ORBITAL STABILITY OF PERIODIC PEAKONS FOR THE
GENERALIZED MODIFIED CAMASSA-HOLM EQUATION

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ABSTRACT. This paper is devoted to studying the dynamical stability of peri-
odic peaked solitary waves for the generalized modified Camassa-Holm equa-
tion. The equation is a generalization of the modified Camassa-Holm equation
and it possesses the Hamiltonian structure shared by the modified Camassa-
Holm equation. The equation admits the periodic peakons. It is shown that
the periodic peakons are dynamically stable under small perturbations in the
energy space.

1. Introduction. In the last 20 years, one of the most famous integrable equations
is the Camassa-Holm (CH) equation [2]

\[ m_t + um_x + 2u_x m = 0, \quad m = u - uu_x. \tag{1.1} \]

The CH equation (1.1) admits both smooth and peaked travelling wave solutions
[2, 3, 21]. Physically, the CH equation (1.1) has attracted a great deal of interest as
an approximate fluid model for two-dimensional water waves propagation over a flat
bed [2, 3, 14, 18, 19, 29, 32, 33, 34]. Interpreted as a fluid model, the solution \( u(x, t) \)
represent the fluid particle velocity induced by the passing wave, or alternatively
as the surface elevation associated with the wave. Moreover, the CH equation (1.1)
constitutes a model for the propagation of nonlinear waves in cylindrical hyper-
estic rods, in which the solution \( u(x, t) \) represent the radial stretching of a rod
relative to the undisturbed state [17]. The CH equation (1.1) has attracted much
attention because of its interesting properties. A particularly striking feature of the
CH equation (1.1) relates to the existence of peaked solutions of the form

\[ u(x, t) = ce^{-|x-ct|} \quad \text{where} \quad \lim_{|x| \to \infty} u(x, t) = 0. \tag{1.2} \]

These peaked solutions (peakons) are weak solutions whose wave crests appear as
peaks [2, 3, 16, 43]. Moreover, the CH equation (1.1) also allows for the existence
of breaking wave solutions, which are realized as solutions which remain bounded
but whose gradient becomes unbounded in a finite time [2, 3, 7, 9, 48]. The CH
equation (1.1) become a highly interesting physical model because the presence of
both peaked and breaking wave solutions. To complement the utility of the system
in modeling a diversity of physical phenomena, the CH equation (1.1) exhibits a rich

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mathematical structure. It is a member of a bi-Hamiltonian hierarchy of equations [23] and it is integrable with a Lax pair representation [2]. A remarkable property of CH equation (1.1) is its formulation as a geodesic flow on the Bott-Birasoro group [13, 30, 44].

Over the last few years, various modifications and generalizations of the CH equation (1.1) have been introduced [4, 5, 27, 39, 45, 55]. So it is of greatest to find such integrable CH-type equations with cubic and higher-order nonlinearity. Indeed, two integrable CH-type equations with cubic nonlinearity which admit peakons have been extensively studied recently. One is the following modified Camassa-Holm (mCH) equation

\[ m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx} \quad (1.3) \]

and another one is the Novikov equation

\[ m_t + u^2m_x + 3uu_xm = 0, \quad m = u - u_{xx}. \quad (1.4) \]

The mCH (or FORQ) equation (1.3) was derived independently in [22, 24, 47, 49]. The mCH equation (1.3) has a bi-Hamiltonian structure [50] and is completely integrable [47]. Moreover, the mCH equation (1.3) exhibits new features, including wave breaking and blowup criteria [25] that do not appear in the CH equation (1.1) [7, 9, 10, 38]. As a similar character of CH equation (1.1), the mCH equation admits peakons of the form [25]

\[ u(x, t) = \sqrt{\frac{3c}{2}} e^{-|x-ct|}, \quad c > 0, \]

and the periodic peakons of the form [52]

\[ u(x, t) = \sqrt{\frac{3c}{1 + 2 \cosh^2 \left( \frac{1}{2} \right)}} \cosh \left( \frac{1}{2} - (x - ct) + [x - ct] \right), \quad c > 0, \]

where \([x]\) denotes the largest integer part of \(x \in \mathbb{R}\). It is worth pointing out that the feature of peakons that their profile is smooth except for a peak at its crest is analogous to the Stokes waves of greatest height, i.e. traveling waves of largest possible amplitude which are solutions to governing equations for water waves [8, 14, 53]. The single (periodic) peakon and train of peakons for the mCH equation (1.3) are shown to be orbitally stable in [41, 52], respectively. The Novikov equation (1.4) was introduced in [46] and their interesting properties such as integrability, well-posedness, blow-up phenomena, global weak solutions, peakons and stability peakons were extensively investigated in [28, 31, 41, 46, 54].

In this paper, we are mainly concerned with the stability of periodic peaked solitary waves of the generalized modified Camassa-Holm (gmCH) equation

\[ m_t + [(u^2 - u_x^2)^2m]_x = 0, \quad m = u - u_{xx}, \quad (1.5) \]

which was introduced as a general family of peakon equations by Recio and Anco [1] that possess the Hamiltonian structure shared by the mCH equation (1.3). The gmCH equation (1.5) has bi-Hamiltonian structures [1] and peakon solutions [1, 26] in the form of

\[ u(x, t) = \sqrt{\frac{15c}{8}} e^{-|x-ct|}, \quad c > 0. \]
and the periodic peakon solution in the form of
\[
u(x,t) = \sqrt{\frac{5c}{3 + 4\cosh^2\left(\frac{x}{2}\right) + 8\cosh^4\left(\frac{x}{2}\right)}} \cosh\left(\frac{1}{2} - (x - ct) + [x - ct]\right), \quad c > 0,
\]
which will be derived explicitly in Section 3 below. Here the notation \([x]\) denotes the largest integer part of the real number \(x \in \mathbb{R}\). It should be pointed out that one of the most relevant motivations look for peaked waves (solitary or periodic) is the fact that the governing equations for irrotational waver waves do admit peaked traveling waves (periodic, as well as solitary), namely the celebrated Stokes waves of greatest height—see the discussion in [6, 8, 11, 53].

Since a small perturbation of a peakon yields another one traveling with a different speed and phase shift, the appropriate notion of stability here is that of orbital stability (i.e., a (periodic) wave starting close to a (periodic) peakon remains close to some translate of it for all later time). This implies that the shape of the (periodic) wave remains approximately the same for all time. It is known above that one of the main features of the CH equation (1.1) and the mCH equation (1.3) is the existence of orbitally stable (periodic) peakons. So, it is natural to extend the results on the stability of (periodic) peakons for CH equation (1.1) and mCH equation (1.3) successfully to the higher-order case (1.5). In Section 4, we will prove the following stability result.

**Theorem 1.1.** The periodic peakons for the generalized modified Camassa-Holm (gmCH) equation (1.5) are orbitally stable in the energy space.

To prove the above main result, we recall an intriguing works for the stability of peakons for the CH equation (1.1) [15, 16, 20, 35, 36] and mCH equation (1.3) [40, 52]. The CH equation (1.1) and the mCH equation (1.3) has the following useful conservation laws:

\[
E_1[u] = \int_S (u^2 + u_x^2) \, dx \quad \text{and} \quad E_2[u] = \int_S (u^3 + uu_x^2) \, dx \quad (\text{CH})
\]
\[
F_1[u] = E_1[u] \quad \text{and} \quad F_2[u] = \int_S \left( u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 \right) \, dx. \quad (\text{mCH})
\]

The following conservation laws of the gmCH equation (1.5) [1]
\[
H_1[u] = E_1[u] \quad \text{and} \quad H_2[u] = \int_S \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5}u_x^6 \right) \, dx, \quad (\text{gmCH})
\]
will play a important role in proving the orbital stability of periodic peakons. Due to the same conservation law \(H_1[u]\) as \(E_1[u]\) and \(F_1[u]\) for CH and mCH equations, we may expect the orbital stability of periodic peakons for the gmCH equation (1.5) in the sense of the energy space \(H^1\)-norm. Note that the conservation law \(H_2[u]\) of the gmCH equation (1.5) is much more complicated than \(E_2[u]\) and \(F_2[u]\) of the CH and mCH equations. Therefore, the analysis of stability issue of the gmCH equation (1.5) is more subtle than that of the CH and mCH equation in the periodic case. Motivated by the works in [35, 52], we shall prove the orbital stability of periodic peakons for the gmCH equation (1.5). The main difficult part arise in the proof of orbital stability result. First difficult part is to construct two auxiliary functionals \(g(x)\) and \(h(x)\), which are connected to the two conservation laws \(H_1[u]\) and \(H_2[u]\). Comparing with the CH and mCH equations we define the
same functional \( g(x) = u_x \pm \sqrt{u^2 - m^2} \), \( m = \min_{x \in \mathbb{S}} u(x) \) but different functional \( h(x) \) as following:

\[
\begin{align*}
\text{(CH)}: & \\
h(x) &= u \\
h(x) &= u^2 \pm \frac{2}{3} u_x \sqrt{u^2 - m^2} - \frac{1}{3} u_x^2 - m^2 \\
h(x) &= u^4 \pm \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{4} u_x^4 \\
& \quad + \left( u^2 \pm \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2 \right) m^2, \quad m = \min_{x \in \mathbb{S}} u(x).
\end{align*}
\]

On the other hand, the functionals \( g(x) \) and \( h(x) \) are required to vanish at the periodic peakons \( \phi_c \). The second difficulty is to derivation of the required polynomial inequality, which is substantially different from the cases for the CH and mCH equation, i.e.,

\[
\begin{align*}
\text{(CH)}: & \\
h(x) &\leq M \\
h(x) &\leq \frac{4}{3} (M^2 - m^2) \\
h(x) &\leq \frac{8}{3} M^4 + \frac{4}{5} M^2 m^2 - \frac{2}{5} m^4.
\end{align*}
\]

Here \( m = \min_{x \in \mathbb{S}} u(x) \) and \( M = \max_{x \in \mathbb{S}} u(x) \). Because of the higher-order nonlinearity and higher-order conservation laws, these two difficulties occur. In order to overcome the above difficulties we carefully determine the coefficients of the function \( h(x) \) and estimate \( h(x) \) by using delicate analytic calculations. Moreover, we use the sign-invariant property of \( m \) which shows that the solution \( m \) is positive and \( u \pm u_x \geq 0 \) when the initial value \( m_0(x) \geq 0 \). This enable us to prove the orbital stability of the gmCH equation (1.5).

The outline of the paper is as follows. Section 2 is a short review on the well-posedness of the generalized modified Camassa-Holm equation. The existence of periodic peakons is demonstrated in Section 3. In Section 4, it is shown that the periodic peakons of the generalized modified Camassa-Holm equation are dynamically stable under small perturbations in the energy space \( H^1 \).

2. Preliminaries. We study the following Cauchy problem for the generalized modified Camassa-Holm equation (gmCH) on the unit circle:

\[
\begin{align*}
\left\{ \begin{array}{ll}
\dot{m}_t + [\left( u^2 - u_x^2 \right) m]_x &= 0, & t > 0, \ x \in \mathbb{S}, \\
u(0, x) &= u_0(x), & m = u - u_{xx}, \ x \in \mathbb{S}, \\
u(t, x + 1) &= \nu(t, x), & t \geq 0, \ x \in \mathbb{S}.
\end{array} \right.
\end{align*}
\]

(2.1)

First, we will give the notions of strong (or classical) and weak solutions for the Cauchy problem (2.1).

**Definition 2.1.** If \( u \in C([0, T), H^s(\mathbb{S}) \cap C^1([0, T), H^{s-1}(\mathbb{S})) \) with \( s > 5/2 \) and some \( T > 0 \) satisfies the Cauchy problem (2.1), then \( u \) is called a strong solution on \([0, T)\). If \( u \) is a strong solution on \([0, T)\) for every \( T > 0 \), then it is called a global strong solution.

The following local well-posedness result and properties for strong solutions on unit circle (up to a slight modification) were established in [26] and [56].
Proposition 1. [56] Let \( u_0 \in H^s(S) \) with \( s > 5/2 \). Then there exists a time \( T > 0 \) such that the Cauchy problem (2.1) has a unique strong solution \( u \in C([0,T),H^s(S)) \cap C^1([0,T),H^{s-1}(S)) \) and the map \( u_0 \mapsto u \) is continuous from a neighborhood of \( u_0 \) in \( H^s(S) \) into \( C([0,T),H^s(S)) \cap C^1([0,T),H^{s-1}(S)) \).

Proposition 2. [1, 26] The Hamiltonian functionals are conserved for the strong solution \( u \) in Proposition 1, that is for all \( t \in [0,T) \),

\[
\frac{d}{dt}H_1[u] = \frac{d}{dt} \int_S (u^2 + u_x^2) \, dx = 0,
\]

\[
\frac{d}{dt}H_2[u] = \frac{d}{dt} \int_S \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5}u_x^6 \right) \, dx = 0.
\]

Furthermore, if \( m_0 = (1-\partial_x^2)u_0 \) does not change sign, then \( m(x,t) \) will not change sign for any \( t \in [0,T) \). It follows that if \( m_0 \geq 0 \), then the corresponding solution \( u(x,t) \) is positive for \( (x,t) \in S \times [0,T) \).

Using \( m = u - u_{xx} \), equation (2.1) can be written as the following fully nonlinear partial differential equation:

\[
u_t + \left( u^4 - \frac{2}{3}u^2u_x^2 + \frac{1}{5}u_x^4 \right) u_x + (1 - \partial_x^2)^{-1}\partial_x \left( \frac{4}{5}u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4 \right) \\
+ (1 - \partial_x^2)^{-1} \left( \frac{2}{3}u^2u_x^3 + \frac{1}{5}u_x^5 \right) = 0. \tag{2.2}
\]

Recall that

\[
u = (1 - \partial_x^2)^{-1}m = G \ast m,
\]

where \( G(x) = \frac{\cosh(\frac{x}{2} - z)}{2\sinh^2(\frac{x}{2})} \) and \( \ast \) denotes the convolution product on \( S \), defined by

\[
(f \ast g)(x) = \int_S f(z)g(x - z) \, dz.
\]

The above formulation allows us to define a weak solution as follows.

Definition 2.2. Given \( u_0 \in W^{1,5}(S) \), the function \( u \in L^\infty([0,T),W^{1,5}(S)) \) is said to a weak solution to the Cauchy problem (2.1) if it holds the following identity:

\[
\int_0^T \int_S \left[ u_0 \partial_t \psi + \frac{1}{5}u^5 \partial_x \psi + \left( \frac{2}{3}u^3u_x^2 - \frac{1}{5}u_x^5 \right) \psi + G(x) \ast \left( \frac{4}{5}u^5 + 2u^3u_x^2 - \frac{1}{3}uu_x^4 \right) \partial_x \psi - G(x) \ast \left( \frac{2}{3}u^2u_x^3 - \frac{1}{5}u_x^5 \right) \psi \right] \, dx \, dt \\
+ \int_S u_0(x) \psi(0,x) \, dx = 0. \tag{2.3}
\]

for any smooth test function \( \psi(x,t) \in C^\infty_c(S \times [0,T)) \). If \( u \) is a weak solution on \([0,T)\) for every \( T > 0 \), then it is called a global weak solution.

Remark 1. From the Sobolev embedding \( W^{1,5}_{loc}(S) \hookrightarrow C^\alpha(S) \) with \( 0 \leq \alpha \leq \frac{3}{5} \), Definition 2.2 does not allow us to have discontinuous shock waves as weak solutions.
3. **Periodic peaked solutions.** In this section, we show that the generalized modified Camassa-Holm (gmCH) equation admits periodic peaked solutions. Recall that the periodic peakon of the Camassa-Holm (CH) equation and modified Camassa-Holm (mCH) equation takes the form

\[ u(x,t) = a \cosh \left( \frac{1}{2} - (x - ct) + [x - ct] \right), \quad (3.1) \]

where the amplitude \( a \) is given by \( \frac{c}{\cosh\left(\frac{1}{2}\right)} \), \( \frac{\sqrt{3}c}{\cosh\left(\frac{1}{2}\right)\sqrt{1+2\cosh^2\left(\frac{1}{2}\right)}} \), respectively. It is worth noting that the periodic peakon of the \( \mu \)-Camassa-Holm equation [37] and the modified \( \mu \)-Camassa-Holm equation [42, 51] admit periodic peakons of the form

\[ u(t,x) = a\varphi(x - ct), \quad c \in \mathbb{R}, \quad (3.2) \]

where \( \varphi(x) = \frac{1}{2} \left( x^2 + \frac{23}{24} \right), \quad x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \), and \( \varphi \) is extended periodically to the real line, the constant \( a \) takes value \( \frac{12c}{13}, \frac{2\sqrt{3}c}{5} \).

**Theorem 3.1.** The periodic peaked functions of the form

\[ u_c(x,t) = a \cosh \left( \frac{1}{2} - (x - ct) + [x - ct] \right), \quad (3.3) \]

with \( a^4 = \frac{15c}{8 \cosh^4\left(\frac{1}{2}\right) + 4 \cosh^2\left(\frac{1}{2}\right) + 3} \) is a global weak solution to the generalized modified Camassa-Holm equation (1.5) in the sense of Definition 2.2.

**Remark 2.** It is noted that all peakons in (3.3) move with positive wave speed, \( c > 0 \). Each positive wave speed has a peakon and anti-peakon of opposite amplitudes

\[ a = \pm \sqrt{\frac{15c}{8 \cosh^4\left(\frac{1}{2}\right) + 4 \cosh^2\left(\frac{1}{2}\right) + 3}}. \]

**Proof of Theorem 3.1.** We identify \( S \) with \([0,1)\) and regard \( u_c \) as periodic functions on \( S \) with period one. Notice that \( u_c \) is continuous on \( S \) with peak at \( x = 0 \). Also, \( u_c \) is smooth on \((0,1)\) and for all \( t \in \mathbb{R}^+ \),

\[ \partial_x u_c(x,t) = -a\sinh(\xi), \quad \xi = \frac{1}{2} - (x - ct) - [x - ct], \quad (3.4) \]

in the sense of periodic distribution \( \mathcal{P}' \). Obviously, (3.4) belongs to \( L^\infty(S) \). Denote \( u_{c,0}(x) = u_c(x,0), \quad x \in S \). Then

\[ \lim_{t \to 0^+} \| u_c(\cdot,t) - u_{c,0}(\cdot) \|_{W^{1,\infty}(S)} = 0. \quad (3.5) \]

As in (3.4), one has

\[ \partial_t u_c(x,t) = a\cosh(\xi) \in L^\infty(S), \quad t \geq 0. \quad (3.6) \]

A simple computation yields the following identity:

\[ u_c^4 \partial_x u_c = -a^5 \cosh^4(\xi) \sinh(\xi) \]
\[ = -a^5 \sinh(\xi) - 2a^5 \sinh^3(\xi) - a^5 \sinh^5(\xi) \quad (3.7) \]
and
\[
\frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 = -\frac{2}{3}a^5\cosh^2(\xi)\sinh^3(\xi) + \frac{1}{5}a^5\sinh^5(\xi) = -\frac{2}{3}a^5\sinh^3(\xi) - \frac{7}{15}a^5\sinh^5(\xi).
\]

Using (3.4)-(3.8) and integration by parts, we infer that for every test function \( \psi(x,t) \in C_c^\infty(S \times [0,\infty)) \),
\[
\int_0^\infty \int_S \left( u_c \partial_t \psi + \frac{1}{5}u_c^5 \partial_x \psi + \left( \frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) \psi \right) dx dt
+ \int_S u_c(x,0)\psi(x,0) dx
= -\int_0^\infty \int_S \psi \left( \partial_t u_c + u_c^4 \partial_x u_c - \frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) dx dt
= -\int_0^\infty \int_S \psi \left( (ac - a^5)\sinh(\xi) - \frac{4}{3}a^5\sinh^3(\xi) - \frac{8}{15}a^5\sinh^5(\xi) \right) dx dt.
\]

On the other hand, using integration by parts, we have
\[
\int_0^\infty \int_S \left[ G(x) \left( \frac{4}{5}u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3}u_c(\partial_x u_c)^4 \right) \partial_x \psi - G(x) \left( \frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) \psi \right] dx dt
= -\int_0^\infty \int_S \psi G(x) \left( \frac{4}{5}u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3}u_c(\partial_x u_c)^4 \right) dx dt
- \int_0^\infty \int_S \psi G_x(x) \left( \frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) dx dt.
\]

We compute from (3.4), (3.8) and the fact \( \sinh^2(x) = \cosh^2(x) - 1 \) that
\[
\frac{4}{5}u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3}u_c(\partial_x u_c)^4
= \frac{4}{5}a^5\cosh^5(\xi) + 2a^5\cosh^3(\xi)\sinh^2(\xi) - \frac{1}{3}a^5\cosh(\xi)\sinh^4(\xi)
= -\frac{1}{3}a^5\cosh(\xi) - \frac{4}{3}a^5\cosh^3(\xi) + \frac{15}{37}a^5\cosh^5(\xi),
\]
which together with (3.10) lead to
\[
\int_0^\infty \int_S \left[ G(x) \left( \frac{4}{5}u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3}u_c(\partial_x u_c)^4 \right) \partial_x \psi - G(x) \left( \frac{2}{3}u_c^2(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) \psi \right] dx dt
= a^5 \int_0^\infty \int_S \psi G(x) \left( \frac{2}{3}\sinh^3(\xi) + \frac{7}{15}\sinh^5(\xi) \right) dx dt
- a^5 \int_0^\infty \int_S \psi G_x(x) \left( -\frac{1}{3}\cosh(\xi) - \frac{4}{3}\cosh^3(\xi) + \frac{15}{37}\cosh^5(\xi) \right) dx dt.
\]

From the definition of \( G(x) \) for the periodic case we see that
\[
G_x(x) = -\frac{\sinh \left( \frac{1}{2} - x + |x| \right)}{2\sinh \left( \frac{1}{2} \right)}, \quad x \in \mathbb{R}.
\]
Thus we obtain
\[ G(x) \ast \left( \frac{2}{3} \sinh^3(\xi) + \frac{7}{15} \sinh^5(\xi) \right)(x,t) \]
\[ = \frac{1}{2 \sinh \left( \frac{1}{2} \right)} \int_S \cosh \left( \frac{1}{2} - (x - z) + [x - z] \right) \cdot \left\{ \frac{2}{3} \sinh^3 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) \right. \]
\[ \left. + \frac{7}{15} \sinh^5 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) \right\} \, dz \] (3.12)
and
\[ G_x(x) \ast \left( -\frac{1}{3} \cosh(\xi) - \frac{4}{3} \cosh^3(\xi) + \frac{15}{37} \cosh^5(\xi) \right)(x,t) \]
\[ = -\frac{1}{2 \sinh \left( \frac{1}{2} \right)} \int_S \sinh \left( \frac{1}{2} - (x - z) + [x - z] \right) \cdot \left\{ -\frac{1}{3} \cosh \left( \frac{1}{2} - (z - ct) + [z - ct] \right) \right. \]
\[ \left. - \frac{4}{3} \cosh^3 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) + \frac{15}{37} \cosh^5 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) \right\} \, dz. \] (3.13)

When \( x > ct \), we split the right-hand side of (3.12) into the following three parts:
\[ G(x) \ast \left( \frac{2}{3} \sinh^3(\xi) + \frac{7}{15} \sinh^5(\xi) \right)(x,t) \]
\[ = \frac{1}{2 \sinh \left( \frac{1}{2} \right)} \left( \int_0^{ct} + \int_{ct}^{x} + \int_x^1 \right) \cosh \left( \frac{1}{2} - (x - z) + [x - z] \right) \]
\[ \cdot \left\{ \frac{2}{3} \sinh^3 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) + \frac{7}{15} \sinh^5 \left( \frac{1}{2} - (z - ct) + [z - ct] \right) \right\} \, dz \]
\[ = I_1 + I_2 + I_3. \] (3.14)

Using the identities
\[ \sinh^3(x) = \frac{1}{4} \sinh(3x) - \frac{3}{4} \sinh(x) \]
and
\[ \sinh^5(x) = \frac{1}{16} \sinh(5x) - \frac{5}{16} \sinh(3x) + \frac{5}{8} \sinh(x), \]
we directly compute \( I_1 \) as follows:
\[ I_1 = \frac{1}{2 \sinh \left( \frac{1}{2} \right)} \int_0^{ct} \cosh \left( \frac{1}{2} - x + z \right) \]
\[ \cdot \left\{ \frac{2}{3} \sinh^3 \left( -\frac{1}{2} + ct - z \right) + \frac{7}{15} \sinh^5 \left( -\frac{1}{2} + ct - z \right) \right\} \, dz \]
\[ = \frac{1}{2 \sinh \left( \frac{1}{2} \right)} \int_0^{ct} \cosh \left( \frac{1}{2} - x + z \right) \cdot \left\{ \frac{7}{240} \sinh \left( -\frac{5}{2} + 5ct - 5z \right) \right. \]
\[ \left. + \frac{1}{48} \sinh \left( -\frac{3}{2} + 3ct - 3z \right) - \frac{5}{24} \sinh \left( -\frac{1}{2} + ct - z \right) \right\} \, dz \]
\[
\begin{align*}
&= \frac{7}{960\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{4} \cosh(-2 - x + ct) + \frac{1}{4} \cosh(-2 - x + 5ct) \\
&\quad - \frac{1}{6} \cosh(3 - x + ct) + \frac{1}{6} \cosh(3 - x - 5ct) \right\} \\
&+ \frac{1}{192\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{2} \cosh(-1 - x + ct) + \frac{1}{2} \cosh(-1 - x + 3ct) \\
&\quad - \frac{1}{4} \cosh(2 - x + ct) + \frac{1}{4} \cosh(2 - x - 3ct) \right\} \\
&- \frac{5}{96\sinh\left(\frac{1}{2}\right)} \left\{ \sinh(ct - x)ct - \frac{1}{2} \cosh(1 - x + ct) + \frac{1}{2} \cosh(1 - x - ct) \right\}. \tag{3.15}
\end{align*}
\]

In a similar manner,
\[
I_2 = \frac{1}{2\sinh\left(\frac{1}{2}\right)} \int_{x}^{x} \cosh\left(\frac{1}{2} - x + z\right) \\
\quad \cdot \left\{ \frac{2}{3} \sinh^{3}\left(\frac{1}{2} + ct - z\right) + \frac{7}{15} \sinh^{5}\left(\frac{1}{2} + ct - z\right) \right\} dz
\]
\[
= \frac{1}{2\sinh\left(\frac{1}{2}\right)} \int_{ct}^{x} \cosh\left(\frac{1}{2} - x + z\right) \cdot \left\{ \frac{7}{240} \sinh\left(\frac{5}{2} + 5ct - 5z\right) \\
\quad + \frac{1}{48} \sinh\left(\frac{3}{2} + 3ct - 3z\right) - \frac{5}{24} \sinh\left(\frac{1}{2} + ct - z\right) \right\} dz
\]
\[
= \frac{7}{960\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{4} \cosh(3 - 5x + 5ct) + \frac{1}{4} \cosh(3 - x + ct) \\
- \frac{1}{6} \cosh(-2 + 5x - 5ct) + \frac{1}{6} \cosh(-2 - x + ct) \right\}
\]
\[
+ \frac{1}{192\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{2} \cosh(2 - 3x + 3ct) + \frac{1}{2} \cosh(2 - x + ct) \\
- \frac{1}{4} \cosh(-1 + 3x - 3ct) + \frac{1}{4} \cosh(-1 - x + ct) \right\}
\]
\[
- \frac{5}{96\sinh\left(\frac{1}{2}\right)} \left\{ \sinh(1 - x + ct)(x - ct) \right\} \tag{3.16}
\]

and
\[
I_3 = \frac{1}{2\sinh\left(\frac{1}{2}\right)} \int_{x}^{1} \cosh\left(\frac{1}{2} + x - z\right) \\
\quad \cdot \left\{ \frac{2}{3} \sinh^{3}\left(\frac{1}{2} + ct - z\right) + \frac{7}{15} \sinh^{5}\left(\frac{1}{2} + ct - z\right) \right\} dz
\]
\[
= \frac{1}{2\sinh\left(\frac{1}{2}\right)} \int_{x}^{1} \cosh\left(\frac{1}{2} - x + z\right) \cdot \left\{ \frac{7}{240} \sinh\left(\frac{5}{2} + 5ct - 5z\right) \\
\quad + \frac{1}{48} \sinh\left(\frac{3}{2} + 3ct - 3z\right) - \frac{5}{24} \sinh\left(\frac{1}{2} + ct - z\right) \right\} dz
\]
\[
= \frac{7}{960\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{4} \cosh(2 + x - 5ct) + \frac{1}{4} \cosh(-2 + 5x - 5ct) 
\right\}
\]
\[-\frac{1}{6}\cosh(-3 + x + 5ct) + \frac{1}{6}\cosh(3 - 5x + 5ct)\]

\[+ \frac{1}{192\sinh\left(\frac{x}{2}\right)} \left\{ \frac{-1}{2}\cosh(1 + x - 3ct) + \frac{1}{2}\cosh(-1 + 3x - 3ct)
\right.

\left. - \frac{1}{4}\cosh(-2 + x + 3ct) + \frac{1}{4}\cosh(2 - 3x + 3ct) \right\} \]

\[-\frac{5}{96\sinh\left(\frac{1}{2}\right)} \left\{ \frac{-1}{2}\cosh(-1 + x + ct) + \frac{1}{2}\cosh(1 - x + ct) - \sinh(x - ct)(1 - x) \right\}.\]  

(3.17)

Plugging (3.15)-(3.17) into (3.14), we obtain that for \(x > ct\),

\[G(x) * \left(\frac{2}{3}\sinh^3(\xi) + \frac{7}{15}\sinh^5(\xi)\right)(x, t)\]

\[= \frac{7}{960\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{12}\cosh(3 - 5x + 5ct) + \frac{1}{12}\cosh(3 - x + ct)
\right.

\left. - \frac{1}{12}\cosh(-2 - x + ct) + \frac{1}{12}\cosh(-2 + 5x - 5ct) \right\} \]

\[+ \frac{1}{192\sinh\left(\frac{1}{2}\right)} \left\{ -\frac{1}{4}\cosh(-1 - x + ct) + \frac{1}{4}\cosh(2 - x + ct)
\right.

\left. - \frac{1}{4}\cosh(2 - 3x + 3ct) + \frac{1}{4}\cosh(-1 + 3x - 3ct) \right\} \]

\[-\frac{5}{96\sinh\left(\frac{1}{2}\right)} \{(x - ct)\sinh(1 - x + ct) - (1 - x + ct)\sinh(x - ct)\}.\]  

(3.18)

On the other hand, when \(x > ct\), the right-hand side of (3.13) can be split into the following three parts:

\[G_x(x) * \left(\frac{-1}{3}\cosh(\xi) - \frac{4}{3}\cosh^3(\xi) + \frac{15}{37}\cosh^5(\xi)\right)(x, t)\]

\[= -\frac{1}{2\sinh\left(\frac{1}{2}\right)} \left( \int_0^ct + \int_{ct}^x + \int_x^1 \sinh\left(\frac{1}{2} - (x - z) + [x - z]\right) \times \left\{ \frac{1}{3}\cosh\left(\frac{1}{2} - (z - ct) + [z - ct]\right) - \frac{4}{3}\cosh^3\left(\frac{1}{2} - (z - ct) + [z - ct]\right)
\right.

\left. + \frac{15}{37}\cosh^5\left(\frac{1}{2} - (z - ct) + [z - ct]\right) \right\}dz \right) \]

\[= J_1 + J_2 + J_3.\]  

(3.19)

For \(J_1 - J_3\), due to the following identities

\[\cosh^3(x) = \frac{1}{4}\cosh(3x) + \frac{3}{4}\cosh(x)\]

and

\[\cosh^5(x) = \frac{1}{16}\cosh(5x) + \frac{5}{16}\cosh(3x) + \frac{5}{8}\cosh(x),\]
a direct calculation yields

\[ J_1 = -\frac{1}{2 \sinh \left( \frac{x}{2} \right)} \int_0^x \sinh \left( \frac{1}{2} - x + z \right) \cdot \left\{ -\frac{1}{3} \cosh \left( \frac{1}{2} - ct + z \right) \right. \]

\[ \left. + \frac{4}{3} \cosh^3 \left( \frac{1}{2} - ct + z \right) + \frac{15}{37} \cosh^5 \left( \frac{1}{2} - ct + z \right) \right\} \, dz \]

\[ = -\frac{37}{960 \sinh \left( \frac{x}{2} \right)} \left\{ -\frac{1}{4} \cosh(-2 - x + ct) + \frac{1}{4} \cosh(-2 - x + 5ct) \right. \]

\[ + \frac{1}{6} \cosh(3 - x + ct) - \frac{1}{6} \cosh(3 - x - 5ct) \} \]

\[ \left. - \frac{7}{64 \sinh \left( \frac{x}{2} \right)} \left\{ -\frac{1}{2} \cosh(-1 - x + ct) + \frac{1}{2} \cosh(-1 - x + 3ct) \right. \right. \]

\[ + \frac{1}{4} \cosh(2 - x + ct) - \frac{1}{4} \cosh(2 - x - 3ct) \} \]

\[ \left. - \frac{5}{96 \sinh \left( \frac{x}{2} \right)} \left\{ \sinh(ct - x) + \frac{1}{2} \cosh(1 - x + ct) - \frac{1}{2} \cosh(1 - x - ct) \right. \right. \right. \]

\[ = -\frac{37}{960 \sinh \left( \frac{x}{2} \right)} \left\{ -\frac{1}{4} \cosh(3 - 5x + 5ct) + \frac{1}{4} \cosh(3 - x + ct) \right. \]

\[ + \frac{1}{6} \cosh(-2 + 5x - 5ct) - \frac{1}{6} \cosh(-2 - x + ct) \} \]

\[ \left. - \frac{7}{64 \sinh \left( \frac{x}{2} \right)} \left\{ -\frac{1}{2} \cosh(2 - 3x + 3ct) + \frac{1}{2} \cosh(2 - x + ct) \right. \right. \]

\[ + \frac{1}{4} \cosh(-1 + 3x - 3ct) - \frac{1}{4} \cosh(-1 - x + ct) \} \]

\[ \left. - \frac{5}{96 \sinh \left( \frac{x}{2} \right)} \left\{ \sinh(1 - x + ct)(x - ct) \right. \right. \}

(3.20)

and

\[ J_3 = -\frac{1}{2 \sinh \left( \frac{x}{2} \right)} \int_x^1 \sinh \left( -\frac{1}{2} - x + z \right) \cdot \left\{ -\frac{1}{3} \cosh \left( \frac{1}{2} + ct - z \right) \right. \]
It follows from (3.10), (3.11), (3.18) and (3.23) that
\[ x > ct, \]
Combining (3.20)-(3.22) with (3.13), we have that for \( x > ct, \)
\[
-\frac{1}{2\sinh\left(\frac{x}{2}\right)} \int_{x}^{1} \sinh\left(\frac{1}{2} - x + z\right) \cdot \left\{ \frac{37}{240} \cosh\left(\frac{5}{2} + 5ct - 5z\right) + \frac{7}{16} \cosh\left(\frac{3}{2} + 3ct - 3z\right) + \frac{5}{24} \cosh\left(\frac{1}{2} + ct - z\right) \right\} dz
\]
\[
= -\frac{37}{960\sinh\left(\frac{x}{2}\right)} \left\{ -\frac{1}{4} \cosh(-2 - x + 5ct) + \frac{1}{4} \cosh(2 - 5x + 5ct) + \frac{1}{6} \cosh(3 - x + 5ct) - \frac{1}{6} \cosh(-3 + 5x - 5ct) \right\}
\]
\[
= -\frac{7}{64\sinh\left(\frac{x}{2}\right)} \left\{ -\frac{1}{2} \cosh(-1 - x + 3ct) + \frac{1}{2} \cosh(1 - 3x + 3ct) + \frac{1}{4} \cosh(2 - x - 3ct) - \frac{1}{4} \cosh(-2 + 3x - 3ct) \right\}
\]
\[
= -\frac{5}{96\sinh\left(\frac{x}{2}\right)} \left\{ \frac{1}{2} \cosh(1 - x - ct) - \frac{1}{2} \cosh(-1 + x - ct) + \sinh(-x + ct)(1 - x) \right\}. \tag{3.22}
\]
Combining (3.20)-(3.22) with (3.13), we have that for \( x > ct, \)
\[
G_x(x) * \left( -\frac{1}{3} \cosh(\xi) - \frac{4}{3} \cosh^3(\xi) + \frac{15}{37} \cosh^5(\xi) \right) (x, t)
\]
\[
= -\frac{37}{960\sinh\left(\frac{x}{2}\right)} \left\{ -\frac{5}{12} \cosh(3 - 5x + 5ct) + \frac{5}{12} \cosh(3 - x + ct) + \frac{5}{12} \cosh(-2 - x + ct) + \frac{5}{12} \cosh(2 - 5x + 5ct) \right\}
\]
\[
= -\frac{7}{64\sinh\left(\frac{x}{2}\right)} \left\{ -\frac{3}{4} \cosh(-1 - x + ct) + \frac{3}{4} \cosh(2 - x + ct) + \frac{3}{4} \cosh(2 - 3x + 3ct) + \frac{3}{4} \cosh(1 - 3x + 3ct) \right\}
\]
\[
= -\frac{5}{96\sinh\left(\frac{x}{2}\right)} \left\{ (x - ct) \sinh(1 - x - ct) - (1 - x + ct) \sinh(x - ct) \right\}. \tag{3.23}
\]
It follows from (3.10), (3.11), (3.18) and (3.23) that
\[
\int_{0}^{\infty} \int_{ct}^{1} \left[ G(x) * \left( \frac{4}{3} u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3} u_c(\partial_x u_c)^4 \right) \partial_x \psi - G(x) * \left( \frac{2}{3} u_c^2(\partial_x u_c)^3 - \frac{1}{5} (\partial_x u_c)^5 \right) \psi \right] dx dt
\]
\[
= a^5 \int_{0}^{\infty} \int_{ct}^{1} \psi G(x) * \left( \frac{2}{3} \sinh^3(\xi) + \frac{7}{15} \sinh^5(\xi) \right) dx dt
\]
\[
- a^5 \int_{0}^{\infty} \int_{ct}^{1} \psi G_x(x) * \left( -\frac{1}{3} \cosh(\xi) - \frac{4}{3} \cosh^3(\xi) + \frac{15}{37} \cosh^5(\xi) \right) dx dt.\]
In a similar way, for the case of which along with (3.24) lead to

\[
\frac{a^5}{60\sinh \left( \frac{1}{2} \right)} \int_0^\infty \int_{ct}^1 \psi \left( \cosh(2 - 5x + 5ct) - \cosh(3 - 5x + 5ct) \\
+ 5\cosh(1 - 3x + 3ct) - 5\cosh(2 - 3x + 3ct) \\
+ \cosh(3 - x + ct) - \cosh(2 + x - ct) \\
+ 5\cosh(2 - x + ct) - 5\cosh(1 + x - ct) \right) dx dt.
\]

Using the identity \( \cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \), we have

\[
\cosh(2 - 5x + 5ct) - \cosh(3 - 5x + 5ct) = -2\sinh \left( \frac{5}{2} - 5x + 5ct \right) \sinh \left( \frac{1}{2} \right),
\]

\[
5\cosh(1 - 3x + 3ct) - 5\cosh(2 - 3x + 3ct) = -10\sinh \left( \frac{3}{2} - 3x + 3ct \right) \sinh \left( \frac{1}{2} \right),
\]

\[
\cosh(3 - x + ct) - \cosh(2 + x - ct) = 2\sinh \left( \frac{1}{2} - x + ct \right) \sinh \left( \frac{5}{2} \right)
\]

and

\[
5\cosh(2 - x + ct) - 5\cosh(1 + x - ct) = 10\sinh \left( \frac{1}{2} - x + ct \right) \sinh \left( \frac{3}{2} \right),
\]

which along with (3.24) lead to

\[
\int_0^\infty \int_{ct}^1 \left[ G(x) * \left( \frac{4}{5}u_c^5 + 2u_c^3(\partial_x u_c)^2 - \frac{1}{3}u_c(\partial_x u_c)^4 \right) \partial_x \psi - G(x) * \left( \frac{2}{3}u_c^3(\partial_x u_c)^3 - \frac{1}{5}(\partial_x u_c)^5 \right) \psi \right] dx dt
\]

\[
= a^5 \frac{1}{60\sinh \left( \frac{1}{2} \right)} \int_0^\infty \int_{ct}^1 \psi \left[ -2\sinh \left( \frac{5}{2} - 5x + 5ct \right) \sinh \left( \frac{1}{2} \right) \\
- 10\sinh \left( \frac{3}{2} - 3x + 3ct \right) \sinh \left( \frac{1}{2} \right) \\
+ \left\{ 2\sinh \left( \frac{5}{2} \right) + 10\sinh \left( \frac{3}{2} \right) \right\} \sinh \left( \frac{1}{2} - x + ct \right) \right] dx dt
\]

\[
= a^5 \int_0^\infty \int_{ct}^1 \psi \left\{ \frac{8}{15}\sinh^4 \left( \frac{1}{2} \right) + \frac{4}{3}\sinh^2 \left( \frac{1}{2} \right) \right\} \sinh \left( \frac{1}{2} - x + ct \right) \\
- \frac{4}{3}\sinh^4 \left( \frac{1}{2} - x + ct \right) - \frac{8}{15}\sinh^5 \left( \frac{1}{2} - x + ct \right) \right\} dx dt.
\]

(3.25)

In a similar way, for the case of \( x < ct \),

\[
G(x) * \left( \frac{2}{3}\sinh^3(\xi) + \frac{7}{15}\sinh^5(\xi) \right) (x, t)
\]

\[
= \frac{7}{960\sinh \left( \frac{1}{2} \right)} \left\{ -\frac{1}{12}\cosh(3 + 5x - 5ct) + \frac{1}{12}\cosh(3 + x - ct) \\
- \frac{1}{12}\cosh(-2 + x - ct) + \frac{1}{12}\cosh(-2 - 5x + 5ct) \right\}
\]
Thus we obtain from (3.25) and (3.28)

\[\int_0^\infty G(x) \ast \left( -\frac{1}{3} \cosh(\xi) - \frac{4}{3} \cosh^3(\xi) + \frac{15}{37} \cosh^5(\xi) \right) \, dx \]

along with (3.10) and (3.11) yield

\[
\int_0^\infty \int_0^ct G(x) \ast \left( \frac{4}{5} u_c^3 + 2 u_c^3 (\partial_x u_c)^2 - \frac{1}{3} u_c (\partial_x u_c)^4 \right) \partial_x \psi dx dt \\
- G(x) \ast \left( \frac{2}{3} u_c^2 (\partial_x u_c)^3 - \frac{1}{5} (\partial_x u_c)^5 \right) \psi dx dt \\
= a^5 \int_0^\infty \int_0^ct \psi G(x) \ast \left( \frac{2}{3} \sinh^3(\xi) + \frac{7}{15} \sinh^5(\xi) \right) dx dt \\
- a^5 \int_0^\infty \int_0^ct \psi G(x) \ast \left( -\frac{1}{3} \cosh(\xi) - \frac{4}{3} \cosh^3(\xi) + \frac{15}{37} \cosh^5(\xi) \right) dx dt \\
= \frac{a^5}{60 \sinh \left( \frac{x}{2} \right)} \int_0^\infty \int_0^ct \psi \left[ \cosh(2 + 5x - 5ct) \\
- \cosh(3 + 5x - 5ct) + 5 \cosh(1 + 3x - 3ct) \\
- 5 \cosh(2 + 3x - 3ct) + \cosh(3 + x - ct) - \cosh(2 + x - ct) \\
+ 5 \cosh(2 + x - ct) - 5 \cosh(1 + x - ct) \right] dx dt \\
= a^5 \int_0^\infty \int_0^ct \psi \left[ \left\{ \frac{8}{10} \sinh^4 \left( \frac{1}{2} \right) + \frac{4}{3} \sinh^2 \left( \frac{1}{2} \right) \right\} \sinh \left( \frac{1}{2} + x - ct \right) \\
- \frac{4}{3} \sinh^3 \left( \frac{1}{2} + x - ct \right) - \frac{8}{15} \sinh^5 \left( \frac{1}{2} + x - ct \right) \right] dx dt. \tag{3.28}
\]

Thus we obtain from (3.25) and (3.28)

\[
\int_0^\infty \int_S G(x) \ast \left( \frac{4}{5} u_c^3 + 2 u_c^3 (\partial_x u_c)^2 - \frac{1}{3} u_c (\partial_x u_c)^4 \right) \partial_x \psi dx dt.
\]
and extends periodically to the entire line. We still identify \( S \) and \( \phi \) such that

\[
\phi(0) = \min_{x \in \mathbb{S}} \phi(x) = \frac{1}{\cosh \left( \frac{1}{2} \right)} = \frac{1}{\cosh \left( \frac{1}{2} \right)}.
\]

By (3.9), (3.10), (3.29) and the definition of \( u_c \), we deduce that

\[
-G(x) \* \left( \frac{2}{3} u_c^3 \left( \partial_x u_c \right)^3 - \frac{1}{5} \left( \partial_x u_c \right)^5 \right) \psi
= a^5 \int_0^\infty \int_\mathbb{S} \left[ \left\{ \frac{8}{15} \sinh^4 \left( \frac{1}{2} \right) + \frac{4}{3} \sinh^2 \left( \frac{1}{2} \right) \right\} \sinh (\xi)
- \frac{4}{3} \sinh^3 (\xi) - \frac{8}{15} \sinh^5 (\xi) \right] dx dt.
\]

(3.29)

\[
\int_0^\infty \int_\mathbb{S} \left( u_c \partial_t \psi + \frac{1}{5} u_c^3 \partial_x \psi + \left( \frac{2}{3} u_c^3 \left( \partial_x u_c \right)^3 - \frac{1}{5} \left( \partial_x u_c \right)^5 \right) \psi
+ (1 - \partial_x^2)^{-1} \left( \frac{4}{5} u_c^5 + 2 u_c^3 \left( \partial_x u_c \right)^2 - \frac{1}{3} u_c \left( \partial_x u_c \right)^4 \right) \partial_x \psi
- (1 - \partial_x^2)^{-1} \left( \frac{2}{3} u_c^3 \left( \partial_x u_c \right)^3 - \frac{1}{5} \left( \partial_x u_c \right)^5 \right) \right) dx dt + \int_\mathbb{S} u_c(x,0) \psi(x,0) dx = 0
\]

for every test function \( \psi(x,t) \in C^\infty_c (\mathbb{S} \times [0, \infty)) \), which completes the proof of Theorem 3.1. \( \square \)

4. Stability of periodic peakons. In this section, we focus on the orbital stability of the periodic peakon (3.3). Let us first give some basic properties of periodic peakons. It is easy to see that the periodic peaked function

\[
u_c(x,t) = \phi_c(x,t) = b \phi(x - ct),
\]

where

\[
b = \sqrt{\frac{15 ccosh^4 \left( \frac{1}{2} \right)}{8cosh^4 \left( \frac{1}{2} \right) + 4cosh^4 \left( \frac{1}{2} \right) + 3}}, \quad c > 0
\]

and \( \phi(x) \) is given for \( x \in [0, 1] \) by

\[
\phi(x) = \frac{\cosh \left( \frac{1}{2} - x \right)}{\cosh \left( \frac{1}{2} \right)}
\]

and extends periodically to the entire line. We still identify \( S \) with the interval \( [0, 1] \) and view functions on \( S \) as periodic functions on the entire line of period one. Note that \( \phi(x) \) is continuous on \( S \) with a peak at \( x = 0 \). A simple computation yields

\[
M_\phi = \max_{x \in \mathbb{S}} \phi(x) = \phi(0) = 1 \quad \text{and} \quad m_\phi = \min_{x \in \mathbb{S}} \phi(x) = \phi \left( \frac{1}{2} \right) = \frac{1}{\cosh \left( \frac{1}{2} \right)}.
\]
Moreover, $\phi(x)$ is smooth on $(0, 1)$, and holds

$$\phi_x(x) = -\frac{\sinh \left(\frac{1}{2} - x\right)}{\cosh \left(\frac{1}{2}\right)}$$

with

$$\phi_x(x) \to -\tanh \left(\frac{1}{2}\right) \quad \text{as} \quad x \to 0 \quad \text{and} \quad \phi_x(x) \to \tanh \left(\frac{1}{2}\right) \quad \text{as} \quad x \to 1.$$  

Using the formulation $\phi_{xx} = \phi - 2\tanh \left(\frac{1}{2}\right) \delta$, we obtain

$$H_1[\phi_c] = b^2 \int_S (\phi^2 + \phi_x^2) dx = b^2 \int_S (\phi^2 - \phi \phi_x) dx$$

$$= b^2 \int_S \left(\phi^2 - \left(\phi - 2\tanh \left(\frac{1}{2}\right) \delta\right) \phi\right) dx$$

$$= 2b^2 \tanh \left(\frac{1}{2}\right) \phi(0) = 2b\tanh \left(\frac{1}{2}\right) M_{\phi_c},$$

where $M_{\phi_c} = \max_{x \in S} \phi_c(x)$ and $\delta$ denotes the Dirac delta distribution. We compute $H_2[\phi_c]$ directly as follows:

$$H_2[\phi_c] = b^6 \int_S \frac{1}{\cosh^6 \left(\frac{1}{2}\right)} \left(\phi^6 + 3\phi^4 \phi_x^2 - \phi^2 \phi_x^4 + \frac{1}{5} \phi_x^6\right) dx$$

$$= \frac{b^6}{\cosh^6 \left(\frac{1}{2}\right)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cosh^6(x) + 3\cosh^4(x)\sinh^2(x) - \cosh^2(x)\sinh^4(x) + \frac{1}{5}\sinh^6(x)\right) dx$$

$$= \frac{b^6}{\cosh^6 \left(\frac{1}{2}\right)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{5}\sinh(3x) + \frac{1}{3}\sinh(2x) + \frac{1}{2}\sinh(x)\right) dx$$

$$= b^6 \left\{\frac{3\sinh(3) + \frac{1}{3}\sinh(2) + \frac{1}{2}\sinh(1)}{\cosh^6 \left(\frac{1}{2}\right)}\right\}. $$

Using the identities $\sinh(2x) = 2\sinh(x)\cosh(x)$, $\cosh(2x) = 2\cosh^2(x) - 1$ and $1 - \tanh^2(x) = \text{sech}^2(x)$, we obtain

$$H_2[\phi_c] = 2b^6 \left[\tanh \left(\frac{1}{2}\right) - \frac{2}{3}\tanh^3 \left(\frac{1}{2}\right) + \frac{1}{5}\tanh^5 \left(\frac{1}{2}\right)\right]. \quad (4.6)$$

We now present the following theorem which shows that the periodic peakons (4.1) of the generalized modified Camassa-Holm equation are orbitally stable.

**Theorem 4.1.** The periodic peakon $\phi_c$ defined in (4.1) traveling with the speed $c > 0$ is orbitally stable in the following sense. Let $u_0 \in H^s(S), s > 5/2$, with $(1 - \partial_x^2)u_0(x) \geq 0$ for any $x \in S$. Let $T > 0$ be the maximal existence time of the corresponding periodic solution $u(x, t) \in C([0, T), H^s(S))$ to (2.1) with the initial data $u_0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that if

$$\|u_0 - \phi_c\|_{H^s(S)} < \delta,$$

then

$$\|u(\cdot, t) - \phi_c(\cdot - \xi(t))\|_{H^s(S)} < \epsilon \quad \text{for} \quad t \in [0, T),$$

where $\xi(t) \in \mathbb{R}$ is any point where the function $u(\cdot, t)$ attains its maximum.
Remark 3. Since the equation in (2.1) is invariant under the inverse transform, that is, \(-u(x, t)\) is the solution of (2.1) with initial data \(-u_0(x)\), if \(u(x, t)\) is the solution of (2.1) with initial data \(u_0(x)\), this implies that the periodic peakon \(-\phi_c\) is also orbitally stable in the sense of Theorem 4.1 with the initial data \(u_0 \in H^s(\mathbb{S})\), \(s > 5/2\) and \((1 - \partial_x^2)u_0(x) \leq 0\), for any \(x \in \mathbb{S}\).

The proof of Theorem 4.1 will be obtained from the following several lemmas. First we expand the conservation law \(H_1[u]\) around the peakon \(\phi_c\) in the \(H^1(\mathbb{S})\)-norm. The proof of the following lemma is similar to that of Lemma 5.1 for the modified Camassa-Holm(mCH) equation in [52]. We omit it here.

**Lemma 4.2.** [52] For any \(u \in H^1(\mathbb{S})\) and \(\xi \in \mathbb{R}\), we obtain

\[
H_1[u] - H_1[\phi_c] = \|u - \phi_c(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 + 4\text{bthn}\left(\frac{1}{2}\right)(u(\xi) - M_{\phi_c}).
\] (4.7)

Remark 4. For a wave profile \(u \in H^1(\mathbb{S})\), the functional \(H_1[u]\) stands for the kinetic energy in [12]. Lemma 4.2 implies that if a wave has energy and height close to the peakon’s energy and height, then the whole shape of this wave is close to that of the peakon. Also, a physical relevance of Lemma 4.2 is that among all waves of fixed energy \(H_1[\cdot]\), the peakon has maximal height.

The following lemma is crucial to prove the Theorem 4.1.

**Lemma 4.3.** For any positive \(u \in H^s(\mathbb{S})\), \(s > \frac{5}{2}\), define a function

\[
F_u : \{(M, m) \in \mathbb{R}^2 | M \geq m > 0\} \rightarrow \mathbb{R}
\]

by

\[
F_u(M, m) = \left(\frac{8}{5}M^4 + \frac{4}{5}M^2m^2 - \frac{2}{5}m^4\right) \\
\times \left(2m^2 \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2M \sqrt{M^2 - m^2} - m^2 + H_1[u]\right) \\
+ \frac{16}{15}M^3(M^2 - m^2)^{3/2} + \frac{8}{5}m^2M(M^2 - m^2)^{3/2} + m^4H_1[u] - H_2[u].
\] (4.8)

Then

\[
F_u(M_u, m_u) \geq 0,
\]

where \(M_u = \max_{x \in \mathbb{S}} u(x)\), \(m_u = \min_{x \in \mathbb{S}} u(x)\).

**Proof.** Let \(0 < u(x) \in H^s(\mathbb{S}) \subset C^2(\mathbb{S})(s > \frac{5}{2})\) and write \(M = M_u = \max_{x \in \mathbb{S}} u(x)\) and \(m = m_u = \min_{x \in \mathbb{S}} u(x)\). Let \(\xi\) and \(\eta\) be such that \(u(\xi) = M\) and \(u(\eta) = m\). Define the same function \(g(x)\) as in [35]

\[
g(x) = \begin{cases} 
    u_x + \sqrt{u^2 - m^2}, & \xi < x \leq \eta, \\
    u_x - \sqrt{u^2 - m^2}, & \eta < x < \xi + 1,
\end{cases}
\] (4.9)

and extended periodically to the entire line, we have

\[
\int_{\mathbb{S}} g^2(x) dx = 2m^2 \ln \left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2M \sqrt{M^2 - m^2} - m^2 + H_1[u].
\] (4.10)
On the other hand, we define the real function $h(x)$ by

\[
h(x) = \begin{cases} 
  u^4 + \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{5} u_x^4 \\
  + \left(u^2 + \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2\right) m^2, & \xi < x \leq \eta, \\
  u^4 - \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{5} u_x^4 \\
  + \left(u^2 - \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2\right) m^2, & \eta \leq x < \xi + 1
\end{cases}
\]

and extend it periodically to the entire line. We compute

\[
\int_S h(x) g^2(x) \, dx = \int_\xi^\eta \left(u^4 + \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{5} u_x^4 \\
  + \left(u^2 + \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2\right) m^2\right)(u_x + \sqrt{u^2 - m^2})^2 \, dx \\
  + \int_\eta^{\xi+1} \left(u^4 - \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{5} u_x^4 \\
  + \left(u^2 - \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2\right) m^2\right)(u_x - \sqrt{u^2 - m^2})^2 \, dx
\]

\[
= I_1 + I_2.
\]

A direct calculation leads to

\[
I_1 = \int_\xi^\eta \left(u^4 + \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u^2 u_x^2 + \frac{1}{5} u_x^4 \\
  + \left(u^2 + \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2\right) m^2\right)(u_x^2 + 2u_x \sqrt{u^2 - m^2} + u^2 - m^2) \, dx
\]

\[
= \int_\xi^\eta \left(u^6 + 3u^4 u_x^2 - u^2 u_x^4 + \frac{1}{5} u_x^6\right) \, dx + \frac{16}{5} \int_\xi^\eta u^4 u_x \sqrt{u^2 - m^2} \, dx \\
  + \frac{8}{5} m^2 \int_\xi^\eta u^2 u_x \sqrt{u^2 - m^2} \, dx - \frac{4}{5} m^4 \int_\xi^\eta u_x \sqrt{u^2 - m^2} \, dx - m^4 \int_\xi^\eta (u^2 + u_x^2) \, dx.
\]

Since

\[
\frac{d}{dx} \left(\frac{1}{6} u^3 (u^2 - m^2)^{3/2}\right) + \frac{1}{2} m^2 u_x^2 \sqrt{u^2 - m^2} = u^4 u_x \sqrt{u^2 - m^2}
\]

and

\[
\frac{d}{dx} \left(\frac{1}{4} u(u^2 - m^2)^{3/2}\right) + \frac{1}{4} m^2 u_x \sqrt{u^2 - m^2} = u^2 u_x \sqrt{u^2 - m^2},
\]

we have

\[
\frac{16}{5} \int_\xi^\eta u^4 u_x \sqrt{u^2 - m^2} \, dx + \frac{8}{5} m^2 \int_\xi^\eta u^2 u_x \sqrt{u^2 - m^2} \, dx - \frac{4}{5} m^4 \int_\xi^\eta u_x \sqrt{u^2 - m^2} \, dx
\]

\[
= \frac{8}{15} \int_\xi^\eta \frac{d}{dx} \left(u^3 (u^2 - m^2)^{3/2}\right) \, dx + \frac{16}{5} m^2 \int_\xi^\eta u^2 u_x \sqrt{u^2 - m^2} \, dx
\]

\[
- \frac{4}{5} m^4 \int_\xi^\eta u_x \sqrt{u^2 - m^2} \, dx.
\]
Using the positivity of the function $u$ which along with (4.13) leads to

$$\frac{8}{15} \int_\xi^\eta \frac{d}{dx} \left( u^3(u^2 - m^2)^{3/2} \right) dx + \frac{4}{5} m^2 \int_\xi^\eta \frac{d}{dx} \left( u(u^2 - m^2)^{3/2} \right) dx$$

$$= -\frac{8}{15} M^3 (M^2 - m^2)^{3/2} - \frac{4}{5} m^2 M (M^2 - m^2)^{3/2}.$$ 

Thus we arrive at

$$I_1 = \int_\xi^\eta \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5} u_x^6 \right) dx - \frac{8}{15} M^3 (M^2 - m^2)^{3/2} 
- \frac{4}{5} m^2 M (M^2 - m^2)^{3/2} - m^4 \int_\xi^\eta (u^2 + u_x^2) dx. \quad (4.13)$$

In the same way, we obtain

$$I_2 = \int_\eta^{\xi+1} \left( u^4 - \frac{2}{5} (3u^2 - u_x^2) u_x \sqrt{u^2 - m^2} - \frac{2}{5} u_x^2 u_x^2 + \frac{1}{5} u_x^4 
+ \left( u^2 - \frac{4}{5} u_x \sqrt{u^2 - m^2} - \frac{3}{5} u_x^2 \right) m^2 \right) (u_x^2 - 2u_x \sqrt{u^2 - m^2} + u^2 - m^2) dx$$

$$= \int_\eta^{\xi+1} \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5} u_x^6 \right) dx - \frac{16}{5} \int_\eta^{\xi+1} u_xu_x \sqrt{u^2 - m^2} dx$$

$$- \frac{8}{5} m^2 \int_\eta^{\xi+1} u^2u_x \sqrt{u^2 - m^2} dx + \frac{4}{5} m^4 \int_\eta^{\xi+1} u_x \sqrt{u^2 - m^2} dx$$

$$- m^4 \int_\eta^{\xi+1} (u^2 + u_x^2) dx$$

$$= \int_\eta^{\xi+1} \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5} u_x^6 \right) dx - \frac{8}{15} M^3 (M^2 - m^2)^{3/2}$$

$$- \frac{4}{5} m^2 M (M^2 - m^2)^{3/2} - m^4 \int_\eta^{\xi+1} (u^2 + u_x^2) dx,$$

which along with (4.13) leads to

$$\int_\gamma^{g(x)g^2(x)dx}$$

$$= -\frac{16}{15} M^3 (M^2 - m^2)^{3/2} - \frac{8}{5} m^2 M (M^2 - m^2)^{3/2} - m^4 H_1[u] + H_2[u]. \quad (4.14)$$

Using the positivity of the function $u$ and Young’s inequality for $x \in S$ we get

$$h(x) = u^4(x) \pm \frac{2}{5} (3u^2(x) - u_x^2(x)) u_x(x) \sqrt{u^2(x) - m^2} - \frac{2}{5} u_x^2(x) u_x^2(x) + \frac{1}{5} u_x^4(x)$$

$$+ \left( u^2(x) \pm \frac{4}{5} u_x \sqrt{u^2(x) - m^2} - \frac{3}{5} u_x^2(x) \right) m^2$$

$$\leq u^4(x) + \frac{3}{5} u^4(x) + \frac{4}{5} u^2(x) m^2 - \frac{2}{5} m^4$$

$$\leq \frac{8}{5} M^4 + \frac{4}{5} M^2 m^2 - \frac{2}{5} m^4.$$
which together with (4.14) yields
\[
- \frac{16}{15} M^3 (2 - m^2)^{3/2} - \frac{8}{5} m^2 M (M^2 - m^2)^{3/2} - m^4 H_1[u] + H_2[u] \leq \left( \frac{8}{5} M^4 + \frac{4}{5} M^2 m^2 - \frac{2}{5} m^4 \right) \int_S g^2(x) dx.
\] (4.15)

Therefore, combining (4.10) with (4.15), we conclude that
\[
0 \leq \left( \frac{8}{5} M^4 + \frac{4}{5} M^2 m^2 - \frac{2}{5} m^4 \right) \left( 2m^2 \ln \left( \frac{M + \sqrt{M^2 - m^2}}{m} \right) - 2M \sqrt{M^2 - m^2} - m^2 + H_1[u] \right) + \frac{16}{15} M^3 (2 - m^2)^{3/2} + \frac{8}{5} m^2 M (M^2 - m^2)^{3/2} + m^4 H_1[u] - H_2[u].
\] (4.16)

This completes the proof of lemma. \(\Box\)

**Remark 5.** When \(u\) is replaced by the periodic peakons \(\phi_c\), then the functionals \(g\) and \(h\) are zero. Note that the function \(F_u\) depends on \(u\) only through the two conservation laws \(H_1[u]\) and \(H_2[u]\).

In the following lemma, we establish some properties of the function \(F_u(M, m)\) related to the peakon \(\phi_c\).

**Lemma 4.4.** For the peakon \(\phi_c\), we have
\[
F_{\phi_c}(M_{\phi_c}, m_{\phi_c}) = 0, \quad \frac{\partial F_{\phi_c}}{\partial M}(M_{\phi_c}, m_{\phi_c}) = 0, \quad \frac{\partial F_{\phi_c}}{\partial m}(M_{\phi_c}, m_{\phi_c}) = 0,
\]
\[
\frac{\partial^2 F_{\phi_c}}{\partial M^2}(M_{\phi_c}, m_{\phi_c}) = -b^4 \tanh \left( \frac{1}{2} \right) \left( \frac{128}{5} + \frac{32}{5} \text{sech}^2 \left( \frac{1}{2} \right) \right),
\]
\[
\frac{\partial^2 F_{\phi_c}}{\partial m^2}(M_{\phi_c}, m_{\phi_c}) = -b^4 \tanh \left( \frac{1}{2} \right) \left( \frac{32}{5} + \frac{168}{5} \text{sech}^2 \left( \frac{1}{2} \right) \right),
\]
\[
\frac{\partial^2 F_{\phi_c}}{\partial M \partial m}(M_{\phi_c}, m_{\phi_c}) = 0,
\]
where \(M_{\phi_c} = \max_{x \in S} \phi_c(x)\) and \(m_{\phi_c} = \min_{x \in S} \phi_c(x)\).

**Proof.** We see from (4.4) that the function \(g(x)\) corresponding to the peakon \(\phi_c\) is identically zero. Thus the inequality (4.16) is an equality in the case of the peakon. This shows that \(F_{\phi_c}(M_{\phi_c}, m_{\phi_c}) = 0\). Let
\[
L_u(M, m) = \int_S g^2(x) dx
\]
\[
= 2m^2 \ln \left( \frac{M + \sqrt{M^2 - m^2}}{m} \right) - 2M \sqrt{M^2 - m^2} - m^2 + H_1[u].
\]

Then we have
\[
F_u(M, m) = \left( \frac{8}{5} M^4 + \frac{4}{5} m^2 M^2 - \frac{2}{5} m^4 \right) L_u(M, m)
\]
\[
+ \frac{16}{15} M^3 (2 - m^2)^{3/2} + \frac{8}{5} m^2 M (M^2 - m^2)^{3/2} + m^4 H_1[u] - H_2[u].
\]
Since

\[ \frac{\partial L_u}{\partial M} = -4\sqrt{M^2 - m^2}, \]
\[ \frac{\partial L_u}{\partial m} = 4m \ln \left( \frac{M + \sqrt{M^2 - m^2}}{m} \right) - 2m, \]
\[ \frac{\partial^2 L_u}{\partial m^2} = 4 \ln \left( \frac{M + \sqrt{M^2 - m^2}}{m} \right) - \frac{4M}{\sqrt{M^2 - m^2}} - 2, \quad (4.17) \]

we obtain

\[ \frac{\partial F_u}{\partial M} = \left( \frac{32}{5} M^3 + \frac{8}{5} m^2 M \right) L_u + \left( \frac{8}{5} M^4 + \frac{4}{5} m^2 M^2 - \frac{2}{5} m^4 \right) \frac{\partial L_u}{\partial M} \]
\[ + \frac{16}{5} M^2 (M^2 - m^2)^{\frac{3}{2}} + \frac{16}{5} M^4 \sqrt{M^2 - m^2} \]
\[ + \frac{8}{5} m^2 (M^2 - m^2)^{\frac{3}{2}} + \frac{24}{5} m^2 M^2 \sqrt{M^2 - m^2} \]
\[ = \left( \frac{32}{5} M^3 + \frac{8}{5} m^2 M \right) L_u - 4\sqrt{M^2 - m^2} \left( \frac{8}{5} M^4 + \frac{4}{5} m^2 M^2 - \frac{2}{5} m^4 \right) \]
\[ + \left( \frac{32}{5} M^4 + \frac{16}{5} M^2 m^2 - \frac{8}{5} m^4 \right) \sqrt{M^2 - m^2} \]
\[ = \left( \frac{32}{5} M^3 + \frac{8}{5} m^2 M \right) L_u, \]
\[ \frac{\partial F_u}{\partial m} = \left( \frac{8}{5} M^2 m - \frac{8}{5} m^3 \right) L_u + \left( \frac{8}{5} M^4 + \frac{4}{5} M^2 m^2 - \frac{2}{5} m^4 \right) \frac{\partial L_u}{\partial m} \]
\[ - 8m^3 M \sqrt{M^2 - m^2} + 4m^3 H_1[u] \]

and

\[ \frac{\partial^2 F_u}{\partial M^2} = \left( \frac{96}{5} M^2 + \frac{8}{5} m^2 \right) L_u + \left( \frac{32}{5} M^3 + \frac{8}{5} M m^2 \right) \frac{\partial L_u}{\partial M} \]
\[ \frac{\partial^2 F_u}{\partial m^2} = \left( \frac{8}{5} M^2 - \frac{24}{5} m^2 \right) L_u + \left( \frac{16}{5} M^2 m - \frac{16}{5} m^4 \right) \frac{\partial L_u}{\partial m} \]
\[ + \left( \frac{8}{5} M^4 + \frac{4}{5} m^2 M^2 - \frac{2}{5} m^4 \right) \frac{\partial^2 L_u}{\partial m^2} \]
\[ + \frac{32m^4 M - 24M^3 m^2}{\sqrt{M^2 - m^2}} + 12m^2 H_1[u], \]
\[ \frac{\partial^2 F_u}{\partial M \partial m} = \frac{16}{5} M m L_u + \left( \frac{32}{5} M^3 + \frac{8}{5} M m^2 \right) \frac{\partial L_u}{\partial m}. \]

Recall that for the peakon \( \phi_c \),

\[ M_{\phi_c} = b, \quad m_{\phi_c} = \frac{b}{\cosh \left( \frac{1}{2} \right)}, \quad H_1[\phi_c] = 2b^2 \tanh \left( \frac{1}{2} \right), \]
\[ H_2[\phi_c] = 2b^6 \left[ \tanh \left( \frac{1}{2} \right) - \frac{2}{3} \tanh^3 \left( \frac{1}{2} \right) + \frac{1}{5} \tanh^5 \left( \frac{1}{2} \right) \right]. \quad (4.18) \]
This yields
\[
\ln \left( \frac{M_{\phi_e} + \sqrt{M_{\phi_e}^2 - m_{\phi_e}^2}}{m_{\phi_e}} \right) = \frac{1}{2} \quad \text{and} \quad \sqrt{M_{\phi_e}^2 - m_{\phi_e}^2} = btanh \left( \frac{1}{2} \right), \quad (4.19)
\]
where we use the identity \( \ln (\sinh(x) + \cosh(x)) = x \). Using (4.18) and (4.19), we get from (4.17)
\[
\begin{align*}
\frac{\partial L_{\phi_e}}{\partial M}(M_{\phi_e}, m_{\phi_e}) &= -4btanh \left( \frac{1}{2} \right), \\
\frac{\partial L_{\phi_e}}{\partial m}(M_{\phi_e}, m_{\phi_e}) &= 0, \\
\frac{\partial^2 L_{\phi_e}}{\partial m^2}(M_{\phi_e}, m_{\phi_e}) &= -4 \tanh \left( \frac{1}{2} \right).
\end{align*}
\quad (4.20)
\]
On the other hand, related to the peakon, \( L_{\phi_e}(M_{\phi_e}, m_{\phi_e}) = 0 \), and hence, to complete the proof of the lemma, it is enough to take \( F_u = F_{\phi_e}, M = M_{\phi_e} \) and \( m = m_{\phi_e} \) in the expressions for the partial derivatives of \( F \) and use (4.18)-(4.20). □

**Lemma 4.5.** [35] It holds that
\[
\max_{x \in \mathbb{S}} |v(x)| \leq \sqrt{\frac{\cosh \left( \frac{1}{2} \right)}{2\sinh \left( \frac{1}{2} \right)}} \|v\|_{H^1(S)}, \quad v \in H^1(S). \quad (4.21)
\]
Moreover, \( \frac{\cosh \left( \frac{1}{2} \right)}{2\sinh \left( \frac{1}{2} \right)} \) is the best constant, and equality holds if and only if \( v = \phi_e(\cdot - \xi) \) for some \( c > 0 \) and \( \xi \in \mathbb{R} \), that is, if and only if \( v \) has the shape of a peakon.

**Remark 6.** The inequality (4.21) implies that the map \( v \mapsto \max_{x \in \mathbb{S}} v(x) \) is continuous from \( H^1(S) \) to \( \mathbb{R} \).

**Lemma 4.6.** [35] If \( u \in C([0,T), H^1(S)) \), then
\[
M_{u(t)} = \max_{x \in \mathbb{S}} u(x,t) \quad \text{and} \quad m_{u(t)} = \min_{x \in \mathbb{S}} u(x,t)
\]
are continuous function of \( t \in [0,T) \).

**Lemma 4.7.** Let \( u \in H^s(S), \quad s > \frac{5}{2} \). Suppose that \( \|u - \phi_e\|_{H^1(S)} < \delta \) for \( 0 < \delta < 1 \). Then
\[
|H_1[u] - H_1[\phi_e]| \leq k_1(b)\delta
\]
and
\[
|H_2[u] - H_2[\phi_e]| \leq k_2(b)\delta,
\]
where \( k_1(b) = \left( 1 + 2b\sqrt{2\tanh \left( \frac{1}{2} \right)} \right), \quad b = \sqrt{\frac{15\cosh^4 \left( \frac{1}{2} \right)}{8\cosh^2 \left( \frac{1}{2} \right) + 4\cosh^2 \left( \frac{1}{2} \right) + 3}}. \)
\[
k_2(b) = \left[ \frac{3k_1(b)}{4\tanh \left( \frac{1}{2} \right)} \right] \left( k_1(b) + 4b^2\tanh \left( \frac{1}{2} \right) \right) \left( k_1(b) + 2b^2\tanh \left( \frac{1}{2} \right) \right) + 3b^4k_1(b) + b^4 \frac{kk_1(b)}{\sqrt{\tanh \left( \frac{1}{2} \right)}} + \frac{\sqrt{2}}{2\tanh \left( \frac{1}{2} \right)} \left( 1 + 2b^2\tanh \left( \frac{1}{2} \right) \right) \left( A + b^8k''^2 \right)^{1/2}
\]
\[
+ \frac{18}{5} \left( A' + b^{10}k''^2 \right)^{1/2},
\]
k, k', k'' are constant and \( A, A' > 0 \) are a constant depending only on the norm \( \|u\|_{H^s(S)} \).
Proof. Using the relation (4.21), we have for any \( v \in H^1(S) \),

\[
\sup_{x \in S} |v(x)| \leq \sqrt{\frac{\cosh \left( \frac{b}{2} \right)}{2 \sinh \left( \frac{b}{2} \right)}} \|v\|_{H^1(S)}.
\] (4.22)

Equality holds if and only if \( v \) is proportional to a translate of \( \phi \). Since \( \|u - \phi_c\|_{H^1} < \delta \), it follows that

\[
|H_1[u] - H_1[\phi_c]| = |(||u||_{H^1} + \|\phi_c\|_{H^1})(||u||_{H^1} - \|\phi_c\|_{H^1})| \\
\leq (||u - \phi_c||_{H^1} + 2\|\phi_c\|_{H^1})\|u - \phi_c\|_{H^1} \\
\leq \delta \left( \delta + 2b \sqrt{2 \tanh \left( \frac{1}{2} \right)} \right) \\
\leq \delta \left( 1 + 2b \sqrt{2 \tanh \left( \frac{1}{2} \right)} \right) := k_1(b)\delta
\]

(4.23)

under the assumption \( 0 < \delta < 1 \). Similarly, we estimate \( |H_2[u] - H_2[\phi_c]| \) as following:

\[
|H_2[u] - H_2[\phi_c]| \\
= \left| \int_S \left( u^6 + 3u^4u_x^2 - u^2u_x^4 + \frac{1}{5}u_x^6 \right) dx \\
- \int_S \left( \phi_c^6 + 3\phi_c^4(\partial_x\phi_c)^2 - \phi_c^2(\partial_x\phi_c)^4 + \frac{1}{5}(\partial_x\phi_c)^6 \right) dx \right| \\
\leq \left| \int_S u^6 dx + \int_S 3u^4u_x^2 dx - \int_S 3u^2u_x^4 dx + \int_S \phi_c^4 dx + \int_S 3\phi_c^2 dx - \int_S 3(\partial_x\phi_c)^2 dx \right| \\
+ \left| \int_S (\partial_x\phi_c)^4 dx + \int_S u^2 dx + \int_S \phi_c^4 dx \right| \\
= J_1 + J_2 + J_3 + J_4 + J_5.
\]

(4.24)

For the term \( J_1 \) and \( J_2 \), we have

\[
J_1 = \left| \int_S (u^4 - \phi_c^4)(u^2 + 3u_x^2) dx \right| \\
\leq 3 \int_S \left( |u^2 + \phi_c^2| \cdot |u + \phi_c| \cdot |u - \phi_c| \cdot (u^2 + u_x^2) dx \right) \\
\leq 3 \left( \|u - \phi_c\|_{L^\infty}^2 + 2\|u - \phi_c\|_{L^\infty} \|\phi_c\|_{L^\infty} + 2\|\phi_c\|_{L^\infty}^2 \right) \cdot (\|u\|_{L^\infty} + \|\phi_c\|_{L^\infty}) \cdot \|u - \phi_c\|_{L^\infty} \int_S (u^2 + u_x^2) dx \\
\leq \frac{3}{4 \tanh^2 \left( \frac{b}{2} \right)} \left( \|u - \phi_c\|_{H^1}^2 + 2\sqrt{2}\|u - \phi_c\|_{H^1} \sqrt{\tanh \left( \frac{1}{2} \right) + 4b^2 \tanh \left( \frac{1}{2} \right)} \right) \\
\cdot \left( \|u - \phi_c\|_{H^1} + 2\sqrt{2}b \sqrt{\tanh \left( \frac{1}{2} \right)} \right) \cdot \|u - \phi_c\|_{H^1} \cdot H_1[u] \\
\leq \frac{3}{4 \tanh^2 \left( \frac{b}{2} \right)} \delta \left( \delta^2 + 2\sqrt{2}\delta b \sqrt{\tanh \left( \frac{1}{2} \right) + 4b^2 \tanh \left( \frac{1}{2} \right)} \right) \\
\left( \delta + 2\sqrt{2}b \sqrt{\tanh \left( \frac{1}{2} \right)} \right)
\]
where we used the fact that
\[
\delta
\]
For the sake of simplicity, we make the following estimates for \( \delta \in (0,1) \),
\[
J_1 + J_2 < \delta \left[ 3 \left( 1 + 2\sqrt{2b} \right) \left( 1 + 2\sqrt{2b} \right) \right]
\]
\[
\cdot \left( 2b^2 \tanh \left( \frac{1}{2} \right) + 1 + 2\sqrt{2b} \right) + 3b^2 \left( 1 + 2\sqrt{2b} \right)
\]
\[
:= \delta k_1 (b) \left[ 3 \left( k_1 (b) + 4b^2 \tanh \left( \frac{1}{2} \right) \right) \right]
\]
\[
\cdot \left( k_1 (b) + 2b^2 \tanh \left( \frac{1}{2} \right) \right) + 3b^4 \right].
\]
(4.27)

On the other hand, for the term \( J_3 \) and \( J_4 \), a direct use of the Hölder’s inequality gives
\[
J_3 = \int_S |(\partial_x \phi_c)^4 (u^2 - \phi_c^2)| \, dx
\]
\[
\leq \left( \|u\|_{L^\infty} + \|\phi_c\|_{L^\infty} \right) \left( \int_S (\partial_x \phi_c)^8 \, dx \right)^{1/2} \left( \int_S (u - \phi_c)^2 \, dx \right)^{1/2}
\]
\[
\leq \frac{1}{\sqrt{2\tanh \left( \frac{1}{2} \right)}} \left( \|u - \phi_c\|_{H^1} + 2\sqrt{2b} \right) \left( 1 + 2\sqrt{2b} \right) \|\partial_x \phi_c\|_{L^8} \|u - \phi_c\|_{H^1}
\]
\[
\leq \frac{1}{\sqrt{2\tanh \left( \frac{1}{2} \right)}} \delta b^4 k \left( \delta + 2\sqrt{2b} \right) \left( 1 + 2\sqrt{2b} \right) < \frac{1}{\sqrt{2\tanh \left( \frac{1}{2} \right)}} \delta b^4 k_1 (b),
\]
(4.28)

where we used the fact that
\[
\|\partial_x \phi_c\|_{L^8}^8
\]
\[
= b^8 \left[ 35 \frac{35}{128} \tanh \left( \frac{1}{2} \right) - 35 \frac{35}{32} \tanh^2 \left( \frac{1}{2} \right) - 377 \frac{35}{192} \tanh^3 \left( \frac{1}{2} \right) + 105 \frac{35}{64} \tanh^4 \left( \frac{1}{2} \right)
\]
\[
+ 577 \frac{377}{384} \tanh^5 \left( \frac{1}{2} \right) - 35 \frac{35}{32} \tanh^6 \left( \frac{1}{2} \right) - 93 \frac{35}{64} \tanh^7 \left( \frac{1}{2} \right) + 35 \frac{35}{128} \tanh^8 \left( \frac{1}{2} \right) \right]
\]
\[
:= b^8 k^2.
\]
By Hölder’s inequality we compute the term $J_4$ as following:

$$J_4 = \int_S \left| u^2(u_x - (\partial_x \phi_c)^4) \right| \,dx$$

$$\leq \|u\|_{L^\infty}^2 \left( \int_S (u_x^2 + (\partial_x \phi_c)^2)^2 (u_x + \partial_x \phi_c)^2 \,dx \right)^{1/2} \left( \int_S (u_x - \partial_x \phi_c)^2 \,dx \right)^{1/2}$$

$$\leq \frac{1}{2tanh\left(\frac{1}{2}\right)} \left( \|u - \phi_c\|_{H^1}^2 + 2b^2 \tanh \left(\frac{1}{2}\right) \right) \times \left( \int_S (u_x^2 + (\partial_x \phi_c)^2)^2 (u_x + \partial_x \phi_c)^2 \,dx \right)^{1/2} \|u - \phi_c\|_{H^1}.$$

Using the Young’s inequality we estimate

$$\int_S (u_x^2 + (\partial_x \phi_c)^2)^2 (u_x + \partial_x \phi_c)^2 \,dx$$

$$= \int_S (u_x^6 + 2u_x^2 \partial_x \phi_c + 3u_c^4(\partial_x \phi_c)^2 + 4u_c^3(\partial_x \phi_c)^3$$

$$+ 3u_c^2(\partial_x \phi_c)^4 + 2u_c(\partial_x \phi_c)^5 + (\partial_x \phi_c)^6) \,dx$$

$$\leq 8 \left( \int_S u_x^6 \,dx + \int_S (\partial_x \phi_c)^6 \,dx \right).$$

Since $u \in H^s(S) \subset H^2(S)$, $s > 5/2$, $\|u\|_{L^6}$ is bounded by $\|u\|_{H^s(S)}$ because of the following Gagliardo-Nirenberg inequality:

$$\|u_x\|_{L^6} \leq C \|u_{xx}\|^{2/3}_{L^2} \|u\|^{1/3}_{L^2}$$

with constant $C > 0$ independent of $u$. Hence, it follows from

$$\|\partial_x \phi_c\|_{L^6}^6$$

$$= b^6 \left[ -\frac{15}{16} + \frac{5}{8} \tanh \left(\frac{1}{2}\right) + \frac{15}{16} \tanh^2 \left(\frac{1}{2}\right) - \frac{5}{3} \tanh^3 \left(\frac{1}{2}\right)$$

$$- \frac{15}{16} \tanh^4 \left(\frac{1}{2}\right) + \frac{11}{8} \tanh^5 \left(\frac{1}{2}\right) + \frac{15}{16} \tanh^6 \left(\frac{1}{2}\right) \right] := b^6 k^{r/2}$$

that

$$J_4 < \frac{\sqrt{2^{s}}}{2tanh\left(\frac{1}{2}\right)} \delta \left( 1 + 2b^2 \tanh \left(\frac{1}{2}\right) \right) \left( C^6 \|u_{xx}\|_{L^2}^4 \|u\|_{L^2}^2 + b^6 k^{r/2} \right)^{1/2}$$

$$< \frac{\sqrt{2^{s}}}{2tanh\left(\frac{1}{2}\right)} \delta \left( 1 + 2b^2 \tanh \left(\frac{1}{2}\right) \right) \left( A(\|u\|_{H^s}) + b^6 k^{r/2} \right)^{1/2}, \quad (4.29)$$

where the constant $A(\|u\|_{H^s}) > 0$ depends only on the norm $\|u\|_{H^s}$. Applying the similar method to treat $J_5$, by the fact that

$$\|\partial_x \phi_c\|_{L^{10}}^{10}$$

$$= b^{10} \left[ -\frac{23}{256} + \frac{297}{320} \tanh \left(\frac{1}{2}\right) + \frac{115}{256} \tanh^2 \left(\frac{1}{2}\right) - \frac{349}{1280} \tanh^3 \left(\frac{1}{2}\right) - \frac{115}{128} \tanh^4 \left(\frac{1}{2}\right)$$

$$+ \frac{507}{128} \tanh^5 \left(\frac{1}{2}\right) + \frac{115}{128} \tanh^6 \left(\frac{1}{2}\right) - \frac{587}{160} \tanh^7 \left(\frac{1}{2}\right) - \frac{115}{256} \tanh^8 \left(\frac{1}{2}\right)$$

$$+ \frac{395}{256} \tanh^9 \left(\frac{1}{2}\right) + \frac{23}{256} \tanh^{10} \left(\frac{1}{2}\right) \right] := b^{10} k^{r/2},$$

...
we obtain
\[ J_5 = \frac{1}{5} \int_S \left| u_x^6 - (\partial_x \phi_c)^6 \right| \, dx \]
\[ \leq \frac{1}{5} \left( \int_S \left( u_x^2 + (\partial_x \phi_c)^2 \right)^2 \left( u_x^2 + u_x \partial_x \phi_c + (\partial_x \phi_c)^2 \right) \, dx \right)^{1/2} \left( \int_S (u_x - (\partial_x \phi_c))^2 \, dx \right)^{1/2} \]
\[ \leq \frac{18}{5} \left( \int_S u_x^2 \, dx + \int_S (\partial_x \phi_c)^2 \, dx \right)^{1/2} \| u - \phi_c \|_{H^1} \]
\[ \leq \frac{18}{5} \delta \left( C^{10} \| u_{xxx} \|_{L^2} \| u \|_{L^2} + b^{10} k^{r/2} \right)^{1/2} \]
\[ < \frac{18}{5} \delta \left( A'(\| u \|_{H^s}) + b^{10} k^{r/2} \right)^{1/2}, \] (4.30)
where the constant \( A'(\| u \|_{H^s}) > 0 \) depends only on the norm \( \| u \|_{H^s} \).

In view of (4.27)-(4.30), we conclude that
\[ |H_2[u] - H_2[\phi_c]| \leq J_1 + J_2 + J_3 + J_4 + J_5 \leq k_2(b) \delta, \]
which completes the proof of the lemma. \( \square \)

**Remark 7.** Lemma 4.7 shows that the functionals \( H_1, H_2 : H^s(S) \to \mathbb{R}, \ s > \frac{5}{2} \)
are continuous with respect to the \( H^1(S) \)-norm.

**Lemma 4.8.** Let \( u \in C([0, T), H^s(S)), \ s > \frac{5}{2}, \) be a solution of the generalized modified Camassa-Holm equation (2.1). Given a small neighborhood \( U \) of \( (M_{\phi_c}, m_{\phi_c}) \in \mathbb{R}^2, \)
there is a \( \delta > 0 \) such that for \( t \in [0, T), \)
\[ (M_{u(t)}, m_{u(t)}) \in U, \ \text{if} \ \| u(\cdot, 0) - \phi_c \|_{H^s(S)} < \delta. \] (4.31)

**Proof.** Since Lemma 4.7, we may suppose \( w \in H^s(S), \ s > 5/2, \) is a small perturbation of \( \phi_c \) in the \( H^1(S) \)-norm such that \( H_1[w] = H_1[\phi_c] + \epsilon_1 \) and \( H_2[w] = H_2[\phi_c] + \epsilon_2. \) Then we have
\[ F_w(M, m) = F_{\phi_c}(M, m) + \left( \frac{8}{5} M^4 + \frac{4}{5} M^2 m^2 + \frac{3}{5} m^4 \right) \epsilon_1 - \epsilon_2. \]
This implies \( F_w \) is a small perturbation of \( F_{\phi_c}. \) By choosing \( \epsilon_1 \) and \( \epsilon_2 \) arbitrarily small, we can make the effect of the perturbation near the point \( (M_{\phi_c}, m_{\phi_c}). \) From Lemma 4.4 we know that \( F_{\phi_c}(M_{\phi_c}, m_{\phi_c}) = 0 \) and \( F_{\phi_c} \) has a critical point with negative definite second derivative at \( (M_{\phi_c}, m_{\phi_c}). \) By continuity of the second derivative, there is a neighborhood around \( (M_{\phi_c}, m_{\phi_c}), \) where \( F_{\phi_c} \) is concave with curvature bounded away from zero. Hence, after a small perturbation, the set \( F_{\phi_c} \geq 0 \) near \( (M_{\phi_c}, m_{\phi_c}) \) will be contained in a neighborhood of \( (M_{\phi_c}, m_{\phi_c}). \)

Let \( U \) be the neighborhood given as in the statement of lemma. Shrinking \( U \) if necessary, we infer that there exists a \( \delta' > 0 \) such that for \( u \in C([0, T), H^s(S), s > 5/2 \) with
\[ |H_1[u] - H_1[\phi_c]| < \delta' \] and
\[ |H_2[u] - H_2[\phi_c]| < \delta', \] (4.32)
which reveals that the set where \( F_{\phi_c} \geq 0 \) near \( (M_{\phi_c}, m_{\phi_c}) \) is contained in \( U \) for each \( t \in [0, T) \) and \( U \) is surrounded by a set where \( F_{\phi_c} < 0. \) It follows from Lemma 4.3 and Lemma 4.6 that \( M_{u(t)} \) and \( m_{u(t)} \) are continuous functions of \( t \in [0, T) \) and \( F_{u(t)}(M_{u(t)}, m_{u(t)}) \geq 0 \) for \( t \in [0, T). \) Thus, for \( u \) satisfying (4.30), we have
\[ (M_{u(t)}, m_{u(t)}) \in U \quad \text{for} \quad t \in [0, T) \quad \text{if} \quad (M_{u(0)}, m_{u(0)}) \in U \] (4.33)
However, the conserved functionals $H_1, H_2 : H^s(S) \to \mathbb{R}, \ s > 5/2$ are continuous with respect to the $H^1(S)$-norm. We conclude that there is a $\delta > 0$ such that (4.30) holds for all $u$ with 
$$
\|u(\cdot, 0) - \phi_c\|_{H^1(S)} < \delta.
$$
Moreover, using Lemma 4.5, choosing a smaller $\delta$ if necessary, we may also assume that 
$$(M_{u(0)}, m_{u(0)}) \in \mathcal{U} \quad \text{if} \quad \|u(\cdot, 0) - \phi_c\|_{H^1(S)} < \delta,$$
which completes the proof of lemma.

We are now in a position to carry out the proof of Theorem 4.1 with all the preparations given above.

**Proof of Theorem 4.1.** Let $u \in C([0, T), H^s(S)), \ s > 5/2$ be a periodic solution of the generalized modified Camassa-Holm equation (2.1) and $\epsilon > 0$ be arbitrary. Choose a neighborhood $\mathcal{U}$ of $(M_{\phi_c}, m_{\phi_c})$ sufficiently small such that 
$$
|M - M_{\phi_c}| < \frac{\epsilon^2}{8b\tanh \left(\frac{1}{2}\right)}, \quad \text{if} \quad (M, m) \in \mathcal{U}.
$$
Let us choose a $\delta$ as in Lemma 4.8 so that (4.29) holds for $t \in [0, T)$. By taking a smaller $\delta$ if necessary, we may also assume that 
$$
|H_1[u] - H_1[\phi_c]| < \frac{\epsilon^2}{2} \quad \text{if} \quad \|u(\cdot, 0) - \phi_c\|_{H^1(S)} < \delta.
$$
Using Lemma 4.2, we conclude that for $t \in [0, T)$,
$$
\|u(\cdot, t) - \phi_c(\cdot, -\xi(t))\|_{H^1(S)}^2 = H_1[u] - H_1[\phi_c] - 4b\tanh \left(\frac{1}{2}\right) (u(\xi(t), t) - M_{\phi_c})
\leq |H_1[u] - H_1[\phi_c]| + 4b\tanh \left(\frac{1}{2}\right) |M_{u(t)} - M_{\phi_c}|
< \epsilon^2,
$$
where $\xi(t) \in S$ is any point where $u(\xi(t), t) = M_{u(t)}$. This completes the proof of Theorem 4.1.

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