Some New Double Sequence Spaces of Fuzzy Numbers Defined by Double Orlicz Functions Using A Fuzzy Metric

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Abstract. This paper presents some new double sequence spaces using a double Orlicz functions and a fuzzy metric. Using the idea that a nonnegative, upper-semi continuous, normal and convex fuzzy number is the distance between two points in a fuzzy metric, we test some basic properties of the new double sequence spaces of fuzzy numbers. We also analyze the relationships between these spaces.

Keywords. Double Orlicz functions, fuzzy number, fuzzy metric, solid space, monotone space, multiplier space.

1. Introduction
Several scholars have explored the principles of fuzzy sets and fuzzy set operations along with aspects of the theory and implementations such as fuzzy topological spaces, fuzzy ordering, fuzzy events measurements, and fuzzy mathematical programming. Matloka [1] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. Later on, sequences of fuzzy numbers were discussed by Diamond and Kloeden [2], Nanda [3], Esi [4], Tripathy and Baruah [5], Tripathy and Sarma [6] and many others.

Tripathy and Mahanta [7] used a general multiplier double sequence \( \Lambda = (\mathcal{T}_{1,s}) \) of nonzero scalars for all \( t, s \in \mathbb{N} \).

Let \( \Lambda = (\mathcal{T}_{1,s}) \) by a double sequence of nonzero scalars. Then for a given double sequence space \( E^2 \), the multiplier double sequence space \( E^2(\Lambda) \) associated with the multiplier double sequence \( \Lambda \) is defined by (see [7])

\[
E^2(\Lambda) = \left\{ (x_{ts}, y_{ts}) : (\mathcal{T}_{1,s} x_{ts}, \mathcal{T}_{1,s} y_{ts}) \in E^2 \right\}.
\]

In this paper, using a double Orlicz functions and a fuzzy metric, we discuss some new double sequence spaces of fuzzy numbers. Using the principles of the fuzzy metric, an inclusion
relationship is established between a double sequence spaces \((c_0)_{\mathcal{H}}(\Lambda), (c)_{\mathcal{H}}(\Lambda)\) and \((l_{\infty})_{\mathcal{H}}(\Lambda, \Lambda)\). Also we show that these spaces are complete metric spaces with a suitable metric. In addition, we study the solid and monotone nature of these spaces. The \(\Lambda\)-invariance of these spaces could also be studied; this is an open problem for researchers.

A double Orlicz function is a function \(\mathcal{H} : [0, \infty) \times [0, \infty) \to \mathbb{R} \) such that \(\mathcal{H}_1(x, y) = (\mathcal{H}_1(x), \mathcal{H}_2(y))\),

\[ \mathcal{H}_1 : [0, \infty) \to [0, \infty) \text{ and } \mathcal{H}_2 : [0, \infty) \to [0, \infty), \text{ such that } \mathcal{H}_1, \mathcal{H}_2 \text{ are Orlicz functions which } \]

continuous, non-decreasing, even, convex and satisfy the following conditions

i) \(\mathcal{H}_1(0) = 0, \mathcal{H}_2(0) = 0 \Rightarrow \mathcal{H}(x, y) = (\mathcal{H}_1(0), \mathcal{H}_2(0)) = (0,0)\),

ii) \(\mathcal{H}_1(x) > 0, \mathcal{H}_2(y) > 0 \Rightarrow \mathcal{H}(x, y) = (\mathcal{H}_1(x), \mathcal{H}_2(y)) > (0,0)\),

for \(x > 0, y > 0\), we mean by \((x, y) = (x_1, y_1) > (0,0)\), that \(\mathcal{H}_1(x) > 0, \mathcal{H}_2(y) > 0\).

iii) \(\mathcal{H}_1(x) \to \infty, \mathcal{H}_2(y) \to \infty \) as \(x, y \to \infty\), then

\(\mathcal{H}(x, y) = (\mathcal{H}_1(x), \mathcal{H}_2(y)) \to (\infty, \infty)\), as \((x, y) \to (\infty, \infty)\), we mean by

\(\mathcal{H}(x, y) \to (\infty, \infty)\), that \(\mathcal{H}_1(x) \to \infty, \mathcal{H}_2(y) \to \infty\) [11].

If the convexity of an Orlicz function \(\mathcal{H}\) is replaced by its subadditivity, i.e. \(\mathcal{H}(x + y) \leq \mathcal{H}(x) + \mathcal{H}(y)\), then this function is called a modulus function.

Battor, Neamah [11] used the concept of Orlicz function to construct the double sequence space \(l_{\mathcal{H}}^2\), we are able to use that concept to construct a double sequence space as follows:

\[ l_{\mathcal{H}}^2 = (2l_{\mathcal{H}_1}, 2l_{\mathcal{H}_2}) = \]

\[ \{ (x, y) \in W^2 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_1 \left( \frac{|x_{i,j}|}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{|y_{i,j}|}{\mu} \right) \right\} < \infty, \text{for some } \mu > 0 \}, \]

where

\[ 2l_{\mathcal{H}_1} = \{ (x, y) \in W^2 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_1 \left( \frac{|x_{i,j}|}{\mu} \right) \right\} < \infty, \text{for some } \mu > 0 \}, \]

\[ 2l_{\mathcal{H}_2} = \{ (x, y) \in W^2 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_2 \left( \frac{|y_{i,j}|}{\mu} \right) \right\} < \infty, \text{for some } \mu > 0 \}, \]

The space \( l_{\mathcal{H}}^2 \), with the norm

\[ \| (x, y) \|_{\mathcal{H}} = \inf \{ \mu > 0 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_1 \left( \frac{|x_{i,j}|}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{|y_{i,j}|}{\mu} \right) \right\} \leq 1 \}, \]

where

\[ \| x \|_{\mathcal{H}_1} = \inf \{ \mu > 0 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_1 \left( \frac{|x_{i,j}|}{\mu} \right) \right\} \leq 1 \}, \]

\[ \| y \|_{\mathcal{H}_2} = \inf \{ \mu > 0 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \mathcal{H}_2 \left( \frac{|y_{i,j}|}{\mu} \right) \right\} \leq 1 \}, \]

becomes a Banach space which is known as a double Orlicz of a double sequence space.

We refer to the set of all closed and bounded intervals \(x = [x_1, x_2]\) on the real line \(\mathcal{R}\) by symbol \(\mathcal{D}\).
For $X = [x_1, x_2] \in \mathbb{D}$ , $Y = [y_1, y_2] \in \mathbb{D}$ and $Z = [z_1, z_2] \in \mathbb{D}$ , $W = [w_1, w_2] \in \mathbb{D}$, define

$$(X,Y) \leq (Z,W) \quad \text{if and only if} \quad x_1 \leq z_1 , x_2 \leq z_2 \quad \text{and} \quad y_1 \leq w_1 , y_2 \leq w_2$$

$$d((X,Y),(Z,W)) = \max \{(|x_1 - y_1|, |x_2 - y_2|), (|z_1 - w_1|, |z_2 - w_2|)\}.$$ 

It is recognized that $(\mathbb{D}, d)$ is a complete metric space. Also the relation $\leq$ is a partial order on $\mathbb{D}$.

A fuzzy number $\Psi$ is a fuzzy combination on the real axis, i.e. a mapping $\Psi : \mathbb{R} \to I$ (= $[0,1]$) associating each real number $u$ with its membership rank $\Psi(u)$, satisfies the following condition:

i) The mapping $\Psi$ is convex if $\Psi(u) \geq \Psi(s) \wedge \Psi(r) = \min\{\Psi(s), \Psi(r)\}$, where $s < u < r$.

ii) The mapping $\Psi$ is normal if there exists $u_0 \in \mathbb{R}$ such that $\Psi(u_0) = 1$.

iii) The mapping $\Psi$ is upper semi continuous if, for each $\epsilon > 0$, $\Psi^{-1}([0, \alpha + \epsilon))$ is open in the topology of $\mathbb{R}$ for all $\alpha \in I$.

iv) The mapping $\Psi$ is called non-negative if $\Psi(u) = 0$, for all $u < 0$. The set of all non-negative fuzzy real numbers is denoted by $\mathcal{R}(I)$ [10].

Let $\mathcal{R}(I)$ denote the set of all upper semi continuous, normal, convex fuzzy numbers and have compact support, i.e. if $\Psi \in \mathcal{R}(I)$, then for any $\alpha \in [0,1]$, $[\Psi]^\alpha$ is compact, where

$$[\Psi]^\alpha = \{u \in \mathcal{R} : \Psi(u) \geq \alpha , \text{if} \; \alpha \in [0,1]\},$$

$$[\Psi]^0 = \text{closure of} \{u \in \mathcal{R} : \Psi(u) > \alpha , \text{if} \; \alpha = 0\}.$$ 

The set of all real number $\mathcal{R}$ is called embedded in $\mathcal{R}(I)$ if we define $\bar{b} \in \mathcal{R}(I)$ by

$$\bar{b}(u) = \begin{cases} 1 & \text{if} \; u = b, \\ 0 & \text{if} \; u \neq b. \end{cases}$$

The multiplicative identity and additive identity of $\mathcal{R}(I)$ are defined by $\bar{1}$ and $\bar{0}$ respectively.

For $b \in \mathcal{R}$ and $\Psi \in \mathcal{R}(I)$, the product $b\Psi$ is defined as follows:

$$b\Psi(u) = \begin{cases} \Psi(b^{-1}u) , & \text{if} \; b \neq 0, \\ 0 , & \text{if} \; b = 0. \end{cases}$$

The absolute value of $\Psi \in \mathcal{R}(I)$, $|\Psi|$ is defined by (see [8])

$$|\Psi|(u) = \begin{cases} \max\{\Psi(u), \Psi(-u)\} , & \text{if} \; u \geq 0, \\ 0 , & \text{if} \; u < 0. \end{cases}$$

Define a mapping $\tilde{d} : \mathcal{R}^2(1) \times \mathcal{R}^2(1) \to \mathcal{R}^+ \cup \{0\}$ by $\tilde{d}((X,Y),(Z,V)) = \sup_{\alpha, \beta = 1} d((|X|^\alpha, [Y]^\alpha), (|Z|^\beta, [V]^\beta))$. It is clear $(\mathcal{R}^2(1), \tilde{d})$ is a complete metric space (see [1]).

A metric on $\mathcal{R}^2(1)$ is called translation invariant if

$$\tilde{d}((\mathcal{A}, \mathcal{B}), (Z, V) + (\mathcal{A}, \mathcal{B})) = \tilde{d}((X, Y), (Z, V)) \quad \text{for all} \quad (X, Y), (Z, V), (\mathcal{A}, \mathcal{B}) \in \mathcal{R}(I).$$
2. Definition and preliminaries

**Definition 2.1:** Let \( \chi = (\chi_{ts}) \), \( \gamma = (\gamma_{ts}) \) be double sequence. A double sequence \( (\chi, \gamma) = (\chi_{ts}, \gamma_{ts}) \) of fuzzy numbers is called converge to a fuzzy number \( (\chi_0, \gamma_0) \) if for every \( \epsilon > 0 \), there is positive integer \( n_0, m_0 \) such that \( d((\chi_{ts}, \gamma_{ts}), (\chi_0, \gamma_0)) < \epsilon \) for all \( t \geq n_0, s \geq m_0 \).

**Definition 2.2:** A double sequence \( (\chi, \gamma) = (\chi_{ts}, \gamma_{ts}) \) of fuzzy numbers is said to be bounded if the set \( \{ (\chi_{ts}, \gamma_{ts}) : t, s \in \mathbb{N} \} \) of fuzzy numbers is bounded.

**Definition 2.3:** A double sequence space \( E_2^f \) is said to be solid if \( (\mathcal{M}_{ts}, \mathcal{K}_{ts}) \in E_2^f \) whenever \( (\chi_{ts}, \gamma_{ts}) \in E_2^f \) and \( (\mathcal{M}_{ts}, \mathcal{K}_{ts}) \leq (\chi_{ts}, \gamma_{ts}) \) for all \( t, s \in \mathbb{N} \).

**Definition 2.4:** A double sequence space \( E_2^f \) is said to be monotone if \( E_2^f \) contains the canonical pre-images of all its step spaces.

Let \( E_2^f \) denote the a double sequence space of fuzzy numbers.

**Remark 2.1:** if A double sequence space \( E_2^f \) is solid, this implies that \( E_2^f \) is monotone[12].

In this paper we define some new classes of double sequences of fuzzy numbers using a fuzzy metric. One may refer to Syau [9] for the notion of a fuzzy metric.

Suppose \( d_F : \mathcal{R}_2(1) \times \mathcal{R}_2(1) \to \mathcal{R}(1) \) and let the mappings \( M, T : [0,1]^2 \to [0,1] \) by symmetric, be nondecreasing in both arguments and satisfy \( M((0,0),(0,0)) = (0,0) \) and \( T((1,1),(1,1)) = (1,1) \) i.e. \( M = \min\{(s,d),(a,b)\} \) and \( T = \max\{(s,d),(a,b)\} \), where \((s,d),(a,b) \in [0,1]\).

Let \( \lambda : \mathcal{R}_2(1) \times \mathcal{R}_2(1) \to \mathcal{R} \) be such that \( \lambda((\chi,\gamma),(\zeta,\nu)) = \sup_{0<\alpha<1}\lambda_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\}) \), where \( \lambda_\alpha : \mathcal{R}_2 \times \mathcal{R}_2 \to \mathcal{R} \) and \( \lambda_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\}) = \min_{i,j \in \mathbb{N}} \{(|X^\alpha_i - Y^\alpha_i|, |Z^\alpha_i - V^\alpha_i|), (|X^\alpha_j - Y^\alpha_j|, |Z^\alpha_j - V^\alpha_j|)\} \).

Similarly, let \( \rho : \mathcal{R}_2(1) \times \mathcal{R}_2(1) \to \mathcal{R} \) be such that \( \rho((\chi,\gamma),(\zeta,\nu)) = \sup_{0<\alpha<1}\rho_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\}) \), where \( \rho_\alpha : \mathcal{R}_2 \times \mathcal{R}_2 \to \mathcal{R} \) and \( \rho_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\}) = \max_{i,j \in \mathbb{N}} \{(|X^\alpha_i - Y^\alpha_i|, |Z^\alpha_i - V^\alpha_i|), (|X^\alpha_j - Y^\alpha_j|, |Z^\alpha_j - V^\alpha_j|)\} \).

Since the distance between two fuzzy numbers is again a fuzzy number, the \( \alpha \)-level set of this distance \( d_F \) between the fuzzy numbers \((\chi,\gamma),(\zeta,\nu)\) is defined as
\[
d_F((\chi,\gamma),(\zeta,\nu)) = \{\lambda_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\}), \rho_\alpha(\{([X]^\alpha,[Y]^\alpha),([Z]^\alpha,[V]^\alpha)\})), 0 < \alpha \leq 1.
\]

The quadruple \((\mathcal{R}_2(1), d_F, M, T)\) is called a fuzzy metric space, and \( d_F \) is a fuzzy metric if:
1) \( d_F((\chi,\gamma),(\zeta,\nu)) = 0,0 \) if and only if \((\chi,\gamma) = (\zeta,\nu)\).
2) \( d_F((\chi,\gamma),(\zeta,\nu)) = d_F((\zeta,\nu),(\chi,\gamma)) \), for all \((\chi,\gamma),(\zeta,\nu) \in \mathcal{R}_2(1)\).
3) for all \((\chi,\gamma),(\zeta,\nu), (\alpha,\beta) \in \mathcal{R}_2(1)\):
(i) \( d_F((X, Y), (Z, V))(s, u) \geq M \left( d_F((X, Y), (A, B))(s), d_F((A, B), (Z, V))(u) \right) \), where \( s \leq \lambda_1((X, Y), (A, B)), u \leq \lambda_1((A, B), (Z, V)) \) and \( s + u \leq \lambda_1((X, Y), (Z, V)) \).

(ii) \( d_F((X, Y), (Z, V))(s, u) \leq T \left( d_F((X, Y), (A, B))(s), d_F((A, B), (Z, V))(u) \right) \), where \( s \geq \lambda_1((X, Y), (A, B)), u \geq \lambda_1((A, B), (Z, V)) \) and \( s + u \geq \lambda_1((X, Y), (Z, V)) \).

Using the concept of double Orlicz functions and the fuzzy metric, we introduce the following double sequence spaces. Let \( \Lambda = (\gamma_{t,s}) \) by a double sequence of nonzero scalars and let \( \mathcal{H} \) be an double Orlicz function and \((X, Y) = (X_{t,s}, Y_{t,s})\) be a double sequence of fuzzy numbers; then we define the following double sequence spaces:

\[
\begin{align*}
(c) \quad & \hat{\mathcal{H}}(\mathcal{H}, \Lambda) = \left\{ (X_{t,s}, Y_{t,s}) \in \mathcal{W}_2^2 : \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}X_{t,s}Y_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}Y_{t,s})}{\mu} \right) \rightarrow 0 \text{ as } (t, s) \rightarrow (0, 0), \text{ and for some } \mu > 0 \right\}, \\
& \text{and} \quad \hat{\mathcal{H}}_0(\mathcal{H}, \Lambda) = \left\{ (X_{t,s}, Y_{t,s}) \in \mathcal{W}_2^2 : \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}X_{t,s}Y_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}Y_{t,s})}{\mu} \right) \rightarrow 0 \text{ as } (t, s) \rightarrow (0, 0), \text{ and for some } \mu > 0 \right\}, \\
& \text{for all } (X_{t,s}, Y_{t,s}) \in \mathcal{W}_2^2 : \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}X_{t,s}Y_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}Y_{t,s})}{\mu} \right) < 0 \text{ for some } \mu > 0 .
\end{align*}
\]

3. Main Results

**Theorem 3.1:** Let \( \Lambda = (\gamma_{t,s}) \) by a double sequence of nonzero scalars and \( \mathcal{H} \) be an double Orlicz function. Then \((c) \hat{\mathcal{H}}(\mathcal{H}, \Lambda), (c_0) \hat{\mathcal{H}}(\mathcal{H}, \Lambda) \) and \((l_\infty) \hat{\mathcal{H}}(\mathcal{H}, \Lambda) \) are metric space with the metric defined by

\[
g_H((X, Y), (M, N)) = \inf \left\{ \mu > 0 : \sup_{t,s} \left[ \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}X_{t,s}Y_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}M_{t,s}N_{t,s})}{\mu} \right) \right] \leq 1 \text{ and } \sup_{t,s} \left[ \mathcal{H}_1 \left( \frac{\rho(y_{t,s}X_{t,s}Y_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\rho(y_{t,s}M_{t,s}N_{t,s})}{\mu} \right) \right] \leq 1 \right\}.
\]

**Proof:** We prove it for the case \((l_\infty) \hat{\mathcal{H}}(\mathcal{H}, \Lambda) \) and the other cases can be established next similar techniques.

Suppose \((X, Y), (M, N) \in (l_\infty) \hat{\mathcal{H}}(\mathcal{H}, \Lambda) \); we have:

1) \( g_H((X, Y), (M, N)) = (0, 0) \). This implies that \( \lambda \left( (y_{t,s}X_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}N_{t,s}) \right) = (0, 0) \) and \( \rho \left( (y_{t,s}X_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}N_{t,s}) \right) = (0, 0) \) for all \( t, s \in \mathbb{N} \). Since \( (\mathcal{H}_1(0), \mathcal{H}_2(0)) = (0, 0) \), which implies that, for all \( \alpha \in (0, 1) \),
\[ \lambda \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = \sup_{0 < \alpha \leq 1} \lambda_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

\[ \Rightarrow \min \left\{ |y_{t,s}x_{t,s} - y_{t,s}y_{t,s}|, |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|, (|y_{t,s}x_{t,s} - y_{t,s}y_{t,s}| - |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|) \right\} \]

Similarly, for all \( \alpha \in (0,1) \),

\[ \rho \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = \sup_{0 < \alpha \leq 1} \rho_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

\[ \Rightarrow \max \left\{ |y_{t,s}x_{t,s} - y_{t,s}y_{t,s}|, |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|, (|y_{t,s}x_{t,s} - y_{t,s}y_{t,s}| - |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|) \right\} \]

From (1) and (2) it follows that, for all \( t, s \in \mathbb{N} \), \((x_{t,s}, y_{t,s}) = (M_{t,s}, N_{t,s}) \Rightarrow (\chi, y) = (M, N)\).

Conversely, assume that \((\chi, y) = (M, N)\). Then using the definitions of \( \lambda \) and \( \rho \), we get

\[ \lambda_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \quad \text{and} \quad \rho_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

for all \( t, s \in \mathbb{N} \) and \( \alpha \in (0,1) \), which implies that

\[ \sup_{0 < \alpha \leq 1} \lambda_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \quad \text{and} \quad \sup_{0 < \alpha \leq 1} \rho_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

for all \( t, s \in \mathbb{N} \).

It follows that

\[ \lambda \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \quad \text{and} \quad \rho \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

Using the continuity of \( \mathcal{H} \), we get \( g_{\mathcal{H}}((\chi, y), (M, N)) = (0,0) \).

Then \( g_{\mathcal{H}}((\chi, y), (M, N)) = (0,0) \) if and only if \((\chi, y) = (M, N)\).

2) We have

\[ g_{\mathcal{H}}((\chi, y), (M, N)) = \inf \left\{ \mu > 0 : \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}x_{t,s}, y_{t,s}y_{t,s})}{\mu} \right) \right) \leq 1 \right\} \]

and

\[ \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(y_{t,s}x_{t,s}, y_{t,s}y_{t,s})}{\mu} \right) \right) \leq 1 \}

From the definition of \( \lambda \) it follows that

\[ \lambda \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = \sup_{0 < \alpha \leq 1} \lambda_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

\[ \rho \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = \sup_{0 < \alpha \leq 1} \rho_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

\[ \Rightarrow \min \left\{ |y_{t,s}x_{t,s} - y_{t,s}y_{t,s}|, |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|, (|y_{t,s}x_{t,s} - y_{t,s}y_{t,s}| - |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|) \right\} \]

Similarly, for all \( \alpha \in (0,1) \),

\[ \rho \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = \sup_{0 < \alpha \leq 1} \rho_\alpha \left( (y_{t,s}x_{t,s}, y_{t,s}y_{t,s}), (y_{t,s}M_{t,s}^\alpha, y_{t,s}N_{t,s}^\alpha) \right) = (0,0) \]

\[ \Rightarrow \max \left\{ |y_{t,s}x_{t,s} - y_{t,s}y_{t,s}|, |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|, (|y_{t,s}x_{t,s} - y_{t,s}y_{t,s}| - |y_{t,s}M_{t,s}^\alpha - y_{t,s}N_{t,s}^\alpha|) \right\} \]
Proceeding in the same way, we get \( \rho \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) = \rho \left( (y_{t,s}X_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) \).

This we get

\[
g_{\mathcal{H}}(\chi, y, (\mathcal{M}, \kappa)) = \inf \left\{ \mu > 0 : \sup \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}x_{t,s}, y_{t,s}Y_{t,s})}{\mu} \right) \right) \leq 1 \text{ and } \sup \left( \mathcal{H}_2 \left( \frac{\rho(y_{t,s}M_{t,s}, y_{t,s}N_{t,s})}{\mu} \right) \right) \leq 1 \right\}
\]

i.e. \( g_{\mathcal{H}}(\chi, y, (\mathcal{M}, \kappa)) = g_{\mathcal{H}}((\mathcal{M}, \kappa), (\chi, y)) \).

3) Let \( (x_{t,s}, y_{t,s}, (\mathcal{M}_{t,s}, \mathcal{N}_{t,s}), (\Omega_{t,s}, \mathcal{P}_{t,s}) \in (\omega, \mathcal{L})(\mathcal{H}, \lambda) \) and let \( \mu_1 > 0, \mu_2 > 0 \) be such that

\[
\sup \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}x_{t,s}, y_{t,s}Y_{t,s})}{\mu_1} \right) \right) \leq 1
\]

and

\[
\sup \left( \mathcal{H}_2 \left( \frac{\rho(y_{t,s}M_{t,s}, y_{t,s}N_{t,s})}{\mu_2} \right) \right) \leq 1.
\]

Let \( \mu = \mu_1 + \mu_2 \). By the definition of \( \lambda \), we get

\[
\lambda \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) = \sup_{0 < \alpha \leq 1} \lambda_{\alpha} \left( (y_{t,s}x_{t,s}^{\alpha}, y_{t,s}Y_{t,s}^{\alpha}), (y_{t,s}M_{t,s}^{\alpha}, y_{t,s}N_{t,s}^{\alpha}) \right),
\]

where

\[
\lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) = \min \left\{ \left| y_{t,s}x_{t,s}^{\alpha} - y_{t,s}Y_{t,s}^{\alpha} \right|, \left| y_{t,s}M_{t,s}^{\alpha} - y_{t,s}N_{t,s}^{\alpha} \right| \right\}.
\]

By the definition of the modulus function, we get

\[
\lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) \leq \lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) + \lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right), \text{ for all } \alpha \in (0, 1].
\]

Taking the supremum throughout \( \alpha \), we get

\[
\sup_{0 < \alpha \leq 1} \lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) \leq \sup_{0 < \alpha \leq 1} \lambda_{\alpha} \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right)
\]

Which implies that

\[
\lambda \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) \leq \lambda \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right) + \lambda \left( (y_{t,s}x_{t,s}, y_{t,s}Y_{t,s}), (y_{t,s}M_{t,s}, y_{t,s}N_{t,s}) \right).
\]
Using the continuity of $\mathcal{H}$, we get

$$\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu} \right) \right) \leq \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu_1 + \mu_2} \right) + \frac{\mu_2}{\mu_1 + \mu_2} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu_1} \right)$$

$$\leq \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu_1 + \mu_2} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{ts}, y_{ts}, y_{ts})}{\mu_1} \right) \right) + \sup_{t,s} \left( \frac{\mu_2}{\mu_1 + \mu_2} \right) \leq 1.$$
Theorem 3.2: The double sequence spaces $(c_0)_{\ell}^S(\mathcal{H}, \Lambda)$ and $(l_{\infty})_{\ell}^S(\mathcal{H}, \Lambda)$ are solid whereas the double sequence space $(c)_{\ell}^S(\mathcal{H}, \Lambda)$ is not solid.

Proof: Consider the space of double sequence $(l_{\infty})_{\ell}^S(\mathcal{H}, \Lambda)$. Let $(X_{ta}, Y_{ta}) \in (l_{\infty})_{\ell}^S(\mathcal{H}, \Lambda)$. Then we have

$$
sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(y_{ta}, Y_{ta})}{\mu} \right) \right) < \infty \quad \text{and} \quad sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\rho(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(y_{ta}, Y_{ta})}{\mu} \right) \right) < \infty, \quad \text{for some } \mu > 0.
$$

Let $(\mathcal{M}_{ta}, \mathcal{N}_{ta})$ be a double sequence of fuzzy numbers with

$$
d \left( (\mathcal{M}_{ta}, \mathcal{N}_{ta}), (0,0) \right) = \left[ \lambda_{\alpha} \left( (y_{ta}, \mathcal{M}_{ta}), \mathcal{N}_{ta}, (0,0) \right), \rho_{\alpha} \left( (y_{ta}, \mathcal{M}_{ta}), \mathcal{N}_{ta}, (0,0) \right) \right],
$$

for all $\alpha \in (0,1]$,

such that $\lambda(y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \leq \lambda(y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0))$ and $\rho(y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \leq \rho(y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0))$ and consequently

$$
\lambda \left( (y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \right) \leq \lambda \left( (y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \right)
$$

and

$$
\rho \left( (y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \right) \leq \rho \left( (y_{ta}, \mathcal{M}_{ta}, \mathcal{N}_{ta}, (0,0)) \right).
$$

Since $\mathcal{H}$ is a nondecreasing continuous function, for some $\mu > 0$ we get

$$
\left( \mathcal{H}_1 \left( \frac{\lambda(y_{ta}, \mathcal{M}_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(y_{ta}, \mathcal{N}_{ta})}{\mu} \right) \right) \leq \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(y_{ta}, Y_{ta})}{\mu} \right) \right)
$$

and

$$
\left( \mathcal{H}_1 \left( \frac{\rho(y_{ta}, \mathcal{M}_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(y_{ta}, \mathcal{N}_{ta})}{\mu} \right) \right) \leq \left( \mathcal{H}_1 \left( \frac{\rho(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(y_{ta}, Y_{ta})}{\mu} \right) \right).
$$

Which implies that

$$
sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ta}, \mathcal{M}_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(y_{ta}, \mathcal{N}_{ta})}{\mu} \right) \right) \leq sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(y_{ta}, Y_{ta})}{\mu} \right) \right) < \infty,
$$

for some $\mu > 0$

and

$$
sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\rho(y_{ta}, \mathcal{M}_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(y_{ta}, \mathcal{N}_{ta})}{\mu} \right) \right) \leq sup_{t \in \mathbb{N}} \left( \mathcal{H}_1 \left( \frac{\rho(y_{ta}, X_{ta})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(y_{ta}, Y_{ta})}{\mu} \right) \right) < \infty,
$$

for some $\mu > 0$.

This implies that
\[
\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(Y_{t,s}, M_{t,s})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\lambda(Y_{t,s}, N_{t,s})}{\mu} \right) \right) < \infty \quad \text{and} \quad \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(Y_{t,s}, M_{t,s})}{\mu} \right) \lor \mathcal{H}_2 \left( \frac{\rho(Y_{t,s}, N_{t,s})}{\mu} \right) \right) < \infty,
\]
for some \( \mu > 0 \),

which implies that \((M_{t,s}, N_{t,s}) \in \mathbb{l}_\infty \). Hence \((l_\infty) \) is solid.

The proof of the second part from the following example.

**Example 3.1:** Let \( \mathcal{H}(x, y) = (\mathcal{H}_1(x), \mathcal{H}_2(y)) = (x, y) \) and \( \gamma_{t,s} = 1 \) for all \( t, s \in \mathbb{N} \). Consider the double sequence \( (\chi_{t,s}, \gamma_{t,s}) \) of fuzzy numbers as follows:

\[
(\chi_{t,s}, \gamma_{t,s})(u) = \begin{cases} 
\left( \frac{1}{2} (1 + u), \frac{1}{2} (1 + u) \right), & \text{for } -1 \leq u \leq 1; \\
\left( \frac{1}{2} (-u + 3), \frac{1}{2} (-u + 3) \right), & \text{for } 1 \leq u \leq 3; \\
(0,0), & \text{otherwise},
\end{cases}
\]

where

\[
(\chi_{t,s})(u) = \begin{cases} 
\frac{1}{2} (1 + u), & \text{for } -1 \leq u \leq 1; \\
\frac{1}{2} (-u + 3), & \text{for } 1 \leq u \leq 3; \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
(\gamma_{t,s})(u) = \begin{cases} 
\frac{1}{2} (1 + u), & \text{for } -1 \leq u \leq 1; \\
\frac{1}{2} (-u + 3), & \text{for } 1 \leq u \leq 3; \\
0, & \text{otherwise}.
\end{cases}
\]

Then \((\chi_{t,s}, \gamma_{t,s}) \in (c) \mathbb{l}_\infty \).

Again consider a double sequence \((M_{t,s}, N_{t,s})\) of fuzzy numbers

\[
(M_{t,s}, N_{t,s})(u) = \begin{cases} 
\left( (1 - u), (1 - u) \right), & \text{for } 0 \leq u \leq 1, \text{ all } (t, s) \text{ even}; \\
\left( \frac{1}{2} (2 - u), \frac{1}{2} (2 - u) \right), & \text{for } 1 \leq u \leq 2, \text{ all } (t, s) \text{ odd}; \\
(0,0), & \text{otherwise},
\end{cases}
\]

where

\[
(M_{t,s})(u) = \begin{cases} 
(1 - u), & \text{for } 0 \leq u \leq 1, \text{ all } (t, s) \text{ even}; \\
\frac{1}{2} (2 - u), & \text{for } 1 \leq u \leq 2, \text{ all } (t, s) \text{ odd}; \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
(N_{t,s})(u) = \begin{cases} 
(1 - u), & \text{for } 0 \leq u \leq 1, \text{ all } (t, s) \text{ even}; \\
\frac{1}{2} (2 - u), & \text{for } 1 \leq u \leq 2, \text{ all } (t, s) \text{ odd}; \\
0, & \text{otherwise}.
\end{cases}
\]
If $\mathcal{I} \subset \mathcal{J}$, for some $\mu > 0$,

$$\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}, u_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}, u_{t,s})}{\mu} \right) \right) \leq \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}, u_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}, u_{t,s})}{\mu} \right) \right)$$

and

$$\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(y_{t,s}, u_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\rho(y_{t,s}, u_{t,s})}{\mu} \right) \right) \leq \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(y_{t,s}, u_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\rho(y_{t,s}, u_{t,s})}{\mu} \right) \right).$$

But $\left( \mathcal{M}_{t,s} \mathcal{N}_{t,s} \right)$ is not convergent.

Hence $\left( c \right)^2(\mathcal{H}, \Lambda)$ is not solid.

Note. By Remark 2.1, we can conclude that the double sequence spaces $\left( l_\infty \right)^2(\mathcal{H}, \Lambda)$ and $\left( c_0 \right)^2(\mathcal{H}, \Lambda)$ are monotone but the double sequence space $\left( c \right)^2(\mathcal{H}, \Lambda)$ is not monotone.

**Theorem 3.3:** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two double Orlicz functions. Then:

1) $\left( c \right)^2(\mathcal{H}_1, \Lambda) \cap \left( c \right)^2(\mathcal{H}_2, \Lambda) \subset \left( c \right)^2(\mathcal{H}_1 + \mathcal{H}_2, \Lambda)$.

2) $\left( c_0 \right)^2(\mathcal{H}_1, \Lambda) \cap \left( c_0 \right)^2(\mathcal{H}_2, \Lambda) \subset \left( c_0 \right)^2(\mathcal{H}_1 + \mathcal{H}_2, \Lambda)$.

3) $\left( l_\infty \right)^2(\mathcal{H}_1, \Lambda) \cap \left( l_\infty \right)^2(\mathcal{H}_2, \Lambda) \subset \left( l_\infty \right)^2(\mathcal{H}_1 + \mathcal{H}_2, \Lambda)$.

Taking $\mathcal{H}_2(x, y) = \mathcal{H}_5(x, \mathcal{H}_6(y)) = (x, y)$ and $\mathcal{H}_3(x, y) = \mathcal{H}_3(x, \mathcal{H}_4(y)) = \mathcal{H}(x, y)$, for all $(x, y) \in [0, \infty) \times [0, \infty)$, we have the following result.

**Corollary 3.1:** $Z \subset Z(\mathcal{H}, \Lambda)$ where $Z = (c)^2, (c_0)^2$ and $(l_\infty)^2$.

**Theorem 3.4:** If $\mathcal{H}_2(x, y) \leq \mathcal{H}_3(x, y)$ for all $(x, y) \in [0, \infty) \times [0, \infty)$, then $Z(\mathcal{H}_2, \Lambda) \subset Z(\mathcal{H}_1, \Lambda)$, for $Z = (c)^2, (c_0)^2$ and $(l_\infty)^2$ where $\mathcal{H}_1 = (\mathcal{H}_3, \mathcal{H}_4)$ and $\mathcal{H}_2 = (\mathcal{H}_5, \mathcal{H}_6)$.

**Theorem 3.5:** Let $\Lambda = (\lambda^t_s)$ be a double sequence of nonzero scalars and $\mathcal{H}$ be an Orlicz function. Then $(\lambda)_{t,s}^2(\mathcal{H}, \Lambda) \subset (\lambda^t_s)_{t,s}^2(\mathcal{H}, \Lambda)$ and the inclusions are proper.

**Proof:** the inclusion $(\lambda^t_s)_{t,s}^2(\mathcal{H}, \Lambda) \subset (\lambda)_{t,s}^2(\mathcal{H}, \Lambda)$ is obvious. So we will only show that $(\lambda)_{t,s}^2(\mathcal{H}, \Lambda) \subset (\lambda^t_s)_{t,s}^2(\mathcal{H}, \Lambda)$.

Define $\mu = 2\mu_1$. Since $\mathcal{H}$ is convex and nondecreasing, we have

$$\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}, x_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}, x_{t,s})}{\mu} \right) \right) \leq \frac{1}{2} \left( \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\lambda(y_{t,s}, x_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\lambda(y_{t,s}, x_{t,s})}{\mu} \right) \right) \right)$$

and

$$\sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(y_{t,s}, x_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\rho(y_{t,s}, x_{t,s})}{\mu} \right) \right) \leq \frac{1}{2} \left( \sup_{t,s} \left( \mathcal{H}_1 \left( \frac{\rho(y_{t,s}, x_{t,s})}{\mu} \right) \vee \mathcal{H}_2 \left( \frac{\rho(y_{t,s}, x_{t,s})}{\mu} \right) \right) \right)$$

for some $\mu > 0$.

Thus we get $\left( \lambda^t_s \right)_{t,s} \in (l_\infty)^2(\mathcal{H}, \Lambda)$. To show that the inclusion is proper, consider the following example.
Example 3.2: Let \( \mathcal{H}(x, y) = (\mathcal{H}_1(x), \mathcal{H}_2(y)) = (x, y) \) and \( y_{ts} = 1 \) for all \( t, s \in \mathbb{N} \). Regard as the a double sequence \( (\chi_{ts}, y_{ts}) \) of fuzzy numbers as follows:

\[
(\chi_{ts}, y_{ts})(u) = \begin{cases} 
\left(\left(\frac{t}{s} u + 1\right), \left(\frac{t}{s} u + 1\right)\right), & -\frac{ts}{2} \leq u \leq 0; \\
\left(\left(-\frac{2}{ts} u + 1\right), \left(-\frac{2}{ts} u + 1\right)\right), & 0 \leq u \leq \frac{ts}{2}; \\
(0,0), & \text{otherwise,}
\end{cases}
\]

if \( (t, s) = (2^i, 2^i), (i = 1, 2, 3 \ldots) \),

where

\[
(\chi_{ts})(u) = \begin{cases} 
\left(\frac{2}{ts} u + 1\right), & -\frac{ts}{2} \leq u \leq 0; \\
\left(-\frac{2}{ts} u + 1\right), & 0 \leq u \leq \frac{ts}{2}; \\
(0,0), & \text{otherwise,}
\end{cases}
\]

\[
(y_{ts})(u) = \begin{cases} 
\left(\frac{2}{ts} u + 1\right), & -\frac{ts}{2} \leq u \leq 0; \\
\left(-\frac{2}{ts} u + 1\right), & 0 \leq u \leq \frac{ts}{2}; \\
(0,0), & \text{otherwise,}
\end{cases}
\]

if \( (t, s) = (2^i, 2^i), (i = 1, 2, 3 \ldots) \).

Then the double sequence \( (\chi_{ts}, y_{ts}) \) of fuzzy numbers is bounded, but this double sequence is not convergent.

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