RECENT MATHEMATICAL DEVELOPMENTS IN
QUANTUM GENERAL RELATIVITY

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ABSTRACT

After a brief chronological sketch of developments in non-perturbative canonical quantum gravity, some of the recent mathematical results are reviewed. These include: i) an explicit construction of the quantum counterpart of Wheeler’s super-space; ii) a rigorous procedure leading to the general solution of the diffeomorphism constraint in quantum geometrodynamics as well as connection dynamics; and, iii) a scheme to incorporate the reality conditions in quantum connection dynamics. Furthermore, there is a new language to formulate the central questions and techniques to answer them. These developments put the program on a sounder footing and, in particular, address certain concerns and reservations about consistency of the overall scheme.

1. Introduction

It is well known that quantum general relativity is perturbatively non-renormalizable. Particle theorists often take this to be a sufficient reason to abandon general relativity and seek an alternative which has a better ultraviolet behavior in perturbation theory. However, one is by no means forced to this route. For, there do exist a number of field theories which are perturbatively non-renormalizable but are exactly soluble. An outstanding example is the Gross-Neveu model in 3 dimensions, \((GN)_3\), which was recently shown\(^1\) to be exactly soluble rigorously. Furthermore, the model does not exhibit any mathematical pathologies. For example, it was at first conjectured that the Wightman functions of a non-renormalizable theory would have a worse mathematical behavior. The solution to \((GN)_3\) showed that this is not the case; as in familiar renormalizable theories, they are tempered distributions. Thus, one can argue that, from a structural viewpoint, perturbative renormalizability is a luxury even in Minkowskian quantum field theories. It does serve as a powerful guiding principle for selecting physically interesting theories since it ensures that the predictions of the theory at a certain length scale are independent of the potential complications at much smaller scales. But it is \textit{not} a consistency check on the mathematical viability of a theory. Furthermore, in quantum gravity, one is interested precisely in
the physics of the Planck scale. The short-distance complications are now the issues of primary interest. Therefore, it seems inappropriate to elevate perturbative renormalizability to a viability criterion.

Even if one accepts this premise, however, one is led to ask: Are there reasons to expect that quantum general relativity may exist at a non-perturbative level? The answer, I believe, is in the affirmative. There are growing indications from a number of different directions—computer simulations, canonical quantization and string theory—that the quantum geometry of space-time would be quite different from classical geometry. Indeed, there are concrete calculations that hint at a discrete structure at the Planck scale. The perturbative treatments, on the other hand, assume the validity of a continuum picture at all scales. The ultraviolet problems one encounters may simply be a consequence of the fact that the true microscopic structure of space-time is captured so poorly in these treatments. Put differently, if the continuum picture is replaced by a more faithful one, the “effective dimension” of space-time could be smaller than four, whence the theory could have a much better behavior non-perturbatively.

In this article, I will accept this premise and consider a non-perturbative quantization of general relativity. This program is based on canonical quantization. More precisely, one casts general relativity in a Hamiltonian form, realizes that it is a dynamically constrained system and uses (an extension of) Dirac’s method for quantization of such systems. It is true that such canonical methods require a 3+1 splitting of space-time and therefore lack manifest covariance. Nonetheless, when applied to diffeomorphism invariant theories such as general relativity, they also have a number of advantages. First, they do not require the use of a background metric or a connection or indeed any background field. In this sense, they respect the diffeomorphism invariance of the theory in an essential way. Second, it seems extremely hard to give a mathematical meaning to other methods such as path integrals in general relativity. The difficulties are not merely the “technical” ones associated with functional analysis. They are also conceptual, having to do with the celebrated “issue of time” in quantum gravity. In particular, while one can extend the key axioms of Osterwalder and Schrader for Euclidean quantum field theory to diffeomorphism invariant theories, because of the absence of a sensible Wick rotation procedure, the transition from Schwinger functions to Wightman ones seems impossible to achieve. Thus, in the

† If one considers general relativity with two Killing fields, one encounters a rather dramatic example of a non-perturbative effect: while the perturbative Hamiltonian is, as one would expect, unbounded above, the exact Hamiltonian is a non-polynomial function of the perturbative Hamiltonian and is in fact bounded above. One would therefore expect that this field theory would be free of the usual ultraviolet difficulties.
Euclidean approach, it seems difficult to extract the physical content of the full quantum theory. Finally, returning to canonical quantization, there are strong indications that, in the final picture, all reference to a continuum space-time would be lost, whence the notion of covariance may not be meaningful at a fundamental level. The final picture may be more combinatorial than geometric. The issue of covariance would be meaningful only in the classical limit where it can be recovered through the Hamiltonian description. The fact that this recovery is not as direct as one would like is then largely an aesthetic question. Furthermore, it is not unreasonable to hope that there may well exist a better way of taking the classical limit – e.g. by exploiting the equivalence\(^7\) between the traditional and covariant Hamiltonian formulations of general relativity – where covariance is manifest.

The approach I will discuss is being pursued by a large number of researchers in about a dozen different research groups. I will not attempt to present a comprehensive or even a systematic survey. Rather, I will focus only on some of the recent mathematical developments and show how these address certain concerns about the viability of this specific program. Since there already exist a number of excellent reviews on the subject, I will only sketch the main ideas and provide references where further details can be found. Finally, readers who are familiar with the program can skip section 2 and proceed directly to sections 3 and 4 where the recent results are discussed.

2. Program

The program I want to discuss here has three conceptual ingredients: i) A reformulation of Hamiltonian general relativity as a dynamical theory of connections, which has close structural similarities with gauge theories\(^8\); ii) A quantization of theories of connections based on loops\(^9,10\); and, iii) An algebraic extension of Dirac’s treatment of constrained systems\(^11,12\). In this section, I will briefly summarize the development of the program, roughly in the chronological order. (For further details, see, e.g., Ref. [13-16].)

The canonical formulation of general relativity was first achieved in the late fifties and early sixites through a series of papers by Bergmann, Dirac and Arnowitt, Deser and Misner (ADM). In this formulation, general relativity arises as a dynamical theory of 3-metrics. The framework was therefore named geometrodynamics by Wheeler and used as a basis for canonical quantization both by him and his associates and by Bergmann and his collaborators. The framework of geometrodynamics has the advantage that classical relativists have a great deal of geometrical intuition and physical insight into the nature of the basic variables – 3-metrics \(q_{ab}\) and extrinsic curvatures \(K_{ab}\).
For these reasons, the framework has played a dominant role, e.g., in numerical relativity. Unfortunately, it also has two important drawbacks. First, it sets the mathematical treatment of general relativity quite far from that of theories of other interactions where the basic dynamical variables are connections rather than metrics. Second, the equations of the theory are rather complicated in terms of metrics and extrinsic curvatures; being non-polynomial, they are difficult to carry over to quantum theory with any degree of mathematical precision. Consequently, as far as full quantum general relativity is concerned, the work in this area remained formal and concrete progress was restricted largely to truncated models –minisuperspaces– where all but a finite number of degrees of freedom are frozen.

In the first step of the new approach, one performs a canonical transformation on the phase space of geometrodynamics. The new configuration variable is a (complex-valued) SU(2) connection, $A_i^a$, the restriction to the 3-slice of the self-dual part of the Lorentz, Spin connection of the 4-dimensional theory. The conjugate momentum variable is a (density weighted) triad, $E_i^a$, on the 3-slice. These are related to geometrodynamical variables non-polynomially. The 3-metric $q_{ab}$ is (apart from certain density weights) the square of the triad $E_i^a$. The connection $A_i^a$ is related to the spin connection $\Gamma^i_a$ of the triad via $A_i^a = \Gamma_i^a - iK_i^a$, where $K_i^a$ is obtained by transforming a spatial index on the extrinsic curvature $K_{ab}$ to an internal index using the triad. This procedure casts general relativity as a dynamical theory of connections $A_i^a$. Indeed, the phase space of the theory is the same as that of Yang-Mills theory and the constraint surface of general relativity is embedded into that of the Yang-Mills theory. Finally, the equations of the theory simplify significantly: they are all low order polynomials in the new canonical variables. Thus, the two drawbacks of geometrodynamics, mentioned above, have been overcome. Furthermore, Einstein evolution is realized as a motion along null geodesics in the infinite dimensional space of connections $A_i^a$.

There is, however, a price paid in this procedure: one has to impose certain “reality conditions” on the canonical variables to ensure that we are dealing with real, Lorentzian general relativity. In geometrodynamics, the reality conditions are trivial: the metrics and the extrinsic curvatures are both required to be real. A self-dual connection, on the other hand can be real only in the Euclidean sector. Therefore, if we demand that the new canonical variables be real, we are led to Euclidean general relativity. If we let both canonical variables be complex, we are led to complex general relativity. The reality conditions required to recover the Lorentzian theory are rather subtle. To give an analogy from particle mechanics, if we let $q$ and $p$ be the analogs of metrics and extrinsic curvatures of geometrodynamics
(which are both real), the new pair of canonical variables, \((A_i^a, E_i^a)\), is analogous to the pair \((z = q - ip, q)\).

The second main ingredient in the program is the use of loop variables for quantization of theories of connections\(^\dagger\). The heuristic idea is as follows. In the Hamiltonian formulation, quantum states of theories of connections arise as suitable functionals \(\Psi[A]\) of gauge equivalence classes of connections. Now, consider the Wilson loop variables, \(W_\alpha[A] := \text{Tr} U_\alpha(A)\), where the group element \(U_\alpha(A)\) is the holonomy of the connection \(A\) around the loop \(\alpha\). Since \(W_\alpha[A]\) is the non-Abelian analog of \(\exp i \oint_\alpha A.dl\), one can attempt to define a generalized Fourier transform:

\[
\psi(\alpha) := \int_{A/G} W_\alpha[A] \Psi[A] d\mu'^{1}
\]

where \(d\mu\) is a measure on \(A/G\) the space of connections modulo gauge transformations. (The quotation marks denote that the expression is formal since we have not specified the measure yet.) One can thus take quantum states to be suitable functions of loops and define operators directly on them. This turns out to be possible. Physically interesting operators have simple actions that involve breaking, re-routing and gluing various loops. This representation seems especially well-suited for diffeomorphism invariant theories. In general relativity, for example, one can impose the diffeomorphism constraint on the loop states \(\psi(\alpha)\) by asking that they depend not on individual loops, but rather only on equivalence classes of loops where two are equivalent if they are related by a diffeomorphism\(^10\). That is, it appears that the general solution of the diffeomorphism constraint would be given by arbitrary functions of (generalized) knot classes on the 3-manifold we began with! This is an appealing idea and generated the initial enthusiasm for loop representations in quantum gravity. Furthermore, it turned out that, on a similar heuristic level, a number of solutions to the Hamiltonian constraints could be found. Some of them\(^17\) have intriguing relations to well-known knot invariants that arise also in the Chern-Simons theory.

The loop quantization methods have been extended to supersymmetric\(^18\) and other matter couplings\(^19\). The “issue of time” in quantum gravity was also analysed by coupling the theory to a massless scalar field\(^20\). More precisely, in a certain truncation, the classical theory was first deparametrized using the scalar field as the time variable and the resulting model was then quantized in the loop representation. This

\(^\dagger\) The loop representation was introduced by Gambini and Trias\(^9\) for gauge theories and, independently, by Rovelli and Smolin\(^10\) for general relativity. Recently, Gambini and collaborators have introduced an extended loop representation. (See [16] and references therein.)
model admits a true Hamiltonian and a number of interesting Dirac observables. In the spatially compact case, for example, it appears that the total volume of space is quantized. Finally, certain approximation methods have been developed to handle the low energy regime\textsuperscript{4,21}. In particular, gravitons arise as approximate notions\textsuperscript{22}, the approximation becoming better at long wave lengths.

This route to quantization has several unfamiliar features. First, the basic canonical variables are hybrid; the connection is complex while the triad is real. Second, the configuration variables of the theory —the Wilson loop variables $W_\alpha$— are overcomplete; there are identities between them. In the Dirac quantization scheme, there is no obvious prescription to handle these identities in the quantum theory. The overcompleteness also holds for momentum variables $P_S$ which are associated with 2-dimensional (ribbons or) strips $S$. (Recall that the Wilson loop variables $W_\alpha$ are associated with loops $\alpha$.)

Third, there are the awkward reality conditions which should be incorporated in quantum theory. Finally, in Minkowskian quantum field theories, one uses the Poincaré group to select the vacuum state and constrain the inner product. In the present case, we do not have a background structure and hence there is no obvious symmetry group. (Note that diffeomorphisms act as gauge and therefore have a trivial action on physical states.) Thus, we need a new principle to constrain the Hilbert space structure.

To handle such issues, an algebraic extension of Dirac’s method of quantization was carried out and applied to a variety of simpler models which share one or more of the above features with the connection formulation of general relativity. These models include, in particular, some minisuperspaces of 4-dimensional general relativity, linearized gravity and 3-dimensional general relativity. In all these cases, the unusual features of connection dynamics could be addressed and the extended program could be completed. Thus, the algebraic extension of the Dirac program provides a consistent framework to incorporate loop quantization of the connection dynamics formulation of general relativity.

Finally, the general framework has had certain applications to high energy physics\textsuperscript{23,24} and to problems in geometry\textsuperscript{25}.

3. Concerns

While the program I just outlined is structurally coherent, there are, nonetheless, a number of issues of mathematical precision that were left open initially. This is of course unavoidable in any initiative —such as loop quantization— which opens up previously unexplored directions. However, a number of years have passed since the initial
burst of activity and it is now appropriate to attempt to weed out the technically unsound results and make the foundations firm so as to build on them further.

Over the past couple of years, several concerns have been expressed about some of the main results of the program. First, there is the issue of the meaning of the loop transform of Eq (1). What is the meaning of the measure used in this transform? If the measure is only formal, will not the operators on loop states be ambiguous? One can take the standpoint that the loop representation is the fundamental one and the transform is only a heuristic devise. That is, one might just begin with a vector space $V_{\text{kin}}$ of “kinematical” loop states $\psi(\alpha)$ and define on it directly operators $\hat{W}_\alpha$ and $\hat{P}_S$ corresponding to the classical configuration and momentum observables, ensuring that ($i\hbar$ times) the classical Poisson brackets go over to commutators. While this strategy is attractive—and was in fact followed— it does not eliminate the problems.

First, classically, the loop-strip observables are subject to a variety of relations (some of which are inequalities). One cannot simply ignore these if, e.g., one wants the theory to have the correct classical limit. (This problem emerges rather clearly in the loop representation of 3-dimensional quantum general relativity.) If the loop representation is constructed using a transform, these relations are automatically incorporated. In the more direct approach to loop states, the task of incorporating all these relations seems quite difficult.

Then there is the question of constraint operators. Since we are dealing with a field theory, these have to be regulated appropriately. Without a good deal of control on the vector space $V_{\text{kin}}$, this is hard to achieve. Therefore, in practice, one proceeded as follows. One simply imposed those regularity conditions on loop states that seemed to be essential for the regularization procedure to be meaningful. Overall, this is a reasonable strategy. One can criticize it on the grounds that the rules of the game are not clear apriori in the sense that there is no underlying conceptual framework that would tell us which regularity conditions are reasonable and which are not. However, this criticism can be made also against many of the regularization procedures used in Minkowskian field theories. Indeed, the overall level of mathematical precision in some of the heuristic treatments of loop quantization is comparable to that used routinely in field theory. However, there is an important difference between the two situations. We have a great deal of theoretical experience with the procedures used in Minkowskian field theories and, more importantly, we have a lot of experimental justification backing them. In non-perturbative quantum gravity, we have neither! We are in completely new territory and it is therefore all the more important that we reduce the number of ad-
hoc steps. For example, in field theory, although we may begin with smooth configurations, quantum states turn out to be functionals of distributional fields. Indeed, the smooth configurations constitute a set of measure zero! So, are we justified even in the assumption that the loop states should be functionals of “nice” (say piecewise analytic) loops? Shouldn’t we consider more general loops? If we do, it would be very hard to impose the required regularity conditions on $V_{\text{kin}}$. Furthermore, some of the regularity conditions require that values $\psi(\alpha)$ of wave functions on individual loops be well-defined. Now, already in non-relativistic quantum mechanics, whether wave functions in the physical Hilbert space take on well-defined values at every point of its argument depends on the choice of representation. In the Schrödinger representation, they need not, while in a holomorphic representation they do. What is the situation for the loop representation?

These concerns in turn raise questions about the solutions to quantum constraints, i.e., about the space $V_{\text{phy}}$ of physical states of quantum gravity. How reliable are the results on solutions to constraints? In solving constraints, no (uncontrolled) infinities were encountered so far. Is this simply because we are ignoring them? Afterall, if we restrict ourselves to smooth configurations for, say QED, there would be no infinities in the potential term of the quantum Hamiltonian. Is something similar happening here? Are we throwing away anomalies by hand? Is the result on the general solution to the diffeomorphism constraint accurate? Can we really have loop states which are characteristic functions of knots?

Finally, there is the important question of reality conditions. In the heuristic treatments, the following strategy is adopted. One simply ignores the reality conditions initially, thereby considering, in effect, complex general relativity. The idea is to impose reality conditions on physical observables at the end, thereby constraining severely the form of the inner product on physical states. Again, the strategy itself is conceptually sound and enabled one to make considerable progress. However, now that these methods have yielded a number of solutions to quantum constraints, before going too far along this road, it is appropriate to ask if there are indications that these solutions refer to the real, Lorentzian sector. And, there are reasons to be concerned that the simplest solutions obtained so far may not refer to this sector. For example, in the classical theory, using connection dynamics, it is easy to write down an infinite family of solutions to all constraints. But these correspond to self-dual solutions and the only ones that satisfy the Lorentzian reality conditions correspond to initial data for Minkowski space! Is something similar happening in the quantum theory as well? Is there a way to analyse this issue now, without having to first solve the full theory? One strategy would be to
incorporate the reality conditions at the kinematic level, i.e., translate to quantum theory the property that the constraint functions are real in real general relativity. Is there a rigorous procedure to implement this strategy?

4. Recent Developments

In this section, I will summarize some recent mathematical developments that provide a sharper focus to the program thereby putting it on a coherent and sound footing. In particular, we will see that most of the concerns raised above have now been addressed.

1. In a systematic treatment, one can begin with geometrodynamics and push it as far as possible. For this, one can use the real-valued, $SU(2)$ spin connection $\Gamma_a^i$ –constructed solely from the triad $E_a^i$– as the configuration variable. This is only a reformulation of geometrodynamics but has the advantage that one can now employ all the mathematical machinery available to deal with theories of connections. This strategy resolves some of the important issues in quantum geometrodynamics. As we will see, one can go quite far along these lines. However, to deal with the Hamiltonian constraint, it will be necessary to go to the (self-dual) complex connections $A_i^a$. (see point 5 below.)

2. Denote by $\mathcal{A}$ the space of all $SU(2)$ connections $\Gamma_a^i$ which are compatible with a non-degenerate triad $E_a^i$. The space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of such connections is now the classical configuration space of the theory. (It is essentially the same as the space of positive definite metrics.) A key question for quantum geometrodynamics is: What is the quantum configuration space? More precisely, in the connection representation, what is the domain space of quantum wave functions? In quantum mechanics –i.e., in the quantum theory of systems with a finite number of degrees of freedom– the classical and the quantum configuration spaces agree. In field theory, by contrast, the quantum configuration space is a substantial enlargement of the classical one; in scalar field theories, for example, although the classical configurations are smooth (say $C^2$) functions on a $t = \text{const}$ slice, the quantum configuration space consists of all tempered distributions. It turns out that there is a systematic way to answer this question. The answer for geometrodynamics is the following: the quantum configuration space is a compact, Hausdorff space $\overline{\mathcal{A}/\mathcal{G}}$. It is a completion of $\mathcal{A}/\mathcal{G}$, obtained by looking at the kinematics of the continuum theory as a precise (namely, projective) limit of that of lattice gauge theories. For a lattice with $n$ independent loops, the (classical as well as the quantum) configuration space is $[SU(2)]^n$. $\overline{\mathcal{A}/\mathcal{G}}$ is a projective limit of $[SU(2)]^n$. As expected, the extended configuration
space does admit generalized connections which are “distributional”.

3. To proceed with quantum theory, one has to specify an inner product on the space of suitable functionals on $A/G$. This can be achieved by specifying measures on $A/G$. Note that $A/G$ does admit regular measures in a rigorous sense; it a compact, Hausdorff space. One can show that regular measures $\mu$ on $A/G$ are in 1-1 correspondence with certain functions $\chi_\mu$ on the space of (based) loops on the given 3-manifold. To see this, note first that the construction of $A/G$ is such that the Wilson loop functions $W_\alpha[A]$ admit natural extensions $W_\alpha[\bar{A}]$ to $A/G$. These functionals are continuous and bounded on $A/G$. Hence, given any regular measure $\mu$ on $A/G$, we can integrate them. The result is the required function $\chi_\mu$ on the loop space:

$$\chi_\mu(\alpha) = \int_{A/G} W_\alpha[\bar{A}] \ d\mu[\bar{A}]$$

(2)

$\chi_\mu$ is called the generating functional for the measure $\mu$. Conversely, a function $\chi$ on the loop space defines a regular measure with $\chi$ as its generating function provided it satisfies the following two conditions:

i) $\sum_{i=1}^n k_i \chi(\alpha_i) = 0$ if $\sum_{i=1}^n k_i W_{\alpha_i}[A] = 0, \forall [A]$; and

ii) $\sum_{i,j} k_i k_j (\chi(\alpha_i \circ \alpha_j) + \chi(\alpha^{-1}_i \circ \alpha_j)) \geq 0$

for all complex numbers $k_i$ and integers $n$. (In the second condition, the loops are composed at the base point.) The first condition ensures that we have handled the overcompleteness of the Wilson loop observables satisfactorily in the quantum theory, while the second ensures that the expectation value of a positive operator is positive. Eq.(2) is the rigorous analog of the heuristic loop transform of Eq.(1). Thus, $\chi_\mu(\alpha)$ are the quantum states in the loop representation. Note that while states in the connection representation have support on generalized (“distributional”) connections, the loop states are functions of ordinary, nice (more precisely, piecewise analytic) loops. Thus, there is no need to consider “distributional loops”. Finally, by construction, $\chi_\mu$ takes on a well-defined value on every loop $\alpha$, whence, it is meaningful to formulate regularity conditions in terms these values.

We can construct measures on $A/G$ explicitly. There is one measure $\mu_o$ in particular which is natural in the sense that (in the projective limit) it is induced simply by the Haar measure on $SU(2)$. It does not require any additional input, is strictly positive and diffeomorphism invariant. It is natural to use it to construct a kinematical Hilbert space $H_{\text{kin}} = L^2(A/G, d\mu_o)$, the precise analog of $V_{\text{kin}}$ of section 3. On this Hilbert space, one can define the configuration and momentum operators $\hat{W}_\alpha$ and $\hat{P}_S$ and show that they are self-adjoint. Since classically, $W_\alpha$ and $P_S$ are real and constitute a complete set of observables, the self-adjointness implies that we have incorporated the classical, kinematical “reality conditions” in quantum kinematics.
4. The next task is to formulate and solve the diffeomorphism constraint. It turns out that the momentum $P_{\alpha}^i$, conjugate to $\Gamma^i_\alpha$ is related to the triad and the extrinsic curvature in a non-local fashion. Nonetheless, one can express each diffeomorphism constraint as an operator on $H_{\text{kin}}$. The commutator algebra of these operators is the same as that in the classical theory. There are no anomalies. As one might expect, the solutions of the diffeomorphism constraint are not elements of $H_{\text{kin}}$. This is a common occurrence even in particle mechanics. There is, however, a precise treatment of the problem and the natural home for solutions is the space of regular, complex-valued (i.e., “signed”) measures $\mu$ on $\mathcal{A}/\mathcal{G}$. Alternatively, the home for solutions is the space of functions $\chi_\mu$ determined via Eq.(2) by a complex-valued measure $\mu$. In this space, we can seek solutions. Not surprisingly, they are simply diffeomorphism invariant measures on $\mathcal{A}/\mathcal{G}$. Alternatively, they are functions $\chi(\alpha)$ on the loop space which depend only on the generalized\(^*\) knot class of $\alpha$ and satisfy the two algebraic conditions which qualify them as generating functions of signed measures.

These conditions are important; contrary to what one might expect at first, not every knot-invariant can solve the diffeomorphism constraint! In particular, we cannot set $\chi(\alpha)$ to be a characteristic function of a regular knot class. One can, however, construct solutions which satisfy all these requirements. In particular, given a suitable invariant $k(\alpha)$ of regular knots, using the fiducial measure $\mu_o$ one can generate a new (complex-valued) measure $\mu_k$ on $\mathcal{A}/\mathcal{G}$ which is also diffeomorphism invariant. The generating function $\chi_k(\alpha)$ of this measure $\mu_k$ is then a solution to the diffeomorphism constraints. However, it differs from $k(\alpha)$; in particular, it does not necessarily vanish on loops with intersections, kinks or overlaps. Thus, while the “obvious” results on the general solution to the diffeomorphism constraint is actually incorrect, its general spirit is realized in the rigorous result.

5. Thus, within quantum geometrodynamics, we have answered three questions: i) What is the domain space of quantum states? (answer: $\mathcal{A}/\mathcal{G}$); ii) Are there anomalies in a rigorous treatment of the diffeomorphism constraint? (answer: no); and, iii) Can one write down the “general solution” rigorously? (answer: yes). The first and the third questions were raised by Wheeler already in the seventies. (There was an implicit assumption that the second question would be answered affirmatively.) However, the subsequent analysis was carried out within the Hamiltonian formulation of the classical theory. Parallel developments in quantum field theory taught us that, in systems with an infinite number of degrees of freedom, the structure of the quan-

\(^*\) We will call a knot regular if its representative loops are smoothly embedded in the 3-manifold and generalized if they are not (e.g., if they have intersections, kinks or overlaps).
quatum configuration space is quite different from that of the classical configuration space and that this difference has to be faced squarely. (This point has been emphasized by Isham). The present treatment takes this lesson seriously. The upshot is that the quantum analog of Wheeler’s superspace is the space of diffeomorphism invariant, signed measures $\mu$ on $\mathcal{A}/\mathcal{G}$, or equivalently, their generating functionals $\chi_\mu$ on the loop space.

From a structural point of view, our main remaining task is to treat the Hamiltonian constraint in a similar fashion. As one might expect, this constraint does not seem manageable within geometrodynamics in the sense that there are no realistic ideas on how one might take it over to the quantum level. It is here that we need to go to the complex (self-dual) connections $A^i_a$: we know that the classical Hamiltonian constraint does simplify considerably in terms of $(A^i_a, E^i_a)$. However, if one works directly in these variables, one has to face the issue of reality conditions all over again. Fortunately, this problem can be avoided by constructing a transform from the real connection $(\Gamma^i_a)$ representation to the complex connection $(A^i_a)$ representation.

The idea is to mimic the celebrated Segal-Bargmann transform which maps one from the Schrödinger representation where the wave functions are arbitrary, complex-valued functions of a real variable $q$ to the coherent state representation where they are holomorphic functions of a complex variable $z = q - ip$. This transform can be constructed using heat kernel methods. Using some key recent results due to Hall, this transform has been extended to the case of general relativity under consideration. The result is a rigorous, quantum field theoretic analog of the canonical transformation that took us from classical geometrodynamics to (self-dual) connection dynamics. Since the two representations are isomorphic, the kinematical reality conditions which were incorporated in the real connection representation, are incorporated also in the holomorphic representation. Thus, what we have is a kinematic arena for dealing with all constraints, where the Wilson loop functions of the complex connection $A^i_a$ act by multiplication.

The discussion of the diffeomorphism constraint extends in a straightforward way to the holomorphic representation and the final results are completely analogous to the one discussed above. Work on the Hamiltonian constraint has only begun. There is a rich body of heuristic results on the treatment of this constraint (see, e.g., Ref. [10, 13-20, 38]). The hope is that they would lead to a precise expression of the constraint in the holomorphic Hilbert space. If this turns out to be the case, one would say that quantum general relativity does exist non-perturbatively. One can then apply suitable approximation techniques to extract the physical content of the theory. Such methods are already being developed at a heuristic level. Indeed, the strength
of the program lies in this simultaneous development of the heuristic techniques and the rigorous framework, each stimulating further developments in the other.

5. Discussion

Let us briefly summarize the overall status.

There now exists a rigorous kinematical framework to deal with quantum gravity non-perturbatively. In terms of physical predictions, this progress is modest. However, these developments have provided us quite a different language to analyze various issues. And the answers have a new degree of precision. Furthermore, even the formulation of some of the basic questions has altered. An outstanding example is: How do you impose constraints? While the quantum constraints $\hat{C}$ arise as operators on the kinematical Hilbert space $H_{\text{kin}}$, one imposes them on measures $\mu$ on $\mathcal{A}/\mathcal{G}$ by demanding:

$$\int_{\mathcal{A}/\mathcal{G}} (\hat{C} \circ W_{\alpha} [\bar{A}]) \ d\mu = 0 , \forall \alpha .$$

The issue of regularization one faces in this formulation is rather different from that in the standard Dirac picture. Similarly, new possibilities for solutions arise. For example, one would expect that, in the holomorphic representation, measures which have support on the moduli space of flat connections would automatically satisfy all constraints. These solutions would carry information about the topology (the first homotopy group) of the manifold. It is difficult to transcribe this idea in the Dirac language of imposing constraints on functions on $\mathcal{A}/\mathcal{G}$. More generally, the language provided by the recent developments enables new constructions and provides new translations of physical questions as well as methods for tackling them. Indeed, the situation has a certain degree of similarity to the emergence of global techniques in classical general relativity which also provided a new language and new ways of thinking of physical issues such as singularities and black holes. The local, coordinate dependent techniques were certainly useful and they did lead to many interesting results. However, their power was restricted. Modern differential geometry added a new degree of precision which in turn resolved certain confusion and led to a variety of powerful results. One would hope that the transition from heuristic methods to the ones discussed in the last section would bring similar fruits.

Although the developments which led to the new framework are mathematical, the various constructions involved do have physical counterparts. The use of loops to probe function spaces of connections, for example, leads one to a theory in which the fundamental
excitations of the gravitational field are “loopy” rather than wave-
like. These in turn lead to a discrete picture of quantum geometry
and a theory in which the fundamental operators act via combinatorics
rather than functional derivatives. The basic interactions will not be
mediated by gravitons; they will correspond to breaking, re-routing
and gluing of loops. Thus, the framework is fundamentally different
from the perturbative one.

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