots, \frac{z_n}{t}) \sim |z_n|^{N-s-1}, \quad |z_n| \to \infty.$$

As a result, the whole integrand is $O(1/|z_n|^2)$ as $|z_n| \to \infty$, and the integration over $z_n$ along a large contour vanishes. It follows that we can deform the integration contour over variable $z_n$ from $C'$, defined in the previous paragraph to a new contour $C''$ enclosing the poles at $z_n = 1/(2\Delta - z_j)$, $j = 1, \ldots, n-1$, and the poles at $z_n = 1/w_l$, $l = 1, \ldots, n$, induced by the factor $\prod_{1 \leq j,k \leq n}(1 - w_j z_k)$ in the denominator of (6.4).

Implementing the same procedure for other integration variables as well, we conclude that for each $z_j$, the corresponding integration contour can be deformed into a contour $C'''$ enclosing only the poles at $z_j = 1/w_l$, $l = 1, \ldots, n$, induced by the denominator in the last line of (6.4), and the poles at $z_j = 1/(2\Delta - z_l)$, $l = 1, \ldots, j-1$.

Now, concerning the poles at $z_n = 1/(2\Delta - z_j)$, $j = 1, \ldots, n$, it appears that their contributions vanishes as well. Indeed, focusing again, for definiteness, on the pole corresponding to $j = 1$, $z_n = 1/(2\Delta - z_1)$, consider the following crucial property of the multivariate polynomial $P_n(w_1, \ldots, w_n; z_1, \ldots, z_n)$,

$$P_n(w_1, \ldots, w_n; z_1, \ldots, z_n) \bigg|_{z_n = \frac{1}{z_1}} \sim \left(1 - \frac{1}{w_k} z_1\right), \quad z_1 \to \frac{1}{w_k}, \quad k = 1, \ldots, n,$$

whose proof goes along the lines of (6.5). This property implies that, after evaluation of the residue of the integrand in (6.4) at $z_n = 1/(2\Delta - z_1)$, the poles at $z_1 = 1/w_k$, $k = 1, \ldots, n$, are all cancelled. It other words, the residue of the integrand at $z_n = 1/(2\Delta - z_j)$, when subsequently integrated with respect to $z_j$ over contour $C_{1/w}$, gives a vanishing contribution. It follows that we can deform the integration contour over variable $z_n$ from $C''$ to shrink it down to $C_{1/w}$.

Implementing the same procedure for other integration variables as well, we have eventually shown that we can deform each of the integration contours for variables $z_1, \ldots, z_n$ from $C_0$ to a new contour $C_{1/w}$ enclosing the poles at $1/w_k$, $k = 1, \ldots, n$. In other words, (6.4) can be rewritten as

$$F_{N}^{(s+n,s)} = \frac{Z_{s+n}Z_{N-s-n}a^{2s(N-s-n)}}{(n!)^2 Z_{N}c^{n}a^{n(n-1)}} \times \oint_{C_0} \cdots \oint_{C_0} \frac{d^n w}{(2\pi)^n} \oint_{C_{1/w}} \cdots \oint_{C_{1/w}} \frac{1}{\prod_{j=1}^{n}(1-tw_j)(w_j z_j)^{N-s}} \times \prod_{j,k=1}^{n} \frac{(w_k - w_j)(z_k - z_j)}{(w_j w_k - 2\Delta w_j + 1)(z_j z_k - 2\Delta z_j + 1)} \times h_{s+n,n} \left(\frac{(2\Delta t - 1)w_1 - t}{t(tw_1 - 1)}, \ldots, \frac{(2\Delta t - 1)w_n - t}{t(tw_n - 1)}\right) \times h_{N-s,n} \left(\frac{z_1}{t}, \ldots, \frac{z_n}{t}\right) \frac{d^n z}{(2\pi)^n}. \quad (6.6)$$
The main benefit of this last expression is evident: the integrations with respect to the variables \( z_1, \ldots, z_n \) involve now only residues at simple poles.

6.3. Performing integrations. The integration with respect to variables \( z_1, \ldots, z_n \) in (6.6) can be easily performed. Indeed, resorting to (5.20), we obtain

\[
F_{N}^{(s+n,s)} = \frac{Z_{s+n} Z_{N-s} a^{2s(N-s-n)}}{n! Z_N c^n a^{n(n-1)}} \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{n} \frac{w_j^{n-2}}{1 - tw_j} \times \prod_{j,k=1 \atop j \neq k}^{n} \frac{(w_k - w_j)}{(w_j w_k - 2\Delta w_j + 1)(w_j w_k - 2\Delta w_j + 1)} \times h_{s+n,n} \left( \frac{(2\Delta t - 1)w_1 - t}{t(tw_1 - 1)}, \ldots, \frac{(2\Delta t - 1)w_n - t}{t(tw_n - 1)} \right) \times P_n(w_1, \ldots, w_n; \frac{1}{tw_1}, \ldots, \frac{1}{tw_n}) \, d^n w \quad \left(\frac{2\pi}{i}\right)^n.
\]

Recalling (5.12), we get

\[
F_{N}^{(s+n,s)} = \frac{Z_{s+n} Z_{N-s} a^{2s(N-s-n)}}{n! Z_N c^n a^{n(n-1)}} \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{n} \frac{1}{w_j(1 - tw_j)} \times \prod_{j,k=1 \atop j \neq k}^{n} \frac{w_k - w_j}{w_j w_k - 2\Delta w_j + 1} h_{N-s,n} \left( \frac{1}{w_1 t}, \ldots, \frac{1}{w_n t} \right) \times h_{s+n,n} \left( \frac{(2\Delta t - 1)w_1 - t}{t(tw_1 - 1)}, \ldots, \frac{(2\Delta t - 1)w_n - t}{t(tw_n - 1)} \right) \, d^n w \quad \left(\frac{2\pi}{i}\right)^n.
\]

Finally, performing the change of variables \( w_j \mapsto 1/(tw_j) \), \( j = 1, \ldots, n \), we arrive at the following expression

\[
F_{N}^{(s+n,s)} = \frac{Z_{s+n} Z_{N-s} a^{2s(N-s-n)} t^{n(n-1)}}{n! Z_N c^n a^{n(n-1)}} \int_{C_1} \cdots \int_{C_1} \prod_{j=1}^{n} \frac{1}{z_j - 1} \times \prod_{j,k=1 \atop j \neq k}^{n} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} h_{N-s,n}(z_1, \ldots, z_n) \times h_{s+n,n} \left( \frac{t^2 z_1 - 2\Delta t + 1}{t^2(z_1 - 1)}, \ldots, \frac{t^2 z_n - 2\Delta t + 1}{t^2(z_n - 1)} \right) \, d^n z \quad \left(\frac{2\pi}{i}\right)^n \quad (6.7)
\]

where we have shrunk each of the \( n \) integration contours from a very large one down to a small contour enclosing only the pole at \( z_j = 1, j = 1, \ldots, n \), thus ignoring the contribution of the poles in the double product. Once again, it can be shown that the total contribution of such poles indeed vanishes.

We consider the multiple integral representation (6.7) as one of the main results of the present paper. At variance with the previously known representation (5.15), the number of integrations is \( n = r - s \), that is the lattice distance of the point \((r, s)\) from the antidiagonal, rather than \( s \), the lattice distance from the top boundary.
This could be useful to investigate the behaviour of the emptiness formation probability in the so-called Hamiltonian limit, of relevance in connection with quantum quenches of the XXZ quantum spin chain [13][16].

The primary ingredients in the derivation of the multiple integral representation (6.7) are: i) the evaluation of two alternative representations for the components of the off-shell Bethe states $Z_{r_1,\ldots,r_s}^{\text{top}}$ and $Z_{r_1,\ldots,r_s}^{\text{bot}}$, that are essentially different from the longly known ‘coordinate wavefunction’ representation, and ii) the symmetrization relation (5.5). Hopefully, such ingredients could turn useful in the derivation of integral representations for more advanced correlation functions, such as polarization.

The availability of two distinct integral representations for the same quantity $F_N^{(r,s)}$ raises the natural question of their mutual relation. Consider for example the quantity $F_N^{(2,1)}$. Evaluating it by means of (5.15), with $r = 2$, $s = 1$, or (6.7), with $s = 1$, $n = 1$, and equating the two results, we get:

$$h_N'(0) = \left[t^2 + (1 - 2\Delta t + t^2)h_{N-1}'(1)\right]h_N(0)$$

relating the first derivatives of function $h_N(z)$, $h_{N-1}(z)$, evaluated at $z = 0$, and $z = 1$. Similarly, considering $F_N^{(3,2)}$, one obtain an identity relating the second derivatives of function $h_N(z)$, $h_{N-1}(z)$, $h_{N-2}(z)$, evaluated at $z = 0$, and $z = 1$. Remarkably, the coefficients appearing in such relations do not depend on $N$. The game can be played further, but the calculations become quite bulky very soon. The existence of such hierachy of identities hints at some nontrivial functional identity for $h_N(z)$, which is essentially Izergin-Korepin partition function with just one inhomogeneity.

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Appendix A. ‘Coordinate wavefunction’ representation

Here we consider an alternative representation for $Z_{r_1,\ldots,r_s}^{\text{top}}$ ensuing from the equivalence of the algebraic and coordinate versions of Bethe ansatz. This equivalence was first explicitly proved, as a side result, in [50] (see appendix D therein); see also book [1], Chap. VII.

In the case where all spectral parameters $\lambda_1, \ldots, \lambda_N$ are taken to the same value $\lambda$, with the remaining spectral parameters, $\nu_1, \ldots, \nu_s$ left free, it reads

$$Z_{r_1,\ldots,r_s}^{\text{top}} = e^\lambda \prod_{k=1}^s \left[a(\lambda, \nu_k)\right]^{N-1} \prod_{1 \leq j < k \leq s} \frac{1}{t_k - t_j} \times \sum_{\sigma} (-1)^{|\sigma|} \prod_{j=1}^s t_{r_j}^{-1} \prod_{1 \leq j < k \leq s} (t_{\sigma_j} t_{\sigma_k} - 2\Delta t_{\sigma_j} + 1), \quad (A.1)$$
where the sum is taken over the permutations \( \sigma : 1, \ldots, s \mapsto \sigma_1, \ldots, \sigma_s \), with \([\sigma]\) denoting the parity of \(\sigma\), and

\[
t_k = \frac{b(\lambda, \nu_k)}{a(\lambda, \nu_k)). \tag{A.2}
\]

Clearly, formula (A.1), modulo the antisymmetric factor \(\prod_{1 \leq j < k \leq s}(t_k - t_j)^{-1}\), coincides with the \(s\)-particle coordinate Bethe ansatz wavefunction. Representation (A.1) has been discussed in details in various contexts, especially in connection with the theory of symmetric polynomials, see, e.g., \[60, 62–64\].

We need now to perform the homogeneous limit \(\nu_1, \ldots, \nu_s \to 0\), which corresponds to sending \(t_1, \ldots, t_s \to t\). For this purpose we implement the procedure explained in section 4.1. Setting \(t_j = tw_j = t(1 + \xi_j)\), we write

\[
Z_{r_1, \ldots, r_s}^\text{top} = c^s \prod_{k=1}^{s} [a(\lambda, \nu_k)]^{N-1} \prod_{j=1}^{s} t_j^{r_j-j} \prod_{1 \leq j < k \leq s} \frac{1}{\xi_k - \xi_j} \times \exp(\xi_1 \partial_{w_1}) \cdots \exp(\xi_1 \partial_{w_s}) \prod_{j=1}^{s} w_j^{r_j-1} \times \prod_{1 \leq j < k \leq s} (t^2w_j w_k - 2\Delta tw_j + 1) \Bigg|_{w_1, \ldots, w_s = 1}.
\]

This expression is still for the inhomogeneous model; it represents an equivalent way of writing the multiple sum in (A.1).

Let us now perform the limit \(\xi_1, \ldots, \xi_s \to 0\). We readily get

\[
Z_{r_1, \ldots, r_s}^\text{top} = \frac{c^sa^{s(N-1)}}{s!} \prod_{j=1}^{s} t_j^{r_j-j} \prod_{1 \leq j < k \leq s} (\partial_{w_k} - \partial_{w_j}) \prod_{j=1}^{s} w_j^{r_j-1} \times \prod_{1 \leq j < k \leq s} (t^2w_j w_k - 2\Delta tw_j + 1) \Bigg|_{w_1, \ldots, w_s = 1} \tag{A.3}
\]

Representation (A.3) can now easily be turned into a multiple integral representation by reexpressing the values of derivatives as residues (or, equivalently, by using relation (4.15) to replace sums with integrals in (A.1) and performing the homogeneous limit). We obtain

\[
Z_{r_1, \ldots, r_s}^\text{top} = c^a s^{(N-1)} \prod_{j=1}^{s} t_j^{r_j-j} \oint_{C_1} \cdots \oint_{C_1} \prod_{j=1}^{s} \frac{w_j^{r_j-1}}{(w_j - 1)^s} \times \prod_{1 \leq j < k \leq s} [(w_j - w_k)(t^2w_j w_k - 2\Delta tw_j + 1)] \frac{dw}{(2\pi i)^s} \tag{A.4}
\]

thus reproducing (1.12). It is worth to mention a rather nontrivial property of representation (A.1): if the integrand is multiplied by a generic function \(f(w_1, \ldots, w_s)\), then the result of the multiple integration is unaltered (modulo a trivial prefactor \(f(1, \ldots, 1)\)) provided that \(f(w_1, \ldots, w_s)\) is symmetric in all its variables, regular and nonvanishing in the vicinity of \(w_1 = \cdots = w_s = 1\). This property follows simply from the fact that in representation (A.1) an arbitrary symmetric function of \(t_1, \ldots, t_s\) can be moved into or out of the sum.
Besides resorting to the equivalence of the algebraic and coordinate Bethe ansatz, formulae (A.1) and (A.3) can be also derived by other methods. For instance, one can start from the vertical monodromy matrix formulation (2.19) for $Z_{r_1,\ldots,r_s}^{\text{top}}$, and use the techniques of paper [65] to evaluate the matrix element. We also mention that expression (A.3), in a slightly different form, and for special values of parameters, $\Delta = \frac{1}{2}$ and $t = 1$, has also been derived in the context of enumerative combinatorics [49].

Let us conclude with a short comment on the relation between representation (4.18), derived in the present paper and the ‘coordinate wavefunction’ representation (A.1). It appears that the latter is given by the residue at the poles due to functions $a(\zeta_j, \nu_k)$ in the denominator of the integrand of representation (4.18). Indeed, let us replace in expression (4.18) all contours $C^{\lambda}$ with simple closed clockwise contours $C_{\nu^\eta} := C_{\nu^1} \cup \cdots \cup C_{\nu^s}$; evaluating the residues, we obtain

$$\prod_{k=1}^{s} [a(\lambda, \nu_k)]^{N-1} \prod_{1 \leq j < k \leq s} d(\nu_j, \nu_k) Z_s(\nu_1 - \eta, \ldots, \nu_s - \eta; \nu_1, \ldots, \nu_s)$$

$$\times \sum_{\sigma} (-1)^{|\sigma|} \prod_{j=1}^{s} t_{\sigma_j}^{\nu_{j}^\eta - 1} \prod_{1 \leq j < k \leq s} 1_{\nu_j^\eta \nu_k^\sigma},$$

with $t_k$ defined by (A.2). To proceed further we need the following identity

$$Z_s(\nu_1 - \eta, \ldots, \nu_s - \eta; \nu_1, \ldots, \nu_s) = e^s \prod_{j,k=1}^{s} e(\nu_j, \nu_k), \quad (A.5)$$

which can be proved by repeated use of the recursion relation for the inhomogeneous partition function obtained in [45]. Using identity (A.5) and reexpressing everything in terms of $t_k$’s, we readily recover the ‘coordinate wavefunction’ representation (A.1).

We have thus seen that, in representation (4.18), the integration contours $C^{\lambda}$ can be deformed into $C_{\nu^\eta}$ without modifying the result of the integration. Taking into account the periodicity of the integrand in (4.18), and its behaviour at infinity, it follows that the contribution arising from the poles due to functions $e(\zeta_k, \zeta_j)$ in (4.18) vanishes. This does not come as a surprise, since, as already observed, such poles are only apparent, being compensated by corresponding zeroes of the integrand. While this fact is evident in the inhomogeneous model, in the homogeneous limit we somehow lose track of it, and its verification requires some work, see discussion in section 6.2 for details.

Appendix B. Dual representations for the ‘top’ and ‘bottom’ partition functions

We collect here various results for $Z_{r_1,\ldots,r_s}^{\text{top}}$, whose derivation is based on relation (3.7). First of all it is convenient to resort to the alternative description of a given configuration of a row, namely in terms of the positions of the $N - s$ down arrows $\bar{r}_1, \ldots, \bar{r}_{N-s}$, rather than of the $s$ up arrows. We can now rewrite (2.19) as

$$Z_{r_1,\ldots,r_s}^{\text{top}} = \langle \psi_{\text{top}} | B_{\text{top}}(\lambda_N) \cdots B_{\text{top}}(\lambda_{r_{N-s}+1}) D_{\text{top}}(\lambda_{r_{N-s}}) B_{\text{top}}(\lambda_{r_{N-s}-1})$$

$$\times \cdots B_{\text{top}}(\lambda_{\bar{r}_1+1}) D_{\text{top}}(\lambda_{\bar{r}_1}) B_{\text{top}}(\lambda_{\bar{r}_1-1}) \cdots B_{\text{top}}(\lambda_1) | \psi_{\text{top}} \rangle. \quad (B.1)$$
To evaluate expression (B.1), we proceed similarly to what we have done in section 3.2 for \( Z_{\text{bot}}^{r_1, \ldots, r_s} \), namely we use (3.7) to move all operators \( D_{\text{top}}(\lambda) \) to the right, and make them act on \( |\uparrow|_{\text{top}} \), exploiting relation (2.17). Iterating \( N - s \) commutation relation (3.11), act on the right on \( |\uparrow|_{\text{top}} \), and multiplying form the left with the state vector \( \langle \psi_{\text{top}} | B_{\text{top}}(\lambda_N) \cdots B(\lambda_{N-s+1}) \rangle \), we obtain

\[
Z_{\text{top}}^{r_1, \ldots, r_s} = \prod_{\alpha=1}^{N} \prod_{k=1}^{r_{\alpha}} a(\lambda_{\alpha}, \nu_k) b(\lambda_{\alpha}, \nu_k) \times \frac{\prod_{j=1}^{N-r_{\alpha}-1} e(\lambda_{\alpha j}, \lambda_{\alpha})}{\prod_{j=1}^{r_{\alpha}} e(\lambda_{\alpha j}, \lambda_{\alpha j})} \det \mathcal{M} \big[ \alpha_1, \ldots, \alpha_{N-s+1}, \ldots, N \big].
\]

The functions \( d(\lambda, \lambda') \) and \( e(\lambda, \lambda') \) are defined in (2.25) and (3.15), respectively, and we have introduced, by analogy with (3.14), the function

\[
\tilde{\vartheta}_{\tau}(\lambda) := \prod_{\alpha=\tau+1}^{N} d(\lambda, \lambda_{\alpha}) \prod_{\alpha=1}^{\tau-1} e(\lambda_{\alpha}, \lambda_{\alpha}) \prod_{j=1}^{r_{\alpha}} a(\lambda_{\alpha j}, \lambda_{\alpha j}).
\]

The function \( \chi(\alpha, \beta) \) is defined in (3.17). We also recall that \( \mathcal{M} \big[ \alpha_1, \ldots, \alpha_{N-s+1}, \ldots, N \big] \) denotes the \( s \times s \) matrix obtained from matrix \( \mathcal{M} \), see (2.3), by removing rows \( \alpha_1, \ldots, \alpha_{N-s+1} \), and the last \( N - s \) columns. The equivalence of representations (B.2) and (3.10) for \( Z_{\text{top}}^{r_1, \ldots, r_s} \) implies the existence of nontrivial identities, which can be recognized as generalizing to \( s \)-fold sums some identities already discussed for the case of a single sum in [33] (see also [30]).

We wish to underline that expression (B.2), can alternatively be obtained directly from the corresponding expression (3.10) for \( Z_{\text{bot}}^{r_1, \ldots, r_s} \), via the crossing symmetry transformation (2.8). Similarly, the orthogonal polynomial representation, analogue to (4.7), can be obtained by direct computation, from expression (B.2), or alternatively, just exploiting the homogeneous version of ‘duality’ relation (2.8). Note that the implementation of this relation on expression (4.7) requires reversing the sign of variables \( \varepsilon_j \), and hence of the corresponding derivatives \( \partial_x \).

In either way, we readily get the following orthogonal polynomial representation:

\[
Z_{\text{top}}^{r_1, \ldots, r_s} = \frac{Z_N}{a^{(N-s)(N-s+1)/2} b^{N-s}} e^{N-s} \prod_{j=1}^{N-s} \left( \frac{b}{a} \right)^{\tilde{\vartheta}_j} \times \left| \begin{array}{ccc} K_0(\partial_{\varepsilon_1}) & \cdots & K_0(\partial_{\varepsilon_{N-s}}) \\ \vdots & \ddots & \vdots \\ K_{N-1}(\partial_{\varepsilon_1}) & \cdots & K_{N-1}(\partial_{\varepsilon_{N-s}}) \end{array} \right| \prod_{j=1}^{N-s} \left( \frac{\tilde{\omega}(\varepsilon_j)}{\tilde{\omega}(\varepsilon_j)} \right)^{N-s-j} \times \prod_{1 \leq j < k \leq N-s} \frac{1}{\tilde{\omega}(\varepsilon_k) \tilde{\omega}(\varepsilon_j)} \big|_{\varepsilon_1, \ldots, \varepsilon_{N-s} = 0}. \tag{B.3}
\]

To verify that representations (1.7) and (B.3) indeed satisfy the duality relation (2.8), a crucial property of orthogonal polynomials \( K_n(x) \) is needed, namely that under a crossing transformation, \( K_n(x; \lambda) \mapsto K_n(x; \pi - \lambda) = (-1)^n K_n(-x; \lambda) \), see
Thus, under the crossing symmetry transformation, one has
\[ K_n(\partial_{\varepsilon_j}) \mapsto (-1)^n K_n(\partial_{\varepsilon_j}). \]

On the other hand, the change of the sign in \( \varepsilon_j \), together with the substitution \( \lambda \mapsto \pi - \lambda \) gives rises to the following substitution rule:
\[ \omega(\varepsilon_j) \mapsto \tilde{\omega}(\varepsilon_j). \]

We recall that, analogously to the corresponding representation for \( Z^\text{bot}_{\text{r}_1, \ldots, \text{r}_s} \), representation (B.3) for \( Z^\text{top}_{\text{r}_1, \ldots, \text{r}_s} \) is valid for arbitrary values of the parameters of the model, independently of the regime.

As a result, we have for \( Z^\text{top}_{\text{r}_1, \ldots, \text{r}_s} \) the following multiple integral representation:
\[
Z^\text{top}_{\text{r}_1, \ldots, \text{r}_s} = Z_N \prod_{j=1}^{N-s} t_j \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{N-s} \frac{1}{z_j} \times 
\prod_{1 \leq j < k \leq N-s} \frac{t^2(z_k - z_j)}{z_j z_k - 2\Delta t z_j + t^2} \hat{h}_{N,N-s}(z_1, \ldots, z_{N-s}) \frac{d^{N-s}z}{(2\pi i)^{N-s}}. \tag{B.4}
\]

Here function \( \hat{h}_{N,s}(z_1, \ldots, z_s) \) is the function resulting from \( h_{N,s}(z_1, \ldots, z_s) \) under exchange of the weights \( a \) and \( b \). Function \( \hat{h}_{N,s}(z_1, \ldots, z_s) \) is defined simply by replacing \( h_{N}(z) \) with \( \hat{h}_{N}(z) \) in expression (4.10), where \( \hat{h}_{N}(z) := z^{N-1}h_{N}(z^{-1}) \). As a consequence, we have
\[
\hat{h}_{N,s}(z_1, \ldots, z_s) = z_1^{N-1} \cdots z_s^{N-1} h_{N,s}(z_1^{-1}, \ldots, z_s^{-1}).
\]

Multiple integral representation (B.3) can be derived along the lines of representation (4.13) using a ‘crossing symmetry transformed’ version of relation (4.14), that is
\[
K_{N-1}(\partial_{\varepsilon}) f(\tilde{\omega}(\varepsilon)) \bigg|_{\varepsilon=0} = \frac{1}{2\pi i} \oint_{C_0} \frac{(1-z)^{N-1}}{z^N} \hat{h}_{N}(z) f(z) \, dz,
\]
or also directly from representation (4.13), by simple application of the set of rules: \( s \leftrightarrow (N-s) \), \( a \leftrightarrow b \), \( t \leftrightarrow 1/t \), and \( h_{N}(z) \leftrightarrow \hat{h}_{N}(z) \), implementing the crossing symmetry transformation.

Finally, for the sake of completeness let us give the representation for \( Z^\text{bot}_{\text{r}_1, \ldots, \text{r}_s} \) that would come out by applying the commutation relation (3.10), rather than (3.9), to the expression (2.20). This is the analogue of representation (4.21) for \( Z^\text{top}_{\text{r}_1, \ldots, \text{r}_s} \), and immediately follows from it, thanks to the crossing symmetry,
\[
Z^\text{bot}_{\text{r}_1, \ldots, \text{r}_s} = Z_N \hat{h}^{(N-s)} \prod_{j=1}^{N-s} t_j \int_{C_1} \cdots \int_{C_1} \prod_{j=1}^{N-s} \frac{(w_j - 2\Delta t + t^2 r_{j-1})}{(w_j - 1)^r_j} \times 
\prod_{1 \leq j < k \leq N-s} \frac{w_k - w_j}{w_j w_k - 2\Delta t w_j + t^2} \times \hat{h}_{N-s,N-s}(w_1, \ldots, w_{N-s}) \frac{d^{N-s}w}{(2\pi i)^{N-s}}. \tag{B.5}
\]

Representations (B.4) and (B.5), just as all other representations, are valid for arbitrary values of the parameters of the model, independently of the regime.
Appendix C. A remarkable identity

We comment here on some remarkable identity stemming from the identification of the two alternative representations resulting for $Z_{r_1,...,r_s}^{\text{top}}$ by applying the QISM machinery in the ‘horizontal’ or in the ‘vertical’ direction. We focus on the situation where all inhomogeneities are turned on.

On the one hand, we have the ‘coordinate wavefunction’ representation, whose homogeneous version has already been discussed in appendix A, and whose fully inhomogeneous version, where all the spectral parameters $\lambda_1,\ldots, \lambda_N$ and $\nu_1,\ldots, \nu_s$ are left free, takes the form [44]:

$$Z_{r_1,...,r_s}^{\text{top}} = c^s \prod_{\beta=1}^{N} \prod_{k=1}^{s} \frac{1}{a(\lambda_\beta, \nu_k)} \prod_{1 \leq j < k \leq s} \frac{1}{d(\nu_j, \nu_k)}$$

$$\times \sum_{\sigma} (-1)^{[\sigma]} \prod_{\beta_1=1}^{r_1-1} \frac{b(\lambda_{\beta_1}, \nu_{\sigma_1})}{a(\lambda_{\beta_1}, \nu_{\sigma_1})} \cdots \prod_{\beta_s=1}^{r_s-1} \frac{b(\lambda_{\beta_s}, \nu_{\sigma_s})}{a(\lambda_{\beta_s}, \nu_{\sigma_s})}$$

$$\times \prod_{j=1}^{s} \frac{1}{a(\lambda_{r_j}, \nu_{\sigma_j})} \prod_{1 \leq j < k \leq s} e(\nu_{r_j}, \nu_{r_k}),$$

see also [66], Eqs. (79) and (80), and discussion in [67].

On the other hand, we have the alternative representation $\mathbb{B}_{18}$. Identification of these two expressions clearly implies the existence of some relation which, after cancellation of some nonrelevant prefactors, reads:

$$c^s \prod_{1 \leq j < k \leq s} \frac{1}{d(\nu_j, \nu_k)} \sum_{\sigma} (-1)^{[\sigma]} \prod_{\beta_1=1}^{r_1-1} \frac{b(\lambda_{\beta_1}, \nu_{\sigma_1})}{a(\lambda_{\beta_1}, \nu_{\sigma_1})} \cdots \prod_{\beta_s=1}^{r_s-1} \frac{b(\lambda_{\beta_s}, \nu_{\sigma_s})}{a(\lambda_{\beta_s}, \nu_{\sigma_s})}$$

$$\times \prod_{j=1}^{s} \frac{1}{a(\lambda_{r_j}, \nu_{\sigma_j})} \prod_{1 \leq j < k \leq s} e(\nu_{r_j}, \nu_{r_k}) =$$

$$= \sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_s=1}^{r_s} \prod_{j=k=1}^{s} \frac{1}{a(\lambda_{\alpha_j}, \nu_k)} Z_{s} (\lambda_{\alpha_1},\ldots, \lambda_{\alpha_s}; \nu_1,\ldots, \nu_s)$$

$$\times \prod_{\beta_1=1}^{r_1} \frac{1}{d(\lambda_{\alpha_1}, \lambda_{\beta_1})} \prod_{\beta_2=1}^{r_2} \frac{1}{d(\lambda_{\alpha_2}, \lambda_{\beta_2})} \cdots \prod_{\beta_s=1}^{r_s} \frac{1}{d(\lambda_{\alpha_s}, \lambda_{\beta_s})}$$

$$\times \prod_{\beta_1=1}^{r_1-1} e(\lambda_{\alpha_1}, \lambda_{\beta_1}) \cdots \prod_{\beta_2=1}^{r_2-1} e(\lambda_{\alpha_2}, \lambda_{\beta_2}) \prod_{1 \leq j < k \leq s} e(\lambda_{\alpha_k}, \lambda_{\alpha_j}). \quad (\text{C.1})$$

The identity holds true by construction. However, a simple and direct proof can be also given, by reexpressing both sides of (\text{C.1}) as contour integrals, and then proceeding along the lines of the discussion provided at the end of appendix A.

Let us consider some particular instance of the identity (\text{C.1}). To start with, let us set $s = 1$. The above relation reduces to

$$\prod_{\beta=1}^{r} \frac{b(\lambda_{\beta}, \nu)}{a(\lambda_{\beta}, \nu)} = \sum_{\alpha=1}^{r} \frac{1}{a(\lambda_{\alpha}, \nu)} \prod_{\beta \neq \alpha}^{\infty} d(\lambda_{\alpha}, \lambda_{\beta}).$$
which is a particular case of equation (45) of [30], see also [53], and readily implies
the identity between (3.18) at \( s = 1 \) and (3.19).

Let us now turn to investigate the identity (C.1) for generic values of \( s \), but
with the lattice coordinates chosen to be \( r_j = j, j = 1, \ldots, s \). We obtain:

\[
c^s \prod_{1 \leq j < k \leq s} \frac{1}{d(\nu_j, \nu_k)} \sum_{\sigma} (-1)^{[\sigma]} \prod_{1 \leq j < k \leq s} a(\lambda_j, \nu_\sigma_j) b(\lambda_j, \nu_\sigma_k) e(\nu_\sigma_j, \nu_\sigma_k) = Z_s(\lambda_1, \ldots, \lambda_s; \nu_1, \ldots, \nu_s), \tag{C.2}
\]

which reproduces essentially the antisymmetrization relation (5.1), that is Proposition C.1 in [20].

Let us now consider the identity (C.1) with the first \( s - 1 \) lattice coordinates
specialized to the values \( r_j = j, j = 1, \ldots, s - 1 \), while the last one is left generic,
\( r_s =: r \). This configuration of \( r_j \)'s appears in the context of the so-called ‘tangent method’ [9]. After some calculation, and use of relation (C.2), we obtain:

\[
\sum_{l=1}^{s} \prod_{\substack{1 \leq j < k \leq s \atop j \neq l}} e(\nu_j, \nu_l) \prod_{\beta \neq l} a(\lambda_\beta, \nu_\beta) b(\lambda_\beta, \nu_l) \prod_{\beta = 1}^{r-1} \frac{1}{a(\lambda_\beta, \nu_l)} Z_{s-1}[\lambda_s; \nu_l] = \sum_{\alpha=1}^{r} \prod_{\substack{1 \leq k \leq s \atop \beta \neq \alpha}} e(\lambda_\alpha, \lambda_\beta) \frac{1}{d(\lambda_\alpha, \lambda_\beta)} e(\lambda_\alpha, \nu_l) Z_s(\lambda_1, \ldots, \lambda_s, \nu_1, \ldots, \nu_s). \tag{C.3}
\]

Note that, unlike both sides of relation (C.2), the left- and right-hand sides of (C.3)
clearly reflect in their structures the application of the fundamental commutation relations (3.2)–(3.8) in the ‘horizontal’ and ‘vertical’ directions, respectively.

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