THE INTERSECTION HOMOLOGY \(D\)-MODULE IN FINITE CHARACTERISTIC.

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Abstract. For \(Y\) a closed normal subvariety of codimension \(c\) of a smooth \(\mathbb{C}\)–variety \(X\), Brylinski and Kashiwara showed that the local cohomology module \(H^c_Y(X,\mathcal{O}_X)\) contains a unique simple \(D_X\)–submodule, denoted by \(L(Y, X)\). In this paper the analogous result is shown for \(X\) and \(Y\) defined over a perfect field of finite characteristic. Moreover, a local construction of \(L(Y, X)\) is given, relating it to the theory of tight closure. From the construction one obtains a criterion for the \(D_X\)–simplicity of \(H^c_Y(X,\mathcal{O}_X)\).

1. Introduction

Let \(Y\) be a closed codimension \(c\) subvariety of the smooth \(\mathbb{C}\) variety \(X\) and let \(Z\) be the singular locus of \(Y\). Denote by \(D_X\) the sheaf or differential operators on \(X\). In [BK81, Proposition 8.5], Brylinski and Kashiwara show the existence (and usefulness) of a unique holonomic \(D_X\) module \(L = L(Y, X)\) satisfying the properties

\[
L|_{X-Z} \cong H^0_Y(X-Z,\mathcal{O}_{X-Z})
\]

\[
H^0_Z(L) = H^0_Z(L^*) = 0,
\]

where the star stands for duality of holonomic \(D\)–modules and \(H^i_Y\) denotes the higher derived sections with support in \(Y\). The proof of this result is rather formal and uses duality theory for holonomic \(D_X\)–modules. Furthermore, they show that \(L(Y, X)\) is the unique simple, selfdual holonomic \(D_X\)–module agreeing with \(H^0_Y(X,\mathcal{O}_X)\) on \(X - Z\). This result is obtained by showing that \(L(Y, X)\) corresponds, via the Riemann–Hilbert correspondence, to the intersection homology complex \(\pi_Y\) of middle perversity, which, by construction, is simple and selfdual. All these constructions, such as holonomicity, duality and the Riemann–Hilbert correspondence completely rely on characteristic zero – on analytic techniques even, if one is strict.

The question answered in this paper is: What is the situation if \(X\) is defined over a field of positive characteristic? Somewhat surprisingly, the existence of a unique simple \(D_X\)–submodule \(L(Y, X)\) can be proved almost independent of the characteristic. The key ingredient – the proof of which is characteristic dependent though – is that \(H^0_Y(X,\mathcal{O}_X)\) has finite length as a \(D_X\)–module. This is guaranteed by holonomicity in characteristic 0 and by [Lyu97, Theorem 5.7] in positive characteristic, respectively.

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We state the result and sketch the simple argument – for a complete proof refer to Theorem 4.1.

**Theorem.** Let $X$ be a smooth $k$–variety and let $Y$ be a closed irreducible subvariety of codimension $c$. Then $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ has a unique simple $\mathcal{D}_X$–submodule $\mathcal{L}(Y, X)$. Furthermore, $\mathcal{L}(Y, X)$ agrees with $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ on $X – \text{Sing} Y$.

**Proof.** (Sketch) Since $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ has finite length as a $\mathcal{D}_X$–module it has some simple non-zero $\mathcal{D}_X$–submodule $\mathcal{L}$. Denote the inclusion $X' \overset{\text{def}}{=} X – \text{Sing} Y \subseteq X$ by $i$ and write $Y'$ for $Y – \text{Sing} Y$. One sees easily that the restriction of $\mathcal{L}$ to $X'$ is nonzero. As the restriction of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ is equal to $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$, and by smoothness of $Y'$, the latter is $\mathcal{D}_X$–simple it follows that $\mathcal{L}|_{X'} = \mathcal{H}^c_{Y'}(X, \mathcal{O}_X)|_{X'}$. Since this holds for any simple submodule of $\mathcal{H}^c_Y(X, \mathcal{O}_X)$ the same argument shows that any two such have nonzero intersection, thus they are equal. This shows the uniqueness. \hfill \Box

This existence proof gives very little information about the concrete structure of $\mathcal{L}(Y, X)$. Even in characteristic zero, to explicitly determine $\mathcal{L}(Y, X)$ is difficult. The best results in this case are due to Vilonen [Vil85] for $Y$ a complete intersection with an isolated singularity. He uses analytic techniques to characterize the sections of $\mathcal{H}^c_Y(X, \mathcal{O}_Y)$ belonging to $\mathcal{L}(Y, X)$. They are precisely the ones vanishing under a certain residue map. Furthermore he gives a canonical generator, the canonical class associated to $Y \subseteq X$, for $\mathcal{L}(Y, X)$ in this case.

To explicitly determine $\mathcal{L}(Y, X)$ in positive characteristic is the main purpose of this paper. The strategy is to use the Frobenius instead of the differential structure. This substitution is justified by the close relationship of so called unit $\mathcal{O}_X[F^c]$–structures and $\mathcal{D}_X$–structures, described in [Lyu97, Bli03, EK00]. Our construction is local in nature. If we denote by $R$ and $A = R/I$ the local rings of $X$ and $Y$ at a point $x \in Y$, we roughly show the following, for precise statements see Section 4.1.

**Theorem.** Let $R$ be regular, local and $F$–finite. Let $A = R/I$ be a normal domain. Then the unique simple $\mathcal{D}_R$–submodule, $\mathcal{L}(A, R)$, of $H^c_Y(R)$ is dual to the unique simple $A[F^c]$–module quotient of $H^d_m(A)$.

The duality we are referring to is an extension of Matlis duality incorporating Frobenius actions. Furthermore, the construction is explicit enough to identify (non canonical) generators for $\mathcal{L}(A, R)$. What we have gained is that the unique simple $A[F^c]$–module quotient of $H^d_m(A)$ is well studied and fairly well understood; it is the quotient of $H^d_m(A)$ by the tight closure of zero, $0^*_m(A)$. The vanishing of $0^*_m(A)$ is governed by $F$–rationality of $A$, which is a positive characteristic analog of rational singularities. As a consequence of this connection we obtain the following $D_R$–simplicity criterion for $H^c_Y(R)$:

**Theorem.** Let $R$ be regular, local and $F$–finite. Let $A = R/I$ be a Cohen–Macaulay domain of codimension $c$. Then, if $A$ is $F$–rational then $H^c_Y(R)$ is $D_R$–simple.

More precise simplicity criteria for $H^c_Y(R)$ are given in Section 4.2.
The paper is structured as follows. In Sections 2 and 3 we recall and further develop some necessary machinery from the theory of \( R[F^e] \)-modules and tight closure. As the techniques used later are local in nature, the notation reflects this and we mainly speak of rings and ideals instead of schemes and their sub-schemes. In these sections we do not concretely deal with the applications to constructing \( \mathcal{L}(Y, X) \) but derive general results which constitute the technical underpinning of what follows. As a notable byproduct we answer a question posed by Lyubeznik showing that minimal roots exist for finitely generated unit \( R[F^e] \)-modules for any regular, local ring \( R \). In [Lyu97] this was only shown in the complete case.

Section 4 contains the main results discussed above and generalizations thereof. Furthermore, as an application to tight closures theory we show that the parameter test module commutes with localization. We finish this section with a complete characterization of \( \mathcal{D}_R \)-simplicity in the case of curves, providing a finite characteristic analog of results of Yekutieli [Yek98] and S. P. Smith [Smi88].

Finally we remark that the substitution of the \( \mathcal{D}_X \)-module structure by a unit \( \mathcal{O}_X[F^e] \)-structure in the study of \( \mathcal{L}(A, R) \), in finite characteristic, in the context of a Riemann–Hilbert type correspondence. That such a correspondence exists is recent work of Emerton and Kisin [EK99, EK00], where an equivalence (on the level of derived categories) of the category of finitely generate unit \( \mathcal{O}_X[F^e] \)-modules and the category of constructible \( \mathbb{F}_p \)-sheaves is developed. Within this correspondence, the simple unit \( \mathcal{O}_X[F^e] \)-module \( \mathcal{L}(Y, X) \) constructed here does indeed correspond to certain middle extensions on the constructible \( \mathbb{F}_p \)-site. These connections will not be discussed here but should appear in the final version of [EK00], and are outlined in [EK03].

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2. Background on \( R[F^e] \)-modules

Throughout this paper, \( R \) denotes a noetherian ring of dimension \( n \) containing a field \( k \) of positive characteristic \( p \), unless stated otherwise. For an ideal \( I \) of height \( c \) we denote the quotient \( R/I \) by \( A \). This is a ring of dimension \( d = n - c \). In general we assume that \( R \) is regular and \( F \)-finite, i.e. \( R \) is a finite module over its subring of \( p \)th powers.

The \textit{(absolute) Frobenius map} on \( R \), i.e. the ring map sending each element to its \( p \)th power, is denoted by \( F = F_R \). The associated map on \( X = \text{Spec} \, R \) is denoted by the same letter \( F = F_X \).

If \( \mathcal{M} \) is an \( R \)-module, then \( \mathcal{M}^e \) denotes the \( R-R \)-bimodule, which, as a left module is just \( \mathcal{M} \), but with right structure twisted by the \( e \)th iterate of the Frobenius, i.e. for \( r \in R \) and \( m \in \mathcal{M} \) one has \( m \cdot r = r^p \cdot m \). With this notation Psekeine and Szpiro’s \textit{Frobenius functor} is defined as \( F^*(\mathcal{M}) = R^1 \otimes \mathcal{M} \). Clearly, \( F^* \) commutes with direct limits and direct sums. If \( R \) is regular, \( F^* \) is flat; therefore it commutes with finite intersections. The
flatness of $F^*$ in the regular case is where the theory draws its power from. The same is valid for higher powers of the Frobenius and clearly we have $(F^*)^r = (F^r)^e$ which we denote by $F^{e^r}$.

We review the definition and basic properties of modules with Frobenius action. Since we are being extremely brief with this here, we advise the reader with no prior exposure to first consult Section 2 of [Blö03]. For a thorough introduction see for example [Blö01], Chapter 2, or [EK99, Lyu97].

**Definition 2.1.** An $R[F^e]$–module is an $R$–module $\mathcal{M}$ together with an $R$–linear map

$$\vartheta^e : F^{e*}\mathcal{M} = R^e \otimes \mathcal{M} \to \mathcal{M}.$$  

If $\vartheta^e$ is an isomorphism, then $(\mathcal{M}, \vartheta^e)$ is called a unit $R[F^e]$–module.

By adjointness these maps $\vartheta^e \in \text{Hom}(F^{e*}\mathcal{M}, \mathcal{M})$ are in one-to-one correspondence with maps $F^e_{\mathcal{M}} \in \text{Hom}(\mathcal{M}, F^e_*\mathcal{M})$ where $F^e_{\mathcal{M}}(m) = \vartheta^e_{\mathcal{M}}(1 \otimes m)$. Therefore, an $R[F^e]$–module is nothing but a module over the non-commutative ring

$$R[F^e] = \frac{R[F^e]}{rb^eF^e - F^e},$$

where $F^e$ acts on $M$ via $F^e_{\mathcal{M}}$. Then the category of $R[F^e]$–modules, $R[F^e]$–mod, is the category of left modules over this ring $R[F^e]$. As the module category over an associative ring, $R[F^e]$–mod is an abelian category. The category of unit $R[F^e]$–modules, $uR[F^e]$–mod, is the full subcategory whose objects are those $R[F^e]$–modules which are unit. Since $R$ is regular, the resulting flatness of $F^{e*}$ implies that this is also an abelian category. If $\mathcal{N}$ is an $R$–submodule of the $R[F^e]$–module $(\mathcal{M}, \vartheta^e, F^e)$, then we denote for convenience $\vartheta^e(F^{e*}\mathcal{N})$ by $F^e(\mathcal{N})$, which is just the $R$–submodule of $\mathcal{M}$ generated by all $F^e(n)$ for $n \in \mathcal{N}$.

**Definition 2.2.** An $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ is called finitely generated if it is a finitely generated module over the ring $R[F^e]$.

Let $\varphi : M \to F^{e*}M$ be an $R$–linear map. Consider the directed limit of the system of Frobenius powers of this map

$$\mathcal{M} = \lim_{\rightarrow}(M \xrightarrow{\varphi} F^{e*}M \xrightarrow{F^{e*}\varphi} F^{2e*}M \to \cdots)$$

which carries a natural unit $R[F^e]$–module structure. If a unit $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ arises in such a fashion one calls $\varphi$ a generator of $(\mathcal{M}, \vartheta^e)$. If $M$ is finitely generated it is called a finite generator, and if, in addition, $\varphi$ is injective, then $\mathcal{M}$ is called a root of $\mathcal{M}$. In this case one identifies $M$ with its isomorphic image in $\mathcal{M} = \lim_{\rightarrow} F^{e*}M$. Thus, a root of a unit $R[F]$–module $\mathcal{M}$ is a finitely generated $R$–submodule $M$, such that $M \subseteq RF^e(M)$ and $\mathcal{M} = \bigcup_{r} RF^{e^r}(M) = RF^e[M]$. A key observation is the following proposition, see [Blö03 Proposition 2.5] or [EK99] for proof:

**Proposition 2.3.** Let $R$ be regular. A unit $R[F^e]$–module $(\mathcal{M}, \vartheta^e)$ is finitely generated if and only if $\mathcal{M}$ has a root.

With this at hand one can easily show that the category of finitely generated unit $R[F^e]$–modules is an abelian subcategory with ACC of the category of $R[F^e]$–modules which is closed under extensions. Significantly more work
The Intersection Homology $D$-Module in Finite Characteristic.

(For the second part) is involved in showing the next important theorem, found as Proposition 2.7 and Theorem 3.2 in [Lyu97].

**Theorem 2.4.** Let $R$ be regular and let $\mathcal{M}$ be a finitely generated unit $R[F^e]$-module. Then $\mathcal{M}$ has ACC in the category of unit $R[F^e]$-modules

If $R$ is also a finitely generated algebra over a regular local ring, then $\mathcal{M}$ has DCC, i.e., $\mathcal{M}$ has finite length as a unit $R[F^e]$-module.

**Examples 2.5.** Standard examples of unit $R[F^e]$-modules are: (1) $R$ itself via the natural isomorphism $R^e \otimes R R \cong R$. (2) A localization $S^{-1} R$ of $R$ via the natural map $R^e \otimes S^{-1} R \rightarrow S^{-1} R$ whose inverse is the map $rs^{-1} \mapsto s^{e-1} r \otimes s^{-1}$. (3) The local cohomology modules, $H^1_I(R)$, of $R$ with support in an ideal $I$ which obtain their unit structure via the Čech complex which consists of localizations of $R$.

Note that any proper nonzero ideal $I \subseteq R$ is an $R[F^e]$-submodule but not unit. Thus $R$ is a simple unit $R[F^e]$-module.

2.1. **Base change.** Let $\pi : R \rightarrow S$ be a map of rings. The base change functor $\pi^* = S \otimes_R -$ extends to a functor from (unit) $R[F^e]$-modules to (unit) $S[F^e]$-modules. For $(\mathcal{M}, \partial^e)$, the (unit) $S[F^e]$-module structure on $S \otimes_R \mathcal{M}$ is given by

$$S^e \otimes_S S \otimes_R \mathcal{M} \cong S \otimes_R R^e \otimes_R \mathcal{M} - \text{id}_S \otimes \partial^e \rightarrow S \otimes_R \mathcal{M}.$$  

Clearly, this is an isomorphism if and only if $\partial^e$ is an isomorphism. For easy reference we record some properties of base change, the easy proofs are left to the reader.

**Proposition 2.6.** Let $R \rightarrow S$ be a map of rings. Let $\mathcal{M}$ be a finitely generated unit $R[F^e]$-module with generator $M$. (1) $S \otimes M$ is a generator of the finitely generated unit $S[F^e]$-module $S \otimes \mathcal{M}$. (2) If $R$ and $S$ are regular, then the image of $S \otimes M$ in $S \otimes \mathcal{M}$ is a root of $S \otimes \mathcal{M}$. (3) If $R \rightarrow S$ is also flat and $M$ is a root of $\mathcal{M}$, then $S \otimes M$ itself is a root of $S \otimes \mathcal{M}$. (4) If $R \rightarrow S$ is faithfully flat, then a submodule $M$ of $\mathcal{M}$ is a root of $\mathcal{M}$ if and only if $S \otimes M$ is a root of $S \otimes \mathcal{M}$.

2.2. **Restriction.** Still fixing the data of a map of rings $\pi : R \rightarrow S$, any $S[F^e]$-module $(\mathcal{N}, \partial^e)$ naturally carries an $R[F^e]$-module structure because $\pi$ induces a ring homomorphism $R[F^e] \rightarrow S[F^e]$. Note that in general, a unit $S[F^e]$-module, viewed as an $R[F^e]$-module (really $\pi_* \mathcal{N}$), is not unit.

What is the case is that restriction of scalars preserves the unit property, if and only if the relative Frobenius $F^e_{S/R} : R^e \otimes_R S \rightarrow S^e$ sending $r \otimes s$ to $\pi(r)s^{e^e}$ is an isomorphism of $R$-$S$-bimodules. The following proposition summarizes some cases where this happens:

**Proposition 2.7.** Let $\pi : R \rightarrow S$ be a map of rings. In the following cases, the relative Frobenius $F^e_{S/R} : R^e \otimes_R S \rightarrow S^e$ is an isomorphism:

1. $S$ is the localization of $R$ at some multiplicative set $T \subseteq R$, and $\pi$ is the localization map.
2. $R \rightarrow S$ is étale.
3. $R$ is regular local and $F$-finite, and $S$ is the $I$-adic completion of $R$ with respect to some ideal $I$ of $R$. 
In all these cases it follows that a unit \( S[F^e] \)-module is also unit as a \( R[F^e] \)-module.

Proof. The case of localization was already observed in Examples \[25\]. If \( R \rightarrow S \) is étale then so is \( R^e \rightarrow R^e \otimes_R S \) and \( R^e \rightarrow S^e \). Thus, by \cite[Corollaire 17.3.4]{Gro67} the relative Frobenius is étale too. It is an isomorphism if it is an isomorphism on fibers, thus we may assume \( R = k \) is a field and \( S \) is a finite product of separable algebraic extensions of \( k \). Then the claim is easy, see \cite[Theorem A.1.4]{Eis95}

Let \( R \rightarrow \hat{R} \) be the \( I \)-adic completion as in (2). By assumption, \( R^e \) is a finitely generated right \( R \)-module. Therefore \( R^e \otimes_R \hat{R} = \varprojlim R^e/R^e I^t \) by \cite[Theorem 7.2]{Eis95}. Using that the sequence \( R^e I^t = I^{[p^e]} \) is cofinal within the powers \( I^t \) of \( I \) we conclude

\[
R^e \otimes_R \hat{R} \cong \varprojlim \frac{R^e}{R^e I^t} = \varprojlim \frac{R^e}{I^{[p^e]} R^e} \cong \hat{R}^e.
\]

\( \Box \)

Generally, the property of being finitely generated is not preserved by restriction. The following is an important exception:

Proposition 2.8. Let \( S \) be finite étale over \( R_x \) with \( x \in R \). Then, a finitely generated unit \( S[F^e] \)-module \( M \) is finitely generated as a unit \( R[F^e] \)-module.

Proof. The module finiteness of \( S \) over \( R_x \) together with Proposition \[24\] shows that \( M \) is finitely generated as a unit \( R_x[F^e] \)-module. Now \cite[Proposition 6.8.1.]{EK99} finishes the proof.

\( \Box \)

Corollary 2.9. Let \( R \) be regular and \( R \rightarrow S \) be one of the cases of Proposition \[24\] that is in particular, \( R^e \otimes S \cong S^e \). Let \( M \) be a finitely generated unit \( R[F^e] \)-module with root \( M \subseteq \mathcal{M} \). Let \( \mathcal{N} \) be a finitely generated unit \( S[F^e] \)-submodule of \( S \otimes_R M \). Then \( \mathcal{N} \cap M \) is a root of the finitely generated unit \( R[F^e] \)-module \( \mathcal{N} \cap \mathcal{M} \).

Proof. By assumption, the finitely generated unit \( S[F^e] \)-modules \( S \otimes_R M \) and \( \mathcal{N} \) are unit \( R[F^e] \)-modules (though quite likely not finitely generated as \( R[F^e] \)-modules). The intersection of the two unit \( R[F^e] \)-submodules \( M \) and \( \mathcal{N} \) is a unit \( R[F^e] \)-module. As it is a submodule of the finitely generated module \( M \), it follows that \( M \cap \mathcal{N} \) is a finitely generated unit \( R[F^e] \)-module since the category of finitely generated unit \( R[F^e] \)-modules is abelian.

To check that the finitely generated module \( \mathcal{N} \) is a root of \( \mathcal{N} \) means that \( \bigcup F^e_S(\mathcal{N}) = \mathcal{N} \) and \( \mathcal{N} \subseteq F^e_S(\mathcal{N}) \). Thus

\[
F^e_R(\mathcal{N} \cap M) = F^e_R(\mathcal{N}) \cap F^e_R(M) = F^e_S(\mathcal{N}) \cap M \supseteq \mathcal{N} \cap M
\]

and

\[
\bigcup F^e_R(\mathcal{N} \cap M) = \bigcup (F^e_S(\mathcal{N}) \cap M) = \mathcal{N} \cap M.
\]

The key point was that for the \( S \)-submodule \( \mathcal{N} \) of \( S \otimes M \) one has

\[
F^e_R(\mathcal{N}) = F^e_S(\mathcal{N})
\]

by assumption. \( \Box \)
It is important to keep in mind that we did not exclude the case that \( M \cap N \) is zero in the last corollary. In particular it follows that \( N \cap M = 0 \) if and only if \( N \cap M = 0 \). Also note that \( N, S \odot M \cap N \) is a root of the \( S[F^e] \)-module \( N \) and naturally \( N \cap M = N \cap M \) is a root of the \( R[F^e] \)-module \( N \cap M \).

2.3. The minimal root. Building on the last proposition and corollary we prove a result on the existence of minimal roots for regular, \( F \)-finite local rings. This was previously only known in the complete case [Lyu97, Theorem 3.5]. First we recall two easily verifiable facts, namely that the intersection of finitely many roots of a finitely generated unit \( R[F^e] \)-module \( M \) is a root of \( M \), and that if \( N \) is a unit \( R[F^e] \)-submodule of \( M \), and \( M \) a root of \( M \), then \( M \cap N \) is a root of \( N \).

Theorem 2.10. Let \( R \) be regular local and \( F \)-finite and let \( M \) be a finitely generated unit \( R[F^e] \)-module. Then \( M \) has a unique minimal root.

Proof. Let \( \hat{R} \) denote the \( m \)-adic completion. By [Lyu97] Theorem 3.5 or [Bli01, Proposition 2.20], the finitely generated unit \( \hat{R}[F^e] \)-module \( \hat{R} \odot M \) has a unique minimal root \( N \). Proposition 2.4 is also a unit \( R[F^e] \)-module.

By Corollary 2.9 \( M \triangleq N \cap M \) is a root of the unit \( R[F^e] \)-module \( M \cap (\hat{R} \odot M) = M \). Clearly, \( \hat{R} \odot M \subseteq N \). Since \( \hat{R} \odot M \) is a root of \( \hat{R} \odot M \), it contains \( N \) by minimality of \( N \). Therefore \( N = \hat{R} \odot M \). Now it follows easily that \( M \) is indeed the unique minimal root of \( M \). If \( M' \subseteq M \) is another root, we have, by minimality of \( N \) the inclusion of roots \( N \subseteq \hat{R} \odot M' \subseteq \hat{R} \odot M \) of \( \hat{R} \odot M \). Since the first and last are equal we have that \( M' \) and \( M \) are equal upon completion. Thus \( M = M' \) by faithfully flat descent. \( \square \)

A consequence of the above proof is the following corollary.

Corollary 2.11. Let \((R, m)\) be regular local and \( F \)-finite. Let \( M \) be a finitely generated unit \( R[F^e] \)-module with minimal root \( M \). Then \( \hat{R} \odot M \) is the minimal root of \( \hat{R} \odot M \).

The question of whether minimal roots exist for not necessarily local rings \( R \) remains open.

2.4. Duality for \( R[F^e] \)-modules. A key tool in local algebra is Matlis Duality. If \((R, m)\) is local then the Matlis dual functor is defined as \( D(\_ \triangleq \text{Hom}(\_ , E_{R/m})) \), where \( E_{R/m} \) denotes the injective hull of \( R/m \). We seek to extend \( D (= D_R) \) to a Functor from \( R[F^e] \)-modules to \( R[F^e] \)-modules. How this can be done is described in [Bli01], Chapter 4, in complete detail, as a consequence of a general investigation on how to extend contravariant functors to incorporate Frobenius action. Here we only give the bare minimum to establish the extension of \( D \), most of the material can already be found in [Lyu97], Section 4.

Proposition 2.12. Let \( R \) be regular complete and local. The natural map

\[ \psi_M : F^e \text{Hom}(M, E_R) \to \text{Hom}(F^e M, F^e E_R) \]

is an isomorphism if \( R \) is \( F \)-finite or if \( M \) is finitely presented or cofinite.
In these cases we have an isomorphism of functors $\Psi : D \circ F^e \cong F^e \circ D$, that is, Matlis duality commutes with Frobenius.

Proof. In the first two cases this follows from the fact that the natural map
$$\psi : S \otimes \text{Hom}_R(M, N) \to \text{Hom}_S(S \otimes M, S \otimes N)$$
is an isomorphism provided that $R \to S$ is a flat map or rings and either
$S$ is module finite over $R$.\footnote{\cite{Hitt}} Proposition 4.9 or $M$ is finitely presented
\cite{Es95} Proposition 2.10. Thus it remains to treat the case that $M$ is a cofinite $R$–module which is treated in \cite{Lyu97} Lemma 4.1.

Fixing a unit structure on $E_R$ by fixing an isomorphism with $H^1_m(R)$ and
combining the above isomorphism with this unit $R[F^e]$–structure on $E_R$ we get a natural (after the unit $R[F^e]$–structure on $E_R/m$ is fixed) isomorphism
$$D(F^e_\ast M) \cong \text{Hom}(F^e_\ast M, F^e_\ast E_R) \cong F^e_\ast \text{Hom}(M, E_R) = F^e_\ast D(M)$$
as desired. \hfill $\square$

Now assume that $R$ is complete. Let $F^e_\ast M \xrightarrow{\vartheta^e} M$ be an $R[F^e]$–module
which is finitely generated or cofinite as an $R$–module or assume that $R$ is $F$–finite.

Applying the Matlis dual functor $D(\underline{\underline{\square}}) = \text{Hom}(\underline{\underline{\square}}, E_R)$ to the structural
morphism of $M$ and composing with the isomorphism of Proposition 2.12
one obtains a map
$$\beta^e : D(M) \xrightarrow{D(\vartheta^e)} D(F^e_\ast M) \xrightarrow{\Psi_M} F^e_\ast (D(M))$$
whose second part is just the isomorphism $\Psi$ form the last Proposition.

Definition 2.13. Let $R$ be complete and $(M, \vartheta^e)$ an $R[F^e]$–module (finitely
generated or cofinite as an $R$–module, if $R$ is not $F$–finite). If $\beta^e \overset{\text{def}}{=} \Psi_M \circ D(\vartheta^e)$, then
$$D(M) \overset{\text{def}}{=} \lim_D(D(M) \xrightarrow{\beta^e} F^e_\ast D(M) \xrightarrow{F^e_\ast \beta^e} F^{2e}_\ast D(M) \to \ldots )$$
is the unit $R[F^e]$–module generated by $\beta^e$. On $R[F^e]$–modules which are
cofinite as $R$–modules this defines an exact functor.

The exactness claim is clear since Matlis duality and direct limits are
exact functors. If $M$ is a unit $R[F^e]$–module then $D(M) = D(M)$, since $\beta^e$
is an isomorphism in this case. If $M$ is cofinite as an $R$–module then $D(M)$ is
a finitely generated $R$–module. Therefore $D(M)$ is a finitely generated
unit $R[F^e]$–module, since $D(M)$, its generator, is a finitely generated $R$–module.
If in addition $\vartheta^e$ is surjective, then $\beta^e$ is injective and therefore
$D(M)$ is a root of $D(M)$.

Notation 2.14. We introduce some notation from \cite{HS77}. An element $m \in M$ of the $R[F^e]$–module $(M, \vartheta^e)$ is called $F$–nilpotent if $F^{re}(m) = 0$ for
some $r$. Then $M$ is called $F$–nilpotent if $F^{re}(M) = 0$ for some $r \geq 0$. It is
possible that every element of $M$ is $F$–nilpotent but $M$ itself is not, since $F$–nilpotency for $M$ requires that all $m \in M$ are killed by the same
power of $F^e$. In particular the sub $F[R^e]$–module consisting of all $F$–nilpotent
elements $M_{\text{nil}}$ need not be nilpotent in general. If $\vartheta^e$ is surjective, then $M$
is called $F$–full. Note that $F$–fullness does not mean $F^e$ is surjective but
merely that the submodule \(F^e(M) = \vartheta^e(F^{er}M)\) is all of \(M\). Finally we say that \(M\) is \(F\)-\textit{reduced} if \(F^e\) acts injectively.

The above notions are the same if we view \(M\) as an \(R[F^{er}]\)-module for some \(r \geq 0\). Therefore they are valid without reference to a specific \(e\).

We are lead to some functorial constructions for \(R[F^e]\)-modules. The \(R[F^e]\)-submodule consisting of all \(F\)-nilpotent elements of \(M\) we denote by \(M_{\text{nil}} = \{m \in M \mid F^{er}(m) = 0 \text{ for some } r\}\). The quotient \(M/M_{\text{nil}}\) is the biggest \(F\)-reduced quotient, we denote it by \(M_{\text{red}}\). The \(R[F^e]\)-submodule \(F^\infty M = \bigcap F^{er}(M)\) is the largest \(F\)-full submodule. If \(M\) is a cofinite \(R\)-module, then the decreasing chain of \(R[F^e]\)-submodules \(F^{er}(M)\) stabilizes and we have \(F^\infty M = F^{er}(M)\) for some \(r > 0\). One can check that the operations \(F^\infty(\_\_)\) and \((\_\_)_{\text{red}}\) mutually commute which makes the \(F\)-full and \(F\)-reduced subquotient \(M_{\text{red}} = (F^\infty M)_{\text{red}} = F^\infty(M_{\text{red}})\) of an \(R[F^e]\)-module \(M\) \(F\)-reduced and \(F\)-full.

The following summary (see [Lyu97, Section 4] for proofs) of the most important properties of the functor \(D\) shows the significance of the just introduced notions in our context.

**Proposition 2.15.** Let \((R, m)\) be a regular, complete \(k\)-algebra and let \(M\) be a \(R[F^e]\)-module that is cofinite as an \(R\)-module. Then

1. \(D(M) = 0\) if and only if \(M\) is \(F\)-nilpotent. If \(N\) is also a cofinite \(R[F^e]\)-module, then \(D(M) \cong D(N)\) if and only if \(M_{\text{red}} \cong N_{\text{red}}\).
2. If \(M\) is \(F\)-full, then \(D(M)\) is a root of \(D(M)\). If \(M\) is also \(F\)-reduced, then \(D(M)\) is the unique minimal root.
3. Every unit \(R[F^e]\)-submodule \(M'\) of \(D(M)\) arises as \(D(N)\) for some \(R[F^e]\)-submodule of \(M\).
4. \(D\) is an isomorphism between the lattice of graded \(R[F^e]\)-modules quotients of \(M\) (up to \((\_\_)_{\text{red}}\)) and the lattice of unit \(R[F^e]\)-submodules of \(D(M)\).

As a final remark we point out that if \(M\) is a non-zero simple \(F\)-full \(R[F^e]\)-module, then \(D(M)\) is non-zero and therefore a simple unit \(R[F^e]\)-module. This follows since a simple \(R[F^e]\)-module is \(F\)-full if and only if it is \(F\)-reduced and therefore by the last Proposition \(D(M)\) is simple (and automatically nonzero). If \(F^e\) had a kernel it would be a nontrivial \(R[F^e]\)-submodule and thus if \(M\) is simple the kernel of \(F^e\) must be all of \(M\). Thus \(F^e(M) = 0\) which contradicts the \(F\)-fullness since this exactly means that \(F^e(M) = M\).

### 2.5. The main example: \(H^d_{m}(A)\). Let \((R, m)\) be complete regular local ring of dimension \(n\), let \(I\) be an ideal of height \(c = n - d\). We denote the quotient \(R/I\) by \(A\).

The top local cohomology module \(H^d_{m}(A)\) is an \(A[F^e]\)-module (cf. Examples 2.3) and, by restriction, also an \(R[F^e]\)-module. As an \(R[F^e]\)-module it is generally not unit, but at least the structural map

\[
R^e \otimes_R H^d_{m}(R/I) \rightarrow H^d_{m}(R/I)
\]

is surjective. This is equivalent to the map induced by the projection

\[
R/I[e] \rightarrow R/I
\]
under the identification of $R^e \otimes_R H^d_m(R/I)$ with $H^d_m(R/I)^{[p^e]}$. Thus, by definition, $\mathcal{D}(H^d_m(R/I))$ is the limit of

$$\begin{align*}
(1) \quad & D(H^d_m(R/I)) \to D(H^d_m(R/I^{[p^e]})) \to D(H^d_m(R/I^{[p^{2e}]}) \to \ldots
\end{align*}$$

Using local duality [BH98] Theorem 3.5.8 for the complete, regular and local ring $R$ this directed sequence is isomorphic to the following

$$\begin{align*}
(2) \quad & \text{Ext}^c_R(R/I, R) \to \text{Ext}^c_R(R/I^{[p^e]}, R) \to \text{Ext}^c_R(R/I^{[p^{2e}]}, R) \to \ldots
\end{align*}$$

where, again, the maps are the ones induced from the natural projections. Since the Frobenius powers of an ideal are cofinite within the normal powers, we get that the limit of this sequence is just $H^c_I(R)$. This is because an alternative definition of $H^c_I(R)$ is as the right derived functor of the functor $\Gamma_I(M) = \lim \text{Hom}(R/I^n, M)$ of sections with support in Spec $R/I$.

The very serious issue on whether the unit $R[F^e]$–structure on $H^c_I(R)$ coming from the computation via Ext’s is the same as the one coming from the Čech complex is dealt with in [Lyu97], Propositions 1.8 and 1.11. Summarizing we get:

**Proposition 2.16.** Let $(R, m)$ be regular, local, complete and $F$–finite. Let $A = R/I$ for some ideal $I$ of $R$ of height $c = n - d$. Then

$$\mathcal{D}(H^d_m(R/I)) \cong H^c_I(R)$$

as unit $R[F^e]$–modules.

By definition of $\mathcal{D}(\_)$, a root for $\mathcal{D}(H^d_m(A))$ is given by

$$\begin{align*}
(3) \quad & \beta^e : \text{Ext}^c_R(R/I, R) \to \text{Ext}^c_R(R/I^{[p^e]}, R) \xrightarrow{\sim} R^e \otimes \text{Ext}^c_R(R/I, R)
\end{align*}$$

where the first part is induced from the surjection $R/I^{[p^e]} \to R/I$, and the second is the isomorphism coming from the natural transformation $\Psi : R^e \otimes \text{Hom}(\_ , R) \cong \text{Hom}(R^e \otimes \_, R)$, cf. Proposition [2.14]. It is straightforward that this natural transformation for Hom induces a natural transformation on its right derived functors, the Ext’s.

If we drop the assumption of completeness in the preceding discussion, and just assume that $(R, m)$ is local we still have that $H^c_I(R)$ arises as the direct limit of

$$\begin{align*}
\text{Ext}^c_R(R/I, R) \to \text{Ext}^c_R(R/I^{[p^e]}, R) \to \text{Ext}^c_R(R/I^{[p^{2e}]}, R) \to \ldots
\end{align*}$$

with maps induced from the natural projections $R/I^{[p^r]} \to R/I^{[p^{(r+1)}]}$. Together with the natural transformation identifying $\text{Ext}^c_R(R/I^{[p^e]}, R)$ with $R^{te} \otimes \text{Ext}^c_R(R/I, R)$ this shows that $\text{Ext}^c_R(R/I, R)$ is a generator for $H^c_I(R)$. Upon completion we get the generator

$$\begin{align*}
\text{Ext}^c_R(R/I \hat{R}, R) \xrightarrow{\beta^e} R^e \otimes \text{Ext}^c_R(R/I \hat{R}, R)
\end{align*}$$

of $H^c_{I \hat{R}}(R) \cong \hat{R} \otimes H^c_I(R)$ where we freely used the identification $\hat{R} \otimes \text{Ext}^c_R(R/I, R) \cong \text{Ext}^c_R(R/I \hat{R}, R)$.  

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1See [BH98] Theorem 3.5.6] for the equivalence with our definition of local cohomology via Čech complexes.
3. Brief tight closure review

Tight closure is a powerful tool in commutative algebra introduced by Mel Hochster and Craig Huneke about fifteen years ago [HH88]. There is a strong connection between the singularities arising in the minimal model program and singularities obtained from tight closure theory [Smi97b]. One of the most significant is the equivalence of the notions of rational singularity and $F$-rational type which was established by Smith [Smi97a] and Hara [Har98], and independently by Mehta and Shrinivas [MS97]. The notion of $F$-rationality arises naturally from tight closure: the local ring $(A, m)$ is called $F$-rational if all ideals $I$ generated by a full system of parameters are tightly closed, i.e. $I = I^*$. In this section we briefly review the tight closure theory needed for our local construction of $L(Y, X)$ given below.

For a more detailed introduction to this beautiful subject we recommend [Smi01, Hun96] and later the more technical original papers [HH90, HH89].

Let $A$ be a noetherian ring. We denote by $A^o$ the subset of elements of $r$ that are not contained in any minimal prime of $A$. Let $N \subseteq M$ be a submodule of $M$. We denote by $N[p^e]$ the image of $F^eM$ in $F^eM$. The tight closure $N^*$ of $N$ inside of $M$ is defined as follows:

**Definition 3.1.** Let $A$ be noetherian and $N \subseteq M$. The tight closure $N^*$ (or just $N^*$ if $M$ is clear from the context) consists of all elements $m \in M$, such that there exists a $c \in A^o$, such that for all $e \gg 0$

$$c \otimes m \in N[p^e].$$

Here $N[p^e]$ denotes the image of $F^eN$ in $F^eM$ and $c \otimes m$ is an element of $F^eM$.

If $N = I$ is just an ideal of $A$, the definition is much more transparent. In this case $r \in A$ is in $I^*$ if and only if there is $c \in A^o$ such that $cr^{p^e} \in I[p^e]$ for all $e \gg 0$. A module is tightly closed if $N^* = N$. We have that $N \subseteq N^*$ as one expects from a decent closure operation. If $N$ is noetherian, then $N^* = (N^*)^*$. There are two related closure operations which are important for us.

**Definition 3.2.** Let $N \subseteq M$ be $A$-modules. The finitistic tight closure of $N$ inside of $M$ consists of all elements $m \in (N \cap M_0)^*_{M_0}$ for some finitely generated $M_0 \subseteq M$. It is denoted by $N^{*fg}_M$.

The **Frobenius closure** $N^F_M$ consists of all elements $m \in M$ such that $1 \otimes m \in N[p^e]$ for some $e \geq 0$.

We immediately see that $N^{*fg} \subseteq N^*$ and that equality holds if $M$ is finitely generated. Clearly, $N^F \subseteq N^*$. For the zero submodule of the top local cohomology module of an excellent, local, equidimensional ring $A$, the finitistic tight closure is equal to the tight closure, i.e. $0^{*fg}_{H^dM}(A) = 0^*_{H^dM}(A)$ (see [Smi93 Proposition 3.1.1]). In general, it is a hard question to decide if the tight closure equals the finitistic tight closure, and it is related to aspects of the localization problem in tight closure theory (cf. [LS01]).

As our focus lies on modules with Frobenius actions we study the above closure operations in this case more closely. The following is an important proposition which is proved in [LS01], Proposition 4.2.
Proposition 3.3. Let $A$ be noetherian and let $(M, \vartheta^e)$ be an $A[F^e]$–module. If $N$ is a $A[F^e]$–submodule, then so are $N^*_M$, $N^*_M[F^g]$ and $N^*_M[F]$. This is checked by observing that $(N^*)[\vartheta^e] \subseteq (N[\vartheta^e])^*$. Then apply $\vartheta$ and use the easily verifiable fact that $\vartheta^e(\_ )^* \subseteq \vartheta^e(\_ )^*$ to see that
\[
F^e(N^*) = \vartheta^e((N^*)[\vartheta^e]) \subseteq \vartheta^e((N[\vartheta^e])^*) \subseteq (F^e(N))^* \subseteq N^*
\]
which finishes the argument. From this we get as an immediate corollary that the tight closure of the zero $A[F^e]$–submodule is a Frobenius stable submodule of any $A[F^e]$–module.

Corollary 3.4. Let $A$ be a ring and let $(M, F^e)$ be an $A[F^e]$–module. Then $0^*_M[F^g]$, $0^*_M$ and $0^*_M = M_{\text{nil}}$ are $A[F^e]$–submodules of $M$.

3.1. Test ideals and test modules. The elements “$c$” occurring in the definition of tight closure play a special role. Those amongst them, that work for all tight closure tests for all submodules of all finitely generated $A$–modules are called the test elements of $A$.

Definition 3.5. An element $c \in A^*$ is called a test element if for all submodules $N \subseteq M$, of every finitely generated $A$–module $M$, we have $cN^*_M \subseteq N$. A test element is called completely stable test element if its image in the completion of every local ring of $A$ is a test element.

It is shown in [HH90] Proposition 8.33], that it is enough to range over all ideals of $A$ in this definition, i.e. $c$ is a test element if and only if for all ideals $I$ and all $x \in I^*$ we have $cx^{\vartheta^e} \in I[\vartheta^e]$ for all $e \geq 0$. Thus, the test elements are those elements $c$ occurring in the definition of tight closure which work for all tight closure memberships of all submodules of all finitely generated $A$–modules. A nontrivial key result is that in most cases, test elements (and even completely stable test elements) exist:

Proposition 3.6. Let $A$ be reduced and of finite type over an excellent local ring. Then $A$ has completely stable test elements. Specifically, any element $c \in A^*$ such that $A_c$ regular has a power which is a completely stable test element.

The proof of this is quite technical and can be found in Chapter 6 of [HH90]. Results in lesser generality (for example, when $A$ is $F$–finite) are obtained fairly easily: for a good account see [Smi01, Hum96].

The ideal $\tau_A$ generated by all test elements is called the test ideal. As remarked, $\tau_A = \bigcap (I :_A I^*)$ where the intersection ranges over all ideals $I$ of $A$. This naturally leads one to consider variants of the test ideal by restricting the class of ideals this intersection ranges over. The parameter test ideal of a local ring $(A, m)$ is the ideal $\overline{\tau}_A = \bigcap (I :_A I^*)$ where the intersection ranges over all ideals generated by a full system of parameters. If $A$ is Cohen-Macaulay, it follows from the definition of $H^d_{m}(A)$ as $\lim A/(x_1, \ldots, x_d)[\vartheta^e]$ that $\overline{\tau}_A = \text{Ann}_A(0_{H^d_{m}}(A))$ [Smi93, Proposition 4.1.4] where $x_1, \ldots, x_d$ is a system of parameters for the local ring $(A, m)$. If $A$ is only an excellent domain, then $\overline{\tau}_A \subseteq \text{Ann}_A(0_{H^d_{m}}(A))$. Further generalizing, the parameter test module is defined as $\tau_{\omega_A} = \text{Ann}_{\omega_A} 0_{H^d_{m}}(A) = \omega_A \cap \text{Ann}_{\omega_A} 0_{H^d_{m}}(A)$ where the
action of $\omega_A$ on $H_m^d(A)$ is the one coming from the Matlis duality pairing $H_m^d(A) \times \omega_A \to E_A$. Of course we require here that $A$ has a canonical module.

**Lemma 3.7.** Let $A$ be reduced, excellent, local and equidimensional with canonical module $\omega_A$. If $c$ is a parameter test element, then $c\omega_A \subseteq \tau_{\omega_A}$. In particular, $\tau_{\omega_A}$ is nonzero.

**Proof.** Let $c$ be a parameter test element. In particular, $c$ annihilates the finitistic tight closure of zero in $H_m^d(A)$. Therefore, for every $\varphi \in \omega_A$ and $\eta \in 0^{*_e}_{H_m^d(A)} = 0^{*_e}_{H_m^d(A)}$ we have $c\varphi \cdot \eta = \varphi \cdot (c\eta) = \varphi \cdot 0 = 0$ where "•" represents the Matlis duality pairing. This shows that $c\omega_A \subseteq \tau_{\omega_A}$. The hypotheses on $A$ ensure by [HH94, Remark 2.2(e)] that the canonical module is faithful, i.e. $c\omega_A \neq 0$. Therefore the last part of the lemma follows from the existence of test elements (Proposition 3.6), since a test element is also a parameter test element. \qed

### 3.2. $F$-rationality and local cohomology

The tight closure of zero in the top local cohomology module $H_m^d(A)$ of a local ring $(A, m)$ plays a role as the obstruction to $F$-rationality of $A$. Its distinguishing property is that it is the maximal proper $A[F^e]$-submodule of $H_m^d(A)$. Precisely the following is the case:

**Theorem 3.8.** Let $(A, m)$ be reduced, excellent and analytically irreducible. Then, the tight closure of zero, $0^{*_e}_{H_m^d(A)}$, in $H_m^d(A)$ is the unique maximal proper $A[F^e]$-submodule of $H_m^d(A)$.

The quotient $H_m^d(A)/0^{*_e}_{H_m^d(A)}$ is a nonzero simple $F$-reduced and $F$-full $A[F^e]$-module.

**Proof.** The case $e = 1$ of the first part was shown by Smith in [Sm93], Theorem 3.1.4. The case $e \geq 1$ can be obtained similarly, see [Bii01], Theorem 5.9. Because $0^{*_e}_{H_m^d(A)}$ is the maximal proper $A[F^e]$-submodule, $H_m^d(A)/0^{*_e}_{H_m^d(A)}$ is a simple $A[F^e]$-module quotient. It remains to show that it is $F$-reduced (a simple $A[F^e]$-module is $F$-full if and only if it is $F$-reduced). For this note that the kernel of $F$ is a $A[F^e]$-submodule and, by simplicity, it must either be zero ($F$-reduced) or all of $H_m^d(A)/0^{*_e}_{H_m^d(A)}$. In the second case, this implies that $F(H_m^d(A)) \subseteq 0^{*_e}_{H_m^d(A)}$. Since $H_m^d(A)$ is a unit $A[F^e]$-module (enough that the structural map $\varphi$ is surjective) we have that $F(H_m^d(A)) = H_m^d(A)$. This contradicts the fact that $0^{*_e}_{H_m^d(A)}$ is a proper submodule. Thus the quotient is $F$-reduced and $F$-full. \qed

To avoid the assumption of analytically irreducible we give a version of the above for the case that $A$ is an excellent equidimensional ring. As the statement is about $H_m^d(A)$ which does not discriminate between $A$ and its completion, we state the result for a complete $A$; in general one has to take the minimal primes of the completion of $A$ in the statement below.

**Corollary 3.9.** Let $A$ be a complete, local, reduced and equidimensional ring of dimension $d$. Let $P_1, \ldots, P_k$ be the minimal primes of $A$. Then the
maximal proper $A[F^e]$–submodules are precisely
\[ M_i \overset{\text{def}}{=} \ker(H_m^d(A) \rightarrow \frac{H_m^d(A/P_i)}{0^e_{H_m^d(A/P_i)}}) \]
where $i = 1 \ldots k$. Furthermore, the tight closure of zero, $0^e_{H_m^d(A)}$, in $H_m^d(A)$
is the intersection of all maximal proper $A[F^e]$–submodules. Even though $H_m^d(A)/0^e_{H_m^d(A)}$
night not be simple as an $A[F^e]$–module, it is still $F$–full and $F$–reduced.

Proof. Since tight closure can be checked modulo minimal primes, the last
statement is immediate.\(^2\) By the last Theorem $0^e_{H_m^d(A/P_i)}$ is the maximal
proper $A[F^e]$–submodule. Thus $H_m^d(A)/M_i \cong H_m^d(A/P_i)/0^e_{H_m^d(A/P_i)}$ is sim-
ple. Thus $M_i$ are all the maximal proper $A[F^e]$–submodule of $H_m^d(A)$ not contained in any of the $M_i$. This implies that for all $i$ the
image of $M$ in $H_m^d(A/P_i) = H_m^d(A)/P_i H_m^d(A)$ is all of $H_m^d(A/P_i)$ (it is an
$A[F^e]$–module not contained in $0^e_{H_m^d(A/P_i)}$) thus must be all of $H_m^d(A/P_i)$ by
last Theorem). But this implies, by the following lemma, that $M = H_m^d(A)$
and we are done.

It remains to remark that a possible kernel of $F^e$ on $H_m^d(A)/0^e_{H_m^d(A)}$
would reduce to all of $H_m^d(A/P_i)/0^e_{H_m^d(A/P_i)}$ for some $i$ and therefore imply
that $F(H_m^d(A/P_i)) \subseteq 0^e_{H_m^d(A/P_i)}$ which is a contradiction to $F$–fullness
of $H_m^d(A/P_i)$.

Lemma 3.10. Let $A$ be a noetherian ring, and let $M \subseteq H$ be an $A$–module
such that for every minimal prime $P$ of $A$ one has $M + PH = H$. Then
$M = H$.

Proof. One immediately reduces to the case $M = 0$. Successive application
of the assumption $H = PH$ implies that
\[ H = (P_1 \ldots P_k)^n H \]
where the $P_i$’s are the minimal primes. But for large enough $n$, a power of
the product of all minimal primes is zero, thus $H = 0$.

If $A$ is Cohen–Macaulay, the vanishing of the tight closure of zero in
$H_m^d(A)$ characterizes $F$–rationality of $A$ by [Smith97a, Theorem 2.6]. By def-
inition, $A$ is called $F$–rational if and only if every ideal that is generated by
a system of parameters is tightly closed.

4. The intersection homology module

First we give a detailed proof of the main existence theorem as sketched
in the introduction.

Theorem 4.1. Let $X$ be an irreducible smooth $k$–scheme, essentially of
finite type over $k$, and let $Y$ be a closed irreducible subscheme of codi-

\(^2\)This is generally proved for the tight closure of ideals in the literature (see [HH90,
Proposition 6.25], but the same proof can be adapted for submodules.
This submodule is also the unique simple $\mathcal{O}_X[F^e]$–module and agrees with $\mathcal{H}^c_{Y'}(X, \mathcal{O}_X)$ on the complement of any closed set containing the singular locus of $Y$.

Proof. Write $Z = \text{Sing} Y$ and $Y' = Y - Z$ and $X' = X - Z$ and denote the open inclusion $X' \subseteq X$ by $i$. First we assume that the characteristic of $k$ is positive; at the end of the proof we indicate how the proof is adapted to characteristic zero.

We first show that $H^c_{Y'}(X', \mathcal{O}_{X'})$ is simple as a unit $\mathcal{O}_X[F^e]$–module: Quite generally we note that, $\mathcal{O}_{Y'}$ is a simple unit $\mathcal{O}_{Y'}[F^e]$–module by observing that a nontrivial ideal $I \subseteq \mathcal{O}_{Y'}$ is never a unit submodule as the containment $I^{[p^e]} \subseteq I$ is strict, cf. Examples 2.5. Using that $Y'$ is smooth and irreducible, it follows that $\mathcal{O}_{Y'}$ is also simple as a $\mathcal{D}_{Y'}$–module. This can be reduced, by étale invariance of $\mathcal{D}_{Y'}$, to the case $Y' = \text{Spec}(k[x_1, \ldots, x_d])$ where one can check it by hand. Under Kashiwara’s equivalence for $\mathcal{D}_{Y'}$–modules ([Haa88], and for unit $\mathcal{O}_X[F^e]$–modules ([EK99], Theorem 5.10.1 or [Lyu97], Proposition 3.1), the module $\mathcal{O}_{Y'}$ corresponds to $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$ (cf. [EK99], Example 5.11.6). Therefore, $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$ is a simple $\mathcal{D}_{Y'}$–module (unit $\mathcal{O}_{X'}[F]$–module, respectively).

Since $\mathcal{H}^c_{Y'}(X, \mathcal{O}_X)$ is a locally finitely generated unit $\mathcal{O}_X[F^e]$–module and therefore, by [Lyu97], Theorem 5.6, it has finite length as a $\mathcal{D}_X$–module. This assures the existence of simple $\mathcal{D}_X$–submodules of the $\mathcal{D}$–module $\mathcal{H}^c_{Y'}(X, \mathcal{O}_X)$ and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two such. Observe the exact sequence (see [Har67], Chapter 1)

$$0 = \mathcal{H}^c_{Y}(X, \mathcal{O}_X) \rightarrow \mathcal{H}^c_{Y'}(X, \mathcal{O}_X) \rightarrow \mathcal{H}^c_{Y'}(X, \mathcal{O}_X) \cong i_* \mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$$

where the last isomorphism is by excision and the vanishing of the first module is because the codimension of $Z$ in $X$ is strictly bigger than $c$. From this it follows that $\mathcal{H}^c_{Y}(X, \mathcal{O}_X)$ and therefore $\mathcal{L}_i$ are submodules of $i_* \mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$. By adjointness of restriction and extension we have

$$0 \neq \text{Hom}_{\mathcal{O}_X}(\mathcal{L}_i, i_* \mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{L}_i|_{X'}, \mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'}))$$

which shows that $\mathcal{L}_1|_{X'}$ are nonzero submodules of $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$. By simplicity of the latter all three have to be equal. In particular, the intersection of $\mathcal{L}_1$ with $\mathcal{L}_2$ is nonzero. As both are simple, this implies that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}(Y, X)$ as claimed. Furthermore, since $F^e(\mathcal{L}(Y, X))$ is also simple, it follows from the uniqueness that $F^e(\mathcal{L}(Y, X)) = \mathcal{L}(Y, X)$ and therefore $\mathcal{L}(Y, X)$ is also the unique simple $\mathcal{O}_X[F^e]$–submodule for all $e$.

Essentially the same proof works in characteristic zero. The key fact then is that $\mathcal{H}^c_{Y'}(X, \mathcal{O}_X)$ is a holonomic $\mathcal{D}_X$–module and that holonomic modules have finite length. Also observe that for the smooth $Y'$ the structure sheaf $\mathcal{O}_{Y'}$ is $\mathcal{D}_{Y'}$–simple which is well known and easy to check by hand for the case $Y' = \text{Spec}(k[x_1, \ldots, x_d])$. Then Kashiwara’s equivalence implies that the corresponding $\mathcal{H}^c_{Y'}(X', \mathcal{O}_{X'})$ is a simple $\mathcal{D}_{X'}$–module. For all of the above statements, see [BGKK+87].

This proof is pretty much identical for zero and positive characteristic. The metaresults which are used though are proved by very different techniques in each case.
If $Y$ is not irreducible, then the above result breaks down since then $H^c_{Y'}(X',\mathcal{O}_{X'})$ is no longer simple. In the case that $Y$ is equidimensional one can give a complete description of the simple submodules of the module $H^c_Y(X,\mathcal{O}_X)$. They correspond to the irreducible components $Y_1,\ldots,Y_k$ of $Y$. For each component we have an inclusion
\[
\mathcal{L}(Y_i, X) \subseteq H^c_Y(X,\mathcal{O}_X) \subseteq H^c_Y(X,\mathcal{O}_X)
\]
which establishes $\mathcal{L}(Y_i, X)$ as simple submodules of $H^c_Y(X,\mathcal{O}_X)$. That the right map is an inclusion uses equidimensionality and follows from [Har67, Proposition 1.9 and Chapter 3]. To see that these are all the simple submodules of $H^c_Y(X,\mathcal{O}_X)$ we show that any submodule $\mathcal{N}$ of $H^c_Y(X,\mathcal{O}_X)$ does contain one of the $\mathcal{L}(Y_i, X)$. At least for one $i$, the restriction of $\mathcal{N}$ to a open subset of $X$ containing $Y_i$ but none of the other components is a nonzero submodule of $H^c_Y(X,\mathcal{O}_X)$ (using excision). But then $\mathcal{N}$ clearly contains $\mathcal{L}(Y_i, X)$ by its uniqueness and simplicity. We get as a corollary:

**Corollary 4.2.** Let $Y$ be an equi-dimensional and reduced sub-scheme of the smooth $k$–variety $X$ of codimension $c$. Let $Y = Y_1 \cup \ldots \cup Y_k$ be its decomposition into irreducible components. Then the simple $D_X$–submodules of $H^c_Y(X,\mathcal{O}_X)$ are precisely the $\mathcal{L}(Y_i, X)$ for $i = 0,\ldots,k$. In this case we denote by $\mathcal{L}(Y, X)$ the (direct) sum of all the $\mathcal{L}(Y_i, X)$. Furthermore, away from the singular locus of $Y$ we have that $\mathcal{L}(Y,X)$ agrees with $H^c_Y(X,\mathcal{O}_X)$.

The similarity of Theorem 3.8 and Corollary 3.9 with Theorem 4.1 and the last corollary was the original motivation which lead to the construction of $\mathcal{L}(Y,X)$ which is given in the next section.

Note that by the uniqueness of $\mathcal{L}(Y,X)$ it is clear that it localizes, *i.e.* if $U$ is a open subset of $X$ then $\mathcal{L}(Y,X)|_U = \mathcal{L}(Y \cap U, U)$.

In the case of positive characteristic, we state the following slightly stronger local version of the last theorem.

**Theorem 4.3.** Let $R$ be a regular local ring of positive characteristic. Let $A = R/I$ be equidimensional of codimension $c$ in $R$. Then the sum of all simple unit submodules of $H^c_f(R)$ is
\[
\mathcal{L}(A, R) = \oplus \mathcal{L}(R/P_i, R)
\]
where the $P_i$ range over the primes minimal over $I$ and $\mathcal{L}(R/P_i, R)$ is the unique simple unit $\mathcal{O}_X[F]$– submodule of $H^c_{P_i}(R)$. Moreover, if $f \in R$ is such that $A_f$ is regular, then $\mathcal{L}(A, R) = H^c_f(R)$. If $R$ is $F$–finite the same holds for the $D_R$–module structure.

*Proof.* The local results of [Lyu97] are not restricted to finitely generated algebras over a field. With the assumption above (sufficient to ensure that a finitely generated unit $R[F^c]$–module has finite length as such) the previous proof goes through verbatim. \qed

4.1. **Local construction of $\mathcal{L}(A, R)$ in positive characteristic.** From now on we assume that $k$ is of positive characteristic and $R$ is $F$–finite. With the last theorem we can dispose of the $D_R$–structure and entirely work with the $R[F^c]$–structure in our investigation $\mathcal{L}(A, R)$. As the construction of $\mathcal{L}(Y,X)$ which we are about to present is local in nature, the language is adjusted accordingly. Moreover, the local construction of $\mathcal{L}(A, R)$ can be
reduced to the complete case with help of the results on minimal roots of Section 2.3. Thus we assume for now that \((R, m)\) is a complete, regular local and \(F\)-finite, and that \(A = R/I\) equidimensional of codimension \(c\). The philosophy behind the description of the simple unit \(R[F^e]\)-module \(\mathcal{L}(A, R)\) is to identify its minimal root. As it turns out, the minimal root of \(\mathcal{L}(A, R)\) is the parameter test module, which, under Matlis duality, corresponds by definition to the tight closure of zero \(0^*_{H^d_m(A)}\) in \(H^d_m(A)\).

**Theorem 4.4.** Let \((R, m)\) be a complete regular local and \(F\)-finite and let \(A = R/I\) be equidimensional and of codimension \(c\). Then, we have that

\[
\mathcal{L}(A, R) = \mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)})
\]

where \(0^*_{H^d_m(A)}\) is the tight closure of zero in \(H^d_m(A)\).

**Proof.** First assume that \(A\) is a domain. By Theorem 3.8 the unique simple \(R[F^e]\)-module quotient of \(H^d_m(A)\) is \(H^d_m(A)/0^*_{H^d_m(A)}\). Therefore

\[
\mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)}) \subseteq \mathcal{D}(H^d_m(A)) \cong H^*_f(R)
\]

is a nonzero simple unit \(R[F^e]\)-module of \(H^d_m(A)\). As \(\mathcal{L}(A, R)\) is the unique submodule of \(H^d_m(A)\). If \(A\) is only equidimensional, let \(P_1, \ldots, P_k\) be its minimal primes. In Corollary 3.9 we show that

\[
0^*_{H^d_m(A)} = \ker(H^d_m(A) \rightarrow \oplus_{i=1}^k H^d_m(A/P_i)/0^*_{H^d_m(A/P_i)})
\]

Applying the functor \(\mathcal{D}\) and using the domain case for \(A/P_i\) as just proved one checks that

\[
\mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)}) = \oplus_{i=1}^k \mathcal{L}(A/P_i, R) = \mathcal{L}(A, R)
\]

where the last equality is by definition. \(\square\)

A more careful investigation of the construction of \(\mathcal{L}(A, R)\) via the duality functor \(\mathcal{D}\) shows its connection with the parameter test module. By definition, \(\mathcal{D}(H^d_m(A)/0^*_{H^d_m(A)}) = \mathcal{L}(A, R)\) is the unit \(R[F^e]\)-module generated by the Matlis dual of the \(R[F^e]\)-module structure on \(H^d_m(A)/0^*_{H^d_m(A)}\). The Matlis dual of \(H^d_m(A)\) is the canonical module \(\omega_A = \text{Ext}^c_R(A, R)\) of \(A\). The dual of \(H^d_m(A)/0^*_{H^d_m(A)}\) is found as the annihilator of \(0^*_{H^d_m(A)}\) under the Matlis duality pairing \(\omega_A \times H^d_m(A) \rightarrow E_{R/m}\). By definition, this annihilator \(\text{Ann}_{\omega_A} 0^*_{H^d_m(A)}\) is the parameter test module \(\tau_{\omega_A}\). Thus the inclusion of unit \(R[F^e]\)-modules \(\mathcal{L}(A, R) \subseteq H^*_f(R)\) arises as the limit of the following map between their generators:

\[
\begin{array}{ccc}
\omega_A & \rightarrow & R^e \otimes \omega_A \\
\uparrow & & \uparrow \\
\tau_{\omega_A} & \rightarrow & R^e \otimes \tau_{\omega_A}
\end{array}
\]

(4)

Since, \(H^d_m(A)/0^*_{H^d_m(A)}\) is \(F\)-reduced and \(F\)-full, the bottom map \(\tau_{\omega_A} \rightarrow R^e \otimes \tau_{\omega_A}\) is, by Proposition 2.15(2), the unique minimal root of \(\mathcal{L}(A, R)\).
Proposition 4.5. Let $\langle R,m \rangle$ be complete regular local and $F$–finite. Let $A = R/I$ be equidimensional of codimension $c$ in $R$. Then the parameter test module $\tau_{\omega_A}$ is the unique minimal root of $\mathcal{L}(A,R)$.

Now we drop the assumption that $R$ be complete and only assume it to be regular, local and $F$–finite.

Theorem 4.6. Let $\langle R,m \rangle$ be regular local and $F$–finite. Let $A = R/I$ be a domain of codimension $c$. Then $\mathcal{L}(A,R)$, the unique simple unit $R[F^e]$–submodule of $H^c_I(R)$, has the parameter test module $\tau_{\omega_A}$ as its minimal root. Furthermore, upon completion, $\hat{R} \otimes \mathcal{L}(A,R) = \mathcal{L}(\hat{A},\hat{R})$.

Proof. By Corollary 2.9, $\mathcal{N} = L(\hat{A},\hat{R}) \cap H^c_I(R)$ is a unit $R[F^e]$–submodule of $H^c_I(R)$. A root of $\mathcal{N}$ is found by intersecting $\mathcal{L}(\hat{A},\hat{R})$ with the root $\omega_A$ of $H^c_I(R)$. Again by Corollary 2.9 this intersection is equal to $\tau_{\omega_A} = \omega_A \cap \tau_{\omega_{\hat{A}}}$ which is the nonzero parameter test module by Lemma 3.7. By simplicity of $\mathcal{L}(A,R)$ it is therefore contained in $L(\hat{A},\hat{R}) \cap H^c_I(R)$, or put differently,

$$(5) \quad \hat{R} \otimes \mathcal{L}(A,R) \subseteq \mathcal{L}(\hat{A},\hat{R}).$$

If $R$ were a domain we would be done. Unfortunately, the completion of a domain might not be a domain, but at least it is equidimensional. If $P_1,\ldots,P_k$ are the primes minimal over $I\hat{R}$, then Theorem 1.3 states that $\mathcal{L}(\hat{A},\hat{R}) = \oplus \mathcal{L}(\hat{R}/P_i,\hat{R})$. It remains to show that $\mathcal{L}(\hat{R}/P_i,\hat{R}) \subseteq \hat{R} \otimes \mathcal{L}(A,R)$. This is, by $\mathcal{L}(\hat{R}/P_i,\hat{R})$ being the unique simple submodule of $H^c_{P_i}(\hat{R})$, equivalent to $\hat{R} \otimes (\mathcal{L}(A,R) \cap H^c_{P_i}(\hat{R})) \neq 0$. To see this let $f \in R - P_i$ such that $A_f$ is regular, then, by the last part of Theorem 1.3

$$\hat{R} \otimes \mathcal{L}(A,R)_f = \hat{R} \otimes H^c_I(R)_f = H^c_{I\hat{R}}(\hat{R})_f \supseteq H^c_{P_i}(\hat{R})_f \neq 0$$

which shows the reverse inclusion of (5). The statement about the parameter test module now follows from the beginning of the proof as we just showed that $\mathcal{L}(A,R) = \mathcal{N}$. □

As done before this proof can be adjusted to work for equidimensional $A$ from the start. For simplicity we treated the domain case and will do so from now on.

Remark 4.7. This construction of $\mathcal{L}(A,R)$ from its minimal root $\tau_{\omega_{\hat{A}}}$ enables one to explicitly construct $D_R$–module generators for $\mathcal{L}(A,R)$: The image of any element of $\tau_{\omega_A}$ in $H^c_I(R)$ is a generator of $\mathcal{L}(A,R)$, in particular, if $c \in R$ is a test element such that $c^n \cdot \eta \neq 0$ for $\eta \in \omega_A$, then $c \cdot \eta$ generates $\mathcal{L}(A,R)$ as a $D_R$–module.

As another consequence of Corollary 2.9 one sees that the minimal root of $\mathcal{L}(A,R)$ is $\tau_{\omega_{\hat{A}}}$, the minimal root of $\mathcal{L}(\hat{A},\hat{R})$, intersected with $\omega_A$, the root of $H^c_I(R)$. By definition, this is the parameter test module $\tau_{\omega_A}$ of $A$. Using Theorem 2.10 it follows immediately that the parameter test module commutes with completion, which was to the best of our knowledge, unknown until now. We state this as a Proposition.

Proposition 4.8. Let $A$ be a domain which is a quotient of a regular local and $F$–finite ring. Then the parameter test module commutes with completion, i.e. $\tau_{\omega_{\hat{A}}} = \hat{A} \otimes \tau_{\omega_A}$. 
4.2. Simplicity criteria for $H_{m}^{d}(R)$. With the connection between the tight closure of zero in $H_{m}^{d}(A)$ and the simple unit $R[F]$–module $L(A, R)$ just derived, the characterization of Smith showing that $0_{H_{m}^{d}(A)}^{∗}$ governs the $F$–rationality of $A$, easily implies a simplicity criteria for $H_{m}^{d}(R)$.

**Theorem 4.9.** Let $R$ be regular local and $F$–finite. Let $I$ be an ideal such that $A = R/I$ is a domain. Then $H_{m}^{d}(R)$ is $D_{R}$–simple if and only if the tight closure of zero in $H_{m}^{d}(A)$ is $F$–nilpotent.

**Proof.** $H_{m}^{d}(R)$ is $D_{R}$–simple if and only if it is equal to $L(A, R)$. Then $\hat{R} \otimes L(A, R) = D(H_{m}^{d}(A)/0_{H_{m}^{d}(A)}^{∗})$ is all of $\hat{R} \otimes H_{m}^{d}(R)$ if and only if $D(0_{H_{m}^{d}(A)}^{∗}) = 0$, by exactness of $D$. This is the case if and only if $0_{H_{m}^{d}(A)}^{∗}$ is $F$–nilpotent by Proposition 2.15. □

**Corollary 4.10.** Let $R$ be regular, local and $F$–finite. Let $A = R/I$ be a domain of codimension $c$. If $A$ is $F$–rational, then $H_{m}^{d}(R)$ is $D_{R}$–simple. If $A$ is $F$–injective (i.e. $F$ acts injectively on $H_{m}^{d}(A)$), then $A$ is $F$–rational if and only if $H_{m}^{d}(R)$ is $D_{R}$–simple.

**Proof.** By [Sm97a], Theorem 2.6, $F$–rationality of $A$ is equivalent to $0_{H_{m}^{d}(A)}^{∗} = 0$. Therefore, by the last theorem, $L(A, R) = H_{m}^{d}(R)$ if $A$ is $F$–rational. Conversely, if $L(A, R) = H_{m}^{d}(R)$ then $0_{H_{m}^{d}(A)}^{∗}$ is $F$–nilpotent. Under the assumption the $H_{m}^{d}(A)$ is $F$–reduced this implies that $0_{H_{m}^{d}(A)}^{∗} = 0$, therefore $A$ is $F$–rational. □

This should be compared to the following characterization of $F$–regularity in terms of $D_{A}$–simplicity due to Smith:

**Proposition 4.11 ([Sm95 2.2(4)])**. Let $A$ be an $F$–finite domain which is $F$–split. Then $A$ is strongly $F$–regular if and only if $A$ is simple as a $D_{A}$–module.

Note that this proposition is a statement about the $D_{A}$–module structure of $A$, i.e. a statement about the differential operators on $A$ itself. This is different from our approach as we work with the differential operators $D_{R}$ of the regular $R$. Nevertheless, the similarity of the result is striking and should be understood from the point of view Kashiwara’s equivalence, i.e. the $D_{A}$–module $A$ should be studied via the corresponding $D_{R}$–module $H_{m}^{d}(R)$.

We reformulate the simplicity criterion of $H_{m}^{d}(R)$ such that it is a criterion solely on $A$, not referring to $H_{m}^{d}(A)$.

**Theorem 4.12.** Let $R$ be regular, local and $F$–finite. Let $I$ be an ideal such that $A = R/I$ is a domain. If for all parameter ideals of $A$ we have $J^{F} = J^{∗}$, then $H_{m}^{d}(R)$ is $D_{R}$–simple.

If $A$ is Cohen–Macaulay, then $H_{m}^{d}(R)$ is $D_{R}$–simple if and only if $J^{∗} = J^{F}$ for all parameter ideals $J$.

**Proof.** We show that if $J^{∗} = J^{F}$ for all parameter ideals, then $0_{H_{m}^{d}(A)}^{∗}$ is $F$–nilpotent, i.e. $0_{H_{m}^{d}(A)}^{∗} = 0_{H_{m}^{d}(A)}^{F}$. Let $\eta \in H_{m}^{d}(A)$ be represented by $z + (x_1, \ldots, x_d)$ for some parameter ideal $J = (x_1, \ldots, x_d)$, thinking of $H_{m}^{d}(A)$
as the limit \( \lim A/J^{[e]} \). Then the colon capturing property of tight closure
shows that \( z \in 0^{\ast}_{A_R}(A) \) if and only if \( z \in J^\ast \) (cf. [Sim93] Proposition 3.1.1).
By our assumption \( J^\ast = J^F \), this implies that \( z^{p^e} \in J^{[e]} \) for some \( e > 0 \).
Consequently, \( F^e(\eta) = z^{p^e} + J^{[p^e]} \) is zero and thus every element of \( 0^{\ast}_{A_R}(A) \) is \( F \)-nilpotent.
Under the assumption that \( A \) is Cohen–Macaulay the same argument can
be reversed using that the limit system defining \( H^c_m(A) \) is injective. \( \square \)

As we were dealing with the domain case above we remark that in most
cases when \( A \) is not analytically irreducible the equality \( L(A, R) = H^c_I(R) \)
cannot hold, in particular \( H^c_I(R) \) cannot be simple.

**Proposition 4.13.** Let \( A = R/I \) be equidimensional, local and satisfy
Serre’s \( S_2 \) condition. Suppose that \( A \) is not analytically irreducible, then
\( L(A, R) \neq H^c_I(R) \).

**Proof.** Equidimensionality and \( S_2 \)-ness implies by [HH91], Corollary 3.7,
that \( H^c_m(A) \) is indecomposable. The properties of the duality functor \( D \)
show now that \( D(H^c_m(A)) = H^c_m(\hat{R}) \) is indecomposable as a unit \( R[F^e] \)
module. If \( \hat{A} \) is not a domain it follows, essentially by definition, that
\( L(\hat{A}, \hat{R}) \) is decomposable, thus it cannot be equal to \( H^c_m(\hat{R}) \). As \( L \)
handles well under completion, it follows that \( L(A, R) \neq H^c_I(R) \). \( \square \)

As an application of Theorem 4.12 we extend the last proposition to
a characterization of the simplicity of \( H^c_I(R) \) for the class of all domains
\( A \) which have only an isolated singularity and whose normalization is \( F \)-
rational. In particular this yields a characterization of the simplicity of
\( H^c_I(R) \) for \( A = R/I \) a one dimensional domain.

**Proposition 4.14.** Let \( R \) be regular local and \( F \)-finite. Let \( A = R/I \) be
a local \( S_2 \) domain with isolated singularity such that the normalization \( \overline{A} \)
is \( F \)-rational. Then \( H^c_I(A) \) is \( D_R \)-simple if and only if \( H^c_I(R) \) is analytically
irreducible.

**Proof.** If \( H^c_I(R) \) is not analytically irreducible then, by Proposition 4.13,
\( H^c_I(R) \) is not \( D_R \)-simple (equiv. unit \( R[F^e] \)-simple).

Since \( R \) is excellent, \( H^c_I(A) \) is analytically irreducible if and only if \( \overline{A} \)
is local by [Gro65], (7.8.31 (vii)). Let \( z \in J^\ast \) for a parameter ideal \( J = \langle y_1, \ldots, y_d \rangle \)
of \( A \). Then, since \( \overline{A} \) is \( F \)-rational and the expansion \( \overline{J} \) of \( J \) to
\( \overline{A} \) is also a parameter ideal, one concludes that \( z \in \overline{J} = \overline{J} \). Let, for some
\( a_i \in \overline{A}, \)
\[ z = a_1 y_1 + \ldots + a_d y_d \]
be an equation witnessing this ideal membership. As we observe in Lemma
4.15 below, for some big enough \( e \), all \( a^{p^e} \) are in \( A \). Therefore \( z^{p^e} = a_1^{p^e} y_1^{p^e} + \ldots + a_d^{p^e} y_d^{p^e} \), which shows that \( z^{p^e} \in J^{[p^e]} \) since all \( a^{p^e} \in A \). Thus \( J^\ast = J^F \)
and Theorem 4.12 implies that \( H^c_I(R) \) is \( D_R \)-simple. \( \square \)

**Lemma 4.15.** Let \( (A, m) \) be a local domain with at worst isolated singularities. Then the normalization \( \overline{A} \) of \( A \) is local if and only if for all \( x \in \overline{A} \) some power of \( x \) lies in \( A \).
Proof. If there were $M_1$ and $M_2$, maximal ideals of $\overline{A}$ lying over $m$, then, by assumption, for some $t \gg 0$ we have $(M_1)^t \subseteq m \subseteq M_2$. Since $M_2$ is prime it follows that already $M_1 \subseteq M_2$. The reverse inclusion follows by symmetry and thus $M_1 = M_2$, therefore $\overline{A}$ is local.

Conversely, if $\overline{A}$ is local with maximal ideal $M$ it follows that $\sqrt{mM} = M$. Therefore, if $x \in M$ it follows that $x^{n_0} \in mM$ for sufficiently big $n_0$. We want to conclude that $x^n \in m$ and thus is in $A$ for big enough $n$. For this note that, by the assumption of isolated singularity, the conductor ideal $C = (A \cap \overline{A})$ is $M$ primary, i.e. sufficiently high powers of $x$ lie in $C$. Now, if $x^{n_0} \in mM$ and $x^n \in C$, then $x^{n_0+n} = x^{n_0}x^n$ is in $m$ itself. Finally, a unit $u$ of $\overline{A}$ can be written as $u = \frac{ux}{x}$ for $x$ a nonunit of $\overline{A}$. Since both, $x$ and $ux$ are not units, sufficiently big powers are in $A$. Thus also sufficiently big powers of $u$ will be in $A$. □

In the case that $R$ is one–dimensional this yields a finite characteristic analog of results of S.P. Smith [Smi88] and Yekutieli [Yek98].

**Corollary 4.16.** Let $A = R/I$ be a one–dimensional local domain with $R$ regular and $F$–finite. Then $H^1_f(R)$ is $D_R$–simple (equiv. unit $R[F^e]$–simple) if and only if $A$ is unibranch.

**Proof.** As remarked in the proof of Proposition 4.14 $A$ is unibranch if and only if $A$ is analytically irreducible. As one dimensional domains have at worst isolated singular points and since the normalization is regular (and thus $F$–rational), Proposition 4.14 applies. □

This last result that for curves $\mathcal{L}(A, R)$ is described in the same way in positive characteristic as it is in characteristic zero is somewhat misleading. In higher dimensions one expects that $\mathcal{L}(A, R)$ behaves significantly different depending on the characteristic. For example, consider the ideal $I = (xy - zw) \subseteq R = k[x, y, z, w]$. Then $A = R/I$ is the coordinate ring of the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with the only singular point being the vertex. The localization of $A$ at the vertex is $F$–rational. Therefore, our results above shows that $H^1_f(R)$ is simple as a $D_R$–module in finite characteristic. Nevertheless, in characteristic zero the module $H^1_f(R)$ is not $D_R$–simple since the Bernstein-Sato polynomial of $xy - zw$ is $(s - 1)(s - 2)$ and therefore has an integral zero of less than $-1$. This shows that the $D_R$–submodule generated by $(xy - zw)^{-1} \in H^1_f(R)$ does not contain $(xy - zw)^{-2}$. Therefore $H^1_f(R)$ has a proper $D_R$–submodule and is therefore not $D_R$–simple.

This is in accordance with the Riemann–Hilbert type correspondences in either characteristic. For zero characteristic, the classical Riemann–Hilbert correspondence relates holonomic $\mathcal{D}_X$–modules to constructible $\mathbb{C}$–vectorspaces by means of a vast generalization of de Rham theory, i.e. to an ultimatively topological theory. In positive characteristic, on the other hand, the Emerton-Kisin correspondence relates finitely generated unit $\mathcal{O}_X[F^e]$–modules to constructible $\mathbb{F}^e$–sheaves on $X$, generalizing Artin–Schreier theory, which ultimately is a coherent theory. This is one reason why there is no surprise for the failure of a complete analogy of the description of the intersection homology module $\mathcal{L}(Y, X)$ in positive and zero characteristic.
References

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, Algebraic D-modules, Academic Press Inc., Boston, MA, 1987.

[BH98] Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge University Press, Cambridge, UK, 1998.

[BK81] J.-L. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, Invent. Math. 64 (1981), no. 3, 387–410.

[Bli01] Manuel Blickle, The intersection homology D–module in finite characteristic, Ph.D. thesis, University of Michigan, 2001.

[Bli03] ________, The D–module structure of R[F]–modules, Trans. Am. Math. Soc. 355 (2003), no. 4, 1647–1668.

[Eis95] David Eisenbud, Commutative algebra, Springer-Verlag, New York, 1995.

[EK99] Mathew Emerton and Mark Kisin, Riemann–Hilbert correspondence for unit F-crystals. I, to appear, 1999.

[EK00] ________, Riemann–Hilbert correspondence for unit F-crystals. II, in preparation, 2000.

[EK03] ________, An introduction to the Riemann-Hilbert correspondence for unit F-crystals, to appear in proceedings of Dwork memorial conference, 2003.

[Gro65] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231.

[Gro67] ———, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.

[Haa88] Burkhard Haastert, On direct and inverse images of d-modules in prime characteristic, Manuscripta Math. 62 (1988), no. 3, 341–354.

[Har67] Robin Hartshorne, Local cohomology, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin, 1967.

[Har98] Nobou Hara, A characterization of rational singularities in terms of the injectivity of the Frobenius maps., American Journal of Mathematics 120 (1998), 981–996.

[HH77] Robin Hartshorne and Robert Speiser, Local cohomological dimension in characteristic p, Annals of Mathematics 105 (1977), 45–79.

[Hun96] Craig Huneke, Tight closure and its applications, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With an appendix by Melvin Hochster.

[LS01] Gennady Lyubeznik and Karen E. Smith, On the commutation of the test ideal with localization and completion, 2001.

[Lyu97] Gennady Lyubeznik, F-modules: an application to local cohomology and D-modules in characteristic p > 0, Journal für reine und angewandte Mathematik 491 (1997), 65–130.

[MS97] V.B. Mehta and V. Srinivas, A characterization of rational singularities, Asian Journal of Mathematics 1 (1997), no. 2, 249–271.

[Smi88] S. P. Smith, The simple D-module associated to the intersection homology complex for a class of plane curves, J. Pure Appl. Algebra 50 (1988), no. 3, 287–294.
[Smi93] Karen E. Smith, *Tight closure of parameter ideals and f-rationality*, Ph.D. thesis, University of Michigan, 1993.

[Smi95] ———, *The D-module structure of F-split rings*, Math. Res. Lett. 2 (1995), no. 4, 377–386.

[Smi97a] ———, *F-rational rings have rational singularities*, Amer. J. Math. 119 (1997), no. 1, 159–180.

[Smi97b] ———, *Vanishing, singularities and effective bounds via prime characteristic local algebra*, Algebraic geometry—Santa Cruz 1995, Amer. Math. Soc., Providence, RI, 1997, pp. 289–325.

[Smi01] ———, *An introduction to tight closure*, Geometric and combinatorial aspects of commutative algebra (Messina, 1999), Dekker, New York, 2001, pp. 353–377.

[Vil85] K. Vilonen, *Intersection homology D-module on local complete intersections with isolated singularities*, Invent. Math. 81 (1985), no. 1, 107–114.

[Yek98] Amnon Yekutieli, *Residues and differential operators on schemes*, Duke (1998), no. 95, 305.

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