NEW BOUNDS FOR $\psi(x)$

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ABSTRACT. In this article we provide new explicit Chebyshev’s bounds for the prime counting function $\psi(x)$. The proof relies on two new arguments: smoothing the prime counting function which allows to generalize the previous approaches, and a new explicit zero density estimate for the zeros of the Riemann zeta function.

1. INTRODUCTION.

1.1. Main Theorem and History. We recall that $\psi(x)$ is the Chebyshev function given by

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \quad \Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ for } k \geq 1, \\
0 & \text{else}.
\end{cases}$$

The Prime Number Theorem (PNT) is equivalent to

$$\psi(x) \sim x \quad \text{as } x \to \infty.$$ 

This estimate is a core tool in solving many problems in number theory and an explicit form of it turns out to be very useful in a wide range of problems. In this article, we investigate explicit bounds (also known as Chebyshev’s bounds) for the error term

$$E(x) = \left| \frac{\psi(x) - x}{x} \right|.$$ 

For instance, the main article of reference [20] in this subject is extensively used in various fields including Diophantine approximation, cryptography, and computer science. Moreover, breakthroughs concerning Goldbach’s conjecture (see the work of Ramaré [18], Tao [25], and Helfgott [6] [7]) rely on sharp explicit bounds for finite sums over primes. We combine a new explicit zero density estimate for $\zeta(s)$ and an optimized smoothing argument to prove

**Theorem 1.1.** Let $b_0 \leq 9963$ be a fixed positive constant. Let $x \geq e^{b_0}$. Then there exists $\epsilon_0 > 0$ such that $E(x) \leq \epsilon_0$, where $\epsilon_0$ is given explicitly in (3.9) and is computed in Table 3.

**Corollary 1.2.** For all $x \geq e^{20}$, $E(x) \leq 5.3688 \cdot 10^{-4}$.

A classic explicit formula that relates prime numbers to non-trivial zeros of $\zeta$ is given by [1] §17, (1):

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}),$$

where $\rho$ ranges over the non-trivial zeros of the Riemann zeta function. The proof of this formula involves an explicit zero density estimate and a smoothing argument.
when $x$ is not a prime power. As the sum over the zeros is not absolutely convergent, it is impossible to directly use this formula to bound the error term $E(x)$. To bypass this problem, the standard argument is to apply an explicit formula to an average of $\psi(x)$ on a small interval containing $[0, x]$.

In 1941 Rosser [22] Theorem 12] provides an explicit version of this proof. In 1962 Rosser and Schoenfeld [23] Theorem 28] improve on this method by introducing further averaging. Later results of Rosser and Schoenfeld [24], Dusart [2] [3], and very recently Nazardonyavi and Yakubovich [14] all use the argument of [23]. They successively obtain smaller bounds for the error term as a consequence of improvements concerning the location of the non-trivial zeros of the Riemann zeta function, namely the verification of the Riemann Hypothesis up to a fixed height $H$, and an explicit zero-free region of the form $\Re s \geq 1 - \frac{1}{R \log|\Im s|}$ and $|\Im s| \geq 2$, where $R$ is a computable constant. On the other hand Theorem 1.1 relies on new arguments. We introduce a smooth weight $f$ and compare $\psi(x)$ to the sum $\mathcal{S}(x) = \sum_{n \geq 1} \Lambda(n) f \left( \frac{n}{x} \right)$. In Section 3.1 we choose $f$ in a close to optimal way so as to make the bound on $E(x)$ as small as possible. We also observe that Rosser and Schoenfeld’s averaging method is a special case of this smoothing method (see Section 3.4 for further discussion). In Theorem 2.3 we establish a general explicit formula for $\mathcal{S}(x)$. A large contribution to the size of $E(x)$ arises from a sum over the non-trivial zeros of the form $\sum_\rho x^{\rho-1} F(\rho)$, where $F$ is the Mellin transform of $f$. This sum is studied in Section 2.3. We split it so as to isolate zeros closer to the 1-line (say of real part larger than a fixed $\sigma_0$) as they contribute the most to the sum. In section 2.3.2 we estimate this contribution by using for the first time explicit estimates for the zero density $N(\sigma_0, T)$ (as given in article [9]). This allows an extra saving over previous methods as they are of size between $\log T$ and $T$ smaller than $N(T)$. Finally Theorem 2.8 provides a general form for the bound of the error term $E(x)$.

We provide here a history of numerical improvements for Theorem 1.1 in the case where $b_0 = 50$. At the same time we mention which height $H$ and constant $R$ were used.

| Authors | $H$ | $R$ | $\epsilon_0$ |
|---------|-----|-----|-------------|
| Rosser [22] | 1.467 [22] | 17.72 [22] | 1.1900 · 10^{-2} |
| Rosser and Schoenfeld [23] | 21.943 [12] [13] | 17.5163... [23] | 1.7202 · 10^{-3} |
| Rosser and Schoenfeld [24] | 1894438 [24] | 9.645908801 [24] | 1.7583 · 10^{-5} |
| Dusart [2] | 545439823 [26] | 9.645908801 [24] | 9.0500 · 10^{-8} |
| Dusart [3] | 2445999556030 [5] | 5.69693 [8] | 1.3010 · 10^{-9} |
| Dusart and Schoenfeld [3] | 2445999556030 [5] | 5.69693 [8] | 1.3055 · 10^{-9} |
| Nazardonyavi and Yakubovich [14] | 30610046000 [17] [16] | 5.69693 [8] | 2.3643 · 10^{-9} |
| Faber and Kadiri | 2445999556030 [5] | | |

(* unpublished)

Note that when we use the same values for $H$ and $R$ than [3] and [14], our bounds for $E(x)$ are consistently smaller than theirs (for all $b_0$ except for $b_0 = 10000$ in the case of [3]).

1.2. **Zeros of the Riemann zeta function.** We use the latest computations of Platt [16] [17] concerning the verification of RH:

**Theorem 1.3.** Let $H = 3.061 \cdot 10^{10}$. If $\zeta(s) = 0$ at $0 \leq \Re(s) \leq 1$ and $0 \leq \Im(s) \leq H$, then $\Re(s) = \frac{1}{2}$.
Table 3 presents values of $\epsilon_0$ computed for this value of $H$. Prior to the work of Platt, Gourdon [5] announced a verification up to $H = 2,445,999,556,030$. We choose to use Platt’s value of $H$ since his verification of RH is the most rigorous to date (he employs interval arithmetic). Since other recent results ([3] and [14]) use Gourdon’s $H$, we also give a version of Theorem 1.1 based on his value (see Table 4).

From [8, Theorem 1.1] we have the zero-free region:

**Theorem 1.4.** Let $R = 5.69693$. Then there are no zeros of $\zeta(s)$ in the region

$$\Re s \geq 1 - \frac{1}{R \log |\Im s|} \text{ and } |\Im s| \geq 2.$$ 

Let $T \geq 2$ and $N(T)$ be the number of non-trivial zeros $\zeta = \beta + i\gamma$ in the region $0 \leq \gamma \leq T$ and $0 \leq \beta \leq 1$. In 1941, Rosser [22, Theorem 19] proved

**Theorem 1.5.** Let $T \geq 2$,

$$P(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, \quad R(T) = a_1 \log T + a_2 \log \log T + a_3,$$

and $a_1 = 0.137$, $a_2 = 0.443$, $a_3 = 1.588$. Then

$$|N(T) - P(T)| \leq R(T).$$

We recall that $N(\sigma_0, T)$ is the number of non-trivial zeros in the region $\sigma_0 \leq \Re s \leq 1$ and $0 \leq \Im s \leq T$. In [9] the second author proved explicit upper bounds for $N(\sigma_0, T)$:

**Theorem 1.6.** Let $3/5 \leq \sigma_0 < 1$. Then there exists constants $c_1, c_2, c_3$ such that, for all $T \geq H$, 

$$N(\sigma_0, T) \leq c_1 T + c_2 \log T + c_3.$$ 

The $c_i$’s depend on various (hidden) parameters and it is possible to choose these so as to make the above bound smaller when $T$ is asymptotically large or when it is close to $H$, the height of the numerical verification of RH. Table 4 at the end of this paper list values for the $c_i$’s in these respective cases. For instance, it gives

$$N(89/100, T) \leq 0.4617T + 0.6644 \log T - 340,272,$$

which provides a saving of about $1/3(\log T)$ compared to Theorem 1.5.

When $T$ is near $H$, Theorem 1.6 yields values for the $c_i$’s which provide a bound for $N(\sigma, T)$ of size about $\log H$. For instance, it gives that $N(99/100, H) \leq 78$ while Rosser’s Theorem gives $5.2 \cdot 10^{10}$.

2. General form of an explicit inequality for $\psi(x)$.

2.1. Introducing a smooth weight $f$.

**Definition 2.1.** Let $0 < a < b$, $m \in \mathbb{N}$ and $m \geq 2$. We define a function $f$ on $[a, b]$ by $f(x) = 1$ if $0 \leq x \leq a$, $f(x) = 0$ if $x \geq b$, and $f(x) = g\left(\frac{x-a}{b-a}\right)$ if $a \leq x \leq b$, where $g$ is a function defined on $[0, 1]$ satisfying

Condition 1: $0 \leq g(x) \leq 1$ for $0 \leq x \leq 1$, 

Condition 2: $g(0) = g(1) = 1$, 

Condition 3: $g(x)$ is continuous on $[0, 1]$. 

Condition 4: $\int_0^1 g(x) dx = 1$, 

Condition 5: $g'(x)$ is non-decreasing on $[0, 1]$. 

Condition 6: $g(x)$ is smooth on $(0, 1)$.
Condition 2: $g$ is an $m$-times differentiable function on $(0, 1)$ such that for all $k = 1, \ldots, m$,

$$g^{(k)}(0) = g^{(k)}(1) = 0,$$

and there exist positive constants $a_k$ such that

$$|g^{(k)}(x)| \leq a_k \text{ for all } 0 < x < 1.$$

We now consider

\begin{equation}
\mathcal{S}(x) = \sum_{n=1}^{\infty} \Lambda(n) f \left( \frac{n}{x} \right) \quad \text{and} \quad E_S(x) = \left| \frac{\mathcal{S}(x) - x}{x} \right|.
\end{equation}

Let $\delta > 0$. We denote $f^-$ and $f^+$ for the function $f$ defined above with the choices $a = 1 - \delta, b = 1$ and $a = 1, b = 1 + \delta$ respectively. We also define $\mathcal{S}^-$ and $\mathcal{S}^+$ the sums $\mathcal{S}$ associated to $f^-$ and $f^+$ respectively. Observe that

\begin{equation}
\mathcal{S}^-(x) \leq \psi(x) \leq \mathcal{S}^+(x) \quad \text{and} \quad E(x) \leq \max(E^-_S(x), E^+_S(x)).
\end{equation}

The Mellin Transform of $f$ is given by

\begin{equation}
F(s) = \int_{0}^{\infty} f(t)t^{s-1}dt.
\end{equation}

We recall the property (see [10, page 80, (3.1.3)]): if there exist $\alpha$ and $\beta$ such that $\alpha < \beta$ and, for every $\epsilon > 0$, $f(x) = \mathcal{O}(x^{-\alpha-\epsilon})$ as $x \to 0$, and $f(x) = \mathcal{O}(x^{-\beta+\epsilon})$ as $x \to +\infty$, then $F$ is analytic in $\alpha < \Re s < \beta$. It follows from our choice of $f$ that $F$ is analytic in $\Re s > 0$. Moreover, we have the inverse Mellin transform formula

\begin{equation}
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)t^{-s}ds.
\end{equation}

Observe that

$$\int_{a}^{b} |f^{(m+1)}(t)|t^{m+1}dt = \frac{1}{(b-a)^m} \int_{0}^{1} |g^{(m+1)}(u)|((b-a)u + a)^{m+1}du.$$

Let $k$ be a non-negative integer. We define

\begin{equation}
M(a, b, k) = \int_{0}^{1} |g^{(k+1)}(u)|((b-a)u + a)^{k+1}du.
\end{equation}

We now record some properties of $F$.

**Lemma 2.2.** Let $0 < a < b, m \in \mathbb{N}, m \geq 2$. Let $f$ and $g$ be functions as in Definition 2.1

(a) The Mellin transform $F$ of $f$ has a single pole at $s = 0$ with residue 1 and is analytic everywhere else.

(b) Let $s \in \mathbb{C}$ such that $\Re s \leq 1$. Then $F$ satisfies

\begin{equation}
F(1) = a + (b - a) \int_{0}^{1} g(u)du,
\end{equation}

\begin{equation}
|F(s)| \leq \frac{M(a, b, k)}{(b-a)^k |s|^{k+1}}, \text{ for all } k = 0, \ldots, m.
\end{equation}
Proof. The identity (2.6) follows immediately from the definition of \( f \).
We now use Condition 1 and Condition 2. We have \( F(s) = \int_0^b f(t) t^{s-1} dt \) with \( f'(x) = 0 \) for \( 0 < x < a \). We integrate by parts once and observe that \( F(s) = \frac{G(s)}{s} \), where
\[
G(s) = -\int_a^b f'(t) t^s dt
\]
is an entire function. The residue of \( F \) at \( s = 0 \) is \( G(0) = 1 \).
Let \( \Re s \leq 1 \) and \( k = 0, \ldots, m \). Inequality (2.7) is obtained by integrating \( F \) by parts \( k+1 \) times:
\[
F(s) = \frac{(-1)^{k+1}}{s(s+1) \ldots (s+k)} \int_a^b f^{(k+1)}(t) t^{s+k} dt.
\]
We consider
\[
G_m(s) = \int_a^b t^{s+m} f^{(m+1)}(t) dt.
\]
Since \( f^{(i)} \) vanishes at both \( a \) and \( b \) for all \( i = k, \ldots, m \), we have
\[
G_m(-k) = (m-k)!(-1)^{m-k} \int_a^b f^{(k+1)}(t) dt = (m-k)!(-1)^{m-k}(f^{(k)}(b) - f^{(k)}(a)) = 0.
\]
Thus \( F \) only has a pole at \( s = 0 \) and is analytic everywhere else. \( \square \)

2.2. An explicit formula for a smooth form of \( \psi(x) \). We use classical techniques to rewrite \( \mathscr{S}(x) \) as a complex integral, shift the integration contour to the left, and collect all the poles of the integrand so as to obtain a smooth analogue of the classical explicit formula (1.1).

**Theorem 2.3.** Let \( 0 < a < b, m \in \mathbb{N}, m \geq 2 \). Let \( f \) be a function satisfying Definition 2.1 and \( F \) its Mellin transform. Then
\[
\mathscr{S}(x) = xF(1) - \sum_\rho x^\rho F(\rho) - \frac{\zeta'}{\zeta}(0) - \sum_{n=1}^\infty x^{-2n} F(-2n),
\]
where \( \rho \) runs through all the non-trivial zeros \( \rho = \beta + i\gamma \) of the Riemann zeta function.

**Proof.** We insert (2.4) in (2.1):
\[
\mathscr{S}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^s F(s) \left( -\frac{\zeta'}{\zeta}(s) \right) ds.
\]
Fix \( k \in \mathbb{R}\setminus2\mathbb{N} \) and \( T \geq 2 \) such that \( T \) does not equal an ordinate of a zero of \( \zeta \). Observe that the integrand has a pole at \( s = 0 \) with residue \( -\frac{\zeta'}{\zeta}(0) \), a pole at \( s = 1 \) with residue \( xF(1) \), poles at the non-trivial zeros of zeta \( \rho = \beta + i\gamma \) with residue \( -x^\rho F(\rho) \), and poles at the trivial zeros of zeta \( s = -2n, n \in \mathbb{N} \), with residue \( -x^{-2n} F(-2n) \). We move the vertical line of integration extending from \( 2 - iT \) to \( 2 + iT \) to the line of integration extending from \( -k - iT \) to \( -k + iT \) so as to form the rectangle \( \mathcal{R} \). Thus
\[
\mathscr{S}(x) = I_1(T, k) + I_2(T, k) - I_3(T, k) - \frac{\zeta'}{\zeta}(0) + F(1)x - \sum_{|\gamma|<T} x^\rho F(\rho) - \sum_{1\leq n \leq \frac{k}{2}} x^{-2n} F(-2n),
\]
where...
where \( I_1, I_2, I_3 \) are respectively integrating along the segments \([-k + iT, 2 + iT], [-k + iT, -k - iT], [-k - iT, 2 - iT]\). It remains to prove that for each \( j = 1, 2, 3 \), \( \lim_{k,T \to +\infty} |I_j(T, k)| = 0 \). We use the classical bounds (see [1] page 108)

\[
|\zeta'(\sigma + iT)| \ll \begin{cases} \log^2 T & \text{if } -1 \leq \sigma \leq 2, \\ \log(|\sigma| + T) & \text{if } -k \leq \sigma \leq -1, \end{cases}
\]

together with inequality (2.7) for \( F \), and obtain

\[
|I_1(T, k)| \ll \frac{\log^2 T}{T^{m+1}} |x^2| + \frac{\log T}{T^{m+1}} \frac{1}{x \log x} + \frac{x^{-T}}{T^{m-1}}.
\]

We conclude that \( \lim_{k,T \to +\infty} |I_1(T, k)| = 0 \). Note that \( I_3(T, k) = I_1(-T, k) \) converges to 0 by a similar argument. For \( I_2(T, k) \), we combine (2.7) with [1] inequality (8):

\[
|F(-k + it)| \ll \frac{-\zeta'(-k + it)}{\zeta} \ll \begin{cases} \frac{\log k}{|k|^{m+1}} & \text{if } |t| \leq \frac{3}{2}, \\ \frac{\log |t|}{|t|^{m+1}} & \text{if } |t| > \frac{3}{2}. \end{cases}
\]

Thus \( |I_2(T, k)| \ll x^{-k} \left( \frac{\log k}{k^{m+1}} + \frac{\log T}{T^{m}} \right) \), and \( \lim_{k,T \to +\infty} |I_2(T, k)| = 0 \). \[\square\]

2.3. A general form of explicit bounds for \( \psi(x) \). We deduce from (2.7) that

\[
\left| \sum_{n=1}^{\infty} x^{-2n} F(-2n) \right| \leq M(a, b, 0) \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} \leq \frac{M(a, b, 0)}{2x^2}.
\]

Together with the above, (2.6), and \( -\frac{\zeta'}{\zeta}(0) = -\frac{\log(2\pi)}{2} \), it follows that

\[
(2.11) \quad E_{\psi}(x) \leq \left| a - 1 + (b - a) \int_{0}^{1} g(u) du \right| + \sum_{\rho} x^{\beta - 1} |F(\rho)| + \frac{\log(2\pi)}{2} x^{-1} + \frac{M(a, b, 0)}{2} x^{-3}.
\]

To study the sum over the zeros, we introduce the notation

\[
* \quad H > 0 \text{ is such that if } \zeta(\beta + i\gamma) = 0 \text{ and } 0 < \gamma < H, \text{ then } \beta = 1/2,
\]

\[
* \quad T_0 > 0 \text{ is such that } \sum_{0 < \gamma < T_0} \gamma^{-1} \text{ can be directly computed},
\]

\[
* \quad T_1 \text{ is a parameter satisfying } T_0 < T_1 < H,
\]

\[
* \quad R \text{ is a constant so that } \zeta(\sigma + iT) \text{ does not vanish in the region}
\]

\[
(2.12) \quad \sigma \geq 1 - \frac{1}{R \log |t|} \text{ and } |t| \geq 2,
\]

\[
* \quad \sigma_0 \text{ is a parameter satisfying } 3/5 \leq \sigma_0 < 1,
\]

\[
* \quad c_1 > 0, c_2 > 0, c_3 < 0 \text{ depend on } \sigma_0 \text{ so that } N(\sigma_0, T) \leq c_1 T + c_2 \log T + c_3, \text{ for all } T \geq H.
\]

Using the symmetry of the zeros of zeta and using the notation \( \sum^* = \frac{1}{2} \sum_{\beta=1/2} + \sum_{1/2 < \beta < 1} \) we have:

\[
(2.13) \quad \sum_{\rho} x^{\beta - 1} |F(\rho)| = \sum_{\gamma > 0}^* \left( x^{\beta - 1} + x^{-\beta} \right) \left( |F(\rho)| + |F(\bar{\rho})| \right).
\]
We now separate the zeros vertically at $H$:

$$
\sum_{\rho} x^{\beta-1} |F(\rho)| = \Sigma_1 + \Sigma_2,
$$

with

$$
\Sigma_1 = x^{-\frac{1}{2}} \sum_{0 < \gamma \leq H} (|F(1/2 + i\gamma)| + |F(1/2 - i\gamma)|), \quad \Sigma_2 = \sum_{\gamma > H}^* (x^{\beta-1} + x^{-\beta}) (|F(\rho)| + |F(\overline{\rho})|).
$$

We split $\Sigma_1$ vertically at $T_1$ and use (2.7) to bound $|F(\rho)|$ with $k = 0$ when $\gamma \leq T_1$, and $k = m$ when $T_1 < \gamma \leq H$ respectively. Thus

$$
\Sigma_1 \leq 2x^{-\frac{1}{2}} \left( M(a, b, 0) \sum_{0 < \gamma \leq T_1} \frac{1}{\gamma} + \frac{M(a, b, m)}{(b - a)^m} \sum_{T_1 < \gamma \leq H} \frac{1}{\gamma^{m+1}} \right).
$$

Moreover, we split the first sum at height $T_0 \leq T_1$ and denote $s_0$ a close upper bound for $\sum_{\gamma \leq T_0} \frac{1}{\gamma}$.

In [24], the authors use $T_0 = 158.84998$ and $s_0 = 0.8113925$. We use here a computation of Darcy Best (personal communication) based on Odlyzko’s list of zeros [15]: $T_0 = 1132.491$ and $s_0 = 11.637732$.

We use (2.7) with $k = m$ for $\Sigma_2$ and split it horizontally at $\sigma_0$. Together with the zero-free region given in Theorem 1.4 and the fact that $x^{\beta-1} + x^{-\beta}$ increases with $\beta$, we obtain

$$
\Sigma_2 \leq 2 \frac{M(a, b, m)}{(b - a)^m} \left( x^{-(1-\sigma_0)} + x^{-\sigma_0} \sum_{\gamma > H} \frac{1}{\gamma^{m+1}} + \sum_{\gamma > H, \sigma_0 < \beta < 1} \frac{x^{-\frac{1}{R \log \gamma}} + x^{-(1-\frac{1}{R \log H})}}{\gamma^{m+1}} \right).
$$

We denote

$$
s_1(T_1) = \sum_{0 < \gamma \leq T_1} \frac{1}{\gamma}, \quad s_2(m, T_1) = \sum_{T_1 < \gamma \leq H} \frac{1}{\gamma^{m+1}}, \quad s_3(m) = \sum_{\gamma > H} \frac{1}{\gamma^{m+1}}, \quad s_4(m, \sigma_0) = \sum_{\gamma > H, \sigma_0 < \beta < 1} \frac{x^{-\frac{1}{R \log \gamma}}}{\gamma^{m+1}}, \quad s_5(x, m, \sigma_0) = \sum_{\gamma > H, \sigma_0 < \beta < 1} \frac{x^{-\frac{1}{R \log \gamma}}}{\gamma^{m+1}}.
$$

We have

$$
\sum_{\rho} x^{\beta-1} |F(\rho)| \leq 2 \left( M(a, b, 0)s_1(T_1) + \frac{M(a, b, m)}{(b - a)^m} s_2(m, T_1) \right) x^{-\frac{1}{2}}
$$

$$
+ 2 \frac{M(a, b, m)}{(b - a)^m} \left( x^{-(1-\sigma_0)} + x^{-\sigma_0} \right) s_3(m) + x^{-(1-\frac{1}{R \log H})} s_4(m, \sigma_0) + s_5(x, m, \sigma_0).
$$

We conclude by inserting (2.18) in (2.11).
Lemma 2.4. Let $0 < a < b, m \in \mathbb{N}$, with $m \geq 2$. Let $f$ be a function satisfying Definition 2.7. Let $H, T_0, T_1, R$, and $\sigma_0$ satisfy (2.12). Then for all $x > 0$, $E_{\gamma}(x) \leq K(x, a, b, m, \sigma_0)$, where

\begin{equation}
K(x, a, b, m, \sigma_0) = \left| a - 1 + (b - a) \int_0^1 g(u)du \right| + \frac{2M(a, b, m)}{(b - a)^m} \left( (x^{-1-\sigma_0} + x^{-\sigma_0})s_3(m) + x^{-1-\frac{1}{m}}s_4(m, \sigma_0) + s_5(x, m, \sigma_0) \right) + 2\left( M(a, b, 0)s_0 + M(a, b, 0)s_1(T_1) + \frac{M(a, b, m)}{(b - a)^m}s_2(m, T_1) \right) x^{-\frac{1}{2}} + \frac{\log(2\pi)}{2}x^{-1} + \frac{M(a, b, 0)}{2}x^{-3},
\end{equation}

and $M(a, b, m)$ and the $s_i$’s are defined in (2.5) and (2.17) respectively.

Note that for $a, b, m, \sigma_0$ fixed constants, $K(x, a, b, m, \sigma_0)$ decreases with $x$. Thus, for all $x \geq x_0$

\begin{equation}
E_{\gamma}(x) \leq K(x_0, a, b, m, \sigma_0).
\end{equation}

2.3.1. Bounding $s_1(T_1)$, $s_2(m, T_1)$, and $s_3(m)$. We apply here a result from Rosser and Schoenfeld [24]. It uses explicit estimates for $N(T)$ as given in Theorem 3.4 to bound certain sums over the zeros of zeta.

Lemma 2.5. [24] Lemma 7] Let $1 < U \leq V$, and let $\Phi(y)$ be nonnegative and differentiable for $U < y < V$. Let $(W - y)\Phi'(y) \geq 0$ for $U < y < V$, where $W$ need not lie in $[U, V]$. Let $Y$ be one of $U, V, W$ which is neither greater than both the others or less than both the others. Choose $j = 0$ or $1$ so that $(-1)^j(V - W) \geq 0$. Then

\begin{equation}
\sum_{U < \gamma \leq V} \Phi(\gamma) \leq \frac{1}{2\pi} \int_U^V \Phi(y) \log \frac{y}{2\pi} dy + (-1)^j \left( a_1 + \frac{a_2}{\log Y} \right) \int_U^V \frac{\Phi(y)}{y} dy + E_j(U, V),
\end{equation}

where the error term $E_j(U, V)$ is given by

\[ E_j(U, V) = (1 + (-1)^j)R(Y)\Phi(Y) + (N(V) - P(V) - (-1)^j R(V))\Phi(V) - (N(U) - P(U) + R(U))\Phi(U). \]

Corollary 2.6. [24] Corollary of Lemma 7] If, in addition, $2\pi < U$, then

\begin{equation}
\sum_{U < \gamma \leq V} \Phi(\gamma) \leq \frac{1}{2\pi} + (-1)^j q(Y)) \int_U^V \Phi(y) \log \frac{y}{2\pi} dy + E_j(U, V), \quad \text{where} \quad q(y) = \frac{a_1 \log y + a_2}{y \log y \log(y/2\pi)}.
\end{equation}

Moreover, if $j = 0$ and $W < U$, then

\begin{equation}
E_0(U, V) \leq 2R(U)\Phi(U).
\end{equation}

We give details on how we apply Corollary 2.6 and (2.21) to $s_1$, $s_2$, and $s_3$. We take respectively

- $\Phi(y) = y^{-1}, U = T_0, V = T_1$,
- $\Phi(y) = y^{-m-1}, U = T_1, V = H$,
- $\Phi(y) = y^{-m-1}, U = H, V = \infty$. 

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In each case, $\Phi'(y) \leq 0$ for all $y$, and we choose $W < U$, $Y = U$, and $j = 0$. Since
\[
\int_{T_0}^{T_1} \log \frac{y}{2\pi} dy = \log(T_1/T_0) \log(\sqrt{T_1T_0}/(2\pi)),
\]
\[
\int_{U}^{V} \log \frac{y}{2\pi} dy = \frac{1 + m \log(U/2\pi)}{m^2 U^m} - \frac{1 + m \log(V/2\pi)}{m^2 V^m},
\]
we obtain:
\[
(2.22)
\]
\[
s_1(T_1) \leq B_1(T_1) = s_0 + \left(\frac{1}{2\pi} + q(T_0)\right) \left(\log(T_1/T_0) \log(\sqrt{T_1T_0}/(2\pi))\right) + \frac{2R(T_0)}{T_0},
\]
\[
(2.23)
\]
\[
s_2(m, T_1) \leq B_2(m, T_1) = \left(\frac{1}{2\pi} + q(T_1)\right) \left(\frac{1 + m \log(T_1/2\pi)}{m^2 T_1^m} \right) - \frac{1 + m \log(H/2\pi)}{m^2 H^m} + \frac{2R(T_1)}{T_1^{m+1}},
\]
\[
(2.24)
\]
\[
s_3(m) \leq B_3(m) = \left(\frac{1}{2\pi} + q(H)\right) \frac{1 + m \log(H/2\pi)}{m^2 H^m} + \frac{2R(H)}{H^{m+1}}.
\]

2.3.2. Bounding $s_4(m, \sigma_0)$ and $s_5(x, m, \sigma_0)$. We assume here that $\Phi(y) = o(y)$ when $y \to \infty$, so as to ensure that $\lim_{y \to \infty} \Phi(y) N(\sigma_0, y) = 0$. Since all non-trivial zeros of zeta have real part $1/2$ when $\gamma \leq H$, then $N(\sigma_0, H) = 0$ and we have the Stieltjes integral
\[
\sum_{\gamma \geq H, \beta > \sigma_0} \Phi(\gamma) = -\int_{H}^{\infty} N(\sigma_0, y) \Phi'(y) dy.
\]

**Lemma 2.7.** Let $H, \sigma_0, c_1, c_2, c_3$ satisfy (2.12). Let $H < U \leq V$, and let $\Phi(y)$ be non-negative and differentiable for $U < y < V$. Assume $\Phi(y) = o(y)$ when $y \to \infty$ and $(W - y)\Phi'(y) \geq 0$ for all $U < y < V$, where $W$ need not lie in $[U, V]$. Let $Y$ be one of $U, V, W$ which is neither greater than both the others or less than both the others. Then
\[
\sum_{U < \gamma < V, \beta > \sigma_0} \Phi(\gamma) \leq (c_1 Y + c_2 \log Y + c_3) \Phi(Y) - (c_1 V + c_2 \log V + c_3) \Phi(V) + \int_{Y}^{V} (c_1 + c_2/y) \Phi(y) dy.
\]

**Proof.** We have $0 \leq N(\sigma_0, y) \leq c_1 y + c_2 \log y + c_3$. Our assumptions ensure us that $\Phi'(y) \geq 0$ if $U \leq y \leq Y$ and that $\Phi'(y) \leq 0$ if $Y \leq y \leq V$. Thus
\[
-\int_{U}^{V} N(\sigma_0, y) \Phi'(y) dy \leq -\int_{Y}^{V} (c_1 y + c_2 \log y + c_3) \Phi'(y) dy,
\]
and we integrate by part to complete the proof. \hfill \Box

For $s_4(m, \sigma_0)$, we take $\Phi(y) = \frac{1}{y^{m+1}}$, $\Phi'(y) = -\frac{m+1}{y^{m+2}}$, $W < U = Y = H$, and $V = \infty$. Thus
\[
(2.25)
\]
\[
s_4(m, \sigma_0) \leq B_4(m, H, \sigma_0) = \left(c_1 \left(1 + \frac{1}{m}\right) + c_2 \log H \frac{1}{H} + (c_3 + \frac{c_2}{m+1}) \frac{1}{H^{m}}\right) \frac{1}{H^{m}}.
\]

For $s_5(x, m, \sigma_0)$, we apply Lemma [2.7] with $U = H$, $V = \infty$, $\Phi(y) = \phi_m(y) = \frac{\log x}{R(\log x)} - (m + 1)\phi_{m}(y)$, $\phi_{m}(y)$ is
\[
\left(\frac{\log x}{R(\log x)} - (m + 1)\right)\frac{\phi_{m}(y)}{y}
\]
and
\[
(2.26)
\]
\[
W = e^{\sqrt{\frac{\log x}{R(\log x)} - (m + 1)}}.
\]
Let \( J_m(Y) \) denote the integral
\[
J_m(Y) = \int_Y^\infty \phi_m(y)dy.
\]
We obtain
\[
s_5(x, m, \sigma_0) \leq (c_1Y + c_2 \log Y + c_3)\phi_m(Y) + c_1J_m(Y) + c_2J_{m+1}(Y),
\]
Let \( z > 0, w \geq 0 \). We appeal to the theory of the following modified Bessel function
\[
K_\nu(z, w) = \frac{1}{2} \int_w^\infty t^{\nu-1} \exp \left( -\frac{z}{2}(t + 1/t) \right) dt.
\]
We do the variable change \( y = e^{z/t} \), take \( z = 2\sqrt{\frac{m}{R}} \), \( w = \sqrt{\frac{mR}{\log z}} \log Y = \frac{2m}{z} \log Y \), and recognize
\[
J_m(Y) = \frac{z}{2m}K_1(z, w).
\]
We use [24, Lemma 4] which asserts that if \( w > 1 \) then
\[
(2.28) \quad K_1(z, w) \leq Q_1(z, w) = \frac{w^2}{z(w^2 - 1)} \exp \left(-z/2(w + 1/w)\right).
\]
We deduce for \( J_m(Y) \) that if \( \log x < mR(\log Y)^2 \), then
\[
(2.29) \quad J_m(Y) \leq \frac{R}{2 \log x} \left( \frac{mR}{\log x} \right)^2 Y^{-m} e^{-\frac{\log x}{R(\log Y)}}.
\]
In this case, we have \( W < H, Y = H \). We insert (2.29) in (2.27) and obtain
\[
s_5(x, m, \sigma_0) \leq (c_1 + c_2 \frac{\log H}{H} + c_3) x^{-\frac{1}{\log H}} + c_1J_m(H) + c_2J_{m+1}(H),
\]
We conclude that if \( \log x < mR(\log H)^2 \) then
\[
(2.30) \quad s_5(x, m, \sigma_0) \leq B_5(x, m, \sigma_0) = \left(c_1 + c_2 \frac{\log H}{H} + c_3 \frac{H}{H} + c_1 \frac{R}{2 \log x} \left( \frac{mR}{\log x} \right)^2 \right) x^{-\frac{1}{\log H}}.
\]

2.3.3. Main Theorem. We deduce a new bound for \( K(x, a, b, m, \sigma_0) \) from (2.22), (2.23), (2.24), (2.25), and (2.30). Lemma 2.4 becomes

**Theorem 2.8.** Let \( 0 < a < b, m \in \mathbb{N} \), with \( m \geq 2 \). Let \( f \) and \( g \) be functions satisfying Definition 2.1 and \( M(a, b, m) \) as defined in (2.5). Let \( H, T_0, T_1, R, \sigma_0, c_1, c_2, c_3 \) satisfy (2.12). Let \( x_0 \) be a positive constant satisfying \( x_0 < \exp(mR(\log H)^2) \). Then for all \( x \geq x_0 \)
\[
(2.31) \quad E_\varphi(x) \leq |a-1+(b-a) \int_0^1 g(u)du| + \frac{2M(a, b, m)B_5(x_0, m, \sigma_0)}{(b-a)^m} x_0^{-(1-\sigma_0)} + \frac{2M(a, b, m)B_3(m)}{(b-a)^m} x_0^{-1} + \frac{2M(a, b, m)B_3(m)}{(b-a)^m} x_0^{-1} + \frac{2M(a, b, m)B_3(m)}{(b-a)^m} x_0^{-1} + \frac{2M(a, b, m)B_3(m)}{(b-a)^m} x_0^{-1} + \frac{2M(a, b, m)B_3(m)}{(b-a)^m} x_0^{-1},
\]
where the \( B_i \)'s are defined in (2.22), (2.23), (2.24), (2.25), and (2.30).
3. New explicit bounds for $\psi(x)$.

3.1. Choosing the smooth function. We want to find a function $g$ satisfying Definition 2.1 and so that the quotient $M(a, b, m)$ is as small as possible. By the Cauchy-Schwarz inequality we have

$$M(a, b, m) \leq \sqrt{\frac{b^{2m+3} - a^{2m+3}}{(b-a)(2m+3)}} \sqrt{\int_0^1 (g^{(m+1)}(u))^2 du}.$$  

It follows from Calculus of Variations (see [4, Chapter 2, §11]) that the function $g$ optimizing the quotient $\sqrt{\int_0^1 (g^{(m+1)}(u))^2 du}$ is given by

$$g(x) = 1 - (2m+1)! \int_0^x t^m(1-t)^m dt.$$  

We observe that our choice of kernel is a primitive of the one used in the context of short intervals containing primes by Ramaré & Saouter [21]. This is not surprising as our object of study is $\sum_{n \geq 1} \Lambda(n) f(n/x)$, while theirs is essentially $\sum_{n \geq 1} \Lambda(n) (f(n/y) - f(n/x))$. Since $y$ is close to $x$, this is approximately $\sum_{n \geq 1} \Lambda(n) f'(n/x)$.

With definition (3.2), we find

$$\int_0^1 g(u) du = 1 - \frac{(2m+1)!}{(m!)^2} \int_0^1 t^m(1-t)^m dt = \frac{1}{2},$$  

and

$$M(a, b, 0) = \frac{a + b}{2}.$$  

We use (3.1) to provide a simple bound for $M(a, b, m)$. Since $g(1) = 0$, $g(0) = 1$, and $g^{(2m+2)}(x) = 0$ for all $0 < x < 1$, integrating by parts $m$ times leads to

$$\int_0^1 (g^{(m+1)}(u))^2 du = (-1)^m \int_0^1 g^{(2m+1)}(u) \cdot g'(u) du = (-1)^m g^{(2m+1)}(0) = \frac{(2m)!(2m+1)!}{(m!)^2}.$$  

Thus (3.1) becomes

$$M(a, b, m) \leq \lambda(a, b, m) = \sqrt{\frac{b^{2m+3} - a^{2m+3}}{(b-a)(2m+3)}} \cdot \frac{\sqrt{(2m)!(2m+1)!}}{m!}.$$  

From (3.2), we recognize that

$$g^{(m+1)}(u) = -\frac{(2m+1)!}{m!} P_m(1-2u),$$  

where $P_m$ is the $m^{th}$ Legendre polynomial as given by Rodrigues’ formula (see [11, formula (0.4)]):

$$P_m(x) = \frac{1}{2^m m! \partial_x^m} (x^2 - 1)^m.$$  

They can be written explicitly (see [11, formula (0.2)]):

$$P_m(x) = \sum_{k=0}^m \binom{m}{k}^2 \left( \frac{x+1}{2} \right)^k \left( \frac{x-1}{2} \right)^{m-k}.$$  

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These polynomials are well-known and are among the built-in functions of PARI/GP. Since the
sign of $P_m$ alternates between its roots, $M(a, b, m)$ can be computed directly from

(3.6) \[ M(a, b, m) = \frac{(2m + 1)!}{m!} \int_0^1 |P_m(1 - 2u)| ((b - a)u + a)^{m+1} du. \]

3.2. New explicit bounds for $\psi(x)$. We rewrite Theorem 2.8 with $g$ as chosen in (3.2):

**Theorem 3.1.** Let $m \in \mathbb{N}$, $m \geq 2$, $\delta > 0$, and the pair $(a, b)$ takes values $(1, 1 + \delta)$ or $(1 - \delta, 1)$. Let $H, T_0, T_1, R, \sigma_0, c_1, c_2, c_3$ satisfy (2.12). Let $b_0 > 0$ be a positive constant satisfying $b_0 < (m + 1)R(\log H)^2$. Then for all $x \geq e^{b_0}$

(3.7) \[ E_\mathcal{S}(x) \leq \frac{\delta}{2} + \frac{2M(a, b, m)B_5(e^{b_0}, m, \sigma_0)}{\delta^m} + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-(1-\sigma_0)b_0} \]
\[ + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-\sigma_0b_0} + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-(1-R/\sigma_0)b_0} \]
\[ + \left( \frac{\delta}{2} B_1(T_1) + \frac{M(a, b, m)B_2(m, T_1)}{\delta^m} \right) e^{-b_0/2} + \frac{\log(2\pi)}{2} e^{-b_0} + \frac{M(a, b, 0)}{2} e^{-3b_0}, \]

where $M(a, b, m)$ is given by (3.6), and the $B_i$’s are defined in (2.22), (2.23), (2.24), (2.25), and (2.30).

3.3. Proof of Theorem 1.1. Let $b_0 \geq 2$ be a fixed constant satisfying $b_0 < 3R(\log H)^2$ (that is $b_0 < 9.963$ for $H = 3.061 \times 10^{10}$ and $b_0 < 13.906$ for $H = 2445999556030$). Let $x \geq e^{b_0}$. We define

(3.8) \[ \epsilon(b_0, a, b, m, \sigma_0, T_1) = \frac{\delta}{2} + \frac{2M(a, b, m)B_5(e^{b_0}, m, \sigma_0)}{\delta^m} + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-(1-\sigma_0)b_0} \]
\[ + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-\sigma_0b_0} + \frac{2M(a, b, m)B_3(m)}{\delta^m} e^{-(1-R/\sigma_0)b_0} \]
\[ + \left( \frac{\delta}{2} B_1(T_1) + \frac{M(a, b, m)B_2(m, T_1)}{\delta^m} \right) e^{-b_0/2} + \frac{\log(2\pi)}{2} e^{-b_0} + \frac{M(a, b, 0)}{2} e^{-3b_0}. \]

The definition for $\epsilon_0$ follows directly from (2.2) and Theorem 3.1.

(3.9) \[ \epsilon_0 = \max \left( \epsilon(b_0, 1, 1 + \delta, m, \sigma_0, T_1), \epsilon(b_0, 1 - \delta, 1, m, \sigma_0, T_1) \right). \]

To compute $\epsilon(b_0, 1, 1 + d, m, \sigma_0, T_1)$, we choose a value for $\sigma_0$ in Table 2\footnote{Table 2 is not included in the text.} an integer value larger than $2$ for $m$, and a value for $\delta$ with up to 4 significant digits. Then we choose a value for $T_1$ which is either $T_0$, $H$ or so that it satisfies

\[ \frac{\delta}{2} B_1(T_1) = \frac{M(1, 1 + \delta, m)B_2(m, T_1)}{\delta^m}. \]

We do the same to compute $\epsilon(b_0, 1 - \delta, 1, m, \sigma_0, T_1)$. All values for $\sigma_0$, $m$, and $\delta$ are chosen to make $\epsilon_0$ as small as possible.

3.4. Comparison with Rosser and Schoenfeld’s method.
3.4.1. The smoothing argument. The first step of their argument consists in studying \( \psi(x) \) on average on a small interval around a large \( x \) value. Let \( x, \delta > 0 \) with \( x \notin \mathbb{N} \). Let \( m \in \mathbb{N} \). It follows from the First Mean Value Theorem for Integrals applied to \( h(z) = \psi(x + z) - (x + z) \) that there exists \( z \in (0, \delta x) \) such that:

\[
h(z) + z \leq \frac{1}{(\delta/m)x^m} \int_0^{\delta x/m} \cdots \int_0^{\delta x/m} (h(y_1 + \ldots + y_m) + (y_1 + \ldots + y_m)) \, dy_1 \ldots dy_m.
\]

(In order to make Rosser and Schoenfeld’s article consistent with our setup, we replace their \( \delta \) with our \( \delta/m \).) Implementing the explicit formula (1.1) in the right integrals together with the fact that \( \psi(x + z) \leq \psi(x) \) leads to [22, Theorems 12 and 14]:

\[
E(x) \leq \frac{\delta}{2} + \Sigma(m, \delta, x) + O(x^{-1}),
\]

with

\[
\Sigma(m, \delta, x) = \left| \sum_\rho x^{\rho-1} I_{m, \delta}(\rho) \right|, \quad \text{and} \quad I_{m, \delta}(\rho) = \frac{\sum_{j=0}^{m}(-1)^{j+m+1} m_j (1 + j \delta/m)^{m+\rho}}{(\delta/m)^{m} \rho(\rho + 1) \ldots (\rho + m)}.
\]

We recall that we obtain (3.10) with

\[
\Sigma(m, \delta, x) = \left| \sum_\rho x^{\rho-1} F(\rho) \right|.
\]

We recognize that \( I_{m, \delta} \) is indeed the Mellin transform of

\[
\nu(t) = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{j+m} \binom{m}{j} \left( \frac{1 + j \delta/m - t}{\delta/m} \right)^m \mathbb{1} \left( \frac{t}{1 + j \delta/m} \right),
\]

where \( \mathbb{1} \) is the indicator function on \((0, 1)\). Instead we use the function \( f \) given by Definition 2.1 and (3.2):

\[
f(x) = 1 - \frac{(2m + 1)!}{(m!)^2} \int_0^{\frac{2+1}{m}} t^m (1 - t)^m \, dt.
\]

We now compare the size of each Mellin transform. Rosser establishes (see [22, Theorem 15]) that

\[
|I_{m, \delta}(\rho)| \leq \frac{(1 + \delta/m)^{m+1} + 1)^m}{(\delta/m)^{m} |\gamma|^{m+1}} = \frac{2^m m^m}{\delta^m |\gamma|^{m+1}} (1 + o(1)),
\]

while we have from (2.7) and (3.5)

\[
|F(\rho)| \leq \frac{M(1, 1 + \delta, m)}{\delta^m |\gamma|^{m+1}} \leq \frac{\sqrt{(2m)!}(2m+1)!}{m!\delta^m |\gamma|^{m+1}} (1 + o(1)).
\]

It follows from Stirling Formula that the quotient \( \frac{|F(\rho)|}{|I_{m, \delta}(\rho)|} = \frac{\sqrt{(2m)!}(2m+1)!}{(2m)^m(m!)^2} \) decreases rapidly to 0 as \( m \) grows. For instance it is 0.0083\ldots when we take \( m = 23 \) for \( b_0 = 50 \).
3.4.2. The new density of zeros. When \( x \) is large enough, the largest contribution to \( \Sigma(m, \delta, x) \) arises from

\[
\sum_{\gamma > H, \sigma_0 < \beta < 1 - \frac{1}{\pi \log \gamma}} x^{-\frac{1}{\pi \log \gamma}} \gamma^m + 1.
\]

Rosser and successive authors took \( \sigma_0 = 1/2 \) since only bounds for \( N(T) \) were available. Rosser and Schoenfeld find (see [24] equations (3.4), (3.16) and (2.4)) that if \( b_0 \leq 2R \log^2 H \) and \( x \geq e^{b_0} \) then

\[
e^{b_0} \frac{H^m}{\pi \log H} \sum_{\gamma > H, 1/2 < \beta < 1 - \frac{1}{\pi \log \gamma}} x^{-\frac{1}{\pi \log \gamma}} \gamma^m + 1 \leq \frac{R(\log H)^3}{2\pi b_0 \left( \frac{mR(\log H)^2}{b_0} - 1 \right)} (1 + o(1)).
\]

We are able to reduce significantly the contribution of the sum by using \( \sigma_0 \) closer to the limit of the zero-free region. We establish that if \( b_0 \leq 3R \log^2 H \) and \( x \geq e^{b_0} \) then the above bound is replaced with

\[
(c_1 + \frac{c_2 \log H}{H} + \frac{c_3}{H}) + \left( c_1 + \frac{c_2}{H} \right) \frac{R}{2b_0} \left( \frac{\log H^2}{\log H^2} - 1 \right).
\]

When \( \left( \frac{mR(b_0)}{b_0} \right)(\log H)^2 - 1 \) is large enough (for instance for \( 45 \leq b_0 \leq 2000 \) and \( m \geq 10 \)), the main contribution arises from the above left expression. We use the values for the \( c_i \)'s from the right column of Table 2, as they make \( c_1H + c_2 \log H + c_3 \) small. Otherwise, we use the values from the left column as they provide the smallest value for \( c_1 + \frac{c_2}{H} \).

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Table 2. For all $T \geq H$, $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$.  

| $\sigma$ | $c_1$ | $c_2$ | $c_3$ | $c_1$ | $c_2$ | $c_3$ |
|----------|-------|-------|-------|-------|-------|-------|
| 0.60     | 4.2288| 2.2841| -81.673| 28.6424| 2.2841| -8.7674 $\cdot 10^{11}$ |
| 0.65     | 2.4361| 1.7965| -97.414| 17.1679| 1.3674| -5.2550 $\cdot 10^{11}$ |
| 0.70     | 1.4934| 1.4609| -136.370| 12.3778| 0.9859| -3.7888 $\cdot 10^{11}$ |
| 0.75     | 1.0031| 1.1442| -169.449| 9.6776| 0.7708| -2.9622 $\cdot 10^{11}$ |
| 0.76     | 0.9355| 1.0921| -176.604| 9.2730| 0.7386| -2.8384 $\cdot 10^{11}$ |
| 0.77     | 0.8750| 1.0437| -184.134| 8.9009| 0.7089| -2.7245 $\cdot 10^{11}$ |
| 0.78     | 0.8205| 0.9986| -192.120| 8.5575| 0.6816| -2.6194 $\cdot 10^{11}$ |
| 0.79     | 0.7714| 0.9566| -200.644| 8.2396| 0.6563| -2.5221 $\cdot 10^{11}$ |
| 0.80     | 0.7269| 0.9176| -209.795| 7.9445| 0.6328| -2.4317 $\cdot 10^{11}$ |
| 0.81     | 0.6864| 0.8812| -219.667| 7.6698| 0.6109| -2.3477 $\cdot 10^{11}$ |
| 0.82     | 0.6495| 0.8473| -230.367| 7.4135| 0.5905| -2.2692 $\cdot 10^{11}$ |
| 0.83     | 0.6156| 0.8157| -242.009| 7.1737| 0.5714| -2.1958 $\cdot 10^{11}$ |
| 0.84     | 0.5846| 0.7862| -254.724| 6.9490| 0.5535| -2.1270 $\cdot 10^{11}$ |
| 0.85     | 0.5561| 0.7586| -268.658| 6.7379| 0.5367| -2.0624 $\cdot 10^{11}$ |
| 0.86     | 0.5297| 0.7327| -283.978| 6.5392| 0.5209| -2.0016 $\cdot 10^{11}$ |
| 0.87     | 0.5053| 0.7085| -300.872| 6.3520| 0.5059| -1.9443 $\cdot 10^{11}$ |
| 0.88     | 0.4827| 0.6857| -319.555| 6.1751| 0.4919| -1.8901 $\cdot 10^{11}$ |
| 0.89     | 0.4617| 0.6644| -340.272| 6.0079| 0.4785| -1.8389 $\cdot 10^{11}$ |
| 0.90     | 0.4421| 0.6443| -363.301| 5.8494| 0.4659| -1.7905 $\cdot 10^{11}$ |
| 0.91     | 0.4238| 0.6253| -388.959| 5.6991| 0.4539| -1.7444 $\cdot 10^{11}$ |
| 0.92     | 0.4066| 0.6075| -417.606| 5.5564| 0.4426| -1.7007 $\cdot 10^{11}$ |
| 0.93     | 0.3905| 0.5906| -449.647| 5.4206| 0.4318| -1.6592 $\cdot 10^{11}$ |
| 0.94     | 0.3754| 0.5747| -485.543| 5.2913| 0.4215| -1.6196 $\cdot 10^{11}$ |
| 0.95     | 0.3612| 0.5596| -525.807| 5.1680| 0.4116| -1.5819 $\cdot 10^{11}$ |
| 0.96     | 0.3478| 0.5452| -571.018| 5.0503| 0.4023| -1.5458 $\cdot 10^{11}$ |
| 0.97     | 0.3352| 0.5316| -621.815| 4.9379| 0.3933| -1.5114 $\cdot 10^{11}$ |
| 0.98     | 0.3232| 0.5187| -678.911| 4.8304| 0.3848| -1.4785 $\cdot 10^{11}$ |
| 0.99     | 0.3118| 0.5063| -743.087| 4.7274| 0.3766| -1.4470 $\cdot 10^{11}$ |

To verify the values for the $c_i$’s, we refer the reader to [9 Section 6]: we choose the parameters from this article to be $H = H_0 - 1, \sigma_0 = 0.522817$ for $\sigma = 0.60$ and $\sigma_0 = 0.5208$ otherwise.
Table 3: Let $H = 3.061 \cdot 10^{10}$ and $b_0 \leq 9963$. For all $x \geq e^{b_0}$, $E(x) \leq \epsilon_0$.

| $b_0$ | $\sigma_0$ | $m$ | $\delta$ | $T_1$ | $\epsilon_0$ |
|-------|------------|-----|-----------|-------|-------------|
| 20    | 0.89       | 4   | $1.363 \cdot 10^{-6}$ | $T_0$ | $5.3668 \cdot 10^{-4}$ |
| 25    | 0.89       | 3   | $7.256 \cdot 10^{-6}$ | $T_0$ | $4.8208 \cdot 10^{-5}$ |
| 30    | 0.89       | 2   | $2.811 \cdot 10^{-6}$ | $T_0$ | $5.6679 \cdot 10^{-6}$ |
| 35    | 0.91       | 3   | $1.751 \cdot 10^{-7}$ | $16739408$ | $7.4457 \cdot 10^{-7}$ |
| 40    | 0.92       | 5   | $2.142 \cdot 10^{-8}$ | $245176468$ | $8.6347 \cdot 10^{-8}$ |
| 45    | 0.92       | 13  | $3.910 \cdot 10^{-9}$ | $4085373679$ | $1.0358 \cdot 10^{-8}$ |
| 50    | 0.93       | 23  | $3.116 \cdot 10^{-9}$ | $9667437397$ | $2.3643 \cdot 10^{-9}$ |
| 55    | 0.93       | 24  | $3.105 \cdot 10^{-9}$ | $10162544235$ | $1.6783 \cdot 10^{-9}$ |
| 60    | 0.93       | 24  | $3.099 \cdot 10^{-9}$ | $10182181286$ | $1.6191 \cdot 10^{-9}$ |
| 65    | 0.94       | 24  | $3.093 \cdot 10^{-9}$ | $10201894453$ | $1.6114 \cdot 10^{-9}$ |
| 70    | 0.94       | 24  | $3.087 \cdot 10^{-9}$ | $10221684178$ | $1.6081 \cdot 10^{-9}$ |
| 75    | 0.94       | 24  | $3.082 \cdot 10^{-9}$ | $10238234420$ | $1.6052 \cdot 10^{-9}$ |
| 80    | 0.95       | 24  | $3.225 \cdot 10^{-9}$ | $10254838399$ | $1.6025 \cdot 10^{-9}$ |
| 85    | 0.95       | 24  | $3.071 \cdot 10^{-9}$ | $10274834474$ | $1.5997 \cdot 10^{-9}$ |
| 90    | 0.95       | 24  | $3.066 \cdot 10^{-9}$ | $10291557599$ | $1.5969 \cdot 10^{-9}$ |
| 95    | 0.95       | 24  | $3.061 \cdot 10^{-9}$ | $10308335305$ | $1.5942 \cdot 10^{-9}$ |
| 100   | 0.95       | 24  | $3.056 \cdot 10^{-9}$ | $10325167860$ | $1.5916 \cdot 10^{-9}$ |
| 200   | 0.97       | 23  | $2.960 \cdot 10^{-9}$ | $10175863512$ | $1.5422 \cdot 10^{-9}$ |
| 300   | 0.97       | 23  | $2.866 \cdot 10^{-9}$ | $10508919281$ | $1.4953 \cdot 10^{-9}$ |
| 400   | 0.98       | 22  | $2.769 \cdot 10^{-9}$ | $10360124846$ | $1.4476 \cdot 10^{-9}$ |
| 500   | 0.98       | 21  | $2.674 \cdot 10^{-9}$ | $10193677612$ | $1.4006 \cdot 10^{-9}$ |
| 600   | 0.98       | 20  | $2.579 \cdot 10^{-9}$ | $10015840574$ | $1.3545 \cdot 10^{-9}$ |
| 700   | 0.98       | 20  | $2.492 \cdot 10^{-9}$ | $10364671352$ | $1.3081 \cdot 10^{-9}$ |
| 800   | 0.98       | 19  | $2.397 \cdot 10^{-9}$ | $10181118220$ | $1.2616 \cdot 10^{-9}$ |
| 900   | 0.98       | 18  | $2.303 \cdot 10^{-9}$ | $9979294107$ | $1.2154 \cdot 10^{-9}$ |
| 1000  | 0.98       | 17  | $2.209 \cdot 10^{-9}$ | $9761696912$ | $1.1695 \cdot 10^{-9}$ |
| 1500  | 0.98       | 14  | $1.753 \cdot 10^{-9}$ | $9882930682$ | $9.3929 \cdot 10^{-10}$ |
| 2000  | 0.99       | 10  | $1.293 \cdot 10^{-9}$ | $9091299627$ | $7.1125 \cdot 10^{-10}$ |
| 2500  | 0.99       | 6   | $8.300 \cdot 10^{-10}$ | $7664220686$ | $4.8137 \cdot 10^{-10}$ |
| 3000  | 0.99       | 2   | $3.000 \cdot 10^{-10}$ | $4992468020$ | $2.2211 \cdot 10^{-10}$ |
| 3500  | 0.99       | 2   | $9.200 \cdot 10^{-11}$ | $14198916944$ | $6.6209 \cdot 10^{-11}$ |
| 4000  | 0.99       | 2   | $2.700 \cdot 10^{-11}$ | $26575655437$ | $1.9689 \cdot 10^{-11}$ |
| 4500  | 0.99       | 2   | $7.810 \cdot 10^{-12}$ | $30196651346$ | $5.8563 \cdot 10^{-12}$ |
| 5000  | 0.99       | 2   | $2.320 \cdot 10^{-12}$ | $30572809972$ | $1.7434 \cdot 10^{-12}$ |
| 6000  | 0.99       | 2   | $2.100 \cdot 10^{-13}$ | $30609694715$ | $1.5457 \cdot 10^{-13}$ |
| 7000  | 0.99       | 2   | $1.826 \cdot 10^{-14}$ | $30609997695$ | $1.3693 \cdot 10^{-14}$ |
| 8000  | 0.99       | 2   | $1.618 \cdot 10^{-15}$ | $3060999985$ | $1.2135 \cdot 10^{-15}$ |
| 9000  | 0.99       | 2   | $1.434 \cdot 10^{-16}$ | $3060999995$ | $1.0755 \cdot 10^{-16}$ |
| 9963  | 0.99       | 2   | $1.390 \cdot 10^{-17}$ | $3060999998$ | $9.5309 \cdot 10^{-18}$ |

For $45 \leq b_0 \leq 2000$ we use the values of $c_i$’s from the right column of Table 3. We use the left values otherwise.
Table 4: Let $H = 2445999556030$ and $b_0 \leq 13906$. For all $x \geq e^{b_0}$, $E(x) \leq \epsilon_0$.

| $b_0$ | $\sigma_0$ | $m$ | $\delta$ | $T_1$ | $\epsilon_0$ |
|-------|----------|----|----------|-------|---------------|
| 20    | 0.88     | 4  | $1.363 \cdot 10^{-6}$ | $T_0$ | $5.3688 \cdot 10^{-4}$ |
| 25    | 0.89     | 3  | $7.256 \cdot 10^{-6}$  | $T_0$ | $4.8208 \cdot 10^{-5}$ |
| 30    | 0.89     | 2  | $2.806 \cdot 10^{-6}$  | $T_0$ | $5.6646 \cdot 10^{-6}$ |
| 35    | 0.90     | 2  | $1.604 \cdot 10^{-7}$  | 11360452 | $7.0190 \cdot 10^{-7}$ |
| 40    | 0.91     | 3  | $1.600 \cdot 10^{-8}$  | 174242715 | $8.0214 \cdot 10^{-8}$ |
| 45    | 0.92     | 4  | $1.613 \cdot 10^{-9}$  | 2393630483 | $8.6997 \cdot 10^{-9}$ |
| 50    | 0.93     | 7  | $2.058 \cdot 10^{-10}$ | 36960925828 | $9.4602 \cdot 10^{-10}$ |
| 55    | 0.96     | 21 | $5.079 \cdot 10^{-11}$ | 532313030046 | $1.1243 \cdot 10^{-10}$ |
| 60    | 0.96     | 28 | $4.807 \cdot 10^{-11}$ | 770935427426 | $3.2156 \cdot 10^{-11}$ |
| 65    | 0.96     | 29 | $4.801 \cdot 10^{-11}$ | 801857986418 | $2.5430 \cdot 10^{-11}$ |
| 70    | 0.96     | 29 | $4.795 \cdot 10^{-11}$ | 802859999396 | $2.4849 \cdot 10^{-11}$ |
| 75    | 0.96     | 29 | $4.789 \cdot 10^{-11}$ | 803864521532 | $2.4773 \cdot 10^{-11}$ |
| 80    | 0.97     | 29 | $4.783 \cdot 10^{-11}$ | 804871562262 | $2.4738 \cdot 10^{-11}$ |
| 85    | 0.97     | 29 | $4.777 \cdot 10^{-11}$ | 805881313075 | $2.4707 \cdot 10^{-11}$ |
| 90    | 0.97     | 29 | $4.771 \cdot 10^{-11}$ | 806893237503 | $2.4677 \cdot 10^{-11}$ |
| 95    | 0.97     | 29 | $4.765 \cdot 10^{-11}$ | 807907891129 | $2.4647 \cdot 10^{-11}$ |
| 100   | 0.97     | 29 | $4.759 \cdot 10^{-11}$ | 808925101582 | $2.4618 \cdot 10^{-11}$ |
| 200   | 0.98     | 28 | $4.647 \cdot 10^{-11}$ | 797441603800 | $2.4065 \cdot 10^{-11}$ |
| 300   | 0.98     | 28 | $4.546 \cdot 10^{-11}$ | 815133603120 | $2.3543 \cdot 10^{-11}$ |
| 400   | 0.98     | 27 | $4.440 \cdot 10^{-11}$ | 802199639823 | $2.3021 \cdot 10^{-11}$ |
| 500   | 0.98     | 26 | $4.334 \cdot 10^{-11}$ | 788664950273 | $2.2506 \cdot 10^{-11}$ |
| 600   | 0.98     | 26 | $4.237 \cdot 10^{-11}$ | 806692808636 | $2.1998 \cdot 10^{-11}$ |
| 700   | 0.99     | 25 | $4.131 \cdot 10^{-11}$ | 792643976191 | $2.1480 \cdot 10^{-11}$ |
| 800   | 0.99     | 25 | $4.032 \cdot 10^{-11}$ | 812075384439 | $2.0969 \cdot 10^{-11}$ |
| 900   | 0.99     | 23 | $3.918 \cdot 10^{-11}$ | 762558970852 | $2.0443 \cdot 10^{-11}$ |
| 1000  | 0.99     | 23 | $3.818 \cdot 10^{-11}$ | 782528018219 | $1.9921 \cdot 10^{-11}$ |
| 1500  | 0.99     | 20 | $3.303 \cdot 10^{-11}$ | 774756126279 | $1.7342 \cdot 10^{-11}$ |
| 2000  | 0.99     | 17 | $2.788 \cdot 10^{-11}$ | 764936897224 | $1.4762 \cdot 10^{-11}$ |
| 2500  | 0.99     | 14 | $2.272 \cdot 10^{-11}$ | 752424086843 | $1.2118 \cdot 10^{-11}$ |
| 3000  | 0.99     | 11 | $1.755 \cdot 10^{-11}$ | 735757894330 | $9.5728 \cdot 10^{-12}$ |
| 3500  | 0.99     | 7  | $1.209 \cdot 10^{-11}$ | 618567513247 | $6.9073 \cdot 10^{-12}$ |
| 4000  | 0.99     | 4  | $6.800 \cdot 10^{-12}$ | 533755825076 | $4.2115 \cdot 10^{-12}$ |
| 4500  | 0.99     | 2  | $3.000 \cdot 10^{-12}$ | 576348240050 | $1.6588 \cdot 10^{-12}$ |
| 5000  | 0.99     | 2  | $8.400 \cdot 10^{-13}$ | 1334194702027 | $6.0522 \cdot 10^{-13}$ |
| 6000  | 0.99     | 2  | $1.036 \cdot 10^{-13}$ | 240190405983 | $7.7686 \cdot 10^{-14}$ |
| 7000  | 0.99     | 2  | $1.332 \cdot 10^{-14}$ | 244525025818 | $9.9890 \cdot 10^{-15}$ |
| 8000  | 0.99     | 2  | $1.713 \cdot 10^{-15}$ | 244598715821 | $1.2845 \cdot 10^{-15}$ |
| 9000  | 0.99     | 2  | $2.202 \cdot 10^{-16}$ | 2445999351095 | $1.6516 \cdot 10^{-16}$ |
| 10000 | 0.99    | 2  | $2.830 \cdot 10^{-17}$ | 2445999552648 | $2.1236 \cdot 10^{-17}$ |
| 13900 | 0.99    | 2  | $9.502 \cdot 10^{-21}$ | $H$ | $7.1265 \cdot 10^{-21}$ |

We only use the values of $c_i$'s from the left column of Table 2.
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