A nonlocal operator method for solving PDEs

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Abstract

We propose a nonlocal operator method for solving PDEs. The nonlocal operator is derived from the Taylor series expansion of an unknown field, and can be regarded as the integral form “equivalent” to the differential form in the sense of nonlocal interaction. The variation of a nonlocal operator is similar to the derivative of shape function in meshless and finite element methods, thus circumvents difficulty in the calculation of shape function and its derivatives. The nonlocal operator method is consistent with the variational principle and the weighted residual method, based on which the residual and the tangent stiffness matrix can be obtained with ease. The nonlocal operator method is equipped with an hourglass energy functional to satisfy the linear consistent of the field. High-order nonlocal operators and high-order hourglass energy functional are generalized. The functional based on the nonlocal operator converts the construction of residual and stiffness matrix into a series of matrix multiplications on the nonlocal operator. The nonlocal strong forms of different functionals can be obtained easily with the aid of support and dual-support, two basic concepts introduced in the paper. Several numerical examples are presented to validate the method.

Keywords: dual-support, nonlocal operators, hourglass energy functional, nonlocal strong form, variational principles, weighted residual method

1. Introduction

In the field of solving PDEs numerically, many methods have been proposed, which include finite element method \cite{1}, Smoothed Particle Hydrodynamics (SPH) \cite{2}, Diffusive Element Method (DEM) \cite{3}, Element-Free Galerkin (EFG) method \cite{4}, Reproducing Kernel

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Particle Method (RKPM) [5], Partition of Unity Methods (PUM) [6], Generalized Finite Element Method (GFEM) [7], hp clouds (HPC) [8], finite point method [9], Generalized Finite Difference Method (GFDM) [10], the reproducing kernel collocation method [11, 12], Peridynamics [13, 14], etc.

Smoothed particle hydrodynamics developed by Lucy [2] in 1977 and Gingold and Monaghan [15] is one of the earliest meshless methods. The basic idea of SPH is the kernel approximation which estimates a function on support domain. While the continuous form of SPH meshless approximation is zeroth-order complete, the discrete SPH form cannot even reproduce constant fields. In order to overcome the difficulty in SPH, some other meshless methods have been proposed, two of the most famous methods are Moving Least Squares (MLS) and RKPM. The objective of MLS is to obtain an approximation based on the nodes in support, but with high accuracy and high order of completeness. The MLS method was first introduced by Lancaster and Salkauskas [16] in curve and surface fitting. Further developments are made in the Element-Free Galerkin method by Belytschko et al. [4]. However, MLS does not satisfy the Kronecker delta criterion and the imposition of essential boundary conditions is more complicated compared with finite element method. The RKPM proposed by Liu et al. [5] improves the continuous SPH approximation by increasing the order of completeness of the approximation with a correction function. While started from a very different point, RKPM turns to be an equivalent procedure with the EFG method. More review of meshless methods, we refer to [17, 18].

Galerkin meshfree methods start from a field value interpolated by the nearby unknown nodal values, where the interpolation is described by shape functions, i.e. \( u_h(x_a) = N_b(x_a) u_b \). Such interpolation often violates the Kronecker delta property, and confronts certain difficulty in satisfying essential or natural boundary conditions. The derivative of a shape function must be calculated implicitly or explicitly.

It is well known that the difficulty in solving PDEs arises from the differential operators, while the handling of other terms beside differential operators is relatively easy. How to describe the different differential operators is the central topic for different methods, including the meshless method, finite element method, finite difference method. The differential operators correspond to the derivatives of the shape function, while the non-differential terms relate to the shape function. For many methods, the construction of the shape function is complicated, let alone the derivatives of the shape function. In this sense, it is desirable to jump into the derivatives directly while ignoring the shape functions.

On the other hand, method based on the local differential operator confronts inconveniences or difficulties when its definition doesn’t exist for problems involving strong/weak discontinuity. For these problems, many meshless methods or finite element methods (i.e. extended finite element method [19]) need special treatment to construct the shape function and to calculate the derivatives of the shape function. To circumvent the difficulties in methods based on local differential operators, many nonlocal theories have been proposed, among which include the nonlocal continuum field theories for different physical fields [20], peridynamics [21], nonlocal integral form for plasticity and damage [22] and nonlocal vector calculus [23], to name a few. The nonlocal theory is based on the integral form with a finite intrinsic length scale, while the definition of a local differential operator is based
on the intrinsic length scale approaching infinitesimal. On the other hand, nonlocal form doesn’t need the shape function nor its derivatives. Peridynamics proposed by Silling [21] reformulates the elasticity theory into the integral form to account for the long range forces, which overcomes the difficulty to define the local derivatives for fractures. Comparing with the local theory, nonlocal theory not only has well-posedness in numerical aspect, but also approaches the real physical process better with an intrinsic length scale [20, 21, 22].

Mathematically, a nonlocal equation is a relation for which the information about the value of the function far from that point is required, in contrast with the differential equations describing relations between the values of an unknown function and its derivatives of different orders. The common scenario for nonlocal equation is the equation involving integral operators, i.e.

$$u_t(t,x) = \int_\Omega (u(t,x+y) - u(t,x))k(x,y)dy,$$

for some kernel $k$, the integral operator is termed as nonlocal operator.

Another example is the nonlocal second-order scalar “elliptic boundary-value” problem,

$$\mathcal{L}(u)(x) := 2\int_\Omega (u(x') - u(x))k(x,x')dx' = b(x) \quad \text{in } \Omega,$$

augmented with nonlocal “Dirichlet” or “Neumann” boundary condition, where $k(x,x')$ is the kernel function [23, 24]. The corresponded local form of Eq.2 is the second-order scalar elliptic boundary-value problem,

$$-\nabla \cdot (D(x) \cdot \nabla u(x)) = b(x) \quad \text{in } \Omega,$$

with Dirichlet or Neumann boundary conditions on the boundary $\partial \Omega$, where $D$ is a symmetric, positive definite, second-order tensor, $b$ a scalar-valued data function. When the length scale decreases to 0, the nonlocal form degenerates to the local form [24]. In the nonlocal equation, a point interacts with another point of finite distance, the intensity of interaction is related to the difference of field values of two points as indicated in Eq.1 and Eq.2. The definition of the differential operator in PDEs resembles to the interaction between two points with finite distance. We take the derivative of a scalar field for example, for vector or tensor field the Fréchet or Gâteaux derivative can be applied. The derivative of scalar field $u(x)$ is defined as

$$u'(x) = \lim_{y \to x} \frac{u(y) - u(x)}{y - x}.$$ 

The derivative is the limit of the difference of two points on their relative distance. Without seeking the limit, the nonlocal form based on the sum of weighted finite difference emerges naturally,

$$\mathcal{L}(u)(x) := \int_{y \in S_x} \frac{u(y) - u(x)}{y - x}k(x,y)dy,$$

where $k(x,y)$ is the weight function or kernel function, $S_x$ is the support. When the nonlocal length scale decreases to zero or $S_x \to 0$, $\mathcal{L}(u)(x) \to u'(x)$, which can be verified by Taylor series expansion. Nonlocal operator provides a direct way to construct the differential
operator, though the concept of “locality” is a special case of “nonlocality”. Based on this basic observation, we construct several nonlocal differential operators to replace the local differential operators to solve PDEs. In the nonlocal operator method, the field value is defined on the node, the shape function based on interpolation method is not required. The differential operator on field value is considered as the nonlocal interactions between the points in support domain.

The purpose of the paper is to propose a nonlocal operator method for solving PDEs based on weighed residual method and variational principles. Though the nonlocal theory is more general than the local theory, we focus on solving the local problems with the nonlocal operator method. The nonlocal operator method constructs the nonlocal operator to represent the nonlocal interaction without shape function or its derivatives in traditional meshless or finite element method. The remainder of the paper is outlined as follows. In §2, the concepts of support and dual-support are introduced. Based on the support, the general nonlocal operator and its variation in continuous form or discrete form are defined. In §3, we discuss the hourglass mode in the nonlocal operator and propose a universal hourglass energy functional to remove the hourglass mode. The high-order nonlocal operators and high-order hourglass energy functional are generalized and obtained in §4. Since the nonlocal operators can be considered as the general templates, we discuss the combination of the nonlocal operator and weighted residual method to solve PDEs in §5. On the other hand, since the differential operator forms the basis of different energy functionals, we study the capabilities of the nonlocal operator based on variational principles in obtaining the strong forms or weak forms of different functionals in §6. Some numerical examples are presented to validate the method in §7. We conclude in §8.

2. Support, dual-support and nonlocal operators

\begin{align*}
\Omega &= \{x_1, x_2, x_4, x_6\}, \quad \mathcal{S}_x = \{x_1, x_2, x_3, x_4\}.
\end{align*}

Figure 1: (a) Domain and notations. (b) Schematic diagram for support and dual-support, all shapes above are support. \(\mathcal{S}_x = \{x_1, x_2, x_3, x_4\}\).
Consider a domain as shown in Fig.1(a), let $x$ be spatial coordinates in the domain $\Omega$; $r := x' - x$ is a Euclidean vector (or a spatial vector, or simply a vector) starts from $x$ to $x'$; $v := v(x, t)$ and $v' := v(x', t)$ are the field values for $x$ and $x'$, respectively; $v_r := v' - v$ is the relative field vector for spatial vector $r$.

**Support** $S_x$ of point $x$ is the domain where any spatial point $x'$ forms spatial vector $r(= x' - x)$ from $x$ to $x'$. Support $S_x$ specifies the range of nonlocal interaction happened with respect to point $x$. The main function of support is to define different nonlocal operators. In mathematics, a specific quantitative measure of the support can be described by a moment (shape tensor). A $n$-order moment of the support is defined as

$$K^n_x = \int_{S_x} w(r) r \otimes r \otimes \cdots \otimes r dV_{x'},$$

where $w(r)$ is the weight function. Two special cases of the shape tensor for $S_x$ are the $0$-order shape tensor (the weighted volume of the support) and the $2$-order shape tensor

$$K_x = \int_{S_x} w(r)r \otimes rdV_{x'}.$$  

**Dual-support** is defined as a union of the points whose supports include $x$, denoted by $S'_x = \{x' | x \in S_{x'}\}$.

The point $x'$ forms dual-vector $r'(= x - x' = -r)$ in $S'_x$. On the other hand, $r'$ is the spatial vector formed in $S_{x'}$. One example to illustrate the support and dual-support is shown in Fig.1(b).

### 2.1. Nonlocal operators in support

The general operators in calculus include the gradient of scalar and vector field, the curl and divergence of vector field. The definitions of some nonlocal operators can be found in reference [24]. These operators have the corresponding nonlocal forms based on the Taylor series expansion. We use $\bar{\nabla}$ to denote the nonlocal operator, while the local operators follow the conventional notations.

The nonlocal gradient of a vector field $v$ for point $x$ in support $S_x$ is defined as

$$\bar{\nabla}v_x = \int_{S_x} w(r)v_r \otimes r dV_{x'} \cdot K^{-1}_x,$$

where $v_r = v_{x'} - v_x$, $K_x$ is the $2$-order shape tensor. One example of the nonlocal gradient is the nonlocal deformation gradient in Peridynamics [13]. The nonlocal curl of a vector field $v$ for point $x$ in support $S_x$ is defined as

$$\bar{\nabla} \times v_x = K^{-1}_x \cdot \int_{S_x} w(r)r \times v_r dV_{x'}.$$
The nonlocal divergence of a vector field \( \mathbf{v} \) for point \( \mathbf{x} \) in support \( S_x \) is defined as
\[
\nabla \cdot \mathbf{v}_x = \int_{S_x} w(\mathbf{r})(K_\mathbf{x}^{-1}\mathbf{r}) \cdot \mathbf{v}_r d\mathbf{V}_x'.
\] (8)

The nonlocal gradient of a scalar field \( v \) for point \( \mathbf{x} \) in support \( S_x \) is defined as
\[
\nabla v_x = \int_{S_x} w(\mathbf{r})v_r \mathbf{r} d\mathbf{V}_x' \cdot K_\mathbf{x}^{-1},
\] (9)

where \( v_r = v_{x'} - v_x \).

In fact, the field value of nearby point \( \mathbf{x}' \) in \( S_x \) is obtained by Taylor series expansion as
\[
v_{\mathbf{x}'} = v_x + \nabla v_x \cdot \mathbf{r} + O(r^2),
\] (10)

where \( O(r^2) \) represents order terms higher than one, and for linear field \( O(r^2) = 0 \). Insert Eq.10 into RHS of Eq.6, Eq.7, Eq.8 and integrate in support \( S_x \), one verifies that the nonlocal operator converges to the local operator. For example,
\[
\nabla v_x = \int_{S_x} w(\mathbf{r})v_r \otimes \mathbf{r} d\mathbf{V}_x' \cdot K_\mathbf{x}^{-1}
\]
\[
= \int_{S_x} w(\mathbf{r})(v_{\mathbf{x}'} - v_x) \otimes \mathbf{r} d\mathbf{V}_x' \cdot K_\mathbf{x}^{-1}
\]
\[
= \int_{S_x} w(\mathbf{r})\nabla v_x \cdot \mathbf{r} \otimes \mathbf{r} d\mathbf{V}_x' \cdot K_\mathbf{x}^{-1}
\]
\[
= \nabla v_x \cdot \int_{S_x} w(\mathbf{r})\mathbf{r} \otimes \mathbf{r} d\mathbf{V}_x' \cdot K_\mathbf{x}^{-1}
\]
\[
= \nabla v_x \cdot K_\mathbf{x} \cdot K_\mathbf{x}^{-1}
\]
\[
= \nabla v_x.
\]

When \( \mathbf{x}' \) is close enough to \( \mathbf{x} \) or when support \( S_x \) is small enough, the nonlocal operator can be considered as the linearization of the field.

The nonlocal operator converges to the local operator in the continuous limit. On the other hand, the nonlocal operator defined by integral form, still holds in the case where strong discontinuity exists and the local operator can’t be defined. The local operator can be viewed as a special case of the nonlocal operator.

2.2. Variation of the nonlocal operator

We present the variation of the general nonlocal operator in continuous form and discrete form. The discrete form is beneficial for the numerical implementation. Different nonlocal operators can be used to replace the local differential operators in PDEs, especially in the framework of weighted residual method and variational principles. We use the \( \delta \) to denote the variation.
2.2.1. Notations for variation

The nonlocal operators defined above are in vector or tensor form. The variation of the nonlocal operators leads to a higher-order tensor form, which is not convenient for implementation. We need to express the high-order tensor into to vector or matrix form. Before we derive the variation of nonlocal operator, some notation to denote the variation and how the variations are related to the first- and second-order derivatives are to be discussed.

Assuming a functional $F(u, v)$, where $u, v$ are unknown functions in unknown vector $[u, v]$, the first and second variation can be expressed as

$$
\delta F(u, v) = \partial_u F \delta u + \partial_v F \delta v = \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}
$$

$$
\delta^2 F(u, v) = \partial_{uu} F \delta u \delta u + \partial_{uv} F \delta u \delta v + \partial_{vu} F \delta v \delta u + \partial_{vv} F \delta v \delta v = \begin{bmatrix} \partial_{uu} F & \partial_{uv} F \\ \partial_{vu} F & \partial_{vv} F \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} \otimes \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}
$$

It can be seen that, the second variation $\delta^2 F(u, v)$ is the double inner product of the hessian matrix and the tensor formed by the variation of the unknowns, while the first variation $\delta F(u, v)$ is inner product of the gradient vector and the variation of the unknowns. The gradient vector and hessian matrix represent the residual vector and tangent stiffness matrix of the functional, respectively, with unknown functions $u, v$ being the independent variables,

$$
R = \nabla [u, v] F(u, v) = [\partial_u F, \partial_v F]
$$

$$
K = \nabla^2 [u, v] F(u, v) = \begin{bmatrix} \partial_{uu} F & \partial_{uv} F \\ \partial_{vu} F & \partial_{vv} F \end{bmatrix}.
$$

The inner product or double inner product indicates that location of an element in the residual or the tangent stiffness matrix corresponds to the location of the unknowns with variation.

In this paper, we use a special variation $\bar{\delta}$, whose function is illustrated by the following examples. The special variations of functional $F(u, v)$ are given as

$$
\bar{\delta} F(u, v) = \partial_u F \bar{\delta} u + \partial_v F \bar{\delta} v = [\partial_u F, \partial_v F]
$$

$$
\bar{\delta}^2 F(u, v) = \partial_{uu} F \bar{\delta} u \bar{\delta} u + \partial_{uv} F \bar{\delta} u \bar{\delta} v + \partial_{vu} F \bar{\delta} v \bar{\delta} u + \partial_{vv} F \bar{\delta} v \bar{\delta} v = \begin{bmatrix} \partial_{uu} F & \partial_{uv} F \\ \partial_{vu} F & \partial_{vv} F \end{bmatrix}
$$

where $\bar{\delta} u$ denotes the index of $\partial_u F$ in residual vector by the index of $u$ in the unknown vector. For example, the term $\partial_v F \bar{\delta} v$ represents $\partial_v F$ be in the second location of the residual vector since $v$ is in the second position of $[u, v]$. The term $\partial_{uv} F \bar{\delta} u \bar{\delta} v$ denotes that the location of $\partial_{uv} F$ is $(1, 2)$, while the term $\partial_{vu} F \bar{\delta} v \bar{\delta} u$ denotes that the location of $\partial_{vu} F$ is $(2, 1)$. 

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Obviously,
\[ \delta F(u, v) = \bar{\delta F}(u, v) \frac{\delta u}{\delta v}, \delta^2 F(u, v) = \bar{\delta^2 F}(u, v) : \begin{bmatrix} \delta u \delta u & \delta u \delta v \\ \delta v \delta u & \delta v \delta v \end{bmatrix} \]
\[ K = \bar{\delta^2 F}(u, v), R = \bar{\delta F}(u, v) \]
The special one-order and two-order variation of a functional lead to the residual and tangent stiffness matrix directly. The traditional variation can be recovered by the inner product of the special variation and the variation of the unknown vector.

2.2.2. Nonlocal divergence operator

The variation of \( \bar{\nabla} \cdot \bar{\mathbf{F}} \) is given by
\[ \bar{\nabla} \cdot \delta \bar{\mathbf{F}} = \int_{s} w(r)(K^{-1}_\mathbf{x} \bar{r}) \cdot (\delta \mathbf{F}_\mathbf{x} - \delta \bar{\mathbf{F}}) dV_\mathbf{x}, \quad (11) \]
The number of dimensions of \( \bar{\nabla} \cdot \delta \bar{\mathbf{F}} \) is infinite, and discretization is required.

After discretization of the domain by particles, the whole domain is represented by
\[ \Omega = \sum_{i=1}^{N_{\text{node}}} \Delta V_i \]
where \( i \) is the global index of volume \( \Delta V_i \), \( N_{\text{node}} \) is the number of particles in \( \Omega \).

Particles in \( S_i \) are represented by
\[ N_i = \{i, j_1, ..., j_k, ..., j_n_i\} \]
where \( j_1, ..., j_k, ..., j_n_i \) are the global indices of neighbors of particle \( i \). The discrete form of \( \bar{\nabla} \cdot \delta \bar{\mathbf{F}} \) can be written as
\[ \bar{\nabla} \cdot \delta \bar{\mathbf{F}} \simeq \sum_{j_k \in S_i} w(r)\Delta V_{j_k}(K^{-1}_\mathbf{x} \bar{r}) \cdot (\delta \mathbf{F}_{j_k} - \delta \bar{\mathbf{F}}_i) = \bar{\nabla} \cdot \delta \bar{\mathbf{F}} \cdot \delta \mathbf{F}_{N_i}, \quad (14) \]
where \( \simeq \) denotes discretization, \( \delta \mathbf{F}_{N_i} \) is all the variations of the unknowns in support \( S_i \),
\[ \delta \mathbf{F}_{N_i} = (\delta \mathbf{F}_i, \delta \mathbf{F}_{j_1}, ..., \delta \mathbf{F}_{j_k}, ..., \delta \mathbf{F}_{j_n_i}) \]
\( \bar{\nabla} \cdot \delta \bar{\mathbf{F}} \) is the coefficient vector with length of \( 3(n_i + 1) \) in 3D case,
\[ \bar{\nabla} \cdot \delta \bar{\mathbf{F}}_i = \sum_{j_k \in S_i} w(r)\Delta V_{j_k}(K^{-1}_\mathbf{x} \bar{r}) \cdot (\delta \mathbf{F}_{j_k} - \delta \bar{\mathbf{F}}_i). \quad (16) \]
Based on the indices of \( \delta \mathbf{F}_{j_k} \) in \( \delta \mathbf{F}_{N_i} \), \( \bar{\nabla} \cdot \delta \bar{\mathbf{F}}_i \) can be obtained by
\[ \bar{\nabla} \cdot \delta \bar{\mathbf{F}}_i[3k, 3k+1, 3k+2] = w(r)\Delta V_{j_k}K^{-1}_\mathbf{x} \bar{r}, \bar{\nabla} \cdot \delta \bar{\mathbf{F}}_i[0, 1, 2] = -\sum_{k=1}^{n_i} w(r)\Delta V_{j_k}K^{-1}_\mathbf{x} \bar{r}, \quad (17) \]
where \( k \) is the index of particle \( j_k \) in \( N_i \). The process to obtain \( \bar{\nabla} \cdot \delta \bar{\mathbf{F}}_i \) on nodal level is sometimes called nodal assembly. In the following section, we mainly discuss the special variation of the nonlocal operator and functional, while the actual variation can be recovered with ease.
2.2.3. Nonlocal curl operator

The variation of $\nabla \times \hat{\mathbf{F}}_i$ in discrete form reads

$$\nabla \times \delta \hat{\mathbf{F}}_i \simeq \sum_{j_k \in S_i} w(r) \Delta V_{j_k} \mathbf{K}_{i}^{-1} \cdot \mathbf{r} \times (\delta \mathbf{F}_{j_k} - \delta \mathbf{F}_i),$$

(18)

where $\Delta V_{j_k}$ is the volume for particle $j_k$. For the 3D case, $\nabla \times \delta \hat{\mathbf{F}}_i$ is a $3 \times 3(n_i + 1)$ matrix, where $n_i$ is the number of neighbors in $S_i$. $N_i$ is given by Eq.13. For each particle $j_k$ in $N_i$ calculating $R_{j_k} = w(r) \Delta V_{j_k} \mathbf{K}_{i}^{-1} \mathbf{r}$, we obtain

$$\nabla \times \delta \hat{\mathbf{F}}_i[1, 3k] = R_{j_k}[2], \quad \nabla \times \delta \hat{\mathbf{F}}_i[2, 3k] = -R_{j_k}[1], \quad \nabla \times \delta \hat{\mathbf{F}}_i[0, 3k + 1] = -R_{j_k}[2]$$

$$\nabla \times \delta \hat{\mathbf{F}}_i[2, 3k + 1] = R_{j_k}[0], \quad \nabla \times \delta \hat{\mathbf{F}}_i[0, 3k + 2] = R_{j_k}[1], \quad \nabla \times \delta \hat{\mathbf{F}}_i[1, 3k + 2] = -R_{j_k}[0]$$

$$\nabla \times \delta \hat{\mathbf{F}}_i[1, 0] = -\sum_{k=1}^{n_i} R_{j_k}[2], \quad \nabla \times \delta \hat{\mathbf{F}}_i[2, 0] = \sum_{k=1}^{n_i} R_{j_k}[1], \quad \nabla \times \delta \hat{\mathbf{F}}_i[0, 1] = \sum_{k=1}^{n_i} R_{j_k}[2],$$

$$\nabla \times \delta \hat{\mathbf{F}}_i[2, 1] = -\sum_{k=1}^{n_i} R_{j_k}[0], \quad \nabla \times \delta \hat{\mathbf{F}}_i[0, 2] = -\sum_{k=1}^{n_i} R_{j_k}[1], \quad \nabla \times \delta \hat{\mathbf{F}}_i[1, 2] = \sum_{k=1}^{n_i} R_{j_k}[0],$$

(19)

where $k$ is the index of particle $j_k$ in $N_i$. The minus sign denotes the reaction from the dual-support, which guarantees the regularity of the stiffness matrix in the absence of external constraints. The nodal assembly for the variation of the vector cross product can be finally obtained by

$$\mathbf{F}^x = \{R_0, R_1, R_2\} \times \{F_0, F_1, F_2\} = \{F_2R_1 - F_1R_2, F_0R_2 - F_2R_0, F_1R_0 - F_0R_1\}$$

(20)

while the gradient of $\mathbf{F}^x$ on $\{F_0, F_1, F_2\}$ is given by

$$\begin{bmatrix}
\frac{\partial F_0^x}{\partial F_0} & \frac{\partial F_0^x}{\partial F_1} & \frac{\partial F_0^x}{\partial F_2} \\
\frac{\partial F_1^x}{\partial F_0} & \frac{\partial F_1^x}{\partial F_1} & \frac{\partial F_1^x}{\partial F_2} \\
\frac{\partial F_2^x}{\partial F_0} & \frac{\partial F_2^x}{\partial F_1} & \frac{\partial F_2^x}{\partial F_2}
\end{bmatrix} = \begin{bmatrix}
0 & -R_2 & R_1 \\
R_2 & 0 & -R_0 \\
-R_1 & R_0 & 0
\end{bmatrix}. $$

(21)

The indices of $R$ correspond to their locations in $\mathbf{F}^x$.

2.2.4. Nonlocal gradient operator for vector field

Similarly, the variation of $\nabla \mathbf{F}_i$ in the discrete form reads

$$\nabla \delta \mathbf{F}_i \simeq \sum_{j_k \in S_i} w(r)(\delta \mathbf{F}_{j_k} - \delta \mathbf{F}_i) \otimes (\mathbf{K}_{i}^{-1} \mathbf{r}) \Delta V_{j_k}$$

(22)

where $\Delta V_{j_k}$ is the volume for particle $j_k$. In 3D, $\nabla \delta \mathbf{F}_i$ is a $9 \times 3(n_i + 1)$ matrix, where $n_i$ is the number of neighbors in $S_i$, $N_i$ is given by Eq.13. For each particle in the neighbor list
with $R_{jk} = w(\mathbf{r})\Delta V_{jk}\mathbf{K}_i^{-1}\mathbf{r}$, the terms in $R_{jk}$ can be added to the $\nabla \bar{\delta} \mathbf{F}_i$ as

\[
\nabla \bar{\delta} \mathbf{F}_i[0, 0] = R_{jk}[0], \quad \nabla \bar{\delta} \mathbf{F}_i[0, 3k] = R_{jk}[1], \quad \nabla \bar{\delta} \mathbf{F}_i[6, 3k] = R_{jk}[2], \\
\nabla \bar{\delta} \mathbf{F}_i[1, 1] = R_{jk}[0], \quad \nabla \bar{\delta} \mathbf{F}_i[4, 3k + 1] = R_{jk}[1], \quad \nabla \bar{\delta} \mathbf{F}_i[7, 3k + 1] = R_{jk}[2], \\
\nabla \bar{\delta} \mathbf{F}_i[2, 3k + 2] = R_{jk}[0], \quad \nabla \bar{\delta} \mathbf{F}_i[5, 3k + 2] = R_{jk}[1], \quad \nabla \bar{\delta} \mathbf{F}_i[8, 3k + 2] = R_{jk}[2],
\]

where $k$ is the index of particle $j_k$ in $N_i$. The sub-index of $\nabla \bar{\delta} \mathbf{F}_i$ can be obtained by the way similar to Eq. (21).

\[2.2.5. \text{Nonlocal gradient operator for scalar field} \]

The variation of $\bar{\nabla} v_i$ reads

\[\nabla \bar{\delta} v_i \simeq \sum_{S_i} w(\mathbf{r})\Delta V_{jk}\mathbf{K}_i^{-1}\mathbf{r}(\bar{\delta} v_{jk} - \bar{\delta} v_i), \quad (24)\]

where $\Delta V_{jk}$ is the volume for particle $x'$. For 3D case, the dimensions of $(\nabla \bar{\delta} v_i)$ are $3 \times (n_i + 1)$, where $n_i$ is the number of neighbors in $S_i$, $N_i$ is given by Eq. (13) For each particle in the neighbor list with $R_{jk} = w(\mathbf{r})\Delta V_{jk}\mathbf{K}_i^{-1}\mathbf{r}$, the terms in $R_{jk}$ can be added to the $\nabla \bar{\delta} \mathbf{F}_i$ as

\[
\nabla \bar{\delta} v_i[0, k] = R_{jk}[0], \quad \nabla \bar{\delta} v_i[1, k] = R_{jk}[1], \quad \nabla \bar{\delta} v_i[2, k] = R_{jk}[2], \\
\n\nabla \bar{\delta} v_i[0, 0] = -\sum_{k=1}^{n_i} R_{jk}[0], \quad \nabla \bar{\delta} v_i[1, 0] = -\sum_{k=1}^{n_i} R_{jk}[1], \quad \nabla \bar{\delta} v_i[2, 0] = -\sum_{k=1}^{n_i} R_{jk}[2],
\]

where $k$ is the index of particle $j_k$ in $N_i$.

It can be seen that the basic element in the nodal assembly is $w(\mathbf{r})\Delta V_{jk}\mathbf{K}_i^{-1}\mathbf{r}$, which is quite similar to the derivative of shape function in meshless or finite element method.

\[3. \text{Hourglass energy functional} \]

The same nonlocal operator can be defined by several configurations, for example, the initial configuration Fig. (2a) with a rigid translation $\Delta u$ for the up and down particles turns into Fig. (2b). It is easy to verify that the nonlocal gradient of $u$ from Eq. (6) is zero, the same as that in the initial configuration. It can be seen that for the same field gradient, the configuration is not unique, where the extra deformation not accounted by the gradient is called the hourglass mode, which has zero energy contribution for the energy functional.
The hourglass mode in the nonlocal operator is due to the deformation vectors counteracting with each other during the summation.

In order to remove the hourglass mode (zero-energy mode), we propose a penalty energy functional to achieve the linear field of the vector field. The penalty energy functional is defined as the weighted difference square between current value of a point and the value predicted by the field gradient. In fact, the vector field in the neighborhood of a particle is required to be linear. Therefore, it has to be exactly described by the gradient of the vector field, which is not described by the vector field gradient. In practice, the difference of current deformed vector $v_r$ and predicted vector by field gradient ($F_r$ in Eq.[6]) is $\Delta u = v_r - F_r$. We formulate the hourglass energy based on the difference in support as follows. Let $\alpha = \frac{\mu}{m_K}$ be a coefficient for the hourglass energy, where $m_K = \text{tr}(K)$, $\mu$ is the penalty coefficient, the functional for zero-energy mode is

$$E^{hg} = \alpha \int_S w(r)(F_r - v_r)^T(F_r - v_r)dV$$

$$= \alpha \int_S w(r)\left(r^T F_r F_r + v_r^T v_r - 2v_r^T F_r \right)dV$$

$$= \alpha \int_S w(r)\left(F_r^T F : r \otimes r + v_r^T v_r - 2F_r^T F_r \right)dV$$

$$= \alpha F_r^T F : K + \alpha \int_S w(r)v_r \cdot v_r dV - 2\alpha F : (FK)$$

$$= \frac{\mu}{m_K} \left( \int_S w(r)v_r \cdot v_r dV - F : FK \right).$$

The above definition of hourglass energy is similar to the variance in probability theory and statistics. In above derivation, we used the relations: $F_r^T F : K = F : (FK), a^T Mb = M : a \otimes b, A : B = \text{tr}(AB^T)$, where capital letter denotes matrix and small letter is column vector. The purpose of $m_K$ is to make the energy functional independent with the support.
since shape tensor $K$ is involved in $F^T F : K$. It should be noted that the zero-energy functional is valid in any dimensions and there is no limitation on the shape of the support.

The variation of $\delta(F : FK)$ can be rewritten as

$$\bar{\delta}(F : FK) = \bar{\delta}(FK : F) = 2FK : \bar{\delta}F$$
$$= 2FK : \int_S w(r)\bar{\delta}v_r \otimes (K^{-1}r) dV$$
$$= 2 \int_S w(r)\bar{\delta}v^T_r FK(K^{-1}r) dV$$
$$= 2 \int_S w(r)\bar{\delta}v^T_r (Fr) dV$$
$$= 2 \int_S w(r)(Fr) \cdot \bar{\delta}v_r dV. \quad (27)$$

Then the variation of $E^{hg}$ is

$$R^{hg} = \bar{\delta} E^{hg}$$
$$= \frac{\mu}{m_K} \left( \int_S w(r)(\bar{\delta}v_r \cdot v_r) dV - \bar{\delta}(F : FK) \right)$$
$$= \frac{\mu}{m_K} \left( \int_S 2w(r)v_r \cdot \bar{\delta}v_r dV - 2 \int_S w(r)(Fr) \cdot \bar{\delta}v_r dV \right)$$
$$= \frac{2\mu}{m_K} \int_S w(r)(v_r - Fr) \cdot (\bar{\delta}v' - \bar{\delta}v) dV. \quad (28)$$

$R^{hg}$ is the residual for hourglass energy. For a consistent field, the residual for hourglass mode is zero.

The hourglass control for individual spatial vector $r$ can be written as

$$T^{hg}_r = \partial_r E^{hg} = \frac{2\mu}{m_K} w(r)(v_r - F_x r). \quad (29)$$

Eq. 28 gives the explicit formula for the hourglass force. The term on $\bar{\delta}v$ is the hourglass term from its support, while the terms on $\bar{\delta}v'$ are the hourglass terms for the dual support $S'_{x'}$ of point $x'$. The second variation of $E^{hg}$ at a point is the tangent stiffness matrix,

$$K^{hg} = \delta^2 E^{hg} = \frac{2\mu}{m_K} \left( \int_S w(r)(\bar{\delta}v' - \bar{\delta}v)^T (\bar{\delta}v' - \bar{\delta}v) dV - \bar{\delta}FK : \bar{\delta}F \right). \quad (30)$$

The second variation of the zero-energy functional is the stiffness matrix, which is constant and can solve the rank deficiency in nodal integration method.

The above equations indicate that when the unknown field is consistent with the field gradient, the hourglass energy residual is zero. In this sense, hourglass energy functional contributes to the linear completeness in meshless method.
4. High-order nonlocal operators and hourglass energy functional

4.1. High-order nonlocal operators

For certain problem like the plate/shell theory, the hessian matrix of scalar field is required. In this case, the 2-order nonlocal hessian operator and its variation are necessary. In this section, we extend the linear nonlocal operator to high-order ones.

For a scalar field \( u \), the Taylor series expansion can be written as

\[
u_r = \nabla u \cdot r + \frac{1}{2!} \nabla^2 u : r^2 + \frac{1}{3!} \nabla^3 u : r^3 + \cdots + \frac{1}{n!} \nabla^n u \cdot (n) r^n, \tag{31}\]

where \( (2) = \cdot, (3) = : \) and \( (n) \) is the generalization of high-order inner product,

\[
\cdot^n := r \otimes r \otimes \cdots \otimes r. \tag{26}
\]

The \( n \)-order shape tensor is rewritten as

\[
K_n = \int_S w(r) r^n dV. \tag{32}
\]

For simplicity, we consider the hessian nonlocal operator

\[
\frac{1}{2!} \nabla^2 u : r^2 = u_r - \nabla u \cdot r \tag{33}
\]

The sum of weighted tensor of Eq. \( 33 \) multiplying \( r^2 \) in support is

\[
\frac{1}{2!} \nabla^2 u : \int_S w(r) r^4 dV = \int_S w(r)(u_r r^2 - \nabla u \cdot r^3) dV. \tag{34}
\]

The weighted sum can be simplified as follows.

\[
\frac{1}{2!} \nabla^2 u : K_4 = \int_S w(r)(u_r r^2 - \nabla u \cdot r^3) dV
\]

\[
= \sum_S w(r) \Delta V' (u_r r^2 - \nabla u \cdot r^3)
\]

\[
= \sum_S w(r) \Delta V' u_r r^2 - \nabla u \cdot \sum_S w(r) \Delta V' r^3
\]

\[
= \sum_S w(r) \Delta V' u_r r^2 - \sum_S w(r) \Delta V' u_r \cdot K_3 \cdot K_2^{-1} \cdot K_3
\]

\[
= \sum_S w(r) \Delta V' u_r r^2 - \sum_S w(r) \Delta V' u_r K_3 K_2^{-1} r
\]

\[
= \sum_S w(r) \Delta V' u_r (r^2 - K_3 K_2^{-1} r)
\]

\[
= \int_S w(r) u_r (r^2 - K_3 K_2^{-1} r) dV. \tag{35}
\]
4.2. High-order hourglass energy functional

Consider a $n$-order hourglass energy functional defined by

$$E_{n}^{hg} = \alpha \int_{S} w(r) \left( \nabla u \cdot r + \frac{1}{2!} \nabla^{2} u : r^{2} + \frac{1}{3!} \nabla^{3} u : r^{3} + \ldots + \frac{1}{n!} \nabla^{n} u .(n) \cdot r^{n} - u_{r} \right)^{2} dV. \quad (38)$$

Let

$$S_{n-1} = u_{r} - (\nabla u \cdot r + \frac{1}{2!} \nabla^{2} u : r^{2} + \frac{1}{3!} \nabla^{3} u : r^{3} + \ldots + \frac{1}{(n-1)!} \nabla^{n-1} u .(n-1) r^{n-1}). \quad (39)$$

In the $n$-order Taylor series expansion of $u$, $S_{n-1} \approx \frac{1}{n!} \nabla^{n} u .(n) r^{n}$.

On the other hand,

$$S_{n} = S_{n-1} - \frac{1}{n!} \nabla^{n} u .(n) r^{n}. \quad (40)$$

$E_{n}^{hg}$ can be simplified as

$$E_{n}^{hg} = \alpha \int_{S} w(r) S_{n}^{2} dV = \int_{S} w(r) (S_{n-1} - \frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} dV$$

$$= \alpha \int_{S} w(r) \left( S_{n-1}^{2} + \frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} - 2S_{n-1} \frac{1}{n!} \nabla^{n} u .(n) r^{n}) \right) dV$$

$$\approx \alpha \int_{S} w(r) \left( S_{n-1}^{2} + \frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} - 2\frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} \right) dV$$

$$= \alpha \int_{S} w(r) \left( S_{n-1}^{2} - \frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} \right) dV$$

$$= E_{n-1}^{hg} - \alpha \left( \frac{1}{n!} \nabla^{n} u .(n) r^{n})^{2} \right) .K_{2n}. \quad (41)$$

Hence, $n$-order hourglass energy functional can be written as

$$E_{n}^{hg} = \alpha \left( \int_{S} w(r) u_{r} u_{r} dV - (\nabla u)^{2} : K_{2} - \left( \frac{1}{2!} \nabla^{2} u \right)^{2} .(4) K_{4} - \ldots - \left( \frac{1}{n!} \nabla^{n} u \right)^{2} .(2n) K_{2n} \right). \quad (41)$$

The $n$-order hourglass energy functional depends on $(1-n)$-order nonlocal operators, and the hourglass residual and hourglass stiffness matrix can be obtained with ease by the variation of the nonlocal operator.
5. Weighted residual method based on nonlocal operators

Many problems in engineering and physics are proposed in the form of the partial differential equations and the boundary conditions which should be satisfied by the unknown field functions. Generally, as shown in Fig. 3(a), the unknown field \( u(x) \) should satisfy

\[
A(u) = \begin{bmatrix}
\Lambda_1(u) \\
\Lambda_2(u) \\
\vdots
\end{bmatrix} = 0, \quad x \in \Omega,
\]  

(42)

where \( \Omega \) can be volume/area domains. Meanwhile, the unknown function \( u \) should satisfy the boundary conditions

\[
B(u) = \begin{bmatrix}
B_1(u) \\
B_2(u) \\
\vdots
\end{bmatrix} = 0, \quad x \in \partial \Omega.
\]  

(43)

The equivalent integral form of Eqs. (42), (43) is

\[
\int_{\Omega} v^T A(u) d\Omega + \int_{\partial \Omega} \bar{v}^T B(u) d\Gamma = 0, \quad \forall v, \bar{v} \in H^1_0
\]  

(44)

The “weak” form of integral form can be obtained by integration by part

\[
\int_{\Omega} C^T(v) D(u) d\Omega + \int_{\partial \Omega} E^T(\bar{v}) F(u) d\Gamma = 0,
\]  

(45)
where $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}$ are the differential operators. The weak form in Eq. 45 decreases the continuity of $\mathbf{u}$ by improving the continuity of $\mathbf{v}$.

For convenience, we assume $\mathbf{C}(\mathbf{v}, \nabla \mathbf{v}), \mathbf{D}(\mathbf{u}, \nabla \mathbf{u}), \mathbf{E}(\mathbf{v}, \nabla \mathbf{v}), \mathbf{D}(\mathbf{u}, \nabla \mathbf{u})$ are functional dependent on the unknowns $\mathbf{v}, \nabla \mathbf{v}, \mathbf{u}, \nabla \mathbf{u}$.

\[
\bar{\nabla} \mathbf{v}_x = \int_{S^u_x} w_v(r) \mathbf{v}_r \otimes rdrV_x(K^u_x)^{-1},
\]

(46)

\[
\bar{\nabla} \mathbf{u}_x = \int_{S^u_x} w_u(r) \mathbf{u}_r \otimes rdrV_x(K^u_x)^{-1}.
\]

(47)

For the same particle $x$, $S^v_x$ and $S^u_x$ can be different, as well as the weight functions $w_v(r), w_u(r)$.

The first variations of $\bar{\nabla} \mathbf{v}_x$ and $\bar{\nabla} \mathbf{u}_x$ are

\[
\bar{\nabla} \delta \mathbf{v}_x = \int_{S^v_x} w_v(r)(\delta \mathbf{v}_x' - \bar{\delta} \mathbf{v}_x) \otimes rdrV_x(K^v_x)^{-1},
\]

(48)

\[
\bar{\nabla} \delta \mathbf{u}_x = \int_{S^u_x} w_u(r)(\delta \mathbf{u}_x' - \bar{\delta} \mathbf{u}_x) \otimes rdrV_x(K^u_x)^{-1}.
\]

(49)

The functional for Eq. 45 can be written as

\[
\Pi = \int_{\Omega} \mathbf{C}^T(\mathbf{v}, \nabla \mathbf{v})\mathbf{D}(\mathbf{u}, \nabla \mathbf{u})d\Omega + \int_{\partial \Omega} \mathbf{E}^T(\bar{\mathbf{v}}, \nabla \bar{\mathbf{v}})\mathbf{F}(\mathbf{u}, \nabla \mathbf{u})d\Gamma
\]

(50)

\[
\simeq \sum_{x \in \Omega} \mathbf{C}^T(\mathbf{v}_x, \nabla \mathbf{v}_x)\mathbf{D}(\mathbf{u}_x, \nabla \mathbf{u}_x) \Delta V_x + \sum_{x \in \partial \Omega} \mathbf{E}^T(\bar{\mathbf{v}}_x, \nabla \bar{\mathbf{v}}_x)\mathbf{F}(\mathbf{u}_x, \nabla \mathbf{u}_x) \Delta V_x.
\]

(51)

Consider the variation of $\Pi$

\[
R_g = \int_{\Omega} \tilde{\delta} \mathbf{C}^T(\mathbf{v}, \nabla \mathbf{v})\mathbf{D}(\mathbf{u}, \nabla \mathbf{u})d\Omega + \int_{\partial \Omega} \tilde{\delta} \mathbf{E}^T(\bar{\mathbf{v}}, \nabla \bar{\mathbf{v}})\mathbf{F}(\mathbf{u}, \nabla \mathbf{u})d\Gamma,
\]

(52)

\[
\simeq \sum_{x \in \Omega} \tilde{\delta} \mathbf{C}^T(\mathbf{v}_x, \nabla \mathbf{v}_x)\mathbf{D}(\mathbf{u}_x, \nabla \mathbf{u}_x) \Delta V_x + \sum_{x \in \partial \Omega} \tilde{\delta} \mathbf{E}^T(\bar{\mathbf{v}}_x, \nabla \bar{\mathbf{v}}_x)\mathbf{F}(\mathbf{u}_x, \nabla \mathbf{u}_x) \Delta V_x,
\]

(53)

where

\[
\tilde{\delta} \mathbf{C}(\mathbf{v}, \nabla \mathbf{v}) = \frac{\partial \mathbf{C}}{\partial \mathbf{v}} \tilde{\delta} \mathbf{v} + \frac{\partial \mathbf{C}}{\partial \nabla \mathbf{v}} \nabla \tilde{\delta} \mathbf{v}
\]

(54)

\[
\tilde{\delta} \mathbf{E}(\mathbf{v}, \nabla \mathbf{v}) = \frac{\partial \mathbf{E}}{\partial \mathbf{v}} \tilde{\delta} \mathbf{v} + \frac{\partial \mathbf{E}}{\partial \nabla \mathbf{v}} \nabla \tilde{\delta} \mathbf{v}.
\]

(55)

It should be noted that, theoretically, $\mathbf{v}$ is not restricted to the nonlocal operators, the other forms such as the collocation method, sub-domain method, least squared method, moment
method, or Galerkin method may work. When $\nabla v$ is defined the same way as $\nabla u$, the Galerkin type is used.

Consider the variation of $R_g$

$$K_g = \int_{\Omega} \delta C^T(v, \nabla v) \delta D(u, \nabla u) d\Omega + \int_{\partial\Omega} \delta E^T(\bar{\nabla}v, \nabla \bar{v}) \delta F(u, \nabla u) d\Gamma,$$

(56)

$$\simeq \sum_{x \in \Omega} \delta C^T(v_x, \nabla v_x) \delta D(u_x, \nabla u_x) \Delta V_x + \sum_{x \in \partial\Omega} \delta E^T(\bar{\nabla}v_x, \nabla \bar{v}_x) \delta F(u_x, \nabla u_x) \Delta V_x,$$

(57)

where

$$\bar{\delta}D(u, \nabla u) = \frac{\partial D}{\partial u} \delta u + \frac{\partial D}{\partial \nabla u} \nabla \delta u$$

(58)

$$\bar{\delta}F(u, \nabla u) = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial \nabla u} \nabla \delta u.$$  

(59)

It can be seen that the weighted residual method based on the dual-support obtains the residual and global tangent stiffness matrix by the matrix multiplication on certain nonlocal operators. When the residual and tangent stiffness matrix of the linear/nonlinear problem are obtained, various of numerical techniques in linear algebra can be used to find the solution.

6. Variational principles based on the nonlocal operator

The problems based on variational principles start from the functional which describes the unknown functions defined in the domain and on the boundary. The residual is the gradient of the functional on the unknown vector, while the tangent stiffness matrix is the Hessian matrix of the functional on the unknown vector. The functional is usually expressed by the local operators such as divergence, curl and gradient. For simplicity, we assume the functional be a function on single local operator. The boundary terms can be handled the similar way. Assuming four general functionals which depend on gradient of a vector field, divergence of a vector field, curl of a vector field and gradient of a scalar field, respectively,

$$\psi_1(\nabla v), \psi_2(\nabla \cdot v), \psi_3(\nabla \times v), \psi_4(\nabla v).$$

(60)

The examples are the strain energy functional in solid mechanics, the volume strain energy functional in solid mechanics, the wave vector form of electromagnetic field, and the thermal conduction, respectively.

The derivatives of the functionals on the operators are

$$P_1 = \frac{\partial \psi_1}{\partial (\nabla v)}, P_2 = \frac{\partial \psi_2}{\partial (\nabla \cdot v)}, P_3 = \frac{\partial \psi_3}{\partial (\nabla \times v)}, P_4 = \frac{\partial \psi_4}{\partial (\nabla v)}.$$  

(61)

In the case of 3D, $P_1$ is a $3 \times 3$ tensor, $P_2$ is a scalar, $P_3, P_4$ are vectors with length of 3. When $\nabla v$ is the deformation gradient with respect to the initial configuration, $P_1$ is the
first Pio-Kirchhoff stress. The second derivatives of the functional on the operators are

\[ D_1 = \frac{\partial P_1}{\partial (\nabla v)} = \frac{\partial^2 \psi_1}{\partial (\nabla v)^T \partial (\nabla v)}, \quad (62) \]
\[ D_2 = \frac{\partial P_2}{\partial (\nabla \cdot v)} = \frac{\partial^2 \psi_2}{\partial (\nabla \cdot v)^T \partial (\nabla \cdot v)}, \quad (63) \]
\[ D_3 = \frac{\partial P_3}{\partial (\nabla \times v)} = \frac{\partial^2 \psi_3}{\partial (\nabla \times v)^T \partial (\nabla \times v)}, \quad (64) \]
\[ D_4 = \frac{\partial P_4}{\partial (\nabla v)} = \frac{\partial^2 \psi_4}{\partial (\nabla v)^T \partial (\nabla v)}. \quad (65) \]

For the case of 3D, \( D_1 \) is a 3\( \times \)3\( \times \)3\( \times \)3 tensor, \( D_2 \) is a scalar, \( D_3, D_4 \) are 3\( \times \)3 matrices. The fourth-order tensor \( D_1 \) can be flattened into a 9\( \times \)9 matrix as long as the \( \nabla v \) is flattened into a vector with length of 9. When \( \nabla v \) is the deformation gradient with respect to the initial configuration, \( D_1 \) is the material tensor in solid mechanics.

6.1. Gradient operator of scalar field

The residual and tangent stiffness matrix at one point are, respectively,

\[ R_{\text{grad}} = \bar{\delta} \psi_4 = \frac{\partial \psi_4}{\partial (\nabla v)} \bar{\nabla} \bar{\delta} v = P_4 \bar{\nabla} \bar{\delta} v \quad (66) \]
\[ K_{\text{grad}} = \bar{\delta}^2 \psi_4 = (\bar{\nabla} \bar{\delta} v)^T D_4 \bar{\nabla} \bar{\delta} v. \quad (67) \]

In order to derive the nonlocal strong form, let’s consider the first variation of all particles, and let \( P_x = \frac{\partial \psi(\nabla v_x)}{\partial (\nabla v)} \).

\[
\delta F(v) = \sum_{\Delta V_x \in \Omega} \Delta V_x \delta \psi_x = \sum_{\Delta V_x \in \Omega} \Delta V_x \nabla \delta v_x \cdot P_x
= \sum_{\Delta V_x \in \Omega} \Delta V_x \left( \sum_{s_x} w(r) \Delta V_x K_x^{-1}(\delta v_x - \delta v_x) \cdot P_x \right)
= \sum_{\Delta V_x \in \Omega} \Delta V_x \left( - \sum_{s_x} w(r) \Delta V_x K_x^{-1} \delta v_x \cdot P_x + \sum_{s'_x} w(r') \Delta V_{x'} K_{x'}^{-1} \delta v_{x'} \cdot P_{x'} \right)
\]

In the second and third step, the dual-support is considered as follows. In the second step, the term with \( \delta v_{x'} \) is the vector from \( \mathbf{x}' \)’s support, but is added to particle \( \mathbf{x} \); since \( \mathbf{x}' \in S_x, \mathbf{x} \) belongs to the dual-support \( S_{x'} \) of \( \mathbf{x}' \). In the third step, all the terms with \( \delta v_x \) are collected from other particles whose supports contain \( \mathbf{x} \) and therefore form the dual-support of \( \mathbf{x} \). The terms with \( \delta v_x \) in the first order variation \( \delta F(v) = 0 \) are

\[
- \sum_{s_x} w(r) \Delta V_x K_x^{-1} \cdot P_x + \sum_{s'_x} w(r') \Delta V_{x'} K_{x'}^{-1} \cdot P_{x'}. \quad (68)
\]
When any particle’s volume $\Delta V_{x'} \to 0$, the continuous form is

$$- \int_{S_x} w(r) K_{x}^{-1} r \cdot P_x dV_x + \int_{S_x'} w(r') K_{x'}^{-1} r' \cdot P_{x'} dV_{x'}.$$  \tag{69}$$

Eq.\(69\) is the nonlocal strong form for energy functional $\psi_4$.

The simplest example for this energy functional is

$$\psi = \frac{1}{2} \kappa \nabla T \cdot \nabla T,$$

where $T$ is the temperature, $\kappa$ is the thermal conductivity. The local strong form corresponding to Eq.\(69\) is $- \nabla \cdot P$, where $P = \kappa \nabla T$.

6.2. Divergence operator

The residual and tangent stiffness matrix at a point are, respectively,

$$R_{\text{div}} = \delta \psi_2 = \frac{\partial \psi (\nabla \cdot v)}{\partial (\nabla \cdot v)} \nabla \cdot \delta v = P_{2} \nabla \cdot \delta v,$$  \tag{70}$$

$$K_{\text{div}} = \delta^2 \psi_2 = (\nabla \cdot \delta v)^T D_2 \nabla \cdot \delta v.$$  \tag{71}$$

Let’s consider the first variation of all particles, and let $P_x = \frac{\partial \psi (\nabla \cdot v)}{\partial (\nabla \cdot v_x)}$.

$$\delta F(v) = \sum_{\Delta V_x \in \Omega} \Delta V_x \delta \psi_x = \sum_{\Delta V_x \in \Omega} \Delta V_x P_x \cdot (\nabla \cdot \delta v_x)$$

$$= \sum_{\Delta V_x \in \Omega} \Delta V_x P_x \cdot \sum_{S_x} w(r) \Delta V_{x'} K_{x}^{-1} r \cdot (\delta v_x - \delta v_{x'})$$

$$= \sum_{\Delta V_x \in \Omega} \Delta V_x \left( - \sum_{S_x} w(r) \Delta V_{x'} K_{x}^{-1} r \cdot \delta v_x \cdot P_x + \sum_{S_x} w(r') \Delta V_{x'} K_{x'}^{-1} r' \cdot \delta v_{x'} \cdot P_{x'} \right)$$

In the second and third step, the dual-support is considered as follows. In the second step, the term with $\delta v_{x'}$ is the vector from $x'$’s support, but is added to particle $x'$; since $x' \in S_x$, $x$ belongs to the dual-support $S_{x'}$ of $x'$. In the third step, all the terms with $\delta v_x$ are collected from other particles whose supports contain $x$ and therefore form the dual-support of $x$. The terms with $\delta v_x$ in the first order variation $\delta F(v) = 0$ are

$$- \sum_{S_x} w(r) \Delta V_{x'} P_x \cdot K_{x}^{-1} r + \sum_{S_x} w(r') \Delta V_{x'} P_{x'} \cdot K_{x'}^{-1} r'.$$  \tag{72}$$

When any particle’s volume $\Delta V_{x'} \to 0$, the continuous form is

$$- \int_{S_x} w(r) P_x \cdot K_{x}^{-1} r dV_{x'} + \int_{S_x} w(r') P_{x'} \cdot K_{x'}^{-1} r' dV_{x'}.$$  \tag{73}$$

Eq.\(73\) is the nonlocal strong form for energy functional $\phi_2$ with the corresponding local strong form is $- \nabla (D_2 \nabla \cdot v)$. The local strong form can be obtained by integration by part of the energy functional.
6.3. Curl operator

The residual and tangent stiffness matrix for one point are

\[ R_{\text{curl}} = \bar{\delta} \psi_3 = \frac{\partial \psi(\nabla \times \mathbf{v})}{\partial(\nabla \times \mathbf{v})} \nabla \times \bar{\delta} \mathbf{v} = P_3 \nabla \times \bar{\delta} \mathbf{v} \]  

(74)

\[ K_{\text{curl}} = \bar{\delta}^2 \psi_3 = (\nabla \times \bar{\delta} \mathbf{v})^T D_3 \nabla \times \bar{\delta} \mathbf{v} = (\nabla \times \bar{\delta} \mathbf{v})^T D_3 \nabla \times \bar{\delta} \mathbf{v}. \]  

(75)

Let’s consider the first variation of all particles, and let \( P_x = \frac{\partial \psi(\nabla \times \mathbf{v})}{\partial(\nabla \times \mathbf{v})} \)

\[
\delta F(\mathbf{v}) = \sum_{\Delta V_x} \Delta V_x \delta \psi_x = \sum_{\Delta V_x} \Delta V_x (\nabla \times \delta \mathbf{v}_x) \cdot P_x \\
= \sum_{\Delta V_x} \Delta V_x \sum_{S_x} w(r) \Delta V_{x'} K_{x'}^{-1} r \times (\delta \mathbf{v}_{x'} - \delta \mathbf{v}_x) \cdot P_x \\
= \sum_{\Delta V_x} \Delta V_x \sum_{S_x} w(r) \Delta V_{x'} P_x \times (K_{x'}^{-1} r) \cdot (\delta \mathbf{v}_{x'} - \delta \mathbf{v}_x) \\
= \sum_{\Delta V_x} \Delta V_x \left( -\sum_{S_x} w(r) \Delta V_{x'} P_x \times (K_{x'}^{-1} r) \cdot \delta \mathbf{v}_x + \sum_{S_{x'}} w(r') \Delta V_{x'} P_{x'} \times (K_{x'}^{-1} r') \cdot \delta \mathbf{v}_x \right). 
\]

The relation \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} \) is used in the third step. In the third and fourth step, the dual-support is considered as follows. In the third step, the term with \( \delta \mathbf{v}_{x'} \) is the vector from \( x' \)’s support, but is added to particle \( x \); since \( x' \in S_x, x \) belongs to the dual-support \( S_{x'} \) of \( x' \). In the fourth step, all the terms with \( \delta \mathbf{v}_x \) are collected from other particles whose supports contain \( x \) and therefore form the dual-support of \( x \). The terms with \( \delta \mathbf{v}_x \) in the first order variation \( \delta F(\mathbf{v}) = 0 \) are

\[ -\sum_{S_x} w(r) \Delta V_{x'} P_x \times (K_{x'}^{-1} r) + \sum_{S_{x'}} w(r') \Delta V_{x'} P_{x'} \times (K_{x'}^{-1} r'). \]  

(76)

When any particle’s volume \( \Delta V_{x'} \rightarrow 0 \), the continuous form is

\[ -\int_{S_x} w(r) P_x \times (K_{x'}^{-1} r) dV_{x'} + \int_{S_{x'}} w(r') P_{x'} \times (K_{x'}^{-1} r') dV_{x'}. \]  

(77)

Eq(77) is the strong form for energy functional \( \psi_3 \), where the corresponding local strong from obtained by integration by part of the energy functional is \(-\nabla \times (D_3 \nabla \times \mathbf{v})\).

6.4. Gradient operator of vector field

The residual and stiffness matrix at one point are, respectively,

\[ R_{\text{grad}} = \bar{\delta} \psi_1 = \frac{\partial \psi(\nabla \mathbf{v})}{\partial(\nabla \mathbf{v})} \nabla \bar{\delta} \mathbf{v} = P_1 \nabla \bar{\delta} \mathbf{v} \]  

(78)

\[ K_{\text{grad}} = \bar{\delta}^2 \psi_1 = (\nabla \bar{\delta} \mathbf{v})^T D_1 \nabla \bar{\delta} \mathbf{v} = (\nabla \bar{\delta} \mathbf{v})^T D_1 \nabla \bar{\delta} \mathbf{v}. \]  

(79)
Let’s consider the first variation of all particles, and let $P_x = \frac{\partial \psi(\nabla v_x)}{\partial (\nabla v_x)}$.

$$\delta F(v) = \sum_{\Delta V_x \in \Omega} \Delta V_x \delta \psi_x = \sum_{\Delta V_x \in \Omega} \Delta V_x P_x \cdot (\nabla \delta v_x)$$

$$= \sum_{\Delta V_x \in \Omega} \Delta V_x P_x \cdot \sum_{S_x} w(r) \Delta V_x K^{-1}_x r \otimes (\delta v_x - \delta v_x')$$

$$= \sum_{\Delta V_x \in \Omega} \Delta V_x \left( - \sum_{S_x} w(r) \Delta V_x K^{-1}_x r \otimes \delta v_x \cdot P_x + \sum_{S'_x} w(r') \Delta V_x' K^{-1}_x' r' \otimes \delta v_x' \cdot P_x' \right).$$

In the second and third step, the dual-support is considered as follows. In the second step, the term with $\delta v_x'$ is the vector from $x'$’s support, but is added to particle $x$; since $x' \in S_x$, $x$ belongs to the dual-support $S'_x$ of $x'$. In the third step, all the terms with $\delta v_x$ are collected from other particles whose supports contain $x$ and therefore form the dual-support of $x$. The terms with $\delta v_x$ in the first order variation $\delta F(v) = 0$ are

$$- \sum_{S_x} w(r) \Delta V_x P_x \cdot K^{-1}_x r + \sum_{S'_x} w(r') \Delta V_x' P_x' \cdot K^{-1}_x r'. \quad (80)$$

When any particle’s volume $\Delta V_x' \to 0$, the continuous form is

$$- \int_{S_x} w(r) P_x \cdot K^{-1}_x r dV_x' + \int_{S'_x} w(r') P_x' \cdot K^{-1}_x r' dV_x'. \quad (81)$$

Eq.81 is the nonlocal strong form of energy functional $\psi_1$, where the local strong form obtained by integration by part of the energy functional is $-\nabla \cdot (D_1 \nabla v)$. If $\nabla v$ denotes the deformation gradient with respect to the initial configuration and $\psi_1$ is the strain energy density, $P_x$ is the first Piola-Kirchhoff stress and the Eq.81 is the key expression in the dual-horizon peridynamics [14, 25].

7. Applications

7.1. One dimensional beam and bar test

The energy functionals of cantilever beam and bar are, respectively,

$$F_{\text{beam}}(u) = \frac{1}{2} \int_0^L (u_{xx}EIu_{xx} - uq) dx \quad (82)$$

$$F_{\text{bar}}(u) = \frac{1}{2} \int_0^L (u_{x}EAu_{x} - uq) dx. \quad (83)$$

We consider the boundary conditions in Eq.84 of the cantilever beam with a concentrated transverse load $P = 1$ is applied on the end.

$$u(0) = 0, \frac{du}{dx}|_{x=0} = 0. \quad (84)$$
For the uniform bar, the left side is fixed and the other side is applied with a load $P = 1$. The theoretical solution for beam and bar are, respectively,

$$u(x) = \frac{P}{6EI} (3Lx^2 - x^3), u(x) = \frac{Px}{EA},$$  \hspace{1cm} (85)

where $EI = 1, EA = 1$ are the stiffness coefficient, $L = 1$ is the length of the beam. The residual and tangent stiffness matrix of Eq.82 and Eq.83 are obtained by simply replacing $\tilde{\delta}u_x$ and $\tilde{\delta}u_{xx}$ with Eq.6 and Eq.A.2 in the first and second variation of the energy functional, respectively. The $L_2$-norm is calculated by

$$\|u\|_{L^2} = \sqrt{\sum_j (u_j - u_j^{\text{exact}}) \cdot (u_j - u_j^{\text{exact}}) \Delta V_j}. \hspace{1cm} (86)$$

The convergence of the $L_2$-norm for the displacement of bar under tension is shown in Fig.4. The convergence of the $L_2$-norm for the deflection of cantilever beam is shown in Fig.5. With the refinement in discretization, the numerical results converge to the theoretical solutions at a rate $r \approx 1$.

### 7.2. The Schrödinger equation in 1D

In this section, we test the accuracy of the eigenvalue problem based on the nonlocal operator. The Schrödinger equation written in adimensional units for a one-dimensional harmonic oscillator is

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x)\right] \phi(x) = \lambda \phi(x), \hspace{1cm} V(x) = \frac{1}{2} \omega^2 x^2.$$  \hspace{1cm} (87)
For simplicity, we use $\omega = 1$. The particles are distributed with constant or variables spacing $\Delta x$ on the region $[-10,10]$.

The exact wave functions and eigenvalues can be expressed as

$$
\phi_n(x) = H_n(x) \exp(\pm \frac{x^2}{2}), \quad \lambda_n = n + \frac{1}{2},
$$

(88)

where $n$ is a non-negative integer. $H_n(x)$ is the $n$-order Hermite polynomial. We calculate the lowest eigenvalue and compare the numerical result with $\lambda_0 = 0.5$. The convergence plot of the error is shown in Fig.6. With the decrease of grid spacing, the numerical result converges to the exact result at a rate of $r \approx 2$. The discretization of the dual-form is given in Fig.7. The first three wave functions are given in Fig.8.

7.3. Poisson equation

In this section, we test the Poisson equation

$$
\nabla^2 u = f(x,y), \quad (x,y) \in (0,1) \times (0,1),
$$

(89)

where $f(x,y) = 2x(y - 1)(y - 2x + xy + 2)e^{x-y}$, and the boundary conditions

$$
\begin{align*}
   u(x, 0) &= u(x, 1) = 0, \quad x \in [0, 1] \\
   u(0, y) &= u(1, y) = 0, \quad y \in [0, 1].
\end{align*}
$$

The analytic solution is

$$
    u(x, y) = x(1-x)y(1-y)e^{x-y}.
$$

(90)
Figure 6: Convergence of the lowest eigenvalue for a one-dimensional harmonic oscillator; $1/r^3$ is the influence function; support radius is selected as $h_n \Delta x$; dual-form with influence function $1/r^3$ uses an inhomogenous discretization in Fig. 7; the particle spacing in dual-form is selected as the minimal particle spacing in the discretization.

Figure 7: The discretization of the dual form based on inhomogeneous discretization.
The corresponded energy functional is

$$\Pi = \int_{\Omega} \left( -\frac{1}{2} \nabla u \cdot \nabla u - f(x,y)u \right) d\Omega. \quad (91)$$

The first and second variation of \( \Pi \) lead to the global residual and stiffness matrix

$$R = \sum_{\Delta V_i \in \Omega} \Delta V_i \left( -\bar{\nabla} u \cdot \bar{\nabla} \delta u - f(x,y)\delta u \right) \quad (92)$$

$$K = \sum_{\Delta V_i \in \Omega} \Delta V_i \left( -\bar{\nabla} \delta u \cdot \bar{\nabla} \delta u \right) \quad (93)$$

The support radius is selected as \( h = 1.2\Delta x \). We test the convergence of the L2 error for \( u \) field under difference discretizations. The convergent plot is given in Fig.9 with convergence rate of \( r = 0.9567 \). The contours of \( u \) field with and without hourglass control are shown in Fig.10. It can be seen that the hourglass control can stabilize and smooth the solution.

7.4. Nonlocal theory for linear small strain elasticity

The elastic energy of a body \( V \) is given by the quadratic functional \cite{26, 22}

$$W = \frac{1}{2} \int_V \int_V \varepsilon^T(x)D_e(x,x')\varepsilon(x')dx'dx = \frac{1}{2} \int_V \varepsilon^T(x)\sigma(x)dx, \quad (94)$$

where \( \varepsilon(x) \)=strain field, \( D_e(x,x') \)=generalized form of the elastic stiffness and

$$\sigma(x) = \int_V D_e(x,x')\varepsilon(x')dx' \quad (95)$$
Figure 9: Convergence of the L2 error of the displacement.

Figure 10: The contour of $u$ with hourglass control $\mu = 0.1$ and without hourglass control for discretization $40 \times 40$. 
is the stress dependent on the strain field in $V$. Only if $\mathbf{D}_e(\mathbf{x}, \mathbf{x'}) = \mathbf{D}_e(\mathbf{x})\delta(\mathbf{x} - \mathbf{x'})$, Eq.\ref{eq:94} reduces to
\begin{equation}
W = \frac{1}{2} \int_V \varepsilon^T(x)\mathbf{D}_e(x)\varepsilon(x)dx = \int_V w[\varepsilon(x), x]dx, \tag{96}
\end{equation}
where $w(\varepsilon, x) = \frac{1}{2}\varepsilon^T\mathbf{D}_e(x)\varepsilon$.

It can be assumed that the interaction effects decay with distance between the two points $\mathbf{x}$ and $\mathbf{x'}$, i.e.,
\begin{equation}
\mathbf{D}_e(\mathbf{x}, \mathbf{x'}) = \mathbf{D}_e(\mathbf{x})\alpha(\mathbf{x}, \mathbf{x'}), \tag{97}
\end{equation}
where $\alpha$ is certain attenuation function satisfying the normalizing condition
\begin{equation}
\int_V \alpha(\mathbf{x}, \mathbf{x'})d\mathbf{x'} = 1. \tag{98}
\end{equation}
$\alpha$ is also called the nonlocal weight function or the nonlocal averaging function, and is often assumed to have the form of Gauss distribution function
\begin{equation}
\alpha_{\infty}(r) = (l\sqrt{2\pi})^{-N_{dim}} \exp\left(-\frac{r^2}{2l^2}\right), \tag{99}
\end{equation}
where $l$ is the parameter with the dimension of length, $N_{dim}$ the number of spatial dimensions. For reasons of computational efficiency, the attenuation function is often selected as the finite support, e.g., the polynomial bell-shaped function,
\begin{equation}
\alpha(r) = c\left(\max(0, 1 - \frac{r^2}{R^2})\right)^2, \tag{100}
\end{equation}
where $c$ is determined by the normalizing condition Eq.\ref{eq:98}

Then the stress-strain law reads
\begin{equation}
\mathbf{\sigma}(\mathbf{x}) = \int_V \mathbf{D}_e(\mathbf{x}, \mathbf{x'})\varepsilon(\mathbf{x'})d\mathbf{x'} = \mathbf{D}_e \int_V \alpha(\mathbf{x}, \mathbf{x'})\varepsilon(\mathbf{x'})d\mathbf{x'} = \mathbf{D}_e\bar{\varepsilon}(\mathbf{x}), \tag{101}
\end{equation}
where
\begin{equation}
\bar{\varepsilon}(\mathbf{x}) = \int_V \alpha(\mathbf{x}, \mathbf{x'})\varepsilon(\mathbf{x'})d\mathbf{x'} \tag{102}
\end{equation}
is the nonlocal strain.

With the aid of nonlocal operator and its variation,
\begin{align*}
\varepsilon &= \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \\
\bar{\delta}\varepsilon &= \frac{1}{2}\left(\nabla \bar{\delta} \mathbf{u} + (\nabla \bar{\delta} \mathbf{u})^T\right),
\end{align*}

where \(\nabla \tilde{\delta} \mathbf{u}\) is the nonlocal gradient operator in Eq.6, the residual and tangent stiffness matrix of nonlocal energy functional Eq.94 are

\[
R = \tilde{\delta}W = \sum_{\mathbf{x} \in V} \sum_{\mathbf{x}' \in V} \epsilon^T(\mathbf{x})D_e(\mathbf{x}, \mathbf{x}')\tilde{\delta} \epsilon(\mathbf{x}')\Delta V_{\mathbf{x}} \Delta V_{\mathbf{x}} \tag{103}
\]
\[
K = \tilde{\delta}^2 W = \sum_{\mathbf{x} \in V} \sum_{\mathbf{x}' \in V} \tilde{\delta} \epsilon^T(\mathbf{x})D_e(\mathbf{x}, \mathbf{x}')\tilde{\delta} \epsilon(\mathbf{x}')\Delta V_{\mathbf{x}} \Delta V_{\mathbf{x}} \tag{104}
\]

It is found that the tangent stiffness matrix for nonlocal elasticity is equivalent to the matrix multiplication on the variational form of the nonlocal operator on each particle. The Neumann boundary conditions and Dirichlet boundary conditions can be applied directly on the residual and stiffness matrix.

### 7.5. Nonhomogeneous Biharmonic Equation

This example tests the nonlocal hessian operator. The nonhomogeneous biharmonic equation reads

\[
\nabla^2 \nabla^2 w = q_0, \quad (x, y) \in (0, 1) \times (-1/2, 1/2) \tag{105}
\]

with boundary conditions

\[
w(x, -1/2) = w(x, 1/2) = 0, \quad x \in [0, 1] \\
w(0, y) = w(1, y) = 0, \quad y \in [-1/2, 1/2].
\]

This biharmonic equation corresponds to the simply support square plate subjected to uniform load with parameters such as length \(a = 1\) m, thickness \(t = 0.01\) m, uniform pressure \(q_0 = -100\) N, Poisson ratio \(\nu = 0\), elastic modulus \(E = 30\) GPa and \(D_0 = \frac{Et^3}{12(1-\nu^2)}\).

The analytic solution for this plate is denoted by \cite{27}

\[
w = \frac{4q_0a^4}{\pi^5D_0} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^5} \left( 1 - \frac{\alpha_m \tanh \alpha_m + 2}{2 \cosh \alpha_m} \cosh \frac{2\alpha_m y}{a} + \frac{\alpha_m^2 y}{2 \cosh \alpha_m} \sinh \frac{2\alpha_m y}{a} \right) \sin \frac{m \pi x}{a}, \tag{106}
\]

where \(\alpha_m = \frac{m \pi}{2}\).

The equivalent energy functional of Eq.105 is

\[
\Pi = \int_{\Omega} \left( \frac{1}{2} (\nabla^2 w)^T (\nabla^2 w) - q_0 w \right) d\Omega.
\]

With the aid of nonlocal hessian operator \(\nabla^2 w\) and its variation \(\nabla^2 \tilde{\delta} w\), the residual and tangent stiffness matrix can be obtained with ease

\[
R = \tilde{\delta} \Pi = \sum_{\Delta V_{\mathbf{x}} \in \Omega} \Delta V_{\mathbf{x}} ((\nabla^2 w)^T (\nabla^2 \tilde{\delta} w) - q_0 \tilde{\delta} w)
\]
\[
K = \tilde{\delta}^2 \Pi = \sum_{\Delta V_{\mathbf{x}} \in \Omega} \Delta V_{\mathbf{x}} (\nabla^2 \tilde{\delta} w)^T (\nabla^2 \tilde{\delta} w).
\]
Figure 11: (a) Deflection of section $y = 0$ under different discretizations. (b) Contour of the deflection $w$ for discretization of 20x20.

The plate is discretized uniformly and the support radius is selected as $h = 2.2\Delta x$. The weight function is $w(r) = \frac{1}{r^2}$. The second-order hourglass control is exploited. The calculation of the nonlocal hessian operator is given in Appendix A. The deflection curves for several discretizations are compared with the analytic solution in Fig.11(b). The contour of the deflection field for discretization of 20 x 20 is shown in Fig.11(b).

8. Conclusions

We propose a nonlocal operator method for solving PDEs. The fundamental elements in nonlocal operator method include the support, dual-support, nonlocal operators and hourglass energy functional. The support is the basis to define the nonlocal operators. The nonlocal operator is a generalization of the conventional differential operators. Under certain conditions such as support decreasing to infinitesimal or linear field, the nonlocal operators converge to the local operators. On the other hand, the nonlocal operator is still valid in the case of field involving discontinuity since the nonlocal operator is defined by integral form. The dual-support as the dual concept of support allows the inhomogeneous discretization of the computational domain. The dual-support contributes to deriving the nonlocal strong discrete or continuous forms of different functionals by means of variational principles.

The nonlocal operator is defined at one point but interacts with any other points in its support domain through nonlocal interactions. In this paper, the continuous form is solved by discretizing the computational domain into particles, and finally results in a discrete system based on nodal integration. Nodal integration method suffers from the rank deficiency and hourglass mode (or zero-energy mode). In order to remove the hourglass mode, the hourglass energy functional is proposed, which can suppress the hourglass modes in implicit/explicit analysis.
The nonlocal operator method is consistent with the weighted residual method and variational principle. The residual and tangent stiffness can be obtained with some matrix multiplication on common terms such as physical constitutions and nonlocal operators with variation. The nonlocal operator can be used to replace the traditional local operator of one-order or high-orders and thus obtains the discrete algebraic system of the PDEs with ease. In the example of nonlocal linear elasticity theory, the nonlocal operator method obtains the residual and tangent stiffness matrix concisely.

Several numerical examples include the the deflection of cantilever beam and plate, the Poisson equation in 2D and eigenvalue problem are presented to illustrate the capabilities of the nonlocal operator method.

Acknowledgments
The authors acknowledge the supports from the COMBAT Program (Computational Modeling and Design of Lithium-ion Batteries, Grant No.615132), the National Basic Research Program of China (973 Program: 2011CB013800) and NSFC (51474157), the Ministry of Science and Technology of China (Grant No.SLDRCE14-B-28, SLDRCE14-B-31).

Appendix A. Nonlocal hessian operator in 1D, 2D

In the case of 1 dimension, all operators are scalar-type. The second derivative and its variation in 1D can be written as

$$\frac{d^2 u}{dx^2} = 2 \int_S w(r) u_r(r^2 - \frac{K_3}{K_2}) r dV \cdot K_4^{-1}$$

(A.1)

$$\bar{\nabla}^2 \delta u = 2 \int_S w(r) (\delta u' - \delta u) (r^2 - \frac{K_3}{K_2}) r dV \cdot K_4^{-1}.$$ (A.2)

For simplicity, we consider hessian operator in two dimensions, i.e. \( r = (x, y)^T \). The 2-order shape tensor is

$$K_2 = \sum_S w(r) \Delta V' \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}. \quad (A.3)$$

The 3-order shape tensor is

$$K_3 = (K_3^x, K_3^y) = \left( \sum_S w(r) \Delta V' x \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}, \sum_S w(r) \Delta V' y \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \right). \quad (A.4)$$

Therefore, the calculation of \( K_3 K_2^{-1} r \) is

$$K_3 K_2^{-1} r = (K_3^x K_2^{-1} r, K_3^y K_2^{-1} r). \quad (A.5)$$

The 4-order shape tensor is

$$K_4 = \begin{bmatrix} K_4^{xx} & K_4^{xy} \\ K_4^{yx} & K_4^{yy} \end{bmatrix} = \sum_S w(r) \Delta V' \begin{bmatrix} x^2 & x^2 & xy & xy \\ xy & xy & y^2 & y^2 \\ x^2 & x^2 & xy & xy \\ xy & xy & y^2 & y^2 \end{bmatrix}. \quad (A.6)$$
It should be noted that the rank of $K$ is 3 since there are only three independent variables in the 2D nonlocal hessian operator where \( \frac{\partial^2 \delta u}{\partial x \partial y} = \frac{\partial^2 \delta u}{\partial y \partial x} \). It is convenient to write \( \nabla^2 \delta u = (\frac{\partial^2 \delta u}{\partial x^2}, \frac{\partial^2 \delta u}{\partial x \partial y}, \frac{\partial^2 \delta u}{\partial y^2}) \).

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