EXPANSIVENESS, SHADOWING AND MARKOV PARTITION FOR ANOSOV FAMILIES

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Abstract. We study Anosov families which are sequences of diffeomorphisms along compact Riemannian manifolds such that the tangent bundle split into expanding and contracting subspaces. In this paper we prove that a certain class of Anosov families: (i) admit canonical coordinates (ii) are expansive, (iii) satisfy the shadowing property, and (iv) exhibit a Markov partition.

1. Introduction

An Anosov family is a (biinfinite) sequence of diffeomorphisms along a sequence of compact Riemannian manifolds, with an invariant sequence of splittings of the tangent bundle into expanding and contracting subspaces, and with a uniform upper bound for the contraction and lower bound for the expansion.

Anosov families (Definition 2.3) were introduced by P. Arnoux and A. Fisher in [7], motivated by generalizing the notion of Anosov diffeomorphisms. The authors concentrated their studies on linear Anosov families on the two-torus. The first goal was to get a natural notion of completion for the collection of the set of all orientation-preserving linear Anosov diffeomorphisms on the two-torus (see [7]). Authors have been study Anosov families. Young [26] proved that families consisting of $C^{1+\epsilon}$ perturbations of an Anosov diffeomorphism of class $C^2$ are Anosov families. In [15] and [23] the authors studied formulas for the entropy of non-stationary subshifts of finite type. Recently, Muentes, studied in his doctoral thesis, [17, 18], the Stable and Unstable Manifold Theorem for Anosov family and the stability structural of Anosov families on compact Riemannian manifolds. In this work we will study some properties related to hyperbolicity in the Anosov families.

From the work done by Walter in [27], many attempts have been made to express the concept of hyperbolicity in topological terms. Notions as shadowing, expansiveness, canonical coordinates, Markov partitions, and others, were essential to achieve results. In this work we will study exactly these properties in Anosov families: expansiveness, canonical coordinates, shadowing and Markov partitions. Unfortunately, we are able to obtain such results only for Anosov families whose sequence are of $C^2$-diffeomorphisms with derivative limited. Under these conditions we get our main four theorems. The first result is about the structure of canonical coordinates for Anosov families (Definition 4.1).

Theorem A. Anosov families admit canonical coordinates.

We also investigate the expansiveness property. In rough terms, the concept of expansiveness means that if two points stay near for forward and backward iterates, then they must be equal. In some sense, expansive systems can be considered chaotic since they exhibit sensitivity to the initial conditions. The appropriate notion of expansiveness for sequences of diffeomorphisms is given in Definition 5.1. Our second result follows:

Theorem B. Anosov families are expansive.
Shadowing was introduced by Anosov and it is central in hyperbolic dynamic. For instance, it is fundamental in the proof of the $C^1$ structural stability of uniformly hyperbolic systems (see [14], [24]). Roughly speaking, it allows us to trace a set of point which looks like an orbit, but with errors, by a true orbit. For practical applications, we can suppose that a map $f$ is viewed as the orbit realized in numerical calculation by computer, or in physical experiments, thus it could have errors. Then shadowing property allow us to “correct” this errors, finding a true evolution which nicely approximates $f$. Many authors has studied this properties and your relation with the hyperbolicity, for example, [19], [20], [22] and [25]. And also, in many contexts, as in [4], [5] and [21]. Thus, to decide which systems possess the shadowing property is an important problem in dynamics. So, we can ask:

Question: How would be shadowing for Anosov families? Would Anosov families have any shadowing notion?

The precise definition of shadowing for Anosov families is in Section 5. We concluded our third result.

**Theorem C.** Anosov families have shadowing property.

Shadowing has many applications and one of them is to get the Markov partition [9]. In Section 3.4 of [2], Arnoux and Fisher, gave a symbolic representation for an Anosov family that admits Markov partition sequence. In this case the symbolic representation is given by a non-stationary subshifts of finite type, which was first investigated in [2] with the motivation to study Anosov families via coding and to deduce properties of adic transformations. In this paper we study Partition Markov (Definition 6.3) for Anosov families. We consider Anosov families whose the sequence of manifold is constant, that is, the manifolds are equal, and for these families we prove our fourth and last result:

**Theorem D.** Anosov family has Markov partition.

This article is organized as follows: In Section 2 we will define precisely an Anosov family and the objects that we will study in this work. We will make important considerations, notations and comments which are relevant in this context. In Section 3, we will mention the Stable and Unstable Manifold Theorem for Anosov families, proved by the author in [17]. Sections 2 and 3 will ease the understanding of the behavior of the Anosov families. We will prove Theorem A in Section 4, which is essential to obtain shadowing for Anosov families. Theorems B and C will be proved in Section 5. In Section 6, we prove Theorem D, that is, in certain contexts there is a Markov partition for Anosov families. We reserve the last section, Section 7, to propose future issues, and further generalizations about Anosov families.

2. Anosov Families and Definitions

In this section we will introduce Anosov family and we will mention the main elements that will be used throughout this work. In addition, we will give some examples and observations of this class of systems. Firstly we will define the objects that are part of the context in which Anosov families are inserted.

Consider a sequence of Riemannian manifolds $M_i$ with a fixed Riemannian metric $\langle \cdot, \cdot \rangle_i$ for $i \in \mathbb{Z}$ with a same injectivity radius $\rho > 0$ (see [18], Remark 2.7). Take the disjoint union

$$M = \bigsqcup_{i \in \mathbb{Z}} M_i = \bigcup_{i \in \mathbb{Z}} M_i \times i.$$
\( M \) will be endowed with the Riemannian metric \( \langle \cdot, \cdot \rangle \) induced by \( \langle \cdot, \cdot \rangle \) on \( TM \), setting \( \langle \cdot, \cdot \rangle |_{M_i} = \langle \cdot, \cdot \rangle \) for \( i \in \mathbb{Z} \). We denote by \( \| \cdot \|_i \) the induced norm by \( \langle \cdot, \cdot \rangle \) on \( TM_i \) and we will take \( \| \cdot \| \) defined on \( M \) as \( \| \cdot \|_{M_i} = \| \cdot \|_i \) for \( i \in \mathbb{Z} \).

**Definition 2.1.** A non-stationary dynamical system \((M, \langle \cdot, \cdot \rangle, F)\) is a map \( F : M \to M \), such that, for each \( i \in \mathbb{Z}, F |_{M_i} = f_i : M_i \to M_{i+1} \) is a diffeomorphism. Sometimes we use the notation \( F = (f_i)_{i \in \mathbb{Z}} \). The composition law is defined to be

\[
F^n_i := \begin{cases} f_{i+n-1} \circ \cdots \circ f_i : M_i \to M_{i+n} & \text{if } n > 0 \\ f_{i-n}^{-1} \circ \cdots \circ f^{-1}_i : M_i \to M_{i-n} & \text{if } n < 0 \\ I_i : M_i \to M_i & \text{if } n = 0, \end{cases}
\]

where \( I_i \) is the identity on \( M_i \).

Non-stationary dynamical systems are classified via topological equiconjugacy.

**Definition 2.2.** A topological equiconjugacy between \( F = (f_i)_{i \in \mathbb{Z}} \) and \( G = (g_i)_{i \in \mathbb{Z}} \) is a map \( \mathcal{H} : M \to M \), such that, for each \( i \in \mathbb{Z}, \mathcal{H}_{M_i} = h_i : M_i \to M_i \) is a homeomorphism, \( (h_i)_{i \in \mathbb{Z}} \) and \((h^{-1}_i)_{i \in \mathbb{Z}}\) are equicontinuous and \( h_{i+1} \circ f_i = g_i \circ h_i \). In that case, we will say the families are equiconjugate.

Now, we have all the elements to rigorously define an Anosov family.

**Definition 2.3.** An Anosov family on \( M \) is a non-stationary dynamical system \((M, \langle \cdot, \cdot \rangle, F)\) such that:

i. the tangent bundle \( TM \) has a continuous splitting \( E^s \oplus E^u \) which is \( DF \)-invariant, i.e., for each \( p \in M, T_pM = E^s_p \oplus E^u_p \) with \( D_pF(E^s_p) = E^s_{F(p)} \) and \( D_pF(E^u_p) = E^u_{F(p)} \),

where \( T_pM \) is the tangent space at \( p \);

ii. there exist constants \( \lambda \in (0, 1) \) and \( c > 0 \) such that for each \( i \in \mathbb{Z}, n \geq 1, \) and \( p \in M_i \), we have:

\[
\|D_p(F^n_i)(v)\| \leq c \lambda^n \|v\| \text{ if } v \in E^s_p \quad \text{and} \quad \|D_p(F^{-n}_i)(v)\| \leq c \lambda^n \|v\| \text{ if } v \in E^u_p.
\]

The subspaces \( E^s_p \) and \( E^u_p \) are called stable and unstable subspaces, respectively.

If we can take \( c = 1 \) we say the family is strictly Anosov.

The next example, which is due to Arnoux and Fisher [2], Example 3, proves that Anosov families are not necessarily sequences of Anosov diffeomorphisms. A random version of the example can be found in [13], Example 2.7. More examples can be found in [2], [17], [18].

**Example 2.4.** For any sequence of positive integers \((n_i)_{i \in \mathbb{Z}}\) set

\[
A_i = \begin{pmatrix} 1 & 0 \\ n_i & 1 \end{pmatrix} \text{ for } i \text{ even} \quad \text{and} \quad A_i = \begin{pmatrix} 1 & n_i \\ 0 & 1 \end{pmatrix} \text{ for } i \text{ odd},
\]

acting on the 2-torus \( M_i = \mathbb{T}^2 \). The family \((A_i)_{i \in \mathbb{Z}}\) is an Anosov family.

**Definition 2.5.** An Anosov family satisfies the property of the angles (or s.p.a.) if the angle between the stable and unstable subspaces are bounded away from zero (see [17], [18]).

**Remark 2.6.** Fix an Anosov diffeomorphism \( \phi \) on a Riemannian manifold \( M \). For each \( i \in \mathbb{Z}, \) we can endow \( M_i = M \) with a suitable Riemannian metric such that if we consider \( f_i = \phi \) for any \( i \in \mathbb{Z}, \) then \((f_i)_{i \in \mathbb{Z}}\) is an Anosov family such that the angle between the unstable and stable subspaces at some point of \( M \) converges to zero as \( i \to \infty \) (see [17], Example 2.4). That is, there exist Anosov families which do not satisfy the property of angles.
Now we define some important sets which we will use throughout this work. Fix \( m \geq 1 \). The set
\[
\mathcal{D}^m(M) = \{ \mathcal{F} = (f_i)_{i \in \mathbb{Z}} : f_i : M_i \to M_{i+1} \text{ is a } C^m\text{-diffeomorphism} \}
\]
can be endowed with the strong topology and the uniform topology \( (13) \). The subset of \( \mathcal{D}^m(M) \) consisting of Anosov families will be denoted by \( \mathcal{A}^m(M) \).

Consider the set
\[
\mathcal{A}_2^p(M) = \{ \mathcal{F} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{A}^2(M) : \mathcal{F} \text{ is Anosov, s.p.a. and } \sup_{i \in \mathbb{Z}} ||Df_i||_{C^2} < \infty \},
\]
where \( ||\phi||_{C^1} = \max \{ ||D\phi||, ||D\phi^{-1}||, ||D^2\phi||, ||D^2\phi^{-1}|| \} \) for a \( C^2 \)-diffeomorphism \( \phi \).

3. Stable and Unstable Manifolds for Anosov Families

In \( [17] \) the author proves the Local Unstable and Stable Manifold Theorem for Anosov family. This theorem is essential to prove Theorem A. In this section we will mention the results that will be used in the next section.

Firstly we note:

Remark 3.1. In this section, we will consider \( \mathcal{F} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{A}_2^p(M) \).

Now we define some sets which will be used throughout the work. Given \( \varepsilon > 0 \) and \( p \in M \), set:
(i) \( B(p, \varepsilon) \subseteq M \) be the ball with radius \( \varepsilon \) and center \( p \);
(ii) \( B(\tilde{0}_p, \varepsilon) \subseteq T_p M \) denote the ball with radius \( \varepsilon \) and center \( \tilde{0}_p \), the zero vector in \( T_p M \);
(iii) \( B^s(\tilde{0}_p, \varepsilon) \subseteq E^s_p \) denote the ball with radius \( \varepsilon \) and center \( \tilde{0}_p \);
(iv) \( B^u(\tilde{0}_p, \varepsilon) \subseteq E^u_p \) denote the ball with radius \( \varepsilon \) and center \( \tilde{0}_p \).

Given two points \( p, q \in M \), set
\[
\Theta_{p,q} = \lim_{n \to \infty} \frac{1}{n} \log d(F^n_i(q), F^n_i(p)) \quad \text{and} \quad \Delta_{p,q} = \lim_{n \to \infty} \frac{1}{n} \log d(F^{-n}_i(q), F^{-n}_i(p)).
\]

Definition 3.2. Let \( \varepsilon > 0 \). Fix \( p \in M \).
(i) \( \mathcal{W}^s(p, \varepsilon) = \{ q \in B(p, \varepsilon) : \Theta_{p,q} < 0 \text{ and } F^n_i(q) \in B(F^n_i(p), \varepsilon) \text{ for } n \geq 1 \} \) := the local stable set at \( p \);
(ii) \( \mathcal{W}^u(p, \varepsilon) = \{ q \in B(p, \varepsilon) : \Delta_{p,q} < 0 \text{ and } F^{-n}_i(q) \in B(F^{-n}_i(p), \varepsilon) \text{ for } n \geq 1 \} \) := the local unstable set at \( p \).

Since \( \mathcal{F} \) satisfies the property of angles, we can suppose that \( \mathcal{F} \) is strictly Anosov and furthermore that \( E^s_p \) and \( E^u_p \) are orthogonal for any \( p \in M \) (see [17]). This is the Lemma of Mather for Anosov families. In Theorems 5.2 and 5.3 of [17] and Theorems 3.7, 3.8, 4.5 and 4.6 of [13] we proved that, for any \( \alpha \in (0, (\lambda^{-1} - 1)/2) \), there exist a small \( \varepsilon > 0 \) and \( \zeta \in (0,1) \) such that follow the next two results:

Theorem 3.3. For each \( p \in M \), \( \mathcal{W}^u(p, \varepsilon) \) is a differentiable submanifold of \( M \) and there exists \( K^u > 0 \) such that:
(i) \( \exp_p^{-1}(\mathcal{W}^u(p, \varepsilon)) = [\phi^u_p(x), x : x \in B^u(\tilde{0}_p, \varepsilon)] \), where \( \phi^u_p : B^u(\tilde{0}_p, \varepsilon) \to B^u(\tilde{0}_p, \varepsilon) \) is an \( \alpha \)-Lipschitz map and \( \phi^u_p(\tilde{0}_p) = \tilde{0}_p \).
(ii) \( T_p \mathcal{W}^u(p, \varepsilon) = E^u_p \).
(iii) \( \mathcal{F}^{-1}(\mathcal{W}^u(p, \varepsilon)) \subseteq \mathcal{W}^u(\mathcal{F}^{-1}(p), \varepsilon) \),
(iv) if \( q \in \mathcal{W}^u(p, \varepsilon) \) and \( n \geq 1 \) we have \( d(F^{-n}(q), F^{-n}(p)) \leq K^n\zeta^n d(q, p) \).
Let \((p_m)_{m \in \mathbb{N}}\) be a sequence in \(M_i\) converging to \(p \in M_i\) as \(m \to \infty\). If \(q_m \in \mathcal{W}^u(p_m, \epsilon)\) converges to \(q \in B(p, \epsilon)\) as \(m \to \infty\), then \(q \in \mathcal{W}^u(p, \epsilon)\).

**Theorem 3.4.** For each \(p \in M\), \(\mathcal{W}^s(p, \epsilon)\) is a differentiable submanifold of \(M\) and there exists \(K^* > 0\) such that:

1. \(\exp_p^{-1}(\mathcal{W}^s(p, \epsilon)) = \{(x, \phi^s_p(x)) : x \in B(\tilde{0}_p, \epsilon)\}\), where \(\phi^s_p : B^s(\tilde{0}_p, \epsilon) \to B^s(\tilde{0}_p, \epsilon)\) is an \(\alpha\)-Lipschitz map and \(\phi^s_p(0_p) = \tilde{0}_p\).
2. \(T_p \mathcal{W}^s(p, \epsilon) = E^s_p\).
3. \(\mathcal{F}(\mathcal{W}^s(p, \epsilon)) \subseteq \mathcal{W}^u(\mathcal{F}(p), \epsilon)\).
4. If \(q \in \mathcal{W}^s(p, \epsilon)\) and \(n \geq 1\) we have \(d(\mathcal{F}^n(q), \mathcal{F}^n(p)) \leq K^* \xi^n d(q, p)\).
5. Let \((p_m)_{m \in \mathbb{N}}\) be a sequence in \(M_i\) converging to \(p \in M_i\) as \(m \to \infty\). If \(q_m \in \mathcal{W}^s(p_m, \epsilon)\) converges to \(q \in B(p, \epsilon)\) as \(m \to \infty\), then \(q \in \mathcal{W}^s(p, \epsilon)\).

Other properties of the invariant manifolds for Anosov families follow in the proposition below.

**Proposition 3.5.** Let \(\beta \in (0, \epsilon/2)\). If \(d(\mathcal{F}^n(p), \mathcal{F}^n(q)) < \beta\) for all \(n \in \mathbb{N}\), then \(q \in \mathcal{W}^s(p, \epsilon)\). On the other hand, if \(d(\mathcal{F}^{-n}(p), \mathcal{F}^{-n}(q)) \leq \beta\) for all \(n \in \mathbb{N}\), then \(q \in \mathcal{W}^u(p, \epsilon)\).

**Proof.** By abuse of notation, we identify \(\mathcal{W}^s(\mathcal{F}_0^n(p), \epsilon) \times \mathcal{W}^u(\mathcal{F}_0^n(p), \epsilon)\) with an open neighborhood of \(\tilde{0} \in \mathcal{T}(\mathcal{F}_0^n(p), \mathcal{M})\) via exponential charts. Suppose that \(q \notin \mathcal{W}^u(p, \epsilon)\). Therefore, since \(\mathcal{F}_0^n(q) = B(\mathcal{F}_0^n(p), \beta)\), we have

\[
\mathcal{F}_0^n(\exp_p^{-1}(q)) = (x_n, y_n) \in \mathcal{W}^s(\mathcal{F}_0^n(p), \epsilon) \times \mathcal{W}^u(\mathcal{F}_0^n(p), \epsilon),
\]

for all \(n \geq 0\), with \(x_n \in \mathcal{W}^s(\mathcal{F}_0^n(p), \epsilon)\) and \(y_n \in \mathcal{W}^u(\mathcal{F}_0^n(p), \epsilon)\), \(n \in \mathcal{W}^u(\mathcal{F}_0^n(p), \epsilon)\setminus \{0\}\). We can obtain from Theorem 3.3 item (iv), and Theorem 3.4 item (iv), that

\[
||(x_n, y_n)|| \geq ||x_n|| - ||y_n|| \geq \frac{1}{K^* \xi^n} ||y_0|| - K^* \xi^n ||x_0||.
\]

We have \(K^* \xi^n ||x_0|| \to 0\) as \(n \to +\infty\). Since \(y_0 \neq 0\), for some \(n \in \mathbb{N}\) we have \(d(\mathcal{F}_0^n(q), \mathcal{F}_0^n(p)) = ||(x_n, y_n)|| > \beta\), which contradicts the assumption. Analogously we can prove the second part of the proposition. \(\square\)

### 4. Canonical Coordinates for Anosov Families

Canonical coordinates were introduced by Bowen who used them to study Axiom A diffeomorphisms [9], [10], [11]. He exploited the fact that an Axiom A diffeomorphism restricted to a basic set has hyperbolic canonical coordinates with respect to some metric. Other results related to canonical coordinates have been developed, such as Fathi’s, which says that an expansive homeomorphism in a compact metric space with canonical coordinates admits a metric compatible with the original topology to which the canonical coordinates are hyperbolic [12]. In our case, we will prove that Anosov families in \(\mathcal{A}_0^2(M)\) admit canonical coordinates.

**Definition 4.1.** An Anosov family \(\mathcal{F}\) has canonical coordinates if given a small \(\epsilon > 0\) there exists a \(\delta > 0\) such that, if \(p, q \in M\) with \(d(p, q) < \delta\), then

\[
\mathcal{W}^s(p, \epsilon) \cap \mathcal{W}^u(q, \epsilon) \neq \emptyset.
\]
**Theorem A.** Let \( \mathcal{F} \in \mathcal{A}_p^2(M) \). Given a small \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( p, q \in M \), and \( d(p, q) < \delta \), then

\[
\mathcal{W}^u(p, \epsilon) \cap \mathcal{W}^u(q, \epsilon)
\]
is a single point in \( M \).

**Proof.** Take \( \epsilon \in (0, \varrho) \), where \( \varrho \) is an injectivity radius of \( M \). Hence, for any \( p \in M \) the exponential map

\[
\exp_p : B(\bar{0}_p, \epsilon) \to B(p, \epsilon)
\]
is a diffeomorphism and \( \|v\| = d(\exp_p(v), p) \), for all \( v \in B(\bar{0}_p, \epsilon) \). By Theorems 3.3 and 3.4 we have that for any \( \alpha \in (0, (\lambda^{-1} - 1)/2) \) there exists an \( \epsilon \in (0, \varrho/4) \) such that

\[
\exp_p^{-1}(\mathcal{W}^u(p, \epsilon)) = \{(x, \phi^1_p(x)) : x \in B^i(\bar{0}_p, \epsilon)\}
\]
and

\[
\exp_p^{-1}(\mathcal{W}^u(p, \epsilon)) = \{(\phi^1_p(x), x) : x \in B^u(\bar{0}_p, \epsilon)\},
\]
where \( \phi^1_p : B^i(\bar{0}_p, \epsilon) \to B^u(\bar{0}_p, \epsilon) \) and \( \phi^u_p : B^u(\bar{0}_p, \epsilon) \to B^i(\bar{0}_p, \epsilon) \) are \( \alpha \)-Lipschitz maps and \( \phi^1_p(\bar{0}_p) = \phi^u_p(\bar{0}_p) = \bar{0}_p \).

Set

\[
K^s_{\alpha, p} = \{(v, w) \in E_p^s \oplus E_p^u : ||w|| \leq \alpha ||v||\}
\]
and

\[
K^u_{\alpha, p} = \{(v, w) \in E_p^s \oplus E_p^u : ||v|| \leq \alpha ||w||\}.
\]

Note that

\[
\exp_p^{-1}(\mathcal{W}^u(p, \epsilon)) \subseteq K^s_{\alpha, p} \quad \text{and} \quad \exp_p^{-1}(\mathcal{W}^u(p, \epsilon)) \subseteq K^u_{\alpha, p}.
\]

Take \( p, q \in M \) with \( d(p, q) < \epsilon/4 \). Thus \( \exp_p^{-1}(q) \in B^u(\bar{0}_p, \epsilon) \times B^u(\bar{0}_p, \epsilon) \). Set \( z = \exp_p^{-1}(q) \),

\[
E^s_q := z + \lambda(\mathcal{W}^u(p, \epsilon)) \subseteq T_pM \quad \text{and} \quad E^u_q := z + D(\exp_p^{-1})_q(K^u_{\alpha, q}) \subseteq T_pM.
\]

Thus \( E^s_q \) is parallel to \( E^s_p \) and \( E^u_q \) is parallel to \( E^u_p \). Hence \( E^s_q \) is perpendicular to \( E^u_q \) and \( E^u_q \) is perpendicular to \( E^s_q \) (remember that \( E^s_p \) and \( E^u_p \) are orthogonal). Consequently, we can choose a \( \delta \in (0, \epsilon/4) \) small enough such that, if \( d(p, q) < \delta \), then any \( (u_q, v_q) \in \exp_p^{-1}(q) + D(\exp_p^{-1})_q(K^u_{\alpha, q}) \) with \( u_q \in B^u(\bar{0}_p, \epsilon) \) belongs to \( B^i(\bar{0}_p, \epsilon) \times B^u(\bar{0}_p, \epsilon) \) (see Figure 4.1).

![Figure 4.1](image)

**Figure 4.1.** \( z = \exp_p^{-1}(q) \); the vertical cone is \( \exp_p^{-1}(q) + D(\exp_p^{-1})_q(K^u_{\alpha, q}) \); the horizontal cone is \( K^u_{\alpha, p} \); the curve inside \( K^u_{\alpha, p} \) is \( \mathcal{W}^u(p, \epsilon) \).

Therefore

\[
[\exp_p^{-1}(q) + D(\exp_p^{-1})_q(K^u_{\alpha, q})] \cap K^u_{\alpha, p}
\]
is not empty and lives inside \( B^i(\bar{0}_p, \epsilon) \times B^u(\bar{0}_p, \epsilon) \). Since \( \exp_q^{-1}(\mathcal{W}^u(q, \epsilon)) \subseteq K^u_{\alpha, q} \), we have that \( \mathcal{W}^i(p, \epsilon) \cap \mathcal{W}^u(q, \epsilon) \) is a single point in \( M \). \( \Box \)
5. Expansiveness and Shadowing

This section is divided into two subsections. In the first subsection we will verify that the Anosov families in $\mathcal{A}^2(M)$ are expansive. In the second subsection we will use the main results from the previous sections, that is, Anosov families in $\mathcal{A}^2(M)$ have canonical coordinates and are expansive, to prove Shadowing Lemma for Anosov families.

5.1. Expansiveness. The notion of expansiveness was introduced by Bowen [8] and a sequence of studies occurred relating expansivity and hyperbolicity. Here, we will appropriately define expansiveness for non-stationary dynamical systems.

**Definition 5.1.** An Anosov family $\mathcal{F}$ is expansive if there exists $\delta > 0$ such that for any distinct points $p, q \in M$ there is some $n \in \mathbb{Z}$ such that

$$d(\mathcal{F}^n_0(p), \mathcal{F}^n_0(q)) \geq \delta.$$ 

In [18], Proposition 7.5, we proved the following result:

**Proposition 5.2.** For any $\mathcal{F} \in \mathcal{A}^2(M)$, there exist $r > 0$ small enough, $\eta > 0$, $\zeta > 0$, with $\eta^{-1} - \zeta > 0$, such that if $p, q \in M_0$ and $d(\mathcal{F}^n_0(p), \mathcal{F}^n_0(q)) < r$ for each $n \in [-N, N]$ for some $N \in \mathbb{N}$, then

$$d(q, p) \leq 2 \sqrt{2}(\eta^{-1} - \zeta)^{-N}r.$$ 

As a consequence we have:

**Theorem B.** Any $\mathcal{F} \in \mathcal{A}^2(M)$ is expansive.

Proof. Take $r > 0$, $\eta > 0$ and $\zeta > 0$ as in Proposition [5.2]. Thus, if $p, q \in M_0$ and $d(f^n_0(p), f^n_0(q)) < r$ for each $n \in \mathbb{Z}$, then $p = q$, which proves the theorem. □

5.2. Shadowing Lemma. In this subsection we use canonical coordinates and the expansiveness of Anosov families, proved above and in Section 4, to prove that elements in $\mathcal{A}^2(M)$ satisfies the shadowing property (see Definition 5.3).

First let us remember the elements we are working on. Consider $\mathcal{F} = (f_i)_{i \in \mathbb{Z}} \in \mathcal{A}^2(M)$. Then $\mathcal{F}$ admits canonical coordinates (Theorem A), and $\mathcal{F}$ is expansive (Theorem B). Take $\epsilon > 0$ and $\delta > 0$ as in Theorem A. For $p, q \in M$ with $d(p, q) < \delta$ set

$$[p, q] = \mathcal{W}^s(p, \epsilon) \cap \mathcal{W}^u(q, \epsilon) \quad \text{and} \quad \mathcal{U}_\delta = \left\{(p, q) \in \bigcup_{i \in \mathbb{Z}} M_i \times M_i : d(p, q) < \delta \right\}.$$ 

Then $[\cdot, \cdot] : \mathcal{U}_\delta \rightarrow M$ is continuous, because $\mathcal{W}^s(p, \epsilon)$ and $\mathcal{W}^u(p, \epsilon)$ vary continuously with $p$.

**Definition 5.3.** Given $\alpha > 0$. A sequence $(x_n)_{n \in \mathbb{Z}}$, where $x_n \in M_n$ for each $n \in \mathbb{Z}$, is an $\alpha$-pseudo orbit for $\mathcal{F}$ if

$$d(f_n(x_n), x_{n+1}) < \alpha \quad \text{for all } n \in \mathbb{Z}.$$ 

A pseudo orbit $(x_n)_{n \in \mathbb{Z}}$ is $\epsilon$-shadowed if there exists $y \in M_0$ such that

$$d(\mathcal{F}^n_0(y), x_n) < \epsilon \quad \text{for each } n \in \mathbb{Z}.$$ 

Note that the first point of the pseudo orbit, and the point of the orbit which shadows the pseudo orbit, do not necessarily have to be in $M_0$, as in the case of the finite pseudo-orbits.

Now we state a version of the Shadowing Lemma for Anosov family.
Theorem C. (Shadowing Lemma for Anosov Family) Let $\mathcal{F} \in \mathcal{A}^2(\mathcal{M})$. Given $\beta > 0$ there is an $\alpha > 0$ such that every $\alpha$-pseudo orbit $(x_n)_{n \in \mathbb{Z}}$ is $\beta$-shadowed for an unique orbit of $\mathcal{F}$ through $y \in \mathcal{M}$.

To prove this result we use the same idea as in the case of Anosov diffeomorphisms, but respecting this class of systems.

Proof. (of Theorem C)

Given $\beta > 0$. In order to simplify the demonstration we will divide the proof into some steps. The first step we define parameters. The second step we define the appropriate $\alpha$ for the pseudo orbit. The third step we prove that all types of pseudo orbit are shadowed. And the last step we prove the uniqueness of the orbit which shadows the pseudo orbit.

(i) Choice of parameters:

Choose $\epsilon > 0$ as in Theorem 3.3. This ensures that $\mathcal{W}^{s}(x, \epsilon)$ and $\mathcal{W}^{u}(x, \epsilon)$ are disks that varies continuously with $x$. Consider the parameters:

- $\lambda \in (0, 1)$ the hyperbolic constant of $\mathcal{F}$.
- $\epsilon_1 < (1-\lambda) \min \{\epsilon, \beta\}$.
- $\eta = \frac{\epsilon_1}{\epsilon}$ (note that $\eta < \epsilon$ and $\eta < \beta$).
- $\delta < \beta - \eta$ positive constant for which $[\cdot, \cdot]_{\epsilon, \delta} : \mathcal{U}_\delta \rightarrow \mathcal{M}$ is well defined, that is,

$$\text{if } d(x, y) < \delta \text{ then } \mathcal{W}^{s}(x, \epsilon_1) \cap \mathcal{W}^{u}(y, \epsilon_1) = [x, y].$$

(ii) Now we choose the $\alpha > 0$ (for the pseudo orbit) appropriately:

Since $[\cdot, \cdot]$ is continuous and the stable and unstable subspaces are orthogonal (see proof of Theorem A, Figure 4.1), we can find an $\alpha > 0$ such that if $d(z, w) < \alpha$ then

$$\mathcal{W}^{s}(z, \epsilon_1) \cap \mathcal{W}^{u}(x, \epsilon_1) \in \mathcal{W}^{s}(z, \epsilon_1), \quad \text{for any } x \in \mathcal{W}^{s}(w, \lambda \epsilon_1).$$

Hence

$$[z, \mathcal{W}^{s}(w, \lambda \epsilon_1)] := \{[z, x] : x \in \mathcal{W}^{s}(w, \lambda \epsilon_1)\} \subseteq \mathcal{W}^{s}(z, \epsilon_1).$$

(iii) We divide the possible types of pseudo orbit in three cases and we show that in any of them the $\alpha$-pseudo orbit is shadowed.

Suppose that we have a finite $\alpha$-pseudo orbit $\bar{x} = [x_0, x_1, \cdots, x_n]$, where $x_i \in M_i$, with $i \in \{0, \ldots, n\}$. We prove that

$$y_0 = x_0, \quad y_1 = [x_1, f_0(y_0)], \quad y_2 = [x_2, f_1(y_1)], \quad \ldots \quad y_n = [x_n, f_{n-1}(y_{n-1})].$$

is a sequence well defined and it shadows the $\alpha$-pseudo orbit $\bar{x}$. In order to prove this fact, we set recursively $y_k = [x_k, f_k(y_k)]$. Suppose that $y_0, \ldots, y_k$ are well defined, for any $k < n$. Since $y_k \in \mathcal{W}^{s}(x_k, \epsilon_1)$ we have $f_k(y_k) \in \mathcal{W}^{s}(f_k(x_k), \lambda \epsilon_1)$. Let us remember that $\lambda$ is the hyperbolicity constant. Thus,

$$d(x_{k+1}, f_k(x_k)) < \alpha \text{ implies } y_{k+1} = [x_{k+1}, f_k(y_k)] \in [x_{k+1}, \mathcal{W}^{s}(f_k(x_k), \lambda \epsilon_1)] \subseteq \mathcal{W}^{s}(x_{k+1}, \epsilon_1).$$

So $y_{k+1}$ is well defined.

Next, we know that $y_k \in \mathcal{W}^{s}(f_k(y_k) - \epsilon_1, \epsilon_1)$ implies $f_k^{-1}(y_k) \in \mathcal{W}^{u}(y_k - \epsilon_1, \lambda \epsilon_1)$. Recursively, $T_{k-j}^{-1}(y_k) \in \mathcal{W}^{u}(y_{k-j}, \theta_j)$ where $\theta_j = \sum_{i=1}^{j} \lambda^i \epsilon_1 < \eta$. Consider

$$y = f_0^{-1} \circ f_1^{-1} \circ \cdots \circ f_{n-1}^{-1}(y_n).$$

Note that

$$T_0^{-j}(y) = T_0^{-(n-j)}(y_n) \in \mathcal{W}^{u}(y_{n-(n-j)}, \theta_{n-j}) = \mathcal{W}^{u}(y_j, \theta_{n-j}).$$
where
\[ \theta_{n-j} = \sum_{i=1}^{n-j} \lambda^i \epsilon_1 < \eta. \]

Thus,
\[ d(F^j_0(y), x_j) \leq d(F^j_0(y), y_j) + d(y_j, x_j) \leq \eta + \delta < \eta + \beta - \eta = \beta. \]

Therefore, we concluded that \( \hat{x} \) is \( \beta \)-shadowed by the orbit of \( y = F^0_n(y_n) \).

The second case, we suppose that \( \hat{x} = [x_{-n}, \ldots, x_0, \ldots, x_n] \), where \( x_i \in M_i \), for \( i \in \{-n, \ldots, n\} \).

Consider the reorganized sequence \( \tilde{x} = [\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{2n}] \), where \( \tilde{x}_i = x_{-n+i} \), with \( i = 0, 1, \ldots, 2n \). As we show in (iii)-first case, \( \tilde{x} \) is shadowed by \( \tilde{y} \in M_{-n} \). So,
\[ d(F^j_{-n}(\tilde{y}), \tilde{x}_j) = d(F^j_{-n}(\tilde{y}), x_{-n+j}) < \beta, \text{ for } j = 0, 1, \ldots, 2n. \]

Therefore, if \( y = F^0_n(\tilde{y}) \), we have \( d(F^j_0(y), x_j) < \beta \), for \( j = \{-n, \ldots, 0, \ldots, n\} \).

Finally, the last case, we consider the infinite \( \alpha \)-pseudo orbit \( \underline{x} = [\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots] \).

For each \( n > 0 \) consider \( x_n = [x_{-n}, \ldots, x_0, \ldots, x_n] \) and \( y_n \in M_0 \) its shadow. As \( M_0 \) is compact, there exists a subsequence \( n_k \in \mathbb{N} \) and \( y \in M_0 \) such that \( \lim_{k \to \infty} y_{n_k} = y \). Since \( d(F^j_0(y_n), x_j) < \beta \) for all \( j \in \{-n, \ldots, n\} \) and \( n \in \mathbb{N} \), when \( k \to \infty \) we have \( d(F^j_0(y), x_j) < \beta \), for all \( j \in \mathbb{Z} \).

(vi) Uniqueness of the orbit which shadows \( \underline{x} = [\ldots, x_{-n}, \ldots, x_0, \ldots, x_n, \ldots] \) follows from the expansiveness of \( \mathcal{F} \). \qed

6. Markov partition for Anosov family

In this section we will prove Theorem D. To prove the theorem we will use ideas of [1] (Chapter 4) and [11] (Theorem 3.12), whose authors used them to prove the existence of Markov partitions for TA-homeomorphism (see the definition in [1], Chapter 1) and Axiom A diffeomorphisms, respectively. The proofs of some of the results that will be presented here can be done as in the singular case and therefore we will omit them.

According to definition of Markov Partition for Anosov Families (Definition 6.3) we will need to consider for each \( i \in \mathbb{Z}, M_i = M \times \{i\} \) for a fixed compact Riemannian manifold \( M \). We will comment on this below.

**Definition 6.1.** A subset \( R \subseteq M_i \) is called a rectangle if, for any \( x, y \in R \), \([x, y]\) is defined and belongs to \( R \). We say that \( R \) is proper if \( R = \text{int } \overline{R} \).

For \( x \in R \) and \( \epsilon > 0 \) small enough, set
(i) \( \mathcal{W}^s(x, R) = \mathcal{W}^s(x, \epsilon) \cap R \),
(ii) \( \partial^s R = \{x \in R : x \notin \text{int}(\mathcal{W}^s(x, R))\} \),
(iii) \( \partial^i R = \{x \in R : x \notin \text{int}(\mathcal{W}^i(x, R))\} \),
where the interior of \( \mathcal{W}^s(x, R) \) and \( \mathcal{W}^i(x, R) \) are taken as subsets of \( \mathcal{W}^s(x, \epsilon) \) and \( \mathcal{W}^i(x, \epsilon) \), respectively.

The following property can be proved as in the single case (see [11]).

**Lemma 6.2.** Let \( R \) and \( T \) be rectangles. Thus:
(i) \( \overline{R} \) is a rectangle.
(ii) If \( \text{int}(R) \neq \emptyset \), then \( \text{int}(R) \) is a rectangle.

(iii) If \( R \cap T \neq \emptyset \), then \( R \cap T \) is a rectangle.

(iv) If \( R \) is a closed rectangle, then \( \partial R = \partial^1 R \cap \partial^2 R \).

**Definition 6.3.** For \( \mathcal{F} = (f_i)_{i \in \mathbb{Z}} \), a Markov partition is a sequence of finite partitions
\[
\mathcal{R}^i = \{ R^i_1, R^i_2, \ldots, R^i_n \}
\]
of \( M_i \), i.e., coverings of \( M_i \) by closed sets with disjoint interiors, such that \( \max_i \text{Card}(\mathcal{R}^i) < \infty \), each partition element is a proper rectangle, and satisfies the Markov condition: for \( R^i_j \in \mathcal{R}^i \) and \( R^{i+1}_k \in \mathcal{R}^{i+1} \), if \( x \in R^i_j \) and \( f_j(x) \in R^{i+1}_k \), then
\[
W^s(f_j(x), R^{i+1}_k) \subseteq \mathcal{W}^s(f_j(x), R^i_j) \quad \text{and} \quad f_j(\mathcal{W}^u(x, R^i_j)) \subseteq \mathcal{W}^u(f_j(x), R^{i+1}_k).
\]

Consider \( \mathcal{F} \in \mathcal{A}_\beta^0(M) \). Let \( \beta > 0 \) be very small and choose \( \alpha > 0 \) small as in Theorem C, that is, every \( \alpha \)-pseudo-orbit in \( M \) is \( \beta \)-shadowed by a orbit through a unique point in \( M \). Since \( \mathcal{F} \in \mathcal{A}_\beta^0(M) \), we can choose \( \gamma \in (0, \min(\beta, \alpha/2)) \) such that
\[
(6.1) \quad d(f^n(x), f^n(y)) < \alpha/2, \text{ when } d(x, y) < \gamma.
\]

In order to satisfy the condition \( \max_i \text{Card}(\mathcal{R}^i) < \infty \) in the Definition 6.3, we will suppose that, for each \( i \in \mathbb{Z}, M_i = M \times [i] \) for a fixed compact Riemannian manifold \( M \). We prove that in this case,

**Theorem D.** Each \( \mathcal{F} \in \mathcal{A}_\beta^0(M) \) admits a Markov partition.

In order to prove Theorem D, first we prove a series of lemmas.

Let \( P = \{p_1, \ldots, p_r\} \) be a \( \gamma \)-dense subset of \( M \). Hence \( P_i = \{(p_1, i), \ldots, (p_r, i)\} \) is a \( \gamma \)-dense subset of \( M_i \). To simplify the notation, we will write \( p_j \) instead of \( (p_j, i) \) for each \( i \in \mathbb{Z}, j = 1, \ldots, r \). Set
\[
\Sigma_0(P) = \left\{ \bar{a} = (\ldots, a_{-1}, a_0, a_1, \ldots), a_i \in M_i, \text{ and } \bar{a} \in \prod_{-\infty}^{\infty} P : d(f^n(a_n), a_{n+1}) < \alpha \text{ for all } n \right\}.
\]
That is, \( \Sigma_0(P) \) is a set consisting of \( \alpha \)-pseudo orbit of \( \mathcal{F}. \) \( \Sigma_0(P) \) will be endowed with the compact topology. It follows from Theorem C that for each \( \bar{a} \in \Sigma_0(P) \) there is a unique \( \theta_0(\bar{a}) \in M_0 \) which shadows the \( \alpha \)-pseudo orbit \( \bar{a} \).

**Lemma 6.4.** \( \theta_0 : \Sigma_0(P) \to M_0 \) is continuous.

**Proof.** Suppose that \( \theta_0 \) is not continuous. Thus, there is a \( \gamma > 0 \) such that for every \( n \in \mathbb{N} \) we can find \( \bar{a}_n, \bar{b}_n \in \Sigma_0(P) \), with \( a_{n,j} = b_{n,j} \) for all \( j \in [-n, n] \), but \( d(\theta_0(\bar{a}_n), \theta_0(\bar{b}_n)) \geq \gamma \). Therefore, for all \( j \in [-n, n] \), we have
\[
d(\mathcal{F}^{-1}_0(\theta_0(\bar{a}_n)), \mathcal{F}^{-1}_0(\theta_0(\bar{b}_n))) \leq d(\mathcal{F}^{-1}_0(\theta_0(\bar{a}_n)), a_{n,j+1}) + d(b_{n,j+1}, \mathcal{F}^{-1}_0(\theta_0(\bar{b}_n))) \leq 2\beta.
\]
We may assume \( \theta(\bar{a}_n) \to a \) and \( \theta(\bar{b}_n) \to b \) as \( n \to \infty \). Hence \( d(\mathcal{F}^{-1}_0(a), \mathcal{F}^{-1}_0(b)) \leq 2\beta \) for all \( j \in \mathbb{Z} \) and \( d(a, b) \geq \gamma \), which contradicts the expansiveness of \( \mathcal{F} \). \( \square \)

**Lemma 6.5.** \( \theta_0 : \Sigma_0(P) \to M_0 \) is surjective.

\[1\] If each \( M_i \) is a different manifold, the set the cardinality of the sequence \( P_i \) could be not bounded.
Proof. Fix \( x_0 \in M_0 \) and set \( x_n = \mathcal{F}^n(x_0) \) for \( n \in \mathbb{Z} \). Since \( P \) is a \( \gamma \)-dense subset, there exists an \( a_n \in P \) such that \( d(x_n, a_n) < \gamma \) for each \( n \in \mathbb{Z} \). By (6.1) we have \( d(f_n(a_n), f_n(a_n)) < \alpha/2 \) for each \( n \in \mathbb{Z} \). Therefore

\[
d(f_n(a_n), a_{n+1}) = d(f_n(a_n), f_n(x_n)) + d(f_n(x_n), a_{n+1}) < \alpha/2 + d(x_{n+1}, a_{n+1}) < \alpha.
\]

Consequently, \( \tilde{a} = (a_n)_{n \in \mathbb{Z}} \in \Sigma_0(P) \) and \( \theta_0(\tilde{a}) = x \), which proves that \( \theta_0 \) is surjective. \( \square \)

For each \( \tilde{a} = (a_n)_{n \in \mathbb{Z}} \in \Sigma_0(P) \) and \( i \in \mathbb{Z} \), let

\[
\sigma(\tilde{a}) = (a_{n+1})_{n \in \mathbb{Z}}, \quad \Sigma_i(P) = \sigma^i(\Sigma_0(P)) \quad \text{and take} \quad \sigma_i := \sigma|_{\Sigma_i(P)} : \Sigma_i(P) \to \Sigma_{i+1}(P).
\]

Let \( \theta_i : \Sigma_i(P) \to M_i \) be inductively defined such that the following diagram commutes:

\[
\begin{array}{c}
\Sigma_{-1}(P) \\ \downarrow \sigma_{-1} & \Sigma_0(P) \downarrow \sigma_0 \\
\vdots & \sigma_1 \downarrow \Sigma_1(P) & \Sigma_2(P) \\
M_{-1} & \downarrow f_{-1} & M_0 \downarrow f_0 \\
M_1 & \downarrow f_1 & M_2
\end{array}
\]

that is, \( \theta_{i+1}(\tilde{a}) = f_i(\theta_i(\sigma_i^{-1}(\tilde{a}))) \) for each \( \tilde{a} \in \Sigma_i(P) \). Since \( \theta_0 : \Sigma_0(P) \to M_0 \) is continuous and surjective, \( \theta_i : \Sigma_i(P) \to M_i \) is continuous and surjective.

Fix \( i \in \mathbb{Z} \). For \( \tilde{a}, \tilde{b} \in \Sigma_i(P) \) with \( a_0 = b_0 \) we define \([\tilde{a}, \tilde{b}]^i \in \Sigma_i(P)\) by

\[
[\tilde{a}, \tilde{b}]^i = \begin{cases} a_j \text{ for } j \geq 0 \\ b_j \text{ for } j \leq 0. \end{cases}
\]

If \( \tilde{c} = [\tilde{a}, \tilde{b}]^i \), we have

\[
d(\mathcal{F}^j_i(\theta_i(\tilde{c})), \mathcal{F}^j_i(\theta_i(\tilde{a}))) \leq 2\beta \quad \text{for } j \geq 0 \quad \text{and} \quad d(\mathcal{F}^{-j}_i(\theta_i(\tilde{c})), \mathcal{F}^{-j}_i(\theta_i(\tilde{b}))) \leq 2\beta \quad \text{for } j \leq 0.
\]

It follows from Proposition 3.5 that \( \theta_i(\tilde{c}) \in \mathcal{W}^s(\theta_i(\tilde{a}), 2\beta) \cap \mathcal{W}^u(\theta_i(\tilde{b}), 2\beta) = [\theta_i(\tilde{a}), \theta_i(\tilde{b})] \).

This fact proves that

(6.2) \[ \theta_i(\tilde{a}, \tilde{b}]^i) = [\theta_i(\tilde{a}), \theta_i(\tilde{b})]. \]

For each \( i \in \mathbb{Z} \) and \( k = 1, \ldots, r \), set

\[
T^i_k = \{ \theta_i(\tilde{a}) : \tilde{a} = (\ldots, a_{-1}, a_0, a_1, \ldots) \in \Sigma_i(P), a_0 = p_k \}.
\]

Lemma 6.6. \( T^i_k \) is a rectangle.

Proof. We prove that if \( x, y \in T^i_k \) then \([x, y] \in T^i_k \). Take \( x = \theta_i(\tilde{a}), y = \theta_i(\tilde{b}) \in T^i_k \) (thus \( a_0 = p_k = b_0 \)). If \( \tilde{c} = [\tilde{a}, \tilde{b}]^i \), then \( c_0 = p_k \), that is, \( \theta_i(\tilde{c}) \in T^i_k \). It follows from (6.2) that \([x, y] = \theta_i(\tilde{c}) \in T^i_k \). \( \square \)

Lemma 6.7. If \( x \in T^i_j \) and \( f_i(x) \in T^{i+1}_k \), then

\[
\mathcal{W}^s(f_i(x), T^{i+1}_k) \subseteq f_i(\mathcal{W}^s(x, T^i_j)) \quad \text{and} \quad f_i(\mathcal{W}^u(x, T^i_j)) \subseteq \mathcal{W}^u(f_i(x), T^{i+1}_k).
\]

Proof. Since \( x \in T^i_j \) and \( f_i(x) \in T^{i+1}_k \), we have \( x = \theta_i(\tilde{a}) \) with \( a_0 = p_j \) and \( a_1 = p_k \), because \( f_i(\theta_i(\tilde{a})) = \theta_{i+1}(\sigma_i(\tilde{a})) \in T^{i+1}_k \). Take \( y \in \mathcal{W}^s(x, T^i_j) = \mathcal{W}^s(x, e) \cap T^i_j \). Then, we can write \( y = \theta_i(\tilde{b}) \), with \( b_0 = p_j \). Therefore

\[
y = [x, y] = \theta_i([\tilde{a}, \tilde{b}]^i) \quad \text{and thus} \quad f_i(y) = f_i(\theta_i(\tilde{b})) = \theta_{i+1}(\sigma_i([\tilde{a}, \tilde{b}]^i)) \in T^{i+1}_k,
\]

because \( a_1 = p_k \). Since \( y \in \mathcal{W}^s(x, e) \), we have \( f_i(y) \in \mathcal{W}^s(f_i(x), e) \). Therefore, \( f_i(y) \in \mathcal{W}^u(f_i(x), T^{i+1}_k) \). We have proved

(6.3) \[ f_i(\mathcal{W}^s(x, T^i_j)) \subseteq \mathcal{W}^u(f_i(x), T^{i+1}_k). \]
Analogously we can prove
\[(6.4) \quad \mathcal{W}^\omega(f_i(x), T_k^{i+1}) \subseteq f_i(\mathcal{W}^\omega(x, T_j^i)),\]
which proves the lemma. \hfill \Box

**Lemma 6.8.** \(T_k^i\) is closed and \(J = \{T_1^i, \ldots, T_k^i\}\) is a covering of \(M_i\).

**Proof.** Since \(T_k^i = \theta_j(P_k^i)\), where \(P_k^i = \{a \in \Sigma(P) : a_0 = p_k\}\) is a closed subset of \(\Sigma(P)\) and \(\theta_i\) is continuous, we have \(T_k^i\) is closed (note that \(\Sigma(P)\) is compact). Furthermore, given that \(\theta_i\) is surjective and \(\Sigma(P) = \bigcup_j \Pi_j^i\), we have \(J = \{T_1^i, \ldots, T_k^i\}\) is a covering of \(M_i\). \hfill \Box

Next, we will build a first refinement of \(J\), since the interiors of the rectangles above could intersect. For \(T_j^i \cap T_k^i \neq \emptyset\), let
\[
\begin{align*}
T_{i,j}^{1,1} &= \{x \in T_j^i : \mathcal{W}^\omega(x, T_j^i) \cap T_k^i \neq \emptyset, \mathcal{W}^\omega(x, T_j^i) \cap T_k^i = \emptyset\} = T_j^i \cap T_k^i \\
T_{i,j}^{1,2} &= \{x \in T_j^i : \mathcal{W}^\omega(x, T_j^i) \cap T_k^i \neq \emptyset, \mathcal{W}^\omega(x, T_j^i) \cap T_k^i = \emptyset\} = T_j^i \cap T_k^i \\
T_{i,j}^{2,3} &= \{x \in T_j^i : \mathcal{W}^\omega(x, T_j^i) \cap T_k^i = \emptyset, \mathcal{W}^\omega(x, T_j^i) \cap T_k^i \neq \emptyset\} = T_j^i \cap T_k^i \\
T_{i,j}^{3,4} &= \{x \in T_j^i : \mathcal{W}^\omega(x, T_j^i) \cap T_k^i = \emptyset, \mathcal{W}^\omega(x, T_j^i) \cap T_k^i \neq \emptyset\} = T_j^i \cap T_k^i.
\end{align*}
\]

**Lemma 6.9.** For \(n = 1, 2, 3, 4\), \(T_{j,k}^{i,n}\) is a rectangle.

**Proof.** Fix \(x, y \in T_{j,k}^{i,n}\). Thus \(x, y \in T_j^i\) and therefore \([x, y] \in T_j^i\) (Lemma 6.6). Given that \([x, y] \in \mathcal{W}^\omega(x, \varepsilon)\), then
\[
\mathcal{W}^\omega([x, y], T_j^i) = \mathcal{W}^\omega([x, y], \varepsilon) \cap T_j^i = \mathcal{W}^\omega(x, \varepsilon) \cap T_j^i = \mathcal{W}^\omega(x, T_j^i)
\]
and since \([x, y] \in \mathcal{W}^\omega(y, \varepsilon)\), then
\[
\mathcal{W}^\omega([x, y], T_j^i) = \mathcal{W}^\omega([x, y], \varepsilon) \cap T_j^i = \mathcal{W}^\omega(y, \varepsilon) \cap T_j^i = \mathcal{W}^\omega(y, T_j^i).
\]
These facts imply that \([x, y] \in T_{j,k}^{i,n}\) and hence \(T_{j,k}^{i,n}\) is a rectangle for \(n = 1, 2, 3, 4\) (see [1], Remark 4.2.3, for more detail in the single case, which work for families). \hfill \Box

For each \(x \in M_i\), set
\[
J'(x) = \{T_j^i \in J_i : x \in T_j^i\}
\]
\[
J'_x = \{T_k^i \in J'_i : T_k^i \cap T_j^i \neq \emptyset\ \text{for some } T_j^i \in J'(x)\}
\]
\[
Z'_i = M_i \setminus \bigcup_{J^i} J'_i
\]
\[
Z'_x = \{x \in M_i : \mathcal{W}^\omega(x, \varepsilon) \cap \partial T_j^i = \emptyset \text{ and } \mathcal{W}^\omega(x, \varepsilon) \cap \partial T_k^i = \emptyset \text{ for all } T_j^i \in J'_x(x)\}.
\]

Since \(J'\) is a closed cover of \(M_i\), we have \(Z'_i\) is an open dense subset of \(M_i\). The proof for single maps works to prove that \(Z'_i\) is open and dense in \(M_i\) (see [1], Lemma 4.2.1). Furthermore, each \(x \in Z'_i\) lies in \(\text{int}(T_{j,k}^{i,n})\) for some \(n\) (see [1], Remark 4.2.5).

For \(x \in Z'_i\) define
\[
R(x) = \bigcap \{\text{int}(T_{j,k}^{i,n}) : x \in T_j^i, T_j^i \cap T_k^i \neq \emptyset \text{ and } x \in T_{j,k}^{i,n}\}.
\]
By Lemma 6.2, we have \(R(x)\) is an open rectangle (\(R(x)\) is a finite intersection of open subsets). Consequently, \(R(x)\) is proper.

**Lemma 6.10.** For any \(y \in R(x) \cap Z'_x\), we have \(J'_x(x) = J'_x(y)\) and \(R(y) = R(x)\).
Proof. See [1], Remark 4.2.6.

Therefore, there are only finitely many distinct $R^i(x)$'s. Let
\[
R^i = \{ R^i(x) : x \in Z_i \} = \{ R_{i_1}^i, \ldots, R_{m_i}^i \} \quad \text{for } i \in \mathbb{Z}.
\]

Finally we prove that:

**Theorem 6.11.** The sequence $R^i$ for $i \in \mathbb{Z}$ is a Markov partition for $\mathcal{F}$.

**Proof.** We obtained that if $z \in Z_i$, then $R^i(z) = R^i(x)$ or $R^i(z) \cap R^i(x) = \emptyset$. Therefore
\[
\overline{(R(x) \setminus R^i(x))} \cap Z_i = \emptyset.
\]
Since $Z_i$ is dense in $M_i$, we have $\overline{R(x) \setminus R^i(x)}$ has no interior in $M_i$ and $R^i(x) = \text{int}(R(x))$. Therefore, for $R^i(x) \neq R^i(z)$, we have
\[
\text{int}(R(x)) \cap \text{int}(R(z)) = R^i(x) \cap R^i(z) = \emptyset.
\]

**Claim:** If $x, y \in Z_i \cap f^{-1}_i(Z_{i+1}^i)$, $R^i(x) = R^i(y)$ and $y \in \mathcal{W}^s(x, \varepsilon)$, then
\begin{enumerate}[(i)]  
  
  \item $J_i^j(f_i(x)) = J_i^j(f_i(y))$.
  \item $R^i_j(f_i(x)) = R^i_j(f_i(y))$.
\end{enumerate}

**Proof.** (of Claim)

For i), assume that $f_i(x) = \theta_{i+1}(\sigma_i(\bar{a})) \in T^i_j$ ($a_1 = p_j$) and $a_0 = p_s$ (that is, $x = \theta_i(\bar{a}) \in T^i_j$). By (6.3) we have
\[
f_i(y) = f_i(W^s(x, T^i_j)) \subseteq W^s(f_i(x), T^{i+1}_j),
\]
therefore $f_i(y) \in T^{i+1}_j$. Similarly, if $f_i(y) \in T^{i+1}_j$, then $f_i(x) \in T^{i+1}_j$, and therefore $J^i_j(f_i(x)) = J^i_j(f_i(y))$.

For ii), we prove that if $T^{i+1}_j \in J^i_j(f_i(x)) = J^i_j(f_i(y))$ and $T^{i+1}_j \cap T^{i+1}_k \neq \emptyset$ for $T^{i+1}_k \in J^i_k$, then $f_i(x), f_i(y)$ belong to the same $T^{i+1}_j$. Since $f_i(y) \in W^s(f_i(x), \varepsilon)$, we have $W^s(f_i(y), T^{i+1}_j) = W^s(f_i(x), T^{i+1}_j)$. Thus $f_i(x)$ and $f_i(y)$ belong to $T^{i+1}_j \cup T^{i+1}_k$. Suppose
\[
W^s(f_i(y), T^{i+1}_j) \cap T^{i+1}_k = \emptyset \quad \text{and} \quad W^s(f_i(x), T^{i+1}_k) \cap T^{i+1}_j \neq \emptyset.
\]
Take $f_i(z) \in W^u(f_i(x), T^{i+1}_j) \cap T^{i+1}_k$. From (6.4) we have $f_i(z) \in f_i(W^u(x, T^i_j))$, that is, $z \in W^u(x, T^i_j)$, since $x \in T^i_j$. Write $f_i(z) = \theta_i(\sigma_i(\bar{b}))$, where $a_1 = p_k$ and $a_0 = p_s$ for some $t = 1, \ldots, k$. Then $z \in T^i_k$ and $f_i(W^s(z, T^i_k)) \subseteq W^s(f_i(z), T^{i+1}_k)$. Hence $z \in T^i_k \cap T^i_j \neq \emptyset$. Since $x \in T^{i+1}_j$, we have $T^{i+1}_j \in J^i(x) = J^i(y)$.

Now, given that $z \in W^u(x, T^i_j) \cap T^i_j$ and $x, y$ are in the same $T^{n,j}_{x,j}$, there exists some $w \in W^u(y, T^i_j) \cap T^i_j$. Hence
\[
v = [z, y] = [z, w] \in W^s(y, T^i_j) \cap W^s(y, T^i_j)
\]
and, since $f_i(z), f_i(y) \in T^{i+1}_j$ and $T^{i+1}_j$ is a rectangle, we have
\[
f_i(v) = [f_i(z), f_i(y)] \in W^s(f_i(z), T^{i+1}_k) \cap W^s(f_i(y), T^{i+1}_k),
\]
which is a contradiction. Therefore $R^{i+1}(f_i(x)) = R^{i+1}(f_i(y))$. □
The rest of the proof, which we present below, is taken from the proof of Bowen for Anosov diffeomorphisms ([11]). All the facts are topological and are valid for our case.

For small \( \delta > 0 \), set

\[
Y'_i = \bigcup \left\{ W^s(z, \delta) : z \in \bigcup_j \partial^s T'_j \right\}
\]

and

\[
Y''_i = \bigcup \left\{ W^u(z, \delta) : z \in \bigcup_j \partial^u T'_j \right\}
\]

\( Y'_i \) and \( Y''_i \) are closed and nowhere dense. Hence \( M_i \setminus (Y'_i \cup Y''_i) \subseteq Z'_i \) is open and dense in \( M_i \). Furthermore, if \( x \notin (Y'_i \cup Y''_i) \cap f^{-1}_i(Y'_i \cup Y''_i) \), then \( x \in Z'_i \cap f^{-1}_i(Z^{+1}_i) \) and hence the set \( \{ z \in W^s(x, R'(x)) : x \in Z'_i \cap f^{-1}_i(Z^{+1}_i) \} \) is open and dense in \( W^s(x, R'(x)) \) (as a subset of \( W^s(x, \mathbb{R}) \cap M_i \)). By the previous claim we have \( R^{i+1}(f_i(y)) = R^{i+1}(f_i(x)) \) for \( y \) in \( \{ z \in W^s(x, R'(x)) : x \in Z'_i \cap f^{-1}_i(Z^{+1}_i) \} \).

By continuity

\[
f_i(W^s(x, R'(x))) \subseteq R^{i+1}(f_i(x))
\]

and since \( f_i(W^s(x, R'(x))) \subseteq W^s(f_i(x), \varepsilon) \), then \( f_i(W^s(x, R'(x))) \subseteq W^s(f_i(x), R^{i+1}(f_i(x))) \).

If \( \text{int}(R^j_i) \cap f^{-1}_i(\text{int}(R^{j+1}_i)) \neq \emptyset \), then there exists some \( x \in \text{int}(R^j_i) \cap f^{-1}_i(\text{int}(R^{j+1}_i)) \) such that \( R^j_k = R^j(x) \) and \( R^{j+1}_k = R^{j+1}(f_i(x)) \). If \( z \in R^j_k \cap f^{-1}_i(R^{j+1}_k) \), then

\[
W^s(z, R^j_k) = \{ [z, y] : y \in W^s(x, R^j_k) \}
\]

and

\[
f_i(W^s(z, R^j_k)) = \{ [f_i(z), f_i(y)] : y \in W^s(x, R^j_k) \} \subseteq \{ [f_i(z), w] : w \in W^s(f_i(x), R_j) \}
\]

\( \subseteq W^s(f_i(x), R^{i+1}_j) \).

Analogously we can prove that \( W^u(f_i(x), R^{i+1}_j) \subseteq f_i(W^u(x, R^j_k)) \), which completes the proof.

\[\square\]

**Proof.** (of Theorem D)

Follows from Theorem 6.11

\[\square\]

### 7. Further Generalizations

There are several directions to pursue the studies of Anosov families and questions that still need to be answered. We leave here some topics of interest, and issues that merit attention in the study of this class of dynamical systems.

(i) Codifying Anosov families, which is useful to code random perturbations of a Anosov diffeomorphism (see [7], [16]).

(ii) Verifying if in the case of non-stationary dynamic systems, we can use the shadowing property to have structural stability.

(iii) Extending the works done in [2], [6] and [7] to the orientation-preserving case, to higher genus surfaces, to higher dimensional tori, and to nonlinear Anosov maps. In [7], Section 1.6, the authors address these issues in detail.

(iv) Generalizing Anosov families to continuous time. In this case we would have a flow families instead of a diffeomorphism families. According to comments and suggestions from [7] Section 1.6, examples of flow families to consider are: (i) the suspension flow of a mapping family and (ii) those given by nonautonomous differential equations, where the orbits are integral curves of time-varying vector fields. The authors note that an interesting fact in the suspension of a multiplicative family is that it models the scenery flow of the transverse irrational circle rotation. See also [2] and [6] for details.
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