THE SINGULARITIES AND BIRATIONAL GEOMETRY OF THE UNIVERSAL COMPACTIFIED JACOBIAN

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Abstract. In this paper we establish that the singularities of the universal compactified Jacobian are canonical if the genus is at least four. As a corollary we determine the Kodaira dimension and the Iitaka fibration of the universal compactified Jacobian for every degree and genus. We also determine the birational automorphism group for every degree if the genus is at least twelve. This extends work of G. Farkas and A. Verra, as well as that of G. Bini, C. Fontanari and the third author.

Introduction

Jacobians of non-singular curves are principally polarized abelian varieties, which from the perspective of birational geometry are among the simplest varieties. On the other hand, for a family of non-singular curves, the relative Jacobian may exhibit more interesting birational behavior, not necessarily reflective of the birational geometry of the base. For instance, over the moduli space of non-singular, genus \( g \geq 2 \), automorphism free curves \( M_g^{\text{nf}} \) there is a universal curve \( C_g^{\text{nf}} \), and consequently a universal Jacobian \( \text{Pic}^0(C_g^{\text{nf}}/M_g^{\text{nf}}) \). In this paper, we investigate the birational geometry of this space and show for instance, that the Kodaira dimension of \( \text{Pic}^0(C_g^{\text{nf}}/M_g^{\text{nf}}) \) can be different from the Kodaira dimension of \( M_g^{\text{nf}} \).

More generally, for any integer \( d \), Caporaso [Cap94] (see also [Pan96]) has constructed a universal compactified Jacobian \( \pi: \overline{J}_{d,g} \to M_g \) over the moduli space of Deligne–Mumford stable curves; this space has fiber over a non-singular, automorphism free curve \( C \) given by the degree \( d \) Jacobian \( J^d C \). In particular \( \overline{J}_{0,g} \) provides a compactification of the universal Jacobian. In this paper we focus on two main problems concerning the birational geometry of these spaces, namely determining the Kodaira dimension, and determining the birational automorphism group. These problems go back at least to Caporaso’s work, and have been investigated recently by Farkas and Verra [FV13] and Bini, Fontanari and the third author [BFV12] in special cases.

Due to the work of [BFV12], the main point needed to answer these questions in full generality is to provide a good description of the local structure of \( \overline{J}_{d,g} \). In this paper, we investigate this question in detail, providing an explicit description of the complete local ring at a point, as well as formulas for various invariants of the ring in terms of the dual graphs of the curves. In particular, we establish that \( \overline{J}_{d,g} \) has canonical singularities.

Theorem A. Assume that \( \text{char}(k) = 0 \). If \( g \geq 4 \), then the universal compactified Jacobian \( \overline{J}_{d,g} \) has canonical singularities for any \( d \in \mathbb{Z} \).

The arguments build on the previous work of the authors in two ways. First, extending the deformation theory in [CMKVe], we are able to reduce the problem to the study of a special class of combinatorial rings, called cographic toric face rings, investigated in [CMKVa]. In full generality, these rings can exhibit poor behavior (see [CMKVe, §5.1]). However, as it turns out, the rings appearing from the deformation theory of the universal compactified Jacobian form a special class of rings with mild singularities. The specific cographic rings appearing in this paper will be denoted by \( U(\Gamma) \) and are defined from the data of a graph \( \Gamma \) (Definition 2.1). Our main result for these rings is the following theorem.

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Theorem B. Let $\Gamma$ be a finite, connected graph and let $k$ be an algebraically closed field. The cographic toric ring $U(\Gamma)$ is a finitely generated, integral $k$-algebra and the singularities of the associated variety $\text{Spec} U(\Gamma)$ are Gorenstein, rational, and terminal.

Using the results in [CMKVa], together with standard results on toric varieties, we are also able to establish a number of further properties of the rings $U(\Gamma)$ (and consequently $J_{d,g}$) in terms of invariants of the graph $\Gamma$, including the dimension (Corollary 4.2), the dimension of the tangent space (Proposition 4.4), and the multiplicity (Theorem 4.11).

From Theorem A and the work of Bini–Fontanari–Viviani, one obtains the following consequence for the birational geometry of $J_{d,g}$.

Corollary C. Assume that $\text{char}(k) = 0$. The Kodaira dimension of the universal Jacobian $\bar{J}_{d,g}$ is given by

$$\kappa(\bar{J}_{d,g}) = \begin{cases} -\infty & \text{if } g \leq 9, \\ 0 & \text{if } g = 10, \\ 19 & \text{if } g = 11, \\ 3g - 3 & \text{if } g \geq 12. \end{cases}$$

Moreover, for $g \geq 10$, the Iitaka fibration of $J_{d,g}$ is given as follows:

1. For $g \geq 12$, the Iitaka fibration is the forgetful morphism $\pi : \bar{J}_{d,g} \to \overline{M}_g$.
2. For $g = 11$, the Iitaka fibration is the rational map $J_{d,11} \dashrightarrow F_{11}$, where $F_g$ is the moduli of K3 surfaces with polarization of degree $2g - 2$, and the rational map takes a general pair $(C, L)$ to the pair $(S, \mathcal{O}_S(C))$, where $S$ is the unique K3 containing $C$ (see [Muk96]).
3. For $g = 10$, the Iitaka fibration is the structure morphism $J_{d,10} \to \text{Spec } k$.

For $g = 22$ and $g \geq 24$ the statement on the Kodaira dimension follows from general results in birational geometry, together with well-known results for $\overline{M}_g$ (see Remark 7.3). In the remaining range, the result was proven by Bini–Fontanari–Viviani [BFV12, Thm. 1.2] under the numerical condition that $\gcd(d + 1 - g, 2g - 2) = 1$ or $g = 23$, and by Farkas–Verra [FV13] in the special case $d = g$. In particular, the case $d = 0$ was not known. We also point out that while we have obtained here a complete classification of the Kodaira dimension for the universal Jacobian, the Kodaira dimension of the moduli of curves is still unknown in the range $17 \leq g \leq 21, g = 23$. Finally, for $10 \leq g \leq 16$, we have $\kappa(J_{d,g}) \neq \kappa(\overline{M}_g)$. We direct the reader to [Fark10] for more details, as well as Remark 7.10 which compares these numerics with the recent work of Farkas–Verra [Far10, FV12, FV14, Far12] on the moduli space of theta characteristics.

Another immediate observation is that the Kodaira dimension is independent of $d$. One might guess the reason for this is that $J_{d,g}$ is birational to $J_{d',g}$ for different $d$ and $d'$. Our next result shows this is not generally the case.

Corollary D. Assume that $\text{char}(k) = 0$ and that $g \geq 12$. If $\eta : J_{d,g} \dashrightarrow J_{d',g}$ is a birational map, then $d' = \pm d + n(2g - 2)$ and $\eta$ is given by the map sending $(C, L) \in J_{d,g}$ into $(C, L^{\pm 1} \otimes \omega_C^n) \in J_{d',g}$. In particular:

1. $J_{d,g}$ is birational to $J_{d',g}$ if and only if $d' \equiv \pm d \mod 2g - 2$.
2. The group $\text{Bir}(J_{d,g})$ of birational automorphisms of $J_{d,g}$ is given by

$$\text{Bir}(J_{d,g}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d = n(g - 1) \text{ for some } n \in \mathbb{Z}, \\ \{\text{Id}\} & \text{otherwise}. \end{cases}$$

Moreover, if $d = n(g - 1)$ for some $n \in \mathbb{Z}$ then the generator of $\text{Bir}(J_{d,g})$ is the birational automorphism sending $(C, L)$ into $(C, L^{-1} \otimes \omega_C^n)$.

This was proven by Bini–Fontanari–Viviani [BFV12, Thm. 1.7] in the special case $\gcd(d + 1 - g, 2g - 2) = 1$ (or $g \geq 22$), and builds on work of Caporaso [Cap94].

The paper is organized as follows. In Section 1 we review terminology concerning graphs, and various constructions with graphs that will appear later. In Section 2 we define the combinatorial rings $U(\Gamma)$ and establish some first properties of the rings. In Section 3 we establish some specific presentations of the ring, which are useful for later computations, and also for connecting the rings with deformations. In Section 4 we discuss the singularities of the rings $U(\Gamma)$. In Section 5 we describe the rings as invariants for a group action, which provides the framework for the connection with deformations of sheaves. In Section 6 we provide some
examples of these rings. In Section 7 we make the connection with the universal compactified Jacobian, and establish the results on the singularities, Kodaira dimension, and birational automorphism group.

The paper ends with an appendix in which we investigate the singularities of finite quotients of toric varieties. More specifically, the focus is on establishing a Reid–Tai–Shepherd-Barron criterion for singular toric varieties; i.e., a numerical condition that can be used to determine when a finite quotient of a singular toric variety has canonical, or terminal singularities. The main result is Proposition A.6, which in conjunction with Theorem A.11 is a direct generalization of the Reid–Tai–Shepherd-Barron criterion. While we expect the generalization is well-known to the experts, we were not aware of a reference, and include proofs here.

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1. Preliminaries on graphs

In this section we introduce some constructions on graphs that will be used in this paper.

1.1. Graph notation. Following Serre [Ser03 § 2.1], a graph $\Gamma$ consists of the data $(\vec{E} \xrightarrow{s} V, \vec{E} \xrightarrow{t} \vec{E})$, where $V$ and $\vec{E}$ are sets, $\iota$ is a fixed-point free involution, and $s$ and $t$ are maps satisfying $s(\vec{e}) = t(\iota(\vec{e}))$ for all $\vec{e} \in \vec{E}$. The maps $s$ and $t$ are called the source and target maps respectively. We call $V := V(\Gamma)$ the set of vertices. We call $\vec{E} := \vec{E}(\Gamma)$ the set of oriented edges.

We define the set of (unoriented) edges to be $E(\Gamma) = E := \vec{E}/\iota$. Given an oriented edge $\vec{e} \in \vec{E}$ we will denote by $\vec{e}$ the class of $\vec{e}$ in $E$. An orientation of an edge $e \in E$ is a representative for $e$ in $\vec{E}$; we use the notation $\vec{e}$ and $\vec{e}$ for the two possible orientations of $e$. An orientation of a graph $\Gamma$ is a section $\phi : E \to \vec{E}$ of the quotient map. An oriented graph consists of a pair $(\Gamma, \phi)$ where $\Gamma$ is a graph and $\phi$ is an orientation. Given an oriented graph, we say that $\phi(e)$ is the positive orientation of the edge $e \in E$. Given a subset $S \subseteq E$, we define $\overline{S} \subseteq \overline{E}$ to be the set of all orientations of the edges in $S$.

We will say that two edges of a graph are parallel if they connect the same (not necessarily distinct) vertices. We say that an edge of a connected graph is a separating edge if removing the edge disconnects the graph. Two edges of a connected graph are a separating pair if they are both non-separating edges and if removing the two edges disconnects the graph.

If $\Gamma$ is connected, then we say that an orientation $\phi$ of $\Gamma$ is totally cyclic if there does not exist a proper non-empty subset $W \subset V(\Gamma)$ such that the edges between $W$ and its complement $V(\Gamma) \setminus W$ all go in the same direction (i.e. either all these edges are oriented from $W$ to $V(\Gamma) \setminus W$ or all are oriented in the opposite direction). If $\Gamma$ is disconnected, then we say that an orientation of $\Gamma$ is totally cyclic if the orientation induced on each connected component of $\Gamma$ is totally cyclic.

A graph $\Gamma$ is called cyclic if it is connected, free from separating edges, and satisfies $b_1(\Gamma) := |E(\Gamma)| - |V(\Gamma)| + 1 = 1$. We will also call a cyclic graph a circuit. A cyclic graph together with a totally cyclic orientation is called an oriented circuit. A loop is a circuit with a single edge.

1.2. Ordinary homology and oriented homology. Given any graph $\Gamma$, we can form its ordinary homology (which coincides with the homology of the underlying topological space) and its oriented homology.

Let $C_0(\Gamma, \mathbb{Z})$ be the free $\mathbb{Z}$-module with basis $V(\Gamma)$, let $C_1(\Gamma, \mathbb{Z})$ be the free $\mathbb{Z}$-module generated by $\vec{E}(\Gamma)$ and consider the boundary map $\partial$ defined as:

\begin{equation}
\partial : C_1(\Gamma, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z})
\end{equation}

\[ \vec{e} \mapsto t(\vec{e}) - s(\vec{e}). \]

We will denote by $\mathbb{H}_*(\Gamma, \mathbb{Z})$ the groups obtained from the homology of $C_*(\Gamma, \mathbb{Z})$ and we will call them the oriented homology groups of $\Gamma$. Let $\langle , \rangle$ be the unique scalar product on $C_1(\Gamma, \mathbb{R}) = C_1(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ (and also its restriction to $\mathbb{H}_1(\Gamma, \mathbb{Z})$) such that the elements of $\vec{E}(\Gamma)$ form an orthonormal basis.
Let \( C_0(\Gamma, \mathbb{Z}) = C_0(\Gamma, \mathbb{Z}) \), let \( C_1(\Gamma, \mathbb{Z}) \) be the quotient of \( C_1(\Gamma, \mathbb{Z}) \) by the relation \( \overrightarrow{e} = -\overrightarrow{e} \) for every \( e \in E(\Gamma) \) and consider the boundary map

\[
\partial : C_1(\Gamma, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z})
\]

(1.2)

where we denote by \([\overrightarrow{e}]\) the class of \( \overrightarrow{e} \) in \( C_1(\Gamma, \mathbb{Z}) \). We will denote by \( H_* \) the groups obtained from the homology of \( C_* \) and we will call them the ordinary homology groups of \( \Gamma \). Note that \( H_* \) is isomorphic to the homology of the underlying topological space of \( \Gamma \). Let \((\ , \ )\) be the unique scalar product on \( C_1(\Gamma, \mathbb{Z}) \) (and also its restriction to \( H_1(\Gamma, \mathbb{Z}) \)) such that

\[
\langle [\overrightarrow{e}], [\overrightarrow{f}] \rangle = -\langle [\overrightarrow{e}], [\overrightarrow{f}] \rangle = 1 \quad \text{for any} \ e \in E,
\]

\[
\langle [\overrightarrow{e}]_1, [\overrightarrow{e}]_2 \rangle = 0 \quad \text{for any} \ \overrightarrow{e}_1, \overrightarrow{e}_2 \in \overrightarrow{E} \text{ such that } |\overrightarrow{e}_1| \neq \pm |\overrightarrow{e}_2|.
\]

For a connected graph \( \Gamma \), the corank of the image of \( \mathbb{D} \) (resp. of \( \partial \)) is one. Consequently, for a connected graph, we have

\[
\text{rank} \ H_1(\Gamma, \mathbb{Z}) = |E(\Gamma)| - |V(\Gamma)| + 1 =: b_1(\Gamma),
\]

\[
\text{rank} \ H_1(\Gamma, \mathbb{Z}) = 2|E(\Gamma)| - |V(\Gamma)| + 1 = b_1(\Gamma) + |E(\Gamma)|.
\]

In order to determine the relationship between ordinary and oriented homology, consider the following commutative diagram

\[
\begin{array}{ccc}
C_1(\Gamma, \mathbb{Z}) & \xrightarrow{\partial} & C_0(\Gamma, \mathbb{Z}) \\
\downarrow \ & \ & \downarrow \\
C_1(\Gamma, \mathbb{Z}) & \xrightarrow{\partial} & C_0(\Gamma, \mathbb{Z})
\end{array}
\]

where the left vertical map send \( \overrightarrow{e} \) into \([\overrightarrow{e}]\). The above diagram (1.4) induces an equality \( H_0(\Gamma, \mathbb{Z}) = H_0(\Gamma, \mathbb{Z}) \) and a surjection \( H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z}) \), whose kernel can be described as follows.

**Lemma 1.1.** The kernel of the natural surjection \( H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z}) \) is generated by \( \{\overrightarrow{e} + \overrightarrow{e}'\}_{e \in E(\Gamma)} \).

**Proof.** From the definition of \( \mathbb{D} \), we have \( \overrightarrow{e} + \overrightarrow{e}' \in H_1(\Gamma, \mathbb{Z}) \). Clearly \( \overrightarrow{e} + \overrightarrow{e}' \) also maps to zero in \( C_1(\Gamma, \mathbb{Z}) \). On the other hand, suppose that \( \sum_{e \in E} (a_e \overrightarrow{e} + b_e \overrightarrow{e}') \in H_1(\Gamma, \mathbb{Z}) \) is in the kernel of the above map. Then by definition \( \sum_{e \in E} (a_e - b_e) \overrightarrow{e} = 0 \), and so \( a_e = b_e \) for all \( e \in E \), since \( \{\overrightarrow{e}\} \) is a basis for \( C_1(\Gamma, \mathbb{Z}) \).

\[ \square \]

### 1.3. Doubled graphs and doubled orientations.

In this section, we introduce a class of graphs, called doubled graphs, together with canonical totally cyclically orientations of them, called doubled orientations, which are obtained from a graph by doubling its edges.

**Definition 1.2.** Let \( \Gamma \) be a connected graph. Define the **doubled graph** of \( \Gamma \), denoted \( \Gamma^d \), to be the graph obtained by doubling the edges of \( \Gamma \); i.e. \( \Gamma^d \) is the graph obtained from \( \Gamma \) by replacing each edge \( e \) of \( \Gamma \) with a pair of parallel edges \( e' \) and \( e'' \) of \( \Gamma^d \) having the same endpoints as \( e \) (see Figure 1). To be precise, \( V(\Gamma^d) = V(\Gamma) \), \( E(\Gamma^d) = \bigcup_{e \in E} \{e', e''\} \), and we define \( s(e') = s(e'') = s(e) \), \( t(e') = t(e'') = t(e) \) and \( \nu(e') = e' \), \( \nu(e'') = e'' \). Note that

\[
E(\Gamma^d) = \bigcup_{e \in E(\Gamma)} \{e', e''\}
\]

where we use the convention that if \( e = \overrightarrow{e} \), then \( e' = \overrightarrow{e} \), \( e'' = \overrightarrow{e} \).

The graph \( \Gamma^d \) drawn with its unoriented edges looks like the graph \( \Gamma \) drawn with its oriented edges (see Figure 1). In this way, choosing an identification of edges gives an orientation \( \phi^d \) of \( \Gamma^d \). In fact, given an orientation \( \phi \) of \( \Gamma \), one obtains an orientation \( \phi^d \) of \( \Gamma^d \) by orienting each edge \( e' \) in the same direction as \( \phi(e) \), and each edge \( e'' \) in the opposite direction (see Figure 1). More precisely:
Definition 1.3. Given an orientation $\phi$ of $\Gamma$, define the doubled orientation

$$
\phi^d : E(\Gamma^d) \rightarrow \vec{E}(\Gamma^d)
$$

$$
\phi^d(e') = \phi(e)' \\
\phi^d(e'') = \iota(\phi(e)'')
$$

Lemma 1.4. The doubled orientation $\phi^d$ on $\Gamma^d$ is canonical, i.e. it does not depend on the choice of $\phi$ up to automorphisms of $\Gamma^d$, and it is totally cyclic.

Proof. Choose an (unoriented) edge $f \in E(\Gamma)$, define a new orientation $\phi^f$ of $\Gamma$ by reversing the orientation on $f$; i.e. setting

$$
\phi^f(e) = \begin{cases} 
\iota(\phi(f)) & \text{if } e = f, \\
\phi(e) & \text{if } e \neq f.
\end{cases}
$$

Define an automorphism $\psi$ of $\Gamma^d$ that is the identity on vertices, exchanges $f'$ and $f''$ and fixes $e'$ and $e''$ for all other edges $e \neq f$ of $\Gamma$. Then clearly $\psi$ will send the orientation $\phi^d$ into $(\phi^f)^d$. Since every other orientation of $\Gamma$ can be obtained from $\phi$ by iteratively applying the above construction, we have shown that $\phi^d$ is canonical.
The fact that \( \phi^d \) is totally cyclic follows easily from the fact that each pair of parallel (unoriented) edges \( e' \) and \( e'' \) of \( \Gamma^d \) associated to an edge \( e \) of \( \Gamma \) are given opposite orientations by \( \phi^d \).

The oriented homology of \( \Gamma \) is canonically isomorphic to the ordinary homology of \( \Gamma^d \). In order to prove this, fix an orientation \( \phi \) of \( \Gamma \) and consider the diagram

\[
\begin{array}{c}
\phi(e) \in C_1(\Gamma, \mathbb{Z}) \\
\downarrow \\
\phi^d(e') \in C_1(\Gamma^d, \mathbb{Z}) \\
\downarrow \\
C_0(\Gamma^d, \mathbb{Z}) \xrightarrow{\partial} C_0(\Gamma, \mathbb{Z})
\end{array}
\]

where the left vertical map is the group isomorphism obtained by, for each \( e \in E(\Gamma) \), sending \( [\phi^d(e')] \in C_1(\Gamma^d, \mathbb{Z}) \) into \( \phi(e) \in C_1(\Gamma, \mathbb{Z}) \) (and \( [\phi^d(e'')] \to i\phi(e) \)). In short, choosing a doubled orientation \( \phi^d \) on \( \Gamma^d \), then \( C_1(\Gamma^d, \mathbb{Z}) \) can be given a basis consisting of the oriented edges determined by \( \phi^d \); these edges are in bijection (including orientation) with the collection of all oriented edges of \( \Gamma \), which form a basis of \( C_1(\Gamma, \mathbb{Z}) \) (see Figure 2).

**Lemma 1.5.** The above diagram (1.5) is commutative and it induces an isomorphism \( H_i(\Gamma^d, \mathbb{Z}) \xrightarrow{\cong} H_i(\Gamma, \mathbb{Z}) \) for \( i = 0, 1 \).

**Proof.** This is straightforward to check and is left to the reader. \( \square \)

### 1.4. The affine semigroup ring \( R(\Gamma, \phi) \) and its associated toric variety \( X_{(\Gamma, \phi)} \)

In this section we review the definitions of the ring \( R(\Gamma, \phi) \) from [CMKVa, §4]. Let \( (\Gamma, \phi) \) be a graph with a totally cyclic orientation. Consider the pointed full-dimensional rational polyhedral cone

\[
\sigma_1(\phi) := \bigcap_{e \in E(\Gamma)} \{ (\cdot, \phi(e)) \geq 0 \} \subseteq H_1(\Gamma, \mathbb{Z}) \otimes \mathbb{R}.
\]

(This was denoted \( \sigma(0, \phi) \) in [CMKVa, §3].) According to Gordan’s Lemma (e.g. [CLS11, Prop. 1.2.17]), the semigroup

\[
C_1(\phi) := \sigma_1(\phi) \cap H_1(\Gamma, \mathbb{Z}) \subseteq H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}^{h_1(\Gamma)}
\]

is a positive, normal, affine semigroup, i.e. a finitely generated subgroup isomorphic to a subsemigroup of \( \mathbb{Z}^d \) for some \( d \in \mathbb{N} \), such that 0 is the unique invertible element and such that if \( m \cdot z \in C_1(\phi) \) for some \( m \in \mathbb{N} \) and \( z \in \mathbb{Z}^d \), then \( z \in C_1(\phi) \).

Recall ([CMKVa, Def. 4.2]) that we define

\[
R(\Gamma, \phi) := k[C_1(\phi)]
\]

to be the affine semigroup ring associated to \( C_1(\phi) \); i.e. the \( k \)-algebra whose underlying vector space has basis \( \{ X^e : e \in C_1(\phi) \} \) and whose multiplication is defined by \( X^e \cdot X^{e'} := X^{e+e'} \). \( R(\Gamma, \phi) \) is a normal, Cohen–Macaulay domain of dimension equal to \( h_1(\Gamma) \)

\[
\text{dim} R(\Gamma, \phi) = \dim \sigma_1(\phi) = b_1(\Gamma).
\]

The affine variety

\[
X_{(\Gamma, \phi)} := \text{Spec } R(\Gamma, \phi)
\]

is the toric variety associated to the fan \( \Sigma_{(\Gamma, \phi)} \) consisting of the dual cone \( \sigma_1(\phi)^\vee \subset H_1(\Gamma, \mathbb{Z})^\vee \otimes \mathbb{R} \) together with all its faces.

### 2. The cographic toric variety \( X_\Gamma \) and the cographic toric ring \( U(\Gamma) \)

Fix a graph \( \Gamma \). Using the notation of (1.2) set \( M_\Gamma := H_1(\Gamma, \mathbb{Z}) \) and \( N_\Gamma := H_1(\Gamma, \mathbb{Z})^\vee \). Consider the pointed rational polyhedral cone

\[
\sigma := \bigcap_{\epsilon \in E} \{ (\cdot, \epsilon) \geq 0 \} \subset M_\Gamma \otimes \mathbb{Z} \mathbb{R},
\]
Lemma 1.5 sends the cone $\sigma$ for a cycle $z$ morphically onto the semigroup with non-zero coefficient in $z$ inducing the isomorphism

$$C(\Gamma) := H_1(\Gamma, \mathbb{Z}) \cap \sigma$$

is a positive, normal, affine semigroup.

**Definition 2.1.**

(i) The **cographic toric ring** $U(\Gamma)$ of $\Gamma$ (over a base field $k$) is the affine semigroup $k$-algebra associated to $C(\Gamma)$, i.e.

$$U(\Gamma) := k[C(\Gamma)].$$

Explicitly, $U(\Gamma)$ is the $k$-algebra whose underlying vector space has basis \{ $X^c : c \in C(\Gamma)$ \} and whose multiplication is defined by $X^c \cdot X^{c'} := X^{c+c'}$.

(ii) The **cographic toric variety** $X_\Gamma$ of $\Gamma$ (over a base field $k$) is the affine variety

$$X_\Gamma := \text{Spec } U(\Gamma) = \text{Spec } k[C(\Gamma)].$$

Observe that $X_\Gamma$ is the (normal) toric variety associated to the rational polyhedral fan $\Sigma_\Gamma$ in $\mathbb{N}^\Gamma \otimes \mathbb{Z} \mathbb{R}$ formed by $\sigma^-_\Gamma$ and all its faces. We describe $\sigma^-_\Gamma$ in more detail in §4.3.

**Example 2.2.** Let $L$ be the loop graph, i.e. the graph with one vertex $v$ and one unoriented edge $e$ which is a loop around $v$. Then $C_1(L, \mathbb{Z})$ is freely generated by $e$ and $\overline{e}$ and the boundary map $\partial$ is trivial; hence $H_1(L, \mathbb{Z}) = \mathbb{C}_1(L, \mathbb{Z}) = \langle e, \overline{e} \rangle$. The cone $\sigma_L$ of \( (\ref{2.2}) \) is the first quadrant in $H_1(L, \mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}^2$ and the semigroup $C(L)$ of \( (\ref{2.2}) \) is isomorphic to $\mathbb{N}^2$, generated by $e$ and $\overline{e}$. Therefore

$$U(L) = k[C(L)] = k[X^e, X^{\overline{e}}] \cong k[X, Y] \text{ and } X_L = \text{Spec } U(L) = k^2.$$ 

The cographic toric ring $U(\Gamma)$ and the cographic toric variety $X_\Gamma$ admit also another presentation in terms of the affine semigroup algebra (and its corresponding affine toric variety) associated to the double graph $\Gamma'$ with its double orientation $\Gamma'$, see \[1.3\] and \[4.3\].

**Proposition 2.3.** There is an isomorphism of $k$-algebras

$$U(\Gamma) \cong R(\Gamma', \phi'\overline{\phi}).$$

**inducing the isomorphism** $X_\Gamma \cong X(\Gamma', \phi'\overline{\phi})$ **of toric varieties.**

**Proof.** Comparing \[1.6\] with \[2.1\], it is easily checked that the isomorphism $H_1(\Gamma', \mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z})$ of Lemma \[1.5\] sends the cone $\sigma_{\Gamma', \phi'\overline{\phi}}$ isomorphically into the cone $\sigma_\Gamma$, and hence the semigroup $C_{\Gamma', \phi'\overline{\phi}}$ isomorphically onto the semigroup $C(\Gamma)$. By taking the associated semigroup algebras we get the isomorphism $R(\Gamma', \phi'\overline{\phi}) \cong U(\Gamma)$ and, by passing to prime spectra, we obtain that $X_{(\Gamma', \phi'\overline{\phi})} \cong X_\Gamma$. 

3. **An explicit presentation of the cographic toric ring $U(\Gamma)$**

The aim of this section is to give an explicit presentation of the cographic toric ring $U(\Gamma)$, which also shows that $U(\Gamma)$ is a deformation of the cographic toric face ring $R(\Gamma)$ introduced and studied in [CMKVa].

To begin, we will define a map

$$\psi : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}^{E(\Gamma)}_{\geq 0}.$$ 

For a cycle $z \in H_1(\Gamma, \mathbb{Z}) \subseteq C_1(\Gamma, \mathbb{Z})$, denote by $\text{Supp}(z)$ (support of $z$) the set of edges of $E(\Gamma)$ that appear with non-zero coefficient in $z$. Then we can write $z$ uniquely as

$$z = \sum_{e \in \text{Supp}(z)} a_e [e]$$

with $a_e > 0$ for all $e \in \text{Supp}(z)$.

Now if

$$z^{(1)} = \sum_{e \in \text{Supp}(z)^{(1)}} a_e^{(1)} [e^{(1)}] \text{ and } z^{(2)} = \sum_{e \in \text{Supp}(z)^{(2)}} a_e^{(2)} [e^{(2)}]$$

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It follows from Lemma 3.5 that there is an explicit bijection between the sets $H_1(\Gamma, \mathbb{Z})$ and $\mathbb{Z}^{E(\Gamma)}$. By tracing the semigroup law on $C(\Gamma)$ via this bijection we ended up exactly with the common cone of the cographic fan $(\Gamma)$ given in Proposition 3.3, we derive the following description of the semigroup $C(\Gamma)$ of $(\Gamma)$.

**Proposition 3.3.** The semigroup $C(\Gamma)$ is isomorphic to the set $H_1(\Gamma, \mathbb{Z}) \times \mathbb{Z}^{E(\Gamma)}$ endowed with the structure of semigroup given by

$$ (z_1, n_1) \times (z_2, n_2) \mapsto (z_1 + z_2, \psi(z_1, z_2) + n_1 + n_2). $$

In order to prove the above proposition, we will need the following two lemmas.

**Lemma 3.4.** Under the natural surjection $\mathbb{H}_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ induced by $[1.4]$, the semigroup $C(\Gamma)$ maps surjectively onto $H_1(\Gamma, \mathbb{Z})$.

**Proof.** We will prove this by constructing a section

$$ H_1(\Gamma, \mathbb{Z}) \to C(\Gamma). $$

Any cycle $z \in H_1(\Gamma, \mathbb{Z})$ can be written uniquely in the form $z = \sum_{e \in \text{Supp}(z)} a_e \overrightarrow{e}$ with $a_e > 0$. Thus

$$ z \mapsto \sum_{e \in \text{Supp}(z)} a_e \overrightarrow{e} $$

gives a well defined map $H_1(\Gamma, \mathbb{Z}) \to C(\Gamma)$. It is clearly a section.

**Lemma 3.5.** $C(\Gamma)$ is the sub-semigroup of $\mathbb{H}_1(\Gamma, \mathbb{Z})$ generated by $\{\overrightarrow{e} + \overrightarrow{e} \}_{e \in E(\Gamma)}$ and the image of the section $H_1(\Gamma, \mathbb{Z}) \to C(\Gamma)$ defined in [1.1] above.

**Proof.** Clearly both $\{\overrightarrow{e} + \overrightarrow{e} \}_{e \in E(\Gamma)}$ as well as the image of the section $H_1(\Gamma, \mathbb{Z}) \to C(\Gamma)$ lie in $C(\Gamma)$.

Now let $z \in C(\Gamma)$. Recall that by definition this means that $z = \sum_{e \in E} a_e \overrightarrow{e}$ with $a_e > 0$ for all $e \in E$. Let $z'$ be the image of $z$ in $H_1(\Gamma, \mathbb{Z})$ and let $z''$ be the image of $z'$ in $C(\Gamma)$ under the section. Then $z - z'' \in \ker(H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z}))$. Thus, using Lemma [1.1] we can write $z = z'' + \sum_{e \in E} b_e (\overrightarrow{e} + \overrightarrow{e})$. But by the construction of $z''$, for all $e \in E$, the coefficient of either $\overrightarrow{e}$ or $\overrightarrow{e}$ in $z''$ is zero. Thus $b_e \geq 0$ for all $e \in E$, and we are done.

**Proof of Proposition 3.3.** It follows from Lemma 3.5 that there is an explicit bijection between the sets $C(\Gamma)$ and $H_1(\Gamma, \mathbb{Z}) \times \mathbb{Z}^{E(\Gamma)}$. By tracing the semigroup law on $C(\Gamma)$ via this bijection we ended up exactly with the semigroup law on $H_1(\Gamma, \mathbb{Z}) \times \mathbb{Z}^{E(\Gamma)}$ given by [1.2], and we are done.

From the explicit description of the semigroup $C(\Gamma)$ given in Proposition 3.3, we derive the following explicit presentation of the cographic toric ring $U(\Gamma)$. 8
Theorem 3.6. Let $\Gamma$ be a connected graph. Consider the $k$-algebra $D(\Gamma)$ whose underlying vector space has basis $\{X^z T^\epsilon : z \in H_1(\Gamma, \mathbb{Z}), \epsilon \in \mathbb{Z}_{\geq 0}^E(\Gamma)\}$ and whose multiplication is defined by the rule $X^z T^\epsilon \cdot X^{z'} T^{\epsilon'} := X^{z+z'} T^{\psi(z,z') + \epsilon + \epsilon'}$. In other words,

$$D(\Gamma) := \frac{k[X^z]_{z \in H_1(\Gamma, \mathbb{Z})} [T^\epsilon]_{\epsilon \in E}}{(X^z X^{z'} - X^{z+z'} T^{\psi(z,z')} )}.$$  

Then we have an isomorphism $U(\Gamma) \cong D(\Gamma)$.

Proof. Observe that $D(\Gamma)$ is the semigroup $k$-algebra associated to the set $H_1(\Gamma, \mathbb{Z}) \times \mathbb{Z}_{\geq 0}^E(\Gamma)$ endowed with semigroup law $[3.2]$. Then the result follows from Proposition $[3.3]$. 

Remark 3.7. From Theorem $3.6$ together with Remark $[3.2]$ it follows that by setting the variables $T_e$ equal to zero we get a surjective morphism of $k$-algebras

$$U(\Gamma) \cong D(\Gamma) \rightarrow R(\Gamma)$$

where $R(\Gamma)$ is the cographic toric face ring introduced in $[CMKVa]$ Def. 1.2]. Thus the cographic toric variety $X_\Gamma$ can be viewed as a deformation of the cographic toric face variety Spec $R(\Gamma)$ ($[CMKVa]$ Def. 1.2]) over the base Spec $k[T_e]_{e \in E}$.

4. Singularities of $X_\Gamma$

The aim of this section is to study the singularities of the cographic toric variety $X_\Gamma$. Recall from Definition $[2.1]$ that $X_\Gamma$ is the (normal) toric variety associated to the rational polyhedral fan $\Sigma_\Gamma$ in $\mathbb{N}_\Gamma \otimes \mathbb{R}$ formed by the rational polyhedral cone $\sigma_\Gamma^\vee (\Sigma_\Gamma)$ and all its faces. The following lemma summarizes the basic properties of the cone $\sigma_\Gamma^\vee$.

Lemma 4.1. Let $\Gamma$ be a connected graph. Set $M_\Gamma = H_1(\Gamma, \mathbb{Z})$.

(i) The cone $\sigma_\Gamma^\vee \subseteq M_\Gamma^\vee = N_\Gamma$ is equal to

$$\sigma_\Gamma^\vee = \left\{ \sum_{\vec{e} \in E(\Gamma)} a_\vec{e} \cdot (, \vec{e}) : a_\vec{e} \geq 0 \right\},$$

where $(, \vec{e})$ denotes the element of $N_\Gamma$ obtained by pairing an element of $M_\Gamma$ with $\vec{e}$ via the scalar product $(, )$ defined in $[1.2]$.

(ii) The cone $\sigma_\Gamma^\vee$ is pointed and of full dimension in $N_\Gamma \otimes \mathbb{R}$.

(iii) The extremal rays of $\sigma_\Gamma^\vee$ are of the form $\langle (, \vec{e}) \rangle := \mathbb{R}_{\geq 0} \cdot (, \vec{e})$ as $\vec{e}$ varies in $E(\Gamma)$. Moreover, given $\vec{e}_1 \neq \vec{e}_2$, we have that

$$\langle (, \vec{e}_1) \rangle = \langle (, \vec{e}_2) \rangle \iff \vec{e}_1 = \vec{e}_2 \text{ is a separating edge of } \Gamma.$$

(iv) For every $\vec{e} \in E(\Gamma)$, the primitive element of the ray $\langle (, \vec{e}) \rangle$ with respect to the lattice $N_\Gamma$ is $(, \vec{e})$, i.e. $(, \vec{e}) \cap N_\Gamma = \mathbb{Z}_{\geq 0} \cdot (, \vec{e})$.

Proof. (i) follows from $[2.1]$, using the definition of a dual cone. Also, the first part of (iii) follows from (i). We will deduce the remaining properties of $\sigma_\Gamma^\vee$ from the properties of its dual cone $\sigma_\Gamma \subseteq M_\Gamma$, which is isomorphic to the cone $\sigma_{\Gamma^d}(\phi^d) \subset H_1(\Gamma^d, \mathbb{Z})$ as shown in the proof of Proposition $[2.3]$.

According to $[CMKVa]$ Prop. 3.1, the cone $\sigma_{\Gamma^d}(\phi^d)$ is a pointed and full-dimensional cone in $H_1(\Gamma^d, \mathbb{Z}) \otimes \mathbb{R}$. By duality, we deduce that (iii) holds.

Observe now that if $e$ is a non-separating edge of $\Gamma$, then the orientation $\phi^d|_{\Gamma^d \setminus \{e\}}$ (resp. $\phi^d|_{\Gamma^d \setminus \{e''\}}$) induced by $\phi^d$ on the graph $\Gamma^d \setminus \{e\}$ (resp. $\Gamma^d \setminus \{e''\}$) is still totally cyclic. On the other hand, if $e$ is a separating edge of $\Gamma$ then the corresponding edges $e'$ and $e''$ of $\Gamma^d$ form a pair of parallel edges; hence the orientation $\phi^d|_{\Gamma^d \setminus \{e',e''\}}$ induced by $\phi^d$ on $\Gamma^d \setminus \{e',e''\}$ is still totally cyclic, while neither the orientation induced by $\phi^d$ on $\Gamma^d \setminus \{e''\}$ nor the one induced on $\Gamma^d \setminus \{e''\}$ is totally cyclic. Therefore, $[CMKVa]$ Prop. 3.1 implies that the codimension one faces of $\sigma_{\Gamma^d}(\phi^d)$ are given by (with the notation of $[1.4]$).
The variety \( \Xi \) does not contain torus factors and it has dimension equal to
\begin{equation}
\dim \Xi = \dim \sigma_1^\Gamma = \text{rank}\ H_1(\Gamma, \mathbb{Z}) = b_1(\Gamma) + |E(\Gamma)|.
\end{equation}

**Proof.** This follows directly from Lemma 4.1 (ii), using [CLS11, Prop. 3.3.9 (c)] and [BH93, Prop. 6.6.1]. \( \square \)

We will want the following result describing the behavior of the cographic toric variety in the presence of separating edges and loops.

**Lemma 4.3.** Let \( \Gamma \) be a connected graph with \( n \) separating edges and \( m \) loops, and let \( \Gamma' \) be the graph obtained from \( \Gamma \) by contracting the separating edges and deleting the loops. Then we have that
\[ \Xi = \mathbb{A}^n_k + m \times \Xi'. \]

**Proof.** Let \( \{f_1, \ldots, f_n\} \) be the separating edges of \( \Gamma \), \( \{e_1, \ldots, e_m\} \) the loops of \( \Gamma \) and set \( \gamma_i := (e_i, e_i) \in H_1(\Gamma, \mathbb{Z}) \). Clearly we have that \( H_1(\Gamma, \mathbb{Z}) = H_1(\Gamma', \mathbb{Z}) \oplus \bigoplus_{i=1}^m \langle \gamma_i \rangle \). Moreover, if we denote by \( \psi' \) the analogous map associated to \( H_1(\Gamma', \mathbb{Z}) \) and by \( \psi \) the map \( \Xi \) associated to \( H_1(\Gamma, \mathbb{Z}) \) and \( \psi' \), then we have that
\[ \psi \left( z^{(1)} + \sum_i n_i^{(1)} \gamma_i, z^{(2)} + \sum_i n_i^{(2)} \gamma_i \right) = \begin{cases} 
\psi'(z^{(1)}, z^{(2)}) & \text{if } e \notin \{e_1, \ldots, e_m\}, \\
0 & \text{if } e = e_i \text{ and } n_i^{(1)} n_i^{(2)} \geq 0, \\
\min(|n_i^{(1)}|, |n_i^{(2)}|) & \text{if } e = e_i \text{ and } n_i^{(1)} n_i^{(2)} < 0,
\end{cases} \]
for any \( z^{(j)} \in H_1(\Gamma', \mathbb{Z}) \) and \( n_i^{(j)} \in \mathbb{Z} \). This implies easily that (using the notation of Theorem 3.6)
\[ D(\Gamma) = D(\Gamma') \otimes_k \frac{k[X_{\gamma_1}, X_{-\gamma_1}, \ldots, X_{\gamma_m}, X_{-\gamma_m}, T_{e_1}, \ldots, T_{e_m}]}{(X_{\gamma_1} X_{-\gamma_1} - T_{e_1}, \ldots, X_{\gamma_m} X_{-\gamma_m} - T_{e_m})} \otimes_k [T_{f_1}, \ldots, T_{f_n}]. \]

By passing to the prime spectra and using Theorem 3.6 we conclude. \( \square \)

**Remark 4.4.** A lengthier, but more elementary argument can be made for Lemma 4.3 directly from the definitions, without using Theorem 3.6.

From the point of view of birational geometry, the singularities of \( \Xi \) are particularly nice:

**Theorem 4.5.** The variety \( \Xi \) is Gorenstein, terminal and has rational singularities.

**Proof.** It is well-known that any (normal) toric variety has rational singularities (e.g. [CLS11, Thm. 11.4.2]) and is Cohen–Macaulay (e.g. [CLS11, Thm. 9.2.9]).

According to [CLS11, Prop. 8.2.12] (see also Proposition 4.3), \( \Xi \) is Gorenstein, i.e. the canonical divisor \( K_\Xi \) is Cartier, if and only if there exists an element \( m \in M_\Gamma \) such that \( \langle m, u_\rho \rangle = 1 \) for any extremal ray \( \rho \) of \( \sigma_1^\Gamma \), where \( \langle , \rangle \) denotes the canonical pairing between \( M_\Gamma \) and \( N_\Gamma = M_\Gamma^\vee \) and \( u_\rho \) denotes the minimal generator of \( \rho \cap N_\Gamma \). Consider now the following element of \( C_1(\Gamma, \mathbb{Z}) \)
\begin{equation}
m_\Gamma := \sum_{e \in \vec{E}} \vec{e} = \sum_{e \in E} (\vec{e} + \vec{e}).
\end{equation}

Since \( D(\vec{e}) = -D(\vec{e}) \), we get that \( D(m_\Gamma) = 0 \), and hence that \( m_\Gamma \in M_\Gamma = H_1(\Gamma, \mathbb{Z}) \). By definition of the scalar product (see [1.2]), we easily get that
\[ (m_\Gamma, \vec{e}) = 1 \text{ for any } \vec{e} \in \vec{E}. \]
For brevity, we will use the notation \( u_{\vec{e}} \) for the element \((\vec{e}, e) \in \mathbb{N}_\Gamma \) determined by \( \vec{e} \in \vec{E} \). The above equality translates into
\[
\langle m_\Gamma, u_{\vec{e}} \rangle = 1. 
\]

By Lemma 4.1, we get that the rays of \( \sigma_\Gamma \) are all of the form \( \langle u_{\vec{e}} \rangle = \mathbb{R}_{\geq 0} \cdot u_{\vec{e}} \) (as \( \vec{e} \) varies in \( \vec{E} \)) and that \( u_{\vec{e}} \) is the primitive element of the ray \( \langle u_{\vec{e}} \rangle \). Therefore we conclude that \( X_\Gamma \) is Gorenstein.

Finally, let us show that \( X_\Gamma \) has terminal singularities. Since we have already proved that \( X_\Gamma \) is Gorenstein, we conclude that \( X_\Gamma \) has canonical singularities by [CLS11, Prop. 11.4.11]. Thus, using [CLS11, Prop. 11.4.12] (see also Proposition 4.5), we conclude that in order to prove that \( X_\Gamma \) has terminal singularities it is (necessary and) sufficient to prove the following:

CLAIM: If \( x \in \sigma_\Gamma \cap \mathbb{N}_\Gamma \) is such that \( \langle m_\Gamma, x \rangle = 1 \), then \( x = u_{\vec{e}} \) for some \( \vec{e} \in \vec{E} \).

By Lemma 4.1 [0], we can write \( x = \sum_{\vec{e} \in \vec{E}} a_{\vec{e}} \cdot u_{\vec{e}} \) for certain \( a_{\vec{e}} \in \mathbb{R}_{\geq 0} \). Note that such a representation may not be unique if the cone \( \sigma_\Gamma \) is not simplicial, but we fix one such representation. By hypothesis, and recalling the definition of \( m_\Gamma \) (4.2), we have that
\[
1 = \langle m_\Gamma, x \rangle = \left( \sum_{\vec{e} = \vec{e}'} \vec{e}' \cdot \sum_{\vec{e} \in \vec{E}} a_{\vec{e}} \cdot u_{\vec{e}} \right) - \sum_{\vec{e} \in \vec{E}} a_{\vec{e}}. 
\]
Consider now, for any \( e \in E(\Gamma) \), the element \( \gamma_e := -\vec{e} + \vec{e} \in C_1(\Gamma, \mathbb{Z}) \). As above, since \( \mathbb{D}(\vec{e}) = -\mathbb{D}(\vec{e}) \), we get that \( \mathbb{D}(\gamma_e) = 0 \); i.e. that \( \gamma_e \in M_\Gamma = H_1(\Gamma, \mathbb{Z}) \). Using (4.4) and the fact that \( a_{\vec{e}} \geq 0 \), we get that
\[
\langle \gamma_e, x \rangle = a_{\vec{e}'} + a_{\vec{e}} \in [0, 1].
\]
Moreover, since \( x \in \mathbb{N}_\Gamma \) and \( \gamma_e \in M_\Gamma \), we get that \( \langle \gamma_e, x \rangle \in \mathbb{Z} \); hence \( \langle \gamma_e, x \rangle \) is equal either to 1 or to 0. In the first case, all the coefficients \( a_{\vec{e}} \) with \( e = e' \) or \( e' \) must vanish because of (4.4); hence \( x = a_{\vec{e}} u_{\vec{e}} + a_{\vec{e}'} u_{\vec{e}'} \). In the second case, i.e. if \( \langle \gamma_e, x \rangle = 0 \), then necessarily \( a_{\vec{e}} = a_{\vec{e}'} = 0 \). We can therefore iterate the argument using all the edges of \( \Gamma \) and, since \( x \neq 0 \), in the end we find that necessarily
\[
x = a_{\vec{e}} u_{\vec{e}} + a_{\vec{e}'} u_{\vec{e}'} \quad \text{for some } e \in E(\Gamma).
\]
By virtue of Lemma 4.3 we may assume that \( \Gamma \) does not have separating edges, so in particular \( e \) is not a separating edge of \( \Gamma \). Using this, it is easy to see that there exists a cycle \( \gamma \in H_1(\Gamma, \mathbb{Z}) \) that contains \( \vec{e} \) but not \( \vec{e}' \). Therefore, from (4.4) and (4.5), we get that
\[
\langle \gamma, x \rangle = a_{\vec{e}} \in [0, 1].
\]
However, since \( x \in \mathbb{N}_\Gamma \) and \( \gamma \in M_\Gamma \), we get that \( \langle \gamma, x \rangle \in \mathbb{Z} \); hence \( a_{\vec{e}} = \langle \gamma, x \rangle \) is equal either to 1 or to 0, which implies that \( x \) is equal either to \( u_{\vec{e}} \) or to \( u_{\vec{e}'} \); the claim is now proved.

We can now give a complete classification of the graphs \( \Gamma \) for which \( X_\Gamma \) is smooth or has finite quotient singularities.

**Proposition 4.6.** Let \( \Gamma \) be a connected graph. The following conditions are equivalent:

(i) \( X_\Gamma = H^1(\Gamma; \mathbb{Q}) \);

(ii) \( X_\Gamma \) is smooth;

(iii) \( X_\Gamma \) has finite quotient singularities;

(iv) \( \Gamma \) is tree-like, i.e. \( \Gamma \) becomes a tree after removing all the loops.

**Proof.** (i) \( \Rightarrow \) (ii) follows from Lemma 4.3.

(ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) are obvious.

(iii) \( \iff \) (iv): First of all, from Lemma 4.3 we get that it is enough to prove the statement under the hypothesis that \( \Gamma \) has no separating edges. Note that, under this assumption, condition (i) now simply becomes that \( \Gamma \) has a unique vertex. According to [CLS11, Thm. 11.4.8], \( X_\Gamma \) has finite quotient singularities if and only if \( \sigma_\Gamma \) is simplicial, i.e. the number of its extremal rays is equal to its dimension. By Lemma 4.1, the dimension of \( \sigma_\Gamma \) is equal to
\[
\dim \sigma_\Gamma = \dim H_1(\Gamma, \mathbb{Z}) = b_1(\Gamma) = |E(\Gamma)| = 2|E(\Gamma)| - |V(\Gamma)| + 1.
\]
and the number of its extremal rays is equal to \(2|E(\Gamma)|\) by Lemma 4.1[13]. Therefore \(\sigma^\circ\) is simplicial if and only if \(\Gamma\) has a unique vertex, and we are done. \(\square\)

Denote by \(\emptyset\) the unique torus fixed point of the affine toric variety \(X_\Gamma\) and let \(\mathfrak{m}\) the maximal ideal of \(U(\Gamma)\) corresponding to it. Explicitly, under the isomorphism \(U(\Gamma) \cong D(\Gamma)\) of Theorem 3.6, the ideal \(\mathfrak{m}\) is generated by the variables \(X^z\) and the variables \(T_e\). The dimension of the tangent space of \(X_\Gamma\) at \(\emptyset\), or equivalently the embedded dimension of \(U(\Gamma)\) at \(\mathfrak{m}\), is easy to determine in terms of the (unoriented) circuits \(\text{Cir}(\Gamma)\) of \(\Gamma\) and of the loops \(\text{Loops}(\Gamma)\) of \(\Gamma\).

**Proposition 4.7.** The zariski tangent space \(T_{\emptyset}(X_\Gamma)\) at \(\emptyset\) has dimension equal to

\[
\dim T_{\emptyset}(X_\Gamma) = 2|\text{Cir}(\Gamma)| + |E(\Gamma)| - |\text{Loops}(\Gamma)|.
\]

**Proof.** By [CMKVa, Thm. 4.15(i), Prop. 5.2], the embedded dimension of \(R(\Gamma^d, \phi^d)\) at \(\mathfrak{m}\) is equal to the cardinality of the set \(\text{Cir}_{\phi^d}(\Gamma^d)\) of oriented circuits compatibly oriented with \(\phi^d\). Therefore, we conclude by applying Proposition 2.2 and the lemma below. \(\square\)

**Lemma 4.8.** The set \(\text{Cir}_{\phi^d}(\Gamma^d)\) of oriented circuits compatibly oriented with \(\phi^d\) is equal to

\[
|\text{Cir}_{\phi^d}(\Gamma^d)| = 2|\text{Cir}(\Gamma)| + |E(\Gamma)| - |\text{Loops}(\Gamma)|
\]

**Proof.** For every \(e \in E(\Gamma) \setminus \text{Loops}(\Gamma)\), we set \(\eta_e := \phi^d(e') + \phi^d(e'') \in \text{Cir}_{\phi^d}(\Gamma^d)\) where \(e'\) and \(e''\) are the two edges of \(\Gamma^d\) corresponding to \(e \in \Gamma\). By taking the image of a circuit of \(\Gamma^d\) under the natural contraction map \(\Gamma^d \to \Gamma\), we get a well-defined map

\[
\text{Cir}_{\phi^d}(\Gamma^d) \setminus \{\eta_e : e \in E(\Gamma) \setminus \text{Loops}(\Gamma)\} \to \text{Cir}(\Gamma).
\]

Since any circuit of \(\Gamma\) can be lifted in exactly two ways to an oriented circuit of \(\Gamma^d\) compatibly oriented with \(\phi^d\), the above map is surjective and \(2:1\). This concludes the proof. \(\square\)

**Remark 4.9.** In terms of Theorem 3.6 Proposition 4.7 reflects the fact that \(D(\Gamma)\) is generated by \(2|\text{Cir}(\Gamma)|\) “\(X\)” variables and \(|E(\Gamma)|\) “\(T\)” variables, and has \(|\text{Loops}(\Gamma)|\) relations involving linear terms.

We now consider the multiplicity of \(X_\Gamma\) at \(\emptyset\). To that aim, we need to recall some definitions. Let \(H_\mathbb{Z}\) be a lattice and let \(\sigma\) be a strongly convex rational polyhedral cone in \(H_\mathbb{R} = H_\mathbb{Z} \otimes \mathbb{R}\). Set \(C(\sigma) := \sigma \cap H_\mathbb{Z}\), \(H_{\mathbb{R}, \sigma} = \langle \sigma \rangle \subseteq H_\mathbb{R}\) to be the linear span of \(\sigma\) in \(H_\mathbb{R}\), and \(H_{\mathbb{Z}, \sigma} := \langle \sigma \rangle \cap H_\mathbb{Z}\); note \(H_{\mathbb{R}, \sigma} = H_{\mathbb{Z}, \sigma} \otimes \mathbb{R}\). We denote by \(\text{vol}_{C(\sigma)}\) the unique translation-invariant measure on \(H_{\mathbb{R}, \sigma}\) such that the volume of a standard unimodular simplex \(\Delta\) is 1 (i.e. \(\Delta\) is the convex hull of a basis of \(H_{\mathbb{Z}, \sigma}\) together with \(0\)). Following [GKZ94 p.184], denote by \(K_+(C(\sigma)) \subseteq H_{\mathbb{R}, \sigma}\) the convex hull of the set \(C(\sigma) \setminus \{0\}\) and by \(K_-(C(\sigma))\) the closure of \(\sigma \setminus K_+(C(\sigma))\). The set \(K_-(C(\sigma))\) is a bounded (possibly not convex) lattice polyhedron in \(H_{\mathbb{R}, \sigma}\) which is called the subdiagram part of \(C(\sigma)\).

**Definition 4.10.** [GKZ94 Ch. 5, Def. 3.8] The subdiagram volume of \(C(\sigma)\) is the natural number

\[
u(C(\sigma)) := \text{vol}_{C(\sigma)}(K_-(C(\sigma))).
\]

Now let \(R(\sigma) = R(C(\sigma))\) be the semigroup ring associated to \(C(\sigma)\). Let \(\mathfrak{m}\) be the maximal ideal generated by the generators of the \(k\)-algebra \(R(\sigma)\). Let \(\emptyset\) be the corresponding point in \(X_{\sigma} := \text{Spec} R(\sigma)\). The multiplicity of \(X_{\sigma}\) at \(\emptyset\) is given by (see e.g. [GKZ94 Ch. 5, Thm. 3.14])

\[
\text{mult}_\emptyset X_{\sigma} = u(C(\sigma)).
\]

**Theorem 4.11.** Let \(\Gamma\) be a graph, and let \(\sigma = \sigma_\Gamma\).

\[
\text{mult}_\emptyset X_{\Gamma} = u(C(\sigma)).
\]

**Remark 4.12.** It would be interesting to have a formula for \(\text{mult}_\emptyset X_{\Gamma}\) in terms of standard invariants of the graph \(\Gamma\) (or \(\Gamma^d\)).
5. The cographic toric ring \( U(\Gamma) \) as a ring of invariants

In this section, we show that the cographic toric ring \( U(\Gamma) \) appears as ring of invariants of a torus acting on a certain polynomial ring. Indeed, this invariant ring will appear in Section 7 in the description of the completed local rings of the universal compactified Jacobian.

Consider the action of the algebraic torus \( T_\Gamma := \prod_{\gamma \in \Gamma} \mathbb{G}_m \) on the polynomial ring

\[
B(\Gamma) := k[X_{\vec{e}} : \vec{e} \in \mathcal{E}]
\]

given by the rule that \( \lambda = (\lambda_{\vec{e}})_{\vec{e} \in \mathcal{E}} \in T_\Gamma \) acts as

\[
\lambda \cdot X_{\vec{e}} = \lambda_{s(\vec{e})}^{\lambda_{t(\vec{e})}} X_{\vec{e}}.
\]

In order to more easily connect the results of this paper to those in [CMKVb], we note the following.

**Remark 5.1.** The ring \( B(\Gamma) \) is isomorphic to

\[
B(\Gamma) \cong \frac{k[X_e, Y_e, T_e : e \in E(\Gamma)]}{(X_e Y_e - T_e)}
\]

and its completion at the maximal ideal \((X_e, Y_e)\) is isomorphic to the ring denoted \( \widehat{B(\Gamma)} \) in [CMKVb] Thm. A. Under this isomorphism, the action of \( T_\Gamma \) on \( B(\Gamma) \) given above, induces the same action of \( T_\Gamma \) on \( \widehat{B(\Gamma)} \) given in [CMKVb] Thm. A.

**Theorem 5.2.** The cographic toric ring \( U(\Gamma) \) is isomorphic to the subring \( B(\Gamma)^T_\Gamma \subset B(\Gamma) \) of \( T_\Gamma \)-invariants on \( B(\Gamma) \).

**Proof.** Using Theorem 3.3, we are going to show that \( B(\Gamma)^T_\Gamma \) is isomorphic to the \( k \)-algebra \( D(\Gamma) \). The proof is essentially identical to the proof of [CMKVa] Thm. 6.1; we first show that the underlying \( k \)-vector spaces agree, and then we show that the multiplication rules agree. In keeping with the notation of the proof of [CMKVa] Thm. 6.1, we first observe (as in Remark 5.1) that \( B(\Gamma) \) can be identified with

\[
k[X_{\vec{e}}, X_{\vec{e}'}, T_e : e \in E(\Gamma)]/(X_e Y_e - T_e).
\]

The key point is then to identify the invariant monomials in this ring. This is made easier by the observation that every monomial has an expression of the form

\[
\prod_{e \in E(\Gamma)} X_{\vec{e}}^{a_e} T_e^{b_e}
\]

with \( a_e, b_e \in \mathbb{Z}_{\geq 0} \), where for each \( e \in E(\Gamma) \) we have that \( \vec{e} \) is one of the two orientations of \( e \). The expression is unique up to replacing \( \vec{e} \) with \( \vec{e}' \) for those \( e \) such that \( a_e = 0 \). The same direct analysis of the action as in the proof of [CMKVa] Thm. 6.1] shows that in order for this monomial to be invariant, \( \sum_{e \in E(\Gamma)} a_e \vec{e} \in H_1(\Gamma, \mathbb{Z}). \) Thus as \( k \)-vector spaces, \( D(\Gamma) \) and \( B(\Gamma)^T_\Gamma \) agree. It remains to check that multiplication agrees. This can be checked at the level of monomials, and \( \psi \) in the definition of \( D(\Gamma) \) (see Remark 3.1) was constructed exactly to make these agree. \( \square \)

The cographic toric ring \( U(\Gamma) \) is related to the cographic toric face ring \( R(\Gamma) \) studied in [CMKVa], as explained in the following remark (see also Remark 3.7).

**Remark 5.3.** The action of \( T_\Gamma \) on \( B(\Gamma) \) defines an action on the quotient

\[
A(\Gamma) := \frac{B(\Gamma)}{(X_{\vec{e}} Y_{\vec{e}} : e \in E(\Gamma))} \cong \frac{k[X_e, Y_e : e \in E(\Gamma)]}{(X_e Y_e)}
\]

which coincides with the action of \( T_\Gamma \) defined in [CMKVa] Thm. A]. Therefore, the natural surjection \( B(\Gamma) \rightarrow A(\Gamma) \) induces, by taking \( T_\Gamma \)-invariants, a map

\[
U(\Gamma) \twoheadrightarrow R(\Gamma)
\]

where \( R(\Gamma) := A(\Gamma)^T_\Gamma \) is the cographic toric face ring of \( \Gamma \) (see [CMKVa] Thm. 6.1)). Indeed, the morphism (5.2) coincides with the morphism (3.3) and in particular it is surjective.
6. Examples

We now include a few examples of cographic toric rings.

6.1. The \( n \)-cycle \( C_n \). Let \( C_n \) be the \( n \)-cycle graph, i.e. the graph formed by \( n \) vertices connected by a closed chain of \( n \) edges, as depicted in Figure 3.

![Figure 3. The \( n \)-cycle \( C_n \) with half of its oriented edges.](image)

The cographic toric ring of \( C_n \) admits the following explicit presentation

\[
U(C_n) = k[T_1, \ldots, T_n] / (XY - T_1 \ldots T_n)
\]

To see this, consider the explicit presentation of the cographic toric ring given in §3. Note that the two oriented circuits of \( C_n \) give rise to the elements \( c := e_1 + \cdots + e_n \) and \(-c\) of \( H_1(G, \mathbb{Z}) \). Then, using Proposition 4.7, we get that the generators of the ring \( U(C_n) \) are

\[
X = c, \quad Y = -c
\]

which is of course in agreement with the formulas obtained in §4.

6.2. The \( n \)-thick edge \( I_n \). Let \( I_n \) be the \( n \)-th thick edge graph, i.e. the graph formed by two vertices joined by \( n \) edges, as depicted in Figure 4. The cographic toric ring of \( I_n \) admits the following explicit presentation

![Figure 4. The \( n \)-thick edge \( I_n \) with half of its oriented edges.](image)

\[
U(I_n) = k[T_{ij}]_{1 \leq i < j \leq n}, \quad (X_{ij} - T_j X_{ik} - T_i X_{jk})
\]

To see this, consider the explicit presentation of the cographic toric ring given in §3. Note that the oriented circuits of \( I_n \) give rise to the elements \( \gamma_{ij} := e_i - e_j \in H_1(I_n, \mathbb{Z}) \) for any \( 1 \leq i < j \leq n \). Then, using Proposition 4.7, we deduce that the generators of the ring \( U(I_n) \) are \( X_{ij} = X^{\gamma_{ij}} \) for \( 1 \leq i \neq j \leq n \) and
The quotient obtained from the local structure of $U$ we relate the local rings of the universal compactified Jacobian to the rings appearing earlier in this paper. J the birational geometry of the universal Jacobians $k$ we get the desired relations among the given generators.

It is easily checked that the cographic toric variety $X_{I_n} = \text{Spec} \ U(I_n)$ satisfies

$$
\begin{align*}
\dim X_{I_n} &= 2n - 1, \\
\dim \mathcal{H}(X_{I_n}) &= n^2, \\
\text{mult}_e X_{I_n} &= \binom{2(n - 1)}{n - 1},
\end{align*}
$$

which is of course in agreement with the formulas obtained in §7.4.

7. The universal compactified Jacobian

The aim of this section is to apply the results of the previous sections in order to study the singularities of the universal compactified Jacobian $\bar{J}_{d,g}$ and eventually prove in Theorem 7.4 that $\bar{J}_{d,g}$ has canonical singularities over a base field $k$ of $\text{char}(k) = 0$, at least if $g \geq 4$. We then deduce some consequences for the birational geometry of the universal Jacobians $J_{d,g}$. The outline of this section is as follows. In §7.1 we relate the local rings of the universal compactified Jacobian to the rings appearing earlier in this paper. The culmination is Theorem 7.4 which essentially reduces the problem to studying finite quotients of the cographic rings $U(\Gamma)$. In order to describe this quotient, it is convenient to compare with an associated quotient obtained from the local structure of $\bar{M}_g$; this comparison is made in §7.2 culminating in Theorem 7.4.

In §7.3 we give the proof of Theorem 7.4. The argument relies on a generalization of the Reid–Tai–Shepherd-Barron criterion to singular toric varieties, which can be found in Appendix A. Consequences for the birational geometry of $\bar{J}_{d,g}$ are given in §7.5.

7.1. The local rings of $\bar{J}_{d,g}$. In this subsection, which is heavily based on our previous work [CMKVb], we obtain an explicit description of the completed local rings of $\bar{J}_{d,g}$ in terms of the cographic toric rings studied in the previous sections.

Fix a point $(C, I) \in \bar{J}_{d,g}$; i.e. $C$ is a stable curve of genus $g$, and $I$ is a rank 1, torsion-free sheaf of degree $d$ on $C$, which is poly-stable with respect to the canonical polarization $\omega_C$. Let $\Sigma_{(C,I)}$ (or simply $\Sigma$ when the pair $(C, I)$ we are dealing with is clear from the context) be the set of nodes of $C$ where $I$ is not locally free. Let $\Gamma_{(C, I)}$ (or simply $\Gamma$ when the pair $(C, I)$ we are dealing with is clear from the context) be the graph obtained from the dual graph of $C$ by contracting the edges corresponding to the nodes that are not in $\Sigma_{(C,I)}$. In particular, the edges of $\Gamma_{(C, I)}$ correspond naturally to the nodes in $\Sigma_{(C,I)}$. Note that $\Gamma_{(C, I)}$ is the dual graph of the curve obtained from $C$ by smoothing the nodes at which $I$ is locally free. For convenience, we fix an arbitrary orientation of $\Gamma_{(C, I)}$ and we denote by $s, t : E(\Gamma_{(C, I)}) \to V(\Gamma_{(C, I)})$ the source and target maps, associating to any edge of $\Gamma_{(C, I)}$ the source and target with respect to the chosen orientation.

We now review the deformation theory of the pair $(C, I)$, referring to [CMKVb] for more details and proofs. As explained in [CMKVb] §3, the deformation functor $\text{Def}_{(C, I)}$ of the pair $(C, I)$ fits into the following sequence

$$
\text{Def}_{(C, I)}^{l.t.} \hookrightarrow \text{Def}_{(C, I)} \rightarrow \prod_{e \in \Sigma} \text{Def}_{(C_e, I_e)} = \text{Def}_{(C, I)}^{\text{loc}},
$$

where $\text{Def}_{(C_\ell, I_e)}$ is the deformation functor of the pair consisting of $C_\ell := \text{Spec} \Hat{O}_{C, \ell}$ and the pull-back $I_\ell$ of $I$ to $C_\ell$, $F$ is the forgetful map mapping taking deformations of $(C, I)$ to local deformations at the set of nodes $e \in \Sigma$ where $I$ fails to be locally free, and $\text{Def}_{(C, I)}^{l.t.}$ is the subfunctor of $\text{Def}_{(C, I)}$ parametrizing locally trivial deformations, i.e. deformations of $(C, I)$ that map to the trivial deformation via the forgetful map $F$. The above three deformation functors are unobstructed and the forgetful map $F$ is formally smooth (see [CMKVb] §3). In particular, we get an exact sequence of tangent spaces

$$
0 \rightarrow T \text{Def}_{(C, I)}^{l.t.} \rightarrow T \text{Def}_{(C, I)} \rightarrow T \text{Def}_{(C, I)}^{\text{loc}} \rightarrow 0.
$$
Define the following $k$-algebra
\[(7.3) \quad R_{(C, I)} := k[T^\vee \text{Def}_{(C, I)}] = \bigoplus_{n \in \mathbb{N}} \text{Sym}^n T^\vee \text{Def}_{(C, I)},\]
where $T^\vee \text{Def}_{(C, I)}$ is the dual of the tangent space $T \text{Def}_{(C, I)}$. Fixing a splitting of the exact sequence \((7.2)\) and using the explicit description of a miniversal deformation ring for $\text{Def}_{(X_e, I_e)}$ obtained in [CMKVb Lemma 3.14], we can write $R_{(C, I)}$ in the following form
\[(7.4) \quad R_{(C, I)} = k[T^\vee \text{Def}^{\text{loc}}_{(C, I)}] \otimes_k k[T^\vee \text{Def}^{\text{lt}}_{(C, I)}] = \bigotimes_{e \in \Sigma} [k[X_e, Y_e, T_e] / (X_e Y_e - T_e)] \otimes_k k[T^\vee \text{Def}^{\text{lt}}_{(C, I)}],\]
where $B(\Gamma)$ is the ring defined in Remark 5.1. As proved in [CMKVb §3.2], the mini-versal deformation ring of the functor $\text{Def}_{(C, I)}$ is given by the completion $\hat{R}_{(C, I)}$ of $R_{(C, I)}$ at the maximal ideal $m_0$ generated by $T^\vee \text{Def}_{(C, I)}$. Geometrically, the variables $X_e$ and $Y_e$ correspond to the deformations of $I$ at the node $e \in E(\Gamma) = \Sigma$ and the variable $T_e$ corresponds to the smoothing of $C$ at $e$. Note also, the completion $\hat{B}(\Gamma)$ of $B(\Gamma)$ at the maximal ideal generated by $T^\vee \text{Def}^{\text{loc}}_{(C, I)}$ was shown to be mini-versal for $\text{Def}^{\text{loc}}_{(C, I)}$; for this reason we will sometimes also write
\[R^{\text{loc}}_{(C, I)} := B(\Gamma) = \bigotimes_{e \in \Sigma} k[X_e, Y_e, T_e] / (X_e Y_e - T_e).\]

Consider now the automorphism group $\text{Aut}(C, I)$ of $(C, I)$, consisting of all the pairs $(\sigma, \tau)$ such that $\sigma : C \xrightarrow{\cong} C$ is an automorphism of $C$ and $\tau : I \xrightarrow{\cong} \sigma^* (I)$ is an isomorphism of sheaves on $C$. We have a natural exact sequence of groups
\[(7.5) \quad \{1\} \to \text{Aut}(I) \xrightarrow{i} \text{Aut}(C, I) \xrightarrow{p} \text{Stab}_{C}(I) \to \{1\},\]
where $\text{Stab}_{C}(I) \subseteq \text{Aut}(C)$ the subgroup of $\text{Aut}(C)$ (which is finite since $C$ is stable) consisting of all the elements $\sigma \in \text{Aut}(C)$ such that $\sigma^*(I) \cong I$. The group $\text{Aut}(I)$ is an algebraic torus, which by [CMKVb Remark 5.9] is naturally isomorphic to
\[(7.6) \quad \text{Aut}(I) = T_{\Gamma} := \prod_{v \in V(\Gamma)} \mathbb{G}_m.\]

The automorphism group $\text{Aut}(C, I)$ acts naturally on $\text{Def}_{(C, I)}$ (see [CMKVb Def. 3.4]); hence it acts also on the tangent space $T \text{Def}_{(C, I)}$ and this action clearly preserves the exact sequence \((7.2)\). Therefore we get a natural linear action of $\text{Aut}(C, I)$ on $R_{(C, I)}$ which preserves the decomposition of $R_{(C, I)}$ given in \((7.4)\). It follows from [CMKVb §5] that the induced action of the subgroup $\text{Aut}(I)$ is trivial on $k[T^\vee \text{Def}^{\text{lt}}_{(C, I)}]$, and coincides with the action of $T_{\Gamma}$ on $B(\Gamma)$ given by \((7.1)\) after the identification of Remark 5.1. Explicitly, an element $\lambda = (\lambda_v)_{v \in V(\Gamma)} \in T_{\Gamma}$ acts on the generators of $B(\Gamma)$ as
\[(7.7) \quad \lambda \cdot X_e = \lambda_{s(e)}^{-1} \lambda_{t(e)} X_e, \quad \lambda \cdot Y_e = \lambda_{s(e)}^{-1} \lambda_{t(e)} Y_e \quad \text{and} \quad \lambda \cdot T_e = T_e.\]

The subring $R^{\text{Aut}(C, I)}(\Gamma) \subseteq R_{(C, I)}$ of invariants for the action of $\text{Aut}(C, I)$ on $R_{(C, I)}$ can be computed in two steps: we first take the subring $R^{\text{Aut}(I)}_{(C, I)} \subseteq R_{(C, I)}$ of invariants for the subgroup $\text{Aut}(I)$; then we take the invariants for the induced action of the finite group $\text{Stab}_{C}(I)$ on $R^{\text{Aut}(I)}_{(C, I)}$. Using Theorem 5.2, the ring of invariants with respect to $\text{Aut}(I)$ is equal to
\[(7.8) \quad R^{\text{Aut}(I)}_{(C, I)} = U(\Gamma) \otimes_k k[T^\vee \text{Def}^{\text{lt}}_{(C, I)}],\]
where $U(\Gamma)$ is the cographic toric ring associated to $\Gamma$. Therefore the subring of invariants with respect to $\text{Aut}(C, I)$ is given by
\[(7.9) \quad R^{\text{Aut}(C, I)}_{(C, I)} = \left( R^{\text{Aut}(I)}_{(C, I)} \right)^{\text{Stab}_{C}(I)} = \left( U(\Gamma) \otimes_k k[T^\vee \text{Def}^{\text{lt}}_{(C, I)}] \right)^{\text{Stab}_{C}(I)}.\]

We show next that the completion of the invariant subring \((7.9)\) at the maximal ideal $m_0 \cap R^{\text{Aut}(C, I)}_{(C, I)}$ gives a description of the completed local ring $\hat{O}_{\tilde{d}_{a,g}(C, I)}$ of the universal compactified Jacobian $\tilde{J}_{d,g}$ at $(C, I)$. 


Theorem 7.1. Notation as above. Assume that Stab\(_C(I)\) does not contain elements of order equal to \(p = \text{char}(k)\). The completion of the invariant subring \(R_{(C,I)}^{\text{Aut}(C,I)}\) at the maximal ideal \(m_0 \cap R_{(C,I)}^{\text{Aut}(C,I)}\) is isomorphic to the completed local ring \(\hat{\mathcal{O}}_{J_{d,g},(C,I)}\) of the universal compactified Jacobian \(\hat{J}_{d,g}\) at \((C,I)\).

Proof. The linear action of Aut\((C,I)\) on \(R_{(C,I)}\) described above induces a unique action on the completion \(\hat{R}_{(C,I)}\) of \(R_{(C,I)}\) at the maximal ideal \(m_0\). Since Aut\((C,I)\) is a linearly reductive group (by our assumption on Stab\(_C(I)\)), the formation of Aut\((C,I)\)-invariants commutes with completion (see e.g. [CMKVb, Lemma 6.7]), or in symbols

\[
(7.10) \quad \left( R_{(C,I)}^{\text{Aut}(C,I)} \right) \cong \left( \hat{R}_{(C,I)} \right)^{\text{Aut}(C,I)}
\]

where on the right hand side the completion is taken with respect to the maximal ideal \(m_0\) of \(\hat{R}_{(C,I)}\) generated by \(T^v \text{Def}_{(C,I)}\) and on the left the completion is taken with respect to the maximal ideal \(m_0 \cap R_{(C,I)}^{\text{Aut}(C,I)}\).

As observed before, the ring \(\hat{R}_{(C,I)}\) is the mini-versal deformation ring of the functor Def\(_{(C,I)}\), which means that there is a formally smooth natural transformation of functors

\[
(7.11) \quad \Phi : \text{Spf} \hat{R}_{(C,I)} \to \text{Def}_{(C,I)}
\]

whose associated map on tangent spaces

\[
(7.12) \quad T\Phi : T\text{Spf} \hat{R}_{(C,I)} \to T\text{Def}_{(C,I)}
\]

is an isomorphism. Explicitly, the isomorphism \(T\Phi\) is obtained by first identifying the tangent space of Spf \(\hat{R}_{(C,I)}\) with the tangent space \(T_{m_0}R_{(C,I)} = (m_0/m_0^2)^v\) of the ring \(R_{(C,I)}\) at \(m_0\) and then by identifying \(T_{m_0}R_{(C,I)}\) with \(T\text{Def}_{(C,I)}\) using the definition \(7.3\) of \(R_{(C,I)}\).

Observe now that our specified linear action of Aut\((C,I)\) on \(R_{(C,I)}\) is defined in such a way that the isomorphism \(T\Phi\) becomes Aut\((C,I)\)-equivariant. Using Rim's arguments (see [Rim80]), the Aut\((C,I)\)-equivariance of \(T\Phi\) implies that also \(\Phi\) is Aut\((C,I)\)-equivariant; hence the specified action of Aut\((C,I)\) on \(\hat{R}_{(C,I)}\) is the unique action that makes \(\Phi\) equivariant, according to Rim's theorem (see [CMKVb, Fact 5.4]). Therefore, we can apply [CMKVb, Thm. 6.1(ii)] in order to conclude that

\[
(7.13) \quad \hat{\mathcal{O}}_{J_{d,g},(C,I)} \cong \hat{R}_{(C,I)}^{\text{Aut}(C,I)}.
\]

The proof of the theorem follows by putting together \((7.10)\) and \((7.13)\). \(\square\)

7.2. The local structure of the morphism \(\pi : \hat{J}_{d,g} \to \overline{\mathcal{M}}_g\). The aim of this subsection is to study the local structure of the morphism \(\pi : \hat{J}_{d,g} \to \overline{\mathcal{M}}_g\) around a point \((C,I) \in \hat{J}_{d,g}\), where we assume as usual that \(I\) is poly-stable with respect to \(\omega_C\).

First of all, there is a natural forgetful morphism \(\Pi : \text{Def}_{(C,I)} \to \text{Def}_C\), from the deformation functor of the pair \((C,I)\) to the deformation functor of \(C\), which is equivariant with respect to the group homomorphism Aut\((C,I) \to \text{Aut}(C)\) and the natural actions of Aut\((C,I)\) on Def\(_{(C,I)}\) and of Aut\((C)\) on Def\(_C\) (see [CMKVb, Def. 3.4]). The forgetful morphism \(\Pi\) fits into the following diagram

\[
(7.14) \quad \begin{array}{ccc}
\text{Def}_{(C,I)} & \longrightarrow & \text{Def}(C) \\
\downarrow & & \downarrow \\
\Pi & & \\
\text{Def}_{C} & \longrightarrow & \text{Def}_C
\end{array}
\]

where \(\text{Def}_{(C,I)}^{\Sigma,\text{loc}}\) is the local deformation functor of \(C\) at the nodes \(\Sigma = \Sigma_{(C,I)}\) of \(C\) where \(I\) is not invertible, and \(\text{Def}_{C}^{\Sigma,\text{loc}}\) is the subfunctor of \(\text{Def}_C\) parametrizing deformations of \(C\) that are locally trivial around the
Observe that the map $T\Pi^{1,1}$ is surjective and its kernel can be naturally identified with the tangent space $T\text{Def}_L$ of the deformation functor $\text{Def}_L$, where $L$ is the line bundle on the partial normalization $g : C_\Sigma \to C$ of $C$ at the nodes of $\Sigma = \Sigma_{(C,L)}$ and $L$ is the unique line bundle on $C_\Sigma$ such that $I = g_*(L)$ (see CMKVb, Lemma 3.16). Fixing a splitting of the second row of (7.15), we define the following $k$-algebra

\begin{equation}
R_C := k[T^\vee \text{Def}_C] = k[T^\vee \text{Def}_C^{\Sigma,\text{loc}}^{1,1}] = \bigotimes_{e \in \Sigma} k[T_e] \otimes_k k[T^\vee \text{Def}_C^{\Sigma,\text{loc}}^{1,1}],
\end{equation}

where the variable $T_e$ corresponds to the smoothing of $C$ at $e$. Observe that the finite group $\text{Aut}(C)$ acts linearly on $R_C$, via its natural action on $T\text{Def}_C$. The diagram (7.15), after choosing compatible splittings of the horizontal rows and of the left vertical column, gives rise to an injective morphism of $k$-algebras

\begin{equation}
R_C = \bigotimes_{e \in \Sigma} k[T_e] \otimes_k k[T^\vee \text{Def}_C^{\Sigma,\text{loc}}^{1,1}] \hookrightarrow R_{(C,I)} = \bigotimes_{e \in \Sigma} k[X_e, Y_e, T_e] \otimes_k k[T^\vee \text{Def}_C^{1,1}] = \bigotimes_{e \in \Sigma} k[X_e, Y_e, T_e] \otimes_k k[T^\vee \text{Def}_C^{\Sigma}] \otimes_k k[T^\vee \text{Def}_L].
\end{equation}

Consider now the action of $\text{Aut}(I)$ on $R_{(C,I)}$ as in (7.1). From (7.17), it follows that each $T_e$ is invariant under the action of $\text{Aut}(I)$ so that the inclusion (7.17) factors through

\begin{equation}
R_C = \bigotimes_{e \in \Sigma} k[T_e] \otimes_k k[T^\vee \text{Def}_C^{\Sigma,\text{loc}}^{1,1}] \hookrightarrow R_{(C,I)}^{\text{Aut}(I)} = U(\Gamma) \otimes_k k[T^\vee \text{Def}_C^{\Sigma,\text{loc}}^{1,1}] \otimes_k k[T^\vee \text{Def}_L].
\end{equation}

Note that the finite subgroup $\text{Stab}_C(I)$ acts in a compatible way on both the above rings, while the bigger finite group $\text{Aut}(C)$ acts only on the ring on the left.

**Theorem 7.2.** Notation as above. Assume that $\text{Aut}(C)$ does not contain elements of order equal to $p = \text{char}(k)$. The inclusion of complete local rings

\[ \hat{O}_{\tilde{M}_g, C} \hookrightarrow \hat{O}_{\tilde{J}_{d,g}, (C,I)} \]

induced by the surjective morphism $\pi : \tilde{J}_{d,g} \to \overline{M}_g$ coincide with the completion of the inclusion

\begin{equation}
\hat{R}_C^{\text{Aut}(C)} \hookrightarrow \hat{R}_{(C,I)}^{\text{Aut}(C)} = \left( \hat{R}_{(C,I)}^{\text{Aut}(I)} \right)^{\text{Stab}_C(I)}
\end{equation}

induced by (7.18), at their maximal ideals $m_0 \cap \hat{R}_C^{\text{Aut}(C)}$ and $m_0 \cap \hat{R}_{(C,I)}^{\text{Aut}(C)}$, respectively.

**Proof.** The assumption on the order of the elements of $\text{Aut}(C)$ implies that $\text{Aut}(C)$ and $\text{Aut}(C,I)$ are linearly reductive groups. Since the formation of invariants under the action of a linear reductive group commutes with completion (see e.g. CMKVb, Lemma 6.7), we get that the completion of the inclusion (7.19) is equal to the inclusion

\begin{equation}
\hat{R}_C^{\text{Aut}(C)} \hookrightarrow \hat{R}_{(C,I)}^{\text{Aut}(C)},
\end{equation}

where the completions, done with respect to the maximal ideals $m_0 \cap \hat{R}_C$ and $m_0$ respectively, are acted upon naturally by $\text{Aut}(C)$ and $\text{Aut}(C,I)$ respectively.
From the discussion in [CMKVb] §3, it follows that the inclusion $\widehat{R}_C \hookrightarrow \hat{R}_{(C, I)}$ induces, by passing to the formal spectrum, a diagram

$$
\begin{array}{ccc}
\text{Spf } \hat{R}_{(C, I)} & \xrightarrow{\Phi} & \text{Def}_{(C, I)} \\
\downarrow & & \downarrow \text{H} \\
\text{Spf } \hat{R}_C & \xrightarrow{\Phi} & \text{Def}_C
\end{array}
$$

where $\Phi$ realizes $\hat{R}_{(C, I)}$ as the mini-versal deformation ring of the functor $\text{Def}_{(C, I)}$ (as discussed in the proof of Theorem 7.1) and $\Phi$ realizes $\hat{R}_C$ as the universal deformation ring of $\text{Def}_C$. Moreover, $\Phi$ is $\text{Aut}(C, I)$-equivariant (as discussed in the proof of Theorem 7.1), $\Phi$ is clearly $\text{Aut}(C)$-equivariant (being an isomorphism of functors) and the two vertical maps in (7.21) are equivariant with respect to the group homomorphism $\text{Aut}(C, I) \rightarrow \text{Aut}(C)$.

Therefore, as an application of Luna’s slice theorem (see [CMKVb] §6), we get a commutative diagram

$$
\begin{array}{ccc}
\widehat{R}^\text{Aut}(C, I)_{(C, I)} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{J_{d,q}(C, I)} \\
\downarrow & & \downarrow \\
\hat{R}^\text{Aut}(C)_{C} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{\hat{M}_{q}, C}
\end{array}
$$

which concludes the proof.

Consider now the graph $\Gamma = \Gamma_{(C, I)}$ obtained from the dual graph of $C$ by contracting the edges corresponding to nodes of $C$ where $I$ is locally free, as in §7.1. It follows from the above discussion that the inclusions $R_C \hookrightarrow R^\text{Aut}(I)_{(C, I)} \hookrightarrow \hat{R}_{(C, I)}$ are given, up to smooth factors, by the following inclusions of $k$-algebras (with the notation of §5)

$$
k[T_e : e \in E(\Gamma)] \hookrightarrow U(\Gamma) = B(\Gamma)^{T_e} \hookrightarrow B(\Gamma) = k[X_e : e \in \overrightarrow{E}(\Gamma)]
$$

where we used that $X_{\overrightarrow{e}} \cdot X_{\overrightarrow{e}} \in B(\Gamma)$ is invariant under the action of $T_e$ given in (5.1). Therefore, we get the following surjective morphism of varieties

$$
\text{Spec } B(\Gamma) = \text{Spec } k[X_{\overrightarrow{e}} : e \in \overrightarrow{E}(\Gamma)] \xrightarrow{f} X_{\Gamma} = \text{Spec } U(\Gamma) \xrightarrow{g} \text{Spec } k[T_e : e \in E(\Gamma)].
$$

The above morphisms are toric morphisms of affine toric varieties, which can be described using toric geometry as follows. With the notation of §3, consider the following injective linear maps

$$
\mathbb{R}\langle e \rangle_{e \in E(\Gamma)} \longrightarrow \mathbb{H}_1(\Gamma, \mathbb{R}) = \ker \mathbb{D} \longrightarrow \mathbb{C}_1(\Gamma, \mathbb{R}),
$$

$$
\sum_{e \in E(\Gamma)} a_e \cdot e \mapsto \sum_{e \in E(\Gamma)} a_e (\overrightarrow{e} + \overleftarrow{e})
$$

which clearly preserve the integral lattices. By taking duals, we get the following surjective lattice-preserving linear maps

$$
\mathbb{C}_1(\Gamma, \mathbb{R})^\vee \xrightarrow{f} \mathbb{H}_1(\Gamma, \mathbb{R})^\vee \xrightarrow{h} \mathbb{R}(e^\vee)_{e \in E(\Gamma)}.
$$

The above three vectors spaces are endowed with standard scalar products that will be denoted with the same symbol $\langle \cdot, \cdot \rangle$ (see §1.2 and §3). Inside the vector space $\mathbb{H}_1(\Gamma, \mathbb{R})^\vee$, we have the cone $\sigma := \sigma^\vee$ introduced in §3. The rational polyhedral fan formed by $\sigma$ and all its faces corresponds to the toric variety $X_{\Gamma}$. Using Lemma 4.1.1, it follows that $\sigma$ is equal to

$$
\sigma = \text{conv}\langle \langle \cdot, \overrightarrow{e} \rangle \rangle_{e \in \overrightarrow{E}(\Gamma)}
$$
where \( \text{conv} \) denotes the convex hull. Set
\[
\tilde{\sigma} := \text{conv}\langle (\cdot, e) \rangle_{e \in E(\Gamma)} \subset C_1(\Gamma, \mathbb{R})^\vee,
\]
\[
\bar{\sigma} := \text{conv}\langle (\cdot, e) \rangle_{e \in E(\Gamma)} = \mathbb{R}_{\geq 0} \langle e^\vee \rangle_{e \in E(\Gamma)} \subset \mathbb{R} \langle e^\vee \rangle_{e \in E(\Gamma)}.
\]
Clearly, the cone \( \tilde{\sigma} \) (resp. \( \bar{\sigma} \)) gives rise to the toric variety \( \text{Spec} k[X_\mathbb{R} : \vec{e} \in \mathbb{R}(\Gamma)] \) (resp. \( \text{Spec} k[T_\mathbb{R} : e \in E(\Gamma)] \)). Moreover, the lattice-preserving linear maps \( (7.25) \) are such that \( l(\tilde{\sigma}) = \sigma \) and \( h(\sigma) = \bar{\sigma} \); hence they induce morphisms \( \text{Spec} k[X_\mathbb{R} : \vec{e} \in \mathbb{R}(\Gamma)] \to X_\Gamma \to \text{Spec} k[T_\mathbb{R} : e \in E(\Gamma)] \) which are easily seen to coincide with the morphisms \( f \) and \( g \) of \( (7.24) \).  

7.3. Singularities of \( \overline{M}_g \). We recall the following result of Harris–Mumford and Ludwig.

**Theorem 7.3** ([HMS2 Thm. 2], [Lud Prop. 4.2.5, Cor. 4.2.6]). Let \( g \geq 4, C \in \overline{M}_g \), and \( \phi \in \text{Aut}(C) \). Set \( R_C \) to be a mini-versal space for \( C \). If \( \phi \) acts as a pseudo-reflection on \( \text{Spec} R_C \), or \( \text{Spec} R_C/\langle \phi \rangle \) does not have canonical singularities, then the following holds:

1. The curve \( C \) has an elliptic tail \( E \subset C \), i.e. an irreducible subcurve of arithmetic genus one that meets the complementary subcurve \( C^c := C \setminus E \) in one point \( p \), and \( \phi \) is an elliptic tail automorphism, i.e. \( \phi|_{C^c} = \text{id}_{C^c} \).
2. The restriction \( \phi|_E \) is an automorphism of \( E \), fixing \( p \), with order \( n = 2, 3, 4 \) or \( 6 \). If \( n = 4 \), then \( E \) is smooth with \( j \)-invariant equal to 1728, and if \( n = 3 \) or \( 6 \), then \( E \) is smooth with \( j \)-invariant equal to 0.
3. If \( E \) is a singular elliptic curve, then \( \phi|_E \) has order \( n = 2 \) and is given as follows: Denote by \( \nu : E^\nu \to E \) the normalization of \( E \) and identify \( E^\nu \) with \( \mathbb{P}^1 \) in such a way that \( \nu^{-1}(\infty) = (1,0) \) and \( \nu^{-1}(q) = \{(1,1), (-1,1)\} \). Then \( \phi|_E \) is induced by the involution of \( \mathbb{P}^1 \) sending \( (x,y) \) into \( (-x,y) \).

Moreover, let \( g \geq 4, C \in \overline{M}_g \) be a curve with an elliptic tail \( E \), and \( \phi \in \text{Aut}(C) \) be an elliptic tail automorphism (with respect to \( E \)). Let \( \{t_1, \ldots, t_{3g-3}\} \) be coordinates of \( T \text{Def}_C \) such that \( t_1 \) corresponds to the smoothing of \( C \) at the node \( p \), and \( t_2 \) corresponds, if \( E \) is smooth, to a coordinate for \( T(E,p)(M_{1,1}) \) (corresponding to the \( j \)-invariant of \( E \)), or if \( E \) is singular, to the smoothing of \( C \) at \( q \). Then the action of \( \phi \) on \( T \text{Def}_C \) on the above coordinates is given by the following matrix (depending on the choice of the primitive \( n \)-th root of unity \( \zeta \)):

\[
M(\phi) = \begin{cases}
\begin{pmatrix}
\zeta^1 \\
\zeta^0 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^5 \\
\zeta^4 \\
\zeta^1
\end{pmatrix} & \text{if } n = 2, \\
\begin{pmatrix}
\zeta^3 \\
\zeta^2 \\
\zeta^1 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1
\end{pmatrix} & \text{if } n = 4, \\
\begin{pmatrix}
\zeta^2 \\
\zeta^1 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^1
\end{pmatrix} & \text{if } n = 3, \\
\begin{pmatrix}
\zeta^1 \\
\zeta^0 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^2 \\
\zeta^1 \\
\zeta^1
\end{pmatrix} & \text{if } n = 6,
\end{cases}
\]

where \( \mathbb{I} \) is the suitable identity matrix.

7.4. Singularities of \( \overline{J}_{d,g} \). The aim of this subsection is to prove that \( \overline{J}_{d,g} \) has canonical singularities if \( g \geq 4 \) and \( \text{char}(k) = 0 \).

**Theorem 7.4.** Assume that \( \text{char}(k) = 0 \), and \( g \geq 4 \). Then the universal compactified Jacobian \( \overline{J}_{d,g} \) has canonical singularities for any \( d \in \mathbb{Z} \).  

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Proof. Since the property of having canonical singularities is invariant under localization and completion (see e.g. [Mat02, Prop. 4-4-4]), it is enough to show, by Theorem 7.1 that the affine variety

\[(7.27) \quad \Spec \left[ (R_{(C,I)})^{\Aut(C,I)} \right] = \Spec \left[ \left( R_{(C,I)}^{\Aut(I)} \right)^{\Stab_C(I)} \right] = \Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\Stab_C(I) \]

has canonical singularities for every \((C,I) \in \bar{J}_{d,g}\).

Roughly speaking, the outline of the argument from this point is as follows. We take the point \((C,I) \in \bar{J}_{d,g}\), and consider its image \(C \in \overline{M}_g\). Then we break the argument into two parts: (1) \(\overline{M}_g\) has canonical singularities near \(C\), and (2) \(\overline{M}_g\) does not have canonical singularities near \(C\). In case (1), we use a generalization of the Reid–Tai criterion that can be applied to singular toric varieties (we review this generalization of Reid–Tai in the appendix), and we obtain that \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\Stab_C(I)\) (and hence \(\bar{J}_{d,g}\)) has canonical singularities at \((C,I)\). In case (2), there is a short list due to Harris–Mumford of possible curves where \(\Spec\) may fail to have canonical singularities (see (7.3)). In these cases, it will turn out that \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)\) is smooth. Thus we can apply the usual Reid–Tai criterion. From the work of Harris–Mumford, and Ludwig (see (7.3), one has an explicit description of the actions needed for the analysis. In the end, for case (2) the argument is very similar to that in [BFV12], and establishes that \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\Stab_C(I)\) (and hence \(\bar{J}_{d,g}\)) also has canonical singularities at \((C,I)\) in this case. Technically, since we are able to focus on one automorphism of \((C,I)\) at a time, the argument is broken into somewhat finer pieces than just described, but this captures the main points.

We now proceed to implement this strategy:

To begin, a standard result (see Theorem A.11) says that \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\Stab_C(I)\) has canonical singularities if and only if for every \(\phi \in \Stab_C(I) \subseteq \Aut(C)\) the quotient

\[\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\langle \phi \rangle\]

has canonical singularities. Thus we proceed by considering the quotients \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\langle \phi \rangle\).

Case 1. The automorphism \(\phi \in \Stab_C(I)\) does not act as a pseudo-reflection on \(\Spec R_C\) and \(\Spec R_C/\langle \phi \rangle\) has canonical singularities.

We will show that \(\Spec \left( R_{(C,I)}^{\Aut(I)} \right)/\langle \phi \rangle\) has canonical singularities. We will apply Lemma A.7 which is essentially a variation on the Reid–Tai criterion tailored to this setting, to the following morphism \(\Psi\) induced by (7.18)

\[
\begin{array}{ccc}
\Spec \left( R_{(C,I)}^{\Aut(I)} \right) & \xrightarrow{\Psi} & \Spec R_C \times \Spec k[T^\vee \Def_L] \\
\Spec \left( U(\Gamma) \otimes k[T^\vee \Def_{C,\Sigma,\Gamma}] \otimes k[T^\vee \Def_L] \right) & \xrightarrow{\Psi} & \Spec \left( k[\Gamma_e : e \in \Sigma] \otimes k[T^\vee \Def_{C,\Sigma,\Gamma}] \otimes k[T^\vee \Def_L] \right)
\end{array}
\]

and the natural action of \(\Z = \langle \phi \rangle\). The added factor of \(\Spec k[T^\vee \Def_L]\) on the right makes the computation more tractable. Let us check the hypothesis of Lemma A.7.

First of all, \(\Psi\) is a toric morphism of affine toric varieties that acts as the identity on the last two factors \(\Spec k[T^\vee \Def_{C,\Sigma,\Gamma}]\) and \(\Spec k[T^\vee \Def_L]\), and coincides with the map \(g : X_\Gamma = \Spec U(\Gamma) \to \Spec k[T_e : e \in E(\Gamma)]\) of (7.24) on the first factor. As explained in (7.2), the morphism \(g\) is induced by the lattice-preserving linear map \(h : \mathbb{H}_1(\Gamma, \mathbb{R})^\vee \to \mathbb{R}^{E(\Gamma)}\) of (7.25) which sends the cone \(\sigma = \sigma^\vee_\Gamma\) associated to the toric variety \(X_\Gamma\) to the cone \(\tilde{\sigma}\) corresponding to the toric variety \(\Spec k[T_e : e \in E(\Gamma)]\). By Lemma 4.1(iii), the extremal rays of \(\sigma^\vee_\Gamma\) are given by \(\langle (\cdot, \tilde{e}) \rangle := \mathbb{R}_{\geq 0} \cdot \tilde{e}\), as \(\tilde{e}\) varies among the oriented edges \(E(\Gamma)\) of \(\Gamma\). As explained in (7.2), the linear map \(h\) sends the extremal ray \(\langle (\cdot, \tilde{e}) \rangle\) of the cone \(\sigma = \sigma^\vee_{\Gamma}\) into the extremal ray \(\langle (\cdot, \tilde{e}) \rangle\) of \(\tilde{\sigma}\), where \(e \in E(\Gamma)\) is the (unoriented) edge underlying \(\tilde{e} \in \tilde{E}(\Gamma)\). Furthermore, by the definition (7.26) it follows that \(h\) sends the primitive element \(\langle \cdot, \tilde{e} \rangle\) of the extremal ray \(\langle (\cdot, \tilde{e}) \rangle\) (see Lemma 4.1(iv)
onto the primitive element $(\cdot, e)$ of the extremal ray $\langle (\cdot, e) \rangle$. This shows that hypothesis \textbf{(i)} and \textbf{(ii)} of Lemma \ref{lem:elliptic} are satisfied.

Consider now the action of $\mathbb{Z}_r = \langle \phi \rangle \subseteq \text{Stab}_C(I)$ on the domain and codomain of $\Psi$. The action preserves the decompositions of the domain and codomain, and the toric structure on the smooth factor $\text{Spec} \left( k[T^\vee \text{Def}_C^{1.1}] \otimes k[T^\vee \text{Def}_L^{1.1}] \right)$ is chosen via an eigen basis for the action of $\phi$. Considering the modular interpretation of the other factors, the two actions preserve the tori inside the domain and codomain, and moreover, as observed in \cite{AS3}, the morphism $\Psi$ is $\mathbb{Z}_r$-equivariant. In addition, the toric variety $\text{Spec} R_C \times \text{Spec} k[T^\vee \text{Def}_L]$ is smooth and $\mathbb{Z}_r$ acts on it without pseudo-reflections since $\phi$ does not act as a pseudo-reflection already on $\text{Spec} R_C$ by assumption. This shows that the hypothesis \textbf{(iii)} and \textbf{(iv)} of Lemma \ref{lem:elliptic} are satisfied.

Finally, the quotient $\text{Spec} R_C / \langle \phi \rangle$ has canonical singularities by assumption. Using the Reid–Tai criterion \ref{lem:elliptic} and the fact that $\phi$ does not act as a pseudo-reflection on $\text{Spec} R_C$, this is equivalent to the fact that the age of $\phi$ on $\text{Spec} R_C$ with respect to any primitive $r$-root of unity is greater than or equal to 1. Of course, this remains true for the age of $\phi$ acting on the space $\text{Spec} R_C \times \text{Spec} k[T^\vee \text{Def}_L]$, which implies that $\langle \text{Spec} R_C \times \text{Spec} k[T^\vee \text{Def}_L] \rangle / \langle \phi \rangle$ has canonical singularities by the Reid–Tai criterion.

We can now apply Lemma \ref{lem:elliptic} in order to conclude that $\text{Spec} \left( R^{\text{Aut}(I)}_{(C,I)} \right) / \langle \phi \rangle$ has canonical singularities, q.e.d. for Case 1.

**Case 2.** The automorphism $\phi \in \text{Stab}_C(I) \subseteq \text{Aut}(C)$ either acts as a pseudo-reflection on $\text{Spec} R_C$ or $\text{Spec} R_C / \langle \phi \rangle$ does not have canonical singularities.

The analysis we are going to perform in this case is similar to the analysis that was performed in \cite{BFV} §4; however, there are two main differences: here we use the Pandharipande \cite{Pan} modular interpretation of $J_{d,g}$ instead of the Caporaso \cite{Cap} modular interpretation of $J_{d,g}$ used in loc. cit.; moreover, we will not restrict ourself to the stable locus, contrary to loc. cit.

To begin, according to the results of Harris–Mumford and Ludwig (see Theorem \ref{thm:elliptic}), Case 2 can occur only if $C$ has an elliptic tail $E \subset C$, i.e. a connected subcurve of arithmetic genus one which meets the complementary subcurve $E^c := C \setminus E$ in one point $p$, and $\phi$ is an elliptic tail automorphism, i.e. $\phi_{|E^c} = \text{id}_{E^c}$.

We now consider two sub-cases:

**Case 2-I:** The sheaf $I$ is not locally free at $p$.

**Case 2-II:** The sheaf $I$ is locally free at $p$.

Note that in either case, if $E$ is a rational elliptic tail with one node $q$, then $I$ could be locally free, or not, at $q$.

Consider now the ring $R^{\text{Aut}(I)}_{(C,I)}$ as in \cite{BFV}. As usual, denote by $\Gamma = \Gamma_{(C,I)}$ the graph obtained from the dual graph of $C$ by contracting all the edges corresponding to nodes of $C$ where $I$ is locally free. Moreover, denote by $\Gamma_E$ (resp. $\Gamma_{E^c}$) the graph obtained from the dual graph of $E$ (resp. of $E^c$) by contracting all the edges corresponding to the nodes of $E$ (resp. of $E^c$) where $I$ is locally free.

In Case 2-II, the graph $\Gamma$ is obtained by joining the graphs $\Gamma_E$ and $\Gamma_{E^c}$ along a common vertex, and in Case 2-I by means of a separating edge corresponding to the node $p$. Therefore, from the explicit presentation of $U(\Gamma) \cong D(\Gamma)$ given in Theorem 3.6 (see also Lemma 4.3), it follows that

\begin{equation}
U(\Gamma) = \begin{cases}
U(\Gamma_{E^c}) \otimes_{k} U(\Gamma_E) \otimes_k k[T_p] & \text{in Case 2-I,} \\
U(\Gamma_{E^c}) \otimes_{k} U(\Gamma_E) & \text{in Case 2-II.}
\end{cases}
\end{equation}

The graph $\Gamma_E$ consists of a vertex with a loop if $E$ is a rational elliptic tail with one node $q$ and $I$ is not locally free at $q$; otherwise, $\Gamma_E$ has one vertex and no edges. Therefore, using Theorem 5.2 (and say Example 2.2), we easily compute

\begin{equation}
U(\Gamma_E) = D(\Gamma_E) = \begin{cases}
k[X, Y] & \text{if } E \text{ has a node } q \text{ and } I \text{ is not locally free at } q, \\
k & \text{otherwise.}
\end{cases}
\end{equation}
Consider now the automorphism $\phi \in \text{Stab}_C(I)$. Clearly, $\phi$ acts on $U(\Gamma)$ by preserving the decomposition (7.28) and moreover, since $\phi_{E^c} = \text{id}_{E^c}$ by assumption, $\phi$ acts trivially on $U(\Gamma_{E^c})$. Therefore, we have that (7.30)

$$\text{Spec}(R^{\text{Aut}(I)}_{(C, I)})/\langle \phi \rangle = \begin{cases} \text{Spec}(U(\Gamma_{E^c}) \times \text{Spec}(U(\Gamma_E) \otimes_k k[T_p] \otimes_k k[T^\vee \text{Def}^{1,1}((C, I))])/\langle \phi \rangle & \text{in Case 2-I,} \\ \text{Spec}(U(\Gamma_{E^c}) \times \text{Spec}(U(\Gamma_E) \otimes_k k[T^\vee \text{Def}^{1,1}((C, I))])/\langle \phi \rangle & \text{in Case 2-II.} \end{cases}$$

Since $\text{Spec}(U(\Gamma_{E^c}))$ has canonical (and even terminal) singularities by Theorem 4.3, it is enough to prove that $\text{Spec}(U(\Gamma_E) \otimes_k k[T_p] \otimes_k k[T^\vee \text{Def}^{1,1}((C, I))])/\langle \phi \rangle$ and $\text{Spec}(U(\Gamma_E) \otimes_k k[T^\vee \text{Def}^{1,1}((C, I))])/\langle \phi \rangle$ have canonical singularities. Taking into account (7.29), we see that in both cases we are dealing with finite quotient singularities so that we can apply the classical Reid–Tai criterion (see Theorem A.1) to check canonicity.

Before applying the criterion, recall from (7.17) the splitting

$$k[T^\vee \text{Def}^{1,1}((C, I))] \cong k[T^\vee \text{Def}^{1,1}((C, I))] \otimes_k k[T^\vee \text{Def}L],$$

where $L$ is the unique line bundle on the partial normalization $g : C_{\Sigma} \to C$ of $C$ at the nodes $\Sigma = \Sigma(C, I)$ with the property that $g_*(L) = I$. We now want to choose a suitable basis of the vector space

$$(7.31)\quad V := \begin{cases} TU(\Gamma_E) \oplus Tk[T_p] \oplus T \text{Def}^{1,1}((C, I)) \oplus T \text{Def}L & \text{in Case 2-I,} \\ TU(\Gamma_E) \oplus T \text{Def}^{1,1}((C, I)) \oplus T \text{Def}L & \text{in Case 2-II,} \end{cases}$$

and compute the matrix $R(\phi)$ of $\phi$ in terms of the chosen basis.

First observe that in both Case 2-I and 2-II, the upper left $2 \times 2$ sub-matrix of $M(\phi)$ from (7.28) appears as a block factor of the matrix $R(\phi)$. Indeed, in Case 2-I we can choose the coordinate $t_1$ of $T \text{Def}_C$ corresponding to the smoothing of $C$ at the node $p$ as a coordinate of $Tk[T_p]$, and in Case 2-II, we can choose $t_1$ as one of the coordinates of $T \text{Def}^{1,1}_C(C, I)$. Moreover, if $n > 2$ (which implies that $E$ is smooth), then we can choose the coordinate $t_2$ of $T \text{Def}_C$ coming from $T(E, p)(M_{1,1})$, as one of the coordinates of $T \text{Def}^{1,1}_C(C, I)$.

We now focus our attention on the action of $\phi$ on $T \text{Def}_L$. Denote by $E^c_\Sigma$ (resp. $E_\Sigma$) the normalization of $E^c$ (resp. $E$) at the nodes belonging to $\Sigma$. The curve $C_\Sigma$ is the disjoint union of $E^c_\Sigma$ and $E_\Sigma$ in Case 2-I, while it is obtained by joined $E^c_\Sigma$ and $E_\Sigma$ at the separating point $p$ in Case 2-II. In any case, $L$ is completely determined by its restrictions $L|_{E^c_\Sigma}$ and $L|_{E_\Sigma}$, and moreover we have a decomposition

$$(7.32)\quad T \text{Def}_L = T \text{Def}_{L|_{E^c_\Sigma}} \oplus T \text{Def}_{L|_{E_\Sigma}}.$$

Since $\phi_{E^c} = \text{id}_{E^c}$ by assumption, we have that $\phi$ acts trivially on $T \text{Def}_{L|_{E^c_\Sigma}}$.

At this point, we have established what we need from the breakdown of Case 2 into Case 2-I and Case 2-II. In short, in all of Case 2, the upper left $2 \times 2$ sub-matrix of $M(\phi)$ from (7.28) will appear as a block factor of the matrix $R(\phi)$, and the action on $T \text{Def}_L$ is determined by the action on $T \text{Def}_{L|_{E^c_\Sigma}} \cong T(L|_{E^c_\Sigma})(\text{Pic}(E_\Sigma))$.

Let us now examine the action of $\phi$ on $T \text{Def}_{E}|_{E^c_\Sigma}$. For this we consider 3 new subcases of Case 2:

**Case 2-i:** $E$ is smooth

**Case 2-ii:** $E$ is a rational elliptic curve with one node $q$ and $I$ is locally free at $q$.

**Case 2-iii:** $E$ is a rational elliptic curve with one node $q$ and $I$ is not locally free at $q$.

We now proceed with a case by case analysis.

**Case 2-i:** We are assuming that $E$ is smooth. Consequently, $E_\Sigma = E$ and $L|_{E_\Sigma} = I_E \in \text{Pic}^{d_E}(E)$. We can identify $E$ with $\text{Pic}^{d_E}(E)$ sending $r \in E$ into $O_E(r + (d_E - 1)p) \in \text{Pic}^{d_E}(E)$. Since $\phi$ acts on $\text{Pic}^{d_E}(E)$ via pull-back, if the action of $\phi$ on $T_E(E)$ is given by the multiplication by a root of unity $\zeta$, then the action of $\phi$ on $T_{E|_{E^c}}(\text{Pic}^{d_E}(E))$ is given by the multiplication by $\zeta^{-1}$. In other words, if the primitive $n$-th root of unity $\zeta$ is chosen for the matrix $M(\phi)$ from (7.28), then here the action is given by the primitive $n$-th root of unity $\zeta^{-1}$. Therefore the matrix $N(\phi)$ of $\phi$ with respect to the decomposition (7.33) is equal to (with respect to
the same choice of the primitive $n$-th root of unity $\zeta$ as in the above matrix $M(\phi)$:

$$
N(\phi) = \begin{cases}
\begin{pmatrix} \zeta^1 \\ \mathbb{I} \end{pmatrix} & \text{if } n = 2, \\
\begin{pmatrix} \zeta^3 \\ \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^1 \\ \mathbb{I} \end{pmatrix} & \text{if } n = 4, \\
\begin{pmatrix} \zeta^2 \\ \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \zeta^1 \\ \mathbb{I} \end{pmatrix} & \text{if } n = 3, \\
\begin{pmatrix} \zeta^1 \text{ or } \zeta^5 \\ \mathbb{I} \end{pmatrix} & \text{if } n = 6,
\end{cases}
$$

(7.33)

where $\mathbb{I}$ is the suitable identity matrix. Note that the first matrix in each row above corresponds to the first matrix in the corresponding row of (7.26). The matrix $R(\phi)$ describing the action of $\phi$ on the vector space $V$ contains the upper left $2 \times 2$ sub-matrix of $M(\phi)$ from (7.26) and the upper left $1 \times 1$ sub-matrix of $N(\phi)$ from (7.33) as block factors. An easy inspection of the matrices $M(\phi)$ and $N(\phi)$ reveals that the condition (A.2) of the Reid–Tai criterion is satisfied, which shows that $V/\langle \phi \rangle$ has canonical singularities, as we wanted.

**Case 2-ii:** In this case we are assuming that $E$ is a rational elliptic curve with one node $q$ and that $I$ is locally free at $q$. Then also in this case $E_\Sigma = E$ and $L|_{E_\Sigma} = I_E \in \text{Pic}^{dE}(E)$. Moreover, we have that $\text{Pic}^{dE}(E) \cong \mathbb{G}_m$. Explicitly, if we consider the normalization morphism $\nu : E^\nu \cong \mathbb{P}^1 \to E$ and let $\nu^{-1}(q) = \{u, v\}$, then any $\lambda \in \mathbb{G}_m$ determines a unique line bundle $L_\lambda \in \text{Pic}^{dE}(E)$ whose local sections are the local sections $s$ of $\mathcal{O}_{\mathbb{P}^1}(d_E)$ such that $s(u) = \lambda s(v)$. Since, as observed before, $\phi|_E$ is induced by an involution of $E^\nu$ that exchanges $u$ and $v$, then clearly $\phi$ will send $L_\lambda$ into $L_{\lambda^{-1}}$. This implies that the action of $\phi$ on $T_{I_E}(\text{Pic}^{dE}(E))$ is given by multiplication by $-1$, hence the matrix $N(\phi)$ is also in this case given by (7.33) with $n = 2$.

Therefore the matrix $R(\phi)$ describing the action of $\phi$ on the vector space $V$ contains the upper left $2 \times 2$ sub-matrix $M(\phi)$ from (7.26) and the upper left $1 \times 1$ sub-matrix of $N(\phi)$ from (7.33) as block factors, and we conclude as in the previous case that $V/\langle \phi \rangle$ has canonical singularities, as we wanted.

**Case 2-iii:** In this case $E$ is a rational elliptic tail with one node $q$, and $I$ is not locally free at $q$. Observe that in this case $E_\Sigma = \mathbb{P}^1$ so that $T \text{Def}_{L|_{E_\Sigma}} = 0$ and hence the action of $\phi$ on $T \text{Def}_L$ is trivial. To proceed in this case we consider instead the action of $\phi$ on $TU(\Gamma_E)$, which is a two-dimensional $k$-vector space since $U(\Gamma_E) = k[X_q, Y_q]$. Geometrically, the variables $X_q$ and $Y_q$ correspond to deforming the sheaf $I$ at $q$ along the two branches of $q$ (see [CMKV] §3 for more details). Since, as observed before, $\phi|_E$ is induced by an involution of the normalization $\nu : E^\nu \to E$ that exchanges the two branches above $q$, then $\phi$ acts on $U(\Gamma_E) = k[X_q, Y_q]$ by exchanging $X_q$ with $Y_q$. Therefore, we can diagonalize the action of $\phi$ on $TU(\Gamma_E) = \langle X_q^\nu, Y_q^\nu \rangle$ by choosing the basis $\{X_q^\nu - Y_q^\nu, X_q^\nu + Y_q^\nu\}$ in such a way that the matrix $P(\phi)$ describing the action of $\phi$ is equal to

$$
P(\phi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(7.34)

Therefore, since the matrix $R(\phi)$ describing the action of $\phi$ on the vector space $V$ contains the upper left $2 \times 2$ sub-matrix of $M(\phi)$ from (7.26) with $n = 2$ and the matrix $P(\phi)$ of (7.34) as block factors, an easy inspection of the matrices $M(\phi)$ and $P(\phi)$ reveals that the condition (A.2) of the Reid–Tai criterion is satisfied also in this case, which shows that $V/\langle \phi \rangle$ has canonical singularities, as we wanted.

Theorem (7.3) was proved by Bini–Fontanari–Viviani in [BFV12] under the assumption that $\gcd(d + 1 - g, 2g - 2) = 1$, which is exactly the numerical condition on $d$ and $g$ that guarantees that $J_{d,g}$ has finite quotient singularities. When this happens, one can prove Theorem (7.3) by a direct application of the Reid–Tai criterion (see [BFV12] Thm. 4.8).

**Remark 7.5.** It follows from Theorem (7.3) that $J_{d,g}$ is $\mathbb{Q}$-Gorenstein. Indeed, more is true: Fontanari showed in [Fon05] that $J_{d,g}$ is $\mathbb{Q}$-factorial.
We end this subsection with a description of the locus where $\bar{J}_{d,g}$ has finite quotient singularities or is smooth.

**Proposition 7.6.** Let $(C, I) \in \bar{J}_{d,g}$ and assume that $\text{Aut}(C)$ does not contain elements of order equal to $p = \text{char}(k)$. Then

(i) $\bar{J}_{d,g}$ has finite quotient singularities at $(C, I)$ if and only if $\Gamma_{(C, I)}$ is tree-like, i.e. it becomes a tree after removing all the loops around its vertices.

(ii) If $g \geq 4$ then $\bar{J}_{d,g}$ is smooth at $(C, I)$ if and only if $\Gamma_{(C, I)}$ is tree-like and $\text{Stab}_C(I) = \{\text{Id}\}$.

**Proof.** Part (i): using the presentation of the complete local ring of $\bar{J}_{d,g}$ at $(C, I)$ given in Theorem 7.3, it is clear that $\bar{J}_{d,g}$ has finite quotient singularities at $(C, I)$ if and only if $X_\Gamma = \text{Spec} U(\Gamma)$ has finite quotient singularities, where $\Gamma = \Gamma_{(C, I)}$. Proposition 4.6 says that this is the case if and only if $\Gamma$ is tree-like, q.e.d.

Part (ii): using Theorem 7.1, the smoothness of $\bar{J}_{d,g}$ at $(C, I)$ is equivalent to the smoothness of the quotient $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right) / \text{Stab}_C(I)$. By part (i), we must have that $\Gamma = \Gamma_{(C, I)}$ is tree-like. In this case, $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right) = X_\Gamma \times \text{Spec} k[T^\vee \text{Def}^{\text{t, f}}_{(C, I)}]$ is smooth by Proposition 4.6.

**Claim:** The finite group $\text{Stab}_C(I)$ acts on $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right)$ without pseudo-reflections.

Indeed, consider the morphism $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right) \rightarrow \text{Spec} R_C$ of smooth varieties. If $1 \neq \phi \in \text{Stab}_C(I)$ acts as a pseudo-reflection on $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right)$ then $\phi$ acts as a pseudo-reflection on $\text{Spec} R_C$. It is well-known that this happens if and only if $C$ has an elliptic tail $E$ and $\phi$ is the elliptic tail involution, i.e. $\phi|_{E^c} = \text{id}_{E^c}$ and $\phi_E$ is the elliptic involution on $E$ (see Theorem 7.3). This situation is a special case of the situation we dealt with in Case II of the proof of Theorem 7.4, where in particular we verified that the age of $\phi$ (with respect to its action on $\text{Spec} \left( R_{(C, I)}^{\text{Aut}(I)} \right)$ and any primitive root of unity) is at least one. This easily implies that $\phi$ is not a pseudo-reflection because clearly any non trivial pseudo-reflection has age less than one since it has a unique eigenvalue different from one.

Using the Claim, we conclude the proof using a classical result of Prill [Pri67], which says that for a finite group $G$ acting on a smooth variety $X$ without pseudo-reflections, the quotient $X/G$ is smooth if and only if $G$ is the trivial group.

Part (ii) of Proposition 7.6 generalizes [BFV12, Prop. 4.7], where the statement is proved under the assumption that $(C, I)$ belongs to the stable locus of $\bar{J}_{d,g}$, i.e. $I$ is stable with respect to $\omega_C$.

**Remark 7.7.** From Proposition 7.6(ii), it follows that the locus where $\bar{J}_{d,g}$ has finite quotient singularities is, in general, strictly bigger than:

- The stable locus of $\bar{J}_{d,g}$, which coincides with the locus of points $(C, I)$ such that $\text{Aut}(I) = \mathbb{G}_m$, or equivalently $\Gamma_{(C, I)}$ has a unique vertex.

- The locus where the fibers of the morphism $\bar{J}_{d,g} \rightarrow \overline{M}_g$ have finite quotient singularities, which coincides with the locus of points $(C, I)$ where $I$ fails to be locally free only at separating nodes of $C$, or equivalently where $\Gamma_{(C, I)}$ is a tree (see [CMKV13, Thm. C]).

### 7.5. Birational geometry of $\bar{J}_{d,g}$

The Kodaira dimension of $\bar{J}_{d,g}$ was computed by Bini–Fontanari–Viviani in [BFV12] under the numerical assumption that $\gcd(d+1-g, 2g-2) = 1$ (or $g \geq 22$; see Remark 7.9). However, the only place where the authors of loc. cit. need the hypothesis that $\gcd(d+1-g, 2g-2) = 1$ is to establish that $\bar{J}_{d,g}$ has canonical singularities, as they observe in the discussion following [BFV12, Thm. 1.4]. Therefore, as a corollary of [BFV12] and Theorem 7.4, we obtain the following result describing the Kodaira dimension of $\bar{J}_{d,g}$.

**Corollary 7.8.** Assume that $\text{char}(k) = 0$. The Kodaira dimension of the universal Jacobian $\bar{J}_{d,g}$ is given by

\[
\kappa(\bar{J}_{d,g}) = \begin{cases} 
-\infty & \text{if } g \leq 9, \\
0 & \text{if } g = 10, \\
19 & \text{if } g = 11, \\
3g - 3 & \text{if } g \geq 12.
\end{cases}
\]
Proof. We sketch the proof for the convenience of the reader. Verra has shown that \( J_{d,g} \) is unirational for \( g \leq 9 \) (Ver03 Thm. 1.2). So let us consider the case where \( g \geq 10 \). Let \( \pi : J_{d,g} \rightarrow \overline{M}_g \) be the natural forgetful map. Using Grothendieck–Riemann–Roch, it is shown in BFV12 Thm. 1.5 that for \( g \geq 4 \), \( K_{J_{d,g}} = \pi^*(14\lambda - 2\delta) \) (\( = \pi^*K_{\overline{M}_g} + \pi^*\lambda \)), agreeing with the naive computation over \( \overline{M}_g \). As \( \pi \) has connected fibers, the Iitaka dimension of \( K_{J_{d,g}} \) and \( 14\lambda - 2\delta \) are the same. The Iitaka dimension of \( 14\lambda - 2\delta \) is by now well known: \( \kappa(14\lambda - 2\delta) = 0 \) if \( g = 10 \), \( \kappa(14\lambda - 2\delta) = 19 \) if \( g = 11 \) and \( \kappa(14\lambda - 2\delta) = 3g - 3 \) if \( g \geq 12 \). (Recall that for \( g \geq 13 \), work of Eisenbud, Harris and Mumford HM82 EH87 shows that the slope of \( \overline{M}_g \) satisfies \( s(\overline{M}_g) < 7 \), and recent work of Cotterill Cot12 shows the same holds for \( g = 12 \). For \( g = 10, 11 \), work of Tan Tan98 and Farkas–Popa FP05 shows that \( s(\overline{M}_g) = 7 \); in these cases \( \kappa(14\lambda - 2\delta) \) is worked out directly in BFV12 §6.) Finally, since in Theorem 7.4 we have shown that \( \bar{J}_{d,g} \) has canonical singularities, we can conclude that \( \kappa(\bar{J}_{d,g}) = \kappa(K_{\bar{J}_{d,g}}) \), completing the proof. \( \square \)

Remark 7.9. From general results of Ueno Uen75 Thm. 6.12 and Kawamata Kaw85 Cor. 1.2, using the fact that the Kodaira dimension of an abelian variety is zero, one obtains the estimate on the Kodaira dimension: \( \kappa(\overline{M}_g) \leq \kappa(\bar{J}_{d,g}) \leq \dim \overline{M}_g \). By virtue of the results of Harris–Mumford, Eisenbud–Harris and Farkas, that \( \overline{M}_g \) is of general type for \( g = 22 \), \( g \geq 24 \), one obtains immediately that \( \kappa(\bar{J}_{d,g}) = \kappa(\overline{M}_g) = 3g - 3 \) for \( g \) in this range.

Remark 7.10. Since the generic fiber of \( \pi : J_{d,g} \rightarrow \overline{M}_g \) has trivial canonical bundle, it is interesting to compare the Kodaira dimensions of the two spaces. For the convenience of the reader, in the table below we compile the current state of the art on the Kodaira dimension of \( \overline{M}_g \) (we refer the reader to Farkas Far09 for references), and compare it with the Kodaira dimension of \( \bar{J}_{d,g} \).

| \( g \leq 7 \) | 8 | 9 | 10 | 11 | 12 \( \leq g \leq 16 \) | 17 \( \leq g \leq 21 \) | 22 | 23 | 24 \( \leq g \) |
|---|---|---|---|---|---|---|---|---|---|
| \( \kappa(\overline{M}_g) \) | \(-\infty\) | \(-\infty\) | \(-\infty\) | \(-\infty\) | \(-\infty\) | unknown | \(3g - 3\) | \(\geq 2\) | \(3g - 3\) |
| \( \kappa(J_{d,g}) \) | \(-\infty\) | \(-\infty\) | \(-\infty\) | unknown | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) |
| \( \kappa(S_g^+ \) | \(-\infty\) | \(-\infty\) | \(-\infty\) | \(-\infty\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) |
| \( \kappa(S_g^-) \) | \(-\infty\) | \(0\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) | \(3g - 3\) |

Remark 7.11. In recent work Farkas–Verra Far10 FV12 Far12 FV14 have computed the Kodaira dimension of the moduli of spin curves; i.e., the moduli of pairs consisting of a curve together with a theta characteristic. For each \( g \geq 2 \), the space has two components, \( S_g^+ \) and \( S_g^- \) corresponding to the even and odd theta characteristics. Since these sit inside \( J_{g-1,g} \), etale over \( M_g \), we find it interesting to compare the Kodaira dimensions of these spaces (7.35). It turns out, for instance, that both \( J_{d,g} \) and \( S_g \) attain “maximal” Kodaira dimension at \( g = 12 \).

In BFV12 Prop. 6.3, Prop. 6.5] the Iitaka fibration of the canonical class \( K_{J_{d,g}} \) is established for \( g \geq 10 \). This provides the Iitaka fibration for \( J_{d,g} \) under the additional hypothesis that \( J_{d,g} \) has canonical singularities. Consequently, BFV12 have determined the Iitaka fibration for \( J_{d,g} \) assuming that \( \gcd(d + 1 - g, 2g - 2) = 1 \) (and also for \( g \geq 22 \) using a different argument; see BFV12 Prop. 3.2)]. As a consequence of Theorem 7.3 we obtain the following result, generalizing those of BFV12.

Corollary 7.12. For \( g \geq 10 \), the Iitaka fibration of \( J_{d,g} \) is given as follows:

1. For \( g \geq 12 \), the Iitaka fibration is the forgetful morphism \( \pi : J_{d,g} \rightarrow \overline{M}_g \).
2. For \( g = 11 \), the Iitaka fibration is the rational map \( J_{d,11} \rightarrow \mathcal{F}_{11} \), where \( \mathcal{F}_g \) is the moduli of K3 surfaces with polarization of degree \( 2g - 2 \), and the rational map takes a general pair \( (C,L) \) to the pair \( (S, O_S(C)) \), where \( S \) is the unique K3 containing \( C \) (see Muk98).
3. For \( g = 10 \), the Iitaka fibration is the structure morphism \( J_{d,10} \rightarrow \text{Spec} \, k \).

Proof. We sketch the proof for the convenience of the reader. For \( g \geq 12 \), this follows from Theorem 7.3 and Uen75 Thm. 6.11]. Indeed, let \( \overline{M}_g \) be a resolution of singularities of \( \overline{M}_g \), and let \( J_{d,g} \) be a resolution of singularities of the fiber product \( J_{d,g} \times \overline{M}_g \overline{M}_g \). Then the morphism \( \tilde{\pi} : J_{d,g} \rightarrow \overline{M}_g \) of smooth projective varieties is an algebraic fiber space such that \( \dim \overline{M}_g = \kappa(J_{d,g}) \) and the generic fiber \( \tilde{\pi}^{-1}(C) = J^{d}C \) is smooth
and irreducible of Kodaira dimension zero. The same argument works for \( g = 10 \), using a desingularization \( J_{d,10} \) of \( J_{d,10} \). For \( g = 11 \), we refer the reader to [BFV12] Prop. 6.5, where it is shown that the rational map \( \bar{J}_{d,11} \to \mathcal{F}_{11} \) is the Iitaka fibration for \( K_{J_{d,11}} \). Since \( \bar{J}_d \) has canonical singularities by Theorem 7.3 it follows that this rational map is the Iitaka fibration for \( J_{d,11} \).

In the last section of [BFV12], the authors investigate the birational maps among the different universal Jacobians \( J_{d,g} \), as \( d \) varies. Using Theorem 7.3 we can relax their hypothesis (see the discussion at the end of [BFV12, §7]).

**Corollary 7.13.** Assume that \( \text{char}(k) = 0 \) and that \( g \geq 12 \). If \( \eta : J_{d,g} \to J_{d',g} \) is a birational map then \( d' = \pm d + n(2g - 2) \) and \( n \) is given by the map sending \((C,L) \in J_{d,g} \) into \((C, L^1 \otimes \omega_C^*) \in J_{d',g} \). In particular:

(i) \( J_{d,g} \) is birational to \( J_{d',g} \) if and only if \( d' \equiv \pm d \mod 2g - 2 \).

(ii) The group \( \text{Bir}(J_{d,g}) \) of birational automorphisms of \( J_{d,g} \) is given by

\[
\text{Bir}(J_{d,g}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d = n(g - 1) \text{ for some } n \in \mathbb{Z}, \\ \{\text{Id}\} & \text{otherwise.} \end{cases}
\]

Moreover, if \( d = n(g - 1) \) for some \( n \in \mathbb{Z} \) then the generator of \( \text{Bir}(J_{d,g}) \) is the birational automorphism sending \((C,L) \) into \((C, L^{-1} \otimes \omega_C^n) \).

**Proof.** We sketch the proof for the convenience of the reader. As established in Corollary 7.12 for \( g \geq 12 \), the morphism \( \pi : \bar{J}_{d,g} \to \bar{M}_g \) is the Iitaka fibration of \( J_{d,g} \). It follows that any birational automorphism \( \eta : J_{d,g} \to J_{d',g} \) induces a commutative diagram of rational maps (see e.g. [Lei75, Ch. II, Thm. 6.11])

\[
\begin{array}{ccc}
\bar{J}_{d,g} & \xrightarrow{\eta} & \bar{J}_{d',g} \\
\downarrow \pi & & \downarrow \pi \\
M_g & \xrightarrow{\xi} & M_g
\end{array}
\]

The rational map \( \xi \) is the identity. Indeed, indeed if \( C \in M_g \) is very general, and \( C' = \xi(C) \), then there is an induced birational map \( J^d C \to J^d C' \). As this is a birational map of abelian varieties, it is an isomorphism, and one concludes that \( C \cong C' \) using the Torelli theorem, and the fact that for a very general curve, the Neron–Severi group of the Jacobian is isomorphic to \( \mathbb{Z} \) (see [BFV12] Lem. 7.4 for more details). Having established that \( \xi \) is the identity, the corollary follows from [Cap] Prop. 3.2.2. Again, we sketch the proof for the convenience of the reader. Let \( U \subseteq M_g^0 \) be an open set over which \( \eta \) is defined. For each \( C \in U \), there is an isomorphism \( \eta|_C : J^d C \to J^d C' \). Since an isomorphism of abelian varieties is given by a translation, followed by a group automorphism, and \( C \) is automorphism free, then \( \eta|_C(L) = (L \otimes L_C)^{\pm 1} \) for some \( L_C \in J_{d',d-d}^* C \), depending only on \( C \) (see [Cap] Lem. 3.2.3, Prop. 3.2.2, p.16 for more details). The assignment \( C \mapsto L_C \) determines a rational section of \( J_{d,d-g} \to M_g \). The Franchetta Conjecture (proven by [Mes87]) asserts that the only such sections are given by pluricanonical bundles.

**Remark 7.14.** It is likely that Corollary 7.13 fails for small values of \( g \), where it is natural to expect that \( J_{d,g} \) is rational for all values of \( d \in \mathbb{Z} \).

**Appendix A. Finite quotients of toric singularities.**

The aim of this appendix is to study when a finite quotient of a toric singularity is Gorenstein, resp. terminal, resp. canonical. We will work over an algebraically closed field \( k \) of characteristic 0. The main focus is to generalize the Reid–Tai–Shepherd-Barron criterion for quotients of smooth varieties by finite groups. We expect these type of results are well-known to the experts, but we were unable to find a reference for the specific results we use, and so we include statements and proofs here.

**A.1. Finite quotient of smooth varieties.** Let us start by recalling the case of finite quotients of smooth varieties which is well-known and attributed to Khinich, Watanabe, Tai, Reid–Shepherd-Barron and Reid (see e.g. [MSS4] Thm. 2.3] and the references therein).
Theorem A.1. Let $G \subseteq \text{GL}_n(k)$ be a finite subgroup and assume that $G$ does not contain pseudo-reflections. Set $X = \mathbb{A}^n_k / G$. For each $g \in G$ of order $r \neq 1$ and each primitive $r$-th root of unity $\zeta$, write the eigenvalues of $g$ as $\zeta^{a_1}, \ldots, \zeta^{a_n}$ with $0 \leq a_i < r$ and define the age of $g$ with respect to $\zeta$ as

$$\text{age}(g, \zeta) := \frac{1}{r} \sum_{i=1}^n a_i.$$ 

(1) (Khinchin and Watanabe) $X$ is Gorenstein if and only if $G \subseteq \text{SL}_n(k)$; i.e.,

$$\text{age}(g, \zeta) \in \mathbb{Z}$$

for each $1 \neq g \in G$ and each (or, equivalently, some) primitive $r$-th root of unity $\zeta$.

(2) (Reid–Shepherd-Barron [Rei87] and Tai [Tai82]) $X$ is canonical if and only if, in the notation above,

$$\text{age}(g, \zeta) \geq 1$$

for each $1 \neq g \in G$ and each primitive $r$-th root of unity $\zeta$.

(3) (Reid [Rei87]) $X$ is terminal if and only if, in the notation above,

$$\text{age}(g, \zeta) > 1$$

for each $1 \neq g \in G$ and each primitive $r$-th root of unity $\zeta$.

Remark A.2. Recall that an element $1 \neq g \in \text{GL}_n(k)$ is a pseudo-reflection if its fixed locus $\text{Fix}(g) := \{ x \in \mathbb{A}^n_k : g \cdot x = x \}$ is a divisor inside $\mathbb{A}^n_k$. Equivalently, $1 \neq g \in \text{GL}_n(k)$ is a pseudo-reflection if and only if $1$ is an eigenvalue of $g$ with multiplicity equal to $n - 1$. In particular, if $1 \neq g \in \text{GL}_n(k)$ is a pseudo-reflection, then $g \not\subseteq \text{SL}_n(k)$. Note that:

(i) In the above theorem, if one removes the hypothesis that $G$ has no pseudo-reflections, the conditions (A.2) and (A.3) still imply canonical and terminal singularities, respectively.

(ii) If $G \subseteq \text{GL}_n(k)$ is a finite group, denote by $G_{\text{ps}}$ be the normal subgroup of $G$ generated by the pseudo-reflections in $G$. Then $\mathbb{A}^n_k / G_{\text{ps}}$ is smooth, i.e. $\mathbb{A}^n_k / G_{\text{ps}} \cong \mathbb{A}^n_k$ for some $m \leq n$, the quotient group $G / G_{\text{ps}}$ acts linearly on $\mathbb{A}^n_k$ without pseudo-reflections and $\mathbb{A}^n_k / G \cong \mathbb{A}^n_k / (G / G_{\text{ps}})$ (see [Kol13 §3.18]).

Therefore, we can always reduce to the case of finite groups acting without pseudo-reflections.

A.2. Notation and background results on toric varieties. We now recall some notation and background results on toric varieties, following [CLS1].

Fix a lattice $N$, i.e. a free $\mathbb{Z}$-module of finite rank, and let $M = N^\vee$ be its dual lattice. Given a (convex, rational polyhedral) cone

$$\sigma \subseteq N \otimes \mathbb{R} := N_\mathbb{R}$$

consider its dual cone (which is still convex, rational polyhedral)

$$\sigma^\vee = \{ \lambda \in M_\mathbb{R} : \langle \lambda, n \rangle \geq 0, \forall n \in \sigma \} \subseteq M \otimes \mathbb{R} =: M_\mathbb{R}.$$ 

The affine toric variety for the torus $T := \text{Spec} k[M] = \mathbb{G}_m \otimes N$ associated to $\sigma \subseteq N_\mathbb{R}$ is given by

$$U_\sigma = U_{\sigma,N} := \text{Spec} k[\sigma^\vee \cap M]$$

where $k[\sigma^\vee \cap M]$ is the affine semigroup $k$-algebra associated to the normal affine semigroup $\sigma^\vee \cap M$ (by Gordon’s Lemma, see [CLS1] Prop. 1.2.17]). Note that the affine toric variety $U_{\sigma,N}$ depends both on the cone $\sigma \subseteq N_\mathbb{R}$ and on the lattice $N \subseteq N_\mathbb{R}$.

In the sequel, we will use the following notation:

- $\sigma(1)$ is the set of one dimensional faces of $\sigma$, i.e. the extremal rays of the cone.
- Given $\rho \in \sigma(1)$, we set $u_\rho = u_{\rho,N}$ to be the primitive element of $\rho \cap N$. That is $u_\rho \in \rho \cap N$, and if $u \in \rho \cap N$, then $u = nu_\rho$ for some $n \in \mathbb{N}$.
- $\Pi_\sigma = \Pi_{\sigma,N} = \{ \text{polytope } \Pi_{\sigma} = \text{Conv}(0, u_{\rho,N})_{\rho \in \sigma(1)} \}$; i.e., the convex hull of $0$ and the primitive elements of the extremal rays of $\sigma$, with respect to the lattice $N$.

Note that the primitive elements associated to the rays of $\sigma$ depend on the lattice $N$ we are considering; therefore, also the polytope $\Pi_{\sigma,N}$ depends upon the lattice $N$.

In what follows, we will be using the following basic results on toric singularities.
Proposition A.3 (Gorenstein Condition [CLS11, Prop. 8.2.12, Prop. 11.4.11]). In the notation above, the affine toric variety $U_\sigma$ is Gorenstein if and only if there exists $m_\sigma \in M$ such that

$$(m_\sigma, u_\rho) = 1 \text{ for all } \rho \in \sigma(1).$$

In this case, $U_\sigma$ has canonical singularities.

Proposition A.4 (Q-Gorenstein Condition [CLS11, Prop. 11.4.12]). In the notation above, the following conditions are equivalent:

(i) $U_\sigma$ is Q-Gorenstein.

(ii) There exists $m_\sigma \in M_\mathbb{Q}$ such that $(m_\sigma, u_\rho) = 1$ for all $\rho \in \sigma(1)$.

(iii) The polytope $\Pi_\sigma$ has a unique facet not containing the origin.

Note that the property of $U_{\sigma,N} = U_\sigma$ being Q-Gorenstein depends both on the cone $\sigma$ and on the lattice $N$ (see Example A.10). This is not the case for the stronger property of $U_{\sigma,N} = U_\sigma$ being Q-factorial, which is equivalent to the cone $\sigma$ being simplicial (see [CLS11, Thm. 11.4.8]), and hence depends only on the cone $\sigma$ and not on the lattice $N$.

Proposition A.5 (Canonical/Terminal Condition [CLS11, Prop. 11.4.12]). In the notation above, assume that $U_\sigma$ is Q-Gorenstein. Then $U_\sigma$ has canonical (resp. terminal) singularities if and only if the only non-zero lattice points in the polytope $\Pi_\sigma$ lie on the unique facet of $\Pi_\sigma$ not containing the origin (resp. the only lattice points of $\Pi_\sigma$ are its vertices).

A.3. The case of cyclic groups. In this subsection, we will consider the special case of a cyclic group $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$ acting on an affine toric variety $U_\sigma$, preserving the torus $T = \text{Spec } k[M]$.

After fixing a primitive $r$-th root of unity $\zeta \in k$, the action of $\mathbb{Z}_r$ on the coordinate ring $k[M]$ of $T$ is given by a linear form $\lambda : M \to \mathbb{Z}$, well defined up to adding an $r$ multiple of a linear form; in other words, the action is uniquely determined by an element $[\lambda] \in N/rN = \text{Hom}(M, \mathbb{Z}/r\mathbb{Z})$. Explicitly, if we choose a primitive $r$-th root of unity $\zeta \in k$, we can identify the group $\mathbb{Z}_r$ with the subgroup of $k^*$ generated by $\zeta$ and the action on $k[M]$ is given by

$$(A.4) \quad \zeta \cdot x^m = \zeta^{\lambda(m)}x^m.$$

Moreover, if we fix an isomorphism $M \cong \mathbb{Z}^n$ so that $k[M] = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then the action of $\mathbb{Z}_r$ on $k[M]$ is given by

$$(A.5) \quad \zeta \cdot x_i = \zeta^{a_i}x_i \quad \text{for some } 0 \leq a_i < r \quad (i = 1, \ldots, n).$$

Proposition A.6. Let $N = \mathbb{Z}^n = \mathbb{Z} (e_1, \ldots, e_n)$, and let $\sigma \subseteq \mathbb{N}^r$ be a (convex, rational polyhedral) cone. Let $\zeta$ be a primitive $r$-th root of unity and suppose that $\zeta_\sigma = (\zeta)$ acts on $U_{\sigma,N}$ preserving the torus $T = \text{Spec } k[M]$ and that the action on the ring $k[M] = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is given by

$$\zeta \cdot x^m = \zeta^{\lambda(m)}x^m \quad \text{for some } [\lambda] \in N/rN,$$

or more explicitly by

$$\zeta \cdot x_i = \zeta^{a_i}x_i \quad \text{for some } 0 \leq a_i < r \quad (i = 1, \ldots, n).$$

Then $U_{\sigma,N}/\mathbb{Z}_r$ is isomorphic to the affine toric variety $U_{\sigma,N'}$, where $N'$ is the super-lattice of $N$ given by

$$N \subseteq N' = N + \mathbb{Z} \left< \frac{1}{r}[\lambda] \right> = \mathbb{Z} \left< e_1, \ldots, e_n, \sum_{i=1}^n a_i e_i \right> \subseteq N_\mathbb{Q}.$$

In particular, $U_{\sigma,N}/\mathbb{Z}_r$ is

(i) Q-Gorenstein if and only if $\Pi_{\sigma,N'}$ has a unique facet not containing the origin;

(ii) canonical if and only if $\Pi_{\sigma,N'}$ has a unique facet not containing the origin and the only non-zero lattice points in $\Pi_{\sigma,N'}$ lie in this facet;

(iii) terminal if and only if $\Pi_{\sigma,N'}$ has a unique facet not containing the origin and the only lattice points of $\Pi_{\sigma,N'}$ are its vertices.
Proof. Let $M' \subseteq M$ be the sub-lattice of invariants; i.e., $k[M'] = k[x_1^{1, \ldots, n}^{1, \ldots, n}]$. Clearly, the quotient $U_{\sigma, N}/\mathbb{Z}_{r}$ is the affine toric variety equal to Spec $k[\sigma \cap M']$. Therefore, in order to prove the first statement, we need to prove that after setting $N' = N + \mathbb{Z} \langle \frac{\lambda_i}{r} \rangle$, we have $(N')' = M' \subseteq M$. Since $N \subseteq N'$, with torsion quotient, we have $M = N' \supseteq (N')'$. Now pick an element $m = \sum_{i=1}^{n} a_i \lambda_i \in M$ (with $a_i \in \mathbb{Z}$). Since $N'$ is obtained from $N$ by adding the element $a_i \lambda_i = \sum_{i=1}^{n} \frac{a_i \lambda_i}{r} \in \mathbb{Q}$, we have that

$$m \in (N')' \iff \sum_{i=1}^{n} \frac{a_i \lambda_i}{r} \in \mathbb{Z} \iff \sum_{i=1}^{n} a_i \lambda_i \equiv 0 \pmod{r} \iff x^m := \prod_{i=1}^{n} x_i^{a_i} \in k[x_1^{1, \ldots, n}^{1, \ldots, n}]$$

$$\iff m \in M'.$$

The assertions (i)-(iii) now follow from this using Propositions A.4 and A.5.

Using the above proposition, we can prove the following criterion that plays a crucial role in the proof of Theorem 7.4.

**Lemma A.7.** For $i = 1, 2$, let $N_i$ be a lattice and let $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ be a (convex, rational polyhedral) cone. Let $\phi : U_{\sigma_i, N_i} \to U_{\sigma_2, N_2}$ be a toric morphism induced by a homomorphism $\bar{\phi} : N_1 \to N_2$ of lattices such that

1. $\rho \in \sigma_2(1) \Rightarrow \bar{\phi}(\rho) \in \sigma_2(1)$;
2. For every $\rho \in \sigma_1(1)$, we have that $\bar{\phi}(u_{\rho, N_1}) = u_{\rho, N_2}$.

Suppose now that the cyclic group $\mathbb{Z}_r$ acts on the $U_{\sigma_i, N_i}$, preserving the torus $\mathbb{T}_1 = \mathbb{G}_m \otimes \mathbb{Z}_r$ for $i = 1, 2$ and assume that

1. $\phi : U_{\sigma_1, N_1} \to U_{\sigma_2, N_2}$ is $\mathbb{Z}_r$-equivariant;
2. $U_{\sigma_2, N_2}$ is smooth and $\mathbb{Z}_r$ acts on $U_{\sigma_2, N_2}$ without pseudo-reflections;

Then $U_{\sigma_1, N_1}/\mathbb{Z}_r$ is $\mathbb{Q}$-Gorenstein. Moreover, if $U_{\sigma_2, N_2}/\mathbb{Z}_r$ has canonical singularities, then $U_{\sigma_1, N_1}$ has canonical singularities.

Proof. Following the above notation, fix a primitive $r$-th root of unity $\zeta$, and suppose that the action of $\mathbb{Z}_r$ on $U_{\sigma_1, N_1}$ is determined by the element $[\lambda_1] \in N_i/rN_i$. Since $\phi$ is $\mathbb{Z}_r$-equivariant by (i), we must have that $\bar{\phi}([\lambda_1]) = [\lambda_2]$ so that the homomorphism $\bar{\phi} : N_1 \to N_2$ extends to a homomorphism (which we will still denote by $\bar{\phi}$)

$$\bar{\phi} : N_1' := N_1 + \mathbb{Z}[\frac{1}{r}\lambda_1] \longrightarrow N_2' := N_2 + \mathbb{Z}[\frac{1}{r}\lambda_2].$$

By Proposition A.6, the toric morphism $\tilde{\phi} : U_{\sigma_1, N_1}' \to U_{\sigma_2, N_2}'$ induced by $\bar{\phi}$ coincides with the quotient map $U_{\sigma_1, N_1}/\mathbb{Z}_r \to U_{\sigma_2, N_2}/\mathbb{Z}_r$ induced by $\phi$.

Fix now an extremal ray $\rho$ of $\sigma_1$ and look at $\bar{\phi}(\rho)$, which is an extremal ray of $\sigma_2$ by (i). Since $N_1 \subseteq N_1'$, the two primitive elements along the ray $\rho$ with respect to the above lattices are related by $u_{\rho, N_1} = c \cdot u_{\rho, N_1}'$ for some $c \in \mathbb{Z}_{>0}$. On the other hand, it follows from (ii) that $u_{\bar{\phi}(\rho), N_2} = u_{\bar{\phi}(\rho), N_2}'$. Moreover, it follows from (ii) that $\bar{\phi}(u_{\rho, N_1}) = u_{\bar{\phi}(\rho), N_2}$. Finally, we will have that $\bar{\phi}(u_{\rho, N_1}) = l \cdot u_{\bar{\phi}(\rho), N_2}'$ for some $l \in \mathbb{Z}_{<0}$. Putting everything together we find that

$$u_{\bar{\phi}(\rho), N_2}' = u_{\bar{\phi}(\rho), N_2}$$

from which we deduce that $c = l = 1$, i.e. that

$$u_{\rho, N_1} = u_{\rho, N_1}' \quad \text{and} \quad \bar{\phi}(u_{\rho, N_1}) = u_{\bar{\phi}(\rho), N_2}'$$

Observe now that, since $U_{\sigma_2, N_2}$ is smooth by (ii), the quotient $U_{\sigma_2, N_2}/\mathbb{Z}_r = U_{\sigma_2, N_2}'$ is $\mathbb{Q}$-factorial, hence in particular $\mathbb{Q}$-Gorenstein. By Proposition A.4 there exists $m_2 \in (M_2)_{\mathbb{Q}} = (M_2)_{\mathbb{Q}} = (N_2')_{\mathbb{Q}}$ such that $\langle m_2, u_{\tau, N_2}' \rangle = 1$ for every extremal ray $\tau$ of $\sigma_2$. Using (i), the element $m_1 = (\bar{\phi})^*(m_2) \in (M_1)_{\mathbb{Q}} = (M_1)_{\mathbb{Q}} = (N_1')_{\mathbb{Q}}$ satisfies (for every extremal ray $\rho$ of $\sigma_1$)

$$\langle m_1, u_{\rho, N_1}' \rangle = \langle [\bar{\phi}]^*(m_2), u_{\rho, N_1}' \rangle = \langle m_2, \bar{\phi}(u_{\rho, N_1}) \rangle = \langle m_2, u_{\bar{\phi}(\rho), N_2}' \rangle = 1,$$

which shows that $U_{\sigma_1, N_1} = U_{\sigma_1, N_1}/\mathbb{Z}_r$ is $\mathbb{Q}$-Gorenstein.
Take now a point \( 0 \neq x \in N_1' \) which belongs to \( \Pi_{\sigma_1,N_1'} \), i.e.
\[
x = \sum_{\rho \in \sigma_1(1)} \alpha_{\rho} \cdot u_{\rho,N_1'} \quad \text{with} \quad \alpha_{\rho} \geq 0 \quad \text{and} \quad 0 < \sum_{\rho \in \sigma_1(1)} \alpha_{\rho} \leq 1.
\]
Using (A.6), we get that
\[
\varphi(x) = \sum_{\rho \in \sigma_1(1)} \alpha_{\rho} \cdot u_{\rho,\varphi(\rho),N_2'} \Rightarrow 0 \neq \varphi(x) \in \Pi_{\sigma_2,N_2'}.
\]

If \( U_{\sigma_2,N_2'} = U_{\sigma_2,N_2}/\mathbb{Z}_r \) has canonical singularities then Proposition (A.5) implies that \( \varphi(x) \) belongs to the unique facet of \( \Pi_{\sigma_2,N_2'} \) not containing the origin. This is equivalent to the fact that \( \sum_{\rho \in \sigma_1(1)} \alpha_{\rho} = 1 \), which then implies that \( x \) also belongs to the unique facet of \( \Pi_{\sigma_1,N_1'} \) not containing the origin, i.e. that \( U_{\sigma_1,N_1'} = U_{\sigma_1,N_1}/\mathbb{Z}_r \) has canonical singularities.

Although we will not use this, just for the sake of completeness, we prove the following criterion for a cyclic quotient of an affine Gorenstein toric variety to be Gorenstein.

**Proposition A.8.** Same notation as in Proposition (A.6). Assume furthermore that \( U_{\sigma,N} \) is Gorenstein, so that there is an \( m_{\sigma} \in M \) such that
\[
\langle m_{\sigma}, u_{\rho} \rangle = 1 \quad \text{for all} \quad \rho \in \sigma(1),
\]
where \( u_{\rho} \) is the primitive element along the ray \( \rho \) with respect to the lattice \( N \). If \( \lambda \) and \( m_{\sigma} \) satisfy
\[
\frac{1}{r} \lambda(m_{\sigma}) \in \mathbb{Z},
\]
then \( U_{\sigma,N}/\mathbb{Z}_r \) is Gorenstein.

**Proof.** We will use the notation of the proof of the above Proposition (A.6). The assumption \( \frac{1}{r} \lambda(m_{\sigma}) \in \mathbb{Z} \) implies that \( m_{\sigma} \in M' = (N')^\vee \). Moreover, the fact that \( \langle m_{\sigma}, u_{\rho} \rangle = 1 \) insures that \( u_{\rho} \) is still a primitive generator of \( \rho \in \sigma(1) \) with respect to \( N' \): indeed if \( u_{\rho} = l \cdot \tilde{u}_{\rho} \) for some \( 2 \leq l \in \mathbb{N} \) and \( \tilde{u}_{\rho} \in N' \), then
\[
1 = \langle m_{\sigma}, u_{\rho} \rangle = l \langle m_{\sigma}, \tilde{u}_{\rho} \rangle \Rightarrow \langle m_{\sigma}, \tilde{u}_{\rho} \rangle \notin \mathbb{Z},
\]
which contradicts the fact that \( m_{\sigma} \in (N')^\vee \). \( \square \)

**Remark A.9.** If we apply the above Propositions (A.6) and (A.8) to the case where \( U_{\sigma,N} = \mathbb{A}^n_k \), we get back one direction of Theorem (A.1) for finite cyclic quotients of smooth varieties.

We warn the reader that, contrary to the fact that finite quotients of \( \mathbb{Q} \)-factorial toric singularities are \( \mathbb{Q} \)-factorial (because the factoriality of \( U_{\sigma,N} \) is equivalent to the fact that the cone \( \sigma \) is simplicial), a finite quotient of a Gorenstein toric singularity need not to be \( \mathbb{Q} \)-Gorenstein, as the following example shows.

**Example A.10.** Let \( N = \mathbb{Z}^3 = \mathbb{Z}(e_1, e_2, e_3) \) and consider the toric variety \( U_{\sigma,N} \) defined by the cone
\[
\sigma = \mathbb{R}_{\geq 0} \langle e_1, e_2, e_3, e_1 + e_2 - e_3 \rangle \subseteq \mathbb{R}^3 = N \otimes \mathbb{R}.
\]
Now let \( \mathbb{Z}_2 \) act by \(-1\) on \( x_1 \) and as \( 1 \) on \( x_2 \) and \( x_3 \). One can check easily using Propositions (A.3) and (A.4) that \( U_{\sigma,N} \) is Gorenstein, while \( U_{\sigma,N}/\mathbb{Z}_2 \) is not \( \mathbb{Q} \)-Gorenstein.

**A.4. Reduction to the Cyclic Case.** In this subsection, we show that in order to detect if a finite quotient \( V/G \) of a normal \( k \)-variety has canonical or terminal singularities, it is enough to check only that the cyclic quotients \( V/C \) are canonical or terminal as \( C \) varies among all the cyclic subgroups of \( G \). The result in the case where \( V \) is smooth appears in a number of places (e.g. [HMS2, p.44], [Kol13, Thm. 3.21]). The argument for singular \( V \) is the same, and while we expect the result is well-known in this case as well, we are unaware of a reference, and so we include the proof here for the convenience of the reader.

**Theorem A.11.** Suppose that \( G \) is a finite group acting on \( V \), a normal scheme of finite type over \( k \). Then \( V/G \) has canonical (resp. terminal) singularities if and only if for every cyclic subgroup \( C \leq G \), the quotient \( V/C \) has canonical (resp. terminal) singularities.
Proof. We will follow the proof of [Kol13, Thm. 3.21], which deals with the case \( V = \mathbb{A}_k^n \). Suppose first that \( X = V/G \) does not have canonical (resp. terminal) singularities. Let \( \bar{X} \rightarrow X \) be a resolution of singularities, and let \( E \subseteq \bar{X} \) be a prime divisor such that the discrepancy \( a(E, X) < 0 \) (resp. \( \leq 0 \)). Let \( p : \bar{V} \rightarrow \bar{X} \) be the normalization of \( \bar{X} \) in the field of fractions of \( V \), and let \( F \subseteq \bar{V} \) be a prime divisor dominating \( E \). We have a commutative diagram

\[
\begin{array}{ccc}
\bar{V} & \longrightarrow & V \\
p \downarrow & & \downarrow \\
\bar{X} & \longrightarrow & X = V/G
\end{array}
\]

where the vertical morphisms are finite and the horizontal ones are birational. It is computed in [Kol13, (2.42.4)] that the discrepancies of \( F \) and \( E \) are related by the formula

\[
a(E, X) + 1 = \frac{a(F, V) + 1}{|C_F|}.
\]

The group \( G \) acts on the field of fractions of \( V \), and one can easily check the action preserves integrality, so \( G \) also acts on \( \bar{V} \) and \( \bar{X} = \bar{V}/G \). Let \( C_F \) be the subgroup of \( G \) acting as the identity on \( F \). Since \( \bar{V} \) is generically smooth along \( F \), the subgroup \( C_F \leq G \) is cyclic. The diagram (A.7) factors as follows

\[
\begin{array}{ccc}
\bar{V} & \longrightarrow & V \\
p \downarrow & & \downarrow \\
\bar{V}/C_F & \longrightarrow & V/C_F \\
\downarrow & & \downarrow \\
\bar{X} = \bar{V}/G & \longrightarrow & X = V/G
\end{array}
\]

where again the vertical morphisms are finite and the horizontal ones are birational. Consider the prime divisor \( E' = q(F) \), which is exceptional over \( V/C_F \). By applying formula (A.8) to the morphism \( q \), we get that

\[
a(E', V/C_F) + 1 = \frac{a(F, V) + 1}{|C_F|},
\]

which together with (A.8) implies that \( a(E', V/C_F) = a(E, X) < 0 \) (resp. \( \leq 0 \)). Consequently, we see that \( V/C_F \) does not have canonical (resp. terminal) singularities.

Conversely, suppose there is a cyclic group \( C \leq G \) such that \( V/C \) does not have canonical (resp. terminal) singularities. Let \( (V/C)^{\sim} \rightarrow V/C \) be a resolution of singularities, and suppose that \( E' \) is an exceptional divisor such that \( a(E', V/C) < 0 \) (resp. \( \leq 0 \)). Let \( \bar{V} \) be the integral closure of \( (V/C)^{\sim} \) in the field of fractions of \( V \), and let \( F \subseteq \bar{V} \) be a prime divisor dominating \( E' \). Again we obtain (A.10). Now using a result of Zariski and Abhyankar [Kol13, Lem. 2.22, p.50] there is a diagram

\[
\begin{array}{ccc}
\bar{V} & \longrightarrow & V \\
p \downarrow & & \downarrow \\
\bar{X} & \longrightarrow & X = V/G
\end{array}
\]

where the bottom morphism is birational, \( p \) is the induced rational map, and \( F \) dominates a prime divisor \( E \) of \( \bar{X} \). The computation of [Kol13, (2.42.4)] holds (see especially the discussion at the end of the proof of [Kol13, Cor. 2.43, p.66]), giving (A.8). Thus we have \( a(E, X) = a(E', V/C) < 0 \) (resp. \( \leq 0 \)), and it follows that \( X \) does not have canonical (resp. terminal) singularities. 

\[\square\]
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