Analytical approximation for the structure of differentially rotating barotropes

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ABSTRACT
Approximate analytical formula for density distribution in differentially rotating stars is derived. Any barotropic EOS and conservative rotation law can be handled with use of this method for wide range of differential rotation strength. Results are in good qualitative agreement with comparison to the other methods. Some applications are suggested and possible improvements of the formula are discussed.

Key words: stars: rotation – methods: analytical

1 INTRODUCTION
Theory of self-gravitating rotating bodies seems to be an unlimited reservoir of difficult problems hardly tractable even under severe simplifications. It is a subject of scientific effort since 1742 when Maclaurin has initiated this field by his studies on incompressible rotating ellipsoids. Development of modern numerical calculations resulted in progress in practical applications nowadays such as e.g. 3D hydrodynamical simulations of rotation of complex objects.

Analytical approach has succeeded for constant density, incompressible bodies. Work of Maclaurin, Jacobi, Poincare, Schwarzschild and many others has explained the behaviour of those objects almost completely. Behaviour of slowly rotating polytropes has been calculated by Chandrasekhar. By applying the differential equation of hydrostatical equilibrium modified by rotation he reduced the problem to an ordinary differential equation. This method however works only for a uniform rotation. This list would be incomplete without the Roche model. It’s simplicity makes it a very powerful tool for understanding behaviour of rotating objects. Present computational methods allow one to handled numerically two- and three-dimensional problems with complicated governing equations.

In this paper we present simple analytical approach which can treat differentially rotating compressible barotropic stars in case of slow or moderately fast rotation. This model could fill a gap between simple analytical methods used for e.g. Maclaurin spheroids or Roche model, and complicated numerical methods such as e.g. HSCF, or those applying straightforward Newton-Raphson technique.

2 FORMULATION OF THE PROBLEM
We attempt to find a density distribution (iso-density contours) of a single self-gravitating object under the following assumptions:

(i) Barotropic EOS \( p = p(\rho) \)
(ii) Simple rotation \( v = r \Omega \mathbf{e}_\phi \) with angular velocity dependent only on the distance from rotation axis \( \Omega = \Omega(r) \)
(iii) Newtonian gravity
(iv) Axisymmetric density distribution
(v) We seek solutions for stationary objects in full mechanical equilibrium, i.e. all quantities are time-independent

With properties (i)–(v) satisfied, the Euler equation becomes, in cylindrical coordinates \((r, z, \phi)\):

\[
r \Omega(r, z)^2 \mathbf{e}_r = \frac{1}{\rho} \nabla p + \nabla \Phi_g
\]

Continuity equation is then fulfilled automatically. Introducing centrifugal potential:

\[
\Phi_c(r) = -r \int_0^r \Omega(\tilde{r})^2 d\tilde{r}
\]

and enthalpy:

\[
h(\rho) = \int \frac{1}{\rho} dp
\]

we get a simple equation:

\[
\nabla [h(\rho) + \Phi_g + \Phi_c] = 0
\]

with a solution

\[
h(\rho) + \Phi_g + \Phi_c = C = \text{const}
\]

Equation \(\Phi_c\) is the most important equation in the study of the structure of rotating stars under conditions (i)–(v). We define the integration constant in eq. \(\Phi_c\) to be such that the enthalpy satisfies the condition \(h(\rho = 0) = 0\). The only term which we haven’t
specified yet is the gravitational potential $\Phi_g$. If we use the Poisson equation:

$$\Delta \Phi_g = 4 \pi G \rho$$

(6)

where $G$ is the gravitational constant, the equation becomes a non-linear second-order differential equation. This form, however, is very inconvenient, because we have to specify boundary conditions at a surface of the star, which is unknown a priori. More powerful is an integral form of eq. (5) obtained by substitution:

$$\Phi_g(r) = -G \int \frac{\rho(\bar{r})}{|\bar{r} - r|} d\bar{r}$$

(7)

This integral form has been used in very successful numerical algorithm developed originally by Ostriker & Mark (1968), and recently improved by Hachisu (1986) and by Eriguchi & Müller (1983). This form will be also used to derive our approximation formula in the next section.

3 APPROXIMATION FOR THE DENSITY DISTRIBUTION

The integral equation form of eq. (5) is:

$$h(\rho) + \mathcal{R}(\rho) + \Phi_c = C$$

(8)

where $\mathcal{R}$ is the integral operator acting on the density performing the integration on the right-hand side of eq. (7) over entire volume of the star. We define the surface of the star to be manifold consisting of points where $\rho = 0, h = 0$. Explicit form of the operator $\mathcal{R}$ in terms of coordinates will not be needed. For a given EOS and for a fixed rotation law i.e. for given functions $h(\rho)$ and $\Phi_c$, the only free parameter is the constant $C$. The values of $C$ label a family of the stellar models with the same EOS and rotation pattern, which differ in total mass and maximum density etc.

Eq. (3) has a form of the Hammerstein non-linear integral equation [Hammerstein 1930] and can be rewritten in a canonical form:

$$f = \mathcal{R}[F(f)]$$

(9)

where:

$$f = C - \Phi_c - h(\rho), \quad F(f) = h^{-1}(f + \Phi_c - C)$$

(10)

In case of linear function $F$, eq. (9) could be easily solved by the von Neumann series. This strongly suggests to try the following iteration scheme:

$$f_1 = \mathcal{R}[F(f_0)],$$

$$f_2 = \mathcal{R}[F(f_1)],$$

$$\cdots$$

(11)

$$f_n = \mathcal{R}[F(f_{n-1})]$$

$$\cdots$$

Indeed, an iteration procedure of this type was successfully applied in the so-called self-consistent field method [Ostriker & Mark 1968, Hachisu 1986]. We have introduced the canonical form to ensure that the first-order approximation is found in a correct order i.e. by using the first line of the sequence (11). When we go back to non-canonical form (8) the first line of (11) takes the form:

$$C - \Phi_c - h(\rho_1) = \mathcal{R}(\rho_0)$$

(12)

From eq. (12) above we can find the first-order deviation from sphericity. It seems impossible at first sight to avoid explicit integration in eq. (12). In case of a general $\rho_0$, this is true. But let us look at equation (5) in case of vanishing centrifugal potential $\Phi_c$, i.e. with no rotation:

$$h(\rho) + \mathcal{R}(\rho) = C$$

(13)

When we use a function which satisfies eq. (12) as zero-order approximation:

$$h(\rho_0) + \mathcal{R}(\rho_0) = C_0$$

(14)

integration in eq. (12) can be easily eliminated:

$$C - \Phi_c - h(\rho_1) = \mathcal{R}(\rho_0) = C_0 - h(\rho_0)$$

(15)

Finally, our formula takes the form:

$$h(\rho_1) = h(\rho_0) - \Phi_c + C - C_0$$

(16)

or simpler, using the enthalpy $h(\rho_0) \equiv h_0, h(\rho_1) \equiv h_1$:

$$h_1 = h_0 - \Phi_c + C - C_0$$

(17)

Functions used as zero-order approximation ($\rho_0$ or $h_0$) are simply density and enthalpy distributions of non-rotating barotropic stars. In case of polytropic EOS, $p = K \rho^\gamma$, these quantities are given by Lane-Emden functions. In more general case we have to find solution of the ordinary differential equation of the hydrostatic equilibrium.

The only unanswered question is what are the value of constants $C$ and $C_0$? It is an essential part of this work, so we decided to explain it in a separate section.

4 ADJUSTING CONSTANTS

When we try to find the enthalpy distribution using the formula (17) we have to find the best zero-order function $h_0$ and the value of $C_0$ given by $h_0$. An equivalent problem is to find the equation for which the function $h_0$ and the value $C_0$ are best zero-order approximations – in this case we seek for $C$. We consider the latter case, as we have to find only one real number. Let us denote:

$$\Delta C = C_0 - C$$

(18)

In our approximation, the equation for the first-order enthalpy distribution $h_1$, in terms of the initial spherical distribution $h_0$ and rotation law is, from (17):

$$h_1 = h_0 - \Phi_c - \Delta C$$

(19)

where $\Delta C$ is still to be determined.

The terms in eq. (19) behave as follows:

(i) $h_0$ is spherical enthalpy distribution, thus it is only a function of the radius, has a maximum at the centre, and goes monotonically to zero, where usually is cut. However, from mathematical point of view, Lane-Emden functions extend beyond the first zero point with negative function values.

3 For a given EOS the function $\rho_0$ gives $C_0$ and vice versa.
When we substitute the formula (19) into our basic equation (5) we get:
\[ h_0 - \Phi_c + C - C_0 + \Phi_g(h_1) + \Phi_c = C \]
(20)
In this formula we have made use of (18). After obvious simplifications, using (14) and denoting \( R(\rho_0) = \Phi_g(h_0) \) we have:
\[ \Phi_g(h_1) = \Phi_g(h_0) \]
(21)
This equality is true only if \( \rho_0 = \rho_1 \). The same holds for the enthalpy:
\[ h_1 = h_0. \]
(22)
Using formula (19) again we finally obtain:
\[ \Delta C = -\Phi_c \]
(23)
Left-hand side of eq. (24) is constant, while the right-hand side is a function of distance from the rotation axis, monotonically decreasing from zero. This equality holds only in trivial case \( \Delta C = 0 \) and \( \Phi_c = 0 \) with no rotation at all. In any other case (24) cannot be fulfilled. So instead we try another possibility and require that
\[ \Delta C = -\widehat{\Phi}_c \]
(24)
where ‘hat’ denotes some mean value of the function \( \Phi_c \). We have chosen
\[ \Delta C = -\widehat{\Phi}_c = \left( \Phi_c \right)_V \]
(25)
Integration is taken over the entire volume \( V_0 = \frac{4}{3}\pi R_0^3 \) of a non-rotating initial star with the radius \( R_0 \). This choice of \( \Delta C \) gives good results. But using the mean value theorem:
\[ \int_{V_0} \Phi_c \, dy = V_0 \, \widehat{\Phi}_c, \]
(26)
where \( \widehat{\Phi}_c \) is some value of \( \Phi_c \) in the integration area, and taking in account monotonicity of the centrifugal potential \( 0 < -\Phi_c < -\Phi_c(R_0) \) we get:
\[ \Delta C = -\widehat{\Phi}_c < -\Phi_c(R_0) \]
(27)
i.e. the value of \( \Delta C \) is in the range \( \Delta C < C^{(3)} \) from Fig. 1. It forces us to use negative values of non-rotating enthalpy. Moreover, in case of polytropic EOS with fractional polytropic index Lane-Emden equation (Kippenhahn & Weigert 1994):
\[ \frac{1}{z^2} \frac{d}{dz} \left( z^2 \frac{dw}{dz} \right) + w^n = 0 \]
(28)
has no real negative values, because of fractional power of negative term \( w^n \). But we can easily write equation, with solution identically equal to solution of Lane-Emden equation for \( w > 0 \), and real solution for \( w < 0 \) e.g.:
\[ \frac{1}{z^2} \frac{d}{dz} \left( z^2 \frac{dw}{dz} \right) + |w|^n = 0 \]
(29)
But, for example, solution of the following equation:
\[ \frac{1}{z^2} \frac{d}{dz} \left( z^2 \frac{dw}{dz} \right) + \left| \frac{w}{w} \right|^{n+1} = 0 \]
(30)
is again identically equal to solution of Lane-Emden equation for

\[4\] Physically interesting cases like degenerate electron gas in non-relativistic case has fractional polytropic index \( n = 3/2 \).
$w > 0$, but differs from eq. (29) for $w < 0$. Fortunately, difference between solution of eq. (29) (Fig. 1 below axis, dot-dashed) and eq. (30) (Fig. 1 solid) for $w < 0$ is small if $|w| \ll 1$. Example from Fig. 1 (for $n = 1$) is representative for other values of $n$. We will use form (29) instead of the original Lane-Emden equation (28) for calculations in this article.

To avoid problems with negative enthalpy we can put simply:

$$\Delta C = -\Phi_c (R_0)$$

(31)

which is strictly boundary value $C^{(1)}$ from Fig. 1. The great advantage of the eq. (31) is the possibility to analytically perform the integration of the centrifugal potential (34) for most often used forms of $\Omega(r)$. In contrast, in formula (25), not only angular velocity profile (3) but also the centrifugal potential have to be analytically integrable function. In both cases (25) (31) however, possibility of analytical integration depends on the form of $\Omega(r)$. The value of $\Delta C$ from eq. (31) also gives reasonable iso-density contours, cf. Fig 11 and 5 but global accuracy is poor (Table 1).

As we noticed, the best value of $\Delta C$ in formula (19) could be found numerically. For example, we can use virial theorem formula for rotating stars (cf. Tassoul 2000):

$$2 E_h - |E_g| + 3 \int p d^3 r = 0,$$

(32)

where $E_h$ and $E_g$ is the rotational kinetic energy and the gravitational energy, respectively. We define, so-called virial test parameter $Z$:

$$Z = \frac{2 E_h - |E_g| + 3(\gamma - 1)U}{|E_g|}$$

(33)

where we introduced internal energy:

$$U = \frac{1}{\gamma - 1} \int p d^3 r$$

(34)

Parameter $Z$ is very common test of the global accuracy for rotating stars models. We may request that our enthalpy satisfy (32), i.e. we choose $\Delta C$ from equation:

$$Z (h_0 - \Phi_c - \Delta C) = 0$$

(35)

We can find $\Delta C$ from equation eq. (35) numerically only.

As it is shown in Fig 2 we can find approximation of the rotating polytrope structure in form (19) satisfying virial theorem (24) up to accuracy limited only by numerical precision.

Values of $\Delta C$ obtained with (31), (35) and from virial test (35) are compared on Fig 4. Some of the global model properties are very sensitive to value of $\Delta C$ (cf. Figs 2 and 5).

Because virial test is unable to check accuracy of our model, we may also try to compare directly eq. (19) with enthalpy distribution from the numerical calculations $h_{num}$ of e.g. Hachisu (1986), Eriszuchi & Muller (1983) and find $\Delta C$ minimizing e.g. the following formula:

$$\int [h_{l}(\Delta C) - h_{num}]^2 d^3 r = \min$$

(36)

This method however, requires numerical results (e.g. enthalpy distribution) in machine-readable form.

5 APPROXIMATE FORMULA ACCURACY

In the above sections, we tried to be as general as possible. Now we give some examples, and test accuracy of approximation.

In case of polytropic EOS $p = K \rho^n$ the enthalpy is:

$$h(\rho) = \frac{K \gamma}{\gamma - 1} \rho^{\gamma - 1}$$

(37)

Zero-order approximation of density (density of non-rotating polytrope, Kippenhahn & Weigert (1992) with n-th $^5$ Lane-Emden function $\omega_n$ is:

$$\rho_0 = \rho_c [w_n(\Omega)]^n, \quad A^2 = \frac{4\pi G}{nK\gamma} \frac{n+1}{\rho_c^n}$$

(38)

and our formula for density becomes:

$$\rho_1 = \left[\rho_c^{1/n} w_n - \frac{1}{nK\gamma} (\Phi_c + \Delta C)\right]^n$$

(39)

where $\Delta C$ is calculated from (25), (31) or (5).

In certain cases Lane-Emden functions are elementary functions as e.g. $w_1$. In cases like this our formula may be expressed even by elementary functions. For example, for $n = 1, K = 1/2, 4\pi G = 1, \rho_c = 1, \Omega(r) = \Omega_0/(1 + r^2/A^2)$ and $\Delta C$ from eq. (31) we get a simple formula:

$$\rho_1(r, z) = \frac{\sin \sqrt{r^2 + z^2}}{\sqrt{r^2 + z^2}} + \frac{1}{2} \frac{\Omega_0^2 A^2 r^2}{1 + r^2/A^2} + \frac{1}{2} \frac{\Omega_0^2 A^2 \pi^2}{1 + \pi^2/A^2}$$

(40)

Functions like this can easily be visualized on a 2D plot. Figure 1 has been made from the formula (40) while figures 3 and 11 from eq. (41).

Now we concentrate on $n = 3/2$ polytrope. In our calculations and figures we will use $4\pi G = 1, \rho_c = 1$ and $K = 2/5$.

Now formula (29) becomes:

$$\rho_1 = (w_n - \Phi_c - \Delta C)^{3/2}$$

(41)

Iso-density contours of $\rho_1$ from (41) are presented on Fig. 11 and Fig. 8.

To test accuracy of approximation we have calculated axis ratio, total energy, kinetic to gravitational energy ratio, and dimensionless angular momentum. Axis ratio is defined as usual as:

$$\text{Axis Ratio} = \frac{R_e}{R_{eq}}$$

(42)

$5 \gamma = 1 + \frac{1}{n}$

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where $R_0$ is distance from centre to pole and $R_{eq}$ is equatorial radius. Total energy $E_{tot}$:

$$E_{tot} = (E_k + E_g + U)/E_0$$

is normalized by:

$$E_0 = (4\pi G)^2 M^3 / J^2$$

and dimensionless angular momentum is defined as:

$$j^2 = \frac{1}{4\pi GM^{10/3}} \rho_{max}^{1/3}$$

where $M$ and $J$ are total mass and angular momentum, respectively; $\rho_{max}$ is maximum density. Quantities $E_k$ and $E_g$ are computed numerically from eq. (41) with $j$-const angular velocity $\Omega(r)$ and chosen $\Delta C$.

### 5.1 Influence of $\Delta C$

We have made detailed comparison of our $n = 3/2$ model with $j$-const rotation law and $A = 0.2R_0$ (middle row of Fig. 8) for different values of $\Delta C$ with results of Eriguchi & Müller (1985, Table 1b). Table I show our results for $\Delta C$ from eq. (45). Value of $\Delta C$ from eq. (25) and corresponding virial test parameter $Z$ is included here for comparison. Table II shows global properties of our approximation with $\Delta C$ equal to the solution of eq. (35), i.e. satisfying virial theorem.

Direct comparison of values from Table I and Table II to Table 1b of Eriguchi & Müller (1985) may be difficult, because our driving parameter is central angular velocity $\Omega_0$, while Eriguchi & Müller (1985), following successful approach of Hachisu (1986), use axis ratio $c$. More convenient in this case is comparison of figures prepared from data found in Table 1b of Eriguchi & Müller (1985) and our tables. This is especially true, because axis ratio isn’t well predicted by our formula (cf. Fig. 5 and Fig. 7), while global properties ($E_{tot}$, $j^2$, $E_k/E_g$), cf. Fig. 4 and virial test $Z$ are in good agreement if $E_k/E_g \ll 0.1$.

Fig. 4 shows that our approximation is valid until $E_k/E_g \simeq 0.05$, and begins to diverge from numerical results strongly for $E_k/E_g \geq 0.1$. Both values of $\Delta C$ give similar behavior here. However, $\Delta C$ from virial test produces better results, and $E_k/E_g$ values are more sensible for the strongest rotation.

In contrast, total energy $E_{tot}$ is very sensitive to $\Delta C$. Value of $\Delta C$ from eq. (25) produces wrong result. $E_{tot}$ begin to increase for $j^2 \geq 0.007$, while numerical results give monotonically decreasing $E_{tot}$. Use of $\Delta C$ from (35) instead, gives correct result, cf. Fig. 5.

While global properties of our model are in good agreement with numerical results for $E_k/E_g \ll 0.1$, axis ratio tends to be
Table 1. Properties of \( n = 3/2 \) polytropic model \(^{11}\) with \( j \)-const rotation law and \( A = 0.2R_0 \). \( \Delta C \) is calculated from eq. \(^{25}\) and \( \Delta \tilde{C} \) from eq. \(^{31}\). Virial test in the latter case is labeled by \( \tilde{Z} \). By little change from \( \Delta \tilde{C} \) to \( \Delta C \) one can notice significant improvement of virial test. In both cases virial test suggest strong deviation from equilibrium, especially for strong rotation. Actually, virial test is very sensitive to \( \Delta C \), cf. Fig. \(^{8}\). We can require virial theorem to be satisfied, by use of another value for \( \Delta C \), solution of eq. \(^{35}\). Results can be improved significantly this way – compare with Table \(^{2}\) and Fig. \(^{5}\).

| \( \Omega_0 \) | Axis Ratio | \( j^2 \) | \( \frac{E_k}{|E_0|} \) | \( \frac{E_\theta+E_k+U}{E_0} \) | Virial test \( \tilde{Z} \) | \( \Delta C \) | \( \Delta \tilde{C} \) |
|-------------|------------|-----------|----------------|-----------------|----------------|--------|--------|
| 0.25        | 1.01       | \( 8.41 \times 10^{-5} \) | 0.004 | \(-2.50 \times 10^{-5}\) | 0.01 | 0.01 | 0.01 |
| 0.50        | 1.05       | \( 3.38 \times 10^{-4} \) | 0.02  | \(-1.00 \times 10^{-5}\) | 0.03 | 0.04 | 0.03 |
| 0.75        | 1.15       | \( 7.64 \times 10^{-4} \) | 0.04  | \(-2.20 \times 10^{-5}\) | 0.07 | 0.10 | 0.07 | 0.08 |
| 1.00        | 1.19       | \( 1.37 \times 10^{-3} \) | 0.07  | \(-3.80 \times 10^{-5}\) | 0.12 | 0.17 | 0.12 | 0.14 |
| 1.25        | 1.31       | \( 2.17 \times 10^{-3} \) | 0.10  | \(-5.56 \times 10^{-5}\) | 0.19 | 0.28 | 0.19 | 0.21 |
| 1.50        | 1.46       | \( 3.12 \times 10^{-3} \) | 0.15  | \(-7.38 \times 10^{-5}\) | 0.27 | 0.43 | 0.27 | 0.31 |
| 1.75        | 1.68       | \( 4.40 \times 10^{-3} \) | 0.20  | \(-8.93 \times 10^{-5}\) | 0.36 | 0.63 | 0.37 | 0.42 |
| 2.00        | 1.98       | \( 5.90 \times 10^{-3} \) | 0.26  | \(-9.89 \times 10^{-5}\) | 0.47 | 0.89 | 0.48 | 0.55 |
| 2.25        | 2.45       | \( 7.55 \times 10^{-3} \) | 0.32  | \(-1.01 \times 10^{-4}\) | 0.57 | 1.22 | 0.60 | 0.69 |
| 2.50        | 3.30       | \( 9.30 \times 10^{-3} \) | 0.38  | \(-9.41 \times 10^{-4}\) | 0.67 | 1.66 | 0.75 | 0.86 |
| 2.75        | 5.80       | \( 1.10 \times 10^{-2} \) | 0.44  | \(-8.81 \times 10^{-4}\) | 0.75 | 2.22 | 0.90 | 1.04 |

Figure 6. Axis ratio versus \( E_k/|E_0| \). We see that our formula (+, ◁) underestimates axis ratio. Choose of \( \Delta C \) satisfying virial theorem (+) improves situation a bit. Solid line again is result of \(^{Eriguchi & Müller 1985}\).

This subsection clearly show importance of constant value \( \Delta C \). Best results are produced with \( \Delta C \) from eq. \(^{35}\), therefore this value will be used in the next subsections to investigate influence of differential rotation parameter \( A \) and type of rotation law on formula accuracy.

5.2 Effects of differential rotation

In addition to the results from previous subsection (\( j \)-const with \( A = 0.2R_0 \)) we have calculated properties of the almost rigidly rotating (\( A = 2R_0 \)) and extremely differentially (\( A = 0.02R_0 \)) rotating model with the same rotation law.

In all three cases we are able to find value of \( \Delta C \) satisfying eq. \(^{35}\). However, this is not enough to find correct solution, because other parameters describing rotating body may be wrong. This is clearly shown on Fig. \(^{9}\) where \( E_k/|E_0| \) versus \( j^2 \) \(^{35}\) is plotted for three cases of differential rotation. Apparent discrepancy for \( A = 2R_0 \) exists. Both \( j \)-const and \( v \)-const angular velocity profiles behaves as rigid rotation in this case. Thus we conclude that our formula is unable to predict correct structure in case of uniform rotation even if rotation is small.

If rotation is concentrated near rotation axis, like in \( A = 0.02R_0 \) case, our and numerical results are of the same order of magnitude. Quantitative agreement is achieved only for very small values of \( \Omega_0 \). Let’s note that in this case \( \Delta C \) required by virial theorem \(^{35}\) is slightly below zero (Table \(^{5}\)). This example shows, that \( \Delta C \) may also be negative. All three cases are summarized on Fig. \(^{9}\).

Results from this section show, that our formula is able to find correct structure of rotating body for differential rotation only. Range of application vary with differential rotation parameters, and best results are obtained in middle range i.e. \( A = 0.2R_0 \). With extremal case (\( A = 0.02R_0 \)) quality of our results is significantly degraded.

In next subsection we examine, if this statement depends on rotation law.
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$\Omega_0$  Axis Ratio  $j^2$  $\frac{E_0}{|E_g|}$  $\frac{E_k + E_g + U}{E_0}$  Virial test $|Z|$  $\Delta C$

| $\Omega_0$ | Axis Ratio | $j^2$ | $\frac{E_0}{|E_g|}$ | $\frac{E_k + E_g + U}{E_0}$ | Virial test $|Z|$ | $\Delta C$ |
|----------|------------|------|----------------|------------------|----------------|--------|
| 0.25     | 1.01       | $8.30 \times 10^{-5}$ | 0.004 | $-2.05 \times 10^{-6}$ | $9 \times 10^{-5}$ | 0.004   |
| 0.50     | 1.05       | $3.22 \times 10^{-4}$ | 0.02  | $-0.98 \times 10^{-5}$ | $3 \times 10^{-4}$ | 0.02    |
| 0.75     | 1.11       | $6.86 \times 10^{-4}$ | 0.03  | $-2.10 \times 10^{-5}$ | $3 \times 10^{-5}$ | 0.04    |
| 1.00     | 1.20       | $1.14 \times 10^{-3}$ | 0.06  | $-3.50 \times 10^{-5}$ | $4 \times 10^{-5}$ | 0.08    |
| 1.25     | 1.33       | $1.65 \times 10^{-3}$ | 0.08  | $-5.14 \times 10^{-5}$ | $6 \times 10^{-4}$ | 0.13    |
| 1.50     | 1.51       | $2.20 \times 10^{-3}$ | 0.11  | $-6.84 \times 10^{-5}$ | $4 \times 10^{-4}$ | 0.19    |
| 1.75     | 1.74       | $2.77 \times 10^{-3}$ | 0.13  | $-8.55 \times 10^{-5}$ | $3 \times 10^{-5}$ | 0.27    |
| 2.00     | 2.04       | $3.32 \times 10^{-3}$ | 0.16  | $-1.00 \times 10^{-4}$ | $1 \times 10^{-3}$ | 0.37    |
| 2.25     | 2.47       | $3.89 \times 10^{-3}$ | 0.18  | $-1.18 \times 10^{-4}$ | $9 \times 10^{-4}$ | 0.48    |
| 2.50     | 3.12       | $4.40 \times 10^{-3}$ | 0.20  | $-1.33 \times 10^{-4}$ | $3 \times 10^{-4}$ | 0.62    |
| 2.75     | 4.34       | $4.93 \times 10^{-2}$ | 0.22  | $-1.50 \times 10^{-4}$ | $1 \times 10^{-6}$ | 0.77    |

Table 2. The same model as in Table 1, but now $\Delta C$ is derived numerically from eq. (35). Virial test show accuracy of solution to eq. (35). Comparison with Table 1 and Table 1b of Eriguchi & Müller (1985) shows significant improvement of the total energy. Axis ratio is also closer to results of numerical calculations, and stability indicator $E_k/|E_g|$ isn’t unreasonably high. See also Figs 4–7.

Figure 8. Examples of the density distributions given by our formula (41). Results for polytropic model with $\gamma = 5/3$ and so-called $j$-const [Eriguchi & Müller, 1985] rotation law $\Omega(r) = \Omega_0/(1 + r^2/A^2)$ are presented. Upper row corresponds to differential rotation with $A = 0.02R_0$. The value of $\Omega_0$ increases from left to the right. Lower row shows behaviour of the almost rigidly rotating star with $A = 2R_0$. For the middle row $A = 0.2R_0$ is adopted. $R_0$ is the radius of non-rotating star. Values of the $\Omega_0$ are, from left: upper row 75, 150, 200, 250; middle row 0.5, 1.0, 1.5, 2.0; bottom row 0.01, 0.02, 0.03, 0.035. Constant $\Delta C$ is calculated from eq. (31).
In addition to previously described cases, we have calculated global properties of our model in case of \( j \)-const angular velocity profile, with parameter \( A = 0.2R_0 \) (Table 5) and \( A = 0.02R_0 \) (Table 6). Results with \( A = 2R_0 \) aren’t presented, because they are similar to \( j \)-const case (cf. Table 4), where both functions \( \Omega(r) \) behave as uniform rotation, and our formula fails in this case.

Figures 10 and 11 show very good agreement of of the global physical quantities \( (E_k/E_g, j^2, E_{tot}) \) with numerical results for entire range of rotation strength covered by both methods. The most extreme case \( (A = 0.02) \) also behaves well. Axis ratio (Fig. 13) however, clearly distinguish between approximation and precise solution. Results are quantitatively correct only for small rotation parameters, e.g. \( j^2 \ll 0.005 \) i.e. \( E_k/E_g \ll 0.1 \).

**5.3 Rotation law effects**

In addition to previously described cases, we have calculated global properties of our model in case of \( v \)-const angular velocity profile, with parameter \( A = 0.2R_0 \). Results with \( A = 2R_0 \) aren’t presented, because they are similar to \( j \)-const case (cf. Table 4), where both functions \( \Omega(r) \) behave as uniform rotation, and our formula fails in this case.

Figures 10 and 12 show very good agreement of of the global physical quantities \( (E_k/E_g, j^2, E_{tot}) \) with numerical results for entire range of rotation strength covered by both methods. The most extreme case \( (A = 0.02) \) also behaves well. Axis ratio (Fig. 13) however, clearly distinguish between approximation and precise solution. Results are quantitatively correct only for small rotation parameters, e.g. \( j^2 \ll 0.005 \) i.e. \( E_k/E_g \ll 0.1 \).

**6 DISCUSSION & CONCLUSIONS**

Comparison of the results obtained with our approximation formula (Fig. 11 – Fig. 13) Table 2 & 3 with other \cite{1985} Fig. 2–5, Fig. 9, Table 1 and 2) shows a correct qualitative
Analytical approximation for rotating barotropes

Ω

Axis Ratio

\( j^2 \)

\( \frac{E_k}{E_0} \)

\( \frac{E_g + E_k + U}{E_0} \)

Virial test \(|Z|\)

\( \Delta C \)

| \( \Omega_0 \) | \( \text{Axis Ratio} \) | \( j^2 \) | \( \frac{E_k}{E_0} \) | \( \frac{E_g + E_k + U}{E_0} \) | \( |Z| \) | \( \Delta C \) |
|---|---|---|---|---|---|---|
| 0.25 | 1.05 | \( 3.29 \times 10^{-4} \) | 0.01 | \(-0.99 \times 10^{-5}\) | 9 \times 10^{-5} | 0.01 |
| 0.50 | 1.21 | \( 1.41 \times 10^{-3} \) | 0.05 | \(-4.26 \times 10^{-5}\) | 2 \times 10^{-4} | 0.05 |
| 0.75 | 1.63 | \( 3.80 \times 10^{-3} \) | 0.11 | \(-1.12 \times 10^{-4}\) | 2 \times 10^{-4} | 0.13 |
| 1.00 | 2.31 | \( 9.22 \times 10^{-3} \) | 0.21 | \(-2.52 \times 10^{-4}\) | 3 \times 10^{-4} | 0.27 |
| 1.25 | 5.26 | \( 1.44 \times 10^{-2} \) | 0.31 | \(-3.87 \times 10^{-4}\) | 5 \times 10^{-4} | 0.47 |

Table 5. Properties of our model with \( v \)-const rotation law and \( A = 0.2R_0 \).

Figure 11. Another example of density distributions given by our formula [41]. The same as in Fig. [8] for so-called \( v \)-const rotation law [Eriguchi & Müller 1985] \( \Omega(r) = \Omega_0/(1 + r/A) \). Values of \( \Omega_0 \) are, from left: in the upper \((A = 0.02R_0)\) row: 1.0, 3.0, 5.0, 7.0; in the bottom \((A = 0.2R_0)\) row: 0.5, 0.75, 1.0 and 1.25.

Figure 12. \( E_{\text{tot}} \) versus square of dimensionless angular momentum \( j^2 \). Symbols description is the same as on previous figure, Fig. [10].

Figure 13. Axis ratio versus \( j^2 \) for \( v \)-const rotation law. Symbols description is the same as on previous figures, Fig. [10] and Fig. [12].

behaviour for even the most simplified version of our approximation formula for a wide range of parameters describing differential rotation and strength of rotation. This make our formula excellent tool for those who are interested in the structure of barotropic, differentially rotating stars, but do not need exact, high precision results. It can be applied for qualitative analysis of structure of rapidly rotating stellar cores (e.g. ‘cusp’ formation, degree of flattening, off-centre maximum density) with arbitrary rotation.

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Table 6. Properties of $v$-const sequence for $A = 0.02 R_0$.

| $\Omega_0$ | Axis Ratio | $j^2$ | $E_k / |E_g|$ | $E_k + E_g + U / \Omega_0$ | Virial test $|Z|$ | $\Delta C$ |
|------------|------------|------|--------------|--------------------------|----------------|--------|
| 1.0        | 1.02       | $9.81 \times 10^{-5}$ | 0.004 | $-0.30 \times 10^{-5}$ | $7 \times 10^{-5}$ | 0.01   |
| 2.0        | 1.09       | $3.94 \times 10^{-4}$ | 0.015 | $-1.20 \times 10^{-5}$ | $1 \times 10^{-4}$ | 0.04   |
| 3.0        | 1.21       | $8.96 \times 10^{-4}$ | 0.03  | $-2.75 \times 10^{-5}$ | $7 \times 10^{-4}$ | 0.09   |
| 4.0        | 1.41       | $1.63 \times 10^{-3}$ | 0.06  | $-5.00 \times 10^{-5}$ | $5 \times 10^{-4}$ | 0.17   |
| 5.0        | 1.73       | $2.65 \times 10^{-3}$ | 0.09  | $-8.09 \times 10^{-5}$ | $1 \times 10^{-4}$ | 0.27   |
| 6.0        | 2.24       | $4.08 \times 10^{-3}$ | 0.13  | $-1.23 \times 10^{-4}$ | $3 \times 10^{-4}$ | 0.39   |
| 7.0        | 3.03       | $6.05 \times 10^{-3}$ | 0.17  | $-1.77 \times 10^{-4}$ | $2 \times 10^{-4}$ | 0.55   |
| 8.0        | 4.60       | $8.46 \times 10^{-3}$ | 0.22  | $-2.38 \times 10^{-4}$ | $3 \times 10^{-4}$ | 0.75   |

law, also for initial guess for numerical algorithms. It can also be used as an alternative for high-quality numerical results for use in a more convenient form as long as $E_k / |E_g| \ll 0.1$ and we are interested mainly in global properties of differentially rotating objects.

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