The exact analytical solution of the dual-phase-lag two-temperature bioheat transfer of a skin tissue subjected to constant heat flux

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This work is dealing with the temperature reaction and response of skin tissue due to constant surface heat flux. The exact analytical solution has been obtained for the two-temperature dual-phase-lag (TTDPL) of bioheat transfer. We assumed that the skin tissue is subjected to a constant heat flux on the bounding plane of the skin surface. The separation of variables for the governing equations as a finite domain is employed. The transition temperature responses have been obtained and discussed. The results represent that the dual-phase-lag time parameter, heat flux value, and two-temperature parameter have significant effects on the dynamical and conductive temperature increment of the skin tissue. The Two-temperature dual-phase-lag (TTDPL) bioheat transfer model is a successful model to describe the behavior of the thermal wave through the skin tissue.

The thermal transfer through a living skin tissue is extremely essential to a wide range of therapies and their applications. Improvements in the microwave, laser, and other technologies also extended bioheat transfer work. Several researchers comment on biothermal transmission analyzes in biological living tissues, including Pennes. He optimized the parabolic model for biological tissues for the first bioheat transfer. Also, the Pennes biological transmission method is used to model the nature and consistency of thermal behavior in biological tissues and living bodies. External similarities to such unpredictable results that display non-Fourier or hyperbolic conduction behavior. Cattaneo and Vernotte provided different modulations of Fourier's law of heat conduction as a linear extension form of the Fourier law to describe the hyperbolic equation type. They suggested a model of thermal waves measure the effect of microwave and thermal flux activity. Different forms of work have been undertaken to address various skin tissue conditions without damaging the balanced tissue surrounding them. Xu et al. solved analytical the Pennes bioheat transfer equation (PBTE), reviewed skin bioheat transfer, skin structure, thermal damage, skin biomechanics, and bio-thermomechanics. The non-Fourier thermomechanical activity of skin tissues under various surface thermal loading limits was analyzed in DPL, hyperbolic and parabolic models of biomass transport, Xu and al. studied and noticed substantial variations between Pennes, thermal wave and DPL anticipations models. Also, Rossmann et al. reviewed the temperature dependence of the dielectric properties, thermal properties, and perfusion of biological tissues at hyper-thermic and ablation temperatures. Poor et al. focused on the temperature disturbance of skin tissue due to time-dependent surface heating. Tzou's model has been extended by the dual-phase-lag principle, with the delayed activity in a high rate of response taken into consideration. Although the process is lagging, the small-scale response is caught in time. Tzou introduced a phase-lag for a temperature gradient. Askarizadeh et al. utilized the dual-phase-lag (DPL) model in treating the transient heat transfer problems in skin tissue. Dutta and Kundu established a two-dimensional thermal model of malignant tissues, focused on the bi-dimensional local thermal non-balance therapy model for bio-heat.
Liu et al. first address the overall bioheat transmission paradigm in living tissue and analyze\(^\text{14}\). Liu et al. are tested during the hyperthermia procedure in the two-stage model for heat transfer problems in the biological tissue\(^\text{15,16}\). The Liu and Chen thermal conductor model (DPL) used the non-Fourier thermal activity for the diagnosis of hyperthermia to describe the thermal transportation occurring in biological tissue\(^\text{17}\). In terms of the properties of blood and tissue and the interphase charge of the heat transfer parameter and perfusion rate, Zhang articulated phase lag or relaxation periods\(^\text{18}\). He observed that the lap periods are very similar together the properties of blood and tissue and the interphase charge of the heat transfer parameter and perfusion rate, leading to parabolic, velocity profiles on the blood flow. Shih et al. investigated the coupled effects of the pulsatile blood flow and thermal relaxation time in living tissues during thermal treatments\(^\text{27}\). Youssef and Alghamdi applied the two-temperature heat conduction as follows\(^\text{28,36,37}\):

\[
K \nabla^2 T = \rho C \frac{\partial T}{\partial t} - \rho_b w_b C_b (T_b - T) - (Q_{\text{met}} + Q_{\text{ext}}),
\]

where \(\rho\), \(C\) and \(K\) are the density, specific heat, and thermal conductivity of the tissue, respectively. \(C_b\), \(w_b\), \(\rho_b\), and \(T_b\) denote the specific heat of the blood, blood perfusion rate, blood density, and blood temperature, respectively. \(T\) is the absolute temperature function. \(Q_{\text{met}}\) denotes the metabolic heat generated by the chemical reaction inside the tissue, and it is constant, and the external heat source is given by \(Q_{\text{ext}} = Q_{\text{ext}}(t)\). \(\nabla^2\) is the well-known Laplace operator.

### Bioheat transfer models for biological tissues

The bioheat transmitting model was developed to measure the time-dependent temperature change as a heat reaction due to the usage of either heat source or thermal heating. The first model of biological tissues was found out by Pennes based on Fourier’s law of heat conduction as follows\(^\text{26,36,37}\):

\[
K \nabla^2 T = \rho C \frac{\partial T}{\partial t} - \rho_b w_b C_b (T_b - T) - (Q_{\text{met}} + Q_{\text{ext}}),
\]

where \(\rho\), \(C\) and \(K\) are the density, specific heat, and thermal conductivity of the tissue, respectively. \(C_b\), \(w_b\), \(\rho_b\), and \(T_b\) denote the specific heat of the blood, blood perfusion rate, blood density, and blood temperature, respectively. \(T\) is the absolute temperature function. \(Q_{\text{met}}\) denotes the metabolic heat generated by the chemical reaction inside the tissue, and it is constant, and the external heat source is given by \(Q_{\text{ext}} = Q_{\text{ext}}(t)\). \(\nabla^2\) is the well-known Laplace operator.

### The modified Fourier’s law of heat conduction

Ventott and Cattaneo modified the classical Fourier thermal drive law by postulating the concept of finite thermal wave propagation speed and taking the following hyperbolic form of the thermal wave\(^\text{26,36,37}\):

\[
K \left(1 + \tau_\alpha \frac{\partial}{\partial t}\right) \nabla^2 T = \rho C \left(1 + \tau_\alpha \frac{\partial}{\partial t}\right) \frac{\partial T}{\partial t} + w_b C_b \rho_b \left(1 + \tau_\alpha \frac{\partial}{\partial t}\right) (T - T_b) - \left(1 + \tau_\alpha \frac{\partial}{\partial t}\right) (Q_{\text{met}} + Q_{\text{ext}}),
\]

where \(\tau_\alpha = \frac{\alpha}{\rho C} > 0\) is a material property and is called the relaxation time parameter, and \(\alpha\) is the thermal fusivity while \(c_0\) is the speed of the thermal wave inside the medium.

### Dual-phase-lag model (DPL) of bioheat transfer

The dual-phase-lag (DPL) model based on the dual reaction between the gradient of the temperature \(\nabla T\) and the heat flux \(q\) which modified the well-known classical Fourier’s law of heat conduction, thus, we have the following heat conduction equation\(^\text{26,36,37}\):

\[
K \left(1 + \tau_\gamma \frac{\partial}{\partial t}\right) \nabla^2 T = \rho C \left(1 + \tau_\gamma \frac{\partial}{\partial t}\right) \frac{\partial T}{\partial t} + w_b C_b \rho_b \left(1 + \tau_\gamma \frac{\partial}{\partial t}\right) (T - T_b) - \left(1 + \tau_\gamma \frac{\partial}{\partial t}\right) (Q_{\text{met}} + Q_{\text{ext}})
\]

For one calming period in deformable structures, Youssef updated Chen and Gurtin’s principle of thermal actions. Two specific temperatures are the thermodynamic temperature and the conductive temperature in this device\(^\text{30,31}\). The disparity in the meaning of these two temperature forms is commensurate with the importance of the material’s thermal supply. Youssef with El-Bary and Alghamdi applied the two-temperature heat conduction in many applications\(^\text{32–37}\).
where $\tau_T \geq 0$ is the second parameter of the relaxation time, which is the phase-lag of the temperature gradient passing through the medium.

**Two-temperature dual-phase-lag (TTDPL) bioheat transfer model on skin tissues.** According to Youssef’s model of heat conduction formulation, we have two equations of heat conduction on the biological tissues as following:

$$K \left(1 + \tau_T \frac{\partial}{\partial t}\right) \nabla^2 T_C = \rho C \left(1 + \tau_q \frac{\partial}{\partial t}\right) \frac{\partial T_D}{\partial t} + w_b C_b \rho_b \left(1 + \tau_q \frac{\partial}{\partial t}\right) T_D - \left(1 + \tau_q \frac{\partial}{\partial t}\right) (Q_{\text{met}} + Q_{\text{ext}})$$  \hspace{1cm} (4)

and

$$T_D = T_C - \beta \nabla^2 T_C$$  \hspace{1cm} (5)

where $\beta$ is a non-negative constant, which is called the two-temperature parameter. $T_D$ and $T_C$ are the dynamical temperature and the conductive temperature, respectively. The value $\beta = 0$ represents the one-temperature model, then, we obtain $T_C = T_D$.

We consider the function of dynamical temperature increment takes the form

$$\theta = T_D - T_b$$  \hspace{1cm} (6)

and the function of conductive temperature increment takes the form

$$\varphi = T_C - T_b$$  \hspace{1cm} (7)

Hence, we have

$$K \left(1 + \tau_T \frac{\partial}{\partial t}\right) \nabla^2 \varphi = \rho C \left(1 + \tau_q \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + w_b C_b \rho_b \left(1 + \tau_q \frac{\partial}{\partial t}\right) \theta - \left(1 + \tau_q \frac{\partial}{\partial t}\right) (Q_{\text{met}} + Q_{\text{ext}})$$  \hspace{1cm} (8)

and

$$\theta = \varphi - \beta \nabla^2 \varphi$$  \hspace{1cm} (9)

**Method**

We consider the region $0 \leq x \leq L$ is filled with skin tissue as in Fig. 1 and obeys the two-temperature dual-phase-lag (TTDPL) as in Eqs. (8) and (9).

The medium is considered to be quiescent initially and without any external heat source ($Q_{\text{ext}} = 0$) while $Q_{\text{met}}$ is constant. The skin is subjected to a constant heat flux $q_0$ on its bounding surface $x = 0$, while the bounding surface $x = L$ has no heat flux (see Fig. 1). Hence, the heat conduction equations take the forms:

$$\left[ \beta \rho C \frac{\partial^2 \varphi}{\partial t^2} + (K \tau_T + \beta \rho C + \beta \tau_q w_b C_b \rho_b) \frac{\partial}{\partial t} + (\beta w_b C_b \rho_b + K) \right] \frac{\partial^2 \varphi}{\partial x^2} = \left( \tau_q \rho C \frac{\partial^2 \varphi}{\partial t^2} + (\rho C + \tau_q w_b C_b \rho_b) \frac{\partial}{\partial t} + w_b C_b \rho_b \right) \varphi - Q_{\text{met}}$$  \hspace{1cm} (10)

and

$$\theta = \varphi - \beta \frac{\partial^2 \varphi}{\partial x^2}$$  \hspace{1cm} (11)
The initial conditions are:
\[ \psi(x, t)|_{t=0} = \theta(x, t)|_{t=0} = \frac{\partial \psi(x, t)}{\partial t} \bigg|_{t=0} = \frac{\partial \theta(x, t)}{\partial t} \bigg|_{t=0} = 0 \] (12)

The boundary conditions are:
\[ -K \frac{\partial \psi(x, t)}{\partial x} \bigg|_{x=0} = q_0, \quad -K \frac{\partial \psi(x, t)}{\partial x} \bigg|_{x=L} = 0 \] (13)

The boundary value problem (10)–(13) consists of non-homogeneous partial differential equations with non-homogeneous boundary conditions on the surface of the skin tissue. Thus, the equations must be formulated in terms of a steady part and a transient part as follows:

\[ \psi(x, t) = \phi_1(x, t) + \phi_2(x) \] (14)

Hence, we obtain the transient part in the form
\[ \left[ \beta \rho C t \frac{d^2}{dt^2} + \left( K \tau_t + \beta \rho C + \beta t_q w_b C_b \rho_b \right) \frac{d}{dt} + \left( \beta w_b C_b \rho_b + K \right) \frac{d^2}{dx^2} \right] \phi_1(x, t) = \psi_1(x, t) = 0 \] (15)

and the steady-state part in the form
\[ \left( \frac{d^2}{dx^2} - \lambda^2 \right) \phi_2(x) = -\gamma \psi \] (16)

where \( \lambda^2 = \frac{w_b C_b \rho_b}{(\beta \rho C t + K)} > 0, \ \gamma = \frac{1}{(\beta \rho C t + K)} > 0, \) and \( \psi = Q_{\text{met}}. \)

The boundary conditions for the steady-state equation are
\[ -K \frac{d \phi_2(x)}{dx} \bigg|_{x=0} = q_0, \quad -K \frac{d \phi_2(x)}{dx} \bigg|_{x=L} = 0 \] (17)

Then, regarding the boundary conditions on (13), the solution of the Eq. (10) is in the form (See "Appendix"):
\[ \phi_2(x) = \frac{q_0 \cosh \lambda (L-x)}{\lambda K \sinh \lambda L} + \frac{\gamma}{\lambda^2} \psi \] (18)

The initial and boundary conditions of the transient Eq. (12) are:
\[ \phi_1(x, t)|_{t=0} = 0, \quad \frac{\partial \phi_1(x, t)}{\partial t} \bigg|_{t=0} = 0 \] (19)

and
\[ \frac{\partial \phi_1(x, t)}{\partial x} \bigg|_{x=0} = 0, \quad \frac{\partial \phi_1(x, t)}{\partial x} \bigg|_{x=L} = 0 \] (20)

To solve the Eq. (15) we expand the function \( \phi_1(x, t) \) in the following Fourier series expansion:
\[ \phi_1(x, t) = \sum_{n=0}^{\infty} \vartheta_n(t) \cos \left( \frac{n\pi}{L} x \right) \] (21)

which satisfies the boundary conditions (20).

Substitute from Eqs. (21) into (15), we get
\[ \left[ \frac{\partial^2}{\partial t^2} + A_{1n} \frac{\partial}{\partial t} + A_{2n} \right] \vartheta_n(t) = 0, \quad n = 0, 1, 2, 3 \ldots \] (22)

where \( A_{1n} = \frac{\alpha_2 (\tau_t + \beta \rho C + 1+\tau_t)}{\tau_t (\rho \omega_n^2 + 1)}, \quad A_{2n} = \frac{\alpha_2 (\beta \rho C + \epsilon)}{\tau_t (\rho \omega_n^2 + 1)}, \ \epsilon = \frac{w_b C_b \rho_b}{\rho C}, \eta = \frac{K}{\rho T} \) and \( \omega_n = \frac{n\pi}{L} \).

The general solutions to (22) take the following forms:
\[ \vartheta_n(t) = a_{0f_1} (k_{1n}, t) + b_{0f_2} (k_{2n}, t), \quad n = 0, 1, 2, \ldots \] (23)

which give
\[ \psi_1(x, t) = \sum_{n=0}^{\infty} \left[ a_{0f_1} (k_{1n}, t) + b_{0f_2} (k_{2n}, t) \right] \cos (\omega_n x) \] (24)
where \( k_{1n}, k_{2n} \) is the solution of the following characteristic equation

\[
k^2 + A_{1n}k + A_{2n} = 0
\]  
(25)

The roots of the characteristic Eq. (25) are in the forms:

\[
k_{1n} = -A_{1n} + \sqrt{\Delta_n}, \quad k_{2n} = -A_{1n} - \sqrt{\Delta_n}
\]  
(26)

where \( \Delta_n = A_{1n}^2 - 4A_{2n} \).

To apply the initial conditions (19), we have to expand \( \phi_2(x) \) in Fourier series expansion to be in the form:

\[
\phi_2(x) = \frac{q_0}{2LKA^2} + \frac{\gamma \psi}{2\lambda^2} + \frac{\varepsilon \tau_q + 1}{2LKA} \sum_{n=1}^{\infty} \frac{1}{\left(\lambda^2 + \omega_n^2\right)} \cos (\omega_n x)
\]  
(27)

Hence, the initial conditions (19) give the following system of algebraic equations:

\[
[a_0 f_1(k_{10}, 0) + b_0 f_2(k_{20}, 0)] \cos (\omega_0 x) + \sum_{n=1}^{\infty} [a_n f_1(k_{1n}, 0) + b_n f_2(k_{2n}, 0)] \cos (\omega_n x) = -\phi_2(x)
\]  
(28)

and

\[
\left[ a_0 \frac{\partial f_1(k_{10}, 0)}{\partial t} + b_0 \frac{\partial f_2(k_{20}, 0)}{\partial t} \right] \cos (\omega_0 x) + \sum_{n=1}^{\infty} \left[ a_n \frac{\partial f_1(k_{1n}, 0)}{\partial t} + b_n \frac{\partial f_2(k_{2n}, 0)}{\partial t} \right] \cos (\omega_n x) = 0
\]  
(29)

For the case of \( \Delta_n > 0 \)

\[
f_1(k_{1n}, t) = e^{k_{1n} t} \quad \text{and} \quad f_2(k_{2n}, t) = e^{k_{2n} t}
\]  
(30)

Thus, when \( t = 0 \) we have

\[
f_1(k_{1n}, 0) = f_2(k_{2n}, 0) = 1, \quad \frac{\partial f_1(k_{1n}, 0)}{\partial t} = k_{1n}, \quad \frac{\partial f_2(k_{2n}, 0)}{\partial t} = k_{2n}, \quad n = 0, 1, 2, 3 \ldots
\]

Then, Eqs. (28) and (29) introduce the following system

\[
[a_0 + b_0] \cos (\omega_0 x) + \sum_{n=1}^{\infty} [a_n + b_n] \cos (\omega_n x) = -\frac{q_0}{2LKA^2} - \frac{\gamma \psi}{2\lambda^2} - \frac{\varepsilon \tau_q - 1}{2LKA} \sum_{n=1}^{\infty} \frac{1}{\left(\lambda^2 + \omega_n^2\right)} \cos (\omega_n x)
\]  
(31)

and

\[
[k_{10} a_0 + k_{20} b_0] \cos (\omega_0 x) + \sum_{n=1}^{\infty} [k_{1n} a_n + k_{2n} b_n] \cos (\omega_n x) = 0
\]  
(32)

when \( n = 0 \) we have

\[
\omega_0 = 0, \quad \cos (\omega_0 x) = 1, \quad A_{10} = \frac{\varepsilon \tau_q + 1}{\tau_q}, \quad A_{20} = \frac{\varepsilon}{\tau_q}, \quad \Delta_0 = \left(\frac{\varepsilon \tau_q - 1}{\tau_q}\right)^2, \quad k_{10} = \frac{-1}{\tau_q}, \quad k_{20} = -\varepsilon.
\]

Hence, the system in Eqs. (31) and (32) will be reduced to the following system

\[
a_0 + b_0 + \sum_{n=1}^{\infty} [a_n + b_n] \cos (\omega_n x) = -\frac{q_0}{2LKA^2} - \frac{\gamma \psi}{2\lambda^2} - \frac{q_0}{LKA} \sum_{n=1}^{\infty} \frac{1}{\left(\lambda^2 + \omega_n^2\right)} \cos (\omega_n x)
\]  
(33)

and

\[
-\frac{a_0}{\tau_q} = \varepsilon b_0 + \sum_{n=1}^{\infty} [k_{1n} a_n + k_{2n} b_n] \cos (\omega_n x) = 0
\]  
(34)

The equations in (33) and (34) lead to the following two systems of algebraic equations

\[
a_0 + b_0 = \left(\frac{q_0}{2LKA^2} + \frac{\gamma \psi}{2\lambda^2}\right), \quad a_0 + \tau_q \varepsilon b_0 = 0
\]  
(35)

and

\[
a_n + b_n = -\frac{q_0}{LKA} \left(\lambda^2 + \omega_n^2\right), \quad k_{1n} a_n + k_{2n} b_n = 0, \quad n = 1, 2, 3, \ldots
\]  
(36)

By solving the above two systems, we get
\[
a_0 = \left( \frac{\varepsilon \tau_q}{1 - \varepsilon \tau_q} \right) \left( \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} \right), \quad b_0 = \left( \frac{-1}{1 - \varepsilon \tau_q} \right) \left( \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} \right)
\]

(37)

and

\[
a_n = \left( \frac{k_{2n}}{k_{1n} - k_{2n}} \right) \frac{q_0}{LK (\lambda^2 + \omega_n^2)}, \quad b_n = \left( \frac{-k_{1n}}{k_{1n} - k_{2n}} \right) \frac{q_0}{LK (\lambda^2 + \omega_n^2)}, \quad n = 1, 2, 3 \ldots
\]

(38)

Now, the final solution of the heat conduction temperature increment, in this case, is in the form

\[
\varphi(x, t) = \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} + \frac{q_0}{KL} \sum_{n=1}^{\infty} \left( \frac{1}{\lambda^2 + \omega_n^2} \right) \cos(\omega_n x) + \left( \frac{1}{1 - \varepsilon \tau_q} \right) \left( \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} \right)
\]

\[
\left[ \varepsilon \tau_q e^{\frac{-t}{\tau_q}} - e^{-\frac{t}{\tau_q}} \right] + \frac{q_0}{KL} \sum_{n=1}^{\infty} \left( \frac{k_{2n} e^{i\omega_n t} - k_{1n} e^{i\omega_n t}}{(k_{1n} - k_{2n})(\lambda^2 + \omega_n^2)} \right) \cos(\omega_n x)
\]

(39)

By using Eqs. (39) and (11), we get the dynamical temperature increment as follows:

\[
\theta(x, t) = \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} + \frac{q_0}{KL} \sum_{n=1}^{\infty} \left( 1 + \beta \omega_n^2 \right) \cos(\omega_n x) + \left( \frac{1}{1 - \varepsilon \tau_q} \right) \left( \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} \right)
\]

\[
\left[ \varepsilon \tau_q e^{\frac{-t}{\tau_q}} - e^{-\frac{t}{\tau_q}} \right] + \frac{q_0}{KL} \sum_{n=1}^{\infty} \left( 1 - \beta \omega_n^2 \right) \left( \frac{k_{2n} e^{i\omega_n t} + k_{1n} e^{i\omega_n t}}{(k_{1n} - k_{2n})(\lambda^2 + \omega_n^2)} \right) \cos(\omega_n x)
\]

(40)

For the case of \( \Delta < 0 \), we have

\[
f_1(k_{1n}, t) = e^{k_{1n} t} \cos\left( \frac{\sqrt{-\Delta_n}}{2} t \right), \quad f_2(k_{2n}, t) = e^{k_{1n} t} \sin\left( \frac{\sqrt{-\Delta_n}}{2} t \right),
\]

(41)

\[
\frac{\partial f_1(k_{1n}, t)}{\partial t} = e^{k_{1n} t} \left[ k_{1n} \cos\left( \frac{\sqrt{-\Delta_n}}{2} t \right) - \frac{\sqrt{-\Delta_n}}{2} \sin\left( \frac{\sqrt{-\Delta_n}}{2} t \right) \right]
\]

(42)

\[
\frac{\partial f_2(k_{2n}, t)}{\partial t} = e^{k_{1n} t} \left[ k_{2n} \sin\left( \frac{\sqrt{-\Delta_n}}{2} t \right) + \frac{\sqrt{-\Delta_n}}{2} \cos\left( \frac{\sqrt{-\Delta_n}}{2} t \right) \right]
\]

(43)

\[
f_1(k_{1n}, 0) = 1, \quad f_2(k_{2n}, 0) = 0, \quad n = 0, 1, 2, 3 \ldots
\]

(44)

\[
\frac{\partial f_1(k_{1n}, 0)}{\partial t} = k_{1n}, \quad \frac{\partial f_2(k_{2n}, 0)}{\partial t} = \frac{\sqrt{-\Delta_n}}{2}, \quad n = 0, 1, 2, 3 \ldots
\]

(45)

\[
A_{10} = \frac{1}{\tau_q}, \quad A_{20} = \frac{\varepsilon}{\tau_q}, \quad \Delta_0 = \left( \frac{\varepsilon \tau_q - 1}{\tau_q} \right)^2, \quad k_{10} = \frac{-1}{\tau_q}, \quad k_{20} = -\varepsilon
\]

Applying the initial conditions (19), we obtain

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) = -\frac{q_0}{2LK \lambda^2} - \frac{\gamma \psi}{2\lambda^2} - \frac{q_0}{KL} \sum_{n=1}^{\infty} \frac{1}{\lambda^2 + \omega_n^2} \cos(\omega_n x)
\]

(46)

and

\[
\left[ \frac{-1}{\tau_q} a_0 + \left( \frac{\tau_q - 1}{2\tau_q} \right) b_0 + \sum_{n=1}^{\infty} \left( a_n k_{1n} + b_n \frac{\sqrt{-\Delta_n}}{2} \right) \cos(\omega_n x) \right] = 0
\]

(47)

Solving the above system, we obtain

\[
a_0 = -\left( \frac{q_0}{2LK \lambda^2} + \frac{\gamma \psi}{2\lambda^2} \right), \quad b_0 = -\left( \frac{1}{\tau_q - 1} \right) \left( \frac{q_0}{LK \lambda^2} + \frac{\gamma \psi}{\lambda^2} \right)
\]

(48)

\[
a_n = -\frac{q_0}{KL (\lambda^2 + \omega_n^2)}, \quad b_n = \frac{2k_{1n} q_0}{KL (\lambda^2 + \omega_n^2) \sqrt{-\Delta_n}}
\]

(49)
Hence, the final solution of the heat conduction temperature increment in this case is:

\[
\psi(x, t) = \frac{q_0}{2LKx^2} + \frac{\psi_0}{kL} \sum_{n=1}^{\infty} \frac{1}{(\omega_n^2 + \alpha_n^2)} \cos(\omega_n x) - \left( \frac{q_0}{2LKx^2} + \frac{\gamma \psi_0}{2Kx^2} \right) e^{-\frac{t}{\tau_T}} \cos\left(\frac{\sqrt{-\Delta_0}}{2} t\right) \\
- \left( \frac{1}{\tau_q - 1} \right) \left( \frac{q_0}{LKx^2} + \frac{\gamma \psi_0}{2Kx^2} \right) e^{-t} \sin\left(\frac{\sqrt{-\Delta_0}}{2} t\right) + \frac{q_0}{kL} \sum_{n=1}^{\infty} e^{(k_n t)} \cos\left(\frac{\sqrt{-\Delta_n}}{2} t\right) \\
+ \frac{2k_1m}{\sqrt{-\Delta_n}} e^{(k_n t)} \sin\left(\frac{\sqrt{-\Delta_n}}{2} t\right) \cos(\alpha_n x) \left(\frac{1}{\lambda^2 + \alpha_n^2}\right) \tag{50}
\]

Table 1. The material properties of the skin tissue.

| Parameter | Unit   | Skin | Parameter | Unit   | Skin |
|-----------|--------|------|-----------|--------|------|
| $K$       | W/m°C  | 0.215| $W_{m}$   | ml/Cm  | 0.0052|
| $\rho$    | kg/m³  | 1000 | $T_b$     | °C     | 37   |
| $\rho_b$  | kg/m³  | 1060 | $\tau_T$  | s      | 10.0 |
| $C$       | J/kg°C | 4187 | $\tau_q$  | s      | 20.0 |
| $C_b$     | J/kg°C | 3800 | $m$       | 0.006  |      |

Figure 2. The temperature increment when $t = 100$ (s), $\tau_T = 10$ (s), $\tau_q = 20$ (s) and $q_0 = 100$ (W/m²).

Figure 3. The temperature increment when $t = 100$ (s), $\tau_T = 10$ (s), $\tau_q = 20$ (s) and various values of the surface heat flux $q_0$ (W/m²) based on the two-temperature model.
By using Eqs. (50) and (11), we get the dynamical temperature increment in this case as follows:

$$\theta(x, t) = \frac{q_0}{2LK^2} + \frac{\psi}{2} + \frac{q_0}{KL} \sum_{n=1}^{\infty} \frac{(1 + \omega_n^2 \beta)}{(\beta^2 + \omega_n^2)} \cos(\omega_n x) - \left( \frac{q_0}{2LK^2} + \frac{\gamma \psi}{L^2} \right) e^{-\frac{\beta}{2L^2}} \cos \left( \frac{\sqrt{-\Delta_0}}{2} t \right) - \left( \frac{1}{\tau_q - 1} \right) \left( \frac{q_0}{LK^2} + \frac{\gamma \psi}{L^2} \right) e^{-t} \sin \left( \frac{\sqrt{-\Delta_0}}{2} t \right) + \frac{q_0}{KL} \sum_{n=1}^{\infty} \left[ e^{(k_1 n)} \cos \left( \frac{\sqrt{-\Delta_n}}{2} t \right) \right] \left( 1 + \omega_n^2 \beta \right) \cos(\omega_n x) \left( \beta^2 + \omega_n^2 \right)$$

$$+ 2k_1 e^{(k_2 n)} \sin \left( \frac{\sqrt{-\Delta_n}}{2} t \right) \left[ (1 + \omega_n^2 \beta) \cos(\omega_n x) \right] \left( \beta^2 + \omega_n^2 \right)$$

Results
In this study, the temperature distribution through skin tissue is investigated for two models (One temperature, two temperature) of bioheat transfer for constant heat flux condition on the skin surface. The values of the relevant thermal parameters used in the present calculations are in Table 1 as follows:

Discussions
Figures 2, 3, 4, 5 represent the conductive and dynamical temperature increment with respect to a wide range of skin distance x (0 ≤ x ≤ 0.006 m) and at constant instance time t = 100(s).

Figure 2 shows that the two temperature parameter β has a significant effect on the temperature increment distribution. The red line represents the conductive temperature distribution, while the blue line represents the dynamical temperature distribution. The conductive temperature increment and dynamical temperature
Figure 6. The temperature increment when $x = 0.003$ (m), $\tau_T = 10$ (s), $\tau_q = 20$ (s) and $q_0 = 100$ (W/m$^2$).

Figure 7. The temperature increment when $x = 0.003$ (m), $\tau_T = 10$ (s), $\tau_q = 20$ (s) and various values of the surface heat flux $q_0$ (W/m$^2$) based on the two-temperature model.

Figure 8. The temperature increment when $x = 0.003$ (m), $q_0 = 100$ (W/m$^2$), $\tau_q = 20$ (s) and various values $\tau_T$ (s) based on the two-temperature model.
Figure 9. The temperature increment when \( x = 0.003 \) (m), \( q_0 = 100 \) (W/m\(^2\)), \( \tau_T = 10 \) (s) and various values \( \tau_q(s) \) based on the two-temperature model.

The temperature increment have the same behavior but with different values because of the effect of the two-temperature parameter. In the context of the one-temperature model \( \beta = 0 \), the conductive and dynamical temperature increments are the same, and they appear in the solid blue curve. It is noted that an increase in the value of the two-temperature parameter leads to an increase in the value of conductive temperature increment and a decrease in the value of dynamical temperature increment. We noticed that thermal waves in the context of the one-temperature model have vanished before the thermal waves in the context of the two-temperature model.

Figure 3 shows that the conductive and dynamical temperature increment distributions when \( t = 100 \) (s), \( \tau_T = 10 \) (s), \( \tau_q = 20 \) (s) and various values of the surface heat flux \( q_0 \) (W/m\(^2\)) = (100, 200) to stand on its effects. It is noted that the value of the heat flux on the surface of the skin tissue has a significant impact on the conductive and dynamical temperature increment distributions. An increase in the value of the heat flux leads to an increase in the value of the conductive and dynamical temperature increment distributions. A smaller value of the surface heat flux makes the temperature more closed to zero value at the end of the skin length.

Figure 4 shows that the conductive and dynamical temperature increment distributions when \( t = 100 \) (s), \( q_0 = 100 \) (W/m\(^2\)), \( \tau_q = 20 \) (s) and various values of the second relaxation time parameter \( \tau_f(s) = (0.0, 10.0) \) to stand on its effects. It is noted that the value of the second relaxation time parameter has a significant impact on the conductive and dynamical temperature increment distributions. An increase in the value of the second relaxation time parameter leads to an increase in the value of the conductive and dynamical temperature increment distributions. The zero value of the first relaxation time parameter \( \tau_T \) (temperature lag-time) makes the thermal waves more closed to the zero value at the end of the skin length.

Figure 5 shows that the conductive and dynamical temperature increment distributions when \( t = 100 \) (s), \( q_0 = 100 \) (W/m\(^2\)), \( \tau_T = 10 \) (s) and various values of the first relaxation time parameter \( \tau_q(s) = (10, 20) \) to stand on its effects. It is noted that the value of the first relaxation time \( \tau_q \) parameter has noticeable effects on the conductive and dynamical temperature increment distributions. An increase in the value of the second relaxation time parameter \( \tau_T \) leads to a decrease in the value of the conductive and dynamical temperature increment distributions. Increasing the value of the second relaxation time parameter \( \tau_q \) (temperature gradient lag-time) makes the thermal waves more closed to the zero value at the end of the skin length.

Figures 6, 7, 8, 9 represent the conductive and dynamical temperature increment with respect to a wide range of time \( x \) \((0 \leq t \leq 100 \) s\) and at a constant distance \( x = 0.003\)(m).

Figure 6 shows that the two temperature parameter \( \beta \) has a significant effect on the temperature increment distribution. The conductive temperature increment and dynamical temperature increment have the same behavior but with different values because of the impact of the two-temperature parameter. In the context of the one-temperature model \( \beta = 0 \), the conductive and dynamical temperature increments are the same, and the solid blue curve represents them. It is noted that an increase in the value of the two-temperature parameter leads to an increase in the value of conductive temperature increment and a decrease in the value of dynamical temperature increment.

Figure 7 shows that the conductive and dynamical temperature increment distributions when \( x = 0.003 \) (m), \( \tau_T = 10 \) (s), \( \tau_q = 20 \) (s) and various values of the surface heat flux \( q_0 \) (W/m\(^2\)) = (100, 200) to stand on its effects. It is noted that the value of the heat flux on the surface of the skin tissue has a significant impact on the conductive and dynamical temperature increment distributions. An increase in the value of the heat flux leads to an increase in the value of the conductive and dynamical temperature increment distributions.

Figure 8 shows that the conductive and dynamical temperature increment distributions when \( x = 0.003 \) (m), \( q_0 = 100 \) (W/m\(^2\)), \( \tau_q = 20 \) (s) and various values of the second relaxation time parameter \( \tau_f(s) = (0.0, 10.0) \) to stand on its effects. It is noted that the value of the second relaxation time parameter has a significant impact on the conductive and dynamical temperature increment distributions. An increase in the value of the second relaxation time parameter leads to an increase in the value of the conductive and dynamical temperature increment distributions.
Figure 10. The conductive and dynamical temperature increment when $q_0 = 100 \, (\text{W/m}^2)$ based on the two-temperature model.

Figure 11. The conductive and dynamical temperature increment when $q_0 = 200 \, (\text{W/m}^2)$ based on the two-temperature model.

Figure 12. The conductive and dynamical temperature increment when $\tau_T(s) = 0.0$ based on the two-temperature model.
Figure 13. The conductive and dynamical temperature increment when $\tau_T(s) = 15$ based on the two-temperature model.

Figure 14. The conductive and dynamical temperature increment when $\tau_q(s) = 10$.

Figure 15. The conductive and dynamical temperature increment when $\tau_q(s) = 20$. 
Figure 9 shows that the conductive and dynamical temperature increment distributions when \( x(m) = 0.003 \), \( q_0(W/m^2) = 100 \), \( \tau_{T}(s) = 10 \) and various values of the second relaxation time parameter \( \tau_q(s) = (0.0, 10.0) \) to stand on its effects. It is noted that the value of the first relaxation time parameter has a significant impact on the conductive and dynamical temperature increment distributions. An increase in the value of the first relaxation time parameter leads to a decrease in the value of the conductive and dynamical temperature increment distributions.

Figures 10 and 11 show the differences between the conductive temperature increment and dynamical temperature increment with respect to a wide range of time \( t(0 \leq t \leq 100(s)) \) and distance \( x(0 \leq x \leq 0.006(m)) \).

Figure 8 represents the conductive temperature increment and dynamical temperature increment when \( q_0(W/m^2) = 100 \) and Fig. 9 when \( q_0(W/m^2) = 200 \). These two figures agree with the results in Figs. 3 and 6 and confirm that the value of the heat flux on the surface of the skin tissue has significant effects on the conductive and dynamical temperature increment distributions.

Figures 12 and 13 show the differences between the conductive temperature increment and dynamical temperature increment with respect to a wide range of time \( t(0 \leq t \leq 100(s)) \) and distance \( x(0 \leq x \leq 0.006(m)) \) for various values of the second relaxation time parameter \( \tau_{T}(s) = (0.0, 10.0) \). This two figures agree with the results in the Figs. 4 and 7 and confirm that the value of the second relaxation time parameter has significant effects on the conductive and dynamical temperature increment distributions.

Figures 14 and 15 show the differences between the conductive temperature increment and dynamical temperature increment with respect to a wide range of time \( t(0 \leq t \leq 100(s)) \) and distance \( x(0 \leq x \leq 0.006(m)) \) for various values of the first relaxation time parameter \( \tau_q(s) = (10, 20) \). These two figures agree with the results in Figs. 5 and 9 and confirm that the value of the first relaxation time parameter has significant effects on the conductive and dynamical temperature increment distributions.

Here, we should refer to the agreement of the results of this research with the results of other researches in order to confirm the validity of the current results and the validity of the proposed mathematical model as well as the method of solution.

The current results agree with the results of Youssef and Alghamdi\(^{28,37}\). Moreover, the current results agree with the result in figure 6 of Kundu and Dewanjee\(^{29}\) for the case of boundary condition of case (b). To verify our results with the results in Kundu and Dewanjee\(^{29}\), we represented Fig. 16 in which we use the same values of the parameters in Kundu and Dewanjee\(^{29}\) as follows:

\[
\rho_b W_b = 0.5 \text{ kg/m}^3 \text{s}, \quad \tau_T = 0, \quad \tau_q = 20 \text{ s}, \quad K = 0.2 \text{ W/m}^\circ\text{C}, \quad t = 160 \text{ s}, \quad q_0 = 500 \text{ W/m}^2
\]

and calculating the absolut values of the dynamical and conductive temperature by adding \( T_b = 37^\circ\text{C} \) to the increments as follows:

\[
T_D = (\theta + T_b), \quad T_C = (\varphi + T_b)
\]
Figure 16 show that the current result of the absolute dynamical and conductive temperature are close to the values of the absolute temperature in Kundu and Dewanjee29. Thus, we have a guarantee that the results of the current research are correct and verified.

**Conclusion**

- The dual-phase lag time parameters $\tau_T$, $\tau_q$ have significant effects on the conductive and the dynamical temperature increment of the skin tissue.
- The initial heat flux applying on the surface of the skin tissue has significant on the conductive and the dynamical temperature increment.
- The two-temperature dual-phase-lag (TTDPL) bioheat transfer model is a successful model to simulate the thermal behavior of the skin tissue.
- The results in the current work agree with the results of Kundu and Dewanjee29. Therefore, the two-temperature dual-phase-lag (TTDPL) bioheat transfer model be a successful model to study the bioheat transfer of a skin tissue and supplies us with exact solutions.

**Appendix**

Substituting from Eqs. (14) into (10), we have

$$\left[ \beta_\rho C_\tau \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} + \left( K_T + \beta_\rho C + \beta_\rho w_b C_b \rho_b \right) \frac{\partial}{\partial t} \frac{\partial}{\partial x^2} + \left( \beta w_b C_b \rho_b + K \right) \frac{\partial^2}{\partial x^2} \right] \phi_1(x,t) + \phi_2(x) = \psi$$

which gives

$$\left[ \beta_\rho C_\tau \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} + \left( K_T + \beta_\rho C + \beta_\rho w_b C_b \rho_b \right) \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} + \left( \beta w_b C_b \rho_b + K \right) \frac{\partial^2}{\partial x^2} \right]$$

The Eq. (53) could be separated into two differential equation, one is a partial differential equation of two variables $x$ and $t$, while the second one is ordinary differential equation of one variable $x$ only as follows:

$$\left[ \beta_\rho C_\tau \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} + \left( K_T + \beta_\rho C + \beta_\rho w_b C_b \rho_b \right) \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} + \left( \beta w_b C_b \rho_b + K \right) \frac{\partial^2}{\partial x^2} \right]$$

and

$$(\beta w_b C_b \rho_b + K) \frac{d^2 \phi_2}{dx^2} = w_b C_b \rho_b \phi_2 - \psi$$

The differentia Eq. (53) is the steady-state part and it takes the form

$$\left( \frac{d^2}{dx^2} - \lambda^2 \right) \phi_2(x) = -\gamma \psi$$

where $\lambda^2 = \frac{w_b C_b \rho_b}{(\beta w_b C_b \rho_b + K)} > 0$, $\gamma = \frac{1}{(\beta w_b C_b \rho_b + K)} > 0$, and $\psi = Q_{me}$. The boundary conditions for the steady-state (17) take the form

$$\frac{d\phi_2(x)}{dx} = \begin{cases} \frac{-\psi}{K} & \text{for } x = 0 \\ 0 & \text{for } x = L \end{cases}$$

The general solution of (56) takes the form

$$\phi_2(x) = C_1 \cosh (\lambda x) + C_2 \sinh (\lambda x) + \frac{\gamma}{\lambda^2} \psi$$

Applying the boundary conditions (57) as follows:

$$\frac{d\phi_2}{dx} \bigg|_{x=0} = C_1 \lambda \sinh (\lambda x) + C_2 \lambda \cosh (\lambda x) \bigg|_{x=0} = -\frac{q_0}{K}$$

and
\[ \frac{d\phi_2}{dx} \bigg|_{x=L} = C_1 \sinh (\lambda x) + C_2 \cosh (\lambda x) \bigg|_{x=L} = 0 \quad (60) \]

Hence, we have

\[ C_2 = -\frac{q_0}{K\lambda} \quad (61) \]

and

\[ C_1 = -\frac{q_0}{K\lambda} \frac{\cosh (\lambda L)}{\sinh (\lambda L)} = \frac{q_0}{K\lambda} \frac{\cosh (\lambda L)}{\sinh (\lambda L)} \quad (62) \]

Finally, we have

\[ \phi_2(x) = \frac{q_0}{K\lambda} \frac{\cosh (\lambda L)}{\sinh (\lambda L)} \cosh (\lambda x) - \frac{q_0}{K\lambda} \frac{\sinh (\lambda L)}{\sinh (\lambda L)} \sinh (\lambda x) + \frac{\gamma}{\lambda^2} \psi \quad (63) \]

The last equation after simplification will take the form

\[ \phi_2(x) = \frac{q_0}{K\lambda} \frac{\cosh (L - x)}{\sinh (\lambda L)} + \frac{\gamma}{\lambda^2} \psi \quad (64) \]

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**Author contributions**

H. Y. investigated the governing equations, obtained the numerical solutions, represented the results in figures, wrote the draft of the paper, and reviewed all the writing. N. A. proposed the point of the work, wrote the abstract, sheared in the numerical solution, wrote the discussions, and reviewed all the paper.

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**Competing interests**

The authors declare no competing interests.

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