Dynamical oscillations in nonlinear optical media

Theodoros P. Horikis\textsuperscript{1} and Hector E. Nistazakis\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, University of Ioannina, Ioannina 45110, Greece
\textsuperscript{2}Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 15784, Greece

Compiled October 13, 2009

OCIS codes: 190.4370, 230.7370, 060.4370, 190.4420, 190.3270

In recent years, great effort has been made in order to explain the behavior of light beams propagating through interfaces separating optical media with different nonlinear refractive indices. This interest has been motivated by different factors that include the analysis of pulse propagation and self-phase modulation effects in fibers, the close connection with the problem of radiation mode propagation in the three-dimensional nonlinear Schrödinger equation, and the persistency in the nonlinear wave propagation regime of some properties found in the corresponding linear problem, such as the occurrence of a parametric instability under modulation of parameters [1, 2].

Nonlinear pulse propagation in graded-index optical waveguides is a much studied problem, both numerically and analytically. In early works, the radial dynamics of pulses in nonlinear fibers were studied by using the paraxial ray approximation. However, this was shown to give inaccurate results for the self-phase modulation of a pulse that propagates in a bulk nonlinear medium (see [3] and references therein) and a variational method was adopted. Using this approach the authors in Ref. [3] studied the resulting dynamics of pulses from the combined effects of spatial diffraction, nonlinearity and parabolic graded index in radially symmetric fibers. More recently, in Ref. [4] the space-time dynamics in nonlinear multimode parabolic index optical fibers were studied in the context of a temporal modulation instability induced by spatial transverse effects. We extend this approach to include the dynamics of the pulse’s center in planar waveguides. In fact, we find that the motion of the center of the pulse does not depend on any other parameter of the pulse and it is only determined by the values of the diffusion and the refractive index.

The nonlinear Schrödinger equation appropriately modified to model beam propagation in graded-index, nonlinear waveguide amplifiers with refractive index \( n(z, x) = n_0 + n_1x^2 + n_2|\psi|^2 \) is

\[
i\psi_z + \frac{d_0}{2}\psi_{xx} + \frac{1}{2}n_1x^2\psi + n_2|\psi|^2\psi = 0
\]  

The parameters, \( d_0, n_1 \) and \( n_2 \) are constant, diffusion \( (d_0) \) can be positive or negative and the medium can be anti-guiding \((n_1 > 0)\) or guiding \((n_1 < 0)\). As in Ref. [5], both cases are considered in order to examine the interplay between self-defocusing of light and guiding or anti-guiding and diffractive effects of the medium.

Interestingly, Eq. (1) is similar to the Gross-Pitaevskii equation that describes the dynamics of confined atomic Bose-Einstein condensates (BECs) [6, 7]. The crucial differences are that, in BECs, \( d_0 \) is always positive and \( n_1 = -\Omega^2 < 0 \), where \( \Omega \) is the normalized harmonic trap strength. As also shown below, this difference in signs is what gives rise to the so-called collective oscillations in the BECs context [8]. Furthermore, a dissipative variant of this equation has also been used to describe the behavior of solitons and self-similar waves in nonlinear systems exhibiting both spatial inhomogeneity and gain or loss at the same time [9, 10].

Equation (1) can be restated in variational form using the Langrangian density

\[
\mathcal{L} = \frac{i}{2}(\psi_\tau \psi^* - \psi^*_\tau \psi) - \frac{d_0}{2}|\psi_x|^2 + \frac{n_1}{2}x^2|\psi|^2 + \frac{n_2}{2}|\psi|^4
\]

If the injected field is a gaussian beam, in the weakly or moderate nonlinear regime \((n_2 \approx 0)\) the field remains gaussian along the propagation distance and can be calculated by means of a variational approach [3, 4]. Thus, we take

\[
\psi(z, x) = A(z)e^{-b(z)[x-x_0(z)]^2}e^{i\phi(z, x)}
\]  

where \( \phi(z, x) = \alpha(z)x^2 + \beta(z)x + \gamma(z) \), describes the pulse’s phase. For a given value of \( z \), Eq. (2) defines a gaussian beam invariant along the \( y \) direction (planar waveguide geometry) displaced by an amount \( x_0 \) along the \( x \)-axis from the origin of the coordinates.

Inserting the gaussian ansatz into the expression for \( \mathcal{L} \) and integrating we obtain the average Langrangian of the problem, \( L = \int_{-\infty}^{\infty} \mathcal{L} \, dx \). Using the Euler-Lagrange equations with variational parameters \( A, b, x_0, \alpha, \beta \) and...
\( \gamma \) we obtain a set of coupled equations describing the evolution of these parameters, namely

\[
\begin{align*}
A_z &= -d_0 \alpha A \\
b_z &= -4d_0 \alpha b \\
x_{oz} &= d_0 (2 \alpha x_0 + \beta) \\
\alpha_z &= \frac{1}{2} (\sqrt{2n_2} A^2 b + 4d_0 b^2 + n_1 - 4d_0 \alpha^2) \\
\beta_z &= \sqrt{2n_2} A^2 b x_0 - 4d_0 b^2 x_0 - 2d_0 \alpha \beta \\
\gamma_z &= \frac{1}{8} (5\sqrt{2n_2} A^2 - 8d_0 b - 4\sqrt{2n_2} A^2 b x_0^2 + 16d_0 b^2 x_0^2 - 4d_0 \beta^2) 
\end{align*}
\]

By dividing the first two equations and integrating, we obtain \( A^4 = E_0 b \), where \( E_0 \) is the energy of the system at \( z = 0 \). This is equivalent to the conservation of energy of the system. Indeed, Eq. (1) is integrable and has an infinite number of conservation laws the first of which characterizes the system’s energy, i.e. \( \int_{-\infty}^{+\infty} |\psi|^2 \, dx = E_0 \). When we substitute for the profile of Eq. (2) we obtain \( A^4 = E_0 b \), as above. In fact, all of Eqs. (3) can also be derived using conservation laws arguments.

In general, the system of Eqs. (3) is coupled and nonlinear. Remarkably, however, differentiating Eq. (3c) and using the rest of the equations we obtain the uncoupled equation

\[
x_{0,zz} - (d_0 n_1) x_0 = 0
\]

The above equation demonstrates that two types of evolution can be observed. If \( d_0 n_1 > 0 \) the location of the center of the pulse moves along an exponential trajectory. The more interesting case is the one with \( d_0 n_1 < 0 \), since it suggests an oscillatory pattern around \( x_0 = 0 \), with frequency \( \omega = \sqrt{d_0 n_1} \). This is illustrated in Fig. 1 where we show how the center of the pulse evolves, with parameters \( d_0 = -1 \), \( n_1 = \pm 1.5 \) (top/bottom) and \( x_0(0) = 1 \). Hereafter, \( n_2 = 1 \) and the relative signs between diffraction and nonlinearity will be controlled by \( d_0 \). Also, note that depending on the initial conditions on \( x_0(z) \) the oscillation will undergo a sine \( (x_0(0) = 0, x_{0,z} \neq 0) \) or cosine \( (x_0(0) \neq 0, x_{0,z} = 0) \) oscillation. This resembles linear propagation of light in a parabolic-index optical fiber, where light is guided if the refractive index is maximal in the center, provided diffraction is positive. A parabolic index with the minimal value of the index in the center would lead to defocusing of the beam. In the case where light is guided, we can expect that different propagation modes will be allowed in the waveguiding structure induced by the graded index. Therefore, if the input profile is not perfectly symmetric, at least one antisymmetric mode will be excited, with a slightly different propagation constant than the fundamental mode. The beating between the main antisymmetric mode and the main symmetric mode will lead to a periodic change in position of the center of the beam along the propagation.

In a similar manner, a breathing behavior may be observed if one is to analyze the propagation of a pulsed beam through the simple nonlinear device composed of a waveguide of certain thickness and refractive index \( n_1 (\neq n_1) \), surrounded by a linear substrate of index \( n_1 \) and a nonlinear cover with Kerr-type nonlinearity of the form \( n = n_1 + n_2 |\psi|^2 \). To demonstrate, we show in Fig. 2 (top) the propagation of a unit gaussian under Eq. (1) with \( d_0 = -1 \) and \( n_1 = 1.5 \). By changing the values of these parameters the qualitati ve behavior of the oscillations are also changing. Moreover, if these two parameters share the same sign no oscillations are observed, as shown in Fig. 2 (bottom). In this case the initial pulse (i.e. unit gaussian with equation parameters \( d_0 = -1 \) and \( n_1 = -1.5 \)) is decaying fast.

To further illustrate this breathing we plot in Fig. 3 the evolution of the amplitude \( A(z) \) of the pulse for two different values of \( n_1 \) starting from a unit gaussian. The analysis for this case is somewhat more involved than in the previous case since the equations do not uncouple in a trivial way. Indeed, differentiating Eq. (3d) and using the rest of the set we obtain a second order nonlinear, coupled equation for \( \alpha(z) \) that reads

\[
\alpha_{zz} + 10d_0 \alpha_z - 8d_0^2 \alpha^3 + 20d_0^2 \alpha^2 + 4d_0^2 b^2 \alpha - 3d_0 n_1 \alpha = 0
\]

The oscillatory nature of the equation becomes apparent.
from its linear part which can be simplified if \( b^2 \ll 1 \) to

\[
\alpha_{zz} - (3d_0n_1)\alpha = 0
\]

Again the oscillations exist if \( d_0n_1 < 0 \) and a good estimate for their frequency is \( \omega' = \sqrt{3|d_0n_1|} \). Notice that \( n_2 \) (nonlinearity) comes into the system in higher-order and the difference with the frequency of the pulse’s center (also apparent in Fig. 4).

We finally briefly discuss the evolution of a unit gaussian originally dislocated at \( x_0 = 1 \), as shown in Fig. 4. Since the equation for \( x_0 \) can be uncoupled the oscillations of the center of the pulse do not effect the breathing of the rest of the pulse’s parameters. This means that the propagation of the center of the pulse is independent of the breathing that may occur due to changes in the refractive index of the medium.

To conclude, we demonstrated that under certain conditions, namely that diffraction and refractive index have opposite signs, pulses propagating in parabolic index optical waveguides exhibit oscillatory patterns. While the pattern for the dislocation of the pulse is simple and described by a linear equation the rest of the pulse’s parameters are described by coupled, nonlinear equations. In the first case, we provided the exact frequency of the oscillation while an estimate based on the linear part of
the equation was provided for the latter.

We wish to thank D.J. Frantzeskakis for many useful discussions while preparing this manuscript and the anonymous reviewers for many clarifying remarks.

References

1. S. Longhi and D. Janner, “Self-focusing and nonlinear periodic beams in parabolic index optical fibres,” J. Opt. B: Quantum Semiclass. Opt. 6, S303–S308 (2004).
2. H. Michinel, “Pulsed nonlinear surface waves and soliton emission at nonlinear graded index waveguides,” Opt. Quant. Elec. 30, 79–97 (1998).
3. M. Karlsson, D. Anderson, and M. Desaix, “Dynamics of self-focusing and self-phase modulation in a parabolic index optical fiber,” Opt. Lett. 17, 22–24 (1992).
4. S. Longhi, “Modulational instability and space time dynamics in nonlinear parabolic-index optical fibers,” Opt. Lett. 28, 2363–2365 (2003).
5. S. Raghavan and G. Agrawal, “Spatiotemporal solitons in inhomogeneous nonlinear media,” Opt. Comm. 180, 377–382 (2000).
6. P. Kevrekidis, D. Frantzeskakis, and R. Carretero-González (eds), Emergent nonlinear phenomena in Bose-Einstein condensates: Theory and experiment (Springer, 2007).
7. R. Carretero-González, D. Frantzeskakis, and P. Kevrekidis, “Nonlinear waves in Bose-Einstein condensates: Physical relevance and mathematical techniques,” Nonlinearity 21, R139–R202 (2008).
8. F. Abdullaev, R. Galimzyanov, and K. Ismatullaev, “Collective oscillations of a quasi-one-dimensional bose condensate under damping,” Phys. Lett. A 357, 48–53 (2006).
9. S. Ponomarenko and G. Agrawal, “Optical similartons in nonlinear waveguides,” Opt. Lett. 32, 1659–1661 (2007).
10. L. Wu, J.-F. Zhang, L. Li, Q. Tian, and K. Porsezian, “Similaritons in nonlinear optical systems,” Opt. Express 16, 6352–6360 (2008).