INFINITE-ENERGY SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS IN A STRIP REVISITED

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Abstract. The paper deals with the Navier-Stokes equations in a strip in the class of spatially non-decaying (infinite-energy) solutions belonging to the properly chosen uniformly local Sobolev spaces. The global well-posedness and dissipativity of the Navier-Stokes equations in a strip in such spaces has been first established in [22]. However, the proof given there contains rather essential error and the aim of the present paper is to correct this error and to show that the main results of [22] remain true.

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1. Introduction

We study the infinite energy solutions of the Navier-Stokes equations

\[
\begin{aligned}
\partial_t u + (u, \nabla_x)u + \nabla_x p &= \Delta_x u + g, \\
\text{div } u &= 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0
\end{aligned}
\tag{1.1}
\]

in a strip \( \Omega = \mathbb{R} \times (-1, 1) \). Note that the case where the solution \( u = (u_1, u_2) \) has the finite energy is well understood now-a-days, see [2, 6, 7, 19, 20] and also references therein. In that case the basic energy estimate can be obtained by multiplication of (1.1) by \( u \), integrating over \( \Omega \) and using the fact that

\[
\int_{\Omega} (u(x), \nabla_x)u(x).u(x) \, dx = 0
\tag{1.2}
\]

for any (square integrable) divergence free function \( u \) satisfying the Dirichlet boundary conditions. However, the most interesting from the physical point of view solutions of problem (1.1) naturally have infinite energy, for instance, it will be so for the classical Poiseille flow

\[
u(x) = \left( \alpha (x_2^2 - 1) \right), \quad \alpha \in \mathbb{R}
\]

as well as for all other solutions bifurcating from it, therefore, exactly the infinite energy solutions look as a relevant class of solutions here from the physical point of view.

The theory of dissipative dynamical systems in unbounded domains and associated infinite energy solutions are intensively developing during the last 20 years starting from the pioneering papers [8, 2, 3].

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see also [14, 10, 24, 15] and references therein. In this theory, the so-called uniformly-local Sobolev spaces defined via
\[
W^{1,p}_b(\Omega) := \{ u \in D'(\Omega), \| u \|_{W^{1,p}_b} := \sup_{s \in \mathbb{R}} \| u \|_{W^{1,p}(\Omega_s)} < \infty \}, \quad \Omega_s := (s, s+1) \times (-1, 1)
\]
are used as the phase spaces for the problems considered. Indeed, on the one hand, in contrast to the usual Sobolev spaces, these spaces contain constants, space-periodic solutions, etc. and look more suitable for the case of unbounded domains. On the other hand, in these spaces one has the regularity theory for the elliptic/parabolic equations which is very similar to the one developed for the usual Sobolev spaces, see e.g., [15] and also Section 2 below. Note also that, in order to obtain the proper estimates for the solutions in the uniformly local spaces, one can use the so-called weighted energy estimates as an intermediate step and utilize the relation
\[
\| u \|_{W_b^2} \sim \sup_{s \in \mathbb{R}} \| u \|_{L^2_{\phi(-s)}}^2,
\]
where \( \phi \) is a properly chosen (square integrable) weight function, see Section 2 for more details.

According to this strategy, in order to obtain the estimates for the solutions of (1.1) in the uniformly local spaces, it is natural to try to multiply equation (1.1) by \( \phi^2 u \), where \( \phi(x_1) \) is a properly chosen weight function. However, there are two principal difficulties arising here. First, we do not have the analogue of (1.2) for the weighted case
\[
\int_{\Omega} \left( u(x), \nabla_x \right) u(x), \phi^2(x_1) u(x) \, dx = -2 \int_{\Omega} \phi(x_1) \phi'(x_1) u_1(x) |u(x)|^2 \, dx \neq 0,
\]
so the non-linear term does not vanish, but produces the extra cubic term which should be somehow estimated (this far is from being straightforward since the "good" terms in the weighted energy inequality are only quadratic). Second, the function \( \phi^2 u \) is no more divergent free
\[
\text{div} \, u = 2\phi\phi' u_1 \neq 0,
\]
so the term containing pressure also survives in the weighted energy estimate and requires to be controlled. The situation is simpler in the case \( \Omega = \mathbb{R}^2 \) or \( \Omega = \mathbb{R} \times (-1, 1) \) with the periodic boundary conditions when the maximum principle can be applied to the vorticity equation which gives an important extra estimate, see [14, 15] and the references therein for more details (see also [13] for some estimates in uniformly local spaces in the 3D case \( \Omega = \mathbb{R}^3 \)).

An effective way to overcome both of the aforementioned problems has been suggested in [22] (see also [21]) where the weighted energy theory for the Navier-Stokes equations in cylindrical domains has been developed. The problem with the extra cubic term has been solved there by using the special weights
\[
\theta_{\varepsilon,s}(x) := \frac{1}{\sqrt{1 + \varepsilon^2 |x - s|^2}}, \quad s \in \mathbb{R},
\]
depending on a small parameter \( \varepsilon > 0 \). Then, since these weights satisfy
\[
|\theta'_{\varepsilon,s}(x)| \leq C\varepsilon |\theta_{\varepsilon,s}(x)|^2,
\]
the nonlinearity produces only the small extra term of the form \( \varepsilon \| u \|_{L^3_{\theta}}^3 \), which can be then controlled by the proper choice of the parameter \( \varepsilon \) depending on the initial condition \( u_0 \), see also Section 3.

To solve the second problem, it was suggested to multiply equation (1.1) by \( \theta^2 u_1 + v_\theta \), where the corrector \( v_\theta \) solves the following linear adjoint problem:
\[
-\partial_t v_\theta + \nabla_x g = \Delta_x v_\theta, \quad \left. v_\theta \right|_{\partial \Omega} = 0, \quad \text{div} \, v_\theta = 2\theta_{\varepsilon,s}\theta'_{\varepsilon,s} u_1.
\]
Then, since \( \text{div}(\theta^2 u - v_\theta) = 0 \), the pressure term vanishes and one has the following weighted energy equality:
\[
\frac{d}{dt} \left( \frac{1}{2} \| u(t) \|_{L^2_{\theta,0}}^2 - \langle u(t), v_\psi(t) \rangle \right) + \langle \nabla_x u(t), \nabla_x (\theta^2 u(t)) \rangle = (g - (u(t), \nabla_x) u(t), \theta^2 u(t) - v_\theta(t)).
\]
Here and below \( \langle u, v \rangle \) stands for the standard inner product in \( L^2(\Omega) \). Moreover, since the data for \( v_\theta \) contains the multiplier \( \theta' \sim \varepsilon^2 \), the corrector \( v_\theta \) should be small (at least, of order \( \varepsilon \)) and, by this reason should not destroy the energy estimate. The realization of this strategy in [22] gave the following result.
Theorem 1.1. For any external force $g \in [L^2(\Omega)]^2$ and any divergent free $u_0 \in [L^2(\Omega)]^2$, $\partial_t u_0|_{\partial \Omega} = 0$, there exists a unique solution $u(t) \in [L^2(\Omega)]^2$ of the Navier-Stokes problem (1.1), satisfying the mean flux condition:
\[
\int_{-1}^{1} u_1(t, x_1, x_2) \, dx_2 = \int_{-1}^{1} u_1(0, x_1, x_2) \, dx_2 = c \in \mathbb{R}
\]
and the following dissipative estimate holds:
\[
\|(u(t))_2 \leq Q(\|u_0\|_{L^2}) e^{-\alpha t} + C(1 + \epsilon^3 + \|g\|^2_{L^2}),
\]
where the positive constants $\alpha$ and $C$ and the monotone function $Q$ are independent of $u_0$, $g$ and $t$.

Clearly, the estimates for the auxiliary problem (1.3) play an important role in the proof of this theorem. Namely, as stated in Theorem 5.1 of [22], the correction $v$ satisfies the following estimate:
\[
\|(v_0)_{C(0,T;W^{1,2}_{\mu,\phi}(\Omega))} + \|v_0\|_{L^2(0,T;W^{2,2}_{\mu,\phi}(\Omega))} \leq C\epsilon(\|u\|_{C(0,T;L^2)} + \|u\|_{L^2(0,T;W^{2,2}_{\mu,\phi}(\Omega))}).
\]

Unfortunately, the proof of this key estimate contains an essential error. Namely, the function $\bar{v}$ involved into equation (5.14) at page 553 of [22] has non-zero boundary conditions (although $v_0|_{\partial \Omega} = 0$, $\Pi v|_{\partial \Omega} \neq 0$ since the Leray projector does not preserve Dirichlet boundary conditions) and, by this reason, the multiplication of this equation by $\text{div}(\varphi_2 \nu \phi \nabla_x (\bar{v} + \Pi v))$ leads to the extra uncontrollable boundary term which is missed in inequality (5.15). Thus, the proof of Theorem 5.1 given in [22] is formally wrong. Moreover, estimate (1.5) is probably wrong as well (at least, we do not know how to prove good estimates for the auxiliary problem (1.3)) in the weighted space $C(0,T;W^{2,2}_{\mu,\phi}(\Omega))$ which are stated in this theorem and only the weaker versions of these estimates, e.g., in the space $C(0,T;L^2_{\phi}(\Omega))$ are available, see Section 3 below.

The aim of the present paper is to correct the aforementioned error and to show that Theorem 1.1 stated above remains true. To this end, we first develop an alternative approach to study the auxiliary equation (1.3) based more on the methods of the analytic semigroup theory rather than on energy estimates and verify a weaker version of Theorem 5.1 from [22]. In a fact, we are unable to establish the control of the corrector $v_\theta$ in $C(0,T;W^{1,2}_{\mu,\phi}(\Omega))$ or in $L^2(0,T;W^{2,2}_{\mu,\phi}(\Omega))$ (as stated in this theorem), but the following weaker version of these estimates hold:
\[
\|v_\theta\|_{C(0,T;L^2_{\mu,\phi}(\Omega))} + \|v_\theta\|_{C(0,T;L^3_{\mu,\phi}(\Omega))} \leq C\epsilon\|u\|_{C(0,T;L^2_{\phi}(\Omega))},
\]
see Section 3 for the details. Then, keeping in mind that estimate (1.7) is essentially weaker than the original estimate (1.5) used in [22], we have to rework most part of proofs given in [22] in order to show that this loss of the regularity of the corrector $v_\theta$ is not crucial and that the main results remain true despite the aforementioned error.

The paper is organized as follows. The definitions of the proper weight functions and associated weighted spaces as well as their basic properties are given in Section 2. Moreover, we briefly recall here the known facts on the regularity of the Leray projector and the Stokes operator in these spaces. The auxiliary linear problem (1.3) is studied in Section 3. In particular, estimate (1.7) as well as the energy identity (1.4) are verified there. The proof of the key estimate (1.5) (in the non-dissipative form with $\alpha = 0$) is verified in Section 4. The existence and uniqueness of a weak solution for the Navier-Stokes problem (1.1) in the uniformly local spaces is verified in Section 5 and, finally, the dissipative version of estimate (1.5) (with $\alpha > 0$) and the parabolic smoothing property for the weak solutions of (1.1) are proved in Section 6.

2. Preliminaries

In that section, we briefly recall the definitions and key properties of weights and weighted Sobolev spaces and state a number of known results on the regularity of the Leray projector and Stokes operator in these spaces which are crucial for what follows, see [7, 22] for the detailed exposition. We start by defining the class of admissible weights and associated weighted spaces adopted to the case of the strip $\Omega = \mathbb{R} \times (-1, 1).$

Definition 2.1. A function $\phi(x)$, $x \in \mathbb{R}$, is a weight function of exponential growth rate $\mu > 0$ if
\[
\phi(x) > 0, \quad \phi(x + y) \leq C e^{\mu|y|} \phi(x)
\]
holds for all \( x, y \in \mathbb{R} \). The weighted Lebesgue space \( L^p_b(\Omega) \), \( 1 \leq p \leq \infty \), is defined as a subspace of \( L^p_{loc}(\Omega) \) for which the following norm is finite:

\[
\|u\|_{L^p_b} := \left( \int_{\Omega} \phi^p(x) |u(x)|^p \, dx \right)^{1/p},
\]

where \( x = (x_1, x_2) \in \Omega \). The uniformly local Lebesgue space \( L^p_b(\Omega) \) is determined by the finiteness of the following norm:

\[
\|u\|_{L^p_b} := \sup_{s \in \mathbb{R}} \|u\|_{L^p(\Omega_s)},
\]

where \( \Omega_s := (s, s+1) \times (0, 1) \). As usual, the weighted \( (W^{l,p}_b(\Omega)) \) and uniformly local \( (W^{l,p}_b(\Omega)) \) Sobolev spaces are defined as spaces of distributions whose derivatives up to order \( l \) belong to \( L^p_b(\Omega) \) (resp. \( L^p(\Omega) \)). This definition works for \( l \in \mathbb{N} \) and, for the non-integer or negative \( l \)th, the corresponding Sobolev spaces can be defined via the interpolation and duality arguments, see [10, 24, 15] for the details.

We will also need the uniformly local spaces for the functions \( u(t, x) \), \( x \in \Omega \), depending also on time \( t \in \mathbb{R} \), so, for every \( 1 \leq p, q \leq \infty \), we define the space \( L^p_b(\mathbb{R} \times \Omega) \) by the following norm:

\[
\|u\|_{L^p_b(\mathbb{R} \times \Omega)} := \sup_{(t, s) \in \mathbb{R}^2} \|u\|_{L^p((t,t+1) \times \Omega_s)},
\]

and the spaces \( L^p_b((A, B) \times \Omega) \) are defined analogously. More general, for \( 1 \leq p, q \leq \infty \), we defined the space \( L^p_b(\mathbb{R}, L^p_b(\Omega)) \) by the following norm:

\[
\|u\|_{L^p_b(\mathbb{R}, L^p_b(\Omega))} := \sup_{(t, s) \in \mathbb{R}^2} \|u\|_{L^q((t,t+1) \times L^p(\Omega_s))}.
\]

The natural choices of the weights of exponential growth rate are the following ones:

\[
\varphi_{\varepsilon,x_0}(x) := e^{-\varepsilon|x-x_0|}, \quad \varphi_{\varepsilon,x_0}(x) := e^{\varepsilon \sqrt{\varepsilon^2|x-x_0|^2+1}}
\]

which, obviously, have the exponential growth rate \( |\varepsilon| \) or the polynomial weights, e.g.,

\[
\theta_{\varepsilon,x_0}(x) := \frac{1}{\sqrt{1 + \varepsilon^2|x-x_0|^2}}
\]

This weight, in addition, to (2.1) (which holds for every positive \( \mu \)), satisfies the following property:

\[
|\theta'_{\varepsilon,x_0}(x)| \leq C\varepsilon \theta_{\varepsilon,x_0}(x)^2 \leq C\varepsilon \theta_{\varepsilon,x_0}(x)
\]

which is crucial for what follows. A bit more general are the weights \( \theta_{\varepsilon,x_0}(x)^N \), \( N \in \mathbb{R}, N \neq 0 \), which are also the weights of exponential growth rate \( \mu \) for any \( \mu > 0 \) and satisfy the analog of (2.6) where the exponent 2 is replaced by \( \frac{N+1}{N} \).

The next proposition which gives the equivalent representation of the weighted Sobolev norms in terms of the non-weighted ones is very useful in many estimates.

**Proposition 2.2.** Let \( \phi \) be the weight of exponential growth rate, \( 1 \leq p < \infty \) and \( l \in \mathbb{R} \). Then

\[
C_2 \|u\|_{W^{l,p}_b(\Omega)}^p \leq \int_{s \in \mathbb{R}} \phi^p(s) \|u\|_{W^{l,p}(\Omega_s)}^p \, ds \leq C_1 \|u\|_{W^{l,p}_b(\Omega)}^p,
\]

where the constants \( C_1 \) depend only on \( \mu \) and constant \( C \) involved in (2.1) and are independent of the concrete choice of the weight \( \phi \).

For the proof of this proposition, see e.g., [10] or [24].

The next proposition which connects the weighted and uniformly local norms is the main technical tool for obtaining the estimates of solutions in uniformly local spaces.

**Proposition 2.3.** Let \( \phi \) be the weight function of exponential growth rate such that \( \phi \in L^p(\mathbb{R}) \) and let \( 1 \leq p < \infty \). Then,

\[
C_1 \|\phi\|_{L^p_b} \|u\|_{W^{l,p}_b(\Omega)} \leq \|u\|_{W^{l,p}_b(\Omega)} \leq C_2 \sup_{s \in \mathbb{R}} \|u\|_{W^{l,p}_b(\Omega_s)},
\]

where the constants \( C_1 \) and \( C_2 \) depend only on \( p \) and \( C \) and \( \mu \) involved in (2.1) and are independent of the concrete choice of the weight \( \phi \).
For the proof of this proposition, see [24].
We will essentially use the particular case of this estimate with \( p = 2 \) and \( \phi = \theta_{\varepsilon,x_0} \) where \( \varepsilon \ll 1 \). Then the left-hand side of (2.8) gives
\[
\|u\|_{L^2_{\Delta \varepsilon,x_0}} \leq C_{\varepsilon}^{-1/2} \|u\|_{L^2_b},
\]
where \( C \) is independent of \( \varepsilon \) and \( x_0 \).
At the next step, we introduce the standard (for the theory of the Navier-Stokes equations) spaces \( \mathcal{H}, \mathcal{V} \) and \( \mathcal{V}^\star \):
\[
\mathcal{H} := [u \in [C_0^\infty(\Omega)], \quad \text{div} u = 0]\|_{L^2(\Omega)}^2, \quad \mathcal{V} := [u \in [C_0^\infty(\Omega)], \quad \text{div} u = 0]\|_{W^{1,2}(\Omega)}^2,
\]
where \([ \cdot ]_V \) means the closure in the space \( V \) and \( \mathcal{V}^\star \) stands for the dual space to \( \mathcal{V} \) (with respect to the inner product in \( \mathcal{H} \)). The spaces \( \mathcal{H}_\phi \) and \( \mathcal{V}_\phi \) are defined analogously (the closure is taken in \([L^2_\phi(\Omega)]^2\) and \([W^{1,2}_\phi(\Omega)]^2\) respectively).

The following proposition describes the structure of the introduced spaces.

**Proposition 2.4.** The spaces \( \mathcal{H} \) and \( \mathcal{V} \) can be described as follows:
\[
(2.10) \quad \mathcal{H} = \{ u \in [L^2(\Omega)]^2, \quad \text{div} u = 0, \quad u_2|_{\partial \Omega} = 0 \}, \quad \mathcal{V} = \mathcal{H} \cap [H^1_0(\Omega)]^2,
\]
where \( \text{div} v(x_1) := \frac{1}{\Omega} \int_{-1}^1 v(x_1,x_2) \, dx_2 \) and \( H^1_0(\Omega) := W^{1,2}(\Omega) \cap \{ u|_{\partial \Omega} = 0 \} \). Moreover, let for any \( u \in \mathcal{H} \) (resp. \( u \in \mathcal{V} \))
\[
(2.11) \quad \psi := \Psi(u) = \int_{-1}^{x_2} u_1(x_1,s) \, ds
\]
be the associated stream function. Then,
\[
u_1 = \partial_{x_2} \psi, \quad u_2 = -\partial_{x_1} \psi
\]
and the operator \( \Psi \) realizes the isomorphism between \( \mathcal{H} \) and \( H^1_0(\Omega) \) (resp. between \( \mathcal{V} \) and \( H^1_0(\Omega) \)). Furthermore, for any weight \( \phi \) of exponential growth rate, the analogous description and the analogous isomorphism works for the weighted spaces \( \mathcal{H}_\phi \) and \( \mathcal{V}_\phi \) as well.

For the proof of this proposition, see e.g., [22].
Analogously to (2.10), we define the uniformly local spaces \( \mathcal{H}_b \) and \( \mathcal{V}_b \) via
\[
(2.12) \quad \mathcal{H}_b = \{ u \in [L^2(\Omega)]^2, \quad \text{div} u = 0, \quad u_2|_{\partial \Omega} = 0 \}, \quad \mathcal{V}_b = \mathcal{H}_b \cap [W^{1,2}_b(\Omega)]^2 \cap \{ u|_{\partial \Omega} = 0 \},
\]
Note that, in contrast to the case of spaces \( \mathcal{H} \) or \( \mathcal{H}_\phi \), the uniformly local spaces \( \mathcal{H}_b \) and \( \mathcal{V}_b \) do not coincide with the closure of \( C_0^\infty(\Omega) \) in the proper uniformly local norms. However, the operator \( \Psi \) is still the isomorphism between the corresponding uniformly local spaces, see [22] for more details.

We now recall that the space \( \mathcal{H} \) is orthogonal to any gradient vector field and, due to the Leray-Helmholtz decomposition, any vector field \( u \in [L^2(\Omega)]^2 \) can be presented in a unique way as a sum
\[
(2.13) \quad u = v + \nabla \psi p, \quad v \in \mathcal{H}.
\]
Therefore, the Leray (ortho)projector \( \Pi : [L^2(\Omega)]^2 \rightarrow \mathcal{H} \) is well defined via \( \Pi u := v \). We also recall that the Stokes operator is defined as the following self-adjoint operator in \( \mathcal{H} \):
\[
(2.14) \quad A := -\Pi \Delta, \quad D(A) = \mathcal{V} \cap [H^2(\Omega)]^2.
\]
The next standard proposition gives the description of domains of its fractional powers. This result will be used in the next section for deriving the weighted energy estimates.

**Proposition 2.5.** Let \( \kappa \in (0,1) \). Then the domain of \( A^\kappa \) possesses the following description:
\[
(2.15) \quad D(A^\kappa) = [D((-\Delta)^{\kappa/2})]^2 \cap \mathcal{H},
\]
where \( D((-\Delta)^{\kappa/2}) \) is the domain of the fractional Laplacian with Dirichlet boundary conditions.

For the proof of this result, see e.g., [3], see also [1, 17] where the analogous result is obtained not only for \( L^2 \), but also for the \( L^p \)-spaces, \( 1 < p < \infty \).
Since, the description of the domains of the fractional Laplacian is well-known, see e.g., \[21\],

\[
D((-\Delta)^{\kappa}) = \begin{cases}
W^{2\kappa,2}(\Omega), & \kappa < 1/4; \\
W^{2\kappa,2}(\Omega) \cap \{u|_{\partial\Omega} = 0\}, & \kappa > 1/4; \\
W^{2\kappa,2}(\Omega) \cap \{\int_{\Omega} \frac{1}{1+|x|^2} |u(x)|^2 \, dx < \infty\}, & \kappa = 1/4,
\end{cases}
\]

Proposition 2.5 gives the description of \(D(A^\gamma)\) in terms of the usual Sobolev spaces.

The next result gives the regularity of the Leray projector and the Stokes operator in weighted and uniformly local Sobolev spaces.

**Proposition 2.6.** Let \(\phi\) be the weight of a sufficiently small exponential growth rate. Then, for any \(l \geq 0\) and \(1 < p < \infty\), the Leray projector \(\Pi\) can be extended in a unique way by continuity to the continuous operator

\[
\Pi : [W^{l,p}_\phi(\Omega)]^2 \to [W^{l,p}_\phi(\Omega)]^2
\]

and the norm of this operator depends on \(l, p\) and the constant \(C\) involved into the inequality \[21\] and is uniformly bounded with respect to the concrete choice of \(\phi\). Furthermore, analogously, the Stokes operator \(A\) can be extended to the isomorphism

\[
A : [W^{l+2,2}_\phi(\Omega)]^2 \cap H_\phi \cap \{u|_{\partial\Omega} = 0\} \to [W^{l,2}_\phi(\Omega)]^2 \cap H_\phi
\]

and the norms of \(A\) and \(A^{-1}\) are uniformly bounded with respect to the concrete choice of \(\phi\). Moreover, the analogous results hold for the uniformly local Sobolev spaces as well.

The proof of this result can be found, e.g., in \[22\], see also \[6\].

We also state the analogue of the regularity result for the Stokes operator in negative Sobolev spaces.

**Proposition 2.7.** Let \(\phi\) be the weight function of sufficiently small exponential growth rate. Then, for every \(g \in [W^{-1,2}_\phi(\Omega)]^2\), there is a unique solution \(u \in V_\phi\) of the Stokes problem

\[
\Delta_x u - \nabla_x p = g, \quad \text{div} \, u = 0
\]

and the following estimate holds:

\[
\|u\|_{V_\phi} \leq C\|g\|_{[W^{-1,2}_\phi(\Omega)]^2},
\]

where the constant \(C\) is independent of the concrete choice of the weight \(\phi\). The analogous result holds also for the uniformly local spaces.

The proof of this result can be found in \[22\] and \[6\].

We conclude this section by given the result on solvability of the non-stationary Stokes problem in weighted spaces for the case of strong solutions.

**Proposition 2.8.** Let \(\phi\) be the weight function of sufficiently small exponential growth rate. Then, for every \(g \in L^2(0,T; L^{2}_\phi(\Omega))\) and every \(u_0 \in V_\phi\), there is a unique solution \(u(t)\) of the problem

\[
\partial_t u - \Delta_x u + \nabla_x p = g(t), \quad \text{div} \, u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0
\]

which satisfies \(\partial_t u, \Delta_x u \in L^2(0,T; L^2_\phi)\) and the following estimate holds:

\[
\|u(t)\|_{L^2_\phi}^2 + \|\partial_t u\|_{L^2(\text{max}(0,t-1),t; L^2_\phi)}^2 + \|\Delta_x u\|_{L^2(\text{max}(0,t-1),t; L^2_\phi)}^2 \leq C\|u_0\|_{L^2_\phi}^2 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\|g(s)\|_{L^2_\phi}^2 \, ds,
\]

where \(C\) and \(\alpha > 0\) are independent of \(t \geq 0\), \(u_0\), \(g\) and the concrete choice of the weight \(\phi\).

The proof of this result also can be found in \[22\] or \[6\].

**Remark 2.9.** Most results of this section will be used in the sequel with the weights \(\phi = \theta_{\epsilon,x_0}(x_1)\) only. However, we will need to control the dependence of all the constants on the parameter \(\epsilon \to 0\) and, by this reason, it is important for us that the constants in the above propositions are "independent of the concrete choice of the weight" and depend only on the constants in \(2.1\) (which are uniform with respect to \(\epsilon \to 0\) and \(x_0 \in \mathbb{R}\)).
Another straightforward observation which will be essentially used in the next section is that, according to Propositions 2.5 and 2.6 and formula (2.10), the Leray projector II maps \( W^{2q,2}(\Omega) \) to \( D(A^\kappa) \):

\[
(2.23) \quad II : \left[ W^{2q,2}(\Omega) \right]^2 \rightarrow D(A^\kappa)
\]

for \( \kappa < 1/4 \) and that is not true for \( \kappa \geq 1/4 \) due to the loss of zero boundary conditions (we recall that, in general, \( \Pi u \big|_{\partial\Omega} \neq 0 \) even if \( u \big|_{\partial\Omega} = 0 \).

Finally, it worth to mention that the dual space \( \mathcal{V}^* \) is not a subspace of distributions \( [D'(\Omega)]^2 \) and the fact that the divergence free vector field \( v \in \mathcal{V}^* \) does not imply in general that its components \( v_1 \) or \( v_2 \) belong to \( H^{-1}(\Omega) \). Indeed, we may add any gradient of a harmonic function to the vector field \( v \) without changing the functional \( v \in \mathcal{V}^* \) and this harmonic function may be not in \( L^2(\Omega) \), say, due to the singularities near the boundary. As will be explained in the next section, this leads to essential difficulties in developing the weighted energy theory for the non-stationary Stokes equations.

3. The linear non-stationary Stokes equation: weighted energy theory

The aim of this section is to derive the so-called weighted energy equality for the linear non-autonomous Stokes problem (2.21) under the assumptions that

\[
(3.1) \quad g \in L^{1/3}_b(\mathbb{R} \times \Omega) \cap L^1_b(\mathbb{R} \times L^{3/2}_b(\Omega)), \quad u_0 \in \mathcal{H}_b
\]

which will be used in the sequel for the study of the nonlinear Navier-Stokes equation.

**Definition 3.1.** A function \( u(t,x) \) is a weak (energy) solution of (2.21) if

\[
(3.2) \quad u \in L^\infty(0,T; \mathcal{H}_b) \cap C(0,T; \mathcal{H}_b), \quad \nabla_x u \in L^2_b([0,T] \times \Omega),
\]

where \( \phi \) is any weight of exponential growth rate such that \( \phi \in L^2(\mathbb{R}) \) and \( u \) solves (2.21) in the sense of distributions, namely, for any \( \varphi \in C_0^\infty([0,T] \times \Omega) \) satisfying \( \text{div} \varphi = 0 \),

\[
- \int_{\mathbb{R}_+} (u, \partial_t \varphi) \, dt - \int_{\mathbb{R}_+} (u, \Delta_x \varphi) \, dt = \int_{\mathbb{R}_+} (g, \varphi) \, dt.
\]

Here and below \( (u, v) \) stands for the standard inner product in \( [L^2(\Omega)]^2 \).

To derive the weighted energy equality for problem (2.21), it would be natural to multiply the equation by \( \phi^2 u \) for some properly chosen weight \( \phi \) of exponential growth rate. However, this does not work in a straightforward way since

\[
\text{div}(\phi^2 u) = 2\phi^2 u_1 \neq 0.
\]

To overcome this difficulty, we introduce (following [22]) the corrector \( v_\phi \) as a solution of the following auxiliary problem:

\[
(3.3) \quad - \partial_t v_\phi - \Delta_x v_\phi + \nabla_x q = 0, \quad \text{div} v_\phi = 2\phi^2 u_1, \quad v_\phi \big|_{\partial\Omega} = 0, \quad v_\phi \big|_{t=T} = (0,2\phi^2 \Psi(u(T))),
\]

where \( T > 0 \) is a parameter and \( \psi = \Psi(u) \) is a stream function of the vector field \( u \). Then, \( \text{div}(\phi^2 u - v_\phi) = 0 \) and we may at least formally multiply equation (2.21) by \( \phi^2 u - v_\phi \) without taking a special care on the pressure term \( \nabla_x p \). Note also that since (3.3) is the adjoint equation to (2.21), we need to solve it backward in time (for \( t \leq T \)) and the unusual initial data at \( t = T \) is chosen in order to satisfy the necessary compatibility condition

\[
\text{div} u \big|_{t=T} = \text{div} u(T) = 2\phi^2 u_1(T).
\]

Then, multiplying formally equation (2.21) by \( \phi^2 u - v_\phi \) and integrating over \( x \), we have

\[
(3.4) \quad (\partial_t u - \Delta_x u - \nabla_x p, \phi^2 u - v_\phi) = \frac{d}{dt} \left( \frac{1}{2} \| u \|_{L^2_\phi}^2 - \langle u, v_\phi \rangle \right) + \langle u, -\partial_t v_\phi - \Delta_x v_\phi \rangle + \\
+ \langle \nabla_x u, \nabla_x(\phi^2 u) \rangle = \frac{d}{dt} \left( \frac{1}{2} \| u \|_{L^2_\phi}^2 - \langle u, v_\phi \rangle \right) + \langle \nabla_x u, \nabla_x(\phi^2 u) \rangle = \\
= \frac{d}{dt} \left( \frac{1}{2} \| u \|_{L^2_\phi}^2 - \langle u, v_\phi \rangle \right) + \langle \nabla_x u, \nabla_x(\phi^2 u) \rangle,
\]
where we have used that \( \text{div } u = \text{div}(\phi^2 u - v_\phi) = 0 \). Thus, we formally end up with the key weighted energy identity

\[
\frac{d}{dt} \left( \frac{1}{2}\|u\|_{L^2_\phi}^2 - (u, v_\phi) \right) + (\nabla_x u, \nabla_x (\phi^2 u)) = (g, \phi^2 u - v_\phi).
\]

The main aim of this section is to justify the weighted energy equality (3.5). To this end, we first need to study the solutions of the auxiliary problem (3.3). For simplicity, we switch back to forward in time solutions by the change of time variable \( t \to T - t \) and consider slightly more general problem

\[
\partial_t v - \Delta_x v + \nabla_x q = 0, \quad \text{div } v = \partial_{x_1} h(t), \quad v|_{\partial\Omega} = 0, \quad v|_{t=1} = (0, h(0))^T,
\]

where the function \( h \in C(0, T; W^{1,2}_\phi(\Omega)) \) for some weight \( \phi \) of exponential growth rate which satisfies the additional assumption

\[
|\phi'(x)| + |\phi''(x)| + |\phi'''(x)| \leq C\varepsilon\phi(x),
\]

and the parameter \( \varepsilon > 0 \) is small enough. Then, the choice

\[
h(t) = 2\epsilon\phi\Psi(u_1(t))
\]

corresponds to the considered auxiliary problem (3.3). The next theorem which gives the estimate for the solution \( v \) in terms of the function \( h \) is crucial for what follows.

**Theorem 3.2.** Let \( \phi \) be a weight of exponential growth rate which satisfies (3.3) for sufficiently small \( \varepsilon \leq \varepsilon_0 \) and let \( h \in C(0, T; W^{1,2}_\phi(\Omega)) \). Then, there exists a unique solution \( v(t) \) of problem (3.3) such that

\[
v \in C(0, T; W^{1/3,2}_\phi(\Omega)), \quad \int_0^T v(t) \, dt \in L^2(0, T; W^{2+1/3,2}_\phi(\Omega)), \quad \mathbb{S}v_1 \equiv 0
\]

and the following estimate holds:

\[
\|v\|_{C(0,T;W^{1/3,2}_\phi(\Omega))} \leq C\|h\|_{C(0,T;W^{1,2}_\phi(\Omega))},
\]

where the constant \( C \) is independent of \( h, T, \varepsilon \) and the choice of the weight \( \phi \).

**Remark 3.3.** The regularity assumptions (3.9) are a bit unusual. Indeed, it would be more natural to expect similar to Proposition 2.8 that \( \partial_t v, \Delta_x v \in L^2(0, T; L^2) \), but this regularity obviously requires that

\[
\partial_t h = \text{div } \partial_t v \in L^2(0, T; H^{-1}_\phi(\Omega)).
\]

However, we do not control the \( H^{-1} \)-norm of \( \partial_t h \) since according to (3.3), we need to control the appropriate norm of \( \partial_t u_1 \), where \( u \) is an energy solution of (2.21). But, from equation (2.21), we know only that

\[
\partial_t u \in L^2(0, T; H^{-1}_\phi) + L^{4/3}(0, T; L^{4/3}_\phi(\Omega))
\]

and, as explained in Remark 2.30 this is not enough to control the reasonable norms of \( \partial_t u_1 \). Thus, in order to avoid the assumptions on \( \partial_t h \), we have to use only the partial regularity, say, of the form (3.9).

The exponent 1/3 in (3.9) can be replaced by 1/2 - \( \kappa \) for every \( \kappa > 0 \) (which is not essential for our purposes), but our method does not allow to take this exponent larger since it utilizes property (2.23). Finally, the condition on the integral \( \int_0^T v(t) \, dt \) is added in order to be able to pose the boundary conditions.

**Proof of the theorem.** We split the proof into several steps.

**Step 1.** At this step we make several equivalent transformations and reduce problem (3.3) to more standard one. First, introducing the new variable \( \bar{v}(t) := v(t) - w(t) \), where \( w(t) := (0, h(t))^T \), we obtain the divergence free problem:

\[
\partial_t \bar{v} - \Delta_x \bar{v} - \nabla_x q = -\partial_t w(t) - \Delta_x w(t), \quad \text{div } \bar{v} = 0, \quad \bar{v}|_{t=0} = 0.
\]

Since, obviously

\[
\|w\|_{C(0,T;W^{1,2}_\phi)} \leq \|h\|_{C(0,T;W^{1,2}_\phi)},
\]
it is enough to verify estimate (3.10) for function $\tilde{v}$ only. Second, we introduce the function $\tilde{w}(t)$ as a solution of the linear Stokes problem

$$\Delta_x \tilde{w}(t) - \nabla_x q = \Delta_x w(t), \quad \text{div} \, \tilde{w} = 0, \quad w|_{\partial \Omega} = 0.$$  

Then, due to Proposition 2.7

(3.12) \quad \|\tilde{w}\|_{C(0,T;V_0^1)} \leq C \|\Delta_x w\|_{C(0,T;W^{-1,2}_0)} \leq C_1 \|h\|_{C(0,T;W^{-1,2}_0)}

and, introducing $\tilde{v} := \tilde{v} - \tilde{w}$, we get

(3.13) \quad \partial_t \tilde{v} - \Delta_x \tilde{v} - \nabla_x \tilde{q} = -\partial_t (w(t) + \tilde{w}(t)), \quad \text{div} \, \tilde{v} = 0, \quad \tilde{v}|_{t=0} = -\tilde{w}(0).

Then, due to (3.12), it is enough to prove (3.10) for the function $\tilde{v}$ only. Finally, we want to get rid of the time derivative in the right-hand side of (3.13). To this end, we introduce the function $V(t)$ via

$$V(t) := \int_0^t e^{-\tau} \tilde{v}(s) \, ds.$$  

Then,

$$\partial_t V(t) + V(t) = \tilde{v}(t)$$

and integrating (3.13) in time, we arrive at

(3.14) \quad \partial_t V - \Delta_x V - \nabla_x Q = H(t), \quad \text{div} \, V = 0, \quad V|_{t=0} = 0,$n

where $H(t) := \tilde{w}(0) + w(0) e^{-t} - \tilde{w}(t) - w(t) + \int_0^t e^{-(t-s)} \tilde{v}(s) + w(s) \, ds$. Then, due to (3.12),

(3.15) \quad \|H\|_{C(0,T;W^{-1,2}_0)} \leq C \|h\|_{C(0,T;W^{-1,2}_0)},

where the constant $C$ is independent of $T$, and, to prove estimate (3.10), it is sufficient to verify that the solutions of the linear Stokes problem (3.14) satisfy the following estimate:

(3.16) \quad \|\partial_t V\|_{C(0,T;W^{1,2}_0)} + \|V\|_{C(0,T;W^{2+1/2,2}_0)} \leq C \|h\|_{C(0,T;W^{-1,2}_0)}

for some constant $C$ independent of $T$ and the concrete choice of the weight $\phi$.

**Step 2.** At this stage, we verify estimate (3.16) for the particular case $\phi = 1$ (the non-weighted case). Namely, assuming that $H \in C(0,T;W^{1,2}_0)$, we want to show that

(3.17) \quad \|\partial_t V\|_{C(0,T;W^{1,2}_0)} + \|V\|_{C(0,T;W^{2+1/2,2}_0)} + \|Q\|_{C(0,T;W^{1+1/2,2}_0)} \leq C \|H\|_{C(0,T;W^{1,2}_0)}

for some constant $C$ independent of $T$. Here and below $\hat{Q} := Q - SQ$.

To verify (3.17), we apply the Leray projector $\Pi$ to both sides of equation (3.14) which gives

(3.18) \quad \partial_t V + AV = \Pi H(t), \quad V|_{t=0} = 0

and, by the variation of constants formula, the solution $V$ can be written as follows:

$$V(t) = \int_0^t e^{-A(t-s)} \Pi H(s) \, ds.$$  

Then, since the Stokes operator $A$ generates an analytic semigroup (recall that it is self adjoint and positive definite in $H$), we have the estimate

$$\|e^{At} v\|_{D(A^{\alpha+\kappa})} \leq C t^{-\kappa} e^{-\alpha t} \|v\|_{D(A^\alpha)}$$

for all $\alpha \in \mathbb{R}$ and $\kappa > 0$, see e.g., [12]. Then, elementary estimates give

$$\|V\|_{C(0,T;D(A^{\alpha+1-\delta}))} \leq C_5 \|\Pi H\|_{C(0,T;D(A^\alpha))},$$

where $\alpha \in \mathbb{R}$, $\delta > 0$ and $C_5$ is independent of $T$. Fixing $\alpha = 1/5$, $\delta = 1/30$ and using the description of the fractional powers of the Stokes operator given in Section 2 as well as (2.23), we have

$$\|AV\|_{C(0,T;W^{1/2,2}_0)} \leq C_4 \|\Pi H\|_{C(0,T;D(A^{1/5}))} \leq C_1 \|H\|_{C(0,T;W^{2/5,2}_0)} \leq C_2 \|H\|_{C(0,T;W^{1,2}_0)}.$$  

Using the maximal regularity of the Stokes operator, see Proposition 2.6 and expressing $\partial_t V$ from equation (3.18), we get

$$\|\partial_t V\|_{C(0,T;W^{1/2,2}_0)} + \|V\|_{C(0,T;W^{2+1/2,2}_0)} \leq C \|H\|_{C(0,T;W^{1,2}_0)}.$$  

After that, from equation (3.14), we obtain the control of the $C(0,T;W^{1,2}_0)$-norm of $\nabla x Q$. The Poincare inequality gives then the desired control of $\hat{Q}$ and proves estimate (3.17).
Step 3. At this step, we deduce the weighted estimate (3.16) from the non-weighted one (3.17) and, thus, finish the proof of the desired estimate (3.10). To this end, we introduce the function $V_0 := \phi V$.

Then, due to assumptions (3.7), for sufficiently small $\varepsilon > 0$, we have

$$C_2 ||V_0||_{W^{r,2}} \leq ||V||_{W^{r,2}} \leq C_1 ||V_0||_{W^{r,2}}$$

for some $C_1$ and $C_2$ which are independent of $\varepsilon$ and $s \in (0, 3]$, see [22]. By this reason, to prove (3.16) it is sufficient to establish the analogous non-weighted estimates for function $V_0$. Multiplying equation (3.14) by $\phi$, after the elementary transformations, we have

$$\partial_t V_0 - \Delta_x V_0 - \phi \nabla_x Q_0 = H_\phi + \varepsilon (M(x)\partial_x V_0 + N(x)V_0), \quad \text{div} \, V_0 = \phi' V_1, \quad V_0|_{t=0} = 0,$$

where $M(x) := -2\varepsilon^{-1}\phi' \phi^{-1}$ and $N(x) := 2\varepsilon^{-1}(\phi' \phi^{-1})^2 - \varepsilon^{-1}\phi'' \phi^{-1}$. Note that, due to assumptions (3.7) on the weight $\phi$,

$$||M(x)||_{L^\infty} + ||N(x)||_{L^\infty} \leq C$$

uniformly with respect to $\varepsilon \to 0$, so the last term in the right-hand side of (3.20) is indeed a small perturbation. To transform the term with pressure, we introduce the function

$$Q_\phi := \phi Q - \int_0^{x_1} \phi'(s)Q(s, x_2) \, ds$$

Then, using that

$$\tilde{Q}_\phi := Q_\phi - SQ_\phi = \phi \tilde{Q}, \quad \phi \nabla_x Q = \nabla_x Q_\phi - \phi' \phi^{-1}(1, 0)^t \tilde{Q}_\phi,$$

we transform (3.20) to

$$\partial_t V_0 - \Delta_x V_0 - \nabla_x Q_\phi = H_\phi, \quad \text{div} \, V_0 = \phi' V_1, \quad V_0|_{t=0} = 0,$$

where $H_\phi := \phi H + \varepsilon (M(x)\partial_x V_0 + N(x)V_0 + \phi' \phi^{-1}(1, 0)^t \tilde{Q}_\phi$. Moreover, due to (3.7),

$$||H_\phi||_{W^{1,2}} \leq ||\phi H||_{W^{1,2}} + C\varepsilon (||V_0||_{W^{2,2}} + ||\tilde{Q}_\phi||_{W^{1,2}}),$$

where the constant $C$ is independent of $\varepsilon$. We want to apply estimate (3.17) to this equation, to this end, similar to Step 1, we need to get rid of the non-zero divergence by introducing the new variables

$$W_0 := (0, 1)^t \phi' \Psi(V), \quad \tilde{V}_0 := V_0 - W_0.$$

Then, it is not difficult to see

$$||W_0||_{W^{r,2}} \leq C\varepsilon ||V_0||_{W^{r-1,2}}, \quad s \in [1, 3]$$

and, therefore, for sufficiently small $\varepsilon > 0$,

$$C_2 ||\tilde{V}_0||_{W^{r,2}} \leq ||V_0||_{W^{r,2}} \leq C_1 ||\tilde{V}_0||_{W^{r,2}}, \quad C_2 ||\partial_t \tilde{V}_0||_{W^{r,2}} \leq ||\partial_t V_0||_{W^{r,2}} \leq C_1 ||\partial_t \tilde{V}_0||_{W^{r,2}},$$

where the constants $C_1$ and $C_2$ are independent of $\varepsilon$. On the other hand, function $\tilde{V}_0$ solves

$$\partial_t \tilde{V}_0 - \Delta_x \tilde{V}_0 - \nabla_x Q_\phi = H_\phi - \partial_t W_0 + \Delta_x W_\phi, \quad \text{div} \, \tilde{V}_0 = 0, \quad \tilde{V}_0|_{t=0} = 0,$$

and, applying estimate (3.17) to this equation, we get

$$||\partial_t V_0||_{C(0,T;W^{1,3/2})} + ||V_0||_{C(0,T;W^{2+1/3,2})} + ||\tilde{Q}_\phi||_{C(0,T;W^{1+1/3,2})} \leq$$

$$\leq C(||\partial_t \tilde{V}_0||_{C(0,T;W^{1,3/2})} + ||\tilde{V}_0||_{C(0,T;W^{2+1/3,2})} + ||\tilde{Q}_\phi||_{C(0,T;W^{1+1/3,2})}) \leq$$

$$\leq C||H_\phi - \partial_t W_0 + \Delta_x W_\phi||_{C(0,T;W^{1,2})} \leq C||\phi H||_{C(0,T;W^{1,2})} + C\varepsilon (||\partial_t \tilde{V}_0||_{C(0,T;L^2)} + ||V_0||_{C(0,T;W^{2,2})} + ||\tilde{Q}_\phi||_{C(0,T;W^{1,2})}).$$

Thus, for sufficiently small $\varepsilon > 0$,

$$||\partial_t V_0||_{C(0,T;W^{1,3/2})} + ||V_0||_{C(0,T;W^{2+1/3,2})} \leq C||\phi H||_{C(0,T;W^{1,2})}$$

which gives the estimate (3.16) which, in turn, gives the desired estimate (3.10).

Step 4. Existence and uniqueness. To construct a solution of (3.6), we approximate the function $h \in C(0, T; W^{1,2})$ by smooth functions $h_n$, which are convergent strongly to $h$ in that space. Let $v_n$ be the solutions of problems (3.30) where $h$ is replaced by $h_n$ (the existence and regularity of $v_n$ follows,
Corollary 3.4. Let \( v \in C(0, T; W^{1/3, 2}_\phi) \) and satisfies indeed estimate (3.10) with the weight \( \phi \). The uniform estimate for the solutions of (3.6) with the same \( C \) holds for almost all \( (0, t_0) \). Then, the function \( v(t) := \int_0^t (v_1(s) - v_2(s)) \, ds \) is a strong solution of the Stokes problem (2.21) with \( u_0 = g = 0 \). By Proposition 2.8, \( v(t) \equiv 0 \). Thus, the uniqueness is proved and Theorem 3.2 is also proved.

We now return to the auxiliary problem (3.3), where \((u_1, u_2)\) solves the non-autonomous Stokes problem (2.21) and \( \phi \) is a weight of exponential growth rate such that \( \phi \in L^2(\mathbb{R}) \) and satisfies (3.7). Then, there exists a unique energy solution \( \Psi(u(t)) \) which also satisfies (3.7), we see that

\[
\|v_\phi\|_{C(0, T; L^3_{\phi^{-1}})} \leq C \|u\|_{C(0, T; L^3_\phi)}.
\]

The following particular choice of the weight \( \phi \) is crucial for what follows.

Corollary 3.5. Let \( u \) be a weak solution of equation (2.21) (in the sense of Definition 3.1) and let \( \theta_\varepsilon(x) = \theta_\varepsilon, s \) be the weight defined via (2.28) where \( \varepsilon > 0 \) is small enough and \( s \in \mathbb{R} \) is a parameter. Then the solution \( v_\phi(t) \) of the auxiliary problem (3.3) with \( \phi = \theta_\varepsilon, s \) satisfies the following estimate:

\[
\|v_\phi\|_{C(0, T; L^3_{\phi^{-2}})} \leq C \varepsilon \|u\|_{C(0, T; L^3_{\phi})},
\]

where the constant \( C \) is independent of \( T, \varepsilon, s \) and \( u \).

Indeed, due to (2.6), we may improve estimate (3.27):

\[
\|v_\phi(t)\|_{W^{1, 2}_{\phi^{-1}}} \leq C \varepsilon \|\Psi(u(t))\|_{W^{1, 2}_\phi} \leq C \varepsilon \|u\|_{L^3_\phi}
\]

and, applying estimate (3.10) with the weight \( \phi = \theta^{-2} \) to equation (3.3), we end up with the desired estimate (3.27).

We now return to energy solutions of the non-autonomous Stokes equation.

Corollary 3.5. Let the weight exponential growth rate \( \phi \in L^{4/3}(\mathbb{R}) \) and satisfy (3.7) with sufficiently small \( \varepsilon > 0 \). Let also \( u_0 \in H_\phi \) and \( g \) satisfy (3.1). Then, there exists a unique energy solution \( u(t) \) of the Stokes problem and this solution satisfies the estimate

\[
\|u\|_{C(0, T; L^3_\phi)} \leq C_T \left( \|u_0\|_{H_\phi} + \|g\|_{L^{4/3}(0, T; L^{4/3}_\phi) \cap L^1(0, T; L^{4/2}_\phi)} \right),
\]

where the constant \( C_T \) may depend on \( T \), but is independent of the concrete choice of the weight. Moreover, the function \( t \to \frac{1}{2} \|u(t)\|_{L^2_\phi}^2 - (u(t), v_\phi(t)) \) is absolutely continuous and the energy identity (3.5) holds for almost all \( t \in (0, T) \).
Proof. We first derive estimate (3.28) assuming that the validity of (3.5) is already verified. Then, for sufficiently small \( \varepsilon > 0 \),

\[
\langle \nabla_x u, \nabla_x (\phi^2 u) \rangle = \| \nabla_x u \|_{L^2}^2 + 2 \| \nabla_x u, \phi\phi' u \| \geq 
\]

\[
\geq \| \nabla_x u \|_{L^2}^2 - C \varepsilon (\phi^2 \| \nabla_x u \|, |u|) \geq \frac{1}{2} \| \nabla_x u \|_{L^2}^2 - C \varepsilon^2 \| u \|_{L^2}^2 \geq \frac{1}{4} \| u \|_{W^{1,2}}^2,
\]

where we have implicitly used the weighted version of the Poincaré inequality.

Moreover, due to (3.25) and the weighted Ladyzhenskaya inequality

\[
\| u \|_{L^2}^2 \leq C \| u \|_{L^2}^2 \| u \|_{W^{1,2}},
\]

together with the Hölder inequality,

\[
[(g, \phi^2 u - v_\phi)] \leq C \| g \|_{L^{4/3}} \| u \|_{L^4}^4 + \| g \|_{L^{4/3}} \| v_\phi \|_{L_{-1}^3}^3 \leq C \| g \|_{L^{4/3}} \| u \|_{L^2}^{1/2} \| u \|_{W^{1,2}}^{1/2} + 
\]

\[
+ C \| g \|_{L^{4/3}} \| u \|_{C(0,T;L^2)}^{2/3} + C \| g \|_{L^2} \| u \|_{C(0,T;L^2)} + 1/8 \| u \|_{W^{1,2}}^2.
\]

Integrating now the energy identity (3.33) in time and using (3.31), (3.29) and the obvious estimate

\[
[(u, v_\phi)] \leq \| u \|_{L^2}^2 \| v_\phi \|_{L_{-1}^2} \leq C \varepsilon \| u \|_{C(0,T;L^2)}^2,
\]

we arrive at

\[
(1 - C \varepsilon) \| u \|_{C(0,T;L^2)}^2 + \| u \|_{L^2(0,T;W^{1,2})}^2 \leq C \| g \|_{L^{4/3}(0,T;L^{4/3})} \| u \|_{C(0,T;L^2)}^{2/3} + 
\]

\[
+ C \| g \|_{L^1(0,T;L^{3/2})} \| u \|_{C(0,T;L^2)} + \| u_0 \|_{L^2}^2
\]

and estimate (3.28) is an immediate corollary of this estimate if \( \varepsilon > 0 \) is small enough.

Note that the uniqueness of a solution \( u \) can be done exactly as in Theorem 3.2. To verify the existence, again similar to the proof of Theorem 3.2, we approximate the initial data \( u_0 \) by the sequence \( u^n_0 \in \mathcal{H}_\phi \) of smooth initial data which is convergent to \( u_0 \) in that space and, analogously, we approximate the external force \( g \) by the sequence \( g_n \) of smooth ones which is convergent in \( L^{4/3}(0,T;L^{4/3}) \cap L^1(0,T;L^{3/2}) \). We note that, since the weight \( \phi \in L^{4/3}(\mathbb{R}) \) then it is not difficult to check, \( \phi \in L^{3/2}(\mathbb{R}) \) as well and, thanks to (2.21)

\[
u_0 \in \mathcal{H}_\phi, \ g \in L^{4/3}(0,T;L^{4/3}) \cap L^1(0,T;L^{3/2})
\]

and, therefore, such approximations exist. Let \( u_n \) be the corresponding solutions of (2.21) which exist due to Proposition 2.8. Then, applying the proved estimate (3.28) to the differences \( u_n - u_m \) of two approximation solutions (since they are smooth, the energy identity hold for them), we have

\[
\| u_n - u_m \|_{C(0,T;L^2)} + \| u_n - u_m \|_{L^2(0,T;W^{1,2})} \leq C_T \left( \| u_0^n - u_0^m \|_{\mathcal{H}_\phi} + \| g_n - g_m \|_{L^{4/3}(0,T;L^{4/3}) \cap L^1(0,T;L^{3/2})} \right).
\]

Thus, \( u_n - u_m \) is a Cauchy sequence in \( C(0,T;L^2) \cap L^2(0,T;W^{1,2}) \) and, passing to the limit \( n \to \infty \), we construct a solution \( u \) of problem (2.21) belonging to this space and justify estimate (3.28). Moreover, applying this estimate with the shifted weights \( \phi(x_1 - s) \), taking the supremum over \( s \in \mathbb{R} \) and using (2.21), we check that \( u \) belongs to the uniformly local spaces (3.2). Thus, the existence of an energy solution is also verified.

It only remains to prove the energy identity. To this end, we write the energy identity for \( u_n \) in the equivalent integral form:

\[
\frac{1}{2} \| u_n(s) \|_{L^2}^2 - (u_n(s), v_\phi^n(s)) \geq \frac{1}{2} \| u_n(\tau) \|_{L^2}^2 + (u_n(\tau), v_\phi^n(\tau)) = 
\]

\[
= \int_\tau^s (g_n(t), \phi^2 u_n(t) - v_\phi^n(t)) - \langle \nabla_x u_n(t), \nabla_x (\phi^2 u_n(t)) \rangle dt,
\]
where \( v^n_\phi \) are the solutions of the auxiliary problem (3.3) which correspond to the solutions \( u_n \). Note that, due to estimate (3.26) applied to \( v^n_\phi - v^m_\phi \), we know that \( v^n_\phi \) converges strongly to \( v_\phi \) in the spaces \( C(0,T;L^2_{\phi^{-1}}) \) and \( C(0,T;L^3_{\phi^{-1}}) \). This allows us to pass to the limit \( n \to \infty \) in (3.3), and verify that the limit function \( u \) also satisfies this integral identity. Since the integral form (3.3) of the energy identity is equivalent to the differential form (3.5), the energy equality is proved and the corollary is also proved. 

\[ \square \]

Remark 3.6. Note that Theorem 3.2 does not give us the control over the \( L^2(0,T;W^{1,2}_{\phi^{-1}}) \)-norm of the corrector \( v_\phi \), so we are not allowed to multiply directly equation (2.21) by \( \phi^2 u - v_\phi \) (the term \((\Delta_x u, v_\phi)\) a priori may have no sense). By this reason, we have to justify this multiplication in a different way based on the approximations and the fact that all bad terms are cancelled out since \( v_\phi \) solves the adjoint equation.

Mention also that the validity of the energy identity (3.5) remains true if we replace the weight function \( \phi \) by the proper cut-off function \( \varphi \) with finite support (the proof just repeats word by word the one given in Corollary 3.5). We will use this observation in the next section for verifying the uniqueness for the non-linear problem.

4. The Navier-Stokes problem: Weighted energy estimates

In this section, we derive the key estimate for the infinite-energy solutions of the Navier-Stokes problem in a strip \( \Omega \):

\[
(4.1) \quad \begin{cases}
\partial_t u + (u, \nabla_x) u - \Delta_x u + \nabla_x p = g, \\
u|_{\partial \Omega} = 0, \quad \text{div } u = 0, \quad u|_{t=0} = u_0.
\end{cases}
\]

We recall that the problem possesses the mean flux first integral:

\[
(4.2) \quad Su_1 = \frac{1}{2} \int_{-1}^1 u_1(t,x_1,s) \, ds = c,
\]

where the constant \( c \) may depend on \( t \) (\( c = c(t) \)), but is independent of \( x_1 \). At the first step, we consider the case of zero flux:

\[
(4.3) \quad c = 0.
\]

The general case will be reduced later to this particular case. Then, similar to Definition 3.1 a function \( u(t) \) is a weak (energy) solution of problem (4.1) if \( u \) satisfies (3.2) for every weight function \( \phi \) of exponential growth rate such that \( \phi \in L^2(\mathbb{R}) \) and solves (1.1) in the sense of distributions. Note that, due to (3.2) and the Ladyzhenskaya inequality,

\[
\|(u, \nabla_x) u\|_{L^3_{\phi}((0,T) \times \Omega)} \leq \|u\|_{L^4((0,T) \times \Omega)} \|\nabla_x u\|_{L^2((0,T) \times \Omega)} \leq C \|u\|_{L^\infty((0,T) \times \Omega)} \|\nabla_x u\|_{L^2((0,T) \times \Omega)}^{3/2} \leq C.
\]

Moreover, due to the embedding \( W^{1,2} \subset L^p \) for all \( p < \infty \), we also have that

\[
\|(u, \nabla_x) u\|_{L^4_{\phi}((0,T) \times \Omega)} \leq C \|\nabla_x u\|_{L^2((0,T) \times \Omega)} \|u\|_{L^4_{\phi}((0,T) \times \Omega)} \leq C \|\nabla_x u\|_{L^2((0,T) \times \Omega)}^2 \leq C.
\]

Thus, the function \( \tilde{g} := g - (u, \nabla_x) u \) satisfies assumption (3.1) and, therefore, treating the non-linear term \( (u, \nabla_x) u \) in equation (4.1) as an external force and using Corollary 3.5 we see that the solution \( u \) of (4.1) satisfies the following energy identity:

\[
(4.4) \quad \frac{d}{dt} \left( \frac{1}{2} \|u(t)\|_{L^2_\phi}^2 - (u(t), v_\phi(t)) \right) + (\nabla_x u(t), \nabla_x (\theta^2 u(t))) = (g - (u(t), \nabla_x) u(t), \theta^2 u(t) - v_\phi(t)),
\]

where the weight function \( \theta = \theta_{\epsilon,s} \) is defined by (2.23) \( (\epsilon > 0 \text{ is small enough and } s \in \mathbb{R}) \) and the corrector \( v_\phi \) solves the auxiliary problem (3.3). The next theorem gives the key energy estimate for the solution \( u \).
Theorem 4.1. Let \( u(t), t \in [0, T] \) be a weak solution of the Navier-Stokes problem (4.1) which satisfies the zero flux condition (4.3). Then, the following estimate holds:

\[
\|u\|_{L^\infty(0,T;\mathbb{H}_0)} + \|\nabla_x u\|_{L^2((0,T) \times \Omega)} \leq C (1 + \|u_0\|_{\mathbb{H}_0} + \|g\|_{L^2_0}^2),
\]

where the constant \( C \) is independent of \( \varepsilon \).

Proof. To estimate the right-hand side of (4.4), we note that, due to the divergence free assumption,

\[
(u, \nabla_x u, \theta^2 u) = -\frac{1}{2} (\text{div} u, \theta^2 |u|^2) - (\theta \theta' u_1, |u|^2) = -(\theta \theta' u_1, |u|^2)
\]

and, therefore, due to (2.6) and the weighted Sobolev embedding theorem,

\[
|u| \leq C \|\nabla_x u\|^2_{L^2_0} \leq C \|u\|^2_{L^2_0} \|\nabla_x u\|^2_{L^2_0},
\]

where the constant \( C \) is independent of \( \varepsilon \). Analogously, using also estimate (3.27), we have

\[
|(u, \nabla_x u, \theta^2 u)| \leq C \|u\|_{L^6_0} \|\nabla_x u\|^2_{L^2_0} \|v_0\|_{L^2_0} \leq C \|u\|_{C(0,T;L^2_0)} \|\nabla_x u\|^2_{L^2_0}.
\]

Inserting these estimates into the right-hand side of (4.4), estimating the terms involving the external force \( g \) by Hölder inequality and using (3.29) and the Poincare inequality, we end up with

\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2_{L^2_0} - (u, v_0) \right) + \alpha \|u\|^2_{L^2_0} + \alpha \|\nabla_x u\|^2_{L^2_0} \left( 1 - C \|u\|_{C(0,T;L^2_0)} \right) \leq C \|g\|^2_{L^2_0} + C \|u\|^2_{C(0,T;L^2_0)},
\]

where the positive constants \( C \) and \( \alpha \) are independent of \( \varepsilon \).

We claim that (4.8) implies the desired estimate (4.5). To show that, we first make an additional assumption that the parameter \( \varepsilon > 0 \) is small enough to guarantee the inequality

\[
C \|u\|_{C(0,T;L^2_0)} \leq \frac{1}{2}.
\]

Then, the last term in the left hand side of (4.8) can be neglected and, applying the Gronwall inequality to the obtained estimate and using that

\[
|(u, v_0)| \leq C \|u\|_{L^6_0} \|v_0\|_{L^2_0} \leq C \|u\|^2_{C(0,T;L^2_0)},
\]

after the elementary transformations, we end up with

\[
\|u(t)\|^2_{L^2_0} \leq C \|u_0\|_{L^2_0} e^{-\alpha t} + C \|g\|^2_{L^2_0} + C \|u\|^2_{C(0,T;L^2_0)}, \quad t \in [0, T],
\]

where the constant \( C \) is independent of \( \varepsilon \). For sufficiently small \( \varepsilon \), this estimate gives

\[
\|u\|_{C(0,T;L^2_0)} \leq C \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right),
\]

where \( C \) is independent of \( \varepsilon \). Moreover, using (2.9), we conclude that

\[
\|u\|_{C(0,T;L^2_0)} \leq C_1 \varepsilon^{-1/2} \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right),
\]

where the constant \( C \) is independent of \( \varepsilon \). Now we are able to justify assumption (4.9). Indeed, if we fix \( \varepsilon \ll 1 \) in a such way that

\[
CC_1 \varepsilon^{1/2} \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right) = \frac{1}{2},
\]

i.e.

\[
\varepsilon : = C_2 \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right)^{-2},
\]

then inequality (4.13) will imply the inequality (4.9). Since (4.9) is satisfied for \( T = 0 \) and the function \( t \rightarrow \|u(t)\|_{L^2_0} \) is continuous (by the definition of an energy solution), the standard continuity arguments show that (4.9) holds for all \( T \). Thus, estimate (4.13) is justified if \( \varepsilon > 0 \) is chosen by (4.14). Integrating now (4.8) in time and using (4.9) and (4.13), we also get the control of the gradient:

\[
\|u\|^2_{C(0,T;L^2_0)} + \int_t^{t+1} \|\nabla_x u(s)\|^2_{L^2_0} ds \leq \leq C \varepsilon^{-1} \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right)^2 \leq C \left( 1 + \|g\|_{L^2_0} + \|u_0\|_{\mathbb{H}_0} \right)^4.
\]
Finally, taking into the account that the weight \( \theta = \theta_{\varepsilon,s} \) depends on the parameter \( s \in \mathbb{R} \) and that \((4.15)\) is uniform with respect to this parameter, we may deduce the desired estimate \((4.5)\) by taking the supremum over \( s \in \mathbb{R} \) and using \((2.8)\). Thus, the theorem is proved.

Our next task is to obtain the analogue of estimate \((4.5)\) for the general case of non-zero flux. For simplicity, we restrict ourselves to the autonomous case where \( c \in \mathbb{R} \) is independent of \( t \) although the generalization to the time dependent fluxes \( c = c(t) \) is straightforward. Following \cite{22}, we reduce the non-zero flux case to the case \( c = 0 \) considered before by introducing the special Poiseuille type velocity profile \( V_c(x) = (v_c(x_2),0)^t \) such that

\[
\text{S}v_c = c.
\]

Then, the difference \( \bar{u} := u - V_c \) will have zero flux and satisfy the perturbed version of equation \((4.1)\)

\[
\partial_t \bar{u} + (\bar{u}, \nabla_x)\bar{u} - \Delta_x \bar{u} + (V_c, \nabla_x)\bar{u} + (\bar{u}, \nabla_x) V_c + \nabla_x p = g + \Delta_x V_c, \quad \text{div} \, \bar{u} = 0, \quad \nabla \bar{u}_1 = 0.
\]

Then, by definition, \( u \) is an energy solution of \((4.1)\) if \( \bar{u} \) is an energy solution of \((4.17)\), see Definition 3.1.

The next lemma specifies the choice of the function \( V_c \).

**Lemma 4.2.** For any \( c \in \mathbb{R} \), there exists \( V_c(x) = (v_c(x_2),0)^t \in H^2(-1,1) \cap H^1(-1,1) \) such that \((4.16)\) is satisfied and, for big \( c \),

\[
\begin{align*}
1. \quad & \|v_c\|_{L^\infty} \sim c, \quad 2. \quad \|v_c\|_{L^2} \sim c^{3/2}, \quad 3. \quad \|v_c\|_{L^{3/2}} \sim c^{7/3}.
\end{align*}
\]

Moreover, the linearized operator \( L_c w := -\Delta_0 V_c + (V_c, \nabla_x) w + (w, \nabla_x) V_c \) is energy stable, i.e., there exists \( \kappa > 0 \) (independent of \( c \)) such that

\[
(L_c w, w) \geq \kappa \|w\|_{H^1}^2, \quad \forall w \in H^1_0(\Omega).
\]

Indeed, the function \( v_c(x_2) \) can be found in the form

\[
v_c(x) = \begin{cases} 
 a, & |x| \leq 1 - \delta, \\
 a(1 - \delta^{-2}(x - 1 + \delta)^2), & x > 1 - \delta, \\
 a(1 - \delta^{-2}(x + 1 - \delta)^2), & x < -1 + \delta,
\end{cases}
\]

where the two parameters \( a \sim c, \delta \sim c^{-1} \) are chosen in such way that \((4.16)\) is satisfied. Then the straightforward calculations show that the other assumptions of the lemma are also satisfied, see \cite{22} for more details.

The next theorem generalizes estimate \((4.5)\) for the case of the non-zero flux.

**Theorem 4.3.** Let \( u \) be an energy solution of equation \((4.1)\) with the mean flux \( c \). Then, the following estimate holds:

\[
\|u\|_{L^\infty(0,T;L^2_0)} + \|\nabla_x u\|_{L^2((0,T) \times \Omega)} \leq C(1 + c^{3/2} + \|u_0\|_{L^2} + \|g\|_{L^2_0})^2,
\]

where the constant \( C \) is independent of \( u_0, g, c, u \) and \( T \).

**Proof.** Applying the weighted energy identity to equation \((4.17)\) (where all terms \( (\bar{u}, \nabla_x)\bar{u}, (V_c, \nabla_x)\bar{u} \) and \( (\bar{u}, \nabla_x) V_c \) are treated as external forces), analogously to \((4.1)\), we have

\[
\frac{d}{dt} \left( \frac{1}{2} \|\bar{u}\|_{L^2_0}^2 - (\bar{u}, v_\theta) \right) + (L_c \bar{u}, \theta^2 \bar{u}) = (g + \Delta_x V_c - (\bar{u}, \nabla_x)\bar{u}, \theta^2 \bar{u} - v_\theta) + ((V_c, \nabla_x)\bar{u} + (\bar{u}, \nabla_x) V_c, v_\theta).
\]

Thus, we only need to estimate the extra terms appearing in this identity due to the presence of \( V_c \). To do that, we assume that \( \varepsilon > 0 \) is small enough to satisfy

\[
\varepsilon(1 + c^3) \leq \mu \ll 1.
\]

Then, the term involving the operator \( L_c \) can be estimated using \((4.15)\) and \((4.18)\):

\[
(L_c \bar{u}, \theta^2 \bar{u}) = (L_c(\theta \bar{u}), \theta \bar{u}) - (|\theta|^2, |\bar{u}|^2) - (v_c, \theta |\bar{u}|^2) \geq \kappa \|\bar{u}\|_{W^{1,2}_0}^2 - C\varepsilon^2 \|\bar{u}\|_{L^2_0}^2 - Cc\varepsilon \|\bar{u}\|_{L^2_0}^2 \geq \frac{\kappa}{2} \|\bar{u}\|_{W^{1,2}_0}^2.
\]
The last term on the right-hand side of (4.21) can be estimated using (1.18) and (3.24):
\begin{equation}
(4.24) \quad \frac{d}{dt} \left( \frac{1}{2} \| \bar{u} \|_{L^2}^2 - (\bar{u}, v_0) \right) + \alpha \| \bar{u} \|_{V^1}^2 + \alpha \| \bar{u} \|_{V^1}^2 \left( 1 - C \| \bar{u} \|_{C(0,T;L^2)} \right) \leq \\
\leq C \varepsilon \| \bar{u} \|_{W^1}^2 \left( \| \bar{u} \|_{C(0,T;L^2)} \right) \leq \\
\leq C \varepsilon\left( 1 + c^3 \right) \varepsilon \| \bar{u} \|_{C(0,T;L^2)} \leq \frac{K}{16} \| \bar{u} \|_{W^1}^2 + C \mu \| \bar{u} \|_{C(0,T;L^2)}^2.
\end{equation}

Finally, the terms involving $\Delta u_c$ can be estimated as follows:
\begin{equation}
(4.25) \quad \| (\Delta u_c, \theta^2 \bar{u}) \| = \| (\nabla u_c, \nabla (\theta^2 \bar{u})) \| \leq C \| \nabla u_c \|_{L^2} \| \nabla \bar{u} \|_{L^2} - C \| \nabla u_c \|_{L^2} \| \bar{u} \|_{L^2} \leq \\
\leq C \| \nabla u_c \|_{L^2}^2 + \frac{K}{16} \| \bar{u} \|_{W^1}^2 \leq C \varepsilon^{-1} (1 + c^3) + \frac{K}{16} \| \bar{u} \|_{W^1}^2.
\end{equation}

and
\begin{equation}
(4.26) \quad \| (\Delta u_c, v_0) \| \leq \| \Delta u_c \|_{L^2} \varepsilon \| v_0 \|_{L^2} \leq C \varepsilon^{-2/3} \| \Delta u_c \|_{L^2} \| \bar{u} \|_{C(0,T;L^2)} \leq \\
\leq C \varepsilon^{-1/2} C \varepsilon^{-1} (1 + c^3) \varepsilon^{-1} \| \bar{u} \|_{C(0,T;L^2)} \leq C \varepsilon^{-1} + \mu \| \bar{u} \|_{C(0,T;L^2)}^2.
\end{equation}

Inserting estimates (4.23), (4.24), (4.25) and (4.26) into the identity (4.21) and estimating the nonlinear term by (4.10) and (4.11), we derive the following analogue of (1.8):
\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} \| \bar{u} \|_{L^2}^2 - (\bar{u}, v_0) \right) + \alpha \| \bar{u} \|_{V^1}^2 + \alpha \| \bar{u} \|_{V^1}^2 \left( 1 - C \| \bar{u} \|_{C(0,T;L^2)} \right) \leq \\
\leq C_1 \left( \varepsilon^{-1} (1 + c^3) + \| u_0 \|_{L^2}^2 + \| g \|_{L^2}^2 \right) + C_1 \mu \| \bar{u} \|_{C(0,T;L^2)}^2,
\end{equation}

where the positive constants $C$, $C_1$, $\alpha$ and $\mu \ll 1$ are independent of $\varepsilon$ and $T$. The rest of the proof repeats word by word the end of the proof of Theorem 4.1. Indeed, under the extra assumption that
\begin{equation}
(4.28) \quad C \varepsilon \| \bar{u} \|_{C(0,T;L^2)} \leq \frac{1}{2},
\end{equation}
estimate (4.24) implies that
\begin{equation}
(4.29) \quad \| \bar{u} \|_{C(0,T;L^2)} \leq C_2 \varepsilon^{-1/2} \left( 1 + c^3/2 + \| u_0 \|_{L^2}^2 + \| g \|_{L^2}^2 \right),
\end{equation}
see the derivation of (4.13). Thus, if we fix
\begin{equation}
(4.30) \quad \varepsilon := \mu \left( 1 + c^3/2 + \| u_0 \|_{L^2}^2 + \| g \|_{L^2}^2 \right)^{-2},
\end{equation}
where $\mu > 0$ is small enough, then both assumptions (4.22) and (4.28) will be satisfied and, therefore, (4.29) is justified. Then, integrating (4.27) in time and using (4.29) together with (4.30), we end up with the analogue of (4.15):
\begin{equation}
(4.31) \quad \| \bar{u} \|_{C(0,T;L^2)}^2 \leq \int_t^{t+1} \| \nabla x \bar{u}(s) \|_{L^2}^2 ds \leq \\
\leq C \varepsilon^{-1} \left( 1 + c^3/2 + \| g \|_{L^2}^2 + \| u_0 \|_{L^2}^2 \right)^2 \leq C \left( 1 + c^3/2 + g \| L^2 \| + \| u_0 \|_{L^2}^2 \right)^4,
\end{equation}
which implies the desired estimate (4.20) and finishes the proof of the theorem. \qed

5. The Navier-Stokes problem: existence, uniqueness and regularity of solutions

In this section, we show the well-posedness of the Navier-Stokes problem (4.1) in the uniformly local spaces. We start with the uniqueness result.

**Theorem 5.1.** Let $u^{(1)}$ and $u^{(2)}$ be two energy solutions of problem (4.1) which satisfy (4.2) with the same constant $C$. Then, the following estimate holds:
\begin{equation}
(5.1) \quad \| u^{(1)} - u^{(2)} \|_{L^2(0,T;L^2)} + \| \nabla x u^{(1)} - \nabla x u^{(2)} \|_{L^2(0,T;\Omega)} \leq C_T \| u^{(1)}(0) - u^{(2)}(0) \|_{L^2}\alpha,
\end{equation}
where the constant $C_T$ depends only on $T$ and on the uniformly local energy norms of $u^{(1)}$ and $u^{(2)}$. \hfill \Box
Proof. Let \( u(t) = u^{(1)}(t) - u^{(2)}(t) \). Then, this function solves
\[
\partial_t u - \Delta u + \nabla \cdot \mathbf{p} + (u^{(2)}, \nabla u) + (u, \nabla u) u^{(1)} = 0, \quad \operatorname{div} u = 0, \quad u|_{t=0} = u^{(1)}(0) - u^{(2)}(0), \quad S u_1 = 0.
\]
Let us also introduce the cut-off function \( \psi \in C_0^\infty(\mathbb{R}) \) such that
\[
1. \quad \psi(x_1) = 1, \quad x_1 \in (0, 1), \quad 2. \quad \psi(x_1) = 0, \quad x_1 \notin (-1, 2)
\]
and let \( \psi_s(x) := \psi(x - s) \). Moreover, we assume that this cut off function depends on a small parameter \( \mu > 0 \) in such way that
\[
3. \quad (\psi^2)'' \leq \mu, \quad 4. \quad \|\psi\|_{L^\infty} + \|\psi\psi'\|_{L^\infty} \leq C,
\]
where \( C \) is independent of \( \mu \). It is not difficult to see that such a cut off function exists, however, the \( L^\infty \)-norm of its derivative must grow as \( \mu \to 0 \):
\[
\|\psi'\|_{L^\infty} \leq C_\mu.
\]
We write down the energy identity \[\eqref{energy_identity}\] with the weight \( \phi \) replaced by the cut off function \( \psi_s \), see Remark \[\eqref{remark}\]
\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2_{L^2} - (\psi, u) \right) + (\nabla u, \nabla (\psi^2 u)) = -((u^{(2)}, \nabla u) u + (u, \nabla u) u^{(1)}, \psi^2 u - v),
\]
where the corrector \( v \) solves \[\eqref{corrector}\] (where \( \phi \) is replaced by \( \psi_s \)) and, since, obviously,
\[
|\psi_s' (x) \psi_s(x)| \leq C \theta_{s,x}(x)^2
\]
uniformly with respect to \( s \), due to Theorem \[\eqref{corrector}\] the corrector \( v \) satisfies the analogue of \[\eqref{corrector}\] for
\[
\|v\|_{L^2(0,T;L^2(\Omega_{x,s}))} + \|v\|_{L^2(0,T;L^3(\Omega_{x,s}))} \leq C \|u\|_{C(0,T;L^2(\Omega_{x,s}))},
\]
where \( \varepsilon > 0 \) is small enough and the constant \( C \) is independent of \( s \in \mathbb{R} \). To simplify the notations, we will write below \( \theta \) and \( \psi \) instead of \( \theta_{x,s} \) and \( \psi_s \).

Our task now is to estimate every term in \[\eqref{energy_identity}\]. First, using that \( \psi_s \) is identically zero outside of \( (s-1,s+2) \) and denoting \( \Omega_{x-1,s+2} := (s-1,s+2) \times \Omega \), we have
\[
\|u, \nabla (\psi^2 u)\|_{L^2} \geq \|\nabla (\psi u)\|_{L^2}^2 - C \|\nabla (\psi u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 - C \|\psi u\|_{L^2}^2 \geq \frac{1}{2} \|\psi u\|_{L^2}^2 - C \|u\|_{L^2(\Omega_{x,s})}^2.
\]
Next, using the Ladyzhenskaya and Hölder inequality,
\[
\|u, \nabla u^{(1)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2 \leq C \|\nabla u^{(1)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2 \leq C \|\nabla u^{(1)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2 \leq \frac{1}{16} \|\nabla (\psi u)\|_{L^2}^2 + C \|\nabla u^{(1)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2.
\]
Integrating by parts and arguing analogously, we also have
\[
\|(u, \nabla u^{(1)}, \psi^2 u)\|_{L^2} \leq C \|(u^{(2)}, |\psi u| \cdot |\psi' u|)\|
\]
and, due to the Hölder inequality,
\[
C \|(u^{(2)}, |\psi u| \cdot |\psi' u|)\| \leq C \|u^{(2)}\|_{L^4(\Omega_{x,s})} \|\psi u\|_{L^4} \|\psi' u\|_{L^2} \leq 
\leq C \|u^{(2)}\|_{L^4(\Omega_{x,s})} \|\psi u\|_{L^4} \|\nabla (\psi u)\|_{L^2} \leq \frac{1}{16} \|\nabla (\psi u)\|_{L^2}^2 + 
+ C \|\nabla u^{(2)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2 + C \|u\|_{L^2(\Omega_{x,s})}^2,
\]
where we have implicitly used the Ladyzhenskaya inequality together with the fact that the uniformly local \( L^2 \)-norm of \( u^{(2)} \) is under the control due to the energy estimate. Combining the obtained estimates, we get
\[
\|(u^{(2)}, \nabla u) + (u, \nabla u^{(1)}), \psi^2 u\| \leq \frac{3}{16} \|\nabla (\psi u)\|_{L^2}^2 + 
+ C \|\nabla u^{(1)}\|_{L^2(\Omega_{x,s})} \|\psi u\|_{L^2}^2 + C \|u\|_{L^2(\Omega_{x,s})}^2.
\]
where the constant $C$ is independent of $s$ and $T$ and the constant $C_\mu$ depends only on the small parameter $\mu$ involved in the weight function $\psi$. It only remains to estimate the terms containing the corrector $v_\psi$. Using the weighted interpolation and Hölder inequalities together with estimate (5.14), we have

$$
(5.10) \quad |(u, \nabla_x u^{(1)}, v_\psi)| \leq C(\|u\| \cdot \|\nabla_x u^{(1)}\|, \theta^{-2}|v_\psi|) \leq \\
\|u\|_{L^2_\theta} \|\nabla_x u^{(1)}\|_{L^2_\theta} |v_\psi|_{L^2_{\theta-2}} \leq \mu \|\nabla_x u^{(1)}\|^2_{L^2_\theta} + C_\mu |u|^2_{L^2_\theta} \leq \\
\leq \mu \|\nabla_x u^{(1)}\|^2_{L^2_\theta} |v_\psi|_{L^2_{\theta-2}} + C_\mu |u|^2_{L^2_\theta}.
$$

$$
(5.11) \quad |(u^{(2)}, \nabla_x u, v_\psi)| \leq \|u^{(2)}\|_{L^2_\theta} \|\nabla_x u\|_{L^2_\theta} |v_\psi|_{L^2_{\theta-2}} \leq \mu \|\nabla_x u\|^2_{L^2_\theta} + \\
+ C_\mu \|u^{(2)}\|^{2/3}_{L^2_\theta} \|\nabla_x u^{(2)}\|^{4/3}_{L^2_\theta} \|u\|_{C(0,T;L^2_\theta)}^2 \leq \mu \|\nabla_x u\|^2_{L^2_\theta} + C_\mu \|u^{(2)}\|_{C(0,T;L^2_\theta)}^2 + C_\mu |u|^2_{C(0,T;L^2_\theta)}.
$$

Integrating now the energy identity (5.3) in time $t \in [0, T]$ and using (5.10), (5.11), (5.13) and (5.3), we end up with

$$
(5.12) \quad \frac{1}{2} \|\psi u(t)\|_{L^2_\theta}^2 - (u(T), v_\psi(T)) + \alpha \int_0^T \|\nabla_x u(t)\|_{L^2_{\theta+2}}^2 \, dt \leq \\
\leq C \int_0^T (\|\nabla_x u^{(1)}(t)\|_{L^2(\Omega_{1-t},s)}^2 + \|\nabla_x u^{(2)}(t)\|_{L^2(\Omega_{1-t},s+2)}^2) |\psi u(t)|_{L^2_\theta}^2 \, dt + \\
+ \mu \int_0^T \|\nabla_x u(t)\|^2_{L^2_\theta} \, dt + C_\mu \int_0^T \|u(t)\|^2_{L^2_\theta} \, dt + \\
+ C_\mu \int_0^T (\|\nabla_x u^{(1)}(t)\|_{L^2_\theta}^2 + \|\nabla_x u^{(2)}(t)\|_{L^2_\theta}^2) \|u\|_{C(0,T;L^2_\theta)}^2 \, dt + \\
+ C_\mu T \|u\|^2_{C(0,T;L^2_\theta)} + C_\mu |\psi u(0)|^2_{L^2_\theta}.
$$

Now, using that $v_\psi(t) = 2\psi'/0(0,1)^2\Psi_u(T)$, where $\Psi_u$ is a stream function of $u$ and our assumptions

$$
(5.13) \quad (u(T), v_\psi(T)) = (u_2(T), (\psi^2)'(u_2(T)) = - (\partial_2 \Psi_u(T), (\psi^2)'(u_2(T)) = \\
= ((\Psi_u(T))^2, (\psi^2)_w) \leq C_\mu \|\Psi_u(T)\|^2_{L^2(\Omega_{1-t},s+2)} \leq C_\mu \|u\|^2_{C(0,T;L^2_\theta)},
$$

where we have implicitly used the obvious inequality

$$
\|u\|_{L^2(\Omega_{1-t},s+2)} \leq C \|u\|^2_{L^2_\theta}.
$$

Using now the fact that the $L^2(0, T; L^2_\theta)$-norms of $\nabla_x u^{(1)}$ and $\nabla_x u^{(2)}$ are bounded for energy solutions, we see that, say, for $T \leq 1$,

$$
(5.14) \quad \|\psi u(T)\|^2_{L^2_\theta} + \alpha \int_0^T \|\nabla_x u(t)\|^2_{L^2(\Omega_{1-t},s)} \, dt \leq C \int_0^T (\|\nabla_x u^{(1)}(t)\|_{L^2_\theta}^2 + \|\nabla_x u^{(2)}(t)\|_{L^2_\theta}^2) |\psi u(t)|_{L^2_\theta}^2 \, dt + \\
+ \mu \int_0^T \|\nabla_x u(t)\|^2_{L^2_\theta} \, dt + C_\mu \int_0^T \|u(t)\|^2_{L^2_\theta} \, dt + (C_\mu + C_\mu T) \|u\|^2_{C(0,T;L^2_\theta)} + C_\mu |\psi u(0)|^2_{L^2_\theta}.
$$

We are now ready to apply the Gronwall inequality with respect to function $T \to \|\psi u(T)\|^2_{L^2_\theta}$. Then, using that the $L^2(0, T; L^2_\theta)$-norms of $\nabla_x u^{(1)}$ and $\nabla_x u^{(2)}$ are bounded, we arrive at

$$
(5.15) \quad \|\psi_\tau u(T)\|^2_{L^2_\theta} + \alpha \int_0^T \|\nabla_x u(t)\|^2_{L^2(\Omega_{1-t},s)} \, dt \leq C_\mu \int_0^T \|\nabla_x u(t)\|^2_{L^2_{\theta,s}} \, dt + \\
+ C_\mu \int_0^T \|u(t)\|^2_{L^2_{\theta,s}} \, dt + (C_\mu + C_\mu T) \|u\|^2_{C(0,T;L^2_{\theta,s})} + C_\mu |\psi_\tau u(0)|^2_{L^2_\theta}.
$$

Now, we multiply this inequality by $\theta_{\tau}(s)$, where $\tau \in \mathbb{R}$ is a parameter, and integrate over $s \in \mathbb{R}$. Then, using (2.7) together with the obvious inequality

$$
(5.16) \quad \int_{\mathbb{R}} \theta_{\tau}(s) \, ds \leq C_\varepsilon \theta_{\varepsilon}(x),
$$
Multiplying this inequality by \( \theta \) (5.20)
\[
\|u(T)\|_{L^2_{\theta,\tau}}^2 + C(\alpha - C\mu) \int_0^T \|\nabla_x u(t)\|_{L^2_{\theta,\tau}}^2 \, dt \leq
\]
\[
\leq C\mu \int_0^T \|u(t)\|_{L^2_{\theta,\tau}}^2 \, dt + (C\mu + C\mu T)\gamma(u, T) + C\mu \|u(0)\|_{L^2_{\theta,\tau}}^2,
\]
where \( \gamma(u, T) := \int_{s \in \mathbb{R}} \theta_{\epsilon, \tau}(s) \|u\|_{C(0,T;L^2_{\theta,\tau})}^2 \). Note that, in contrast to the other term, we cannot simplify \( \gamma(u, T) \) since we cannot change the order of integration and supremum. Assuming that \( \mu \) is small enough, we have
\[
\|u(T)\|_{L^2_{\theta,\tau}}^2 + \alpha_1 \int_0^T \|\nabla_x u(t)\|_{L^2_{\theta,\tau}}^2 \, dt \leq
\]
\[
\leq C\mu \int_0^T \|u(t)\|_{L^2_{\theta,\tau}}^2 \, dt + (C\mu + C\mu T)\gamma(u, T) + C\mu \|u(0)\|_{L^2_{\theta,\tau}}^2.
\]
We apply once more the Gronwall inequality, now with respect to function (5.18)
\[
\|u(T)\|_{L^2_{\theta,\tau}}^2 \leq (C\mu + C\mu T)\gamma(u, T) + C\mu \|u(0)\|_{L^2_{\theta,\tau}}^2.
\]
Multiplying this inequality by \( \theta_{\epsilon, \tau}(\tau) \) and using (5.16) again, we get
\[
\gamma_\tau(u, T) \leq (C\mu + C\mu T)\gamma_\tau(u, T) + C\mu \|u(0)\|_{L^2_{\theta,\tau}}^2.
\]
Finally, fixing \( \mu \) and \( T = T(\mu) \) being so small that \( C\mu + C\mu T < \frac{1}{2} \), we see that
\[
\gamma_\tau(u, T) \leq C\|u(0)\|_{L^2_{\theta,\tau}}^2.
\]
Inserting this estimate into the right-hand side of (5.19), we prove that
\[
\|u(t)\|_{C(0,T;L^2_{\theta,\tau})}^2 + \alpha_2 \int_0^T \|\nabla_x u(t)\|_{L^2_{\theta,\tau}}^2 \, dt \leq C\|u(0)\|_{L^2_{\theta,\tau}}^2
\]
holds for small \( T \). Iterating this estimate, we verify that it actually holds for all \( T \geq 0 \) (of course, with the constant \( C \) depending on \( T \)). Taking the supremum with respect to \( \tau \in \mathbb{R} \) from both sides of this inequality and using (2.23), we get the desired estimate (5.1) and finish the proof of the theorem.

**Remark 5.2.** Arguing in a similar way, one shows that if \( u(1) \) and \( u(2) \) are two solutions of the Navier-Stokes equation which correspond to different non-autonomous external forces \( g_1(t) \) and \( g_2(t) \), then the following analogue of estimate (5.21) holds for \( u = u(1) - u(2) \):
\[
\|u\|_{C(0,T;L^2_{\theta,\tau})}^2 + \alpha_2 \int_0^T \|\nabla_x u(t)\|_{L^2_{\theta,\tau}}^2 \, dt \leq C\|u(1)(0) - u(2)(0)\|_{L^2_{\theta,\tau}}^2 + \|g_1 - g_2\|_{L^2(0,T;L^2_{\theta,\tau})}^2,
\]
where the constant \( C \) depends only on the uniformly local energy norms of \( u(1) \) and \( u(2) \). Indeed, to derive this estimate, one only need to estimate the additional term \((g_1 - g_2, \psi^2 u - v_\psi)\) which is done in a standard way with the help of Cauchy Schwartz inequality and estimate (5.4).

We are now ready to prove the existence of a solution for the Navier-Stokes problem (1.1).

**Proposition 5.3.** For every \( c \in \mathbb{R} \) and initial data \( u_0 \in V_c + H_b \), there exist a unique global solution \( u(t) \) of problem (1.1) with the mean flux \( c \) and, therefore, the solution semigroup
\[
S(t) : V_c + H_b \rightarrow V_c + H_b, \quad S(t)u_0 \rightarrow u(t), \quad t \geq 0
\]
is well-defined.
where

\[ (5.28) \]

We approximate the initial data \( u_0 \in H_b \) and the external forces \( g \in [L^2_b(\Omega)]^2 \) by the sequences \( u_0^n \in H \) and \( g_n \in [L^2(\Omega)]^2 \) in such way that

\[ (5.29) \]

Obviously such approximations exist. We denote by \( u_n(t) \) the solutions of the Navier-Stokes problem \( (4.1) \) where the initial data \( u_0 \) and the external force \( g_n \) are replaced by \( u_0^n \) and \( g_n \) respectively. Since \( u_0^n \) and \( g_n \) are square integrable, \( u_n(t) \) is a usual energy solution of the Navier-Stokes problem and the existence of such solutions is well-known, see [19, 20] or [7]. Moreover, due to estimate \( (4.20) \) and the second assertion of \( (5.24) \), we have

\[ (5.30) \]

where \( C \) is independent of \( n \). Let us now fix some weight \( \theta_{\varepsilon, \tau} \) with sufficiently small \( \varepsilon > 0 \). Then, \( (5.31) \) implies that \( u_0^n \to u_0 \) and \( g_n \to g \) strongly in \( L^2_{\theta_{\varepsilon, \tau}} \), see e.g., [23] and consequently, estimate \( (5.32) \) applied to solutions \( u_n \) and \( u_m \) shows that \( u_n \) is a Cauchy sequence in \( C(0, T; L^2_b) \cap L^2(0, T; W^{1,2}_b) \).

Let \( u(t) \) be the limit of this sequence. Then, due to \( (5.33) \), \( u \in L^\infty(0, T; L^2_b) \) and \( \nabla_x u \in L^2(0, T \times \Omega) \). Moreover, passing to the limit \( n \to \infty \) in the equations for \( u_n(t) \), we see that \( u(t) \) solves the Navier-Stokes problem \( (4.1) \) (the passage to the limit in the nonlinear terms is straightforward since we have the strong convergence in \( C(0, T; L^2_b) \) and in \( L^2(0, T; W^{1,2}_b) \). Thus, the desired solution \( u(t) \) is constructed and the proposition is proved.

We conclude this section by stating the preliminary result on the higher regularity of solutions which will be improved in the next section.

**Proposition 5.4.** Let the assumptions of Theorem 4.3 hold and let, in addition, the initial data \( u_0 \in [W^{2,2}_b(\Omega)]^2 \). Then, the solution \( u(t) \in [W^{2,2}_b(\Omega)]^2 \) for all \( t \geq 0 \) and the following estimate holds:

\[ (5.34) \]

where the function \( Q_T \) depends on \( T \), but is independent of \( u_0 \) and \( g \).

**Proof.** We give only the formal derivation of estimate \( (5.34) \) which can be justified in a standard way (e.g., by approximating the solution \( u \) by the square integrable solutions which regularity is well known) and also consider for simplicity only the case \( c = 0 \). We differentiate equation \( (4.1) \) in time and denote \( v = \partial_t u \). Then this function solves the equation

\[ (5.35) \]

This equation has the form of \( (5.32) \), so repeating word by word the arguments given in the proof of Theorem 5.1 we derive that

\[ (5.36) \]

where \( \theta_{\varepsilon, \tau} \) is a weight function, \( \varepsilon \ll 1 \) and \( \tau \in \mathbb{R} \). Now, applying the Leray projector \( P \) to equation \( (4.1) \) we have

\[ (5.37) \]

Using the regularity of the Leray projector in \( L^2 \) and the embedding \( W^{2,2}_b \subset C_b \), we see that

\[ (5.38) \]

and taking the supremum with respect to \( \tau \in \mathbb{R} \) from both sides of \( (5.38) \), we have

\[ (5.39) \]

To derive the desired estimate \( (5.34) \) from \( (5.39) \), we need the following lemma which also has an independent interest.
Lemma 5.5. Let \( g \in [L^2_b(\Omega)]^2 \) and \( u \in V_b \) be the solution of the stationary Navier-Stokes problem
\[
\Delta_x u + (u, \nabla_x)u + \nabla_x p = g, \quad \text{div} u = 0, u|_{\partial \Omega} = 0.
\]
Then, \( u \in [W^{1,2}_b(\Omega)]^2 \) and the following estimate holds:
\[
\|u\|_{W^{2,2}_b} \leq Q(\|g\|_{L^2_b}) + Q(\|u\|_{L^2_b})
\]
for some monotone function which is independent of \( u \) and \( g \).

Proof of the lemma. Indeed, since the stationary solution \( u \) formally satisfies the non-autonomous equation \([4.1]\), estimate \([5.30]\) is formally applicable to it and gives that
\[
\|\nabla_x u\|_{L^2_b} \leq C(1 + \|g\|_{L^2_b}^2 + \|u\|_{L^2_b}^2)^{3/2}.
\]
Moreover,
\[
\|(u, \nabla_x)u\|_{L^2_b} \leq C\|u\|_{L^\infty} \|\nabla_x u\|_{L^2_b} \leq C\|\nabla_x u\|_{L^2_b}^{3/2}\|u\|_{L^{1/2}_b}^{1/2} \leq \mu\|u\|_{W^{2,2}_b} + C\|\nabla_x u\|_{L^2_b}^3,
\]
where \( \mu > 0 \) can be arbitrarily small. Using now the maximal regularity of the Stokes operator in \( L^2_b \) and interpreting the nonlinear term in \([5.31]\) as a part of the external forces, we derive the desired estimate \([5.32]\) and finish the proof of the lemma. \( \square \)

Now it is not difficult to finish the proof of the proposition. Indeed, interpreting the Navier-Stokes equation \([4.1]\) as stationary equation \([5.31]\) at every fixed point \( t \) with the external forces \( g(t) = g - \partial_t u(t) \) and using estimate \([5.32]\) together with \([4.5]\) and \([5.30]\), we end up with \([5.26]\) and finish the proof of the proposition. \( \square \)

6. Dissipativity and smoothing property

In this concluding section, we prove the dissipative estimate for the infinite-energy solutions of the Navier-Stokes problem \([1.1]\) as well as the so-called smoothing property. These properties are crucial, e.g., for the attractor theory, see e.g., \([15]\) and references therein. We start with the dissipativity. Next theorem refines the result of Theorem \([4.3]\).

Theorem 6.1. Let the assumptions of Theorem \([4.3]\) hold. Then, the energy solution \( u(t) \) satisfies the following estimate:
\[
\|u(t)\|_{L^2_b} \leq Q(\|u_0\|_{L^2_b})e^{-\alpha t} + C(1 + c^{3/2} + \|g\|_{L^2_b}^2),
\]
where the positive constants \( \alpha \) and \( C \) and the monotone function \( Q \) are independent of \( t, c \) and \( u_0 \).

Proof. The proof given below follows \([22], [10, 25]\) for the alternative method based on using the time dependent weights \( \theta(t,s) \). We start with estimate \([4.27]\) which is proved under assumption \([4.22]\) on the parameter \( \varepsilon \). Applying the Gronwall estimate to it assuming that \([4.28]\) holds, analogously to \([4.11]\), we end up with
\[
\|u(t)\|_{L^2_{b,s}}^2 \leq C_1\|u_0\|_{L^2_{b,s}}^2 e^{-\alpha t} + C_1\varepsilon^{-1}(1 + c^{3/2} + \|g\|_{L^2_b}^2)^2 + C_1\mu\|u\|_{C(0,T;L^2_{b,s})}^2,
\]
where the constants \( C_1, \alpha > 0 \) are independent of \( T \) and \( s \in \mathbb{R} \). Taking the supremum over \( t \in [0, T] \) from both sides of this inequality and utilizing that \( \mu \ll 1 \), we obtain the estimate for \( \|u\|_{C(0,T;L^2_{b,s})} \) and, inserting it into the right-hand side of the above derived inequality, we get
\[
\|u(t)\|_{L^2_{b,s}}^2 \leq C_1(\mu + \varepsilon^{-\alpha})\|u_0\|_{L^2_{b,s}}^2 + C_1\varepsilon^{-1}K, \quad K := (1 + c^{3/2} + \|g\|_{L^2_b}^2)^2.
\]
This estimate is the main technical tool to verify the dissipativity. We recall that, for its validity, the parameter \( \varepsilon \) should satisfy two inequalities: \([4.22]\) and \([4.28]\). Actually, we will use in the sequel only such \( \varepsilon \) which satisfy
\[
\varepsilon \leq \varepsilon_0 := \mu \left(1 + c^{3/2} + \|g\|_{L^2_b}^2\right)^2,
\]
where \( \mu \ll 1 \) is fixed, so \([4.22]\) is automatically satisfied. To verify \([4.28]\), it is sufficient to note that, since the right-hand side of \([5.2]\) is monotone decreasing in time, so \([4.28]\) will be formally satisfied if
\[
\varepsilon^2(C_1(\mu + 1)\|u_0\|_{L^2_{b,s}}^2 + C_1\varepsilon^{-1}K) \leq \frac{1}{4C^2}.
\]
and this can be justified using the standard continuity arguments. Finally, using (6.3) and the fact that \( \mu < 1 \), we see that the last inequality will be satisfied for all \( t \geq 0 \) if
\[
(6.4) \quad \varepsilon^2 \|u_0\|_{L^2_{\theta, s}}^2 \leq \frac{1}{8C^2C_1}.
\]
Thus, estimate (6.2) holds for all \( t \geq 0 \) if \( \varepsilon \) and \( u_0 \) satisfy inequalities (6.3) and (6.4).

At the next step, we get rid of the "non-dissipative" parameter \( \mu \) in the right-hand side of (6.2). To this end, we iterate it with a sufficiently large timestep \( T \):
\[
\|u(nT)\|_{L^2_{\theta, s}}^2 \leq C_1(\mu + e^{-\alpha T})\|u((n-1)T)\|_{L^2_{\theta, s}}^2 + C_1\varepsilon^{-1}K
\]
which after the summations of the geometric progressions gives
\[
\|u(nT)\|_{L^2_{\theta, s}}^2 \leq C_2e^{-\beta nT}\|u_0\|_{L^2_{\theta, s}}^2 + C_2\varepsilon^{-1}K
\]
for the appropriate positive constants \( \beta \) and \( C_2 \) which are independent of \( n \). Combining this estimate with (6.2) (in order to estimate \( u(t) \) for \( t \in ((n-1)T, nT) \)), we end up with the desired estimate:
\[
(6.5) \quad \|u(t)\|_{L^2_{\theta, s}}^2 \leq C_3e^{-\gamma t}\|u_0\|_{L^2_{\theta, s}}^2 + C_3\varepsilon^{-1}K
\]
with positive \( \gamma \) and \( C_3 \) which are independent of \( t \).

Estimate (6.3) looks similar to (6.1), but there is still an essential difference, namely, the parameter \( \varepsilon \) still depends on the initial value \( u_0 \) through condition (6.4) (in order to initialize the estimates, we need to take \( \varepsilon \) satisfying (1.3) or something similar). Thus, in order to finish the proof of the theorem, we need to show that, for large \( t \), the parameter \( \varepsilon \) can be increased and finally made independent of \( u_0 \).

Let \( \varepsilon = \varepsilon_1 \) be such that
\[
(6.6) \quad \mu(1 + \|u_0\|_{L^2_{\theta}} + e^{3/2} + \|g\|_{L^2_{\theta}})^{-2} \leq \varepsilon_1 \leq \varepsilon_0 = \mu K^{-1}
\]
and assumption (6.3) be satisfied. Then, according to (6.3), there exists \( T = T(\|u_0\|_{L^2_{\theta}}) \) such that
\[
(6.7) \quad \|u(t)\|_{L^2_{\theta, s}} \leq 2C_3\varepsilon_1^{-1}K, \quad t \geq T.
\]
Assume now that \( \varepsilon_1 \leq \varepsilon_0/2 \). Then, the new \( \varepsilon_2 = 2\varepsilon_1 \) satisfies
\[
\varepsilon_2^2 \|u(T)\|_{L^2_{\theta, s}}^2 \leq 4\varepsilon_1^2 \|u(T)\|_{L^2_{\theta, s}}^2 \leq 8C_3\varepsilon_1 K = 8C_3\mu\varepsilon_1 \varepsilon_0^{-1} \leq 4C_3\mu
\]
and, for sufficiently small \( \mu \), condition (6.4) is satisfied. Thus, we may apply estimate (6.5) starting from \( t = T \) and using \( \varepsilon = \varepsilon_2 = 2\varepsilon_1 \). Repeating this procedure finitely many times if necessary, we finally prove that, for every \( u_0 \), there exists \( T = T(\|u_0\|_{L^2_{\theta}}) \) such that
\[
(6.8) \quad \|u(t)\|_{L^2_{\theta, s}}^2 \leq 2C_3\varepsilon_1^{-1}K, \quad \varepsilon_0/2 \leq \varepsilon \leq \varepsilon_0, \quad t \geq T.
\]
The fact that \( T \) is also independent of \( s \in \mathbb{R} \) is guaranteed by the left inequality of (6.6). Taking the supremum over \( s \in \mathbb{R} \) from both sides of (6.8) and using (2.5) and the definition of \( \varepsilon_0 \), we see that
\[
\|u(t)\|_{L^2_{\theta}}^2 \leq C(1 + e^{3/2} + \|g\|_{L^2_{\theta}})^4, \quad t \geq T.
\]
This estimate, which establishes the existence of an absorbing ball for the solution semigroup \( S(t) \) in \([L^2_{\theta}(\Omega)]^2 \) is equivalent to the desired dissipative estimate (6.1). Thus, Theorem 6.1 is proved. \( \square \)

At the next step, we establish the uniformly local analogue of the standard \( L^2 \to H^1 \) smoothing property for the solutions of the Navier-Stokes problem.

**Theorem 6.2.** Let the assumptions of Theorem 4.3 hold and let \( u(t) \) be the energy solution of the Navier-Stokes problem (4.1). Then, the following estimate holds:
\[
(6.9) \quad t^{1/2}\|u(t)\|_{W^{1,2}} + \|t^{1/2}\Delta x u\|_{L^2(b(0,1)\times\Omega)} \leq Q(\|u_0\|_{L^2_{\theta}}), \quad t \in [0, 1]
\]
for some monotone function depending on \( c \), but independent of \( t \) and \( u_0 \). Moreover, if in addition \( u_0 \in [W^{1,2}_b(\Omega)]^2 \), then the following dissipative estimate holds:
\[
(6.10) \quad \|u(t)\|_{W^{1,2}} \leq Q(\|u_0\|_{W^{1,2}})e^{-\alpha t} + Q(\|g\|_{L^2_{\theta}}),
\]
where the positive constant $\alpha$ and the monotone function $Q$ are independent of $t$ and $u_0$.

**Proof.** The proof of this theorem is identical to the one given in [22]. Since, in contrast to many results given above, the auxiliary equation (6.10) is not used there, this proof is not affected by the mistake related with the auxiliary equation mentioned in the introduction and, therefore, remains correct. Nevertheless, to the convenience of the reader, we give a sketch of this proof below. To this end, we need the following lemma.

**Lemma 6.3.** Let $\psi_s$, $s \in \mathbb{R}$ be the cut-off functions introduced in the proof of Theorem 5.4 (with some fixed $\mu > 0$ which is not important here) and let also $u \in H_b \cap [W^{2,2}_0(\Omega)]^2$. Then, for sufficiently small $\varepsilon > 0$, the following estimate holds:

$$
\| \psi_s u \|_{W^{2,2}}^2 \leq C \| \psi_s \Delta u \|_{L^2}^2 + \delta \| u \|_{W^{2,2}_{\Theta_\varepsilon,s}}^2 + C_\delta \| u \|_{L^2_{\Theta_\varepsilon,s}}^2,
$$

where $\delta > 0$ can be arbitrarily small and the constants $C$ and $C_\delta$ are independent of $s$ and $u$. Moreover, for every $u \in [L^2_0(\Omega)]^2$, the following commutation relation holds:

$$
\| (\psi_s \circ \Pi - \Pi \circ \psi_s) u \|_{W^{1,2}_{\Theta_\varepsilon,s}} \leq C \| u \|_{L^2_{\Theta_\varepsilon,s}}^2,
$$

where the constant $C$ is also independent of $s$.

The proof of this lemma is given [22].

We will derive below estimates (6.9) and (6.10) only for the case $c = 0$ (the general case of non-zero flux can be reduced to it exactly as in Theorem 4.3). To this end, following [22], we multiply equation (6.11) by

$$
\nabla_x (\psi_s^2 \nabla_x u) = \psi_s^2 \Delta_x u + 2 \psi_s \psi_s' \partial_{x_1} u
$$

and integrate over $x \in \Omega$. This gives

$$
\frac{d}{dt} \| \nabla_x (\psi_s u) \|_{L^2}^2 + (\Pi \Delta_x \psi_s \partial_{x_1} u, \psi_s u) = (\Pi [(u, \nabla_x) u], \nabla_x (\psi_s^2 \nabla_x u)) - (\Pi g, \nabla_x (\psi_s^2 \nabla_x u)).
$$

We will derive below the estimates of the most complicated terms which contain $\psi_s^2 \Delta_x u$ in the right-hand side of (6.13). The remaining terms which contain $2 \psi_s \psi_s' \partial_{x_1} u$ are lower order and can be estimated in a similar but simpler way (we leave the details to the reader, see also [22]). First, denoting $Lu := (\psi_s^2 \circ \Pi - \Pi \circ \psi_s^2) \Delta_x u$ and integrating by parts, we get

$$
(\Pi \Delta_x \psi_s \partial_{x_1} u, \psi_s u) = \| \psi_s \Pi \Delta_x u \|_{L^2}^2 - (\Delta_x u, \Pi Lu) = \| \psi_s \Pi \Delta_x u \|_{L^2}^2 + (\nabla_x \psi_s, \nabla_x (\Pi Lu)) - (\partial_{n} u, \Pi Lu)_{\partial \Omega},
$$

where $(u, v)_{\partial \Omega} = \int_{\partial \Omega} u v \, dS$. Using the regularity of the Leray projector (see (2.17)), the weighted trace theorem $H^1_0(\Omega) \subset L^2(\partial \Omega)$ for $s > \frac{1}{2}$ (see e.g., [24] or [22]), estimate (6.12) and the weighted interpolation inequality, we get

$$
(\| \nabla_x (\psi_s u) \|_{L^2_{\Theta_\varepsilon,s}}^2 + C \| u \|_{W^{1,2}_{\Theta_\varepsilon,s}}^2 + C_\delta \| u \|_{L^2_{\Theta_\varepsilon,s}}^2) \leq \delta \| u \|_{W^{2,2}_{\Theta_\varepsilon,s}}^2 + C_\delta \| u \|_{L^2_{\Theta_\varepsilon,s}}^2.
$$

This estimate, together with (6.11) gives

$$
(\Pi \Delta_x \psi_s \partial_{x_1} u, \psi_s u) \geq \kappa \| u \|_{W^{2,2}_{\Theta_\varepsilon,s}}^2 - \delta \| u \|_{W^{2,2}_{\Theta_\varepsilon,s}}^2 - C_\delta \| u \|_{L^2_{\Theta_\varepsilon,s}}^2,
$$

where $\delta > 0$ can be arbitrarily small and the positive constants $\kappa$ and $C_\delta$ are independent of $s$.

Second, we transform the nonlinear term as follows:

$$
(\Pi [(u, \nabla_x) u], \psi_s \Pi \Delta_x u) = ((u, \nabla_x) (\psi_s u), \psi_s \Pi \Delta_x u) - ((u, \nabla_x) u, Lu) - (u, \psi_s' u, \psi_s \Pi \Delta_x u).
$$

Using estimate (6.11), the first term in the right-hand side can be estimated as follows:

$$
|((u, \nabla_x) (\psi_s u), \psi_s \Pi \Delta_x u)| \leq C |((u, \nabla_x) (\psi_s u))|_{L^2}^2 + \frac{\kappa}{4} \| \psi_s u \|_{W^{2,2}}^2 + \delta \| u \|_{W^{2,2}_{\Theta_\varepsilon,s}}^2 + C_\delta \| u \|_{L^2_{\Theta_\varepsilon,s}}^2
$$

and the first term in the right-hand side of this inequality is controlled by the Ladyzhenskaya inequality:

$$
C \| u \|_{L^2_{\Theta_\varepsilon,s}}^2 \leq C \| u \|_{L^1(\Omega_{-1,s+2})}^2 \| \nabla_x (\psi_s u) \|_{L^4}^2 \leq \leq C \| u \|_{L^2_{\Theta_\varepsilon,s}}^2 \| \nabla_x u \|_{L^2(\Omega_{-1,s+2})}^2 \| \nabla_x (\psi_s u) \|_{L^2}^2 + \frac{\kappa}{4} \| \psi_s u \|_{W^{2,2}}^2.
$$
The last two terms in the right-hand side of (6.15) are lower order and easier to estimate:

\[ |(u, \nabla_x)u, L\nu) + |(u_1^t u, \psi_1 \Delta_x) | \leq C|u|^2_{L^2_{b, \tau_o}} + \|u\|^2_{W^{2,2}_{b, \tau_o}} \leq C \|u\|^2_{L^2_{b, \tau_o}} + \delta \|u\|^2_{W^{2,2}_{b, \tau_o}} \]

and, finally, we end up with the following estimate of the nonlinear term:

\[
(6.16) \quad |(\Pi[u, \nabla_x)u, \psi_s^2 \Delta_x u)| \leq C\|u\|^2_{L^2_{b, \tau_o}} \|\nabla_x u\|^2_{L^2(\Omega_{-1, +1})} + \nabla_x (\psi_s u)\|^2_{L^2_{b, \tau_o}} + \frac{\kappa}{4}\|\psi_s u\|^2_{W^{2,2}_{b, \tau_o}} + \delta \|u\|^2_{W^{2,2}_{b, \tau_o}} (1 + \|u\|^4_{L^2_{b, \tau_o}}).
\]

Inserting the estimates (6.14) and (6.16) (together with the analogous estimates for the lower order terms) to (6.13), we arrive at

\[
(6.17) \quad \frac{d}{dt}\|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}} = C\|u\|^2_{L^2_{b, \tau_o}} \|\nabla_x u\|^2_{L^2(\Omega_{-1, +1})} \|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}} + \frac{\kappa}{4}\|\psi_s u(t)\|^2_{W^{2,2}_{b, \tau_o}} + \delta \|u(t)\|^2_{W^{2,2}_{b, \tau_o}} (1 + \|u(t)\|^4_{L^2_{b, \tau_o}}).
\]

Multiplying this estimate by \(t\) and using the interpolation inequality \(\|v\|_{H^1}^2 \leq C\|v\|_{L^2} \|v\|_{H^2}\), we have

\[
(6.18) \quad \frac{d}{dt}(t\|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}}) + \kappa t\|\psi_s u(t)\|^2_{W^{2,2}_{b, \tau_o}} \leq C\|u\|^2_{L^2_{b, \tau_o}} \|\nabla_x u\|^2_{L^2(\Omega_{-1, +1})} (t\|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}}) + \delta t\|u(t)\|^2_{W^{2,2}_{b, \tau_o}} + C\|u(t)\|^2_{W^{2,2}_{b, \tau_o}} (1 + \|u(t)\|^4_{L^2_{b, \tau_o}}).
\]

Applying the Gronwall inequality (with respect to the function \(t \rightarrow t\|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}}\) to this estimate and using that \(\|u(t)\|_{L^2_{b, \tau_o}}\) and \(\|\nabla_x u\|_{L^2((0,1) \times \Omega)}\) are controlled by the energy norm of the initial data (due to estimate (4.20)), we have

\[
(6.19) \quad t\|\nabla_x (\psi_s u(t))\|^2_{L^2_{b, \tau_o}} + \int_0^t t\|u(t)\|^2_{W^{2,2}_{b, \tau_o}} dt \leq \delta \int_0^t t\|u(t)\|^2_{W^{2,2}_{b, \tau_o}} dt + Q_\delta(\|u(0)\|_{L^2_{b, \tau_o}}), \quad t \in [0, 1],
\]

where \(\delta > 0\) is arbitrary and the monotone function \(Q_\delta\) is independent of \(t\) and \(u\). Multiplying this inequality by \(\theta_{\epsilon, r}(s), \tau \in \mathbb{R},\) integrating over \(s \in \mathbb{R}\) and using (2.7) and (5.10), we see that, for sufficiently small \(\delta > 0\),

\[
(6.20) \quad t\|\nabla_x u(t)\|^2_{L^2_{b, \tau_o}} + \int_0^t t\|u(t)\|^2_{W^{2,2}_{b, \tau_o}} dt \leq Q(\|u(0)\|_{L^2_{b, \tau_o}}), \quad t \in [0, 1]
\]

and, taking the supremum over \(\tau \in \mathbb{R}\) from both sides of this inequality, we prove the desired estimate (6.20).

To verify (6.10), we observe that if we know that \(u_0 \in [W^{1,2}_b(\Omega)]^2\), we need not to multiply (6.17) by \(t\) and may apply the Gronwall inequality directly to (6.17). Then, arguing as before, we end up with

\[
\|\nabla_x u(t)\|^2_{L^2_{b, \tau_o}} + \int_0^t \|u(t)\|^2_{W^{2,2}_{b, \tau_o}} dt \leq Q(\|u(0)\|_{L^2_{b, \tau_o}}), \quad t \in [0, 1]
\]

which proves (6.10) for \(t \in [0, 1]\). To verify it for \(t \geq 1\), it is sufficient to combine the smoothing estimate (6.10) of the form

\[
\|u(t)\|_{W^{1,2}_b} \leq Q(\|u(t - 1)\|_{L^2_{b, \tau_o}}), \quad t \geq 1
\]

with the dissipative estimate (4.20) for \(u(t - 1)\). Thus, estimate (6.10) is proved and the theorem is also proved.

We conclude the section by establishing the analogue of Theorem 6.2 for more regular space \(W^{2,2}_b(\Omega)\).

**Corollary 6.4.** Let the assumptions of Theorem 5.3 hold. Then, the energy solution \(u(t)\) belongs to the space \([W^{2,2}_b(\Omega)]^2\) and the following estimate holds:

\[
(6.21) \quad \|u(t)\|_{W^{2,2}_b} \leq Q(\|u(0)\|_{L^2_{b, \tau_o}}), \quad t \in [0, 1],
\]

where the monotone function \(Q\) is independent of \(u\) and \(t\). Moreover, if in addition \(u_0 \in [W^{2,2}_b(\Omega)]^2\), then the following dissipative estimate holds:

\[
(6.22) \quad \|u(t)\|_{W^{2,2}_b} \leq Q(\|u(0)\|_{W^{2,2}_b}) e^{-\alpha t} + Q(\|g\|_{L^2_{b, \tau_o}}),
\]
where the positive constant $\alpha$ and the monotone function $Q$ are independent of $t \geq 0$ and $u$.

**Proof.** Indeed, arguing exactly as at the end the proof of Theorem 5.2 we see that the dissipative estimate (6.22) is a formal corollary of the smoothing estimate (6.21) and estimates (4.20) and (5.26). Thus, we only need to prove the smoothing property (6.21). To this end, we differentiate equation (5.29) and denote $v(t) = \partial_t u(t)$. Then, estimate (6.23) (applied from the initial time moment $t = \tau$ instead of $t = 0$ and used on the time interval $t \in (\tau, T, T \leq 1)$ gives

$$
(6.23) \quad \|u(T)\|_{L^2_{\theta_t,s}}^2 + \int_\tau^T \|\nabla_x v(t)\|_{L^2_{\theta_t,s}}^2 \, dt \leq C\|\partial_t u(\tau)\|_{L^2_{\theta_t,s}}^2,
$$

where $C$ depends on the $L^2$-norm of the initial data $u_0$, but is independent of $\tau$, $T$ and $s$. Multiplying this estimate by $t$ and integrating over $t \in [0, T]$, we arrive at

$$
(6.24) \quad T^2\|v(T)\|_{L^2_{\theta_t,s}}^2 + \int_0^T t^2\|\nabla_x v(t)\|_{L^2_{\theta_t,s}}^2 \, dt \leq 2C \int_0^T t\|\partial_t u(t)\|_{L^2_{\theta_t,s}}^2 \, dt.
$$

To estimate the integral in the right-hand side, we use (5.29). This gives

$$
\int_0^1 t\|\partial_t u(t)\|_{L^2_{\theta_t,s}}^2 \leq C \int_0^1 t\|u(t)\|_{W^{2,2}_{\theta_t,s}}^2 + C \int_0^1 t\|u(t, \nabla_x)u(t)\|_{L^2_{\theta_t,s}}^2 \, dt.
$$

Using the Ladyzhenskaya inequality together with (2.27), the smoothing property (6.20) and the energy estimate (6.23), we get

$$
\int_0^1 t\|u(t, \nabla_x)u(t)\|_{L^2_{\theta_t,s}}^2 \, dt \leq C \int_0^1 \int_{s \in \Omega} \theta_s(y)\|u(t)\|_{L^2_{\theta_t,s}}^2 \|\nabla_x u(t)\|_{L^2_{\theta_t,s}}^2 \, dy \, dt \leq C_1\|u\|_{C(0,T;L^2_t)}^2 \int_{s \in \Omega} \theta_s(y)\|u(t)\|_{L^2_{\theta_t,s}}^2 \|\nabla_x u(t)\|_{L^2_{\theta_t,s}}^2 \, dy \, dt \leq C_2 \int_0^1 (t\|u(t)\|_{L^2_{\theta_t,s}}^2 + t\|u(t)\|_{L^2_{\theta_t,s}}^2) \, dt \leq Q(\|u_0\|_{L^2_t}).
$$

Thus, due to (6.24), we have the smoothing property for the time derivative $\partial_t u(t)$, namely,

$$
(6.25) \quad t\|\partial_t u(t)\|_{L^2_t} \leq Q(\|u_0\|_{L^2_t}), \quad t \in [0, 1]
$$

and we only need to derive from this estimate the desired estimate for the $W^{2,2}_b$-norm of $u(t)$. To this end, we consider equation (5.29) as a stationary Navier-Stokes problem (with the right-hand side $g - \partial_t u(t)$) for every fixed $t$ and use the maximal regularity of the Stokes operator in the uniformly local spaces as well as the Ladyzhenskaya inequality. This gives

$$
\int_0^1 t\|u(t)\|_{W^{2,2}_b}^2 \leq C\|g\|_{L^2_b}^2 + t\|\partial_t u(t)\|_{L^2_b}^2 + C\|u(t)\|_{L^2_b}^2 + C\|u(t)\|_{L^2_b}^2 \frac{1}{2} t\|u(t)\|_{W^{2,2}_b}^2 + C\|u(t)\|_{L^2_b}^2 \frac{1}{2} t\|u(t)\|_{W^{2,2}_b}^2 + C\|u(t)\|_{L^2_b}^2 + Q(\|g\|_{L^2_b}) \leq \frac{1}{2} t\|u(t)\|_{W^{2,2}_b}^2 + Q(\|u_0\|_{L^2_t}^2)
$$

This estimates give the desired control on $t\|u(t)\|_{W^{2,2}_b}$ and finishes the proof of the corollary. \qed

**Remark 6.5.** The standard bootstrapping arguments show that the actual regularity of a weak solution $u(t)$ is restricted by the regularity of the external forces $g$ and the initial data only and, for instance if $g \in C^\infty(\Omega)$, the solution $u(t) \in C^\infty(\Omega)$ for all $t > 0$.

Moreover, the obtained in this section results (dissipative and smoothing estimates) allow us to verify the existence of the so-called locally compact global attractor $A$ for that equation and derive the estimate for its size in $L^2_b(\Omega)$:

$$
\|A\|_{L^2_b} \leq C(1 + c^3 + \|g\|_{L^2_b}^2)
$$

which is exactly the same as in [22]. Thus, we see that despite the inaccuracy with estimating the solutions of the auxiliary problem, main results of [22] remain correct.
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