The hierarchy problem, radion mass, localization of gravity and 4D effective Newtonian potential in string theory on $S^1/Z_2$

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In this paper, we present a systematical study of brane worlds of string theory on $S^1/Z_2$. In particular, starting with the toroidal compactification of the Neveu-Schwarz/Neveu-Schwarz sector in (D+d) dimensions, we first obtain an effective D-dimensional action, and then compactify one of the $(D - 1)$ spatial dimensions by introducing two orbifold branes as its boundaries. We divide the whole set of the gravitational and matter field equations into two groups, one holds outside the two branes, and the other holds on them. By combining the Gauss-Codacci and Lanczos equations, we write down explicitly the general gravitational field equations on each of the two branes, while using distribution theory we express the matter field equations on the branes in terms of the discontinuities of the first derivatives of the matter fields. Afterwards, we address three important issues: (i) the hierarchy problem; (ii) the radion mass; and (iii) the localization of gravity, the 4-dimensional Newtonian effective potential and the Yukawa corrections due to the gravitational high-order Kaluza-Klein (KK) modes. The mechanism of solving the hierarchy problem is essentially the combination of the large extra dimension and warped factor mechanisms together with the tension coupling scenario. With very conservative arguments, we find that the radion mass is of the order of $10^{-2}$ GeV. The gravity is localized on the visible brane, and the spectrum of the gravitational KK modes is discrete and can be of the order of TeV. The corrections to the 4-dimensional Newtonian potential from the higher order of gravitational KK modes are exponentially suppressed and can be safely neglected in current experiments. In an appendix, we also present a systematical and pedagogical study of the Gauss-Codacci equations and Israel’s junction conditions across a (D-1)-dimensional hypersurface, which can be either spacelike or timelike.

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I. INTRODUCTION

Superstring and M-theory all suggest that we may live in a world that has more than three spatial dimensions. Because only three of these are presently observable, one has to explain why the others are hidden from detection. One such explanation is the so-called Kaluza-Klein (KK) compactification, according to which the size of the extra dimensions is very small (often taken to be on the order of the Planck length). As a consequence, modes that have momentum in the directions of the extra dimensions are excited at currently inaccessible energies.

Recently, the braneworld scenarios has dramatically changed this point of view and, in the process, received a great deal of attention. At present, there are a number of proposed models (See, for example, and references therein.) In particular, Arkani-Hamed et al (ADD) pointed out that the extra dimensions need not necessarily be small and may even be on the scale of millimeters. This model assumes that Standard Model fields are confined to a three (spatial) dimensional hypersurface (a 3-brane) living in a larger dimensional bulk while the gravitational field propagates in the bulk. Additional fields may live only on the brane or in the bulk, provided that their current undetectability is consistent with experimental bounds. One of the most attractive features of this model is that it may potentially resolve the long standing hierarchy problem, namely the large difference in magnitudes between the Planck and electroweak scales, $M_{pl}/M_{EW} \simeq 10^{16}$, where $M_{pl}$ denotes the four-dimensional Planck mass with $M_{pl} \sim 10^{16}$ TeV, and $M_{EW}$ the electroweak scale with $M_{EW} \sim$ TeV.

Considering a N-dimensional spacetime and assuming that the extra dimensions are homogeneous and finite, we find

$$S_{g}^{(N)} \sim -M_{N+2} \int dx^{4}dz^{n}\sqrt{-g^{(N)}}R^{(N)}$$

$$= -M_{N+2}V_{n} \int d^{4}x\sqrt{-g^{(4)}}R^{(4)}$$

$$\simeq -M_{pl}^{2} \int d^{4}x\sqrt{-g^{(4)}}R^{(4)},$$

where $V_{n}$ denotes the volume of extra dimensions, $n \equiv N - 4$, and $M$ the N-dimensional fundamental Planck mass, which is related to $M_{pl}$ by

$$M = (M_{pl}^{2}/V_{n})^{1/(2+n)}.$$  

Clearly, for any given extra dimensions, if $V_{n}$ is large enough, $M$ can be as low as the electroweak scale, $M \simeq M_{EW} \simeq$ TeV. Therefore, if we consider $M$ as...
the fundamental scale and $M_{pl}$ the deduced one, we can see that the hierarchy between the two scales is exactly due to the dilution of the spacetime in high dimensions, whereby the hierarchy problem is resolved. Table top experiments show that Newtonian gravity is valid at least down to the size $R \sim 44 \mu m$. From the above we can see that for $n \geq 2$ the N-dimensional Planck mass $M$ can be lowered down to the electroweak scale from the four-dimensional Planck scale.

In a different model, Randall and Sundrum (RS1) showed that if the self-gravity of the brane is included, gravitational effects can be localized near the Planck brane at low energy and the 4D Newtonian gravity is reproduced on this brane. In this model, the extra dimensions are not homogeneous, but warped. One of the most attractive features of the model is that it will soon be fully explored by LHC. In the RS1 scenario, the mechanism to solve the hierarchy problem is completely different. Instead of using large dimensions, RS used the warped factor, for which the mass $m_0$ measured on the invisible (Planck) brane is related to the mass $m$ measured on the visible (TeV) brane by $m = e^{-ky_c}m_0$, where $e^{-ky_c}$ is the warped factor. Clearly, by properly choosing the distance $y_c$ between the two branes, one can lower $m$ to the order of TeV, even $m_0$ is of the order of $M_{pl}$. It should be noted that the five-dimensional Planck mass $M_5$ in the RS1 scenario is still in the same order of $M_{pl}$.

In fact, the 5-dimensional action $S_g^{(5)}$ can be written as

$$S_g^{(5)} \sim -M^3 \int dx^4 d\phi \sqrt{-g^{(5)}}R^{(5)}$$

$$= -M^3 \int_{-\pi}^{\pi} r_c e^{-2kr_c} |\phi| \sqrt{d^4 x \sqrt{-g^{(4)}}} R^{(4)}$$

$$\simeq -M^2_{pl} \int d^4 x \sqrt{-g^{(4)}} R^{(4)}, \quad (1.3)$$

where now we have

$$M^2_{pl} = M^3 k^{-1} (1 - e^{-2ky_c}) \simeq M_5^2, \quad (1.4)$$

for $k \simeq M_5$ and $ky_c \simeq 35$.

Another long-standing problem is the cosmological constant problem: Its theoretical expectation values from quantum field theory exceed its observational limits by 120 orders of magnitude. Even if such high energies are suppressed by supersymmetry, the electroweak corrections are still 56 orders higher. This problem was further sharpened by recent observations of supernova (SN) Ia, which reveal the revolutionary discovery that our universe has lately been in its accelerated expansion phase. Cross checks from the cosmic microwave background radiation and large scale structure all confirm it. In Einstein’s theory of gravity, such an expansion can be achieved by a tiny positive cosmological constant, which is well consistent with all observations carried out so far. Because of this remarkable result, a large number of ambitious projects have been aimed to distinguish the cosmological constant from dynamical dark energy models. Since the problem is intimately related to quantum gravity, its solution is expected to come from quantum gravity, too. At the present, string/M-Theory is our best bet for a consistent quantum theory of gravity, so it is reasonable to ask what string/M-Theory has to say about the cosmological constant.

In the string landscape, it is expected there are many different vacua with different local cosmological constants. Using the anthropic principle, one may select the low energy vacuum in which we can exist. However, many theorists still hope to explain the problem without invoking the existence of ourselves. In addition, to have a late time accelerating universe from string/M-Theory, Townsend and Wohlforth invoked a time-dependent compactification of pure gravity in higher dimensions with hyperbolic internal space to circumvent Gibbons’ non-go theorem. Their exact solution exhibits a short period of acceleration. The solution is the zero-flux limit of spacelike branes. If non-zero flux or forms are turned on, a transient acceleration exists for both compact internal hyperbolic and flat spaces. Other accelerating solutions by compactifying more complicated time-dependent internal spaces can be found in.

Recently, we studied brane cosmology in the framework of both string theory and the Horava-Witten (HW) heterotic M Theory on $S^1/Z_2$. From a pure numerology point of view, we found that the 4D effective cosmological constant can be cast in the form,

$$\rho_A = \frac{\Lambda_4}{8\pi G_4} = 3 \left( \frac{R}{l_{pl}} \right)^{\alpha_R} \left( \frac{M}{M_{pl}} \right)^{\alpha_M} M^4_{pl}, \quad (1.5)$$

where $R$ denotes the typical size of the extra dimensions, $M$ is the energy scale of string or M theory, and $(\alpha_R, \alpha_M) = (10, 16)$ for string theory and $(\alpha_R, \alpha_M) = (12, 18)$ for the HW heterotic M Theory. In both cases, it can be shown that for $R \simeq 10^{-22} m$ and $M_{10} \simeq 1$ TeV, we obtain $\rho_A \sim \rho_{A, ob} \simeq 10^{-47} GeV^4$.

When orbifold branes are concerned, a critical ingredient is the radion stability. Using the mechanism of Goldberger and Wise, we showed that the radion is stable. Such studies were also generalized to the HW heterotic M Theory, and found that, among other things, the radion is stable and has a mass of order of $10^{-2} GeV$. In this paper, we shall give a systematical study of brane worlds of string theory on $S^1/Z_2$. Similar studies in 5-dimensional spacetimes have been carried out in the framework of both string theory and M Theory. However, to have this paper as much independent as possible, it is difficult to avoid repeating some of our previous materials, although we would try our best to keep it to its minimum. The rest of the paper is organized as follows: In Sec. II, we consider the toroidal compactification of the Neveu-Schwarz/Neveu-Schwarz (NS-NS) sector in $(D + d)$ dimensions, and obtain an effective $D$-dimensional action. Then, we compactify one of the
(D−1) spatial dimensions by introducing two orbifold branes as the boundaries along this compactified dimension. In Sec. III, we divide the whole set of the gravitational and matter field equations into two groups, one holds outside the two branes, and the other holds on each of them. Combining the Gauss-Codacci and Lanczos equations, we write down explicitly the general gravitational field equations on the branes, while using distribution theory we are able to express the matter field equations on the branes in terms of the discontinuities of the first derivatives of the matter fields. In Sec. IV, we study the hierarchy problem, while in Sec. V, we consider the radion mass by using the Goldberger-Wise mechanism [20]. In Sec. VI we study the localization of gravity, the 4-dimensional effective potential and high order Yukawa corrections. In Sec. VII, we present our main conclusions with some discussing remarks. We also include an appendix, in which we present a systematical and pedagogical study of the Gauss-Codacci equations and Israel’s junction conditions across a surface, where the metric coefficients are only continuous, i.e., C0, in higher dimensional spacetimes. To keep such a treatment as general as possible, the surface can be either spacelike or timelike.

Before turning to the next section, we would like to note that in 4-dimensional spacetimes there exists Weinberg’s no-go theorem for the adjustment of the cosmological constant [6]. However, in higher dimensional spacetimes, the 4-dimensional vacuum energy on the brane does not necessarily give rise to an effective 4-dimensional cosmological constant. Instead, it may only curve the brane still flat [24], whereby Weinberg’s no-go theorem is evaded. It was exactly in this vein, the cosmological constant problem was studied in the framework of brane worlds in 5-dimensional spacetimes [25] and 6-dimensional supergravity [26]. However, it was soon found that in the 5-dimensional case hidden fine-tunings are required [27]. In the 6-dimensional case such fine-tunings may not be needed, but it is still not clear whether loop corrections can be as small as expected [28].

II. THE MODEL

In this section, we first consider the toroidal compactification of the NS-NS sector in (D+d) dimensions, and obtain an effective D-dimensional action. Then, we compactify one of the (D−1) spatial dimensions by introducing two orbifold branes as the boundaries along this compactified dimension.

A. Toroidal Compactification of the NS-NS sector

Let us consider the NS-NS sector in (D+d) dimensions, ̂M_{D+d} = M_D × T_d, where T_d is a d-dimensional torus. Topologically, it is the Cartesian product of d circles, T_d = S^1 × S^1 × ... × S^1. Then, the action takes the form [29 31].

\[ \hat{S}_{D+d} = -\frac{1}{2\kappa_{D+d}^2} \int d^{D+d}x \sqrt{|g_{D+d}|} e^{-\Phi} \left\{ \hat{R}_{D+d}[\hat{g}] + \hat{g}^{AB} \left( \nabla_A \Phi \right) \left( \nabla_B \Phi \right) - \frac{1}{12} \hat{H}^2 \right\}, \]  

where \( \nabla \) denotes the covariant derivative with respect to ̂g^{AB} with A, B = 0, 1, ..., D+d−1, and \( \Phi \) is the dilaton field. The NS three-form field \( \hat{H}_{ABC} \) is defined as

\[ \hat{H}_{ABC} = 3\partial_{[A} \hat{B}_{BC]} = \partial_A \hat{B}_{BC} + \partial_B \hat{B}_{CA} + \partial_C \hat{B}_{AB}, \]  

where the square brackets imply total antisymmetrization over all indices, and

\[ \hat{B}_{CD} = -\hat{B}_{DC}, \quad \partial_A \hat{B}_{CD} = \frac{\partial \hat{B}_{CD}}{\partial x^A}. \]  

The constant \( \kappa_{D+d}^2 \) denotes the gravitational coupling constant, defined as

\[ \kappa_{D+d}^2 = 8\pi G_{D+d} = \frac{1}{(M_{D+d})^{D+d-2}}, \]  

where \( G_{D+d} \) and \( M_{D+d} \) denote, respectively, the (D+d)-dimensional Newtonian constant and Planck mass.

In this paper we consider the (D+d)-dimensional spacetimes described by the metric,

\[ \hat{s}_{D+d}^2 = \hat{g}_{ab} dx^a dx^b \]  

\[ = \hat{g}_{ab}(x) dx^a dx^b + h_{ij}(x) dz^i dz^j, \]  

where \( \hat{g}_{ab}(x) \) is the metric on \( M_D \), parametrized by the coordinates \( x^a \) with \( a, b, c = 0, 1, ..., D - 1 \), and \( h_{ij}(x) \) is the metric on the compact space \( T_d \) with periodic coordinates \( z^i \), where \( i, j = D, D+1, ..., D + d - 1 \).

We assume that all the matter fields, similar to the metric coefficients, are functions of \( x^a \) only,

\[ \hat{\Phi} = \hat{\Phi}(x^a), \quad \hat{B}_{CD} = \hat{B}_{CD}(x^a). \]  

This implies that the compact space \( T_d \) is Ricci flat,

\[ R_d[h] = 0, \]  

and that

\[ \hat{H}_{ij} = 3\partial_i \hat{B}_{jk} = 0. \]  

For the sake of simplicity, we also assume that the flux \( \hat{B} \) is block diagonal,

\[ \left( \hat{B}_{MN} \right) = \left( \begin{array}{cc} \hat{B}_{ab}(x) & 0 \\ 0 & B_{ij}(x) \end{array} \right). \]  

Then, it can be shown that

\[ \hat{H}_{abc} = \hat{H}_{abc} = 3\partial_a \hat{B}_{bc}, \]  

\[ \hat{H}_{aij} = \hat{\nabla}_a B_{ij}, \quad \hat{H}_{abi} = 0. \]  

(2.10)
where $\tilde{\nabla}_a$ denotes the covariant derivative with respect to $\tilde{g}^{ab}$.

On the other hand, we also have
\[
\tilde{R}_{D+d}[\tilde{g}] = R_D[\tilde{g}] + 4 \left( \tilde{\nabla}_a h^{ij} \left( \tilde{\nabla}^a h_{ij} \right) \right) \\
+ \tilde{\nabla}_a \left( \ln \sqrt{|h|} \right) \tilde{\nabla}^a \left( \ln \sqrt{|h|} \right) \\
- \frac{2}{|h|} \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \left( \sqrt{|h|} \right). \tag{2.11}
\]

Inserting Eq. (2.11) into Eq. (2.1) and then integrating the internal part, we obtain the effective $D-$dimensional action,
\[
S_D = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|g_D|} e^{-\tilde{\phi}} \left\{ \tilde{R}_D[\tilde{g}] + \left( \tilde{\nabla}_a \tilde{\phi} \right) \left( \tilde{\nabla}^a \tilde{\phi} \right) \right. \\
+ \frac{1}{4} \left( \tilde{\nabla}_a \hat{h}^{ij} \right) \left( \tilde{\nabla}^a \hat{h}_{ij} \right) - \frac{1}{12} \tilde{H}_{abc} \tilde{H}^{abc} \\
- \frac{1}{4} \hat{h}^{ik} \hat{h}^{jl} \left( \tilde{\nabla}_a B_{ij} \left( \tilde{\nabla}^a B_{kl} \right) \right), \tag{2.12}
\]
where
\[
\tilde{\phi} = \hat{\Phi} - \frac{1}{2} \ln |h|, \tag{2.13}
\]
\[
\kappa^2_D = \frac{\kappa^2_{D+d}}{V_0}, \tag{2.14}
\]
with the $d-$dimensional internal volume given by
\[
V_d(x^a) = \int d^d x \sqrt{|h|} = |h|^{1/2} V_0. \tag{2.15}
\]

Action (2.12) is usually referred to as that written in the string frame.

To go to the Einstein frame, we make the following conformal transformations,
\[
g_{ab} = \Omega^2 h_{ab}, \\
\Omega^2 = \exp \left( -\frac{2}{D-2} \tilde{\phi} \right), \\
\tilde{\phi} = \sqrt{\frac{2}{D-2}} \hat{\tilde{\phi}}, \tag{2.16}
\]

Then, the action (2.12) takes the form
\[
S^{(E)}_D = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|g_D|} \left\{ R_D[g] - \frac{1}{2} (\nabla \tilde{\phi})^2 \\
+ \frac{1}{4} \left( \nabla_a \hat{h}^{ij} \right) \left( \nabla^a \hat{h}_{ij} \right) \\
- \frac{1}{12} e^{-\sqrt{\frac{2}{D-2}} \hat{\phi}} H_{abc} H^{abc} \\
- \frac{1}{4} \hat{h}^{ik} \hat{h}^{jl} \left( \nabla_a B_{ij} \left( \nabla^a B_{kl} \right) \right) \right\}, \tag{2.17}
\]
where $\nabla_a$ denotes the covariant derivative with respect to $g_{ab}$. It should be noted that, since the definition of the three-form $\hat{H}_{ABC}$ given by (2.22) is independent of the metric, it is conformally invariant. In particular, we have
\[
H_{abc} = \hat{H}_{abc}, \quad B_{ab} = \hat{B}_{ab}. \tag{2.18}
\]

However, we do have
\[
H^{abc} = g^{ad} g^{be} g^{cf} H_{def} = \Omega^{-6} \tilde{H}^{abc}, \\
H_{abc} \tilde{H}^{abc} = \Omega^{-6} \tilde{H}_{abc} \tilde{H}^{abc}. \tag{2.19}
\]

Considering the addition of a potential term $\Phi_0$, in the string frame we have
\[
S^{m}_{D+d} = -\int d^{D+d} x \sqrt{|g_{D+d}|} V^s_{D+d}, \tag{2.20}
\]
Then, after the dimensional reduction we find
\[
S_{D,m} = -V_0 \int d^D x \sqrt{|g_D|} |h|^{1/2} V^s_{D+d}, \tag{2.21}
\]
where
\[
\tilde{g}_D = \exp \left( \sqrt{\frac{2D^2}{D-2}} \phi \right) g_D. \tag{2.22}
\]

Changed to the Einstein frame, the action (2.21) becomes
\[
S^{(E)}_{D,m} = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|g_D|} V^s_D, \tag{2.23}
\]
where
\[
V_D \equiv 2\kappa^2_D V_0 \left( \frac{D}{\sqrt{2(D-2)}} \phi \right) |h|^{1/2}. \tag{2.24}
\]
If we further assume that
\[
h_{ij} = -\exp \left( -\sqrt{\frac{2}{D}} \psi \right) \delta_{ij}, \\
h^{ij} = -\exp \left( -\sqrt{\frac{2}{D}} \psi \right) \delta^{ij}, \tag{2.25}
\]
we find that
\[
S^{(E)}_D + S^{(E)}_{D,m} = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|g_D|} \left\{ R_D[g] \\
- \frac{1}{2} \left[ (\nabla \phi)^2 + (\nabla \psi)^2 - 2V_D \right] \\
- \frac{1}{4} e^{-\sqrt{\frac{2}{D}} \phi} \psi \left( \nabla_a B_{ij} \right) \left( \nabla^a B^{ij} \right) \\
- \frac{1}{12} e^{-\sqrt{\frac{2}{D}} \phi} \hat{H}_{abc} \tilde{H}^{abc} \right\}, \tag{2.26}
\]
where $B^{ij} \equiv \delta^{ik} \delta^{jl} B_{kl}$, and the effective $D-$dimensional potential (2.22) now is given by
\[
V_D \equiv 2\kappa^2_D V_0 \left( \frac{D}{\sqrt{2(D-2)}} \phi + \sqrt{\frac{2}{D}} \psi \right). \tag{2.27}
\]
B. $S^1/Z_2$ Compactification of the D-Dimensional Sector

We shall compactify one of the $(D - 1)$ spatial dimensions by putting two orbifold branes as its boundaries. The brane actions are taken as,

$$S^{(E, I)}_{D-1,m} = - \int_{M^{(I)}_{D-1}} \sqrt{|g^{(I)}_{D-1}|} \left( \epsilon_I V^{(I)}_{D-1} (\phi, \psi) + g^{(I)}_{\omega} \right) \times d^{D-1} \xi_I + \int_{M^{(I)}_{D-1}} d^{D-1} \xi_I \sqrt{|g^{(I)}_{D-1}|} \times \mathcal{L}^{(I)}_{D-1,m} (\phi, \psi, B, \chi),$$

(2.28)

where $I, J = 1, 2$, $V^{(I)}_{D-1} (\phi, \psi)$ denotes the potential of the scalar fields $\phi$ and $\psi$ on the branes, and $\xi^\mu_I$'s are the intrinsic coordinates of the branes with $\mu, \nu = 0, 1, 2, ..., D - 2$, and $\epsilon_1 = -\epsilon_2 = 1$. $\chi$ denotes collectively the matter fields, and $g^{(I)}_{\omega}$ denotes the tension of the $I$-th brane. As to be shown below, it is directly related to the $(D - 1)$-dimensional Newtonian constant $G^{(I)}_{D-1}$ [32]. The two branes are localized on the surfaces,

$$\Phi_I (x^a) = 0,$$

(2.29)

or equivalently

$$x^a = x^a \left( \xi^{\mu}_I \right).$$

(2.30)

$g^{(I)}_{\omega}$ denotes the determinant of the reduced metric $g^{(I)}_{\mu \nu}$ of the $I$-th brane, defined as

$$g^{(I)}_{\mu \nu} = g_{a b} (\xi^{\mu}_I) (\xi^{\nu}_I) \big|_{M^{(I)}_{D-1}},$$

(2.31)

where

$$\epsilon^{(I)}_{\mu} \equiv \frac{\partial x^a}{\partial \xi^{\mu}_I}.$$  

(2.32)

Then, the total action is given by,

$$S^{(E)}_{\text{total}} = S^{(E)}_{D} + S^{(E)}_{D, m} + \sum_{I=1}^{2} S^{(E, I)}_{D-1,m}.$$  

(2.33)

III. FIELD EQUATIONS BOTH OUTSIDE AND ON THE ORBIFOLD BRANES

Variation of the total action (2.33) with respect to the metric $g_{a b}$ yields the field equations,

$$G^{(D)}_{a b} = \kappa^2_D T^{(D)}_{a b} + \kappa^2_D \sum_{I=1}^{2} T^{(I)}_{a b} \epsilon^{(I)}_{\mu} \epsilon^{(I)}_{\nu} \times \sqrt{\left| \frac{g^{(I)}_{D-1}}{g_D} \right|} \delta (\Phi_I),$$

(3.1)

where $\delta(x)$ denotes the Dirac delta function, normalized in the sense of [33], and the energy-momentum tensors $T^{(D)}_{a b}$ and $T^{(I)}_{a b}$ are defined as,

$$\kappa^2_D T^{(D)}_{a b} = \frac{1}{2} \left[ \left( \nabla_a \phi \right) \left( \nabla_b \phi \right) + \left( \nabla_a \psi \right) \left( \nabla_b \psi \right) + \frac{1}{4} \nabla^D_\omega \left( \nabla_a \psi \right) \left( \nabla_b \psi \right) \nabla^D_\omega \left( \nabla_a \phi \right) \left( \nabla_b \phi \right) \right] - \frac{1}{4} \nabla^D_\omega \left( \nabla_\omega \psi \right)^2 - 2 \kappa^2_D \nabla_\omega \nabla^D_\omega \left( \nabla_\omega \phi \right)^2,$$

(3.2)

$$\kappa^2_D T^{(I)}_{a b} = \frac{1}{2} \kappa^2_D \sum_{I=1}^{2} \left( \nabla^D_\omega \left( \nabla_\omega \phi \right) \nabla^D_\omega \left( \nabla_\omega \psi \right) \right) - \kappa^2_D \left( \nabla^D_\omega \left( \nabla_\omega \phi \right) \nabla^D_\omega \left( \nabla_\omega \psi \right) \right),$$

(3.3)

where

$$\tau^{(I)}_\omega \equiv \frac{\epsilon^{(I)}_{\mu}}{\sqrt{g^{(I)}_{D-1} \delta (\Phi_I)}},$$

(3.4)

Variation of the total action (2.33), respectively, with respect to $\phi$, $\psi$, $B_{i j}$, and $B_{a b}$, yields the following equations of the matter fields,

$$\nabla^a \delta (\phi) = \frac{1}{12} \sqrt{\frac{8}{D-2}} \left( \nabla^a \nabla^a \phi \right) \delta (\Phi_I),$$

(3.5)

$$\nabla^a \delta (\psi) = \frac{1}{2} \sqrt{\frac{1}{2d}} \left( \nabla^a \nabla^a \psi \right) \delta (\Phi_I),$$

(3.6)

$$\nabla^a \delta (B_{i j}) = \frac{\sqrt{8}}{\sqrt{d}} \left( \nabla^a \nabla^a B_{i j} \right) \delta (\Phi_I),$$

(3.7)

$$\nabla^c H_{c a b} = \sqrt{\frac{8}{D-2}} H_{c a b} \nabla^c \phi.$$
express the delta function parts in the left-hand sides of Eqs. (3.11) and (3.20)- (3.28) in terms of the discontinuities of the first derivatives of the metric coefficients and matter fields, and then equal the corresponding delta function parts in the right-hand sides of these equations, as shown systematically in [34]. (2) The second approach is to use the Gauss-Codacci and Lanczos equations to write down the \((D-1)\)-dimensional gravitational field equations on the branes [32]. It should be noted that these two approaches are equivalent and complementary one to the other. In this paper, we shall follow the second approach to write down the gravitational field equations on the two branes, and the first approach to write the matter field equations on the two branes.

1. Gravitational Field Equations on the Two Branes

For timelike branes, their normal vectors are spacelike. Then, setting \(\epsilon(n) = -1\) in (2.11), we obtain,

\[
G_{\mu\nu}^{(D-1)} = G_{\mu\nu}^{(D)} + E_{\mu\nu}^{(D)} + \mathcal{F}^{(D-1)}_{\mu\nu},
\]

with

\[
G_{\mu\nu}^{(D)} \equiv \frac{D - 3}{(D - 2)} \left\{ C^{(D)}_{ab} c^a_{(\mu)} c^b_{(\nu)} - \frac{1}{D - 1} G^{(D)} g_{\mu\nu} \right\},
\]

\[
E_{\mu\nu}^{(D)} \equiv C_{abcd} n^a c^b_{(\mu)} n^c d_{(\nu)},
\]

\[
\mathcal{F}^{(D-1)}_{\mu\nu} \equiv K_{\mu\lambda} K_{\nu}^{\lambda} - K K_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( K_{\lambda\beta} K^{\alpha\beta} - K^2 \right),
\]

where \(n^a\) denotes the normal vector to the brane, \(G^{(D)} = g^{ab} G_{ab}^{(D)}\), and \(C_{abcd}\) the Weyl tensor. The extrinsic curvature \(K_{\mu\nu}\) is defined as

\[
K_{\mu\nu} \equiv \epsilon^a_{(\mu)} c^b_{(\nu)} \nabla_a n_b. \tag{3.17}
\]

A crucial step of this approach is the Lanczos equations [30],

\[
\left[ K^{(I)}_{\mu\nu} \right] - g^{(I)}_{\mu\nu} \left[ K^{(I)} \right] = -\kappa_D^2 T^{(I)}_{\mu\nu}, \tag{3.18}
\]

where

\[
K^{(I)}_{\mu\nu} \equiv \lim_{\phi_i \to 0^+} K^{(I)}_{\mu\nu} + \lim_{\phi_i \to 0^-} K^{(I)}_{\mu\nu},
\]

\[
\left[ K^{(I)} \right] - g^{(I)}_{\mu\nu} \left[ K^{(I)}_{\mu\nu} \right] = 0. \tag{3.19}
\]

Assuming that the branes have \(Z_2\) symmetry, we can express the intrinsic curvatures \(K^{(I)}_{\mu\nu}\) in terms of the effective energy-momentum tensor \(T^{(I)}_{\mu\nu}\) through the Lanczos equations (3.18). Setting

\[
S^{(I)}_{\mu\nu} = \tau^{(I)}_{\mu\nu} + g^{(I)}_{\mu\nu} j^{(I)}_{\mu\nu}, \tag{3.20}
\]
where $g_{\kappa}^{(l)}$ is a constant, which will be uniquely determined by the $(D + d)$- and $(D - 1)$-dimensional gravitational coupling constants $\kappa_{D+d}$ and $\kappa_{D-1}$ via Eqs. (2.14) and (3.24), we find that
\[
\mathcal{T}_{\mu\nu}^{(l)} = (\tau_{\mu\nu}^{(l)} + (g_{\kappa}^{(l)} + \tau_{(\phi,\psi)}^{(l)}) g_{\mu\nu}^{(l)}).
\]
(3.21)

Then, $G_{\mu\nu}^{(D-1)}$ given by Eq. (3.15) can be cast in the form [cf. Eq. (3.10)],
\[
G_{\mu\nu}^{(D-1)} = G_{\mu\nu}^{(D)} + \mathcal{E}_{\mu\nu}^{(D-1)} + \kappa_{D}^{2} \pi_{\mu\nu} + \kappa_{D-1}^{2} \mathcal{E}_{\mu\nu} + \Lambda_{D-1} g_{\mu\nu},
\]
(3.22)
where
\[
\pi_{\mu\nu} = \frac{1}{4} \left[ \tau_{\mu\nu} \tau_{\gamma}^{\lambda} - \frac{1}{D-2} \tau_{\mu\nu} \right.
\]
\[
- \frac{1}{2} g_{\mu\nu} \left( \tau_{\alpha\beta} \tau_{\alpha\beta} - \frac{1}{D-2} \tau_{\mu\nu} \right),
\]
(3.23)
and
\[
\mathcal{E}_{\mu\nu}^{(D-1)} = \frac{\kappa_{D}^{2} (D - 3)}{4(D - 2)} \tau_{(\phi,\psi)}
\]
\[
\times \left[ \tau_{\mu\nu} + (g_{\kappa} + \frac{1}{2} \tau_{(\phi,\psi)}) g_{\mu\nu} \right],
\]
(3.24)

For a perfect fluid,
\[
\tau_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - pg_{\mu\nu},
\]
(3.25)

where $u_{\mu}$ is the four-velocity of the fluid, we find that
\[
\pi_{\mu\nu} = \frac{D-3}{4(D-2)} \rho
\]
\[
\times \left[ (\rho + p) u_{\mu} u_{\nu} - \left( p + \frac{1}{2} \rho \right) g_{\mu\nu} \right].
\]
(3.26)

Note that in writing Eqs. (3.22) - (3.26), without causing any confusion, we had dropped the super indices ($\mu$).

It should be noted that in writing Eqs. (3.22) - (3.26), we implicitly assumed that only the brane tension has contribution to the (D-1)-dimensional Newtonian constant. However, it was argued that when the scalar field does not vanish, it also contributes to it [57]. While this seems reasonable, considering the fact that the tension $g_{\kappa}$ has the same contribution to $G_{D-1}$, as one can see from Eqs. (3.21), there are several disadvantages for such an inclusion: (i) The resulted Newtonian constant usually depends not only on time but also on space, $G_{D-1} = G_{D-1}(\phi(t,x^{i}))$, which is highly constrained experimentally [58]. (ii) It is model-dependent. Different potentials of the scalar field on the brane will give different $G_{D-1}$. (iii) It is not unique, even after the potential is fixed. In fact, one can always redefine the energy-momentum tensor $\tau_{\mu\nu}$ so that $\tau_{\mu\nu}^{(l)} = \tau_{\mu\nu}^{(l)} + \lambda^{(l)} g_{\mu\nu}^{(l)}$, where the $\lambda^{(l)}$ term in Eq. (3.21) takes the same form as $g_{\kappa}$ and $\tau_{(\phi,\psi)}$ do. Then, since both $\lambda^{(l)}$ and $\tau_{(\phi,\psi)}$ are due to matter fields on the branes, there is no reason to assume that $\lambda^{(l)}$ has no contribution to $G_{D-1}$ but $\tau_{(\phi,\psi)}$ does. Therefore, in this paper, we shall take the point of view of [59], and assume that only brane tension couples with $G_{D-1}$. With such an assumption, it can be seen that $G_{D-1}$ is uniquely defined once the brane tension is specified.

2. Matter Field Equations on the Two Branes

On the other hand, the I-th brane, localized on the surface $\Phi_{I}(x) = 0$, divides the spacetime into two regions, one with $\Phi_{I}(x) > 0$ and the other with $\Phi_{I}(x) < 0$ [cf. Fig. 1]. Since the field equations are the second-order differential equations, the matter fields have to be at least continuous across this surface, although in general their first-order directives are not. Introducing the Heaviside function, defined as
\[
H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}
\]
(3.27)
in the neighborhood of $\Phi_{I}(x) = 0$ we can write the matter fields in the form,
\[
F(x) = F^{+}(x) H(\Phi_{I}) + F^{-}(x) [1 - H(\Phi_{I})],
\]
(3.28)
where $F \equiv \{ \phi, \psi, B \}$, and $F^{+}$ ($F^{-}$) is defined in the region $\Phi_{I} > 0$ ($\Phi_{I} < 0$). Then, we find that
\[
F_{\alpha}(x) = F_{\alpha}^{+}(x) H(\Phi_{I}) + F_{\alpha}^{-}(x) [1 - H(\Phi_{I})]
\]
\[
+ [F_{\alpha}]^{-} \frac{\partial \Phi_{I}(x)}{\partial x^{\mu}} \delta (\Phi_{I}),
\]
(3.29)
where $[F_{\alpha}]^{-}$ is defined as that in Eq. (3.19). Projecting $F_{\alpha}$ into $n^{\alpha}$ and $e_{(\mu)}^{a}$ directions, we find
\[
F_{\alpha} = F_{\alpha}^{+} e_{(\mu)}^{a} - F_{n} n_{a},
\]
(3.30)
where
\[
F_{n} \equiv n^{a} F_{a}, \quad F_{\mu} \equiv e_{(\mu)}^{a} F_{a}.
\]
(3.31)

Then, we have
\[
[F_{a}]^{-} n^{a} = [F_{n}]^{-},
\]
\[
[F_{a}]^{-} e_{(\mu)}^{a} = 0.
\]
(3.32)

Inserting Eqs. (3.24) - (3.26) into Eq. (3.29), we find
\[
F_{\alpha\beta}(x) = F_{\alpha\beta}^{+}(x) H(\Phi_{I}) + F_{\alpha\beta}^{-}(x) [1 - H(\Phi_{I})]
\]
\[
- [F_{n}]^{-} n_{a} n_{b} N_{I} \delta (\Phi_{I}),
\]
(3.33)
where $N_I \equiv \sqrt{\dot{\Phi}_I \cdot \dot{\Phi}_I}$, and
\begin{equation}
\frac{1}{N_I} \frac{\partial \Phi_I(x)}{\partial x^a} . \tag{3.34}
\end{equation}

Substituting Eq. (3.33) into Eqs. (3.35)-(3.38), we find that
\begin{align*}
\left[ \phi_n^{(I)} \right] &= -\Xi^{(I)} \left( 2\kappa_5^2 \epsilon_I \frac{\partial V^{(I)}_{D-1}}{\partial \phi} + \sigma_\phi^{(I)} \right), \\
\left[ \psi_n^{(I)} \right] &= -\Xi^{(I)} \left( 2\kappa_5^2 \epsilon_I \frac{\partial V^{(I)}_{D-1}}{\partial \psi} + \sigma_\psi^{(I)} \right), \\
\left[ B_{ij,n}^{(I)} \right] &= -\Xi^{(I)} \sigma_{ij}^{(I)}, \\
\left[ H_{nab}^{(I)} \right] &= -\Xi^{(I)} \sigma_{nab}^{(I)},
\end{align*}

where
\begin{equation}
H_{nab} \equiv H_{cab,n} \equiv \frac{1}{N_I} \left[ \frac{g^{(I)}_{D-1}}{g_D} \right] . \tag{3.39}
\end{equation}

IV. GRAVITATIONAL COUPLING IN 4-DIMENSIONAL EFFECTIVE THEORY AND THE HIERARCHY PROBLEM

One of the main motivations of the brane worlds is to resolve the long standing hierarchy problem, namely the large difference in magnitudes between the Planck and electroweak scales \[^{[12]}\]. In this section, we are going to show explicitly how the problem is solved in our current setup. We first note that in deriving the relation between the two scales $M_D$ and $M_{pl}$, given by Eqs. (1.2) and (1.3), it was implicitly assumed that the 4-dimensional effective Einstein-Hilbert action $S_g^{eff}$ couples with matter directly in the form,
\begin{equation}
S_g^{eff} + S_m = \int \sqrt{-g} d^4 \left( -\frac{1}{2\kappa_4^2} R + L_m \right) , \tag{4.1}
\end{equation}

from which one obtains the Einstein field equations, $G_{\mu\nu} = \kappa_4^2 \tau_{\mu\nu}$. In the weak field limit, one arrives at $\kappa_4^2 = 8\pi G/c^4$ \[^{[3]}\]. However, in the brane-world scenarios, the coupling between the effective Einstein-Hilbert action and matter is much more complicated than that given by Eq. (4.1). In particular, the gravitational field equations on the branes are given by Eqs. (3.22)-(3.24), which are a second-order polynomial in terms of the energy-momentum tensor $\tau_{\mu\nu}$ of the brane. In the weak-field regime, the quadratic terms are negligible, and the term linear to $\tau_{\mu\nu}$ dominates. Then, under the weak-field limit, one can show that $\kappa_4^2$ defined by Eq. (3.24) is related to the Newtonian constant exactly by $\kappa_4^2 = 8\pi G/c^4$, from which we find that
\begin{equation}
g_e = \frac{6\kappa_4^2}{\kappa_5^4} . \tag{4.2}
\end{equation}

Note that this result is quite general, and applicable to a large class of brane-world scenarios \[^{[3]}\]. In the present case, we have $\kappa_4^2 = M_5^3 = 1/(M_{pl} R^5)$, where $R$ is the typical size of the extra dimensions \[^{[22]}\]. Then, one finds that $g_e \simeq 10^{-47} \text{GeV}^4$, that is, to solve the hierarchy problem in the framework of string theory on $S^1/Z_2$, the tension of the brane has to be in the same order of the observational cosmological constant $\rho_{obs}^\Lambda$.

V. RADION MASS

In \[^{[18]}\], we studied the radion stability using the Goldberger-Wise mechanism \[^{[20]}\], and found that the radion is stable. To show this claim, we considered the 5-dimensional static metric with a 4-dimensional Poincaré symmetry,
\begin{align*}
&ds_5^2 = e^{2\sigma(y)} \left( \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 \right) , \\
&\sigma(y) = \frac{1}{9} \ln \left( \frac{|y| + y_0}{L} \right) , \\
&\phi(y) = -\sqrt{\frac{25}{54}} \ln \left( \frac{|y| + y_0}{L} \right) + \phi_0 ,
\end{align*}

where $y = x/D$. This completes our general description for $(D + d)$-dimensional spacetimes of string theory with two orbifold branes. Setting $D = d = 5$, we shall obtain the results presented in \[^{[18]}\], from now on we shall restrict ourselves to this case.
The gravitational and matter field equations both outside and on the branes, for any given potentials of the branes for $\tau^{(l)}_{\mu\nu} = 0$. For the detail, we refer readers to [18].

To study the radion stability and mass, it is found to be very large and $mY_0 \gg 1$ [20], we found

$$V_\Phi (Y_c) = \left( \frac{10Y_0}{9L} \right)^{2/5} \frac{M}{\sinh (z_c - z_0)} \left\{ -2v_1 v_2 + (v_1^2 + v_2^2) \cosh (z_c - z_0) \right\},$$

from which we find that

$$\frac{\partial V_\Phi (Y_c)}{\partial Y_c} = \left( \frac{10Y_0}{9L} \right)^{2/5} \frac{2v_1 v_2 M}{\sinh^2 (z_c - z_0)} \left\{ \cosh (z_c - z_0) - \frac{v_1^2 + v_2^2}{2v_1 v_2} \right\},$$

where $z_c - z_0 = MY_c$. Figs. 3 shows the potential for $(z_0, v_1, v_2) = (10, 1.0, 0.1)$.

Then, it can be shown that the above solution satisfies the gravitational and matter field equations both outside and on the branes, for any given potentials of the branes for $\tau^{(l)}_{\mu\nu} = 0$. For the detail, we refer readers to [18].

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from which we find that

$$\frac{\partial V_\Phi (Y_c)}{\partial Y_c} = \left( \frac{10Y_0}{9L} \right)^{2/5} \frac{2v_1 v_2 M}{\sinh^2 (z_c - z_0)} \left\{ \cosh (z_c - z_0) - \frac{v_1^2 + v_2^2}{2v_1 v_2} \right\},$$

where $z_c - z_0 = MY_c$. Figs. 3 shows the potential for $(z_0, v_1, v_2) = (10, 1.0, 0.1)$. Clearly, $V_\Phi (Y_c)$ has a minimum at

$$Y_{c \text{min}} = \frac{1}{M} \cosh^{-1} \left( \frac{v_1^2 + v_2^2}{2v_1 v_2} \right),$$

for which we have

$$\frac{\partial^2 V_\Phi (Y_c)}{\partial Y_c^2} \bigg|_{Y_c = Y_{c \text{min}}} = \left( \frac{10Y_0}{9L} \right)^{2/5} \frac{4v_1 v_2 M^3}{|v_1^2 - v_2^2|},$$

As shown in [20, 23], the radion field $\varphi$ is related to the proper distance $Y_c$ between the two branes by

$$\varphi (Y_c) = \sqrt{12f (Y_c)},$$

where

$$f \equiv \frac{1}{\kappa_5^3} \int_0^{Y_c} e^{-2A (Y)} dY = \frac{5L}{6\kappa_5} \frac{10}{9} \left( \frac{Y_c}{L} \right)^{1/5} \times \left\{ \frac{Y_c + Y_0}{L} \right\}^{6/5} - \left( \frac{Y_0}{L} \right)^{6/5}.$$
Then, we find that

$$m_{\phi}^2 = \frac{1}{2} \frac{\partial^2 V_\phi (Y_c)}{\partial \phi^2} \bigg|_{Y_c = Y_{c, \min}} = \left( \frac{10Y_0}{9L} \right)^{1/5} \frac{2M^5}{3M_5^3} \times \frac{h_i \tilde{v}_1 \tilde{v}_2}{v_i^2 - v_2^2},$$

(5.15)

where $v_i = M^{3/2} \tilde{v}_i$. Since $v_i$ has the dimension $[m]^{3/2}$, we can see that $\tilde{v}_i$ is dimensionless. In addition, $M$ and $v_i$ are all 5-dimensional quantities, we expect that $M \sim M_5$ and $\tilde{v}_i \sim O(1)$. Without introducing new hierarchy, we also expect that $(Y_0/L)^{1/5} \sim O(1)$ and $Y_c/Y_0 \sim O(1)$. Then, from Eq. (5.15) we find

$$m_{\phi} \simeq M_5 = \left( \frac{M_{10}}{M_{pl}} \right)^{8/3} \left( \frac{R}{l_{pl}} \right)^{5/3} M_{pl}. \quad (5.16)$$

For $M_{10} \sim TeV$ and $R \sim 10^{-22} m$, we find that $m_{\phi} \simeq 10^{-2} GeV$, which is much larger than the experimental limit $m_{\phi} > 10^{-3} eV$ [2].

VI. LOCALIZATION OF GRAVITY AND 4D EFFECTIVE NEWTONIAN POTENTIAL

To study the localization of gravity and the four-dimensional effective gravitational potential, in this section let us consider small fluctuations $h_{ab}$ of the 5-dimensional static metric with a 4-dimensional Poincaré symmetry, given by Eqs. (5.1) and (5.2) in its conformally flat form.

A. Tensor Perturbations and the KK Towers

Since such tensor perturbations are not coupled with scalar ones [10], without loss of generality, we can set the perturbations of the scalar fields $\phi$ and $\psi$ to zero, i.e., $\delta \phi = 0 = \delta \psi$. We shall choose the gauge

$$h_{ab} = 0, \quad h_\lambda^\lambda = 0 = \partial^i h_\mu^\lambda. \quad (6.1)$$

Then, it can be shown that [41]

$$\delta G_{ab}^{(5)} = -\frac{1}{2} \Box_5 h_{ab} - \frac{3}{2} \left( (\partial_c \sigma) (\partial^c h_{ab}) - 2 [\Box_5 \sigma + (\partial_i \sigma) (\partial^i h_{ab})] \right),$$

$$\kappa_5^2 \delta T_{ab}^{(5)} = \frac{1}{4} \left( \partial^2 + \psi^2 + 2e^{2\sigma} v_5 \right) h_{ab},$$

$$\delta T_{\mu\nu}^{(4)} = \left( \tilde{\tau}^{(4)} \right) e^{2\sigma(y_i)} h_{\mu\nu}(x, y_i), \quad (6.2)$$

where $\Box_5 \equiv \eta^{ab} \partial_a \partial_b$ and $(\partial_i \sigma) (\partial^i h_{ab}) \equiv \eta^{ab} (\partial_i \sigma) (\partial^i h_{ab})$, with $\eta^{ab}$ being the five-dimensional Minkowski metric. Substituting the above expressions into the Einstein field equations [31] with $D = 5$, and noticing that

$$\left| \frac{g^{(I)}}{g_b} \right|^{1/2} = e^{-\sigma(y_i)}, \quad (6.3)$$

we find that in the present case there is only one independent equation, given by

$$\Box_5 h_{\mu\nu} + 3 (\partial_\sigma)(\partial^c h_{\mu\nu}) = 0, \quad (6.4)$$

which can be further cast in the form,

$$\Box_5 \tilde{h}_{\mu\nu} + 3 \left( 2\sigma'' + 3 \frac{\sigma'}{\sigma} \right) \tilde{h}_{\mu\nu} = 0, \quad (6.5)$$

where $h_{\mu\nu} \equiv e^{-3\sigma/2} \tilde{h}_{\mu\nu}$. Setting

$$\tilde{h}_{\mu\nu}(x, y) = \tilde{h}_{\mu\nu}(x) \psi(y),$$

$$\Box_5 \tilde{h}_{\mu\nu} = 3 (\partial_\sigma)(\partial^c \tilde{h}_{\mu\nu}) = 0,$$

$$\Box_5 \tilde{h}_{\mu\nu}(x) = -m^2 \tilde{h}_{\mu\nu}(x), \quad (6.6)$$

we find that Eq. (6.3) takes the form of the schrödinger equation,

$$(-\nabla_y^2 + V) \psi = m^2 \psi, \quad (6.7)$$

where

$$V = \frac{3}{2} \left( \sigma'' + \frac{3}{2} \sigma'^2 \right) = -\frac{5}{36 (|y_c| + y_0)} + \frac{\delta(y)}{3y_0} - \frac{\delta(y - y_c)}{3(y_c + y_0)}, \quad (6.8)$$

From the above expression we can see clearly that the potential has a delta-function well at $y = y_c$, which is responsible for the localization of the graviton on this brane. In contrast, the potential has a delta-function barrier at $y = 0$, which makes the gravity delocalized on
the $y = 0$ brane. Fig. 4 shows the potential schematically.

Introducing the operators,

$$Q \equiv \nabla_y - \frac{3}{2}s', \quad Q^\dagger \equiv -\nabla_y - \frac{3}{2}s', \quad (6.9)$$

Eq. (6.17) can be written in the form of a supersymmetric quantum mechanics problem,

$$Q^\dagger \cdot Q \psi = m^2 \psi. \quad (6.10)$$

It should be noted that Eq. (6.10) itself does not guarantee that the operator $Q^\dagger \cdot Q$ is Hermitian, because now it is defined only on a finite interval, $y \in [0, y_c]$. To ensure its Hermiticity, in addition to writing the differential equation in the Shr"odinger form, one also needs to show that it has Hermitian boundary conditions, which can be formulated as [12]

$$\psi'_n(0) \psi'_m(0) - \psi_n(0) \psi'_m(0) = \psi'_n (y_c) \psi_m (y_c), \quad (6.11)$$

for any two solutions of Eq. (6.10). To show that in the present case this condition is indeed satisfied, let us consider the boundary conditions at $y = 0$ and $y = y_c$. Integration of Eq. (6.7) in the neighbourhood of $y = 0$ and $y = y_c$ yields, respectively, the conditions,

$$\lim_{y \to y_c^-} \psi'(y) = \frac{1}{6(y_c + y_0)} \lim_{y \to y_c^-} \psi(y), \quad (6.12)$$

$$\lim_{y \to y_0^+} \psi'(y) = \frac{1}{6y_0} \lim_{y \to y_0^+} \psi(y). \quad (6.13)$$

Note that in writing the above equations we had used the $Z_2$ symmetry of the wave function $\psi$. Clearly, any solution of Eq. (6.7) that satisfies the above boundary conditions also satisfies Eq. (6.11). That is, the operator $Q^\dagger \cdot Q$ defined by Eq. (6.11) is indeed a positive definite Hermitian operator. Then, by the usual theorems we can see that all eigenvalues $m_n^2$ are non-negative, and their corresponding wave functions $\psi_n(y)$ are orthogonal to each other and form a complete basis. Therefore, the background is gravitationally stable in our current setup.

1. Zero Mode

The four-dimensional gravity is given by the existence of the normalizable zero mode, for which the corresponding wavefunction is given by

$$\psi_0(y) = N_0 \left( \frac{|y| + y_0}{L} \right)^{1/6}, \quad (6.14)$$

where $N_0$ is the normalization factor, defined as

$$N_0 \equiv 2 \left\{ 3L \left[ \left( \frac{y_c + y_0}{L} \right)^{4/3} - \left( \frac{y_0}{L} \right)^{4/3} \right] \right\}^{-1/2}. \quad (6.15)$$

Eq. (6.14) shows clearly that the wavefunction is increasing as $y$ increases from 0 to $y_c$. Therefore, the gravity is indeed localized near the $y = y_c$ brane.

2. Non-Zero Modes

In order to have localized four-dimensional gravity, we require that the corrections to the Newtonian law from the non-zero modes, the KK modes, of Eq. (6.7), be very small, so that they will not lead to contradiction with observations. To solve Eq. (6.7) outside of the two branes, it is found convenient to introduce the quantities,

$$\psi(y) \equiv z^{1/2} u(z), \quad z \equiv m (y + y_0). \quad (6.16)$$

Then, in terms of $z$ and $u(z)$, Eq. (6.7) takes the form,

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + \left( z^2 - \nu^2 \right) u = 0. \quad (6.17)$$

but now with $\nu = 1/3$. Eq. (6.17) is the standard Bessel equation [12], which have two independent solutions $J_\nu(z)$ and $Y_\nu(z)$. Therefore, the general solution of Eq. (6.7) are given by

$$\psi = z^{1/2} \left\{ c J_\nu(z) + d Y_\nu(z) \right\}, \quad (6.18)$$

where $c$ and $d$ are the integration constants, which will be determined from the boundary conditions given by Eqs. (6.12) and (6.13). Setting

$$\Delta_{11} \equiv 2J_\nu (z_c) - 3z_c J_{\nu+1} (z_c), \quad \Delta_{12} \equiv 2Y_\nu (z_c) - 3z_c Y_{\nu+1} (z_c),$$

$$\Delta_{21} \equiv 2J_\nu (z_0) - 3z_0 J_{\nu+1} (z_0), \quad \Delta_{22} \equiv 2Y_\nu (z_0) - 3z_0 Y_{\nu+1} (z_0), \quad (6.19)$$

we find that Eqs. (6.12) and (6.13) can be cast in the form,

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0. \quad (6.20)$$

It has no trivial solutions only when

$$\Delta \equiv \det (\Delta_{ij}) = 0. \quad (6.21)$$
whose roots are given by

$$\tan (z_c - z_0) = - \frac{6 (z_c - z_0)}{4 + 9z_0 z_c}.$$  

(6.24)

From this equation, we can see that \(m_n\) satisfies the bounds

$$\left(n - \frac{1}{2}\right) \frac{\pi}{y_c} < m_n < n\pi \frac{y_c}{y_c} \quad (n = 1, 2, 3, \ldots).$$  

(6.25)

Combining the above expression with Table I, we find that \(m_n\) is well approximated by

$$m_n \approx n\pi \left(\frac{c}{y_c}\right) M_{pl},$$  

(6.26)

For \(z_0 \gg 1\), in particular, we have

$$m_1 \approx 3.14 \left(10^{-19} \frac{m}{y_c}\right) \text{TeV}$$

$$\approx 3.14 \begin{cases} 1 \text{TeV}, & y_c \approx 10^{-19} \text{m}, \\ 10^{-2} \text{eV}, & y_c \approx 10^{-5} \text{m}, \\ 10^{-4} \text{eV}, & y_c \approx 10^{-3} \text{m}. \end{cases}$$  

(6.27)

It should be noted that the mass \(m_n\) calculated above is measured by the observer with the metric \(\eta_{\mu\nu}\). However, since the warped factor \(e^{\sigma(y)}\) is different from one at \(y = y_c\), the physical mass on the visible brane should be given by \(2\)

$$m_n^{\text{obs}} = e^{-\sigma(y_c)} m_n = \left(\frac{y_c + y_0}{L}\right)^{-1/9} m_n.$$  

(6.28)

Without introducing any new hierarchy, we expect that \([y_c + y_0/L]^{-1/9} \approx \mathcal{O}(1)\). As a result, we have

$$m_n^{\text{obs}} \approx \left(\frac{y_c + y_0}{L}\right)^{-1/9} m_n \approx m_n.$$  

(6.29)

For each \(m_n\) that satisfies Eq.(6.21), the wavefunction \(\psi_n(z)\) is given by

$$\psi_n(z) = N_n z^{1/2} \left\{ \Delta_{12} (m_n, y_c) J_\nu (z) - \Delta_{11} (m_n, y_c) Y_\nu (z) \right\},$$  

(6.30)

where \(N_n \equiv N_n (m_n, y_c)\) is the normalization factor, so that

$$\int_0^{y_c} |\psi_n(z)|^2 dy = 1.$$  

(6.31)

### B. 4D Newtonian Potential and Yukawa Corrections

To calculate the four-dimensional effective Newtonian potential and its corrections, let us consider two point-like sources of masses \(M_1\) and \(M_2\), located on the brane at \(y = y_c\). Then, the discrete eigenfunction \(\psi_n(z)\) of mass \(m_n\) has an Yukawa correction to the four-dimensional gravitational potential between the two particles \(\Delta_{12}\)

$$U(r) = G_4 \frac{M_1 M_2}{r} + \frac{M_1 M_2}{M_5^3 r} \sum_{n=1}^{\infty} e^{-m_n r} |\psi_n(z_c)|^2,$$  

(6.32)
where $\psi_n(z_c)$ is given by Eq. (6.30). When $z_0 = m_n y_0 \gg 1$, from Eqs. (6.22), (6.30) and (6.31) we find that

$$N_n \simeq \sqrt{\frac{\pi^2}{18z_c y_c}},$$

$$\psi_n(z_c) \simeq \sqrt{x}. \quad (6.33)$$

Then, we obtain

$$\delta_1(r) \simeq \left(\frac{10^{28} \text{m}}{y_c}\right) e^{-\frac{r}{\pi y_c}}. \quad (6.34)$$

Clearly, for $y_c \simeq 10^{-19} \text{ m}$ and $r \simeq 10 \mu \text{m}$, we have $\delta_1(r) \ll 1$, and the corresponding Yukawa corrections are negligible.

**VII. CONCLUSIONS**

In this paper, we have systematically studied the brane worlds of string theory on $S^1/Z_2$. Starting with the toroidal compactification of the Neveu-Schwarz/Neveu-Schwarz sector in $(D+d)$ dimensions, in Sec. II.A we have first obtained an effective $D$-dimensional action given by Eq. (2.24) for non-vanishing dilaton field and flux with an effective potential given by Eq. (2.27). Then, in Sec. II.B we have compactified one of the $(D-1)$ spatial dimensions by adding two orbifold branes as the boundaries of the spacetime along the compactified dimension.

Variations of the total action with the metric and matter field yields, respectively, the gravitational and matter field equations. This has been done in Sec. III and given by Eqs. (3.1)-(3.9). Dividing the whole set of the field equations into two groups, one holds outside the two branes, and the other holds on them, in Sec. III.A we have first written down the field equations outside the two branes, Eqs. (3.10)-(3.14), while in Sec. III.B, we have written down explicitly the general gravitational field equations on each of the two branes, Eqs. (3.22)-(3.24), by combining the Gauss-Codacci and Lanczos equations. On the other hand, by using the distribution theory, we have also been able to write down the matter field equations on the branes in terms of the discontinuities of the first derivatives of the matter fields, Eqs. (3.35)-(3.39).

In the study of orbifold branes, one of the most attractive features is that it may resolve the long standing hierarchy problem. In Sec. IV, we have shown explicitly how it can be solved in the current setup. The mechanism is essentially the combination of the ADD large extra dimension [1] and RS warped factor [2] mechanisms together with the tension coupling scenario [32]. In order to solve the hierarchy problem in the current setup, the tensions of the branes are required to be in the order of the cosmological constant.

Another important issue in brane worlds is the radion stability and radion mass [3]. Previously, we showed that the radion is stable [18]. In this paper, we have devoted Sec. V to study the radion mass. With some very conservative arguments, we have found that the radion mass is of the order of $10^{-2} \text{ GeV}$, which is by far beyond its current observational constraint, $m_\varphi > 10^{-3} \text{ eV}$.

In Sec. VI we have also shown that the gravity is localized on the visible (TeV) brane, in contrast to the RS1 model in which the gravity is localized on the Planck (hidden) brane [2]. In addition, the spectrum of the gravitational KK modes is discrete, and given explicitly by Eq. (6.26), which can be of the order of TeV. The corrections to the 4D Newtonian potential from the higher order gravitational KK modes are exponentially suppressed and can be safely neglected [cf. Eq. (6.32)].

In Appendix, we have also presented a systematical and pedagogical study of the Gauss-Codacci equations and Israel’s junction conditions across a surface, which can be either spacelike or timelike, in higher dimensional spacetimes.

It should be noted that, when studied the radion stability, we have ignored the backreaction of the perturbations. Although it is expected that the main results obtained here will be continuously valid even after taking such backreaction into account, as what exactly happened in the Randall-Sundrum model [45], it would be very interesting to show explicitly that this is indeed the case.

Other important issues that have not been addressed in this paper include the constraints from the solar system tests [46], and linear perturbations in the current setup.

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**Appendix: Gauss-Codacci Equations and Israel’s Junction Conditions in Higher Dimensional Spacetimes**

In this appendix, we shall present a systematic and pedagogical study of the Gauss-Codacci equations and Israel’s junction conditions across a surface, where the metric coefficients are only continuous, i.e., $C^0$ in higher dimensional spacetimes.
A. Notations and Conventions

We shall closely follow notations and conventions of d’Inverno [39]. The metric is given by
\[ ds^2 = g_{ab}(x^c)\, dx^a dx^b, \]  
with the signature [47],
\[ \text{sign}(g_{ab}) = \{+, - , - , ..., - \}. \]

We shall use the lowercase Latin indices, such as, \( a, b, c, \) to run from 0 to \( D - 1, \) and the Greek indices, such as, \( \mu, \nu, \lambda, \) to run from 0 to \( D - 2. \) The Riemann tensor is defined by [48],
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = (D) R^a_{bcd} X^d, \]  
where \( \nabla_a \) denotes the covariant derivative with respect to \( g_{ab}. \) In terms of the Christoffel symbols, it is given by
\[ (D) R^a_{bcd} = (D) \Gamma^a_{bd,c} - (D) \Gamma^a_{bc,d} + (D) \Gamma^a_{ce} (D) \Gamma^e_{bd} - (D) \Gamma^a_{de} (D) \Gamma^e_{bc}, \]  
where
\[ (D) \Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{dc,b} + g_{bd,c} - g_{bc,d}), \]
and \( g_{ab, c} \equiv \partial g_{ab} / \partial x^c, \) etc. The Ricci and Einstein tensors are defined as
\[ R_{ab} \equiv (D) R_{abc} = (D) \Gamma^c_{ab,c} - (D) \Gamma^c_{ac,b} + (D) \Gamma^c_{ce} (D) \Gamma^e_{ab} - (D) \Gamma^c_{de} (D) \Gamma^e_{ab}, \]

\[ C_{ab}^{(D)} \equiv R_{ab} - \frac{1}{2} g_{ab} R^{(D)}, \]
where
\[ R^{(D)} \equiv R_{ab}^{(D)} g^{ab}. \]

The Weyl tensor is defined as
\[ C_{abcd}^{(D)} = R_{abcd}^{(D)} + \frac{1}{D - 2} \left( g_{ad} R_{bc}^{(D)} + g_{bc} R_{ad}^{(D)} - g_{ac} R_{bd}^{(D)} - g_{bd} R_{ac}^{(D)} \right) \frac{1}{(D - 1)(D - 2)} \left( g_{ac} g_{bd} - g_{ad} g_{bc} \right) R^{(D)}. \]

In this paper, we also use the convention,
\[ (D) X \equiv X^{(D)}. \]

B. Gauss and Codacci Equations

Assume that \( M_{D-1} \) is a hypersurface in \( M_D \) given by
\[ M_{D-1} = \{ x^a : \Phi(x^c) = 0 \}. \]

If we choose the intrinsic coordinates of \( M_{D-1} \) as
\[ \{ \xi^\mu \} = \{ \xi^0, \xi^2, ..., \xi^{D-2} \}, \]
we find that the hypersurface \( M_{D-1} \) can be also written in the form,
\[ x^a = x^a(\xi^c). \]

Then, we have
\[ d\Phi(x^c) = \frac{\partial \Phi(x^c)}{\partial x^a} \frac{\partial x^a(\xi^c)}{\partial \xi^\mu} d\xi^\lambda = 0. \]

Since \( d\xi^\lambda \)'s are linearly independent, we must have
\[ N_a e^a_{(\mu)} = 0, \]
where
\[ N_a \equiv \frac{\partial \Phi(x^c)}{\partial x^a}, \]
\[ e^a_{(\mu)} \equiv \frac{\partial x^a(\xi^c)}{\partial \xi^\mu}, \]
and \( N_a \) denotes the normal vector to the hypersurface \( \Phi(x^c) = 0, \) and \( e^a_{(\mu)} \)'s are the tangent vectors.

When \( N_a N^a \neq 0, \) a condition that we shall assume in this section, we define the unit normal vector \( n_a \) as
\[ n_a = \frac{N_a}{|N_a N_c|^{1/2}}, \]
with
\[ n_a n_b g^{ab} = \epsilon(n), \]
where \( \epsilon(n) = \pm 1. \) When \( \epsilon(n) = +1 \) the normal vector \( n_a \) is timelike, and the corresponding hypersurface \( M_{D-1} \) is spacelike; when \( \epsilon(n) = -1 \) the normal vector \( n_a \) is spacelike, and the corresponding hypersurface \( M_{D-1} \) is timelike.

On the hypersurface \( M_{D-1}, \) the metric \((A.1)\) reduces to
\[ ds^2|_{M_{D-1}} = g_{ab}(x^c(\xi^\lambda)) \frac{\partial x^a(\xi^\rho)}{\partial \xi^\mu} \frac{\partial x^b(\xi^\sigma)}{\partial \xi^\nu} d\xi^\mu d\xi^\nu = g_{\mu\nu}(\xi^\lambda) d\xi^\mu d\xi^\nu, \]
where \( g_{\mu\nu} \) is the reduced metric on \( M_{D-1} \) and defined as
\[ g_{\mu\nu}(\xi^\lambda) \equiv g_{ab}(x^c(\xi^\lambda)) e^a_{(\mu)} e^b_{(\nu)}. \]

On the other hand, introducing the projection operator, \( h_{ab}, \) by
\[ h_{ab} = g_{ab} - \epsilon(n)n_a n_b, \]
we find the following useful relations,
\[ g^{ab} = g^{\mu\nu} e^a_{(\mu)} e^b_{(\nu)} + \epsilon(n) n^a n^b, \]
\[ g_{\mu\nu} = g_{ab} e^a_{(\mu)} e^b_{(\nu)}, \]
\[ h_{ab} = g_{ab} - \epsilon(n) n_a n_b = g^{\mu\nu} e_{(\mu)} a e_{(\nu)} b, \] (B.12)
where \( e_{(\mu)} a \equiv g_{ab} e_{(\mu)} b \).

For a tangent vector \( A \) of \( M_{D-1} \), we have
\[ A_\mu = e_{(\mu)} \cdot A = e^c_{(\mu)} A_c, \quad A = A^\mu e_{(\mu)}, \] (B.13)
with \( A \cdot n = 0 \), and
\[ A^\mu \equiv g^{\mu\nu} A_\nu. \] (B.14)

The intrinsic covariant derivative of \( A \) with respect to \( \xi^\mu \) is defined as the projection of the vector \( \nabla A / \nabla \xi^\mu \) onto \( M_{D-1} \),
\[ A_{\mu;\nu} \equiv e_{(\mu)} \cdot \nabla_{\xi^\nu} A = e^c_{(\mu)} \partial_{\xi^\nu} b A_c = \partial_{\xi^\nu} \left[ b \left( e_{(\mu)} c A_c \right) - A \nabla_{\xi^\nu} b c_{(\mu)} \right] = \partial_{\xi^\nu} b \left( c_{(\mu)} A_c \right) \nabla_{\xi^\nu} e_{(\mu)} - A \cdot \nabla_{\xi^\nu} e_{(\mu)}. \] (B.15)

Since
\[ \nabla_{\xi^\nu} \left( e_{(\mu)} \cdot A \right) = \partial_{\xi^\nu} c_{(\mu)} A_c + \partial_{\xi^\nu} b_{(\mu)} A_b = \partial_{\xi^\nu} A_{\mu}; \]
\[ A \cdot \nabla_{\xi^\nu} \left( e_{(\mu)} \right) = A^\sigma \cdot \nabla_{\xi^\nu} \left( e_{(\mu)} \right), \] (B.16)
we find that Eq. (B.15) can be written as
\[ A_{\mu;\nu} = e_{(\mu)} \cdot \nabla_{\xi^\nu} A = A_{\mu;\nu} - \Lambda_{\mu} \Gamma^\lambda_{\mu\nu} \] (B.17)
where
\[ \Gamma^\lambda_{\mu\nu} \equiv g^{\lambda\sigma} e_{(\sigma)} \cdot \nabla_{\xi^\nu} e_{(\mu)}. \] (B.18)

After tedious but simple calculations, we finally arrive at
\[ \Gamma^\lambda_{\mu\nu} = g^{\lambda\sigma} e_{(\sigma)} \cdot \frac{\nabla_{\xi^\nu} e_{(\mu)}}{\nabla_{\xi^\nu}} = \frac{1}{2} g^{\lambda\sigma} \left( g_{\sigma\nu\mu} + g_{\mu\sigma\nu} - g_{\nu\lambda\mu} \right). \] (B.19)

Properties of a non-intrinsic character enter when we consider the way in which \( M_{D-1} \) bends in \( M_D \). This is measured by the variations of \( \nabla n_{a \lambda} / \nabla \xi^\mu \) of the normal vector. Since each of these \( (D-1) \) vectors is perpendicular to \( n_a \), we can write
\[ \nabla_{\xi^\nu} n^a_{\lambda} = K^a_{\lambda} e_{(\lambda)}^a, \] (B.20)
thus defining the extrinsic curvature \( K_{\mu\nu} \) of the hypersurface \( M_D \). From Eqs. (B.12) and (B.20) we obtain that
\[ K_{\mu\nu} = g_{\mu\lambda} K^\lambda_{\nu} = e_{(\mu)} \cdot \nabla_{\xi^\nu} e_{(\lambda)} \nabla_{\xi^\lambda} n^a_{\nu} = e_{(\mu)} \nabla_{\xi^\nu} e_{(\lambda)} \nabla_{\xi^\lambda} n^a_{\nu}. \] (B.21)

Because \( n_a e_{(\mu)}^a = 0 \), we find that
\[ K_{\mu\nu} = e_{(\mu)} e_{(\nu)} \nabla_{\xi^\nu} n^a_{\nu} = -n_a e_{(\mu)} e_{(\nu)} \nabla_{\xi^\nu} \left( e_{(\nu)}^a \right) = -n_a \left( e_{(\mu)} - e_{(\nu)} \Gamma^a_{\mu\nu} \right), \]
\[ = -n_a \left( \frac{\partial^2 x^a}{\partial \xi^\mu \partial \xi^\nu} + (D) \Gamma^a_{\mu\nu} \Gamma^c_{\mu\nu} \right) = K_{\mu\nu}. \] (B.22)

Assuming
\[ \frac{\nabla e_{(\mu)}}{\nabla \xi^\nu} = \alpha_{\mu\nu} n + \beta_{\mu\nu} e_{(\sigma)}, \] (B.23)
we find that
\[ \nabla e_{(\mu)} = \alpha_{\mu\nu} e_{(\nu)} = \beta_{\mu\nu} \] (B.24)
\[ = \beta_{\mu\nu} \Gamma^\sigma_{\mu\nu}, \] (B.25)
\[ = \alpha_{\mu\nu} e_{(\nu)} = -K_{\mu\nu}, \]
\[ = \Gamma_{\mu\nu} \] (B.26)
\[ \frac{\nabla e_{(\mu)}}{\nabla \xi^\nu} = -\epsilon(n) K_{\mu\nu}, \]
\[ = \Gamma_{\mu\nu}, \] (B.27)
\[ \frac{\nabla A}{\nabla \xi^\nu} = A_{\mu;\nu} - \epsilon(n) K_{\mu\nu} n, \]
\[ = A_{\mu;\nu} - \epsilon(n) A^\mu K_{\mu\nu} n. \] (B.28)

Operating on Eq. (B.20) with \( \nabla / \nabla \xi^\nu \) and using Eq. (B.20), we find that
\[ \frac{\nabla}{\nabla \xi^\nu} \left( \frac{\nabla e_{(\mu)}^a}{\nabla \xi^\nu} \right) = \frac{\nabla}{\nabla \xi^\nu} \left( -\epsilon(n) K_{\mu\nu} n^a + \Gamma^a_{\mu\nu} e_{(\sigma)} \right). \]
\[
\begin{align*}
\epsilon_\rho e^\rho &= -\epsilon(n) \frac{\nabla K_{\mu\nu}}{\nabla \xi^\lambda} a^a - \epsilon(n) K_{\mu\nu} \frac{\nabla n^\sigma}{\nabla \xi^\lambda} \\
&\quad + \frac{\nabla \Gamma_{\mu\nu}}{\nabla \xi^\lambda} e^a_{(\rho)} + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} \epsilon^a_{(\delta)} \\
&= -\epsilon(n) K_{\mu\nu,\lambda} a^a - \epsilon(n) K_{\mu\nu} K_\lambda a^a_{(\sigma)} \\
&\quad + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} \epsilon^a_{(\sigma)} \\
&\quad + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} \left( -\epsilon(n) K_{\mu\nu,\lambda} a^a + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} \epsilon^a_{(\delta)} \right) \\
&= \left( \Gamma_{\mu\nu,\lambda} + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} \Gamma_{\delta\lambda} \right) a^a_{(\sigma)} - \epsilon(n) \left( K_{\mu\nu,\lambda} + \frac{\Gamma_{\mu\nu}}{\nabla \xi^\lambda} K_{\delta\lambda} \right) a^a. 
\end{align*}
\]
Thus, we have
\[
\left( \frac{\nabla^2}{\nabla \xi^\lambda \nabla \xi^\nu} - \frac{\nabla^2}{\nabla \xi^\nu \nabla \xi^\lambda} \right) e^a_{(\mu)} = (D^{-1}) R_{\mu\lambda\nu} e^a_{(\sigma)} + \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right) n^a. 
\] (B.28)

where
\[
(D^{-1}) R_{\mu\lambda\nu} \equiv \Gamma_{\mu\lambda,\nu} - \Gamma_{\lambda\nu,\mu} - \frac{\Gamma_{\mu\lambda}}{\nabla \xi^\lambda} \Gamma_{\delta\lambda} - \frac{\Gamma_{\mu\nu}}{\nabla \xi^\nu} \Gamma_{\delta\nu}. 
\] (B.30)

On the other hand, we have
\[
\frac{\nabla^2 e^a_{(\mu)}}{\nabla \xi^\lambda \nabla \xi^\nu} = \frac{\nabla}{\nabla \xi^\lambda} \left( \frac{\nabla e^a_{(\mu)}}{\nabla \xi^\nu} \right) = e^c_{(\lambda)} \left( e^b_{(\nu)} \nabla_b e^a_{(\mu)} \right) \\
= e^c_{(\lambda)} e^b_{(\nu)} \left( \nabla_c \nabla_b e^a_{(\mu)} \right) \\
+ e^c_{(\lambda)} \left( \nabla_b e^a_{(\mu)} \right) \left( \nabla_b e^a_{(\nu)} \right) \\
= e^c_{(\lambda)} e^b_{(\nu)} \left( \nabla_c \nabla_b e^a_{(\mu)} \right) + \left( \nabla_b e^a_{(\nu)} \right) \\
\times \left( \frac{\partial^2 x^b}{\partial \xi^\lambda \partial \xi^\nu} + (D) \frac{\partial^a}{\partial \xi^\lambda} \frac{\partial^b}{\partial \xi^\nu} \right). 
\] (B.31)

and
\[
\left( \frac{\nabla^2}{\nabla \xi^\lambda \nabla \xi^\nu} - \frac{\nabla^2}{\nabla \xi^\nu \nabla \xi^\lambda} \right) e^a_{(\mu)} = \left[ \nabla_c \nabla_b - \nabla_b \nabla_c \right] e^a_{(\mu)} \\
= (D) R^{a}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)}. 
\] (B.32)

Then, the combination of Eqs. (B.29) and (B.32) yields,
\[
(D) R^{a}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} = \left( D^{-1} \right) R^{a}_{\rho\mu\nu} e^c_{(\beta)} + \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right) e^a_{(\sigma)} \\
\quad + \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right) n^a. 
\] (B.33)

Multiplying Eq. (B.33) by \( e_{(\rho)} a \) we obtain the Gauss equation,
\[
R^{a}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} = R^{(D-1)}_{\rho\mu\nu} + \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right). 
\] (B.34)

Similarly, multiplying Eq. (B.33) with \( n_a \) we obtain the Codacci equation,
\[
R^{a}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} = K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda}. 
\] (B.35)

Multiplying Eq. (B.34) by \( g^{ab} g^\mu\nu \), and noting
\[
\epsilon^c_{(\mu)} e^b_{(\nu)} = g^{ab} - \epsilon(n) n^a n^b, 
\] (B.36)

we find that
\[
R^{a}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} g^{ab} g^\mu\nu = R^{(D)}_{\rho\sigma} g^{ac} g^{bd} - \epsilon(n) n^a n^c g^{bd} - \epsilon(n) n^b n^d g^{ac} \\
\quad = R^{(D)}_{\rho\sigma} g^{ac} g^{bd} - \epsilon(n) n^a n^b g^{ac} g^{bd} - \epsilon(n) n^b n^d g^{ac} \\
\quad = R^{(D)} - 2\epsilon(n) R_{\rho\sigma} n^a n^b \\
\quad = R^{(D-1)} + \epsilon(n) \left( K^{\sigma}_{\lambda} K^{\lambda}_{\sigma} - K^2 \right), 
\] (B.37)

this is,
\[
-2\epsilon(n) G^{(D)}_{\rho\sigma} n^a n^b = R^{(D-1)} + \epsilon(n) \left( K^{\sigma}_{\lambda} K^{\lambda}_{\sigma} - K^2 \right). 
\] (B.38)

From the Gauss equation Eq. (B.34), we find that
\[
R^{(D-1)}_{\mu\lambda\nu} = R^{(D)}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} - \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right), 
\] (B.39)

from which we obtain
\[
R^{(D-1)}_{\mu\lambda\nu} = R^{(D)}_{\rho\sigma} e^c_{(\mu)} e^b_{(\lambda)} e^a_{(\nu)} - \epsilon(n) \left( K_{\mu\lambda,\nu} - K_{\mu\nu,\lambda} \right). 
\] (B.40)

Then, from Eq. (A.8) we find that
\[
R^{(D)}_{\rho\mu\nu} e^c_{(\rho)} e^a_{(\sigma)} + \frac{1}{D-2} \left\{ R^{(D)}_{\rho\sigma} n^a n^b - \frac{\epsilon(n)}{D-1} R^{(D)}_{\rho\sigma} n^a n^b \right\} g^{\mu\nu} + E^{(D)}_{\mu\nu}, 
\] (B.41)
where
\[ E^{(D)}_{\mu\nu} \equiv C^{(D)}_{\alpha\beta\gamma\delta} n^\alpha e^\beta_{(\mu)} n^\gamma e^\delta_{(\nu)}. \] (B.42)

From the definition of the Einstein tensor, we find that
\[ R^{(D)}_{ab} e^a_{(\mu)} e^b_{(\nu)} = G^{(D)}_{ab} e^a_{(\mu)} e^b_{(\nu)} - \frac{1}{D-2} g_{\mu\nu} G^{(D)}, \]
\[ R^{(D)}_{ab} n^a b = G^{(D)}_{ab} n^a b - \frac{\epsilon(n) c(D)}{D-2} c(D), \]
\[ R^{(D)} = -\frac{2}{D-2} g^{(D)}. \] (B.43)

Then, combining Eqs. (B.34)–(B.43), we obtain
\[ G^{(D-1)}_{\mu\nu} = \frac{D-3}{D-2} \left\{ G^{(D)}_{ab} e^a_{(\mu)} e^b_{(\nu)} + \epsilon(n) G^{(D)}_{ab} n^a b g_{\mu\nu} - \frac{1}{D-1} G^{(D)} g_{\mu\nu} \right\} - \epsilon(n) (K_{\mu\nu - K}^a K_{\nu}^a - K K_{\mu\nu}) + \frac{\epsilon(n)}{2} (K_{\alpha\beta - K}^a K_{\beta}^a - K^2) g_{\mu\nu} - \epsilon(n) E^{(D)}_{\mu\nu}. \] (B.44)

C. Surface Layers

Assume that the hypersurface \( M_{D-1} \) divides the whole spacetime \( M_D \) into two regions \( M^+_D \), where
\[ M^+_D := \{ x^a, \Phi \geq 0 \}, \quad M^-_D := \{ x^a, \Phi \leq 0 \}. \] (C.1)

In terms of \( x^a \), the hypersurface \( M_{D-1} \) is given by
\[ x^a = x^a (\xi^\mu), \quad x^a = x^a (\xi^\mu), \] (C.2)

or equivalently
\[ \Phi^+ (x^+ b) = 0, \quad \Phi^- (x^- b) = 0. \] (C.3)

From the above equations we find that
\[ n^+_a = \frac{N^+_a}{\sqrt{N^+_a N^+_b c}}, \quad n^-_a = \frac{\partial \Phi^+ (x^+)}{\partial x^a}, \]
\[ e^a_{(\mu)} = \frac{\partial \xi^\mu (\xi^\lambda)}{\partial x^{a(\lambda)}}, \]
\[ n^-_a = \frac{N^-_a}{\sqrt{N^-_a N^-_b c}}, \quad n^-_a = \frac{\partial \Phi^- (x^- c)}{\partial x^a}, \]
\[ e^a_{(\mu)} = \frac{\partial x^a (\xi^\lambda)}{\partial \xi^\mu}. \] (C.4)

Then, it is easy to see that in each of the two regions, the Gauss and Codacci equations take the form of Eqs. (B.34) and (B.35), from which Eqs. (B.37) and (B.38) result. On the hypersurface \( M_{D-1} \), the reduced metric from each side of \( M_{D-1} \) should be the same, so we must have
\[ g^{+}_{\mu\nu} (\xi^\mu) |_{\Sigma^+} = g^{-}_{\mu\nu} (\xi^\mu) |_{\Sigma^-} \equiv g_{\mu\nu} (\xi^\mu). \] (C.5)

On the other hand, from the Lanczos equations (36),
\[ [K_{\mu\nu}^- - g_{\mu\nu} [K]^t = -\kappa^2 D T_{\mu\nu}, \] (C.6)

one defines the symmetric tensor \( T_{\mu\nu} \) as the effective surface energy-momentum tensor, where
\[ [K_{\mu\nu}^- \equiv \lim_{\Phi \to 0} K_{\mu\nu}^+- \lim_{\Phi \to 0} K_{\mu\nu}^-, \]
\[ [K]^- \equiv g_{\mu\nu} [K]^- \]. (C.7)

Combining Eq. (B.38) with Eq. (C.6), we obtain that
\[ \left[ G^{(D)}_{ac} n^a e^c_{(\mu)} \right] = -\kappa D T_{\mu\nu}^L, \] (C.8)

which serves as the conservation law for the surface EMT. Assuming reflection symmetry of the brane, we have
\[ K_{\mu\nu}^- = -K_{\mu\nu}^- = -K_{\mu\nu}. \] (C.9)

Then, from the Lanczos equations (36), we find that
\[ K_{\mu\nu} - g_{\mu\nu} K = \frac{\kappa^2}{2} T_{\mu\nu}. \] (C.10)

Considering the case where
\[ T_{\mu\nu} = \tau_{\mu\nu} + \lambda^{\text{total}} g_{\mu\nu}, \] (C.11)

we find that
\[ K = -\frac{\kappa^2}{2(D-2)} [(D-1) \lambda^{\text{total}} + \tau], \]
\[ K_{\mu\nu} = \frac{\kappa^2}{2} \left[ \tau_{\mu\nu} - \frac{1}{D-2} (\tau + \lambda^{\text{total}}) g_{\mu\nu} \right]. \] (C.12)

where \( \tau \equiv g_{\mu\nu} \tau_{\mu\nu} \), and
\[ \lambda^{\text{total}} \equiv \lambda + \tau_p. \] (C.13)

Then, we obtain
\[ \mathcal{F}^{(D-1)}_{\mu\nu} \equiv (K_{\mu\lambda} K_{\lambda\nu} - K K_{\mu\nu}) \]
\[ -\frac{1}{2} g_{\mu\nu} (K_{\alpha\beta} K_{\alpha\beta} - K^2) \]
\[ = -\epsilon(n) \left\{ \kappa^2_{D-1} \tau_{\mu\nu} + \lambda^{c\mu\nu} g_{\mu\nu} + \kappa^4 D \pi_{\mu\nu} \right\} \]
\[ \quad + \kappa^4 D \frac{(D-3)}{4(D-2)} \tau_p \left\{ \tau_{\mu\nu} + \frac{1}{2} (2\lambda + \tau_p) g_{\mu\nu} \right\}, \] (C.14)

where
\[ \tau_{\mu\nu} = -\frac{\epsilon(n)}{4} \left\{ \tau_{\mu\lambda} \tau_{\nu}^\lambda \right\} \]
\[ -\frac{1}{D-2} \tau_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( \tau_{\alpha\beta} \tau_{\alpha\beta} - \frac{1}{D-2} \tau^2 \right) \}, \]
\[ \kappa^2_{D-1} = -\epsilon(n) \frac{D-3}{4(D-2)} \lambda^{2} \kappa^4 D, \]
\[ \lambda^{c\mu\nu} = -\epsilon(n) \frac{D-3}{8(D-2)} \lambda^{2} \kappa^4 D. \] (C.15)
Then, Eq. (C.44) takes the form,
\[ G^{(D-1)}_{\mu\nu} = -\epsilon(n) \left( G^{(D)}_{\mu\nu} + E^{(D)}_{\mu\nu} \right) \]
\[ -\epsilon(n) \frac{\kappa^{2}_{D} (D-3)}{4(D-2)} \tau_{\mu\nu} \left\{ \tau_{\mu\nu} + \frac{1}{2} (2 \lambda + \tau_{\mu\nu}) g_{\mu\nu} \right\} \]
\[ + \kappa^{2}_{D-1} \tau_{\mu\nu} + \Lambda^{D} \epsilon(n) g_{\mu\nu} + \kappa^{4}_{D} \pi_{\mu\nu}, \]  \hspace{1cm} (C.16)
where
\[ G^{(D)}_{\mu\nu} = -\epsilon(n) \frac{D-3}{D-2} \left\{ G^{(D)}_{ab} \epsilon^{(\mu)}(n) \epsilon^{(\nu)}(n) \right\} \]
\[ + \epsilon(n) \left[ G^{(D)}_{ab} n^{a} n^{b} - \frac{\epsilon(n)}{D-1} G^{(D)} \right] g_{\mu\nu} \] .

For a perfect fluid,
\[ \tau_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}, \]  \hspace{1cm} (C.18)
where \( u_{\mu} \) is the four-velocity of the fluid, we find that
\[ \pi_{\mu\nu} = -\epsilon(n) \frac{D-3}{4(D-2)} \rho \left\{ (\rho + p) u_{\mu} u_{\nu} - \frac{1}{2} (\rho + 2p) g_{\mu\nu} \right\} . \]  \hspace{1cm} (C.19)

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