Deciding football sequences

Dedicated to the memory of Antal Bege (1962–2012)

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Abstract. An open problem posed by the first author [36, 53, 54, 59, 100] is the complexity to decide whether a sequence of nonnegative integer numbers can be the final score of a football tournament. In this paper we propose polynomial time approximate and exponential time exact algorithms which solve the problem.

1 Introduction

Let a, b and n be nonnegative integers (b ≥ a ≥ 0, n ≥ 1), \( T(a, b, n) \) be the set of directed multigraphs \( T = (V, E) \), where \( |V| = n \), and each pair of different vertices \( u, v \in V \) are connected with at least \( a \) and at most \( b \) arcs [56, 57]. \( T \in T(a, b, n) \) is called \((a, b, n)\)-tournament. \((1, 1, n)\)-tournaments are the usual tournaments, and \((0, 1, n)\)-tournaments are also called oriented graphs or simple directed graphs [45, 93]. The set \( T \) is defined by

\[
T = \bigcup_{b \geq a \geq 0, \ n \geq 1} T(a, b, n).
\]

The definition of (undirected) \((a, b, n)\)-graphs is similar. The \((0, 1, n)\)-graphs are the usual simple graphs.

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An \((a, b, n)\)-tournament is called complete, if the set of permitted results is \(\{0 : c, 1 : c - 1, \ldots, c : 0\}\) for all possible \(c\) \((a \leq c \leq b)\). If some of these results are prohibited, then the tournament is called incomplete \([55, 56, 57]\).

For example football is an incomplete \((2,3,n)\)-tournament since the permitted results are \(0 : 3, 1 : 1\) and \(3 : 0\), while \(0 : 2, 1 : 2, 2 : 0\), and \(2 : 1\) are prohibited.

According to this definition \(\mathcal{T}\) is the set of the finite directed loopless multigraphs. We remark, that if \(a' \leq a \leq b \leq b'\) then an \((a, b, n)\)-tournament is also an \((a', b', n)\)-tournament. The outdegree sequence of an \((a, b, n)\)-tournament we call the score sequence of the tournament \([45, 93, 95]\).

Let \(l, u,\) and \(m\) be integer numbers with \(u \geq l\) and \(m \geq 1\). The sequence \(s = (s_1, \ldots, s_m)\) of integer numbers with \(l \leq s_1 \leq \cdots \leq s_m \leq u\) is called \((l,u,m)\)-regular. It is well-known that the number of \((l,u,m)\)-regular sequences is

\[
R(l, u, m) = \left(\frac{u - l + m}{m}\right).
\]

In this paper we consider only the graph theoretical aspects of the investigated problems, although they have many applications \([1, 16, 17, 68, 76, 88, 108]\). We analyze only sequential algorithms. The Reader can find parallel results e.g. in \([2, 29, 92, 102, 104]\).

The structure of the paper is as follows. After this introduction in Section 2 we deal with the filtering of potential complete sequences, then in Section 3 describe incomplete sequences. Section 4 contains filtering and Section 5 reconstruction algorithms of potential football sequences. Finally in Section 6 we deal with the enumeration of football sequences.

2 Filtering of potential complete sequences

We are seeking football sequences. Taking into account that a score sequence of an incomplete \((a, b, n)\)-tournament is at the same time a score sequence of the complete \((a, b, n)\)-tournament, the properties of score sequences of complete tournaments allow some filtering among the regular sequences.

In 1953 Landau \([75]\) proved the following popular theorem. About ten proofs are summarized by Reid \([95]\). Further proofs are in \([3, 18, 19, 20, 22, 44, 46, 101, 106, 110]\).

**Theorem 1** (Landau \([75]\)) \(A (0, n - 1, n)\)-regular sequence \(s = (s_1, \ldots, s_n)\) is...
the outdegree sequence of some \((1, 1, n)\)-tournament if and only if

\[
\sum_{i=1}^{k} s_i \geq \frac{k(k - 1)}{2}, \quad 1 \leq k \leq n,
\]

with equality when \(k = n\).

**Proof.** See [64, 75, 86].

Moon [85] proved the following generalization of Landau’s theorem (we present it in reformulated form). Later Takahashi [107] reproved the theorem.

**Theorem 2** (Moon [85]) A \((0, b(n - 1), n)\)-regular sequence \(s = (s_1, \ldots, s_n)\) is the score sequence of some \((b, b, n)\)-tournament if and only if

\[
\sum_{i=1}^{k} s_i \geq \frac{bk(k - 1)}{2}, \quad 1 \leq k \leq n,
\]

with equality when \(k = n\).

**Proof.** See [85].

We define a point-loss function \(P_k\) \((k = 0, \ldots, n)\) by the following recursion: \(P_0 = 0\) and if \(1 \leq k \leq n\), then

\[
P_k = \max \left( P_{k-1}, \frac{bk(k - 1)}{2} - \sum_{i=1}^{k} s_i \right).
\]

Now \(P_k\) gives a lower bound for the number of lost points in the matches among the teams \(T_1, \ldots, T_k\) (not the exact value since the teams \(T_1, \ldots, T_k\) could win points against \(T_{k+1}, \ldots, T_n\)).

**Theorem 3** (Iványi [56]) A \((0, b(n - 1), n)\)-regular sequence \(s = (s_1, \ldots, s_n)\) is the score sequence of some complete \((a, b, n)\)-tournament if and only if

\[
\frac{ak(k - 1)}{2} \leq \sum_{i=1}^{k} s_i \leq \frac{bn(n - 1)}{2} - P_k - (n - k)s_k \quad (1 \leq k \leq n).
\]

**Proof.** See [56].
3 Incomplete tournaments

We know only the following three results on the score sequences of incomplete tournaments.

Semicomplete digraphs (semicomplete tournaments) are defined as $(1, 2, n)$-digraphs in which if two vertices are connected with two arcs then these arcs have different directions.

**Theorem 4** (Reid, Zhang [96]) A $(0, n-1, n)$-regular sequence $s = (s_1, \ldots, s_n)$ is the score sequence of some semicomplete tournament if and only if

$$\sum_{i=1}^{k} s_i \geq \frac{k(k-1)}{2} \quad \text{and} \quad s_k \leq n - 1, \quad 1 \leq k \leq n.$$  

(3)

**Proof.** See [96].

Antal Bege asked in 1999 [7] how many wins are necessary in a football tournament of $n$ teams to get a strictly monotone score sequence. If $n = 2$ then 1, if $n = 3$ then 1, and if $n = 4$ then 2 are sufficient and necessary. The following assertion gives the general answer.

**Theorem 5** (Iványi [55]) If $N(n)$ denotes the minimal number of necessary and sufficient wins for different scores in a football tournament of $n$ teams then

$$N(n) = \left(\frac{3}{2} - \sqrt{2}\right)n^2 + \Theta(n).$$  

(4)

Recently Berger [10] published the following criterion for special incomplete $(0, 2, n)$-tournaments.

**Theorem 6** (Berger [10]) Sequence $\sigma = \left(\left(a_1, b_1\right), \ldots, \left(a_n, b_n\right)\right)$ with $a_1 \geq \cdots \geq a_n$ is the score sequence of special incomplete $(0, 2, n)$-tournaments—in which $0 : 0$, $0 : 1$, $1 : 0$, and $1 : 1$ are the permitted results—if and only if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \min(b_i, k) + \sum_{i=k+1}^{n} \min(b_i, k) \quad \text{for all} \quad k = 1, \ldots, n, \quad \text{with equality for} \quad n.$$  

(5)

**Proof.** See [10].

Earlier (weaker) results can be found in [23, 41, 42, 99].
4 Filtering of potential football sequences

There are many exact results deciding whether a given sequence is the degree/outdegree sequence of a given type of undirected (e.g. \([24, 25, 30, 31, 32, 33, 34, 35, 43, 46, 47, 48, 50, 51, 61, 69, 74, 83, 84, 90, 116, 118]\)) or directed (e.g. \([12, 13, 75, 85, 56, 57, 94, 115]\)) graphs. Several authors studied the case when the indegree and outdegree sequences are together prescribed \([9, 10, 14, 33, 48, 91]\).

The score sequences of the football tournaments we call football sequences.

A \((0, 3n - 3, n)\)-regular sequence \(s = (s_1, \ldots, s_n)\) is called good if there exists a football tournament whose score sequence is \(s\), and \(s\) is called bad otherwise. We denote the football sequences by \(f = (f_1, \ldots, f_n)\).

In this section we present approximate algorithms which filter only some part of the bad sequences. Since these filtering algorithms have short running time they help to reduce the expected running time of the exact algorithms.

The filtering algorithms are classified according to their worst running time as constant, linear, and other polynomial type ones.

4.1 Constant time filtering algorithms

The expected running time can be substantially decreased if we can filter some part of the investigated sequences in constant time.

Let \(n \geq 2\) and \(f = (f_1, \ldots, f_n)\) a football sequence.

**Lemma 7** (C1 test) \(f_n \neq 3n - 4\).

**Proof.** If a team wins all matches then its score is \(3n - 3\). If not, then it loses at least two points making a draw, so its score is at most \(3n - 5\). \(\square\)

**Lemma 8** (C2 test) If \(f_n = 3n - 3\) then \(f_{n-1} \leq 3n - 6\).

**Proof.** \(f_n\) can be \(3n - 3\) only so, that \(T_n\) wins all matches. Then the score \(T_{n-1}\) is at most \(3n - 6\). \(\square\)

**Lemma 9** (C3 test) If \(f_1 = 0\) then \(f_2 \geq 3\).

**Proof.** If \(f_1 = 0\) then \(T_1\) lost all matches therefore \(T_2\) has at least one win and so \(f_2\) is at least 3. \(\square\)

**Lemma 10** (C4 test) If \(f_1 = f_2 = 1\) then \(f_3 \geq 6\).
Proof. If \( f_1 = f_2 = 1 \) then the match of \( T_1 \) and \( T_2 \) ended with a draw implying that \( T_3 \) has at least two wins and so at least six points. \( \square \)

**Lemma 11** (C5 test) If \( f_1 = f_{n-1} = 3n - 5 \), then \( f_{n-2} \leq 3n - 9 \).

**Proof.** If the joint score of \( T_n \) and \( T_{n-1} \) is \( 3n - 5 \) then the result of their match has to be a draw. In this case \( T_{n-2} \) lost at least two matches and so \( f_{n-2} \leq 3n - 9 \). \( \square \)

**Lemma 12** (C6 test) If \( f_n = 3n - 3 \) and \( f_{n-1} = 3n - 6 \), then \( f_{n-2} \leq 3n - 9 \).

**Proof.** If \( f_n = 3n - 3 \), then \( T_n \) won all matches. In this case the score of \( T_{n-1} \) can be \( 3n - 6 \) only then if \( T_{n-1} \) loses against \( T_n \) but wins all remaining matches. Then \( T_{n-2} \) lost at least two matches and so \( f_{n-2} \leq 3n - 9 \). \( \square \)

**Lemma 13** (C7 test) If \( f_1 = 0 \) and \( f_2 = 3 \) then \( f_3 \geq 6 \).

**Proof.** See the proof of Lemma 12. \( \square \)

**Lemma 14** (C8 test) If \( f_1 = 1 \) and \( f_2 = 2 \) then \( f_3 \geq 4 \).

**Proof.** Since \( T_1 \) and \( T_2 \) gathered points only with draws their match ended with a draw. Therefore \( T_3 \) won against \( T_1 \) and either won against \( T_2 \) or they made a draw, so \( T_3 \) has at least 4 points. \( \square \)

**Lemma 15** (C9 test) If \( f_n = 3n - 5 \) and \( f_{n-1} = 3n - 7 \) then \( f_{n-2} \leq 3n - 8 \).

**Proof.** If \( f_n = 3n - 5 \) then \( T_n \) has a draw and \( n - 2 \) wins. If \( f_{n-1} = 3n - 7 \) then \( T_{n-1} \) has two draws and \( n - 3 \) wins, and the match between \( T_n \) and \( T_{n-1} \) ended with a draw. In this case \( T_{n-2} \) has at least a loss and a draw implying \( f_{n-2} \leq 3n - 8 \). \( \square \)

The following program \texttt{CONSTANT} realizes the tests of the previous 9 lemmas. This and later programs are written using the pseudocode conventions described in [27]. In this and in the further pseudocodes input variables are \( n \): the length of the investigated sequence \( (n \geq 3) \); \( s = (s_1, \ldots, s_n) \): a \((0, 3n - 3, n)\)-regular sequence; output variable is \( L \): \( L = 0 \) means that the investigated input is bad, \( L = 1 \) means that it is good while \( L = 2 \) shows that the given algorithm could not decide.
CONSTANT((n, s))
01 \text{L} = 0 \quad \text{// line 01: initialization of L}
02 \text{if } s_n == 3n - 4 \quad \text{// line 02-03: C1}
03 \quad \text{return } \text{L}
04 \text{if } s_n == 3n - 3 \text{ and } s_{n-1} \geq 3n - 5 \quad \text{// line 04-05: C2}
05 \quad \text{return } \text{L}
06 \text{if } s_1 == 0 \text{ and } s_2 \leq 2 \quad \text{// line 06-07: C3}
07 \quad \text{return } \text{L}
08 \text{if } s_1 == 1 \text{ and } s_2 == 1 \text{ and } s_3 \leq 5 \quad \text{// line 08-09: C4}
09 \quad \text{return } \text{L}
10 \text{if } s_n == 3n - 5 \text{ and } s_{n-1} == 3n - 5 \text{ and } s_{n-2} \geq 3n - 8 \quad \text{// line 10-11: C5}
11 \quad \text{return } \text{L}
12 \text{if } s_n == 3n - 3 \text{ and } s_{n-2} == 3n - 6 \text{ and } s_{n-3} \geq 3n - 8 \quad \text{// line 12-13: C6}
13 \quad \text{return } \text{L}
14 \text{if } s_1 == 0 \text{ and } s_2 == 3 \text{ and } s_3 \leq 5 \quad \text{// line 14-15: C7}
15 \quad \text{return } \text{L}
16 \text{if } s_1 == 1 \text{ and } s_2 == 2 \text{ and } s_3 \leq 3 \quad \text{// line 16-17: C8}
17 \quad \text{return } \text{L}
18 \text{if } s_n == 3n - 5 \text{ and } s_{n-1} == 3n - 7 \text{ and } s_{n-2} \geq 3n - 8 \quad \text{// line 18-19: C9}
19 \quad \text{return } \text{L}
20 \text{L} = 2 \quad \text{// line 20-21: these tests can not decide}
21 \quad \text{return } 2

Tables 1, 2, and 3 show the filtering results of CONSTANT. The numbers in the tables show how many sequences are accepted from the sequences accepted by the previous filtering algorithm. The exact results in these tables are printed with bold font (such emphasizing will be used in the later tables too).

The programs are written in C by Loránd Lucz and run on an Inter Core i7 processor (3.4 GHz) with optimization level O3. The running times are given in seconds.

Table 2 shows the filtering results of C4, C5, C6 and C7.

Table 3 shows the filtering results of algorithms C8 and C9, further the number of football sequences (F) and the running time of LINEAR for $n = 1, \ldots, 15$ teams. Column $R$ in Table 1 and column $t$ in Table 3 show that the running time is approximately proportional with the number of the regular sequences.

For example if $n = 2$ then C1, C2 and C3 filter $80\%$ of the regular and $100\%$ of the bad sequences. If $n = 3$ then they filter 54 from the 84 regular sequences while C1, \ldots, C9 filter 70 sequences which represent $90.90\%$ of the
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Table 1: Number of \((0, 3n - 3, n)\)-regular sequences accepted by C1, C2, and C3 for \(n = 1, \ldots, 15\) teams.

| \(n\) | \(R\) | C1 | C2 | C3 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 10 | 7 | 4 | 2 |
| 3 | 84 | 63 | 45 | 30 |
| 4 | 715 | 550 | 414 | 311 |
| 5 | 6,188 | 4,823 | 3,718 | 2,911 |
| 6 | 54,264 | 42,636 | 33,320 | 26,650 |
| 7 | 480,700 | 379,753 | 299,421 | 242,624 |
| 8 | 4,292,145 | 3,404,115 | 2,700,775 | 2,207,800 |
| 9 | 38,567,375 | 30,678,375 | 24,452,220 | 20,116,030 |
| 10 | 348,330,136 | 277,722,676 | 222,146,496 | 183,629,160 |
| 11 | 3,159,461,960 | 2,523,716,572 | 2,024,386,180 | 1,679,655,640 |
| 12 | 28,760,021,745 | 23,008,017,396 | 18,498,140,232 | 15,394,304,500 |
| 13 | 262,596,783,764 | 210,345,382,913 | 169,436,070,190 | 141,355,053,635 |
| 14 | 2,403,979,904,200 | 1,927,719,734,500 | 1,555,302,958,664 | 1,300,210,775,786 |
| 15 | 22,057,981,462,440 | 17,704,432,899,590 | 14,303,680,429,990 | 11,978,596,958,384 |

Table 2: Number of \((0, 3n - 3, n)\)-regular sequences accepted by C4, C5, C6, and C7 for \(n = 1, \ldots, 15\) teams.

| \(n\) | C4 | C5 | C6 | C7 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 |
| 3 | 26 | 22 | 19 | 17 |
| 4 | 281 | 255 | 237 | 222 |
| 5 | 2,691 | 2,501 | 2,374 | 2,271 |
| 6 | 24,000 | 23,373 | 22,302 | 21,596 |
| 7 | 227,770 | 215,227 | 207,042 | 200,609 |
| 8 | 2,700,775 | 2,207,800 | 2,097,803 | 1,972,783 |
| 9 | 19,155,258 | 18,065,694 | 17,460,916 | 16,989,609 |
| 10 | 175,138,885 | 165,526,269 | 160,206,767 | 156,070,967 |
| 11 | 1,591,808,376 | 1,518,358,621 | 1,471,133,714 | 1,434,460,309 |
| 12 | 14,605,778,836 | 13,947,629,921 | 13,524,714,862 | 13,196,925,716 |
| 13 | 134,230,657,710 | 128,305,394,396 | 124,497,616,840 | 121,549,435,860 |
| 14 | 1,235,669,598,354 | 1,181,962,750,733 | 1,147,511,569,252 | 1,208,609,923,538 |
| 15 | 11,391,620,617,874 | 10,903,053,416,141 | 10,590,098,238,918 | 10,348,178,700,655 |

If \(n = 15\) then the nine constant time algorithms filter 54.73\% of the bad sequences. This is surprisingly high efficiency but smaller than the sum of the individual asymptotic efficiency of the 9 algorithms. The reason is simple: e. g. the sequence \(s = (0, 0, 5)\) would be filtered by C1 and C3 too.
Table 3: Number of \((0, 3n - 3, n)\)-regular sequences accepted by C8 and C9, the number of football sequences (F), and the running time (t) of C9 for \(n = 1, \ldots, 15\) teams.

| n  | C8     | C9     | F     | t     |
|----|--------|--------|-------|-------|
| 2  | 2      | 2      | 2     | 0.000 |
| 3  | 15     | 14     | 7     | 0.000 |
| 4  | 200    | 203    | 40    | 0.000 |
| 5  | 2175   | 2133   | 355   | 0.000 |
| 6  | 20039  | 20510  | 3678  | 0.000 |
| 7  | 194333 | 191707 | 37263 | 0.016 |
| 8  | 1795074| 1772842| 361058| 0.062 |
| 9  | 16524335| 16332091| 3403613| 0.499 |
| 10 | 15436149| 150288309| 31653777| 4.602 |
| 11 | 1398051547| 1383099467| 292547199| 41.771 |
| 12 | 12870899770| 12737278674| 2696619716| 380.984 |
| 13 | 118612802828| 117411184292| 3489299| 34079.254 |
| 14 | 1094282911155| 1083421567482| 316965954| 316965954 |

4.2 Efficiency of the constant time testing algorithms

Using (1) we give the efficiency of the nine constant time filtering algorithms.

Lemma 16 (efficiency of C1) The ratio of sequences with \(s_n = 3n - 4\) among \((0, 3n - 3, n)\)-regular sequences is

\[
\frac{\binom{4n-5}{n-1}}{\binom{4n-3}{n}} = \frac{n(3n-3)}{(4n-4)(4n-3)} = \frac{3}{16} + \frac{9}{16(4n-3)} = \frac{3}{16} + o(1). \quad (6)
\]

Proof. The sequences satisfying the given condition are such \((0, 3n - 3, n)\)-regular ones, whose lower bound is \(l = 0\), upper bound is \(u = 3n - 4\), and contain \(m = n - 1\) elements. So according to (1) the required ratio is

\[
\frac{R(0, 3n - 4, n - 1)}{R(0, 3n - 3, n)} = \frac{n(3n-3)}{(4n-4)(4n-3)} = \frac{3}{16} + o(1). \quad (7)
\]

Lemma 17 (efficiency of C2) The ratio of the sequences satisfying the conditions \(s_n = 3n - 3\) and \(s_{n-1} \geq 3n - 5\) among the \((0, 3n - 3, n)\)-regular sequences is

\[
\frac{37}{256} + o(1). \quad (8)
\]
Proof. Since \( R(0, 3n - 3, n - 2) \) sequences satisfy the conditions \( s_n = 3n - 3 \) and \( s_{n-1} = 3n - 3 \), the corresponding ratio is

\[
\frac{R(0, 3n - 3, n - 2)}{R(0, 3n - 3, n)} = \frac{n(n - 1)}{(4n - 4)(4n - 3)} = \frac{1}{16} + o(1). \tag{9}
\]

\( R(0, 3n - 4, n - 2) \) sequences satisfy \( s_n = 3n - 3 \) and \( s_{n-1} = 3n - 4 \), so the corresponding ratio is

\[
\frac{R(0, 3n - 4, n - 2)}{R(0, 3n - 3, n)} = \frac{n(n - 1)(3n - 3)}{(4n - 3)(4n - 4)(4n - 5)} = \frac{3}{64} + o(1). \tag{10}
\]

\( R(0, 3n - 5, n - 2) \) sequences have the properties \( s_n = 3n - 3 \) and \( s_{n-1} = 3n - 5 \), so the corresponding ratio is

\[
\frac{R(0, 3n - 5, n - 2)}{R(0, 3n - 3, n)} = \frac{n(n - 1)(3n - 3)(3n - 4)}{(4n - 3)(4n - 4)(4n - 5)(4n - 6)} = \frac{9}{256} + o(1). \tag{11}
\]

Summing up the right sides (9), (10), and (11) we get the value (8). □

Lemma 18 (efficiency of C3) The ratio of the sequences satisfying the conditions \( s_1 = 0 \) and \( s_2 \leq 2 \) among the \((0, 3n - 3, n)\)-regular sequences is

\[
\frac{37}{256} + o(1). \tag{12}
\]

Proof. Similar to the proof of Lemma 8. □

Lemma 19 (efficiency of C4) The ratio of the sequences satisfying the conditions \( s_1 = 1 \) and \( s_2 = 1 \) and \( s_3 \leq 5 \) among the \((0, 3n - 3, n)\)-regular sequences is

\[
\frac{2343}{48} + o(1). \tag{13}
\]

Proof. Since \( R(1, 3n - 3, n - 3) \) sequences satisfy the conditions \( s_1 = s_2 = s_3 = 1 \) the corresponding ratio is

\[
\frac{R(1, 3n - 3, n - 3)}{R(0, 3n - 3, n)} = \frac{n(n - 1)(n - 2)(3n - 3)}{(4n - 3)(4n - 4)(4n - 5)(4n - 6)} = \frac{3}{4^4} + o(1). \tag{14}
\]

The sequences with \( s_1 = s_2 = 1 \) and \( s_3 = 2 \), \( s_1 = s_2 = 1 \) and \( s_3 = 3 \), \( s_1 = s_2 = 1 \) and \( s_3 = 4 \), and \( s_1 = s_2 = 1 \) and \( s_3 = 5 \) have the asymptotic ratio \( 3/4^5, 3/4^6, 3/4^7, \) and \( 3/4^8 \) resp.
The sum of the received five ratios is
\[
\frac{3}{4^2} + \frac{3^2}{4^5} + \frac{3^3}{4^6} + \frac{3^4}{4^7} + \frac{3^5}{4^8} = \frac{2343}{4^8},
\]
(15)
implying (13).

**Lemma 20** (efficiency of C5) *The ratio of the sequences satisfying the conditions of* \(s_n = s_{n-1} = 3n - 5\) *and* \(s_{n-3} \geq 3n - 8\) *among the* \((0, 3n-3, n)\)-regular *sequences is*
\[
\frac{1575}{4^8} + o(1).
\]
(16)

**Proof.** We have to sum the contributions of \(R(0, 3n-5, n-2), R(0, 3n-6, n-2), R(0, 3n-7, n-2),\) and \(R(0, 3n-8, n-2)\) sequences:
\[
\frac{3^2}{4^5} + \frac{3^3}{4^6} + \frac{3^5}{4^7} + \frac{3^6}{4^8} = \frac{1575}{4^8},
\]
(17)
implying (16).

**Lemma 21** (efficiency of C6) *The ratio of the sequences satisfying the conditions of* \(s_n = 3n-3, s_{n-1} = 3n-6,\) and \(s_{n-2} \geq 3n-8\) *among the* \((0, 3n-3, n)\)-regular *sequences is*
\[
\frac{999}{4^8} + o(1).
\]
(18)

**Proof.** In this case we sum the contributions of \(R(0, 3n-6, n-3), R(0, 3n-7, n-1),\) and \(R(0, 3n-8, n-1)\) sequences:
\[
\frac{3^3}{4^6} + \frac{3^4}{4^7} + \frac{3^5}{4^8} = \frac{999}{4^8},
\]
(19)
implying (18).

**Lemma 22** (efficiency of C7) *The ratio of the sequences satisfying the conditions of* \(s_1 = 0, s_2 = 3,\) and \(s_3 \leq 5\) *among the* \((0, 3n-3, n)\)-regular *sequences is*
\[
\frac{999}{4^8} + o(1).
\]
(20)

**Proof.** Similar to the proof of Lemma 21.
Lemma 23 (efficiency of C8) The ratio of the sequences satisfying the conditions of \( s_1 = 1 \), \( s_2 = 2 \), and \( s_3 \leq 3 \) among the \((0, 3n - 3, n)\)-regular sequences is
\[
\frac{63}{4^6} + o(1).
\] (21)

**Proof.** We sum the contributions of \( R(2, 3n - 3, n - 3) \) and \( R(3, 3n - 3, n - 3) \) sequences:
\[
\frac{3^2}{4^5} + \frac{3^3}{4^5} = \frac{63}{4^6},
\] (22)
implying (21).

Lemma 24 (efficiency of C9) The ratio of the sequences satisfying the conditions of \( s_n = 3n - 5 \), \( s_{n-1} = 3n - 7 \), and \( s_{n-2} \geq 3n - 7 \) among the \((0, 3n - 3, n)\)-regular sequences is
\[
\frac{3^4}{4^7} + o(1).
\] (23)

**Proof.** Similar to the proof of Lemma 23.

The cumulated asymptotic efficiency of the constant time algorithms is
\[
\frac{3}{16} + \frac{2 \cdot 37}{4^4} + \frac{2343}{4^8} + \frac{1575}{4^8} + \frac{2 \cdot 999}{4^8} + \frac{63}{4^6} + \frac{3^4}{4^7} = \frac{38480}{4^8}.\] (24)

The cumulated efficiency of the nine constant time algorithms is about 58.72 %. According to Table 1 the practical joint efficiency of C1, C2 and C3 is 64.28 % for \( n = 3 \) and 45.91 % for \( n = 14 \). According to Table 3 the total practical efficiency of the nine constant time algorithms is 91.67 % for \( n = 3 \) and 54.93 % for \( n = 14 \).

The practical cumulated efficiency is smaller than the theoretical one, since some part of the sequences is filtered by several algorithms: e.g. the sequence \( s = (0, 0, 5) \) is filtered by C1 and C3 too.

We remark that the algorithms of CONSTANT are sorted on the base of their nonincreasing asymptotic efficiency. We get the same order of the practical efficiency of these algorithms shown on the small values of \( n \).

### 4.3 Filtering algorithms with linear running time

We investigate the following filtering algorithms whose worst running time is linear: \textsc{Complete} = L1, \textsc{Point-Losses} = L2, \textsc{Reduction0} = L3, \textsc{Reduction1} = L4, \textsc{Draw-Unique} = L5, \textsc{Balanced} = L6, \textsc{Draw-Uniform} = L7, \textsc{Draw-Sorted-Unique} = L8.
4.3.1 Linear filtering algorithm \( L_1 = \text{COMPLETE} \)

The first linear time filtering algorithm \( L_1 = \text{COMPLETE} \) is based on the following special case of Lemma 3 in [56].

**Corollary 25** ((2,3,n)-complete test, [56]) If \( n \geq 1 \) and \((f_1, \ldots, f_n)\) is a football sequence then

\[
2 \binom{k}{2} \leq \sum_{i=1}^{k} f_i \leq 3 \binom{n}{2} - (n - k)f_k \quad (k = 1, \ldots, n).
\]  

(25)

Basic parameters of \( \text{COMPLETE} \) are the usual ones, further \( S \): the current sum of the first \( i \) elements of \( s \).

\[ \text{COMPLETE}(n, s) \]

01 \( S = 0 \) \quad // line 01: initialization of \( S \)
02 for \( i = 1 \) to \( n \) \quad // line 02–06: test
03 \( S = S + s_i \)
04 if \((S < 2\binom{i}{2}) \lor (S > 3\binom{n}{2} - (n - i)s_i) = \text{true}\)
05 \( L = 0 \)
06 return \( L \)
07 \( L = 2 \) \quad // line 07–08: \( s \) is undecided
08 return \( L \)

4.3.2 Linear filtering algorithm \( L_2 = \text{POINT-LOSSES} \)

The second linear time filtering algorithm \( L_2 = \text{POINT-LOSSES} \) is based on the following assertion which is an extension of Lemma 3 in [56]. The basic idea is, that the small sums of the prefixes of \( s \) and the mod 3 remainders of the elements of \( s \) signalize lost points.

**Lemma 26** If \((f_1, \ldots, f_n)\) is a football sequence then

\[
2 \binom{k}{2} \leq \sum_{i=1}^{k} f_i \leq 3 \binom{n}{2} - (n - k)f_k - P_k \quad (k = 1, \ldots, n),
\]  

(26)

where \( P_0 = 0 \) and

\[
P_k = \max \left( P_{k-1}, 3 \binom{k}{2} - \sum_{i=1}^{k} f_i, \left\lceil \frac{\sum_{i=1}^{k} |f_i - 3[f_i/3]|}{2} \right\rceil \right).
\]  

(27)
Proof. The sum of the $k$ smallest scores is at least $2 \binom{k}{2}$ and at most $3 \binom{n}{2}$ minus the following point-losses:

1. the sum of the remaining scores, which is at least $(n - k)f_k$;
2. the point-losses due to draws documented by the mod 3 remainders;
3. the point-losses documented by differences $3 \binom{k}{2} - \sum_{i=1}^{k} f_i$;

□

Basic parameters of POINT-LOSSES are the usual ones, further $S$: the current sum of the first $i$ elements of $s$, and $P$: the current value of the point-losses.

POINT-LOSSES$(n, s)$
01 $S = P = L = 0$ // line 01: initialization of $S$, $P$, and $L$
02 for $k = 1$ to $n$ // line 02–06: filtering
03 $S = S + s_k$
04 $P = \max \left( P_{k-1}, 3\binom{k}{2} - S, \left\lceil \frac{\sum_{i=1}^{k} (s_i - 3\left\lfloor s_i/3 \right\rfloor)}{2} \right\rceil \right)$
05 if $S > 3\binom{n}{2} - (n - k)s_k - P$
06 return $L$
07 $L = 2$
08 return $L$ // line 08: $s$ is undecided

4.3.3 Linear filtering algorithm $L3 = \text{Reduction0}$

The third linear test is based on the observation that if the sum of the $k$ smallest scores is minimal then all matches among the first $k$ teams ended by a draw and if the sum of the $k$ largest scores is maximal then the corresponding scores are multiples of 3 and further if $k < n$ then $f_{n-k} \leq 3(n - k - 1)$.

Lemma 27 If $n \geq 2$, $1 \leq k \leq n$, and $f = (f_1, \ldots, f_n)$ is a football sequence then

1) if the sum of the first $k$ scores is $k(k-1)$ then $f_1 = \cdots = f_k = k - 1$ and if further $k < n$ then $f_{k+1} \geq 3k$;
2) if the sum of the last $k$ scores is $3(n - k)k + 3\binom{k}{2}$ then $f_{n-k+1}, \ldots, f_n$ are multiples of 3 and if further $k < n$ then $f_{n-k} \leq 3(n - k - 1)$.

Proof. If $f_1 + \cdots + f_k = k(k-1)$ then all matches among $T_1, \ldots, T_k$ ended with a draw and these teams lost all matches against the remaining teams implying assertions 1).
If \( f_{k+1} + \cdots + f_n = 3(n - k)k + 3\binom{k}{2} \) then \( T_{k+1}, \ldots, T_n \) won all matches against the remaining teams and have no draws implying assertion 2. \( \square \)

Parameters of \( \text{REDUCTION0} \) are the usual ones, further \( S \): the current sum of the \( i \) smallest scores; \( Q \): the current sum of the \( i \) largest scores; \( B \) is a logical variable characterizing the remainders mod 3 of the \( i \) largest scores.

\[ \begin{align*}
\text{REDUCTION0}(n, s) \\
01 & L = B = S = Q = 0 & // line 01: initialization of \( L, B, S, \) and \( Q \\
02 & \text{for } i = 1 \text{ to } n - 1 & // line 02–12: test of the small scores \\
03 & S = S + s_i \\
04 & \text{if } S == i(i-1) \\
05 & \quad \text{return } L \\
06 & \quad \text{return } L \\
07 & S = S + s_n \\
08 & \text{if } S == n(n-1) \\
09 & \quad \text{if } s_1 < n - 1 \\
10 & \quad \quad \text{return } L \\
11 & \text{for } i = n \text{ down to } 2 & // line 13–25: test of the large scores \\
12 & Q = Q + s_i \\
13 & \text{if } s_{i-1} > 3(n - i - 1) \\
14 & \quad \text{return } L \\
15 & \quad \text{if } s_i - 3\lfloor s_i/3 \rfloor > 0 \\
16 & \quad \quad \text{B} = 1 \\
17 & \quad \quad \text{if } \text{B} == 1 \\
18 & \quad \quad \quad \text{return } L \\
19 & \quad \text{Q} = Q + s_1 \\
20 & \text{if } s_1 - \lfloor s_1/3 \rfloor > 0 \\
21 & \quad \quad \text{B} = 1 \\
22 & \quad \quad \text{if } \text{B} == 1 \\
23 & \quad \quad \quad \text{return } L \\
24 & \text{L} = 2 & // line 26–27: \( s \) is undecided \\
25 & \text{return } L
\end{align*} \]

Even this simple filtering algorithm finds a football sequence: if the condition of line 11 does not hold then the sum of all scores is minimal therefore all matches ended with draw. For the sake of the simplicity of the program we left this sequence undecided.
4.3.4 Linear filtering algorithm $L_4 = \text{Reduction1}$

The fourth linear test is based on the observations that if the sum of the $i$ smallest scores is $i(i-1) + 1$ then either zero or one match among the first $i$ teams ended with a win and if the sum of the $i$ largest scores has near the maximal $3i(n-i) + 3(i-1)/2$ value then among the $i$ maximal scores $i-2$ are multiples of 3 and 2 give 1 as remainder mod 3.

**Lemma 28** If $n \geq 3$, $f = (f_1, \ldots, f_n)$ is a football sequence, $1 \leq k \leq n$ then

1) if

$$\sum_{i=1}^{k} f_i = k(k-1) + 1$$

then

a) either $f_1 = \cdots = f_{k-1} = k-1$, $f_k = k$, and if $k+1 \leq n$, and $f_{k+1} \geq 3k-2$;

b) or $f_1 = k-2$, $f_2, \ldots, f_{k-1} = k-1$, $f_k = k+1$, and $f_{k+1} \geq 3k$;

2) if

$$\sum_{i=1}^{k} f_{n-i+1} = 3k(n-k) + 3\left(\frac{k}{2}\right) - 1$$

then

a) $\sum_{i=1}^{k} f_{i-3\lfloor f_i/3\rfloor} = 0$,

b) $\sum_{i=1}^{k} f_{i-3\lfloor f_i/3\rfloor} = 1$,

c) $\sum_{i=1}^{k} f_{i-3\lfloor f_i/3\rfloor} = 2$,

d) if $n-k > 0$ then $f_{n-k} \leq 3(n-k-1)$.

**Proof.** 1) If $f_1 + \cdots + f_k = k(k-1) + 1$ then either all matches among $T_1$, $\ldots$, $T_k$ ended with a draw and these teams lost all but one match against the remaining teams and $T_k$ made a draw with one of the teams $T_k$ implying assertions a) or $T_k$ won against $T_1$, the remaining matches among $T_1$, $\ldots$, $T_k$ ended with a draw and the teams $T_{k+1}, \ldots, T_n$ has no draw and won all matches against the first $n-k$ teams implying assertions b).

2) In case 2) of the lemma the teams $T_{k+1}, \ldots, T_n$ won all matches against the first $n-k$ teams, and made exactly one draw.

Parameters of $\text{Reduction1}$ are the usual ones, further $S$: the current sum of the first $i$ scores; $Q$: the current sum of the last $i$ scores; $L_1$ and $L_2$: logical variables; $B$ is the number of scores giving remainder 1 mod 3; $C$ is the number of scores giving remainder 0 mod 3.

$\text{Reduction1}(n, s)

01 L = B = C = S = Q = 0 \quad // line 01: initialization of L, B, C, S, and Q
for i = 1 to n - 1  // line 02–12: test of the small scores
  S = S + s_i
  if S == i(i - 1) + 1
    L_1 = (s_1 == i - 1) \land (s_{i-1} == i - 1) \land (s_i == i) \land (s_{i+1} \geq 3i - 2)
    L_2 = (s_1 == i - 2) \land (s_2 == i - 1) \land (s_{i-1} == i - 1)
          \land (s_i == i + 1) \land (s_{i+1} \geq 3i)
  if (L_1 == \text{FALSE}) \land (L_2 == \text{FALSE}) == \text{TRUE}
    return L  // line 07–08: s is not good
  S = S + s_n
  if S == n(n - 1) + 1
    if (s_1 < n - 2) \land (s_2 == n - 1) \land (s_{n-1} == n - 1) \land (s_n == n + 1)
      == \text{FALSE}
      return L
  for i = n downto 2  // line 13–35: test of the large scores
    Q = Q + s_i
    if s_i - 3 \lfloor s_i/3 \rfloor == 2
      return L
    if s_i - 3 \lfloor s_i/3 \rfloor == 1
      B = B + 1
    if s_i - 3 \lfloor s_i/3 \rfloor > 0
      C = C + 1
    if Q == 3(n - i)i + 3i(i - 1)/2 - 1
      if s_{n-i} > 3(n - i - 1)
        return L
    if (B == 2) \land (C == i - 2) == \text{FALSE}
      return L
  Q = Q + s_1
  if s_1 - 3 \lfloor s_i/3 \rfloor == 2
    return L
  if s_1 - 3 \lfloor s_i/3 \rfloor == 1
    B = B + 1
  if s_1 - 3 \lfloor s_i/3 \rfloor > 0
    C = C + 1
  if Q == 3n(n - 1)/2 - 1
    if (B == 2) \land (C == i - 2) == \text{FALSE}
      return L
  L = 2  // line 36–37: s is undecided
return L
4.3.5 Linear filtering algorithm \( L_5 = \text{Draw-Unique} \)

A draw sequence \( d(s) = (d_1, \ldots, d_n) \) belonging to a \((0, 3(n - 1), n)\)-regular sequence \( s \) accepted by \( L_4 \) is defined as a sequence of nonnegative integers having the following properties for \( i = 1, \ldots, n \):

1. \( 0 \leq d_i \leq 2 \);
2. \( d_i = s_i \mod 3 \);
3. \( d_i \leq \min(s_i, n - 1) \);
4. \( d_i + 3(n - 1 - d_i) \geq s_i \),

further
\[
\sum_{i=1}^{n} d_i = 2 \left( 3 \left( \begin{array}{c} n \\ 2 \end{array} \right) - \sum_{i=1}^{n} s_i \right). \tag{30}
\]

A draw sequence \( d = (d_1, \ldots, d_n) \) is called \((0, 1, n)\)-graphic (or simply graphic or good), if there exists a \((0, 1, n)\)-graph whose degree sequence is \( d \).

The fifth linear filtering algorithm is based on the following assertion.

**Lemma 29** If a \((2, 3, n)\)-regular sequence \( s \) has only a unique draw sequence \( d(s) \) which is not graphical then \( s \) is not football sequence.

**Proof.** Since the football sequences have at least one graphical draw sequence, the regular sequences without graphical draw sequence are not football sequences. \( \square \)

Basic parameters of \text{Draw-Unique} are the usual ones, further \( S \): the current sum of the elements of \( s \); \( R \): the number of obligatory draws; \( D_n \): the number of draws in the investigated potential tournament; \( d = (d_1, \ldots, d_n) \): \( d_i \) is the number of draws allocated to \( T_i \); \( r = (r_1, \ldots, r_n) \): \( r_i \) is the remainder of \( s_i \mod 3 \); \( y \): is the current number of allocated draws; \( x \): is the current maximal number of draw packets acceptable by \( T_i \).

\text{Draw-Unique}(n, s)

\begin{verbatim}
01 S = R = L = x = y = 0        // line 01: initialization of S, R, L, x and y
02 for i = 1 to n               // line 02-03: computation of S
03     S = S + s_i             
04 Dn = 3(n/2) - S            // line 04: computation of Dn
05 for i = 1 to n              // line 05-17: allocation of draws
06     r_i = s_i - 3[3/2]     
\end{verbatim}
$R = R + r_i$

$x = \min\left(\frac{s_i - r_i}{3}, \left\lfloor \frac{n - 1 - r_i}{3} \right\rfloor, \left\lfloor \frac{3(n - 1) - 2r_i - s_i}{6} \right\rfloor\right)$

d_i = r_i + 3x

if $R > 2Dn$

return $L$, d

if $y < 2Dn$

return $L$, d

if $y \geq 2Dn$

$L = 2$

return $L$, d

sort d in decreasing order by COUNTING-SORT resulting $d'$

HHL($d'$)

return $L$, d

Procedure HHL (HAVEL-HAKIMI-LINEAR) is described in [60]. We remark that the original Havel-Hakimi algorithm requires in worst case $\Theta(n^2)$ time. Recently Király [70] published a version which uses the data structure proposed by van Emde Boas [71, 114] and requires $O(n \log \log n)$ time. Our algorithm is linear and works also for some multigraphs.

A natural requirement is $d_i \leq n - 1$ but $d_i > n - 1$ can occur only in the cases $s = (0,2)$ and $s = (1,2)$ which are filtered by the constant time algorithms.

We get a stronger filtering algorithm DRAW-SORTED-UNIQUE using the definition of the uniqueness of the sorted draw sequence. For example in the case of the sequence $s = (3,3,3,5)$ we have three possibilities to allocate two draw packets but only the teams having 3 points can accept a packet therefore we get in each case the bad draw sequence $(3,3)$.

We remark that the problem of unicity of graphs determined in a unique way by their degree sequences was studied for some graph classes (see e.g. the papers of Tetali [109], Tyskevich [113], and Barrus [6]).

4.3.6 Linear filtering algorithm L6 = BALANCED-LIN

The sixth linear filtering algorithm L6 = BALANCED-LIN is based on the observation that if the draw sequence is unique, then the victory sequence $w = (w_1, \ldots, w_n)$ and the loss sequence $l = (l_1, \ldots, l_n)$ are also unique. The following assertion gives a necessary condition for the reconstructability of the sequence pair $(w, l)$.
Lemma 30 (Lucz [77]) If \( n \geq 2 \), \( w = (w_1, \ldots, w_n) \) is the win sequence and 
\( l = (l_1, \ldots, l_n) \) is the loss sequence of a football sequence 
\( f = (f_1, \ldots, f_n) \) with 
\( \sum_{i=1}^{n} f_i > n(n - 1) \) then let

\[
 w_i = \max_{1 \leq j \leq n} w_j \quad \text{and} \quad l_j = \max_{1 \leq i \leq n} l_i.
\] 

(31)

In this case

\[
 w_i \leq \sum_{j=1}^{i-1} \left\lceil \frac{l_j}{n-1} \right\rceil + \sum_{j=i+1}^{n} \left\lceil \frac{l_j}{n-1} \right\rceil
\] 

(32)

and

\[
 l_j \leq \sum_{i=1}^{j-1} \left\lceil \frac{w_i}{n-1} \right\rceil + \sum_{i=j+1}^{n} \left\lceil \frac{w_i}{n-1} \right\rceil
\] 

(33)

for \( n = 1, \ldots, n \).

Proof. The wins (losses) of the team \( T_i \) (\( T_j \)) having the maximal number of
wins (losses) can be paired with losses (wins) only if there are at least \( w_i \) (\( l_j \))
teams having at least one loss (win). \( \Box \)

4.3.7 Linear filtering algorithm \( L7 = \text{Sport-Uniform} \)

The seventh linear filtering algorithm \( L7 = \text{Sport-Uniform} \) is connected
with a popular concept called in the world of sport sport matrix. It is an \( n \times 5 \)
sized matrix containing the basic data of the teams of a tournament. We use
the following formal definition of sport matrix for \( n \) teams.

Definition 31 Let \( n \geq 1 \) and \( s = (s_1, \ldots, s_n) \) be a \((0,3(n - 1),n)\)-regular
sequence. Then the sport matrices \( S(s) \) corresponding to \( s \) are defined by the
following properties:

1. the size of the matrix is \( n \times 5 \), its elements are nonnegative integers;
2. \( w_i + d_i + l_i = n - 1 \) for \( i = 1, \ldots, n \);
3. \( 3w_i + d_i = s_i \) for \( i = 1, \ldots, n \);
4. \( \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} s_i - n(n - 1) \);
5. \( \sum_{i=1}^{n} d_i = 2 \left(3(n - 1) - \sum_{i=1}^{n} s_i \right)\).
We remark that the \( i \)th row of the sport matrices contains data of \( T_i \) for \( i = 1, \ldots, n \): index \( i \), number of wins \( w_i \), number of draws \( d_i \), number of losses \( l_i \) and number of points \( s_i \) (\( w_i, d_i \) and \( l_i \) are estimated values). These formal requirements are only necessary for \( S \) to contain the basic characteristics of some football tournament.

A sequence \( s = (s_1, \ldots, s_n) \) is called sport sequence if there exists at least one sport matrix corresponding to \( s \).

Another useful concept is the obligatory sport matrix belonging to given regular sequence \( s \).

**Definition 32** Let \( n \geq 1 \) be and \( s = (s_1, \ldots, s_n) \) be a \((0,3(n-1),n)\)-regular sequence. Then the obligatory sport matrix \( O(s) \) corresponding to \( s \) is defined by the following properties:

1. the size of the matrix is \( n \times 5 \), its elements are nonnegative integers;
2. \( w_{o_i} = \max \left( 0, \left\lfloor \frac{s_i - (n-1)}{2} \right\rfloor \right) \) for \( i = 1, \ldots, n \);
3. \( d_{o_i} = s_i - 3 \left\lfloor \frac{s_i}{3} \right\rfloor \) for \( i = 1, \ldots, n \);
4. \( l_{o_i} = \max(0, n - 1 - s_i) \) for \( i = 1, \ldots, n \).

The \( i \)th row of the matrix contains the (partially estimated) data of \( T_i \) for \( i = 1, \ldots, n \): index \( i \), number of obligatory wins \( w_{o_i} \), number of obligatory draws \( d_{o_i} \), number of obligatory losses \( l_{o_i} \) and number of points \( s_i \) (the obligatory values are lower bounds for the correct values, the index and the number of points are exact values).

**Definition 33** We say that the obligatory sport matrix \( O(s) \) of \( s \) is extendable to a sport matrix \( S(s) \) corresponding to \( s \) if

1. \( O(s) \) is a sport matrix belonging to \( s \) or
2. we can increase some \( w_i, d_i \) and \( l_i \) values so that the result will be a sport matrix \( S(s) \).

According to the following assertion we get a linear filtering algorithm using the obligatory sport matrix.

**Lemma 34** The obligatory sport matrix \( O(s) \) belonging to a \((0,3(n-1),n)\)-regular sequence \( s \) is unique. If \( O(s) \) is not extendable to a sport matrix \( S(s) \) then \( s \) is not a football sequence.
Deciding football sequences

Proof. The obligatory sport matrix is defined by unique formulas therefore it is unique. If \( s \) is a football sequence then its obligatory sport matrix \( O(s) \) contains lower bounds for \( w_i, d_i, \) and \( l_i \) of any sport matrix \( S(s) \) therefore any sport matrix \( S(s) \) can be constructed by the extension of \( O(s) \). □

The following Sport-Uniform is a draw-based algorithm which at first constructs the obligatory sport matrix belonging to \( s \) then tries to extend it to a sport matrix so that it allocates the draw packets in a greedy way as uniformly as possible. If the so received draw sequence is not graphic then the investigated sequence is not good.

The base of the uniform allocation of the draws is the following assertion.

Lemma 35 If \( n \geq 1 \), \( d = (d_1, \ldots, d_n) \) is graphical and \( d_i < d_j \) then the sequence \( d' \)—received increasing \( d_i \) by 1 and decreasing \( d_j \) by 1—is also graphical.

Proof. Let \( G \) be a \((0,1,n)\)-graph on vertices \( V_1, \ldots, V_n \) having the degree sequence \( d = (d_1, \ldots, d_n) \) in which \( d_i < d_j \). Then there exists a vertex \( V_k \) which is connected with \( V_j \) and not connected with \( V_i \). In \( G \) delete the edge between \( V_j \) and \( V_k \) and add the edge between \( V_i \) and \( V_k \). Then the received new graph is graphical with the required degree sequence. □

This lemma has a useful corollary.

Corollary 36 If \( n \geq 1 \), \( s = (s_1, \ldots, s_n) \) is a \((2,3,n)\)-regular sequence, and its uniform draw sequence \( u(s) = (u_1, \ldots, u_n) \) is not graphical, then \( s \) is not a football sequence.

Proof. By the recursive application of Lemma 35 we get that if \( s \) has a graphical draw sequence then its uniform draw sequence is also graphical. □

We remark that the problem of the pairing of the draws has a reach bibliography as the problem of degree sequences of simple graphs [24, 32, 47, 50, 51, 62, 80, 89, 107, 110, 111, 112].

Basic parameters of Sport-Uniform are the usual ones further \( S \); the sum of the elements of \( s \); \( S_0 \): auxiliary variable; \( wo = (wo_1, \ldots, wo_n) \): \( wo_i \) is the number of obligatory wins of \( T_i \); \( do = (do_1, \ldots, do_n) \): \( do_i \) is the number of obligatory draws of \( T_i \); \( lo = (lo_1, \ldots, lo_n) \): \( lo_i \) is the number of obligatory losses of \( T_i \); \( WO = (WO_0, \ldots, WO_n) \): \( WO_i \) is the total number of wins of the first \( i \) teams; \( DO = (DO_0, \ldots, DO_n) \): \( DO_i \) is the total number of draws of the first \( i \) teams; \( LO = (LO_0, \ldots, LO_n) \): \( LO_i \) is the total number of the first \( i \) teams; \( wm = (wm_1, \ldots, wm_n) \): \( wm_i \) is the maximal number of wins of \( T_i \);
\( \text{dm} = (\text{dm}_1, \ldots, \text{dm}_n) \): \( \text{dm}_i \) is the maximal number of draw packets of \( T_i \);

\( \text{lm} = (\text{lm}_1, \ldots, \text{lm}_n) \): \( \text{lm}_i \) is the maximal number of losses of \( T_i \);

\( W_n \): the number of the wins in the tournament;

\( L_n \): the number of the losses in the tournaments;

\( D_n \): the number of draws in the tournament;

\( D \): the current number of yet not allocated draws;

\( \text{da} = (\text{da}_1, \ldots, \text{da}_n) \): \( \text{da}_i \) is the number of allocated to \( T_i \) draw packets;

\( \text{wa} = (\text{wa}_1, \ldots, \text{wa}_n) \): \( \text{wa}_i \) is the number of allocated wins of \( T_i \);

\( \text{la} = (\text{la}_1, \ldots, \text{la}_n) \): \( \text{la}_i \) is the number of allocated losses of \( T_i \);

\( R = (R_0, R_1, R_2) \): \( R_i \) is the number of elements of \( s \) giving remainder \( i \mod 3 \);

\( c \): average number of draw packets to allocate for a team.

**SPORT-Uniform** \((n, s)\)

01 \( S_0 = W_0 = D_0 = L_0 = W_0 = R_0 = R_1 = R_2 = L = 0 \) // line 01: initialization

02 **for** i = 1 to \( n \) // line 02–03: computation of the parameters

03 \( S = S + s_i \)

04 \( \text{wo}_i = \max \left( 0, \left\lfloor \frac{s_i}{3} \right\rfloor \right) \)

05 \( W_0 = W_0 + \text{wo}_i \)

06 \( \text{do}_i = s_i - 3\left\lfloor \frac{s_i}{3} \right\rfloor \)

07 \( D_0 = D_0 + \text{do}_i \)

08 \( R_{\text{do}_i} = R_{\text{do}_i} + 1 \)

09 \( \text{lo}_i = \max (n - 1 - s_i, 0) \)

10 \( L_0 = L_0 + \text{lo}_i \)

11 \( \text{dm}_i = \min \left( \frac{s_i - \text{do}_i}{3}, n - 1 - \text{do}_i, \left\lfloor \frac{3(n - 1) - 2\text{do}_i - s_i}{6} \right\rfloor \right) \)

12 \( \text{wa}_i = s_i - 3\text{da}_i \)

13 \( \text{la}_i = \left\lfloor \frac{3(n - 1) - s_i}{3} \right\rfloor \)

14 \( W_n = L_n = S_n - n(n - 1) \) // line 14: computation of \( W_n, L_n \)

15 \( D_n = D = 3n(n - 1)/2 - S \) // line 15: computation of \( D_n, D \)

16 **if** \( \frac{D - D_0}{3} > \left\lfloor \frac{D - D_0}{3} \right\rfloor \) // line 16–43: allocation of draw packets

17 **return** \( L \)

18 **while** \( D > 0 \)

19 \( c = \left\lfloor \frac{D}{R_0 + R_1 + R_2} \right\rfloor \)

20 **while** \( c \geq 1 \)

21 \( R_0 = R_1 = R_2 = 0 \)

22 **for** i = 1 to \( n \)

23 \( \text{da}_i = \min \left( \frac{s_i - \text{di}}{3}, c \right) \)

24 \( \text{di} = \text{do}_i + 3\text{da}_i \)

25 \( D = D - 3\text{da}_i \)

26 **if** \( \text{di} < \text{dm}_i \)
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\[ R_{do_i} = R_{do_i} + 1 \]

\[ c = \left\lceil \frac{D}{R_0 + R_1 + R_2} \right\rceil \]

\[ \text{if } 0 < \frac{D}{3} \leq R_0 \]

\[ \text{for } i = 1 \text{ to } n \]

\[ \text{if } (D > 0 \land do_i == 0) == \text{true} \]

\[ d_i = d_i + 3 \]

\[ D = D - 3 \]

\[ \text{if } R_0 < \frac{D}{3} \leq R_0 + R_1 \]

\[ \text{for } i = 1 \text{ to } n \]

\[ \text{if } (D > 0 \land do_i == 0 \lor do_i == 1) == \text{true} \]

\[ d_i = d_i + 3 \]

\[ D = D - 3 \]

\[ \text{if } R_0 + R_1 > \frac{D}{3} \]

\[ \text{for } i = 1 \text{ to } n \]

\[ \text{if } D > 0 \]

\[ d_i = d_i + 3 \]

\[ D = D - 3 \]

sort \( d \) in decreasing order resulting \( d' \)

\[ \text{HHL}(d') \quad // \text{line 44–45: sorting of the draw sequence} \]

\[ \text{return } L, d \quad // \text{line 46: } s \text{ is undecided (if } L = 2) \text{ or bad (if } L = 0) \]

4.3.8 Linear filtering algorithm \( \text{L8 = Draw-Sorted-Unique} \)

The fifth linear filtering algorithm \( \text{DRAW-UNIQUE} \) exploits the fact that some football sequences have unique sport matrix implying the uniqueness of the draw sequence. The eighth linear algorithm \( \text{L8 = DRAW-Sorted-Unique} \) exploits that the uniqueness of the sport matrix is not necessary to have a unique sorted draw sequence.

Sorted version of a sport matrix \( S(s) \) is denoted by \( \overline{S}(s) \) and is defined by the following property: if \( 1 \leq i < j \leq n \) then either \( d_i' < d_j' \) or \( d_i' = d_j' \) and \( w_i' < w_j' \) or \( d_i' = d_j' \) and \( w_i' = w_j' \) and \( i' < j' \) (\( d_i' \) is the draw value in the \( i \)-th row of the sorted matrix and \( i' \) is the original index belonging to \( d_i' \)).

\( \text{DRAW-Sorted-Unique} \) is based on the following assertion.

Lemma 37 If \( n \geq 1, s = (s_1, \ldots, s_n) \) is a \((2, 3, n)\)-regular sequence, the sorted versions of the sport matrices \( S(s) \) are identical and their joint draw sequence is not graphical, then \( s \) is not a football sequence.

Basic parameters of \( \text{DRAW-Sorted-Unique} \) are the usual ones, further \( S \): the current sum of the elements of \( s \); \( D \): the number of the draws in
the investigated potential tournament; \(d = (d_1, \ldots, d_n)\): \(d_i\) is the number of draws allocated to \(T_i\); \(do = (do_1, \ldots, do_n)\): \(do_i\) is the number of obligatory draws of \(T_i\); DO: the number of the obligatory draws in the tournament; \(lo = (lo_1, \ldots, lo_n)\): \(lo_i\) is the number of obligatory losses of \(T_i\); LO is the number of obligatory losses in the tournament; \(wo = (wo_1, \ldots, wo_n)\): \(wo_i\) is the number of obligatory wins of \(T_i\); WO is the number of obligatory wins in the tournament; \(dm = (dm_1, \ldots, dm_n)\): \(dm_i\) is the maximal number of draw packets which can be accepted by \(T_i\); DM: the sum of the \(dm_i\)'s; \(lm = (lm_1, \ldots, lm_n)\): \(lm_i\) is of the maximal number of losses of \(T_i\); LM: the sum of the \(lm_i\)'s; \(wm = (wm_1, \ldots, wm_n)\): \(wm_i\) is the maximal number of wins of \(T_i\); WM: the sum of the \(wm_i\)'s; \(wn\): the number of the wins in the tournament; \(Ln\): the number of the losses in the tournaments; \(Dm\): the number of draws in the tournament; \(D\): the number of yet not allocated draws; \(da = (da_1, \ldots, da_n)\): \(da_i\) is the number of allocated to \(T_i\) draw packets; \(wa = (wa_1, \ldots, wa_n)\): \(wa_i\) is the number of allocated to \(T_i\) wins; \(la = (la_1, \ldots, la_n)\): \(la_i\) is the number of allocated to \(T_i\) losses; \(h\): the maximal number of draw packets assigned to a team; \(R = R_{i,j}\): a \(3 \times h\) sized matrix, where \(R_{i,j}\) gives the number of teams which are able at most \(i\) draw packets and having score of form \(3k + j\); \(A = (A_0, A_1, A_2)\): \(A_j\) is the number of scores giving \(i \mod 3\); \(B = (B_0, \ldots, B_h)\): \(B_i\) is the number of teams which are able to accept at most \(i\) draw packets; \(z\): number of draw packets which the program tries to allocate to all teams; \(fs\): first score among the scores receiving maximal number of draw packets; \(Rm\): critical value of the remainder (mod 3) of the scores.

**Draw-Sorted-Unique**\((n, s)\)

01 \(S = WO = DO = LO = A_0 = A_1 = A_2 = L = 0\) \hspace{1em} // line 01: initialization
02 **for** \(i = 1\) **to** \(n\) \hspace{1em} // line 02–29: test of the obligatory sport matrix
03 \(S = S + s_i\)
04 \(do_i = s_i - 3\lfloor s_i/3\rfloor\)
05 \(DO = DO + do_i\)
06 \(wo_i = \max(0, \lceil s_i -(n-1)/2\rceil)\)
07 \(WO = WO + wo_i\)
08 \(lo_i = \max(0, n - 1 - s_i)\)
09 \(LO = LO + lo_i\)
10 \(dm_i = \min\left(\frac{s_i-do_i}{3}, \frac{n-1-do_i}{3}, \frac{3(n-1)-2do_i-s_i}{6}\right)\)
11 \(DM = DM + dm_i\)
12 \(wm_i = \min\left(\frac{s_i-da_i}{3}, (N - 1) - DO_i\right)\)
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13 \[ WM = WM + \text{wm}_i \]
14 \[ \text{lm}_i = \min \left( \left\lfloor \frac{3(n-i-1)}{3} \right\rfloor, n - 1 - \text{do}_i \right) \]
15 \[ LM = LM + \text{lm}_i \]
16 \[ D_n = 3 {\binom{n}{2}} - S \]
17 \[ W_n = L_n = S - 2 {\binom{n}{2}} \]
18 if DO > 2Dn
19 \quad \text{return } L
20 if 3DM < 2Dn
21 \quad \text{return } L
22 if WO > Wn
23 \quad \text{return } L
24 if WM < Wn
25 \quad \text{return } L
26 if LO > Ln
27 \quad \text{return } L
28 if LM < Ln
29 \quad \text{return } L
30 \quad h = \left\lfloor \frac{(n - 1)/3}{3} \right\rfloor \quad // \text{line 30–45: preparation of the allocation}
31 \quad \text{for } i = 0 \text{ to } h
32 \quad \quad B_i = 0
33 \quad \quad \text{for } j = 0 \text{ to } 2
34 \quad \quad \quad R_{j,i} = 0
35 \quad \quad \text{for } i = 1 \text{ to } n
36 \quad \quad \quad R_{do_i, dm_i} = R_{do_i, dm_i} + 1
37 \quad \quad \text{for } i = 1 \text{ to } h
38 \quad \quad \quad \text{for } j = 0 \text{ to } 2
39 \quad \quad \quad \quad A_{i,j} = A_{i,j} + R_{i,j}
40 \quad \quad \quad B_i = B_i + R_{i,j}
41 \quad q = 0
42 \quad A = A_0 + A_1 + A_2
43 \quad D = 2D_n - DO
44 \quad c = \left\lfloor \frac{D}{A} \right\rfloor
45 \quad q = q + c
46 \quad \text{while } c \geq 1 \quad // \text{line 46–78: allocation of the draws}
47 \quad z = 0
48 \quad \text{for } i = q - c + 1 \text{ to } q
49 \quad \quad z = z + iB_i
50 \quad D = D - 3z
for $i = 0$ to 2
   for $j = q - c + 1$ to $q$
      $A_i = A_i - R_{i,j}$
      $A = A_0 + A_1 + A_2$
      $c = \lfloor \frac{D}{A} \rfloor$
if $q > 0$
   for $i = 1$ to $n$
      $d_i = d_i - 3 \min(z, dm_i)$
if $D == 0$
go to 79
$fs = -1$
$Rm = 2$
if $D \leq A_1 + A_2$
   $Rm = 1$
if $D \leq A_1$
   $Rm = 0$
for $i = 1$ to $n$
   if $(dm_i > q) \land (do_i \leq Rm) == TRUE$
      if $fs == -1$
         $fs = s_i$
   else if $s_i \neq fs$
      return $L$, $d$
      if $do_i < Rm$
      $d_i = d_i + 3$
      $D = D - 3$
      if $(do_i == Rm) \land (D > 0) == TRUE$
      $d_i = d_i + 3$
      $D = D - 3$
sort $d$ in nonincreasing order resulting $d'$  // line 79–80: sorting of $d$
HHL($d'$)
return $L$, $d$  // line 81: return the result of HHL

Procedure HHL (Havel-Hakimi-Linear) is described in [60]. We remark that the original Havel-Hakimi algorithm requires in worst case $\Theta(n^2)$ time. Recently Király [70] published a quicker algorithm which uses the data structure proposed by van Emde Boas [27, 71, 114] and requires only $O(n \log \log n)$ time. Our algorithm is linear and works also for some multigraphs.

A natural requirement is $d_i \leq n - 1$ but $d_i > n - 1$ can occur only in the cases $s = (0, 2)$ and $s = (1, 2)$ which are filtered by the constant time algorithms.
4.3.9 Efficiency of linear time filtering algorithms

\textsc{Linear} is the union of the described linear time algorithm.

\texttt{Linear}(n, s)

\begin{verbatim}
01  L = 0  // line 01: initialization of L
02  L1(n, s)  // line 02-04: filtering by \textsc{Complete}
03  if L = 0
04    \textbf{return} L
05  L2(n, s)  // line 05-07: filtering by \textsc{Losses}
06  if L = 0
07    \textbf{return} L
08  L3(n, s)  // line 08-10: filtering by \textsc{Reduction0}
09  if L = 0
10    \textbf{return} L
11  L4(n, s)  // line 11-13: filtering by \textsc{Reduction1}
12  if L = 0
13    \textbf{return} L
14  L5(n, s)  // line 14-16: filtering by \textsc{Draw-Unique}
05  if L = 0
16  \textbf{return} L
17  L6(n, s)  // line 17-19: filtering by \textsc{Balanced}
18  if L = 0
19  \textbf{return} L
20  L7(n, s)  // line 20-22: filtering by \textsc{Draw-Uniform}
21  if L = 0
22  \textbf{return} L
23  L8(n, s)  // line 23-25: filtering by \textsc{Draw-Sorted-Unique}
24  if L = 0
25  \textbf{return} L
26  return 1  // line 26-27: the linear time algorithms can not decide
27  \textbf{return} L
\end{verbatim}

Since all included algorithms have linear worst case running time, the total running time of \textsc{Linear} is also $\mathcal{O}(n)$. Since the best running time of \textsc{L1} is $\mathcal{O}(1)$, therefore the best running time of \textsc{Linear} is also $\mathcal{O}(1)$.

Tables 4 and 5 show the concrete filtering results of the linear time filtering algorithms. Table 4 contains the number of regular sequences ($R$), the number of sequences, accepted by \textsc{C9, L1 = Complete-Test, L2 = Losses} and \textsc{L3 = Reduction0}. 
Table 4: Results of filtering by linear tests tests L1, L2, and L3 + L4 for \( n = 1, \ldots, 14 \) teams.

| \( n \) | C9 | L1 | L2 | L3 + L4 |
|-------|----|----|----|---------|
| 1     | 1  | 1  | 1  | 1       |
| 2     | 2  | 2  | 2  | 2       |
| 3     | 14 | 12 | 10 | 10      |
| 4     | 203| 134| 94 | 87      |
| 5     | 2133 | 1230 | 901 | 814    |
| 6     | 20518 | 10947 | 8348 | 7526   |
| 7     | 191707 | 97427 | 76526 | 69349  |
| 8     | 1772842 | 872234 | 699344 | 637735 |
| 9     | 16332091 | 7851193 | 6387443 | 5859125 |
| 10    | 150288309 | 71001641 | 58367243 | 53817029 |
| 11    | 1383099467 | 644668154 | 538591486 | 494427384 |
| 12    | 12737278674 | 5873396400 | 4888701306 | 4544762304 |
| 13    | 117411184292 | 53669099755 | 44823480671 | 41804695971 |
| 14    | 1083421567402 | 491669304392 | 411496549436 | 384847810936 |

Table 5 contains the number of sequences accepted by \( L4 = \text{Reduction1} \), \( L5 = \text{Draw-Unique} \), \( L6 = \text{Balanced} \), \( L7 = \text{Sport-Uniform} \) and \( L8 = \text{Inner-Draw} \), further the number of the football sequences (\( F \)) and the cumulated running time and (the exact values of \( L7 \) are bold).

### 4.4 Quadratic filtering algorithms

In this section the quadratic recursive filtering algorithms \( Q1 = \text{Balanced-Quad} \), \( Q2 = \text{Reduction-Rec-Small} \), and \( Q3 = \text{Reduction-Rec-Large} \) are described.

#### 4.4.1 Quadratic filtering algorithm \( Q1 = \text{Balanced-Quad} \)

The filtering algorithm \( Q1 = \text{Balanced-Quad} \) is based on the observation that if the draw sequence is unique, then the victory sequence \( w = (w_1, \ldots, w_n) \) and the corresponding loss sequence \( l = (l_1, \ldots, l_n) \) are also unique, further that the wins (losses) of any subset of teams have to be paired with inner and outer losses (wins). The following assertion gives a necessary condition for the reconstructability of the sequence pair \((v, l)\).
Table 5: Results of filtering by linear tests L5 + L6 and L7 + L8, further the number of football sequences (F) and the running time of L8 (t) for 1, ..., 14 teams.

| n  | L5 + L6 | L7 + L8 | F    | t    |
|----|---------|---------|------|------|
| 1  | 1       | 1       | 1    | 0.000|
| 2  | 2       | 2       | 2    | 0.000|
| 3  | 7       | 7       | 7    | 0.000|
| 4  | 46      | 40      | 40   | 0.000|
| 5  | 475     | 365     | 355  | 0.000|
| 6  | 4459    | 4086    | 3678 | 0.015|
| 7  | 47867   | 44657   | 37263| 0.047|
| 8  | 460153  | 451213  | 361058| 0.437|
| 9  | 4371783 | 4348655 | 3403613| 4.196|
| 10 | 41261057| 41166157| 31653777| 40.217|
| 11 | 387821927| 387416935| 292547199| 393.280|
| 12 | 3635039265| 3633749149| 2696619716| 3828.002|
| 13 | 34011137972| 33821636274| 37611.185|
| 14 | 317827900632| 316291028902| 364978.049|

Lemma 38 If \((a_1, \ldots, a_n)\) is the monotone nonincreasing win sequence and \((b_1, \ldots, b_n)\) is the corresponding loss sequence of a football tournament then

\[
\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \min(b_i, k-1) + \sum_{i=k+1}^{n} \min(b_i, k) \quad (34)
\]

for all \(k = 1, \ldots, n\), with equality for \(n\).

Proof. The wins included in the sum of the left side of (34) have to be paired with the "inner losses" (losses among \(T_1, \ldots, T_k\)) and "outer losses" (losses of \(T_1, \ldots, T_k\) in the matches against the remaining teams). □

We remark that this lemma is a consequence of Theorem 3 of the recent paper due to Berger [10] containing a necessary and sufficient condition for some incomplete \((0, 2, n)\)-tournaments. As the sequence \((1, 1, 8, 9, 9)\) satisfying 34 shows, in our case (34) is only a necessary condition, since \(s\) has a unique sport matrix shown in Table 6 which is not reconstructable.

The paper [33] contains an algorithm for our problem but the algorithm does not terminate for some inputs.
Table 6: Unique sport matrix belonging to the sequence \( s = (1, 1, 8, 9, 9) \).

The following natural implementation \textsc{Balanced-Quad} of Lemma 38 requires quadratic time.

Parameters of \textsc{Balanced-Quad} are the usual ones, further \( w = (w_1, \ldots, w_n) \): \( w_i \) is the number of wins allocated to \( T_i \) \((0 \leq w_i \leq n-1)\); \( l = (l_1, \ldots, l_n) \): \( l_i \) is the number of losses allocated to \( T_i \) \((0 \leq l_i \leq n - 1)\); \( Sw \): the current number of the necessary wins; \( Ss \): the maximal number of pairable losses of teams having small indices; \( Sl \): the maximal number of pairable losses of the teams having large indices.

\textsc{Balanced-Quad}(\( n, w, l \))

```plaintext
01 Sw = L = 0  // line 01: initialization of Sw and L
02 sort \((w, l)\) nonincreasingly in \( w \) using \textsc{Counting-Sort}
03 for \( i = 1 \) to \( n \)  // line 03–13: counting of wins and losses
04 \( Ss = Sl = 0 \)
05 \( Sw = Sw = w_i \)
06 for \( j = 1 \) to \( i \)  // line 06–07: small indices
07 \( Ss = Ss + \min(w_j, i - 1) \)
08 for \( j = i + 1 \) to \( n \)  // line 08–09: large indices
09 \( Sl = Sl + \min(w_j, i) \)
10 if \( Sw > Ss + Sl \)  // line 10–13: \((w, l)\) is not pairable
11 return \( L \)
12 if \( Sw < Ss + Sl \)
13 return \( L \)
14 else \( L = 2 \)  // line 14–15: \( s \) is undecided
15 return \( L \)
```

We yet did not implemented \textsc{Balanced-Quad}.
4.4.2 Quadratic filtering algorithm Q2 = Reduction-Rec-Small

Algorithm Q2 = Reduction-Rec-Small is based on the recursive application of Recursive0 and Recursive1. Using Q2 and the next Q3 we shorten the input sequences and often can filter them.

Parameters are the usual ones, further \( e = (e_1, \ldots, e_N) \): work version of the investigated sequence; \( n_t \): smallest index of not deleted elements of \( s \).

**Reduction-Rec-Small** \((n, s)\)

01 \( L = S = 0 \) // line 01–04: initialization of \( L \), \( S \), \( n_t \), and \( e \)
02 for \( i = 1 \) to \( n \)
03 \( e_i = s_i \)
04 \( n_t = 1 \)
05 while \( n_t \leq n \)
06 \( S = 0 \)
07 for \( i = n_t \) to \( n \)
08 \( S = S + e_i \) // line 09–21: \( S \) is minimal
09 if \( S = i(i-1) \) if \( i < n \)
10 if \( (e_{n_t} \neq i-1) \lor (e_{n_t+i-1} \neq i-1) \) return \( L \)
11 if \( e_{n_t+i} < 3i \)
12 return \( L \)
13 if \( i = n \)
14 if \( (e_{n_t} \neq i-1) \lor (e_n \neq i-1) \) return \( L \)
15 else \( L = 1 \)
16 return \( L \)
17 \( n_t = n_t + i \)
18 for \( j = n_t \) to \( n \)
19 \( e_j = e_j - 3i \)
20 go to 05
21 if \( S = i(i-1) + 1 \) // line 22–35: \( S \) is minimum plus one
22 if \( i < n \)
23 \( L_1 = (e_{n_t} = i-1) \land (e_{n_t+i-2} = i-1) \land (e_{n_t+i-1} = i) \land (e_{n_t+i} \geq 3i-2) \)
24 \( L_2 = (e_{n_t} = i-2) \land (e_{n_t+i-2} = i-1) \land (e_{n_t+i-1} = i+1) \)
25 if \( \neg (L_1 \land L_2) \land \neg (L_1 \land L_2) \land \) return \( L \) // line 28–29: \( s \) is not football sequence

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if \(i == n\)

\[L_2 = (e_{n_l} = n - 2) \land (e_{n_l+1} = n - 1) \land (e_{n-1} = n + 1)\]

if \(L_2 == \text{FALSE}\)  // line 32–33: \(s\) is not football sequence

\[\text{return } L\]

\[n_l = n_l + i\]

for \(j = n_l\) to \(n\)

\[e_j = e_j - 3i\]

go to 05

\[\text{Reduction-Rec-Large}(n - n_l + 1, e)\]

if \(L == 0\)

\[\text{return } L\]

if \(n_u > 0\)

\[\text{Filter}(n_u, e)\]

if \(L == 0\)

\[\text{return } L\]

\[L = 2\]  // line 45–46: \(s\) is undecided

\[\text{return } L\]

\[\text{Reduction-Rec-Small}\]
calls \(\text{Filter}\) which is a union of the constant and linear time filtering algorithms and \(\text{Reduction-Rec-Large}\) which is the next quadratic filtering algorithm.

\[\text{Filter}(n, e)\]

\[\text{Constant}(n, e)\]  // line 01–03: filtering by the constant time algorithms

if \(L == 0\)

\[\text{return } L\]

\[\text{Linear}(n, e)\]  // line 04–06: filtering by the linear time algorithms

if \(L == 0\)

\[\text{return } L\]

\[L = 2\]  // line 07–08: \(s\) is undecided

\[\text{return } L\]

\[\text{4.4.3 Quadratic filtering algorithm Q3}\]

Algorithm Q3 = \(\text{Reduction-Rec-Large}\) is based on the recursive application of \(\text{Recursive0}\) and \(\text{Recursive1}\).

Parameters are the usual ones, further \(e = (e_1, \ldots, e_N)\): work version of the investigated sequence; \(n_u\): smallest index of the not deleted elements of \(s\); \(Q\): the sum of the \(i\) largest scores; \(B\): the number of investigated scores giving
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0 remainder mod 3; C: the number of investigated scores giving remainder 1 mod 3; D: the number of investigated scores giving remainder 2 mod 3.

REDUCTION-REC-LARGE(n, e)
01 L = Q = B = C = D = 0 // line 01–02: initialization of L, Q, B, C, D, n_u
02 n_u = n
03 while n_u ≤ 1 // line 03–25: recursive reduction
04 for i = n_u downto 1 // line 04–09: preparing of the filtering
05 Q = Q + e_i
06 if e_i − 3⌈e_i/3⌉ == 0
07 B = B + 1
08 if e_i − 3⌈e_i/3⌉ == 1
09 C = C + 1
10 if e_i − 3⌈e_i/3⌉ == 2
11 D = D + 1
12 if Q == 3i(n_u − i) + 3i(i − 1)/2 // line 12–17: Q is maximal
13 if B ≠ i
14 return L, n_u
15 if i > 1
16 if e_{n_u−i} > 3(n_u − i − 1)
17 return L, n_u
18 n_u = n_u − i
19 go to 03
20 if Q == 3i(n_u − i) + 3i(i − 1)/2 − 1
21 if (B == i − 2) ∧ (C == 2) == false
22 return L, n_u
23 if i > 1 // line 20–25: Q is maximum minus 1
24 if e_{n_u−i} > 3(n_u − i − 1)
25 return L, n_u
26 L = 2 // line 26–27: s is not decided
27 return L, n_u

The following Table 7 contains the results of quadratic filtering algorithms.

5 Reconstruction of potential football sequences

In this part we investigate polynomial reconstruction algorithms, as R1 = REDUCTION, R2 = DRAW-UNIFORM-REC, and R3 = DRAW-INNER-REC.
Table 7: Results of filtering by Linear and quadratic algorithms Q2 + Q3, further the number of football sequences (F) and the running time of Q3 (t) for n = 1, ..., 13 teams.

| n  | Linear | Q2 + Q3 | F   | t     |
|----|--------|---------|-----|-------|
| 1  | 1      | 1       | 1   | 0.000 |
| 2  | 2      | 2       | 2   | 0.000 |
| 3  | 7      | 7       | 7   | 0.000 |
| 4  | 40     | 40      | 40  | 0.000 |
| 5  | 365    | 355     | 355 | 0.000 |
| 6  | 4086   | 3760    | 3678| 0.015 |
| 7  | 44657  | 39417   | 37273| 0.109 |
| 8  | 451213 | 393072  | 361058| 1.264 |
| 9  | 434855 | 3804485 | 3403613| 15.226 |
| 10 | 387416935 | 344012885 | 292547199| 179.249 |
| 11 | 3633749149 | 3246651763 | 2696619716| 23429.877 |
| 12 | 33821636274 | 30405902165 | 2696619716| 23429.877 |

5.1 Reconstruction algorithm R1 = Reduction

R1 = Reduction is based on filtering algorithms Reduction0 and Reduction1.

5.2 Reconstruction algorithm R2 = Draw-Uniform-Rec

R2 = Draw-Uniform-Rec is based on filtering algorithms: it tries—using the degree sequence d produced by Sport-Uniform or Draw-Sorted-Unique and using a greedy pairing algorithm “largest wins with largest losses”—to pair the wins and losses.

Parameters of R2 are the usual ones further S: sport matrix computed using the output draw sequence d of Sport-Uniform or Draw-Sorted-Unique and sorted its rows so that either \( w_i > w_{i+1} \) or \( w_i = w_{i+1} \) and \( l_i \leq l_{i+1} \); d = \((d_1, \ldots, d_n)\): draw sequence of S; \( M_{n \times n} \) (result matrix): \( M_{i,j} \) is the number of points received by \( T_i \) in the match against \( T_j \); w = \((w_1, \ldots, w_n)\): \( w_i \) is the number of wins of \( T_i \); l = \((l_1, \ldots, l_n)\): \( l_i \) is the number of losses of \( T_i \).

Draw-Uniform-Rec(\( n, s, d \))

01 for \( i = 1 \) to \( n \) // line 01–03: initialization of \( M \)
R2 uses a special version of Havel-Hakimi algorithm called HAVEL-HAKIMI-DRAWS (or shortly HHD). While for the classical Havel-Hakimi algorithm the equal scores are equivalent, in this application we have to distinguish them.

Additional parameters are d = (d_1,\ldots, d_n): a draw sequence produced by DRAW-REC; M: n \times n sized matrix where M_{ij} is the number of points received by T_i in the match with T_j; E = (E_1,\ldots, E_n) = ((e_1, h_1),\ldots, (e_n, h_n)): current extended and sorted version of d; H = (h_1,\ldots, h_n): h_i is the index of e_i in d; n_l: lower index of the essential part of E; n_r: upper index of the essential part of E; c = (c_0,\ldots, c_n): c_i is the number of i's among e_{n_l},\ldots, e_{n_r}; C = (C_0,\ldots, C_n): C_i is the cumulated number of i's among e_{n_l},\ldots, e_{n_r}.

**HAVEL-HAKIMI-DRAWS(n, d, M)**

01 n_l = 1 // line 01–05: initialization of n_l, n_r, and E;
02 n_r = n
03 for i = n_l to n_r // line 03–07: initialization of G and n_r;
04 \quad e_i = d_i
05 \quad h_i = i
06 \quad n_r = n
07 for i = 1 to n // line 08–15: pairing of the draws;
08 \quad COUNTING-SORT-DRAWS(n, i, n_r, E) // line 09: sorting
if \( e_i = 0 \)

return \( M \)

for \( k = 1 \) to \( e_i \)

\[ M_{h_i, h_i+k} = M_{h_i+k, h_i} = 1 \]  
// line 13: a draw is fixed

\( e_{i+k} = e_{i+k} + 1 \)

while \( n_u == 0 \)

\( n_u = n_u - h_i \)

return \( M \)  
// line 17: return the matrix containing the paired draws

\[ \text{Counting-Sort-Draws} \]

is a modified version of the well-known linear time sorting algorithm \text{Counting-Sort} [27].

Additional parameters are \( d = (d_1, \ldots, d_n) \): a draw sequence produced by \text{Draw-Uniform-Rec}; \( n_l \): lower index of the essential part of \( E \); \( n_u \): upper index of the essential part of \( E \); \( M \): \( n \times n \) sized matrix where \( M_{ij} \) is the number of points received by \( T_i \) in the match with \( T_j \); \( E = (E_1, \ldots, E_n) = ((g_{11}, g_{12}), \ldots, (g_{1n}, g_{2n})) \): current extended and sorted version of \( d \) with the corresponding indices; \( G \): the working version of \( E \); \( n_l \): lower index of the essential part of \( E \); \( n_u \): upper index of the essential part of \( E \); \( c = (c_0, \ldots, c_{n-1}) \): \( c_i \) is the number of \( i \)'s among \( g_{1,n_l}, \ldots, g_{1,n_u} \); \( C_n = 0 \) working variable; \( C = (C_0, \ldots, C_{n-1}) \): \( C_i \) is the number of investigated scores larger or equal with \( i \).

\text{Counting-Sort-Draws}(n, d, n_l, n_u, E)

01 for \( i = n_l \) to \( n_u \)  
// line 01–05: initialization of \( G \) and \( c \);
02 \( g_{1,i} = e_{1,i} \)
03 \( g_{2,i} = e_{2,i} \)
04 for \( i = 0 \) to \( n - 1 \)
05 \( c_1 = 0 \)
06 for \( i = n_l \) to \( n_u \)  
// line 06-10: computation of the counters
07 \( c_{g_{1i}} = c_{g_{1i}} + 1 \)
08 \( C_n = 1 \)
09 for \( n - 1 \) downto \( 0 \)
10 \( C_i = C_{i+1} + c_i \)
11 for \( i = n_l \) to \( n_u \)  
// line 11-16: computation of the new \( E \)
12 \( x = C_{g_{1,i}} + 1 \)
13 \( e_{1,x} = g_{1,i} \)
14 \( e_{2,x} = g_{2,i} \)
15 \( C_x = C_x + 1 \)
16 return \( E \)

The running time of \text{Counting-Sort-Draw} is \( \Theta(n) \), of \text{Havel-Hakimi-Draw} is \( O(n^2) \) and the one of \text{Draw-Uniform-Rec} is also \( O(n^2) \).
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As an example let \( s = (1,1,7,7) \). Then \( s \) has a unique draw sequence \((1,1,1,1)\) and unique sport matrix shown in Table 8.

\[
\begin{array}{|c|c|c|c|c|}
\hline
i & w_i & d_i & l_i & s_i \\
\hline
1 & 0 & 1 & 2 & 1 \\
2 & 0 & 1 & 2 & 1 \\
3 & 2 & 1 & 0 & 7 \\
4 & 2 & 1 & 0 & 7 \\
\hline
\end{array}
\]

Table 8: Unique sport matrix belonging to the sequence \( s = (1,1,7,7) \).

According to the relatively quick version HAVEL-HAKIMI-SHIFTING [62] \( T_1 \) plays a draw with \( T_4 \) and \( T_2 \) with \( T_3 \) resulting the partial result matrix shown in Table 9.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
i & T_1 & T_2 & T_3 & T_4 & s_i \\
\hline
1 & 1 & ? & ? & 1 & 1 \\
2 & ? & 1 & ? & 1 & ? \\
3 & ? & 1 & ? & 7 & ? \\
4 & 1 & ? & ? & 7 & ? \\
\hline
\end{array}
\]

Table 9: Partial result matrix belonging to the draws of \( s = (1,1,7,7) \).

The partial result matrix containing the draws in Table 9 is not reconstructible since no acceptable result for the match between \( T_1 \) and \( T_2 \).

If we use the classical Havel-Hakimi algorithm then the draws are between \( T_1 \) and \( T_2 \), further between \( T_3 \) and \( T_4 \) and our greedy algorithm DRAW-UNIFORM-REC reconstructs the received partial result matrix.

Another example let \( s = (1,1,8,8,10,13) \). Then \( s \) has a unique draw sequence \((1,1,2,2,1,1)\) and a unique sport matrix shown in Table 10.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
i & w_i & d_i & l_i & s_i \\
\hline
1 & 0 & 1 & 4 & 1 \\
2 & 0 & 1 & 4 & 1 \\
3 & 2 & 2 & 1 & 8 \\
4 & 2 & 2 & 1 & 8 \\
5 & 3 & 1 & 1 & 10 \\
6 & 4 & 1 & 0 & 13 \\
\hline
\end{array}
\]

Table 10: Unique sport matrix belonging to the sequence \( s = (1,1,8,8,10,13) \).
In this case at first $E = ((2, 3), (2, 4), (1, 1), (1, 2), (1, 5), (1, 6))$. The draws allocated by HHD are shown in Table 11.

| i | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $s_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | ? | 1 | ? | ? | 1 |
| 2 | ? | ? | 1 | ? | ? | 1 |
| 3 | 1 | ? | ? | 1 | ? | ? |
| 4 | ? | 1 | ? | ? | ? | 8 |
| 5 | ? | ? | ? | ? | ? | 8 |
| 5 | ? | ? | ? | ? | ? | 10 |
| 6 | ? | ? | ? | ? | 1 | ? |

Table 11: Partial result matrix belonging to the draws of $s = (1, 1, 8, 8, 10, 13)$.

The partial result matrix in Table 11 is not reconstructible since no acceptable result for the match between $T_1$ and $T_2$.

### 5.3 Reconstruction algorithm $R_3 = \text{Draw-Inner-Rec}$

Reconstruction algorithm $R_3 = \text{Draw-Inner-Rec}$ is an improved version of $R_2$: it takes into account the obligatory inner draws.

The base of $\text{INNER-DRAWS}$ is the following lemma.

**Lemma 39** If $n \geq 1$, $f = (f_1, \ldots, f_n)$ is a football sequence, $1 \leq k \leq n$ and

$$
\sum_{i=1}^{k} f_i < 3 \binom{k}{2}, \tag{35}
$$

then among the teams $T_1, \ldots, T_k$ there are at least

$$
\left\lfloor 3 \binom{k}{2} - \sum_{i=1}^{k} f_i \right\rfloor / 2 \tag{36}
$$

draws.

**Proof.** If

$$
2 \left(3 \binom{k}{2} - \sum_{i=1}^{k} f_i \right) / 2 = q > 0, \tag{37}
$$

then the first $k$ teams lost at least $q$ points due to inner draws (or even more, if they gathered points in the matches against the remaining teams). \qed
Trying to reconstruct the sequence \( s = (1, 1, 8, 8, 10, 13) \) which was the last example of the previous Section 5.2 \textsc{Draw-Inner-Rec} (see Table 10 and 11) recognizes that \( s_1 + s_2 = 2 \) therefore according to Lemma 39 the obligatory result between \( T_1 \) and \( T_2 \) is a draw. Then \textsc{Draw-Inner-Rec} finishes the allocation of the draws as it is shown in Table 12.

|   | 1 | - | 1 | ? | ? | ? | 1 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | - | ? | ? | ? | ? | 1 |
| 3 | ? | ? | 1 | ? | 1 | ? | 8 |
| 4 | ? | ? | 1 | 1 | ? | 1 | 8 |
| 5 | ? | ? | 1 | ? | - | ? | 10 |
| 6 | ? | ? | 1 | 1 | ? | - | 13 |

Table 12: Partial result matrix belonging to the draws of \( s = (1, 1, 8, 8, 10, 13) \) allocated by \textsc{Draw-Inner-Rec}.

Using the matrix of the allocated draws shown in Table 12 \textsc{Draw-Uniform-Rec} produces the complete result matrix shown in Table 13. proving that \( s = (1, 1, 8, 8, 10, 13) \) is a football sequence.

|   | \( i \) | \( T_1 \) | \( T_2 \) | \( T_3 \) | \( T_4 \) | \( T_5 \) | \( T_6 \) | \( s_i \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | - | 1 | 0 | 0 | 0 | 0 | 1 |
| 2 | 1 | - | 0 | 0 | 0 | 0 | 1 |
| 3 | 3 | 3 | - | 1 | 1 | 0 | 8 |
| 4 | 3 | 3 | 1 | - | 0 | 1 | 8 |
| 5 | 3 | 3 | 1 | 3 | - | 0 | 10 |
| 6 | 3 | 3 | 3 | 1 | 3 | - | 13 |

Table 13: Partial result matrix belonging to the draws of \( s = (1, 1, 8, 8, 10, 13) \) allocated by \textsc{Draw-Inner-Rec}.

The algorithm based on this lemma yet is is not implemented.

6  

**Enumeration of football sequences**

There are many publications connected with the generation [5, 52, 58, 100] and enumeration of degree sequences of graphs, e.g. [4, 5, 8, 21, 26, 49, 62, 63, 67, 72, 78, 79, 81, 87, 92, 97, 98, 105, 117]. The problems connected with directed graphs sometimes are considered as problems of orientation of undirected graphs [37, 36, 38, 39].
The enumeration of degree \([4, 21, 40, 62, 63]\) and score \([49, 53]\) sequences also has a reach literature.

The first published enumeration results connected with football score sequences belong to Gábor Kovács, Norbert Pataki, Zoltán Hernyák and Tamás Hegyessy \([73]\) who computed \(F(n)\) for \(n = 1, \ldots, 8\) in 2002. N. J. A. Sloane in May 2007 determined \(F(9)\), then in June 2008 Min Li computed \(F(10)\). The newest results were received by J. E. Schoenfield who computed \(F(11)\) in September of 2008 and \(F(12)\) in December of 2008 \([100]\).

Connected problems are the listing of all degree sequences and sampling of degree sequences \([11, 15, 28, 65, 66, 82]\).

Our basic method is similar as we enumerated the degree sequences of simple graphs \([62, 103]\).

On one side we try to test the elements of the possible smallest set, and from the other side we try to use quickly as possible testing and reconstruction algorithms.

A natural idea is to investigate only the nonincreasing sequences of integers having 0 as lower bound and \(3(n - 1)\) as upper bound. Paul Erdős and Tibor Gallai called such sequences regular \([32]\). The number of such sequences is given by (1).

### 6.1 Decreasing of the number of the investigated sequences

A useful tool of the enumeration of the number of football sequences is the decreasing of the number of the considered sequences.

In Section 4 we proposed and analyzed filtering of regular sequences with constant, linear and quadratic time algorithms. For 14 teams we excluded more than the half of the regular sequences by the constant time algorithms. For 13 teams the linear and quadratic algorithms left less than 10.58 percent of the regular sequences. In Section 5 the polynomial reconstruction algorithms decreased the fraction of the undecided regular sequences to 4.68 percent of the regular sequences.

### 6.2 Backtrack filtering and accepting test

This method is due to Antal Iványi \([54, 73]\).

The results of the filtering algorithms are summarized in Table 14.

The running time of the filtering algorithms are presented in Table 15. The times are cumulated and contain the time necessary for the generation of the sequences too.
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| n  | Constant | Linear | Quad | Backtrack = F |
|----|----------|--------|------|---------------|
| 1  | 1        | 1      | 1    | 1             |
| 2  | 2        | 2      | 2    | 2             |
| 3  | 14       | 7      | 7    | 7             |
| 4  | 203      | 40     | 40   | 40            |
| 5  | 2 133    | 365    | 355  | 355           |
| 6  | 20 518   | 4 086  | 3 760| 3 678         |
| 7  | 191 707  | 44 657 | 39 417| 27 263        |
| 8  | 1 772 442| 451 213| 393 072| 361 058      |
| 9  | 16 332 091| 4 348 655| 3 804 485| 3 403 613   |
| 10 | 150 288 309| 41 166 157| 36 302 148| 31 653 777 |
| 11 | 1 383 099 467| 387 416 935| 344 012 885| 292 547 199|
| 12 | 12 737 278 674| 3 633 749 149| 3 246 651 763| 2 696 619 716|
| 13 | 117 411 154 292| 33 821 636 274| 30 405 902 165| ...
| 14 | 1 083 421 567 482| ...

Table 14: Numbers of sequences accepted by constant, linear and quadratic time and Backtrack filtering algorithms for n = 1, ..., 14 teams.

| n  | Constant | Linear | Quad | Backtrack = F |
|----|----------|--------|------|---------------|
| 1  | 0.000    | 0.000  | 0.000| 0.000         |
| 2  | 0.000    | 0.000  | 0.000| 0.000         |
| 3  | 0.000    | 0.000  | 0.000| 0.000         |
| 4  | 0.000    | 0.000  | 0.000| 0.000         |
| 5  | 0.000    | 0.000  | 0.000| 0.000         |
| 6  | 0.000    | 0.000  | 0.000| 0.015         |
| 7  | 0.016    | 0.031  | 0.042| 0.172         |
| 8  | 0.046    | 0.375  | 0.577| 52.603        |
| 9  | 0.468    | 3.572  | 5.772|               |
| 10 | 4.134    | 34.632 | 54.741|              |
| 11 | 37.612   | 329.816| 525.752|            |
| 12 | 343.575  | 3 145.494| 4 998.831|          |
| 13 | 3 142.469| 30 541.260| 49 035.625|        |
| 14 | 29 438.094|          |           |            |

Table 15: Running times of constant, linear and quadratic time filtering algorithms for n = 1, ..., 14 teams.

The individual results of the reconstruction algorithms are summarized in Table 16.
The running times of the reconstruction algorithms are shown in Table 17.
| n   | R1   | R2 + R3 | Backtrack | F   |
|-----|------|---------|-----------|-----|
| 1   | 1    | 0       | 0         | 1   |
| 2   | 2    | 0       | 0         | 2   |
| 3   | 6    | 1       | 0         | 7   |
| 4   | 18   | 22      | 0         | 40  |
| 5   | 50   | 305     | 0         | 355 |
| 6   | 137  | 3 460   | 81        | 3 678 |
| 7   | 375  | 33 993  | 2 895     | 37 263 |
| 8   | 1 023| 304 349 | 56 909    | 361 058 |
| 9   | 2 776| 2 576 124| 3 403 613 |
| 10  | 7 498| 21 453 751| 31 653 777 |
| 11  | 20 177| 177 819 555| 292 547 199 |
| 12  | 54 127| 1 476 661 425| 2 696 619 716 |
| 13  | 144 708| 12 300 060 430|            |

Table 16: Number of \((0,3n - 3, n)\)-regular sequences reconstructed by reconstruction algorithms R1, R2 + R3 and Backtrack for \(n = 1, \ldots, 14\) teams.

| n   | R1     | R3     | Backtrack |
|-----|--------|--------|-----------|
| 2   | 0.000  | 0.000  | 0.000     |
| 3   | 0.000  | 0.000  | 0.000     |
| 4   | 0.000  | 0.000  | 0.000     |
| 5   | 0.000  | 0.000  | 0.000     |
| 6   | 0.000  | 0.015  | 0.015     |
| 7   | 0.063  | 0.109  | 0.172     |
| 8   | 0.546  | 1.264  | 52.603    |
| 9   | 5.491  | 15.226 |           |
| 10  | 53.880 | 179.249|           |
| 11  | 522.386| 2 066.323|        |
| 12  | 4 998.831| 23 429.877|      |
| 13  | 49 035.625| 261 904.750|     |

Table 17: Running times of the R1, R3 and Backtrack reconstructing algorithms for \(n = 1, \ldots, 13\) teams.
6.3 Recursive accepting test

This method is due to Schoenfield [100]. According to this method we compare the sequences of length \( n \) passed through the filtering and accepting tests with the good sequences of length \( n - 1 \) whether they can be derive from them.

Since if we omit a team with its results from a football matrix of size \( n \times n \), then we get a football matrix of size \( (n - 1) \times (n - 1) \), therefore we regularly delete the first elements of the investigated \( n \)-length sequences.

Let \( n \geq 2 \). We suppose that when we enumerate the \( n \)-length good sequences then we know the \( F(n - 1) \times (n - 1) \) sized matrix \( M \) containing the \( (n - 1) \)-length good sequences in lexicographically increasing order, and also know the vector \( (P_0, \ldots, P_k) \), where \( k = \lfloor 3(n - 1)/2 \rfloor \) and \( P_i \) gives the number of \( (n - 1) \)-length good sequences starting with \( i \).

Let start the recursion with \( n = 2 \). Matrix \( M_1 \) contains only one row \( (0) \) and \( P \) contains one element \( P(1) = 1 \).

The constant time filtering algorithms accept only the sequences \( (0,3) \) and \( (1,1) \). At first we omit 0 from the first sequence and state that the remaining sequence \( (3) \) can be derived from \( (0) \) only if the team having zero points in the shorter sequence wins against the omitted player. So the omitted player has to have zero points. Since the omitted score is exactly zero, \( (0,3) \) is a good sequence.

Then we delete the first element from the sequence \( (1,1) \) and compare the remaining \( (1,1) \) with the known good sequences. There are thee possibilities: the first team of the good sequence received 3, 1 or 0 points against the omitted one. If 3, then the good sequence has to start with 0. There is only one sequence \( (0,3) \) requiring two losses for the omitted team. Since the omitted element is exactly zero, \( (0,3,6) \) is a good sequence.

Now let \( n = 3 \). Then \( M_2 \) contains two rows: \( (0,3) \) and \( (1,1) \). In this case the filtering algorithms accept only the seven good sequences: \( (0,3,6) \), \( (0,4,4) \), \( (1,1,4) \), \( (1,2,4) \), \( (1,3,4) \), \( (2,2,2) \) and \( (3,3,3) \).

At first we delete 0 from \( (0,3,6) \) and compare the remaining \( (3,6) \) with the known good sequences. There are thee possibilities: the first team of the good sequence received 3, 1 or 0 points against the omitted one. If 3, then the good sequence has to start with 0. There is only one sequence \( (0,3) \) requiring two losses for the omitted team. Since the omitted element is exactly zero, \( (0,3,6) \) is a good sequence.

The second accepted sequence is \( (0,4,4) \). Omitting 0 and comparing \( (4,4) \) with the good sequences we get, that \( (1,1) \) is the only potential ancestor requiring zero points for the deleted team. Since it has exactly zero points, \( (0,4,4) \) is also a good sequence.

In a similar way we can prove that the remaining five accepted sequences are also good.
When \( n = 4 \) then \( M \) contains seven elements and \( P = (1, 3, 6, 7) \).

Reconstruct executes this recursive step. Its additional parameters are \( F(n-1) \): the number of \((n-1)\)-length good sequences; \( M_{F(n-1)\times(n-1)} \): matrix of good sequences of length \( n - 1 \) (this matrix consists of submatrices containing the good sequences having identical first element; \( P = (P_0, \ldots, P_k) \), where \( k = \lfloor 3(n-1)/2 \rfloor \) and \( P_i \) is the number of \( n-1 \) length football sequences starting with \( i \); \( N_{F(n)\times n} \): matrix of good sequences of length \( n \); \( m = (m_1, \ldots, m_{n-1}) \): the current reduced version of \( s \); \( d \): the current score of the deleted team.

**Reconstruct** \((n, s, F, M, P)\)

01 \( L = 1 \) line 01–02: initialization of \( L \) and \( u \)
02 \( u = \lfloor 3(n-1)/2 \rfloor \)
03 **if** \( s_2 \leq u \) **//** line 03-21: omitted element starts with a loss
04 \hspace{1em} \( j \leftarrow P_{s_2} \)
05 **while** \( M_{j,1} == s_2 \)
06 \hspace{1em} \( d \leftarrow 0 \)
07 \hspace{1em} \( k \leftarrow 2 \)
08 \hspace{2em} **while** \( k \leq n \)
09 \hspace{3em} **if** \( s_k - M_{j,k} == 3 \)
10 \hspace{4em} \( d = d + 0 \)
11 \hspace{4em} **go to** 19
12 \hspace{3em} **if** \( s_k - M_{j,k} == 1 \)
13 \hspace{4em} \( d = d + 1 \)
14 \hspace{4em} **go to** 19
15 \hspace{3em} **if** \( s_k - M_{j,k} == 0 \)
16 \hspace{4em} \( d = d + 3 \)
17 \hspace{4em} **go to** 19
18 \hspace{3em} **go to** 22
19 \hspace{1em} \( k \leftarrow k + 1 \)
20 **if** \( d == s_1 \)
21 \hspace{1em} **return** \( L \)
22 **if** \( 0 \leq s_2 - 1 \) **//** line 22-40: omitted element starts with a draw
23 \hspace{1em} \( j \leftarrow P_{s_2-1} \)
24 **while** \( M_{j,1} == s_2 - 1 \)
25 \hspace{1em} \( d \leftarrow 1 \)
26 \hspace{1em} \( k \leftarrow 2 \)
27 **while** \( k \leq n \)
28 \hspace{2em} **if** \( s_k - M_{j,k} == 1 \)
29 \hspace{3em} \( d = d + 1 \)
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Table 18 shows the number of regular sequences \((R(n))\), the number of football sequences \((F(n))\), the ratio \((R(n + 1)/R(n))\), the ratio \(F(n + 1)/F(n)\), and the ratio \((F(n)/R(n))\) for \(n = 1, \ldots, 12\). In this table if \(n \geq 2\) then \(R(n)\) is decreasing.

**Lemma 40** If \(n\) tends to infinity then \(R(n + 1)/R(n)\) tends to \(256/27\).
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
n & R(n) & R(n+1) & F(n) & F(n+1) \\
\hline
1 & 1 & 10.000 & 1 & 2.000 \\
2 & 1 & 8.400 & 2 & 3.500 \\
3 & 84 & 8.512 & 7 & 5.714 \\
4 & 715 & 8.655 & 40 & 8.875 \\
5 & 6188 & 8.769 & 355 & 10.361 \\
6 & 54264 & 8.859 & 3678 & 10.131 \\
7 & 480700 & 8.929 & 37263 & 9.689 \\
8 & 4292145 & 8.986 & 361058 & 9.427 \\
9 & 38567100 & 9.032 & 3403613 & 9.300 \\
10 & 348330136 & 9.070 & 31653777 & 9.242 \\
11 & 3159461968 & 9.103 & 292547199 & 9.217 \\
12 & 28760021745 & 9.131 & 2696619716 & 9.242 \\
13 & 262596783864 & 9.155 \\
14 & 240397990420 \\
\hline
\end{tabular}
\end{center}
\caption{Number of regular and football sequences and the ratio of these numbers for neighboring numbers of teams}
\end{table}

Proof. According to (1)

\[
\frac{R(n+1)}{R(n)} = \frac{(4n+1)(4n)(4n-1)(4n-2)}{(n+1)(3n)(3n-1)(3n-2)} = \frac{256}{27} + o(1),
\]

implying the required limit. \(\square\)

If \(n \geq 1\) then in Table 18 \(F(n+1)/F(n)\) is nondecreasing. We suppose that it tends to 1.

If \(5 \leq n \leq 12\) then \(F(n)/R(n)\) is increasing. It is easy to see that

\[
\lim_{n \to \infty} \frac{F(n+1)}{F(n)} \leq \frac{R(n+1)}{R(n)}.
\]

The behavior of \(F(n)/R(n)\) is a bit surprising since the similar relative density of tournaments score sequences tends to zero (see [21]). We suppose that \(F(n)/R(n)\) also tends to zero but the convergence is slow.

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