Quantum frustration of dissipation by a spin bath

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Abstract. We investigate the evolution of a central spin coupled to a spin bath without internal dynamics. We compare the case where a single bath couples to one component of the spin with that of two competing baths coupling symmetrically to orthogonal spin components. It is found that the central spin dynamics is enhanced in the latter case, which may be interpreted as a frustration of dissipation. The quantum purity of the spin decays in both cases, exponentially for a single bath and algebraically for a double bath. We conclude that the symmetric double coupling frustrates dissipation and slows down decoherence.
1. Introduction

The study of the role of a dissipative environment is of central importance to the field of quantum computation and for the fundamental understanding of the transition from quantum to classical behavior. In that context, the dissipative two-level system (TLS) is a well-studied paradigm [1, 2]. The generic TLS, which in particular can be a spin-$\frac{1}{2}$ particle, experiences dissipation due to its coupling to a bath of harmonic oscillators. In the resulting spin–bath problem, an external magnetic field interacts with one component of the spin operator (e.g. $S_z$), whereas a second component (e.g. $S_x$) couples to an oscillator bath. Depending on the relative strength of the two interactions, one can switch over between underdamped and overdamped behavior, as may become manifest through quantities such as the average spin energy or various correlation functions. Recently, Castro Neto et al [3] and Novais et al [4] have shown that if two components ($S_x$ and $S_y$) of the effective spin are coupled to two different baths, then the competing effects of those baths can reduce the effect of dissipation. In particular, they have found that, for a given coupling strength, symmetric coupling to the two spin components is less decoherent than coupling to a single component. Moreover, in the case of symmetric coupling with $S_x$ and $S_y$, coherent behavior is preserved for arbitrarily strong coupling. These two properties are remarkable because one would naively expect that coupling to a higher number of bath oscillators would increase the effect of dissipation. Rather, they have shown that the competition between the baths contributes to protecting the TLS energy gap. The reduction of dissipation stems from the non-commutative character of the spin operators coupled to the different baths. Logically, if the two baths interact with the same spin component, then no dissipation reduction is observed. In [3, 4], it is argued that this feature arises from the lack of a preferred basis to which the TLS may relax at long times. This new phenomenon has been coined quantum frustration of decoherence, since it is interpreted as the frustrated attempt of the two environments to ‘measure’ simultaneously two non-commuting observables. The study of [3, 4] has been restricted to equilibrium properties such as the transverse susceptibility, which was evaluated using the numerical renormalization group.

Features of quantum frustration were also reported in [5], where it was noted that the phase-number variable of a superconducting Josephson junction is coupled simultaneously to
two different dissipative environments: through the phase to the quasiparticle field and through the Cooper pair number to the quantum electromagnetic field. The result is that the uncertainty in the macroscopic phase has contributions from both baths which tend to cancel each other. They never cancel completely because, in that particular case, the sources of dissipation differ widely both in nature and strength. Frustration of decoherence in Josephson networks has also been investigated [6].

In [7, 8] the dynamics of an oscillator coupled through its position and momentum to two different oscillator baths was studied. The problem can be shown to be equivalent to that of a large (quasiclassical) spin impurity in a ferromagnetic environment. It was noted that the dissipative oscillator may be driven from overdamped to underdamped behavior by the symmetrical addition of a second bath. This surprising effect was found only for the case where the momentum operator \( (p) \) and the position operator \( (q) \) are coupled to two different baths. Like in [3, 4], these effects were investigated through equilibrium properties such as the position–position response function. However, some dynamical aspects were also analyzed, noting that the purity of the quantum oscillator decays faster in the presence of two baths than in the presence of a single bath. This last result is important because it reveals that the competition between two baths coupling to non-commuting observables is not a universal panacea to suppress decoherence. Our present study is motivated by the need for a more detailed understanding of the equilibrium and dynamical properties of a quantum system in the presence of competing environments.

In the studies of quantum frustration made earlier [3]–[8], the bath has been modeled by a set of non-interacting oscillators. However, if the interpretation is correct that the essence of quantum frustration stems from the canonically conjugate character of the two observables which couple to separate baths, then one should expect a similar behavior to appear when the dissipative environment is formed by a bath of spins acting on a central spin impurity. We note that, because of the vector nature of the spin, the spin bath can by itself be viewed as formed by several baths. Thus a single spin bath may exhibit features of quantum frustration. Spin baths have been studied [9]–[16] as an alternative to the conventional oscillator models of quantum dissipation [1, 2]. They are known to give rise to non-Markovian evolution, with the system evolution showing a strong dependence on the polarization of the initial state [17].

In this work, we shall evaluate the dynamical properties of a TLS coupled to a bath of spins. Similar to the work of Castro Neto \textit{et al} [3] and Novais \textit{et al} [4], we shall consider the situation where the components of the spin are coupled to two different baths and study the effects of frustration arising due to the non-commuting nature of the spin components. By assuming that the bath has no internal dynamics of its own, we are able to perform an analytical study. Comparing the case of coupling to a single component to that of symmetric coupling to both components, we find that the spectral function behaves similarly to the study of [3, 4]. Namely, the spectral function develops a peak in the symmetric case which is absent in the case of single-component coupling. In [3, 4], this was interpreted as the preservation of decoherence arising from the frustrated attempt of the two environments to measure two non-commuting observables. However, we find that the emergence of the peak as the second bath intervenes is compatible with a slower decay of the quantum purity of the central spin. Specifically, purity decays exponentially fast for a single bath and only algebraically for a double bath. This suggests that the frustration induced by the two competing environments has more to do with dissipation than with decoherence. Energy relaxation is indeed inhibited by the presence of a second bath coupled to the other spin component, whereas the quantum purity still decays...
in the presence of a second bath, with only a minor form of frustration revealed by a short-lived revival, followed by a slower, algebraic decay.

Section 2 is devoted to the presentation of the model of a central spin coupled to a spin bath. In section 3, we study the time evolution analytically, deriving expressions for the expectation values and the quantum purity of the central spin. Sections 4 and 5 focus on the density of states and the response function of the central spin. Section 6 deals with the tailoring of the spin–bath properties which mimics the behavior of a spin coupled to a conventional bath of harmonic oscillators. Finally, the main conclusions of this work are summarized and discussed in section 7.

2. Central spin model

We consider the dynamics of a TLS which is linearly coupled through two non-commuting observables to two independent environments of TLSs. We will refer indistinctively to both the central impurity and the constituents of the bath as particles of spin-$\frac{1}{2}$ or TLSs. If the spins of the environment carry their own dynamics, in general there are no conserved quantities other than energy and the system cannot be treated analytically without approximations. However, in most of the solid-state spin systems where the spin–bath interaction is a dominant source of mechanism for the dissipation of a TLS, the internal bath dynamics is generally very slow (for example, in quantum dot systems, where the bath spins are nuclear spin-half particles and the TLS is the electronic spin, [18]). We therefore assume that both environments carry no dynamics of their own, i.e. that their Hamiltonians are zero. Thus the bath dynamics is exclusively due to its interaction with the central spin [15]. The total Hamiltonian of system and bath is then given by

$$ H = H_S + H_{SB}, $$

where

$$ H_S = \omega_0 S^z, $$

$$ H_{SB} = g_1 S^x \sum_{k=1}^{N} I^x_k + g_2 S^y \sum_{l=1}^{N} J^y_l, $$

where $S^i, i = x, y, z$ are the components of the spin operator of the central spin and $I^i_k$ and $J^i_l$ are spin operators of the bath spins. We assume homogeneous interaction between the central spin and the baths. Moreover, we assume the number of spins of each environment to be the same. The strength of the coupling to each environment is thus described by one parameter $g_i$ ($i = 1, 2$) only. For a single bath environment the case of non-homogeneous coupling ($g_1, g_2$ dependent on index $k, l$, respectively) was solved explicitly in [10]. However, it was shown in [10, 15] that the only effect of inhomogeneous interaction is that of destroying certain revival effects. All other features can be captured within the homogeneous interaction between system and bath. The main advantage of the homogeneous interaction approximation is that exact, closed-form expressions for the expectation values and correlation functions can be obtained.

The Hamiltonian (1) is similar to that employed in [3, 4] in that two different environments couple to the two perpendicular components of the central spin. The main difference from
the model of [3, 4] is the non-dynamic character of the bath which we consider, which contrasts with the oscillator bath there considered. We shall examine whether our simpler model (1) can yield frustration effects similar to those obtained from the more complex model of [3, 4].

In the particular case where the spin bath couples to only one component of the central spin, a number of nontrivial effects are known to appear, despite its apparent simplicity. One instance is the crossover from overdamped to underdamped behavior as the coupling strength increases, similar to the spin-boson model where the TLS is coupled to an oscillator bath [1, 2]. For a more complete account of the dynamics of a spin coupled to a single bath we refer to [9, 15].

3. Time evolution

In the absence of bath dynamics the x- and y-components of the total spin of the respective environments, \( I^x_{\text{tot}} = \sum_k I^x_k \) and \( J^y_{\text{tot}} = \sum_k J^y_k \), are conserved quantities. We write the total Hilbert space as a tensor product \( \mathcal{H}_S \otimes \mathcal{H}_B \). The total 2\( ^N \) dimensional Hilbert space of the baths \( \mathcal{H}_B \) decomposes into invariant subspaces \( \mathcal{H}_{B(m_1,m_2)} \). These are labeled by the eigenvalues \( m_1, m_2 \) of the total spin operators \( I^x_{\text{tot}} \) and \( J^y_{\text{tot}} \). Each \( m_i \) (\( i = 1, 2 \)) runs from \(-N/2\) to \( N/2\). The Hamiltonian (1) acts on the subspace \( \mathcal{H}_S \otimes \mathcal{H}_{B(m_1,m_2)} \) as

\[
H(\sigma) |\alpha_1, \alpha_2; m_1, m_2\rangle = (\omega_0 S^z + S^x m_1 g_1 + S^y m_2 g_2) |\sigma\rangle |\alpha_1, \alpha_2; m_1, m_2\rangle,
\]

for \( |\alpha_1, \alpha_2; m_1, m_2\rangle \in \mathcal{H}_{B(m_1,m_2)} \) and \( |\sigma\rangle \in \mathcal{H}_S \). Here \( \alpha_i \), which is not important in the following, labels the irreducible representation. We therefore can write \( H \) most conveniently as a direct sum

\[
H = \bigoplus_{m_1,m_2=-N/2}^{N/2} \mathcal{H}_{m_1,m_2}
\]

where

\[
H_{m_1,m_2} = (\omega_0 S^z + g_1 m_1 S^x + g_2 m_2 S^y) \otimes I_{\lambda m_1} \otimes I_{\lambda m_2}
\]

\[
= S^z \cdot \vec{\Omega}_{m_1,m_2} \otimes I_{\lambda m_1} \otimes I_{\lambda m_2}.
\]

Here \( I_\lambda \) is the \( \lambda \times \lambda \) unit matrix and the parameters

\[
\lambda_m = \left( \frac{N}{N/2 - m} \right)
\]

measure the dimension of the invariant subspace \( \mathcal{H}_{B(m_1,m_2)} \). From equation (5) it is clear that the effect of the environment is to give rise to an effective magnetic field \( \vec{\Omega}_{m_1,m_2} = (m_1 g_1, m_2 g_2, \omega_0) \). However, this effective magnetic field is different from the static magnetic field pointing in the \( z \)-direction, since it does not take a single value but rather is a distribution characterized by the degeneracy coefficients \( \lambda_m \).

Since \( H_{m_1,m_2} \) acts trivially on the subspace of the environment, we will often write \( H_{m_1,m_2} = \vec{S}_z \cdot \vec{\Omega}_{m_1,m_2} \) for short. The eigenvalues of \( H_{m_1,m_2} \) are \( \pm \Omega_{m_1,m_2}/2 \), where we introduced the frequency

\[
\Omega_{m_1,m_2} = |\vec{\Omega}_{m_1,m_1}| = (\omega_0^2 + g_1^2 m_1^2 + g_2^2 m_2^2)^{1/2}
\]

We denote the eigenstates of \( H_{m_1,m_2} \) by \( |\pm, m_1, m_2\rangle \). They are related to the eigenstates \( |\uparrow\rangle, |\downarrow\rangle \) of the non-interacting system Hamiltonian \( H_S \) by the unitary transformation

\[
\begin{pmatrix}
  |+, m_1, m_2\rangle \\
  |-, m_1, m_2\rangle
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta_{m_1,m_2} & \sin \theta_{m_1,m_2} e^{i \phi_{m_1,m_2}} \\
  -\sin \theta_{m_1,m_2} e^{-i \phi_{m_1,m_2}} & \cos \theta_{m_1,m_2}
\end{pmatrix}
\begin{pmatrix}
  |\uparrow\rangle \\
  |\downarrow\rangle
\end{pmatrix},
\]

\[\text{New Journal of Physics 10} (2008) 115017 (http://www.njp.org/)\]
where the angles $\phi_{m_1 m_2}$ and $\theta_{m_1 m_2}$ are given by

\[
\phi_{m_1 m_2} = \arctan\left( \frac{m_2 g_2}{m_1 g_1} \right),
\]

\[
\cos^2 \theta_{m_1 m_2} = \frac{\Omega_{m_1 m_2} + \omega_0}{2\Omega_{m_1 m_2}}.
\]

The case of a single environment is recovered by setting $g_2$ or, equivalently, $\phi_{m_1 m_2}$ equal to zero.

The time evolution operator is straightforwardly derived from equation (5). We obtain

\[
U(t) = \oplus_{m_1, m_2 = -N/2}^{N/2} U_{m_1, m_2}(t)
\]

with

\[
U_{m_1, m_2}(t) = \cos\left( \frac{t}{2} \Omega_{m_1 m_2} \right) \mathbb{I}_2 + 2i \frac{\sin\left( \frac{t}{2} \Omega_{m_1 m_2} \right)}{\Omega_{m_1 m_2}} \vec{S} \cdot \vec{\Omega}_{m_1 m_2}.
\]

We can decompose an arbitrary system (central spin) operator $O$ as $O = \oplus_{m_1, m_2 = -N/2}^{N/2} O_{m_1 m_2}$. In particular, we are interested in the Heisenberg spin operator $\vec{S}(t)$ and its commutators and anticommutators at different times. Using (10), we find

\[
\vec{S}_{m_1 m_2}(t) = \cos(\Omega_{m_1 m_2} t) \vec{S}(0) - \sin(\Omega_{m_1 m_2} t)(\vec{S}(0) \times \vec{n}_{m_1 m_2})
\]

\[
+ \left[1 - \cos(\Omega_{m_1 m_2} t)\right](\vec{S}(0) \cdot \vec{n}_{m_1 m_2}) \vec{n}_{m_1 m_2},
\]

\[
-i[S^i_{m_1 m_2}(t), S^j_{m_1 m_2}(0)] = \cos(\Omega_{m_1 m_2} t) \epsilon_{ijk} S^k(0)
\]

\[
+ \sin(\Omega_{m_1 m_2} t) \left[\delta_{ij} \vec{n}_{m_2 m_1} \vec{S}(0) - S^i(0) n^j_{m_1 m_2}\right)
\]

\[
+ \left[1 - \cos(\Omega_{m_1 m_2} t)\right] n^i_{m_1 m_2} \left(\vec{S}(0) \times \vec{n}_{m_1 m_2}\right),
\]

\[
2[S^i_{m_1 m_2}(t), S^j_{m_1 m_2}(0)] = \cos(\Omega_{m_1 m_2} t) \delta_{ij} - \sin(\Omega_{m_1 m_2} t) \epsilon_{ijk} n^k_{m_1 m_2}
\]

\[
+ \left[1 - \cos(\Omega_{m_1 m_2} t)\right] n^i_{m_1 m_2} n^j_{m_1 m_2}.
\]

The vector $\vec{n}_{m_1 m_2}$ is a unit vector pointing in the direction of the effective magnetic field $\vec{\Omega}_{m_1 m_2}$.

If the density matrix $\rho(t)$ of the total system is initially invariant under rotations within a subspace $\mathcal{H}_B^{(m_1, m_2)}$ due to the trivial action of the Hamiltonian in this subspace, this invariance will persist at all times. In particular, the density matrix can be written for all times as $\rho(t) = \oplus_{m_1, m_2 = -N/2}^{N/2} \rho_{m_1 m_2}(t)$. This means that $\rho(t)$ shares for all times the block structure of the Hamiltonian. If $\rho(0)$ fulfills this condition, the expectation value of an arbitrary system operator $O(t)$ with respect to $\rho(0)$ can then be written as

\[
\langle O(t) \rangle = \text{Tr} \left[ \rho(0) O(t) \right]
\]

\[
= \sum_{m_1, m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \text{Tr}_S \left[ O_{m_1 m_2}(t) \rho_{m_1 m_2}(0) \right],
\]

where $\text{Tr}_S$ denotes the trace over the central system. In the following, we will analyze the expectation values of the operators (11) to (13) with respect to an initially unpolarized bath. Since the magnetic field applied along the $z$-direction only affects the central spin, we take this
to be initially in the ground state determined by $H_S$ and consequently choose the initial density matrix as

$$\rho(0) = \frac{1}{2^N} | \downarrow \rangle \langle \downarrow | \otimes \mathbb{I}_{2^N} \otimes \mathbb{I}_{2^N}. \quad (15)$$

We immediately see that, in this state, $\langle S_i(0) \rangle = -\delta_{i3}/2$.

Using the above formalism for evaluating the dynamical properties of the TLS, we shall now calculate various quantities for the TLS operators and study the effects brought about by the coupling to two different baths.

### 3.1. Expectation values

The expectation values $\langle S_i(t) \rangle$ can now be calculated using equations (11) and (15). We find

$$\langle S_z(t) \rangle = \frac{-1}{2} + \sum_{m_1, m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \left( \frac{g_1^2 m_1^2 + g_2^2 m_2^2}{\Omega_{m_1 m_2}^2} \cos(\Omega_{m_1 m_2} t) + \frac{\omega_0^2}{\Omega_{m_1 m_2}^2} \right), \quad (16)$$

$$\langle S_x(t) \rangle = \langle S_y(t) \rangle = 0. \quad (17)$$

In the absence of an external field $\omega_0 = 0$, we would expect that the initial polarization of the TLS would decay faster in comparison to the single bath case, since the total number of spins with which the TLS is interacting is doubled. In figure 1, we have plotted the time variation of the $\langle S_z(t) \rangle$ for various values of $g_1, g_2$ keeping $g = \sqrt{g_1^2 + g_2^2}$ constant and for $\rho(0)$ given by equation (15). As one can see, for a single bath the polarization decays to zero very fast, whereas in the presence of the second bath, the decay is comparatively slow. In contrast to the case of a single bath, one observes a change of sign in the time-dependent behavior of the polarization of the TLS, indicating the presence of a nonzero field.

In figure 2 the time variation of the $\langle S_z(t) \rangle$ is plotted for non-vanishing external field $\omega_0 \neq 0$. We find that the polarization saturates to a finite value at long times with faster oscillations in the case of a symmetric double bath, which is consistent with the spectral properties discussed later in the text. Inspection of figures 1 and 2 reveals that the change of sign occurs in the single bath case only if there is a nonzero field, whereas it is observed for both zero and nonzero field in the case of a symmetric double bath. Since the baths are completely unpolarized ($\lambda_m$ peaks at $m = 0$), it is clear that the effective field responsible for this change of sign can only stem from the competing effect of two baths coupled to non-commuting components of the central spin.

### 3.2. Quantum purity

For an arbitrary density matrix $\rho$, purity is defined as $P = \text{Tr} \rho^2$. Purity is a convenient, basis-independent measure of the degree of coherence, if $\rho$ is the reduced density matrix of the central spin (that which results from tracing out the bath degrees of freedom in the total density matrix). In our case the decay of purity is directly related to the relaxation of the spin expectation values to equilibrium,

$$P(t) = \frac{1}{2} + 2 \sum_{i=1}^{3} \langle S_i(t) \rangle^2. \quad (18)$$
Figure 1. The magnetization of the TLS is plotted with time for various values of the system–bath interaction, in zero-field. In the presence of two different baths a change of sign arises during the time evolution. The sign reversal reaches its maximum value when the two baths are identical. The total number of spins in each bath is taken to be $N = 100$. The system–bath couplings given in the inset are dimensionless ($g_i/\sqrt{g_1^2 + g_2^2}$).

Figure 2. Same as figure 1 but for a nonzero field such that $\omega_0/\sqrt{g_1^2 + g_2^2} = 2$. In the presence of the external field the magnetization for the single bath case decays non-monotonically and saturates to a nonzero value.
Figure 3. The purity of TLS is plotted against time for various values of the system–bath interaction. Though the initial decay rate strongly depends on the combined interaction strength \( \sqrt{g_1^2 + g_2^2} \) (which is same for all the three cases), the TLS loses its initial polarization faster in the presence of a single bath in comparison with the case of two baths. The external field is set to zero and the number of spins in each bath are taken to be equal with \( N = 100 \). The system–bath couplings given in the inset are dimensionless \( (g_i/\sqrt{g_1^2 + g_2^2}) \).

The result is plotted in figure 3. We notice that, both in the single and double symmetric bath cases, the purity decays quickly to its minimum value 1/2. In the symmetric case, we notice a small, short-lived revival that may be interpreted as a weak form of decoherence frustration which however does not affect the long time behavior of the central spin.

4. Density of states

To get further analytical insight we introduce the function

\[
D(\omega) = \frac{1}{2^{2N+1}} \sum_{m_1, m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \left[ \delta(\omega - \Omega_{m_1 m_2}) + \delta(\omega + \Omega_{m_1 m_2}) \right],
\]

which is essentially the density of states, normalized to fulfill the sum rule \( \int d\omega D(\omega) = 1 \). The expectation value \( \langle S^z(t) \rangle \) is related to \( D(\omega) \) by

\[
\langle S^z(t) \rangle = -\frac{1}{2} \int d\omega D(\omega) \left( \cos(\omega t) \frac{\omega^2 - \omega_0^2}{\omega^2} + \frac{\omega_0^2}{\omega^2} \right).
\]

\( D(\omega) \) can be further evaluated by using the approximation

\[
\frac{1}{2^N} \sum_m \lambda_m \approx \sqrt{\frac{2}{\pi N}} \int_{-\infty}^{\infty} dm \ e^{-2m^2/N},
\]

New Journal of Physics 10 (2008) 115017 (http://www.njp.org/)
Figure 4. Density of states as a function of frequency in normalized units.

for the binomials $\lambda_m$, which is known as Laplace–de Moivre formula in probability theory [19] and which is valid only for large $N$. We find

$$D(\omega) = \frac{4|\omega|\theta(\omega - \omega_0)}{Ng^2 \sin(2\theta_g)}I_0\left(\frac{4\cot(2\theta_g)(\omega^2 - \omega_0^2)}{Ng^2 \sin(2\theta_g)}\right) \exp\left(-\frac{4(\omega^2 - \omega_0^2)}{Ng^2 \sin^2(2\theta_g)}\right),$$

where the total coupling strength $g$ and the angle $\theta_g$ are defined by

$$g_1 = g \cos \theta_g, \quad g_2 = g \sin \theta_g,$$

and $I_0$ is the modified Bessel function. The case $\theta_g = 0$ corresponds to the single bath and the case $\theta_g = \pi/4$ corresponds to two identical baths. In equation (22), the limit of a single bath can be taken by using the asymptotic expansion of the modified Bessel function $\lim_{z \to \infty} I_0(z) = e^z/\sqrt{2\pi z} + \cdots$. The results are plotted in figure 4. As the coupling of the TLS with the baths becomes symmetric, i.e. $\theta_g \to \pi/4$, the density of states peaks at a frequency away from $\omega_0$. As $\theta_g \to 0$, this peak shifts toward $\omega_0$, which is expected for the case of TLS coupling to a single bath. Thus the density of states can by itself reveal the frustrating effects of decoherence more elegantly.

For the particular case of $H_S = 0$ ($\omega_0 = 0$), one can obtain simplified expressions for the density of states. We obtain

$$D(\omega) \simeq \frac{1}{g\sqrt{N}}e^{-\omega^2/g^2N}, \quad g_1 = g, \quad g_2 = 0,$$

$$D(\omega) \simeq \frac{\omega}{g^2N}e^{-\omega^2/g^2N}, \quad g_1 = g_2 = g.$$

In the case of a single bath the peak is at $\omega = 0$, whereas for two baths, the peak is shifted to $\omega = \sqrt{N}/2g$. We note in this respect that, if the two baths were coupled to the same spin component, then, trivially, the behavior would be the same as that of a single-bath with $2N$
spins coupled to one spin component. In such a case the peak in $D(\omega)$ would remain at $\omega = 0$. Thus, the emergence of a peak at $\omega \neq 0$ may be viewed as a frustration of dissipation due to the competition between two environments coupled to non-commuting spin components.

We end by noting that $D(\omega)$ is the Fourier transform of $\langle S^z(t) S^z(0) \rangle$ and is a measure of the strength of the transitions induced by a periodic perturbation $\sim \cos(\omega t)$.

5. Correlation functions and purity evolution

We now investigate the spin–spin correlation functions defined by $C_{ij}(t) = \langle S^i(t) S^j(0) \rangle$. Specifically, we focus on its symmetrized and antisymmetrized versions, $S_{ij}(t) \equiv \frac{1}{2}\langle [S^i(t), S^j(0)] \rangle$ and $A_{ij}(t) \equiv -i\langle [S^i(t), S^j(0)] \rangle$.

Since the system is initially in an eigenstate of $S^z$ such that $\langle S^z(0) \rangle = -\frac{1}{2}$, the symmetrized autocorrelation function in the $z$-direction is simply $S_{zz}(t) = -\frac{1}{2} \langle S^z(t) \rangle$. Using the general formulae of section 3, we find for the transversal symmetrized auto–correlation functions

$$S_{xz}(t) = \frac{-1}{2 N^2} \sum_{m_1,m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \left( \frac{\omega_0^2 + g_2^2 m_1^2}{\Omega_{m_1 m_2}^2} \cos(\Omega_{m_1 m_2} t) + \frac{g_1^2 m_1^2}{\Omega_{m_1 m_2}^2} \right)$$

and a similar result is obtained for $S_{xy}(t)$ by exchanging indices $m_1$ and $m_2$. All symmetrized cross-correlation functions $S_{ij}(t), (i \neq j)$ are zero.

We look at the autocorrelation function in the $z$-direction in the case that there is no external magnetic field applied $\omega_0 = 0$. Since $S_{zz}(t)$ is proportional to $\langle S^z(t) \rangle$, we can use the integral representation (20). In general, i.e. for intermediate values of $\theta_g$ and for non-zero frequency $\omega_0$, the integral (20) becomes quite difficult and cannot be solved analytically. However, in some limits closed expressions can be derived. For $\omega_0 = 0$, we obtain

$$4S_{zz}(t) = \exp(-Ng^2 t^2/8), \quad \theta_g = 0,$$

$$4S_{zz}(t) = 1 - \sqrt{\frac{\pi Ng^2}{8}} t \exp \left( -\frac{Ng^2 t^2}{8} \right) \text{Erfi} \left( \sqrt{\frac{Ng^2 t^2}{8}} \right)$$

$$= -\frac{4}{Ng^2 t^2} + \mathcal{O}(t^{-4}), \quad t \to \infty, \quad \theta_g = \pi/4. \quad (26)$$

We see that, as a result of the coupling to an additional bath, the long time behavior of the autocorrelation function changes from exponential decay for one bath to an algebraic decay for two baths. Since in the present context the decay of the autocorrelation functions is directly related to the decoherence of the system as measured by purity, this implies that due to the competing effects of the two baths the long time coherence decay is not exponential but algebraic. Specifically, noting the relations (17) and (18), we obtain

$$P(t) - \frac{1}{2} = \frac{1}{2} \exp(-Ng^2 t^2/8), \quad all \ t, \quad \theta_g = 0$$

$$\simeq \frac{8}{N^2 g^4 t^4}, \quad t \to \infty, \quad \theta_g = \pi/4. \quad (27)$$

We conclude that the symmetric presence of a second bath slows down decoherence.

The antisymmetrized correlation functions are related to the dynamical susceptibilities, defined as

$$\chi_{ij}(\omega) = \int_0^\infty \frac{dt}{2\pi} e^{i\omega t} A_{ij}(t). \quad (28)$$
In particular, the imaginary part of the susceptibility can be used as a measure of the energy dissipated from the system to the bath. Using equations (12) and (14), we find
\[ A_{zz}(t) = 0, \]
\[ A_{xx}(t) = -\frac{1}{2^{N+1}} \sum_{m_1, m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \frac{\omega_0}{\Omega_{m_1 m_2}} \sin(\Omega_{m_1 m_2} t), \]
and \( A_{yy}(t) = A_{xx}(t) \), where \( D(\omega) \) has been analyzed in the previous section. All antisymmetrized cross-correlation functions but \( A_{xy}(t) \) are zero. For \( A_{xy}(t) \) we find
\[ A_{xy}(t) = -\frac{1}{2^{N+1}} \sum_{m_1, m_2 = -N/2}^{N/2} \lambda_{m_1} \lambda_{m_2} \cos(\Omega_{m_1 m_2} t), \]
In the second lines of equations (29) and (30), we used an integral representation in terms of \( D(\omega) \). In this form the dynamical susceptibilities are readily evaluated
\[ \chi''_{xx}(\omega) = \frac{\omega_0}{2\omega} \int d\omega' \frac{D(\omega')}{\omega^2 - \omega'^2 + \text{sgn}(\omega)0^+}. \]
Splitting \( \chi_{ij}(\omega) \) into its real and its imaginary part, \( \chi_{ij}(\omega) = \chi''_{ij}(\omega) + i\chi'_{ij}(\omega) \), one obtains the relation (22)
\[ \chi''_{xx}(\omega) = \frac{\omega_0}{2\omega} D(\omega), \]
which holds for \( \omega \geq \omega_0 \). Moreover \( \chi''_{xx}(\omega) = 0 \) for \( |\omega| < \omega_0 \), and \( \chi''_{xx}(-\omega) = -\chi''_{xx}(\omega) \).
We can use the approximation (21) and obtain
\[ \chi''_{xx}(\omega) = \frac{2\omega_0}{N g^2} \exp\left(-\frac{4(\omega^2 - \omega_0^2)}{N g^2}\right), \quad \theta_g = \frac{\pi}{4}, \]
\[ \chi''_{xx}(\omega) = \frac{\omega_0}{\sqrt{2\pi N g^2(\omega^2 - \omega_0^2)}} \exp\left(-\frac{2(\omega^2 - \omega_0^2)}{N g^2}\right), \quad \theta_g = 0. \]
From the above equations it can be seen that there is a strong singularity at \( \omega = \omega_0 \), in addition to the Gaussian spread arising due to the interaction with the bath. For the symmetric coupling, this singularity is removed and only a Gaussian spread peaked at \( \omega = \omega_0 \) remains. Both functions are peaked at \( \omega = \omega_0 \), and hence one can say that, since there is no peak shifting there is no frustration. If we try to remove the singularity for the single bath case by multiplying \( \sqrt{\omega^2 - \omega_0^2} \) by \( \chi''_{xx}(\omega) \) then one can immediately see that the peak for \( \chi''_{xx}(\omega) \) is shifted away from \( \omega = \omega_0 \) for the symmetric case. Scalings of such kind can be avoided by considering other kinds of distributions for the bath spins. In the next section, we consider bath spin distributions \( \lambda_m \) with a Gaussian cutoff.
6. Tailoring the density of states

In [3, 4] respectively, [5, 8] similar expressions were obtained for $\chi''_{ss}(\omega)$ in the first case and for the Fourier transform of the antisymmetrized position–position correlation function in the second case. In both cases, the function under consideration is of the form

$$\chi''_{ss}(\omega) = \frac{Z\omega}{(\omega^2 - \tilde{\omega}_0^2)^2 + g_{\text{eff}} \omega^2},$$

(35)

where $\tilde{\omega}_0$ is the renormalized frequency of the system (Larmor frequency, respectively oscillator frequency), $g_{\text{eff}}$ is the effective damping coefficient. For the detailed expressions of $\tilde{\omega}_0$, $g_{\text{eff}}$ and $Z$ see equation (26) of [4] and equation (20) of [8]. The functional form (35), and in particular the linear behavior for small values of $\omega$ is typical for Ohmic type of dissipation.

We would like to model our system to best mimic the behavior that would result from an Ohmic oscillator bath. The spectrum of the unperturbed spin bath cannot be tuned since, due to the lack of internal dynamics, it can only be $\propto \delta(\omega)$. However, density of states or susceptibility with a different structure is obtained for the global interacting system (see, e.g. equations (19) and (32)). The distribution $\lambda_m$ can be chosen to yield a given form of $D(\omega)$ or $\chi''_{ss}(\omega)$, which are related through the simple relation (32). We use a general $\lambda_m$ of the form

$$\lambda_m = \frac{1}{2} \left( \frac{2}{N} \right)^{(\alpha+1)/2} \frac{|m|^\alpha}{\Gamma((\alpha+1)/2)} \exp \left( -\frac{2m^2}{N} \right).$$

(36)

For small values of $m$ the behavior of $\lambda_m$ is dominated by the power $m^\alpha$ with a characteristic exponent $\alpha$. In equation (36), we took a Gaussian cutoff for large values of $\omega$ with a cutoff frequency chosen as $\sqrt{N}/2$ in order to make contact with the former results. It is a well-known fact in the theory of open quantum systems that the specific form of the cutoff function is not relevant [2]. The function $\lambda_m$ is normalized such that $\int dm \lambda(m) = 1$ holds. We see that the form of $\lambda_m$ described in equation (6) is just a special case of equation (36) corresponding to $\alpha = 0$ (see also equation (21)).

In a calculation which is similar to that performed in section 4, we obtain for the density of states

$$D(\omega) = \frac{4|\omega|\sqrt{\pi}}{2\Gamma((\alpha+1)/2)N g^2 \sin(2\theta_g)} \left( \frac{2}{Ng^2 \cos(2\theta_g)} \right)^{\alpha/2} (\omega^2 - \omega_0^2)^{\alpha/2} \theta(\omega - \omega_0)$$

$$\times I_{\alpha/2} \left( 4 \cot 2\theta_g (\omega^2 - \omega_0^2) \right) \exp \left( -\frac{4(\omega^2 - \omega_0^2)}{Ng^2 \sin^2(2\theta_g)} \right),$$

(37)

where, as before, $g^2 = g_s^2 + g_g^2$ and the angle $\theta_g$ is defined in equation (23). Moreover, we have introduced the modified Bessel function $I_{\alpha/2}$ of order $\alpha/2$. For the transverse susceptibility we find in the two limiting cases $\theta_g = 0$ and $\theta_g = \pi/4$ the expressions

$$\chi''_{ss}(\omega) = \frac{1}{2} \left( \frac{\omega^2 - \omega_0^2}{\omega_0} \right)^{\alpha+1} \theta(\omega - \omega_0) \exp \left( -\frac{4(\omega^2 - \omega_0^2)}{Ng^2 \theta_0} \right), \quad \theta_g = \frac{\pi}{4},$$

(38)

$$\chi''_{ss}(\omega) = \frac{1}{2} \left( \frac{\omega^2 - \omega_0^2}{\omega_0} \right)^{(\alpha-1)/2} \theta(\omega - \omega_0) \exp \left( -\frac{4(\omega^2 - \omega_0^2)}{Ng^2 \theta_0} \right), \quad \theta_g = 0,$$

(39)
Figure 5. Plot of the transverse susceptibility $\chi''_{xx}(\omega)$ for an Ohmic type of density of states $\alpha = 1$ and for three different angles $\theta_g = 0, \pi/8, \pi/4$. The other parameters are $\omega_0 = 10$, $g = 1$ and $N = 1000$. The offset at which the function is normalized is $\epsilon = 0.1$.

Figure 6. Plot of the transverse susceptibility $\chi''_{xx}(\omega)$ for the symmetric case $\theta_g = \pi/4$ for the values $\alpha = 0$ (dotted), $\alpha = 1$ (dashed) and $\alpha = 2$ (full). The parameters are $\omega_0 = 10$, $g = 1$ and $N = 1000$. The offset at which the function is normalized is $\epsilon = 0.5$.

which satisfy the general properties given after equation (32). We note that, unlike in the case of an oscillator bath [3, 4, 7, 8], it is not possible to choose a spin bath which yields the same low $\omega$ dependence for $\chi''_{xx}(\omega)$ in the symmetric ($\theta_g = \pi/4$) and single bath ($\theta_g = 0$) cases.
The transverse susceptibility is zero for $\omega = \omega_0$ if $\alpha > 1$. This zero value is found because the distribution $\lambda_m$ is centered at $m = 0$. If the distribution is shifted to be centered at a nonzero $m$ value, then the value of $\chi''_{xx}(\omega)$ will be nonzero at $\omega = \omega_0$. In figure 5, we have plotted the transverse susceptibility for the symmetric ($\theta_g = \pi/4$) and single-bath ($\theta_g = 0$) cases, as well as for an intermediate situation. In order to compare with other results, in particular with the curves obtained in [4], we have normalized $\chi''_{xx}(\omega)$ so that $\chi''_{xx}(\omega_0 + \epsilon) = 1$, where $\epsilon$ is a convenient small offset which is chosen for proper scaling and comparison. We note that making $\epsilon$ too small shoots the peak to infinity in those cases where $\lim_{\epsilon \to 0} \chi''_{xx}(\omega_0 + \epsilon) = 0$. Though the natural sum rule $\int d\omega \omega \chi''_{xx}(\omega) = \omega_0$ is spoiled by adding this $\epsilon$, the essential underlying physics is unaffected. When there is no interaction with the bath, $\chi''_{xx}(\omega)$ is a delta function peaked at $\omega = \omega_0$. In the presence of one bath, the peak broadens with the maximum still located at $\omega = \omega_0$.

Surprisingly, the peak at $\omega = \omega_0$ disappears when we introduce a second bath which couples to a different component of the central spin. A similar shift was reported in [3, 4]. It results from a pure frustration effect due to the non-commuting nature of the spin operators.

In figure 6, we have plotted the transverse susceptibility for three different types of infrared behavior $\alpha = 0, 1, 2$, mimicking a subohmic, an Ohmic, respectively, a superohmic bath. Dissipation decreases as the power $\alpha$ increases, as expected for general dissipative quantum systems [1]. On the other hand, we find that the frustration effect of an additional bath increases with increasing power $\alpha$ (not shown).

7. Conclusions

We have analyzed the spectral properties of a TLS which is coupled to one or two dissipative baths through non-commuting observables. A peak in the spectrum at a nonzero frequency reveals the existence of an effective magnetic field experienced by the central spin. We have seen that the coupling to a second bath enhances rather than diminishes that effective field and, with it, the dynamics of the central spin. In the extreme case of a zero external field, the nonzero field is generated solely from the competition of two environments coupling to non-commuting spin components. This fact is remarkable if one notes that the baths are assumed to be initially unpolarized.

These physical effects arising from the non-commuting nature of the spin operators are a general feature of the dynamics which does not depend on details such as the Markovian or non-Markovian character of the reduced system dynamics or the strength of the system–bath interactions. Frustration of dissipation, as signaled by the emergence of a peak in the susceptibility for the symmetric case, seems to be a universal feature shared by widely differing models such as those of [3, 4], [6]–[8] and that considered here. Frustration of decoherence seems to be less robust by contrast. We have explicitly proved that the appearance of a peak in the spectral function can be compatible with a decay of the quantum purity, although the decay is slower for the case of a double bath (algebraic) than for the single bath case (exponential); i.e. while dissipation is frustrated by the competition between the two baths, decoherence is only slowed down. Our present results are consistent with the results of [7, 8] for a harmonic oscillator (equivalent to a large or quasiclassical spin) coupled to a dissipative bath of oscillators. There, a decay of purity was found to coexist with a weak form of suppression of dissipation. Here, we have proved that frustration of dissipation not accompanied by frustration of decoherence also takes place for a dissipative TLS which, given its reduced dimensionality, is much more quantum in nature than the harmonic oscillator.
The intuitive idea that two competing environments attempt to measure non-commuting observables and thus fail to generate decoherence sounds appealing but may be misleading. The statement would be true if the only possible result of a quantum measurement were to select a narrow distribution of eigenstates of the measured observable, since two non-commuting observables cannot be simultaneously well defined. However, a possible outcome of the coupling to a dissipative environment is that the reduced density matrix, while becoming diagonal in the representation of the eigenstates of the measured observable, may display a broad probability distribution in that representation. In the limit in which that distribution is very broad, the reduced density matrix approaches the identity matrix, which is invariant under a change of basis. Thus a reduced density matrix may be simultaneously diagonal in the representations of two non-commuting observables, provided it is close to the identity matrix. This is what actually happens to our central spin-$\frac{1}{2}$, as is clearly revealed by the quantum purity tending to its minimum value $1/2$ at long times, both for a single and a symmetric double bath. As we have seen, this feature is compatible with the reinforcement of the central spin dynamics resulting from the competition of the two environments. The upshot of the present study on the effect of competing environments is that, at least for dissipative TLS, it may be misleading to speak of quantum frustration of decoherence and it is more appropriate to introduce the concept of quantum frustration of dissipation.

The question remains of how to reconcile the coexistence of decoherence and dynamics enhancement by two competing environments found here and in [7, 8] with the results of [3, 4]. For one thing, the absence of bath internal dynamics might be yielding an overestimation of the decoherence experienced by the spin-$\frac{1}{2}$. Perhaps more importantly, the weak coupling flow found in [3, 4] may be describing a dressed spin interacting residually with the dissipative bath. This contrasts with our focus on a bare spin which quickly suffers decoherence. A possible reconciliation between the two pictures might depict a bare spin which experiences decoherence and renormalization by the environment while forming a dressed spin that (in the symmetric case) couples weakly to the remaining bath degrees of freedom.

A firmer conclusion on the existence or absence of decoherence frustration will require a deeper understanding of the behavior of genuinely quantum properties such as purity or pair entanglement in the presence of competing environments with internal dynamics.

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