String Theory in Magnetic Monopole Backgrounds

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We discuss string propagation in the near-horizon geometry generated by Neveu-Schwarz fivebranes, Kaluza-Klein monopoles and fundamental strings. When the fivebranes and KK monopoles are wrapped around a compact four-manifold $\mathcal{M}$, the geometry is $AdS_3 \times S^3/\mathbb{Z}_N \times \mathcal{M}$ and the spacetime dynamics is expected to correspond to a local two dimensional conformal field theory. We determine the moduli space of spacetime CFT’s, study the spectrum of the theory and compare the chiral primary operators obtained in string theory to supergravity expectations.
1. Introduction

It is currently believed that many (perhaps all) vacua of string theory have the property that their spacetime dynamics can be alternatively described by a theory without gravity \[1,2,3,4\]. This theory is in general non-local, but in certain special cases it is expected to become a local quantum field theory (QFT). It is surprising that string dynamics can be equivalent to a local QFT. A better understanding of this equivalence would have numerous applications to strongly coupled gauge theory, black hole physics and a non-perturbative formulation of string theory.

An important class of examples for which string dynamics is described by local QFT is string propagation on manifolds that include an anti-de-Sitter spacetime \(AdS_{p+1}\) [3,5,6]. In this case the corresponding theory without gravity is a \(p\) dimensional conformal field theory (CFT). In general, solving the string equations of motion on \(AdS_{p+1}\) requires turning on Ramond-Ramond (RR) backgrounds, which are not well understood. This makes it difficult to study the \(AdS/CFT\) correspondence in string theory and most of the work on this subject is restricted to situations where the curvature on \(AdS_{p+1}\) is small and the low energy supergravity approximation is reliable.

String theory on \(AdS_3\) is special in several respects. First, in this case the “dual” CFT is two dimensional and the corresponding conformal symmetry is infinite dimensional. In general, two dimensional CFT’s are better understood than their higher dimensional analogs and one may hope that this will also be the case here. Second, string theory on \(AdS_3\) can be defined without turning on RR fields and thus should be more amenable to traditional worldsheet methods. Perturbative string theory on \(AdS_3\) was studied in [7]. The purpose of this paper is to continue this study and to apply it to some additional examples that are of interest in the different contexts mentioned above. Another application of the results of [7] appears in [8].

In section 2 we introduce the brane configuration whose near-horizon geometry will serve as the background for string propagation later in the paper. The configuration of interest includes \(NS5\)-branes, Kaluza-Klein (KK) monopoles and fundamental strings. We describe the supergravity solution and its near-horizon limit, and review some earlier results on chiral primary operators that are visible in supergravity. We also determine the moduli space of vacua in this geometry; by the \(AdS/CFT\) correspondence this gives the moduli space of dual CFT’s. The resulting moduli space, which can be thought of as the moduli space of M-theory on \(AdS_3 \times S^2 \times T^6\), is given in (2.17). The duality group is \(F_{4(4)}(\mathbb{Z})\), a discrete, non-compact version of the exceptional group \(F_4\).
In section 3 we review the work of [7] on string theory on \( \text{AdS}_3 \times S^3 \times T^4 \), and elaborate on some aspects of it. We find the spectrum of chiral primaries in string theory and compare it to supergravity. We also comment on a proposed identification of string theory on \( \text{AdS}_3 \times S^3 \times \mathcal{M} \) with CFT on the symmetric product \( \mathcal{M}^n/S_n \), and show that the spectra of \( U(1)^4 \) affine Lie algebras in string theory and in CFT on the symmetric product disagree. We also briefly describe the extension of the work of [7] to heterotic string theory and show that the resulting spacetime CFT is “heterotic” as well (i.e. its left and right central charges are different).

In sections 4, 5 we discuss string propagation in the magnetic monopole background of section 2. The near-horizon geometry includes a Lens space \( S^3/\mathbb{Z}_N \); therefore we start in section 4 with a description of CFT on Lens spaces. In section 5 we turn to string theory on such spaces and discuss in turn bosonic, heterotic and type II strings. We discuss the spectrum, obtain the moduli space of vacua, and compare the resulting structure to the supergravity analysis of section 2. We show that the set of chiral operators in string theory on \( \text{AdS}_3 \times S^3/\mathbb{Z}_N \times T^4 \) is much larger than that in the corresponding low energy supergravity theory. In particular, it includes an exponentially large density of perturbative string states which carry momentum and winding around an \( S^1 \) in \( S^3/\mathbb{Z}_N \). We also show that string theory in a monopole background exhibits an effect familiar from quantum mechanics, the shift of angular momenta of electrically charged particles in the background of a magnetic monopole. Two appendices contain conventions, results and derivations used in the text.

2. Supergravity Analysis

2.1. Brane Configuration

Consider M-theory compactified on a six dimensional manifold \( \mathcal{N} \) parametrized by \((x^4, x^5, x^6, x^7, x^8, x^{11})\), down to five non-compact dimensions \((x^0, x^1, x^2, x^3, x^9)\). We will concentrate on the case \( \mathcal{N} = T^6 \), but will comment briefly on the cases where \( \mathcal{N} \) is \( K3 \times T^2 \) or a Calabi-Yau manifold. Since we would like to use weak coupling techniques, we identify \( x^{11} \) with the M-theory direction, and send its radius to zero. In the resulting weakly coupled string theory we consider the following brane configuration [8]:

(a) \( N \) KK monopoles wrapped around the \( T^4 \) labeled by \((x^5, \cdots, x^8)\), infinitely extended in \( x^9 \) and charged under the gauge field \( A_\mu = G_{\mu 4} \) (\( \mu = 0, 1, 2, 3 \)).

(b) \( N' \) NS5-branes wrapped around the above \( T^4 \) and extended in \( x^9 \).
(c) $p$ fundamental strings infinitely stretched in $x^9$.

The configuration (a) – (c) preserves four supercharges which form a chiral $(4, 0)$ supersymmetry algebra in the $1+1$ dimensional non-compact spacetime $(x^0, x^9)$ shared by all the branes. It will be useful later to note that all the unbroken supercharges originate from the same worldsheet chirality. At low energies the theory on the branes decouples from bulk string dynamics and approaches a $(4, 0)$ superconformal field theory.

M-theory on $T^6$ has a large U-duality group which can be used to relate the above brane configuration to many others, such as that of three $M5$-branes wrapped around different four-cycles in $T^6$ and intersecting along the $x^9$ direction [10]. The specific realization (a) – (c) is special in that only Neveu-Schwarz (NS) sector fields are excited; therefore we will be able to use the results of [7] to study the near-horizon dynamics.

The classical supergravity fields around the above collection of NS5-branes, KK monopoles and fundamental strings are as follows. The string frame metric is

\[
\begin{align*}
\text{ds}^2 = & H_5^{-1} (dx_4 + P_K (1 - \cos \theta) d\phi)^2 + H_K (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)) + \\
& + F^{-1} (-dt^2 + dx_9^2) + \sum_{i=5}^{8} dx_i^2
\end{align*}
\]

(2.1)

where the harmonic functions

\[
\begin{align*}
H_5 = & 1 + \frac{P_5}{r} \\
H_K = & 1 + \frac{P_K}{r} \\
F = & 1 + \frac{Q}{r}
\end{align*}
\]

(2.2)

are associated with NS5-branes, KK monopoles, and fundamental strings, respectively, and we have parametrized the space transverse to the branes $(x^1, x^2, x^3)$ by spherical coordinates $(r, \theta, \phi)$. The dilaton and NS $B_{\mu\nu}$ field are:

\[
\begin{align*}
B_{t9} &= F \\
B_{\phi 4} &= P_5 (1 - \cos \theta)
\end{align*}
\]

(2.3)

\[
e^{-2[\Phi_{10}(r) - \Phi_{10}(\infty)]} = \frac{F}{H_5}
\]

The charges $P_5$, $P_K$, $Q$ in (2.2) are related to the numbers of branes $N'$, $N$ and $p$ via:

\[
\begin{align*}
P_5 &= \frac{\alpha'}{2R} N' \\
P_K &= \frac{R}{2} N \\
Q &= \frac{\alpha'^3 g_s^2}{2RV} p
\end{align*}
\]

(2.4)
where \( R \) is the radius of \( x^4 \) (asymptotically far from the branes), \( g_s \equiv \exp\Phi_{10}(\infty) \) and \((2\pi)^4V\) is the volume of the \( T^4 \). Note that:

(a) From the point of view of the 3 + 1 dimensional non-compact spacetime labeled by \( x^\mu, \mu = 0,1,2,3 \), the vacuum (2.1) – (2.4) is magnetically charged under two gauge fields, \( G_{\mu 4}, B_{\mu 4} \). The magnetic charges are \( N \) and \( N' \), respectively.

(b) One can make the vacuum (2.1) – (2.4) arbitrarily weakly coupled everywhere by sending \( g_s \to 0 \) and \( p \to \infty \) with \( Q \) fixed.

(c) T-duality in \( x^4 \) exchanges NS5-branes and KK monopoles. Thus it exchanges \( N \) and \( N' \) as well as type IIA and IIB.

(d) The fields that are excited in (2.1) – (2.4) exist in all closed string theories, including the bosonic string, the heterotic string, and the type II superstring. Therefore, one can study this solution in all these theories.

2.2. The near-horizon limit

A dual description of the decoupled low energy dynamics on the branes is obtained by studying string dynamics in the background (2.1) – (2.4) in the near-horizon limit \( r \to 0 \). In this limit the metric reduces to:

\[
\begin{align*}
 ds^2 &= \frac{P_5}{P_K} [dx_4 + P_K(1-\cos \theta)d\phi]^2 + P_5P_K/(\sin^2 \theta d\phi^2) \\
 &\quad + \frac{P_5P_K}{r^2}dr^2 + \frac{r}{Q}(-dt^2 + dx_5^2) + \sum_{i=5}^8 dx_i^2 
\end{align*}
\]

The \( B_{\mu \nu} \) field is given by (2.3). The dilaton, which far from the branes is arbitrary, becomes a fixed scalar in the near-horizon geometry (2.5):

\[
e^{-2\Phi_{\text{hor}}} = e^{-2\Phi_{10}} \frac{V}{\alpha'^2} = \frac{Q}{P_5} \frac{V}{\alpha'^2} e^{-2\Phi_{10\infty}} = \frac{p}{\alpha' NN'}
\]  

The three dimensional space parametrized by \( t, r, \) and \( x^9 \) becomes after the coordinate change \( \rho^2 = 4P_5P_Kr/Q \):

\[
 ds^2 = \frac{l^2}{\rho^2}d\rho^2 + \frac{\rho^2}{l^2}(-dt^2 + dx_5^2)
\]

where

\[
l^2 = 4P_5P_K = \alpha' NN'
\]

\footnote{Many similar examples are discussed in \[11\] .}
The metric (2.7) is that of $AdS_3$ with curvature $\Lambda = -1/l^2$.

We next turn to the three dimensional space parametrized by $\theta$, $\phi$, and $x_4$. Far from the branes, the radius of $x^4$, $R$, is a free parameter, the expectation value of a massless scalar field. In the near-horizon geometry (2.7) $R$ is fixed:

$$R_{\text{hor}}^4 = \sqrt{G_{44}} R = R \sqrt{\frac{P_5}{P_K}} = \sqrt{\frac{N'\alpha'}{N}}$$  \hspace{1cm} (2.9)

The corresponding scalar field is massive. The radius of $x^4$ (2.9) is typically of string size (if $N$ and $N'$ are comparable). Hence, at low energies we can dimensionally reduce along $x^4$. This leaves an $S^2$ labeled by $\theta$, $\phi$, which can be interpreted as the horizon of a four dimensional black hole, or a five dimensional black string. In the full theory $x^4$ is retained and the resulting three dimensional space can be identified with the manifold $S^3/\mathbb{Z}_N$. Altogether, we are led to study string theory on

$$AdS_3 \times S^3/\mathbb{Z}_N \times T^4$$  \hspace{1cm} (2.10)

Another difference between the asymptotic and near-horizon geometries is that while, as we saw, the asymptotic background (2.1) – (2.4) is magnetically charged under both $G_{\mu^4}$ and $B_{\mu^4}$ (with different magnetic charges $N$, $N'$), in the near-horizon geometry (2.3) $G_{\phi^4} - B_{\phi^4} = 0$. Thus, the vacuum (2.3) is magnetically charged under $G_{\mu^4} + B_{\mu^4}$ only,

$$G_{\phi^4} + B_{\phi^4} = 2P_5(1 - \cos \theta)$$  \hspace{1cm} (2.11)

Recalling that $G + B$ and $G - B$ couple to different chiralities on the worldsheet, we see that the background in question is $S^2 \times S^1$ with the $SO(3)$ isometry of the two-sphere arising from the worldsheet chirality that does not couple to $G + B$, which we will refer to as right-moving. We conclude that the $\mathbb{Z}_N$ orbifold in (2.10) is asymmetric.

2.3. Black Holes and Spacetime CFT

One of the motivations for the present work is the relation to black holes. The configuration given in (2.1) can be generalized by adding one more charge, corresponding to momentum along the 9th direction. If $x^9$ is compact, the resulting metric is that of a regular black hole in four dimensions, the Cvetić-Youm dyon [12]. Such black holes can

\footnote{The foregoing discussion is valid for large $N$, $N'$, when supergravity is reliable. We will see later that when $N$ or $N'$ are \leq 2, a second $SO(3)$ arises from the other worldsheet chirality.}
be interpreted as excitations of the configuration (2.1). The black hole entropy follows from the large degeneracy of these excitations. Understanding the spacetime dynamics of strings on (2.10) involves understanding precisely these excitations, and thus is important for black hole physics.

Brown and Henneaux [13] have shown that any theory of gravity on $\text{AdS}_3$ has a large symmetry algebra containing two copies of the Virasoro algebra with central charge

$$c_{st} \simeq \frac{3l}{2l_p},$$

(2.12)

where $l$ is the radius of curvature of $\text{AdS}_3$ (2.8), and $l_p$ is the three dimensional Newton constant. The calculation of [13] is semiclassical and is expected in general to receive both quantum gravity corrections which are suppressed by powers of $l_p/l$, and string corrections suppressed by powers of $l_s/l$ ($l_s^2 \equiv \alpha'$). We will see examples of such corrections below. In our case, the semiclassical computation gives

$$c_{st} = 6N N' p$$

(2.13)

Strominger [14] pointed out that if the spacetime dynamics of a theory of gravity on $\text{AdS}_3$ corresponds to a unitary and modular invariant CFT with the central charge (2.12) (and satisfies certain additional mild assumptions), then the standard CFT degeneracy of states agrees with the Bekenstein-Hawking entropy of the corresponding black holes. For the case of interest here the details were discussed in [15]. Thus in our study of string theory on (2.10) we will be interested in computing the central charge (2.13) and any stringy corrections to it, and understanding the spectrum of operators that contribute to the BH$^3$ entropy.

2.4. The Spectrum of Perturbations

Later we will study the spectrum of perturbations of string theory in the background (2.10). Some of these perturbations are visible already in the supergravity approximation. In this subsection we summarize the results of [16,17] regarding the spectrum of chiral operators in supergravity.

Consider eleven dimensional supergravity compactified on

$$\text{AdS}_3 \times S^2 \times \mathcal{N}$$

(2.14)

$^3$ = Black Hole, Bekenstein-Hawking, Brown-Henneaux
where $\mathcal{N}$ is a Calabi-Yau manifold, $K3 \times T^2$ or $T^6$. We will assume that the manifold $\mathcal{N}$ has Planck (or string) scale size, and dimensionally reduce all supergravity fields on it. The size of $S^2$ is assumed to be large; therefore we will keep Kaluza–Klein harmonics on the sphere. The theory preserves $(4, 0)$ supersymmetry and, as explained in the previous subsection, is equivalent to a two dimensional CFT.

States are labeled by the quantum numbers $h, \bar{h}, \bar{j}$, where $h$ is the scaling dimension with respect to the left-moving Virasoro algebra of $[13]$, while $\bar{h}$ and $\bar{j}$ are the quantum numbers under the right-moving $N = 4$ superconformal algebra (i.e. under right-moving Virasoro and $SU(2)_R$). Note that an $SL(2)_R \times SL(2)_L$ subalgebra of Virasoro is identified with the isometry of $AdS_3$ in (2.14) while the $SU(2)_R$ symmetry is the $SO(3)$ isometry of the two-sphere.

Reducing the eleven dimensional supergravity fields on the manifold (2.14) gives rise to short representations of the $N = 4$ superconformal algebra. To analyze the resulting spectrum it is convenient to perform the reduction in two steps, first reducing to five dimensions on the manifold $\mathcal{N}$, and then further reducing on $AdS_3 \times S^2$. After reducing on $\mathcal{N}$, one finds the following spectrum:

(a) A gravity multiplet, whose bosonic components are the five dimensional graviton and a gauge field (the graviphoton). This multiplet contains eight bosonic and eight fermionic degrees of freedom.

(b) $n_H$ hypermultiplets, consisting of two real scalars and fermions. Each multiplet thus has two bosonic and two fermionic degrees of freedom.

(c) $n_V$ vectormultiplets consisting of a vector field, a scalar and fermions ($4 + 4$ physical degrees of freedom).

(d) $n_S$ gravitino multiplets consisting of two vectors and fermions ($6 + 6$ degrees of freedom).

The values of $n_H$, $n_V$, and $n_S$ for different choices of the manifold $\mathcal{N}$ are [18]:

(a) On a Calabi-Yau manifold, at a generic point in moduli space, $n_H = 2(h_{21} + 1)$, $n_V = h_{12} - 1$, and $n_S = 0$.

(b) On $K3 \times T^2$, generically $n_V = 22$, $n_H = 2(n_V - 1)$, and $n_S = 2$. At points in moduli space where the gauge symmetry is enhanced, $n_V$ increases accordingly. Note that this problem is dual to the heterotic string on a torus, a case we will discuss below.

(c) On $T^6$, $n_H = n_V = 14$ and $n_S = 6$.

The further reduction on $AdS_3 \times S^2$ gives the following spectrum of chiral primaries:
$$h - \bar{h} \quad \text{degeneracy} \quad \text{range of } \bar{h} = \bar{j}$$

|   |   |   |   |
|---|---|---|---|
| 1/2 | $n_H$ | $1/2, 3/2, \ldots$ |
| 0   | $n_V$  | $1, 2, \ldots$   |
| 1   | $n_V$  | $1, 2, \ldots$   |
| $-1/2$ | $n_S$ | $3/2, 5/2, \ldots$ |
| $1/2$ | $n_S$ | $3/2, 5/2, \ldots$ |
| $3/2$ | $n_S$ | $1/2, 3/2, \ldots$ |
| $-1$ | 1     | $2, 3, \ldots$   |
| 0   | 1     | $2, 3, \ldots$   |
| 1   | 1     | $1, 2, \ldots$   |
| 2   | 1     | $1, 2, \ldots$   |

**Table 1: Spectrum of chiral primaries for $\text{AdS}_3 \times S^2 \times \mathcal{N}$.

In section 5 we will find a stringy generalization of the spectrum of Table 1 for toroidally compactified heterotic and type II string theories, and will see that the above table is indeed obtained in the supergravity limit.

### 2.5. Moduli Space of Vacua

The purpose of this subsection is to determine the moduli space of vacua of M-theory on the manifold (2.14) for the case $\mathcal{N} = T^6$. The moduli space of M-theory on $T^6 \times \mathbb{R}^{4,1}$ is

$$E_{6(6)}(\mathbb{Z}) \backslash E_{6(6)}/USp(8)$$  \hspace{1cm} (2.15)

$E_{6(6)}$ is a non-compact form of $E_6$ with maximal compact subgroup $USp(8)$. Black strings in the five non-compact dimensions are charged under the various $B_{\mu\nu}$ fields (which in five dimensions are dual to gauge fields). There are 27 independent strings transforming in the 27 of the $E_{6(6)}(\mathbb{Z})$ U-duality group (2.13): the $M5$-branes with four of their dimensions wrapped on a $T^4$ (15 possibilities), the $M2$-branes with one dimension wrapped on an $S^1$ (6 of these), and the KK monopoles charged under the six gauge fields $G_{\mu,i}$ and wrapped around the remaining $T^5$. It is convenient to organize these 27 charges into an 8 $\times$ 8 symplectic-traceless antisymmetric matrix (utilizing the maximal $USp(8)$ subgroup of $E_6$):

$$27 = \begin{pmatrix} b J_{(1)} & Z \\ -Z^T & -\frac{1}{3} b J_{(3)} + A_{ij} T^{ij} \end{pmatrix}$$  \hspace{1cm} (2.16)
where \( J_{(i)} \) are the symplectic invariants of \( USp(2i) \), and \( T^{ij} \) are a basis of traceless antisymmetric \( 6 \times 6 \) matrices. One can choose the \( 6 \times 2 \) charges \( Z \) in (2.16) to correspond to the \( M2 \)-brane and KK monopole charges described above, while \( A_{ij} \) and \( b \) parametrize the \( M5 \)-brane charges.

Eq. (2.16) makes manifest the decomposition of the 27 of \( E_6 \) in terms of representations of its \( USp(2) \times USp(6) \) subgroup: \( b(1,1) + Z(2,6) + A(1,14) \). The near-horizon geometry (2.3) is U-dual to a configuration of three \( M5 \)-branes with charges: \( \langle b \rangle \neq 0, \langle Z \rangle = \langle A_{ij} \rangle = 0 \) [16]. To find the subgroup of \( E_6(6) \) that is left unbroken by \( \langle b \rangle \), one notes that \( E_6 \) has a maximal \( F_4 \) subgroup, under which the 27 decomposes as \( 26 + 1 \). The \( USp(2) \times USp(6) \) discussed above is the maximal compact subgroup of \( F_4 \). Therefore, \( b \) is a singlet under \( F_4 \), and the subgroup of \( E_6(6) \) U-duality preserved by the near-horizon geometry is \( F_4(4) \). The moduli space is the coset:

\[
F_4(4) \backslash F_4(4)/USp(2) \times USp(6) \tag{2.17}
\]

The adjoint of \( F_4 \), the 52, decomposes under \( USp(2) \times USp(6) \) as \( [(3,1) + (1,21)] + (2,14) \). The noncompact form thus has signature \( (28,24) \), and the coset has dimension 28. The \( 2 \times 14 \) moduli correspond to the \( n_H = 14 \) hypermultiplets with \( h = 1, \bar{h} = 1/2 \) in Table 1. In Section 5 we will reproduce aspects of the moduli space (2.17) in string theory.

3. String Theory on \( AdS_3 \times S^3 \times T^4 \)

The near-horizon geometry of a system of \( k \) \( NS5 \)-branes and \( p \) fundamental strings in type II string theory on a four-manifold \( \mathcal{M} (\mathcal{M} = T^4 \) or \( K3) \) is

\[
AdS_3 \times S^3 \times \mathcal{M} \tag{3.1}
\]

Type II string theory on the manifold (3.1) has \( (4,4) \) superconformal symmetry when \( \mathcal{M} = T^4 \) and in the type IIB case also when \( \mathcal{M} = K3 \). It has \( (4,0) \) superconformal symmetry for the heterotic string on \( \mathcal{M} = T^4, K3 \), and for type IIA on \( \mathcal{M} = K3 \) (which is dual to the heterotic theory on \( T^4 \)). String theory on manifolds including an \( AdS_3 \) factor was discussed in [7], and the case of type II string propagation on (3.1) with \( \mathcal{M} = T^4 \) was described in detail. In this section we will review this construction, as a warmup for the case (2.10). We will also make some comments on the spectrum of the theory, and briefly discuss the heterotic case. Some of the conventions and other details appear in Appendix A.
3.1. Symmetries of string theory on $AdS_3 \times S^3 \times T^4$

The worldsheet and spacetime symmetries of string theory on (3.1) act separately on the left and right-movers on the worldsheet. Therefore in this subsection we will discuss the chiral (holomorphic) symmetry structure, both on the worldsheet and in spacetime. This will also be useful when we turn to string theory on (2.10) whose (anti-) holomorphic structure is identical to that discussed here.

The affine worldsheet symmetry of string theory on $AdS_3 \times S^3 \times T^4$ is $\hat{SL}(2) \times \hat{SU}(2) \times \hat{U}(1)_4$. It is realized as follows. There are three bosonic currents $j^A$, realizing $\hat{SL}(2)$ at level $k+2$, and three fermions $\psi^A$, forming an $\hat{SL}(2)$ at level $-2$. Similarly, there are three bosonic currents $K^a$, realizing an $\hat{SU}(2)$ at level $k-2$, and three fermions $\chi^a$, giving an $\hat{SU}(2)$ at level 2. The total currents

$$J^A = j^A - \frac{i}{k} \epsilon^{ABC} \psi^B \psi^C, \quad A, B = +, -, 3$$
$$K^a = k^a - \frac{i}{k} \epsilon^{abc} \chi^b \chi^c, \quad a, b = +, -, 3$$

thus have the same level $k$. The total worldsheet central charge of the $SL(2) \times SU(2)$ theory is identical to its flat space value ($c = 9$), for all values of $k$.

The $\hat{U}(1)_4$ is realized in terms of free bosonic currents $i\partial Y^j$ and free fermions $\lambda^j$, $j = 1, 2, 3, 4$. The worldsheet theory is superconformal; the supercurrent is

$$T_F(z) = \frac{2}{k} (\eta_{AB} \psi^A \psi^B - \frac{i}{3k} \epsilon_{ABC} \psi^A \psi^B \psi^C) + \frac{2}{k} (\chi^a k^a - \frac{i}{3k} \epsilon_{abc} \chi^a \chi^b \chi^c) + \lambda^i \partial Y^i$$

Generally in string theory affine symmetries on the worldsheet give rise to gauge symmetries in spacetime. Contour integrals of the worldsheet generators give global charges, corresponding to gauge transformations that do not vanish rapidly enough at infinity. The analysis of [13] shows that in gravity on $AdS_3$ one should allow a rich set of non-trivial behaviors of the metric at infinity. This leads to a large spacetime global symmetry group, the 2d conformal group. Similarly, allowing non-trivial behaviors of gauge fields at the boundary of $AdS_3$ leads to an enhancement of global spacetime gauge symmetries to the corresponding affine Lie algebras [7].

To obtain a worldsheet description of the above infinite spacetime symmetry algebras it is convenient to use the Wakimoto representation of the conformal $\sigma$-model on $AdS_3$ [13]. This representation, which is summarized in Appendix A, parametrizes the manifold by the coordinates $(\phi, \gamma, \bar{\gamma})$. The radial direction is $\phi$, with the boundary of $AdS_3$ corresponding
to $\phi = \infty$. The Wakimoto description is particularly useful for studying the structure of string theory on $AdS_3$ in the limit $\phi \to \infty$. In this limit the following two important simplifications occur:

(a) The worldsheet $\sigma$-model becomes free.
(b) The string coupling goes to zero.

Thus the full string theory becomes weakly coupled both on the worldsheet and in spacetime in this limit, regardless of the fixed coupling in the original description. This allows one to study all aspects of string theory on $AdS_3$ which are observable at $\phi \to \infty$ using free field theory on the worldsheet. This includes the infinite spacetimes symmetries, since these correspond to gauge transformations which are completely specified by the behavior at infinity. As shown in [7], the form of the Virasoro ($L_n$), $SU(2)$ ($T^a_n$) and $U(1)^4$ ($\alpha^i_n$) charges as $\phi \to \infty$ is:

$$L_n = \oint dz \left[ (1-n^2) J^3 \gamma^n + \frac{n(n-1)}{2} J^+ \gamma^{n+1} + \frac{n(n+1)}{2} J^- \gamma^{n-1} \right]$$

$$T^a_n = \oint dz \{ G_{-1/2}, \chi^a(z) \gamma^n(z) \}$$

$$\alpha^i_n = \oint dz \{ G_{-1/2}, \lambda^i(z) \gamma^n(z) \}$$

The algebra satisfied by (3.4) is:

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c_{st}}{12} (n^3 - n) \delta_{n+m,0}$$

$$[T^a_n, T^b_m] = i \epsilon^{abc} T^c_{n+m} + \frac{k_{st}}{2} n \delta^{ab} \delta_{n+m,0}$$

$$[L_n, T^a_m] = -n T^a_{n+m}$$

$$[\alpha^i_n, \alpha^j_m] = pn \delta^{ij} \delta_{n+m,0}$$

$$[L_n, \alpha^i_m] = -n \alpha^i_{n+m}$$

where

$$c_{st} = 6kp; \quad k_{st} = kp$$

and $p$ is a certain winding number that characterizes the embedding of the worldsheet into spacetime, reflecting the presence of $p$ fundamental strings in the vacuum,

$$p \equiv \oint \frac{dz}{2\pi i} \frac{\partial z \gamma(z)}{\gamma(z)}$$
As mentioned above, string theory on (3.1) exhibits $N = 4$ superconformal invariance in spacetime. The superconformal generators are given by

$$Q = \oint dz e^{-\frac{\phi}{2}} S(z) ; \quad S(z) = e^{\frac{i}{2} \sum_I \epsilon_I H_I(z)} \quad (3.8)$$

where $H_I, I = 1, \cdots, 5$ are the scalar fields, defined in Appendix A, which are obtained by bosonizing the fermions, and $\epsilon_I = \pm 1$. As discussed in [7], mutual locality of the supercharges and BRST invariance lead to the constraints:

$$\prod_{I=1}^5 \epsilon_I = \prod_{I=1}^3 \epsilon_I = 1 \quad (3.9)$$

These projections leave eight spacetime supercharges, which together with $L_{\pm 1}, L_0$ and $T^a_0$ form the global $N = 4$ superconformal algebra [7].

3.2. Comments on the spectrum

The construction of string excitations in the background (3.1) is very similar to that corresponding to superstrings in flat spacetime. The plane wave zero mode wave function familiar from flat spacetime is replaced by $V_j; m, \bar{m} V'_j; m', \bar{m}' \exp(i\vec{p} \cdot \vec{Y} + i\bar{\vec{p}} \cdot \bar{\vec{Y}})$ where $V, V'$ are the wave functions on $AdS_3, S^3$ respectively. Their transformation properties under the worldsheet affine Lie algebra are described in Appendix A. $(\vec{p}, \bar{\vec{p}})$ is a vector in an even, self-dual Narain lattice $\Gamma_{4,4}$. The towers of string states are obtained by multiplying this zero mode wave function by a polynomial in the fermionic oscillators $\psi^A, \chi^a, \lambda^j$, and the bosonic oscillators $j^A, k^a, \partial Y^j$ (and their derivatives), and a similar polynomial in the antiholomorphic oscillators, and restricting to the BRST cohomology. The most general state in the $(-1, -1)$ picture of the NS-NS sector has the form

$$V_{NS} = e^{-\phi - \bar{\phi}} P_N(\psi^A, \partial \psi^A, \cdots, j^A, \partial j^A, \cdots) \bar{P}_N(\bar{\psi}^A, \cdots) V_j; m, \bar{m} V'_j; m', \bar{m}' \exp(i\bar{\vec{p}} \cdot \bar{\vec{Y}} + i\vec{p} \cdot \vec{Y}) \quad (3.10)$$

---

4 More precisely, the supercharges $Q$ form the global $N = 4$ superconformal algebra. As explained in [8], the full infinite superconformal symmetry can be obtained by acting on the global supercharges with the generators (3.4). This has been analyzed in [20].

5 For simplicity we restrict here to NS-NS sector excitations. The generalization to other sectors is straightforward. We also work at generic values of the momenta; for special values there might be physical states at other ghost numbers.
where $P_N$ is a polynomial in the bosonic and fermionic worldsheet fields and their derivatives with scaling dimension $N$, and similarly for $\bar{P}_{\bar{N}}$. BRST invariance is the requirement that the matter part of (3.10) is a lower component of an $N = 1$ worldsheet superfield with $(\Delta, \bar{\Delta}) = (\frac{1}{2}, \frac{1}{2})$. Thus, one must have:

$$N - \frac{j(j + 1)}{k} + \frac{j'(j' + 1)}{k} + \frac{|\vec{p}|^2}{2} = \frac{1}{2}$$

(3.11)

and a similar relation with $N \to \bar{N}$ and $\vec{p} \to \bar{\vec{p}}$ coming from the other chirality. One can think of (3.11) as determining $j$ (the “energy”) in terms of $\vec{p}$, $j'$ (the “momentum”) and $N$, the excitation level of the string. $V_{NS}$ is null if it can be written as the higher component of a worldsheet superfield with $(\Delta, \bar{\Delta}) = (\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$.

The free spectrum can be summarized by writing down the torus partition sum. The partition sum of CFT on $AdS_3$ (more precisely on its Euclidean version $H_3^+$) appears in [21]; the rest is standard. Denoting the $SU(2)$ characters with spin $j$ at level $k$ by $\chi_j^{(k)}(\tau)$, the string partition sum is ($q \equiv \exp(2\pi i \tau)$):

$$Z(\tau, \bar{\tau}) = \frac{1}{\sqrt{\eta(\tau)|\bar{\eta}(\tau)|^2}} \sum_{j, \bar{j}} \mathcal{N}_{j, \bar{j}} \chi_j^{(k-2)}(\tau) \chi_{\bar{j}}^{(k-2)}(\bar{\tau}) \frac{1}{|\eta(\tau)|^8} \sum_{(\vec{p}, \bar{\vec{p}}) \in \Gamma^4} q^{\frac{1}{2} \vec{p}^2} q^{\frac{1}{2} \bar{\vec{p}}^2} \left| \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right|^2$$

(3.12)

where the matrix $\mathcal{N}_{j, \bar{j}}$ parametrizes the bosonic $SU(2)_{k-2}$ modular invariant (see e.g. [22]); we will mainly discuss the simplest case, the $A$-series modular invariant for which $\mathcal{N}_{j, \bar{j}} = \delta_{j, \bar{j}}$. We have factored out an infinite overall constant from the $AdS_3$ partition sum, which has a clear physical meaning – it is the infinite volume of the $\phi$ direction (see Appendix A).

The transformation properties of physical states such as (3.10) under the spacetime symmetries are obtained by computing the commutators of the generators (3.4), (3.8) with the vertices (3.10). As an example, under the spacetime Virasoro algebra, the states (3.10) transform either as primaries or as descendants. The primaries, $V_{phys}(h; m, \bar{m})$ satisfy

$$[L_n, V_{phys}(h; m, \bar{m})] = (n(h - 1) - m) V_{phys}(h; m + n, \bar{m})$$

(3.13)

where $h$ is the scaling dimension of $V_{phys}$. A state of the form (3.10) is primary under spacetime Virasoro if it is primary (in the sense of Appendix A) under $SL(2)$, however
the latter is not a necessary condition. A large class of Virasoro primaries that are also $SL(2)$ primaries is obtained by taking the polynomial $P_N$ in (3.10) to be independent of $\psi^A, j^A$ (and their derivatives). The resulting operators have spacetime scaling dimension $h = j + 1$ with $j$ determined by (3.11) [7].

Similarly, one can study the transformation properties of physical states under $SU(2), \hat{U}(1)^4$ (see [7] for details). We would like to point out an interesting property of the spectrum of the $U(1)$ symmetry in this model. From the form of the generators $\alpha^i_n$ (3.4) it is clear that physical states which carry right-moving momentum $\vec{p}$ along the $T^4$, such as (3.10), and are primary under affine $U(1)^4$, transform as:

$$[\alpha^i_n, V_{\text{phys}}^\vec{p}(j; m, \bar{m})] = p^i V_{\text{phys}}^\vec{p}(j; m + n, \bar{m})$$

(3.14)

\textit{i.e.} the charges of primaries under $U(1)^4$ in spacetime and on the worldsheet are the same. However, we saw in eq. (3.3) that the spacetime $U(1)^4$ generators are not normalized canonically. In terms of canonically normalized generators, the charges of physical states are in fact $p^i/\sqrt{p}$, where $p$ is given in (3.7). For example, if the worldsheet $T^4$ on which the theory lives is a product of four circles with radii $R_i l_s$ and the $B$ field on $T^4$ vanishes, the spectrum of $U(1)^4$ charges becomes

$$Q^i = \frac{1}{\sqrt{p}} \left( \frac{n}{R_i} + \frac{mR_i}{2} \right)$$

(3.15)

It has been proposed that the spacetime SCFT corresponding to string theory on (3.1) is a blowing up deformation of the $(4,4)$ superconformal $\sigma$–model on the symmetric product $M_L/S_L$ with $L = kp$. For $M = T^4$ the symmetric product SCFT has a $\hat{U}(1)^4_R \times \hat{U}(1)^4_L$ affine symmetry, like string theory on (3.1). The spectrum of charges under this affine symmetry in the symmetric product SCFT is independent of the blowing up deformations. Therefore, it is interesting to compare it to the spectrum (3.15) in string theory. As there, we will for simplicity take the spacetime $T^4$ to be a product of four circles with radii $r_i$ ($i = 1, \cdots, 4$) (which may in general be different from $R_i$), and set the $B$ field to zero. The generalization to an arbitrary $T^4$ is straightforward.

To compare to (3.15) we have to normalize the $\hat{U}(1)^4$ generators in the symmetric product canonically, and then compute their spectrum at the orbifold point. The resulting spectrum of charges $q_i$ is:

$$q^i = \frac{1}{\sqrt{L}} \left( \frac{n}{r_i} + \frac{m r_i}{2} \right)$$

(3.16)
Since $L = kp$ and both in (3.15) and in (3.16) states with all possible values of the integers $n, m$ appear, we conclude that the string theory spectrum is incompatible with the symmetric product at any point in its moduli space (for $k > 1$). We view this as evidence that the spacetime SCFT relevant for string theory on (3.1) is not on the moduli space of the symmetric product SCFT (at least for $M = T^4$ and $k > 1$).

We next turn to a discussion of the chiral primaries. Unitarity of the spacetime $N = 4$ superconformal algebra implies that the scaling dimension $h$, and $SU(2)$ spin $j'$ satisfy the inequality $h \geq j'$. Chiral primaries are states that saturate this bound. One can prove that the spectrum (3.10) satisfies the $N = 4$ unitarity bound. The proof uses the worldsheet unitarity bounds $j < k/2, j' \leq k/2$. One also finds that the states that saturate the bound are those with $N = 1/2$ and $\bar{p} = 0$ in (3.10). There are analogous statements in the Ramond sector.

The chiral primaries can be analyzed holomorphically; then the two chiralities are combined in all possible ways consistent with modular invariance. Consider first the NS sector. For $N = 1/2, \bar{p} = 0$, (3.10), (3.11) imply that $j = j'$. For generic $j$ there are eight physical states (with given $m, m'$):

$$V_i^j = e^{-\phi} \chi^i \chi_j V_j V_j'$$
$$W_j^\pm = e^{-\phi} (\psi V_j)_{j \pm 1} V_j'$$
$$X_j^\pm = e^{-\phi} V_j (\chi V_j')_{j \pm 1}$$

where we have used results and notations from Appendix A for combining fermions and bosons into representations of the total current algebras, and suppressed the indices $m, m', \cdots$. One can show that the states (3.17) are BRST invariant and thus physical while the two remaining combinations, which involve $(\psi V_j)_j$ and $(\chi V_j')_j$ are not physical (one combination is not BRST invariant, and the other is null). The four operators $V_j$ have $h = j + 1, j' = j$; $W^\pm$ have $h = j + 1 \pm 1, j' = j$; and $X^\pm$ have $h = j + 1, j' = j \pm 1$. The chiral operators are thus $W_j^-, X_j^+$. The former are physical for $j \geq 1/2$ (for $j = 0$ the corresponding operator is null); the latter exist for all $j \geq 0$.

One can similarly analyze the spectrum of chiral primaries in the Ramond sector. It is convenient to split the spinors into six dimensional spinors $S$, which transform in the $(2, 2)$ of $SL(2) \times SU(2)$, and spinors on the $T^4$, $\exp^{i \frac{\epsilon_4}{2}(\epsilon_4 H_4 + \epsilon_5 H_5)}$ (see Appendix A). Equating the spacetime scaling dimension and $SU(2)$ spin leads to two BRST invariant chiral primaries with $h = j' = j + 1/2, j = 0, 1/2, \cdots$:

$$V_j^\pm = e^{-\frac{\phi}{2}} e^{\pm\frac{i}{2}(H_4 - H_5)} (SV_j V_j')_{j - \frac{1}{2}, j + \frac{1}{2}}$$

(3.18)
By applying the supercharges (3.8) one finds that the upper component of these superfields are $V^i_j$ (3.17).

To summarize, the holomorphic analysis leads to the following spectrum of chiral primaries:

| $\mathcal{V}$ | $h$ | range of $j$ |
|----------|-----|--------------|
| $X^+_j$  | $j + 1$ | 0, 1/2, ... |
| $W^-_j$  | $j$   | 1/2, 1, ...  |
| $Y^\pm_j$| $j + 1/2$| 0, 1/2, ... |

Table 2: Holomorphic chiral primary operators.

Tensoring Table 2 with its antiholomorphic analog leads to the following list of chiral primaries:

| $(h, \bar{h})$ | degeneracy |
|----------------|------------|
| $\left( l + \frac{3}{2}, l + \frac{1}{2} \right)$ | 1          |
| $\left( l + \frac{1}{2}, l + \frac{3}{2} \right)$ | 1          |
| $\left( l + \frac{1}{2}, l + \frac{1}{2} \right)$ | 5          |
| $\left( l + \frac{2}{2}, l + \frac{2}{2} \right)$ | 1          |
| $\left( l + \frac{2}{2}, l + \frac{1}{2} \right)$ | 4          |
| $\left( l + \frac{1}{2}, l + \frac{2}{2} \right)$ | 4          |

Table 3: Spectrum of chiral primaries for $AdS_3 \times S^3 \times T^4$.

where the index $l = 0, 1, \ldots$ up to an upper bound that is determined by the fact that in Table 2, $j$ satisfies the constraint $j \leq k/2 - 1$.

The five $(h, \bar{h}) = \left( \frac{1}{2}, \frac{1}{2} \right)$ chiral primaries in Table 3 give rise to twenty moduli parametrizing the space

$$SO(5, 4; \mathbb{Z}) \backslash SO(5, 4)/SO(5) \times SO(4)$$

(3.19)

Sixteen of the moduli arise from the NS-NS sector on the worldsheet; they correspond to the metric and $B$ field on the $T^4$. The remaining four are RR moduli.
The spectrum of spacetime chiral primaries, Table 3, is in agreement with expectations from supergravity \[23,24,16,17\]. In fact, Table 3 is identical to a table that appears in \[10\]. The upper bound on \(h, \tilde{h}\) that arises from string theory cannot be seen in the supergravity analysis since it is due to string scale physics. Of course, the operators in Table 3 are only those that correspond to the single particle chiral states, and there are many additional chiral operators that correspond to multiparticle states. Finally, general considerations lead one to expect the spectrum of chiral primaries to be truncated at \(h, \tilde{h}\) of order the central charge \[23\], but checking this directly may be beyond the reach of our perturbative analysis.

3.3. Heterotic strings on \(AdS_3 \times S^3 \times T^4\)

The discussion of the previous subsections generalizes easily to the heterotic case. The right-moving worldsheet theory is the same as before, while the left-movers are purely bosonic and live on the manifold \(AdS_3 \times S^3 \times T^4 \times \Gamma^{16}\), where \(\Gamma^{16}\) is the \(E_8 \times E_8\) or \(\text{Spin}(32)/\mathbb{Z}_2\) torus. The symmetry structure in the right-moving sector is as before, while the left-movers give rise to non-supersymmetric conformal symmetry, an \(SU(2)\) affine Lie algebra from the sphere, and some additional affine symmetries from the \(T^4\) and \(\Gamma^{16}\). At generic points in the Narain moduli space there is a \(U(1)^{20} \times SU(2)\) left-moving affine symmetry. At points with enhanced gauge symmetry, the affine symmetry in spacetime is enhanced.

The spacetime theory is thus a \((4, 0)\) superconformal field theory. As we saw \((3.4) - (3.6)\), the spacetime central charge is determined by the total \(SL(2)\) level on the worldsheet, which is \(k = (k + 2) - 2\) for the right-movers, and \(k + 2\) for the left-movers. Therefore, the central charges of the spacetime theory are: \(\tilde{c}_{\text{st}} = 6pk\) and \(c_{\text{st}} = 6p(k + 2)\). The difference

\[
c_{\text{st}} - \tilde{c}_{\text{st}} = 12p
\]

between the left and right central charges is an example of a stringy correction to the semiclassical results of \[13, \text{2.12}\]. In the supergravity limit \(c_{\text{st}} = \tilde{c}_{\text{st}}\); \[3.20\] is suppressed relative to the leading contribution by two powers of \(l_s/l\). Similarly, the levels of the left and right-moving \(SU(2)\) algebras in spacetime differ: \(\tilde{k}_{\text{st}} - k_{\text{st}} = 2p\).

One can repeat the discussion of the previous subsections rather closely; we will not describe the details here. As mentioned above, the theory described in this subsection is S-dual to type IIA on \(AdS_3 \times S^3 \times K3\). It might be interesting to study this duality further using the techniques of \[6\] and this paper.

\[6\] Of course, as usual, one can turn on moduli that mix the \(T^4\) with \(\Gamma^{16}\) and give rise to the standard Narain moduli space of vacua.
4. Conformal Field Theory on $S^3/\mathbb{Z}_N$

As we saw in section 2, the near-horizon geometry of $NS5$-branes and KK monopoles includes a Lens space. In order to study string propagation in this geometry we need to construct worldsheet CFT on $SU(2)_R \times SU(2)_L/\mathbb{Z}_N$, where the $\mathbb{Z}_N$ orbifold acts on the left-movers only. These CFT’s were studied in [25,26]; in this section we will review their properties, first in the bosonic and then in the supersymmetric case.

4.1. Bosonic CFT on Lens spaces

Consider $SU(2)$ WZW theory at level $k$. The theory has two affine $SU(2)$ symmetries acting on the left and right-movers. Denoting the left-moving currents by $K_a(z)$, and the right-moving ones by $\bar{K}_a(\bar{z})$, the action of the $\mathbb{Z}_N$ symmetry by which we mod out to get the Lens space is as follows. We define a $2 \times 2$ (hermitian) matrix of currents $K \equiv K_a\sigma^a$, $\bar{K} \equiv \bar{K}_a\sigma^a$ and $G \equiv \exp\left(\frac{2\pi i}{N}\sigma_3\right)$. The $\mathbb{Z}_N$ then acts on the currents as:

\[
\begin{align*}
K & \rightarrow GKG^{-1}; \quad \bar{K} \rightarrow \bar{K} \\
K^\pm & \rightarrow e^{\pm \frac{4\pi i}{N}k}\pm\, K^\pm, \quad K^3 \rightarrow K^3; \quad \bar{K}^a \rightarrow \bar{K}^a
\end{align*}
\]

Thus, for $N > 2$ the projection breaks $SU(2)_L \rightarrow U(1)$. One can think of the resulting asymmetric orbifold CFT in the following way. It is well known (see e.g. [22]) that one can decompose $SU(2)$ WZW CFT into a product of the $SU(2)/U(1)$ parafermion coset CFT and a free scalar field $\xi$, normalized as $\langle \xi(z)\xi(0) \rangle = -\frac{2}{k} \ln z$, which is related to $K^3$ via:

\[
K^3 = \frac{ik}{2} \partial \xi
\]

Since the eigenvalues of $K^3$ are half-integer, $\xi$ is a compact scalar field,

\[
\xi \sim \xi + 4\pi
\]

More precisely, $\xi$ has period $2\pi$, but wavefunctions with $j \in \mathbb{Z}+1/2$ go to minus themselves as $\xi \rightarrow \xi + 2\pi$. Under the above decomposition, the $SU(2)$ primaries can be written as:

\[
V_{j; m, \bar{m}}' = f_{j; m, \bar{m}}e^{im\xi}
\]

7 Whose OPE is given in Appendix A.
\( m, \bar{m} \) take values in the range \(|m|, |\bar{m}| \leq j \), with \( j - m, j - \bar{m} \in \mathbb{Z} \) and \( k \geq 2j \in \mathbb{Z}_+ \). The chiral parafermion fields \( f_{j;m,\bar{m}} \) commute with the current \( K^3 \) (but not with \( \bar{K}^3 \)) and have scaling dimensions

\[
\Delta = \frac{j(j+1)}{k+2} - \frac{m^2}{k} \quad \Delta = \frac{j(j+1)}{k+2}
\]  

(4.5)

The left-moving parts of the operators \( f_{j;m,\bar{m}} \) (4.4) with \(|m| = j \) are further identified, \((j, m) \simeq \left( \frac{k}{2} - j, m \pm \frac{k}{2} \right)\). The currents \( K^\pm \) take the form:

\[
K^\pm = e^{\pm i \xi} \psi^\pm_{\text{para}}
\]  

(4.6)

where \( \psi^\pm_{\text{para}} \) are parafermionic fields with scaling dimension \((k - 1)/k\). Comparing (4.6) to (4.4) we see that the \( \mathbb{Z}_N \) acts on the chiral scalar field \( \xi \) as \( \xi \rightarrow \xi + 4\pi/N \), decreasing the radius of \( \xi \) (4.3) by \( N \) units. Out of the original operators in the \( SU(2) \) WZW model (4.4) only those with \( 2m \in N\mathbb{Z} \) survive the \( \mathbb{Z}_N \) projection. These form the untwisted sector of the orbifold.

The twisted sector of the orbifold includes operators of the form

\[
V_{j;m,m',\bar{m}}' = f_{j;m,\bar{m}} e^{im'\xi}
\]  

(4.7)

with \( m \neq m' \) (compare to (4.4)). These must have the same periodicity in \( \xi \) as the untwisted ones: as \( \xi \rightarrow \xi + 2\pi \) they must be multiplied by \((-)^{2j} \). Therefore, they must satisfy:

\[
m - m' \in \mathbb{Z}
\]  

(4.8)

Modular invariance requires the inclusion in the theory of operators of the form (4.7) that satisfy level matching and are mutually local with respect to all operators in the untwisted sector (4.4) that survive the \( \mathbb{Z}_N \) projection. Requiring mutual locality of (4.4) (with \( 2m \in N\mathbb{Z} \)) and (4.7) leads to the constraint

\[
m - m' \in \frac{k}{N}\mathbb{Z}
\]  

(4.9)

Since completeness of the OPE requires us to include all operators (4.7) that satisfy (4.9), comparing to (4.8) we conclude that the asymmetric orbifold we are studying can only be consistent when \( \frac{k}{N} \equiv N' \in \mathbb{Z} \). Level matching of the operators (4.7) further leads to the requirement,

\[
\frac{m^2}{k} - \frac{m'^2}{k} \in \mathbb{Z}
\]  

(4.10)
which together with (4.9) means that \( m, m' \) satisfy the constraints:

\[
\begin{align*}
  m' + m & \in N \mathbb{Z} \\
  m' - m & \in N' \mathbb{Z}
\end{align*}
\]

(4.11)

Clearly, the constraint (4.11) must hold for all operators in the orbifold theory, since we can generate all such operators by multiplying operators of the form (4.7). As a check on (4.11) one can compute the resulting torus partition sum and check its modular invariance. This is done in Appendix B.

4.2. \( N = 1 \) SCFT on Lens spaces

The starting point of the construction is the \( N = 1 \) superconformal \( SU(2) \) WZW model (discussed around eq. (3.2)). The total left-moving \( SU(2) \) current \( K^a \) has level \( k \); the \( \mathbb{Z}_N \) orbifold acts on \( K^a \) as before (4.1). To preserve the worldsheet \( N = 1 \) superconformal algebra (3.3) the orbifold must also act on the left-moving fermions \( \chi^a \) in a similar way to (4.1):

\[
\chi^\pm \rightarrow e^{\pm \frac{4\pi i}{N}} \chi^\pm, \chi^3 \rightarrow \chi^3
\]

(4.12)

and, as before, the right-movers are invariant. To perform the orbifold (4.1), (4.12), we decompose the \( N = 1 \) superconformal \( SU(2) \) WZW model into \( SU(2)/U(1) \times U(1) \). The first factor is well known to have an accidental \( N = 2 \) superconformal symmetry; it gives rise to an \( N = 2 \) minimal model (see e.g. [27,28] for reviews). The second factor corresponds to a free superfield \( (\xi, \chi^3) \) where \( \xi \) is related to \( K^3 \) by (4.2). The analogs of the spectrum generating operators (4.7) in this case are:

\[
V'_{j;m,m',\bar{m}} = \frac{V_{j;m,\bar{m}}}{N} e^{im'\xi}
\]

(4.13)

where \( m, m' \) satisfy (4.11), and \( V_{j;m,\bar{m}}^{N=2} \) are minimal model primaries on the left and \( SU(2) \) primaries on the right. Focusing on the left-moving structure, there are two kinds of such operators: NS sector operators which are mutually local with respect to the supercurrent (3.3), and Ramond sector fields which have branch cuts with respect to \( G \). They have the spectrum of scaling dimensions (\( \Delta \)) and \( U(1) \) charges (\( Q \)):

\[
\begin{align*}
  \Delta_{NS} &= \frac{j(j + 1)}{k} - \frac{m^2}{k}; \quad Q_{NS} = \frac{2m}{k} \\
  \Delta_{R} &= \frac{j(j + 1)}{k} - \frac{m^2}{k} + \frac{1}{8}; \quad Q_{R} = \frac{2m}{k} + \frac{1}{2}
\end{align*}
\]

(4.14)
where the range of the indices \( j, m \) is as in (4.5) in the NS sector, while in the Ramond sector \(|m \mp \frac{1}{2}| \leq j\) (with the same identifications as before). The two kinds of Ramond operators in (4.14) are related by application of \( G^\pm_0 \), the zero modes of the \( N = 2 \) superconformal generators. One can again verify using the appropriate characters that the resulting theory is modular invariant. This is discussed in Appendix B.

5. String Theory on \( AdS_3 \times S^3/\mathbb{Z}_N \times T^4 \)

Armed with an understanding of string theory on \( AdS_3 \) (section 3), and CFT on \( S^3/\mathbb{Z}_N \) (section 4) we are now ready to tackle the problem of interest, string theory on the manifold (2.10). Some interesting aspects of the problem appear already in the bosonic case, which we therefore discuss first.

5.1. Bosonic string theory on \( AdS_3 \times S^3/\mathbb{Z}_N \times T^{20} \)

The right-movers are described by CFT on \( AdS_3 \times S^3 \times T^{20} \); in the left-moving sector we replace the \( S^3 \) by the monopole background described in section 4.1. To conform with the supersymmetric case, we take the level of \( S\tilde{L}(2) \) to be \( k + 2 \) and that of \( S\tilde{U}(2) \) to be \( k - 2 \), so that the total worldsheet central charge is \( k \) independent. The spacetime central charge (3.6) is in this case \( c_{st} = \bar{c}_{st} = 6(k+2)p \). The torus partition sum from which one can read-off the spectrum of physical states is (see (3.12) for the notation):

\[
Z(\tau, \bar{\tau}) = \frac{1}{\sqrt{2|\eta(\tau)|^2}} \sum_{j, \bar{j} \leq \frac{k}{2} - 1 \atop \vec{p}, \bar{\vec{p}} \in \Gamma_{20,20}} \mathcal{N}_{j, \bar{j}} \chi_j^{(k-2)}(\tau) \bar{\chi}_{\bar{j}}^{(k-2)}(\bar{\tau}) \frac{1}{|\eta(\tau)|^{40}} \sum_{(\vec{p}, \bar{\vec{p}}) \in \Gamma_{20,20}} q^{\frac{1}{2} \vec{p}^2} \bar{q}^{\frac{1}{2} \bar{\vec{p}}^2} \quad (5.1)
\]

where the left-moving characters \( \chi_j^{(k-2)} \) are the bosonic monopole characters \( \chi_j^{\text{monopole}} \) described in Appendix B, while the right-moving ones \( \bar{\chi}_{\bar{j}}^{(k-2)} \) are \( SU(2) \) WZW characters. Of course, the bosonic theory is tachyonic, but it is still useful for studying in a simpler setting aspects of the theory which survive in the supersymmetric examples.

One such aspect is the transformation of string excitations under the \( SO(3) \) rotation symmetry of the two-sphere discussed in section 2. The sum over \( j \) in the partition sum (5.1) runs over both half-integer and integer values; at the same time the theory only has bosonic excitations – all states contribute with positive sign to the vacuum energy (5.1). Naively this implies that this background violates the spin-statistics theorem.

To resolve this puzzle we need to recall some facts about the behavior of magnetic monopoles and dyons in quantum mechanics (see [29] for a discussion). Consider a quantum
mechanical particle with spin zero and electric charge $e$ in the background of a magnetic monopole with magnetic charge $g$. Dirac quantization is the statement that $eg \in \mathbb{Z}/2$. The system has a conserved angular momentum, but the allowed values that this angular momentum can take are:

$$j = eg, eg + 1, \cdots$$

(5.2)

Thus, the spin $j$ is bounded from below by $eg$ and can even take half-integer values (if $eg \in \mathbb{Z} + 1/2$). This is consistent with spin-statistics since, as explained in [29], dyons with electric charge $e$ and magnetic charge $g$ with half integer $eg$ indeed behave as fermions. In particular the wavefunction of the system is antisymmetric under interchange of two such objects.

This seems to explain half of the puzzle raised above. The fact that the partition sum (5.1) has contributions with half-integer $j$ despite the fact that we expect all excitations of a bosonic string to have integer spin must be due to the effect of the monopole field (5.2). In essence, some of the angular momentum $j$ is due to the electromagnetic field of the dyon; it is thus a property of the string vacuum and is not intrinsic to a particular excitation with charge $e$. In fact, we can use this interpretation to compute the electric charge of the different string excitations. Recall (2.11) that the magnetic charge of the vacuum is $P_5$. The angular momentum $j$ of string states is bounded by the quantum number $m$ (4.5), (4.7) by $j \geq m$. Comparing to (5.2) we see that the electric charge $e$ of states like (4.4), (4.7) is $e = m/P_5$.

There still seems to be some tension between the fact that dyons with half integer $eg$ are fermions, and the fact that excitations with half integer $j$ contribute with positive sign to the string partition sum (5.1). The resolution is that string theory on $AdS_3 \times S^3/\mathbb{Z}_N$ is not really quantizing these dyons. Instead it is quantizing bosonic particles in the background of a monopole with a given magnetic charge. Thus, the relevant statistics is not that associated with exchange of dyons, but rather the one that has to do with exchanging fluctuations of the bosonic fields that correspond to string modes in the fixed background of a monopole. The latter statistics is that of the basic quanta of the string field, which are bosonic. Hence, there is no contradiction between (5.1) and the spin-statistics theorem.
5.2. Heterotic string theory on $AdS_3 \times S^3 / \mathbb{Z}_N \times T^4$

We next turn to the heterotic theory. Our main task will be to reproduce and extend the supergravity result, Table 1, for the spectrum of chiral primaries.

The basic structure is very similar to that of section 3.3. The worldsheet right-movers are described by a superconformal $\sigma$-model on $AdS_3 \times S^3 \times T^4$ and give rise to $N = 4$ superconformal symmetry in spacetime. The left-movers are described by a bosonic worldsheet $\sigma$-model on $AdS_3 \times S^3 / \mathbb{Z}_N \times T^{20}$ and give rise to conformal symmetry in spacetime. The lattice of momenta corresponding to $T^4_R \times T^{20}_L$ is an even, self-dual Narain lattice $\Gamma^4_{20}$. The left and right moving spacetime central charges are again determined by the total levels of $SL(2)$ and are equal to: $c_{\text{st}} = 6p(k + 2)$, $\bar{c}_{\text{st}} = 6pk$. To determine the relation between $k$ and the number of branes in the background, we recall that according to section 4.1 the level of the bosonic $SU(2)$, $k - 2$, must be a product of two integers

$$k - 2 = NN'$$

in order for the asymmetric $\mathbb{Z}_N$ orbifold to be consistent. We readily identify $N$ with the number of KK monopoles, $N'$ with the number of NS5-branes (see (2.1) – (2.4)). Thus, the central charge is in this case

$$c_{\text{st}} = 6p(NN' + 4)$$
$$\bar{c}_{\text{st}} = 6p(NN' + 2)$$

Comparing to the Brown and Henneaux semiclassical gravity analysis (2.13) we see that this is another example of stringy corrections modifying the semiclassical formula (2.12).

We next turn to the spectrum of chiral primaries under the $(4,0)$ superconformal spacetime symmetry. The spacetime supersymmetry in this model arises purely from the right-movers. As in section 3, the most convenient way to analyze the spectrum of chiral operators is to construct it holomorphically and then combine the two worldsheet chiralities. Short representations of the right-moving $N = 4$ superconformal algebra are given in Table 2. What remains to do is to analyze the left-moving Virasoro primaries that can be tensored with Table 2 to give physical states. We will next do that for the case of the $A$-series modular invariant, $N_{j,\bar{j}} = \delta_{j,\bar{j}}$. The generalization to other modular invariants is straightforward.
Since we want to compare to Table 1, we start with chiral primaries arising from the untwisted sector \((4.4)\). The right-moving structure constrains \(j = j'\) (the bosonic \(SL(2)\) and \(SU(2)\) spins are the same). One finds the following physical operators:

\[
\begin{align*}
(J^A V_j)_{j \pm 1} V'_j, & \quad 2m \in N \mathbb{Z}; \quad h = j + 1 \pm 1 \\
V_j (K^a V'_j)_{j \pm 1, m}, & \quad 2m \in N \mathbb{Z}; \quad h = j + 1 \\
B^a V_j V'_j, & \quad 2m \in N \mathbb{Z}, \quad \alpha = 1, \ldots, n_V - 2; \quad h = j + 1
\end{align*}
\]

The operators with \(h = j\) in the first line of \((5.5)\) start only at \(j = 1/2\). The same is true for those with \(SU(2)\) spin \(j - 1\) in the second line. The rest of the towers of operators start at \(j = 0\). The notation in the last line of \((5.5)\) is as follows. \(B^a\) are purely left-moving worldsheet currents. At generic points in the Narain moduli space of \(\Gamma^{4,20}\) there are twenty such currents, corresponding to \(\partial X^i\), with \(i\) running over the directions of the \(T^4\), and the sixteen Cartan subalgebra generators of the heterotic gauge group in ten dimensions. Thus, generically, \(n_V = 22\). At points in the Narain moduli space where the gauge symmetry is enhanced to a non-abelian group \(G\), \(n_V\) increases accordingly. At such points the spacetime theory has a current algebra \(\hat{G}\) at level \(p\).

The holomorphic operators \((5.5)\) can be arranged into the following towers:

\[
\begin{array}{ccc}
\hline
h & \text{degeneracy} & \text{range of } j \\
\hline
j + 2 & 1 & 0, 1, \ldots \\
\hline
j & 1 & 1, 2, \ldots \\
\hline
j + 1 & n_V - 1 & 0, 1, \ldots \\
\hline
j + 1 & 1 & 1, 2, \ldots \\
\hline
\end{array}
\]

**Table 4: Operators of bosonic sector of heterotic string.**

The spectrum of chiral primaries of the heterotic string is now obtained by tensoring Table 4 with Table 2. Furthermore, to compare to supergravity we put the index \(m\) in Table 4 to zero, since the states with \(m \neq 0\) are not low energy states in the supergravity limit \(N, N' \to \infty\). This implies that the index \(j\) (which is the same for left and right-movers) is integer. It is not difficult to see that the resulting spectrum is precisely of the form indicated in Table 1 with \(n_H = 2(n_V - 1)\) and \(n_S = 2\).

Moduli of the spacetime CFT arise from chiral primaries with \(\bar{h} = 1/2, h = 1\). In Table 1 they arise from the \(n_H = 42\) hypermultiplets that exist generically in moduli.
space; these give rise to an 84-dimensional moduli space. In the string description the spacetime moduli are directly related to the worldsheet moduli $\partial x^a \bar{\partial} \bar{x}^\bar{b}$, where $a$ runs over the sixteen chiral scalars which exist in the heterotic string already in ten dimensions, the four scalars parametrizing the $T^4$, and $\xi$ defined in (4.2). The index $\bar{b}$ runs over the four antiholomorphic scalars parametrizing the right-movers on $T^4$. From the string point of view it is obvious that the moduli space of spacetime SCFT’s is

$$SO(4,21; \mathbb{Z}) \backslash SO(4,21)/SO(4) \times SO(21)$$

(5.6)

since this is the moduli space of worldsheet theories. Note that the moduli space (5.6) is consistent with the fact that, as explained in section 2, the radius of $x^4$, $R_4$ (2.9), is a fixed scalar in the near-horizon geometry (2.5).

While the string theory analysis reproduces the supergravity result of Table 1 in the region where the latter is expected to be applicable, it also generalizes it significantly. We have already seen an example of this above: the states visible in supergravity have $m = m' = 0$ in the notation of (4.7) (and thus integer $j$) while the full string spectrum has in addition states with $m = m' \in N^2\mathbb{Z}$, some of which have half-integer $j$ if $N$ is odd. Since T-duality in $x^4$ exchanges $N$ and $N'$ (see section 2), we also expect to find similar states in the twisted sector with $m = -m' \in N^2\mathbb{Z}$.

In fact, the twisted sectors have in this case a rich spectrum of stringy chiral operators. Consider, for example, vertex operators of the general form:

$$(\bar{\psi} V_j)_{j-1} V_{j;m,m',\bar{m}} P_n(\psi^A, J^A, \cdots)$$

(5.7)

In the right-moving sector this is an operator of the type $W^-_j$, as in Table 2; hence it is chiral under the right-moving spacetime $N = 4$ superconformal algebra. $V'$ is given by (4.7). The left-moving worldsheet scaling dimension $n$ of the polynomial $P_n$ is determined by level matching:

$$n - 1 = \frac{m^2 - m'^2}{NN'}$$

(5.8)

Recall that the r.h.s. of (5.8) is integer (4.10), since $m$, $m'$ must satisfy (4.11). For every solution of the level matching condition (5.8) one finds an operator in a short multiplet of the spacetime $(4,0)$ superconformal symmetry. For large $n$ the growth of the number of such solutions is exponential (in $\sqrt{n}$). These states are curved spacetime analogs of Dabholkar-Harvey states [30]. Clearly, there are other states of the general form (5.7);
their construction is a straightforward application of the methods of this paper and we will not discuss them in detail.

The string construction also allows one to study in a simple way the transformation properties of various operators under the full Virasoro algebra. The supergravity analysis that leads to Table 1 is only sensitive to the transformation properties of the different states under the global superconformal algebra. While on the supersymmetric side this is not a serious restriction – an operator with $h = j$ must be a primary under the full superconformal algebra since (at least in a unitary SCFT) it cannot be a descendant – on the left-moving bosonic side one could ask whether the states in Table 1 are primary under the full Virasoro algebra or only under its $SL(2)$ subgroup; the latter are known as quasi-primaries in $2d$ CFT.

This question can be answered in string theory by applying the Virasoro generators (3.4) to the vertex operators (5.5) and asking whether (3.13) is satisfied for all $n$ or only for $n = 0, \pm 1$. One finds that all operators are primary under the full Virasoro algebra except for those on the first line of (5.3) that involve $(J^AV_j)_{j+1}$; these are quasi-primary. This is not surprising given the fact that for $j = 0$ this combination appears in the Virasoro generator itself, and the stress tensor is a quasi-primary descendant of the identity in $2d$ CFT.

5.3. Type II string theory on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$

The main difference with respect to the analysis of the previous subsection is that the left-movers now live on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ and are described by a superconformal worldsheet theory. The spacetime central charge is in this case given exactly by the semiclassical results (2.13), $c_{st} = \bar{c}_{st} = 6pk = 6pNN'$.

In order to define the worldsheet theory we have to specify the GSO projection on the worldsheet. Performing a chiral projection, which acts separately on left and right-movers, $(-)^F_L = (-)^F_R = 1$, leads to spacetime supersymmetry, which is still coming only from the right-moving sector (in agreement with the brane picture of section 2). The fact that the left-moving supercharges (3.8) are projected out can be seen by noting that their values of $m, m' = \pm \frac{1}{2}$ are inconsistent with the projection (4.11) when $N, N' > 1$. To see the problem in more detail, trying to enforce mutual locality of (3.8) with the twisted NS sector states (4.13) leads to the condition $(m - m')/k \in \mathbb{Z}$ or, using (4.11), $1/N \in \mathbb{Z}$, which implies $N = 1$ ($N' = 1$ also has enhanced SUSY, arising from the twisted sector).
It is not difficult to repeat the analysis of chiral primaries performed in the previous subsection for the type II case. We start again with states that survive in the supergravity limit. Their holomorphic, left-moving structure is closely related to (3.18), (5.5). In the NS sector one has (in the $-1$ picture):

$$e^{-\phi}(\psi^AV_j)_{j\pm 1 V_{j;m}}, \ 2m \in N\mathbb{Z}; \ h = j + 1 \pm 1$$

$$e^{-\phi}V_j(\chi^AV'_{j})_{j\pm 1; m}, \ 2m \in N\mathbb{Z}; \ h = j + 1$$

$$e^{-\phi}\lambda^iV_jV'_{j;m}, \ 2m \in N\mathbb{Z}, \ i = 1, \cdots, 4; \ h = j + 1$$

where $V'$ is given by (4.13) with $m = m'$, and we are temporarily suppressing the right-moving index $\bar{m}$ (since the analysis is chiral). As in the previous subsection, one of the towers on the first line of (5.9) and one of the towers on the second line (the ones corresponding to spin $j-1$) start at $j = \frac{1}{2}$. The other towers exist for all $j \geq 0$ (bounded from above by the worldsheet unitarity bound). The Ramond sector states are (in the $-1/2$ picture):

$$e^{-\phi}e^{\pm \frac{i}{2}(H_4-H_5)}(SV_jV'_{j})_{j\pm \frac{1}{2} j\pm \frac{1}{2}; m}, \ 2m \in N\mathbb{Z}$$

(5.10)

Altogether, the quantum numbers and degeneracies of the left-moving parts of the chiral operators which survive in the supergravity ($m = 0$) limit are:

| $h$ | degeneracy | range of $j$ |
|-----|------------|--------------|
| $j + 2$ | 1 | 0, 1, ... |
| $j$ | 1 | 1, 2, ... |
| $j + 1$ | 5 | 0, 1, ... |
| $j + 1$ | 1 | 1, 2, ... |
| $j + 1/2$ | 4 | 1/2, 3/2, ... |
| $j + 3/2$ | 4 | 1/2, 3/2, ... |

Table 5: Left-moving primaries for $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$.

The physical spectrum of chiral primaries is obtained by tensoring Table 5 with Table 2. The result is precisely the one given in Table 1 with $n_V = n_H = 14$ and $n_S = 6$, in agreement with supergravity.

In section 2.5 we saw that the moduli space of vacua of string theory on $AdS_3 \times S^3/\mathbb{Z}_N \times T^4$ is the twenty eight dimensional coset (2.17). It is interesting to ask how
this space arises in the worldsheet construction. The twenty eight moduli are the upper components of chiral primaries with \( h = 1, \bar{h} = \frac{1}{2} \). Twenty of them arise from the NS sector. These are the obvious moduli of the worldsheet theory; they parametrize the coset (3.19). The remaining eight moduli come from the RR sector, and enlarge the moduli space from (3.19) to (2.17). As a check, note that \( SO(5, 4) \) is a subgroup of \( F_{4(4)} \) and \( SO(5) \times SO(4) \) is a subgroup of \( USp(2) \times USp(6) \).

The situation is in fact very similar to that in type II string theory on \( T^5 \times \mathbb{R}^{4,1} \), where the full moduli space is given by the coset (2.17) but only the subspace \( SO(5, 5; \mathbb{Z})/SO(5, 5)/SO(5) \times SO(5) \) comes from NS fields (the Narain moduli space) while the rest of the moduli come from the RR sector. \( SO(5, 5) \) is a subgroup of \( E_6(6) \) and \( SO(5) \times SO(5) \) a subgroup of \( USp(8) \).

Comments:
(a) As in the previous subsection, the string analysis produces a rich spectrum of stringy chiral operators in addition to those seen in supergravity.
(b) Both here and in the two previous subsections we have implicitly assumed that \( N, N' > 2 \). When either \( N \) or \( N' \) is two, the theory in fact has more symmetry, both on the worldsheet and in spacetime. When either \( N \) or \( N' \) is one, we get back the theory discussed in [7].

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Appendix A. Some Properties of String Theory on \( AdS_3 \times S^3 \times T^4 \)

In this Appendix we collect some conventions and results about CFT and string theory on \( AdS_3 \times S^3 \times T^4 \).

1. Worldsheet fermions: The fermions \( \psi^A, \chi^a, \lambda^i \) defined in the text are normalized as:

\[
\langle \psi^A(z) \psi^B(w) \rangle = \frac{k \eta^{AB}}{z - w}, \quad A, B = +, -, 3 \\
\langle \chi^a(z) \chi^b(w) \rangle = \frac{k \delta^{ab}}{z - w}, \quad a, b = +, -, 3 \\
\langle \lambda^i(z) \lambda^j(w) \rangle = \frac{\delta^{ij}}{z - w}, \quad i, j = 1, 2, 3, 4
\]  

(A.1)
The metrics are $\eta^{33} = -1, \eta^{+-} = 2; \delta^{33} = 1, \delta^{+-} = 2$.

To study the Ramond sector of the worldsheet theory one needs to construct the spin fields for $\psi^A, \chi^a, \lambda^j$. It is convenient to choose a complex structure by pairing the ten free fermions into five complex ones, and bosonizing them into canonically normalized scalar fields, $H_I$, (normal ordering implied):

$$
-i \partial H_1 = \frac{1}{k} \psi^+ \psi^-
$$
$$
-i \partial H_2 = \frac{1}{k} \chi^+ \chi^-
$$
$$
i \partial H_3 = \frac{2}{k} \psi^3 \chi^3
$$
$$
\partial H_4 = \lambda^1 \lambda^2
$$
$$
\partial H_5 = \lambda^3 \lambda^4
$$

2. Worldsheet current algebra: The currents (3.2) satisfy the OPE’s:

$$
J^A(z)J^B(w) = \frac{k \eta^{AB}}{2(z-w)^2} + \frac{i \epsilon^{ABc} J^C}{z-w} + \cdots, \quad A, B, C = +, -, 3
$$

$$
K^a(z)K^b(w) = \frac{k \delta^{ab}}{2(z-w)^2} + \frac{i \epsilon^{abc} K^c}{z-w} + \cdots; \quad a, b, c = +, -, 3
$$

where $i \epsilon^{+-3} = 2$ in both $SL(2)$ and $SU(2)$ (thus, lowering a 3 index gives a relative minus sign between the two).

As in the text, we denote primaries of the bosonic $\widetilde{SL}(2)$ by $V_{j;m,\bar{m}}$, and those of the bosonic $\widetilde{SU}(2)$ by $V'_{j';m,\bar{m}}$. In both cases $j$ is the quadratic Casimir, while $(m, \bar{m})$ are the eigenvalues of $(j^3, \bar{j}^3)$ or $(k^3, \bar{k}^3)$, as appropriate. The normalizations of the operators $V, V'$ are such that:

$$
j^\pm(z)V_{j;m,\bar{m}}(w) = (m \mp j)V_{j;m\pm1,\bar{m}}(w)/(z-w) + \cdots
$$
$$
j^3(z)V_{j;m,\bar{m}}(w) = mV_{j;m,\bar{m}}(w)/(z-w) + \cdots
$$

$$
k^\pm(z)V'_{j';m,\bar{m}}(w) = (j \mp m)V'_{j;m\pm1,\bar{m}}(w)/(z-w) + \cdots
$$
$$
k^3(z)V'_{j';m,\bar{m}}(w) = mV'_{j;m,\bar{m}}(w)/(z-w) + \cdots
$$

The scaling dimensions corresponding to $SL(2)_{k+2}$ and $SU(2)_{k-2}$ are:

$$
\Delta(V_j) = -\frac{j(j+1)}{k}; \quad \Delta(V'_{j'}) = \frac{j'(j'+1)}{k}
$$
The operators \( V_j \) are left-right symmetric; thus their left and right scaling dimensions are equal. For \( SU(2) \), there are non-diagonal modular invariants with \( j' \neq j' \).

The fermions \( \psi^A, \chi^a \) transform in the spin \((1,0)\) and \((0,1)\) representations of \( SL(2) \times SU(2) \). It is useful to decompose the products of fermions with bosonic primaries \( V, V' \) into representations of the total currents \((3.2)\). The relevant decompositions are:

\[
(\psi V_j)_{j+1;m,\bar{m}} = (j + 1 - m)(j + 1 + m)\psi^3 V_{j;m,\bar{m}} + \frac{1}{2}(j + m)(j + 1 + m)\psi^+ V_{j;m-1,\bar{m}} + \frac{1}{2}(j - m)(j + 1 - m)\psi^- V_{j;m+1,\bar{m}}
\]

\[
(\psi V_j)_{j;m,\bar{m}} = m\psi^3 V_{j;m,\bar{m}} - \frac{1}{2}(j + m)\psi^+ V_{j;m-1,\bar{m}} + \frac{1}{2}(j - m)\psi^- V_{j;m+1,\bar{m}}
\]

\[
(\psi V_j)_{-1;m,\bar{m}} = m\psi^3 V_{j;m,\bar{m}} - \frac{1}{2}\psi^+ V_{j;m-1,\bar{m}} - \frac{1}{2}\psi^- V_{j;m+1,\bar{m}}
\]

\[
(\chi V_j')_{j+1;m,\bar{m}} = (j + 1 - m)(j + 1 + m)\chi^3 V'_{j;m,\bar{m}} - \frac{1}{2}(j + m)(j + 1 + m)\chi^+ V'_{j;m-1,\bar{m}} + \frac{1}{2}(j - m)(j + 1 - m)\chi^- V'_{j;m+1,\bar{m}}
\]

\[
(\chi V_j')_{j;m,\bar{m}} = m\chi^3 V'_{j;m,\bar{m}} + \frac{1}{2}(j + m)\chi^+ V'_{j;m-1,\bar{m}} + \frac{1}{2}(j - m)\chi^- V'_{j;m+1,\bar{m}}
\]

\[
(\chi V_j')_{j-1;m,\bar{m}} = m\chi^3 V'_{j;m,\bar{m}} + \frac{1}{2}\chi^+ V'_{j;m-1,\bar{m}} - \frac{1}{2}\chi^- V'_{j;m+1,\bar{m}}
\]

(A.6)

Similarly, the eight spin fields \( S = \exp_2(\epsilon_1 H_1 + \epsilon_2 H_2 + \epsilon_3 H_3) \) associated with the fermions \( \psi^A, \chi^a \) transform as two copies (corresponding to \( \epsilon_1 \epsilon_2 \epsilon_3 = \pm 1 \)) of \((1/2,1/2)\) under \( SL(2) \times SU(2) \). The product \( SV_j V'_j \) can be decomposed into representations of the total currents \((3.2)\). This gives rise to the four representations \((j \pm 1/2, j' \pm 1/2)\) for each of the two chiralities of \( S \) above.

3. **Wakimoto representation of \( SL(2) \) CFT:** A convenient representation of CFT on \( AdS_3 \) is given by the Wakimoto representation \([19]\). The worldsheet Lagrangian is:

\[
\mathcal{L} = \partial \phi \partial \phi - \frac{2}{\alpha_+} \tilde{R}^{(2)} \phi + \beta \partial \gamma + \beta \tilde{\partial} \gamma - \beta \tilde{\beta} \exp \left( - \frac{2}{\alpha_+} \phi \right) \tag{A.7}
\]

where\(^8\) \( \alpha_+ = 2k \) is related to \( l \), the radius of curvature of \( AdS_3 \), via:

\[
l^2 = l_s^2 k. \tag{A.8}
\]

Integrating out \( \beta, \tilde{\beta} \) leads to the standard description of CFT on \( AdS_3 \). The description with \( \beta, \tilde{\beta} \) \((A.7)\) is convenient since it simplifies in the limit \( \phi \to \infty \):

\(^8\) Recall that the level of the bosonic \( SL(2) \) algebra is \( k + 2 \).
(a) The interaction term \( \beta \bar{\beta} \exp \left( -\frac{2\phi}{\alpha_+} \right) \) goes to zero; thus the worldsheet theory becomes weakly coupled.

(b) The linear dilaton term implies that the string coupling goes to zero exponentially in \( \phi \); thus the spacetime theory becomes weakly coupled.

In the free field region \( \phi \to \infty \) the propagators that follow from (A.7) are:

\[
\langle \phi(z)\phi(0) \rangle = -\log |z|^2, \quad \langle \beta(z)\gamma(0) \rangle = 1/z.
\]

The current algebra is represented by (normal ordering is implied):

\[
\begin{align*}
  j^3 &= \beta \gamma + \frac{\alpha_+}{2} \partial \phi \\
  j^+ &= \beta \gamma^2 + \alpha_+ \gamma \partial \phi + k \partial \gamma \\
  j^- &= \beta.
\end{align*}
\]

(B.9)

Bosonic primary vertex operators are given by:

\[
V_{j m \bar{m}} = \gamma^{j+m} \bar{\gamma}^{j+\bar{m}} \exp \left( \frac{2j}{\alpha_+} \phi \right).
\]

(A.10)

States (A.10) with \( j > -1/2 \) correspond to wavefunctions on AdS\(_3\) that are exponentially supported at \( \phi \to \infty \). Thus many of their properties, such as the transformation properties under the infinite symmetry algebras (3.4), (3.5) can be studied using the above weakly coupled representation.

### Appendix B. Modular properties of monopole characters

In this Appendix we discuss the modular transformation properties of the characters of the monopole CFT discussed in section 4. We start with the bosonic theory and then turn to the fermionic one.

**1. Bosonic case:** The \( SU(2)_k \) character in the representation with spin \( j \), \( \chi_j^{(k)}(\tau) \), is decomposed in terms of parafermion characters \( \Omega_{j m}^{(k)}(\tau) \) and \( U(1) \) characters \( L_{m}^{(k)}(\tau) \) as:

\[
\chi_j^{(k)}(\tau) = \frac{1}{2} \sum_{2m = -k+1}^{k} \Omega_{j m}^{(k)}(\tau)L_{m}^{(k)}(\tau)
\]

(B.1)

where it is understood that \( j - m \in \mathbb{Z} \), and we have extended the range of \( m \) for convenience to \( m = -k + 1, \cdots, k \). By using the identification

\[
\Omega_{j,m}^{(k)} = \Omega_{\frac{k}{2} - j, m + \frac{k}{2}}^{(k)}
\]

(B.2)

More precisely \( L_{m}^{(k)} = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{k(n+m/k)^2} \) is the character of an extended algebra that exists in \( c = 1 \) CFT for rational \( R^2/2 \). See [22] for a more detailed discussion.
one can always map \((j, m)\) to the fundamental range discussed around eq. (4.5). Thus each distinct state contributes twice to the r.h.s. of (B.1); hence the \(1/2\). The monopole characters obtained after performing the chiral \(\mathbb{Z}_N\) twist (4.1) can be written as

\[\chi_j^{\text{monopole}}(\tau) = \frac{1}{2} \sum_{m'} \sum_{m} \Omega_{jm}^{(k)}(\tau) L_{m}^{(k)}(\tau)\]  

(B.3)

where (4.11) \(m' + m \in N\mathbb{Z}, \ m' - m \in N'\mathbb{Z}\).

We wish to determine the modular properties of the monopole characters (B.3). Since we have imposed level matching, the transformation under \(\tau \to \tau + 1\) is the same as that of the corresponding \(SU(2)_k\) character. We next show that the same is true for the transformation \(\tau \to -1/\tau\).

The \(SU(2)_k\) characters satisfy \[\chi_j^{(k)}(-\frac{1}{\tau}) = \sqrt{\frac{2}{k + 2}} \sum_{j'=0}^j \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k + 2} \right] \chi_j^{(k)}(\tau)\]  

(B.4)

while for the \(U(1)\) characters:

\[L_m^{(k)}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2k}} \sum_{l = -k+1}^{k} e^{-\frac{4\pi i ml}{k}} L_l^{(k)}(\tau)\]  

(B.5)

These transformation properties, and the definition of the parafermionic characters (B.1) give

\[\Omega_{jm}^{(k)}(-\frac{1}{\tau}) = \frac{1}{\sqrt{k(k+2)}} \sum_{j'=0}^j \sum_{m' = -k+1}^{k} e^{\frac{4\pi i j'm}{k}} \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k + 2} \right] \Omega_{j'm}^{(k)}(\tau)\]  

(B.6)

With these formulae in hand it is straightforward to compute the transformation properties of the monopole characters from their definition (B.3). After performing the various sums we find

\[\chi_j^{\text{monopole}}(-\frac{1}{\tau}) = \sqrt{\frac{2}{k + 2}} \sum_{2j'=0}^k \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k + 2} \right] \chi_j^{\text{monopole}}(\tau)\]  

(B.7)

Comparing to (B.4) we see that the transformation properties of the monopole characters at a given level are identical to those of the \(SU(2)\) characters at the same level. Therefore, we can take any modular invariant theory with \(SU(2)_R \times SU(2)_L\) affine symmetry, and replace
the $SU(2)$ characters by monopole characters for one of the two worldsheet chiralities without spoiling modular invariance.

2. **Supersymmetric case:** In the supersymmetric case, characters depend on the spin structure and $(-)^F$ projection for the fermions. They can be distinguished by an index $s = 0, 1, 2, 3$; $s = 0, 2$ correspond to the Ramond sector with an insertion of $(-)^{sF/2}$. $s = 1, 3$ correspond to the NS sector with an insertion of $(-)^{(s-1)F/2}$. The monopole characters are

$$
\chi_{j,s}^{\text{monopole}}(\tau) = \frac{1}{2} \sum_{m, m'} \chi_{j,ms}^{N=2}(\tau) L_{m'}^k(\tau) \left( \frac{\Theta_s}{\eta} \right) \frac{1}{2} \tag{B.8}
$$

with the sum over $m, m'$ running over the same range (and with the same identifications) as in (B.3). $\chi_{j,ms}^{N=2}$ is the $N = 2$ minimal model character with the appropriate boundary conditions. $\Theta_s(\tau)$ are related to the standard Jacobi $\theta$-functions by relabeling of indices: $s = 0, 2$ correspond to $\theta_2, \theta_1$, while $s = 1, 3$ correspond to $\theta_3, \theta_4$. The last two terms on the r.h.s. of (B.8) are the contributions of $K^3$ and $\chi^3$. Again, level matching implies that (B.8) transforms under $\tau \to \tau + 1$ in the same way as the untwisted theory. The transformation under $\tau \to -1/\tau$ is discussed next.

In the untwisted $SU(2)$ WZW theory the characters are

$$
\chi_{j,s}^{SU(2)}(\tau) = \chi_j^{(k-2)}(\tau) \chi_s^{(2)}(\tau) \tag{B.9}
$$

where $\chi_j^{(k-2)}$ are the $SU(2)_{k-2}$ characters and

$$
\chi_s^{(2)}(\tau) = \left( \frac{\Theta_s}{\eta} \right)^{3/2} \tag{B.10}
$$

Using (B.4) and the transformation of the $\Theta$-functions

$$
\frac{\Theta_s}{\eta}(-\frac{1}{\tau}) = \sum_{s'} C_{ss'} \frac{\Theta_{s'}}{\eta}(\tau), \tag{B.11}
$$

where the matrix elements of the symmetric matrix $C$ are $C_{ss'} = 1$ for $(s, s') = (0, 3), (2, 2), (1, 1)$ and $C_{ss'} = 0$ otherwise, one finds for the untwisted theory

$$
\chi_{j,s}^{SU(2)}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2k}} \sum_{j', s'} C_{ss'} \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k} \right] \chi_{j',s'}^{SU(2)}(\tau). \tag{B.12}
$$

\[^{10}\text{In the literature on } N = 2 \text{ models (see e.g. \cite{27,28}) sectors labeled by } s \text{ have insertions of } 1 \pm (-)^F; \text{ they are obvious linear combinations of ours.}\]
Returning to the monopole characters (B.8), the $N = 2$ minimal model characters transform as

\[ \chi_{jms}^{N=2}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}k} \sum_{j'm's'} \mathcal{C}_{ss'} e^{\frac{4\pi im'm'}{k}} \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k} \right] \chi_{j'm's'}^{N=2}(\tau). \]  

(B.13)

Combining this with (B.5), (B.11) one finds that

\[ \chi_{js}^{\text{monopole}}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}k} \sum_{j's'} \mathcal{C}_{ss'} \sin \left[ \frac{\pi(2j + 1)(2j' + 1)}{k} \right] \chi_{j's'}^{\text{monopole}}(\tau) \]  

(B.14)

in agreement with the transformation of the untwisted characters (B.12).

Therefore, just like in the bosonic case, in any SCFT with an $\hat{SU}(2)$ symmetry we can replace the left-moving supersymmetric affine $SU(2)$ sector with a monopole SCFT without spoiling modular invariance.
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