Non-linear IV characteristics in two-dimensional superconductors:
Berezinskii-Kosterlitz-Thouless physics vs inhomogeneity

G. Venditti, 1 J. Biscaras, 2 S. Hurand, 3, 4 N. Bergeal, 3, 5 J. Lesueur, 3, 5 A. Dogra, 6 R. C. Budhani, 7
Mintu Mondal, 8, 9 John Jesudasan, 9 Pratap Raychaudhuri, 9 S. Caprara, 1 and L. Benfatto 1

1 ISCN-CNR and Dep. of Physics, Sapienza University of Rome, P.le A. Moro 5, 00185 Rome, Italy
2 Sorbonne Université, CNRS, MNHN, Institut de Minéralogie de Physique des Matériaux et de Cosmochimie, IMPMC, F-75005 Paris, France
3 Laboratoire de Physique et d’Etude des Matériaux, ESPCI Paris, PSL Research University, CNRS, 10 Rue Vauquin - 75005 Paris, France
4 Institute Pprime, UPR 3346 CNRS, Université de Poitiers, ISAE-ENSSMA, BP 30179, 86962 Futuroscope-Chasseneuil Cedex, France
5 Université Pierre and Marie Curie, Sorbonne-Universités, 75005 Paris, France
6 National Physical Laboratory, New Delhi, 110012, India
7 Department of Physics, Morgan State University, Baltimore, Maryland 21251, USA
8 School of Physical Sciences, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, India
9 Tata Institute of Fundamental Research, Homi Bhabha Rd, Colaba, Mumbai 400005, India

(Dated: May 6, 2019)

One of the hallmarks of the Berezinskii-Kosterlitz-Thouless (BKT) transition in two-dimensional (2D) superconductors is the universal jump of the superfluid density, that can be indirectly probed via the non-linear exponent of the current-voltage IV characteristics. Here, we compare the experimental measurements of IV characteristics in two cases, namely NbN thin films and SrTiO3-based interfaces. While the former display a paradigmatic example of BKT-like non-linear effects, the latter do not seem to justify a BKT analysis. Rather, the observed IV characteristics can be well reproduced theoretically by modelling the effect of mesoscopic inhomogeneity of the superconducting state. Our results offer an alternative perspective on the spontaneous fragmentation of the superconducting background in confined 2D systems.

The progress in material science has made nowadays available a wide class of systems with thickness ranging from a few nanometers down to the atomic-layer limit. The possibility to engineer these effectively two-dimensional (2D) materials in field-effect devices opens also the exciting possibility to tune their quantum-mechanical ground state by changing the electron density. In some remarkable cases, including transition-metal dichalcogenides [1], SrTiO3-based oxide interfaces [2], such as LaAlO3/SrTiO3 and LaTiO3/STO, and the recently discovered twisted graphene [3], the ground state can be continuously tuned from metallic/insulating to superconducting (SC). How the reduced dimensionality influences both phases is still a largely open question, which challenges our basic understanding of the collective fluctuations in 2D systems.

A particularly interesting issue about 2D SC materials regards the very nature of the SC transition, that is expected to be described by the Berezinskii-Kosterlitz-Thouless (BKT) theory [4-6]. This expectation is based on symmetry arguments, since the SC transition in these systems belongs to the same universality class of the 2D XY model, that can be viewed as an effective coarse-grained model for phase of the SC order parameter, especially when the system is thin enough to avoid the screening effects of charged supercurrents [7]. The relevant excitations in this case are topological vortex-like configurations of the phase, and the energy scale is set by the superfluid stiffness $J_s = \hbar^2 n_s / 4 m = \hbar^2 c^2 d / 16 \pi e^2 \lambda^2$, where $n_s$ is the 2D superfluid density, $\lambda$ the penetration depth, and $d$ the film thickness. Within the BKT scenario, the transition to the normal state is driven by the thermal unbinding of vortex-antivortex pairs, that leads to specific signatures, the most striking being the discontinuous jump of $J_s$ from a finite value right below $T_{BKT}$ to zero above it, with an universal ratio $J_s(T_{BKT}) / T_{BKT} = 2/\pi$. This feature is in principle observable via direct measurements of $\lambda(T)$, or it can be inferred from the measurements of the non-linear exponent of the IV characteristics [9], that is ruled by the breaking of vortex-antivortex pairs induced by a large enough current.

In practice, the experimental observation of the BKT transition in real systems is far from being straightforward. The most crucial limitation comes from the fact that in clean thick films $J_s$ is much larger than the critical temperature, so that the temperature $T_{BKT}$ where $J_s(T_{BKT}) \simeq T_{BKT}$ is indistinguishable from the $T_c$ at which pairing disappears. In few-nanometer thick films of conventional superconductors, like NbN or MoGe, the BKT scale becomes accessible, thanks to the fact that in such ultrathin films $n_s$ (and then $J_s$) is strongly suppressed by disorder [1] [2] [10][15] [18] [20]. A similar condition can be reached in STO-based interfaces, where an extremely fragile SC condensate was recently reported [21] [23]. However, in both cases the increase of disorder comes along with an increasing inhomogeneity of the SC background, on length scales that can be different in dif-
In this Letter we analyze the role of SC inhomogeneity in the non-linear IV characteristics of 2D superconductors. We compare two paradigmatic systems: NbN thin films and STO-based interfaces. In the former case we show that the superfluid-stiffness behavior extracted from the measurements of the IV characteristics is consistent with the direct measurements of $\lambda^{-2}$, and both are compatible with a BKT transition, even if the BKT universal jump is smeared by disorder. In contrast, for STO-based interfaces the non-linearity of the IV characteristics cannot be simply ascribed to vortex-antivortex unbinding triggered by a large current, as it happens within the BKT scheme, since this would lead to dramatically overestimate the BKT transition temperature. We then argue that in these systems the non-linearity of the IV characteristics is due to the pair-breaking effect in the weaker SC regions, as the driving current increases, see Fig. 1. By modelling this mechanism within an effective medium (EM) theory, we can reproduce an IV non-linearity in qualitative agreement with the experiments, suggesting that mesoscopic inhomogeneity can essentially hinder the observation of BKT effects at these interfaces.

Let us start with the case of a 3nm NbN thin film, whose IV characteristics are shown in Fig. 2a (see [49] for details of the measurements). As mentioned above, within the BKT scenario the IV characteristics acquire a non-linear dependence near $T_c$, since a large enough current can unbind the vortex-antivortex pairs present below $T_c$. This effect generates a voltage $V \propto n_V(I)$, where the equilibrium density of free vortices $n_V(I)$ scales with a power-law of the applied current, with an exponent proportional to $J_s$ [9]:

$$V \propto I^{a(T)}, \quad a(T) = 1 + \frac{\pi J_s(T)}{T}. \quad (1)$$

In the ideal BKT case [4, 6, 8] the discontinuous jump of $J_s(T)$ at the transition translates into the jump of the IV exponent from $a(T_{BKT}) = 3$ to $a = 1$. In real 2D superconductors, as NbN thin films, the $J_s$ obtained by direct measurements of $\lambda^{-2}$ by means of two-coil mutual inductance technique [1–3] displays a rapid but smeared downturn, see Fig. 2b, that can be explained accounting for a moderate inhomogeneity of the sample, and for the small vortex-core energy [1, 19, 27]. As a consequence, the real BKT temperature is not at the intersection with the universal BKT line $2T/\pi$, but it is the scale $T_c$ where $J_s = 0$. In Fig. 2, we show that $J_s(T) \equiv (a - 1)T/\pi$, extracted from the IV exponent $a(T)$, closely matches below $T_c$, and vanishes at a slightly larger $T$. This phenomenon can be ascribed to finite-size effects, since the current used to estimate $J_s$ from Eq. 1 sets a finite length scale which rounds off the vanishing of the stiffness above $T_{BKT}$ [5]. This is the same effect usually seen while measuring the stiffness at finite microwave frequencies [14, 15, 19–20]. Thus, the critical temperature $T_c^a$ identified by the vanishing of $J_s^a$, i.e., $a(T_c^a) = 1$, is only few percent larger than the true $T_c$ set by dc transport, $R(T_c) = 0$, or by the vanishing of $J_s^a$. We also notice that the temperature $T_{BKT}^a$ where $a(T_{BKT}^a) = 3$ has no particular significance in the realistic case of a smeared jump, but it is still expected to be lower than the real $T_c$. 

![Diagram of BKT physics and inhomogeneity](image-url)
We now turn to the case of STO-based interfaces. Fig. 2 shows the IV characteristics of a LTO/STO sample. The first observation is the presence of a persistent non-linear behavior over a wide temperature range above $T_c$, which is identified by the vanishing of the dc resistivity. This has to be contrasted with the case of NbN, where at $T \approx 1.1 T_c$ the IV characteristics display a full linear behavior, as indeed expected in the metallic case where vortices are already thermally unbound. While this simple observation should already suggest that at different mechanism is at play here, we can nonetheless pursue the BKT analysis based on Eq. 1, and extract the $\alpha(T)$ exponent, see Fig. 2a. The corresponding $J_s^c(T)$, shown in Fig. 2b, vanishes at a temperature $T_{c}^s$ twice as large as the SC critical temperature $T_c$, and even the temperature $T_{c}^BKT$, where $J_s^c$ intersects the BKT line is above $T_c$, highlighting the failure of the BKT analysis. Notice that similar findings for the IV characteristics are very common in the literature in STO-based interfaces and other gated 2D superconductors.

To explain the IV non-linearity in LTO/STO we then propose a simple model, starting from the basic idea that in these systems transport is dominated by percolation through a strongly inhomogeneous background emerging at mesoscopic length scales. By closer inspection of Fig. 2, one sees that the non-linear regime connects smoothly to the linear one, while for NbN in Fig. 2a the non-linear regime is followed by an abrupt jump at
the critical current where normal-state resistance is recovered. Such a difference is due to the fact that in LTO/STO one is in practice analysing non-linear characteristics above $T_c$, where no SC critical current exists but the resistivity is strongly temperature dependent. As already observed before \[7\] \[25\] \[43\], such a broadening of the resistive SC transition cannot be ascribed to usual paraconductivity effects due to SC fluctuations. Instead, it can be well captured by assuming that the metal-to-superconductor transition can be mapped onto a random-resistor-network problem. We then consider a set of local resistances $R_i$ which switch off from the normal-state value $R_N$ to zero at a local temperature $T_{c,i}$, whenever the driving current $I$ is below a threshold $I_{c,i}$. The local $T_{c,i}$ are distributed with a probability $P(T_{c,i})$, with overall weight $w_s = \int dT_{c,i} P(T_{c,i})$. The SC transition can be well understood already in the EM approximation \[7\] \[43\], where the sample resistance $R_{em}(T, I)$ is a solution of the self-consistency equation \[3\] \[9\] \[30\]

$$\sum_i \frac{R_i - R_{em}}{R_i + R_{em}} = 0, \quad (2)$$

where each $R_i$ has a probability $w(T) = \int_0^\infty dT_{c,i} P(T_{c,i})$ of being zero \[49\]. Even though the EM approach neglects spatial correlations, nonetheless it gives insight about the qualitative behaviour of the system. At $I = 0$ the condition $R_{em} = 0$ requires that the fraction $w(T)$ of SC links has reached the percolation threshold $w^*$ (in two dimensions, $w^* = 0.5$, see \[19\] for more details). The shape of $R_{em}(T, I = 0)$ depends on the width of the $P(T_{c,i})$ distribution, that sets the width of the paraconductivity regime, and on the total fraction $w_s$ of SC links. When $w_s$ is smaller than one, i.e., part of the system remains metallic, and slightly larger than the percolation threshold, i.e., $w_s \gtrsim 0.5$, one finds \[7\] that $R_{em}$ has a marked tail above $T_c$, as shown by the numerical solution of Eq. (2) in Fig. 2, in agreement with the experiments. Here, we assumed that the $T_{c,i}$ distribution is gaussian, with average $T_c = 0.24$ K and standard deviation $\sigma = 0.029$ K, and we used $w_s = 0.52$. As a consequence, when the temperature decreases below $T_c + 3\sigma \approx 0.29$ K the condition $T < T_{c,i}$ is fulfilled for a progressively larger fraction of local resistors $R_i$, which then switch off to zero, leading to a suppression of $R_{em}(T, I = 0)$.

A finite driving current is then able to break the weak links between the good SC regions having mesoscopic length scales. Even though we lack a precise information on the nature of the microscopic weak links, we checked \[19\] that the experimental data can be well reproduced by a temperature-dependent critical current following the Ambegaokar-Baratoff formula \[9\]:

$$I_{c,i} = I_{c,0} \frac{\Delta_i(T)}{\Delta_i(0)} \tanh \left( \frac{\Delta_i(T)}{2k_B T} \right), \quad (3)$$

where $I_{c,0}$ at $T = 0$ is independent of the resistor, and the $T = 0$ value of the local gap scales with the local $T_c$ as $\Delta_i(0)/k_B T_c \approx 1.76$. We notice that the temperature dependence of $I_c$ following from Eq. (3) is also in good agreement with a recent analysis of the critical current at STO-based interfaces \[8\]. From Eq. (3) we see that, at a given temperature, only the resistors having $I_{c,i}$ larger than the driving current $I$ can be SC. This leads to a shift of the $R_{em}(T, I)$ curves towards lower temperatures, for increasing current, as shown in Fig. 2. The same effect is also responsible for the observed non-linearity of the IV characteristics shown in Fig. 2. Here we used $I_{c,0} = 5 \mu A$ and a somehow larger width $\sigma = 0.06$ K of the $P(T_{c,i})$ distribution. Despite the simplifications implicit in our model, with this set of parameters we can very well reproduce the experimental curves. Below $T_c$ all the $I_{c,i}$ rapidly collapse towards $I_{c,0}$, which essentially identifies the real critical current in the SC state (see also \[49\] for further details). On the other hand, above $T_c$, in the whole regime of temperatures where $R_{em}(T, I = 0) \ll R_N$ because of the sample inhomogeneity, the non-linear behavior is due to the current-induced breaking of the SC links. As $I$ increases a larger fraction of the SC link becomes normal, and the global resistivity progressively crosses over towards its normal-state value. Finally, the wider distribution of $T_{c,i}$ found in the analysis of IV characteristics may be ascribed to the possible occurrence of avalanche effects that are not easily captured by our simple model. Indeed, as soon as the first weak link breaks down, more current is expected to go through the remaining SC links, which then will be easier to break and so on. As a consequence, the distribution of local SC links can get broader at larger applied currents, as indeed we found while comparing the theoretical simulation with the experiments.

In summary, we analyzed the IV characteristics in two paradigmatic examples of 2D superconductors: NbN thin films, and STO-based oxide interfaces. In the former case we observed a non-linear behavior that is well consistent with the BKT physics. Indeed, even if nanoscopic inhomogeneity of the SC background can partly hinder the manifestation of the BKT signatures, as the universal BKT jump, its essential features remain visible. In contrast, in STO-based interfaces the non-linear IV characteristics cannot be ascribed to a BKT phenomenon, rather to the existence of a strong fragmentation of the SC properties on mesoscopic length scales. By modelling the SC transition with a percolative model, where the fraction of SC regions depends both on the temperature and on the driving current, we can well reproduce the observed non-linearity of the transport. A behavior of the IV characteristics similar to the one observed in our LTO/STO sample is very common in the literature, especially for gated superconductors. Indeed, it has been seen in other STO-based interfaces \[46\] \[48\], in 2D transition-metal dichalcogenides \[51\] \[53\] and also in the recently discovered twisted bilayer graphene \[2\]. As a consequence, while our results question the possibility to observe a
BKT physics in this extremely confined 2D electron gas, they also suggest that non-linear IV characteristics can be used as a benchmark for emergent inhomogeneity in a wide class of superconductors.

Acknowledgements

The work was supported by talia-India collaborative project SuperTop (Italian MAECI PGRO4879 and Indian Department of Science and Technology No. INT/Italy/P-21/2016 (SP)), by the Sapienza University via Ateneo 2017 (prot. RM11715C642E8370) and Ateneo 2018 (prot. RM11816431DBA5AF), by the Delegation Générale à l’Armement (which supported the PhD grant of SH), and by the Nano-SO2DEG project of the JCJC program of the ANR.

Authors contribution

MM, JJ and PR synthesized the NbN sample and performed the measurements on it, AD and RCB provided the LTO/STO sample and JB, SH and NJ performed the measurements on it, GV, SC and LB elaborated the theoretical model and GV performed the numerical calculations. LB conceived the project together with SC, NB and PR. LB wrote the manuscript with inputs from all the coauthors.

* lara.benfatto@roma1.infn.it
F. Debontridder, B. Vignolle, W. Tabis, D. Demaille, L. Largeau, K. Ilin, M. Siegel, D. Roditchev, and B. Leridon, Phys. Rev. B 93, 144509 (2016).

[38] C. Brun, T. Cren and D. Roditchev, Supercond. Sci. Technol. 30, 013003 (2017)

[39] J. Biscaras, N. Bergeal, S. Hurand, C. Feuillet-Palma, A. Rastogi, R. C. Budhani, M. Grilli, S. Caprara, and J. Lesueur, Nat. Mater. 12, 542 (2013).

[40] Gopi Nath Daptary, Shelender Kumar, Pramod Kumar, Anjana Dogra, N. Mohanta, A. Taraphder, and Aveek Bid, Phys. Rev. B 94, 085104 (2016).

[41] G. E. D. K. Prawiroatmodjo, F. Trier, D. V. Christensen, Y. Chen, N. Pryds, and T. S. Jespersen Phys. Rev. B 93, 184504 (2016).

[42] N. Scopigno, D. Bucheli, S. Caprara, J. Biscaras, N. Bergeal, J. Lesueur, and M. Grilli, Phys. Rev. Lett. 116, 026804 (2016).

[43] S. Hurand, A. Jouan, E. Lesne, G. Singh, C. Feuillet-Palma, M. Bibes, A. Barthélémy, J. Lesueur, and N. Bergeal, Phys. Rev. B 99, 104515 (2019).

[44] S. Caprara, M. Grilli, L. Benfatto, C. Castellani, Phys. Rev. B 84, 014514 (2011).

[45] S. Caprara, D. Bucheli, N. Scopigno, N. Bergeal, J. Biscaras, S. Hurand, J. Lesueur and M. Grilli, Supercond. Sci. Technol. 28 044002 (2015).

[46] N. Reyren, S. Thiel, A. Caviglia, L. F. Kourkoutis, G. Hammerl, C. Richter, C. Schneider, T. Kopp, A.-S. Rueetschi, D. Jaccard, M. Gabay, D. Müller, J.-M. Triscone, and J. Mannhart, Science 317, 1196 (2007).

[47] Y.-L. Han, S.-C. Shen, J. You, H.-O. Li, Z.-Z. Luo, C.-J. Li, G.-L. Qu, C.-M. Xiong, R.-F. Dou, L. He, D. Naugle, G.-P. Guo, and J. Nie, Appl. Phys. Lett. 105, 192603 (2014).

[48] A. M. R. V. L. Monteiro, D. J. Groenendijk, I. Groen, J. de Bruijcere, R. Gaudenzi, H. S. J. van der Zant, and A. D. Caviglia, Phys. Rev. B 96, 020504(R) (2017).

[49] See Supplementary Material at...

[50] Indranil Roy, Prashant Chauhan, Harkirat Singh, Sanjeev Kumar, John Jesudasan, Pradnya Parab, Rajdeep Sensarma, Sangita Bose, and Pratap Raychaudhuri, Phys. Rev. B 95, 054513 (2017).

[51] J. M. Lu, O. Zheliuk, I. Leermakers, N. F. Q. Yuan, U. Zeitler, K. T. Law, J. T. Ye, Science 350, 1353 (2015)

[52] L. J. Li, E. C. T. O’Farrell, K. P. Loh, G. Eda, B. Ozyilmaz, and A. H. Castro Neto, Nature 529, 185 (2016).

[53] A. W. Tsen, B. Hunt, Y. D. Kim, Z. J. Yuan, S. Jia, R. J. Cava, J. Hone, P. Kim, C. R. Dean and A. N. Pasupathy, Nature Physics 12, 208 (2016).

[54] R. Landauer, in Electrical Transport and Optical Properties of Inhomogeneous Media, edited by J. C. Garland and D. B. Tanner (American Institute of Physics, New York, 1978), p. 2.

[55] S. Kirkpatrick, Rev. Mod. Phys. 45, 574 (1973).

[56] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. 10, 486 (1962); erratum, 11, 104 (1963).
Supplementary Material for Non-linear IV characteristics in two-dimensional superconductors: Berezinskii-Kosterlitz-Thouless physics vs inhomogeneity

EXPERIMENTAL DETAILS

NbN sample The measurement of $1/\lambda^2$ by means of two-coils mutual inductance were performed on a 3 nm thick NbN film grown on single crystalline MgO substrate. Details of sample preparation are given in [S1], and details of the two-coils technique can be found in [S1][S3]. As shown in Ref. [S1], the corresponding superfluid stiffness $J_\lambda$ closely follows at low temperatures the BCS temperature dependence. However, as $T_c$ is approached one observes a rapid deviation from the BCS fit. By accounting for a moderated inhomogeneity of the sample, and for the small vortex-core energy, one can indeed identify this downturn with the universal BKT transition in all $J_\sigma c$ superconductors: Berezinskii-Kosterlitz-Thouless transition.

The $IV$ measurements were performed by means of a standard 4-probe technique, by using a current source and a nanovoltmeter in a conventional $^4$He cryostat to minimize heating effects. The temperature variation in all $IV$ scans was less than 30 mK. To improve sensitivity, the film was patterned into a 20 µm wide stripe using ion-beam milling with large current contacts and narrow voltage contacts. In Fig. S1a we report the full data set of the $IV$ characteristics, along with the fit done with Eq. (1) of the manuscript. The resulting $a(T)$ coefficient is shown in Fig. S1b, along with the resistivity curve.

STO-based sample For what concerns the case of STO-based interfaces, in this experiment we used a 10 u.c thick LaTiO$_3$ epitaxial layer grown on a TiO$_2$-terminated SrTiO$_3$ single by Pulsed Laser Deposition [S4]. The 3×3 mm LTO/STO sample was thermally anchored to the last stage of a dilution refrigerator and standard four probes resistivity measurements were performed in a Van der Pauw geometry.

THEORETICAL MODEL

The effective medium approximation for the random-resistor network

As explained in the main text, to simulate the mesoscopic inhomogeneity in STO-based samples we described the inhomogeneous SC background by means of a random resistor network (RRN) model. In this picture, every bond represents a resistor $R_i$, made by a mesoscopic region of electrons, with a specific local critical temperature $T_c^i$ randomly distributed. The global resistance $R_{em}$ of the system is given, within the effective-medium approximation (EMA), as a solution of the following self-consistent equation in two dimensions [S5][S6]:

$$\sum_i R_{em} - R_i \frac{R_{em}}{R_{em} + R_i} = 0, \quad (S1)$$

where the sum is carried over all the bonds. An equivalent way to rewrite Eq. (S1) is to sum instead over all possible values $\rho$ attained by the local resistors, weighted with the corresponding probability distribution $p(\rho)$:

$$\int p(\rho) R_{em} - \rho \frac{R_{em}}{R_{em} + \rho} = 0. \quad (S2)$$

Suppose now that the each resistor can take only two constant values: $R_i = R_N$ if the link is in the normal-state, and $R_i = 0$ if the temperature is lowered below the bond critical temperature $T_c^i$, so the temperature dependence in each bond will be $R_i = R_N \theta(T - T_c^i)$, where $\theta(x)$ is the Heavyside step function. If we denote with $P(T_c^i)$ the probability distribution of the local critical temperatures, the probability distribution of resistivity in Eq. (S2) is then $p(\rho) = w(T) \delta(\rho) + [1 - w(T)] \delta(\rho - R_N)$, where $w(T) \equiv \int_T^{+\infty} P(T_c^i) dT_c^i$ is the statistical weight of the superconducting fraction. Eq. (S2) then reduces to:

$$w + (1 - w) \frac{R_{em} - R_N}{R_{em} + R_N} = 0. \quad (S3)$$

The critical temperature $T_c$ of the network, i.e. the temperature where $R_{em} \to 0$, is then defined by Eq. (S3) as the temperature where the SC fraction reaches the percolation threshold of 1/2, as expected in two dimensions [S7].

$$w(T_c) \equiv \int_{T_c}^{+\infty} P(T_c) dT_c \equiv \frac{1}{2}. \quad (S4)$$

For the distribution of local critical temperatures we assume a Gaussian distribution

$$P(T_c^i) = \frac{w_s}{\sqrt{2\pi} \sigma} e^{-\frac{(T_c^i - \overline{T_c})^2}{2\sigma^2}} \quad (S5)$$

with average value $\overline{T_c}$ and variance $\sigma$, $w_s$ representing the total fraction of SC regions in the material. To determine numerically the EMA solution we will resort to the form [S1], by randomly sampling the local $T_c^i$ of each resistor according to the distribution [S4]. At each temperature
the absence of a full microscopic model for the SC puddles, we analyzed different critical-current schemes for the relation $I_c^i = f(T_c^i, T)$ and compared them with the data, in order to get an insight on the physical mechanism at play. The simplest relation one can guess is the Ginzburg-Landau (GL) relation for the critical current:

$$I_c^i = I_0^i(T)(I_c^i - T)^{3/2}.$$  \hfill (S8)

Here $I_0^i(T)$ sets the magnitude of the current, depending on the microscopic structure of the material; in principle, it can be a function of the external temperature $T$ and it can depend on the single resistor. As a starting point, we consider the easiest case $I_0^i(T) = I_0^i$ so the function is analytically invertible and therefore, for the $i$-th resistor to be superconducting, the condition to be fulfilled is $T_c^i \geq T + (I/I_0)^{2/3}$. We thus have

$$R_i = \begin{cases} 1, & \text{if } T_c^i < T_{c_{eff}}, \\ 0, & \text{if } T_c^i \geq T_{c_{eff}}, \end{cases}$$  \hfill (S9)

where $T_{c_{eff}} = T + (I/I_0)^{2/3}$ is the effective temperature perceived by the resistors. In this situation the $R_{em}$ depends on the applied current and the $IV$ characteristics will be in general non-linear.

In Fig. S2, we show the resistivity curve and the $IV$ characteristics at different $T$ in the GL case. The effective resistivity $R_{em}$ (solid red curve in fig. [S2a]) fits well the experimental data at vanishing driving current, using parameters $w = 0.5$, $\sigma = 0.029$ K, $T_c = 0.24$ K. At finite current, using $I_0 = 80 \mu A$, we obtain the

Figure S1. (a) Measurements of the $IV$ characteristics for our 3nm NbN thin film. The same data at selected temperatures are shown in Fig. 2a of the main text. Solid lines are fit with the Eq. (1) of the main manuscript. (b) Temperature dependence of the correspondence $a(T)$ parameter, compared with the resistivity. (c) Temperature dependence of the superfluid stiffness $J_s^i$ extracted from the $a(T)$ exponent, compared to the stiffness $J_s^{em}$ obtained by means of direct two-coils mutual inductance technique.
\( R_{em}(T, I) \) displayed in Fig. S2b, with dashed lines. Despite the fact that one obtains in general an increasing of \( R_{em} \) as \( I \) increases for a fixed temperature, the agreement with the experimental IV curves is very poor. In fig. S2b, we compare the experimental IV characteristics of our LTO/STO sample with the EMA numerical calculations. The experimental data display a tendency to recover the ideal behaviour of a homogeneous superconductor as the temperature decreases, i.e. \( V \propto I(I - I_c) \) when \( T \to 0^+ \). This trend is not captured by the numerical calculation presented in the right panel of fig. S2b, which provide very broad IV characteristics, even at temperatures much lower than the percolation temperature \( T_{perc} \simeq 0.19 \text{K} \). To understand the origin of such drawback, we computed the probability distribution \( P_I(I_c) \) of the critical currents, that is directly related to \( P(T_c^i) \) by \( P_I(I_c) = \int_0^L \delta(I_c - f(T_c^i))P(T_c^i)dT_c^i \), where \( I_c = f(T_c) \) is the functional relation between the local critical current and the local critical temperature. Given its inverse function \( T_c = g(I_c) \) one simply gets

\[
P_I(I_c) = \frac{P_I(g(I_c))}{|f'(g(I_c))|}.
\]

where \( P(x) \) is the distribution given in Eq. S5.

For the GL model of the critical current we showed above that \( f(T_c) = I_0(T_c - T)^{3/2} \) and \( g(I_c) = T_{eff} \), so that \( P_I(I_c) \) takes the following form:

\[
P_I(I_c) = \frac{2w}{3\sigma\sqrt{2\pi}I_0^{3/2}} e^{-\frac{(I_c/I_0)^{3/2}}{2\sigma^2}} I_c^{-1/3}.
\]

The main result is that in this case \( P_I(I_c) \) does not depend on the external temperature \( T \). This is also evident looking at the resistivity at finite \( I \) in fig. S2a, where all curve are obtained by shifting of the resistivity at \( I = 0 \). This is a consequence of the fact that in the GL case the effect of the finite current is just to redefine the effective temperature of the system, as given by Eq. (S9). In contrast, the experimental data shown in the left panel of fig. S2b, suggest that while above \( T_c \) the system recovers smoothly the normal-state resistivity as \( I \) increases, i.e. a wide distribution of local \( I_c^i \) is present, as \( T \) decreases the \( V \) jumps almost suddenly to the normal-state value as \( I \) increasing, signalling that the distribution of local \( I_c \) values should progressively shrink towards a critical value \( I_{c,0} \) that is the same for all the mesoscopic resistors. These observations suggest that a different modelling for \( I_c^i(T) \), able to satisfy two requirements: (i) the zero temperature critical current must be independent on the single resistor \( I_{c,0}^i = \text{const} \), (ii) the critical current should saturate pretty fast to its zero-temperature value in order to recover the behaviour of IV curves at low temperature. The second item is also suggested by recent measurements in an other STO-based sample of the critical current distribution below \( T_c \). We then explored the outcomes of the Ambegaokar and Baratoff \[S9\] formulas, describing the critical current for a weak link between two SC electrodes

\[
I_c R_N = \frac{\pi \Delta(T)}{2e} \tanh \left( \frac{\Delta(T)}{2k_BT} \right).
\]

According to Eq. (S12), the critical current through a
constriction scales with the superfluid density, that is expected to follow a BCS-like relation with \( J_S(T) = J_S(0) \frac{\Delta(T)}{\Delta(0)} \tanh \left( \frac{\Delta(T)}{2k_B T} \right) \). To mimic the BCS temperature dependence of the gap \( \Delta(T) \) in each resistor we use a simple approximated formula that reproduces well the BCS behavior (see inset of Fig. S3a):

\[
f(\tau) = \frac{\Delta_i(T)}{\Delta_i(0)} = \left( 1 - \frac{\tau^4}{3} \right) \sqrt{1 - \tau^4}, \quad \frac{\Delta_i(0)}{k_B T_c} \simeq 1.76
\]

where \( \tau = T/T_c \). The resulting temperature dependence of \( I_c(T) \) from Eq. (S12) is shown in Fig. S3a. As mentioned above, the experimental data suggest that all resistors have the same critical current as \( T \to 0 \). We then assume for each local resistor the following temperature-dependent critical current:

\[
I_c^i(T) = I_{c,0} f(\tau^i) \tanh \left( \frac{1.76 f(\tau^i)}{\tau^i} \right), \quad \text{(S14)}
\]

so that all the local link have the same \( I_{c,0} \) as \( T \to T_c \) but their behavior is different as \( T \) approaches the local transition temperature \( T_c^i \). The \( IV \) characteristics obtained from the model (S14) are shown in Fig. 2g of the main manuscript. As one can see, they reproduce very well the experimental findings. In particular, the model (S14) accounts for the sharpening of the RRN critical current as \( T \) is lowered below \( T_c \), as one can see in Fig. S3b, where we show the \( P_T(I_c) \) obtained by inverting numerically the \( T_c^i \) vs \( I_c^i \) relation from Eq. (S14). Here one recovers a narrowing of the critical-current distribution as \( T \) is lowered below \( T_c \), and already for \( T \simeq 0.06 \) K \( P_T(I_c) \) tends to a delta function centered at \( I_{c,0} \).

---

* lara.benfatto@roma1.infn.it

[S1] M. Mondal, S. Kumar, M. Chand, A. Kamlapure, G. Saraswat, G. Seibold, L. Benfatto, and P. Raychaudhuri, Phys. Rev. Lett. 107, 217003 (2011).
[S2] A. Kamlapure, M. Mondal, M. Chand, A. Mishra, J. Jesudasan, V. Bagwe, L. Benfatto, V. Tripathi, and P. Raychaudhuri, Appl. Phys. Lett. 96, 072509 (2010).
[S3] Indranil Roy, Prashant Chauhan, Harkirat Singh, Sanjeev Kumar, John Jesudasan, Pradnya Parab, Rajdeep Sensarma, Sangita Bose, and Pratap Raychaudhuri, Phys. Rev. B 95, 054513 (2017).
[S4] J. Biscaras, N. Bergeal, A. Kushwaha, T. Wolf, A. Rastogi, R.C. Budhani and J. Lesueur, Nat. Comm. 1, 89 (2010).
[S5] R. Landauer, in Electrical Transport and Optical Properties of Inhomogeneous Media, edited by J. C. Garland and D. B. Tanner (American Institute of Physics, New York, 1978), p. 2.
[S6] S. Kirkpatrick, Rev. Mod. Phys. 45, 574 (1973).
[S7] S. Caprara, M. Grilli, L. Benfatto, C. Castellani, Phys. Rev. B 84, 014514 (2011).
[S8] S. Hurand, A. Jouan, E. Lesne, G. Singh, C. Feuillet-Palma, M. Bibes, A. Barthélémy, J. Lesueur, and N. Bergeal, Phys. Rev. B 99, 104515 (2019).
[S9] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. 10, 486 (1962); erratum, 11, 104 (1963).
Figure S3. a) Temperature dependence of the critical current according to the Ambegaokar-Baratoff model \( S_{14} \). Inset: approximated expression for the BCS-like temperature dependence of the gap, as given by Eq. \( S_{13} \). (b) Probability distribution of the critical current for Ambegaokar-Baratoff model \( S_{14} \), computed from Eq. \( S_{10} \). The critical temperatures are distributed with the non-normalised Gaussian in Eq. \( S_{5} \), using the fitting parameters \( \omega = 0.02 \), \( T_c = 0.24 \) K, \( \sigma = 0.06 \) K and \( I_{c,0} = 5 \mu A \).