IMPROVED HIGHER ORDER POINCARÉ INEQUALITIES ON THE HYPERBOLIC SPACE VIA HARDY-TYPE REMAINDER TERMS

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ABSTRACT. The paper deals about Hardy-type inequalities associated with the following higher order Poincaré inequality:

\[
\left( \frac{N-1}{2} \right)^{2(k-l)} := \inf_{u \in C_0^\infty(H^N) \setminus \{0\}} \frac{\int_{H^N} |\nabla^k_{H^N} u|^2 d\sigma_{H^N}}{\int_{H^N} |\nabla^l_{H^N} u|^2 d\sigma_{H^N}},
\]

where 0 \leq l < k are integers and H^N denotes the hyperbolic space. More precisely, we improve the Poincaré inequality associated with the above ratio by showing the existence of k Hardy-type remainder terms. Furthermore, when k = 2 and l = 1 the existence of further remainder terms are provided and the sharpness of some constants is also discussed. As an application, we derive improved Rellich type inequalities on upper half space of the Euclidean space with non-standard remainder terms.

1. Introduction. Let H^N denote the hyperbolic space and let k, l be non-negative integers such that l < k. The following higher order Poincaré inequality [23, Lemma 2.4] holds

\[
\int_{H^N} |\nabla^k_{H^N} u|^2 d\sigma_{H^N} \geq \left( \frac{N-1}{2} \right)^{2(k-l)} \int_{H^N} |\nabla^l_{H^N} u|^2 d\sigma_{H^N}, \tag{1}
\]

for all u \in C_0^\infty(H^N), where

\[
\nabla^j_{H^N} := \begin{cases} \Delta_{H^N}^{j/2} & \text{if } j \text{ is an even integer,} \\ \nabla_{H^N} \Delta_{H^N}^{(j-1)/2} & \text{if } j \text{ is an odd integer} \end{cases}
\]

and \nabla_{H^N} denotes the Riemannian gradient while \Delta_{H^N} denotes the j
- th iterated Laplace-Beltrami operator. The present paper takes the origin from the basic observation that the inequality in (1) is strict for u ≠ 0, namely the following infimum
is never achieved
\[
\left( \frac{N - 1}{2} \right)^{2(k-l)} = \inf_{u \in C^\infty_0(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^k u|^2 \, d\nu_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^l u|^2 \, d\nu_{\mathbb{H}^N}}.
\]
It becomes then a natural problem to look for possible remainder terms for (1). In this direction, when \( k = 1 \) and \( l = 0 \), a remainder term of Sobolev type has been determined in [26]. The aim of our study is to deal with Hardy remainder terms, namely to determine improved Hardy inequalities for higher order operators, where the improvement is meant with respect to the higher order Poincaré inequality (1). More precisely, settled \( r := \varrho(x,x_0) \), where \( \varrho \) denotes the geodesic distance and \( x_0 \in \mathbb{H}^N \) denotes the pole, we wish to answer the question

Does there exist positive constants \( C \) and \( \gamma \) such that the following Poincaré-Hardy inequality

\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^k u|^2 \, d\nu_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^l u|^2 \, d\nu_{\mathbb{H}^N} \geq C \int_{\mathbb{H}^N} \frac{u^2}{r^{2\gamma}} \, d\nu_{\mathbb{H}^N} \tag{2}
\]
holds for all \( u \in C^\infty_0(\mathbb{H}^N) \)?

The literature on improved Hardy and Rellich inequalities in the Euclidean setting dates back to the seminal works of Brezis-Vazquez [11] and Brezis-Marcus [10]. Without claiming of completeness, we also recall [1, 4, 5, 6, 14, 17, 18, 19, 20, 27, 30, 32] and references therein. The reason of such a great interest is surely due to the fact that Hardy inequalities and their improved versions have various applications in the theory of partial differential equations and nonlinear analysis, see for instance [11, 34, 35]. Further generalizations to Riemannian manifolds are quite recent and a subject of intense research after the work of Carron [12]. We enlist few important recent works [8, 9, 13, 15, 21, 22, 24, 29, 33] and references therein. Most of these works deals with classical Hardy inequalities and their improvement on Riemannian manifolds. Namely, differently from (2), the optimal Hardy constant is taken as fixed and one looks for bounds of the constant in front of other remainder terms. The main motivation of our study initiated in [3] on improved Poincaré inequalities comes from a paper of Devyver-Fraas-Pinchover [15], which deals with optimal Hardy inequalities for general second order operators. In particular, the existence of at least one Hardy-type remainder term for (1) with \( k = 1 \) and \( l = 0 \) follows as an application of their results. Nevertheless, their weight is given in terms of the Green’s function of the associated operator and does not imply the validity of an inequality like (2). See [3] for further details. The same can be said for the inequality in [7, Example 5.3] where \( N = 3 \). The above mentioned goal was achieved in [2], without optimality issues, and in [3] where, developing a suitable construction of super solution, the following inequality was shown

\[ \text{• Case } k = 1 \text{ and } l = 0. \] For \( N > 2 \) and for all \( u \in C^\infty_0(\mathbb{H}^N) \) there holds

\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, d\nu_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} u^2 \, d\nu_{\mathbb{H}^N} \geq \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, d\nu_{\mathbb{H}^N}, \tag{3}
\]
where the constants \( \left( \frac{N-1}{2} \right)^2 \) and \( \frac{1}{4} \) are sharp.

Unfortunately, the super solution construction applied in the proof of (3) seems not applicable to the higher order case. Nevertheless, by exploiting a completely different technique based on spherical harmonics, in [3] the following second order analogue of (3) was obtained
• **Case** \(k = 2\) and \(l = 0\). For \(N > 4\) and for all \(u \in C^\infty_0(\mathbb{H}^N)\) there holds

\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^4 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} \geq \frac{(N - 1)^2}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N},
\]

where the constant \((\frac{N - 1}{2})^4\) and \((\frac{N - 1}{8})^2\) are sharp.

It is clear that (3) and (4) do not give a complete proof of (2). The aim of the present paper is to either generalize to the higher order (3) and (4) and to investigate all the remaining cases when \(l \neq 0\). A first step in this direction is represented by the proof of the validity of (2) when \(k = 2\) and \(l = 1\). This case is not covered by (3) and (4) and its proof requires some effort. A clever transformation which uncovers the Poincaré term and spherical harmonics technique are the main tools applied, see Sections 2 and 5. Also we note that when \(k = 2\) and \(l = 1\) further singular remainder terms, involving hyperbolic functions, are provided and some optimality issues are proved. Namely, we have

**Theorem 1.1. (Case \(k = 2\) and \(l = 1\))** Let \(N > 4\). For all \(u \in C^\infty_0(\mathbb{H}^N)\) there holds

\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq \frac{(N - 1)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} + \frac{(N - 1)(N - 3)(N^2 - 2N - 7)}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} - \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N}.
\]

The constant \((\frac{N - 1}{2})^2\) is sharp by construction and sharpness of the other constants is discussed in Section 2.

Theorem 1.1 turns out to be one of the key ingredients in our strategy to get the arbitrary case, i.e. inequality (2) for every \(l < k\). Furthermore, from Theorem 1.1 we derive improved Rellich type inequalities on upper half space of the Euclidean space having their own interest. See Corollary 2.2 for the details. The technique adopted relies on the so-called “Conformal Transformation” to the Euclidean space.

As concerns the general case \(l < k\), a fine combination of the previous results and some technical inequalities allow us to finally derive the following family of inequalities

**Theorem 1.2. (Case \(0 \leq l < k\))** Let \(k, l\) be integers such that \(0 \leq l < k\) and let \(N > 2k\). There exist \(k\) positive constants \(\alpha_{k,l} = \alpha_{k,l}(N)\) such that the following inequality holds

\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq \sum_{j=1}^{k} \alpha_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^{2j}} \, dv_{\mathbb{H}^N}
\]

for all \(u \in C^\infty_0(\mathbb{H}^N)\). Furthermore, the constant \((\frac{N - 1}{2})^{2(k-l)}\) is sharp and the leading terms as \(r \to 0\) and \(r \to +\infty\), namely \(\alpha_{k,l}\) and \(\alpha_{k,l}^\prime\), are given explicitly in Theorems 3.1 and 4.1 below.
In view of possible applications to differential equations, we point out that the strategy of our proofs basically allows to determine explicitly all the constants $\alpha_{k,l}^j$ in Theorem 1.2. Nevertheless, for the sake of simplicity, we prefer to focus on the leading terms $\alpha_{k,l}^j$ and $\alpha_{k,l}^k$. This choice is also justified by the fact that our interest is devoted to the non-Euclidean behavior of inequalities and the constant highlighting this aspect is exactly $\alpha_{k,l}^j$, i.e., the constant in front of the leading term as $r \to +\infty$. As a matter of example, here below we specify our family of inequalities for some particular choices of $k$ and $l$.

**Corollary 1.3. (Case $0 = l < k$)** Let $k$ be a positive integer and let $N > 2k$.

If $k = 2m$ for some positive integer $m$, there holds

$$
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} - \left(\frac{N - 1}{2}\right)^{4m} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
\geq \sum_{j=1}^{m} \frac{(N - 1)^{4m - 2j}}{2^{4m - j}} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{9}{2^{4m}} \prod_{j=1}^{m-1} (N + 4j)^2 (N - 4j - 2)^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
$$

for all $u \in C_0^\infty(\mathbb{H}^N)$, where we use the convention $\prod_{j=1}^{0} = 1$.

If $k = 2m + 1$ for some positive integer $m$, there holds

$$
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(\Delta_{\mathbb{H}^N}^m u)|^2 \, dv_{\mathbb{H}^N} - \left(\frac{N - 1}{2}\right)^{4m+2} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
\geq \sum_{j=1}^{m} \frac{(N - 1)^{4m - 2j + 2}}{2^{4m+1}} + \frac{(N - 1)^{2m}}{2^{4m+2}} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
+ \frac{1}{2^{4m+2}} \prod_{j=1}^{m} (N + 4j - 2)^2 (N - 4j - 2)^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
$$

for all $u \in C_0^\infty(\mathbb{H}^N)$.

**Corollary 1.4. (Case $k - 1 = l < k$)** Let $k$ be a positive integer and let $N > 2k$.

If $k = 2m$ for some positive integer $m$, there holds

$$
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} - \left(\frac{N - 1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(\Delta_{\mathbb{H}^N}^{m-1} u)|^2 \, dv_{\mathbb{H}^N} \geq
\frac{(N - 1)^{2m}}{2^{4m}} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{9}{2^{4m}} \prod_{j=1}^{m-1} ((N + 4j)(N - 4j - 2))^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
$$

for all $u \in C_0^\infty(\mathbb{H}^N)$, where we use the convention $\prod_{j=1}^{0} = 1$.

If $k = 2m + 1$ for some positive integer $m$, there holds

$$
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}(\Delta_{\mathbb{H}^N}^m u)|^2 \, dv_{\mathbb{H}^N} - \left(\frac{N - 1}{2}\right)^2 \int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} \geq
\frac{(N - 1)^{2m}}{2^{4m+2}} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{1}{2^{4m+2}} \prod_{j=1}^{m} ((N + 4j - 2)(N - 4j - 2))^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}
$$

for all $u \in C_0^\infty(\mathbb{H}^N)$. 
The article is organized as follows. Section 2 is devoted to the precise statement and discussion of results for the case $k = 2$ and $l = 1$. The complete proof of the results discussed in Section 2 is postponed to Section 5. Section 3 and Section 4 are devoted to discussions and proofs of the results for $0 = l < k$ and for $0 \neq l < k$. The statements of the results given in these sections will contain the precise constants for the leading terms mentioned in the statement of Theorem 1.2.

2. Case $k = 2$ and $l = 1$. We start by restating Theorem 1.1 in its complete form. The proof of the results given in this section will be postponed to Section 5.

**Theorem 2.1.** Let $N > 4$. For all $u \in C^\infty_0(\mathbb{H}^N)$ there holds

$$
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \\
\geq \frac{(N - 1)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} \\
+ \frac{(N - 1)(N - 3)(N^2 - 2N - 7)}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} \\
+ \frac{(N - 1)(N - 3)(N^2 - 4N - 3)}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^4 r} \, dv_{\mathbb{H}^N} .
$$

(5)

The constant $\left( \frac{N - 1}{2} \right)^2$ is sharp by construction, namely cannot be replaced by a larger one. Furthermore, the constant $\frac{(N - 1)^2}{16}$ is sharp in the sense that no inequality of the form

$$
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq c \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}
$$

holds for all $u \in C^\infty_0(\mathbb{H}^N)$ when $c > \frac{(N - 1)^2}{16}$.

**Remark 2.1.** Inequality (5) does not follow directly from (3) and (4) but requires an independent proof which is achieved by means of a suitable modification of the proof of (4) as given in [3]. As already remarked in the Introduction, the main tools exploited are a suitable transformation which uncovers the Poincaré term and spherical harmonic analysis. Recently, spherical harmonics technique has been successfully exploited in the context of Weighted Calderón-Zygmund and Rellich inequalities [28].

**Remark 2.2.** As already explained in the introduction, the leading term of inequality (5) is the one in front of $1/r^2$ for functions supported outside a large ball. Hence, it is particularly important to determine the sharp constant in front of such a term to highlight the non-Euclidean behavior of the inequality. Nevertheless, as happens for inequality (4), the problem of finding the best constant in front of the term $1/r^4$ is still open. See also [3, Remark 6.1].

**Remark 2.3.** It’s worth noting that, as happens for inequality (4), the constants appearing in front of the terms $1/r^4$ and $1/\sinh^4 r$ are jointly sharp, in the sense that the inequality

$$
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} \\
\geq a \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + b \int_{\mathbb{H}^N} \frac{u^2}{\sinh^4 r} \, dv_{\mathbb{H}^N}
$$

holds.
\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} \geq \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} + \frac{(N-1)(N-3)(N^2 - 4N - 3)}{16} \int_{\mathbb{R}^N} \frac{v^2}{\sinh^4 r} \, dv_{\mathbb{R}^N}
\]

where
\[
\frac{9}{16} + \frac{(N-1)(N-3)(N^2 - 4N - 3)}{16} = \frac{N^2(N-4)^2}{16}
\]

and \(\frac{N^2(N-4)^2}{16}\) is the best constant (namely, the largest) for the standard \(N\) dimensional Euclidean Rellich inequality, both on the whole \(\mathbb{R}^N\) or in any open set containing the origin.

Consider the upper half space model for \(\mathbb{H}^N\), namely \(\mathbb{R}_+^N = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}^+\}\) endowed with the Riemannian metric \(\frac{dy^2}{y^2}\). We set
\[
d := d((x, y), (0, 1)) := \cosh^{-1} \left( 1 + \frac{(|y| - 1)^2 + |x|^2}{2|y|} \right).
\]

It is readily seen that \(d \sim \log(1/y)\) as \(y \to 0\). By exploiting the transformation
\[
v(x, y) := y^\alpha u(x, y), \quad x \in \mathbb{R}^{N-1}, y \in \mathbb{R}^+,
\]

with \(\alpha = -\frac{N-2}{2}\) or \(\alpha = -\frac{N-4}{2}\), from (5) we derive the following statements

**Corollary 2.2.** Let \(N > 4\) and \(d\) as defined in (7). For all \(v \in C^\infty_c(\mathbb{R}_+^N)\) the following inequalities hold
\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left( y^2 (\Delta v)^2 + \frac{(N^2 - 2N - 1)}{4} |\nabla v|^2 \right) \, dx \, dy \geq \frac{N(N-2)}{16} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^2} \, dx \, dy + \frac{(N-1)^2}{16} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^2 \, d^2} \, dx \, dy
\]

and
\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left( (\Delta v)^2 + \frac{(N^2 - 2N - 9)}{4} |\nabla v|^2 \right) \, dx \, dy \geq \frac{9}{16} (N + 2)(N - 4) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^4} \, dx \, dy + \frac{(N-1)^2}{16} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^4 \, d^2} \, dx \, dy.
\]

Furthermore, we have:
- no inequality of the form
\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} (y^2 (\Delta v)^2 + c |\nabla v|^2) \, dx \, dy \geq \frac{N(N-2)}{16} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^2} \, dx \, dy
\]
holds for all \(v \in C^\infty_c(\mathbb{H}^N)\) when \(c < \frac{(N^2 - 2N - 1)}{4}\);
• no inequality of the form
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \left( y^2 (\Delta v)^2 + \frac{(N^2 - 2N - 1)}{4} |\nabla v|^2 \right) \, dx \, dy \geq c \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \frac{v^2}{y^2} \, dx \, dy \]
holds for all \( v \in C_c^\infty(\mathbb{H}^N) \) when \( c > \frac{N(N-2)}{16} \);
• no inequality of the form
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \left( y^2 (\Delta v)^2 + \frac{(N^2 - 2N - 1)}{4} |\nabla v|^2 \right) \, dx \, dy \geq \frac{N(N-2)}{16} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \frac{v^2}{y^2} \, dx \, dy + \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \frac{v^2}{y^2} \, dx \, dy \]
holds for all \( v \in C_c^\infty(\mathbb{H}^N) \) when \( c > \frac{(N-1)^2}{16} \).
Similar conclusions hold for the constants \( \frac{(N^2-2N-9)}{2} \), \( \frac{9}{16} (N+2)(N-4) \) and \( \frac{(N-1)^2}{16} \) in (9).

3. Case \( k \) arbitrary and \( l = 0 \). In this section we restate and prove Theorem 1.2 for \( 0 = l < k \).

**Theorem 3.1.** Let \( k \) be a positive integer and \( N > 2k \). There exist \( k \) positive constants \( c_k^l = c_k(N) \) such that the following inequality holds
\[ \int_{\mathbb{H}^N} \left| \nabla_{\mathbb{H}^N} u \right|^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^2 k \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} \geq \sum_{i=1}^{k} c_k \int_{\mathbb{H}^N} \frac{u^2}{r_{\mathbb{H}^N}} \, dv_{\mathbb{H}^N}, \quad (10) \]
for all \( u \in C_0^\infty(\mathbb{H}^N) \). Furthermore, the leading terms as \( r \to 0 \) and \( r \to \infty \) are explicitly given by
\[ c_k^1 = d_k := \begin{cases} \frac{1}{2^m} \sum_{j=1}^{m} \frac{(N-1)^{4m-2j}}{2^m-j} & \text{if } k = 2m, \\ \frac{1}{2^m} \sum_{j=1}^{m} \frac{(N-1)^{4m-2j+2}}{2^m-j+1} + \frac{(N-1)^{2m}}{2^{4m+2}} & \text{if } k = 2m+1, \end{cases} \]
\[ c_k^k = e_k := \begin{cases} \frac{9}{2^m} \prod_{j=1}^{m-1} (N+4j)^2 (N-4j-4)^2 \prod_{j=1}^{m} (N+4j-2)^2 (N-4j-2)^2 & \text{if } k = 2m, \\ \frac{1}{2^m} \prod_{j=1}^{m} (N+4j-2)^2 (N-4j-2)^2 & \text{if } k = 2m+1, \end{cases} \]
where we use the conventions: \( \sum_{j=1}^{0} = 0 \) and \( \prod_{j=1}^{0} = 1 \).

**Proof.** Here and after, for shortness we will write \( \Delta_{\mathbb{H}^N} = \Delta \). In the proof we will repeatedly exploit the following inequality from [33, Theorem 4.4]:
\[ \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^\beta} \, dv_{\mathbb{H}^N} \geq \frac{(N+\beta)^2 (N-\beta-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{4+\beta}} \, dv_{\mathbb{H}^N} + \frac{(N-2-\beta)(N-2+\beta)(N-1)}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^{2+\beta}} \, dv_{\mathbb{H}^N} + \frac{(N-1)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{3}} \, dv_{\mathbb{H}^N}, \quad (11) \]
for all \( u \in C_0^\infty(\mathbb{H}^N) \) and \( 0 \leq \beta < N-4 \).

We prove separately the case \( k \) even and \( k \) odd. First we assume \( k = 2m \) even. If \( m = 1 \), (10) follows directly from (4). When \( m = 2 \), by (4) and (11) with \( N > 8 \),
we have
\[
\int_{\mathbb{H}^N} (\Delta (\Delta u))^2 \, dv_{\mathbb{H}^N} \geq \frac{(N - 1)^4}{16} \int_{\mathbb{H}^N} (\Delta u)^2 \, dv_{\mathbb{H}^N} + \frac{(N - 1)^2}{8} \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^4} \, dv_{\mathbb{H}^N}
\]
Next, by (11), for \( \lambda > 0 \) \( \geq \]
\[
\int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{(N - 1)^2}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N}
\]
\[
+ \frac{(N - 1)^2}{8} \left[ \frac{(N + 2)^2(N - 6)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^6} \, dv_{\mathbb{H}^N} + \frac{N(N - 1)(N - 4)}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^8} \, dv_{\mathbb{H}^N} \right]
\]
\[
+ \frac{(N - 1)^2}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{9}{16} \left[ \frac{(N + 2)^2(N - 8)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^8} \, dv_{\mathbb{H}^N} \right]
\]
\[
= \left( \frac{N - 1}{2} \right)^8 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^4 c_i^4 \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}},
\]
where \( c_1 = \frac{(N-1)^2}{2^6} + \frac{(N-1)^4}{2^7} \), \( c_4 = \frac{9}{2^7} (N + 4)^2 (N - 8)^2 \). Hence, (10) is proved for \( k = 4 \).

Next we assume (10) holds for \( k = 2m \) with \( m > 2 \), namely
\[
\int_{\mathbb{H}^N} (\Delta^m u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^{4m} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} \geq \sum_{j=1}^m \frac{(N - 1)^{4m-2j}}{2^{4m-1}} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}
\]
\[
+ \sum_{i=2}^{2m-1} c^2_m \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N} + \frac{9}{2^{4m+1}} \prod_{j=1}^{m-1} (N + 4j)^2 (N - 4(j + 1))^2 \int_{\mathbb{H}^N} \frac{u^2}{r^{4m}} \, dv_{\mathbb{H}^N},
\]
where, for \( 2 \leq i \leq 2m - 1 \), the \( c^2_m \) are suitable positive constants and \( N > 4m \). Inequality (12) yields
\[
\int_{\mathbb{H}^N} (\Delta^{m+1} u)^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\Delta^m (\Delta u))^2 \, dv_{\mathbb{H}^N}
\]
\[
\geq \left( \frac{N - 1}{2} \right)^{4m} \int_{\mathbb{H}^N} (\Delta u)^2 \, dv_{\mathbb{H}^N} + \sum_{j=1}^m \frac{(N - 1)^{4m-2j}}{2^{4m-1}} \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^2} \, dv_{\mathbb{H}^N}
\]
\[
+ \sum_{i=2}^{2m-1} c^2_m \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^{2i}} \, dv_{\mathbb{H}^N} + \frac{9}{2^{4m+1}} \prod_{j=1}^{m-1} (N + 4j)^2 (N - 4(j + 1))^2 \int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^{4m}} \, dv_{\mathbb{H}^N}.
\]
Next, by (11), for \( N > 4m + 4 \) we have
\[
\int_{\mathbb{H}^N} \frac{(\Delta u)^2}{r^2} \, dv_{\mathbb{H}^N} \geq \frac{(N + 2)^2(N - 6)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^6} \, dv_{\mathbb{H}^N}
\]
\[
+ \frac{N(N - 1)(N - 4)}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^8} \, dv_{\mathbb{H}^N} + \frac{(N - 1)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{10}} \, dv_{\mathbb{H}^N},
\]
\[
\sum_{i=2}^{2m-1} c^2_i \frac{(\Delta u)^2}{r^{2i}} \, dv_{\mathbb{H}^N} \geq \sum_{i=2}^{2m-1} c^2_i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N},
\]
\[
\geq \sum_{i=2}^{2m-1} c^2_i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N} \geq \sum_{i=2}^{2m-1} c^2_i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N},
\]
where, for $2 \leq i \leq 2m+1$, $\hat{c}_{2m}$ are suitable positive constants. The above inequalities and (4), finally yield

\[
\int_{\mathbb{H}^n} (\Delta^{m+1} u)^2 - \left( \frac{N-1}{2} \right)^{4(m+1)} \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n} \geq \left( \frac{N-1}{2} \right)^{4(m+1)} \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n} \left( \frac{N-1}{2} \right)^{4(m+1)} \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n} + \sum_{i=2}^{2(m+1)-1} \hat{c}_{2m} \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n}
\]

where, for $2 \leq i \leq 2m+1$, $\hat{c}_{2m}$ are suitable positive constants. By induction, this completes the proof of (10) for $k$ even.

Next we turn to the case $k = 2m+1$ odd. For $m = 1$ and $N > 6$, by (3), (4) and (11), we deduce

\[
\int_{\mathbb{H}^n} |\nabla (\Delta u)|^2 \, dv_{\mathbb{H}^n} \geq \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{H}^n} (\Delta u)^2 \, dv_{\mathbb{H}^n} + \frac{1}{4} \int_{\mathbb{H}^n} (\Delta u)^2 \, dv_{\mathbb{H}^n} + \frac{1}{4} \left( \frac{N-1}{2} \right)^2 \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n} + \frac{N(N-1)(N-4)}{8} \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n}
\]

Hence we obtain,

\[
\int_{\mathbb{H}^n} |\nabla (\Delta u)|^2 \, dv_{\mathbb{H}^n} \geq \left( \frac{N-1}{2} \right)^6 \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n} \geq \sum_{i=1}^{3} c_i \int_{\mathbb{H}^n} u^2 \, dv_{\mathbb{H}^n},
\]

where $c_1 = \frac{(N-1)^2}{2}$ and $c_3 = \frac{1}{16} (N+2)^2 (N-6)^2$ hence, (10) with $m = 1$ is verified. For general $m + 1$ the proof follows very similar to the case $k$ even, we skip the details for brevity. This completes the proof. \(\square\)

4. **Case $k > l > 0$ arbitrary.** In this section we restate and prove Theorem 1.2 for $l > 0$, the case $l = 0$ has already been dealt with in Section 3.
Theorem 4.1. Let \( k > l \) be positive integers and \( N > 2k \). There exist \( k \) positive constants \( \alpha^i_{k,l}(N) \) such that the following inequality holds
\[
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq \sum_{i=1}^{k} \alpha^i_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N},
\]
for all \( u \in C^\infty_0(\mathbb{H}^N) \). Furthermore, the leading terms as \( r \to 0 \) and \( r \to \infty \), namely \( \alpha^1_{k,l} \) and \( \alpha^k_{k,l} \) are explicitly given as follows
\[
\alpha^1_{k,l} := \begin{cases} 
0 & \text{if } k = 2m, \quad l = 2h; \\
2^4(m-h) a_h & \text{if } k = 2m, \quad l = 2h+1, \quad h \neq m-1; \\
\alpha_1 a_m & \text{if } k = 2m, \quad l = 2m-1; \\
4a_1 a_h d_{2(m-h)} + \frac{1}{4} a_m & \text{if } k = 2m+1, \quad l = 2h; \\
\frac{1}{4} a_m + 4a_1 a_{m-1} & \text{if } k = 2m+1, \quad l = 2m-1; \\
\end{cases}
\]
\[
\alpha^k_{k,l} := \begin{cases} 
\frac{c_{2(m-h)} b_{h,4(m-h)}}{9} & \text{if } k = 2m, \quad l = 2h; \\
\frac{c_{2(m-h-1)} b_{h+1,4(m-h-1)}}{16} & \text{if } k = 2m, \quad l = 2h+1, \quad h \neq m-1; \\
\frac{1}{4} b_{m-1,4} & \text{if } k = 2m, \quad l = 2m-1; \\
\frac{1}{4} b_{m,2} & \text{if } k = 2m+1, \quad l < k; \\
\end{cases}
\]
where \( a_0 = 1, d_0 = 0 \) and, for any \( \gamma \) and \( \beta \) positive integers, \( a_\gamma = \frac{(N-1)^{2\gamma}}{2^{4\gamma}}, \; b_{\gamma,\beta} = \prod_{j=0}^{\gamma-1} \left( \frac{N+(\beta+4j)}{16} \right)^{2(\gamma-j+4j-4)}, \) \( d_\gamma \) and \( e_\gamma \) are the constants defined in Theorem 3.1.

The proof is achieved by considering separately four cases. In each proof we will exploit the following technical lemma whose proof can be obtained by induction, iterating (11). Notice that, except for the main statements, for shortness we will always write \( \Delta_{\mathbb{H}^N} = \Delta \).

Lemma 4.2. Let \( \gamma \) be a positive integer. For all \( u \in C^\infty_0(\mathbb{H}^N) \) and \( 0 \leq \beta < N-4\gamma \), the following inequality holds
\[
\int_{\mathbb{H}^N} \left( \frac{\Delta u}{r^\beta} \right)^2 \, dv_{\mathbb{H}^N} \geq a_\gamma \int_{\mathbb{H}^N} \frac{u^2}{r^\beta} \, dv_{\mathbb{H}^N} + \sum_{j=1}^{2\gamma-1} a_{j,\beta} \int_{\mathbb{H}^N} \frac{u^2}{r^{2j+\beta}} \, dv_{\mathbb{H}^N} + b_{\gamma,\beta} \int_{\mathbb{H}^N} \frac{u^2}{r^{4\gamma+\beta}} \, dv_{\mathbb{H}^N},
\]
where \( b_{\gamma,\beta} = \prod_{j=0}^{\gamma-1} \left( \frac{N+(\beta+4j)}{16} \right)^{2(\gamma-j+4j-4)} \), \( a_\gamma = \frac{(N-1)^{2\gamma}}{2^{4\gamma}}, \) and \( a_{m,\beta} \) are suitable positive constants.
4.1. Case \( k = 2m \) even and \( l = 2h \) even.

**Theorem 4.3.** Let \( m, h \) be integers such that \( 0 < h < m \) and \( N > 4m \). There exist \( 2m \) positive constants \( \alpha^i = \alpha^i(N, m, h) \) such that the following inequality holds

\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^h u)^2 \, dv_{\mathbb{H}^N} \geq \sum_{i=1}^{2m} \alpha^i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N},
\]

(14)

for all \( u \in C_0^\infty(\mathbb{H}^N) \). Furthermore, the leading terms as \( r \to 0 \) and \( r \to \infty \) are explicitly given by

\[
\alpha^1 := d_{2(m-h)} \, a_h \quad \text{and} \quad \alpha^{2m} := e_{2(m-h)} \, b_{h,4(m-h)},
\]

where, for any \( \gamma \) and \( \beta \) positive integers, \( b_{\gamma,\beta} = \prod_{j=0}^{\gamma-1} \frac{(N+(\beta+4j))^2(N-(\beta+4j)-4)^2}{16} \), \( a_{\gamma} = \frac{(N-1)^{2\gamma}}{2^{\gamma}}, \) \( d_{\gamma} \) and \( e_{\gamma} \) are the constants defined in Theorem 3.1.

**Proof.** By applying (10) with \( k = 2(m-h) \) and (13) with \( \gamma = h \) and \( \beta = 2i \) we deduce

\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^{m-h}(\Delta_{\mathbb{H}^N}^h u))^2 \, dv_{\mathbb{H}^N}
\]

\[
\geq \left( \frac{N-1}{2} \right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^h u)^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2(m-h)} c_{2(m-h)}^i \int_{\mathbb{H}^N} \left( \frac{\Delta_{\mathbb{H}^N}^h u}{r^{2i}} \right)^2 \, dv_{\mathbb{H}^N}
\]

\[
\geq \left( \frac{N-1}{2} \right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^h u)^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2(m-h)} c_{2(m-h)}^i \left( a_h \int_{\mathbb{H}^N} \frac{u^2}{r^{4i}} \, dv_{\mathbb{H}^N} + \frac{2h-1}{j} a_{h,2i} \int_{\mathbb{H}^N} \frac{u^2}{r^{4j+2i}} \, dv_{\mathbb{H}^N} + b_{h,2i} \int_{\mathbb{H}^N} \frac{u^2}{r^{4h+2i}} \, dv_{\mathbb{H}^N} \right),
\]

where all the constants are positive. Setting \( g(j, i) := 2j + 2i \), for \( 0 \leq j \leq 2h \) and \( 1 \leq i \leq 2(m-h) \) it is readily verified that \( g \) has a unique global minimum \( g(0, 1) = 1 \) and a unique global maximum \( g(2h, 2(m-h)) = 4m \). Furthermore, by the fact that \( g(j, 1) \) goes monotonically from 2 to \( 4h+2 \) and \( g(2h, i) \) goes monotonically from \( 4h+2 \) to \( 4m \), we deduce the existence of \( 2m \) positive constants \( \alpha^i = \alpha^i(N, m, h) \) such that (14) holds. Moreover,

\[
\alpha^1 = d_{2(m-h)} \, a_h \quad \text{and} \quad \alpha^{2m} := e_{2(m-h)} \, b_{h,4(m-h)}.
\]

\( \square \)

4.2. Case \( k = 2m \) even and \( l = 2h + 1 \) odd.

**Theorem 4.4.** Let \( m, h \) be integers such that \( 0 \leq h < m \) and \( N > 4m \). There exist \( 2m \) positive constants \( \bar{\alpha}^i = \bar{\alpha}^i(N, m, h) \) such that the following inequality holds

\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N}^m u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^{4(m-h)-2} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} \Delta_{\mathbb{H}^N}^h u|^2 \, dv_{\mathbb{H}^N}
\]

(15)

for all \( u \in C_0^\infty(\mathbb{H}^N) \). Furthermore, the leading terms as \( r \to 0 \) and \( r \to \infty \) are explicitly given by:

\[
\sum_{i=1}^{2m} \bar{\alpha}^i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N},
\]
if \( 0 \leq h < m - 1 \)
\[
\tilde{\alpha}^1 := 2^{4(m-h)-2} a_{2(m-h)} a_h + d_{2(m-h-1)} a_{h+1} \quad \text{and} \quad \tilde{\alpha}^k := e_{2(m-h-1)} b_{h+1,4(m-h-1)}
\]
if \( h = m - 1 \)
\[
\tilde{\alpha}^1 := a_1 a_{m-1} \quad \text{and} \quad \tilde{\alpha}^k := \frac{9}{16} b_{m-1,4}.
\]

where \( a_0 = 1 \) and, for any \( \gamma \) and \( \beta \) positive integers, \( a_\gamma = \frac{(N-1)^{\gamma}}{2^{4\gamma}} \), \( b_{\gamma, \beta} = \prod_{j=0}^{\gamma-1} \frac{(N+(\beta+4j))^2(N-(\beta+4j)-4)^2}{16} \), \( d_\gamma \) and \( e_\gamma \) are the constants defined in Theorem 3.1.

Proof. Let \( 0 < h < m - 1 \), by applying first (10) with \( k = 2(m-h-1) \), then (5) and finally (13) with \( \gamma = h, h + 1 \) and \( \beta = 2, 4, 2i \), we deduce
\[
\int_{\mathbb{H}^N} (\Delta^m u)^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\Delta^{m-1}(\Delta^{h+1} u))^2 \, dv_{\mathbb{H}^N}
\]
\[
\geq \left( \frac{N-1}{2} \right)^{4(m-h)-2} \int_{\mathbb{H}^N} (\nabla \Delta^h u)^2 \, dv_{\mathbb{H}^N} + \frac{1}{4} \left( \frac{N-1}{2} \right)^{4(m-h)-2} \int_{\mathbb{H}^N} (\frac{\Delta^h u}{r^2})^2 \, dv_{\mathbb{H}^N}
\]
\[
+ \frac{9}{16} \left( \frac{N-1}{2} \right)^{4(m-h)-2} \int_{\mathbb{H}^N} (\frac{\Delta^h u}{r^2})^2 \, dv_{\mathbb{H}^N}
\]
\[
+ \sum_{i=1}^{2(m-h-1)} \frac{\frac{1}{2} a_{i,2(m-h-1)} \int_{\mathbb{H}^N} (\Delta^{h+1} u)^2 \, dv_{\mathbb{H}^N}}{r^{2i}}
\]
\[
\geq \left( \frac{N-1}{2} \right)^{4(m-h)-2} \int_{\mathbb{H}^N} (\nabla \Delta^h u)^2 \, dv_{\mathbb{H}^N} + \frac{1}{4} \left( \frac{N-1}{2} \right)^{4(m-h)-2} \left( a_h \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \sum_{j=1}^{2h-1} \frac{a_{j,2} u^2}{r^{2j+2}} \, dv_{\mathbb{H}^N}
\]
\[
+ \frac{9}{16} \left( \frac{N-1}{2} \right)^{4(m-h)-2} \left( a_h \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \sum_{j=1}^{2h-1} \frac{a_{j,4} u^2}{r^{2j+4}} \, dv_{\mathbb{H}^N}
\]
\[
+ b_{h,4} \int_{\mathbb{H}^N} \frac{u^2}{r^{4h+4}} \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2h-1} \frac{e_{i,2} (m-h-1)}{r^{2i}} \left( a_{h+1} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N} + \sum_{j=1}^{2h-1} \frac{a_{j,1,2} u^2}{r^{2j+2i}} \, dv_{\mathbb{H}^N}
\]
\]

Hence, with an argument similar to that applied in the proof of Theorem 4.3, it’s readily deduced the existence of \( 2m \) positive constants \( \tilde{\alpha}^i = \alpha^i(N, m, h) \) such that (15) holds. Furthermore,
\[
\tilde{\alpha}^1 = \frac{1}{4} \left( \frac{N-1}{2} \right)^{4(m-h)} a_h + d_{2(m-h-1)} a_{h+1} \quad \text{and} \quad \tilde{\alpha}^k = e_{2(m-h-1)} b_{h+1,4(m-h-1)}.
\]

When \( h = 0 \) the above computations may be slightly modified to show the validity of (15). Furthermore, by setting \( a_0 = 0 \), the leading terms are still given as above.
When \( h = m - 1 \), the existence of \( 2m \) positive constants \( \alpha^i = \alpha^i(N, m, h) \) such that (15) holds follows similarly by applying first (5) and then (13), with \( \gamma = m - 1 \) and \( \beta = 2, 4 \). Finally, the leading terms turn out to be

\[
\alpha^1 = \frac{(N - 1)^2}{16} a_{m-1} \quad \text{and} \quad \alpha^k = \frac{9}{16} b_{m-1,4}.
\]

\[\square\]

4.3. Case \( k = 2m + 1 \) odd and \( l = 2h \) even.

**Theorem 4.5.** Let \( m, h \) be integers such that \( 0 < h \leq m \) and \( N > 4m + 2 \). There exist \( 2m + 1 \) positive constants \( \delta^i = \delta^i(N, m, h) \) such that the following inequality holds

\[
\int_{\mathbb{R}^N} |\nabla (\Delta^m u)|^2 \, dv_{\mathbb{R}^N} - \left( \frac{N - 1}{2} \right)^4 \int_{\mathbb{R}^N} (\Delta^h u)^2 \, dv_{\mathbb{R}^N} + \sum_{i=1}^{2m+1} \delta^i \int_{\mathbb{R}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{R}^N},
\]

for all \( u \in C_0^\infty(\mathbb{H}^N) \). Furthermore, the leading terms as \( r \to 0 \) and \( r \to \infty \) are explicitly given by:

\[
\delta^1 := 4a_1 a_h d_{2(m-h)} + \frac{1}{4} a_m \quad \text{and} \quad \delta^{2m+1} := \frac{1}{4} b_{m,2}
\]

where \( d_0 = 0 \), and, for any \( \gamma \) and \( \beta \) positive integers, \( a_\gamma = \frac{(N-1)^\gamma}{2^{\gamma m}} \), \( b_{\gamma, \beta} = \prod_{j=0}^{l-1} \frac{(N+(\beta+4j)^2(N-(\beta+4j)-4)^2}{16} \), \( d_\gamma \) and \( e_\gamma \) are the constants defined in Theorem 5.1.

**Proof.** From (3) we know

\[
\int_{\mathbb{R}^N} |\nabla (\Delta^m u)|^2 \, dv_{\mathbb{R}^N} \geq \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{R}^N} (\Delta^m u)^2 \, dv_{\mathbb{R}^N} + \frac{1}{4} \int_{\mathbb{R}^N} (\Delta^m u)^2 \, dv_{\mathbb{R}^N}.
\]

If \( 0 < h < m \), from (14) and (13) we readily get

\[
\int_{\mathbb{R}^N} |\nabla (\Delta^m u)|^2 \, dv_{\mathbb{R}^N} \\
\geq \left( \frac{N - 1}{2} \right)^2 \left( \frac{N - 1}{2} \right)^{4(m-h)} \int_{\mathbb{R}^N} (\Delta^h u)^2 \, dv_{\mathbb{R}^N} + \sum_{i=1}^{2m} \alpha^i \int_{\mathbb{R}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{R}^N} + \sum_{j=1}^{2m-1} a_{m,j} \int_{\mathbb{R}^N} \frac{u^2}{r^{2j+2}} \, dv_{\mathbb{R}^N} + b_{m,2} \int_{\mathbb{R}^N} \frac{u^2}{r^{4m+2}}
\]

by which the existence of \( 2m + 1 \) positive constants \( \delta^i = \delta^i(N, m, h) \) such that (16) holds follows. Furthermore, one has

\[
\delta^1 := \left( \frac{N - 1}{2} \right)^2 a_h d_{2(m-h)} + \frac{1}{4} a_m \quad \text{and} \quad \delta^{2m+1} := \frac{1}{4} b_{m,2}.
\]

When \( h = m \) the same proof may be adopted without applying (14). In this case, the leading terms are defined as above by assuming \( d_0 = 0 \). \(\square\)
4. Case $k = 2m + 1$ odd and $l = 2h + 1$ odd.

**Theorem 4.6.** Let $m, h$ be integers such that $0 \leq h < m$ and $N > 4m + 2$. There exist $2m + 1$ positive constants $\delta^i = \delta^i(N, m, h)$ such that the following inequality holds

$$
\int_{\mathbb{H}^N} |\nabla u|^{2(4m-h)} \, dv_{\mathbb{H}^N} \geq \left( \frac{N-1}{2} \right)^4 \int_{\mathbb{H}^N} |\Delta u|^{2(4m-h)} \, dv_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}
$$

for all $u \in C_0^\infty(\mathbb{H}^N)$. Furthermore, the leading terms as $r \to 0$ and $r \to \infty$ are explicitly given by:

if $0 \leq h < m - 1$

$$
\delta^1 := \frac{1}{4}a_m + 4^{2(m-h)}a_{2(m-h)}a_{h}a_1 + 4d_2(m-h-1)a_{h+1}a_1 \quad \text{and} \quad \delta^{2m+1} := \frac{1}{4}b_{m,2},
$$

if $h = m - 1$

$$
\delta^1 := \frac{1}{4}a_m + 4a_{2m-1} \quad \text{and} \quad \delta^{2m+1} = \frac{1}{4}b_{m,2},
$$

where $a_0 = 1$ and, for any $\gamma$ and $\beta$ positive integers, $a_\gamma = \frac{(N-1)^{2\gamma}}{2^{2\gamma}}$, $b_{\gamma,\beta} = \prod_{j=0}^{\gamma-1} \frac{(N+\beta+s_1)(N-\beta+s_1)-4s_1^2}{4s_1^2}$, $d_{\gamma}$ and $e_{\gamma}$ are the constants defined in Theorem 3.1.

**Proof.** From (3) we know

$$
\int_{\mathbb{H}^N} |\Delta u|^2 \, dv_{\mathbb{H}^N} \geq \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla u|^2 \, dv_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}.
$$

Now, by applying (15) and (13) we deduce

$$
\int_{\mathbb{H}^N} |\nabla (\Delta u)|^2 \, dv_{\mathbb{H}^N} \geq \left( \frac{N-1}{2} \right)^2 \left( \left( \frac{N-1}{2} \right)^4 \int_{\mathbb{H}^N} |\Delta (\Delta u)|^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2m} \tilde{a}_i \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} \right) + \frac{1}{4} \left( a_m \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \sum_{j=1}^{2m-1} a_{m,2} \int_{\mathbb{H}^N} \frac{u^2}{r^{2j+2}} \, dv_{\mathbb{H}^N} + b_{m,2} \int_{\mathbb{H}^N} \frac{u^2}{r^{2m+2}} \right),
$$

by which the existence of $2m + 1$ positive constants $\tilde{\delta}^i = \delta^i(N, m, h)$ such that (17) holds follows. Furthermore, one has

$$
\tilde{\delta}^1 := \left( \frac{N-1}{2} \right)^2 \tilde{a}_1 + \frac{1}{4} a_m \quad \text{and} \quad \tilde{\delta}^{2m+1} := \frac{1}{4} b_{m,2}
$$

and the thesis follows by recalling the definition of $\tilde{a}_1$ in Theorem 4.4 for $h = m - 1$ and $h \neq m - 1$. $\square$
5. **Proof of Theorem 2.1 and Corollary 2.2.** This section is devoted to the proofs of Theorem 2.1 and Corollary 2.2. The proof of Theorem 2.1 mainly relies on the transformation \( u \mapsto (\sinh r)^{\frac{(N-1)}{2}} u \), which uncovers the Poincaré term, and spherical harmonics technique. Before entering the proof we recall some facts on spherical harmonics.

The Laplace-Beltrami operator on hyperbolic space in spherical coordinates is given by

\[
\Delta_{HN} = \frac{\partial^2}{\partial r^2} + (N - 1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{S^{N-1}},
\]

where \( \Delta_{S^{N-1}} \) is the Laplace-Beltrami operator on the unit sphere \( S^{N-1} \). If we write \( u(x) = u(r, \sigma) \in C^\infty_c(\mathbb{H}^N), \ r \in [0, \infty), \ \sigma \in S^{N-1} \), then by [31, Ch.4, Lemma 2.18] we have that

\[
u(x) := u(r, \sigma) = \sum_{n=0}^{\infty} d_n(r) P_n(\sigma)
\]

in \( L^2(\mathbb{H}^N) \), where \( \{P_n\} \) is a complete orthonormal system of spherical harmonics and

\[
d_n(r) = \int_{S^{N-1}} u(r, \sigma) P_n(\sigma) \, d\sigma.
\]

We note that the spherical harmonic \( P_n \) of order \( n \) is the restriction to \( S^{N-1} \) of a homogeneous harmonic polynomial of degree \( n \). Now we recall the following

**Lemma 5.1.** [28, Lemma 2.1] Let \( P_n \) be a spherical harmonic of order \( n \) on \( S^{N-1} \). Then for every \( n \in \mathbb{N}_0 \)

\[
\Delta_{S^{N-1}} P_n = -(n^2 + (N - 2)n) P_n.
\]

The values \( \lambda_n := n^2 + (N - 2)n \) are the eigenvalues of the Laplace-Beltrami operator \( -\Delta_{S^{N-1}} \) on \( S^{N-1} \) and enjoy the property \( \lambda_n \geq 0 \) and \( \lambda_0 = 0 \). The corresponding eigenspace consists of all the spherical harmonics of order \( n \) and has dimension \( d_n \) where \( d_0 = 1 \), \( d_1 = N \) and

\[
d_n = \binom{N + n - 1}{n} - \binom{N + n - 3}{n - 2},
\]

for \( n \geq 2 \).

From Lemma 5.1 it is easy to see that

\[
\Delta_{HN} u(r, \sigma) = \sum_{n=0}^{\infty} \left( d''_n(r) + (N - 1) \coth r d'_n(r) - \frac{\lambda_n d_n(r)}{\sinh^2 r} \right) P_n(\sigma).
\]

In the sequel we will also exploit the following 1-dimensional Hardy-type inequality from [3]:

**Lemma 5.2.** For all \( u \in C^\infty_c(0, \infty) \) there holds

\[
\int_0^\infty \frac{u'^2}{\sinh^2 r} \, dr \geq \frac{9}{4} \int_0^\infty \frac{u^2}{\sinh^2 r} \, dr + \int_0^\infty \frac{u^2}{\sinh^2 r} \, dr.
\]

**Proof of Theorem 2.1.**

The proof is divided in several steps.

**Step 1.** For \( u \in C^\infty_c(\mathbb{H}^N) \) we define

\[
v(x) = (\sinh r)^{\frac{N-1}{2}} u(x), \text{ where } r = \rho(x, x_0)
\]
Then the following relation holds for \( x = (r, \sigma) \in (0, \infty) \times S^{N-1} \),

\[
|\nabla_{HN} u|^2 = (\sinh r)^{1-N} \left( |\nabla_{HN} v|^2 + \frac{(N-1)^2}{4} \coth^2 r v^2 - (N-1) \coth r \frac{\partial v}{\partial r} \right).
\]  

(18)

**Proof of Step 1.** A straightforward computation gives

\[
|\nabla_{HN} v|^2 = \left( \frac{\partial v}{\partial r} \right)^2 + \frac{1}{\sinh^2 r} |\nabla_{SN-1} v|^2
\]

\[
= \frac{(N-1)^2}{4} (\sinh r)^{N-1} \coth^2 r u^2 + |\nabla_{HN} u|^2 (\sinh r)^{N-1}
\]

\[
+ (N-1) (\sinh r)^{N-1} \coth r (\sinh r)^{\frac{N-1}{2}} \frac{\partial u}{\partial r}
\]

\[
= \frac{(N-1)^2}{4} \coth^2 r v^2 + |\nabla_{HN} u|^2 (\sinh r)^{N-1} - \frac{(N-1)^2}{2} \coth^2 r v^2
\]

\[
+ (N-1) \coth r \frac{\partial v}{\partial r}.
\]

Now by rearranging the terms above we conclude the proof of Step 1.

**Step 2.** In this step we compute,

\[
\Delta_{HN} v = \left( \frac{\partial^2}{\partial r^2} + (N-1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{SN-1} \right) (\sinh r)^{\frac{N-1}{2}} u.
\]

\[
\Delta_{HN} v = \frac{(N-1)(N-3)}{4} (\sinh r)^{\frac{N-1}{2}} \coth^2 r u + (N-1)(\sinh r)^{\frac{N-1}{2}} \cosh r \frac{\partial u}{\partial r}
\]

\[
+ \frac{(N-1)}{2} (\sinh r)^{\frac{N-1}{2}} u + (\sinh r)^{\frac{N-1}{2}} \frac{\partial^2 u}{\partial r^2} + \frac{(N-1)^2}{2} \coth^2 r (\sinh r)^{\frac{N-1}{2}} u
\]

\[
+ (N-1) \coth r (\sinh r)^{\frac{N-1}{2}} \frac{\partial u}{\partial r} + (\sinh r)^{\frac{N-1}{2}} \frac{1}{\sinh^2 r} \Delta_{SN-1} u
\]

\[
= (\sinh r)^{\frac{N-1}{2}} \left[ \frac{\partial^2 u}{\partial r^2} + (N-1) \coth r \frac{\partial u}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{SN-1} u \right]
\]

\[
+ \left( \frac{(N-1)(N-3)}{4} + \frac{(N-1)^2}{2} \right) \coth^2 r (\sinh r)^{\frac{N-1}{2}} u + \frac{(N-1)}{2} (\sinh r)^{\frac{N-1}{2}} u
\]

\[
+ (N-1)(\sinh r)^{\frac{N-3}{2}} \cosh r \left[ \frac{\partial}{\partial r} ((\sinh r)^{\frac{(N-1)}{2} v}) \right]
\]

\[
= (\sinh r)^{\frac{N-1}{2}} (\Delta_{HN} u) + \left( \frac{(N-1)(N-3)}{4} + \frac{(N-1)^2}{2} \right) \coth^2 r v
\]

\[
+ \frac{(N-1)}{2} v - \frac{(N-1)^2}{2} \coth^2 r v + (N-1) \coth r \frac{\partial v}{\partial r}
\]

\[
= (\sinh r)^{\frac{N-1}{2}} (\Delta_{HN} u) + \frac{(N-1)(N-3)}{4} \coth^2 r v + \frac{(N-1)}{2} v + (N-1) \coth r \frac{\partial v}{\partial r}.
\]
Hence, we get
\[
\Delta_{H^N} u = \frac{1}{(\sinh r)^\frac{N-1}{2}} \left[ \Delta_{H^N} v - \left( \frac{(N-1)(N-3)}{4} \coth^2 r + \frac{N-1}{2} \right) v - (N-1) \coth r \frac{\partial v}{\partial r} \right]
\]
\[
= \frac{1}{(\sinh r)^\frac{N-1}{2}} \left[ \frac{\partial^2 v}{\partial r^2} - \left( \frac{(N-1)(N-3)}{4} \coth^2 r + \frac{N-1}{2} \right) v + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{N-1}} v \right].
\]

(19)

**Step 3.** We recall that in hyperbolic polar coordinate, the volume element is given by:
\[
dv_{H^N} := (\sinh r)^{N-1} \, dr \, d\sigma, \quad \text{where} \ (r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}.
\]

Expanding \(v\) in spherical harmonics
\[
v(x) := v(r, \sigma) = \sum_{n=0}^{\infty} d_n(r) P_n(\sigma),
\]
we observe
\[
\int_{H^N} (\sinh r)^{-(N-1)} |\nabla_{H^N} v|^2 \, dv_{H^N} = \int_0^\infty \int_{\mathbb{S}^{N-1}} \left[ \left( \frac{\partial v}{\partial r} \right)^2 + \frac{1}{\sinh^2 r} |\nabla_{\mathbb{S}^{N-1}} v|^2 \right] \, d\sigma \, dr
\]
\[
= \sum_{n=0}^{\infty} \int_0^\infty \left( (d'_n(r))^2 + \lambda_n \frac{d_n^2(r)}{\sinh^2 r} \right) \, dr.
\]

(20)

Further expanding in spherical harmonics and using (19), we have
\[
\int_{H^N} |\Delta_{H^N} u|^2 \, dv_{H^N}
\]
\[
= \int_0^\infty \int_{\mathbb{S}^{N-1}} \left[ \frac{\partial^2 v}{\partial r^2} - \left( \frac{(N-1)(N-3)}{4} \coth^2 r + \frac{N-1}{2} \right) v + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{N-1}} v \right]^2 \, d\sigma \, dr
\]
\[
= \int_0^\infty \int_{\mathbb{S}^{N-1}} \sum_{n=0}^{\infty} \left( d''_n(r) - \frac{(N-1)(N-3)}{4} \coth^2 r d_n(r) - \frac{(N-1)}{2} d_n(r) - \lambda_n \frac{d_n^2(r)}{\sinh^2 r} \right)^2 P_n \, d\sigma \, dr
\]
\[
= \sum_{n=0}^{\infty} \int_0^\infty \left[ d''_n(r) - \frac{(N-1)(N-3)}{4} \coth^2 r d_n(r) - \frac{(N-1)}{2} d_n(r) - \lambda_n \frac{d_n^2(r)}{\sinh^2 r} \right]^2 \, dr.
\]

(21)

where the eigenvalues \(\lambda_n\) are repeated according to their multiplicity. Here we have used: \(\int_{\mathbb{S}^{N-1}} P_n(\sigma) d\sigma = 0\) for \(n \neq m\), \(\int_{\mathbb{S}^{N-1}} P_n^2(\sigma) \, d\sigma = 1\) and \(\Delta_{\mathbb{S}^{N-1}} P_n = -\lambda_n P_n\).
Let us write using (20) and (21),

\[
\int_{\mathbb{R}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{R}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \\
= \sum_{n=0}^{\infty} \left[ \int_0^{\infty} \left( d_n^2 - \frac{(N-1)(N-3)}{4} \coth^2 r d_n - \frac{(N-1)}{2} d_n - \frac{\lambda_n}{\sinh^2 r} d_n \right) \, dr \\
- \frac{(N-1)}{2} \int_0^{\infty} \left( (d_n')^2 + \frac{(N-1)^2}{4} \coth^2 r d_n^2 - (N-1) \coth r d_n d_n' \right) \, dr \right].
\]

Considering each term separately and simplifying further, for detail see the proof of [3, Theorem 3.1]), we get

\[
\int_0^{\infty} \left( d_n^2 - \frac{(N-1)(N-3)}{4} \coth^2 r d_n - \frac{(N-1)}{2} d_n - \frac{\lambda_n}{\sinh^2 r} d_n \right) \, dr \\
= \int_0^{\infty} \left( (d_n')^2 + \frac{(N-1)^2}{2} (d_n')^2 + \left( \frac{(N-1)(N-3)}{2} + 2\lambda_n \right) \frac{1}{\sinh^2 r} (d_n')^2 \right) \, dr \\
+ \frac{(N-1)^4}{16} \int_0^{\infty} (d_n')^2 \, dr + \left( \frac{(N-1)^2 (N-3)^2}{8} + \frac{(N-1)^2 (N-3)}{4} \right) \int_0^{\infty} \frac{1}{\sinh^2 r} (d_n')^2 \, dr \\
+ \frac{(N-1)(N-3)}{2} \lambda_n + (N-5) \lambda_n - (N-1)(N-3) \int_0^{\infty} \frac{1}{\sinh^2 r} (d_n')^2 \, dr (22)
\]

and

\[
\left( \frac{N-1}{2} \right)^2 \int_{\mathbb{R}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \left( \frac{N-1}{4} \right)^2 \int_0^{\infty} (d_n')^2 \, dr + \frac{(N-1)^4}{16} \int_0^{\infty} d_n^2 \, dr \\
+ \frac{(N-1)^3 (N-3)}{16} \int_0^{\infty} \frac{1}{\sinh^2 r} d_n^2 \, dr. \\
+ \frac{(N-1)^2}{4} \lambda_n \int_0^{\infty} \frac{d_n^2}{\sinh^2 r} \, dr (23)
\]

By (22) and (23), using Lemma 5.2, the 1-dimensional Hardy inequality:

\[
\int_0^{\infty} (d_n')^2 \, dr \geq \frac{1}{4} \int_0^{\infty} \frac{d_n^2}{r^2} \, dr
\]

and the 1-dimensional Rellich inequality:

\[
\int_0^{\infty} (d_n')^2 \, dr \geq \frac{9}{16} \int_0^{\infty} \frac{d_n^2}{r^4} \, dr,
\]

we obtain

\[
\int_{\mathbb{R}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N-1}{2} \right)^2 \int_{\mathbb{R}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq \frac{9}{16} \int_0^{\infty} \frac{d_n^2}{r^4} \, dr \\
+ \frac{(N-1)^2}{16} \int_0^{\infty} \frac{d_n^2}{r^2} \, dr + A_n \int_0^{\infty} \frac{d_n^2}{\sinh^2 r} \, dr + B_n \int_0^{\infty} \frac{d_n^2}{\sinh^2 r} \, dr,
\]

where

\[
A_n = \lambda_n^2 + \frac{N(N-4)}{2} \lambda_n + \frac{(N-1)(N-3)^2}{16} - \frac{3}{8} (N-1)(N-3)
\]

and

\[
B_n = \frac{(N^2 - 2N - 7)}{4} \lambda_n + \frac{(N-1)^2 (N-3)}{4} + \frac{(N-1)^2 (N-3)(N-5)}{16} - \frac{(N-1)(N-3)}{2}.
\]
We note that
\[
\min_{n \in \mathbb{N}_0} A_n = \frac{(N - 1)(N - 3)(N^2 - 4N - 3)}{16} \quad \text{and} \quad \min_{n \in \mathbb{N}_0} B_n = \frac{(N - 1)(N - 3)(N^2 - 2N - 7)}{16}
\]
for \( N \geq 5 \) and hence they are both positive. Also we have
\[
\int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} v^2 (\sinh r)^{-(N-1)} \, dv_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \int_0^{\infty} d_n^2 \, dr,
\]
similarly
\[
\int_{\mathbb{H}^N} u^2 r^2 \, dv_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d_n^2}{r^2} \, dr
\]
and so on. Now, using all these facts, we obtain
\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq 9 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{(N - 1)(N - 3)(N^2 - 4N - 3)}{16} \int_0^{\infty} \frac{u^2}{\sinh^4 r} \, dv_{\mathbb{H}^N}
\]
and hence the proof of inequality (5).

**Step 4.** Next we show the optimality of the constant \( \frac{(N-1)^2}{16} \). Let us suppose that the \( \frac{(N-1)^2}{16} \) is not optimal, i.e., there exist \( C > \frac{(N-1)^2}{16} \) such that there holds,
\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left( \frac{N - 1}{2} \right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} \geq C \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N},
\]
using [3, Theorem 2.1] and above we obtain
\[
\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} \geq C \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{(N - 1)^2}{4} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N}
\]
\[
= \left( C + \frac{(N - 1)^2}{16} \right) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{(N - 1)^4}{16} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N},
\]
comparing (24) with [3, Theorem 3.1], we conclude that \( C \leq \frac{(N-1)^2}{16} \) which gives a contradiction and hence \( \frac{(N-1)^2}{16} \) is the best constant. \( \square \)

**Proof of Corollary 2.2.** By considering the upper half space model \( \mathbb{R}^N_+ \) for \( \mathbb{H}^N \) and using the explicit expression of the gradient in these coordinates, namely
\[ \nabla_{HN} = y^2 \nabla, \text{ we obtain} \]
\[ \int_{\mathbb{H}^N} |\nabla_{HN} u(x,y)|^2 \, dv_{HN} \]
\[ = \int_{R^+} \int_{\mathbb{R}^{N-1}} y^{2\alpha+2-N} |\nabla v|^2 \, dx \, dy + \alpha^2 \int_{R^+} \int_{\mathbb{R}^{N-1}} y^{2\alpha-N} v^2 \, dx \, dy \]
\[ - \alpha (2\alpha + 1 - N) \int_{R^+} \int_{\mathbb{R}^{N-1}} y^{2\alpha-N} v^2 \, dx \, dy \]  
(25)

and also using Laplacian expression, \( \Delta_{HN} = y^2 \Delta - (N-2)y \), we get
\[ \Delta_{HN} u = y^{\alpha+2} \Delta v + (2\alpha - (N-2)) y^\alpha \frac{\partial v}{\partial y} + \alpha (\alpha - (N-1)) y^\alpha v, \]  
(26)

where and \( u(x,y) := y^\alpha v(x,y) \). With \( \alpha = \frac{N-2}{2} \), we get
\[ \int_{\mathbb{H}^N} (\Delta_{HN} u(x,y))^2 \, dv_{HN} \]
\[ = \int_{R^+ \times \mathbb{R}^{N-1}} \left( y^{N+1} (\Delta v)^2 + \frac{N^2(N-2)^2}{16} y^{N-2} v^2 - \frac{N(N-2)}{2} y^N v \Delta v \right) \, dx \, dy \]
\[ = \int_{\mathbb{R}^{N-1}} \left( y^2 (\Delta v)^2 + \frac{N^2(N-2)^2}{16} \frac{v^2}{y^2} + \frac{N(N-2)}{2} y^N |\nabla v|^2 \right) \, dy \, dx. \]

Similarly with \( \alpha = \frac{N-4}{2} \), and by denoting \( \frac{\partial v}{\partial y} := v_y \), we get
\[ \int_{\mathbb{H}^N} (\Delta_{HN} u(x,y))^2 \, dv_{HN} \]
\[ = \int_{R^+ \times \mathbb{R}^{N-1}} \left( (\Delta v)^2 + \frac{v_y^2}{y^2} + \frac{(N-4)(N+2)^2 v^2}{16} \frac{v^2}{y^2} \right) \, dy \, dx \]
\[ - 4 v_y \Delta v - \frac{(N-4)(N+2)}{2} \frac{v \Delta v}{y^2} + (N-4)(N+2) \frac{v y v_y}{y^3} \, dx \, dy \]
\[ = \int_{R^+ \times \mathbb{R}^{N-1}} \left( (\Delta v)^2 + \frac{v_y^2}{y^2} + \frac{(N-4)^2(N+2)^2 v^2}{16} \frac{v^2}{y^2} \right) \, dy \, dx \]
\[ - 4 \frac{v_y^2}{y^2} + 2 \frac{|\nabla v|^2}{y^2} + \frac{(N-4)(N+2) |\nabla v|^2}{2} \frac{v^2}{y^2} \right) \, dy \, dx \]
\[ = \int_{R^+ \times \mathbb{R}^{N-1}} \left( (\Delta v)^2 + \frac{(N-4)^2(N+2)^2 v^2}{16} \frac{v^2}{y^2} + 2 \frac{|\nabla v|^2}{y^2} + \frac{(N-4)(N+2) |\nabla v|^2}{2} \frac{v^2}{y^2} \right) \, dy \, dx. \]

Now the proof follows by inserting (25), (26) and the above computations in (5) with \( \alpha = \frac{N-2}{2} \) and \( \alpha = \frac{N-4}{2} \) successively.

Next we turn to the optimality issues. Assume by contradiction that the following inequality holds
\[ \int_{R^+} \int_{\mathbb{R}^{N-1}} (y^2 (\Delta v)^2 + c |\nabla v|^2) \, dx \, dy \geq \frac{N(N-2)}{16} \int_{R^+} \int_{\mathbb{R}^{N-1}} v^2 \, dx \, dy \]
for all \( u \in C^\infty_c(\mathbb{H}^N) \) with \( c < \frac{N^2-2N-1}{4} \). The above inequality, jointly with (1) with \( k = 1, l = 0 \) and Hardy-Maz'ya inequality:
\[ \int_{R^+} \int_{\mathbb{R}^{N-1}} |\nabla v|^2 \, dx \, dy \geq \frac{1}{4} \int_{R^+} \int_{\mathbb{R}^{N-1}} \frac{v^2}{y^2} \, dx \, dy, \]
where $\frac{1}{4}$ is the best constant, see [25] and [16, 18], yields

$$\int_{\mathbb{R}^N} |\nabla H^N u|^2 \, dv_{H^N} \geq \frac{(N-1)^2}{4} \int_{\mathbb{R}^N} |\nabla^2 H^N u|^2 \, dv_{H^N} + \left( \frac{N^2 - 2N - 1}{4} - c \right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \, dy$$

$$\geq \frac{(N-1)^4}{16} \int_{\mathbb{R}^N} u^2 \, dv_{H^N} + \frac{1}{4} \left( \frac{N^2 - 2N - 1}{4} - c \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N-1} \frac{u^2}{y^2} \, dx \, dy$$

$$= \left( \frac{(N-1)^4}{16} + \frac{1}{4} \left( \frac{N^2 - 2N - 1}{4} - c \right) \right) \int_{\mathbb{R}^N} u^2 \, dv_{H^N}$$

a contradiction with (1) with $k = 2$ and $l = 0$. The optimality of the other constants follows straightforwardly from what remarked above. \( \Box \)

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**REFERENCES**

[1] S. Agmon, *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations: Bounds on Eigenfunctions of N-body Schrödinger Operators*, Math. Notes, vol. 29, Princeton University Press, Princeton, 1982.

[2] K. Akutagawa, H. Kumura, *Geometric relative Hardy inequalities and the discrete spectrum of Schrödinger operators on manifolds*, Calc. Var. Part. Diff. Eq., 48 (2013), 67–88.

[3] E. Berchio, D. Ganguly and G. Grillo, *Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hyperbolic space*, Preprint 2015, arXiv:1507.02550v2.

[4] G. Barbatis, S. Filippas and A. Tertikas, *A unified approach to improved $L^p$ Hardy inequalities with best constants*, Trans. Amer. Soc., 356 (2004), 2169–2196.

[5] G. Barbatis, S. Filippas and A. Tertikas, *Series expansion for $L^p$ Hardy inequalities*, Indiana Univ. Math. J., 52 (2003), 171–190.

[6] G. Barbatis and A. Tertikas, *On a class of Rellich inequalities*, J. Comput. Appl. Math., 194 (2006), 156–172.

[7] B. Bianchini, L. Mari and M. Rigoli, *Yamabe type equations with a sign-changing nonlinearity, and the prescribed curvature problem*, J. Funct. Anal., 268 (2015), 1–72.

[8] Y. Bozhkov and E. Mitidieri, *Conformal Killing vector fields and Rellich type identities on Riemannian manifolds*, I Lecture Notes of Seminario Interdisciplinare di Matematica, 7 (2008), 65–80.

[9] Y. Bozhkov and E. Mitidieri, *Conformal Killing vector fields and Rellich type identities on Riemannian manifolds II*, Mediterr. J. Math., 9 (2012), 1–20.

[10] H. Brezis and M. Marcus, *Hardy’s inequalities revisited*, Ann. Scuola Norm. Sup. Cl. Sci., 25 (1997), 217–237.

[11] H. Brezis and J. L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid, 10 (1997), 443–469.

[12] G. Carron, *Inegalités de Hardy sur les varietes Riemanniennes non-compactes*, J. Math. Pures Appl., 76 (1997), 883–891.

[13] L. D’Ambrosio and S. Dipierro, *Hardy inequalities on Riemannian manifolds and applications*, Ann. Inst. H. Poinc. Anal. Non Lin., 31 (2014), 449–475.

[14] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L^p(\Omega)$*, Math. Z., 227 (1998), 511–523.
[15] B. Devyver, M. Fraas and Y. Pinchover, Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon, *J. Funct. Anal.*, **266** (2014), 4422–4489.

[16] S. Filippas, L. Moschini and A. Tertikas, Sharp trace Hardy-Sobolev-Maz’ya inequalities and the fractional Laplacian, *Arch. Ration. Mech. Anal.*, **208** (2013), 109–161.

[17] S. Filippas and A. Tertikas, Optimizing improved Hardy inequalities, *J. Funct. Anal.*, **192** (2002), 186–233.

[18] S. Filippas, A. Tertikas and J. Tidblom, On the structure of Hardy-Sobolev-Maz’ya inequalities, *J. Eur. Math. Soc.*, **11** (2009), 1165–1185.

[19] F. Gazzola, H. Grunau and E. Mitidieri, Hardy inequalities with optimal constants and remainder terms, *Trans. Amer. Math. Soc.*, **356** (2004), 2149–2168.

[20] N. Ghoussoub and A. Moradifam, Bessel pairs and optimal Hardy and Hardy-Rellich inequalities, *Math. Ann.*, **349** (2011), 1–57.

[21] I. Kombe and M. Ozaydin, Improved Hardy and Rellich inequalities on Riemannian manifolds, *Trans. Amer. Math. Soc.*, **361** (2009), 6191–6203.

[22] I. Kombe and M. Ozaydin, Rellich and uncertainty principle inequalities on Riemannian manifolds, *Trans. Amer. Math. Soc.*, **365** (2013), 5035–5050.

[23] D. Karmakar and K. Sandeep, Adams Inequality on the Hyperbolic space, *J. Funct. Anal.*, **270** (2016), 1792–1817.

[24] P. Li and J. Wang, Weighted Poincaré inequality and rigidity of complete manifolds, *Ann. Sci. École Norm. Sup.*, **39** (2006), 921–982.

[25] V. G. Maz’ya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.

[26] G. Mancini and K. Sandeep, On a semilinear equation in \( \mathbb{R}^n \), *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **5** vol.VII (2008), 635–671.

[27] M. Marcus, V. J. Mizel and Y. Pinchover, On the best constant for Hardy’s inequality in \( \mathbb{R}^n \), *Trans. Am. Math. Soc.*, **350** (1998), 3237–3255.

[28] G. Metafune, M. Sobajima and C. Spina, Weighted Calderón-yngmund and Rellich inequalities in \( L^p \), *Math. Ann.*, **361** (2015), 313–366.

[29] E. Mitidieri, A simple approach to Hardy inequalities, *Mat. Zametki*, **67** (2000), 563–572.

[30] F. Rellich, Halbbeschrankte differential operatoren herer Ordnung, *Proceedings of the International Congress of Mathematicians III* (1954), 243-250.

[31] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 32, Princeton (1971).

[32] A. Tertikas and N. B. Zographopoulos, Best constants in the Hardy-Rellich inequalities and related improvements, *Adv. Math.*, **209** (2007), 407–459.

[33] Q. Yang, D. Su and Y. Kong, Hardy inequalities on Riemannian manifolds with negative curvature, *Commun. Contemp. Math.*, **16** (2014), 1350043.

[34] J. L. Vazquez, Fundamental solution and long time behaviour of the Porous medium equation in hyperbolic space, *J. Math. Pures Appl.*, **104** (2015), 454–484.

[35] J. L. Vazquez and E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.*, **173** (2000), 103–153.

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