Ion-Acoustic Rogue Waves in Double Pair Plasma Having Non-Extensive Particles

Sharmin Jahan 1,*, Mohammad Nurul Haque 1, Nure Alam Chowdhury 2, Abdul Mannan 1,3 and Abdullah Al Mamun 1

1 Department of Physics, Jahangirnagar University, Savar, Dhaka 1342, Bangladesh; mnhaque2460@gmail.com (M.N.H.); abdulmannan@juniv.edu (A.M.); mamun_phys@juniv.edu (A.A.M.)
2 Plasma Physics Division, Atomic Energy Centre, Dhaka 1000, Bangladesh; nurealam1743phy@gmail.com
3 Institut für Mathematik, Martin Luther Universität Halle-Wittenberg, D-06099 Halle, Germany

* Correspondence: jahan88phy@gmail.com

Abstract: The modulational instability (MI) of ion-acoustic (IA) waves (IAWs) and associated IA rogue waves (IARWs) are studied in double-pair plasma containing inertial positive and negative ions, inertialless non-extensive electrons and iso-thermal positrons. A standard nonlinear Schrödinger equation (NLSE) is derived by employing reductive perturbation method. It can be seen from the numerical analysis that the plasma system supports both modulationally stable (unstable) parametric regime in which the dispersive and nonlinear coefficients of the NLSE have opposite (same) sign. It is also found that the basic features of IAWs (viz., MI criteria of IAWs, amplitude, and width of the IARWs, etc.) are rigorously changed by the plasma parameters such as mass, charge state, and number density of the plasma species. The outcomes of our present investigation should be useful in understanding the propagation of nonlinear electrostatic IAWs and associated IARWs in astrophysical and laboratory plasmas.

Keywords: ion-acoustic waves; NLSE; modulational instability; rogue waves

1. Introduction

Pair-ion (PI) plasma is characterised as fully ionized gas, having electrons, positive and negative ions, and is believed to exist in astrophysical environments such as Van Allen radiation belt and near the polar cap of fast rotation neutron stars [1], solar atmosphere [2], upper regions of Titan’s atmosphere [3–10], cometary comae [11], (H+, O−2) and (H+, H−) plasmas in the D and F-regions of Earth’s ionosphere [4–9], and also in laboratory experiments namely, (Ar+, F−) plasma [12], (K+, SF6−) plasma [13,14], neutral beam sources [15], plasma processing reactors [16], (Ar+, SF6−) plasma [17–20], combustion products [21], plasma etching [21], (Xe+, F−) plasma [22], (Ar+, O−2) plasma, Fullerene (C60+, C60−) plasma [23–25], RF glow discharges [16], and Tokamak [26], etc. A number of authors studied ion-acoustic (IA) waves (IAWs) and associated nonlinear electrostatic structures, namely, solitons, shocks, rogue waves, and double layers in the presence of positrons in PI plasma [27–29].

The Maxwellian distribution function is one of the most widely used velocity distribution functions for describing the dynamics of iso-thermal particles. Highly energetic particles have been observed in the galaxy clusters [30], Earth’s bow-shock [31], upper ionosphere of Mars [32], vicinity of the Moon [33], and magnetospheres of Jupiter and Saturn [34], etc. So, to describe the dynamics of these highly energetic particles, Renyi [35] first recognized the modification of Maxwellian distribution, and finally, Tsallis [36] generalized the non-extensive q -distribution. It is noted that the parameter q in the non-extensive q -distribution characterizes the degree of non-extensivity of these highly energetic particles [37–40]. Shalini et al. [38] studied IAWs in non-extensive plasma having two-temperature electrons, and observed that the width of the first and second-order IA rogue
waves (IARWs) associated with IAWs decreases with increasing the value of $q$ but the amplitude of the first and second-order IARWs associated with IAWs remain constant. Tribeche et al. [39] investigated electrostatic solitary waves in the presence of non-extensive electrons, and found that the amplitude of the potential increases with non-extensive parameter. Hafez and Talukder [40] examined the propagation of nonlinear electrostatic waves in a three-component non-extensive plasma having inertialess non-extensive electrons and positrons, and inertial ions, and reported that the amplitude of the soliton increases with increasing temperature of non-extensive electron.

The investigation of modulational instability (MI) of electrostatic waves is one of the most important research areas for plasma physicists [41,42]. It may be noted that the MI of electrostatic waves is considered to be the primary reason for the formation of massive and gigantic rogue waves (RWs) [43–48]. Rogue wave, which is the rational solution of the standard nonlinear Schrödinger equation (NLSE), is a short-lived phenomenon that emerges from nowhere and disappears without a trace [43–48]. A number of authors have investigated the MI of IAWs by considering the non-extensive particles [49–53]. Bains et al. [51] studied the MI of IAWs in the presence of non-extensive electrons, and demonstrated that the critical wave number ($k_c$), at which the instability sets in, increases with the increase in the value of $q$ ($q > 0$). Bouzit et al. [52] investigated the stability conditions of IAWs in the presence of non-extensive non-thermal electrons. Eslami et al. [53] investigated the MI of IAWs in electron-positron-ion plasma having non-extensive electrons and positrons, and found that the amplitude of the potential increases with non-extensive parameter. Hafez and Talukder [40] indicated the negative (positive) ion fluid, and is normalized by the IA wave speed $C_{-i} = \frac{\lambda_4}{\beta}$.

2. Model Equations

We consider the propagation of IAWs in a collisionless, fully ionized, unmagnetized plasma system consisting of warm negative ions (denoted by $n_{-i}$; charge $q_{-i} = -Z_i e$; mass $m_{-i}$), warm positive ions (denoted by $n_{+i}$; charge $q_{+i} = Z_i e$; mass $m_{+i}$), non-extensive $q$-distributed electrons (denoted by $n_e$; charge $q_e = -e$; mass $m_e$), and iso-thermal positrons (denoted by $n_p$; charge $q_p = +e$; mass $m_p$); where $Z_{-i}$ ($Z_{+i}$) is the charge state of the negative (positive) ion, and $e$ being the magnitude of the charge of electron. The charge neutrality condition of our present model can be written as $n_{p0} + Z_{+i}n_{+i0} = n_{e0} + Z_{-i}n_{-i0}$.

Now, the normalized governing equations can be written as

\[
\frac{\partial n_{-i}}{\partial t} + \frac{\partial}{\partial x}(n_{-i}u_{-i}) = 0,
\]

\[
\frac{\partial u_{-i}}{\partial t} + u_{-i}\frac{\partial u_{-i}}{\partial x} + \lambda_1 n_{-i}\frac{\partial n_{-i}}{\partial x} = \frac{\partial \phi}{\partial x},
\]

\[
\frac{\partial n_{+i}}{\partial t} + \frac{\partial}{\partial x}(n_{+i}u_{+i}) = 0,
\]

\[
\frac{\partial u_{+i}}{\partial t} + u_{+i}\frac{\partial u_{+i}}{\partial x} + \lambda_3 n_{+i}\frac{\partial n_{+i}}{\partial x} = -\lambda_2 \frac{\partial \phi}{\partial x},
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = \lambda_4 n_e + n_{-i} - (1 + \lambda_4 - \lambda_5)n_p - \lambda_5 n_{+i},
\]

where $n_{-i}$, $n_{+i}$, $n_e$, and $n_p$ are normalized by $n_{-i0}$, $n_{+i0}$, $n_{e0}$, and $n_{p0}$, respectively; $u_{-i}$ ($u_{+i}$) indicates the negative (positive) ion fluid, and is normalized by the IA wave speed $C_{-i} = \frac{\lambda_4}{\beta}$. 

where \( \Pi \) denotes the electrostatic wave potential, and is normalized by \( k_B T_e / e \); the time and space variables are, respectively, normalized by \( \omega_{-1}^{-1} = (m_{-1}/4\pi e^2 Z_{-1}^2 n_{-1,0})^{1/2} \) and \( \lambda_{-1} = (k_B T_e/4\pi e^2 Z_{-1} n_{-1,0})^{1/2} \). The pressure term of the negative ion can be represented as \( P_{-i} = P_{-i} n_{-i}(N_{-i}/n_{-i})^\gamma \) with \( P_{-i} = n_{-i} k_B T_{-i} \), where \( P_{-i} \) \( (T_{-i}) \) being the equilibrium pressure \( (temperature) \) of negative ion; and \( P_{+i} = P_{+i}(N_{+i}/n_{+i})^\gamma \) with \( P_{+i} = n_{+i} k_B T_{+i} \), where \( P_{+i} \) \( (T_{+i}) \) being the equilibrium pressure \( (temperature) \) of the positive ion, and \( \gamma = (N + 2)/N \) (where \( N \) represents the degree of freedom and for one-dimensional case \( N = 1 \), and then \( \gamma = 3 \)). Other parameters are \( \lambda_1 = 3T_{-1}/Z_{-1}T_e, \lambda_2 \) \( = Z_{+1} m_{-1}/Z_{-1} m_{+1}, \lambda_3 \) \( = 3T_{+1} m_{-1}/Z_{-1} m_{+1} \), \( \lambda_4 \) \( = n_{+0}/Z_{-1} n_{-0} \), and \( \lambda_5 \) \( = Z_{+1} n_{+0}/Z_{-1} n_{-0} \). Now, the number densities of non-extensive electron [54] and iso-thermal positron can be represented, respectively, by the following normalized equations

\[
\begin{align*}
n_e &= [1 + (q - 1)\phi]^{\lambda_{-1}^2 / \gamma - 1}, \\
n_p &= \exp(-\lambda_6 \phi),
\end{align*}
\]

where \( \lambda_6 = T_e/T_p \) (with \( T_p \) being the temperature of the iso-thermal positron and \( T_e > T_p \)). The parameter \( q \), generally known as entropic index which defines the degree of non-extensivity. It can be noted that when \( q = 1 \), the entropy reduces to standard Maxwell–Boltzmann distribution. On the other hand, in the limits \( q > 0 \) \( (q < 0) \), the entropy shows sub-extensivity \( (super-extensivity) \). Now, by substituting Equations (6) and (7) into Equation (5) and expanding up to third order in \( \phi \), we can write

\[
\frac{\partial^2 \phi}{\partial x^2} + 1 + \lambda_5 n_{-i} = \lambda_5 + n_{-i} + M_1 \phi + M_2 \phi^2 + M_3 \phi^3 + \cdots,
\]

where

\[
\begin{align*}
M_1 &= [\lambda_4(q + 1) + 2\lambda_6(1 + \lambda_4 - \lambda_5)]/2, \\
M_2 &= [\lambda_4(q + 1)(3 - q) - 4\lambda_6^2(1 + \lambda_4 - \lambda_5)]/8, \\
M_3 &= [\lambda_4(q + 1)(q - 3)(3q - 5) + 8\lambda_6^2(1 + \lambda_4 - \lambda_5)]/48.
\end{align*}
\]

It may be noted here that the terms containing \( M_1, M_2, \) and \( M_3 \) in Equation (8) are due to the contribution of non-extensive electrons and iso-thermal positrons.

3. Derivation of the NLSE

To study the MI of the IAWs, first we want to derive the NLSE by employing the reductive perturbation method. In that case, the stretched co-ordinates can be written in the following form [54,55]

\[
\begin{align*}
\xi &= \epsilon(x - v_g t), \\
\tau &= \epsilon^2 t,
\end{align*}
\]

where \( v_g \) is the group speed and \( \epsilon \) is a small parameter. After that the dependent variables can be represented as [54,55]

\[
\Pi(x, t) = \Pi_0 + \sum_{m=1}^{\infty} e^m \sum_{l=-\infty}^{\infty} \Pi_{l}^{(m)}(\xi, \tau) \exp(\imath l(kx - \omega t)),
\]

where \( \Pi_{l}^{(m)} = [n_{m}^{l} u_{m}^{l}, n_{m+1}^{l} u_{m+1}^{l}, n_{m+2}^{l} u_{m+2}^{l}, 0, 0]^{T} \), \( \Pi_0 = [1, 0, 1, 0, 0]^{T} \), and \( k \) \( (\omega) \) is real variables representing the carrier wave number \( (frequency) \). The derivative operators can be written as [54,55]
\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \epsilon v_s \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau},
\]
\[
\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \xi}.
\]

(12)

Now, by substituting Equations (9)–(13) into Equations (1)–(4) and (8), and collecting the power terms of \( \epsilon \), the first order \((m = 1\) with \( l = 1\)) reduced equations can be written as

\[
n^{(1)}_{-i} = \frac{k^2}{S} \phi_1^{(1)},
\]
\[
u^{(1)}_{-i} = \frac{\lambda \omega}{S} \phi_1^{(1)},
\]
\[
n^{(1)}_{i} = \frac{\lambda_2 k^2}{A} \phi_1^{(1)},
\]
\[
u^{(1)}_{i} = \frac{\lambda_2 \omega k}{A} \phi_1^{(1)},
\]

(14)

(15)

(16)

(17)

where \( S = \lambda_1 k^2 - \omega^2 \) and \( A = \omega^2 - \lambda_3 k^2 \). These equations provide the dispersion relation of IAWs in the following form

\[
\omega^2 \equiv \omega_f^2 = \frac{I + \sqrt{I^2 - 4UJ}}{2U},
\]
\[
\omega^2 \equiv \omega_s^2 = \frac{I - \sqrt{I^2 - 4UJ}}{2U},
\]

(18)

(19)

where \( I = (\lambda_1 k^2 + \lambda_3 k^2 + \lambda_1 M_1 + \lambda_3 M_1 + \lambda_2 \lambda_5 + 1), \ U = (k^2 + M_1)/k^2, \) and \( J = k^2(\lambda_1 \lambda_3 k^2 + \lambda_1 \lambda_3 M_1 + \lambda_3 + \lambda_1 \lambda_2 \lambda_5). \) However, to obtain the real and positive values of \( \omega_f, \) the conditions \( I^2 > 4UJ \) must be satisfied. It is noted that \( \omega_f (\omega_s) \) represents the fast (slow) IA modes. The second-order \((m = 2\) with \( l = 1\)) equations and with the compatibility condition, we can write the group velocity of IAWs in the DPP

\[
\nu^*_s = \frac{(\lambda_2 \lambda_3 \lambda_5 k^2 S^2 + \lambda_2 \lambda_5 \omega^2 S^2 + \lambda_2 \lambda_5 S^2 + \lambda_1 A^2 k^2 + A^2 \omega^2 - 2A^2 S^2 - SA^2)}{2\omega k(A^2 + \lambda_2 \lambda_5 S^2)},
\]

(20)

Now, the co-efficient of \( \epsilon \) (when \( m = 2\) with \( l = 2\)) yields the second-order harmonic amplitudes, is found to be proportional to \( |\phi_1^{(1)}|^2 \)

\[
n^{(2)}_{-i} = M_4 |\phi_1^{(1)}|^2,
\]
\[
u^{(2)}_{-i} = M_5 |\phi_1^{(1)}|^2,
\]
\[
n^{(2)}_{i} = M_6 |\phi_1^{(1)}|^2,
\]
\[
u^{(2)}_{i} = M_7 |\phi_1^{(1)}|^2,
\]
\[
\phi_2 = M_8 |\phi_1^{(1)}|^2,
\]

(21)

(22)

(23)

(24)

(25)

where

\[
M_4 = \frac{2M_8 k^2 S^2 - 3\omega^2 k^4 - \lambda_1 k^6}{2S^3}, \quad M_5 = \frac{\omega M_4 S^2 - \omega k^4}{kS^2},
\]
\[
M_6 = \frac{2\lambda_2 M_8 A^2 k^2 + 3\lambda_2 \omega^2 k^4 + \lambda_3 \lambda_5 k^6}{2A^3}, \quad M_7 = \frac{\omega M_6 A^2 - \omega \lambda_2 k^4}{kA^2},
\]
\[
M_8 = \frac{2M_2 A^3 S^3 - 3\lambda_2 \lambda_5 \omega^2 S^3 k^4 - 3\omega^2 A^3 k^4 - \lambda_3 \lambda_5 \lambda_2^2 S^3 k^6 - \lambda_1 A^3 k^6}{2A^2 S^3(5\lambda_2 \lambda_5 k^2 - 4k^2 AS - M_1 AS - Ak^2)}.
\]
Now, we consider the expression for \((m = 3\) with \(l = 0\)) and \((m = 2\) with \(l = 0\)), which leads the zeroth harmonic modes. Thus, we obtain
\[
\begin{align*}
  n_{-i0}^{(2)} & = M_9|\phi_1^{(1)}|^2, \\
  u_{-i0}^{(2)} & = M_{10}|\phi_1^{(1)}|^2, \\
  n_{+i0}^{(2)} & = M_{11}|\phi_1^{(1)}|^2, \\
  u_{+i0}^{(2)} & = M_{12}|\phi_1^{(1)}|^2, \\
  \phi_0^{(2)} & = M_{13}|\phi_1^{(1)}|^2.
\end{align*}
\]  

where
\[
\begin{align*}
  M_9 & = \frac{2\omega \nu g k^3 + \lambda_1 k^4 + k^2 \omega^2 - M_{13} S^2}{S^2(\nu_g^2 - \lambda_1)}, \\
  M_{10} & = \frac{\lambda_2 M_{13} A^2 + \lambda_3 \lambda_4^2 k^4 + \lambda_5^2 \omega^2 k^2 + 2\omega \nu_g \lambda_2^2 k^3}{A^2(\nu_g^2 - \lambda_3)}, \\
  M_{11} & = \frac{M_{11} \nu_g A^2 - 2\omega \lambda_2^2 k^3}{A^2}, \\
  M_{12} & = \frac{A^2(\nu_g^2 - \lambda_3)(2\omega \nu_g k^3 + \lambda_1 k^4 + \omega^2 k^2) - M_{14}}{A^2 S^2 \nu_g^2 + A^2 S^2[2\lambda_5 \omega \nu_g \lambda_2^2 k^3 + \lambda_3 \lambda_5 \lambda_2^2 k^4 + \lambda_5 \lambda_2^2 \omega^2 k^2] - 2M_2 A^2 S^2(\nu_g^2 - \lambda_1)(\nu_g^2 - \lambda_3)}. \\
  M_{13} & = \frac{(\nu_g^2 - \lambda_1) S^2(2\lambda_5 \omega \nu_g \lambda_2^2 k^3 + \lambda_3 \lambda_5 \lambda_2^2 k^4 + \lambda_5 \lambda_2^2 \omega^2 k^2)}{A^2 S^2 \nu_g^2 + A^2 S^2[2\lambda_5 \omega \nu_g \lambda_2^2 k^3 + \lambda_3 \lambda_5 \lambda_2^2 k^4 + \lambda_5 \lambda_2^2 \omega^2 k^2] - 2M_2 A^2 S^2(\nu_g^2 - \lambda_1)(\nu_g^2 - \lambda_3)}.
\end{align*}
\]

Finally, the third harmonic modes \((m = 3)\) and \((l = 1)\), with the help of Equations (14)–(30), give a set of equations, which can be reduced to the following NLSE:
\[
\frac{i}{\hbar}\frac{\partial \Phi}{\partial t} + P \frac{\partial^2 \Phi}{\partial \xi^2} + Q|\Phi|^2\Phi = 0,
\]
where \(\Phi = \phi_1^{(1)}\) for simplicity. In Equation (31), \(P\) is the dispersion co-efficient, which can be written as
\[
P = \frac{4\lambda_2 \lambda_5 \omega^2 k^2 \nu_g^2 S^3 + 4\lambda_1 \omega \nu_g A^3 k^3 + 4k^2 \nu_g^3 A^3 + M_{15}}{2\omega A S^2(\nu_g^2 + \lambda_2 \lambda_5 S^2)},
\]
\[
M_{15} = 2\lambda_2 \lambda_5 \omega^2 k^2 S^3 + \lambda_2 \lambda_5 \lambda_2^2 S^3 k^4 + \lambda_1 S^2 A^3 + \lambda_2 \lambda_5 S^3 \omega^4 + \lambda_2 \lambda_5 \lambda_5 A^2 S^3 - 4\lambda_2 \lambda_5 k^2 \nu_g^2 A^3 - 4\lambda_2 \lambda_5 \lambda_5 \omega \nu_g k^2 S^3 - 4k^2 \nu_g^2 \lambda_2^2 A^3 - \lambda_2^2 A^3 k^4 - 2\lambda_2 \lambda_5 k^2 A^3 - \nu_g^2 \lambda_2^2 A^3 - A^3 \omega^4 - \lambda_2 \lambda_5 A^2 k^2 \nu_g^2 S^3 - A^3 S^3,
\]
and \(Q\) is the nonlinear co-efficient, which can be written as
\[
Q = \frac{3M_3 A^2 S^2 + 2M_2 M_6 A^2 S^2 + 2M_2 M_{15} A^2 S^2 - M_{16}}{2\omega k^2(\nu_g^2 + \lambda_2 \lambda_5 S^2)},
\]
where
\[
M_{16} = 2\omega M_5 A^2 k^3 + 2\lambda_2 \lambda_5 \omega M_7 S^2 k^3 + 2\omega M_{10} A^2 k^3 + 2\lambda_2 \lambda_5 \omega M_{12} S^2 k^3 + \lambda_1 M_4 A^2 k^4 + M_4 \omega^2 A^2 k^2 + \lambda_2 \lambda_5 M_6 S^2 k^4 + \lambda_2 \lambda_5 M_6 S^2 k^4 + M_9 \omega^2 A^2 k^2 + \lambda_1 M_6 A^2 k^4 + \lambda_2 \lambda_5 M_{11} S^2 k^4 + \lambda_2 \lambda_5 M_{11} S^2 k^4
\]
It may be noted here that both $P$ and $Q$ are directly depend on different parameters namely, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, q, k$, and are indirectly depend on the mass, number density, temperature, and charge state of the different plasma species.

4. Modulational Instability of the IAWs

The propagation of IAWs is modulational stable when $P$ and $Q$ have the opposite sign (i.e., $P/Q < 0$), and is modulationaly unstable when both $P$ and $Q$ have same sign (i.e., $P/Q > 0$) \[54,55\]. The point, at which the transition of $P/Q$ curve intersects with the $k$-axis in the “$P/Q$ versus $k$” graph, is known as threshold or critical wave number $k_c$ \[54,55\]. We have depicted the variation of $P/Q$ with $k$ for different values of $\lambda_5$ under the consideration of fast ($\omega_f$) and slow ($\omega_s$) IA modes, respectively, in Figures 1 and 2, and it can be seen from these two figures that (a) both stable and unstable parametric regimes are allowed by the plasma system; (b) the $k_c$ increases (decreases) with increasing of positive (negative) ion number density when their charge state remains constant; (c) the stable parametric regime (i.e., $P/Q < 0$) increases (decreases) with positive (negative) ion charge state when their mass remains constant.

The effects of sub-extensivity and super-extensivity of electrons on the stability condition of IAWs can be seen in Figures 3 and 4, respectively. It is obvious from these two figures that the sub-extensive property (i.e., $q > 0$) of the electrons allows the IAWs to be stable for large wave number while the super-extensive property (i.e., $q < 0$) of the electrons allows the IAWs to be stable for small a wavenumber, and this result agrees with the result of Eslami et al. \[53\].

![Figure 1](image1.png)

Figure 1. The variation of $P/Q$ with $k$ for different values of $\lambda_5$ when $\lambda_1 = 0.007, \lambda_2 = 1.2, \lambda_3 = 0.07, \lambda_4 = 1.8, \lambda_6 = 1.7, q = 1.4$, and $\omega \equiv \omega_f$.

![Figure 2](image2.png)

Figure 2. The variation of $P/Q$ with $k$ for different values of $\lambda_5$ when $\lambda_1 = 0.007, \lambda_2 = 1.2, \lambda_3 = 0.07, \lambda_4 = 1.8, \lambda_6 = 1.7, q = 1.4$, and $\omega \equiv \omega_s$. 
5. Rogue Waves

The instability induced from a small perturbation on top of a plane wave leads to an increase in the perturbation up to its highest amplitude and then to a decay so that it finally “disappears without a trace” [43–48]. The NLSE has a variety of solutions, from which the rogue wave/rational solution (developed by Darboux Transformation Scheme) is localized in both the $\xi$ and $\tau$ [43–48]. The first-order rational rogue wave solution of the NLSE in the unstable parametric regime (i.e., $P/Q > 0$) of the IAWs can be written as [43–48]

$$\Phi(\xi, \tau) = \sqrt{\frac{2P}{Q}} \left[ \frac{4 + 16i\tau P}{1 + 4\xi^2 + 16\tau^2 P^2} - 1 \right] \exp(2i\tau P).$$ (32)

It can be seen from the literature that the PI plasma system can support these conditions: $m_{-i} > m_{+i}$ (i.e., $H^+ - O_2^-$ [4–9], $Ar^+ - SF_6^-$ [17–20], and $Xe^+ - SF_6^-$ [17–20]), $m_{-i} = m_{+i}$ (i.e., $H^+ - H^-$ [4–9] and $C_{60}^+ - C_{60}^-$ [23–25]), and $m_{-i} < m_{+i}$ (i.e., $Ar^+ - F^-$ [5,6]). So, in our present investigation, we have graphically observed the variation of electrostatic potential with $\lambda_2$ under the consideration of $m_{-i} > m_{+i}$ (i.e., $\lambda_2 > 1$) in Figure 5, and it is obvious from this figure that (a) the amplitude and width of IARWs decrease with an increase in the value of the negative ion mass but increase with an increase in the value of the positive ion mass for a fixed value of their charge state; (b) the height of the IARWs increases (decreases) with negative (positive) ion charge state for a constant mass of positive and negative ion species. So, the mass and charge state of the PI play an opposite role to the formation of IARWs in PI plasma. Figure 6 describes the nature of the electrostatic IARWs with the variation of $\lambda_2$ under the consideration of $m_{-i} < m_{+i}$ (i.e., $\lambda_2 < 1$). It is clear from this figure that (a) the amplitude and width of IARWs increases (decreases) with negative (positive) ion mass when other plasma parameters are constant;
and (b) the existence of the heavy positive ion change the dynamics of the plasma system. So, the dynamics of the DPP rigourously changes with these conditions \( m_{-i} > m_{+i} \) (i.e., \( \lambda_2 > 1 \)) and \( m_{-i} < m_{+i} \) (i.e., \( \lambda_2 < 1 \)).

The nature of IARWs may also be affected by the electron and positron temperature which can be observed in Figure 7. This figure reveals that as we increase the value of electron (positron) temperature, then the IARWs associated with IAWs are taller (smaller). The physics behind this result is that the nonlinearity of the plasma medium as well as the amplitude of the IARWs decreases (increases) with positron (electron) temperature.
6. Conclusions

We have investigated the MI of IAWs and associated IARWs in a four-component DPP having inertial positive and negative ions, and inertialess non-extensive electrons and iso-thermal positrons by deriving a standard NLSE. The results that have been found from our investigation can be summarized as follows:

- Both stable and unstable parametric regimes of IAWs can be observed.
- The sub-extensive property of the electrons allows the IAWs to be stable for large wave number while the super-extensive property of the electrons allows the IAWs to be stable for small wave number.
- The dynamics of the DPP rigourously changes with these conditions \( m_{-i} > m_{+i} \) (i.e., \( \lambda_2 > 1 \)) and \( m_{-i} < m_{+i} \) (i.e., \( \lambda_2 < 1 \)).
- The nonlinearity of the plasma medium as well as the amplitude of the IARWs decreases (increases) with positron (electron) temperature.

The results of our present investigation are applicable in understanding the process of MI of IAWs and associated IARWs in both astrophysical environments, viz., Van Allen radiation belt and near the polar cap of fast rotation neutron stars [1], solar atmosphere [2], upper regions of Titan’s atmosphere [3–10], cometary comae [11], \((\text{H}^+, \text{O}^+_2, \text{O}_2^-)\) and \((\text{H}^+, \text{H}^-)\) plasmas in the D and F-regions of Earth’s ionosphere [4–9], and also in laboratory experiments namely, \((\text{Ar}^+, \text{F}^-)\) plasma [12], \((\text{K}^+, \text{SF}_6^-)\) plasma [13,14], neutral beam sources [15], plasma processing reactors [16], \((\text{Ar}^+, \text{F}^-)\) plasma [17–20], combustion products [21], plasma etching [21], \((\text{Ar}^+, \text{F}^-)\) plasma [22], \((\text{Ar}^+, \text{O}_2^-)\) plasma, Fullerene \((\text{C}_60^+, \text{C}_60^-)\) plasma [23–25], RF glow discharges [16], and Tokamak [26], etc.

Author Contributions: All authors contributed equally to complete this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing not applicable—no new data generated.

Acknowledgments: The authors are also grateful to the anonymous reviewers for their constructive suggestions which have significantly improved the quality of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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