Topological Bogoliubov excitations in inversion-symmetric systems of interacting bosons

G. Engelhardt and T. Brandes
Institut für Theoretische Physik, Technische Universität Berlin, Hardenbergstr. 36, 10623 Berlin, Germany

On top of the mean-field analysis of a Bose-Einstein condensate, one typically applies the Bogoliubov theory to analyze quantum fluctuations of the excited modes. Therefore, one has to diagonalize the Bogoliubov Hamiltonian in a symplectic manner. In our article, we investigate the topology of these Bogoliubov excitations in inversion-invariant systems of interacting bosons in one dimension. We analyze, how the condensate influences the topology of the Bogoliubov excitations. Analogously to the fermionic case, we here establish a symplectic extension of the polarization characterizing the topology of the Bogoliubov excitations and link it to the eigenvalues of the inversion operator at the inversion-invariant momenta. We also demonstrate at an instructive, but experimentally feasible example that this quantity is also related to edge states in the excitation spectrum.

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I. INTRODUCTION

Since the discovery of Bloch bands with non-trivial topological structure, the field of topological insulators and superconductors has been rapidly growing [1, 2]. The most prominent example is the integer quantum Hall effect, where one can link the Hall conductance of the ground state with the Chern number of the occupied bands [3]. This strict quantization is due to the fermionic character of the particles, forcing all states within a band to be equally occupied. For this reason, a system consisting of bosons does not exhibit such quantized observables.

In addition, there are topologically protected edge states as a consequence of the bulk-boundary relation [1, 2]. These edge states do not rely on the fermionic character, but are an implication of the single-particle wave function. So they can also appear in the spectrum of noninteracting bosons in systems of cold atoms [4–7].

In recent years, there occurred a great effort for creating non-trivial topological structures of fractional quantum Hall states in strongly interacting bosonic systems [8–13]. In contrast, the investigations of the topology of bosons in the weakly interacting regime has obtained insignificant attention. An interesting approach is discussed in Ref. [10], where the edge states of the Su-Schrieffer-Heeger model get dynamically unstable by properly preparing the condensate in an excited transverse mode. In such a setup one considers the excitations above the Bose-Einstein-Condensate (BEC), which are effectively single-particle like due to a expansion of the Hamiltonian in orders of the condensate density [17]. The resulting Bogoliubov Hamiltonian couples pure particle excitations and hole excitations and has to be diagonalized in a symplectic manner due to the bosonic commutation relation. Therefore, the definition of a topological invariant for these Bogoliubov-Bloch bands is a priori not clear. As a result, the condensed part of the atoms has a substantial influence on the topology of the Bogoliubov excitations, which has not been discussed in the previous literature. To the best of our knowledge, there are only few articles about the definition of a Chern number for bosonic Bogoliubov bands, yet, in the context of magnonic systems [18, 19].

In contrast, here we focus on the treatment of the topology in inversion-invariant systems of weakly interacting bosons in one dimension. In fermionic systems, the topological invariant of inversion-invariant system is given by the macroscopic polarization [20, 21], which is a geometric phase of the occupied bands [22]. Although its strict quantization is due to the fact that all orbitals of the bands below the Fermi energy are equally occupied, one can also consider it to be a quantity describing the structure of the Bloch bands independent of the fermionic character of the particles. For this reason, for bosonic systems it can be considered to be a topological invariant not directly connected to a bulk observable, but predicting the existence of edge states.

The article is organized as follows: In Sec. II, we introduce a model considered as an instructive, but experimentally feasible example throughout the article. This system has the property that the condensate influences the topology of its Bogoliubov excitations. In Sec. III, we recall Bogoliubov theory. In Sec. IV, we investigate the topology of the Bogoliubov excitations. A main result of our article is Eq. (20) defining an extension of the macroscopic polarization for bosonic Bogoliubov excitations. We also discuss the problems for the definition of the polarization caused by the Goldstone mode appearing in the lowest band. In Sec. V, we apply our findings to the instructive model and show the existence of edge states.
In position space the Hamiltonian reads sketched in Fig. 1a). There, we consider an ensemble of particles. In Sec. IV F, we show that topologically protected by the condensate due to the interactions between the particles. In the main text, one can distinguish the topology of the Bogoliubov dispersion relations in panel d). The corresponding parameters are marked with arrows in the phase diagram. As explained in the main text, one can distinguish the topology of the Bogoliubov excitations depending on the symplectic polarization $P_z$. For this reason, one can split phase II in a trivial phase IIa and a topological phase IIb. The phase boundary is strongly influenced by the condensate due to the interactions between the particles. In Sec. IV F we show that topologically protected edge states appear for the system in the topological phase IIb.

II. MODEL SYSTEM

A. Hamiltonian

We consider systems of weakly interacting bosons in a periodic potential. An instance of such a system is sketched in Fig. 1b). There, we consider an ensemble of bosonic atoms with internal degree of freedom (spin) confined in an array of wells created by an optical lattice. In position space the Hamiltonian reads

$$H = \sum_{m=-M/2+1}^{M/2} H_m + V_m. \quad (1)$$

where

$$H_m = \omega \left( \hat{a}^\dagger_{\downarrow,m} \hat{a}_{\downarrow,m} - \hat{a}^\dagger_{\uparrow,m} \hat{a}_{\uparrow,m} \right) - \nu_0 \left( \hat{a}^\dagger_{\uparrow,m} \hat{a}_{\uparrow,m+1} - \hat{a}^\dagger_{\downarrow,m+1} \hat{a}_{\downarrow,m} + \text{h.c.} \right) - \nu_{so} \left( \hat{a}^\dagger_{\downarrow,m} \hat{a}_{\downarrow,m-1} - \hat{a}^\dagger_{\uparrow,m} \hat{a}_{\uparrow,m+1} + \text{h.c.} \right) \quad (2)$$

$$V_m = \sum_{\sigma,\sigma'=\uparrow,\downarrow} \chi_{\sigma,\sigma'} \hat{a}^\dagger_{\sigma,m} \hat{a}_{\sigma',m} \hat{a}_{\sigma,m} \hat{a}_{\sigma',m}. \quad (3)$$

Here, $m$ denotes the position of the wells and $\uparrow, \downarrow$ may denote the internal degree of freedom of the bosons. The two states have a level splitting of $\omega$. The atoms can jump between the wells, described by the parameters $\nu_0$ and $\nu_{so}$. The modulus of the hopping integral $\nu_0$ is equal for the two spin components but differs in sign. The term proportional to $\nu_{so}$ denotes a spin-orbit coupling, which can be generated within current technology [24]. Additionally, we have a state-dependent interaction which is local in position space. Therefore, we assure that this is not in conflict with the inversion symmetry of the Hamiltonian.

The single-particle Hamiltonian corresponds to the systems in [24] discussing fermionic systems of cold-atoms which have a non-trivial topology. These articles suggest possible experimental implementations for the single-particle Hamiltonian. These could be also applied to bosonic systems. Additionally, one can control the interactions between the particles using Feshbach resonances.

B. Mean-field expansion

We shift the operators

$$\hat{a}_{\uparrow/\downarrow,m} \rightarrow \sqrt{N_0} \frac{\zeta_{\uparrow/\downarrow,m}}{M} + \hat{a}_{\uparrow/\downarrow,m}, \quad (4)$$

where $\zeta_{\uparrow/\downarrow,m} \in \mathbb{C}$ and $N_0$ denotes the number of particles in the condensate which is assumed to be macroscopically occupied. The bosonic operators now account for quantum fluctuations on top of the condensate. We expand the Hamiltonian as

$$H = \rho_0 E_{GP} + \sqrt{\rho_0} H^{(L)} + H^B + O(\rho_0^{-1/2}), \quad (5)$$

where $\rho_0 = \sqrt{\frac{N_0}{M}}$ is the density of the condensed particles. $H^{(L)} (H^B)$ depends on $\{\zeta_{\uparrow/\downarrow,m}\}$ and contains terms which are linear (quadratic) in bosonic operators.

$E_{GP}$ is a function of $\{\zeta_{\uparrow/\downarrow,m}\}$ and denotes the Gross-Pitaevskii functional. It exactly reads as (1) with the operators replaced by $\zeta_{\uparrow/\downarrow,m}$ and $\chi_{\sigma,\sigma'} \rightarrow \chi_{\sigma,\sigma'} \rho_0$. To find the mean-field ground state, we minimize it by an appropriate choice of $\zeta_{\uparrow/\downarrow,m}$. The minimization procedure is performed by using a modified ansatz of Ref. [20]. Details can be found in Appendix A.
The result is depicted in the phase diagram in Fig. 1(b) for the special choice $\chi_{\uparrow\uparrow} = \chi_{\downarrow\downarrow}$ and $\chi_{\uparrow\downarrow} = 0$. There, one can find three phases. In phase I, appearing for $\rho_0 \chi < 0$, we find that the atoms condense within a small area as a localized wave function [27, 28]. In phase II, we find a condensation at $k = 0$ and $\langle \zeta_{\uparrow,m}, \zeta_{\downarrow,m} \rangle = (1, 0)$. Due to the spin-orbit coupling and the interactions, the atoms condense at a finite momentum $k > 0$ in phase III. Thus, only in phase II there is a mean-field wave function, which is invariant under inversion.

At a stationary point of the Gross-Pitaevskii functional, the linear part in [5] vanishes and the excitations are solely governed by the quadratic Bogoliubov Hamiltonian. To respect that we work at constant particle number we consider $N_0$ to be an operator and replace [17]

$$\hat{N}_0 = N - \sum_{m,s=\uparrow,\downarrow} \hat{a}^\dagger_{m,s} \hat{a}_{m,s}. \quad (6)$$

This leads to the appearing of an effective chemical potential $\mu_{eff}$ in $H_B$ which reads

$$\mu_{eff} = \frac{1}{M} \sum_m \omega \left( \zeta_{\uparrow,m}^* \zeta_{\downarrow,m} - \zeta_{\uparrow,m}^* \zeta_{\downarrow,m} \right)$$

$$- \nu_0 \left( \zeta_{\uparrow,m}^* \zeta_{\downarrow,m+1} - \zeta_{\uparrow,m+1}^* \zeta_{\downarrow,m} + c.c. \right)$$

$$\nu_0 \left( \zeta_{\uparrow,m}^* \zeta_{\downarrow,m+1} + \zeta_{\uparrow,m+1}^* \zeta_{\downarrow,m} + c.c. \right)$$

$$+ 2 \rho \sum_{s,s'} \chi_{s,s'} \zeta_{s,m}^* \zeta_{s',m} \zeta_{s,m} \zeta_{s',m}, \quad (7)$$

where $\zeta_{\uparrow/\downarrow,m}$ denotes now the stationary point of the Gross-Pitaevskii function $E_{GP}$. We also defined $\rho = N/M$ which is the density of all particles.

### C. Bogoliubov Hamiltonian in momentum space

We proceed to work in phase II where $\langle \zeta_{\uparrow,m}, \zeta_{\downarrow,m} \rangle = (1, 0)$. As we have there a translational-invariant Bogoliubov Hamiltonian, we can perform a Fourier transformation and obtain a Bogoliubov Hamiltonian of the form [17]

$$\mathcal{H}^{(B)} = \frac{1}{2} \sum_k \left( \hat{a}_{k,\uparrow}^\dagger \hat{a}_{-k}^\dagger \right) \mathbf{H}_k \left( \hat{a}_{k,\uparrow} \hat{a}_{k,\uparrow}^\dagger \right)$$

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{H}_k^{(0)} & \mathbf{H}_k^{(1)} \\ \mathbf{H}_k^{(2)} & \mathbf{H}_k^{(1)} \end{pmatrix}, \quad (8)$$

where $\hat{a}_{k,\uparrow}^\dagger = \left( \hat{a}_{k,\uparrow}^\dagger, \hat{a}_{k,\downarrow}^\dagger \right)$ is a vector of bosonic creation operators and

$$\mathbf{H}_k^{(0)} = \begin{pmatrix} -\omega - 2 \nu_0 \cos k & 2 \nu_0 \sin k \\ -2 \nu_0 \sin k & \omega + 2 \nu_0 \cos k \end{pmatrix} \quad (9)$$

$$\mathbf{H}_k^{(1)} = \begin{pmatrix} 4 \chi \rho - \mu_{eff} & 0 \\ 0 & 2 \chi_{\uparrow\downarrow} - \mu_{eff} \end{pmatrix} \quad (10)$$

$$\mathbf{H}_k^{(2)} = \begin{pmatrix} 2 \chi \rho & 0 \\ 0 & 0 \end{pmatrix} \quad (11)$$

The chemical potential reduces to $\mu_{eff} = -\omega - 2 \nu_0 + 2 \rho \chi$. The Bogoliubov Hamiltonian determines the topological properties of the excitations. In order to investigate this, one first has to diagonalize the Bogoliubov Hamiltonian. Importantly, one can not diagonalize the Bogoliubov Hamiltonian by a simple unitary transformation as the resulting quasi-particles would not fulfill bosonic commutation relations. In contrast, the diagonalization has to be performed in a symplectic manner. Consequently, one can not apply the definitions of topological invariants for usual noninteracting systems directly, but has to respect the symplectic nature of the diagonalization procedure. In the next section, we shortly recall this procedure. Afterwards we then can define a symplectic extension of the macroscopic polarization constituting a topological invariant which characterizes the Bogoliubov-Bloch bands.

### III. BOGOLIUBOV THEORY

A generic Bogoliubov Hamiltonian can be written in the form $\mathcal{H}^{(B)} = \frac{1}{2} \left( \hat{a}_{\lambda}^\dagger \hat{a}_{\lambda} \right) \mathbf{H} \left( \hat{a}_{\lambda}^\dagger \hat{a}_{\lambda} \right)^T$, where

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}^{(0)} & \mathbf{H}^{(\beta)} \\ \mathbf{H}^{(\beta)^*} & \mathbf{H}^{(0)^*} \end{pmatrix} \quad (12)$$

and $\hat{a}_{\lambda} = \left( \hat{a}_{\lambda=1}^\dagger, \ldots, \hat{a}_{\lambda=N}^\dagger \right)$. The label $\lambda$ may denote the position, momentum or an internal degree of spinor bosons. The matrix $\mathbf{H}^{(\alpha)}$ is hermitian and $\mathbf{H}^{(\beta)}$ is symmetric.

This Hamiltonian can be diagonalized with the ansatz $\left( \hat{a}_{\lambda}^\dagger, \hat{a}_{\lambda} \right) = \left( \hat{b}_{\lambda}^\dagger, \hat{b}_{\lambda} \right) \mathbf{T}^\dagger$, where $\mathbf{T}$ denotes a $2N \times 2N$ paramatry matrix [17] [18] [20]. The new operators $\hat{b}$ shall also full bosonic commutation relations. To this end, one has to require that

$$\mathbf{T}^\dagger \sigma_z \mathbf{T} = \sigma_z \quad \mathbf{T} \sigma_z \mathbf{T}^\dagger = \sigma_z, \quad (13)$$

with the diagonal matrix $(\sigma_z)_{\ell\ell'} = \delta_{\ell\ell'} \sigma_1$ and $\sigma_1 = 1$ for $\ell \leq N$ and $\sigma_1 = -1$ else. After inserting the ansatz into the Bogoliubov Hamiltonian, one easily sees, that the Hamiltonian is diagonalized, if

$$\mathbf{H} \mathbf{T} = \mathbf{T} \sigma_z \begin{pmatrix} \mathbf{E} & \mathbf{-E} \end{pmatrix}, \quad (14)$$

where $\mathbf{E}$ denotes a diagonal matrix $\mathbf{E} = \text{diag}[E_1, \ldots, E_N]$. As a consequence of Eq. (14) and of the symmetic structure of $\mathbf{H}$, the paramatry matrix $\mathbf{T}$ can be written in the form

$$\mathbf{T} = \begin{pmatrix} \mathbf{U} & \mathbf{V}^* \\ \mathbf{V} & \mathbf{U}^* \end{pmatrix} \quad (15)$$

with $\mathbf{U}, \mathbf{V}$ being $N \times N$ matrices. In the following we denote $\mathbf{U}(\mathbf{V})$ as the particle (hole) part of the excitations. Consequently, only the first $N$ columns of $\mathbf{T}$ contain independent solutions. The other $N$ columns resemble exactly the same Hamiltonian as the first one, also having a
positive energy. Thus, there are only positive excitation energies.

Finally we remark, that the Hamiltonian in momentum space $H_k$ in Eq. [8] does not necessarily have the form [12]. This problem can be solved by formally combining the entries of $H_k$ and $H_{-k}$ in an enlarged matrix.

IV. TOPOLOGY OF BOGOLIUBOV EXCITATIONS

A. Symmetry considerations

Let us assume that there are symmetries $S_j$ transforming a noninteracting Hamiltonian as $S_j H_k^{(0)} S_j^{-1} = \alpha H_{\beta k}^{(0)}$ with $\alpha, \beta = \pm 1$. Depending on the value of $\alpha, \beta$, the properties of $S_j$ and the dimension of the system one finds different topological classes [30, 31]. For, e.g., $\alpha = -\beta = 1$ and $S_j = P = P^{-1} = P^\dagger$, the single particle Hamiltonian is invariant under inversion which is the focus of our article. For the noninteracting Hamiltonian [9] the inversion operator can be written as $P = \sigma_z$, where $\sigma_z$ denotes the usual $2 \times 2$ Pauli matrix.

Next, we turn our attention to the symmetry relations of $H_k$. Thereby, our approach is to consider the symmetry operator $S_j^{(0)} \equiv 1_2 \otimes S_j$, where $S_j$ denotes a symmetry of $H_k^{(0)}$ and $1_2$ is the $2 \times 2$ identity matrix.

It is natural that the symmetry of the Bogoliubov Hamiltonian has a block structure as otherwise the symmetry would relate a pure particle-like excitation with a particle-hole-like one. Due to the appearing of the additional matrices $H_k^{(1,2)}$, the symmetries $S_j$ commuting with $H_k^{(0)}$ do not necessarily create symmetries of $H_k$, so that this symmetry can be lost. Consequently, the interaction of the particles can change the topological classification.

However, let us assume, that the Bogoliubov Hamiltonian commutes with the symplectic extension of the inversion symmetry $P^B = 1_2 \otimes P$, thus

$$P^B H_k P^B = H_{-k}. \quad (16)$$

This condition is fulfilled for the Hamiltonian [9]. For most $k$ values, this does not impose a constraint as the symmetry just connects the Hamiltonian at $k$ with that at $-k$. Due to the periodicity in position space the moment is only defined within the Brillouin zone which we assume to have length $2\pi$. Therefore, the momentum $k_{inv} = 0$ and the boundary of the Brillouin zone $k_{inv} = \pi$ are invariant under inversion as $-k_{inv} \equiv k_{inv} \mod 2\pi$.

For these momenta relation (16) exhibit a strict constraint for the Hamiltonians $H_{k_{inv}}$.

B. Symplectic Polarization

For a single-particle Hamiltonian the so-called macroscopic polarization constitutes a topological invariant [20]. We now want to formulate a symplectic generalization of the macroscopic polarization determining the topology.

For the following derivations we consider systems with a discrete basis. The extension to continuous systems works analogously, but one has to be careful with the dimension of the basis.

Let $T$ be the solution of Eq. [14] in position space. The rows of $T$ can be labeled with $(m, l)$ where $m \in \{-M/2 + 1, ..., M/2\}$ denotes the position and $l$ an internal degree of freedom within the unit cell. The columns of $T$ contain the eigenstates of $[14]$. As we consider periodic systems, they can be labeled with the indices $(k, \lambda)$, where $k = 2\pi\tilde{n}/M$ with $\tilde{n} \in \{-M/2 + 1, ..., M/2\}$ denotes a quasi momentum within the Brillouin zone and $\lambda \in \{1, ..., 2L\}$ denotes the band index. Let us further denote the $(k, \lambda)$-th column of $U$ ($V$) as $T^u_{(k,\lambda)}$ ($T^v_{(k,\lambda)}$) being $N = K \times 2L$ dimensional vectors.

The entries of $T$ can be expressed in terms of Bloch functions $T^c_{(m,l),(k,\lambda)} = \frac{1}{\sqrt{M}} e^{ikm}t^c_{(k,\lambda)}$ where $c \in \{u, v\}$.

We note that, the periodic part are the solutions of the Bogoliubov equation in momentum space

$$H_k t_k = \sigma_z t_k \left( E_k - E_{-k} \right), \quad (17)$$

where $H_k$ is the matrix in [9]. The paraunitary matrix $t_k$ has dimension $2L \times 2L$. It also fulfills $t^i_{(k,\lambda)} \sigma_z t_k = \sigma_z$ and $t_k \sigma_z t_k^\dagger = \sigma_z$. More precisely, for $\lambda \leq L$ the relation reads $(t_k)_{l,\lambda} = t^v_{(l,\lambda)}$ with $c = u$ for $l \leq L$ and $c = v$ else. For $\lambda > L$ we have $(t_k)_{l,\lambda} = (t^v_{(l,\lambda-L-1)})^\dagger$ with $c = v$ for $l \leq L$ and $c = u$ else [17].

Analogously to the noninteracting case we define the corresponding Wannier functions for $M \rightarrow \infty$ as

$$w^c_{\lambda, m,l} = \frac{1}{2\pi} \int_{BZ} dk e^{ikm} t^c_{(k,\lambda)}, \quad (18)$$

where $BZ$ denotes the Brillouin zone. Here we explicitly distinguish between particle $c = u$ and hole $c = v$ contribution to the Wannier function. Before defining the polarization, we have to sort the bands $\lambda$. As can be seen in Eq. [14], we have always pairs of energies $\pm E$. We here consider only the columns with $\lambda < L$ corresponding to positive $E$. Due to [15], the columns corresponding to negative energies contain only copies of $\lambda < L$. We consider the bands up to an energy $E_{max}$ which shall be in a band gap. Sorting the columns with $E > 0$ by energy, we denote the band with the largest energy $E_1 < E_{max}$ with $\lambda_{max}$. Thus we consider the bands $\lambda \leq \lambda_{max}$ so that there is an energy gap between $\lambda_{max}$ and $\lambda_{max} + 1$. For the Hamiltonian [9], we depict some dispersion relations in Fig. [11]. As $L = 2$, we have two bands and the spectrum is gapped between $\lambda = 1$ and $\lambda = 2$. For $c = u, v$, we separately define the corresponding contributions to the macroscopic polarization to be

$$P_c = \lim_{M \rightarrow \infty} \sum_{m = -M/2 + 1}^{M/2} \sum_{\lambda \leq \lambda_{max}} (w^c_{\lambda, m,l})^* m w^c_{\lambda, m,l}, \quad (19)$$
With these definitions we can define the symplectic polarization as the difference of the particle and hole polarization contribution:

\[ P_s = P_u - P_v = \frac{1}{2\pi} \int_{BZ} dk A(k), \]  

where we introduced the Berry potential vector

\[ A(k) = i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ \Gamma_{\lambda} \sigma_z^t \sigma_z \left( \frac{\partial}{\partial k} t_k \right) \right]. \]  

We define the matrix \((\Gamma_{\lambda})_{j,j'} = \delta_{j,j'} \delta_{j,\lambda}\) being a \(2\mathcal{L} \times 2\mathcal{L}\) matrix. The symplectic polarization of the bands \(\lambda \leq \lambda_{\text{max}}\) determines, if there is an edge state between the bands \(\lambda_{\text{max}}\) and \(\lambda_{\text{max}} + 1\) or not.

For the noninteracting case the symplectic polarization reduces to \(P_s \rightarrow P_u\) and coincides with the macroscopic polarization of Ref. [21]. Equation (21) agrees with the vector potential of Ref. [18] found in the context of a bosonic Chern number. Yet, in that article there is no interpretation in terms of the symplectic polarization (20). A proof of the last step in (20) is given in Appendix B. 1.  

We also prove in Appendix B. 2 that the Berry potential is real valued.

C. Topological invariant

The symplectic polarization is strictly quantized to the values \(P_s = m, \frac{1}{2} + m\) with \(m \in \mathbb{Z}\) as the one for noninteracting systems [20]. The proof also works essentially as in the noninteracting case, yet one has to respect the symplectic structure of the eigenstates. We first define the sewing matrix connecting the state at \(k\) with the one at \(-k\). If we have a solution of the Bogoliubov equation (17) in momentum space \(t_k\), then \(\mathcal{P}^B t_k \) diagonalizes \(H_{-k}\). Thus, one can connect the paraunitary matrices at \(k\) and \(-k\) as

\[ B_k = t_k^\dagger \sigma_z^B t_k \quad \Leftrightarrow \quad t_{-k} = \mathcal{P}^B t_k \sigma_z^B t_k^\dagger, \]  

where \(B_k\) denotes the sewing matrix. Importantly, it can only mix states being degenerate. When there are no degeneracies, the sewing matrix reduces to a diagonal matrix with elements of unit modulus. As we assumed that our system is gapped between \(\lambda_{\text{max}}\) and \(\lambda_{\text{max}} + 1\), the sewing matrix \(B_k\) has a block-diagonal structure. We denote the block referring to the band below the gap with \(B_{<,k}\). For the inversion-invariant momenta \(k_{\text{inv}}\), its determinant is of the eigenvalues of \(\mathcal{P}^B\) regarding the eigenstates \(t_{k_{\text{inv}}, \lambda}\),

\[ \det [B_{<,k_{\text{inv}}}] = \prod_{\lambda < \lambda_{\text{max}}} \eta_\lambda(k_{\text{inv}}). \]  

The sewing matrix obeys the same transformation rules as \(t_k\),

\[ B_k^\dagger \sigma_z^B B_k = t_k^\dagger \mathcal{P}^B \sigma_z^B t_{-k} \sigma_z^B t_{-k} \sigma_z^B \mathcal{P}^B t_k = \sigma_z^B. \]  

Using the sewing matrix, we link the symplectic polarization \(P_s\) to the eigenvalues of the symplectic inversion operator. Analogously to the noninteracting case (20), we need to relate the Berry potential at \(k\) and \(-k\), but respecting the symplectic structure of the eigenstates. We find

\[ A(-k) = -A(k) + i \partial_k \log [\det (B_{<,k})], \]  

which we prove in Appendix B. 3. Using this we finally find

\[ P_s = \frac{1}{2\pi} \int_0^\pi dk \left[ A(k) + A(-k) \right] \]  

Thus, one can connect the paraunitary matrices at \(k\) and \(-k\) as

\[ B_k = t_k^\dagger \sigma_z^B t_k \quad \Leftrightarrow \quad t_{-k} = \mathcal{P}^B t_k \sigma_z^B t_k^\dagger, \]  

where \(B_k\) denotes the sewing matrix. Importantly, it can only mix states being degenerate. When there are no degeneracies, the sewing matrix reduces to a diagonal matrix with elements of unit modulus. As we assumed that our system is gapped between \(\lambda_{\text{max}}\) and \(\lambda_{\text{max}} + 1\), the sewing matrix \(B_k\) has a block-diagonal structure. We denote the block referring to the band below the gap with \(B_{<,k}\). For the inversion-invariant momenta \(k_{\text{inv}}\), its determinant is of the eigenvalues of \(\mathcal{P}^B\) regarding the eigenstates \(t_{k_{\text{inv}}, \lambda}\),

\[ \det [B_{<,k_{\text{inv}}}] = \prod_{\lambda < \lambda_{\text{max}}} \eta_\lambda(k_{\text{inv}}). \]  

In the last step we have used that the eigenvalues of the inversion operator are \(\eta_\lambda(k_{\text{inv}}) = \pm 1\). Representing the eigenvalues in the form \(\eta_\lambda(k_{\text{inv}}) = 1 = e^{i\pi m}\) or \(\eta_\lambda(k_{\text{inv}}) = -1 = e^{i(\pi + 2\pi m)}\) finally proves that \(P_s = m, \frac{1}{2} + m\).

D. Polarization of the lowest band

The Bogoliubov excitations of a BEC typically exhibit a Goldstone mode in the lowest band denoted here with \(\lambda = 1\). This means a linear dispersion for small \(E_{k,1} \propto |k|\). This can be seen in the panels of Fig. [1]. The solution at \(k = 0\) resembles the mean-field solution \(\Psi_0\) obtained by the Gross-Pitaevskii equation in the form

\[ t_{0,1} = (u_{0,1}, v_{0,1})^T = (\Psi_0, \Psi_0)^T. \]  

Yet, this solution is not normalizable according to (13) as \(t_{0,1}^\dagger \sigma_z^B t_{0,1} = 0\).

The fact that \(t_{0,1}\) is not normalizable is an obstruction for defining the vector potential in Eq. (21) at \(k = 0\). Nevertheless, we here argue how to circumvent this obstacle. We use a slightly modified definition for the symplectic polarization to respect the difficulties of the lowest band, namely

\[ P_s = \frac{1}{2\pi} \lim_{\delta \to 0} \left[ \int_{-\pi}^{-\delta} dk A(k) + \int_{\delta}^{\pi} dk A(k) \right]. \]  

Of course, relation (26) for \(k \neq 0\) is not affected and therefore one can adopt the derivation up to line (29) by...
including a limit operation so that

\[ P_s = \frac{i}{2\pi} \left\{ \log \det (\sigma_z B_\pi) - \lim_{\delta \to 0} \log \det (\sigma_z B_\delta) \right\}. \]

(32)

The crucial point is to discuss the limit. We also assume that the lowest band is non-degenerate in a finite region around \( k = 0 \). Consequently, the reduced sewing matrix has the form

\[ B_{<,k} = \left( \begin{array}{cc} b_{k,1} & 0 \\ \tilde{B}_{<,k} & 0 \end{array} \right). \]

(33)

The submatrix \( \tilde{B}_{<,k} \) behaves regularly for \( k \to 0 \) and does not cause any problems. So we have just to discuss the implications of \( b_{k,1} \). To this end, for a given Gross-Pitaevskii solution \( \Psi_0 \), we define a normalization function \( f_k \) so that

\[ \lim_{k \to 0} f_k t_{k,1} = (\Psi_0, \Psi_0)^T. \]

(34)

The exact shape of \( f_k \) is not crucial for our discussion, but in agreement with inversion symmetry we require \( f_k = f_{-k} \). For \( k \to 0 \), the relation between the solutions at inversion-symmetric momenta reads according to (23)

\[ t_{-k,1} = e^{ib_{0,1}} P B t_{k,1}. \]

(35)

This relation is not well-defined at \( k = 0 \). Therefore, we multiply \( f_k \), so that the limit \( k \to 0 \) exists on both sides of the equation. Consequently,

\[ e^{ib_{0,1}} P B = 1 \]

(36)

and this constrains \( b_{0,1} = 2\pi m \) or \( b_{0,1} = \pi(2m + 1) \) depending on the eigenvalue of \( P B \) regarding \( (\Psi_0, \Psi_0)^T \). Conclusively one can say, that, although the limit \( k \to 0 \) of the state is not well-defined, the phase is up to \( 2\pi m \), which does not affect the final outcome in (30).

E. Application

Here we return to the Hamiltonian [8] and investigate its topology. As the Bogoliubov Hamiltonian and the matrix \( \tau_z \) fulfill

\[ \tau_z H_k \tau_z = H_{-k}, \]

(37)

the operator \( \tau_z \) is an inversion symmetry. In Fig. 1d) we depict some dispersion relations of the system.

To determine the topology of our model we have to consider the inversion-invariant momenta \( k_{inv} = 0, \pi \). Let us consider the lower band. As the mean-field wave function \( \langle \zeta_{m,\uparrow}, \zeta_{m,\downarrow} \rangle = (1,0) \) is constant for the parameter regime in our model, the eigenvalue under inversion is \( \eta_{k=0,1} = 1 \). The gap closes just at the boundary of the Brillouin zone so that the symplectic polarization \( P_s \) can only change there. To investigate this, we consider the eigenstates at \( k = \pi \), which read

\[ E_{\pi,1} = 4\sqrt{\nu_0(\rho \chi + \nu_0)} \quad E_{\pi,2} = 2(\omega - \rho \chi + \rho \chi \uparrow) \]

(38)

\[ t_{\pi,1} \propto \begin{pmatrix} g + 2\nu_0 - 2\sqrt{\nu_0(\rho \chi + \nu_0)} & 0 \\ 0 & g \end{pmatrix} \quad t_{\pi,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(39)

Obviously, they fulfill \( \tau_z t_{\pi,\lambda} = (-1)^\lambda t_{\pi,\lambda} \). Consequently, the topological invariant [29] changes at the degeneracy point \( E_{\pi,1} = E_{\pi,2} \). Thus, the boundary between the topological phases is

\[ \nu_{0,\text{tpt}} = -\rho \chi + \sqrt{2(\rho \chi)^2 - 2\rho \chi(\omega + \rho \chi \uparrow) + (\omega + \rho \chi \uparrow)^2}. \]

(40)
We depict this topological phase boundary also in the phase diagram in Fig. 1b) for χ_{tt} = 0. One can see, that the product χ_{t} has a strong impact on the topology of the system. The topological invariant for the lower band is P_s = 0 for ν_0 < ν_{0,tpt} and changes to P_s = \frac{1}{2} for ν_0 > ν_{0,tpt}. Accordingly, the system is in a topologically trivial or a non-trivial phase, respectively.

F. Edge states

Although we have defined a topological invariant, there is still the question about the physical consequences of it. In contrast to fermionic systems, where the polarization is an actual physical quantity, in bosonic systems this is not the case as not all momenta of a band are equally occupied. However, the symplectic polarization (20) determines the existence of edge states of a chain with finite length and fixed-boundary conditions.

As a demonstration, we consider our model with fixed-boundary conditions in Fig. 2a). For this illustration, we do not assume an additional harmonic confining potential. Due to its topological origin, the edge states are robust in the presence of moderate perturbations [32]. We also refer to Ref. [33] for the creation of sharp boundaries.

Importantly, we have to determine first the mean-field wave function by minimizing the Gross-Pitaevskii functional. Here the Gross-Pitaevskii mean-field has no uniform density at all sites due to the boundaries. We depict the part of the condensate close to the left boundary in the insets of Fig. 2a) and b). One can see, that the density is smaller close to the edges. Away from the boundaries, the mean-field is approximately constant so that the results derived in Sec. IV are still valid.

On top of the mean-field wave function, we perform a Bogoliubov diagonalization in position space and depict its spectrum in Fig. 2a) and b). In Fig. 2a), the system is in the trivial phase IIa so that no midgap states appear. In contrast, one clearly identifies two midgap states within the spectrum in panel b) depicting the spectrum for parameters in phase IIb. In Fig. 2b), we plot the wave function of one of these states. We find, that it is strongly localized on the edges. The interpretation of these edge states works analogously as in the fermionic case [20]. There, each edge state contributes half an electron to each boundary. Thus, one particle splits up in two half particles. In the bosonic case correspondingly, each edge mode can be considered to consist of two “half modes” at the boundaries.

V. CONCLUSION

We investigated the topology of Bogoliubov excitations in inversion-invariant systems of interacting bosons. To characterize the topology, we extended the definition of the macroscopic polarization in a symplectic manner. We called this new quantity symplectic polarization which is defined in Eq. (20) as the difference of the particle- and hole-like polarization contribution. As in noninteracting systems with inversion symmetry, this quantity can be expressed by the inversion eigenvalues of the states at inversion-invariant momenta. In an instructive example we showed, that the topological invariant strongly depends on the condensate density so that the interaction between the particles modifies the topology of the excitations.

The definition of the symplectic polarization can also be applied to analyze inversion-invariant insulators in higher dimensions. In this case we expect that an invariant defined as the product of the inversion eigenvalues of the states at the inversion-invariant momenta predicts edge states [20]. Furthermore, one can link also the symplectic polarization of the one-dimensional system to the Chern number in two dimensions [18].

Importantly, the symplectic polarization discussed here can be used to define the symplectic generalization of the time-reversal polarization. This can be used to analyze the topology of time-reversal invariant systems of interaction bosons.

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Appendix A: Minimization of the Gross-Pitaevskii functional

The Gross-Pitaevskii functional reads

\[
E_{GP} = \sum_{m=-M/2+1}^{M/2} E_{0,m} + E_{V,m}, \tag{A1}
\]

where

\[
E_{0,m} = \omega (\zeta_{\downarrow,m}^* \zeta_{\downarrow,m} - \zeta_{\uparrow,m}^* \zeta_{\uparrow,m}) - \nu_0 (\zeta_{\uparrow,m}^* \zeta_{\uparrow,m+1} - \zeta_{\downarrow,m}^* \zeta_{\downarrow,m+1} + \text{h.c.}) - \nu_{so} (\zeta_{\uparrow,m}^* \zeta_{\downarrow,m+1} - \zeta_{\uparrow,m+1}^* \zeta_{\downarrow,m} + \text{h.c.}) \tag{A2}
\]

\[
E_{V,m} = \sum_{s,s'=\uparrow,\downarrow} \rho_0 \chi_{s,s'} \zeta_{s,m}^* \zeta_{s',m} \zeta_{s',m} \zeta_{s,m}. \tag{A3}
\]

Motivated by Ref. [26] we use the modified ansatz

\[
\zeta_{\uparrow,m}^* = C_1 \left( \cos \theta \right) e^{ikm} + C_2 \left( \cos \theta \right) e^{-ikm}. \tag{A4}
\]

The variational parameters are \( C_1 \), \( C_2 \), \( \theta \) and \( k \). As we work at a fixed particle number, we have to respect the constraint \( |C_1|^2 + |C_2|^2 = 1 \). Due to this ansatz the noninteracting part of the energy functional reads

\[
\sum_m E_{0,m} = M \cos 2\theta (-w - 2\nu_0 \cos k) + M 2\nu_{so} \sin 2\theta \sin k \tag{A5}
\]

Importantly, the noninteracting part does not depend on the coefficients \( C_1 \) and \( C_2 \) which reflects the inversion symmetry of the system. Accordingly, the interaction terms turn out to be

\[
\sum_m E_{V,m} = M \rho_0 (1 + 2\beta) (\chi_{\uparrow\uparrow} \cos^4 \theta + \chi_{\downarrow\downarrow} \sin^4 \theta) + M \rho_0 2\chi_{\uparrow\downarrow} (1 - 2\beta) \cos \theta^2 \sin^2 \theta, \tag{A6}
\]

where we define \( \beta = |C_1|^2 / |C_2|^2 \) with \( \beta \in (0, 1/4) \). We immediately see that \( E_{GP} \) is a linear function of \( \beta \). Therefore the minimum can only be located at \( \beta = 0 \) or \( \beta = 1/4 \).

Next, we derive a relation between \( k \) and \( \theta \). To this end we take the derivative of \( E_{GP} \) with respect to \( k \). After a short algebraic manipulation, we obtain

\[
\tan 2\theta = -\frac{\nu_0}{\nu_{so}} \tan k. \tag{A8}
\]

Having done these analytical preparations, we now can numerically minimize the Gross-Pitaevskii functional, which is now just a function of essentially one variable, thus \( E_{GP} = E_{GP}(\theta, \beta) \) as \( \beta \in \{0, 1/4\} \). The result is depicted in Fig[1b]. For comparison, we also directly minimized the Gross-Pitaevskii functional numerically, where we also found the localized ground-state wave function for \( \chi_0 \rho_0 < 0 \).

Appendix B: Berry vector potential

1. Details of the derivation

We start with the final expression of the vector potential and perform the proof from the end. First, we insert
the representation (15) for $t_k$ and perform the multiplications with $\sigma_z$ so that we obtain

$$A(k) = i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ \Gamma_\lambda \sigma_z t_k^\dagger \sigma_z (\partial_k t_k) \right]$$

(B1)

$$= i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ \Gamma_\lambda \left( \begin{array}{cc} u^\dagger_k & -v^\dagger_k \\ -v_k & u_k \end{array} \right) \frac{\partial}{\partial k} \left( \begin{array}{c} v^*_k \\ u^*_k \end{array} \right) \right]$$

(B2)

$$= i \sum_{\lambda \leq \lambda_{\text{max}}} \left( u^\dagger_k \partial_k u_k - v^\dagger_k \partial_k v_k \right)_{\lambda, \lambda}$$

(B3)

We evaluate the matrix product by inserting a complete 1 of the basis states $l$ of the unit cell so that

$$A(k) = i \sum_l \sum_{\lambda \leq \lambda_{\text{max}}} \sigma_l (t^c_{l,(k,\lambda)})^* \partial_k t^c_{l,(k,\lambda)},$$

using $\sigma(u) = +1$ and $\sigma(v) = -1$. We have also used that the columns of $u_k$ and $v_k$ are the periodic part of the Bloch function, namely $t^u_{l,(k,\lambda)}$ and $t^v_{l,(k,\lambda)}$, respectively. We continue by inserting a unity so that

$$\frac{1}{2\pi} \int_{BZ} dk A(k) = \frac{i}{2\pi} \int_{BZ} dk dk' \sum_{\lambda \leq \lambda_{\text{max}}} \sum_{\ell \in \text{U}, \ell' \in \text{V}} \sigma_l (t^c_{\ell, (k', \lambda)})^* \delta (k - k') \partial_k t^c_{\ell, (k, \lambda)}$$

(B4)

$$= \frac{i}{(2\pi)^2} \lim_{M \to \infty} \int_{BZ} dk dk' \sum_{\lambda \leq \lambda_{\text{max}}} \sum_{m = -M/2 + 1}^{M/2} \sigma_l (t^c_{\ell, (k, \lambda)})^* e^{i(k-k')m} \partial_k t^c_{\ell, (k, \lambda)}$$

(B5)

$$= \frac{i}{(2\pi)^2} \lim_{M \to \infty} \int_{BZ} dk dk' \sum_{m, l \leq \lambda_{\text{max}}} \sum_{\ell \in \text{U}, \ell' \in \text{V}} \sigma_l (t^c_{\ell, (k, \lambda)})^* e^{i(k-k')m} \partial_k t^c_{\ell, (k, \lambda)}$$

(B6)

$$= -\frac{i}{(2\pi)^2} \lim_{M \to \infty} \int_{BZ} dk dk' \sum_{m, l \leq \lambda_{\text{max}}} \sum_{\ell \in \text{U}, \ell' \in \text{V}} \sigma_l (t^c_{\ell, (k, \lambda)})^* e^{-ik'm} i m t^c_{\ell, (k, \lambda)} e^{ikm}$$

(B7)

$$= P_u - P_v$$

(B8)

### 2. Real valueness of the Berry potential

To prove that the Berry potential (21) is real-valued, we calculate

$$A^*(k) = -i \sum_{\lambda \leq \lambda_{\text{max}}} \left\{ \text{Tr} \left[ \Gamma_\lambda \sigma_z t^\dagger_k \sigma_z (\partial_k t_k) \right] \right\}^*$$

(B9)

$$= -i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ (\partial_k t_k)^\dagger \sigma_z t_k \sigma_z \Gamma_\lambda \right]$$

(B10)

$$= -i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ (\partial_k t_k)^\dagger \sigma_z t_k \sigma_z \Gamma_\lambda \right]$$

(B11)

$$= i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ t_k^\dagger \sigma_z (\partial_k t_k) \sigma_z \Gamma_\lambda \right]$$

(B12)

$$= A(k)$$

(B13)

3. Symmetry relation

Here we prove relation (26). By definition we have

$$A(-k) = i \sum_{\lambda \leq \lambda_{\text{max}}} \text{Tr} \left[ \Gamma_\lambda \sigma_z t_{-k}^\dagger \sigma_z (\partial_{-k} t_{-k}) \right]$$

(B14)

$$= i \sum_{\lambda} \text{Tr} \left[ \Gamma_\lambda B_{\lambda} \sigma_z t_{-k}^\dagger \sigma_z \left( \partial_{-k} B_{\lambda} \right) \sigma_z \right]$$

(B15)

$$= i \sum_{\lambda} \text{Tr} \left[ \Gamma_\lambda B_{\lambda} \sigma_z t_{-k}^\dagger \sigma_z B_{\lambda} \right]$$

(B16)

$$+ i \sum_{\lambda} \text{Tr} \left[ \Gamma_\lambda B_{\lambda} \sigma_z t_{-k}^\dagger \sigma_z \left( \partial_{-k} B_{\lambda} \right) \sigma_z \right]$$

(B17)

$$= -A(k) - i \sum_{\lambda} \text{Tr} \left[ \Gamma_\lambda \sigma_z B_{\lambda} \sigma_z \left( \partial_{-k} B_{\lambda} \right) \sigma_z \right]$$

(B18)

Next, we recognize that

$$\sigma_z B_{\lambda} \sigma_z B_{\lambda}^\dagger = 1$$

(B20)

due to Eq. (29). As the sewing matrix has a block diagonal structure, we find

$$B_{<,k} B_{<,k}^\dagger = 1$$

(B21)
This relation means that $B_{<,k}$ is unitary. We expand it in terms of its eigenstates $|i\rangle$ such that

$$B_{<,k} = \sum_i e_i |i\rangle \langle i| \leftrightarrow B_{<,k}^\dagger = \sum_i e_i^{-1} |i\rangle \langle i|$$  \hspace{1cm} (B22)

Both, the eigenstates as well as the eigenvalues depend on $k$. For a notational reason we suppress the index in the following. Now we can prove relation (26):

$$\text{Tr} \left[ B_{<,k} \partial_k B_{<,k}^\dagger \right] = \text{Tr} \left[ \sum_i e_i |i\rangle \langle i| \left( \partial_k \sum_j e_j^{-1} |j\rangle \langle j| \right) \right]$$  \hspace{1cm} (B23)

$$= \text{Tr} \left[ \sum_i e_i |i\rangle \langle i| \sum_j (\partial_k e_j^{-1}) |j\rangle \langle j| \right]$$  \hspace{1cm} (B24)

$$+ \text{Tr} \left[ \sum_i e_i |i\rangle \langle i| \sum_j e_j^{-1} (\partial_k |j\rangle \langle j|) \right]$$  \hspace{1cm} (B25)

$$+ \text{Tr} \left[ \sum_i e_i |i\rangle \langle i| \sum_j e_j^{-1} |j\rangle (\partial_k \langle j|) \right]$$  \hspace{1cm} (B26)

Evaluating the traces in the eigenbasis of $B_{<,k}$ we find

$$= \sum_i \partial_k \log e_i^{-1} + \sum_i \langle i| \partial_k |i\rangle + (\partial_k \langle i| i\rangle)$$  \hspace{1cm} (B28)

$$= -\partial_k \log \prod_i e_i + \sum_i (\partial_k \langle i| i\rangle)$$  \hspace{1cm} (B29)

$$= -\partial_k \det (B_{<,k})$$  \hspace{1cm} (B30)

In the derivation we have also used that $|e_i| = 1$ so that $\partial_k e_i^{-1} = -\partial_k e_i$. 

\hspace{1cm}