STEADY FLOW FOR SHEAR THICKENING FLUIDS WITH ARBITRARY FLUXES

GILBERLANDIO J. DIAS AND MARCELO M. SANTOS

Abstract. We solve the stationary Navier-Stokes equations for non-Newtonian incompressible fluids with shear dependent viscosity in domains with unbounded outlets, in the case of shear thickening viscosity, i.e. the viscosity $\mu$ is given by the power law $\mu = |D(v)|^{p-2}$, where $|D(v)|$ is the shear rate and $p > 2$. The flux assumes arbitrary given values and the Dirichlet integral of the velocity field grows at most linearly in the outlets of the domain. Under some smallness conditions on the “energy dispersion” we also show that the solution of this problem is unique. Our results are an extension of those obtained in [15] for Newtonian fluids ($p = 2$).

1. Introduction

The Navier-Stokes system for stationary incompressible flows in a domain with unbounded straight outlets, with the velocity field converging to parallel flows (Poiseuille flow) in the ends of the outlets, was solved first by C. Amick [2] in the 1970s. This problem is known as Leray problem, cf. [2, p. 476]. Amick’s solution assumes the fluxes of the fluid in the outlets to be sufficiently small, which turns out to be a sufficient condition to deal with the convective (nonlinear) term in Navier-Stokes equations. It is an open problem to solve Leray problem for arbitrary fluxes. Alternately, Ladyzhenskaya and Solonnikov [15] considered the stationary Navier-Stokes equations not demanding the fluid to be parallel in the ends of the outlets, but instead having arbitrary fluxes. In this case, the outlets do not need to be straight and they solved this new problem for domains having arbitrary uniformly bounded cross sections and with the fluid having arbitrary fluxes. Besides, their solution has the property that the Dirichlet’s integral of the velocity field of the fluid grows at most linearly with the direction of each outlet, and they also proved that this solution is unique under some additional smallness condition.

1991 Mathematics Subject Classification. 76D05, 76D03, 35Q30, 76D07.

Key words and phrases. Power law fluids, Ladyzhenskaya-Solonnikov problem, non-Newtonian fluids, shear thickening fluids, Ostwald-De Waele law, Leray problem.

Supported partly by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brasil, under grant 307192/2007-5.
In this paper we extend the Ladyzhenskaya-Solonnikov’s theorem, i.e. “Theorem 3.1” in [15], for power-law shear thickening fluids, i.e. incompressible non-Newtonian fluids obeying the power law

\[(1.1) \quad S = |D(v)|^{p-2}D(v),\]

when \(p > 2\). Here, \(S\) is the viscous stress tensor, \(v\) is the velocity field of the fluid and \(D(v)\) is the symmetric part of velocity gradient \(\nabla v\) (i.e. \(D_{ij}(v) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\) for \(v = (v_1, \cdots, v_n)\), \(i, j \in \{1, \cdots, n\}\), \(n \in \mathbb{N}\)). For \(p = 2\), the fluid is Newtonian. If \(1 < p < 2\), the fluid is called shear thinning (or plastic and pseudo-plastic) and if \(p > 2\), shear thickening (or dilatant). In engineering literature the power law (1.1) is also known as Ostwald-De Waele law (see e.g. [6]). Corresponding to (1.1) we have the following system of equations modelling the flow of an incompressible fluid in a stationary regime:

\[(1.2) \quad \begin{cases} -\text{div}(|D(v)|^{p-2}D(v)) + (v \cdot \nabla)v + \nabla P = 0 \\ \text{div} v = 0, \end{cases} \]

where \(P\) is the pressure function of the fluid (and \(v\) is the velocity field, as already indicated above). This model equations are also referred to as Smagorinsky model, due to [22], or Ladyzhenskaya model, due to [12, 13, 14]. A related model where the viscosity is given by \(|v|^{p-2}\), instead of \(|D(v)|^{p-2}\), is considered in [16]. For this case, it is shown in Remark 5.5 in Chap.2, §5.2 the existence of a (weak) solution for system (1.2) in a bounded domain with homogeneous Dirichlet boundary condition, for \(p \geq \frac{3n}{n+2}\). There are many results concerning the solution of (1.2) in bounded domains. For instance, in [9] the existence of a solution for (1.2) is obtained under the weaker condition that \(p \geq \frac{2n}{n+2}\).

In unbounded domains there are not so many results. For parallel fluids we can identify \(v\) with a scalar function \(v\) and the system (1.2) reduces to the \(p\)-Laplacian equation

\[(1.3) \quad -\text{div}(|\nabla v|^{p-2}\nabla v) = c \]

for some constant \(c\) (related to the “pressure drop”). So, we can consider the Leray problem for (1.2), i.e. the solution of (1.2) in a domain with straight outlets with the velocity field tending to the solution of (1.3) in the ends of the outlets. This problem was solved by E. Marušić-Paloka [17] under the condition that the fluxes are sufficiently small and \(p > 2\), thus extending Amick’s theorem [2] for power fluids with \(p \geq 2\). As far as we know, the Leray problem for (1.2) when \(p < 2\) (with small fluxes) is an open problem.

In this paper, as we mentioned above, we extend Ladyzhenskaya-Solonnikov’s theorem [15, Theorem 3.1] for (1.2) when \(p > 2\). More precisely, we obtain the existence of a solution \(v\) to the system (1.2) for \(n = 2, 3\), and \(p \geq 2\), in a domain \(\Omega\) with unbounded outlets, specified
in the next Section, for any given fluxes in the outlets and homogeneous Dirichlet boundary condition $v|\partial \Omega = 0$. The “Dirichlet integrals” $\int |\nabla v|^p$ of our solution grows at most linearly with the direction of the outlets (see (2.1) in Section 2). Besides, we observe that these integrals over portions of the outlets with a fixed ‘length’ are bounded by a constant that tends to zero with the flux (see Proposition 4.2 and Remark 4). Under this condition and some additional one, we have uniqueness of solution (see Theorem 4.4). All these facts were obtained in [15] for the case $p = 2$, but the power-law model ((1.2) with $p \neq 2$) was not treated in [15]. In the next two paragraphs we look at some facts relating to the case $p \neq 2$.

First, to deal with the nonlinear term $\text{div}(|D(v)|^{p-2} D(v))$ one can use the monotone method of Browder-Minty. Secondly, we extend the technique employed in [15] to obtain the existence of a solution, which, in particular, consists in first solving the problem in a bounded truncated domain and then taking the limit when the parameter of the truncation tends to infinity, to obtain a solution in the whole domain. To take this limit we need first uniform estimates with respect to the truncation parameter for the solution in the truncated domain, and this is obtained by integrating by parts the equation times the solution in some fixed bounded domain. Then we need the regularity of the solution in bounded domains, more precisely, that the solutions have velocity field at least in the Sobolev space $W^{2,l}$ and pressure in $W^{1,l}$, for some positive number $l$, due to the boundary terms that comes from the integration by parts. However this regularity is not expected for the weak solutions of (1.2), if $p \neq 2$. To overcome this difficult, when dealing with (1.2) in a truncated bounded domain we modify it to

$$\begin{cases}
- \text{div}\left\{ \left( \frac{1}{T} + |D(v)|^{p-2} \right) D(v) \right\} + (v \cdot \nabla)v + \nabla P = 0 \\
\text{div}v = 0,
\end{cases}$$

where $T > 0$ is the truncation parameter. See Proposition 4.1 in Section 4

As in [23] and [15], and in several subsequent papers, here the velocity field $v$ is sought in the form $v = u + a$, where $u$ is the new unknown with zero flux and $a$ is a constructed vector field carrying the given fluxes in the outlets (i.e. if the given flux in an outlet with cross section $\Sigma$ is $\alpha$ then $\int_{\Sigma} a \cdot n = \alpha$ and $\int_{\Sigma} u \cdot n = 0$, where $n$ is the unit normal vector to $\Sigma$ pointing toward infinity). This vector field $a$ depends on the geometry of the domain and, in the aforementioned papers, its construction is very tricky and makes use of the Hopf cutoff function (see [23, 15]). In the case of power-law fluids (1.2) with $p > 2$ we found out that the construction of $a$ can be quite simplified. Indeed, a key point in the construction, in any case, is to obtain a vector field $a$ that controls the quadratic nonlinear term $(u \nabla u)a$, which appears after substituting $v = u + a$ in (1.2) and multiplying it by $u$. That is, to obtain a priori
estimates, one multiplies the first equation in (1.2) by $u$ and try to bound all the resulting terms by the ‘leading’ term $|D(u)|^p$. In [15] it is shown that for any positive number $\delta$ there is a vector field $a$ which, in particular, satisfies the estimate
\[
\int_{\Omega_t} |u|^2 |a|^2 \leq c\delta^2 \int_{\Omega_t} |\nabla u|^2
\]
for some constant $c$ independent of $\delta$, $u$ and $\Omega_t$, where $\Omega_t$ is any truncated portion of the domain with a length of order $t$. Looking at their construction and using Korn’s inequality it is possible to show that
\[
(1.5) \quad \int_{\Omega_t} |u|^p |a|^{p'} \leq c\delta t^{(p-2)/(p-1)} \left( \int_{\Omega_t} |D(u)|^p \right)^{p'/p},
\]
where $p'$ is the conjugate exponent of $p$, i.e. $p' = p/(p-1)$. When $p = 2$ this estimate reduces to $|\int_{\Omega_t} (u \nabla u) a| \leq c\delta \int_{\Omega_t} |\nabla u|^2$. With this estimate we can estimate the integral of $(u \nabla u) a$ in the truncated domain $\Omega_t$, by using Hölder inequality:
\[
(1.6) \quad |\int_{\Omega_t} (u \nabla u) a| \leq \left( \int_{\Omega_t} |\nabla u|^2 \right)^{1/2} \left( \int_{\Omega_t} |u|^{2'} |a|^{2'} \right)^{1/2'} \leq c\delta \int_{\Omega_t} |\nabla u|^2.
\]
Thus we can control the nonlinear term $(u \nabla u) a$ by taking necessarily $\delta$ sufficiently small. When $p > 2$, proceeding similarly and using also Korn’s inequality, we obtain
\[
(1.7) \quad |\int_{\Omega_t} (u \nabla u) a| \leq c\delta t^{(p-2)/p} \left( \int_{\Omega_t} |D(u)|^p \right)^{2/p}.
\]
Then, by Young inequality with $\epsilon$, we have
\[
|\int_{\Omega_t} (u \nabla u) a| \leq \epsilon \int_{\Omega_t} |D(u)|^p + C_\epsilon t,
\]
for some new constant $C_\epsilon$. From this estimate, we can control the nonlinear term $(u \nabla u) a$ by taking $\epsilon$ sufficiently small, and so we do not need to construct the vector field $a$ satisfying the estimate (1.5) for a sufficiently small $\delta$. See Section 4 for the details. In fact, if $a$ is only a (smooth) bounded divergence free vector field vanishing on $\partial \Omega$, then, by Poincaré, Hölder and Korn inequalities, and the fact that our domain has uniformly bounded cross sections and $p/p' = p - 1 > 1$.
(p > 2), we have
\[
\int_{\Omega_t} |u|^p |a|^p \leq c \int_{\Omega_t} |u|^{p'} \leq c \int_{\Omega_t} |\nabla u|^{p'}/p \\
\leq ct^{1-p'/p} \left( \int_{\Omega_t} |\nabla u|^p \right)^{p'/p} \\
= ct^{(p-2)/(p-1)} \left( \int_{\Omega_t} |\nabla u|^p \right)^{p'/p} \\
\leq ct^{(p-2)/(p-1)} \left( \int_{\Omega_t} |D(u)|^p \right)^{p'/p}
\]
(1.8)
which is (1.5) for \( \delta = 1 \).

The plan of this paper is the following. Besides this introduction, in Section 2 we introduce the main notations and set precisely the problem we will solve, state a lemma about the existence of the vector field \( a \), carrying the flux of the fluid, and state our main theorem (Theorem 2.2). In Section 3 we state some preliminaries results we need to prove our main results. In Section 4 we prove our main theorem, make some remarks and prove a result about the uniqueness of our solution.

2. Ladyzhenskaya-Solonnikov Problem for power-law fluids

In this section we set notations and the problem we are concerned with and state a lemma and our main theorem.

We denote by \( \Omega \) a domain in \( \mathbb{R}^n \), \( n = 2, 3 \), with a \( C^\infty \) boundary, of the following type:
\[
\Omega = \bigcup_{i=0}^{2} \Omega_i ,
\]
where \( \Omega_0 \) is a bounded subset of \( \mathbb{R}^n \), while, in different cartesian coordinate system,
\[
\Omega_1 = \{ x \equiv (x_1, x') \in \mathbb{R}^n; x_1 < 0, x' \in \Sigma_1(x_1) \}
\]
and
\[
\Omega_2 = \{ x \equiv (x_1, x') \in \mathbb{R}^n; x_1 > 0, x' \in \Sigma_2(x_1) \},
\]
with \( \Sigma_i(x_1) \) being \( C^\infty \) simply connected domains (open sets) in \( \mathbb{R}^{n-1} \), and such that, for constants \( l_1, l_2, 0 < l_1 < l_2 < \infty \), they satisfy
\[
\sup_{(-1)^i x_1 > 0} \text{diam } \Sigma_i(x_1) \leq l_2
\]
and contain the cylinders
\[
C_i = \{ x \in \mathbb{R}^n; (-1)^i x_1 > 0 \text{ and } |x'| < \frac{l_1}{2} \} \subset \Omega_i ;
\]
i = 1, 2.
For simplicity, we will denote by $\Sigma$ any of the cross sections $\Sigma_i \equiv \Sigma_i(x_1)$ or, more generally, any cross section of $\Omega$, i.e., any bounded intersection of $\Omega$ with a $(n-1)$-dimensional plane. We will denote by $n$, the orthonormal vector to $\Sigma$ pointing from $\Omega_1$ toward $\Omega_2$, i.e., in the above local coordinate systems, we have $n = (1, 0)$ (where $0 \in \mathbb{R}^{n-1}$) in both outlets $\Omega_1$ and $\Omega_2$. With these notations, the flux through any cross section $\Sigma$ of $\Omega$ of an incompressible fluid in $\Omega$ with velocity field $v$ vanishing on $\partial \Omega$, is given by the 'surface' integral $\int_{\Sigma} v \cdot n$ (notice that by the divergence theorem applied to the region bounded by $\partial \Omega$, $\Sigma_1$ and $\Sigma_2$, we have $\int_{\Sigma_1} v \cdot n = \int_{\Sigma_2} v \cdot n$, for any cross sections $\Sigma_1$ and $\Sigma_2$ of $\Omega_1$ and $\Omega_2$, respectively).

We remark that we take our domain $\Omega$ with only two outlets $\Omega_i$, $i = 1, 2$, just to simplify the presentation, i.e. we can take $\Omega$ with any finite number of outlets with no significant change in the notations, results and proofs given in this paper.

We shall use the further notations, where $U$ is an arbitrary subdomain of $\Omega$, $s > t > 0$ and $1 \leq q < \infty$:

$$\begin{align*}
\Omega_{i,t} &= \{ x \in \Omega_i ; (-1)^i x_1 < t \}, \ i = 1, 2 \\
\Omega_{i,t,s} &= \Omega_{i,s} \setminus \Omega_{i,t} \\
\Omega_t &= \Omega_0 \cup \Omega_{1,t} \cup \Omega_{2,t} \\
\Omega_{t,s} &= \Omega_s \setminus \Omega_t \\
\| v \|_{q,U} &= \left( \int_U |v|^q \right)^{1/q} \\
\| v \|_{1,q,U} &= \left( \int_U |v|^q + |\nabla v|^q \right)^{1/q} \\
|v|_{1,q,U} &= \left( \int_U |\nabla v|^q \right)^{1/q} \\
(u, v)_U &= \int_U u \cdot v \\
\mathcal{D}(U) &= \{ \varphi \in C^\infty_c(U; \mathbb{R}^n); \nabla \cdot \varphi = 0 \} \\
\mathcal{D}^1_{0,q}(U) &= \mathcal{D}(U)^{1,q}_{\text{loc}}
\end{align*}$$

In these notations, the set $\Omega_t$ - a bounded cut of $\Omega$ with a “length” of order $t$ - will be taken usually for large $t$, so this notation will not cause confusion with the (unbounded) outlets $\Omega_i$, where $i = 1, 2$.

By $W^{1,q}(U)$ and $W^{1,q}_0(U)$ we stand for the usual Sobolev spaces, consisting of vector or scalar valued functions, and $W^{1,q}_{\text{loc}}(U)$ is the set of functions in $W^{1,q}(V)$ for any bounded open set $V \subset U$. Often when it is clear from the context we will omit the domain of integration in the notations.

The notation $|E|$ will stand for the Lebesgue measure of a Lebesgue measurable set $E$ in the dimension which is clear in the context. Finally, the same symbol $C, c, C'$ or $c'$ will denote many different constants.

In this paper, we are concerned with the following problem: given $\alpha \in \mathbb{R}$, find a velocity field $v$ and a pressure $P$ such that
Formally, multiplying (2.2) by \( \varphi = (\varphi_1, \cdots, \varphi_n) \in D(\Omega) \) and noticing that

\[
\sum_{i,j=1}^{n} D(u)_{ij} \frac{\partial \varphi_i}{\partial x_j} = \sum_{i,j=1}^{n} D(u)_{ij} D(\varphi)_{ij}
\]
and \( \int_{\Omega} \nabla P \cdot \varphi = -\int_{\Omega} P \nabla \cdot \varphi = 0 \), after integration by parts we get

\[
\int_{\Omega} |D(u) + D(a)|^{p-2} (D(u) + D(a)) : D(\varphi) = - (u \cdot \nabla u, \varphi) - (u \cdot \nabla a, \varphi) - (a \cdot \nabla u, \varphi) - (a \cdot \nabla a, \varphi),
\]

for all \( \varphi \in \mathcal{D}(\Omega) \), where for \( n \times n \) matrices \( A = (a_{ij}) \), \( B = (b_{ij}) \) we use the notation \( A : B = \sum_{i,j=1}^{n} a_{ij} b_{ij} \). Thus, the following definition for a weak solution to the problem (2.2) is in order.

**Definition 1.** A vector field \( u \) is said to be a weak solution to the problem (2.2) if it has the following properties:

i) \( u \in W^{1,p}_{loc}(\Omega) \);

ii) \( u \) satisfies (2.4) for every \( \varphi \in \mathcal{D}(\Omega) \);

iii) \( u \) satisfies (2.2)_2 - (2.2)_5.

Similarly, a vector field \( v \) is said to be a weak solution to the problem (2.1) if \( v \in W^{1,p}_{loc}(\Omega) \) and satisfies (2.1)_2 - (2.1)_5 and (2.4) with \( u + a \) replaced by \( v \), i.e.

\[
\int_{\Omega} |D(v)|^{p-2} D(v) : D(\varphi) = - (v \cdot \nabla v, \varphi)
\]

for all \( \varphi \in \mathcal{D}(\Omega) \).

**Remark 1.** The use of divergence free test functions \( \varphi \) in (2.4) eliminates the pressure \( P \), but it is a standard fact that it can be recovered due to ‘De Rham’s lemma’ (cf. e.g. [10, Lemma IV.1.1]).

We end this Section stating our main theorem, which we prove in Section 4.

**Theorem 2.2.** Let \( p \geq 2 \). Then, for any \( \alpha \in \mathbb{R} \), problem (2.1) has a weak solution \( v \), in the sense of Definition 1.

### 3. Preliminary results

In this Section we give some preliminary facts we shall need to prove our main results in Section 4. We begin with Lemma 3.1 below, which is due to Ladyzhenskaya and Solonnikov [15, Lemma 2.3]. Our statement below differs slightly from [15] and, for convenience of the reader, we present its proof, which essentially can be found in [15] and [20] [21].

**Lemma 3.1.** Let \( \Psi : \mathbb{R} \to \mathbb{R} \) be a strictly increasing function, \( \delta \) be a number in the interval \( (0, 1) \) and \( t_0 < T \).

i) If \( z \) and \( \varphi \) are differentiable functions in the interval \( [t_0, T] \) satisfying the inequalities

\[
\begin{align*}
    z(t) &\leq \Psi(z'(t)) + (1 - \delta) \varphi(t), \\
    \varphi(t) &\geq \delta^{-1} \Psi(\varphi'(t)),
\end{align*}
\]

for all \( t \in [t_0, T] \), and \( z(T) \leq \varphi(T) \), then

\[
z(t) \leq \varphi(t), \quad \forall t \in [t_0, T].
\]
ii) Suppose \( \Psi(0) = 0 \). If \( z : [t_0, \infty) \rightarrow [0, \infty) \) is a non-identically zero and non-decreasing differentiable function, and satisfies the inequality \( z(t) \leq \Psi(z'(t)), \) for all \( t \geq t_0, \) then \( \lim_{t \to \infty} z(t) = \infty. \) Besides, if \( \Psi(\tau) \leq \lambda t^m \) for all \( \tau \geq \tau_1, \) for some constants \( m > 1, \lambda > 0 \) and \( \tau_1 > 0, \) then
\[
\lim_{t \to \infty} t^{-\frac{m}{m-1}} z(t) > 0;
\]
\( If, \) however, \( \Psi(\tau) \leq c \tau, \) for \( \tau \geq \tau_1, \) then
\[
\lim_{t \to \infty} e^{-c/t} z(t) > 0.
\]

**Proof.** i) Suppose that \( \varphi(t_1) < z(t_1) \) for some \( t_1 \in [t_0, T) \). Then, by the first inequality, we have \( z(t_1) < \delta^{-1} \Psi(z'(t_1)) \), and so, using the second inequality, we have also \( \delta^{-1} \Psi(\varphi'(t_1)) \leq \varphi(t_1) < z(t_1) < \delta^{-1} \Psi(z'(t_1)) \), then, \( \Psi(\varphi'(t_1)) < \Psi(z'(t_1)) \). Since \( \Psi \) is strictly increasing, it follows that \( z'(t_1) > \varphi'(t_1) \). Consequently, \( z(t) > \varphi(t) \) for all \( t \) on a neighborhood on the right of \( t_1 \), and so, taking \( t_2 \) to be the supremum of these points in \( (t_1, T) \), we have \( t_1 < t_2 < T \) and, by the previous reasoning, we have \( z'(t) > \varphi'(t) \) for all \( t \) in \( (t_1, t_2) \), but this yields a contradiction, since \( z(t) - \varphi(t) > 0 \) is strictly positive at \( t = t_1 \) and must be zero at \( t = t_2 \).

ii) Let \( t_1 \geq t_0 \) such that \( z(t_1) > 0 \) and \( \lambda = \Psi^{-1}(z(t_1)) \). Notice that \( \lambda > 0 \), since \( \Psi(0) = 0 \) and \( \Psi \) is strictly increasing. As \( z \) is a non-decreasing function, we have that \( z(t) \geq z(t_1) \) for all \( t \geq t_1 \). Then we claim that \( z(t) \geq z(t_1) + \lambda(t - t_1) \) for all \( t \geq t_1 \). Indeed, the inequalities \( z(t) \geq z(t_1) \) and \( z(t) \leq \Psi(z'(t)) \) imply \( z'(t) \geq \psi^{-1}(z(t)) \geq \psi^{-1}(z(t_1)) = \lambda. \) Thus, we have shown the first statement in part 2) of the Lemma. For the remainder, notice that, since \( \lim_{t \to \infty} z(t) = \infty \), there exists a \( r \) such that \( z(t) \geq \tau_1 \) for all \( t > r \), so from \( \Psi(\tau) \leq c \tau \) and \( z(t) \leq \Psi(z'(t)) \) we have \( z(t) \leq c(z'(t))^m \) for all \( t > r, \) and the results then follow by direct integrating this inequality.

In the next lemma we collect three very useful inequalities. The first can be found in many texts, as for instance in [5] and [3], Lemma 2.1, p. 526. The third inequality contains Korn’s inequality (see [13]). The last one is a classical Poincaré type inequality; see e.g. [10] p.56. In these inequalities, \( c_1, c_2 \) are positive constants depending only on \( p \) and, for the last two, on the domain \( U \).

**Lemma 3.2.**

i) \[
(\|x\|^{p-2}x - |y|^{p-2}y, x - y) \geq c_1 \|x - y\|^2 \left( \|x\|^{p-2} + |y|^{p-2} \right) \geq c_2 \|x - y\|^p
\]
for all \( x, y \in \mathbb{R}^n \) and \( p \geq 2. \)

\*In [13], Korn’s inequality is stated for dimension three. The result in dimension two can be obtained from the one in dimension three by extending the domain \( U \subset \mathbb{R}^2 \) to \( U \times (0, 1) \) and the vector field \( \mathbf{v} : U \to \mathbb{R}^2 \) to \( (\mathbf{v}, 0) : U \times (0, 1) \to \mathbb{R}^3 \).
\[ c_1 |v|_{1,p,U} \leq \| D(v) \|_{p,U} \leq c_2 |v|_{1,p,U}, \]

for all \( v \in D^{1,p}(U) \) such that \( v|_{\Gamma} = 0 \).

\[ \| v \|_{q,U} \leq c_1 \left( |v|_{1,q,U} + \| v \|_{1,\Gamma} \right), \]

for all \( v \in W^{1,q}(U) \). In ii) and iii), \( U \) is an arbitrary bounded domain of \( \mathbb{R}^n \), \( n = 2, 3 \), with a smooth boundary, \( \Gamma \) is any Lebesgue measurable subset of \( \partial U \) with positive measure, and \( 1 \leq p < \infty \).

Next, we state a corollary of Brouwer fixed point theorem.

**Lemma 3.3.** Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous map such that for some \( \rho > 0 \), \( F(\xi) \cdot \xi \geq 0 \) for all \( \xi \in \mathbb{R}^n \) with \( |\xi| = \rho \). Then, there is a \( \xi_0 \in \mathbb{R}^n \) with \( |\xi_0| \leq \rho \) such that \( F(\xi_0) = 0 \).

For a proof, see [10, Lemma VIII.3.1] or [8, p. 493].

The next lemma yields a solution \( v \) of the equation \( \text{div} \ v = f \) satisfying a nice estimate. This result is an important step in the proof of our main theorem.

**Lemma 3.4.** Let \( U \) be a locally Lipschtzian and bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), and \( 1 < q < \infty \). Then there is a constant \( c \) such that, for any \( f \in L^q(U) \) satisfying \( \int_U f = 0 \), there is a vector field \( v \in W^{1,q}_0(U) \) such that \( \nabla \cdot v = f \) and \( \| v \|_{1,q,U} \leq c \| f \|_{q,U} \).

See [10, Teorema III.3.2].

The final result of this Section regards the regularised distance function to the boundary of a domain (an open connected set) in \( \mathbb{R}^n \).

**Lemma 3.5.** Let \( V \) be a domain in \( \mathbb{R}^n \) and \( d(x) = \text{dist}(x, \partial V) \), \( x \in V \). Then, there is a function \( \rho \in C^\infty(V) \) such that for every \( x \in V \) and any derivative \( \partial^\beta \), \( \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}_+ \), we have

\begin{equation}
\begin{aligned}
d(x) &\leq \rho(x) \\
|\partial^\beta \rho(x)| &\leq k_\beta (d(x))^{1-|\beta|},
\end{aligned}
\end{equation}

where \( k_\beta \) is a constant depending only on \( \beta \) and \( n \).

See [24, Theorem VI.2].

4. **Proof of Theorem 2.2 and other results**

In this section we prove Lemma 2.1 and our main theorem - Theorem 2.2. Besides, we make some remarks, prove a Proposition on the ‘uniform’ distribution of energy dissipation (Proposition 4.2) and a Theorem regarding the uniqueness of solution of problem (2.1).

We begin by proving Lemma 2.1. As we observed in the Introduction, the proof of this lemma (the construction of \( a \)) is simpler in this paper (i.e. for the case \( p > 2 \) than for the classical one for newtonian fluids.
(p = 2). For the construction in the case p = 2, see [15, p.744] and references therein; see also [10, Lemma XI.7.1, p. 272] and [19, p. 46].

**Proof of Lemma 2.1.** Suppose we have a vector field ̂a as in Lemma 2.1. Then the statements with respect to a = ̂a follow, with c depending on p, sup|x|^2(Σ), sup t | ̂a| and supΩ |∇ ̂a|. Indeed, for property Lemma 2.1, see (1.8). For property ii), we have

\[ \int_{\Omega_{i, t-1, t}} |\nabla a|^p \leq (\sup |\nabla ̂a|^p)(\sup |\Sigma|)|\alpha|^p \]

and iii) follows from ii):

\[ \int_{\Omega_{i, t-1, t}} |\nabla a|^p = \int_{\Omega_0} |\nabla a|^p + \sum_{i=1,2} \int_{\Omega_{i, t}} |\nabla a|^p \]

\[ \leq |\Omega_0|(\sup |\nabla ̂a|^p)|\alpha|^p + ((\sup |\nabla ̂a|^p)(\sup |\Sigma|)) |\alpha|^p t. \]

To construct a vector field ̂a with the properties in the statement of Lemma 2.1, first we observe that it is enough to construct in each outlet Ω_i a vector field a' satisfying these properties in Ω_i. Indeed, if we have this, then we can obtain the desired vector field ̂a defined in Ω by using appropriate cutoff functions. We omit this part of the proof and refer to [10, cap.VI] for a similar procedure in a domain with straight outlets and Poiseuille flows in place of the vector fields a', to be constructed below.

We first construct ̂a in the case n = 2. By what we observed above, it is enough to construct the vector field ̂a in an arbitrary outlet Ω_i, which we shall denote by Ω in this proof. Without loss of generality, we take Ω = \{x = (x_1, x_2) ∈ \mathbb{R}^2; f_1(x_1) < x_2 < f_2(x_1)\} for smooth functions f_1, f_2 such that f_1(x_1) ≤ -\frac{1}{2}, \frac{b}{2} ≤ f_2(x_1) and f_2(x_1) - f_1(x_1) ≤ l_2, for all x_1 ∈ \mathbb{R}. (l_1 < l_2 are positive numbers introduced in Section 2.) Then we set

\[ ̂a = \nabla \zeta \cdotp \zeta \equiv (\partial_{x_2} \zeta, -\partial_{x_1} \zeta), \]

for \( \zeta(x_1, x_2) = \psi(x_2/\rho(x)) \), where \( \rho(x) \) is the regularised distance to \( \partial \Omega \) (see Lemma 3.5) and \( \psi : \mathbb{R} \to \mathbb{R} \) is a smooth nondecreasing function such that \( \psi(s) = 0 \) if \( s < 0 \) and \( 1 \), if \( s > 1 \). We notice that \( \zeta \) is identically zero in the ‘lower band’ \( \{x ∈ \Omega; f_1(x_1) < x_2 < 0\} \) and identically one in a neighborhood of the ‘upper boundary’ \( \{x ∈ \partial \Omega; x_2 = f_2(x_1)\} \).

In particular, ̂a is a divergence free bounded vector field vanishing on a neighborhood of \( \partial \Omega \) and

\[ \int_{\Sigma} ̂a \cdot n = \int_{\Sigma} \zeta_{x_2} dx_2 = \zeta(x_1, f_2(x_1)) - \zeta(x_1, f_1(x_1)) = 1. \]

Now, because \( \zeta \) is constant in a neighborhood of each of the two components of \( \partial \Omega \), we have that any derivative of \( \zeta \) is zero in this neighborhood and, thus, bounded in \( \Omega \). Then ̂a and its derivatives are bonded function in \( \Omega \).

In the case n = 3, we take \( \zeta(x_1, x') = \psi(|x'|/\rho(x)), x' \equiv (x_2, x_3) ∈ \mathbb{R}^2 \), where \( \rho(x) \) is the regularised distance to \( \partial \Omega \) (see Lemma 3.5), \( \psi \) is
as above, but \( \psi(s) = 0 \) if \( s < 1 \) and \( 1 \), if \( s > 2 \). Then we set
\[
\tilde{a} = \nabla \times (\zeta b) = (\nabla \zeta) b,
\]
where \( b \) is the angle form in \( \mathbb{R}^2 \), i.e. \( b(x_2, x_3) = \frac{1}{2x_3 + x_2^2}(-x_3, x_2) \). Notice that \( \zeta \) constant for \( x' \) close to zero and equal to one in a neighborhood of \( \partial \Omega \) (i.e. \( \rho(x) \) close to zero), and thus, \( \zeta \) is a smooth function with bounded derivatives, vanishing in neighborhoods of \( x' = 0 \) and \( \partial \Omega \). Therefore, \( \tilde{a} \) is a smooth function vector with bounded derivatives. Beside, it is divergence free, and, by Stokes theorem in the plane, we have
\[
\int_\Sigma \tilde{a} \cdot n = \int_{\partial \Omega} b d\sigma = 1.
\]

To solve problem (2.2), first we shall solve the truncated modified problem, \( T > 0 \):
\[
\begin{cases}
\text{div}\{\left( \frac{1}{p} + |D(u) + D(a)|^{p-2} \right)(D(u) + D(a)) \} = u \cdot \nabla u + u \cdot \nabla u + a \cdot \nabla u + \nabla a \cdot \nabla P & \text{in } \Omega_T \\
\nabla \cdot u = 0 & \text{in } \Omega_T \\
u = 0 & \text{on } \partial \Omega_T
\end{cases}
\]
(4.1)

Then we will use Lemma 3.1 to obtain a weak solution of (2.2) by taking the limit, when \( T \to \infty \), in the solution \( u^T \) of (4.1), extended by zero outside \( \Omega_T \).

**Proposition 4.1.** Let \( p \geq 2 \) and \( T > 0 \). Then problem (4.1) has a solution \( (u^T, P) \) in \( D_0^{1,p}(\Omega_T) \times L^p(\Omega_T) \cap W^{2,l}(\Omega_T) \times W^{1,l}(\Omega_T) \), for any \( t \in (0, T) \), where \( l = 2q/(p + q - 2) \), being \( q = 2p + 2 \) if \( n = 3 \) and any number in \( [1, \infty) \) if \( n = 2 \).

**Proof.** The regularity part, i.e. \( (u^T, P) \in W^{2,l}(\Omega_T) \times W^{1,l}(\Omega_T) \), for any \( t \in (0, T) \), is a corollary of the proof of Theorem 1.2 in [4]. Notice that if \( (u^T, P) \) is a weak solution with \( u^T \) in \( D_0^{1,p} \) then \( v = u^T + a \) is a weak solution in \( W^{1,p}(\Omega_T) \) of
\[
\begin{cases}
\text{div}\{\left( \frac{1}{p} + |D(v)|^{p-2} \right)D(v) \} + v \cdot \nabla v + \nabla P = 0 & \text{in } \Omega_T \\
v = a & \text{on } \partial \Omega_T
\end{cases}
\]
(4.2)
The fact that we do not have here the homogeneous Dirichlet boundary condition \( v = 0 \) here in the whole boundary \( \partial \Omega_T \) does not affect the method given in [4] because \( a = 0 \) in \( (\partial \Omega_T) \cap (\partial \Omega) \) and the remaining part of \( \partial \Omega_T \), i.e., \( (\partial \Omega_T) \setminus (\partial \Omega) \), is interior to \( \Omega_T \).

Then we have only to show the existence of a weak solution for (4.1).

For simplicity, most of the time in this proof we shall write \( \Omega_T = \Omega \) and \( u^T = u \). Also we keep the notation \((\cdot, \cdot)\) with the integration over \( \Omega = \Omega_T \) in this proof, i.e. for (vector) functions \( v, w \) such that \( v \cdot w \in L^1(\Omega_T) \), \( (v, w) = \int_{\Omega_T} v \cdot w \). We will apply the Galerkin method and the monotonicity method of Browder-Minty (cf. [8], Remark, p. 497)). The Browder-Minty method is used due to the nonlinear term in the left hand side of (4.1).
Let \( \{ \varphi^j; j = 1, 2, \ldots \} \subset D(\Omega) \) be a denumerable and linearly independent set of functions whose linear hull is dense in \( D_0^{1,p}(\Omega) \). We shall write for \( m = 1, 2, \ldots \),

\[
(4.3) \quad u^m = \sum_{j=1}^{m} c_{jm} \varphi^j,
\]

where \( (c_{1m}, \ldots, c_{mm}) \in \mathbb{R}^m \) solves the algebraic system

\[
(4.4) \quad \frac{1}{p} \int_{\Omega} (D(u^m) + D(a)) : D(\varphi^j)
+ \int_{\Omega} |D(u^m) + D(a)|^{p-2} (D(u^m) + D(a)) : D(\varphi^j)
+ (u^m \cdot \nabla u^m, \varphi^j) + (u^m \cdot \nabla a, \varphi^j) + (a \cdot \nabla u^m, \varphi^j) = 0,
\]

\( j = 1, \ldots, m \). To see that system (4.4) has a solution \( (c_{1m}, \ldots, c_{mm}) \), let \( F = (F_1, \ldots, F_m) : \mathbb{R}^m \to \mathbb{R}^m \) be the map such that, for \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \), \( F_j(\xi) \) is defined by the left hand side of (4.4) with \( u^m = \sum_{j=1}^{m} \xi_j \varphi^j \). By Lemma 3.3 it is enough to show that there is a \( \rho > 0 \) such that \( F(\xi) \cdot \xi \geq 0 \) for all \( |\xi| = \rho \). Since \( (u^m \cdot \nabla u^m, u^m) = (a \cdot \nabla u^m, u^m) = 0 \), we have

\[
(4.5) \quad F(\xi) \cdot \xi = \frac{1}{p} \int_{\Omega} (D(u^m) + D(a)) : D(u^m)
+ \int_{\Omega} |D(u^m) + D(a)|^{p-2} (D(u^m) + D(a)) : D(u^m)
+ (u^m \cdot \nabla a, u^m) + (a \cdot \nabla a, u^m).
\]

By lemmas 2.1 and 3.2 and Hölder and Young inequalities, we obtain the following estimates (for some small positive numbers \( \varepsilon_i \), and some constants \( C_{\varepsilon_i}, c_i \), which may depend on \( m \):

\[
|u^m \cdot \nabla a, u^m| = |(u^m \cdot \nabla u^m, a)|
\leq |u^m|_{1,p} \left( \int_{\Omega} |a|^{p'} |u^m|^{p'/p} \right)^{1/p} \leq c |u^m|_{1,p} \leq \varepsilon_1 |u^m|_{1,p}^p + C_{\varepsilon_1};
\]

\[
|a \cdot \nabla a, u^m| \leq \varepsilon_2 |u^m|_{1,p}^p + C_{\varepsilon_2};
\]

\[
\int_{\Omega} |D(u^m) + D(a)|^{p-2} (D(u^m) + D(a)) - |D(a)|^{p-2} D(a) : D(u^m)
\geq c_p \int_{\Omega} |D(u^m)|^p \geq c_1 |u^m|_{1,p}^p,
\]

then

\[
(4.6) \quad \int_{\Omega} D(u^m) + D(a) |D(u^m) + D(a)|^{p-2} D(u^m) + D(a) : D(u^m)
\geq c_1 |u^m|_{1,p}^p + \int_{\Omega} |D(a)|^{p-2} D(a) : D(u^m),
\]

\[
(4.7) \quad \int_{\Omega} |D(u^m)|^p \geq c_1 |u^m|_{1,p}^p - \varepsilon_3 |u^m|_{1,p}^p + C_{\varepsilon_3};
\]

\[
(4.8) \quad \int_{\Omega} (D(u^m) + D(a)) : D(u^m) = \int_{\Omega} |D(u^m)|^2 + D(a) : D(u^m)
\geq \frac{1}{2} \int_{\Omega} |D(u^m)|^2 - \frac{1}{2} \int_{\Omega} |D(a)|^2 \geq c_1 |u^m|_{1,2}^2 - \frac{1}{2} \int_{\Omega} |D(a)|^2.
\]
Then, taking $\varepsilon_i$, $i = 1, 2, 3$, sufficiently small, and noticing that $|u^m|_{1,q}^p \geq c|u^m|_{1,2}^2 \geq c_1|\xi|^2$ (notice that $|\xi| = |u^m|_{1,2}$ is a norm in $\mathbb{R}^m$), from (4.5) we get

$$F(\xi) \cdot \xi \geq c_1|\xi|^2 - c_2 \geq 0$$

for all $\xi \in \mathbb{R}^m$ such that $|\xi| \geq \sqrt{c_2/c_1}$, for some positive constants $c_1$, $c_2$.

Next, we notice that $|u^m|_{1,p}$ is uniformly bounded with respect to $m$. Indeed, multiplying (4.4) by $\xi_j$ and summing in $j$ from 1 to $m$, we obtain, as in (4.5),

$$\frac{1}{2} \int_\Omega |D(u^m)|^2 + \frac{1}{q} \int_\Omega D(u^m) : D(a)
+ \int_\Omega |D(u^m) + D(a)|^{p-2}(D(u^m) + D(a)) : D(u^m)
+ (u^m \cdot \nabla a, u^m) + (a \cdot \nabla a, u^m) = 0,$$

and then, proceeding with similar estimates to obtain (4.6), (4.8) and (4.9), we arrive at

$$\frac{1}{2} \int_{\Omega_T} |D(u^m)|^2 + |u^m|_{1,p}^p \leq c,$$

for some constant $c$. Thus, there exists a subsequence of $\{u^m\}$, which we still shall denote by $\{u^m\}$, and a vector field $u \in \mathcal{D}^{1,p}_0(\Omega_T)$ such that

$$u^m \rightharpoonup u \quad \text{in} \quad \mathcal{D}^{1,p}_0(\Omega_T)
\quad u^m \to u \quad \text{in} \quad L^q(\Omega_T)$$

when $m \to \infty$, where $q \geq 1$ is any number less than the critical Sobolev exponent $p^* := \frac{np}{n-p} = \frac{3p}{3-p}$, if $n = 3$ and $p < 3$, and $1 \leq q < \infty$ is arbitrary, if $p \geq n$ ($n = 2, 3$). In particular, $1 \leq q < \infty$ is arbitrary for $n = 2$, since $p > 2$.

Now we want to pass to the limit in (4.4) when $m \to \infty$ and obtain it with $u$ in place of $u^m$ and with any $\varphi \in \mathcal{D}(\Omega)$ in place of $\varphi^j$. We begin by defining the operators

$$A(w) = -\text{div}\{|D(w) + D(a)|^{p-2}(D(w) + D(a))\},$$

and

$$C(w) = -\frac{1}{T}\text{div}\{D(w) + D(a)\},$$

for $w \in \mathcal{D}^{1,p}_0(\Omega)$. More precisely, $A$ and $C$ are operators from $\mathcal{D}^{1,p}_0(\Omega)$ into $\mathcal{D}^{1,p}_0(\Omega)'$, defined by

$$\langle A(w), \varphi \rangle = \int_{\Omega_T} |D(w) + D(a)|^{p-2}(D(w) + D(a)) : D(\varphi),$$

and

$$\langle C(w), \varphi \rangle = \int_{\Omega_T} (D(w) + D(a)) : D(\varphi).$$

*Here we write explicitly $\Omega_T$, instead of $\Omega$, for future reference.
Notice that $D(w) + D(a) \in L^p(\Omega)$ because $p > 2 \Rightarrow p' < p$ and $\Omega = \Omega_T$ is a bounded domain. We also write

$$B(w) = -(w \cdot \nabla w + w \cdot \nabla a + a \cdot \nabla w + a \cdot \nabla a).$$

So we want to show that $(A(u) + C(u), \varphi) = (B(u), \varphi)$ for every $\varphi$ in $\mathcal{D}(\Omega)$, or, equivalently, for every $\varphi$ in $\mathcal{D}_0^{1,p}(\Omega)$.

By (4.12), we have

$$(4.12) \quad \langle A(u^m) + C(u^m), \varphi \rangle = (B(u^m), \varphi)$$

for all $\varphi \in \mathcal{D}_0^{1,p}(\Omega)$ and all $m = 1, 2, \ldots$.

Since $|u^m|_{1,p}$ is uniformly bounded, by H"older inequality we have that the sequence $\{A(u^m)\}$ is a bounded sequence in $\mathcal{D}_0^{1,p}(\Omega)'$, so there is a $\chi \in \mathcal{D}_0^{1,p}(\Omega)'$ and a further subsequence $\{u^{m_k}\}$ such that

$$(4.13) \quad \langle A(u^{m_k}), \varphi \rangle \to (\chi, \varphi)$$

for all $\varphi \in \mathcal{D}_0^{1,p}(\Omega)$. Next, we show that

$$(4.14) \quad (B(u^{m_k}), \varphi) \to (B(u), \varphi), \quad \forall \varphi \in \mathcal{D}_0^{1,p}(\Omega).$$

First we notice that

$$\|(u^m \cdot \nabla u^m, \varphi) - (u \cdot \nabla u, \varphi)\| \leq \|u^m - u\|_p \|\nabla u^m\|_p \|\varphi\|_q + \|u\|_q \|\nabla \varphi\|_p \|u^m - u\|_q \to 0,$$

where $q$ is large enough such that $\frac{2}{q} + \frac{1}{p} \leq 1$ and less than $p^* := \frac{np}{n-p}$ if $p < n$. Notice that if $p < n$ then $n = 3$ in this paper and $p > 2$ and, since $p > 2$, we have $\frac{2}{p^*} + \frac{1}{p} < 5/6$. Similarly, and more easily, we also have

$$\|(a \cdot \nabla u^m, \varphi) - (a \cdot \nabla u, \varphi)\| \to 0.$$

Thus we have shown (4.14). From (4.11) and the fact that $p > 2$ and $\Omega = \Omega_T$ is bounded, we also have $\lim \langle C(u^m), \varphi \rangle = \langle C(u), \varphi \rangle$ for all $\varphi \in \mathcal{D}_0^{1,p}(\Omega)$. Then, from (4.12)-(4.14), we have $\chi + C(u) = B(u)$ in $\mathcal{D}_0^{1,p}(\Omega)'$. Then, to conclude the proof, it remains to show that $\chi = A(u)$. To see this, it is enough now to show that $\langle A(u^m), u^m \rangle$ converges to $\langle \chi, u \rangle$, since, by Lemma 3.2, the operator $A$ is monotone. Indeed, we have the following classical argument for monotone operators. From

$$\langle A(u^m) - A(w), u^m - w \rangle \geq 0,$$ i.e.

$$\langle A(u^m), u^m \rangle - \langle A(u^m), w \rangle - \langle A(w), u^m \rangle + \langle A(w), w \rangle \geq 0,$$

if $\langle A(u^m), u^m \rangle$ converges to $\langle \chi, u \rangle$ then, by (4.11) and (4.13), we can take the limit in this inequality when $m \to \infty$ and obtain $\langle \chi - A(w), u - w \rangle \geq 0$, for all $w \in \mathcal{D}_0^{1,p}(\Omega)$. Now replacing $w$ by $u - \lambda w$, for $\lambda \in \mathbb{R}^+$, we arrive at $\langle \chi - A(u - \lambda w), w \rangle \geq 0$ for all $w \in \mathcal{D}_0^{1,p}(\Omega)$ and
all $\lambda \in \mathbb{R}^+$. Then the desired result follows, once one shows that
\[
\lim_{\lambda \to 0^+} \langle A(u - \lambda w), w \rangle = \langle A(u), w \rangle.
\] Here, we can show this using the Lebesgue’s dominated convergence theorem, since the integrand in
\[
\langle A(u - \lambda w), w \rangle
\] is dominated, for any $\lambda \in (0, 1)$, by some constant times the function $|D(u)|^{p-1} + |D(w)|^{p-1} + |D(a)|^{p-1}|D(w)|$, which belongs to $L^1(\Omega)$.

To show that $\langle A(u^m), u^m \rangle = (B(u^m), u^m) - \langle C(u^m), u^m \rangle$ converges
to $\langle \chi, u \rangle$ which is equal to $(B(u), u) - \langle C(u), u \rangle$, we write
\[
(B(u^m), u^m) - (B(u), u) = [(B(u^m), u^m) - (B(u), u^m)] + (B(u), u^m - u)
\]
and notice that the two last terms converge to zero, when $m \to \infty$, by the estimates above we used to obtain (4.14). It is easy to see, using again (4.11) and the fact that $p > 2$ and $\Omega = \Omega_T$ is bounded, that we have also $\lim C(u^m, u^m) = \langle C(u), u \rangle$.

Next, given any $t > 0$, we show that the solution $u^T$ of (1.1) is uniformly bounded in $D_0^{1,p}(\Omega_t)$, with respect to $T$, for $T \geq t + 1$. Proceeding similarly to (4.8), we introduce the function
\[
y(t) = \frac{1}{T} |u^T|^2_{1,2,\Omega_t} + |u^T|^p_{1,p,\Omega_t}, \quad t > 0, \quad T \geq t + 1.
\]

In the sequel we write $u^T = u$ and often $u + a = v$. Multiplying (1.1) by $u$ and integrating by parts, using that $u|\partial \Omega = 0$, we have
\[
y(t) = \frac{1}{T} \int_{\Omega_t} |D(u)|^2 + \int_{\Omega_t} |D(v)|^{p-2} D(v) : D(u)
\] 
\[
= -\frac{1}{2} \int_{\Omega_t} D(a) : D(u) + \int_{\Omega_t} (u \cdot \nabla u \cdot a - a \cdot \nabla u \cdot u - a \cdot \nabla a \cdot u)
\] 
\[
+ \int_{\Omega_t} (u \cdot (D(v)n) + u \cdot ((D(v))^{p-2} D(v)n)) - \int_{\Omega_t} \left(\frac{1}{2} |u|^2 (u \cdot n) + (u \cdot a)(u \cdot n) + (u \cdot n) P\right),
\]
where $\Gamma_t = \Sigma(t) \cup \Sigma(-t)$. First we estimate the ‘interior’ integrals $\int_{\Omega_t} \cdots$. Using Young inequality and Lemma (2.1) we get
\[
\int_{\Omega_t} D(a) : D(u) \leq \varepsilon \int_{\Omega_t} |D(u)|^2 + C_\varepsilon,
\]
and
\[
\int_{\Omega_t} (u \cdot \nabla u \cdot a - a \cdot \nabla u \cdot u - a \cdot \nabla a \cdot u)
\]
\[
\leq |u|_{1,p,\Omega_t} \left(\int_{\Omega_t} \left|a|^{p'}|u|^{p'}\right)^{1/p'} + \int_{\Omega_t} (a \cdot \nabla u \cdot u - a \cdot \nabla a \cdot u)\right)
\]
\[
\leq \varepsilon |u|^p_{1,p,\Omega_t} + C_\varepsilon t,
\]
where $\varepsilon > 0$ is fixed below. Besides, proceeding as in (4.8), we get
\[
\int_{\Omega_t} |D(v)|^{p-2} D(v) : D(u) \geq c_p |u|^p_{1,p,\Omega_t} + \int_{\Omega_t} |D(a)|^{p-2} D(a) : D(u)\]
and
\[
\left| \int_{\Omega_t} |D(a)|^{p-2} D(a) : D(u) \right| \leq \varepsilon |u|_{1,p,\Omega_t}^p + C\varepsilon t.
\]

Then, from (4.16)-(4.19) and taking \( \varepsilon \ll 1 \), we obtain
\[
y(t) \leq c_1 t + I,
\]
where
\[
I = \int_1^{\eta} \left[ \frac{1}{T} \mathbf{u} \cdot D(v) \mathbf{n} + \mathbf{u} \cdot |D(v)|^{p-2} D(v) \mathbf{n} - \frac{1}{2} |\mathbf{u}|^2 (\mathbf{u} \cdot \mathbf{n}) - (\mathbf{u} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{n}) - (\mathbf{u} \cdot \mathbf{n})P \right].
\]

Now the idea is to control the boundary integral \( I \) by the interior integral \( y(t) \), but if for instance one tries to apply the trace theorem then higher order derivatives arise. To achieve that purpose we use the clever idea given in [15] for the case \( p = 2 \), that is, to integrate \( I \equiv I(t) \) from \( \eta - 1 \) to \( \eta \), for \( \eta > 1 \), or better, integrate the estimate (4.21). Thus we introduce the function
\[
z(\eta) = \int_{\eta - 1}^{\eta} y(t) dt.
\]

Notice that since \( y \) is a nondecreasing function we have \( y(\eta - 1) \leq z(\eta) \leq y(\eta) \) for all \( \eta > 1 \), thus estimating \( y \) is the same as estimating \( z \). Another interesting feature of the function \( z \) is that
\[
z'(\eta) = y(\eta) - y(\eta - 1) = \frac{1}{T} \left[ |\mathbf{u}|_{1,2,\Omega_{\eta - 1},\eta}^2 + |\mathbf{u}|_{1,p,\Omega_{\eta - 1},\eta}^p \right]
\]
then if we estimate \( \int_{\eta - 1}^{\eta} I(t) dt \) in terms of \( |\mathbf{u}|_{1,p,\Omega_{\eta - 1},\eta}^p \) and \( \frac{1}{T} |\mathbf{u}|_{1,2,\Omega_{\eta - 1},\eta}^2 \), in the end, in virtue of (4.21), we shall obtain a estimate for \( z(\eta) \) in terms of \( z'(\eta) \). Then we shall use Lemma 3.1 to get the desired estimate for \( z(\eta) \). Let’s do the details.

By (4.21) and (4.23), and the fact that \( \int_{\eta - 1}^{\eta} \int_{\Gamma_t} \cdot = \int_{\Omega_{\eta - 1},\eta} \cdot \), we have
\[
z(\eta) \equiv \int_{\eta - 1}^{\eta} y(t) dt \leq c_1 \eta + \frac{1}{T} I_1 + I_2 + I_3 + I_4 + I_5,
\]
where
\[
\begin{align*}
I_1 &= \int_{\Omega_{\eta - 1},\eta} \mathbf{u} \cdot D(v) \mathbf{n} \\
I_2 &= \int_{\Omega_{\eta - 1},\eta} \mathbf{u} \cdot |D(v)|^{p-2} D(v) \mathbf{n} \\
I_3 &= -\int_{\Omega_{\eta - 1},\eta} \frac{1}{2} |\mathbf{u}|^2 (\mathbf{u} \cdot \mathbf{n}) \\
I_4 &= -\int_{\Omega_{\eta - 1},\eta} (\mathbf{u} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{n}) \\
I_5 &= -\int_{\Omega_{\eta - 1},\eta} P(\mathbf{u} \cdot \mathbf{n}).
\end{align*}
\]
Using Hölder inequality, Lemma (2.11ii), Poincaré inequality (Lemma (3.2.iii)) and Young inequality, we have

\[ |I_2| \leq \int_{\Omega_{\eta-1,\eta}} |D(v)|^{p-1}|u| \]
\[ \leq c \left( \int_{\Omega_{\eta-1,\eta}} |D(u)|^{p-1}|u| + \int_{\Omega_{\eta-1,\eta}} |D(a)|^{p-1}|u| \right) \]
\[ \leq c \left( |u|_{L^p(\Omega_{\eta-1,\eta})}^p + |a|_{L^p(\Omega_{\eta-1,\eta})}^p \right) \]
\[ \leq c \left( |u|_{L^p(\Omega_{\eta-1,\eta})}^p + |u|_{L^p(\Omega_{\eta-1,\eta})}^p \right), \]

so

\[ \hat{I}_2 \leq c \left( z'(\eta) + z'(\eta)^{1/p} \right). \]

Analogously,

\[ \hat{I}_1 \leq c \left( z'(\eta) + z'(\eta)^{1/2} \right). \]

Regarding \( I_3 \) and \( I_4 \), using Sobolev embedding, we get

\[ |I_3| + |I_4| \leq \int_{\Omega_{\eta-1,\eta}} \frac{1}{2}|u|^3 + \int_{\Omega_{\eta-1,\eta}} c|u|^2 \]
\[ \leq c \left( |u|_{L^p(\Omega_{\eta-1,\eta})}^3 + |u|_{L^p(\Omega_{\eta-1,\eta})}^2 \right) \]
\[ = c \left( z'(\eta)^{3/p} + z'(\eta)^{2/p} \right). \]

To estimate \( I_5 \), we use Lemma (3.3). Let \( w \) be a vector field in \( W_0^{1,p}(\Omega_{\eta-1,\eta}) \) such that \( \nabla \cdot w = u \cdot n \) and \( |w|_{L^p(\Omega_{\eta-1,\eta})} \leq c|u|_{L^p(\Omega_{\eta-1,\eta})} \), where \( c \) is some constant, independent of \( w \) and \( u \). Then, using the equation (1.1) \( 1 \), we can write

\[ I_5 = -\int_{\Omega_{\eta-1,\eta}} P(u \cdot n) = \int_{\Omega_{\eta-1,\eta}} \nabla \cdot \nabla P w \]
\[ = \int_{\Omega_{\eta-1,\eta}} |D(v)|^{p-2}D(v) : D(w) + \int_{\Omega_{\eta-1,\eta}} (u \cdot \nabla u) \cdot w + \int_{\Omega_{\eta-1,\eta}} (a \cdot \nabla a) \cdot w. \]

Thus, proceeding with similar estimates to those used to obtain (4.27)-(4.28), we arrive at

\[ |I_5| \leq c \left( z'(\eta) + z'(\eta)^{1/p} + z'(\eta)^{2/p} + z'(\eta)^{3/p} \right). \]

From (4.25)-(4.29), we have

\[ z(\eta) \leq c_1 \eta + \Psi(z'(\eta)), \]

for all \( \eta \geq 1 \), with \( \Psi(\tau) = c_2 \left( \tau + \tau^{1/2} + \tau^{1/p} + \tau^{2/p} + \tau^{3/p} \right) \). Now, from (4.10) and the weak convergence (4.11), we have \( y(T) \leq c \) for some constant \( c \) (independent of \( T \)), so by \( z(T) \leq y(T) \) and by assuming that \( c_1 \geq c \), without loss of generality, we have

\[ z(T) \leq c_1 T. \]

Next, let \( c_3 > 0 \) be a constant such that

\[ 2c_1 + c_3 \geq 2\Psi(2c_1). \]

Then, by (4.30)-(4.32), we have the conditions of Lemma (3.3) satisfied, with \( \varphi(\eta) = 2c_1 \eta + c_3, \delta = 1/2, t_0 = 1, T > 1 \) (arbitrary). Therefore,
This lead us to localize the arguments and operators used in that proof, as follows.

Proof of Theorem \ref{thm:existence}. Let \(u^k\) be the solution of (4.1) in \(\Omega_k\), \(k = 3, 4, \cdots\), whose existence is assured by Proposition \ref{prop:existence} and set \(u^k = 0\) in \(\Omega / \Omega_k\). By (4.33), for each \(j = 2, 3, \cdots\), the sequence \(\{u^k\}_{k \geq j+1}\) is weakly compact in \(W^{1,p}(\Omega_j)\), thus, by a diagonalization process we obtain a subsequence, which we also denote by \(\{u^k\}\), and an \(u\) in \(W^{1,p}_{\text{loc}}(\overline{\Omega})\) such that

\[
\begin{align*}
\lim_{k \to \infty} u^k &\rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega_t), \\
\lim_{k \to \infty} u^k &\to u \quad \text{in} \quad L^q(\Omega_t),
\end{align*}
\]

for any \(t > 0\), where \(q \geq 1\) is arbitrary, if \(p \geq n\), and less than \(p^*: = \frac{3p}{3-p}\) if \(n = 3\) and \(p < 3\). (Cf. (1.11)). Besides, by (4.34), the estimate (4.33) and the fact that \(u^k \in D_0^{1,p}(\Omega)\), we have that the limit \(u\) satisfies (2.2). Then, to conclude the proof of Theorem \ref{thm:existence} it remains to prove that \(u\) satisfies the equation (2.4), in the weak sense (2.4). Again, we shall use the Browder-Minty method, due to the shear dependent viscosity. The idea here is to mimic the proof of Proposition \ref{prop:existence} paying attention that now \(\Omega\) is not a bounded domain and \(D(u)\) is only locally integrable in \(\overline{\Omega}\). This lead us to localize the arguments and operators used in that proof, as follows.

Given \(\varphi \in D(\Omega)\), letting \(k_0 \in \mathbb{N}\) such that \(\text{supp} \, \varphi \subset \Omega_{k_0-1}\), we have

\[
\int_{\Omega_{k_0}} S_k(u^k) : D(\varphi) = \int_{\Omega_{k_0}} B(u^k) \cdot \varphi,
\]

for all \(k \geq k_0\), where

\[
S_k(w) = \left( \frac{1}{k} + |D(w) + D(a)|^{p-2} \right) (D(w) + D(a))
\]

and

\[
B(w) = - (w \cdot \nabla w + w \cdot \nabla a + a \cdot \nabla w + a \cdot \nabla a).
\]

Then, we want to pass to the limit in (4.35) when \(k \to \infty\) and obtain (2.4). Let \(\zeta: \Omega \to \mathbb{R}_+\) be a smooth function such that \(\zeta = 1\) in \(\text{supp} \, \varphi\) and \(\zeta = 0\) in \(\Omega \setminus \Omega_{k_0}\) and \(A_\zeta, A_\zeta,k\) be the operators defined by

\[
\begin{align*}
\langle A_\zeta,w_1, w_2 \rangle &= \int_{\Omega_{k_0}} (S_k(w_1) : D(w_2))
\end{align*}
\]

on the space

\[
V_0 \equiv W^{1,p}(\Omega_{k_0}, \partial \Omega \cap \partial \Omega_{k_0}) := \{ w \in W^{1,p}(\Omega_{k_0}); w = 0 \text{ in } \partial \Omega \cap \partial \Omega_{k_0} \},
\]
where
\[ S(w) = |D(w) + D(a)|^{p-2}(D(w) + D(a)). \]

Thus, (4.35) becomes
\[ (4.36) \quad \langle A_{\zeta,k}(u^k), \varphi \rangle = (B(u^k), \varphi) \]
and (2.4) becomes
\[ (4.37) \quad \langle A_\zeta(u), \varphi \rangle = (B(u), \varphi). \]

We notice, as \( \zeta \) is a nonnegative function, that \( A_{\zeta,k} \) is still a monotone operator. Besides, \( \{A_{\zeta,k}(u^k)\} \) is a bounded sequence in \( V_0' \), then, up to a subsequence, we have \( A_{\zeta,k}(u^k) \xrightarrow{\text{a.e.}} \chi_\zeta \) for some \( \chi_\zeta \) in \( V_0' \). As in (4.14), we also have
\[ (4.38) \quad \langle B(u^k), \varphi \rangle \to (B(u), \varphi). \]

Then, by (4.36), we obtain \( \langle \chi_\zeta, \varphi \rangle = (B(u), \varphi) \), so it remains to show that \( \chi_\zeta = A_\zeta(u) \). To obtain this, from the monotonicity of \( A_{\zeta,k} \), it is enough to prove that \( \langle A_{\zeta,k}(u^k), u^k \rangle \) converges to \( \langle \chi_\zeta, u \rangle \). Indeed,
\[ (4.39) \quad \langle A_{\zeta,k}(u^k), u^k \rangle - \langle A_{\zeta,k}(u^k), w \rangle - \langle A_{\zeta,k}(u^k), u^k \rangle + \langle A_{\zeta,k}(w), w \rangle \geq 0, \]
for all \( w \in V_0 \) and, by (4.34), \( \langle A_{\zeta,k}(w), u^k \rangle \) and \( \langle A_{\zeta,k}(u^k), w \rangle \) tend, respectively, to \( \langle A_{\zeta,k}(w), u \rangle \) and \( \langle A_{\zeta,k}(u), w \rangle \), when \( k \to \infty \). Then, once we have
\[ \lim_{k \to \infty} \langle A_{\zeta,k}(u^k), u^k \rangle = \langle \chi_\zeta, u \rangle, \]
we shall have \( \langle \chi_\zeta - A_\zeta(u - \lambda w), w \rangle \geq 0 \)
for all \( w \in V_0 \) and all \( \lambda \geq 0 \), and by Lebesgue’s dominated convergence theorem, \( \lim_{\lambda \to 0^+} \langle A_\zeta(u - \lambda w), w \rangle = \langle A_\zeta(u), w \rangle \), hence \( \chi_\zeta = A_\zeta(u) \).

Let us show then that \( \lim_{k \to \infty} \langle A_{\zeta,k}(u^k), u^k \rangle = \langle \chi_\zeta, u \rangle \). We compute \( \langle \chi_\zeta, u \rangle \) and \( \lim_{k \to \infty} \langle A_{\zeta,k}(u^k), u^k \rangle \) using directly the equation (4.11), with \( T = k \). Multiplying this equation by \( \zeta u \) and integrating by parts in \( \Omega_{\varepsilon_0} \), we arrive at
\[ (4.40) \quad \frac{\langle A_{\zeta,k}(u^k), u \rangle}{\zeta} = \int_{\Omega_{\varepsilon_0}} B(u^k) : \zeta u - \int_{\Omega_{\varepsilon_0}} \mathcal{P}^k u : \nabla \zeta = \int_{\Omega_{\varepsilon_0}} u : S(u^k) : \nabla \zeta \]
\[ - \frac{1}{2} \left( \int_{\Omega_{\varepsilon_0}} u : D(u^k) : \nabla \zeta + \int_{\Omega_{\varepsilon_0}} u : D(a) : \nabla \zeta \right), \]
where \( \mathcal{P}^k \) is the pressure function associated with \( u^k \). From (4.33), we have
\[ (4.41) \quad \frac{1}{k} \left( \int_{\Omega_{\varepsilon_0}} u : D(u^k) : \nabla \zeta + \int_{\Omega_{\varepsilon_0}} u : D(a) : \nabla \zeta \right) \leq \frac{c_{\varepsilon_0}}{k} \to 0. \]
and that \( \{S(u^k)\} \) is uniformly bounded in \( L^p(\Omega_{\varepsilon_0}) \), so there is a \( \chi_{p'} \in L^p(\Omega_{\varepsilon_0}) \) such that
\[ (4.42) \quad \lim_{k \to \infty} \int_{\Omega_{\varepsilon_0}} u : S(u^k) : \nabla \zeta = \int_{\Omega_{\varepsilon_0}} u : \chi_{p'} : \nabla \zeta. \]
Similarly to the proof of (4.14), we also have

\[(4.43) \quad \lim_{k \to \infty} \int_{\Omega_{k_0}} B(u^k) \cdot \zeta u = \int_{\Omega_{k_0}} B(u) \cdot \zeta u.\]

Next, we show that

\[(4.44) \quad \int_{\Omega_{k_0}} \mathcal{P}^k u \cdot \nabla \zeta = \int_{\Omega_{k_0}} \mathcal{P} u \cdot \nabla \zeta,\]

for some further subsequence of \(k \to \infty\), where, up to a constant, \(\mathcal{P}\) is the pressure function associated with \(u\). For this, it is enough to show that there is a \(\mathcal{P} \in L^{p'}(\Omega_{k_0})\) \((p' = p/(p - 1))\) such that \(\mathcal{P}^k \to \mathcal{P}\) in \(L^{p'}(\Omega_{k_0})\), i.e. \(\{\mathcal{P}^k\}\) is uniformly bounded in \(L^{p'}(\Omega_{k_0})\). Let us assume, without loss of generality, \(\int_{\Omega_{k_0}} \mathcal{P}^k dx = 0\). Writing

\[g = |\mathcal{P}^k|^{-2} - |\mathcal{P}_{k_0}|^{-1} \int_{\Omega_{k_0}} |\mathcal{P}^k|^{-2} \mathcal{P}^k dx,\]

by Lemma 3.4 there exist a constant \(c\) (independent of \(k\)) and a vector field \(\psi \in W^{1,p}_{0}(\Omega_{k_0})\) such that

\[(4.45) \quad \left\{ \begin{array}{l}
\nabla \cdot \psi = g \\
\|\psi\|_{1,p,\Omega_{k_0}} \leq c \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{1-p}.
\end{array} \right.\]

Notice that \(\int_{\Omega_{k_0}} g dx = 0\), \(g \in L^{p}(\Omega_{k_0})\) and \(\|g\|_{p,\Omega_{k_0}} \leq 2 \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{1-p} \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{p-1} \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{p-1} dx\),

Then,

\[(4.46) \quad \int_{\Omega_{k_0}} |\mathcal{P}^k|^{p'} = \int_{\Omega_{k_0}} \left( |\mathcal{P}^k|^{p'-2} \mathcal{P}^k \right) \mathcal{P}^k = \int_{\Omega_{k_0}} g \mathcal{P}^k dx + |\mathcal{P}_{k_0}|^{-1} \left( \int_{\Omega_{k_0}} |\mathcal{P}^k|^{p'-2} \mathcal{P}^k \right) \int_{\Omega_{k_0}} \mathcal{P}^k dx = \int_{\Omega_{k_0}} \mathcal{P}^k \nabla \cdot \psi = \int_{\Omega_{k_0}} \mathcal{P}^k \nabla \cdot \psi = \int_{\Omega_{k_0}} \mathcal{P}^k \nabla \cdot \psi = \int_{\Omega_{k_0}} \mathcal{P}^k \mathcal{P}^k \cdot \psi,\]

where, for the last inequality, we used equation (3.11). Using again (4.45) and previous estimates, it follows that

\[(4.47) \quad \int_{\Omega_{k_0}} S_k(u^k) : D(\psi) + \int_{\Omega_{k_0}} B(u^k) \cdot \psi \leq c \left( \|u^k\|_{1,p,\Omega_{k_0}} + \|a\|_{1,p,\Omega_{k_0}} \right) \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{1-p}.\]

Therefore,

\[
\|\mathcal{P}^k\|_{p',\Omega_{k_0}} \leq c \left( \|u^k\|_{1,p,\Omega_{k_0}} + \|a\|_{1,p,\Omega_{k_0}} \right) \|\mathcal{P}^k\|_{p',\Omega_{k_0}}^{1-p} \leq C,
\]

as we wished.

From (4.40)–(4.44), we obtain

\[(4.48) \quad \langle \chi \zeta, u \rangle = \int_{\Omega_{k_0}} B(u) \cdot (\zeta u) - \int_{\Omega_{k_0}} P \nabla \cdot \nabla \zeta - \frac{1}{2} \int_{\Omega_{k_0}} u \cdot \chi p' \nabla \zeta.\]
Now, replacing $u$ by $u^k$ in (4.40), we have

$$
\langle A_{\zeta,k}u^k, u^k \rangle = \int_{\Omega_k} B(u^k) \cdot (\zeta u^k) - \int_{\Omega_k} P^k u^k \cdot \nabla \zeta - \frac{1}{2} \int_{\Omega_k} u^k \cdot S(u^k) \cdot \nabla \zeta
- \frac{1}{k} \left( \int_{\Omega_k} u \cdot D(u^k) \cdot \nabla \zeta + \int_{\Omega_k} u \cdot D(a) \cdot \nabla \zeta \right),
$$

and taking the limit when $k \to \infty$ in the right hand side here, analogously to the steps we did to obtain (4.48), we get the right hand side of (4.48), i.e.

$$
\lim_{k \to \infty} \left\{ \int_{\Omega_k} B(u^k) \cdot (\zeta u^k) - \int_{\Omega_k} P^k u^k \cdot \nabla \zeta - \frac{1}{2} \int_{\Omega_k} u^k \cdot S(u^k) \cdot \nabla \zeta
- \frac{1}{k} \left( \int_{\Omega_k} u \cdot D(u^k) \cdot \nabla \zeta + \int_{\Omega_k} u \cdot D(a) \cdot \nabla \zeta \right) \right\} = \int_{\Omega_1} B(u) \cdot (\zeta u) - \int_{\Omega_1} P u \cdot \nabla \zeta - \frac{1}{2} \int_{\Omega_1} u \cdot \chi_{p'} \cdot \nabla \zeta.
$$

Then, combining (4.48) and (4.49), we have $
\lim_{k \to \infty} \langle A_{\zeta,k}u^k, u^k \rangle = \langle \chi_{p'}, u \rangle$, and thus conclude the proof of Theorem 2.2. □

Next, we make some remarks and prove two additional results, one on the rate of dissipation of energy of the solution obtained for problem (2.1) and another on the uniqueness of solution.

**Remark 2.** Dropping the convective term $v \cdot \nabla v$ in (2.1), we obtain the Ladyzhenskaya-Solonnikov problem for Stokes’ system with a power law. The solution of this problem can be obtained as in the proof of Theorem 2.2, with obviously much less computations.

The solution of problem (2.1) has energy dissipation uniformly distributed along the outlets. More precisely, we have the following result, which generalizes Theorem 3.2 in [15] for power law shear thickening fluids.

**Proposition 4.2.** Let $v$ be a solution of problem (2.1) with $p \geq 2$, obtained by the proof of Theorem 2.2. Then there exists a constant $\kappa$ such that

$$
\int_{\Omega_{i,t-1,t}} |\nabla v|^p \leq \kappa, \quad \forall t \geq 1,
$$

where $i = 1, 2$.

**Proof.** Let $u = v - a$. By the proof of Theorem 2.2, $u$ is the weak limit in $W^{1,p}_{loc}(\Omega)$ of a sequence $\{u^k\}_{k=1}^{\infty}$, where $u^k$ is a solution of problem (4.1) with $T = k$. Now, for $\tau \geq \max\{t, 2\}$ we define the function

$$
z_{\tau}(\eta) = \int_{\eta-1}^{\eta} y_{\tau}(t) dt, \quad \eta \geq 1,
$$

where

$$
y_{\tau}(t) := \frac{1}{k} \left| u^k \right|^2_{1,2,\Omega_{i,t-\tau+1,t+\tau}} + \left| u^k \right|^p_{1,p,\Omega_{i,t-\tau+1,t+\tau}}.
$$
Similarly to the proof of (4.33), it is possible to show that
\[ z_{\tau}(\eta) \leq \varphi(\eta), \quad \forall \eta \in [1, \tau], \]
where \( \varphi(\eta) = c_2 \eta + c_3 \), for some constants \( c_2, c_3 \). Since
\[ y_{\tau}(1/2) = \int_{1/2}^{3/2} y_{\tau}(1/2) \, dt \leq \int_{1/2}^{3/2} y_{\tau}(t) \, dt = z_{\tau}(3/2) \leq \varphi(3/2) \equiv c, \]
we have
\[ \int_{\tau - 1/2}^{\tau + 1/2} \int_{\Sigma_i} |\nabla u|^p \leq y_{\tau}(1/2) \leq c \]
and, consequently, by the weak convergence of \( u_k \) to \( u \), we also have
\[ \int_{\tau - 1/2}^{\tau + 1/2} \int_{\Sigma_i} |\nabla u|^p \leq c, \]
which is (4.50) with \( u \) in place of \( v \). Since, by Lemma 2.1, the vector field \( a \) also satisfies this property, this ends the proof of Proposition 4.2.

In [17, p.1437], Marušić-Paloka observes the difficult of obtaining uniqueness results for Navier-Stokes system with a power law. In particular, this is an open question even in bounded domains. We can prove an uniqueness result for problem (2.1) under some conditions, which we specify precisely in Theorem 4.4 below. One of these conditions is motivated by Proposition 4.2 and another, by the following proposition, which was inspired by the solution of Leray problem given by Marušić-Paloka; cf. [17, Lemma 4.2/(4.24)].

**Proposition 4.3.** For \( i \) either equal to 1 or 2, let \( v = (v_1, \ldots, v_n) \) be a divergence free vector field in \( W^{1,p}_{\text{loc}}(\Omega_i) \) vanishing on \( \partial \Omega_i \) and having property (4.50). If for some \( j \in \{1, \ldots, n\} \) and some positive number \( c \),
\[ \left| \frac{\partial v_j}{\partial x_j}(x) \right| \geq c|x'|^{1/(p-1)} \]
for all \( x = (x_1, x') \in \Omega_i \), then there is a constant \( C \) such that
\[ (w \cdot \nabla v, w)_{\Omega_{i,t}} \leq C_k \|D(v)(p-2)/2 D(w)\|_{2,\Omega_{i,t}}^2, \]
for all \( w \in D^{1,p}_{\text{loc}}(\Omega_i) \) and \( t > 0 \).

**Proof.** Denote \( \Omega = \Omega_i \). By Hölder inequality and (4.50), we obtain
\[ |(w \cdot \nabla v, w)_{\Omega_{i-1,t}}| \leq C_k \|w\|_{2p',\Omega_{i-1,t}}^2. \]
By Sobolev embedding and Poincaré inequality (Lemma 3.2), we have
\[ \|w\|_{2p',\Omega_{i-1,t}} \leq C \|w\|_{1,q,\Omega_{i-1,t}}, \]
for any \( r \in (1, 2) \) such that \( 2p' \leq \frac{rn}{n-r} \). Now, by Korn inequality (Lemma 3.2), Hölder inequality and (4.51), we obtain
\[
|w|_{1,r;\Omega_{t-1},t} = \int_{\Omega_{t-1},t} |\nabla w|^r \leq C \int_{\Omega_{t-1},t} |D(v)|^{(p-2)r/2} |D(w)|^{1/2} \frac{1}{|D(v)|^{(p-2)r/2}} \leq C \||D(v)|^{(p-2)/2} D(w)\|_{2,\Omega_{t-1},t}.
\]

Then, chosen \( r \leq \frac{2(n-1)(p-1)}{np(n+1)} \), it follows that
\[
(4.55) \quad |w|_{1,r;\Omega_{t-1},t} \leq C \||D(v)|^{(p-2)/2} D(w)\|_{2,\Omega_{t-1},t}.
\]

Thus, from (4.53) and (4.55), we have
\[
\|w\|_{2p';\Omega_{t-1},t} \leq C \||D(v)|^{(p-2)/2} D(w)\|_{2,\Omega_{t-1},t}
\]
and from (4.53), we get
\[
(4.56) \quad |\langle w \cdot \nabla v, w\rangle_{\Omega_{t-1},t}| \leq C \kappa \|D(v)|^{(p-2)/2} D(w)\|_{2,\Omega_{t-1},t}^2.
\]

Finally, writing \( \Omega_t \) as a finite union of domains \( \Omega_{t-j,1}, \cdots, \Omega_{t-j,t-j} \), \( j = 0, \cdots, m < \infty \), and adding inequality (4.56) with \( \Omega_{t-j,1}, \cdots, \Omega_{t-j,t-j} \) in place of \( \Omega_{t-1},t \) with respect to \( j \), we obtain (4.52).

**Remark 3.** An example of a solution satisfying property (4.51) when \( \Omega_i \) is a straight outlet (i.e. the cross sections \( \Sigma(x_i) \) are constant, with respect to \( x_i \)) is the Poiseuille flow in \( \Omega_i \). See [17] §3.

We now state and prove our uniqueness result.

**Theorem 4.4.** Let \( \kappa > 0 \) be sufficiently small and \( l \) be some positive number. Then there is no more than one weak solution of problem (2.1) in \( W^{2,l}_{\text{loc}}(\Omega) \) and satisfying (4.50) and property (4.52) in \( \Omega \), i.e. for some constant \( C \),
\[
(4.57) \quad \langle w \cdot \nabla v, w\rangle_{\Omega_t} \leq C \kappa \|D(v)|^{(p-2)/2} D(w)\|_{2,\Omega_t}^2
\]
for all \( w \in D^{1,p}_{\text{loc}}(\Omega) \) and \( t > 0 \).

**Proof.** Let \( v_1 \) and \( v_2 \) be solutions of (2.1) satisfying the assumptions in Theorem 4.4. Denote \( w = v_1 - v_2 \). Then
\[
-\text{div}\{D(w) + |D(v_2)|^{p-2}[D(w) + D(v_2)] - |D(v_2)|^{p-2}D(v_2)\} + w \cdot \nabla w + w \cdot \nabla v_2 + v_2 \cdot \nabla w + \nabla (P_1 - P_2) = 0,
\]
where \( P_1, P_2 \in W^{1,\frac{l}{2}}_{\text{loc}}(\Omega) \). Multiplying this equation by \( w \) and integrating by parts over \( \Omega_t \), similarly to derivation of (4.10), we obtain
\[
\int_{\Omega_t} \{D(w) + D(v_2)|^{p-2}[D(w) + D(v_2)] - |D(v_2)|^{p-2}D(v_2)\} : D(w) = -\langle w \cdot \nabla v_2, w\rangle_{\Omega_t} - I,
\]
where
\[
I = -\int_{\partial \Omega_t} \frac{|w|^2}{2} (w \cdot n + v_2 \cdot n - (P_1 - P_2)(w \cdot n) + \int_{\partial \Omega} w \cdot \{D(w) + D(v_2)|^{p-2}[D(w) + D(v_2)] - |D(v_2)|^{p-2}D(v_2)\} n.
\]
Estimating some terms in the above equation by using Lemma 3.2 and assumption (4.57), it follows that
\[
c_1 \int_{\Omega_t} |\nabla w|^p + c_2 \int_{\Omega_t} |D(v_2)|^{p-2} |D(w)|^2 \leq c_3 |\kappa| \int_{\Omega_t} |D(v_2)|^{p-2} |D(w)|^2 + I,
\]
for some positive constants \( c_1, c_2, c_3 \). Thus, if \( |\kappa| < c_1 / c_2 \), we have
\[
y(t) := |w|^p_{1,p,\Omega_t} y(t) \leq c_I,
\]
(for some constant \( c \)). Next, integrating \( y(t) \) from \( \eta - 1 \) to \( \eta, \eta \geq 1 \), and proceeding similarly to the proof of (4.30), but using (4.50) instead of Lemma 2.1 ii), we obtain
\[
z(\eta) \leq c \Psi (z'(\eta)),
\]
with \( \Psi(\tau) = \tau + \tau^{1/p} + \tau^{2/p} + \tau^{3/p} \). Now suppose \( z \) is not identically zero. Then, by Lemma 3.1, we have
\[
\lim_{t \to \infty} z(t) = \infty.
\]
Besides, since for \( \tau \geq \tau_1 \) (for some \( \tau_1 > 0 \))
\[
\Psi(\tau) \leq \begin{cases} 
\tau, & \text{if } p \geq 3 \\
3^{3/(3-p)}, & \text{if } p < 3
\end{cases}
\]
by Lemma 3.1 again, we also have
\[
\liminf_{t \to \infty} e^{-t} z(t) > 0, \quad \text{if } p \geq 3 \\
\liminf_{t \to \infty} t^{-3/(3-p)} z(t) > 0, \quad \text{if } p < 3.
\]
This contradicts (2.1)5. Therefore, \( z \equiv 0 \) and so, \( v_1 = v_2 \).

Remark 4. By tracking all the estimates we did to obtain (4.33), similarly, to obtain (4.50), we can see that the constant \( \kappa \) in (4.50) depends on the flux \( \alpha \) so that \( \kappa = \mathcal{O}(|\alpha|^{\gamma}) \), for some positive number \( \gamma \). In particular, \( \kappa \) tends to zero when \( \alpha \) tends to zero.

Remark 5. For an example where condition (4.57) is accomplished, see [17, p.1437/§4.2].

Remark 6. Regarding the Stokes system with a power law, i.e. system (1.2) discarding \( (v \cdot \nabla)v \), we have uniqueness of solution for the corresponding Ladyzhenskaya-Solonnikov problem for any flux \( \alpha \), as occurs in the case \( p = 2 \) [15, Corollary 2.1, p. 739].

References

[1] Adams, R.A. Sobolev Spaces. Academic Press, New York, 1975.
[2] Amick, C.J. Steady solutions of the Navier-Stokes Equations in Unbounded Channels and Pipes. Ann. Scuola Norm. Sup. Pisa Cl.Sci. (4), 4(3) (1977) 473-513.
[3] Barrett, J.W. and Liu, W.B. Finite element approximation of the p-Laplacian. Mathematics of Computation, 61(204), (1993) 523-537.
[4] Beirão da Veiga, H., Kaplický, H. and Růžička, M. Boundary regularity of shear thickening flows. J. Math. Fluid Mech., to appear; published online, doi:10.1007/s00021-010-0025-y.

[5] Beirão da Veiga, H., Kaplický, H. and Růžička, M. Regularity theorems, up to the boundary, for shear thickening flows. C. R. Math. Acad. Sci. Paris 348 (2010), no. 9-10, 5415-544.

[6] Bird, R., Stewart, W. and Lightfoot, E. Transport Phenomena. Jogh Wiley & Sons, Inc. 2007.

[7] DiBenedetto, E. Degenerate Parabolic Equations. Springer-Verlag, Berlin, 1994.

[8] Evans, L.C. Partial Differential Equations. Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 1998.

[9] Frehse, J., Málek, J. and Steinbauer, M. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. SIAM J. Math. Anal., 34 (5), (2003) 1064-1083.

[10] Galdi, G.P. An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I e II. Springer-Verlag, Berlin, 1994.

[11] Kaplický, P., Málek and J., Stará, J. $C^{1,\alpha}$-solutions to a class of nonlinear fluids in two dimensions stationary Dirichlet problem. (English, Russian summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 259 (1999), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 30, 89–121, 297; translation in J. Math. Sci. (New York) 109 (2002), no. 5, 1867-193

[12] Ladyzhenskaya, O.A. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. Proc. Stek. Inst. Math. 102, 95118 (1967)

[13] Ladyzhenskaya, O.A. On some modifications of the Navier-Stokes equations for large gradients of velocity. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 7, 126154 (1968)

[14] Ladyzhenskaya, O. A. The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[15] Ladyzhenskaya, O.A. and Solonnikov, V.A. Determination of the solutions of boundary value problems for steady-state Stokes and Navier-Stokes equations in domains having an unbounded Dirichlet integral. Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI), 96 (1980) 117-160. English Transl.: J. Soviet Math., 21 (1983) 728-761.

[16] Lions, J.L. Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires. Dunod, Gauthier-Villars, 1969.

[17] Marušić-Paloka, E. Steady flow of a non-Newtonian fluid in unbounded channels and pipes. Math. Models Methods Appl. Sci. 10 (2000), no. 9, 1425-1445.

[18] Neff, P. On Korn’s first inequality with non-constant coefficients. Proc. Roy. Soc. Edinburgh Sect. A 132(1) (2002) 221-243.

[19] Passerini, A., Patria and M.C., Thater, G. Steady flow of a viscous incompressible fluid in an unbounded “funnel-shaped” domain. Ann. Mat. Pura Appl. (4) 173 (1997) 43-62.

[20] Silva, F.V. On a Lemma due to Ladyzhenskaya and Solonnikov and some applications. Nonlinear Analysis 64 (2006) 706-725.

[21] Silva, F.V., Os problemas de Leray e de Ladyzhenskaya-Solonnikov para fluidos micropolares (Leray and Ladyzhenskaya-Solonnikov problems for micropolar fluids), doctoral thesis, in portuguese; IMECC-Unicamp-Brazil (2004). http://cutter.unicamp.br/document/?code=vtls000316769
[22] Smagorinsky, J. S. General circulation experiments with the primitive equations. I. The basic experiment, Mon. Weather Rev., 91 (1963), 99164.
[23] Solonnikov, V. A. and Pileckas, K. I. Certain spaces of solenoidal vectors, and the solvability of a boundary value problem for a system of Navier-Stokes equations in domains with noncompact boundaries. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 73 (1977) 136-151. English Transl.: J. Soviet Math. 34 (1986) 2101-2111.
[24] Stein, E. M. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, 1970.

Colegiado de Matemática, Universidade Federal do Amapá-UNIFAP, Rodovia Juscelino Kubitschek de Oliveira, s/n, Jardim Marco Zero, Caixa Postal 261, Macapá, AP 68902-280, Brazil
E-mail address: gjd@unifap.br

Departamento de Matemática, IMECC, Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, Universidade Estadual de Campinas - UNICAMP, Campinas, SP 13083-859, Brazil
E-mail address: msantos@ime.unicamp.br