We define the idea of real path quantum theory, a realist generalisation of quantum theory in which it is postulated that the configuration space path actually followed by a closed quantum system is probabilistically chosen. This is done by a postulate defining probabilities for paths, which we propose are determined by an expression involving path amplitudes and a distance function that quantifies path separation. We suggest a possible form for a path probability postulate and explore possible choices of distance function, including choices suitable for Lorentz or generally covariant versions of real path quantum theory. We set out toy models of quantum interferometry and show that in these models the probability postulate and specific distance functions do indeed give a physically sensible path ontology. These functions can be chosen so as to predict quantum interference for interference of microscopic quantum systems and the failure of interference for macroscopic quantum systems. More generally, they predict interference when the beams are close, and its failure when they are far apart, as determined by the distance function. If taken seriously in its present relatively unconstrained form, real path quantum theory thus motivates experimental tests of quantum interference in all unexplored regimes defined by potentially physically interesting parameters, including the mass of the beam object, the beam separation distance, the beam separation time, and many others. We discuss open questions raised by these ideas.

INTRODUCTION

Feynman’s path integral formulation \cite{feynman1948} of quantum theory is widely seen as an elegant and beautiful unifying principle that may yet turn out to be the fundamentally correct way to define quantum field theory and quantum gravity. It also motivates Hartle and Hawking’s intriguing no-boundary proposal \cite{hartle1983} and other theories of the cosmological initial conditions, giving rise to some hope of a unified theory of dynamics and boundary conditions. Rigorously defining path integrals in physically relevant quantum theories poses formidable and unresolved technical difficulties. Path integrals are nonetheless often said to give an elegant and intuitively appealing explanation of the relationship between classical and quantum theories, and specifically to allow a simple derivation of the classical principle of stationary action from quantum theory. While the technical problems in rigorously defining path integrals are generally acknowledged to be formidable, the conceptual and logical problems in the path integral account of the relationship between classical and quantum theories have received, surprisingly, relatively little attention.

In this paper we first discuss why, even if we had a mathematically rigorous path integral for some preferred choice of variables, we could not use it to explain why macroscopic objects approximately follow classical trajectories. We then explore a new way of understanding quantum path integrals, defined by a new path probability postulate that involves a relatively simple modification to the standard path integral. The proposal has some clear conceptual advantages compared to the standard path integral. It gives a clear physical meaning to the paths and to probabilities associated with them. It also suggests a clear and conceptually unproblematic way of justifying from first principles the appearance of quasiclassical trajectories.

Like the standard quantum path integral, our modified path integrals are not presently rigorously defined. However, the formal path integrals we consider do at least tend to suppress contributions from “pathological” paths – that is, paths that are very far from intuitively sensible representations of the physics of the relevant system. This perhaps offers some grounds for hope that more rigorous definitions might be achievable, at least for a wider range of physically interesting models than those that have rigorously defined standard path integrals. Another unresolved issue is that our path probability postulate requires a choice for a distance function between paths, and at present we see no unique natural choice. However, some simple and interesting possibilities suggest themselves.

We illustrate the ideas of real path quantum theory in toy models. In these models, the generalized path integral rules we consider give a simple and physically reasonable ontology. In general, they make different experimental predictions to those made by standard quantum theory. These differences can be small enough to be undetectable in microscopic interference experiments but large enough to predict that macroscopic objects follow definite classical trajectories, even in experiments where quantum theory predicts they should display interference.

We focus throughout on position space path integrals. This follows a venerable tradition \cite{weinberg1972} in being
willing to pay the price of singling out some particular variable or variables in order to define equations structures that address the problem of the appearance of quasiclassicality within quantum theory and the broader quantum reality problem. It also follows the mainstream of that tradition in seeing position as a natural choice. However, we certainly do not mean to exclude other choices from consideration; it would be interesting to explore the various possibilities. And indeed, of course, if our ideas can be successfully applied to path integral formulations of quantum gravity, some more fundamental choice – perhaps involving paths of geometries – would need to be made.

It will be evident that the research program we outline here is incomplete: this paper begins to explore ideas and raises some difficult questions for which we presently do not have answers. That said, at the technically unrigorous level of argument used in standard path integral discussions, our proposed axioms do suggest a potentially conceptually satisfactory unified explanation for the quasiclassical behaviour of macroscopic objects and the observation of quantum interference in microscopic systems. They also suggest a way of defining a large class of generalizations of quantum theory – real path quantum theories – that are equipped with a natural realist ontology and that have experimentally testable consequences. While there are certainly many important and potentially daunting unresolved technical issues, there seems no intrinsic conceptual or logical obstacle to defining this ontology in a way that respects Lorentz invariance, general covariance, and other symmetries.

In summary then, we have some new ideas that motivate a research program with the ambitious aim of generalising the quantum path integral to give a unified description of microscopic and macroscopic physics that is consistent with special and general relativity and applicable to quantum field theory and quantum gravity. We comment briefly on the relationship of these ideas to other work on finding realist versions of standard or generalised quantum theory, at the end of the paper.

PATH INTEGRALS AND THE PRINCIPLE OF STATIONARY ACTION

“Before we go on making the mathematics more complete, we shall compare this quantum law with the classical rule. At first sight, from Eq. (2.15), all paths contribute equally, although their phases vary, so it is not clear how, in the classical limit, some particular path becomes most important. The classical approximation, however, corresponds to the case that the dimensions, masses, times, etc., are so large that \( S \) is enormous in relation to \( \hbar \). Then the phase of the contribution \( S/\hbar \) is some very, very large angle. The real (or imaginary) part of \( \phi \) is the cosine (or sine) of this angle. This is as likely to be plus as minus. Now if we move the path as shown in Fig 2-1 by a small amount \( \delta x \), small on the classical scale, the change in \( S \) is likewise small on the classical scale, but not when measured in the tiny units of \( \hbar \). These small changes in path will, generally, make enormous changes in phase, and our cosine or sine will oscillate exceedingly rapidly between plus and minus values. The total contribution will then add to zero; for if one path makes a positive contribution, another infinitesimally close (on a classical scale) makes an equal negative contribution, so that no net contribution arises.

Therefore, no path really needs to be considered if the neighbouring path has a different action; for the paths in the neighbourhood cancel out the contribution. But for the special path \( \bar{x}(t) \), for which \( S \) is an extremum, a small change in path produces, in the first order at least, no change in \( S \). All the contributions from the paths in this region are nearly in phase, at phase \( S_{cl}/\hbar \), and do not cancel out. Therefore, only for paths in the vicinity of \( \bar{x}(t) \) can we get important contributions, and in the classical limit we need only consider this particular trajectory as being of importance. In this way the classical laws of motion arise from the quantum laws.

We may note that trajectories which differ from \( \bar{x}(t) \) contribute as long as the action is still within about \( \hbar \) of \( S_{cl} \). The classical trajectory is indefinite to this slight extent, and this rule serves as a measure of the limitations of the precision of the classically defined trajectory.”

(Feynman and Hibbs. [10]; italics original, bold face added)

Some version of this argument is still propagated in many quantum theory lecture courses to this day. However, it is conceptually and logically confused, and the main conclusion – which we have highlighted in bold face above – does not follow.
Decoherence and the appearance of quasiclassicality: the Feynman-Hibbs lacuna in context

Feynman and Hibbs start from a true statement – or, at least, one that could be true if rigorous definitions were available – that notes that two different situations are related mathematically. Namely, we can calculate as if the approximately classical trajectories were the only relevant ones, even though actually they are not. They add the premise that quantum theory is fundamentally correct – which superficially may seem reasonable enough, since we don’t have a better theory. They also add the empirical observation that we see classical systems approximately following classical equations of motion. But then, rather than testing whether we can actually derive a fully consistent explanation of the appearance of quasi-classical physics from quantum theory, they effectively assume the answer. That is, they assume the appearance of quasiclassicality must be directly derivable from path integral quantum theory. Given that, the calculational result must be a derivation of the empirical observation, since it is essentially the only relevant equation that the theory gives us. But, of course, being able to calculate as if something were true isn’t the same as showing that it is true from first principles.

Another version of this error arises in attempts to explain the appearance of a quasiclassical world from quantum decoherence, starting from the correct observation that decoherence models show that one can calculate as if an initially pure quantum system is represented by a proper probabilistic mixed state after interacting with an apparatus, although in fact it is represented by the reduced density matrix of an entangled state. Extended discussions of this point can be found in Refs. [14, 28] and elsewhere.

Not only do these attempts to derive the appearance of quasiclassical world from unitary quantum theory fail, but – in the view of many physicists – all such attempts have failed. Unitary quantum theory can only be made sense of via many-worlds ideas, and there is, after 55 years, no consensus even among proponents of those ideas as to how they can be made rigorous and can give a scientific theory with explanatory power [28].

This motivates exploring ways to go beyond standard quantum theory, for example by adding extra mathematical structure (as in de Broglie-Bohm theory [5, 6]) or new dynamical laws (as in GRWP models [13, 14]). We explore a new idea in this direction below, adapting the existing path integral formalism by adding new postulates.

That said, as we have indicated, there is of course not currently a complete consensus on whether quantum theory can explain the appearance of quasiclassicality. For those still unpersuaded that it cannot, a more conservative motivation is that, whatever the status of standard quantum theory, natural-looking additions or alterations should be explored, since they might either give a valuable new perspective on standard quantum theory, or interesting new generalizations of quantum theory that can be tested.

Examining the Feynman-Hibbs argument in a toy model

To see the problem with the Feynman-Hibbs argument more clearly, it is very helpful to separate conceptual questions from the problems of rigorously defining any path integral. To this end we define a toy discrete path integral model (which we will call $M1$) for the centre of mass motion of a single massive object in position space, involving some large finite number of paths from $A$ to $B$. Because the number of paths is finite, we can rigorously define the path integral and related quantities. We define the model to have a set of paths that have mathematical properties analogous to those of quasiclassical trajectories – those in the neighbourhood of the stationary action path – in the standard quantum path integral. This allows us to focus on the conceptual question of what conclusions about quasiclassical physics do or do not follow from the path integral.

For simplicity, we define the model $M1$ so that each path has phase $\pm 1$, and we take the paths to have some natural ordering $P_1, \ldots, P_N$, in which paths $P_i$ and $P_{i+1}$ are supposed to be physically adjacent. This is not generally entirely realistic, even in the simple versions of path integrals defined in discrete models of $1 + 1$ dimensional space-time. However, it simplifies the model while still allowing it to illustrate a key point that can be replicated in more geometrically realistic models.

We suppose also that we can identify paths $P_{M}, \ldots, P_{M+K}$ that correspond approximately to the quasiclassical trajectories for the particle, where $M$ is odd, $N - M - K$ is even, $1 < M < M + K < N$ and $K \ll M, N$. We call these the quasiclassical paths in the model. Finally, we suppose that the path amplitudes $A(P_i)$ obey

$$A(P_i) = (-1)^{i-1} \text{ for } 1 \leq i \leq M - 1,$$

$$A(P_i) = 1 \text{ for } M \leq i \leq M + K,$$

$$A(P_i) = (-1)^{i-M-K} \text{ for } M + K < i \leq N.$$  

In other words, the amplitudes alternate in pairs before and after the quasiclassical paths, which all have amplitude
+1. Listed in order they are

$$1, -1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, 1, -1, 1.$$ 

This is intended to model the features of the quantum path integral relevant to Feynman and Hibbs’ argument. The quasiclassical paths in the quantum path integral are close to one another; in our discrete model they are all adjacent. The quasiclassical path amplitudes in the quantum path integral are approximately constant; in our model they are precisely equal, taking the value +1. The amplitudes of paths away from the quasiclassical paths oscillate rapidly in the quantum path integral; in our discrete model they oscillate as rapidly as possible, alternately taking the values ±1.

Since the path amplitudes either side of the quasiclassical paths cancel in pairs, we have the arithmetical identity

$$\sum_{i=1}^{N} A(P_i) = \sum_{i=M}^{M+K} A(P_i).$$

So we can indeed calculate the total sum – our discrete version of the path integral – by summing the amplitudes of the quasiclassical paths and ignoring the rest. But notice that this property per se does not single out the quasiclassical paths in our model as special. For example, we could also write

$$\sum_{i=1}^{N} A(P_i) = \sum_{i=1}^{K+1} A(P_{2i-1}).$$

More generally,

$$\sum_{i=1}^{N} A(P_i) = \sum_{i \in I} A(P_i)$$

for any size $(K + 1)$ subset $I$ of the set

$$\{1, 3, \ldots, M, M + 1, \ldots, M + K, M + K + 2, \ldots, N\} = \{i : A(P_i) = +1\},$$

that is, the set of paths with amplitude +1.

It is true that the quasiclassical paths are all adjacent in our ordering, while the other paths are not. But nothing in the definition of the path integral or any standard presentation of its physical implications gives a special ontological status to subsets of adjacent paths. To derive classical laws of motion, we need to be able to make a statement about the actual trajectories of macroscopic objects. In particular, here, we need to be able to derive that the object follows one of the quasiclassical paths from $A$ to $B$. This does not follow from the rules of standard path integral quantum theory, as set out in Feynman and Hibbs or elsewhere.

This point is worth elaborating. The standard treatment of the quantum path integral only defines a transition probability from $A$ to $B$. It does not supply a rule that tells us that the system actually follows any path. In particular, it gives no rule that ensures the system will follow one path from among a set of adjacent paths with similar phases and amplitudes, or even that we can make some more coarse-grained statement about its behaviour characterized by that set.

In fact, when path integrals are being discussed in contexts where no quasiclassical dynamics is expected to emerge, authors often suggest an intuitive picture according to which, since every path amplitude has modulus one, in some loose sense they are all equally significant: the quantum system “follows all possible paths”.

Of course, this intuition isn’t properly justified either. It’s not even clear what it is really intended to mean. Sometimes some form of Everettian many-worlds picture seems to be intended. While it is difficult to criticize so underdeveloped an intuition, three comments are worth making here. First, there have been many different attempts to define an ontologically sensible and scientifically useful Everettian picture of quantum theory [28], and in the view of many (e.g. [1, 21, 22, 27]), none of them succeed. Second, elementary paths in the path integral cannot be identified with the quasiclassical branching worlds that are normally thought to be crucial in Everettian explanations. Third, in any case, without an ontological rule, we can’t use the path integral to say anything about what the system does between $A$ and $B$.

Notice also that this intuition about the meaning of the path integral for microscopic quantum systems directly conflicts with the intuition discussed above for macroscopic objects following quasiclassical trajectories. According to this folk intuition, they don’t follow all possible paths, but instead follow some quasiclassical path.
As this conflict of intuitions highlights, it is not the case that there is some tacit rule, generally familiar to experts but unaccountably omitted from textbooks, that unifies the path integral treatments of the microscopic and the macroscopic. We have here a genuine conceptual problem that needs to be resolved.

Fortified by the inescapability of this conclusion, while also admiring the beauty and generality of the path integral formalism and suspecting that despite its present flaws the Feynman-Hibbs argument may contain the germ of a key insight about the relationship between quasiclassical and quantum physics, we now look for fresh inspiration in the form of alternative ways of thinking about quantum path integrals that might make more conceptual and physical sense.

**REAL PATH QUANTUM THEORY**

**Unphysicality of path probabilities in the standard path integral**

Consider the position space path integral for a single non-relativistic particle transition from point $A = (x_A, t_A)$ to point $B = (x_B, t_B)$. Assigning probabilities to individual paths in this integral makes no evident sense, as noted above. Consider again the naive rule that the probability of following any given path $P$ is proportional to the square of the associated amplitude: $\text{Prob}(P) = C|A(P)|^2$. Each path $P$ has amplitude $A(P) = \exp(iS(P))$, so adopting this rule would make all paths have equal probability weight. In realistic models, these weights are unnormalisable, so that we cannot obtain a path probability distribution from this rule. Nor, even if we could somehow solve this problem, would adopting this rule help explain the origins of quasiclassicality, since if any sensible definition of measure existed, the approximately classical paths of a macroscopic object should have measure zero among the set of all paths.

**The path probability postulate**

If we can’t extract a sensible explanation of quasiclassical physics (or indeed do anything more than calculate transition probabilities) from the path integral in its present form, then perhaps we need to change the definition of the path integral, or add additional postulates, or both. The difficulty in making physical sense of the path integral seems to be connected with the fact that it hints at an interpretation in which paths have probabilities, while at the same time suggesting conflicting intuitions about these physical path probabilities. We thus propose to explore the implications of explicitly assigning probabilities to paths via a new postulate. The idea here is that we define a quantity $\text{Prob}(P)$ that has the standard properties of a probability:

$$\text{Prob}(P) \geq 0 \quad \int_{\text{paths } P} dP \text{Prob}(P) = 1.$$ 

This quantity represents the probability that the given path $P$ was actually followed. Physical reality – in an experiment, or, in principle, in the evolution of the cosmos from initial to final state – is given by the chosen path. Specifically, we will consider a postulate of the form:

$$\text{Prob}(P) = C \left| \int dQ \exp(-iS(Q)) \exp(-d(P, Q)) \right|^2 \left( \int dQ \exp(-d(P, Q)) \right)^{-1}.$$ 

(2)

Here and below we take $\hbar = 1$. For the moment we take the integrals in this expression to be over all paths $Q$ that have the same endpoints ($A$ and $B$) as $P$. (Note that we would need to allow larger classes of paths to obtain an effective description of experiments with an extended initial wave function or to discuss the general possibilities allowed in cosmology.) We take $d(P, Q)$ to be some distance measure defined between paths $P$ and $Q$. This measure $d$ is supposed in some natural sense (to be elaborated) to say how distinct the paths are.

Note that at present we have no compelling reason to believe that there is a unique physically sensible choice for either the form of (2) or the distance function $d$. We aim to show that the simple path probability rule (2) does lead to physically interesting conclusions for some choices of distance function. We find this encouraging, since it is not a priori obvious that there are any modifications of the quantum path integral that give physically sensible results in agreement with empirical evidence for both quasiclassical and microscopic quantum systems. For concreteness, we focus on (2) here, and explore various distance functions. However, it is certainly also interesting to explore the range of physically sensible alternatives to (2).
In the form just given, the path probability postulate assigns probabilities for paths between $A$ and $B$ conditioned on the fact that $A$ and $B$ are the initial and final states respectively. Under this conditioning assumption, the normalisation factor $C$ is defined by

$$\int_{\text{paths: } A \rightarrow B} dP \operatorname{Prob}(P) = 1.$$ 

Without the conditioning assumption on the final state, the path probability postulate also implies a new rule for the total probability for arriving at the final point $B$ from the initial point $A$:

$$\operatorname{Prob}(B|A) = \int_{\text{paths: } A \rightarrow B} dP \operatorname{Prob}(P).$$

In this context $C$ is defined by normalising over a complete basis of final states:

$$\int_{\text{final states } B} dB \operatorname{Prob}(B|A) = 1.$$ 

For example, for a single particle in Minkowski space, the basis $B$ could be taken to be the final position $x$ on any given spacelike hypersurface intersecting the future light cone of the initial position.

We now consider what constraints our notion of naturality could imply on $d$. If $d$ were to define a metric on the space of paths it would satisfy the following conditions:

1. $d(P, Q) \geq 0$ for all $P$ and $Q$ (non-negativity),
2. $d(Q, P) = d(Q, P)$ (symmetry),
3. $d(P, Q) = 0$ if and only if $P = Q$ (identity of indiscernibles),
4. $d(P, R) \leq d(P, Q) + d(Q, R)$ (triangle inequality).

Of these, only non-negativity is strictly needed for our present discussion. The symmetry postulate also seems very natural at first sight, but there turns out to be some motivation to consider asymmetric distances when considering paths in Minkowski space or other fixed background Lorentzian spacetimes. We will not adopt the identity of indiscernibles as an axiom here for two reasons. First, it is useful in simple toy models to allow distinct neighbouring paths to have zero separation rather than very small separation. Second, some interesting candidates for Lorentz invariant distance functions between paths in Minkowski space have the property $d(P, P) > 0$ for some non-causal paths $P$. The distance functions in our toy models also violate the triangle inequality. While this too seems a plausible candidate postulate for a fundamental theory, it is also easy to find simple distance measures that violate it. We thus do not impose the last three postulates as axioms at present, but keep them in mind as natural possibilities to consider adopting in a fundamental formulation of a probabilistic path theory.

In summary, then, in what follows below, $d$ is supposed to give some intuitively sensible measure of path separation, and to be non-negative, but is not necessarily a metric.

We also require a physically motivated condition on $d$: that $d(P, Q) \approx 0$ when the difference between $P$ and $Q$ is "microscopic" and $d(P, Q) \gg 1$ when the difference is "macroscopic". The motivating idea here is that the path distance $d$ is ultimately defined by a new fundamental theory generalising quantum theory, and that this definition of $d$ is what ultimately allows the theory to determine the boundary between the "microscopic" and "macroscopic".

If we are to produce a generalization of quantum theory that has not already been falsified, $d$ should be chosen so that the predictions are consistent with experiment and observation to date. So, paths $P$ and $Q$ in experiments that demonstrate the path interference predicted by standard quantum theory should be microscopically separated: $d(P, Q) \approx 0$. However, if we see an object following an approximately classical trajectory, consistent with a path $P$, and there is another path $Q$ describing a trajectory that we can distinguish from $P$ by observation, so that we also see that the object does not follow $Q$, then $P$ and $Q$ should be macroscopically separated: $d(P, Q) \gg 1$.

In a truly fundamental formulation of a new theory, the probability rule should apply to paths between possible initial and final states of the universe, and so should presumably be formulated within some quantum theory of gravity. We will be rather less ambitious initially in exploring the idea, by making various simplifying assumptions.

First, we will suppose the probability rule makes sense for paths in the appropriate configuration spaces in quantum mechanics or quantum field theory, for finite time intervals or between finitely separated space-like hypersurfaces. This does not necessarily conflict with the idea that the fundamental formulation should be for paths between initial and
final cosmological states. The intuition, rather, is that the restricted application of the probability rule should be derivable as an approximation from the fundamental version. Similarly, the intuition is that the application to quantum mechanics should be derivable from that to quantum field theory, which in turn should be derivable from that to some underlying unified theory that includes gravity.

Second, we simplify considerably further by considering discrete toy model versions of path integrals, in which there are only finitely many relevant paths. In the simplest toy models we label these paths numerically, with a distance function depending on the abstract label, rather than defining an underlying path geometry and a distance function based on that geometry. We assign the phases of the paths in these toy models by fiat rather than deriving them from a specified Lagrangian and action. As in our earlier toy model, these phases are chosen so as to mimic the essential features of the sort of phase distribution one might expect from a realistic path integral calculation, but are not directly derived from any path integral. Our aim is to illustrate the possibility of extracting new physical insight from the path probability postulate \(^2\) for some reasonably sensible choices of the distance function \(d(P,Q)\), without worrying about questions of rigour. We leave for future exploration the scope for physically sensible choices of \(d(P,Q)\) for which the path probability postulate may be rigorously defined in realistic models.

REAL PATH QUANTUM THEORY IN TOY MODELS

In principle, the path probability postulate is intended to be an interesting possibility to explore in any physical theory, including quantum gravity theories with no fixed background space-time. In the first instance, though, we are mainly interested in exploring the cases with a fixed background space-time. In the non-relativistic case, we focus here on the example of Galilean spacetime with trivial topology \(R \times R^3\). In the relativistic case, we focus here on Minkowski space and on other fixed background Lorentzian space-times with the same topology. Of course, more general space-times with non-trivial topologies are also interesting. We focus on these examples because they are sufficient to illustrate the generality of real path quantum theory ideas, without complicating the discussion by considering non-trivial topologies.

The toy models we consider next are abstract enough that they could apply to each of these cases and to others. To be concrete, it may be helpful to think of them as models in Galilean space-time, and so we will assume this for now. We make some comments later about possible choices of distance function that give potentially physically interesting versions of the path probability postulate in Minkowski and other Lorentzian space-times.

Modelling single particle beams: Real path quantum theory in the toy model M1

We now return to our toy path integral model M1 above, and develop it as a model for real path quantum theory, by adding a choice of distance function. Recall that we previously proposed M1 as a discrete model of paths of the centre of mass motion of a massive object between two specified points \(A\) and \(B\) in space-time, where \(B\) is in the causal future of \(A\). Intuitively, we expect such an object approximately to follow the least action path from \(A\) to \(B\), even though we cannot rigorously justify this intuition from within standard quantum theory.

The same model M1 can also be thought of as a model for a single quantum particle in a beam between a source \(A\) and a detector \(B\) that – in the ordinary intuitive but unrigorous language generally used about particle beams in experiments – approximately defines a single definite path from \(A\) to \(B\).

We will use the model to consider both cases. In either case, the point is to show that, if we apply real path quantum theory to the model, we can rigorously justify the intuitive expectations.

For the moment we are interested in calculating path probabilities conditioned on fixed initial and final points, \(A\) and \(B\). As above, then, we will suppose there are \(N\) possible paths between \(A\) and \(B\), where \(N\) is a large positive integer, the paths are labelled \(P_j\) for \(1 \leq j \leq N\), and they have corresponding phases \(S_j = S(P_j)\) and amplitudes \(A_j = \exp(-iS_j)\). The label \(j\) is supposed to correspond to the geometric location of the path in space-time, in such a way that paths \(P_i, P_{i+1}\) with adjacent labels are in some sense neighbouring, and the difference between labels is a measure of the separation between paths, and so we will assume that \(d(P_i, P_j) = f(|i - j|)\) for some function \(f\). We note again that this is not entirely realistic. If we think of the paths as living in some discretized Galilean or Minkowski space, these properties do not generally hold for interesting sets of paths if we use choices of the distance function that are naturally defined via the underlying geometry. However, it simplifies our model and allows us to derive some interesting features. The essential points we make also hold in more geometrically realistic models.

We again suppose some set of adjacent paths \(P_{M+1}, \ldots, P_{M+K}\) lie in a region in path space where the path phase is essentially constant, while for the remaining paths the path phase oscillates. Here \(M\) is odd and we take \(N = M - K\)
Here we require $1 < M < M + K < N$ and $K \ll N$. We again take the path amplitudes in the constant region to all be $+1$, and take the amplitudes for the paths outside the region to be alternately $\pm 1$. So we have

$$
A(P_i) = \begin{cases} 
(-1)^{i-1} & \text{for } 1 \leq i \leq M - 1, \\
1 & \text{for } M \leq i \leq M + K, \\
(-1)^{i-M-K} & \text{for } M + K < i \leq N,
\end{cases}
$$

and thus the path amplitudes listed in order from $A_1$ to $A_N$ are

$$1, -1, \ldots, 1, -1, 1, \ldots, 1, -1, 1, \ldots, -1, 1.$$

We now take the distance function to be

$$
d(P_i, P_j) = \begin{cases} 
0 & \text{if } |i - j| < D, \\
\infty & \text{if } |i - j| > D, \\
\log(1/2) & \text{if } |i - j| = D.
\end{cases}
$$

This infinite step function is meant as a simplifying approximation to something more natural, such as

$$d(P_i, P_j) = \exp(|i - j|/D).$$

Here we require $D \ll N, M > 2D + 1$, and $N - M - K > 2D + 1$.

Since we are now working within real path quantum theory, we can apply (2) to obtain an explicit expression for the probability of the particle following any given path:

$$
\text{Prob}(P_i) = C(2D)^{-1/2} \left( \sum_{|j - i| < D} A(P_j) + \sum_{|j - i| = D} \frac{1}{2} A(P_j) \right)^2,
$$

where $C$ is the normalisation factor ensuring that

$$
\sum_{i=1}^{N} \text{Prob}(P_i) = 1.
$$

### Modelling a single particle beam

If we think of $M1$ as modelling a single microscopic particle beam between source $A$ and detector $B$, the most immediately interesting case for us is $2D > K$ (in particular $2D \gg K$, though the calculations depend only on whether $2D$ or $K$ is larger). This parameter choice captures the intuition that any model that alters the predictions of quantum theory by introducing some intrinsic decoherence should ensure that paths lying within a single beam of a microscopic quantum particle are very far from decohering. (We see no decoherence for microscopic particles even in interference experiments involving multiple separated beams – a scenario we will model later.)

For $2D > K$, and for $i$ further than $D$ from the ends of the list of paths, i.e. in the range $D < i < N - D$, this gives

$$
\text{Prob}(P_i) = \begin{cases} 
0 & \text{if } D + 1 < i < M - D \text{ or } N - (D + 1) > i > M + K + D, \\
C(2D)^{-1}|K|^2 & \text{if } M + K - D < i < M + D, \\
C(2D)^{-1}|(i + D - M)|^2 & \text{if } M < i + D < M + K, \\
C(2D)^{-1}|(M + K - i + D)|^2 & \text{if } M < i - D < M + K.
\end{cases}
$$

For completeness we also consider the case $2D \leq K$. Here, again for paths $P_i$ away from the ends of list, with $i$ lying in the range $D < i < N - D$, we find

$$
\text{Prob}(P_i) = \begin{cases} 
0 & \text{if } D + 1 \leq i < M - D \text{ or } N - (D + 1) \geq i > M + K + D, \\
C(2D)^{-1}|2D|^2 = 2CD & \text{if } M + D < i < M + K - D, \\
C(2D)^{-1}|(i + D - M)|^2 & \text{if } i - D < M < i + D, \\
C(2D)^{-1}|(M + K - i + D)|^2 & \text{if } i - D < M + K < i + D.
\end{cases}
$$
So, in either case, aside from the paths near the ends of the path list, all paths with non-zero probability are close to the paths in the region of constant phase, in the sense that their index $i$ is within $D$ of that region. Paths closer to the region are likelier, while the probabilities fall off towards zero for paths further away.

In both cases boundary effects mean our model also gives slightly nonzero path probabilities for paths near the ends of the path list:

$$
\text{Prob}(P_i) = C(1/4)((i + D - 1)/2)^{-1} \quad \text{for } 1 \leq i \leq D,
$$
$$
\text{Prob}(P_i) = C(1/4)((N - i) + D - 1)/2)^{-1} \quad \text{for } N - D \leq i \leq N.
$$

These are artefacts, which would be eliminated if we took the path list to be infinite or imposed periodic boundary conditions; we ignore them as irrelevant to our discussion.

This toy model thus gives a realist ontology that tells us that, if the particle goes from $A$ to $B$, then it follows a definite path. It respects physical intuition, in that the realised path will be close to the region of constant phase in path space. How close depends on the parameter $D$ that characterizes our chosen distance function in this model.

Although our realist ontology is not part of standard quantum theory, its dependence on $K$ in this model is consistent with standard intuitions. Our parameter $K$ here models (in a loose intuitive sense, since we are not considering measures on the set of realistic paths here) the size of the set of paths around the stationary path for which $(S/\hbar)$ is approximately constant in standard path integral quantum theory. The Feynman-Hibbs argument discussed earlier also aims to select a set of roughly equally relevant paths of similar phase around the stationary point of the action.

However, a new feature of our models is that the ontology and hence the physical predictions also depend on the distance function $d$ – in this case via the parameter $D$. The physically relevant paths in our ontology – those with nonzero probability – are not only those in the stationary phase region, but also those $d$-close to that region. A more fundamental new feature of our models, of course, is that they we have a realist ontology: as already noted, the standard Feynman-Hibbs intuitions have no logical justification in standard path integral quantum theory.

### Modelling classical and quasiclassical trajectories

As we noted above, the same toy model can be used to describe the centre of mass motion of a macroscopic object, which we expect to follow a quasiclassical trajectory. Whether we should take $2D > K$ or $2D < K$ here is less clear, since we do not have strong experimental constraints on quantum interference of general macroscopic objects. Since both ranges give qualitatively similar results when modelling a single beam, this does not immediately matter. As shown above, our model suggests that, ignoring boundary artefacts, all the trajectories with non-zero probability are approximately quasiclassical, in the (newly defined) sense that their index $i$ is within $D + K$ of the path at the centre of the quasiclassical set (which we take to represent the stationary action path in our model). Trajectories closer to the quasiclassical set are likelier, and the probabilities tail off towards zero for paths further away.

Our realist ontology thus says that, if a classical object’s centre of mass goes from $A$ to $B$, then it follows a definite trajectory, and this trajectory will be approximately the classical one. How good the approximation is depends on the size $K$ of the set of paths close enough to the classical trajectory that their phase is essentially the same as its, and on the parameter $D$ that characterizes our distance function. As already noted, the $K$-dependence is in line with intuitions based on standard quantum theory. In particular, it supports the conclusions of the Feynman-Hibbs argument in the context where it originally was meant to apply, namely selecting a set of roughly equally relevant paths of similar phase around a classical trajectory that is (necessarily) a stationary point of the action. Again, we have the new feature of dependence on the distance function $d$, parameterized in our model by $D$.

### Modelling beam interferometry in real path quantum theory

We next consider a toy model, which we call $M2$, in which there are two different regions in path space where the path phase is constant. Since we want to model quantum interference of beams with general complex phases we now let the path amplitudes in these regions take any complex value of modulus one. However, to keep the model simple we still for the moment suppose the amplitudes outside the regions are alternately $\pm 1$. We now have

$$
A(P_i) = (-1)^{i-1} \text{ for } 1 \leq i \leq M_0 - 1,
$$

where $M_0$ is the number of paths in the middle region.
paths of constant phase \( \exp(i\theta_0) \) for \( M_0 \leq i \leq M_0 + K_0 \),
\[ A(P_i) = (-1)^{i-M_0-K_0} \text{ for } M_0 + K_0 < i \leq M_1 - 1 , \]
\[ A(P_i) = \exp(-i\theta_1) \text{ for } M_1 \leq i \leq M_1 + K_1 , \]
\[ A(P_i) = (-1)^{i-M_1-K_1} \text{ for } M_1 + K_1 < i \leq N , \]

where we take \( M_0 \) to be odd, \( M_1 - M_0 - K_0 \) to be odd, \( N - M_1 - K_1 \) to be even, and assume the inequalities
\( 1 < M_0 < M_0 + K_0 < M_1 < M_1 + K_1 < N, \ K_0 \ll N, \ K_1 \ll N, \ M_0 > 2D + 1 \) and \( N - M_1 - K_1 > 2D + 1 \). The path amplitudes listed in order from \( A_1 \) to \( A_N \) are thus now

\[ 1, -1, \ldots, 1, -1, \exp(-i\theta_0), \exp(-i\theta_0), \ldots, \exp(-i\theta_0), 1, -1, \ldots, \]
\[ \ldots, 1, -1, \exp(-i\theta_1), \exp(-i\theta_1), \ldots, \exp(-i\theta_1), 1, -1, \ldots, 1, -1 . \]  

(13)

The most interesting cases for our discussion are (i) \( D \gg K_0, K_1 \) and \( D \gg M_1 + K_1 - M_0 \) and (ii) \( D \gg K_0, K_1 \) and \( 2D + 1 < M_1 - M_0 - K_0 \).

In case (i), if \( i + D > M_1 + K_1 \) and \( i - D < M_0 \) we have

\[ \text{Prob}(P_i) \approx C\left| (K_0 + 1) \exp(-i\theta_0) + (K_1 + 1) \exp(-i\theta_1) \right|^2 (2D)^{-1} . \]  

(14)

In other words, these path probabilities are defined by an interference term, which is essentially the term that would arise from two beams with respective amplitudes \((K_0 + 1)\exp(-i\theta_0)\) and \((K_1 + 1)\exp(-i\theta_1)\).

For \( i \) in the range where \( i - D < M_0 \) and \( M_0 \leq i + D < M_1 + K_1 \), the paths \( P_i \) also have significantly nonzero probabilities representing partial interference; similarly for \( i \) in the range where \( i + D > M_1 + K_1 \) and \( M_0 \leq i - D \leq M_1 + K_1 \).

As in the previous model, we also have slightly nonzero probabilities for paths near the ends of the list, which we again ignore as artefacts.

In case (ii), if \( i - D < M_0 \) and \( i + D > M_0 + K_0 \), we have

\[ \text{Prob}(P_i) \approx C\left| (K_0 + 1) \exp(-i\theta_0) \right|^2 (2D)^{-1} = C(K_0 + 1)^2 (2D)^{-1} . \]  

(15)

Similarly, if \( i - D < M_1 \) and \( i + D > M_1 + K_1 \), we have

\[ \text{Prob}(P_i) \approx C\left| (K_1 + 1) \exp(-i\theta_1) \right|^2 (2D)^{-1} = C(K_1 + 1)^2 (2D)^{-1} . \]  

(16)

Outside these ranges, ignoring boundary artefacts, the path probabilities fall to zero.

So, in case (ii), paths with significantly nonzero probability are associated with (and \( d \)-close to) either the region of paths with constant amplitude \( \exp(-i\theta_0) \) or the region of paths with constant amplitude \( \exp(-i\theta_1) \). However, in this case, the values of these probabilities are given solely by the corresponding “beam strengths” \((K_i + 1)^2\), with no interference term.

In this model, then, we represent particle beam amplitude phases by path amplitude phases. The beam strengths correspond to the number of adjacent paths with the same amplitude, and these paths collectively represent the beam. The model gives a real path ontology according to which any path with significantly nonzero probability is \( d \)-close to at least one beam. If the beams themselves are \( d \)-close, these probabilities display the familiar quantum interference. However, if the beams are widely separated by the \( d \) measure, even though they ultimately recombine at the same point, the probabilities have no interference term. The particle follows a path \( d \)-close to one beam or the other, with probabilities proportional to the respective beam strengths.

Multiple beam interference

Consider now a quantum multiple beam interferometry experiment in which – according to the standard intuitive but unrigorous language used to describe beam interferometry – a particle leaves a source \( A \), follows one of \( n \) linear beam paths \( BP_0, BP_1, \ldots, BP_{n-1} \) to one of \( n \) different slits \( S_0, S_2, \ldots, S_{n-1} \), and then follows a linear path from the relevant slit to a point \( B \) on a detecting screen. Suppose that, in a realistic description of the experiment, the beam path \( BP_i \) has action \( S_i = S(BP_i) \), with \( \exp(is_i) = \exp(i\theta_i) \), where the phases \( \theta_i \) obey \( 0 \leq \theta_i < 2\pi \), and that the beam along \( BP_i \) has strength \( \alpha_i > 0 \), with \( \sum_i \alpha_i^2 = 1 \).

We can extend our two beam model \( M2 \) to model multiple beams by introducing multiple regions of \( K_i \) adjacent paths of constant phase \( \exp(i\theta_i) \), where \((K_i + 1)\) is proportional to \( \alpha_i \). We obtain the same qualitative features. If all beams are \( d \)-close, our model will reproduce standard interference; if each pair of beams is \( d \)-distant, it predicts none.
In particular, to recover the standard quantum interference predictions, we need to suppose that the beam path separations are microscopic: $d(BP_i, BP_j) \approx 0$ for each $i, j$. We can achieve this with a generalization of $M2$ (call it $M3$) in which the paths $P_1, \ldots, P_N$ have amplitudes

$$A(P_i) = (-1)^{i-1} \text{ for } 1 \leq i \leq M_0 - 1,$$
$$A(P_i) = \exp(-i\theta_0) \text{ for } M_0 \leq i \leq M_0 + K_0,$$
$$A(P_i) = (-1)^{i-M_0-K_0} \text{ for } M_0 + K_0 < i \leq M_1 - 1,$$
$$A(P_i) = \exp(-i\theta_1) \text{ for } M_1 \leq i \leq M_1 + K_1,$$
$$A(P_i) = (-1)^{i-M_1-K_1} \text{ for } M_1 + K_1 < i \leq M_2 - 1,$$

$$\ldots$$
$$A(P_i) = (-1)^{i-M_{n-2}-K_{n-2}} \text{ for } M_{n-2} + K_{n-2} < i \leq M_{n-1} - 1,$$
$$A(P_i) = \exp(-i\theta_{n-1}) \text{ for } M_{n-1} \leq i \leq M_{n-1} + K_{n-1},$$
$$A(P_i) = (-1)^{i-M_{n-1}-K_{n-1}} \text{ for } M_{n-1} + K_{n-1} < i \leq N,$$

where we take $M_0, M_1 - M_0 - K_0, M_2 - M_1 - K_1, \ldots, M_{n-1} - M_{n-2} - K_{n-2},$ to be odd and $N - M_{n-1} - K_{n-1}$ to be even. The path amplitudes listed in order from $A_1$ to $A_N$ are thus now

$$1, -1, \ldots, 1, -1, \exp(-i\theta_0), \exp(-i\theta_0), \ldots, \exp(-i\theta_0), 1, -1, \ldots,$$
$$\ldots, 1, -1, \exp(-i\theta_1), \exp(-i\theta_1), \ldots, \exp(-i\theta_1), 1, -1, \ldots, 1, -1$$

$$\ldots$$
$$\ldots, 1, -1, \exp(-i\theta_{n-1}), \exp(-i\theta_{n-1}), \ldots, \exp(-i\theta_{n-1}), 1, -1, \ldots, 1, -1.$$

Here the beam path $BP_j$ is modelled by paths $P_i$ in the range $M_j \leq i \leq M_j + K_j$. We need to take $1 < M_0 < M_0 + K_0 < M_1 < M_1 + K_1 < \ldots < M_{n-1} + K_{n-1} < N$, to ensure the beams in our model do not overlap. $K_i \ll N$. We also want to take $M_{n-1} + K_{n-1} - M_0 \ll 2D + 1$, to ensure the beam paths are all $d$-close. Finally, we take $M_0 \gg 2D + 1$ and $N - M_{n-1} - K_{n-1} \gg 2D + 1$, ensuring that there are many more paths “outside” the region in which the interferometry beams lie than “inside” that region.

Real path quantum theory, via (2), then tells us that the probability distribution of real paths is dominated by $\approx 2D$ paths around the beam paths, each of which has probability proportional to

$$(2D)^{-1} \sum_i (K_i + 1) \exp(-i\theta_i)^2.$$

Note that, if we take this model seriously as a guide to ordinary quantum interferometry experiments, it suggests the real path in such experiments is generally very unlikely to be a path lying within any beam $BP_j$ – i.e. to be one of the beam paths in our model. In our model the beam paths are not only all $d$-close to one another, but considerably closer than this constraint requires. Our model suggests that a real path can be $d$-close to them while following some exotic non-beam path through the region in which the beam paths lie, or while going well outside that region.

Of course, our toy model relies on many simplifying assumptions. A full path integral description would include infinitely many paths with phases close to $\theta_i$ in the neighbourhood of each $P_i$, and infinitely many more exotic paths that are not piecewise linear and have rapidly fluctuating phases. Moreover, even in $1 + 1$ dimensions, these paths are not geometrically related in a way that allows the sort of one dimensional representation that our toy model and choice of distance function assume. Nor is it evident which choice of distance function one should make for the full set of quantum paths, even for a single particle travelling between two specified points in space-time. These issues clearly ultimately need to be addressed, and we discuss some of them further below. Our present aim, though, is to extract intuitions from and explore the range of possibilities suggested by discrete toy models. We consider next ways of modelling multi-particle systems, beginning by considering models of interferometry experiments that include the measuring apparatus as well as the interfering particle.

**Real paths in multi-particle configuration space**

We now extend our toy model discussion to consider an interference experiment with interfering beams in which the particle is emitted by a source at $A$ and may arrive at any of various points $B_j$ (for $1 \leq j \leq l$) on the screen.
A parsimonious way of doing this is to continue to model the experiment by considering only paths of the interfering particle. Thus, we could consider sets of paths \( L_j = \{ P^j_1, \ldots, P^j_{N_j} \} \) from \( A \) to \( B_j \), with corresponding amplitudes of the form \( \psi_{ij} \), now characterized by \( j \)-dependent parameters \( M^j_k, K^j_k, \theta^j_k, n_j, N_j \) (where \( 0 \leq k \leq n_j - 1 \)). We can then apply the path probability postulate to the union of these sets of paths – i.e. calculate path probabilities conditioned on the particle arriving at any of the points \( B_j \). In the regime considered above, with \( M^j_{n_j-1} + K^j_{n_j-1} - M^j_l \ll 2D + 1, M^j_0 \gg 2D + 1 \) and \( N_j - M_{n_j-1} - K_{n_j-1} \gg 2D + 1 \) for each \( j \), the model reproduces quantum interference for paths from \( A \) to any given \( B_j \). It gives the probability of the particle arriving at \( B_j \) as approximately proportional to

\[
| \sum_k (K^j_k + 1) \exp(-i\theta^j_k)|^2,
\]

and so the quantum predictions for the observed detection ratios \( \text{Prob}(A \rightarrow B_j)/\text{Prob}(A \rightarrow B_k) \) are recovered, as expected.

To explore a bit further, we want to extend the model further to consider paths in the multi-particle configuration space of the interfering particle together with constituent particles of the screen, and continue the model from the time (say \( t = 0 \)) at which the particle is emitted up to some time \( (t = T_f) \) significantly after the time \( (t = T_h) \) at which it hits the screen.

We take there to be \( Z \) constituent particles of the screen. For the purposes of this model we suppose that, in a quasiclassical description of the physics, a subset of \( Z_j \) of these are appreciably disturbed by an impact at \( B_j \). We make the further simplifying assumptions that the possible impact points \( B_j \) are discrete, and the relevant subsets of \( Z_j \) disturbed particles corresponding to distinct \( B_j \) are disjoint. Rather than considering paths for each individual particle, though, we first model the screen path in multi-particle configuration space by essentially the same toy model considered above. Between time \( t = 0 \) and \( T_h \), we consider the beam particle and screen paths separately; after \( T_h \) we use a single set of paths to model both. The idea here is that on impact the beam particle becomes part of the screen, indistinguishable from the other screen particles: this is not an essential assumption but simplifies the model.

We expect screen particles to follow a quasiclassical trajectory at all times, and we model this by adapting model M1. Between times \( 0 \) and \( T_h \), we take there to be \( N' \) possible configuration space paths in our discrete model of the \( Z \)-particle screen’s configuration space, listed in order as \( P'_1, P'_2, \ldots, P'_{N'} \). Of these, there are \( K' + 1 \) adjacent paths with phases \( +1 \), beginning with \( P'_M \), and the rest have alternating phase. We suppose there are \( N'' \) possible configuration space paths describing the \((Z + 1)\) screen particles (now including the absorbed beam particle) after time \( T_h \), listed in order as \( P''_1, P''_2, \ldots, P''_{N''} \). We suppose that the phases of the screen particle paths after the beam particle is absorbed are effectivelty determined by the absorption point \( B_j \) via some interaction Hamiltonian. Specifically, if we consider paths in which the interferometry particle arrives at \( B_j \), we suppose the \( K'' + 1 \) adjacent screen particle paths \( P''_{M''}, P''_{M'' + 1}, \ldots, P''_{M'' + K''} \) to have phases \( +1 \), while the other screen path paths have alternating phases \( \pm 1 \). We assume the quasiclassical paths corresponding to the distinct absorption points are ordered and separated by more than \( (2D + 1) \) from each other and the endpoints, so that \( 2D + 1 < M''_l < M''_l + K'' + 2D + 1 < M''_m < \ldots < M''_{n_j - 1} + K'' + 2D + 1 < M''_{n_j} < M''_l + K'' + 2D + 1 < N'' \). We also assume \( K'' \ll N'' \). Note that we take \( K'' \) to be a constant, independent of the impact point \( B_j \). This is to ensure that the probability of observing an impact at point \( B_j \) is not retroactively affected by the dynamics of the screen particles after impact. This is required so that our model reflects quantum unitarity in the quantum limit in which \( d(P_j, P_j') = 0 \) for all paths \( P_j, P_j' \). Allowing \( K'' \) to depend on the impact point \( B_j \) would mean that the post-interaction dynamics of screen particles would affect the probabilities of detecting the beam particle at given locations on the screen, strongly violating both unitarity and the no-signalling principle.

The possible complete paths from \( t = 0 \) to \( t = T_f \), including an absorption at the screen point \( B_j \) at \( t = T_h \), then take the form

\[
\{(\text{particle path } P^j_i) \otimes (\text{screen path } P''_m)\} \oplus \{(\text{post-absorption screen path } P''_m)\},
\]

where \( 1 \leq i \leq N_j, 1 \leq k \leq N' \), and \( 1 \leq m \leq N'' \).

Here \( \otimes \) denotes the path in \((Z + 1)\)-particle configuration space given by the product of the relevant particle and screen paths, running from \( t = 0 \) to \( t = T_h \), and \( \oplus \) denotes the composition of a path of this type with a post-absorption screen path that runs from \( t = T_h \) to \( t = T_f \).

We need to define distance functions for products of paths and compositions of paths. For the moment let us take

\[
d(P \otimes P', Q \otimes Q') = \max((d(P, Q), d(P', Q'))), \quad d(P \oplus Q, P' \oplus Q') = \max(d(P, P'), d(Q, Q')).
\]
This ensures that (as usual neglecting boundary artefacts in the model) the non-zero probability paths take the form

$$ (P'_i \otimes P'_k) \oplus P''_m, $$

where $P'_i$ is any nonzero probability particle path in the single-particle interferometry model with endpoint $B_j$, $P'_k$ is any screen path up to time $T_h$ within $D$ of the quasiclassical paths, and $P''_m$ is any post-absorption screen path within $D$ of those quasiclassical paths that describe the screen after an absorption at $B_j$.

It also ensures that we again recover the quantum predictions for detection probability ratios

$$ \text{Prob}(A \to B_j)/\text{Prob}(A \to B_k) $$

in this extended model.

While it is encouraging that we can find a simple model in which the one-particle path probability postulate follows from applying the path probability postulate to a model that includes a measuring apparatus or environment (the screen) as well as the measured particle, the assumptions used in this model raise many questions. To list just a few:

Could we obtain similar results from a more realistic model in which paths for all $(Z+1)$ particles are treated on an equal footing? In which paths live in discretized Galilean or Minkowski space-time, with three spatial dimensions, and the distance function depends (only) on the space-time geometry? Or in which we consider the full set of paths in ordinary continuous Galilean or Minkowski space-time? Is there a compelling reason to assume that $d(P \otimes P', Q \otimes Q') = \max((d(P, Q), d(P', Q') \text{ and } d(P \otimes Q, P' \otimes Q') = \max(d(P, P'), d(Q, Q'))$, or are there other interesting options? What form do we expect the distance function $d$ to take in a realistic quantum path integral treatment of real systems? Is it necessarily reasonable to take the distinct quasiclassical outcomes of measurements to be necessarily $d$-distant, as we did in our model? If so, is this because of the number of particles whose typical quasiclassical trajectories are distinct for distinct outcomes, or because of the total mass of these particles? Or is it also relevant that the quasiclassical trajectories remain distinct indefinitely: is the length of time (or in the relativistic case, proper path time) for which paths are distinguishable as (or even more) relevant as their spatial separation? How should particle interactions be handled in real path quantum theory? Can we find prescriptions that do not rely on the approximation in which they are treated by introducing interaction potentials in the Schrödinger equation for each pair of particles, and assuming that the particle number is fixed? Could and should the distance function $d(P, Q)$ depend on interactions – for instance on the relationships between the vertices in Feynman diagrams corresponding to paths $P$ and $Q$ – in a more fundamental field-theoretic treatment?

We believe the first two of these questions can be satisfactorily (and positively) answered by somewhat more sophisticated toy models. As the remaining questions illustrate, though, there is a limit to what toy models can persuasively establish. Fundamental issues need to be addressed in more realistic settings. In what follows we set out some intuitions and possibilities to explore, in the hope of both clarifying the vision underlying real path quantum theory and encouraging wider interest in the research program.

### POSSIBLE PATH DISTANCE FUNCTIONS AND THEIR PHYSICAL IMPLICATIONS

#### Choices of distance function for single particle paths

What if we apply (2) to realistic path integral models? A full path integral description of an interferometry experiment (whether of a microscopic or macroscopic object) involves uncountably many paths, most of which are not close to being the piecewise linear paths that we normally consider as the interfering beams.

If real path quantum theory does make sense in realistic models, it has one immediately clear empirical prediction. Standard quantum interference should be observed for interferometry experiments where the beam paths are close, as determined by the distance function $d$. However, quantum interference should not be observed when the beam paths are widely $d$-separated. For some choices of $d$, these predictions are very broadly qualitatively similar to those made by dynamical collapse models and other intrinsic decoherence models, although the underlying models are conceptually and ontologically quite different. For other choices of $d$, the predictions are qualitatively very different from those made by any existing intrinsic decoherence model.

For a microscopic object, given that the beam paths are microscopically separated from one another, and that there are infinitely many non-beam paths between the beam paths, we expect most paths that are microscopically separated from the beam paths to be non-beam paths. If (2) gives a well-defined probability measure on the paths, we thus expect a very large set $\{P_\lambda\}_{\lambda \in \Lambda}$ of possible paths from $A$ to $B$, and a probability measure on $\Lambda$ that is roughly
uniform on a large subset. We also expect that the total probability of finding the microscopic object at $B$ is, to very good approximation, the quantum probability. However, it isn’t evident that (2) does give a well-defined probability measure on paths, for interesting choices of the distance $d$.

At best, then, we get a rather more complicated picture than toy models suggest, which needs to be worked out carefully, but which at first sight is not evidently inconsistent nor evidently in contradiction with experiment or observation. At worst, we have a prescription that as yet makes no rigorous sense. However, either way, there are interesting ways to alter (2) so as to simplify – and, one might hope, rigorize – the picture. We turn to these next.

Is there a unique natural choice of $d$? Even for single particle paths in Galilean space-time, there seem to be many possible candidates. Consider two paths

$$P = \{ (x_P(t), t) : 0 \leq t \leq T \}, \quad Q = \{ (x_Q(t), t) : 0 \leq t \leq T \},$$

for a particle of mass $m$ between points $A = (x_P(0), 0) = (x_Q(0), 0)$ and $B = (x_P(T), T) = (x_Q(T), T)$. Some simple possible definitions of $d$ that capture arguably natural notions of path separation include

$$d(P, Q) = \max_{0 \leq t \leq T} |x_P(t) - x_Q(t)|,$$

$$d(P, Q) = m \max_{0 \leq t \leq T} |x_P(t) - x_Q(t)|,$$

$$d(P, Q) = \int_0^T dt |x_P(t) - x_Q(t)|,$$

$$d(P, Q) = T^{-1} \int_0^T dt |x_P(t) - x_Q(t)|,$$

$$d(P, Q) = m \int_0^T dt |x_P(t) - x_Q(t)|,$$

$$d(P, Q) = \left( \int_0^T dt |x_P(t) - x_Q(t)|^2 \right)^{1/2}.$$

One might also explore definitions sensitive to the first and/or higher path derivatives, for example

$$d(P, Q) = \int_0^T dt |x'_P(t) - x'_Q(t)|.$$

Of course, many other options, or combinations of these options, could be explored.

It seems then, even for thinking about real path quantum theory in the context of single particle interferometry, that we either need some new compelling theoretical reason for picking out some particular distance function, or empirical guidance. A theoretical preference could perhaps come either from a new theoretical idea purporting to explain why nature might follow the path probability postulate for some specific choice(s) of distance function, or conceivably from establishing that the path probabilities are rigorously definable only for some specific choice(s). But, absent further help from theory, if we take the idea of real path quantum theory seriously, it seems we need to continue exploring empirically whether quantum interference fails, and to be open to the possibility that the transition between interference and effective decoherence could be governed by almost any physical parameter or combination of parameters: maximum beam separation, average beam separation, separation time, particle mass, and so on. This is no doubt discouraging for those hoping for a precise prediction of how and where quantum theory should break down. It may, however, point to the appropriately scientifically open-minded strategy of encouraging experimental tests of quantum interference in every possibly interesting new physical parameter range. If real path quantum theory is relatively theoretically underconstrained, by the same token it points to previously unconsidered theoretical possibilities. It suggests our theoretical understanding of the range of plausible empirical implications of unified models of quasiclassical and quantum physics has been too limited and should be broadened.

Many particles and composition rules

These comments apply even more strongly when considering multi-particle systems. We have already noted interesting choices of distance function that violate the path sum rule

$$d(P \oplus Q, P' \oplus Q') = \max(d(P, P'), d(Q, Q')).$$
proposed in [19]. The path product rule
\[ d(P \otimes P', Q \otimes Q') = \max((d(P, Q), d(P', Q')) , \]
is equally open to question. Other seemingly mathematically natural possibilities for \( n \) distinguishable particles include
\[
d(P_1 \otimes P_2 \otimes \ldots \otimes P_n, Q_1 \otimes Q_2 \otimes \ldots \otimes Q_n) = \sum_i d(P_i, Q_i), \tag{22}
\]
and of course many others could be considered.

Even more generally, even for distinguishable particles, in principle the distance function for a product of paths need not be expressible as a function of individual path distance functions at all.

For indistinguishable particles, clearly, the distance function should respect the permutation symmetry. One possibility would be to frame a definition of a symmetric distance function \( d_{symm} \) in terms of one of the distance functions above:
\[
d_{symm}(P_1 \otimes P_2 \otimes \ldots \otimes P_n, Q_1 \otimes Q_2 \otimes \ldots \otimes Q_n) = \min_{\rho} d(P_1 \otimes P_2 \otimes \ldots \otimes P_n, Q_{\rho(1)} \otimes Q_{\rho(2)} \otimes \ldots \otimes Q_{\rho(n)}) ,
\]
where the minimum is taken over all permutations \( \rho \).

One further significant theoretical constraint here arises from the fact that, if real path quantum theory is fundamentally correct, it should in principle be applied to the entire universe, while if it is to be of any empirical use, it must be applicable to small subsystems. An effective \( d \) for single or few particle subsystems must be derivable from the definition of \( d \) for a many-particle system, even if the latter is not directly defined in terms of the former. But this still leaves many possibilities.

**Lorentz covariant rules for paths in Minkowski space**

A major reason for optimism that the quantum path integral may be fundamental is that – formally at least – it can naturally incorporate Lorentz and other symmetries. Encouragingly for real path quantum theory, we can also find relatively simple and seemingly natural Lorentz invariant measures of distance defined on reasonably general classes of paths.

Consider points \( A \) and \( B \) in Minkowski space, where \( B \) is in the causal future of \( A \). Let \( S_A \) and \( S_B \) be spacelike hyperplanes through \( A \) and \( B \) respectively.

Let \( P = (X_P(\lambda) : 0 \leq \lambda \leq 1) \) be a parametrised path in Minkowski space between points \( A \) and \( B \), where \( X_P(0) = A \), \( X_P(1) = B \), and the four-vector \( X_P \) is a continuous function of \( \lambda \). Take the Minkowski metric with the convention that spacelike vectors have positive length and \( c = 1 \), i.e. \( \Delta(x, y, z, t) = x^2 + y^2 + z^2 - t^2 \). We say \( P \) is a causal path if \( X_P(\lambda') \) is in the causal future of \( X_P(\lambda) \) whenever \( \lambda' > \lambda \). Clearly, if \( P \) is causal, it lies between \( S_A \) and \( S_B \). We say \( P \) is non-causal otherwise. We say \( P \) is anti-causal if there exists \( \lambda' > \lambda \) such that \( X_P(\lambda') \) is in the causal past of \( X_P(\lambda) \).

Let \( Q \) be a path that is not anti-causal (but not necessarily causal) path also lying between \( S_A \) and \( S_B \).

One simple candidate measure of distance for two such paths \( P, Q \) between points \( A \) and \( B \) in Minkowski space-time is
\[
d_1(P, Q) = \max_{\lambda, \lambda'} (\Delta(x_P(\lambda) - x_Q(\lambda')) ,
\]
the maximum spacelike separation between any pair of points on the two paths.

Note that if \( Q \) is not causal, it includes two spacelike separated points, and so if we extended this definition to pairs of non-causal paths we would have \( d_1(Q, Q) > 0 \), i.e. \( d_1 \) would violate the identity of indiscernibles. The distance function thus distinguishes causal paths \( P \), for which \( d_1(P, P) = 0 \), from non-causal paths \( Q \), for which \( d_1(Q, Q) > 0 \). Given this distinction, one arguably natural prescription for real path quantum theory in Minkowski space is then
to allow only causal paths to be realised, while allowing amplitudes from non-causal but not anticausal paths to contribute to their probabilities via the path probability postulate \(^2\). (Anticausal paths are ignored altogether in this prescription.) This would have the intuitively satisfactory consequence that only causal paths can be physically realised. It is also interesting to explore whether allowing all paths to be realised has physically sensible consequences, i.e. whether the fact that \(d_1(Q, Q) > 0\) for non-causal paths \(Q\), and the highly oscillatory variation of the phase in the neighbourhood of such paths, in any case suppresses the probability of highly non-causal paths being realised.

Another interesting Lorentz invariant distance function for pairs of paths \(P, Q\) with \(P\) causal and \(Q\) not anti-causal, is

\[
d_2(P, Q) = \int_P d\tau(\lambda) \max_{\lambda'} \Delta(x_P(\lambda) - x_Q(\lambda')) ,
\]

where \(\tau(\lambda)\) is the proper time along \(P\) from \(A\) to \(x_P(\lambda)\). This measure is sensitive not only to the space-like separations between points on the paths, but to the the proper time interval along \(P\) for there is any given space-like separation between \(P\) and \(Q\). Note that in this form this measure is not defined if \(P\) is not causal, and is not symmetric for causal \(P\) and \(Q\). It also does not satisfy the identity of indiscernibles: if \(P\) is an everywhere null causal path, then we have \(d_3(P, Q) = 0\) for all paths \(Q\).

(The definition could, of course, be extended in various ways. For example, for non-causal but not anti-causal \(P, Q\), we could define

\[
d'_2(P, Q) = \int_P d\tau(\lambda) \max_{\lambda'} \Delta(x'_P(\lambda) - x_Q(\lambda')) ,
\]

where \(P'\) is the not necessarily connected sub-path of \(P\) that is the maximal sub-path comprising causal segments, and \(\tau\) is now the proper time along each such segment. We could also symmetrise the definition by hand:

\[
d''_2(P, Q) = \frac{1}{2}(d'(P, Q) + d'(Q, P)) .
\]

As in the Galilean case, we could generalise in other ways too, for example by using monotonically increasing functions of \(\Delta(x_P(\lambda) - x_Q(\lambda'))\) in \(d_1\) or \(d_2\), by defining a proper time average version of \(d_2\) for non-null causal paths \(P\) by taking \(\frac{1}{\tau_P}d_2\) (where \(\tau_P\) is the total proper time along \(P\)) and so forth.

We stress that, as in the Galilean case, the suggestion here is not that any one of these distance functions (or some combination), or the path probability postulate \(^2\), or the prescription that only causal paths can be realised, must necessarily be right. There are many other possibilities. What we find encouraging is that the existence of Lorentz invariant distance functions transforms a conceptual problem (is there any conceivable realist Lorentz invariant generalisation of quantum theory?) into a technical problem (can we find a well-defined version of real path quantum theory for some choice(s) of Lorentz invariant distance function and path probability postulate?). Although the technical problem is formidable and we are far from understanding whether it is solvable, the existence of Lorentz invariant path distance functions suggests that there is no purely conceptual no-go result preventing the possibility of Lorentz invariant path-based solutions to the quantum reality problem.

**Generally covariant path distance functions**

We can also find relatively simple generally covariant definitions of the distance for a reasonably general class of paths in fixed Lorentzian background space-times other than Minkowski space. For simplicity we focus here on space-times with no closed time-like curves and trivial spatial topology.

For two non-causally separated points \(X, Y\) in such a space-time we define a distance function, \(\tilde{\Delta}(X, Y)\) to be the minimum spacelike distance along a space-like geodesic from \(X\) to \(Y\).

Now consider points \(A\) and \(B\) in our spacetime, where \(B\) is in the causal future of \(A\). Let \(S_A\) and \(S_B\) be spacelike hypersurfaces through \(A\) and \(B\) respectively, with the property that their proper time separation is bounded. That is, if we define \(\tau(X, Y)\) to be the maximum proper time along any causal path from a point \(X\) to some point \(Y\) in its causal future, we have \(\tau(X, Y) \leq \tau_0\) for all \(X \in S\) and \(Y \in S'\).

Now, defining causal and anti-causal paths as before let \(P\) be a causal path between \(A\) and \(B\), which (necessarily) lies between \(S_A\) and \(S_B\), and let \(Q\) be a path that is not anti-causal (but not necessarily causal) between \(A\) and \(B\) that also lies between \(S_A\) and \(S_B\).
We can define
\[ d(P, Q) = \max_{\lambda, \lambda'}(\hat{\Delta}(x_P(\lambda), x_Q(\lambda'))), \]
where as before \( \lambda, \lambda' \) define parametrisations of \( P, Q \) respectively and we use the definition of \( \hat{\Delta}(X, Y) \) just given.

As in the Minkowski case, extending this definition to pairs of non-causal paths would imply \( d(Q, Q) > 0 \) for non-causal \( Q \). Again, one arguably natural option for real path quantum theory is to postulate that only causal paths can be realised, while allowing amplitudes from non-causal but not anticausal paths to contribute to their probabilities via the path probability postulate \( \langle 2 \rangle \).

The other definitions considered in the Minkowski case can similarly be extended. Once again we stress that, as for the Galilean and Minkowski space cases, we presently see no compelling reason for singling out any of these particular definitions of covariant path distance or real path probability prescriptions: the encouraging point is that covariant definitions exist.

**Suppressing unphysical paths**

“Pathological” paths – which may traverse many regions of space-time very far from the stationary path and from each other, and may be very rapidly varying or even undifferentiable – are problematic in standard path integral quantum theory. Technically, they make it hard to define a computable path integral in realistic models. Physically, they seem to make it hard to assign a sensible meaning to the path integral, even at the level of intuition: does one really want to say that the real behaviour of a system is, in some sense, dominated by pathological paths?

The path probability postulate \( \langle 2 \rangle \), together with the scope for choices of the distance function, offer some hope of ameliorating or even eliminating these problems. A distance function sensitive to spatial separation would mean that a pathological path that travels far from the stationary paths makes little contribution to the probability of the latter being realised. Distance functions sensitive to first or higher derivatives can also suppress the contributions of rapidly varying or undifferentiable paths. If the intuition that the Feynman-Hibbs argument can be rigorized in real path quantum theory is justified, then the rapid oscillations of phase of paths in the neighbourhood of pathological paths tend to suppress the probability of these paths being realised themselves, since their probabilities will be close to zero.

At the risk of multiplying hypotheses, it is worth mentioning that another strategy for suppressing the probabilities and amplitudes of undesirably pathological paths could also be used. Real path quantum theory is an example of – in Bell’s terminology – a beable theory, that is, a theory with a sample space of possible realised ontologies (in this case, possible paths in the appropriate configuration space) and a probability distribution on that sample space. A quite general way \( \langle 24 \rangle \) of producing potentially interesting generalisations of a beable ontology is to allow the probability of any given configuration of “beables” being realised to depend, via simple rules, directly on the properties of that configuration, as well as on the Hamiltonian and boundary conditions of the underlying quantum theory.

In the case of real path quantum theory, one could further modify the path probability postulate \( \langle 2 \rangle \), for example by adding a prefactor in the form of a non-negative weight function \( w(P) \) that suppresses the probability of pathological paths \( P \) being realised. Thus, we could take
\[
\text{Prob}(P) = C' w(P) \left| \int dQ \exp(-iS(Q)) \exp(-d(P, Q)) \right|^2 \left( \int dQ \exp(-d(P, Q)) \right)^{-1},
\]
where \( w(P) \geq 0 \) is significantly nonzero for physically reasonable paths and small for pathological paths, and the constant \( C' \) normalises this new probability distribution. (We have already considered a version of this in suppressing non-causal paths in Lorentzian space-times.) In principle, one can consider any choice of \( w(P) \): for example, it could depend on the maximum or typical curvature of the path \( P \), or on some measure(s) of its derivative(s). In Galilean or Minkowski space-time, one could even ensure that any realised \( P \) must be piecewise linear with a characteristic linear scale, by setting \( w(P) = 0 \) unless \( P \) comprises linear segments of given length \( \delta \) or given proper time \( \delta \tau \), if one wished to introduce new fundamental scales into the ontology.

**Comments on double ontologies and the problem of tails**

Superficially, real path quantum theory (in models where it is rigorously defined) has some resemblance to de Broglie-Bohm theory. In the simplest scenario, that of models of \( N \) distinguishable particles, both approaches produce a probability distribution on trajectories in \( N \) particle configuration space.
In fact, there are major differences. The probability distributions are different, for a given initial quantum state and Hamiltonian. Real path quantum theory generally predicts different physical outcomes from those of standard quantum theory, whereas standard de Broglie-Bohm theory predicts the same outcomes. The basic postulates of real path quantum theory extend naturally to relativistic settings, whereas de Broglie-Bohm theory is hard to relativise.

There is also a significant difference in the ontologies. Standard de Broglie-Bohm theory offers the possibility of two separate although related ontologies, defined by taking the beables to be respectively the quantum wave function (or some derivative thereof, such as the mass density) and the de Broglie-Bohm trajectories. That the first of these choices is identical to the one proposed (albeit in many different versions with many different interpretative strategies) by many Everettians lead to Deutsch's well-known charge that "pilot-wave theories are parallel-universe theories in a state of chronic denial". Of course, one logically consistent defence for de Broglie-Bohm theorists is observe that in any physical theory one has to make choices about which parts of the mathematical formalism are beables – i.e. define the ontology – and which parts are auxiliary, and to decide to declare by fiat that the particle trajectories are beables while the wave function is not.

Whether one is entirely happy with this defence is ultimately a matter of metaphysical taste. Since in practice many are not, it seems worth noting that real path quantum theory does not seem vulnerable to the same charge of a potential or actual double ontology. Paths are fundamental in the formulation of this approach to quantum theory. It produces a probability distribution on paths. The ontology is as expected from any straightforwardly probabilistic physical theory: one element of the sample space (in this case one path) is randomly chosen from the probability distribution, and it is (only) this element that is physically realised. The wave function is not an ontological competitor. It plays no fundamental role, emerging only – and only in some situations – as an approximately defined quantity that gives a convenient alternative approximate mathematical description of expected experimental outcomes.

Real path quantum theory also has some resemblance to GRWP and other dynamical collapse models. Like these models, it can preserve quantum interference for suitably microscopic interference experiments, while predicting an intrinsic decoherence that suppresses interference for suitably macroscopic interference experiments. Collapse models, like de Broglie-Bohm models, offer the possibility of at least two distinct types of ontology. One is the "flash ontology" originally defined by Bell for the discrete GRW model, or its analogue for continuous dynamical collapse models. The other is some ontology defined by or derived from the collapse model wave function (again, for example, via the mass density).

Worries about a double ontology perhaps carry less force for GRW models than for de Broglie-Bohm theory, for various reasons. First, some might perhaps argue that the "flash ontology" is less compellingly natural than the trajectory ontology. Second, the two ontologies are in any case roughly aligned – in the sense that, at least on a (perhaps overly) superficial reading they tell similar stories. In a standard quantum measurement setting, the collapse model wave function tends towards a description of system and apparatus corresponding to one measurement outcome, while the component corresponding to the other outcome is swiftly and exponentially suppressed; similarly, the flashes congregate around apparatus particle position locations corresponding to one outcome, and swiftly and with increasingly high probability tend to avoid the other. There is no straightforwardly Everettian competitor ontology here.

Nonetheless, these double ontology worries are not entirely eliminated. In particular, a feature that many have found problematic is that, although the components of the wave function corresponding to "unselected" outcomes decay rapidly after measurement, they never entirely disappear, and the physical description of the alternative outcome remains encoded in these small but nonvanishing components. This "problem of tails" opens up the possibility of a subtle but persistent residual Everettian ontology. Whether it is a potential concern depends on whether one is justified in neglecting small amplitude components of the wave function as essentially irrelevant. However, this very issue lies at the heart of the problem of probability in many-worlds quantum theory. Those (many) who believe the problem of probability has no satisfactory solution thus find it hard to dismiss the problem of tails completely. On the other hand, those (also many) who believe the problem of probability in many-worlds quantum theory does have a satisfactory solution tend also to take many-worlds quantum theory as a satisfactory answer to the quantum reality problem, and are correspondingly less motivated to take dynamical collapse models seriously in the first place.

Again, then, it seems worth noting that real path quantum theory does not suffer from a problem of tails. The wave function plays no fundamental role. One path is randomly chosen to be physically realised. The other paths are not chosen, and so do not form part of the ontology.
DISCUSSION

All approaches to quantum theory and its generalisations – whether or not they are motivated by the quantum reality problem – currently have deep problems, if not in the eyes of their proponents, then certainly in the view of most dispassionate outsiders. This makes the task of the theorist interested in these fundamental questions very challenging. We have to strive for conceptual clarity and mathematical rigour. We have to try to understand which approaches might at least potentially ultimately produce a mathematically and conceptually complete description of nature, and which are likely dead ends that are bound to fall short of this goal. At the same time, so long as no approach offers a fully satisfactory solution, we should not neglect potentially useful insights that incompletely developed models can provide. To list some examples:

Could it be literally true that unitary quantum theory is fundamental and the wave function represents reality, as Everett first suggested? Can we even make conceptual sense of the idea and recover ordinary science? Many physicists think not, but few would deny that the idea has been a theoretically and practically useful way of thinking about quantum theory and quantum information theory, even if it is best thought of as an unrealistic limiting case. In particular, the idea makes a clear prediction that could not so unambiguously be extracted from pre-Everettian quantum theory. Namely that, however large and complex a physical system may be, in principle, provided it can be suitably controlled while effectively isolated from the environment, it will display quantum interference.

Is wave function collapse a real and fundamental physical phenomenon? Perhaps, perhaps not. But the idea has motivated the theoretically and scientifically fertile dynamical collapse model program, including speculative but intriguing ideas relating collapse to gravity [8, 25]. Again, these ideas suggest clear predictions: that quantum interference will fail in a regime where a well-defined collapse model suggests that the superposed states collapse to a component.

Is Bell’s notion of beables [2, 3] a good way of thinking about fundamental physics, and the right language in which to address the quantum reality problem? Again, many, perhaps most, physicists presently think not. But nonetheless the idea motivates new generalisations of quantum theory [21, 24, 34], and new ways of thinking about the relationship of quantum theory and gravity, that can be scientifically useful and suggest new experimental tests – spin-offs which might lead to new physics, even if that physics turns out not to be written in the language of beables.

We would thus tentatively suggest that, while further development is of course urgently needed, real path quantum theory has some scientific yield, even in its presently undeveloped state. This is the motivation to explore empirically whether quantum interference fails, with an open mind about what governs the transition between interference and effective decoherence. If we take the idea of real path quantum theory seriously, then until further theoretical and/or mathematical constraints are uncovered, this transition could be governed by almost any physically interesting parameter or combination of parameters: maximum beam separation, average beam separation, separation time, particle mass, and so on.

As we remarked earlier, in one important sense this is scientifically quite discouraging. In an ideal universe, we might prefer any idea for a generalisation of quantum theory to be strongly constrained (or, even better, to offer a unique alternative) and to make precise predictions of how and where quantum theory should break down. It would be pleasing to identify some critical, and preferably soon feasible, experiment that distinguishes between quantum theory and the generalisation, so that we can refute or confirm the idea once and for all.

However, we do not get to choose the universe we live in, nor the size or structure of the class of mathematically consistent theories that generalise our best current theories of that universe. Our theoretical understanding of the class of unified models of quasiclassical and quantum physics and the range of plausible empirical implications of these models may indeed – as the idea of real path quantum theory presently suggests – have been too limited. It appears there may be a very large class of consistent generalisations of quantum theory (as Refs. [21, 24] also suggest, using different ideas and different classes of generalisations from those considered here). Unless and until our theoretical understanding advances further, it may be – as the preliminary version of real path quantum theory presented here suggests – that the appropriately scientifically open-minded and curious strategy is to test quantum interference experimentally in every possibly interesting new physical parameter range, rather than for example – focussing only on those picked out by simple dynamical collapse models, well-motivated though tests of dynamical collapse models certainly are.

One significant theoretical constraint whose implications for real path quantum theory need to be properly understood is the no-signalling principle. Practically speaking, for models in which a physically sensible multi-particle or field configuration distance function allows real path quantum theory for small subsystems of the universe to be approximately derived from a fundamental real path quantum theory for the universe, we do not expect agents’ choices about actions on one small subsystem to have a significant effect on the real path probability distribution of
another distant small subsystem. So, for such models, we expect no-signalling to be an excellent approximation by this operational definition. However, the possibility of enforcing no-signalling precisely and in all circumstances, and the implications of such a constraint, needs further investigation. A failure of no-signalling need not be a reason for rejecting any form of real path quantum theory, particularly if the observational consequences are negligible. There are consistent Lorentz invariant generalisations of quantum theory that in principle allow superluminal signalling but evade causal paradoxes\cite{22}. However, the no-signalling principle remains, at the very least, a very interesting theoretical constraint on generalizations of quantum theory\cite{15, 26}.

The ultimate vision of those who take path integral quantum theory as fundamental to all of physics is a path integral formulation of quantum gravity and quantum cosmology. It hardly needs saying that it would be very interesting to explore the possibilities opened up by real path quantum theory in this context, the range of potentially natural distance functions between cosmological paths, and the possibility of using the path probability postulate and the properties of distance functions to allow rigorous definitions and calculations.

Finally, we wish to emphasize that the appeal and generality of the path integral formalism and its consequent possible value in generalising quantum theory and/or addressing the quantum reality problem has of course also been recognised by others. In particular – to list just those of which we are aware – Sorkin\cite{29, 30}, Dowker-Johnston-Sorkin\cite{9}, Gell-Mann and Hartle\cite{11, 17}, Stamp\cite{32}, and Spekkens\cite{31} have ongoing research programs exploring these questions from various perspectives. The independent work described here starts from different premises, and as far as we presently understand things it seems to suggest a different research agenda and qualitatively (as well as quantitatively) different ontological and experimental conclusions. It would, however, certainly be interesting to explore possible connections between, or combinations of ideas from, these programs.

ACKNOWLEDGEMENTS

This work was partially supported by a Leverhulme Research Fellowship, a grant from the John Templeton Foundation, and by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. I thank Bei Lok Hu for a helpful conversation.

References

* Electronic address: A.P.A.Kent@damtp.cam.ac.uk

[1] D. Albert. Probability in the Everett picture. Many worlds?: Everett, quantum theory, and reality, Saunders, S., Barrett, J., Kent, A. and Wallace, D. (eds.), Oxford Univ. Press, pages 355–368, 2010.
[2] J.S. Bell. The theory of local beables. Epistemological Letters, 9:11, 1976.
[3] J.S. Bell. Beables for quantum field theory. Quantum implications: Essays in honour of David Bohm, pages 227–234, 1987.
[4] J.S. Bell. Are there quantum jumps? John S. Bell on the foundations of quantum mechanics, page 172, 2001.
[5] David Bohm. A suggested interpretation of the quantum theory in terms of “hidden” variables. 1. Phys. Rev., 85:166–179, Jan 1952.
[6] L. de Broglie. in Solvay Congress (1927). Electrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique tenu Bruxelles du 24 au 29 Octobre 1927 sous les Auspices de l’Institut International de Physique Solvay, 1928.
[7] D. Deutsch. Comment on Lockwood. The British Journal for the Philosophy of Science, 47(2):222–228, 1996.
[8] L. Diosi. Models for universal reduction of macroscopic quantum fluctuations. Phys. Rev. A, 40:1165, 1989.
[9] F. Dowker, S. Johnston, and R. Sorkin. Hilbert spaces from path integrals. Journal of Physics A: Mathematical and Theoretical, 43(27):275302, 2010.
[10] R. Feynman, A. Hibbs, and D. Styer. Quantum mechanics and path integrals, volume 2. McGraw-Hill New York, 1965.
[11] M. Gell-Mann and J. Hartle. Decoherent histories quantum mechanics with one real fine-grained history. Physical Review A, 85(6):062120, 2012.
[12] R. Geroch. The everett interpretation. Noûs, 18(4):617–633, 1984.
[13] G.C. Ghirardi, P. Pearle, and A. Rimini. Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles. Phys. Rev. A, 42:78–89, Jul 1990.
[14] G.C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and macroscopic systems. Physical Review D, 34(2):470, 1986.
[15] N. Gisin. Weinberg's non-linear quantum mechanics and supraluminal communications. *Physics Letters A*, 143(1-2):1–2, 1990.

[16] D. Giulini, E. Joos, C. Kieffer, J. Kupsch, I.O. Stamatescu, and H.D. Zeh. *Decoherence and the appearance of a classical world in quantum theory*. Springer, Berlin (Germany), 1996.

[17] J. Hartle. Quantum mechanics with extended probabilities. *Physical Review A*, 78(1):012108, 2008.

[18] J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28:2960–2975, Dec 1983.

[19] A. Kent. “Quantum jumps” and indistinguishability. *Modern Physics Letters A*, 4:1839–1845, 1989.

[20] A. Kent. Against many-worlds interpretations. *International Journal of Modern Physics*, page 1745, 1990.

[21] A. Kent. Beyond boundary conditions: General cosmological theories. In *Particle Physics and the Early Universe, Proceedings of COSMO-97*, L. Roszkowski (ed.), pages 562–564. World Scientific, 1998; arXiv:0905.0632.

[22] A. Kent. Causality in time-neutral cosmologies. *Phys. Rev. D*, 59:043505, 1999.

[23] A. Kent. One world versus many: the inadequacy of Everettian accounts of evolution, probability, and scientific confirmation. *Many worlds?: Everett, quantum theory, and reality*, Saunders, S., Barrett, J., Kent, A. and Wallace, D. (eds.), *Oxford Univ. Press*, pages 307–354; arXiv:0905.0624, 2010.

[24] A. Kent. Beable-guided quantum theories: Generalizing quantum probability laws. *Physical Review A*, 87(2):022105, 2013.

[25] R. Penrose. *The Emperor’s New Mind*. Oxford University Press, 1999.

[26] J. Polchinski. Weinberg’s nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox. *Physical Review Letters*, 66(4):397–400, 1991.

[27] H. Price. Decisions, decisions, decisions: Can Everett salvage Savage? *Many worlds?: Everett, quantum theory, and reality*, Saunders, S., Barrett, J., Kent, A. and Wallace, D. (eds.), *Oxford Univ. Press*, pages 369–390, 2010.

[28] S. Saunders, J. Barrett, A. Kent, and D. Wallace. *Many worlds?: Everett, quantum theory, and reality*. Oxford University Press, 2010.

[29] R. Sorkin. Quantum measure theory and its interpretation. *Arxiv preprint gr-qc/9507057*, 1995.

[30] R. Sorkin. Quantum dynamics without the wavefunction. *Journal of Physics A: Mathematical and Theoretical*, 40(12):3207, 2007.

[31] R. Spekkens. unpublished.

[32] P. Stamp. Environmental decoherence versus intrinsic decoherence. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 370(1975):4429–4453, 2012.

[33] R. Tumulka. A relativistic version of the Ghirardi–Rimini–Weber model. *Journal of Statistical Physics*, 125(4):821–840, 2006.

[34] A. Valentini. Signal-locality, uncertainty, and the subquantum H-theorem. I. *Physics Letters A*, 156(1-2):5–11, 1991.

[35] We exclude this possibility in our models, since we intend the set of allowed paths in our toy models to reflect physically significant features of the quantum path integral as faithfully as possible. Our discrete toy models are meant to illustrate plausible consequences of our real path ontology and path probability postulate when applied to the full set of paths in realistic models. Our interest is in modifying standard quantum theory via the real path ontology and path probability postulate. New features that arise solely because of artefacts of a particular discrete path integral model are not interesting for this project. Although, as we comment later, real path quantum theory in its present form is not guaranteed to respect the no-signalling principle, it makes no sense to introduce a further model-dependent and ad hoc violation.