Diagrams of ⋆-trisections
José Román Aranda and Jesse Moeller
November 2019

Abstract

In this note we provide a generalization for the definition of a trisection of a 4-manifold with boundary. We demonstrate the utility of this more general definition by finding a trisection diagram for the Cacime Surface, and also by finding a trisection-theoretic way to perform logarithmic surgery. In addition, we describe how to perform 1-surgery on closed trisections. The insight gained from this description leads us to a classification of the Farey trisection diagrams.

Contents

1 Introduction 2

2 ⋆-Trisections of 4-manifolds 4
  2.1 The standard pieces ................................................. 5
    2.1.1 Construction 1 .................................................. 6
    2.1.2 Construction 2 .................................................. 6
    2.1.3 The boundary of Z ............................................ 7
  2.2 Trisection diagrams ................................................. 7
    2.2.1 New Relative Diagrams from Old .............................. 8

3 Pasting ⋆-trisections 9

4 The Complement of a Simple Closed Curve 12
  4.1 Loops in genus one trisections .................................. 13

5 Trisections of genus 3 16
  5.1 Farey Trisections are standard .................................. 18

6 Surface surgery 18
  6.1 Embedded surfaces and their complements ........................ 19
  6.2 Cacime Surface .................................................... 19
  6.3 Knot Surgery ....................................................... 23
  6.4 Torus Surgery ...................................................... 26

A Classic Diagrams 27
  A.1 Relative trisections from Kirby diagrams ........................ 27
  A.2 The monodromy induced on ∂X .................................. 31
1 Introduction

In [9] Gay and Kirby proved that every closed smooth 4-manifold admits a trisection. A trisection of a closed 4-manifold $X$ is a decomposition $X = Z_1 \cup Z_2 \cup Z_3$ into three 4-dimensional 1-handlebodies so that the pairwise intersections are 3-dimensional 1-handlebodies $H_i = Z_{i-1} \cap Z_i$, and the triple intersection is a closed surface $\Sigma = X_1 \cap X_2 \cap X_3$. This surface is called the trisection surface, or the central surface. The minimal genus among all trisection surfaces of $X$ is called the genus of $X$. In the last section of their paper, Gay and Kirby also define trisections of 4-manifolds with boundary called relative trisections. When $X$ is a trisected 4-manifold with boundary, the central trisection surface is not necessarily closed. Naturally, the relative trisection of $X$ induces structure on $\partial X$. If the relative trisection surface has non-empty boundary, the trisection restricts to $\partial X$ as an open book decomposition. The definition and diagrammatic of relative trisections in this case can be found in [4] and [5]. When the trisection surface is closed, the binding link is empty and thus the trisection restricts to the boundary of $X$ as a fibration of closed surfaces over $S^1$.

The goal of this paper is to introduce a generalization of trisections of 4-manifolds, called $\ast$-trisections, and to develop the diagramatics of this new theory. The utility of this more general definition is apparent when taking complements of embedded submanifolds. In Sections 4 and 6 we show that the naive diagrammatic approach to removing neighborhoods of embedded submanifolds yields $\ast$-trisection diagrams. Additionally, motivated by [6], we prove a pasting lemma in Section 3 which allows us to glue two $\ast$-trisections along connected boundary components. In Section 6 we use this pasting lemma explicitly to produce a closed trisection diagram for the Cacime Surface. In addition, we use the pasting lemma to describe how to trisect the Fintushel-Stern knot surgery [7] and Logarithmic transforms in the spirit of [2].

In [20] Meier and Zupan proved that the only closed irreducible genus two trisection is $S^2 \times S^2$ and genus 1 trisections are classified easily enough. It is therefore natural to seek a classification for genus three trisections. As a proving ground, we now introduce an infinite family of genus three trisection diagrams. Consider three rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{p}{q}$ in reduced form. Let $\alpha_1$ be the top curve and $\alpha_2$ be bottom curve in the middle of the diagram in Figure 1 and let $\alpha_3$ be the $\frac{a}{b}$ torus knot in the torus obtained by compressing along $\alpha_1$ and $\alpha_2$. Take $\alpha$ to be the union of these curves and define $\beta$ and $\gamma$ similarly using $\frac{c}{d}$ and $\frac{p}{q}$, respectively. Observe that $(\Sigma, \alpha, \beta)$ is a stabilization of a genus one Heegaard splitting. So, when $|ad - bc| \leq 1$, $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^1 \times S^2$ or $S^3$. When this condition is satisfied for each pair of fractions, $(\Sigma, \alpha, \beta, \gamma)$ is a genus three trisection diagram and we call it a Farey diagram or a Farey trisection.

![Figure 1: The Farey diagram $D(\frac{1}{1}, \frac{1}{2}, \frac{2}{3})$.](image)

In [20] Meier and Zupan proved that the only closed irreducible genus two trisection is $S^2 \times S^2$ and genus 1 trisections are classified easily enough. It is therefore natural to seek a classification for genus three trisections. As a proving ground, we now introduce an infinite family of genus three trisection diagrams. Consider three rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{p}{q}$ in reduced form. Let $\alpha_1$ be the top curve and $\alpha_2$ be bottom curve in the middle of the diagram in Figure 1 and let $\alpha_3$ be the $\frac{a}{b}$ torus knot in the torus obtained by compressing along $\alpha_1$ and $\alpha_2$. Take $\alpha$ to be the union of these curves and define $\beta$ and $\gamma$ similarly using $\frac{c}{d}$ and $\frac{p}{q}$, respectively. Observe that $(\Sigma, \alpha, \beta)$ is a stabilization of a genus one Heegaard splitting. So, when $|ad - bc| \leq 1$, $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^1 \times S^2$ or $S^3$. When this condition is satisfied for each pair of fractions, $(\Sigma, \alpha, \beta, \gamma)$ is a genus three trisection diagram and we call it a Farey diagram or a Farey trisection.
The problem of understanding Farey diagrams was proposed during the first day of the 2019 Spring Trisectors Meeting at UGA. By this time, progress had already been made on this problem in [17], where Meier showed that \( D(\frac{q}{p}, \frac{r}{p}, \frac{q}{p}) \) is the diagram of a spun lens space \( L(p, q) \). Seeking a classification for genus three trisections, he conjectured the following.

**Conjecture 1** (Meier [17]). Every irreducible 4-manifold with trisection genus three is either the spin of a lens space, or a Gluck twist on a specific 2-knot in the spin of a lens space.

In Section 5, we show that every Farey diagram is the result of pasting together two \( \ast \)-trisections. In particular, a Farey diagram represents 1-surgery on some genus one manifold. When this genus one manifold is simply connected, this surgery is determined only up to the two choices of framing, we demonstrate a procedure to slide the curves and find a reducing curve on the Farey diagram. By the classification of genus one and two trisection diagrams, reducible genus three trisections are standard. We therefore obtain the following theorem.

**Theorem 5.2.** Let \( \{\frac{a}{b}, \frac{c}{d}, \frac{p}{q}\} \subset \mathbb{Q} \cup \{\frac{1}{b}\} \) with \( d(x, y) \leq 1 \) for each \( x, y \in \{\frac{a}{b}, \frac{c}{d}, \frac{p}{q}\} \). If at least two of these fractions are distinct, then \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) is equivalent to the standard diagram for \( T \# S \) where \( T \in \{S^4, \mathbb{CP}^2, \mathbb{HP}^2\} \) and \( S \in \{S^2 \times S^2, S^2 \times \mathbb{CP}^2, S^2 \times \mathbb{HP}^2\} \).

This theorem, in conjunction with Meier’s results on spun lens spaces, proves his conjecture for the family of genus three Farey trisections.

To end, it is worthy of note that the current work on trisections of 4-manifolds with boundary is restricted to the case of the trisection surface having non-negative boundary. Besides a few remarks in the original trisections paper, not much has been said in the closed case. In order to complete the discussion of the basic theory of trisections of manifolds with boundary, in Appendix A we offer adaptations of proofs of the main theorems in [4] and [5]. For example, we prove that the algorithm which recovers the monodromy \( \phi : P \to P \) of the induced circular structure \( \partial X = P \times_\phi S^1 \) also works in the case where \( P \) is a closed surface.

**Acknowledgements**

The first author would like to thank Jeff Meier and the topology group of the University of Iowa for helpful conversations. The second author would like to thank the Max Planck Institute for Mathematics, and Steve Hamborg, for their hospitality.
2 **-Trisections of 4-manifolds**

For integers \(a, b \in \mathbb{Z}_{\geq 0}\), we denote by \(F_{a,b}\) a connected orientable surface of genus \(a\) with \(b\) boundary components. A 3-dimensional compression body \(C\) is a cobordism from a connected surface \(F_+ = F_{k,b}\) to a (possibly disconnected or empty) surface \(F_-\) built as follows.

\[
\begin{align*}
C &= F_+ \times [0, 1] + (2\text{-handles along } F_+ \times \{1\}) + (3\text{-handles}) \\
&= \text{(some 3-balls) } \cup (F_- \times [0, 1]) + (1\text{-handles along } F_- \times \{0\})
\end{align*}
\]  

(1)

(2)

The boundary of \(C\) decomposes in \(\partial_+ C = F_+ \times \{0\}, \partial_- C = F_- \times \{1\}\) and \(\partial_0 C = (\partial F_+) \times [0, 1]\). If \(\partial_- C\) is empty, \(C\) is a well-known handlebody. The 2-handles in (1) are attached along a collection of pairwise disjoint and possibly boundary parallel simple closed curves \(\delta \subset F_+\). The 3-handles are attached along the sphere components of the surface \(F_+\) after compression along the 2-handles. Notice that, up to diffeomorphism of \(F_+\), the smallest set of such loops \(\delta\) is determined by the diffeomorphism type of \(F_+\) and \(F_-\).

Let \(\tilde{C}\) be a compression body with positive boundary \(F_{k,b_0}\). Define \(Z\) to be the 4-manifold obtained from \(\tilde{C} \times [0, 1]\) by attaching 1-handles along its boundary. Equivalently, \(Z\) is obtained by removing tubular neighborhoods of a collection of simultaneously boundary parallel properly embedded 2-dimensional disks \(D\) in \(\tilde{C} \times [0, 1]\). Consider \(\tilde{C}^0, \tilde{C}^1 \subset \tilde{C}\) two sub-compression bodies spanning \(\tilde{C}\) with common part a sub-compression body \(\tilde{C}_{alt}\); i.e., there is a collection of simple closed curves \(\tilde{\delta} \subset F_{k,b_0}\) determining \(\tilde{C}\) such that \(\tilde{\delta} = \delta^0 \cup \delta^1, \delta_{alt} = \delta^0 \cap \delta^1\) with \(\tilde{C}^i\) determined by \(\delta^i\) and \(\tilde{C}_{alt}\) determined by \(\delta_{alt}\). Then

\[
\tilde{C} \times [0, 1] = \tilde{C}^0 \times [0, 1/2] \cup_{\tilde{C}^0 \times \{1/2\}} \tilde{C}^1 \times [1/2, 1].
\]

The link \(U = D \cap \partial(\tilde{C} \times [0, 1])\) is an unlink of unknots given by \(D\). Isotope \(U\) in bridge position with respect to the Heegaard splitting

\[
\begin{align*}
\left[ \left( \tilde{C}^0 \times \{0\} \right) \cup (F_{k,b_0} \times [0, 1/2]) \right] \cup_{F_{k,b_0} \times \{1/2\}} \left[ (F_{k,b_0} \times [1/2, 1]) \cup (\tilde{C}^1 \times \{1\}) \right].
\end{align*}
\]

Set \(C^i = (\tilde{C}^i \times \{i\}) - \eta(U)\), and let \(C_{alt}\) be the common sub-compression body of \(C^0\) and \(C^1\).

By construction \(C^0, C^1\) and \(C_{alt}\) are compression bodies with positive boundary \(F_{k,b}\) with \(b = b_0 + |U \cap (F_{k,b_0} \times \{1/2\})|\). It follows from Theorem 1.1 of [13] that \(Z\) can be written as follows

\[
Z := Z_{C^0,C^1} = C^0 \times [0, 1/2] \cup_{C_{alt} \times \{1/2\}} C^1 \times [1/2, 1].
\]

Part of \(\partial Z_{C^0,C^1}\) is

\[
Y_{C^0,C^1} := (C^0 \times \{0\}) \cup (F_{k,b} \times [0, 1]) \cup (C^1 \times \{1\}),
\]

which admits a standard Heegaard splitting \(Y_{C^0,C^1} = Y^+_{C^0,C^1} \cup Y^-_{C^0,C^1}\) as follows

\[
Y^+_{C^0,C^1} := (F_{k,b} \times [1/2, 1]) \cup (C^1 \times \{1\}),
\]

\[
Y^-_{C^0,C^1} := (C^0 \times \{0\}) \cup (F_{k,b} \times [0, 1/2]).
\]

For \(g \geq k\), let \(Y_{C^0,C^1} = Y^+_{C^0,C^1,g} \cup Y^-_{C^0,C^1,g}\) be the splitting above stabilized \(g - k\) times.
**Definition 2.1.** A $*$-trisection of a connected smooth 4-manifold $X$ is a decomposition $X = X_1 \cup X_2 \cup X_3$ with connected compression bodies $C_i^0, C_i^1$ as above and a integer $g \geq g(\partial X, C_i^j)$ for $i = 1, 2, 3$, $j = 0, 1$, such that each $X_i$ is diffeomorphic to $Z_{C_i^0, C_i^1}$ via a map $\varphi_i : X_i \to Z_{C_i^0, C_i^1}$ for which

$$\varphi_i(X_i \cap X_{i+1}) = Y_{C_i^0, C_i^1; g}^+ \text{ and } \varphi_i(X_{i-1} \cap X_i) = Y_{C_i^0, C_i^1; g}^-.$$  

The triple intersection is a connected surface $\Sigma$ of genus $g$ with $b \geq 0$ boundary components called the $*$-trisection surface.

**Remark 2.2.** If all of the compression bodies $C_i^j$ have empty negative boundary, each $C_i^j$ is a handlebody and it follows that $X$ is closed. Here, the definition above agrees with the original definition of a trisection when $X$ is closed [9]. If for all $i$ we have that $C_i^0 = C_i^1$ are compression bodies with non-empty negative boundary, then this decomposition is the same as a trisection of a 4-manifold with boundary in [9]. Such trisections induce an open book decomposition on $\partial X$ with binding a $b$-component link. We will refer to all the above as classical trisections, among them we refer to the ones with $\partial X \neq \emptyset$ as relative trisections. We will sometimes wish to distinguish whether or not the binding of the open book $\partial X$ is empty; in these cases we will simply write $b = 0$ or $b > 0$, referring to the number of boundary components on the relative trisection diagram. The most general trisection, or $*$-trisection, is when $C_i^0$ is not the same as $C_i^1$ for some $i$. Figure 2 contains a diagram from each level of generality mentioned so far. The rest of this section is dedicated to presenting different ways of thinking about the sectors of $*$-trisections.

![Figure 2: (left to right) A trisection of $\mathbb{C}P^2$, a relative trisection of the spin of $S^1 \times D^2$, a relative trisection of a disk bundle over $S^2$ with $e = -1$, a $*$-trisection of $\mathbb{C}P^2 \setminus \eta(S^1)$](image)

### 2.1 The standard pieces

In this subsection, we will describe two useful ways to build $Z_{C_0, C_1}$ which we will use to have a better understanding of $*$-trisections. The casual reader may skip these computations and read the conclusions in Section 2.2.

Let $\tilde{\delta}$ be a collection of non-isotopic simple closed curves in a connected surface $F_{k, b_0}$. Let $\tilde{\delta} = \tilde{\delta}^0 \cup \tilde{\delta}^1$ with $\tilde{\delta}_{alt} = \tilde{\delta}^0 \cap \tilde{\delta}^1$. Denote by $\tilde{C}$, $\tilde{C}^0$ and $\tilde{C}_{alt}$ the 3-dimensional compression bodies determined by $\tilde{\delta}$, $\tilde{\delta}^0$ and $\tilde{\delta}_{alt}$, respectively. Let $U$ be an unlink of unknots in bridge position with respect to the Heegaard splitting $\tilde{C}^0 \cup_{F_{k, b_0}} \tilde{C}^1$. Let $C^i = (\tilde{C}^i \times \{i\}) - \eta(U)$. By construction $C^0$ and $C^1$ are compression bodies with positive boundary $F_{k, b}$ with $b = b_0 + |U \cap (F_{k, b} \times \{1/2\})|$. Take two collections of non-isotopic simple closed curves $\delta^0$, $\delta^1 \subset F_{k, b}$ such that $C^0$ and $C^1$ are given by $\delta^0$ and $\delta^1$, respectively, and let $\delta_{alt} = \delta^0 \cap \delta^1$ be the common curves spanning $C_{alt}$. Let $\delta_0 = \delta^0 - \delta_{alt}$ and $\delta_1 = \delta^1 - \delta_{alt}$. We can always consider $\delta^0$ and $\delta^1$ satisfying the condition that no sphere...
component of the surface obtained by compressing $F_{k,b}$ along $\delta_{all}$ contains loops in $\delta_i$ for $i = 1, 2$. $F_{all}$ will denote the negative boundary of $C_{all}$. Notice that we can identify $\delta^i$ as a subset of $\delta^j$ and so $\delta_{all} = \delta_{all}$.

Denote by $C_i = \overline{C_i - C_{all}}$, the compression body obtained from $(\partial_- C_{all}) \times [0, 1]$ by attaching 3-dimensional 2-handles along $\delta_i \times \{1\}$. In particular $C_i = C_{all} \cup F_{all} C_i$ for $i = 1, 2$ and

$$Z_{C^0, C_1} = C_0 \times [0, 1/2] \bigcup_{F_{all} \times [0, 1/2]} C_{all} \times [0, 1] \bigcup_{F_{all} \times [1/2, 1]} C_1 \times [1/2, 1].$$

2.1.1 Construction 1

Take a collar neighborhood $F_{all} \times [1 - \varepsilon, 1]$ of $F_{all} \subset C_{all}$. We have the following decomposition.

$$Z_{C_0, C_1} = (C_0 \times [0, 1/2]) \cup_{F_{all} \times [0, 1/2]} (C_{all} \times [0, 1]) \cup_{F_{all} \times [1/2, 1]} (C_1 \times [1/2, 1])$$

$$= \left((C_0 \times [0, 1/2]) \cup_{F_{all} \times \{1\} \times [0, 1/2]} (F_{all} \times [1 - \varepsilon, 1] \times [0, 1]) \cup_{F_{all} \times \{1\} \times [1/2, 1]} (C_1 \times [1/2, 1])\right)$$

$$\cup_{F_{all} \times \{1 - \varepsilon\} \times [0, 1]} (C_{all} \times [0, 1])$$

$$= ((C_0 \cup F_{all} C_1) \times [0, 1]) \cup_{F_{all} \times [0, 1]} (C_{all} \times [0, 1])$$

$$= ((C_0 \cup F_{all} C_1) \times [0, 1]) + ((\delta_{all}) \text{ 1-handles})$$

$$= ((C_0 \cup F_{all} C_1) \times [0, 1]) + \left(\frac{1}{2}(\chi(\partial_- C_0) + \chi(\partial_- C_1) - 2\chi(F_{all}))\right) \text{ 1-handles}$$

It is important to mention that in the above equality, if $F_{all} = \emptyset$ then $((C_0 \cup F_{all} C_1) \times [0, 1])$ is replaced by one 0-handle. Additionally, by our construction of $\delta^0$ and $\delta^1$, when all of the components of $\partial_- C_0$ and $\partial_- C_1$ have boundary, the standard piece $Z_{C^0, C_1}$ is a 4-dimensional 1-handlebody.

2.1.2 Construction 2

From Equation (1), we can build the 4-manifold $Z_{C^0, C_1}$ as follows.

$$Z_{C^0, C_1} = F_{k,b} \times [0, 1]^2 + \left(\text{2-handles along } (\delta_{all} \cup \delta_1) \times \{1\} \times \{1\}\right) + \left(\text{3-handles} + \text{4-handles}\right).$$

Some 3-handles are attached along the spheres components of $F_{k,b}$ compressed along $(\delta_{all} \cup \delta_1) \times \{1\} \times \{1\}$, and $(\delta_{all} \cup \delta_0) \times \{1\} \times \{0\}$. The rest of the 3-handles are attached along the spheres

$$core(\sigma \times \{1\} \times \{1\}) \cup (\sigma \times \{1\} \times [0, 1]) \cup core(\sigma \times \{1\} \times \{0\}),$$

where $\sigma$ is a loop in $\delta_{all}$. The 4-handles correspond to sphere components of $\delta_{all}$. By adding 1/2-canceling pairs of handles we get the following decomposition

$$Z_{C^0, C_1} = F_{g,b} \times [0, 1]^2 + \left(\text{2-handles along } \alpha \times \{1\} \times \{1\} \text{ and } \beta \times \{1\} \times \{0\}\right) + \left(\text{3-handles} + \text{4-handles}\right).$$

Here, $\alpha, \beta \subset F_{g,b}$, $g \geq k$ are two collections of pairwise disjoint loops which we decompose into three sets $\alpha = \alpha_{all} \cup \alpha_{stab} \cup \alpha_0$, $\beta = \beta_{all} \cup \beta_{stab} \cup \beta_0$ where loops in $\alpha_{stab}$ and $\beta_{stab}$ correspond to the stabilizations of $F_{k,b}$ to get $F_{g,b}$, and the rest loops correspond to $\delta_{all}$ and $\delta_i$. 
2.1.3 The boundary of \( Z \)

Fix \( i = 0, 1 \). From Equation (2) we get,

\[
C^i \times [0, 1/2] = (0\text{-handles}) \cup (\partial C^i) \times [0, 1] \times (|\delta|^1 \text{ handles along disks in } \partial C^i) \times \{0\} \times [0, 1]) = \#(\partial C^i \times D^2) \cup (|\delta|^1 \text{ handles along } D^2) \times (\partial C^i) \times \{0\} \times [0, 1] \times \{0\}.
\]

Thus \( \partial (C^i \times [0, 1/2]) = \#(\partial C^i \times S^1) \#(\#(\delta)^1 S^1 \times S^2) \).

Notice that \( \partial [(\partial C^i) \times D^2] \) has a natural open book decomposition with binding \( B := (\partial F_{k,b}) \times D^2 \). If \( b = 0 \) then the binding is empty and \( \partial [(\partial C^i) \times D^2] \) is just a fibration over \( S^1 \). Hence, \( (C^i \times [0, 1/2]) - B \) admits an circular-like handle decomposition described by,

\[
\partial (C^i \times [0, 1/2]) - B = F_{k,b} \times [0, 1] + (2\text{-handles along } \delta^i \times \{1\} \text{ and } \delta^i \times \{0\}) + (3\text{-handles})
\]

\[
\left\langle \text{Top } = \text{id Bottom} \right\rangle
\]

\[
= F_{k,b} \times [0, 1] + (2\text{-handles } \delta_{alt} \times \{1\}, \delta_i \times \{0\}) + (3\text{-handles})
\]

\[
+ (0\text{-handles}) + (1\text{-handles dual to } \delta_{alt} \times \{1\} \text{ and } \delta_i \times \{0\}) \left\langle \text{Top } = \text{id Bottom} \right\rangle
\]

The 3-handles above are attached along the sphere components of the compressed surfaces, and the 0-handles are dual to them. Notice that the submanifold \( F_{k,b} \times [0, 1] + (2\text{-handles } \delta \times \{1\}) + (3\text{-handles}) \) is a copy of \( C_{alt} \). Since \( C^0 \times [0, 1/2] \) and \( C^1 \times [1/2, 1] \) have common intersection \( C_{alt} \times \{1/2\} \) in \( Z_{C^0,C^1} \), it follows that the boundary of \( Z_{C^0,C^1} \) admits the following decomposition.

\[
\partial Z_{C^0,C^1} - B = F_{k,b} \times [0, 1] + \left( \begin{array}{c}
2\text{-handles } (\delta_{alt} \cup \delta_1) \times \{1\}, (\delta_{alt} \cup \delta_0) \times \{0\}
\end{array}\right) + (3\text{-handles})
\]

\[
+ (0\text{-handles}) + (1\text{-handles dual to } \delta_1 \times \{1\} \text{ and } \delta_0 \times \{0\}) \left\langle \text{Top } = \text{id Bottom} \right\rangle
\]

\[
= \left( (F_{k,b} \times [1/2, 1]) \cup (C^1 \times \{1\}) \right) \cup \left( (C^0 \times \{0\}) \cup (F_{k,b} \times [0, 1/2]) \right)
\]

\[
+ (0\text{-handles}) + (1\text{-handles dual to } \delta_1 \times \{1\} \text{ and } \delta_0 \times \{0\}) \left\langle \text{Top } = \text{id Bottom} \right\rangle
\]

\[
= C_1 \cup \partial_{-C_1} \left( Y^+_{C^0,C^1,g} \right) \cup F_{k,b} \left( Y^+_{C^0,C^1,g} \right) \cup \partial_{-C_0} C_0 \left\langle \text{Top } = \text{id Bottom} \right\rangle
\]

Thus, the boundary of \( X \) intersects each standard piece \( X_i \approx Z_{C^0,C^1} \) in a copy of \( C_0 \cup F_{alt} C_1 \). In particular if \( C^0 = C^1 \) then \( C_0 \) and \( C_1 \) become collars of \( F_{alt} \) and \( (\partial X) \cap X_i = F_{alt} \times [0, 1] \). When all the compression bodies \( C^0_i = C^1_i \) are the same, this exhibits the existance of an open book decomposition on \( \partial X \) with binding \( B \).

2.2 Trisection diagrams

Let \( X = X_1 \cup X_2 \cup X_3 \) be a \( \star \)-trisected connected 4-manifold. Denote by \( \Sigma = X_1 \cap X_2 \cap X_3 \), this is a connected orientable surface of genus \( g \) with \( b \) boundary components. Let \( \alpha, \beta, \gamma \subset \Sigma \) be three collections of simple closed curves each determining the compression bodies \( H_\alpha = X_1 \cap X_2, H_\beta = X_3 \cap X_1, \) and \( H_\gamma = X_2 \cap X_3 \). By definition, each pair \((\alpha, \beta), (\beta, \gamma), (\gamma, \alpha)\) is handle slide equivalent to a pair of collections of pairwise disjoint simple closed curves \((\delta^0, \delta^1)\) with common curves \( \delta_{alt} \) and decompositions \( \delta^i = \delta_{alt} \cup \delta^i_{\text{stab}} \cup \delta_i \) such that \((\Sigma; \delta^0, \delta^1)\) is an stabilization of
$(F; \delta_{all} \cup \delta_0, \delta_{all} \cup \delta_1)$. In other words, there is a bijection between the sets $\delta^{0}_{stab}$ and $\delta^{1}_{stab}$ where corresponding loops intersect in exactly one point and are disjoint from the rest of the curves in $\delta^{0} \cup \delta^{1}$, and the surface $F$ is obtained from replacing small neighborhoods of $\delta^{j} \cup \delta^{0}$ in $\Sigma$ by disks. If the given trisection is classical then each pair satisfies $\delta_0 = \delta_1 = \emptyset$. In general, the pairs $(\delta_0, \delta_1)$ further decompose into $\delta_i = \tilde{\delta}_i \cup \hat{\delta}_i$ where each curve $\tilde{\delta}_i$ is disjoint and non-isotopic to the curves in $\delta_j$ ($i \neq j$); and there are some loops $c$ in $F$ separating $\tilde{\delta}_0 \cup \tilde{\delta}_1$ from $\hat{\delta}_0 \cup \hat{\delta}_1$ such that the components $\hat{F}$ of $F$ after a compression along $c$ that contain $\hat{\delta}_0 \cup \hat{\delta}_1$ are all planar with $(\hat{F}; \hat{\delta}_0, \hat{\delta}_1)$ being a Heegaard splitting for the complement of an unlink of unknots in the union of $|\hat{F}|$ copies of $S^3$.

Let $p_j = e^{2\pi i/3} \in S^1$, $j = 1, 2, 3$. The 4-manifold $X$ can be built from $\Sigma \times D^2$ by attaching 2-handles along the loops with surface framing $\alpha \times \{p_1\}$, $\beta \times \{p_2\}$, and $\gamma \times \{p_3\}$; attaching 3-handles along the 2-spheres arising from the common curves between each pair $(\alpha, \beta)$, $(\beta, \gamma)$, $(\gamma, \alpha)$, and attaching the 4-handles along $S^3$ boundary components arising from sphere components of $\Sigma$ compressed along the common loops of each pair as in Construction 1 of Section 2.1. The tuple $(\Sigma, \alpha, \beta, \gamma)$ is called a trisection diagram. Figure 2 shows examples of various $\star$-trisection diagrams.

From the work in Section 2.1.3 we see that the boundary of $X$ can be decomposed as the union of six compression bodies glued along the positive and negative boundaries. This induces a circular handle decomposition intercalating between components of the pages $P_{\alpha}$, $P_{\alpha \cap \beta}$, $P_{\beta}$, $P_{\beta \cap \gamma}$, $P_{\gamma}$, and $P_{\gamma \cap \alpha}$, where each of these are obtained from the trisection surface $\Sigma$ by compressing along the indicated set of curves.

In the case of relative trisections, all the surfaces $P = P_{\varepsilon} = P_{\varepsilon \cap \rho}$ will be the same and such $P$ will foliate the complement of the binding $\partial \Sigma$ in $\partial X$. In particular, the number of components of $P$ will correspond with the boundary components of $X$. If a component of $P$ is closed, the corresponding component of $\partial X$ will be an $S^1$-fibration over the surface. One can calculate the corresponding monodromies of the fibrations using an algorithm of Castro, Gay and Pinzon-Caicedo [4] which we will discuss in Appendix A.2.

### 2.2.1 New Relative Diagrams from Old

In this section we briefly discuss the relationship between the classical relative trisection diagrams and $\star$-trisection diagrams for manifolds with boundary. To begin, observe that each $\tau_i$ in Figure 3 is a trisection diagram for $S^2 \times D^2$.

![Figure 3](image.png)

Figure 3: (left) A $\star$-trisection diagram of $S^2 \times D^2$. (right) A relative trisection diagram of $S^2 \times D^2$.

Given two relative trisection diagrams $(\Sigma_i, \alpha_i, \beta_i, \gamma_i)$ $i = 1, 2$ for two 4-manifolds with non-empty boundary; the connected sum of their diagrams is a relative trisection diagram for the “fiber sum” of $X_1$ and $X_2$ along fibered tori in the boundaries coming from disks $B_i \subset \Sigma_i$ disjoint from the
trisection curves. More precisely, we obtain a relative trisection diagram for
\[ X' = \left[ X_1 - (B_1 \times D^2) \right] \cup_{\text{id}_{S^1 \times D^2}} \left[ X_2 - (B_2 \times D^2) \right]. \]

Since \( S^2 \times D^2 \) behaves like the identity under such operation, it follows that connect summing the relative trisection diagram with the diagram \( \mathcal{T}_2 \) from Figure 8 will give a relative trisection diagram for the same 4-manifold with two more binding components in the open book decomposition induced on \( \partial X \). The new open book decomposition is given by taking a solid torus neighborhood of a regular circle fiber and replacing it with the Seifert fibered space \( S(0,1;+1,-1) \). For relative trisection diagrams, the new monodromy is given by composing one positive and one negative Dehn twist along the boundary components, respectively (see Section 3.1 of [10]). With the same philosophy as above, the following lemma implies that, by poking the trisection surface enough, we can always consider \( * \)-trisections where all the pages are surfaces with non-empty boundary.

**Lemma 2.3 (Poking Lemma).** Let \( \tau \) be a \( * \)-trisection of a 4-manifold \( X \) with non-empty boundary. Then \( \tau \# \tau_1 \) is also a \( * \)-trisection diagram for \( X \). In particular, if \( \partial X \) is connected and \( \tau \) is a relative trisection with closed trisection surface, then every page of the circular handle decomposition induced by \( \tau \# \tau_1 \) is a surface with boundary.

### 3 Pasting \( * \)-trisections

Motivated by work of Castro and Ozbagci [3, 6], we want to describe a simple method to paste two compatible \( * \)-trisections along their boundaries. The main technical observation that makes Theorem [3, 1] work is that when we paste two standard pieces \( Z_{C_0,C_1} \) along some connected components of \( C_0 \cup F_{alt} C_1 \), the resulting 4-manifold is also a standard piece whenever all the glued surfaces in \( F_{alt} \) have boundary.

**Theorem 3.1 (Pasting Lemma for \( * \)-relative trisections).** Let \( W = W_1 \cup W_2 \cup W_3 \) and \( W' = W'_1 \cup W'_2 \cup W'_3 \) be two \( * \)-trisection 4-manifolds with boundary. Let \( Y \subset W \) and \( Y' \subset W' \) be connected components of the boundary of each 4-manifold and suppose \( f : Y \to Y' \) is a homeomorphism satisfying \( f(W_i \cap Y) = W'_i \cap Y' \) and \( f(W_j \cap Y) = W'_j \cap Y' \) for all \( i, j \). Suppose that each component of the negative boundary of the compression bodies \( W_i \cap W_j \cap Y \) and \( W'_i \cap W'_j \cap Y' \) has boundary for all \( i \neq j \). Then \( X = W \cup_f W' \) admits a \( * \)-trisection with pieces given by \( W_i \cup_f W'_i \) (\( i = 1,2,3 \)), and \( * \)-trisection surface the result of gluing the trisection surface of \( W \) and \( W' \) along the boundary components corresponding to \( Y \) and \( Y' \).

**Proof.** For each \( i = 1,2,3 \) let \( X_i = W_i \cup_f W'_i \). As stated in the theorem, we will show that \( X = X_1 \cup X_2 \cup X_3 \) is a \( * \)-trisection.

We will first focus on the pairwise intersection \( X_i \cap X_j \). Fix \( i \neq j \) and let \( C = W_1 \cap W_2 \) and \( C' = W'_1 \cap W'_2 \). By definition \( C \) and \( C' \) are compression bodies with positive boundary the connected \( * \)-trisection surfaces \( \Sigma \) and \( \Sigma' \), respectively. Let \( P_Y = \partial_- C \cap Y \) and \( P_{Y'} = \partial_- C' \) be the (negative) boundary components of \( C \) and \( C' \) lying inside the gluing regions. Since each component of \( P_Y \) and \( P_{Y'} \) is a surface with boundary, one can check that \( \tilde{C} = C \cup P_Y = P_{Y'} \) is also a compression body with positive boundary \( \tilde{\Sigma} = \Sigma \cup \Sigma' \). The curves in \( \tilde{\Sigma} \) bounding curves in \( \tilde{C} \) are given by (1) meridian disks for \( C \) and \( C' \); and (2) disks obtained from pairwise disjoint arcs filling the compressed page \( P_Y \), glued along their boundaries to their images on \( P_{Y'} \) under \( f \). To end notice

\footnote{Actually any diagram of \( S^2 \times D^2 \).}
that $X_i \cap X_j = (W_i \cap W_j) \cup (W'_i \cap W'_j) = C \cup_{P_Y = P_Y'} C'$ which we have shown is a compression body as desired. From the above, we can also conclude that the triple intersection $X_1 \cap X_2 \cap X_3$ is a copy of $\tilde{\Sigma}$.

We now show that each piece $X_i$ is diffeomorphic to a standard piece for some compression bodies. Fix $i \in \{1, 2, 3\}$, by definition $X_i = Z_{C^0, C^1}$ and $X_i' = Z_{C^{0'}, C^{1'}}$. We use the notation of Section 2.1, i.e., $C_{all} = C^0 \cap C^1$ is the common sub-compression body, $C_i = \overline{C^i - C}$ and $F_{all} = \partial C_{all} = \partial_+ C_i$.

By Construction 1 of Section 2.1, we can build $Z$ and $Z'$ as follows.

$$Z_{C^0, C^1} = \left((C_0 \cup_{F_{all}} C_1) \times [0, 1]\right) + \left(\frac{1}{2}(\chi(\partial_- C_0) + \chi(\partial_- C_1) - 2\chi(F_{all}))\right) \text{1-handles}.$$ 

Similarly for $Z_{C^{0'}, C^{1'}}$. Recall that $C_i$ and $C'_i$ might be disconnected and denote by $D_i = C_i - Y$, $G_{all} = F_{all} - Y$ (similarly $D'_i$ and $G'_{all}$) be the connected components inside $Y$. From the fact that $W_i \cap \partial W = C_0 \cup_{F_{all}} C_1$, we obtain the following decomposition of $X_i$.

$$X_i = W_i \cup W_i' = ((C_0 \cup_{F_{all}} C_1) \times [0, 1]) \cup ((C_0 \cup_{F_{all}} C'_1) \times [0, 1]) + (\text{1-handles})$$

$$= ((D_0 \cup_{G_{all}} D_1) \times [0, 1]) \cup ((D'_0 \cup_{G'_{all}} D'_1) \times [0, 1]) + (\text{1-handles}) + (1\text{-handles})$$

Since all connected components of $\partial_- C_i \cap Y$ and $\partial_- C'_i \cap Y'$ are surfaces with boundary, the discussed in Construction 1 of Section 2.1 states that $((C_0 \cup_{F_{all}} C_1) \times [0, 1])$ is a 1-handlebody. Thus, in the last two equalities some of the 1-handles come from this extra piece. To end, recall that in the previous paragraph we showed that $C^i \cup_f C'^i$ is a compression body with positive boundary $\tilde{\Sigma}$, and notice that $D_i \cup D'_i$ is a sub-compression body of $C^i \cup_f C'^i$. Hence $X_i = Z_{\tilde{C}^0, \tilde{C}^1}$ with $\tilde{C}_i = C^i \cup_f C'^i$, and $X_1 \cup X_2 \cup X_3$ is a $*$-trisection.

**Remark 3.2.** In practice, one might need to apply Lemma 2.3 enough times to ensure that every connected component $\partial_-(W_i \cap W_j)$ is a surface with boundary in order to apply Theorem 3.1. In particular, for relative trisections with $b = 0$, it is sufficient to poke the trisection surface three times.

**Corollary 3.3** (Pasting Lemma for relative trisections with empty binding). Let $W = W_1 \cup W_2 \cup W_3$ and $W' = W'_1 \cup W'_2 \cup W'_3$ be two trisected 4-manifolds with non-empty connected boundary and **closed** trisection surfaces $\Sigma$ and $\Sigma'$, respectively. Let $P$ and $P'$ be the pages of the fibration over $S^1$ on $\partial W$ and $\partial W'$ induced by the trisections, respectively. Let $f : \partial W \to \partial W'$ be a homeomorphism between the boundaries respecting the pages; i.e., $f(P) = P'$. Then the glued closed 4-manifold $X = W \cup_f W'$ admits a $(G; K)$-trisection where

$$G = g(\Sigma) + g(\Sigma') + 2 \text{ and } K_i = k_i + k'_i + 2g(P).$$

Here $k_i$ denote the number of common curves in the trisection diagrams.

**Remark 3.4** (The diagrams). Take two relative trisection diagrams (with empty binding) for $W$, $W'$ with a diffeomorphism between connected components of their boundaries $f : Y \to Y'$ satisfying $f(W_i \cap Y) = W'_i \cap Y'$ and $f(W_i \cap W_j \cap Y) = W'_i \cap W'_j \cap Y'$ for all $i, j$. We get a trisection diagram for $X = W \cup_f W'$ with trisection surface

$$\tilde{\Sigma} = (\Sigma - \cup_i B_i) \cup_{\partial B_i = \partial B'_i} (\Sigma' - \cup_i B'_i).$$

Here $B_1, B_2, B_3$ are disks disjoint from the trisection diagram for $W$ such that $B'_i = f(B_i)$ are also disjoint from the diagram of $W'$. The curves for the handlebody $H_{ij} = X_i \cap X_j$ are:
1. Meridian disks for $C_{ij}$ and $C'_{ij}$.

2. One disk corresponding to $B_k$.

3. Disks obtained from pairwise disjoint arcs filling the compressed page $P_{ij} - (B_i \cup B_j)$, glued along their boundaries to their images on $P'_{ij} - (B'_i \cup B'_j)$ under $f$.

If we were given $\star$-trisections instead and $f : Y \to Y'$ is a homeomorphism between connected components of the boundaries, then the curves for the compression body $X_i \cap X_j$ are given by:

1. Meridian disks for $C$ and $C'$.

2. Disks obtained from pairwise disjoint arcs filling the compressed page $P_Y$, glued along their boundaries to their images on $P_Y'$ under $f$.

![Figure 4: Two distinct ways of gluing a pair of thickened spheres. The map $\tau$ twists the $S^2$ fiber once while traversing the $S^1$ direction.](image)

![Figure 5: Trisection diagram for $T^2 \times S^2$. One can see two thrice punctured tori (left and right) corresponding to each copy of $T^2 \times D^2$.](image)

**Example 3.5** (Sphere bundles over $RP^2$). During the last day of the 2019 Spring Trisectors Meeting at UGA, the pair of trisection diagrams of Figure 6 was discussed. Work of Gay and Meier in [8] shows that $B$ is a Gluck twist of $A$ along some embedded 2-sphere. We can use Theorem 3.1 and Remark 4.2 to decompose the 4-manifold $A$ as the union $A = (S^2 \times D^2) \cup_f X$ where $X$ is the complement of a 2-loop in $S^1 \times S^3$ and the pasting map does not twist the $S^2$ component (see Figure 7). It is a nice exercise to see that $X$ is diffeomorphic to the product $S^2 \times M^2$ where $M^2$ is a Mobius band. Hence $A$ is a trisection for the product $S^2 \times RP^2$ and $B = S^2 \times RP^2$. 

11
Figure 6: A pair of genus 3 trisection diagrams that differ by Gluck twist.

Figure 7: Decomposing $A$ as $A = (S^2 \times D^2) \cup_f X$. Note that $X$ is a trisection diagram for the complement of a 2-loop in $S^1 \times S^3$ (see Section 4.1).

4 The Complement of a Simple Closed Curve

Let $X$ be a compact 4-manifold with $\star$-trisection $X = X_1 \cup X_2 \cup X_3$. Let $\Sigma$ be the trisection surface and consider $c \subset X$ a simple closed curve in $X$. Since $\pi_1(\Sigma) \twoheadrightarrow \pi_1(X)$, we can represent $c$ with an immersion $S^1 \hookrightarrow \Sigma$. Given an immersed curve as such, we are interested in finding a $\star$-trisection $X - \eta(c)$. To accomplish this, we decompose the immersed curve into a union of embedded arcs, push the arcs into the handlebodies, and then remove the tubular neighborhood of each arc.

Definition 4.1. Given a trisection $(\Sigma, \alpha_1, \alpha_2, \alpha_3)$, we say that an immersed curve $c \subset \Sigma$ is decomposed if $c$ is the union of three collections of embedded arcs $c = a_1 \cup a_2 \cup a_3$ with the property that $a_1 \cap \alpha_1 = a_2 \cap \alpha_2 = a_3 \cap \alpha_3 = \emptyset$ and that each arc in $a_i$ is connected to one arc from each of $a_{i-1}$ and $a_{i+1}$. Denote the discrete set of points $a_{i-1} \cap a_{i+1}$ by $b_i$.

Starting with a decomposed curve $c$, push each arc of $a_i$ into $H_i$, leaving the endpoints fixed. We claim that the result $X - \eta(c) = \cup_{i=1}^3 X_i - \eta(c)$ is a $\star$-trisection. Since $c \cap X_i = a_{i-1} \cup a_i \subset \partial X_i$ is a collection of disjoint arcs in the boundary of $X_i$, the complement $\tilde{X}_i := X_i - \eta(c)$ is diffeomorphic to $X_i$. By construction, the arcs $a_i$ are simultaneously parallel to the boundary of $H_i$, thus $\tilde{X}_i \cap \tilde{X}_{i+1}$ is also a compression body. Hence we have a $\star$-trisection of $X - \eta(c)$.

We now describe the $\star$-trisection diagram for $X - \eta(c)$ resulting from this procedure. The trisection surface $\tilde{\Sigma} = \cap_i \tilde{X}_i$ is a copy of $\Sigma$ with open disks removed around the endpoints of the arcs. Start by drawing a trisection diagram $(\Sigma, \alpha_1, \alpha_2, \alpha_3)$ for $X$ together with the immersed decomposed curve. Let $\tilde{\Sigma}$ be the punctured surface $\Sigma - \cup_{i=1}^3 \eta(b_i)$. Then the compression body $\tilde{H}_i = \tilde{X}_i \cap \tilde{X}_{i+1}$ can be built from $\tilde{\Sigma}$ by attaching 2-handles along the following curves (see Figure 8).

1. The original curves $\alpha_i$,
2. the boundary parallel curves $\partial \eta(b_i)$
3. the non-boundary parallel components of $\partial (\eta(a_i))$.

![Figure 8](image1.png)

Figure 8: How a ⋆-trisection diagram changes when taking the complement of a decomposed curve in $\Sigma$.

**Remark 4.2.** If $X$ is closed and we can decompose the curve $c$ so that $|a_i| = 1$, then the ⋆-trisection diagram given by the procedure above introduces an unnecessary curve which we can remove, as suggested by Figure 9. To see why this is true, compress $\tilde{\Sigma}$ along all of the original $\alpha$ curves. What remains is a three punctured surface. The two new $\alpha$ curves introduced by the procedure above become parallel, thus we can remove one of them.

![Figure 9](image2.png)

Figure 9: How the ⋆-trisection diagram changes if each $|a_i| = 1$ and $\partial X = \emptyset$.

### 4.1 Loops in genus one trisections

A careful observer might remark that all simple closed curves in a 4-manifold $X$ representing $[c] \in \pi_1(X)$ are isotopic. It is therefore natural to wonder if decomposed curves representing $[c] \in \pi_1(X)$ are slide equivalent to representatives of the same class $[c] \in \pi_1(\Sigma)$ in the trisection surface. We prove that this is the case for embedded curves in genus one trisections.

**Proposition 4.3.** Let $(\Sigma, \alpha, \beta, \gamma)$ be a genus one trisection diagram for a closed 4-manifold. Let $c$ be an embedded decomposed curve in $\Sigma$. Let $\{\mu, \lambda\}$ be a basis for $\pi_1(\Sigma)$ with $[\alpha] = [\mu]$. If $[c] = [m\mu + n\lambda]$, then by sliding the arcs $a_i$ over the boundaries $b_i = a_{i+1} \cap a_{i-1}$, and sometimes sliding $a_1$ over $\alpha$, $c$ is slide equivalent to an immersed curve representing $[n\lambda]$ with $a_1$ twisting around $b_3$ a total of $m$ times.

*Proof.* Throughout this argument $\beta$ and $\gamma$ will be pushed around as needed. If $\gamma$ or $\beta$ are in the way of sliding $a_i$ over $b_i$ for $i = 1, 2$, simply include them in the slide as in the left of Figure 10. Since $c$ is embedded and $a_1 \cap \alpha = \emptyset$, the $n$ intersections of $c$ with $\alpha$ occur on $a_2 \cup a_3$. Isotope $c$ such that $a_2 \cap \alpha = \emptyset$. Since $a_1$ and $a_2$ both miss $\alpha$, we may isotope them such that $a_1$ is a small
segment leaving $b_2$, $a_2$ is a small segment leaving $b_3$, and $a_3$ represents all but $a_1 \cup a_2$ of the loop $m\mu + n\lambda$ in $\pi_1(X, b_2)$. However, since $c$ is embedded, the word in $\pi_1(X, b_2)$ given by traversing $a_3$ is ordered in such a way as to ensure that $a_3$ is an embedded arc also. Thus we may assume from the beginning that the trisection as well as the embedded decomposed curve is equivalent to the model in the right of Figure 10 where we have suppressed the uniquely determined $\beta$ and $\gamma$ curves.

Let $w$ be the word in the alphabet \{\mu, \lambda\} which represents $[c]$ and decompose $w$ into three subwords $w_1, w_2, w_3$ such that $w = w_1w_2w_3$ and each $w_i$ records the path traversed by the arc $a_i$. Initially, as described above, we have that $w_1$ and $w_2$ are empty words and $w_3$ is a certain permutation of the multiset \{m\mu, n\lambda\} which allows $a_3$ to be an embedded arc. We claim that a decomposed $c$ representing $[m\mu + n\lambda]$, $m > 0$, in such a way that $w_1$ and $w_2$ are empty words, can be slid to be a representative of $[(m-1)\mu + n\lambda]$ with $w_1$ and $w_2$ being empty words.

Suppose that $c$ represents $[m\mu + n\lambda]$, $m > 0$ with $w_1$ and $w_2$ empty. Then since $m > 0$, let $j \geq 0$ such that $\lambda^j\mu$ is a prefix of $w_3$. The endpoint of $a_2$ connected to $a_3$ is $b_1$. We can make $b_1$ “move past” $\alpha$ and $a_1$ by performing the local move in Figure 11.

We can therefore move $b_1$ along $a_3$ until $w_2$ reads $\lambda^j\mu$, Figure 12. Using the fact that this is a genus one trisection, and the fact that $\lambda^j\mu$ is embedded, commute $\mu$ past $\lambda^j$ by sliding $a_2$ over $b_2$ when necessary as in Figure 13. Now we are free to grow $a_1$ so that $w_1 = \mu$ and $w_2 = \lambda^j$ and from here, we can slide $a_1$ against $\alpha$ to remove $\mu$ completely at the expense of adding a single twist of $a_1$ around $b_3$, see Figure 14. Now it is possible to commute $\mu$ past $\lambda^j$ by sliding $a_2$ over $b_2$ when necessary.

After completing this slide to eliminate $\mu$, we can shrink $a_2$, removing $\lambda^j$ from $w_2$ and appending $\lambda^j$ to the front of $w_3$. The decomposed curve at this stage is a representative of $[(m-1)\mu + n\lambda]$ with $w_1 = w_2$ empty, so the claim is proved. By repeating this process, we can slide $c$ to the model representative of $[n\lambda]$ below with $w_1 = w_2$ being empty and $w_3 = \lambda^j$.
On the other hand, if \((\Sigma, \alpha, \beta, \gamma)\) is a simply connected genus one trisection with \([\lambda] = [\beta]\) in the above argument, then we can say even more. Since \(w = \lambda^n\), we may extend \(a_2\) so that \(w_2 = \lambda^n\) and \(w_1 = w_3\) are empty words. Putting \(\beta\) back into the picture, it is now clear that we can slide \(\lambda\) off as well, see Figure 16. This proves the following.

**Corollary 4.4.** Let \((\Sigma, \alpha, \beta, \gamma)\) be a genus one trisection diagram for a simply connected closed 4-manifold. Let \(c\) be an embedded decomposed curve in \(\Sigma\). Let \(\{\mu, \lambda\}\) be a basis for \(\pi_1(X)\) with \([\alpha] = [\mu]\) and \([\beta] = [\lambda]\). If \(c = m\mu + n\lambda\), then by sliding the arcs \(a_i\) over the boundaries \(b_i = a_{i+1} \cap a_{i-1}\), sometimes sliding \(a_1\) over \(\alpha\), and sometimes sliding \(a_2\) over \(\beta\), \(c\) is slide equivalent to an immersed curve representing \(1 \in \pi_1(\Sigma, b_2)\) with \(a_1\) twisting around \(b_3\) a total of \(m\) times and \(a_2\) twisting around \(b_1\) a total of \(n - 1\) times.
5 Trisections of genus 3

For two irreducible fractions \( \frac{a}{b}, \frac{c}{d} \in \mathbb{Q} \cup \{ \frac{1}{0} \} \), define \( d(\frac{a}{b}, \frac{c}{d}) := \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \). Given an ordered triple of rational numbers \( \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \in \mathbb{Q} \cup \{ \frac{1}{0} \} \), we can consider the diagram \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) as in Figure 17. Here one curve of each \( \alpha, \beta, \gamma \) set has slope \( \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \), respectively, in the torus obtained by compressing the genus three surface along the two central curves of the same color as in the figure below. The diagram \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) is a trisection diagram for some closed smooth 4-manifold if and only if each pair \( x, y \in \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) satisfies the inequality \(|d(x, y)| \leq 1\). If the three numbers in the triplet are all distinct with \( d(\frac{a}{b}, \frac{c}{d}) = \pm 1 \) for all \( x \neq y \in \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \), then we call \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) a Farey triplet. In this case, \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) corresponds to a triangle in the Farey graph.

![Figure 17: (left) The longitude (l) and meridian (m) for each of the three tori. (right) The diagram \( D(\frac{2}{3}, \frac{1}{1}, \frac{1}{2}) \).](image)

The question we will discuss now is what 4-manifolds the diagrams \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) represent. Jeffrey Meier proved in [17] that \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) is the diagram of a spun lens space \( L(p, q) \). He conjectured that the only 4-manifolds admitting genus three trisections are spun lens spaces and certain connected sums of combinations of \( S^1 \times S^3, S^2 \times S^2, \mathbb{CP}^2, \) and \( \overline{\mathbb{CP}^2} \). We will call the latter combinations standard manifolds. Note that \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) is simply connected whenever \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) contains two or three distinct numbers. Thus, such diagrams must represent standard manifolds if we expect the conjecture to be true. We prove that this is indeed the situation.

**Theorem 5.1.** Let \( \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \in \mathbb{Q} \cup \{ \frac{1}{0} \} \) satisfying \(|d(x, y)| \leq 1\) for all \( x, y \in \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \). Then \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) describes a trisection diagram for

1. either \( \mathbb{CP}^2 \# \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \) or \( \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2} \) if \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) is a Farey triplet,

2. either \( S^2 \times S^2 \) or \( S^2 \times S^2 \) if \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} = \{ x, y \} \) with \( d(x, y) = \pm 1 \),

---

Figure 16: An immersed curve representing \( 1 \in \pi_1(\Sigma, b_2) \) with twists around boundary points.
3. A spun lens space if \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} = \{ x \} \).

To find out the specific 4-manifold the above diagrams represent, it is enough to compute its intersection matrix using \[11\] or \[12\].

**Proof.** The third part was done by Jeffrey Meier in \[17\]. Denote by \( X \) the 4-manifold represented by the diagram \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \). Notice that we can decompose the genus three surface in Figure 17 into a thrice puncured sphere and a thrice puntured torus glued together along their boundaries, see Figure 18. Theorem 3.3 implies that \( X \) decomposes as the union \( X = (S^2 \times D^2) \cup \partial Y \) for some 4-manifold \( Y \).

![Figure 18: Decomposing \( X \) as \( (S^2 \times D^2) \cup \partial Y \) glued via some map \( f : S^1 \times S^2 \to \partial Y \).](image)

If \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) is a Farey triplet, then \( Y \) is the complement of some \( S^1 \times B^3 \) in \( \mathbb{CP}^2 \). Since \( \mathbb{CP}^2 \) is simply connected, we can take such circle to be unknotted and \( Y = \mathbb{CP}^2 \# (S^2 \times D^2) \); hence \( X \) is the connected sum of \( \mathbb{CP}^2 \) with a sphere bundle over the sphere. If \( \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \) with \( d(\frac{a}{b}, \frac{c}{d}) = \pm 1 \), then \( Y = S^4 - (S^1 \times B^3) \) and \( X \) is a copy of \( S^2 \times S^2 \) or \( S^2 \widetilde{\times} S^2 \). This proves that the manifolds associated to the trisection diagrams \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) are diffeomorphic to standard ones. To figure out which ones specifically, it is sufficient to compute the intersection form \( Q_X \).

The intersection form of \( X \), computed from \( D(\frac{a}{b}, \frac{c}{d}, \frac{p}{q}) \) using \[11\], is given by

\[
Q_X = \begin{bmatrix}
\frac{bd}{ad-bc} & -1 & \frac{b(cq-dp)}{bc-ad} \\
-1 & 0 & 0 \\
\frac{b(cq-dp)}{bc-ad} & 0 & \frac{(bp-aq)(cq-dp)}{bc-ad}
\end{bmatrix}
\]

In the case that \( \frac{c}{d} = \frac{p}{q} \), notice the third column and third row become zero and we are left with the intersection form \( Q_X = \begin{bmatrix} \frac{bd}{ad-bc} & -1 \end{bmatrix} \) which is equivalent to the intersection form for \( S^2 \times S^2 \) when \( bd \) is even and to \( S^2 \widetilde{\times} S^2 \) when \( bd \) is odd. If all rationals are distinct, then without loss of generality suppose that \( d(\frac{a}{b}, \frac{c}{d}) = 1 \). Because \( \{ \frac{a}{b}, \frac{c}{d}, \frac{p}{q} \} \) is a Farey triple, we know that \( \frac{p}{q} = \frac{a+c}{b+d} \). This gives

\[
Q_X = \begin{bmatrix}
bd & -1 & b \\
-1 & 0 & 0 \\
b & 0 & \mp 1
\end{bmatrix}
\]

By inspection, one determines that \( Q_X \) is equivalent to \( \langle 1 \rangle \oplus \langle \mp 1 \rangle \oplus \langle -1 \rangle \). \qed
5.1 Farey Trisections are standard

We will now demonstrate that in cases 1 or 2 of Theorem 5.1 the diagrams are actually reducible and thus standard.

**Theorem 5.2.** Let \( \{ \frac{a}{p}, \frac{b}{q}, \frac{c}{r} \} \subset \mathbb{Q} \cup \{ \frac{1}{0} \} \) with \( d(x, y) \leq 1 \) for each \( x, y \in \{ \frac{a}{p}, \frac{b}{q}, \frac{c}{r} \} \). If at least two of these fractions are distinct, then \( D(\frac{a}{p}, \frac{b}{q}, \frac{c}{r}) \) is equivalent to the standard diagram for \( T \# S \) where \( T \in \{ S^4, \mathbb{C}P^2, \overline{\mathbb{C}P^2} \} \) and \( S \in \{ S^2 \times S^2, S^2 \times \overline{S^2} \} \).

**Proof.** Decompose \( D(\frac{a}{p}, \frac{b}{q}, \frac{c}{r}) \) into two pieces as suggested by Figure 18, let \( D' \) denote the thrice punctured torus component of this decomposition and let \( P \) denote the thrice punctured sphere component. Notice that this \( \star \)-trisection \( D' \) is a diagram for the complement of an embedded curve in a simply connected 4-manifold. Specifically, \( D' \) is the result of taking the complement of the \( c = \lambda \) curve in a genus 1 trisection with \( \alpha = a\lambda + b\mu \), \( \beta = c\lambda + d\mu \), and \( \gamma = p\lambda + q\mu \) where the curve \( c \) has been decomposed as suggested by the left side of Figure 18. By Corollary 4.4, the decomposed curve \( c \) is slide equivalent to an immersed decomposed curve \( c' \) representing the trivial curve where some of the arcs twist around the boundary points, as in the last frame of Figure 16. In particular, \( \alpha \) and \( \beta \) are disjoint from \( c' \). Using the fact that \( c' \) represents the trivial loop, we can slide \( \gamma \) against \( b_3 \) until \( \gamma \) is disjoint from \( c' \) also. Thus there is a curve \( \delta \) separating \( c' \) from \( \alpha, \beta, \gamma \). By surgering \( D' \) along \( \delta \) we get two components. Let \( Q \) be the component coming from the side of \( \delta \) containing \( c' \) and let \( T \) be the torus component containing \( \alpha, \beta, \gamma \). Notice that \( S = Q \cup_f P \), with the attaching map \( f \) given by the \( a_i \), is a genus two trisection of a closed 4-manifold with an intersection form of full rank. By the work of Meier and Zupan [20], \( S \) is either \( S^2 \times S^2 \) or \( S^2 \times \overline{S^2} \). The component \( T \) is a genus one trisection of a simply connected closed 4-manifold: \( S^4, \mathbb{C}P^2 \) or \( \overline{\mathbb{C}P^2} \). \( \square \)

**Remark 5.3 (Spin Lens spaces).** Similar to Theorem 5.2, Corollary 4.4 implies that the diagrams \( D(\frac{a}{p}, \frac{b}{q}, \frac{c}{r}) \) and \( D(\frac{1}{q}, \frac{1}{q}, \frac{1}{q}) \) are equivalent to diagrams which are identical outside of the regular neighborhood of a \( \gamma \) curve, say \( \gamma_0 \). In this annulus \( \nu(\gamma_0) \), the diagrams differ by their \( \alpha \) curves, where one twists \( q \) times around this annulus and the other twists once (see Figure 15). This motivates the following question about uniqueness of trisection diagrams for 1-surgeries.

**Question 5.1.** Let \( c \) be an embedded loop in a 4-manifold \( X \) represented by a (possibly immersed) decomposed curve in the trisection surface. Is the trisection diagram for \( (X - \eta(c)) \cup (S^2 \times D^2) \), with a specific choice of framing in \( \mathbb{Z}/2\mathbb{Z} \), unique up to isotopy?

6 Surface surgery

In Section 6.1 we \( \star \)-trisect the complement of surfaces in bridge position. Then we will use the Pasting Lemma to give descriptions of how to perform surgery along codimension two submanifolds. More explicitly, we trisect the Cacime Surface and explain how to perform Fintushel-Stern knot surgery and torus surgeries such as Logarithmic transforms and Luttinger transforms. The careful reader might observe that the diagrams for these transformations change by concatenating a fixed picture or by changing some loops in a high enough stabilization of the original trisection diagram. Thus to study the behavior of 4-manifold invariants under surface surgery, it could be worthwhile to explore these local modifications in detail.
6.1 Embedded surfaces and their complements

Let $X = X_1 \cup X_2 \cup X_3$ be a $\ast$-trisected 4-manifold and let $F \subset X$ be an embedded closed surface. Following Meier and Zupan [19], we say that $F$ is in bridge position with respect to the $\ast$-trisection if, for each $i \neq j$, $D_i = F \cap X_i$ is a collection of trivial disks in $X_i$, and the arcs $D_i \cap D_j$ form a trivial tangle in the compression body $C_{ij} = X_i \cap X_j$. Given a $\ast$-trisection diagram $(\Sigma, \alpha, \beta, \gamma)$, we can decode a bridge trisection of $F$ by three sets of embedded arcs $s_\alpha, s_\beta, s_\gamma$ in $\Sigma$ corresponding to the shadows of the trivial tangles $F \cap C_\varepsilon, \varepsilon \in \{\alpha, \beta, \gamma\}$. The shadow arcs have common endpoints $t = F \cap \Sigma$. We consider the arcs in $s_\varepsilon$ to be disjoint from the loops in $\varepsilon$. Thus, isotopy of the arcs $F \cap C_\varepsilon$ relative to their boundaries corresponds to sliding the shadows $s_\varepsilon$ over $\varepsilon$. For more details and examples of bridge trisections see [18, 19, 16].

Given a bridge trisected surface $F \subset X$, there is an obvious $\ast$-trisection for the complement $X - \eta(F)$ given by $\tilde{X}_i = X_i - \eta(D_i), i = 1, 2, 3$. Let $s_\alpha, s_\beta, s_\gamma$ be a set of shadows for $F$ in the $\ast$-trisection diagram $(\Sigma, \alpha, \beta, \gamma)$. A diagram for the $X - \eta(F)$ is given by $(\tilde{\Sigma}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ where $\tilde{\Sigma} = \Sigma - \eta(t)$ is copy of $\Sigma$ with $|t|$ disks removed, and for each $\varepsilon \in \{\alpha, \beta, \gamma\}, \tilde{\varepsilon} = \varepsilon \cup \varepsilon'$ where the extra loops in $\varepsilon'$ are obtained from the non-boundary parallel components of $\partial\eta(s_\varepsilon \cup \eta(t))$. See Figure 19 for a concrete example. In the rest of the section we will see how we can use $\ast$-trisection diagrams to draw diagrams for surgeries along bridge trisected surfaces.

Figure 19: (left) A bridge trisection for $T^2 \times \{pt\}$ inside $T^2 \times D^2$. (right) A $\ast$-trisection for $T^2 \times S^1 \times [0,1] = T^3 \times [0,1]$.

Remark 6.1. It is important to mention that this decomposition was previously discussed by Kim and Miller in [14]. The authors of [14] observed that the above $\ast$-trisection is a classic trisection only when $F$ is a 2-sphere. To obtain an open book decomposition on the boundary, Kim and Miller performed a sequence of “boundary-stabilizations” to the $\ast$-trisection above. This procedure increases the complexity of the trisection surface in a controlled way, and in principle can also be used to perform surgery along surfaces following the methods in this section.

6.2 Cacime Surface

Let $F_2, F_3$ be oriented surfaces of genus two and three, respectively. Define $\tau_i : F_i \to F_i$ to be involutions as in Figure 20. Define the Cacime Surface to be the quotient $C = F_2 \times F_3 / \tau_2 \times \tau_3$. 

\(^2\text{See Definition 2.1}\)
Following Chapter 4.2 of [1], $C$ is diffeomorphic to a fiber sum of $F_2$ bundles over $T^2$

$$C = (F_2 \times T^2) \natural (F_2 \times S^1 \times [0,1]) / \tau_2,$$

here $\tau_2(x,t) = (\tau_2(x),t)$ is a diffeomorphism of $F_2 \times S^1$. To obtain a trisection diagram for $C$ we follow the following three steps drawn in figures 21, 22 and 24.

1. Draw a Heegaard splitting for $F_2 \times S^1$ and perform Dale Koening’s algorithm [15] to draw a genus 21 trisection diagram for $X_f = (F_2 \times S^1) \times [0,1]/f$ for $f \in \{id, \tau_2\}$.

2. Notice that $F_2 \times \{pt\}$ can be seen in the Heegaard splitting for $F_2 \times S^1$ and use this to draw a system of shadows for a bridge position for $F_2 \times \{pt\} \times \{pt\}$.

3. Apply the method in Section 6.1 to trisect the complements $X_f - \eta(F_2)$ and tube the corresponding boundaries using the Pasting Lemma to obtain a genus 51 trisection diagram for $C$.

Koening showed in [15] that his algorithm always results in a trisection which can be destabilized. Here, we can destabilize at least ten times (five on each $X_f$). This picture resembles the handle diagram in Figure 4.17 of [1]. Thus we think of the Pasting Lemma as the trisection analog of the Roping Method for handle decompositions.

**Question 6.1.** Is there an interpretation for the phrase “upside-down trisection”? If so, is there a different method of gluing two $\ast$-trisected 4-manifolds?
Figure 21: Genus five Heegaard splitting for $F_2 \times S^1$ and a genus 21 trisection diagram for $X_{id} = F_2 \times T^2$. Both diagrams are drawn in punctured surfaces with the correct identifications on the boundaries. The bottom left annulus is a diagrammatic representation of the trisection for $X_{id}$. The colored arcs in the core of the annulus correspond to thickened punctured Heegaard surfaces, and the rest arcs are copies of the 3-dimensional handlebodies of the original Heegaard splitting. For more details see [15].

Figure 22: (left) A genus 21 trisection diagram for $X_{\tau_2}$. Notice the twist on some of the blue arcs is decoding the action of $\tau_2$ in the Heegaard splitting of $F^2 \times S^1$. (right) A local model for the bridge trisection of $F^2 \times \{pt\} \times \{pt\}$. 
Figure 23: Genus 51 trisection diagram of the Cacime surface.

Figure 24: A close-up of the above trisection. The disks with common labels are identified as shown. Notice that some curves can be erased as in Figure 29.
### 6.3 Knot Surgery

Let $K$ be a knot in $S^3$, and let $m$ denote a meridian of $K$. Let $M_K$ be the 3-manifold obtained by 0-surgery along $K$. Notice that $m$ can be viewed as a circle in $M_K$ and that the torus $T_m = m \times S^1 \subset M_K \times S^1$ has self-intersection zero. Let $X$ be a 4-manifold containing an embedded torus $T$ with self-intersection zero. Denote by $X_K$ the fiber sum

$$X_K = X_{\partial T=m}(M_K \times S^1).$$

Here, we glue the complement of the corresponding thickened tori along a diffeomorphism preserving $\{pt\} \times \partial D^2$. Fintushel and Stern introduced the knot surgery operation in [7] to build exotic copies of smooth 4-manifolds by controlling the change of the Seiberg-Witten invariants using the Alexander polynomial of $K$. We will describe how to draw trisection diagrams for $X_K$.

Let $K$ be a knot in $S^3$. Find a Heegaard splitting for $S^3$ such that $K$ can be isotoped to be a subset of the core of one of the handlebodies. In order to do this one can consider a tunnel system for $K$ as in Figure [25]. Draw $K$ in such a way that $K$ is embedded in the Heegaard surface $F$ and the framing induced by the surface is the 0-framing on $K$. By construction we can find a Heegaard diagram $(F,a,b)$ such that an isotopic copy of $m$ belongs to $a$ and $K$ is disjoint from all other elements of $a$. A Heegaard diagram for $M_K$ is given by $(F,a',b)$ where $a' = (a - m) \cup K$. Furthermore, the loop $m$ as a subset of $F$ corresponds to the meridian of $K$ inside $M_K$. Now perform Dale Koening’s algorithm to draw a trisection diagram $(\Sigma, \alpha, \beta, \gamma)$ for $M_K \times S^1$ using $(F,a',b)$. Our choice of $m$ as a subset of $F$ allows us to see a bridge position for $T_m = m \times S^1$. To see this recall that $\Sigma$ is obtained by four copies of $F$ tubed as in Figure [26]. Draw $m$ on each copy of $\Sigma$ and pick four distinct points on each circle. Then push-off $T_m$ away from $F \times S^1$ fixing the 16 selected points. This procedure gives us the bridge trisection of $T_m$ with 8 bridges as in Figure [26].

Now let $T$ be a torus with self-intersection zero embedded in a 4-manifold $X$. Suppose $T$ is in bridge position with respect to some trisection of $X$. There are two approaches we can take in order to draw a trisection diagram for $X_K$. The first approach is to perturb both bridge trisections for $T_m$ and $T$ until the new bridge trisections induce the same cell decomposition on both $T_m$ and $T$. Then to draw a trisection for $X_K$ we have to draw the $*$-trisection diagrams for the corresponding surface complements following Section 6.1 and tube them using the Pasting Lemma as we did for the Cacime Surface.

The second approach is to glue a copy of $T^3 \times [0,1]$ in such a way that the new boundary has a nice $S^1$-fibration with fiber a copy of the surface $T_m \times \{pt\}$ (similarly for $T$). In order to do this, draw the cell decomposition induced by the bridge trisection on the torus $T_m$ (see Figure [27]). This picture can be thought as a bridge trisection for $T_m \times \{0\}$ inside $T_m \times D^2$. Thus we can draw a $*$-trisection for $T_m \times S^1 \times [0,1]$ with one boundary having the same handle decomposition as the $*$-trisection in Figure [26] and other boundary a $S^1$-fibration with fiber $T_m \times \{pt\}$. This new trisection is drawn in Figure [27]. Now tube this new $*$-trisection with the trisection for the complement of $T_m$ in $M_K \times S^1$ to obtain a classical relative trisection (with empty binding) with a copy of $T_m$ as the fiber on its boundary, as desired. Notice the appearance of sphere components in the compressed surfaces $\Sigma_\alpha$, $\Sigma_\beta$ and $\Sigma_\gamma$, thus some curves are redundant (see Figure [28]). The final trisection diagram is depicted in Figure [29]. After performing a similar process to the bridge trisection of $T$ in $X$ one, in theory, can perform Pasting Lemma one last time to draw a trisection for $X_K$.

The advantage of the second method is that any diffeomorphism of the form $f \times id_{S^1} : T_m \times \partial D^2 \rightarrow T \times \partial D^2$ can be used to perform the fiber sum.  

---

\textsuperscript{4}See [18].
Figure 25: Heegaard diagram for $M_K$ with $K$ a trefoil knot.

Figure 26: Bridge trisection diagram for $T_m \subset M_K \times S^1$. After an isotopy, we draw a $\star$-trisection diagram for the complement $(M_K \times S^1) - (T_m \times D^2)$.

Figure 27: (left and middle) The cell decomposition induced by the bridge trisection on $T_m$, notice that also describes a bridge trisection for $T_m \times \{0\} \subset T_m \times D^2$. (right) The associated $\star$-trisection for the complement of this bridge trisected surface.
Figure 28: Red loops after performing the pasting lemma. Notice that they are some redundancies.

Figure 29: A trisection diagram for the complement of $T_m \times S^1$ inside $M_K \times S^1$ with boundary admitting a $S^1$-foliation with fiber $T_m \times \{pt\}$. To perform knot surgery, we must attach this diagram to the complement of a torus in $X$. 
6.4 Torus Surgery

Let $F$ be an embedded torus with trivial tubular neighborhood in a $\ast$-trisected 4-manifold $X$. In this subsection we are interested in drawing trisection diagrams for $(X - \eta(F)) \cup_g (T^2 \times D^2)$ for some homeomorphism $g : \partial \eta(F) \to T^3$. In principle, there are $SL_3(\mathbb{Z})$ many such maps.

Recall the construction of a $\ast$-trisection for the complement of a surface in Section 6.1. Observe that the components of the pages $\partial(X - \eta(F)) \cap \tilde{X_i} \cap \tilde{X_j}$ corresponding to the neighborhood of the surface $F$ are annuli. Furthermore, one can check that there is a unique way (up-to trisection surface diffeomorphism) to draw the arcs to perform the Pasting Lemma. Thus there is only one diffeomorphism $\partial \eta(F) \to F \times S^1$ preserving the structure on the boundary induced by the $\ast$-trisection: the identity map. To overcome this problem, we can paste the diagram we obtain for $X - \eta(F)$ with a diagram for $T^3 \times [0, 1]$ like in figures 19 and 27. We will refer to such trisections by $\tau_0$. Gluing $\tau_0$ maintains the diffeomorphism type of the complement fixed, and replaces the restrictive decomposition in the boundary by the $S^1$-foliation $F \times S^1$ with fiber $F \times \{pt\}$ induced by the new glued trisection. We can now apply Pasting Lemma to $X - \eta(F)$ using diffeomorphisms of the form $f \times \text{id}_{S^1}$ for some $f : F \to F$.

Denote by $\tau_{23}$ the trisection diagram of Example A.3. $\tau_{23}$ is a relative trisection for $T^3 \times [0, 1]$ such that the $S^1$-foliation induced in $\partial(T^3 \times [0, 1])$ has page $S^1 \times S^1 \times \{pt\}$ in $T^3 \times \{0\}$ and page $S^1 \times \{pt\} \times S^1$ in $T^3 \times \{1\}$. We think of $\tau_{23}$ as acting on the standard basis for $H_1(T^3, \mathbb{Z})$ via a permutation matrix $\sigma_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. By modifying the labelings in $\tau_{23}$, we can draw trisections diagrams $\tau_{31}$ and $\tau_{23}$ of $T^3 \times [0, 1]$ corresponding to 2-cycles $(3, 1)$ and $(2, 3)$ respectively.

Equipped with the trisections $\tau_0$ and $\tau_{ij}$ the procedure of performing torus surgery becomes a linear algebra problem. We will explain this explicitly by showing methods to trisect Logarithmic transforms and Luttinger Transforms.

**Logarithmic Transform.** Let $F$ be an embedded torus in $X$ with self-intersection number zero. Fix a basis for $H_1(F, \mathbb{Z})$ and a trivialization $\eta(F) \cong T^2 \times D^2$. Following [2], given a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(\mathbb{Z})$, we will denote as $A$-Logarithmic transform the 4-manifold $X_{F,A} = (X - \eta(F)) \cup_g (T^2 \times D^2)$, where $g : T^2 \times D^2 \to T^2 \times D^2$ is a homeomorphism given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a_{21} & a_{22} \end{pmatrix}$.

Suppose that $F$ is in bridge position with respect to some trisection $\tau$ of $X$. Notice that

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{22} & a_{21} & 0 \\ a_{12} & a_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus to perform $A$-logarithmic transform along $F$ we can take the $\ast$-trisection $\tilde{\tau}$ for the complement of $F$, and concatenate it with specific trisections as follows,

$$X_{F,A} : \tau \cup \tau_0 \cup_{\text{id}} \tau_{31} \cup_{A \times \text{id}_{S^1}} \tau_{31} \cup_{\text{id}} \tau_0.$$

Here $\tau_0$ is the relative trisection for $T^2 \times D^2$ given by the empty diagram on a closed torus. Integral logarithmic transform is given by the matrix $A_p = \begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix}$. In particular 0-logarithmic transform is given by the map gluing map corresponding with the permutation $(2, 3)$ and so the trisection diagram for the 0-logarithmic transform $X_{F,0}$ can be simplified as follows:

$$X_{F,0} : \tau \cup \tau_0 \cup_{\text{id}} \tau_{23} \cup_{\text{id}} \tau_0.$$
**Luttinger surgery.** For an embedded torus $F$ in $X$ with self-intersection zero, a Luttinger surgery is an operation $X \mapsto X_{m,n}$ where $X_{m,n}$ is torus surgery along $F$ via a homeomorphism given by the matrix $A_{m,n} = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$. Since $A_{m,n}$ factors as follows,

$$A_{m,n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  

We can preform Luttinger surgery along $F$, after fixing a trivialization $\eta(F) \cong T^2 \times D^2$ and a basis for $H_1(F,\mathbb{Z})$, by concatenating the following trisections.

$$X_{F,m,n} : \bar{\tau} \cup \tau_0 \cup \id_{T^2} \cup (1_{m}) \times \id_{T^2} \cup (1_{n}) \times \id_{T^2} \cup \tau_{\emptyset}.$$  

Here, $\bar{\tau}$ is the $\ast$-trisection for the complement of $F$ and $\tau_{\emptyset}$ is the empty trisection diagram for $T^2 \times D^2$.

## A Classic Diagrams

Let $X$ be a $\ast$-trisected 4-manifold with non-empty boundary. If the trisection is a classic relative trisection, the compression bodies given by the pairwise intersection satisfy $C_0^i = C_1^i$ for all $i = 1, 2, 3$. In particular, any diagram $(\Sigma, \alpha, \beta, \gamma)$ of such trisections satisfies that each pair of loops is slide equivalent to the loops like in Figure 30. In this case, there is a open book decomposition on $\partial X$ induced by the trisection with binding having $b = |\partial \Sigma|$ components. In the existing literature algorithms have only been developed when the trisection surface has boundary ($b > 0$). The following appendix extends these results to the case $b = 0$.

![Figure 30: The standard picture for “classic” relative trisection diagrams.](image)

### A.1 Relative trisections from Kirby diagrams

Let $X$ be a connected 4-manifold with connected boundary. In [5] the authors showed how to draw a trisection diagram from a Kirby diagram of $X$ if a page $P$ for an open book decomposition of $\partial X$ is given in the Kirby diagram. As expected, the proper modification of this result holds if $P$ is the page of a fibration of $\partial X$ over $S^1$; i.e., if $\partial P = \emptyset$. We will state the result.

**Theorem A.1 (Adaptation from Main Theorem of [5]).** Take a handle decomposition of $X$ with one 0-handle, some 1-handles, 2-handles and 3-handles described explicitly in the form of a Kirby diagram. Let $P$ be the page of an open book decomposition or a fibration over $S^1$ of $\partial X$. Suppose that $P$ is explicitly drawn in the Kirby diagram. Then there is an algorithm to draw a trisection diagram for $X$ described as follows:
1. Isotope $P$ in the diagram so that $P$ has a 2-dimensional handle decomposition induced by the 0-handle and some 1-handles and 2-handles of $X$. You might need to add $1/2$-cancelling pairs to do so.

2. If not all the 1-handles of $X$ were used to build $P$, add genus to $P$ by tubing it as in Figure 31. Call this new surface $\tilde{\Sigma}$.

![Add tube to tubing](image)

Figure 31: Tubing the page.

3. Project the attaching regions of the rest of the 2-handles onto $\tilde{\Sigma}$. With the help of Reidemeister I moves ensure that the framing of the handles is given by the surface; and use Reidemeister II moves to ensure that every loop has at least one overcrossing in the link projection.

4. Stabilize $\tilde{\Sigma}$ so that the link above has no crossing following Figure 32. Call this new surface $\Sigma$. Let $\gamma$ be the loops in $\Sigma$ arising from the link projection, let $\alpha$ and $\beta$ be the red and blue curves in $\Sigma$ coming from the stabilizations.

![Stabilization](image)

Figure 32: How to fix a crossing.

5. By construction $|\alpha| = |\beta| \geq |\gamma|$. If the inequality is strict, we do the following: For each component $\gamma_i$, by construction we can pick a loop $\beta_i$ intersecting $\gamma_i$ transversely in one point and disjoint from other $\gamma$ curves. Take a $\beta_j$ not in the selected set $\{\beta_i\}_i$; $\beta_j$ intersects a unique $\gamma$ curve in one point, say $\gamma_{i_0}$. Slide $\beta_j$ over $\beta_{i_0}$ using an arc of $\gamma_{i_0}$; denote the resulting curve by $\gamma_j$.

The tuple $(\Sigma; \alpha, \beta, \gamma)$ is a relative trisection diagram for $X$ inducing the given fibration on the boundary.

Proof. The decomposition of $X$ will be given as follows: Divide the 2-handles of $X$ by $h^2 = h^2_P \cup h^2_r$ where $h^2_P$ are the ones used to build $P$ and $h^2_r$ the rest of the 2-handles. Define $X_1 = B^3[h^1 \cup h^2_P]$. We can see $\Sigma$ as embedded in $\partial X_1$ by stabilizing the standard circular decomposition in $\partial X_1$ as in Subsection 2.1.3 say

$$\partial X_1 = \left( C_\alpha \cup_{\Sigma} C_\beta \right) / \left( \partial_- C_\alpha = \text{id} \partial_- C_\beta \right).$$
Define $X_2 = \eta_X(C_\beta)[h^2]$ and $X_3 = X - \text{int}(X_1 \cup X_2)$. The proof that $X = X_1 \cup X_2 \cup X_3$ is indeed a relative trisection is the same as in Theorem 1 of [5].

**Example A.2.** Figures 33, 34 and 35 describe how to draw a relative trisection diagram for the complement of an unknotted torus in $S^4$.

![Figure 33](image1)

Figure 33: Left: A Kirby diagram for the complement of the unknotted torus in $S^4$. Right: The shaded surface is an embedding of a torus page for the fibration of $T^3$.

![Figure 34](image2)

Figure 34: After sliding, notice that the page has a handle decomposition induced by the 0-handle, the pair of 1-handles and one 2-handle of the 4-manifold.

![Figure 35](image3)

Figure 35: By resolving the crossings as in Step 4, since we have the same number of loops of each color, we obtain a relative trisection diagram for $X$ (left). We get the diagram in the right by a diffeomorphism of the surface.

**Example A.3.** Figures 36 and 37 show how to trisect the thickened 3-torus $T^3 \times [0, 1]$ in such a way that on one side the $S^1$-foliation has fiber $S^1 \times S^1 \times \{pt\}$ and in the other boundary the fiber is $S^1 \times \{pt\} \times S^1$. 

29
Figure 36: A Kirby diagram for \( T^3 \times [0, 1] \) obtained by thickening a Heegaard diagram for the 3-torus. After adding a 1/2-cancelling pair, you can see two embedded tori corresponding to \( S^1 \)-fibers \( T_{12} = S^1 \times S^1 \times \{pt\} \times \{0\} \) and \( T_{13} = S^1 \times \{pt\} \times S^1 \times \{1\} \). Notice that the pages \( T_{12} \) and \( T_{13} \) have handle decompositions induced by the 0-handle, the 1-handles and some 2-handle of the 4-manifold.

Figure 37: After tubing the two tori, we draw a link diagram for the rest of the 2-handles in a genus two surface. By resolving the crossings as in Step 4, we obtain a diagram with fewer \( \gamma \)-curves (green). We perform Step 5 in order to find \( \gamma_3 \) (brown). This final result is a relative trisection diagram for \( T^3 \times [0, 1] \) with \( S^1 \)-fibers on its boundary given by \( T_{12} = S^1 \times S^1 \times \{pt\} \times \{0\} \) and \( T_{13} = S^1 \times \{pt\} \times S^1 \times \{1\} \).
A.2 The monodromy induced on \( \partial X \)

In [4] the authors described an algorithm to compute the monodromy of an open book decomposition induced by a trisection when the diagram has boundary. If the trisection surface is closed, the trisection will induce a fibration over \( S^1 \) and the monodromy can also be computed following a suitable modification of the original algorithm. We now describe the algorithm in general. The key idea is to take properly embedded 1-manifolds in the trisection surface that cut a page into a disk and traverse the boundary of the trisection using the correct handle slides.

**Theorem A.4** (Adaptation from Theorem 5 of [4]). A relative trisection diagram encodes an open book decomposition or a fibration over \( S^1 \) on \( \partial X \) with page given by \( \Sigma_\alpha \), the surface resulting from \( \Sigma \) by compressing along the \( \alpha \) curves, and monodromy \( \mu : \Sigma_\alpha \to \Sigma_\alpha \) determined as follows:

1. Choose an ordered collection of properly embedded arcs or simple closed curves \( a \) on \( \Sigma \), disjoint from \( \alpha \) and such that the corresponding 1-manifolds in \( \Sigma_\alpha \) cut \( \Sigma_\alpha \) into a disk.

2. There exists a collection of properly embedded 1-manifolds \( a_1 \) and simple closed curves \( \beta' \) in \( \Sigma \) such that \( (\alpha, a_1) \) is handle slide equivalent to \( (\alpha, a) \), \( \beta' \) is handle slide equivalent to \( \beta \), and \( a_1 \) and \( \beta' \) are disjoint. We claim that in this step we do not need to slide \( \alpha \) curves over \( \alpha \) curves, only \( a \) 1-manifolds over \( \alpha \) curves and \( \beta \) curves over \( \beta \) curves. Choose such an \( a_1 \) and \( \beta' \).

3. There exists a collection of properly embedded 1-manifolds \( a_2 \) and simple closed curves \( \gamma' \) in \( \Sigma \) such that \( (\beta', a_2) \) is handle slide equivalent to \( (\beta', a_1) \), \( \gamma' \) is handle slide equivalent to \( \gamma \), and \( a_2 \) and \( \gamma' \) are disjoint. Again we claim that we do not need to slide \( \beta' \) curves over \( \beta' \) curves. Choose such an \( a_2 \) and \( \gamma' \).

4. There exists a collection of properly embedded 1-manifolds \( a_3 \) and simple closed curves \( \alpha' \) in \( \Sigma \) such that \( (\gamma', a_3) \) is handle slide equivalent to \( (\gamma', a_2) \), \( \alpha' \) is handle slide equivalent to \( \alpha \), and \( a_3 \) and \( \alpha' \) are disjoint. Again we claim that we do not need to slide \( \gamma' \) curves over \( \gamma' \) curves. Choose such an \( a_3 \) and \( \alpha' \).

5. The pair \( (\alpha', a_3) \) is handle slide equivalent to \( (\alpha, a_*) \) for some collection of 1-manifolds. Choose such an \( a_* \). Note that now \( a \) and \( a_* \) are both disjoint from \( \alpha \) and thus we can compare the corresponding 1-manifolds in \( \Sigma_\alpha \).

6. The monodromy \( \mu \) is the unique map (up to isotopy) such that

\[
\mu(\varphi_\alpha(a)) = \varphi_\alpha(a_*),
\]

respecting the ordering of the 1-manifolds.

**Proof.** The proof is the same as in Theorem 5 on [4]. The only observation is that the proof of Lemma 13 of [4], a key lemma for this result, does not apply when \( \Sigma_\alpha \) is closed. This problem can be solved by considering the annulus \( a \times [-1, 1] \) for any loop in \( P \) instead of the disk in the proof of Lemma 13. The proof then works.

Figures 38, 39 and 40 show how to run the algorithm for the monodromy in the concrete case of the trisection of the complement of the unknotted torus in \( S^4 \).

---

4 We could have both arcs and curves simultaneously.

5 As many disks as boundary components of \( X \).
Figure 38: Left: The outer pair of curves (yellow and pink) correspond to the 1-manifolds $a = a_1$ disjoint from the $\alpha$ (red) and $\beta$ (blue) loops. Right: After switching to the $\beta$, $\gamma$ (green) pair, we need to slide $a_1$ over $\beta$ to get 1-manifolds disjoint from $\gamma$, we call those $a_2$.

Figure 39: After isotopy of $a_2$ and drawing now the pair $(\gamma, \alpha)$, we slide $a_2$ over $\gamma$ to get 1-manifolds disjoint from $\alpha$, we call those $a_3 = a_\ast$.

Figure 40: The orientation preserving monodromy is defined in the torus obtained by compressing along the $\alpha$ loops and its determined by $a \mapsto a_\ast$. Notice that in this case we obtained the identity map, as expected.
References

[1] Akbulut, S. (2016). *4-manifolds* (Vol. 25). Oxford University Press.

[2] Baykur, R. I., & Sunukjian, N. (2013). *Round handles, logarithmic transforms and smooth 4-manifolds*. Journal of Topology, 6(1), 49-63.

[3] Castro, N. A. (2017). *Trisecting smooth 4-dimensional cobordisms*. arXiv preprint arXiv:1703.05846

[4] Castro, N., Gay, D., & Pinzn-Caicedo, J. (2018). *Diagrams for relative trisections*. Pacific Journal of Mathematics, 294(2), 275-305.

[5] Castro, N. A., Gay, D. T., & Pinzn-Caicedo, J. (2018). *Trisections of 4-manifolds with boundary*. Proceedings of the National Academy of Sciences, 115(43), 10861-10868.

[6] Castro, N. A., & Ozbaci, B. (2019). *Trisections of 4-manifolds via Lefschetz fibrations*. Mathematical Research Letters 26, no.2, 383-420.

[7] Fintushel, R., & Stern, R. J. (1998). Knots, links, and 4-manifolds. Inventiones mathematicae, 134(2), 363-400.

[8] Gay, D., & Meier, J. (2018). *Doubly pointed trisection diagrams and surgery on 2-knots*. arXiv preprint arXiv:1806.05351

[9] Gay, D., & Kirby, R. (2016). *Trisecting 4-manifolds*. Geometry & Topology, 20(6), 3097-3132.

[10] Etgü, T., & Ozbagci, B. (2006). *Explicit horizontal open books on some plumbings*. International Journal of Mathematics, 17(09), 1013-1031.

[11] Feller, P., Klug, M., Schirmer, T., & Zemke, D. (2018). *Calculating the homology and intersection form of a 4-manifold from a trisection diagram*. Proceedings of the National Academy of Sciences, 115(43), 10869-10874.

[12] Florens, V., & Moussard, D. (2019). *Torsions and intersection forms of 4-manifolds from trisection diagrams*. arXiv preprint arXiv:1901.04734

[13] Hayashi, C., & Shimokawa, K. (1998). *Heegaard splittings of the trivial knot*. Journal of Knot Theory and Its Ramifications, 7, 1073-1086.

[14] Kim, S., & Miller, M. (2018). *Trisections of surface complements and the Price twist*. arXiv preprint arXiv:1805.00129

[15] Koenig, D. (2017). *Trisections of 3-manifold bundles over S1*. arXiv preprint arXiv:1710.04345

[16] Lambert-Cole, P., & Meier, J. (2018). *Bridge trisections in rational surfaces*. arXiv preprint arXiv:1810.10450

[17] Meier, J. (2018). *Trisections and spun four-manifolds*. Math. Res. Lett, 25(5), 1497-1524.

[18] Meier, J., & Zupan, A. (2017). *Bridge trisections of knotted surfaces in S4*. Transactions of the American Mathematical Society, 369(10), 7343-7386.

[19] Meier, J., & Zupan, A. (2018). *Bridge trisections of knotted surfaces in 4-manifolds*. Proceedings of the National Academy of Sciences, 115(43), 10880-10886.

[20] Meier, J., & Zupan, A. (2017). *Genus-two trisections are standard*. Geometry & Topology, 21(3), 1583-1630.

[21] Kim, S., & Miller, M. (2018). *Trisections of surface complements and the Price twist*. arXiv preprint arXiv:1805.00129

email: jesse.moeller@huskers.unl.edu
email: jose-arandacuevas@uiowa.edu