Metastable states of a ferromagnet on random thin graphs

A. Lefevre and D. S. Dean

IRSAMC, Laboratoire de Physique Quantique, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4.

Abstract:
We calculate the mean number of metastable states of an Ising ferromagnet on random thin graphs of fixed connectivity $c$. We find, as for mean field spin glasses that this mean increases exponentially with the number of sites, and is the same as that calculated for the $\pm J$ spin glass on the same graphs. An annealed calculation of the number $\langle N_{MS}(E) \rangle$ of metastable states of energy $E$ is carried out. For small $c$, an analytic result is obtained. The result is compared with the one obtained for spin glasses in order to discuss the role played by loops on thin graphs and hence the effect of real frustration on the distribution of metastable states.

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1 Introduction

The nature of the spin glass phase is still a subject which is widely debated. Two possible scenarios have been proposed, one inspired from mean field models which shows that in the spin glass phase one has an extensive number of pure states, this phenomenon appears as replica symmetry breaking (RSB) in mean field models [1, 2]. The other scenario is the droplet picture where there is a non extensive number of pure states as in a ferromagnet [3, 4]. If the RSB image is correct then an extensive number of pure states must also show up in the number of metastable states in a system, thought the relevance of metastable states or inherent states, as they are referred to in the literature on glassy systems, to pure states is not obvious or even justified [5, 6].

There has been a considerable amount of effort to analyze the metastable states in mean field model [7, 8, 9, 10, 11, 12] and also the number of solutions of the TAP mean field equations for this model (the generalization of metastable states to finite temperature) [13, 14]. Calculations on the Sherrington Kirkpatrick (SK) totally connected spin glass demonstrate the existence of an exponentially large (in terms of the number of spins \( N \)) number of metastable states and the continuing existence of a macroscopic entropy of metastable states even at arbitrarily high values of a uniform magnetic field (in agreement with the divergence of the Almeida Thouless line at zero temperature) [15]. This latter fact is clearly a pathology of the totally connected geometry of the SK model.

In the SK model each spin is connected to all the other spins and the existence of the thermodynamic limit is ensured by scaling the couplings by a factor \( 1/\sqrt{N} \) in the case of symmetric distributions. This scaling of the interaction strength with the system size is clearly undesirable when one wishes to make a connection with the finite dimensional analogue. Corrections to order \( 1/c \), where \( c \) is the lattice connectivity, about mean field theory [7] seem to suggest an enhancement of the number of metastable states when the dimension is reduced. Analytic studies of finite dimensional spin glasses are extremely difficult given that the complexity of the starting point of any perturbative analysis, that is to say the Parisi replica symmetry breaking scheme.

Recently there has been renewed interest in spin glasses on random graphs of finite connectivity [16, 17, 18, 19, 20], the advantage with such systems is that while a mean field analysis is still possible, these systems mimic the finite connectivity of real finite dimensional spin glasses. It has been shown that the replica symmetric solution in such systems is not stable [21]. Unfortunately no exact treatment of the RSB solution has been achieved, there are however approximate treatments which yield promising results [17, 22]. Additionally one may carry out a perturbative replica symmetry breaking in some cases, such as close to the critical temperature or in the limit of large connectivity [23, 24]. Interestingly it can be shown that the replica symmetric solution on thin graphs (random graphs where each site has a fixed connectivity \( c \)) is equivalent to the solution for a spin glass on a Cayley tree with branching ratio \( c - 1 \) (see [17] and references therein), that is to say the graph one would obtain roughly if one eliminated all the loops present in the corresponding random thin graph. It is well known that the fraction of loops in such graphs goes as \( \ln(N) \) where \( N \) is the number of sites. Therefore
one can see, that despite the scarcity of such loops, their effect is extremely important and that they make a replica symmetric system become RSB. Of course it is only through loops that one can have real frustration [25], without loops one may construct local gauge transformations that make the system equivalent to a ferromagnetic one. In a recent paper [26], the number average metastable states of $\pm J$ (where each bond is taken to be $\pm J$ with probability $1/2$) spin glasses on random thin graphs has been calculated. At zero temperature the number of metastable states is defined to be the number of spin configurations stable to single spin flips. It was shown that this number decreases as the connectivity is increased and in the limit $c \to \infty$ the result for the Sherrington Kirkpatrick mean field spin glass was recovered. In this paper we consider the problem of purely ferromagnetic systems on such graphs. Here there is clearly no real frustration even with loops. We find that the average total number of such metastable states on the ferromagnet is equal to average total number on the corresponding $\pm J$ spin glass. However one finds that when one calculates the average number of metastable states of fixed energy $E$, $N_{MS}(E)$ there exists a critical energy $E^*$ such that $\langle N_{MS}^{SG}(E) \rangle = \langle N_{MS}^{F}(E) \rangle$ for $E \geq E^*$. (here the superscripts $F$ and $SG$ denote ferromagnet and spin glass respectively and $\langle \cdot \rangle$ denotes the disorder average) but $\langle N_{MS}^{SG}(E) \rangle < \langle N_{MS}^{F}(E) \rangle$ for $E < E^*$. Hence the rather surprising result that at lower energies the ferromagnet has more metastable states than the spin glass. We show that this difference is due to the effect of loops and moreover that $E^*$ is the energy at which the metastable states of the ferromagnetic system acquire a non zero magnetisation. In addition we show that for $E > E^*$, $\ln \left( \langle N_{MS}^{SG}(E) \rangle \right) / N = \ln \left( \langle N_{MS}^{F}(E) \rangle \right) / N$ is a concave function of $E$ whilst for $E < E^*$, $\ln \left( \langle N_{MS}^{SG}(E) \rangle \right) / N$ remains concave but $\ln \left( \langle N_{MS}^{F}(E) \rangle \right) / N$ becomes convex. Thus suggesting that the concavity of $\ln \left( \langle N_{MS}(E) \rangle \right) / N$ at low energies and and hence temperature, may be an indication of replica symmetry breaking.

2 Analysis

The model we shall consider has the Hamiltonian

$$H = -\frac{1}{2} \sum_{j \neq i} J_{ij} n_{ij} S_i S_j$$

(1)

where the $S_i$ are Ising spins, $n_{ij}$ is equal to one if the sites $i$ and $j$ are connected. In the spin glass case considered in [10] the $J_{ij}$ are taken from a binary distribution where $J_{ij} = -1$ with probability half and $J_{ij} = 1$ with probability half. In the ferromagnetic case we consider here one has $J_{ij} = 1$. A metastable state is defined to be a configuration where if one changes the sign of any given spin the energy does not decrease, for the purposes of this paper we shall include the marginal, case where the energy does not change, as being metastable. With this definition number of metastable states is given
The fact that we include the marginal case implies that here $\theta(x)$ the Heaviside step function is taken such that $\theta(0) = 1$. In the spin glass case one may exploit the parity of the distribution of the $J_{ij}$ by making a gauge transformation $J_{ij} \rightarrow J_{ij}S_i S_j$ to obtain

$$\langle N_{MS} \rangle = 2^N \langle \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} J_{ij} n_{ij} \right) \rangle \quad (3)$$

However this is not possible in the ferromagnetic case, and thus renders the ferromagnetic problem more difficult than the spin glass.

We shall use the method of construction of the thin graphs used in [26]. Another method to generate these graphs by considering planar Feynman diagrams was used in [20, 18, 19]. The random graphs are constructed as follows: any two points are connected with probability $p/N$. Hence $n_{ij}$ is equal to one with probability $p/N$ and zero with probability $1 - p/N$. Here $p$ is some arbitrary number of order one and we shall see that the results one obtains are independent of the choice of $p$. If we denote the average on a random graph (with a specified value of $p$) by $\langle \cdot \rangle_p$ then the induced average over the subset of thin graphs of connectivity $c$ is given by

$$\langle F \rangle = \frac{\langle \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} J_{ij} n_{ij} \right) \rangle_p}{M(N, c, p)}$$

where

$$M(N, c, p) = \langle \prod_{i=1}^{N} \delta_{\sum_{j \neq i} n_{ij}, c} \rangle_p$$

is the average number of thin graphs of connectivity $c$ generated by the random graph ensemble for a given $p$. Here, as opposed to the spin glass case, this is the only disorder average. It was shown in [26] that

$$\ln(M(N, c, p)/N) = \frac{c}{2} \left( \ln c - \ln p - 1 \right) - \ln(c!) - \frac{p}{2}$$

With this averaging we therefore find that

$$\langle N_{MS}^F \rangle = \frac{D(N, c, p)}{M(N, c, p)}$$

where

$$D(N, c, p) = \langle \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} J_{ij} n_{ij} \right) \delta_{\sum_{i \neq j} n_{ij}, c} \rangle_p$$

by [14, 15, 16]
To compute $D(N, c, p)$ we introduce Fourier representations of the Heaviside and Kro-

nnecker delta functions to obtain:

$$
D(N, c, p) = \int d\lambda_i d\lambda_j d\lambda_k \mathcal{L} \sum_{i \neq j} n_{ij} \left( \lambda_i + \lambda_j + y_i S_i S_j \right)_{p}
$$

as $N$ goes to infinity. Here the integration ranges are $\lambda \in [0, 2\pi]$, $x \in [0, \infty]$ and $y \in [-\infty, \infty]$. Now, we use the useful identity:

$$
e^{-\lambda - \lambda' - \frac{y+y'}{2}} S S' = e^{-\lambda - \lambda'} \left( \cosh(\frac{y}{2}) \cosh(\frac{y'}{2}) + \sinh(\frac{y}{2}) \sinh(\frac{y'}{2}) \right)
$$

which allows us to write the term into brackets as

$$
\exp \left[ \cdots \right] = \exp \left[ -\frac{Np}{2} + \frac{p}{2N} \left( \sum_i e^{-\lambda_i} \cosh(\frac{y_i}{2}) \right)^2 + \frac{p}{2N} \left( \sum_i e^{-\lambda_i} \sinh(\frac{y_i}{2}) \right)^2 + \frac{p}{N} \left( \sum_i e^{-\lambda_i} S_i \cosh(\frac{y_i}{2}) \right) \left( \sum_j S_j e^{-\lambda_j} \sinh(\frac{y_j}{2}) \right) \right]
$$

One can now decouple the sums by introducing two real Hubbard-Stratonovich fields $u$ and $v$ and a complex field $z$ giving

$$
D(N, c, p) = \int dz \bar{\pi} du dv \left( \frac{\bar{\pi}^2 + u^2 + |z|^2}{-4\pi} \right)^N \mathcal{L} \sum_{i \neq j} n_{ij} \left( \lambda_i + \lambda_j + y_i S_i S_j \right)_{p}
$$

where the trace above is over a single spin. By using the following identity:

$$
\int \frac{d\lambda}{2i\pi} e^{\lambda c + \alpha e^{-\lambda}} = \frac{\alpha c}{c!}
$$

we get:

$$
\int \frac{d\lambda}{-4\pi^2} \mathcal{L} \sum_{i \neq j} n_{ij} \left( \lambda_i + \lambda_j + y_i S_i S_j \right)_{p}
$$
\[
= \frac{1}{2i\pi c!} \text{Tr}_S \int dx \, dy \left[ \sqrt{p} (u + Sz) \cosh(\frac{y}{2}) + (v + Sz) \sinh(\frac{y}{2}) \right]^c
\]
\[
= \frac{p^c}{2i\pi c!} \int dx \, dy \sum_{S=\pm 1} e^{iwx} \left[ \frac{e^{\frac{y}{2}}}{2} (u + Sz + v + Sz) + \frac{e^{-\frac{y}{2}}}{2} (u + Sz - v - Sz) \right]^c
\]
Now, introducing \( A = \frac{u + z + v + Sz}{2} \), \( B = \frac{u - z + v - Sz}{2} \) and \( C = \frac{u + z - v - Sz}{2} \), this term becomes
\[
\frac{p^c}{2c!} \left( C^c f\left( \frac{A}{C} \right) + \overline{C^c} f\left( \frac{B}{C} \right) \right),
\]
where:
\[
f(x) = \sum_{\frac{c}{4} \leq n \leq c} \left( \frac{c}{n} \right) x^n. \tag{10}
\]
On calculating the ratio \( D(N, c, p)/M(N, c, p) \) we find, as it should, that the dependence on \( p \) disappears and one obtains via a saddle point calculation in the large \( N \) limit
\[
\ln \left( \langle N_{MS}^F \rangle / N \right) = \max_{A, B, C} S^*(A, B, C) \tag{11}
\]
where
\[
S^*(A, B, C) = -\frac{A^2 + B^2}{2} - C\overline{C} + \ln \left( C^c f\left( \frac{A}{C} \right) + \overline{C^c} f\left( \frac{B}{C} \right) \right) - \frac{c}{2}(\ln(c) - 1) \tag{12}
\]
We again change variables: \( u = \frac{A}{C} \), \( v = \frac{B}{C} \) and \( t = \frac{C}{\overline{C}} \), solving the saddle point equation for \( C \) and substituting in this solution yields
\[
\ln \left( \langle N_{MS}^F \rangle / N \right) = \max_{u, v, t} S_F(u, v, t) \tag{13}
\]
where
\[
S_F(u, v, t) = -\frac{c}{2} \ln \left( u^2 + t^2 v^2 + 2t \right) + \ln \left( f(u) + t^c f(v) \right). \tag{14}
\]
We notice \( S_F(u, v, t) \) is invariant under the transformations \( u \to v, \ v \to u, \ t \to \frac{1}{t} \). There is consequently a saddle point solution at the fixed point of this transformation \( u = v \) with \( t = 1 \), this leads to exactly the saddle point obtained for spin glasses where
\[
S_{SG}(u) = -\frac{c}{2} \ln(1 + u^2) + \ln(f(u)) + (1 - \frac{c}{2}) \ln 2, \tag{15}
\]
In general one must solve the remaining saddle point equations numerically. For the case \( c = 1 \) (dimers) and \( c = 2 \), one dimensional chains the solution is identical to that for the spin glass case [23]. To continue we will focus on the \( c = 3 \) case, which can also be computed analytically. In this case, the stationarity conditions are:
\[
\begin{align*}
u + 2 &= t(v + 2) \\
t &= \Psi(v) \\
u^2 &= t^3v^2
\end{align*}
\]

where \(\Psi(v) = \frac{v + 2}{v^2}\). These equations imply the following:

\[
\begin{align*}
U &\equiv u + 2 \\
V &\equiv v + 2 \\
U &= \varphi(V) \\
V &= \varphi(U)
\end{align*}
\]

where \(\varphi(U) = \frac{U^2}{(U - 2)^2}\). This implies that \(U\) and \(V\) are solutions of \(U = \varphi \circ \varphi(U)\), which has two kind of solutions:

- \(U \neq V\);
- \(U = V = \varphi(U)\), which is the one found for the spin-glass.

The first solution gives for the action the value \(\frac{1}{2} \ln \frac{8}{7}\), whereas the spin-glass one is \(\frac{1}{2} \ln (\frac{2}{5})\). This shows that for \(c = 3\), the logarithm of the average number of metastable states for the Ising model on random thin graphs is the same as the one obtained for the \(\pm 1\) spin-glass on the same graphs with an annealed calculation. Carrying out a numerical investigation for \(c > 3\) we find that \(\langle N_{MS}^F \rangle = \langle N_{MS}^{SG} \rangle\). One may understand this result heuristically if one considers that \(\langle N_{MS}^F \rangle\) and \(\langle N_{MS}^{SG} \rangle\) are dominated by the metastable states at an energy level where the effect of loops is not important, then one may write

\[
\langle N_{MS}^F \rangle = \langle \text{Tr}_{S_i} \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} n_{ij} S_i S_j \right) \rangle \tag{16}
\]

and

\[
\langle N_{MS}^{SG} \rangle = \langle \text{Tr}_{J_{ij}} \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} n_{ij} J_{ij} \right) \rangle \tag{17}
\]

where \(\text{Tr}_{J_{ij}}\) indicates a trace over independent dynamical variables \(J_{ij}\) taking the values \(\pm 1\) on the bonds of the graph and the \(J_{ij}\). In the ferromagnetic case one may take the variables \(S_i S_j\) to be independent variables taking the values \(\pm 1\) if one neglects the effects of loops which would introduce correlations between these bond variables. Hence one expects that \(\langle N_{MS}^F \rangle = \langle N_{MS}^{SG} \rangle\) if loops are not important at the energy level where the metastable states are concentrated. In the case \(c = 1\) and \(c = 2\) it is clear
that loops cannot play a thermodynamically important role. One can make nonlocal
gauge transformations that in fact demonstrate that $\ln(N_{\text{MS}}^{SG})/N = \ln(N_{\text{MS}}^{F})/N$ with
probability 1 – in the thermodynamic limit the two models are equivalent up to a gauge
transformation. We confirm this picture in the next section.

3 Metastable states of fixed energy

Here we define the average number of metastable states of fixed energy $NE$

$$N_{\text{MS}}(E) = \text{Tr} \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} J_{ij} n_{ij} S_{i} S_{j} \right) \delta(H - NE).$$

To achieve this, we need to introduce a Lagrange multiplier $\alpha$ to fix the energy and
carry out a calculation almost identical to that of the previous section.

We find

$$\frac{\ln \left( \langle N_{\text{MS}}^{F}(E) \rangle \right)}{N} = \max_{u,v,t,\alpha} S(u,v,t,\alpha; E)$$

(19)

where

$$S(u,v,t,\alpha; E) = -\frac{c}{2} \ln(u^{2} + t^{2}v^{2} + 2t) + \ln \left( f(ue^{-\alpha}) + t f(ve^{-\alpha}) \right) + \alpha \left( \frac{c}{2} - E \right)$$

At the saddle-point, the energy is:

$$E = \frac{c}{2} \frac{2t - u^{2} - t^{2}v^{2}}{2t + u^{2} + t^{2}v^{2}}.$$  

We call $x = \frac{2E}{c}$, so we have the relation:

$$\frac{u^{2} + t^{2}v^{2}}{2t} = \frac{1 - x}{1 + x}.$$  

(20)

For $c = 2$, the result is the same as for the spin-glass as expected. Let us show briefly
how to recover this with a transfer matrix method on the one dimensional model.
First, one makes the gauge transformation $S_{i} S_{i+1} \rightarrow S_{i}$. One introduces a Lagrange
multiplier $\alpha$ to fix the energy and we get:

$$N_{\text{MS}}(E) = \max_{\alpha} \text{Tr} \mathcal{M}_{\alpha}^{N},$$

where

$$\mathcal{M}_{\alpha}(S,S') = e^{\alpha E + \frac{s_{i} s'}{2}} \theta(S + S').$$
So \( \frac{\ln(N_{MS}(E))}{N} = \max_\alpha \ln(\mu_\alpha) \), where \( \mu_\alpha \) is the largest eigenvalue of \( M_\alpha \). We find:

\[
\mu_\alpha = \frac{e^\alpha + \sqrt{e^{2\alpha} + 4}}{2}
\]

So at the maximum: \( \alpha^* = \ln \frac{-2E}{\sqrt{1-E^2}} \) and we recover the result of \cite{26} for the \( c = 2 \) spin glass \( \pm J \) on a thin graph

\[
\frac{\ln(N_{MS}(E))}{N} = \frac{1 - E}{2} \ln(\frac{1 - E}{2}) - \frac{1 + E}{2} \ln(\frac{1 + E}{2}) + E \ln(-2E),
\]

confirming the above assertion. For generic values of the local connectivity, the saddle-point equations can be solved numerically. In fig.\( (1) \), we have plotted the result for \( c = 4 \). The curve corresponding to \( c = 3 \) has also been calculated numerically and agrees perfectly with the following calculation. Let us focus on the \( c = 3 \) case. For convenience, we introduce new variables:

\[
U = ue^{-\alpha} + 2 \quad (21)
V = ve^{-\alpha} + 2 \quad (22)
\]
\[
a = \frac{e^{-2\alpha}}{2} \quad (23)
\]

In this case, the stationarity conditions lead to:

\[
U = t V \quad (24)
\]
\[
t = \frac{ve^{-\alpha} + 2}{v^2} = \frac{u^2}{ue^{-\alpha} + 2}
\]

and the function \( \phi \) is now \( \frac{U^2 e^{-2\alpha}}{(U - 2)^2} \). The equation \( \phi \circ \phi(U) = U \) is of degree four and can be factorised by the second degree equation \( \phi(U) = U \). There are two solutions with \( U \neq V \) obeying

\[
\frac{U + V}{2} = \frac{2 - a}{(1 - a)^2} \quad (25)
\]

\[
UV = \frac{4}{(1 - a)^2}
\]

Moreover, by using (24) in (24) one obtains

\[
\frac{u^2 + t^2 v^2}{2t} = \frac{U + V}{2} = \frac{1 - x}{1 + x}
\]

yielding two different values for \( a \):
\[ a_\pm = \frac{1 - 3x \pm \sqrt{(1 + x)(5 - 3x)}}{2(1 - x)}, \]
yielding two possible solutions \( \alpha_+ \) and \( \alpha_- \) from equation (23). In fig. (2), we have plotted
the value of the action obtained by solving the remaining stationary conditions for the
two different values of \( \alpha \) and the annealed calculation for the \( \pm 1 \) spin-glass (there is
only one real solution for \( U = V \)). The curve corresponding to \( \alpha_- \) gives defined values
for \( x \) between -1 and \(-\frac{1}{3}\), and reaches a maximum at \( x^* = -\frac{5}{7} \) which value is \( \frac{1}{2} \ln \frac{8}{7} \)
and corresponds to the value obtained for \( U \neq V \) in the previous calculation of the
total complexity. This solution however always has an action of lower value than that
coming from the spin glass solution. The solution coming from \( \alpha_+ \) is more pathological.
Above the value \( x^* = -\frac{5}{7} \) the solution corresponding to \( \alpha_+ \) does not exist (this value
of \( x^* \) is the value over which \( U \) and \( V \) obtained from \( \alpha_+ \) become imaginary and so the
corresponding value of the action \( S \) not real). The value of \( E \) corresponding to this \( x^* \) is
shown by the vertical dotted line on fig. (2). For \( x < x^* \) the action corresponding to the
solution with \( \alpha_+ \) is greater than that coming from the spin glass saddle point and hence
dominates in the thermodynamic limit. One should also note that this action becomes
equal to zero at \( E = -\frac{3}{2} \) the ground state for the ferromagnet, as it should. Hence
we see that the energy level with the largest number of metastable states, and thus
dominating the average total number, occurs at any energy higher than that where the
difference between the spin glass and ferromagnetic calculations yields different results
and above this energy level the effect of loops is negligible. However, below \( x^* \) the
number of metastable states is larger in the ferromagnet than in the spin glass. In
addition we shall see that \( E^* \) is the energy below which the metastable states acquire
a non-zero global magnetisation.

We now continue the computation of the number of metastable states of fixed energy
\( E \) but with fixed magnetisation \( m = \frac{1}{N} \sum S_i \). The average number of metastable states
of energy \( E \) and magnetisation \( M \) is then given by

\[
N_{MS}(E, m) = \text{Tr} \prod_{i=1}^{N} \theta \left( \sum_{j \neq i} J_{ij} n_{ij} S_i S_j \right) \delta(H - NE) \delta(Nm - \sum_i S_i)
\]

One introduces another Lagrange multiplier \( h \) to fix with \( m \). The resulting action is now

\[
S^*_F(u, v, t, h, \alpha; E, m) = -\frac{c}{2} \ln(u^2 + t^2v^2 + 2t) + \ln(f(ve^{-\alpha}) e^h + t e^{-h})
+ \alpha \left( \frac{c}{2} - E \right) - mh.
\]

\[
\frac{\ln (N_{MS}(E, m))}{N} = \max_{u,v,t,h,\alpha} S^*_F(u, v, t, h, \alpha; E, m)
\]
The stationarity condition with respect to $h$ gives

$$m = \frac{f(ue^{-\alpha}) e^h - t^c f(ve^{-\alpha}) e^{-h}}{f(ue^{-\alpha}) e^h + t^c f(ve^{-\alpha}) e^{-h}}$$

(26)

substituting this values for $h$ in the action yields the reduced action

$$S^*_F(u, v, t, \alpha; E, m) = -\frac{c}{2} \ln (u^2 + t^2 v^2 + 2t) + \frac{1 + m}{2} \ln (f(ue^{-\alpha})) + \frac{1 - m}{2} \ln (f(ve^{-\alpha}))$$

$$+ \frac{1 - m}{2} \ln t + \alpha \left( \frac{c}{2} - E \right) - \frac{1 + m}{2} \ln \left( \frac{1 + m}{2} \right) - \frac{1 - m}{2} \ln \left( \frac{1 - m}{2} \right).$$

The remaining stationarity conditions are

$$U = \frac{2(m + 1)}{1 + x} - 1$$

$$V = \frac{2(1 - m)}{1 + x} - 1$$

$$\alpha = \frac{1}{4} \ln \left( \frac{UV}{(U - 2)^2 (V - 2)^2} \right)$$

$$t = \frac{u^2}{U} = \frac{V}{v^2}$$

For a fixed energy greater than the ground state, the number of metastable states must go to zero before $m^2 = 1$. Indeed, when $m + 1 = \frac{3}{2}(1 + x)$ and $1 - m = \frac{3}{2}(1 + x)$, $u$ and $v$ are respectively zero, $S^*_F$ exhibits a singularity hence values of $|m| > \frac{1 + \sqrt{2}}{2}$ are excluded. Now, if we fix the energy and plot the number of metastable states for $m$ going from $-1$ to $1$, we get two kinds of configurations:

- if $x \geq x^*$, then the maximum is at $m = 0$, that is the magnetisation is zero;
- if $x \leq x^*$, then the point $m = 0$ is a local minimum, and there are two local maxima of opposite non zero magnetisations.

These results are demonstrated in the different regimes in fig. (3). The stationarity condition on $m$ leads to:

$$m = \frac{f(ue^{-\alpha}) - t^c f(ve^{-\alpha})}{f(ue^{-\alpha}) + t^c f(ve^{-\alpha})},$$

which leads to a second order transition in the value of $m$ at $x^*$.

One finds therefore, by comparison at the same energy $E$ with the spin glass, that the possibility of a non zero magnetisation paradoxically increases the metastability of the system.
4 Conclusion

In conclusion, we have seen that the mean number of metastable states for the Ising ferromagnet on thin graphs increases exponentially with the size of the system. Moreover, for the total average number, the result is the same as the one obtained for the corresponding $\pm J$ spin glass. The complexity does change in the low energy phase where it becomes convex in the case of the ferromagnet. This shows that at high energy, the metastability is mainly due to the local geometry of the graphs, and the relevance of loops seems not to be significant. At low energy, the presence of some non-zero magnetisation for $c > 2$ seems to be responsible of a complexity bigger than the one computed (in an annealed calculation) for the spin glass.

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Figure 1: Numerical calculation of $S_F(E) = \ln(\langle N_{MS}(E) \rangle)/N$ of the number of metastable states of fixed energy for $c = 4$ (a). The dashed line (b) is the corresponding solution for the spin glass.
\[ S_F(E) = \ln \left( \langle N_{MS}(E) \rangle \right) / N \text{ for } c = 3 \text{ shown by the solid curve (d). Also plotted is the solution corresponding to } \alpha_- \text{ (a), the corresponding spin glass solution in the low energy region (b). The vertical dotted line (c) represents the crossover point where the spin glass and ferromagnetic solutions start to differ.} \]
Figure 3: $S_F(E, m) = \ln \left( \langle N_{MS}(E, m) \rangle \right) / N$ for $c = 3$ as a function of the magnetisation $m$ for different values of the energy: $-0.765 > E^* (a), -1.071 = E^* (b), -1.245 < E^* (c)$. The arrows indicate the local maximum which gives the dominant contribution to $S_F(E)$ at fixed energy. At $E \approx E^*$, the maximum corresponding to the spin glass solution splits into two maxima and the $Z_2$ symmetry is spontaneously broken.