Unification of versions of photon quantum mechanics through Clifford spacetime algebra

Margaret Hawton
Department of Physics, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1

The Clifford spacetime algebraic description of Maxwell’s equations is reviewed and shown to give a unified picture of recently published versions of photon quantum mechanics. Photon wave equations and a conserved four-current are derived from the complexified standard Lagrangian. The equations of motion and scalar product are found to be in good agreement with those obtained from Fourier transformation of momentum space wave function and scalar product [Phys. Rev. A 102, 042201 (2020)].

I. INTRODUCTION

Photons are important particles in applications and tests of the foundations of quantum mechanics. The Reimann-Silberstein (RS) vector is a popular choice for the first quantized description of single photon states and entangled pairs. We will show here that in $Cl_{1,3}$ space-time algebra the RS vector reflects the geometry of space-time. The Clifford algebraic description of electromagnetism is concise, but the primary advantage of this approach that will be exploited here is to make a distinction between the trivector, $i$, and the unit imaginary, $j$. This makes it possible to separate the RS vector into its electric and magnetic field parts, $F = E + icB$, while reserving $j$ for expressions such as the circularly polarized plane wave $(e_1 + j\lambda e_2)\exp[-j(\omega_k t - k \cdot x)]$ where in this example $k = ke_3$, $\lambda = \pm 1$ and $e = +$.

In Section II the $Cl_{1,3}$ space-time algebraic description of electromagnetism will be reviewed and the equations of motion and scalar product for relativistic photon quantum mechanics will be derived from the complexified standard Lagrangian. This is complimentary to the derivation by Bialynicki-Birula and Bialynicka-Birula in which the $k$-space photon wave function and scalar product are Fourier transformed to configuration space. In Section III we will examine the relationship of space-time algebra to the recently published work on photon quantum mechanics and compare the equations derived in Section II with the results of In Section IV we will Conclude. SI units are used throughout.

II. PHOTON QUANTUM MECHANICS IN CLIFFORD SPACETIME ALGEBRA

Paravectors and their reversals will be written as $U = U^0 + U$ and $\overline{U} = U^0 - U$ respectively. The (noncommutative) multiplication rule in $Cl_{1,3}$ Clifford algebra is

$$UV = (U^0V^0 + U \cdot V) + (U^0V + UV^0 + iU \times V) \quad (1)$$

where $U_0V + UV_0$ and $U \times V$ are vectors and $iU \times V$ is a bivector. Here $U \cdot V$ and $U \times V$ are the usual dot and cross products. In an orthonormal 3D Cartesian basis $\{e_1, e_2, e_3\}$ according to [1] the unit vectors satisfy $e_i e_j = \delta_{ij} + i e_i \times e_j$ so that $e_i e_j = 1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. The trivector is $i = e_1 e_2 e_3$. Using these relations it is straightforward to verify that $i^2 = -1$ and $i e_3 = e_1 e_2 e_3 e_1 = e_1 e_2$ is a bivector [2]. This last relation can be permuted in cyclic order. If $U^0$ is a scalar and $U$ is a vector, in relativistic notation the paravector $U$ is equivalent to the four-vector $U = (U^0,\mathbf{U})$. In tensor notation $U = U^\mu$ is a contravariant four-vector, $\overline{U} = U_\mu = g_{\mu\nu}U^\nu$ is the corresponding covariant four-vector and $g_{\mu\nu} = g^{\mu\nu}$ is a 4x4 matrix with diagonal $(1, -1, -1, -1)$. Repeated Greek indices are summed over so, for example, $U \overline{V} = U^\mu V_\mu = U^0V_0 - \mathbf{U} \cdot \mathbf{V}$ is an invariant scalar.

Writing the space-time coordinates, four-gradiant and four-potential $x = (ct, \mathbf{x})$, $\partial = (\partial_{ct} - \nabla)$ and $A = (\phi/c, \mathbf{A})$ as the paravectors $x = ct + \mathbf{x}$, $\partial = \partial_{ct} - \nabla$ and $A = \phi/c + \mathbf{A}$ respectively, (1) gives

$$c \partial \overline{A} = cA + E + icB \quad (2)$$
$$cA = c^{-2}\partial \phi + \nabla \cdot \mathbf{A}, \quad E = -\nabla \phi - \partial_{ct} \mathbf{A}, \quad B = \nabla \times \mathbf{A}, \quad (3)$$

where $\Lambda = c^{-2}\partial \phi + \nabla \cdot \mathbf{A}$.

$c$ is the speed of light in vacuum, $\partial_{ct} = \frac{1}{ct} \frac{\partial}{\partial x^0}$, $\Lambda$ determines the gauge, $\mathbf{E}$ and $\mathbf{B}$ are vectors and $ic\mathbf{B}$ is a bivector. The paravector (2) and the RS vector $\mathbf{F}$ will be written as

$$F = c \partial \overline{A} = cA + \mathbf{F}, \quad (5)$$
$$\mathbf{F} = \mathbf{E} + ic\mathbf{B}. \quad (6)$$

Note especially that $i$ in this expression is a space-time trivector not the unit imaginary. This separation of $F$ into its scalar, vector, bivector and trivector parts is Lorentz reference frame dependent [4].

The wave equation and scalar product should be derivable from a Lagrangian. To pass from a classical to a quantum mechanical interpretation of Maxwell’s equations and allow for the positive frequency wave functions in the literature it will be assumed that the four-potential, $A$, and and its complex conjugate, $A^*$, are independent. This is equivalent to treating the real and imaginary parts of $A$ as independent. The standard,
Fermi, covariant photon and matter-photon interaction Lagrangian densities are then

$$\mathcal{L}_{\text{std}} = -\epsilon_0 \left( E^\ast \cdot E - c^2 B^\ast \cdot B \right) = -\epsilon_0 F^\mu_\nu F_{\mu\nu},$$

$$\mathcal{L}_{\text{Fermi}} = \mathcal{L}_{\text{std}} - \epsilon_0 c^2 \Lambda^\ast,$$  

$$\mathcal{L}_{\text{cov}} = -\epsilon_0 c^2 \left( \partial^\mu A^\ast_\nu \right) \left( \partial_\mu A^\nu \right),$$  

$$\mathcal{L}_{\text{int}} = -J_m^\mu A_\nu - J_m^\nu A_\mu,$$

where $\epsilon_0$ is the dielectric permittivity,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

is the antisymmetric Faraday tensor in which $F^{00} = F^{\mu0} = 0$, $F^{0\nu} = -F^{\nu0} = \epsilon_0 E_\nu$ and $F^{ij} = -F^{ji} = \epsilon_{ijk} c B_k$ and $\epsilon_{ijk}$ is the Levi-Civita symbol. The Fermi Lagrangian density differs only by the four-divergence $-\epsilon_0 c^2 \partial_\lambda \left( A^\mu \partial_\lambda A^{\ast\nu} - A^{\ast\nu} \partial_\lambda A^\mu \right)$ so they give identical equations of motion. The matter current density $J_m$ is included for generality and to allow discussion of emission of a photon by an atom and some recent papers on the dressed photon wave function. Variation of the action leads to the Lagrange equation of motion

$$\frac{\partial^\mu}{\partial (\partial_\nu A^\mu_\nu)} - \frac{\partial^\nu}{\partial A^\mu_\mu} = 0$$

and its complex conjugate for any $\mathcal{L}(A_\mu, A^\ast_\nu, \partial^\mu A_\nu, \partial^\mu A^\ast_\nu)$. With the conjugate to $A_\nu^\ast$ defined as

$$\Pi^{\mu\nu} = \partial \mathcal{L}/\partial (\partial_\nu A^\mu_\nu) \text{ where }$$

$$\Pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{std}}}{\partial (\partial_\nu A^\mu_\nu)} = \epsilon_0 F^{\mu\nu},$$

$$\Pi^{\mu\nu}_{\text{cov}} = \frac{\partial \mathcal{L}_{\text{cov}}}{\partial (\partial_\nu A^\mu_\nu)} = -\epsilon_0 \partial^\mu A^\nu,$$

the Lagrange equation becomes $\partial_\nu \Pi^{\mu\nu} = \partial \mathcal{L}/\partial A^\mu_\nu$. The global phase change $A \rightarrow e^{j \epsilon_0 A}$ and $A^\ast \rightarrow e^{-j \epsilon_0 A^\ast}$ is a symmetry of $\mathcal{L}_{\text{std}}, \mathcal{L}_{\text{Fermi}}$ and $\mathcal{L}_{\text{cov}}$. Since $\delta A \approx j \epsilon_0$ and $\delta A^\ast \approx -j \epsilon_0$, for infinitesimal $\alpha$ the Noether current II\#A generated by this phase change is $J^\mu \propto j \Pi^{\mu\nu} \epsilon_{\nu} - j \Pi^{\mu\nu} \epsilon_{\nu} A^\nu$. This four-current will be discussed later in this section. The operator $\tilde{\epsilon} = j \left( \nabla^2 \right)^{-1/2} \partial_\epsilon$ and the positive definite number density $\epsilon^0$ were introduced in [13, 20]. This operator extracts the sign of the frequency, $\epsilon = \pm$ from $A$ and plays the same role as the Paul-Dirac matrix $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ where $I$ is the $2 \times 2$ unit matrix. In a basis of positive and negative frequency waves $\tilde{\epsilon}$ can be replaced with its eigenvalues, $\epsilon = \pm$.

The Lagrange equation with $\mathcal{L} = \mathcal{L}_{\text{std}} + \mathcal{L}_{\text{int}}$ gives the wave equation

$$\nabla^2 \Lambda = \mathcal{F}(c \Lambda + F) = c \nabla^2 \Lambda + Z_0 \nabla J_m$$

where $\mu_0 = \left( \epsilon_0 c^2 \right)^{-1}$ is the magnetic permeability, $Z_0 = \sqrt{\mu_0 / \epsilon_0}$ is the impedance of vacuum and $\nabla^2 = \nabla \nabla = \partial^2 \lambda - \partial^2 \lambda = \partial^2 \lambda - \partial^2 \lambda$. In the Coulomb gauge $\nabla \cdot A = 0$, there are no longitudinal modes and the scalar field satisfying $\nabla^2 \phi = -\rho / \epsilon_0$ responds instantaneously to changes in charge density. Only the transverses modes $\lambda = \pm 1$ propagate at the speed of light and are second quantized in QED to allow creation and annihilation of physical photons. Use of the covariant Lagrangian $\mathcal{L}_{\text{cov}}$ or insertion of the Lorenz gauge condition $\Lambda = 0$ into (14) gives $\square A = \mu_0 J_m$. In the Lorenz gauge all four components of $A$ describing scalar, longitudinal and transverse $\lambda = \pm 1$ photon modes propagate at the speed of light and are second quantized in QED.

The equation of motion for $A$ is gauge dependent but the $c\partial_\epsilon A$ terms on the left and right sides of (14) cancel to give a gauge independent equation for the RS vector,

$$\nabla F = \nabla F + \partial_\epsilon F + i \nabla \times F = Z_0 J_m.$$  

When written as $F = E + ic B$, (15) becomes

$$\nabla \times E + ic \nabla \times B + \partial_\epsilon E - c^2 \nabla \times B + i \partial_\epsilon B + i \nabla \times E = Z_0 J_m.$$  

Since the scalar, trivector, vector and bivector terms are independent (15) is equivalent to the Maxwell equations

$$\nabla \cdot B = 0, \quad \partial_\epsilon B + \nabla \times E = 0,$$

$$\nabla \cdot E = \frac{1}{\epsilon_0} \rho_m, \quad \partial_\epsilon E - c^2 \nabla \times B = \frac{1}{\epsilon_0} J_m.$$  

The fields are in general complex and the complex conjugates of Eqs. (12) to (17) are also valid.

A basis of positive and negative frequency waves, $\epsilon = \pm$, and both helicities, $\lambda = \pm 1$, will be used. This allows for the possibility that the wave function is real or restricted to positive frequencies. In the first quantized theory described here $\epsilon = -$ represents a negative energy photon. After second quantization a negative energy photon is replaced with its positive energy antiphoton and annihilation of an $\epsilon = -$ photon with wave vector $k$ and helicity $\lambda$ appears as creation of an antiphoton with wave vector $-k$ and helicity $-\lambda$. To avoid confusion, $\epsilon$ will be referred to as the sign of the frequency rather than the sign of the energy. Photons and antiphotons are indistinguishable and $\epsilon$ and $\lambda$ are Lorentz invariants.

In vacuum the electric field describing a plane wave with helicity $\lambda$ and frequency $\omega_k$ will be written as

$$E_{k\lambda}(t, x) = E_{\lambda}(k) e^{-j \omega_k t + j k \cdot x}$$

where

$$E_{\lambda}(k) = \frac{1}{\sqrt{2}} (e_\theta + j \lambda e_\phi),$$

$$\omega_k = ck \quad \text{with} \quad k = |k| \text{ and the } k\text{-space unit vectors } \{e_\theta, e_\phi, e_k\} \text{ form an orthonormal triad. It can be verified by expansion that the real and imaginary parts of this field rotate about the } k \text{ direction, clockwise if } \epsilon \lambda = +1 \text{ and counterclockwise if } \epsilon \lambda = -1. \text{ It follows from } \partial_\epsilon B + \nabla \times E = 0 \text{ that }$$

$$c B_{k\lambda} = -j \epsilon \lambda E_{k\lambda}. $$
for any \( \mathbf{k} \) and hence the RS vector is
\[
\mathbf{F}_k^\lambda = (1 - i\epsilon \lambda) \mathbf{E}_k^\lambda. \tag{21}
\]
Eqs. (18) and (15) then give \( \nabla \times \mathbf{F}_k^\lambda = \lambda \kappa \mathbf{F}_k^\lambda \) and
\[
i \partial_\lambda \mathbf{F}_k^\lambda = -i\epsilon \kappa (1 - i\epsilon \lambda) \mathbf{E}_k^\lambda = \lambda \kappa \mathbf{F}_k^\lambda \]
so individually and in linear combination the plane waves \( \mathbf{F}_k^\lambda \) satisfy
\[
i \partial_\lambda \mathbf{F}(t, \mathbf{x}) = \nabla \times \mathbf{F}(t, \mathbf{x}), \tag{22}
\]
as they must for consistency with [15] in the absence of a matter-four-current. Thus any free space solution to (22) can be expanded in the \( \epsilon \lambda \)-plane wave basis as the Fourier series
\[
\mathbf{F}(t, \mathbf{x}) = \sum_{\epsilon, \lambda = \pm 1} \int \, dk \alpha_\lambda^\epsilon (\mathbf{k}) \mathbf{F}_k^\lambda(t, \mathbf{x}). \tag{23}
\]
In the notation used here in which \( i \) is a trivector and \( j \) is the unit imaginary, the single equation (22) describes positive and negative frequency photons of both helicities. Eq. (23) allows us to add the positive and negative frequency waves describing the indistinguishable photons and antiphotons to give real waves and take sums and differences of circularly polarized waves to give linearly polarized waves. When second quantized the expansion coefficient \( \alpha_\lambda^\epsilon (\mathbf{k}) \) in (24) is replaced with the annihilation operator \( \alpha_\lambda^\epsilon (\mathbf{k}) \) and \( a^-_\lambda (-\mathbf{k}) \) becomes the creation operator \( \alpha^-_\lambda (\mathbf{k}) \) that can act on the vacuum state \( \ket{0} \) to give the Schrödinger picture (SP) one-photon plane wave state
\[
|1_{k\lambda}\rangle = \alpha^-_\lambda (\mathbf{k}) \ket{0}. \tag{24}
\]
The Noether current generated by the phase change \( A' = e^{i\lambda \sigma^a A'^a} A' \) is \( \mathcal{J}^{\mu}_{\text{cov}} = \epsilon \partial^\mu A'^\nu - \epsilon \partial^\nu A'^\mu \), with \( \Pi_{\text{std}} \) and \( \Pi_{\text{cov}} \) given by (13). The covariant Lagrangian density \( \mathcal{L}_{\text{cov}} \) gives the covariant photon four-current density
\[
\mathcal{J}^{\mu}_{\text{cov}}(x) = -\frac{\epsilon_0 c}{\hbar} \sum_{\epsilon, \lambda = \pm 1} \epsilon \alpha^\epsilon_{\Lambda} (x) \partial^\mu \mathcal{A}^\nu (x) \tag{25}
\]
in the \( \epsilon = \pm \) basis. In this expression \( \frac{\epsilon}{\hbar} \nabla \partial_{\tau} \mu \mathcal{A}^\nu - (\partial^\mu \mathcal{A}^\nu) \). In the Lorenz gauge the scalar and longitudinal contributions to \( \mathcal{J}^{\mu}_{\text{cov}} \) cancel, leaving only the contributions of the transverse modes \( \hat{S} \). If the standard Lagrangian density \( \mathcal{L}_{\text{std}} \) in the Coulomb gauge is used instead the Noether current is
\[
\mathcal{J}^{\mu}(x) = \frac{j \epsilon_0 c}{\hbar} \sum_{\epsilon, \lambda = \pm 1} \epsilon (E^\epsilon_{\lambda} (x) \cdot A^\lambda (x)) \tag{26}
\]
where \( c.c. \) is the complex conjugate. Since \( E^\lambda (x) = -\partial_{\lambda} A^\lambda (x) \) for the transverse modes \( \lambda = \pm 1 \), \( J^0(x) = J^{0}_{\text{cov}}(x) \). In a general gauge longitudinal modes are not excluded and \( \mathcal{J}^{\mu} \propto \sum_{\epsilon, \lambda = \pm 1} \epsilon (E^\epsilon \cdot A^\lambda - \mathbf{B}^\epsilon \times A^\lambda + E^\epsilon \phi^\lambda (x)) + c.c. \). The four-current density \( \mathcal{J}^{\mu}(x) \propto \mathcal{F}^{\mu\nu}(x) A^\nu(x) \) was first obtained in [7] where it was used to derive a Hermitian number density operator. A continuity equation with a charged matter source can be derived by evaluating \( \partial_{\tau} \mathcal{J}^{\mu} \) and substituting the equation of motion (14) to give
\[
\partial_{\tau} \mathcal{J}^{\mu}(x) = -\frac{j \mu_0}{\hbar c} \mathcal{A}^\nu (x) J_{\mu} + c.c. \tag{27}
\]
In the absence of a charged matter source (27) verifies that the four-currents (25) and (26) satisfy continuity equations and photon number is conserved.

For vector potentials \( A_1 \) and \( A_2 \) with \( \epsilon \) and \( \lambda \) components \( A^\epsilon_{1\lambda} \) and \( A^\epsilon_{2\lambda} \), the scalar product at a fixed time \( t \) derived from the spatial integral of the zeroth component of the four-current (26) is
\[
(A_1, A_2) = \int \frac{dk}{(2\pi)^3} \epsilon \int dt \mathbf{e}_\lambda \mathbf{k} \alpha^\epsilon_{\Lambda}(t, \mathbf{k}) (e^{-j\omega_{\Lambda} t + j\mathbf{k} \cdot \mathbf{x}} \tag{28}
\]
Substitution of \( E = -\partial_{\tau} \mathbf{A} \) and the one-photon amplitude \( E = \frac{1}{(2\pi)^3} \sqrt{\frac{\hbar}{\epsilon_0}} \) in (13) gives the one-photon SP four-potential
\[
A^\mu_{\lambda}(t, \mathbf{x}) = j \int \frac{dt}{\hbar} \int \frac{dk}{(2\pi)^3} \mathbf{e}_\lambda \mathbf{k} \alpha^\epsilon_{\Lambda}(t, \mathbf{k}) e^{-j\omega_{\Lambda} t + j\mathbf{k} \cdot \mathbf{x}}, \tag{29}
\]
and the scalar product (28) as
\[
(A_1, A_2) = \int \frac{dt}{(2\pi)^3} \sum_{\epsilon, \lambda = \pm 1} \int \frac{dk}{(2\pi)^3} \alpha^\epsilon_{\Lambda}(t, \mathbf{k}) \alpha^\epsilon_{\Lambda}(t, \mathbf{k}). \tag{30}
\]
In addition to a Hilbert space, standard quantum mechanics requires operators representing the relevant physical observables. The momentum, Hamiltonian, angular momentum and Lorentz reference frame are symmetries of the Lagrangian density [3] [3] [11]. The corresponding operators are the generators of translations in space, translation in time, rotations and boosts that should satisfy the commutation relations\( [\hat{J}_i, \hat{J}_j] = jh \varepsilon_{ijk} \hat{J}_k, \quad [\hat{J}_i, \hat{K}_j] = jh \varepsilon_{ijk} \hat{K}_k, \quad [\hat{K}_i, \hat{K}_j] = jh \delta_{ij} \hat{H}, \quad [\hat{K}_i, \hat{H}] = -jh \hat{P}_i, \quad [\hat{J}_i, \hat{H}] = \hat{P}_i, \quad [\hat{P}_i, \hat{H}] = 0 \) for \( i, j = 1, 2, 3 \). In the SP these operators are time independent. In configuration space \( \hat{P} = -jh\nabla, \quad \hat{P} = \hat{h} \sqrt{-\nabla^2}, \quad \hat{J} = jh \mathbf{x} \times \mathbf{v} + \hat{S} \) as in \( \hat{F} \) compliment \( B_0 \) where \( \hat{S} \) is the spin operator. In momentum space \( \hat{P} = \hat{h} \mathbf{k}, \quad \hat{P} = \hat{h} \mathbf{k}, \quad \hat{J} = -jh \mathbf{k} \times \nabla + \hat{S} \) and \( \hat{K} = jh (\mathbf{k} \nabla + \hat{S}) \) is the helicity operator. A Hamiltonian operator is also needed, but [22] is not of Schrödinger form required for description of unitary time evolution since its left hand side is multiplied by the trivector rather than the unit imaginary. This can be easily be remedied by multiplying [22] by \(-ji\). Using \( a \times b = -j (a \cdot \hat{S}) \) to write \( \nabla \times \mathbf{F} \) as \(-j (\hat{S} \cdot \nabla) \mathbf{F} \), [22] multiplied by \(-ji\)
becomes $jh\partial_t \mathbf{F} = ec\mathbf{p}\mathbf{F}$ which is of the Schrödinger form $jh\partial_t \mathbf{F} = \hat{H}\mathbf{F}$ for the Hamiltonian $\hat{H} = ec\mathbf{p}$. This Hamiltonian operator describes the time development of both positive and negative frequency waves.

III. RELATIONSHIP TO PREVIOUS WORK ON CONFIGURATION SPACE PHOTON QUANTUM MECHANICS

In this Section the BB-Sipe photon wave function as extended by Smith and Raymer and a number of recent proposals for photon quantum mechanics are discussed in approximate chronological order [1, 3, 11, 22]. This work was motivated by a desire to understand the relationship of photon quantum mechanics as formulated in [8, 19, 20] to the Dirac-like formulation in [1, 14, 21], leading to a study of spacetime algebra as presented in [3, 4], an appreciation of the fundamental geometrical significance of the RS-vector, and the realization that the trivector, $i$, need not be equated to the unit imaginary, $j$.

Motivated by the potential for applications in quantum optics, Bialynicki-Birula and Sipe [3, 11] independently proposed that the positive energy part of the six-component RS vector normalized to the average energy of a single photon is a good wave function "for practical purposes". Smith and Raymer state that "Maxwell unknowingly discovered a correct relativistic, quantum theory for the light quantum". They extended the BB-Sipe formalism to include a biorthogonal scalar product and modes that form an orthonormal set. Their scalar product is of the form $\langle \mathbf{F} | \mathbf{B} \rangle$. It is proportional to the spatial integral of the field multiplied by its dual. In $k$-space the potential is reduced by a factor $k^{-1}$ relative to the field so in configuration space the potential is nonlocal with a kernel proportional to $|\mathbf{x} - \mathbf{x}'|^{-2}$. This is a feature of any biorthogonal scalar product of the form $\langle \mathbf{F} | \mathbf{B} \rangle$. The authors note that this provides a link to the photon counting operator derived in [22].

Newton and Wigner solved the position eigenvector problem for spin $\frac{1}{2}$ electrons and spin 0 Klein-Gordon particles, but their method failed in the case of spin 1, that is for photons [27]. Following their method but with omission of their spherical symmetry assumption, a photon position operator with commuting components, $\hat{x}$, was derived in [15] and it was found in [9] that the biorthogonal photon position eigenvectors are cylindrically symmetrical. This cylindrical symmetry reflects the $e(2)$ symmetry of the photon little group [10]. To take this little group symmetry into account, "invariant under rotations about the origin" in NW’s assumptions should be generalized to "invariant under symmetry operations of the Wigner little group". For massive particles these operations are just those considered by NW, while for photons the little group consists of rotations about some conveniently chosen 3-axis and two boosts with commuting rotations [6, 10]. It was proved that $\{ \hat{J}_3, \hat{x}_1, \hat{x}_2 \}$ is a realization of the algebra. The photon position eigenvectors derived in [15] satisfy these modified postulates and describe the physics of optical beams [2, 10] that have a corresponding axis of symmetry.

Mostafazadeh and collaborators [16, 21] derived a positive definite number density by defining the conjugate field $A_c = j\hat{D}^{-1/2}\partial_\mathbf{a}A = c\mathbf{A}$ with $\hat{D} = -\nabla^2$. In the $\epsilon = \pm$ basis used here $\mathbf{A}^\epsilon = \hat{c}\mathbf{A}^\pm = c\mathbf{A}^\epsilon$. It can be verified by substitution that if $\mathbf{A}$ satisfies Maxwell’s wave equation, $\mathbf{A}_c$ also satisfies this equation and that the photon four-current $J^\mu(x) = \frac{-4\epsilon A_\nu(x)}{\hat{\partial}^\nu A^\nu(x)}$ satisfies a continuity equation. The photon density $J^0(x)$ is positive definite. Based on this number density they define a general scalar product for which all state vectors have a real positive norm.

Tamburini and Vicino [14] write (22) in covariant form using the Pauli matrices that are a representation of the $Cl_{1,3}$ spacetime algebra. They conclude that the photon wave function approach can lead only to results described by standard QED. Standard QED will be obtained by second quantization of the theory described here in Section II.

Refs. [8, 19, 20] are based on the second order wave equation for $\mathbf{A}$ and a scalar product that can be written in the form $\langle \mathbf{F} | \mathbf{B} \rangle$. They include a position operator and give an expression for the probability density as a function of $\mathbf{x}$ on the $t$-hyperplane. Babaei and Mostafazadeh derived Hamiltonian, position, helicity, momentum and chirality operators. Their Hamiltonian in [20] is equivalent to $\hat{H}$ derived in Section II and their position operator is derived in the Heisenberg Picture.

The Newton Wigner position eigenvectors are nonlocal in configuration space and they are not Lorentz covariant. Hawton and Debiere [8, 19] proved that the NW factor $k^{1/2}$ is a consequence of non-covariant normalization of the plane wave basis and that it can be eliminated if invariant plane wave normalization is used [29]. The position eigenvectors and their duals are potentials that transforms as Lorentz four-vectors and fields proportional to $\mathbf{E}_{\gamma \lambda}$ that at $t = 0$ is a pulse of twisted light localizable in an arbitrarily small region. They derive an expression for a positive and negative frequency position probability amplitudes. These functions have a nonlocal imaginary part instantaneously masked by destructive interference at $t = 0$ and that is nonzero everywhere for $t \neq 0$, consistent with the Hegerfeldt theorem [26]. True rather than apparent localization is achieved by adding positive and negative frequency waves to give a probability amplitude that is localized at $t = 0$ and propagates causally [8, 19].

Kiessling and Tahvildar-Zadeh [21] write the photon wave equation in first order Dirac form using gamma matrices and derive a rank-two bi-spinor photon wave function, $\psi_{ph}$, from a Lagrangian. Their $\sigma(\xi)$ and $\sigma(a)$ in $\psi_{ph}$ are Pauli representations of the paravectors $\xi$ and $a$. They identify $\mathbf{e} + i\mathbf{b}$ in $\xi_+ = \xi_- = (0, \mathbf{e} + i\mathbf{b})^T$ with the RS vector $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$. Using [2] and writing $\xi_+$ as
with $\Pi$ defined as a projection operator onto the diagonal of $\psi_{ph}$, can be written as

$$\left( \frac{m\hbar c}{\sqrt{i\hbar c + m\sqrt{\omega^2 - c^2}}} \left( \frac{c\partial A^+ - F^+}{\partial F^+} \right) - \frac{m\hbar c}{\sqrt{i\hbar c - m\sqrt{\omega^2 - c^2}}} \left( \frac{c\partial A^- - F^-}{\partial F^-} \right) \right) = 0$$

(32)

where the factor $\frac{m\hbar c}{\sqrt{i\hbar c + m\sqrt{\omega^2 - c^2}}}$ ensures that all the elements of (32) have the same dimensions. Then, according to (17), their $\xi_+ = F^+$ and $\xi_- = F^-$ satisfy Maxwell’s equations and are four-gradients of $A^\pm$. Eq. (32) incorporates (2) and (15) of Section II into a single first order matrix equation. Kiessling and Tahvildar-Zadeh derive a non-negative Born-rule-type quantum probability whose absolute square has dimensions of number density. Their equation (31) makes the doubled-up self-dual $F$ cannot be a wave function in a complex Hilbert space. If $i$ were to be identified with $j$ in (21), $F = 0$ in the case $c\lambda = -\frac{1}{3}$ so (15) here would have to be doubled-up to give a six-component vector describing photons of both helicities. The need for doubling is avoided in Section II by making a distinction between $i$ and $j$ so that the single 3-vector equation (22) describes photons and antiphotons of both helicities and $F$ is a good wave function. Their equation (31) makes explicit the relationship of photon quantum mechanics to the Dirac theory of electrons and positrons.

Wharton gives an insightful discussion of the difficulties encountered when imposing two initial/boundary conditions on the solutions to the KG equation (21). In his formalism boundary conditions are imposed at two different times. The complex Maxwell wave equation for $A$ is also second order and requires two boundary conditions. It is conventional in classical electromagnetism to reject the advanced Green’s function and keep only the retarded solution but careful examination of the mathematics shows that the advanced component determines the past history of the field, while the retarded component determines its future history (28). The device of the retarded solution is illusory and boundary condition are required to determine contributions from homogeneous solutions to the wave equation. Understanding of the relationships amongst boundary conditions, causality and emission by a localized source is still work in progress.

In the Landau-Peierls (LP) formalism the RS vector is divided by a factor proportional to $\sqrt{k}$ to give a function whose absolute square has dimensions of number density. This approach was found to be not very useful, since the LP wave function is non-locally connected to the classical electromagnetic field and the current density (13). Recently Sebens (22) postulated a Dirac-like equation of this type and again concluded that this approach is unsatisfactory because the probabilities derived from it do not always transform properly under Lorentz transformations. It appears that the LP function is not an acceptable photon wave function.

The photon wave function approach has been applied to emission of a photon by an atom and the propagation of electromagnetic waves in non-absorptive continuous media. Bialynicki-Birula derived the RS vector in a medium and in curved space (1). Sipe (11) and Debiere (18) calculate the photon wave function emitted by an atom in an excited state. Their positive frequency wave functions are intrinsically nonlocal, and this leads to apparent violations of Einstein causality. Keller provides an extensive discussion of the localization problem with an emphasis on the near field regime (12). In (22) the dressed photon wave function was second quantized and the quantum state of the photons generated in parametric down conversion was found to be intuitive but in agreement with previous QED based treatments. This approach was later applied to Raman scattering in which a phonon created in a Stokes process is absorbed by a second anti-Stokes photon to yield a highly correlated photon pair.

To sum up, all of the theories discussed this Section are consistent with the spacetime formulation in Section II and together they provide a complete picture of configuration space photon quantum mechanics. Bialynicki-Birula and Sipe recognized the importance of the RS vector. Hawton and Baylis derived a photon position operator with commuting components and investigated its properties (8, 15). Tamburini and Vicino derived (22) from the standard Lagrangian density and came to the correct conclusion that any first quantized theory of the photon could only lead to QED. Mostafazadeh and co-workers extended the Hilbert space to negative frequencies by deriving a number density that is positive definite for both positive and negative frequency waves (10, 20). Hawton and Debiere used biorthogonality to derive covariant position eigenvectors and concluded that only the real part of the positive and negative frequency position eigenvectors are truly localized and propagate causally (8, 19). Kiessling and Tahvildar-Zadeh derive a Dirac type wave equation from a Lagrangian (21). Refs. (8, 19, 21) include a position eigenvectors, while in (20, 21) a complete theory of quantum mechanics was emphasized.

Photon quantum mechanics is well established in k-space. Bialynicki-Birula and Bialynicka-Birula (2) Fourier transformed the k-space wave function and scalar product to configuration space to give a wave function and scalar product that are in excellent agreement with those obtained in Section II. The equation of motion (15) or (22) with $\nabla \cdot F = 0$ here is equivalent to (7) in (2). The Fourier expanded RS wave function (23) reduces to (10).
in §2 if the trivector $i$ is equated to the unit imaginary $j$ in (24). In the notation used here their k-space scalar product (2) is the biorthogonal form $(k^{-1}f_1|f_2)$ and its Fourier transformed configuration space equivalent (20) is also biorthogonal and equivalent to (28) here.

IV. CONCLUSION

In Section II equations of motion for $A^a$ and the RS vector $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$ and a continuity equation for a positive definite photon number density were derived from the complexified standard Lagrangian. The reverse of this derivation in which the k-space photon wave function and scalar product are transformed to configuration space compliments and supports the results obtained here. So what do these recent formulations of photon quantum mechanics contribute to our understanding of electromagnetism and QED? Their very existence implies that the photon is an elementary particle like any other and hence the photon is a good candidate for tests of the foundations of quantum mechanics. The photon wave function has been found to be useful in quantum optics for practical purposes, so perhaps this extension of the BB-Sipe wave function to full photon quantum mechanics and to non-dispersive and inhomogeneous media will find additional applications. These first quantized theories are consistent with QED as required for agreement with all experiments performed to date. It appears that a quantum interpretation of "classical" electromagnetic theory does indeed exist, as hinted at in Chapter 2 of Photons and Atoms where transverse angular momentum is written as the expectation value of $\hat{J}$ [3].

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