DIAMETRAL DIAMETER TWO PROPERTIES IN
BANACH SPACES

JULIO BECERRA GUERRERO, GINES LÓPEZ-PÉREZ AND ABRAHAM RUEDA ZOCA

Abstract. The aim of this note is to provide several variants of the
diameter two properties for Banach spaces. We study such properties
looking for the abundance of diametral points, which holds in the setting
of Banach spaces with the Daugavet property, for example, and we intro-
duce the diametral diameter two properties in Banach spaces, showing
for these new properties stability results, inheritance to subspaces and
characterizations in terms of finite rank projections.

1. Introduction

We recall that a Banach space $X$ satisfies the strong diameter two pro-
property (SD2P), respectively diameter two property (D2P), slice-diameter two
property (LD2P), if every convex combination of slices, respectively every
nonempty relatively weakly open subset, every slice, in the unit ball of $X$ has
diameter 2. The weak-star slice diameter two property ($w^*$-LD2P), weak-
star diameter two property ($w^*$-D2P) and weak-star strong diameter two
property ($w^*$-SD2P) for a dual Banach space are defined as usual, changing
slices by $w^*$-slices and weak open subsets by $w^*$- open subsets in the unit
ball. It is known that the above six properties are extremely different as it
is proved in [5].

Even though diameter two property theory is a very recent topic in ge-
ometry of Banach spaces, a lot of nice results have appeared in the last few
years (e.g. [1] [5] [6] [4] [8]). Moreover, it turns out that there are quite lot of
examples of Banach spaces with such properties as infinite-dimensional $C^*$-
algebras [4], non-reflexive $M$-embedded spaces [9] or Daugavet spaces [11].
Last example is quite important because Banach spaces with the Daugavet
property actually satisfy the diameter two properties in a stronger way. In-
deed, as it was pointed out in [10], Banach spaces with Daugavet property

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verify that each slice of the unit ball $S$ has diameter two and each norm-one element of $S$ is diametral, i.e. given $x \in S \cap S_X$ it follows that

$$diam(S) = \sup_{y \in S} \|y - x\|.$$  

We will say that a Banach space $X$ has the diametral local diameter two property (DLD2P) whenever $X$ verifies the above condition. It is known that this property is stable by taking $\ell_p$ sums \[10\] and that is inherited to almost isometric ideals \[2\]. Moreover, this property is different to the Daugavet property (see again \[10\]).

The aim of this note is to provide extensions of the diameter two properties in the way exposed above and make an intensive study of such properties. Indeed, in sections 2 and 3, we shall analyze extensions of the D2P and SD2P, respectively, by the existence of diametral points. Whilst we shall define the diametral diameter two property by the obvious generalization in view of the diametral slice diameter two property, we provide a natural extension of the SD2P in terms of diametrality in some different way. Given a Banach space $X$ we will say that $X$ has the diametral strong diameter two property (DSD2P) whenever given $C$ a convex combination of non-empty relatively weakly open subsets of $B_X$, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ we can find $y \in C$ such that $\|y - x\| > 1 + \|x\| - \varepsilon$. This alternative definition is given because a convex combination of non-empty relatively weakly open subsets of the unit ball of a Banach space does not have to intersect to the unit sphere and it is quite clear that (1.1) implies $\|x\| = 1$. We will get some results of stability of diametral diameter two properties in terms of $\ell_p$ sums and inheritance to subspaces. Moreover, we will exhibit some characterizations of such properties in terms of finite-rank projections or weakly convergent nets. Finally, section 4. is devoted to exhibit some open problems and remarks.

We shall introduce some notation. We consider real Banach spaces, $B_X$ (resp. $S_X$) denotes the closed unit ball (resp. sphere) of the Banach space $X$. If $Y$ is a subspace of a Banach space $X$, $X^*$ stands for the dual space of $X$. A slice of a bounded subset $C$ of $X$ is a set of the form

$$S(C, f, \alpha) := \{x \in C : f(x) > M - \alpha\},$$

where $f \in X^*$, $f \neq 0$, $M = \sup_{x \in C} f(x)$ and $\alpha > 0$. If $X = Y^*$ is a dual space for some Banach space $Y$ and $C$ is a bounded subset of $X$, a $w^*$-slice of $C$ is a set of the form

$$S(C, y, \alpha) := \{f \in C : f(y) > M - \alpha\},$$

where $y \in Y$, $y \neq 0$, $M = \sup_{f \in C} f(y)$ and $\alpha > 0$. $w$ (resp. $w^*$) denotes the weak (resp. weak-star) topology of a Banach space.

It is proved in \[3\] Corollary 2.2\] that a Banach space $X$ has the SD2P if, and only if, $X^*$ has an octahedral norm.
Moreover, it is proved in [8, Theorems 3.2 and 3.4] that a Banach space $X$ has the LD2P (respectively the D2P) if, and only if, $X^*$ has a locally octahedral (respectively weakly octahedral) norm.

Let $X$ be a Banach space and $Y \subseteq X$ a closed subspace. According to [3], we will say that $Y$ is an almost isometric ideal in $X$ if for each $\varepsilon > 0$ and $E \subseteq X$ a finite-dimensional subspace there exists a linear and bounded operator $T : E \to Y$ satisfying the following conditions:

1. $T(e) = e$ for each $e \in E \cap Y$.
2. For each $e \in E$ one has
   \[
   \frac{1}{1 + \varepsilon} \|e\| \leq \|T(e)\| \leq (1 + \varepsilon) \|e\|.
   \]

In spite of the fact that almost isometric ideals in Banach spaces do not have to be closed, by a perturbation argument it follows that a non-closed subspace is an almost isometric ideal if, and only if, its closure is also an almost isometric ideal. Hence, we will consider only closed almost isometric ideals.

In [3] is proved that each diameter two property as well as Daugavet property are inherited to almost isometric ideals from the whole space. This is a consequence of the following

**Theorem 1.1.** [3, Theorem 1.4] Let $X$ be a Banach space and $Y \subseteq X$ an almost ideal in $X$. Then there exists $\varphi : Y^* \to X^*$ a Hahn-Banach operator such that, for each $\varepsilon > 0$, for each $E \subseteq X$ finite-dimensional subspace and each $F \subseteq Y^*$ finite dimensional subspace, there exists $T : E \to Y$ verifying the following:

1. $T(e) = e$ for each $e \in E \cap Y$.
2. For each $e \in E$ one has
   \[
   \frac{1}{1 + \varepsilon} \|e\| \leq \|T(e)\| \leq (1 + \varepsilon) \|e\|.
   \]
3. For each $e \in E$ and $f \in F$ it follows
   \[
   \varphi(f)(e) = f(T(e)).
   \]

We shall also exhibit the following known result which will be used several times in the following. A proof can be found in [10, Lemma 2.1].

**Lemma 1.2.** Let $X$ be a Banach space. Consider $x^* \in S_{X^*}, \varepsilon \in \mathbb{R}^+$ and $x \in S(B_X, x^*, \varepsilon) \cap S_X$. Then, given $0 < \delta < \varepsilon$, there exists $y^* \in S_{X^*}$ such that

\[
x \in S(B_X, y^*, \delta) \subseteq S(B_X, x^*, \varepsilon).
\]

Similarly, a dual version of Lemma above can be stated as follows.

**Lemma 1.3.** Let $X$ be a Banach space. Consider $x \in S_X, \varepsilon \in \mathbb{R}^+$ and $x^* \in S(B_{X^*}, x, \varepsilon) \cap S_{X^*}$. Then, given $0 < \delta < \varepsilon$, there exists $y \in S_X$ such that

\[
x^* \in S(B_{X^*}, y, \delta) \subseteq S(B_{X^*}, x, \varepsilon).
\]
2. DIAMETRAL DIAMETER TWO PROPERTY AND STABILITY RESULTS

We shall start by giving the following

**Definition 2.1.** Let $X$ be a Banach space.

We will say that $X$ has the diametral diameter two property (DD2P) if given $W$ a non-empty relatively weakly open subset of $B_X$, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ there exists $y \in W$ such that

$$\|x - y\| > 2 - \varepsilon. \quad (2.1)$$

If $X$ is a dual Banach space we will say that $X$ has the weak-star diametral diameter two property ($w^*$-DD2P) if given $W$ a non-empty relatively weakly-star open subset of $B_X$, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ there exists $y \in W$ satisfying (2.1).

From [11, Lemma 2.3] we get that each Banach space enjoying to have Daugavet property satisfies DD2P. However, there are Banach spaces with the DD2P which do not enjoy to have the Daugavet property.

**Example 2.2.** Let $X$ be the renorming of $C([0,1])$ given in [2] satisfying that $X$ is MLUR, has the DLD2P and $X$ fails SD2P. Then $X$ fails the Daugavet property. However, $X$ has the DD2P because $X$ has the DLD2P, applying the well known Choquet lemma [?, Lemma 3.40].

It is known that a Banach space $X$ has the D2P if, and only if, $X^{**}$ has the $w^*$-D2P. However, this fact is far from being true for the DD2P. Indeed, applying the weak-star lower semicontinuity of a bidual norm is easy to get the following

**Proposition 2.3.** Let $X$ be a Banach space. If $X^{**}$ has the $w^*$-DD2P, then $X$ has the DD2P.

**Remark 2.4.** The converse of Proposition 2.3 is not true. Indeed consider $X := C(K)$, for an infinite compact Hausdorff and perfect topological space $K$. Now $X$ has the DD2P as being a Daugavet space. However, $B_X^*$ has denting points, so $X^*$ fails the DLD2P and, consequently, $X^{**}$ fails the $w^*$-DLD2P [2, Theorem 3.6].

Now we shall provide several characterizations of the DD2P. First of all, we shall show a useful characterization of the DD2P in terms of weakly convergent nets which will be used in order to prove the stability of the DD2P by $\ell_p$ sums.

**Proposition 2.5.** Let $X$ be a Banach space. The following assertions are equivalent:

1. $X$ has DD2P.
2. For each $x \in S_X$ there exists a net $\{x_s\} \subset B_X$ which converges weakly to $x$ and such that

$$\{|\|x - x_s\||\} \to 2.$$
Proof. (1)⇒(2). Pick $U$ a neighborhood system of $x$ in the weak topology relative to $B_X$. Now, for each $U \in \mathcal{U}$ and every $\varepsilon \in \mathbb{R}^+$, choose $x_{(U,\varepsilon)} \in U$ such that

$$\|x - x_{(U,\varepsilon)}\| \geq 2 - \varepsilon.$$  

Such $x_{(U,\varepsilon)}$ exists because $X$ has the DD2P. Now, considering in $\mathcal{U} \times \mathbb{R}^+$ the partial order given by the reverse inclusion in $\mathcal{U}$ and the inverse natural order in $\mathbb{R}$ we conclude that $\{x_{(U,\varepsilon)}\}_{(U,\varepsilon) \in \mathcal{U} \times \mathbb{R}^+} \to x$ in the weak topology of $B_X$. It is also clear that $\{\|x - x_{(U,\varepsilon)}\|\}_{(U,\varepsilon) \in \mathcal{U} \times \mathbb{R}^+} \to 2$.

(2)⇒(1). Pick $W$ a non-empty relatively weakly open subset of $B_X$, $x \in W \cap S_X$ and $\varepsilon \in \mathbb{R}^+$ and let us prove that there exists $y \in W$ such that $\|x - y\| > 2 - \varepsilon$. By assumption there exists $\{x_s\}$ a net in $B_X$ such that

$$\|x - x_s\| \to 2,$$

and

$$\{x_s\} \overset{w}{\to} x.$$  

From both convergences then there exists $s$ such that $x_s \in W$ and $\|x - x_s\| > 2 - \varepsilon$. Now (1) follows choosing $y := x_s$. □

Now, for dual Banach spaces we have the following characterization of the $w^*$-DD2P, as the above one.

**Corollary 2.6.** Let $X$ be a dual Banach space. The following assertions are equivalent:

1. $X$ has $w^*$-DD2P.
2. For each $x \in S_X$ there exists a net $\{x_s\}$ in $B_X$ which converges to $x$ in the weak-star topology such that

$$\{\|x - x_s\|\} \to 2.$$  

**Remark 2.7.** In view of Proposition 2.5 Daugavet property can also be easily characterized in terms of weakly convergent nets. Indeed it is straightforward to prove from [11, Lemma 2.3] that a Banach space $X$ has the Daugavet property if, and only if, given $x, y \in S_X$ there exists $\{y_s\}$ a net in $B_X$ weakly convergent to $y$ such that

$$\{\|x - y_s\|\} \to 2.$$  

In [10] it is proved a characterization of DLD2P in terms of the behavior of rank-one projections in a Banach space. It turns out to be also true that DD2P can be characterized regarding the behaviour of the rank-one projections. In fact, we have the following characterization of the DD2P.

**Proposition 2.8.** Let $X$ be a Banach space. The following assertions are equivalent:

1. $X$ has the DD2P.
(2) For each \( x_1^*, \ldots, x_n^* \in S_{X^*} \) and \( x \in X \) such that \( x_i^*(x) \neq 0 \), if we define

\[
p_i := x_i^* \otimes \frac{x}{x_i^*(x)} \quad \forall i \in \{1, \ldots, n\}
\]

one has that, for each \( \varepsilon \in \mathbb{R}^+ \), there exists \( y \in B_X \) such that

\[
\|y - p_i(y)\| > 2 - \varepsilon \quad \forall i \in \{1, \ldots, n\}
\]

and

\[
\frac{x_i^*(y)}{x_i^*(x)} \geq 0 \quad \forall i \in \{1, \ldots, n\}
\]

(3) Given \( S := S(B_X, x, \delta) \) a weak-star slice of \( B_{X^*} \) and \( x_1^*, \ldots, x_n^* \in S \cap S_{X^*} \) there exist \( y^* \in S \) and \( y \in S_X \) such that

\[
(x_i^* - y^*)(y) > 2 - \delta \quad \forall i \in \{1, \ldots, n\}.
\]

Proof. (1)\( \Rightarrow \) (2).

Consider \( x_1^*, \ldots, x_n^* \in S_X, x \in X \) and \( p_i := x_i^* \otimes \frac{x}{x_i^*(x)} \) for each \( i \in \{1, \ldots, n\} \).

Consider \( \varepsilon > 0 \) such that \( \varepsilon < 2 \). Note that

\[
\frac{x}{\|x\|} \in W := \left\{ y \in B_X : \left| \frac{x_i^*(y)}{x_i^*(x)} - \frac{1}{\|x\|} \right| \|x\| < \frac{\varepsilon}{2} \right\},
\]

where \( W \) is a relatively weakly open subset of \( B_X \). Moreover \( \frac{x}{\|x\|} \in S_X \). As \( X \) has the DD2P we can assure the existence of an element \( y \in W \) such that

\[
\|y - \frac{x}{\|x\|}\| > 2 - \frac{\varepsilon}{2}.
\]

Now, on the one hand, as \( y \in W \), given \( i \in \{1, \ldots, n\} \), one has

\[
\left| \frac{x_i^*(y)}{x_i^*(x)} - \frac{1}{\|x\|} \right| \|x\| < \frac{\varepsilon}{2} \Rightarrow \frac{x_i^*(y)}{x_i^*(x)} > \frac{1}{\|x\|} - \frac{\varepsilon}{2\|x\|} = \frac{1 - \frac{\varepsilon}{2\|x\|}}{\|x\|} \geq 0.
\]

On the other hand, given \( i \in \{1, \ldots, n\} \), it follows

\[
\|y - p_i(y)\| \geq \left\| y - \frac{x}{\|x\|} \right\| - \left\| \frac{x}{\|x\|} - p_i(y) \right\| > 2 - \frac{\varepsilon}{2} - \left\| \frac{x}{\|x\|} - \frac{x_i^*(y)}{x_i^*(x)} \right\| > 2 - \varepsilon
\]

since \( y \in W \). So (2) follows.

(2)\( \Rightarrow \) (1).

Consider \( W := \bigcap_{i=1}^n S(B_X, y_i^*, \varepsilon_i) \) a non-empty relatively weakly open subset of \( B_X \) and pick \( x \in W \cap S_X \). In order to prove that \( X \) has the DD2P choose \( 0 < \varepsilon < \min_{1 \leq i \leq n} \varepsilon_i \). By Lemma \ref{lemma12} we can find, for each \( i \in \{1, \ldots, n\} \), a functional \( x_i^* \in S_{X^*} \) such that

\[
x \in S(B_X, x_i^*, \varepsilon) \subseteq S(B_X, y_i^*, \varepsilon_i) \forall i \in \{1, \ldots, n\},
\]
and so \( x \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \varepsilon) \subseteq W \). Consider \( \eta \in \mathbb{R}^+ \) small enough to satisfy \( x_i^*(x) (1 - \eta) > 1 - \varepsilon \) for each \( i \in \{1, \ldots, n\} \). For each \( i \in \{1, \ldots, n\} \) define
\[
p_i := x_i^* \otimes \frac{x}{x_i^*(x)}.
\]
From the hypothesis we can find \( y \in B_X \) such that
\[
\|y - p_i(y)\| > 2 - \eta \quad \forall i \in \{1, \ldots, n\}
\]
and
\[
x_i^*(y) \geq x_i^*(x) > 0.
\]
Now, on the one hand, one has
\[
1 - \eta < \|p_i(y)\| = \left\| \frac{x_i^*(y)}{x_i^*(x)} \right\| = \frac{x_i^*(y)}{x_i^*(x)} \quad \forall i \in \{1, \ldots, n\}.
\]
So \( x_i^*(y) > (1 - \eta) x_i^*(x) > 1 - \varepsilon \) for each \( i \in \{1, \ldots, n\} \) and, consequently, \( y \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \varepsilon) \subseteq W \). Moreover, chosen \( i \in \{1, \ldots, n\} \), it follows
\[
\|y - x\| \geq \|y - p_i(y)\| - \|p_i(y) - x\| > 2 - \eta - \frac{|x_i^*(y) - x_i^*(x)|}{x_i^*(x)} > 2 - \eta - \varepsilon.
\]
As \( 0 < \varepsilon < \min_{1 \leq i \leq n} \varepsilon_i \) was arbitrary we conclude the desired result.

(1) \( \Rightarrow \) (3). Let \( S \) and \( x_1^*, \ldots, x_n^* \) be as in the hypothesis and pick \( 0 < \eta < \delta \). Now given \( i \in \{1, \ldots, n\} \) one has
\[
x_i^* \in S \iff x_i^*(x) > 1 - \delta \iff x \in S(B_X, x_i^*, \delta).
\]
So \( x \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \delta) \cap S_X \). As \( X \) has the DD2P then there exists \( y \in \bigcap_{i=1}^{n} S(B_X, x_i^*, \delta) \cap S_X \) such that
\[
\|x - y\| > 2 - \eta \Rightarrow \exists y^* \in S_X^* : y^*(x) - y^*(y) > 2 - \eta \Rightarrow \left\{ y^*(x) > 1 - \eta \quad \right. \left. y^*(y) < -1 + \eta \right\}.
\]
So \( y^*(x) > 1 - \delta \) and thus \( y^* \in S \). In addition, given \( i \in \{1, \ldots, n\} \), it follows
\[
(x_i^* - y^*)(y) = x_i^*(y) - y^*(y) > 1 - \delta + 1 - \eta = 2 - \delta - \eta.
\]
From the arbitrariness of \( 0 < \eta < \delta \) we get the desired result by a perturbation argument, if necessary.

(3) \( \Rightarrow \) (1). Let \( W := \bigcap_{i=1}^{n} S(B_X, y_i^*, \varepsilon_i) \) be a non-empty relatively weakly open subset of \( B_X \) and consider \( x \in W \cap S_X \). Pick \( 0 < \delta < \min_{1 \leq i \leq n} \varepsilon_i \). From
Lemma 1.2 we can find, for each $i \in \{1, \ldots, n\}$, an element $x_i^* \in S_X$ such that

$$x \in S(B_X, x_i^*, \delta) \subseteq S(B_X, y_i^*, \varepsilon_i)$$

holds for each $i \in \{1, \ldots, n\}$. Now $x_1^*, \ldots, x_n^* \in S(B_{X^*}, x, \delta)$. From assumptions we can find $y^* \in S(B_{X^*}, x, \delta)$ and $y \in S_X$ such that

$$(x_i^* - y^*)(y) > 2 - \delta$$

holds for each $i \in \{1, \ldots, n\}$. Now, on the one hand

$$x_i^*(y) > 1 - \delta \Rightarrow y \in \bigcap_{i=1}^n S(B_X, x_i^*, \delta) \subseteq W.$$

Moreover, as $y^* \in S(B_{X^*}, x, \delta)$, it follows

$$\|x - y\| \geq y^*(x) - y^*(y) > 1 - \delta + 1 - \delta = 2(1 - \delta).$$

From the arbitrariness of $0 < \delta < \min_{1 \leq i \leq n} \varepsilon_i$ we have that $X$ has the DD2P, as desired. \qed

Remark 2.9. Note that given $p_1, \ldots, p_n$ rank one projections as in above Proposition one has

$$\|I - p_i\| \geq 2$$

whenever $X$ enjoys to have the DLD2P. However, if $X$ also satisfies the DD2P these projections can be “normed” by a common point of the space.

A dual version of above Proposition is the following

Proposition 2.10. Let $X$ be a Banach space. The following assertions are equivalent:

1. $X^*$ has the $w^*$-DD2P.
2. For each $x_1, \ldots, x_n \in S_X$ and $x^* \in X^*$ such that $x^*(x_i) \neq 0$, if we define

$$p_i := \frac{x^*}{x^*(x_i)} \otimes x_i \forall i \in \{1, \ldots, n\}$$

one has that, for each $\varepsilon \in \mathbb{R}^+$, there exists $y^* \in B_{X^*}$ such that

$$\|y^* - p_i(y^*)\| > 2 - \varepsilon \forall i \in \{1, \ldots, n\}$$

and

$$\frac{y^*(x_i)}{x^*(x_i)} \geq 0 \forall i \in \{1, \ldots, n\}$$

3. Given $S := S(B_X, x^*, \delta)$ a slice of $B_X$ and $x_1, \ldots, x_n \in S \cap S_X$ there exist $y \in S$ and $y^* \in S_{X^*}$ such that

$$y^*(x_i - y) > 2 - \delta \forall i \in \{1, \ldots, n\}.$$
It is known that DLD2P is stable under taking $\ell_p$-sums. Indeed, given two Banach spaces $X$ and $Y$ and $1 \leq p \leq \infty$, the Banach space $X \oplus_p Y$ has the DLD2P if, and only if, $X$ and $Y$ enjoy to have the DLD2P [10, Theorem 3.2].

Our aim is to establish the same result for the DD2P. We shall begin with the stability result

**Theorem 2.11.** Let $X, Y$ be Banach spaces which satisfy the DD2P and let $1 \leq p \leq \infty$. Then $X \oplus_p Y$ enjoys to have the DD2P.

**Proof.** Define $Z := X \oplus_p Y$, pick $(x_0, y_0) \in S_Z$ and let us apply Proposition 2.5.

On the one hand, if $p = \infty$, then either $\|x_0\| = 1$ or $\|y_0\| = 1$. Assume, with no loss of generality, that $\|x_0\| = 1$. As $X$ has the DD2P then there exists $\{x_s\}$ a net in $B_X$ such that

$$\{x_s\} \to x_0$$

in the weak topology of $X$ and

$$\|x - x_s\| \to 2.$$

Then we have that

$$\{(x_s, y_0)\} \to (x_0, y_0)$$

in the weak topology of $B_Z$ (note that, from the definition of the norm on $Z$ we have that each term of the above net belongs to $B_Z$). In addition, given $s$ one has

$$2 \geq \|(x_0, y_0) - (x_s, y_0)\|_\infty = \max\{\|x - x_s\|, \|y_0\|\} \geq \|x - x_s\|.$$

As $\{\|x - x_s\|\} \to 2$ we conclude that $\{(x_0, y_0) - (x_s, y_s)\} \to 2$.

On the other hand, assume $p < \infty$. As $(x_0, y_0) \in S_Z$ we have that

$$\left(\|x_0\|^p + \|y_0\|^p\right)^{\frac{1}{p}} = 1.$$

Now $x_0$ is an element of $\|x_0\|S_X$. As $X$ has the DD2P then by Proposition 2.5 there exists $\{x_s\}_{s \in S}$ a net in $\|x_0\|B_X$ such that

$$\{x_s\} \to x_0$$

in the weak topology of $X$ and

$$\{\|x_0 - x_s\|\} \to 2\|x_0\|.$$

In addition, as $Y$ also has the DD2P, then there exists a net $\{y_t\}_{t \in T}$ in $\|y_0\|B_Y$ such that

$$\{y_t\}_{t \in T} \to y_0$$

in the weak topology of $Y$ and such that

$$\{\|y_0 - y_t\|\}_{t \in T} \to 2\|y_0\|.$$

Now we have $\{(x_s, y_t)\}_{(s, t) \in S \times T} \to (x_0, y_0)$ in the weak topology of $Z$. Moreover, given $s \in S, t \in T$ one has

$$\|(x_s, y_t)\|_p = \left(\|x_s\|^p + \|y_t\|^p\right)^{\frac{1}{p}} \leq \left(\|x_0\|^p + \|y_0\|^p\right)^{\frac{1}{p}} = 1,$$
so \((x_s, y_t) \in B_Z\) for each \(s \in S, t \in T\). Finally, given \(s \in S, t \in T\) it follows
\[
\| (x_0, y_0) - (x_s, y_t) \|_p = \left( \| x_0 - x_s \|^p + \| y_0 - y_t \|^p \right)^{\frac{1}{p}} \rightarrow \left( (2 \| x_0 \|)^p + (2 \| y_0 \|)^p \right)^{\frac{1}{p}} = 2.
\]

Now let us prove the converse of the above result.

**Proposition 2.12.** Let \(X, Y\) be Banach space and define \(Z := X \oplus_p Y\) for 

\(1 \leq p \leq \infty\). If \(X\) fails to have DD2P so does \(Z\).

**Proof.** As \(X\) fails the DD2P then there exists \(U\) a non-empty relatively weakly open subset of \(B_X\), \(x_0 \in U \cap S_X\) and \(\varepsilon_0 \in \mathbb{R}^+\) such that
\[
\| x_0 - y \| \leq 2 - \varepsilon_0 \ \forall \ y \in U.
\]

Now we shall argue by cases:

(1) If \(p = \infty\) define the weak open subset of \(B_Z\) given by
\[
W := \{ (x, y) \in B_Z : x \in U \cap B_X \},
\]
and pick \((x_0, 0) \in W\). Then for each \((x, y) \in W\) one has
\[
\| (x_0, 0) - (x, y) \| = \max\{ \| x_0 - x \|, \| y \| \} \leq \max\{ 2 - \varepsilon_0, 1 \} < 2,
\]
as \(x \in U \cap B_X\).

(2) If \(p < \infty\), given \(\varepsilon \in \mathbb{R}^+\), there exists \(\delta > 0\) such that
\[
1 - \delta < |r| \leq 1, |s| \leq 1, 
\]
\[
\left( |r|^p + |s|^p \right)^{\frac{1}{p}} \leq 1 \Rightarrow |s|^p < \varepsilon.
\]

Define
\[
W := \{ (x, y) \in B_Z : x \in U \cap B_X \text{ and } \| x \| > 1 - \delta \},
\]
which is a weakly open subset of \(B_Z\) from the lower weakly semi-continuity of the norm on \(X\). Consider \((x_0, 0) \in W\). Now, given \((x, y) \in W\) we have from (2.2) that \(|y|^p \leq \varepsilon\). In addition, as \(x \in U \cap B_X\) we conclude \(\| x - x_0 \| \leq 2 - \varepsilon_0\). Hence
\[
\| (x_0, 0) - (x, y) \| = \left( \| x_0 - x \|^p + \| y \|^p \right)^{\frac{1}{p}} \leq \left( (2 - \varepsilon_0)^p + \varepsilon \right)^{\frac{1}{p}}.
\]
So, taking \(\varepsilon\) small enough, we conclude that \(\sup_{(x, y) \in W} \| (x_0, 0) - (x, y) \| < 2\), so we are done.

Even though Example 2.2 shows that Daugavet property and DD2P are different, above results provide us more examples of such Banach spaces. Indeed, given \(1 < p < \infty\), \(Z := X \oplus_p Y\) has the DD2P and fails to have Daugavet property (actually, \(Z\) fails to have the strong diameter two property \([1, \text{ Theorem 3.2}]\)) whenever \(X, Y\) are Banach spaces with the DD2P (in particular, Daugavet spaces).
Finally we shall study the following problem: when a subspace of a Banach space having the DD2P inherits DD2P? In order to give a partial answer, it has been recently proved in [7] that D2P is hereditary to finite-codimensional subspaces. Bearing in mind the ideas of the proof of that result, we can prove the following

**Theorem 2.13.** Let $X$ be a Banach space which satisfies the DD2P. If $Y$ is a closed subspace of $X$ such that $X/Y$ is finite-dimensional then $Y$ has the DD2P.

**Proof.** Consider

$$W := \{ y \in Y : |y_i^*(y - y_0)| < \varepsilon_i \forall i \in \{1, \ldots, n\} \},$$

for $n \in \mathbb{N}, \varepsilon_i \in \mathbb{R}^+, y_i^* \in Y^*$ for each $i \in \{1, \ldots, n\}$ and $y_0 \in Y$ such that $W \cap B_Y \neq \emptyset$.

Pick $y \in W \cap S_Y$ and let us find, for each $\delta \in \mathbb{R}^+$, a point $z \in W \cap B_Y$ such that $\|y - z\| > 2 - \delta$. To this aim pick an arbitrary $\delta \in \mathbb{R}^+$. Assume that $y_i^* \in X^*$ for each $i \in \{1, \ldots, n\}$. Observe that there is no loss of generality from the Hahn-Banach theorem.

Define

$$U := \{ x \in X : |y_i^*(x - y_0)| < \varepsilon_i \forall i \in \{1, \ldots, n\} \},$$

which is a weakly open set in $X$ such that $U \cap B_X \neq \emptyset$.

Let $p : X \rightarrow X/Y$ be the quotient map, which is a $w - w$ open map. Then $p(U)$ is a weakly open set in $X/Y$. In addition

$$\emptyset \neq p(U \cap B_X) \subseteq p(U) \cap p(B_X) \subseteq p(U) \cap B_{X/Y}.$$ 

Defining $A := p(U) \cap B_{X/Y}$, then $A$ is a non-empty relatively weakly open and convex subset of $B_{X/Y}$ which contains to zero. Hence, as $X/Y$ is finite-dimensional, we can find a weakly open set $V$ of $X/Y$, in fact a ball centered at 0, such that $V \subset A$ and that

$$ \text{(2.3)} \quad \text{diam}(V \cap p(U) \cap B_{X/Y}) = \text{diam}(V) < \frac{\delta}{16}. $$

As $V \subset A$ then $B := p^{-1}(V) \cap U \cap B_X \neq \emptyset$. Hence $B$ is a non-empty relatively weakly open subset of $B_X$. Moreover $y \in p^{-1}(V)$ because $p(y) = 0 \in V$, so $y \in B \cap S_X$. Using that $X$ satisfies the DD2P we can assure the existence of $v \in B$ such that

$$ \text{(2.4)} \quad \|v - y\| > 2 - \frac{\delta}{16}. $$

Note that $v \in B$ implies $p(v) \in V = V \cap p(U) \cap B_{X/Y}$. In view of (2.3) it follows

$$ \|p(v)\| \leq \text{diam}(V \cap p(U) \cap B_{X/Y}) < \frac{\delta}{16}. $$
Hence there exists $u \in Y$ such that $||u - v|| < \frac{\delta}{16}$ and so $||u|| < 1 + \frac{\delta}{16}$. Letting $z = \frac{u}{||u||}$, we have that

$$||v - z|| \leq ||v - u|| + \left|\frac{u}{||u||}\right| < \frac{\delta}{16} + ||u||(||u|| - 1) < \frac{\delta}{16} + \left(1 + \frac{\delta}{16}\right) \frac{\delta}{16} = \frac{\delta}{16} \left(2 + \frac{\delta}{16}\right).$$

So

$$\text{(2.5)} \quad ||v - z|| < \frac{\delta}{4}.$$

Note that given $i \in \{1, \ldots, n\}$ and bearing in mind (2.5) one has

$$|y_i^*(z - y_0)| \leq |y_i^*(z - v)| + |y_i^*(v - y_0)| \leq ||y_i^*|| \frac{\delta}{4} + \varepsilon_i,$$

using that $v \in U$. Thus, if we define

$$W_\delta := \left\{ y \in Y : |y_i^*(y - y_0)| < \varepsilon_i + ||y_i^*|| \frac{\delta}{4} \forall i \in \{1, \ldots, n\} \right\}$$

it follows that $v \in W_\delta \cap B_Y$. On the other hand, in view of (2.4) and (2.5) we can estimate

$$||y - z|| \geq ||y - v|| - ||v - z|| > 2 - \frac{\delta}{16} - \frac{\delta}{4} > 2 - \delta.$$

From here we can conclude the desired result. Indeed, for each $i \in \{1, \ldots, n\}$ we can find $\widehat{\varepsilon}_i \in \mathbb{R}^+$ and $\delta_0 \in \mathbb{R}^+$ such that

$$\widehat{\varepsilon}_i + \delta_0 ||y_i^*|| < \varepsilon_i \forall i \in \{1, \ldots, n\},$$

and that

$$y \in \widehat{W} := \{ z \in Y : |y_i^*(z - y_0)| < \widehat{\varepsilon}_i \forall i \in \{1, \ldots, n\} \}.$$

For $0 < \delta < \delta_0$ one has

$$\widehat{W}_\delta := \left\{ y \in Y : |y_i^*(y - y_0)| < \widehat{\varepsilon}_i + ||y_i^*|| \frac{\delta}{4} \forall i \in \{1, \ldots, n\} \right\} \subseteq W.$$

The arbitrariness of $\delta$ in the above argument allow us to conclude the desired result. ■

As it is done in [7] for the $w^*$-D2P, we can conclude a stability result for the $w^*$-DD2P.

**Corollary 2.14.** Let $X$ be a Banach space and let $Y \subseteq X$ a closed subspace. If $X^*$ has the $w^*$-DD2P and $Y$ is finite-dimensional, then $(X/Y)^*$ has the $w^*$-DD2P.
Proof. Consider $W$ a weakly-star open subset of $Y^* = (X/Y)^*$ such that

$$W \cap B_{Y^*} \neq \emptyset,$$

and pick $z^* \in W \cap S_{Y^*}$. Now we can extend $W$ to a weak-star open subset of $X^*$, say $U$, as it is done in Theorem 2.13 satisfying $z^* \in U \cap S_{X^*}$.

Let $p : X^* \to X^*/Y^*$ be the quotient map, which is a $w^*-w^*$ open map. Then $p(U)$ is a weak-star open set of $X^*/Y^*$ which meets with $B_{X^*/Y^*}$.

If we define $A := p(U) \cap B_{X^*/Y^*}$, then we have that $A$ is a relatively weak-star open and convex subset of $B_{X^*/Y^*}$ which contains to zero.

From here, it is straightforward to check that computations of Theorem 2.13 work and allow us to conclude that

$$\sup_{x^* \in W \cap B_{Y^*}} \|z^* - x^*\| = 2,$$

so $Y^* = (X/Y)^*$ has the $w^*$-DD2P as desired.

As we have pointed out in the Introduction, the D2P is inherited to almost isometric ideals from the whole space [3, Proposition 3.2]. Now, following similar ideas, we get the following

**Proposition 2.15.** Let $X$ be a Banach space and let $Y \subseteq X$ a closed almost isometric ideal. If $X$ has the DD2P, so does $Y$.

**Proof.** Take $n = 1$ in the proof of Proposition 3.12.

### 3. DIAMETRAL STRONG DIAMETER TWO PROPERTY AND STABILITY RESULTS

Now we shall introduce the natural extension of the SD2P in the same way the DD2P is defined.

**Definition 3.1.** Let $X$ be a Banach space. We will say that $X$ has the diametral strong diameter two property (DSD2P) if given $C$ a convex combination of non-empty relatively weakly open subsets of $B_X$, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ then there exists $y \in C$ such that

$$\|x - y\| > 1 + \|x\| - \varepsilon. \quad (3.1)$$

If $X$ is a dual space, we will say that $X$ has the weak-star diametral strong diameter two property ($w^*$-DSD2P) if given $C$ a convex combination of non-empty relatively weakly-star open subsets of $B_X$, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ then there exists $y \in C$ satisfying (3.1).

**Remark 3.2.** On the one hand, note that the above definition extends the strong diameter two property from the Bourgain lemma [?].
On the other hand, the condition (1.1) is replaced with (3.1) to get the implication \( DSD2P \Rightarrow SD2P \). Indeed, consider \( X \) the Banach space of Example 2.2 and \( C := \sum_{i=1}^{n} \lambda_i W_i \) a convex combination of non-empty relatively weakly open subsets of \( B_X \). If \( C \cap S_X \neq \emptyset \) then there exists \( x := \sum_{i=1}^{n} \lambda_i x_i \in C \cap S_X \). As \( X \) is a strictly convex space we conclude that \( x_1 = x_2 = \ldots = x_n \). Consequently \( x \in \bigcap_{i=1}^{n} W_i \subseteq C \) and, as \( X \) has the DD2P, we can find, for each \( \varepsilon > 0 \), an element \( y \in \bigcap_{i=1}^{n} W_i \subseteq X \) such that \( \| y - x \| > 2 - \varepsilon \). However, \( X \) fails to have the SD2P.

As in the DD2P, the first example of Banach space with the DSD2P comes from Daugavet spaces.

**Example 3.3.** Daugavet Banach spaces enjoy to have DSD2P.

**Proof.** Consider \( X \) to be a Banach space enjoying to have the Daugavet property. From the proof of [11, Lemma 2.3] it follows that given \( C \) a convex combination of non-empty relatively weakly open subsets of \( B_X \), \( x \in S_X \) and \( \varepsilon \in \mathbb{R}^+ \) we can find \( y \in C \) such that \( \| x + y \| > 2 - \varepsilon \).

From here let us prove that \( X \) enjoys to have the DSD2P. To this aim pick \( C := \sum_{i=1}^{n} \lambda_i W_i \) a convex combination of non-empty relatively weakly open subsets of \( B_X \). Let \( x \in C \) such that \( x \neq 0 \). From Daugavet property we can find \( y \in \sum_{i=1}^{n} \lambda_i (\neg W_i) \) such that

\[
\left\| \frac{x}{\|x\|} + y \right\| > 2 - \varepsilon.
\]

Now \( -y \in C \). Moreover

\[
\left\| x - (-y) \right\| \geq \left\| \frac{x}{\|x\|} + y \right\| - \left\| \frac{x}{\|x\|} - x \right\| > 2 - \varepsilon - \| x \| \cdot \frac{1}{\|x\|} - 1 = 2 - \varepsilon - \| x \| - 1 + \| x \| = 1 + \| x \| - \varepsilon.
\]

In order to conclude the proof assume that \( 0 \in C \). As \( \text{diam}(C) = 2 \) (see the proof of [6, Lemma 2.3]) we can find \( x, y \in C \) such that

\[
\left\{ \begin{array}{l}
\| x - y \| > 2 - \varepsilon \\
\| x \| \leq 1
\end{array} \right\} \Rightarrow \| y - 0 \| = \| y \| > 1 - \varepsilon = 1 + \| 0 \| - \varepsilon.
\]

From the arbitrariness of \( C \) we conclude that \( X \) has the DSD2P.

Given a Banach space \( X \), it is true that \( X \) has the DSD2P whenever \( X^{**} \) has the \( w^* \)-DSD2P by a similar argument to the one given in Proposition 2.3. Again, the converse is not true, because the example exhibited in Remark 2.4 also works for the DSD2P.

Moreover, DSD2P admits a characterization in terms of weakly convergent nets as DD2P does. Indeed, we have the following
Proposition 3.4. Let $X$ be a Banach space. The following assertions are equivalent:

1. $X$ has the DSD2P.
2. For each $x_1, \ldots, x_n \in B_X$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n \lambda_i = 1$ it follows that, for each $i \in \{1, \ldots, n\}$, there exists a net in $B_X$ weakly convergent to $x_i$ such that

$$\left\{ \left\| \sum_{i=1}^n \lambda_i (x_i - x_s^i) \right\| \right\} \to 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\|.$$

Proof. (1)⇒(2). Pick $U$ a system of neighborhoods of 0. Now, for each $U \in \mathcal{U}$ and $\varepsilon \in \mathbb{R}^+$, pick $x_{i,U,\varepsilon}$ for each $i \in \{1, \ldots, n\}$ such that $x_{i,U,\varepsilon} \in (x_i + U) \cap B_X$

and

$$\left\| \sum_{i=1}^n \lambda_i (x_i - x_{i,U,\varepsilon}) \right\| > 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\| - \varepsilon,$$

which can be done because $X$ has the DSD2P.

Now it is quite clear that, given $i \in \{1, \ldots, n\}$, then

$$\{x_{i,U,\varepsilon}\}_{(U,\varepsilon) \in \mathcal{U} \times \mathbb{R}^+} \to x_i$$

in the weak topology of $X$. Moreover, it is clear that

$$\left\{ \left\| \sum_{i=1}^n \lambda_i (x_i - x_{i,U,\varepsilon}) \right\| \right\} \to 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\|$$

from triangle inequality.

(2)⇒(1). Is similar to Proposition 2.5.\$

Now we can establish a dual version for the result above.

Proposition 3.5. Let $X$ be a dual Banach space. The following assertions are equivalent:

1. $X$ has the $w^*$-DSD2P.
2. For each $x_1, \ldots, x_n \in B_X$ and each $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^n \lambda_i = 1$ it follows that, for each $i \in \{1, \ldots, n\}$, there exists a net in $B_X$ convergent to $x_i$ in the weak-star topology of $X$ such that

$$\left\{ \left\| \sum_{i=1}^n \lambda_i (x_i - x_{i,s}^i) \right\| \right\} \to 1 + \left\| \sum_{i=1}^n \lambda_i x_i \right\|.$$

As we have checked, DLD2P and DD2P have strong links with the rank one projections. This fact turns out to be also true for the DSD2P when we consider finite-rank projections.
Proposition 3.6. Let $X$ be a Banach space. Assume that $X$ has the DSD2P. Then for each $p := \sum_{i=1}^n x_i^* \otimes x_i$ projection we have
\[ \|I - p\| \geq 1 + \left\| \sum_{i=1}^n \frac{1}{n} x_i \right\|. \]

Proof. Pick $p := \sum_{i=1}^n x_i^* \otimes x_i$ a finite rank projection and let $\varepsilon \in \mathbb{R}^+$. Then
\[ \sum_{i=1}^n \frac{1}{n} \frac{x_i}{\|x_i\|} \in \sum_{i=1}^n \frac{1}{n} \left\{ y \in B_X : \left| x_i^*(y) - \frac{1}{\|x_i\|} \|x_i\| \right| < \frac{\varepsilon}{4} \right\} \]
As $X$ has the DSD2P then, for each $i \in \{1, \ldots, n\}$, there exists $y_i \in \left\{ y \in B_X : \left| x_i^*(y) - \frac{1}{\|x_i\|} \|x_i\| \right| < \frac{\varepsilon}{4} \right\}$ such that
\[ \left\| \sum_{i=1}^n \frac{1}{n} \left( y_i - \frac{x_i}{\|x_i\|} \right) \right\| > 1 + \left\| \sum_{i=1}^n \frac{1}{n} \frac{x_i}{\|x_i\|} \right\| - \frac{\varepsilon}{4} \]
Then
\[ \|I - p\| \geq \left\| \sum_{i=1}^n \frac{1}{n} y_i - p \left( \sum_{i=1}^n \frac{1}{n} y_i \right) \right\| \geq \left\| \sum_{i=1}^n \frac{1}{n} \left( y_i - \frac{x_i}{\|x_i\|} \right) \right\| \]
\[ - \left\| \sum_{i=1}^n \frac{1}{n} \left( \frac{x_i}{\|x_i\|} - p \left( \sum_{i=1}^n \frac{1}{n} y_i \right) \right) \right\|. \]
Now
\[ \left\| \sum_{i=1}^n \frac{1}{n} \left( \frac{x_i}{\|x_i\|} - p \left( \sum_{i=1}^n \frac{1}{n} y_i \right) \right) \right\| \leq \sum_{i=1}^n \frac{1}{n} \left\| \frac{x_i}{\|x_i\|} - \sum_{j=1}^n x_j^*(y_i)x_j \right\| \leq \sum_{i=1}^n \frac{1}{n} \left( \left\| \frac{1}{\|x_i\|} - x_i^*(x_i) \right\| + \sum_{j \neq i} \|x_j^*(y_i)\| \|x_j\| \right) < \frac{\varepsilon}{4}. \]
Thus
\[ \|I - p\| \geq 1 + \left\| \sum_{i=1}^n \frac{1}{n} x_i \right\|. \]

Now, as we have done in Theorem 2.11 for the DD2P, we will focus on analysing the DSD2P in the $\ell_p$ sum of two Banach spaces. As every Banach space enjoying to have the DSD2P has the strong diameter two property, we conclude that the $\ell_p$ sum of two Banach spaces does not have the DSD2P whenever $1 < p < \infty$ [1 Theorem 3.2]. Nevertheless, we will prove that, as well as happens with Daugavet spaces, DSD2P has a nice behavior in the case $p = \infty$. We shall begin proving the following
Proposition 3.7. Let $X, Y$ be Banach spaces and assume that $X \oplus_p Y$ has the DSD2P for $p \in \{1, \infty\}$. Then $X$ and $Y$ enjoy to have the DSD2P.

Proof. In order to prove the Proposition, assume that $X$ does not satisfy the DSD2P. Then there exists $C := \sum_{i=1}^{n} \lambda_i \cap_{j=1}^{n_i} S(B_X, x_{ij}^*, \eta_{ij})$ a convex combination of non-empty relatively weakly open subsets of $B_X$, an element $\sum_{i=1}^{n} \lambda_i x_i \in C$ and $\varepsilon \in \mathbb{R}^+$ satisfying that

\begin{equation}
\left\| \sum_{i=1}^{n} \lambda_i (x_i - y_i) \right\| \leq 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon \forall \sum_{i=1}^{n} \lambda_i y_i \in C.
\end{equation}

(3.2)

Obviously we will assume the non-trivial case (i.e. $\sum_{i=1}^{n} \lambda_i x_i \neq 0$), so we can assume, taking $\delta < \varepsilon$ if necessary, that $\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon \geq 0$.

Using Lemma 1.2 as much times as necessary we can assume that each number $\eta_{ij}$ are equal (say $\eta$) and that $\eta < \frac{\varepsilon}{2} \forall i \in \{1, \ldots, n\}$. Define

$C := \sum_{i=1}^{n} \lambda_i \cap_{j=1}^{n_i} S(B_{X \oplus_p Y}, (x_{ij}^*, 0), \eta)$.

If $p = 1$ we have from [1] Theorem 3.1, equation (3.1)] that

\begin{equation}
C \subseteq C \times \eta B_Y.
\end{equation}

(3.3)

Now on the one hand, from (3.2), we have the inequality

\begin{equation}
\left\| \sum_{i=1}^{n} \lambda_i (x_i - x_i') \right\| \leq 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon.
\end{equation}

On the other hand we have from (3.3) the following

\begin{equation}
\left\| \sum_{i=1}^{n} \lambda_i y_i' \right\| < \eta.
\end{equation}

So combining both previous inequalities and keeping in mind that $\eta < \frac{\varepsilon}{2}$ we conclude

\begin{equation}
\left\| \sum_{i=1}^{n} \lambda_i ((x_i, 0) - (x_i', y_i')) \right\| \leq 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon.
\end{equation}

From the arbitrariness of $\sum_{i=1}^{n} \lambda_i (x_i', y_i') \in C$ we conclude that $X \oplus_1 Y$ fails the DSD2P, so we are done in the case $p = 1$.

The case $p = \infty$ is quite easier than the above one. Indeed, pick $\sum_{i=1}^{n} \lambda_i (x_i', y_i') \in C$. Then

\begin{equation}
\left\| \sum_{i=1}^{n} \lambda_i ((x_i, 0) - (x_i', y_i')) \right\| = \max \left\{ \left\| \sum_{i=1}^{n} \lambda_i (x_i - x_i') \right\|, \left\| \sum_{i=1}^{n} \lambda_i y_i \right\| \right\}
\end{equation}

Diametral diameter two properties in Banach spaces.

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Proposition 3.4. As one has $Z$ let

$$\sum_{i=1}^{n} \lambda_i x_i$$

so does $Z$. X/Y that ($\Sigma_{i=1}^{n} \lambda_i x_i$) = 1 + $\sum_{i=1}^{n} \lambda_i x_i$ 1 - $\epsilon$, where the last inequality holds from the assumption $\|\sum_{i=1}^{n} \lambda_i x_i\| - \epsilon \geq 0$. Hence, $X \oplus_{\infty} Y$ does not have the DSD2P, so we are done.

Now we shall establish the converse of the result above for $p = \infty$.

Theorem 3.8. Let $X, Y$ be a Banach spaces. If $X$ and $Y$ have the DSD2P so does $Z := X \oplus_{\infty} Y$.

Proof. Pick $n \in \mathbb{N}$, $(x_1, y_1), \ldots, (x_n, y_n) \in B_Z$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{n} \lambda_i = 1$. In order to prove that $Z$ has the DSD2P we shall use Proposition 3.4. As

$$\max_{i=1}^{n} \lambda_{i}(x_i, y_i)_{\infty} = \max \left\{\left\| \sum_{i=1}^{n} \lambda_i x_i \right\|, \left\| \sum_{i=1}^{n} \lambda_i y_i \right\| \right\},$$

then either $\|\sum_{i=1}^{n} \lambda_i (x_i, y_i)\|_{\infty} = \|\sum_{i=1}^{n} \lambda_i x_i\|_{\infty} = \|\sum_{i=1}^{n} \lambda_i y_i\|_{\infty}$.

We shall assume, with no loss of generality, that $\|\sum_{i=1}^{n} \lambda_i (x_i, y_i)\|_{\infty} = \|\sum_{i=1}^{n} \lambda_i x_i\|$. Now, as $X$ has the DSD2P, we have from Proposition 3.4 that, for each $i \in \{1, \ldots, n\}$, there exists $\{x_i^s\}$ a net weakly convergent to $x_i$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i (x_i - x_i^s) \right\|_{\infty} \rightarrow 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_{\infty}.$$

Now we have that $\{(x_i^s, y_i)\} \rightarrow (x_i, y_i)$ in the weak topology of $Z$ for each $i \in \{1, \ldots, n\}$. Moreover, from the definition of the norm on $Z$, we have that $(x_i^s, y_i) \in B_Z$ for each $i \in \{1, \ldots, n\}$ and for each $s$. Finally, given $s$ one has

$$1 + \left\| \sum_{i=1}^{n} \lambda_i (x_i, y_i) \right\|_{\infty} \geq \left\| \sum_{i=1}^{n} \lambda_i ((x_i, y_i) - (x_i^s, y_i)) \right\|_{\infty} \geq \left\| \sum_{i=1}^{n} \lambda_i (x_i - x_i^s) \right\|_{\infty} \rightarrow 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_{\infty} = 1 + \left\| \sum_{i=1}^{n} \lambda_i (x_i, y_i) \right\|_{\infty}.$$

So $\|\sum_{i=1}^{n} \lambda_i ((x_i, y_i) - (x_i^s, y_i))\|_{\infty} \rightarrow 1 + \|\sum_{i=1}^{n} \lambda_i (x_i, y_i)\|_{\infty}$. Consequently, $Z$ has the DSD2P applying Proposition 3.4 so we are done.

Finally we will analyze the inheritance of DSD2P to subspaces. Again in [7] it is proved that given $X$ a Banach space with the SD2P and $Y \subseteq X$ a closed subspace such that $X/Y$ is strongly regular, then $Y$ has the SD2P. Following similar ideas we have the following

Theorem 3.9. Let $X$ be a Banach space and $Y \subseteq X$ be a closed subspace. If $X$ has the DSD2P and $X/Y$ is strongly regular then $Y$ also has the DSD2P.
Proof. Let
\[ C := \sum_{i=1}^{n} \lambda_i W_i = \sum_{i=1}^{n} \lambda_i \{ y \in B_Y : |y_{ij}^* (y - y_0^i)| < \eta_{ij} \ 1 \leq j \leq n_i \} \]
be a convex combination of non-empty relatively weakly open subsets of \( B_Y \), where \( y_{ij}^* \in B_{Y^*} \) for each \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\} \) and \( y_0^i \in Y \) for each \( i \in \{1, \ldots, n\} \). Pick \( \sum_{i=1}^{n} \lambda_i x_i \in C \), \( \varepsilon \in \mathbb{R}^+ \) and let us prove that there exists \( \sum_{i=1}^{n} \lambda_i y_i \in C \) such that
\[ \left\| \sum_{i=1}^{n} \lambda_i (x_i - y_i) \right\| > 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon. \]
To this aim pick \( 0 < \delta \) such that
\[ |y_{ij}^* (x_i - y_0)| \geq \frac{\delta}{32} \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right) < \eta_{ij} \ \forall i \in \{1, \ldots, n\} \]
and
\[ \frac{\delta}{16} + 2 \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right) + \frac{\delta}{4} < \varepsilon. \]

Let \( \pi : X \to X/Y \) the quotient map. We have no loss of generality, by Hahn-Banach theorem, if we assume that \( y_{ij}^* \in B_{X^*} \) for each \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\} \). Consider \( W_i \) the non-empty relatively weakly open subset of \( B_X \) defined by \( W_i \) for each \( i \in \{1, \ldots, n\} \).

For each \( i \in \{1, \ldots, n\} \) consider \( A_i := \pi(W_i) \), which is a convex subset of \( B_{X/Y} \) containing to zero. By [7, Proposition III.6] then \( A_i \) is equal to the closure of the set of its strongly regular points. As a consequence, for each \( i \in \{1, \ldots, n\} \), there exists \( a_i \) a strongly regular point of \( A_i \) such that
\[ \|a_i\| < \frac{\delta}{32}. \]

For every \( i \in \{1, \ldots, n\} \) we can find \( m_i \in \mathbb{N}, \mu_1, \ldots, \mu_{m_i} \in [0,1] \) such that
\[ \sum_{j=1}^{m_i} \mu_j = 1 \]
and \( (a_1)^*, \ldots, (a_{m_i})^* \in S_{(X/Y)^*}, \alpha_j^* \in \mathbb{R}^+ \) satisfying that
\[ a_i \in \sum_{j=1}^{m_i} \mu_j \{ S(B_{X/Y}, (a_j^*)^*, \alpha^*_j) \cap A_i \} \]
and
\[ \text{diam} \left( \sum_{j=1}^{m_i} \mu_j \{ S(B_{X/Y}, (a_j^*)^*, \alpha^*_j) \cap A_i \} \right) < \frac{\delta}{32}. \]

It is clear that, for \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m_i\} \), one has
\[ S(B_{X/Y}, (a_j^*)^*, \alpha^*_j) \cap A_i \neq \emptyset \Rightarrow S(B_X, \pi^*((a_j^*)^*), \alpha^*_j) \cap W_i \neq \emptyset. \]

Check that we can not still apply the hypothesis because we do not know whether \( \sum_{i=1}^{n} \lambda_i x_i \in C := \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m_i} \mu_j (W_i \cap S(B_X, \pi^*((a_j^*)^*), \alpha^*_j)) \).
Now, in order to finish the proof, we need to find points in $C$ close enough to $\sum_{i=1}^{n} \lambda_i x_i$. This will be done in the following

**Claim 3.10.** We can find, for each $i \in \{1, \ldots, n\}$, an element $z_i \in B_X$ such that $\sum_{i=1}^{n} \lambda_i z_i \in C$ and that

$$
\left\| \sum_{i=1}^{n} \lambda_i (x_i - z_i) \right\| < \frac{\delta}{32} + \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right).
$$

**Proof.** Pick $i \in \{1, \ldots, n\}$. As $\| \pi(x_i) - a_i \| = \| a_i \| < \frac{\delta}{32}$ we can find $z_i \in X$ such that $\pi(z_i) = a_i$ and such that

$$
\| x_i - z_i \| < \frac{\delta}{32}.
$$

Now $a_i \in \sum_{j=1}^{m_i} \mu_j \left( S(B_{X/Y}, (a_j^*) \cap \overline{A}_j) \right)$ so, for each $j \in \{1, \ldots, m_i\}$, we can find $b_{ij} \in S(B_{X/Y}, (a_j^*) \cap \overline{A}_j)$ such that $a_i = \sum_{j=1}^{m_i} \mu_j b_{ij}$. For each $j \in \{1, \ldots, m_i\}$ we can find, considering a perturbation argument if necessary, an element $z_{ij} \in B_X$ such that $\pi(z_{ij}) = b_{ij}$. Finally, as $\pi(z_i) - \sum_{j=1}^{m_i} \mu_j \pi(z_{ij}) = 0$ we can find, by definition of the norm on $X/Y$, an element $y_i \in Y$ such that

$$
z_i = \sum_{j=1}^{m_i} \mu_j z_{ij} + y_i,
$$

and

$$
\| y_i \| < \frac{\delta}{32}.
$$

We shall prove that $\sum_{i=1}^{n} \lambda_i \frac{z_i}{1 + \frac{\delta}{32}}$ works. First of all we have

$$
\left\| \sum_{i=1}^{n} \lambda_i z_i \right\| \leq \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| + \sum_{i=1}^{n} \lambda_i \| x_i - z_i \| < 1 + \frac{\delta}{32}.
$$

So $\frac{z_i}{1 + \frac{\delta}{32}} \in B_X$ for each $i \in \{1, \ldots, n\}$.

Moreover, given $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}$, one has

$$
\left| y_{ij}^* \left( \frac{z_i}{1 + \frac{\delta}{32}} - y_0^* \right) \right| \leq \left| y_{ij}^* (x_i - y_0^*) \right| + \| x_i - z_i \| + \left| z_i - \frac{z_i}{1 + \frac{\delta}{32}} \right| \leq \left| y_{ij}^* (x_i - y_0^*) \right| + \frac{\delta}{32} + \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right) < \eta_{ij}.
$$

Finally, pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$. Then by (3.10) one has

$$
z_i = \sum_{j=1}^{m_i} \mu_j^i (z_{ij} + y_i).
$$

In addition

$$
\pi^* (a_{ij}^*) (z_{ij} + y_i) = a_{ij}^* (b_{ij}) + a_{ij}^* (\pi(y_i)).
$$
On the one hand, as \( y_i \in Y \) then \( \pi(y_i) = 0 \). On the other hand \( a_{ij}'(b_{ij}) > 1 - \alpha_{ij}' \). Now, up to consider a smaller positive number in \((3.9)\) (check that the choice of \( b_{ij} \) does not depend on the one of \( z_1, \ldots, z_n \)), we can assume that \( a_{ij}'(b_{ij}) > (1 - \alpha_{ij}') (1 + \frac{1}{32}) \), so \( \sum_{i=1}^{n} \lambda_i \frac{z_i}{1 + \frac{1}{32}} \in C \). Now the claim follows just considering \( \frac{z_i}{1 + \frac{1}{32}} \) instead of \( z_i \).

Now, as \( X \) has the DSD2P, we can find \( \sum_{i=1}^{n} \lambda_i z_i' \in C \) such that

\[
\left\| \sum_{i=1}^{n} \lambda_i (z_i - z_i') \right\| > 1 + \left\| \sum_{i=1}^{n} \lambda_i z_i \right\| - \frac{\delta}{32}.
\]

Given \( i \in \{1, \ldots, n\} \) we have that

\[
\pi(z_i') = \sum_{j=1}^{m_i} \mu_{ij}^i (S(B_{X/Y}, (a_j^i)^*, \alpha_j^i) \cap A_i)
\]

\[
\Rightarrow \left\| \pi(z_i') \right\| \leq \left\| a_i \right\| + \text{diam} \left( \sum_{j=1}^{m_i} \mu_{ij}^i (S(B_{X/Y}, (a_j^i)^*, \alpha_j^i) \cap A_i) \right) < \frac{\delta}{16}.
\]

Now, as it is done in Proposition \(2.13\) we can find \( y_i \in B_Y \) such that

\[
\left\| y_i - z_i' \right\| < \frac{\delta}{4}.
\]

Now, on the one hand, given \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, n_i\} \), one has

\[
|y_{ij}'(y_i - y_0)| \leq |y_{ij}'(z_i' - y_0)| + |y_{ij}'(y_i - z_i)| < \eta_{ij} + \frac{\delta}{4}.
\]

On the other hand,

\[
\left\| \sum_{i=1}^{n} \lambda_i (x_i - y_i) \right\| \geq \left\| \sum_{i=1}^{n} \lambda_i (z_i - z_i') \right\| - \left\| \sum_{i=1}^{n} \lambda_i (x_i - z_i) \right\| - \left\| \sum_{i=1}^{n} \lambda_i (y_i - z_i') \right\|
\]

\[
> 1 + \left\| \sum_{i=1}^{n} \lambda_i z_i \right\| - \frac{\delta}{32} \sum_{i=1}^{n} \lambda_i \left\| y_i - z_i' \right\| - \left\| \sum_{i=1}^{n} \lambda_i (z_i - x_i) \right\|
\]

\[
> 1 + \left\| \sum_{i=1}^{n} \lambda_i z_i \right\| - \frac{\delta}{32} \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right)
\]

\[
> 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \frac{\delta}{16} \left( 1 + \frac{\delta}{32} \right) \left( 1 - \frac{1}{1 + \frac{\delta}{32}} \right)
\]

\[
> 1 + \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \varepsilon.
\]

From the arbitrariness of \( \varepsilon \) we conclude that \( Y \) has the DSD2P by a perturbation argument similar to the one done in Proposition \(2.13\).
Now we have a weak-star version of Theorem 3.9.

**Corollary 3.11.** Let $X$ be a Banach space and let $Y \subseteq X$ be a closed subspace. If $X^*$ has the $w^*\text{-DSD2P}$ and $Y$ is reflexive then $(X/Y)^*$ has the $w^*\text{-DSD2P}$.

**Proof.** Consider $C := \sum_{i=1}^{n} \lambda_i W_i$ a convex combination of non-empty relatively weakly-star open subsets of $B_{Y^*}$ and pick $\sum_{i=1}^{n} \lambda_i z_i^* \in C$.

Define $W_i$ to be the weak-star open subset of $B_{X^*}$ define by $W_i$ for each $i \in \{1, \ldots, n\}$.

Let $\pi: X^* \rightarrow X^*/Y^*$ the quotient map and define $A_i := \pi(W_i)$.

As $X^*/Y^* = Y^*$ is reflexive, then $X^*/Y^*$ is strongly regular, so we can find, for each $i \in \{1, \ldots, n\}$, a point of strong regularity point of $A_i$ whose norm is as close to zero as desired. Given $i \in \{1, \ldots, n\}$, as $a_i$ is a point of strong regularity, we can find convex combination of slices containing $a_i$ and whose diameter is as small as wanted. In addition, because of reflexivity of $X^*/Y^*$, convex combination of slices are indeed convex combination of weak-star slices, so we can actually find convex combination of weak-star slices containing $a_i$ and whose diameter is a closed to zero as desired for each $i \in \{1, \ldots, n\}$.

Using the previous ideas, the result can be concluded following word by word the proof of Theorem 3.9.

Now we shall prove the inheritance of the DSD2P to almost isometric ideals.

**Proposition 3.12.** Let $X$ be a Banach space and let $Y \subseteq X$ a closed almost isometric ideal. If $X$ has the DSD2P, so does $Y$.

**Proof.** Pick $C := \sum_{i=1}^{n} \lambda_i \bigcap_{j=1}^{n_i} S(B_{Y^*}, y_{ij}^*, \alpha_{ij})$ a convex combination of non-empty relatively weakly open subsets of $B_Y$, choose $\sum_{i=1}^{n} \lambda_i y_i \in C$ and pick $\varepsilon > 0$. Our aim is to find $\sum_{i=1}^{n} \lambda_i z_i \in C$ such that $\|\sum_{i=1}^{n} \lambda_i (y_i - z_i)\| > 1 + \|\sum_{i=1}^{n} \lambda_i y_i\| - \varepsilon$. Assume, with no loss of generality, that $\max_{1 \leq i \leq n} \max_{1 \leq j \leq n_i} y_{ij}^i(y_i) < 1$.

Choose $\mu_0 > 0$ such that

\begin{equation}
0 < \mu < \mu_0 \Rightarrow \frac{y_{ij}^i(y_i)}{1 + \mu} > 1 - \alpha_{ij} \quad \forall i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}.
\end{equation}

and

\begin{equation}
0 < \mu < \mu_0 \Rightarrow \frac{1}{1 + \mu} (1 + \|\sum_{i=1}^{n} \lambda_i y_i\| - \mu) - \mu > 1 + \left\|\sum_{i=1}^{n} \lambda_i y_i\right\| - \varepsilon.
\end{equation}
Now consider $0 < \mu < \mu_0$ and $\varphi : Y^* \to X^*$ a Hahn-Banach operator satisfying the properties described in Theorem 1.1. Define

$$\hat{C} := \sum_{i=1}^{n} \lambda_i \bigcap_{i=1}^{n} S(BX, \varphi(y^*_ij), 1 - y^*_ij(y_j)).$$

As $X$ has the DSD2P and clearly $\sum_{i=1}^{n} \lambda_i y_i \in \hat{C}$ we can conclude the existence of an element $\sum_{i=1}^{n} \lambda_i x_i \in \hat{C}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i(y_i - x_i) \right\| > 1 + \left\lVert \sum_{i=1}^{n} \lambda_i y_i \right\| - \mu.$$

Now for $\mu, E := \text{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n\} \subseteq E$ and $F := \text{span}\{y^*_ij / i \in \{1, \ldots, n\}, j \in \{1, \ldots, n_i\}\} \subseteq Y^*$ consider $T$ the operator satisfying the properties described in Theorem 1.1. Given $i \in \{1, \ldots, n\}$ one has

$$\|T(x_i)\| \leq (1 + \mu)\|x_i\| \leq 1 + \mu.$$

So, if we define $z := \sum_{i=1}^{n} \lambda_i \frac{T(x_i)}{1 + \mu}$, it is clear that $z \in BY$. We will prove that indeed $z \in C$. To this aim, pick $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n_i\}$. Hence

$$y^*_ij \left( \frac{T(x_i)}{1 + \mu} \right) = \frac{y^*_ij(T(x_i))}{1 + \mu} = \frac{\varphi(y^*_ij)(x_i)}{1 + \mu} > \frac{1 - (1 - y^*_ij(y_j))}{1 + \mu} = \frac{y^*_ij(y_j)}{1 + \mu}$$

Thus $z \in C$. Finally, we have that

$$\left\| \sum_{i=1}^{n} \lambda_i y_i - z \right\| = \left\| \sum_{i=1}^{n} \lambda_i \left( y_i - \frac{T(x_i)}{1 + \mu} \right) \right\| = \frac{\left\| \sum_{i=1}^{n} \lambda_i(y_i - T(x_i) + \mu y_i) \right\|}{1 + \mu} \geq \frac{\left\| T \left( \sum_{i=1}^{n} \lambda_i(y_i - x_i) \right) - \mu \left( \sum_{i=1}^{n} \lambda_i y_i \right) \right\|}{1 + \mu} > \frac{\frac{1}{1 + \mu} \left( \left\| \sum_{i=1}^{n} \lambda_i y_i \right\| - \mu \right)}{1 + \mu} \geq 1 + \left\| \sum_{i=1}^{n} \lambda_i y_i \right\| - \varepsilon.$$

As $\varepsilon$ was arbitrary we conclude that $Y$ has the DSD2P, so we are done.

4. SOME REMARKS AND OPEN QUESTIONS.

Let us consider the following diagram

$$
\begin{align*}
DP & \xRightarrow{(1)} DSD2P \xrightarrow{(2)} DD2P \xrightarrow{(3)} DLD2P \\
\downarrow{(4)} & \downarrow{(5)} \downarrow{(6)} \\
w^* - DSD2P & \xRightarrow{(7)} w^* - DD2P \xrightarrow{(8)} w^* - DLD2P
\end{align*}
$$
where the last row only make sense in dual Banach spaces. By Example 2.2 or Theorem 2.11, neither the converse implication of (2) nor the one of (7) holds. In addition, there are Banach spaces with the Daugavet property whose dual unit ball have denting points (e.g. $C([0,1])$). Consequently, the converse of (4),(5) or (6) is not true.

However, it remains open the following

**Question 1.** Does the converse of (1),(3) or (8) hold?

It is known that a Banach space $X$ has the DLD2P if, and only if, $X^*$ has the $w^*$-DLD2P. This fact arise two questions.

**Question 2.** Let $X$ be a Banach space. Is it true that $X$ has the DD2P if, and only if, $X^*$ has the $w^*$-DD2P?

A similar question remains open for the DSD2P.

**Question 3.** Let $X$ be a Banach space. Is it true that $X$ has the DSD2P if, and only if, $X^*$ has the $w^*$-DSD2P?

Check that a positive answer to the above question would provide, by Proposition 3.5 and a similar argument to the one done in Theorem 3.8 to the dual space, a positive answer to the following

**Question 4.** Let $X, Y$ Banach space. Does $X \oplus_1 Y$ enjoy to have the DSD2P whenever $X$ and $Y$ have the DSD2P?

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Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)

E-mail address: juliobg@ugr.es, glopezp@ugr.es arz0001@correo.ugr.es