Leveraging second-order information for tuning of inverse optimal controllers

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Abstract—We leverage second-order information for tuning of inverse optimal controllers for a class of discrete-time nonlinear input-affine systems. For this, we select the input penalty matrix, representing a tuning knob, to yield the Hessian of the Lyapunov function of the closed-loop dynamics. This draws a link between second-order methods known for their high speed of convergence and the tuning of inverse optimal stabilizing controllers to achieve a fast decay of the closed-loop trajectories towards a steady state. In particular, we ensure quadratic convergence, a feat that is otherwise not achieved with a constant input penalty matrix. To balance trade-offs, we suggest a practical implementation of the Hessian and validate this numerically on a network of phase-coupled oscillators that represent voltage source controlled power inverters.

I. INTRODUCTION

In nonlinear programming, the rate of convergence of first- and second-order methods has been extensively studied in the literature. For a fixed stepsize, gradient methods have a sublinear convergence rate for a convex and differentiable function. Linear convergence rates of first-order methods can only be obtained for strongly convex functions, under specific conditions on the search direction [1], [2, Theorem 3.3]. It is known that Newton’s method finds the global minimum of a positive quadratic function in a single iteration [3, Ch1]. More generally, Newton’s method converges very fast asymptotically and does not exhibit the zig-zagging of gradient descent. In fact, gradient descent can have an acceptably slow rate of convergence, even when the Hessian is reasonably well conditioned [2]. In the vicinity of a steady state, such that the Hessian is positive definite, Newton’s method is well defined and converges quadratically [2, Ch 3]. Superlinear convergence results can be found also for an adaptive stepsize [3, Ch1].

In inverse optimal control, we seek to determine all performance criteria for which a given control law is optimal [4], instead of asking for a control law corresponding to a given performance criterion. In particular, we start from a known and stabilizing controller associated with a control Lyapunov function. By a posteriori retrieving or reverse engineering the cost, the controller is optimal. The optimal control problem is called inverse because the cost is determined after a feedback controller has been designed [5], [6]. In fact, establishing that every Lyapunov function is a meaningful value function [7] has a handful of implications on the study of optimal nonlinear controllers. First, inverse optimal control makes optimal feedback solutions intuitive and accessible, i.e., without analytical and computational burden due to the curse of dimensionality, so named by Bellman. Second, choosing a cost function which accurately reflects the functional objectives of the system and at the same time yields an optimal control law that is simple, concise and in closed-form is a cumbersome task that requires a trade off between complexity of the physical structures to implement, minimal (e.g., monetary) budget and input effort. Third, inverse optimal stabilizing controllers can be intuitive to tune and inherit the nonlinear analog of desirable robustness margins similar to multivariable linear quadratic regulators controllers (for a diagonal input penalty matrix). Inverse optimal control formulations have been studied also in discrete time as in [5], [8, Ch14] and the most recent applications of inverse optimal control problems are pronounced at the interface of data-driven control [9], [10, Ch3].

Contributions: Our contributions are threefold and can be summarized as follows. First, inspired by the improved direction of second-order methods, we bring into light the second-order information, i.e., the Hessian and its role in speeding up the convergence of inverse optimal controllers for a class of nonlinear input-affine systems, where the input matrix is assumed to be the identity. Second, motivated by the key idea of inverse optimal control [4]–[7], [11] revolving around a priori knowledge of a value function consisting of a control Lyapunov function for the system dynamics, we encode the Hessian in the input penalty matrix to improve the rate of convergence towards a steady state from sublinear to quadratic. In this way, we leverage the convergence properties of the first-second-order methods in optimization to tune inverse optimal stabilising controllers. Third, we suggest practical implementations for tuning inverse optimal controller that accommodates large scale networks and less noise sensitivity based on the approximation of the inverse of the Hessian. We illustrate this through an example of integrator dynamics of phase-coupled oscillators representing a power inverter network.

The paper unfurls as follows: Section II presents the setup and in particular the class of inverse optimal control problems under study. Section III links inverse optimal control to second-order methods by choosing the input penalty matrix...
to be the Hessian. Section IV suggests a numerical implementation on a phase-coupled oscillator network representing power inverters. Finally, Section V concludes the paper.

**Notation:** For a matrix $P = P^\top > 0$ and a vector $v \in \mathbb{R}^n$, let $\|v\|_P = \sqrt{v^\top P v}$. Let $\text{diag}(v)$ be the diagonal matrix with elements $v_i$, $i = 1, \ldots, n$, and $\sin(v)$ and $\cos(v)$ be the vector-valued sine and cosine functions. Given a twice differentiable function $V(x)$, let $V V(x) = \frac{\partial^2 V}{\partial x^2}$ be the gradient of $V$ with respect to $x$ and $V^2 V(x) = \frac{\partial^4 V}{\partial x^4}$ its Hessian matrix. For $p \in \mathbb{N}$, let $I_p$ be the $p \times p$ identity matrix and $I_p$ be the $p \times 1$ vector of all ones. Let $u_{[0, \infty]} = \{u(0), u(1), \ldots\}$ denote the sequence of the control input. Given a discrete-time dynamical system, $x(k+1) = f(x(k)) + u(k)$, $k \geq 0$, $x(0) = x_0$, where $k \in \mathbb{Z}_+$, is the time step, we consider the system to be time-invariant throughout and eventually drop the time step dependence of the state vector in the notation.

**II. DISCRETE-TIME INVERSE OPTIMAL CONTROL**

We depart from the following class of nonlinear discrete-time dynamical systems,

$$x(k+1) = f(x(k)) + u(k), \quad k \geq 0,$$

$$x(0) = x_0,$$

where $k \in \mathbb{Z}_+$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^n$ is the input vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally continuous function, i.e., in a neighborhood around the origin with $f(0) = 0$. Furthermore, given $R(x) = R^\top(x) > 0$, we consider the following class of cost functions with infinite horizons,

$$\sum_{k=0}^{\infty} q(x(k)) + \|u(k)\|_R^2,$$

where $q(x(k)) > 0$, $q(0) = 0$ and $k = 0, 1, \ldots$. The function $q(x(k)) + \|u(k)\|_R^2$ describes the stage or running cost associated with the states and controls at time step $k$.

The following result is the discrete-time version of Theorem 1 in [11]. See also [5, Ch14].

**Theorem II.1.**

Consider the discrete-time system (1) with cost function (2). Assume that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$. Suppose that,

$$V(f(x(k)) + u^\star(k)) - V(x(k)) \leq -\|u^\star(k)\|_R^2,$$

(3)

Furthermore, let

$$q(x(k)) = V(x(k)) - V(f(x(k) + u^\star(k))) - \|u^\star(k)\|_R^2,$$

(4)

and a stabilizing feedback control law be given by

$$u^\star(k) = -\frac{1}{2} \alpha_k R^{-1}(x(k)) V V(x(k)),$$

(5)

where $\alpha_k > 0$ is the step size and $R(x) = R^\top(x) > 0$, for all $k = 0, 1, \ldots$. Then, the optimal control problem

$$\min_{u(0)} \sum_{k=0}^{\infty} q(x(k)) + \|u(k)\|_R^2,$$

(6)

has the minimal value $V(x_0)$.

**Remark 1.**

A state dependent matrix $R(x)$ in the formulation of the discrete-time inverse optimal control problem (6) has been considered in continuous time in [6, Ch.3]. Indeed, the results obtained for a constant positive definite input penalty matrix (e.g., in [11]) can be one-to-one extended for a state-dependent positive definite matrix.

**III. SECOND-ORDER INFORMATION IN INVERSE OPTIMAL CONTROL**

**A. The rate of convergence**

The rate of convergence is a principle means to opt for a certain algorithm out of a large array of candidate algorithms [3]. Note here that, by Theorem II.1, the closed-loop trajectories provably converge to the origin. In practice, though, they may converge very slowly. Without a good understanding of the rate of convergence properties of the closed-loop trajectories, it may be difficult to avoid costly numerical and computational experimentation.

The steepest descent is one of the simplest but also the slowest methods to converge to a steady state solution. Newton’s method is on the other hand of the spectrum and arguably more complex but also the fastest gradient method. Given a twice differentiable function $f(x)$ to minimize with a positive definite Hessian, Newton’s method follows the direction of the descent,

$$d_k = -\left(\nabla^2 f(x(k))\right)^{-1} \nabla f(x(k)),$$

where at each iteration $k = 0, 1, \ldots$,

$$x(k+1) = x(k) + d_k.$$  

(7)

Keeping this in mind, we aim in the remainder to leverage the knowledge on the convergence properties of second-order methods to suggest a tuning for the inverse optimal controller (5), where $R(x) > 0$ is a degree of freedom. We start with the following assumption.

**Assumption 1.**

Assume that $V$ is twice differentiable and $V^2 V$ is Lipschitz continuous and positive definite in a neighborhood around the origin. That is,

$$\nabla^2 V(x(k)) > 0,$$

$$(\nabla^2 V(x(k)) - \nabla^2 V(y(k))) \leq L \|x(k) - y(k)\|,$$

for all $k = 0, 1, \ldots$ and $\|x(k)\| < \varepsilon_x, \|y(k)\| < \varepsilon_y$, where $L, \varepsilon_x, \varepsilon_y$ are positive constants and $\varepsilon_x, \varepsilon_y$ are sufficiently small.

Local convergence deals with the behavior of the trajectories near a non-singular local minimum. Our main result specifies the rate of local convergence of the trajectories of (1) towards zero in closed-loop with (5) for two different choices of the input penalty matrix $R(x)$ and is summarized in the following Proposition.

**Proposition III.1.**

Consider the optimal control problem (6) together with
Proof. 1) Let $\lambda > 0$ be the minimum and maximum eigenvalues of $R_1$, i.e., it holds that,
$$\lambda \|z\|^2 \leq z^T R_1 z \leq \bar{\lambda} \|z\|^2,$$
for all $k = 0, 1, \ldots$. The inequality (1) translates to
$$V(x(k + 1)) - V(x(k)) \leq \frac{1}{\lambda} \|\nabla V(x(k))\|^2,$$
where $\alpha_k = 2$. This leads to,
$$V(x(k + 1)) - V(x(k)) \leq -M_k,$$
where $M_k = \frac{1}{2} \|\nabla V(x(k))\|^2$. By taking the sum over $h = 0, \ldots, k$, we arrive at,
$$\sum_{h=0}^{k} M_h \leq V(x_0).$$
We define $M_k := \min_{h=0,1,\ldots,k} M_h$ and conclude that,
$$M_k \leq \frac{\sum_{h=0}^{k} M_h}{k+1} \leq \frac{V(x_0)}{k+1}.$$
This shows $O(1/k)$ sublinear convergence to the origin.

2) By selecting the input matrix to be the Hessian, i.e.,
$$R(x) = \nabla^2 V(x),$$
the input (5) can be written as,
$$u^*(k) = - (\nabla^2 V(x(k)))^{-1} \nabla V(x(k)),$$
where $\alpha_k = 2$ and the inverse of the Hessian exists, in the vicinity of the origin under Assumption 1. The closed-loop system dynamics are given by,
$$x(k + 1) = f(x(k)) - (\nabla^2 V(x(k)))^{-1} \nabla V(x(k)),$$
$$x(0) = x_0.$$
Along the lines of the proof of Theorem 3.9 in [2, Ch.3], it holds that,
$$x(k + 1) = x(k) - (\nabla^2 V(x(k)))^{-1} \nabla V(x(k)) + f(x(k)) - x(k),$$
$$= (\nabla^2 V(x(k)))^{-1} \left[ \nabla^2 V(x(k)) x(k) - \nabla V(x(k)) \right] + f(x(k)) - x(k),$$
where we added and subtracted $x(k)$. We apply Taylor’s Theorem [2, Theorem 2.1] to write,
$$\nabla V(x(k)) - \nabla V(0) = \int_{0}^{1} \nabla^2 V((1 - t) x(k)) x(k) \, dt.$$
Thus, it holds that,
$$\|\nabla^2 V(x(k)) x(k) - \nabla V(x(k))\|$$
$$= \left\| \int_{0}^{1} \left[ \nabla^2 V(x(k)) - \nabla^2 V((1 - t) x(k)) \right] x(k) \, dt \right\|$$
$$\leq \int_{0}^{1} \left\| \nabla^2 V(x(k)) - \nabla^2 V((1 - t) x(k)) \right\| \|x(k)\| \, dt$$
$$\leq \int_{0}^{1} tL \|x(k)\|^2 \, dt \leq \frac{1}{2} L \|x(k)\|^2,$$
where we use local Lipschitz continuity of $\nabla^2 V$ by Assumption 1. Therefore, we arrive at
$$\|x(k + 1)\| \leq \|\nabla^2 V(x(k))\| + \|f(x(k) - x(k))\| + \|f(x(k)) + \|x(k)\|\|
$$
where we used the triangle inequality of the norm. By local continuity of the vector field $f$ and for sufficiently small $\varepsilon > 0$, there exists $\delta > 0$, s.t.,
$$\|x(k)\| < \delta \implies \|f(x(k))\| < \varepsilon.$$
Additionally, $\nabla^2 V(0)$ is non-singular. Hence, there is a radius $r > 0$ such that $\|\nabla^2 V(x(k))\| \leq 2\|\nabla^2 V(0)\|^{-1}$, for all $x(k)$ satisfying $\|x(k)\| \leq r$. Let $\bar{L} = L\|\nabla^2 V(0)\|^{-1}$. Then,
$$\|x(k + 1)\| \leq \bar{L} \|x(k)\|^2 + \varepsilon + \min\{\delta, r\}.$$
Choosing the initial condition $x_0$ so that $\|x_0\| \leq \min(r, \delta, 1/(2\bar{L}))$ yields quadratic convergence.

Remark 2.

- Without loss of generality, we consider the origin to be a steady state. Our results can be one-to-one extended to any non-zero steady state.
- Suppose that the Lyapunov function $V$ satisfies,
$$V(x(k + 1)) - \rho V(x(k)) < -Z_k,$$
where $Z_k = \|u^*(k)\|^2_{R(x)}$ and $\rho \in [0, 1)$. Then, linear or quadratic convergence rates (respectively) can be proven, even for a constant input penalty matrix, i.e., $R(x) = R_1 > 0$. Note though that the inequalities (8) or (9) imply (3). This restricts the class of nonlinear input-affine systems (1) and Lyapunov functions $V$ that enjoy such a decay. By selecting $R(x) = \nabla^2 V(x)$, we obtain a quadratic (and thus linear) rate of convergence in Proposition 3.1 without additional restrictions on the decay of $V$. 

B. Discussions

1) Integrator dynamics: If
\[ f(x(k)) = x(k), \quad k = 0, 1, \ldots, \]
then the system dynamics \([1]\) reduces to the following equation,
\[ x(k + 1) = x(k) + u(k), \]

with \(x(0) = x_0\). Given the optimal controller \(u^*(k)\) in \([5]\) with \(\alpha_k = 2\), we arrive at the closed-loop system \([7]\). The vanilla Newton’s method \([7]\) with \(I(x) = V(x)\) illustrates the usefulness of the choice of the input matrix \(R(x) = \nabla^2 V(x)\). In fact, Newton’s method has a convergence rate known to be quadratic [3, Ch1] and thus guaranteeing a fast decay of the closed-loop trajectories towards a steady state.

2) Feasible implementation via quasi second-order method: In this section, quasi second-order methods are motivated by two observations. First, Newton’s second method is often restricted from small to medium scale problems due to the computational cost associated with the calculation of the inverse of the Hessian. Second, the trade-off between convergence rate and robustness has been studied both in control [6], [12] and optimization [13]. In particular, the noise sensitivity for high frequencies (corresponding to fast convergence) for single-input-single-output systems is well-known and requires a trade-off between sensitivity and complementary sensitivity [12]. For nonlinear systems, the optimal stabilization of \([1]\) with respect to the cost function \([2]\) achieves a disk margin of \(1/2\) for \(R(x) = I_n\), a sector margin \((\frac{1}{2}, \infty)\) (for their definitions, see [6, p.75]) if \(R(x)\) is diagonal and no guarantees on stability margin for a general positive definite \(R(x)\) in [6, p.103]. Choosing the input penalty matrix as the Hessian might result in poor robustness margin and high noise sensitivity.

To overcome these restrictions and balance the different trade-offs, namely to solve also large-scale problems and account for the noise (e.g., static uncertainties), while still achieving a fast speed of convergence, we follow the variation of Newton’s method presented in [3] that involves estimating the inverse of the Hessian using a diagonal positive definite matrix given by,
\[ (\nabla^2 V(x(k)))^{-1} \approx \begin{bmatrix} g_1^k & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & g_n^k \end{bmatrix}, \quad (10) \]

where,
\[ g_i^k = \left( \frac{\partial^2 V(x(k))}{(\partial x_i)^2} \right)^{-1}, \quad i = 1, \ldots, n. \]

Observe the effect of estimating the inverse Hessian \([10]\) on the convergence rate of the algorithm. In fact, the results of Proposition III.1 are relaxed and a superlinear convergence can be achieved [2, Theorem 3.6, Theorem 3.7].

3) Step size: Different choices of the step size \(\alpha_k > 0\) are possible. Since a constant stepsize is the simplest to implement, we adopt in this work a constant stepsize, i.e., \(\alpha_k = 2\). We note though the general trade-off, that if the stepsize is too small, the rate of convergence might be too slow and if the stepsize is too large, the closed-loop trajectories might diverge [3].

IV. Numerical Example

In this section, we numerically study the effect of choosing the Hessian, as the input penalty matrix, on the error decay rate. In [14], we suggest an optimal control law that stabilizes the phase angles of a network of inverters towards an induced steady state angle. In particular, we consider a connected graph, where the set of nodes consists in \(n\)—phase-coupled oscillators, representing controllable voltage sources (with constant voltage magnitude, i.e., one per unit), whose (virtual) angle dynamics are described (in a rotating frame at a nominal frequency) by the following integrator dynamics, written here in discrete-time,
\[ \theta(k + 1) = \theta(k) + u(k), \quad \theta(0) = \theta_0. \quad (11) \]

where \(\theta(k) = [\theta_1(k), \ldots, \theta_n(k)]^\top \in \mathbb{R}^n\) is the angle vector at time step \(k\). The set of edges consists in inductive (i.e., lossless) \(m\)—transmission lines. The coupling strengths \(b_{ij} > 0\) are collected in the diagonal matrix \(\Xi = \text{diag}(b_{ij})\).

An example of three voltage source controlled inverters that illustrates our setup is shown in Figure 1. The active power deviation from the nominal is given by,
\[ P_i(\theta) - P^*_i = \sum_{j \in N_i} b_{ij} \left( \sin(\theta_i(k) - \theta_j(k)) - \sin(\theta_i^* - \theta_j^*) \right), \quad (12) \]

where \(N_i\) denotes the neighborhood of the \(i\)–th inverter, \(P_i(\theta)\) is its active power injected into the network and \(P^*_i\) is its nominal active power.

Remark 3.
The discrete-time representation of the inverter model \([11]\) follows discrete-time phase-coupled oscillator models, e.g., in [15]–[17].

Let \(\theta^* := \lim_{k \to \infty} \theta(k)\) be an induced steady state angle of \([11]\). We make the following assumption stating that neighboring steady state angle differences are contained in an arc of length \(\pi\).

Assumption 2 (14).
The induced steady state angle vector \(\theta^* = \{\theta_i^*\}_{i=1}^n\) satisfies \(B^\top \theta^* \in (-\frac{\pi}{2}, \frac{\pi}{2})^m\), where \(B \in \mathbb{R}^{n \times m}\) is the incidence matrix of the underlying graph.

Next, let \(R(\theta) = \nabla^2 \theta(\theta) > 0\) be the input penalty matrix and consider the following optimal control problem,
\[ \min_{u \in L_{\infty}} \sum_{k=0}^\infty g(\theta(k)) + \|u(k)\|_{R(\theta)}^2 \]
\[ \text{s.t.} \quad \theta(k + 1) = \theta(k) + u(k), \quad \theta(0) = \theta_0. \quad (13) \]
where \( q(\theta(k)) \) is assumed to be positive and \( q(0) = 0 \). We introduce \( \Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_n\} \) and the following function,

\[
V(\theta) = \frac{1}{2} \|\theta - \theta^*\|^2 + \sum_{i=1}^{n} \sum_{j \in N_i} b_{ij} (\cos(\theta_{ij}) - \cos(\theta_{ij}^*) - (\theta_{ij} - \theta_{ij}^*)\sin(\theta_{ij}^*)) .
\]

(14)

Let \( \varepsilon > 0 \) and \( \Omega = \{ \theta \in \mathbb{R}^n | \|\theta - \theta^*\| < \varepsilon \} \) be a neighborhood of \( \theta^* \). Under Assumption \([2]\) it holds that

\[
\nabla^2 V(\theta) = \Gamma + \mathcal{B} \Sigma \mathcal{B}^\top > 0 ,
\]

(15)

where \( \mathcal{B}^\top = \text{diag}(\cos(\mathcal{B}^\top \theta)) \), for all \( \theta \in \Omega \). This shows that Assumption \([1]\) is satisfied and \( V(\theta) \) in (14) is locally positive definite. In \([14]\), we have found that the following choice of the cost function,

\[
q(\theta) = \frac{1}{4} \nabla^\top V(\theta) R^{-1}(\theta) \nabla V(\theta) ,
\]

where \( \nabla V = \Gamma(\theta - \theta^*) + P(\theta) - P^* \) and \( P(\theta) - P^* = [P_1(\theta) - P_1^* , \ldots , P_n(\theta) - P_n^*]^\top \), yields the locally (i.e., in a neighborhood of \( \theta^* \)) inverse optimal stabilizing controller given by,

\[
u^*(\theta) = -\frac{1}{2} \alpha_k R^{-1}(\theta) \nabla V(\theta) ,
\]

(16)

with \( \alpha_k > 0 \) is the stepsize and \( V(\theta_0) \) is the initial value of \([13]\).

Next, for the initial condition \( \theta_0 = [0.02, 0.1, 0.3]^\top \), we select the input penalty matrix \( R(\theta) \) to be,

i) a constant matrix \( r \cdot I_3 \), where \( r \) is a positive constant,
ii) the exact Hessian matrix \([13]\),
iii) the approximation of the Hessian \([10]\) consisting of only the diagonal elements of \( \nabla^2 V(\theta) \).

Let \( \Gamma = \Xi = I_3 \). We investigate the effect of the aforementioned choices of the input matrix \( R(\theta) \) on the rate of convergence of the angle vector \( \theta \) towards an induced steady state \( \theta^* \) and discuss the control implementation for large-scale networks. Our observations are as follows:

- Once plotted on a logarithmic scale, it becomes apparent that the exact Hessian ii) outperforms all other choices of \( R(\theta) \) and yields the fastest convergence rate. Independent of the allowed control effort, the convergence rate of the trajectories following the approximate inverse of the Hessian iii) improves upon that of a constant input matrix i). An example for \( r = 3 \) is shown in Fig. 2.
- The control implementation of (16) with an input penalty matrix \( R(\theta) \) in i) is distributed as highlighted in \([14]\) and is thus feasible, also for large-scale networks. With \( R(\theta) \) in ii), each controller \([16]\) possibly requires full state measurements and is thus not suitable for large scale systems. The controller tuned with the approximate Hessian in iii) enjoys a distributed and thus feasible implementation, while improving the rate of convergence towards a steady state.
- We note the well-known trade-off between control effort/energy and the speed of convergence from multi-variable linear quadratic regulators as can be seen in Fig. 3, where the Hessian implementation requires the highest control effort compared to the other choices of \( R(\theta) \).

Fig. 2: Simulations of the 2-norm of the angle error \( \|\theta(k) - \theta^*\|_2 \), \( k = 0, 1, \ldots \) (in rad.) on a logarithmic scale with \( \theta_0 = [0.02, 0.1, 0.3]^\top \) in closed-loop with the input given in \([16]\) of the dynamical system described in Figure 1 in closed-loop with the optimal control \([16]\) where \( R(\theta) \) is chosen according to i) for \( r = 3 \), ii), iii), respectively. The angles are stabilized at the steady state \( \theta^* = [0.451, 0.92, 0.967]^\top \) satisfying Assumption \([2]\). The angle error decay rate improves significantly with a choice of the input penalty matrix according to the Hessian ii) as well as the approximated Hessian iii).

V. Conclusion

We studied an approach to tune the input penalty matrix of a class of inverse optimal control problems using the
Fig. 3: Simulations of the input vector $u = [u_1, u_2, u_3]^{\top}$ corresponding to the angle error decay in Fig. 2 where $R(\theta)$ is chosen according to i) for $r = 3$, ii) and iii) respectively. The Hessian implementation iii) yields the fastest error decay at the cost of high control effort/energy compared to the other choices.

Hessian information of the Lyapunov function. We leverage the knowledge from second-order methods in optimization to improve the speed of convergence towards a steady state, from sublinear to quadratic. Our future work aims to study the rate of convergence for a generalization to any nonlinear input-affine dynamics and the effect of an adaptive stepsize.

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