The general approach of weak-to-strong measurement transition for Fock state based pointers

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Recently, the measurement transition between von Neumann’s projective strong measurement and Aharonov’s weak measurement is investigated theoretically and experimentally in many studies. In this work, the general formulas related to the weak-to-strong measurement transition are presented for Fock state based pointers, and as an example the coherent state based pointer case is illustrated. The possible ways to realize this proposal using the trapped ion system are also discussed.

I. INTRODUCTION

Measurement as a fundamental issue in physics, plays very important role in our understanding around the world. In microscopic field, the measurement problems are promoting the development of quantum theory and its applications in other related sciences[1–8]. Whenever the quantum measurement problem is mentioned, it usually reminds us the projective measurement model which formulated by the von Neumann [9]. The Stern-Gerlach experiment illustrated in any standard quantum mechanics textbooks is a typical example of this kind of measurement [10]. Since in this strong measurement model the measurement strength between the pointer (measuring device) and measured system is strong enough, we can obtain the wanted information of the measured system after a single trial but the system is will be collapsed to the corresponding eigenstates of the system observable. However, there is another type of measurement model proposed in 1988 by Aharonov et al [11]. This new measurement model also follows the basic requirements of von Neumann measurement theory but is only valid in weak coupling regimes. Compared to the strong measurement model, in this weak measurement procedure, since the coupling strength is too weak, the initially prepared system state will not be destroyed after a single trial and we still can get the required information statistically after many repetition [12]. In this measurement model, the value of the measured system observable is occurred in the form of weak value caused by the pre- and post-selection processes. This weak value, in general, is a complex number and different from the eigenvalue or expectation value of the corresponding system observable. An other important feature of the this weak value is that it could exceed normal eigenvalue regimes of the observable. This feature of the weak value is usually considered as weak signal amplification effect and is found to be very helpful in the solution of many related problems in physics and related sciences [13–26]. For details, the reader is referred to [27, 28] and the references therein.

Furthermore, a recent study [29] showed that the nature of the weak value is different from the nature of the expectation value of the system observable, and the weak value describes the interaction in the same way as the eigenvalue does. As mentioned above, in general, the system observable has three kinds of values such as eigenvalue, (conditional) expectation value and weak value. The system information we want to get can be readout from the pointer after measurement which is related to the pointer’s shifts. In order to get these values, we rely on different types of measurement models. The eigenvalue and expectation value of the system observable are usually related to the position shift of the pointer after strong measurement. On the other hand, in weak measurement procedures [30], the real and imaginary parts of the weak value can be readout from the position and momentum shifts of the pointer, respectively. One of the interesting points in von Neumann measurement model is that the coupling strength between the measured system and pointer can take arbitrary value so that we can connect the weak and strong measurements by adjusting it. Recently, the weak-to-strong measurement transition problems are investigated experimentally [31] and theoretically [32, 33] for some specific pointer states. However, the general approach of weak-to-strong measurement transition applicable for arbitrary pointers has not been studied yet.

In this paper, we studied the general approach of weak-to-strong measurement transition problem for Fock state based pointers. We know that the Fock state (photon number state) $|n\rangle$ is the eigenstate of photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$, i.e., $\hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$. Thus, we can expand any quantum state $|\phi\rangle$ in the basis of $|n\rangle$ as $|\psi\rangle = \sum_n c_n |n\rangle$ with $c_n = \langle n | \psi \rangle$. We consider the internal and external degrees of freedom of the system as the measured system and pointer (measuring device), respectively, and give the general expressions of pointer shifts corresponding to the position and momentum operators after measurement. Since these expressions are derived without any approximation and valid in all coupling regimes, we can make transitions from Aharonov’s weak measurement to von Neumann’s projective strong measurement by modifying the coupling strength for all quantum state pointers. By using our general approach, the two extreme cases in previous studies can be recovered, in which the shifted values of the pointers are directly associated with the conditional expectation value (strong
measurement regime) and weak value (weak measurement regime), respectively. In particular, we firstly derive a very simple formula of position shift of the pointer in weak measurement regime and verify it by comparing with the prior studies. We also check our formulas for coherent state pointer, and discuss the potential feasibility of our general measurement transition model in trapped ion systems.

The rest of this paper is organized as follows. In Sec. II, we give a brief introduction to the von Neumann measurement considering the weak and strong interaction cases. In Sec. IV, we give the final pointer state of our weak-to-strong measurement transition model based on Fock state pointers. As a major result we present general expressions of position and momentum shifts of the Fock state pointers. As a major result we present general weak-to-strong measurement transition model based on coherent state based pointers, and realize the weak-to-strong measurement transition by modifying the coupling strengths between measured system and pointer. We also verify our new formula by comparing it with the previous studies. In Sec. IV, as an example of our general approach, we investigate the coherent state based weak-to-strong measurement transition. In Sec. V, we discuss the possible experimental implementation of our model in trapped ion systems. In Sec. VI, we provide a brief summary of this paper. Throughout this paper, we use the unit \( \hbar = 1 \).

II. BRIEF INTRODUCTION OF QUANTUM MEASUREMENT

The total Hamiltonian of the measurement problems consisted of three parts, i.e., \( H = H_s + H_p + H_{int} \). Here, \( H_s \) and \( H_p \) represent the Hamiltonians of the measured system and pointer (measuring device), respectively, and \( H_{int} \) is the interaction Hamiltonian between the system and pointer. Since, in quantum measurement, the read-out of the system information after measurement is directly related to coupling between system and pointer, we are only interested in the interaction Hamiltonian \( H_{int} \). According to the von Neumann measurement theory, the general form of the interaction Hamiltonian can be written as \([9]\)

\[
H_{int} = g(t) \hat{A} \otimes \hat{P}, \quad \int_{t_0}^{t} g(t)dt = g(t-t_0).
\]

Here, \( \hat{A} \) is the system observable we want to measure, and \( \hat{P} \) represents the conjugate canonical momentum operator to the position operator \( \hat{X} \) of the pointer, \( [\hat{X}, \hat{P}] = i \). The \( g(t) \) is a nonzero function in a finite interaction time interval, and \( g \) can characterize the coupling strength between the system and pointer. If we assume \( |a_i\rangle \) is the eigenstate of the observable \( \hat{A} \) with the eigenvalue \( a_i \), i.e., \( \hat{A}|a_i\rangle = a_i|a_i\rangle \), then the observable \( \hat{A} \) can be written as \([10]\)

\[
\hat{A} = \sum_i a_i|a_i\rangle\langle a_i|.
\]

In general, the observable \( \hat{A} \) has three kinds of values including eigenvalue, (conditional) expectation value and weak value. According to the measurement theory, these values can be obtained by a pointer’s shift after measurement. Since the eigenvalue is a special case of (conditional) expectation value and weak value of the observable, next we give a brief introduction about the readout procedures of (conditional) expectation value and weak value in the related measurements.

1. (Conditional) Expectation value. If we suppose that the initial state is \( |\Phi\rangle \) with the wave function \( \Phi(x) \), and measured system state is prepared in a superposition relative to \( \hat{A} \), i.e., \( |\psi_i\rangle = \sum_i \alpha_i|a_i\rangle \) with \( \sum_i |\alpha_i|^2 = 1 \), under the action of the unitary evolution operator \( e^{-ig\hat{A}\otimes\hat{P}} \), the total system state (unnormalized) evolves into

\[
|\Psi\rangle = e^{-ig\hat{A}\otimes\hat{P}}|\psi_i\rangle \otimes |\Phi(x)\rangle = \sum_i \alpha_i|a_i\rangle \otimes |\Phi(x- ga_i)\rangle.
\]

As we can see, the wave function of the pointer after strong measurement becomes as \( \Phi(x- ga_i) \) and its center is displaced in the amount of \( ga_i \). After some algebra, the final position shift of the pointer can be obtained as

\[
\delta x_{st} = \frac{\langle \Psi | \hat{X} | \Psi \rangle}{\langle \Psi | \Psi \rangle} - \langle \Phi | \hat{X} | \Phi \rangle = g \langle \hat{A} \rangle.
\]

This result can be explained very clearly if we express the initial state \( |\psi_i\rangle \) in terms of density matrix as

\[
\rho = |\psi_i\rangle\langle \psi_i| = \sum_i |\alpha_i|^2|a_i\rangle\langle a_i| + \sum_i \alpha_i \alpha_i^* |a_i\rangle \langle a_j|.
\]

It is clear that, \( |\alpha_i|^2 \) is the measuring probability of eigenvalue \( a_i \) of the observable \( \hat{A} \) corresponding to the eigenstate \( |a_i\rangle \) if the system is prepared in \( |\psi_i\rangle \). In order to obtain a determined outcome (i.e., an eigenvalue of the measured observable corresponding to a given eigenstate \( i \)), the second term (off-diagonal part which represented the coherence) of \( \rho \) has to vanish after the measurement. This means that, after the measurement the system is a mixture of the eigenstates of the measured observable, i.e., \( \sum_j P_j \rho P_j \). Here, \( P_j = |a_j\rangle \langle a_j| \) are the projectors on the different eigenstates of \( \hat{A} \), and the above mixture is as commonly known as projection postulate.

Furthermore, in time-symmetric quantum mechanics \([34–36]\), if we take a postselection with the state \( |\psi_f\rangle = |\psi_f\rangle \otimes |\Phi(x- ga_i)\rangle \) as

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\]
\[ \sum_i \beta_i |a_i\rangle \text{ onto } |\Psi\rangle, \text{ then the unnormalized final state of the total system becomes as} \]
\[ |\Psi'\rangle = \sum_i \alpha_i \beta_i^* |a_i\rangle \otimes |\Phi(x - ga_i)\rangle. \tag{8} \]

Here, \( \sum_i |\beta_i|^2 = \sum_i \langle a_i |\psi_f \rangle|^2 = 1. \) For this case, after the measurement the position shift of the pointer is proportional to the average value of the position operator
\[ \delta x_{st} = \frac{\langle \Psi' |\hat{X} |\Psi'\rangle}{\langle \Psi' |\Psi'\rangle} - \langle \Phi |\hat{X} |\Phi \rangle = g \langle A_c \rangle, \tag{9} \]
where \( \langle A_c \rangle \) is the conditional expectation value of the observable \( A \), and defined as \([34]\)
\[ \langle A_c \rangle = \frac{\sum_j a_j \langle \psi_f |a_j\rangle \langle a_j |\psi_i\rangle}{\sum_k \langle \psi_f |a_k\rangle \langle a_k |\psi_i\rangle} = \frac{\sum_i a_i |\alpha_i \beta_i^*|^2}{\sum_i |\alpha_i |\beta_i|^2}. \tag{10} \]

This value is also called the postselected strong value of \( A \). In the above processes we assume that the coupling between measured system and pointer is strong enough so that the sub-wavepackets corresponding to the different eigenvalues of the observable are distinguishable, i.e., \( g\Delta a \gg \sigma \). Here, \( \Delta a = a_i - a_{i-1} \) and \( \sigma \) represent the differences of neighboring eigenvalues and the width of the sub-wavepackets, respectively.

2. Weak value. Contrary to the above distinguishable condition of different wavepackets, if \( g\Delta a < \sigma \), i.e., system-pointer coupling is sufficiently weak, we can’t get the needed information after a single trial. In this case, we can only consider the expansion of the unitary operator up to the first order term, then the first line of Eq. (3) can be written as
\[ |\Psi'\rangle \approx (1 - ig\hat{A} \otimes \hat{P}) |\psi_i\rangle \otimes |\Phi(x)\rangle. \tag{11} \]

Just as the conditional strong measurement process, if we take a postselection with the state \( |\psi_f\rangle \), then the above total system state (unnormalized) becomes as
\[ |\Psi'\rangle \approx e^{-ig\langle A \rangle_w \hat{P}} |\psi_f\rangle \otimes |\Phi(x)\rangle \]
\[ \propto |\psi_f\rangle \otimes |\Phi(x - g Re \langle A \rangle_w)\rangle. \tag{12} \]

After the measurement, the position and momentum shifts of the pointer are given as [30]
\[ \delta x_w \propto g Re \langle A \rangle_w, \tag{13} \]
and
\[ \delta p_w = 2g Im \langle A \rangle_w Var(\hat{P}), \tag{14} \]
respectively, with \( Var(\hat{P}) = \langle \Phi |\hat{P}^2 |\Phi \rangle - \langle \Phi |\hat{P} |\Phi \rangle \) is the variance of the momentum operator \( \hat{P} \) under the initial pointer state, and \( \langle A \rangle_w \) is the weak value which is defined as
\[ \langle A \rangle_w = \langle \psi_i |\hat{A} |\psi_f\rangle = \frac{\sum_j a_j \beta_j^* \langle a_k |\hat{A} |a_j\rangle}{\sum_j \sum_k a_j \beta_k^* \langle a_k |a_j\rangle} = \frac{\sum_i a_i \alpha_i \beta_i^*}{\sum_i |\alpha_i |\beta_i|^2}. \tag{15} \]

It can be seen that, in general, the conditional expectation value \( \langle A \rangle_c \) and weak value \( \langle A \rangle_w \) are different (please see the Eq. (10) and Eq. (15) ), and thus can be characterized by different measurement strengths [37]. That is to say, the (conditional) expectation value of system observable is related to the (postselected) strong measurement, while the weak value is caused by the postselected weak measurement. In can be easily seen that if \( \beta_i = \alpha_i \), the above introduced conditional expectation value and weak value can be reduced to the expectation value given in Eq. (6). As shown above, these values can be given by the position shifts of the pointer after measurement.

### III. GENERAL APPROACH OF WEAK-TO-STONG MEASUREMENT TRANSITION

In recent studies, the weak-to-strong measurement transition is investigated theoretically and experimentally by using different pointers [31, 32]. However, the general approach for this issue has not been studied yet. In this section, we try to give a general method to this problem. The schematics of our measurement transition showed in Fig. 1. We know that the Fock state (photon number state) is the eigenvalue of photon number operator \( \hat{n} = \hat{a}^\dagger \hat{a} \), i.e., \( \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle \). One can expand any state in the basis of \( |n\rangle \) since the photon number operator is Hermitian. Thus, the any state \( |\phi\rangle \) into a series with respect to \( |n\rangle \) can be written as:
\[ |\phi\rangle = \sum_n c_n |n\rangle, \tag{16} \]
where the expansion coefficients \( c_n \) are determined by \( c_n = \langle n |\phi \rangle \). If we suppose that the initial states of the pointer and measured system are prepared in \( |\phi\rangle \) and \( |\psi_i\rangle \), respectively, then the total initial state can be written as \( |\phi\rangle \otimes |\psi_i\rangle \). The time evolution of the total system described by the interaction Hamiltonian given in Eq. (1) reads as
\[ |\Psi_{evol}\rangle = \exp \left( -i \int_0^t H(\tau) d\tau \right) |\psi_{ini}\rangle \otimes |\phi\rangle \]
\[ = \sum_i \alpha_i D \left( \Gamma \frac{1}{2} a_i \right) |a_i\rangle \otimes |\phi\rangle \]
\[ = \sum_n \sum_{n,i} \alpha_i c_n |a_i\rangle \otimes |\Gamma \frac{1}{2} a_i, n\rangle, \tag{17} \]
where
\[ D\left(\frac{\Gamma}{2}a_i\right) = \exp\left(\frac{g}{2\sigma}a_i(a^\dagger - a)\right), \quad \Gamma = \frac{g}{\sigma} \quad (18) \]
is the displacement operator, and
\[ \left| \frac{\Gamma}{2}a_i, n \right> = D\left(\frac{\Gamma}{2}a_i\right) |n\rangle \quad (19) \]
is called the displaced Fock state. In the derivation of Eq. (17), we write the momentum operator \( P \) of the pointer in terms of annihilation and creation operators \( \hat{a} \) and \( \hat{a}^\dagger \) as
\[ \hat{P} = \frac{i}{2\sigma} (\hat{a}^\dagger - \hat{a}) \quad (20) \]
The parameter \( \Gamma = g/\sigma \) characterizes the coupling strength between the measured system and pointer. If \( \Gamma \gg 1 \) (\( \Gamma \ll 1 \)), the measurement is in the strong (weak) measurement regime. After a postselection with the state \( |\psi_i\rangle \) onto Eq. (17), the normalized final state of the pointer reads as
\[ |\Phi\rangle = N \sum_i \sum_n \alpha_i \beta_i^* c_n \left| \frac{\Gamma}{2}a_i, n \right> \quad (21) \]
where \( N \) is the normalization coefficient given by
\[ N^{-2} = \sum_{i,j,n,m} \alpha_i \beta_j^* c_m \alpha_i^* \beta_j \alpha_i^* \beta_j^* L_n \left( \frac{\Gamma^2}{4} |a_i - a_j|^2 \right) \times \exp \left[ -\frac{\Gamma^2}{8} |a_i - a_j|^2 \right] \times \sum_i \sum_n \alpha_i \beta_i^* c_n \left| \frac{\Gamma}{2}a_i - a_j \right|^2 \times \exp \left[ -\frac{\Gamma^2}{8} |a_i - a_j|^2 \right], \quad (22) \]
where \( L_n(x) \) is the Laguerre polynomials. The Eq. (21) contains the information of the system observable \( A \) we want to measure, and holds for all Fock state based pointers. In the next subsection, we derive the general formulas of pointer shifts and discuss the weak-to-strong measurement transition.

**A. Position shift**

Here, we give the general expression of the position shift of the Fock state based pointers after postselected von Neumann measurement. The position operator \( \hat{X} \) of the pointer can be written in term of annihilation and creation operators \( \hat{a} \) and \( \hat{a}^\dagger \) as
\[ \hat{X} = \sigma (\hat{a}^\dagger + \hat{a}) \quad (23) \]

Since the interaction causes the shifts of the pointer, we can read the system information by comparing the changes of the apparatus’s scales before and after the measurement. For our scheme, the position shift of the pointer after the postselected von Neumann measurement reads as
\[ \delta x = \langle \Phi|\hat{X}|\Phi\rangle - \langle \phi|\hat{X}|\phi\rangle \]
\[ = |N|^2 \sigma \sum_{i,j,n,m} \alpha_i \beta_j^* c_m \alpha_i^* \beta_j \alpha_i^* \beta_j^* L_n \left( \frac{\Gamma^2}{4} |a_i - a_j|^2 \right) \times \exp \left[ -\frac{\Gamma^2}{8} |a_i - a_j|^2 \right] \times \sum_i \sum_n \alpha_i \beta_i^* c_n \left| \frac{\Gamma}{2}a_i - a_j \right|^2 \times \exp \left[ -\frac{\Gamma^2}{8} |a_i - a_j|^2 \right] \times \Gamma a_i \langle m| \left( \frac{\Gamma}{2} (a_i - a_j) \right)^n \rangle - \langle \phi|\hat{X}|\phi\rangle, \quad (24) \]
where
\[ \langle \phi|\hat{X}|\phi\rangle = 2\sigma \sum_n \sqrt{n+1} Re \left[ c_n c_{n+1}^* \right] \quad (25) \]
is the initial average value of the position operator under the initial pointer state \( |\phi\rangle \). The explicit forms of the matrix elements of displacement operator in the above formula can be given by the below corresponding expression
\[ \langle m|D\left(\frac{\Gamma}{2} (a_i - a_j)\right) |n\rangle \]
\[ = e^{-|\alpha|^2} \left\{ \begin{array}{ll} \sqrt{\frac{m!}{m^n} \frac{m-n}{m^n} \frac{|\alpha|^2}{m^n}} & , \quad m \geq n \\ \sqrt{\frac{m!}{m^n} \frac{m-n}{m^n} \frac{|\alpha|^2}{m^n}} & , \quad m \leq n \end{array} \right\} \quad (26) \]
Here, the generalized Laguerre polynomials are defined as
\[ L_n^{(n)}(x) = \sum_{k=0}^{n} \binom{n+\eta}{n-k} (-1)^k \frac{(-1)^k}{k!} x^k \quad (27) \]
The Eq. (24) is the general expression of position shift in Fock state based pointer measurement schemes, and it holds for whole measurement regimes.

If one wants to know the position shift of the pointer in the postselected strong measurement regime, then one can consider the larger value of the coupling strength parameter $\Gamma$. In order to do that, a limit $\Gamma \to \infty$ should be taken to Eq. (24) which gives

$$\frac{\delta x}{\sigma} = \lim_{\Gamma \to \infty} \delta x = \frac{\sum_{i} |\alpha_i\beta_i|^2}{\sum_{m} c_m^* n_m |\alpha_i\beta_i|^2} \times \left( \sqrt{n + 1/\Gamma} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1} + \Gamma a_i \delta_{m,n} \right)$$

$$- \langle \phi | \hat{X} | \phi \rangle = g \sum_{i} a_i |\alpha_i\beta_i|^2 = g \langle A \rangle_c.$$  

Here, $\delta_{m,n}$ is Kronecker delta function and its value is determined by

$$\delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Eq. (28) is the general position shift formula for the postselected strong measurement we introduced in Sec. II. On the other hand, if we are interested in the postselected weak measurement, the $\Gamma$ can be set infinitely small. For this extreme case, the position shift of our pointer reads as

$$\langle \delta x \rangle_w = \lim_{\Gamma \to 0} \delta x = g \text{Re} \left[ \langle A \rangle_w \right] + 2g \text{Im} \left[ \langle A \rangle_w \right] \times \left\{ \text{Im} \left[ \sum_{n} \sqrt{n + 1}(n + 2)c_n^* c_{n+2} \right] \right.$$ 

$$\left. - \text{Im} \left[ \sum_{n} \sqrt{n} c_n^* c_{n+1} \right] \right\} = g \text{Re} \left[ \langle A \rangle_w \right] + 2g \text{Im} \left[ \langle A \rangle_w \right] \text{Im} \left[ \langle a^2 \rangle - \langle a \rangle^2 \right].$$  

(30)

For this case the position shift, Eq. (30), is given as

$$\langle \delta x \rangle_{w,coh} = g \text{Re} \left[ \langle A \rangle_w \right].$$  

This result is presented in Ref. [38, 39].

(2) Coherent squeezed state. Suppose we choose the coherent squeezed state as a pointer, we also get the same result as Ref. [32] by using Eq. (30). The squeezed coherent state is defined as $|\phi\rangle = |\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle$ with

$$D(\alpha) = \exp \left( \alpha a^\dagger - \alpha^* a \right), S(\xi) = \exp \left( \frac{1}{2} \xi^2 a^2 - \frac{1}{2} a^2 \xi^2 \right).$$  

Here, $D(\alpha)$ and $S(\xi)$ are the displacement and squeezed operator, respectively, with $\alpha = |\alpha|e^{i\phi_\alpha}$ and $\xi = re^{i\phi_\xi}$. This state also can be written in terms of Fock state basis as [40]

$$|\alpha, \xi\rangle = \sum_{n} c_n |n\rangle,$$  

where

$$c_n = \frac{1}{\sqrt{\cosh r}} \exp \left[ -\frac{|\alpha|^2}{2} - \frac{1}{2} \alpha^* e^{i\phi_\xi} \tanh r \right] \times \sum_{n=0}^{\infty} \frac{[\frac{1}{2} e^{i\phi_\xi} \tanh r]^{n/2}}{\sqrt{n!}} H_n \left[ \gamma \left( e^{i\phi_\xi} \sinh(2r) \right)^{-\frac{1}{2}} \right]$$  

with $\gamma = \alpha \cosh r + \alpha^* e^{i\phi_\xi} \sinh r$, and $H_n(x)$ being the the order $n$ Hermite polynomial. The expectation value of $\langle a \rangle$ and $\langle a^2 \rangle$ under the state $|\alpha, \xi\rangle$ are given as

$$\langle a \rangle = \alpha, \quad \langle a^2 \rangle = \alpha^2 - e^{i\phi_\xi} \sinh r \cosh r.$$  

For this case, after the postselected weak measurement, the position shift of the pointer reads as

$$\langle \delta x \rangle_{w,sc} = g \text{Re} \left[ \langle A \rangle_w \right] - g \text{Im} \left[ \langle A \rangle_w \right] \sin 2r \sin \phi_\xi.$$  

This is the very result derived in Ref. [32] for squeezed coherent pointer.

(3) Single-photon-added coherent state(SPAC). The SPAC state is defined as [41]

$$|1, \alpha\rangle = \sum_{n} c_n |n + 1\rangle$$  

where

$$c_n = \frac{e^{-|\alpha|^2}}{\sqrt{1 + |\alpha|^2}} \sqrt{n + 1/\sqrt{n!}}.$$  

(39)

The $\langle a \rangle$ and $\langle a^2 \rangle$ can be calculated under the state $|1, \alpha\rangle$ as

$$\langle a \rangle = \frac{\alpha(2 + |\alpha|^2)}{1 + |\alpha|^2},$$  

(40)
and
\[
(a^2) = \frac{\alpha^2(3 + |a|^2)}{1 + |a|^2},
\]  
\( \text{(41)} \)

respectively. Thus, if we take the SPAC state as a pointer, its position shift after the postselected weak measurement is given as
\[
(\delta x)_{w,\text{spac}} = g \text{Re}[\langle A \rangle_w] - 2g|a|^2 \sin 2\phi_a \frac{|a|^2}{(1 + |a|^2)^2} \text{Im}[\langle A \rangle_w].
\]  
\( \text{(42)} \)

This is the exact same result as given in Ref. [33, 39]. From the above examples it can be confirmed that the Eq. (30) can be used for any pointer states.

**B. Momentum shift**

In this same way, the momentum shifts of the Fock state based pointers after postselected von Neumann measurement is given by

\[
(\delta p) = (\Phi|\hat{P}|\Phi) - \langle \phi|\hat{P}|\phi \rangle
\]
\[
= i 2 \Delta \sum_{i,j} n,m c^*_m c_n \alpha^*_j \beta^*_j \alpha^*_i \beta^*_i
\]
\[
\times \{ \sqrt{n} + 1 \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n + 1 \} - \sqrt{n} \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n - 1 \} - \langle \phi|\hat{P}|\phi \rangle.
\]  
\( \text{(43)} \)

Here,
\[
\langle \phi|\hat{P}|\phi \rangle = \frac{1}{\sigma} \text{Im} \left[ \sum_n \sqrt{n + 1} c^*_{n+1} c_n \right]
\]  
\( \text{(44)} \)

is the expectation value of the momentum operator under the initial pointer state |φ⟩. Since we didn’t use any approximation during the derivation of Eq. (43), it can represent the general momentum shift of the pointer in all measurement regimes. Going to two extremes, we can get momentum shifts corresponding to the strong and weak measurement, respectively: (i) For the very strong coupling (Γ ≫ 1) corresponding to the strong measurement regime, the momentum shift becomes as

\[
(\delta p)_s = \lim_{\Gamma \to \infty} \delta p
\]
\[
= 2\sigma \sum_{i,j} n,m c^*_m c_n \alpha^*_j \beta^*_j \alpha^*_i \beta^*_i
\]
\[
\times \{ \sqrt{n} + 1 \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n + 1 \} - \sqrt{n} \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n - 1 \} - \langle \phi|\hat{P}|\phi \rangle
\]
\[
= 2g \text{Im}[\langle A \rangle_w] (\langle \hat{P} \rangle^2 - \langle \hat{P} \rangle^2)
\]  
\( \text{(45)} \)

(ii) On the contrary, in the weak measurement regime (Γ ≪ 1), we get

\[
(\delta p)_w = \lim_{\Gamma \to 0} \delta p
\]
\[
= 2\sigma \sum_{i,j} n,m c^*_m c_n \alpha^*_j \beta^*_j \alpha^*_i \beta^*_i
\]
\[
\times \{ \sqrt{n} + 1 \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n + 1 \} - \sqrt{n} \langle m|D [\Gamma \frac{n}{2} (a_i - a_j)] |n - 1 \} - \langle \phi|\hat{P}|\phi \rangle
\]
\[
= 2g \text{Im}[\langle A \rangle_w] \text{Var}(P).
\]  
\( \text{(46)} \)

where Var(P) is the variance of the momentum operator \( \hat{P} \) under the initial pointer state |ϕ⟩. The final results of Eq. (45) and Eq. (46) are confirmed in many related studies [30, 32, 33], and holds for any pointer states.

**IV. EXAMPLE: COHERENT STATE POINTER CASE**

**A. Measurement transition**

As an example of the above general method to the weak-to-strong measurement for the Fock state based pointers, in this section we introduce the coherent state based weak-to-strong measurement transition. We assume that the interaction Hamiltonian between the system and pointer take the same form as Eq. (1), and our system observable \( \hat{A} \) is the Pauli-x operator, i.e., \( \hat{A} = \hat{\sigma}_x = |+\rangle\langle +| + |\rangle\langle -| \). Here, |±⟩ ≡ (|↑⟩ ± |↓⟩)/√2 are the eigenstates of \( \hat{\sigma}_x \) with eigenvalues ±1, respectively. If we assume that initially the system state is prepared in |ψinit⟩ = |↓⟩, and pointer is prepared in coherent state \( |\phi(z)\rangle = |β⟩ \) with \( |β⟩ = re^{iφ} \), then the initial total state of the system can be written as |↓⟩ ⊗ |β⟩. Under the unitary operator \( U = e^{-ig\sigma_y \otimes \hat{P}} \), the total system state given in Eq. (17) changed to
\[ |\Theta\rangle = \frac{1}{\sqrt{2}} \left[ |+\rangle D \left( \frac{\Gamma}{2} \right) |\beta\rangle - |-\rangle D^\dagger \left( \frac{\Gamma}{2} \right) |\beta\rangle \right], \]

with \( \Gamma = \frac{\pi}{\sigma} \). If we take a postselection with the state \( |\psi_f\rangle = \cos \theta |\uparrow\rangle - \sin \theta |\downarrow\rangle \), then the final state of the pointer which is presented in Eq. (21) is reduced to

\[ |\Phi\rangle = \sin \left( \frac{\pi}{4} - \theta \right) e^{-i\frac{\pi}{2} r \sin \varphi} |\beta + \frac{\Gamma}{2}\rangle - \cos \left( \frac{\pi}{4} - \theta \right) e^{i\frac{\pi}{2} r \sin \varphi} |\beta - \frac{\Gamma}{2}\rangle \sqrt{1 - \cos (2\theta) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \Gamma^2}}. \]

As we can see, this is a superposition state of two coherent states with different coherent amplitudes. In x-representation the state \( |\Phi\rangle \) can be expressed as

\[ \Phi(x) = \langle x | \Phi \rangle = \sin \left( \frac{\pi}{4} - \theta \right) e^{-i\frac{\pi}{2} r \sin \varphi} \psi_+ (x) - \cos \left( \frac{\pi}{4} - \theta \right) e^{i\frac{\pi}{2} r \sin \varphi} \psi_- (x) \sqrt{1 - \cos (2\theta) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \Gamma^2}}, \]

where

\[ \psi_{\pm} (x) = |x | x \pm \frac{\Gamma}{2} \rangle = \left( \frac{1}{2\pi \sigma^2} \right)^{\frac{1}{4}} e^{-|x |^2 \pm \frac{\Gamma}{4} \sigma^2} \times \exp \left( \frac{x^2}{4\sigma^2} \right) \exp \left[ - \frac{(x - \sigma (x | x \pm \frac{\Gamma}{2} \rangle))^2}{2 \sigma^2} \right]. \]

are the Gaussian wave packets after measurement with central position shifts \( \pm \frac{\Gamma}{2} \) compared to the initial Gaussian caused by the interaction with the system. For our pre- and postselected states \( |\downarrow\rangle \) and \( \cos \theta |\uparrow\rangle - \sin \theta |\downarrow\rangle \), the corresponding conditional expectation value and weak value defined in Eq. (10) and Eq. (15) can be written for the Pauli \( x \)-operator as

\[ \langle \sigma_x \rangle_c = -\sin (2\theta), \]

and

\[ \langle \sigma_x \rangle_w = -\cot (\theta), \]

respectively. As we show in the below, the conditional expectation value \( \langle \sigma_x \rangle_c \) and weak value \( \langle \sigma_x \rangle_w \) directly related with the pointer shifts of strong postselected measurement and postselected weak measurement, respectively.

\textit{Position shift.} We can check the weak-to-strong measurement transition by investigating the position and momentum shifts of the pointer after measurement. As mentioned in the Sec. II, in our scheme the the parameter \( \Gamma \) can characterize the measurement strength and it can takes all values through weak to strong regimes. The general expression of the position shift after postselected von Neumann measurement for Fock state based pointers, Eq. (24), can be reduced to the coherent state pointer case as

\[ \delta x_{coh} = \langle x | f_i - \langle x \rangle_{ini} \rangle = -\frac{g \sin (2\theta)}{1 - \cos (2\theta) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \Gamma^2}}. \]

It is straightforward from this expression that the measurement transition can be controlled by adjusting \( \Gamma \) in \( e^{-\frac{1}{2} \Gamma^2} \). To get the position shift of the coherent state pointer after the strong measurement, we can take a limit of Eq. (53) for \( \Gamma \) goes to infinity, i.e.,

\[ (\delta x)_{strong} = \lim_{\Gamma \to \infty} \delta x_{coh} = -g \sin (2\theta) = g \langle \sigma_x \rangle_c. \]

This result perfectly matches with the Eq. (28). At the other extreme limit, i.e., \( \Gamma \ll 1 \), the transition from strong to weak measurement regime could be made resulting in the pointer’s position shift as

\[ (\delta x)_{weak} = \lim_{\Gamma \to 0} \delta x_{coh} = -g \cot (\theta) = g \langle \sigma_x \rangle_w. \]

This result validates again the formula Eq. (30). In order to explore the weak-to-strong measurement transition characterized by the position shift of the coherent pointer state, the numerical simulations of the \( \delta x_{coh} \) as a function of postselected system state parameter \( \theta \) for various coupling strengths \( \Gamma \) are plotted in Fig. 2. As showed in Fig. 2 (a) and (b), the position shift curves changed from the weak measurement case to the strong one with the increasing of the coupling strengths \( \Gamma \), which fits very well to our theoretical predictions.

\textit{Momentum shift.} The momentum shift of the coherent pointer state also can be obtained by substituting related quantities onto the Eq. (43), and its explicit expression reads as

\[ \delta p_{coh} = \langle \Phi | \hat{P} | \Phi \rangle - \langle \beta | \hat{P} | \beta \rangle = -\frac{1}{2 \sigma} \frac{\Gamma}{1 - \cos (2\theta) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \Gamma^2}} \times \left[ 2i \text{Im}(\beta^*) - i \sin (2\Gamma r \sin \varphi) \cos (2\theta) e^{-\frac{1}{2} \Gamma^2} \right] - 2i \text{Im}(\beta^*) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \Gamma^2} \right] + \frac{1}{2 \sigma} 2i \text{Im}(\beta^*). \]

It is obvious from the above expression that, the factor \( e^{-\frac{1}{2} \Gamma^2} \) can be used to control the measurement regimes.
in the same way as with the position shift. Similarly, we can obtain

\[(\delta p)_{\text{weak}} = \lim_{\Gamma \to 0} \delta p_{\text{coh}} = 0\]  \hspace{1cm} (57)

and

\[(\delta p)_{\text{st}} = \lim_{\Gamma \to \infty} \delta p_{\text{coh}} = 0, \hspace{1cm} (58)\]

These results also support well our theoretical derivations given in Sec. II. We need to mention that for our current scheme the weak value is real, but in general theory the momentum shift of the pointer in postselected weak measurement proportional to the imaginary part of the weak value [see Eq.(46)]. Thus, for our current case the momentum shift of the pointer for weak measurement also zero. In Fig. 3, we showed the numerical results of the momentum shift [see Eq. (56)] as a function of parameter \(\theta\). As indicated in Fig. 3, the momentum shifts also fit well to our theoretical predictions.

In Ref. [31], the authors experimentally studied the weak-to-strong measurement transition problem in 

\[40 \text{Ca}^+ \text{ trapped ion system by considering the mean zero Gaussian as a pointer. The zero Gaussian profile corresponds to the ground state (}\ n = 0\text{) of the coherent pointer state. Thus, our discussion in this section can be seen as an extension of the theoretical part of Ref. [31].} \]

### B. The phase space function—– the Husimi-Kano Q function

In this subsection, to further investigate the weak-to-strong transition of coherent state pointer case, we check the phase space distribution—– Q function. In quantum mechanics, however, Heisenberg’s uncertainty principle prevents the notion of a system being characterized by a point in phase space. But, since the coherent states minimize the uncertainty relation for the two orthogonal quadrature operators and the uncertainties of the two quadratures are equal and these operators are dimensionless scaled position and momentum operators, we can write the quantum state’s phase space distribution by using the coherent state. There are three typical phase space quasi-probability distribution functions [40] such as Glaubier-Sudarshan P-function, the Husimi-Kano Q function and Wigner function. Among them, the Q function characterizes the expectation value of the density operator in a coherent state. As Milburn showed [42] the interference effect in phase space can be interpreted by considering the properties of Q function

\[Q_\psi(\alpha) = \frac{1}{\pi} |\langle \alpha | \psi \rangle|^2 \hspace{1cm} (59)\]

for the state \(|\psi\rangle\) under study. This expression also can be rewritten as

\[\langle \alpha | \psi \rangle = \sqrt{\pi Q_\psi(\alpha)} e^{i \phi_\psi(\alpha)}. \hspace{1cm} (60)\]
Here, if $|\alpha\rangle$ and $|\psi\rangle$ represent two different quantum states, then their inner product $\langle\alpha|\psi\rangle$ is related with the overlapping area of the states in the phase space, and $\pi Q_\psi(\alpha)$ and $\phi_\psi(\alpha)$ denote the overlapping area and its assigned phase, respectively. If we take $|\psi\rangle = |\Phi\rangle$, its $Q$ function can be calculated as

$$Q_\psi(\alpha) = \frac{1}{\pi} |\langle\alpha|\Phi\rangle|^2$$

$$= \frac{1}{\pi} \frac{1}{1 - \cos (2\theta) \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{4} \left|\alpha - \beta\right|^2}}$$

$$\times \left\{ \sin^2 \left( \frac{\pi}{4} - \theta \right) e^{-\left|\alpha - \beta - \frac{\Gamma}{2}\right|^2} + \cos^2 \left( \frac{\pi}{4} - \theta \right) e^{-\left|\alpha - \beta + \frac{\Gamma}{2}\right|^2} - \cos 2\theta \cos (2\Gamma r \sin \varphi) e^{-\frac{1}{2} \left|\alpha - \beta - \frac{\Gamma}{2}\right|^2} e^{-\frac{1}{2} \left|\alpha - \beta + \frac{\Gamma}{2}\right|^2} \right\}$$

$$= \frac{1}{\pi} e^{-|\alpha - \beta|^2}.$$  

(62)

The $Q$-function of our initial coherent pointer state $|\beta\rangle$ defined as

$$Q_{coh}(\alpha) = \frac{1}{\pi} |\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi} e^{-|\alpha - \beta|^2}.$$  

(63)

By comparing the Eqs. (61) and (62), we can see that the $Q_\psi(\alpha)$ contains three terms: the first and second terms represent the $Q$-functions of two different coherent states with amplitudes $|\beta + \frac{\Gamma}{2}|$ and $|\beta - \frac{\Gamma}{2}|$, respectively, and the third term describes their interference caused by overlapping. The extra $\pm \frac{\Gamma}{2}$ in the exponential function part of the first and second terms of Eq.(61) compared to the Eq. (62) are caused by the interaction between the measured system and the pointer. After measurement one wave-packet separated to two sub-wave-packets corresponding to the two eigenvalues of Pauli-$x$ operator.

We also noticed that in this expression the transition factor $e^{-\frac{1}{2} \Gamma^2}$ still exists. If we fixed the coherent state parameter $\beta$, with the increasing of the coupling parameter $\Gamma$, the overlapping areas gradually diminished and the associated third term became zero. To observe this measurement transition phenomena, in Fig. 4 we have plotted the $Q_\psi(\alpha)$ function in the phase space for different coupling strengths $\Gamma$. As showed in Fig.4 (a), the $Q$-function of the coherent state $|\beta\rangle$ is a Gaussian bell located at $\alpha_c = \sqrt{3}/2$ and $\alpha_i = 1/2$. This Gaussian bell is symmetric, and the below contour plot corresponding to the $1/e$ decay of the $Q_{coh}(\alpha)$-function. When the interaction strength is weak, there are two overlapping Gaussian wave-packets and thus there appears interference due to the third term in Eq.(61) [see the Fig. 4 (b)-(e)]. However, as Fig. 4 (f) indicated, if the coupling between the system and pointer is strong enough, the initially overlapping two sub-wave-packets are separated completely, and the measurement transition from weak to strong measurement regimes is completed. In this paper, we take $\beta = \beta_r + i\beta_i = \sqrt{3}/2 + 0.5i$, but we need to mention that the $Q$-function of the coherent state is always Gaussian independent of the parameters $\beta_r$ or $\beta_i$. By comparing the Fig.4(a) with Fig.4(e) and (f), we find that the radius of the circular contour plot is indeed independent of $\beta$. Since the coherent state is a displaced ground state of a harmonic oscillator, the fluctuations of the electric field operator in a coherent state are independent of the displacement $\beta$. Thus, its fluctuations are only determined by the properties of the harmonic oscillator and not by the parameters of the displacement.

V. POSSIBLE EXPERIMENTAL REALIZATION

Here, based on the previous theoretical and experimental works, we illustrate the possible implementation of our scheme in trapped ion system. As previous studies showed [8], the laser cooled trapped ion systems are very good platform in quantum state preparation and manipulation processes [43] since its unwanted dissipation time much longer than the typical times in which the experiment takes places. Furthermore, the motion of generated states in trapped ion systems can be completely determined in the sense of tomographic measurements [44, 45]. Up to date, variety of schemes have been proposed and experimentally observed for the preparation of various motional nonclassical states of a trapped ion [46–49], such as Fock [50], coherent [51–54], squeezed [43, 55, 56], Schrödinger cat state [2, 57–60], pair coherent [61], and pair cat [62] states. For the possible realization of our model, we consider an ion trapped in a harmonic potential and driven by two laser beams interacting resonantly with the system tuned to the lower (red) and upper (blue) vibrational sidebands, respectively. By taking the Lamb-Dicke regime [63] into account, in the interaction picture the total system Hamiltonian reads as

$$H = \eta \Omega (\hat{\sigma}_z \sin \phi_+ + \hat{\sigma}_y \cos \phi_+)$$

$$\otimes \left( \hat{\sigma} \sin \phi_+ \hat{P} - \frac{\hbar}{2\sigma} \cos \phi_+ \hat{X} \right),$$

(63)

where $\eta$ is the Lamb-Dicke parameter, $\Omega$ is the Rabi frequency, and $\phi_{\pm} = \frac{1}{2} (\phi_{red} \pm \phi_{blue})$ are the phases related to the lower and upper sideband laser phase $\phi_{red}$ and $\phi_{blue}$, respectively. Here, $\hat{X} = \hat{\sigma} (\hat{a} + \hat{a}^\dagger)$ and $\hat{P} = \frac{\hbar}{2\sigma} (\hat{a}^\dagger - \hat{a})$ are the position and momentum operators for the external vibration part of an ion, and $\sigma = \sqrt{\hbar/2m\nu}$ characterizes the size of the motional state that depends on the mass $m$ and vibrational frequency $\nu$ of the ion. Another important point is that here the ion is considered as a two level system. In the above Hamiltonian, the $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the Pauli-$x$ and -$y$ operators which can be written in terms of ion’s ground ($|\downarrow\rangle$) and
optically excited ($|↑⟩$) states as $\hat{\sigma}_x = |↑⟩⟨↓| + |↓⟩⟨↑|$ and $\hat{\sigma}_y = i(|↑⟩⟨↓| - |↓⟩⟨↑|)$, respectively. The Lamb-Dicke parameter $\eta$ is related to the wave vector $k$ and defined as $\eta = k/\sqrt{2\nu}m$, and in the derivation of the above Hamiltonian we assumed $\eta \ll 1$.

If we take the external and internal parts of the two level trapped ion as the pointer and measured system, respectively, Eq. (63) describes the typical von Nuemann type measurement by adjusting some related parameters. In general, the system observable $\hat{A}$ satisfies $\hat{A}^2 = \hat{A}$ or $\hat{A}^2 = \hat{I}$, and the $\hat{A}^2 = \hat{I}$ case is very usual in measurement problems. If we set $\phi_- = \frac{\pi}{2}$, $\phi_+ = 0$, or $\theta$, the above interaction Hamiltonian takes the form

$$H = g\hat{\sigma}_x \otimes \hat{P},$$

(64)

or

$$H = g\hat{\sigma}_y \otimes \hat{P},$$

(65)

respectively, with $g = \eta\sigma\Omega$. These are the typical von Nuemann type measurement Hamiltonians introduced in Sec. II. In recent studies [31, 32], they investigated the weak-to-strong measurement transition problems by taking zero mean Guassian and coherent squeezed states as pointers in $^{40}\text{Ca}^+$ trapped ion system. Since the state preparation with trapped ion system is mature in experiment and generated states possess high stability in long time, we anticipate that our present general approach of weak-to-strong measurement transition could be implemented by using Eq. (64) or Eq. (65) type measurement Hamiltons for Fock [65–68], SPACS [48] and Schodinger cat pointer states [69].

VI. CONCLUSION REMARKS

In conclusion, we gave the general expressions of the position and momentum shifts which holds for any Fock state based pointers in all allowed coupling strengths between the measured system and pointer. We firstly derived the general formula of the position shift holds for all pointers to the postselected weak measurements [see Eq. (30)], and verified it by comparing with the previous related studies. We also found that by adjusting the coupling strength parameter $\Gamma$, we can connect the weak value and conditional expectation value of system observable obtained by the shifts of the pointer in weak and strong measurement regimes, respectively. As a typical example of our general approach, we illustrated the weak-to-strong measurement transition for coherent pointer state, and this phenomena is also analyzed in phase space distribution by using the $Q$- function. Finally, we also discussed the possible experimental realization of weak-to-strong measurement transition for Fock state based pointers in trapped ion system.

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