Coherent states for rational extensions and ladder operators related to infinite-dimensional representations

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Abstract.

The systems we consider are rational extensions of the harmonic oscillator, the truncated oscillator and the radial oscillator. The wavefunctions for the extended states involve exceptional Hermite polynomials for the oscillator and truncated oscillator and exceptional Laguerre polynomials for the radial oscillator. In all cases it is possible to construct ladder operators that have infinite-dimensional representations of their polynomial Heisenberg algebras and couple all levels of the systems. We construct Barut-Girardello coherent states in all cases, eigenvectors of the respective annihilation operators with complex eigenvalues. Then we calculate their physical properties to look for classical or non-classical behaviour.

1. Introduction

This conference paper is a selection of results from three published papers [1, 2, 3] and a work in progress. It is concerned with supersymmetric rational extensions of the harmonic oscillator, the truncated oscillator and the radial oscillator. In addition, we construct special ladder operators associated with infinite-dimensional representations of their polynomial Heisenberg algebras. Then we form the coherent states associated with the annihilation operators and investigate their physical properties.

After a brief summary of supersymmetric quantum mechanics, in Section 2, we present results from [2, 3] on one-step rational extensions of the harmonic oscillator, in Section 3. Then, in Section 4, we present results from [1] on a multi-step rational extension of the truncated oscillator. In Section 5, we present unpublished results on a one-step rational extension of the radial oscillator.

In these papers, we constructed coherent states for special ladder operators that have infinite-dimensional representations. Then we investigated the physical properties of these coherent states using a variety of measures. In all cases, we looked for evidence of classical or non-classical behaviour.

Conclusions follow in Section 6.
2. Supersymmetric quantum mechanics

Supersymmetric quantum mechanics (SUSY QM) [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] has been widely used to create partner Hamiltonians for a given, exactly solvable, Hamiltonian that have almost all of their spectrum in common with the original. Typically there are one or more energy levels of the partner Hamiltonian below the ground state energy of the original Hamiltonian. We consider only one-dimensional bound-state systems with discrete spectra. The methods can be applied to higher-dimensional systems [17, 22].

Suppose a Hamiltonian with an exact solution can be written in factorized form, 
\[ H^{(+)} = A \dagger A, \]

in terms of the supercharge, \( A \), and its Hermitian conjugate. It is is Hermitian and positive semi-definite. Then 
\[ H^{(-)} = AA \dagger \]
is another candidate Hamiltonian, Hermitian and positive semi-definite. The result of supersymmetric quantum mechanics is that these two Hamiltonians have the same energy spectra, except for additional eigenvector(s), \( |0_i \rangle \), of \( H^{(-)} \) satisfying 
\[ A |0_i \rangle = 0. \]

Notice that the partner Hamiltonians satisfy the intertwining relation

\[ A H^{(+)} = H^{(-)} A. \] (1)

In higher-order supersymmetry [23, 22] the \( A \) operator is a product of first-order supercharges, such as

\[ A = A^{(4)} A^{(3)} A^{(2)} A^{(1)} \] (2)

that we will use in Section 4, with the \( A^{(i)} \) defined in Eqs. (18,19). Then the simple forms for \( H^{(+)} \) and \( H^{(-)} \) do not hold. Instead, one requires an intertwining relation of the form of Eq. (1) to hold. It is often solved using symbolic manipulation software. The partial isospectrality of the partner Hamiltonians is then guaranteed.

3. One-step rational extensions of the harmonic oscillator

These papers [2, 3] are concerned with rational extensions of the harmonic oscillator. These have been studied by many authors [24, 25, 10, 21].

Of interest is the choice of ladder operators used to connect the basis states. If \( A \) and \( A \dagger \) are the supercharges used in the supersymmetric transformation and \( a \) and \( a \dagger \) are the ladder operators for the unmodified harmonic oscillator, one class of ladder operators is the natural [26],

\[ b = A a A \dagger, \quad b \dagger = A a \dagger A \dagger. \] (3)

The second class is the intrinsic class [26], that is a linearized version of the natural, with ladder operators of the form

\[ \tilde{b} = f(H^{(-)})b, \quad \tilde{b} \dagger = b \dagger f(H^{(-)}), \] (4)

where \( f \) is chosen so that these ladder operators then satisfy the Heisenberg algebra, like the operators \( a \) and \( a \dagger \). Here \( H^{(-)} \) is the supersymmetric partner Hamiltonian of the harmonic oscillator \( H^{(+)} \).

In general, the natural and intrinsic ladder operators will have finite-dimensional as well as infinite-dimensional representations in the state space of \( H^{(-)} \). Ladder operators have been constructed [27, 22, 28, 29] with no finite-dimensional representations of their polynomial Heisenberg algebras. This is done using combinations of supercharges of Darboux-Crum and Krein-Adler type, allowing more than one possible path between the \( H^{(+)} \) and \( H^{(-)} \) states. Such ladder operators have been studied by the present authors [3, 2, 1] and by other authors [30, 29].

Polynomial Heisenberg algebras arise in many systems, such as the harmonic oscillator [24, 3, 2, 1], the Morse potential [31], the Scarf potential [9] and the infinite well [32]. These have
been used in the study of superintegrable systems, Painlevé transcendents and coherent states [33, 16, 34, 35].

We note that we will find that the position wavefunctions for the modified states will involve exceptional Hermite orthogonal polynomials. These are defined in [36, 22, 23]. Their use has been connected with new potentials, shape invariance and position-dependent mass [37, 38, 39, 40].

For a one-step rational extension of the harmonic oscillator of order $m$ (even), the supercharges are [22]

$$A = -\frac{d}{dx} + x + \frac{H_m'}{H_m}, \quad A^\dagger = +\frac{d}{dx} + x + \frac{H_m'}{H_m}. \quad (5)$$

The modified Hermite polynomials are $H_m(x) = (-i)^m H_m(ix)$, which are positive definite for even $m$ and satisfy the second-order differential equation

$$H''_m + 2xH'_m - 2mH_m = 0. \quad (6)$$

Then we find

$$A^\dagger A = -\frac{d^2}{dx^2} + x^2 + 2m + 1 = H^{(+)} \quad \text{and} \quad AA^\dagger = -\frac{d^2}{dx^2} + V_m^{(-)} = H^{(-)}, \quad (7)$$

with partner potential

$$V_m^{(-)} = x^2 - 1 - \frac{H_m H_m'' - 2xH_m H_m' - 2H_m^2}{H_m^2}. \quad (8)$$

For $m = 2$, $H_2(x) = 4x^2 + 2$, so

$$V_2^{(-)} = x^2 + 3 + \frac{16(4x^2 - 2)}{(4x^2 + 2)^2}. \quad (9)$$

The spectra of the partner Hamiltonian and the harmonic oscillator Hamiltonian are the same except for the unpartnered ground state of the partner Hamiltonian

$$E^{(+m)}_\nu = 2(\nu + m + 1) \quad \text{for} \quad \nu = 0, 1, 2, \ldots$$

$$E^{(-m)}_\nu = 2(\nu + m + 1) \quad \text{for} \quad \nu = -m - 1, 0, 1, 2, \ldots. \quad (10)$$

The ladder operators, $c(m)$, are constructed and their matrix elements derived in [22], with

$$c(m) = A_m \ldots A_2 A_1 A^\dagger, \quad c^\dagger(m) = AA_1 A_2 \ldots A_m, \quad (11)$$

and

$$A_i = +\frac{d}{dx} + x + \frac{H_{i-1}'}{H_{i-1}}, \quad A_i^\dagger = -\frac{d}{dx} + x + \frac{H_{i-1}'}{H_{i-1}} - \frac{H_i'}{H_i}. \quad (12)$$

We see that $c(m)$ involves a path from $H^{-}$ to $H^{+}$ through the supercharge $A^\dagger$ then a return to $H^{-}$ in $m$ steps through intermediate Hamiltonians that may be singular.

The matrix elements of $c(m)$ are

$$\langle \nu - m - 1 (-) | c(m) | \nu (-) \rangle = -[2m+1(\nu-1)(\nu-2)\ldots(\nu-m)(\nu+m+1)]^{\frac{1}{2}} \quad (13)$$

for all $\nu = -m - 1, 0, 1, 2, \ldots$. The operator $c(m)$ lowers the index by $m + 1$ and separates the spectrum into $m + 1$ distinct ladders with lowest weights $\mu = -(m + 1), 1, 2, 3, \ldots m$. 

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The operators $H^{(-m)}, c(m)$ and $c^\dagger(m)$ satisfy a polynomial Heisenberg algebra [22], with polynomials of order $m + 1$ in $H^\dagger$.

The position wavefunctions involve type III Hermite exceptional orthogonal polynomials [37, 38, 39, 40].

Then we construct Barut-Girardello [41] coherent states as eigenvectors of $c(m)$ with complex eigenvalue $z$,

$$c(m) | z, c, m, \mu \rangle = z | z, c, m, \mu \rangle. \quad (14)$$

The time-dependent position probability densities for these coherent states involve the exceptional Hermite polynomials. For $m = 2, \mu = -3$ and $z = 15$, we construct the cat state superpositions

$$| \pm \rangle = \frac{1}{\sqrt{2}} \{ | + z \rangle \pm | - z \rangle \}. \quad (15)$$

Their time-dependent position probability densities (Figure 1) show distinct wavepackets, oscillating and interfering semi-classically.

![Fig 1](Image)

**Figure 1.** Cat state density for $c(2), \mu = -3$ coherent states, (a) even and (b) odd, for $z = 15$.

Then for $m = 6$ and a large value of $z$, we see a great deal of wavepacket structure in the cat state densities (Figure 2), again semi-classical.

### 4. A multi-step rational extension of the truncated oscillator

The truncated oscillator has the Hamiltonian

$$\tilde{H}^{(+)} = \begin{cases} -\frac{d^2}{dx^2} + x^2 & 0 \leq x < \infty, \\ \infty & x < 0. \end{cases} \quad (16)$$

The wavefunctions must vanish at the origin, which selects only the odd states of the usual harmonic oscillator. Again, these involve exceptional Hermite polynomials. We choose a particular model, a four-step supersymmetric transformation with steps $\{m_1, m_2, m_3, m_4\} = \{1, 2, 3, 4\}$ [28, 2]. Then the special ladder operators are defined by

$$\tilde{C} = \tilde{A}^{(5)} A^{(4)} A^{(1)\dagger} A^{(2)\dagger} A^{(3)\dagger} A^{(4)\dagger}, \quad \tilde{C}^\dagger = A^{(1)} A^{(2)} A^{(3)} A^{(4)} \tilde{A}^{(4)\dagger} \tilde{A}^{(5)\dagger}. \quad (17)$$

Here the supercharges for the state adding approach are

$$A^{(i)} = \frac{d}{dx} - \frac{d}{dx} \ln \varphi^{(i)}(x), \quad A^{(i)\dagger} = -\frac{d}{dx} - \frac{d}{dx} \ln \varphi^{(i)}(x), \quad (18)$$
Figure 2. Time-dependent probability densities for (a) even and (b) odd cat states, for \( c(6) \), \( \mu = -7 \) and \( z = 10^8 \).

with

\[
\varphi^{(i)}(x) = \phi_1(x), \quad \varphi^{(i)}(x) = \frac{W(\phi_1, \phi_2, \ldots, \phi_i)}{W(\phi_1, \phi_2, \ldots, \phi_{i-1})},
\]

and \( \phi_m(x) = H_m(x) e^{x^2/2} \). The \( W(\phi_1, \phi_2, \ldots, \phi_n) \) are Wronskian determinants. The supercharges for the state deleting approach are

\[
\bar{A}^{(i)} = \frac{d}{dx} - \frac{d}{dx} \ln \bar{Q}^{(i)}, \quad \bar{A}^{(i)\dagger} = \frac{d}{dx} - \frac{d}{dx} \ln \bar{Q}^{(i)},
\]

with

\[
\bar{Q}^{(4)} = \psi_4, \quad \bar{Q}^{(5)} = \frac{W(\psi_4, \psi_5)}{\psi_4}
\]

and \( \psi_\nu(x) = H_\nu(x) e^{-x^2/2} \).

The action of these operators separates the spectrum into three infinite ladders with lowest weights \( \mu = -5, -3, 5 \).

The matrix elements of \( \tilde{C} \), as given in [28, 2], are labelled

\[
\langle \mu + 6i - 6 \mid \tilde{C} \mid \mu + 6i \rangle = a_{\mu+6i}.
\]

There are two particular cases

\[
a_{-5+6-1} = [645120]^{\frac{1}{2}} \quad \text{and} \quad a_{-3+6-1} = [387072]^{\frac{1}{2}}.
\]

The remaining matrix elements are

\[
a_\nu = [2^\nu(\nu+6)! (\nu-1)! \nu + 3 \nu + 4 \nu + 5]^{\frac{1}{2}}, \quad \text{for} \quad \nu = 6 = 1, 3, 5, \ldots.
\]

We construct the coherent states associated with these ladder operators for the three lowest weights.

We model placing a coherent state on one arm of an optical beamsplitter [2], with the excitations corresponding to photons. The beamsplitter entangles the state with the vacuum from the other arm. Then we calculate the linear entropy of the output state, an approximation to the von Neumann entropy. Small values indicate a low degree of entanglement. The maximum possible value is one. We see a moderate degree of entanglement, \( S \sim 0.4 \), a nonclassical feature, for \( \mu = -5, -3 \).
5. One-step rational extension of the radial oscillator

For the three-dimensional spherically symmetric harmonic oscillator, the radial equation is (with \( \hbar = 1, m = \frac{1}{2}, \omega = 1 \))

\[
\left( -\frac{d^2}{dx^2} + \frac{1}{4} x^2 + \frac{l(l+1)}{x^2} \right) y_l(x) = E y_l(x). \tag{25}
\]

We considered an \( m = 2 \) rational extension as in Section 3 [28, 3, 2], with the choice \( l = 2 \). The position wavefunctions of the partner Hamiltonian involve exceptional Laguerre polynomials [28].

The special ladder operators are defined by

\[
c = \tilde{A} \tilde{A}^\dagger \quad \text{and} \quad c^\dagger = \tilde{A} \tilde{A}^\dagger \tilde{A}^\dagger,
\]

where \( A \) and \( A^\dagger \) use a state adding approach and \( \tilde{A} \) and \( \tilde{A}^\dagger \) use a state deleting approach. An intermediate transformation, enacted by \( \tilde{A} \) and \( \tilde{A}^\dagger \), is needed to shift the spectrum [28].

The lowest weights for the special ladder operators are \( \mu = -3, 1, 2 \). The matrix elements of \( c \) are

\[
\langle -3 \mid c \mid 0 \rangle = 8 \left[ \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha - 1)} \right]^{\frac{1}{2}} \tag{26}
\]

and

\[
\langle \nu - 3 \mid c \mid \nu \rangle = 2^{\frac{3}{2}} \left[ \frac{(\nu - 1)! \Gamma(\nu + \alpha + 2)}{(\nu - 3)! \Gamma(\nu + \alpha - 1)} \right]^{\frac{1}{2}}, \quad \text{for} \ \nu \geq 3, \tag{27}
\]

with \( \alpha = l + 1/2 \).

The Wigner function for a coherent state is defined by

\[
W(x, p; z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{dy} \ \langle z \mid x - y \rangle \langle x + y \mid z \rangle e^{-ipy}, \tag{28}
\]

is everywhere real and acts as a distribution in phase space. For the harmonic oscillator, it is everywhere positive and peaked at \( x = \text{Re} \ z, p = \text{Im} \ z \).

We calculated the Wigner function for our three coherent states and found areas of negativity (the white areas in Figure 3) in all three cases, considered evidence of non-classicality.

![Figure 3](image.png)

Figure 3. Wigner functions for \( z = 500, l = 2 \) and (a) \( \mu = -3 \) (b) \( \mu = 1 \) and (c) \( \mu = 2 \).

6. Conclusions

The coherent states of the harmonic oscillator have essentially classical behaviour [42]. The mean values of position and momentum follow classical trajectories. For rational extensions of the harmonic oscillator, much of this classical behaviour is lost. Instead, we find distinctly quantum
behaviour such as wavepacket interference, significant entropy in the beamsplitter model and Wigner functions taking on negative values.

We have seen exceptional orthogonal polynomials in the position wavefunctions of the extended states. We have constructed special ladder operators with only infinite-dimensional representations of their polynomial Heisenberg algebras.

Future work will involve more results on the rational extensions of the radial oscillator and investigations of other potentials such as the Morse potential [31], the Scarf potential [9] and the infinite well [32].

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