Normal numbers with given limits of multiple ergodic averages

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Abstract
We are interested in the set of normal sequences in the space \( \{0, 1\}^\mathbb{N} \) with a given frequency of the pattern 11 in the positions \( k, 2k \). The topological entropy of such sets is determined.

1 Introduction and statement of results

Let \( \Sigma = \{0, 1\}^\mathbb{N} \). In [K12, FLM12], the authors proposed to calculate the topological entropy spectrum of level sets of multiple ergodic averages. Here, the topological entropy means Bowen’s topological entropy (in the sense of [B73]) which can be defined for any subset, not necessarily invariant. Among other questions, they asked for the topological entropy of

\[ A_\alpha := \left\{ \omega_{k}^\mathbb{N} \in \Sigma : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \omega_{k} \omega_{2k} = \alpha \right\} \quad (\alpha \in [0, 1]). \]

As a first step to solve the question, they also suggested to study a subset of \( A_0 \):

\[ A := \left\{ \omega_{k}^\mathbb{N} \in \Sigma : \omega_{k} \omega_{2k} = 0 \quad \text{for all } k \geq 1 \right\}. \]

The topological entropy of \( A \) was later given by Kenyon, Peres and Solomyak [KPS12].

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Theorem 1.1 (Kenyon-Peres-Solomyak). We have
\[ h_{\text{top}}(A) = -\log(1 - p) = 0.562399..., \]
where \( p \in [0, 1] \) is the unique solution of
\[ p^2 = (1 - p)^3. \]

Enlightened by the idea of [KPS12], the question about \( A_\alpha \) was finally answered by Peres and Solomyak [PS12], and then in higher generality by Fan, Schmeling and Wu [FSW16].

Theorem 1.2 (Peres-Solomyak, Fan-Schmeling-Wu). For any \( \alpha \in [0, 1] \), we have
\[ h_{\text{top}}(A_\alpha) = -\log(1 - p) - \frac{\alpha}{2} \log \frac{q(1 - p)}{p(1 - q)}, \]
where \( (p, q) \in [0, 1]^2 \) is the unique solution of the system
\[
\begin{align*}
    p^2(1 - q) &= (1 - p)^3, \\
    2pq &= \alpha(2 + p - q).
\end{align*}
\]
In particular, \( h_{\text{top}}(A_0) = h_{\text{top}}(A) \).

Another, interesting, related set is
\[ B := \left\{ (\omega_k)_{k=1}^\infty \in \Sigma : \omega_k = \omega_{2k} \quad \text{for all } k \geq 1 \right\}. \]

The sequence \( x \in \{0, 1\}^\mathbb{N} \) is said to be simple normal if the frequency of the digit 0 in the sequence is 1/2. It is said to be normal if for all \( n \in \mathbb{N} \), each word in \( \{0, 1\}^n \) of length \( n \) has frequency \( 1/2^n \). We denote the set of normal sequences by \( \mathcal{N} \).

We are interested in the intersection of \( \mathcal{N} \) with the set \( A_\alpha \) of given frequency of the pattern 11 in \( w_kw_{2k} \). For the usual ergodic (Birkhoff) averages the normal numbers all belong to one set in the multifractal decomposition – the situation for multiple ergodic averages turns out to be very different.

Our results are as follows:

Theorem 1.3. For \( \alpha \leq 1/2 \) we have
\[ h_{\text{top}}(\mathcal{N} \cap A_\alpha) = \frac{1}{2} \log 2 + \frac{1}{2} H(2\alpha), \]
where \( H(x) = -x \log x - (1 - x) \log(1 - x) \). For \( \alpha > 1/2 \) the set \( \mathcal{N} \cap A_\alpha \) is empty.

Further,
\[ h_{\text{top}}(\mathcal{N} \cap A) = h_{\text{top}}(\mathcal{N} \cap A_0) = \frac{1}{2} \log 2. \]
Moreover, \( \mathcal{N} \cap B \subset A_{1/2} \) and
\[ h_{\text{top}}(\mathcal{N} \cap B) = h_{\text{top}}(\mathcal{N} \cap A_{1/2}) = h_{\text{top}}(B) = \frac{1}{2} \log 2. \]
The last statement of Theorem 1.3 was recently proved, in higher generality, in [ABC].

Let us now define the set of sequences with prescribed frequency of 0's and 1's:

\[ E_\theta := \{ x \in [0,1] : \lim_{n \to \infty} \frac{\omega_1(x) + \cdots + \omega_n(x)}{n} = \theta \}. \]

In particular, \( E_{1/2} \) is the set of simple normal sequences.

**Theorem 1.4.** We have

\[
h_{\text{top}}(E_\theta \cap A_\alpha) = (1 - \frac{\theta}{2})H\left(\frac{2\theta - \alpha}{2 - \theta}\right) + \frac{\theta}{2}H\left(\frac{\theta - \alpha}{\theta}\right)
\]

for \( \alpha \leq \theta \leq \frac{(2 + \alpha)}{3} \), otherwise \( E_\theta \cap A_\alpha = \emptyset \). Further,

\[
h_{\text{top}}(E_\theta \cap A) = h_{\text{top}}(E_\theta \cap A_0) = \frac{2 - \theta}{2}H\left(\frac{2\theta}{2 - \theta}\right).
\]

Note that

\[
h_{\text{top}}(E_{1/2} \cap A) = \frac{3}{4}H\left(\frac{2}{3}\right) > h_{\text{top}}(N \cap A).
\]

**Remark.** Applying the results of [PS12] one can show that

\[
h_{\text{top}}(E_\theta \cap A_\alpha) = h_{\text{top}}(A_\alpha)
\]

if and only if \( \alpha, \theta \) satisfy the relation

\[
(2\theta - \alpha)^2(\theta - \alpha)(2 - \theta) = \theta(2 - 3\theta + \alpha)^3.
\]

In particular, when

\[
\theta = \frac{2}{3} \left(1 + \left(\frac{2}{23}\right)^{2/3} \sqrt[3]{3\sqrt{69} - 23} - \left(\frac{2}{23}\right)^{2/3} \sqrt[3]{3\sqrt{69} + 23}\right) = 0.354...,\]

i.e., the unique real solution of the equation \( 4\theta^3(2 - \theta) = (2 - 3\theta)^3 \), we have

\[
\dim_H E_\theta \cap A = \dim_H A.
\]

We omit the details.
2 Proof of Theorem 1.3

Given \( \alpha \in [0, 1] \), let \( \mu_\alpha \) be a probability measure on \( \Sigma \) given by

- if \( k \) is odd then \( \omega_k = 1 \) with probability \( 1/2 \),
- if \( k \) is even and \( \omega_{k/2} = 1 \) then \( \omega_k = 1 \) with probability \( 2\alpha \),
- if \( k \) is even and \( \omega_{k/2} = 0 \) then \( \omega_k = 1 \) with probability \( 1 - 2\alpha \),

with the events \( \{ \omega_k = 1 \} \) and \( \{ \omega_\ell = 1 \} \) independent except when \( k/\ell \) is a power of 2. Precisely, let \((p_0, p_1) := (1/2, 1/2)\) and let

\[
\begin{pmatrix}
p_{00} & p_{01} \\
p_{10} & p_{11}
\end{pmatrix} := \begin{pmatrix} 2\alpha & 1 - 2\alpha \\ 1 - 2\alpha & 2\alpha \end{pmatrix}.
\]

Let \( C_n(\omega_1, \ldots, \omega_n) \) be the set of sequences beginning with the word \( \omega_1 \cdots \omega_n \in \{0, 1\}^n \). Such sets are called cylinders of order \( n \). The measure \( \mu_\alpha \) of a cylinder is given by

\[
\mu_\alpha(\omega_1 \cdots \omega_n) = \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_{2k}} = \frac{1}{2^{\lfloor n/2 \rfloor}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_k \omega_{2k}},
\]

where \( \lceil \cdot \rceil, \lfloor \cdot \rfloor \) denote the ceiling function and the integer part function correspondingly.

We will prove that the measure \( \mu_\alpha \) is supported on the set \( \mathcal{N} \cap A_\alpha \).

**Lemma 2.1.** We have \( \mu_\alpha(\mathcal{N} \cap A_\alpha) = 1 \).

**Proof.** Denote

\[
x_n(\omega) = \frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k.
\]

For a \( \mu_\alpha \)-typical \( \omega \), the Law of Large Numbers implies

\[
x_{2n}(\omega) = \frac{1}{4} + \frac{x_n(\omega)}{2} 2\alpha + \frac{1 - x_n(\omega)}{2} (1 - 2\alpha) + o(1).
\]

Noting that \( |\frac{4n-1}{2}| < 1 \), we have as \( k \to \infty \),

\[
x_{2k}(\omega) \to \frac{1}{2}.
\]

By [PS12, Lemma 5], this implies that \( \mu_\alpha \)-almost surely

\[
\lim_{n \to \infty} x_n(\omega) = \frac{1}{2}.
\]

(2.1)
Then, for \(\mu_\alpha\)-a.e. \(\omega\),
\[
\frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k \omega_{2k} = x_n(\omega)(2\alpha + o(1)) \to \alpha.
\]
Thus \(\mu_\alpha(A_\alpha) = 1\).

Now, we show \(\mu(N) = 1\). We can divide the set of natural numbers into infinitely many subsets of the form \(A_k = \{2k - 1, 4k - 2, \ldots, 2^\ell(2k - 1), \ldots\}\) \((k \geq 1)\). Let \(B_k\) be the \(\sigma\)-field generated by events \(\{\omega_{2^\ell(2k-1)} = 1\}\), \(\ell \in \mathbb{N}\).

Observe that for the measure \(\mu\) the \(\sigma\)-fields \(B_k\) are independent.

Observe further that \(\mu(\omega_{2^\ell(2k-1)} = 1) = 1/2\) for every \(k, \ell\). Indeed, for \(\ell = 0\) it follows from the definition of \(\mu\), and then it is proved by induction:
\[
\begin{align*}
\mu(\omega_{2^\ell(2k-1)} = 1) &= \mu(\omega_{2^\ell(2k-1)} = 1) + \mu(\omega_{2^{\ell+1}(2k-1)} = 0) \\
&= 2\alpha \cdot 1/2 + (1 - 2\alpha) \cdot 1/2 = 1/2.
\end{align*}
\]

Consider now, for any \(n\), the sequence \(\omega_{m+1}, \ldots, \omega_{m+n}\). If \(m \geq n\) then positions \(m+1, \ldots, m+n\) come all from different \(A_k\)’s, thus \(\omega_{m+1}, \ldots, \omega_{m+n}\) are independent and each of them takes values 0, 1 with probability 1/2 respectively. That is, the measure \(\mu\) restricted to such subset of positions is \((1/2, 1/2)\)-Bernoulli, and for any word \(\eta \in \{0, 1\}^n\) with \(n \leq m\), the probability that we have \(\omega_{m+i} = \eta_i\) for \(i = 1, \ldots, n\) equals \(2^{-n}\). Thus, for a given word \(\eta \in \{0, 1\}^n\) we can divide \(\mathbb{N}\) into intervals \([2^j + 1, 2^{j+1}]\), inside all except initial finitely many of them (with \(j < \log_2 n\)) for any \(\mu\)-generic sequence \(\omega\) the frequency of appearance of \(\eta\) equals \(2^{-n} + O(2^{-j/2}j \log j)\), and this means that the \(\mu\)-generic sequence \(\omega\) is normal.

Next, we will calculate the local dimension of the measure \(\mu_\alpha\) with the help of Mass Distribution Principle, [2, 7]. We denote for \(x \in [0, 1]\)
\[
H(x) = -x \log x - (1 - x) \log(1 - x)
\]
with convention \(H(0) = H(1) = 0\).

**Lemma 2.2.** We have
\[
h_{\mu_\alpha} = \frac{1}{2} \log 2 + H(2\alpha).
\]

**Proof.** For \(\omega \in \Sigma\) denote
\[
C_n(\omega) = \{\tau \in \Sigma; \tau_k = \omega_k \forall k \leq n\}.
\]
Let
\[
h_n(\omega) := \log \mu_\alpha(C_{2n}(\omega)) - \log \mu_\alpha(C_n(\omega)).
\]
By the Law of Large Numbers, for $\mu_\alpha$-typical $\omega$ and for big enough $n$ we have
\[
\frac{2}{n} h_n(\omega) = -\log 2 + (1 - x_n(\omega))((2\alpha \log(2\alpha) + (1 - 2\alpha) \log(2\alpha)) \\
+ x_n(\omega)((1 - 2\alpha) \log(1 - 2\alpha)) + (2\alpha) \log(2\alpha)) + o(1).
\]
Thus,
\[
\lim_{n \to \infty} -\frac{1}{n} h_n(\omega) = \frac{1}{2} \log 2 + \frac{1}{2} H(2\alpha) \quad \mu_\alpha - \text{a.e.}
\]
Note that for all $k, n \in \mathbb{N}$
\[
\frac{1}{k2^n} \log \mu_\alpha(C_{k2^n}(\omega)) = \frac{1}{k2^n} \sum_{i=1}^{n-1} h_{2i}.
\]
Then for all $k \in \mathbb{N}$
\[
\lim_{n \to \infty} -\frac{1}{k2^n} \log \mu_\alpha(C_{k2^n}(\omega)) = \frac{1}{2} \log 2 + \frac{1}{2} H(2\alpha) \quad \mu_\alpha - \text{a.e.}
\]
Hence, by [PS12, Lemma 5],
\[
h_{\mu_\alpha} = \liminf_{n \to \infty} -\frac{1}{n} \mu_\alpha(C_n(\omega)) = \frac{1}{2} \log 2 + \frac{1}{2} H(2\alpha) \quad \mu_\alpha - \text{a.e.}
\]
Applying the Mass Distribution Principle ends the proof.

To finish the proof of the lower bound we note that $A \subset A_0$ but the measure $\mu_0$ is actually supported on $A$, that the measure $\mu_{1/2}$ is supported on $B$, and that the relation $\mathcal{N} \cap B \subset A_{1/2}$ follows from
\[
\frac{1}{n} \{ n + 1 \leq j \leq 2n : \omega_j = \omega_{2j} = 1 \} = \frac{1}{n} \{ n + 1 \leq j \leq 2n : \omega_j = 1 \} \to \frac{1}{2}
\]
being satisfied for every $\omega \in \mathcal{N} \cap B$.

For the upper bound, let us first observe that
\[
\frac{1}{n} \sum_{k=1}^{n} \omega_k \omega_{2k} \leq \frac{1}{n} \sum_{k=1}^{n} \omega_k
\]
and the right hand side converges to $1/2$ for every normal sequence $\omega$. Thus, the set $\mathcal{N} \cap A_\alpha$ is empty for all $\alpha > 1/2$.

We will now need the following lemma

**Lemma 2.3.** Let $\omega$ be a normal sequence and let $(n_k = k \ell_2)$ be an arithmetic subsequence of $\mathbb{N}$. Then $\omega$ restricted to the positions $(n_k)$ is normal.

**Proof.** This is a well-known result of Kamae [K73].
Let us fix some \( m > 0 \). For \( N > m \) and \( i = 0, 1, \ldots, m \) denote by \( R(N, i) \) the set \( \{ 2^i(2k - 1), k \leq 2^{N-i-1} \} \) (for example, \( R(N, 0) \) is the set of odd numbers smaller than \( 2^N \)). Further, let \( R(N, i, I) = R(N - 2, i) \), \( R(N, i, II) = R(N - 1, i) \setminus R(N - 2, i) \), and \( R(N, i, III) = R(N, i) \setminus R(N - 1, i) \). Note here obvious relations

\[
2R(N, i, I) = R(N, i + 1, I) \cup R(N, i + 1, II),
\]

\[
2R(N, i, II) = R(N, i + 1, III),
\]

\[
2R(N, i, III) \cap R(N, i + 1) = \emptyset.
\]

We denote by \( \mathcal{N}(N, m, \varepsilon) \) the set of sequences \( \omega \) such that for all \( n \geq N \) in each \( R(n, i, *) \), \( i = 0, \ldots, m \), \( * \in \{ I, II, III \} \) the frequency of 1’s is between \( 1/2 - \varepsilon \) and \( 1/2 + \varepsilon \). By Lemma 2.3,

\[
\mathcal{N} \subset \bigcap_{\varepsilon > 0} \bigcap_{m = 1}^{\infty} \bigcup_{N = m+1} \mathcal{N}(N, m, \varepsilon).
\]

Similarly, let us denote by \( A(\alpha, N, \varepsilon) \) the set of sequences \( \omega \) such that for all \( n \geq N \) we have

\[
\alpha - \varepsilon < 2^{-n+1} \sum_{j=1}^{2^{n-1}} \omega_j \omega_{2j} < \alpha + \varepsilon.
\]

We have

\[
A_{\alpha} = \bigcap_{\varepsilon > 0} \bigcup_{N = 1}^{\infty} A(\alpha, N, \varepsilon).
\]

To obtain the upper bound, we will estimate from above the number of cylinders \( [\omega_1, \ldots, \omega_{2^N}] \) needed to cover the set \( \mathcal{N}(N, m, \varepsilon) \cap A(\alpha, N, \varepsilon) \). Let us fix \( N, m, \varepsilon \). For \( i = 1, \ldots, m \), \( k_1, k_2 \in \{ 0, 1 \} \), and \( * \in \{ I, II \} \) we denote

\[
X_{k_1, k_2, *}^{i}(\omega) = \#\{ n \in R(N, i - 1, *); \omega_n = k_1, \omega_{2n} = k_2 \}.
\]

For example, \( X_{01, I}^i(\omega) \) denotes the number of odd positions smaller than \( 2^{N-2} \) such that \( \omega_n = 0, \omega_{2n} = 1 \). Similarly, let

\[
X_{k_1, *}^{i}(\omega) = \#\{ n \in R(N, i, *); \omega_n = k_1 \}.
\]

We have obvious relations: for any \( i \)

\[
X_{10, I}^i + X_{11, I}^i = X_{1, I}^{i-1}
\]

\[
X_{00, I}^i + X_{01, I}^i = X_{0, I}^{i-1}
\]

\[
X_{10, II}^i + X_{11, II}^i = X_{1, II}^{i-1}
\]
\[ X_{00,III} + X_{01,III} = X_{0,III} \]
\[ X_{01,I} + X_{11,I} = X_{1,I} + X_{1,II} \]
\[ X_{00,I} + X_{10,I} = X_{0,I} + X_{0,II} \]
\[ X_{01,II} + X_{11,II} = X_{1,II} \]
\[ X_{00,II} + X_{10,II} = X_{1,III} \]

Note that for a sequence \( \omega \in N(N, m, \varepsilon) \) the right hand sides in all those relations is in range \( 2^{N-3-i} \cdot (1 - \varepsilon, 1 + \varepsilon) \). In particular,

\[ |X_{11,I} - X_{00,I}| \leq \varepsilon \cdot 2^{N-2-i}. \]

We can now start the counting. The values of \( \{\omega_i; n \in R(N, 0)\} \) can be chosen in no more than \( 2^{2N-1} \) ways. After we have chosen \( \{\omega_i; n \in R(N, i - 1)\} \), we can choose \( \{\omega_i; n \in R(N, i)\} \) in no more than

\[ \left( \frac{X_{i-1}}{X_i} \right)_{I} \cdot \left( \frac{X_{i-1}}{X_0} \right)_{II} \cdot \left( \frac{X_{i-1}}{X_1} \right)_{III} \cdot \left( \frac{X_{i-1}}{X_0} \right)_{III} \]

ways. Finally, after we have chosen \( \{\omega_i; n \in R(N, i)\} \) for all \( i \leq m \), we will still have \( 2^{N-m-1} \) positions left, which we can cover in no more than \( 2^{2N-m-1} \) ways. Thus, for any choice of \( (X_{00,II}, X_{11,I}, X_{00,II}, X_{11,I}) \), the logarithm of total number of cylinders needed \( Z \) is not larger than

\[
\log Z((X_{00,II}, X_{11,I}, X_{00,II}, X_{11,I})) \leq (2^{N-1} + 2^{N-m-1}) \log 2 \\
+ \sum_{i=1}^{m} \left( 2 \log \left( \frac{2^{N-3-i}}{X_{11,I}} \right) + 2 \log \left( \frac{2^{N-3-i}}{X_{11,I}} \right) + 2^{N-3-i} O(\varepsilon) \right)
\]

and there are no more than \( \prod_{i=1}^{m} 2^{4(N-i-3)} < 2^{4mN} \ll 2^{2N} \) such choices.

We estimate

\[ \log \left( \frac{n}{k} \right) \approx n \left( -\frac{k}{n} \log \frac{k}{n} - \frac{n-k}{n} \log \frac{n-k}{n} \right) = nH\left( \frac{k}{n} \right) \]

and observe that \( H \) is a concave function, thus we can apply Jensen inequality. We get

\[
\log Z((X_{00,II}, X_{11,I}, X_{00,II}, X_{11,I})) \leq (2^{N-1} + 2^{N-m-1}) \log 2 \\
+ \sum_{i=1}^{m} 2^{N-i-1} \cdot H\left( \frac{m}{2^{N-i-1}} \right) \left( \sum_{i=1}^{m} \frac{1}{2^{N-i-1}} \cdot \sum_{i=1}^{m} 2^{N-i-2} \frac{X_{11,I} + X_{11,I}}{2^{N-i-3}} \right) \\
+ \sum_{i=1}^{m} 2^{N-i-3} \cdot O(\varepsilon).
\]
Hence,
\[
\log Z((X_{00,I}^i, X_{11,I}^i, X_{00,II}^i, X_{11,II}^i)) \leq 2^{N-1} \log 2 + 2^{N-1} H\left(\sum_{i=1}^{m} (X_{11,I}^i + X_{11,II}^i)\right) + 2^N \cdot (O(\varepsilon + 2^{-m})).
\]

On the other hand, for all \(\omega \in A(\alpha, N, \varepsilon)\),
\[
\left|2^{-N+1} \sum_{i=1}^{m} (X_{11,I}^i + X_{11,II}^i) - 2\alpha\right| < \varepsilon.
\]

Passing with \(m, N\) to infinity and with \(\varepsilon\) to 0, we finish the proof of the upper bound.

\[\square\]

3 Proof of Theorem 1.4

Given \(p, q \in [0, 1]\), let \(\mu_{p,q}\) be a probability measure on \(S\) given by
- if \(k\) is odd then \(\omega_k = 1\) with probability \(p\),
- if \(k\) is even and \(\omega_{k/2} = 0\) then \(\omega_k = 1\) with probability \(p\),
- if \(k\) is even and \(\omega_{k/2} = 1\) then \(\omega_k = 1\) with probability \(q\),

with events \((\omega_k = 1)\) and \((\omega_k = 1)\) independent except when \(k/\ell\) is a power of 2. Precisely, let \((p_0, p_1) := (1-p, p)\) and let
\[
\begin{pmatrix}
  p_{00} & p_{01} \\
  p_{10} & p_{11}
\end{pmatrix} := \begin{pmatrix}
  1-p & p \\
  1-q & q
\end{pmatrix}.
\]

Then the measure \(\mu_{p,q}\) of a cylinder is given by
\[
\mu_{p,q}([\omega_1 \cdots \omega_n]) = \prod_{k=1}^{[n/2]} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{[n/2]} p_{\omega_k \omega_{2k}}.
\]

where \([\cdot], [\cdot]\) denote the ceiling function and the integer part function correspondingly.

For positive integers \(m < n\), write \(\omega_m^n\) for the word \(\omega_m \omega_{m+1} \cdots \omega_n\). For \(i, j \in \{0, 1\}\) and \(\omega \in \Sigma\), denote
\[
N_i(\omega_m^n) = \#\{m \leq k \leq n : \omega_k = i\},
\]
and
\[
N_{ij}(\omega_m^n) = \#\{m \leq k \leq n : \omega_k \omega_{2k} = ij\}.
\]

We also denote
\[
N_{i,\text{odd}}(\omega_m^n) = \#\{m \leq k \leq n : k \text{ odd}, \omega_k = i\}.
\]
Then we have
\[ \mu_{p,q}(C_n(\omega)) = (1-p)^{N_{0,\text{odd}}(\omega)}p^{N_{1,\text{odd}}(\omega)}(1-p)^{N_{00}}p^{N_{01}}(1-q)^{N_{10}}q^{N_{11}}, \]
with \( N_{i,\text{odd}} = N_{i,\text{odd}}(\omega_n) \), and \( N_{ij} = N_{ij}(\omega_n) \). Thus
\[ -\log \mu_{p,q}(C_n(\omega)) = -\frac{1}{2} \left( N_{0,\text{odd}}(\omega_n) \log(1-p) + N_{1,\text{odd}}(\omega_n) \log p \right) \]
\[ + \frac{N_{00}}{n/2} \log(1-p) + \frac{N_{01}}{n/2} \log p \]
\[ + \frac{N_{10}}{n/2} \log(1-q) + \frac{N_{11}}{n/2} \log q. \] (3.1)

**Lemma 3.1.** If \( p = (2\theta - \alpha)/(2 - \theta) \) and \( q = \alpha/\theta \), then
\[ \mu_{p,q}(E_{\theta} \cap A_{\alpha}) = 1. \]

**Proof.** Denote
\[ x_n(\omega) = N_1(\omega_{n/2+1}) = \frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k. \]
By the Law of Large Numbers, for \( \mu_{p,q} \)-almost all \( \omega \)
\[ x_{2n}(\omega) = \frac{p}{2} + \frac{x_n(\omega)}{2}q + \frac{1-x_n(\omega)}{2}p + o(1) = \frac{x_n(\omega)}{2} \cdot \frac{q - p}{2} + o(1). \]
Note that \( \frac{q - p}{2} \) < 1. Then, as \( k \to \infty \),
\[ x_{2n}(\omega) \to \frac{2p}{2 + p - q}. \]
By [PS12, Lemma 5], it implies that \( \mu_{p,q} \)-almost surely
\[ \lim_{n \to \infty} x_n(\omega) = \frac{2p}{2 + p - q} = \theta, \] (3.2)
where the last equality comes from the choices of \( p \) and \( q \). Thus \( \mu_{p,q}(E_{\theta}) = 1. \)
On the other hand, by applying the Law of Large Numbers again, for \( \mu_{p,q} \)-a.e. \( \omega \),
\[ \frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k \omega_{k} = x_n(\omega)(q + o(1)) \to q\theta = \alpha. \]
By [PS12, Lemma 5], we conclude \( \mu_{p,q}(A_{\alpha}) = 1. \)

**Lemma 3.2.** For \( p = (2\theta - \alpha)/(2 - \theta) \) and \( q = \alpha/\theta \), we have
\[ h_{\mu_{p,q}} = (1 - \frac{\theta}{2})H(\frac{2\theta - \alpha}{2 - \theta}) + \frac{\theta}{2}H(\theta - \frac{\alpha}{\theta}). \]
Proof. By (3.1), we have for \( \mu_{p,q} \) almost all \( w \in \Sigma \),

\[
h_{\mu_{p,q}} = \lim_{n \to \infty} -\frac{\log \mu_{p,q}(C_n(w))}{n} = -\frac{1}{2} \left( (1 - p) \log(1 - p) + p \log p + (1 - \theta)(1 - p) \log(1 - p) + (1 - \theta)p \log p + \theta(1 - q) \log(1 - q) + \theta q \log q \right)
\]

\[= \frac{1}{2} \left( (2 - \theta)H(p) + \theta H(q) \right) = (1 - \frac{\theta}{2})H \left( \frac{2\theta - \alpha}{2 - \theta} \right) + \frac{\theta}{2} H \left( \frac{\theta - \alpha}{\theta} \right).
\]

\( \Box \)

**Lemma 3.3.** If \( \theta \notin [\alpha, (2 + \alpha)/3] \) we have \( E_\theta \cap A_\alpha = \emptyset \), otherwise for \( p = (2\theta - \alpha)/(2 - \theta) \) and \( q = \alpha/\theta \), we have for all \( x \in E_\theta \cap A_\alpha \),

\[
\lim_{n \to \infty} -\frac{\log \mu_{p,q}(C_n(w))}{n} = (1 - \frac{\theta}{2})H \left( \frac{2\theta - \alpha}{2 - \theta} \right) + \frac{\theta}{2} H \left( \frac{\theta - \alpha}{\theta} \right).
\]

Proof. Observe that for any \( x \in E_\theta \cap A_\alpha \), for any small \( \varepsilon > 0 \), for \( n \) large enough, we have

\[
N_1(\omega_{n/2}^n) \in \left[ \frac{\theta n}{2}(1 - \varepsilon), \frac{\theta n}{2}(1 + \varepsilon) \right]
\]

\[
N_1(\omega_{n}^n) \in \left[ \theta n(1 - \varepsilon), \theta n(1 + \varepsilon) \right]
\]

\[
N_{11}(\omega_{n/2}^n) \in \left[ \frac{\alpha n}{2}(1 - \varepsilon), \frac{\alpha n}{2}(1 + \varepsilon) \right].
\]

The obvious inequalities \( N_{11}(\omega_{n/2}^{2n}) \leq N_1(\omega_{n/2}^n) \) and \( N_1(\omega_{n}^{2n}) - N_{11}(\omega_{n/2}^{2n}) \leq n/2 + N_0(\omega_{n/2}^n) = n - N_1(\omega_{n/2}^n) \) imply \( \theta \in [\alpha, (2 + \alpha)/3] \). Furthermore, we have

\[
\log \mu_{p,q}(C_{2n}(\omega)) - \log \mu_{p,q}(C_n(\omega)) = N_{11}(\omega_{n}^{2n}) \log q + (N_1(\omega_{n}^n) - N_{11}(\omega_{n/2}^{2n})) \log(1 - q)
\]

\[+ (N_1(\omega_{n/2}^n) - N_{11}(\omega_{n/2}^{2n})) \log p \]

\[+ (n - N_1(\omega_{n/2}^n) - N_1(\omega_{n}^{2n}) + N_{11}(\omega_{n/2}^{2n})) \log(1 - p)
\]

\[= n \left( (1 - \frac{\theta}{2})H \left( \frac{2\theta - \alpha}{2 - \theta} \right) + \frac{\theta}{2} H \left( \frac{\theta - \alpha}{\theta} \right) + \varepsilon \cdot O(1) \right).
\]

Hence by the same argument of the proof of Lemma 2.2, we have for all \( x \in E_\theta \cap A_\alpha \),

\[
\lim_{n \to \infty} -\frac{\log \mu_{p,q}(C_n(w))}{n} = (1 - \frac{\theta}{2})H \left( \frac{2\theta - \alpha}{2 - \theta} \right) + \frac{\theta}{2} H \left( \frac{\theta - \alpha}{\theta} \right).
\]

\( \Box \)
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