DIAGONAL COMPLEXES FOR PUNCTURED SURFACES AND SURFACES WITH INVOLUTION

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Abstract. Two related constructions are studied:
— The diagonal complex $\mathcal{D}$ and its barycentric subdivision $\mathcal{B}\mathcal{D}$ related to a punctured oriented surface $F$ equipped with a number of labeled marked points.
— The symmetric diagonal complex $\mathcal{D}^{\text{inv}}$ and its barycentric subdivision $\mathcal{B}\mathcal{D}^{\text{inv}}$ related to a symmetric (=with an involution) oriented surface $F$ equipped with a number of (symmetrically placed) labeled marked points.

Eliminating a puncture gives rise to a bundle whose fibers are homeomorphic to a surgery of the surface $F$. The bundle can be viewed as the "universal curve with holes".

The symmetric complex is shown to be homotopy equivalent to the complex of a punctured surface obtained by a surgery of the initial symmetric surface.

1. Introduction

Let us start with two elementary motivations:

(*) It is known that the poset of the collections of non-crossing diagonals in an $n$-gon is combinatorially isomorphic to a convex polytope called associahedron (see Appendix B).

(**) The poset of the collections of non-crossing diagonals in a punctured $n$-gon is combinatorially isomorphic to a convex polytope called cyclohedron (see Example 1).

A natural question which proved to be meaningful is: what happens if one replaces the $n$-gon by an arbitrary closed surface equipped with a number of labeled marked points (=vertices)?

The motivation (*) has led to [3], where we introduced and studied the complex of pairwise non-intersecting diagonals on an oriented surface (with no punctures so far). If the surface is closed, the complex is homotopy equivalent to the space of metric ribbon graphs $\mathcal{RG}^{\text{met}}_{g,n}$, or, equivalently, to the decorated moduli space $\tilde{\mathcal{M}}_{g,n}$. For bordered

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surfaces, we proved in \cite{3} the following: (1) Contraction of a boundary edge does not change the homotopy type of the support of the complex. (2) Contraction of a boundary component to a new marked point yields a forgetful map between two diagonal complexes which is homotopy equivalent to the Kontsevich’s tautological circle bundle $L_i$. (3) In the same way, contraction of several boundary components corresponds to Whitney sum of the tautological bundles.

The paper \cite{3} deals with the case when each of the boundary components contains at least one marked point. Motivation (**) suggests us to relax this condition. In the present paper we allow boundary components without marked points, or punctures. Elimination of a puncture gives rise to a bundle whose fiber is a surface $\overline{F}$ obtained from $F$ by cutting a hole around each of the marked points, see Section \ref{sec:3}. This bundle can be viewed as the combinatorial model to the "universal curve with holes".

In Section \ref{sec:4} we consider symmetric surfaces (that is, surfaces with a distinguished involution), symmetric diagonal arrangements, and associated diagonal complex. The latter is shown to be homotopy equivalent to the diagonal complex of some punctured surface. Oversimplifying, the relation reads as follows: cut the surface $F$ through the "symmetry axis", take one half of $F$, and replace the cut by a puncture. This naive surgery leads to a map between diagonal complexes which is a homotopy equivalence, see Theorem \ref{thm:3}.

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**2. Diagonal complex related to punctured surfaces: construction and introductory examples**

Assume that an oriented surface $F$ of genus $g$ with $b + f$ labeled boundary components $B_i$ is fixed. We mark $n$ distinct labeled points on $F$ not lying on the boundary. Besides, for each $i = 1, \ldots, b$ we fix $n_i > 0$ distinct labeled points on the boundary component $B_i$.

Unlike \cite{3}, we allow $f$ boundary components with no marked points. Let us turn them to punctures $P_i$. We assume that the punctures are labeled.

We assume that $F$ can be triangulated (decomposed into triangles, possibly punctured ones) with vertices at the marked points. That is, we exclude all "small" cases (like a sphere with one marked point and one puncture).
Altogether we have $N = n + \sum_{i=1}^{b} n_i$ marked points; let us call them vertices of $F$. The vertices not lying on the boundary are called free vertices. The vertices that lie on the boundary split the boundary components into edges.

A pure diffeomorphism $F \to F$ is an orientation preserving diffeomorphism which maps marked points to marked points, maps punctures to punctures, and preserves the labeling of marked points and punctures. Therefore, a pure diffeomorphism maps each boundary component to itself. The pure mapping class group $\text{PMC}(F)$ is the group of isotopy classes of pure diffeomorphisms.

A diagonal is a simple (that is, not self-intersecting) smooth curve $d$ on $F$ whose endpoints are some of the (possibly the same) vertices such that

1. $d$ contains no vertices (except for the endpoints).
2. $d$ does not intersect the boundary (except for its endpoints).
3. $d$ is not homotopic to an edge of the boundary.
   
   Here and in the sequel, we mean homotopy with fixed endpoints in the complement of the vertices $F \setminus \text{Vert}$. In other words, a homotopy never hits a vertex.
4. $d$ is non-contractible.

An admissible diagonal arrangement (or an admissible arrangement, for short) is a collection of diagonals $\{d_j\}$ with the properties:

1. Each free vertex is an endpoint of some diagonal.
2. No two diagonals intersect (except for their endpoints).
3. No two diagonals are homotopic.
4. The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of open (possibly punctured) disks. We allow any number of punctures in a single disc.

We say that a tuple $(g, b, n, f)$ is stable if no admissible arrangement has a non-trivial automorphism (that is, each pure diffeomorphism which maps an arrangement to itself, maps each germ of each of the diagonals $d_i$ to itself).

Tuples with $b > 1$ are stable since a boundary component allows to set a linear ordering on the germs of diagonals emanating from each of its vertices. It is known\footnote{This follows from Lefschetz fixed point theorem, as explained by Bruno Joyal in personal communications.} that any tuple with $n > 2g + 2$ is stable. Throughout the paper we assume that all the tuples are stable.
Definition 1. Two arrangements $A_1$ and $A_2$ are strongly equivalent whenever there exists a homotopy taking $A_1$ to $A_2$.

Two arrangements $A_1$ and $A_2$ are weakly equivalent whenever there exists a composition of a homotopy and a pure diffeomorphism of $F$ which maps bijectively $A_1$ to $A_2$.

Poset $\tilde{D}$ and cell complex $\tilde{D}$. Strong equivalence classes of admissible arrangements are partially ordered by reversed inclusion: we say that $A_1 \leq A_2$ if there exists a homotopy that takes the arrangement $A_2$ to some subarrangement of $A_1$.

Thus for the data $(g, b, n, f; n_1, ..., n_b)$ we have the posets of all strong equivalence classes of admissible arrangements $\tilde{D} = \tilde{D}_{g,b,n,f;n_1,...,n_b}$.

Theorem 1. ([15], [16]). If $F$ is a polygon with $n_1 > 0$ number of marked points on its boundary, $f$ punctures, and no free marked points, then the poset $\tilde{D} = \tilde{D}_{0,1,0,f;n_1}$ is a combinatorial ball $B^{n_1+2f-3}$. The cellulation has the unique biggest cell that corresponds to the empty diagonal arrangement.

The following example (and its generalizations) is well-known and is used in the cluster algebras world [4], [5]. However we put it here with a proof for the sake of completeness.

Example 1. The complex $D_{0,1,0,1;n}$ (that is, the diagonal complex associated with once-punctured $n$-gon) is combinatorially isomorphic to the cyclohedron.

Proof. Given an admissible arrangement, cut the polygon by a path connecting the puncture with the boundary of the polygon. We assume that the cut does not cross the diagonals. Take the copy of the polygon with the same arrangement and with the same cut, and glue the two copies together. We get a (non-punctured) $2n$-gon together with a centrally symmetric diagonal arrangement, see Fig. 1. This construction can be reversed, and therefore establishes a combinatorial isomorphism with the cyclohedron (see Appendix B).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Building a symmetric diagonal arrangement. The dashed line denotes the cut.}
\end{figure}
Theorem 4 implies that the poset $\tilde{D}$ can be realized as the poset of some (uniquely defined) regular cell complex $\tilde{D}$. Indeed, let us build up $\tilde{D}$ starting from the cells of maximal dimension. Each such cell corresponds to cutting of the surface $F$ into a single polygon with $f$ punctures. Adding more diagonals reduces to Theorem 1. In other words, $\tilde{D}$ is a patch of combinatorial balls that arise in the theorem.

For the most examples, $\tilde{D}$ has infinitely many cells. Our goal is to factorize $\tilde{D}$ by the action of the pure mapping class group. For this purpose consider the defined below barycentric subdivision of $\tilde{D}$.

**Poset $\overline{BD}$ and cell complex $\overline{BD}$**. We apply now the construction of the order complex [21] of a poset, which yields the barycentric subdivision. Each element of the poset $\overline{BD}_{g,b,n,f;n_1,\ldots,n_b}$ is (the strong equivalence class of) some admissible arrangement $A = \{d_1, \ldots, d_m\}$ with a linearly ordered partition $A = \bigsqcup S_i$ into some non-empty sets $S_i$ such that the first set $S_1$ in the partition is an admissible arrangement.

The partial order on $\overline{BD}$ is generated by the following rule: $(S_1, \ldots, S_p) \leq (S'_1, \ldots, S'_p)$ whenever one of the two conditions holds:

1. We have one and the same arrangement $A$, and $(S'_1, \ldots, S'_p)$ is an order preserving refinement of $(S_1, \ldots, S_p)$.
2. $p \leq p'$, and for all $i = 1, 2, \ldots, p$, we have $S_i = S'_i$. That is, $(S_1, \ldots, S_p)$ is obtained from $(S'_1, \ldots, S'_p)$ by removal $S'_{p+1}, \ldots, S'_{p'}$.

Let us look at the incidence rules in more details. Given $(S_1, \ldots, S_p)$, to list all the elements of $\overline{BD}$ that are smaller than $(S_1, \ldots, S_p)$ one has (1) to eliminate some (but not all!) of $S_i$ from the end of the string, and (2) to replace some consecutive collections of sets by their unions.

Examples:

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3, d_1, d_6\}, \{d_4, d_7\}).$$

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}).$$

$$(\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7\}, \{d_8\}) > (\{d_5, d_2\}, \{d_3\}, \{d_1, d_6\}, \{d_4\}, \{d_7, d_8\}).$$

Minimal elements of $\overline{BD}$ correspond to admissible arrangements. Maximal elements correspond to maximal arrangements $A$ together with some minimal admissible subarrangement $A' \subset A$ and a linear ordering on the set $A \setminus A'$.

By construction, the complex $\overline{BD}$ is combinatorially isomorphic to the barycentric subdivision of $\tilde{D}$.

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2 A cell complex $K$ is regular if each $k$-dimensional cell $c$ is attached to some subcomplex of the $(k-1)$-skeleton of $K$ via a bijective mapping on $\partial c$. 

We are mainly interested in the quotient complex:

**Definition 2.** For a fixed data \((g, b, n, f; n_1, \ldots, n_b)\), the diagonal complex \(BD_{g, b, n, f; n_1, \ldots, n_b}\) is defined as

\[
BD = \frac{BD_{g, b, n, f; n_1, \ldots, n_b}}{PMC(F)}.
\]

We define also

\[
D = \frac{D_{g, b, n, f; n_1, \ldots, n_b}}{PMC(F)}.
\]

Alternative definition reads as:

**Definition 3.** Each cell of the complex \(BD_{g, b, n, f; n_1, \ldots, n_b}\) is labeled by the weak equivalence class of some admissible arrangement \(A = \{d_1, \ldots, d_m\}\) with a linearly ordered partition \(A = \bigsqcup S_i\) into some non-empty sets \(S_i\) such that the first set \(S_1\) is an admissible arrangement.

The incidence rules are the same as the above rules for the complex \(BD\).

**Proposition 1.** The cell complex \(BD\) is regular. Its cells are combinatorial simplices.

Proof. If \((S_1, \ldots, S_r) \leq (S'_1, \ldots, S'_{r'})\) then there exists a unique (up to isotopy) order-preserving pure diffeomorphism of \(F\) which embeds \(A = S_1 \cup \cdots \cup S_r\) in \(A' = S'_1 \cup \cdots \cup S'_{r'}\). Indeed, If \(S_1 = S'_1\), the arrangement \(S_1\) maps identically to itself since it has no automorphisms by stability assumption. The rest of the diagonals are diagonals in polygons, and are uniquely defined by their endpoints. Assume that \(S_1 \subset S'_1\). For the rest of the cases it suffices to take \(A = S_1, A' = S'_1 \bigcup S_2\). If \(A\) embeds in \(A'\) in different ways, then \(A\) has a non-trivial isomorphism, which contradicts stability assumption. \(\square\)

**Remark.** A reader may imagine each (combinatorial) simplex in \(BD\) as a (Euclidean) equilateral simplex and to define the support, or geometric realization of the complex \(|BD| = |D|\) as the patch of these simplices.

3. **Diagonal Complexes Related to Punctured Surfaces: Main Theorems**

Contraction (or removal) of a puncture gives rise to a natural forgetful projection

\[
\pi : BD_{g, b, n, f+1; n_1, \ldots, n_b} \to BD_{g, b, n, f; n_1, \ldots, n_b}.
\]

It is defined as follows. An element of \(BD_{g, b, n, f+1; n_1, \ldots, n_b}\) corresponds to some admissible arrangement together with a partition \((S_1, \ldots, S_r)\).
Eliminate the puncture \( f + 1 \). We obtain a collection of diagonals on the surface with \( f \) punctures. Some of the diagonals may become either contractible or homotopic to an edge of \( F \). Eliminate them. Some of the diagonals may become pairwise homotopy equivalent. In each class we leave exactly one that belongs to \( S_i \) with the smallest index \( i \). Eventually some of the sets \( S_i \) may become empty in the process. Eliminate all the empty sets keeping the order of the rest. We obtain an element from \( BD_{g,b,n,f;m_1,...,n_b} \). It is easy to check that \( A < A' \) implies \( \pi(A) \leq \pi(A') \), so the map is indeed a poset morphism.

The poset morphism extends to a piecewise linear map (we denote it by the same letter \( \pi \)).

We shall need the new surface \( \overline{F} \) which is obtained from \( F \) by eliminating (or, contracting) all the punctures, and replacing each of the free marked points \( v_i \) by a hole \( H_i \).

**Theorem 2.** (1) The above defined forgetful projection

\[
\pi : BD_{g,b,n,f+1;m_1,...,n_b} \rightarrow BD_{g,b,n,f;m_1,...,n_b}
\]

is homotopy equivalent to a locally trivial bundle over \( BD_{g,b,n,f;m_1,...,n_b} \)
whose fibers are homeomorphic to the (above defined) surface \( \overline{F} \).

(2) Each of the holes \( H_i \) gives rise to a circle bundle over \( BD_{g,b,n,f;m_1,...,n_b} \).
It is isomorphic to the tautological circle bundle \( L_i \).

Proof. Let us examine the preimage \( \pi^{-1}(x) \) of an inner point \( x \) of a simplex of \( BD_{g,b,n,f;m_1,...,n_b} \) labeled by \( (S_1, ..., S_r) \). The label corresponds to an arrangement \( A = \bigcup S_i \) on \( F \) with \( f \) punctures. We shall show that \( \pi^{-1}(x) \) is a cell complex homeomorphic to \( \overline{F} \).

The explicit construction consists of two steps. On the first step we understand what diagonals are added to \( A \) in the labels of the cells that intersect \( \pi^{-1}(x) \).

On the second step we analyze the partition on the extended set \( A \).

**Step 1.** Analyze the "new" curves in the preimage.

Remove all the free boundary components (or punctures) from \( F \). Now the arrangement \( A \) cuts \( F \) into some polygons. A corner is a vertex with two germs of incident edges \( g_1 \) and \( g_2 \) such that there are no other germs between \( g_1 \) and \( g_2 \). For each of the corners, we blow up its vertex, that is, replace it by an extra edge, as is shown in Fig. 2. So each free vertex turns to a new boundary component. We get a two-dimensional cell complex \( \overline{F}(A) \) homeomorphic to \( \overline{F} \).

Each cell \( \sigma \) of \( \overline{F}(A) \) gives rise to a new diagonal arrangement \( A(x, \sigma) \supset A \) and specifies the place where the boundary component \( P_{f+1} \) should
be inserted. This is illustrated in Figures 3 and 4 and is described by the following rules:

(1) If $\sigma$ is a 2-dimensional cell, keep the arrangement $A$ as it is, and add the new puncture $P_{f+1}$ in the cell $\sigma$.
(2) If the cell $\sigma$ is an edge $e$ of $F$, add one more diagonal which is parallel to $e$, and add the new puncture $P_{f+1}$ between the two copies of $e$.
(3) If the cell $\sigma$ is a blow-up of one of the corners, add $P_{f+1}$ in a cell which is adjacent to the corner, and add one loop diagonal embracing $P_{f+1}$. The loop starts and ends at the corner.
(4) If the cell is one of the diagonals $d \in A$, duplicate $d$, and put $P_{f+1}$ between $d$ and its copy.
(5) If the cell is one of the new vertices, that is, corresponds to a corner and a diagonal (or to a corner and an edge) of $F$, we combine either (3) and (2), or (3) and (4). That is, add both a loop and a double of a diagonal (or a double of an edge).

**Step 2.** Analyze the partition in the preimage.

First, construct a new cell complex $\tilde{F} = \tilde{F}(S_1, ..., S_r)$ whose support is homeomorphic to $F$. This complex is more sensitive: it "knows" the partition $A = \bigcup S_i$, whereas $F$ depends on $A$ only.

(1) Start with the complex $\overline{F}$. Replace each boundary edge of $\overline{F}$ by $r+1$ parallel lines. We call the area between the lines the *grid related to the edge*.
(2) Replace each of the internal edges of \( F \), that is, each of diagonals \( d \in A \) by \( 2(r-i) + 3 \) parallel lines, where \( i \) is defined by condition \( d \in S_i \). The area between the lines is called the grid related to the edge.

(3) Each of the vertices is thus replaced by a rectangular grid \( r \times (2(r-i)+3) \). Add diagonals to some of the squares as is shown in Figure 7.

(4) Contract the strips corresponding to edges and diagonals along their lengths, so each of the strips becomes a segment. The rectangular grids survive unchanged.

(5) The complement of the union of all the grid areas is a number of "big disks". They bijectively correspond to two-cells of \( \overline{F} \). Contract each of them to a point.
Remark. In Figures 5,7 we depict a piece of $\widehat{\mathcal{F}}$ without contractions for the sake of better visualization. Keeping the contractions in mind, one should remember that the contracted cells have a different dimension.

The intersection of $\pi^{-1}(x)$ with the cells of the complex $\mathcal{BD} = \mathcal{BD}_{g,b,n,f+1,n_1,...,n_b}$ yields a structure of a cell complex on $\pi^{-1}(x)$. Let us show that the cell structure is isomorphic to $\widehat{\mathcal{F}}$ by presenting a bijection between the cells of $\widehat{\mathcal{F}}$ and the cells of $\mathcal{BD}$ intersecting $\pi^{-1}(x)$.

We add new diagonals to $(S_1, ..., S_r)$ and $P_{f+1}$ in the same way as we did on Step 1. But this time we specify the partition on the new
set of diagonals. That is, we decide in addition where to put the new diagonals (is any). Possible options are: we either put them in one of $S_i$, or create separate sets coming right after one of $S_i$. The above described grid tells us the choice of an option.

(1) If $\sigma$ is a 0-dimensional cell coming from a big disk, keep the arrangement $A$, its partition $(S_1, ..., S_r)$ and add the new puncture $P_{f+1}$ in the corresponding two-cells of $\mathcal{F}$.

(2) If the cell $\sigma$ is an edge of a grid related to one of the edges of $F$, we keep $(S_1, ..., S_r)$, duplicate the edge, and add $P_{f+1}$ between the edge and its double. The new diagonal (that is, the double of the edge) we put in one of $S_i$ where $i$ is determined by the number of the line in the grid. The lines of the grid are numbered starting from the edge.

(3) If the cell $\sigma$ is a strip of a grid related to one of the edges of $F$, we add the same diagonal as above, but now we put it in a singleton coming after $S_i$. The strips of the grid are numbered also starting from the edge.

(4) We apply the same strategy when adding a loop diagonal embracing the new $P_{f+1}$.

(5) If the cell corresponds to one of the diagonals $d \in S_k$, we duplicate $d$, put $P_{f+1}$ between $d$ and its copy. The new and the old diagonals now are undistinguished. At least one of them should be in $S_k$, the other one should be either in $S_k$ (this corresponds to the central line of the grid) or to the right of $S_k$. Altogether we have $(S_1, S_2, ..., S_r, \{d\}), (S_1, S_2, ..., S_r \cup \{d\}), (S_1, S_2, ..., \{d\}, S_r), ..., (S_1, S_2, ..., S_k \cup \{d\}, ..., S_r), ... (S_1, S_2, ..., S_k \setminus \{d'\} \cup \{d\}, ..., S_r \cup \{d'\}), (S_1, S_2, ..., S_k \setminus \{d'\} \cup \{d\}, ..., S_r, \{d'\})$.

(6) If the cell comes from the grid at some vertex we combine previous construction.

Example.

Figure 7 depicts a piece of $\hat{F}$ with grids. This fragment corresponds to a (vertical) diagonal $d$ plus two adjacent corners, right and left. In this particular example $r = 3$, that is, we have $(S_1, S_2, S_3)$. Here we also have $k = 2$, that is, $d \in S_2$. The arrows in the figure point to some cells of $\hat{F}$.

The cells correspond to adding loops $e_l$ or $e_r$, and duplicating the diagonal $d$, see Figure 8. The double of the diagonal we denote by $d'$. If the diagonal is duplicated, the left and right loops are indistinguishable.

We list the labels of the corresponding cells in $BD$:

(a) corresponds to $(S_1, \{e_l\}, S_2, S_3)$. 
Figure 6. Illustration to the Example: right loop \( e_r \), left loop \( e_l \), duplicated diagonal, loop + duplicated diagonal. On the last figure one sees that right and left loops are indistinguishable whenever the edge is duplicated.

Figure 7. This is a fragment of \( \tilde{\mathcal{F}} \) with grids but without contractions. The arrows refer to the above example.

(b) corresponds to \((S_1, S_2 \cup \{d'\}, S_3)\).
(c) corresponds to \((S_1 \cup \{e'\}, S_2 \cup \{d'\}, S_3)\).
(d) corresponds to \((S_1, S_2, \{e_r\} \cup \{d'\}, S_3)\).
(e) corresponds to \((S_1, S_2, S_3, \{d'\})\).

Remark.
One can rephrase the above theorem as: the triple

\[
\pi : \mathcal{BD}_{g,b,n,f+1;m_1,\ldots,m_b} \to \mathcal{BD}_{g,b,n,f;m_1,\ldots,m_b}
\]

is homotopy equivalent to the universal curve where each of the marked points is replaced by a hole.
4. Symmetric diagonal complexes: main construction and introductory examples

We are going to repeat the construction of Section 2 for symmetric arrangements on a symmetric $F$. Assume that an oriented surface $F$ of genus $2g$ with $2b$ labeled boundary components $B_1, ..., B_{2b}$ is fixed.

Let us assume that an involution $\text{inv} : F \to F$ is such that the set of fixed points is a separating circle $C_{\text{fix}} \subset F$.

We mark $2n$ distinct labeled points on $F$ not lying on the boundary. Besides, for each $i = 1, ..., b$ we fix $n_i > 0$ distinct labeled points on each of the boundary components $B_i$ and $B_{b+i}$ assuming that:

1. no marked point lies on $C_{\text{fix}}$;
2. the involution $\text{inv}$ maps marked points to marked points;
3. the involution $\text{inv}$ maps $B_i$ to $B_{b+i}$ for each $i = 1, ..., b$.

We assume that $F$ can be tiled by polygons with vertices at the marked points such that: (1) each polygon has at least three vertices, and (2) the tiling is symmetric with respect to $\text{inv}$. That is, we exclude all ”small” cases (like sphere with two marked points).

Altogether we have $2N = 2n + 2 \sum_{i=1}^{b} n_i$ marked points; let us call them vertices of $F$.

A pure diffeomorphism $F \to F$ is an orientation preserving diffeomorphism which commutes with $\text{inv}$ and maps each labeled point to itself. Therefore, a pure diffeomorphism maps each boundary component to itself. The pure mapping class group $\text{PMC}_{\text{inv}}(F)$ is the group of isotopy classes of pure diffeomorphisms.

A diagonal is a simple (that is, not self-intersecting) smooth curve $d$ on $F$ whose endpoints are some of the (possibly the same) vertices such that

1. $d$ contains no vertices (except for the endpoints).
2. $d$ does not intersect the boundary (except for its endpoints).
3. $d$ is not homotopic to an edge of the boundary.
4. $d$ is non-contractible.

Lemma 1. Assume that two homotopic diagonals $d_1$ and $d_2$ connect $v$ and $\text{inv}(v)$. Assume also that both $d_1$ and $d_2$ are symmetric: $\text{inv}$ maps each of $d_1$ and $d_2$ to itself. Then there exists a symmetric (=commuting with $\text{inv}$) homotopy taking $d_1$ to $d_2$. \hfill $\square$

A symmetric admissible diagonal arrangement (or a symmetric admissible arrangement, for short) is a non-empty collection of diagonals $\{d_j\}$ with the properties:

1. Each free vertex is an endpoint of some diagonal.
(2) No two diagonals intersect (except for their endpoints).
(3) No two diagonals are homotopic.

Two remarks are necessary: (a) As in the previous sections, we mean homotopy with fixed endpoints in the complement of the vertices $F \setminus Vert$. In other words, a homotopy never hits a vertex.

(b) The condition "The homotopy commutes with $\text{inv}$" is relaxed due to Lemma 1.

(4) The complement of the arrangement and the boundary components $(F \setminus \bigcup d_j) \setminus \bigcup B_i$ is a disjoint union of open disks.

(5) $\text{inv}$ takes the arrangement to itself.

Lemma 2. Let $F$ and $\text{inv}$ be as above.

(1) If a diagonal $d$ belongs to an admissible arrangement, then it intersects $C^{\text{fix}}$ at at most one point.

(2) If a disk in $F \setminus \{\text{diagonals}\}$ is bounded by diagonals $d_1, \ldots, d_k$, then all the diagonals taken together intersect $C^{\text{fix}}$ at at most two points.

\[ \square \]

Definition 4. Two symmetric arrangements $A_1$ and $A_2$ are strongly equivalent whenever there exists a symmetric homotopy taking $A_1$ to $A_2$.

Two symmetric arrangements $A_1$ and $A_2$ are weakly equivalent whenever there exists a composition of a symmetric homotopy and a pure diffeomorphism of $F$ which maps bijectively $A_1$ to $A_2$.

We assume that all the tuples are stable: no admissible arrangement has a non-trivial automorphism (that is, each pure diffeomorphism which maps an arrangement to itself, maps each germ of each of $d_i$ to itself).

Poset $\tilde{D}^{\text{inv}}$ and cell complex $\tilde{D}^{\text{inv}}$. Strong equivalence classes of symmetric admissible arrangements are partially ordered by reversed inclusion: we say that $A_1 \leq A_2$ if there exists a symmetric homotopy that takes the arrangement $A_2$ to some symmetric subarrangement of $A_1$.

Thus for the data $(g, b, n; n_1, \ldots, n_b)$ we have the poset of all strong equivalence classes of symmetric admissible arrangements $\tilde{D}^{\text{inv}} = \tilde{D}^{\text{inv}}_{g, b, n; n_1, \ldots, n_b}$.

Minimal elements of the poset correspond to maximal (by inclusion) symmetric cuts of $F$. 
into triangles and quadrilaterals such that each quadrilateral is symmetric under \( \text{inv} \).

Maximal elements of the poset correspond to minimal symmetric admissible arrangements, that is, to symmetric cuts of \( F \) into a single disc.

**Proposition 2.** Take a planar regular \( 2k \)-gon with a fixed symmetry axis which contains no vertices. Consider the poset of collections of its non-crossing diagonals that are symmetric with respect to the axis. The poset is isomorphic to the face poset of the associahedron \( As_{k+1} \).

Proof. Cut the polygon through the symmetry axis and contract the cut to a new vertex. This yields a \((k + 1)\)-gon. Pairs of symmetric diagonals of the \( 2k \)-gon correspond bijectively to diagonals of the \((k + 1)\)-gon. Single symmetric diagonals correspond to diagonals emanating from the new vertex. We therefore arrive to a poset isomorphism. \( \square \)

The poset \( \tilde{D}^{inv} \) can be realized as the poset of some (uniquely defined) cell complex. Indeed, let us build up \( \tilde{D}^{inv} \) starting from the cells of maximal dimension. Each such cell corresponds to cutting of the surface \( F \) into a single polygon. Adding more diagonals reduces to Example 2. In other words, \( \tilde{D}^{inv} \) is a patch of associahedra.

As in [3], and in Section 2, we factorize \( \tilde{D}^{inv} \) by the action of the pure mapping class group. For this purpose consider the defined below barycentric subdivision of \( \tilde{D}^{inv} \).

**Poset \( \tilde{BD}^{inv} \) and cell complex \( \tilde{BD}^{inv} \).** We apply now the construction of the order complex of a poset, which gives us barycentric subdivision. Each element of the poset \( \tilde{BD}^{inv}_{q,b,n_1,...,n_k} \) is (the strong equivalence class of) some symmetric admissible arrangement \( A = \{ d_1, ..., d_m \} \) with a linearly ordered partition \( A = \bigsqcup S_i \) into some non-empty sets \( S_i \) such that the first set \( S_1 \) in the partition is an admissible arrangement, and all the sets \( S_i \) are invariant under \( \text{inv} \).

The partial order on \( \tilde{BD}^{inv} \) verbatim repeats the constructions of [3] and Section 2. Namely, it is generated by the following rule:

\((S_1, ..., S_p) \leq (S'_1, ..., S'_p)\) whenever one of the two conditions holds:

1. We have one and the same arrangement \( A \), and \((S'_1, ..., S'_p)\) is an order preserving refinement of \((S_1, ..., S_p)\).
2. \( p \leq p' \), and for all \( i = 1, 2, ..., p \), we have \( S_i = S'_i \). That is, \((S_1, ..., S_p)\) is obtained from \((S'_1, ..., S'_p)\) by removal \( S'_{p+1}, ..., S'_{p'} \).

By construction, the complex \( \tilde{BD}^{inv} \) is combinatorially isomorphic to the barycentric subdivision of \( \tilde{D}^{inv} \).
We are mainly interested in the quotient complex:

**Definition 5.** For a fixed data \((g, b, n; n_1, \ldots, n_b, \text{inv})\), the diagonal complex \(\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}\) is defined as

\[
\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} = \mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} := \mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} / P M C_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}(F).
\]

We define also

\[
\mathcal{D}_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} = \mathcal{D}_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} := \mathcal{D}_{g,b,n; n_1,\ldots,n_b}^{\text{inv}} / P M C_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}(F).
\]

Alternative definition reads as:

**Definition 6.** Each cell of the complex \(\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}\) is labeled by the weak equivalence class of some symmetric admissible arrangement \(A = \{d_1, \ldots, d_m\}\) with a linearly ordered partition \(A = \bigcup S_i\) into some non-empty sets \(S_i\) such that the first set \(S_1\) is an admissible arrangement, and all \(S_i\) are invariant under \(\text{inv}\).

The incidence rules are the same as the above rules for the complex \(\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}\).

**Proposition 3.** The cell complex \(\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}\) is regular. Its cells are combinatorial simplices.

Proof. If \((S_1, \ldots, S_r) \leq (S'_1, \ldots, S'_r)\) then there exists a unique (up to isotopy) order-preserving pure diffeomorphism of \(F\) which embeds \(A = S_1 \cup \ldots \cup S_r\) in \(A' = S'_1 \cup \ldots \cup S'_r\). Indeed, If \(S_1 = S'_1\), the arrangement \(S_1\) maps identically to itself since it has no automorphisms by stability assumption. The rest of the diagonals are diagonals in polygons, and are uniquely defined by their endpoints. Assume that \(S_1 \subset S'_1\). For the rest of the cases it suffices to take \(A = S_1\), \(A' = A = S'_1 \cup S'_2\). If \(A\) embeds in \(A'\) in different ways, then \(A\) has a non-trivial isomorphism, which contradicts stability assumption. \(\Box\)

5. Symmetry vs puncture

In our setting, the surface \(F\) is patched of two copies of an orientable surface with a distinguished boundary component \(C^{\text{fix}}\). Contract \(C^{\text{fix}}\) to a new vertex \(v\) and denote the resulted surface by \(\frac{1}{2}F\). It inherits from the initial surface \(F\) one half of its vertices. Now replace the vertex \(v\) by a puncture. This gives \(\frac{1}{2}F^{\text{punc}}\).

Take a simplex in \(\mathcal{B}D_{g,b,n; n_1,\ldots,n_b}^{\text{inv}}\) (starting from now, let us omit subscripts). It is labeled by an admissible arrangement \(A = S_1 \sqcup S_2 \ldots \sqcup S_k\). Cut the surface \(F\) with the arrangement through \(C^{\text{fix}}\) and take one half. Contract \(C^{\text{fix}}\) to a new vertex \(v\). A moment’s reflection reveals that
one gets the surface $\frac{1}{2}F$ together with an admissible arrangement such that removal of $v$ with all the incident diagonals leaves an admissible arrangement. Next, replace $v$ by a puncture and remove all the incident diagonals. We obtain two well-defined maps:

$$\mathcal{BD}^{\text{inv}} \xrightarrow{\pi_1} \mathcal{BD}\left(\frac{1}{2}F\right),$$

and

$$\text{Im } \pi_1 \xrightarrow{\pi_2} \mathcal{BD}\left(\frac{1}{2}F_{\text{punc}}\right).$$

**Theorem 3.** (1) The map

$$\mathcal{BD}^{\text{inv}} \xrightarrow{\pi_1} \mathcal{BD}\left(\frac{1}{2}F\right)$$

which takes one half of $F$ and contracts $C^{\text{fix}}$ to a new vertex $v$, is a combinatorial isomorphism on its image.

(2) The triple

$$\text{Im } \pi_1 \xrightarrow{\pi_2} \mathcal{BD}\left(\frac{1}{2}F_{\text{punc}}\right)$$

is a homotopy equivalence. Here the map $\pi_2$ removes all diagonals incident to $v$ and replaces $v$ by a puncture.

(3) Altogether,

$$\mathcal{BD}^{\text{inv}} \xrightarrow{\pi_2 \circ \pi_1} \mathcal{BD}\left(\frac{1}{2}F_{\text{punc}}\right)$$

is a homotopy equivalence.

**Proof.**

The claim (1) is clear by construction and Lemma 2.

Assume that a simplex $\sigma$ belongs to $\mathcal{BD}\left(\frac{1}{2}F\right)$. As we know, it is labeled by some $(S_1, \ldots, S_k)$, and the diagonals from $S_1$ cut $\frac{1}{2}F$ into disks. The simplex $\sigma$ belongs to $\text{Im } \pi_1$ iff none of the discs has more than one corner incident to $v$.

The statement (3) follows directly from (1),(2).

Now prove (2). Due to [18], Theorem A, it suffices to prove that the preimage of each closed simplex is contractible. So let us take a simplex $\sigma$ in $\mathcal{BD}\left(\frac{1}{2}F_{\text{punc}}\right)$. It is labeled by some $(S_1, \ldots, S_k)$, where $S_1$ and $A = S_1 \sqcup \ldots \sqcup S_k$ are admissible arrangements. Turn the puncture to a vertex $v$. The vertex $v$ lies in the minimal polygon $P$ whose edges belong to $S_1$ or are the edges of boundary components. Denote the vertices (with possible repetitions) of $P$ by $p_1, \ldots, p_q$, see Fig. 8(a). A repeated vertex is given different indices, although it is one and the same marked point.
Figure 8. (a) We depict a fragment of the label 
\((S_1, \ldots, S_k)\) of one of the simplices \(\sigma\). The polygon \(P\) is the bold one. (b) The label of one of the simplices lying in the preimage \(\pi^{-1}(\sigma)\).

The labels of the simplices of the preimage \(\pi^{-1}(\text{Cl} \sigma)\) of the closure 
are obtained by: (1) eliminating some (might be none) of the entries 
from the end of the string \((S_1, \ldots, S_k)\), (2) replacing some consecutive 
entries by their union, and (3) adding a number of new diagonals 
in the polygon \(P\) together with assigning numbers to the new diagonals. 
Each new diagonal should emanate from \(v\). At least one of the new 
diagonals should be assigned number 1, see Fig. 8(b).

We shall prove that \(\pi^{-1}(\text{Cl} \sigma)\) is contractible by presenting a discrete 
Morse function with exactly one critical simplex (see Appendix A for 
necessary backgrounds).

Step 1. "Moving new diagonals with numbers greater than 1". Each new diagonal connects \(v\) with some of \(p_i\). Since it is uniquely 
determined by \(p_i\), let us call it \(d_i\).

1. Assume that a simplex in the preimage \(\pi^{-1}(\text{Cl} \sigma)\) is labeled by 
\((Q_1, \ldots, (Q_i \cup \{d_1\}), \ldots, Q_m)\) such that \(i > 1\) and \(Q_i \neq \emptyset\). Match 
it with \((Q_1, \ldots, \{d_1\}, Q_i, \ldots, Q_m)\).

The unmatched simplices are labeled by \((Q_1, \ldots, Q_m)\) such 
that either (a) \(d_1 \in Q_1\), or (b) \(d_1\) is missing, or (c) \(\{d_1\}\) is a 
singleton at the end.

2. Take the simplices in \(\pi^{-1}(\text{Cl} \sigma)\) that are unmatched on the previous step. Assume that a (nonmatched) simplex is labeled
by \((Q_1, \ldots, (Q_i \cup \{d_2\}), \ldots, Q_m)\) such that \(i > 1\), \(Q_i \neq \emptyset\), and \(Q_i \neq \{d_1\}\). Match it with \((Q_1, \ldots, \{d_2\}, Q_i, \ldots, Q_m)\).

After this series of matchings the unmatched simplices have labels of the following four types:

1. \((Q_1, \ldots, Q_m)\) such that none of \(Q_i\) contains \(d_1\) or \(d_2\) if \(i > 1\).
2. \((Q_1, \ldots, Q_m, \{d_1\})\) such that none of \(Q_i\) contains \(d_2\) if \(i > 1\).
3. \((Q_1, \ldots, Q_m, \{d_2\})\) such that none of \(Q_i\) contains \(d_1\) if \(i > 1\).
4. \((Q_1, \ldots, Q_m, \{d_2\}, \{d_1\})\).

(3) Proceed the same way with all the other diagonals.

Finally, the unmatched simplices are labeled by \((Q_1, \ldots, Q_m, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\) such that (a) none of \(Q_i\) contains new diagonals if \(i > 1\), and (b) \(i_1 > i_2 > \ldots > i_r\).

In other words, for an unmatched simplex, the new diagonals either sit in the set \(Q_1\), or come as singletons at the very end of the label in the decreasing order.

**Step 2. "Getting rid of old diagonals with numbers greater than 1".** Take a simplex in \(\pi_1^{-1}(\text{Cl}\sigma)\) labeled by \((Q_1, \ldots, Q_m, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\) which is not matched on the Step 1. Denote the set of all the new diagonals lying in \(Q_1\) by \(NEW\). Recall also that \(Q_2, \ldots, Q_m\) contain no new diagonals. Assuming that \(Q_1 \setminus NEW \neq S_1\), match \((Q_1, \ldots, Q_m, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\) with \((S_1 \cup NEW, Q_1 \setminus NEW \setminus S_1, Q_2, \ldots, Q_m, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\).

After Step 2, the unmatched simplices are labeled by \((Q_1, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\) such that (a) \(Q_1 = S_1 \cup NEW\), and (b) \(i_1 > i_2 > \ldots > i_r\).

**Step 3. "Final contractions. Getting rid of singletons with new diagonals."**

1. Take an unmatched (after Steps 1 and 2) simplex \((Q_1, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\). If \(i_r = 1\), match it with \((Q_1, \{d_{i_1}\}, \ldots, \{d_{i_{r-1}}\})\). The unmatched simplices are those with \(d_1 \in Q_1\).
2. Take an unmatched simplex \((Q_1, \{d_{i_1}\}, \ldots, \{d_{i_r}\})\). If \(i_r = 2\), match it with \((Q_1, \{d_{i_1}\}, \ldots, \{d_{i_{r-1}}\})\). The unmatched simplices are those without \(d_1, d_2 \in Q_1\).
3. Proceed the same way for \(d_3, d_4, \text{etc.}\)

Let us show that the matching is a discrete Morse function. The first two axioms follow straightforwardly from the construction. The aciclicity will be proved a bit later.

So we arrive at a discrete Morse function with a unique unmatched (that is, unique critical) simplex labeled by \((S_1 \cup \{d_1, \ldots, d_r\})\). By the basic discrete Morse theory (see Appendix A), the preimage \(\pi_1^{-1}(\text{Cl}\sigma)\) is contractible.
Before we prove the acyclicity, let us look at an example of a gradient path:

\[(Q_1, Q_2 \cup \{d_2\}, Q_3 \cup \{d_1\}), \ (Q_1, Q_2 \cup \{d_2\}, \{d_1\}, Q_3), \]
\[(Q_1, Q_2 \cup \{d_2\} \cup \{d_1\}, Q_3), \ (Q_1, \{d_1\}, Q_2 \cup \{d_2\}, Q_3)\]
\[(Q_1 \cup \{d_1\}, Q_2 \cup \{d_2\}, Q_3), \ (Q_1 \cup \{d_1\}, \{d_2\}, Q_2, Q_3)\]

Assume there exists a closed path.

(a) For a closed path, no new diagonal enters \(Q_1\), since it can never leave \(Q_1\).

(b) For each entry \(\beta_i^{p+1}\) or \(\alpha_i^p\) of the path, denote by \(NEW\) the set of new diagonals not lying in \(Q_1\). For a closed path, the set \(NEW\) cannot decrease (if it contains some \(d_i\), it never disappears on consequent steps of the path). Since the path is closed, \(NEW\) does not change during the path. Therefore, in a closed path the Step 3 is missing.

(c) Assume that \(d_i, d_j \in NEW\) with \(i < j\). Then if \(d_i\) is positioned to the left of \(d_j\), it never appears to the right of \(d_j\) in a closed path. We conclude that in a closed path, all the entries of \(NEW\) appear as singletons coming in the decreasing order at the end of the label. In other words, in a closed path, Step 1 matchings are missing.

(d) Finally, a closed path cannot have Step 2 matchings only. This follows from a simple case analysis. \(\square\)

APPENDIX A. DISCRETE MORSE THEORY \[6, 7\]

Assume we have a regular cell complex. By \(\alpha^p\), \(\beta^p\) we denote its \(p\)-dimensional cells, or \(p\)-cells, for short.

A discrete vector field is a set of pairs

\[(\alpha^p, \beta^{p+1})\]

such that:

(1) each cell of the complex is matched with at most one other cell, and
(2) in each pair, the cell \(\alpha^p\) is a facet of \(\beta^{p+1}\).

Given a discrete vector field, a path is a sequence of cells

\[\alpha_0^p, \beta_0^{p+1}, \alpha_1^p, \beta_1^{p+1}, \alpha_2^p, \beta_2^{p+1}, \ldots, \alpha_m^p, \beta_m^{p+1}, \alpha_{m+1}^p,\]

which satisfies the conditions:

(1) Each \((\alpha_i^p\) and \(\beta_i^{p+1}\)) are matched.
(2) Whenever \(\alpha\) and \(\beta\) are neighbors in the path, \(\alpha\) is a facet of \(\beta\).
(3) \(\alpha_i \neq \alpha_{i+1}\).
A path is a closed path if $\alpha^p_{m+1} = \alpha^p_0$.

A discrete Morse function on a regular cell complex is a discrete vector field without closed paths. It gives a way of contracting all the cells of the complex that are matched: if a cell $\sigma$ is matched with its face $\sigma'$, then these two can be contracted by pushing $\sigma'$ inside $\sigma$. Acyclicity guarantees that if we have many matchings at a time, one can consequently perform the contractions. The order of contractions does not matter, and eventually one arrives at a complex homotopy equivalent to the initial one.

In the paper we use the following fact: if a regular cell complex has a discrete Morse function with exactly one critical cell, then the complex is contractible.

Appendix B. Associahedron and cyclohedron

Associahedron and cyclohedron, [19] and [1]. Assume that $n > 2$ is fixed. We say that two diagonals in a convex $n$-gon are non-intersecting if they intersect only at their endpoints (or do not intersect at all). Consider all collections of pairwise non-intersecting diagonals in the $n$-gon. This set is partially ordered by reverse inclusion, and it was shown by John Milnor, that the poset is isomorphic to the face poset of some convex $(n-3)$-dimensional polytope $As_n$ called associahedron.

In particular, the vertices of the associahedron $As_n$ correspond to the triangulations of the $n$-gon, and the edges correspond to edge flips in which one of the diagonals is removed and replaced by a (uniquely defined) different diagonal. Single diagonals are in a bijection with facets of $As_n$, and the empty set corresponds to the entire $As_n$.

Analogously, centrally symmetric collections of diagonals in a $2n$-gon give rise to a convex polytope called cyclohedron, or Bott–Taubes polytope. Its first definition is the compactification of the configuration space of $n$ points on the circle.

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3that is, a cell of dimension $\dim(\sigma) - 1$ lying on the boundary of $\sigma$.

4It is important that the vertices of the polygon are labeled, and therefore we do not identify collections of diagonals that differ on a rotation.
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