SUBELLITIC PSEUDO-DIFFERENTIAL OPERATORS AND
FOURIER INTEGRAL OPERATORS ON COMPACT LIE
GROUPS

DUVÁN CARDONA AND MICHAEL RUZHANSKY

ABSTRACT. In this work we extend the theory of global pseudo-differential operators on compact Lie groups to a subelliptic context. More precisely, given a compact Lie group $G$, and the sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \ldots, X_k\}$ satisfying the Hörmander condition, we introduce a (subelliptic) pseudo-differential calculus associated to $\mathcal{L}$, based on the matrix-valued quantisation process developed in [107]. This theory will be developed as follows. First, we will investigate the singular kernels of this calculus, estimates of $L^p-L^p$, $H^1-L^1$, $L^\infty-BMO$ type and also the weak (1,1) boundedness of these subelliptic Hörmander classes. Between the obtained estimates we prove subelliptic versions of the celebrated sharp Fefferman $L^p$-theorem and the Calderón-Vaillancourt theorem. The obtained estimates will be used to establish the boundedness of subelliptic operators on subelliptic Sobolev and Besov spaces. We will investigate the ellipticity, the construction of parametrices, the heat traces and the regularisation of traces for the developed subelliptic calculus. A subelliptic global functional calculus will be established. This subelliptic functional calculus will be used to prove a subelliptic version of the Garding inequality, which we also use to study the global solvability for a class of subelliptic pseudo-differential problems. Finally, by using both, the matrix-valued symbols and also the notion of matrix-valued phases we study the $L^2$-boundedness of global Fourier integral operators. The approach established in characterising our subelliptic Hörmander classes (by proving that the definition of these classes is independent of certain parameters) also will be applied in order to characterise the global Hörmander classes on arbitrary graded Lie groups developed in [65].

CONTENTS

1. Introduction and historical remarks 3
2. Outline and main results 6
  2.1. Notation 6
  2.2. Main results 8
3. Preliminaries: sub-Laplacians and pseudo-differential operators on compact Lie groups 13

2010 Mathematics Subject Classification. Primary 22E30; Secondary 58J40.

Key words and phrases. Sub-Laplacian, Compact Lie group, Pseudo-differential operator, Fourier analysis.

The first author was supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations. The second author was supported in parts by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151.
3.1. Pseudo-differential operators via localisations 13
3.2. The positive sub-Laplacian and pseudo-differential operators via
   global symbols 14
3.3. Calderón-Zygmund type estimates for multipliers 17
3.4. $L^p$-multipliers and $L^p$-boundedness of pseudo-differential operators
   on compact Lie groups 18
3.5. The subelliptic spaces $H^1$ and $BMO$ on compact Lie groups 20
4. Subelliptic pseudo-differential operators 22
4.1. Subelliptic symbols on compact Lie groups 22
4.2. Singular kernels of subelliptic pseudo-differential operators 38
4.3. Calderón-Vaillancourt Theorem for subelliptic classes 44
4.4. The formal adjoint of subelliptic operators 54
4.5. Composition of subelliptic pseudo-differential operators 56
5. Weak (1,1) type and $L^p$-boundedness of subelliptic operators with
   non-smooth symbols 60
6. Boundedness of subelliptic pseudo-differential operators with smooth
   symbols 67
6.1. $L^2$-$BMO$ boundedness for subelliptic Hörmander classes 67
6.2. $L^p(G)$, Sobolev and Besov boundedness for subelliptic Hörmander
   classes 79
7. Ellipticity in the context of the subelliptic calculus: construction of
   parametrices and regularisation of traces 80
7.1. Construction of parametrices 81
7.2. Parameter $\mathcal{L}$-ellipticity with respect to an analytic curve in the
   complex plane 82
7.3. Asymptotic expansions for regularised traces of $\mathcal{L}$-elliptic global
   pseudo-differential operators 85
8. Subelliptic global functional calculus and applications 94
8.1. Functions of symbols vs functions of operators 95
8.2. Gårding inequality 98
8.3. Dixmier traces 102
8.4. Subelliptic operators in Schatten classes in $L^2(G)$ 107
8.5. Compactness and Gohberg lemma 108
9. Global solvability for evolution problems associated to subelliptic
   operators 109
10. Global Fourier Integral operators on compact Lie groups 113
10.1. Matrix-valued phase functions on compact Lie groups 113
10.2. $L^p$-boundedness of Fourier integral operators 116
11. Appendix I: Sub-Laplacians on $S^3 \cong SU(2)$, SO(4), SU(3), and
    Spin(4) $\cong$ SU(2) $\times$ SU(2) 126
12. Appendix II: Subelliptic Besov spaces 129
13. Appendix III: A characterisation for global Hörmander classes on
    graded Lie groups 133
13.1. Homogeneous and graded Lie groups 133
13.2. Fourier analysis on nilpotent Lie groups 134
13.3. Homogeneous linear operators and Rockland operators 135
1. Introduction and historical remarks

This work is devoted to the development of the pseudo-differential calculus for subelliptic pseudo-differential operators on arbitrary compact Lie groups and its applications. For instance, the theory developed here could remains valid in several compact non-commutative structures with the presence of symmetries (see [107, Part IV]).

In modern mathematics, the theory of pseudo-differential operators is a powerful branch in the analysis of linear partial differential operators due to its interactions with several areas of mathematics. For instance, from the point of view of the theory of partial differential equations, pseudo-differential operators are used e.g. to study the global/local solvability of several partial differential problems, to understand the mapping properties of certain singular integral operators, to understand the propagation of singularities in distribution theory, and in the construction of fundamental solutions and parametrices. Also, in the interplay between differential geometry and algebraic topology, pseudo-differential operators are used to compute some geometric invariants arising in the index theory. That is the case of analytical expressions for the Euler characteristic, the Hirzebruch signature and, in a more general context, the Atiyah-Singer index theorem (see e.g. Atiyah and Singer [6, 7, 8, 9, 10, 11], Kohn and Nirenberg [83], the fundamental book by Hörmander [80] and references therein). On the other hand, in the microlocal analysis, the theory of Fourier integral operators becomes a prominent generalisation of pseudo-differential operators, to study the spectral function for elliptic operators on vector bundles and in solving hyperbolic differential equations (see Duistermaat and Hörmander [56] and Hörmander [79]).

In this work we develop a subelliptic pseudo-differential calculus on compact Lie groups and some of its applications, by contributing with the notions and results of harmonic analysis on compact Lie groups, building up on the monograph [107] by V. Turunen and the second author, which was devoted to the development of the general theory of global pseudo-differential operators (with matrix-valued symbols) on spaces with symmetries. Starting our work, we investigate the action of the subelliptic calculus on \( L^p \), subelliptic Sobolev, and subelliptic Besov spaces and in the final part of the paper, we study the \( L^2 \)-boundedness of global Fourier integral operators. We will follow the notion of global symbol on every compact Lie group \( G \) introduced in [107] which is a non-commutative extension of the classical Konh-Nirenberg quantisation [83], instead of the notion of a symbol via localisations (see Hörmander [80]) where the argument of such a symbols is
defined in points \((x, \xi)\) of the cotangent bundle over \(G\), \(T^*G \cong G \times \mathfrak{g}^*\). The pseudo-differential calculus associated to the usual \((\rho, \delta)\)-classes and this global notion are equivalent for \(1 - \rho \leq \delta < \rho \leq 1\), (see [61, 108]) with the global notion allowing also for the range \(0 \leq \delta < \rho \leq 1/2\), where the pseudo-differential calculus associated with the notion of symbol via localisations is not operable\(^2\). So, in particular, the resultant global calculus for the range \(0 \leq \delta < \rho \leq 1/2\), can be applied to the treatment of inverses of complex vector fields, sub-Laplacians and a wide variety of pseudo-differential problems (see [107] for a complete description and also e.g. the references of this work).

From its beginning, the theory of pseudo-differential operators was closely related to the theory of singular integral operators developed by Mihlin [86], Calderón and Zygmund [22]. However, in the case of \(\mathbb{R}^n\), and other manifolds with symmetries, we can use the Fourier transform (as Kohn and Nirenberg in [83], and the works of the second author with Turunen, Fischer, and Măntoiu [107, 65, 87, 88]) to define pseudo-differential operators by using global symbols. In the case of compact Lie groups as the theory developed in [107], these global symbols are matrix-valued with the size of the matrix growing according to the size of the representation spaces. In the other general cases, for example on graded Lie groups [65] and on general Lie groups of type I [87, 88], the symbols become operator-valued and densely defined on the possibly infinitely dimensional representation spaces. In the spirit of the theory of singular integrals of Coifman and Weiss [42], we will follow the criterion given by Coifman and De Guzmán [41] and the approach developed by Fefferman [59] to establish the mapping properties for subelliptic pseudo-differential operators. The classical Cotlar-Stein Lemma will also applied to obtain the Calderón-Vaillancourt theorem and the \(L^2\)-boundedness of global Fourier integral operators (see Section 2 for details).

Because the subelliptic pseudo-differential calculus developed here is a parallel theory to the ones developed in [107] and [65], we will exploit that our operators have singular kernels in order to study their mapping properties and other spectral properties arising in the spectral geometry, specifically from the regularisation of traces. The singularity orders for the kernels of the obtained subelliptic calculus can be classified in terms of the Hausdorff dimension of a compact Lie group \(G\), \(Q \geq \dim(G)\), associated with the Carnot-Carathéodory distance induced by the sub-Laplacian under consideration, so that in local coordinates we obtain more singular kernels that those obtained by the Hörmander calculus by using charts in the case of compact Lie groups [61, 108].

Let us consider on the compact Lie group \(G\), the positive sub-Laplacian

\[
\mathcal{L} = -(X_1^2 + \cdots + X_k^2),
\]

\(^1\)Here, \(\mathfrak{g}\) denotes the Lie algebra of \(G\), and we denote by \(\hat{G}\) its unitary dual. The global symbols according to the theory developed in [107] are defined on the non-commutative phase space \(G \times \hat{G}\), however, the idea of studying pseudo-differential operators on Lie groups as a generalisation of multipliers of the Fourier transform can be traced back to Taylor [120].

\(^2\) Indeed, in this case the \((\rho, \delta)\)-classes on arbitrary \(C^\infty\)-manifolds are not stable under coordinate changes.
which is considered in such a way that the system of vector fields \( X = \{X_i\}_{i=1}^k \) satisfies the Hörmander condition of order \( \kappa \). In this paper we develop a subelliptic pseudo-differential calculus associated with \( \mathcal{L} \), by defining certain Hörmander type classes \( \mathcal{S}^{m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G}) = \mathcal{S}^{m,\mathcal{L}}_{\rho,\delta}(G), 0 \leq \rho, \delta \leq 1 \), (stable under compositions and adjoints). We have opted for using the matrix-valued (global) quantisation process developed by the second author and V. Turunen in [107, 109]. This viewpoint has shown to be a versatile tool in the analysis of pseudo-differential operators for describing their analytic and spectral properties (see e.g. [102, 46, 61, 70, 71, 107, 108, 109, 111, 112]) and in the treatment of some problems for PDE on compact Lie groups. Our motivation for using the matrix-valued quantisation came from the fact that the global symbols obtained with this procedure together with a suitable difference structure of the unitary dual \( \hat{G} \), of \( G \), characterise the Hörmander classes of pseudo-differential operators defined by charts [108].

The sub-Laplacian \( \mathcal{L} \), associated with a system of vector fields \( X = \{X_i\}_{i=1}^k \) satisfying the Hörmander condition, endows \( G \) with a natural sub-Riemannian structure. The sub-bundle \( \mathcal{H} = \text{span}(X) \), of the tangent bundle \( TG \) generated by the system \( X \), provides a natural setting to study subelliptic operators as \( \mathcal{L} \), which is, in fact, hypoelliptic by an application of the theorem of sum of squares of Hörmander. This kind of sub-Riemannian structures appears in many areas, to say, describing constrained systems in mechanics, or as limiting classes of Riemannian geometries (see e.g. Gordina and Laetsch [73] and reference therein). We also refer the reader to Bramanti [14], where the applications of these kind of sub-Riemannian manifolds are discussed, as its relation with the Kolmogorov-Fokker-Plank equation, the \( \mathcal{B} \)-Neumann problem, the tangencial Cauchy-Riemann Complex, the Kohn-Laplacian \( \nabla_b \), and other differential problems.

When we review the criteria obtained in terms of the matrix-valued quantization developed in [107, 109], we observe that some of them are given in terms of the decay of the matrix-valued symbol and its derivatives (or its differences) which is measured compared to the spectrum of the positive Laplace operator

\[
\mathcal{L}_G = -(X_1^2 + \cdots + X_n^2), \quad n = \dim(G).
\]

Consequently, the symbolic calculus on compact Lie groups developed with the matrix-valued symbols enjoys good properties when we look at its action on function spaces associated to the Laplacian. From this view point, if we measure the decay of global symbols used in the symbolic calculus associated to the matrix-valued quantisation, with respect to the spectrum of the sub-Laplacian \( \mathcal{L} \) (instead of using the spectrum of the Laplacian), we could provide sharp estimates for subelliptic problems without loss of regularity (see e.g. [70], for the case of the wave equation associated to sub-Laplacians) on subelliptic function spaces by exploiting in this case the sub-Riemannian structure of \( G \).

This analysis is organised as follows.

\footnote{This means that their iterated commutators of length \( \leq \kappa \) span the Lie algebra \( \mathfrak{g} \) of \( G \).}
In Section 2 we will present and we will discuss the main results of this work, and its contribution in relation with the existent literature for global operators on compact Lie groups.

In Section 3, we present the preliminaries used throughout this work. For instance, we will follow the original exposition in [107].

In Section 4 we define and develop the subelliptic pseudo-differential calculus on compact Lie groups, in terms of the matrix-valued quantisation. By using the Calderón-Zygmund estimates of Coifman and de Guzmán [41] (see also [112]), we prove Theorem 5.2 and Theorem 5.3 in Section 5.

By using the Littlewood-Paley theory and some estimates for commutators, Theorems 6.6 and 6.9 will be proved in Section 6. In Section 7 we study the notion of ellipticity associated with the developed subelliptic calculus. We provide the construction of subelliptic parametrices and we also study the heat traces and the regularised traces in the subelliptic context.

A subelliptic global functional calculus will be developed in Section 8.

Applications of this global functional calculus include the (subelliptic) Gårding inequality which will be used to study the global solvability for subelliptic evolution problems in Section 9.

Finally, we will study the \( L^2 \)-boundedness of global Fourier integral operators in Section 10.

2. OUTLINE AND MAIN RESULTS

2.1. Notation. In order to explain the main results of this work we will present some preliminaries on the matrix-valued quantisation. Indeed, by following [107, 109], we associate to a continuous linear operator \( A : C^\infty(G) \to C^\infty(G) \), the global symbol \( \sigma_A \) defined on the phase space \( G \times \hat{G} \) (here \( \hat{G} \) denotes the unitary dual of \( G \)) by the identity

\[
a(x, \xi) = \xi(x)^* A \xi(x), \quad [\xi] \in \hat{G}. \quad (2.1)
\]

Then, the operator \( A \) can be written in terms of this global symbol as

\[
Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} [\xi(x) a(x, \xi) \hat{f}(\xi)], \quad f \in C^\infty(G),
\]

where \( \hat{f} \) denotes the group Fourier transform of \( f \),

\[
\hat{f}(\xi) := \int_G f(x) \xi(x)^* dx,
\]

where \( dx \) is the (normalised) Haar measure on \( G \). We denote by \( \Psi^m_{\rho,\delta}(G, \text{loc}) \), \( \delta < \rho, \rho \geq 1 - \delta \), the Hörmander class of order \( m \) and of type \( (\rho, \delta) \). Then (see

\[\text{4} \text{ Strictly speaking from every equivalence class } [\xi] \text{ we choose one and only one matrix-valued representation so that } a(x, [\xi]) := a(x, \xi).\]

\[\text{5} \text{ i.e. the class of operators which in all local coordinate charts give operators in } \Psi^m_{\rho,\delta}(\mathbb{R}^n).\]
\[ A \in \Psi_{\rho,\delta}^m(G, \text{loc}) \quad \text{if and only if} \quad \|\Delta_\xi^\alpha \partial_x^\beta a(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}, \]

for all multi-indices \(\alpha, \beta\). Here \(\langle \xi \rangle = (1 + \lambda_\xi)^{\frac{1}{2}}\), and \(\{\lambda_\xi\}_{\xi \in \hat{G}}\) is the positive spectrum of the Laplacian \(\mathcal{L}_G\) which can be enumerated by the unitary dual and \(\{\Delta_\xi^\alpha : \alpha \in \mathbb{N}_0^\nu\}\) is the collection of the difference operators introduced in [107] that provide a difference structure on the unitary dual \(\hat{G}\). Furthermore, in [107], the Hörmander classes \(\Psi_{\rho,\delta}^m(G)\) defined by

\[ A \in \Psi_{\rho,\delta}^m(G), \quad 0 \leq \delta, \rho \leq 1, \quad \text{if and only if} \quad \|\Delta_\xi^\alpha \partial_x^\beta a(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad (2.2) \]

admit the complete range \(0 \leq \delta \leq 1\) and a symbolic calculus for these classes is possible for \(0 \leq \delta < \rho \leq 1\) without the standard restriction \(\rho \geq 1 - \delta\). The symbol classes

\[ \mathcal{S}_{\rho,\delta}^m(G \times \hat{G}) := \{a(x, \xi) : A = \text{Op}(a) \in \Psi_{\rho,\delta}^m(G)\}, \quad (2.3) \]

are useful tools in the analysis on compact Lie groups, where there appear operators with symbols in local coordinates belonging to Hörmander classes of type \((\rho, \delta)\) and satisfying the condition \(\rho \leq 1 - \delta\), under which these classes are not invariant under coordinate changes, and the classical methods, where one uses local symbols (see Hörmander [80]), could not be applicable. The subelliptic classes of pseudo-differential operators will be defined in Section 4. They will be denoted by

\[ S_{\rho,\delta}^{m,\ell,\ell'}(G \times \hat{G}) = \mathcal{S}_{\rho,\delta}^{m,\ell,\ell'}(G), \quad 0 \leq \rho, \delta \leq 1, \]

by indicating that the symbols there have order \(m\), satisfying the \(\rho\)-type (subelliptic) conditions up to order \(\ell \in \mathbb{N}\), and \(\delta\)-type (subelliptic) conditions up to order \(\ell'\). So,

\[ S_{\rho,\delta}^{m,\ell}(G \times \hat{G}) := \bigcap_{\ell', \ell} S_{\rho,\delta}^{m,\ell,\ell'}(G \times \hat{G}), \quad 0 \leq \rho, \delta \leq 1, \]

denotes the contracted class of subelliptic smooth symbols of order \(m\). We also define in Section 4 the class of symbols for subelliptic Fourier multipliers

\[ \mathcal{S}_{\rho,\delta}^{m,\ell}(\hat{G}) = \mathcal{S}_{\rho,\delta}^{m,\ell}(G), \quad 0 \leq \rho \leq 1. \]

The corresponding classes of operators associated with these symbols classes are denoted by

\[ \text{Op}(S_{\rho,\delta}^{m,\ell,\ell'}(G \times \hat{G})), \quad \text{Op}(S_{\rho,\delta}^{m,\ell}(G \times \hat{G})), \quad 0 \leq \rho, \delta \leq 1, \quad (2.4) \]

and

\[ \text{Op}(S_{\rho,\delta}^{m,\ell}(G \times \hat{G})) := \bigcap_{\ell'} \text{Op}(S_{\rho,\delta}^{m,\ell,\ell'}(G \times \hat{G})), \quad 0 \leq \rho, \delta \leq 1. \quad (2.5) \]

The reason for defining the contracted classes is that with such definition we have, concerning the orders, \(\mathcal{M}_s := (1 + \mathcal{L})^{\frac{s}{2}} \in \text{Op}(S_{\rho,\delta}^{m,\ell}(G \times \hat{G}))\) for any \(s \in \mathbb{R}\), (see Example 4.6 and Remark 4.35 for \(s > 0\), and Example 4.19 for \(s < 0\)), where \(\mathcal{M}_s\) is defined by the functional calculus.

\[ ^6 \text{e.g. these classes appear with symbols in the class } \mathcal{F}_{\rho,\delta}^{-1}(G) \text{ where we have symbols of pseudo-differential parametrices of sub-Laplacians, or the parametrix of the heat operator } D_3 - D_1^2 - D_2^2 \text{ on } SU(2) \text{ (see [112])}. \]
We will reserve as usually, the notation
\[ \text{Op}(\mathcal{S}^{m,\ell,\ell'}_{\rho,\delta}(G \times \mathcal{G})), \text{Op}(\mathcal{S}^{m,\ell}_\rho(G)), \quad 0 \leq \rho, \delta \leq 1, \] 
and
\[ \Psi^m_{\rho,\delta}(G) \equiv \text{Op}(\mathcal{S}^m_{\rho,\delta}(G \times \mathcal{G})) := \bigcap_{\ell,\ell'} \text{Op}(\mathcal{S}^{m,\ell,\ell'}_{\rho,\delta}(G \times \mathcal{G})), \quad 0 \leq \rho, \delta \leq 1, \] 
for the corresponding classes of limited regularity and of smooth symbols. Here, \( \kappa \) will be defined as the smallest even integer larger than \( \dim(G)/2 \). It is interesting to note that in stark contrast to graded Lie groups, the symbol classes here may depend on the choice of a sub-Laplacian (see Remarks 4.20 and 14.1).

2.2. Main results. In terms of the notations that we fix above, the main results of this paper are, the symbolic calculus developed in Section 4 and Section 7, the subelliptic global functional calculus developed in Section 8, and the following subelliptic boundedness results/low bounds/applications which we describe as follows and in the subsequent remarks.

- (Subelliptic Marcinkiewicz multiplier Theorem). Every \( A \in \text{Op}(S^0_{\rho,\delta}(\mathcal{G})) \), extends to an operator of weak type \((1,1)\) and is bounded on \( L^p(G) \) for all \( 1 < p < \infty \) (see Theorem 5.2). Moreover, if \( A \in \text{Op}(S^{-(1-\rho),\kappa,\ell,\ell}_\rho(G)) \), \( 0 \leq \rho < 1 \), then \( A \) extends to an operator of weak type \((1,1)\) and compact on \( L^p(G) \) for all \( 1 < p < \infty \), (see Theorem 5.2 for values of \( \kappa \)).

- (Subelliptic Calderón-Vaillancourt Theorem). For \( 0 \leq \delta \leq \rho \leq 1/2\kappa \), or \( 0 \leq \delta < \rho \leq 1 \), let us consider a continuous linear operator \( A : C^\infty(G) \to \mathcal{D}'(G) \) with symbol \( \sigma \in S^{0,\ell}_\rho(G \times \mathcal{G}) \). Then \( A \) extends to a bounded operator from \( L^2(G) \) to \( L^2(G) \). The case \( \rho = \delta = 0 \) can be deduced from Theorem 10.5.5 of [107]. We observe that a similar \( L^2 \)-theorem for \( \delta = \rho = 0 \), can be proved for global Fourier integral operators. But, we will return to this point in detail after presenting the mapping properties of the subelliptic calculus (see Theorem 10.11).

- Let \( A \in \text{Op}(S^{-(1-\rho),\kappa,\ell,\ell+1}_\rho(G)) \), \( 0 \leq \rho \leq 1 \). For \( \rho = 1 \), \( A \) extends to a bounded operator on \( L^p(G) \), and for \( 0 \leq \rho < 1 \), \( A \) extends to a compact operator on \( L^p(G) \), in both cases for all \( 1 < p < \infty \), (Theorem 5.3).

- (Subelliptic Fefferman Theorem). For any compact Lie group \( G \), let us denote by \( Q \) its Hausdorff dimension associated to the control distance associated to the sub-Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{X_1, \cdots, X_k\} \) is a system of vector fields satisfying the Hörmander condition of order \( \kappa \). For \( 0 \leq \delta < \rho \leq 1 \), \( \delta < 1/\kappa \), let us consider a continuous linear operator \( A : C^\infty(G) \to \mathcal{D}'(G) \) with symbol \( \sigma \in S^{-m,\ell,\ell}_\rho(G \times \mathcal{G}) \). If \( m = \frac{Q(1-\rho)}{2} \), then \( A \) extends to a bounded operator from \( L^\infty(G) \) to BMO\( \mathcal{L}(G) \), from

---

7 The main point in this situation is the weak (1,1) estimate because the compactness is straightforward on \( L^2(G) \) and by using the interpolation with the weak (1,1) estimate the compactness on \( L^p(G) \) also follows.

8 Again, the main point here is the boundedness estimate because the compactness is straightforward from the argument of interpolation.
the subelliptic Hardy space $H^{1,\mathcal{L}}(G)$ to $L^1(G)$, and bounded on $L^p(G)$, for all $1 < p < \infty$. Moreover, for $1 < p < \infty$, and
\[
m \geq m_p := Q(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|,
\]
the linear operator $A$ extends to a bounded operator on $L^p(G)$. (Theorems 6.6 and 6.9).

• (Subelliptic Gårding inequality). Let $a(x, D) : C^\infty(G) \to \mathcal{D}'(G)$ be an operator with symbol $a \in \mathcal{S}_{\rho,\delta}^{m,\mathcal{L}}(G \times \hat{G})$, $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, and $m > 0$. Let $\mathcal{M} = (1 + \mathcal{L})^\sharp$ be defined by the functional calculus and let $\{\hat{\mathcal{M}}(\xi)\}_{\xi \in \hat{G}}$ be its corresponding global symbol. Let us assume that $a(x, D)$ is strongly $\mathcal{L}$-elliptic which means that,
\[
A(x, \xi) := \frac{1}{2}(a(x, \xi) + a(x, \xi)^*) \quad (x, \xi) \in G \times \hat{G}, \quad a \in \mathcal{S}_{\rho,\delta}^{m,\mathcal{L}}(G \times \hat{G}),
\]
satisfies
\[
\|\hat{\mathcal{M}}(\xi)^m A(x, \xi)^{-1}\|_{\text{op}} \leq C_0.
\]
Then, there exist $C_1, C_2 > 0$, such that the lower bound
\[
\text{Re}(a(x, D)u, u) \geq C_1\|u\|_{L^2,\mathcal{L}}^2(G) - C_2\|u\|_{L^2(G)}^2,
\]
holds true for every $u \in C^\infty(G)$. Here, $H^{2,\mathcal{L}}_m(G) \equiv L^2_{m,\mathcal{L}}(G)$, is the subelliptic Sobolev space associated to $\mathcal{L}$ with regularity order $m/2$. This subelliptic version of the Gårding inequality will be proved in Theorem 8.7.

• We will use Gårding inequality to study the well posedness for the Cauchy problem
\[
(PVI) : \begin{cases} \frac{\partial v}{\partial t} = K(t, x, D)v + f, & v \in \mathcal{D}'((0, T) \times G), \\ v(0) = u_0, \end{cases}
\]
where $K(t) = K(t, x, D)$ is strongly $\mathcal{L}$-elliptic. The simplest case $K(t) = \mathcal{L}$, corresponds to the heat equation for the sub-Laplacian. In particular we can take $K(t) = a(t, x)\mathcal{L}^s$ or $K(t) = a(t, x)(1 + \mathcal{L})^\sharp$, where $a(t, x) \in C^\infty([0, T] \times G)$ is such that $|\text{Re}(a(t, x))| \geq a_0 > 0$, and $s > 0$. We refer the reader to Section 9 for details.

Now, we describe some $L^2$-estimates for global Fourier integral operators on compact Lie groups which appear as continuous linear operators of the form
\[
Af(x) := \sum_{[\xi] \in \hat{G}} \text{Tr}(e^{i\phi(x, \xi)}\sigma(x, \xi)\hat{f}(\xi)), \quad f \in C^\infty(G), \quad x \in G,
\]
where $\phi : G \times \hat{G} \to \cup_{[\xi] \in \hat{G}} \mathbb{C}^{d_{\xi} \times d_{\xi}}$ is the matrix-valued phase function of $A$. Global Fourier integral operators (FIOS) appear as useful extensions of pseudodifferential operators (see Remark 10.3) and arise in solutions of some differential problems (see e.g. Remark 10.4). We study essentially two kinds of symbol conditions:
• First, to study the $L^2$-boundedness of global FIO we need to impose reasonable conditions on the symbol $\sigma(x, \xi)$ and also on the matrix-valued phase function $\phi(x, \xi) = \text{diag}(\phi_{jj}(x, \xi))$. To do so, if $X = \{X_1, \cdots, X_n\}$ a basis for the Lie algebra $\mathfrak{g}$, and the corresponding gradient $\nabla_X$ is defined by

$$\nabla_X \psi(x) = (X_1 \psi, \cdots, X_n \psi) \in C^\infty(G) \times \cdots \times C^\infty(G), \quad \psi \in C^\infty(G),$$

in Theorem 10.11 we show that the conditions

$$\sup_{(x, [\xi]) \in G \times \hat{G}} \| X_\alpha \sigma (x, \xi) \|_{\text{op}} < \infty,$$  \quad (2.12)

for all $|\alpha| \leq 5n/2$, and

$$|\nabla_X \phi_{jj}(x, \xi) - \nabla_X \phi_{jj'}(x, \xi')| = |\lambda_{[\xi]} - \lambda_{[\xi']}|,$$  \quad (2.13)

uniformly in $([\xi], [\xi']) \in \hat{G} \times \hat{G}$ for some $0 < \tau < 1$, imply the existence of a bounded extension of $A$ (defined in (2.11)) on $L^2(G)$.

• For a global Fourier integral operator of the form

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} (\xi(x) e^{i\Phi(\xi)} \sigma(x, \xi) \hat{f}(\xi)), \quad f \in C^\infty(G),$$  \quad (2.14)

where the function $\Phi : \hat{G} \to \bigcup_{[\xi] \in \hat{G}} \mathbb{C}^{d_\xi \times d_\xi}$, is such that $\Phi(\xi) = \Phi(\xi)^*$ for all $[\xi] \in \hat{G}$, we will prove in Theorem 10.5 that under the symbol inequalities

$$\sup_{(x, [\xi]) \in G \times \hat{G}} \| X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k} \sigma(x, \xi) \|_{\text{op}} < \infty, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k,$$  \quad (2.15)

$|\alpha| \leq \kappa_Q$, if $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition, the operator $A$ extends to a bounded linear operator on $L^2(G)$. Moreover, if $1 < p < \infty$ and $0 < \rho \leq 1$, in Theorem 10.6 we will prove that under the following conditions:

$$\sup_{(x, [\xi]) \in G \times \hat{G}} \| \hat{M}(\xi)^{m + \rho|\gamma|} \Delta^{\rho}_{\xi} (e^{i\Phi(\xi)} X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k} \sigma(x, \xi)) \|_{\text{op}} < \infty, \quad \gamma \in \mathbb{N}_0^n,$$  \quad (2.16)

for all $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k$, and $|\alpha| \leq \kappa_Q,^10$ the Fourier integral operator $A$ extends to a bounded linear operator on $L^p(G)$, provided that

$$m \geq m_p := Q(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|. \quad (2.17)$$

• In particular, for $\rho = \frac{1}{2}$, Theorem 10.6 implies that under the condition

$m \geq (Q - 1) \left| \frac{1}{p} - \frac{1}{2} \right|$, the operator $A \equiv \text{FIO}(\sigma, \phi) : C^\infty(G) \to \mathcal{D}'(G)$ associated with the symbol $\sigma$ satisfying the family of inequalities in (2.16), extends to a bounded operator from $L^p(G)$ to itself where $1 < p < \infty$.

---

9 Here, $\kappa_Q$ is the smallest even integer larger than $Q/2$.

10 For all $1 < p < \infty$, $\kappa_Q,^p$ is the smallest even integer larger than $Q/p$. In particular $\kappa_Q := \kappa_{Q,^2}$. 
Remark 2.1. Let us explain the properties of the developed subelliptic calculus. In Definition 4.5 we define the (contracted) subelliptic Hörmander classes for the sub-Laplacian $\mathcal{L}$, $S_{\rho,\delta}^{m,\mathcal{L}}(G \times \hat{G})$, $0 \leq \rho, \delta \leq 1$, (and the dilated classes in Definition 4.3). These classes are closed under compositions and adjoints, and we prove in Proposition 4.23, that the operators associated to these classes have Calderón-Zygmund kernels in some sense.

Remark 2.2. The singular order for the right-convolution kernels of subelliptic operators can be classified in terms of the Hausdorff dimension $Q$ of the Lie group $G$ with respect to $\mathcal{L}$. Indeed, if $A : C^\infty(G) \to \mathcal{P}'(G)$ is a continuous linear operator with symbol $\sigma \in S_{\rho,\delta}^{m,\mathcal{L}}(G \times \hat{G})$, then the right-convolution kernel of $A$, $x \mapsto k_x : G \to C^\infty(G \setminus \{e_G\})$, defined by $k_x := \mathcal{F}^{-1}\sigma(x, \cdot)$, satisfies the following estimates for $|y| < 1$ (see Proposition 4.23):

$$
\begin{align*}
|k_x(y)| & \lesssim_m \|\sigma\|_{\ell,S_{\rho,\delta}^{m,\mathcal{L}}} |y|^{-\frac{Q+m}{p}}, & \text{if } m > -Q \\
|k_x(y)| & \lesssim_m \|\sigma\|_{\ell,S_{\rho,\delta}^{m,\mathcal{L}}} \log |y|, & \text{if } m = -Q \\
|k_x(y)| & \lesssim_m \|\sigma\|_{\ell,S_{\rho,\delta}^{m,\mathcal{L}}}, & \text{if } m < -Q.
\end{align*}
$$

where $\ell \in \mathbb{N}$, independent of $\sigma$, is large enough.

Remark 2.3. The condition (2.8) is sharp for connected and simply connected Lie groups $G$ in the following sense. If we replace the sub-Laplacian by the Laplace operator on $G$, $\mathcal{L}_G$, we recover Theorem 4.15 of [46] and then (see Remark 3.11) the critical order $m_p$ in (2.8) is the best possible in order to assure the $L^p(G)$-boundedness for operators in the class $\text{Op}(S_{\rho,\delta}^{m,\mathcal{L}}(G \times \hat{G}))$. Indeed, in this case (2.8) with $\kappa = 1$, and $Q = \dim(G)$, is a necessary and sufficient condition for the $L^p(G)$-boundedness of $A$. The condition (2.8) is an analogy of Theorem 1.2 in [34] about the boundedness of Hörmander classes on arbitrary graded groups which is an extension of the sharp theorem due to C. Fefferman [59].

Remark 2.4. Theorems 6.6 and 6.9 are analogues on compact Lie groups of the boundedness theorems due to Fefferman [59] and Hirschman [77] for the classical Hörmander classes on $\mathbb{R}^n$, extensions of the classical Wainger $L^p$-estimates for oscillating multipliers on the torus [125] and also extensions of the $L^p$-estimates for oscillating central multipliers for the Laplacian on compact connected and simply connected Lie groups proved in Chen and Fan [40, Theorem 1].

Remark 2.5. Theorem 5.2 and Theorem 5.3 are extensions of the weak (1,1) boundedness theorem proved by the second author and J. Wirth in [111, 112]. Our main $L^p$-subelliptic estimates can be used to prove estimates for pseudo-differential operators on subelliptic Sobolev and subelliptic Besov espaces (see Corollaries 5.5, 5.6 and 6.10).

Remark 2.6. The condition (2.13) for the $L^2$-boundedness of Fourier integral operators is a non-commutative version of the usual local graph condition for Fourier
integral operators, necessary for the local $L^2$-boundedness for Fourier integral operators on $\mathbb{R}^n$ (see e.g. Eskin [58] and Hörmander [79]). Theorem 10.11 is the non-commutative extension of Theorem 4.14.2 for Fourier integral operators on the torus [107] (see Theorem 10.8) and also extends the $L^2$-boundedness Theorem 10.5.5 in [107] for pseudo-differential operators on compact Lie groups.

For the general aspects of the theory of Fourier integral operators we refer the reader to Hörmander [79] and Duistermaat and Hörmander [56]. The problem of the boundedness of Fourier integral operators has been treated in Fujiwara [68], Asada and Fujiwara [3], Miyachi [89], Peral [98], Seeger, Sogge, and Stein [115], Tao [118], and also the references [104], [37], [38], [113]. Results on the boundedness of Fourier integral operators on the torus can be found in [107, Theorem 4.14.2] and [31].

Remark 2.7. We will consider a suitable notion of ellipticity associated to the sub-Laplacian $\mathcal{L}$ which we call $\mathcal{L}$-ellipticity. We found this notion consistent with the classical notion of ellipticity from the point of view of construction of parametrices (see Section 7).

Remark 2.8 (Heat traces and regularised traces for subelliptic operators). We will study the asymptotic behaviour of the heat traces and also of other regularised traces for $\mathcal{L}$-elliptic pseudo-differential operators. Indeed, we will prove under reasonable conditions on an operator $A \in S^{\rho, \delta}_{\mathcal{L}}(G \times \hat{G}), 0 \leq \delta, \rho \leq 1$, that

$$\text{Tr}(A e^{-t(1+\mathcal{L})^2}) \sim t^{-\frac{Q+\rho}{q}} \sum_{k=0}^{\infty} a_k t^k - \frac{b_0}{q} \log(t), \quad t \to 0^+, \quad (2.18)$$

for $m \geq -Q$. If $m = -Q$, then $a_k = 0$ for every $k$, and for $m > -Q$, $b_0 = 0$. If we consider the case of the Laplacian (see Corollary 7.12), and we restrict our attention to the case $(\rho, \delta) = (1, 0)$ and $m = -n$, $n := \dim(G)$, it is known that the term $b_0$ in the asymptotic expansion (2.18) agrees with the Wodzicki residue of $A$ (see e.g. Wodzicki [127] and Lesch [85]) and consequently with the Dixmier trace of $A$, which is a consequence of a celebrated theorem due to A. Connes [35]. We will also prove asymptotic expansions of the form (see Theorem 7.14),

$$\text{Tr}(A \psi(tE)) = t^{-\frac{Q+m}{q}} \left( \sum_{k=0}^{\infty} a_k t^k \right) + \frac{c_Q}{q} \int_0^\infty \psi(s) \times \frac{ds}{s}, \quad t \to 0^+, \quad (2.19)$$

for $m \geq -Q$, where $E$ is an $\mathcal{L}$-elliptic left-invariant pseudo-differential operator of order $q > 0$, and $A \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G}), 0 \leq \delta, \rho \leq 1$, is a suitable operator in the subelliptic calculus. Again, if we consider the case of the Laplacian on $G$, we replace $Q$ by $n$, and for $(\rho, \delta) = (1, 0)$ and $m = -n$, it was proved e.g. in [63], that in the constant component

$$\frac{c_n}{q} \int_0^\infty \psi(s) \times \frac{ds}{s},$$

the term $c_n$ agrees with the Wodzicki residue of $A$. Other kind of traces on compact manifolds can be found e.g. in the seminal paper of Kontsevich and Vishik...
where the canonical trace was introduced, and other complete references on the subject as Fedosov, Golse, and Leichtnam [60], Grubb and Schrohe [75], Scott [114], and Paycha [95]. We refer the reader to [62, 63] for the treatment of regularised traces (in whose expansions appear the Wodzicki residue and the canonical trace) using the matrix-valued quantisation. A complete investigation about the spectral trace of global operators on compact Lie groups can be found in [47, 48, 49, 50].

The second source of applications came from the functional calculus for subelliptic operators developed in Section 8. Indeed, as application of the subelliptic functional calculus we will deduce a subelliptic version of the Gårding inequality and we will study the Dixmier traceability of subelliptic operators.

Remark 2.9. One of main features of the developed subelliptic calculus is that if we replace the role of the sub-Laplacian by the Laplace operator on the group G, we recover the known properties for the global Hörmander classes on compact Lie groups [61, 107, 108] which makes our calculus parallel to others existing in the literature, where the global symbols are used e.g. to develop the calculus on graded Lie groups by using Rockland operators [65].

3. Preliminaries: sub-Laplacians and pseudo-differential operators on compact Lie groups

3.1. Pseudo-differential operators via localisations. In this subsection we describe the well-known formulation of pseudo-differential operators on compact manifolds (and so on compact Lie groups) via local symbols (see Hörmander [80] and e.g. the book of M. Taylor [119]). If U is an open topological subset of \( \mathbb{R}^n \), we say that \( a : U \times \mathbb{R}^n \to \mathbb{C} \), belongs to the Hörmander class \( S^{m}_{\rho,\delta}(U \times \mathbb{R}^n) \), 0 \( \leq \rho, \delta \leq 1 \), if for every compact subset \( K \subset U \), the symbol inequalities,

\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\rho,\delta}(K) (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},
\]

hold true uniformly in \( x \in K \) and \( \xi \in \mathbb{R}^n \). Then, a continuous linear operator \( A : C^\infty_0(U) \to C^\infty(U) \) is a pseudo-differential operator of order \( m \), of \( (\rho, \delta) \)-type, if there exists a function \( a \in S^{m}_{\rho,\delta}(U \times \mathbb{R}^n) \), satisfying

\[
Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi)(\mathfrak{F}_{\mathbb{R}^n} f)(\xi) d\xi,
\]

for all \( f \in C^\infty_0(U) \), where

\[
(\mathfrak{F}_{\mathbb{R}^n} f)(\xi) := \int_{U} e^{-i2\pi x \cdot \xi} f(x) dx,
\]

is the Euclidean Fourier transform of \( f \) at \( \xi \in \mathbb{R}^n \). The class \( S^{m}_{\rho,\delta}(U \times \mathbb{R}^n) \) on the phase space \( U \times \mathbb{R}^n \), is invariant under coordinate changes only if \( \rho \geq 1 - \delta \), while a symbolic calculus (closed for products, adjoints, parametrices, etc.) is only possible for \( \delta < \rho \) and \( \rho \geq 1 - \delta \). In the case of a \( C^\infty \)-manifold \( M \), a linear continuous operator \( A : C^\infty_0(M) \to C^\infty(M) \) is a pseudo-differential operator of
order $m$, of $(\rho, \delta)$-type, $\rho \geq 1 - \delta$, if for every local coordinate patch $\omega : M_\omega \subset M \to U \subset \mathbb{R}^n$, and for every $\phi, \psi \in C_0^\infty(U)$, the operator

$$Tu := \psi(\omega^{-1})^* A \omega^*(\phi u), \ u \in C^\infty(U),$$

is a pseudo-differential operator with symbol $a_T \in S^m_{\rho, \delta}(U \times \mathbb{R}^n)$. In this case we write that $A \in \Psi^m_{\rho, \delta}(M, \text{loc})$.

### 3.2. The positive sub-Laplacian and pseudo-differential operators via global symbols

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$. Under the identification $\mathfrak{g} \simeq T_{e_G} G$, where $e_G$ is the identity element of $G$, let us consider a system of $C^\infty$-vector fields $X = \{X_1, \cdots, X_k\} \in \mathfrak{g}$. For all $I = (i_1, \cdots, i_\omega) \in \{1, 2, \cdots, k\}^\omega$, of length $\omega \geq 2$, denote

$$X_I := [X_{i_1}, [X_{i_2}, \cdots [X_{i_{\omega-1}}, X_{i_{\omega}}], \cdots],$$

and for $\omega = 1$, $I = (i)$, $X_I := X_i$. Let $V_\omega$ be the subspace generated by the set $\{X_I : |I| \leq \omega\}$. That $X$ satisfies the Hörmander condition, means that there exists $\kappa' \in \mathbb{N}$ such that $V_{\kappa'} = \mathfrak{g}$. Certainly, we consider the smallest $\kappa'$ with this property and we denote it by $\kappa$ which will be later called the step of the system $X$. We also say that $X$ satisfies the Hörmander condition of order $\kappa$. Note that the sum of squares

$$\mathcal{L} \equiv \mathcal{L}_X := -(X_1^2 + \cdots + X_k^2),$$

is a subelliptic operator which by following the usual nomenclature is called the subelliptic Laplacian associated with the family $X$. For short we refer to $\mathcal{L}$ as the sub-Laplacian. In view of the Hörmander theorem on sums of the squares of vector fields (see Hörmander [78]) it is a hypoelliptic operator (i.e. if $\mathcal{L}u \in C^\infty(G)$ with $u \in \mathcal{D}'(G)$ then $u \in C^\infty(G)$, and also locally at all points). For other aspects on the analysis of the sub-Laplacian we refer the reader to Agrachev et al. [1], Bismut [18], Domokos et al. [54] as well as to the fundamental book of Montgomery [90].

A central notion in the analysis of the sub-Laplacian is that of the Hausdorff dimension, in this case, associated to $\mathcal{L}$. Indeed, for all $x \in G$, denote by $H^\omega_x G$ the subspace of the tangent space $T_x G$ generated by the $X_I$’s and all the Lie brackets

$$[X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}, [X_{j_2}, \cdots, , X_{j_\omega}]],$$

with $\omega \leq \kappa$. The Hörmander condition can be stated as $H^\kappa_x G = T_x G$, $x \in G$. We have the filtration

$$H^0_x G \subset H^1_x G \subset H^2_x G \subset \cdots \subset H^{\kappa-1}_x G \subset H^\kappa_x G = T_x G, \ x \in G.$$

In our case, the dimension of every $H^\omega_x G$ does not depend on $x$ and we write $\dim H^\omega_x G := \dim H^\omega_x G$, for any $x \in G$. So, the Hausdorff dimension can be defined as (see e.g. [76, p. 6]),

$$Q := \dim(H^{i+1} G) - \dim H^i G) - \dim H^i G). \tag{3.1}$$

Explicit examples of sub-Laplacians in some compact Lie groups are given in Section 11 for $S^3 \cong \text{SU}(2)$, $\text{SU}(3)$, $\text{SO}(4)$ and $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$.

---

11As usually, $\omega^*$ and $(\omega^{-1})^*$ are the pullbacks induced by the maps $\omega$ and $\omega^{-1}$, respectively.
In this work we are interested in developing a pseudo-differential calculus associated to the sub-Laplacian $\mathcal{L}$. We will use the quantisation process developed by the second author and V. Turunen in [107]. We explain it as follows. First, let us record the notion of the unitary dual of a compact Lie group $G$, $\hat{G}$. So, let us assume that $\xi$ is a continuous, unitary and irreducible representation of $G$, this means that,

- $\xi \in \text{Hom}(G, U(H_\xi))$, for some finite-dimensional vector space $H_\xi \cong \mathbb{C}^{d_\xi}$, i.e. $\xi(xy) = \xi(x)\xi(y)$ and for the adjoint of $\xi(x)$, $\xi(x)^* = \xi(x^{-1})$, for every $x, y \in G$.
- The map $(x, v) \mapsto \xi(x)v$, from $G \times H_\xi$ into $H_\xi$ is continuous.
- For every $x \in G$, and $W_\xi \subset H_\xi$, if $\xi(x)W_\xi \subset W_\xi$, then $W_\xi = H_\xi$ or $W_\xi = \emptyset$.

Let $\text{Rep}(G)$ be the set of unitary, continuous and irreducible representations of $G$. The relation,

$\xi_1 \sim \xi_2$ if and only if, there exists $A \in \text{End}(H_{\xi_1}, H_{\xi_2})$, such that $A\xi_1(x)A^{-1} = \xi_2(x)$, for every $x \in G$, is an equivalence relation and the unitary dual of $G$, denoted by $\hat{G}$ is defined via

$$\hat{G} := \text{Rep}(G)/\sim.$$  

By a suitable changes of basis, we always can assume that every $\xi$ is matrix-valued and that $H_\xi = \mathbb{C}^{d_\xi}$. If a representation $\xi$ is unitary, then

$$\xi(G) := \{\xi(x) : x \in G\}$$

is a subgroup (of the group of matrices $\mathbb{C}^{d_\xi \times d_\xi}$) which is isomorphic to the original group $G$. Thus the homomorphism $\xi$ allows us to represent the compact Lie group $G$ as a group of matrices. This is the motivation for the term ‘representation’.

Now, let us follow [107, Chapter 10] to introduce the analysis of operators on the phase space $G \times \hat{G}$. Indeed, if $A$ is a continuous linear operator on $C^\infty(G)$, there exists a function

$$a : G \times \hat{G} \to \cup_{\ell \in \mathbb{N}} \mathbb{C}^{d_\xi \times \ell},$$

(3.2)

such that for every equivalence class $[\xi] \in \hat{G}$, $a(x, \xi) := a(x, [\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}$, (where $d_\xi$ is the dimension of the continuous, unitary and irreducible representation $\xi : G \to U(\mathbb{C}^{d_\xi})$) and satisfying

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)a(x, \xi)\hat{f}(\xi)], \quad f \in C^\infty(G).$$

(3.3)

Here

$$\hat{f}(\xi) \equiv (\mathcal{F} f)(\xi) := \int_G f(x)\xi(x)^*dx \in \mathbb{C}^{d_\xi \times d_\xi}, \quad [\xi] \in \hat{G},$$

is the matrix-valued Fourier transform of $f$ at $\xi = (\xi_{ij})_{i,j=1}^{d_\xi}$. The function $a$ in (3.2), satisfying (3.3) is unique and satisfies the identity,\(^\text{12}\)

$$a(x, \xi) = \xi(x)^*(A\xi)(x), \quad A\xi := (A\xi_{ij})_{i,j=1}^{d_\xi}, \quad [\xi] \in \hat{G}.$$
In general, we refer to the function $a$ as the (global or matrix) symbol of the operator $A$.

**Remark 3.1.** Let us denote by $\mathcal{H}(\hat{G}) := \mathcal{F}(C^\infty(G))$ the Schwartz space on the unitary dual. Then the Fourier transform on the group $\mathcal{F}$ is a bijective mapping from $C^\infty(G)$ into $\mathcal{H}(\hat{G})$ (see [107, Page 541]), and in terms of the Fourier transform we have

$$Af(x) = \mathcal{F}^{-1}[a(x,\cdot)(\mathcal{F}f)](x),$$

for every $f \in C^\infty(G)$. In particular, if $a(x,\xi) = I_{d_c}$ is the identity matrix in every representation space, $A \equiv I$ is the identity operator on $C^\infty(G)$, and we recover the Fourier inversion formula

$$f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)\hat{f}(\xi)], \ f \in C^\infty(G).$$

To classify symbols in the Hörmander classes developed in [107], the notion of difference operators on the unitary dual, by endowing $\hat{G}$ with a difference structure, is an instrumental tool. By following [112], a difference operator $Q_\xi$ of order $k$, is defined by

$$Q_\xi \hat{f}(\xi) = \hat{q}(\xi)\hat{f}(\xi), \ [\xi] \in \hat{G},$$

for all $f \in C^\infty(G)$, for some function $q$ vanishing of order $k$ at the identity $e = e_G$. We will denote by $\text{diff}^k(\hat{G})$ the set of all difference operators of order $k$. For a fixed smooth function $q$, the associated difference operator will be denoted by $\Delta_q := Q_\xi$. We will choose an admissible collection of difference operators (see e.g. [46, 112]),

$$\Delta_\xi^\alpha := \Delta_{q^{(1)}} \cdots \Delta_{q^{(n)}}, \ \alpha = (\alpha_j)_{1 \leq j \leq i},$$

where

$$\text{rank}\{\nabla q_{(j)}(e) : 1 \leq j \leq i\} = \dim(G), \ \text{and} \ \Delta_{q_{(j)}} \in \text{diff}(\hat{G}).$$

We say that this admissible collection is strongly admissible if

$$\bigcap_{j=1}^i \{x \in G : q_{(j)}(x) = 0\} = \{e_G\}.$$

**Remark 3.2.** Difference operators can be defined by using the representation theory on the group $G$. Indeed, if $\xi_0$ is a fixed irreducible representation, a special matrix-valued difference operator is given by $D_{\xi_0} = (D_{\xi_0, i, j})_{i,j=1}^{d_c} = \xi_0(\cdot) - I_{d_c}$. If the representation is fixed we omit the index $\xi_0$ so that, from a sequence $D_1 = D_{\xi_0, j_1, i_1}, \ldots, D_n = D_{\xi_0, j_n, i_n}$ of operators of this type we define $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $\alpha \in \mathbb{N}^n$.

**Remark 3.3** (Leibniz rule for difference operators). The difference structure on the unitary dual $\hat{G}$, induced by the difference operators acting on the momentum variable $[\xi] \in \hat{G}$, implies the following Leibniz rule

$$\Delta_{q}(a_1a_2)(x_0, \xi) = \sum_{|\gamma|,|\delta| < |\eta| < |\eta| + |\epsilon|} C_{\xi,\eta}(\Delta_{q, a_1})(x_0, \xi)(\Delta_{q, a_2})(x_0, \xi), \ (x_0, [\xi]) \in G \times \hat{G},$$

for $a_1, a_2 \in C^\infty(G, \mathcal{H}(\hat{G}))$. For details we refer the reader to [107, 108].
Remark 3.4. Every $X \in \mathfrak{g}$, can be identified with the differential operator $X : C^\infty(G) \to C^\infty(G)$ defined by
\[
(X_x f)(x) := \frac{d}{dt} f(x \exp(tX))|_{t=0}, \quad x \in G.
\]

If $A \in \Psi^m_{\rho,\delta}(G, \text{loc})$, $\rho \geq 1 - \delta$, the matrix-valued symbol $\sigma_A$ of $A$ satisfies (see [107, 108]),
\[
\|X^\beta_{x} \Delta_{q_0} \sigma_A(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\gamma|+\delta|\beta|} \tag{3.5}
\]
for all $\beta$ and $\gamma$ multi-indices and all $(x, [\xi]) \in G \times \hat{G}$. Now, if $0 \leq \delta, \rho \leq 1$, we say that $\sigma_A \in \mathcal{S}^m_{\rho,\delta}(G \times \hat{G})$, if the global symbol inequalities (3.5) hold true. So, for $\sigma_A \in \mathcal{S}^m_{\rho,\delta}(G \times \hat{G})$ we write $A \in \Psi^m_{\rho,\delta}(G) \equiv \text{Op}(\mathcal{S}^m_{\rho,\delta}(G \times \hat{G}))$. As we mentioned early in the introduction,
\[
\text{Op}(\mathcal{S}^m_{\rho,\delta}(G \times \hat{G})) = \Psi^m_{\rho,\delta}(G, \text{loc}), \quad 0 < \delta < \rho \leq 1, \quad \rho \geq 1 - \delta.
\]

3.3. Calderón-Zygmund type estimates for multipliers. In order to provide $L^p$-estimates for multipliers in the subelliptic context, we will use the techniques developed by the second author and J. Wirth in [112], where a special case (compatible with the notion of difference operators) of a statement of Coifman and de Guzmán ([41], Theorem 2) was established. We record it as follows (see [112, p. 630]).

Criterion 3.5. Assume that $A : L^2(G) \to L^2(G)$ is a left-invariant operator on $G$ satisfying
\[
\|A \psi_r\|_{L^2(G, \rho(x)^n(1+\varepsilon)dx)} := \left( \int_G |A \psi_r(x)|^2 \rho(x)^{n(1+\varepsilon)}dx \right)^{\frac{1}{2}} \leq C r^{\frac{\varepsilon}{2}}, \tag{3.6}
\]
for some constants $C > 0$ and $\varepsilon > 0$, uniformly in $r$. Then $A$ is of weak type $(1, 1)$ and bounded on $L^p(G)$, for all $1 < p < \infty$.

The family $\{\psi_r\}_{r>0}$ that appears in Criterion (3.5) is defined by $\psi_r = \phi_r - \phi_{r/2}$, where the functions in the net $\{\phi_r\}_{r>0}$, satisfy, among other things, the following properties (see [41, p. 140]):
\begin{itemize}
  \item $\int_G \phi_r(x)dx = 1$,
  \item $\int_G \phi_r^2(x)dx = O(\frac{1}{r})$,
  \item $\phi_r * \phi_s = \phi_s * \phi_r$, $r, s > 0$.
\end{itemize}

The function $\rho : x \mapsto \rho(x)$, appearing in (3.6), is a suitable pseudo-distance defined on $G$. If $G$ is semi-simple (this means that the centre of $G$, $Z(G)$ is trivial), it is defined by
\[
\rho(x)^2 := \dim(G) - \text{Tr}(\text{Ad}(x)) = \sum_{\xi \in \Delta_0} (d_\xi - \text{Tr}(\xi(x))), \quad x \in G, \tag{3.7}
\]
where $\text{Ad} : G \to \text{U}(\mathfrak{g})$, and $\Delta_0$ is the system of positive roots. It can be decomposed into irreducible representations as,

$$\text{Ad} = [\text{rank}(G)e_G] \oplus \left( \bigoplus_{\xi \in \Delta_0} \xi \right),$$

where $e_G$ is the trivial representation. With the consideration on the centre $Z(G) = \{e_G\}$, it can be shown (see Lemma 3.1 of [112]) that

- $\rho^2(x) \geq 0$ and $\rho(x) = 0$ if and only if $x = e_G$. 
- $\Delta_{\rho^2} \in \text{diff}^2(\hat{G})$.

If $G$ is not semi-simple, we refer the reader to [112, Remark 3.2] for the modifications in the definition of $\rho$, in this particular case. For our further analysis, we will use the following lemma which exploits the properties of the functions $\psi_r$, (see Lemma 3.4 of [112, p. 630]).

**Lemma 3.6.** Let $q \in C^\infty(G)$ be a smooth function vanishing to order $\tilde{l} \in \mathbb{R}$, at $e_G$. Then

$$\|q(x)\psi_r\|_{H^{-s}(G)} \leq C_{q,s}r^{\frac{\tilde{l}}{2} - \frac{s}{2}},$$

for all $0 \leq s \leq 1 + \frac{n}{2}$.

### 3.4. $L^p$-multipliers and $L^p$-boundedness of pseudo-differential operators on compact Lie groups

We record the $L^p$-estimates for multipliers on compact Lie groups through the methods developed by the second author and J. Wirth in [112] by using Criterion 3.5. We will denote by $\Sigma(\hat{G} \times G)$ and $\Sigma(\hat{G})$ the space of matrix-valued functions,

$$\Sigma(\hat{G} \times G) := \{\sigma : \hat{G} \times G \to \cup_{[\xi] \in \hat{G}} \mathbb{C}^{d_{\xi} \times d_{\xi}}\},$$

$$\Sigma(\hat{G}) := \{\sigma : \hat{G} \to \cup_{[\xi] \in \hat{G}} \mathbb{C}^{d_{\xi} \times d_{\xi}}\}.$$

**Theorem 3.7.** Let $G$ be a compact Lie group and let $\mathfrak{z} \in 2\mathbb{N}$ be such that $\mathfrak{z} > \frac{n}{2}$. Let $a \in \Sigma(\hat{G})$ be a symbol, satisfying

$$\|D^\alpha a(\xi)\|_{\text{op}} \leq C_{\alpha} \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \mathfrak{z}.$$

Then $A = \text{Op}(a)$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for all $1 < p < \infty$. Moreover, if $0 \leq \rho < 1$, and a satisfies

$$\|D^\alpha a(\xi)\|_{\text{op}} \leq C_{\alpha} \langle \xi \rangle^{-\rho|\alpha|}, \quad |\alpha| \leq \mathfrak{z},$$

then $A$ extends to a bounded operator from $L^p_r(G)$ into $L^p(G)$ for all $1 < p < \infty$ and $r = \mathfrak{z}(1 - \rho)\left[\frac{1}{p} - \frac{1}{2}\right]$. Here $L^p_r(G)$ denotes the Sobolev space of order $r$ over $L^p(G)$.

From the proof of Corollary 5.1 of [112], one has the following version of Theorem 3.7.

**Theorem 3.8.** Let us assume that $G$ is a compact Lie group of dimension $n = 2d$ or $n = 2d + 1$, and that $d$ is odd. Let $0 < \rho \leq 1$, and $a \in \Sigma(\hat{G})$ be a symbol satisfying

$$\|D^\alpha a(\xi)\|_{\text{op}} \leq C_{\alpha} \langle \xi \rangle^{-\mathfrak{z}(1 - \rho) - \rho|\alpha|}, \quad |\alpha| \leq \mathfrak{z} := d + 1,$$
then $A = \text{Op}(a)$ extends to a linear operator of weak type $(1, 1)$. Moreover, if the dimension of the group is $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, and $d$ is even, the conclusion on $A$ is the same provided that
\[
\|D^\alpha a(\xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-\rho(\alpha)}, \quad |\alpha| \leq \kappa := d + 2.
\]

The argument developed in [112] using the Sobolev embedding theorem for extending the $L^p$-estimates from multipliers to pseudo-differential operators, allows us to present the following consequence of Theorem 3.8 (see Theorem 5.2 of [112]).

**Theorem 3.9.** Let us assume that $G$ is a compact Lie group of dimension $n = 2d$ or $n = 2d + 1$, and that $d$ is odd. Let $a \in \Sigma(G \times \hat{G})$ be a non-invariant symbol satisfying
\[
\|X_{x}^\beta D^\alpha a(x, \xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \kappa := d + 1, \quad |\beta| \leq \left\lfloor \frac{n}{p} \right\rfloor + 1,
\]
then $A = \text{Op}(a)$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. Moreover, if the dimension of the group is $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, and $d$ is even, the conclusion on $A$ is the same provided that
\[
\|X_{x}^\beta D^\alpha a(x, \xi)\|_{\text{op}} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}, \quad |\alpha| \leq \kappa := d + 2, \quad |\beta| \leq \left\lfloor \frac{n}{p} \right\rfloor + 1.
\]

The following theorem records the action of the Hörmander classes $\mathcal{S}_{\rho,\delta}^{m}(G \times \hat{G})$ (see (2.3)) on $L^p(G)$ spaces (see [46]).

**Theorem 3.10.** Let $G$ be a compact Lie group of dimension $n$. Let $0 \leq \delta < \rho \leq 1$ and let
\[
0 \leq \nu < \frac{n(1 - \rho)}{2},
\]
for $0 < \rho < 1$ and $\nu = 0$ for $\rho = 1$. Let $\sigma \in \mathcal{S}_{\rho,\delta}^{-\nu}(G \times \hat{G})$. Then $A \equiv \sigma(x, D)$ extends to a bounded operator on $L^p(G)$ provided that
\[
\nu \geq n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|.
\]  

**Remark 3.11** (Sharpness of Theorem 3.10). Let $G$ be a connected, simply connected, compact semi-simple Lie group of dimension $n$. Theorem 3.10 is sharp in the following sense. Let $\Lambda$ be the lattice of the highest weights, $\Delta_0$ be the system of positive roots and $\delta$ be the half sum of all positive roots. There exists a correspondence $\lambda \mapsto [\xi_\lambda]$ between $\Lambda$ and the unitary dual $\hat{G}$. For $0 < \beta < 1$, $\alpha > 0$, let us define the oscillating Fourier multiplier,
\[
T_{\alpha,\beta,\Omega}f(x) = \sum_{\lambda + \delta \in \Lambda \setminus \{0\}} d_{\xi_\lambda} \text{Tr}[\xi_\lambda(x)\sigma_{\alpha,\beta}(\xi_\lambda)\hat{f}(\xi_\lambda)],
\]
with symbol
\[
\sigma_{\alpha,\beta}(\xi_\lambda) := \frac{e^{i|\lambda + \delta|\beta} \Omega(\frac{\lambda + \delta}{|\lambda + \delta|})}{\|\lambda + \delta\|^\alpha} I_{d_{\xi_\lambda}},
\]
3.5. The subelliptic spaces $H^1$ and BMO on compact Lie groups. Let $G$ be a compact Lie group. Let us consider a sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$ on $G$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of step $\kappa$. For every point $g \in G$, let us denote $X_g = \{X_i\}_{i=1}^k$, $\mathcal{H}_g = \text{span}\{X_g\}$. We say that a curve $\gamma : [0,1] \to G$ is horizontal if

$$\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}, \text{ for a.e. } t \in (0,1).$$

The Carnot-Carathéodory distance associated to the sub-Riemannian structure induced by $X$, is defined by

$$d_s(g_0, g_1) := \inf_{\gamma \text{ horizontal}} \{l(\gamma) := \int_0^1 |\dot{\gamma}(t)|dt : \gamma(0) = g_0, \gamma(1) = g_1\}, \quad g_0, g_1 \in G.$$
Then subelliptic $BMO$ space on $G$, $BMO^\mathcal{L}(G)$, is the space of locally integrable functions $f$ satisfying

$$
\|f\|_{BMO^\mathcal{L}(G)} := \sup_{B} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty, \text{ where } f_B := \frac{1}{|B|} \int_B f(x) dx,
$$

and $B$ ranges over all balls $B(x_0, r)$, with $(x_0, r) \in G \times (0, \infty)$. The subelliptic Hardy space $H^{1,\mathcal{L}}(G)$ will be defined via the atomic decomposition. Thus, $f \in H^{1,\mathcal{L}}(G)$ if and only if $f$ can be expressed as

$$
f = \sum_{j=1}^{\infty} c_j a_j,
$$

where $\{c_j\}_{j=1}^{\infty}$ is a sequence in $\ell^1(\mathbb{N})$, and every function $a_j$ is an atom, i.e., $a_j$ is supported in some ball $B_j$, $(a_j$ satisfies the cancellation property)

$$
\int_{B_j} a_j(x) dx = 0,
$$

and

$$
\|a_j\|_{L^\infty(G)} \leq \frac{1}{|B_j|}.
$$

The norm $\|f\|_{H^{1,\mathcal{L}}(G)}$ is the infimum over all possible series $\sum_{j=1}^{\infty} |c_j|$. Furthermore $BMO^\mathcal{L}(G)$ is the dual of $H^{1,\mathcal{L}}(G)$. This can be understood in the following sense:

(a). If $\phi \in BMO^\mathcal{L}(G)$, then

$$
\Phi : f \mapsto \int_G f(x) \phi(x) dx,
$$

admits a bounded extension on $H^{1,\mathcal{L}}(G)$.

(b). Conversely, every continuous linear functional $\Phi$ on $H^{1,\mathcal{L}}(G)$ arises as in (a) with a unique element $\phi \in BMO^\mathcal{L}(G)$.

The norm of $\phi$ as a linear functional on $H^{1,\mathcal{L}}(G)$ is equivalent with the $BMO^\mathcal{L}(G)$-norm. Important properties of the $BMO^\mathcal{L}(G)$ and the $H^{1,\mathcal{L}}(G)$ norms are the following,

$$
\|f\|_{BMO^\mathcal{L}(G)} = \sup_{|g|_{H^{1,\mathcal{L}}(G)} = 1} \left| \int_G f(x)g(x) dx \right|,
$$

$$
\|g\|_{H^{1,\mathcal{L}}(G)} = \sup_{|f|_{BMO^\mathcal{L}(G)} = 1} \left| \int_G f(x)g(x) dx \right|.
$$

If we replace $\mathcal{L}$ by the Laplacian $\mathcal{L}_G$ in the definitions above, we will write $BMO(G)$ and $H^{1}(G)$, defined by the distance induced by the usual bi-invariant Riemannian metric on $G$. The subelliptic Fefferman-Stein interpolation theorem in this case can be stated as follows (see Carbonaro, Mauceri and Meda [23]).
Theorem 3.13. Let $G$ be a compact Lie group. Let us consider a sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$ on $G$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of step $\kappa$. For every $\theta \in (0, 1)$, we have,

- If $p_\theta = \frac{2}{1-\theta}$, then $(L^2, \text{BMO}^2)_{[\theta]}(G) = L^{p_\theta}(G)$.
- If $p_\theta = \frac{2}{2-\theta}$, then $(H^{1,\mathcal{L}}, L^2)_{[\theta]}(G) = L^{p_\theta}(G)$.

4. Subelliptic pseudo-differential operators

4.1. Subelliptic symbols on compact Lie groups. In order to establish the basic properties of the subelliptic symbolic calculus, and as it was pointed out in [107], we will use as in the Euclidean case, the expansion of smooth functions in the Taylor series. Although it has been constructed in [107, Section 10.6] for arbitrary compact Lie groups, and we apply it in further sections, we will explain this notion in the case of compact connected Lie groups.

Remark 4.1 (Local Taylor series on compact connected Lie groups). If $V$ is an open and convex subset of $\mathbb{R}^n$, and $h \in C^\infty(V)$, the Taylor polynomial of order $N$ at $y_0 \in V$, is given by

$$h(y) = (p_{y_0,N}h)(y) + (R_Nh)(y, y_0), \quad (p_{y_0,N}h)(y) := \sum_{|\alpha| \leq N} \frac{(y - y_0)^\alpha}{\alpha!} \frac{\partial^\alpha h}{\partial y^\alpha}(y_0),$$

for all $y \in V$, where

$$(R_Nh)(y, y_0) := (N + 1) \sum_{|\alpha| = N + 1} \frac{(y - y_0)^\alpha}{\alpha!} \int_0^1 (1 - t)^N \frac{\partial^\alpha h}{\partial y^\alpha}(y_0 + t(y - y_0))dt.$$ 

Now, let $G$ be a connected compact Lie group with Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and of dimension $n$. The exponential map

$$\exp : \mathfrak{g} \rightarrow G, \tag{4.1}$$

is a local diffeomorphism from an open neighbourhood of $0_\mathfrak{g}$ into an open set containing the identity $e_G$ of $G$. Moreover, both hypothesis, connectedness and compactness, assure the completeness of $G$ and hence the exponential map is surjective according to the Hopf-Rinow Theorem (and the fact that the Riemannian exponential map agrees with the exponential map). So, if $f \in C^\infty(G)$, and $x \in G$, $f(x) = f(\exp(X))$ for some $X \in \mathfrak{g}$. By defining the function $\tilde{f} := f \circ \exp$, and under the identification $\mathfrak{g} \simeq \mathbb{R}^n$, we can think that $\tilde{f}$ is defined on $\mathbb{R}^n$ and consequently, we have the expansion at $0_\mathfrak{g} \sim 0 \in \mathbb{R}^n$,

$$\tilde{f}(X) = (p_{0,N}\tilde{f})(X) + (R_{0,N}\tilde{f})(X, 0), \quad X \in \mathfrak{g}.$$ 

Returning to the coordinates of $G$ with the inverse exponential map, we have

$$f(x) = \sum_{|\alpha| \leq N} \frac{x^\alpha}{\alpha!} \frac{\partial^\alpha (f \circ \exp)}{\partial y^\alpha}(0)$$

$$+ (N + 1) \sum_{|\alpha| = N + 1} \frac{x^\alpha}{\alpha!} \int_0^1 (1 - t)^N \frac{\partial^\alpha (f \circ \exp)}{\partial y^\alpha}(t \exp^{-1}(x))dt,$$
where, for a basis $B = \{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$ we have used the multi-index notation $x^\alpha := \omega(X_1)^{a_1} \cdots \omega(X_n)^{a_n} = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, again using the identification between $\mathfrak{g}$ and $\mathbb{R}^n$, under the diffeomorphism $\omega: \mathfrak{g} \to \mathbb{R}^n$, $y_i = \omega(X_i)$, where $y_i$, $1 \leq i \leq n$, are the coordinate functions on $\mathbb{R}^n$. We will use the notation

$$\partial_{x_i}^\alpha f := \frac{\partial^{\alpha_i}(f \circ \exp)}{\partial y_i^{\alpha_i}}, \quad \partial_x^\alpha f := \frac{\partial^{\alpha}(f \circ \exp)}{\partial y^{\alpha}} \equiv \partial_{x_1}^\alpha \partial_{x_2}^\alpha \cdots \partial_{x_n}^\alpha f, \quad (4.2)$$

for the local differential operators appearing in the local Taylor series. However, in order to introduce our subelliptic classes, we need a suitable Taylor expansion associated with a suitable system of vector fields. So, we present it in the following Lemma (see Lemma 7.4 in [61]).

**Lemma 4.2** (Global Taylor Series on compact Lie groups). Let $G$ be a compact Lie group of dimension $n$. Let us consider an strongly admissible admissible collection of difference operators $\mathcal{D} = \{\Delta_{q(j)}\}_{1 \leq j \leq n}$, which means that

$$\text{rank}\{\nabla q(j)(e) : 1 \leq j \leq n\} = n, \quad \bigcap_{j=1}^n \{x \in G : q(j)(x) = 0\} = \{e_G\}.$$ 

Then there exists a basis $X_\mathcal{D} = \{X_{1,\mathcal{D}}, \ldots, X_{n,\mathcal{D}}\}$ of $\mathfrak{g}$, such that $X_{j,\mathcal{D}} q(k)(\cdot^{-1})(e_G) = \delta_{jk}$. Moreover, by using the multi-index notation $\partial_X^{(\beta)} = \partial_{X_1}^{\beta_1} \cdots \partial_{X_n}^{\beta_n}$, $\beta \in \mathbb{N}_0^n$, where

$$\partial_{X_{i,\mathcal{D}}} f(x) := \frac{d}{dt} f(x \exp(tX_{i,\mathcal{D}}))|_{t=0}, \quad f \in C^\infty(G),$$

and denoting for every $f \in C^\infty(G)$

$$R^{f}_{x,N}(y) := f(xy) - \sum_{|\alpha| < N} q^{\alpha}_{(1)}(y^{-1}) \cdots q^{\alpha}_{(n)}(y^{-1}) \partial_X^{(\alpha)} f(x),$$

we have that

$$|R^{f}_{x,N}(y)| \leq C|y|^N \max_{|\alpha| \leq N} \|\partial_X^{(\alpha)} f\|_{L^\infty(G)}.$$ 

The constant $C > 0$ is dependent on $N$, $G$ and $\mathcal{D}$, but not on $f \in C^\infty(G)$. Also, we have that $\partial_X^{(\beta)} |_{x=1} R^{f}_{x,N} = R^{f^{(\beta)}}_{x,N}$ and

$$|\partial_X^{(\beta)} |_{y=1} R^{f}_{x,N}(y_1) | \leq C|y|^{N-|\beta|} \max_{|\alpha| \leq N-|\beta|} \|\partial_X^{(\alpha+\beta)} f\|_{L^\infty(G)},$$

provided that $|\beta| \leq N$.

Now with the notation above and the following one $\Delta_X^\alpha := \Delta_{q(1)}^{a_1} \cdots \Delta_{q(n)}^{a_n}$, we introduce the subelliptic Hörmander class of symbols of order $m \in \mathbb{R}$, in the $(\rho, \delta)$-class.

**Definition 4.3** (Dilated subelliptic Hörmander classes). Let $G$ be a compact Lie group and let $0 \leq \delta, \rho \leq \kappa$. Let us consider a sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$ on $G$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of step $\kappa$. The dilated class $\mathcal{S}_{\rho,\delta}^m(G)$ of subelliptic Hörmander order $m$
and of type \((\rho, \delta)\), consists of those functions \(\sigma \in \Sigma(G \times \hat{G})\), satisfying the symbol inequalities

\[
p_{\alpha, \beta, \rho, \delta, m, \text{left}}(a) := \sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)} \partial_X^{(\beta)} \Delta_\xi^\sigma a(x, \xi)\|_{\text{op}} < \infty, \quad (4.3)
\]

and

\[
p_{\alpha, \beta, \rho, \delta, m, \text{right}}(a) := \sup_{(x, [\xi]) \in G \times \hat{G}} \|\partial_X^{(\beta)} \Delta_\xi^\sigma a(x, \xi) \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)}\|_{\text{op}} < \infty. \quad (4.4)
\]

Here, \(\hat{\mathcal{M}}\) is the matrix-valued symbol of the operator \(\mathcal{M} := (1 + \mathcal{L})^\frac{1}{2}\), and for every \([\xi] \in \hat{G}\),

\[
\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)} := \text{diag}([(1 + \nu_{ii}(\xi)\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)]_{1 \leq i \leq d_\xi},
\]

where \(\hat{\mathcal{L}}(\xi) := \text{diag}[\nu_{ii}(\xi)]_{1 \leq i \leq d_\xi}\) is the symbol of the sub-Laplacian \(\mathcal{L}\) at \([\xi]\).

**Remark 4.4.** Note that in contrast with the usual conditions \(0 \leq \rho, \delta \leq 1\), appearing in Section 3 for Hörmander classes on manifolds (by using charts), in the definition of dilated subelliptic Hörmander classes, we allow \(0 \leq \rho, \delta \leq \kappa\), in view of the normalisation factor \(\frac{1}{2}\) in (4.3) and (4.4). To establish a calculus with the usual conditions on \(\rho\) and \(\delta\) we will define the following contracted classes. The reason for their definition will be clear in Example 4.6.

**Definition 4.5** (Contracted subelliptic Hörmander classes). Let \(G\) be a compact Lie group and let \(0 \leq \delta, \rho \leq 1\). Let us consider a sub-Laplacian \(\mathcal{L} = -(X_1^2 + \cdots + X_k^2)\) on \(G\), where the system of vector fields \(X = \{X_i\}_{i=1}^k\) satisfies the Hörmander condition of step \(\kappa\). A symbol \(\sigma\) belongs to the contracted class \(\mathcal{S}_{\rho, \delta}^{m, \mathcal{L}}(G \times \hat{G})\) if \(\sigma \in \mathcal{S}_{\rho, \delta, \kappa}^{m, \mathcal{L}}(G)\). This means that \(\sigma\) satisfies the symbol inequalities

\[
p_{\alpha, \beta, \rho, \delta, m, \text{left}}(\sigma)' := \sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)} \partial_X^{(\beta)} \Delta_\xi^\sigma(x, \xi)\|_{\text{op}} < \infty, \quad (4.5)
\]

and

\[
p_{\alpha, \beta, \rho, \delta, m, \text{right}}(\sigma)' := \sup_{(x, [\xi]) \in G \times \hat{G}} \|\partial_X^{(\beta)} \Delta_\xi^\sigma(x, \xi) \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha| - |\delta| - |\beta| - m)}\|_{\text{op}} < \infty. \quad (4.6)
\]

By following the usual nomenclature, we define:

\[
\text{Op}(\mathcal{S}_{\rho, \delta}^{m, \mathcal{L}}(G)) := \{A : C^\infty(G) \to \mathcal{D}'(G) : \sigma_A \equiv \hat{A}(x, \xi) \in \mathcal{S}_{\rho, \delta}^{m, \mathcal{L}}(G)\},
\]

with

\[
Af = \sum_{[\xi] \in G} d_\xi \text{Tr}(\xi \cdot \hat{A}(\cdot, \xi) \hat{f}(\xi)), \quad f \in C^\infty(G).
\]

We also define

\[
\text{Op}(\mathcal{S}_{\rho, \delta}^{m, \mathcal{L}}(G \times \hat{G})) := \text{Op}(\mathcal{S}_{\rho, \delta, \kappa}^{m, \mathcal{L}}(G)).
\]

To provide operators in these classes, we show in the following example that real powers of \((1 + \mathcal{L})^\frac{1}{2}\), have subelliptic symbols.
**Example 4.6** (Positive powers of \((1 + \mathcal{L})^{\frac{1}{2}}\) in subelliptic Hörmander classes). We will apply the inequality
\[
\langle \xi \rangle^{\frac{1}{2}} \leq (1 + \nu_{ii}(\xi)^2)^{\frac{1}{2}} \leq \langle \xi \rangle,
\]
proved in Proposition 3.1 of [71]. Let us observe that for \(0 < s < \infty\),
\[
\mathcal{M}_s = (1 + \mathcal{L})^{\frac{s}{2}} \in \text{Op}(\mathcal{S}_{s,0}^{\mathcal{E}}(G)) \equiv \text{Op}(\mathcal{S}_{s,0}^{\mathcal{E}}(G \times \hat{G})).
\]
Indeed, we can prove (4.8) as follows. For \(0 < s < 1\), if \(m = m_s\) is the subelliptic Hörmander order of \(m_s\), according to (4.3), if we set \(|\alpha| = 0\), then we have that
\[
\sup_{[\xi] \in G} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-m_s)}\hat{\mathcal{M}}_s(\xi)\|_{\text{op}} = \sup_{1 \leq i \leq d_{\xi}} (1 + \nu_{ii}(\xi)^2)^{\frac{1}{2}(-m_s+s)} < \infty,
\]
if and only if \(m_s \geq \kappa s\). This suggests that perhaps \(m_s = \kappa s\). Indeed, this is the case. Because, for \(0 < s < 1\), \(\mathcal{M}_s \in \text{Op}(\mathcal{S}_{s,0}(G))\), we have the estimates,
\[
\|\Delta^0_{\xi} \hat{\mathcal{M}}_s(\xi)\|_{\text{op}} \leq C_{\alpha} (\langle \xi \rangle)^{s-|\alpha|}.
\]
So, in order to deduce that \(\mathcal{M}_s \in \text{Op}(\mathcal{S}_{s,0}(G))\), it remains to show that
\[
\Gamma' := \sup_{[\xi] \in G} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(|\alpha|-\kappa s)}\hat{\Delta}^0_{\xi} \hat{\mathcal{M}}_s(\xi)\|_{\text{op}} < \infty,
\]
and
\[
\Pi' := \sup_{[\xi] \in G} \|\Delta^0_{\xi} \hat{\mathcal{M}}_s(\xi)\|_{\text{op}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(|\alpha|-\kappa s)}\|_{\text{op}} \leq C_{\alpha} \sup_{1 \leq i \leq d_{\xi}} (1 + \nu_{ii}(\xi)^2)^{\frac{1}{2}(|\alpha|-\kappa s|\alpha|+\kappa s)} \leq \infty,
\]
for all \(|\alpha| \geq 1\). However, in this case \(s - |\alpha| < 1 - |\alpha| \leq 0\), and the estimate
\[
\langle \xi \rangle^{s-|\alpha|} \leq (1 + \nu_{ii}(\xi)^2)^{\frac{1}{2}(s-|\alpha|)}
\]
leads to
\[
\|\Delta^0_{\xi} \hat{\mathcal{M}}_s(\xi)\|_{\text{op}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(|\alpha|-\kappa s)}\|_{\text{op}} \leq C_{\alpha} \sup_{1 \leq i \leq d_{\xi}} (1 + \nu_{ii}(\xi)^2)^{\frac{1}{2}(|\alpha|-\kappa s|\alpha|+\kappa s)} \leq C_{\alpha}.
\]
This analysis proves that \(\Gamma', \Pi' \leq C_{\alpha}\). In the general case \(s \geq 1\), we also have \(\mathcal{M}_s \in \text{Op}(\mathcal{S}_{s,0}(G))\). This will be proved in Remark 4.35 as a consequence of the subelliptic symbolic calculus.

For symbols of Fourier multipliers we will use the following notation,
\[
\mathcal{S}_{\rho}^{m,\mathcal{E}}(\hat{G}) = \{ \sigma \in \Sigma(\hat{G}) : \sigma \in \mathcal{S}_{\rho,0}^{m,\mathcal{E}}(G) \}.
\]

We also define the class of symbols of order \(m\), satisfying the \(\rho\)-type conditions up to order \(\ell \in \mathbb{N}\), \(\mathcal{S}_{\rho}^{m,\ell,\mathcal{E}}(\hat{G})\), by those symbols satisfying,
\[
\sup_{[\xi] \in G} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|-m)} \Delta^0_{\xi} a(\xi)\|_{\text{op}} < \infty, \ |\alpha| \leq \ell,
\]
and
\[
\sup_{[\xi] \in G} \|\Delta^0_{\xi} a(\xi) \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|-m)}\|_{\text{op}} < \infty, \ |\alpha| \leq \ell.
\]

\(^{13}\) One of the reasons to introduce contracted classes \(S_{\rho,\delta}^{m,\mathcal{E}}(G \times \hat{G})\) is that we want \(\mathcal{M}_s\) to be an operator of order \(s\). We will use the notation \(\mathcal{S}_{\rho,\delta}^{m,\mathcal{E}}(G)\) just to differentiate the dilated classes of the (elliptic) Hörmander classes \(\mathcal{S}_{\rho,\delta}^{m,\mathcal{E}}(G \times \hat{G})\). However, in both cases the matrix-valued symbols are defined in the phase space \(G \times \hat{G}\).
Finally, there exists $i$

On the other hand, let us estimate Lemma 4.8.

Let $a \in S_{\rho, \delta}^{m, \ell, \mathcal{L}}(G)$, by those symbols satisfying,

$$\sup_{[\xi] \in \mathcal{G}} \| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) c^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op} < \infty, \quad |\alpha| \leq \ell, \quad |\beta| \leq \ell',$$

and

$$\sup_{[\xi] \in \mathcal{G}} \| \hat{c}^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \|_{op} < \infty, \quad |\alpha| \leq \ell, \quad |\beta| \leq \ell'.$$

Lemma 4.7. Let $G$ be a compact Lie group and let $0 \leq \delta, \rho \leq 1$. If $a \in S_{\rho, \delta}^{m, \mathcal{L}}(G)$ then for every $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{\alpha, \beta} > 0$ satisfying the estimates

$$\| \hat{c}^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op} \leq C_{\alpha, \beta} \sup_{1 \leq i \leq d_\xi} (1 + \nu_{ii}(\xi))^\frac{m - \rho|\alpha| + |\delta| |\beta|}{\kappa},$$

uniformly in $(x, [\xi]) \in G \times \hat{G}$.

Proof. Let us assume that $a \in S_{\rho, \delta}^{m, \mathcal{L}}(G)$. Then for every $\alpha, \beta \in \mathbb{N}_0^n$, we have

$$\| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) c^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op} \leq C_{\alpha, \beta} < \infty.$$

On the other hand, let us estimate

$$\| \hat{c}^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op} = \| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \hat{c}^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op}$$

$$\leq \| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \|_{op} \| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \hat{c}^{(\beta)}_X \Delta_\xi^\alpha a(x, \xi) \|_{op}$$

$$\leq C_{\alpha, \beta} \| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \|_{op}.$$

Finally, there exists $i \in \{1, 2, \ldots, d_\xi\}$ depending on $[\xi]$, such that

$$\| \hat{M}(\xi) \frac{1}{2} (\rho|\alpha| - |\delta| - m) \|_{op} = (1 + \nu_{ii}(\xi)^2)^\frac{m - \rho|\alpha| + |\delta| |\beta|}{2\kappa} \leq \sup_{1 \leq i \leq d_\xi} (1 + \nu_{ii}(\xi)^2)^\frac{m - \rho|\alpha| + |\delta| |\beta|}{2\kappa}$$

$$= \sup_{1 \leq i \leq d_\xi} (1 + \nu_{ii}(\xi))^\frac{m - \rho|\alpha| + |\delta| |\beta|}{\kappa}.$$

The proof is complete. □

The following Lemma is straightforward.

Lemma 4.8. Let $G$ be a compact Lie group and $0 \leq \delta, \rho \leq 1$. Let us consider a sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_\ell^2)$ on $G$, where the system of vector fields $X = \{X_i\}_{i=1}^\ell$ satisfies the Hörmander condition of step $\kappa$. Let us consider $\sigma \in S_{\rho, \delta}^{m, \mathcal{L}}(G \times \hat{G})$ and let $k_x$ be its associated right-convolution kernel. Then,

1. If $\Delta_q \in \text{diff}^\ell(\hat{G})$, then the right convolution kernel associated with the symbol $c^{(\beta)}_X \Delta_q \sigma(x, \xi) \in S_{\rho, \delta}^{m - \rho\ell + |\delta|, \mathcal{L}}(G \times \hat{G})$ is given by $q(\cdot) \hat{c}^{(\beta)}_X k_x$.

2. If $0 \leq \delta \leq \delta' \leq 1$ and $0 \leq \rho \leq \rho' \leq 1$, then

$$S_{\rho, \delta}^{m, \mathcal{L}}(G \times \hat{G}) \subset S_{\rho, \delta'}^{m, \mathcal{L}}(G \times \hat{G}) \subset S_{\rho', \delta'}^{m, \mathcal{L}}(G \times \hat{G}),$$

with continuous inclusions. In particular,

$$S_{1,0}^{m, \mathcal{L}}(G \times \hat{G}) \subset S_{0,1}^{m, \mathcal{L}}(G \times \hat{G}) \subset S_{0,1}^{m, \mathcal{L}}(G \times \hat{G}).$$
In the following proposition we compare some subelliptic classes with (elliptic) classes on compact Lie groups (we will use (2.4) and (2.6) for the corresponding classes of matrix symbols of limited regularity).

**Proposition 4.9.** Let $G$ be a compact Lie group and $0 \leq \delta, \rho \leq 1$. Let us consider a sub-Laplacian $L = -(X_1^2 + \cdots + X_k^2)$ on $G$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of step $\kappa$. Let us assume that $\nu \geq 0$. Then, for every $\ell \in \mathbb{N}$ and all $0 \leq \rho \leq 1$, we have

$$
\mathcal{S}^{-\nu,\ell,\mathcal{L}}_{\rho,0}(G) \subset \mathcal{S}^{-\frac{\nu}{\kappa},\ell}(G \times \widehat{G}), \quad \mathcal{S}^{-\nu,\ell}_{\rho,0}(G \times \widehat{G}) \subset \mathcal{S}^{-\nu,\ell,\mathcal{L}}_{\rho,0}(G)
$$

with continuous inclusions.

**Proof.** Because we are considering symbols with $\delta = 0$, it is sufficient to prove Proposition 4.9 for classes of invariant symbols. Let us assume that $a \in \mathcal{S}^{-\nu,\ell,\mathcal{L}}(\widehat{G})$ where $0 \leq \rho \leq \kappa$. Then we have

$$
\sup_{[\xi] \in \widehat{G}} \| \Delta^\alpha_{\xi} a(\xi) \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \|_{\text{op}}, \sup_{[\xi] \in \widehat{G}} \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} < \infty, \ |\alpha| \leq \ell.
$$

Now, let us note that

$$
\| \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} = \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} \\
\leq \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \|_{\text{op}} \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} \\
= \sup_{1 \leq i \leq d} (1 + \nu \iota_i(\xi)^2)^{\frac{1}{2}(\rho|\alpha|+\nu)} \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} \\
\leq \langle \xi \rangle^{-\frac{1}{\kappa}(\rho|\alpha|+\nu)} \sup_{[\xi] \in \widehat{G}} \| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\rho|\alpha|+\nu)} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}},
$$

where in the last line we have used (4.7). So, we have proved that

$$
\mathcal{S}^{-\nu,\ell,\mathcal{L}}(\widehat{G}) \subset \mathcal{S}_{\frac{\kappa}{\nu},\ell}(\widehat{G}), \ 0 \leq \rho \leq \kappa.
$$

(4.9)

So, if we replace $\rho$ in (4.9) by $\rho \kappa$, where $0 \leq \rho \leq 1$, we obtain the inclusion,

$$
\mathcal{S}^{-\nu,\ell,\mathcal{L}}(\widehat{G}) \subset \mathcal{S}_{\frac{\kappa}{\nu},\ell}(\widehat{G}), \ 0 \leq \rho \leq 1.
$$

The inclusion $\mathcal{S}^{-\nu,\ell}(\widehat{G}) \subset \mathcal{S}^{-\nu,\ell,\mathcal{L}}(\widehat{G})$ for $0 \leq \rho \leq 1$, can be proved in an analogous way. Indeed, if $a \in \mathcal{S}^{-\nu,\ell}(\widehat{G})$, we have

$$
\| \hat{\mathcal{M}}(\xi)^{-\nu} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} = \| \hat{\mathcal{M}}(\xi)^{-\nu} \langle \xi \rangle^{-\rho|\alpha|+\nu} \langle \xi \rangle^{\rho|\alpha|+\nu} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} \\
\leq \langle \xi \rangle^{-\nu} \| \hat{\mathcal{M}}(\xi)^{-\nu} \langle \xi \rangle^{-\rho|\alpha|+\nu} \langle \xi \rangle^{\rho|\alpha|+\nu} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}}.
$$

Because $a \in \mathcal{S}^{-\nu,\ell}(\widehat{G})$,

$$
\sup_{[\xi] \in \widehat{G}} \| \langle \xi \rangle^{\rho|\alpha|+\nu} \Delta^\alpha_{\xi} a(\xi) \|_{\text{op}} < \infty, \ |\alpha| \leq \ell.
$$
So, we only need to check that \( \| \widehat{M}(\xi)^{\frac{1}{2}(\rho|\alpha|+\kappa\nu)}\langle \xi \rangle^{-\rho|\alpha|-\nu} \|_{op} \) is uniformly bounded in \( [\xi] \in \hat{G} \). For this, we can estimate
\[
\| \widehat{M}(\xi)^{\frac{1}{2}(\rho|\alpha|+\kappa\nu)}\langle \xi \rangle^{-\rho|\alpha|-\nu} \|_{op} = \sup_{1 \leq i \leq d_v} (1 + \nu_i(\xi)^2)^{\frac{1}{2}(\rho|\alpha|+\nu)}\langle \xi \rangle^{-\rho|\alpha|-\nu} \leq 1,
\]
because \( (1 + \nu_i(\xi)^2)^{\frac{1}{2}(\rho|\alpha|+\nu)} \leq \langle \xi \rangle^{\rho|\alpha|+\nu} \) in view of (4.7). So, we finish the proof.

**Corollary 4.10.** Let us assume that \( \nu \geq 0 \). Then, for every \( \ell \in \mathbb{N} \) and all \( 0 \leq \rho \leq 1 \), we have
\[
S_{-\nu,\ell}^{-\rho}(G \times \hat{G}) \subset \mathcal{J}_{\frac{\nu}{\rho},0}(G \times \hat{G}), \mathcal{J}_{\rho,0}^{-\nu,\ell}(G \times \hat{G}) \subset S_{-\nu,\ell}^{-\rho}(G \times \hat{G})
\]
with continuous inclusions.

Now, we will prove some useful identities in order to characterise the subelliptic Hörmander classes, by showing that (4.3) and (4.4) are equivalent in some sense. To do so, we will use the following version of the Corach-Porta-Recht inequality (see Corach, Porta, and Recht [36] and Seddik [116, Theorem 2.3] for (4.10) and Andruchow, Corach, and Stojanoff [2, page 297] for (4.11)).

**Proposition 4.11.** Let \( H \) be a complex Hilbert space and let \( A, P, Q, X \in \mathcal{B}(H) \) be bounded operators on \( H \). Let us assume that \( P \) and \( Q \) are positive and invertible operators with \( PQ = QP \), and that \( A \) is self-adjoint. Then we have the norm inequalities
\[
2\|X\|_{op} \leq \max\{\|XPX^{-1} + Q^{-1}XQ\|_{op}, \|PX*P^{-1} + Q^{-1}X*Q\|_{op}\}, \tag{4.10}
\]
and
\[
\|X\|_{op} \leq \|AXA + (1 + A^2)^{\frac{1}{2}}X(1 + A^2)^{\frac{1}{2}}\|_{op}. \tag{4.11}
\]

**Remark 4.12.** It was proved also in Andruchow, Corach, and Stojanoff [2, page 302] that the inequalities (4.10) and (4.11) are equivalent and 2 is the best constant in (4.10) if \( P = Q \).

So, we are ready to prove the following characterization of dilated subelliptic Hörmander classes.

**Theorem 4.13.** Let \( G \) be a compact Lie group and let \( 0 \leq \delta, \rho \leq \kappa \). The following conditions are equivalent.

A. For every \( \alpha, \beta \in \mathbb{N}_0^d \),
\[
p_{\alpha,\beta,\rho,\delta,m,\text{left}}(a) := \sup_{(x,\xi) \in G \times \hat{G}} \| \widehat{M}(\xi)^{\frac{1}{2}(\rho|\alpha|+\delta|\beta|)|\alpha|-\nu} \langle \xi \rangle^\delta \Delta^\alpha_a(x,\xi) \|_{op} < \infty. \tag{4.12}
\]

B. For every \( \alpha, \beta \in \mathbb{N}_0^d \),
\[
p_{\alpha,\beta,\rho,\delta,m,\text{right}}(a) := \sup_{(x,\xi) \in G \times \hat{G}} \| \langle \Delta^\alpha_a(x,\xi) \rangle \widehat{M}(\xi)^{\frac{1}{2}(\rho|\alpha|+\delta|\beta|)|\alpha|-\nu} \|_{op} < \infty. \tag{4.13}
\]

C. For all \( r \in \mathbb{R}, \alpha, \beta \in \mathbb{N}_0^d \),
\[
p_{\alpha,\beta,\rho,\delta,m,\tau}(a) := \sup_{(x,\xi) \in G \times \hat{G}} \| \widehat{M}(\xi)^{\frac{1}{2}(\rho|\alpha|-\nu)\Delta^\alpha_a(x,\xi)\langle \xi \rangle^\delta} \|_{op} < \infty. \tag{4.14}
\]
D. There exists \( r_0 \in \mathbb{R} \), such that for every \( \alpha, \beta \in \mathbb{N}_0^p \),
\[
P_{\alpha,\beta,\rho,\delta,\sigma,\tau}(a) := \sup_{(x,\xi) \in G \times \hat{G}} \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (\rho|\alpha| - \sigma|\beta| - m - r_0) \hat{c}^{(\beta)}_x \Delta \xi^a a(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{\tau}{\pi} \|_{\text{op}} < \infty.
\] (4.15)

E. \( a \in \mathcal{S}^{m,\mathcal{E}}(G) \).

**Proof.** We only need to prove that \( D \implies C \). Let us assume that (4.15) holds true for some \( r_0 \in \mathbb{R} \) and let \( r \in \mathbb{R} \) be a real number. Let us assume first that \( r > r_0 \). Let us note that the operator \( A(x, \xi) := \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \) is self-adjoint. Let us denote
\[
X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi) = \widehat{\mathcal{M}}(\xi) \frac{1}{2} (\rho|\alpha| - \sigma|\beta| - m - r_0) \hat{c}^{(\beta)}_x \Delta \xi^a a(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{\tau}{\pi}.
\]

From the Corach-Porta-Recht inequality (4.11), we have
\[
\| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (\rho|\alpha| - \sigma|\beta| - m - r_0) \hat{c}^{(\beta)}_x \Delta \xi^a a(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{\tau}{\pi} \|_{\text{op}}
\]
\[
= \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (\rho|\alpha| - \sigma|\beta| - m - r_0) \hat{c}^{(\beta)}_x \Delta \xi^a a(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{\tau}{\pi} \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \|_{\text{op}}
\]
\[
= \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \|_{\text{op}}
\]
\[
\leq \| A(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) A(x, \xi)
\]
\[
+ (1 + A(x, \xi)^2) \frac{1}{2} \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) (1 + A(x, \xi)^2) \frac{1}{2} \|_{\text{op}}
\]
\[
= \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi)
\]
\[
+ (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) X_{\alpha,\beta,\rho,\delta,\tau}(x, \xi) \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \|_{\text{op}}.
\]

Taking into account that \( r_0 - r < 0 \), and the following facts (see [33, Remark 5.9])
\[
\widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \in \mathcal{S}_{\frac{2(r_0 - r)}{\pi},0}^{\frac{2(r_0 - r)}{\pi}} (G \times \hat{G}), \quad (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \in \mathcal{S}_{\frac{2(r_0 - r)}{\pi},0}^{\frac{2(r_0 - r)}{\pi}} (G \times \hat{G}),
\]
\[
\widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \in \mathcal{S}_{\frac{2(r_0 - r)}{\pi},0}^{\frac{2(r_0 - r)}{\pi}} (G \times \hat{G}),
\]
we deduce that
\[
\mathcal{D}_1 := \sup_{[\xi] \in \hat{G}} \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \|_{\text{op}}, \quad \mathcal{D}_2 := \sup_{[\xi] \in \hat{G}} \| (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) \|_{\text{op}} < \infty,
\]
and
\[
\mathcal{D}_3 := \sup_{[\xi] \in \hat{G}} \| \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r) (1 + \widehat{\mathcal{M}}(\xi) \frac{1}{2} (r_0 - r)) \frac{1}{2} \|_{\text{op}} < \infty.
\]
Consequently,
\[
\| \hat{M}(\xi)^{\frac{1}{2} (\rho|\alpha| - |\beta| - m - r)} \phi^{(\beta)}_X a(x, \xi) \hat{M}(\xi)^{\frac{1}{2}} \|_{op} \\
\leq \| \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r - r_0)} (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \|_{op} \\
\leq \| \hat{M}(\xi)^{\frac{1}{2} (r - r_0)} \|_{op} \| X_{a, \beta, r_0}(x, \xi) \|_{op} \\
+ \| (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \hat{M}(\xi)^{\frac{1}{2} (r - r_0)} \|_{op} \| X_{a, \beta, r_0}(x, \xi) \|_{op} \\
\times \| \hat{M}(\xi)^{\frac{1}{2} (r - r_0)} (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \|_{op} \\
\leq (\mathcal{D}_1 + \mathcal{D}_2 \times \mathcal{D}_3) \| X_{a, \beta, r_0}(x, \xi) \|_{op}.
\]

The previous argument shows that $D \implies C$ for $r > r_0$. In the case where $r < r_0$, we can define $A(x, \xi) = \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}$. By repeating the argument above we can deduce that $D \implies C$ for $r < r_0$. Indeed, by using again the Corach-Porta-Recht inequality (4.11), we have

\[
\| \hat{M}(\xi)^{\frac{1}{2} (\rho|\alpha| - |\beta| - m - r)} \phi^{(\beta)}_X a(x, \xi) \hat{M}(\xi)^{\frac{1}{2}} \|_{op} \\
= \| \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \hat{M}(\xi)^{\frac{1}{2} (\rho|\alpha| - |\beta| - m - r)} \phi^{(\beta)}_X \Delta_{\xi}^\alpha a(x, \xi) \hat{M}(\xi)^{\frac{1}{2}} \|_{op} \\
= \| \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \|_{op} \\
\leq \| A(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \|_{op} \\
+ \| (1 + A(x, \xi)^2)^\frac{1}{2} \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} (1 + A(x, \xi)^2)^\frac{1}{2} \|_{op} \\
= \| X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \|_{op} \\
+ \| (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} X_{a, \beta, r_0}(x, \xi) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \|_{op}.
\]

Since $r - r_0$ is negative, we have the following facts (see [33, Remark 5.9])

\[
\hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \in \mathcal{S}^{\frac{1}{2} (r_0 - r)}_{\frac{1}{2} + 0} (G \times \hat{G}), \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \frac{1}{2} \in \mathcal{S}^{\frac{1}{2} (r_0 - r)}_{\frac{1}{2} + 0} (G \times \hat{G}),
\]

\[
(1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \in \mathcal{S}^{0}_{\frac{1}{2} + 0} (G \times \hat{G}),
\]

we deduce that

\[
\mathcal{D}'_1 := \sup_{[\xi] \in \hat{G}} \| \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \|_{op}, \quad \mathcal{D}_2' := \sup_{[\xi] \in \hat{G}} \| (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} \|_{op} < \infty,
\]

and

\[
\mathcal{D}_3' := \sup_{[\xi] \in \hat{G}} \| \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)} (1 + \hat{M}(\xi)^{\frac{1}{2} (r_0 - r)}) \|_{op} < \infty.
\]
Consequently,
\[
\|\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\xi^{(\beta)}X^{(\alpha)}\Delta_{\xi}^{\alpha}a(x,\xi,\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}
\leq\|X_{\alpha,\beta,0}(x,\xi,\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}
+ (1 + \hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}
\leq\|\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}\|X_{\alpha,\beta,0}(x,\xi,\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}
+ (1 + \hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}\|X_{\alpha,\beta,0}(x,\xi,\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}
\leq (D' + D' \times D')\|X_{\alpha,\beta,0}(x,\xi,\hat{M}(\xi)\frac{1}{\pi^{(\alpha|\delta|\beta|m-\rho)}}\|_{op}.
\]

The previous argument shows that $D \implies C$ for $r_0 > r$. The proof is complete. \hfill \Box

Theorem 4.13 implies the following characterization for the contracted subelliptic classes.

**Corollary 4.14.** Let $G$ be a compact Lie group and let $0 < \delta, \rho < 1$. The following conditions are equivalent.

A. For every $\alpha, \beta \in \mathbb{N}_0^n$,
\[
p_{\alpha,\beta,\rho,\delta,m,\text{left}}(a)' := \sup_{(x,\xi)G \times \hat{G}} \|\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\xi^{(\beta)}X^{(\alpha)}\Delta_{\xi}^{\alpha}a(x,\xi,\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\|_{op} < \infty. \tag{4.16}
\]

B. For every $\alpha, \beta \in \mathbb{N}_0^n$,
\[
p_{\alpha,\beta,\rho,\delta,m,\text{right}}(a)' := \sup_{(x,\xi)G \times \hat{G}} \|\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\xi^{(\beta)}X^{(\alpha)}\Delta_{\xi}^{\alpha}a(x,\xi,\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\|_{op} < \infty. \tag{4.17}
\]

C. For all $r \in \mathbb{R}, \alpha, \beta \in \mathbb{N}_0^n$,
\[
p_{\alpha,\beta,\rho,\delta,m,r}(a)' := \sup_{(x,\xi)G \times \hat{G}} \|\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\xi^{(\beta)}X^{(\alpha)}\Delta_{\xi}^{\alpha}a(x,\xi,\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\|_{op} < \infty. \tag{4.18}
\]

D. There exists $r_0 \in \mathbb{R}$, such that for every $\alpha, \beta \in \mathbb{N}_0^n$,
\[
p_{\alpha,\beta,\rho,\delta,m,r_0}(a)' := \sup_{(x,\xi)G \times \hat{G}} \|\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\xi^{(\beta)}X^{(\alpha)}\Delta_{\xi}^{\alpha}a(x,\xi,\hat{M}(\xi)^{(\rho|\delta|\beta|m-\rho)}\|_{op} < \infty. \tag{4.19}
\]

E. $a \in S_{\rho,\delta}^{m,L}(G \times \hat{G})$.

**Remark 4.15.** We will prove an analogy of Theorem 4.13 for arbitrary graded Lie groups in Theorem 13.16 extending Theorem 5.5.20 of [65]. For this we will use the same approach that in the proof of Theorem 4.13.

To study, for example, the classification of negative powers of $(1 + \mathcal{L})^\frac{1}{2}$ we need to study the inversion of symbols in the dilated subelliptic Hörmander classes. So, we have the following theorem.
Theorem 4.16. Let $m \in \mathbb{R}$, and let $0 \leq \delta < \rho \leq \kappa$. Let $a = a(x, \xi) \in \mathcal{F}_{\rho, \delta}^{m, L}(G)$. Assume also that $a(x, \xi)$ is invertible for every $(x, [\xi]) \in G \times \hat{G}$, and satisfies
\begin{equation}
\sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{m}{2}} a(x, \xi)^{-1}\|_{\text{op}} < \infty. \tag{4.20}
\end{equation}

Then, $a^{-1} := a(x, \xi)^{-1} \in \mathcal{F}_{\rho, \delta}^{-m, L}(G)$.

Proof. Let us estimate $\hat{\partial}_{x}^{(\beta)} a^{-1}$ first. Suppose we have proved that
\begin{equation}
I := \sup_{|\beta| \leq \ell} \sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta|+m)} \hat{\partial}_{x}^{(\beta)} a(x, \xi)^{-1}\|_{\text{op}} < \infty,
\end{equation}
for some $\ell \in \mathbb{N}$. We proceed by mathematical induction. Let us analyse the cases $|\beta| = \ell + 1$. If we write $\hat{\partial}_{x}^{(\beta)} = \hat{\partial}_{x}^{(\beta_1)} \hat{\partial}_{x}^{(\beta_2)}$ where $|\beta| \leq \ell$, then $\hat{\partial}_{x}^{(\beta)} a^{-1} = \hat{\partial}_{x}^{(\beta_1)} \hat{\partial}_{x}^{(\beta_2)} a^{-1}$.

From the identity $a(x, \xi) a(x, \xi)^{-1} = I_{d_x}$ we have
\begin{equation}
\hat{a}(x, \xi) \hat{\partial}_{x}^{(\beta)} a^{-1}(x, \xi) = -\sum_{\beta_1 + \beta_2 = \beta + e_j, |\beta_2| \leq |\beta|} C_{\beta_1, \beta_2}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi)) (\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi)).
\end{equation}

Consequently,
\begin{equation}
\hat{\partial}_{x}^{(\beta)} a^{-1}(x, \xi) = -a(x, \xi)^{-1} \sum_{\beta_1 + \beta_2 = \beta + e_j, |\beta_2| \leq |\beta|} C_{\beta_1, \beta_2}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi)) (\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi)).
\end{equation}

We want to prove the estimate
\begin{equation}
\sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta|+1)+m} \hat{\partial}_{x}^{(\beta)} a(x, \xi)^{-1}\|_{\text{op}} < \infty.
\end{equation}

For this, we only need to show that for every $\beta_1$ and $\beta_2$ such that $\beta_1 + \beta_2 = \beta + e_j, |\beta_2| \leq |\beta|$,
\begin{equation}
\sup_{(x, [\xi]) \in G \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta|+1)+m} a(x, \xi)^{-1}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}} < \infty.
\end{equation}

Observe that
\begin{equation}
\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta|+1)+m} a(x, \xi)^{-1}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi))
\end{equation}
\begin{equation}
= \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta_1|+|\beta_2|)+m} a(x, \xi)^{-1}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi)).
\end{equation}

First, let us prove that
\begin{equation}
\|\hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta_1|+|\beta_2|)} \hat{\mathcal{M}}(\xi)^{\frac{m}{2}} a(x, \xi)^{-1}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\end{equation}
\begin{equation}
\leq \|(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}.
\end{equation}

Indeed, by using (4.11) with $A = \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta_1|+|\beta_2|)}$ and
\begin{equation}
X = \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(-|\beta_1|+|\beta_2|)} \hat{\mathcal{M}}(\xi)^{\frac{m}{2}} a(x, \xi)^{-1}(\hat{\partial}_{x}^{(\beta_1)} a(x, \xi))(\hat{\partial}_{x}^{(\beta_2)} a^{-1}(x, \xi)),
\end{equation}
we obtain
\[
\|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}} a(x, \xi)^{-1}(\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\leq \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}} a(x, \xi)^{-1}(\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
+ (1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2} X (1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}
\]
\[
\leq \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}} a(x, \xi)^{-1}(\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
+ \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}\|X (1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}
\]
\[
\leq \|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
+ \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}\|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\leq \|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
+ \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}\|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
Observe that we have estimated
\[
\|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}} = O(1),
\]
because of
\[
\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2} \in \mathcal{S}^{0}_{\frac{\gamma}{2},0}(G \times \hat{G}).
\]
Again, by using Theorem 4.13, we have
\[
\|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\leq \|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\times \|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(1 + \widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}})^\frac{1}{2}\|_{\text{op}}
\]
\[
\leq \|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\|\partial_X^{(\beta_1)} a(x, \xi))(\partial_X^{(\beta_2)} a^{-1}(x, \xi))\|_{\text{op}}
\]
\[
\leq 1,
\]
uniformly in \((x, [\xi])\). A similar analysis using the Leibniz rule for difference operators can be used in order two estimate the differences \(\Delta_\chi^\alpha a^{-1}\). For this, we need the following two estimates,
\[
\|\widehat{\mathcal{M}}(\xi)^{-\frac{\gamma}{2}}(\partial_X^{(\beta_1)} a(x, \xi))\|_{\text{op}} = O(1),
\]
(4.21)
\[ \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))a(x, \xi)^{-1}\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|)}\|_{op} = O(1). \] (4.22)

For the proof, let us use (4.11), observing that
\[
\|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))a(x, \xi)^{-1}\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|)-m}\|_{op} \\
\leq \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|-m)}\|_{op} \\
\leq \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|-m)}\|_{op} = O(1).
\]

On the other hand,
\[
\|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))a(x, \xi)^{-1}\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|)}\|_{op} \\
= \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|)}\|_{op} \\
\leq \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|-m)}\|_{op} \\
\leq \|\hat{\Delta}^{\gamma}_{\xi} \hat{\partial}^{(\beta)}_{X}(a(x, \xi))\hat{\mathcal{M}}(\xi)^{\frac{1}{\kappa}(\rho|\gamma|-\delta|\beta|-m)}\|_{op} = O(1).
\]

Now, we will estimate in \(\frac{1}{\kappa}(m + \rho\ell)\) the subelliptic order for the differences \(\Delta_{q_{\ell}} a^{-1}\), in the dilated classes. To do so, we will use mathematical induction. The case \(\ell = 0\) holds true from the hypothesis of Theorem 4.16. To study the differences of higher order we will use the Leibniz rule (see Remark 3.3),
\[
\Delta_{q_{\ell}}[a_{1}a_{2}](x, \xi) = \sum_{|\gamma|,|\varepsilon|\geq |\gamma|+|\varepsilon|} C_{\varepsilon,\gamma}(\Delta_{q_{\ell}} a_{1})(x, \xi)(\Delta_{q_{\ell}} a_{2})(x, \xi),
\]
for \(a_{i} \in C^{\infty}(G) \times \mathscr{C}(\hat{G})\). From the identity, \(a(x, \xi)a(x, \xi)^{-1} = I_{d_{\xi}}\), we deduce that
\[
(\Delta_{q_{\ell}} a)(x, \xi)a(x, \xi)^{-1} + a(x, \xi)(\Delta_{q_{\ell}} a^{-1})(x, \xi) \\
= -\sum_{1=1}^{\nu,\nu'} C_{\nu,\nu'}(\Delta_{q_{(\nu)}} a)(x, \xi)(\Delta_{q_{(\nu')} a^{-1}})(x, \xi),
\]
and consequently
\[
(\Delta_{q_{\ell}} a^{-1})(x, \xi) = -a(x, \xi)^{-1}(\Delta_{q_{\ell}} a)(x, \xi)a^{-1}(x, \xi) \\
-\sum_{1=1}^{\nu,\nu'} C_{\nu,\nu'}a(x, \xi)^{-1}(\Delta_{q_{(\nu)}} a)(x, \xi)(\Delta_{q_{(\nu')} a^{-1}})(x, \xi).
\]

The differences of higher order \(\Delta_{q_{\ell+1}} a^{-1}\), can be estimated e.g. from difference operators of the form
\[
\Delta_{q_{\ell+1}} = \Delta_{q_{\ell}} \Delta_{q_{1}}.
\]
Then, by applying the difference operator \(\Delta_{q_{\ell}}\) to \((\Delta_{q_{1}} a^{-1})(x, \xi)\) we essentially obtain linear combinations of terms of the following kind as a consequence of the Leibniz rule:
\[
\begin{align*}
IV_1(x, \xi) &:= \Delta_{q_1}a(x, \xi)^{-1}(\Delta_{q_1}a(x, \xi) \times a^{-1}(x, \xi)) \\
IV_2(x, \xi) &:= a(x, \xi)^{-1} \times \Delta_{q_1}a(x, \xi) \times a^{-1}(x, \xi) \\
IV_3(x, \xi) &:= a(x, \xi)^{-1} \times \Delta_{q_1}a(x, \xi) \times \Delta_{q_1}a^{-1}(x, \xi) \\
IV_4(x, \xi) &:= \Delta_{q_1}a(x, \xi)^{-1}(\Delta_{q_{1+\ell}}a(x, \xi)^{-1}(x, \xi) \times \Delta_{q_{1+\ell}}a^{-1}(x, \xi)) \times \Delta_{q_{1+\ell}}a^{-1}(x, \xi) \times \Delta_{q_{1+\ell}}a^{-1}(x, \xi), 1 \leq \ell_1, \ell_2, \ell_3 \leq \ell \leq \ell_1 + \ell_2 + \ell_3,
\end{align*}
\]

which we can estimate as follows. First, assume that for every \( \ell \geq 1, \ell \in \mathbb{N} \), we have

\[
\sup_{(x, \xi)} \| \hat{M}(\xi)^{1\over 2}a^{-1}(x, \xi) \|_{op} < \infty.
\]

We need to prove that

\[
\sup_{(x, \xi)} \| \hat{M}(\xi)^{1\over 2}(m+\rho)\delta)(\Delta_{q_{1+\ell}}a^{-1})(x, \xi) \|_{op} < \infty.
\]

For this, it is enough to prove that

\[
\sup_{(x, \xi)} \| \hat{M}(\xi)^{1\over 2}(m+\rho)\delta)(IV_i(x, \xi) \|_{op}, \sup_{(x, \xi)} \| \hat{M}(\xi)^{1\over 2}(m+\rho)\delta)(V_j(x, \xi) \|_{op} < \infty,
\]
for all $1 \leq i, j \leq 4$. Next, let us omit the argument $(x, \xi)$ in order to simplify the notation. So, we have

$$
\| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} IV(1) \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} IV(2) (x, \xi) \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} IV(3) (x, \xi) \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} V(1) (x, \xi) \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} V(2) (x, \xi) \|_{op}
$$

$$
\lesssim \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} \Delta_{q_{\ell}} a^{-1} \times (\Delta_{q_{1}} a) \times a^{-1} \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} \Delta_{q_{t+1}} a \times a^{-1} \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} \Delta_{q_{1}} a \times \Delta_{q_{t}} a^{-1} \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} \Delta_{q_{t+1}} a \times (\Delta_{q_{t+1}} a) \times \Delta_{q_{t+1}} a^{-1} \|_{op} + \| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} \Delta_{q_{t+1}} a \times \Delta_{q_{t+1}} a^{-1} \|_{op}
$$

$$
\lesssim \| \Delta_{q_{1}} a \times a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} \|_{op} + \| \Delta_{q_{t+1}} a \times a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} \|_{op} + \| \Delta_{q_{1}} a \|_{op} + \| (\Delta_{q_{t+1}} a) \times a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} \|_{op} + \| \Delta_{q_{t+1}} a \|_{op} \lesssim 1,
$$

where we have used the estimates

$$
\| (\Delta_{q_{t+1}} a) \times \Delta_{q_{t+1}} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} \|_{op} \lesssim 1, \tag{4.23}
$$

and

$$
\| \widehat{M}(\xi)^{\frac{1}{2}(m+\rho(\ell+1))} a^{-1} \cdot \Delta_{q_{t+1}} a \cdot \Delta_{q_{t+1}} a^{-1} \|_{op} \lesssim \| \widehat{M}(\xi)^{\frac{1}{2}(\rho(\ell+1))} a^{-1} \cdot \Delta_{q_{t+1}} a \|_{op}. \tag{4.24}
$$
Indeed, for the proof of (4.23) observe that from the induction hypothesis, we have
\[
\| (\Delta_{q(v)} a) \times \Delta_{q(v')} a^{-1} \widehat{M}(\xi) \|_{\text{op}}
\leq \| \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho)(\Delta_{q(v)} a) \times \Delta_{q(v')} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho) \|_{\text{op}}
= \| \Delta_{q(v)} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho) \|_{\text{op}} \| \Delta_{q(v')} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho) \|_{\text{op}}
\leq 1.
\]

In order to prove (4.24), observe that
\[
\| \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho(\ell+1)) a^{-1} \times \Delta_{q(v)} a \times \Delta_{q(v')} a^{-1} \|_{\text{op}}
\leq \| \widehat{M}(\xi)^{\frac{1}{2}} (\rho(\ell+1)) a^{-1} \times \Delta_{q(v)} a \times \Delta_{q(v')} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} \|_{\text{op}}
\leq \| \widehat{M}(\xi)^{\frac{1}{2}} (\rho(\ell+1)) a^{-1} \times \Delta_{q(v)} a \|_{\text{op}} \| \Delta_{q(v')} a^{-1} \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho) \|_{\text{op}} \| \widehat{M}(\xi)^{\frac{1}{2}} (m+\rho) \|_{\text{op}}
\leq 1.
\]

A similar analysis can be used to study $IV_{(4)}(x, \xi), V_{(3)}(x, \xi)$ and $V_{(4)}(x, \xi)$. Thus, we end the proof. □

Theorem 4.16 implies the following result for the contracted subelliptic classes.

Corollary 4.17. Let $m \geq 0$, and let $0 \leq \delta < \rho \leq 1$. Let $a = a(x, \xi) \in S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G})$. Assume also that $a(x, \xi)$ is invertible for every $(x, [\xi]) \in G \times \widehat{G}$, and satisfies
\[
\sup_{(x, [\xi]) \in G \times \widehat{G}} \| \widehat{M}(\xi)^{m} a(x, \xi)^{-1} \|_{\text{op}} < \infty. \tag{4.25}
\]
Then, $a^{-1} := a(x, \xi)^{-1} \in S_{\rho, \delta}^{-m, \mathcal{L}}(G \times \widehat{G})$.

Definition 4.18 ($\mathcal{L}$-ellipticity). In view of Theorem 4.16 and Corollary 4.17, it is justified to say that the symbols $a \in S_{\rho, \delta}^{m, \mathcal{L}}(G \times \widehat{G}) = \mathcal{F}_{\rho, \delta}^{m, \mathcal{L}}(G)$ satisfying (4.25) are $\mathcal{L}$-elliptic of order $m$ and of type $(\rho, \delta)$, $0 \leq \delta, \rho \leq 1$, in the subelliptic contracted classes (or $\mathcal{L}$-elliptic of order $m\kappa$ and of type $(\rho\kappa, \delta\kappa)$, $0 \leq \delta, \rho \leq 1$, in the subelliptic dilated classes).

Example 4.19 (Negative powers of $(1 + \mathcal{L})^{\frac{1}{2}}$). For $s > 0$, set $\mathcal{M}_{-s} := (1 + \mathcal{L})^{-\frac{s}{2}}$. We will discuss the reasons justifying the following fact,
\[
\mathcal{M}_{-s} \in \text{Op}(\mathcal{F}_{1,0}^{-s\kappa, \mathcal{L}}(G)) = \text{Op}(S_{\kappa, 0}^{-s, \mathcal{L}}(G \times \widehat{G})). \tag{4.26}
\]
Indeed, we have from Example 4.6 that $\mathcal{M}_{s} \in \text{Op}(\mathcal{F}_{1,0}^{s\kappa, \mathcal{L}}(G))$. Since
\[
\sup_{[\xi] \in \widehat{G}} \| \widehat{M}(\xi)^{\frac{s}{2}} \widehat{M}_{-s}(\xi) \|_{\text{op}} = \sup_{[\xi] \in \widehat{G}} \| I_{d_{x}} \times d_{\xi} \|_{\text{op}} = 1,
\]
we have proved (4.20) for $m = s\kappa$. So, from Theorem 4.16, we deduce (4.26).

Remark 4.20 (Dependence of the subelliptic Hörmander classes on the choice of a sub-Laplacian). In general, if we take two sub-Laplacians $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$ on $G$, associated with two systems of vector fields satisfying the Hörmander condition of order $\kappa$, the corresponding subelliptic Hörmander classes $S_{\rho, \delta}^{m, \mathcal{L}_{1}}(G \times \widehat{G})$ and
\[ S_{\rho,\delta}^{m,\mathcal{L}_2}(G \times \widehat{G}) \] do not agree necessarily. This implies that the subelliptic classes may in general depend on the choice of the sub-Laplacian as we will see in Remark 14.1, even for collections of vector fields of the same step.

4.2. Singular kernels of subelliptic pseudo-differential operators. In order to study the behaviour of the kernel of a subelliptic operator near the origin we will consider a strongly admissible collection of difference operators \{\Delta_{q(j)}\} and \( \gamma_0 \in \mathbb{N}_0^i, |\gamma_0| = \ell \), such that
\[
\Delta_{q_0} = \Delta_{q_1} = \cdots = \Delta_{q_{i(1)}} \Delta_{q_{i(j)}}, \quad \gamma_0 = (\gamma_j)_{1 \leq j \leq i}.
\]
The strong admissibility means that
\[
\text{rank}\{\nabla q_{(j)}(e) : 1 \leq j \leq i\} = \dim(G), \quad \Delta_{q(j)} \in \text{diff}^1(\widehat{G}),
\]
and that \( x = e_G \) is the only common zero of the functions \( q(j) \), i.e.
\[
\bigcap_{j=1}^i \{x \in G : q(j)(x) = 0\} = \{e_G\}.
\]

Remark 4.21. For our further analysis we will use that in view of the Weyl eigenvalue counting formula for the sub-Laplacian, we have
\[
N(\lambda) := \sum_{[\xi] \in \hat{G} : \langle \xi \rangle \leq \lambda} d_\xi^2 = O(\lambda^{\frac{\rho}{2}}),
\]
for every \( \lambda > 0 \), and \( s \in \mathbb{N} \) (following from Hassannezhad and Kokarev [76, Theorem 3.5]).

Remark 4.22. Let us note that from Remark 4.21, we can deduce that \( d_\xi^2 \leq (1 + \nu_i(\xi)^2)\frac{\rho}{2} \), for \( \langle \xi \rangle \to \infty \), which also implies that
\[
d_\xi \leq (1 + \nu_i(\xi)^2)\frac{\rho}{2} = (1 + \nu_i(\xi))^\frac{\rho}{2}, \quad \langle \xi \rangle \to \infty. \tag{4.27}
\]

We summarise the properties for the kernels of subelliptic operators in the following theorem. We will denote \(|\cdot|\) the metric induced by the geodesic distance on \( G \times G \), measuring the distance from the identity element \( e_G \).

Proposition 4.23. Let \( G \) be a compact Lie group of dimension \( n \) and let \( 0 \leq \delta, \rho \leq 1 \). Let \( A : \mathcal{C}^\infty(G) \to \mathcal{D}'(G) \) be a continuous linear operator with symbol \( \sigma \in S_{\rho,\delta}^{m,\mathcal{L}}(G \times \widehat{G}) \) in the contracted subelliptic Hörmander class of order \( m \) and of type \( (\rho, \delta) \). Then, the right-convolution kernel of \( A \), \( x \mapsto k_x : G \to \mathcal{C}^\infty(G \setminus \{e_G\}) \), defined by \( k_x := \mathcal{F}^{-1} \sigma(x, \cdot) \), satisfies the following estimates for \( |y| < 1 \):

(i) if \( m > -Q \), there exists \( \ell \in \mathbb{N} \), independent of \( \sigma \), such that
\[
|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S_{\rho,\delta}^{m,\mathcal{L}}} |y|^{-\frac{Q+m}{\rho}}.
\]

(ii) If \( m = -Q \), there exists \( \ell \in \mathbb{N} \), independent of \( \sigma \), such that
\[
|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S_{\rho,\delta}^{m,\mathcal{L}}} \log |y|.
\]

(iii) If \( m < -Q \), there exists \( \ell \in \mathbb{N} \), independent of \( \sigma \), such that
\[
|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S_{\rho,\delta}^{m,\mathcal{L}}}.
\]
The proof of Proposition 4.23 requires some preliminary results.

**Remark 4.24.** Let us consider \( k \in \mathcal{D}'(G) \), and let \( s > Q/2 \). Then,

\[
\|k\|_{L^2(G)} \lesssim_s \sup_{[\xi] \in \hat{G}} \|\hat{\mathcal{M}}(\xi)^s \hat{k}(\xi)\|_{\text{op}},
\]

(4.28)
in the sense that \( k \in L^2(G) \) when the right hand side is finite. Indeed, let us observe that

\[
\|k\|_{L^2(G)} = \|\hat{k}\|_{L^2(\hat{G})} = \|\hat{\mathcal{M}}(\xi)^{-s} \hat{\mathcal{M}}(\xi)^s \hat{k}\|_{L^2(\hat{G})} \\
\leq \sup_{[\xi] \in \hat{G}} \|\hat{\mathcal{M}}(\xi)^s \hat{k}(\xi)\|_{\text{op}} \|\hat{\mathcal{M}}(\xi)^{-s}\|_{L^2(\hat{G})}.
\]

Observe that

\[
\|\hat{\mathcal{M}}(\xi)^{-s}\|_{L^2(\hat{G})}^2 = \sum_{[\xi] \in \hat{G}} d_{\xi} \|\hat{\mathcal{M}}(\xi)^{-s}\|_{\text{HS}}^2 \\
= \sum_{j=0}^{\infty} \sum_{[\xi]: 2^j \leq (1 + \nu_{\xi}(\xi)^2)^{1/2} < 2^{j+1}, \ \forall 1 \leq k \leq d_{\xi}} d_{\xi} \sum_{i=1}^{d_{\xi}} (1 + \nu_{\xi}(\xi)^2)^{-s} \\
\leq \sum_{j=0}^{\infty} \sum_{[\xi]: 2^j \leq (1 + \nu_{\xi}(\xi)^2)^{1/2} < 2^{j+1}, \ \forall 1 \leq k \leq d_{\xi}} d_{\xi}^2 2^{-2js} \\
\leq \sum_{j=0}^{\infty} \sum_{[\xi]: 2^j \leq (1 + \nu_{\xi}(\xi)^2)^{1/2} < 2^{j+1}, \ \forall 1 \leq k \leq d_{\xi}} d_{\xi}^2 2^{-2js} \\
= \sum_{j=0}^{\infty} N(2^{j+1}) 2^{-2js},
\]

where \( N(\lambda) \) denotes the Weyl function for the sub-Laplacian. From Remark 4.21, we have that \( N(2^{j+1}) \approx N(2^j) \approx 2^{jQ} \). Consequently \( \|\hat{\mathcal{M}}(\xi)^{-s}\|_{L^2(\hat{G})}^2 < \infty \), for \( s > Q/2 \). Thus, we conclude the proof.

**Remark 4.25.** Let us observe that for \( s > Q/2 \), we have the embedding

\[
H^{s, \mathcal{L}}(G) \equiv L^2_s(\mathcal{L}, G) \hookrightarrow C(G).
\]
This is well known, but we will provide a proof for completeness. If \( f \in C^\infty(G) \), then we have
\[
|f(x)| \leq \sum_{[\xi] \in \mathcal{G}} d_\xi |\text{Tr}[\xi(x)\hat{f}(\xi)]| = \sum_{[\xi] \in \mathcal{G}} d_\xi |\text{Tr}[\xi(x)\hat{M}(\xi)^{-s}\hat{M}(\xi)^s\hat{f}(\xi)]|
\]
\[
\leq \sum_{[\xi] \in \mathcal{G}} d_\xi ^{2} \|\xi(x)\hat{M}(\xi)^{-s}\|_{\text{HS}} d_\xi ^{2} \|\hat{M}(\xi)^s\hat{f}(\xi)\|_{\text{HS}}
\]
\[
\leq \left( \sum_{[\xi] \in \mathcal{G}} d_\xi \|\xi(x)\hat{M}(\xi)^{-s}\|_{\text{HS}} \right)^{1/2} \left( \sum_{[\xi] \in \mathcal{G}} d_\xi \|\hat{M}(\xi)^s\hat{f}(\xi)\|_{\text{HS}}^2 \right)^{1/2}
\]
\[
= \|\hat{M}(\xi)^{-s}\|_{L^2(\mathcal{G})} \|f\|_{L^2,\mathcal{G}(G)}.
\]

The condition \( s > Q/2 \), implies that \( \|\hat{M}(\xi)^{-s}\|_{L^2(\mathcal{G})} < \infty \). Now, if \( f_i \to f \) in \( H^s(G) \), the previous inequality shows that \( f_i \to f \) uniformly on \( G \), and it shows that \( f \) is continuous.

**Lemma 4.26.** Let \( G \) be a compact Lie group of Hausdorff dimension \( Q \) and let \( 0 \leq \delta, \rho \leq 1 \). Let \( A : C^\infty(G) \to \mathscr{D}'(G) \) be a continuous linear operator with symbol \( \sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \mathcal{G}) \) in the contracted subelliptic Hörmander class of order \( m \) and of type \( (\rho, \delta) \). Let \( k_x \) be the right-convolution kernel of \( A \), and let us define the function
\[
f_{\beta',\beta,\alpha}(z) := (1 + L_x)^{i|\beta|}([q^{\alpha(1)}_i \cdots q^{\alpha(i)}_i] \partial_z^{\beta'} k_x(z))
\]
where \( \{\Delta_{q(i)}\} \) is an admissible family of difference operators with \( x = e_G \) being the only common zero of the functions \( q(i) \). Then, \( f_{\beta',\beta,\alpha} \) is continuous and bounded provided that
\[
Q + 2|\beta| + m + \delta|\beta'| < \rho|\alpha|.
\]
Moreover,
\[
\sup_{z \in G} |f_{\beta',\beta,\alpha}(z)| \leq \|\sigma\|_{|\alpha|+|\beta'|,S^{m,\mathcal{L}}_{\rho,\delta}}.
\]

**Proof.** Let us fix \( s > Q/2 \), and let \( a = |\alpha| \). By using the subelliptic Sobolev embedding theorem (see Remark 4.25), if we prove that there exists \( C_s > 0 \), such that
\[
\|f_{\beta',\beta,\alpha}(z)\|_{H^s,\mathcal{G}(G)} \leq C_s,
\]
then \( f_{\beta',\beta,\alpha}(z) \) is a continuous and bounded function, and
\[
\sup_{z \in G} |f_{\beta',\beta,\alpha}(z)| \leq s \|f_{\beta',\beta,\alpha}(z)\|_{H^s,\mathcal{G}(G)}.
\]

Observe that
\[
\|f_{\beta',\beta,\alpha}(z)\|_{H^s,\mathcal{G}(G)} = \|(1 + L)^{\frac{s+2|\beta|}{2}}[q^{\alpha(1)}_i \cdots q^{\alpha(i)}_i] \partial_z^{\beta'} k_x(z)\|_{L^2(G)}.
\]
Fix \( s' > Q/2 \). From (4.28), we deduce
\[
\left\| (1 + \mathcal{L})^{\frac{s+2|\beta|}{2}} [q^0_{\alpha(1)} \cdots q^0_{\alpha(s)}] \mathcal{C}^\beta_{x} k_x(z) \right\|_{L^2(G)} \\
\leq \sup_{[\xi] \in \hat{G}} \left\| \mathcal{M}(\xi) s' \mathcal{F} [(1 + \mathcal{L})^{\frac{s+2|\beta|}{2}} [q^0_{\alpha(1)} \cdots q^0_{\alpha(s)}] \mathcal{C}^\beta_{x} k_x(z)] \right\|_{op} \\
= \sup_{[\xi] \in \hat{G}} \left\| \mathcal{M}(\xi) s' + 2|\beta| \Delta_{q_o} \mathcal{C}^\beta_{x} \sigma(x, \xi) \right\|_{op}.
\]
Since \( \sigma \in S^m_{\rho, \delta} (G \times \hat{G}) = S^m_{\rho, \delta} (G \times \hat{G}) \), we can write,
\[
\left\| \mathcal{M}(\xi) s' + 2|\beta| \Delta_{q_o} \mathcal{C}^\beta_{x} \sigma(x, \xi) \right\|_{op} \\
= \left\| \mathcal{M}(\xi) s' + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \mathcal{M}(\xi) - m + \rho|\alpha| - \delta|\beta| \Delta_{q_o} \mathcal{C}^\beta_{x} \sigma(x, \xi) \right\|_{op} \\
\leq \sup_{[\xi] \in \hat{G}} \left\| \mathcal{M}(\xi) s' + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \mathcal{M}(\xi) - m + \rho|\alpha| - \delta|\beta| \Delta_{q_o} \mathcal{C}^\beta_{x} \sigma(x, \xi) \right\|_{op} \\
\leq \sup_{[\xi] \in \hat{G}} \left\| \mathcal{M}(\xi) s' + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \mathcal{M}(\xi) - m + \rho|\alpha| - \delta|\beta| \Delta_{q_o} \mathcal{C}^\beta_{x} \sigma(x, \xi) \right\|_{op}.
\]
To conclude the proof, we need only that
\[
I_{s, s'} := \sup_{[\xi] \in \hat{G}} \left\| \mathcal{M}(\xi) s' + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \right\|_{op} \leq \infty.
\]
Indeed, in (4.29), we can take \( C_{s} := I_{s, s'} \| \sigma \|_{\alpha + |\beta|, S^m_{\rho, \delta}} \). But, in order to assure that \( I_{s, s'} \leq \infty \), we only need to impose that \( s' + s + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \leq 0 \), restricted to \( s + s' > \frac{Q}{2} + \frac{Q}{2} = Q \). The inequality, \( Q + 2|\beta| < s' + s + 2|\beta| \), shows that the required inequality holds if, \( Q + 2|\beta| < m + \rho|\alpha| - \delta|\beta| < 0 \). In this case, we can assure the existence of \( s \), and \( s' \) close enough to \( Q/2 \), as above such that
\[
Q + 2|\beta| + m - \rho|\alpha| + \delta|\beta| < s' + s + 2|\beta| + m - \rho|\alpha| + \delta|\beta| \leq 0.
\]
Thus, we finish the proof.

From the previous lemma we deduce the following immediate consequence.

**Corollary 4.27.** Let \( G \) be a compact Lie group of Hausdorff dimension \( Q \) and let \( 0 \leq \delta, \rho \leq 1 \). Let \( A : C^\infty_c (G) \to \mathcal{D}'(G) \) be a continuous linear operator with symbol \( \sigma \in S^m_{\rho, \delta} (G \times \hat{G}) \) in the contracted subelliptic Hörmander class of order \( m \) and of type \( (\rho, \delta) \). Let \( k_x \) be the right-convolution kernel of \( A \), and let us define the function
\[
f(z) := k_x(z).
\]
Then, \( f \) is continuous and bounded provided that
\[
Q + m < 0.
\]

Finally we need the following Lemma (see Lemma 6.8 of [61]).

**Lemma 4.28.** Let \( \sigma \in S^m_{\rho, \delta} (G \times \hat{G}) \), with \( 0 \leq \delta \leq \rho \leq 1 \). Let \( \eta \in C^\infty_0 (\mathbb{R}) \). For every \( t \in (0, 1) \), we define the symbol \( \sigma_t \) by \( \sigma_t(x, \xi) = \sigma(x, \xi) \eta(t/\xi) \). Then, for every \( m_1 \in \mathbb{R} \), we have
\[
\| \sigma_t \|_{a + b, S^m_{\rho, \delta}} \leq C \| \sigma \|_{a + b, S^m_{\rho, \delta}} t^{m_1 - m/2},
\]
where \( C = C_{m, m_1, a, b, \eta} \) does not depend on \( \sigma \) and \( t \in (0, 1) \).
Proof of Proposition 4.23. In the light of Remark 4.24 and Lemma 4.26 we will follow the approach in the proof of Proposition 6.8 in [61], adapted to the subelliptic case. Note that for $m < -Q$, Proposition 4.23 follows from Corollary 4.27.

So, we will assume that $m \geq -Q$. In order to prove the theorem in this case, we will use a Littlewood-Paley decomposition. Let us choose $\eta_0$ and $\eta_1$, supported in $[-1, 1]$ and $[\frac{1}{2}, 2]$ respectively, with $0 \leq \eta_0, \eta_1 \leq 1$, such that

$$\forall t \geq 0, \sum_{\ell \in \mathbb{N}} \eta_\ell(t) = 1, \quad \text{where for all } \ell \in \mathbb{N}, \eta_\ell(t) := \eta_1(2^{-(\ell - 1)}t).$$

For every $\ell \in \mathbb{N}_0$, let us denote $\sigma_\ell(x, \xi) := \sigma(x, \xi)\eta_\ell(\langle \xi \rangle)$. Let us denote by $k_x$ and $k_{x, \ell}$ the right-convolutions kernels associated with $A$ and $T_{\sigma_\ell} = \text{Op}(\sigma_\ell)$, respectively. We have the (possibly unbounded summation),

$$|k_x(y)| \leq \sum_{\ell = 0}^{\infty} |k_{x, \ell}(y)|.$$

By Lemma 4.26, we have

$$\sup_{z \in \mathcal{G}} |(q_1^{0(1)} \cdots q_i^{0(i)}) k_{x, \ell}(z)| \lesssim \|\sigma_\ell\|_{[\alpha], S_{\rho, \delta}^{m, \mathcal{E}}}.$$

Although we can take an arbitrary $m_1 \in \mathbb{R}$ because $k_{x, \ell}$ is a smooth function in view that $\sigma_\ell \in \mathcal{S}_{1, 0}^{-\infty} := \cap_{\nu \in \mathbb{R}} \mathcal{S}_{1, 0}^{\nu}$, the hypothesis in Lemma 4.26 suggests us to fix the following condition

$$m_1 + Q < \rho|\alpha|.$$

From (6.5) in Remark 6.5, applied with $s = 2^{-(\ell - 1)}$ and $\omega = (m_1 - m)/2$, we obtain the following estimate for the subelliptic seminorm $\| \cdot \|_{[\alpha], S_{\rho, \delta}^{m, \mathcal{E}}}$ of $\sigma_\ell$,

$$\|\sigma_\ell\|_{[\alpha], S_{\rho, \delta}^{m, \mathcal{E}}}, \lesssim \|\sigma\|_{[\alpha], S_{\rho, \delta}^{m, \mathcal{E}}} 2^{-(\ell - 1)\frac{(m_1 - m)}{2}} \lesssim \|\sigma\|_{[\alpha], S_{\rho, \delta}^{m, \mathcal{E}}} 2^{-\ell\frac{(m_1 - m)}{2}}.$$

As is Lemma 5.6 of [61], the fact that (see [61, page 20])

$$\text{rank}\{\nabla q_{(j)}(e) : 1 \leq j \leq i\} = \dim(G), \Delta q_{(j)} \in \text{diff}(\tilde{G}),$$

and by using that $x = e_G$ is the only common zero of the functions $q_{(j)}$, implies that (see [61, page 20])

$$|z|^a \lesssim \sum_{|\alpha| = a} |q_1^{0(1)} \cdots q_i^{0(i)}|.$$

Consequently, for every $a \in 2\mathbb{N}_0$, and $m_1 \in \mathbb{R}$ with $m_1 + Q < \rho a$, we have proved the estimate

$$|z|^a |k_{x, \ell}(z)| \lesssim \|\sigma\|_{a, S_{\rho, \delta}^{m, \mathcal{E}}} 2^\frac{(m - m_1)}{2}.$$

Because our analysis is local, and $G$ is compact, we only need to consider the case $|z| < 1$. So, let us choose $\ell_0 \in \mathbb{N}$, such that $|z| \sim 2^{-\ell_0}$. So, we consider the following two situations.

- Case 1. $m + Q > 0$.
- Case 2. $m + Q = 0$. 

In the first case, for $\ell \leq \ell_0$, we can choose $m_1 \in \mathbb{R}$, such that

$$\frac{m + Q}{\rho} > a \geq \frac{m + Q}{\rho} - 2, \text{ and, } \frac{m - m_1}{2} = \frac{m + Q}{\rho} - a.$$  

Because $m + Q > \rho a$, $\frac{m + Q}{\rho} - a = \frac{m - m_1}{2} > 0$. So, $m > m_1$, and

$$\sum_{\ell=0}^{\ell_0} |k_{x,\ell}(z)| \lesssim \sum_{\ell=0}^{\ell_0} |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell (\frac{m-m_1}{2})}$$

$$\lesssim \sum_{\ell=0}^{\ell_0} |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell_0 (\frac{m-m_1}{2})}$$

$$= |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} \ell_0 2^{\ell_0 (\frac{m-m_1}{2})}$$

$$\lesssim \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell_0 (\frac{m+Q}{\rho})} \approx |z|^{-\frac{m+Q}{\rho}}.$$

On the other hand, if $\ell > \ell_0$, we choose $m_1$ satisfying

$$\frac{m + Q}{\rho} > a \geq \frac{m + Q}{\rho} - 2, \text{ and, } \frac{m - m_1}{2} = \frac{m + Q}{\rho} - a - 2.$$  

So, $m < m_1$, and again we have

$$\sum_{\ell=\ell_0+1}^{\infty} |k_{x,\ell}(z)| \lesssim \sum_{\ell=\ell_0+1}^{\infty} |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell (\frac{m-m_1}{2})}$$

$$\lesssim |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell_0 (\frac{m-m_1}{2})}$$

$$= |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |z|^{-(\frac{m-m_1}{2}+a)} = \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |z|^{-(\frac{m-m_1}{2}+a)}$$

$$= \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |z|^{-(\frac{m+Q}{\rho}-2)}$$

$$\lesssim \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |z|^{-\frac{m+Q}{\rho}}.$$

So, we have proved the statement for Case 1. Now, if we consider $m = -Q$, then we choose $a = 0$, $m_1 = m$ and we proceed as above,

$$\sum_{\ell=0}^{\ell_0} |k_{x,\ell}(z)| \lesssim \sum_{\ell=0}^{\ell_0} |z|^{-a} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} 2^{\ell (\frac{m-m_1}{2})} = \sum_{\ell=0}^{\ell_0} \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}}$$

$$= \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} \cdot \ell_0$$

$$= \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |\log(2^{-\ell_0})|$$

$$= \|\sigma\|_{a,\mathcal{S}_{p,\delta}^{m,c}} |\log |z||.$$
On the other hand, if \( \ell > \ell_0 \), we choose \( m_1 = m + 4 \), and \( a = 2 \). Then, \( \frac{m_1 - m}{2} = 2 = a \), and we can estimate

\[
\sum_{\ell = \ell_0 + 1}^{\infty} |k_{x, \ell}(z)| \lesssim \sum_{\ell = \ell_0 + 1}^{\infty} |z|^{-a} \|\sigma\|_{a, \rho, \delta} 2^{-2 \ell} = \|\sigma\|_{a, \rho, \delta} 2^{2\ell_0} \sum_{\ell = \ell_0 + 1}^{\infty} 2^{-2\ell} = \|\sigma\|_{a, \rho, \delta} 2^{2\ell_0} \left( \frac{2^{-2(\ell_0 + 1)}}{1 - 2^{-2}} \right) \lesssim \|\sigma\|_{a, \rho, \delta} \ell_0 \]

Thus, we finish the proof. \( \square \)

4.3. Calderón-Vaillancourt Theorem for subelliptic classes. The aim of this subsection is to prove the following subelliptic version of the Calderón-Vaillancourt theorem.

**Theorem 4.29.** Let \( G \) be a compact Lie group and let us consider the sub-Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{ X_1, \ldots, X_k \} \) is a system of vector fields satisfying the Hörmander condition of order \( \kappa \). For \( 0 \leq \delta < \rho \leq 1 \), let us consider a continuous linear operator \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) with symbol \( \sigma \in S^{0, \rho}(G \times \hat{G}) \). Then \( A \) extends to a bounded operator from \( L^2(G) \) to \( L^2(G) \). Moreover,

\[
\|A\|_{\mathcal{B}(L^2(G))} \leq C \|\sigma\|_{\rho, \sigma, \delta},
\]

for \( \ell \in \mathbb{N} \) large enough. In the case where \( \delta \leq 1/2 \kappa \), and \( \rho \leq 1/2 \kappa \), the condition \( \delta < \rho \) can be improved to \( \delta \leq \rho \) in order to obtain the \( L^2(G) \)-boundedness of \( A \).

In order to prove Theorem 4.29, we start with the following subelliptic bilinear estimate. For every \( \eta \in \hat{G} \), we will denote by \( H_{\lambda(\eta)}(G) \) the eigenspace of the Laplacian \( \mathcal{L}_G \) associated with the eigenvalue \( \lambda(\eta) \).

**Lemma 4.30.** Let \( f \in H_{\lambda[\xi]}(G) \), \( g \in H_{\lambda[\xi']} (G) \), and let \( \gamma \in \mathbb{R} \), \( s > Q/2 \), be such that \( 2\gamma + s \leq 0 \). Then, the following subelliptic bilinear estimate

\[
\| (1 + \mathcal{L})^{\gamma} (fg) \|_{L^2(G)} \leq C_{s, \gamma} (1 + |\lambda_{[\xi]} - \lambda_{[\xi']}|)^{\frac{1}{n} \left( \gamma + \frac{s}{2} \right) + \frac{n}{2} \delta} \|g\|_{L^2(G)} \|f\|_{L^2(G)}
\]

holds true uniformly in \( \xi, \xi' \in \hat{G} \). Here \( n \) is the topological dimension of \( G \).

**Proof.** Let us consider \( f \in H_{\lambda[\xi]}(G) \) and \( g \in H_{\lambda[\xi']} (G) \). Without loss of generality let us consider \( \lambda_{[\xi]} \gg \lambda_{[\xi']} \). From the Fourier inversion formula we have,

\[
(fg)(x) := f(x)g(x) = \sum_{[\eta] \in \hat{G} : \lambda_{[\eta]} = \lambda_{[\xi']}} d_{\eta} \text{Tr}[\eta(x)f(x)\hat{g}(\eta)].
\]

So, if the apply \( (1 + \mathcal{L})^{\gamma} \) to both sides, we have

\[
(1 + \mathcal{L})^{\gamma} (fg)(x) = \sum_{[\eta] \in \hat{G} : \lambda_{[\eta]} = \lambda_{[\xi']}} d_{\eta} \text{Tr}[(1 + \mathcal{L})^{\gamma} (\eta(x)f(x))\hat{g}(\eta)].
\]
The Cauchy-Schwarz inequality gives,
\[ |(1 + \mathcal{L})^\gamma (fg)(x)| \leq \sum_{[\eta]\in\hat{G} : \lambda_{[\eta]} = \lambda_{[\eta']}} d_\eta |\text{Tr}[(1 + \mathcal{L})^\gamma (\eta(x)f(x))\widehat{g}(\eta)]| \]
\[ \leq \|g\|_{L^2(G)} \left( \sum_{[\eta]\in\hat{G} : \lambda_{[\eta]} = \lambda_{[\eta']}} d_\eta \| (1 + \mathcal{L})^\gamma (\eta(x)f(x)) \|_{\text{HS}}^2 \right)^{\frac{1}{2}} , \]
and consequently
\[ \| (1 + \mathcal{L})^\gamma (fg) \|_{L^2(G)}^2 \leq \|g\|_{L^2(G)}^2 \sum_{[\eta]\in\hat{G} : \lambda_{[\eta]} = \lambda_{[\eta']}} d_\eta \int_G \| (1 + \mathcal{L})^\gamma (\eta(x)f(x)) \|_{\text{HS}}^2 dx \] (4.31)
To estimate the integral on the right-hand side we can consider a basis \( \{ e_{\tau,i} \}_{1 \leq i \leq d_\tau} \) of every representation space \( \mathbb{C}^{d_\tau} \), where the symbol of operator \( \mathcal{M}_{2\gamma} := (1 + \mathcal{L})^\gamma \) is diagonal
\[ \widehat{\mathcal{M}}_{2\gamma}(\tau)e_{\tau,i} = \mathcal{M}(\tau)^{2\gamma}e_{\tau,i} = (1 + \nu_{\tau}(\tau)^2)^\gamma e_{\tau,i}, \ \forall 1 \leq i \leq d_\tau, \ [\tau] \in \hat{G} . \]
So, from the Plancherel formula, we have
\[ \int_G \| (1 + \mathcal{L})^\gamma (\eta(x)f(x)) \|_{\text{HS}}^2 dx = \sum_{1 \leq k, \ell \leq d_\eta} \int_G \| (1 + \mathcal{L})^\gamma (f(x)\eta_{\ell,k}(x)) \|_{\text{HS}}^2 dx \]
\[ = \sum_{1 \leq k, \ell \leq d_\eta} \sum_{[\tau] \in \hat{G}} d_\eta \| \widehat{\mathcal{M}}(\tau)^{2\gamma} \mathcal{F}[\eta_{\ell,k}](\tau) \|_{\text{HS}}^2 \]
\[ = \sum_{1 \leq k, \ell \leq d_\eta} \sum_{[\tau] \in \hat{G}} \sum_{\ell', k' = 1} d_\tau (1 + \nu_{\tau}(\tau)^2)^\gamma \mathcal{F}[\eta_{\ell,k}](\tau_{\ell'k'})^2 . \]
Observe that
\[ \mathcal{F}[\eta_{\ell,k}](\tau_{\ell'k'}) = \int_G f(y)\eta_{\ell,k}(y)\tau_{\ell'k'}(y) dy = \int_G f(y^{-1})\tau_{\ell'k'}(y)\eta_{\ell,k}^*(y) dy = \mathcal{F}[\tilde{f}\tau_{\ell'k'}](\eta_{\ell,k}) , \]
where \( \tilde{f}(-\cdot) := f(\cdot)^{-1} \). So, by changing the summation order we have,
\[ \sum_{1 \leq k, \ell \leq d_\eta} \sum_{[\tau] \in \hat{G}} \sum_{\ell', k' = 1} d_\tau (1 + \nu_{\tau}(\tau)^2)^\gamma \mathcal{F}[\eta_{\ell,k}](\tau_{\ell'k'})^2 \]
\[ = \sum_{[\tau] \in \hat{G}} d_\tau \sum_{\ell', k' = 1} (1 + \nu_{\tau}(\tau)^2)^2 \sum_{1 \leq k, \ell \leq d_\eta} |\mathcal{F}[\tilde{f}\tau_{\ell'k'}](\eta_{\ell,k})|^2 \]
\[ = \sum_{[\tau] \in \hat{G}} d_\tau \sum_{\ell', k' = 1} (1 + \nu_{\tau}(\tau)^2)^2 \mathcal{F}[\tilde{f}\tau_{\ell'k'}](\eta_{\ell,k})^2 . \]
Returning to (4.31), and using the Plancherel theorem, we can estimate
\begin{align*}
\|(1 + \mathcal{L})^\gamma (fg)\|_{L^2(G)}^2 & \leq \|g\|_{L^2(G)}^2 \sum_{[\eta]\in \hat{G}: \lambda_{[\eta]} = \lambda_{[\xi]}} d_\eta \int_G \|(1 + \mathcal{L})^\gamma (\eta(x)f(x))\|_{\text{HS}}^2 \, dx \\
& \leq \|g\|_{L^2(G)}^2 \sum_{[\eta]\in \hat{G}: \lambda_{[\eta]} = \lambda_{[\xi]}} d_\eta \sum_{[\tau]\in \hat{G}} d_\tau \sum_{e', \kappa' = 1}^{d_\tau} (1 + \nu e' e'' (\tau)^2)^2 \gamma \sum_{[\eta]\in \hat{G}: \lambda_{[\eta]} = \lambda_{[\xi]}} d_\eta \|\mathcal{F}[\tilde{f}_\tau \tau_{e''}] (\eta)\|_{\text{HS}}^2 \\
& = \|g\|_{L^2(G)}^2 \sum_{[\tau]\in \hat{G}} d_\tau \sum_{e', \kappa' = 1}^{d_\tau} (1 + \nu e' e'' (\tau)^2)^2 \gamma \|P_{\xi'}[\tilde{f}_\tau \tau_{e''}]\|_{L^2(G)}^2,
\end{align*}

where \(P_{\xi'} : L^2(G) \to H_{\lambda_{[\xi']}}(G)\), denotes the orthogonal projection on the subspace \(H_{\lambda_{[\xi']}}(G)\), with \(\lambda_{[\eta]} = \lambda_{[\xi']}\). By using that \(f \in H_{\lambda_{[\xi]}}(G)\), and \(\tau_{e''} \in H_{\lambda_{[\tau]}(G)}\). It was proved in [61, page 45], that \(\tau_{e''} \in \bigoplus_{\lambda_{[\xi]} \leq \max\{\lambda_{[\xi]}, \lambda_{[\tau]}\}} H_{\lambda_{[\xi]}}\). So, if \(\lambda_{[\xi]} + \lambda_{[\tau]} < \lambda_{[\xi']}\), then \(P_{\xi'}[\tilde{f}_\tau \tau_{e''}] = 0\). From this analysis we deduce

\begin{align*}
\|(1 + \mathcal{L})^\gamma (fg)\|_{L^2(G)}^2 & \leq \|g\|_{L^2(G)}^2 \sum_{[\eta]\in \hat{G}: \lambda_{[\eta]} = \lambda_{[\xi]}} d_\eta \int_G \|(1 + \mathcal{L})^\gamma (\eta(x)f(x))\|_{\text{HS}}^2 \, dx \\
& \leq \|g\|_{L^2(G)}^2 \sum_{[\tau]\in \hat{G}: \lambda_{[\tau]} > \lambda_{[\xi]} - \lambda_{[\xi]}} d_\tau \sum_{e', \kappa' = 1}^{d_\tau} (1 + \nu e' e'' (\tau)^2)^2 \gamma \sup_{1 \leq \tau' \leq d_\tau} \sum_{k' = 1}^{d_\tau} \|P_{\xi'}[\tilde{f}_\tau \tau_{e''}]\|_{L^2(G)}^2.
\end{align*}

Since \(\|P_{\xi'}\|_{\mathcal{B}(L^2(G))} = 1\), and \(\|\tau_{e''}(x)\|_\text{op} = 1\), we have

\begin{align*}
\sum_{k' = 1}^{d_\tau} \|P_{\xi'}[\tilde{f}_\tau \tau_{e''}]\|_{L^2(G)}^2 & \leq d_\tau \sup_{k' = 1, \ldots, d_\tau} \|\tau_{e''}(x)\|_\text{op}^2 \|\tilde{f}_\tau\|_{L^2(G)}^2 \leq d_\tau \|\tilde{f}_\tau\|_{L^2(G)}^2,
\end{align*}

Consequently,

\begin{align*}
\|(1 + \mathcal{L})^\gamma (fg)\|_{L^2(G)}^2 & \leq \|g\|_{L^2(G)}^2 \sum_{[\eta]\in \hat{G}: \lambda_{[\eta]} = \lambda_{[\xi]}} d_\eta \int_G \|(1 + \mathcal{L})^\gamma (\eta(x)f(x))\|_{\text{HS}}^2 \, dx \\
& \leq \|g\|_{L^2(G)}^2 \sum_{[\tau]\in \hat{G}: \lambda_{[\tau]} > \lambda_{[\xi]} - \lambda_{[\xi]}} d_\tau \sum_{1 \leq \tau' \leq d_\tau} (1 + \nu e' e'' (\tau)^2)^2 \gamma \times d_\tau \|f\|_{L^2(G)}^2 \\
& = \|g\|_{L^2(G)}^2 \|f\|_{L^2(G)}^2 \sum_{[\tau]\in \hat{G}: \lambda_{[\tau]} > \lambda_{[\xi]} - \lambda_{[\xi]}} d_\tau^2 \sum_{1 \leq \tau' \leq d_\tau} (1 + \nu e' e'' (\tau)^2)^2 \gamma.
\end{align*}
Hence, we have obtained the following estimate,
\[
\| (1 + \mathcal{L})^\gamma (fg) \|_{L^2(G)}^2 \leq \| g \|_{L^2(G)}^2 \| f \|_{L^2(G)}^2 I,
\]
where,
\[
I := \sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} d_r^2 (1 + \nu_{0 r}^2 (\tau)^2)^{2 \gamma} < \infty.
\]
In order to finish the proof we need to show that \( I < \infty \). Let us fix \( \gamma \) and \( s \) such that \( 2 \gamma + s + \frac{Q}{4} < 0 \), and \( s > Q/2 \). Observe that
\[
\sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} d_r^2 (1 + \nu_{0 r}^2 (\tau)^2)^{2 \gamma}
\]
\[
= \sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} d_r \times d_r (1 + \nu_{0 r}^2 (\tau)^2)^{2 \gamma + s} (1 + \nu_{0 r}^2 (\tau)^2)^{-s}
\]
\[
\leq \sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} \sum_{1 \leq t' \leq d_r} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{2 \gamma + s} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{-s}
\]
and by using the estimate \( d_r \leq (1 + \nu_{0 r}^2 (\tau)^2)^{\frac{Q}{4}} \) (see (4.27)) we have
\[
\sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} \sup_{1 \leq t' \leq d_r} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{2 \gamma + s} \sum_{1 \leq t' \leq d_r} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{-s}
\]
\[
\leq \sum_{[r] \in \mathcal{G} : \lambda r \geq \lambda [q] - \lambda [l]} (1 + \lambda [q] - \lambda [l])^{\frac{1}{2} (2 \gamma + s + \frac{Q}{4})} \sup_{1 \leq t' \leq d_r} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{-s}
\]
\[
\leq (1 + \lambda [q] - \lambda [l])^{\frac{1}{2} (2 \gamma + s + \frac{Q}{4})} \sum_{[r] \in \mathcal{G}} \sum_{1 \leq t' \leq d_r} d_r (1 + \nu_{0 r}^2 (\tau)^2)^{-s}
\]
\[
= (1 + \lambda [q] - \lambda [l])^{\frac{1}{2} (2 \gamma + s + \frac{Q}{4})} \| \hat{\mathcal{M}}_{-\gamma} (\xi) \|_{L^2(G)}^2 < \infty,
\]
where we have used that \( \| \hat{\mathcal{M}}_{-\gamma} (\xi) \|_{L^2(G)} < \infty \) for \( s > Q/2 \). Thus, we conclude that
\[
\| (1 + \mathcal{L})^\gamma (fg) \|_{L^2(G)}^2 \leq \| g \|_{L^2(G)}^2 \| f \|_{L^2(G)}^2 (1 + \lambda [q] - \lambda [l])^{\frac{1}{2} (2 \gamma + s + \frac{Q}{4})}.
\]
So, we finish the proof.

\[ \square \]

**Proof of Theorem 4.29.** Let us fix \( 0 < \delta \leq \rho \leq 1/2 \kappa, \delta \neq 1/2 \kappa \). Indeed, Theorem 4.29 for \( \rho = 0 \) follows from Theorem 10.5.5 of [107]. We will give a additional proof for the case \( 0 \leq \delta < \rho \leq 1 \) in Corollary 6.7. Let us choose \( \eta_0 \) and \( \eta_1 \), supported in \([-1,1]\) and \([\frac{1}{2},2]\) respectively, with \( 0 \leq \eta_0, \eta_1 \leq 1 \), such that
\[
\forall t \geq 0, \sum_{\ell=0}^{\infty} \eta_\ell(t) = 1, \text{ where for all } \ell \in \mathbb{N}, \eta_\ell(t) := \eta_\ell (2^{-\ell} t),
\]
For every $\ell \in \mathbb{N}_0$, let us denote $\sigma_\ell(x, \xi) := \sigma(x, \xi)\eta_\ell(\langle \xi \rangle)$. Let us denote by $k_x$ and $k_{x,\ell}$ the right-convolutions kernels associated with $A$ and $T_{\sigma_\ell} = \text{Op}(\sigma_\ell)$. From the properties of the dyadic decomposition we have (see e.g. [61, page 30])

$$\|A\|_{\mathcal{B}(L^2(G))}^2 \leq \sup_{\ell \in \mathbb{N}_0} \|T_{\sigma_\ell}\|_{\mathcal{B}(L^2(G))}^2 + \sum_{\ell \neq \ell', \ell, \ell' \in 2\mathbb{N}_0} \|T^*_{\sigma_\ell}T_{\sigma_{\ell'}}\|_{\mathcal{B}(L^2(G))} + \sum_{\ell \neq \ell', \ell, \ell' \in 2\mathbb{N}_0+1} \|T^*_{\sigma_\ell}T_{\sigma_{\ell'}}\|_{\mathcal{B}(L^2(G))}
= I + II + III,$$
where $I := \sup_{\ell \in \mathbb{N}_0} \|T_{\sigma_\ell}\|_{\mathcal{B}(L^2(G))}$, and $II := \sum_{\ell \neq \ell', \ell, \ell' \in 2\mathbb{N}_0} \|T^*_{\sigma_\ell}T_{\sigma_{\ell'}}\|_{\mathcal{B}(L^2(G))}$. In order to prove that $I < \infty$, we will use that the exponential map

$$\exp : v \subset g \simeq \mathbb{R}^n \to B(e_G, \varepsilon_0),$$

is a local diffeomorphism from an open neighbourhood $\mathfrak{v}$ of $0 \in \mathbb{R}^n$, $n = \dim(G)$, into an ball $B(e_G, \varepsilon_0)$ containing the identity $e_G$ of $G$. The ball $B(e_G, \varepsilon_0)$ is defined by the geodesic distance on $G$ and $\varepsilon_0 > 0$. By the compactness of $G$, there exists a finite number of elements $x_0 = e_G, x_i, 1 \leq i \leq N_0$, in $G$, and some smooth functions $\chi_j \in C^\infty(G, [0, 1])$, supported in $B(e_G, \varepsilon_0/2)$, such that

$$G = \bigcup_{j=1}^{N_0} B(x_j, \varepsilon_0/4), \quad \text{and} \quad \sum_{j=0}^{N_0} \chi_j(x_j^{-1}x) = 1, \quad x \in G.$$

For every $0 < r \leq 1$, let us define the local dilation $D_r : \mathfrak{v} \to B(e_G, \varepsilon_0)$ given by

$$D_r(x) \equiv r \cdot x := \exp(r \exp^{-1}(x)).$$

Observe that,

$$\|T_\ell\|_{\mathcal{B}(L^2(G))} = \|\text{Op}(\sigma_\ell(x, \xi))\|_{\mathcal{B}(L^2(G))} \leq \sum_{j=0}^{N_0} \|\text{Op}(\sigma_\ell(x, \xi)\chi_j(x_j^{-1}x))\|_{\mathcal{B}(L^2(G))}
= \sum_{j=0}^{N_0} \|\text{Op}(\sigma_{\ell j}(x_j x, \xi)\chi_j(x))\|_{\mathcal{B}(L^2(G))},$$

with the possibly unbounded sequence in $\ell \in \mathbb{N}_0$ in the right-hand side of the inequalities. The idea, is to show that this is not the case, and we will estimate the operator norm $\|\text{Op}(\sigma_{\ell j}(x_j x, \xi)\chi_j(x))\|_{\mathcal{B}(L^2(G))}$. To simplify the notation, let us write $\sigma_{j,\ell}(x, \xi) := \sigma_{\ell j}(x_j x, \xi)\chi_j(x)$. So, we have

$$\sup_{\ell \in \mathbb{N}_0} \|T_\ell\|_{\mathcal{B}(L^2(G))} \leq \sum_{j=0}^{N_0} \sup_{\ell \in \mathbb{N}_0} \|\text{Op}(\sigma_{j,\ell})\|_{\mathcal{B}(L^2(G))}.$$

Because the family of symbols are compactly supported in both variables, $x$ and $[\xi]$, the fact that $\sigma \in \mathcal{S}_\rho^0,\xi (G \times \hat{G})$, also implies that $\sigma_{j,\ell} \in \mathcal{S}_{\rho,\delta}^0,\xi (G \times \hat{G})$.

Let us define some dilations of every dyadic part $\sigma_{j,\ell}$ in the spirit of the classical proof of the Calderón-Vaillancourt Theorem on $\mathbb{R}^n$ (see [20, 21]). The support of

---

14 The exponential of $X \in g$ is given by $\exp(X) = \gamma_X(1)$, where $\gamma_X : \mathbb{R} \to G$, is the unique one parameter subgroup of $G$ whose tangent vector at the identity is equal to $X$. 
\( \sigma_{j, \ell} \) is contained in the set of \((x, [\xi])\), such that \( \langle \xi \rangle \sim 2^\frac{\ell}{n} \) and \( x \in B(e_G, \varepsilon_0) \), so we can define the following dilation of \( \sigma_{j, \ell} \),

\[
\tilde{\sigma}_{j, \ell}(x, \xi) := \sigma_{j, \ell}(2^{-\frac{\ell}{n}} \cdot x, \xi) \times 1_{B(e_G, \varepsilon_0)},
\]

Taking into account that \( \tilde{\sigma}_{j, \ell} \), keep fixed the \([\xi]\)-variables of the symbol \( \sigma_{j, \ell} \), we deduce that \( \tilde{\sigma}_{j, \ell} \) satisfies the subelliptic conditions of type \( \rho \). We claim that \( \tilde{\sigma}_{j, \ell} \in S^{0, \xi}_{\rho, \delta}(G \times \hat{G}) \). So, we need to check that \( \tilde{\sigma}_{j, \ell} \) satisfies the subelliptic conditions of type \( \delta \), for \( \delta \leq \rho \). For this, observe that,

\[
\begin{align*}
\hat{c}^{(1)}_{X_i} \tilde{\sigma}_{j, \ell}(x, \xi) &= \hat{c}^{(1)}_{X_i} \sigma_{j, \ell}(2^{-\frac{\ell}{n}} \cdot x, \xi) = \hat{c}^{(1)}_{X_i} \sigma_{j, \ell}(\exp(2^{-\frac{\ell}{n}} \exp^{-1}(x)), \xi) \\
&= (\hat{c}^{(1)}_{X_i} \sigma_{j, \ell})(\exp(2^{-\frac{\ell}{n}} \exp^{-1}(x)), \xi) \cdot (\hat{c}^{(1)}_{X_i} \exp(2^{-\frac{\ell}{n}} \exp^{-1}(x)) \\
&\quad \times 2^{-\frac{\ell}{n}} (\hat{c}^{(1)}_{X_i} \exp^{-1}(x)) \\
&= 2^{-\frac{\ell}{n}} (\hat{c}^{(1)}_{X_i} \sigma_{j, \ell})(\exp(2^{-\frac{\ell}{n}} \exp^{-1}(x)), \xi) \cdot (\hat{c}^{(1)}_{X_i} \exp(2^{-\frac{\ell}{n}} \exp^{-1}(x)) \\
&\quad \times (\hat{c}^{(1)}_{X_i} \exp^{-1}(x)).
\end{align*}
\]

Now, from the fact that \( \sigma_{j, \ell} \in S^{0, \xi}_{\rho, \delta}(G \times \hat{G}) \), we deduce

\[
\begin{align*}
\| &\hat{\mathcal{M}}(\xi)\|^{- \delta} \| \hat{c}^{(1)}_{X_i} \tilde{\sigma}_{j, \ell}(z'', \xi) \|_\text{op} \\
\leq &\ 2^{-\frac{\ell}{n}} \sup_{z'' \in G} \| (\hat{c}^{(1)}_{X_i} \exp(z) \hat{c}^{(1)}_{X_i} \exp^{-1}(z')) \| \| \hat{\mathcal{M}}(\xi)\|^{- \delta} \| \hat{c}^{(1)}_{X_i} \tilde{\sigma}_{j, \ell}(z'', \xi) \|_\text{op} \\
\leq &\ 2^{-\frac{\ell}{n}} \| \hat{\mathcal{M}}(\xi)\|^{- \delta} \| \hat{c}^{(1)}_{X_i} \tilde{\sigma}_{j, \ell}(z'', \xi) \|_\text{op} \\
\leq &\ 2^{-\frac{\ell}{n}} \| \hat{\mathcal{M}}(\xi)\|^{- \delta} \| \hat{c}^{(1)}_{X_i} \tilde{\sigma}_{j, \ell}(z'', \xi) \|_\text{op}.
\end{align*}
\]

A similar argument can help us to deduce the following estimate for higher derivatives

\[
\begin{align*}
\sup_{(x, \xi) \in G \times \hat{G}} \| &\hat{\mathcal{M}}(\xi)\|^{- \delta|\beta|} \| \hat{c}^{(\beta)}_{X_i} \tilde{\sigma}_{j, \ell} \|_\text{op} \leq 2^{-\frac{\ell}{n}|\beta|} \sup_{(x, \xi) \in G \times \hat{G}} \| \hat{\mathcal{M}}(\xi)\|^{- \delta|\beta|} \| \hat{c}^{(1)}_{X_i} \sigma(z'', \xi) \|_\text{op} < \infty.
\end{align*}
\]

The symbol \( \sigma_{j, \ell} \) and its convolution kernel \( k_{j, \ell, x} \) are supported in \( x \) in \( B(0, \varepsilon_0) \), and dilating the \( x \)-argument to \( x \in B(e_G, 2^\frac{\ell}{n} \varepsilon_0) \), implies the identity

\[
\begin{align*}
(\text{Op}(\sigma_{j, \ell}) f)(2^{-\frac{\ell}{n}} \cdot x) &= f \ast k_{j, \ell, 2^{-\frac{\ell}{n}} \cdot x} = f \ast \tilde{k}_{j, \ell, x}(2^{-\frac{\ell}{n}} \cdot x) = (\text{Op}(\tilde{\sigma}_{j, \ell}) f)(2^{-\frac{\ell}{n}} \cdot x),
\end{align*}
\]

where we have denoted by \( \tilde{k}_{j, \ell, x} = \mathcal{F}^{-1} \tilde{\sigma}_{j, \ell} \) the right-convolution kernel associated with \( \tilde{\sigma}_{j, \ell} \). From the usual Sobolev embedding theorem we have

\[
\| (\text{Op}(\sigma_{j, \ell}) f)(2^{-\frac{\ell}{n}} \cdot x) \|_{L^2(G, dx', dz')} \leq \sum_{|\beta| \leq \lceil \frac{\ell}{2} \rceil + 1} \| \hat{c}^{(\beta)}_{Z'} (f \ast \tilde{k}_{j, \ell, x'})(2^{-\frac{\ell}{n}} \cdot x) \|_{L^2(G, dx' dz')},
\]

and we deduce

\[
\begin{align*}
\| (\text{Op}(\sigma_{j, \ell}) f)(2^{-\frac{\ell}{n}} \cdot x) \|_{L^2(B(e_G, 2^\frac{\ell}{n} \varepsilon_0), dx)} &\leq \sum_{|\beta| \leq \lceil \frac{\ell}{2} \rceil + 1} \| \hat{c}^{(\beta)}_{Z'} (f \ast \tilde{k}_{j, \ell, x'})(2^{-\frac{\ell}{n}} \cdot x) \|_{L^2(B(e_G, 2^\frac{\ell}{n} \varepsilon_0) \times G, dx' dz')}.
\end{align*}
\]
By the change of variables, for the transformation $x' = 2^{-\frac{\mu}{n}} \cdot x$, we have

$$
\| (\text{Op}(\sigma_{j,\ell}) f)(2^{-\frac{\mu}{n}} \cdot x) \|_{L^2(B(e_G, 2^{-\frac{\mu}{n}} \varepsilon_0), dx)} = \left( \int_{B(e_G, 2^{-\frac{\mu}{n}} \varepsilon_0)} |(\text{Op}(\sigma_{j,\ell}) f)(2^{-\frac{\mu}{n}} \cdot x)|^2 dx \right)^{\frac{1}{2}}
$$

$$
= \left( \int_{B(e_G, \varepsilon_0)} |(\text{Op}(\sigma_{j,\ell}) f)(x')|^2 \left| \det \left( \frac{\partial x}{\partial x'} \right) \right| dx' \right)^{\frac{1}{2}}
\approx 2^{\frac{\mu}{2}} \left( \int_{B(e_G, \varepsilon_0)} |(\text{Op}(\sigma_{j,\ell}) f)(x')|^2 dx' \right)^{\frac{1}{2}}.
$$

A similar argument implies that

$$
\sum_{|\beta| \in [\frac{\mu}{2}] + 1} \| \hat{\mathcal{C}}^{(\beta)}_{Z'} (f * \tilde{k}_{j,\ell,z'})(2^{-\frac{\mu}{n}} \cdot x) \|_{L^2(B(e_G, 2^{-\frac{\mu}{n}} \varepsilon_0) \times G, dx dz')} = 2^{\frac{\mu}{2}} \sum_{|\beta| \in [\frac{\mu}{2}] + 1} \| \hat{\mathcal{C}}^{(\beta)}_{Z'} (f * \tilde{k}_{j,\ell,z'})(x') \|_{L^2(B(e_G, \varepsilon_0), dx' dz')}.
$$

and consequently we have proved that,

$$
\| (\text{Op}(\sigma_{j,\ell}) f)(x') \|_{L^2(B(e_G, \varepsilon_0), dx')} \lesssim \sum_{|\beta| \in [\frac{\mu}{2}] + 1} \| \hat{\mathcal{C}}^{(\beta)}_{Z'} (f * \tilde{k}_{j,\ell,z'})(x') \|_{L^2(B(e_G, \varepsilon_0), dx' dz')}.
$$

Now, let us estimate the right-hand side of the previous inequality,

$$
\| \hat{\mathcal{C}}^{(\beta)}_{Z'} (f * \tilde{k}_{j,\ell,z'})(x') \|_{L^2(B(e_G, \varepsilon_0) \times G, dx dz')} = \| f * \hat{\mathcal{C}}^{(\beta)}_{Z'} \tilde{k}_{j,\ell,z'}(x') \|_{L^2(B(e_G, \varepsilon_0) \times G, dx dz')}
\leq \sup_{z' \in G} \| f * \hat{\mathcal{C}}^{(\beta)}_{Z'} \tilde{k}_{j,\ell,z'}(x') \|_{L^2(B(e_G, \varepsilon_0)) dx'}
= \sup_{z' \in G} \| \text{Op}(\hat{\mathcal{C}}^{(\beta)}_{Z'}, \tilde{\sigma}_{j,\ell}) f(x') \|_{L^2(B(e_G, \varepsilon_0)) dx'}
\leq \sup_{z' \in G} \| \text{Op}(\hat{\mathcal{C}}^{(\beta)}_{Z'}, \tilde{\sigma}_{j,\ell}) \|_{\mathcal{B}(L^2(G))} \| f \|_{L^2(G)},
$$

Observe that from (4.34), for $|\beta| \geq 1$, we have

\[
\sup_{z' \in G} \| \text{Op}(\partial_{z'}^{(\beta)} \sigma_j, \ell) \|_{\mathcal{B}(L^2(G))} = \sup_{z' \in G, \langle \xi \rangle \sim 2^M} \| \partial_{z'}^{(\beta)} \sigma_j, \ell (z', \xi) \|_{\text{op}}
\]

\[
= \sup_{z'' \in G, \langle \xi \rangle \sim 2^M} \| \hat{\mathcal{M}}(\xi)^{\delta|\beta|} \hat{\mathcal{M}}(\xi)^{-\delta|\beta|} \partial_{z'}^{(\beta)} \sigma_j, \ell (z', \xi) \|_{\text{op}}
\]

\[
\lesssim \sup_{z'' \in G, \langle \xi \rangle \sim 2^M} \| \hat{\mathcal{M}}(\xi)^{\delta|\beta|} \|_{\text{op}} 2^{-\frac{\rho|\beta|}{n}} | \hat{\mathcal{M}}(\xi)^{-\delta|\beta|} \partial_{X}^{(\beta)} \sigma(z'', \xi) \|_{\text{op}}
\]

\[
= \sup_{z'' \in G} \sup_{1 \leq i \leq d_x} (1 + \nu_i(\xi)^2) 2^{-\frac{\rho|\beta|}{n}} \| \hat{\mathcal{M}}(\xi)^{-\delta|\beta|} \partial_{X}^{(\beta)} \sigma(z'', \xi) \|_{\text{op}}
\]

where we have used that $\delta - \rho \leq 0$. A similar argument applied for $|\beta| = 0$, allows us to deduce that

\[
\| (\text{Op}(\sigma_j, \ell) f)(x') \|_{L^2(B(eG, s_0), dx')} \leq \sum_{|\beta| \leq \frac{d}{2} + 1} \sup_{z'' \in G, \langle \xi \rangle \sim 2^M} \| \hat{\mathcal{M}}(\xi)^{-\delta|\beta|} \partial_{X}^{(\beta)} \sigma(z'', \xi) \|_{\text{op}} < \infty.
\]

So, we have proved that

\[
I := \sup_{\ell \in \mathbb{N}} \| T_\ell \|_{\mathcal{B}(L^2(G))} \leq \sup_{\ell, \ell' \in \mathbb{N}} \sum_{j=1}^{N_0} \| \text{Op}(\sigma_j, \ell) \|_{\mathcal{B}(L^2(G))} \leq \sum_{|\beta| \leq \frac{d}{2} + 1} \sup_{z'' \in G, \langle \xi \rangle \sim 2^M} \| \hat{\mathcal{M}}(\xi)^{-\delta|\beta|} \partial_{X}^{(\beta)} \sigma(z'', \xi) \|_{\text{op}} < \infty.
\]

In order to finish the proof, we need to show that $II$ and $III$ are finite. To do so, we will use the bilinear estimate in Lemma 4.30. Let $k_\ell$ be the convolution kernel of $T_\ell$ and let us denote by $K_{\ell, \ell'}$, the integral kernel of $T_\ell^* T_{\ell'}$. From the Schur Lemma, we can estimate

\[
\| T_\ell^* T_{\ell'} \|_{\mathcal{B}(L^2(G))} \lesssim \sup_{(x, y) \in G \times G} | K_{\ell, \ell'}(x, y) |.
\]

Let us fix $N \in \mathbb{N}$, and let $s_0 > \frac{Q}{4}$. Note that

\[
K_{\ell, \ell'}(x, y) = \int_G k_{\ell, z}(x^{-1} z) k_{\ell', z_1}(y^{-1} z_2) dz.
\]
Taking into account the following equivalence of norms

So, from the subelliptic Sobolev embedding theorem (see Remark 4.25), and the continuous inclusion $H^s(G) \subset H^{s,L}(G)$ for $s \geq 0$, we have,

$$ |K_{\ell',\ell}(x, y)| $$

$$ = \left| \int_G (1 + \mathcal{L})^{-\frac{N}{2}}(1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)]dz \right| $$

$$ \leq \int_G \sup_{z_2 \in G} \left| (1 + \mathcal{L})^{-\frac{N}{2}}(1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right| dz_2 $$

$$ \leq \int_G \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}(1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)} dz_2 $$

$$ \leq \left( \int_G \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}(1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{H^{2N+2q}(G, dz_2)} dz_2 \right)^\frac{1}{2} $$

$$ = \left( \int_G \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{H^{2N+2q}(G, dz_2)} dz_2 \right)^\frac{1}{2} $$

$$ \leq \left( \int_G \left\| (1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{H^{2N+2q}(G, dz_2)} dz_2 \right)^\frac{1}{2}. $$

Taking into account the following equivalence of norms

$$ \left\| (1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{H^{2N+2q}(G, dz_2)} $$

$$ = \sum_{|\alpha_1| + |\alpha_2| \leq 2Q(N+6)} \left\| (1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)}, $$

from Lemma 4.30, and by using that $k_{\ell',\ell} \in H_{|\ell'|}(G)$, $k_{\ell,\ell} \in H_{|\ell|}(G)$, (see Lemma 4.30 for this notation) with $\lambda_{|\ell'|} \sim 2^{\ell'}$ and $\lambda_{|\ell|} \sim 2^{\ell}$, we estimate for $s > Q/2$,

$$ \left\| (1 + \mathcal{L})^{-\frac{N}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)} $$

$$ \leq (1 + |2^{\ell'} - 2^{\ell}|) \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)} $$

$$ \leq 2^{-\frac{Q}{4}(\max(|\ell|, |\ell'|) + \frac{1}{2}N + 2q)} \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)} $$

$$ \leq 2^{-\frac{Q}{4}(\max(|\ell|, |\ell'|) + \frac{1}{2}N + 2q)} \left\| (1 + \mathcal{L})^{-\frac{N+4q}{2}}[\mathcal{K}_{\ell',\ell}(x^{-1}z_2)k_{\ell',\ell}(y^{-1}z_2)] \right\|^2_{L^2(G, dz_2)}. $$

From (4.28), for $s > Q/2$ we have

$$ \left\| \mathcal{K}_{\ell',\ell}^\alpha(x^{-1}z_2) \right\|_{L^2(G)} \lesssim_{s} \sup_{|\xi| \in G} \left\| \mathcal{M}(\xi)^s \mathcal{K}_{\ell',\ell}^\alpha(\sigma_{\ell}(z_1, \xi)) \right\|_{op} $$

$$ \leq \sup_{|\xi| \sim 2^n} \left\| \mathcal{M}(\xi)^s \mathcal{K}_{\ell',\ell}^\alpha(\sigma_{\ell}(z_1, \xi)) \right\|_{op}. $$
Thus, we have obtained that
\[ \mathcal{M}(\xi)^{s+\delta|\alpha_1|}\sigma_\ell(z_1,\xi) \leq \mathcal{M}(\xi)^{s+\delta|\alpha_1|}\sigma_\ell(z_1,\xi) \leq \sup_{\langle \xi \rangle \sim 2^j} \| \mathcal{M}(\xi)^{-\delta|\alpha_1|}\sigma_\ell(z_1,\xi) \|_{op} \]
\[ \leq \sup_{\langle \xi \rangle \sim 2^j} \sup_{1 \leq j \leq d_\ell} (1 + \nu_j(\xi)^{s+\delta|\alpha_1|})^{1/2} \| \mathcal{M}(\xi)^{-\delta|\alpha_1|}\sigma_\ell(z_1,\xi) \|_{op} \]
\[ \leq \sup_{\langle \xi \rangle \sim 2^j} \langle \xi \rangle^{s+\delta|\alpha_1|} \| \mathcal{M}(\xi)^{-\delta|\alpha_1|}\sigma_\ell(z_1,\xi) \|_{op}. \]

we have
\[ \| \partial_{z_1}^{(\alpha_1)} k_{\ell,z_1}(x^{-1}z_2) \|_{L^2(G)} \leq 2^{\ell(s+\delta|\alpha_1|)/n} \]
\[ \| \partial_{z_1}^{(\alpha_2)} k_{\ell',z_1}(y^{-1}z_2) \|_{L^2(G)} \leq 2^{\ell(s+\delta|\alpha_2|)/n} \]

Thus, the condition \(|\alpha_1| + |\alpha_2| \leq 2(N + s_0)\) implies
\[ \| \partial_{z_1}^{(\alpha_1)} k_{\ell,z_1}(x^{-1}z_2) \|_{L^2(G)} \| \partial_{z_1}^{(\alpha_2)} k_{\ell',z_1}(y^{-1}z_2) \|_{L^2(G)} \leq 2^{\max(\ell,\ell') (s+\delta(2N+2s_0))/n}. \]

Now, we have the estimate
\[ \| (1 + \mathcal{L})^{-N/2} \partial_{z_1}^{(\alpha_1)} k_{\ell,z_1}(x^{-1}z_2) \partial_{z_1}^{(\alpha_2)} k_{\ell',z_1}(y^{-1}z_2) \|_{L^2(G,dz_1)} \]
\[ \leq 2^{1-n/2} \max(\ell,\ell')(N - \frac{s}{2} - \frac{\alpha}{2}) \times 2^{\max(\ell,\ell') (s+\delta(2N+2s_0))/n} \]
\[ = 2^{\frac{n}{2} \max(\ell,\ell')(N - \frac{s}{2} - \frac{\alpha}{2}) - s - 2N\delta\kappa - 2s_0\delta\kappa} \]
\[ = 2^{\frac{n}{2} \max(\ell,\ell')(N - 2\delta\kappa) - s(k + \frac{1}{2}) - \frac{Q}{8} - 2\delta\kappa s_0}. \]

Thus, we have proved
\[ |K_{\ell,\ell'}(x,y)| \leq 2^{\frac{n}{2} \max(\ell,\ell')(N - 2\delta\kappa) - s(k + \frac{1}{2}) - \frac{Q}{8} - 2\delta\kappa s_0}. \]

This shows that \(II, III < \infty\), for \(N(1 - 2\delta\kappa) - s(k + \frac{1}{2}) - \frac{Q}{8} - 2\delta\kappa s_0 > 0\). So, we only need to choose
\[ N > \frac{s(k + \frac{1}{2}) + \frac{Q}{8} + 2\delta\kappa s_0}{1 - 2\delta\kappa}, \]
where we have used that \(\delta < 1/2\kappa\). Thus, the proof is complete, with the remaining case of \(0 \leq \delta < \rho \leq 1\) proved in Corollary 6.16. \(\square\)
4.4. The formal adjoint of subelliptic operators. In Theorems 4.31 and 4.33 we will show that the subelliptic classes introduced before are closed under compositions and adjoints. If $A : C^\infty(G) \to C^\infty(G)$ is a continuous operator, its formal adjoint is the operator $A^*$, defined by

$$(Af, g)_{L^2(G)} = (f, A^*g)_{L^2(G)}, \ f, g \in C^\infty(G).$$

Next, we study the formal adjoint of subelliptic pseudo-differential operators.

**Theorem 4.31.** Let $0 \leq \delta < \rho \leq 1$. If $A : C^\infty(G) \to C^\infty(G)$ is a continuous operator, $A \in \operatorname{Op}(S^{m,\mathcal{E}}_{p,\delta}(G \times \hat{G}))$, then $A^* \in \operatorname{Op}(S^{m,\mathcal{E}}_{p,\delta}(G \times \hat{G}))$. The symbol of $A^*$, $\widehat{A^*}(x, \xi)$ satisfies the asymptotic expansion,

$$\widehat{A^*}(x, \xi) \sim \sum_{|\alpha| = 0}^{\infty} \Delta_{\xi_0} \partial_X^{(\alpha)}(\hat{A}(x, \xi)^*). \quad (4.35)$$

This means that, for every $N \in \mathbb{N}$, and all $\ell \in \mathbb{N},$

$$\Delta_\xi^{\alpha_\ell} \partial_X^{(\beta)} \left( \widehat{A^*}(x, \xi) - \sum_{|\alpha| \leq N} \Delta_\xi^{\alpha_0} \partial_X^{(\alpha)}(\hat{A}(x, \xi)^*) \right) \in S^{m-(\rho-\delta)(N+1)-\rho \ell + \delta \ell, \mathcal{E}}_{p,\delta}(G \times \hat{G}),$$

where $|\alpha_\ell| = \ell$.

**Proof.** We note that the formula (4.35) was established in [107, Theorem 10.7.8] so that we only need to prove the remainder estimate. By following [107, p. 569], the right kernel of $A^*$, $k^*_x(\cdot)$, satisfies the identity,

$$k^*_x(v) = k_{xv^{-1}}(v^{-1}), \ x, v \in G,$$

where $k(\cdot)$ is the right-kernel associated to $A$. Let us prove that for any multi-index $\beta, \beta_0, \alpha_0 \in \mathbb{N}_0$, there exists $N_0 \in \mathbb{N}_0$ such that for any integer $N > N_0$, we have the estimate

$$\sup_{x \in G} \| \partial_Y^{(\beta)} \partial_X^{(\beta_0)} (q^{\alpha_0}(y))(k^*_x(y)) - \sum_{|\alpha| < N} q^{\alpha}(y) \partial_X^{(\alpha)} k^*_x(y) \|_{L^1(G)} \leq C \| \sigma \|_{\ell', S_{p,\delta}^{m,\mathcal{E}}}, \quad (4.36)$$

for $\ell'$ large enough. Later, we will conclude the proof using this estimate. In the notation of Lemma 4.2, we have

$$R^{k^*_x}(y^{-1}) = k^*_x(y) - \sum_{|\alpha| < N} q^{\alpha}(y) \partial_X^{(\alpha)} k^*_x(y),$$
where $q^\alpha := q^{\alpha_1}_1 \cdots q^{\alpha_n}_n$. By using the remainder estimates in Lemma 4.2 and the Leibniz rule, we have

$$|\partial^{(\beta)}_X (\partial^{(\beta_0)}_Y (q^{\alpha_0}(y))(k_\ast^{(\beta_0)}(y)) - \sum_{|\alpha| < N} q^\alpha(y)\partial^{(\alpha)}_X k_\ast^{(\beta_0)}(y))|$$

$$= |\partial^{(\beta)}_Y (R^{(\alpha_0)}_{X,N}(q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))(y^{-1}))|$$

$$\lesssim \sum_{|\beta_1| + |\beta_2| = \beta} |\partial^{(\beta_2)}_Y (R^{(\alpha_0)}_{X,N}(q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))(y^{-1}))|$$

$$\lesssim \sum_{|\beta_1| + |\beta_2| = \beta} |y|^{(N - |\beta_2|)} \max_{|\alpha| \leq N} \|\partial^{(\alpha)}_X \partial^{(\beta_1)}_Y (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))\|_{L^\infty(G)},$$

where we have used the notation $(r)_+ := \max\{r, 0\}$. Now, by using the kernel estimates in Proposition 4.23, and the fact that $\partial^{(\alpha + \beta_0)}_X (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))$ is the right-convolution kernel of some symbol $\sigma$ with subellitic order equal to $m + \delta(|\beta_0| + |\alpha|) + |\beta_1| - \rho|\alpha_0|$, we have that there exists $\ell \in \mathbb{N}$, such that

(i) If $s > 0$,

$$\|\partial^{(\alpha + \beta_0)}_X (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))\|_{L^\infty(G)} \lesssim_m \|\sigma\|_{L^s_{\rho,\delta}} |y|^{-\frac{s}{2}}.$$

(ii) If $s = 0$,

$$\|\partial^{(\alpha + \beta_0)}_X (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))\|_{L^\infty(G)} \lesssim_m \|\sigma\|_{L^s_{\rho,\delta}} \log |y|.$$

(iii) If $s < 0$,

$$\|\partial^{(\alpha + \beta_0)}_X (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))\|_{L^\infty(G)} \lesssim_m \|\sigma\|_{L^s_{\rho,\delta}}.$$

with $s = Q + m + \delta(|\beta_0| + N) + |\beta_1| - \rho|\alpha_0|$. Observe that

$$s = Q + m + \delta(|\beta_0| + |\alpha|) + |\beta_1| - \rho|\alpha_0|$$

$$= Q + m + \delta(|\beta_0| + |\alpha|) + |\beta_1| - \rho|\alpha_0| - \rho|\alpha| + \rho|\alpha|$$

$$= Q + m + \delta|\beta_0| + (\delta - \rho)|\alpha| + |\beta_1| - \rho|\alpha_0| + \rho|\alpha| =: s' + (\delta - \rho)|\alpha|.$$

Consequently,

$$\int_G \|\partial^{(\alpha + \beta_0)}_X (q^{\alpha_0}(y)k_\ast^{(\beta_0)}(y))\|_{L^\infty(G)} dy \lesssim_m \|\sigma\|_{L^s_{\rho,\delta}} \int_G |y|^{-\frac{s}{2}} |y|^{(N - |\beta_2|)} dy$$

$$\lesssim \|\sigma\|_{L^s_{\rho,\delta}} \int_G |y|^{\frac{s' + (\delta - \rho)|\alpha|}{\rho}} dy < \infty,$$

provided that $s' < n\rho + (\rho - \delta)|\alpha|$. To assure this condition, observe that $n\rho + (\rho - \delta)|\alpha| \leq n\rho + (\rho - \delta)N$, so we only need to choose $N_0 \in \mathbb{N}$ such that

$$N \geq N_0 > \frac{s' - n\rho}{\rho - \delta},$$
where we have used that \( \delta < \rho \). So, we conclude the proof of the estimate (4.36). Now, in order to conclude the proof of Theorem 4.31, let us define

\[
\hat{A}_N(x, \xi) := \hat{A}^*(x, \xi) - \sum_{|\alpha| \leq N} \Delta_\xi^{\alpha} \hat{c}_X^{(\alpha)} (\hat{A}(x, \xi)^*). \tag{4.37}
\]

We need to prove that \( \Delta_\xi^{\alpha_0} c_X^{(\beta)} \hat{A}_N(x, \xi) \in S_{\rho, \delta}^{m-(\rho-\delta)(N+1)-\rho|\alpha_0|+\delta|\beta|, \ell, L}(G \times \hat{G}) \). Set \( m' := m - (\rho - \delta)(N + 1) - \rho|\alpha_0| + \delta|\beta| \). Observe that for any \( M' > m' \), with \( M' = 0 \text{ mod } 2 \), we have

\[
\| \Delta_\xi^{\alpha_0} c_X^{(\beta)} \hat{A}_N(x, \xi) \|_{op} \leq \| \Delta_\xi^{\alpha_0} c_X^{(\beta)} \hat{A}_N(x, \xi) \|_{op} \\
\lessapprox \| (1 + L_\gamma)^{M'} \hat{R}_{x,N+1}^{c_X^{(\beta)}(q^{\alpha_0}(y)k_{i_j}(y))(y^{-1})} \|_{L^1(G_\gamma)} \\
\leq \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k, |\alpha| \leq M'} \| X_{i_1}^{\alpha_{i_1}} \cdots X_{i_k}^{\alpha_{i_k}} \hat{R}_{x,N+1}^{c_X^{(\beta)}(q^{\alpha_0}(y)k_{i_j}(y))(y^{-1})} \|_{L^1(G_\gamma)}.
\]

By Lemma 4.2, and using that \( X_\Delta = \{ X_{1,\Delta}, \ldots, X_{n,\Delta} \} \) is a basis of the Lie algebra \( \mathfrak{g} \), we can express every vector field \( X_{i_j, y} \) as a linear combination of elements in \( X_\Delta \), so that we can estimate

\[
\| \Delta_\xi^{\alpha_0} c_X^{(\beta)} \hat{A}_N(x, \xi) \|_{op} \lessapprox \sum_{|\alpha| \leq M'} \| c_X^{(\alpha)} \hat{R}_{x,N+1}^{c_X^{(\beta)}(q^{\alpha_0}(y)k_{i_j}(y))(y^{-1})} \|_{L^1(G_\gamma)} \\
\leq C\| \sigma \|_{\ell' S_{\rho, \delta}^{m, L}},
\]

for \( \ell' \) large enough, where in the last line we have used (4.36). The proof is complete. \( \square \)

**Corollary 4.32.** Let \( 0 \leq \delta < \rho \leq \kappa \). If \( A : C^\infty(G) \to C^\infty(G) \) is a continuous operator, \( A \in \text{Op}(\mathcal{S}_{\rho, \delta}^{m, L}(G)) \), then \( A^* \in \text{Op}(\mathcal{S}_{\rho, \delta}^{m, L}(G)) \). The symbol of \( A^* \), \( \hat{A}^*(x, \xi) \) satisfies the asymptotic expansion,

\[
\hat{A}^*(x, \xi) \sim \sum_{|\alpha| = 0}^{\infty} \Delta_{\hat{\xi}}^{\alpha} c_X^{(\alpha)} (\hat{A}(x, \xi)^*).
\]

This means that, for every \( N \in \mathbb{N} \), and all \( \ell \in \mathbb{N} \),

\[
\Delta_\xi^{\alpha} c_X^{(\beta)} \left( \hat{A}^*(x, \xi) - \sum_{|\alpha| \leq N} \Delta_\xi^{\alpha} c_X^{(\alpha)} (\hat{A}(x, \xi)^*) \right) \in \mathcal{S}_{\rho, \delta}^{m-(\rho-\delta)(N+1)-\rho\ell+\delta|\beta|, \ell, L}(G \times \hat{G}),
\]

where \( |\alpha_\ell| = \ell \).

**4.5. Composition of subelliptic pseudo-differential operators.** Next we prove the stability of the \((\rho, \delta)\)-classes subelliptic classes by taking compositions.

**Theorem 4.33.** Let \( 0 \leq \delta < \rho \leq 1 \). If \( A_i \in \text{Op}(\mathcal{S}_{\rho, \delta}^{m, L}(G \times \hat{G})) \), \( A_i : C^\infty(G) \to C^\infty(G) \), \( i = 1, 2 \), then the composition operator \( A := A_1 \circ A_2 : C^\infty(G) \to C^\infty(G) \)
belongs to the subelliptic class $\text{Op}(S^m_{\rho,\delta} + m^2, \mathcal{L})(G \times \hat{G})$. The symbol of $A$, $\hat{A}(x, \xi)$, satisfies the asymptotic expansion,

$$\hat{A}(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} (\Delta_{\xi}^\alpha \hat{A}_1(x, \xi)) (\partial_{x}^{(\alpha)} \hat{A}_2(x, \xi)),$$

(4.38)

this means that, for every $N \in \mathbb{N}$, and all $\ell \in \mathbb{N}$,

$$\Delta_{\xi}^{\alpha_{\ell}} \partial_{x}^{(\beta)} \left( \hat{A}(x, \xi) - \sum_{|\alpha| \leq N} (\Delta_{\xi}^\alpha \hat{A}_1(x, \xi)) (\partial_{x}^{(\alpha)} \hat{A}_2(x, \xi)) \right) \in S^m_{\rho,\delta} - (\rho - \delta)(N + 1) - \rho \ell + \delta|\beta|, \mathcal{L}(G \times \hat{G}),$$

for every $\alpha_{\ell} \in \mathbb{N}_0^n$ with $|\alpha_{\ell}| = \ell$.

Proof. As in Theorem 10.7.8 in [107, p. 569] we have the formula 4.38, so we only need to prove the remainder estimate. If $K_A \in C^\infty(G) \mathcal{S}'(G)$ is the Schwartz kernel of $A$, we denote by $k_x(y) := K_A(x, xy^{-1})$ and $k_{x,i}$ the right-convolution kernels of $A$ and $A_i$. Therefore,

$$\hat{A}(x, \xi) = \int_G k_x(y) \xi(y)^* dy$$

$$= \int_G \int_G k_{x,1}(x, z^{-1}) \xi(z^{-1})^* k_{x,2}(xz, yz) \xi(yz)^* dz dy,$$

where we have used the identity

$$Af(x) = \int_G A_2 f(xz) k_x(z^{-1}) dz = \int_G f(xy^{-1}) \int_G k_{x,2}(yz) k_{x,1}(z^{-1}) dz dy.$$

So, we have

$$k_x(y) := \int_G k_{x,1,2}(yz^{-1}) k_{x,1}(z) dz.$$

Observe that,

$$k_x(y) - \sum_{|\alpha| \leq N} \left( \partial_{x}^{(\alpha)} k_{x,2} \right) * (q^\alpha k_{x,1})(y)$$

$$= \int_G \left( k_{2,xz^{-1}}(yz^{-1}) - \sum_{|\alpha| \leq N} q^\alpha(z) \partial_{x}^{(\alpha)} k_{2,x}(z^{-1}) \right) k_{1,x}(z) dz$$

$$= \int_G R_{x,N} k_{2}(yz^{-1}) (z^{-1}) k_{1,x}(z) dz.$$

So, by applying the group Fourier transform we obtain

$$\hat{A}(x, \xi) - \sum_{|\alpha| \leq N} \Delta_{\xi}^\alpha \hat{A}_1(x, \xi) (\partial_{x}^{(\alpha)} \hat{A}_2(x, \xi)) = \int_G k_{x,1}(z) \xi(z)^* R_{x,N}^{\alpha}(z^{-1}) dz,$$
where we have denoted $R_{x,N} \hat{A}_2(\cdot, \xi) = [R_{x,N} \hat{A}_2(\cdot, \xi)]_{i,j=1}$. Now, the central part in the proof is to show that there exists $N_0 \in \mathbb{N}$ such that the following estimate holds true for all $N \geq N_0$ and all $b_1 > 0$. Indeed, if we assume (4.39), by defining
\begin{equation}
\hat{A}_N(x, \xi) := \hat{A}(x, \xi) - \sum_{|\alpha| < N} \Delta_\alpha \hat{A}_1(x, \xi)(\partial_X^{(\alpha)} \hat{A}_2(x, \xi)),
\end{equation}
and taking $b_1 = -m_1 - m_2 + (\rho - \delta)N + \rho|\alpha_0| - \delta|\beta_0| \geq -m_1 - m_2 + (\rho - \delta)N_0 + \rho|\alpha_0| - \delta|\beta_0|$, the condition $\rho > \delta$ and the choice of $N_0$ large enough implies that $b_1 > 0$, and from (4.39) the statement of Theorem 4.33 follows. So, let us fix $b \in \mathbb{N}$, with $b \equiv 0 \mod (2)$. By using the Leibniz rule we can write
\begin{align*}
\partial_X^{(\beta_0)} \Delta_\alpha & [\int_G k_{x,1}(z) \xi(z) * R_{x,N} \hat{A}_2(\cdot, \xi)(z^{-1})dz] \\
& = \sum_{|\alpha| \leq |\alpha_1| + |\beta_0|} \int_G \partial_X^{(\beta_0,1)} q^{\alpha_1}(z) k_{x,1}(z) \xi(z) * R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz,
\end{align*}
and from the identity $\xi(z) = \hat{\mathcal{M}}^{-b}(\xi)(1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^*$, by writing $I = \{\alpha_1, \beta_{0,i} : \alpha_1 + |\alpha_2| \leq 2|\alpha_0|, |\beta_{0,1}| + |\beta_{0,2}|, i = 1, 2\}$, we can estimate
\begin{align*}
\|\partial_X^{(\beta_0)} \Delta_\alpha & [\int_G k_{x,1}(z) \xi(z) * R_{x,N} \hat{A}_2(\cdot, \xi)(z^{-1})dz] \hat{\mathcal{M}}(\xi)^{b_1} \|_{op} \\
\leq & \sum_I \int_G \partial_X^{(\beta_0,1)} q^{\alpha_1}(z) k_{x,1}(z) \hat{\mathcal{M}}(\xi)^{-b}(1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^* R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz \hat{\mathcal{M}}(\xi)^{b_1} \|_{op} \\
= & \sum_I \| \hat{\mathcal{M}}(\xi)^{-b_1} \| G \int_G \partial_X^{(\beta_0,1)} q^{\alpha_1}(z) k_{x,1}(z) (1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^* R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz \|_{op} \\
\approx & \sum_I \| \hat{\mathcal{M}}(\xi)^{b_1-b} \int_G \partial_X^{(\beta_0,1)} q^{\alpha_1}(z) k_{x,1}(z) (1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^* R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz \|_{op}.
\end{align*}

From the equality
\begin{align*}
\int_G \partial_X^{(\beta_0,1)} q^{\alpha_1}(z) k_{x,1}(z) (1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^* R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz \\
= \int_G [(1 + \mathcal{L}_z) \hat{\mathcal{L}} \xi(z)^*]^{\partial_X^{(\beta_0,1)}} q^{\alpha_1}(z) k_{x,1}(z) R_{x,N} \Delta_\alpha \hat{A}_2(\cdot, \xi)(z^{-1})dz,
\end{align*}
we have
\[
\|\hat{M}(\xi)^{b-b} \int_G \frac{\partial}{\partial x} q^{a_1}(z)k_{x,1}(z) (1 + L_z) \frac{\partial}{\partial z} [\mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} \\
= \|\hat{M}(\xi)^{b-b} \int_G \xi(z)^*(1 + L_z) \frac{\partial}{\partial z} [\mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} \\
\leq \int_G \|\hat{M}(\xi)^{b-b} (1 + L_z) \frac{\partial}{\partial z} [\mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} dz \\
\leq \sum_{|x|+|y| \leq b} \int_G \|\hat{M}(\xi)^{b-b} X^{(\beta_1)} [\mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] X^{(\beta_2)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} dz,
\]
where $J = \{(i_1, i_2, \cdots, i_k) : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k\}$. By Lemma 4.2, and using that $X_D = \{X_{1,D}, \cdots, X_{n,D}\} \text{ is a basis of the Lie algebra g}$, as in the proof of the asymptotic expansion for the adjoint, we can express every vector field $X_{ij}$ as a linear combination of elements in $X_D$. So, we have
\[
\sum_{|x|+|y| \leq b} \int_G \|\hat{M}(\xi)^{b-b} X^{(\beta_1)} [\mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] X^{(\beta_2)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} dz \\
\leq \sum_{|x|+|y| \leq b} \int_G \|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] \mathcal{X}^{(\beta_2)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} dz \|_{op} dz.
\]
Observe that from Lemma 4.2 we have
\[
\|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} \|_{op} \\
\leq |z|^{-|N|-|\beta_2|} \sup_{x \in G, |x| \leq N} \|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} \|_{op} \\
\leq |z|^{-|N|-|\beta_2|} \|\hat{M}(\xi)^{b-b+m_2+|\beta_1|+\rho|\alpha_2|} \mathcal{X}^{(\beta_1)} R^q_{x,N}[\mathcal{X}^{(\beta_1)}] z^{-1} \|_{op}.
\]
By Proposition 4.23 applied to $\sigma = \mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)$, there exists $\ell' \in \mathbb{N}$, such that
\begin{enumerate}
  \item if $s > 0$, 
  \[
  \|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] \| \lesssim m_1 \|\hat{A}_2\|_{\ell', S^m_{\rho, \delta}} |z|^{-s}. 
  \]
  \item if $s = 0$, 
  \[
  \|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] \| \lesssim m_1 \|\hat{A}_2\|_{\ell', S^m_{\rho, \delta}} \log |z|. 
  \]
  \item if $s < 0$, 
  \[
  \|\hat{M}(\xi)^{b-b} \mathcal{X}^{(\beta_1)} q^{a_1}(z)k_{x,1}(z)] \| \lesssim m_1 \|\hat{A}_2\|_{\ell', S^m_{\rho, \delta}}.
  \]
\end{enumerate}
where $s = Q + m_1 + \delta|\beta_0| + |\beta_1| - \rho|\alpha_1|$. First, let us assume that
\[
b > \max\{b_1 + m_2 + \delta(N + |\beta_0| - \rho|\alpha_2|)\}.
\]
This choice of \( b \) implies that \( \| \mathcal{M} (\xi)^{b_1 + m_2 + \delta(N + |\beta_0|) - \rho} \|_{op} \leq 1 \). In particular,

\[
b > b_1 + m_2 + \delta(N + |\beta_0|) - \rho|\alpha_0|.
\]

Finally, a similar analysis as in the final part of the proof of Theorem 4.31, and the hypothesis \( \delta < \rho \), can be used to guarantee the existence of \( N_0 \) such that for any \( N \geq N_0 \) the integral \( \int_{\Omega} I_s(z) \, dz \) converges, where \( I_s(z) := |z|^{N - |\beta_2|} \) if \( s > 0 \), and \( I_s(z) := |z|^{N - |\beta_2|} \) if \( s < 0 \). Thus, we end the proof.

\[ \square \]

**Corollary 4.34.** Let \( 0 \leq \delta < \rho \leq \kappa \). If \( A_i \in \text{Op}(\mathcal{S}_{\rho,\delta}^{m_1,1}(G)), A_i : C^\infty(G) \to C^\infty(G), i = 1, 2 \), then the composition operator \( A := A_1 \circ A_2 : C^\infty(G) \to C^\infty(G) \) belongs to the subelliptic class \( \text{Op}(\mathcal{S}_{\rho,\delta}^{m_1,m_2,1}(G)) \). The symbol of \( A, \hat{A}(x, \xi) \), satisfies the asymptotic expansion,

\[
\hat{A}(x, \xi) \sim \sum_{|\alpha| = 0}^\infty (\Delta_\xi^\alpha \hat{A}_1(x, \xi))(\hat{\partial}_X^\alpha \hat{A}_2(x, \xi)),
\]

this means that, for every \( N \in \mathbb{N} \), and all \( \ell \in \mathbb{N} \),

\[
\Delta_\xi^\alpha \hat{\partial}_X^\beta \left( \hat{A}(x, \xi) - \sum_{|\alpha| \leq N} (\Delta_\xi^\alpha \hat{A}_1(x, \xi))(\hat{\partial}_X^\alpha \hat{A}_2(x, \xi)) \right)
\]

\[
\in \mathcal{S}_{\rho,\delta}^{m_1,m_2,0}(G) (G),
\]

for every \( \alpha \in \mathbb{N} \) with \( |\alpha| = \ell \).

**Remark 4.35.** Now, we return to the last part of Example 4.6 where we claimed that \( \mathcal{M}_s \in \text{Op}(\mathcal{S}_{k,0}^{s,k,1}(G)) \), for all \( s > 0 \). For \( 1 \leq s < \infty \), if \( N_0 \in \mathbb{N} \) satisfies \( 1 \leq s < N_0 \), then, from Example 4.6, \( (1 + L)^{s/k_0} \in \text{Op}(\mathcal{S}_{k_0}^{s,k,0}(G)) \). From Theorem 4.33, we deduce that

\[
\mathcal{M}_s = \left( 1 + L \right)^{s/k_0} \cdots \left( 1 + L \right)^{s/k_0} \in \text{Op}(\mathcal{S}_{k_0}^{s,k,0}(G \times \hat{G})).
\]

Thus, \( \mathcal{M}_s \in \text{Op}(\mathcal{S}_{k,0}^{s,k,1}(G)) = \text{Op}(\mathcal{S}_{k,0}^{s,k}(G \times \hat{G})) \).

5. **Weak \((1,1)\) Type and \(L^p\)-boundedness of subelliptic operators with non-smooth symbols**

In this section we extend the results in [112] for the Laplacian, to the subelliptic setting. For our further analysis we will apply the following lemma (see Persson [99]):

**Lemma 5.1.** Let us assume that \( \Omega \) is a locally compact topological space and \( \mu \) a positive measure on \( \Omega \). If \( A \) extends to a bounded operator on \( L^r(\Omega, \mu) \) and to a compact operator on \( L^q(\Omega, \mu) \), then \( A \) extends to a compact operator on \( L^p(\Omega, \mu) \) for all \( p \) between \( q \) and \( r \), (i.e such that \( 1/p = \theta/q + (1 - \theta)/r \) for some \( \theta \in (0,1) \)).
For $0 < p < \infty$, the subelliptic $L^p$-Sobolev space of order $s \in \mathbb{R}$, is defined by the family of distribution $f \in \mathcal{D}'(\mathbb{G})$ such that

$$\|f\|_{L^p_s(G)} := \|\mathcal{M}_s f\|_{L^p(G)} < \infty.$$  

We will develop an $L^p$-multiplier theorem for invariant operators in the dilated subelliptic classes with a limited number of regularity conditions. In particular, we will impose conditions of the type

$$\sup_{(x, \xi) \in \mathbb{G} \times \hat{\mathbb{G}}} \|\hat{c}^{(\beta)}(\Delta_\xi^\alpha a(x, \xi)) \hat{\mathcal{M}}(\xi) \frac{1}{\kappa} (\rho |\alpha| - \delta |\beta| + \kappa (\kappa - \rho))\|_{\text{op}} < \infty,$$

which from Theorem 4.13, is equivalent to the following ones

- For all $\epsilon \in \mathbb{R}$,
  $$\sup_{(x, \xi) \in \mathbb{G} \times \hat{\mathbb{G}}} \|\hat{\mathcal{M}}(\xi) \frac{1}{\kappa} (\rho |\alpha| - \delta |\beta| + \kappa (\kappa - \rho) - \epsilon) \hat{c}^{(\beta)}(\Delta_\xi^\alpha a(x, \xi) \hat{\mathcal{M}}(\xi) \frac{\kappa}{\epsilon})\|_{\text{op}} < \infty.$$  

- There exists $r_0 \in \mathbb{R}$, such that
  $$\sup_{(x, \xi) \in \mathbb{G} \times \hat{\mathbb{G}}} \|\hat{\mathcal{M}}(\xi) \frac{1}{\kappa} (\rho |\alpha| - \delta |\beta| + \kappa (\kappa - \rho) - r_0) \hat{c}^{(\beta)}(\Delta_\xi^\alpha a(x, \xi) \hat{\mathcal{M}}(\xi) \frac{\kappa}{r_0})\|_{\text{op}} < \infty.$$  

The next theorem will be proved using the difference operators $\mathbb{D}_\xi^\alpha$ in Remark 3.2.

**Theorem 5.2.** Let us assume that $G$ is a compact Lie group of dimension $n = 2d$ or $n = 2d + 1$, and that $d$ is odd. Let $a \in \Sigma(\hat{\mathbb{G}})$ be a symbol satisfying

$$\|\mathbb{D}_\xi^\alpha a(\xi) \hat{\mathcal{M}}(\xi) \frac{\rho |\alpha| + \kappa (\kappa - \rho)}{\kappa}\|_{\text{op}} \leq C_\alpha, \quad |\alpha| \leq \kappa := d + 1.$$  

Then $A = \text{Op}(a)$ extends to an operator of weak type $(1, 1)$. Moreover, if the dimension of the group is $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, and $d$ is even, the conclusion on $A$ is the same provided that

$$\|\mathbb{D}_\xi^\alpha a(\xi) \hat{\mathcal{M}}(\xi) \frac{\rho |\alpha| + \kappa (\kappa - \rho)}{\kappa}\|_{\text{op}} \leq C_\alpha, \quad |\alpha| \leq \kappa := d + 2.$$  

In both cases, if $\rho = \kappa$, $A$ extends to a bounded operator on $L^p(G)$, while for $0 \leq \rho < \kappa$, $A$ extends to a compact bounded operator on $L^p(G)$ for all $1 < p < \infty$.

**Proof.** Let us consider the difference operator associated to $q_m(x) := \rho(x)^{2m}$, $\Delta_{q_m} \in \text{diff}^{2m}(\hat{\mathbb{G}})$, where $\kappa := 2m \in \mathbb{N}$, and $\rho(x)$ is as in (3.7). Let us define $\varepsilon > 0$ by the equality

$$n(1 + \varepsilon) = 4m = 2\kappa.$$  

The main step of the proof is to observe that the condition (3.6), is equivalent to showing that

$$\|\Delta_{q_m}(a(\xi) \hat{\psi}_r)\|_{L^2(\hat{\mathbb{G}})} \leq r^{\frac{\varepsilon}{2}} = r^{\frac{2m}{n} - \frac{1}{2}},$$

(5.1)
where we have used that \( \varepsilon = \frac{4m}{n} - 1 \). The Leibniz rule applied for every \([\xi] \in \hat{G}\), gives

\[
\Delta_{q_m}(a(\xi)\psi_r(\xi)) = \Delta_{q_m}(a(\xi)\psi_r(\xi)) + a(\xi)\Delta_{q_m}(\psi_r(\xi)) + \sum_{\ell=1}^{m-1} \sum_j (Q_{\ell,j} a(\xi))(\hat{Q}_{\ell,j}\psi_r(\xi)),
\]

for some differences operators \( Q_{\ell,j} \in \text{diff}(\hat{G}) \) and \( \hat{Q}_{\ell,j} \in \text{diff}^{2m-\ell}(\hat{G}) \). Because for every \( \nu \geq 0 \), we have

\[
\Delta_{q_m}(a(\xi)\psi_r(\xi)) = \Delta_{q_m}(a(\xi))\mathcal{M}(\xi)^{2m+\nu}\mathcal{M}(\xi)^{-2m-\nu}\psi_r(\xi) + a(\xi)\mathcal{M}(\xi)^{\nu}\mathcal{M}(\xi)^{-\nu}\Delta_{q_m}(\psi_r(\xi)) + \sum_{\ell=1}^{2m-1} \sum_j (Q_{\ell,j} a(\xi))\mathcal{M}(\xi)^{\rho\ell+\nu}\mathcal{M}(\xi)^{-\rho\ell-\nu}(\hat{Q}_{\ell,j}\psi_r(\xi)),
\]

taking the norm inequalities with \( \nu := \varepsilon(k - \rho) \), we have

\[
\|\Delta_{q_m}(a(\xi)\psi_r(\xi))\|_{\mathcal{H}S} \leq \|\Delta_{q_m}(a(\xi))\|_{\mathcal{H}S} \mathcal{M}(\xi)^{-\nu} \|\Delta_{q_m}(\psi_r(\xi))\|_{\mathcal{H}S} \\
+ \|a(\xi)\mathcal{M}(\xi)^{\nu}\|_{\mathcal{H}S} \|\mathcal{M}(\xi)^{-\nu}\Delta_{q_m}(\psi_r(\xi))\|_{\mathcal{H}S} \\
+ \sum_{\ell=1}^{2m-1} \sum_j \|\sum_{j} (Q_{\ell,j} a(\xi))\|_{\mathcal{H}S} \mathcal{M}(\xi)^{\rho\ell+\nu} \|\mathcal{M}(\xi)^{-\rho\ell-\nu}(\hat{Q}_{\ell,j}\psi_r(\xi))\|_{\mathcal{H}S} \\
\leq \|\mathcal{M}(\xi)^{-\nu}\psi_r(\xi)\|_{\mathcal{H}S} + \sum_{\ell=0}^{2m-1} \sum_j \|\Pi_{\ell,j}(\xi)\|_{\mathcal{H}S},
\]

where \( \Pi_{\ell,j}(\xi) = \mathcal{M}(\xi)^{-\rho\ell-\nu}(\hat{Q}_{\ell,j}\psi_r(\xi)) \), and \( \Pi_{2m,j}(\xi) = \mathcal{M}(\xi)^{-2m-\nu}\psi_r(\xi) \), with \( j = 0 \) being the unique possible index for \( \ell = 0, 2m \). Now, we study the terms \( \Pi_{\ell,j}(\xi) \). Let \( \hat{Q}_{2m-\ell,j} \) be the associated function to the difference operator \( \hat{Q}_{\ell,j} \) vanishing with order \( 2m - \ell \) at \( e_G \). Then,

\[
\|\hat{Q}_{2m-\ell,j}\psi_r\|_{H^{-\rho\ell-\nu,\mathcal{L}}(G)} = \left( \sum_{[\xi] \in \hat{G}} d_{\xi} \|\mathcal{M}(\xi)^{-\rho\ell-\nu}(\hat{Q}_{\ell,j}\psi_r(\xi))\|_{\mathcal{H}S}^2 \right)^{\frac{1}{2}}.
\]

Let us fix \( \nu = \varepsilon(k - \rho) \). By the embedding \( H^{-\varepsilon}\mathcal{E}(G) \hookrightarrow H^{-s,\mathcal{L}}(G) \), (see Proposition 3.1 of [71]), we have the following inequality,

\[
\|\hat{Q}_{2m-\ell,j}\psi_r\|_{H^{-\rho\ell-\nu,\mathcal{L}}} \lesssim \|\hat{Q}_{2m-\ell,j}\psi_r\|_{H^{-\rho\ell+\nu}} \lesssim r^{\frac{2m-\ell}{n} - \frac{1}{2}}.
\]
where we have used Lemma 3.6, with $\tilde{\ell} = 2m - \ell$ and $s = \frac{\rho + \nu}{\kappa}$, with $0 \leq \frac{\rho + \nu}{\kappa} \leq 1 + \frac{n}{2}$. Indeed, note that $\frac{\rho + \nu}{\kappa} \leq \frac{\rho + \nu}{\kappa}$, and the condition $0 \leq \frac{\rho + \nu}{\kappa} \leq 1 + \frac{n}{2}$, is equivalent to prove that: $\kappa \leq 1 + \frac{n}{2}$, which holds true for $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, with $d$ odd. For $0 < r < 1$, and $\frac{1}{2} \varepsilon_{\ell} := \frac{2m - \ell - \frac{\rho + \nu}{\kappa} - \frac{1}{2}}{n}$, the condition $\nu = \kappa(\kappa - \rho)$ assures that $\frac{1}{2} \varepsilon_{\ell} \leq \frac{1}{2} \varepsilon_{\ell}$, and consequently, $r \frac{1}{2} \varepsilon_{\ell} \leq r \frac{1}{2} \varepsilon_{\ell}$.

Hence, we conclude that

$$
\sum_{\ell=0}^{2m-1} \sum_{j} \|\Pi_{\ell,j}\|_{L^2(G)} = \sum_{\ell=0}^{2m-1} \sum_{j} \|\tilde{q}_{2m-\ell,j} \check{\psi}_{\ell,j}\|_{H^{-\frac{1}{2},\kappa}} \leq r^\frac{\rho}{2}.
$$

(5.2)

Now, if $r \geq 1$, we observe that

$$
\|\tilde{q}_{2m-\ell,j} \check{\psi}_{\ell,j}\|_{H^{-\frac{1}{2},\kappa}} \leq \|\tilde{q}_{2m-\ell,j} \check{\psi}_{\ell,j}\|_{H^{-\frac{1}{2},\kappa}} \leq \|\tilde{q}_{2m-\ell,j} \check{\psi}_{\ell,j}\|_{L^2(G)}
$$

\begin{align*}
&\leq \|\tilde{q}_{2m-\ell,j} \check{\psi}_{\ell,j}\|_{L^2(G)} = \|\phi_{\ell} - \phi_{\ell,j}\|_{L^2(G)} \leq \|\phi_{\ell}\|_{L^2(G)} + \|\phi_{\ell,j}\|_{L^2(G)} \\
&\leq r^{-1/2} + \left(\frac{r}{2}\right)^{-1} \leq 3r^{-\frac{1}{2}},
\end{align*}

(5.1) for $r \geq 1$. Now, we end the proof of the weak (1,1) inequality by observing (according to the proof of Theorem 2.1 of [112]) that the difference operator $\Delta_{q_m}$ is a linear combination of operators of the form $D_\xi^\gamma$, with $|\gamma| = 2m := \kappa$. The proof of the weak (1,1) inequality for $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, $d$ even, can be analysed in a similar way. Let us observe that in view of Lemma 4.7 we have the estimate

$$
\|a(\xi)\|_{op} \leq (1 + \nu_{ii}(\xi)^2)^{-\frac{n(\kappa - \rho)}{2\kappa}},
$$

for some $1 \leq i \leq d_\xi$. So, if $\rho = \kappa$, $A$ extends to a bounded operator on $L^2(G)$, while for $0 \leq \rho < \kappa$, we deduce that $A$ extends to a compact operator on $L^2(G)$\footnote{Indeed, let us observe that the fact that $\lim_{\xi \to \infty} \|a(\xi)\|_{op} \leq \lim_{\xi \to \infty} \max_{1 \leq i \leq d_\xi} (1 + \nu_{ii}(\xi)^2)^{-\frac{n(\kappa - \rho)}{2\kappa}} \leq \lim_{\xi \to \infty} \langle \xi \rangle^{-\frac{n(\kappa - \rho)}{2\kappa}} = 0$, implies the existence of a compact extension of $A$ on $L^2(G)$ (see [45]).}. If $\rho = \kappa$, in view of the Marcinkiewicz interpolation theorem and the duality argument we have that $A$ extends to a bounded operator on $L^p(G)$. However, for $0 \leq \rho < \kappa$, the compactness of $A$ on $L^2(G)$, the weak (1,1) type of $A$, the Marcinkiewicz interpolation theorem, and the compactness interpolation theorem (Theorem 5.1) allow us to conclude that $A$ extends to a compact operator on $L^p(G)$ for all $1 < p < \infty$. \hfill \Box

The argument used in the proof of Theorem 3.9 via the Sobolev embedding theorem can be easily adapted to prove the following result for non-invariant operators, but, in the subelliptic context.
Theorem 5.3. Let us assume that $G$ is a compact Lie group of dimension $n = 2d$ or $n = 2d + 1$, with $d$ odd. Let $0 \leq \rho \leq \kappa$, and $a \in \Sigma(G \times \hat{G})$ be a symbol satisfying

$$\| (\partial_x^\rho \partial_\xi^\alpha a(x, \xi)) \hat{\mathcal{M}}(\xi)^{\rho(\alpha+\kappa(\kappa-\rho))} \| \leq C_\alpha, \ |\alpha| \leq \kappa := d + 1, \ |eta| \leq \left( \frac{n}{p} \right) + 1.$$ 

Then $A = \operatorname{Op}(a)$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$ for $\rho = \kappa$, and to a compact linear operator on $L^p(G)$ for all $1 < p < \infty$ when $0 \leq \rho < \kappa$. Moreover, if the dimension of the group is $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, with $d$ even, the conclusion on $A$ is the same provided that

$$\| (\partial_x^\rho \partial_\xi^\alpha a(x, \xi)) \hat{\mathcal{M}}(\xi)^{\rho(\alpha+\kappa(\kappa-\rho))} \| \leq C_\alpha, \ |\alpha| \leq \kappa := d + 2, \ |eta| \leq \left( \frac{n}{p} \right) + 1.$$ 

Proof. Now, for every $z \in G$, let us define the family of invariant operators $\{A_z\}_{z \in G}$, by

$$A_z f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \operatorname{Tr}[\xi(x) a(z, \xi) \hat{f}(\xi)], \ f \in C^{\kappa}(G).$$

By the identity $A_z f(x) = A f(x), \ x \in G$, the Sobolev embedding Theorem gives

$$\sup_{z \in G} |A_z f(x)| \lesssim \sum_{|\beta| \leq \left[ \frac{n}{p} \right] + 1} \| \partial_x^\beta A_z f(\cdot) \|_{L^p(G)} = \sum_{|\beta| \leq \left[ \frac{n}{p} \right] + 1} \left( \int_G |\partial_x^\beta A_z f(x)|^p \, dz \right)^{\frac{1}{p}}.$$ 

Every operator $A_{z, \beta} := \partial_x^\beta A_z$ has an invariant symbol $a_{z, \beta} := \partial_x^\beta a(z, \cdot)$, and the estimates

$$\| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\kappa(\kappa-\rho)+\rho|\alpha|)} (\partial_x^\beta \partial_\xi^\alpha a(x, \xi)) \| \leq C_\alpha, \ |\alpha| \leq \kappa, \ |eta| \leq \left[ \frac{n}{p} \right] + 1,$$

are equivalent to the following ones,

$$\| \hat{\mathcal{M}}(\xi)^{\frac{1}{2}(\kappa(\kappa-\rho)+\rho|\alpha|)} \partial_\xi^\alpha a_{z, \beta}(\xi) \| \leq C_\alpha, \ |\alpha| \leq \kappa, \ |eta| \leq \left[ \frac{n}{p} \right] + 1.$$ 

Consequently the family of operators $\{A_{z, \beta}\}_{z \in G, |\beta| \leq \left[ \frac{n}{p} \right] + 1}$, satisfies the conclusions in Theorem 5.2. Moreover, for every $|\beta| \leq \left[ \frac{n}{p} \right] + 1$, the function $z \mapsto A_{z, \beta}$, is a continuous function from $G$ into $\mathcal{B}(L^p(G))$, the set of bounded linear operators on $L^p(G)$, for all $1 < p < \infty$. The compactness of $G$ implies that

$$\sup_{z \in G} \| A_{z, \beta} \|_{\mathcal{B}(L^p(G))} = \sup_{z \in G} \| \partial_x^{\beta} A_z \|_{\mathcal{B}(L^p(G))} = \| \partial_x^{\beta} A_{z_0, \beta} \|_{\mathcal{B}(L^p(G))} = \| \partial_x^{\beta} A_{z_0, \beta} \|_{\mathcal{B}(L^p(G))} = \| \partial_x^{\beta} A_{z_0, \beta} \|_{\mathcal{B}(L^p(G))}.$$
for some \( z_{0,\beta} \in G \). Consequently, we can estimate the \( L^p(G) \)-norm of \( Af, f \in C^\infty(G) \), by

\[
\|Af\|_{L^p(G)}^p = \int_G |Af(x)|^p \, dx \leq \int_G |f(x)|^p \, dx
\]

\[
\leq \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \int_G |\partial_Z^\beta f(x)|^p \, dx \, dz
\]

\[
= \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \int_G |\partial_Z^\beta f(x)|^p \, dx \, dz
\]

\[
\leq \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \sup_{x \in G} \|\partial_Z^\beta f\|_{L^p(\mathcal{R}(G))} \|f\|_{L^p(G)}^p
\]

\[
= \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \|\partial_Z^\beta f_{z_{0,\beta}}\|_{L^p(G)} \|f\|_{L^p(G)}^p.
\]

So, we have the boundedness of \( A \) on \( L^p(G) \), for all \( 1 < p < \infty \), and all \( 0 \leq \rho \leq \kappa \). However, for \( 0 \leq \rho < \kappa \), \( A \) extends to a compact operator on \( L^p(G) \). For the proof, we can take a sequence \( \{f_j\} \) in \( L^p(G) \), converging weakly to zero. We need to show that \( Af_j \) converges to zero in the \( L^p(G) \)-norm. The previous analysis gives us the inequality,

\[
\|Af_j\|_{L^p(G)}^p \leq \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \int_G |\partial_Z^\beta f_j(x)|^p \, dx \, dz = \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \int_G |\partial_Z^\beta f_j|_{L^p(G)}^p \, dz.
\]

Because, every weakly convergent sequence is a bounded sequence,\(^\text{16}\) we have that \( \sup_j \|f_j\|_{L^p(G)} < \infty \), and the estimate

\[
\|\partial_Z^\beta f_j\| \leq \|\partial_Z^\beta f_{z_{0,\beta}}\|_{L^p(\mathcal{R}(G))} \sup_j \|f_j\|_{L^p(G)}
\]

allows us to use the dominated convergence Theorem in order to conclude that

\[
\lim_{j \to \infty} \|Af_j\|_{L^p(G)}^p \leq \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \lim_{j \to \infty} \int_G |\partial_Z^\beta f_j|_{L^p(G)}^p \, dz
\]

\[
= \sum_{|\beta| \leq \left\lfloor \frac{d}{p} \right\rfloor + 1} \int_G \lim_{j \to \infty} |\partial_Z^\beta f_j|_{L^p(G)}^p \, dz.
\]

Now, for every \( z \in G \), \( \partial_Z^\beta A_z \) is a Fourier multiplier satisfying the hypothesis in Theorem 5.2, and consequently they admit compact extensions on \( L^p(G) \), for all \( 1 < p < \infty \), provided that \( 0 \leq \rho < \kappa \). This, and our assumption that \( \{f_j\} \) in \( L^p(G) \) converges weakly to zero, implies that \( \lim_{j \to \infty} \|\partial_Z^\beta A_z f_j\|_{L^p(G)} = 0 \). So, we conclude that \( \lim_{j \to \infty} \|Af_j\|_{L^p(G)} = 0 \). The proof is complete. \(\square\)

\(^\text{16}\) this is a known fact, indeed, it can be proved by using the Uniform Boundedness Principle.
Remark 5.4. As we can see in Theorem 5.2, the class of symbols with limited regularity \( S_\rho^{\kappa(\kappa-\rho)}(G) \) begets operators of weak type \((1,1)\). The same conclusion is valid for operators with smooth symbols in the class \( S_\rho^{-\kappa(\kappa-\rho)}(G) \). Also, for \( \rho = \kappa \), we have
\[
\text{Op}(S_{\kappa,0}^0(G)) \subset \text{Op}(S_{\kappa,0}^{0,\kappa,\kappa}[\tfrac{n}{p}]^{+1,L}(G)) \subset \mathcal{B}(L^p(G)), \quad 1 < p < \infty,
\]
or equivalently,
\[
\text{Op}(S_{1,0}^{0,L}(G \times \hat{G})) \subset \text{Op}(S_{1,0}^{0,\kappa,\kappa}[\tfrac{n}{p}]^{+1,L}(G)) \subset \mathcal{B}(L^p(G)), \quad 1 < p < \infty.
\]

**Corollary 5.5.** Let \( \kappa \) be the smallest even integer larger than \( \frac{n}{2} \), \( n := \text{dim}(G) \). Let \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) be a continuous operator such that its matrix symbol \( \sigma_A \) satisfies
\[
\| (\partial_X^{(\beta)} \Delta \gamma \sigma_A(x, \xi)) \hat{M}(\xi)^{\alpha} \|_{\text{op}} \leq C_{\gamma, \beta}, \quad |\gamma| \leq \kappa, \quad |\beta| \leq \left[ \frac{n}{p} \right] + 1. \tag{5.3}
\]
If \( s \in \mathbb{R} \), then \( A \) extends to a bounded operator from \( L_s^p(G) \) into \( L_s^p(G) \) for all \( 1 < p < \infty \).

**Proof.** Note that
\[
\|Af\|_{L_s^p(G)} = \|\mathcal{M}_s \mathcal{A}_s \mathcal{M}_{-s} \mathcal{M}_s f\|_{L_s^p(G)}.
\]
By taking into account Remarks 4.6 and 4.19, we have that \( \mathcal{M}_s \in \text{Op}(S_{1,0}^{0,L}(G)) \subset \text{Op}(S_{\kappa,0}^{0,\kappa,\kappa}[\tfrac{n}{p}]^{+1,L}(G)) \), \( A \in \text{Op}(S_{\kappa,0}^{0,\kappa,\kappa}[\tfrac{n}{p}]^{+1,L}(G)) \), and \( \mathcal{M}_{-s} \text{Op}(S_{0,\kappa}^{-\kappa,L}(G)) \subset \text{Op}(S_{\kappa,0}^{-\kappa,L}(G)) \), and we conclude that
\[
A_s := \mathcal{M}_s \mathcal{A}_s \mathcal{M}_{-s} \in \text{Op}(S_{\kappa,0}^{0,\kappa,\kappa}[\tfrac{n}{p}]^{+1,L}(G)).
\]
Moreover, from Theorem 5.3 we deduce that \( A_s \) extends to a bounded operator on \( L^p(G) \) for all \( 1 < p < \infty \). Consequently we deduce the estimate
\[
\|Af\|_{L_s^p(G)} \leq \|A_s\|_{\mathcal{B}(L^p(G))} \|f\|_{L_s^p(G)}.
\]
So, \( A \) extends to a bounded operator from \( L_s^p(G) \) into \( L_s^p(G) \). Thus, we finish the proof.

Corollary 5.5 implies the following result on the boundedness of pseudo-differential operators on Besov spaces (we refer the reader to Appendix 12 for the definition and to [33] for the interpolation properties of subelliptic Besov spaces).

**Corollary 5.6.** Let \( \kappa \) be the smallest even integer larger than \( \frac{n}{2} \), \( n := \text{dim}(G) \). Let \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) be a continuous operator such that its matrix symbol \( \sigma_A \) satisfies
\[
\| (\partial_X^{(\beta)} \Delta \gamma \sigma_A(x, \xi)) \hat{M}(\xi)^{\gamma} \|_{\text{op}} \leq C_{\gamma,\beta}, \quad |\gamma| \leq \kappa, \quad |\beta| \leq \left[ \frac{n}{p} \right] + 1. \tag{5.4}
\]
If \( s \in \mathbb{R} \), then \( A \) extends to a bounded operator from \( B_{p,q}^{s,L}(G) \) into \( B_{p,q}^{s,L}(G) \) for all \( 1 < p < \infty \), and \( 0 < q < \infty \).
Proof. We will present a proof based on the real interpolation analysis. Observe that Corollary 5.5, shows that $A$ extends to a bounded operator from $L^p_s(G)$ into $L^p_{s-1}(G)$ for every $1 < p < \infty$. In particular, if $1 < p_0 < p_1 < \infty$ and $\theta \in (0,1)$ satisfies $1/p = \theta/p_0 + (1-\theta)/p_1$, then from the boundedness of the following bounded extensions of $A$,

$$A : L^p_{s-1}(G) \rightarrow L^p_{s-1}(G), \quad A : L^p_{s-1}(G) \rightarrow L^p_{s-1}(G)$$

by the real interpolation, we deduce

$$A : (L^p_{s-1}(G), L^p_{s-1}(G))_{(\theta,q)} \rightarrow (L^p_{s-1}(G), L^p_{s-1}(G))_{(\theta,q)}, \quad 0 < q < \infty.$$  

Because $(L^p_{s-1}(G), L^p_{s-1}(G))_{(\theta,q)} = B^p_{\theta,q}(G)$ for every $r \in \mathbb{R}$, (see [33, Theorem 6.2]) we conclude that $A$ extends to a bounded operator from $B^p_{\theta,q}(G)$ into $B^p_{\theta,q}(G)$. The proof is complete. 

6. BOUNDEDNESS OF SUBELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS WITH SMOOTH SYMBOLS

In this section we develop a Fefferman type $L^p$-estimate for subelliptic classes on compact Lie groups. In contrast with the previous subsection, we will use the calculus for subelliptic operators e.g. in the duality argument. First, we will prove an estimate of $L^\infty$-BMO type and we will provide the $H^{-1}-L^1$ estimate by the duality argument. Finally, by using the Fefferman-Stein interpolation theorem we will obtain the $L^p$-estimate for subelliptic operators. This gives an extension of the results in [46] for the Laplacian, to the subelliptic setting.

6.1. $L^\infty$-BMO boundedness for subelliptic Hörmander classes.

Remark 6.1 ($L^\infty(G)$-boundedness of pseudo-differential operators). Let $x \mapsto k_x$, the right convolution kernel of A, this means that for every smooth function $f$ on $G$,

$$Af(x) = (f \ast k_x)(x), \quad x \in G.$$  

If $\sigma(x, \xi)$ is the matrix-valued symbol of $A$, we have that $k_x = F^{-1}_G \sigma(x, \cdot)$. If we assume the following condition

$$\sup_{x \in G} \|F^{-1}_G \sigma(x, \cdot)\|_{L^1(G)} = \sup_{x \in G} \|k_x\|_{L^1(G)} < \infty,$$

then

$$|Af(x)| \leq \int_G |k_x(y^{-1}x)||f(y)|dy \leq \sup_{x \in G} \|k_x\|_{L^1(G)} \|f\|_{L^\infty(G)}.$$

Consequently,

$$\|A\|_{\mathcal{B}(L^\infty(G))} \leq \sup_{x \in G} \|k_x\|_{L^1(G)}.$$  

Remark 6.2. For every smooth function $f \in C^\infty(0, \infty)$, we will define

$$f(\hat{\mathcal{M}})(\xi) \equiv f(\hat{\mathcal{M}}(\xi)) := \text{diag}[f((1+\nu_i^2)^{d_i})]_{1 \leq i \leq d}, \quad (f((1+\nu_i^2)^{d_i}) \delta_{ij})_{1 \leq i,j \leq d}$$  

for every $[\xi] \in \hat{G}$. According to the symbolic calculus developed in [111], (6.1) defines the matrix-symbol $\{f(\hat{\mathcal{M}})(\xi)\}_{[\xi] \in \hat{G}}$ of the operator $f(\mathcal{M})$ defined by the functional calculus.
Lemma 6.3. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$. For $0 \leq \delta < \rho \leq 1$, let $\sigma \in S^{\frac{Q(1-\rho)}{2}}_{\rho, \delta}(G \times \hat{G})$ be a symbol satisfying
\[
\sigma(x, \xi) \psi_j(\hat{\mathcal{M}}(\xi)) = \sigma(x, \xi),
\]
where $\psi_j(\lambda) = \psi_0(2^{-j} \lambda)$, for some test function $\psi_0 \in C_0^\infty(0, \infty)$, and some fixed integer $j \in \mathbb{N}_0$. Then $A = \text{Op}(\sigma)$ extends to a bounded operator from $L^\infty(G)$ to $L^\infty(G)$, and for $\ell := \left[\frac{Q}{2}\right] + 1$, we have
\[
\|A\|_{\mathscr{S}(L^\infty(G))} \leq \left( \sup_{(x, [\xi]) \in \hat{G}, |\alpha| \leq \ell} \|\Delta_\xi^\alpha \sigma(x, \xi) [\hat{\mathcal{M}}(\xi)]^{\frac{Q(1-\rho)}{2} + \rho|\alpha|}\|_{op} \right).
\]
with the positive constant $C$ independent of $j$ and $\sigma$.

Proof. Let us fix $j \in \mathbb{N}_0$, and allow us to define $a := 1 - \rho$. Let $b = R^{a-1}$, $R = 2^j$. Let us consider $\sigma \in S^{\frac{Qa}{2}}_{\rho, \delta}(G \times \hat{G}) = \mathcal{S}^{\frac{Qa}{2}}_{\rho, \delta}(G)$. In view of Remark 6.1, we only need to prove that
\[
\sup_{x \in G} \|k_x\|_{L^1(G)} \leq C,
\]
where $C$ is a positive constant independent of $R$, and $k_x$, as usually, is the right-convolution kernel of $A$. Let us denote by $|\cdot|$ the seminorm induced by the Carnot Carathéodory distance on $G$. First, let us split the $L^1(G)$-norm of $k_x$ as,
\[
\int_G |k_x(z)|dz = \int_{|z| < b} |k_x(z)|dz + \int_{|z| > b} |k_x(z)|dz.
\]
By using the Hölder inequality we estimate the first integral as follows,
\[
\int_{|z| < b} |k_x(z)|dz \leq \left( \int_{|z| < b} dz \right)^\frac{1}{\gamma} \left( \int_{|z| < b} |k_x(z)|^2dz \right)^{\frac{1}{2}}
\]
\[
= R^{\frac{Q(a-1)}{2}} \|k_x\|_{L^2(G)}
\]
\[
= 2^{\frac{Q(a-1)a}{2}} \|k_x\|_{L^2(G)}.
\]
The Plancherel theorem and the definition of the right-convolution kernel: $k_x = \mathcal{F}_G^{-1}(\sigma(x, \cdot))$, for every $x \in G$, together with (6.2) imply
\[ \|k_x\|_{L^2(G)} = \left( \sum_{[\xi] \in \hat{G}} d_\xi \| \sigma(x, \xi) \|^2_{HS} \right)^{\frac{1}{2}} \]
\[ = \left( \sum_{[\xi] \in \hat{G}} d_\xi \| \sigma(x, \xi) \hat{\mathcal{M}}(\xi) \frac{\partial}{\partial \xi} \hat{\mathcal{M}}(\xi)^{-\frac{Q_0}{2}} \|^2_{HS} \right)^{\frac{1}{2}} \]
\[ = \left( \sum_{[\xi] \in \hat{G}} d_\xi \| \sigma(x, \xi) \hat{\mathcal{M}}(\xi) \frac{\partial}{\partial \xi} \hat{\mathcal{M}}(\xi)^{-\frac{Q_0}{2}} \|^2_{HS} \right)^{\frac{1}{2}} \]
\[ \leq \left( \sup_{[\xi] \in \hat{G}} \| \sigma(x, \xi) \hat{\mathcal{M}}(\xi) \frac{\partial}{\partial \xi} \|^2_{op} \right) \left( \sum_{[\xi] \in \hat{G}} d_\xi \| \psi_j(\hat{\mathcal{M}}(\xi)) \hat{\mathcal{M}}(\xi)^{-\frac{Q_0}{2}} \|^2_{HS} \right)^{\frac{1}{2}}. \]

Let us denote, for every \( j \geq 1 \),
\[ I_j := \left( \sum_{[\xi] \in \hat{G}} d_\xi \| \psi_j(\hat{\mathcal{M}}(\xi)) \hat{\mathcal{M}}(\xi)^{-\frac{Q_0}{2}} \|^2_{HS} \right)^{\frac{1}{2}}. \]

So, we can estimate \( I_j \) for every \( j \geq 1 \). Indeed, the Weyl eigenvalue counting formula for the sub-Laplacian (4.21), gives
\[ I_j^2 \leq \sum_{[\xi] \in \hat{G}} d_\xi \| \psi_j(\hat{\mathcal{M}}(\xi)) \hat{\mathcal{M}}(\xi)^{-\frac{Q_0}{2}} \|^2_{HS} \]
\[ \leq \sum_{[\xi] \in \hat{G}} d_\xi \sum_{i=1}^{d_\xi} (1 + \nu_{ik}(\xi)^2)^{-\frac{Q_0}{2}} \]
\[ \leq \sum_{[\xi] \in \hat{G}} d_\xi^2 2^{-jQ_0} \lesssim 2^{jQ-jQ_0}. \]

Consequently,
\[ I_j \lesssim 2^{\frac{jQ(1-a)}{2} - \frac{jQ(1-a)}{2}}. \]

This analysis and the fact that \( 0 < a \leq 1 \), allows us to deduce that
\[ \left( \int_{|z| \leq b} \left( \int_{|z| \leq b} |k_x(z)|^2 dz \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} = 2^{\frac{jQ(1-a)}{2} - \frac{jQ(1-a)}{2}} = 1, \]
and consequently we estimate
\[ \int_{|z| \leq b} |k_x(z)| dz = O(1). \]

On the other hand, if \( \Delta_q \) is a difference operator associated to \( q \) that vanishes at \( e_G \) of order \( \ell \) and \( e_G \) is an isolated zero, observe that
\[
\int_{|z| > b} |k_x(z)| dz \leq \left( \int_{|z| > b} q(z)^{-2} dz \right)^{\frac{1}{2}} \left( \int_{|z| > b} |q(z)k_x(z)|^2 dz \right)^{\frac{1}{2}} \lesssim \left( \int_{|z| > b} |z|^{-2\ell} dz \right)^{\frac{1}{2}} \left( \int_{G} |q(z)k_x(z)|^2 dz \right)^{\frac{1}{2}} = b^{\frac{Q-2\ell}{2}} \left( \sum_{[\xi] \in G} d_\xi \|\Delta_q \sigma(x, \xi)\|_{L^2(G)}^2 \right)^{\frac{1}{2}} = b^{\frac{Q-2\ell}{2}} \|\Delta_q \sigma(x, \xi)\|_{L^2(G)}^2.
\]

We observe that the condition \( \ell > \frac{Q}{2} \) is a necessary and sufficient condition in order that \( \int_{|z| > b} |z|^{-2\ell} dz < \infty \), which can be deduced from an argument using polar coordinates in order to estimate
\[
\left( \int_{|z| > b} |z|^{-2\ell} dz \right)^{\frac{1}{2}} \sim b^{\frac{Q-2\ell}{2}}.
\]

In order to control the \( L^2(G) \)-norm of \( \Delta_q \sigma(x, \xi) \) we proceed as follows,
\[
\|\Delta_q \sigma(x, \xi)\|_{L^2(G)}^2 \leq \sum_{[\xi]: 2^{j-1} \leq (1 + \nu_{k, k} \xi^2)^{\frac{1}{2}} < 2^{j+1}, \forall 1 \leq k \leq d_\xi} d_\xi \|\Delta_q \sigma(x, \xi)\|_{L^2(G)}^2 \lesssim \sum_{[\xi]: 2^{j-1} \leq (1 + \nu_{k, k} \xi^2)^{\frac{1}{2}} < 2^{j+1}, \forall 1 \leq k \leq d_\xi} d_\xi \|\Delta_q \sigma(x, \xi)\|_{L^2(G)}^2 \lesssim \sum_{[\xi]: 2^{j-1} \leq (1 + \nu_{k, k} \xi^2)^{\frac{1}{2}} < 2^{j+1}, \forall 1 \leq k \leq d_\xi} d_\xi \sum_{i=1}^{d_\xi} (1 + \nu_{i, i} \xi^2)^{-(\frac{Qa}{2} + (1-a)\ell)} \lesssim \sum_{[\xi]: 2^{j-1} \leq (1 + \nu_{k, k} \xi^2)^{\frac{1}{2}} < 2^{j+1}, \forall 1 \leq k \leq d_\xi} d_\xi^2 2^{-2j\frac{Qa}{2} + (1-a)\ell} \lesssim 2^{-2j\frac{Qa}{2} + (1-a)\ell} \times 2jQ = 2^{j(1-a)(Q-2\ell)},
\]
where we have used again the Weyl eigenvalue counting formula for the sub-Laplacian (Remark 4.21). So, we have

$$\|\Delta_\eta \sigma(x, \xi)\|_{L^2(\hat{G})} \leq 2^{j(1-a)(\frac{d}{2}-\ell)}.$$  

The preceding analysis allows us to conclude that for \( \ell := [\frac{Q}{2}] + 1 > \frac{Q}{2} \),

$$\int_{|z| > b} |k_z(z)| dz \leq 2^{j(a-1)(\frac{d}{2}+\ell)} \|\Delta_\eta \sigma(x, \xi)\|_{L^2(\hat{G})} \leq 2^{j(a-1)(\frac{d}{2}+\ell)}2^{j(1-a)(\frac{d}{2}-\ell)} = 1.$$  

Thus, the proof is complete. \( \square \)

**Lemma 6.4.** Let \( G \) be a compact Lie group and let us denote by \( Q \) the Hausdorff dimension of \( G \) associated to the control distance associated to the sub-Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{X_1, \ldots, X_k\} \) is a system of vector fields satisfying the Hörmander condition of order \( \kappa \). For \( 0 \leq \delta < \rho \leq 1 \), and \( \varepsilon \in \mathbb{R} \), let us consider a continuous linear operator \( A : C^\infty(G) \to \mathcal{P}(\hat{G}) \) with symbol \( \sigma \in S^{c,\mathcal{L}}(G \times \hat{G}) \).

Let \( \eta \) be a smooth function supported in \( \{\lambda : R \leq \lambda \leq 3R\} \), for some \( R > 1 \). Then for all \( \alpha \in \mathbb{N}^n \) with \( |\alpha| \leq \ell \), there exists \( C > 0 \), such that for every \( \omega \geq 0 \),

$$\sup_{(x,[\xi]) \in \mathcal{G} \times \hat{G}} \|\hat{\mathcal{M}}(\xi)^{-\varepsilon+\rho|\alpha|} \Delta^{\alpha}_\xi [\sigma(x, \xi) \eta(s\langle \xi \rangle)]\|_{\text{op}} \leq C\|\sigma\|_{L^{\rho,\delta,c}(\eta(s\langle \xi \rangle))^{-\omega}},$$

with the positive constant \( C \) independent of \( s > 0 \), \( R \) and \( \sigma \).

**Proof.** Let us consider \( \sigma \in S^{c,\mathcal{L}}(G \times \hat{G}) = \mathcal{S}_{\kappa,\rho,\delta}(G) \). The Leibniz rule allows us to write

$$\Delta^{\alpha}_\xi [\sigma(x, \xi) \eta(s\langle \xi \rangle)I_{d_\xi}] = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}[\Delta_{\alpha_1}\sigma](x, \xi)[\Delta_{\alpha_2}\eta(s\langle \xi \rangle)I_{d_\xi}],$$

because \( \eta \) has support in \( \{\lambda : R \leq \lambda \leq 3R\} \), we can estimate (see Lemma 6.8 of [61]),

$$\|\Delta_{\alpha_2}\eta(s\langle \xi \rangle)I_{d_\xi}\|_{\text{op}} \leq C_\omega(s\langle \xi \rangle)^{-\omega}, \quad (6.3)$$

for every \( \omega \in \mathbb{R}_0^+ \). So, we deduce

$$\|\Delta^{\alpha}_\xi [\sigma(x, \xi) \eta(s\langle \xi \rangle)I_{d_\xi}]\|_{\text{op}} \leq \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}[\hat{\mathcal{M}}(\xi)^{-\varepsilon+\rho|\alpha|}][\Delta_{\alpha_1}\sigma](x, \xi)\|_{\text{op}}\|\Delta_{\alpha_2}[\eta(s\langle \xi \rangle)I_{d_\xi}]\|_{\text{op}}$$

$$\leq \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2}[\hat{\mathcal{M}}(\xi)^{-\varepsilon+\rho|\alpha|}][\Delta_{\alpha_1}\sigma](x, \xi)\|_{\text{op}}\|\eta(s\langle \xi \rangle)^{-\omega}$$

$$\leq \|\sigma\|_{L^{\rho,\delta,c}}(s\langle \xi \rangle)^{-\omega},$$

proving Lemma 6.4. \( \square \)

**Remark 6.5.** Let us mention that for \( s \in (0,1) \), (instead of the general case \( s > 0 \) considered in the previous lemma), from [61, page 3434], (6.3) can be replaced by the following estimate,

$$\|\Delta_{\alpha_2}\eta(s\langle \xi \rangle)I_{d_\xi}\|_{\text{op}} \leq C_{\omega,\eta}s^{\omega}, \quad (6.4)$$
for all \( \omega \in \mathbb{R} \). So, from the proof of Lemma 6.4, we deduce the following estimate which is interesting by itself,

\[
\sup_{(x, \xi) \in G \times \hat{G}} \| \hat{M}(\xi)^{-\varepsilon + \rho/\alpha} \Delta_\xi^\alpha [\sigma(x, \xi) \eta(s \langle \xi \rangle)]\|_{\text{op}} \leq C \| \sigma \|_{\ell, S_{p, \delta}^{\varepsilon-\rho} S^\omega}, \quad 0 < s < 1.
\]  

(6.5)

This estimate is a subelliptic analogy of Lemma 4.28.

Now, we will present one of our fundamental theorems in this paper. The subelliptic BMO space and the subelliptic Hardy space \( \text{BMO}^\mathcal{L} \) and \( \text{H}^{1, \mathcal{L}} \) were defined in Subsection 3.5.

**Theorem 6.6.** Let \( G \) be a compact Lie group and let us denote by \( Q \) the Hausdorff dimension of \( G \) associated to the control distance associated to the sub-Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{X_1, \ldots, X_k\} \) is a system of vector fields satisfying the Hörmander condition of order \( \kappa \). For \( 0 \leq \delta < \rho \leq 1 \), \( \delta < 1/\kappa \), let us consider a continuous linear operator \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) with symbol \( \sigma \in S_{p, \delta}^{\frac{-\rho(1-\rho)}{2}} \mathcal{L}(G \times \hat{G}) \). Then \( A \) extends to a bounded operator from \( L^\infty(G) \) to \( \text{BMO}^\mathcal{L}(G) \), and from \( \text{H}^{1, \mathcal{L}}(G) \) to \( L^1(G) \). Moreover,

\[
\|A\|_{\mathcal{S}(L^\infty(G), \text{BMO}^\mathcal{L}(G))} \leq C \max \{ \|\sigma\|_{\ell, S_{p, \delta}^{\frac{q(1-\rho)}{2}} \mathcal{L}}, \|\sigma(\cdot, \cdot) \hat{M}(\xi)^{Q(1-\rho)}\|_{\ell, S_{p, \delta}^{0, \mathcal{L}}} \}
\]  

(6.6)

and

\[
\|A\|_{\mathcal{S}(\text{H}^{1, \mathcal{L}}(G), L^1(G))} \leq C \max \{ \|\sigma^*\|_{\ell, S_{p, \delta}^{\frac{q(1-\rho)}{2}} \mathcal{L}}, \|\sigma^*(\cdot, \cdot) \hat{M}(\xi)^{Q(1-\rho)}\|_{\ell, S_{p, \delta}^{0, \mathcal{L}}} \}
\]  

(6.7)

for some integer \( \ell \), where \( \sigma^* \) denotes the matrix-valued symbol of the formal adjoint \( A^* \) of \( A \).

**Proof.** Let us consider \( 0 \leq \alpha := (1 - \rho) \leq 1 \), and let \( f \in L^\infty(G) \). Let us fix a ball \( B(x_0, r) \) where \( x_0 \in G \), and \( r > 0 \). We will prove that there exists a constant \( C > 0 \), independent of \( f \) and \( r \), such that

\[
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |Af(x) - (Af)_{B(x_0, r)}| \eta(s \langle \xi \rangle) dx \leq C \| \sigma \|_{\ell, S_{p, \delta}^{\frac{q(1-\rho)}{2}}} \| f \|_{L^\infty(G)}
\]

where

\[
(Af)_{B(x_0, r)} := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} Af(x) dx.
\]

We will prove the existence of two symbols \( \sigma^0 \) and \( \sigma^1 \) such that (see (6.10))

\[
\sigma(x, \xi) = \sigma^0(x, \xi) + \sigma^1(x, \xi), \quad \sigma^j(x, \xi) \in \mathbb{C}^{d_x \times d_\xi}, \quad j = 0, 1,
\]

in a such way that both, \( \sigma^0(x, \xi) \) and \( \sigma^1(x, \xi) \), satisfy the estimate

\[
\|\sigma^j\|_{\ell, S_{p, \delta}^{\frac{q(1-\rho)}{2}}} \leq C_{j, \ell} \|\sigma\|_{\ell, S_{p, \delta}^{\frac{q(1-\rho)}{2}}}, \quad j = 0, 1, \ell \geq 1.
\]  

(6.8)
Now, if \( A^j := \text{Op}(\sigma^j) \), for \( j = 0, 1 \), then \( A = A^0 + A^1 \) on \( C^\infty(G) \), and we only need to prove that
\[
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^j f(x) - (A^j f)_B(x_0, r)| \, dx \leq C \|\sigma^j\|_{L^p_{\nu, \delta}} \frac{Q(1 - \rho)}{\epsilon} \|f\|_{L^\infty(G)}. \tag{6.9}
\]

Then, if we prove (6.8), we can deduce (6.6) and by the duality argument we could obtain (6.7) proving Theorem 6.6. Now, we proceed to prove the existence of \( \sigma^0 \) and \( \sigma^1 \) satisfying the requested properties. Let us define
\[
\sigma^0(x, \xi) = \sigma(x, \xi) \tilde{\gamma}(zz\xi), \quad \sigma^1(x, \xi) = \sigma(x, \xi) - \sigma^0(x, \xi), \quad (x, \xi) \in G \times \hat{G}, \tag{6.10}
\]
where \( \tilde{\gamma}(\xi) := \gamma(r\langle \xi \rangle) \), and \( \gamma \in C^\infty_0(\mathbb{R}, \mathbb{R}^+_{\nu}) \), is a smooth function supported in \( \{ t : |t| \leq 1 \} \), satisfying \( \gamma(t) = 1 \), for \( |t| \leq \frac{1}{2} \). To estimate the integral
\[
I_0 := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^0 f(x) - (A^0 f)_B(x_0, r)| \, dx,
\]
we will use the Mean value Theorem. If \( d_G(x, y) \) is the geodesic distance between \( x \) and \( y \), observe that
\[
|A^0 f(x) - A^0 f(y)| \leq C_0 \sum_{k=1}^n \|X_k A^0 f\|_{L^\infty(G)} d_G(x, y) \leq r \sum_{k=1}^{\dim(G)} \|X_k A^0 f\|_{L^\infty(G)}, \tag{6.11}
\]
where \( \{X_k\}_{k=1}^n \) is a basis for the Lie algebra \( \mathfrak{g} \) of \( G \). In order to estimate the \( L^\infty \)-norm of \( X_k A^0 f \), first let us observe that the matrix-valued symbol of \( X_k A^0 = \text{Op}(\sigma^k) \) is given by
\[
\sigma^k(x, \xi) := \sigma_{X_k}(\xi) \sigma^0(x, \xi) + (X_k \sigma^0(x, \xi)). \tag{6.12}
\]
Indeed, the Leibniz law gives
\[
X_k A^0 f(x) = \sum_{[\xi] \in \hat{G}} \text{Tr}(X_k(\xi(x) \sigma^0(x, \xi)) \hat{f}(\xi))
= \sum_{[\xi] \in \hat{G}} \text{Tr}([X_k(\xi(x)) \sigma^0(x, \xi) + \xi(x) X_k \sigma^0(x, \xi)] \hat{f}(\xi)).
\]
The identity \( \sigma_{X_k}(\xi) = \xi(x)^* X_k \xi(x) \), implies \( X_k \xi(x) = \xi(x) \sigma_{X_k}(\xi) \), and we obtain
\[
X_k A^0 f(x) = \sum_{[\xi] \in \hat{G}} \text{Tr}([\xi(x) \sigma_{X_k}(\xi) \sigma^0(x, \xi) + \xi(x) X_k \sigma^0(x, \xi)] \hat{f}(\xi)),
\]
which proves (6.12). By using a suitable partition of the unity we will decompose the matrix \( \sigma^k(x, \xi) \) in the following way,
\[
\sigma^k(x, \xi) = \sum_{j=1}^{\infty} \rho_{j,k}(x, \xi).
\]
To construct the family of operators $\rho_{j,k}(x, \xi)$ we will proceed as follows. We choose a smooth real function $\eta$ satisfying $\eta(t) \equiv 0$ for $|t| \leq 1$ and $\eta(t) \equiv 1$ for $|t| \geq 2$. Set

$$\rho(t) = \eta(t) - \eta\left(\frac{t}{2}\right).$$

The support of $\rho$ satisfies $\text{supp}(\rho) \subset [1, 4]$. One can check that

$$1 = \eta(t) + \sum_{j=1}^{\infty} \rho(2^j t), \quad \text{for all } t \in \mathbb{R}.$$

Indeed,

$$\eta(t) + \sum_{j=1}^{\ell} (2^j t) = \eta(t) + \sum_{j=1}^{\ell} \eta(2^j t) - \eta(2^{j-1} t) = \eta(2^\ell t) \to 1, \ \ell \to \infty.$$

For $t = r\langle \xi \rangle$ we have

$$1 = \eta(r\langle \xi \rangle) + \sum_{j=1}^{\infty} \rho(2^j r\langle \xi \rangle).$$

Observe that

$$\sigma'_k(x, \xi) = \eta(r\langle \xi \rangle)\sigma'_k(x, \xi) + \sum_{j=1}^{\infty} \rho(2^j r\langle \xi \rangle)\sigma'_k(x, \xi).$$

Because $\eta(r\langle \xi \rangle)\sigma'_k(x, \xi) = 0$, in view that the support of $\tilde{\gamma}$ and $\eta$ are disjoint sets, we have

$$\sigma'_k(x, \xi) = \sum_{j=1}^{\infty} \rho_{j,k}(x, \xi), \quad \rho_{j,k}(x, \xi) := \rho(2^j r\langle \xi \rangle)\sigma'_k(x, \xi).$$

From the Mean value Theorem we have,

$$|A^0 f(x) - A^0 f(y)| \leq C_0 \sum_{k=1}^{\dim(G)} |y^{-1} x| \cdot \|X_k A^0 f\|_{L^p(G)} \leq r \sum_{k=1}^{\dim(G)} \|X_k A^0 f\|_{L^p(G)}.$$
Now, in order to prove (6.9) for $j = 0$, we proceed by using Lemma 6.4 with $\omega = 1$, Lemma 6.3 and (6.11). So, we have,

$$I_0 := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^0 f(x) - (A^0 f)_{B(x_0, r)}| \, dx$$

$$= \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} (A^0 f(x) - A^0 f(y)) \, dy \, dx$$

$$\leq \frac{1}{|B(x_0, r)|^2} \int_{B(x_0, r)} \int_{B(x_0, r)} |A^0 f(x) - A^0 f(y)| \, dy \, dx \lesssim r \sup_{1 \leq k \leq \dim(G)} \|X_k A^0 f\|_{L^\infty(G)}$$

$$= r \sup_{1 \leq k \leq \dim(G)} \|\text{Op}(\sigma_k') f\|_{L^\infty(G)} \leq r \sup_{1 \leq k \leq \dim(G)} \sum_{j=1}^\infty \|\text{Op}(\rho_{j,k}) f\|_{L^\infty(G)}$$

$$\leq r \sum_{j=1}^\infty r^{-1} 2^{-j} \|\sigma\|_{L^\infty} \frac{q_\rho}{q_\delta} \|f\|_{L^\infty(G)}$$

$$\leq r r^{-1} \|\sigma\|_{L^\infty} \frac{q_\rho}{q_\delta} \|f\|_{L^\infty(G)}$$

$$= \|\sigma\|_{L^\infty} \frac{q_\rho}{q_\delta} \|f\|_{L^\infty(G)}.$$

In order to finish the proof, we need to prove (6.9) for $j = 1$. In order to obtain a similar $L^\infty(G)$-BMO$^\mathcal{L}(G)$ estimate for $A^1$, we will proceed as follows. Let $\phi$ be a smooth function compactly supported in $B(x_0, 2r)$ satisfying that

$$\phi(x) = 1, \text{ for } x \in B(x_0, r), \text{ and } 0 \leq \phi \leq 10.$$ 

Note that

$$|B(x_0, r)| \leq \int_{B(x_0, r)} \phi(x)^2 \, dx \leq \int_{B(x_0, 2r)} \phi(x)^2 \, dx = \|\phi\|^2_{L^2(G)} \leq 100|B(x_0, 2r)|,$$

and consequently

$$|B(x_0, r)|^{\frac{1}{2}} \leq \|\phi\|_{L^2(G)} \leq 10|B(x_0, 2r)|^{\frac{1}{2}} \leq 10C|B(x_0, r)|^{\frac{1}{2}}, \quad (6.13)$$

where in the last inequality we have used that the measure on the group satisfies the doubling property. Taking into account that

$$\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1 f(x) - (A^1 f)_{B(x_0, r)}| \, dx \leq \frac{2}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1 f(x)| \, dx,$$
we will estimate the right-hand side. Indeed, let us observe that
\[
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1 f(x)| dx \leq \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\phi(x) A^1 f(x)| dx
\]
\[
\leq \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1 [\phi f](x)| dx + \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |[M_\phi, A^1] f(x)| dx
\]
\[= I + II,
\]
where \(M_\phi\) is the multiplication operator by \(\phi\). To estimate \(I\), observe that, in view of the Cauchy-Schwartz inequality, we have
\[
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1 [\phi f](x)| dx \leq \frac{1}{|B(x_0, r)|} \left( \int_{B(x_0, r)} |A^1 [\phi f](x)|^2 dx \right)^{\frac{1}{2}}.
\]
For \(0 < \rho \leq 1\), let \(L := (1 + \mathcal{L})^{\frac{Q_2}{2}} \in \mathcal{S}^{\frac{Q_2}{2} \min(\rho, 1/\kappa), 0}(G \times \hat{G}) \subset \mathcal{S}^{\frac{Q_2}{2} \min(\rho, 1/\kappa), 0}(G \times \hat{G}) \subset \mathcal{S}^{\frac{Q_2}{2} \min(\rho, 1/\kappa), \delta}(G \times \hat{G})\), where \(\delta := 1 - \rho\) and \(0 < \varepsilon < 1\). For a moment, allow us to assume that we have
\[
\text{Op}(S^{-m, \mathcal{E}}_{\rho, \delta'}(G \times \hat{G})) \subset \mathcal{B}(L^2(G)),
\]
(6.14)
for \(m \geq 0\) and \(0 \leq \delta' < \rho' \leq 1\). Because \(A^1 \in \mathcal{S}^{\frac{Q_2}{2}, \mathcal{E}}(G \times \hat{G})\), the subelliptic symbolic calculus gives
\[
A^1 L \in \mathcal{S}^{\frac{Q_2}{2} \min(1-\varepsilon), \mathcal{E}}_{\rho, \delta}(G \times \hat{G}).
\]
Because \(m = \frac{Q_2}{2} (1-\varepsilon) > 0\), it follows from (6.14) and the condition \(\delta < 1/\kappa\), (this in order that \(\delta < \min\{\rho, 1/\kappa\}\)) that \(A^1 L\) is bounded on \(L^2(G)\). Consequently,
\[
\frac{1}{|B(x_0, r)|} \left( \int_{B(x_0, r)} |A^1 L[L^{-1}(\phi f)](x)|^2 dx \right)^{\frac{1}{2}} \leq \frac{\|A^1 L[L^{-1}(\phi f)]\|_{L^2(G)}}{|B(x_0, r)|^{\frac{1}{2}}}.
\]
By observing that
\[
\|L^{-1}(\phi f)\|_{L^2(G)} = \|\phi f\|_{H^{-\frac{Q_2}{2}, \mathcal{E}}(G)},
\]
where \(H^{-\frac{Q_2}{2}, \mathcal{E}}(G)\) is the Sobolev space of order \(-\frac{Q_2}{2}\), associated with \(\mathcal{L}\), the embedding \(L^2(G) \hookrightarrow H^{-\frac{Q_2}{2}, \mathcal{E}}(G)\), (see [71]) implies that
\[
\|L^{-1}(\phi f)\|_{L^2(G)} = \|\phi f\|_{H^{-\frac{Q_2}{2}, \mathcal{E}}(G)} \leq \|\phi f\|_{L^2(G)}.
\]
Moreover, from (6.13), we deduce the inequality
\[
\|\phi f\|_{L^2(G)} \leq \|f\|_{L^\infty(G)} \|\phi\|_{L^2(G)} \leq 10 \|f\|_{L^\infty(G)} |B(x_0, r)|^{\frac{1}{2}}.
\]
we deduce,

\[ I := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |A^1[\phi f](x)| \, dx \leq C \|\sigma_{A^1L}\|_{\ell^\rho, \ell^\rho} \|f\|_{L^\infty(G)}, \]

which is the desired estimate for \( I \). Now, we will estimate \( II \). For this, observe that the symbol of \([M_\phi, A^1] = \text{Op}(\theta)\), is given by

\[ \theta(x, \xi) = \int_G (\phi(x) - \phi(xy^{-1}))k_x(y)\xi(y)^* \, dy, \quad (6.15) \]

where \( x \mapsto k_x \), is the right-convolution kernel of \( A^1 \). Indeed, the equality \((6.15)\) was shown in \([46, \text{page 554}]\). Using the Taylor expansion we obtain

\[ \phi(xy^{-1}) = \phi(x) + \sum_{|\alpha|=1} (X_x^\alpha \phi)(x)\bar{q}_\alpha(y), \]

where, every \( \bar{q}_\alpha \) is a smooth function vanishing with order 1 at \( x_0 \). So, we can write

\[ \theta(x, \xi) = \sum_{|\alpha|=1} X_x^\alpha \phi(x)\Delta_{\bar{q}_\alpha} \sigma(x, \xi). \]

The hypothesis \( \sigma \in S^{-Q(1-\rho)}_{\rho, \delta}(G \times \hat{G}) \), implies that \( \Delta_{\bar{q}_\alpha} \sigma(x, \xi) \in S^{-Q(1-\rho)}_{\rho, \delta}(G \times \hat{G}) \), and the fact that \( \phi \) is of compact support, implies the same conclusion for

\[ \theta := \theta(x, \xi) \in S_{\rho, \delta}^{Q(1-\rho)-\rho, \mathcal{L}}(G \times \hat{G}). \]

From Lemma 6.3, one has

\[ \|\text{Op}(\theta)\|_{\mathcal{B}(L^\infty(G))} \lesssim \left( \sup_{(x, \xi) \in G \times \hat{G}, |\alpha| \leq \ell} \|\Delta_x^\alpha \theta(x, \xi)\|_{\mathcal{M}(\xi, Q_n + \rho + |\alpha|)} \right), \]

\[ \lesssim \sup_{|\alpha| \leq \ell} \|X_x^\alpha \phi\|_{L^\infty(G)} \left( \sup_{(x, \xi) \in G \times \hat{G}, |\alpha| \leq \ell} \|\Delta_x^\alpha \sigma(x, \xi)\|_{\mathcal{M}(\xi, Q_n + \rho + |\alpha|)} \right), \]

\[ \lesssim \|\sigma\|_{\ell^\rho + 1, S^{-Q(1-\rho)}_{\rho, \delta}} \cdot \]

Observing that

\[ \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\sigma_{M_\phi, A^1}[f](x)| \, dx \leq \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \|\sigma_{M_\phi, A^1}[f]\|_{L^\infty(G)} \]

\[ = \|\sigma_{M_\phi, A^1}[f]\|_{L^\infty(G)}, \]

we deduce,

\[ \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\sigma_{M_\phi, A^1}[f](x)| \leq \|\text{Op}(\theta)f\|_{L^\infty(G)} \]

\[ \lesssim \|\sigma\|_{\ell^\rho + 1, S^{-Q(1-\rho)}_{\rho, \delta}} \cdot \|f\|_{L^\infty(G)}. \]
Thus, we obtain
\[ II := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} ||[M_{\phi}, A^1]f(x)|| dx \leq C\|\sigma_A\|_{\ell^{r+1}_m} \sup_{\rho \in \mathbb{C}_m} \|f\|_{L^\infty(G)}. \]

Now, in order to finish the proof, we only need to prove (6.14). We will follow the classical argument of Hörmander. Indeed, assume first that \( p(x, \xi) \in S^{m_0, \xi}(G \times \hat{G}) \), where \( m_0 > 0 \). The kernel of \( p(x, D) = \text{Op}(p) \), \( K_p(x, y) \), belong to \( L^\infty(G \times G) \) for \( m_0 \) large enough. Hence,
\[
\sup_{x \in G} |K_p(x, y)| dy, \sup_{y \in G} |K_p(x, y)| dx < \infty,
\]
and the \( L^2(G) \)-continuity of \( p(x, D) \) follows from Schur lemma. Next we prove by induction that \( p(x, D) \) is \( L^2 \)-bounded if \( p(x, \xi) \in S^{m_0, \xi}(G \times \hat{G}) \), for \( m_0 < m \leq -(\rho' - \delta') \). To do so we form for \( u \in C^\infty(G) \)
\[
\|p(x, D)u\|_{L^2(G)} = (p(x, D)u, p(x, D)u)_{L^2(G)} = (p^*(x, D)p(x, D)u, u)_{L^2(G)} = (b(x, D)u, u)_{L^2(G)},
\]
where \( b(x, D) = p^*(x, D)p(x, D) \) has symbol in \( S^{2m, \xi}(G \times \hat{G}) \), for \( 0 \leq \delta' < \rho' \leq 1 \). From the induction hypothesis the continuity of \( p(x, D) \) for all \( p \in S^{2m, \xi} \) now follows successively for \( m \leq -\frac{m_0}{2}, \ldots, -\frac{m_0}{4}, \ldots, -\frac{m_0}{2n}, \ldots, \ell_0 \in \mathbb{N} \), and hence for \( m \leq -\frac{m_0}{2\ell_0} \) where \( \frac{m_0}{2\ell_0} < \rho' - \delta' \), after a finite number of steps. Now, assume that \( p(x, \xi) \in S^{\nu, \xi}(G \times \hat{G}) \) and chose
\[
M > 2 \sup_{(x,[\xi])} \|p(x, \xi)\|_{\text{op}}^2.
\]
Then, in view of the subelliptic functional calculus, in particular by Corollary 8.4, we have that \( c(x, \xi) = (M\delta^\epsilon - p(x, \xi)p(x, \xi)^*)^{1/2} \in S^{\nu, \xi}(G \times \hat{G}) \). Now,
\[
c(x, D)^*c(x, D) = M - p^*(x, D)p(x, D) + r(x, D)
\]
where \( r \in S^{-(\rho' - \delta')}(G \times \hat{G}) \). Hence, \( \|p(x, D)\|_{\mathfrak{L}(L^2)} \leq M + \|r(x, D)\|_{\mathfrak{L}(L^2)} \). Thus, the proof is complete.

In view of (6.14) we have the following \( L^2(G) \)-estimate.

**Corollary 6.7.** Let us assume that \( G \) is a compact Lie group. Then,
\[
\text{Op}(S^{\nu, \xi}(G \times \hat{G})) \subset \mathfrak{B}(L^2(G)),
\]
(6.16)
for all \( 0 \leq \delta' < \rho' \leq 1 \).
6.2. $L^p(G)$, Sobolev and Besov boundedness for subelliptic Hörmander classes.

Remark 6.8. Under those hypothesis in Theorem 6.6, every continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho, \delta}^{-m/2}(G \times \hat{G})$ extends to a bounded operator from $L^\infty(G)$ to $\text{BMO}^\mathcal{L}(G)$, and from $H^{1, \mathcal{L}}(G)$ to $L^1(G)$. The simple argument of interpolation gives that $A$ also extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. However, an argument via the Fefferman-Stein interpolation theorem gives a more precise result that we will now present in Theorem 6.9.

Theorem 6.9. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$. For $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, let us consider a continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho, \delta}^{-m/2}(G \times \hat{G})$, $m \geq 0$. Then $A$ extends to a bounded operator on $L^p(G)$ provided that

$$m \geq m_p := Q(1 - \rho) \left\lvert \frac{1}{p} - \frac{1}{2} \right\rvert.$$  

Proof. Now, having proved Theorem 6.6, the proof is verbatim the proof of Theorem 4.15 of [46]. Let us write $a := 1 - \rho$. We will present the argument here, for completeness. We will use the complex Fefferman-Stein interpolation theorem. We only need to prove the theorem for $m = m_p$ in view of the inclusion $S_{\rho, \delta}^{-m, \mathcal{L}}(G \times \hat{G}) \subset S_{\rho, \delta}^{-m_p, \mathcal{L}}(G \times \hat{G})$ for $m > m_p$. Let us consider the complex family of operators indexed by $z \in \mathbb{C}$, $\mathfrak{Re}(z) \in [0, 1]$, given by

$$T_z := \text{Op}(\sigma_z), \quad \sigma_z(x, \xi) := e^{z^2} \sigma(x, \xi)\hat{\mathcal{M}}(\xi)^{Q\rho}(z^{-1}).$$

The family of operators $\{T_z\}$ defines an analytic family of operator from $\mathfrak{Re}(z) \in (0, 1)$, (resp. continuous for $\mathfrak{Re}(z) \in [0, 1]$), into the algebra of bounded operators on $L^2(G)$. Let us observe that $\sigma_0(x, \xi) = \sigma(x, \xi)\hat{\mathcal{M}}(\xi)^m\mathcal{M}^{Q\rho}$, and $\sigma_1(x, \xi) = e\sigma(x, \xi)\hat{\mathcal{M}}(\xi)^m$. Because $T_0$ is bounded from $L^\infty(G)$ into $\text{BMO}^\mathcal{L}(G)$ and $T_1$ is bounded on $L^2(G)$, the Fefferman-Stein interpolation theorem implies that $T_t$ extends to a bounded operator on $L^p(G)$, for $p = \frac{2}{t}$ and all $0 < t \leq 1$. Because $0 \leq m \leq \frac{Q\rho}{2}$, there exist $t_0 \in (0, 1)$ such that $m = m_p = \frac{Q\rho}{2}(1 - t_0)$. So, $T_{t_0} = e^{t_0}A$ extends to a bounded operator on $L^\frac{2}{t_0}$. The Fefferman-Stein interpolation theorem, the $L^2(G)$-boundedness and the $L^\frac{2}{t_0}$-boundedness of $A$ give the $L^p(G)$-boundedness of $A$ for all $2 \leq p \leq \frac{2}{t_0}$, and interpolating the $L^\frac{2}{t_0}(G)$-boundedness with the $L^\infty(G)$-$\text{BMO}^\mathcal{L}(G)$ boundedness of $A$ we obtain the boundedness of $A$ on $L^p(G)$ for all $\frac{2}{t_0} \leq p < \infty$. So, $A$ extends to a bounded operator on $L^p(G)$ for all $2 \leq p < \infty$. The $L^p(G)$-boundedness of $A$ for $1 < p \leq 2$ now follows by the duality argument.  

Corollary 6.10. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying
the Hörmander condition of order $\kappa$. For $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, let us consider a continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G})$, $m \geq 0$. If $s \in \mathbb{R}$, then $A$ extends to a bounded operator from $L^p_s(G)$ into $L^p_s(G)$, and also from $B^s_{p,q}(G)$ into $B^s_{p,q}(G)$, for all $1 < p < \infty$, and $0 < q < \infty$, provided that
\[ m \geq m_p := Q(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|. \]

**Proof.** Let us assume that $s \in \mathbb{R}$. Note that
\[ \|Af\|_{L^p_s(G)} = \|\mathcal{M}_s A \mathcal{M}_{-s} \mathcal{M}_s f\|_{L^p(G)}. \]

Because $\mathcal{M}_s \in \text{Op}(S^{s,\mathcal{L}}_{1/\kappa,0}(G \times \hat{G}))$, $A \in \text{Op}(S^{-m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G}))$, and $\mathcal{M}_{-s} \text{Op}(\mathcal{S}^{-s,\mathcal{L}}_{1/\kappa,0}(G \times \hat{G}))$, we have that
\[ A_s := \mathcal{M}_s A \mathcal{M}_{-s} \in \text{Op}(S^{-m,\mathcal{L}}_{\min\{\rho,1/\kappa\},\delta}(G \times \hat{G})), \]

and from Theorem 6.6, we deduce that $A_s$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. Consequently we deduce the estimate
\[ \|Af\|_{L^p_s(G)} \leq \|A_s\|_{\mathcal{B}(L^p(G))} \|f\|_{L^p_s(G)}. \]
So, $A$ extends to a bounded operator from $L^p_s(G)$ into $L^p_s(G)$. In a similar way it can be proved that $A$ extends to a bounded operator from $L^p_s(G)$ into $L^p_{-s}(G)$ for all $1 < p < \infty$. The interpolation argument as in Corollary 5.6 allows us the deduce the corresponding boundedness result on Besov spaces. Thus, we finish the proof. \(\square\)

**Corollary 6.11.** Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$. For $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, let us consider a continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G})$, $m \in \mathbb{R}$. If $s \in \mathbb{R}$, then $A$ extends to a bounded operator from $L^2_s(G)$ into $L^2_{-s}(G)$.

**Proof.** Observe that $\mathcal{M}_{-m}A$ extends to a bounded operator on $L^2(G)$ in view of the subelliptic Calderón-Vaillancourt Theorem (Theorem 4.29). So, from Corollary 6.11 applied to $p = 2$ we have $A \mathcal{M}_{-m}$ extends to a bounded operator from $L^2_s(G)$ into $L^2_s(G)$ which is equivalent to the boundedness of $A$ from $L^2_s(G)$ into $L^2_{-s}(G)$. Thus, we finish the proof. \(\square\)

7. **Ellipticity in the context of the subelliptic calculus:**

**Construction of parametrices and regularisation of traces**

In this section we will study the notion of ellipticity associated to the subelliptic calculus. As in the theory of pseudo-differential operators on compact manifolds (see Hörmander [80]) the ellipticity notion can be applied to study some singularity order appearing in heat traces and regularisation of traces (see Wodzicki [127] and Kontsevich and Vishik [84]). So, we will study the analogy of such kind of traces for subelliptic operators.
7.1. Construction of parametrices. Now, we will present a technical result about the existence of parametrices for $L$-elliptic operators (see Definition 4.18) in the subelliptic calculus. We denote $S^{-\infty,L}(G \times \hat{G}) = \cap_{m \in \mathbb{R}} S^{m,L}_{\rho,\delta}(G \times \hat{G})$.

**Proposition 7.1.** Let $m \in \mathbb{R}$, and let $0 \leq \delta < \rho \leq 1$. Let $a = a(x, \xi) \in S^{m,L}_{\rho,\delta}(G \times \hat{G})$. Assume also that $a(x, \xi)$ is invertible for every $(x, [\xi]) \in G \times \hat{G}$, and satisfies

$$\sup_{(x,[\xi]) \in G \times \hat{G}} \| \hat{M}(\xi)^m a(x, \xi)^{-1} \|_{\text{op}} < \infty. \quad (7.1)$$

Then, there exists $B \in S^{-m,L}_{\rho,\delta}(G \times \hat{G})$, such that $AB - I, BA - I \in S^{-\infty,L}(G \times \hat{G})$. Moreover, the symbol of $B$ satisfies the following asymptotic expansion

$$\hat{B}(x, \xi) \sim \sum_{N=0}^{\infty} \hat{B}_N(x, \xi), \quad (x, [\xi]) \in G \times \hat{G}, \quad (7.2)$$

where $\hat{B}_N(x, \xi) \in S^{-m-(\rho-\delta)N,L}_{\rho,\delta}(G \times \hat{G})$ obeys to the recursive formula

$$\hat{B}_N(x, \xi) = -a(x, \xi)^{-1} \left( \sum_{k=0}^{N-1} \left( \sum_{|\gamma|=N-k} (\Delta^\gamma a(x, \xi))(\partial^\gamma X) \hat{B}_k(x, \xi) \right) \right), \quad N \geq 1, \quad (7.3)$$

with $\hat{B}_0(x, \xi) = a(x, \xi)^{-1}$.

**Proof.** The idea is to find a symbol $\hat{B}$ such that if $\mathcal{I} = AB$, then $\mathcal{I} - I$ is a smoothing operator, where

$$\hat{I}(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} (\Delta_{\hat{q}_\alpha} a(x, \xi))(\partial^\alpha X) \hat{B}(x, \xi).$$

The asymptotic expansion means that for every $N \in \mathbb{N}$,

$$\Delta_{\hat{x}_\alpha} \hat{X}^\beta (\hat{I}(x, \xi) - \sum_{|\alpha| \leq N} (\Delta_{\hat{q}_\alpha} a(x, \xi))(\partial^\alpha X) \hat{B}(x, \xi)) \in S^{-\rho-(\rho-\delta)N,L}_{\rho,\delta}(G \times \hat{G}),$$

for every $\alpha \in \mathbb{N}_0$ of order $\ell \in \mathbb{N}_0$, where $\hat{B}$ is requested to satisfy the asymptotic expansion (7.2). So, formally we can write

$$\hat{I}(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} (\Delta_{\hat{q}_\alpha} a(x, \xi))(\partial^\alpha X) \hat{B}(x, \xi) = \sum_{|\alpha|=0}^{\infty} \sum_{N=0}^{\infty} (\Delta_{\hat{q}_\alpha} a(x, \xi))(\partial^\alpha X) \hat{B}_N(x, \xi).$$

Observe that the fact that $\hat{B}_0(x, \xi) \in S^{-m,L}_{\rho,\delta}(G \times \hat{G})$ follows from Corollary 4.17. Now, one can check easily that $\hat{B}_N(x, \xi) \in S^{-m-(\rho-\delta)N,L}_{\rho,\delta}(G \times \hat{G})$, for all $N \geq 1$ by using induction. Consequently,

$$\hat{B}(x, \xi) = \sum_{j=0}^{N-1} \hat{B}_j(x, \xi) \in S^{-m-(\rho-\delta)N,L}_{\rho,\delta}(G \times \hat{G}).$$
This analysis allows us to deduce that
\[
\hat{T}(x, \xi) - \sum_{k=0}^{N-1} \sum_{|\gamma|<N} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) \in S_{\rho, \delta}^{-(p-\delta)N, L}(G \times \widehat{G}).
\]

On the other hand, observe that
\[
\sum_{k=0}^{N-1} \sum_{|\gamma|<N} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) = I_{d_{k}} + \sum_{k=1}^{N-1} \left( a(x, \xi) \widehat{B}_k(x, \xi) + \sum_{|\gamma|\leq N, |\gamma|\geq 1} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) \right)
\]
\[
= I_{d_{k}} + \sum_{k=1}^{N-1} \left( a(x, \xi) \widehat{B}_k(x, \xi) + \sum_{|\gamma|=N-j, j<k} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) \right)
\]
\[
+ \sum_{|\gamma|+j\geq N, |\gamma|<N, j<N} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)),
\]
where we have used that
\[
a(x, \xi) \widehat{B}_k(x, \xi) + \sum_{|\gamma|=k-j, j<k} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) = 0,
\]
in view of (7.3). Because, for $|\gamma|+j \geq N$, $(\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) \in S_{\rho, \delta}^{-(p-\delta)N, L}(G \times \widehat{G})$, it follows that
\[
\sum_{k=0}^{N-1} \sum_{|\gamma|<N} (\Delta^\gamma_x a(x, \xi))(\partial^\gamma_x \widehat{B}_k(x, \xi)) - I_{d_{k}} \in S_{\rho, \delta}^{-(p-\delta)N, L}(G \times \widehat{G}),
\]
and consequently, \( \hat{T}(x, \xi) - I_{d_{k}} \in S_{\rho, \delta}^{-(p-\delta)N, L}(G \times \widehat{G}) \), for every $N \in \mathbb{N}$. So, we have proved that $AB - I \in S^{-\infty, L}(G \times \widehat{G})$. A similar analysis can be used to prove that $BA - I \in S^{-\infty, L}(G \times \widehat{G})$. \(\square\)

7.2. Parameter $L$-ellipticity with respect to an analytic curve in the complex plane. To develop the subelliptic functional calculus we need a more wide notion of ellipticity. By following the approach in [111], we present in our subelliptic context the notion of parameter $L$-ellipticity.

**Definition 7.2.** Let $m > 0$, and let $0 \leq \delta < \rho \leq 1$. Let $\Lambda = \{\gamma(t) : t \in I\}$

\[
\text{be an analytic curve in the complex plane } \mathbb{C}. \text{ If } I \text{ is a finite interval we assume that } \Lambda \text{ is a closed curve. For simplicity, if } I \text{ is an infinite interval we assume that } \Lambda \text{ is homotopy equivalent to the line } \Lambda_{x_{\mathbb{R}}} := \{iy : -\infty < y < \infty\}. \text{ Let } a = a(x, \xi) \in S_{\rho, \delta}^{m, L}(G \times \widehat{G}). \text{ Assume also that } R_{\Lambda}(x, \xi)^{-1} := a(x, \xi) - \lambda \text{ is}
\]

\[\text{where } I = [a, b], -\infty < a \leq b < \infty, I = [a, \infty), I = (-\infty, b) \text{ or } I = (-\infty, \infty).\]

\[\text{We have denoted } a(x, \xi) - \lambda := a(x, \xi) - \lambda I_{d_{k}} \text{ to simplify the notation.}\]
invertible for every \((x, [\xi]) \in G \times \hat{G}\), and \(\lambda \in \Lambda\). We say that \(a\) is parameter \(\mathcal{L}\)-elliptic with respect to \(\Lambda\), if
\[
\sup_{\lambda \in \Lambda} \sup_{(x, [\xi]) \in G \times \hat{G}} \| (|\lambda|^\frac{1}{m} \hat{\mathcal{M}}(\xi) + \hat{\mathcal{M}}(\xi))^{m} R_{\lambda}(x, \xi) \|_{\text{op}} < \infty.
\]

**Remark 7.3.** Observe that for \(a = b = 0\), \(I = \{0\}\), and for the trivial curve \(\gamma(t) = 0\), that \(a\) is parameter \(\mathcal{L}\)-elliptic with respect to \(\Lambda = \{0\}\), is equivalent to say that \(a\) is \(\mathcal{L}\)-elliptic.

The following theorem classifies the matrix resolvent \(R_{\lambda}(x, \xi)\) of a parameter \(\mathcal{L}\)-elliptic symbol \(a\).

**Theorem 7.4.** Let \(m > 0\), and let \(0 \leq \delta < \rho \leq 1\). If \(a\) is parameter \(\mathcal{L}\)-elliptic with respect to \(\Lambda\), the following estimate
\[
\sup_{\lambda \in \Lambda} \sup_{(x, [\xi]) \in G \times \hat{G}} \| (|\lambda|^\frac{1}{m} + \hat{\mathcal{M}}(\xi))^{m(k+1)} \hat{\mathcal{M}}(\xi)^{\rho|\alpha| - \delta|\beta|} \partial_{\lambda}^{k} \hat{\mathcal{S}}(\beta) \Delta_{\xi}^{\alpha} R_{\lambda}(x, \xi) \|_{\text{op}} < \infty,
\]
holds true for all \(\alpha, \beta \in \mathbb{N}_{0}^{n}\) and \(k \in \mathbb{N}_{0}\).

**Proof.** We will split the proof in the cases \(|\lambda| \leq 1\), and \(|\lambda| > 1\), where \(\lambda \in \Lambda\). It is possible however that one of these two cases could be trivial in the sense that \(\Lambda_{1} := \{\lambda \in \Lambda : |\lambda| \leq 1\}\) or \(\Lambda_{1}^{c} := \{\lambda \in \Lambda : |\lambda| > 1\}\) could be empty sets. In such a case the proof is self-contained in the situation that we will consider where we assume that \(\Lambda_{1}\) and \(\Lambda_{1}^{c}\) are not trivial sets. For \(|\lambda| \leq 1\), observe that
\[
\| (|\lambda|^\frac{1}{m} + \hat{\mathcal{M}}(\xi))^{m(k+1)} \hat{\mathcal{M}}(\xi)^{\rho|\alpha| - \delta|\beta|} \partial_{\lambda}^{k} \hat{\mathcal{S}}(\beta) \Delta_{\xi}^{\alpha} R_{\lambda}(x, \xi) \|_{\text{op}} = \| (|\lambda|^\frac{1}{m} \hat{\mathcal{M}}(\xi))^{m(k+1)} \hat{\mathcal{M}}(\xi)^{\rho|\alpha| - \delta|\beta|} \partial_{\lambda}^{k} \hat{\mathcal{S}}(\beta) \Delta_{\xi}^{\alpha} R_{\lambda}(x, \xi) \|_{\text{op}}.
\]

Note that
\[
\| (|\lambda|^\frac{1}{m} + \hat{\mathcal{M}}(\xi))^{m(k+1)} \hat{\mathcal{M}}(\xi)^{m(k+1)} \|_{\text{op}} = \| (|\lambda|^\frac{1}{m} \hat{\mathcal{M}}(\xi))^{m(k+1)} \|_{\text{op}} \leq \| |\lambda|^\frac{1}{m} \hat{\mathcal{M}}(\xi) \|_{\text{op}}^{m(k+1)} \leq \sup_{|\lambda| \leq 1} \sup_{1 \leq j \leq d_{\xi}} \| |\lambda|^\frac{1}{m} \hat{\mathcal{M}}(\xi) \|_{\text{op}}^{m(k+1)} \leq O(1).
\]

On the other hand, we can prove that
\[
\| \hat{\mathcal{M}}(\xi)^{m(k+1)} \partial_{\lambda}^{k} \hat{\mathcal{S}}(\beta) \Delta_{\xi}^{\alpha} R_{\lambda}(x, \xi) \|_{\text{op}} = O(1).
\]

For \(k = 1\), \(\partial_{\lambda} R_{\lambda}(x, \xi) = R_{\lambda}(x, \xi)^{2}\). This can be deduced from the Leibniz rule, indeed,
\[
0 = \partial_{\lambda}(R_{\lambda}(x, \xi)(a(x, \xi) - \lambda)) = (\partial_{\lambda} R_{\lambda}(x, \xi))(a(x, \xi) - \lambda) + R_{\lambda}(x, \xi)\partial_{\lambda}(a(x, \xi) - \lambda) = (\partial_{\lambda} R_{\lambda}(x, \xi))(a(x, \xi) - \lambda) + R_{\lambda}(x, \xi)(-1)
\]
implies that
\[-\partial_\lambda (R_\lambda(x, \xi))(a(x, \xi) - \lambda) = R_\lambda(x, \xi)(-1).\]
Because \((a(x, \xi) - \lambda) = R_\lambda(x, \xi)^{-1}\) the identity for the first derivative of \(R_\lambda\), \(\partial_\lambda R_\lambda\) it follows. So, from the chain rule we obtain that the term of higher order expanding the derivative \(\partial^k_\lambda R_\lambda\) is \(R_\lambda^{k+1}\). From Corollary 4.17 we deduce that \(R_\lambda \in S^{-m,\mathcal{L}}(G \times \hat{G})\). The subelliptic calculus implies that \(R_\lambda^{k+1} \in S^{-m(k+1),\mathcal{L}}(G \times \hat{G})\).

This fact, and the compactness of \(\Lambda_{\mathcal{L}}\), provide us the uniform estimate
\[
\sup_{\lambda \in \Lambda_{\mathcal{L}}} \sup_{(x,\xi) \in G \times \hat{G}} \|\hat{M}(\xi)^{m(k+1)+\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi R_\lambda(x, \xi)\|_{\text{op}} < \infty.
\]

Now, we will analyse the situation for \(\lambda \in \Lambda_{\mathcal{L}}\). We will use induction over \(k\) in order to prove that
\[
\sup_{\lambda \in \Lambda_{\mathcal{L}}} \sup_{(x,\xi) \in G \times \hat{G}} \|\hat{M}(\xi)^m(a(x, \xi) - \lambda)^{-1}\|_{\text{op}} < \infty.
\]

For \(k = 0\) notice that
\[
\|\hat{M}(\xi)^m(a(x, \xi) - \lambda)^{-1}\|_{\text{op}} = \|\hat{M}(\xi)^m\hat{M}(\xi)^{m(\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi R_\lambda(x, \xi)\|_{\text{op}}.
\]

and denoting \(\theta = \frac{1}{|\lambda|}, \omega = \frac{1}{|\Lambda_{\mathcal{L}}|},\) we have
\[
\|\hat{M}(\xi)^m(a(x, \xi) - \lambda)^{-1}\|_{\text{op}} \leq \|\hat{M}(\xi)^m\hat{M}(\xi)^{m(\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi R_\lambda(x, \xi)\|_{\text{op}}.
\]

Because \((1 + \theta \hat{M}(\xi)^m)\hat{M}(\xi)^{m-\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi R_\lambda(x, \xi)\|_{\text{op}} is uniformly bounded in \(\theta \in [0, 1]\). The same argument can be applied to the operator norm
\[
\|\hat{M}(\xi)^{m(\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi R_\lambda(x, \xi)\|_{\text{op}},
\]

by using that \((\theta \lambda a(x, \xi) - \omega)^{-1} \in S_{\rho,\delta}^{-m,\mathcal{L}}(G \times \hat{G})\), with \(\theta \in [0, 1]\) and \(\omega\) being an element of the complex circle. The case \(k \geq 1\) for \(\lambda \in \Lambda_{\mathcal{L}}\) can be proved in a analogous way.

Combining Proposition 7.1 and Theorem 7.4 we obtain the following corollaries.

**Corollary 7.5.** Let \(m > 0\), and let \(0 \leq \delta < \rho \leq 1\). Let \(a\) be a parameter \(\mathcal{L}\)-elliptic symbol with respect to \(\Lambda\). Then there exists a parameter-dependent parametrix of \(A - \lambda I\), with symbol \(a^{-\#}(x, \xi, \lambda)\) satisfying the estimates
\[
\sup_{\lambda \in \Lambda} \sup_{(x,\xi) \in G \times \hat{G}} \|\hat{M}(\xi)^{m(k+1)}\hat{M}(\xi)^{m(\rho|\alpha|-\delta|\beta|\partial^k_\lambda \sigma(\beta) \Delta^\alpha_\xi a^{-\#}(x, \xi, \lambda)\|_{\text{op}} < \infty,
\]
for all $\alpha, \beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$.

**Corollary 7.6.** Let $m > 0$, and let $a \in S_{\rho,\delta}^{m,L}(G \times \hat{G})$ where $0 \leq \delta < \rho \leq 1$. Let us assume that $\Lambda$ is a subset of the $L^2$-resolvent set of $A$, $\text{Resolv}(A) := \mathbb{C} \setminus \text{Spec}(A)$. Then $A - \lambda I$ is invertible on $\mathcal{D}'(G)$ and the symbol of the resolvent operator $\mathcal{R}_\lambda := (A - \lambda I)^{-1}$, $\mathcal{R}_\lambda(x, \xi)$ belongs to $S_{\rho,\delta}^{m,L}(G \times \hat{G})$.

### 7.3. Asymptotic expansions for regularised traces of $\mathcal{L}$-elliptic global pseudo-differential operators

In this subsection we will study the trace for the heat semigroup associated with $\mathcal{L}$-elliptic positive left-invariant operators and also regularised traces of subelliptic operators. So, we start with the following Pleijel type formula.

**Theorem 7.7.** Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. For $0 \leq \rho \leq 1$, let us consider a positive left-invariant $\mathcal{L}$-elliptic continuous linear operator $A : C_c^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho,\delta}^{m,L}(\hat{G})$, $m > 0$. If $A$ commutes with $\mathcal{L}$, then the heat trace of $A$ has an asymptotic behaviour of the form

$$\text{Tr}(e^{-tA}) \sim c_{m,Q} t^{-\frac{Q}{m}} \times \int_{t^{\frac{1}{m}}}^\infty e^{-s^m} s^{-1} ds, \quad \forall t > 0. \quad (7.4)$$

Moreover, we have the following asymptotic expansion,

$$\text{Tr}(e^{-sA}) = s^{-\frac{Q}{m}} \left( \sum_{k=0}^\infty a_k s^{\frac{k}{m}} \right), \quad s \to 0^+. \quad (7.5)$$

**Proof.** Note that $A$ is densely defined and positive on $L^2(G)$, so it admits a self-adjoint extension. At the level of symbols, if $A$ commutes with $\mathcal{L}$, for every $[\xi] \in \hat{G}$, $\sigma(\xi)$ commutes with $\hat{\mathcal{L}}(\xi)$ and consequently, $\sigma(\xi)$ and $\hat{\mathcal{L}}(\xi)$ are simultaneously diagonalisable on every representation space. So, in a suitable basis of the representation space we can write,

$$\sigma(\xi) = \text{diag}[\sigma_{jj}(\xi)]_{j=1}^{d_\xi}, \quad \hat{\mathcal{L}}(\xi) = \text{diag}[(1 + \nu_{jj}(\xi)^2)^{\frac{1}{2}}]_{j=1}^{d_\xi},$$

where $\sigma_{jj}(\xi), 1 \leq k \leq d_\xi$, is the system of positive eigenvalues of $\sigma(\xi), [\xi] \in \hat{G}$. The spectral mapping theorem implies that

$$\text{spectrum}(e^{-tA}) = \{e^{-t\sigma_{jj}(\xi)} : 1 \leq j \leq d_\xi, \quad [\xi] \in \hat{G}\}.$$

So, we have

$$\text{Tr}(e^{-tA}) = \sum_{[\xi] \in \hat{G}} \sum_{j=1}^{d_\xi} e^{-t\sigma_{jj}(\xi)}.$$

The $\mathcal{L}$-ellipticity of $A$, implies that,

$$\sup_{1 \leq j \leq d_\xi} \sigma_{jj}(\xi)^{-1}(1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}} = \|\sigma(\xi)^{-1} \hat{\mathcal{L}}(\xi)\|_{\text{op}} \leq \sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{-1} \hat{\mathcal{L}}(\xi)\|_{\text{op}} < \infty.$$
Consequently,
\[
\inf_{1 \leq j \leq d_\xi} \sigma_{jj}(\xi)(1 + \nu_{jj}(\xi)^2)^{-\frac{m}{2}} \geq \sup_{[\xi] \in G} \| \sigma(\xi)^{-1}\mathcal{M}(\xi)^m \|_{op}^{-1}.
\]

Now, observe that from the hypothesis $\sigma \in S^m_{p,\mathcal{L}}(\mathcal{G})$ we have,
\[
\sup_{1 \leq j \leq d_\xi} \sigma_{jj}(\xi)(1 + \nu_{jj}(\xi)^2)^{-\frac{m}{2}} \leq \sup_{[\xi] \in G} \| \sigma(\xi)^{-1}\mathcal{M}(\xi)^m \|_{op}.
\]

These inequalities reduce the problem of computing the trace $\text{Tr}(e^{-tA})$ to compute $\text{Tr}(e^{-t(1+\mathcal{L})\frac{m}{2}})$. Indeed,
\[
\text{Tr}(e^{-tA}) = \sum_{[\xi] \in G} \sum_{j=1}^{d_\xi} e^{-t\sigma_{jj}(\xi)} = \sum_{[\xi] \in G} \sum_{j=1}^{d_\xi} e^{-t\sigma_{jj}(\xi)(1 + \nu_{jj}(\xi)^2)^{-\frac{m}{2}}(1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}}}
\]
\[
= \sum_{[\xi] \in G} \sum_{j=1}^{d_\xi} e^{-t(1+\nu_{jj}(\xi)^2)^{\frac{m}{2}}} = \text{Tr}(e^{-t(1+\mathcal{L})\frac{m}{2}}).
\]

Now, we will use the Weyl-law for the sub-Laplacian (see e.g. Remark 4.21). Observe that,
\[
\text{Tr}(e^{-t(1+\mathcal{L})\frac{m}{2}}) = \sum_{k=0}^{\infty} \sum_{[\xi]: 2^k \leq (1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}} < 2^{k+1}, \forall 1 \leq j \leq d_\xi} \sum_{j=1}^{d_\xi} e^{-t(1+\nu_{jj}(\xi)^2)^{\frac{m}{2}}}.
\]

Because,
\[
\sum_{[\xi]: 2^k \leq (1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}} < 2^{k+1}, \forall 1 \leq j \leq d_\xi} \sum_{j=1}^{d_\xi} e^{-t(1+\nu_{jj}(\xi)^2)^{\frac{m}{2}}} = \sum_{[\xi]: 2^k \leq (1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}} < 2^{k+1}, \forall 1 \leq j \leq d_\xi} d_\xi e^{-2km},
\]
we have
\[
\text{Tr}(e^{-t(1+\mathcal{L})\frac{m}{2}}) = \sum_{k=0}^{\infty} e^{-2km} \sum_{[\xi]: 2^k \leq (1 + \nu_{jj}(\xi)^2)^{\frac{m}{2}} < 2^{k+1}, \forall 1 \leq j \leq d_\xi} \sum_{j=1}^{d_\xi}
\]
\[
= \sum_{k=0}^{\infty} e^{-2km} N(2^k) = \sum_{k=0}^{\infty} e^{-2km} 2^k Q
\]
\[
= \sum_{k=0}^{\infty} e^{-2km} 2^{k(Q-1)2^k}.
\]

Observe that
\[
\sum_{k=0}^{\infty} e^{-2km} 2^{k(Q-1)2^k} = \int_{1}^{\infty} e^{-t\lambda^m} \lambda^{Q-1} d\lambda = t^{-\frac{Q}{m}} \int_{t^{-\frac{Q}{m}}}^{\infty} e^{-s^m} s^{Q-1} ds.
\]
The condition $m > 0$, implies that $g(t) := \int_{t}^{\infty} e^{-s^m} s^{q-1} ds$, is smooth and real-analytic on $\mathbb{R}^+ := (0, \infty)$, admitting a Taylor expansion of the form

$$g(s) = \sum_{k=0}^{\infty} a_k s^k \quad s \to 0^+.$$ 

So, we have the estimate $\text{Tr}(e^{-sA}) \sim c_{m,Q} s^{-\frac{q}{m}} g(s)$, for some positive constant $c_{m,Q}$. On the other hand, we deduce that $F(s) := s^{\frac{q}{m}} \text{Tr}(e^{-sA})$ is a real-analytic function and its Taylor expansion at $s = 0$, has the form: $\sum_{k=0}^{\infty} a'_ks^k$, which implies the following expansion,

$$\text{Tr}(e^{-sA}) = s^{-\frac{q}{m}} \left( \sum_{k=0}^{\infty} a_k s^k \right), \quad s \to 0^+.$$ 

Thus, we end the proof. \qed

Remark 7.8. Observe that under the conditions of Theorem 7.7, we have

$$\text{Tr}(e^{-tA}) \sim c_{m,Q,t} t^{-\frac{q}{m}}, \quad \forall t > 0, \quad (7.6)$$

where $c_{m,Q,t} := c_{m,Q} \int_{t^m}^{\infty} e^{-s^m} s^{q-1} ds$. For $t \to \infty$, $c_{m,Q,t} \to 0^+$, and in general,

$$0 < c_{m,Q,t} \leq \int_{0}^{\infty} e^{-s^m} s^{q-1} ds = o(1), \quad 0 \leq t < \infty.$$ 

So, (7.4) implies the following estimate

$$\text{Tr}(e^{-tA}) \sim c_{m,Q} t^{-\frac{q}{m}}.$$ 

Now, we study other kind of singularities appearing in traces of the form $\text{Tr}(A e^{-t(1+L)})^{\frac{q}{m}}$. To illustrate the importance in computing such traces let us recall an interesting situation that comes from spectral geometry. If $M$ is an orientable and compact manifold without boundary, and if $E$ is a positive elliptic pseudo-differential operator of order $q > 0$, for every elliptic and positive pseudo-differential operator $A$ with order $m$, $m \geq -\dim(M)$, we have

$$\text{Tr}(A e^{-tL}) \sim t^{-\frac{m+\dim(M)}{q}} \sum_{k=0}^{\infty} a_k t^k - \frac{b_0}{q} \log(t) + O(1). \quad (7.7)$$

If $m > -\dim(M)$, $b_0 = 0$, and for $m = -\dim(M)$, $a_k = 0$ for every $k$, and $b_0 = \text{res}(A)$ is the Wodzicki residue of $A$, see e.g. Wodzicki [127] and Lesch [85].

Let us recall that a matrix $M \in \mathbb{K}^{\ell \times \ell}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is positive if

$$(Mv, v)_{\mathbb{K}^\ell} \geq 0, \quad \forall v \in \mathbb{K}^\ell.$$ 

Now, we will compute an analogy of (7.7) for subelliptic operators.
Theorem 7.9. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $L = L_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition of order $\kappa$. For $0 \leq \delta, \rho \leq 1$, let us consider an $\mathcal{L}$-elliptic continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G})$, $m \in \mathbb{R}$. Let $\mathcal{M}_q = (1 + \mathcal{L})^{\frac{q}{2}}$ be the subelliptic Bessel potential of order $q > 0$. If $\sigma(x, [\xi]) \geq 0$, for every $(x, [\xi]) \in G \times \hat{G}$, then

$$\text{Tr}(A e^{-t\mathcal{M}_q}) \sim c_{m, q} t^{-\frac{Q - m}{q}} \times \int_1^\infty e^{-s^m} s^{Q + m - 1} ds, \ \forall t > 0. \quad (7.8)$$

In particular, for $m = -Q$, we have

$$\text{Tr}(A e^{-t\mathcal{M}_q}) \sim -\frac{CQ}{q} \log(t), \ \forall t \in (0, 1), \quad (7.9)$$

while for $m > -Q$, we have the asymptotic expansion

$$\text{Tr}(A e^{-t\mathcal{M}_q}) = t^{-\frac{Q - m}{q}} \left( \sum_{k=0}^\infty a_k t^k \right), \quad t \to 0^+. \quad (7.10)$$

Proof. We will follow the same approach as in Theorem 7.7. Because the trace of $A e^{-t\mathcal{M}_q}$ is the integral of its Schwartz kernel over the diagonal (this is a consequence of the main results in [50]), we have

$$\text{Tr}(A e^{-t\mathcal{M}_q}) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}[\sigma(x, \xi) e^{-t\hat{M}(\xi)^q}] dx$$

$$= \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}[\sigma(x, \xi) \hat{M}(\xi)^{-m} \hat{M}(\xi)^m e^{-t\hat{M}(\xi)^q}] dx. \quad (7.11)$$

In a suitable basis of the representation space we can diagonalise the operator $\sigma(x, \xi) \hat{M}(\xi)^{-m}$, and we can write in a such basis,

$$\sigma(x, \xi) \hat{M}(\xi)^{-m} = \text{diag}[\lambda_{jj}(x, \xi)]_{j=1}^{d_\xi}, \quad \hat{M}(\xi)^m e^{-t\hat{M}(\xi)^q} = [\Omega_{ij,t}(\xi)]_{i,j=1}^{d_\xi}. \quad (7.11)$$

Now, we can write

$$\text{Tr}(A e^{-t\mathcal{M}_q}) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}[\sigma(x, \xi) \hat{M}(\xi)^{-m} \hat{M}(\xi)^m e^{-t\hat{M}(\xi)^q}] dx$$

$$= \sum_{[\xi] \in \hat{G}} \sum_{j,j'=1}^{d_\xi} d\xi \int_G \lambda_{jj'}(x, \xi) dx \Omega_{jj',t}(\xi).$$
The $\mathcal{L}$-ellipticity of $A$ and the positivity of its symbol, imply that

$$
\sup_{(x,\xi)\in G} \|\sigma(x,\xi)^{-1}\widehat{\mathcal{M}}(\xi)^{m}\|_{op}^{-1} \leq \inf_{x\in G} \inf_{j,j',\xi\in G} \lambda_{j,j'}(x,\xi)
$$

$$
\leq \sup_{x\in G} \sup_{j,j',\xi\in G} \lambda_{j,j'}(x,\xi)
$$

$$
= \sup_{(x,\xi)\in G} \|\sigma(x,\xi)\widehat{\mathcal{M}}(\xi)^{-m}\|_{op},
$$

from which we deduce the following estimate,

$$
\text{Tr}(Ae^{-t\mathcal{M}}) = \sum_{[\xi]\in \hat{G}} \sum_{j,j'=1}^{d_\xi} d_\xi \Omega_{j,j',\xi}(\xi).
$$

Because $\widehat{\mathcal{M}}(\xi)^{m}e^{-t\widehat{\mathcal{M}}(\xi)^{m}} = [\Omega_{i,j,\xi}(\xi)]_{i,j=1}^{d_\xi}$ is a symmetric matrix written in the basis that allows to write in a diagonal form the operator $\sigma(x,\xi)\widehat{\mathcal{M}}(\xi)^{-m}$, we can find a matrix $P(\xi)$ such that

$$
\widehat{\mathcal{M}}(\xi)^{m}e^{-t\widehat{\mathcal{M}}(\xi)^{m}} = P(\xi)^{-1}\text{diag}[(1 + \nu_{jj}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{jj}(\xi)^{2})^{\frac{2}{2}}}]_{j=1}^{d_\xi} P(\xi),
$$

and

$$
\sum_{j,j'=1}^{d_\xi} \Omega_{j,j',\xi}(\xi) = \sum_{j,j'=1}^{d_\xi} [P(\xi)^{-1}\text{diag}[(1 + \nu_{ss}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{ss}(\xi)^{2})^{\frac{2}{2}}}]_{s=1}^{d_\xi} P(\xi)]_{j,j'}
$$

$$
= \sum_{j,j'=1}^{d_\xi} P(\xi)^{-1}(1 + \nu_{jj}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{jj}(\xi)^{2})^{\frac{2}{2}}} P(\xi)_{j,j'}
$$

$$
= \text{Tr}[P(\xi)^{-1}\text{diag}[(1 + \nu_{jj}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{jj}(\xi)^{2})^{\frac{2}{2}}}]_{j=1}^{d_\xi} P(\xi)] = \text{Tr}[\widehat{\mathcal{M}}(\xi)^{m}e^{-t\widehat{\mathcal{M}}(\xi)^{m}}].
$$

Now, as above, we will use the Weyl-law for the sub-Laplacian (see e.g. Remark 4.21). Observe that,

$$
\text{Tr}((1 + \mathcal{L})^{\frac{m}{2}} e^{-t(1+\mathcal{L})^{\frac{2}{2}}})
$$

$$
= \sum_{k=0}^{\infty} \sum_{[\xi]:2^{k} \leq (1 + \nu_{j,j'}(\xi)^{2})^{\frac{2}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{jj}(\xi)^{2})^{\frac{2}{2}}}.
$$

Because,

$$
\sum_{[\xi]:2^{k} \leq (1 + \nu_{j,j'}(\xi)^{2})^{\frac{2}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^{2})^{\frac{m}{2}} e^{-t(1+\nu_{jj}(\xi)^{2})^{\frac{2}{2}}}
$$

$$
= \sum_{[\xi]:2^{k} \leq (1 + \nu_{j,j'}(\xi)^{2})^{\frac{2}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi} d_\xi 2^{km} e^{-t2^{kq}},
$$
we can write

$$\text{Tr}(A e^{-tM_q}) = \sum_{k=0}^{\infty} 2^{km} e^{-t2^{k+q}} \sum_{\xi: 2^k \leq (1 + \nu_j' j'') \xi < 2^{k+1}, \forall 1 \leq j' \leq d} d\xi$$

\[= \sum_{k=0}^{\infty} 2^{km} e^{-t2^{k+q}} N(2^k) = \sum_{k=0}^{\infty} 2^{km} e^{-t2^{k+q}} 2^{kQ} \]

\[= \sum_{k=0}^{\infty} e^{-t2^{k+q}} 2^k (Q+m-1) 2^k. \]

From the estimate

$$\sum_{k=0}^{\infty} e^{-t2^{k+q}} 2^k (Q+m-1) 2^k = \int_1^{\infty} e^{-t\lambda^q} \lambda^{Q+m-1} d\lambda = t^{-\frac{Q+m}{q}} \int_0^{\frac{1}{t^{\frac{1}{q}}}} e^{-s^q} s^{Q+m-1} ds.$$ 

So, we have proved the first part of the theorem. Now, in particular, for $m = -Q$, we have

$$\text{Tr}(A e^{-tM_q}) \sim \int_{t^{\frac{1}{q}}}^{\infty} e^{-s^q} s^{-1} ds.$$ 

Observe that for $0 < t < 1$, the main contribution in the integral $\int_{t^{\frac{1}{q}}}^{\infty} e^{-s^q} s^{-1} ds$ is the integral of $G(s) := e^{-s^q} s^{-1}$, on the interval $[t^{\frac{1}{q}}, 1]$. Indeed, $\int_1^{\infty} e^{-s^q} s^{-1} ds = o(1)$ for $q > 0$. Now, we can compute

$$\int_{t^{\frac{1}{q}}}^{\infty} e^{-s^q} s^{-1} ds \sim \int_{t^{\frac{1}{q}}}^{1} s^{-1} ds = -\frac{1}{q} \log(t).$$

In the case $m > -Q$, we have that the function $g(t) = \int_t^{\infty} e^{-s^q} s^{Q+m-1} ds < \infty$, is real analytic in $[0, \infty)$, and for $t \to 0^+$, $g(t) = \sum_{k=0}^{\infty} b_k t^k$, which implies

$$\text{Tr}(A e^{-tM_q}) \sim t^{-\frac{Q+m}{q}} \left( \sum_{k=0}^{\infty} b_k' t^{\frac{k}{q}} \right), \ t \to 0^+.$$ 

So, we end the proof.

\[\square\]

Remark 7.10. Observe that we can summarise (7.9) and (7.10) by writing

$$\text{Tr}(A e^{-t(1+L)^{\frac{q}{2}}}) \sim t^{-\frac{m+Q}{q}} \sum_{k=0}^{\infty} a_k t^{\frac{k}{q}} - \frac{b_0}{q} \log(t), \ t \to 0^+, \quad (7.12)$$

for $m \geq -Q$. If $m = -Q$, then $a_k = 0$ for every $k$, and for $m > -Q$, $b_0 = 0$.\[\square\]
Example 7.11. Let us assume that \( a(x) \) is an integrable function over \( G \). Let 
\[ P = a(x)A, \]
where \( A \in \mathcal{S}^{m,l}_\rho (\hat{G}), \) \( 0 \leq \rho \leq 1 \), is a positive pseudo-differential operator of order \( m \geq -Q \). Let us assume that \( A \) is an \( \mathcal{L} \)-elliptic operator which 
commutes with \( \mathcal{L} \). Because the \( L^2 \)-trace of \( P \) is the integral of its kernel on the 
diagonal (see [50]), we have
\[
\text{Tr}(Pe^{-t(1+L)^2}) = \frac{1}{\pi} \int \sum_{\xi \in \hat{G}} \text{Tr}(a(x)\sigma(\xi)e^{-t\hat{L}_0(\xi)})dx = \int a(g)dg \times \text{Tr}(Ae^{-t(1+L)^2}).
\]
This implies that \( P \) also admits an asymptotic expansion of the form
\[
\text{Tr}(Pe^{-t(1+L)^2}) \sim t^{-\frac{m+Q}{q}} \sum_{k=0}^\infty a_k t^{k} - \frac{b_0}{q} \int a(x)dx \log(t), \quad t \to 0^+.
\]

Now, if in Theorem 7.9 we replace the role of the sub-Laplacian \( \mathcal{L} \) by using the 
Laplacian \( L_G \) on \( G \), we obtain the following corollary.

Corollary 7.12. Let \( G \) be a compact Lie group and let us denote by \( n \) its 
dimension. For \( 0 < \delta, \rho \leq 1 \), let us consider an elliptic continuous linear operator 
\( A : C^\infty(G) \to \mathcal{D}'(G) \) with symbol \( \sigma \in \mathcal{S}^{m,0}_\rho (G \times \hat{G}), \) \( m \in \mathbb{R} \). Let \( B_q = (1 + L_G)^\frac{q}{2} \) 
be the Bessel potential of order \( q > 0 \). If \( \sigma(x, [\xi]) \geq 0 \), for every \( (x, [\xi]) \in G \times \hat{G}, \) then
\[
\text{Tr}(Ae^{-tB_q}) \sim c_{m,n} t^{-\frac{n+m}{q}} \times \int e^{-s^m s^{n+m-1}} ds, \quad \forall t > 0.
\]
(7.13)

In particular, for \( m = -n \), we have
\[
\text{Tr}(Ae^{-tB_q}) \sim -\frac{c_n}{q} \log(t), \quad \forall t \in (0, 1),
\]
(7.14)
while for \( m > -n \), we have the asymptotic expansion
\[
\text{Tr}(Ae^{-tB_q}) = t^{-\frac{n+m}{q}} \left( \sum_{k=0}^\infty a_k t^{k} \right), \quad t \to 0^+.
\]
(7.15)

Remark 7.13. It is obvious that Corollary 7.12 follows from (7.7) in the case 
\( (\rho, \delta) = (1, 0) \). However, the notion of ellipticity in the Hörmander classes allows 
us to extend this kind of asymptotic expansions in the complete range \( 0 \leq \delta, \rho \leq 1 \), 
without the natural assumption \( \delta < \rho \).

Now, we will study regularised traces of the form \( \text{Tr}(A\psi(tE)) \) where \( t \in \mathbb{R}, \)
\( \psi \) is a compactly supported real-valued function and \( E \) is an \( \mathcal{L} \)-elliptic positive 
left-invariant operator of order \( q > 0 \).

Theorem 7.14. Let \( G \) be a compact Lie group and let us denote by \( Q \) the Hausdorff 
dimension of \( G \) associated to the control distance associated to the sub-
Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{ X_1, \cdots, X_k \} \) is a system of vector fields satisfying 
the Hörmander condition. For \( 0 \leq \delta, \rho \leq 1 \), let us consider an \( \mathcal{L} \)-elliptic continuous linear operator 
\( A : C^\infty(G) \to \mathcal{D}'(G) \) with symbol \( \sigma \in \mathcal{S}^{m,0}_\rho (G), m \in \mathbb{R}. \)
Let $E$ be positive $\mathcal{L}$-elliptic left-invariant operator of order $q > 0$. If $\sigma(x, [\xi]) \geq 0$, for every $(x, [\xi]) \in G \times \hat{G}$, then

$$
\text{Tr}(A\psi(tE)) \sim \frac{1}{q} \int_0^\infty \psi(s) \times \frac{ds}{s}, \; \forall t > 0,
$$

(7.16)

provided that $\psi \in L^1(\mathbb{R}_0^+, \frac{ds}{s}) \cap C_0^\infty(\mathbb{R}_0^+)$, and $m = -Q$. On the other hand, for $m > -Q$ and $\psi \in C_0^\infty(\mathbb{R}_0^+)$, we have

$$
\text{Tr}(A\psi(tE)) \sim \frac{t^{-\frac{Q+m}{q}}}{q} \int_0^\infty \psi(s) \frac{q^{m}}{s} \times \frac{ds}{s}, \; \forall t > 0.
$$

(7.17)

So, we have the asymptotic expansion

$$
\text{Tr}(A\psi(tE)) = t^{-\frac{Q+m}{q}} \left( \sum_{k=0}^\infty a_k t^k \right) + \frac{CQ}{q} \int_0^\infty \psi(s) \frac{ds}{s}, \; t \to 0^+,
$$

(7.18)

for $m \geq -Q$.

Proof. By writing the trace of $A\psi(tE)$ as the integral of its Schwartz kernel at the diagonal (see [50]), we have

$$
\text{Tr}(A\psi(tE)) = \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\sigma(x, \xi)\psi(t\hat{E}(\xi))] dx
$$

$$
= \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\sigma(x, \xi)\hat{\mathcal{M}}(\xi)^{-m}\hat{\mathcal{M}}(\xi)^m \psi(t\hat{E}(\xi))] dx
$$

$$
= \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\hat{\mathcal{M}}(\xi)^m \psi(t\hat{E}(\xi))],
$$

where the last line will be justified by using the $\mathcal{L}$-ellipticity of $A$ and the positivity of its symbol. Indeed, as in the proof of Theorem 7.9, in a suitable basis of the representation space we can diagonalise the operator $\sigma(x, \xi)\hat{\mathcal{M}}(\xi)^{-m}$, and we can write in a such basis,

$$
\sigma(x, \xi)\hat{\mathcal{M}}(\xi)^{-m} = \text{diag}[\lambda_{ij}(x, \xi)]_{j,t=1}^{d_\xi}, \; \hat{\mathcal{M}}(\xi)^m \psi(t\hat{E}(\xi)) = [\Omega_{ij,t}(\xi)]_{i,j=1}^{d_\xi}.
$$

(7.19)

Now, we can write

$$
\text{Tr}(A\psi(tE)) = \sum_{[\xi] \in \hat{G}} \sum_{j,j'=1}^{d_\xi} d_\xi \int_G \lambda_{j,j'}(x, \xi) dx \times \Omega_{j,j',t}(\xi).
$$

The $\mathcal{L}$-ellipticity of $A$ and the positivity of its symbol, implies that

$$
1 \leq \sup_{(x, [\xi]) \in G} \|\sigma(x, \xi)^{-1} \hat{\mathcal{M}}(\xi)^m\|_{op}^{-1} \leq \inf_{x \in G} \inf_{j,j', [\xi] \in \hat{G}} \lambda_{j,j'}(x, \xi)
$$

$$
\leq \sup_{(x, [\xi]) \in G} \|\sigma(x, \xi)\hat{\mathcal{M}}(\xi)^{-m}\|_{op} = \sup_{x \in G} \sup_{j,j' \in \hat{G}} \lambda_{j,j'}(x, \xi) \leq 1,
$$
and we consequently deduce the estimate,

\[ \text{Tr}(A\psi(tE)) = \sum_{[\xi] \in \hat{G}} \sum_{j,j'=1}^{d_\xi} d_\xi \Omega_{j,j',\xi}(\xi). \]

Because \( \hat{M}(\xi)^m \psi(t\hat{E}(\xi)) = [\Omega_{ij,t}(\xi)]_{i,j=1}^{d_\xi} \) is a symmetric matrix written in the basis that allows to write in a diagonal form the operator \( \sigma(x,\xi)\hat{M}(\xi)^{-m} \), we can find a matrix \( P(\xi) \) such that

\[ \hat{M}(\xi)^m \psi(t\hat{E}(\xi)) = P(\xi)^{-1} \text{diag}[\Lambda_{t,ij}(\xi)]_{j=1}^{d_\xi} P(\xi), \]

where \( \Lambda_{t,ij}(\xi) \) is the sequence of eigenvalues of the matrix \( \hat{M}(\xi)^m \psi(t\hat{E}(\xi)) \). Observe that

\[
\sum_{j,j'=1}^{d_\xi} \Omega_{j,j',\xi}(\xi) = \sum_{j,j'=1}^{d_\xi} [P(\xi)^{-1} \text{diag}[\Lambda_{t,ss}(\xi)]_{s=1}^{d_\xi} P(\xi)]_{j,j'}
\]

\[
= \sum_{j,j',s=1}^{d_\xi} P(\xi)^{-1}_{j,j'} \times \Lambda_{t,jj'}(\xi) \times P(\xi)_{j,s}
\]

\[
= \text{Tr}[P(\xi)^{-1} \text{diag}[\Lambda_{t,ij}(\xi)]_{j=1}^{d_\xi} P(\xi)] = \text{Tr}[\hat{M}(\xi)^m \psi(t\hat{E}(\xi))].
\]

Now, we will use the Weyl-law for the sub-Laplacian. Observe that in a suitable basis of the representation spaces the operator \( \hat{E}(\xi) \) is diagonal and from the \( \mathcal{L} \)-ellipticity of \( E \) and its positivity, we have

\[ t\hat{E}_{jj}(\xi) \sim t(1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}}, \quad \forall 1 \leq j \leq d_\xi. \]

So, we get,

\[
\text{Tr}(\hat{M}(\xi)^m \psi(t\hat{E}(\xi)))
\]

\[
= \sum_{k=0}^{\infty} \sum_{[\xi]:2^k \leq (1 + \nu_{jj'}(\xi)^2)^{\frac{3}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi}^{[\xi]} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}} \psi(t\hat{E}_{jj}(\xi))
\]

\[
= \sum_{k=0}^{\infty} \sum_{[\xi]:2^k \leq (1 + \nu_{jj'}(\xi)^2)^{\frac{3}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi}^{[\xi]} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}} \psi(t(1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}}).
\]

Now, we can deduce that

\[
\sum_{[\xi]:2^k \leq (1 + \nu_{jj'}(\xi)^2)^{\frac{3}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi}^{[\xi]} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}} \psi(t(1 + \nu_{jj}(\xi)^2)^{\frac{3}{2}})
\]

\[
= \sum_{[\xi]:2^k \leq (1 + \nu_{jj'}(\xi)^2)^{\frac{3}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi}^{[\xi]} d_\xi 2^{km} \psi(t2^kq),
\]
and consequently,

\[
\text{Tr}(\widehat{\mathcal{M}}(\xi)^m \psi(t\hat{E}(\xi))) = \sum_{k=0}^{\infty} 2^{km} \psi(t2^k) \sum_{[\xi]:2^k \leq 1 + \nu_{j',j}(\xi)^2 \leq 2^{k+1}, \forall 1 \leq j' \leq d} d\xi
\]

\[
= \sum_{k=0}^{\infty} 2^{km} \psi(t2^k) N(2^k) = \sum_{k=0}^{\infty} 2^{km} \psi(t2^k) 2^{kQ}
\]

\[
= \sum_{k=0}^{\infty} \psi(t2^k) 2^{k(Q+m-1)} 2^k.
\]

Estimating the sums in \(k\) as an integral, we have

\[
\sum_{k=0}^{\infty} \psi(t2^k) 2^{k(Q+m-1)} 2^k = \int_0^{\infty} \psi(t\lambda) \lambda^{Q+m-1} d\lambda
\]

\[
= \frac{t^{-\frac{Q}{q}(Q+m)}}{q} \int_0^{\infty} \psi(s) s^{\frac{Q+m}{q}} ds.
\]

In particular, for \(m = -Q\), we have

\[
\text{Tr}(\widehat{\mathcal{M}}(\xi)^m \psi(t\hat{E}(\xi))) \sim \frac{1}{q} \int_0^{\infty} \psi(s) \frac{ds}{s},
\]

provided that the compactly supported function \(\psi\) on \(\mathbb{R}_0^+\) belongs to \(L^1(\mathbb{R}_0^+, \frac{ds}{s})\).

Observe that the integral \(\int_0^{\infty} \psi(s) s^{\frac{Q+m}{q}} \times \frac{ds}{s}\) makes sense if \(\psi\) is smooth and it has compact support in \((0, \infty)\). However if \(\psi(0) \neq 0\), in order to assure that

\[
\int_0^{\infty} \psi(s) s^{\frac{Q+m}{q}} \times \frac{ds}{s} < \infty, \quad \psi \in C_0^\infty(\mathbb{R}_0^+),
\]

we require the condition \(1 - \frac{Q+m}{q} < 1\), or equivalently that \(Q + m > 0\). So, in such a situation, the function

\[
G(s) := s^{\frac{Q+m}{q}} \text{Tr}(\widehat{\mathcal{M}}(\xi)^m \psi(s\hat{E}(\xi))), \quad s > 0,
\]

is real-analytic and we can deduce the asymptotic formula (7.18). Thus, we end the proof.

\[
\square
\]

8. Subelliptic global functional calculus and applications

In this section we develop the global functional calculus for subelliptic operators. The calculus will be applied to obtaining a subelliptic Gårding inequality and for studying the Dixmier trace of subelliptic operators.
8.1. Functions of symbols vs functions of operators. Let \( a \in S_{\rho,\delta}^{m,L}(G \times \hat{G}) \) be a parameter \( L \)-elliptic symbol of order \( m > 0 \) with respect to the sector \( \Lambda \subset \mathbb{C} \). For \( A = \mathrm{Op}(a) \), let us define the operator \( F(A) \) by the (Dunford-Riesz) complex functional calculus

\[
F(A) = -\frac{1}{2\pi i} \int_{\partial \Lambda} F(z)(A - zI)^{-1}dz,
\]

where

- (CI). \( \Lambda_{\varepsilon} := \Lambda \cup \{ z : |z| \leq \varepsilon \}, \varepsilon > 0 \), and \( \Gamma = \partial \Lambda_{\varepsilon} \subset \text{Resolv}(A) \) is a positively oriented curve in the complex plane \( \mathbb{C} \).
- (CII). \( F \) is an holomorphic function in \( \mathbb{C} \setminus \Lambda_{\varepsilon} \), and continuous on its closure.
- (CIII). We will assume decay of \( F \) along \( \partial \Lambda_{\varepsilon} \) in order that the operator (8.1) will be densely defined on \( C^\infty(G) \) in the strong sense of the topology on \( L^2(G) \).

Now, we will compute the matrix-valued symbols for operators defined by this complex functional calculus.

**Lemma 8.1.** Let \( a \in S_{\rho,\delta}^{m,L}(G \times \hat{G}) \) be a parameter \( L \)-elliptic symbol of order \( m > 0 \) with respect to the sector \( \Lambda \subset \mathbb{C} \). Let \( F(A) : C^\infty(G) \to \mathcal{D}'(G) \) be the operator defined by the analytical functional calculus as in (8.1). Under the assumptions (CI), (CII), and (CIII), the matrix-valued symbol of \( F(A) \), \( \sigma_{F(A)}(x, \xi) \) is given by,

\[
\sigma_{F(A)}(x, \xi) = -\frac{1}{2\pi i} \int_{\partial \Lambda} F(z)\hat{\mathcal{R}}_z(x, \xi)dz,
\]

where \( \mathcal{R}_z = (A - zI)^{-1} \) denotes the resolvent of \( A \), and \( \hat{\mathcal{R}}_z(x, \xi) \in S_{\rho,\delta}^{-m,L}(G \times \hat{G}) \) its symbol.

**Proof.** From Corollary 7.6, we have that \( \hat{\mathcal{R}}_z(x, \xi) \in S_{\rho,\delta}^{-m,L}(G \times \hat{G}) \). Now, observe that

\[
\sigma_{F(A)}(x, \xi) = \xi(x)^*F(A)\xi(x) = -\frac{1}{2\pi i} \int_{\partial \Lambda} F(z)\xi(x)^*(A - zI)^{-1}\xi(x)dz.
\]

We finish the proof by observing that \( \hat{\mathcal{R}}_z(x, \xi) = \xi(x)^*(A - zI)^{-1}\xi(x) \), for every \( z \in \text{Resolv}(A) \).

Assumption (CIII) will be clarified in the following theorem where we show that the subelliptic calculus is stable under the action of the complex functional calculus.

**Theorem 8.2.** Let \( m > 0 \), and let \( 0 \leq \delta < \rho \leq 1 \). Let \( a \in S_{\rho,\delta}^{m,L}(G \times \hat{G}) \) be a parameter \( L \)-elliptic symbol with respect to \( \Lambda \). Let us assume that \( F \) satisfies the estimate \( |F(\lambda)| \leq C|\lambda|^s \) uniformly in \( \lambda \), for some \( s < 0 \). Then the symbol of \( F(A) \), \( \sigma_{F(A)} \in S_{\rho,\delta}^{m,s,L}(G \times \hat{G}) \) admits an asymptotic expansion of the form

\[
\sigma_{F(A)}(x, \xi) \sim \sum_{N=0}^{\infty} \sigma_{B_N}(x, \xi), \quad (x, [\xi]) \in G \times \hat{G},
\]

(8.2)
where \( \sigma_{\mathcal{R}}(x, \xi) \in S_{p, \delta}^{m, -s}(G \times \hat{G}) \) and

\[
\sigma_{B_0}(x, \xi) = -\frac{1}{2\pi i} \int_{\partial \Lambda_z} F(z)(a(x, \xi) - z)^{-1} dz \in S_{p, \delta}^{m, s}(G \times \hat{G}).
\]

Moreover,

\[
\sigma_{F(A)}(x, \xi) \equiv -\frac{1}{2\pi i} \int_{\partial \Lambda_z} F(z)a^{-\#}(x, \xi, \lambda) dz \text{ mod } S^{-x}G \times \hat{G}),
\]

where \( a^{-\#}(x, \xi, \lambda) \) is the symbol of the parametrix to \( A - \lambda I \), in Corollary 7.5.

**Proof.** First, we need to prove that the condition \( |F(\lambda)| \leq C|\lambda|^s \) uniformly in \( \lambda \), for some \( s < 0 \), is enough in order to guarantee that

\[
\sigma_{B_0}(x, \xi) := -\frac{1}{2\pi i} \int_{\partial \Lambda_z} F(z)(a(x, \xi) - z)^{-1} dz,
\]

is a well defined matrix-symbol. From Theorem 7.4 we deduce that \( (a(x, \xi) - z)^{-1} \) satisfies the estimate

\[
\|(|z|^{\frac{1}{m}} + \hat{M}(\xi))^{m(k+1)}\hat{M}(\xi)^{\rho|\alpha| - |\delta|/\beta} \hat{\rho}^k \hat{r}^l \chi_{\nu,\lambda}(a(x, \xi) - z)^{-1}\|_{op} < \infty.
\]

Observe that

\[
\| (a(x, \xi) - z)^{-1} \|_{op} = \| (|z|^{\frac{1}{m}} + \hat{M}(\xi))^{-m}(|z|^{\frac{1}{m}} + \hat{M}(\xi))^{m}(a(x, \xi) - z)^{-1}\|_{op}
\]

\[
\leq \sup_{1 \leq j \leq d_\xi} (|z|^{\frac{1}{m}} + (1 + \nu_j(\xi)^{\frac{1}{2}})^{-m})^{-1}
\]

\[
\leq |z|^{-1},
\]

and the condition \( s < 0 \) implies

\[
\left| -\frac{1}{2\pi i} \int_{\partial \Lambda_z} F(z)(a(x, \xi) - z)^{-1} dz \right| \leq \int_{\partial \Lambda_z} |z|^{-1+s}|dz| < \infty,
\]

uniformly in \( (|\xi|) \in G \times \hat{G} \). In order to check that \( \sigma_{B_0} \in S_{p, \delta}^{m, s}(G \times \hat{G}) \) let us analyse the cases \(-1 < s < 0 \) and \( s \leq -1 \) separately. So, let us analyse first the situation of \(-1 < s < 0 \). We observe that

\[
\| \hat{M}(\xi)^{-ms + \rho|\alpha| - |\delta|/\beta} \hat{\rho}^k \hat{r}^l \chi_{\nu,\lambda}(a(x, \xi) - z)^{-1}\|_{op} \leq \frac{C}{2\pi} \int_{\partial \Lambda_z} |z|^s |\hat{M}(\xi)^{-ms + \rho|\alpha| - |\delta|/\beta} \hat{\rho}^k \hat{r}^l \chi_{\nu,\lambda}(a(x, \xi) - z)^{-1}|_{op}|dz|.
\]

Now, we will estimate the operator norm inside of the integral. Indeed, the identity

\[
\| (|z|^{\frac{1}{m}} + \hat{M}(\xi))^{-m}(|z|^{\frac{1}{m}} + \hat{M}(\xi))^{m}\hat{M}(\xi)^{-ms + \rho|\alpha| - |\delta|/\beta} \hat{\rho}^k \hat{r}^l \chi_{\nu,\lambda}(a(x, \xi) - z)^{-1}\|_{op} =
\]

\[
\|
\]
implies that
\[
\left\| \hat{\mathcal{M}}(\xi)^{-m}s + \rho|\alpha|^{-\beta}c_X(\xi)^{-\beta} \Delta_\xi^\alpha (a(x, \xi) - z)^{-1} \right\|_{\text{op}} \lesssim \left\| \left( \frac{1}{m} + \hat{\mathcal{M}}(\xi) \right)^{-m} \hat{\mathcal{M}}(\xi)^{-ms} \right\|_{\text{op}}
\]
where we have used that
\[
\sup_{z \in \partial \Lambda_\xi(x, \xi)} \left( \left\| \frac{1}{m} + \hat{\mathcal{M}}(\xi) \right\|_{\text{op}} \hat{\mathcal{M}}(\xi)^{-m}s \right) < \infty.
\]
Consequently, by using that \( s < 0 \), we deduce
\[
\begin{align*}
\frac{C}{2\pi} \int_{\partial \Lambda_\xi} |z|^s \left( \left\| \hat{\mathcal{M}}(\xi)^{m}s + \rho|\alpha|^{-\beta}c_X(\xi)^{-\beta} \Delta_\xi^\alpha (a(x, \xi) - z)^{-1} \right\|_{\text{op}} \right) dz \\
\lesssim \frac{C}{2\pi} \int_{\partial \Lambda_\xi} |z|^s \left( \left\| \frac{1}{m} + \hat{\mathcal{M}}(\xi) \right\|_{\text{op}} \hat{\mathcal{M}}(\xi)^{-ms} \right) dz \\
= \frac{C}{2\pi} \int_{\partial \Lambda_\xi} |z|^s \sup_{1 \leq j \leq d_\xi} \left( \left\| \frac{1}{m} + (1 + \nu_{jj}(\xi)^2)^{\frac{1}{2}} \right\|_{\text{op}} \hat{\mathcal{M}}(\xi)^{-ms} \right) dz.
\end{align*}
\]
To study the convergence of the last contour integral we only need to check the convergence of \( \int_1^\infty r^s \left( \frac{1}{r^{m}} + \kappa \right)^{-m} \kappa^{-ms} dr \), where \( \kappa > 1 \) in a parameter. The change of variable \( r = \kappa^{-m}t \) implies that
\[
\begin{align*}
\int_1^\infty r^s \left( \frac{1}{r^{m}} + \kappa \right)^{-m} \kappa^{-ms} dr &= \int_{\kappa^{-m}}^\infty \kappa^{ms}t^s \left( \kappa t^{\frac{1}{m}} + \kappa \right)^{-m} \kappa^{-ms} dt \\
&= \int_{\kappa^{-m}}^\infty t^s \left( t^{\frac{1}{m}} + 1 \right)^{-m} dt \\
&\lesssim \int_{\kappa^{-m}}^\infty t^s dt + \int_1^\infty t^{-1+s} dt < \infty.
\end{align*}
\]
Indeed, for \( t \to \infty \), \( t^s \left( t^{\frac{1}{m}} + 1 \right)^{-m} \lesssim t^{-1+s} \) and we conclude the estimate because \( \int_1^\infty t^{-1+s} dt < \infty \), for all \( s' < 0 \). On the other hand, the condition \( -1 < s < 0 \) implies that
\[
\int_{\kappa^{-m}}^1 t^s dt = \frac{1}{1+s} - \frac{\kappa^{-m(1+s)}}{1+s} = O(1).
\]
In the case where \( s \leq -1 \), we can find an analytic function \( \tilde{G}(z) \) such that it is a holomorphic function in \( \mathbb{C} \setminus \Lambda_\xi \), and continuous on its closure and additionally satisfying that \( F(\lambda) = \tilde{G}(\lambda)^{1+[s]} \).\(^{19}\) In this case, \( \tilde{G}(A) \) defined by the complex functional calculus
\[
\tilde{G}(A) = -\frac{1}{2\pi i} \int_{\partial \Lambda_\xi} \tilde{G}(z)(A-zI)^{-1} dz, \quad (8.3)
\]
\(^{19}\) \([s]\) denotes the integer part of \(-s\).
has symbol belonging to $\mathcal{S}_{\rho,\delta}^{\alpha m}(G \times \hat{G})$ because $\hat{G}$ satisfies the estimate $|G(\lambda)| \leq C|\lambda|^{\frac{s}{1+[\alpha s]}}$, with $-1 < \frac{s}{1+[\alpha s]} < 0$. By observing that

$$\sigma_{F(A)}(x, \xi) = -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} F(z) \hat{\mathcal{R}}_z(x, \xi) dz = -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} \hat{G}(z)^{1+[\alpha s]} \hat{\mathcal{R}}_z(x, \xi) dz = \sigma_{\mathcal{G}(A)^{1+[\alpha s]}}(x, \xi),$$

and computing the symbol $\sigma_{\mathcal{G}(A)^{1+[\alpha s]}}(x, \xi)$ by iterating $1+[\alpha s]$-times the asymptotic formula for the composition in the subelliptic calculus (see Corollary 4.34), we can see that the term with higher order in such expansion is $\sigma_{\mathcal{G}(A)}(x, \xi)^{1+[\alpha s]} \in \mathcal{S}_{\rho,\delta}^{m,s,\mathcal{L}}(G \times \hat{G})$. Consequently we have proved that $\sigma_{F(A)}(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m,s,\mathcal{L}}(G \times \hat{G})$. This completes the proof for the first part of the theorem. For the second part of the proof, let us denote by $a^{-\#}(x, \xi, \lambda)$ the symbol of the parametrix to $A - \lambda I$, in Corollary 7.5. Let $P_\lambda = \text{Op}(a^{-\#}(\cdot, \cdot, \lambda))$. Because $\lambda \in \text{Resolv}(A)$ for $\lambda \in \partial \Lambda_\varepsilon$, $(A - \lambda)^{-1} - P_\lambda$ is an smoothing operator. Consequently, from Lemma 8.1 we deduce that

$$\sigma_{F(A)}(x, \xi) = -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} F(z) \hat{\mathcal{R}}_z(x, \xi) dz = -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} F(z)a^{-\#}(x, \xi, z) dz - \frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} F(z)(\hat{\mathcal{R}}_z(x, \xi) - a^{-\#}(x, \xi, z)) dz \equiv -\frac{1}{2\pi i} \int_{\partial \Lambda_\varepsilon} F(z)a^{-\#}(x, \xi, z) dz \mod \mathcal{S}^{-\infty,\mathcal{L}}(G \times \hat{G}).$$

The asymptotic expansion (8.2) came from the construction of the parametrix in the subelliptic calculus (see Proposition 7.1). \hfill \Box

8.2. Gårding inequality. In this section we will prove the Gårding inequality for the subelliptic calculus. To do so, we need some preliminary propositions.

**Proposition 8.3.** Let $0 \leq \delta < \rho \leq 1$. Let $a \in \mathcal{S}_{\rho,\delta}^{m,s,\mathcal{L}}(G \times \hat{G})$ be an $\mathcal{L}$-elliptic matrix-valued symbol where $m \geq 0$ and let us assume that $a$ is positive definite. Then $a$ is parameter-elliptic with respect to $\mathbb{R}_- := \{z = x + i0 : x < 0\} \subset \mathbb{C}$. Furthermore, for any number $s \in \mathbb{C}$,

$$\hat{B}(x, \xi) \equiv a(x, \xi)^s := \exp(s \log(a(x, \xi))), \ (x, [\xi]) \in G \times \hat{G},$$

defines a symbol $\hat{B}(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m,\text{Re}(s),\mathcal{L}}(G \times \hat{G})$.

**Proof.** From the estimates

$$\sup_{(x, [\xi])} \|\widehat{\mathcal{M}}(\xi)^{-m} a(x, \xi)\|_{\text{op}} < \infty, \quad \sup_{(x, [\xi])} \|\widehat{\mathcal{M}}(\xi)^m a(x, \xi)^{-1}\|_{\text{op}} < \infty,$$

we deduce that

$$\sup_{x \in G} \|a(x, \xi)\|_{\text{op}} = \|\widehat{\mathcal{M}}(\xi)^m\|_{\text{op}}, \quad \sup_{x \in G} \|a(x, \xi)^{-1}\|_{\text{op}} = \|\widehat{\mathcal{M}}(\xi)^{-m}\|_{\text{op}}.$$
Consequently we have
\[ \|\hat{M}(\xi)^m\|_{op}^{-1} \sup_{x \in G} \|a(x, \xi)\|_{op} = 1, \quad \|\hat{M}(\xi)^{-m}\|_{op}^{-1} \sup_{x \in G} \|a(x, \xi)^{-1}\|_{op} = 1, \]
from which we deduce that
\[ \|\hat{M}(\xi)^{-m}\|_{op}^{-1}\text{Spectrum}(a(x, \xi)) \subset [c, C], \]
where \(c, C > 0\) are positive real numbers. The matrix \(a(x, [\xi])\) is normal, so that for every \(\lambda \in \mathbb{R}_-\) we have
\[
\|(|\lambda|^m + \hat{M}(\xi))a(x, \xi) - \lambda\|_{op}
\]
\[
= \|(|\lambda|^m + \hat{M}(\xi))a(x, \xi) - \lambda\|_{op}
\]
\[
\leq \|(|\lambda|^m + \hat{M}(\xi))a(x, \xi) - \lambda\|_{op} \|(\hat{M}(\xi)^m - \lambda)(a(x, \xi) - \lambda)^{-1}\|_{op}
\]
\[
\leq \|(|\lambda|^m + \hat{M}(\xi))a(x, \xi) - \lambda\|_{op} \|(\hat{M}(\xi)^m - \lambda)(a(x, \xi) - \lambda)^{-1}\|_{op}.
\]
Let us note that the condition \(m \geq 0\), implies that \(\|\hat{M}(\xi)^{-m}\|_{op} \leq 1\). So, if \(|\lambda| \in [0, 1/2]\), then
\[ |\lambda| \|\hat{M}(\xi)^{-m}\|_{op} \leq 1/2, \]
which implies that for all \(0 \leq |\lambda| \leq 1/2\), \((1 - \lambda\hat{M}(\xi)^{-m})\) is invertible and from the first von-Neumann identity,
\[
\|(1 - \lambda\hat{M}(\xi)^{-m})\|_{op} \leq (1 - |\lambda|\hat{M}(\xi)^{-m})_{op}^{-1} = (1 - |\lambda|\hat{M}(\xi)^{-m})_{op}^{-1} \leq 2.
\]
Now, fixing again \(|\lambda| \in \mathbb{R}_-\) observe that from the compactness of \([0, 1/2]\) we deduce that
\[
\sup_{0 \leq |\lambda| \leq 1/2} \|(|\lambda|^m + \hat{M}(\xi))^m(\hat{M}(\xi)^m - \lambda)^{-1}\|_{op} = \sup_{0 \leq |\lambda| \leq 1/2} \|\hat{M}(\xi)^m(\hat{M}(\xi)^m - \lambda)^{-1}\|_{op}
\]
\[
\leq \sup_{0 \leq |\lambda| \leq 1/2} \|(I_{d_{\xi}} - \lambda\hat{M}(\xi)^{-m})^{-1}\|_{op}
\]
\[
\leq 1,
\]
where in the last line we have used the continuity of the function \(U(\lambda) := \|(I_{d_{\xi}} - \lambda\hat{M}(\xi)^{-m})^{-1}\|_{op}\), and the fact that it is bounded on \([0, 1/2]\). On the other hand,
\[
\sup_{|\lambda| \geq 1/2} \|(|\lambda|^m + \hat{M}(\xi))^m(\hat{M}(\xi)^m - \lambda)^{-1}\|_{op}
\]
\[
= \sup_{|\lambda| \geq 1/2} \|(|\lambda|^m + \hat{M}(\xi)^{-1} + I_{d_{\xi}})^m(I_{d_{\xi}} - \hat{M}(\xi)^{-m}\lambda)^{-1}\|_{op}
\]
\[
= \sup_{|\lambda| \geq 1/2} \|(\hat{M}(\xi)^{-1} + |\lambda|^{-1}mI_{d_{\xi}})^m|\lambda|(I_{d_{\xi}} - \hat{M}(\xi)^{-m}\lambda)^{-1}\|_{op}
\]
\[
\leq \sup_{|\lambda| \geq 1/2} \|\hat{M}(\xi)^{-m}|\lambda|(-\lambda)^{-1}\hat{M}(\xi)^m\|_{op}
\]
\[
= 1.
\]
So, we have proved that \(a\) is parameter-elliptic with respect to \(\mathbb{R}_-\). To prove that \(\hat{B}(x, \xi) \in S^{m, Re(s)}_{\delta, \delta}(G \times \hat{G})\), we can observe that for \(Re(s) < 0\), we can apply Theorem 8.2. If \(Re(s) \geq 0\), we can find \(k \in \mathbb{N}\) such that \(Re(s) - k < 0\) and
consequently from the spectral calculus of matrices we deduce that \( a(x, \xi)^{\text{Re}(s)-k} \in S_{\rho, \delta}^{m \times \text{Re}(s)-k, L}(G \times \hat{G}) \). So, from the calculus we conclude that
\[
a(x, \xi)^s = a(x, \xi)^{s-k}a(x, \xi)^k \in S_{\rho, \delta}^{m \times \text{Re}(s), L}(G \times \hat{G}).
\]
Thus the proof is complete. \( \square \)

**Corollary 8.4.** Let \( 0 \leq \delta < \rho \leq 1 \). Let \( a \in S_{\rho, \delta}^{m, L}(G \times \hat{G}) \), be an \( L \)-elliptic symbol where \( m \geq 0 \) and let us assume that \( a \) is positive definite. Then \( \hat{B}(x, \xi) \equiv a(x, \xi)^{\frac{1}{2}} := \exp\left(\frac{1}{2} \log(a(x, \xi))\right) \in S_{\rho, \delta}^{m, \frac{2}{L}}(G \times \hat{G}) \).

Now, let us assume that
\[
A(x, \xi) := \frac{1}{2} (a(x, \xi) + a(x, \xi)^*), (x, [\xi]) \in G \times \hat{G}, \ a \in S_{\rho, \delta}^{m, L}(G \times \hat{G}),
\]
satisfies
\[
\|\hat{\mathcal{M}}(\xi)^m A(x, \xi)^{-1}\|_{\text{op}} \leq C_0. \tag{8.4}
\]
Observe that (8.4) implies that
\[
\lambda(x, \xi) := \inf \{ \tilde{\lambda}(x, [\xi])^{-1} : \text{det}(\hat{\mathcal{M}}(\xi)^{-m} A(x, \xi) - \tilde{\lambda}(x, \xi) I_{d_x}) = 0 \} \leq C_0.
\]
So, \( \lambda(x, \xi)^{-1} \geq \frac{1}{C_0} \) and consequently
\[
\hat{\mathcal{M}}(\xi)^{-m} A(x, \xi) \geq \frac{1}{C_0} I_{d_x}.
\]
This implies that
\[
A(x, \xi) \geq \frac{1}{C_0} \hat{\mathcal{M}}(\xi)^m,
\]
and for \( C_1 \in (0, \frac{1}{C_0}) \) we have that
\[
A(x, \xi) - C_1 \hat{\mathcal{M}}(\xi)^m \geq \left( \frac{1}{C_0} - C_1 \right) \hat{\mathcal{M}}(\xi)^m > 0.
\]
If \( 0 \leq \delta < \rho \leq 1 \), from Corollary 8.4, we have that
\[
q(x, \xi)q(x, \xi)^* = A(x, \xi) - C_1 \hat{\mathcal{M}}(\xi)^m + r(x, \xi), \ r(x, \xi) \in S_{\min[\rho, 1/\kappa], \delta}^{m-(\rho-\delta), L}(G \times \hat{G}).
\]
From the symbolic calculus we obtain
\[
q(x, \xi)q(x, \xi)^* = A(x, \xi) - C_1 \hat{\mathcal{M}}(\xi)^m + r(x, \xi), \ r(x, \xi) \in S_{\min[\rho, 1/\kappa], \delta}^{m-(\rho-\delta), L}(G \times \hat{G}).
\]
Now, let us assume that \( u \in C^{\infty}(G) \). Then we have
\[
\text{Re}(a(x, D)u, u) = \frac{1}{2}((a(x, D) + \text{op}(a^*))u, u)
\]
\[
= C_1 (\mathcal{M}_n u, u) + (q(x, D)q(x, D)^*u, u) + (r(x, D)u, u)
\]
\[
= C_1 (\mathcal{M}_n u, u) + (q(x, D)^*u, q(x, D)^*u) - (r(x, D)u, u)
\]
\[
\geq C_1 \|u\|_{L^2_{\mathcal{M}}(G)}^2 - (r(x, D)u, u)
\]
\[
= C_1 \|u\|_{L^2_{\mathcal{M}}(G)}^2 - (\mathcal{M}_{m-(\rho-\delta)} r(x, D)u, \mathcal{M}_{m-(\rho-\delta)} u).
\]
Observe that
\[
(\mathcal{M}_{\frac{m}{2},(p-\delta)} r(x, D) u, \mathcal{M}_{\frac{m}{2},(p-\delta)} u) \leq \| \mathcal{M}_{\frac{m}{2},(p-\delta)} r(x, D) u \|_{L^2(G)} \| u \|_{L^2,G}(G) \\
= \| r(x, D) u \|_{L^2,G}(G) \| u \|_{L^2,G}(G) \\
\leq C_1 \| u \|_{L^2,G}(G) \| u \|_{L^2,G}(G),
\]
where in the last line we have used the subelliptic Sobolev boundedness of \( r(x, D) \) from \( L^2_{m-(p-\delta)}(G) \) into \( L^2_{m-(p-\delta)}(G) \), in view of Corollary 6.11. Consequently, we deduce the lower bound
\[
\text{Re}(a(x, D) u, u) \geq C_1 \| u \|_{L^2,G}(G) - C \| u \|_{L^2,G}(G).
\]
If we assume for a moment that for every \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \), such that
\[
\| u \|_{L^2,G}(G) \leq \varepsilon \| u \|_{L^2,G}(G) + C_\varepsilon \| u \|_{L^2,G}(G), \tag{8.5}
\]
for \( 0 < \varepsilon < C_1 \) we have
\[
\text{Re}(a(x, D) u, u) \geq (C_1 - \varepsilon) \| u \|_{L^2,G}(G) - C_\varepsilon \| u \|_{L^2,G}(G).
\]
So, with the exception of the proof of (8.5) we have deduced the following estimate which is the main result of this subsection.

**Theorem 8.5 (Subelliptic Gårding inequality).** Let \( G \) be a compact Lie group and let us denote by \( Q \) the Hausdorff dimension of \( G \) associated to the control distance associated to the sub-Laplacian \( \mathcal{L} = \mathcal{L}_X \), where \( X = \{X_1, \ldots, X_k\} \) is a system of vector fields satisfying the Hörmander condition. For \( 0 \leq \delta < \rho \leq 1 \), \( \delta < 1/\kappa \), let \( a(x, D) : C^\infty(G) \rightarrow \mathcal{D}'(G) \) be an operator with symbol \( a \in S^m_{\rho,\delta}(G \times \hat{G}) \), \( m \in \mathbb{R} \). Let us assume that
\[
A(x, \xi) := \frac{1}{2}(a(x, \xi) + a(x, \xi)^*), (x, [\xi]) \in G \times \hat{G}, \ a \in S^m_{\rho,\delta}(G \times \hat{G}),
\]
satisfies
\[
\| \hat{\mathcal{M}}(\xi)^m A(x, \xi)^{-1} \|_{\text{op}} \leq C_0.
\]
Then, there exist \( C_1, C_2 > 0 \), such that the lower bound
\[
\text{Re}(a(x, D) u, u) \geq C_1 \| u \|_{L^2,G}(G) - C_2 \| u \|_{L^2,G}(G), \tag{8.6}
\]
holds true for every \( u \in C^\infty(G) \).

In view of the analysis above, for the proof of Theorem 8.5 we only need to proof (8.5). However we will deduce it from the following more general lemma.

**Lemma 8.6.** Let us assume that \( s \geq t \geq 0 \) or that \( s, t < 0 \). Then, for every \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
\| u \|_{L^2,G}(G) \leq \varepsilon \| u \|_{L^2,G}(G) + C_\varepsilon \| u \|_{L^2,G}(G), \tag{8.7}
\]
holds true for every \( u \in C^\infty(G) \).
Proof. Let \( \varepsilon > 0 \). Then, there exists \( C_\varepsilon > 0 \) such that
\[
(1 + \nu_{ii}(\xi))^t - \varepsilon (1 + \nu_{ii}(\xi))^s \leq C_\varepsilon,
\]
uniformly in \( [\xi] \in \hat{G} \). Then (8.7) it follows from the Plancherel theorem. Indeed,
\[
\|u\|_{L^2_{\hat{G}}(G)}^2 = \sum_{[\xi] \in \hat{G}} \sum_{i,j=1}^{d_\xi} (1 + \nu_{ii}(\xi))^t |\hat{u}_{ij}(\xi)|^2
\]
\[
\leq \sum_{[\xi] \in \hat{G}} \sum_{i,j=1}^{d_\xi} (\varepsilon (1 + \nu_{ii}(\xi))^s + C_\varepsilon |\hat{u}_{ij}(\xi)|^2
\]
\[
= \varepsilon \|u\|_{L^2_{\hat{G}}(G)}^2 + C_\varepsilon \|u\|_{L^2(G)}^2,
\]
completing the proof. \( \square \)

**Corollary 8.7.** Let \( G \) be a compact Lie group and let us denote by \( Q \) the Hausdorff dimension of \( G \) associated to the control distance associated to the sub-Laplacian \( L = L_X \), where \( X = \{X_1, \ldots, X_k\} \) is a system of vector fields satisfying the Hörmander condition. Let \( a(x, D) : C^\infty(G) \to \mathcal{D}'(G) \) be an operator with symbol \( a \in S^{m,L}_\rho,\delta \) \( \hat{G} \times \hat{G} \), \( m \in \mathbb{R}, 0 \leq \delta < \rho \leq 1, \delta < 1/\kappa \). Let us assume that
\[
a(x,\xi) \geq 0, (x,[\xi]) \in G \times \hat{G},
\]
satisfies
\[
\| \hat{M}(\xi)^m a(x,\xi)^{-1} \|_{\text{op}} \leq C_0.
\]
Then, there exist \( C_1, C_2 > 0 \), such that the lower bound
\[
\text{Re}(a(x, D)u,u) \geq C_1 \|u\|_{L^2_{\hat{G}}(G)}^2 - C_2 \|u\|_{L^2(G)}^2,
\]
holds true for every \( u \in C^\infty(G) \).

### 8.3. Dixmier traces

Now, we will apply the subelliptic functional calculus to study the membership of subelliptic operators in the Dixmier ideal on \( L^2(G) \).

By following Connes [35], if \( H \) is a Hilbert space (we are interested in \( H = L^2(G) \) for instance), the class \( L^{(1,\infty)}(H) \) consists of those compact linear operators \( A \in \mathcal{L}(H) \) satisfying
\[
\sum_{1 \leq n \leq N} s_n(A) = O(\log(N)), \quad N \to \infty,
\]
where \( \{s_n(A)\} \) denotes the sequence of singular values of \( A \), i.e. the square roots of the eigenvalues of the non-negative self-adjoint operator \( A^*A \). So, \( L^{(1,\infty)}(H) \) is endowed with the norm
\[
\|A\|_{L^{(1,\infty)}(H)} = \sup_{N \geq 2} \frac{1}{\log(N)} \sum_{1 \leq n \leq N} s_n(A).
\]

We define the functional
\[
\text{Tr}_\omega(A) := \lim_{N \to \infty} \frac{1}{\log(N)} \sum_{1 \leq n \leq N} s_n(A),
\]
for the family of bounded operators $A$ in $\mathcal{L}^{(1,\infty)}(H)$ (see [35] or [51]). Our starting point is the following lemma.

**Lemma 8.8.** Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \ldots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. For $0 \leq p \leq 1$, let us consider a positive left-invariant $\mathcal{L}$-elliptic continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_p^m(\hat{G})$, $m \in \mathbb{R}$. Let us assume that $A$ commutes with $\mathcal{L}$. Then, $A$ belongs to the Dixmier ideal $\mathcal{L}^{1,\infty}(L^2(G))$ if and only if $m \leq -Q$. If $A \neq 0$, $\text{Tr}_w(A) = \frac{1}{Q}$ for $m = -Q$, and for $m < -Q$, $\text{Tr}_w(A) = 0$.

**Proof.** Let us use the positivity of $A$ computing the Dixmier trace of $A$ from the identity (see e.g. Sukochev and Usachev [117, page 35])

$$\text{Tr}_w(A) = \lim_{p \to 1^+} (p - 1)\text{Tr}(A^p).$$

At the level of symbols, if $A$ commutes with $\mathcal{L}$, for every $[\xi] \in \hat{G}$, $\sigma(\xi)$ commutes with $\hat{\mathcal{L}}(\xi)$ and consequently, $\sigma(\xi)$ and $\hat{\mathcal{L}}(\xi)$ are simultaneously diagonalisable on every representation space. So, in a suitable basis of the representation space we can write,

$$\sigma(\xi) = \text{diag}[\sigma_{jj}(\xi)]_{j=1}^{d_\xi}, \quad \hat{\mathcal{L}}(\xi) = \text{diag}[(1 + \nu_{jj}(\xi)^2)^{\frac{1}{2}}]_{j=1}^{d_\xi},$$

where $\sigma_{jj}(\xi), 1 \leq k \leq d_\xi$, is the system of positive eigenvalues of $\sigma(\xi), [\xi] \in \hat{G}$. The spectral mapping theorem implies that

$$\text{spectrum}(A^p) = \{\sigma_{jj}(\xi)^p : 1 \leq j \leq d_\xi, [\xi] \in \hat{G}\}.$$ 

So, we have

$$\text{Tr}(A^p) = \sum_{[\xi] \in \hat{G}} \sum_{j=1}^{d_\xi} \sigma_{jj}(\xi)^p,$$

The $\mathcal{L}$-ellipticity of $A$, implies that,

$$\sup_{1 \leq j \leq d_\xi} (1 + \nu_{jj}(\xi)^2)^\frac{m}{2} = \|\sigma(\xi)^{-1}\hat{\mathcal{M}}(\xi)\|_{\text{op}} \leq \sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{-1}\hat{\mathcal{M}}(\xi)\|_{\text{op}} < \infty.$$

Consequently,

$$\inf_{1 \leq j \leq d_\xi} \sigma_{jj}(\xi)(1 + \nu_{jj}(\xi)^2)^{-\frac{m}{2}} \geq \sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{-1}\hat{\mathcal{M}}(\xi)^m\|_{\text{op}}^{-1}.$$ 

Now, observe that from the hypothesis $\sigma \in S_p^m(\hat{G})$ we have,

$$\sup_{1 \leq j \leq d_\xi} \sigma_{jj}(\xi)(1 + \nu_{jj}(\xi)^2)^{-\frac{m}{2}} \leq \sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{-1}\hat{\mathcal{M}}(\xi)^{-m}\|_{\text{op}}.$$
These inequalities imply that

\[
\text{Tr}(A^p) = \sum_{[\xi]\in G} \sum_{j=1}^{d_\xi} \sigma_{jj}(\xi)^p (1 + \nu_{jj}(\xi)^2)^{-\frac{km}{Q}} (1 + \nu_{jj}(\xi)^2)^{\frac{mp}{Q}}
\]

\[
= \sum_{[\xi]\in G} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{mp}{Q}}
\]

\[
= \text{Tr}((1 + \mathcal{L})^{\frac{mp}{Q}}).
\]

Now, as in the previous section, we will use the Weyl-law for the sub-Laplacian (see Remark 4.21). Observe that,

\[
\text{Tr}(e^{-t(1+\mathcal{L})^{\frac{mp}{Q}}}) = \sum_{k=0}^{\infty} \sum_{[\xi]:2^k \leq (1+\nu_{j'j}(\xi)^2)^{\frac{1}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi} \sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{mp}{Q}}.
\]

Because,

\[
\sum_{j=1}^{d_\xi} (1 + \nu_{jj}(\xi)^2)^{\frac{mp}{Q}} = d_\xi 2^{kmp},
\]

we have

\[
\text{Tr}((1 + \mathcal{L})^{\frac{mp}{Q}}) = \sum_{k=0}^{\infty} 2^{kmp} \sum_{[\xi]:2^k \leq (1+\nu_{j'j}(\xi)^2)^{\frac{1}{2}} < 2^{k+1}, \forall 1 \leq j' \leq d_\xi} d_\xi
\]

\[
= \sum_{k=0}^{\infty} 2^{kmp} N(2^k) = \sum_{k=0}^{\infty} e^{2^{kmp}} 2^k Q
\]

\[
= \sum_{k=0}^{\infty} 2^{kmp} 2^k (Q-1) 2^k.
\]

Observe that

\[
\sum_{k=0}^{\infty} 2^{k(Q+mp-1)} 2^k = \int_{1}^{\infty} \lambda^{Q+mp-1} d\lambda < \infty,
\]

for all \( p > 1 \), if and only if \( m \leq -Q \). So, from the identity

\[
\int_{1}^{\infty} \lambda^{Q+mp-1} d\lambda = -\frac{1}{Q + mp},
\]

we deduce that

\[
\text{Tr}_w(A) = \lim_{p \to 1^+} (p - 1) \times \frac{(-1)}{Q + mp} = \delta_{m,Q} \times \frac{1}{Q}, \quad m \leq -Q,
\]

where \( \delta_{m,Q} \) is the Kronecker delta. Thus, we end the proof. \( \Box \)
Lemma 8.9. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $L = L_X$, where $X = \{X_1, \ldots , X_k\}$ is a system of vector fields satisfying the Hörmander condition. For $0 \leq \rho \leq 1$, let us consider a positive left-invariant $L$-elliptic continuous linear operator $A : C^\infty (G) \to \mathcal{D}' (G)$ with symbol $\sigma \in S^m_{\rho} (\hat{G})$, $m \in \mathbb{R}$. Then, $A$ belongs to the Dixmier ideal $L^{1, \infty} (L^2 (G))$ if and only if $m \leq -Q$. If $A \neq 0$, $\text{Tr}_w (A) = \frac{1}{Q}$ for $m = -Q$, and for $m < -Q$, $\text{Tr}_w (A) = 0$.

Proof. Let us fix $p > 1$. We will compute the trace of $A^p$ using the formula,

$$\text{Tr}(A^p) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\widehat{A^p}(\xi)].$$

From the positivity of $A$, in a suitable basis of the representation space we can diagonalise the operator $\sigma(\xi)\widehat{M}(\xi)^{-m}$, and we can write in a such basis,

$$\sigma(\xi)^p\widehat{M}(\xi)^{-mp} = \text{diag}[\lambda_{jj}(\xi)]^{d_\xi}_{j=1}, \quad \widehat{M}(\xi)^{mp} = [\Omega_{ij}(\xi)]^{d_\xi}_{i,j=1}. \quad (8.11)$$

Now, we can write

$$\text{Tr}(A^p) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\sigma(\xi)^p\widehat{M}(\xi)^{-mp}\widehat{M}(\xi)^{mp}]dx$$

$$= \sum_{[\xi] \in \hat{G}} d_\xi \sum_{j,j'=1}^{d_\xi} d_\xi \lambda_{jj'}(\xi)\Omega_{jj'}(\xi).$$

The $L$-ellipticity of $A$ implies the $L$-ellipticity of $A^p$ and the positivity of its symbol, implies that

$$\sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{-p}\widehat{M}(\xi)^{mp}\|_{\text{op}}^{-1} \leq \inf_{j,j',[\xi] \in \hat{G}} \lambda_{jj'}(\xi)$$

$$\leq \sup_{j,j',[\xi] \in \hat{G}} \lambda_{jj'}(\xi)$$

$$= \sup_{[\xi] \in \hat{G}} \|\sigma(\xi)^{p}\widehat{M}(\xi)^{-pm}\|_{\text{op}},$$

from which we deduce the following estimate,

$$\text{Tr}(A) = \sum_{[\xi] \in \hat{G}} \sum_{j,j'=1}^{d_\xi} d_\xi \Omega_{jj'}(\xi).$$

Because $\widehat{M}(\xi)^{mp} = [\Omega_{ij}(\xi)]^{d_\xi}_{i,j=1}$ is a symmetric matrix written in the basis that allows to write in a diagonal form the operator $\sigma(\xi)^p\widehat{M}(\xi)^{-mp}$, we can find a matrix $P(\xi)$ such that

$$\widehat{M}(\xi)^{mp} = P(\xi)^{-1}\text{diag}[(1 + \nu_{jj}(\xi)^2)^{\frac{mp}{2}}]^{d_\xi}_{j=1} P(\xi),$$
and
\[
\sum_{j,j'=1}^{d_\xi} \Omega_{j,j'}(\xi) = \sum_{j,j'=1}^{d_\xi} [P(\xi)^{-1} \text{diag} ((1 + \nu_{ss}(\xi)^2)^{m_p} P(\xi)_{j,j'}]_{j'=1}^{d_\xi} P(\xi)
\]
\[
= \sum_{j,j',s=1}^{d_\xi} P(\xi)_{j,j'}^{-1} (1 + \nu_{j,j'}(\xi)^2)^{m_p} P(\xi)_{j,s}
\]
\[
= \text{Tr}[P(\xi)^{-1} \text{diag} ((1 + \nu_{jj}(\xi)^2)^{m_p} P(\xi)]_{j=1}^{d_\xi} P(\xi) = \text{Tr}[\widehat{M}(\xi)^{m_p}].
\]

Consequently, from Lemma 8.8 we deduce that
\[
\text{Tr}(A^p) = \text{Tr}((1 + \mathcal{L})^{m_p}),
\]
from which we deduce that
\[
\text{Tr}_w(A) = \text{Tr}_w((1 + \mathcal{L})^{m_p}) = \frac{1}{Q} \delta_{m,Q}, \quad m \leq -Q.
\]

Thus, we end the proof. \qed

Corollary 8.10. For $0 \leq \delta < \rho \leq 1$, $\delta < \frac{1}{2}$, let us consider a continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho,\delta}^{m,E}(G \times \widehat{G})$, with $m < -Q$. Then $\text{Tr}_w(A) = 0$.

Proof. We will use the notation $s_n(T)$, $n \in \mathbb{N}_0$, for the sequence of singular values of a compact operator $T$ on a Hilbert space $H$. Then, the following inequality holds (see [12, Page 75]): $s_n(CB) \leq \|C\|_{\text{op}} s_n(B)$, for $C$ a bounded linear operator and $B$ a compact linear operator. From the definition of the functional $\text{Tr}_w$, we conclude easily that $0 \leq \text{Tr}_w(CB) \leq \|C\|_{\text{op}} \text{Tr}_w(B)$. Now, let us use this inequality in our setting. From the subelliptic Calderón-Vaillancourt Theorem, we have that $\mathcal{M}^{-m} \in S_{0,\min(\rho,1/\delta)}^{m,E}(G \times \widehat{G})$ extends to a bounded operator on $L^2(G)$, where $\mathcal{M} := (1 + \mathcal{L})^{\frac{1}{2}}$. Consequently,
\[
0 \leq \text{Tr}_w(\mathcal{M}^{-m}) \leq \|\mathcal{M}^{-m}\|_{\mathcal{B}(L^2(G))} \text{Tr}_w(\mathcal{M}^m) = 0,
\]
where we have used that $\text{Tr}_w(\mathcal{M}^m) = 0$ in view of Lemma 8.9. This implies that $\text{Tr}_w(A) = 0$. The proof is complete. \qed

Theorem 8.11. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. For $0 \leq \delta < \rho \leq 1$, $\delta < \frac{1}{2}$, let us consider an $\mathcal{L}$-elliptic continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma \in S_{\rho,\delta}^{m,E}(G \times \widehat{G})$, $m \in \mathbb{R}$. Let us assume that $\sigma(x, [\xi]) \geq 0$, for every $x, [\xi] \in G \times \widehat{G}$. If $A \neq 0$, $\text{Tr}_w(A) = \frac{1}{Q}$, for $m = -Q$, and for $m < -Q$, $\text{Tr}_w(A) = 0$.

Proof. For every $z \in G$, let us consider the Fourier multiplier associated to the symbol $\sigma(z, \cdot)$, $A_z$ which satisfies the hypothesis in Lemma 8.9. Indeed, $\sigma(z, [\xi]) \geq 0$, for every $[\xi]$ implies that $A_z$ is also positive and the $\mathcal{L}$-ellipticity of $A$ implies the $\mathcal{L}$-ellipticity of $A_z$ for every $z \in G$. Observe that from the functional calculus
\[
A^p = \text{Op}[(x, \xi) \mapsto \sigma(x, \xi)^p] + R_p,
\]
(8.12)
where $R_p$ is a subelliptic pseudo-differential operator of order $mp - 1$. Because $m \leq -Q$, and $p \to 1^+$, from Corollary 8.10, we deduce that $\text{Tr}_w(R_p) = 0$. Indeed, the subelliptic order of $R_p$ is $mp - 1 < -Q$. So, note that

$$\text{Tr}_w(A^p) = \text{Tr}_w(\text{Op}[(x, \xi) \mapsto \sigma(x, \xi)^p]) + \text{Tr}_w(R_p)$$

$$= \text{Tr}_w(\text{Op}[(x, \xi) \mapsto \sigma(x, \xi)^p]).$$

Observe that, integrating on the diagonal of the Schwartz kernel of $\text{Op}[(x, \xi) \mapsto \sigma(x, \xi)^p]$, (see e.g. [50]), we have

$$\text{Tr}_w(\text{Op}[(x, \xi) \mapsto \sigma(x, \xi)^p]) = \lim_{p \to 1^+} (p - 1) \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}[\sigma(z, \xi)^p]dz$$

$$= \int_G \lim_{p \to 1^+} (p - 1) \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}[\sigma(z, \xi)^p]dz = \int_G \text{Tr}_w(\text{Op}[(\xi) \mapsto \sigma(z, \xi)])dz$$

$$= \int_G \frac{1}{Q} \delta_{m,Q}dz = \frac{1}{Q} \delta_{m,Q}.$$

Thus, we end the proof. 

**Remark 8.12.** Other results about the classification of the Dixmier trace of pseudo-differential operators on compact manifolds, with or without boundary (or on the lattice $\mathbb{Z}^n$) can be found in [28, 29] and [30].

### 8.4. Subelliptic operators in Schatten classes in $L^2(G)$

As a consequence of our analysis on the Dixmier traceability of subelliptic operators we will explain its consequences in the classification of subelliptic operators in Schatten classes. Let us record that, if $A$ is a compact operator on a Hilbert space $H$ and $\{s_n(A)\}$ denotes the sequence of its singular values, the Schatten von-Neumann class of order $r > 0$, $S_r(H)$, consists of all compact operators $A$ on $H$ such that

$$\|A\|_{S_r(H)} := \left( \sum_{n=0}^{\infty} s_n(A)^r \right)^{\frac{1}{r}} < \infty.$$

Let us recall that the following inequality holds (see e.g. [12, Page 75]): $s_n(CB) \leq \|C\|_{op}s_n(B)$, for a bounded linear operator $C$ and a compact linear operator $B$.

**Corollary 8.13.** Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. Let $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, (or let $0 \leq \delta \leq \rho < 1/2\kappa$) and let $r > 0$. If $T \in \text{Op}(S_p^{-m,\mathcal{L}}(G \times \hat{G}))$, then $T \in S_r(L^2(G))$ if $m > Q/r$, and this condition on the order is sharp.

**Proof.** Indeed, the inequality

$$s_n(T) = s_n(TM_m\mathcal{M}_m) \leq \|TM_m\|_{\mathcal{B}(L^2(G))} s_n(\mathcal{M}_m),$$

implies that

$$\|T\|_{S_r(L^2(G))} \leq \|TM_m\|_{\mathcal{B}(L^2(G))} \|\mathcal{M}_m\|_{S_r(L^2(G))} = \|TM_m\|_{\mathcal{B}(L^2(G))} \|\mathcal{M}_m\|_{S_r(L^2(G))} \|T\|_{S_r(L^2(G))}^{\frac{1}{r}}.$$
We can use the subelliptic Calderón-Vaillancourt Theorem (see Theorem 4.29) to deduce that \(\|T \mathcal{M}_m\|_{\mathcal{B}(L^2(G))} < \infty\). Because of Corollary 8.10, \(\text{Tr}_w(\mathcal{M}_{-\,r}) = 0\), for \(mr > Q\), which means that \(m > Q/r\). But, it is well known that the Dixmier functional \(\text{Tr}_w\) vanishes on the Schatten class of order \(r = 1\), (see e.g. Connes [35]). This fact, that \(m > r/Q\) and Corollary 8.10 implies that \(\|\mathcal{M}_{-\,mr}\|_{S_r(L^2(G))} < \infty\), which implies that \(\|T\|_{S_r(L^2(G))} < \infty\). The sharpness of the order \(m = Q/r\), can be explained as follows. For \(T = \mathcal{M}_{-\,Q/r} \in \text{Op}(S^{-Q/r}_{\mathcal{D}}(G \times \hat{G}))\), (see Remark 4.35) then from Theorem 8.11, \(\text{Tr}_w(T^\gamma) = 1/Q\), which certainly implies (see Connes [35]) that, \(\|\mathcal{M}_{-\,Q}\|_{S_r(L^2(G))} = \|\mathcal{M}_{-\,Q/r}\|_{S_r(L^2(G))} = \infty\). \(\square\)

A complete investigation about the spectral trace of global operators on compact Lie groups can be found in [47, 48, 49, 50].

### 8.5. Compactness and Gohberg lemma

In this subsection, the compactness of subelliptic pseudo-differential operators is discussed. So, we will deduce necessary conditions for the compactness of subelliptic pseudo-differential operators by using the Gohberg Lemma proved for arbitrary compact Lie groups in [45]. Gohberg lemmas are results about the compactness of continuous linear operators. In the case of a compact Lie group \(G\), a left-invariant operator with subelliptic negative order is compact on \(L^2(G)\), (as a consequence of the Plancherel theorem). We will explain why a similar result is valid for pseudo-differential operators. The original Gohberg Lemma was proved by Israel Gohberg in his investigation of integral operators with kernels (see [72]), and its version on \(L^2(S^1)\) was proved by [91] and it was extended to \(L^p(S^1), 1 < p < \infty\), in [124]. The Gohberg Lemma for compact Lie groups was obtained in [45] and it was generalised for \(L^p\)-spaces on compact manifolds (with or without boundary), for the pseudo-differential calculus developed by the second author and Tokmagambetov (see [105, 106]), in [110]. Let us record (see Theorem 3.1 in [45], but for the following statement see Remark 4.2 of [45]) the following theorem.

**Theorem 8.14.** Let \(A : C^\infty(G) \to \mathcal{D}'(G)\) be a continuous linear operator admitting a bounded extension on \(L^2(G)\). If the matrix valued symbol of \(A\), \(a(x, \xi)\) satisfies the following conditions

\[
\|a(x, \xi)\|_{\text{op}} + \|\check{c}^{(\alpha)}(x, \xi)\|_{\text{op}} \leq C, \quad \|\Delta_\xi^\beta a(x, \xi)\|_{\text{op}} \leq C\langle \xi \rangle^{-\epsilon},
\]

where \(\alpha, \beta \in \mathbb{N}_0, |\alpha| = |\beta| = 1\) and for some \(\epsilon \in (0, 1)\), then for all compact operators \(K\) on \(L^2(G)\),

\[
\|A - K\|_{\mathcal{B}(L^2(G))} \geq d_{\text{min}},
\]

where

\[
d_{\text{min}} := \limsup_{\langle \xi \rangle \to \infty} \sup_{x \in G} \|a(x, \xi) a(x, \xi)^*\|_{\text{min}} \leq \limsup_{\langle \xi \rangle \to \infty} \sup_{x \in G} \|a(x, \xi)\|_{\text{op}},
\]

with \(\|a(x, \xi) a(x, \xi)^*\|_{\text{min}} := \min\{\lambda \geq 0 : \det(a(x, \xi) a(x, \xi)^* - \lambda I_{d_\xi}) = 0\}\).

As a consequence of Theorem 8.14 we have:
Corollary 8.15. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a continuous linear operator admitting a bounded extension on $L^2(G)$. If the matrix valued symbol of $A$, $a(x, \xi)$ satisfies the following conditions

$$\|a(x, \xi)\|_{\text{op}} + \|\partial^{(\alpha)}_\xi a(x, \xi)\|_{\text{op}} \leq C, \quad \|\widehat{\mathcal{M}(\xi)^\tau \Delta^\beta_\xi a(x, \xi)}\|_{\text{op}} \leq C,$$

where $\alpha, \beta \in \mathbb{N}_0$, $|\alpha| = |\beta| = 1$ and for some $\tau \in (0, 1)$, then for all compact operators $K$ on $L^2(G)$,

$$\|A - K\|_{\mathcal{B}(L^2(G))} \geq d_{\text{min}}.$$

Proof. In view of Theorem 8.14, we only need to prove that the condition

$$\|\widehat{\mathcal{M}(\xi)^\tau \Delta^\beta_\xi a(x, \xi)}\|_{\text{op}} \leq C,$$

implies the existence of $\epsilon > 0$, such that the second inequality in the right hand side of (8.13) holds true. Indeed, observe that for $|\beta| = 1$,

$$\|\Delta^\beta_\xi a(x, \xi)\|_{\text{op}} = \|\widehat{\mathcal{M}(\xi)^{-\tau} \mathcal{M}(\xi)^\tau \Delta^\beta_\xi a(x, \xi)}\|_{\text{op}}$$

$$\leq C\|\widehat{\mathcal{M}(\xi)^{-\tau}}\|_{\text{op}} \leq \langle \xi \rangle^{-\frac{\tau}{2}},$$

so, we can pick $\epsilon = \frac{\tau}{2}$ in order to use Theorem 8.14. The proof is complete. \qed

Remark 8.16. If $A$ satisfying the hypothesis in Corollary 8.15 admits a compact linear extension on $L^2(G)$, then $d_{\text{min}} = 0$. By observing the approach in \cite{110} we expect that the condition $\limsup_{(\xi) \to \infty} \sup_{x \in \hat{G}} \|a(x, \xi)\|_{\text{op}} = 0$ implies that $A$ is compact on $L^p(G)$ for all $1 < p < \infty$. However, the characterisation of the $L^p$-compactness of subelliptic pseudo-differential operators is still an open problem.

9. Global solvability for evolution problems associated to subelliptic operators

The subelliptic pseudo-differential calculus developed for every subelliptic Laplacian will be applied in this section in a relevant problem of PDE, the global solvability of parabolic and hyperbolic/parabolic problems, in this case associated to subelliptic pseudo-differential operators. More precisely, we will study the existence and uniqueness for the following Cauchy problem

$$(\text{PVI}): \begin{cases} \frac{\partial v}{\partial t} = K(t, x, D)v + f, & v \in \mathcal{D}'((0, T) \times G), \\ v(0) = u_0, \end{cases}$$

where the initial data $u_0 \in L^2(G)$, $K(t) := K(t, x, D) \in C([0, T], S^{m, L}_{\rho, \delta}(G \times \hat{G}))$, $f \in L^2([0, T] \times G) \simeq L^2([0, T], L^2(G))$, $m > 0$, and a suitable positivity condition is imposed on $K$. Indeed, we will assume that

$$\text{Re}(K(t)) := \frac{1}{2}(K(t) + K(t)^*), \quad 0 \leq t \leq T,$$

is $L$-elliptic. Under such assumptions we will prove the existence and uniqueness of a solution $v \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, L}(G))$. We will start with the following energy estimate.

\footnote{This means that $A = K(t)$ is strongly $L$-elliptic.}
Theorem 9.1. Let $K(t) = K(t, x, D) \in C([0, T], S_{\rho, \delta}^{m, L}(G \times \hat{G}))$, $0 \leq \delta < \rho \leq 1$, $\delta < 1/\kappa$, be a subelliptic pseudo-differential operator of order $m > 0$, and let us assume that $\text{Re}(K(t))$ is $L$-elliptic, for every $t \in [0, T]$ with $T > 0$. If $v \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, L}(G))$, then there exist $C, C' > 0$, such that

$$\|v(t)\|_{L^2(G)} \leq \left( C\|v(0)\|_{L^2(G)}^2 + C'\int_0^T \|{\partial}_t - K(\tau)v(\tau)\|_{L^2(G)}^2 \, d\tau \right),$$  

(9.2)

holds for every $0 \leq t \leq T$. Moreover, we also have the estimate

$$\|v(t)\|_{L^2(G)} \leq \left( C\|v(T)\|_{L^2(G)}^2 + C'\int_0^T \|{\partial}_t - K(\tau)v(\tau)\|_{L^2(G)}^2 \, d\tau \right).$$  

(9.3)

Proof. Let $v \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, L}(G))$. Let us start by observing that $v \in C^1([0, T], H^{m, L}(G))$ because of the embedding $H^{m, L} \hookrightarrow H^{\frac{m}{2}, L}$. This fact will be useful later because we will use the Gårding inequality applied to the operator $\text{Re}(K(t))$. So, $v \in \text{Dom}({\partial}_x - K(\tau))$ for every $0 \leq \tau \leq T$. In view of the embedding $H^{m, L} \hookrightarrow L^2(G)$, we also have that $v \in C([0, T], L^2(G))$. Let us define $f(\tau) := Q(\tau)v(\tau)$, $Q(\tau) := (\partial_x - K(\tau))$, for every $0 \leq \tau \leq T$. Observe that

$$\frac{d}{dt}\|v(t)\|_{L^2(G)}^2 = \frac{d}{dt}(v(t), v(t))_{L^2(G)}$$

$$= \left( \frac{dv(t)}{dt}, v(t) \right)_{L^2(G)} + \left( v(t), \frac{dv(t)}{dt} \right)_{L^2(G)}$$

$$= (K(t)v(t) + f(t), v(t))_{L^2(G)} + (v(t), K(t)v(t) + f(t))_{L^2(G)}$$

$$= ((K(t) + K(t)^*)v(t), v(t))_{L^2(G)} + 2\text{Re}(f(t), v(t))_{L^2(G)}$$

$$= (2\text{Re}K(t)v(t), v(t))_{L^2(G)} + 2\text{Re}(f(t), v(t))_{L^2(G)}.$$  

Now, from the subelliptic Gårding inequality,

$$\text{Re}(-K(t)v(t), v(t)) \geq C_1\|v(t)\|_{H^{m, L}(G)} - C_2\|v(t)\|_{L^2(G)}^2,$$  

(9.4)

and the parallelogram law, we have

$$2\text{Re}(f(t), v(t))_{L^2(G)} \leq 2\text{Re}(f(t), v(t))_{L^2(G)} + \|f(t)\|_{L^2(G)}^2 + \|v(t)\|_{L^2(G)}^2$$

$$= \|f(t) + v(t)\|^2_{L^2(G)} \leq \|f(t) + v(t)\|^2 + \|f(t) - v(t)\|^2$$

$$= 2\|f(t)\|_{L^2(G)}^2 + 2\|v(t)\|_{L^2(G)}^2,$$

and consequently

$$\frac{d}{dt}\|v(t)\|_{L^2(G)}^2 \leq 2\left( C_2\|v(t)\|_{L^2(G)}^2 - C_1\|v(t)\|_{H^{m, L}(G)} \right) + 2\|f(t)\|_{L^2(G)}^2 + 2\|v(t)\|_{L^2(G)}^2.$$  

So, we have proved that

$$\frac{d}{dt}\|v(t)\|_{L^2(G)}^2 \leq \|f(t)\|_{L^2(G)}^2 + \|v(t)\|_{L^2(G)}^2.$$
By using Gronwall’s Lemma we obtain the energy estimate

\[ \|v(t)\|_{L^2(G)}^2 \leq \left( C\|v(0)\|_{L^2(G)}^2 + C' \int_0^T \|f(\tau)\|_{L^2(G)}^2 d\tau \right), \quad (9.5) \]

for every \( 0 \leq t \leq T, \) and \( T > 0. \) To finish the proof, we can change the analysis above with \( v(T - \cdot) \) instead of \( v(\cdot) \), \( f(T - \cdot) \) instead of \( f(\cdot) \) and \( Q^* = -\partial_t - K(t)^* \), (or equivalently \( Q = \partial_t - K(t) \)) instead of \( Q^* = -\partial_t + K(t)^* \) (or equivalently \( Q = \partial_t - K(t) \)) using that \( \text{Re}(K(T - t)^*) = \text{Re}(K(T - t)) \) to deduce that

\[ \|v(T - t)\|_{L^2(G)}^2 \leq \left( C\|v(T)\|_{L^2(G)}^2 + C' \int_0^T \|(-\partial_t + K(T - t)^*)v(T - \tau)\|_{L^2(G)}^2 d\tau \right) \]

\[ = \left( C\|v(T)\|_{L^2(G)}^2 + C' \int_0^T \|(-\partial_t - K(t)^*)v(s)\|_{L^2(G)}^2 ds \right). \]

So, we conclude the proof. \( \Box \)

**Theorem 9.2.** Let \( K(t) = K(t, x, D) \in C([0, T], \mathcal{S}_{\rho, \delta}^m \mathcal{L}(G \times \hat{G})) \), \( 0 \leq \delta < \rho \leq 1, \delta < 1/\kappa, \) be a subelliptic pseudo-differential operator of order \( m > 0, \) and let us assume that \( \text{Re}(K(t)) \) is \( \mathcal{L} \)-elliptic, for every \( t \in [0, T] \) with \( T > 0. \) Let \( f \in L^2([0, T], L^2(G)) \), and let \( u_0 \in L^2([0, T], L^2(G)). \) Then there exists a unique \( v \in C^1([0, T], L^2(G)) \cap C([0, T], H^m \mathcal{L}(G)) \) solving (9.1). Moreover, \( v \) satisfies the energy estimate

\[ \|v(t)\|_{L^2(G)}^2 \leq \left( C\|u_0\|_{L^2(G)}^2 + C'\|f\|_{L^2([0, T], L^2(G))}^2 \right), \quad (9.6) \]

for every \( 0 \leq t \leq T. \)

**Proof.** Let us denote by \( Q = \partial_t - K(t), \) and let us introduce the spaces

\[ E := \{ \phi \in C^1([0, T], L^2(G)) \cap C([0, T], H^m \mathcal{L}(G)) : \phi(T) = 0 \}, \]

and \( Q^* E := \{ Q^* \phi : \phi \in E \}. \) Let us define the linear form \( \beta \in (Q^* E)' \) by

\[ \beta(Q^* \phi) := \int_0^T (f(\tau), \phi(\tau)) d\tau + \frac{1}{t}(u_0, \phi(0)). \]

From (9.3) we deduce for every \( \phi \in E \) that,

\[ \|\phi(t)\|_{L^2(G)}^2 \leq \left( C\|\phi(T)\|_{L^2(G)}^2 + C' \int_0^T \|\partial_t - K(\tau)^* \phi(\tau)\|_{L^2(G)}^2 d\tau \right). \]
So, we have
\[
|\beta(Q^*\phi)| \leq \int_0^T \|f(\tau)\|_{L^2(G)} \|\phi(\tau)\|_{L^2(G)} d\tau + \|u_0\|_{L^2(G)} \|\phi(0)\|_{L^2(G)}
\]
\[
\leq \|f\|_{L^2([0,T],L^2(G))} \|\phi\|_{L^2([0,T],L^2(G))} + \|u_0\|_{L^2(G)} \|\phi\|_{L^2(G)}
\]
\[
\leq_T \left(\|f\|_{L^2([0,T],L^2(G))} + \|u_0\|_{L^2(G)}\right) \|Q^*\phi(\tau)\|_{L^2([0,T],L^2(G))},
\]
which shows that $\beta$ is a bounded functional on $\mathcal{T} := Q^*E \cap L^2([0, T], L^2(G))$, with $\mathcal{T}$ endowed with the topology induced by the norm of $L^2([0, T], L^2(G))$. By using the Hahn-Banach extension theorem, we can extend $\beta$ to a bounded functional $\tilde{\beta}$ on $L^2([0, T], L^2(G))$, and by using the Riesz representation theorem, there exists $v \in (L^2([0, T], L^2(G))^\prime = L^2([0, T], L^2(G))$, such that
\[
\tilde{\beta}(\psi) = (v, \psi), \quad \psi \in L^2([0, T], L^2(G)).
\]
In particular, for $\psi = Q^*\phi \in \mathcal{T}$, we have
\[
\tilde{\beta}(Q^*\phi) = \beta(Q^*\phi) = (v, Q^*\phi),
\]
Because, we can identify $C^\infty_0((0, T), \mathcal{D}(G))$ as a subspace of $E$
\[
C^\infty_0((0, T), C^\infty(G)) \subset E = \{\phi \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, \mathcal{L}}(G)) : \phi(T) = 0\},
\]
we have the identity
\[
(f, \phi) = \int_0^T (f(\tau), \phi(\tau)) d\tau = \int_0^T (f(\tau), \phi(\tau)) d\tau + \frac{1}{i}(u_0, \phi(0)) = (v, Q^*\phi),
\]
for every $\phi \in C^\infty_0((0, T), C^\infty(G))$. So, this implies that $v \in \text{Dom}(Q^{**})$. Because $Q^{**} = Q$, we have that
\[
(v, Q^*\phi) = (Qv, \phi) = (f, \phi), \quad \forall \phi \in C^\infty_0((0, T), C^\infty(G)),
\]
which implies that $Qv = f$. A routine argument of integration by parts shows that $v(0) = u_0$. Now, in order to show the uniqueness of $v$, let us assume that $u \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, \mathcal{L}}(G))$ is a solution of the problem
\[
\begin{cases}
\frac{\partial u}{\partial t} = K(t, x, D)u + f, & u \in \mathcal{D}'((0, T) \times G), \\
u(0) = u_0.
\end{cases}
\]
Then $\omega := v - u \in C^1([0, T], L^2(G)) \cap C([0, T], H^{m, \mathcal{L}}(G))$ solves the problem
\[
\begin{cases}
\frac{\partial \omega}{\partial t} = K(t, x, D)\omega, & \omega \in \mathcal{D}'((0, T) \times G), \\
\omega(0) = 0,
\end{cases}
\]
and from Theorem 9.1, $\|\omega(t)\|_{L^2(G)} = 0$, for all $0 \leq t \leq T$, and consequently, from the continuity in $t$ of the functions we have that $v(t, x) = u(t, x)$ for all $t \in [0, T]$ and a.e. $x \in G$. \hfill \Box
10. Global Fourier Integral operators on compact Lie groups

In this section we will study the $L^2$-boundedness of Fourier integral operators on compact Lie groups. The motivation to include a study of the $L^2$-boundedness of Fourier integral operators is to make a generalisation of Theorem 10.5.5 in [107], where it has been proved that the following condition

$$
\sup_{(x, [\xi]) \in \mathbb{G}} \| X^\alpha_x \sigma(x, \xi) \|_{op} < \infty, \quad |\alpha| \leq \left[ \frac{n}{2} \right] + 1, \quad n = \dim(G),
$$

implies the $L^2(G)$-boundedness of a densely defined extension of $A \equiv \text{Op}(\sigma)$ on $C^\infty(G)$. We will use additional to the notion of a global matrix-valued symbol, that of matrix-valued phase function. To do so, we will use Cotlar Lemma, by exploiting in our case the almost-orthogonality of the decomposition of a Fourier integral operator in several pieces induced by every representation space.

10.1. Matrix-valued phase functions on compact Lie groups. First, we introduce a global definition of Fourier integral operators on every compact Lie group $G$.

**Definition 10.1 (FIO).** A continuous linear operator $A : C^\infty(G) \rightarrow \mathcal{D}'(G)$ with Schwartz kernel $K_A \in C^\infty(G) \otimes \mathcal{D}'(G)$, is a global Fourier integral operator, if

- there exists a symbol $\sigma \equiv \{ \sigma(x, \xi) \in \mathbb{C}^{d_x \times d_\xi} \}_{(x, [\xi]) \in \mathbb{G}}$ such that

  $$
  \forall [\xi] \in \hat{\mathbb{G}}, \quad \sigma(\cdot, [\xi]) = \sigma(\cdot, \xi) \in C^\infty(G, \mathbb{C}^{d_x \times d_\xi}),
  $$

- and a phase function $\phi \equiv \{ \phi(x, \xi) \in \mathbb{C}^{d_x \times d_\xi} \}_{(x, [\xi]) \in \mathbb{G} \times \hat{\mathbb{G}}}$ depending (possibly) on $x \in G$, and satisfying

  $$
  \forall [\xi] \in \hat{\mathbb{G}}, \quad \phi(\cdot, [\xi]) = \phi(\cdot, \xi) \in C^\infty(G, \mathbb{C}^{d_x \times d_\xi}), \quad \phi(x, \xi) = \phi(x, \xi)^*,
  $$

such that, the Schwartz kernel of $A$ is defined by the distribution,

$$
K_A(x, y) = \sum_{[\xi] \in \hat{\mathbb{G}}} d_\xi \text{Tr}(\xi(y)^* e^{i\phi(x, \xi)} \sigma(x, \xi)),
$$

and the operator $A$ is defined via

$$
Af(x) = \sum_{[\xi] \in \hat{\mathbb{G}}} d_\xi \text{Tr}(e^{i\phi(x, \xi)} \sigma(x, \xi) \hat{f}(\xi)), \quad (10.1)
$$

for every $f \in C^\infty(G)$.

**Remark 10.2.** To motivate the definition of the Schwartz Kernel of a Fourier integral operator, observe that under suitable conditions on $\phi$ and $\sigma$, for $f \in$
$C^\infty(G)$, we can use Fubini theorem to write

$$Af(x) := \int_G K_A(x, y) f(y) dy = \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(y)*e^{i\phi(x, \xi)} \sigma(x, \xi)) f(y) dy$$

\[= \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} \left( \int_G f(y) \xi(y)*dy e^{i\phi(x, \xi)} \sigma(x, \xi) \right) \]

\[= \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} \left( \hat{f}(\xi) e^{i\phi(x, \xi)} \sigma(x, \xi) \right) \]

\[= \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr} \left( e^{i\phi(x, \xi)} \sigma(x, \xi) \hat{f}(\xi) \right).\]

We will use the notation $A = \text{FIO}(\sigma, \phi)$, to denote the Fourier integral operator associated with the symbol $\sigma$ and with the phase function $\phi$.

To motivate the definition of global Fourier integral operators we will review how the definition given in the Definition 10.1 extends that of pseudo-differential operators and how it does appear to express solutions of some differential problems. We will make some remarks explaining both situations in detail.

**Example 10.3 (Pseudo-differential operators).** Let us assume that $A : C^\infty(G) \to \mathcal{S}'(G)$ is a continuous linear operator with symbol $\sigma$,

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x) \sigma(x, \xi) \hat{f}(\xi)), \quad f \in C^\infty(G).$$

Since the matrix $\xi(x) \in \mathbb{C}^{d_\xi \times d_\xi}$ is unitary for every $x \in G$, there exists a self-adjoint operator $\psi(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ such that $\xi(x) = e^{i\psi(x, \xi)}$. Consequently, we have

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(e^{i\psi(x, \xi)} \sigma(x, \xi) \hat{f}(\xi)), \quad f \in C^\infty(G). \quad (10.2)$$

Moreover, since $\text{Spectrum}(\xi(x)) \subset \{z \in \mathbb{C} : |z| = 1\}$, in some basis $B_\xi$ of the representation space $\mathbb{C}^{d_\xi}$, $\xi(x) = \text{diag}[e^{i\phi_{jj}(x, \xi)}]_{1 \leq j \leq d_\xi}$ is a diagonal matrix with complex entries, and in this case on the same basis we have

$$\psi(x, \xi) \equiv \text{diag}[\phi_{jj}(x, \xi)]_{1 \leq j \leq d_\xi}, \quad (x, \xi) \in G \times \hat{G}.$$

To give a more precise description for the functions $\phi_{jj}(x, \xi)$, $1 \leq j \leq d_\xi$, let us consider $X = \{X_1, \ldots, X_n\}$ a basis for the Lie algebra $\mathfrak{g}$. Every $X_k$ is a left-invariant operator. Observe that

$$X_k \xi(x) = \text{diag}[i e^{i\phi_{jj}(x, \xi)} (X_k \phi_{jj})(x, \xi)]_{j=1}^{d_\xi},$$

and

$$X_k^2 \xi(x) = \text{diag}[-e^{i\phi_{jj}(x, \xi)} (X_k \phi_{jj})(x, \xi)^2 + i e^{i\phi_{jj}(x, \xi)} (X_k^2 \phi_{jj})(x, \xi)]_{j=1}^{d_\xi}.$$
Taking into account that \( \xi(\epsilon_G) = I_{d_x \times d_x} = (\delta_{ij})_{1 \leq i, j \leq d_{\xi}} \), it follows that \( e^{i\phi_{jj}(e_G, \xi)} = 1 \), for all \( 1 \leq j \leq d_{\xi} \). Consequently,

\[
\hat{X}_k^2(\xi) = X_k^2(\epsilon_G) \\
= \text{diag}[-(X_k\phi_{jj}(e_G, \xi))^2 + i(X_k^2\phi_{jj})(e_G, \xi)]_{j=1}^{d_{\xi}} \\
= \hat{X}_k(\xi)\hat{X}_k(\xi) = X_k(\epsilon_G)X_k(\epsilon_G) \\
= \text{diag}[-(X_k\phi_{jj}(e_G, \xi))^2]_{j=1}^{d_{\xi}},
\]

which implies that

\[
(X_k^2\phi_{jj})(e_G, \xi) = 0, \; 1 \leq j \leq d_{\xi}. \tag{10.3}
\]

A similar analysis implies that

\[
(X_k^\ell\phi_{jj})(e_G, \xi) = 0, \; 1 \leq j \leq d_{\xi}, \; \ell \geq 2. \tag{10.4}
\]

So, the Taylor expansion at \( x = e_G \), gives that for some smooth function \( q_j \) vanishing with order 1 at \( x = e_G \), we have

\[
\phi_{jj}(x, \xi) = \phi_{jj}(e_G, \xi) + \sum_{k=1}^{n} (X_k\phi_{jj})(e_G, \xi)q_k(x). \tag{10.5}
\]

Also, the condition \( e^{i\phi_{jj}(e_G, \xi)} = 1 \), implies that for some \( n_{\xi,j} \in \mathbb{Z} \), \( \phi_{jj}(e_G, \xi) = 2\pi n_{\xi,j} \). This analysis implies the following representation for the pseudo-differential operator \( A \),

\[
Af(x) = \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr} \left( \text{diag} \left[ e^{i(\xi_{2\pi n_{\xi,j} + \sum_{k=1}^{n} (X_k\phi_{jj})(e_G, \xi)q_k(x))]} \right]_{j=1}^{d_{\xi}} \sigma(x, \xi) \hat{f}(\xi) \right) \\
= \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr} \left( \text{diag} \left[ e^{i(\sum_{k=1}^{n} (X_k\phi_{jj})(e_G, \xi)q_k(x))]} \right]_{j=1}^{d_{\xi}} \sigma(x, \xi) \hat{f}(\xi) \right).
\]

**Example 10.4** (Cauchy problem for the wave equation). Let us consider the differential problem,

\[
\partial_t^2 u(x, t) + Au(x, t) = 0, \; (x, t) \in G \times [T_0, \infty), \; T_0 \geq 0, \tag{10.6}
\]

with initial conditions \( u(x, 0) = f_0 \in C^\infty(G) \) and \( u_t(x, t) = f_1 \in C^\infty(G) \), where \( A \) is a positive left-invariant operator. In this case, if \( \lambda = 0 \) is an isolated point of the spectrum of \( A \), the solution \( u(x, t) \) of (10.6) can be written according to the representation,

\[
u(x, t) = A_{t,0}f_+(x) + A_{t,1}f_-(x), \; (x, t) \in G \times [T_0, \infty),
\]

where

\[
f_+(x) = \frac{1}{2}(f_0 - iA^{-\frac{1}{2}}f_1), \; f_-(x) = \frac{1}{2}(f_0 + iA^{-\frac{1}{2}}f_1),
\]

and the operators

\[
A_{t,j}f_\pm(x) = \int_G \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr} \left( \xi(y^{-1}x)e^{\pm i\sqrt{\lambda(\xi)}\hat{A}(\xi)^{-\frac{1}{2}}} \hat{f}_\pm(y)dy \right), \; j = 0, 1,
\]

are global Fourier integral operators.
10.2. $L^p$-boundedness of Fourier integral operators. Now we will study the $L^p$-boundedness of Fourier integral operators. Motivated by the solution for the Cauchy problem for the wave equation, let us consider the case of Fourier integral operators where the phase function admits a factorisation of the form

$$e^{i\phi(x,\xi)} = \xi(x)e^{i\Phi(\xi)},$$

where $\Phi(\xi) = \Phi(\xi)^*$, $\forall [\xi] \in \hat{G}$.

Observe that the matrix $\xi(x)e^{i\Phi(\xi)}$ is unitary for every $[\xi]$, and the existence of such $\phi$ follows. More explicitly, for a Fourier integral operator of the form

$$Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x)e^{i\Phi(\xi)}\sigma(x,\xi)\hat{f}(\xi)), \quad f \in C^\infty(G), \quad (10.7)$$

we have the following $L^2$-boundedness theorem. We will denote by $\sQ$ the smallest even integer larger than $Q/2$.

**Theorem 10.5.** Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $L = L_X$, where $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. Let us consider the Fourier integral operator $A \equiv \text{FIO}(\sigma, \phi) : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma$ defined as in (10.7), where the function $\hat{\Phi} : \hat{G} \to \cup_{[\xi] \in \hat{G}} \mathbb{C}^{d_\xi \times d_\xi}$, is such that $\Phi(\xi) = \Phi(\xi)^*$ for all $[\xi] \in \hat{G}$. Let us assume that $\sigma$ satisfies the following conditions,

$$\sup_{(x, [\xi]) \in G \times \hat{G}} \|X_{i_1}^{r_1} \cdots X_{i_k}^{r_k}\sigma(x, \xi)\|_{op} < \infty, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k, \quad (10.8)$$

for all $|\alpha| \leq \sQ$. Then $A$ extends to a bounded linear operator on $L^2(G)$.

**Proof.** Let us write $\sQ = 2\ell$. Note that if $\sigma$ does not depend on $x \in G$, then the statement follows from the Plancherel theorem. Now, in the general case where $\sigma$ depends on $x$, the idea is to use the subelliptic Sobolev embedding theorem. For every $z \in G$, let us define the family of invariant operators $\{A_z\}_{z \in G}$, by

$$A_z f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)e^{i\Phi(\xi)}\sigma(z,\xi)\hat{f}(\xi)], \quad f \in C^\infty(G).$$

By the identity $A_z f(x) = Af(x), x \in G$, the subelliptic Sobolev embedding Theorem (see Remark 4.25) gives

$$\sup_{z \in G} |A_z f(x)| \leq \sup_{z \in G} \|M_{2\ell} A_z f\|_{L^2(G)} \leq \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k, |\alpha| \leq 2\ell} \|X_{i_1, z}^{r_1} \cdots X_{i_k, z}^{r_k} A_z f\|_{L^2(G)}.$$

Every operator $A_{z, \alpha} := X_{i_1, z}^{r_1} \cdots X_{i_k, z}^{r_k} A_z$ has an invariant symbol given by

$$a_{z, \alpha}(\xi) := e^{i\Phi(\xi)} X_{i_1, z}^{r_1} \cdots X_{i_k, z}^{r_k}\sigma(z, \cdot),$$

and the estimates

$$\sup_{(z, [\xi]) \in G \times \hat{G}} \|e^{i\Phi(\xi)} X_{i_1, z}^{r_1} \cdots X_{i_k, z}^{r_k}\sigma(z, \xi)\|_{op} < \infty, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k,$$

are equivalent, indeed, to the following ones,

$$\sup_{(z, [\xi]) \in G \times \hat{G}} \|X_{i_1, z}^{r_1} \cdots X_{i_k, z}^{r_k} a_{z, \alpha}(\xi)\|_{op} < \infty, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k.$$
because, using that $e^{i\Phi(\xi)}$ is unitary for all $[\xi] \in \hat{G}$, one has that
\[
\|e^{i\Phi(\xi)} X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k} a_{z, \alpha}(\xi)\|_{op} = \|X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k} a_{z, \alpha}(\xi)\|_{op}.
\]
Consequently the family of left-invariant operators $\{A_{z, \alpha}\}_{z \in G, |\alpha| \leq 2\ell}$, are uniformly bounded on $L^2(G)$. Moreover, for every $|\alpha| \leq 2\ell$, the function $z \mapsto A_{z, \alpha}$, is a continuous function from $G$ into $\mathscr{B}(L^2(G))$. The compactness of $G$ implies that
\[
\sup_{z \in G} \|A_{z, \alpha}\|_{\mathscr{B}(L^2(G))} = \|A_{z_0, \alpha}\|_{\mathscr{B}(L^2(G))} = \sup_{[\xi] \in \hat{G}} \|X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k}\|_{op},
\]
for some $z_{0, \alpha} \in G$. Consequently, we can estimate the $L^2(G)$-norm of $Af, f \in C^\infty(G)$, by,
\[
\|Af\|^2_{L^2(G)} = \int_G |Af(x)|^2 dx \leq \|A_{z_0, \alpha}\|_{\mathscr{B}(L^2(G))} \|f\|^2_{L^2(G)} = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq k, |\alpha| \leq 2\ell} \int_{\hat{G}} \|X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k}\|^2_{op} dz dx
\]
So, we have proved the boundedness of $A$ on $L^2(G)$. \hfill \square

Now, we extend Theorem 10.5 to the $L^p$-case for $p \neq 2$, (although the following theorem also absorbs the case $p = 2$). We will denote by $\chi_{Q,p}$ the smallest even integer larger than $Q/p$, for $1 < p < \infty$.

**Theorem 10.6.** Let $G$ be a compact Lie group and let us denote by $\mathcal{L}$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = \mathcal{L}_X$, where $X = \{X_1, \cdots, X_k\}$ is a system of vector fields satisfying the Hörmander condition. Let us consider the Fourier integral operator $A \equiv \text{FIO}(\sigma, \phi) : C^\infty(G) \to \mathscr{B}(G)$ with symbol $\sigma$ defined as in (10.7), where $\hat{\Phi} : \hat{G} \to \cup_{[\xi] \in \hat{G}} \mathbb{C}^{d_e \times d_e}$, is a matrix-valued function. Let $1 < p < \infty$ and let $0 < \rho \leq 1$. Let us assume that $\sigma$ satisfies the following conditions,
\[
\sup_{(x, [\xi]) \in G \times \hat{G}} \|\widetilde{M}(\xi)^{m+|\gamma|} \Delta^\gamma (e^{i\Phi(\xi)} X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k}\sigma(x, \xi))\|_{op} < \infty, \quad \gamma \in \mathbb{N}_0^n, \quad (10.9)
\]
for all $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k$, and $|\alpha| \leq \chi_{Q,p}$. Then $A$ extends to a bounded linear operator on $L^p(G)$, provided that
\[
m \geq m_p := Q(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|. \quad (10.10)
\]
Remark 10.7. Let us observe that for \( \rho = \frac{1}{q} \), Theorem 10.6 implies that under the condition \( m \geq (Q - 1) \left| \frac{1}{p} - \frac{1}{2} \right| \), the Fourier integral operator \( A \equiv \text{FIO}(\sigma, \phi) : \mathcal{C}^{\infty}(G) \to \mathcal{D}'(G) \) with symbol \( \sigma \) defined as in (10.7), and satisfying the family of inequalities in (10.9), extends a bounded operator from \( L^p(G) \) to itself.

Proof. Let us proceed as in the proof of Theorem 10.5 and let us write \( \mathcal{M}_{Q,p} = 2\ell \). Note that if \( \sigma \) does not depend on \( x \in G \), then the statement follows from Theorem 6.9, because in that case, \( A \) is a Fourier multiplier whose symbol \( e^{i\Phi} \sigma \) belongs to \( \mathcal{S}^{-m,\mathcal{L}}_{\rho,\delta}(G \times \hat{G}) \) with \( m = m_p \). Now, in the general case where \( \sigma \) depends on \( x \), the idea is to use the subelliptic \( L^p \)-Sobolev embedding theorem. For every \( z \in G \), let us define the family of invariant operators \( \{A_z\}_{z \in G} \), by

\[
A_z f(x) = \sum_{[\ell] \in \hat{G}} d_\ell \text{Tr}[\xi(x)e^{i\Phi(\xi)}\sigma(z, \xi)\hat{f}(\xi)], \quad f \in \mathcal{C}^{\infty}(G).
\]

By the identity \( A_z f(x) = Af(x), \quad x \in G \), the subelliptic \( L^p \)-Sobolev embedding Theorem (see Coulhon, Russ and Tardivel-Nachef [39, Page 288]) gives

\[
\sup_{z \in G} |A_z f(x)| \lesssim \sup_{z \in G} \| M_{2\ell} A_z f \|_{L^p(G)} \lesssim \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k, |\alpha| \leq 2\ell} \| X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k} A_z f \|_{L^p(G)}.
\]

Every operator \( A_{z, \alpha} := X_{i_1, z}^{\alpha_1} \cdots X_{i_k, z}^{\alpha_k} A_z \) has an invariant symbol given by

\[
a_{z, \alpha}(\xi) := e^{i\Phi(\xi)} X_{i_1, \xi}^{\alpha_1} \cdots X_{i_k, \xi}^{\alpha_k} \sigma(z, \cdot),
\]

and the estimates

\[
\sup_{(z, [\xi]) \in G \times \hat{G}} \| \hat{M}(\xi)^{m+|\gamma|} \Delta_\xi^2 (e^{i\Phi(\xi)} X_{i_1, \xi}^{\alpha_1} \cdots X_{i_k, \xi}^{\alpha_k} \sigma(z, \xi)) \|_{\text{op}} < \infty,
\]

are equivalent, indeed, to the following ones,

\[
\sup_{(z, [\xi]) \in G \times \hat{G}} \| \hat{M}(\xi)^{m+|\gamma|} \Delta_\xi^2 (e^{i\Phi(\xi)} X_{i_1, \xi}^{\alpha_1} \cdots X_{i_k, \xi}^{\alpha_k} a_{z, \alpha}(\xi)) \|_{\text{op}} < \infty.
\]

Consequently the family of left-invariant operators \( \{A_{z, \alpha}\}_{z \in G, |\alpha| \leq 2\ell} \), are uniformly bounded on \( L^p(G) \). Moreover, for every \( |\alpha| \leq 2\ell \), the function \( z \mapsto A_{z, \alpha} \), is a continuous function from \( G \) into \( \mathcal{B}(L^p(G)) \). The compactness of \( G \) implies that

\[
\sup_{z \in G} \| A_{z, \alpha} \|_{\mathcal{B}(L^p(G))} = \| A_{z_0, \alpha} \|_{\mathcal{B}(L^p(G))} < \infty,
\]

for some \( z_0, \alpha \in G \). Consequently, we can estimate the \( L^p(G) \)-norm of \( Af \), \( f \in \mathcal{C}^{\infty}(G) \), by,

\[
\| Af \|_{L^p(G)}^p = \int_G |Af(x)|^p dx \lesssim \int_G \sup_{z \in G} |A_z f(x)|^p dx \lesssim \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq k, |\alpha| \leq 2\ell} \| A_{z_0, \alpha} \|_{\mathcal{B}(L^p(G))}^p \| f \|_{L^p(G)}^p.
\]

So, we have proved the boundedness of \( A \) on \( L^p(G) \). \( \square \)
In order to motivate Theorem 10.11, let us recall the following $L^2$-estimate proved for Fourier integral operators on the torus in [107, Theorem 4.14.2, page 415]. Here, $\{e_j\}_{j=1}^n$ is the canonical basis of $\mathbb{R}^n$.

**Theorem 10.8.** Let us consider the Fourier integral operator $A \equiv \text{FIO}(\sigma, \phi) : C^\infty(T^n) \to \mathcal{D}'(T^n)$ with symbol $\sigma$ and phase function $\phi$, satisfying the following conditions

$$
\sup_{(x,\xi) \in T^n \times \mathbb{Z}^n} |\partial_x^\alpha \sigma(x,\xi)| < \infty, \quad \sup_j \sup_{(x,\xi) \in T^n \times \mathbb{Z}^n} |\partial_x^\alpha (\phi(x+e_j) - \phi(x,\xi))| < \infty
$$

for all $|\alpha| \leq 2n + 1$, and

$$
|\nabla_x \phi(x,\xi) - \nabla_x \phi(x,\xi')| \geq C|\xi - \xi'|.
$$

(10.11)

Then, $A$ extends to a bounded linear operator on $L^2(T^n)$.

The second condition in the right hand side of (10.11) together with (10.12) implies the following condition (see [107, page 417]),

$$
|\nabla_x \phi(x,\xi) - \nabla_x \phi(x,\xi')| \asymp |\xi - \xi'|.
$$

(10.13)

The $L^p$-boundedness of Fourier integral operators on the torus has been treated in [31]. We will require a similar condition in the context of compact Lie groups. In the proof of Theorem 10.11 we will use the following version of Cotlar-Stein Lemma due to Comech [43].

**Lemma 10.9** (Cotlar-Stein Lemma). Let $E, F$ Hilbert spaces and let $\{T_i : E \to F\}_{i \in \mathbb{Z}}$, be a sequence of bounded operators satisfying the conditions of almost-orthogonality, this means that

$$
\|T^*_i T_i\|_{\mathcal{B}(E,F)} \leq a(i,j), \quad \|T_j T^*_i\|_{\mathcal{B}(F,E)} \leq b(i,j),
$$

where where $a(i,j)$ and $b(i,j)$ are non-negative symmetric functions on $\mathbb{Z} \times \mathbb{Z}$, which satisfy

$$
M := \sup_i \sum_j a(i,j)^{1-\varepsilon}, \quad N := \sup_i \sum_j b(i,j)^{\varepsilon}, \quad 0 < \varepsilon < 1.
$$

Then, $T = \sum_i T_i : E \to F$, extends to a bounded operator and

$$
\|T\|_{\mathcal{B}(E,F)} \leq (MN)^{1/2}.
$$

**Assumption 10.10** (On the gap between the eigenvalues of the Laplacian). To analyse the $L^2$-estimate for Fourier integral operators on compact Lie groups in Theorem 10.11, we will assume a condition about the gap between the eigenvalues for the Laplacian. So, we will assume that for every $\tau \in (0,1)$, there exists $\varepsilon_0 \in (0,1)$, such that for every $[\xi], [\xi'] \in \hat{G}$, with $\lambda_{[\xi]} \leq \lambda_{[\xi']}$, we have

$$
1 - \left( \frac{\lambda_{[\xi]}}{\lambda_{[\xi']} \tau} \right)^\tau \geq \varepsilon_0.
$$

(10.14)

This condition is easily verifiable for many classes of compact Lie groups (e.g. the torus $T^n$, $\text{SU}(2)$, $\text{SO}(3)$, etc).
We will denote by $\mathbb{X} = \{X_1, \cdots, X_n\}$ a basis for the Lie algebra $\mathfrak{g}$, and the corresponding gradient $\nabla_{\mathbb{X}}$, defined by

$$\nabla_{\mathbb{X}} f(x) = (X_1 f, \cdots, X_n f) \in C^\infty(G) \times \cdots \times C^\infty(G), \ f \in C^\infty(G).$$

**Theorem 10.11.** Let $G$ be a compact Lie group of topological dimension $n$, and let us consider the Fourier integral operator $A \equiv \text{FIO}(\sigma, \phi) : C^\infty(G) \to \mathcal{D}'(G)$ with symbol $\sigma$ and phase function $\phi$, satisfying the following conditions

$$\sup_{(x,[\xi]) \in \mathbb{G}} \|X_\xi^\alpha \sigma(x, \xi)\|_{\text{op}} < \infty,$$

(10.15)

for all $|\alpha| \leq 5n/2$, and

$$|\nabla_{\mathbb{X}} \phi_{jj}(x, \xi) - \nabla_{\mathbb{X}} \phi_{jj'}(x, \xi')| = |\lambda_{[\xi]}^\tau - \lambda_{[\xi']}^\tau|, \ 1 \leq j \leq d_{\xi},$$

(10.16)

uniformly in $([\xi], [\xi']) \in \hat{G} \times \hat{G}$, for some $0 < \tau < 1$. Let us assume the condition in (10.14). Then, $A$ extends to a bounded linear operator on $L^2(G)$.

**Proof.** To start the proof observe that by Plancherel Theorem, it is enough to prove that the operator $S : \mathcal{S}(\hat{G}) \subset L^2(\hat{G}) \rightarrow L^2(\hat{G})$, defined by

$$Sw(x) = \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr}(e^{i\phi(x, \xi)} \sigma(x, \xi)w(\xi)), \ w \in \mathcal{S}(\hat{G}) := \mathcal{F}C^\infty(\hat{G}),$$

admits a bounded extension. Because $\phi(x, \xi)$ is a self-adjoint matrix for every $(x, [\xi])$, in some basis $B_{\xi}$ of the representation space $\mathbb{C}^{d_{\xi}}$, $e^{i\phi(x, \xi)} = \text{diag}[e^{i\phi_{jj}(x, \xi)}]_{j=1}^{d_{\xi}}$ is a diagonal matrix where

$$\phi_{jj}(x, \xi), \ 1 \leq j \leq d_{\xi}, \ (x, \xi) \in G \times \hat{G},$$

are the real eigenvalues of $\phi(x, \xi)$. Now, by using the representation

$$Sw(x)$$

$$= \sum_{[\xi] \in \hat{G}} d_{\xi} \sum_{i,j,k} [e^{i\phi(x, \xi)}]_{ij} \sigma(x, \xi)_{jk} w(\xi)_{ki} = \sum_{[\xi] \in \hat{G}} d_{\xi} \sum_{j,k=1}^{d_{\xi}} e^{i\phi_{jj}(x, \xi)} \sigma(x, \xi)_{jk} w(\xi)_{kj},$$

we can decompose the operator $S$ as,

$$Sw(x) = \sum_{[\xi] \in \hat{G}} d_{\xi} \sum_{j,k=1}^{d_{\xi}} S_{[\xi], j,k} w(x), \quad S_{[\xi], j,k} w(x) := d_{\xi} e^{i\phi_{jj}(x, \xi)} \sigma(x, \xi)_{jk} w(\xi)_{kj}.$$

In order to use Cotlar Lemma (Lemma 10.9), we will compute the adjoint of every operator $S_{[\xi], j,k}$. So, from the identities

$$(S_{[\xi], j,k} w, v)_{L^2(G)} := \int_{G} d_{\xi} e^{i\phi_{jj}(x, \xi)} \sigma(x, \xi)_{jk} w(\xi)_{kj} \overline{v(x)} dx$$

$$= d_{\xi} w(\xi)_{kj} \int_{G} e^{-i\phi_{jj}(x, \xi)} \sigma(x, \xi)_{jk} v(x) dx$$

$$= d_{\xi} w(\xi)_{kj} \langle S_{[\xi], j,k}^* v(\xi) \rangle,$$
where
\[
(S^*_{[\xi],j,k}v)(\eta)_{u,v} := \int_G e^{-i\phi_{jj}(x,\xi)\sigma(x,\xi)_{jk}v(x)} dx \cdot \delta([\eta],u,v).([\xi],k,j),
\]
we have
\[
(w, S^*_{[\xi],j,k}v)_{L^2(\hat{G})} := \sum_{[\eta] \in \hat{G}} d_\eta \text{Tr}(w(\eta)(S^*_{[\xi],j,k}v(\eta))^*) = \sum_{[\eta] \in \hat{G}} d_\eta w(\eta)_{uv}(S^*_{[\xi],j,k}v(\eta))^*_{v,u}
\]
\[
= \sum_{[\eta] \in \hat{G}} d_\eta w(\eta)_{uv}(S^*_{[\xi],j,k}v(\eta))_{u,v} \cdot \delta([\eta],u,v).([\xi],k,j)
\]
\[
= d_\xi w(\xi)_{k,j}(S^*_{[\xi],j,k}v)(\xi)
\]
which shows that \( S^*_{[\xi],j,k} \) is the adjoint operator of \( S_{[\xi],j,k} \). The next step is to estimate the operators norms,
\[
\|S^*_{[\xi],j,k}S_{[\xi],j,k}\|_{\mathcal{B}(L^2(\hat{G}))}, \quad \|S_{[\xi],j,k}S^*_{[\xi'],j',k'}\|_{\mathcal{B}(L^2(\hat{G}))}.
\]
Observe that for every \([\eta] \in \hat{G}\), and \(1 \leq u, v \leq d_\eta\),
\[
(S^*_{[\xi],j,k}S_{[\xi'],j',k'}w)(\eta)_{u,v} = \int_G e^{-i\phi_{jj}(x,\xi)\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)} dx \cdot \delta([\eta],u,v).([\xi'],k',j')
\]
\[
= d_\xi' e^\frac{i\phi_{jj}(x,\xi)\sigma(x,\xi)_{jk}e^\frac{i\phi_{jj}(x,\xi)\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)}}{i\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)} dx \cdot \delta([\eta],u,v).([\xi'],k',j')
\]
\[
= d_\xi' e^\frac{i\phi_{jj}(x,\xi)-\phi_{jj}(x,\xi)\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)}{i\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)} dx \cdot \delta([\eta],u,v).([\xi'],k',j')
\]
\[
= d_\eta w(\eta)_{u,v} \int_G e^{i(\phi_{jj}(x,\xi)-\phi_{jj}(x,\xi))\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)} dx \cdot \delta([\eta],u,v).([\xi'],k',j')
\]
\[
= d_\eta w(\eta)_{u,v} S_{[\xi'],j',k',k'}(\eta, \eta)_{vu},
\]
where we have defined
\[
K_{[\xi],[\xi'],j',k',k'}(\eta, \eta)_{vu} := \int_G e^{i(\phi_{jj}(x,\xi')-\phi_{jj}(x,\xi))\sigma(x,\xi)_{jk}S_{[\xi'],j',k'}w(x)} dx \cdot \delta([\eta],u,v).([\xi'],k',j').
\]
Estimating the \( L^2(\hat{G}) \)-norm of \( S^*_{[\xi],j,k}S_{[\xi'],j',k'}w \), we have
Because,

$$d_e |w(\xi')_{k'}|^2 \leq \sum_{\eta \in G} d_\eta |w(\eta)|^2 = \|w\|_{L^2(G)}^2,$$

we deduce that

$$\|S_{[\xi]}^{*},j,kS_{[\xi'],j',k'}w\|_{L^2(G)}^2 \leq d_e^2 \|K_{[\xi],\xi'],j,j',k,k'}(\xi',\xi')_{j,k'}\|_{L^2(G)},$$

or equivalently,

$$\|S_{[\xi]}^{*},j,kS_{[\xi'],j',k'}\|_{L^2(G)} \leq d_e \|K_{[\xi],\xi'],j,j',k,k'}(\xi',\xi')_{j,k'}|.$$ (10.17)

To estimate $|K_{[\xi],\xi'],j,j',k,k'}(\xi',\xi')_{j,k'}|$, we will use the argument of integration by parts. To do so, using the chain rule, observe that for every $X_k \in X$,

$$X_k e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))} = i \cdot e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))} X_k (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi)).$$

Consequently, the action of the gradient field $\nabla_x$ on the function $e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}$, is given by

$$\nabla_x (e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}) = i \cdot e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))} \nabla_x (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi)).$$

By considering the operator,

$$\mathbb{L} := \frac{1}{i} \sum_{r=1}^n \frac{\nabla_x (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}{|X_r(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))|^2} \cdot \nabla_x,$$

defined by

$$i \mathbb{L} f(x) := \frac{\sum_{r=1}^n X_r (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))(X_s f(x))}{\sum_{r=1}^n |X_r(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))|^2}, \quad f \in C^\infty(G),$$

we have the following identity

$$\mathbb{L} (e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))})$$

$$= \frac{1}{i} \sum_{r=1}^n \frac{\nabla_x (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}{|X_r(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))|^2} \cdot \nabla_x (e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))})$$

$$= \frac{1}{i} \sum_{r=1}^n \frac{|\nabla_x (\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))|^2}{|X_r(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))|^2} \cdot i \cdot e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}$$

$$= e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi)).}$$
Now, by using integration by parts with the operator $L$, for every $N \in \mathbb{N}$, we have,

$$
\int_G e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))} \overline{\sigma(x,\xi)_{jk}} \sigma(x,\xi')_{j'k'} \, dx
$$

$$
= \int_G e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))} \overline{\sigma(x,\xi)_{jk}} \sigma(x,\xi')_{j'k'} \, dx
$$

$$
= \int_G L^N (e^{i(\phi_{j',j}(x,\xi')-\phi_{jj}(x,\xi))}) \overline{\sigma(x,\xi)_{jk}} \sigma(x,\xi')_{j'k'} \, dx
$$

$$
= \int_G \overline{\sigma(x,\xi)_{jk}} \sigma(x,\xi')_{j'k'} \, dx.
$$

Note that the transpose of $L$, $L'$ is given by,

$$
L' := \frac{1}{i} \cdot \nabla_{X'} (\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi)) \nabla_{X'}
$$

$$
= \frac{1}{i} \cdot \frac{\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))}{|\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))|^2} \cdot \nabla_{X'}
$$

where $\nabla_{X'}$ is defined by

$$
\nabla_{X'} f(x) := (-X_1 f, \ldots, -X_n f) \in C^\infty(G) \times \cdots \times C^\infty(G), \ f \in C^\infty(G).
$$

In order to estimate (10.17), using (10.16) we observe that

$$
|\overline{(L')}^N [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}]| \lesssim \frac{1}{|\lambda_{[\xi]}^\tau - \lambda_{[\xi]}^\tau|^N}.
$$

(10.18)

So, from (10.17) and (10.18) we would claim that

$$
\|S_{[\xi],j,k}^* S_{[\xi'],j',k'}\|_{\mathcal{B}(L^2(G))} \lesssim \frac{d_{k'}}{|\lambda_{[\xi]}^\tau - \lambda_{[\xi]}^\tau|^N}.
$$

(10.19)

Indeed, for $N = 1$, by using the Cauchy-Schwarz inequality on $\mathbb{C}^n$ we have,

$$
|\overline{(L')} [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}]|
$$

$$
= \left|\frac{1}{i} \cdot \frac{\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))}{|\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))|^2} \cdot \nabla_{X'} [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}] \right|
$$

$$
\lesssim \frac{\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))}{|\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))|^2} \cdot |\nabla_{X'} [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}]|
$$

$$
= \frac{\nabla_{X}(\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'})}{|\nabla_{X}(\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'})|^2}
$$

$$
\lesssim \left( \sum_{r=1}^n \left| \overline{X_r} [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}] \right|^2 \right)^{\frac{1}{2}}
$$

$$
= \frac{\left( \sum_{r=1}^n \left| \overline{X_r} [\sigma(x,\xi)_{jk} \sigma(x,\xi')_{j'k'}] \right|^2 \right)^{\frac{1}{2}}}{|\nabla_{X}(\phi_{j',j}(x,\xi') - \phi_{jj}(x,\xi))|}.
$$
By the Leibniz rule, and using (10.15), we can estimate
\[
\left( \sum_{r=1}^{n} \left| X_{\sigma(x, \xi)} \sigma(x, \xi')_{jk} \right|^2 \right)^{\frac{1}{2}}
= \left( \sum_{r=1}^{n} \left| X_{\sigma(x, \xi)} \sigma(x, \xi')_{jk} + [\sigma(x, \xi)_{jk} X_{\sigma(x, \xi)} \sigma(x, \xi')]_{jk} \right|^2 \right)^{\frac{1}{2}}
\leq \left( \sum_{r=1}^{n} \left[ \|X_{\sigma(x, \xi)}\|_{op} \|\sigma(x, \xi')\|_{op} + \|\sigma(x, \xi)\|_{op} \|X_{\sigma(x, \xi)} \sigma(x, \xi')\|_{op} \right]^2 \right)^{\frac{1}{2}}
\leq (2n)^{\frac{1}{2}},
\]
and consequently, we have
\[
\left| (L') \left[ \sigma(x, \xi) \right]_{jk} \sigma(x, \xi')_{jk} \right| \leq \frac{1}{|\lambda_{\xi}^r - \lambda_{\xi}^q|},
\]
(10.20)
which proves (10.18) for \( N = 1 \). The general case \( N > 1 \), can be proved using induction. In order to use Lemma 10.9, we need symmetry in the upper bound for the norm of \( S_{[\xi], j, k}^{*} S_{[\xi'], j', k'} \|_{\mathcal{B}(L^2(\hat{G}))} \). So, we trivially can deduce the following inequality
\[
\|S_{[\xi], j, k}^{*} S_{[\xi'], j', k'} \|_{\mathcal{B}(L^2(\hat{G}))} \leq \frac{d_{\xi'} + d_{\xi}}{|\lambda_{\xi}^r - \lambda_{\xi}^q| N},
\]
(10.21)
from (10.19). Let us assume for a moment the inequality
\[
\|S_{[\xi'], j', k'} S_{[\xi], j, k}^{*} \|_{\mathcal{B}(L^2(\hat{G}))} \leq \frac{d_{\xi'} + d_{\xi}}{|\lambda_{\xi}^r - \lambda_{\xi}^q| N}.
\]
(10.22)
So, from Lemma 10.9, it follows the \( L^2(G) \)-boundedness of \( A \), if we prove that
\[
\sup_{[\xi], [\xi'] \neq [\xi'], j, k} \sum_{\xi', j', k' \neq [\xi], j, k} \frac{(d_{\xi'} + d_{\xi})^{\frac{1}{2}}}{|\lambda_{\xi}^r - \lambda_{\xi}^q|^N} = \sup_{[\xi], j, k} \sum_{[\xi'] \neq [\xi], j', k'} \frac{(d_{\xi'} + d_{\xi})^{\frac{1}{2}}}{|\lambda_{\xi}^r - \lambda_{\xi}^q|^N} < \infty.
\]
For this, we will use the Weyl-eigenvalue counting Formula for the Laplacian (see e.g. in Remark 4.21, that in the case of the Laplacian, \( Q = n, s = 1 \) in order to deduce that \( N(\lambda) = O(\lambda^n) \)). First, we can split the sums as follows
\[
\sum_{[\xi], [\xi'] \neq [\xi'], j, k} \frac{(d_{\xi'} + d_{\xi})^{\frac{1}{2}}}{|\lambda_{\xi}^r - \lambda_{\xi}^q|^N} = \sum_{\lambda_{\xi} |< \lambda_{\xi'}^{s, j, k}} \frac{(d_{\xi'} + d_{\xi})^{\frac{1}{2}}}{|\lambda_{\xi}^r - \lambda_{\xi}^q|^N} + \sum_{\lambda_{\xi} |> \lambda_{\xi'}^{s, j, k}} \frac{(d_{\xi'} + d_{\xi})^{\frac{1}{2}}}{|\lambda_{\xi}^r - \lambda_{\xi}^q|^N}\]
\[
:= I + II.
\]
To estimate $I$, observe that

$$
I = \sum_{\lambda_{[e]}<\lambda_{[e']},j,k} \frac{(d_{e'} + d_{e})^{\frac{1}{2}}}{\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}} = \sum_{\lambda_{[e]}<\lambda_{[e']},j,k} \frac{(d_{e'} + d_{e})^{\frac{1}{2}}}{(\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}})^{\frac{n}{2}}}
$$

$$
\leq \sum_{\lambda_{[e]}<\lambda_{[e']},j,k} \frac{(\langle \xi' \rangle^{\frac{n}{2}} + \langle \xi \rangle^{\frac{n}{2}})^{\frac{1}{2}}}{(\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}})^{\frac{n}{2}}} = \sum_{\lambda_{[e]}<\lambda_{[e']},j,k} \frac{(\langle \xi' \rangle^{\frac{n}{2}} + \langle \xi \rangle^{\frac{n}{2}})^{\frac{1}{2}}}{(\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}})^{\frac{n}{2}}}
$$

$$
= \lambda_{[e']}^{\frac{n}{2}} \sum_{\lambda_{[e]}<\lambda_{[e']},j,k} \frac{1}{(1 - \lambda_{[e]}^{\frac{n}{2}} \lambda_{[e']}^{\frac{n}{2}})^{\frac{n}{2}}}
$$

$$
\leq \varepsilon_0^{-\frac{n}{2}} \lambda_{[e']}^{\frac{n}{2}} \sum_{\lambda_{[e]}<\lambda_{[e']}} d_{e}^{\frac{1}{2}}
$$

$$
= O(\varepsilon_0^{-\frac{n}{2}} \lambda_{[e']}^{\frac{n}{2}} \lambda_{[e]}^{\frac{n}{2}}) < \infty,
$$

where we have used $\varepsilon_0$ from (10.14), and also that $\frac{n}{4} - \frac{N}{2} + n = \frac{5n}{4} - \frac{N}{2} \geq 0$, or equivalently that $N \geq \frac{5n}{2}$. Now, we will estimate $II$ as follows,

$$
\sum_{\lambda_{[e]}} \frac{(d_{e'} + d_{e})^{\frac{1}{2}}}{|\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}|^{\frac{n}{2}}} \leq \sum_{\lambda_{[e]}}>\lambda_{[e']},j,k \frac{d_{e}^{\frac{1}{2}}}{|\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}|^{\frac{n}{2}}}
$$

$$
\leq \sum_{\lambda_{[e]}>\lambda_{[e']},j,k} \frac{d_{e}^{\frac{1}{2}}}{|\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}|^{\frac{n}{2}}} \leq \sum_{\lambda_{[e]}>\lambda_{[e']},j,k} \frac{\lambda_{[e]}^{\frac{n}{2}}}{|\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}|^{\frac{n}{2}}}
$$

Observe that

$$
\sum_{\lambda_{[e]}>\lambda_{[e']},j,k} \frac{\lambda_{[e]}^{\frac{n}{2}}}{|\lambda_{[e']}^{\frac{n}{2}} - \lambda_{[e]}^{\frac{n}{2}}|^{\frac{n}{2}}} \leq \varepsilon_0^{-\frac{n}{2}} \sum_{\lambda_{[e]}>\lambda_{[e']},j,k} \lambda_{[e]}^{\frac{n}{2}} \leq \varepsilon_0^{-\frac{n}{2}} \sum_{\lambda_{[e]} \notin \mathcal{G}} d_{e}^{2} \lambda_{[e]}^{\frac{n}{2}} \lambda_{[e]}^{\frac{n}{2}} < \infty,
$$

provided that $N \geq \frac{5n}{2}$. In order to finish the proof we will estimate the norm $\|S_{[e],j,k}S_{[e']}',j',k'\|_{\mathcal{B}(L^2(\mathcal{G}),L^2(\mathcal{G}))}$ as we have assumed in (10.22). However, observe that

$$
S_{[e],j,k}(S_{[e']}',j',k' \sigma(x,\xi)_{j,k} \left[ S_{[e']}',j',k' \sigma(y,\xi')_{j,k} \right]_{\mathcal{G}} \cdot \delta_{([e],j,k),([e'],j',k')}
$$

$$
= d_{e} e^{i\phi_{j,k}(x,\xi)} \sigma(x,\xi)_{j,k} \int_{\mathcal{G}} e^{-i\phi_{j,k}(y,\xi')} \sigma(y,\xi')_{j,k} \sigma(y,\xi')_{j,k} dy \cdot \delta_{([e],j,k),([e'],j',k')}.
$$
So, we can estimate
\[
\|s_{(\xi),j,k}^{*} s_{(\xi'),j',k'}^{*}\|_{L^2(G)} \leq \|d_{\xi} e^{-i\phi_{j,k}^{\xi}(\xi')} \|_{L^1(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
\[
 \leq \sup_{x \in G, u, v} |\sigma(x, \xi)_{uv}| d_{\xi} v \|_{L^2(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
\[
 \leq \sup_{x \in G} \|\sigma(x, \xi)\|_{L^2(G)} \|d_{\xi} v \|_{L^2(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
\[
 \leq \sup_{x \in G} \|\sigma(x, \xi)\|_{L^2(G)} \|d_{\xi} v \|_{L^2(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
\[
 \leq \sup_{x \in G} \|\sigma(x, \xi)\|_{L^2(G)} \|d_{\xi} v \|_{L^2(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
\[
 \leq \sup_{x \in G} \|\sigma(x, \xi)\|_{L^2(G)} \|d_{\xi} v \|_{L^2(G)} \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
So, from the estimate
\[
\|s_{(\xi),j,k}^{*} s_{(\xi'),j',k'}^{*}\|_{L^2(G)} \leq \sigma \left( d_{\xi} + d_{\xi'} \right) \cdot \delta_{(\xi),j,k}.(\xi',j',k')
\]
we can deduce (10.22). The proof is complete.

Remark 10.12. In Theorem 10.11, we can replace the conditions
\[
\sup_{(x, [\xi]) \in G \times G} \|x^{\alpha} \sigma(x, \xi)\|_{L^2(G)} < \infty, \quad |\nabla_{x} \phi_{jj}^{(\xi)}(x, \xi) - \nabla_{x} \phi_{j'j'}^{(\xi)}(x, \xi')| = |\lambda_{\xi}^{j} - \lambda_{\xi}^{j'}|
\]
Example 11.2 (The sub-Laplacian on SO(4)). Consider the Lie group SO(4) = \{g \in GL(3, \mathbb{R}) : gg^t = I_4 \equiv (\delta_{ij})_{1 \leq i, j \leq 4}, \det(g) = 1\}, with Lie algebra \mathfrak{so}(4). The latter consists of all skew-symmetric matrices. This Lie algebra is generated by matrices of the form
\[ B^{ij} := e_i e^t_j - e_j e^t_i, \]
where \{e_i\}_{i=1}^4 are the canonical vectors in \mathbb{R}^4. Let us define
\[ X_1 := B^{12}, X_2 := B^{14}, X_3 := B^{24}, X_4 := B^{34}. \]
By writing
\[ Z_1 := -[X_2, X_4] = B^{13}, \text{ and } Z_2 := -[X_3, X_4] = B^{23}, \]
from Table 1 the system of vector fields \( X = \{X_1, X_2, X_3, X_4\} \) satisfies the Hörmander condition of step \( \kappa = 2 \) (see Berge and Grong [15]). So, the Hausdorff dimension associated to the control distance associated to the sub-Laplacian
\[ \mathcal{L} = -X_1^2 - X_2^2 - X_3^2 - X_4^2, \]
can be computed from (3.1) as follows.
\[ Q : = \dim(H^1G) + \sum_{i=1}^2 (i + 1)(\dim H^{i+1}G - \dim H^iG) = 4 + 2(6 - 4) = 8. \]

| Table 1. Commutators in SO(4) |
|-----------------------------|
| \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( Z_1 \) | \( Z_2 \) |
| \( X_1 \) | 0 | \(-X_3\) | \(X_2\) | 0 | \(-Z_2\) | \(Z_1\) |
| \( X_2 \) | \(X_3\) | 0 | \(-X_1\) | \(-Z_1\) | \(X_4\) | 0 |
| \( X_3 \) | \(-X_2\) | \(X_1\) | 0 | \(-Z_2\) | 0 | \(X_4\) |
| \( X_4 \) | 0 | \(Z_1\) | \(Z_2\) | 0 | \(-X_2\) | \(-X_3\) |
| \( Z_1 \) | \(Z_2\) | \(-X_4\) | 0 | \(X_2\) | 0 | \(-X_1\) |
| \( Z_2 \) | \(-Z_1\) | 0 | \(-X_4\) | \(X_3\) | \(X_1\) | 0 |

Example 11.3 (The sub-Laplacian on SU(3)). The special unitary group of 3 \times 3 complex matrices is defined by
\[ \text{SU}(3) = \{g \in GL(3, \mathbb{C}) : gg^* = I_3 \equiv (\delta_{ij})_{1 \leq i, j \leq 3}, \det(g) = 1\}, \]
and its Lie algebra is given by
\[ \mathfrak{su}(3) = \{g \in GL(3, \mathbb{C}) : g + g^* = 0, \text{ Tr}(g) = 0\}. \]
The inner product is defined by a multiple of the Killing form on \( \mathfrak{su}(3) \) given by
\[ B(X, Y) = -1/2\text{Tr}[XY]. \]
The torus
\[ \mathbb{T}_{\text{SU}(3)} = \text{diag}[e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}] : \theta_1 + \theta_2 + \theta_3 = 0, \theta_i \in \mathbb{R} \]
is a maximal torus of SU(3), and its Lie algebra is given by
\[ \mathfrak{t}_{\text{su}(3)} = \text{diag}[i\theta_1, i\theta_2, i\theta_3] : \theta_1 + \theta_2 + \theta_3 = 0, \theta_i \in \mathbb{R}. \]
The following vectors

\[ T_1 = \text{diag}[-i, i, 0], \quad T_2 = \text{diag}[-i/\sqrt{3}, -i/\sqrt{3}, 2i/\sqrt{3}] \]

provide a basis for \( t_{\text{su}(3)} \). Completing this basis with the following vectors

\[
X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \\
X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\]

we obtain the Gell-Mann system, which forms an orthonormal basis of \( \text{su}(3) \). The system of vector fields \( X = \{X_1, X_2, X_3, X_4, X_5, X_6\} \) satisfies the Hörmander condition at step \( \kappa = 2 \), (see Domokos and Manfredi [55]). Indeed, it can be deduced if we write

\[
X_7 = -[X_1, X_2] = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
X_8 = -[X_3, X_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix},
\]

from Table 2. Observe that the Hausdorff dimension associated to the control

\[
\text{Table 2. Commutators in SU}(3)
\]

|     | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 \) | \( X_8 \) |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( X_1 \) | 0         | \(-X_7\)   | \( X_5\)  | \(-X_6\)  | \(-X_3\)  | \( X_4\)  | 4\( X_2\) | 2\( X_2\) |
| \( X_2 \) | \( X_7\)  | 0         | \( X_6\)  | \( X_5\)  | \(-X_4\)  | \(-X_3\)  | 4\( X_1\) | 2\( X_1\) |
| \( X_3 \) | \(-X_5\) | \(-X_6\)  | 0         | \(-X_8\)  | \( X_1\)  | \( X_2\)  | 2\( X_4\) | 4\( X_4\) |
| \( X_4 \) | \( X_6\)  | \(-X_5\)  | \( X_8\)  | 0         | \( X_2\)  | \(-X_1\)  | \(-2X_3\) | \(-4X_3\) |
| \( X_5 \) | \( X_3\)  | \( X_4\)  | \(-X_1\)  | \(-X_2\)  | \( 0\)    | \( X_8\)  | \(-X_7\)  | \( 2X_6\) |
| \( X_6 \) | \(-X_4\) | \( X_3\)  | \(-X_2\)  | \( X_1\)  | \( X_7\)  | \(-X_8\)  | 0         | \(-2X_5\) |
| \( X_7 \) | \(-4X_2\) | \( 4X_1\) | \(-2X_4\) | \( 2X_3\) | \(-2X_6\) | \( 2X_5\) | \( 0\)     | \( 0\)     |
| \( X_8 \) | \( 2X_2\) | \( 2X_1\) | \(-4X_4\) | \( 4X_3\) | \( 2X_6\) | \(-2X_5\) | \( 0\)     | \( 0\)     |

distance associated to the sub-Laplacian

\[ \mathcal{L} = -X_1^2 - X_2^2 - X_3^2 - X_4^2 - X_5^2 - X_6^2, \]

can be computed from (3.1) as follows.

\[ Q = \dim(H^1G) + 2(\dim H^2G - \dim H^1G) = 6 + 2(8 - 6) = 10. \]
Example 11.4 (The sub-Laplacian on Spin(4) ≃ SU(2) × SU(2)). Let us consider the Lie algebra of SU(2), \( \mathfrak{su}(2) \) spanned by the following matrices

\[
A = (1/\sqrt{2}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B = (1/\sqrt{2}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = (1/\sqrt{2}) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

On the Lie group Spin(4) ≃ SU(2) × SU(2) with Lie algebra \( \mathfrak{spin}(4) \equiv \mathfrak{su}(2) \times \mathfrak{su}(2) \), let us define the following vector fields,

\[
X^\pm = A_1 \pm A_2, \quad Y^\pm = B_1 \pm B_2, \quad Z^\pm = C_1 \pm C_2,
\]

where

\[
A_1 = A \otimes I, \quad A_2 = I \otimes A, \quad B_1 = B \otimes I, \quad B_2 = I \otimes B, \quad C_1 = C \otimes I, \quad C_2 = I \otimes C.
\]

For every real-number \( c \in (\infty, \infty) \), let

\[
X^c := X^- + cX^+.
\]

Table 3. Commutators in Spin(4) ≃ SU(2) × SU(2)

| \( X^c \) | \( Y^- \) | \( Z^- \) | \( X^+ \) | \( Y^+ \) | \( Z^+ \) |
|---|---|---|---|---|---|
| \( X^c \) | 0 | \( Z^+ + cZ^- \) | \( -Y^+ - cY^- \) | 0 | \( Z^- + cZ^+ \) | \( -Y^- - cY^+ \) |
| \( Y^- \) | \( -Z^+ - cZ^- \) | 0 | \( X^+ - Z^- \) | 0 | \( X^c - cX^+ \) |
| \( Z^- \) | \( Y^+ + cY^- \) | \( -X^+ \) | 0 | \( Y^- - cX^+ - X^c \) | 0 |
| \( X^+ \) | 0 | \( Z^- \) | \( -Y^- \) | 0 | \( -Z^+ \) | \( -Y^+ \) |
| \( Y^+ \) | \( -Z^- - cZ^+ \) | 0 | \( X^c - cX^+ \) | \( Z^+ \) | 0 | \( X^+ \) |
| \( Z^+ \) | \( Y^- + cY^+ \) | \( cX^+ - X^c \) | 0 | \( Y^+ \) | \( -X^+ \) | 0 |

The system \( X = \{ X^c, Y^-, Z^- \} \) satisfies the Hörmander condition, and we can consider the associated sub-Laplacian associated with \( X \),

\[
\mathcal{L} = -(X^c)^2 - (Y^-)^2 - (Z^-)^2.
\]

12. Appendix II: Subelliptic Besov spaces

In this appendix we present the description of subelliptic Besov spaces as in [33]. On \( \mathbb{R}^n \), Besov spaces appear in [16, 17]. We refer the reader to Triebel [122, 123], Furioli, Melzi, and Veneruso, [69], Nursultanov, Ruzhansky and Tikhonov [93, 94], Peetre [96, 97], and [26], for the analytic aspects of the theory of Besov spaces on \( \mathbb{R}^n \) and other Lie groups.

In order to define subelliptic Besov spaces we will use the notion of dyadic decompositions. Here, the sequence \( \{ \psi_\ell \}_{\ell \in \mathbb{N}_0} \) is a dyadic decomposition, defined as follows: we choose a function \( \psi_0 \in C_0^\infty(\mathbb{R}) \), \( \psi_0(\lambda) = 1 \), if \( |\lambda| \leq 1 \), and \( \psi_0(\lambda) = 0 \), for \( |\lambda| \geq 2 \). For every \( j \geq 1 \), let us define \( \psi_j(\lambda) = \psi_0(2^{-j}\lambda) - \psi_0(2^{-j+1}\lambda) \). For \( \psi(\lambda) := \psi_0(\lambda) - \psi_0(2\lambda) \), \( \psi_j(\lambda) = \psi(2^{-j}\lambda) \). In particular, we have

\[
\sum_{\ell \in \mathbb{N}_0} \psi_\ell(\lambda) = 1, \quad \text{for every} \quad \lambda > 0.
\]
We define the operator
\[
\psi_{\ell}(D)f(x) := \sum_{[\xi] \in \hat{G}} d_{\xi} \text{Tr}[\xi(x)\psi_{\ell}((I_{d_{\xi}} + \hat{\mathcal{L}}(\xi))^{1/2})\hat{f}(\xi)], \quad f \in C^\infty(G).
\] (12.2)

So, for \( s \in \mathbb{R}, 0 < q < \infty \), the subelliptic Besov space \( B^s_{p,q}(G) \) consists of those functions/distributions satisfying
\[
\|f\|_{B^s_{p,q}(G)} = \left( \sum_{\ell \in \mathbb{N}_0} 2^{\ell q s} \|\psi_{\ell}(D)f\|_{L^p(G)}^q \right)^{1/q} < \infty,
\]
for \( 0 < p \leq \infty \), with the following modification
\[
\|f\|_{B^s_{p,q}(G)} = \sup_{\ell \in \mathbb{N}_0} 2^{\ell q s} \|\psi_{\ell}(D)f\|_{L^p(G)} < \infty,
\]
when \( q = \infty \). In [33], the authors have described the subelliptic Besov spaces in terms of the matrix-valued quantization. We record it as follows (see Remark 4.2 in [33]).

**Remark 12.1 (Fourier description for subelliptic Besov spaces).** If we write \( \hat{\mathcal{M}}(\xi) := (\mathcal{M}\xi)(\epsilon_G) \) for the symbol of the operator \((1 + \mathcal{L})^{1/2}\), which is given by,
\[
\hat{\mathcal{M}}(\xi) = \begin{bmatrix}
(1 + \nu_1(\xi)^2)^{1/2} & 0 & 0 & \ldots & 0 \\
0 & (1 + \nu_{22}(\xi)^2)^{1/2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (1 + \nu_{d_{\xi}d_{\xi}}(\xi)^2)^{1/2}
\end{bmatrix},
\]
so that \( \hat{\mathcal{M}}(\xi) = \text{diag}[(1 + \nu_{ii}(\xi)^2)^{1/2}]_{1 \leq i \leq d_{\xi}} \), then \( \psi_{\ell}(\xi) \) denotes the symbol of the operator \( \psi_{\ell}(D) \), and we have
\[
\psi_{\ell}(\xi) = \text{diag}[\psi_{\ell}((1 + \nu_{ii}(\xi)^2)^{1/2})]_{1 \leq i \leq d_{\xi}}, \quad \ell \in \mathbb{N}_0, \ [\xi] \in \hat{G}.
\]
Then the subelliptic Besov spaces can be re-written as
\[
\|f\|_{B^s_{p,q}(G)} = \sum_{\ell \in \mathbb{N}_0} 2^{\ell q s} \left\| \text{Tr}[\xi(x)\text{diag}[\psi_{\ell}((1 + \nu_{ii}(\xi)^2)^{1/2})]\hat{f}(\xi)] \right\|_{L^p(G)}^q < \infty,
\] (12.3)
for \( 0 < p \leq \infty \), and
\[
\|f\|_{B^s_{p,q}(G)} = \sup_{\ell \in \mathbb{N}_0} 2^{\ell q s} \left\| \text{Tr}[\xi(x)\text{diag}[\psi_{\ell}((1 + \nu_{ii}(\xi)^2)^{1/2})]\hat{f}(\xi)] \right\|_{L^p(G)} < \infty,
\] (12.4)
for \( q = \infty \).

**Remark 12.2.** Subelliptic Sobolev and Besov spaces may depend on the choice of a sub-Laplacian on a compact Lie group (see Remark 14.2). This is one of the contrasts to the case of graded Lie groups (see [65, Chapter 4]).

The following are the embedding properties proved in Theorem 4.3 of [33].
Theorem 12.3. Let $G$ be a compact Lie group and let us denote by $Q$ the Hausdorff dimension of $G$ associated to the control distance associated to the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition. Then

1. $B^{p+\varepsilon,\mathcal{L}}_{p,q_1}(G) \hookrightarrow B^{p,\mathcal{L}}_{p,q_1}(G) \hookrightarrow B^{p,\mathcal{L}}_p(G), \varepsilon > 0, 0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty$.

2. $B^{p+\varepsilon,\mathcal{L}}_{p,q_1}(G) \hookrightarrow B^{p,\mathcal{L}}_{p,q_1}(G), \varepsilon > 0, 0 < p \leq \infty, 1 \leq q_2 < q_1 < \infty$.

3. $B^{p,\mathcal{L}}_{p_1,q}(G) \hookrightarrow B^{p_2,\mathcal{L}}_{p_2,q}(G), 1 \leq p_1 \leq p_2 \leq \infty, 0 < q_2 < \infty, r_1 \in \mathbb{R},$ and $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$.

4. $H^{r,\mathcal{L}}(G) = B^{r,\mathcal{L}}_{2,2}(G)$ and $B^{r,\mathcal{L}}_{p,p}(G) \hookrightarrow L^p(G) \hookrightarrow B^{r,\mathcal{L}}_{p,2}(G), 1 < p \leq 2$.

5. $B^{p,\mathcal{L}}_{p_1}(G) \hookrightarrow L^q(G), 1 \leq p \leq q \leq \infty, r = Q(\frac{1}{p} - \frac{1}{q})$ and $L^q(G) \hookrightarrow B^{0,\mathcal{L}}_{q,\infty}(G)$ for $1 < q \leq \infty$.

In order to compare subelliptic Besov spaces with Besov spaces associated to the Laplacian, we start by presenting the problem for Sobolev spaces (see Theorem 5.9 of [33]). Here,

$$\kappa := [n/2] + 1,$$

is the smallest integer larger than $n/2$, $n = \dim(G)$.

Theorem 12.4. Let $G$ be a compact Lie group and let us consider the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$, where the system of vector fields $X = \{X_i\}_{i=1}^k$ satisfies the Hörmander condition of order $\kappa$. Then we have the continuous embeddings

$$L^p_s(G) \hookrightarrow B^{p,\kappa,\mathcal{L}}_s(G) \hookrightarrow L^p_{\frac{\kappa}{\kappa - \kappa(1 - \frac{1}{\kappa}) \left\| \frac{1}{2} - \frac{1}{p} \right\|}(G).$$

More precisely, for every $s \geq 0$ there exist constants $C_a > 0$ and $C_b > 0$ satisfying,

$$C_a \| f \|_{B^{p,\kappa,\mathcal{L}}_s(G)} \leq \| f \|_{L^p_s(G)}, f \in L^p_s(G),$$

where $s_{\kappa,\kappa} := \frac{s}{\kappa} - \kappa(1 - \frac{1}{\kappa}) \left\| \frac{1}{2} - \frac{1}{p} \right\|$, and

$$\| f \|_{L^p_s(G)} \leq C_b \| f \|_{L^p_s(G)}, f \in L^p_s(G).$$

Consequently, we have the following embeddings

$$L^p_{\frac{\kappa}{\kappa + \kappa(1 - \frac{1}{\kappa}) \left\| \frac{1}{2} - \frac{1}{p} \right\|}(G) \hookrightarrow L^p_{-\kappa s}(G) \hookrightarrow L^p_{-s}(G).$$

The following theorem (see Theorem 5.11 [33]) shows some embedding properties between subelliptic Besov spaces and the Besov spaces associated to the Laplacian.

Theorem 12.5. Let $G$ be a compact Lie group and let us consider the sub-Laplacian $\mathcal{L} = -(X_1^2 + \cdots + X_k^2)$, where the system of vector fields $X = \{X_i\}_{i=1}^k$
satisfies the Hörmander condition of order \( \kappa \). Let \( s \geq 0 \), \( 0 < q \leq \infty \), and \( 1 < p < \infty \). Then we have the continuous embeddings
\[
B^s_{p,q}(G) \hookrightarrow B^{s',\mathcal{L}}_{p,q}(G) \hookrightarrow B_{p,q}^{s,\mathcal{L}}(G) \hookrightarrow B_{p,q}^{s,\mathcal{L}}(G).
\]
More precisely, for every \( s \geq 0 \) there exist constants \( C_a > 0 \) and \( C_b > 0 \) satisfying,
\[
C_a \| f \|_{B^s_{p,q}(G)} \leq \| f \|_{B^{s,\mathcal{L}}_{p,q}(G)}, \quad f \in B^{s,\mathcal{L}}_{p,q}(G),
\]
where \( s_{\kappa,\nu} := \frac{s}{\kappa} - \nu(1 - \frac{1}{\kappa}) \frac{1}{2} - \frac{1}{p} \), and
\[
\| f \|_{B_{p,q}^{s,\mathcal{L}}(G)} \leq C_b \| f \|_{B^s_{p,q}(G)}, \quad f \in B^{s}_{p,q}(G).
\]
Consequently, we have the following embeddings
\[
B_{p,q}^{-\frac{s}{\kappa} + \nu(1 - \frac{1}{\kappa}) \frac{1}{2} - \frac{1}{p}}(G) \hookrightarrow B^{-s,\mathcal{L}}_{p,q}(G) \hookrightarrow B^{-s,\mathcal{L}}_{p,q}(G).
\]

Finally we summarise the action of the Hörmander classes on subelliptic Sobolev and Besov spaces (See Theorems 1.2 and 1.3 in [33]).

**Theorem 12.6** (Fefferman Subelliptic Sobolev Theorem). Let \( G \) be a compact Lie group of dimension \( n \). Let us assume that \( \sigma \in \mathcal{S}_{\rho,\delta}^{-\nu}(G \times \hat{G}) \) and let \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \). Then \( A \equiv \sigma(x, D) \) extends to a bounded operator from \( L^{p,\mathcal{L}}_{\vartheta}(G) \) to \( L^p(G) \) provided that
\[
n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| - (\vartheta/\kappa) \leq \nu.
\]
In particular, if \( \sigma \in \mathcal{S}_{\rho,\delta}^0(G) \), the operator \( A \equiv \sigma(x, D) \) extends to a bounded operator from \( L^{p,\mathcal{L}}_{\vartheta}(G) \) to \( L^p(G) \) with
\[
n\kappa(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \vartheta.
\]

**Theorem 12.7** (Fefferman Subelliptic Besov Theorem). Let us assume that \( \sigma \in \mathcal{S}_{\rho,\delta}^{-\nu}(G \times \hat{G}) \) and let \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \). Then \( A \equiv \sigma(x, D) \) extends to a bounded operator from \( B^{s+\vartheta,\mathcal{L}}_{p,q}(G) \) to \( B^s_{p,q}(G) \) provided that
\[
n(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| - (\vartheta/\kappa) \leq \nu.
\]
In particular, if \( \sigma \in \mathcal{S}_{\rho,\delta}^0(G) \), the operator \( A \equiv \sigma(x, D) \) extends to a bounded operator from \( B^{s+\vartheta,\mathcal{L}}_{p,q}(G) \) to \( B^s_{p,q}(G) \) with
\[
n\kappa(1 - \min\{\rho, 1/\kappa\}) \left| \frac{1}{p} - \frac{1}{2} \right| \leq \vartheta.
\]

Observe that Theorem 12.6 and Theorem 12.7 are analogies of the ones obtained in this work for the subelliptic Hörmander classes (see Theorems 6.6 and 6.9). The mapping properties for the global calculus developed in [107] on \( L^p(G) \), Sobolev spaces \( L^p(G) \), Besov spaces \( B^r_{p,q}(G) \) and subelliptic Sobolev and Besov spaces can be found in the references [107, Chapter 10], the works of the second author and J. Wirth [111, 112], [24, 25, 27, 33] and the work of the second author and J. Delgado [46].
13. Appendix III: A characterisation for global Hörmander classes on graded Lie groups

It was proved in Theorem 4.13 and consequently in Corollary 4.14 that the following family of seminorms,

$$p'_{\alpha,\beta,\rho,\delta,m}(a) := \sup_{(x,\xi) \in G \times \hat{G}} \|\widehat{\mathcal{M}(\xi)} [\partial^{[\alpha]} - \partial^{[\beta]} - m - r] \mathcal{G}^{(\beta)} \Delta^{\alpha}_x a(x,\xi) \mathcal{M}(\xi)^r \|_{op} < \infty,$$

(13.1)
can be used to define the subelliptic Hörmander class $S^{m,L}_{\rho,\delta}(G \times \hat{G})$. More precisely, we have proved that the following conditions are equivalent:

(A'). $\forall \alpha, \beta \in \mathbb{N}^n_0, \forall r \in \mathbb{R}, p'_{\alpha,\beta,r,m}(a) < \infty$.
(B'). $\forall \alpha, \beta \in \mathbb{N}^n_0, p'_{\alpha,\beta,0,m}(a) < \infty$.
(C'). $\forall \alpha, \beta \in \mathbb{N}^n_0, p'_{\alpha,\beta,m+\delta,\rho(\alpha),m}(a) < \infty$.
(D'). $\forall \alpha, \beta \in \mathbb{N}^n_0, \exists r_0 \in \mathbb{R}, p'_{\alpha,\beta,r_0,m}(a) < \infty$.
(E'). $a \in S^{m,L}_{\rho,\delta}(G \times \hat{G})$.

The main goal of this appendix is to prove an analogy of Theorem 4.13 for the global Hörmander classes on arbitrary graded Lie groups developed by the second author and V. Fischer in [65]. The main result of this appendix is Theorem 13.16 where we prove that the following seminorm inequalities are equivalent: (see the next subsections for the notations and definitions)

(A). $\forall \alpha, \beta \in \mathbb{N}^n_0, \forall \gamma \in \mathbb{R}, p_{\alpha,\beta,\gamma,m}(\sigma) < \infty$.
(B). $\forall \alpha, \beta \in \mathbb{N}^n_0, p_{\alpha,\beta,0,m}(\sigma) < \infty$.
(C). $\forall \alpha, \beta \in \mathbb{N}^n_0, p_{\alpha,\beta,m+\delta,\rho(\alpha),m}(\sigma) < \infty$.
(D). $\forall \alpha, \beta \in \mathbb{N}^n_0, \exists r_0 \in \mathbb{R}, p_{\alpha,\beta,r_0,m}(\sigma) < \infty$.
(E). $\sigma \in S^{m,L}_{\rho,\delta}(G \times \hat{G})$.

In the case of the Heisenberg group it was proved in Theorem 6.5.1 of [65, Page 479] that (A), (B), (C) and (D) are equivalent conditions and, in Section 5.5 of [65, Page 479], it was proved that on a arbitrary graded Lie group only (A), (B) and (C) are equivalent seminorms. Condition (D), is certainly, most useful. We will prove Theorem 13.16 by following the arguments in Section 4.

13.1. Homogeneous and graded Lie groups. The notation and terminology of this appendix on the analysis of homogeneous Lie groups are mostly taken from Folland and Stein [67]. For the theory of pseudo-differential operators we will follow the setting developed in [65] through the notion of (operator-valued) global symbols. If $E, F$ are Hilbert spaces, $\mathcal{B}(E, F)$ denotes the algebra of bounded linear operators from $E$ to $F$, and also we will write $\mathcal{B}(E) = \mathcal{B}(E, E)$.

Let $G$ be a homogeneous Lie group. This means that $G$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $D^\theta_r$, $r > 0$, which are automorphisms on $\mathfrak{g}$ satisfying the following two conditions:

- For every $r > 0$, $D^\theta_r$ is a map of the form $D^\theta_r = \text{Exp}(rA)$ for some diagonalisable linear operator $A \equiv \text{diag}[\nu_1, \cdots, \nu_n]$ on $\mathfrak{g}$.
\begin{itemize}
    \item $\forall X, Y \in \mathfrak{g}$, and $r > 0$, $[D^\mathbb{R}_r X, D^\mathbb{R}_r Y] = D^\mathbb{R}_r [X, Y]$. We call the eigenvalues of $A, \nu_1, \nu_2, \ldots, \nu_n$, the dilations weights or weights of $G$. The homogeneous dimension of a homogeneous Lie group $G$ is given by
    \[ Q = \text{Tr}(A) = \nu_1 + \cdots + \nu_n. \]
    The dilations $D^\mathbb{R}_r$ of the Lie algebra $\mathfrak{g}$ induce a family of maps on $G$ defined via
    \[ D_r := \exp_G \circ D^\mathbb{R}_r \circ \exp_G^{-1}, \quad r > 0, \]
    where $\exp_G : \mathfrak{g} \to G$ is the usual exponential mapping associated to the Lie group $G$. We refer to the family $D_r$, $r > 0$, as dilations on the group. If we write $r x = D_r(x)$, $x \in G$, $r > 0$, then a relation on the homogeneous structure of $G$ and the Haar measure $dx$ on $G$ is given by
    \[ \int_G (f \circ D_r)(x) dx = r^{-Q} \int_G f(x) dx. \]
    A Lie group is graded if its Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$. Examples of such groups are the Heisenberg group $\mathbb{H}^n$ and more generally any stratified groups where the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$. Here, $n$ is the topological dimension of $G$, $n = n_1 + \cdots + n_s$, where $n_k = \dim \mathfrak{g}_k$. A Lie algebra admitting a family of dilations is nilpotent, and hence so is its associated connected, simply connected Lie group. The converse does not hold, i.e., not every nilpotent Lie group is homogeneous (see Dyer [53]) although they exhaust a large class (see Johnson [82, page 294]). Indeed, the main class of Lie groups under our consideration is that of graded Lie groups. A graded Lie group $G$ is a homogeneous Lie group equipped with a family of weights $\nu_j$, all of them positive rational numbers. Let us observe that if $\nu_i = \frac{a_i}{b_i}$ with $a_i, b_i$ integer numbers, and $b$ is the least common multiple of the $b_i$'s, the family of dilations
    \[ D^\mathbb{R}_r = \text{Exp}(\log(r^b) A) : \mathfrak{g} \to \mathfrak{g}, \]
    have integer weights, $\nu_i = \frac{a_i}{b_i}$. So, here we always assume that the weights $\nu_j$, defining the family of dilations are non-negative integer numbers which allow us to assume that the homogeneous dimension $Q$ is a non-negative integer number. This is a natural context for the study of Rockland operators (see Remark 4.1.4 of [65]).
\end{itemize}

13.2. Fourier analysis on nilpotent Lie groups. Let $G$ be a simply connected nilpotent Lie group. Let us assume that $\pi$ is a continuous, unitary and irreducible representation of $G$, this means that,
\begin{itemize}
    \item $\pi \in \text{Hom}(G, \mathbb{U}(H_\pi))$, for some separable Hilbert space $H_\pi$, i.e. $\pi(xy) = \pi(x)\pi(y)$ and for the adjoint of $\pi(x)$, $\pi(x)^* = \pi(x^{-1})$, for every $x, y \in G$.
    \item The map $(x, v) \mapsto \pi(x)v$, from $G \times H_\pi$ into $H_\pi$ is continuous.
    \item For every $x \in G$, and $W_\pi \subset H_\pi$, if $\pi(x)W_\pi \subset W_\pi$, then $W_\pi = H_\pi$ or $W_\pi = \emptyset$.
\end{itemize}
Let $\text{Rep}(G)$ be the set of unitary, continuous and irreducible representations of $G$. The relation,
$\pi_1 \sim \pi_2$ if and only if, there exists $A \in \mathbb{B}(H_{\pi_1}, H_{\pi_2})$, such that $A\pi_1(x)A^{-1} = \pi_2(x)$,
for every \( x \in G \), is an equivalence relation and the unitary dual of \( G \), denoted by \( \hat{G} \) is defined via \( \hat{G} := \text{Rep}(G)/\sim \). Let us denote by \( d\pi \) the Plancherel measure on \( \hat{G} \). The Fourier transform of \( f \in \mathcal{S}(G) \), (this means that \( f \circ \exp_G \in \mathcal{S}(g) \), with \( g \simeq \mathbb{R}^{\dim(G)} \)) at \( \pi \in \hat{G} \), is defined by

\[
\hat{f}(\pi) = \int_{G} f(x)\pi(x)^* dx : H_\pi \to H_\pi, \text{ and } \mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\hat{G}) := \mathcal{F}_G(\mathcal{S}(G)).
\]

If we identify one representation \( \pi \) with its equivalence class, \( [\pi] = \{\pi' : \pi \sim \pi'\} \), for every \( \pi \in \hat{G} \), the Kirillov trace character \( \Theta_\pi \) defined by

\[
(\Theta_\pi, f) := \text{Tr}(\hat{f}(\pi)),
\]

is a tempered distribution on \( \mathcal{S}(G) \). In particular, the identity \( f(e_G) = \int\!(\Theta_\pi, f)d\pi \), implies the Fourier inversion formula \( f = \mathcal{F}_G^{-1}(\hat{f}) \), where

\[
(\mathcal{F}_G^{-1}\sigma)(x) := \int_{\hat{G}} \text{Tr}(\pi(x)\sigma(\pi))d\pi, \quad x \in G, \quad \mathcal{F}_G^{-1} : \mathcal{S}(\hat{G}) \to \mathcal{S}(G),
\]
is the inverse Fourier transform. In this context, the Plancherel theorem takes the form \( \|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(\hat{G})} \), where

\[
L^2(\hat{G}) := \int_{\hat{G}} H_\pi \otimes H_\pi^* d\pi,
\]
is the Hilbert space endowed with the norm: \( \|\sigma\|_{L^2(\hat{G})} = (\int_{\hat{G}} \|\sigma(\pi)\|^2_{HS} d\pi)^{\frac{1}{2}} \).

We will fix a homogeneous quasi-norm \( |\cdot| \) on \( G \). This means that \( |\cdot| \) is a non-negative function on \( G \), satisfying

\[
|x| = |x^{-1}|, \quad r|x| = |D_r(x)|, \quad \text{and } |x| = 0 \text{ if and only if } x = e_G,
\]
where \( e_G \) is the identity element of \( G \). It satisfies a triangle inequality with a constant: there exists a constant \( \gamma \geq 1 \) such that \( |xy| \leq \gamma(|x| + |y|) \).

13.3. **Homogeneous linear operators and Rockland operators.** A linear operator \( T : C^\infty(G) \to \mathcal{D}'(G) \) is homogeneous of degree \( \nu \in \mathbb{C} \) if for every \( r > 0 \) the equality

\[
T(f \circ D_r) = r^\nu (Tf) \circ D_r
\]
holds for every \( f \in \mathcal{D}(G) \). If for every representation \( \pi \in \hat{G} \), \( \pi : G \to U(\mathcal{H}_\pi) \), we denote by \( \mathcal{H}_\pi^\infty \) the set of smooth vectors, that is, the space of elements \( v \in \mathcal{H}_\pi \) such that the function \( x \mapsto \pi(x)v, \ x \in \hat{G} \), is smooth, a Rockland operator is a left-invariant differential operator \( \mathcal{R} \) which is homogeneous of positive degree \( \nu = \nu_\mathcal{R} \) and such that, for every unitary irreducible non-trivial representation \( \pi \in \hat{G} \), \( \pi(\mathcal{R}) \) is injective on \( \mathcal{H}_\pi^\infty \); \( \sigma_\mathcal{R}(\pi) = \pi(\mathcal{R}) \) is the symbol associated to \( \mathcal{R} \). It coincides with the infinitesimal representation of \( \mathcal{R} \) as an element of the universal enveloping algebra. It can be shown that a Lie group \( G \) is graded if and only if there exists a differential Rockland operator on \( G \). If the Rockland operator is formally self-adjoint, then \( \mathcal{R} \) and \( \pi(\mathcal{R}) \) admit self-adjoint extensions on \( L^2(G) \).
and \(H_\pi\), respectively. Now if we preserve the same notation for their self-adjoint extensions and we denote by \(E\) and \(E_\pi\) their spectral measures, we will denote by

\[
f(\mathcal{R}) := \int_{-\infty}^{\infty} f(\lambda)dE(\lambda), \quad \text{and} \quad \pi(f(\mathcal{R})) \equiv f(\pi(\mathcal{R})) := \int_{-\infty}^{\infty} f(\lambda)dE_\pi(\lambda),
\]

the operators defined by the functional calculus. In general, we will reserve the notation \(dE_A(\lambda)_{0 < \lambda < \infty}\) for the spectral measure associated with a positive and self-adjoint operator \(A\) on a Hilbert space \(H\).

We now recall a lemma on dilations on the unitary dual \(\hat{\mathcal{G}}\), which will be useful in our analysis of spectral multipliers. For the proof, see Lemma 4.3 of [65].

**Lemma 13.1.** For every \(\pi \in \hat{\mathcal{G}}\) let us define \(D_r(\pi)(x) = \pi(rx)\) for every \(r > 0\) and \(x \in \mathcal{G}\). Then, if \(f \in L^\infty(\mathbb{R})\) then \(f(\pi(r)(\mathcal{R})) = f(r^\nu \pi(\mathcal{R}))\).

13.4. **Symbols and quantization of pseudo-differential operators.** In order to present a consistent definition of pseudo-differential operators one developed in [65] (see the quantisation formula (13.4)), a suitable class of spaces on the unitary dual \(\hat{\mathcal{G}}\) acting in a suitable way with the set of smooth vectors \(H_\pi^\infty\), on every representation space \(H_\pi\). Let now recall the main notions.

**Definition 13.2** (Sobolev spaces on smooth vectors). Let \(\pi_1 \in \text{Rep}(\mathcal{G})\), and \(a \in \mathbb{R}\). We denote by \(H_\pi^a\), the Hilbert space obtained as the completion of \(H_\pi^\infty\) with respect to the norm\n
\[
\|v\|_{H_\pi^a} = \|\pi_1(1 + \mathcal{R})^{\frac{a}{\nu}}v\|_{H_{\pi_1}},
\]

where \(\mathcal{R}\) is a positive Rockland operator on \(\mathcal{G}\) of homogeneous degree \(\nu > 0\).

In order to introduce the general notion of a symbol as the one developed in [65], we will use a suitable notion of operator-valued symbols acting on smooth vectors. We introduce it as follows.

**Definition 13.3.** A \(\hat{\mathcal{G}}\)-field of operators \(\sigma = \{\sigma(\pi) : \pi \in \hat{\mathcal{G}}\}\) defined on smooth vectors is defined on the Sobolev space \(H_\pi^a\) when for each representation \(\pi_1 \in \text{Rep}(\mathcal{G})\), the operator \(\sigma(\pi_1)\) is bounded from \(H_\pi^a\) into \(H_{\pi_1}\) in the sense that\n
\[
\sup_{\|v\|_{H_{\pi_1}^a} = 1} \|\sigma(\pi_1)v\| < \infty.
\]

We will consider those \(\hat{\mathcal{G}}\)-fields of operators with ranges in Sobolev spaces on smooth vectors. We recall that the Sobolev space \(L^2_s(\mathcal{G})\) is defined by the norm (see [65, Chapter 4])

\[
\|f\|_{L^2_s(\mathcal{G})} = \|(1 + \mathcal{R})^{\frac{s}{\nu}}f\|_{L^2(\mathcal{G})},
\]

for \(s \in \mathbb{R}\).

**Definition 13.4.** A \(\hat{\mathcal{G}}\)-field of operators defined on smooth vectors with range in the Sobolev space \(H_\pi^a\) is a family of classes of operators \(\sigma = \{\sigma(\pi) : \pi \in \hat{\mathcal{G}}\}\) where

\[
\sigma(\pi) := \{\sigma(\pi_1) : H_\pi^\infty \to H_{\pi_1}^a, \pi_1 \in \pi\},
\]
for every $\pi \in \hat{G}$ viewed as a subset of $\text{Rep}(G)$, satisfying for every two elements $\sigma(\pi_1)$ and $\sigma(\pi_2)$ in $\sigma(\pi)$:

$$\text{If } \pi_1 \sim \pi_2 \text{ then } \sigma(\pi_1) \sim \sigma(\pi_2).$$

The following notion will be useful in order to use the general theory of non-commutative integration (see e.g. Dixmier [52]).

**Definition 13.5.** A $\hat{G}$-field of operators defined on smooth vectors with range in the Sobolev space $H^s_\pi$ is measurable when for some (and hence for any) $\pi_1 \in \pi$ and any vector $v_{\pi_1} \in H^s_{\pi_1}$, as $\pi \in \hat{G}$, the resulting field $\{\sigma(\pi)v_{\pi_1} : \pi \in \hat{G}\}$, is $d\pi$-measurable and

$$\int_{\hat{G}} \|v_{\pi_1}\|^2_{H^s_\pi} d\pi = \int_{\hat{G}} \|\pi(1 + \mathcal{R})^{\frac{\nu}{2}}v_{\pi_1}\|^2_{H^s_\pi} d\pi < \infty.$$ 

**Remark 13.6.** We always assume that a $\hat{G}$-field of operators defined on smooth vectors with range in the Sobolev space $H^s_\pi$ is $d\pi$-measurable.

The $\hat{G}$-fields of operators associated to Rockland operators can be defined as follows.

**Definition 13.7.** Let $L^2_a(\hat{G})$ denote the space of fields of operators $\sigma$ with range in $H^s_\pi$, that is,

$$\sigma = \{\sigma(\pi) : H^s_\pi \rightarrow H^s_\pi\}, \text{ with } \{\pi(1 + \mathcal{R})^{\frac{\nu}{2}}\sigma(\pi) : \pi \in \hat{G}\} \in L^2(\hat{G}),$$

for one (and hence for any) Rockland operator of homogeneous degree $\nu$. We also denote

$$\|\sigma\|_{L^2(\hat{G})} := \|\pi(1 + \mathcal{R})^{\frac{\nu}{2}}\sigma(\pi)\|_{L^2(\hat{G})}.$$ 

With the notation above, we will introduce some natural spaces which arise as spaces of $\hat{G}$-fields of operators.

**Definition 13.8** (The spaces $\mathcal{L}_L(L^2_a(G), L^2_b(G))$, $\mathcal{K}_{a,b}(G)$ and $L^\infty_{a,b}(\hat{G})$).

- The space $\mathcal{L}_L(L^2_a(G), L^2_b(G))$ consists of all left-invariant operators $T$ such that $T : L^2_a(G) \rightarrow L^2_b(G)$ extends to a bounded operator.
- The space $\mathcal{K}_{a,b}(G)$ is the family of all right convolution kernels of elements in $\mathcal{L}_L(L^2_a(G), L^2_b(G))$, i.e. $k = T\delta \in \mathcal{K}_{a,b}(G)$ if and only if $T \in \mathcal{L}_L(L^2_a(G), L^2_b(G))$.
- We also define the space $L^\infty_{a,b}(\hat{G})$ by the following condition: $\sigma \in L^\infty_{a,b}(\hat{G})$ if

$$\|\pi(1 + \mathcal{R})^{\frac{\nu}{2}}\sigma(\pi)(1 + \mathcal{R})^{-\frac{\nu}{2}}\|_{L^\infty(\hat{G})} := \sup_{\pi \in \hat{G}} \|\pi(1 + \mathcal{R})^{\frac{\nu}{2}}\sigma(\pi)(1 + \mathcal{R})^{-\frac{\nu}{2}}\|_{\mathcal{B}(H^s_\pi)} < \infty.$$ 

In this case $T_\sigma : L^2_a(G) \rightarrow L^2_b(G)$ extends to a bounded operator with

$$\|\sigma\|_{L^\infty_{a,b}(\hat{G})} = \|T_\sigma\|_{\mathcal{L}(L^2_a(G), L^2_b(G))},$$

and $\sigma \in L^\infty_{a,b}(\hat{G})$ if and only if $k := \mathcal{F}_G^{-1}\sigma \in \mathcal{K}_{a,b}(G)$. 

With the previous definitions, we will introduce the type of symbols that we will use further and under which the quantization formula make sense.

**Definition 13.9** (Symbols and right-convolution kernels). A symbol is a field of operators \( \{ \sigma(x, \pi) : H^\infty_\pi \to H_\pi, \, \pi \in \hat{G} \} \), depending on \( x \in G \), such that
\[
\sigma(x, \cdot) = \{ \sigma(x, \pi) : H^\infty_\pi \to H_\pi, \, \pi \in \hat{G} \} \in L^\infty_{a,b}(\hat{G})
\]
for some \( a, b \in \mathbb{R} \). The right-convolution kernel \( k \in C^\infty(G, \mathcal{S}'(G)) \) associated with \( \sigma \) is defined, via the inverse Fourier transform on the group by
\[
x \mapsto \sigma(x, \cdot) = \mathcal{F}_G^{-1}(\sigma(x, \cdot)) : G \to \mathcal{S}'(G).
\]

Definition 13.9 in this section allows us to establish the following theorem, which gives sense to the quantization of pseudo-differential operators in the graded setting (see Theorem 5.1.39 of \cite{ElstRobinson}).

**Theorem 13.10.** Let us consider a symbol \( \sigma \) and its associated right-convolution kernel \( k \). For every \( f \in \mathcal{S}(G) \), let us define the operator \( A \) acting on \( \mathcal{S}(G) \), via
\[
Af(x) = (f \ast k_x)(x), \quad x \in G.
\]
Then \( Af \in C^\infty \), and
\[
Af(x) = \int_{\hat{G}} \text{Tr}(\pi(x)\sigma(x, \pi)\hat{f}((\pi)))d\pi.
\]

Theorem 13.10 motivates the following definition.

**Definition 13.11.** A continuous linear operator \( A : C^\infty(G) \to \mathcal{D}'(G) \) with Schwartz kernel \( K_A \in C^\infty(G) \hat{\otimes}_\pi \mathcal{D}'(G) \), is a pseudo-differential operator, if there exists a symbol, which is a field of operators \( \{ \sigma(x, \pi) : H^\infty_\pi \to H_\pi, \, \pi \in \hat{G} \} \), depending on \( x \in G \), such that
\[
\sigma(x, \cdot) = \{ \sigma(x, \pi) : H^\infty_\pi \to H_\pi, \, \pi \in \hat{G} \} \in L^\infty_{a,b}(\hat{G})
\]
for some \( a, b \in \mathbb{R} \), such that, the Schwartz kernel of \( A \) is given by
\[
K_A(x, y) = \int_{\hat{G}} \text{Tr}(\pi(y^{-1}x)\sigma(x, \pi))d\pi = k_x(y^{-1}x).
\]

Let \( \mathcal{R} \) be a positive Rockland operator on a graded Lie group. Then \( \mathcal{R} \) and \( \pi(\mathcal{R}) := d\pi(\mathcal{R}) \) (the infinitesimal representation of \( \mathcal{R} \)) are symmetric and densely defined operators on \( C^\infty_0(G) \) and \( H^\infty_\pi \subset H_\pi \). We will denote by \( \mathcal{R} \) and \( \pi(\mathcal{R}) := d\pi(\mathcal{R}) \) their self-adjoint extensions to \( L^2(G) \) and \( H_\pi \) respectively (see Proposition 4.1.5 and Corollary 4.1.16 of \cite{ElstRobinson}, page 178).

**Remark 13.12.** Let \( \mathcal{R} \) be a positive Rockland operator of homogeneous degree \( \nu \) on a graded Lie group \( G \). Every operator \( \pi(\mathcal{R}) \) has discrete spectrum (see ter Elst and Robinson \cite{ElstRobinson}) admitting, by the spectral theorem, a basis contained in its domain. In this case, \( H^\infty_\pi \subset \text{Dom}(\pi(\mathcal{R})) \subset H_\pi \), but in view of Proposition 4.1.5 and Corollary 4.1.16 of \cite{ElstRobinson}, page 178, every \( \pi(\mathcal{R}) \) is densely defined and
symmetric on $H^\infty_\pi$, and this fact allows us to define the (restricted) domain of $\pi(\mathcal{R})$, as
\[ \text{Dom}_{\text{rest}}(\pi(\mathcal{R})) = H^\infty_\pi. \] (13.5)
Next, when we mention the domain of $\pi(\mathcal{R})$ we are referring to the restricted domain in (13.5). This fact will be important, because, via the spectral theorem we can construct a basis for $H_\pi$, consisting of vectors in $\text{Dom}_{\text{rest}}(\pi(\mathcal{R})) = H^\infty_\pi$, where the operator $\pi(\mathcal{R})$ is diagonal. So, if $B_\pi = \{e_{\pi,k}\}_{k=1}^\infty \subset H^\infty_\pi$, is a basis such that $\pi(\mathcal{R})$ satisfies
\[ \pi(\mathcal{R})e_{\pi,k} = \lambda_{\pi,k}e_{\pi,k}, \quad k \in \mathbb{N}, \quad \pi \in \hat{G}, \]
for every $x \in G$, the function $x \mapsto \pi(x)e_{\pi,k}$, is smooth and the family of functions
\[ \pi_{ij} : G \to \mathbb{C}, \quad \pi(x)_{ij} := (\pi(x)e_{\pi,i}, e_{\pi,j})_{H_\pi}, \quad x \in G, \] (13.6)
are smooth functions on $G$. Consequently, for every continuous linear operator $A : C^\infty(G) \to C^\infty(G)$ we have
\[ \{\pi_{ij}\}_{i,j=1}^\infty \subset \text{Dom}(A) = C^\infty(G), \]
for every $\pi \in \hat{G}$.

In view of Remark 13.12 we have the following theorem where we present the formula of a global symbol in terms of its corresponding pseudo-differential operator in the graded setting. The proof can be found in [34, Section 3].

**Theorem 13.13.** Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu$ on a graded Lie group $G$. For every $\pi \in \hat{G}$, let $B_\pi = \{e_{\pi,k}\}_{k=1}^\infty \subset H^\infty_\pi$, be a basis where the operator $\pi(\mathcal{R})$ is diagonal, i.e.,
\[ \pi(\mathcal{R})e_{\pi,k} = \lambda_{\pi,k}e_{\pi,k}, \quad k \in \mathbb{N}, \quad \pi \in \hat{G}. \]
For every $x \in G$, and $\pi \in \hat{G}$, let us consider the functions $\pi(\cdot)_{ij} \in \mathcal{C}^\infty(G)$ in (13.6) induced by the coefficients of the matrix representation of $\pi(x)$ in the basis $B_\pi$. If $A : C^\infty(G) \to C^\infty(G)$ is a continuous linear operator with symbol
\[ \sigma := \{\sigma(x, \pi) \in \mathcal{L}(H^\infty_\pi, H_\pi) : x \in G, \pi \in \hat{G}\}, \] (13.7)
such that
\[ Af(x) = \int_{\hat{G}} \text{Tr}(\pi(x)\sigma(x, \pi)\hat{\pi}(\pi))d\pi, \] (13.8)
for every $f \in \mathcal{S}(G)$, and a.e. $(x, \pi)$, and if $A\pi(x)$ is the densely defined operator on $H^\infty_\pi$, via
\[ A\pi(x) \equiv ((A\pi(x)e_{\pi,i}, e_{\pi,j}))_{i,j=1}^\infty, \quad (A\pi(x)e_{\pi,i}, e_{\pi,j}) =: (A\pi_{ij})(x), \] (13.9)
then we have
\[ \sigma(x, \pi) = \pi(x)^*A\pi(x), \] (13.10)
for every $x \in G$, and a.e. $\pi \in \hat{G}$. 

13.5. **Global Hörmander classes** $S^m_{\rho,\delta}$ **of pseudo-differential operators on graded Lie groups.** The main tool in the construction of global Hörmander classes is the notion of difference operators. Indeed, for every smooth function $q \in C^\infty(G)$ and $\sigma \in L^\infty_{a,b}(G)$, where $a, b \in \mathbb{R}$, the difference operator $\Delta_q$ acts on $\sigma$ according to the formula (see Definition 5.2.1 of [65]),

$$\Delta_q \sigma(\pi) \equiv [\Delta_q \sigma](\pi) := \mathcal{F}_G(qf)(\pi), \quad \text{for a.e. } \pi \in \hat{G}, \text{ where } f := \mathcal{F}_G^{-1} \sigma.$$  

We will reserve the notation $\Delta^\alpha$ for the difference operators defined by the functions $q_\alpha$ and $\tilde{q}_\alpha$ defined by $q_\alpha(x) := x^\alpha$ and $\tilde{q}_\alpha(x) = (x^{-1})^\alpha$, respectively. In particular, we have the Leibnitz rule,

$$\Delta^\alpha(\sigma \tau) = \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1, \alpha_2} \Delta^{\alpha_1}(\sigma) \Delta^{\alpha_2}(\tau), \quad \sigma, \tau \in L^\infty_{a,b}(\hat{G}). \quad (13.11)$$

For our further analysis we will use the following property of the difference operators $\Delta^\alpha$, (see e.g. [64, page 20]),

$$\Delta^\alpha(\sigma_r)(\pi) = r^{\alpha}(\Delta^\alpha \sigma)(r \cdot \pi), \quad r > 0 \quad \pi \in \hat{G}, \quad (13.12)$$

where we have denoted

$$\sigma_r := \{\sigma(r \cdot \pi) : \pi \in \hat{G}\}, \quad r \cdot \pi(x) := \pi(D_r(x)), \quad x \in G. \quad (13.13)$$

**Definition 13.14.** In terms of difference operators, the global Hörmander classes introduced in [65] can be defined as follows. Let $0 \leq \delta, \rho \leq 1$, and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu > 0$. If $m \in \mathbb{R}$, we say that the symbol $\sigma \in L^\infty_{a,b}(\hat{G})$, where $a, b \in \mathbb{R}$, belongs to the $(\rho, \delta)$-Hörmander class of order $m$, $S^m_{\rho,\delta}(G \times \hat{G})$, if for all $\gamma \in \mathbb{R}$, the following conditions

$$p_{\alpha,\beta,\gamma,m}(\sigma) = \text{ess sup} \| \pi(1 + \mathcal{R})^{\rho[|\alpha|+|\beta|] - m - \gamma} [X^\beta \Delta^\alpha \sigma(x, \pi)]\pi(1 + \mathcal{R})^{\frac{\delta}{2}} \|_{\text{op}} < \infty, \quad (13.14)$$

hold true for all $\alpha$ and $\beta$ in $\mathbb{N}_0^n$.

**Remark 13.15.** The resulting class $S^m_{\rho,\delta}(G \times \hat{G})$, does not depend on the choice of the Rockland operator $\mathcal{R}$ (see [65, Page 306]). Moreover (see Theorem 5.5.20 of [65]), the following facts are equivalents:

- $\forall \alpha, \beta \in \mathbb{N}_0^n, \forall \gamma \in \mathbb{R}, \; p_{\alpha,\beta,\gamma,m}(\sigma) < \infty.$
- $\forall \alpha, \beta \in \mathbb{N}_0^n, \; p_{\alpha,\beta,0,m}(\sigma) < \infty.$
- $\forall \alpha, \beta \in \mathbb{N}_0^n, \; p_{\alpha,\beta,m+|\beta|-\rho|\alpha|,m}(\sigma) < \infty.$
- $\sigma \in S^m_{\rho,\delta}(G \times \hat{G}).$

We will denote,

$$\|\sigma\|_{k,\rho,m} = \max_{|\alpha|+|\beta| \leq k} \{p_{\alpha,\beta,0,m}(\sigma)\}. \quad (13.15)$$

By keeping in mind Remark 13.15, we will improve Theorem 5.5.20 of [65] proving a characterization of Hörmander classes on graded Lie groups.

**Theorem 13.16.** Let $G$ be a graded Lie group of homogeneous dimension $Q$, and let $0 \leq \delta, \rho \leq 1$. The following conditions are equivalent:

(A). $\forall \alpha, \beta \in \mathbb{N}_0^n, \forall \gamma \in \mathbb{R}, \; p_{\alpha,\beta,\gamma,m}(\sigma) < \infty.$

(B). \(\forall \alpha, \beta \in \mathbb{N}_0^n, p_{\alpha, \beta, 0, m}(\sigma) < \infty\).
(C). \(\forall \alpha, \beta \in \mathbb{N}_0^n, p_{\alpha, \beta, m+|\beta| - |\alpha|, m}(\sigma) < \infty\).
(D). \(\forall \alpha, \beta \in \mathbb{N}_0^n, \exists \gamma_0 \in \mathbb{R}, p_{\alpha, \beta, \gamma_0, m}(\sigma) < \infty\).
(E). \(\sigma \in S^m_{\rho, \delta}(G \times \hat{G})\).

Proof. We only need to prove that \(D \implies C\). Let us assume that \(D\) holds true for some \(\gamma_0 \in \mathbb{R}\) and let \(\gamma \in \mathbb{R}\) be a real number. Let us assume first that \(\gamma > \gamma_0\). Let us define the operator \(Q\) by the functional calculus in the following way

\[
Q := (1 + \mathcal{R})^{\frac{1}{2}} \equiv \int_0^\infty (1 + \lambda)^{\frac{1}{2}} dE_{\mathcal{R}}(\lambda),
\]

where \(\{dE_{\mathcal{R}}(\lambda)\}_{\lambda \geq 0}\) denotes the spectral resolution associated with \(\mathcal{R}\). Let us denote by \(\{\pi(Q)\}\) the symbol of \(Q\), indexed by \(\pi \in \hat{G}\) except possibly on a subset of \(\hat{G}\) of null Plancherel measure. Let us note that the operator \(A(\pi) := \pi(Q)^{(\gamma_0 - \gamma)}\) is self-adjoint and bounded. Let us denote

\[
X_{\alpha, \beta, \gamma_0}(x, \pi) := \pi(Q)^{(|\alpha| - |\beta| - m - \gamma_0)} X_2^x \Delta^\alpha \sigma(x, \pi) \pi(Q)^{\gamma_0},
\]

which is a bounded operator on \(H_\pi\). From the Corach-Porta-Recht inequality (4.11), we have

\[
\begin{align*}
\| \pi(Q)^{(|\alpha| - |\beta| - m - \gamma_0)} X_2^x \Delta^\alpha \sigma(x, \pi) \pi(Q)^{\gamma_0} \|_{\text{op}} & = \| \pi(Q)^{(\gamma_0 - \gamma)} \pi(Q)^{(|\alpha| - |\beta| - m - \gamma_0)} X_2^x \Delta^\alpha \sigma(x, \pi) \pi(Q)^{\gamma_0} \pi(Q)^{(\gamma - \gamma_0)} \|_{\text{op}} \\
& = \| A(\pi) \pi(Q)^{(\gamma_0 - \gamma)} X_{\alpha, \beta, \gamma_0}(x, \pi) \pi(Q)^{(\gamma - \gamma_0)} \|_{\text{op}} \\
& \leq \| A(\pi) \pi(Q)^{(\gamma_0 - \gamma)} X_{\alpha, \beta, \gamma_0}(x, \pi) \pi(Q)^{(\gamma - \gamma_0)} A(\pi) \|_{\text{op}} \\
& + (1 + A(\pi)^2)^{\frac{1}{2}} \| \pi(Q)^{(\gamma_0 - \gamma_0)} X_{\alpha, \beta, \gamma_0}(x, \pi) \pi(Q)^{(\gamma_0 - \gamma_0)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}} \\
& = \| \pi(Q)^{2(\gamma_0 - \gamma)} X_{\alpha, \beta, \gamma_0}(x, \pi) \|_{\text{op}} \\
& + (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \| \pi(Q)^{(\gamma_0 - \gamma_0)} X_{\alpha, \beta, \gamma_0}(x, \pi) \pi(Q)^{(\gamma_0 - \gamma_0)} (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \|_{\text{op}}.
\end{align*}
\]

Taking into account that \(\gamma_0 - \gamma < 0\), and the functional calculus for real powers of \(\mathcal{R}\) and \(I + \mathcal{R}\) (see [65, Page 319]) imply

\[
\pi(Q)^{2(\gamma_0 - \gamma)} \in S_{1,0}^{2(\gamma_0 - \gamma)}(G \times \hat{G}), \quad (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \pi(Q)^{(\gamma_0 - \gamma_0)} \in S_{1,0}^{2(\gamma_0 - \gamma)}(G \times \hat{G}),
\]

\[
\pi(Q)^{(\gamma_0 - \gamma_0)} (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \pi(Q)^{2(\gamma_0 - \gamma)} \in S_{1,0}^{0}(G \times \hat{G}),
\]

from which we deduce that

\[
D_1 := \sup_{\pi \in G} \| \pi(Q)^{2(\gamma_0 - \gamma)} \|_{\text{op}}, \quad D_2 := \sup_{\pi \in G} \| (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \pi(Q)^{2(\gamma_0 - \gamma)} \|_{\text{op}} < \infty,
\]

and

\[
D_3 := \sup_{\pi \in G} \| \pi(Q)^{(\gamma_0 - \gamma_0)} (1 + \pi(Q)^{2(\gamma_0 - \gamma)})^{\frac{1}{2}} \|_{\text{op}} < \infty.
\]
Consequently,
\[
\|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} X^{\beta} \Delta^\alpha \sigma(x, \pi) \pi(Q)^\gamma \|_{\text{op}} \\
\leq \|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} X_{\alpha,\beta,\gamma_0}(x, \pi) \\
+ (1 + \pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} \hat{x} \pi(Q)^{(\gamma \to \gamma)} X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)})^{\frac{1}{2}} \|_{\text{op}} \\
\leq \|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} \|_{\text{op}} \|X_{\alpha,\beta,\gamma_0}(x, \pi)\|_{\text{op}} \\
+ \| (1 + \pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} \hat{x} \pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}} \|X_{\alpha,\beta,\gamma_0}(x, \pi)\|_{\text{op}} \\
\times \|\pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)})^{\frac{1}{2}} \|_{\text{op}} \\
\leq (\mathcal{D}_1 + \mathcal{D}_2 \times \mathcal{D}_3) \|X_{\alpha,\beta,\gamma_0}(x, \pi)\|_{\text{op}}.
\]

This argument shows that $D \implies C$ for $\gamma > \gamma_0$. In the case where $\gamma < \gamma_0$, we can define $A(\pi) = \pi(Q)^{(\gamma \to \gamma)}$. By repeating the argument above we can deduce that $D \implies C$ for $\gamma < \gamma_0$. Indeed, by using again the Corach-Porta-Recht inequality \[(4.11), \]
we have
\[
\|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} X^{\beta} \Delta^\alpha \sigma(x, \pi) \pi(Q)^\gamma \|_{\text{op}} \\
= \|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} X^{\beta} \Delta^\alpha \sigma(x, \pi) \pi(Q)^\gamma \pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}} \\
= \|\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}} \\
\leq \|A(\pi) \pi(Q)^{(\gamma \to \gamma)} X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}} \\
+ (1 + A(\pi)^2)^{\frac{3}{2}} \pi(Q)^{(\gamma \to \gamma)} X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}} \\
= \|X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}} \\
+ (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}}.
\]

Since $\gamma - \gamma_0$ is negative, and by using again the functional calculus for real powers of $\mathcal{R}$ and $I + \mathcal{R}$ (see [65, Page 319]) we have that
\[
\pi(Q)^{(\rho|\alpha|\delta|\beta|\gamma \to \gamma)} \in S^{2(\gamma \to \gamma)}_{1,0}(G \times \hat{G}), \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}} \in S^{2(\gamma \to \gamma)}_{1,0}(G \times \hat{G}),
\]
and consequently we deduce that
\[
\mathcal{D}_1 := \sup_{\pi \in G} \|\pi(Q)^{(\gamma \to \gamma)} \|_{\text{op}}, \mathcal{D}_2 := \sup_{\pi \in G} \| (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}} < \infty,
\]
and
\[
\mathcal{D}_3 := \sup_{\pi \in G} \|\pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + \pi(Q)^{(\gamma \to \gamma)} (1 + A(\pi)^2)^{\frac{1}{2}} \|_{\text{op}} < \infty.
\]
Consequently,
\[
\| \pi(Q)^{(\rho|\alpha|-\delta|\beta|-m-\gamma)} X^\beta \Delta^\alpha \sigma(x, \pi) \pi(Q)^\gamma \|_{\text{op}} \\
\leq \| X_{\alpha,\beta,\gamma_0}(x, \pi) \pi(Q)^{2(\gamma-\gamma_0)} \\
+ (1 + \pi(Q)^{2(\gamma-\gamma_0)}) \frac{\hat{2}}{\hat{2}} \pi(Q)^{\gamma_0}(1 + \pi(Q)^{2(\gamma-\gamma_0)}) \|_{\text{op}} \\
\leq \| \pi(Q)^{2(\gamma-\gamma_0)} \|_{\text{op}} \| X_{\alpha,\beta,\gamma_0}(x, \pi) \|_{\text{op}} \\
+ \| (1 + \pi(Q)^{2(\gamma-\gamma_0)}) \frac{\hat{2}}{\hat{2}} \pi(Q)^{\gamma_0} \|_{\text{op}} \| X_{\alpha,\beta,\gamma_0}(x, \pi) \|_{\text{op}} \\
\times \| \pi(Q)^{\gamma_0}(1 + \pi(Q)^{2(\gamma-\gamma_0)}) \|_{\text{op}} \\
\leq (\mathcal{D}_1 + \mathcal{D}_2 \times \mathcal{D}_3) \| X_{\alpha,\beta,\gamma_0}(x, \pi) \|_{\text{op}}.
\]

This argument shows that \( D \implies C \) for \( \gamma_0 > \gamma_0 \). The proof is complete. \( \square \)

Remark 13.17. In the case of the \( n \)-dimensional Heisenberg group \( G = \mathbb{H}_n \), Theorem 13.16 was proved in [65, Page 479], relying on the description of these Hörmander classes in terms of the Shubin calculus. For general graded Lie groups, the equivalence of other conditions to condition (D) remained open.

We finish this subsection by also noting the following Theorem 13.18, which is an extension of the classical Fefferman theorem to an arbitrary graded Lie group \( G \) of homogeneous dimension \( Q \) (see Theorem 1.2 in [34]).

**Theorem 13.18.** Let \( G \) be a graded Lie group of homogeneous dimension \( Q \). Let \( A : C^\infty(G) \to \mathcal{D}'(G) \) be a pseudo-differential operator with symbol \( \sigma \in S_{\rho,\delta}^{-m}(G \times \hat{G}) \), \( 0 \leq \delta \leq \rho \leq 1 \), \( \delta \neq 1 \). Then,

- (a) if \( m = \frac{Q(1-\rho)}{2} \), then \( A \) extends to a bounded operator from \( L^\infty(G) \) to \( \text{BMO}(G) \), from the Hardy space \( H^1(G) \) to \( L^1(G) \), and from \( L^p(G) \) to \( L^p(G) \) for all \( 1 < p < \infty \).

- (b) If \( m \geq m_p := Q(1-\rho) \left| \frac{1}{p} - \frac{1}{2} \right| \), \( 1 < p < \infty \), then \( A \) extends to a bounded operator from \( L^p(G) \) into \( L^p(G) \).

**Remark 13.19.** Theorem 13.18 is an analogue of Theorems 6.6 and 6.9.

**14. Appendix IV: Dependence of the subelliptic Hörmander calculus on the choice of sub-Laplacians**

We have mentioned in Remark 13.15, that in the case of a graded Lie group \( G \), the resulting class \( S_{\rho,\delta}^{-m}(G \times \hat{G}) \), does not depend on the choice of the Rockland operator \( \mathcal{R} \). However, in the case of a compact Lie group we have mentioned without proof in Remark 4.20 that for two sub-Laplacians, the corresponding subelliptic classes may not agree as we can see in the following remark for the case of \( G = \text{SU}(2) \). This may happen even when two sub-Laplacians are made from Hörmander collections of vector fields of the same step.

**Remark 14.1.** Let us consider the positive sub-Laplacians \( \mathcal{L}_1 = -X_1^2 - X_2^2 \) and \( \mathcal{L}_2 = -X_2^2 - X_3^2 \) on \( G = \text{SU}(2) \cong \mathbb{S}^3 \), defined in Example 11.1. The unitary dual
of SU(2) can be identified as, (see [107, Chapter 12])

$$\widehat{\text{SU}(2)} \equiv \{[t_1] : 2l \in \mathbb{N}, d_l := \dim t_l = (2l + 1)\}. \quad (14.1)$$

There are explicit formulae for $t_l$ as functions of Euler angles in terms of the so-called Legendre-Jacobi polynomials, see [107, Chapter 11]. In terms of the representations $t_l$, it was shown in [109] that (by considering the positive Laplacian $\mathcal{L}_{\text{SU}(2)} = -X_1^2 - X_2^2 - X_3^2$ on SU(2)),

$$\sigma \mathcal{L}_{\text{SU}(2)}(t_l) = \text{diag}[l(l+1)\delta_{mn}]_{m,n=-l}^l, \quad \sigma X_3(t_l) = \text{diag}[-in\delta_{mn}]_{m,n=-l}^l, \quad (14.2)$$

where $\delta_{mn}$ is the Kronecker-Delta. If a-priori we assume that $S_{\rho,\delta}^m\mathcal{L}_1(SU(2) \times \frac{1}{2}\mathbb{N}_0) = S_{\rho,\delta}^m\mathcal{L}_2(SU(2) \times \frac{1}{2}\mathbb{N}_0)$, with $m \in \mathbb{R}$, and $0 \leq \rho < \delta \leq 1$, then we would have that $1 + \mathcal{L}_2 \in S_{1,0}^2(SU(2) \times \frac{1}{2}\mathbb{N}_0) = S_{1,0}^2(SU(2) \times \frac{1}{2}\mathbb{N}_0)$, which from the definition of the subelliptic classes implies that

$$\sup_{l \in \frac{1}{2}\mathbb{N}_0} \| (1 + \sigma \mathcal{L}_1(t_l))^{-1}(1 + \sigma \mathcal{L}_2(t_l)) \|_{\text{op}} < \infty. \quad (14.3)$$

As a consequence of the Plancherel theorem, the previous inequality implies that $(1 + \mathcal{L}_1)^{-1}(1 + \mathcal{L}_2)$ is bounded on $L^2(SU(2))$. Let us note that

$$(1 + \mathcal{L}_1)^{-1}(1 + \mathcal{L}_2) = (1 + \mathcal{L}_1)^{-1}(1 - X_2^2) - (1 + \mathcal{L}_1)^{-1}X_3^2. \quad (14.4)$$

So, from the positivity of $(1 + \mathcal{L}_1)^{-1}$, $(1 - X_2^2)$ and $-(1 + \mathcal{L}_1)^{-1}X_3^2$ we have that

$$\| (1 + \mathcal{L}_1)^{-1}(1 + \mathcal{L}_2) \|_{\mathcal{B}(L^2(SU(2)))} = \sup_{f \in L^2(SU(2))} \| (1 + \mathcal{L}_1)^{-1}(1 + \mathcal{L}_2)f, f \|_{L^2(SU(2))} \geq \sup_{f \in L^2(SU(2))} \| (1 + \mathcal{L}_1)^{-1}X_3^2f, f \|_{L^2(SU(2))} = \| (1 + \mathcal{L}_1)^{-1}X_3^2 \|_{\mathcal{B}(L^2(SU(2)))},$$

which implies that $(1 + \mathcal{L}_1)^{-1}X_3^2$, is bounded on $L^2(SU(2))$, which indeed, is equivalent to say that

$$\sup_{l \in \frac{1}{2}\mathbb{N}_0} \| (1 + \sigma \mathcal{L}_1(t_l))^{-1}\sigma X_3(t_l) \|_{\text{op}} = \sup_{l \in \frac{1}{2}\mathbb{N}_0} \| (1 + \sigma \mathcal{L}_{SU(2)}(t_l) + \sigma X_3(t_l))^{-1}\sigma X_3(t_l) \|_{\text{op}} < \infty. \quad (14.5)$$

In terms of (14.2), (14.5) implies that

$$\sup_{l \in \frac{1}{2}\mathbb{N}_0} \| (1 + \sigma \mathcal{L}_{SU(2)}(t_l) + \sigma X_3(t_l))^{-1}\sigma X_3(t_l)^2 \|_{\text{op}} = \sup_{l \in \frac{1}{2}\mathbb{N}_0} \| \text{diag}[l(l+1) - n^2]^{-1}n^2\delta_{mn}]_{m,n=-l}^l \|_{\text{op}} = 0 \to \infty, \text{ when } l \to \infty.$$

This shows that (14.3) does not hold. In consequence, $S_{1,0}^2(SU(2) \times \frac{1}{2}\mathbb{N}_0) \neq S_{1,0}^2(SU(2) \times \frac{1}{2}\mathbb{N}_0)$, which shows that the subelliptic calculus may depend on the choice of the sub-Laplacian.

\textsuperscript{21}Let $H$ be a Hilbert space. Then an operator $A : \text{Dom}(A) \subset H \to H$ admitting a self-adjoint extension has a bounded extension, if and only if, $\|A\|_{\mathcal{B}(H)} = \sup_{f \parallel \|f\|_H = 1}(Af, f)_H < \infty$, where $(\cdot, \cdot)_H$ is the inner product of $H$, and $\| \cdot \|_H$ the induced norm (see e.g. Weidmann [126]).
Remark 14.2. As a consequence of the argument in Remark 14.1 also subelliptic Sobolev and Besov spaces may depend on the choice of a sub-Laplacian on a compact Lie group. Indeed, let us consider the case of $G = SU(2)$, the sub-Laplacians $L_1$ and $L_2$ in Example 11.1 and the subelliptic Sobolev spaces $L_{2}^{2\mathcal{L}_1}(SU(2))$ and $L_{2}^{2\mathcal{L}_2}(SU(2))$. Let us define $f := \mathcal{F}^{-1}_{SU(2)}[\mathcal{M}_{2,2} \hat{f}(t)],$ where $\mathcal{M}_{2,2} := 1 + L_2$. Because

$$\sup_{t \in \frac{1}{2}\mathbb{N}_0} \|\mathcal{M}_{2,2}^{-1}(t)\hat{f}(t)\|_{\text{op}} = 1,$$

we have that $f \in L_{2}^{2\mathcal{L}_1}(SU(2))$. If we assume that $L_{2}^{2\mathcal{L}_1}(SU(2)) = L_{2}^{2\mathcal{L}_2}(SU(2))$, then we could have that

$$\sup_{t \in \frac{1}{2}\mathbb{N}_0} \|\mathcal{M}_{1,2}^{-1}(t)\hat{f}(t)\|_{\text{op}} < \infty,$$

(14.6)

where $\mathcal{M}_{1,2} := 1 + L_1$. However, (14.6) is equivalent to saying that $\|(1 + L_1)^{-1}(1 + L_2)\|_{\mathcal{L}(L^2(SU(2)))} < \infty$, which certainly, from Remark 14.1 is not possible. This analysis implies that $L_{2}^{2\mathcal{L}_1}(SU(2)) \neq L_{2}^{2\mathcal{L}_2}(SU(2))$. Because subelliptic Besov spaces can be obtained from the real interpolation between subelliptic Sobolev spaces (see Theorem 6.2 of [33]) a similar argument as the one done in this remark, shows that subelliptic Besov spaces may depend on the choice of a sub-Laplacian on a compact Lie group.

Acknowledgements. The authors thank J. Delgado, A. Cardona, S. Federico, D. Rottensteiner, V. Kumar, and J. Wirth for discussions.

References

1. Agrachev, A., Boscaiu, U., Gauthier, J. P., Rossi, F. The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups, J. Funct. Anal. 255 (9), pp. 2190–2232, (2008).
2. Andrucho, E., Corach, G., Stojanoff, D. Geometric operator inequalities, Linear Algebra Appl., 258, pp. 295–310, (1997).
3. Asada, K., Fujiwara, D. On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$, J. Math. (N.S.), Japan, 4 (2), pp. 299–361, (1978).
4. Atiyah, M. F., Bott, R. The index problem for manifolds with boundary. Differential Analysis, Bombay Colloq., pp. 175–186 Oxford Univ. Press, London, (1964).
5. Atiyah, M., Bott, R., Patodi, V. K. On the heat equation and the index theorem. Invent. Math. 19, pp. 279–330, (1973).
6. Atiyah, M. F., Singer, I. M. The index of elliptic operators on compact manifolds. Bull. Amer. Math. Soc. 69, pp. 422–433, (1963).
7. Atiyah, M. F., Singer, I. M. The index of elliptic operators. I. Ann. of Math. (2) 87, pp. 484–530, (1968).
8. Atiyah, M. F., Segal, G. The index of elliptic operators. II. Ann. of Math. (2) 87, pp. 531–545, (1968).
9. Atiyah, M. F., Singer, I. M. The index of elliptic operators. III. Ann. of Math. (2) 87, pp. 546–604, (1968).
10. Atiyah, M. F., Singer, I. M. The index of elliptic operators. IV. Ann. of Math. (2) 93, pp. 119–138, (1971).
11. Atiyah, M. F., Singer, I. M. The index of elliptic operators. V. Ann. of Math. (2) 93, 139–149, (1971).
12. Bhatia, R. Matrix Analysis. Springer-Verlag, New York, 1997.
13. Bleecker, D., Booss-Bavnbek, B. Index theory with applications to mathematics and physics. International Press, Somerville, MA, 2013.

14. Bramanti, M. An invitation to hypoelliptic operators and Hörmander’s vector fields. Springer Briefs in Mathematics. Springer, Cham, 2014. xii+150 pp. ISBN: 978-3-319-02086-0; 978-3-319-02087-7.

15. Berge, S. M., Grong, E. A Lichnerowicz estimate for the spectral gap of a sub-Laplacian. (English summary) Proc. Amer. Math. Soc. 147 , no. 12, pp. 5153–5166, (2019).

16. Besov, O. V. On a family of function spaces. Embeddings theorems and applications [in Russian] Dokl. Akad. Nauk. SSSR., 126, pp. 1163–1165, (1959).

17. Besov, O. V. On a family of function spaces in connection with embeddings and extensions, [in Russian] Trudy. Mat. Inst. Steklov. 60, pp. 42–81, (1961).

18. Bismut, J. M. The hypo elliptic Laplacian on a compact Lie group, J. Funct. Anal. 255, 2190–2232, (2008).

19. Brezis, H. Analyse fonctionnelle: théorie et applications. Collection Mathématiques appliquées pour la maîtrise. Ciarlet, P.G. and Lions, J.L. (Eds). Dunod, 1999.

20. Calderón, A. P., Vaillancourt, R. On the boundedness of pseudo-differential operators, J. Math. Soc. Japan 23, pp. 374–378, (1971).

21. Calderón, A. P., Vaillancourt, R. A class of bounded pseudo-differential operators, Proc. Nat. Acad. Sci. USA 69, pp. 1185–1187, (1972).

22. Calderón, A. P., Zygmund, A. On singular integrals, .Amer. J. Math., Vol. 78. 1956.

23. Carbonaro, A., Mauceri, G., Meda, S. $H^1$ and $BMO$ for certain locally doubling metric measure spaces, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)8, pp. 543–582, (2009).

24. Cardona, D. Besov continuity for Multipliers defined on compact Lie groups. Palest. J. Math., 5(2), 35–44, (2016).

25. Cardona, D. Besov continuity of pseudo-differential operators on compact Lie groups revisited, C. R. Math. Acad. Sci. Paris Vol. 355(5), pp. 533–537, (2017).

26. Cardona, D. Nuclear pseudo-differential operators in Besov spaces on compact Lie groups. J. Fourier Anal. Appl. 23(5), pp. 1238–1262, (2017).

27. Cardona, D. Continuity of pseudo-differential operators on Besov spaces on compact homogeneous manifolds, J. Pseudo-Differ. Oper. Appl., 9(4), pp. 861–880, (2018).

28. Cardona, D., Del Corral, C. The Dixmier trace and the Wodzicki residue for pseudo-differential operators on compact manifolds, Rev. Integr. Temas. Mat. Vol. 38 (1). pp. 67–79, (2020).

29. Cardona, D., Del Corral, C. The Dixmier trace and the non-commutative residue for multipliers on compact manifolds, submitted. arXiv:1703.07453.

30. Cardona, D., Kumar, V., Del Corral, C. Dixmier traces for discrete pseudo-differential operators. To appear in J. Pseudo-Differ. Oper. Appl. arXiv:1911.03924

31. Cardona, D., Messiouene, R., Senoussaoui, A., $L^p$-bounds for Fourier integral operators on the torus. arXiv:1807.09892

32. Cardona, D., Ruzhansky, M. Multipliers for Besov spaces on graded Lie groups. C. R. Math. Acad. Sci. Paris. 355(4), pp. 400–405, (2017).

33. Cardona, D., Ruzhansky, M. Boundedness of pseudo-differential operators in subelliptic Sobolev and Besov spaces on compact Lie groups. arXiv:1901.06825.

34. Cardona, D., Delgado, J., Ruzhansky, M. $L^p$-bounds for pseudo-differential operators on graded Lie groups. arXiv:1911.03397

35. Connes A. Noncommutative geometry. Academic Press Inc., SanDiego, CA (1994).

36. Corach, G., Porta, H., Recht, L. An operator inequality. Linear Algebra Appl. 142, 153–158, (1990).

37. Coriasco, S., Ruzhansky, M. On the boundedness of Fourier integral operators on $L^p(\mathbb{R}^n)$, C. R. Math. Acad. Sci. Paris, 348(15–16), pp. 847–851, (2010).

38. Coriasco, S., Ruzhansky, M. Global $L^p$ continuity of Fourier integral operators, Trans. Amer. Math. Soc, 366(5), pp. 2575–2596, (2014).
39. Coulhon, T., Russ, E., Tardivel-Nachef, V., Sobolev algebras on Lie groups and Riemannian manifolds. Amer. J. Math. 123(2), pp. 283–342, (2001).
40. Chen, J., Fan, D. Central oscillating multipliers on compact Lie groups, Math. Z. 267, 235-259, (2011).
41. Coifman, R.R., De Guzmán, M. Singular integrals and multipliers on homogeneous spaces. Rev. un. Mat. Argentina, pp. 137–143, (1970).
42. Coifman, R., Weiss, G. Analyse harmonique non-commutative sur certains espaces homogénes. (French) Étude de certaines intégrales singuliéres. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. v+160 pp.
43. Comech, A. Cotlar-Stein Almost Orthogonality Lemma, lecture note. Columbia University, New York, 1997.
44. Christ, M. Estimates for fundamental solutions of second-order subelliptic differential operators. Proc. Amer. Math. Soc. 105, pp. 166–172, (1989).
45. Dasgupta, A., Ruzhansky, M. The Gohberg lemma, compactness, and essential spectrum of operators on compact Lie groups. J. Anal. Math. 128, pp. 179–190, (2016).
46. Delgado, J., Ruzhansky M., $L^p$-bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18, no. 3, pp. 531–559, (2019).
47. Delgado, J., Ruzhansky, M. Fourier multipliers, symbols, and nuclearity on compact manifolds. J. Anal. Math. 135, no. 2, pp. 757–800, (2018).
48. Delgado, J., Ruzhansky, M. Schatten classes and traces on compact groups. Math. Res. Lett. 24, no. 4, pp. 979–1003, (2017).
49. Delgado, J., Ruzhansky, M. Kernel and symbol criteria for Schatten classes and $r$-nuclearity on compact manifolds. C. R. Math. Acad. Sci. Paris 352, no. 10, pp. 779–784, (2014).
50. Delgado, J., Ruzhansky, M. $L^p$-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups. J. Math. Pures Appl. (9) 102, no. 1, pp. 153–172, (2014).
51. Dixmier, J. Existence de traces non normales, C. R. Acad. Sci. Paris. Series B, 262: 1107A–1108A, (1966).
52. Dixmier, J. Formes linéaires sur un anneau d’opérateurs. Bull. Soc. Math. France, 81, 9–39, (1953).
53. Dyer, J. L. A nilpotent Lie algebra with nilpotent automorphism group. Bull. Amer. Math. Soc., 76, pp. 52–56, (1970).
54. Domokos, A., Esquerra, R., Jaffa, B., Schulte, T. Subelliptic estimates on compact semisimple Lie groups. Nonlinear Analysis: Theory, Methods and Applications, 74(14), pp. 4642–4652, (2011).
55. Domokos, A., Manfredi, J. $C^{1,\alpha}$-subelliptic regularity on SU(3) and compact, semi-simple Lie groups. Anal. Math. Phys. 10, no. 1, Art. 4, 32 pp, (2020).
56. Duistermaat, J. J., Hörmander, L. Fourier integral operators. II, Acta Math, 142, 1972, no 1, pp 1–31. DOI 10.1007/BFb0070950; zbl 0877.58005; MR1419415.
57. Duandikoetxea, J. Fourier Analysis. 29, American Mathematical Society, Providence (2000)
58. Eskin, G.I. Degenerate elliptic pseudodifferential equations of principal type, Mat. Sb. (N.S.), 82(124), pp. 585–628, (1970).
59. Fefferman, C., $L^p$-bounds for pseudo-differential operators, Israel J. Math. 14, pp. 413–417, (1973).
60. Fedosov, B. V., Golse, F., Leichtnam, E., Schrohe, E., The noncommutative residue for manifolds with boundary, J. Funct. Anal., 142, 1996, no 1, pp 1–31. DOI 10.1006/jfan.1996.0142; zbl 0877.58005; MR1419415.
61. Fischer, V. Intrinsic pseudo-differential calculi on any compact Lie group. J. Funct. Anal., 268, pp. 3404–3477, (2015).
62. Fischer, V. Real trace expansions. Doc. Math. 24, pp. 2159–2202, (2019).
63. Fischer, V. Local and global symbols on compact Lie groups. J. Pseudo-Differ. Oper. Appl. doi:10.1007/s11868-019-00299-x., (2019).
64. Fischer, V. Fermanian-Kammerer, C. Defect measures on graded Lie groups. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. arXiv:1707.04002.
65. Fischer V., Ruzhansky M., Quantization on nilpotent Lie groups, Progress in Mathematics, Vol. 314, Birkhauser, 2016. xiii+557pp.
66. Fischer, V. Differential structure on the dual of a compact Lie group, arXiv:1610.06348.
67. Folland, G., Stein, E. Hardy Spaces on Homogeneous Groups, Princeton University Press, Princeton, N.J., 1982.
68. Fujiwara, A. Construction of the fundamental solution for the Schrödinger equations, Proc. Japan Acad. Ser. A Math. Sc, 55(1), pp. 10–14, (1979).
69. Furioli, G., Melzi, C., Veneruso, A. Littlewood-Paley decompositions and Besov spaces on Lie groups of polynomial growth. Math. Nachr. 279(9-10), pp. 1028–1040, (2006).
70. Garetto, C., $L^p$ and Sobolev boundedness of pseudodifferential operators with non-regular symbol: A regularisation approach. J. Math. Anal. Appl. 381, pp. 328–343, (2011).
71. Garetto, C., Ruzhansky, M. Wave equation for sum of squares on compact Lie groups, J. Differential Equations, 258, pp. 4324–4347, (2015).
72. Golberg, I. On the theory of multidimensional singular integral equations,Soviet Math. Dokl. 1, 960–963, (1960).
73. Gordina, M., Laetsch, T. Sub-Laplacians on sub-Riemannian manifolds. Potential Anal. 44, no. 4, pp. 811–837, (2016).
74. Grothendieck, A. Produits tensoriels topologiques et espaces nucléaires, Memoirs Amer. Math. Soc. 16, Providence, 1955 (Thesis, Nancy, 1953).
75. Grubb, G., Schrohe, E. Traces and quasi-traces on the Boutet de Monvel algebra. Ann. Inst. Fourier(Grenoble) 54(5), pp. 1641–1696 xvii, xxii., (2004).
76. Hassannezhad, A., Kokarev, G. Sub-Laplacian eigenvalue bounds on sub-Riemannian manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci., XVI(4):1049–1092, (2016).
77. Hirschman, I. I., Multiplier transformations I, Duke Math. J., pp. 222–242, (1956).
78. Hörmander, L. Hypoelliptic second order differential equations, Acta Math. 119, 147–171, (1967).
79. Hörmander, L. Fourier integral operators. I, Acta Math, 127(1-2), pp. 79–183, (1971).
80. Hörmander, L. The Analysis of the linear partial differential operators Vol. III. Springer-Verlag, (1985)
81. Hirzebruch, F. Neue topologische methoden in der algebraischen Geometrie. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.
82. Johnson, J. W. Homogeneous Lie algebras and expanding automorphisms. Proc. Amer. Math. Soc. 48, 292–296, (1975).
83. Kohn, J.J., Nirenberg, L. Commun. Pure and Appl. Math., An algebra of pseudodifferential operators. 18, 269–305, (1965).
84. Kontsevich, M., Vishik, S. Geometry of determinants of elliptic operators. In: Functional Analysis on the Eve of the 21st Century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math., vol. 131, pp. 173–197.Birkhäuser, Boston (1995).
85. Lesch, M. On the Noncommutative Residue for Pseudodifferential Operators with log-Polyhomogeneous Symbols, Annals of Global Analysis and Geometry, 17, pp. 151–187, (1999).
86. Mihlin, S. G., Singular integral equations, Uspehi Mat. Nauk, Vol. 3, So. 25, 1948, pp. 29–112; New York Univ., Courant Inst. Math. Sci., 1963. Amer. Math. Soc. translation Vol. 24, 1950.
87. Măntoiu, M., Ruzhansky, M. Quantizations on nilpotent Lie groups and algebras having flat coadjoint orbits. J. Geom. Anal. 29, no. 3, pp. 2823–2861, (2019).
88. Măntoiu, M., Ruzhansky, M. Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups. Doc. Math. 22, pp. 1539–1592, (2017).
89. Miyachi, A. On some estimates for the wave equation in $L^p$ and $H^p$, J. Fac. Sci. Univ. Tokyo Sect. IA Math, 27(2), pp. 331–354, (1998).
90. Montgomery, R. A Tour of Sub-Riemannian Geometries, Their Geodesics and Applications, Math. Surveys Monogr., vol. 91, Amer. Math. Soc., Providence, RI, 2002.
91. Molahajloo, S. A characterization of compact pseudo-differential operators on $S^1$, in Pseudo-differential Operators: Analysis, Applications and Computations, Birkhäuser/Springer Basel AG, Basel, pp. 25–29, (2011).
92. Noether, F. Über eine Klasse singulärer Integralgleichungen. Math. Ann. 82, 42–63, (1921).
93. Nursultanov, E., Ruzhansky, M., Tikhonov, S. Nikol’skii inequality and functional classes on compact Lie groups, Funct. Anal. Appl. 49, pp. 226–229, (2015).
94. Nursultanov, E., Ruzhansky, M., Tikhonov S. Nikol’skii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci., XVI, pp. 981–1017, (2016).
95. Paycha, S. Regularised Integrals, Sums and Traces, University Lecture Series, 59. American Mathematical Society, Providence, RI, An analytic point of view, 2012.
96. Peetre, J. Sur les espaces de Besov, C. R. Acad. Sci. Paris. 264, pp. 281–283, (1967).
97. Peetre, J. Remarques sur les espaces de Besov. Le cas $0 < p < 1$, C. R. Acad. Sci. Paris. 277, pp. 947–950, (1973).
98. Peral, J. C. $L^p$-estimates for the wave equation, J. Funct. Anal, 36(1), pp. 114–145, (1980).
99. Persson, A. Compact linear mappings between interpolation spaces. Ark. Mat. 5, pp. 215–219, (1964).
100. Pietsch, A. Operator ideals. Mathematische Monographien, 16. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
101. Pietsch, A. History of Banach spaces and linear operators. Birkhäuser Boston, Inc., Boston, MA, 2007.
102. Rodríguez Torijano, C. A., Ruzhansky, M. Subelliptic wave equations with log-Lipschitz coefficients, arXiv:2007.09396.
103. Rothschild, L. P., Stein, E. M. Hypoelliptic differential operators and nilpotent groups. Acta Math., 137(3-4), pp. 247–320, (1976).
104. Ruzhansky, M. Regularity theory of Fourier integral operators with complex phases and singularities of affine fibrations, Volume 131 of CWI Tract, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 2001.
105. Ruzhansky, M., Tokmagambetov, N., Nonharmonic analysis of boundary value problems, Int. Math. Res. Notices, (2016) 2016 (12), 3548–3615.
106. Ruzhansky, M., Tokmagambetov, N. Nonharmonic analysis of boundary value problems without WZ condition, Math. Model. Nat. Phenom., 12 (2017), 115–140.
107. Ruzhansky, M., Turunen, V. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010.
108. Ruzhansky, M., Turunen, V., Wirth J., Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity, J. Fourier Anal. Appl. 20, pp. 476–499, (2014).
109. Ruzhansky, M., Turunen, V. Global quantization of pseudo-differential operators on compact Lie groups, SU(2) and 3-sphere, Int. Math. Res. Not. IMRN. 11, pp. 2439–2496, (2013).
110. Ruzhansky, M., Velasquez-Rodriguez, J. P., Non-harmonic Gohberg’s lemma, Gershgorin theory and heat equation on manifolds with boundary. Math. Nachr., to appear. arXiv:1902.00920v2.
111. Ruzhansky, M., Wirth, J. Global functional calculus for operators on compact Lie groups, J. Funct. Anal., 267, 144–172, (2014).
112. Ruzhansky, M., Wirth, J. $L^p$ Fourier multipliers on compact Lie groups, Math. Z., 280, pp. 621–642, (2015).
113. Ruzhansky, M., Sugimoto, M. Global $L^2$-boundedness theorems for a class of Fourier integral operators. Comm. Partial Differential Equations, 31(4–6), pp. 547–569, (2006).
114. Scott, S. Traces and determinants of pseudodifferential operators, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2010. zbl 1216.35192; MR2683288.
115. Seeger, A., Sogge, C. D., Stein, E. M. Regularity properties of Fourier integral operators, Ann. of Math. (2), 134(2), pp. 231–251, (1991).
116. Seddik, A. Some results related to the Corach-Porta-Recht inequality. Proc. Amer. Math. Soc. 129, no. 10, pp. 3009–3015, (2001).
117. Sukochev, F., Usachev, A. Dixmier traces and non–commutative analysis. J. Geom. Phys. 105, pp. 102–122, (2016).
118. Tao, T. The weak-type \((1, 1)\) of Fourier integral operators of order \(-\(n - 1\)/2\), J. Aust. Math. Soc, 76(1), pp. 1–21, (2004).
119. Taylor, M. Pseudodifferential Operators, Princeton Univ. Press, Princeton, N.J., 1981.
120. Taylor, M. Noncommutative Harmonic Analysis. Mathematical Surveys and Monographs, vol. 22. American Mathematical Society, Providence, RI, 1986.
121. ter Elst, A. F. M., Robinson, D. W. Spectral estimates for positive Rockland operators, in Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser., 9, 195–213, Cambridge Univ. Press, 1997.
122. Triebel, H. Theory of function spaces, vol. 78 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 1983.
123. Triebel, H. Theory of function spaces. III, volume 100 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 2006.
124. Velasquez-Rodriguez, J. P. On some spectral properties of pseudo-differential operators on T. J. Fourier Analysis and Appl. https://link.springer.com/article/10.1007/s00041-019-09680-2.
125. Wainger, S. Special trigonometric series in \(k\)-dimensions, Mem. Amer. Math. Soc. 59, (1965).
126. Weidmann, J. Linear operators in Hilbert spaces. Translated from the German by Joseph Szücs. Graduate Texts in Mathematics, 68. Springer-Verlag, New York-Berlin, 1980.
127. Wodzicki, M. Noncommutative Residue. I. Fundamentals, K-theory, Lecture Notes in Math., 1289, Springer, Berlin, 320–399, (1987).
128. Zhang, Y. Strichartz estimates for the Schrödinger flow on compact Lie groups. arXiv:1703.07548.

Duván Cardona Sánchez:
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
E-mail address duvanc306@gmail.com

Michael Ruzhansky:
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
AND
SCHOOL OF MATHEMATICS
QUEEN MARY UNIVERSITY OF LONDON
UNITED KINGDOM
E-mail address ruzhansky@gmail.com