Hypercontractive inequalities for weighted Bergman spaces

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1 | INTRODUCTION

In his paper on hypercontractive inequalities for multipliers on orthogonal polynomials ([7]), Janson posed a question on the hypercontractivity of the operator $P_r f(z) := f(rz)$ on spaces of analytic functions. Weissler ([16]) earlier proved the result for Hardy space, while Janson proved it for Fock spaces (using the estimates from [13]) and Bergman space with very specific weight. We will further investigate these estimates for the weighted Bergman spaces:

$$\left( \int_{\mathbb{D}} |f(rz)|^q dA_\alpha (z) \right)^{1/q} \leq \left( \int_{\mathbb{D}} |f(z)|^p dA_\alpha (z) \right)^{1/p}, \tag{1}$$

where $dA_\alpha (z) = \frac{\pi^{\frac{\alpha-1}{2}}}{\alpha-1} (1 - |z|^2)^{\alpha-2} dA(z)$ and $0 < p < q < +\infty$ and $\alpha > 1$. Here $\mathbb{D}$ stands for the unit disk. It can be easily seen, as in the setting of Hardy spaces, that $r$ cannot be greater than $\sqrt{\frac{p}{q}}$. This value of $r$ comes from considering the family $f_\varepsilon(z) = 1 + \varepsilon z$, for small $\varepsilon$'s.

Weissler and Janson, in fact, proved (1) for $\alpha = 1$ (Hardy space) and $\alpha = \frac{3}{2}$. In [1], using Beckner’s estimates for Poisson semigroup on the unit sphere in $\mathbb{R}^n$ ([2]), Bayart, Brevig, Haimi, Ortega-Cerda, and Perfekt proved these inequalities for $\alpha = \frac{n+2}{2}$, with $n \in \mathbb{N}$ and any
In this note, we will prove those estimates for \( q \geq 2 \) and all \( \alpha > 1 \) and give some partial results for \( q < 2 \). More precisely, we prove the following theorem.

**Theorem 1.** Let 0 < \( p < q < + \infty \) and \( \alpha > 1 \). For \( f \in A^p_\alpha \), that is, an analytic function such that the right-hand side of the inequality (1) is finite, we have:

- For \( q \geq 2 \) — the estimate (1) holds with \( r = \sqrt{\frac{p}{q}} \).
- For \( q < 2 \) — (1) holds with \( r^2 = \max\{\frac{p}{2}, \frac{p\alpha-p}{q\alpha-p}\} \) for an arbitrary holomorphic function, while for the zero-free function, this estimate holds with \( r = \sqrt{\frac{p}{q}} \).

By Janson’s paper and very well-known argument on logarithmic Sobolev inequalities from [6], we have the following.

**Corollary 1.** There holds the logarithmic Sobolev inequality

\[
\int_D |f(z)|^p \log |f(z)| dA_\alpha(z) \leq \frac{1}{2} \Re \int_D |f(z)|^{p-2} f(z) f'(z) dA_\alpha(z)
+ \frac{1}{p} \int_D |f(z)|^p dA_\alpha(z) \log \left( \int_D |f(z)|^p dA_\alpha(z) \right),
\]

for \( p \geq 2 \) and holomorphic function \( f \), where \( dA_\alpha(z) = \frac{\alpha-1}{\pi} (1 - |z|^2)^{\alpha-2} dA(z) \).

Moreover, by [6], this inequality is equivalent to (1).

## 2 THE METHOD OF A PROOF AND THE MAIN RESULT

The main tools in our approach are recent inequalities due to Kulikov ([10]) and the lemma on convexity of certain integral means of analytic functions ([15]). Let us say that, in [10], Kulikov has proved a conjecture of Pavlović from [14] and [15] and Bayart, Brevig, Haimi, Ortega-Cerda, and Perfekt from [1], where several results that support this and some other conjectures are provided. Moreover, he proved a more general conjecture of Lieb and Solovej from [11]. Some interesting partial results are given in [4, 9] and [12]. See also [8] and [5] for some generalizations and extensions of these results. Now, we will formulate Kulikov’s theorem from [10] (in the appropriate form) and then the lemma from [15], including its proof for the sake of completeness.

**Theorem 2.** For 0 < \( p < q < + \infty \) and \( f \in A^p_\alpha(D) \), there holds the inequality

\[
\left( \int_D |f(z)|^q dA_p(z) \right)^{\frac{1}{q}} \leq \left( \int_D |f(z)|^p dA_\alpha(z) \right)^{\frac{1}{p}},
\]

where \( dA_\alpha \) is defined as before and \( \frac{\alpha}{p} = \frac{\beta}{q} \). The inequality is sharp and the extremizers for the fixed \( p, q, \alpha, \beta \) are the functions given by \( f(z) = C(1 - az)^{-\frac{2\alpha}{p}} \) with \( C \in \mathbb{C}, a \in \mathbb{D} \).
Lemma 1. For $f$ analytic in the unit disk $\mathbb{D}$ and $p \geq 2$, the function

$$\Phi_p(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(\sqrt{re^{i\theta}})|^p \, d\theta$$

is convex on $r$. The same function is convex on $r$ for $0 < p < 2$ for zero-free function $f$.

Proof. We start from the formula

$$\Phi_p(r) = |f(0)|^p + \frac{p^2}{2\pi} \int_0^{\sqrt{r}} \frac{1}{\rho} \left(\int_{|z|<\rho} |f(z)|^{p-2} |f'(z)|^2 \, dA(z)\right) d\rho,$$

from whom we easily find

$$\Phi'_p(r) = \frac{p^2}{4\pi r} \int_{|z|<\sqrt{r}} |f(z)|^{p-2} |f'(z)|^2 \, dA(z)$$

and

$$\Phi''_p(r) = \frac{p^2}{4\pi r^2} \left(\frac{r}{2} \int_0^{2\pi} |f(\sqrt{re^{i\theta}})|^{p-2} |f'((\sqrt{re^{i\theta}})|^2 \, d\theta \right.$$

$$\left. - \int_0^{\sqrt{r}} \rho \left(\int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2} |f'(\rho e^{i\theta})|^2 \, d\theta\right) d\rho\right).$$

If we denote $F(\rho) = \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2} |f'(\rho e^{i\theta})|^2 \, d\theta$, then from its subharmonicity, we would have

$$\int_0^{\sqrt{r}} \rho F(\rho) \, d\rho \leq \int_0^{\sqrt{r}} \rho F(\rho) \, d\rho = \frac{r}{2} F(\sqrt{r})$$

and, by the last formula, we can conclude $\Phi''_p(r) \geq 0$. However, we easily find

$$\Delta(|f(z)|^{p-2} |f'(z)|^2)$$

$$= |f(z)|^{p-4} \left((p-2)^2 |f'(z)|^4 + 4 |f(z)|^2 |f''(z)|^2 \right) - 4(p-2) \text{Re} \left( f(z) f''(z) \bar{f}(z) \right) \right)$$

$$= |f(z)|^{p-4} (p-2)^2 |f'(z)|^2 - 2f(z)f''(z) \geq 0,$$

and the conclusion follows. 

Now we prove our Theorem 1.

Proof. First, we estimate the right-hand side from below using Kulikov’s inequality (2):

$$\left(\frac{\beta - 1}{\pi} \int_{\mathbb{D}} |f(z)|^q (1-|z|^2)^{\beta-2} \, dA(z)\right)^{\frac{1}{q}} \leq \left(\frac{\alpha - 1}{\pi} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{\alpha-2} \, dA(z)\right)^{\frac{1}{p}}.$$
Our estimate will follow from the inequality:

\[
\frac{\alpha - 1}{\pi} \int_{\mathbb{D}} |f(rz)|^q (1 - |z|^2)^{\alpha - 2} dA(z) \leq \frac{\beta - 1}{\pi} \int_{\mathbb{D}} |f(z)|^q (1 - |z|^2)^{\beta - 2} dA(z).
\] (3)

Using polar coordinates, we get that the last inequality is equivalent to

\[
(\alpha - 1) \int_{0}^{1} \rho(1 - \rho^2)^{\alpha - 2} \left( \int_{0}^{2\pi} |f(\rho e^{i\theta})|^q d\theta \right) d\rho
\]

\[
\leq (\beta - 1) \int_{0}^{1} \rho(1 - \rho^2)^{\beta - 2} \left( \int_{0}^{2\pi} |f(\rho e^{i\theta})|^q d\theta \right) d\rho.
\]

After the change of variable \( \rho = \sqrt{y} \) and introducing the function \( \Phi(y) = \int_{0}^{2\pi} |f(\sqrt{y}e^{i\theta})|^q d\theta \), we get:

\[
(\alpha - 1) \int_{0}^{1} (1 - y)^{\alpha - 2} \Phi(r^2y) dy \leq (\beta - 1) \int_{0}^{1} (1 - y)^{\beta - 2} \Phi(y) dy.
\]

Integrating by parts gives

\[
r^2 \int_{0}^{1} (1 - y)^{\alpha - 1} \Phi'(r^2y) dy \leq \int_{0}^{1} (1 - y)^{\beta - 1} \Phi'(y) dy.
\] (4)

Integrating by parts once again, we get

\[
r^2 \int_{0}^{1} (1 - y)^{\alpha} \Phi''(r^2y) dy \leq \frac{r^2}{\alpha} \Phi'(0) + \frac{1}{\alpha} \Phi'(0) \int_{0}^{1} (1 - y)^{\beta} \Phi''(y) dy,
\]

or, by using that \( r^2 = \frac{L}{q} = \frac{\alpha}{\beta} \):

\[
r^2 \int_{0}^{1} (1 - y)^{\alpha} \Phi''(r^2y) dy \leq \int_{0}^{1} (1 - y)^{\beta} \Phi''(y) dy.
\]

Since

\[
r^2 \int_{0}^{1} (1 - y)^{\alpha} \Phi''(r^2y) dy = \int_{0}^{r^2} (1 - \frac{y}{r^2})^{\alpha} \Phi''(y) dy
\]

and Lemma 1 gives the convexity of \( \Phi(y) \), that is, \( \Phi''(y) \geq 0 \), the result follows from the inequality

\[
\int_{0}^{r^2} (1 - \frac{y}{r^2})^{\alpha} \Phi''(y) dy \leq \int_{0}^{r^2} (1 - y)^{\beta} \Phi''(y) dy,
\]

which is implied by the inequality

\[
(1 - \frac{y}{r^2})^{\alpha} \leq (1 - y)^{\beta} \quad \text{for} \quad 0 \leq y \leq \frac{\alpha}{\beta} = r^2.
\] (5)

But \( g(y) = (1 - y)^{\frac{\beta}{\alpha}} \) is convex and therefore not smaller than \( g(0) + g'(0)y = 1 - \frac{y}{r^2} \). This finishes the proof for the case \( q \geq 2 \).
For an arbitrary holomorphic function \( f \), the function \( \Phi(y) \) need not be convex, but it is still monotone increasing (since \( |f(z)|^q \) is subharmonic for holomorphic \( f \) and \( q > 0 \)) and we can prove (4) with \( r^2 = \frac{\alpha - 1}{\beta - 1} \) using the inequality (5) with \( \alpha - 1 \) and \( \beta - 1 \) instead of \( \alpha \) and \( \beta \), respectively. The second estimate for \( r \) in the case \( q < 2 \) follows from the first part of the theorem with \( q = 2 \) and Jensen’s inequality \( \|f(\sqrt{\frac{E}{2}}z)\|_A^q \leq \|f(\sqrt{\frac{E}{2}}z)\|_A^{2} \).

For the zero-free function \( f \) and \( 0 < p < q \) with \( q < 2 \), we can use the first part of this theorem for the function \( f^{\frac{1}{n}} \) and \( np \) and \( nq \) instead of \( p \) and \( q \), respectively, where \( n \) is such that \( nq \geq 2 \). □

Remark. The careful reader can easily deduce that the statement of our theorem also holds for functions with the zeros whose multiplicities are all big enough depending on \( p \). For example, if a function \( f \) has all zeros in \( \mathbb{D} \) of multiplicities 2, than its square root function is well defined and the desired estimate (1) holds for \( q \geq 1 \). Also, it can be easily verified that the estimate (3) is not valid for \( f(z) = z \) and \( q < 2 \). However, using log-Sobolev inequality from Corollary 1, one can see from some estimates of Gamma function and its derivatives that (1) holds for \( f(z) = z \) for all \( \alpha > 1 \) and \( 0 < p < q < +\infty \).

After submitting this paper, we are informed that Adrian Llinares and Alejandro Mas have proved the similar result using different techniques.

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