Nonlinear self-interaction of plane gravitational waves

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Recently Mendonça and Cardoso [Phys. Rev. D 66, 104009 (2002)] considered nonlinear gravitational wave packets propagating in flat space-time. They concluded that the evolution equation—to third order in amplitude—takes a similar form to what arises in nonlinear optics. Based on this equation, the authors found that nonlinear gravitational waves exhibit self-phase modulation and high harmonic generation leading to frequency up-shifting and spectral energy dilution of the gravitational wave energy. In this Brief Report we point out the fact—a possibility that seems to have been overlooked by Mendonça and Cardoso—that the nonlinear terms in the evolution equation cancels and, hence, that there is no amplitude evolution of the pulse. Finally we discuss scenarios where these nonlinearities may play a role.

PACS numbers: 04.30.Nk, 42.65.-k, 95.30.Sf

As discussed by Mendonça and Cardoso in [1] the Einstein field equations are highly nonlinear and it is an interesting question whether gravitational waves share any similarities with other types of more well-known nonlinear waves, for instance those occurring in nonlinear optics. Assuming the presence of a number of wave perturbations

\[ h_{ab}(n) = \epsilon_{ab}(n)a(n) \exp[iq_{c}(n)x^{c}] + \text{c.c.}, \tag{1} \]

it was—based on the structure of the vacuum gravitational field equations—argued by Mendonça and Cardoso that the wave amplitude evolution equations for four-wave mixing processes take the form

\[ iq_{a}(1) \frac{\partial}{\partial x_{a}} a(1) = w(1)a^{*}(2)a(3)a(4) \exp[i\Delta q_{b}x^{b}], \tag{2} \]

where \((n)\) labels the wave perturbations, \(\epsilon_{ab}(n)\) is the unit polarization tensor for wave \((n)\), \(a(n)\) is the slowly varying amplitude (* stands for complex conjugate), \(q_{c}(n)\) is the wave four-vector, c.c. denotes the complex conjugate of the preceding term, \(\Delta q_{b} = q_{b}(3) + q_{b}(4) - q_{b}(1) - q_{b}(2)\) and the coupling coefficient \(w(1)\) is determined by the Ricci tensor cubic in the wave amplitudes (not explicitly computed in [1]). The background spacetime was assumed flat. The authors then considered the implications of Eq. (2), which indeed is similar to what arises in nonlinear optics. The analysis was restricted to the case when all present waves propagate in the same direction. In particular they focused on (i) self-phase modulation (when the four waves have identical polarization and wave vector) resulting in that a gravitational wave pulse will be continuously frequency up-shifted as it propagates through space and (ii) harmonic cascading through which gravitational wave energy is spectrally diluted.

In this Brief Report we present the results of detailed calculations of the amplitude evolution equation for plane gravitational waves propagating along \(x^{3} \equiv z\). The perturbed metric \(g_{ab}\) is

\[ g_{ab} = \eta_{ab} + h_{ab}^{(1)} + h_{ab}^{(2)} + h_{ab}^{(3)}, \tag{3} \]

where \(\eta_{ab} = \text{diag}(-1, 1, 1, 1)\), \(h_{ab}^{(2)}\) and \(h_{ab}^{(3)}\) are the second and third order nonlinear response to the linear wave perturbation \(h_{ab}^{(1)}\). As it turns out, there are no third order response terms that couple back to the evolution equation of \(h_{ab}^{(1)}\), therefore \(h_{ab}^{(3)}\) is omitted from here on. The vacuum Einstein field equations reads

\[ R_{ab}^{(1)} + R_{ab}^{(2)} + R_{ab}^{(3)} = 0, \tag{4} \]

where the Ricci tensor has been divided into \(R_{ab}^{(1)}\) (linear in \(h_{ab}^{(1)}\), \(R_{ab}^{(2)}\) (quadratic in \(h_{ab}^{(1)}\) and linear in \(h_{ab}^{(2)}\)) and \(R_{ab}^{(3)}\) (cubic in \(h_{ab}^{(1)}\) and with terms of order \(h_{ab}^{(1)}h_{ab}^{(2)}\)). Having wave perturbations of the type given by Eq. (1) in mind we note that in \(R_{ab}^{(3)}\), but not in \(R_{ab}^{(2)}\), there are terms that are resonant with \(R_{ab}^{(1)}\). Thus, we identify

\[ R_{ab}^{(1)} = -R_{ab}^{(3)}, \tag{5} \]

with the amplitude evolution equation, where it should be understood that only the part of \(R_{ab}^{(3)}\) resonant with \(R_{ab}^{(1)}\) should be taken into account. The second order nonlinear response terms, \(h_{ab}^{(2)}\), are related to the \(h_{ab}^{(1)}\) by

\[ R_{ab}^{(2)} = 0. \tag{6} \]

We are free to choose a particular gauge for \(h_{ab}^{(1)}\) and we choose the TT gauge such that the only nonzero components are \(h_{11}^{(1)} = -h_{22}^{(1)} \equiv h^{(1)}\). With this choice \(R_{ab}^{(2)}\)
reads
\[
R^{(2)}_{ab} = \frac{1}{2} \left[ -\partial^2 h_{ab}^{(2)} + \alpha \partial \partial_3 h_{ab}^{(3)} \right], \tag{7a}
\]
\[
R^{(3)}_{ab} = \frac{1}{4} (\partial_3 h_{ab}) - \frac{1}{4} (\partial_3 h_{ab}), \tag{7b}
\]
\[
R^{(3)}_{33} = \frac{1}{2} \left[ \lambda - \partial^2 H \right] + \frac{1}{2} (\partial_3 h)^2 + h \partial^2 h, \tag{7c}
\]
and
\[
R^{(2)}_{0a} = \frac{1}{2} \left[ -\partial^2 h_{0a}^{(2)} + \alpha_1 \partial_3 h_{0a}^{(3)} \right], \tag{8a}
\]
\[
R^{(3)}_{3a} = \frac{1}{4} (\partial_3 h_{3a}) - \partial_3 h_{0a}^{(2)}, \tag{8b}
\]
\[
R^{(2)}_{\alpha\beta} = \frac{1}{2} [\partial_3 h_{\alpha\beta}^{(2)} - \partial_3 h_{\alpha\beta}^{(2)} ] - \frac{1}{2} (\partial_3 h)^2 \delta_{\alpha\beta}, \tag{8c}
\]
where \(\alpha, \beta = 1, 2\) and
\[
H \equiv h_{11}^{(2)} + h_{22}^{(2)}, \tag{9}
\]
\[
\lambda \equiv \partial^2 h_{00}^{(2)} - 2 \partial_3 h_{03}^{(3)} + \partial_3 h_{33}^{(2)}. \tag{10}
\]
When computing \(R^{(2)}_{ab}\) and \(R^{(3)}_{ab}\), one should make use of the approximation \(\partial_{a} \approx -\partial_{a}\). Applying this to \(R^{(2)}_{ab}\), it follows that all the components of \(h_{ab}^{(2)}\) may be set to zero except \(h_{11}^{(2)}\) and \(h_{22}^{(2)}\) (the final result does not depend on whether this choice, corresponding to a freedom in gauge, is made or not). Making this choice Eq. (9) becomes
\[
\partial_3^2 H = - (\partial_3 h)^2 + 2 \partial_3^2 (h^2). \tag{11}
\]
The nonzero components of \(R^{(3)}_{ab}\) are
\[
R^{(3)}_{11} = \frac{1}{4} (\partial_1 h)(\partial_1 H) - \frac{1}{4} (\partial_1 h)(\partial_1 H), \tag{12a}
\]
\[
R^{(3)}_{22} = - \frac{1}{4} (\partial_2 h)(\partial_2 H) + \frac{1}{4} (\partial_2 h)(\partial_2 H), \tag{12b}
\]
and with the approximation \(\partial_{a} \approx -\partial_{a}\) we see that \(R^{(3)}_{ab} = 0\). The amplitude evolution equation (8) thus reduces to
\[
(\partial_3^2 - \partial_3^2) h = 0, \tag{13}
\]
and we conclude that—up to third order in amplitude—plane gravitational waves propagate with no change in the amplitude modulation, i.e., \(w(1)\) in Eq. (8) is zero. To some extent this may seem counterintuitive, since a gravitational wave-packet propagating in flat space-time transports energy and momentum as described by the pseudo energy-momentum tensor which is of order \(h^2\). This produces a local curvature of space-time of the same order, and one could expect a coupling to the original wave perturbation, leading to cubic nonlinearities in the amplitude evolution equation. However, this picture is only partially true. The pseudo energy-momentum tensor does produce curvature quadratic in the amplitude \(h\) (essentially described by Eq. (11)), and this couples back to the original perturbation leading to cubic nonlinearities in the Riemann tensor. But as follows from Eq. (12a) and Eq. (12b) these terms cancels when computing the Ricci tensor and thus does not enter the amplitude evolution equation.

It is instructive to compare this result to existing exact plane wave solutions of the vacuum Einstein field. One particular exact plane wave solution is given by the metric (see, e.g., Ref. [4])
\[
ds^2 = -dt^2 + a(\xi)^2 dx^2 + b(\xi)^2 dy^2 + dz^2, \tag{14}
\]
where \(a\) and \(b\) satisfy the equation
\[
ba^2 - a^2 = 0, \tag{15}
\]
where \(\xi = z - t\). Focusing on weak waves, we take \(a = (1 + h + 2h)^2\) and \(b = (1 - h + 2h)^2\) where \(1 \gg h\) and \(H \sim h^2\). Expanding Eq. (15) up to third order in amplitude one find precisely the relation (11) between \(H\) and \(h\) and that the third order terms cancels identically. The fact that the metric components of the gravitational wave perturbations in Eq. (11) (with the gauge choices made above) coincide with those of the exact wave solution when expanded in powers of amplitude suggests that Eq. (11) will remain even if one relaxes the approximations made in this Brief Report.

It should be pointed out, however, that gravitational four-wave mixing processes are not ruled out in general. Firstly, we have here only considered parallel propagation, and it is well-known that anti-parallel gravitational waves can interact (see e.g., Ref. [4] and references therein). Secondly, the presence of background curvature and matter modifies the properties of gravitational waves, e.g., giving rise to wave dispersion and new types of nonlinearities. To what extent this increases the possibilities of nonlinear self interaction is an open question.

The equations presented here provides a starting point for future investigations of these possibilities.

[1] J. T. Mendonça and V. Cardoso, Phys. Rev. D 66, 104009 (2002).
[2] Due to the assumption of slowly varying amplitudes \(h_{11}^{(1)}\) and \(h_{22}^{(1)}\) are not the only nonzero components. The other components are, however, of higher order and these terms can therefore be absorbed in \(h_{ab}^{(3)}\).
[3] One might think that the use of \(\partial_{a} \approx -\partial_{a}\) is too restrictive. However, improving this approximation produces necessarily terms in the amplitude evolution equation that are of higher order than cubic.
[4] H. Stephani, General Relativity (Cambridge University
[5] V. Faraoni and R. M. Dumse, Gen. Rel. Grav. 31, 91 (1999).