Force Percolation Transition of Jammed Granular Systems

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The mechanical and transport properties of jammed materials originate from an underlying percolating network of contact forces between the grains. Using extensive simulations we investigate the force-percolation transition of this network, where two particles are considered as linked if their interparticle force overcomes a threshold. We show that this transition belongs to the random percolation universality class, thus ruling out the existence of long-range correlations between the forces. Through a combined size and pressure scaling for the percolative quantities, we show that the continuous force percolation transition evolves into the discontinuous jamming transition in the zero pressure limit, as the size of the critical region scales with the pressure.

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Amorphous particulate systems such as foams and granular materials jam and acquire mechanical rigidity when subjected to an external pressure [1–5]. In the jammed state, a network of interparticle forces determines resistance to shear [6, 7], sound and heat transport [8–11], as well as electrical conductivity [12]. The distribution of the magnitude of the interparticle forces has been the subject of numerous studies [13–23], and it is now ascertained that this decays exponentially at large forces, while exhibiting a pressure dependent power law behavior at small forces. Large forces organize along chains [24–29], which suggests the existence of a large scale structure one might identify through statistical physics or network–based tools [30–37]. In this line of research, the main open question concerns the spatial organization of the force network, and the possible existence of long range correlations between the forces. These issues are conveniently investigated studying a force based bond percolation transition in which two particles are assumed to belong to a cluster if the magnitude of their interparticle force is larger than a threshold \( f_c \). In the jammed phase, when \( f_c = 0 \) all contacting particles belong to the same cluster, while conversely in the \( f_c \to \infty \) limit there are no clusters. Thus, a percolation transition occurs when the threshold overcomes a critical value \( f_c \). Ostoja et al. [38] numerically investigated this force percolation transition in frictionless and frictional systems of disks packings prepared at constant pressure, finding a universal critical behavior and exponents not compatible with those of the random universality class. A recent experimental and numerical investigation of the force percolation transition of jammed disks packings at constant density [39] found different critical exponents, also not compatible with the random universality class. These results point towards the existence of long-range correlations between the forces. However, direct numerical and experimental investigations of the spatial correlation between the forces [15, 40] failed to observe long correlation lengths. Accordingly, it is currently unclear whether the correlations between the forces of jammed packings are truly long–ranged. The answer to this question might depend on the pressure/density of the system, that controls the percolation threshold \( f_c \), as this must vanish at the jamming transition, as illustrated Fig. 1, where all forces vanish. Thus, it is important to understand how the continuous force percolation transition in the zero pressure limit relates to the discontinuous jamming transition.

In this Letter, we investigate force correlations in

![FIG. 1. A percolative analysis of jammed configurations is introduced by considering particles as connected if interacting with a force whose magnitude is greater than a threshold \( f_c \). This schematic phase diagram illustrates the existence of a continuous percolation line that ends at the jamming volume fraction \( \Phi_J \), where the percolation transition becomes discontinuous. The panels illustrate the percolative analysis of a \( \mathcal{N} = 10^4 \) particle system across the transition. Lines connect particles belonging to the same cluster. For clarity, not all of clusters are shown.](image-url)
jammed granular packings via the numerical study of the force percolation transition of Harmonic and Hertzian particles, in both two and three spatial dimensions. First we show that the force percolation transition does actually belong to the random universality class, regardless of the distance from the jamming threshold, thus ruling out the presence of long-range force correlations. Then we clarify how the features of the force percolation transition depend on pressure, thus rationalizing how the continuous force percolation transition evolves into the discontinuous jamming transition in the zero pressure limit.

Model systems – We numerically [41] investigate systems of particles interacting via purely repulsive potential, \( v(r) = \frac{1}{\alpha} \varepsilon \left( \frac{\sigma - r}{\sigma r} \right)^{\alpha} \) for \( r < \sigma \), and \( v(r) = 0 \) for \( r \geq \sigma \), where \( \varepsilon \) is the characteristic energy scale, \( \sigma \) is the average diameter of the interacting particles and \( r \) the distance between their centers. We study the system for two potentials, \( \alpha = 2 \) (Harmonic) and \( \alpha = 2.5 \) (Hertzian), focusing on a 50 : 50 binary mixture of particles with diameters \( \sigma_1 = 1 \) and \( \sigma_2 = \sigma_1/1.4 \). We choose \( \sigma_1 \) and \( \varepsilon \) as our units of length and energy. We have considered various systems with number of particles, \( N = 4 \times 10^3, 10^4, 2 \times 10^4 \), and \( 4 \times 10^4 \), and various values of the pressure, \( p = 10^{-2}, 5 \times 10^{-3}, 2.5 \times 10^{-3}, 5 \times 10^{-4}, 5 \times 10^{-5}, \) and \( 5 \times 10^{-6} \). To prepare the system at the desired value of pressure, we first randomly distribute particles in a square/cubic box, with periodic boundary conditions. The size of the box is then repeatedly changed via a divide and conquer algorithm, where we minimize the elastic energy of the system using conjugate gradient algorithm after every change in simulation box size. This iterative procedure continues until the pressure equals the desired value with a tolerance of \( |dp|/p < 10^{-6} \). For each dimensionality \( d \), potential and pressure value, our results are averaged over 500 independent jammed configurations. In this main text, we present results for two dimensional harmonic systems. Analogous results for Hertzian potential and the three dimensional system are presented in supplementary materials [42].

Force percolation – The force percolation transition is a bond percolation transition occurring on a disordered lattice whose nodes correspond to the particles, and whose bonds correspond to the interparticle forces. The percolation is induced by the removal of all bonds associated to interparticle forces \( f \) lower than the threshold force \( f_t \), as the threshold \( f_t \) is varied. At each value of the pressure (\( p > 0 \)) a percolation transition occurs as \( f_t \) varies, as schematically illustrated in Fig. 1. The percolative properties of this transition reflect those of the forces because the bonds that are retained are not randomly chosen, but correspond to interparticle forces greater than \( f_t \).

We have determined the critical exponents governing the critical behavior of the strength of the percolating cluster \( P_\infty \), of the mean cluster size \( S \), and of the percolation correlation length \( \xi \), \( P_\infty \sim |f - f_c|^\beta, S \sim |f - f_c|^{-\gamma}, \xi \sim |f - f_c|^{-\nu} \), performing a standard finite size scaling analysis. Indeed, in finite systems the above critical behaviors are replaced by crossovers satisfying the scaling relations

\[
P_\infty(N, f) = N^{-\frac{\nu}{\beta}} m_1 \left[ N^{\frac{\beta}{\tau}} (f - f_c) \right],
\]

\[
S(N, f) = N^{\frac{\gamma}{\nu}} m_2 \left[ N^{\frac{\nu}{\gamma}} (f - f_c) \right],
\]

with \( m_1 \) and \( m_2 \) universal scaling functions. To improve numerical accuracy we have performed a size scaling analysis of the fraction of particles in largest cluster \( C_1 \), that scales as \( P_\infty \) but is of easier investigation as it does not depend on the percolation threshold. The mean cluster size is defined as \( S = \sum s^2 n(s)/\sum s n(s) \), where \( s \) and \( n(s) \) refer to the size and number of clusters, and the summation excludes the percolating cluster. The size \( s \)
of a cluster equals its number of bonds.

Figure 2 illustrates the finite size scaling investigation of the force percolation transition of a two dimensional system of disks at pressure $p = 5 \times 10^{-3}$. This investigation strongly suggests the percolation transition to belong to the random percolation universality class. First, panels a-d show our scaling analysis for $C_1$ and $S$. Data nicely collapse when rescaled using the random percolation universality class exponents [43]. Second, panels e and f show the normalized cluster size distribution $n(s)$ and the radial distribution function of particles belonging to the percolating cluster $g_{pc}(r)$ respectively, at the percolation threshold for the $N = 4 \times 10^4$ system. The distribution $n(s)$ fits very well to the power-law decay $n(s) \sim s^{-\tau}$, with $\tau$ Fischer exponent of the random percolation in two dimensions. Similarly, for large $r$ the pair-connected correlation function is well described by a power-law decay $g_{pc}(r) \sim r^{-d+2-\eta}$, with the random percolation value of the anomalous dimension $\eta$. We have found the same results for all values of the pressure we have considered. In addition, analogous findings occur for the Hertzian potential, and in three dimensions, as we illustrate in the supplementary material [42]. Summarizing, these results clarify that the force percolation transition is a continuous percolation transition in the random percolation universality class. As a consequence, the correlations between the interparticle forces of jammed packings are finite ranged.

It is worth noticing that these results contrast with those of Refs. [38, 39], which suggested the force percolation transition to be correlated. First, Ref. [38] reported two exponents, $\phi = 0.89 \pm 0.01$, and $\nu = 1.6 \pm 0.1$. Of the two exponents, $\phi$ is compatible with the random percolation expectation, $43/48 = 0.895$, while $\nu$, which is estimated with lesser accuracy, is not compatible with the random value, $\nu = 4/3$. We speculate that the difference is due to numerical errors arising from using small system sizes, as pointed in Ref. [39]. Second, Ref. [39] reported $\phi = 0.77 - 0.85$, and $\nu = 1.04 - 1.58$ depending on the packing fraction and polydispersity. Our speculation is that the differences from the random percolation values are due to using volume fraction as control parameter. Indeed, the pressure of jammed packings of given volume fraction has strong finite size effects.

We now consider how the force percolation transition, that occurs at fixed pressure as the threshold force $f_t$ varies, is related to the geometrically discontinuous jamming transition that occurs at $f_t = 0$, as the pressure/density varies. First, we note that the critical threshold $f_c$ decreases with the pressure as $f_c \sim p^q$, with $q \approx 0.98$, as illustrated in Fig. 3. This is the same scaling we observe for the average force, that is commonly used [44] to decompose the force network in subnetworks with different mechanical properties. Next, in Fig. 4, we show the pressure dependence of cluster statistics. We observe in Fig. 4a that the size of the largest cluster as function of the distance from the percolation threshold $f - f_c$ is pressure dependent, and becomes more abrupt when approaching the zero pressure limit. This tendency is quantified by investigating the derivative of $C_1$ at the inflection point, $dC_1/df|_{f=f_c}$. Figure 4b shows that this derivative increases in modulus as the pressure decreases,

![Figure 3](image-url)

**FIG. 3.** Pressure dependence of the force percolation threshold, $f_c$, and of the average force, $\langle f \rangle$. These two forces are proportional, and scale as $f_c = \alpha_c p^q$ and as $\langle f \rangle = \alpha p^q$, with $q \approx 0.98(1)$, $\alpha_c = 1.00(4)$, and $\alpha = 0.67(3)$.

![Figure 4](image-url)

**FIG. 4.** Dependence of the size of the largest cluster, $C_1$, on $f - f_c$, for different values of pressure (a), and pressure dependence of its inflection point (b). The derivative diverges as $p^{q-3}$ in the $p \rightarrow 0$ limit. Panel (c) clarifies that $C_1$ scales as $N^{-\beta/d\nu}$ for all values of the pressure, ruling out the presence of a discontinuous transition in the thermodynamic limit, for $p \rightarrow 0$. Panel (d) shows that the $C_1$ data of (a) collapse when plotted versus $(f - f_c)/p^q$, thus clarifying that the size of the critical region scales as $p^q$. In panels a,b and d, $N = 2 \times 10^4$. 

diverging as a power-law $\sim p^{-k}$, with $k \approx q$, in the zero pressure limit. This might suggest that the transition becomes discontinuous in the zero pressure limit. However, we show in Fig. 4b that $C_1$ at the inflection point has weak dependence on pressure, and that it scales as $N^{-\frac{q}{d}}$. Thus, $C_1$ does not exhibit a jump of finite size in the $p \to 0$ limit. Thus, in the zero pressure limit the force percolation transition remains a continuous transition, and the only effect of the pressure appears that of controlling the size of the critical region, which is expected to scale as $p^q$. We confirm this speculation in Fig. 4d, where we illustrate that the data of panel a collapse when plotted as a function of $(f-f_c)/p^q$. Overall, these results suggest a combined size and pressure scaling for the strength of the percolating cluster, and, similarly, for the mean cluster size,

$$P_\infty(N, p, f) = N^{-\frac{q}{d}} m_1 \left[ N^{-\frac{q}{d}} p^{-q} (f-f_c) \right],$$

$$S(N, p, f) = N^{-\frac{q}{d}} m_2 \left[ N^{-\frac{q}{d}} p^{-q} (f-f_c) \right],$$

(2)

where the exponent $q$ is that controlling the dependence of the critical threshold on the pressure (see Fig. 3). The validity of the proposed scaling relations is confirmed by the good data collapse obtained for various pressure and system size, as shown in Fig. 5. From Eq. 2, and the scaling of the percolation threshold on the pressure, Fig. 3, we understand that the correlation percolation length, which measures the typical size of the cluster of forces larger than $f$, scales as $\xi = l[p^{-q}(f-f_c)]^{-\nu} = l(\alpha f/(f-f_c))^{-\nu}$, with $l$ pressure independent length scale. Since in the $p \to 0$ limit the force percolation transition does not become discontinuous, we understand that the order of the limits $p \to 0$ and $f \to 0$ matters. If the $p \to 0$ limit is carried out first, then the continuous force percolation transition is observed at $f = 0$. Conversely, if the $p \to 0$ limit is carried out first, the one observes the jamming transition at $f = 0$.

While these results prove that there are not long-range force correlations, forces have short range correlations as revealed by the common observation of force chains. In the percolation setting, short-range correlations can be revealed comparing the actual percolation threshold $f_c(p)$ to that obtained after removing all correlations, $f^R_c(p)$. The relation between $f^R_c(p)$ and $f_c(p)$ depends on the short-range correlation length, as well as on anisotropy of the correlations [45–48]. We have performed this investigation removing the force correlations by randomly swapping the forces associated to the different contacts, treating them as labels, and have found $f^R_c(p) > f_c(p)$. Importantly, we have found $f^R_c(p)/f_c(p) \approx 0.6$ regardless of the pressure, in the $p \to 0$ limit, consistently with our proposed scenario according to which the pressure only fixes the size of the critical region.

**Conclusions** – We have shown that the force percolation transition of jammed granular packings belongs to the random percolation universality class, and thus demonstrated the absence of long-ranged force correlations, contrary to earlier speculations [38]. This result occurs regardless of the distance from the jamming transition, as proved by a combined size and pressure scaling. The main peculiarity of this force percolation transition is the dependence of the width of the critical region on the pressure, and thus on percolation threshold, as in the jamming zero pressure limit the critical region disappear. While we do expect this scenario to hold regardless of the protocol used to prepare the jammed packings, and thus regardless of the jamming volume fraction [49, 50], this is certainly an interesting avenue of research. Similarly, it would be of interest to investigate force correlations on the unjammed side of the transition, where interparticle forces can be defined in hard sphere systems from the collisional momentum exchange [51].

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