EQUIVARIANT HYPERBOLIZATION OF 3-MANIFOLDS
VIA HOMOLOGY COBORDISMS

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Abstract. The main result of this paper is that any 3-dimensional manifold with a finite group action is equivariantly, invertibly homology cobordant to a hyperbolic manifold; this result holds with suitable twisted coefficients as well. The following two consequences motivated this work. First, there are hyperbolic equivariant corks (as defined in previous work of the authors) for a wide class of finite groups. Second, any finite group that acts on a homology 3-sphere also acts on a hyperbolic homology 3-sphere. The theorem has other applications, including establishing the existence of an infinite number of hyperbolic homology spheres with a free \( \mathbb{Z}_p \) action that does not extend to any contractible manifold. A non-equivariant version yields an infinite number of hyperbolic integer homology spheres that bound integer homology balls but do not bound contractible manifolds. In passing, it is shown that the invertible homology cobordism relation on 3-manifolds is antisymmetric.

1. Introduction

The 3-dimensional smooth homology cobordism group \( \Theta_3^H \) is rather complicated, and not fully understood despite many advances coming from 4-dimensional gauge theory; see e.g. [17, 20]. It appears in the theory of higher-dimensional manifolds, and also features prominently in the study of smooth 4-manifolds. The Rohlin invariant gives an epimorphism from this cobordism group to \( \mathbb{Z}_2 \), and for a while this was all that was known about it. With the advent of gauge theory techniques [13] it was shown that \( \Theta_3^H \) is infinite [17] (e.g. it is an easy consequence of Donaldson’s diagonalization theorem [14] that the Poincare homology sphere represents an element of infinite order), indeed infinitely generated [20, 18]. There have been many results since on the structure of this group, and on its applications, including Manolescu’s spectacular resolution of the triangulation conjecture [33].

It is interesting to explore how homology cobordism interacts with geometric structures on 3-manifolds. For example, there exist homology 3-spheres that are not homology cobordant to Seifert fibered homology spheres; see [15], although the question of whether Seifert fibered spaces generate the homology cobordism group is still unsolved. In contrast, Myers [39] proved that every 3-manifold is homology cobordant to a hyperbolic manifold, and this result was later refined by Ruberman [36] to show that such cobordisms can be taken to be invertible; the latter result has been applied to construct exotic smooth structures on contractible 4-manifolds [1].

In this paper it is shown that any 3-manifold with a finite group action is equivariantly invertibly homology cobordant, with twisted coefficients, to a hyperbolic manifold. Even if the group is trivial, this refines the earlier work of Myers and Ruberman since it applies simultaneously to all covering spaces. As will be seen, this result has applications to 4-dimensional smooth topology, to the 3-dimensional space form problem, and may also be of interest in spectral geometry (cf. [5]).

Throughout we work implicitly in the category of smooth, compact, oriented manifolds; all group actions will be assumed to be effective and to preserve the given orientations. To state our main result, recall that a homology cobordism is a cobordism whose inclusions from the ends induce isomorphisms on integral homology. For non-simply connected manifolds there is a stronger notion of homology cobordism with twisted coefficients in any module over the group ring of the fundamental

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group. Also recall (see e.g. [49, 50, 46]) that a cobordism $\mathcal{P}$ from $M$ to $N$ is invertible if there is a cobordism $\mathcal{Q}$ from $N$ to $M$ with $\mathcal{P} \cup_\mathcal{Q} \mathcal{Q} \cong M \times I$; see Section 2 for details, and the Appendix for a proof that invertible homology cobordism is a partial order on 3-manifolds.

**Theorem A.** Any closed 3-manifold $M$ with an action of a finite group $G$ is equivariantly invertibly $\mathbb{Z}[\pi_1(M)]$-homology cobordant to a hyperbolic 3-manifold $N$ with a $G$-action by isometries.

We were led to this theorem by a question in 4-dimensional smooth topology. Consider the family of all finite groups that can act effectively on the boundary of some compact contractible 4-dimensional submanifold of $\mathbb{R}^4$; these include all finite subgroups of $SO(4)$. In a recent paper [1] we constructed for any such group $G$ a compact contractible 4-manifold $C$ with a $G$-action on its boundary and an embedding of $C$ in a closed 4-manifold $X$ such that removing $C$ from $X$ and regluing by distinct elements of $G$ produces distinct smooth 4-manifolds; related results were obtained by Tange [51] for $G$ finite cyclic and Gompf [21] for $G$ infinite cyclic. We call such a gadget a $G$-cork. In our construction $\partial C$ is reducible, and it was natural to ask if there are $G$-corks with irreducible or even hyperbolic boundaries. We refer to the latter as hyperbolic $G$-corks. Tange [52] has recently shown that his cyclic corks have irreducible boundaries, and (by computer calculations with HIKMOT [27]) that some are hyperbolic. As a consequence of Theorem A we will deduce:

**Corollary B.** There exist hyperbolic $G$-corks for any finite group $G$ that acts on the boundary of some compact contractible 4-dimensional submanifold of $\mathbb{R}^4$.

The proof will be given in Section 6 along with the following applications to low dimensional topology. We start with a hyperbolic version of a non-extension result for group actions due to Anvari and Hambleton [2].

**Corollary C.** For any Brieskorn homology sphere $\Sigma(a, b, c)$ and prime $p$ not dividing $abc$, there is a hyperbolic homology sphere $N(a, b, c)$ with a free action of $\mathbb{Z}_p$ such that $N(a, b, c)$ and $\Sigma(a, b, c)$ are $\mathbb{Z}_p$-equivariantly homology cobordant, and the action of $\mathbb{Z}_p$ does not extend over any contractible 4-manifold that $N(a, b, c)$ might bound.

We apply Theorem A in a non-equivariant setting to show that the difference between bounding an acyclic and contractible 4-manifold occurs for hyperbolic homology spheres.

**Corollary D.** There are an infinite number of hyperbolic integer homology spheres that bound integer homology balls but do not bound any contractible manifold.

The class of groups $\mathcal{G}$ that can act on some homology 3-sphere – hyperbolic or not – include the finite subgroups of $SO(4)$, but also a infinite proper subclass of the generalized quaternion groups of period 4 (as shown by Milgram [35] and Madsen [32]; see also [12, p.xi], [29]). It has been an open question since the early 1980s to determine exactly which groups lie in $\mathcal{G}$, and to say something about the geometric nature of the homology spheres on which they act. Theorem A sheds light on this last question, especially for free actions. The constructive part of this corollary is related to recent work of Bartel and Page [5].

**Corollary E.** Any finite group that acts on a homology 3-sphere also acts on a hyperbolic homology 3-sphere with equivalent fixed-point behavior. In particular, there exist infinitely many finite groups that are not subgroups of $SO(4)$, and so by geometrization do not act freely on the 3-sphere, but that do act freely on some hyperbolic homology 3-sphere.

In outline, the proof of Theorem A is similar to the proofs of the analogous theorems in [39] and [46]. Start with a Heegaard splitting of $M$ (of genus $\geq 2$) with gluing map $h$. Then replace each handlebody, viewed as the exterior of a trivial tangle in the 3-ball, with the exterior of an invertibly null-concordant hyperbolic tangle. To build the cobordism, glue the two concordances together by the map $h \times id$. The top of the cobordism will be hyperbolic by Thurston’s gluing theorem.
To make this construction $G$-equivariant requires some modifications of this argument, even in the case of a free action. In any case it is natural to start with a Heegaard splitting of the orbit space $M/G$, and then to replace the handlebodies with copies of the tangle exterior as in the outline above. One thus obtains an invertible cobordism from $M/G$ to a hyperbolic 3-manifold. Now if the action is free, then the induced $G$ cover is an invertible cobordism $P$ from $M$ to a hyperbolic 3-manifold $N$ that is equivariant with respect to the $G$-action. However, there is no reason that $P$ should be a homology cobordism, or indeed that $N$ should have the same homology as $M$. The issue is that while the tangle exteriors are $\mathbb{Z}$-homology equivalent to handlebodies, they are not necessarily homology equivalent with arbitrary (in this case $\mathbb{Z}[G]$) coefficients. This is of course familiar from knot theory; a covering space of a homology circle such as a knot complement need not be a homology circle. The resolution of this issue is to further decompose each handlebody into 0 and 1-handles. These handles will be replaced with ‘fake’ 0 and 1-handles that will be hyperbolic tangle exteriors. These are no longer homology handles, but rather homology handlebodies, but now one has control over their lifts.

We will begin with some standard tangle exteriors, referred to as atoms, then glue these together by a bonding process to make the fake handles, and finally glue these fake handles together to make fake handlebodies and relative cobordisms. This localization will ensure that the replacement is homology cobordant to a handlebody $H$ (with coefficients in $\mathbb{Z}[[\pi_1(H)]]$) and again Thurston’s gluing theorem will be used to create a closed hyperbolic manifold. With some additional work, this argument extends to the case when $G$ has some fixed points. In this setting the quotient $M/G$ will be an orbifold, and we will essentially be working with an orbifold Heegaard splitting.

In our proof we need tangles that are doubly slice and simple (a.k.a. hyperbolic). Furthermore the tangles should retain these properties as they are suitably glued together. These notions will be made precise in the next section. An elementary four-component tangle in the 3-ball with these properties, denoted $R_4$, is displayed in Figure 1a. Its $n$-component generalization $R_n$ is the lift of the generating arc $\alpha$ in the 3-ball shown in Figure 1b to the $n$-fold branched cover along the diameter $\delta$ perpendicular to the page. We will refer to the $R_n$ as atomic tangles; they were the key players in the last author’s construction of invertible homology cobordisms in [46]. The following technical proposition is extracted from Theorem 2.6 of [46].

![Atomic tangles](image)

**Figure 1.** Atomic tangles

**Proposition 1.1.** The atomic tangles $R_n$ are doubly slice for all $n$, and simple for $n \geq 3$.  

The proof is reviewed in the next section in the process of analyzing the more complicated tangles that arise in our constructions.
Tangles.

In this paper, a tangle refers to a union $T$ of finitely many disjoint arcs properly embedded in a 3-manifold $M$; closed loops are not allowed. Two tangles (necessarily with the same endpoints) are equivalent if they are isotopic rel boundary. A marking of $T$ is a collection of disjoint arcs in $\partial M$ joining the endpoints of each of its strands. Note that markings are not generally unique; two markings for $R_4$ are shown in Figure 2, where the part of the tangle inside the ball is drawn in tastefully muted tones. In subsequent pictures of tangles $T \subset M$, the part inside $M$ will be drawn schematically or omitted entirely, but a marking may be drawn to indicate which strands are paired up inside.

A tangle $T \subset M$ is trivial if it is boundary parallel, meaning the strands of $T$ together with the arcs in a suitable marking $A$ of $T$ bound disjoint disks in $M$ meeting $\partial M$ in the markings. More generally, a tangle $T$ with $n$ components is a boundary tangle if the strands of $T$ together with a marking bound $n$ disjoint surfaces in $M$ meeting $\partial M$ in the markings. The union of these surfaces will be called a Seifert surface for $T$, with outer boundary $A$ and inner boundary $T$; it will be a trivial Seifert surface if all components are disks. For example, the atomic tangles $R_n$ introduced above (and all other tangles we construct in this paper) are boundary tangles, with the obvious Seifert surfaces consisting of $n$ genus one surfaces. In fact, these Seifert surfaces satisfy an additional condition, captured in the following definition of an ‘elementary tangle’.

**Definition 2.1.** Let $T \subset M$ be a boundary tangle that has a Seifert surface $F$ with a geometric symplectic basis (embedded curves $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ representing a basis for $H_1(F)$ with $|\alpha_i \cap \beta_i| = \delta_{ij}$) satisfying the two conditions

a) the $\alpha$ curves bound disjoint disks in $M$ that intersect $F$ only in arcs transverse to the $\beta$ curves,

b) the $\beta$ curves bound disjoint disks in $M$ that intersect $F$ only in arcs transverse to the $\alpha$ curves.

Equivalently, the $\alpha$ curves should have zero linking number with their pushoffs in $F$, and should bound disjoint disks in $M$ that lie in the complement of a collection of arcs $\gamma_i$ in $F$ from $\alpha_i \cap \beta_i$ to the outer boundary of $F$, and similarly for the $\beta$ curves. Then we say that the tangle $T \subset M$ is elementary. For example, as noted above, the atomic tangles $R_n \subset B^3$ are elementary.

Tangle Sums.

Tangles can be added together in a variety of ways. For the present purposes, the following notion of a ‘tangle sum’ will suffice.

**Definition 2.2.** Given a pair of tangles $T_i \subset M_i$ for $i = 0, 1$, choose gluing disks $D_i \subset \partial M_i$ containing an equal number of tangle endpoints, all at interior points of the $D_i$. Then glue $M_1$ to $M_0$ by a diffeomorphism $h: D_1 \to D_0$ that identifies these endpoints without creating any loops. The result is the tangle sum $T_0 \cup_h T_1$, a tangle in the boundary connected sum $M_0 \cup_h M_1$. The common image of the $D_i$ under the gluing is a properly embedded disk $D \subset M_0 \cup_h M_1$ called the
splitting surface for the sum. More generally we allow the $D_i$, and thus $D$, to be unions of more than one disk.

To propagate hyperbolic structures on tangles to their tangle sums (see Proposition 2.12 below), we will use a restricted class of ‘simple’ tangle sums.

**Definition 2.3.** A tangle sum $T = T_0 \cup_h T_1$ as defined above is simple if each gluing disk in $D_i$ contains at least two tangle endpoints, and each component of $\partial M_i - D_i$ contains at least two tangle endpoints if it is a sphere and one if it is a disk. In other words, $D_i - T_i$ contains no disk components, and $(\partial M_i - D_i) - T_i$ contains no sphere or disk components.

An example is shown in Figure 3, where the thickened arcs indicate both the pairing of tangle endpoints inside the 3-manifold and the new marking of the tangle sum. Note that in general, the marking is only well-defined up to Dehn twists about the boundary of the splitting disk. In the tangle sums performed in Section 4, this issue does not arise, as we have a preferred gluing.

**Figure 3.** A simple tangle sum $T_0 \cup_h T_1 \subset M = M_0 \cup_h M_1$

**Invertible cobordisms and doubly slice tangles.**

Invertible cobordisms of manifolds and knots have been studied since the 1960s; see for example [19, 23, 49, 50]. Recall first that if if $M$ and $N$ are manifolds of the same dimension (with boundaries identified by a diffeomorphism $h$ if they are nonempty), then a cobordism from $M$ to $N$ is a manifold $P$ with $\partial P = -M \sqcup N$ (or in the bounded case, $\partial P = -M \cup_h N$, in which case $P$ can be viewed as a relative cobordism from $M$ to $N$ so that the vertical part of $\partial P$ is diffeomorphic to $\partial M \times I$, extending $h$ at the top). This cobordism $P$ is said to be invertible if there is a cobordism $Q$ from $N$ to $M$ such that $P \cup N Q \cong M \times I$. We then say that $M \times I$ is split along $N$, and call $Q$ an inverse of $P$; this inverse need not be unique, nor invertible (see Remark 2.4 below). Familiar examples of 3-dimensional invertible cobordisms arise from homology 3-spheres $M$ that bound contractible 4-manifolds $W$ whose doubles are the 4-sphere; the complement in $W$ of an open 4-ball is then an invertible cobordism from the 3-sphere to $M$.

Similar language applies to concordances of knots, links, and tangles, where for tangles the concordance is required to be a product along the boundary. In particular, a concordance from $S$ to $T$ is invertible if it can be followed by a concordance from $T$ to $S$ to produce a product concordance from $S$ to itself. If $S$ is an unknot, unlink or trivial tangle, then $T$ is said to be invertibly null-concordant or doubly slice.

**Remark 2.4.** The relations of invertible cobordism and concordance are clearly reflexive and transitive, but generally not symmetric. For example, any sphere is invertibly cobordant to a disjoint union of two spheres, but not conversely, and analogously an unknot is invertibly concordant to a two component unlink, but not conversely. In fact, for closed manifolds of dimension 3 or less, invertible cobordism is an antisymmetric relation, and thus a partial order. For hyperbolic 3-manifolds this follows from degree and volume considerations (cf. [6, Theorem C.5.5]) and a general proof for 3-manifolds is given in the appendix (where it is also noted that antisymmetry fails in higher dimensions).
The focus here is on tangles. The first part of the following proposition is straightforward from the definitions, while the second part is a relative version of a well known result of Terasaka and Hosokawa [54]; compare [46, Proof of 2.6].

Proposition 2.5. a) Tangle sums of doubly slice tangles are doubly slice.
b) Elementary tangles (and in particular all atomic tangles \( \mathcal{R}_n \)) are doubly slice.

Proof of b). Let \( T \subset M \) be an \( n \)-stranded elementary tangle, and \( F \) be a Seifert surface for \( T \) with outer boundary \( A \) and geometric symplectic basis \( \{ \alpha_i, \beta_i \} \) as in Definition 2.1. View \( F \) as \( n \) disjoint disks with bands attached along the geometric basis. Removing the \( \beta \) bands from \( F \) yields a surface \( F_0 \subset M \) that can be capped off with (parallel copies of) the disks bounded by the \( \alpha \) curves, provided by [22]b, to form a trivial Seifert surface \( E_0 \subset M \) for a trivial tangle \( U_0 \). Similarly form \( F_1 \subset M \) by removing the \( \alpha \) bands from \( F \), and then cap off with disks bounded by the \( \beta \) curves to produce another trivial Seifert surface \( E_1 \subset M \) for a trivial tangle \( U_1 \). By construction, \( E_0 \) and \( E_1 \) have the same outer boundary \( A \) as \( F \).

Now build a 3-dimensional cobordism \( \mathbb{P} \subset M \times [0, 1/2] \) from \( E_0 \subset M \times 0 \) to \( F \subset M \times 1/2 \), with outer lateral boundary \( A \times [0, 1/2] \), as follows. Start with \( \mathbb{P} = F \times [0, 1/2] \) with 2-handles attached ambiently in \( F \times [-1/2, 0] \) along the \( \alpha_i \subset F \times 0 \). To arrange for \( \mathbb{P} \) to lie in \( M \times [0, 1/2] \), push it up from its bottom level \( E_0 \). After this adjustment, a top down movie of the inner lateral boundary \( P \) of \( \mathbb{P} \) (the closure of the complement in \( \partial \mathbb{P} \) of the union of \( A \times [0, 1/2] \), \( E_0 \) and \( T \)) is described as follows: Start with \( T \). Then perform saddle moves along the cocores of the \( \alpha \) bands, tracing out a genus zero cobordism from \( T \) to \( \partial F_0 \). Finish the movie by capping off the \( \alpha_i \) with disjoint disks. Note that \( P \) is a concordance from \( U_0 \) to \( T \).

Similarly build a cobordism \( \mathbb{Q} \subset M \times [1/2, 1] \) from \( F \subset M \times 1/2 \) to \( E_1 \subset M \times 1 \) with outer lateral boundary \( A \times [1/2, 1] \) and inner lateral boundary \( Q \), a concordance in \( M \times [1/2, 1] \) from \( T \) to \( U_1 \). Then \( \mathbb{P} \cup \mathbb{Q} \) is a product cobordism. Indeed, since \( |\alpha_i \cap \beta_j| = \delta_{ij} \), the 1-handles (upside down 2-handles) in \( \mathbb{P} \) are cancelled by the 2-handles in \( \mathbb{Q} \), and so \( \mathbb{P} \cup \mathbb{Q} \) is in fact a union of 3-balls. It follows that \( P \) is the desired null-concordance of \( T \), with inverse \( Q \). □

Remark 2.6. Note that each tangle component has a preferred meridian, defined as the boundary of a normal disk. Each component \( J \) of a marked boundary tangle also has a longitude, determined by a Seifert surface and consisting of two arcs \( u \cup v \) with \( u \) lying along the marking and \( v \) running along \( J \). Now suppose that two marked components \( T_0 \) and \( T_1 \) are concordant in \( M \times I \), via a tangle concordance that is homologous (rel boundary) to the outer lateral boundary \( A \times I \). Fix a component \( J_0 \) of \( T_0 \) and its corresponding component \( J_1 \) in \( T_1 \). Note that the longitude \( u_0 \cup v_0 \) for \( J_0 \) is freely homotopic to the longitude \( u_1 \cup v_1 \) for \( J_1 \) along a cylinder part of which is parallel to the concordance and the other part of which travels along the boundary. In particular, a longitude on one end of the concordance determines a preferred longitude on the other end.

A key consequence of this remark, to be used in the proof of Theorem [A] in Section 5 is that for concordant tangles \( T_0 \) and \( T_1 \) in \( M \), as above, there is a canonical identification between the boundary of \( M - \text{int} N(T_0) \) and that of \( M - \text{int} N(T_1) \).

Homology cobordisms.

Recall that a homology cobordism is a cobordism for which the inclusions from the ends induce homology isomorphisms. It is a standard and very useful observation that a concordance between knots or links induces a homology cobordism between their complements; see for instance [22]. We note a somewhat stronger property for the concordances constructed in the preceding subsection. Let \( X \) be the exterior of a tangle \( T \) in a 3-manifold \( M \). The inclusion of \( X \hookrightarrow M \) induces a homomorphism \( \pi_1(X) \to \pi_1(M) \). Thus any module \( V \) over \( \mathbb{Z}[\pi_1(M)] \) is also a module over \( \mathbb{Z}[\pi_1(X)] \), so we can consider the twisted homology \( H_* (X; V) \).
Lemma 2.7. Let $T_0$ and $T_1$ be tangles in a compact 3-manifold $M$, with exteriors $X_0$ and $X_1$, and $C$ be an invertible concordance in $M \times I$ from $T_0$ to $T_1$, with exterior $X$. Then $X$ is an invertible homology cobordism from $X_0$ to $X_1$, with twisted coefficients in any module $V$ over $\mathbb{Z}[\pi_1(M)]$.

In particular, any cobordism from a handlebody $H$ to the exterior of a doubly slice tangle in the 4-ball induced by an invertible null-concordance of the tangle is an invertible homology cobordism with twisted coefficients in any $\mathbb{Z}[\pi_1(H)]$-module.

Proof. The idea is implicit in [9], but here is a quick proof for the reader’s convenience. By hypothesis, $X = M \times I - \text{int}(N)$ and $X_i = X \cap (M \times i)$ for some tubular neighborhood $N$ of $C$. Set $N_i = N \cap (M \times i)$, and note that the restriction of the coefficient system $V$ to $\partial N$ (and similarly for the $\partial N_i$) is trivial, because it extends over $N$. Then for $i = 0$ and 1, there are relative Mayer-Vietoris sequences

$$\cdots \rightarrow H_*(X \cap N, X_i \cap N_i) \rightarrow H_*(X, X_i) \oplus H_*(N, N_i) \rightarrow H_*(M \times I, M_i) \rightarrow \cdots,$$

with $V$-coefficients understood throughout, in which all the groups except $H_*(X, X_i)$ are obviously zero. Thus $H_*(X, X_i) = 0$ as well, and the result follows. $\square$

Thurston’s hyperbolization and simple tangles.

To show that the 3-manifolds we produce are hyperbolic, we will use Thurston’s hyperbolization theorem for Haken 3-manifolds [55, 28, 40, 41] and standard techniques for checking that 3-manifolds obtained by gluing satisfy the hypotheses of his theorem.

To state Thurston’s theorem and the relevant gluing results in a unified way, we will use the phrase essential surface in a 3-manifold $M$ to mean a compact, connected, incompressible, nonboundary-parallel, properly embedded surface in $M$ (recall that all manifolds are assumed oriented). With this language, a 3-manifold that is compact, irreducible and boundary-irreducible (see Waldhausen [58]) is Haken if it contains an essential surface, and simple if it contains no essential tori, annuli or disks; we also refer to a tangle as simple or Haken if its exterior has that property. Thurston’s theorem for closed 3-manifolds asserts that these conditions together, i.e. the existence of an essential surface of negative euler characteristic, imply the existence of a hyperbolic structure.

Theorem 2.8. (Thurston) Any closed simple Haken manifold admits a complete hyperbolic metric.

The hyperbolization result of [46] relied on proving that the atomic tangles $R_\alpha$ described in §1 are simple (Proposition 1.1). We need a way to see that certain manifolds built from these atoms are also simple. The necessary gluing results can be found in Myer’s work [38, 39].

† Here is a sketch of the argument from [46] (it would also be nice to have a direct proof using Lemma 2.10 below): Recall from the end of §1 that $R_\alpha$ is a branched cover of a solid torus $V = B^3 - \delta$ branched along a knotted arc $\alpha$. Let $X = V - \text{int}(N)$, where $N$ is a tubular neighborhood of $\alpha$, and set $P = \partial X \cap N$. Gluing results from [46] can be
Definition 2.9. Let $M$ be a compact, irreducible 3-manifold and $F$ be a compact surface in $\partial M$. The pair $(M, F)$ is simple if it satisfies the following three properties:

a) $F$ and $\partial M - F$ are incompressible in $M$.

b) $F$ contains no essential tori, annuli or disk component.

c) $M$ contains no essential tori, annuli disjoint from $\partial F$, or disks intersecting $F$ in a single arc.

The pair is very simple if it satisfies a), b) and d), where d) extends c) by also disallowing essential disks in $M$ that intersect $F$ in two disjoint arcs. Note that $M$ is simple if and only if $(M, \emptyset)$ is simple, or equivalently $(M, \emptyset)$ is very simple.

These notions of simple and very simple are exactly Myers’ Properties $B'$ and $C'$, and feature in the following result [39, Lemma 2.5], proved in [38]:

Lemma 2.10. If $(M_0, F_0)$ is very simple, $(M_1, F_1)$ is simple, and $h: F_1 \to F_0$ is a homeomorphism, then $M_0 \cup_h M_1$ is simple and Haken.

To identify new gluing regions in the boundary of a simple manifold, we will use the following:

Lemma 2.11. If $M$ is simple and $F$ is a compact surface in $\partial M$, then $(M, F)$ is very simple if and only if $F$ has no torus, annulus or disk components, and $\partial M - F$ has no disk components.

Proof. For the forward implication, note that a disk component in $\partial M - F$ gives a compression of $F$ in $\partial M$, and thus in $M$, contradicting property a) in Definition 2.9 of ‘very simple’. For the converse we first verify 2.9b, so let $D$ be a properly embedded disk in $M$ with boundary in $F$ or in $\partial M - F$. Since $M$ is simple, $D$ is inessential in $M$, so $\partial D$ bounds a disk $E$ in $\partial M$. But then $E$ must lie entirely in $F$ or $\partial M - F$, since neither has disk components. Thus $F$ and $\partial M - F$ are incompressible in $M$. It remains to verify 2.9d, which is immediate from the hypotheses, and 2.9c, which follows from the fact that $M$ is simple, precluding the existence of any essential tori, annuli or disks whatsoever. □

We illustrate the use of these gluing techniques in two situations, the first when the gluing surfaces have boundary, and the second when they are closed. For the bounded case, consider a simple tangle sum $T = T_0 \cup_h T_1$ of a pair of simple tangles $T_i \subset M_i$, as defined above, with gluing disks $D_i \subset \partial M_i$. Write $X_i$ for the exterior of $T_i$ in $M_i$, and set $Y_i = D_i \cap X_i$. By Definition 2.3 this means that $Y_i$ has no annulus or disk components, and $\partial X_i - Y_i$ has no disk components. Thus both pairs $(X_i, Y_i)$ are very good by Lemma 2.11 and so the exterior $X_0 \cup_h X_1$ of in $M_0 \cup_h M_1$ is simple (and Haken) by Lemma 2.10. This proves the first part of the following result; the second part is an immediate consequence of Lemmas 2.10 and 2.11 and Thurston’s theorem.

Proposition 2.12. a) Any simple sum of simple tangles is a simple tangle.

b) Any closed 3-manifold obtained by gluing together a pair of simple 3-manifolds is hyperbolic. □

Our proof of Theorem A will rely on this proposition in the following way: Starting with a Heegaard splitting $H_0 \cup_h H_1$ of a 3-manifold $M$, we will apply Propositions 2.12a and 2.12b repeatedly to construct simple, doubly slice ‘molecular’ tangles in $H_0$ and $H_1$, with an equal number of strands. Then gluing their exteriors $H_0$ and $H_1$ together by a natural map $h_{hyp}$ induced by $h$ will yield a hyperbolic manifold $N$, by Proposition 2.12b. If the Heegaard splitting of $M$ is equivariant with respect to the action of a finite group $G$ on $M$ (in a strong sense explained in the next section), and the simple tangles in the $H_i$ are suitably chosen, then $M$ and $N$ will be invertibly homology cobordant. The details of this construction will be explained in the next three sections.

used to show that $X$ is hyperbolic in a certain sense, in particular the pair $(X, P)$ is a ‘pared manifold’. The result then follows by standard arguments about incompressible surfaces in branched covers, using equivariant versions of the loop, sphere, annulus and torus theorems (see Theorem 2.10 in [36]).
To complete the proof of Theorem A we will need to show that the orbifold \( M/G \) is hyperbolic. There are two routes to this: one could either expand the discussion above to include a definition of simple orbifold pairs, and argue that the gluing results hold in this more general setting, or one could make use of Thurston’s orbifold theorem \([8, 11]\). We follow the latter route. In fact, we need only the following special case (see for example \([6, \text{Theorem C.5.6}]\)).

**Theorem 2.13.** Any action of a finite group \( G \) on a closed hyperbolic 3-manifold \( M \) is conjugate to an action by isometries, and so \( M/G \) is a hyperbolic orbifold.

### 3. Equivariant Heegaard splittings

Given a closed 3-manifold \( M \) with an action of a finite group \( G \), we seek to replace \( M \) with a hyperbolic manifold with a \( G \)-action. We assume without loss of generality that \( M \) is connected. The strategy is to find a \( G \)-equivariant Heegaard splitting \( H_0 \cup H_1 \) of \( M \) (the goal of this section), and then to replace each handlebody \( H_i \) with a fake handlebody \( \mathcal{H}_i \) with a \( G \)-action, chosen so that the glued up manifold \( H_0 \cup \mathcal{H}_1 \) is hyperbolic (treated in the next section). This replacement process will require a further decomposition of the \( H_i \) into 0 and 1-handles that will be regarded as part of the structure of the Heegaard splitting. Our exposition will be facilitated by passing back and forth between \( M \) and its quotient \( M/G \), and so for clarity and notational economy we henceforth denote the image of any subset \( K \) of \( M \) under the quotient map \( M \to M/G \) by \( \overline{K} \). In particular \( \overline{M} = M/G \).

If \( G \) acts freely, then we could simply lift a Heegaard splitting of the quotient manifold \( \overline{M} \) with an arbitrary handle structure on the two sides. When \( G \) has fixed points, the quotient \( \overline{M} \) is an orbifold, albeit a good one and so still a 3-manifold. In this case we will need the Heegaard splitting of \( \overline{M} \) and the associated handle structures of the sides to be adapted to the orbifold structure, cf. \([34, 59]\) for a related discussion of orbifold handlebodies. This splitting is constructed as follows.

Observe that the \( G \)-action on \( M \) is locally linear (since it is smooth) and orientation preserving (by hypothesis). Thus the stabilizer \( G_x \) of any point \( x \in M \) is isomorphic to a finite subgroup of \( \text{SO}(3) \), so is either cyclic, dihedral, or one of the three symmetry groups of the Platonic solids, acting linearly on a 3-ball about \( x \). It follows that the *singular set* \( \Delta \) of all points in \( M \) with nontrivial stabilizers forms a graph in \( M \), which may include edges with endpoints identified and circle components with no vertices. The vertices of \( \Delta \) are the points with noncyclic stabilizers, and each (open) edge is made up of points with the same nontrivial cyclic stabilizer. To record this fact more precisely, we assign labels to these vertices and edges. Since the noncyclic finite subgroups of \( \text{SO}(3) \) are all triangle groups (the dihedral group \( D_{2n} \) is \( \Delta(2, 2, n) \), while the tetrahedral, octahedral and icosahedral groups are respectively \( \Delta(2, 3, 3) \approx A_4 \), \( \Delta(2, 3, 4) \approx S_4 \) and \( \Delta(2, 3, 5) \approx A_5 \)), assign the integer triple \((p, q, r)\) to each vertex \( x \) of \( \Delta \) with \( G_x \approx \Delta(p, q, r) \), and assign the integer \( n \) to any edge whose stabilizer is isomorphic to \( C_n \).

Now consider the image \( \overline{\Delta} \) of \( \Delta \) in the quotient orbifold \( \overline{M} \). This is also a graph, with labels inherited from \( \Delta \). Since it is locally the singular set of a finite linear quotient of the 3-ball, \( \overline{\Delta} \) is in fact a *trivalent* graph, with each vertex labeled by the triple of labels on the edges incident to it. We call \( \overline{\Delta} \) the *branch locus* as \( M \) is the branched cover of \( \overline{M} \) along \( \overline{\Delta} \), with branching indices given by the labels. The quotient map \( \Delta \to \overline{\Delta} \) is illustrated in Figure 5 near a tetrahedral vertex in \( \Delta \).

To build the Heegaard splitting of \( \overline{M} \), first extend the branch locus \( \overline{\Delta} \) to a larger trivalent graph \( \overline{\Delta}_0 \subset \overline{M} \) whose complement is an open handlebody. To accomplish this, add new 1-labeled edges to \( \overline{\Delta} \) corresponding to all the 1-handles of a relative handlebody structure of the complement of a regular neighborhood of \( \overline{\Delta} \), with endpoints chosen to lie at interior points of the edges in \( \overline{\Delta} \). Of course some edges of \( \overline{\Delta} \) may be subdivided in this process. If \( e \) is such an edge, with label \( n \), then label each new edge of \( \overline{\Delta}_0 \) lying in \( e \) with \( n \), and each new vertex lying on \( e \) with \((1, n, n)\). Note that \( M \) is still a branched cover of \( \overline{M} \) along \( \overline{\Delta}_0 \), so we call \( \overline{\Delta}_0 \) the *extended branch locus*.
Figure 5. The picture of $\Delta \rightarrow \Xi$ near a point with tetrahedral stabilizer

Now let $\overline{\Pi}_0$ be a regular neighborhood of the extended branch locus $\overline{\Sigma}_0$, built as a handlebody with 0 and 1-handles corresponding in the usual way to the vertices and edges of $\Sigma_0$. The closure $\overline{\Pi}_1$ of the complement of $\overline{\Pi}_0$ in $\overline{M}$ is another handlebody of the same genus, which we can decompose into 0 and 1-handles using an arbitrarily chosen trivalent ‘trivially’ labeled spine (label all the edges with 1 and all the vertices with $(1,1,1)$). This gives an orbifold Heegaard splitting

$$\overline{M} = \overline{\Pi}_0 \cup \overline{\Pi}_1$$

with $\overline{\Sigma}_0 \subset \overline{\Pi}_0$, in which each of the handlebodies is equipped with a specific handle structure reflecting the orbifold structure on $\overline{M}$. The lifts of the $\overline{H}_i$ will then be equivariant handlebodies $H_i$, equipped with their lifted handle structures, giving the desired equivariant Heegaard splitting

$$M = H_0 \cup H_1.$$ 

In the next section, the orbifold Heegaard splitting will be used as a template to build a stabilized, equivariant ‘fake’ Heegaard splitting $\mathcal{H}_0 \cup \mathcal{H}_1$ of the desired hyperbolic 3-manifold.

4. Replacement Handlebodies

In this section we describe how to insert equivariant doubly slice hyperbolic tangles $T_0$ and $T_1$ into the handlebodies in the equivariant Heegaard splitting $M = H_0 \cup H_1$ constructed in §3. These ‘molecular’ tangles will be built up using tangle sums from ‘atomic’ tangles placed in the 0-handles and 1-handles in the decompositions of $H_0$ and $H_1$ described in the last section. We view the exteriors of the tangles in each of $H_0$ and $H_1$ as a replacement for those handlebodies (as in [46]). Each exterior is a simple and therefore hyperbolic homology handlebody that comes equipped with an equivariant invertible cobordism from a genuine handlebody. In §5 we will show how to glue the two complements together, and complete the proof of Theorem A.

To achieve equivariance, it is convenient to work downstairs in the orbifold $\overline{M}$ and then lift all of the constructions back up to $M$. To this end we first describe how to place atomic tangles in the orbifold 1-handles, then explain the somewhat more complicated tangle sums that are inserted in the orbifold 0-handles, next show how to assemble the orbifold tangles into a single tangle $\overline{T}_i$ in each orbifold handlebody $\overline{\Pi}_i$, and finally lift these tangles up to the handlebodies in $M$.

Orbifold 1-handles.

From the discussion above, an orbifold 1-handle of degree $n$ is a pair $(D^2 \times I, D^2 \times \partial I)$ where the orbifold singularity is the $n$-labeled arc $\{0\} \times I$. For any such 1-handle, insert a copy of the tangle $\mathcal{R}_4$ (defined in the introduction) so that the singular set corresponds to the diameter $\delta$ in Figure 4, and so that the endpoints of two of the tangle components lie on the disk $D^2 \times \{0\}$ of the attaching region while the endpoints of the other two components lie on $D^2 \times \{1\}$. Thus any lift of this 1-handle in $H_0$ or $H_1$ has a copy of $\mathcal{R}_{4n}$ inserted with half its endpoints on each disk of the attaching region of the handle.
Orbifold 0-handle.

An orbifold 0-handle is a 3-ball neighborhood of a singularity corresponding to a triangle group, as described above. The intersection of the extended branch locus (see §3) with the sphere on the boundary is 3 points, which form the vertices of a triangle \( \overline{K} \) on the sphere. The edges of \( \overline{K} \) are simply 3 arcs joining these points, as drawn below in Figure 6. We remark that a similar construction can be done with an arbitrary graph \( K \) on the boundary of a 3-manifold, and will make use of this generalization when we glue handles below.

![Figure 6. The triangle \( \overline{K} \) lying on the surface to an orbifold 0-handle.](image)

Now insert a 12-stranded tangle \( R_{\overline{K}} \) in a boundary collar of the 0-handle, as follows. Place a copy of \( R_4 \) near each vertex of \( \overline{K} \). The ball in which \( R_4 \) lies is drawn as a prism over a bigon, as shown on the left side Figure 7, with its top face on the surface of the 0-handle, and its bottom face on the inner boundary of the collar. The tangle is placed in this prism so that the diameter \( \delta \) coincides with the singular set, and so that its strands run from the top bigon to the vertical sides inside the 0-handle, two to each side. Similarly, each edge is surrounded by a long thin rectangular prism, with a copy of \( R_4 \) positioned so that its strands run from the top rectangle to the short sides, two ending on each side to match up with the ends of the vertex tangles. This is drawn on the right side of Figure 7, where the markings on the surface designate as usual how the strands pair up inside the prism.

![Figure 7. \( R_4 \) tangles in a neighborhood of a vertex and edge.](image)

The resulting tangle \( R_{\overline{K}} \), shown schematically in Figure 8, has twelve strands, two joining each vertex prism to each of its adjacent edge prisms.

![Figure 8. Assembling tangles along the triangle \( \overline{K} \).](image)

**Lemma 4.1.** The tangle \( R_{\overline{K}} \) created in this fashion is doubly slice and simple.
Proof. This follows from Proposition 2.5a) and Proposition 2.12a. In particular, to show that $R_{\mathcal{H}}$ is simple, first order the six simplices in $\mathcal{K}$ (three vertices and three open edges) so that the union $\mathcal{K}_i$ of the first $i$ of them is connected for each $i = 1, \ldots, 6$. Then for $i > 1$, the tangle $R_{\mathcal{K}_i}$, defined in the obvious way, is obtained from $R_{\mathcal{K}_{i-1}}$ by a simple tangle sum. The result follows by applying Proposition 2.12a repeatedly.

Assembling the orbifold tangles $\mathcal{T}_i \subset \mathcal{P}_i$.

The tangles $\mathcal{T}_i \subset \mathcal{P}_i$ are now formed by gluing together the tangles in their 0 and 1-handles. Every orbifold 0-handle has 3 attaching bigon regions. Each 1-handle will then be a product of a bigon with an interval, containing an $R_4$ tangle as above. When we attach it to the 0-handles, we are performing a simple tangle sum. The final result is thus a pair of doubly slice simple tangles $\mathcal{T}_i$ in $\mathcal{P}_i$ for $i = 0, 1$, by Propositions 2.5 and 2.12. The exterior of $\mathcal{T}_i$ in $\mathcal{P}_i$, denoted $\mathcal{H}_i$, is the homology handlebody that replaces $\mathcal{P}_i$.

Remark 4.2. Each 0-handle in $\mathcal{P}_i$ contributes 6 components to $\mathcal{T}_i$, since the 12 components of $R_{\mathcal{K}_i}$ are glued up in pairs in the 1-handles. If the $\mathcal{P}_i$ have genus $g$, then there are $2g - 2$ such 0-handles, corresponding to the vertices of the trivalent graph whose thickening is $\mathcal{H}_i$. Thus $\mathcal{T}_i$ has $12g - 12$ components.

Lifting the orbifold tangles to $\mathcal{T}_i \subset H_i$.

When the orbifold tangles $\mathcal{T}_i \subset \mathcal{P}_i$ are lifted to equivariant tangles $\mathcal{T}_i \subset H_i$, the picture is exquisitely embellished, as in the creation of folded paper sculptures. Fortunately, the proof that these lifted tangles are doubly slice and simple is essentially the same as in the orbifold case. The model for the 1-handles above a degree $n$ orbifold 1-handle is now a prism over a 2$n$-gon, with a copy of the atomic tangle $R_{4n}$ inserted so that half the strands end on the top and half on the bottom of the prism. For the 0-handles, the triangle $\mathcal{K}$ lifts to a 1-complex $K$ (namely the 1-skeleton of the first barycentric subdivision of the corresponding dihedron, tetrahedron, octahedron or icosahedron) and the proof that the tangle $R_K$ is simple proceeds exactly as in the proof of Lemma 4.1. These tangles now assemble into a pair of doubly slice simple tangles $\mathcal{T}_i \subset H_i$ for $i = 0, 1$, whose exteriors $\mathcal{H}_i$ are the homology handlebodies that replace $H_i$.

Remark 4.3. We could have used many other tangles in place of the atomic tangle $R_4$ as the basis of our construction. The only properties that were needed for a tangle $\mathcal{T}$ in the 3-ball to give rise to doubly slice, simple, equivariant tangles in the handlebodies of $M$, are that it should be doubly slice and simple, and that all its cyclic branched covers (along a suitable diameter of the 3-ball) should also be doubly slice and simple.

5. GLUING REPLACEMENT HANDLEBODES AND THE PROOF OF THEOREM A

Proof of Theorem A. Starting from an action of $G$ on $M$, we constructed in the last section an equivariant handlebody decomposition $M = H_0 \cup H_1$ by lifting an orbifold handle decomposition of the quotient $\overline{M} = \overline{H}_0 \cup \overline{H}_1$. Then we removed neighborhoods of doubly slice simple tangles $\mathcal{T}_i \subset \overline{\mathcal{P}}_i$ and their lifts $\mathcal{T}_i \subset H_i$ to obtain the replacement homology handlebodies $\overline{\mathcal{H}}_i \subset \overline{\mathcal{P}}_i$ covered by $\mathcal{H}_i \subset H_i$. The remaining step is to describe how to glue these replacement handlebodies together to create the equivariant homology cobordism that proves the theorem.

We begin by working in the quotient orbifold. Recalling that $\mathcal{T}_i$ is doubly slice, choose a boundary parallel tangle $\mathcal{U}_i$ in $\overline{\mathcal{P}}_i$ that is invertibly concordant to $\mathcal{T}_i$. Removing a neighborhood of $\mathcal{U}_i$ from $\overline{\mathcal{P}}_i$ has the effect of stabilizing $\overline{\mathcal{P}}_i$, i.e. adding 1-handles to $\overline{\mathcal{P}}_i$. For our purposes, and in particular to properly specify how to glue $\overline{\mathcal{H}}_0$ and $\overline{\mathcal{H}}_1$ together to form a hyperbolic manifold homology cobordant to $\overline{M} = M/G$, we need to make this more precise.
From this data and an ordering of the $n = 12g - 12$ (see Remark \[12\]) components of $\mathcal{U}_i$ and $\mathcal{T}_i$ consistent with their identification by the concordance $\mathcal{C}_i$, Remark \[26\] gives rise to preferred decompositions

$$\partial \mathcal{H}_i = \partial \mathcal{H}_i \# n(S^1 \times S^1)$$

where the $k$th torus summand is chosen so that the first $S^1$ factor is identified with the preferred longitude of the $k$th component of $\mathcal{T}_i$, while the second $S^1$ factor is the meridian of that component. From this, the boundary of the vertical part of the exterior of the tangle concordance in $\mathcal{H}_i \times I$ acquires a preferred diffeomorphism with

$$\partial \mathcal{H}_i \#_I (S^1 \times S^1 \times I)$$

where $\#_I$ denotes the connected sum along a vertical arc.

Now we glue $\mathcal{H}_0$ to $\mathcal{H}_1$ via a diffeomorphism of their boundaries that identifies corresponding tori in such a way that their meridians and longitudes are interchanged, but that is otherwise the identity. This yields a hyperbolic orbifold $\mathcal{N}$. Because of our choice of the meridian/longitude pair, a similar construction with $\mathcal{T}_i$ replaced by $\mathcal{U}_i$ simply stabilizes the orbifold Heegaard splitting of $\mathcal{M}$, and does not change the resulting orbifold. Gluing the exterior of the concordance $\mathcal{C}_0$ in $\mathcal{H}_0 \times I$ to the exterior of $\mathcal{C}_1$ in $\mathcal{H}_1 \times I$ then gives an orbifold homology cobordism $\mathcal{P}$ from $\mathcal{M}$ to $\mathcal{N}$. By repeating the construction with the inverses of the concordances $\mathcal{C}_i$ to obtain the inverse orbifold homology cobordism $\mathcal{Q}$ and applying Lemma \[27\] we see that $\mathcal{P}$ is in fact an invertible cobordism.

Finally we pass to the orbifold covers. The cobordism that has been constructed is automatically invertible and equivariant, and so it remains to show that the orbifold cover $N$ is hyperbolic, with $G$ acting by isometries, and to check the homological properties of the invertible cobordisms $P$ and $Q$. That $N$ is hyperbolic follows from the fact that the tangles $T_i \subset H_i$ are simple, as noted at the end of §4, together with Proposition \[2.12\]. By Theorem \[2.13\] we can in fact assume that $G$ acts by isometries. The decomposition \[1\] lifts to a similar decomposition of the boundary of the vertical part of the exterior of the preimage of the tangle concordance in $H_i \times I$. Because of the interchange of meridian and longitude, this implies that both $P$ and $Q$ are $\mathbb{Z}[\pi_1(M)]$ homology cobordisms.

The final statement about the fundamental group is a general property of maps induced on the fundamental group of invertible cobordisms. Its proof is postponed to Appendix \[A\] where we make use of similar arguments about invertible cobordisms of 3-manifolds.

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6. Applications of hyperbolization

In this section, we supply proofs for the corollaries of Theorem \[A\] listed in the introduction.

**Hyperbolic G-corks.**

As mentioned in the introduction, our original motivation was to show the existence of (effective) hyperbolic $G$-corks, and we start there.

**Proof of Corollary \[13\]** The main result of our earlier paper \[4\] asserts that if $G$ is a finite group that acts smoothly on the boundary of some compact contractible 4-dimensional submanifold of $\mathbb{R}^4$, then there exists a 4-manifold $X$ and a compact contractible submanifold $C \subset X$, with a $G$-action on its boundary, such that the 4-manifolds $X_{C,g} = (X - \text{int}(C)) \cup_g C$ for $g \in G$ are all smoothly distinct. We say that $C$ (with its boundary $G$-action) is an effective $G$-cork in $X$. Now by Theorem \[A\] there is a $G$-equivariant invertible homology cobordism $P$ from $\partial C$ to a hyperbolic homology sphere $N$, with inverse the equivariant homology cobordism $Q$.

**Claim.** $C' = C \cup_{\partial C} P$ is an effective $G$-cork in $X$, with boundary $\partial C' = N$.

To see this, note first that

$$C' = (C \cup_{\partial C} P) \subset (C \cup_{\partial C} P \cup N Q) \cong C,$$
and this induces an embedding of $C'$ in $X$. Now the $G$-equivariance of $P$ and $Q$ implies that $X_{C',g} \cong X_{C,g}$ for every $g \in G$. Since the manifolds $X_{C,g}$ are smoothly distinct as $g$ runs over $G$, the same is true for $X_{C',g}$. \hfill \qed

**Non-extendible group actions.**

The hyperbolization results in $[39, 46]$ have been used to show that results proved about homology cobordisms and knot concordance can apply to hyperbolic examples; the next application is an equivariant version of this principle. Building on work of Kwasik-Lawson $[30]$, Anvari-Hambleton $[2]$ have shown that for any Brieskorn sphere $\Sigma(a,b,c)$ and prime $p \mid abc$, the natural $\mathbb{Z}_p$ action on $\Sigma(a,b,c)$ does not extend over any contractible manifold that it bounds. (Note that while not all Brieskorn spheres bound contractible manifolds—see for instance $[17, 20]$, there are infinite families $[10, 47, 10]$ that do.) We now show that Theorem A gives non-extension results examples with hyperbolic boundaries.

**Corollary C.** For any Brieskorn homology sphere $\Sigma(a,b,c)$ and prime $p$ not dividing $abc$, there is a hyperbolic homology sphere $N(a,b,c)$ with a free action of $\mathbb{Z}_p$ such that $N(a,b,c)$ and $\Sigma(a,b,c)$ are $\mathbb{Z}_p$-equivariantly homology cobordant, and the action of $\mathbb{Z}_p$ does not extend over any contractible 4-manifold that $N(a,b,c)$ might bound.

We remark that for many choices of $(a,b,c)$, the conclusion can be strengthened to say that the action of $\mathbb{Z}_p$ does not extend over any acyclic 4-manifold that $N(a,b,c)$ might bound. This is shown via the method of Kwasik and Lawson taking into account that Donaldson’s definite manifolds theorem applies to non-simply connected manifolds; see $[14]$. Kwasik and Lawson $[30]$, Proposition 12 give a list of examples to which this method applies.

**Proof of Corollary C.** The condition that $p$ does not divide $abc$ implies that the action of $\mathbb{Z}_p$ on $\Sigma(a,b,c)$ is free. By Theorem A there is a $\mathbb{Z}_p$-equivariant invertible homology cobordism $P$ from $\Sigma(a,b,c)$ to a hyperbolic manifold $N(a,b,c)$. By construction, the action of $\mathbb{Z}_p$ on both cobordisms is free. If the action on $N$ extends over a contractible manifold $W$, then the manifold $W \cup_{N} P$ is a homology ball over which the $\mathbb{Z}_p$ action on $\Sigma(a,b,c)$ extends. By Proposition A.2, this homology ball is simply-connected, and hence contractible, contradicting $[2]$.

**Acyclic versus contractible.**

An important consequence of Taubes’ periodic ends theorem $[53]$, observed by Akbulut is that there are reducible homology spheres that bound homology balls, but do not bound contractible manifolds; the original example was $\Sigma(2,3,5) \# -\Sigma(2,3,5)$. We show that one can in fact choose the homology sphere to be hyperbolic.

**Corollary D.** There are an infinite number of hyperbolic integer homology spheres that bound integer homology balls but do not bound any contractible manifold.

**Proof.** Let $\Sigma$ be any integer homology sphere that bounds a smooth 4-manifold $X$ with non-standard negative definite intersection form. (For example the Poincaré homology sphere.) We first describe how to generate one example, and then describe the modifications required to detect infinitely many distinct examples. By our main theorem, there is an invertible homology cobordism $P$ from $\Sigma \# -\Sigma$ to a hyperbolic 3-manifold $N$. Furthermore, by Proposition A.2 the fundamental group of $P$ is normally generated by the fundamental group of $N$. One sees that

$$W = I \times (\Sigma - \text{int}(B^3)) \cup_{\Sigma \# -\Sigma} P$$

is an integer homology ball with boundary $N$.

Now assume that $N$ bounds a contractible manifold $Z$. Adding a 3-handle to $Z \cup_N P$ along the sphere separating $\Sigma$ and $-\Sigma$ results in a simply-connected (since $\pi_1(P)$ is normally generated
Equivariant hyperbolization

by $\pi_1(N)$ acyclic 4-manifold $V$ with boundary $\Sigma \cup -\Sigma$. This contradicts [53, Proposition 1.7], a consequence of Taubes’ periodic ends theorem.

To show that there are infinitely many distinct examples, we iterate this process. Add a copy of $P$ to $X$ to get a smooth 4-manifold $Y$ with non-standard negative definite intersection form and boundary $N$. Let $Y_k$ denote the boundary connected sum of $k$ copies of $Y$, with boundary equal to $N_k = \#^kN$. Note that the intersection form of $Y_k$ is also non-standard; this is readily verified using Elkies’ criterion for diagonalizability of a unimodular form [15]. Repeating the argument above with $Y_k$ in place of $X$, we obtain a series of hyperbolic manifolds $M_k$ that bound acyclic 4-manifolds but not contractible ones. By construction, there are homology cobordisms from $N_k$ to $M_k$, and hence degree one maps from $M_k$ to $N_k$. In particular the Gromov norm of $M_k$ is at least that of $N_k$, which is in turn $k$ times the (non-zero) norm of $M$. It follows that the Gromov norms of the $M_k$ are unbounded, so an infinite sequence of them are distinct. □

Finite groups acting on homology spheres.

Finally, we give an application of Theorem A to an aspect of the classical spherical space form problem; see [12] and the discussion of problem 3.37 in [29].

Corollary E. Any finite group that acts on a homology 3-sphere also acts on a hyperbolic homology 3-sphere with equivalent fixed-point behavior. In particular, there exist infinitely many finite groups that are not subgroups of $SO(4)$, and so by geometrization do not act freely on the 3-sphere, but that do act freely on some hyperbolic homology 3-sphere.

Proof of Corollary [2]. The first part is a direct corollary of Theorem A replacing an action of a group $G$ on a homology sphere by an action on a hyperbolic homology sphere. The second part, constructing free actions on homology spheres by groups that cannot act freely on the 3-sphere, requires results on the topological spherical space form problem dating to the 1970s and 1980s. The underlying principle is that there are homotopy-theoretic (finiteness) and surgery-theoretic obstructions, depending only on $n$ modulo 8, for a finite group $G$ to act freely on a sphere of dimension $n$. If $n$ is greater than 4, the vanishing of these obstructions is sufficient for the existence of such an actions. In dimension 3, the vanishing of the obstructions implies only that $G$ acts freely on a homology 3-sphere; see for example [26, Remark 8.2].

Work of Madsen [32], Milgram [35], and Bentzen [7] evaluated the finiteness and surgery obstructions in number theoretic terms. Their results show that infinitely many generalized quaternionic groups $Q(8p, q)$ (the smallest being $Q(24, 313)$ [7]) act freely on spheres in dimensions $8k+3$ ($k > 0$) and hence on homology 3-spheres. However, the geometric nature of any such homology sphere was unknown. The geometrization theorem [42, 43, 44, 37] implies that it cannot be $S^3$; presumably one could also verify that it cannot be Seifert-fibered. We deduce directly from Theorem A that such a homology sphere can be taken to be hyperbolic. □

Corollary [C] is related to a recent paper of Bartel and Page [5] that constructs an action of a finite group $G$ on a hyperbolic 3-manifold $M$ such that the induced action on $H_1(M; \mathbb{Q})$ realizes any given finitely generated $\mathbb{Q}[G]$ module. This result and the main results of the current paper are related to some degree, as both construct actions of finite groups on 3-manifolds with prescribed homological action. However, neither paper implies the results of the other; for instance [5] deals only with the action on rational homology, and does not provide a homology cobordism. On the other hand, our hyperbolization requires the existence of an action on some 3-manifold as a starting point. It would be of interest to establish a sharper result realizing a given $\mathbb{Z}[G]$ module (even one with $\mathbb{Z}$ torsion) by an action on some 3-manifold; our hyperbolization procedure would then show that this action is realized on a hyperbolic manifold.
APPENDIX A. ANTISYMMETRY OF INVERTIBLE HOMOLOGY COBORDISM OF 3-MANIFOLDS

We show that for closed oriented 3-manifolds, invertible homology cobordism is an antisymmetric relation, and thus a partial order. This is false in higher odd dimensions, as seen from the existence of h-cobordisms $X$ with non-trivial Whitehead torsion, for which $-X$ is the inverse cobordism; compare \[15\], Lemma 7.8.

**Theorem A.1.** Let $M$ and $N$ be closed 3-manifolds. If there is an invertible homology cobordism from $M$ to $N$, and one from $N$ to $M$, then $M$ and $N$ are homeomorphic.

*Proof.* Let $P$ be the cobordism from $M$ to $N$, and $Q$ be the inverse cobordism from $N$ to $M$, so that $P \cup_N Q = M \times I$. This gives a map $f : N \to M$, the composition of the inclusion $N \hookrightarrow M \times I$ followed by the projection $M \times I \to M$. There is also another pair of cobordisms $Q'$ from $N$ to $M$ and $P'$ from $M$ to $N$, so that $Q' \cup_M P' = N \times I$, and this gives a map $g : M \to N$. Both of these maps have degree one, so their induced maps on $\pi_1$ are surjective. Thus the composition

$$g_* \circ f_* : \pi_1(N) \to \pi_1(N)$$

is surjective. But 3-manifold groups are Hopfian \[3\], which means that in fact this composition is an isomorphism. It follows that $f_*$ is injective, so it is an isomorphism. We write $\pi$ for $\pi_1(M) \cong \pi_1(N)$.

Now computing the fundamental group of $M \times I = P \cup_N Q$ by van Kampen’s theorem yields a pushout diagram:

$$\begin{array}{ccc}
\pi_1(Q) & \xrightarrow{j_Q} & \pi_1(M) \\
\pi_1(N) & \xrightarrow{f_*} & \pi_1(M) \cong \pi_1(M \times I) \\
\pi_1(P) & \xleftarrow{i_P} & \pi_1(P) \end{array}$$

Since $f_*$ is an isomorphism, $i_P$ and $i_Q$ are injective, and $j_P$ and $j_Q$ are surjective. A standard result about pushouts says that in fact $j_P$ and $j_Q$ are injective, so all of these maps are isomorphisms.

One can make a similar argument in homology, with arbitrary twisted coefficients, with the Mayer-Vietoris sequence replacing the van Kampen pushout diagram. In particular, the inclusions of $M$ and $N$ into $P$ and $Q$ induce isomorphisms on homology with coefficients in $\mathbb{Z}[\pi]$, so by Whitehead’s theorem, those maps are homotopy equivalences. In particular, $P$ is an h-cobordism. Now a theorem of Kwasik-Schultz \[31\] implies that the Whitehead torsions of $(P, N)$ and $(P, M)$ both vanish. (Their theorem was proved under the hypothesis that both $M$ and $N$ are geometric, which is now a consequence of the geometrization theorem.) In particular, $M$ and $N$ are simple homotopy equivalent. By a theorem of Turaev (see \[56\] \[57\] as well as \[31\] Theorem 1.1]) $M$ and $N$ are homeomorphic. \(\square\)

We remark that if $M$ and $N$ are hyperbolic manifolds, then there is an alternate (and perhaps simpler) route to this conclusion, based on the Gromov-Thurston proof of Mostow’s rigidity theorem \[24\] \[25\]. This proof implies directly that if there are degree one maps from $M$ to $N$ and from $N$ to $M$, then $M$ and $N$ are homeomorphic.

Finally, we establish the following general property of maps induced on the fundamental group of invertible cobordisms that was used in Corollaries \[\text{C}\] and \[\text{D}\].

**Proposition A.2.** Let $P$ be an invertible cobordism from $M$ to $N$, with inverse cobordism $Q$. Then the image of the map $i_P$ induced by the inclusion of $N$ into $P$ normally generates $\pi_1(P)$, and likewise the image of $i_Q$ normally generates $\pi_1(Q)$.
Proof. We continue the notation from above. It suffices to show that the quotient groups
\[ G_P = \pi_1(P)/\langle \text{im}(i_P) \rangle \quad \text{and} \quad G_Q = \pi_1(Q)/\langle \text{im}(i_Q) \rangle \]
are trivial, where \( \langle \rangle \) denotes the normal closure. The natural maps \( k_P : \pi_1(P) \to G_P \ast G_Q \) and \( k_Q : \pi_1(Q) \to G_P \ast G_Q \) induce a unique map \( h : \pi_1(M) \to G_P \ast G_Q \) such that \( h \circ j_P = k_P \) and \( h \circ j_Q = k_Q \), which must be trivial since \( f_* \) is onto. This forces \( j_P \) and \( j_Q \) to be trivial, which implies that \( G_P \) and \( G_Q \) are trivial. \( \square \)

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Equivariant hyperbolization

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