On fractional boundary value problems involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions

Mohammed S. Abdo, Thabet Abdeljawad, Saeed M. Ali and Kamal Shah

Abstract

In this paper, we consider two classes of boundary value problems for nonlinear implicit differential equations with nonlinear integral conditions involving Atangana–Baleanu–Caputo fractional derivatives of orders \(0 < \vartheta \leq 1\) and \(1 < \vartheta \leq 2\). We structure the equivalent fractional integral equations of the proposed problems. Further, the existence and uniqueness theorems are proved with the aid of fixed point theorems of Krasnoselskii and Banach. Lastly, the paper includes pertinent examples to justify the validity of the results.

MSC: 34A08; 34A12; 47H10

Keywords: Implicit fractional differential equations; Atangana–Baleanu–Caputo fractional derivative; Integral conditions; Fixed point theorems

1 Introduction

Fractional calculus [1–3] has continued to attract the attention of many authors in the past three decades. Recently, new fractional derivatives (FDs) which interpolate the Riemann–Liouville, Caputo, Hilfer, Hadamard, and generalized FDs have appeared, see [4–9]. Some investigators have recognized that innovation for novel FDs with various nonsingular or singular kernels is necessary to address the need to model more realistic problems in various areas of engineering and science. Caputo and Fabrizio [10] introduced a new kind of FDs where the kernel is based on the exponential function. Losada and Nieto [11] studied some properties of this new operator. In [12, 13], the authors presented new interesting FDs where the kernel relies on Mittag-Leffler function, the so-called Atangana–Baleanu–Caputo (AB–Caputo) which is basically a generalization of the Caputo FD. Then in [14, 15], the authors deliberated the discrete versions of those new operators. For modeling in the framework of nonsingular kernels and fractal-fractional derivatives, we refer to [16–18]. There are many works pertinent to ABC problem in medical science and engineering. Hence we highlight medical, as well as engineering, applications by referring to [19–21].
On the other hand, the fixed point theory is a collection of results saying that a mapping $T$ will have at least one fixed point (i.e., $T(x) = x$), under some conditions on $T$. Results of this kind are of paramount importance in many areas of mathematics, other sciences, and engineering. So, some recent articles which are pertinent to the fixed point theory can found in [22–29]. The existence and uniqueness of solutions for different classes of fractional differential equations (FDEs) with initial or boundary conditions have been studied by several researchers; see [30–38] and the references therein. Some recent contributions on FDEs involving ABC-FDs can be found in the following articles series: [39–49]. For instance, AB–Caputo fractional IVP is one of the studied problems by Jarad et al. [39], and has the form

$$\begin{cases}
ABC_{\alpha}^{\gamma} \zeta(t) = f(t, \zeta(t)), & t \in [a, T], 0 < \gamma \leq 1, \\
\zeta(a) = \zeta_a.
\end{cases}$$

The BVP of AB-Caputo FD, presented by Abdeljawad in [40], is also one of the recent problems through which the higher fractional orders are addressed:

$$\begin{cases}
ABC_{\alpha}^{\gamma} \zeta(t) + q(t) \zeta(t) = 0, & t \in [a, T], 1 < \gamma \leq 2, \\
\zeta(a) = \zeta(T) = 0.
\end{cases}$$

Motivated by the above arguments, the intent of this work is to investigate two AB–Caputo-type implicit FDEs with nonlinear integral conditions described by

$$\begin{cases}
ABC_{\alpha}^{\gamma} \zeta(t) = f(t, \zeta(t), ABC_{\alpha}^{\gamma} \zeta(t)), & t \in [a, T], 0 < \gamma \leq 1, \\
\zeta(a) - \zeta'(a) = \int_a^T g(s, \zeta(s)) \, ds
\end{cases}$$

and

$$\begin{cases}
ABC_{\alpha}^{\gamma} \zeta(t) = f(t, \zeta(t), ABC_{\alpha}^{\gamma} \zeta(t)), & t \in [a, T], 1 < \gamma \leq 2, \\
\zeta(a) = 0, \quad \zeta(T) = \int_a^T g(s, \zeta(s)) \, ds
\end{cases}$$

where $ABC_{\alpha}^{\gamma}$ is the AB–Caputo FD of order $\gamma$, while $f : [a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [a, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Some fixed point theorems (FPTs) are applied to establish the existence and uniqueness theorems for the problems (1.1) and (1.2). The proposed problems are more general, and the results generalize those obtained in recent studies; we also provide an extension of the development of FDEs involving this new operator. Moreover, the analysis of the results was limited to the minimum assumptions.

Many other recent works have investigated similar topics using the same concepts; one can see [50–55].

The rest of the paper is structured as follows. In Sect. 2, we give some useful preliminaries related to main consequences. Section 3 is devoted to obtaining formulas of solution to the proposed problems. Moreover, the existence and uniqueness theorems for the problems at hand are proved by means of various techniques for FPTs. Ultimately, illustrative examples are offered in Sect. 4.
2 Background materials and preliminaries

Here we recollect some requisite definitions and preliminary concepts related to our work.

Let $\mathcal{Z} = [a, T] \subset \mathbb{R}, C(\mathcal{Z}, \mathbb{R})$ be the space of continuous functions $\varsigma : \mathcal{Z} \to \mathbb{R}$ with the norm

$$\|\varsigma\| = \max\{ |\varsigma(t)| : t \in \mathcal{Z} \},$$

Clearly, $C(\mathcal{Z}, \mathbb{R})$ is a Banach space with the norm $\|\varsigma\|$.

Definition 2.1 ([12, 13]) Let $\vartheta \in (0, 1]$ and $p \in H^1(\mathcal{Z})$. Then the AB–Caputo and AB–Riemann–Liouville FDs of order $\vartheta$ for a function $p$ are described by

$$ABCD_{t a}^\vartheta p(t) = \frac{\eta(\vartheta)}{1 - \vartheta} \int_a^t E_{\vartheta} \left( -\frac{\vartheta}{\vartheta - 1} (t - s)^\vartheta \right) p'(s) ds, \quad t > a,$$

and

$$ABRD_{t a}^\vartheta p(t) = \frac{\eta(\vartheta)}{1 - \vartheta} \frac{d}{dt} \int_a^t E_{\vartheta} \left( -\frac{\vartheta}{\vartheta - 1} (t - s)^\vartheta \right) p(s) ds, \quad t > a,$$

respectively, where $E_{\vartheta}$ is called the Mittag-Leffler function and described by

$$E_{\vartheta} (p) = \sum_{k=0}^{\infty} \frac{p^k}{\Gamma(\vartheta + 1)^k}, \quad \text{Re}(\vartheta) > 0, \quad p \in \mathbb{C}.$$

The associated AB fractional integral is specified by

$$ABI_{t a}^\vartheta p(t) = \frac{1 - \vartheta}{\eta(\vartheta)} p(t) + \frac{\vartheta}{\eta(\vartheta) \Gamma(\vartheta)} \int_a^t (t - s)^{\vartheta - 1} p(s) ds, \quad t > a,$$

where $\eta(\vartheta) > 0$ is a normalization function satisfying $\eta(0) = \eta(1) = 1$.

Definition 2.2 ([13]) In particular, if $a = 0$, the Laplace transform of AB–Caputo FD of $p(t)$ is specified by

$$\mathcal{L}[ABCD_{t a}^\vartheta p(t)] = \frac{\eta(\vartheta)}{s^\vartheta (1 - \vartheta) + \vartheta s^\vartheta} \mathcal{L}[p(t)] - s^{\vartheta - 1} p(0)].$$

Lemma 2.1 ([14]) Let $\vartheta \in (0, 1]$ and $p \in H^1(\mathcal{Z})$, if AB–Caputo FD exists, then we have

$$ABR_{t a}^\vartheta ABC_{t a}^\vartheta p(t) = p(t)$$

and

$$AB_{t a}^\vartheta ABC_{t a}^\vartheta p(t) = p(t) - p(a).$$

Definition 2.3 ([40]) Let $\vartheta \in (n, n + 1]$ and $p$ be such that $p^n \in H^1(\mathcal{Z})$. Set $\nu = \vartheta - n$ where $\nu \in (0, 1]$. Then the AB–Caputo and AB–Riemann–Liouville FDs of order $\vartheta$ for a function $p$ are described by

$$ABC_{t a}^\vartheta p(t) = ABC_{t a}^{\vartheta - \nu} p^{(n)}(t)$$
and

\[ ABR^\vartheta_{a^\ast} p(t) = AB^\vartheta D^\vartheta_{a^\ast} p^{(n)}(t), \]

respectively. The associated AB fractional integral is specified by

\[ AB^\vartheta_{a^\ast} p(t) = \int^\vartheta_{a^\ast} AB^\vartheta D^\vartheta_{a^\ast} p(t). \]

**Remark 2.1** If \( \vartheta \in (0, 1] \), we have \( \vartheta = v \). Hence

\[ AB^{\vartheta}_{a^\ast} p(t) = AB^{\vartheta}_{a^\ast} p(t), \]

\[ ABC^{\vartheta}_{a^\ast} p(t) = ABC^{\vartheta}_{a^\ast} p(t), \]

\[ ABR^{\vartheta}_{a^\ast} p(t) = ABR^{\vartheta}_{a^\ast} p(t). \]

**Definition 2.4** ([40]) The relation between the AB–Riemann–Liouville and AB–Caputo FDs is

\[ ABC^{\vartheta}_{a^\ast} p(t) = ABR^{\vartheta}_{a^\ast} p(t) - N(\vartheta) \frac{p(a) E_\vartheta}{\vartheta - 1} (t - a)^\vartheta. \] (2.1)

**Lemma 2.2** ([40]) For \( n - 1 < \vartheta \leq n, n \in \mathbb{N}_0 \), and \( p(t) \) defined on \( \mathfrak{r} \), we have:

(i) \( ABR^{\vartheta}_{a^\ast}, AB^{\vartheta}_{a^\ast} p(t) = p(t) \);

(ii) \( ABR^{\vartheta}_{a^\ast}, ABC^{\vartheta}_{a^\ast} p(t) = p(t) - \sum_{k=0}^{n-1} \frac{p^{(k)}(a)}{k!} (t - a)^k \);

(iii) \( ABR^{\vartheta}_{a^\ast}, ABR^{\vartheta}_{a^\ast} p(t) = p(t) - \sum_{k=0}^{n-1} \frac{p^{(k)}(a)}{k!} (t - a)^k \).

**Remark 2.2** With the help of (2.1), for any \( \vartheta \), it can be shown that

\[ ABC^{\vartheta}_{a^\ast}, AB^{\vartheta}_{a^\ast} p(t) = p(t) - p(a). \] (2.2)

Hence, under the condition that \( p(a) = 0 \), we get the identity

\[ ABC^{\vartheta}_{a^\ast}, AB^{\vartheta}_{a^\ast} p(t) = p(t). \] (2.3)

**Lemma 2.3** ([40]) Let \( n < \vartheta \leq n + 1 \). Then \( ABR^{\vartheta}_{a^\ast} p(t) = 0 \), if \( p(t) \) is constant function.

**Lemma 2.4** ([13]) Let \( \vartheta > 0 \). Then \( AB^{\vartheta}_{a^\ast} \) is bounded from \( C(\mathfrak{r}, \mathbb{R}) \) into \( C(\mathfrak{r}, \mathbb{R}) \).

**Lemma 2.5** Let \( n < \vartheta \leq n + 1 \). Then \( ABC^{\vartheta}_{a^\ast} (t - a)^k = 0 \), for \( k = 0, 1, \ldots, n \).

**Proof** Let \( p(t) = (t - a)^k \). By Definition 2.3, we have

\[ ABC^{\vartheta}_{a^\ast} p(t) = ABC^{\vartheta}_{a^\ast} p^{(n)}(t) \]
\[ = ABC^{\vartheta}_{a^\ast} [(t - a)^k]^{(n)} \]
\[ = ABC^{\vartheta}_{a^\ast} \left( \frac{d}{dt} \right)^n (t - a)^k. \]
Since \( k < n \in \mathbb{N} \), we have \( \left( \frac{d}{dt} \right)^n (t-a)^k = 0 \). It follows from Lemma 2.3 that

\[
ABC^n_{a^\vartheta} p(t) = 0. 
\]

\( \square \)

**Lemma 2.6** ([39]) Let \( \vartheta \in (0,1] \) and \( \varpi \in C(\mathfrak{A}, \mathbb{R}) \) with \( \varpi(a) = 0 \). Then the solution of the following problem

\[
ABC^n_{a^\vartheta} p(t) = \varpi(t), \quad t \in \mathfrak{A},
\]

\[ p(a) = c \]

is given by

\[
p(t) = c + \frac{1 - \vartheta}{\Gamma(\vartheta)} \varpi(t) + \frac{\vartheta}{\Gamma(\vartheta)} \int_a^t (t-s)^{\vartheta-1} \varpi(s) \, ds. \]

**Lemma 2.7** ([40]) Let \( \vartheta \in (1,2] \) and \( \varpi \in C(\mathfrak{A}, \mathbb{R}) \) with \( \varpi(a) = 0 \). Then the solution of the following problem

\[
\begin{aligned}
ABC^n_{a^\vartheta} p(t) &= \varpi(t), \quad t \in \mathfrak{A}, \\
p(a) &= c_1, \quad p'(a) = c_2
\end{aligned}
\]

is given by

\[
p(t) = c_1 + c_2(t-a) + \frac{2 - \vartheta}{\Gamma(\vartheta-1)} \int_a^t \varpi(s) \, ds + \frac{\vartheta - 1}{\Gamma(\vartheta-1) \Gamma(\vartheta)} \int_a^t (t-s)^{\vartheta-1} \varpi(s) \, ds. \]

**Definition 2.5** ([56]) Let \( J \) be a Banach space. The operator \( \mathcal{B} : J \rightarrow J \) is a contraction if

\[
\| \mathcal{B} x_1 - \mathcal{B} x_2 \| \leq p \| x_1 - x_2 \|, \quad \text{for all } x_1, x_2 \in J, \quad 0 < p < 1.
\]

**Theorem 2.1** (Banach FPT, [56]) Let \( J \) be a Banach space, and \( \mathcal{K} \) be a nonempty closed subset of \( J \). If \( \mathcal{B} : \mathcal{K} \rightarrow \mathcal{K} \) is a contraction, then there exists a unique fixed point of \( \mathcal{B} \).

**Theorem 2.2** (Krasnoselskii FPT, [56]) Let \( \mathcal{K} \) be a nonempty, closed, convex subset of a Banach space \( J \). Let \( \mathcal{B}_1, \mathcal{B}_2 \) be two operators such that (i) \( \mathcal{B}_1 u + \mathcal{B}_2 v \in \mathcal{K}, \forall u, v \in \mathcal{K} \); (ii) \( \mathcal{B}_1 \) is compact and continuous; (iii) \( \mathcal{B}_2 \) is a contraction mapping. Then, there exists \( w \in \mathcal{K} \) such that \( \mathcal{B}_1 w + \mathcal{B}_2 w = w \).

### 3 Main results

This section is devoted to obtaining formulas of solutions to linear problems corresponding to (1.1) and (1.2). Moreover, we prove the existence and uniqueness theorems to suggested problems by applying Theorems 2.1 and 2.2.

#### 3.1 Solution formulas

**Theorem 3.1** Let \( 0 < \vartheta \leq 1 \), and let \( \varpi, g \in C(\mathfrak{A}, \mathbb{R}) \) with \( \varpi(a) = \varpi'(a) = 0 \). A function \( \varsigma \in C(\mathfrak{A}, \mathbb{R}) \) is a solution of the fractional integral equation (FIE)

\[
\varsigma(t) = \int_a^t g(s) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} \varpi(t) + \frac{\vartheta}{\Gamma(\vartheta)} \int_a^t (t-s)^{\vartheta-1} \varpi(s) \, ds, \quad t \in \mathfrak{A},
\]
if and only if $\varsigma$ is a solution of the ABC-problem

$$\begin{align*}
\text{ABC}_a^{\vartheta} \varsigma(t) &= \varpi(t), \quad t \in \mathcal{I}, \\
\varsigma(t) &= \int_a^T g(s) \, ds.
\end{align*}$$

(3.2)

**Proof** Assume $\varsigma$ satisfies the first equation of (3.2). From Lemma 2.6, we have

$$\begin{align*}
\varsigma(t) &= \varsigma(a) + \frac{1 - \vartheta}{\Gamma(\vartheta)} \varpi(t) + \frac{\vartheta}{\Gamma(\vartheta)} \int_a^t (t - s)^{\vartheta - 1} \varpi(s) \, ds,
\end{align*}$$

(3.3)

Also,

$$\begin{align*}
\varsigma'(t) &= \frac{1 - \vartheta}{\Gamma(\vartheta)} \psi(t) + \frac{1}{\Gamma(\vartheta - 1)} \int_a^t (t - s)^{\vartheta - 2} \varpi(s) \, ds.
\end{align*}$$

(3.4)

Taking $t \to a$ on both sides of (3.4), we have

$$\varsigma'(a) = \frac{1 - \vartheta}{\Gamma(\vartheta)} \psi(a).$$

Using the integral condition, we obtain

$$\begin{align*}
\varsigma(a) &= \varsigma'(a) + \int_a^T g(s) \, ds = \frac{1 - \vartheta}{\Gamma(\vartheta)} \psi(a) + \int_a^T g(s) \, ds.
\end{align*}$$

(3.5)

From (3.3) and (3.5), and from fact that $\varpi'(a) = 0$, we get

$$\begin{align*}
\varsigma(t) &= \int_a^T g(s) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} \psi(t) + \frac{1}{\Gamma(\vartheta - 1)} \int_a^t (t - s)^{\vartheta - 2} \varpi(s) \, ds, \\
& \quad t \in \mathcal{I}.
\end{align*}$$

Thus (3.1) is satisfied.

Conversely, suppose that $\varsigma$ satisfies equation (3.1). Applying $\text{ABC}_a^{\vartheta}$ on both sides of (3.1), then using Remark 2.2 and Lemma 2.3, we find that

$$\begin{align*}
\text{ABC}_a^{\vartheta} \varsigma(t) &= \text{ABC}_a^{\vartheta} \int_a^T g(s) \, ds + \text{ABC}_a^{\vartheta} \int_a^T \varpi(t) \\
& = \varpi(t).
\end{align*}$$

Thus, we can simply infer that

$$\begin{align*}
\varsigma(a) - \varsigma'(a) &= \int_a^T g(s) \, ds.
\end{align*}$$

\[
\square
\]

**Theorem 3.2** Let $1 < \vartheta \leq 2$, and let $\varpi, g \in C(\mathcal{I}, \mathbb{R})$ with $\varpi(a) = 0$. A function $\varsigma \in C(\mathcal{I}, \mathbb{R})$ is a solution of the FIE

$$\begin{align*}
\varsigma(t) &= \frac{(t - a)}{T} \int_a^T g(s) \, ds + \frac{\vartheta - 2}{\Gamma(\vartheta - 1)} \int_a^T \varpi(t) \\
& \quad + \frac{1 - \vartheta}{\Gamma(\vartheta - 1)} \int_a^T \varpi(s) \, ds
\end{align*}$$

Thus, $\varsigma$ is a solution of the ABC-problem.
(3.6) if and only if $\varsigma$ is a solution of the ABC-problem

$$A^{\mathbb{C}}D^\psi_{a^*} \varsigma(t) = \varpi(t), \quad \tau \in \mathbb{R},$$

$$\varsigma(a) = 0, \quad \varsigma(T) = \int_a^T g(s) \, ds. \tag{3.7}$$

Proof. Assume $\varsigma$ satisfies the first equation of (3.7). From Lemma 2.7, we have

$$\varsigma(t) = c_1 + c_2 (T - a) + \frac{2 - \theta}{\mathbb{N}(\theta - 1)} \int_a^T \varpi(s) \, ds + \frac{\theta - 1}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds, \tag{3.8}$$

for some $c_1, c_2 \in \mathbb{R}$. Since $\varsigma(a) = 0$, we get $c_1 = 0$. Hence

$$\varsigma(T) = c_2 (T - a) + \frac{2 - \theta}{\mathbb{N}(\theta - 1)} \int_a^T \varpi(s) \, ds + \frac{\theta - 1}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds. \tag{3.9}$$

Using the integral condition $\varsigma(T) = \int_a^T g(s) \, ds$, we get

$$c_2 = \frac{1}{(T - a)} \int_a^T g(s) \, ds - \frac{2 - \theta}{\mathbb{N}(\theta - 1)} \frac{1}{(T - a)} \int_a^T \varpi(s) \, ds$$

$$- \frac{\theta - 1}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \frac{1}{(T - a)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds. \tag{3.10}$$

Substituting the values of $c_1$ and $c_2$ into (3.8), we obtain

$$\varsigma(t) = \frac{(t - a)}{(T - a)} \int_a^T g(s) \, ds + \frac{\theta - 2}{\mathbb{N}(\theta - 1) (T - a)} \int_a^T \varpi(s) \, ds$$

$$+ \frac{1 - \theta}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \frac{(t - a)}{(T - a)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds$$

$$+ \frac{2 - \theta}{\mathbb{N}(\theta - 1)} \int_a^T \varpi(s) \, ds + \frac{\theta - 1}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds.$$

Thus (3.6) is satisfied.

Conversely, assume that $\varsigma$ satisfies (3.6). Applying $A^{\mathbb{C}}D^\psi_{a^*}$ on both sides of (3.6), then using Lemmas 2.2, 2.3, and 2.5, we find that

$$A^{\mathbb{C}}D^\psi_{a^*} \varsigma(t) = \frac{1}{(T - a)} \int_a^T g(s) \, ds A^{\mathbb{C}}D^\psi_{a^*} (t - a)$$

$$+ \frac{\theta - 2}{\mathbb{N}(\theta - 1) (T - a)} \int_a^T \varpi(s) \, ds A^{\mathbb{C}}D^\psi_{a^*} (t - a)$$

$$+ \frac{1 - \theta}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \frac{1}{(T - a)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds A^{\mathbb{C}}D^\psi_{a^*} (t - a)$$

$$+ \frac{\theta - 1}{\mathbb{N}(\theta - 1) \Gamma(\theta)} \frac{1}{(T - a)} \int_a^T (T - s)^{\theta - 1} \varpi(s) \, ds A^{\mathbb{C}}D^\psi_{a^*} (t - a)$$

$$= \varpi(t).$$
Clearly, \( \zeta(a) = 0 \). Thus, we can simply infer that
\[
\zeta(T) = \int_a^T g(s) \, ds + \frac{\vartheta - 2}{\Gamma(\vartheta - 1)} \int_a^T \sigma(s) \, ds
\]
\[
+ \frac{1 - \vartheta}{\Gamma(\vartheta - 1) \Gamma(\vartheta)} \int_a^T (T - s)^{\vartheta - 1} \sigma(s) \, ds
\]
\[
+ \frac{2 - \vartheta}{\Gamma(\vartheta - 1)} \int_a^T \sigma(s) \, ds + \frac{\vartheta - 1}{\Gamma(\vartheta - 1) \Gamma(\vartheta)} \int_a^T (T - s)^{\vartheta - 1} \sigma(s) \, ds
\]
\[
= \int_a^T g(s) \, ds. \quad \Box
\]

Before proceeding with the main findings, we are obligated to provide the following assumptions:

\((A_1)\) \( f : \mathcal{A} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and there exist \( L_f > 0 \) and \( 0 < K_f < 1 \) such that
\[
|f(t, u, \overline{v}) - f(t, v, \overline{v})| \leq L_f|u - v| + K_f|\overline{u} - \overline{v}|, \quad t \in \mathcal{A} \text{ and } u, v, \overline{u}, \overline{v} \in \mathbb{R};
\]

\((A_2)\) \( g : \mathcal{A} \times \mathbb{R} \to \mathbb{R} \) is continuous and there exist constant \( L_g > 0 \) such that
\[
|g(t, u) - g(t, v)| \leq L_g|u - v|, \quad t \in \mathcal{A} \text{ and } u, v \in \mathbb{R};
\]

\((A_3)\)
\[
\left[ L_g(T - a) + \left( \frac{1 - \vartheta}{\Gamma(\vartheta)} + \frac{(T - a)^\vartheta}{\Gamma(\vartheta) \Gamma(\vartheta)} \right) \frac{L_f}{1 - K_f} \right] < 1.
\]

### 3.2 Existence and uniqueness theorems for (1.1)

As a result of Theorem 3.1, we have the following theorem:

**Theorem 3.3** Let \( 0 < \vartheta \leq 1 \), and let \( f : \mathcal{A} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathcal{A} \times \mathbb{R} \to \mathbb{R} \) be continuous with \( f(a, \zeta(a), \text{ABC}D^\vartheta a, \zeta(a)) = f'(a, \zeta(a), \text{ABC}D^\vartheta a, \zeta(a)) = 0 \). If \( \zeta \in C(\mathcal{A}, \mathbb{R}) \) then \( \zeta \) satisfies (1.1) if and only if \( \zeta \) fulfills
\[
\zeta(t) = \int_a^T g(s, \zeta(s)) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} f(t, \zeta(t), \text{ABC}D^\vartheta t, \zeta(t))
\]
\[
+ \frac{\vartheta}{\Gamma(\vartheta) \Gamma(\vartheta)} \int_a^T (t - s)^{\vartheta - 1} f(s, \zeta(s), \text{ABC}D^\vartheta s, \zeta(s)) \, ds.
\]

**Theorem 3.4** Let \((A_1)\) and \((A_3)\) be fulfilled. Then the ABC-problem (1.1) has a unique solution.

**Proof** Set
\[
\mathcal{D} = \{ \zeta \in C(\mathcal{A}, \mathbb{R}) : \text{ABC}D^\vartheta a \zeta \in C(\mathcal{A}, \mathbb{R}) \}.
\]

By Theorem 3.3, we define the operator \( T : \mathcal{D} \to \mathcal{D} \) by
\[
(T \zeta)(t) = \int_a^T g(s, \zeta(s)) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} f(t, \zeta(t), \text{ABC}D^\vartheta t, \zeta(t))
\]
\[
+ \frac{\vartheta}{\Gamma(\vartheta) \Gamma(\vartheta)} \int_a^T (t - s)^{\vartheta - 1} f(s, \zeta(s), \text{ABC}D^\vartheta s, \zeta(s)) \, ds.
\]
This $T$ is well defined, that is, $T(\Omega) \subseteq \Omega$. Indeed, for any $\zeta \in C(\mathbb{J}, \mathbb{R}), f(\cdot, \zeta(\cdot), ABC\mathbb{D}^{\omega}_{\omega, \omega}, \zeta(\cdot))$ is continuous. Besides, by Lemma 2.4, $T\zeta \in C(\mathbb{J}, \mathbb{R})$. Also, by Lemma 2.1 and Remark 2.1, we end up with

$$
ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega}(T\zeta)(t) = \int_{a}^{T} g(s, \zeta(s)) \, ds ABC\mathbb{D}^{\omega}_{\omega, \omega}. \, (1) + ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) = ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) = f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)).
$$

Since $f(t, \cdot, \cdot)$ is continuous on $[a, T]$, one has $ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega}(T\zeta)(t) \in C(\mathbb{J}, \mathbb{R})$.

Now, we need to prove that $T$ is a contraction. Let $\zeta, \bar{\zeta} \in \Omega$ and $t \in \mathbb{J}$. Then

$$
| (T\zeta)(t) - (T\bar{\zeta})(t) |
\leq \int_{a}^{T} | g(s, \zeta(s)) - g(s, \bar{\zeta}(s)) | \, ds
+ \frac{1 - \theta}{\eta(\theta)} | f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) - f(t, \bar{\zeta}(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(t)) |
+ \frac{\theta}{\eta(\theta)} \cdot \frac{1}{\Gamma(\theta)} \int_{a}^{t} (t - s)^{\theta - 1} | g(s, \zeta(s), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(s)) - g(s, \bar{\zeta}(s), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(s)) | \, ds.
$$

Using (A1) and the fact that $ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t) = f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t))$, we obtain

$$
| f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) - f(t, \bar{\zeta}(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(t)) |
\leq L_{f} | \zeta(t) - \bar{\zeta}(t) | + K_{f} | ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t) - ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(t) |
= L_{f} | \zeta(t) - \bar{\zeta}(t) | + K_{f} | f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) - f(t, \bar{\zeta}(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(t)) |,
$$

which implies

$$
| f(t, \zeta(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \zeta(t)) - f(t, \bar{\zeta}(t), ABC\mathbb{D}^{\omega}_{\omega, \omega, \omega} \bar{\zeta}(t)) | \leq \frac{L_{f}}{1 - K_{f}} | \zeta(t) - \bar{\zeta}(t) |. \tag{3.11}
$$

By (A2) and (3.11), for $t \in \mathbb{J}$,

$$
| (T\zeta)(t) - (T\bar{\zeta})(t) |
\leq L_{g} \int_{a}^{T} | \zeta(s) - \bar{\zeta}(s) | \, ds
+ \frac{1 - \theta}{\eta(\theta)} L_{f} \frac{1}{1 - K_{f}} | \zeta(t) - \bar{\zeta}(t) |
+ \frac{\theta}{\eta(\theta)} L_{f} \frac{1}{1 - K_{f}} \cdot \frac{1}{\Gamma(\theta)} \int_{a}^{t} (t - s)^{\theta - 1} \, ds
\leq \left[ L_{g} (T - a) + \frac{1 - \theta}{\eta(\theta)} \cdot \frac{(T - a)^{\theta}}{\eta(\theta) \Gamma(\theta)} + \frac{L_{f}}{1 - K_{f}} \right] \| \zeta - \bar{\zeta} \|.
$$

Condition (A3) shows that $T$ is a contraction. Hence, by Theorem 2.1, $T$ has a unique fixed point. \qed
Theorem 3.5 Suppose \((A_1)\) and \((A_3)\) are fulfilled. Then there exists at least one solution of the problem (1.1).

Proof Choose \((T\xi)(t) = (T_1\xi)(t) + (T_2\xi)(t)\), where

\[
(T_1\xi)(t) = \int_a^T g(s, \xi(s)) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t))
\]  

(3.12)

and

\[
(T_2\xi)(t) = \frac{\vartheta}{\Gamma(\vartheta)} \int_a^t (t - s)^{\vartheta - 1} f(s, \xi(s), \frac{\partial}{\partial s} \mathcal{D}_a^\vartheta, \xi(s)) \, ds.
\]

(3.13)

Set \(\mu_f := \max\{|f(t, 0, 0)|; t \in \mathcal{I}\} < \infty\) and \(\mu_\xi := \max\{|g(t, 0)|; t \in \mathcal{I}\} < \infty\). Let

\[
B_\xi = \{\xi \in \mathcal{D} : \|\xi\| \leq \xi\}
\]

(3.14)

with the radius

\[
\xi \geq \frac{\mu_\xi (T - a) + \left( \frac{\vartheta}{\Gamma(\vartheta)} + \frac{(T - a)^\vartheta}{\Gamma(\vartheta)(T - a)} \right) \mu_f}{1 - (L_2(T - a) + \left( \frac{\vartheta}{\Gamma(\vartheta)} + \frac{(T - a)^\vartheta}{\Gamma(\vartheta)(T - a)} \right) L_2)}. \tag{3.15}
\]

We will complete the proof in several steps.

Step 1. We show that \(T_1\xi \in T_2\nu\), for all \(\xi, \nu \in B_\xi\).

By (3.12),

\[
|T_1\xi(t)| \leq \int_a^T |g(s, \xi(s))| \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} |f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t))|.
\]

(3.16)

From \((A_1)\) and \((A_2)\), we have

\[
|f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t))| \leq |f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t)) - f(t, 0, 0)| + |f(t, 0, 0)|
\]

\[
\leq L_2|\xi(t)| + K_f|\frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t)| + \mu_f
\]

\[
= L_2|\xi(t)| + K_f|f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t))| + \mu_f,
\]

which gives

\[
|f(t, \xi(t), \frac{\partial}{\partial t} \mathcal{D}_a^\vartheta, \xi(t))| \leq \frac{L_2|\xi(t)| + \mu_f}{1 - K_f} \tag{3.17}
\]

and

\[
|g(t, \xi(t))| = |g(t, \xi(t)) - g(t, 0)| + |g(t, 0)|
\]

\[
\leq L_2|\xi(t)| + \mu_\xi. \tag{3.18}
\]

By substituting (3.17) and (3.18) into (3.16), we have for \(\xi \in B_\xi\),

\[
|T_1\xi(t)| \leq \int_a^T \left( L_2 \|\xi\| + \mu_\xi \right) \, ds + \frac{1 - \vartheta}{\Gamma(\vartheta)} \frac{L_2 \|\xi\| + \mu_f}{1 - K_f}.
\]
\[
\leq (L_g \xi + \mu_g)(T - a) + \frac{1 - \theta}{\eta(\vartheta)} \frac{L_f \xi + \mu_f}{1 - K_f}
\]
\[
= \left( L_g(T - a) + \frac{1 - \theta}{\eta(\vartheta)} \frac{L_f}{1 - K_f} \right) \xi + \mu_g(T - a) + \frac{1 - \theta}{\eta(\vartheta)} \frac{\mu_f}{1 - K_f}.
\] (3.19)

Also, by (3.13),

\[
|T_2 \varphi(t)| = \frac{\vartheta}{\eta(\vartheta) \Gamma(\vartheta)} \int_a^t (t - s)^{\vartheta - 1} |f(s, \varphi(s), ABCD_a^\vartheta \varphi(s))| \, ds.
\]

From (3.17), then for \( \varphi \in B_\xi \),

\[
|T_2 \varphi(t)| \leq \frac{\vartheta}{\eta(\vartheta) \Gamma(\vartheta)} \int_a^t (t - s)^{\vartheta - 1} L_f \| \varphi \| + \frac{\mu_f}{1 - K_f} \, ds
\]
\[
\leq \frac{(T - a)^\vartheta L_f \xi + \mu_f}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f}
\]
\[
= \frac{(T - a)^\vartheta \mu_f}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} + \frac{(T - a)^\vartheta L_f}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} \xi.
\] (3.20)

Inequalities (3.19) and (3.20) give

\[
\left| (T_1 \varphi + T_2 \varphi) \right| \leq \left| T_1 \varphi \right| + \left| T_2 \varphi \right|
\]
\[
\leq \left( L_g(T - a) + \frac{1 - \theta}{\eta(\vartheta) 1 - K_f} \right) \xi + \mu_g(T - a) + \frac{1 - \theta}{\eta(\vartheta) 1 - K_f} \mu_f
\]
\[
+ \frac{(T - a)^\vartheta \mu_f}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} + \frac{(T - a)^\vartheta L_f}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} \xi
\]
\[
= \left( L_g(T - a) + \left( \frac{1 - \theta}{\eta(\vartheta) \Gamma(\vartheta)} + \frac{(T - a)^\vartheta}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} \right) \mu_f \right) \xi
\]
\[
+ \mu_g(T - a) + \left( \frac{1 - \theta}{\eta(\vartheta) \Gamma(\vartheta)} + \frac{(T - a)^\vartheta}{\eta(\vartheta) \Gamma(\vartheta) 1 - K_f} \right) \frac{\mu_f}{1 - K_f}.
\]

Using (A3) and (3.15), for \( t \in \mathcal{F} \) and \( \varphi, \psi \in B_\xi \),

\[
\| T_1 \varphi + T_2 \psi \| \leq \xi.
\]

Thus, \( T_1 \varphi + T_2 \psi \in B_\xi \), for all \( \varphi, \psi \in B_\xi \).

**Step 2.** We prove that \( T_1 \) is a contraction.

From (A1), we have

\[
|f(t, \varphi(t), \psi(t)) - f(t, \varphi^*(t), \psi^*(t))| \leq \frac{L_f}{1 - K_f} |\varphi(t) - \psi(t)|.
\] (3.21)

From (A2) and (3.21), for \( \varphi, \psi \in B_\xi \),

\[
\left| (T_1 \varphi)(t) - (T_1 \psi^*)(t) \right| \leq \int_a^t \left| g(s, \varphi(s)) - g(s, \psi^*(s)) \right| \, ds
\]
\[
+ \frac{1 - \theta}{\eta(\vartheta)} \left| f(t, \varphi(t), \psi(t)) - f(t, \psi^*(t), \psi^*(t)) \right|
\]
Using (3.17), for $\xi \in B_\varepsilon$, one has

$$
|T_2\xi(t_2) - T_2\xi(t_1)| \\
\leq \frac{\phi}{\Gamma(\phi)} \int_{t_1}^{t_2} (t_2 - s)^{\phi - 1} \left( \frac{L_f |\xi(s)| + \mu_f}{1 - K_f} \right) ds \\
+ \frac{\phi}{\Gamma(\phi)} \int_{t_1}^{t_1} |(t_2 - s)^{\phi - 1} - (t_1 - s)^{\phi - 1}| \left( \frac{L_f \xi + \mu_f}{1 - K_f} \right) ds \\
= \frac{2(L_f \xi + \mu_f)}{\Gamma(\phi)(1 - K_f)} (t_2 - t_1)^{\phi}.
$$

Using (3.17), for $\xi \in B_\varepsilon$,

$$
|T_2\xi(t_2) - T_2\xi(t_1)| \\
\leq \frac{\phi}{\Gamma(\phi)} \int_{t_1}^{t_2} (t_2 - s)^{\phi - 1} \left( \frac{L_f |\xi(s)| + \mu_f}{1 - K_f} \right) ds \\
+ \frac{\phi}{\Gamma(\phi)} \int_{t_1}^{t_1} |(t_2 - s)^{\phi - 1} - (t_1 - s)^{\phi - 1}| \left( \frac{L_f \xi + \mu_f}{1 - K_f} \right) ds \\
= \frac{2(L_f \xi + \mu_f)}{\Gamma(\phi)(1 - K_f)} (t_2 - t_1)^{\phi}.
$$

Observe that $|T_2\xi(t_2) - T_2\xi(t_1)| \to 0$ as $t_2 \to t_1$. In light of the former steps, together with Arzela–Ascoli theorem, we derive that $(T_2 B_\varepsilon)$ is relatively compact, and hence $T_2$ is completely continuous. So, Theorem 2.2 shows that (1.1) has at least one solution. □
3.3 Existence and uniqueness theorems for (1.2)

As a result of Theorem 3.2, we have the following theorem:

**Theorem 3.6** Let $1 < \theta \leq 2$, and let $f : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ be continuous with $f(a, \varsigma(a), ABCD^\theta_a \varsigma(a)) = 0$. If $\varsigma \in C(\mathbb{J}, \mathbb{R})$, then $\varsigma$ satisfies (1.2) if and only if $\varsigma$ fulfills

$$
\varsigma(t) = \frac{(t-a)}{(T-a)} \int_a^T \frac{\vartheta - 2}{\vartheta \Gamma(\vartheta)} \left( \frac{(T-s)}{T-a} \right)^{\vartheta-2} \left( \frac{T}{T-a} \right)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds + \frac{1-\vartheta}{\Gamma(\vartheta)} \int_a^T \frac{(T-s)}{T-a} \left( \frac{T}{T-a} \right)^{\vartheta-1} g(s, \varsigma(s)) \, ds
$$

$$
+ \frac{2-\vartheta}{\Gamma(\vartheta)} \int_a^T \frac{(T-s)}{T-a} \left( \frac{T}{T-a} \right)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds
$$

$$
+ \frac{\vartheta - 1}{\Gamma(\vartheta)} \int_a^T (T-s)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds.
$$

**Theorem 3.7** Suppose $(A_1)$ and $(A_2)$ are satisfied. If

$$
L_a(T-a) < 1,
$$

then the problem (1.2) has a unique solution.

**Proof** In view of Theorem 3.6, we consider $T^* : \mathcal{D} \to \mathcal{D}$ defined by

$$
(T^* \varsigma)(t) = \frac{(t-a)}{(T-a)} \int_a^T \frac{\vartheta - 2}{\vartheta \Gamma(\vartheta)} \left( \frac{(T-s)}{T-a} \right)^{\vartheta-2} \left( \frac{T}{T-a} \right)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds + \frac{1-\vartheta}{\Gamma(\vartheta)} \int_a^T \frac{(T-s)}{T-a} \left( \frac{T}{T-a} \right)^{\vartheta-1} g(s, \varsigma(s)) \, ds
$$

$$
+ \frac{2-\vartheta}{\Gamma(\vartheta)} \int_a^T \frac{(T-s)}{T-a} \left( \frac{T}{T-a} \right)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds
$$

$$
+ \frac{\vartheta - 1}{\Gamma(\vartheta)} \int_a^T (T-s)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds,
$$

From the continuity of $g$ and $f$, $T^*$ is well defined, that is, $T^*(\mathcal{D}) \subseteq \mathcal{D}$.

Now, let $\varsigma, \overline{\varsigma} \in \mathcal{D}$ and $t \in \mathbb{J}$.

Then

$$
\left| (T^* \varsigma)(t) - (T^* \overline{\varsigma})(t) \right|
$$

$$
\leq \frac{(t-a)}{(T-a)} \int_a^T \left| g(s, \varsigma(s)) - g(s, \overline{\varsigma}(s)) \right| \, ds
$$

$$
+ \frac{\vartheta - 2}{\vartheta \Gamma(\vartheta)} \left( \frac{(T-s)}{T-a} \right)^{\vartheta-2} \left( \frac{T}{T-a} \right)^{\vartheta-1} \int_a^T \left| f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) - f(s, \overline{\varsigma}(s), ABCD^\theta_a \overline{\varsigma}(s)) \right| \, ds
$$

$$
+ \frac{1-\vartheta}{\Gamma(\vartheta)} \left( \frac{T}{T-a} \right)^{\vartheta-1} \int_a^T \left| g(s, \varsigma(s)) - g(s, \overline{\varsigma}(s)) \right| \, ds
$$

$$
+ \frac{2-\vartheta}{\Gamma(\vartheta)} \int_a^T \left( \frac{T-s}{T-a} \right)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds
$$

$$
+ \frac{\vartheta - 1}{\Gamma(\vartheta)} \int_a^T (T-s)^{\vartheta-1} f(s, \varsigma(s), ABCD^\theta_a \varsigma(s)) \, ds.
$$
\[ \frac{\theta - 1}{\mathcal{N}(\theta - 1)\Gamma(\theta)} \times \int_a^t (t - s)^{\theta-1} \left| f(s, \zeta(s), ABCD_{a}^{\theta} \zeta(s)) - f(s, \bar{\zeta}(s), ABCD_{a}^{\theta} \bar{\zeta}(s)) \right| ds. \]

Using (A\textsubscript{1}) and same arguments used to get (3.11), we obtain
\[ |f(t, \zeta(t), ABCD_{a}^{\theta} \zeta(t)) - f(s, \bar{\zeta}(s), ABCD_{a}^{\theta} \bar{\zeta}(s))| \leq \frac{L_f}{1-K_f} |\zeta(t) - \bar{\zeta}(t)|. \] (3.23)

By (A\textsubscript{2}) and (3.23), for \( t \in \mathbb{R} \),
\[ |(T^* \zeta)(t) - (T^*_a \zeta)(t)| \]
\[ \leq \frac{(t-a)}{(T-a)} L_g \int_a^T |\zeta(s) - \bar{\zeta}(s)| ds \]
\[ + \frac{\theta - 2}{\mathcal{N}(\theta - 1)} \frac{(t-a)}{(T-a)} L_f \int_a^T |\zeta(s) - \bar{\zeta}(s)| ds \]
\[ + \frac{1 - \theta}{\mathcal{N}(\theta - 1)\Gamma(\theta)} L_f \int_a^T (T - s)^{\theta-1} |\zeta(s) - \bar{\zeta}(s)| ds \]
\[ + \frac{2 - \theta}{\mathcal{N}(\theta - 1)} L_f \int_a^T |\zeta(s) - \bar{\zeta}(s)| ds \]
\[ = L_g(T-a) \| \zeta - \bar{\zeta} \| + \frac{\theta - 2}{\mathcal{N}(\theta - 1)} \frac{L_f(T-a)}{1-K_f} \| \zeta - \bar{\zeta} \| \]
\[ + \frac{1 - \theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} L_f(T-a)^\theta \| \zeta - \bar{\zeta} \| + \frac{2 - \theta}{\mathcal{N}(\theta - 1)} \frac{L_f(T-a)}{1-K_f} \| \zeta - \bar{\zeta} \| \]
\[ \leq L_g(T-a) \| \zeta - \bar{\zeta} \|. \]

Condition (3.22) shows that \( T^* \) is a contraction. Hence, by Theorem 2.1, \( T^* \) has a unique fixed point.

**Theorem 3.8** Suppose that (A\textsubscript{1}) and (A\textsubscript{2}) are satisfied. If
\[ \left( L_g(T-a) + \left( \frac{(\theta - 2)(T-a)}{\mathcal{N}(\theta - 1)} + \frac{(1 - \theta)(T-a)^\theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \right) \frac{L_f}{1-K_f} \right) < 1, \] (3.24)
then there exists at least one solution of the problem (1.2).

**Proof** Choose \((T^*_a \zeta)(t) = (T^*_a \zeta)(t) + (T^*_a \bar{\zeta})(t)\) where
\[ (T^*_a \zeta)(t) = \frac{(t-a)}{(T-a)} \int_a^T g(s, \zeta(s)) ds + \frac{\theta - 2}{\mathcal{N}(\theta - 1)} \frac{(t-a)}{(T-a)} \int_a^T f(s, \zeta(s), ABCD_{a}^{\theta} \zeta(s)) ds \]
\[ + \frac{1 - \theta}{\mathcal{N}(\theta - 1)\Gamma(\theta)} \frac{(t-a)}{(T-a)} \int_a^T (T - s)^{\theta-1} f(s, \zeta(s), ABCD_{a}^{\theta} \zeta(s)) ds \] (3.25)
and

\[
(T_2 \zeta)(t) = \frac{\theta - 1}{\mathcal{N}(\theta - 1)\Gamma(\theta)} \int_a^t (t-s)^{\theta-1} f(s, \zeta(s), ABCD_a^\theta, \zeta(s)) \, ds \\
+ \frac{2 - \theta}{\mathcal{N}(\theta - 1)} \int_a^t f(s, \zeta(s), ABCD_a^\theta, \zeta(s)) \, ds.
\] (3.26)

Let \( B_\xi \) be defined by (3.14) with the radius

\[
\xi \geq \frac{\mu_\xi(T-a)}{1 - L_\xi(T-a)},
\] (3.27)

where \( \mu_\xi \) is as in Theorem 3.5. The proof will be complete in several steps:

**Claim 1.** \( T_1 \zeta + T_2 \nu \in B_\xi \), for all \( \zeta, \nu \in B_\xi \).

By (3.25),

\[
|T_1 \zeta(t)| \leq \frac{(t-a)}{(T-a)} \int_a^T |g(s, \zeta(s))| \, ds
\]

\[
+ \frac{\theta - 2}{\mathcal{N}(\theta - 1)(T-a)} \int_a^T |f(s, \zeta(s), ABCD_a^\theta, \zeta(s))| \, ds
\]

\[
+ \frac{1 - \theta}{\mathcal{N}(\theta - 1)\Gamma(\theta)(T-a)} \int_a^T (T-s)^{\theta-1} |f(s, \zeta(s), ABCD_a^\theta, \zeta(s))| \, ds.
\]

From (A1), (A2), and for \( \zeta \in B_\xi \), we get \( |f(t, \zeta(t), ABCD_a^\theta, \zeta(t))| \leq \frac{L_f\xi + \mu_f}{1 - K_f} \) (where \( \mu_f \) is as in Theorem 3.5) and \( |g(s, \zeta(s))| \leq (L_\xi \xi + \mu_\xi) \). Hence,

\[
|T_1 \zeta(t)| \leq \frac{(t-a)(L_\xi \xi + \mu_\xi) + (\theta - 2)(t-a)(L_f \xi + \mu_f)}{\mathcal{N}(\theta - 1)} \frac{1}{1 - K_f}
\]

\[
+ \frac{(1 - \theta)(t-a)}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \frac{(T-a)^{\theta-1}(L_f \xi + \mu_f)}{1 - K_f}
\]

\[
\leq \left( L_\xi(t-a) + \left( \frac{(\theta - 2)}{\mathcal{N}(\theta - 1)} + \frac{(1 - \theta)}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \right) \frac{L_f(t-a)}{1 - K_f} \right) \xi
\]

\[
+ \mu_\xi(t-a) + \left( \frac{(\theta - 2)}{\mathcal{N}(\theta - 1)} + \frac{(1 - \theta)}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \right) \frac{\mu_f(t-a)}{1 - K_f}.
\] (3.28)

Also, by (3.26),

\[
|T_2 \nu(t)| \leq \frac{\theta - 1}{\mathcal{N}(\theta - 1)\Gamma(\theta)} \int_a^t (t-s)^{\theta-1} |f(t, \nu(s), ABCD_a^\theta, \nu(s))| \, ds
\]

\[
+ \frac{2 - \theta}{\mathcal{N}(\theta - 1)} \int_a^t |f(t, \nu(s), ABCD_a^\theta, \nu(s))| \, ds.
\]

For \( \nu \in B_\xi \),

\[
|T_2 \nu(t)| \leq \frac{\theta - 1}{\mathcal{N}(\theta - 1)\Gamma(\theta)} \frac{L_f \xi + \mu_f}{1 - K_f} \int_a^t (t-s)^{\theta-1} \, ds
\]

\[
+ \frac{2 - \theta}{\mathcal{N}(\theta - 1)} \frac{L_f \xi + \mu_f}{1 - K_f} \int_a^t \, ds
\]
\[
\phi(T) = \frac{(\theta - 1)(T-a)^\theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} L_f \xi + \mu_f + \frac{(2-\theta)(T-a)}{\mathcal{N}(\theta - 1)} L_f \xi + \frac{\mu_f}{1-K_f} \\
= \left( \frac{(\theta - 1)(T-a)^\theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} + \frac{(2-\theta)(T-a)}{\mathcal{N}(\theta - 1)} \right) \frac{\mu_f}{1-K_f} \\
+ \left( \frac{(\theta - 1)(T-a)^\theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} + \frac{(2-\theta)(T-a)}{\mathcal{N}(\theta - 1)} \right) \frac{L_f}{1-K_f} \xi.
\]

(3.29)

From (3.28), (3.29), and for \( \tau \in \mathfrak{Z} \), we get
\[
\| T_1^* \xi + T_2^* \upsilon \| \leq \| T_1 \xi \| + \| T_2^* \upsilon \| \\
= \max_{\tau \in \mathfrak{Z}} \| (T_1^* \xi)(\tau) \| + \max_{\tau \in \mathfrak{Z}} \| (T_2^* \upsilon)(\tau) \| \\
\leq \left( L_\phi(T-a) + \frac{(\theta - 2)(T-a)}{\mathcal{N}(\theta - 1)} + \frac{(1-\theta)(T-a)}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \right) \frac{L_f}{1-K_f} \xi \\
+ \mu_\phi(T-a) + \left( \frac{(\theta - 2)(T-a)}{\mathcal{N}(\theta - 1)} + \frac{(1-\theta)(T-a)}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} \right) \frac{\mu_f}{1-K_f} \\
+ \left( \frac{(\theta - 1)(T-a)^\theta}{\mathcal{N}(\theta - 1)\Gamma(\theta + 1)} + \frac{(2-\theta)(T-a)}{\mathcal{N}(\theta - 1)} \right) \frac{L_f}{1-K_f} \xi \\
< L_\phi(T-a) \xi + \mu_\phi(T-a).
\]

Here we used fact that \( (T-a) < (T-a)^\theta \) for all \( 1 < \theta \leq 2 \). By (3.24), we conclude that \( L_\phi(T-a) < 1 \), it follows from (3.27) that
\[
\| T_1^* \xi + T_2^* \upsilon \| \leq \xi.
\]

Thus, \( T_1^* \xi + T_2^* \upsilon \in B_\xi \), for all \( \xi, \upsilon \in B_\xi \).

**Claim 2.** \( T_1^* \) is a contraction.

From (A1) and (A2). Then for each \( \xi, \xi^* \in B_\xi \) and \( \tau \in \mathfrak{Z} \),
\[
| (T_1^* \xi)(\tau) - (T_1^* \xi^*)(\tau) | \\
\leq \frac{(t-a)}{(T-a)} \int_t^T \left| g(s, \xi(s)) - g(s, \xi^*(s)) \right| ds \\
+ \frac{\theta - 2}{\mathcal{N}(\theta - 1) (T-a)} \int_t^T \left| f(s, \xi(s), ABC D_a^\phi \xi(s)) - f(s, \xi^*(s), ABC D_a^\phi \xi^*(s)) \right| ds \\
+ \frac{(t-a)}{\mathcal{N}(\theta - 1) \Gamma(\theta) (T-a)} \\
\times \int_t^T (T-s)^{\theta-1} \left| f(s, \xi(s), ABC D_a^\phi \xi(s)) - f(s, \xi^*(s), ABC D_a^\phi \xi^*(s)) \right| ds \\
\leq \frac{L_\phi(t-a)}{(T-a)} \int_t^T \left| \xi(s) - \xi^*(s) \right| ds \\
+ \frac{\theta - 2}{\mathcal{N}(\theta - 1) (T-a)} \frac{L_f}{1-K_f} \int_t^T \left| \xi(s) - \xi^*(s) \right| ds
\]
Condition (3.24) shows that \( T_1^* \) is a contraction.

Claim 3. \( T_2^* \) is compact and continuous.

The map \( T_2^* : B_\ell \to B_\ell \) is continuous due to the continuity of \( f \). Next, let \( \zeta \in B_\ell \) and \( \tau \in \mathcal{J} \). Then by using (3.29), we have

\[
\|T_2^*\| \leq \left( \frac{\mu_f + L_f \zeta}{1 - K_f} \right) \left( \frac{(\theta - 1)(T - a)^\theta}{\Gamma(\theta + 1)} + \frac{(2 - \theta)(T - a)^\theta}{\Gamma(\theta - 1)} \right).
\]

This leads to the conclusion that \( T_2^* \) is uniformly bounded on \( B_\ell \).

Now, we show that \( T_2^*(B_\ell) \) is equicontinuous. Let \( \zeta \in B_\ell \) and \( a \leq \tau_1 < \tau_2 \leq T \). Then

\[
\left| (T_2^*)^*(\tau_2) - (T_2^*)^*(\tau_1) \right| \leq \left| \frac{\vartheta - 1}{\Gamma(\theta - 1)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\theta - 1} f(s, \zeta(s), ABC_{\alpha}, \zeta(s)) \, ds \right|
\]

\[
\leq \left| \frac{\vartheta - 1}{\Gamma(\theta - 1)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\theta - 1} f(s, \zeta(s), ABC_{\alpha}, \zeta(s)) \, ds \right|
\]

Since \( |f(s, \zeta(s), ABC_{\alpha}, \zeta(s))| \leq \frac{L_f \xi + \mu_f}{1 - K_f} \), for \( \zeta \in B_\ell \), we have

\[
\left| (T_2^*)^*(\tau_2) - (T_2^*)^*(\tau_1) \right| \leq \left| \frac{\vartheta - 1}{\Gamma(\theta - 1)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\theta - 1} \left( \frac{L_f \xi + \mu_f}{1 - K_f} \right) \, ds \right|
\]

\[
+ \left| \frac{\vartheta - 1}{\Gamma(\theta - 1)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\theta - 1} - (\tau_1 - s)^{\theta - 1} \left( \frac{L_f \xi + \mu_f}{1 - K_f} \right) \, ds \right|
\]

\[
+ \left| \frac{\vartheta - 1}{\Gamma(\theta - 1)} \int_{\tau_1}^{\tau_2} (\mu_f + \mu_f) \, ds \right|
\]

\[
= \frac{\vartheta - 1}{\Gamma(\theta - 1)} \frac{L_f \xi + \mu_f}{1 - K_f} \left[ 2(\tau_2 - \tau_1)^\theta + (\tau_1 - a)^\theta - (\tau_2 - a)^\theta \right]
\]

\[
+ \left( \frac{\vartheta - 1}{\Gamma(\theta - 1)} \right) (\tau_2 - \tau_1)^\theta
\]

\[
\leq \left( \frac{2(\vartheta - 1)}{\Gamma(\theta - 1)} (\tau_2 - \tau_1)^\theta + \frac{2 - \vartheta}{\Gamma(\theta - 1)} (\tau_2 - \tau_1) \right) \frac{L_f \xi + \mu_f}{1 - K_f}.
\]
Observe that \(|(T^2_2\zeta)(t_2) - (T^2_2\zeta)(t_1)| \to 0\) as \(t_2 \to t_1\). In view of the preceding claims, together with Arzela–Ascoli theorem, we infer that \((T^2_2B\zeta)\) is relatively compact. Hence, Claim 3 holds. So, Theorem 2.2 shows that (1.2) has at least one solution. \(\square\)

4 Examples

Example 4.1 In this example, we justify the validity of Theorem 3.4. For \(\vartheta \in (0, 1)\), we consider the following ABC fractional problem:

\[
\begin{cases}
\mathcal{A}^\vartheta_{\vartheta} \zeta(t) = \frac{c^2}{\vartheta^2 + 1} \left( \frac{|c(t)|}{1 + |c(t)|} + \frac{|ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|}{1 + |ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|} \right), & t \in [0, 1], \\
\zeta(0) - \zeta'(0) = \int_0^r \frac{|a|}{10 + |a|} \, ds.
\end{cases}
\]  

(4.1)

Set \(f(t, \zeta(t), \overline{\zeta}(t)) = \frac{c^2}{\vartheta^2 + 1} \left( \frac{|c(t)|}{1 + |c(t)|} + \frac{|ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|}{1 + |ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|} \right)\) and \(g(t, \zeta(t)) = \frac{c(t)}{10 + |c(t)|}\), for \(t \in [0, 1]\), \(\zeta, \overline{\zeta} \in \mathbb{R}\). Clearly, \(f(0, \zeta(0), \overline{\zeta}(0)) = g(0, \zeta(0)) = 0\). Let \(t \in [0, 1]\) and \(\zeta, \overline{\zeta}, \nu, \overline{\nu} \in \mathbb{R}\). Then

\[
|f(t, \zeta, \overline{\zeta}) - f(t, \nu, \overline{\nu})| \leq \frac{c^2}{\vartheta^2 + 1} \left( \frac{|c(t)|}{1 + |c(t)|} + \frac{|ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|}{1 + |ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|} \right)
\]

\[
\leq \frac{1}{9} |\zeta - \overline{\zeta}| + \frac{1}{9} |\nu - \overline{\nu}|
\]

and

\[
|g(t, \zeta) - g(t, \nu)| = \left| \frac{c(t)}{10 + |c(t)|} - \frac{\nu}{10 + |\nu|} \right| \leq \frac{10 |\zeta - \nu|}{(10 + |\zeta|)(10 + |\nu|)} \leq \frac{1}{10} |\zeta - \nu|.
\]

Therefore, the hypotheses (A1) and (A2) hold with \(L_f = K_f = \frac{1}{9}\) and \(L_g = \frac{1}{10}\). We shall examine that the condition (A3) holds too, with \(\vartheta = \frac{1}{2}\) and \(\mathfrak{N}(\vartheta) = 1\). Indeed,

\[
\left[ L_g (T - a) + \left( 1 + \frac{T - a}{\mathfrak{N}(\vartheta) \Gamma(\vartheta)} \right) \frac{L_f}{1 - K_f} \right] \approx 0.23 < 1.
\]

Thus by Theorem 3.4, the problem (4.1) has a unique solution on [0, 1].

Example 4.2 The following example validates Theorem 3.7. For \(\vartheta \in (1, 2]\), we consider the following ABC fractional problem:

\[
\begin{cases}
\mathcal{A}^\vartheta_{\vartheta} \zeta(t) = \frac{c^2}{\vartheta^2 + 1} \left( \frac{|c(t)|}{1 + |c(t)|} + \frac{|ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|}{1 + |ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|} \right), & t \in [1, 2], \\
\zeta(1) - \zeta(2) = \int_1^r \frac{|a|}{100 + |a|} \, ds.
\end{cases}
\]  

(4.2)

Set \(f(t, \zeta(t), \overline{\zeta}(t)) = \frac{c^2}{\vartheta^2 + 1} \left( \frac{|c(t)|}{1 + |c(t)|} + \frac{|ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|}{1 + |ABC \mathcal{D}^\vartheta_{\vartheta} \zeta(t)|} \right)\) and \(g(t, \zeta(t)) = \frac{c(t)}{100 + |c(t)|}\), for \(t \in [1, 2]\), \(\zeta, \overline{\zeta} \in \mathbb{R}\). Clearly, \(f(1, \zeta(1), \overline{\zeta}(1)) = 0\). Let \(t \in [1, 2]\) and \(\zeta, \overline{\zeta}, \nu, \overline{\nu} \in \mathbb{R}\). Then

\[
|f(t, \zeta, \overline{\zeta}) - f(t, \nu, \overline{\nu})| \leq \frac{1}{10} |\zeta - \overline{\zeta}| + \frac{1}{10} |\nu - \overline{\nu}|
\]

and

\[
|g(t, \zeta) - g(t, \nu)| \leq \frac{1}{100} |\zeta - \nu|.
\]
Therefore, the hypotheses \((A_1)\) and \((A_2)\) hold with \(L_f = K_f = \frac{1}{9}\) and \(L_g = \frac{1}{100}\). Also, for \(1 < \theta \leq 2\), \(a = 1\), \(T = 2\), and \(\mathcal{N}(\theta) = 1\), the condition \((A_3)\) holds, that is, \(L_a(T - a) = \frac{1}{100} < 1\). Thus by Theorem 3.7, the problem (4.2) has a unique solution on \([1, 2]\).

Remark 4.1 In Theorems 3.3, 3.4, and 3.5, if \(f'(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) \neq 0\), then the formula of solution of the problem (1.1) becomes

\[
\varsigma(t) = \int_a^T g(s, \varsigma(s)) \, ds + \frac{1 - \vartheta}{\mathcal{N}(\vartheta)} \left[ f'(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) + f(t, \varsigma(t), ABCD^\vartheta_a, \varsigma(t)) \right] \\
+ \vartheta \frac{1}{\mathcal{N}(\vartheta)} \int_a^t \frac{(t - s)^{\vartheta - 1}}{\Gamma(\vartheta + 1)} \left[ f(s, \varsigma(s), ABCD^\vartheta_a, \varsigma(s)) \right] \, ds,
\]

for \(t \in \mathcal{A}\). Accordingly, the analysis of the results remains valid with the addition of the Lipschitz-type condition on \(f'\), that is,

\((A_4)\) \(f' : \mathcal{A} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous and there exist \(L^* > 0\) and \(0 < K^* < 1\) such that

\[
|f'(t, u, \overline{u}) - f'(t, v, \overline{v})| \leq L^*|u - v| + K^*|\overline{u} - \overline{v}|, \quad t \in \mathcal{A} \quad \text{and} \quad u, v, \overline{u}, \overline{v} \in \mathbb{R}.
\]

5 Conclusions

The theory of fractional operators containing nonsingular kernels is novel and of considerable recent interest, thus there is a need to study the qualitative properties of differential equations involving such operators. In this work, we have considered two classes of boundary value problems for fractional implicit differential equations with nonlinear integral conditions in the framework of Atangana–Baleanu–Caputo derivatives. Krasnosel’skii and Banach fixed point theorems were applied to establish the existence and uniqueness theorems for the problems (1.1) and (1.2). For problem (1.1), we realized that one must always have the necessary conditions \(f(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) = 0\) and \(f'(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) = 0\) to guarantee a unique solution, whereas for problem (1.2) we needed \(f'(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) = 0\) to confirm the initial data for the solution. To avoid the condition \(f'(a, \varsigma(a), ABCD^\vartheta_a, \varsigma(a)) = 0\) in Theorem 3.3, one can use condition \((A_4)\) mentioned in Remark 4.1 to obtain the same results. The proposed problems are more general, also the results obtained generalize the recent studies and offer an extension of the development of FDEs that involve this new operator. Moreover, the analysis of the results was limited to the minimum assumptions. The problems scrutinized include some special cases, in other words, they could be reduced to the corresponding problems that contain Caputo–Fabrizio operator. We are certain that the communicated results here are rather interesting, and will have a positive effect on the development of more applications in engineering and sciences.

Acknowledgements

The authors are very thankful to the reviewers for their useful suggestions. Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through the research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM), group number RG-DES-2017-01-17.

Funding

Not applicable.

Availability of data and materials

Not applicable.
Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to this article. All authors read and approved the final manuscript.

Author details
1Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen. 2Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. 3Department of Medical Research, China Medical University, Taichung 40402, Taiwan. 4Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. 5Department of Basic Engineering Sciences, College of Engineering, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam, 34151, Saudi Arabia. 6Department of Mathematics, University of Malakand, Chakdara Dir (Lower), Khyber Pakhtunkhwa, Pakistan.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
1. Diethelm, K.: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics, vol. 2004. Springer, Berlin (2010)
2. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
3. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
4. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus Models and Numerical Methods. World Scientific, New York (2012)
5. Gambo, Y.Y., Jarad, F., Baleanu, D., Abdeljawad, T.: On Caputo modification of the Hadamard fractional derivatives. Adv. Differ. Equ. 2014, 10 (2014)
6. Jarad, F., Abdeljawad, T.: Generalized fractional derivatives and Laplace transforms. Discrete Contin. Dyn. Syst., Ser. S 13(3), 709–722 (2020). https://doi.org/10.3934/dcdss.2020039
7. Katugampola, U.: A new approach to a generalized fractional integral. Appl. Math. Comput. 218, 860–865 (2011)
8. Sousa, J.V.C., Oliveira, E.C.: On the $\psi$-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 60, 72–91 (2018)
9. Almeida, R.: A Caputo fractional derivative of a function with respect to another function. Commun. Nonlinear Sci. Numer. Simul. 44, 460–481 (2017)
10. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 73–85 (2015)
11. Losada, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. Prog. Fract. Differ. Appl. 1(2), 87–92 (2015)
12. Abdeljawad, T., Baleanu, D.: Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. J. Nonlinear Sci. Appl. 9, 1098–1107 (2016)
13. Atangana, A., Baleanu, D.: New fractional derivative with non-local and non-singular kernel. Therm. Sci. 20(2), 757–763 (2016)
14. Abdeljawad, T., Baleanu, D.: Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels. Adv. Differ. Equ. 2016, 232 (2016). https://doi.org/10.1186/s13662-016-0490-5
15. Abdeljawad, T., Baleanu, D.: On fractional derivatives with exponential kernel and their discrete versions. Rep. Math. Phys. 80(1), 11–27 (2017)
16. Singh, J., Kumar, D., Hammouch, Z., Atangana, A.: A fractional epidemiological model for virus infections pertaining to a new fractional derivative. Appl. Math. Comput. 316, 504–515 (2018)
17. Uçar, S., Uçar, E., Özdemir, N., Hammouch, Z.: Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative. Chaos Solitons Fractals 118, 300–306 (2019)
18. Atangana, A.: Modelling the spread of COVID-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination? Chaos Solitons Fractals 136, 109860 (2020)
19. Abro, K.A., Atangana, A.: A comparative analysis of electromechanical model of piezoelectric actuator through Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. Math. Methods Appl. Sci. 43(17), 9681–9691 (2020)
20. Ghorbani, B., Atangana, A.: A new application of fractional Atangana–Baleanu derivatives: designing ABC-fractional masks in image processing. Physica A 542, 123516 (2020)
21. Khan, M.A., Atangana, A.: Modeling the dynamics of novel coronavirus (2019-nCoV) with fractional derivative. Alex. Eng. J. 59(4), 2379–2389 (2020)
22. Abdeljawad, T., Agarwal, RP, Karapinar, E., Kumar, PS.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry 11(5), 686 (2019)
23. Abdeljawad, T., Karapinar, E., Panda, S.K., Mlaiki, N.: Solutions of boundary value problems on extended-Branciari b-distance. J. Inequal. Appl. 2020(1), 130 (2020)
24. Panda, S.K., Abdeljawad, T., Ravichandran, C.: A complex valued approach to the solutions of Riemann–Liouville integral, Atangana–Baleanu integral operator and non-linear telegraph equation via fixed point method. Chaos Solitons Fractals 130, 109439 (2020)
25. Panda, S.K., Abdeljawad, T., Ravichandran, C.: Novel fixed approach to Atangana–Baleanu fractional and $Lp$-Fredholm integral equations. Alex. Eng. J. 59(4), 1959–1970 (2020)
26. Panda, S.K., Abdeljawad, T., Swamy, K.K.: New numerical scheme for solving integral equations via fixed point method using distinct (w-F)-contractions. Alex. Eng. J. 59(4), 2015–2026 (2020)

27. Panda, S.K., Karapinar, E., Atangana, A.: A numerical schemes and comparisons for fixed point results with applications to the solutions of Volterra integral equations in dislocated extended b-metric space. Alex. Eng. J. 59(2), 815–827 (2020)

28. Panda, S.K., Tarasдавд, A., Agarwal, R.P.: A new approach to the solution of non-linear integral equations via various FBe-contractions. Symmetry 11(2), 206 (2019)

29. Ravichandran, C., Logeswar, K., Panda, S.K., Nisar, K.S.: On new approach of fractional derivative by Mittag-Leffler kernel to neutral integro-differential systems with impulsive conditions. Chaos Solitons Fractals 139, 110012 (2020)

30. Abdeljawad, T., Baleanu, D.: On fractional derivatives with generalized Mittag-Leffler kernels. Adv. Differ. Equ. 2018(1), 468 (2018)

31. Abdo, M.S., Panchal, S.K.: Fractional integro-differential equations involving ψ-Hilfer fractional derivative. Adv. Appl. Math. Mech. 11, 338–359 (2019)

32. Abdo, M.S., Saeed, A.M., Panchal, S.K.: Fractional boundary value problem with ψ-Caputo fractional derivative. Proc. Math. Sci. 129(5), 65 (2019). https://doi.org/10.1007/s12044-019-0514-8

33. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Sequential fractional differential equations and inclusions with nonlocal and nonlocal integro-multipooint boundary conditions. J. King Saud Univ., Sci. 31, 184–193 (2019)

34. Benchohra, M., Benmezoula, M., Hammouch, Z.: A new approach of modeling of a fractional order model of a fractional order epidemic model. Chaos Solitons Fractals 552–559 (2016)

35. Mei, Z.: A fractional differential equation involving ψ-Hilfer fractional derivative with nonlocal and singular boundary conditions. Adv. Differ. Equ. 2017(1), 130 (2017)

36. Benchohra, M., Bouriah, S., Nieto, J.J.: Terminal value problem for differential equations with ψ-Hilfer–Katugampola fractional derivative. Symmetry 11(5), 672 (2019)

37. Mei, Z.D., Peng, J.G., Gao, J.H.: Existence and uniqueness of solutions for nonlinear general fractional differential equations in Banach spaces. Indag. Math. 26, 669–678 (2015)

38. Srivastava, H.M.: Remarks on some families of fractional-order differential equations. Integral Transforms Spec. Funct. 28, 560–564 (2017)

39. Jarad, F., Abdeljawad, T., Hammouch, Z.: On a class of ordinary differential equations in the frame of Atangana–Baleanu fractional derivative. Chaos Solitons Fractals 117, 16–20 (2018)

40. Abdeljawad, T.: A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel. J. Inequal. Appl. 2017(1), 130 (2017)

41. Abdo, M.S., Panchal, S.K., Shah, K., Abdeljawad, T.: Existence theory and numerical analysis of three species prey–predator model under Mittag-Leffler power law. Adv. Differ. Equ. 2020(1), 249 (2020). https://doi.org/10.1186/s13662-020-02709-7

42. Abdo, M.S., Shah, K., Wahash, H.A., Panchal, S.K.: On a comprehensive model of the novel coronavirus (COVID-19) under Mittag-Leffler derivative. Chaos Solitons Fractals 135, 109867 (2020). https://doi.org/10.1016/j.chaos.2020.109867

43. Abdo, M.S., Abdeljawad, T., Shah, K., Jarad, F.: Study of impulsive problems under Mittag-Leffler power law. Hellyon 6(10), e05109 (2020)

44. Redhwan, S.S., Abdo, M.S., Shah, K., Abdeljawad, T., Dawood, S., Abdo, H.A., Shaikh, S.L.: Mathematical modeling for the outbreak of the coronavirus (COVID-19) under fractional nonlocal operator. Results Phys. 19, 103610 (2020). https://doi.org/10.1016/j.rinp.2020.103610

45. Kucche, K.D., Sutar, S.T.: Analysis of nonlinear fractional differential equations involving Atangana–Baleanu–Caputo derivative. Preprint. arXiv:2007.09132 (2020)

46. Algaba, N.J.: Comparing the Atangana–Baleanu and Caputo–Fabricio derivative with fractional order: Allen Cahn model. Chaos Solitons Fractals 89, 552–559 (2016)

47. Atangana, A., Koca, I.: Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. Chaos Solitons Fractals 89, 447–454 (2016)

48. Ijazmir, A., Baleanu, D.: A new fractional analysis on the interaction of HIV with CD4+ T-cells. Chaos Solitons Fractals 113, 221–239 (2018)

49. Yavuz, M., Özdemir, N., Bağkun, H.M.: Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel. Eur. Phys. J. B 133(6), 215 (2018)

50. Danane, J., Allali, K., Hammouch, Z.: Mathematical analysis of a fractional differential model of HBV infection with antibody immune response. Chaos Solitons Fractals 136, 109787 (2020)

51. Ullah, S., Khan, M.A., Farooq, M., Hammouch, Z., Baleanu, D.: A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative. Discrete Contin. Dyn. Syst., Ser. B 25(3), 975 (2020)

52. Atangana, A., Goufo, E.F.D.: Some misinterpretations and lack of understanding in differential operators with no singular kernels. Open Phys. 18(1), 594–612 (2020)

53. Goufo, E.F.D.: Evolution equations with a parameter and application to transport-convection differential equations. Turk. J. Math. 41(3), 636–654 (2017)

54. Goufo, E.F.D., Khan, Y., Mugisha, S.: Control parameter & solutions to generalized evolution equations of stationarity, relaxation and diffusion. Results Phys. 9, 1502–1507 (2018)

55. Goufo, E.F.D., Mbehou, M., Pene, M.M.K.: A peculiar application of Atangana–Baleanu fractional derivative in neuroscience: chaotic burst dynamics. Chaos Solitons Fractals 115, 170–176 (2018)

56. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)