Negotiation as Concurrency Primitive

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This paper introduces negotiations, a model of concurrency close to Petri nets, with multi-party negotiations as concurrency primitive. We study two fundamental analysis problems. The soundness problem consists of deciding if it is always possible for a negotiation to terminate successfully, whatever the current state is. Given a sound negotiation, the summarization problem aims at computing an equivalent one-step negotiation with the same input/output behavior. The soundness and summarization problems can be solved by means of simple algorithms acting on the state space of the negotiation, which however face the well-known state explosion problem. We study alternative algorithms that avoid the construction of the state space. In particular, we define reduction rules that simplify a negotiation while preserving the sound/non-sound character of the negotiation and its summary. In a first result we show that our rules are complete for the class of weakly deterministic acyclic negotiations, meaning that they reduce all sound negotiations in this class, and only them, to equivalent one-step negotiations. This provides algorithms for both the soundness and summarization problems that avoid the construction of the state space. We then study the class of deterministic negotiations. Our second main result shows that the rules are also complete for this class, even if the negotiation contains cycles. Moreover, we present an algorithm that completely reduces all sound deterministic negotiations, and only them, in polynomial time.

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1. Introduction

Negotiation has long been identified as a paradigm for process interaction [DS83]. It has been applied to different problems (see e.g. [WSJ00, AAS00]), and studied on its own [JFL+01]. The purpose of this paper is to initiate the study of negotiations from a concurrency-theoretic, observational point of view. Observationally, a negotiation is an interaction in which several partners come together to agree on one out of a number of possible outcomes (a synchronized nondeterministic choice). While such an interaction can be modelled in any standard process algebra as a combination of parallel composition and nondeterministic choice, or as a small Petri net, we argue that much can be gained by studying formal models with atomic negotiation as concurrency primitive. In particular, we show that the negotiation point of view reveals new classes of systems with polynomial analysis algorithms.

Atomic negotiations can be combined into distributed negotiations. For example, a distributed negotiation between a buyer, a seller, and a broker consists of one or more rounds of atomic negotiations involving the buyer and the broker, or the seller and the broker, followed by a concluding atomic negotiation between the buyer and the seller.

We introduce a formal model for distributed negotiations, inspired by van der Aalst’s workflow Petri nets [dA98]. A negotiation atom, or just atom, involves a set of parties (for instance, buyer and broker), and has a set of possible outcomes (for example, buy and sell). Each party has a set of possible internal states, and each outcome has associated a state-transformer; if the parties agree on a given outcome, then its associated transformer determines the possible “exit states” of the parties after executing the atom as a function of their “entry states” before executing the atom. Atoms are combined into distributed negotiations, by means of a next-atoms function that determines for each atom, each party, and each outcome, the set of atoms the party is ready to engage in next if the atomic negotiation ends with that outcome. We assume that a distributed negotiation always starts with an initial atom and ends with a final atom, both of which involve all parties of the complete distributed negotiation.

Like workflow nets, distributed negotiations can be unsound because of deadlocks or livelocks. The soundness problem consists of deciding if a given negotiation is sound. Further, a sound negotiation is equivalent to one single atom whose state-transformer determines the possible final internal states of all parties as a function of their initial internal states. The summarization problem consists of computing such an atomic negotiation, called a summary. The set of reachable global states can be finite or infinite because each party can have infinitely many internal states. If it is finite, then the soundness and summarization problems can be solved by means of well-known algorithms based on the exhaustive exploration of the state space. However, this approach badly suffers from the state-explosion problem: the state-space of distributed negotiations grows exponentially in the size of the negotiation itself, even if all atoms had only one single local state.

In this paper we first show that the state-explosion problem cannot be avoided in full generality, because the soundness problem is PSPACE-complete for arbitrary negotiations, and a decision problem related to the summarization problem is PSPACE-complete.
even for negotiations in which each party has only two internal states. We then provide reduction algorithms for the soundness and summarization problems. Reduction algorithms exhaustively apply syntactic reduction rules that simplify the system while preserving some aspects of the behavior, like absence of deadlocks. The rules are exhaustively applied, until the system is either a trivial one, or a non-trivial system, but with a smaller state space. This approach has been extensively applied to Petri nets and workflow nets, but most of this work has been devoted to the liveness or soundness problems [Ber86, Had88, HPP06]. For these problems many reduction rules are known, and they have been proved complete for certain classes of systems [GT84, Des92, DE95], meaning that they reduce all live or sound systems in the class, and only those, to a trivial system (in our case, to a single atom). However, many of these rules, like the linear dependency rule of [DE95], cannot be applied to the summarization problem, because they do not preserve the summary, only the soundness property.

We provide a solution to the summarization problem for deterministic negotiations, based on a complete set of reduction rules. In deterministic negotiations all involved agents are deterministic, meaning that they are never ready to engage in more than one atomic negotiation. We provide a reduction strategy guaranteeing that a sound deterministic negotiation will be summarized by means of a polynomial number of applications of the rules.

Intuitively, nondeterministic agents may be ready to engage in several atomic negotiations, and which one takes place is decided by the deterministic parties, which play thus the role of negotiation leaders. We also introduce weakly deterministic negotiations (roughly speaking, in weakly deterministic negotiations, each atomic negotiation has a deterministic party), and provide a complete set of reduction rules for acyclic weakly deterministic negotiations.

This paper is the definitive version of the results presented in [ED13, ED14]. It contains detailed proofs of all theorems. In particular, it corrects an error of the algorithm of [ED14], which in a certain special case might not terminate. We also conduct a better complexity analysis, leading to an algorithm of cubic complexity, instead of the $O(n^4)$ analysis of [ED14].

**Related work.** Specific distributed negotiation protocols have been modeled with the help of Petri nets or process calculi (see e.g. [SFC04, JTL05, BMMvdA11, Cap12]). However, these papers do not address the issue of negotiation as concurrency primitive.

The reduction rules introduced in Petri net theory by Berthelot and Haddad [Ber86, Had88] have more recently been applied to the analysis of workflow nets in [DvdAV05, VWvdAtH10]. These works do not address completeness issues. Complete rule sets have been proposed for several Petri net classes [GT84, Des92, DE95]. However, the rules only preserve a property close to soundness, and so cannot be used for summarization.

**Applications.** In work published after [ED13, ED14] we have applied the reduction algorithms of this paper to the analysis of workflow Petri nets. Deterministic negotiations are very tightly related to free-choice workflow nets [DE15], and the reduction algorithms
can be adapted. In [EH16] we report on experimental results on the analysis of a collection of almost 2000 workflows from industrial sources.

**Structure.** The paper is structured as follows. Section 2 introduces the syntax and semantics of negotiations, and the classes of deterministic and weakly deterministic negotiations. Section 3 presents the soundness and summarization problems, and analyzes their computational complexity. Section 4 introduces our set of reduction rules. The rest of the paper presents completeness and complexity results of our set of rules. Section 5 studies acyclic negotiations. It shows that our rules summarize all sound and acyclic weakly-deterministic negotiations, and presents a simple reduction strategy that reduces all acyclic sound deterministic negotiations in polynomial time. Sections 6 and 7 present the main result of the paper: a polynomial reduction algorithm for arbitrary deterministic negotiations. Several lengthy proofs are contained in a number of appendices.

### 2. Negotiations: Syntax and Semantics

We fix a finite set $A$ of *agents* representing potential parties of negotiations. Each agent $p \in A$ has a (possibly infinite) nonempty set $Q_p$ of *internal states*. We denote by $Q_A = \prod_{p \in A} Q_p$, for a subset $P \subseteq A$ we similarly define $Q_P = \prod_{p \in P} Q_p$.

A *transformer* is a left-total relation $\tau \subseteq Q_A \times Q_A$, representing a nondeterministic state transforming function. Given $P \subseteq A$, we say that a transformer $\tau$ is a $P$-transformer if, for each $p_i \notin P$, $\left\{(q_{p_1},\ldots,q_{p_i},\ldots,q_{p_{|A|}}),(q'_{p_1},\ldots,q'_{p_i},\ldots,q'_{|A|})\right\} \in \tau$ implies $q_{p_i} = q'_{p_i}$. This means that a $P$-transformer only transforms the internal states of agents in $P$.

**Definition 1.** A negotiation atom, or just an atom, is a triple $n = (P_n, R_n, \delta_n)$, where $P_n \subseteq A$ is a nonempty set of parties, $R_n$ is a finite, nonempty set of results, and $\delta_n$ is a mapping assigning to each result $r$ in $R_n$ a $P_n$-transformer $\delta_n(r)$. For each result $r \in R_n$, we call the pair $(n,r)$ an outcome.

Intuitively, if the respective states of the agents before an atom $n$ are given by a tuple $q$ and the result of the negotiation is $r$, then the agents change their states to another tuple $q'$ for some $(q,q') \in \delta_n(r)$.

For a simple example, consider a negotiation atom $n$ with parties $F$ (Father) and $D$ (teenage Daughter). The goal of the negotiation is to determine whether $D$ can go to a party, and the time at which she must return home. The possible results are \{yes, no, ask mother\}. Both sets $Q_F$ and $Q_D$ contain a state *angry* plus a state $t$ for every time $T_1 \leq t \leq T_2$ in a given interval $[T_1,T_2]$ (we assume that the time values are appropriately ordered, choose for example $\{9,10,11,12\}$). Initially, $F$ is in state $t_f$ and $D$ in state $t_d$.

If the state of $F$ or of $D$ is *angry*, then both will be *angry* after negotiating, no matter what the result of the negotiation atom is. The remaining transformations of the mapping $\delta_n$ are given by
\[\delta_n(\text{yes}) \supset \{(t_f, t_d), (t, t) \mid t_f \leq t \leq t_d \lor t_d \leq t \leq t_f\}\]

\[\delta_n(\text{no}) \supset \{(t_f, t_d), (\text{angry, angry})\}\]

\[\delta_n(\text{ask\_mother}) \supset \{(t_f, t_d), (t_f, t_d)\}\]

That is, if the result is yes, then F and D agree on a time \(t\) which is not earlier and not later than both suggested times. If it is no, then there is a quarrel and both parties get angry. If the result is ask\_mother, then the parties keep their previous times.

### 2.1. Combining Atomic Negotiations

A negotiation is a composition of atoms. We add a transition function \(X\) that assigns to every triple \((n, p, r)\) consisting of an atom \(n\), a participant \(p\) of \(n\), and a result \(r\) of \(n\) a set \(X(n, p, r)\) of atoms. Intuitively, this is the set of atomic negotiations agent \(p\) is ready to engage in after the atom \(n\), if the result of \(n\) is \(r\).

Negotiations can be graphically represented as shown in Figure 1 (ignore the black dots on the arrows for now). For each atom \(n \in N\) we draw a black bar; for each party \(p \in P_n\) we draw a white circle on the bar, called a port. For each triple \((n, p, r)\), we draw a hyperarc leading from the port of \(p\) in \(n\) to all the ports of \(p\) in the atoms of \(X(n, p, r)\), and label it by \(r\). (If \(X(n, p, r)\) contains only one atom, then the hyperarc is actually an arc.) If \(X(n, p, r) = X(n, p, r')\), i.e., if the next possible atoms of an agent \(p\) after the negotiation atom \(n\) are the same for two results \(r\) and \(r'\), then we draw a single hyperarc and label it by \(r\) and \(r'\), and similarly for more than two results. In later examples, we omit the labels whenever we are only interested in the structure of the negotiation.

![Figure 1: Two negotiations between three agents.](image)

Figure 1 shows two Father-Daughter-Mother negotiations, omitting, for the sake of brevity, the state transformers associated to the results. On the left, after an initial atomic negotiation \(n_0\) involving all three agents, Daughter and Father negotiate (atom \(n_1\)) with possible results yes (\(y\)), no (\(n\)), and ask\_mother (\(am\)). If the result is ask\_mother, then Daughter and Mother negotiate (atom \(n_2\)) with possible results yes and no. The negotiation ends with a final atom \(n_f\).
In the negotiation on the right, Father, Daughter and Mother initially negotiate with possible results yes and no. If the result is yes, then Father and Daughter negotiate a time for the Daughter to return home (atom \( n_1 \)) and propose it to Mother (atom \( n_2 \)). If Mother approves (result yes), then the negotiation terminates with \( n_f \), otherwise (result \( r \)) Daughter and Father renegotiate the time.

Before defining negotiations, we need a more general structure that we call pre-negotiation:

**Definition 2.** Given a finite set of atoms \( N \), let \( T(N) \) denote the set of triples \( (n, p, r) \) such that \( n \in N \), \( p \in P_n \), and \( r \in R_n \). A pre-negotiation is a tuple \( N = (N, n_0, n_f, X) \), where \( n_0, n_f \in N \) are the initial atom and the final atom, and \( X: T(N) \to 2^N \) is the transition function.

Abstracting from ports leads to the notion of graph of a pre-negotiation:

**Definition 3.** Let \( N = (N, n_0, n_f, X) \) be a pre-negotiation. The graph of \( N \) is the directed graph with \( N \) as set of vertices and an edge from \( n \) to \( n' \) labeled by \((p, r)\) if \( n' \in X(n, p, r) \).

A path of \( N \) is a sequence \((n_1, p_1, r_1)(n_2, p_2, r_2)\cdots(n_k, p_k, r_k)\) such that, for \( 1 \leq i \leq k - 1 \), \( n_{i+1} \in X(n_i, p_i, r_i) \). If \( n_1 \in X(n_k, p_k, r_k) \) then the path is also a cycle.

\( N \) is acyclic if its graph has no cycles, otherwise it is cyclic.

The graphs of the negotiations in Figure 1 are shown in Figure 2. In the graph of the negotiation on the right we have omitted the labels. The negotiation on the left of Figure 1 is acyclic, while the one the right is cyclic. Observe that in acyclic pre-negotiations and negotiations every agent can reach an atom at most once.

Now we are ready to formally define negotiations.

**Definition 4.** A pre-negotiation \( N = (N, n_0, n_f, X) \) is a negotiation if it satisfies the following properties:

1. every agent of \( A \) participates in both \( n_0 \) and \( n_f \) (i.e., \( P_{n_0} = P_{n_f} = A \));
(2) for every \((n, p, \mathbf{r}) \in T(N)\): \(\mathcal{X}(n, p, \mathbf{r}) = \emptyset\) iff \(n = n_f\).
(3) for every atom \(n \in N\) there is a path \((n_1, p_1, \mathbf{r}_1) \cdots (n_k, p_k, \mathbf{r}_k)\) such that \(n_0 = n_1, n_f = n_k,\) and \(n = n_i\) for some \(1 \leq i \leq k\).

We call a result \(\mathbf{r}\) of an atom \(n\) final result for party \(p\) if \(\mathcal{X}(n, p, \mathbf{r}) = \emptyset\). Condition (2) of the above definition states that only the final atom has final results and that all results of the final atom are final results for all its parties, which is – by (1) – the set of all agents.

If \(|N| = 1\) then the negotiation consists of a single atom, which is both the initial atom \(n_0\) and the final atom \(n_f\); further, in this case every result is final.

If \(|N| > 1\) then \(n_0\) and \(n_f\) are distinct; otherwise, in the graph of \(N\) the vertex \(n_0\) (\(= n_f\)) has no successor and so no other vertex can be on a path starting with \(n_0\).

Figure 1 shows two negotiations. The initial atoms have no ingoing arc (which is not necessarily the case in general), whereas the final atoms have no outgoing arcs by definition. Therefore, there is no graphical representation of the final results. Since \(\mathcal{X}(n_f, p, \mathbf{r}) = \emptyset\) holds for every final result \(\mathbf{r}\) and every agent \(p\), any visual representation holds no further information other than the existence of that result.

Condition (3) of the definition of negotiations is inspired by a corresponding property of workflow Petri nets [dA98]. Paths of the graph represent possible subsequent atomic negotiations of an agent. Therefore, if for some vertex there was no path from the initial vertex (representing the initial atom) to that vertex, then the corresponding atom can never take place (remember that all participants of all atoms participate in the initial atom). As will become clear in the following subsection, we are mainly interested in distributed negotiations which can always eventually terminate (represented by the final atom); hence the requirement of a path from each vertex to the vertex representing the final atom (which also involves all agents).

### 2.2. Semantics

A marking of a negotiation \(N = (N, n_0, n_f, \mathcal{X})\) is a mapping \(\mathbf{x}: A \rightarrow 2^N\). Intuitively, \(\mathbf{x}(p)\) is the set of atoms that agent \(p\) is currently ready to engage in next. The initial and final markings, denoted by \(\mathbf{x}_0\) and \(\mathbf{x}_f\) respectively, are given by \(\mathbf{x}_0(p) = \{ n_0 \}\) and \(\mathbf{x}_f(p) = \emptyset\) for every \(p \in A\).

A marking \(\mathbf{x}\) enables an atom \(n\) if \(n \in \mathbf{x}(p)\) for every \(p \in P_n\), i.e., if every party of \(n\) is currently ready to engage in it. If \(\mathbf{x}\) enables \(n\), then \(n\) can take place and its parties agree on a result \(\mathbf{r}\); we say that the outcome \((n, \mathbf{r})\) occurs. The occurrence of \((n, \mathbf{r})\) produces a next marking \(\mathbf{x}'\) given by \(\mathbf{x}'(p) = \mathcal{X}(n, p, \mathbf{r})\) for every \(p \in P_n\), and \(\mathbf{x}'(p) = \mathbf{x}(p)\) for every \(p \in A \setminus P_n\). We write \(\mathbf{x} \xrightarrow{(n, \mathbf{r})} \mathbf{x}'\) to denote this, and call it a small step.

We write \(\mathbf{x}_1 \xrightarrow{\sigma} \mathbf{x}_{k+1}\) to denote that there is a sequence

\[
\mathbf{x}_1 \xrightarrow{(n_1, \mathbf{r}_1)} \mathbf{x}_2 \xrightarrow{(n_2, \mathbf{r}_2)} \cdots \xrightarrow{(n_{k-1}, \mathbf{r}_{k-1})} \mathbf{x}_k \xrightarrow{(n_k, \mathbf{r}_k)} \mathbf{x}_{k+1}\]

of small steps such that \(\sigma = (n_1, \mathbf{r}_1) \cdots (n_k, \mathbf{r}_k)\). We call \(\sigma\) an occurrence sequence from marking \(\mathbf{x}_1\). If \(\sigma\) is finite and ends with \((n_k, \mathbf{r}_k)\), then we write \(\mathbf{x}_1 \xrightarrow{\sigma} \mathbf{x}_{k+1}\) and
say that $x_{k+1}$ is reachable from $x_1$. If $x_1$ is the initial marking, then we call $\sigma$ an initial occurrence sequence. If moreover $x_{k+1}$ is the final marking, then $\sigma$ is a large step.

Given an agent $p$, we always have that either $x(p) = \{n_0\}$ or $x(p) = X(n,p,r)$ for some atom $n$ and result $r$. The marking $x_f$ can only be reached by the occurrence of an outcome $(n_f,r)$, where $n_f$ is the final atom and thus $r$ is a final result. It does not enable any atom. Any other marking that does not enable any atom is a deadlock. A marking which is reachable from itself but from which the final marking is not reachable is a livelock.

Reachable markings are graphically represented by placing tokens (black dots) on the forking points of the hyperarcs (or, if the hyperarc consists of just one arc, in the middle of the arc). Figure 1 shows on the left a marking in which all agents are ready to engage in $n_f$ and $M$ moreover is ready to engage in $n_2$. So the only enabled outcomes are $(n_f,r_f)$, where $r_f$ is a final result. The figure on the right shoes a marking in which $F$ and $D$ are ready to engage in $n_1$ and $M$ is ready to engage in $n_2$. Since $n_1$ involves only $F$ and $D$ and has only the result $tm$, the only enabled outcome is $(n_2, tm)$.

### 2.3. Determinism and Weak Determinism

We introduce deterministic and weakly deterministic negotiations. Intuitively, an agent of a negotiation is deterministic if it is never ready to engage in more than one atom; graphically, an agent is deterministic if all hyperarcs of the agent are normal arcs. Formally:

**Definition 5.** An agent $p \in A$ is deterministic if for every $(n,p,r) \in T(N)$ such that $n \neq n_f$ there exists one atom $n'$ such that $X(n,p,r) = \{n'\}$.

Consider again Figure 1. In the negotiation on the left, Father and Daughter are deterministic agents, but Mother is not, because she has a proper hyperarc from $n_0$ to $n_2$ and $n_f$. After $n_0$, Mother is ready to engage in the atoms $n_2$ and $n_f$.

The fundamental property of deterministic agents is that, loosely speaking, they “force” atoms to occur. Assume that a deterministic agent is currently ready to engage in atom $n$. Then the agent is not ready to engage in any other atom, and the only way to change this is by executing $n$. Therefore, any extension of the current execution to a large step must necessarily execute $n$.

**Definition 6.** A negotiation is deterministic if all its agents are deterministic.

A negotiation is weakly deterministic if for every $(n,p,r) \in T(N)$ there is a deterministic agent $b$ that is a party of every atom in $X(n,p,r)$, i.e., $b \in P_{n'}$ for every $n' \in X(n,p,r)$.

Observe that every deterministic negotiation is also weakly deterministic. Consider again Figure 1. The right negotiation is deterministic, while the left negotiation is not, as Mother has a proper hyperarc from $n_0$ to $n_2$ and $n_f$. However, the left negotiation is weakly deterministic, as Daughter is deterministic and participates in every atom.
3. Analysis Problems

We introduce a notion of well-behavedness of a negotiation which we call soundness. It is similar to the soundness property of workflow Petri nets [dA98].

Definition 7. A negotiation is sound if
(a) every atom is enabled at some reachable marking, and
(b) every occurrence sequence from the initial marking is either a large step or can be extended to a large step.

Intuitively, (a) captures that there are no useless atoms, and (b) that the negotiation can never reach a state from which it cannot terminate. In particular, sound negotiations can reach neither a deadlock nor a livelock.

The negotiations of Figure 1 are sound. However, if we set in the left negotiation $X(n_0, M, \text{st}) = \{ n_2 \}$ instead of $X(n_0, M, \text{st}) = \{ n_2, n_f \}$, then the occurrence sequence $(n_0, \text{st})(n_1, \text{yes})$ leads to a deadlock.

We now introduce the notion of summary transformer, and summary of a negotiation. Intuitively, the summary transformer gives for each final outcome $(n_f, r)$ and for each tuple of initial states $q_0$ the possible resulting tuples of states, after the negotiation finishes with the result $r$.

Definition 8. Given a negotiation $N = (N, n_0, n_f, X)$, we attach to each final result $r$ a summary transformer $\langle N, r \rangle \subseteq Q_A \times Q_A$ as follows.

Given two transformers $\tau_1, \tau_2 \subseteq Q_A \times Q_A$, we define their concatenation as the transformer

$$\tau_1 \tau_2 = \{ (q, q') \in Q_A \times Q_A \mid (q, q'') \in \tau_1 \text{ and } (q'', q') \in \tau_2 \text{ for some } q'' \in Q_A \}$$

For every finite occurrence sequence $\sigma = (n_1, r_1)(n_2, r_2)\ldots(n_k, r_k)$ of $N$, define $\langle \sigma \rangle = \delta_{n_1}(r_1) \delta_{n_2}(r_2)\ldots\delta_{n_k}(r_k)$. Let $L_r$ be the set of large steps of $N$ that end with $(n_f, r)$. We define $\langle N, r \rangle = \bigcup_{\sigma \in L_r} \langle \sigma \rangle$.

Finally, we introduce the notion of equivalent negotiations and summary.

Definition 9. Two negotiations $N_1$ and $N_2$ over the same set of agents are equivalent ($N_1 \equiv N_2$) if

1. they are either both sound, or both unsound;
2. they have the same final results; and
3. $\langle N_1, r \rangle = \langle N_2, r \rangle$ for every final result $r$.

If $N_1$ and $N_2$ are equivalent and $N_2$ consists of a single atom, then $N_2$ is the summary of $N_1$.

We collect some easy consequences of the definition.
A negotiation has a summary iff it is sound. Indeed, if \( N \) is sound, then the negotiation with only one atom, all final results of \( N \) as results of this atom, and with transformers \( \langle N, \mathbf{r} \rangle \) for each of these final results \( \mathbf{r} \) is a summary. Conversely, if \( N \) has a summary \( N' \), then, since negotiations consisting of one single atom are sound, \( N' \) is sound. Since \( N \equiv N' \), by condition (1) \( N \) is sound.\(^1\)

Summaries are unique up to the identity of the single atom.

A negotiation with one single atom \( n \) is completely determined by the transformers of \( n \) (observe that if two atoms have the same transformers then they necessarily have the same sets of agents and results). So two one-atom negotiations whose atoms have the same transformers are equivalent, although the two single atoms need not be identical.

Equivalence of negotiations is a congruence with respect to substitution: if in a negotiation \( N \) we replace a subnegotiation \( M \) by an equivalent negotiation \( M' \), then the resulting negotiation \( N' \) is equivalent to \( N \).

The formal definitions of subnegotiation and “replacing a negotiation by an equivalent one” are the expected ones, and the proof is easy but laborious, and we omit it. Without condition (2) in Definition 9 the congruence property does not hold. Indeed, without condition (2) the notion of substituting a negotiation by an equivalent one is not even well defined.

While Definition 9 preserves soundness in a way that makes substitutions possible, there are behavioral aspects that are not preserved under equivalence. In particular, a negotiation with infinite behaviors may be equivalent to a negotiation having none.

### 3.1. Deciding soundness

The soundness problem consists of deciding if a given negotiation is sound. It can be solved with the help of the reachability graph.

The reachability graph of a negotiation \( N \) has all markings reachable from \( x_0 \) as vertices, and an edge from \( x \) to \( x' \) whenever \( x \xrightarrow{(n,\mathbf{r})} x' \). To decide soundness we can (1) compute the reachability graph of \( N \) and (2a) check that every atom appears at some arc, and (2b) that, for every reachable marking \( x \), there is an occurrence sequence \( \sigma \) such that \( x \xrightarrow{\sigma} x_f \).

Step (1) needs exponential time in the number of atoms, and steps (2a) and (2b) are polynomial in the size of the reachability graph. So the algorithm is single exponential in the number of atoms. Appendix A shows that this cannot be easily avoided, because the problem is PSPACE-complete, and co-NP-hard and in DP for acyclic negotiations.

\(^1\) Alternatively, we could also define summaries of unsound negotiations by introducing unreliable atoms that, intuitively, may “get stuck”. The summary of an unsound negotiation would then be an unreliable atom with the same summary transformers as the original negotiation. In this paper we do not further investigate this possibility.
(a language \( L \) is in the class DP if there exist languages \( L_1 \) in NP and \( L_2 \) in co-NP such that \( L = L_1 \cap L_2 \) [PY82]).

**Theorem 10.** The soundness problem is PSPACE-complete. For acyclic negotiations, the problem is co-NP-hard and in DP (and so at level \( \Delta_2^P \) of the polynomial hierarchy).

*Proof. See the Appendix.  

3.2. A summarization algorithm

The summarization problem consists of computing a summary of a given negotiation, if it is sound. We show that the summary can be computed from the labeled reachability graph of the negotiation.

Let \( \langle n, r \rangle \) denote the transformer of the outcome \( (n, r) \). The labeled reachability graph of a negotiation \( N \) is defined as its reachability graph, but labeling the edge corresponding to a step \( x \xrightarrow{(n,r)} x' \) with the transformer \( \langle n, r \rangle \). In other words, the labeled reachability graph has an edge from \( x \) to \( x' \) labeled with \( \langle n, r \rangle \) for every step \( x \xrightarrow{(n,r)} x' \).

Observe that the labeled reachability graph is in fact a multi-graph, since there may be more than one edge between two nodes. Figure 3 shows a negotiation and its labeled reachability graph. If all the results of a negotiation have different names, we can identify an outcome \( (n, r) \) with the result \( r \) and further shorten \( \langle n, r \rangle \) to \( \langle r \rangle \).

![Labeled Reachability Graph](image)

Figure 3: A negotiation and its labeled reachability graph. We assume that the only outcome of the final atom is \( (n_f, f) \).

The concatenation of two transformers was already introduced in Definition 8. Similarly we define the union \( \tau_1 \cup \tau_2 \) of two transformers, and the Kleene star \( \tau^* \) of a
By definition, the summary transformer \( \langle N, f \rangle \) is the union over all large steps \( \sigma \) ending with \((n_f, f)\) of the transformers \( \langle \sigma \rangle \). We recall a well-known algorithm for the computation of this union based on state elimination (see e.g. [HMU06]). The algorithm iteratively reduces the graph to smaller graphs with the same summary transformers, until the graph contains only the nodes \( x_0 \) and \( x_f \), and one edge \( x_0 \xrightarrow{\tau} x_f \) for each final outcome \((n_f, f)\) of \( N \). Then we have \( \tau_r = \langle N, f \rangle \). At each step, the algorithm applies one of the following three reduction rules:

1. Replace a pair \( x_1 \xrightarrow{\tau} x_2, x_1 \xrightarrow{\tau'} x_2 \) of distinct edges, where \( x_2 \neq x_f \), by an edge \( x_1 \xrightarrow{\tau \cup \tau'} x_2 \).

2. Given a self-loop \( x \xrightarrow{\tau} x \), replace every edge \( x \xrightarrow{\tau'} x' \) such that \( x' \neq x \) by an edge \( x \xrightarrow{\tau' \tau} x' \), and then remove the self-loop.

3. Given an edge \( x' \xrightarrow{\tau'} x \) such that \( x' \neq x \) and \( x \) has at least one successor, add for every edge \( x \xrightarrow{\tau''} x'' \) a shortcut edge \( x' \xrightarrow{\tau' \tau''} x'' \), and then remove the edge \( x' \xrightarrow{\tau'} x \). Further, if after removing this edge the node \( x \) has no other incoming edges, then remove \( x \), together with all its outgoing edges.

The algorithm applies rules (1)-(3) in phases. At each phase, it follows this strategy:

- Apply rule (1) as long as possible.
- Apply rule (2) as long as possible.
- When neither rule (1) nor (2) are applicable, select a node \( x \) different from \( x_0 \) and \( x_f \), and apply rule (3) to all its ingoing edges, that is, to all edges of the form \( x' \xrightarrow{\tau'} x \). (Observe that this step necessarily ends with the removal of \( x \) and its outgoing edges.)

It is easy to see that the application of any of these rules does not change the summary of the graph. In particular, the idea behind rule (3) is that, in every large step, a step \( x' \xrightarrow{\tau'} x \) must necessarily be followed by one of the steps \( x \xrightarrow{\tau''} x'' \). After all shortcut edges have been added, the edge \( x' \xrightarrow{\tau'} x \) becomes redundant: the two graphs with and without the edge have the same summary.

After each phase the number of nodes decreases by one. If the original negotiation is sound (which implies that every node of the reachability graph lies on a path starting at the initial marking \( x_0 \) and ending at the final marking \( x_f \)), then the algorithm can
only terminate with a graph containing exactly the nodes $x_0$ and $x_f$, and an edge from $x_0$ to $x_f$ for each final result. If the original negotiation is not sound because from some reachable marking the final marking cannot be reached, then the algorithm terminates with a graph containing additional nodes. If the original negotiation is not sound only because some atom can never be executed, then the algorithm terminates with a graph as in the sound case.

When applied to the labeled reachability graph shown in Figure 3, the algorithm reduces the graph completely because the negotiation is sound. Figure 4 shows some intermediate steps of the algorithm.

![Figure 4: Intermediate steps in the reduction of the graph of Figure 3.](image)

This algorithm is simple and elegant, but it suffers from the state-explosion problem: the size of the reachability graph can be exponential in the size of the negotiation, and so the algorithm has exponential worst-case time complexity. While the complexity results of Section 3 show that this is unavoidable unless PSPACE=P, the main problem is that the algorithm takes exponential time for any family of negotiations whose reachability graphs exhibit exponential growth, even if the negotiations have a very simple structure.

In the next section we present reduction rules that act directly on the negotiation diagram, not on the reachability graph, and thus can be applied without constructing it. The rules can be applied to any negotiation, until no reduction is possible anymore. If the resulting negotiation consists of one single atom, the summary can be read out directly from it. Otherwise the algorithm above can be applied, with the advantage that, since the negotiation is now smaller, the reachability graph is smaller, too.
4. Reduction Rules for Negotiations

We introduce four reduction rules acting on negotiation diagrams. The rules are correct in the following sense: The negotiation obtained after applying a rule is equivalent to the negotiation before applying it. In particular, this implies that the rules preserve soundness, unsoundness and the summary.

The first two rules are straightforward generalizations of the rules applied in steps (1) and (2) of the summarization algorithm of Section 3.2.

The third rule deals with a characteristic of negotiation diagrams that is not present when we consider their reachability graphs. Finally, the last rule is a generalization of the one applied in step (3), but it is far from straightforward, and we will discuss it in detail.

4.1. Reduction Rules

A reduction rule, or just a rule, is a binary relation on the set of negotiations. Given a rule \( R \), we write \( N_1 R \rightarrow N_2 \) for \((N_1, N_2) \in R\). Notice that a rule \( R \) is not necessarily applicable to a given negotiation \( N_1 \). If it is applicable, then the resulting negotiation \( N_2 \) is not necessarily unique.

We describe rules as pairs of a guard and an action; \( N_1 R \rightarrow N_2 \) holds if \( N_1 \) satisfies the guard and \( N_2 \) is a possible result of applying the action to \( N_1 \).

A rule \( R \) is correct if its application to a negotiation always yields an equivalent negotiation, i.e., if \( N_1 R \rightarrow N_2 \) implies \( N_1 \equiv N_2 \). In particular, this implies that \( N_1 \) is sound iff \( N_2 \) is sound.

A finite sequence \( R_1 \ldots R_k \) of rules is a reduction sequence for a negotiation \( N \) if there are \( N_1, \ldots, N_k \) such that \( N \xrightarrow{R_1} N_1 \xrightarrow{R_2} \cdots \xrightarrow{R_k} N_k \). We say that the sequence reduces \( N \) to \( N_k \). Infinite reduction sequences are defined similarly\(^2\).

A set of correct rules \( \mathcal{R} \) is complete with respect to a class of negotiations if every sound negotiation in the class can be reduced to a negotiation consisting of a single atom by a finite sequence of reductions in \( \mathcal{R} \).

In the following we introduce the reduction rules for negotiations. For convenience, we assume in this section that a rule is applied to a negotiation \( N = (N, n_0, n_f, X) \).

The actions are formulated as assignments to the components of \( N \), so that the reduced negotiation is the one obtained after performing these assignments.

4.2. Merge and iteration rules

Merge rule. The merge rule merges results with identical transition functions into one.

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\(^2\)Infinite reduction sequences are of course undesirable. We define them, but show how to avoid them.
Definition 11. Merge rule

Guard: \( N \) contains an atom \( n \), \( n \neq n_f \) with two distinct results \( r_1, r_2 \in R_n \) such that \( \mathcal{X}(n, p, r_1) = \mathcal{X}(n, p, r_2) \) for every \( p \in P_n \).

Action: (1) \( R_n \leftarrow (R_n \setminus \{r_1, r_2\}) \cup \{r\} \), where \( r \) is a fresh name.
(2) For all \( p \in P_n \): \( X(n, p, r) \leftarrow X(n, p, r_1) \).
(3) \( \delta_n(r) \leftarrow \delta_n(r_1) \cup \delta_n(r_2) \).

An example is sketched in Figure 5. The results \( a \) and \( b \) are merged into one result \( c \) with an appropriate transformer.

Observe that the guard forbids application of the merge rule to the final atom; by the definition of equivalence, merging results of the final atom yields a non-equivalent negotiation. Notice further that the result of applying the rule is always a negotiation, according to Definition 4.

Iteration rule. Loosely speaking, the iteration rule removes self-loops, i.e., it removes results of an atom after which all parties are ready to take part only in the same atom again. The rule changes the transformers of all other results of that atom.

Definition 12. Iteration rule

Guard: \( N \) contains an outcome \( (n, r) \) such that \( \mathcal{X}(n, p, r) = \{n\} \) for every party \( p \) of \( n \).

Action: (1) \( R_n \leftarrow R_n \setminus \{r\} \).
(2) For every \( r' \in R_n \): \( \delta_n(r') \leftarrow \delta_n(r)^* \delta_n(r') \).

For an example, consider Figure 6. The result \( a \) for which all agents stay in \( n \) is removed and the transformer of result \( b \) is changed.
Observe that the application of the rule always yields a negotiation. Indeed, before applying the rule every atom lies on some path of the negotiation graph leading from the initial to the final atom: since the rule only removes a self-loop of this graph, the same property holds after the rule is applied.

The correctness of the merge and iteration rules is an immediate consequence of the definitions, and we state it without proof.

**Theorem 13.** The merge rule and the iteration rule are correct.

### 4.3. Useless arc rule

If the graphical representation of a negotiation contains a hyperarc for some agent leading from an atom $n$ to atoms $\{n_1, \ldots, n_k\}$, we say that this hyperarc consists of $k$ different arcs.

**Definition 14.** An arc of the negotiation $N$ is a tuple $(n, p, r, n')$ such that $(n, p, r) \in T(N)$ and $n' \in X(n, p, r)$.

Some of these arcs may be “useless”. Intuitively, an arc is useless if no occurrence sequence makes a token flow along it. More precisely, an arc $(n, p, r, n'')$ is useless if in every initial occurrence sequence, atom $n''$ is never enabled after the occurrence of the outcome $(n, r)$, and $n''$ remains disabled unless another atom involving agent $p$ has occurred.

Consider as an example the negotiation in Figure 7. The arc for agent A from $n$ to $n''$ can never be “used”, that is, after outcome $(n, r)$ agent A is ready to engage in $n'$ and also in $n''$, but $n''$ can not happen directly after $n$ because after $(n, r)$ agent B is only ready to engage in $n'$. Thus $n'$ must happen before $n''$, and agent A is also required for $n'$. 

![Figure 6: Illustration of the iteration rule](image-url)
The useless arc rule deletes useless arcs, which obviously preserves the occurrence sequences of the negotiation. However, in general useless arcs may be very difficult to identify. So, instead of requiring in the guard of the rule that an arc is useless, we require a stronger, but easier to check condition.

**Definition 15.** Useless arc rule

**Guard:**
- There are three distinct arcs \((n,p,r,n'), (n,p,r,n''), (n,q,r,n')\), such that \(X(n,q,r) = \{n'\}\).
- The pre-negotiation obtained by removing the arc \((n,p,r,n'')\) (see the action of this rule) is a negotiation, i.e., satisfies the conditions of Definition 4.

**Action:** \(X(n,p,r) \leftarrow X(n,p,r) \setminus \{n''\}\).

Figure 7: Example of a useless arc

Figure 8: Illustration of the useless arc rule
Intuitively (see Figure 8), after choosing \( r \) at atom \( n \), agent \( p \) is ready to engage in both \( n' \) and \( n'' \). However, another party \( q \) of both \( n' \) and \( n'' \) is only willing to engage in \( n' \). Then agent \( p \) can never engage directly in \( n'' \) after \( n \), and so this part of the hyperarc can be removed without changing the behavior.

To see why the second part of the guard is necessary, recall that in a negotiation every atom lies on some path from the initial to the final atom. Without the second part, the result of applying the rule may no longer satisfy this condition. For example, it is easy to define a negotiation in which all paths from the initial to the final atom containing \( n'' \) traverse the arc \( (n, p, r, n') \). We will show later that for the negotiations of the classes of negotiations considered in this paper, the second part of the guard is always true and can be omitted.

For the correctness proof we first specify what it means for an arc to occur in an occurrence sequence.

**Definition 16.** An arc \((n, p, r, n')\) of a negotiation occurs in an occurrence sequence \(\sigma\) if \(\sigma = \sigma_1(n, r)\sigma_2(n', r')\sigma_3\) for some outcome \(r'\) of \(n'\), and agent \(p\) does not participate in any outcome of \(\sigma_2\).

**Lemma 17.** Let \((n, p, r, n')\) be an arc of a negotiation \(N_1\) that does not occur in any initial occurrence sequence and belongs to a hyperarc with more than one arc. If the result of removing this arc is a negotiation \(N_2\) then \(N_2\) is equivalent to \(N_1\).

**Proof.** Assume that \(N_2\) is a negotiation. We proceed by showing that \(N_2\) is equivalent to \(N_1\).

We claim that \(N_1\) and \(N_2\) have the same occurrence sequences. Since \(X_2(n'', p'', r'') \subseteq X_1(n'', p'', r'')\) for every \((n'', p'', r'') \in T(N_1)\), every occurrence sequence of \(N_2\) is also an occurrence sequence of \(N_1\). Since the transition functions only differ in \((n, p, r)\), and the arc \((n, p, r, n')\) never occurs in an initial occurrence sequence of \(N_1\), every initial occurrence sequence of \(N_1\) is an initial occurrence sequence of \(N_2\), and the claim is proved.

A first immediate consequence of the claim is that an atom can occur in \(N_1\) iff it can occur in \(N_2\). It remains to show that every occurrence sequence of \(N_1\) can be extended to a large step iff the same holds for \(N_2\). Since both negotiations have the same occurrence sequences, we only have to show that an occurrence sequence is a large step of \(N_1\) iff it is a large step of \(N_2\). For this we observe that the markings \(x_1\) and \(x_2\) of \(N_1\) and \(N_2\) reached by executing an occurrence sequence \(\sigma\) in both \(N_1\) and \(N_2\) can only differ with respect to agent \(p\) and atom \(n''\): If \(x_1 \neq x_2\), then \(n', n'' \in x_2(p)\), whereas \(n' \in x_1(p)\) but \(n'' \notin x_2(p)\). If \(\sigma\) is a large step of \(N_1\) or \(N_2\) then \(x_1(p) = \emptyset\) or \(x_2(p) = \emptyset\), and so \(x_1 = x_2\).

**Theorem 18.** The useless arc rule is correct.

**Proof.** We adopt the notations of Definition 15 and show that an arc removed by the rule satisfies the conditions of Lemma 17. After every occurrence of \((n, r)\), agent \(q\) is only ready to engage in \(n'\). Therefore, after an occurrence of \((n, r')\) the atom \(n''\) must occur before \(n'\) can occur. Thus the arc \((n, p, r, n')\) can never occur in any initial occurrence sequence.

\[\square\]
4.4. Shortcut rule

The shortcut rule is inspired by the rule used in step (3) of the summarization algorithm presented in Section 3.2. Intuitively, the goal of the rule is to remove an outcome \((n, r)\) such that, for every large step \(\sigma\) containing \((n, r)\) of the original negotiation, there is a large step \(\sigma'\) of the reduced negotiation that does not contain \((n, r)\), such that \(\langle \sigma \rangle = \langle \sigma' \rangle\). Moreover, sometimes the rule must allow us to remove some successor atom of \((n, r)\); otherwise we never reduce the number of atoms. Finally, this must be achieved while preserving the sound or the unsound character of a negotiation.

As a warm-up we first define the rule for the simple case of negotiations with only one agent. Then we illustrate the problems involved in extending the rule to arbitrary negotiations. Finally, we introduce the rule for the general case and provide its correctness proof.

In the one-agent case we can consider atoms and outcomes as nodes and edges of a graph (see e.g. Figure 9a). We write \(n \xrightarrow{d} n'\) instead of \(X(n, p, r) = \{n'\}\), where \(p\) is the single agent of the negotiation. Informally, the rule states:

1. If \(n \xrightarrow{d} n'\) and \(n' \neq n_f\), then
   - add for every \(n''\) such that \(n' \xrightarrow{r'} n''\) a new shortcut \(n \xrightarrow{r''} n''\) with transformer \(\langle n, r'' \rangle = \langle n, r \rangle \langle n', r' \rangle\), and then remove \(n \xrightarrow{d} n'\);
   - if \(n'\) has no other incoming edges, then remove \(n'\), together with its outgoing edges.

2. If \(n \xrightarrow{d} n_f, r\) is the only result of \(n\) and \(n_f\) has no other incoming edges, then
   - add for every result \(r'\) of \(n_f\) a new result \(r''\) of \(n\) with \(X(n, p, r'') = \emptyset\) and transformer \(\langle n, r'' \rangle = \langle n, r \rangle \langle n', r' \rangle\), and then remove \(n \xrightarrow{d} n'\);
   - remove \(n_f\);
   - consider \(n\) as the new final atom.

For example, in the negotiation of Figure 9a we can apply the rule to \(n_2 \xrightarrow{d} n_3\), obtaining the negotiation of Figure 9b, and then to \(n_1 \xrightarrow{c} n_3\), yielding the negotiation of Figure 9c.

To see why we require \(n' \neq n_f\) in the guard of part (1) of the rule, consider Figure 9c. Without this condition, the application of the rule to \(n_1 \xrightarrow{c'} n_f\) replaces the result \(c'\) by a new result, say \(c''\), such that \(X(n_1, p, c'') = \emptyset\) (graphically, the arc from \(n_1\) to \(n_f\) disappears). The result is not even a negotiation, because in a negotiation all atoms but the final one must have at least one outgoing edge.

Part (2) of the rule allows us to apply the rule to \(n_3 \xrightarrow{c} n_f\) in Figure 9a. Since \(n_f\) has no other incoming edge, it is removed. Atom \(n_3\) loses its only outgoing edge, and becomes the new final atom. To preserve equivalence, the results of \(n_f\) are transferred to the new final atom \(n_3\).
In the case of multiple agents, defining a correct shortcut rule is more complicated. Consider the negotiation of Figure 10. The large steps of this negotiation are

\((n_0, a), (n_1, c), (n_2, d), (n_f, f)\)

where \(f\) is a final result, and

\((n_0, b), (n_2, d), (n_f, f)\).

So, if \((n_1, c)\) occurs, it is succeeded by \((n_2, d)\). However, there is no obvious way to remove the outcome \((n_1, c)\) by shortcutting it with its “successor” \((n_2, d)\), in particular because the latter has an additional participant. For this reason, we apply the shortcut rule only to outcomes \((n, r)\) such that, after executing them, an atom \(n'\) is enabled, and remains enabled until it occurs. This condition apparently depends on the markings enabling \((n, r)\). Since there is no efficient way to identify the markings reachable from the initial marking and since the conditions for applying the rule must be easy to verify, we require that any marking, reachable or not, that enables \((n, r)\), subsequently enables...
\( n' \) until it occurs. For example, in Figure 10 we can shortcut \((n_0, a)\), because after its occurrence the atom \( n_1 \) is enabled, and can become disabled only by the occurrence of \( n_1 \). However, we cannot shortcut \((n_1, c)\), because the marking that places a token on each of the ingoing arcs of \( n_1 \) does enable \( n_1 \), but after \((n_1, c)\) occurs no other atom is enabled.

The following simple structural condition formulates an obvious sufficient condition of the property mentioned before.

**Definition 19** (Unconditionally enables). An outcome \((n, r)\) unconditionally enables an atom \( n' \) if \( P_n \supseteq P_{n'} \) and \( X(n, p, r) = \{n'\} \) for every \( p \in P_{n'} \).

We now consider a second problem. If a negotiation is sound before application of the shortcut rule to an outcome \((n, r)\) and an atom \( n' \), and after the shortcut the atom \( n' \) is not enabled at any reachable marking, then we must remove \( n' \), otherwise the negotiation becomes unsound. For example, if in the sound negotiation of Figure 9 we do not remove \( n_3 \) after shortcutting \( n_1 \xrightarrow{c} n_3 \), we are left with an unsound negotiation. In the one-agent case, the atom \( n' \) can be enabled after the shortcut iff it still has incoming edges. However, in the case of multiple agents, deciding whether \( n' \) must be removed or not can be much harder. Consider the sound negotiation of Figure 11. The only result of \( n_2 \) enables \( n_3 \) unconditionally. After shortcutting this result the atom \( n_3 \) can never become enabled, and must be removed. However, after the shortcut all ports of \( n_3 \) still have incoming edges, and so this criterion does not suffice to decide whether the atom must be removed. In general, deciding if \( n' \) must be removed is an intractable problem (observe that it is equivalent to deciding whether these arcs are useless). Fortunately, it suffices to consider two cases for which the decision is simple, and to apply the shortcut rule only in these cases.

First, as in the one-agent case, if after the shortcut no edge leads to a port of \( n' \), then \( n' \) cannot be enabled, and must be removed. This is the case if all ports have, before applying the rule, only one ingoing arc, namely the one from \( n \) labeled by \( r \). In this case we say that \((n, r)\) has exclusive access to \( n' \).

**Definition 20** (Exclusive access). An outcome \((n, r)\) has exclusive access to an atom \( n' \) if, for each \( p \in P_{n'} \), \( n' \in X(n, p, r) \) and \( n' \not\in X(n'', p, r'') \) for \((n, r) \neq (n'', r'')\).

So, if after the shortcut all ports of \( n' \) have no ingoing arc, we can remove \( n' \). Assume now that after the shortcut some deterministic arc leads to a port of \( n' \) (that is, an arc which is not a proper hyperarc). Say this arc corresponds to an outcome \((n'', r'')\). As we shall see, if the negotiation was sound before the shortcut, then \( n'' \) can still be enabled after it, the outcome \((n'', r'')\) can occur, and \( n' \) is the only atom removing the token on the above deterministic arc. So keeping \( n' \) preserves the soundness in this case. If the negotiation was unsound before, then keeping \( n' \) will obviously not make the resulting negotiation sound. So in both cases, if \( n' \) is not removed the sound/unsound character of the negotiation is preserved.

If \((n, r)\) unconditionally enables \( n' \) and has exclusive access to \( n' \) then a shortcut is possible and \( n' \) has to be removed.
Definition 21 (Commits to). An outcome \((n'', r'')\) commits to an atom \(n'\) if \(\{n'\} = X(n'', p, r'')\) for some \(p \in P_{n''}\).

As in the case of a single agent, we can only have \(n' = n_f\) if \(n'\) is removed and \(n\) qualifies as a new final atom. Therefore, we require \(n' \neq n_f\) unless \((n, r)\) has exclusive access to \(n'\) and moreover \(r\) is the only result of \(n\).

Equipped with Definitions 19, 20 and 21 we can now formally define the general shortcut rule, illustrated in Figure 12.

![Figure 11: Problematic case for a shortcut](image)

Definition 22. Shortcut rule

**Guard:** \(N\) contains an outcome \((n, r)\) that unconditionally enables an atom \(n'\) distinct from \(n\). Moreover:

- \(n' \neq n_f\) and \((n, r)\) has exclusive access to \(n'\), or
- \(n' \neq n_f\) and some outcome \((n'', r'')\) \(\neq (n, r)\) commits to \(n'\), or
- \(n' = n_f\), the outcome \((n, r)\) has exclusive access to \(n'\), and \(r\) is the only result of \(n\).

**Action:**
1. \(R_n \leftarrow (R_n \setminus \{r\}) \cup \{r' \mid r' \in R_{n'}\}\), where \(r'\)s are fresh names.
2. For all \(p \in P_{n'}\) and all \(r' \in R_{n'}\): \(X(n, p, r') \leftarrow X(n', p, r')\).
   For all \(p \in P_n \setminus P_{n'}\) and all \(r' \in R_{n'}\): \(X(n, p, r') \leftarrow X(n, p, r)\).
3. For all \(r' \in R_{n'}\): \(\delta_n(r'_s) \leftarrow \delta_n(r)\delta_{n'}(r')\).
4. If \((n, r)\) has exclusive access to \(n'\), then remove \(n'\).
5. If \(n'\) was the final atom before applying the rule and is removed, then \(n\) is the new final atom.

Consider the negotiation of Figure 10. We can shortcut result \(a\) of \(n_0\) because it unconditionally enables \(n_1\) and has exclusive access to it. We can also shortcut result \(b\), because it unconditionally enables \(n_2\) and result \(a\) of \(n_0\) (or result \(c\) of \(n_1\)) commits to \(n_2\).
Consider now Figure 11. We cannot shortcut the results of \( n_0 \) and \( n_1 \) because they do not unconditionally enable any node. The result of \( n_2 \) unconditionally enables \( n_3 \), but we cannot shortcut it either: it does not have exclusive access to \( n_3 \), and no other result commits to \( n_3 \). The result of \( n_3 \) unconditionally enables \( n_f \) with exclusive access, and so we can shortcut it. The application of the rule removes \( n_f \), and \( n_3 \) becomes the new final atom of the negotiation.

**Theorem 23.** The shortcut rule is correct.

**Proof.** Let \( N_2 \) be the result of applying the shortcut rule to \( N_1 \). Assume the atoms \( n \) and \( n' \) and the result \( r \in R_n \) are as in the shortcut rule (Definition 22). We first show that \( N_2 \) is in fact a negotiation.

- \( X(n'''', p, r''') = \emptyset \) if and only if \( n''' = n_f \) in \( N_2 \). This is obviously the case if \( n' \) was not the final atom before the reduction. If it was the final atom, then we only apply the rule if we also remove \( n' \), and if moreover \( n \) has only the single result \( r \). So it remains to show that in this case \( n \) satisfies the criteria of a final atom.

  We have \( P_{n'} \subseteq P_n \) because \((n, r)\) unconditionally enables \( n' \). We have \( P_{n'} = A \) because \( n' = n_f \), and therefore \( P_n = A \). By definition of the rule, \( X(n, p, r) = \emptyset \) after the rule was applied, for all agents \( p \). Since \( r \) is the only result, we obtain \( X(n, p, r'') = \emptyset \) for all results \( r'' \). So \( n \) qualifies as new final atom. The transition function \( X \) is not changed by the rule application for any other atom.

- Every atom \( n''' \) of \( N_2 \) is on a path from the initial atom to the final atom in the graph of the negotiation. Consider first the case \( n'' = n_f \). A path of \( N_2 \) that visits \( n'' \) is either a path of \( N_1 \), or it is obtained from a path of \( N_1 \) by replacing...
\((n, p, r)(n', p', r')\) by \((n, p', r')\). So, no matter if \(n'\) is the final atom or not, \(n_2\) lies on some path of \(N_2\) from \(n_0\) to \(n_f\).

Assume now \(n'' = n'\). Then \(n'\) has not been removed by the rule, and so \((n, r)\) does not have exclusive access to \(n'\). This implies that some party \(p\) of \(n'\) has an additional input arc, i.e., that \(p\) is in \(X(n'', p, r'')\) for some \((n, r) \neq (n'', r'')\). There is a path from \(n_0\) via \(n''\) to \(n'\), before and after application of the rule. Moreover, the path from \(n'\) to \(n_f\) remains unchanged by the rule application (assuming that \(n'\) does not occur again in the path). So, also after application of the rule, \(n'\) is on a path from \(n_0\) to \(n_f\).

We continue by studying the relation between occurrence sequences in \(N_1\) and \(N_2\). We introduce the concept of corresponding sequences which we then use to show that soundness of \(N_1\) implies soundness of \(N_2\) and vice versa. We say that an atom \(\pi\) occurs in an occurrence sequence \(\sigma\) if \((\pi, r)\) occurs in \(\sigma\) for some \(r \in R\).

For each initial occurrence sequence \(\sigma_2\) of \(N_2\), replacing all occurrences of \((n, r')\) by \((n, r, r')\) yields an initial occurrence sequence \(\sigma_1\) of \(N_1\). We call \(\sigma_1\) the occurrence sequence in \(N_1\) corresponding to \(\sigma_2\). The markings reached by these two sequences are the same.

For each initial occurrence sequence \(\sigma_1\) of \(N_1\), between each two occurrences of \((n, r)\) there is an occurrence of \((n', r')\) (with possibly varying \(r'\)) because \(n\) unconditionally enables \(n'\). Thus only the last occurrence of \((n, r)\) might not be followed by an occurrence of \(n'\), which may happen only if \(n'\) is still enabled. We transform \(\sigma_1 = \sigma_1(n, r)\sigma_1'(n', r_a)\sigma_1(n, r)\sigma_1'(n', r_b)\ldots\) to a sequence \(\sigma_2 = \sigma_1(n, r_f)\sigma_1'(n, r_f)\sigma_1'(n, r_b)\ldots\) of \(N_2\). Should, in \(\sigma_1\), the last occurrence of \((n, r)\) not be followed by an occurrence of \(n'\), we instead apply the transformation to the extended sequence \(\sigma_1(n', r')\) for some \(r'\). In both cases, this yields an occurrence sequence \(\sigma_2\) of \(N_2\), which we call the corresponding sequence to \(\sigma_1\), and which leads to the same marking as \(\sigma_1\) (or \(\sigma_1(n', r')\)).

(1) Assume that \(N_1\) is sound. We prove that \(N_2\) is sound.

First we show that every atom \(\tilde{n}\) of \(N_2\) can be enabled by some initial occurrence sequences, and then that every initial occurrence sequence \(\sigma_2\) in \(N_2\) can be extended to a large step. For the first part, we consider two cases.

Case \(\tilde{n} \neq n'\). Since \(N_1\) is sound, \(\tilde{n}\) can be enabled by some initial occurrence sequence \(\sigma_1\) of \(N_1\). Since the corresponding sequence \(\sigma_2\) in \(N_2\) leads to the same marking as either \(\sigma_1\) or \(\sigma_1(n', r')\) for some \(r'\), it also enables \(\tilde{n}\). (Note that the addition of \((n', r')\) to the occurrence sequence cannot influence that \(\tilde{n}\) is enabled: We only add \((n', r')\) if it was unconditionally enabled by some prior occurrence of \((n, r)\), but then the agents of \(n'\) were only ready to engage in \(n'\) and thus do not participate in \(\tilde{n}\).)

Case \(\tilde{n} = n'\). Since \(n'\) is still an atom of \(N_2\) and by the definition of the guard of the rule, some outcome \((n'', r'')\) \(\neq (n, r)\) of \(N_1\) commits to \(n'\), i.e., there is \((n'', p'', r'')\) in \(T(N_1)\) such that \((n'', r'') \neq (n, r)\) and \(\{n'\} = X(n'', p'', r'')\). Since \((n'', r'') \neq (n, r)\), the outcome \((n'', r'')\) is unchanged in \(N_2\). By soundness of \(N_1\), some initial occurrence sequence \(\sigma_1\) of \(N_1\) enables \(n''\). We extend it by an occurrence of \((n'', r'')\) and then to a large step \(\sigma_1(n'', r'')\rho_1\) in \(N_1\), which is possible since \(N_1\) is sound. Since after \((n'', r'')\) agent \(p''\) is only ready to engage in \(n''\), the sequence \(\rho_1\) contains an occurrence.
of \( n' \). During the construction we replace occurrences of \((n, r)\) and those occurrences of \( n' \) that are direct consequences of the occurrence of some \((n, r)\). We will however not replace the occurrence of \( n' \) that was caused by \((n'', r'')\), and so the corresponding sequence in \( N_2 \) will also contain an occurrence of \( n' \). Thus \( n' \) can be enabled in \( N_2 \).

We now prove that every initial occurrence sequence \( \sigma_2 \) in \( N_2 \) can be extended to a large step. Take the corresponding occurrence sequence \( \sigma_1 \) in \( N_1 \) and extend it to a large step \( \tau_1 = \sigma_1 \rho_1 \) in \( N_1 \) (possible by soundness of \( N_1 \)). The corresponding sequence in \( N_2 \) is \( \tau_2 = \sigma_2 \rho_2 \), which is an extension of \( \sigma_2 \) (by construction of corresponding sequences) to a large step.

(2) Assume that \( N_2 \) is sound. We prove that \( N_1 \) is sound.

Since every atom in \( N_2 \) can be enabled by some initial occurrence sequence, using the corresponding sequence in \( N_1 \) we see that the same is true for all atoms but \( n' \). Further, it is easy to show that also \( n' \) can be enabled in \( N_1 \): Take the initial occurrence sequence that enables \( n \) and extend it by \((n, r)\), which unconditionally enables \( n' \).

For an initial occurrence sequence \( \sigma_1 \) in \( N_1 \), the corresponding occurrence sequence \( \sigma_2 \) in \( N_2 \) can be extended to a large step \( \tau_2 = \sigma_2 \rho_2 \) in \( N_2 \), because \( N_2 \) is sound. Let \( \tau_1 \) be the occurrence sequence of \( N_1 \) that corresponds to \( \tau_2 \). Then \( \tau_1 = \sigma'_1 \rho'_1 \) where \( \sigma'_1 \) corresponds to \( \sigma_2 \), by the definition of “corresponds”. Now either \( \sigma_1 \) or \( \sigma_1(n', r') \) leads to the same marking as \( \sigma_2 \), and \( \sigma_2 \) leads to the same marking as \( \sigma'_1 \). Therefore, either \( \sigma_1 \rho'_1 \) or \( \sigma_1(n', r') \rho'_1 \) is a large step of \( N_1 \).

\[ \square \]

4.5. Rules Preserve (Weak-)Determinism

We conclude the section with a simple but important observation.

**Proposition 24.** If a negotiation \( N_1 \) is deterministic (weakly deterministic) and the application of one of the rules above yields negotiation \( N_2 \), then \( N_2 \) is also deterministic (weakly deterministic, respectively). Further, if \( N_1 \) is acyclic, then so is \( N_2 \).

**Proof.** For the merge rule and the iteration rule the proposition is obvious. The useless-arc rule removes an arc and thus may even lead to a deterministic negotiation starting from a weakly deterministic one (but never the other way round). For the shortcut rule, observe that, by the definition of the rule, for every \((n_2, a_2, r_2) \in T(N_2)\) there is \((n_1, a_1, r_1) \in T(N_1)\) such that \( \mathcal{X}(n_2, a_2, r_2) = \mathcal{X}(n_1, a_1, r_1) \) (in fact, we can always take \( a_1 = a_2 \)). Since whether a negotiation \( N \) is deterministic or weakly deterministic only depends on the set \( \{\mathcal{X}(n, a, r) \mid (n, a, r) \in T(N)\} \), we are done. Finally, the definitions of the rules imply immediately that acyclicity is preserved (but not cyclicity).

\[ \square \]

4.6. Reducible outcomes

In the rest of the paper we present reduction algorithms for different classes of negotiations. All of them repeatedly choose a reducible outcome, and apply the corresponding rule. Formally:

**Definition 25.** Let \( N \) be a negotiation. An outcome \((n, r)\) of \( N \) is reducible if
• it satisfies the guard of the iteration or the shortcut rule; or
• it satisfies, together with another outcome, the guard of the merge rule; or
• \((n, p, r, n')\) is a useless arc for some agent \(p\) and atom \(n'\).

The set of reducible outcomes of \(N\) is denoted \(R(N)\).

As we shall see, a nondeterministic reduction procedure that can choose any reducible outcome may not terminate, or terminate in exponentially many steps. In order to obtain polynomial algorithms we will constrain the choices of the procedure.

5. Summarizing Acyclic Weakly Deterministic Negotiations

Recall that if a set of rules is complete for a class of negotiations, then it completely reduces all (and only) sound negotiations in the class to an atomic negotiation. Therefore, a reduction procedure that transforms every negotiation in the class into an equivalent irreducible negotiation allows us to check soundness and compute a summary without constructing the reachability graph of the negotiation. Indeed, the negotiation is sound iff its corresponding irreducible negotiation is atomic.

**Definition 26.** Let \(R\) be a set of rules and let \(N\) be a negotiation.

A negotiation is irreducible with respect to \(R\) if no rule of \(R\) can be applied to it.

A finite or infinite reduction sequence \(R_1 R_2 R_3 \ldots\) for \(N\) over \(R\) is maximal if it is, applied to \(N\), either infinite or if it is finite and cannot be extended, i.e. leads to an irreducible negotiation.

In this section we show that the rules presented in the previous section are complete for acyclic, weakly deterministic negotiations. This class contains for example the Father / Daughter / Mother negotiation from the beginning of this paper. Further, we show in Section 5.1 that if the negotiation is deterministic, then there is a reduction algorithm that reaches an irreducible negotiation after a polynomial number of rule applications, and thus runs in polynomial time. Whether such an algorithm also exists in the weakly deterministic case is an open problem.

Acyclicity is a severe restriction, which in particular implies that every atom occurs at most once. However, the results on acyclic negotiations will be reused as lemmas in the following sections, where we consider cyclic deterministic negotiations.

We start with a first lemma, showing that for acyclic negotiations the guard of the useless arc rule can be simplified.

**Lemma 27.** Let \(N\) be an acyclic negotiation. \(N\) can be transformed into \(N'\) by the useless arc rule iff it can be transformed into \(N'\) by the following rule:

**Guard:**

- There are three distinct arcs \((n, p, r, n')\), \((n, p, r, n'')\), \((n, q, r, n')\), such that \(X(n, q, r) = \{n'\}\).
- The arc \((n, p, r, n'')\) is not the only arc leading to a port of \(n''\).

**Action:** \(X(n, p, r) \leftarrow X(n, p, r) \setminus \{n'\}\).
If moreover $N$ is sound, then $N$ can be transformed into $N'$ by the useless arc rule iff it can be transformed into $N'$ by the following rule:

**Guard:** There are three distinct arcs $(n,p,r,n'), (n,p,r,n'')$, $(n,q,r,n')$, such that $X(n,q,r) = \{n'\}$.

**Action:** $X(n,p,r) \leftarrow X(n,p,r) \setminus \{n'\}$.

**Proof.** Assume that $N$ is acyclic.

$(\Leftarrow)$ If $N$ can be transformed into $N'$ by the useless arc rule, then after removing the useless arc $(n,p,r,n'')$ the node $n''$ still lies on a path from the initial to the final node, and so $N$ also contains another arc leading to $n''$. So $N$ can also be transformed into $N'$ by the new rule.

$(\Rightarrow)$ Roughly speaking, we have to show that after applying the new rule to an acyclic negotiation, the resulting pre-negotiation is actually a negotiation, i.e., every node is on some path from $n_0$ to $n_f$ which does not contain such an arc.

Consider an arbitrary atom $\tilde{n}$. There is a path of $N$ from $n_0$ to $\tilde{n}$. By assumption there is an arc leading from some atom $n_1$ to $n''$. If the path from $n_0$ to $\tilde{n}$ contains an arc from $n$ to $n''$, then we can replace the prefix up to $n$ by a path from $n_0$ to $n_1$, yielding a path from $n_0$ to $\tilde{n}$ without the arc. By acyclicity, this path does not contain $n''$. If a path from $\tilde{n}$ to $n_f$ contains an arc from $n$ to $n''$ then we can replace the suffix from $n''$ by a path from $n'$ to $n_f$. Again by acyclicity, this path does not contain $n$. Summarizing, after the rule application the node $\tilde{n}$ is again on a path from $n_0$ to $n_f$, and we are done.

Assume now that $N$ is acyclic and moreover sound.

$(\Leftarrow)$ Obvious.

$(\Rightarrow)$ If the negotiation is sound, then the atom $n''$ definitely has more ingoing arcs than the useless one, and we can apply the first part of the lemma.

**Theorem 28.** Let $N$ be an acyclic and weakly deterministic negotiation, and let $\rho$ be a maximal reduction sequence for $N$ using only the shortcut, merge and useless arc rules. Then $\rho$ is finite, and it reduces $N$ to an atomic negotiation iff $N$ is sound.

**Proof.** We first prove that $\rho$ is finite. It suffices to show that none of the shortcut, merge, and useless arc rules can be applied infinitely often in a reduction sequence. Let $N_3$ be the result of applying any of the rules to a negotiation $N_1$. For every large step $\sigma$ of $N_2$, let $\phi(\sigma)$ be defined as follows: if the useless arc rule or the merge rule has been applied, then $\phi(\sigma) = \sigma$; if the shortcut rule has been applied to atoms $n,n'$, then let $\phi(\sigma)$ be the result of replacing every occurrence of $(n,x'_f)$ by the sequence $(n,x)(n',x')$. We have $|\sigma| \leq |\phi(\sigma)|$ for every large step $\sigma$. Moreover, if the shortcut rule is applied, then $|\sigma'| < |\phi(\sigma')|$ for at least one large step $\sigma'$, indeed for all large steps containing $(n,x'_f)$.

None of the rules destroy acyclicity, i.e., all considered negotiations are acyclic. In an acyclic negotiation, the length of a large step is bounded by the number of atoms,
and so the set of large steps is finite. Therefore, no infinite sequence of applications of the rules can contain infinitely many applications of the shortcut rule. So any infinite sequence of applications must, from some point on, only apply the useless arc rule and the merge rule. But this is also not possible as both rules reduce the number of arcs. This completes the proof that $\rho$ is finite.

Assume that $N$ is not sound. Since the three rules are correct, $\rho$ reduces $N$ to an unsound negotiation $N'$. By definition of soundness, atomic negotiations are sound. Therefore, $N'$ is not atomic.

Assume now that $N$ is sound. We prove that $\rho$ reduces $N$ to an atomic negotiation. The proof has two parts:

1. If $N$ has more than two atoms, then the shortcut rule or the useless arc rule can be applied to it.

Since $N$ is acyclic, its graph induces a partial order on atoms in the obvious way ($n < n'$ if there is a path from $n$ to $n'$). Clearly $n_0$ and $n_f$ are the unique minimal and maximal elements, respectively. We choose an arbitrary linearization of this partial order.

Since $N$ has more than two atoms, this linearization begins with $n_0$ and has some second element, say $n_1 \neq n_f$.

Since $N$ is sound and since $n_1$ does not depend on any prior atom except $n_0$, some occurrence sequence begins with an occurrence of $n_0$ and a subsequent occurrence of $n_1$. So $n_0$ has a result $r_0$ such that $(n_0, r_0)$ unconditionally enables $n_1$.

Consider two cases:

- $\{n_1\} \subseteq X(n_0, p, r_0)$ for some party $p$ of $n_1$.
  Then $n'_1, n_2 \in X(n_0, p, r_0)$ for some atom $n_2 \neq n_1$.

  Since $N$ is weakly deterministic, some deterministic agent $q$ must be a party of all atoms in $X(n_0, p, r_0)$, and so in particular of $n_1$ and $n_2$.

  Since $(n_0, r_0)$ unconditionally enables $n_1$ and $q$ is deterministic, $X(n_0, q, r_0) = \{n_1\}$.

  So we have three distinct arcs $(n_0, p, r_0, n_1)$, $(n_0, p, r_0, n_2)$, and $(n_0, q, r_0, n_1)$ such that $X(n_0, q, r_0) = \{n_1\}$. Since $N$ is sound, by Lemma 27 the useless arc rule can be applied with $n := n_0$, $n' := n_2$, $n'' := n_1$.

- $\{n_1\} = X(n_0, p, r_0)$ for every party $p$ of $n_1$.

  We claim that the shortcut rule or the useless arc rule can be applied. Assume $N$ does not satisfy the guard of the shortcut rule. Then, since $(n_0, r_0)$ unconditionally enables $n_1$ and $n_1 \neq n_f$, we have that (a) $(n_0, r_0)$ does not have exclusive access to $n_1$, and (b) no outcome distinct from $(n_0, r_0)$ commits to $n_1$.

  By (a), there exists an arc $(n, p, x, n_1)$ such that $(n, x) \neq (n_0, r_0)$. Since $n_1$ is the second atom in the linearization, we have $n = n_0$ and $x \neq r_0$. So there exists an arc $(n_0, p, x, n_1)$.

  By (b), there is an arc $(n_0, p, x, n_2)$ for some atom $n_2 \neq n_1$.

  Since $N$ is weakly deterministic, some deterministic agent $q$ is a party of all atoms in $X(n_0, p, x')$. Moreover, by soundness, the unique atom in $X(n_0, q, x)$ must be
one of them. We can assume without loss of generality that the atom is either \( n_1 \) or \( n_2 \).

If the atom is \( n_1 \), then we have three distinct arcs \((n_0, p, \sigma, n_1)\), \((n_0, p, \sigma, n_2)\), and \((n_0, q, \sigma, n_1)\) such that \( X(n_0, q, \sigma) = \{n_1\} \). By Lemma 27, the useless arc rule can be applied with \( n := n_0, n' := n_2, \) and \( n'' := n_1 \).

If the atom is \( n_2 \), then we have three distinct arcs \((n_0, p, \sigma, n_1)\), \((n_0, p, \sigma, n_2)\), and \((n_0, q, \sigma, n_2)\) such that \( X(n_0, q, \sigma) = \{n_2\} \). Further, since \((n_0, \sigma_0)\) unconditionally enables \( n_1 \), there is a path from \( n_0 \) to \( n_1 \) that does not contain the arc \((n_0, p, \sigma, n_1)\). By Lemma 27 the useless arc rule can be applied with \( n := n_0, n' := n_1, \) and \( n'' := n_2 \).

(2) If \( N \) has exactly two atoms, it can be summarized (i.e., reduced to an equivalent atomic negotiation).

In this case, the atoms of \( N \) are \( n_0 \) and \( n_f \). Since the negotiation is sound, all results of \( n_0 \) have the same transition function. So the merge rule can be applied until \( n_0 \) only has one result. Then the shortcut rule can be applied with \( n = n_0 \) and \( n' = n_f \) to obtain an equivalent atomic negotiation.

We can now conclude the proof. By the maximality of \( \rho \) and (1), \( \rho \) reduces \( N \) to a negotiation containing at most two atoms. By (2), \( \sigma \) reduces \( N \) to an atomic negotiation.

As a corollary of Theorem 28 we obtain the completeness result:

**Corollary 29.** The shortcut rule, merge rule and useless arc rules are complete for acyclic, weakly deterministic negotiations.

Figure 13 shows a weakly deterministic negotiation, which is a slightly modified version of the Father-Daughter-Mother negotiation shown in Figure 13a: Daughter now has the choice in \( n_1 \) whether to ask her Father (and possibly later her mother) or to directly ask her mother. Father thus has a nondeterministic edge to \( n_f \) for the latter case. This negotiation is sound, acyclic and weakly deterministic, and can therefore be summarized by Theorem 29. The figure also illustrates the summarization of the negotiation.

We apply the merge rule twice to merge the results \( y \) and \( n \) of the atoms \( n_2 \) and \( n_3 \). We call the merged results “\( yn \)” The resulting negotiation is shown in Figure 13b.

Now we shortcut the result \( st \) of \( n_0 \), which unconditionally enables \( n_1 \) and has exclusive access to it. The result \( st \) and the node \( n_1 \) are removed, and two new results \( af \) and \( am \) are added to \( n_0 \), yielding the negotiation in Figure 13c.

In this negotiation, applying the useless arc rule to \((n_0, F, af)\) and \((n_0, D, af)\) removes the arc \((n_0, F, af, n_f)\). Similarly, applying it to \((n_0, M, am)\) and \((n_0, D, am)\) removes the arc \((n_0, M, am, n_f)\). We obtain the negotiation of Figure 13d.

Now the result \( am \) of \( n_0 \) unconditionally enables \( n_3 \), and the result \( am \) of \( n_2 \) commits to \( n_3 \). So we can apply the shortcut rule, removing the result \( am \) of \( n_0 \), and adding a new result \( am \cdot yn \). This yields the negotiation in Figure 13e.
Figure 13: Example of the summary procedure
The useless arc rule can be applied to \((n_0, F, \text{am}_yn)\) and \((n_0, D, \text{am}_yn)\), removing the arc \((n_0, F, \text{am}_yn, n_2)\). After that, the result \(\text{af}\) of \(n_0\) unconditionally enables \(n_2\) and has exclusive access to it. So we can apply the shortcut rule and remove \(n_2\). As \(n_2\) has two results, \(yn\) and \(am\), two new results \(\text{af}_yn\) and \(\text{af}_am\) are added to \(n_0\). We obtain the negotiation of Figure 13f.

We can now apply the useless arc rule twice, once to \((n_0, M, \text{af}_yn)\) and \((n_0, F, \text{af}_yn)\), and once to \((n_0, M, \text{af}_am)\) and \((n_0, F, \text{af}_am)\). The result is shown in Figure 13g.

Outcome \(\text{af}_am\) of \(n_0\) unconditionally enables \(n_3\), and has exclusive access to it. After the shortcut we obtain the negotiation of Figure 13h. Applying the merge rule to \(\text{am}_yn\), \(\text{af}_yn\) and \(\text{af}_am\) yields a single result \(r\) of \(n_0\) (result in Figure 13i). A final application of the shortcut rule to result \(r\) of \(n_0\), which unconditionally enables \(n_f\) with exclusive access, yields the summary of the initial negotiation.

5.1. Acyclic deterministic negotiations: Runtime analysis

Theorem 28 shows that every maximal reduction sequence is finite, but does not provide any bound on its length. The family of negotiations shown at the top of Figure 14 shows that the length of maximal reduction sequences can grow exponentially in the size of the negotiation, even in the sound and deterministic case. Indeed, consider a sequence of applications of the shortcut rule that always give priority to applications involving the initial atom. The second negotiation of the figure shows the result of shortcutting \((n_0, a)\) and \((n_1, a)\) into a new outcome \((n_0, a a_1)\), shortcutting \((n_0, a)\) and \((n_1, b_1)\) into a new outcome \((n_0, a b_1)\), and removing \(n_1\). The third negotiation shows the result of shortcutting \((n_0, a a_1)\) and \((n_0, a b_1)\) with the two outcomes of \(n_2\), after which \(n_0\) has four outcomes. Proceeding in this way we generate \(2^{k-1}\) different results for \(n_0\).

However, there is another maximal reduction sequence with only a polynomial number of rule applications. At the second negotiation of the figure we first apply the shortcut rule to \((n_0, a a_1)\) and \((n_1, a)\), and then to \((n_0, a b_1)\) and \((n_1, b_1)\). This yields two outcomes that can be merged with the help of the merge rule, yielding a negotiation in which \(n_0\) has again only one outcome. Proceeding in this way we completely reduce the negotiation after only \(5k + 1\) rule applications.

The reason why the first reduction sequence “blows up” the negotiation is that an application of the shortcut rule may actually increase the size of the negotiation. Indeed, when the rule is applied to an outcome \((n, r)\) that unconditionally enables an atom \(n'\) with \(l\) results for \(l > 1\), then the rule removes the result \(r\) but adds \(l\) new results, and so the total number of results increases. This problem is avoided by applying the shortcut rule to an outcome only if the atom unconditionally enabled by it has one single result.

Definition 30. The deterministic shortcut rule, or d-shortcut rule, is defined by adding to the guard of the shortcut rule in Definition 22 a new condition: \(n'\) has at most one result. (The action of the d-shortcut rule is identical to that of the shortcut rule.)

This reduction strategy corresponds to Algorithm 1, where \(R(N)\) denotes the set of reducible outcomes of \(N\) (see Definition 25). The algorithm also gives priority to the merge rule over the d-shortcut rule.
Figure 14: An exponentially long reduction sequence
Algorithm 1 Summarization algorithm for an acyclic negotiation $N$

1: while $R(N) \neq \emptyset$ do
2:     if possible then apply the merge rule
3:     else if possible then apply the d-shortcut rule
4:     else stop and answer “unsound”
5:     end if
6: $N :=$ negotiation obtained after the application of the rule
7: end while

In the rest of the section we prove that for an acyclic deterministic negotiation $N$ Algorithm 1 terminates after at most length $K \cdot L$ rule applications, where $K$ and $L$ are the number of atoms and the number of outcomes of $N$, respectively. Moreover, if $N$ is sound, then the algorithm returns an atomic negotiation. Whether such an efficient reduction algorithm also exists for the weakly deterministic case is still open.

We first prove that Algorithm 1 terminates after at most $K \cdot L$ rule applications for arbitrary acyclic negotiations. For this we introduce a measure that strictly decreases when a rule is applied.

**Definition 31.** Let $N$ be a negotiation, and let $(n, r)$ be an outcome of $N$. A $(n, r)$-sequence is a finite occurrence sequence that starts with $(n, r)$ and satisfies $P_{n'} \subseteq P_n$ for every atom occurring in $\sigma$. A $(n, r)$-sequence is maximal if it cannot be extended to a longer $(n, r)$-sequence.

**Fact 1.** Let $x_n$ be the marking given by $x_n(p) = \{n\}$ if $p \in P_n$ and $x_n(p) = \emptyset$ otherwise. A sequence $\sigma$ is an $(n, r)$-sequence iff it starts with $(n, r)$ and is enabled at $x_n$. Moreover, $\sigma$ is maximal iff $x_n \xrightarrow{\sigma} x$ for a marking $x$ that enables no atom.

**Definition 32.** Let $N$ be a negotiation and let $(n, r)$ be an outcome of $N$. The index of $(n, r)$ in $N$ is the length of a longest maximal $(n, r)$-sequence of $N$ minus 1, if such sequence exists, or $\infty$, if $N$ has arbitrarily long $(n, r)$-sequences. The index of $N$, denoted by $I(N)$, is the sum of the indices of all outcomes $(n, r)$ such that $n \neq n_f$.

**Lemma 33.** Let $N$ be an acyclic negotiation, and let $N'$ be the result of applying the merge or the d-shortcut rule to $N$. Then $I(N') < I(N)$.

**Proof.** If the merge rule is applied to merge two outcomes $(n, r_1)$ and $(n, r_2)$, then $(n, r_1)$ and $(n, r_2)$ have the same index $\ell$, and $I(N') = I(N) - \ell$. Assume now that the d-shortcut rule is applied to some outcome $(n, r)$ of $N$, yielding a new outcome $(n, \hat{r})$. Since $(n, \hat{r})$ is a shortcut, for every outcome $(n', r') \neq (n, \hat{r})$ appearing in both $N$ and $N'$, the index of $(n', r')$ in $N'$ is smaller than or equal to the index of $(n', r')$ in $N$. Moreover, we have $\hat{\ell} = \ell - 1$, where $\ell$ and $\hat{\ell}$ are the indices of $(n, r)$ in $N$ and $(n, \hat{r})$ in $N$, respectively. So $I(N') \leq I(N) - \ell + \hat{\ell} < I(N)$, and we are done.

**Proposition 34.** Let $N$ be an acyclic negotiation with $K$ atoms and $L$ outcomes. Every maximal reduction sequence of $N$ containing only applications of the merge and d-shortcut
rules has length at most $K \cdot L$. In particular, Algorithm 1 terminates after at most $K \cdot L$ iterations of the while-loop.

**Proof.** By Lemma 33, and since $I(N) \geq 0$ by definition, it suffices to show $I(N) \leq K \cdot L$. Since $N$ is acyclic, every occurrence sequence fires an atom at most once. So the index of any outcome of $N$ is at most $K - 1$, and hence $I(N) \leq K \cdot L$. \[ \square \]

In the rest of the section we prove that if $N$ is acyclic, deterministic, and sound, then Algorithm 1 completely reduces $N$, i.e., returns an atomic negotiation. The proof, which is rather involved, proceeds in three steps. First, we show a technical lemma showing that sound and deterministic acyclic negotiations can be reduced so that all agents participate in every atom with more than one result (Lemma 35). Then we use this result to prove that, loosely speaking, every sound and deterministic acyclic negotiation can be reduced to a “replication” of a negotiation with only one agent (Lemma 37). Intuitively, a replication is a one-agent negotiation in disguise: Even if the negotiation has more than one agent, after each outcome they all move synchronously to the same new atom. Figure 15 shows a one-agent negotiation and its replication to two agents. In the third step, we show that the algorithm reduces replications to atomic negotiations.

The following lemma is proved in Appendix B:

**Lemma 35.** Let $N$ be an irreducible sound and deterministic acyclic negotiation and let $n \neq n_f$ be an atom of $N$ with more than one result. Then every agent participates in $n$.

Now we formally define replications.

**Definition 36.** An outcome $(n, r)$ is uniform if $X(n, p, r) = X(n, p', r)$ for every two parties $p, p' \in P_n$. A negotiation is a replication if all atoms have the same parties and every outcome is uniform.

Observe that final outcomes $(n_f, r)$ are uniform because $X(n_f, p, r) = \emptyset$ for every agent $p$ and result $r$. Moreover, if $(n, r)$ is uniform and $n \neq n_f$ then there is a unique atom $n'$ such that $X(n, p, r) = \{n'\}$ for every party $p$.  

![Figure 15: A one-agent negotiation and its replication to two agents.](image-url)
Lemma 37. Let \( N \) be an irreducible sound and deterministic acyclic negotiation. Then \( N \) is a replication.

**Proof.** We first show that every agent participates in every atom. By Lemma 35, it suffices to prove that every atom \( n \neq n_f \) has more than one result. Assume the contrary, i.e., some atom different from \( n_f \) has only one result. Since, by soundness, every atom can occur, there is an initial occurrence sequence \((n_0, r_0)(n_1, r_1) \cdots (n_k, r_k)\) such that \( n_k \) has only one result and all of \( n_0, \ldots, n_{k-1} \) have more than one result. By Lemma 35, all agents participate in all of \( n_0, n_1, n_{k-1} \). It follows that \((n_i, r_i)\) unconditionally enables \((n_{i+1}, r_{i+1})\) for every \( 0 \leq i \leq k-1 \). In particular, \((n_{k-1}, r_{k-1})\) unconditionally enables \((n_k, r_k)\). But then, since \( n_k \) has only one result, the d-shortcut rule can be applied to \( n_{k-1}, n \), contradicting the hypothesis that \( N \) is irreducible.

For the second part, assume that some outcome \((n, r)\) is not uniform. Then \( n \neq n_f \), and there are two distinct agents \( p_1, p_2 \) such that \( X(n, p_1, r) = \{ n_1 \} \neq \{ n_2 \} = X(n, p_2, r) \). By the first part, every agent participates in \( n, n_1 \) and \( n_2 \). Since \( N \) is sound, some reachable marking \( x \) enables \( n \). Moreover, since all agents participate in \( n \), and \( N \) is deterministic, the marking \( x \) only enables \( n \). Let \( x' \) be the marking given by \( x \xrightarrow{(n,r)} x' \). Since \( p_1 \) participates in all atoms, no atom different from \( n_1 \) can be enabled at \( x' \). Symmetrically, no atom different from \( n_2 \) can be enabled at \( x' \). So \( x' \) does not enable any atom, contradicting that \( N \) is sound.

**Proposition 38.** Let \( N \) be an irreducible sound and deterministic acyclic negotiation. Then \( N \) is atomic.

**Proof.** Assume \( N \) contains more than one atom. For every atom \( n \neq n_f \), let \( l(n) \) be the length of the longest path from \( n \) to \( n_f \) in the graph of \( N \). Let \( n_{\text{min}} \) be any atom such that \( l(n_{\text{min}}) \) is minimal, and let \( r \) be an arbitrary result of \( n_{\text{min}} \). By Lemma 37 there is an atom \( n \) such that \( X(n_{\text{min}}, a, r) = \{ n \} \) for every party \( p \) of \( n_{\text{min}} \). If \( n \neq n_f \), then by acyclicity we have \( l(n) < l(n_{\text{min}}) \), contradicting the minimality of \( l(n_{\text{min}}) \). So \( n = n_f \), and therefore \( X(n_{\text{min}}, p, r) = \{ n_f \} \) for every result \( r \) of \( n_{\text{min}} \) and every party \( p \). If \( n_{\text{min}} \) has more than one result, then the merge rule is applicable. If \( n_{\text{min}} \) has one result, then, since it unconditionally enables \( n_f \), the d-shortcut rule is applicable. In both cases we get a contradiction to irreducibility.

Proposition 38 proves that every maximal reduction sequence that uses the merge and d-shortcut rules leads to the same result: a summary of \( N \).

Putting Proposition 34 and Proposition 38 together, we obtain our result:

**Theorem 39.** Let \( N \) be an acyclic, deterministic negotiation. Algorithm 1 terminates after at most \( K \cdot L \) iterations of the while-loop, and returns an atomic negotiation iff \( N \) is sound.
6. Summarizing Deterministic Negotiations: The One-agent Case

We have shown that any maximal reduction sequence for a sound and deterministic acyclic negotiation that only applies the d-shortcut rule leads to a summary after a finite, polynomial number of steps. The following examples show that this does not hold for cyclic deterministic negotiations, not even for the degenerate case of negotiations with one single agent.

Figure 16: A cyclic negotiation where every atom has two outcomes

A first problem is that in the cyclic case we can no longer restrict ourselves to the d-shortcut rule. Consider the sound negotiation of Figure 16. Every atom has two results, and so the d-shortcut rule cannot be applied. Since the merge and iteration rules are not applicable either, the negotiation cannot be summarized unless we allow to apply the shortcut rule in more generality.

Figure 17: Shortcutting and cycles

A second problem is that in the cyclic case infinite reduction sequences are possible. Consider the negotiation on the left of Figure 17. If we apply the shortcut rule to \( n_0 \) and the result leading to \( n_1 \), we obtain the negotiation on the right. Applying the shortcut

---

3Since in the examples of this section the names of the results are not important, in all the figures we omit them for clarity.
rule to \( n_0 \) and the result leading to \( n_2 \), and then the merge rule, we obtain again the negotiation on the left. This process can be repeated arbitrarily often.

![Figure 18: A removed result is produced again later on](image)

To solve this problem, one could try the following policy: do not apply the shortcut rule if it produces an arc that has already been removed by an earlier rule (intuitively, we do not apply rules that “undo” reductions by earlier rule applications.) However, this policy prevents the summarization of the sound negotiation on the left of Figure 18. We start by applying the d-shortcut rule to \( n_0 \) and the result b, which yields the negotiation in the middle of the figure. It is easy to see that this negotiation cannot be summarized without generating again a result of \( n_0 \) leading to \( n_3 \). Indeed, after shortcutting \( c \) and applying the iteration rule we get the negotiation on the right, where after removing \( n_1 \) and \( n_2 \) with the shortcut rule we get a negotiation where \( n_0 \) contains again a result leading to \( n_3 \). Other reduction sequences starting at the negotiation in the middle also generate such a result. We therefore need a more sophisticated approach for the summarization of cyclic deterministic negotiations. In this section we consider the one-agent case, which already illustrates the main ideas. The general case is studied in Section 7.

Let \( N = (N, n_0, n_f, \mathcal{X}) \) be an arbitrary deterministic negotiation with only one agent. We first describe our summarization procedure, and then discuss its steps in more detail. We use the negotiation of Figure 19 as example.

The summarization procedure is parameterized by a fixed but arbitrary total order \( \prec \) on the atoms of the negotiation. In Figure 19 the order is \( 1 \prec 2 \prec \cdots \prec 6 \).

Using this order, we classify outcomes into backward and forward outcomes, and define a total order on outcomes.

**Definition 40.** Let \( N \) be a negotiation with one single agent \( p \). Let \( n \xrightarrow{r} n' \) denote \( \mathcal{X}(n,p,r) = \{n'\} \). Abusing language, we call \( n \xrightarrow{r} n' \) an outcome of \( N \) (instead of \( (n,r) \)).

An outcome \( n \xrightarrow{r} n' \) is backward if \( n' \neq n_f \) and \( n' \prec n \). Otherwise it is forward. We extend the order \( \prec \) to a total order on outcomes, also denoted by \( \prec \), as follows:

\[
(n_1 \xrightarrow{r_1} n'_1) \prec (n_2 \xrightarrow{r_2} n'_2) \iff n'_1 \prec n'_2 \text{ or } n'_1 = n'_2 \text{ and } n_1 \prec n_2.
\]
A backward outcome is minimal if no other backward outcome is strictly smaller with respect to $\prec$.

In Figure 19 the only backward outcome is $n_4 \rightarrow n_1$, which is also minimal. The classification of outcomes into forward and backward is only defined for the one-agent case; we will generalize it later.

Observe that every backward outcome $n \rightarrow n'$ is reducible, because $n' \neq n_f$ and thus the shortcut rule can be applied to it. The summarization procedure is shown in Algorithm 2. It gives priority to the merge and iteration rules, that is, they are applied whenever possible. If neither of them can be applied, then the algorithm selects a minimal backward outcome and applies the shortcut rule to it. If this is also not possible, then the algorithm applies the d-shortcut rule to another outcome. We will see that in this case the d-shortcut rule is necessarily applicable, and so that the algorithm never gets stuck.

Notice that the algorithm never answers “not sound”. The reason is that all one-agent negotiations are sound. Indeed, by definition every atom of a negotiation lies on a path between the initial and final atoms, which in the one-agent case implies soundness.

Let us apply Algorithm 2 to the negotiation of Figure 19. In the first iteration of the loop the algorithm shortcuts the only backward outcome $n_4 \rightarrow n_1$, yielding the negotiation on the left of Figure 20. This negotiation has two backward outcomes, $n_4 \rightarrow n_2$ and $n_4 \rightarrow n_3$. Since $n_3 \prec n_2$, the minimal backward outcome is $n_4 \rightarrow n_3$. After applying the shortcut rule to $n_4 \rightarrow n_3$, a merge yields the negotiation in the middle of the figure. In the third iteration, the algorithm shortcuts the only backward result $n_4 \rightarrow n_2$, yielding the negotiation on the right. An application of the iteration rule yields an acyclic negotiation, which is reduced by applying the merge and d-shortcut rules.


Algorithm 2 Summarization algorithm for an one-agent negotiation $N$

1: fix an arbitrary total order $\prec$ on the atoms of $N$
2: while $R(N) \neq \emptyset$ do
3:  if possible then apply the merge rule
4:  else if possible then apply the iteration rule
5:  else if possible then apply the shortcut rule to a minimal backward outcome
6:  else apply the d-shortcut rule
7:  end if
8:  $N :=$ negotiation obtained after the application of the rule
9: end while

Figure 20: Reduction of the negotiation of Figure 19.

6.1. Termination and Runtime Analysis

We show that Algorithm 2 always terminates after at most $2K^3 + K^2 + L$ rule applications, where $K$ and $L$ are the number of atoms and outcomes of the negotiation, respectively. Moreover, the algorithm always returns a summary of the negotiation.

Let $N$ be a one-agent negotiation, and let $N_1, N_2, \ldots$ be the sequence of negotiations produced by Algorithm 2 starting from $N_1 := N$, where each $N_{i+1}$ is obtained from $N_i$ by applying one of the rules. We call this sequence the computation of the algorithm on $N$.

We start the proof with a lemma:

Lemma 41. Let $N_1, N_2, \ldots$ be the computation of Algorithm 2 on a one-agent negotiation $N$. Let $N_{t_1}$ and $N_{t_2}$ be two negotiations of the computation to which Algorithm 2 applies the shortcut rule at line 5, and such such that $t_1 < t_2$. Let $n_1 \xrightarrow{n_1'} n_2$ and
$n_2 \xrightarrow{r_2} n_2'$ be the outcomes of $N_{t_1}$ and $N_{t_2}$ to which the shortcut rule is applied. Then $(n_1 \xrightarrow{r_1} n_1') \prec (n_2 \xrightarrow{r_2} n_2')$.

**Proof.** By the transitivity of $\prec$, it suffices to prove the result for the case in which the algorithm does not apply the shortcut rule to a negotiation $N_i$ with $t_i < i < t_2$.

If $N_{t_1}$ contains some outcome $n_2 \xrightarrow{r_2} n_2'$, then, since $n_2 \xrightarrow{r_2} n_2'$ is a backward outcome, $n_2 \xrightarrow{r_2} n_2'$ is a backward outcome too. Since $n_1 \xrightarrow{r_1} n_1'$ is the minimal backward outcome of $N_{t_1}$, we have $(n_1 \xrightarrow{r_1} n_1') \prec (n_2 \xrightarrow{r_2} n_2')$, and thus $(n_1 \xrightarrow{r_1} n_1') \prec (n_2 \xrightarrow{r_2} n_2')$.

Assume now that $N_{t_1}$ does not contain any outcome from $n_2$ to $n_2'$. Since between $t_1 + 1$ and $t_2 - 1$ the algorithm only applies the merge and iteration rules, and these rules do not create any additional outcomes, $n_2 \xrightarrow{r_2} n_2'$ is created by the application of the shortcut rule to the outcome $n_1 \xrightarrow{r_1} n_1'$ of $N_{t_1}$. By the definition of the shortcut rule, it follows that $n_2 = n_1$ and that $N_{t_1}$ contains an outcome $n_1' \xrightarrow{r_1} n_2$. If $n_1' = n_2'$ then, since the algorithm gives priority to the iteration rule over the shortcut rule, it would not have applied the shortcut rule to $n_1 \xrightarrow{r_1} n_1'$. So we have $n_1' \neq n_2'$.

If $n_2' \prec n_1'$ then $n_1' \xrightarrow{r_1} n_2$ is a backward outcome and by definition $(n_1' \xrightarrow{r_1} n_2') \prec (n_1 \xrightarrow{r_1} n_1')$. This contradicts the assumption that $(n_1 \xrightarrow{r_1} n_1')$ is the minimal backward outcome of $N_{t_1}$. So $n_1' < n_2'$, and thus $(n_1 \xrightarrow{r_1} n_1') \prec (n_2 \xrightarrow{r_2} n_2')$. \hfill $\square$

**Theorem 42.** If $N$ has $K$ atoms and $L$ outcomes, then Algorithm 2 terminates after at most $2K^3 + K^2 + L$ rule applications, and it summarizes $N$.

**Proof.** Let $N_1, N_2, \ldots$ be the computation of Algorithm 2 on $N$. We divide it into two phases. Phase I terminates immediately before the first execution of line 6. Phase II starts with the first execution of line 6 and continues until the end of the computation. We show that phase I and phase II terminate after at most $K^3 + K^2 + L$ and $K^2$ rule applications, respectively.

By Lemma 41, if during phase I of the computation the algorithm applies the shortcut rule to two outcomes $(n_1 \xrightarrow{r_1} n_1')$ and $(n_2 \xrightarrow{r_2} n_2')$, then $(n_1 \xrightarrow{r_1} n_1') \prec (n_2 \xrightarrow{r_2} n_2')$. In particular, this implies $n_1 \neq n_1'$ or $n_2 \neq n_2'$. So the number of applications of the shortcut rule during phase I is bounded by the number of pairs of atoms, i.e., by $K^2$.

Since the algorithm gives priority to the merge and iteration rules over the shortcut rule, it only applies the shortcut rule to negotiations having at most one result between any pair of nodes. Therefore, if the shortcut rule is applied to $n \xrightarrow{r} n'$, then the atom $n'$ has at most $K$ outcomes, and so the rule generates at most $K$ new outcomes. It follows that in phase I the merge and iteration rules are applied at most $K$ times between two consecutive applications of the shortcut rule, and also after the last application of the shortcut rule. Further, since every application removes one outcome, before the first application of the shortcut rule merge and iteration are applied together at most $L$ times. So the total number of applications of the merge and iteration rules during phase I is bounded by $L + K^2 \cdot K = L + K^3$, and the overall total number of applications is bounded by $K^3 + K^2 + L$.

Let $N_i$ be the first negotiation of phase II of the computation, i.e., the first negotiation to which the $d$-shortcut rule is applied at line 6. By the definition of the algorithm, $N_i$
has only forward outcomes. We prove that during phase II the algorithm applies at most \(K^3\) rules. We need two preliminary claims.

**Claim 1.** \(N_t\) is acyclic.

If \(n \rightarrow n\), then the iteration rule can be applied to \(N_t\), a contradiction. If \(N_t\) contains a cycle with at least two atoms, then the cycle contains at least one result \(n \rightarrow n'\) such that \(n \succ n'\). Since \(n_f\) does not belong to any cycle, we have \(n' \neq n_f\), and so \(n \rightarrow n'\) is backward, again a contradiction.

**Claim 2.** During phase II the algorithm only applies the merge, iteration, and d-shortcut rules.

Follows easily from the fact that the application of the shortcut rule to a negotiation that has only forward outcomes leads to another negotiation satisfying the same property. So \(N_{t'}\) contains only forward results for \(t' > t\). By the definition of the algorithm, line 5 is never executed, and we are done.

These two claims allow us to apply Proposition 34. We conclude that the length of the sequence of rules applied by the algorithm from \(N_t\) is bounded by \(K_t \cdot L_t\), where \(K_t\) and \(L_t\) are the number of atoms and non-final outcomes of \(N_t\). Clearly we have \(K_t \leq K\). Further, since neither the merge nor the iteration rule can be applied to \(N_t\), for every two atoms \(n, n'\) of \(N\) there is at most one outcome \(n \rightarrow n'\), and so \(L_t \leq K^2\). So \(K_t \cdot L_t \leq K^3\), and we are done.

It remains to show that Algorithm 2 summarizes \(N\), i.e., that it eventually reaches an atomic negotiation. It suffices to look at the suffix of the sequence starting with \(N_t\). By Claim 1, \(N_t\) is acyclic. Further, since \(N_t\) has one agent and every atom of \(N_t\) lies on a path between the initial and the final atoms, \(N_t\) is sound. By Proposition 38, the algorithm summarizes \(N_t\).

6.1.1. **Extension to Replications**

Recall that a replication is a negotiation whose atoms have all the same parties, and whose outcomes are all uniform, that is, for every result all parties move together to the same atom (see Definition 36). Algorithm 2 also works for replications. Indeed, in order to summarize a replication, we just apply the same sequence of rules that we would apply if all atoms had only one party. By Theorem 42, we obtain:

**Theorem 43.** Algorithm 2 summarizes a replication \(N\) with \(K\) atoms and \(L\) outcomes after at most \(2K^3 + K^2 + L\) rule applications.

While this extension is straightforward, it is important because such negotiations will appear at intermediate stages of our general summarization algorithm.

7. **Summarizing Deterministic Negotiations with Multiple Agents**

We present a reduction algorithm for deterministic negotiations and show that it summarizes every sound deterministic negotiation with \(K\) atoms and \(L\) outcomes after at
most $2K^3 + K^2 + KL + L$ rule applications.

Figure 21: A sound deterministic negotiation.

7.1. Informal description

We give a first overview of the summarization procedure using the sound deterministic negotiation of Figure 21 as example.

The procedure is based on the notion of the fragment of a negotiation generated by an atom. A fragment is almost a negotiation, with the only difference that the final atom may be missing, and so there may be “dangling outcomes”. For example, in the negotiation of Figure 21 the fragment generated by $n_1$ is highlighted in red, and $(n_4, a)$ is a “dangling outcome”. Intuitively, the fragment generated by an atom $n$ is the part of the negotiation that can occur from the marking $x$ satisfying $x(p) = \{ n \}$ for every party of $n$, and $x(p) = \emptyset$ otherwise.

The procedure first summarizes all one-agent fragments, i.e., all fragments generated by atoms with one party, then all two-agent fragments, and so on. Figure 22 shows on the left the one-agent fragments of the negotiation, and on the right the result of the reduction: The fragments generated by $n_2$ and $n_5$ are already summarized, and thus do not change, while the fragment generated by $n_6$ (which coincides with the fragment of $n_7$) is summarized into atom $n_{67}$. In the second stage the procedure summarizes the fragments of the atoms $n_1$ and $n_8$ (see Figure 23, where the fragments are drawn using different colors for clarity). Notice that the fragments generated by $n_1$, $n_3$, and $n_4$ are identical. The fragment of $n_6$ is atomic, while the fragment for $n_1$ is summarized into $n_{15}$. The only three-agent fragment of the resulting negotiation is the negotiation itself, and after the third step the negotiation is completely summarized.
The procedure to reduce a fragment is illustrated in Figure 24. On the left we see the fragment of the negotiation of Figure 21 generated by the atom $n_1$, with a “mock” final atom $\hat{n}_1$ for the dangling outcome. The procedure first removes all atoms with fewer agents than $n_1$, in this case $n_2$ and $n_5$. In this example this is achieved by two applications of the shortcut rule, and it yields the negotiation in the middle of the figure. This negotiation is nothing but the replication of the one-agent negotiation shown on the right of the figure: The two agents behave identically. We reduce this replication by means of the same sequence of rules we would use to reduce its one-agent counterpart.

Observe that, while Figure 24 shows the fragment separated from the rest of the negotiation, the sequence of rules applied to summarize the fragment can also be applied to the negotiation of Figure 21. The sequence produces the same effect, and eventually replaces the fragment by one single atom. It has no side-effects.

What happens if the summarization procedure is applied to an unsound deterministic negotiation? Since the rules preserve unsoundness, the procedure does not summarize the negotiation. It may not terminate, or terminate after a very large number of rule applications. Notice, however, that we can still use the algorithm to decide soundness in polynomial time. Indeed, if after $2K^3 + K^2 + KL + L$ rule applications the procedure has not summarized the negotiation, then we know that it will never do, and that the initial negotiation is unsound. We discuss this point in more detail later.

In Appendix C.1 we obtain some preliminary results. We study loops, which are occurrence sequences leading from a marking to itself. We prove a lemma which turns out to be fundamental for the sequel: The loops of sound deterministic negotiations contain a “synchronizing” atom involving all parties of all atoms in the loop. In Section 7.2 we formally define the fragment of a negotiation generated by an atom. We also define the segment generated by an outcome, and prove a second fundamental result: The fragments and segments of sound and deterministic negotiations are also sound negotiations. Section 7.3 describes the summarization algorithm for the sound case.
Sections 7.4 and 7.5 analyze its behavior. Section 7.6 provides the algorithm for the general case in which the negotiation can be sound or unsound.

7.2. Fragments and Segments

We show that every atom $n$ and every outcome $(n, r)$ of a sound deterministic negotiation induces a “sound subnegotiation” of $N$, called the $n$-fragment and the $(n, r)$-segment of $N$, respectively.

In Definition 31 we introduced $(n, r)$-sequences. Loosely speaking, a $(n, r)$-sequence is a finite occurrence sequence that starts with $(n, r)$ and can be executed by the parties of $n$ on their own, without involving any other agent. We now come back to this notion.

**Definition 44.** A sequence of outcomes is a $n$-sequence if it is a $(n, r)$-sequence for some result $r$ of $n$. A $n$-sequence is maximal if it cannot be extended to a longer $n$-sequence.
A \((n,r)\)-sequence is strict if every atom \(n'\) that appears in it after the first outcome \((n,r)\) satisfies \(P_{n'} \subset P_n\). A strict \((n,r)\)-sequence is maximal if it cannot be extended to a longer strict \((n,r)\)-sequence.

Let \(x_n\) be the marking given by \(x_n(p) = \{n\}\) if \(p \in P_n\) and \(x_n(p) = \emptyset\) otherwise. The target of a \((n,r)\)-sequence \(\sigma\) is the marking reached by firing \(\sigma\) from \(x_n\).

The sequences \((n_6,a)(n_7,a)\) and \((n_6,a)(n_7,b)(n_6,b)\) of the negotiation shown in Figure 21 are maximal \(n_6\)-sequences, but not strict \((n_6,a)\)-sequences. Calling the agents of the negotiation \(A, B,\) and \(C\), the target of all these sequences is the marking \(x\) given by \(x(A) = x(B) = \emptyset\) and \(x(C) = \{n_8\}\). The sequence \((n_1,a)(n_2,a)\) is a maximal strict \((n_1,a)\)-sequence, but not a maximal \(n_1\)-sequence. The sequence \((n_3,a)(n_4,a)\) is a maximal \(n_3\)-sequence, and its target is the marking \(x'\) given by \(x'(A) = \{n_f\}, x'(B) = \{n_8\}\), and \(x'(C) = \emptyset\).

Appendix C provides a proof of the proposition below, stating that, for every atom \(n\) of a sound deterministic negotiation, all maximal \(n\)-sequences have the same target. In our example, the markings \(x\) and \(x'\) above are the targets of all maximal \(n_6\)-sequences and all maximal \(n_2\)-sequences, respectively. The same property holds for the strict \((n,r)\)-sequences too.

**Proposition 45.** Let \(N\) be a sound and deterministic negotiation, and let \(n\) be an atom of \(N\).

(a) All maximal \(n\)-sequences have the same target.
That is: there is a unique marking \(x\) such that \(x_n \xrightarrow{\sigma} x\) for every maximal \(n\)-sequence \(\sigma\). We call \(x\) the target of \(n\).

(b) For every outcome \((n,r)\), all maximal strict \((n,r)\)-sequences have the same target.
That is: there is a unique marking \(x\) such that \(x_n \xrightarrow{\sigma} x\) for every maximal strict \((n,r)\)-sequence \(\sigma\). We call \(x\) the target of \(n,r\).

While the proof of this proposition is surprisingly non-trivial, there is a clear intuition for it. After a maximal \(n\)-sequence, the agents of \(P_n\) have to interact with agents of \(A \setminus P_n\) in order to reach the final marking, and these agents have to be ready for the interaction, otherwise a deadlock would arise. If maximal \(n\)-sequences can lead to different targets, then, since the agents of \(A \setminus P_n\) do not know which target will be chosen by the agents of \(P_n\), they must be ready to engage in different atoms. But this is not possible, because in a deterministic negotiation an agent is always committed to at most one atom.

Proposition 45 allows us to define the fragments of sound deterministic negotiations.

**Definition 46.** Let \(N = (N, n_0, n_f, \mathcal{X})\) be a sound and deterministic negotiation. Let \(n\) be an atom of \(N\), and let \(\text{tgt}(n)\) be its target. The \(n\)-fragment of \(N\) is the tuple \(\mathcal{F}_n = (F_n, n, \hat{n}, \mathcal{X}_n)\) defined as follows:

- \(F_n\) contains the atoms occurring in the \(n\)-sequences of \(N\), plus a fresh atom \(\hat{n}\). This atom \(\hat{n}\) has the same parties as \(n\), and it has one single result \(\hat{r}\).
• For every atom \( n' \in F_n \), party \( p \) and result \( r \):

\[
X_n(n', p, r) = \begin{cases} 
\emptyset & \text{if } n' = \hat{n} \text{ and } r = \hat{r} \\
\hat{n} & \text{if } X(n', p, r) = \text{tgt}(n) \\
X(n', p, r) & \text{otherwise}
\end{cases}
\]

The \( n \)-fragment of \( N \) is atomic if \( F_n = \{ n, \hat{n} \} \) and \( n \) has exactly one result.

Let \((n, r)\) be an outcome of \( N \), and let \( \text{tgt}(n, r) \) be its target. The \((n, r)\)-segment of \( N \) is the tuple \( S_{(n, r)} = (S_{(n, r)}, n, \hat{n}, X_{(n, r)}) \) defined exactly as the \( n \)-fragment above, but replacing the \( n \)-sequences by the strict \((n, r)\)-sequences of \( N \). Atomic segments are defined like atomic fragments.

The \( n_1 \)-fragment of the negotiation of Figure 21 is shown on the left of Figure 24. Observe that this is also the \( n_3 \)-fragment and the \( n_4 \)-fragment. The segments of the outcomes contained in the \( n_1 \)-fragment are all atomic, except the segments of \((n_1, a)\) and of \((n_4, b)\).

We show that the fragments and segments of sound deterministic negotiations are also sound. This result allows us to summarize sound negotiations “inside-out”: first summarize the fragments for agents with one party, then for agents with two parties, and so on.

**Proposition 47.** Let \( N = (N, n_0, n_f, X) \) be a sound and deterministic negotiation.

1. For every atom \( n \), the \( n \)-fragment \( F_n \) of \( N \) is a sound and deterministic negotiation.
2. For every outcome \((n, r)\), the \((n, r)\)-segment of \( N \) is a sound and deterministic negotiation.

**Proof.** We prove the result for the \( n \)-fragment \( F_n \), the proof for the \((n, r)\)-segment is analogous. Observe first that, by the definition of \( n \)- and \((n, r)\)-sequences, every agent of an atom of \( F_n \) is also an agent of \( n \), and so \( F_n \) is indeed a negotiation.

For soundness, observe first that every atom of \( F_n \) is executable: If \( n' \) belongs to \( F_n \), then by definition some \( n \)-sequence contains \( n \), and this sequence is also a sequence of \( F_n \).

Now let \( \sigma \) be an occurrence sequence of \( F_n \). Then \( \sigma \) is also an \( n \)-sequence of \( N \), and it can be extended to a maximal \( n \)-sequence \( \sigma \tau \). By Proposition 45 we have \( x_n \xrightarrow{\sigma \tau} x_n \) in \( F_n \), and so \( x_n \xrightarrow{\sigma \tau(\hat{n}, \hat{r})} x_f \).

**Corollary 48.** A deterministic negotiation is sound iff all its fragments and segments are sound.

**Proof.** One direction follows immediately from Proposition 47. For the other, it suffices to observe that, since every agent is a party of the initial atom, the fragment of the initial atom is the complete negotiation.
Algorithm 3: Summarization algorithm for a sound deterministic negotiation $\mathcal{N}$

1: fix an arbitrary order $\prec$ on the atoms of $\mathcal{N}$
2: for $k = 1$ to $A$ do
3:     while $R(\mathcal{N}, k) \neq \emptyset$ do
4:         if possible then
5:             apply the merge rule to some outcome of $R(\mathcal{N}, k)$
6:         else if possible then
7:             apply the iteration rule to some outcome of $R(\mathcal{N}, k)$
8:         else if possible then
9:             apply the d-shortcut rule to some non-uniform outcome of $R(\mathcal{N}, k)$
10:         else if possible then
11:             apply the shortcut rule to a backward uniform outcome of $R(\mathcal{N}, k)$
12:                 which is minimal w.r.t. $\prec$
13:         else
14:             apply the d-shortcut rule to some outcome of $R(\mathcal{N}, k)$
15:         end if
16:     end while
17: end for

7.3. The Summarization Algorithm

We first present Algorithm 3, an algorithm that summarizes a deterministic negotiation under the premise that it is sound. The general case is considered in Section 7.6. Algorithm 3 is a generalization of Algorithm 2 for one-agent negotiations. We start by explaining the meaning of $R(\mathcal{N}, k)$ in line 3 and backward outcome in line 11.

In the one-agent case all atoms have only one and thus the same number of parties. Now this is no longer the case, and to cope with this the new algorithm proceeds in stages. During the $k$-th stage the algorithm only applies rules to outcomes $(n, r)$ such that $n$ has $k$ parties.

Definition 49. An outcome $(n, r)$ is $k$-reducible if it is reducible and $n$ has $k$ parties. The set of $k$-reducible outcomes of $\mathcal{N}$ is denoted $R(\mathcal{N}, k)$.

Backward outcomes were introduced in Definition 40 for one-agent negotiations: An outcome $n \rightarrow n'$ is backward if $n' \neq n_f$ and $n' \prec n$. In the many-agent case we may have $X(n, p_1, r) = \{n'\}$ and $X(n, p_2, r) = \{n''\}$ for $n'' \neq n'$, and so not even the notation $n \rightarrow n'$ makes sense. But it does make sense when $X(n, p, r) = \{n'\}$ for every party $p$ of $n$, i.e., for uniform outcomes (recall Definition 36). So we write $n \leftrightharpoons n'$ to denote that $(n, r)$ is uniform and that $X(n, p, r) = \{n'\}$ holds for each $p \in P_n$, and generalize Definition 40 as follows:

Definition 50. A uniform outcome $n \leftrightharpoons n'$ is backward if $n' \neq n_f$ and $n' \prec n$. We extend the order $\prec$ to an order on uniform outcomes as in Definition 40.
There is another difference between Algorithm 3 and Algorithm 2. The if-command of lines 4-13 contains a new clause, which concerning the shortcut rule gives preference to non-uniform outcomes over uniform ones. To understand the reason for this, consider the fragment on the left of Figure 24. The non-uniform outcomes are \((n_1, a)\) and \((n_4, b)\). By giving them priority, we eliminate from this \(n_1\)-fragment all atoms with strictly fewer parties than \(n_1\) and “make the fragment uniform”.

The algorithm indeed summarizes the negotiation of Figure 21 in the way sketched at the beginning of the section. We choose the order \(\preceq\) given by the numbering of the atoms. During the execution of the while-loop for \(k = 1\) the algorithm executes lines 5, 7, and 11 to yield the negotiation on the right of Figure 22. The execution for \(k = 2\) begins with applications of the d-shortcut rule to the non-uniform outcomes \((n_1, a)\), and \((n_4, b)\) in line 9, which removes the atoms \(n_2\) and \(n_5\). After that, lines 5, 7, and 11 are executed to yield the negotiation on the right of Figure 23. Finally, the execution for \(k = 3\) applies the d-shortcut rule in line 9 several times, removing the atoms \(n_{15}\), \(n_{67}\), and \(n_8\), and leaving a negotiation with only the atoms \(n_0\) and \(n_f\). A final execution of line 13 finishes the summarization.

We conclude the section with a small lemma. Observe that the algorithm forbids to first apply a reduction to a \(k\)-reducible outcome and then to a \(k'\)-reducible outcome where \(k' < k\). This may seem dangerous: In principle, rule applications to \(k\)-reducible outcomes might produce new \(k'\)-reducible outcomes, and the algorithm would not be able to “use” these outcomes for further reductions. The following lemma shows that this is not the case.

**Lemma 51.** Let \(N\) be a negotiation without \(k\)-reducible outcomes for any \(k \leq K\), and let \(N'\) be obtained from \(N\) by an arbitrary sequence of reductions. Then \(N'\) also has no \(k\)-reducible outcomes for any \(k \leq K\).

**Proof.** Clearly, it suffices to prove the result for sequences of length 1. Assume the sequence applies a rule to a reducible outcome \((n, r)\). Then \(n\) has more than \(k\) parties. If the rule is the iteration rule, then every reducible outcome of \(N'\) was already a reducible outcome of \(N\), and we are done. In the case of the merge and shortcut rules, the only outcomes that might be reducible in \(N'\) but not in \(N\) are those added as a consequence of the rule. But these are outcomes of \(n\), which has more than \(k\) parties. \(\square\)

This lemma already can be used to show that Algorithm 3 summarizes all sound and deterministic acyclic negotiations. Assume this is not the case. By Proposition 34 such a negotiation is reducible by any maximal sequence of applications of the merge and d-shortcut rules, in particular by any in which we always choose a reducible outcome with a minimal number of parties for the next rule application, and within those we give preference to the merge over the shortcut rule. By Lemma 51, if \(k < k'\) then such sequences never choose a \(k'\)-reducible outcome and then later a \(k\)-reducible one, and so the algorithm executes one of them.
7.4. Completeness

We prove a proposition at the core of our completeness result which states that Algorithm 3 summarizes all sound deterministic negotiations. Intuitively, the proposition shows that the algorithm summarizes the negotiation “inside-out”, meaning that after $k$ iterations of the for-loop all fragments for atoms with $k$ parties have been summarized. Intuitively, the proof shows that before the $k$th iteration all the segments of these fragments are acyclic, and that the $k$-iteration performs the following two steps:

- Summarize all these segments using the merge and d-shortcut rules. This transforms the fragments into replications.
- Summarize the replications using Algorithm 2.

**Proposition 52.** Let $N$ be a sound deterministic negotiation, and assume Algorithm 3 terminates for input $N$. For every $k \geq 0$, let $N^k$ be the negotiation produced by Algorithm 3 after $k$ iterations of the for-loop.

1. For every atom $n$ of $N$ with at most $k$ parties, the $n$-fragment of $N^k$ is atomic.
2. For every outcome $(n, r)$ of $N$ with at most $k+1$ parties, the $(n, r)$-segment of $N^k$ is acyclic.

**Proof.** We prove (1) and (2) simultaneously by induction on $k$. If $k = 0$ then (2) is vacuously true. For (1) just observe that if $n$ has one party then the only $(n, r)$-sequence is $(n, r)$, and so the $(n, r)$ segment is acyclic.

Assume now $k > 0$, and let $n$ be an atom of $N^k$ with $\ell \leq k$ parties. We prove (1) in four steps.

**Claim 1.** All segments of the $n$-fragment are acyclic.

By the definition of a fragment, every atom of the $n$-fragment of $N^{k+1}$ has at most $\ell$ parties. By induction hypothesis on (2), all segments of the $n$-fragment of $N^k$ are acyclic, and therefore the same holds for $N^{k+1}$.

**Claim 2.** All segments of the $n$-fragment are atomic.

By Theorem 47, the segments are sound and deterministic. Moreover, since every atom of a segment has at most $\ell$ parties and no outcome with at most $k-1$ parties is reducible, all segments are irreducible. By Claim 1 and Proposition 38, all segments are atomic.

**Claim 3.** The $n$-fragment is a replication.

Since every segment is atomic, all atoms of the $n$-fragment have the same set of parties as $n$. Now let $(n', r')$ be an outcome of the $n$-fragment. By Claim 2 the $(n', r')$-segment is atomic, and so in the negotiation $N^k$ we have $x_{n'} \xrightarrow{(n', r')} x$ for a marking $x$ such that either $x = x_{n''}$ for some atom $n''$ with the same parties as $n'$, or $x$ does not enable any atom. In the first case we have $n' \xrightarrow{r'} n''$, and we are done. In the second case, by the definition of a segment we have $n' \xrightarrow{r'} \hat{n}$.

**Claim 4.** The $n$-fragment is atomic.

Assume the $n$-fragment is not atomic. Then by Claim 3 and Theorem 43 the $n$-fragment
has at least one reducible outcome, and the outcome has \( \ell \leq k \) parties. This contradicts the hypothesis that the minimal reducible outcome has \( k + 1 \) parties.

Now we proceed to prove (2). Let \((n, r)\) be an outcome of \( N \) with at most \( k + 1 \) parties. Assume that the \((n, r)\)-segment is cyclic. By Theorem 47, the segment is sound and deterministic. By Proposition 62, the segment has a loop, and by Proposition 64, proved in the Appendix, the loop has a synchronizer \( s \). By the definition of a segment, \( s \) is neither the initial atom \( n \) nor the final atom of the segment. Since, by definition, every atom of the \((n, r)\)-segment different from the initial and final atoms has strictly fewer parties than \( n \), we get \( |P_s| \leq k \). By induction hypothesis on (1), the \( s \)-fragment is atomic. But this contradicts that \( s \) is a synchronizer.

**Theorem 53.** Algorithm 3 summarizes every sound deterministic negotiation \( N \).

**Proof.** Let \( K \) be the number of agents of \( N \). By Theorem 47, after termination the \( n_0 \)-fragment is atomic. It follows that \( N^K \) is atomic, and we are done.

### 7.5. Runtime Analysis

It is easy to give a \( O(K^4 + KL) \) bound for the number of rule applications of Algorithm 3. Indeed, since each \( n \)-fragment can be summarized using at most \( 2K^3 + K^2 + L \) applications and the negotiation has \( K \) atoms, the bound follows.

We obtain a better, \( O(K^3 + KL) \) bound. The key observation is that, for a given number \( k \), the fragments of the atoms with \( k \) parties may share nodes. Consider for instance the negotiation of Figure 25, and take \( k = 1 \). The three atoms with one party are \( n_1, n_2, \) and \( n_5 \). The fragments \( \mathcal{F}_{n_1} \) and \( \mathcal{F}_{n_2} \) share the nodes \( n_3 \) and \( n_4 \). Therefore, adding the number of rule applications needed to summarize the three fragments separately gives us a very crude bound.

We define the \( k \)-fragment of \( N \) as, loosely speaking, the union of all the \( n \)-fragments for all atoms \( n \) with exactly \( k \) parties. In Figure 25 the 1-fragment of \( N \) is shown at the bottom, on the right. We now formally define the \( k \)-fragment of a sound deterministic negotiation, and give a bound on the number of rule applications needed to summarize it.

**Definition 54.** Let \( N = (N, n_0, n_f, \mathcal{X}) \) be a deterministic sound negotiation. Let \( N^i \) be the set of atoms of \( N \) with exactly \( i \) parties. For every \( n \in N_i \), let \( \mathcal{F}_n = (F_n, n, \hat{n}, X_n) \) be the \( n \)-fragment, with the fresh final atoms \( \{\hat{n} \mid n \in N_i\} \) chosen in such a way that \( \hat{n}_1 = \hat{n}_2 \) holds for any two atoms \( n_1, n_2 \in N^1 \) satisfying \( P_{n_1} = P_{n_2} \) (i.e., the fragments of two atoms have the same final atom iff they have the same parties).

The \( i \)-fragment of \( N \) is the pre-negotiation \( \mathcal{F}^i = (F, \mathcal{X}^i) \) defined as follows

- \( F = \bigcup_{n \in N^i} F_n \)
- for every \( n \in F, \ p \in P_n, \ r \in R_n: \mathcal{X}^i(n, p, r) = \mathcal{X}_{n'}(n, p, r), \) where \( n' \) is any atom with \( i \) parties such that \( n \in F_{n'} \).

The \( k \)-fragment is summarized if \( \mathcal{X}^i(n, p, r) \neq \emptyset \) implies that \( n \in N_i \) and \( \mathcal{X}^i(n, p, r) = \{\hat{n}\} \).
Figure 25: A sound deterministic negotiation, its three one-party fragments, and their combination into the 1-fragment.

**Lemma 55.** Let $N$ be a sound deterministic negotiation. Let $F^i$ be the $i$-fragment of $N$, and let $K$ and $L$ be the number of nodes and outcomes of $F^1$. If $F^i$ is a replication, then Algorithm 2 summarizes $N^i$ after at most $2K^3 + K^2 + L$ rule applications.

**Proof.** By the correspondence between replications and one-agent (pre-)negotiations, it suffices to prove the result for $i = 1$. The proof is exactly as the proof of Theorem 42. Indeed, an inspection shows that the proof does not use the fact that a negotiation has one single initial atom. The only point that has to be slightly adapted appears at the very end of the proof: A negotiation with only the initial and final atom can be reduced to an atomic negotiation. On the contrary, a pre-negotiation with multiple initial atoms and one final atom cannot, because the shortcut rule cannot be applied. However, this is only a technicality, because the final atoms are atoms added artificially to give a destination to “dangling outcomes”. □
Theorem 56. Let \( N \) be a sound deterministic negotiation with \( K \) atoms, and \( L \) outcomes. Algorithm 3 terminates after at most \( 2K^3 + K^2 + KL + L \) rule applications.

Proof. For every \( i \geq 1 \) let \( k_i \) and \( \ell_i \) be the number of atoms and outcomes of the atoms \( n \) and outcomes \( (n, r) \) such that \( n \) has exactly \( i \) parties, respectively. Further, let \( K_i = \sum_{j=1}^i k_j \) and \( L_i = \sum_{j=1}^i \ell_j \). We have \( K_{i+1} = K_i + k_{i+1} \) and \( L_{i+1} = L_i + \ell_{i+1} \) for every \( i \geq 1 \).

We prove that for every \( i \geq 1 \) the \( i \)-th iteration of the for-loop terminates after at most \( 2K_i^3 + K_i^2 + K_i L_i + L_i \) rule applications. We proceed by induction on \( i \).

Case \( i = 1 \). During the first iteration of the loop the algorithm summarizes the 1-fragment of \( N \), and so the result follows by Lemma 55.

Case \( i > 1 \). By induction hypothesis, the \((i-1)\)-th iteration of the for-loop has terminated after at most \( 2K_{i-1}^3 + K_{i-1}^2 + K_{i-1} L_{i-1} + L_{i-1} \) rule applications. Let \( N^{i-1} \) be the negotiation obtained after the \((i-1)\)-th iteration. By Proposition 52, the fragments of \( N^{i-1} \) for atoms with at most \( i - 1 \) parties are atomic, and the segments of outcomes with at most \( i \) parties are acyclic.

Claim. We claim that after further \( K_i \ell_i \) rule applications all segments for atoms with at most \( i \) parties are not only acyclic but also atomic.

Consider a non-atomic \((n, r)\)-segment such that \( n \) has at most \( i \) parties. If \( n \) has less than \( i \) parties, then the \( n \)-fragment is atomic, and the claim follows. So assume that \( n \) has exactly \( i \) parties. By definition all atoms of the \((n, r)\)-segment different from \( n \) have less than \( i \) parties. We have:

1. \((n, r)\) unconditionally enables some atom with fewer parties than \( n \), that is, \((n, r)\) is non-uniform.
   Otherwise the \((n, r)\)-segment is not sound, contradicting the soundness of \( N \).

2. The index of the \((n, r)\)-segment is equal to or smaller than its number of atoms. Recall that the index of an outcome \((n', r')\) in the \((n, r)\)-segment is the length of a longest maximal \((n', r')\)-sequence, and the index of the \((n, r)\)-segment is the sum of the indices of its atoms (Definition 32). Since the \((n, r)\)-segment is acyclic, the index of \((n, r)\) is at most the number of atoms of the segment minus 1. Consider now any other outcome \((n', r')\) of the \((n, r)\)-segment. Then \( n' \neq n \), and so \( n' \) has fewer parties than \( n \). But then the \((n', r')\)-segment is atomic, which implies that the index of \((n', r')\) is 0. So the total index of the segment is at most equal to its number of atoms.

By (1), the algorithm keeps choosing non-uniform outcomes of the segments with \( i \) parties, until there are no more non-uniform atoms. At this point, all \((n, r)\)-segments are irreducible, and by Proposition 38 atomic. Since there at most \( \ell_i \) segments, each of them with at most \( K_i \) atoms, by (2) and Lemma 33 the segments become atomic after at most \( K_i \cdot \ell_i \) rule applications. This finishes the proof of the claim.

Let \( \tilde{N}^{i-1} \) be the negotiation obtained after the sequence of \( K_i \cdot \ell_i \) rule applications. By the claim, the fragments of atoms of \( \tilde{N}^{i-1} \) with \((i-1)\) parties and the segments of atoms
with *i* parties are atomic. It follows that the outcomes of \( \hat{N}^{i-1} \) that have exactly *i* parties are uniform. As shown in Claim 3 in the proof of Proposition 52, this implies that all fragments for atoms with *i* parties are replications. So the complete \( i \)-fragment of \( \hat{N}^{i-1} \) has at most \( k_i \) atoms and \( \ell_i \) outcomes. By Lemma 55, the \( i \)-fragment is summarized after at most \( 2k_i^3 + k_i^2 + \ell_i \) rule applications.

So we obtain that the \( i \)-th iteration of the for-loop terminates after at most

\[
(2K_i^3 + K_i^2 + K_i - 1 + L_i - 1) + K_i \ell_i + (2k_i^3 + k_i^2 + \ell_i)
\]

iterations. Now, using \( K_i = K_{i-1} + k_i \) and \( L_i = L_{i-1} + \ell_i \), we get:

\[
(2K_{i-1}^3 + K_{i-1}^2 + K_{i-1} - 1 + L_{i-1} - 1) + K_{i-1} \ell_{i-1} + (2k_i^3 + k_i^2 + \ell_i)
\]

\[
= 2(K_{i-1}^3 + k_i^3) + (K_{i-1}^2 + k_i^2) + (K_{i-1} - 1 + K_{i-1} \ell_{i-1}) + (L_{i-1} + \ell_{i-1})
\]

\[
\leq 2(K_i^3 + k_i^3) + (K_i - 1 + k_i)^2 + K_i(L_i - 1 + \ell_i) + (L_i - 1 + \ell_i)
\]

\[
= 2K_i^3 + K_i^2 + K_i L_i + L_i
\]

and we are done. \( \square \)

### 7.6. The Unsound Case

Algorithm 4 assumes that its input is a sound deterministic negotiation. It is easy to transform it into an algorithm for arbitrary deterministic negotiations, with the same complexity. The algorithm is shown as Algorithm 4. If the input negotiation is sound, Algorithm 4 algorithm summarizes it, and otherwise it answers “unsound”. The algorithm counts the number of rule applications and answers “unsound” if the counter exceeds \( 2K^3 + K^2 + KL + L \). This is correct by Theorem 56. The algorithm also reports “unsound” if it reaches an irreducible but non-atomic net, or if it can only apply the shortcut rule to non-uniform or uniform forward outcomes.

### 7.7. A Correction

In [ED14] we claimed that the merge, iteration, and shortcut rule were complete for sound deterministic negotiations, and presented a reduction algorithm. While the completeness result holds, the algorithm of [ED14] is unfortunately incorrect. The algorithm is based on a lemma (lemma number 3 in [ED14]), which, rewritten in the terminology of this paper, states:

**Lemma 57** (Incorrect Lemma 3 of [ED14]). A cyclic sound deterministic negotiation \( N \) contains an atom \( n \) such that all loops of the \( n \)-fragment of \( N \) contain \( n \).

This lemma was used in [ED14] to prove the correctness of the summarization procedure that iterates the following three steps. First: identify an \( n \)-fragment satisfying the property stated in the lemma. Second: use the procedure for summarizing acyclic negotiations to reduce this \( n \)-fragment to a single atom, with possibly some self-loop outcomes (i.e., outcomes \((n, r)\) such that \( X(n, a, r) = n \) for every party \( a \) of \( n \)). Third, use the iteration rule to remove such outcomes.
Algorithm 4 Summarization algorithm for arbitrary deterministic negotiations

1: fix an arbitrary order $\prec$ on the atoms of $N$
2: $counter := 0$
3: for $k = 1$ to $A$ do
4:  while $R(N, k) \neq \emptyset$ and $counter \leq 2K^3 + K^2 + KL + L$ do
5:    if possible then
6:      apply the merge rule to some outcome of $R(N, k)$
7:    else if possible then
8:      apply the iteration rule to some outcome of $R(N, k)$
9:    else if possible then
10:       apply the d-shortcut rule to some non-uniform outcome of $R(N, k)$
11:    else if possible then
12:       apply the shortcut rule to a backward uniform outcome of $R(N, k)$
13:       minimal w.r.t. $\prec$
14:    else if possible then
15:       apply the d-shortcut rule to some outcome of $R(N, k)$
16:    else stop and answer “unsound”
17:  end if
18:  $N :=$ negotiation obtained after the application of the rule to $N$
19:  $counter := counter + 1$
20: end while
21: end for
22: if $N$ is not atomic then answer “unsound”
23: end if

However, the lemma is wrong. Figure 26 shows a one-agent negotiation in which, for every atom $n$, the $n$-fragment contains a loop which does not contain $n$. As a consequence, the algorithm of [ED14] does not summarize this negotiation.

In this paper we have corrected this mistake. The procedure of Section 6 correctly summarizes the negotiation of Figure 26.

8. Conclusion

We have introduced negotiations, a formal model of concurrency with atomic negotiations (a form of synchronized choice) as primitive. We have defined and studied two important analysis problems: soundness, which coincides with the soundness notion for workflow nets, and the new summarization problem. We have provided a complete set of reduction rules for sound deterministic negotiations, and we have shown that, when applied following a simple strategy, the rules allow one to compute a summary of a sound deterministic negotiations with a polynomial number of rule applications.

The reduction algorithm is based on several results about the structure of deterministic negotiations that have independent interest. Proposition 64 and its syntactic counter-
part, Proposition 68, are important results about the structure of loops. Proposition 45 shows that every sound deterministic negotiation is necessarily organized compositionally.

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A. Appendix: Complexity of Soundness

Theorem 10. The soundness problem is PSPACE-complete. For acyclic negotiations, the problem is co-NP-hard and in DP (and so at level $\Delta^P_2$ of the polynomial hierarchy).

Proof. The soundness problem is in PSPACE.
Membership in PSPACE could be proved by observing that the soundness problem can be formulated in CTL, and then applying the PSPACE algorithm for CTL and 1-safe Petri nets of [Esp96]. This algorithm only assumes that, given a marking, one can compute a successor marking in polynomial time, which is the case both for Petri nets and for negotiations. However, since we only need a very special case of the CTL algorithm, we provide a self-contained proof.

We show that both conditions for soundness can be checked in nondeterministic polynomial space. The result then follows from Savitch's theorem (NPSPACE=PSPACE).

The first condition is: every atom is enabled at some reachable marking. For this we consider each atom $n$ in turn, and guess step by step an occurrence sequence ending with an occurrence of $n$. This only requires to store the marking reached by the sequence executed so far.

The second condition is: every occurrence sequence from the initial marking is either a large step or can be extended to a large step. This case is a bit more involved. Let $S$ denote the problem of checking this second condition. We prove $S \in$ PSPACE.

(1) The following problem is in PSPACE: given some marking $x$, check that no occurrence sequence starting at $x$ ends with the final atom.
Let us call this problem NO-OCC. We have NO-OCC $\in$ NPSPACE, because we can nondeterministically guess an occurrence sequence starting at $x$ that ends with the final atom (we guess one step at a time). Since NPSPACE=PSPACE=co-PSPACE, we get NO-OCC $\in$ PSPACE.

(2) $\overline{S} \in$ NPSPACE.
$\overline{S}$ consists of checking the existence of a sequence $\sigma$, fireable from the initial marking, that is neither a large step nor can be extended to it. For this we guess a sequence $\sigma$ step by step that does not end with the final atom. Then we consider the marking $x$ reached by the occurrence of $\sigma$. Clearly, we have $\sigma \in \overline{S}$ iff $x \in$ NO-OCC. So it suffices to apply our deterministic polynomial-space algorithm for NO-OCC (see (1)).

(3) $S \in$ PSPACE.
Follows from (2) and NPSPACE=PSPACE=co-PSPACE.

The soundness problem is PSPACE-hard.
For PSPACE-hardness, we reduce the problem of deciding if a deterministic linearly bounded automaton (DLBA) recognizes an input to the soundness problem. Let $A =$
(Q, Σ, δ, q0, F) be a DLBA, and consider an input $w = a_1 \ldots a_k \in \Sigma^*$. The construction is very similar to that of [Esp96] for proving PSPACE-hardness of the reachability problem for 1-safe Petri nets, and so we do not provide all details. The negotiation $N_A$ has a control agent $C$, a head agent $H$, and a cell agent $T_i$ for every tape cell (i.e., $1 \leq i \leq k$). All agents have only one internal state, i.e., the internal states are irrelevant. The negotiation has an atom $n[q, h, a]$ (with only one result) for every state $q$, every head position $1 \leq h \leq k$, and every $a \in \Sigma$, plus an initial atom $n_0$ and a final atom $n_f$. The parties of $n[q, h, a]$ are $C$, $H$, and $T_h$. The transition function $X$ is defined so that $N_A$ simulates $A$ in the following sense: If $A$ is currently in state $q$ with the head at position $h$, and the contents of the tape are $b_1 \ldots b_k$, then the current marking $x$ of the negotiation satisfies the following properties:

- if $q \neq q_f$, then $x(C)$ is the set of atoms $n[h', q', a]$ such that $q' = q$, and both $h'$ and $a$ are arbitrary; if $q = q_f$, then $x(C) = \{n_f\}$;
- $x(H)$ is the set of atoms $n[h', q', a]$ such that $h' = h$ and $q', a$ are arbitrary, plus the final atom;
- $x(T_i)$ is the set of atoms $n[h', q', a]$ such that $h' = i$, $q'$ is arbitrary, and $a = b_i$, plus the final atom.

Intuitively, agent $C$ is only ready to engage in atoms for the state $q$; agent $H$ is only ready to engage in atoms for the position $h$; and $T_h$ is only ready to engage in atoms for the letter $b_i$. These properties guarantee that the only atom enabled by $x$ is $n[h, q, b_k]$ if $q \neq q_f$, or the atom $n_f$ if $q = q_f$. So the negotiation $N_A$ has only one initial occurrence sequence, which corresponds to the execution of $A$ on $w$.

It remains to define $X$ so that it satisfies these properties. For the initial atom we take
(recall that the input of the DLBA $A$ is the word $w = a_1 \ldots a_k$):

\[
X(n_0, C, \text{step}) = \{n[h', q_0, a'] \mid 1 \leq h' \leq k, a' \in \Sigma\}
\]

\[
X(n_0, H, \text{step}) = \{n[1, q', a'] \mid q' \in Q, a' \in \Sigma\}
\]

\[
X(n_0, T_i, \text{step}) = \{n[i, q_0, a_i]\}
\]

For the transition function of an atom $n[q, h, a]$ we must consider the three possible cases of the transition relation (head moves to the right, to the left, or stays put). We only deal with the case in which the machine moves to the right, the others being analogous. Assume $δ(q, a) = (q', a_R)$. Then we take

\[
X(n[h, q, a], C, \text{step}) = \{n[h', q, a'] \mid 1 \leq h' \leq k, a' \in \Sigma\}
\]

\[
X(n[h, q, a], H, \text{step}) = \{n[h + 1, q', a'] \mid q' \in Q, a' \in \Sigma\}
\]

\[
X(n[h, q, a], T_h, \text{step}) = \{n[h, q', a_i] \mid q' \in Q\}
\]

Since $A$ is deterministic, $N_A$ has only one maximal occurrence sequence, which is a large step iff $A$ accepts. So $N_A$ is sound iff $A$ accepts.

The soundness problem for acyclic negotiations is in DP.
We first observe that no occurrence of an acyclic negotiation contains an atom more than once (loosely speaking, once the tokens of the parties of the atom have "passed" beyond it, they cannot return). It follows that the length of an occurrence sequence is at most equal to the number of atoms. It also follows that there are no livelocks, but there may be deadlocks. To check soundness we must check that (1) every atom can be enabled, and that (2) every occurrence sequence can be extended to a large step. Checking (1) can be done by guessing in polynomial time enabling sequences for all atoms, and so (1) is in NP. Checking the negation of (2) can be done by guessing in polynomial time an occurrence sequence that cannot be extended to a large step because it leads to a deadlock, and so (2) is in coNP. So the conjunction of (1) and (2) is in DP.

The soundness problem for acyclic negotiations is co-NP-hard.

We reduce 3-CNF-UNSAT to soundness. Given a boolean formula \( \phi \) with variables \( x_i \), \( 1 \leq i \leq n \) and clauses \( c_j \), \( 1 \leq j \leq m \), we construct a negotiation \( N_\phi \) with an agent \( X_i \) for each \( x_i \), and an agent \( J \) (for judge). W.l.o.g. we assume that no clause of \( \phi \) is a tautology. For each variable \( x_i \), \( N_\phi \) has an atom \( Set_x \) with \( X_i \) as only party and results true and false. For each clause \( c_j \), the negotiation \( N_\phi \) has an atom \( False \) whose parties are the variables appearing in \( c_j \) and the judge \( J \). The atom has only one result false.

After the initial atom, agent \( X_i \) engages in \( Set_x \) and sets \( x_i \) to a value \( b \in \{\text{true, false}\} \) by choosing the appropriate result. After that, \( X_i \) is ready to engage in the atoms \( False \) satisfying the following condition: the clause \( c_j \) is not made true by setting \( x_i \) to \( b \); moreover, it is also ready to engage in the final atom. As a consequence, \( False \) becomes enabled iff the assignment chosen by the \( X_i \)'s makes \( c_j \) false. Finally, after the occurrence of a \( False \), its parties are only ready to engage in the final atom.

After the initial atom, the judge \( J \) is ready to engage in all atoms \( False \), and then, if any of them occurs, in the final atom.

We argue that \( N_\phi \) is sound iff \( \phi \) is unsatisfiable. Notice first that, since by assumption no clause is a tautology, every \( False \) atom is enabled by some occurrence sequence. So all atoms but perhaps the final atom can be enabled by some sequence. So \( N_\phi \) is sound iff every occurrence sequence can be extended to a large step, and therefore it suffices to show that \( \phi \) is unsatisfiable iff every occurrence sequence of \( N_\phi \) can be extended to a large step.

If \( \phi \) is unsatisfiable then, whatever the assignment determined by the outcome of the \( Set_x \)'s, some clause is false, and so at least one of the \( False \) atoms is enabled. After some \( False \) occurs, the final atom becomes enabled, and so the computation can be extended to a large step.

If \( \phi \) is satisfiable, then consider an initial occurrence sequence in which the atoms \( Set_x \) occur, and then choose the outcomes corresponding to a satisfying assignment. This way none of the \( False \) atoms become enabled. Moreover, the final atom is not enabled either, because the judge \( J \) is not ready to engage in it. So the occurrence sequence cannot be extended to a large step.
B. Appendix: A Lemma on Irreducible Acyclic Negotiations

Lemma 35. Let \( N \) be an irreducible sound and deterministic acyclic negotiation and let \( n \neq n_f \) be an atom of \( N \) with more than one result. Then every agent participates in \( n \).

Proof. We proceed in two steps.

(a) The atom \( n \) has a result \( r \) such that either \((n, r)\) unconditionally enables \( n_f \), or \((n, r)\) unconditionally enables some atom with more than one result.

We first prove a preliminary claim: if some outcome \((n, r)\) unconditionally enables some atom, then (a) holds. Indeed: if \((n, r)\) unconditionally enables some atom \( n' \), then either \( n' = n_f \) or \( n' \) has more than one result, because otherwise the d-shortcut rule can be applied to \( n \) and \( n' \), contradicting the irreducibility of \( N \). This proves the claim.

It remains to show that some outcome \((n, r)\) unconditionally enables some atom. For this, we assume the contrary, and prove that \( N \) contains a cycle, contradicting that \( N \) is acyclic.

Since the merge rule is not applicable to \( N \), the atom \( n \) has two results \( r_1, r_2 \) such that \( X(n, a, r_1) \neq X(n, p, r_2) \) for some party \( a \). We proceed in three steps.

(a1) For every reachable marking \( x \) that enables \( n \) there is a sequence \( \sigma \) such that \( x \xrightarrow{(n,r_1)\sigma} x_1 \) and \( x \xrightarrow{(n,r_2)\sigma} x_2 \) for some markings \( x_1, x_2 \) such that the sets \( N_1 \) and \( N_2 \) of atoms enabled by \( x_1, x_2 \) are nonempty and disjoint.

Let \( \sigma \) be a longest occurrence sequence such that \( x \xrightarrow{(n,r_1)\sigma} x_1 \) and \( x \xrightarrow{(n,r_2)\sigma} x_2 \) for some markings \( x_1, x_2 \) (notice that \( \sigma \) exists, because all occurrence sequences of \( N \) are finite by acyclicity). We have \( N_1 \cap N_2 = \emptyset \), because otherwise we can extend \( \sigma \) with the occurrence of any atom enabled by both markings. We prove that, furthermore, \( N_1 \neq \emptyset \neq N_2 \). Assume w.l.o.g. \( N_1 = \emptyset \). Then, since \( N \) is sound, we have \( x_1 = x_f \), which means that the last step of \( \sigma \) is of the form \((n_f, r_f)\). So \( x_2 \) is also a marking obtained after the occurrence of \((n_f, r_f)\). Since every agent participates in \( n_f \) and \( X(n_f, p, r_f) = \emptyset \) for every agent \( p \) and result \( r_f \), we also have \( x_2 = x_f \). So \( x \xrightarrow{(n,r_1)} x'_1 \xrightarrow{\sigma} x_f \) and \( x \xrightarrow{(n,r_2)} x'_2 \xrightarrow{\sigma} x_f \), which implies \( x'_1 = x'_2 \). Since \( N \) is deterministic, we then have \( X(n, p, r_1) = X(n, p, r_2) \), contradicting the hypothesis.

(a2) For every \( n_1 \in N_1 \) there is a path leading from some \( n_2 \in N_2 \) to \( n_1 \), and for every \( n_2 \in N_2 \) there is a path leading from some \( n_1 \in N_1 \) to \( n_2 \).

By symmetry it suffices to prove the first part. Since \( N_1 \) and \( N_2 \) are disjoint, \( n_1 \) is enabled at \( x_1 \) but not at \( x_2 \). Moreover, since \( N \) is acyclic, every atom can occur at most once in an occurrence sequence, and so neither \( n_1 \) nor \( n_2 \) appear in \( \sigma \). Since, furthermore, the sequences \((n, r_1)\sigma\) and \((n, r_2)\sigma\) only differ in their first element, there is an agent \( p \) such that \( X(n, p, r_1) = \{n_1\} \) and \( X(n, p, r_2) = \{n'_2\} \) for some \( n'_2 \neq n_1 \) (\( n'_2 \) is not necessarily the \( n_2 \in N_2 \) we are looking for). So we have \( x_1(p) = \{n_1\} \) and \( x_2(p) = \{n'_2\} \) (see Figure 27).
We first show that there is a path from \( n'_2 \) to \( n_1 \). By assumption, no outcome of \( n \) unconditionally enables any atom, and so \( (n, r_1) \) does not unconditionally enable \( n_1 \). So \( n_1 \) has a participant \( q \neq p \) such that either \( q \) is not a participant of \( n \) or \( X(n, q, r_1) \neq n_1 \). Since \( x_1 \) enables \( n_1 \) we have \( x_1(q) = \{n_1\} \), and since \( q \) is not a participant of \( n \) or \( X(n, q, r_1) \neq n_1 \), we have \( x_2(q) = \{n_1\} \) as well. Since \( x_2(p) \neq n_1 \) and \( N \) is a sound and deterministic acyclic negotiation, there is an occurrence sequence \( \tau \) such that \( x_2 \xrightarrow{\tau} x'_2 \) and \( x'_2 \) enables \( n_1 \) (intuitively, the white token on the arc to the \( p \)-port can only leave the arc through the occurrence of \( n_1 \)). Since \( x_2(p) \neq n_1 \), there is a path from \( n'_2 \) to \( n_1 \) (intuitively, the white token on the arc leading to the \( p \)-port of \( n'_2 \) has to travel to some arc leading to the \( p \)-port of \( n_1 \), and by determinism it can only do so through a path of \( p \)-ports that crosses \( n'_2 \)).

We now prove that there is a path from some \( n_2 \in N_2 \) to \( n'_2 \). If \( n'_2 \) is enabled at \( x_2 \), then \( n'_2 \in N_2 \) and we are done. If \( n'_2 \) is not enabled at \( x_2 \) (as in the figure), then, since \( x_2(p) \neq n_2 \) and \( N \) is a sound and deterministic acyclic negotiation, there is a sequence \( \tau \) such that \( x_2 \xrightarrow{\tau} x'_2 \) and \( x'_2 \) enables \( n'_2 \) (again, by soundness the white token on the arc to the \( p \)-port of \( n'_2 \) can eventually leave the arc to move towards \( n_f \), and by determinacy it can only leave the arc through the occurrence of \( n'_2 \)). Since \( N_2 \) is the set of transitions enabled at \( x_2 \), we have \( \tau = (n_2, x) \tau' \) for some \( n_2 \in N_2 \). So some subword of \( \tau \) is a path from some transition of \( N_2 \) to \( n'_2 \).

(a3) \( N \) contains a cycle.

Follows immediately from (a2) and the finiteness of \( N_1 \) and \( N_2 \).

(b) Every agent participates in \( n \).

By repeated application of (a) we find a chain \((n_1, r_1) \ldots (n_k, r_k)\) such that \( n_1 = n \), \( n_k = n_f \), and \( (n_i, r_i) \) unconditionally enables \( n_{i+1} \) for \( 1 \leq i \leq k - 1 \). By the definition of an unconditionally enabled atom we have \( P_1 \supseteq P_2 \supseteq \cdots \supseteq P_k = P_f \). Since \( P_f = A \), we obtain \( P_1 = A \).
C. Appendix: Unique Targets of Maximal Sequences

The proof of Proposition 45 is very involved, and requires some preliminaries. In Section C.1 we introduce the notions of loop of a negotiation, and synchronizer of a loop. Loosely speaking, a loop is an occurrence sequence leading from a marking to itself, and a synchronizer of a loop is an atom having as parties all the agents that participate in any of the atoms of the loop. We prove a fundamental result stating that every loop of a sound deterministic negotiation has a synchronizer (Proposition 62). In Section C.1 we use this result to prove Proposition 68, stating that every cycle of a negotiation has a dominating atom: An atom having as parties all the agents that participate in any of the atoms of the cycle.

Equipped with these propositions, in Section C.3 we proceed to prove Proposition 45.

C.1. Loops and Synchronizers

Intuitively, a loop is an occurrence sequence leading back to the marking from which it started.

**Definition 60.** A loop of a negotiation $N$ is a nonempty sequence $\sigma$ of outcomes such that $x \xrightarrow{\sigma} x$ for some marking $x$ of $N$. We say that $x$ enables $\sigma$. Given a loop $\sigma$, we denote $N_\sigma$ the set of atoms of $N$ that appear in $\sigma$, and $A_\sigma$ the set of agents that participate in at least one of these atoms. A loop $\sigma$ is minimal if no loop $\sigma'$ satisfies $N_{\sigma'} \subset N_\sigma$.

The sequences

\[
\sigma_1 = (n_1, a) (n_2, a) (n_4, b) (n_5, a) \\
\sigma_2 = (n_6, a) (n_7, b) \\
\sigma_3 = (n_1, b) (n_3, a) (n_4, c) (n_3, a) (n_4, b) (n_5, a)
\]

are loops of the negotiation of Figure 21. We have $N_{\sigma_3} = \{n_1, n_3, n_4, n_5\}$. Numbering the agents of the negotiation from left to right, the set $A_{\sigma_2}$ contains only the third agent, while $A_{\sigma_1}$ and $A_{\sigma_3}$ contain the first and second agents.

**Definition 61.** Let $\sigma$ be a loop of a negotiation. An atom $n \in N_\sigma$ is a synchronizer of $\sigma$ if $P'_n \subseteq P_n$ for every atom $n' \in N_\sigma$. We say that $\sigma$ is synchronized by $n$. An atom is a synchronizer if it synchronizes at least one loop of $N$.

The loop $\sigma_1$ above is synchronized by $n_1$ and $n_4$, but not by $n_2$ or $n_6$. The atoms $n_1, n_3, n_4, n_6, n_7$ are synchronizers of the negotiation, while $n_0, n_2, n_5, n_8$, and $n_f$ are not. Indeed, the atoms $n_0, n_8$, and $n_f$ do not belong to any loop. Atoms $n_2$ and $n_5$ do belong to loops, but only to loops that contain atoms with strictly more parties.

We prove two important properties of a sound cyclic deterministic negotiation $N$: Such a negotiation always has at least one loop, and every loop has at least one synchronizer.

**Proposition 62.** Every sound cyclic deterministic negotiation has a loop.
Proof. Let $\pi$ be a cycle of the graph of a sound deterministic negotiation $N$. Let $n_1$ be an arbitrary atom occurring in $\pi$, and let $n_2$ be its successor in $\pi$. Observe that $n_1 \neq n_f$ because $n_f$ has no successor, and hence no cycle contains $n_f$. By soundness some reachable marking $x_1$ enables $n_1$, and so $n_2 \in X(n_1, p, x)$ for at least one party $p$ and one result $r$. By determinism, we have $n_2 \in X(n_1, p, x) = \{n_2\}$. Let $x_1 \xrightarrow{(n_1, r)} x_1'$. Again by soundness, some occurrence sequence leads from from $x_1'$ to the final marking. This sequence must contain an occurrence of $n_2$, because this is the only atom agent $p$ is ready to engage in. In particular, some prefix of this sequence leads to a marking $x_2$ that enables $n_2$.

Repeating this argument for the nodes $n_1, n_2, n_3, \ldots, n_k = n_1$ of the cycle $\pi$, we conclude that $N$ has an infinite occurrence sequence containing infinitely many occurrences of atoms of $\pi$. Since the set of reachable markings is finite, this sequence contains a loop.

Now we prove that every minimal loop has a synchronizer. We start with a technical lemma. Intuitively, it states that firing an outcome of a loop $\sigma$ never decreases the set of agents ready to engage in atoms of $N_\sigma$.

Lemma 63. Let $\sigma$ be a loop of a negotiation, let $(n, x)$ be an outcome appearing in $\sigma$, and let $x_1, x_2$ be arbitrary markings such that $x_1 \xrightarrow{(n,r)} x_2$. For every agent $p$: if $x_1(p) \cap N_\sigma \neq \emptyset$, then $x_2(p) \cap N_\sigma \neq \emptyset$.

Proof. Let $x_1 \xrightarrow{(n,r)} x_2$ and assume $x_1(p) \cap N_\sigma \neq \emptyset$. If $p$ does not participate in $n$, then $x_1(p) = x_2(p)$, and so $x_2(p) \cap N_\sigma \neq \emptyset$. So assume that $p$ participates in $n$. Then, since $x_1 \xrightarrow{(n,r)} x_2$, we have $n \in x_1(p)$. By the definition of a loop, there is a sequence $\tau$ and reachable markings $x, x'$ such that $x \xrightarrow{(n,r)} x' \xrightarrow{\tau} x$ (the sequence $(n,r) \tau$ is a circular permutation of $\sigma$). Since we have $x_2(p) = x'(p)$, it suffices to show $x'(p) \cap N_\sigma \neq \emptyset$. If $n \in x'(p)$, then we are done. If $n \notin x'(p)$ then, since $n \in x(p)$ and $x' \xrightarrow{\tau} x$, the sequence $\tau$ contains an outcome $(n', x')$ such that $n' \in x'(p)$. Since, by definition, only atoms of $N_\sigma$ occur in $\tau$, we have $n' \in N_\sigma$, and so $x'(p) \cap N_\sigma \neq \emptyset$.

Proposition 64. Every minimal loop of a sound deterministic negotiation is synchronized by at least one of its atoms.

Proof. Let $\sigma$ be a minimal loop enabled at a reachable marking $x$. Since $N$ is sound, there is an occurrence sequence $x \xrightarrow{\sigma_f} x_f$. Let $\sigma_f = (n_1, x_1), \ldots, (n_k, x_k)$. Choose an arbitrary agent $\hat{p}$ of $A_\sigma$, and let $n_{j_0}$ be the atom of $N_\sigma$ with $\hat{p}$ as party that appears last in $\sigma_f$, i.e. $n_{j_0} \in N_\sigma$ and $n_{j_0+1}, \ldots, n_k \notin N_\sigma$. We claim that $n_{j_0}$ is a synchronizer of $\sigma$.

We iteratively construct a path $\pi$ of the graph of $N$ containing only atoms having $\hat{p}$ as party. The first atom of the path is $n_{j_0}$. Assume now that the path we have constructed so far is $(n_{j_0}, \hat{p}, x_0) (n_{j_1}, \hat{p}, x_1) \ldots (n_{j_i}, \hat{p}, x_i)$ for some $j_0 < j_1 < \cdots < j_i$. We choose $n_{j_i+1}$ as the last occurrence in $\sigma_f$ of an atom that has $\hat{p}$ as party and is a successor of $n_{j_i}$, meaning that $\{n_{j_i+1}\} = X(n_{j_i}, \hat{p}, x_i)$ for some result $x_i$. This guarantees that the only
atom of $\pi$ that belongs to $N_\sigma$ is $n_{j_0}$, and that all atoms of $\pi$ are distinct. Since all agents participate in $n_f$ the path ends with $n_f$.

In the rest of the proof we rename $n_{j_0}$ as $n_\pi$. Since $x$ enables the loop $\sigma$ and since $n_\pi \in N_\sigma$, after some prefix of $\sigma$ a marking $x_\pi$ is reached which enables $n_\pi$. The loop $\sigma$ continues with some outcome $(n_\pi, r_1)$, where $r_1$ is one possible result of $n_\pi$. By construction of $\pi$, there is another result $r_2$ of $n_\pi$ such that $X(n_\pi, \hat{p}, r_2)$ is the second atom of $\pi$, and this atom does not belong to $N_\sigma$.

Let $x'_\pi$ be the marking such that $x_\pi (n_\pi, r_2) \rightarrow x'_\pi$. We iteratively construct an occurrence sequence enabled at $x'_\pi$ as follows. Let $\tau$ be the sequence constructed so far and let $x$ be the marking reached by $\tau$ (initially $\tau = \epsilon$ and $x = x'_\pi$):

(0) If $x$ is the final marking, stop.

(1) Else, if $x$ enables an atom $n$ of $N_\sigma$, then at least one outcome $(n, r)$ occurs in $\sigma$. Let $\tau := \tau (n, r)$.

(2) Else, if $x$ enables an atom $n$ of $\pi$, then let $r$ be the result such that $X(n, \hat{p}, r)$ is the successor of $n$ in $\pi$. Let $\tau := \tau (n, r)$.

(3) Else, let $\rho$ be a shortest occurrence sequence that either leads to the final marking or enables an atom that appears in $\sigma$, or enables an atom that appears in $\pi$ (so that after this sequence either (1) or (2) can be applied). Such an occurrence sequence exists because $N$ is sound. Further, $\rho$ is not empty because otherwise we would have taken branch (0). Let $\tau := \tau \rho$.

For the rest of the proof let $\tau$ be the sequence generated by this procedure. We claim that $\tau$ is finite (i.e., that the procedure terminates) and leads to the final marking. For this we prove that the procedure takes branches (1)-(3) only finitely often.

We first prove that the procedure takes branch (2) only finitely often. Since every time (2) is taken $\tau$ is extended with an outcome of an atom of $\pi$, it suffices to show that these atoms occur only finitely often in $\tau$. Recall that $\pi$ is a finite path ending with $n_f$, and that all atoms of $\pi$ involve $\hat{p}$. Let $\pi = (n_1, \hat{a}, r_1) (n_2, \hat{a}, r_2) \ldots (n_k, \hat{p}, r_k)$, where $n_k = n_f$. By determinism, $\hat{p}$ is always ready to engage in at most one atom of $\pi$. By construction, after the $i$-th occurrence in $\tau$ of an atom of $\pi$ the agent $\hat{p}$ is ready to engage in $n_{i+1}$. Therefore, the atoms of $\pi$ occur at most $k$ times in $\tau$.

We now prove that the procedure takes branch (3) only finitely often. Since branch (2) is taken only finitely often, some suffix $\tau'$ of $\tau$ does not contain any occurrence of atoms of $\pi$. For every marking $x$ reached along the execution of $\tau'$, let $A_{\sigma}(x)$ be the set of agents $p$ such that $x(p) \in N_\sigma$ (that is, the agents that at $x$ are ready to engage in an atom of $N_\sigma$). We show that along the execution of $\tau'$ these sets never decrease, and strictly increase each time the procedure takes branch (3); since the set of agents is finite, this concludes the proof.

Let $x_1$ be a marking of $\tau'$ at which the procedure chooses either branch (1) or branch (3). If the procedure takes branch (1), then, by the definition of (1), the procedure selects a result $(n, r)$ that occurs in $\sigma$, and extends the current sequence with the step
Let Definition 66. Arbitrarily often, and never executes any other outcome of the atoms in the circuit.

We find an arbitrarily long occurrence sequence that executes the outcomes of the circuit has a synchronizer. Indeed, a loop can be seen as a circuit in the marking graph of the atom. This result is a syntactic counterpart to Proposition 64, stating that every loop be a path of Definition 65. Loosely speaking, an atom \( n \) of a path of a negotiation dominates the path if every agent that participates in some atom of the path also participates in \( n \).

**Definition 65.** Let \( \mathcal{N} \) be a negotiation and let \( \pi = (n_1, p_1, x_1) (n_2, p_2, x_2) \cdots (n_k, p_k, x_k) \) be a path of \( \mathcal{N} \). An atom \( n_i \) dominates \( \pi \) if \( P_{n_i} \subseteq P_{n_j} \) for \( 1 \leq j \leq k \).

We prove that every circuit of a sound deterministic negotiation has a dominating atom. This result is a syntactic counterpart to Proposition 64, stating that every loop has a synchronizer. Indeed, a loop can be seen as a circuit in the marking graph of the negotiation. The proof shows that every circuit can be “executed”, meaning that one can find an arbitrarily long occurrence sequence that executes the outcomes of the circuit arbitrarily often, and never executes any other outcome of the atoms in the circuit.

**Definition 66.** Let \( \pi = (n_1, p_1, x_1) (n_2, p_2, x_2) \cdots (n_k, p_k, x_k) \) be a path of a negotiation. An execution of \( \pi \) is an occurrence sequence \( x \xrightarrow{\sigma} x' \) such that \( x \) is a reachable marking, \( \sigma = \sigma_0(n_1, x_1) \sigma_1(n_2, x_2) \sigma_2 \cdots \sigma_{k-1}(n_k, x_k) \), and for every atom \( n \) of \( \pi \), if \( (n, x) \) occurs in \( \sigma \) then \( (n, p, x) \) occurs in \( \pi \).

A path \( \pi \) of \( \mathcal{N} \) is executable if it has an execution \( x \xrightarrow{\sigma} x' \).

**Lemma 67.** Every path of a sound deterministic negotiation is executable.
Proof. Let $N$ be a sound deterministic negotiation, and let $\pi = (n_1, p_1, r_1) \cdots (n_k, p_k, r_k)$ be a path of $N$.

We construct a sequence $\sigma$ that executes $\pi$, by induction on $k$. If $k = 1$ then, by soundness, some reachable marking enables $n_1$, and so we can take $\sigma = (n_1, r_1)$. If $k > 1$ then, by induction hypothesis, there exists a reachable marking $x$ and an occurrence sequence $\sigma'$ that executes $\pi' = (n_1, p_1, r_1) \cdots (n_{k-1}, p_{k-1}, r_{k-1})$. Let $x \xrightarrow{\sigma'} x'$. Since $N$ is sound, there exists an occurrence sequence $x' \overset{\tau}{\rightarrow} x_f$. Since $N$ is deterministic and $n_k \in X(n_i, a_{k-1}, r_{k-1})$, we have $X(n_{k-1}, p_{k-1}, r_{k-1}) = \{n_k\}$, and so $x'(p_{k-1}) = \{n_k\}$. Since $x_f(p_{k-1}) = \emptyset$, we have $\tau = \tau'(n, x) \tau''$ for some sequence $\tau'$ that contains any atom having $p_{k-1}$ as party and some atom $n$ such that $p_{k-1} \in \mathcal{P}_n$. Let $x' \overset{\tau'}{\rightarrow} x''$. We have $x''(p_{k-1}) = x'(p_{k-1}) = \{n_1\}$, and so $n = n_k$. So we can take $\sigma = \sigma' \tau'(n_k, r_k)$. \qed

**Proposition 68.** Every cycle of a sound deterministic negotiation has a dominating atom.

Proof. Let $N$ be a sound deterministic negotiation and let $\pi = (n_1, p_1, r_1) \cdots (n_k, p_k, r_k)$ be a cycle of $N$. Then $\pi^i$ (the result of concatenating $i$ copies of $\pi$) is a path of $N$ for every $i \geq 1$. By Lemma 67, $\pi^i$ is executable for every $i > 1$, and so for every number $\ell$ there is a firing sequence containing at least $\ell$ occurrences of each of the outcomes $(n_1, r_1), \ldots, (n_k, r_k)$, and no occurrence of any other outcome of the atoms $n_1, \ldots, n_k$. Since $N$ has only finitely many reachable markings, we can extract from the firing sequence for a sufficiently large $\ell$ a loop $x \overset{\sigma}{\rightarrow} x$ of $N$ containing all of $(n_1, r_1), \ldots, (n_k, r_k)$, and no other outcome of $n_1, \ldots, n_k$. By Proposition 64, $\sigma$ has a synchronizer $n$.

We claim that $n = n_i$ for some $1 \leq i \leq k$, which proves the proposition. Assume the contrary, and let $y$ be a marking of the loop that enables $n$. Since $n$ is a synchronizer, we have $y(p) = \{n\}$ for every party $p$ of $n_1, \ldots, n_k$. Let $x \overset{(n_i, r_i)}{\rightarrow} x' \overset{\tau}{\rightarrow} y$ be the unique segment of the loop such that no outcome of $\pi$ occurs in $\tau$. Since $(n_i, r_i)$ is the only outcome of $n_i$ that appears in $\sigma$, no outcome of $n_i$ appears in $\tau$. Since $N$ is deterministic, we have $x'(p) = y(p)$ for every party $p$ of $n_i$. But, by the definition of the cycle $\pi$, we have $x'(p_i) = \{n_{i+1}\}$. So we get $\{n_{i+1}\} = x'(p_i) = y(p_i) = \{n\}$, which implies $n = n_{i+1}$. \qed

### C.3. Proof of Proposition 45

We first prove a lemma.

**Lemma 69.** Let $N$ be a sound deterministic negotiation with a set of agents $A$. Let $B, C$ be a partition of $A$, i.e., $A = B \cup C$ and $B \cap C = \emptyset$. Let $x_1, x_2$ be two reachable markings such that

- $x_1(B) = x_2(B)$, and
- every atom enabled at $x_1$ or $x_2$ has at least one party in $B$, and at least one party in $C$.

Then $x_1(C) = x_2(C)$.
Claim 1: $N_1 \cap N_2 \neq \emptyset$.
We show that if $N_1 \cap N_2 = \emptyset$ then $N$ contains a cycle without a dominating atom, contradicting Proposition 68. Let $n_1 \in N_1$. By soundness, there is a shortest sequence $x_2 \xrightarrow{a} x'_2$ such that $x'_2$ enables $n_1$. We prove two subclaims.

Claim 1a: No atom of $\sigma$ dominates $n_1$.
Since $n_1$ is enabled at $x_1$ and at least one of its parties belongs to $B$, we have $x_1(b) = \{n_1\}$ for some agent $b \in B$, and so, by the definition of $B$, also $x_2(b) = \{n_1\}$. Since $\sigma$ is shortest, it does not contain any occurrence of $n_1$. So, since $N$ is deterministic, we also have $y(b) = \{n_1\}$ for every intermediate marking $y$ reached during the execution of $\sigma$. It follows that $b$ does not participate in any atom occurring in $\sigma$. So no atom of $\sigma$ dominates $n_1$.

Claim 1b: There exists $m_0 \in N_2$ and a path $\pi = (m_0, p_0, x_0) \ldots (m_k, p_k, x_k)$ such that $X(m_k, p_k, x_k) = \{n_1\}$ and none of $m_0, \ldots, m_k$ dominates $n_1$.
By Claim 1, it suffices to construct a path such that $m_0 \in N_2, X(m_k, p_k, x_k) = \{n_1\}$, and all of $m_0, \ldots, m_k$ occur in $\sigma$. Let $\sigma = (n, r)\sigma'$. We proceed by induction on the length of $\sigma$.

If $|\sigma| = 1$ then $\sigma = (n, r)$, and so $x_2 \xrightarrow{(n,r)} x'_2$. Since $x_2$ does not enable $n_1$ but $x'_2$ does, we have $X(n, p, r) = \{n_1\}$ for some $p \in P_n$. Choose $\pi = (n, p, r)$. Since $n$ is enabled at $x_2$, we have $n \in N_2$, and we are done.

If $|\sigma| > 1$, let $y$ be the marking given by $x_2 \xrightarrow{(n,r)} y \xrightarrow{\sigma'} x'_2$. Then $\sigma'$ is a shortest sequence enabling $n_1$ from $y$ and so, by induction hypothesis, there is a path $\sigma'' = (m_1, p_1, x_1) \ldots (m_k, p_k, x_k)$ such that $m_1$ is enabled at $y$, $X(m_k, p_k, x_k) = \{n_1\}$ and all of $m_1, \ldots, m_k$ occur in $\sigma''$. If $m_1 \in N_2$ then we can take $\pi := \sigma''$. If $m_1 \notin N_2$, then $m_1$ is not enabled at $x_2$. Since $m_1$ is enabled at $y$, we have $x_2(p) = \{n\}$ and $y(p) = \{m_1\}$ for some party $p$ of both $n$ and $m_0$. It follows $X(n, p, r) = \{m_1\}$. So we can take $\pi = (n, p, r)\sigma''$, which concludes the proof.

Observe that Claim 1b holds for every atom of $N_1$. By symmetry, for every $n_2 \in N_2$ there is $m_0 \in N_1$ and a path $\pi = (m_0, p_0, x_0) \ldots (m_k, p_k, x_k)$ such that $X(m_k, p_k, x_k) = \{n_2\}$ and none of $m_0, \ldots, m_k$ dominates $n_2$. Since $N$ has only finitely many atoms, there are $n_1, \ldots, n_{1k} \in N_1, n_{21}, \ldots, n_{2k} \in N_2$ and a cycle that visits the sequence of atoms $n_{11}, n_{21}, n_{12}, n_{22}, \ldots, n_{1k}, n_{2k}$ in that order, and such that no atom of the cycle dominates all others. So the cycle does not contain a dominating atom, contradicting Proposition 68. This proves Claim 1.

By Claim 1, there is an atom $n \in N_1 \cap N_2$. Let $c \in C \cap P_n$, which exists by assumption, and let $B' = B \cup \{c\}$ and $C' = C \setminus \{c\}$. Since $n$ is enabled at $x_1$ and $x_2$ we have $x_1(c) = \{n\} = x_2(c)$. Since $|C'| = |C| - 1$ we can apply the induction hypothesis to $B'$ and $C'$. So we have $x_1(C') = x_2(C')$, and therefore $x_1(C) = x_2(C)$.

\[\Box\]
We are now ready to prove that all maximal \(n\)-sequences have the same target, and that the same holds for maximal strict \((n,r)\)-sequences.

**Proposition 45.** Let \(N\) be a sound and deterministic negotiation, and let \(n\) be an atom of \(N\).

(a) All maximal \(n\)-sequences have the same target.

That is: there is a unique marking \(x\) such that \(x_n \xrightarrow{\sigma} x\) for every maximal \(n\)-sequence \(\sigma\). We call \(x\) the **target** of \(n\).

(b) For every outcome \((n,r)\), all maximal strict \((n,r)\)-sequences have the same target.

That is: there is a unique marking \(x\) such that \(x_n \xrightarrow{\sigma} x\) for every maximal strict \((n,r)\)-sequence \(\sigma\). We call \(x\) the **target** of \((n,r)\).

**Proof.** (a) Let \(\sigma_1\) and \(\sigma_2\) be two maximal \(P_n\)-sequences and \(x_0 \xrightarrow{\sigma} y\) be an occurrence sequence that enables \(n\) for the first time. We then have

\[
x_0 \xrightarrow{\sigma} y \xrightarrow{\sigma_1'} x_1 \quad \text{and} \quad x_0 \xrightarrow{\sigma} y \xrightarrow{\sigma_2'} x_2
\]

for two markings \(x_1, x_2\) such that \(x_1(a) = x_2(a)\) for every \(a \notin P_n\). By the maximality of \(\sigma_1\) and \(\sigma_2\), neither \(x_1\) nor \(x_2\) enable any atom \(n'\) such that \(P_{n'} \subseteq P_n\).

We prove \(x_1 = x_2\), which shows that \(\sigma_1\) and \(\sigma_2\) have the same target. Let \(B, C\) be the partition of the agents of \(N\) defined by \(p \in B\) iff \(x_1(p) = x_2(p)\). We have \(C \subseteq P_n\).

By soundness there exists a firing sequence \(x_1 \xrightarrow{\tau} x_f\). Let \(\tau\) be the longest \(B\)-prefix of \(\rho\), and let \(x_1 \xrightarrow{\tau} x_1'\). Since \(x_1(B) = x_2(B)\) (meaning \(x_1(p) = x_2(p)\) for every \(p \in B\)), we have \(x_2 \xrightarrow{\tau} x_2'\) for some marking \(x_2'\) such that \(x_2'(B) = x_2(B)\).

Let \(n'\) be an atom enabled at \(x_1'\). We claim that \(P_{n'}\) intersects both \(B\) and \(C\). Since \(B, C\) is a partition, it suffices to show that \(P_{n'}\) is not contained in \(B\) and is not contained in \(C\). That \(P_{n'}\) is not contained in \(B\) follows from the maximality of \(\tau\). Assume \(P_{n'} \subseteq C\). Since \(C \subseteq P_n\) we get \(P_{n'} \subseteq P_n\). Since \(\tau\) does not contain any atom with a party in \(C\), we have \(x_1'(C) = x_2(C)\), and so \(n'\) is also enabled at \(x_1\), contradicting the maximality of \(\sigma_1\). This proves the claim.

So every atom enabled at \(x_1'\) has parties in both \(B\) and \(C\). By symmetry, the same holds for \(x_2'\). Since \(x_1'(B) = x_2'(C)\) we can apply Lemma 69 and conclude \(x_1' = x_2'\).

Since \(x_1'\) and \(x_2'\) are reached from \(x_1\) and \(x_2\) by means of the same sequence \(\tau\), we get \(x_1 = x_2\).

(b) The proof is exactly as in (a). In this case we even have \(C \subseteq P_n\).