GEOMETRY OF CLASSICAL HIGGS FIELDS

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In gauge theory, Higgs fields are responsible for spontaneous symmetry breaking. In classical
gauge theory on a principal bundle $P$, a symmetry breaking is defined as the reduction of a
structure group of this principal bundle to a subgroup $H$ of exact symmetries. This reduction
takes place iff there exists a global section of the quotient bundle $P/H$. It is a classical Higgs field.
A metric gravitational field exemplifies such a Higgs field. We summarize the basic facts on the
reduction in principal bundles and geometry of Higgs fields. Our goal is the particular covariant
differential in the presence of a Higgs field.

1 Introduction

Gauge theory deals with the three types of classical fields. These are gauge potentials, mat-
ter fields and Higgs fields. Higgs fields are responsible for spontaneous symmetry breaking.
Spontaneous symmetry breaking is a quantum phenomenon. In classical gauge theory on a
principal bundle $P \rightarrow X$, a symmetry breaking is defined as the reduction of the structure
Lie group $G$, $\dim G > 0$, of this principal bundle to a closed (consequently, Lie) subgroup
$H$, $\dim H > 0$, of exact symmetries [1, 2, 3, 4, 5, 6]. From the mathematical viewpoint, one
speaks on the Klein–Chern geometry or a reduced $G$-structure [7, 8, 9]. By virtue of the
well-known theorem (see Theorem 2 below), the reduction of the structure group of a prin-
cipal bundle takes place iff there exists a global section of the quotient bundle $P/H \rightarrow X$.
It is treated as a classical Higgs field. In the gauge gravitation theory, a pseudo-Riemannian
metric exemplifies such a Higgs field, associated to the Lorentz reduced structure [1, 10].

This article aims to summarize the relevant material on the reduction in principal bun-
dles and geometry of Higgs fields. It is geometry on the composite fiber bundle

$$P \rightarrow P/H \rightarrow X,$$  \hspace{1cm} (1)

where

$$P_{\Sigma} = P \xrightarrow{\pi_{\Sigma}} P/H$$  \hspace{1cm} (2)

is a principal bundle with the structure group $H$ and

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma}} X$$  \hspace{1cm} (3)

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is a $P$-associated fiber bundle with the typical fiber $G/H$ on which the structure group $G$ acts on the left. Let $Y \to \Sigma$ be a vector bundle associated to the $H$-principal bundle $P\Sigma$ (2). Then sections of the fiber bundle $Y \to X$ describe matter fields with the exact symmetry group $H$ in the presence of Higgs fields [6]. A problem is that $Y \to X$ fails to be a $P$-associated bundle with a structure group $G$ and, consequently, it need not admit a principal connection. Our goal is the particular covariant differential (23) on $Y \to X$ defined by a principal connection on the $H$-principal bundle $P \to P/H$, but not $P \to X$. For instance, this is the case of the covariant differential of spinor fields in the gauge gravitation theory [11, 12].

2 Reduced structures

We start with a few Remarks summarizing the relevant facts on principal and associated bundles [13].

**Remark 1.** By a fiber bundle associated to the principal bundle $P \to X$ is usually meant the quotient

$$Y = (P \times V)/G,$$

where the structure group $G$ acts on the typical fiber $V$ of $Y$ on the left. The quotient (4) is defined by identification of the elements $(p, v)$ and $(pg, g^{-1}v)$ for all $g \in G$. By $[p]$ is further denoted the restriction of the canonical morphism

$$P \times V \to (P \times V)/G$$

to $\{p\} \times V$, and we write $[p](v) = (p, v) \cdot G$, $v \in V$. Then, by definition of $Y$, we have $[p](v) = [pg](g^{-1}v)$. Strictly speaking, $Y$ (4) is a fiber bundle canonically associated to a principal bundle $P$. Recall that a fiber bundle $Y \to X$, given by the triple $(X, V, \Psi)$ of a base $X$, a typical fiber $V$ and a bundle atlas $\Psi$, is called a fiber bundle with a structure group $G$ if $G$ acts effectively on $V$ on the left and the transition functions $\rho_{\lambda,\beta}$ of the atlas $\Psi$ take their values into the group $G$. The set of these transition functions form a cocycle. Atlases of $Y$ are equivalent iff cocycles of their transition functions are equivalent. The set of equivalent cocycles are elements of the first cohomology set $H^1(X; G)$. Fiber bundles $(X, V, G, \Psi)$ and $(X, V', G, \Psi')$ with the same structure group $G$, which may have different typical fibers, are called associated if the transition functions of the atlases $\Psi$ and $\Psi'$ belong to the same element of the the cohomology set $H^1(X; G)$. Any two associated fiber bundles with the same typical fiber are isomorphic to each other, but their isomorphism is not canonical in general. A fiber bundle $Y \to X$ with a structure group $G$ is associated to some $G$-principal bundle $P \to X$. If $Y$ is canonically associated to $P$ as in (4), then every atlas of $P$ determines canonically the associated atlas of $Y$, and every automorphism
of a principal bundle $P$ yields the corresponding automorphism of the $P$-associated fiber bundle (4).

**Remark 2.** Recall that an automorphism $\Phi_P$ of a principal bundle $P$, by definition, is equivariant under the canonical action $R_g \circ \Phi_P = \Phi_P \circ R_g$, $g \in G$, of the structure group $G$ on $P$. Every automorphism of $P$ yields the corresponding automorphisms

$$\Phi_Y : (p, v) \cdot G \mapsto (\Phi_P(p), v) \cdot G, \quad p \in P, \quad v \in V;$$

of the $P$-associated bundle $Y$ (4). For the sake of brevity, we will write

$$\Phi_Y : (P \times V)/G \to (\Phi_P(P) \times V)/G.$$ 

Every automorphism of a principal bundle $P$ is represented as

$$\Phi_P(p) = pf(p), \quad p \in P,$$

where $f$ is a $G$-valued equivariant function on $P$, i.e., $f(pg) = g^{-1}f(p)g$, $g \in G$.

One says that the structure group $G$ of a principal bundle $P$ is reducible to a Lie subgroup $H$ if there exists a $H$-principal subbundle $P^h$ of $P$ with the structure group $H$. This subbundle is called a reduced $G^h$-structure. Two reduced $G^h$-structures $P^h$ and $P^{h'}$ on a $G$-principal bundle are said to be isomorphic if there is an automorphism $\Phi$ of $P$ which provides an isomorphism of $P^h$ and $P^{h'}$. If $\Phi$ is a vertical automorphism of $P$, reduced structures $P^h$ and $P^{h'}$ are called equivalent.

**Remark 3.** Note that, in [8, 9] (see also [16]), the reduced structures on the principle bundle $LX$ of linear frames in the tangent bundle $TX$ of $X$ are only considered, and a class of isomorphisms of such reduced structures is restricted to holonomic automorphisms of $LX$, i.e., the canonical lifts onto $LX$ of diffeomorphisms of the base $X$.

Let us recall the following two theorems [14].

**Theorem 1.** A structure group $G$ of a principal bundle $P$ is reducible to its closed subgroup $H$ iff $P$ has an atlas $\Psi_P$ with $H$-valued transition functions.

Given a reduced subbundle $P^h$ of $P$, such an atlas $\Psi_P$ is defined by a family of local sections $\{z_\alpha\}$ which take their values into $P^h$.

**Theorem 2.** There is one-to-one correspondence $P^h = \pi_{P^h}^{-1}(h(X))$ between the reduced $H$-principal subbundles $P^h$ of $P$ and the global sections $h$ of the quotient fiber bundle $P/H \to X$.

In general, there are topological obstructions to the reduction of a structure group of a principal bundle to its subgroup. One usually refers to the following theorems [15].
Theorem 3. Any fiber bundle \( Y \to X \) whose typical fiber is diffeomorphic to \( \mathbb{R}^m \) has a global section. Its section over a closed subset \( N \subset X \) is always extended to a global section.

By virtue of Theorem 3, the structure group \( G \) of a principal bundle \( P \) is always reducible to its closed subgroup \( H \) if the quotient \( G/H \) is diffeomorphic to a Euclidean space.

Theorem 4. A structure group \( G \) of a principal bundle is always reducible to its maximal compact subgroup \( H \) since the quotient space \( G/H \) is homeomorphic to a Euclidean space.

For instance, this is the case of \( G = GL(n, \mathbb{C}) \), \( H = U(n) \) and \( G = GL(n, \mathbb{R}) \), \( H = O(n) \).

It should be emphasized that different \( H \)-principal subbundles \( P^h \) and \( P^{h'} \) of a \( G \)-principal bundle \( P \) need not be isomorphic to each other in general.

Proposition 5. Every vertical automorphism \( \Phi \) of a principal bundle \( P \to X \) sends an \( H \)-principal subbundle \( P^h \) onto an equivalent \( H \)-principal subbundle \( P^{h'} \). Conversely, let two reduced subbundles \( P^h \) and \( P^{h'} \) of a principal bundle \( P \) be isomorphic to each other, and \( \Phi : P^h \to P^{h'} \) be an isomorphism over \( X \). Then \( \Phi \) is extended to a vertical automorphism of \( P \).

Proof. Let

\[
\Psi^h = \{(U_\alpha, z^h_\alpha), \rho^h_{\alpha\beta}\}, \quad z^h_\alpha(x) = z^h_\beta(x)\rho^h_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta,
\]

be an atlas of the reduced subbundle \( P^h \), where \( z^h_\alpha \) are local sections of \( P^h \to X \) and \( \rho^h_{\alpha\beta} \) are the transition functions. Given a vertical automorphism \( \Phi \) of \( P \), let us provide the reduced subbundle \( P^{h'} = \Phi(P^h) \) with the atlas \( \Psi^{h'} = \{(U_\alpha, z^{h'}_\alpha), \rho^{h'}_{\alpha\beta}\} \) determined by the local sections \( z^{h'}_\alpha = \Phi \circ z^h_\alpha \) of \( P^{h'} \to X \). Then it is readily observed that \( \rho^{h'}_{\alpha\beta}(x) = \rho^h_{\alpha\beta}(x) \), \( x \in U_\alpha \cap U_\beta \). Conversely, any isomorphism \( \Phi \) of reduced structures \( P^h \) and \( P^{h'} \) on \( P \) determines a \( G \)-valued function \( f \) on \( P^h \) given by the relation \( pf(p) = \Phi(p) \), \( p \in P^h \). Obviously, this function is \( H \)-equivariant. Its prolongation to a \( G \)-equivariant function on \( P \) is defined to be \( f(pg) = g^{-1}f(p)g \), \( p \in P^h \), \( g \in G \). In accordance with the relation (6), this function defines a vertical automorphism of \( P \) whose restriction to \( P^h \) coincides with \( \Phi \).

Proposition 6. If the quotient \( G/H \) is diffeomorphic to a Euclidean space \( \mathbb{R}^k \), all \( H \)-principal subbundles of a \( G \)-principal bundle \( P \) are equivalent to each other [15].

Given a reduced subbundle \( P^h \) of a principal bundle \( P \), let

\[
Y^h = (P^h \times V)/H
\]
be the associated fiber bundle with a typical fiber $V$. Let $P^h'$ be another reduced subbundle of $P$ which is isomorphic to $P^h$, and

$$Y^{h'} = (P^h' \times V)/H$$

The fiber bundles $Y^h$ and $Y^{h'}$ are isomorphic, but not canonically isomorphic in general.

**Proposition 7.** Let $P^h$ be a $H$-principal subbundle of a $G$-principal bundle $P$. Let $Y^h$ be the $P^h$-associated bundle (7) with a typical fiber $V$. If $V$ carries a representation of the whole group $G$, the fiber bundle $Y^h$ is canonically isomorphic to the $P$-associated fiber bundle

$$Y = (P \times V)/G.$$

**Proof.** Every element of $Y$ can be represented as $(p, v) \cdot G$, $p \in P^h$. Then the desired isomorphism is

$$Y^h \ni (p, v) \cdot H \iff (p, v) \cdot G \in Y.$$

It follows that, given a $H$-principal subbundle $P^h$ of $P$, any $P$-associated fiber bundle $Y$ with the structure group $G$ is canonically equipped with a structure of the $P^h$-associated fiber bundle $Y^h$ with the structure group $H$. Briefly, we can write

$$Y = (P \times V)/G \simeq (P^h \times V)/H = Y^h.$$

However, if $P^h \neq P^{h'}$, the $P^h$- and $P^{h'}$-associated bundle structures on $Y$ need not be equivalent. Given bundle atlases $\Psi^h$ of $P^h$ and $\Psi^{h'}$ of $P^{h'}$, the union of the associated atlases of $Y$ has necessarily $G$-valued transition functions between the charts of $\Psi^h$ and $\Psi^{h'}$.

3 **Classical Higgs fields**

In accordance with Theorem 2, the set of reduced $H$-principal subbundles $P^h$ of $P$ is in bijective correspondence with the set of Higgs fields $h$. Given such a subbundle $P^h$, let $Y^h$ (7) be the associated vector bundle with a typical fiber $V$ which admits a representation of the group $H$ of exact symmetries, but not the whole symmetry group $G$. Its sections $s_h$ describe matter fields in the presence of the Higgs fields $h$ and some principal connection $A_h$ on $P^h$. In general, the fiber bundle $Y^h$ (7) is not associated or canonically associated (see Remark 1) to other $H$-principal subbundles $P^{h'}$ of $P$. It follows that, in this case, $V$-valued matter fields can be represented only by pairs with Higgs fields. The goal is to
describe the totality of these pairs \((s_h, h)\) for all Higgs fields \(h\). We refer to the following theorems [13].

**Theorem 8.** Given an arbitrary composite fiber bundle
\[
Y \overset{\pi_Y}{\longrightarrow} \Sigma \overset{\pi_{\Sigma}}{\longrightarrow} X, \tag{8}
\]
let \(h\) be a global section of the fiber bundle \(\Sigma \to X\). Then the restriction
\[
Y_h = h^* Y
\]
(9)
of the fiber bundle \(Y \to \Sigma\) to \(h(X) \subset \Sigma\) is a subbundle \(i_h : Y_h \hookrightarrow Y\) of the fiber bundle \(Y \to X\).

In the case of a principal bundle \(Y = P\) and \(\Sigma = P/H\), the restriction \(h^* P_{\Sigma}\) (9) of the \(H\)-principal bundle \(P_{\Sigma}\) (2) to \(h(X) \subset \Sigma\) is a \(H\)-principal bundle over \(X\), which is equivalent to the reduced subbundle \(P^h\) of \(P\).

**Theorem 9.** Given a section \(h\) of the fiber bundle \(\Sigma \to X\) and a section \(s_{\Sigma}\) of the fiber bundle \(Y \to \Sigma\), their composition
\[
s = s_{\Sigma} \circ h
\]
(10)
is a section of the composite fiber bundle \(Y \to X\) (8). Conversely, every section \(s\) of the fiber bundle \(Y \to X\) is the composition (10) of the section \(h = \pi_Y \circ s\) of the fiber bundle \(\Sigma \to X\) and some section \(s_{\Sigma}\) of the fiber bundle \(Y \to \Sigma\) over the closed submanifold \(h(X) \subset \Sigma\).

Let us consider the composite fiber bundle (1) and the composite fiber bundle
\[
Y \overset{\pi_Y}{\longrightarrow} P/H \overset{\pi_{P/H}}{\longrightarrow} X \tag{11}
\]
where \(Y \to \Sigma = P/H\) is a vector bundle
\[
Y = (P \times V)/H
\]
associated to the corresponding \(H\)-principal bundle \(P_{\Sigma}\) (2). Given a global section \(h\) of the fiber bundle \(\Sigma \to X\) (3) and the \(P^h\)-associated fiber bundle (7), there is the canonical injection
\[
i_h : Y^h = (P^h \times V)/H \hookrightarrow Y
\]
over \(X\) whose image is the restriction
\[
h^* Y = (h^* P \times V)/H
\]
of the fiber bundle \(Y \to \Sigma\) to \(h(X) \subset \Sigma\), i.e.,
\[
i_h(Y^h) \cong \pi_{Y\Sigma}^{-1}(h(X)) \tag{12}
\]
(see Theorem 8). Then, by virtue of Theorem 9, every global section \( s_h \) of the fiber bundle \( Y^h \) corresponds to the global section \( i_h \circ s_h \) of the composite fiber bundle \( (11) \). Conversely, every global section \( s \) of the composite fiber bundle \( (11) \) which projects onto a section \( h = \pi_{Y^h} \circ s \) of the fiber bundle \( P/H \to X \) takes its values into the subbundle \( i_h(Y^h) \subset Y \) in accordance with the relation \( (12) \). Hence, there is one-to-one correspondence between the sections of the fiber bundle \( Y^h \) \( (7) \) and the sections of the composite fiber bundle \( (11) \) which cover \( h \).

Thus, it is precisely the composite fiber bundle \( (11) \) whose sections describe the above-mentioned totality of pairs \( (s_h, h) \) of matter fields and Higgs fields in gauge theory with broken symmetries. For instance, this is the case of spinor fields in the presence of gravitational fields [11]. A problem is that the typical fiber of the fiber bundle \( Y \to X \) fails to admit a representation of the group \( G \), unless \( G \to G/H \) is a trivial bundle. It follows that \( Y \to X \) is not associated to \( P \) and, it does not admit a principal connection in general. If \( G \to G/H \) is a trivial bundle, there exists its global section whose values are representatives of elements of \( G/H \). In this case, the typical fiber of \( Y \to X \) is \( V \times G/H \), and one can provide it with an induced representation of \( G \). Of course, this representation is not canonical, unless \( V \) itself admits a representation of \( G \).

4 Composite and reduced connections

Since the reduction in a principal bundle leads to the composite fiber bundle \( (1) \), we turn to the notion of a composite connection [13].

Let us consider the composite bundle \( (8) \) provided with bundle coordinates \( (x^\lambda, \sigma^m, y^i) \), where \( (x^\mu, \sigma^m) \) are bundle coordinates on \( \Sigma \to X \) and the transition functions of \( \sigma^m \) are independent of the coordinates \( y^i \). Let us consider the jet manifolds \( J^1 \Sigma, J^1_\Sigma Y, \) and \( J^1 Y \) of the fiber bundles \( \Sigma \to X, Y \to \Sigma \) and \( Y \to X \), respectively. They are endowed with the adapted coordinates

\[
(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \bar{y}^i, \bar{y}^i_\lambda), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda). \tag{13}
\]

There is the canonical map [17]

\[
\varrho : J^1 \Sigma \times J^1_\Sigma Y \to J^1 Y, \quad y^i_\lambda \circ \varrho = y^i_m \sigma^m_\lambda + \bar{y}^i_\lambda. \tag{14}
\]

With this map, one can obtain the relations between connections on the fiber bundles \( Y \to X, Y \to \Sigma \) and \( \Sigma \to X \) as follows. Let

\[
A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i), \tag{13}
\]

\[
\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m) \tag{14}
\]
be connections on the fiber bundles $Y \to \Sigma$ and $\Sigma \to X$, respectively. They define the composite connection

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_\lambda + A^i_m \Gamma^m_\lambda) \partial_i)$$

(15)
on $Y \to X$ in accordance with the diagram

$$\xymatrix{J^1\Sigma \times J^1Y_{\Sigma} \ar[r]_{\theta} \ar[d]_{(\Gamma,A_\Sigma)} & J^1Y \ar[d]^\gamma \\
\Sigma \times Y \ar[u]_X & Y \ar[l]_Y}$$

In brief, we will write

$$\gamma = A_\Sigma \circ \Gamma. \quad (16)$$

In particular, let us consider a vector field $\tau$ on the base $X$, its horizontal lift $\Gamma \tau$ onto $\Sigma$ by means of the connection $\Gamma$ and, in turn, the horizontal lift $A_\Sigma(\Gamma \tau)$ of $\Gamma \tau$ onto $Y$ by means of the connection $A_\Sigma$. Then $A_\Sigma(\Gamma \tau)$ coincides with the horizontal lift $\gamma \tau$ of $\tau$ onto $Y$ by means of the composite connection $\gamma$ (16).

**Remark 4.** Recall the notions of a pull-back connection and a reducible connection. Given a fiber bundle $Y \to X$, let $f : X' \to X$ be a map and $f^*Y \to X'$ the pull-back of $Y$ by $f$. Written as a vertical-valued form

$$\Gamma = (dy^i - \Gamma^i_\lambda dx^\lambda) \otimes \partial_i,$$

any connection $\Gamma$ on $Y \to X$ yields the pull-back connection

$$f^*\Gamma = (dy^i - (\Gamma \circ f_\Sigma)^i_\lambda \partial f^\lambda_{x^\mu} dx^\mu) \otimes \partial_i$$

(17)on $f^*Y \to X'$. In particular, let $P$ be a principal bundle and $f^*P$ the pull-back principal bundle with the same structure group. If $A$ is a principal connection on $P$, then the pull-back connection $f^*A$ (17) on $f^*P$ is also a principal connection [14]. Let $i_Y : Y \to Y'$ be a subbundle of a fiber bundle $Y' \to X$ and $\Gamma'$ a connection on $Y' \to X$. If there exists a connection $\Gamma$ on $Y \to X$ such that the diagram

$$\xymatrix{Y' \ar[r]^{\Gamma'} & J^1Y \\
Y \ar[r]^{\Gamma} \ar[u]_{i_Y} & J^1i_Y \ar[u]_{J^1i_Y}}$$

commutes, we say that $\Gamma'$ is reducible to the connection $\Gamma$. The following conditions are equivalent:
(i) $\Gamma'$ is reducible to $\Gamma$;
(ii) $T i_Y(HY) = HY'|_{i_Y(Y)}$, where $HY \subset TY$ and $HY' \subset TY'$ are the horizontal subbundles determined by $\Gamma$ and $\Gamma'$, respectively;
(iii) for every vector field $\tau \in T(X)$, the vector fields $\Gamma \tau$ and $\Gamma' \tau$ are $i_Y$-related, i.e.,

$$Ti_Y \circ \Gamma = \Gamma' \circ i_Y.$$  \hspace{1cm} (18)

Let $h$ be a section of the fiber bundle $\Sigma \to X$ and $Y_h$ the subbundle (9) of the composite fiber bundle $Y \to X$, which is the restriction of the fiber bundle $Y \to \Sigma$ to $h(X)$. Every connection $A_{\Sigma}$ (13) induces the pull-back connection

$$A_h = i_h^* A_{\Sigma} = dx^\lambda \otimes [\partial_\lambda + ((A_m^i \circ h) \partial h^m + (A \circ h)^i)] \partial_i]$$ \hspace{1cm} (19)
on $Y_h \to X$. Now, let $\Gamma$ be a connection on $\Sigma \to X$ and let $\gamma = A_{\Sigma} \circ \Gamma$ be the composition (16). Then it follows from (18) that the connection $\gamma$ is reducible to the connection $A_h$ iff the section $h$ is an integral section of $\Gamma$, i.e., $\Gamma^m \circ h = \partial(h^m)$. Such a connection $\Gamma$ always exists.

Given a composite fiber bundle $Y$ (8), there is the following exact sequences

$$0 \to V_{\Sigma}Y \hookrightarrow VY \to \Sigma \times V_{\Sigma} \to 0,$$ \hspace{1cm} (20)
of vector bundles over $Y$, where $V_{\Sigma}Y$ denotes the vertical tangent bundle of the fiber bundle $Y \to \Sigma$. Every connection $A$ (13) on the fiber bundle $Y \to \Sigma$ determines the splitting

$$VY = V_{\Sigma}Y \oplus A_{\Sigma}(\Sigma \times V_{\Sigma}),$$ \hspace{1cm} (21)
$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A_m^i \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i),$$
of the exact sequence (20). Using this splitting, one can construct the first order differential operator

$$D : J^1 Y \to T^* X \otimes V_{\Sigma} Y,$$ \hspace{1cm} (22)
called the vertical covariant differential on the composite fiber bundle $Y \to X$. It possesses the following important property. Let $h$ be a section of the fiber bundle $\Sigma \to X$ and $Y_h$ the subbundle (9) of the composite fiber bundle $Y \to X$, which is the restriction of the fiber bundle $Y \to \Sigma$ to $h(X)$. Then the restriction of the vertical covariant differential $D$ (22) to $J^1 h(J^1 Y_h) \subset J^1 Y$ coincides with the familiar covariant differential on $Y_h$ relative to the pull-back connection $A_h$ (19).

Turn now to the properties of connections compatible with a reduced structure of a principal bundle. Recall the following theorems [14].
**Theorem 10.** Since principal connections are equivariant, every principal connection $A_h$ on a reduced $H$-principal subbundle $P^h$ of a $G$-principal bundle $P$ gives rise to a principal connection on $P$.

**Theorem 11.** A principal connection $A$ on a $G$-principal bundle $P$ is reducible to a principal connection on a reduced $H$-principal subbundle $P^h$ of $P$ iff the corresponding global section $h$ of the $P$-associated fiber bundle $P/H \to X$ is an integral section of the associated principal connection $A$ on $P/H \to X$.

**Theorem 12.** Let the Lie algebra $\mathfrak{g}(G)$ of $G$ is the direct sum $\mathfrak{g}(G) = \mathfrak{g}(H) \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}(H)$ of $H$ and a subspace $\mathfrak{m}$ such that $ad(g)(\mathfrak{m}) \subset \mathfrak{m}$, $g \in H$. Let $\overline{A}$ be a $\mathfrak{g}(G)$-valued connection form on $P$. Then, the pull-back of the $\mathfrak{g}(H)$-valued component of $\overline{A}$ onto a reduced subbundle $P^h$ is the connection form of a principal connection on $P^h$.

**Remark 5.** Given a $G$-principal bundle $P$, let $P^h$ be its reduced $H$-principal subbundle. Let $A_\Sigma$ be a principal connection on the $H$-principal bundle $P \to P/H$, and $i_h^*A_\Sigma$ (19) the pull-back connection $i_h^*A$ onto a reduced subbundle $P^h$. With this fact, we come to the following feature of the dynamics of field systems with symmetry breaking.
pull-back principal connection on $P^h$. In accordance with Theorem 10, it gives rise to a principal connection on $P$. For different $h$ and $h'$, the connections $i^*_h A_\Sigma$ and $i^*_{h'} A_\Sigma$ however yield different principal connections on $P$.

References

[1] D. Ivanenko and G. Sardanashvily, The gauge treatment of gravity, *Phys. Rep.* 94 (1983) 1.

[2] A.Trautman, *Differential Geometry for Physicists* (Bibliopolis, Naples, 1984).

[3] L.Nikolova and V.Rizov, Geometrical approach to the reduction of gauge theories with spontaneous broken symmetries, *Rep. Math. Phys.* 20 (1984) 287.

[4] R.Percacci, *Geometry on Nonlinear Field Theories* (World Scientific, Singapore, 1986).

[5] M.Keyl, About the geometric structure of symmetry breaking, *J. Math. Phys.* 32 (1991) 1065.

[6] G.Sardanashvily, On the geometry of spontaneous symmetry breaking, *J. Math. Phys.* 33 (1992) 1546.

[7] R.Zulanke, P.Wintgen, *Differentialgeometrie und Faserbündel*, Hochschulbucher fur Mathematik, 75 (VEB Deutscher Verlag der Wissenschaften, Berlin, 1972).

[8] S.Kobayashi, *Transformation Groups in Differential Geometry* (Springer-Verlag, Berlin, 1972).

[9] F.Gordejuela and J.Masqué, Gauge group and $G$-structures, *J. Phys. A* 28 (1995) 497.

[10] G.Sardanashvily and O.Zakharov, *Gauge Gravitation Theory* (World Scientific, Singapore, 1992).

[11] G.Sardanashvily, Covariant spin structure, *J. Math. Phys.* 39 (1998) 4874; *E-print arXiv* gr-qc/9711043.

[12] G.Sardanashvily, Classical gauge theory of gravity, *Theor. Math. Phys.* 132 (2002) 1163; *E-print arXiv* gr-qc/0208054.

[13] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).
[14] S.Kobayashi and K.Nomizu, *Foundations of Differential Geometry, Vol.1* (Interscience Publ., N.Y., 1963).

[15] N.Steenrod, *The Topology of Fibre Bundles* (Princeton Univ. Press, Princeton, 1972).

[16] L.Cordero, C.Dodson and M de León, *Differential Geometry of Frame Bundles* (Kluwer Acad. Publ., Dordrecht, 1988).

[17] D.Saunders, *The Geometry of Jet Bundles* (Cambr. Univ. Press, Cambridge, 1989).