Continuity equation and vacuum regions in compressible flows

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Abstract. We investigate the creation and properties of eventual vacuum regions in the weak solutions of the continuity equation, in general, and in the weak solutions of compressible Navier–Stokes equations, in particular. The main results are based on the analysis of renormalized solutions to the continuity and pure transport equations and their inter-relations which are of independent interest.

1. Introduction

In this paper, we consider evolution of the couple \((\varrho, u) = (\varrho(t, x), u(t, x))\) — (density, velocity) of the compressible fluid—over the time interval \(I, I = (0, T), T > 0, t \in \overline{I}\) in a bounded domain \(\Omega \subset \mathbb{R}^d, d \geq 2, x \in \Omega\). We concentrate on the question of the creation of vacuum regions \(\{x \in \Omega | \varrho(t, x) = 0\}\) in this flow. This is one of important open questions in the mathematical fluid mechanics of compressible fluids. It is closely connected to the question of regularity of solutions to the compressible Navier–Stokes equations. If the density is initially bounded away from zero, for weak solutions, it is not excluded that the vacuum may appear in finite time.

We show that if this happens it must happen in a sense smoothly. More precisely, the measure of the set, where the density may be equal to zero, is continuous in time, or, in the other words, the vacuum (if any) creates and evolves continuously in time and the vacuum of positive measure cannot appear instantaneously. The exact formulation of this result is presented in Theorem 1.

More interesting and intriguing is the second result. It translates as follows: Assume that \((\varrho, u)\) is a (standard) weak solution to the compressible Navier–Stokes equations. Then whatever distributional solution \(R\) with a small additional regularity (specified in (26)) of the continuity equation with the same velocity \(u\) we take (whatever arbitrary its initial data are!), \(R\) must develop at any time \(t\) a vacuum region \(\{x | R(t, x) = 0\}\) that includes the vacuum of \(\varrho(t)\), i.e., \(\{x | \varrho(t, x) = 0\}\) is contained in the vacuum set of the function \(R\). This result definitely pleads for a non-existence of vacuum in compressible fluids.

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flows at least in many physically reasonable situations. The exact formulation of this result is given in Theorem 2 and its Corollaries 1, 2, 3.

On the other hand, it is important to recall that if the velocity field $u \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d))$ (this is the generic situation for flows of Newtonian fluids with constant viscosities), there is no direct way of constructing solutions $R$ to the continuity equation with the given velocity unless $\text{div } u \in L^1(I; L^\infty(\Omega))$—cf. DiPerna–Lions [8, Proposition II.1]. Indeed, existence of solutions to the continuity equation with the transporting velocity fields in spaces $L^1(I; W^{1,p}(\Omega; \mathbb{R}^d))$, $p \in [1, \infty)$ only, is, in general, an open problem.

The conclusions of our paper described above are based on nowadays classical results and techniques for the continuity and transport equation that have been forged within the process of the development of the existence theory for weak solutions to the compressible Navier–Stokes equations and recently also for the mixtures of compressible fluids. They are all inspired by the classical regularization technique implemented to the investigation of transport equations with transport coefficients in Sobolev spaces in the seminal work of DiPerna–Lions [8]. (The spaces needed for the results in [8] are those needed in the Friedrichs lemma about commutators with $\alpha = \infty$, $p = 1$, cf. Lemma 3.) Some of them are valid only within the functional setting of the transport theory [8] (namely those dealing with extension of distributional solutions to weak solutions (up to the boundary), time integration of weak or distributional solutions and passage from distributional or weak solutions to renormalized distributional or weak solutions). They are formulated in Sect. 3.2.2 in Theorems 5 and 6. Some of them, namely those valid for the renormalized solutions, must go beyond the transport theory [8] (in the sense that the transporting velocity belongs still to Sobolev spaces but one requires less summability of the solution then the summability required in [8])—in order to get stronger results with respect to the constitutive laws of pressure in the applications to compressible Navier–Stokes equations. (This is notably the case of Theorems 3 and 4 in Sect. 3.2.1). Indeed, all available constructions of weak solutions to the compressible Navier–Stokes equations provide a couple $(\rho, u)$ which satisfies the continuity equation in the renormalized sense. The latter results are often formulated in the mathematical literature in a particular functional setting applicable to the concrete situation without ambition to full generality, see Lions [18], Feireisl [10,12,24] if we limit ourselves to the monographs only. Our aim is to provide generalization and synthesis of the results we need and prove them in their full generality, either for the sake of completeness or if we could not find a reliable exhausting reference.

A new approach to the compactness in the compressible Navier–Stokes equations allowing to treat some other physically different situations then [10,12,13] has been introduced by Bresch, Jabin [5], deriving, in particular a “log log estimate” for the Friedrichs type commutator, [6, Theorem 2.3.6] in the DiPerna–Lions functional

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1The various notions of solutions used in the above text are rigorously defined in Sect. 2.
framework. This theory does not allow to go beyond the DiPerna–Lions functional setting and seems at the time being so far in-exploitable for our purpose.

Among the main auxiliary questions which has to be answered in order to apply the theory of transport equations to the compressible fluid dynamics in general, and to the investigation of the vacuum states, in particular, are the following:

1. What are the least conditions imposed on the transporting velocity $u$ (in terms of Sobolev spaces) and solution $\varrho$ of the continuity equation (in terms of Lebesgue spaces) allowing to pass from renormalized distributional or weak solutions to time integrated weak solutions? The answers to these questions are subject of Theorems 3 and 4.

2. What are the least conditions on the couple $(\varrho, u)$ (in the same functional setting) to pass from distributional solutions of the continuity or pure transport equations to the weak (up to the boundary) solutions (eventually to the renormalized weak solutions), and from distributional or weak solutions to their time integrated counterparts (eventually to the renormalized time integrated counterparts)? The answer to these questions are given in Theorems 5 and 6.

3. How are interconnected solutions of pure transport equation and continuity equations? and what does this interconnection imply for the formation of vacuum in the compressible flows? The first question is treated in Theorem 7. The last question is object of Theorems 1, 2 and their Corollaries 1, 2.

It is to be noticed that the conditions mentioned in Items 1.-3. determine in large extend the admissible constitutive laws in the theory of weak solutions to compressible Navier–Stokes equations, [5, 9, 12, 13]. The usefulness of the subject of Item 4 was firstly discovered in connection with the investigation of weak solutions of systems describing compressible mixtures, see [19, 22, 23, 27].

Our approach is exclusively Eulerian. The Lagrangian approach (dealing with characteristics of the vector field $u$ rather than with the transport equation, and translating them afterward to the Eulerian vocabulary) introduced in seminal paper of Ambrosio [1] allows to extend some results of [8] (namely those related to existence, uniqueness and passage from distributional or weak to renormalized distributional or weak solutions) to $L^1(I; BV(\Omega; R^d))$ vector fields$^2$ with divergence always in $L^1(I; L^\infty(\Omega))$.

It was extended and generalized in several papers by Ambrosio, Crippa, De Lellis [2, 7] and others. Further, deep generalization of this approach consisting in replacing the condition imposed on the divergence of $u$ by a weaker condition postulating that “$u$ is weakly incompressible” is due to Bianchini, Bonicatto [3]. The latter result (which is essentially about the properties of the flow of the vector field $u$) implies as a corollary the uniqueness for the pure transport equation under “weak incompressibility” condition. (It is not without interest, that a stronger form of this corollary can be obtained within the Sobolev functional setting quite easily by the purely Eulerian approach [22, Proposition 5].) In contrast with conservation laws, where the $BV(\Omega)$

$^2$The space $BV(\Omega)$ is the space of functions with bounded variations.
Theory found many applications, it has not so far appeared to be exploitable in the theory of compressible Navier–Stokes equations.

The paper is organized as follows. In Sect. 2, we introduce various notions of solutions to the continuity and transport equations that will be used in the sequel. Section 3 is devoted to the formulation of the main results, and of the auxiliary results needed for their proofs, which are of independent interest. Theorems 1 and 2 (and Corollary 4 in Sect. 3.1) deal with the properties of vacuum in any renormalized time integrated weak solution of the continuity equation. This implies immediately the same properties of vacuum in any renormalized weak solution to the compressible Navier–Stokes equations. This issue is discussed in Sect. 3.3 (see namely Corollary 4 and Remark 3). Theorems 1–2 and Corollaries 1, 2 and 4 are main results of the paper. Their proofs require a good understanding of the relation between various types of solutions introduced in Sect. 2. This issue of independent interest is treated in Sect. 3.2. The matters of time integration of renormalized distributional of weak solutions are treated in Sect. 3.2.1 (see Theorems 3, 6). The passage from distributional to renormalized weak solution is handled in Sect. 3.2.2 (see Theorems 5, 7). The passage from continuity and pure transport equation to a continuity equation is formulated in Sect. 3.2.3 (see Theorem 7). The remaining part of the paper is devoted to the proof of Theorems (1–7). Section 4 collects three preliminary classical results whose conclusions will be frequently used throughout the proofs. Section 5 is devoted to the proof of Theorems 3–4, Sect. 6 to the proof of Theorems 5–6 and Sect. 7 to the proof of Theorem 7. Finally in the last section, we combine the results of Theorems 3–7 to prove the main theorems: Theorems 1 and 2.

We finish this section by introducing the functional spaces and some notations. In what follows, we use standard notation for the Lebesgue and Sobolev spaces \( L^p(\Omega) \) and \( W^{1,p}(\Omega) \) with the corresponding norms \( \|u\|_{L^p(\Omega)} \) and \( \|u\|_{W^{1,p}(\Omega)} \), respectively. We do not distinguish the notation for the norms for scalar- and vector-valued functions. However, the vector-valued functions are printed boldface and we write \( \mathbf{u} \in L^p(\Omega; \mathbb{R}^d) \) instead of \( \mathbf{u} \in L^p(\Omega) \), similarly for other functions spaces. For function spaces of time and space-dependent function, we use the standard notation for the Bochner spaces \( L^p(I; L^q(\Omega)) \) or \( L^p(I; W^{1,q}(\Omega)) \), respectively. We also use the notation \( C([0, T]; L^p(\Omega)) \) for continuous functions on interval \([0,T]\) with values in \( L^p(\Omega) \) and \( C_{\text{weak}}([0, T]; L^p(\Omega)) \) a vector subspace of \( L^\infty(0, T; L^p(\Omega)) \) of functions continuous on \([0, T]\) with respect to the weak topology of \( L^p(\Omega) \). More exactly, a function \( f: [0, T] \mapsto L^p(\Omega) \) (defined on \([0, T]\)) belongs to \( C_{\text{weak}}([0, T]; L^p(\Omega)) \) iff \( f \in L^\infty(0, T; L^p(\Omega)) \) and for all \( \eta \in L^p(\Omega) \) the map \( \tau \mapsto \int_\Omega f(\tau)\eta \, dx \) is continuous on interval \([0, T]\). For the norms in Bochner spaces, we use the function space as full index, as e.g., \( \|u\|_{L^p(I; L^q(\Omega))} \) or \( \|u\|_{L^p(I; W^{1,q}(\Omega))} \). Throughout the paper, the constants are denoted by \( C \) and their value may change even in the same formula.
2. Various notions of solutions to continuity and pure transport equations

The main results of this paper will largely rely on various notions of (weak) solutions to the continuity and pure transport equations and their inter-relations. We shall introduce these notions in this section.

We consider the equations on the time-space cylinder $Q = I \times \Omega$, $\Omega$ a bounded open set in $\mathbb{R}^d$, $d \geq 2$, and $I = (0, T)$, $T > 0$ a time interval. The equations read:

1. Continuity equation

$$\partial_t \varrho + \text{div} \ (\varrho u) = 0 \ \text{in} \ (0, T) \times \Omega$$

with initial condition

$$\varrho(0, \cdot) = \varrho_0(\cdot) \ \text{in} \ \Omega.$$  

2. Pure transport equation

$$\partial_t s + u \cdot \nabla s = 0 \ \text{in} \ (0, T) \times \Omega$$

with initial condition

$$s(0, \cdot) = s_0(\cdot) \ \text{in} \ \Omega.$$  

We shall consider several different notions of solutions to these equations.

**Definition 1.** *(Continuity equation)* Let

$$u \in L^1(I \times \Omega; \mathbb{R}^d), \ \text{div} \ u \in L^1(I \times \Omega).$$

(3)

We say that function

$$\varrho \in L^1(I \times \Omega) \ \text{such that} \ \varrho u \in L^1(I \times \Omega; \mathbb{R}^d)$$

is:\

1. **Distributional solution** to the continuity equation (1) iff it satisfies (1) in the sense of distributions over the time-space, namely iff

$$\int_0^T \int_{\Omega} (\varrho \partial_t \varphi + \varrho u \cdot \nabla \varphi) \, dx \, dt = 0$$

holds for arbitrary $\varphi \in C^\infty_c(I \times \Omega)$.  

2. **Weak solution** to the continuity equation (1) iff

equation(5) holds with arbitrary $\varphi \in C^\infty_c(I \times \Omega)$.  

\[ \text{In some cases, it would be enough to assume} \ u \in L^1_{\text{loc}}(I \times \Omega; \mathbb{R}^d), \ \text{div} \ u \in L^1_{\text{loc}}(I \times \Omega) \ \text{and condition (4) could be weaken to} \ \varrho \in L^1_{\text{loc}}(I \times \Omega), \ \text{such that} \ \varrho u \in L^1_{\text{loc}}(I \times \Omega; \mathbb{R}^d). \ \text{We do not consider this situation since it is irrelevant from the point of view of the present paper.} \]
3. **Time integrated distributional solution** to the continuity equation (1) iff \( \varrho \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and it holds
\[
\int_{\Omega} (\varrho \varphi)(\tau, \cdot) \, dx - \int_{\Omega} (\varrho \varphi)(0, \cdot) \, dx = \int_0^\tau \int_{\Omega} (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi) \, dx \, dt
\] (7)
for any \( \varrho \in C_c^\infty(\overline{T} \times \Omega) \) and any \( \tau \in \overline{T} \).

4. **Time integrated weak solution** to the continuity equation (1) iff \( \varrho \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and
\[
\text{equation (7) holds with arbitrary } \varphi \in C_c^\infty(\overline{T} \times \Omega) \text{ and any } \tau \in \overline{T}.
\] (8)

5. **Renormalized distributional solution** to the continuity equation (1) iff in addition to (5),
\[
\int_0^T \int_{\Omega} \left( b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla \varphi - (b'(\varrho) \varrho - b(\varrho)) \text{div} \mathbf{u} \varphi \right) \, dx \, dt = 0
\] (9)
holds with all \( \varphi \in C_c^\infty(I \times \Omega) \) and all renormalizing functions
\[ b \in C^1([0, \infty)), \quad b' \in C_c([0, \infty)). \] (10)

6. **Renormalized weak solution** to the continuity equation (1) iff in addition to (6),
\[
\text{equation (9) holds with all } \varphi \in C_c^\infty(I \times \Omega) \text{ and all } b \text{ in (10)}.
\] (11)

7. **Renormalized time integrated distributional solution** to the continuity equation (1) iff \( b(\varrho) \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and in addition to (7),
\[
\int_{\Omega} (b(\varrho) \varphi)(\tau, \cdot) \, dx - \int_{\Omega} (b(\varrho) \varphi)(0, \cdot) \, dx \\
= \int_0^T \int_{\Omega} \left( b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla \varphi - (b'(\varrho) \varrho - b(\varrho)) \text{div} \mathbf{u} \varphi \right) \, dx \, dt
\] (12)
holds with all \( \varphi \in C_c^\infty(\overline{T} \times \Omega) \), all \( \tau \in \overline{T} \) and all renormalizing functions \( b \) in the class (10).

8. **Renormalized time integrated weak solution** to the continuity equation (1) iff \( b(\varrho) \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and in addition to (8),
\[
\text{equation (12) holds with all } \varphi \in C_c^\infty(\overline{T} \times \Omega),
\] (13)
all \( \tau \in \overline{T} \) and all \( b \) in (10).

Due to the presence of term containing \( s \text{div} \mathbf{u} \) in the weak formulation of the pure transport equation, the definition of weak solutions/renormalized weak solutions in this case asks for better summability of the quantity \( s \) (compared to the summability required for \( \varrho \) expressed through assumption (4) in the case of the continuity equation).

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\(^4\text{Conditions (10), (3) and (4) immediately ensure that } b(\varrho), (b(\varrho) \mathbf{u} + (\varrho b'(\varrho) - b(\varrho)) \text{div} \mathbf{u}) \in L^1(I \times \Omega). \) As will be seen later, in fact \( b(\varrho) \in C([0, T]; L^1(\Omega)) \), too.
Definition 2. (Pure transport equation) Let \( u \) satisfy (3). We say that function
\[
s \in L^1(I \times \Omega) \quad \text{such that} \quad su \text{ and } s \text{ div } u \in L^1(I \times \Omega)
\] is\(^5\):

1. **Distributional solution** to the pure transport equation (2) iff it satisfies (2) in the sense of distributions over the time-space, namely iff
\[
\int_0^T \int_\Omega (s \partial_t \varphi + su \cdot \nabla \varphi + s \text{ div } u \varphi) \, dx \, dt = 0
\]
holds for arbitrary \( \varphi \in C_c^\infty(I \times \Omega) \).

2. **Weak solution** to the pure transport equation (2) iff equation (15) holds with arbitrary \( \varphi \in C_c^\infty(I \times \Omega) \).

3. **Time integrated distributional solution** to the pure transport equation (2) iff
\[
s \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \quad \text{and it holds}
\]
\[
\int_\Omega (s\varphi)(\tau, \cdot) \, dx - \int_\Omega (s\varphi)(0, \cdot) \, dx = \int_0^\tau \int_\Omega (s \partial_t \varphi + su \cdot \nabla \varphi) \, dx \, dt
\]
for any \( \varphi \in C_c^\infty(\overline{T} \times \Omega) \) and any \( \tau \in \overline{T} \).

4. **Time integrated weak solution** to the pure transport equation (2) iff \( s \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and
\[
equation (17) \text{ holds with arbitrary } \varphi \in C_c^\infty(\overline{T} \times \Omega) \text{ and any } \tau \in \overline{T}.
\]

5. **Renormalized distributional solution** to the pure transport equation (2) iff in addition to (15),
\[
\int_0^\tau \int_\Omega \left( b(s) \partial_t \varphi + b(s)Bu \cdot \nabla \varphi + b(s) \text{ div } u \varphi \right) \, dx \, dt = 0
\]
holds with all \( \varphi \in C_c^\infty(I \times \Omega) \) and all renormalizing functions \( b \) belonging to class (10).

6. **Renormalized weak solution** to the pure transport equation (2) iff in addition to (16),
\[
equation (19) \text{ holds with all } \varphi \in C_c^\infty(\overline{T} \times \Omega) \text{ and all } b \text{ in (10).}
\]

7. **Renormalized time integrated distributional solution** to the pure transport equation (2) iff \( b(\varphi) \in C_{\text{weak}}(\overline{T}; L^1(\Omega)) \) and in addition to (17),
\[
\int_\Omega (b(s)\varphi)(\tau, \cdot) \, dx - \int_\Omega (b(s)\varphi)(0, \cdot) \, dx
\]
\[
\quad = \int_0^\tau \int_\Omega \left( b(s) \partial_t \varphi + b(s)Bu \cdot \nabla \varphi + b(s) \text{ div } u \varphi \right) \, dx \, dt
\]
\[
equation (21)
\]
\(^5\)In some cases, it would be enough to assume \( u \in L^1_{\text{loc}}(I \times \Omega; \mathbb{R}^d) \), \( \text{div } u \in L^1_{\text{loc}}(I \times \Omega) \) and condition (14) could be weaken to \( s \in L^1_{\text{loc}}(I \times \Omega) \), such that \( su, s \text{ div } u \in L^1_{\text{loc}}(I \times \Omega) \). We do not consider this situation since it is irrelevant from the point of view of the present paper.
holds with all \( \varphi \in C_c^\infty(\overline{I} \times \Omega) \), all \( \tau \in \overline{I} \) and all renormalizing functions \( b \) in the class (10).

8. **Renormalized time integrated weak solution** to the pure transport equation (2) iff \( b(s) \in C_{\text{weak}}(\overline{I}; L^1(\Omega)) \) and in addition to (8),

\[
\text{equation (21) holds with all } \varphi \in C_c^\infty(\overline{I} \times \Omega), \quad \text{all } \tau \in \overline{I} \text{ and all } b \text{ in (10).} (22)
\]

3. Main results

The primal goal of this paper is the investigation of the vacuum formation in the weak solution \((\varrho, \mathbf{u})\)—in the compressible Navier–Stokes equations. We shall prove that the volume of eventual vacuum set evolves continuously in time and, more surprisingly, if there is no vacuum at time 0 and there is a vacuum of non-zero measure at some time \( \tau \in (0, T) \), then any distributional solution \( R \) (with certain reasonable summability properties) to the continuity equation (with the same transporting velocity \( \mathbf{u} \))—if it exists—admits at time \( \tau \) a larger vacuum set \( \{ x \in \Omega | R(\tau) = 0 \} \) than the vacuum set of \( \varrho \). This property does not imply the absence of vacuum but pleads in favor of the sparseness of the event of creation of vacuum in compressible flows.

All these properties rely exclusively on the properties of continuity and transport equations. We shall therefore formulate them as such in Sect. 3.1, postponing the formulation in the context of Navier–Stokes equations to Sect. 3.3.

The proofs of results in Sect. 3.1 rely essentially on the properties and inter-relation of various types of weak/renormalized solutions to the continuity and transport equations and their combinations, which are of independent interest. Bits of pieces of some of these results (all of them having ground in the seminal work by DiPerna and Lions [8]) are non systematically spread through the mathematical literature in several (mostly recent) papers dealing with the existence of weak solutions to the compressible Navier–Stokes equations and compressible mixtures as auxiliary tools, [10,12,13,23,24,27]. We will state in Sect. 3.2 those of these properties needed in this paper in their full generality and provide their detailed proofs.

3.1. Properties of vacuum in the weak solution of the continuity equation

The first theorem dealing with vacuum sets in the continuity equation reads.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain. Let \( 1 \leq q, p \leq \infty \) and \( \mathbf{u} \in L^p(0, T; W^{1,q}(\Omega; \mathbb{R}^d)) \). Let

\[
0 \leq \varrho \in C_{\text{weak}}(\overline{I}; L^\gamma(\Omega)), \quad \gamma > 1
\]

be a renormalized time integrated weak solution to the continuity equation (1) with transporting velocity \( \mathbf{u} \) (i.e., it belongs to class (4), satisfies equation (8) and equation (13) with the renormalizing functions \( b \) from (10)).
Then, the map \( t \mapsto s_\varrho(t, \cdot) := 1_{\{x \in \Omega | \varrho(t, x) = 0\}}(\cdot) \) belongs to \( C([0, T]; L^r(\Omega)) \) with any \( 1 \leq r < \infty \), and it is a time integrated renormalized weak solution of the pure transport equation (2) with transporting velocity \( u \). In particular,

\[
\| \{ x \in \Omega | \varrho(t, x) = 0 \} \|_d \in C([0, T]).
\]

In the above, \( |A|_d \) denotes the \( d \)-dimensional Lebesgue measure of the set \( A \).

The second theorem about the vacuum issue reads.

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain. Let

\[
1 \leq q, p, \alpha, \beta \leq \infty, \quad (q, \beta) \neq (1, \infty), \quad \frac{1}{\beta} + \frac{1}{q} \leq 1, \quad \frac{1}{\alpha} + \frac{1}{p} \leq 1.
\]

Let \( \varrho \) from class (23) be a renormalized time integrated weak solution to the continuity equation (1) with transporting velocity \( u \in L^p(0, T; W^{1,q}_0(\Omega; \mathbb{R}^d)) \) (i.e., it belongs to class (4), satisfies equation (5) and equation (9) with renormalizing functions \( b \) from (10)).

Let

\[
0 \leq R \in L^\infty(0, T; \tilde{L}^\gamma(\Omega)) \cap L^\alpha(0, T; L^\beta(\Omega)), \quad \tilde{\gamma} > 1
\]

be a distributional solution to the continuity equation (1) with the same transporting velocity \( u \). Then

1. Function \( R \) belongs to

\[
R \in C_{\text{weak}}([0, T]; \tilde{L}^\gamma(\Omega)) \cap C([0, T]; L^r(\Omega)), \quad 1 \leq r < \tilde{\gamma}
\]

and it is a renormalized time integrated weak solution of the continuity equation (1).

2. The map \( t \mapsto (s_\varrho R)(t) \) belongs to \( C([0, T]; L^r(\Omega)) \) with any \( 1 \leq r < \tilde{\gamma} \) and it is a renormalized time integrated weak solution of the continuity equation (1) (with the same transporting velocity). In particular,

\[
\int_\Omega (s_\varrho R)(t, \cdot) \, dx = \int_\Omega (s_\varrho R)(0, \cdot) \, dx
\]

for all \( t \in [0, T] \).

3. If further \( \varrho(0, \cdot) > 0 \ a.e. \ in \ \Omega \), then, up to sets of \( d \)-dimensional Lebesgue measure zero, for all \( t \in (0, T] \)

\[
\{ x \in \Omega | \varrho(t, x) = 0 \} \subset \{ x \in \Omega | R(t, x) = 0 \}.
\]

The second theorem has the following immediate consequences:

**Corollary 1.** Let \( q, p, \alpha, \beta \) verify conditions (25) and \( \tilde{\gamma}, \gamma > 1 \). Let \( \Omega, \varrho, u \) verify assumptions of Theorem 2, where \( \varrho(0, x) > 0 \). (In particular, \( \varrho \) is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity \( u \).)
Let $\tau \in (0, T)$. Suppose that continuity equation (1) with transporting velocity $u$ admits at least one distributional solution $R$ belonging to class (26) which does not admit in $\Omega$ a vacuum at time $\tau$, i.e., $R(\tau) > 0$ a.e. in $\Omega$.

Then, $\varrho$ does not admit a vacuum at time $\tau$, i.e.,

$$||x \in \Omega | \varrho(\tau, x) = 0||_d = 0.$$ 

**Corollary 2.** Let $(q, p, \alpha, \beta)$ and $(q, p, \tilde{\alpha}, \tilde{\beta})$ verify conditions (25) and $\tilde{\gamma}$, $\gamma > 1$. Let $\Omega$, $u$ be as in Theorem 2. Suppose that $\varrho$ belongs to class $(27)_{\alpha, \beta, \gamma}$, while $R$ belongs to class $(27)_{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}}$, and that each of $\varrho$ and $R$ represents a distributional solution to the continuity equation (1) with the transporting velocity $u$. Then, $\varrho \in C_{\text{weak}}([0, T]; L^\gamma(\Omega))$, $R \in C_{\text{weak}}([0, T]; L^{\tilde{\gamma}}(\Omega))$, and they are both renormalized time integrated solutions of the continuity equation (1). Moreover, if $\varrho(0, \cdot) > 0$ and $R(0, \cdot) > 0$ a.e. in $\Omega$ then

$$\forall t \in [0, T], \ {x | \varrho(t, x) = 0} = {x | R(t, x) = 0}.$$ 

**Corollary 3.** Let $q, \alpha, \beta, \gamma, \tilde{\gamma}$ verify assumptions of Corollary 1 with $p = \infty$.

Let $\Omega$, $\varrho$, $u$ verify assumptions of Corollary 1. (In particular, $0 \leq \varrho$ is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity $u$ and $\varrho(0, x) > 0$.) We assume that $u$ is time independent, i.e., $u = u(x)$, $u \in W_{0}^{1, q}(\Omega; R^d)$.

Suppose that continuity equation (1) with transporting velocity $u$ admits at least one (local in time) distributional solution $R$ on $(0, T') \times \Omega$ with some $T' > 0$ belonging to class $(26)_{T = T'}$ which does not admit in $\Omega$ a vacuum at time $\tau \in (0, T')$, i.e., there exists $\tau \in (0, T')$ such that $R(\tau) > 0$ a.e. in $\Omega$.

Then, $\varrho$ does not admit a vacuum at any time in $[0, T]$, i.e.,

$$\forall t \in [0, T], \ ||x \in \Omega | \varrho(t, x) = 0||_d = 0.$$ 

**Remark 1.**

1. In practice, if $u \in L^p(0, T; W^{1, q}(\Omega; R^d))$, condition (4) in Theorems 1 and 2 is ensured by assumption

$$1 < \gamma \leq \infty, \ \frac{1}{\gamma} + \frac{1}{q} \leq 1 + \frac{1}{d}. \quad (29)$$

Alternatively, condition (4) can be achieved by requiring that $u \in L^p(0, T; W^{1, q}(\Omega; R^d))$, $\varrho \in L^p(0, T; L^{\beta}(\Omega))$, where $p, q, \alpha, \beta$ verifies (25). In the theory of weak solutions to compressible Navier–Stokes equations, the former setting provides stronger results, cf. Sect. 3.3.

2. Condition $u|_{I \times \partial \Omega} = 0$ in Theorem 2 and Corollaries 1–3 can be replaced by $u \cdot n|_{I \times \partial \Omega} = 0$.

3. We notice that Theorem 1 holds independently of the boundary condition imposed on $u$ at the boundary (since it deals with weak solutions in the sense of Definition 1. This is not the case of Theorems 2 and Corollaries 1, 2. Nevertheless, they continue to hold if we replace $W_{0}^{1, q}(\Omega)$ by $W^{1, q}(\Omega)$ provided we
suppose that $R$ is a renormalized time integrated weak solution (instead of a renormalized time integrated distributional solution). Anyway, however, in all these cases the condition $\rho \mathbf{u} \cdot \mathbf{n}|_{I \times \partial \Omega} = 0$ must always be satisfied at least in the weak sense; it is implicitly required in the weak formulation of the equation through the fact that the test functions do not vanish on the boundary.

3.2. Relations between various types of solutions to continuity and pure transport equations

The proofs of Theorems 1 and 2 are based on the systematic study of relations and properties of the various types of weak solutions to the continuity and pure transport equations and their inter-relations. In this section, we formulate the adequate results. They are, indeed, of independent interest.

3.2.1. Time integration of renormalized distributional/weak solutions

The main message of this subsection is the observation that any renormalized distributional (or weak) solution of the continuity equation/pure transport equation (introduced in Definitions 1–2) admits—under certain reasonable conditions—a representative that is continuous on the time interval $[0, T]$ with values in $L^1(\Omega)$, and that both continuity/pure transport and renormalized continuity/pure transport equations can be integrated up to the end-points of any time interval $[0, \tau], \tau \in [0, T]$.

**Theorem 3.** (Continuity equation) Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be a bounded domain with Lipschitz boundary. Let $\mathbf{u} \in L^p(I; W^{1,q}(\Omega; \mathbb{R}^d)), 1 \leq p, q \leq \infty$. Suppose that

$$0 \leq \rho \in L^\infty(I; L^\gamma(\Omega)), \gamma > 1.$$  (30)

Then, the following statements are true:

1. If $\rho$ is a renormalized distributional solution of the continuity equation with transporting velocity $\mathbf{u}$ (i.e., it belongs to class (4) and satisfies (5), (9) with any renormalizing function $b$ from (10)), then function $\rho$ and functions $b(\rho)$ with any $b$ from (10) belong to the class $(27)_{\gamma'} = \gamma$ and $\rho$ is a renormalized time integrated distributional solution of the continuity equation with transporting velocity $\mathbf{u}$ (i.e., it belongs to class (4) and satisfies identities (7) and (12) with any renormalizing function $b$ from (10)).

2. If $\rho$ is a renormalized weak solution of the continuity equation with transporting velocity $\mathbf{u}$ (i.e., it belongs to class (4) and it satisfies equations (6), (11) with any $b$ from (10)), then function $\rho$ and functions $b(\rho)$ with any $b$ from (10) belong to the class $(27)_{\gamma'} = \gamma$ and it is a renormalized time integrated weak solution of the continuity equation with transporting velocity $\mathbf{u}$ (i.e., it belongs to class (4) and it satisfies identities (8) and (13) with any renormalizing function $b$ in class (10)).

3. Particularly, in both cases, $\rho \in C(I; L^r(\Omega)), 1 \leq r < \gamma$. 
The same statement holds for the pure transport equation. The theorem reads:

**Theorem 4.** (Pure transport equation) Let \( \Omega \) and \( u \) satisfy assumptions of Theorem 3 and let \( s \) fulfill (30). Then, the following statements are true:

1. If \( s \) is a renormalized distributional solution of the pure transport equation with transporting velocity \( u \) (i.e., it belongs to class (14) and satisfies identities (15), (19)), then \( s \) and \( b(s) \) with any \( b \) from (10) belong to class (27) \( \widetilde{\gamma} = \gamma \) and \( s \) is a time integrated renormalized distributional solution of the pure transport equation (i.e., it belongs to class (14) and it satisfies identities (17) and (21) with any renormalizing function \( b \) from (10)).

2. If \( s \) is a renormalized weak solution of the continuity equation with transporting velocity \( u \) (i.e., it belongs to class (14) and it satisfies identities (16) and (20)), then \( s \) and \( b(s) \) with any \( b \) from (10) belong to class (27) \( \widetilde{\gamma} = \gamma \) and \( s \) is renormalized time integrated weak solution (i.e., it belongs to class (14) and it satisfies identities (18) and (22) with any renormalizing function \( b \) from (10)).

3. Particularly, in both cases, \( s \in C(\overline{I}; L^r(\Omega)) \), \( 1 \leq r < \gamma \).

**Remark 2.**

1. Concerning the continuity equation: In practice, if we have \( u \in L^p(0, T; W^{1,q}(\Omega; \mathbb{R}^d)) \), condition (4) in Theorem 3 can be ensured by assumption (30) with \( \gamma \) from (29). If it is so, then the class of admissible renormalizing functions in Theorem 3 can be extended from (10) to \(^6\)

\[
\begin{align*}
    b &\in C^1([0, \infty)), \quad b(\varrho) \leq c(1 + s^{\gamma/q^*_q}), \quad \varrho b'(\varrho) - b(\varrho) \leq c(1 + \varrho^{\gamma/q'_q}).
\end{align*}
\]

This is the setting that allows to get the strongest results in applications to weak solutions to compressible fluids, see Sect. 3.3.

Alternatively, condition (4) can be achieved by requiring that \( u \in L^p(0, T; W^{1,q}(\Omega; \mathbb{R}^d)) \), \( \varrho \in L^\alpha(0, T; L^\beta(\Omega)) \), where \( p, q, \alpha, \beta \) verify (25), as mentioned in Remark 1. In this case, one can take the true condition (30) with any \( \gamma > 1 \). Condition (25) is however more restrictive than (29) from the point of view of applications to compressible fluids. This setting is merely used only at the level of approximations of underlying compressible systems during the process of construction of weak solutions. Note finally that part of the first two claims of Theorem 3 hold without the requirement that the solutions is renormalized, i.e., if \( \varrho \) is a distributional solution, then under the assumptions of this theorem it is a time integrated distributional solution, similarly in the case of weak solution. On the other hand, Item 3 requires that the solution is renormalized.

2. Concerning the transport equation: In practice, if the transporting velocity \( u \in L^p(0, T; W^{1,q}(\Omega; \mathbb{R}^d)) \) and \( s \in L^q(0, T; L^\beta(\Omega)) \), it is condition (25) with \( s \) replaced by \( \varrho \) which guarantees satisfaction of condition (14) in Theorem 4. In this situation, the class of admissible renormalizing functions in Theorem 4 can

---

\(^6\) Here and in the sequel, the exponent \( q' \) is the Hölder conjugate exponent for \( q \), \( q^*_q \) is the Sobolev exponent for \( q \) (and \( q'_q \) is the Hölder conjugate exponent for \( q^*_q \)).
be extended from (10) to
\[
b \in C^1([0, \infty)), \quad b(s) \leq c(1 + s^{\nu/q'}).
\] (32)

Note further that part of the first two claims of Theorem 4 hold without the requirement that the solutions is renormalized; i.e., if \( s \) is a distributional solution, then under the assumptions of this theorem it is a time integrated distributional solution, similarly in the case of weak solution. On the other hand, Item 3. requires that the solution is renormalized.

3. It appears that condition (25) coincides with the conditions in the assumptions in the Friedrichs commutator lemma (see Lemma 3 later) which is the basic tool in the passage from distributional solutions to renormalized distributional solutions. The same condition is needed in the passage from distributional to weak solutions in order to allow the application of the Hardy inequality near the boundary, cf. Theorem 5 for both features. This makes of the setting (25) an universal setting convenient for general transport equations (including continuity and pure transport). This setting in the context of general transport equations has been introduced and fully exploited in the seminal DiPerna–Lions’ paper [8].

3.2.2. Passage from distributional to renormalized weak solutions

The main message of this section is the observation that under certain assumptions (which are, in general, slightly stronger than assumptions in the previous section), any distributional solution (time integrated distributional solution) of the continuity equation/pure transport equation (introduced in Definitions 1–2) is a renormalized weak solution.

**Theorem 5.** (Continuity equation) Let \( \Omega \subset \mathbb{R}^d, d \geq 2 \) be a bounded domain with Lipschitz boundary. Further, let \( u \in L^p(I; W^{1,q}(\Omega; \mathbb{R}^d)), 0 \leq \rho \in L^\alpha(I; L^\beta(\Omega)), \) where \( p, q, \alpha, \beta \) satisfy condition (25).

1. Assume that \( \rho \) is a distributional solution of the continuity equation with transporting velocity \( u \) (i.e., it satisfies (5)). Then, the following statements are true:
   (a) 1.1 \( \rho \) is a renormalized distributional solution, i.e., it satisfies, in addition to equation (5), also equation (9) with any renormalizing function \( b \) in class (10).
   (b) 1.2 If moreover
   \[
u \in L^p(I; W^{1,q}_0(\Omega; \mathbb{R}^d)),
\] (33)
   then \( \rho \) is a renormalized weak solution of the continuity equation, i.e., \( \rho \) satisfies continuity equation (6) and its renormalized counterpart (11) with any renormalizing function \( b \) belonging to class (10).

2. Assume that \( \rho \) belongs to class
\[
\rho \in C_{\text{weak}}(\bar{I}; L^\gamma(\Omega)) \quad \text{with some} \ \gamma > 1
\] (34)
and is a time integrated distributional solution of the continuity equation with transporting velocity \( u \) (i.e., it satisfies (4) and (7)). Then, the following statements are true:

2.1 Function \( \varrho \) belongs to \((27)_{\gamma=q_s'}\) and functions \( b(\varrho) \) with any \( b \in (31) \) belong to class \((27)_{\gamma=q_s'}\). Moreover, \( \varrho \) is a renormalized time integrated distributional solution and it satisfies equation (12) with any renormalizing function \( b \) belonging to (31).

2.2 If moreover \( u \) has zero traces (i.e., \( u \) satisfies (33)), then \( \varrho \) is a renormalized time integrated weak solution of the continuity equation and it satisfies equations (8) and (13) with any renormalizing function \( b \) belonging to (31).

**Theorem 6.** Exactly the same statement—only with minor modifications—is valid for the pure transport equation. The modifications are the following:

1. In assumptions of Statement 1., equation (5) must be replaced by (15), and further:

   In Statement 1.1, equation (9) must be replaced by (19). In Statement 1.2, equations (7), (12) must be replaced (17), (21) and condition (31) by (32).

2. In assumptions of Statement 2., equations (4) and (7) must be replaced by (14) and (17) and condition (31) by (32), and further:

   In Statement 2.1, equations (12) must be replaced by (21). In Statement 2.2, equations (8), (13) must be replaced by (18), (22) and relation \((27)_{\gamma=q_s'}\) must be replaced by \((27)_{\gamma=q_s'}\).

3.2.3. From pure transport equation to continuity equation

**Theorem 7.** Let \( \Omega \) be a bounded domain with Lipschitz boundary.\(^7\) Suppose that

\[
1 \leq q, p, \alpha, \beta, \alpha_s, \beta_s \leq \infty, \quad (q, \beta) \neq (1, \infty), \quad (q, \beta) \neq (1, \infty),
\]

\[
\frac{1}{\alpha} + \frac{1}{\alpha_s} + \frac{1}{p} \leq 1, \quad \frac{1}{r} + \frac{1}{rs} + \frac{1}{q} \leq 1,
\]

where

\[
r = \beta \quad \left\{ \begin{array}{l}
\in [1, \infty) \quad \text{if } q > 1 \text{ and } \beta = \infty \\
\in [1, \infty) \quad \text{otherwise}
\end{array} \right.
\]

\[
r_s \in [1, \infty) \quad \text{if } q > 1 \text{ and } \beta_s = \infty
\]

\[
r_s \in [1, \infty) \quad \text{otherwise}
\]

Let

\[
\varrho \in L^{\alpha}(I; L^{\beta}(\Omega)), \quad s \in L^{\alpha_s}(I; L^{\beta_s}(\Omega)), \quad u \in L^p(I; W^{1,q}(\Omega; \mathbb{R}^d)).
\]

Then there holds:

---

\(^7\) As a matter of fact, the assumptions is important only in case of weak solutions. The result dealing with distributional solutions holds for arbitrary domain \( \Omega \).
1. Assume additionally that
\[ \frac{1}{t_\varrho} + \frac{1}{t_s} + \frac{1}{p} \leq 1, \]
where
\[ t_\varrho \begin{cases} \in [1, \infty) & \text{if } p > 1 \text{ and } \alpha_\varrho = \infty \\ = \alpha_\varrho & \text{otherwise} \end{cases} \]
\[ t_s \begin{cases} \in [1, \infty) & \text{if } p > 1 \text{ and } \alpha_s = \infty \\ = \alpha_s & \text{otherwise} \end{cases} \].

If \( \varrho \) is a distributional (resp. weak) solution of the continuity equation (1) and \( s \) a distributional (resp. weak) solution of the pure transport equation (2) with transporting velocity \( u \), then \( \varrho s \) is a renormalized distributional (resp. weak) solution of the continuity equation with the same transporting velocity \( u \).

2. If \( \varrho \in C_{\text{weak}}(\overline{T}; L^{\gamma_\varrho}(\Omega)) \) is a time integrated distributional (resp. weak) solution of the continuity equation (1) and \( s \in C_{\text{weak}}(\overline{T}; L^{\gamma_s}(\Omega)) \) a time integrated distributional (resp. weak) solution of the pure transport equation (2) with transporting velocity \( u \) (where \( 1 < \gamma_\varrho, \gamma_s \leq \infty \), \( \frac{1}{\gamma_\varrho} + \frac{1}{\gamma_s} := \frac{1}{\gamma} < 1 \)), then \( \varrho s \in C(\overline{T}; L^{r}(\Omega)) \), \( 1 \leq r < \gamma \) is a renormalized distributional (resp. weak) solution of the continuity equation with the same transporting velocity \( u \).

3.3. Application to compressible Navier–Stokes equations

For simplicity, let us first recall the compressible Navier–Stokes equations in barotropic regime:
\[ \partial_t \varrho + \text{div} (\varrho \mathbf{u}) = 0 \]
\[ \partial_t (\varrho \mathbf{u}) + \text{div} (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \text{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} \] (35)

which we consider in \((0, T) \times \Omega\), together with the initial conditions in \( \Omega \)
\[ \varrho(0, \cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0 \] (36)

and so called no-slip boundary condition on \((0, T) \times \partial \Omega \)
\[ \mathbf{u}(t, x) = \mathbf{0}. \] (37)

The homogeneous boundary condition (37) can be replaced by Navier (slip) boundary conditions or by periodic boundary conditions if \( \Omega \) is a periodic cell.

In the above, \( \mathbb{S} \) is the viscous stress tensor, which reads
\[ \mathbb{S}(\nabla \mathbf{u}) = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{d} \text{div} \mathbf{u} \mathbb{I} \right) + \lambda \text{div} \mathbf{u} \mathbb{I}. \] (38)
The viscosity coefficients are assumed to be constant: \(\mu > 0\) and \(\lambda \geq 0\). Function \(\varrho \mapsto p(\varrho)\) denotes the pressure. One supposes that 
\[
p \in C^1([0, \infty)).
\]

The classical (or strong) solutions, in general, may not exist. (We can prove their existence either if the data are smooth and the time interval is sufficiently short or if the data are in some sense additionally sufficiently small.) We therefore consider the weak solutions. They are defined as follows:

**Definition 3.** Let \(\varrho_0 \in L^\gamma(\Omega), 0 \leq \varrho_0 \in L^\gamma(\Omega)\) a.e. in \(\Omega\), \(\gamma > 1\), \(r > 1\), \((\varrho u)(0, \cdot) = m_0 \in L^1(\Omega; R^d)\) and \(f \in L^\infty((0, T) \times \Omega; R^d)\). A couple \((\varrho, u)\) is a renormalized weak solution to the initial boundary value problem (35–37) if

1. The couple \((\varrho, u)\) belongs to functional spaces

\[
\varrho \in C_{\text{weak}}(\tilde{T}; L^\gamma(\Omega)), \quad u \in L^2(I; W_0^{1,2}(\Omega; R^d)), \quad p(\varrho) \in L^1(Q),
\]

\[
\varrho u \in C_{\text{weak}}(\tilde{T}; L^r(\Omega; R^d)), \quad \varrho(u \otimes u), p(\varrho) \in L^1((0, T) \times \Omega; R^{d \times d}).
\]

2. \(\varrho\) is a time integrated renormalized weak solution to the continuity equation (35)1 with transporting velocity \(u\).

3. The couple \((\varrho, u)\) verifies the momentum equation (35)2 in the sense of distributions.

If Navier or periodic conditions are considered, the functional spaces and test functions in the above definition must be accordingly modified, see [12, 24] or [14].

**Corollary 4.** Let \(\gamma\) verify condition (29) with \(q = 2\) (in particular \(\gamma \geq 6/5\) if \(d = 3\)). Then, the claims of Theorems 1 and 2 (and Corollaries 1, 2 and 3) hold for any renormalized weak solution to the compressible Navier–Stokes equations specified in Definition 3 (with \(p = 2\) in Theorem 1).

**Remark 3.**

1. Note that renormalized weak solutions to the Navier–Stokes equations with the regularity properties stated above (and, additionally, fulfilling the energy inequality) can be constructed with any of no-slip, Navier (slip) or periodic boundary conditions provided \(\gamma > d/2\) and

\[
p(0) = 0, \quad p'(\varrho) \geq a_1 \varrho^\gamma - b, \quad p(\varrho) \leq a_2 \varrho^\gamma + b, \quad \text{with some} \ a_1, a_2, b > 0,
\]

[13] (for monotone pressure), [9] (for non monotone pressure) and sufficiently regular domains, and [16, 21] or [25] for a generalization to Lipschitz domains.

2. The above condition for pressure allows pressure functions which are non monotone on a compact portion of \([0, \infty)\). In the case of periodic boundary conditions and provided \(\gamma \geq 9/5\), this condition can be generalized allowing pressure functions non-monotone up to infinity and, also, another generalization allows small anisotropic perturbations of the isotropic stress tensor (38), see Bresch, Jabin [5, Theorems 3.1 and 3.2].
3. Theorems 1, 2 and Corollaries 1, 2 also apply to a couple \((\varrho, \mathbf{u})\), where \((\varrho, \mathbf{u}, \vartheta)\)—(density, velocity, temperature)—is a weak solution of the full Navier–Stokes–Fourier system, constructed (according to different definitions of weak solutions under different physical assumptions on constitutive laws and transport coefficients) either in Feireisl [10, Definition 7.1 and Theorem 7.1] or in [12, Theorem 3.1] or in [11, 15].

4. Theorems 1, 2 and Corollaries 1, 2 do not, in general, directly apply to a couple \((\varrho, \mathbf{u})\) of weak solutions of Navier–Stokes equations with degenerate density-dependent viscosities unless it cannot be guaranteed that \(\nabla \mathbf{u}\) belongs to a Sobolev space of type \(L^p(I; W^{1,q}(\Omega; \mathbb{R}^d))\). In fact, in this situation, typically, \(\nabla \mathbf{u}\) belongs to a Lebesgue space weighted by a positive power of \(\varrho\) (cf. Bresch, Desjardins [4], Mellet and Vasseur [20], Vasseur and Yu [26], Li and Xin [17] for non exhausting relevant references).

4. Basic preliminaries

Let us mention some standard preliminary tools. We shall use several times the theorem on Lebesgue points in the following form.

**Lemma 1.** Let \(f \in L^1(0, T; L^\gamma(\Omega))\), \(1 \leq \gamma < \infty\). Then, there exists \(N \subset (0, T)\) of zero Lebesgue measure such that for all \(\tau \in (0, T) \setminus N\),

\[
\lim_{h \to 0+} \frac{1}{h} \int_{\tau-h}^{\tau} \left\| f(t, \cdot) - f(\tau, \cdot) \right\|_{L^\gamma(\Omega)} dt \to 0,
\]

\[
\lim_{h \to 0+} \frac{1}{h} \int_{\tau}^{\tau+h} \left\| f(t, \cdot) - f(\tau, \cdot) \right\|_{L^\gamma(\Omega)} dt \to 0.
\]

Moreover, if \(f \in C_{\text{weak}}([0, T]; L^\gamma(\Omega))\), then for any \(\eta \in L^\gamma(\Omega)\)

\[
\forall \tau \in [0, T), \int_{\Omega} f(\tau, \cdot) \eta dx = \lim_{h \to 0+} \frac{1}{h} \int_{\tau}^{\tau+h} \left( \int_{\Omega} f(t, \cdot) \eta dx \right) dt
\]

and

\[
\sup_{\tau \in [0, T]} \| f(\tau, \cdot) \|_{L^\gamma(\Omega)} \leq \| f \|_{L^\infty(0, T; L^\gamma(\Omega))}.
\]

We shall also frequently use mollifiers. For the sake of completeness, we recall the basic facts. We denote by \(j\) a function on \(\mathbb{R}^d, \ d \geq 1\), satisfying the following requirements: \(j \in C^\infty_c(\mathbb{R}^d), \ \text{supp}(j) = B(0, 1), \ j(x) = j(-x), \ j \geq 0\) on \(\mathbb{R}^d, \ \int_{\mathbb{R}^d} j(x) dx = 1\). Next, for \(\epsilon > 0\), we denote by \(j_\epsilon\) the function \(j_\epsilon(x) := \frac{1}{\epsilon^d} j(\frac{x}{\epsilon})\). For a given function \(f \in L^1_{\text{loc}}(\mathbb{R}^d)\), we finally define mollified \(f\) as follows: \([f]_\epsilon := f \ast j_\epsilon(x) = \int_{\mathbb{R}^d} j_\epsilon(x - y) f(y) dy\).

Let us recall the classical properties of these approximations.
Lemma 2.  
1. If \(1 \leq p < \infty\), then for any \(f \in L^p(R^d)\)
\[
[f]_e \in C^\infty(R^d) \cap L^p(R^d), \| [f]_e \|_{L^p(R^d)} \leq \| f \|_{L^p(R^d)}
\]
and
\[
f_e \to f \quad \text{in} \ L^p(R^d).
\]
2. If \(p = \infty\), then
\[
[f]_e \in C^\infty(R^d) \cap L^\infty(R^d), \| [f]_e \|_{L^\infty(R^d)} \leq \| f \|_{L^\infty(R^d)}.
\]
 Moreover, if \(f\) is uniformly continuous on \(R^d\), then
\[
[f]_e \to f \quad \text{in} \ C_b(R^d).
\]
3. Let \(1 \leq p \leq \infty\). For all \(f \in L^p(R^d), g \in L^{p'}(R^d)\),
\[
\int_{R^d} [f]_e g \, dx = \int_{R^d} f[g]_e \, dx.
\]

The next lemma is the well-known Friedrichs lemma on commutators. It deals with the regularization of the quantity \(u \cdot \nabla f\) defined in the sense of distributions as

\[
u \cdot \nabla f := \text{div}(fu) - f \text{div}u.
\]

The lemma reads.

Lemma 3. (Friedrichs commutator lemma) Let \(I \subset R\) be an open bounded interval and \(f \in L^\alpha(I; L^\beta_{\text{loc}}(R^d)), u \in L^p(I; W^{1,q}_{\text{loc}}(R^d; R^d))\). Let \(1 \leq q, \beta \leq \infty, (q, \beta) \neq (1, \infty), \frac{1}{q} + \frac{1}{\beta} \leq 1, 1 \leq \alpha \leq \infty\) and \(\frac{1}{\alpha} + \frac{1}{p} \leq 1\). Then
\[
[u \cdot \nabla f]_e - u \cdot \nabla [f]_e \to 0
\]

strongly in \(L^t(I; L^r_{\text{loc}}(R^d))\), where
\[
\frac{1}{t} \geq \frac{1}{\alpha} + \frac{1}{p}, \quad t \in [1, \infty)
\]
and
\[
r \in [1, q) \quad \text{for} \ \beta = \infty, \ q \in (1, \infty],
\]
while \(\frac{1}{\beta} + \frac{1}{q} \leq \frac{1}{r} \leq 1\) otherwise.
5. Proof of Theorems 3–4

The proof of Theorems 3–4 is based on the following two lemmas. The first lemma deals with distributional (or weak) solutions to conservation laws (39) and claims that their solutions admit, under certain conditions, $C_{\text{weak}}([0, T]; L^1(\Omega))$ representatives, and can be therefore integrated up to the endpoints of any time interval $[0, \tau] \subset [0, T]$.

**Lemma 4.** Let $d \in L^\infty(I, L^Y(\Omega)), \gamma > 1$ and $F \in L^1(Q; \mathbb{R}^d)$, $G \in L^1(Q)$. 

1. Suppose that $d \in L^\infty(I, L^Y(\Omega))$, $\gamma > 1$ and $\mathbf{F} \in L^1(Q; \mathbb{R}^d)$, $G \in L^1(Q)$.

2. Suppose that (39) holds up to the boundary, i.e.

$$\int_Q \left( d \partial_t \varphi + \mathbf{F} \cdot \nabla \varphi - G \varphi \right) \, dt = 0 \quad \text{for all } \varphi \in C^1_c((0, T) \times \overline{\Omega}).$$

Then, there exists a representative of $d$ such that it belongs to $C_{\text{weak}}([0, T]; L^Y(\Omega))$ and equation (39) can be integrated up to any time $\tau \in (0, T)$, i.e., $\forall \tau \in (0, T)$ and $\forall \eta \in C^1_c(\Omega)$, there holds

$$\int_0^\tau \int_\Omega \left( d(t, x) \partial_t \xi(t) + \mathbf{F}(t, x) \cdot \nabla \eta(x) \xi(t) - G(t, x) \xi(t) \eta(x) \right) \, dx \, dt = 0.$$

**Proof.** We shall show only Statement 1. of Lemma 4. Statement 2. can be obtained repeating word by word the proof of Statement 1. with minor modifications.

We take in equation (39) test functions $\varphi(t, x) = \psi(t) \eta(x)$, where $\eta \in C^1_c(\Omega)$, and

$$\psi(t) = \psi_{t,h}^+ = \begin{cases} \frac{1}{h} t & \text{if } t \in [0, h) \\ 1 & \text{if } t \in [h, \tau] \\ 1 - \frac{t - \tau}{h} & \text{if } t \in [\tau, \tau + h] \\ 0 & \text{if } t \in (\tau + h, +\infty). \end{cases}$$

Under assumptions on $d$, $\mathbf{F}$ and $G$, it is a folklore to show that this is an admissible test function in equation (39).

We obtain by direct calculation,

$$\frac{1}{h} \int_\tau^{\tau + h} \int_\Omega d(t, x) \eta(x) \, dx \, dt - \frac{1}{h} \int_0^h \int_\Omega d(t, x) \eta(x) \, dx \, dt = \int_0^{\tau + h} \psi(t) \int_\Omega \mathbf{F}(t, x) \cdot \nabla \eta(x) \, dx \, dt - \int_0^{\tau + h} \psi(t) \int_\Omega G(t, x) \eta(x) \, dt \, dx.$$

(41)
This identity leads to the following observations:

1. According to the theorem on Lebesgue points (cf. Lemma 1), there is a set 
   \( N \subset (0, T) \) of zero Lebesgue measure \(|N| = 0\), such that for all \( \tau \in (0, T) \setminus N \), the limit \( h \to 0^+ \) of the first expression exists. Since the limit of the right-hand
   side as \( h \to 0^+ \) exists as well, we deduce that 
   \[
   \forall \eta \in C^1_c(\Omega), \quad \lim_{h \to 0^+} \frac{1}{h} \int_0^h \int_\Omega d(t, x) \eta(x) \, dx \, dt := \vartheta_\eta(0+) \in R.
   \]
   The map \( C^1_c(\Omega) \ni \eta \mapsto \vartheta_\eta(0+) \in R \) is evidently linear. Moreover, since 
   \( d \in L^\infty(I, L^\gamma(\Omega)) \), we have estimate 
   \[
   \sup_{0 < h < T} \left| \frac{1}{h} \int_0^h \int_\Omega d(t, x) \eta(x) \, dx \, dt \right| \leq \|d\|_{L^\infty(0, T; L^\gamma(\Omega))} \|\eta\|_{L^\gamma(\Omega)}
   \]
   by virtue of the Hölder inequality. In view of the Riesz representation theorem, we deduce that there exists \( d(0+) \in L^\gamma(\Omega) \) such that 
   \[
   \forall \eta \in C^1_c(\Omega), \quad \vartheta_\eta(0+) = \int_\Omega d(0+) \eta \, dx.
   \]

2. Now, we take an arbitrary \( \tau \in (0, T) \) and calculate limit \( h \to 0^+ \) in equation 
   \( (41) \). We already know that for all \( \eta \in C^1_c(\Omega) \) the limits of the second term at
   the left-hand side and the limit of the right-hand side exist and belong to \( R \). We deduce from this fact that 
   \[
   \lim_{h \to 0^+} \frac{1}{h} \int_\tau^{\tau + h} \int_\Omega d(t, x) \eta(x) \, dx \, dt := \vartheta_\eta(\tau+),
   \]
   where by the same token as in the previous step,
   \[
   \forall \eta \in C^1_c(\Omega), \quad \vartheta_\eta(\tau+) = \int_\Omega d(\tau+) \eta \, dx \quad \text{with} \quad d(\tau+) \in L^\gamma(\Omega).
   \]

3. We test equation \( (39) \) by functions \( \phi(t, x) = \psi(t) \eta(x) \), where 
   \[
   \psi(t) = \psi_{\tau, h} \begin{cases}
   \frac{1}{h} t & \text{if } t \in [0, h] \\
   1 & \text{if } t \in [h, \tau] \\
   1 - \frac{t - \tau + h}{h} & \text{if } t \in [\tau - h, \tau] \\
   0 & \text{if } t \in (\tau, +\infty).
   \end{cases}
   \]
   It reads 
   \[
   \frac{1}{h} \int_{\tau - h}^{\tau} \int_\Omega d(t, x) \eta(x) \, dx \, dt - \frac{1}{h} \int_0^h \int_\Omega d(t, x) \eta(x) \, dx \, dt
   = \int_0^\tau \psi(t) \int_\Omega F(t, x) \nabla \eta(x) \, dx \, dt - \int_0^\tau \psi(t) \int_\Omega G(t, x) \eta(x) \, dt \, dx. \quad (42)
   \]
4. By the same token as in Items 1. and 2. we define $\varrho(\tau-)$ and $\varrho(\tau-) \in L^\gamma(\Omega)$ for all $\tau \in (0, T]$. Subtracting (41) and (42) and effectuating limit $h \to 0^+$, we obtain

$$\forall \tau \in (0, T), \varrho(\tau) := \varrho(\tau+) = \varrho(\tau-).$$

We define

$$\varrho(0) := \varrho(0+), \quad \varrho(\tau) := \varrho(\tau+), \quad \tau \in (0, T), \quad \varrho(T) := \varrho(T-).$$

We easily verify that $\varrho$ satisfies equation (40). Subtracting (40) with $\tau = \tau_1$ and $\tau = \tau_2$, $\tau_1, \tau_2 \in [0, T]$ we readily verify that

$$\forall \eta \in C^1_c(\Omega), \text{ the map } \tau \mapsto \int_\Omega \varrho(\tau) \eta \, dx \text{ is continuous on } [0, T].$$

Since $C^1_c(\Omega)$ is dense in $L^\gamma'(\Omega)$, we finally conclude that

$$\varrho \in C_{weak}([0, T]; L^\gamma(\Omega)).$$

5. According to theorem on Lebesgue points (cf. Proposition 1), we have

$$d(\tau+) = d(\tau-) = d(\tau) \text{ a.e. in } (0,T).$$

This completes the proof of the fact that there exists a representative of $d$ such that $d \in C_{weak}([0, T]; L^\gamma(\Omega))$.

6. It remains to show equation (40). To this end, we can repeat the whole procedure consisting of Items 1.–5. with test functions $\varphi(t, x) = \psi(t)\xi(t)\eta(x)$, where $\psi = \psi_{\tau, h}, \xi \in C^1([0, T])$ and $\eta \in C^1_c(\Omega)$.

Lemma 4 is thus proved. \hfill \Box

The continuity and pure transport equations are particular cases of equations investigated in Lemma 4. If we additionally know that their solutions are renormalized, we can show that they not only belong to the class $C_{weak}([0, T]; L^1(\Omega))$ but even to the class $C([0, T]; L^r(\Omega))$. This is subject of the second lemma.

**Lemma 5.**  Let $u \in L^p(I; W^{1,q}(\Omega; R^d)), 1 \leq p \leq \infty, 1 < q \leq \infty$,

$$\varrho \in L^\infty(I; L^\gamma(\Omega)), \gamma > 1, \quad \frac{1}{\gamma} + \frac{1}{q} \leq 1 + \frac{1}{d} \quad (43)$$

or

$$\varrho \in L^\infty(I; L^\gamma(\Omega)) \cap L^{p'}(I; L^{q'}(\Omega)), \gamma > 1. \quad (44)$$

Suppose that $\varrho$ is a renormalized distributional solution of the continuity equation (i.e., it satisfies (5), (9) with renormalizing function $b$ in the class (10)). Then, there exists a representative of $\varrho$ such that

$$\varrho \in C(\overline{I}; L^r(\Omega)), 1 \leq r < \gamma.$$
2. The same statement, under the same assumptions on \( u \) and under assumption (44) holds for any renormalized distributional solution to the pure transport equation (satisfying (15), (19) with renormalizing function \( b \) in the class (10)).

**Proof.** Again, it is enough to prove Statement 1. dealing with the continuity equation. The proof of Statement 2. for the pure transport equation requires only minor modifications and is, therefore, left to the reader as an exercise. It is to be noticed that, due to the presence of term \( s \text{div} u \) in the weak formulation of the pure transport equation, Statement 2. is not true under assumption (43) unless \( \gamma \geq q' \).

Employing Lemma 4 (with \( d = \varrho, F = \varrho u, G = 0 \)) we may suppose that \( \varrho \in C_{\text{weak}}([0, T]; L^p(\Omega)) \).

Since \( \varrho \) is a renormalized distributive solution of the continuity equation, it satisfies
\[
\partial_t T_k(\varrho) + \text{div} (T_k(\varrho)u) + (\varrho T'_k(\varrho) - T_k(\varrho))\text{div} u = 0 \quad \text{in} \ D'(Q),
\]
where for any \( k > 1 \)
\[
T_k(\varrho) = kT\left(\frac{\varrho}{k}\right) \quad \text{with} \ T \in C^1([0, \infty)),
\]
with
\[
T(s) = \begin{cases} 
  s & \text{if } 0 \leq s \leq 1 \\
  2 & \text{if } s \geq 3.
\end{cases}
\]

According to Lemma 4 applied to (45) with \( d := T_k(\varrho), F := T_k(\varrho)u \) and \( G := (\varrho T'_k(\varrho) - T_k(\varrho))\text{div} u \), there exists
\[
T_k(\varrho) \in C_{\text{weak}}([0, T]; L^p(\Omega)), \ \forall 1 \leq p < +\infty, \\
(T_k(\varrho))(t) = T_k(\varrho(t)) \quad \text{a.a. in} \ \Omega \ \text{for a.a.} \ t \in (0, T),
\]
such that
\[
\partial_t T_k(\varrho) + \text{div} (T_k(\varrho)u) + (\varrho T'_k(\varrho) - T_k(\varrho))\text{div} u = 0 \quad \text{in} \ D'(Q).
\]

We can extend \( T_k(\varrho) \) by 0 outside \( \Omega \) and regularize it by using standard mollifiers over the space variables. The equation for mollified functions \( [T_k(\varrho)]_\varepsilon \) reads
\[
\partial_t [T_k(\varrho)]_\varepsilon + \text{div} ([T_k(\varrho)]_\varepsilon u) + \left[ (\varrho T'_k(\varrho) - T_k(\varrho))\text{div} u \right]_\varepsilon = r_\varepsilon
\]
a.e. in
\[
Q_\varepsilon = I \times \Omega_\varepsilon, \ \Omega_\varepsilon = \{ x \in \Omega \mid \text{dist}(x, R^d \setminus \Omega) > \varepsilon \},
\]
where
\[
r_\varepsilon := r_\varepsilon(T_k(\varrho), u) = \text{div} ([T_k(\varrho)]_\varepsilon u) - \text{div} [T_k(\varrho)u]_\varepsilon \to 0 \text{ as } \varepsilon \to 0
\]
in \( L^p(I; \tilde{L}^q(K)) \), with any compact \( K \subset \Omega, \tilde{q} < q \) by virtue of the Friedrichs lemma on commutators (cf. Lemma 3).

Due to the standard properties of mollifiers

\[
\left[ \rho \mathcal{T}'_k(\rho) - \mathcal{T}_k(\rho) \right] \rightarrow \left[ \rho \mathcal{T}'_k(\rho) - \mathcal{T}_k(\rho) \right]
\]

in \( L^p(I; L^q(K)) \), \( K \subset \Omega \), compact.

On the other hand, since \( \mathcal{T}_k(\rho)(t, \cdot) \in L^r(\Omega) \) for all \( t \in [0, T] \), \( 1 \leq r < +\infty \), we get by the same token, in particular,

\[
\forall t \in [0, T] \quad [\mathcal{T}_k(\rho)(t, \cdot)]_\varepsilon \rightarrow \mathcal{T}_k(\rho)(t, \cdot) \quad \text{in} \quad L^2(K) \quad \text{with any compact} \quad K \subset \Omega.
\]

(49)

Moreover, since \( \mathcal{T}_k(\rho) \in C_{\text{weak}}([0, T]; L^p(\Omega)) \), we infer that the mapping \( t \mapsto [\mathcal{T}_k(\rho)]_\varepsilon(\cdot, x) \) belongs to \( C([0, T]) \) for all \( x \in \Omega_\varepsilon \) and hence \( t \mapsto [\mathcal{T}_k(\rho)]_{\varepsilon}(\cdot, x) \in C[0, T] \) for all \( x \in \Omega_\varepsilon \).

Consequently,

\[
\left( t \mapsto \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2(t, x) \eta(x) \, dx \right) \in C([0, T])
\]

for all \( \eta \in C^1_c(\Omega) \) and \( 0 < \varepsilon < \text{dist}(\text{supp} \eta, R^d \setminus \Omega) \). We deduce from estimate

\[
\sup_{t \in [0, T]} \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2(t, x) \eta(x) \, dx \leq \sup_{t \in [0, T]} \| \mathcal{T}_k(\rho(t, \cdot)) \|_{L^2(\Omega)} \| \eta \|_{L^2(\Omega)}
\]

\[
\leq \| \mathcal{T}_k(\rho) \|_{L^\infty(0, T; L^2(\Omega))} \| \eta \|_{L^2(\Omega)} \leq C
\]

that the family of maps

\[
\left\{ t \mapsto \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2(t, x) \eta(x) \, dx \mid 0 < \varepsilon < \text{dist}(\text{supp} \eta, R^d \setminus \Omega) \right\}
\]

(50)

is for any \( k > 1 \) and any \( \eta \in C^1_c(\Omega) \) equi-bounded in \( C([0, T]) \).

We multiply (48) by \( 2[\mathcal{T}_k(\rho)]_{\varepsilon} \), in order to get

\[
\partial_t [\mathcal{T}_k(\rho)]_{\varepsilon}^2 + \text{div} ( [\mathcal{T}_k(\rho)]_{\varepsilon}^2 \mathbf{u} ) + [\mathcal{T}_k(\rho)]_{\varepsilon}^2 \text{div} \mathbf{u} + 2[\mathcal{T}_k(\rho)]_{\varepsilon} \left[ \rho \mathcal{T}'_k(\rho) - \mathcal{T}_k(\rho) \right] \text{div} \mathbf{u} \eta \, dx = 2[\mathcal{T}_k(\rho)]_{\varepsilon} \eta \quad \text{a.e. in} \quad Q_\varepsilon.
\]

(51)

Now, we take \( \eta \in C^1_c(\Omega) \), multiply equation (51) by \( \eta \) and integrate over \( \Omega \). We get, after an integration by parts,

\[
\partial_t \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2 \eta \, dx - \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2 \mathbf{u} \cdot \nabla \eta \, dx + \int_\Omega [\mathcal{T}_k(\rho)]_{\varepsilon}^2 \text{div} \mathbf{u} \eta \, dx
\]

\[
+ \int_\Omega 2[\mathcal{T}_k(\rho)]_{\varepsilon} \left[ \rho \mathcal{T}'_k(\rho) - \mathcal{T}_k(\rho) \right] \text{div} \mathbf{u} \eta \, dx = \int_\Omega 2[\mathcal{T}_k(\rho)]_{\varepsilon} \eta \, dx,
\]
where \( 0 < \varepsilon < \text{dist}(\text{supp} \eta, \mathbb{R}^d \setminus \Omega) \).

We may integrate (51) between \( t_1, t_2 \), where \( t_i \in [0, T] \), by virtue of Lemma 4, in order to obtain,

\[
\begin{align*}
\left| \int_{\Omega} [T_k(\varphi)]^2_{\varepsilon}(t_2, \cdot) \eta(x) \, dx - \int_{\Omega} [T_k(\varphi)]^2_{\varepsilon}(t_1, \cdot) \eta(x) \, dx \right| \\
\leq C \left( \| u \|_{L^p(t_1, t_2; W^{1, q}(\Omega))} + \| r_{\varepsilon} \|_{L^p(t_1, t_2; L_{\tilde{q}}(\Omega))} \| \eta \|_{C^1(\Omega)} \right) (t_2 - t_1)^{1/p'},
\end{align*}
\]

where \( C \) may depend on \( k \) but is independent of \( 0 < \varepsilon < \text{dist}(\text{supp} \eta, \mathbb{R}^d \setminus \Omega) \). The latter inequality shows in view of Lemmas 2, 3 that the family of maps (50) is for any \( k > 1 \) and \( \eta \in C^1_c(\Omega) \) equi-continuous in \( C([0, T]) \).

Now, we denote \( J(\Omega) \subset C^1_c(\Omega) \) a countable dense subset of \( L^2(\Omega) \). Using Arzelà–Ascoli theorem and countability of \( J(\Omega) \) (in order to employ a diagonalization procedure) we may show that there is a subsequence of \( \varepsilon \to 0 \) and \( Z^{(k)}_{\eta} \in C([0, T]) \) such that \( \forall \eta \in J(\Omega) \)

\[
\int_{\Omega} [T^2_k(\varphi)(t, x)]^2_{\varepsilon} \eta(x) \, dx \mapsto Z^{(k)}_{\eta} \text{ in } C[0, T] \text{ as } \varepsilon \to 0 + .
\]

By virtue of (49)

\[
Z^{(k)}_{\eta}(t) = \int_{\Omega} T^2_k(\varphi)(t, x) \eta(x) \, dx.
\]

Now, we use density of \( J(\Omega) \) in \( L^2(\Omega) \) and the uniform bound with respect to \( \varepsilon \) of \( \sup \tau \in [0, T] \| [T_k(\varphi)(t, x)]_{\varepsilon} \|_{L^2(\Omega)} \) (cf. the last inequality in Lemma 1 and Item 2 in Lemma 2) to show that

\[
\int_{\Omega} [T_k(\varphi)(t, x)]^2_{\varepsilon} \eta(x) \, dx \mapsto \int_{\Omega} T^2_k(\varphi)(t, x) \eta(x) \, dx
\]

in \( C([0, T]) \) for all \( \eta \in L^2(\Omega) \). In particular,

\[
\forall k > 1, \quad \left( t \mapsto \int_{\Omega} T_k(\varphi)(t, x)^2 \, dx \right) \in C([0, T]). \tag{52}
\]

Resuming: According to (46) \( T_k(\varphi(t')) \to T_k(\varphi(t)) \) weakly in \( L^2(\Omega) \) as \( t' \to t \) and according to (52),

\[
\| T_k(\varphi(t')) \|_{L^2(\Omega)} \to \| T_k(\varphi(t)) \|_{L^2(\Omega)} \text{ as } t' \to t.
\]

Since weak convergence and convergence in norms in \( L^2(\Omega) \) imply strong convergence, we have

\[
T_k(\varphi) \in C([0, T]; L^2(\Omega)) \text{ for any } k > 1.
\]
It remains to show that the latter formula implies \( \varrho \in C([0, T]; L^r(\Omega)), 1 \leq r < \gamma \).
To this end, we write
\[
\sup_{t \in [0, T]} \| (T_k(\varrho) - \varrho)(t) \|_{L^r(\Omega)} \leq \| T_k(\varrho) - \varrho \|_{L^\infty(0, T; L^r(\Omega))},
\]
where we have used the last inequality in Lemma 1. Consequently, for all \( t \in [0, T] \),
\[
\| (T_k(\varrho) - \varrho)(t) \|_{L^r(\Omega)} \leq \sup_{t \in [0, T]} \left( \int_{|\varrho| \geq k} 2^{r'} |\varrho|^{r'} \, dx \right)^{\frac{1}{r'}} \| \{ |\varrho| \geq k \} \|_{\gamma}^\frac{1}{\gamma} \| \varrho \|_{L^\gamma(\Omega)}.
\]
Whence,
\[
\forall t \in [0, T], \| T_k(\varrho) - \varrho \|_{L^r(\Omega)} \to 0 \text{ as } k \to \infty.
\]
With this information, writing,
\[
\| \varrho(t) - \varrho(t') \|_{L^r(\Omega)} \leq \| \varrho(t) - T_k(\varrho)(t) \|_{L^r(\Omega)} + \| T_k(\varrho)(t') - T_k(\varrho)(t) \|_{L^r(\Omega)} + \| T_k(\varrho)(t') - \varrho(t') \|_{L^r(\Omega)},
\]
we conclude that \( \varrho \in C([0, T]; L^r(\Omega)) \). □

6. Proof of Theorems 5–6

It is enough to outline the proof only in the case of Theorem 5. The proof of Theorem 6 follows the same lines.

The proof of Statements 1.1 and 2.1 of Theorem 5 is based on regularization of the equation via mollifiers, cf. Lemma 2. The regularized equation
\[
\partial \varrho + \text{div}(\varrho \mathbf{u}) = r(\varrho, \mathbf{u}), \quad r(\varrho, \mathbf{u}) = \text{div}(\varrho \mathbf{u}) - \text{div}[\varrho \mathbf{u}],
\]
is satisfied almost everywhere in \( I \times \Omega \), \( \Omega = \{ x \in \Omega \mid \text{dist}(x, R^d \setminus \Omega) > \varepsilon \} \) and can be therefore multiplied by \( b'([\varrho]) \). The Friedrichs commutator lemma (cf. Lemma 3) ensures that the term \( \| b - b' \|_{L^1(I; L^1(\Omega))} \) is the main property which allows to conclude at the first stage for \( b \in \text{class (10)} \), and consequently, for any \( b \in \text{class (31)} \), by using a convenient approximation of the function \( b \) in class (31) and the dominated Lebesgue convergence theorem. This is the standard procedure introduced in the same context in the seminal work [8].

Concerning the proof of Statements 1.2 and 2.2 of Theorem 5, we shall show solely the latter. Furthermore, it is enough to deal only with the “integrability up to \( \partial \Omega \)” in the case of Statement 2.2.
We define a function $\xi_n$ as follows:

$$
\xi_n(x) := \chi_n(\text{dist}(x, \partial \Omega))
$$

with

$$
\chi_n(s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq \frac{1}{4} \\
1 & \text{if } s \geq \frac{1}{2}.
\end{cases}
$$

Recall that $\text{dist}(\cdot, \partial \Omega)$ is a 1-Lipschitz function.

Notice that it can be deduced from the above $\xi_n \in C^\infty([0, \infty))$, $\xi'_n(x) \leq Cn$, $\xi_n(x) = \begin{cases} 
0 & \text{if } \text{dist}(x, \partial \Omega) \leq \frac{1}{4n} \\
1 & \text{if } \text{dist}(x, \partial \Omega) \geq \frac{1}{2n},
\end{cases}$

with some $C > 0$ ($C$ depends on the choice of $\chi$).

We calculate for $\eta \in C^\infty(\Omega)$

$$
\begin{align*}
&\int_\Omega \varrho(\tau, x)\psi(\tau)\eta(x) \, dx - \int_\Omega \varrho(0, x)\psi(0)\eta(x) \, dx - \int_\Omega \varrho(t, x)\partial_t \psi(t)\eta(x) \, dx \, dt \\
&\quad - \int_Q \varrho(t, x)u(t, x) \cdot \nabla \eta(x)\psi(t) \, dx \, dt \\
&= \int_\Omega \varrho(\tau, x)\psi(\tau)\eta(x)\xi_n(x) \, dx - \int_\Omega \varrho(0, x)\psi(0)\eta(x)\xi_n(x) \, dx \\
&\quad - \int_\Omega \varrho(t, x)\partial_t \psi(t)\eta(x)\xi_n(x) \, dx \, dt \\
&\quad - \int_\Omega \varrho(t, x)u(t, x) \cdot \nabla (\eta(x)\xi_n(x)) \psi(t) \, dx \, dt \\
&\quad + \int_\Omega \varrho(\tau, x)\psi(\tau)\eta(x)(1 - \xi_n(x)) \, dx - \int_\Omega \varrho(0, x)\psi(0)\eta(x)(1 - \xi_n(x)) \, dx \\
&\quad - \int_\Omega \varrho(t, x)\partial_t \psi(t)\eta(x)(1 - \xi_n(x)) \, dx \, dt \\
&\quad - \int_Q \psi(t)\varrho(t, x)u(t, x) \cdot \nabla (\eta(x)(1 - \xi_n(x))) \, dx \, dt.
\end{align*}
$$

We easily verify due to the above formulas for $\xi_n$ that $\eta\xi_n \in W^{1, p}_0(\Omega)$ with any $1 \leq p < +\infty$. Since $C^1_c(\Omega)$ is dense in $W^{1, p}_0(\Omega)$, it is an admissible test function for equation (5). Consequently, the sum of first four terms at the right-hand side (terms containing $\eta\xi_n$) is equal to 0.

To complete the proof we would like to show that the limit $n \to +\infty$ of the sum of the last four terms at the right-hand side of identity (53) is zero. To this aim, we have to assume that all functions are integrable up to the boundary of $\Omega$. 


We set $A_n \coloneqq \{ x : \text{dist}(x, \partial \Omega) \leq \frac{1}{2n} \}$. Since $\Omega$ is a bounded Lipschitz domain, $|A_n| \to 0$. In the sequel, we will systematically use this fact.

We have

1.

$$\int_Q |\varrho(t, x)\partial_t \psi(t)\eta(x)(1 - \xi_n(x))| \, dx \, dt = \int_0^T \int_{A_n} |\varrho(t, x)\partial_t \psi(t)\eta(x)| \, dx \, dt \leq C \left| \varrho \right|_{L^\beta(0,T;L^\beta(A_n))} \left\| \partial_t \psi \right\|_{L^\infty((0,T))} \left\| \eta \right\|_{L^\infty(\Omega)} \left| A_n \right|^{1 - \frac{1}{\beta}} \to 0, \quad n \to \infty.$$ 

2.

$$\int_Q |\varrho(t, x)u(t, x) \cdot \nabla \left( \eta(x)(1 - \xi_n(x)) \right) \psi(t)| \, dx \, dt \to 0, \quad n \to \infty,$$

where we have used the fact that $u \in L^p(I, W^{1,q}_0(\Omega; \mathbb{R}^d))$. Indeed,

$$\lim_{n \to \infty} \int_Q |\varrho(t, x)u(t, x) \cdot \nabla \left( \eta(x)(1 - \xi_n(x)) \right) \psi(t)| \, dx \, dt \leq \lim_{n \to \infty} \int_Q |\varrho(t, x)u(t, x) \cdot \nabla \eta(x)(1 - \xi_n(x)) \psi(t)| \, dx \, dt$$

$$+ \lim_{n \to \infty} \int_Q |\varrho(t, x)u(t, x) \cdot \nabla \xi_n(x) \psi(t) \eta(x)| \, dx \, dt = \lim_{n \to \infty} \int_Q |\varrho(t, x)u(t, x) \cdot \nabla \xi_n(x) \psi(t) \eta(x)| \, dx \, dt$$

$$\leq \lim_{n \to \infty} C \int_0^T \int_{A_n} \left| \varrho(t, x) \frac{u(t, x)}{\text{dist}(x, \partial \Omega)} \cdot \nabla \text{dist}(x, \partial \Omega) \psi(t) \eta(x) \right| \, dx \, dt$$

$$\leq \lim_{n \to \infty} C \int_0^T \int_{A_n} \left| \varrho(t) \right|_{L^\beta(A_n)} \left\| \psi \right\|_{L^\infty((0,T))} \left\| \eta \right\|_{L^\infty(A_n)} \, dx \, dt$$

$$\leq \lim_{n \to \infty} C \int_0^T \left| \varrho \right|_{L^\beta(A_n)} \left\| \psi \right\|_{L^\infty((0,T))} \left\| \eta \right\|_{L^\infty(A_n)} \left\| \nabla u \right\|_{L^q(A_n)} \, dt$$

$$\leq \lim_{n \to \infty} C \left| \varrho \right|_{L^\beta(0,T;L^\beta(A_n))} \left\| \psi \right\|_{L^\infty((0,T))} \left\| \eta \right\|_{L^\infty(A_n)} \left\| \nabla u \right\|_{L^p(0,T;L^q(\Omega; \mathbb{R}^d \times \mathbb{R}^d))} = 0,$$

after employing the Hardy inequality (hence $\Omega$ must have Lipschitz boundary).

Similarly, we treat also the first two integrals over $\Omega$, where we use the fact that $\varrho \in C_{\text{weak}}(T; L^\gamma(\Omega))$ and the product $\eta(1 - \xi_n)$ is bounded uniformly in $L^\infty(\Omega)$. This finishes the proof of Statement 2.2 and thus Theorem 5 as well as Theorem 6 are proved.
7. Proof of Theorem 7

We present the proof for distributional solutions only. The case of weak solutions follows more or less the same lines. Due to the fact that $\Omega$ is Lipschitz, we may extend the function $u$ to the whole $\mathbb{R}^d$ in such a way that it belongs to $L^p(I; W^{1,q}(\mathbb{R}^d; \mathbb{R}^d))$ and either $q$ or $s$ by zero outside $\Omega$. Then, clearly, the extended $q$ resp. $s$ solve the continuity resp. transport equation in the whole $I \times \mathbb{R}^d$ with the transporting velocity the extended $u$. We can therefore apply the mollification in $\mathbb{R}^d$ and then equations (54) hold a.e. in $I \times \mathbb{R}^d$. Hence, we may repeat the whole proof given below in $I \times \mathbb{R}^d$.

Let us start with Statement 1. Since both $u \cdot \nabla q$ and $u \cdot \nabla s$ fulfill assumptions of the Friedrichs commutator lemma (Lemma 3), we see that $[q]_\varepsilon$ and $[s]_\varepsilon$, the corresponding mollifications in the spatial variable satisfy a.e. in $I \times \Omega_\varepsilon$, where $\Omega_\varepsilon$ is defined in the proof of Lemma 5,

\[
\partial_t [s]_\varepsilon + u \cdot \nabla [s]_\varepsilon = r^1_\varepsilon,
\]

\[
\partial_t [q]_\varepsilon + \text{div} ([q]_\varepsilon u) = r^2_\varepsilon,
\]

where $r^1_\varepsilon \rightarrow 0$ in $L^\tau_1(I; L^\sigma_{\text{loc}}(\Omega))$, $\sigma_1 \in [1, q)$ if $\beta_q = \infty$, $\frac{1}{\tau_1} \geq \frac{1}{\tau_2} + \frac{1}{q}$ otherwise, and $\frac{1}{\tau_1} \geq \frac{1}{\sigma_0} + \frac{1}{p}$, $\tau_1 < \infty$. Similarly $r^2_\varepsilon \rightarrow 0$ in $L^\tau_2(0, T; L^\sigma_{\text{loc}}(\Omega))$, $\sigma_2 \in [1, q)$ if $\beta_s = \infty$, $\frac{1}{\tau_2} \geq \frac{1}{\beta_0} + \frac{1}{q}$ otherwise, and $\frac{1}{\tau_2} \geq \frac{1}{\sigma_s} + \frac{1}{p}$, $\tau_2 < \infty$. We may multiply (54)1 by $[q]_\varepsilon$ and (54)2 by $[s]_\varepsilon$. Thus, a.e. in $I \times \Omega_\varepsilon$,

\[
\partial_t ([s]_\varepsilon [q]_\varepsilon) + \text{div} ([s]_\varepsilon [q]_\varepsilon u) = r^1_\varepsilon [q]_\varepsilon + r^2_\varepsilon [s]_\varepsilon,
\]

i.e.

\[
\int_0^T \int_\Omega ([s]_\varepsilon [q]_\varepsilon \partial_t \varphi + [s]_\varepsilon [q]_\varepsilon u \cdot \nabla \varphi + (r^1_\varepsilon [q]_\varepsilon + r^2_\varepsilon [s]_\varepsilon) \varphi) \, dx \, dt = 0
\]

for all $\varphi \in C_c^\infty((0, T) \times \Omega_\varepsilon)$.

We now intend to let $\varepsilon \rightarrow 0^+$. We need to verify that the first two terms converge to the corresponding counterparts while the last two terms converge to zero.

First, since the sequence $[s]_\varepsilon$ is bounded in $L^{a_s}(I; L^{\beta_s}(\Omega_\varepsilon))$, the term $r^2_\varepsilon [s]_\varepsilon \rightarrow 0$ in $L^1((0, T) \times \Omega_\varepsilon)$. Similarly, since $[q]_\varepsilon$ is bounded in the space $L^{a_q}(I; L^{\beta_q}(\Omega_\varepsilon))$, the other term also goes to zero.

Next, we consider the first and the second term. Indeed, the second term is more restrictive than the first one. Since $[s]_\varepsilon \rightarrow s$ in $L^\beta(I; L^\sigma_{\text{loc}}(\Omega))$, $[q]_\varepsilon \rightarrow q$ in $L^\beta(I; L^\sigma_{\text{loc}}(\Omega))$ and $u \in L^p(I; W^{1,q}(\Omega; \mathbb{R}^d))$, we easily see $[q]_\varepsilon [s]_\varepsilon u \rightarrow q s u$ in $L^1(I; L^\sigma_{\text{loc}}(\Omega; R^d))$. This finishes the proof of Statement 1.

In the case of Statement 2 we first proceed as above and verify that $\varphi s$ is a distributional (weak) solution to the continuity equation. Only in the limit passage of $[q]_\varepsilon [s]_\varepsilon u$ we have to employ additionally the Sobolev embedding theorem for $u$ in the spatial variable together with the $L^\infty$ bound in time for $[q]_\varepsilon$ and $[s]_\varepsilon$, if some of the exponents is equal to $\infty$, and interpolate these bounds. Next, we apply Theorem 5.
Statement 1.1, to see that $\varrho s$ is a renormalized distributional solution to the continuity equation.

Furthermore, since $\varrho s \in C_{\text{weak}}(\bar{T}; L^r(\Omega))$ and $\gamma > 1$, we may employ Theorem 3, Statement 3., to verify that $\varrho s \in C(\bar{T}; L^r(\Omega))$ for any $1 \leq r < \gamma$. Theorem 7 is proved.

8. Proof of the main results

8.1. Proof of Theorem 1

To prove Theorem 1, we first use the fact that $\varrho$ is a renormalized time integrated weak solution of the transport equation and use $b_{\delta}(\varrho) := \frac{\delta}{\delta + \varrho}$ with $\delta > 0$ in the renormalized formulation. As we know that $\varrho \geq 0$ a.e. in $(0, T) \times \Omega$, the function $b_{\delta}$ is an appropriate renormalizing function. We get

$$\int_{\Omega} \frac{\delta}{\delta + \varrho(t, \cdot)} \varphi(t, \cdot) \, dx - \int_{\Omega} \frac{\delta}{\delta + \varrho(0, \cdot)} \varphi(0, \cdot) \, dx - \int_{0}^{T} \int_{\Omega} \frac{\delta}{\delta + \varrho} \varphi \, dx \, d\tau$$

for all $\varphi \in C_\infty([0, T] \times \bar{\Omega})$. We may let $\delta \to 0^+$ in (55) to get (we use the Lebesgue dominated convergence theorem; recall that $\frac{\delta}{\delta + \varrho(t, x)} = 1$ provided $\varrho(t, x) = 0$)

$$\int_{\Omega} s_{\varrho}(t, \cdot) \varphi(t, \cdot) \, dx - \int_{\Omega} s_{\varrho}(0, \cdot) \varphi(0, \cdot) \, dx - \int_{0}^{T} \int_{\Omega} s_{\varrho} \varphi \, dx \, d\tau$$

for all $\varphi$ as above. Here, $s_{\varrho}$ denotes the characteristic function of the set, where $\varrho = 0$. Hence, $s_{\varrho}$ is a time integrated weak solution to the transport equation with the function $u$. Moreover, repeating the argument above with $\tilde{b}(\varrho) := b\left(\frac{\delta}{\delta + \varrho}\right)$, where $b$ belongs to the class (10), we also get that $s_{\varrho}$ is a renormalized time integrated weak solution.

Since $\int_{\Omega} s_{\varrho}(\tau, \cdot) \, dx = |\{x \in \Omega; \varrho(\tau, x) = 0\}|_d$, we may subtract equations (56) with $\varphi = 1$ for $t := \tau_1$ and $t := \tau_2$ and it is easy to see that

$$\left| \int_{\tau_1}^{\tau_2} \int_{\Omega} s_{\varrho} \, d\text{div}u \, dx \, d\tau \right| \to 0 \quad \text{for} \ \tau_1 \to \tau_2.$$ 

Hence

$$|\{x \in \Omega; \varrho(\tau, x) = 0\}|_d \in C([0, T]).$$

8Strictly speaking, function $b_{\delta}$ does not satisfy the second condition (10). Nevertheless, the map $\varrho \mapsto \varrho b_{\delta}(\varrho) - b_{\delta}(\varrho)$ remains bounded. We can thus take instead of $b_{\delta}$ a convenient approximation (e.g., $j_\varepsilon \max\{b_{\delta}(\cdot + \varepsilon), 1/\varepsilon\}$, $\varepsilon \in (0, \delta)$, see Lemma 2 for the notation) which satisfies (10), and then let $\varepsilon \to 0$ in order to get (55).
Note further that repeating the argument to get (56) with a test function only space
dependent, we get \( s_\varrho \in C_{\text{weak}}([0, T]; L^r(\Omega)) \) for any \( 1 \leq r < \infty \) and thus, by
Lemma 5,
\[
s_\varrho \in C([0, T]; L^r(\Omega)), \quad 1 \leq r < \infty.
\]
The theorem is proved.

8.2. Proof of Theorem 2 and Corollaries 1–4

The first claim of Theorem 2 is a direct consequence of Theorems 3 and 5. The
second claim follows directly from Theorem 7, Statement 2. The third claim is a direct
consequence of formula (28).

Corollary 1 follows immediately from Theorem 2. Interchanging the role of \( R \) and \( \varrho \) we immediately get Corollary 2.

We now consider Corollary 3. We aim at proving that \( |\{ x \in \Omega | \varrho(t_0, x) = 0 \}|_d = 0 \)
for any \( t_0 \in (0, T] \). First, for \( t_0 \in (0, \tau] \), we define \( \tilde{R}(t) := R(t - t_0 + \tau) \).
Since \( u \) is
time independent, the function \( \tilde{R} \) is a distributional solution to the continuity equation
on \( (t_0 - \tau, T' + t_0 - \tau) \) and we may apply Theorem 2, in particular formula (28)
with \( \tilde{R} \) instead of \( R \) and \( t_0 \) instead of \( t \). Hence, Corollary 3 holds in the time interval
\( (0, \tau] \). Next, we consider \( t_0 \in (\tau, 2\tau] \). We redefine \( \tilde{R} \) as \( \tilde{R}(t) := R(t - \tau) \) and apply
formula (28) with \( t := t_0 + \tau \) on the left-hand side and \( \tau \) instead of \( 0 \) on the right-hand
side. Hence, \( |\{ x \in \Omega | \varrho(t, x) = 0 \}|_d = 0 \) for \( t \in [0, 2\tau] \). Proceeding similarly, after
finite number of steps, we cover the whole interval \( (0, T) \). Corollary 3 is proved.

Note finally that due to our definition of the weak solution to the compressible
Navier–Stokes equations Corollary 4 follows directly, since all assumptions of Theorems
1 and 2, as well as of Corollaries 1, 2 and 3, are fulfilled.

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