Probing Noncommutativity with Inflationary Gravitational Waves

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In this paper we study the behaviour of gravitational wave background (GWB) generated during inflation in the noncommutative field approach. From this approach we derive one additional term, and then we find that the dispersion relation of the gravitational wave would be modified and the primordial gravitational wave would obtain an effective mass. Therefore it breaks local Lorentz symmetry. Moreover, this additional term suppresses the energy spectrum of the GWB greatly at low energy scale where the wave length is near the current horizon. Due to this, a sharp peak is formed in the energy spectrum at the low frequency band. This peak should be a key criterion in detecting the spacetime noncommutativity and a critical test for Lorentz symmetry breaking in local field theory.

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I. INTRODUCTION

Under the standard model of cosmology plus the theory of inflation\cite{1, 2, 3}, it is very natural to predict the existence of the gravitational wave background. Scalar type and tensor type primordial perturbations are generated during inflation. The primordial scalar perturbations can be tested in current observations of cosmic microwave background\cite{15, 16}. And they provide seeds for the large scale structure (LSS), which then gradually forms today's galaxies\cite{1, 7, 8, 9, 10}. The tensor perturbations escape the horizon during inflation, and then they keep conserved and form the relic GWB which carries the information of the very early universe. There have been a number of detectors operating for the signals of primordial GWB, e.g. Big Bang Observer (BBO)\cite{11}, Planck\cite{12}. Besides, there have been evidences from the indirect detections of the next generation of CMB observations, see related analysis\cite{13, 14, 15}. The basic mechanism for the generation of primordial GWB in cosmology has been discussed in Refs. \cite{16, 17}. See \cite{18, 19} for the gravitational waves generated during the epoch of inflation; see \cite{20} for that in Pre Big Bang scenario; see \cite{21} for that in cyclic universe; see \cite{22} for that from phantom super-inflation (also see \cite{23, 24}).

Generally speaking, the GWB was generated in the epoch when the energy scale of our universe is extremely high. Thus we should take into consideration on more fundamental theories in logic, namely, the string theory. In string theory it is commonly realized that the space-time coordinates of a Dp-brane is noncommutative under certain external fields, and this can be described by the language of noncommutative geometry\cite{25, 26, 27, 28, 29}. Accordingly, there are a lot of new properties, opening up new challenges for current theories and experiments. For example, causality is possibly violated as a consequence of the noncommutative behaviour (see Refs. \cite{30, 31}), and thus the Lorentz invariance\cite{32, 33, 34} is also probably broken. When the noncommutativity is taken into account during the very early universe, it would provide large corrections to inflation, the primordial fluctuations\cite{35, 36, 37, 38}, black-body spectra of CMB\cite{39}, and also primordial magnetic field\cite{40}. Furthermore, the paper \cite{41} suggested another mechanism, named noncommutative field approach, which can also lead to the Lorentz invariance breaking under noncommutative fields while leaving spacetime commutative. The key idea of this mechanism is to construct an effective action which contains noncommutative terms partly obtained from noncommutative geometry. Later, this noncommutative field approach has been generalized to various models and has made many interesting predictions (see Refs. \cite{42, 43, 44, 45}). If Lorentz symmetry is indeed broken in our world, it would result in the breaking of the CPT symmetry, which is possible to be detected in experiments. The paper \cite{46} has already found some evidences to support that cosmic CPT symmetry may not be conserved from the analysis of data fitting for cosmological observations.

In our paper, using the noncommutative field approach, we investigate the gravitational wave background during inflation. The outline of this paper is as follows. In section II, we introduce the additional term by applying the noncommutative field approach to the equation of motion for the gravitational perturbations in the flat Friedmann-Robertson-Walker (FRW) background. Then we quantize the noncommutative tensor perturbations and discuss the consistent initial conditions for primordial gravitational waves with the noncommutative terms. After that, we derive the classical solution of primordial tensor perturbation with noncommutativity, and analyze the behaviour of the primordial power spectrum and the corresponding spectral index. In section III, from considering the influence of possible factors on the energy spectrum, we discuss the transfer function for the noncommutative GWB in order to relate the current GWB we

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may probe with the primordial one. Finally, we give the implications of the results from calculating the noncommutative GWB in section IV, and compare them with the future development of the cosmological observations. Then we summarize the fruitful behaviors of GWB possessing noncommutativity which may be detected in the future probe.

In this paper, we will see that, the dispersion relation of the primordial gravitational wave will be modified, and its power spectrum will exhibit difference from the normal one around the CMB scale due to the noncommutative term. Furthermore, we can obtain the same viewpoint from the relationship between the tensor spectral index and its frequency. From the modified primordial power spectrum, we can show that there will be a sharp peak in the CMB scale of the energy spectrum observed today. In order to add the contribution of the transfer function, we discuss some leading corrections to the primordial GWB during the evolution of the universe. Among these corrections, we mainly consider the suppression of the GWB from the redshift, the impression of the background equation of state of the universe when the GWB re-enters the horizon, and the damping effects from the anisotropic stress generated from some particles' freely streaming. After considering all these contributions, we can clearly find that the effects of noncommutative terms still play a significant role in the low frequency range. So the results and predictions in the above sections are important for future observations.

II. NONCOMMUTATIVE TENSOR PERTURBATIONS DURING INFLATION

A. Bases and Conventions

In this section, we take a brief review of the standard theory of tensor perturbation during inflation (for more details, see the review article Ref. 51). In the flat FRW background, the tensor perturbation is given in the metric,

$$ds^2 = a(\tau)^2[-d\tau^2 + (\delta_{ij} + \tilde{h}_{ij})dx^i dx^j] ,$$

where $$g_{\mu\nu} = diag(-a^2, a^2, a^2, a^2)$$ is the background metric, $$\tau$$ is the conformal time, $$a(\tau)$$ is the scale factor, and the Latin indexes represent spatial coordinates. Here the perturbation $$\tilde{h}_{ij}$$ satisfies the following constraints:

$$\tilde{h}_{ij} = \tilde{h}_{ji} ; \quad \tilde{h}_{ii} = 0 ; \quad \tilde{h}_{iij} = 0 .$$

Therefore, $$\tilde{h}_{ij}$$ only have two degrees of freedom corresponding to two polarizations of gravitational waves.

To start from Einstein’s field equation, one can deduce the action of free tensor perturbation as follows:

$$(2)^{\text{st}} S_g = \int d\tau d^3 x \frac{1}{64\pi G} a^2 \tilde{h}_{ij}^2 - (\delta \tilde{h}_{ij})^2 ,$$

where prime represents the derivative with respect to the conformal time. The interaction part of the action with other matter sources is of the form

$$(2)^{\text{nd}} S_m = \int d\tau d^3 x \frac{1}{24} \sigma_{ij} \tilde{h}_{ij} ,$$

where $$\sigma_{ij}$$ is the anisotropic part of the stress tensor, constructed by the spatial components of the perturbed energy-momentum tensor $$T^i_j$$,

$$\sigma_{ij} = T^i_j - \frac{1}{2} \delta^i_j \rho .$$

Then one can derive the general equation of motion for tensor perturbation:

$$\ddot{\tilde{h}}_{ij} + 2 \frac{a'}{a} \dot{\tilde{h}}_{ij} - \nabla^2 \tilde{h}_{ij} - 16\pi G a^2 \sigma_{ij} = 0 .$$

The Fourier transformation of the tensor perturbation and the anisotropic stress tensor takes the form,

$$\tilde{h}_{ij}(\tau, \mathbf{x}) = \sqrt{16\pi G} \int \frac{d^3 k}{(2\pi)^{3}} H_{ij}(\tau, \mathbf{k}) e^{i\mathbf{kx}} ,$$

$$\sigma_{ij}(\tau, \mathbf{x}) = \sqrt{16\pi G} \int \frac{d^3 k}{(2\pi)^{3}} \Sigma_{ij}(\tau, \mathbf{k}) e^{i\mathbf{kx}} ,$$

where we leave the Latin indices in order to keep the following progresses to be general. In cosmology, what we care about is the distribution of the spectra of gravitational waves and the corresponding spectral index. Based on the above formalism, the tensor power spectrum can be written as,

$$P_T(k, \tau) \equiv \frac{d\langle |\tilde{h}_{ij}(\tau, 0)|^2 \rangle}{dk} = 32\pi G \frac{k^3}{(2\pi)^2} |H_{ij}(\tau, \mathbf{k})|^2 ,$$

and the definition of tensor spectral index $$n_T$$ is given by

$$n_T \equiv \frac{d\ln P_T}{d\ln k} .$$

The GWB we observed today is characterized by the energy spectrum,

$$\Omega_{GW}(k, \tau) \equiv \frac{1}{\rho_c(\tau)} \frac{d\langle |\rho_{GW}(\tau)|^2 \rangle}{dk} ,$$

where $$\rho_{GW}(\tau)$$ indicates the energy density of gravitational waves, and the parameter $$\rho_c(\tau)$$ is the critical density of the universe. Make use of the Friedmann equation

$$H^2(\tau) = \frac{8\pi G}{3} \rho_c(\tau) ,$$

the energy spectrum of GWB can be written as,

$$\Omega_{GW}(k, \tau) = \frac{8\pi G}{3H^2(\tau)} \frac{k^3}{2(2\pi)^2 a^2} \frac{1}{|H_{ij}|^2 + k^2 |H_{ij}|^2} .$$

In respect that the GWB we observed has already re-entered the horizon, its mode should oscillate in the form
of sinusoidal function. Accordingly, we can deduce the relation between the power spectrum and the energy spectrum we care about as follows

$$\Omega_{GW}(k, \tau) \simeq \frac{1}{12} a^2(\tau) H^2(\tau) P_T(k, \tau). \quad (14)$$

Note that, the equations \[13\] and \[14\] can be extended into the case of noncommutative gravitational waves because in the framework of noncommutative GWB the Hamiltonian is not modified, we will see the detailed analysis in the next section.

B. Quantization and Noncommutativity

In this section, we will quantize the tensor perturbation, and then introduce the noncommutative term in the canonical commutation relation. Based on this, we will obtain a modified theory of gravity in weak gravity approximation and finally provide the equation of motion of primordial gravitational waves in this theory. Since the action of tensor perturbation in noncommutative field approach will break the Lorentz symmetry in local Minkowski spacetime which can be exhibited in the dispersion relation of gravitons\[45\], we can see the corresponding behaviour of gravitons in the flat FRW universe background.

Firstly, for simplicity, we redefine the tensor perturbation as follows

$$H_{ij}(\tau, k) \equiv \frac{h_{ij}(\tau, k)}{a(\tau)}. \quad (15)$$

Therefore, in the momentum space the action of tensor perturbation including the minimal coupling with matter is given by

$$\mathcal{S} = \int d\tau d^3k \left[ \frac{1}{4} (h''_{ij} + \frac{a''}{a} h''_{ij} - k^2 h''_{ij}) + 8\pi G a^3 \Sigma_{ij} h_{ij} \right], \quad (16)$$

with the equation of motion,

$$h''_{ij} - \frac{a''}{a} h_{ij} + k^2 h_{ij} - 16\pi G a^3 \Sigma_{ij} = 0. \quad (17)$$

Secondly, from the standard canonical quantization, the conjugate momentum can be written as,

$$p_{ij} = \frac{\delta^{(2)} S}{\delta h''_{ij}} = \frac{1}{2} h_{ij}^\prime, \quad (18)$$

and the canonical Hamiltonian density in Fourier space is given by

$$H = p_{ij} h_{ij}^\prime - \mathcal{L}$$

$$= \frac{1}{4} p_{ij}^2 - \frac{1}{4} \frac{a''}{a} h_{ij}^2 + \frac{1}{4} k^2 h_{ij}^2 - 8\pi G a^3 \Sigma_{ij} h_{ij}. \quad (19)$$

Converting the fields $h_{ij}$ and the conjugated momenta $p_{ij}$ into operators, we obtain the equal-time commutation relations from the Poisson algebra,

$$[h_{ij}(\tau, k), h_{kl}(\tau, k')] = 0, \quad (20)$$

$$[p_{ij}(\tau, k), p_{kl}(\tau, k')] = 0, \quad (21)$$

$$[h_{ij}(\tau, k), p_{kl}(\tau, k')] = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \times \delta^{(3)}(k - k'). \quad (22)$$

Finally, following the similar formalism in \[45\], we make a deformation of the commutation relation of the conjugated momenta into the noncommutative form

$$[p_{ij}(\tau, k), p_{kl}(\tau, k')] = \alpha_{ijkl} \delta^{(3)}(k - k'), \quad (23)$$

where $\alpha_{ijkl}$ is a constant matrix. In order to preserve the properties of tensor perturbation, we require that $\alpha_{ijkl}$ satisfies the following relation,

$$\alpha_{ijkl} = \alpha_{jikl} = -\alpha_{klij}. \quad (24)$$

This is satisfied if

$$\alpha_{ijkl} = \alpha_{ik} \delta_{jl} + \alpha_{il} \delta_{jk} + \alpha_{jk} \delta_{il} + \alpha_{jl} \delta_{ik},$$

with $\alpha_{jk}$ a constant antisymmetric matrix. One can check that $\alpha_{ijkl}$ keeps the traceless property of tensor perturbation. Moreover, in order to simplify the calculation, $\alpha_{jk}$ can be further expressed as,

$$\alpha_{jk} = -\epsilon_{ijk} a^j, \quad (26)$$

where $\epsilon_{ijk}$ is a totally antisymmetric tensor, and we use the convention that $\epsilon_{0123} = -1$ and $\epsilon^{0123} = +1$ in our paper.

In order to recover a canonical system quantized in normal commutative relations like eq. \[21\], we construct a new conjugated momentum $\pi_{ij} = p_{ij} - \frac{1}{2} \alpha_{ijkl} h_{kl}$ for the tensor fields $h_{ij}$. Since in the derivation above we have discarded the components of metric perturbations containing the time index, the origin form of lagrangian is not covariant. We modify the lagrangian by adding an additional term of covariant form and then leave physical degrees of freedom in the following,

$$\Delta \mathcal{L} = -2 \epsilon_{\mu
u\sigma\rho} \alpha_{\sigma} h_{\nu \sigma} \partial_{\rho} h_{\mu}^\prime, \quad (27)$$

$$\rightarrow -2 \alpha_0 \epsilon_{ijkl} h_{kl} \partial_{\tau} h_{ij} - 2 \alpha_m \epsilon_{ijklm} h_{kl} h_{ij}^\prime, \quad (28)$$

where we call $\alpha_0$ as noncommutative parameters.(NP).

Note that after making the Fourier transformation, the term containing $\alpha_0$ disappears because of the symmetry of exchanging indices “$k$” and “$j$”. Here one can check that Hamiltonian still keep invariant since the term containing $\alpha_m$ can be cancelled by modified conjugated momentum and the term including $\alpha_0$ also disappears due to the symmetry of exchanging indices. Therefore, the modified Einstein-Hilbert action of tensor perturbations is given by

$$\Delta \mathcal{L} = -2 \epsilon_{\mu
u\sigma\rho} \alpha_{\sigma} h_{\nu \sigma} \partial_{\rho} h_{\mu}^\prime, \quad (27)$$

$$\rightarrow -2 \alpha_0 \epsilon_{ijkl} h_{kl} \partial_{\tau} h_{ij} - 2 \alpha_m \epsilon_{ijklm} h_{kl} h_{ij}^\prime, \quad (28)$$
and the corresponding equation of motion is of form

\[ h''_{ij} + k^2 h_{ij} - \frac{a''}{a} h_{ij} + 8\alpha_m e^{\eta_{im}} h_{ij} = 16\pi G a^3 \Sigma_{ij} h_{ij} . \]  

(30)

Compared with eq. (17), one can already find the difference appeared in each one’s dispersion relation roughly.

C. Noncommutative Tensor Perturbations During Inflation

In order to analyze the noncommutative tensor perturbations explicitly, we need to obtain the classical solution of each mode of tensor perturbations under the slow roll inflation background. Here we choose NP of form: \( \alpha_m = \{ 0, 0, \alpha_3 \} \) and substitute this into Eq. (30).

Besides, we neglect the anisotropic tensor fluctuations of matters and thus \( \Sigma_{ij} \approx 0 \) here. According to the qualities of gauge invariance, we can obtain only two independent modes which can be expressed as

\[ v_1 = \frac{1}{\sqrt{2}} (h_{11} + i h_{12}) , \quad v_2 = \frac{1}{\sqrt{2}} (h_{11} - i h_{12}) , \]  

(31)

and in our note we call them left-handed and right-handed respectively. These two modes satisfy their own equations:

\[ v_1'' + 8i\alpha_3 v_1' + k^2 v_1 - \frac{a''}{a} v_1 = 0 , \]  

\[ v_2'' - 8i\alpha_3 v_2' + k^2 v_2 - \frac{a''}{a} v_2 = 0 . \]  

(33)

To solve these two equations explicitly, we need to know their initial conditions. In the noncommutative field approach the dispersion relation of gravitational waves has been modified, thus it seems that the initial conditions should be different from those in commutative case. Thus we need to reconsider the choice of initial conditions. Here we fix the initial conditions by requiring the initial state to be the lowest energy state(more general initial conditions of modified dispersion relation, see Ref. 52). Since the noncommutative term does not contribute to energy density, we can use the original form appeared in Eq. (13), and then give the expression of the energy density:

\[ \rho_{GW} = \int_0^\infty dk \frac{k^2}{8\pi^2a^4} \sum_{i=1,2} \left[ v_i' v_i'' + \frac{a''}{a} v_i' v_i' - \frac{a'^2}{a^2} v_i v_i' \right] , \]  

(34)

where we sum ‘i’ over the left-handed and right-handed modes. Note that, the energy density can be divided into two part, for one only containing \( v_1 \) and for the other only containing \( v_2 \). Accordingly, we can analyze each mode separately and then obtain the initial conditions for each one. Now we define a ratio \( \frac{v_1}{v_2} = x(\tau) + iy(\tau) \), and for convenience we convert a Wronskian \( W \equiv v_1' v_2' - v_1 v_2' \). Then at the initial conformal time \( \tau_0 \) the energy density of the left-handed mode is given by

\[ \rho_L = \int_0^\infty \frac{dk k^2 W}{16i\pi^2a^4} (x_0^2 + y_0^2 - 2\frac{a'}{a} x_0 + \frac{a'^2}{a^2} + k^2) . \]  

(35)

where \( x_0 = x(\tau_0) \) and \( y_0 = y(\tau_0) \) which are the ratio parameters at the initial conformal time.

Commonly, it is sufficient to obtain the initial conditions by calculating the variation of the energy density with respect to \( x_0 \) and \( y_0 \). From the vanishing variations we obtain the “initial” conditions \( x_0 = \frac{a'}{a} \) and \( y_0 = k^2 \). However, if these two conditions are satisfied in Eq. (32) and (33), we have to require \( \alpha_3 \) to be zero which violates the noncommutativity. Therefore, the lowest energy density of noncommutative primordial tensor perturbations is not an extremal point where the variations with respect to \( x_0 \) and \( y_0 \) are zero. Instead, the lowest energy density can only be chosen on the boundary value of the range we care about. Moreover, this boundary value should be positive and independent of any time-dependent phase. Therefore, we deduce that only if \( |y_0| \) approaches \( k \) as closely as possible, it is satisfied that the energy state is the lowest in the range we allow. Consequently, we obtain the correct initial condition \( y_0 = -l - 4\alpha_3 \) for the left-handed in our note; similarly, we also obtain the corresponding initial condition for the right-handed.

Thus when \( |l| \gg a\dot{H} \), \( v_1 \) and \( v_2 \) are given by

\[ v_1 \rightarrow \frac{1}{\sqrt{2l}} e^{-i(4\alpha_3)\tau} , \quad v_2 \rightarrow \frac{1}{\sqrt{2l}} e^{-i(l+4\alpha_3)\tau} , \]  

(36)

where \( l \) denote the effective co-moving wave numbers of the modified tensor perturbations, and its form is given by

\[ l = (k^2 + 16\alpha_3^2)^\frac{1}{2} . \]  

(37)

Thus we see that both \( v_1 \) and \( v_2 \) display a well oscillating behaviour inside of horizon, and are identical with the standard primordial tensor perturbations if we neglect the noncommutative terms. During inflation, in the slow roll approximation, we have the relation

\[ \frac{a''}{a} = \frac{v_2 - \frac{1}{2}}{\tau^2} . \]  

(38)
where \( \nu = \frac{3}{2(1-\epsilon)} \) and \( \epsilon \equiv -\frac{\dot{H}}{H^2} \) is the slow roll parameter. Thus with the initial conditions \((32)\), the solutions of Eqs. \((32)\) and \((33)\) can be given as

\[
v_1(k, \tau) = \sqrt{2} e^{i(\nu+\frac{1}{2})H_\nu^{(1)}(-\tau)} e^{-4i\alpha_3 \tau} H_\nu^{(1)}(-2\tau) , \quad (39) \\
v_2(k, \tau) = \sqrt{2} e^{i(\nu+\frac{1}{2})H_\nu^{(1)}(-\tau)} e^{-4i\alpha_3 \tau} H_\nu^{(1)}(-2\tau) , \quad (40)
\]

where \( H_\nu^{(1)} \) is the \( \nu \)th Hankel function of the first kind. It is interesting to notice that the tensor perturbations obtain their asymptotic forms:

\[
v_1 \rightarrow e^{i(\nu+\frac{1}{2})H_\nu^{(1)}(-2\tau)} e^{-4i\alpha_3 \tau} \frac{\Gamma(\nu) (-2\tau)^{\nu-1}}{\Gamma(\nu + \frac{1}{2})} , \quad (41) \\
v_2 \rightarrow e^{i(\nu+\frac{1}{2})H_\nu^{(1)}(-2\tau)} e^{-4i\alpha_3 \tau} \frac{\Gamma(\nu) (-2\tau)^{\nu-1}}{\Gamma(\nu + \frac{1}{2})} , \quad (42)
\]

which are the solutions of primordial gravitational waves after adding noncommutative terms. These solutions are used in the next section.

III. WHAT CAN WE SEE FROM GWB TODAY?

A. Features of Primordial GWB

In this section we focus on the connections of theory and observations, and hence we need to know the detailed features of primordial gravitational waves. Since the noncommutative terms would modify the dispersion relation of gravitational waves, it is interesting to see what would be brought about in the propagations and spectra of primordial tensor perturbations.

For convenience we define

\[
u_1(k, \tau) \equiv \frac{v_1(k, \tau)}{a} , \quad \nu_2(k, \tau) \equiv \frac{v_2(k, \tau)}{a} .
\]

Combining the definition of the power spectrum, we have the expression:

\[
PT(k, \tau) = 32\pi G \frac{k^3}{2\pi^2} ((\nu_1(k, \tau))^2 + |\nu_2(k, \tau)|^2) . \quad (44)
\]

Moreover, the power spectra of tensor perturbations at different time can be connected by a parameter called transfer function. Therefore, we define the transfer function as follows:

\[
PT(k, \tau) = T(k, \tau) P_T(k, \tau) , \quad (45)
\]

where the index \((\tau)\) indicates the end of inflation. To substitute eqs. \((31)\) and \((32)\) into the definition of tensor power spectrum in Eq. \((14)\), we obtain the analytic solution of primordial power spectrum \( P_T(k) \):

\[
P_T(k) \equiv P_T(k, \tau_i) = 64\pi G \frac{2^{2\nu-3}}{4\pi^2 a_i^2} \frac{\Gamma^2(\nu)}{\Gamma^2(\nu+\frac{3}{2})} (1-\epsilon) a_i H_i^{2\nu} \times k^3 l^{-2\nu}(k) . \quad (46)
\]

Furthermore, there have been a number of papers to investigate the mass of today’s graviton, and they provide an upper limit around \( 6.395 \times 10^{-32} \) eV theoretically \[53\, 54, 52, 56\]. Taking into account this limit, we easily find that \( \alpha_3 \) has to be less than the frequency of \( 10^{-16} \) Hz, whose Compton wavelength is in equal order of the size of supercluster. Besides, to require that the slow-roll parameter \( \epsilon \) is close to zero, we then derive the form of primordial tensor power spectrum,

\[
P_T(k) \simeq \frac{128\pi}{3M_{pl}^3 V_{inf}} \frac{k^3}{(k^2 + 16\alpha_3^2)^2} . \quad (47)
\]

If the momentum \( k \) is much larger than \( \alpha_3 \), one can see that this primordial tensor power spectrum will return to the standard form \( \frac{128\pi}{3M_{pl}^3 V_{inf}} k^3 \) from the equation above. However, if \( k \) is tuned around the value of NP, the amplitude of the power spectrum starts to decrease. In this theory \( P_T \) would vanish when we tune \( k \) to be zero which is greatly different from the typical knowledge of GWB.

Eventually, to make the analysis more specific, we give a scenario of primordial tensor power spectrum in the inflation of nearly constant potential in Fig. 1. Comparing the one from the standard theory, our primordial

![FIG. 1: The black solid curve represents the primordial power spectrum of noncommutative tensor perturbations \( P_T(k, \tau) \) at the end time of inflation. The red dash curve represents the primordial power spectrum without noncommutative contributions while in the same inflation model. In this figure, we adopt the value of NP as \( \alpha_3 = 10^{-18} \), and the potential of inflation as \( V_{inf} \sim M^4 \) in which \( M = 5 \times 10^{15} \) Gev.](image-url)
Moreover, once the effective co-moving wave number is larger than $aH$, the perturbations begin to oscillate like plane wave. In the following, we will establish the relation to relate the power spectrum observed today to the primordial one. The method we used here is similar to [57].

In order to make clear every possible ingredient affecting the evolvement of the GWB, it is suitable and reasonable to decompose the transfer function into three parts as,

$$T(k, \tau) = F_1 F_2 F_3$$

$$= \left| \frac{\bar{u}_{1(2)}(k, \tau)}{u_{1(2)}(k, \tau)} \right|^2 |\frac{\tilde{u}_{1(2)}(k, \tau)}{u_{1(2)}(k, \tau)}|^2 |\frac{u_{1(2)}(k, \tau)}{\tilde{u}_{1(2)}(k, \tau)}|^2.$$  (48)

Here $u_{1(2)}(k, \tau)$ is the exact solution of Eq. [50]; $\tilde{u}_{1(2)}(k, \tau)$ is an approximate solution of Eq. [53] by neglecting the anisotropic stress tensor $\Sigma_{ij}$; and $\bar{u}_{1(2)}(k, \tau)$ is an even more crude solution which is equal to $u_{1(2)}(k, \tau)$ if $k < aH$ while equal to plane wave if $k > aH$. Note that those two polarizations are only different on their phases which does not affect the expression of the transfer function. Consequently, we only need to investigate one polarization in the following.

Firstly, from Eq. [30] one can see that after horizon re-entering, gravitational waves begin to oscillate with a decaying amplitude proportional to $a^{-1}(\tau)$. Therefore, from the definition of $u_1$ we get

$$\bar{u}_1(k, \tau) = \begin{cases} u_{1\max}(k) \cos[(l - \tau - \phi_1)], & l > aH \\ u_1(k, \tau), & l < aH \end{cases}$$  (49)

where $\phi_1$ depends on the initial condition, $u_{1\max}(k)$ is the maximum of the amplitude of oscillation, and $\tau$ is the conformal time when $l = aH$. Since we require this function to be continuous, there must be a matching relation that $u_1(k, \tau) = [u_{1\max}(k) \cos \phi_1]/a(\tau)$. Based on these relations one can get the first factor $F_1$ as follows

$$F_1 = \left(1 + \frac{z(\tau)}{1 + z_1}\right)^2 \cos^2[l(\tau - \tau) + \phi_1]/\cos^2 \phi_1, $$  (50)

where we introduce the redshift $1 + z = a_0/a(\tau)$ in aim of showing the effects of the suppression as a result of redshift. The index “0” indicates today, and $z_1$ is the redshift when the modes re-entered the horizon $l = aH$. Note that the relation of $z_1$ and $k$ can be given by the following equation

$$\left(\frac{k}{k_0}\right)^2 = \sum_i \Omega_i^{(0)} (1 + z_1) \exp \left[3 \int_0^{z_1} \frac{\Omega_i(z)}{1 + z} d\tilde{z} \right], $$  (51)

where the sum over $i$ includes all components in the universe. Since the contributions from dark energy and the fluctuations in radiation are very small, here we ignore...
them and then solve out

\[ 1 + z_l = \frac{1 + z_{eq}}{2} \left[ -1 + \sqrt{1 + \frac{4(l/k_0)^2}{(1 + z_{eq})\Omega_m^{(0)}}} \right], \quad (52) \]

where \( z_{eq} \equiv -1 + \Omega_m^{(0)}/\Omega_r^{(0)} \). The factor \( F_1 \) describes the redshift-suppressing effect on the primordial gravitational waves. Since this factor shows strongly oscillating behaviour which is inconspicuous to be observed in the GWB, we usually average the term \( \cos^2[l(\tau - \tau_1) + \phi_l] \) and instead it with \( \frac{1}{2} \).

Secondly, when considering the influence of the background state of universe on the re-entry of horizon, we focus on analyzing the factor \( F_2 \). Since the background equation of state \( w \) varies very slowly, it is profitable to assume that the evolution of the scale factor is of form \( a = a_0 \left( \frac{\tau}{\tau_0} \right)^\alpha \) with \( \alpha = \frac{2}{1 + 3w} \). Then solving Eq. (50) again and ignoring \( \Sigma_{ij} \), we have \( \tilde{u}_1(k, \tau) = u_1(k, \tau) \Gamma(\alpha + \frac{1}{2}) \left( -\frac{l}{k_0} \right)^{2-\alpha} J_{\alpha-4}(-l\tau) \), where \( \Gamma \) is the Gamma function and \( J_\nu \) is the \( \nu \)th Bessel function. If \(|l\tau| \gg 1\), there is such a relation that \( |\tilde{u}_1(k, \tau)|^2 = \frac{l^{2(\alpha+\frac{1}{2})}}{\pi} \left( -\frac{l}{k_0} \right)^{2-2\alpha} \cos^2(l\tau + \frac{\pi}{4}) \). To match with Eq. (50), considering that the phase should be continuous, hence we have the solution that when GWB re-enters the horizon the conformal time \( \tau_1 = -\frac{\pi}{4} \). Therefore, the second factor \( F_2 \) is given by

\[ F_2 = \frac{\Gamma^2(\alpha + \frac{1}{2})}{\pi} \left( \frac{2}{\alpha} \right)^{2\alpha} \cos^2 \phi_l. \quad (53) \]

The second factor shows that, when the gravitational waves re-enter the horizon, there is a "wall" lying on the horizon which affects the tensor power spectrum.

Thirdly, during the evolution of tensor perturbations, the nonzero anisotropic stress tensor \( \Sigma_{ij} \) would more or less bring some effects on the GWB. This effect is proposed by Steven Weinberg[47], and usually the primary ingredients are the freely streaming neutrinos which damp the amplitude of the tensor power spectrum. However, this damping effect just make \( P_T \) times a constant but do not change the behaviour of the GWB’s evolving. In our paper, we adopt \( F_3 = 0.80313 \).

Ultimately, we have discussed three kinds of leading corrections in the transfer function which make contributions in the evolution of the GWB. Using this transfer function, we are able to connect the primordial gravitational waves with what we observe today.

C. Analysis of Today’s GWB

In this section, we investigate the power spectrum of the current GWB and the corresponding energy spectrum. To substitute eqs. (50), (53), and the damping factor \( F_3 \) into (49), we can give today’s tensor power spectrum as follows,

\[ P_T(k, \tau_0) = 64\pi G_0^2 \frac{0.80313}{(1 + z_l)^2} \frac{\Gamma^2(\alpha + \frac{1}{2})}{2\pi} \left( \frac{2}{\alpha} \right)^{2\alpha} \]

\[ \times \frac{2^{2\nu-3}}{4\pi^2 a_i^3} \frac{\Gamma^2(\nu)}{\Gamma^2(\frac{3}{2})} \left[ (1 - \varepsilon) a_i H_i \right]^{2\nu-1} \frac{k^3}{2\nu}, \quad (54) \]

and from Eq. (14), the present energy spectrum is given by

\[ \Omega_{GW}(k, \tau_0) = \frac{1}{120 \langle t_{inf} \rangle^2} P_T(k, \tau_0). \]

When the frequency of GWB is large enough, one can see that today’s noncommutative tensor perturbation power spectrum would agree with the standard theory very well. Consequently, there will be very few signals in the high-frequency range. However, if the frequency reaches the value near NP where the corresponding wave length is around the size of the current horizon, the noncommutative term begins to affect the behaviour of the GWB. Again we require the slow-roll parameter \( \varepsilon \) to tend forwards to zero, and assume the potential of inflation to be nearly constant of which the scale is \( 5 \times 10^{15} \text{Gev} \). Then we give the semi-analytical form of the present energy spectrum of the noncommutative tensor perturbations as follows

\[ \Omega_{GW}(k, \tau_0) h^2 = 2ek^5 l^{-3} \left( -1 + \sqrt{1 + f l^2} \right)^{-2}, \quad (55) \]

where \( e = 2.68563 \times 10^{14} \) and \( f = 3.10475 \times 10^{32} \). In order to make a comparison, we give the corresponding energy
of the primordial tensor perturbations in the noncommu-

select approaches 0, the commutative one is recovered. In fact we

of NP. This is consistent with the case that when

noncommutativity in the GWB with very minor values

That is to say, it is most possible to detect the features of

groups of NP to see the differences among them. We

periments(see CMB Pol [58]). In Fig. 4 we choose three

be in the detecting range of next generation of CMB ex-

FIG. 3 and see that if tuning NP felicitously the peak can

gin beginning. To be more specifically, we show this feature in

spectrum without noncommutative term $\Omega_{GW}^{\text{normal}} h^2 = 2\epsilon k^2 (-1 + \sqrt{1 + f k^2})^{-2}$. Note that, when the frequency

of GWB is near NP, the term after the parameter \(f\) determines the behaviour of the energy spectrum and the term in the brackets (...) of Eq. (55) will never vanish even \(k\) approaches zero. Due to that, the energy spec-

trum of noncommutative GWB form a peak in low fre-

quency and then decay rapidly as mentioned in the begin-

ning. To be more specifically, we show this feature in

Fig. 3 and see that if tuning NP felicitously the peak can be in the detecting range of next generation of CMB ex-

periments(see CMB Pol [58]). In Fig. 4 we choose three groups of NP to see the differences among them. We

find that, smaller the NP is, more manifest the peak is. That is to say, it is most possible to detect the features of noncommutativity in the GWB with very minor values of NP. This is consistent with the case that when $\alpha_3$ ap-

proaches 0, the commutative one is recovered. In fact we select $\alpha_3 = 10^{-19}$ in Fig. 3 for the same consideration.

IV. CONCLUSIONS AND DISCUSSIONS

As a conclusion, we have investigated the key features of the primordial tensor perturbations in the noncommu-

tative field approach, and discussed the possibility of detecting the corresponding GWB in the experiments. Due to the noncommutative effects, the dispersion relation for the primordial gravitational waves is modified and the solution of the the tensor perturbations is different from the commutative case. Therefore, it brings about a lot of exciting phenomena which is brand-new and valuable for us to investigate. To study these new features, we investigated the transfer function to obtain the spectrum of noncommutative gravitational waves that we are ob-

serving today. Since the noncommutative term would bring CPT violation and produce effective mass for the graviton, it is reasonable to require this term to be small enough. In our note, this has already been discussed that it is allowed to set the values of NP lower than $10^{-16}$ due to the requirements of both experiments and theories. From the calculations in this note, one can see that one most intriguing effect of noncommutative GWB is that it would generate a peak on its energy spectrum where the frequency may be lower than $10^{-16}$. As a result, on one hand, this phenomenon provides a much more stronger limit on the graviton mass, since we can check the po-

sition of this possible peak in the energy spectrum; on the other hand, we expect that the signals of noncom-

mutativity can be found in the next generation of CMB observations if the noncommutativity in the relic tensor perturbations is hidden in the range near current hori-

zont. Eventually, the noncommutativity of gravitational waves definitely go beyond the knowledge of Einstein's gravity and therefore should be an important subject for us to investigate.

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