Sobolev Inequalities in Manifolds with Nonnegative Curvature

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Abstract

We prove a sharp Sobolev inequality on manifolds with nonnegative Ricci curvature. Moreover, we prove a Michael-Simon inequality for submanifolds in manifolds with nonnegative sectional curvature. Both inequalities depend on the asymptotic volume ratio of the ambient manifold. © 2022 Wiley Periodicals LLC.

1 Introduction

Let $M$ be a complete noncompact manifold of dimension $k$ with nonnegative Ricci curvature. The asymptotic volume ratio of $M$ is defined as

$$\theta := \lim_{r \to \infty} \frac{|\{ p \in M : d(p, q) < r \}|}{|B^k| r^k},$$

where $q$ is some fixed point in $M$ and $B^k$ denotes the unit ball in $\mathbb{R}^k$. The Bishop-Gromov relative volume comparison theorem implies that the limit exists, and that $\theta \leq 1$. Note that $\theta$ does not depend on the choice of the point $q$.

Our first result gives a sharp Sobolev inequality on manifolds with nonnegative Ricci curvature.

THEOREM 1.1. Let $M$ be a complete noncompact manifold of dimension $k$ with nonnegative Ricci curvature. Let $D$ be a compact domain in $M$ with boundary $\partial D$, and let $f$ be a positive smooth function on $D$. Then

$$\int_D |\nabla f| + \int_{\partial D} f \geq n |B^k|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \int_D f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

where $\theta$ denotes the asymptotic volume ratio of $M$.

Moreover, we are able to characterize the case of equality in Theorem 1.1:

THEOREM 1.2. Let $M$ be a complete noncompact manifold of dimension $n$ with nonnegative Ricci curvature. Let $D$ be a compact domain in $M$ with boundary $\partial D$, and let $f$ be a positive smooth function on $D$. Suppose that

$$\int_D |\nabla f| + \int_{\partial D} f = n |B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \int_D f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} > 0,$$
where $\theta$ denotes the asymptotic volume ratio of $M$. Then $f$ is constant, $M$ is isometric to Euclidean space, and $D$ is a round ball.

Putting $f \equiv 1$ in Theorem 1.1, we obtain a sharp isoperimetric inequality.

**Corollary 1.3.** Let $M$ be a complete noncompact manifold of dimension $n$ with nonnegative Ricci curvature. Let $D$ be a compact domain in $M$ with boundary $\partial D$. Then

$$|\partial D| \geq n |B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} |D|^{\frac{n-1}{n}},$$

where $\theta$ denotes the asymptotic volume ratio of $M$.

Corollary 1.3 is similar in spirit to the Lévy-Gromov inequality for manifolds with Ricci curvature at least $n-1$ (cf. [10], app. C). The Lévy-Gromov inequality was recently generalized in [13] and [8].

In the three-dimensional case, Corollary 1.3 was proved in a recent work of V. Agostiniani, M. Fogagnolo, and L. Mazzieri (cf. [1, theorem 6.1]). The proof of [1, theorem 6.1] builds on an argument due to G. Huisken [12] and uses mean curvature flow.

We will present the proof of Theorem 1.1 in Section 2. The proof of Theorem 1.1 uses the Alexandrov-Bakelman-Pucci method and is inspired in part by an elegant argument due to X. Cabré [5] (see also [4, 6, 15–17] for related work). The proof of Theorem 1.2 will be discussed in Section 3.

We next turn to Sobolev inequalities for submanifolds. In a recent paper [3], we proved a Michael-Simon-type inequality for submanifolds in Euclidean space. While the classical Michael-Simon inequality (cf. [2, 14]) is not sharp, our inequality is sharp if the codimension is at most 2. In particular, the results in [3] imply a sharp isoperimetric inequality for minimal submanifolds in Euclidean space of codimension at most 2, answering a question first studied by Torsten Carleman [7] in 1921.

The following theorem generalizes the main result in [3] to the Riemannian setting.

**Theorem 1.4.** Let $M$ be a complete noncompact manifold of dimension $n + m$ with nonnegative sectional curvature. Let $\Sigma$ be a compact submanifold of $M$ of dimension $n$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$. If $m \geq 2$, then

$$\int_{\Sigma} \sqrt{\nabla^2 \Sigma f^2 + f^2 |H|^2} + \int_{\partial \Sigma} f \geq n \left( \frac{(n + m) |B^{n+m}|}{m |B^m|} \right)^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \int_{\Sigma} f \frac{n+1}{n+2} \right)^{\frac{n-1}{n}},$$

where $\theta$ denotes the asymptotic volume ratio of $M$ and $H$ denotes the mean curvature vector of $\Sigma$.

Note that $(n + 2)|B^{n+2}| = 2|B^2||B^n|$. Hence, we obtain a sharp Sobolev inequality for submanifolds of codimension 2.
Corollary 1.5. Let $M$ be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let $\Sigma$ be a compact submanifold of $M$ of dimension $n$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$. Then
\[
\int_{\Sigma} \sqrt{|\nabla_{\Sigma} f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f \geq n |B^n| \frac{1}{n} \theta \left( \int_{\Sigma} f^\frac{n}{n-1} \right)^\frac{n-1}{n},
\]
where $\theta$ denotes the asymptotic volume ratio of $M$ and $H$ denotes the mean curvature vector of $\Sigma$.

Moreover, we can characterize the case of equality in Corollary 1.5:

Theorem 1.6. Let $M$ be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let $\Sigma$ be a compact submanifold of $M$ of dimension $n$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$. Suppose that
\[
\int_{\Sigma} \sqrt{|\nabla_{\Sigma} f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f = n |B^n| \frac{1}{n} \theta \left( \int_{\Sigma} f^\frac{n}{n-1} \right)^\frac{n-1}{n} > 0,
\]
where $\theta$ denotes the asymptotic volume ratio of $M$ and $H$ denotes the mean curvature vector of $\Sigma$. Then $f$ is constant, $M$ is isometric to Euclidean space, and $\Sigma$ is a flat round ball.

By putting $f = 1$ in Corollary 1.5, we obtain an isoperimetric inequality for minimal submanifolds of codimension 2, generalizing the result in [3].

Corollary 1.7. Let $M$ be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let $\Sigma$ be a compact minimal submanifold of $M$ of dimension $n$ with boundary $\partial \Sigma$. Then
\[
|\partial \Sigma| \geq n |B^n| \frac{1}{n} \theta \left| \Sigma \right| \frac{n-1}{n},
\]
where $\theta$ denotes the asymptotic volume ratio of the ambient manifold $M$.

Finally, the inequalities in Corollary 1.5 and Corollary 1.7 also hold in the codimension 1 setting. Indeed, if $\Sigma$ is an $n$-dimensional submanifold of an $(n+1)$-dimensional manifold $M$, then we can view $\Sigma$ as a submanifold of the $(n+2)$-dimensional manifold $M \times \mathbb{R}$. Note that the product $M \times \mathbb{R}$ has the same asymptotic volume ratio as $M$ itself.

The proof of Theorem 1.4 will be presented in Section 4. This argument extends our earlier proof in the Euclidean case (cf. [3]), and relies on the Alexandrov-Bakelman-Pucci technique. Moreover, the proof shares some common features with the work of E. Heintze and H. Karcher [11] concerning the volume of a tubular neighborhood of a submanifold. Finally, the proof of Theorem 1.6 will be discussed in Section 5.
2 Proof of Theorem 1.1

Throughout this section, we assume that \((M, g)\) is a complete noncompact manifold of dimension \(n\) with nonnegative Ricci curvature. Moreover, we assume that \(D\) is a compact domain in \(M\), and \(f\) is a positive smooth function on \(D\). Let \(R\) denote the Riemann curvature tensor of \((M, g)\).

It suffices to prove the assertion in the special case when \(D\) is connected. By scaling, we may assume that

\[
\int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{n-1}. 
\]

Since \(D\) is connected, we can find a function \(u : D \to \mathbb{R}\) with the property that

\[
\text{div}(f \nabla u) = n f^{\frac{n-1}{n-1}} - |\nabla f|
\]
on \(D\) and \(\langle \nabla u, \eta \rangle = 1\) at each point on \(\partial D\). Here, \(\eta\) denotes the outward-pointing unit normal to \(\partial D\). Standard elliptic regularity theory implies that the function \(u\) is of class \(C^{2,\gamma}\) for each \(0 < \gamma < 1\) (cf. [9, theorem 6.30]).

We define

\[
U := \{x \in D \setminus \partial D : |\nabla u(x)| < 1\}. 
\]

For each \(r > 0\), we denote by \(A_r\) the set of all points \(\bar{x} \in U\) with the property that

\[
ru(x) + \frac{1}{2} d(x, \exp_{\bar{x}}(ru(\bar{x})))^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 |\nabla u(\bar{x})|^2
\]

for all \(x \in D\). Moreover, for each \(r > 0\), we define a map \(\Phi_r : D \to M\) by

\[
\Phi_r(x) = \exp_x(r \nabla u(x))
\]

for all \(x \in D\). Note that the map \(\Phi_r\) is of class \(C^{1,\gamma}\) for each \(0 < \gamma < 1\).

**Lemma 2.1.** Assume that \(x \in U\). Then \(\Delta u(x) \leq n f(x)^{\frac{1}{n-1}}\).

**Proof.** Using the inequality \(|\nabla u(x)| < 1\) and the Cauchy-Schwarz inequality, we obtain

\[
-\langle \nabla f(x), \nabla u(x) \rangle \leq |\nabla f(x)|.
\]

Moreover, \(\text{div}(f \nabla u) = n f^{\frac{n}{n-1}} - |\nabla f|\) by definition of \(u\). This implies

\[
f(x) \Delta u(x) = n f(x)^{\frac{n}{n-1}} - |\nabla f(x)| - \langle \nabla f(x), \nabla u(x) \rangle \leq n f(x)^{\frac{n}{n-1}}.
\]

From this, the assertion follows. \(\square\)

**Lemma 2.2.** The set

\[
\{p \in M : d(x, p) < r \text{ for all } x \in D\}
\]
is contained in the set

\[
\{\Phi_r(x) : x \in A_r\},
\]

for all
PROOF. Fix a point $p \in M$ with the property that $d(x, p) < r$ for all $x \in D$. Since $\langle \nabla u, \eta \rangle = 1$ at each point on $\partial D$, the function $x \mapsto ru(x) + \frac{1}{2}d(x, p)^2$ cannot attain its minimum on the boundary of $D$. Let us fix a point $\bar{x} \in D \setminus \partial D$ where the function $x \mapsto ru(x) + \frac{1}{2}d(x, p)^2$ attains its minimum. Moreover, let $\bar{\gamma} : [0, r] \to M$ be a minimizing geodesic such that $\bar{\gamma}(0) = \bar{x}$ and $\bar{\gamma}(r) = p$. Clearly, $r|\bar{\gamma}'(0)| = d(\bar{x}, p)$. For every smooth path $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = p$, we obtain

$$ru(\gamma(0)) + \frac{1}{2} r \int_0^r |\gamma'(t)|^2 dt \geq ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), p)^2$$

$$\geq ru(\bar{x}) + \frac{1}{2} d(\bar{x}, p)^2$$

$$= ru(\bar{\gamma}(0)) + \frac{1}{2} r^2 |\bar{\gamma}'(0)|^2$$

$$= ru(\bar{\gamma}(0)) + \frac{1}{2} r \int_0^r |\bar{\gamma}'(t)|^2 dt.$$

In other words, the path $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2 dt$ among all smooth paths $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = p$. Hence, the formula for the first variation of energy implies

$$\nabla u(\bar{x}) = \bar{\gamma}'(0).$$

From this, we deduce that

$$\Phi_r(\bar{x}) = \exp_{\bar{x}}(r \nabla u(\bar{x})) = \exp_{\bar{\gamma}(0)}(r \bar{\gamma}'(0)) = \bar{\gamma}(r) = p.$$ 

Moreover,

$$r|\nabla u(\bar{x})| = r|\bar{\gamma}'(0)| = d(\bar{x}, p).$$

By assumption, $d(\bar{x}, p) < r$. This implies $|\nabla u(\bar{x})| < 1$. Therefore, $\bar{x} \in U$. Finally, for each point $x \in D$, we have

$$ru(x) + \frac{1}{2} d(x, \exp_{\bar{x}}(r \nabla u(\bar{x})))^2 = ru(x) + \frac{1}{2} d(x, p)^2$$

$$\geq ru(\bar{x}) + \frac{1}{2} d(\bar{x}, p)^2$$

$$= ru(\bar{x}) + \frac{1}{2} r^2 |\nabla u(\bar{x})|^2.$$ 

Thus, $\bar{x} \in A_r$. This completes the proof of Lemma 2.2. \qed

**Lemma 2.3.** Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(t \nabla u(\bar{x}))$ for all $t \in [0, r]$. If $Z$ is a smooth vector field along $\bar{\gamma}$ satisfying $Z(r) = 0$, then

$$(D^2 u)(Z(0), Z(0)) + \int_0^r \left( |D_t Z(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t)) \right) dt \geq 0.$$
PROOF. Let us consider an arbitrary smooth path \( \gamma : [0, r] \to M \) satisfying \( \gamma(0) \in D \) and \( \gamma(r) = \overline{\gamma}(r) \). Since \( \overline{\gamma} \in A_r \), we obtain
\[
ru(\gamma(0)) + \frac{1}{2} r \int_0^r |\gamma'(t)|^2dt \geq ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), \gamma(r))^2
\]
\[
= ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), \exp_{\overline{\gamma}}(r \nabla u(\overline{\gamma})))^2
\]
\[
\geq ru(\overline{\gamma}) + \frac{1}{2} r^2 |\nabla u(\overline{\gamma})|^2
\]
\[
= ru(\overline{\gamma}(0)) + \frac{1}{2} r \int_0^r |\overline{\gamma}'(t)|^2dt.
\]
In other words, the path \( \overline{\gamma} \) minimizes the functional \( u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2dt \) among all smooth paths \( \gamma : [0, r] \to M \) satisfying \( \gamma(0) \in D \) and \( \gamma(r) = \overline{\gamma}(r) \). Hence, the assertion follows from the formula for the second variation of energy.

\[\square\]

**Lemma 2.4.** Assume that \( \overline{\gamma} \in A_r \), and let \( \overline{\gamma}(t) := \exp_{\overline{\gamma}}(t \nabla u(\overline{\gamma})) \) for all \( t \in [0, r] \). Moreover, let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_{\overline{\gamma}}M \). Suppose that \( W \) is a Jacobi field along \( \overline{\gamma} \) satisfying \( \langle D_t W(\overline{\gamma}), e_j \rangle = (D^2 u)(W(0), e_j) \) for each \( 1 \leq j \leq n \). If \( W(\tau) = 0 \) for some \( \tau \in (0, r) \), then \( W \) vanishes identically.

**Proof.** Suppose that \( W(\tau) = 0 \) for some \( \tau \in (0, r) \). By assumption,
\[
\langle D_t W(\overline{\gamma}), W(0) \rangle = (D^2 u)(W(0), W(0)).
\]
Since \( W \) is a Jacobi field, we obtain
\[
\int_0^\tau \left( |D_t W(t)|^2 - R(\overline{\gamma}'(t), W(t), \overline{\gamma}'(t), W(t)) \right)dt = \langle D_t W(\tau), W(\tau) \rangle - \langle D_t W(0), W(0) \rangle
\]
\[
= -(D^2 u)(W(0), W(0)).
\]
Let us define a vector field \( \tilde{W} \) along \( \overline{\gamma} \) by
\[
\tilde{W}(t) = \begin{cases} W(t) & \text{for } t \in [0, \tau] \\ 0 & \text{for } t \in [\tau, r]. \end{cases}
\]
Clearly, \( \tilde{W}(r) = 0 \). Moreover,
\[
\int_0^r \left( |D_t \tilde{W}(t)|^2 - R(\overline{\gamma}'(t), \tilde{W}(t), \overline{\gamma}'(t), \tilde{W}(t)) \right)dt = -(D^2 u)(\tilde{W}(0), \tilde{W}(0)).
\]
Using Lemma 2.3, we conclude that
\[
\int_0^R \left( |D_t Z(t)|^2 - R(\nabla(t), Z(t), \nabla'(t), Z(t)) \right) dt \\
\geq \int_0^R \left( |D_t \tilde{W}(t)|^2 - R(\nabla(t), \tilde{W}(t), \nabla'(t), \tilde{W}(t)) \right) dt
\]
for every smooth vector field $Z$ along $\nabla$ satisfying $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. By approximation, this inequality holds for every vector field $Z$ which is piecewise $C^1$ and satisfies $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. In other words, the vector field $\tilde{W}$ minimizes the index form among all vector fields which are piecewise $C^1$ and have the same boundary values as $\tilde{W}$. Consequently, $\tilde{W}$ must be of class $C^1$. This implies $D_t W(\tau) = 0$. Since $W(\tau) = 0$ and $D_t W(\tau) = 0$, standard uniqueness results for ODE imply that $W$ vanishes identically. \hfill \Box

**Proposition 2.5.** Assume that $x \in A_r$. Then the function
\[
t \mapsto (1 + t f(x))^{\frac{1}{n-1}} |\det D\Phi_t(x)|
\]
is monotone decreasing for $t \in (0, r)$.

**Proof.** Fix an arbitrary point $\overline{x} \in A_r$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\overline{x}} M$, and let $(x_1, \ldots, x_n)$ be a system of geodesic normal coordinates around $\overline{x}$ such that $\frac{\partial}{\partial x_i} = e_i$ at $\overline{x}$. Let $\nabla(t) := \exp_\overline{x}(t \nabla t(\overline{x}))$ for all $t \in [0, r]$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of $e_i$ along $\nabla$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\nabla$ satisfying $X_i(0) = e_i$ and
\[
\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j)
\]
for all $1 \leq j \leq n$. It follows from Lemma 2.4 that $X_1(t), \ldots, X_n(t)$ are linearly independent for each $t \in (0, r)$.

Let us define an $n \times n$-matrix $P(t)$ by
\[
P_{ij}(t) = \langle X_i(t), E_j(t) \rangle
\]
for $1 \leq i, j \leq n$. Moreover, we define an $n \times n$-matrix $S(t)$ by
\[
S_{ij}(t) = R(\nabla'(t), E_i(t), \nabla'(t), E_j(t))
\]
for $1 \leq i, j \leq n$. Clearly, $S(t)$ is symmetric. Moreover, since $M$ has non-negative Ricci curvature, we know that $\text{tr}(S(t)) \geq 0$. Since the vector fields $X_1(t), \ldots, X_n(t)$ are Jacobi fields, we obtain
\[
P''(t) = -P(t)S(t).
\]
Moreover,
\[
P_{ij}(0) = \delta_{ij}
\]
and
\[
P_{ij}'(0) = (D^2 u)(e_i, e_j).
\]
In particular, the matrix \( P'(0)P(0)^T \) is symmetric. Moreover, the matrix
\[
\frac{d}{dt}(P'(t)P(t)^T) = P''(t)P(t)^T + P'(t)P'(t)^T
\]
\[
= -P(t)S(t)P(t)^T + P'(t)P'(t)^T
\]
is symmetric for each \( t \). Thus, we conclude that the matrix \( P'(t)P(t)^T \) is symmetric for each \( t \).

Since \( X_1(t), \ldots, X_n(t) \) are linearly independent for each \( t \in (0, r) \), the matrix \( P(t) \) is invertible for each \( t \in (0, r) \). Since \( P'(t)P(t)^T \) is symmetric for each \( t \in (0, r) \), it follows that the matrix \( Q(t) := P(t)^{-1}P'(t) \) is symmetric for each \( t \in (0, r) \). The matrix \( Q(t) \) satisfies the Riccati equation
\[
Q'(t) = P(t)^{-1}P''(t) - P(t)^{-1}P'(t)P(t)^{-1}P'(t) = -S(t) - Q(t)^2
\]
for all \( t \in (0, r) \). Moreover, since \( Q(t) \) is symmetric, we obtain \( \text{tr}(Q(t)^2) \geq \frac{1}{n} \text{tr}(Q(t))^2 \) for all \( t \in (0, r) \). Since \( \text{tr}(S(t)) \geq 0 \), it follows that
\[
\frac{d}{dt} \text{tr}(Q(t)) \leq -\text{tr}(Q(t)^2) \leq -\frac{1}{n} \text{tr}(Q(t))^2
\]
for all \( t \in (0, r) \). Clearly,
\[
\lim_{t \to 0} Q_{ij}(t) = (D^2 u)(e_i, e_j).
\]
Using Lemma 2.1, we obtain
\[
\lim_{t \to 0} \text{tr}(Q(t)) = \Delta u(\mathbf{x}) \leq n f(\mathbf{x})^{\frac{1}{n-1}}.
\]
Hence, a standard ODE comparison principle implies
\[
\text{tr}(Q(t)) \leq \frac{n f(\mathbf{x})^{\frac{1}{n-1}}}{1 + t f(\mathbf{x})^{\frac{1}{n-1}}}
\]
for all \( t \in (0, r) \).

We next consider the determinant of \( P(t) \). Clearly, \( \det P(t) > 0 \) if \( t \) is sufficiently small. Since \( P(t) \) is invertible for each \( t \in (0, r) \), it follows that \( \det P(t) > 0 \) for all \( t \in (0, r) \). Using the estimate for the trace of \( Q(t) = P(t)^{-1}P'(t) \), we obtain
\[
\frac{d}{dt} \log \det P(t) = \text{tr}(Q(t)) \leq \frac{n f(\mathbf{x})^{\frac{1}{n-1}}}{1 + t f(\mathbf{x})^{\frac{1}{n-1}}}
\]
for all \( t \in (0, r) \). Consequently, the function
\[
t \mapsto (1 + t f(\mathbf{x})^{\frac{1}{n-1}})^{-n} \det P(t)
\]
is monotone decreasing for \( t \in (0, r) \).

Finally, we observe that
\[
\frac{\partial \Phi_i}{\partial x_i} = X_i(t)
\]
for $1 \leq i \leq n$. Consequently, $|\det D\Phi_t(x)| = \det P(t)$ for all $t \in (0, r)$. Putting these facts together, the assertion follows. \hfill \Box

**Corollary 2.6.** The Jacobian determinant of $\Phi_r$ satisfies

$$|\det D\Phi_r(x)| \leq (1 + rf(x)\frac{1}{n-1})^n$$

for all $x \in A_r$.

**Proof.** Since $\lim_{t \to 0} |\det D\Phi_t(x)| = 1$, the assertion follows from Proposition 2.5. \hfill \Box

After these preparations, we now complete the proof of Theorem 1.1. Using Lemma 2.2 and Corollary 2.6, we obtain

$$|\{p \in M : d(x, p) < r \text{ for all } x \in D\}| \leq \int_{A_r} |\det D\Phi_r(x)| d\text{vol}(x) \leq \int_U (1 + rf(x)\frac{1}{n-1})^n d\text{vol}(x)$$

for all $r > 0$. Finally, we divide by $r^n$ and send $r \to \infty$. This gives

$$|B^n| \theta \leq \int_U f^{\frac{n}{n-1}} \leq \int_D f^{\frac{n}{n-1}}.$$ 

Thus,

$$\int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{\frac{n}{n-1}} \geq n |B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \int_D f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} > 0.$$ 

This completes the proof of Theorem 1.1.

**3 Proof of Theorem 1.2**

Let $(M, g)$ be a complete noncompact manifold of dimension $n$ with nonnegative Ricci curvature. Let $D$ be a compact domain in $M$ with boundary $\partial D$, and let $f$ be a positive smooth function on $D$ satisfying

$$\int_D |\nabla f| + \int_{\partial D} f = n |B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \int_D f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} > 0,$$

where $\theta$ denotes the asymptotic volume ratio of $M$.

If $D$ is disconnected, we may apply Theorem 1.1 to each connected component of $D$, and take the sum over all connected components. This will lead to a contradiction. Therefore, $D$ must be connected.

By scaling, we may assume that

$$\int_D |\nabla f| + \int_{\partial D} f = n |B^n| \theta$$

and

$$\int_D f^{\frac{n}{n-1}} = |B^n| \theta.$$
In particular, 
\[ \int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{\frac{n}{n-1}}. \]
Since \( D \) is connected, we can find a function \( u : D \to \mathbb{R} \) such that
\[ \text{div}(f \nabla u) = n f^{\frac{n}{n-1}} - |\nabla f| \]
on \( D \) and \( \langle \nabla u, \eta \rangle = 1 \) at each point on \( \partial D \). Moreover, \( u \) is of class \( C^{2;\gamma} \) for each \( 0 < \gamma < 1 \). Let us define \( U, A_r, \) and \( \Phi_r \) as in Section 2.

**Lemma 3.1.** Assume that \( x \in U \). Then \( |\det D \Phi_t(x)| \geq (1 + t f(x)^{\frac{1}{n-1}})^n \) for each \( t > 0 \).

**Proof.** Let us fix a point \( \overline{x} \in U \). Suppose that \( |\det D \Phi_t(\overline{x})| < (1 + t f(\overline{x})^{\frac{1}{n-1}})^n \) for some \( t > 0 \). Let us fix a real number \( \varepsilon \in (0, 1) \) such that
\[ |\det D \Phi_t(\overline{x})| < (1 - \varepsilon) (1 + t f(\overline{x})^{\frac{1}{n-1}})^n. \]
By continuity, we can find an open neighborhood \( V \) of the point \( \overline{x} \) such that
\[ |\det D \Phi_t(x)| \leq (1 - \varepsilon) (1 + t f(x)^{\frac{1}{n-1}})^n \]
for all \( x \in V \). Using Proposition 2.5, we conclude that
\[ |\det D \Phi_r(x)| \leq (1 - \varepsilon) (1 + r f(x)^{\frac{1}{n-1}})^n \]
for all \( r > t \) and all \( x \in A_r \cap V \). Using this fact together with Lemma 2.2 and Corollary 2.6, we obtain
\[ |\{ p \in M : d(x, p) < r \text{ for all } x \in D \}| \]
\[ \leq \int_{A_r} |\det D \Phi_r(x)| d\text{vol}(x) \]
\[ \leq \int_U (1 - \varepsilon \cdot 1_V(x)) (1 + r f(x)^{\frac{1}{n-1}})^n d\text{vol}(x) \]
for all \( r > t \). Finally, we divide by \( r^n \) and send \( r \to \infty \). This gives
\[ |B^n|_\theta \leq \int_U (1 - \varepsilon \cdot 1_V) f^{\frac{1}{n-1}} < \int_D f^{\frac{1}{n-1}} = |B^n|_\theta. \]
This is a contradiction. \( \square \)

**Lemma 3.2.** Assume that \( x \in U \). Then \( D^2 u(x) = \frac{1}{2} f(x)^{\frac{1}{n-1}} g \).

**Proof.** Let us fix a point \( \overline{x} \in U \). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_{\overline{x}} M \). We define \( \overline{\gamma}(t) := \exp_{\overline{x}}(t \nabla u(\overline{x})) \) for all \( t \geq 0 \). For each \( 1 \leq i \leq n \), we denote by \( E_i(t) \) the parallel transport of \( e_i \) along \( \overline{\gamma} \). Moreover, for each \( 1 \leq i \leq n \), we denote by \( X_i(t) \) the unique Jacobi field along \( \overline{\gamma} \) satisfying \( X_i(0) = e_i \) and
\[ \langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j) \]
for all $1 \leq j \leq n$. Finally, we define an $n \times n$-matrix $P(t)$ by
\[
P_{ij}(t) = \langle X_i(t), E_j(t) \rangle
\]
for $1 \leq i, j \leq n$.

By Lemma 3.1, we know that $|\det P(t)| \geq (1 + t f(\overline{x})^{\frac{1}{n-1}})^n$ for all $t > 0$. Since $\det P(t) > 0$ if $t > 0$ is sufficiently small, we conclude that
\[
\det P(t) \geq (1 + t f(\overline{x})^{\frac{1}{n-1}})^n
\]
for all $t > 0$. In particular, $P(t)$ is invertible for each $t > 0$.

We next define $Q(t) := P(t)^{-1} P'(t)$ for all $t > 0$. As in Section 2, we can show that the matrix $Q(t)$ is symmetric for each $t > 0$. The Riccati equation for $Q(t)$ gives
\[
\frac{d}{dt} \text{tr}(Q(t)) \leq -\text{tr}(Q(t)^2) \leq -\frac{1}{n} \text{tr}(Q(t))^2
\]
for all $t > 0$. Moreover,
\[
\lim_{t \to 0} \text{tr}(Q(t)) = \Delta u(\overline{x}) \leq n f(\overline{x})^{\frac{1}{n-1}}
\]
by Lemma 2.1. This implies
\[
\text{tr}(Q(t)) \leq \frac{n f(\overline{x})^{\frac{1}{n-1}}}{1 + t f(\overline{x})^{\frac{1}{n-1}}},
\]
hence
\[
\frac{d}{dt} \log \det P(t) \leq \frac{n f(\overline{x})^{\frac{1}{n-1}}}{1 + t f(\overline{x})^{\frac{1}{n-1}}}
\]
for all $t > 0$. Integrating this ODE gives
\[
\det P(t) \leq (1 + t f(\overline{x})^{\frac{1}{n-1}})^n
\]
for all $t > 0$.

Putting these facts together, we conclude that $\det P(t) = (1 + t f(\overline{x})^{\frac{1}{n-1}})^n$ for all $t > 0$. Differentiating this identity with respect to $t$, we obtain
\[
\text{tr}(Q(t)) = \frac{n f(\overline{x})^{\frac{1}{n-1}}}{1 + t f(\overline{x})^{\frac{1}{n-1}}}
\]
for all $t > 0$. Using the Riccati equation for $Q(t)$, we conclude that $\text{tr}(Q(t)^2) = \frac{1}{n} \text{tr}(Q(t))^2$ for all $t > 0$. Consequently, the trace-free part of $Q(t)$ vanishes for each $t > 0$. Therefore,
\[
Q_{ij}(t) = \frac{f(\overline{x})^{\frac{1}{n-1}}}{1 + t f(\overline{x})^{\frac{1}{n-1}}} \delta_{ij}
\]
for all $t > 0$. In particular,
\[
(D^2u)(e_i, e_j) = \lim_{t \to 0} Q_{ij}(t) = f(\overline{x})^{\frac{1}{n-1}} \delta_{ij},
\]
This completes the proof of Lemma 3.2.

**LEMMA 3.3.** Assume that \( x \in U \). Then \( \nabla f(x) = 0 \).

**PROOF.** Let us consider an arbitrary point \( x \in U \). Using the definition of \( \mathcal{U} \), we obtain

\[
f(x) \Delta u(x) = n \frac{f(x)^n}{n-1} - |\nabla f(x)| - \langle \nabla f(x), \nabla u(x) \rangle.
\]

On the other hand, Lemma 3.2 implies \( \Delta u(x) = n \frac{f(x)^n}{n-1} \). Putting these facts together, we conclude that \( \langle \nabla f(x), \nabla u(x) \rangle = -|\nabla f(x)|. \) Since \( |\nabla u(x)| < 1 \), it follows that \( \nabla f(x) = 0 \). This completes the proof of Lemma 3.3. \( \square \)

**LEMMA 3.4.** The set \( U \) is dense in \( D \).

**PROOF.** Suppose that \( U \) is not dense in \( D \). Arguing as in Section 2, we obtain

\[
|B^n| \theta \leq \int_U f \frac{n}{n-1} < \int_D f \frac{n}{n-1} = |B^n| \theta.
\]

This is a contradiction. This completes the proof of Lemma 3.4. \( \square \)

Since \( U \) is a dense subset of \( D \), we conclude that \( \nabla f = 0 \) and \( D^2 u = f^{1/(n-1)} g \) at each point on \( D \). Since \( D \) is connected, it follows that \( f \) is constant. This implies \( |\partial D| = n |B^n|^{1/n} \theta^{1/n} |D|^{(n-1)/n} \). Note that \( \mathcal{U} \) is a smooth function on \( D \). Each critical point of \( u \) lies in the interior of \( D \) and is nondegenerate with Morse index 0. In particular, the function \( u \) has at most finitely many critical points.

We next consider the flow on \( \Sigma \) generated by the vector field \(-\nabla \Sigma \mathcal{U}\). Since the vector field \(-\nabla \Sigma \mathcal{U}\) points inward along the boundary \( \partial \Sigma \), the flow is defined for all nonnegative times. This gives a one-parameter family of smooth maps \( \psi_s : \Sigma \rightarrow \Sigma \), where \( s \geq 0 \). Since \( \Sigma \) is connected, standard arguments from Morse theory imply that the function \( u \) has exactly one critical point, and \( \mathcal{U} \) attains its global minimum at that point. It follows that the diameter of \( \psi_s(\Sigma) \) converges to 0 as \( s \rightarrow \infty \).

Since \( D^2 \mathcal{U} \) is a constant multiple of the metric, the isoperimetric ratio is unchanged under the flow \( \psi_s \). This implies \( |\psi_s(\partial \Sigma)| = n |B^n|^{1/n} \theta^{1/n} |\psi_s(\Sigma)|^{(n-1)/n} \) for each \( s \geq 0 \). If \( \theta < 1 \), this contradicts the Euclidean isoperimetric inequality when \( s \) is sufficiently large. Thus, we conclude that \( \theta = 1 \). Consequently, \( \mathcal{M} \) is isometric to Euclidean space.

Once we know that \( \mathcal{M} \) is isometric to Euclidean space, it follows that \( D \) is a round ball. This completes the proof of Theorem 1.2.

### 4 Proof of Theorem 1.4

Throughout this section, we assume that \((\mathcal{M}, \mathcal{g})\) is a complete noncompact manifold of dimension \( n + m \) with nonnegative sectional curvature. Moreover, we assume that \( \Sigma \) is a compact submanifold of \( \mathcal{M} \) of dimension \( n \) (possibly with boundary \( \partial \Sigma \)), and \( f \) is a positive smooth function on \( \Sigma \). Let \( \overline{\mathcal{D}} \) denote the Levi-Civita
connection on the ambient manifold \((M, \overline{g})\), and let \(\overline{R}\) denote the Riemann curvature tensor of \((M, \overline{g})\). We denote by \(II\) the second fundamental form of \(\Sigma\). For each point \(x \in \Sigma\), \(II\) is a symmetric bilinear form on \(T_x \Sigma\) that takes values in the normal space \(T_x^\perp \Sigma\). If \(X\) and \(Y\) are tangent vector fields on \(\Sigma\), and \(V\) is a normal vector field along \(\Sigma\), then \(\langle II(X, Y), V \rangle = \langle \overline{D}_X Y, V \rangle = -\langle \overline{D}_X V, Y \rangle\).

It suffices to prove the assertion in the special case when \(\Sigma\) is connected. By scaling, we may assume that

\[
\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2|H|^2} + \int_{\partial \Sigma} f = n \int_\Sigma f \frac{\mu}{n-1}.
\]

Since \(\Sigma\) is connected, we can find a function \(u : \Sigma \to \mathbb{R}\) with the property that

\[
\text{div}_\Sigma(f \nabla^\Sigma u) = n f \frac{\mu}{n-1} - \sqrt{|\nabla^\Sigma f|^2 + f^2|H|^2}
\]

on \(\Sigma\) and \(\langle \nabla^\Sigma u, \eta \rangle = 1\) at each point on \(\partial \Sigma\). Here, \(\eta\) denotes the conormal to \(\partial \Sigma\). Standard elliptic regularity theory implies that the function \(u\) is of class \(C^{2,\gamma}\) for each \(0 < \gamma < 1\) (cf. [9, theorem 6.30]).

We define

\[
\Omega := \{x \in \Sigma \setminus \partial \Sigma : |\nabla^\Sigma u(x)| < 1\},
\]

\[
U := \{(x, y) : x \in \Sigma \setminus \partial \Sigma, y \in T_x^\perp \Sigma, |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}.
\]

For each \(r > 0\), we denote by \(A_r\) the set of all points \((\overline{x}, \overline{y}) \in U\) with the property that

\[
u u(x) + \frac{1}{2} d^2 \left( x, \exp_\Sigma(r \nabla^\Sigma u(\overline{x}) + r \overline{y}) \right) \geq u(\overline{x}) + \frac{1}{2} r^2 (|\nabla^\Sigma u(\overline{x})|^2 + |\overline{y}|^2)
\]

for all \(x \in \Sigma\). Moreover, for each \(r > 0\), we define a map \(\Phi_r : T_x^\perp \Sigma \to M\) by

\[
\Phi_r(x, y) = \exp_\Sigma(r \nabla^\Sigma u(x) + ry)
\]

for all \(x \in \Sigma\) and \(y \in T_x^\perp \Sigma\). Note that the map \(\Phi_r\) is of class \(C^{1,\gamma}\) for each \(0 < \gamma < 1\).

**Lemma 4.1.** Assume that \(x \in \Omega\) and \(y \in T_x^\perp \Sigma\) satisfy \(|\nabla^\Sigma u(x)|^2 + |y|^2 \leq 1\). Then \(\Delta u(x) - \langle H(x), y \rangle \leq n f(x)^{1/(n-1)}\).

**Proof.** Using the inequality \(|\nabla^\Sigma u(x)|^2 + |y|^2 \leq 1\) and the Cauchy-Schwarz inequality, we obtain

\[
-\langle \nabla^\Sigma f(x), \nabla^\Sigma u(x) \rangle - f(x) \langle H(x), y \rangle \leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2|H(x)|^2} \sqrt{|\nabla^\Sigma u(x)|^2 + |y|^2}
\]

\[
\leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2|H(x)|^2}.
\]

Moreover, \(\text{div}_\Sigma(f \nabla^\Sigma u) = n f \frac{\mu}{n-1} - \sqrt{|\nabla^\Sigma f|^2 + f^2|H|^2}\) by definition of \(u\). Consequently,

\[
f(x) \Delta u(x) - f(x) \langle H(x), y \rangle
\]
SOBOLEV INEQUALITIES

\[ = nf(x) \frac{\partial}{\partial t} - \sqrt{|\nabla u |^2 + f(x)^2|H(x)|^2} \]

\[ - \langle \nabla u, \nabla u(x) \rangle - f(x) \langle H(x), y \rangle \]

\[ \leq nf(x) \frac{\partial}{\partial t}. \]

From this, the assertion follows. \qed

**Lemma 4.2.** For each \( 0 \leq \sigma < 1 \), the set

\[ \{ p \in M : \sigma r < d(x, p) < r \text{ for all } x \in \Sigma \} \]

is contained in the set

\[ \{ \Phi_r(x, y) : (x, y) \in A_r, |\nabla^1 u(x)|^2 + |y|^2 > \sigma^2 \}. \]

**Proof.** Let us fix a real number \( 0 \leq \sigma < 1 \) and a point \( p \in M \) with the property that \( \sigma r < d(x, p) < r \) for all \( x \in \Sigma \). Since \( \langle \nabla^1 u, \eta \rangle = 1 \) at each point on \( \partial \Sigma \), the function \( x \mapsto ru(x) + \frac{1}{2} d(x, p)^2 \) cannot attain its minimum on the boundary of \( \Sigma \). Let us fix a point \( \overline{x} \in \Sigma \setminus \partial \Sigma \) where the function \( x \mapsto ru(x) + \frac{1}{2} d(x, p)^2 \) attains its minimum. Moreover, let \( \overline{y} : [0, r] \rightarrow M \) be a minimizing geodesic such that \( \overline{y}(0) = \overline{x} \) and \( \overline{y}(r) = p \). Clearly, \( r|\overline{y}'(0)| = d(\overline{x}, p) \). For every smooth path \( \gamma : [0, r] \rightarrow M \) satisfying \( \gamma(0) \in \Sigma \) and \( \gamma(r) = p \), we obtain

\[ ru(\gamma(0)) + \frac{1}{2} r \int_0^r |\gamma'(t)|^2 dt \geq ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), p)^2 \]

\[ \geq ru(\overline{x}) + \frac{1}{2} d(\overline{x}, p)^2 \]

\[ = ru(\overline{y}(0)) + \frac{1}{2} r^2 |\overline{y}'(0)|^2 \]

\[ = ru(\overline{y}(0)) + \frac{1}{2} r \int_0^r |\overline{y}'(t)|^2 dt. \]

In other words, the path \( \overline{y} \) minimizes the functional \( u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2 dt \) among all smooth paths \( \gamma : [0, r] \rightarrow M \) satisfying \( \gamma(0) \in \Sigma \) and \( \gamma(r) = p \).

Hence, the formula for the first variation of energy implies

\[ \nabla^1 u(\overline{x}) - \overline{y}'(0) \in T_{\overline{x}} \Sigma. \]

Consequently, we can find a vector \( \overline{y} \in T_{\overline{x}} \Sigma \) such that

\[ \nabla^1 u(\overline{x}) + \overline{y} = \overline{y}'(0). \]

From this, we deduce that

\[ \Phi_r(\overline{x}, \overline{y}) = \exp_{\overline{x}}(r \nabla^1 u(\overline{x}) + r \overline{y}) = \exp_{\overline{y}(0)}(r \overline{y}'(0)) = \overline{y}(r) = p. \]

Moreover,

\[ r^2 (|\nabla^1 u(\overline{x})|^2 + |\overline{y}|^2) = r^2 |\nabla^1 u(\overline{x}) + \overline{y}|^2 = r^2 |\overline{y}'(0)|^2 = d(\overline{x}, p)^2. \]
By assumption, $\sigma r < d(\overline{x}, p) < r$. This implies $\sigma^2 < |\nabla^\Sigma u(\overline{x})|^2 + |\overline{y}|^2 < 1$. In particular, $(\overline{x}, \overline{y}) \in U$. Finally, for each point $x \in \Sigma$, we have

$$ru(x) + \frac{1}{2} d(x, \exp_\overline{x}(r \nabla^\Sigma u(\overline{x}) + r \overline{y}))^2 \geq ru(x) + \frac{1}{2} d(x, p)^2$$

$$= ru(x) + \frac{1}{2} r^2 (|\nabla^\Sigma u(\overline{x})|^2 + |\overline{y}|^2).$$

Thus, $(\overline{x}, \overline{y}) \in A_r$. This completes the proof of Lemma 4.2. \hfill \Box

**Lemma 4.3.** Assume that $(\overline{x}, \overline{y}) \in A_r$, and let $\overline{y}(t) := \exp_\overline{x}(t \nabla^\Sigma u(\overline{x}) + t \overline{y})$ for all $t \in [0, r]$. If $Z$ is a smooth vector field along $\overline{y}$ satisfying $Z(0) \in T_{\overline{x}}\Sigma$ and $Z(r) = 0$, then

$$(D^2 u)(Z(0), Z(0)) - \langle II(Z(0), Z(0), \overline{y} \rangle$$

$$+ \int_0^r (|\overline{D}_t Z(t)|^2 - \overline{R}(\overline{y}'(t), Z(t), \overline{y}'(t), Z(t)))dt \geq 0.$$

**Proof.** Let us consider an arbitrary smooth path $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = \overline{y}(r)$. Since $(\overline{x}, \overline{y}) \in A_r$, we obtain

$$ru(\gamma(0)) + \frac{1}{2} r \int_0^r |\gamma'(t)|^2 dt \geq ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), \gamma(r))^2$$

$$= ru(\gamma(0)) + \frac{1}{2} d(\gamma(0), \exp_\overline{x}(r \nabla^\Sigma u(\overline{x}) + r \overline{y}))^2$$

$$\geq ru(\overline{x}) + \frac{1}{2} r^2 (|\nabla^\Sigma u(\overline{x})|^2 + |\overline{y}|^2)$$

$$= ru(\overline{y}(0)) + \frac{1}{2} r \int_0^r |\overline{y}'(t)|^2 dt.$$

In other words, the path $\overline{y}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2 dt$ among all smooth paths $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = \overline{y}(r)$. Using the formula for the second variation of energy, we obtain

$$(D^2 u)(Z(0), Z(0)) - \langle II(Z(0), Z(0), \overline{y}'(0) \rangle$$

$$+ \int_0^r (|\overline{D}_t Z(t)|^2 - \overline{R}(\overline{y}'(t), Z(t), \overline{y}'(t), Z(t)))dt \geq 0.$$

On the other hand, the identity $\overline{y}'(0) = \nabla^\Sigma u(\overline{x}) + \overline{y}$ implies

$$\langle II(Z(0), Z(0), \overline{y}'(0) \rangle = \langle II(Z(0), Z(0), \overline{y} \rangle.$$

Putting these facts together, the assertion follows. \hfill \Box

**Lemma 4.4.** Assume that $(\overline{x}, \overline{y}) \in A_r$. Then $g + rD^2 u(\overline{x}) - r\langle II(\overline{x}, \overline{y}) \rangle \geq 0.$
PROOF. As above, we define $\overline{y}(t) := \exp_{\overline{x}}(t \nabla u(\overline{x}) + t \overline{y})$ for all $t \in [0, r]$. Let us fix an arbitrary vector $w \in T_{\overline{x}} \Sigma$, and let $\overline{W}(t)$ denote the parallel transport of $w$ along $\overline{y}$. Applying Lemma 4.3 to the vector field $Z(t) := (r - t) \overline{W}(t)$ gives

$$rg(w, w) + r^2 (D^2_{\Sigma} u)(w, w) - r^2 \langle II(w, w), \overline{y} \rangle$$

$$- \int_0^r (r - t)^2 \overline{R}(\overline{y}'(t), W(t), \overline{y}'(t), W(t)) dt \geq 0,$$

Since $M$ has nonnegative sectional curvature, it follows that

$$rg(w, w) + r^2 (D^2_{\Sigma} u)(w, w) - r^2 \langle II(w, w), \overline{y} \rangle \geq 0,$$

as claimed. □

**Lemma 4.5.** Assume that $(\overline{x}, \overline{y}) \in A_r$, and let $\overline{y}(t) := \exp_{\overline{x}}(t \nabla u(\overline{x}) + t \overline{y})$ for all $t \in [0, r]$. Moreover, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\overline{x}} \Sigma$. Suppose that $W$ is a Jacobi field along $\overline{y}$ satisfying $W(0) \in T_{\overline{x}} \Sigma$ and $\langle \overline{D}_t W(0), e_j \rangle = (D^2_{\Sigma} u)(W(0), e_j) - \langle II(W(0), e_j), \overline{y} \rangle$ for each $1 \leq j \leq n$. If $W(t) = 0$ for some $t \in (0, r)$, then $W$ vanishes identically.

**Proof.** Suppose that $W(\tau) = 0$ for some $\tau \in (0, r)$. By assumption,

$$\langle \overline{D}_t W(0), W(0) \rangle = (D^2_{\Sigma} u)(W(0), W(0)) - \langle II(W(0), W(0)), \overline{y} \rangle.$$

Since $W$ is a Jacobi field, we obtain

$$\int_0^\tau (|\overline{D}_t W(t)|^2 - \overline{R}(\overline{y}'(t), W(t), \overline{y}'(t), W(t))) dt$$

$$= \langle \overline{D}_t W(\tau), W(\tau) \rangle - \langle \overline{D}_t W(0), W(0) \rangle$$

$$= -(D^2_{\Sigma} u)(W(0), W(0)) + \langle II(W(0), W(0)), \overline{y} \rangle.$$

Let us define a vector field $\overline{W}$ along $\overline{y}$ by

$$\overline{W}(t) = \begin{cases} W(t) & \text{for } t \in [0, \tau], \\ 0 & \text{for } t \in [\tau, r]. \end{cases}$$

Clearly, $\overline{W}(0) = W(0) \in T_{\overline{x}} \Sigma$ and $\overline{W}(r) = 0$. Moreover,

$$\int_0^\tau (|\overline{D}_t \overline{W}(t)|^2 - \overline{R}(\overline{y}'(t), \overline{W}(t), \overline{y}'(t), \overline{W}(t))) dt$$

$$= -(D^2_{\Sigma} u)(\overline{W}(0), \overline{W}(0)) + \langle II(\overline{W}(0), \overline{W}(0)), \overline{y} \rangle.$$

Using Lemma 4.3, we conclude that

$$\int_0^\tau (|\overline{D}_t Z(t)|^2 - \overline{R}(\overline{y}'(t), Z(t), \overline{y}'(t), Z(t))) dt$$

$$\geq \int_0^\tau (|\overline{D}_t \overline{W}(t)|^2 - \overline{R}(\overline{y}'(t), \overline{W}(t), \overline{y}'(t), \overline{W}(t))) dt$$
for every smooth vector field $Z$ along $\mathcal{P}$ satisfying $Z(0) = \widetilde{W}(0)$ and $Z(r) = \widetilde{W}(r)$. By approximation, this inequality holds for every vector field $Z$ which is piecewise $C^1$ and satisfies $Z(0) = \widetilde{W}(0)$ and $Z(r) = \widetilde{W}(r)$. In other words, the vector field $\widetilde{W}$ minimizes the index form among all vector fields which are piecewise $C^1$ and have the same boundary values as $\widetilde{W}$. Consequently, $\widetilde{W}$ must be of class $C^1$. This implies $\overline{\mathcal{D}}_t W(\tau) = 0$. Since $W(\tau) = 0$ and $\overline{\mathcal{D}}_t W(\tau) = 0$, standard uniqueness results for ODE imply that $W$ vanishes identically. □

**Proposition 4.6.** Assume that $(x, y) \in A_r$. Then the function
\[ t \mapsto t^{-m} (1 + t f(x, y))^{-n} |\det D \Phi(x, y)| \]
is monotone decreasing for $t \in (0, r)$.

**Proof.** Fix an arbitrary point $(\overline{x}, \overline{y}) \in A_r$. Let us choose an orthonormal basis \{\(e_1, \ldots, e_n\)\} of $T_{\overline{x}}M$ such that the $n \times n$-matrix
\[ (D_\Sigma^2 u)(e_i, e_j) - \langle II(e_i, e_j), \overline{y} \rangle \]
is diagonal. Let $(x_1, \ldots, x_n)$ be a system of geodesic normal coordinates on $\Sigma$ around the point $\overline{x}$. We can arrange that $\frac{\partial}{\partial x_i} = e_i$ at $\overline{x}$. Let $\{v_{n+1}, \ldots, v_{n+m}\}$ be a local orthonormal frame for the normal bundle, chosen so that $\langle \overline{\mathcal{D}}_{e_i} v_\alpha, v_\beta \rangle = 0$ at $\overline{x}$. We write a normal vector $y$ as $y = \sum_{\alpha = n+1}^{n+m} v_\alpha v_\alpha$. With this understood, $(x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m})$ is a local coordinate system on the total space of the normal bundle $T_{\Sigma}$. Let $\overline{y}(t) := \exp_{\overline{x}}(t \nabla^\Sigma u(\overline{x}) + t \overline{y})$ for all $t \in [0, r]$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of $e_i$ along $\overline{y}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\overline{y}$ satisfying $X_i(0) = e_i$ and
\[ \langle \overline{\mathcal{D}}_t X_i(0), e_j \rangle = (D_\Sigma^2 u)(e_i, e_j) - \langle II(e_i, e_j), \overline{y} \rangle, \]
\[ \langle \overline{\mathcal{D}}_t X_i(0), v_\beta \rangle = \langle II(e_i, \nabla^\Sigma u), v_\beta \rangle \]
for all $1 \leq j \leq n$ and all $n + 1 \leq \beta \leq n + m$. For each $n + 1 \leq \alpha \leq n + m$, we denote by $N_\alpha(t)$ the parallel transport of $v_\alpha$ along $\overline{y}$. Moreover, for each $n + 1 \leq \alpha \leq n + m$, we denote by $Y_\alpha(t)$ the unique Jacobi field along $\overline{y}$ satisfying $Y_\alpha(0) = 0$ and $\overline{\mathcal{D}}_t Y_\alpha(0) = v_\alpha$. It follows from Lemma 4.5 that $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, r)$.

Let us define an $(n+m) \times (n+m)$-matrix $P(t)$ by
\[ P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad P_{i\beta}(t) = \langle X_i(t), N_\beta(t) \rangle, \]
\[ P_{\alpha j}(t) = \langle Y_\alpha(t), E_j(t) \rangle, \quad P_{\alpha\beta}(t) = \langle Y_\alpha(t), N_\beta(t) \rangle, \]
for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$. Moreover, we define an $(n+m) \times (n+m)$-matrix $S(t)$ by
\[ S_{ij}(t) = \overline{R}(\overline{y}(t), E_i(t), \overline{y}(t), E_j(t)), \quad S_{i\beta}(t) = \overline{R}(\overline{y}(t), E_i(t), \overline{y}(t), N_\beta(t)), \]
\[ S_{\alpha j}(t) = \overline{R}(\overline{y}(t), Y_\alpha(t), \overline{y}(t), E_j(t)), \quad S_{\alpha\beta}(t) = \overline{R}(\overline{y}(t), Y_\alpha(t), \overline{y}(t), N_\beta(t)) \]
for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$. Clearly, $S(t)$ is symmetric. Moreover, $S(t) \geq 0$ since $M$ has nonnegative sectional curvature. Since the vector fields $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are Jacobi fields, we obtain

$$P''(t) = -P(t)S(t).$$

Moreover,

$$P(0) = \begin{bmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$P'(0) = \begin{bmatrix} (D^2_{\Sigma}u)(e_i, e_j) - \langle II(e_i, e_j), \bar{\gamma} \rangle & \langle II(e_i, \nabla_{\Sigma}u, \nu_{\beta} \rangle \\ 0 & \delta_{\alpha\beta} \end{bmatrix}.$$ 

In particular, the matrix $P'(0)P(0)^T$ is symmetric. Moreover, the matrix

$$\frac{d}{dt}(P'(t)P(t)^T) = P''(t)P(t)^T + P'(t)P'(t)^T$$

is symmetric for each $t$. Thus, we conclude that the matrix $P'(t)P(t)^T$ is symmetric for each $t$.

Since $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, r)$, the matrix $P(t)$ is invertible for each $t \in (0, r)$. Since $P'(t)P(t)^T$ is symmetric for each $t \in (0, r)$, it follows that the matrix $Q(t) := P(t)^{-1}P'(t)$ is symmetric for each $t \in (0, r)$. The matrix $Q(t)$ satisfies the Riccati equation

$$Q'(t) = P(t)^{-1}P''(t) - P(t)^{-1}P'(t)P(t)^{-1}P'(t) = -S(t) - Q(t)^2$$

for all $t \in (0, r)$. Since $S(t) \geq 0$, it follows that

$$Q'(t) \leq -Q(t)^2$$

for all $t \in (0, r)$. Using the asymptotic expansion

$$P(t) = \begin{bmatrix} \delta_{ij} + O(t) & O(t) \\ O(t) & t \delta_{\alpha\beta} + O(t^2) \end{bmatrix},$$

we obtain

$$P(t)^{-1} = \begin{bmatrix} \delta_{ij} + O(t) & O(1) \\ O(1) & t^{-1} \delta_{\alpha\beta} + O(1) \end{bmatrix}$$

as $t \to 0$. Moreover,

$$P'(t) = \begin{bmatrix} (D^2_{\Sigma}u)(e_i, e_j) - \langle II(e_i, e_j), \bar{\gamma} \rangle + O(t) & O(1) \\ O(t) & \delta_{\alpha\beta} + O(t) \end{bmatrix}$$

as $t \to 0$. Consequently, the matrix $Q(t) = P(t)^{-1}P'(t)$ satisfies the asymptotic expansion

$$Q(t) = \begin{bmatrix} (D^2_{\Sigma}u)(e_i, e_j) - \langle II(e_i, e_j), \bar{\gamma} \rangle + O(t) & O(1) \\ O(1) & t^{-1} \delta_{\alpha\beta} + O(1) \end{bmatrix}$$

as $t \to 0$. 
By our choice of \(\{e_1, \ldots, e_n\}\), the matrix \((D^2 u_i)(e_i, e_j) - \langle H(e_i, e_j), \nabla \rangle\) is diagonal. Let us write
\[
(D^2 u_i)(e_i, e_j) - \langle H(e_i, e_j), \nabla \rangle = \lambda_i \delta_{ij}
\]
for \(1 \leq i, j \leq n\). It follows from Lemma 4.4 that \(1 + r \lambda_i \geq 0\) for each \(1 \leq i \leq n\). Since
\[
Q(\tau) = \begin{bmatrix}
\lambda_i \delta_{ij} & O(\tau) & O(1)
\end{bmatrix}
\]
\[\tau^{-1} \delta_{\alpha \beta} + O(1)
\]
as \(\tau \to 0\), we can find a small number \(\tau_0 \in (0, r)\) such that
\[
Q(\tau) < \begin{bmatrix}
(\lambda_i + \sqrt{\tau}) \delta_{ij} & 0
\end{bmatrix}
\]
for all \(\tau \in (0, \tau_0)\). A standard ODE comparison principle implies
\[
Q(t) \leq \begin{bmatrix}
\frac{\lambda_i + \sqrt{\tau}}{t + (t-\tau)(\lambda_i + \sqrt{\tau})} \delta_{ij} & 0
\end{bmatrix}
\]
\[t - \frac{\tau}{2} \delta_{\alpha \beta}
\]
for all \(\tau \in (0, \tau_0)\) and all \(t \in (\tau, r)\). Passing to the limit as \(\tau \to 0\), we conclude that
\[
Q(t) \leq \begin{bmatrix}
\frac{\lambda_i}{t + t\lambda_i} \delta_{ij} & 0
\end{bmatrix}
\]
\[t - t \delta_{\alpha \beta}
\]
for all \(t \in (0, r)\). In particular, the trace of \(Q(t)\) satisfies
\[
\text{tr}(Q(t)) \leq \frac{m}{t} + \sum_{i=1}^{n} \frac{\lambda_i}{1 + t \lambda_i}
\]
for all \(t \in (0, r)\). It follows from Lemma 4.1 that
\[
\sum_{i=1}^{n} \lambda_i = \Delta \Sigma u(\nabla) - \langle H(\nabla), \nabla \rangle \leq nf(\nabla) \frac{1}{n-1}.
\]
Using the arithmetic-harmonic mean inequality, we obtain
\[
\sum_{i=1}^{n} \frac{1}{1 + t \lambda_i} \geq \frac{n^2}{\sum_{i=1}^{n} (1 + t \lambda_i)} \geq \frac{n}{1 + t \frac{f(\nabla)}{n-1}},
\]
hence
\[
\sum_{i=1}^{n} \frac{\lambda_i}{1 + t \lambda_i} = \frac{1}{t} \left( \frac{n}{\sum_{i=1}^{n} \frac{1}{1 + t \lambda_i}} \right) \leq \frac{nf(\nabla) \frac{1}{n-1}}{1 + t \frac{f(\nabla)}{n-1}}
\]
for all \(t \in (0, r)\). Putting these facts together, we conclude that
\[
\text{tr}(Q(t)) \leq \frac{m}{t} + \frac{nf(\nabla) \frac{1}{n-1}}{1 + tf(\nabla) \frac{1}{n-1}}
\]
for all \(t \in (0, r)\).
We next consider the determinant of $P(t)$. Clearly, $\lim_{t \to 0} t^{-m} \det P(t) = 1$. In particular, $\det P(t) > 0$ if $t > 0$ is sufficiently small. Since $P(t)$ is invertible for each $t \in (0, r)$, it follows that $\det P(t) > 0$ for all $t \in (0, r)$. Using the estimate for the trace of $Q(t) = P(t)^{-1} P'(t)$, we obtain

$$\frac{d}{dt} \log \det P(t) = \text{tr}(Q(t)) \leq \frac{m}{t} + \frac{nf(\overline{x}) n^{-1}}{1 + tf(\overline{x}) n^{-1}}$$

for all $t \in (0, r)$. Consequently, the function

$$t \mapsto t^{-m} (1 + tf(\overline{x}) n^{-1})^{-n} \det P(t)$$

is monotone decreasing for $t \in (0, r)$.

Finally, we observe that

$$\frac{\partial \Phi_t}{\partial x_i}(\overline{x}, \overline{y}) = X_i(t), \quad \frac{\partial \Phi_t}{\partial y_\alpha}(\overline{x}, \overline{y}) = Y_\alpha(t)$$

for $1 \leq i \leq n$ and $n + 1 \leq \alpha \leq n + m$. Consequently, $|\det D \Phi_t(\overline{x}, \overline{y})| = \det P(t)$ for all $t \in (0, r)$. Putting these facts together, the assertion follows.

**Corollary 4.7.** The Jacobian determinant of $\Phi_t$ satisfies

$$|\det D \Phi_t(x, y)| \leq r^m (1 + rf(x) n^{-1})^n$$

for all $(x, y) \in A_r$.

**Proof.** Since $\lim_{t \to 0} t^{-m} |\det D \Phi_t(x, y)| = 1$, the assertion follows from Proposition 4.6. □

After these preparations, we now complete the proof of Theorem 1.4. Using Lemma 4.2 and Corollary 4.7, we obtain

$$|\{p \in M : \sigma r < d(x, p) < r \text{ for all } x \in \Sigma_t\}|$$

$$\leq \int_{\overline{\Omega}} \left( \int_{\{y \in T^+_\Lambda : \Sigma 2 < |\nabla \Sigma u(x)|^2 + |y|^2 < 1\}} |\det D \Phi_t(x, y)| I_{A_r}(x, y) dy \right) d\text{vol}(x)$$

$$\leq \int_{\overline{\Omega}} \left( \int_{\{y \in T^+_\Lambda : \Sigma 2 < |\nabla \Sigma u(x)|^2 + |y|^2 < 1\}} \sigma^m r^n (1 + rf(x) n^{-1})^n \right) d\text{vol}(x)$$

$$= |B^n| \int_{\overline{\Omega}} \left[ \left( 1 - |\nabla \Sigma u(x)|^2 \right)^{\frac{m}{2}} - (\sigma^2 - |\nabla \Sigma u(x)|^2)^{\frac{m}{2}} \right]$$

$$\cdot r^m (1 + rf(x) n^{-1})^n d\text{vol}(x)$$

for all $r > 0$ and all $0 \leq \sigma < 1$. Since $m \geq 2$, the mean value theorem implies $b^{m/2} - a^{m/2} \leq m(b - a)/2$ for $0 \leq a \leq b \leq 1$. Hence, we have the pointwise inequality

$$(1 - |\nabla \Sigma u(x)|^2)^{\frac{m}{2}} - (\sigma^2 - |\nabla \Sigma u(x)|^2)^{\frac{m}{2}}$$

$$\leq \frac{m}{2} \left[ (1 - |\nabla \Sigma u(x)|^2) - (\sigma^2 - |\nabla \Sigma u(x)|^2)^+ \right] \leq \frac{m}{2} (1 - \sigma^2)$$
for all $x \in \Omega$ and all $0 \leq \sigma < 1$. Therefore,
\[
|\{ p \in M : \sigma r < d(x, p) < r \text{ for all } x \in \Sigma \}| \leq \frac{m}{2} |B^m|(1 - \sigma^2) \int_{\Omega} r^m (1 + r f(x) \frac{1}{n-1})^n d\text{vol}(x)
\]
for all $r > 0$ and all $0 \leq \sigma < 1$.

In the next step, we divide by $r^{n+m}$ and send $r \to \infty$ while keeping $\sigma$ fixed. This gives
\[
|B^{n+m}|(1 - \sigma^{n+m}) \theta \leq \frac{m}{2} |B^m|(1 - \sigma^2) \int_{\Omega} f^\frac{n}{n-1}
\]
for all $0 \leq \sigma < 1$. Finally, if we divide by $1 - \sigma$ and send $\sigma \to 1$, we obtain
\[
(n + m) |B^{n+m}| \theta \leq m |B^m| \int_{\Omega} f^\frac{n}{n-1} \leq m |B^m| \int_{\Sigma} f^\frac{n}{n-1}.
\]
Thus,
\[
\int_{\Sigma} \sqrt{|\nabla^\Sigma f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f = n \int_{\Sigma} f^\frac{n}{n-1} \geq n \left( \frac{(n + m) |B^{n+m}|}{m |B^m|} \right) \frac{1}{\theta} \left( \int_{\Sigma} f^\frac{n}{n-1} \right)^{\frac{n-1}{n}}.
\]
This completes the proof of Theorem 1.4.

5 Proof of Theorem 1.6

Let $(M, \bar{g})$ be a complete noncompact manifold of dimension $n + 2$ with nonnegative sectional curvature. Let $\Sigma$ be a compact submanifold of $M$ of dimension $n$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$ satisfying
\[
\int_{\Sigma} \sqrt{|\nabla^\Sigma f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f = \int_{\Sigma} |B^n| \frac{1}{\theta} \left( \int_{\Sigma} f^\frac{n}{n-1} \right)^{\frac{n-1}{n}} > 0,
\]
where $\bar{\theta}$ denotes the asymptotic volume ratio of $M$.

If $\Sigma$ is disconnected, we may apply Corollary 1.5 to each connected component of $\Sigma$, and take the sum over all connected components. This will lead to a contradiction. Therefore, $\Sigma$ must be connected.

By scaling, we may assume that
\[
\int_{\Sigma} \sqrt{|\nabla^\Sigma f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f = n |B^n| \bar{\theta}
\]
and
\[
\int_{\Sigma} f^\frac{n}{n-1} = |B^n| \bar{\theta}.
\]
In particular,
\[ \int_{\Sigma} \sqrt{|\nabla \Sigma f|^2 + f^2 |H|^2} + \int_{\partial \Sigma} f = n \int_{\Sigma} f \frac{n-1}{n-1}. \]
Since \( \Sigma \) is connected, we can find a function \( u : \Sigma \to \mathbb{R} \) such that
\[ \text{div}_\Sigma (f \nabla \Sigma u) = nf \frac{n-1}{n-1} - \sqrt{|\nabla \Sigma f|^2 + f^2 |H|^2} \]
on \( \Sigma \) and \( (\nabla \Sigma u, \eta) = 1 \) at each point on \( \partial \Sigma \). Moreover, \( u \) is of class \( C^2, \gamma \) for each \( 0 < \gamma < 1 \). Let us define \( \Omega, U, A_r \), and \( \Phi_r \) as in Section 4.

**Lemma 5.1.** Assume that \( x \in \Omega \) and \( y \in T^d_x \Sigma \) satisfy \( |\nabla \Sigma u(x)|^2 + |y|^2 = 1 \). Then \( |\det D\Phi_t(x, y)| \geq t^2 (1 + t f(x))^{\frac{1}{n-1}} \) for each \( t > 0 \).

**Proof.** Let us fix a point \( \tilde{x} \in \Omega \) and a vector \( \tilde{y} \in T^d_x \Sigma \) satisfying \( |\nabla \Sigma u(\tilde{x})|^2 + |\tilde{y}|^2 = 1 \). Suppose that \( |\det D\Phi_t(\tilde{x}, \tilde{y})| < t^2 (1 + t f(\tilde{x}))^{\frac{1}{n-1}} \) for some \( t > 0 \). Let us fix a real number \( \varepsilon \in (0, 1) \) such that
\[ |\det D\Phi_t(\tilde{x}, \tilde{y})| < (1 - \varepsilon) t^2 (1 + t f(\tilde{x}))^{\frac{1}{n-1}}. \]
By continuity, we can find an open neighborhood \( \mathcal{V} \) of the point \( (\tilde{x}, \tilde{y}) \) such that
\[ |\det D\Phi_t(x, y)| \leq (1 - \varepsilon) t^2 (1 + t f(x))^{\frac{1}{n-1}} \]
for all \( (x, y) \in \mathcal{V} \). Using Proposition 4.6, we conclude that
\[ |\det D\Phi_r(x, y)| \leq (1 - \varepsilon) r^2 (1 + r f(x))^{\frac{1}{n-1}} \]
for all \( r > t \) and all \( (x, y) \in A_r \cap \mathcal{V} \). Using this fact together with Lemma 4.2 and Corollary 4.7, we obtain
\[ |\{ \rho \in M : \sigma \rho < d(x, \rho) < r \text{ for all } x \in \Sigma \}| \]
\[ \leq \int_{\Omega} \left( \int_{\{y \in T^d_x \Sigma : \sigma^2 < |\nabla \Sigma u(x)|^2 + |y|^2 < 1\}} |\det D\Phi_r(x, y)| 1_{A_r}(x, y) \, d\text{vol}(x) \right) \, d\text{vol}(x) \]
\[ \leq \int_{\Omega} \left( \int_{\{y \in T^d_x \Sigma : \sigma^2 < |\nabla \Sigma u(x)|^2 + |y|^2 < 1\}} (1 - \varepsilon \cdot 1_{\mathcal{V}}(x, y)) \cdot r^2 (1 + r f(x))^{\frac{1}{n-1}} \, d\text{vol}(x) \right) \, d\text{vol}(x) \]
\[ \leq |B^2|(1 - \sigma^2) \int_{\Omega} r^2 (1 + r f(x))^{\frac{1}{n-1}} \, d\text{vol}(x) \]
\[ - \varepsilon \int_{\Omega} \left( \int_{\{y \in T^d_x \Sigma : \sigma^2 < |\nabla \Sigma u(x)|^2 + |y|^2 < 1\}} 1_{\mathcal{V}}(x, y) r^2 (1 + r f(x))^{\frac{1}{n-1}} \, d\text{vol}(x) \right) \, d\text{vol}(x) \]
for all $r > t$ and all $0 \leq \sigma < 1$. We now divide by $r^{n+2}$ and send $r \to \infty$, while keeping $\sigma$ fixed. This implies

$$|B^{n+2}|(1 - \sigma^{n+2})\theta$$

$$\leq |B^2|(1 - \sigma^2) \int_{\Omega} f(x) \pi^\frac{n}{n-1} \text{vol}(x)$$

$$- \varepsilon \int_{\Omega} \left( \int_{y \in T_x^+ \Sigma : \sigma^2 - |\nabla \Sigma u(x)|^2 + |y|^2 < 1} 1_{V}(x, y) f(x) \pi^\frac{n}{n-1} \text{d}y \right) \text{vol}(x)$$

for all $0 \leq \sigma < 1$. Dividing by $1 - \sigma$ and taking the limit as $\sigma \to 1$ gives

$$(n + 2)|B^{n+2}|\theta < 2|B^2| \int \pi^{\frac{n}{n-1}} \leq 2|B^2||B^n|\theta.$$  

This contradicts the fact that $(n + 2)|B^{n+2}| = 2|B^2||B^n|$.  

\textbf{Lemma 5.2.} Assume that $x \in \Omega$ and $y \in T_x \Sigma$ satisfy $|\nabla \Sigma u(x)|^2 + |y|^2 = 1$. Then $D_{\Sigma}^2 u(x) - \langle II(x), y \rangle = f(x) \pi^\frac{n-1}{n-2} g$.

\textbf{Proof.} Let us fix a point $\overline{x} \in \Omega$ and a vector $\overline{y} \in T_{\overline{x}} \Sigma$ satisfying $|\nabla \Sigma u(\overline{x})|^2 + |\overline{y}|^2 = 1$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{T}_{\overline{x}} M$ with the property that the $n \times n$-matrix

$$D_{\Sigma}^2 u(e_i, e_j) - \langle II(e_i, e_j), \overline{y} \rangle$$

is diagonal, and let $\{v_{n+1}, v_{n+\frac{1}{2}}\}$ be an orthonormal basis of $T_{\overline{x}}\Sigma$. We define $\overline{y}(t) := \exp(\overline{x}(t \nabla u(\overline{x}) + t \overline{y}))$ for all $t \geq 0$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of $e_i$ along $\overline{y}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\overline{y}$ satisfying $X_i(0) = e_i$ and

$$\langle \overline{D}_{\overline{t}} X_i(0), e_j \rangle = (D_{\Sigma}^2 u)(e_i, e_j) - \langle II(e_i, e_j), \overline{y} \rangle,$$

$$\langle \overline{D}_{\overline{t}} X_i(0), v_\beta \rangle = \langle II(e_i, \nabla \Sigma u), v_\beta \rangle,$$

for all $1 \leq j \leq n$ and all $n + 1 \leq \beta \leq n + 2$. For each $n + 1 \leq \alpha \leq n + 2$, we denote by $N_{\alpha}(t)$ the parallel transport of $v_{\alpha}$ along $\overline{y}$. Moreover, for each $n + 1 \leq \alpha \leq n + 2$, we denote by $Y_{\alpha}(t)$ the unique Jacobi field along $\overline{y}$ satisfying $Y_{\alpha}(0) = 0$ and $\overline{D}_{\overline{t}} Y_{\alpha}(0) = v_{\alpha}$.

Finally, we define an $(n + 2) \times (n + 2)$-matrix $P(t)$ by

$$P_{ij}(t) = (X_i(t), E_j(t)), \quad P_{i\beta}(t) = (X_i(t), N_\beta(t)),$$

$$P_{\alpha j}(t) = (Y_\alpha(t), E_j(t)), \quad P_{\alpha \beta}(t) = (Y_\alpha(t), N_\beta(t)),$$

for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + 2$.

By Lemma 5.1, we know that $|\det P(t)| \geq t^2 (1 + t f(\overline{x}) \pi^\frac{1}{n-1})^n$ for all $t > 0$. Since $\det P(t) > 0$ if $t > 0$ is sufficiently small, we conclude that

$$\det P(t) \geq t^2 (1 + t f(\overline{x}) \pi^\frac{1}{n-1})^n$$

for all $t > 0$. In particular, $P(t)$ is invertible for each $t > 0$.  


We next define \( Q(t) := P(t)^{-1} P'(t) \) for all \( t > 0 \). Moreover, we write

\[
(D^2_x u)(e_i, e_j) - \langle II(e_i, e_j), v \rangle = \lambda_i \delta_{ij}
\]

for \( 1 \leq i, j \leq n \). Arguing as in Section 4, we obtain

\[
\text{tr}(Q(t)) \leq \frac{2}{t} + \sum_{i=1}^{n} \frac{\lambda_i}{1 + t \lambda_i}
\]

for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \). Moreover,

\[
\sum_{i=1}^{n} \lambda_i \leq nf(\overline{x})^{\frac{1}{n-1}}
\]

by Lemma 4.1. As above, the arithmetic-harmonic mean inequality implies

\[
\sum_{i=1}^{n} \frac{\lambda_i}{1 + t \lambda_i} \leq \frac{nf(\overline{x})^{\frac{1}{n-1}}}{1 + tf(\overline{x})^{\frac{1}{n-1}}}
\]

for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \). Therefore,

\[
\text{tr}(Q(t)) \leq \frac{2}{t} + \frac{nf(\overline{x})^{\frac{1}{n-1}}}{1 + tf(\overline{x})^{\frac{1}{n-1}}},
\]

hence

\[
\frac{d}{dt} \log \det P(t) \leq \frac{2}{t} + \frac{nf(\overline{x})^{\frac{1}{n-1}}}{1 + tf(\overline{x})^{\frac{1}{n-1}}}
\]

for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \). Integrating this ODE gives

\[
\det P(t) \leq t^2 \left(1 + tf(\overline{x})^{\frac{1}{n-1}}\right)^n
\]

for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \).

Putting these facts together, we conclude that \( \det P(t) = t^2(1 + tf(\overline{x})^{1/(n-1)})^n \) for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \). Differentiating this identity with respect to \( t \), we obtain

\[
\text{tr}(Q(t)) = \frac{2}{t} + \frac{nf(\overline{x})^{\frac{1}{n-1}}}{1 + tf(\overline{x})^{\frac{1}{n-1}}}
\]

for all \( t > 0 \) satisfying \( \min_{1 \leq i \leq n} (1 + t \lambda_i) > 0 \). Consequently, we must have equality in the arithmetic-harmonic mean equality, and furthermore \( \sum_{i=1}^{n} \lambda_i = nf(\overline{x})^{1/(n-1)} \). Therefore, \( \lambda_i = f(\overline{x})^{1/(n-1)} \) for each \( 1 \leq i \leq n \). This completes the proof of Lemma 5.2.

\[\]
PROOF. By Lemma 5.2, $D^2_{\Sigma} u(x) = \langle II(x), y \rangle = f(x) \frac{1}{|\Sigma|} g$ for all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Replacing $y$ by $-y$ gives $D^2_{\Sigma} u(x) + \langle II(x), y \rangle = f(x) \frac{1}{|\Sigma|} g$ for all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Therefore,
$$D^2_{\Sigma} u(x) = f(x) \frac{1}{|\Sigma|} g \quad \text{and} \quad \langle II(x), y \rangle = 0$$
for all $y \in T_x^\perp \Sigma$ that satisfy $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. From this, the assertion follows easily. □

**LEMMA 5.4.** Assume that $x \in \Omega$. Then $\nabla^\Sigma f(x) = 0$.

**PROOF.** Let us consider an arbitrary point $x \in \Omega$. Using the definition of $u$, we obtain
$$f(x) \Delta^\Sigma u(x) = nf(x) \frac{1}{|\Sigma|} - \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2 |H(x)|^2}$$
$$- \langle \nabla^\Sigma f(x), \nabla^\Sigma u(x) \rangle.$$  
On the other hand, Lemma 5.3 implies $\Delta^\Sigma u(x) = nf(x)^{(n-1)}$ and $H(x) = 0$. Putting these facts together, we conclude that $\langle \nabla^\Sigma f(x), \nabla^\Sigma u(x) \rangle = -|\nabla^\Sigma f(x)|$. Since $|\nabla^\Sigma u(x)| < 1$, it follows that $\nabla^\Sigma f(x) = 0$. This completes the proof of Lemma 5.4. □

**LEMMA 5.5.** The set $\Omega$ is dense in $\Sigma$.

**PROOF.** Suppose that $\Omega$ is not dense in $\Sigma$. Arguing as in Section 4, we obtain
$$(n + 2) |B^{n+2}| \theta \leq 2 |B^2| \int_{\Omega} f \frac{1}{|\Sigma|} < 2 |B^2| \int_{\Sigma} f \frac{1}{|\Sigma|} = 2 |B^2||B^n| \theta.$$  
This contradicts the fact that $(n + 2) |B^{n+2}| = 2 |B^2||B^n|$. This completes the proof of Lemma 5.5. □

Since $\Omega$ is a dense subset of $\Sigma$, we conclude that $\nabla^\Sigma f = 0$, $D^2_{\Sigma} u = f \frac{1}{|\Sigma|} g$, and $II = 0$ at each point on $\Sigma$. Since $\Sigma$ is connected, it follows that $f$ is constant. This implies $|\partial \Sigma| = n |B^n| (1/n) \theta^{1/n} |\Sigma|^{(n-1)/n}$. Note that $u$ is a smooth function on $\Sigma$. Each critical point of $u$ lies in the interior of $\Sigma$ and is nondegenerate with Morse index 0. In particular, the function $u$ has at most finitely many critical points.

We next consider the flow on $\Sigma$ generated by the vector field $-\nabla^\Sigma u$. Since the vector field $-\nabla^\Sigma u$ points inward along the boundary $\partial \Sigma$, the flow is defined for all nonnegative times. This gives a one-parameter family of smooth maps $\psi_s : \Sigma \rightarrow \Sigma$, where $s \geq 0$. Since $\Sigma$ is connected, standard arguments from Morse theory imply that the function $u$ has exactly one critical point, and $u$ attains its global minimum at that point. It follows that the diameter of $\psi_s(\Sigma)$ converges to 0 as $s \rightarrow \infty$.

Since $D^2_{\Sigma} u$ is a constant multiple of the metric, the isoperimetric ratio is unchanged under the flow $\psi_s$. This implies $|\psi_s(\partial \Sigma)| = n |B^n| (1/n) \theta^{1/n} |\psi_s(\Sigma)|^{(n-1)/n}$ for each $s \geq 0$. If $\theta < 1$, this contradicts the Euclidean isoperimetric inequality.
when \( s \) is sufficiently large. Thus, we conclude that \( \theta = 1 \). Consequently, \( M \) is isometric to Euclidean space.

Once we know that \( M \) is isometric to Euclidean space, the arguments in [3] imply that \( \Sigma \) is a flat round ball. This completes the proof of Theorem 1.6.

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