SUBQUANTUM MODELS:
BASIC PRINCIPLES, EFFECTS AND TESTS

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Abstract. We present models in which the indeterministic feature of Quantum Mechanics is represented in the form of definite physical mechanisms. Our way is completely different from so-called hidden parameter models, namely, we start from a certain variant of QM – deterministic QM – which has most features similar to QM, but the evolution in this theory is deterministic. Then we introduce the subquantum medium composed of so-called space-like objects. The interaction of a deterministic QM-particle with this medium is represented by the random force, but it is the random force governed by the probability amplitude distribution. This is the quantum random force and it is very different from classical random force. This implies that in our models there are no Bell’s inequalities and that our models (depending on a certain parameter \( \tau \)) can be arbitrarily close to QM. The parameter \( \tau \) defines a relaxation time and on time intervals shorter than \( \tau \), the evolution violates Heisenberg’s uncertainty principle and it is almost deterministic – spreading of the wave packet is much slower than in QM. Such type of short-time effects form the bases of proposed tests, which can, in principle, define limits of validity of QM. The proposed experiments are related to the behavior of quantum objects on short time intervals, where we expect the behavior different from QM. The main proposed feature is violation of uncertainty relations on short time intervals.

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Introduction

The main goal of this research is to study the indeterministic feature of the Quantum Mechanics (QM). We construct models describing possible mechanisms of this indeterminism. We call them subquantum models, since
(i) these models contain more details with respect to QM, but QM is the limit of these models,
(ii) these models are part of the general quantum theory (defined by Feynman’s rules on the probability amplitude); they do not contain any classical concept like the probability distribution.

Our subquantum models are strictly different from so-called hidden parameter models. The main difference is the following. If some model contains certain random element, say something like a random force, then we postulate the probability amplitude distribution of these random forces (and not the probability distribution postulated in the hidden parameter models). The reason is that we consider these random forces as a quantum phenomenon and not as a classical phenomenon. Our subquantum models are based on the general quantum principle – the probability amplitude.

The hidden parameter models are, from our point of view, inconsistent mixtures of quantum and classical concepts. They attempt to explain the quantum phenomenon of the indeterminism by using the classical concept of the probability distribution. This is impossible and Bell’s inequalities prove that this mixture of classical and quantum concepts is inconsistent. Our subquantum models contain no analogs of Bell’s inequalities.

A rational model of any indeterminism phenomena must contain two elements:
(i) the underlying ”deterministic” model,
(ii) the ”random” elements which model the indeterminism.

These two parts together must give a model close to QM. Both elements must belong to the general quantum theory.

The general quantum theory contains the following Feynman’s principles:

(a) to each possible trajectory of the system, there is associated a probability amplitude of the form $A = \exp(iS/\hbar)$, where $S$ is an action associated to this trajectory,
(b) for principally indistinguishable alternatives the probability amplitudes are summed,

(c) for principally distinguishable alternatives the probabilities $P = |A|^2$ are summed. The probability $P$ of an elementary (principally distinguishable) event is given by $P = |A|^2$.

The last Feynman’s postulate:

(d) all possible trajectories contribute with the same weight, does not make part of the general quantum theory. This postulate is a basis of the indeterminism of QM. We study mainly the general quantum models without this postulate.

The ”deterministic” QM model satisfies all postulates (a)–(c), but it does not satisfy (d). In this model the system moves along (contrarily to (d)) classical trajectories but the state is defined by the wave function (more generally, by the density matrix) and standard quantum rules (a)–(c) are applied. Only the evolution operator is different from QM, mainly the wave packets do not disperse.

The random element is introduced into the deterministic QM by postulating the existence of a certain medium which we call SLO-vacuum. This is a medium composed from space-like objects. Space-like objects are hypothetical new objects which are not directly observable.

The main feature of a space-like object is its non-localizability. The trajectory of a freely moving space-like object is the hyperplane in $\mathbb{R}^4$ given by the equation

$$t = t_0 + \vec{w}.\vec{x},$$

where $\vec{w} = (w_1, w_2, w_3)$ (called the space-like velocity) has the physical dimension second/meter. We shall assume that

$$|w| = (w_1^2 + w_2^2 + w_3^2)^{1/2} < c^{-1},$$

where $c$ is the velocity of light.

The trajectory is the 3-dimensional hyperplane

$$h_{\vec{w}, t_0} := \{ (t, \vec{x}) \in \mathbb{R} | \ t = t_0 + \vec{w}.\vec{x}, \ \vec{x} \in \mathbb{R}^3 \}$$
and the principal non-locality of the space-like object is clear. The most typical space-like object is the zero (space-like) velocity object with the trajectory

$$h_{0,t_0} := \{(t_0, \vec{x}) | \vec{x} \in \mathbb{R}^3\},$$

i.e. the slice of the space-time given by $t = t_0$, i.e. by fixing time.

This feature makes space-like objects completely different from standard (time-like or light-cone) objects and also from so-called tachyons.

The main consequence of non-localizability of a space-like object is the impossibility to observe any particular space-like object. Otherwise, this does not imply that the system of ("infinitely") many space-like objects cannot have observable consequences.

We suppose that there exists "vacuum" composed from space-like objects – the SLO-vacuum. We assume that the DetQM-particles interact with this SLO-vacuum. This is our basic general subquantum model. This model describes the physical bases of our subquantum models.

To make this model mathematically more simple, we assume that the interaction of DetQM-particles with space-like objects (from SLO-vacuum) can be modelled by the concept of the random force. We assume the simplest possible probability amplitude distribution for these random forces and this gives the basic SubQM$_{RF}$-model.

The basic subquantum model is parametrized by a certain constant which defines how close the given subquantum model is to QM. This basic constant of our subquantum model can be interpreted as a relaxation constant $\beta = 1/\tau$, where $\tau$ is a relaxation time. The meaning of $\tau$ is that on time intervals $\Delta t \gg \tau$ the subquantum behavior approaches the QM-behavior.

On the other hand, on short time intervals $\Delta t \ll \tau$, the subquantum effects can happen. In particular, so-called concentrated states can exist, which do not satisfy the uncertainty relations. We can prepare states for which

$$\Delta p \cdot \Delta x \ll \hbar.$$ 

These concentration subquantum effects can exist and make the observational differences between subquantum models and standard QM.
These subquantum effects can happen only under specific circumstances described below.

We show that in the long-time ($\Delta t \gg \tau$) limit the subquantum models reduce to QM, while in the short-time ($\Delta t \ll \tau$) limit the subquantum effects happen (the concentration effect, the correlation effect) under specific circumstances.

Our models represent the non-locality of QM by the concept of SLO-vacuum. This must be explained in more details. The first thing is to note that concepts of causality and locality are completely independent. It is possible to reformulate the electro-dynamics as a theory without electro-magnetic field but with a force acting-on-distance (this was done e.g. by Feynman). Such a theory is non-local but causal. Non-locality is resolved by introducing an electro-magnetic field.

Non-locality of QM (consider EPR-pairs, the teleportation etc.) is clearly of a space-like character (and not of light-cone character typical for electro-dynamics) and it is completely natural that the resolution of non-locality of QM is given by using space-like objects.

For the exact prediction of SubQM behavior it is necessary to know trajectories of all space-like objects. But any particular space-like object is not observable. Thus only the (probability amplitude) stochastic features of space-like objects can be assumed.

A certain fine point is that the resulting effects of SLO-vacuum are well-observed. These are mainly the indeterminism of QM or, equivalently, Feynman’s postulate (d), that any trajectory contributes to the total amplitude. At this level, our subquantum models are something like the quantum mechanical Brownian motion. In fact, this analogy is not correct: QM is the quantum analog of the Brownian motion, while subquantum models are analogs of Ornstein-Uhlenbeck stochastic process (see [6]).

One can ask clearly: why any trajectory contributes to the resulting probability amplitude? Our answer is clear: the particle is subjected to the random force (originated from the interaction with space-like objects) and this makes any trajectory possible.

Subquantum effects happen on short time intervals where relaxation phenomena are not yet realized. So the proposed tests concern the behavior of quantum particles during short time intervals.
A typical effect is the concentration effect where the uncertainty relation is explicitly broken, or the correlation effect which is especially interesting, showing directly the existence of the subquantum medium.

In Section 1, we introduce the concept of a space-like object. This concept was originally introduced during the study of the quaternionic quantum theory of tachyons ([1], [3], [5], [6]). One consequence of this theory is the classical approximation to this quaternionic quantum tachyon and this is exactly our space-like object. Here we introduce also the simplest model for SLO-vacuum.

In Section 2, we introduce the concept of the deterministic QM. This means to change Feynman’s assumption (d) to its opposite, that only classical trajectories contribute to the total transition amplitude. A rudimentary form of this idea was presented in [4] and also in [2] (and it is implicitly assumed in [6]). The existence of some type of the deterministic QM is clearly necessary if we want to introduce an explicit random mechanism in SubQM. The random element (like SLO-vacuum) must be working inside certain deterministic situation. Of course, other models are also possible, but we think, our DetQM has a certain mathematically appealing form.

In Section 3, the proper subquantum models are introduced. The general model

$$\text{SubQM} := \text{DetQM} + \text{SLO-vacuum}$$

forms the physical basis. The basic random model $\text{SubQM}_{RF}$ is the model where the interaction of the DetQM-particle with SLO-vacuum is represented by quantum random force. We call this random force as quantum force since it is governed by the probability amplitude distribution. This is the main model studied in this paper and it was introduced in [6].

In Section 4, we study the unitarity of the evolution of the state in $\text{SubQM}_{RF}$, especially the simplest form of the relaxation phenomena. We follow ideas from [6].

In Section 5, we study a possible interpretation of SubQM models from the point of view of QM-approximation. We observe that the simplest probability interpretation is inconsistent with the right QM-limit. The second proposed interpretation seems to be quite
reasonable. It means that states with different (particle) velocities are principally indistinguishable alternatives in the sense of Feynman’s approach to QM. The content of this section is new, the interpretation approved here was implicitly assumed in [6].

In Section 6, we study the long-time (with respect to the relaxation time $\tau$) approximation to SubQM$_{RF}$ and we show that, in a certain sense, the standard QM is obtained. The passage

$$\text{SubQM}_{RF} \rightarrow \text{QM}$$

corresponds to the passage

$$\text{Ornstein-Uhlenbeck} \rightarrow \text{Brownian motion;}$$

this was especially made clear in [6]. Here we do not repeat this argument and an interested reader can find details in [6]. The ideas of this long-time approximation were already presented in [6].

In Section 7, we study the short-time approximation. We show that on short time intervals (relatively to the relaxation time $\tau$), the SubQM-behavior is completely different from QM-behavior. We study quantitative details also by calculating spreading of Gaussian wave packets after passing through repeated slits.

We calculate quantitatively the breaking out of uncertainty relations and we define and study the corresponding effect called the concentration effect. This concentration effect was implicitly (but not quantitatively) mentioned in [6] before studying the more involved "coherence effect".

Here we introduce the important concepts: the particle’s momentum $\vec{p}$ and the QM-momentum $p^{(QM)}$. In the relaxed state the mean value of $\vec{p}$ is infinite (like the mean actual velocity of a Brownian particle) and only the QM-momentum $p^{(QM)}$, defined by the Fourier transform, can be used.

On the other hand, in the non-relaxed state (typically in the concentrated state) the mean value of $p$ is finite (like the mean actual velocity of an Ornstein-Uhlenbeck particle can be finite before approaching the thermal equilibrium). We have only the particle’s momentum $p$, while the QM-momentum is defined only for the relaxed states. The difference between $p$ and $p^{(QM)}$ lies in the heart of the subquantum models.
In Section 8, we introduce the correlated random force model \( \text{SubQM}_{CRF} \), where the random forces acting on different particles are inter-correlated. This is a refined form of \( \text{SubQM}_{RF} \) and it was introduced in [6]. The correlation effect was proposed in [6] under the name "subquantum coherence effect". The ideas can be found in [6] but without any calculations.

In Section 9, we propose possible experiments based on the concentration effect or on the correlation effect. All proposed experiments need to study the system during short time intervals. Proposed physical values characterizing the experiments are, in certain cases, calculated. The main correlation experiment was already proposed in [6], but without any explicit calculation.

In Conclusions we discuss some general consequences of subquantum models and we also present our point of view on the interpretation questions of QM (which were already briefly presented in [6], where an interested reader can find some details not repeated here).

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1. **Space-like objects and SLO-vacuum**

Here we show that assuming the Einsteinian locality but without so-called "causality" assumption leads to new phenomena.

Dynamical state of an elementary object can be characterized by the energy-momentum vector

\[(E, \vec{p})\].

This vector lies in the Minkowski space and it can belong to one of three possible types (assuming \(c = 1\)):

(i) \((E, \vec{p})\) is a time-like vector, \(E^2 - \vec{p}^2 > 0\) – the object is a time-like object, i.e. a standard massive particle;

(ii) \((E, \vec{p})\) lies on the light-cone, \(E^2 - \vec{p}^2 = 0\) – the object is a light-cone object, i.e. a standard mass-less particle;

(iii) \((E, \vec{p})\) is a space-like vector, \(E^2 - \vec{p}^2 < 0\) – the object is a space-like object (SLO) and this is a new type of objects.

Space-like objects are completely different from so-called tachyons, as will be clearly seen below. Space-like object is a new concept of an "object" and it was proposed in [1], [2].

The trajectory of the freely moving time-like object is given by

\[\vec{x} = \vec{x}(t), \ t \in \mathbb{R},\]

with the constraint \(|\dot{\vec{x}}(t)|^2 < c^2, \forall t \in \mathbb{R}\).

This constraint is a consequence of the tacit assumption of the conservation of the type – time-like, light-cone, space-like. The velocity of a time-like object is given by

\[\vec{v}(t) = \dot{\vec{x}}(t)\]

and the constraint is \(|\vec{v}|^2 < c^2\).

The acceleration is given by

\[\vec{a}(t) = \ddot{\vec{x}}(t) = \vec{x}(t), \ t \in \mathbb{R},\]

and Newton’s law says

\[m.\vec{a}(t) = \vec{F}(\vec{x}(t), t).\]
Now for the space-like object we shall assume that its trajectory is a hypersurface
\[ t = f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3. \]

This form is natural for the space-like object since it is lying locally at each point outside the light-cone. We shall define the (space-like) velocity of a space-like object as the local approximation by the gradient
\[ \vec{w}(\vec{x}) = \delta_x f(\vec{x}) = \nabla_{\vec{x}} f(\vec{x}), \quad \vec{x} \in \mathbb{R}^3. \]

Clearly, the linear approximation has the form
\[ t = \vec{w} \cdot \vec{x} + t_0. \]

This is the trajectory of a freely moving space-like object.

The condition to be outside the light-cone requires that
\[ |\vec{w}(\vec{x})| < 1/c, \quad \vec{x} \in \mathbb{R}^3 \]
(remember that this \( \vec{w} \) is the space-like velocity with the physical dimension \((\text{second})(\text{meter})^{-1})\).

Then the acceleration matrix is given by
\[ \hat{b}(\vec{x}) = \delta_x \vec{w}(\vec{x}) = \delta_x \delta_x f(\vec{x}) = \left( f_{x_\alpha x_\beta} \right)_{\alpha, \beta = 1}^3 \in M^{3 \times 3}_{\text{sym}}. \]

It is a symmetric \( 3 \times 3 \) matrix. The analogy of the Newton’s law for a space-like object is the following
\[ m\hat{b}(\vec{x}) = m\delta_x \delta_x f(\vec{x}) = \hat{F}(t(\vec{x}), \vec{x}), \quad \vec{x} \in \mathbb{R}^3. \]

Here, clearly, \( \vec{x} \) is a ”time variable” and it is a 3-dimensional quantity.

This type of a theory with the multi-dimensional time was considered by Carathéodory [7]. It follows that the force \( \hat{F}(t(\vec{x}), \vec{x}) \in M^{3 \times 3}_{\text{sym}} \) must satisfy the integrability condition presented in the book of Carathéodory.

The main consequence of these assumptions is non-locality of a space-like object. Typically, the trajectory of a freely moving space-like object is
\[ t = \vec{w} \cdot \vec{x} + t^0. \]
and for a fixed $t = t_0$ we obtain the 2-dimensional plane (assuming $\vec{w} \neq 0$)

$$\{\vec{x} \in \mathbb{R}^3 | \vec{w}.\vec{x} = t_0 - t^0\} = \vec{w} \cdot \frac{t_0 - t^0}{|\vec{w}|^2} + \vec{w}^\perp,$$

where $\vec{w}^\perp = \{\vec{u} | \vec{u}.\vec{w} = 0\}$.

Such an object cannot be localized in a given laboratory. In fact, if $L_{lab}$ is the linear estimation of the laboratory and $L_{cosm}$ is the linear estimation of the cosmos, we obtain that only

$$(L_{lab}/L_{cosm})^2 \text{ - part of the space-like object}$$

is inside the laboratory.

From the fact that $L_{lab}/L_{cosm} \to 0$ we obtain that every particular space-like object is non-localizable and then non-observable.

Such a conclusion allows us to make the following hypotheses:

(i) there exists large number of space-like objects,
(ii) the collective effect of all these space-like objects is observable.

Of course, this means that a particular space-like object cannot be observed, but the collection of a very large number (of order of $L_{cosm}/L_{lab}$) of them may be observable.

This allows us to assume the existence of many space-like objects. In other words: non-observability of a single space-like object does not imply that the system of many space-like objects cannot cause certain observable effects.

In fact, if there exist $L_{cosm}/L_{lab}$ space-like objects for each second on the time axis, then there may exist observable collective effects of these space-like objects.

Later we shall show that the possible main collective effect of SLO-vacuum is the non-deterministic behavior of particles in Quantum Mechanics (QM).

We shall assume the following hypotheses. There exists freely moving space-like object $O_\alpha$ for each $\alpha \in \mathbb{Z}$ ($\mathbb{Z}$ = integers) such that each object $O_\alpha$ has a trajectory

$$t = t^0_\alpha + \vec{w}^\alpha_\alpha.\vec{x} = f_\alpha(\vec{x}), \; \alpha \in \mathbb{Z}.$$ 

We assume that parameters $t^0_\alpha$ and $\vec{w}^\alpha_\alpha$ are randomly distributed in the following way.
Let $\tau_1 > \tau_0 > 0$ be the two fixed times and let for each $\alpha \in \mathbb{Z}$, $\xi_\alpha$ be a random number in the interval $(0, 1)$ and $\eta_\alpha$ be a random vector in the unit ball in $\mathbb{R}^3$, $|\eta_\alpha| < 1$. We shall assume that $\xi_\alpha$ and $\eta_\alpha$, $\alpha \in \mathbb{Z}$, are independent random variables with the uniform distribution in $(0, 1)$ and $\mathbb{B}^3$ (the open unit ball in $\mathbb{R}^3$), respectively.

Then we shall assume that for $\alpha \in \mathbb{Z}$

$$t^0_\alpha = \alpha.\tau_0 + \xi_\alpha.\tau_1,$$

$$\vec{w}_\alpha = \eta_\alpha.c^{-1}$$

where $c$ is the velocity of light (numerically $c = 1$).

As a consequence we obtain that for a fixed $\vec{x}^0 \in \mathbb{R}^3$, $t^0 \in \mathbb{R}$ we have the density of space-like objects at a given $\vec{x}^0$ equal to

$$\lim_{T \to \infty} \frac{1}{T} \# \{ \alpha \mid f_\alpha(\vec{x}^0) \in (t^0, t^0 + T) \} = \frac{1}{\tau_0}$$

and this density does not depend on $t_0$.

Note that we could assume the general form of $O_\alpha$:

$$t = f_\alpha(\vec{x}),$$

but we can simply assume the freely moving space-like objects, because the collective effect depends only very weakly on the detailed form of the trajectories of space-like objects.

In the most simple form we can suppose that $w^\alpha = 0, \forall \alpha \in \mathbb{Z}$. (The precise distribution of velocities should be invariant with respect to the Lorentz group.) Nevertheless, this simplest form is often sufficient.

The Newton’s equations for the system of $n$ space-like objects with trajectories $f^i(\vec{x}), i = 1, \ldots, n$ are given by

$$m_i \delta_x \delta_x f^i(\vec{x}) = \hat{F}^i(f^1(\vec{x}), \ldots, f^n(\vec{x}), \vec{x})$$

or in the expanded form

$$m_i \delta_{x_\alpha} \delta_{x_\beta} f(\vec{x}) = F^i_{\alpha\beta}(f^1(\vec{x}), \ldots, f^n(\vec{x}), \vec{x})$$

$$\alpha, \beta = 1, 2, 3, \ i = 1, \ldots, n$$
where
\[
\hat{F}^i(t^1, \ldots, t^n, \vec{x}) = \left( F_{\alpha\beta}^i(t^1, \ldots, t^n, \vec{x}) \right)_{\alpha,\beta} \in M^{3 \times 3}_{sym}.
\]

It is clear that this system of equations is over-determined and that there should exist well-defined integrability conditions. The method to obtain these integrability conditions on \( \hat{F}^i \) can be found in Carathéodory’s book [7].
2. Deterministic Quantum Mechanics

We shall consider in details the two Feynman principles mentioned already in the Introduction:
(PA) Probability Amplitude,
(ID) Indeterminism.

The Probability Amplitude Principle says that for each observable event there exists a complex number $A_{\text{event}}$ such that the probability of this event (being observed) is given by

$$P_{\text{event}} = |A_{\text{event}}|^2.$$ 

Let $(q_1, \ldots, q_n)$ be generalized Lagrangian coordinates on the configuration manifold $M$.

For $t^1 < t^2$, $t^1, t^2 \in \mathbb{R}$, and $q^1, q^2 \in M$, we shall suppose that we have given the transition amplitude

$$A_{t^1, t^2}(q^1; q^2) \in \mathbb{C}$$

for the transition of the system from the initial state $q^1$ at the time $t^1$ to the state $q^2$ at the final time $t^2$. The probability of this transition is given by

$$P_{t^1, t^2}(q^1; q^2) = |A_{t^1, t^2}(q^1; q^2)|^2.$$

Then we shall use the principle of the additivity of probability amplitudes of the transition along different possible paths

$$A_{t^1, t^2}(q^1; q^2) = \sum A_{t^1, t^2}[q]$$

(BC)

$q(t^1) = q^1, \quad q(t^2) = q^2$,

where (BC) are boundary conditions for the allowed trajectories and

$$A_{t^1, t^2}[q]$$

is the probability amplitude for a given trajectory $q(t)$, $t \in [t^1, t^2]$. This probability amplitude is given by the formula using the classical action

$$A_{t^1, t^2}[q] = \exp \left\{ \frac{i}{\hbar} S_{t^1, t^2}[q] \right\},$$

$$S_{t^1, t^2}[q] = \int_{t^1}^{t^2} L(q, \dot{q}) dt.$$
Here $L(q, \dot{q})$ is the Lagrange function of the system.

We shall reformulate all this on the phase space $\mathbf{P}$ with the canonical coordinates

$$(q_1, \ldots, q_n, p_1, \ldots, p_n).$$

Here

$$p_i = \partial_{\dot{q}_i} L(q, \dot{q}), \ i = 1, \ldots, n,$$

and we express (assuming that this can be done) $\dot{q}$ as a function

$$\dot{q}_i = \dot{q}_i(q, p).$$

The Hamilton function is then given by

$$H(q, p) = \sum \dot{q}_i(q, p)p_i - L(q, \dot{q}(q, p)).$$

For the trajectory

$$[q, p] = [q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t)]_{t_1}^{t_2}$$

in the phase space, the action can be expressed as

$$\bar{S}_{t_1, t_2}[q, p] = \int_{t_1}^{t_2} (\dot{q}(q, p).p - H(q, p))dt,$$

where we use the bar $\bar{S}$ for the action expressed on the phase space.

Then we have

$$\bar{A}_{t_1, t_2}[q, p] = \exp \left\{ \frac{i}{\hbar} \bar{S}_{t_1, t_2}[q, p] \right\},$$

$$\bar{A}_{t_1, t_2}(q^1; q^2) = \sum_{q(t_1) = q^1, q(t_2) = q^2} \bar{A}_{t_1, t_2}[q, p]$$

and

$$P_{t_1, t_2}(q^1; q^2) = |\bar{A}_{t_1, t_2}(q^1; q^2)|^2.$$
It is a well-known fact that this phase space approach gives the same resulting transition amplitude as for the configuration approach mentioned before, i.e.

$$\tilde{\mathcal{A}}_{t^1,t^2}(q^1; q^2) = \mathcal{A}_{t^1,t^2}(q^1; q^2).$$

In fact, the standard way to obtain the Feynman formulation of QM starts from the Schrödinger equation, its propagator is then expressed using the Trotter’s formula as a phase space Feynman integral. It is then transformed by integrating out all momentum variables to the configuration space Feynman integral.

Our starting point will be the Feynman formulation on the phase space. On the phase space we can set (or, at least, we can try to set) finer boundary conditions fixing also the momentum variables, i.e.

$$\tilde{\mathcal{A}}_{t^1,t^2}(q^1, p^1; q^2, p^2) = \sum_{(BC)} \tilde{\mathcal{A}}_{t^1,t^2}[q, p]$$

where boundary conditions are

(BC) \quad q(t^1) = q^1, \quad p(t^1) = p^1, \quad q(t^2) = q^2, \quad p(t^2) = p^2.

This refinement is the main point in what follows. This is fundamental in considering wave functions.

The standard wave function at the time \( t = t^1 \) is a complex function \( \psi(q; t^1) \) on the configuration space \( \mathbf{M} \) normalized by the condition

$$\int_{\mathbf{M}} |\psi(q; t^1)|^2 dq = 1.$$ 

We can think of \( \psi \) as a probability amplitude distribution on the configuration space \( \mathbf{M} \). Using the transition amplitude \( \mathcal{A} \) we obtain the evolution of a PA-distribution \( \psi \):

$$\psi(q^2; t^2) = \int_{\mathbf{M}} \mathcal{A}_{t^1,t^2}(q^1; q^2)\psi(q^1; t^1)dq^1.$$
On the phase space we can consider similarly the PA-distribution (or phase space wave function)

$$\psi(q, p; t)$$

with the evolution

$$\psi(q^2, p^2; t^2) = \int \tilde{A}_{t^1, t^2}(q^1, p^1; q^2, p^2) \psi(q^1, p^1; t^1) \frac{dq^1 dp^1}{\hbar}.$$  

Of course, there are degenerate cases, where this type of the PA-distribution (wave function) depending on the position and momentum variables is meaningless. This will be considered in more details below.

In fact, QM is a degenerate case in this sense, while SubQM models introduced here are not degenerate.

The second Feynman principle – we call it the Indeterminism – the ID-principle – specifies which trajectories contribute to the transition amplitude and what is the weight with which a given trajectory contributes.

In fact, this specification was left in a complete dark above expressing the total amplitude as a sum over trajectories. But one must specify which trajectories are allowed and what are their weights.

There are many possible specifications and the Feynman ID-principle defines the extreme case by setting:

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each trajectory contributes with the equal weight.
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The exact meaning of the phrase ”with equal weight” is the content of many studies, but here we do not need to consider these details.

The second Feynman principle is expressed as a Feynman integral

$$\tilde{A}_{t^1, t^2}(q^1; q^2) = \int_{q(t^1) = q^1, i = 1, 2} \tilde{A}_{t^1, t^2}[q, p] \prod_{t = t^1, i = 1}^{t^2, n} \frac{dq_i(t) dp_i(t)}{\hbar}.$$  

These two principles together give the same transition amplitudes as in the cannonical QM.
The uncountable product $\prod_{t \in [t^1, t^2]}$ expresses the so-called Feynman measure, which exists only formally. We shall use Feynman integrals in a formal sense.

Of course, on the phase space we can also write

$$\tilde{A}_{t^1, t^2}(q^1, p^1; q^2, p^2) = \int_{q(t^i) = q^i, \ p(t^i) = p^i, \ i = 1, 2} \tilde{A}_{t^1, t^2}[q, p] \prod_{t, i} \frac{d{q}_i(t) \ d{p}_i(t)}{\hbar},$$

where the short notation of $\prod$ over $t, i$ means, as above, that $t \in [t^1, t^2], \ i = 1, \ldots, n.$

If for the amplitude $\tilde{A}_{t^1, t^2}[q, p]$ the standard QM holds, then the resulting transition amplitude $\tilde{A}_{t^1, t^2}(q^1, p^1; q^2, p^2)$ does not effectively depend on $p^1$ and $p^2$. We then obtain the independence of the PA-distribution $\psi(q, p; t)$ on $p$ and thus we have to return to the PA-distribution $\psi(q; t)$, which does not depend on the momentum variable $p$.

As a consequence of the ID-principle we shall obtain the standard Feynman integral for the transition amplitude

$$A_{t^1, t^2}(q^1, q^2) = \int_{q(t^i) = q^i, \ i = 1, 2} \tilde{A}_{t^1, t^2}[q, p] \prod_{t, i} \frac{d{q}_i(t) \ d{p}_i(t)}{\hbar}.$$

This ID-principle implies then the Heisenberg commutation relations, the uncertainty principle and so on.

The indeterminism of QM follows from the fact that each possible trajectory contributes (with the equal weight) to the resulting transition amplitude. This implies, for example, spreading out of wave packets.

In fact, both Feynman principles are very mysterious and both are confirmed by an infinite amount of observational data.

The first PA-principle says that instead of summing up probabilities, one has to sum up probability amplitudes and it forms the basis of all interference phenomena in the quantum physics. This principle cannot be explained, it can be only considered as an axiom.

We shall assume this PA-principle as a fundamental law and we shall not try to modify it in any way.
The second Feynman principle, the indeterminacy, is less fundamental and we shall look for modifications of it. Subquantum models considered in this paper start with modifying the second Feynman principle.

Our first subquantum model, the Deterministic Quantum mechanics, denoted as DetQM, starts with assuming the PA-principle unchanged, but the second ID-principle to the opposite extreme case. Instead of assuming that all trajectories contribute to the transition amplitude, we postulate that only a classical trajectory (maybe, a finite number of classical trajectories) contributes to the transition amplitude.

So, instead of ID-principle we shall postulate the Det-principle.

For the transition on the phase space:

\[(q^1, p^1) \text{ at } t^1 \rightarrow (q^2, p^2) \text{ at } t^2,\]

only the classical trajectory contributes.

I.e., only such trajectory that is a solution of the Hamilton equations.

Let

\[q_i(t) = q_i(q^1, p^1, t^1; t),\]
\[p_i(t) = p_i(q^1, p^1, t^1; t)\]

be the classical evolution of the system in the phase space starting from the point \((q^1, p^1)\) at the time \(t^1\).

On the phase space the phase volume is conserved

\[\prod_i dq_i(t)dp_i(t) = \prod_i dq^1_i dp^1_i, \forall t \in [t^1, t^2].\]

We have inverse maps for \(t > t^1\)

\[q^1_i = \hat{q}_i(q(t), p(t), t; t^1),\]
\[p^1_i = \hat{p}_i(q(t), p(t), t; t^1)\]

which give the inverse map to the evolution map

\[(q^1, p^1) \mapsto (q(t), p(t)), \ t \geq t^1.\]
Following the PA-principle we obtain the amplitude for the classical trajectory

\[
\tilde{A}_{t_1, t_2}(q^1, p^1; q^2, p^2) = \exp \left\{ \frac{i}{\hbar} \tilde{S}_{t_1, t_2}(q^1, p^1; q^2, p^2) \right\} \cdot \hbar \prod_i \delta(q_i(q^1, p^1, t^1; t^2) - q_i^2) \delta(p_i(q^1, p^1, t^1; t^2) - p_i^2).
\]

Here

\[
\tilde{S}_{t_1, t_2}(q^1, p^1; q^2, p^2) = \int_{t_1}^{t_2} \left( \sum_{i=1}^n p_H p_{p_i} - H \right) (q, p) dt
\]

is the action along the classical trajectory.

At this moment it is completely natural to consider the amplitude distribution

\[
\psi(q, p; t)
\]

which will be called the wave function.

Then the evolution of the wave function is given by

\[
\psi(q^2, p^2; t^2) = \int \tilde{A}_{t_1, t_2}(q^1, p^1; q^2, p^2) \psi(q^1, p^1; t^1) \frac{dq^1 dp^1}{\hbar}.
\]

It is now very important that the phase volume \(dq dp\) is conserved during the evolution.

This gives

\[
\psi(q^2, p^2; t^2) = \exp \left\{ \frac{i}{\hbar} \tilde{S}_{t_1, t_2}(q^1, p^1; q^2, p^2) \right\} \psi(q^1, p^1; t^1)
\]

where \(q^1\) and \(p^1\) are defined by

\[
q^1 = \hat{q}(q^2, p^2, t^2; t^1), \quad p^1 = \hat{p}(q^2, p^2, t^2; t^1).
\]

Here \(q_i(t), p_i(t)\) satisfy the Hamilton equations

\[
\dot{q}_i(t) - H_{p_i}(q(t), p(t); t) = 0, \\
\dot{p}_i(t) + H_{q_i}(q(t), p(t); t) = 0.
\]
Expanding the evolution into the Feynman integral we obtain

$$\psi(q^2, p^2; t^2) = \int \mathcal{A}_{t^1, t^2}[q, p]. \prod_{t, i} \delta(\dot{q}_i(t) - H_{p_i}(t))\delta(\dot{p}_i(t) + H_{q_i}(t)).$$

$$\hbar^2 \prod_i \delta(q_i(t^1) - q_i^1)\delta(p_i(t^1) - p_i^1)\delta(q_i(t^2) - q_i^2)\delta(p_i(t^2) - p_i^2).$$

$$\cdot \psi(q^1, p^1; t^1) \prod_{t, i} \frac{dq_i(t)dp_i(t)}{\hbar} \cdot \prod_i \frac{dq_i^1dp_i^1}{\hbar}.$$ 

It is clear that the wave packets in DetQM do not spread out. If the wave function is supported in the certain phase volume, then the evolved wave function is supported in the region of the same volume.

We shall find the evolution equation by assuming the infinitesimal step $\varepsilon \rightarrow 0$ in the time variable. Let $t^2 = t^1 + \varepsilon$. Then

$$q_i^2 = q_i^1 + \varepsilon q_i = q_i^1 + \varepsilon H_{p_i},$$

$$p_i^2 = p_i^1 + \varepsilon p_i = p_i^1 - \varepsilon H_{q_i},$$

where all terms $o(\varepsilon)$ were neglected.

We also have

$$S_{t^1, t^2} = \int_{t^1}^{t^2} (pH_p - H)dt = \varepsilon(pH_p - H).$$

For the infinitesimal evolution of the wave function we have

$$\psi(q^2, p^2; t^2) = \psi(q^1, p^1; t^1) \left(1 + \frac{i\varepsilon}{\hbar}(pH_p - H)\right).$$

On the other hand we have

$$\psi(q^2, p^2; t^2) = \psi(q^1, p^1; t^1) + \sum_i \psi_{q_i}\varepsilon H_{p_i} - \psi_{p_i}\varepsilon H_{q_i} + \varepsilon \psi_t|_{q^1, p^1, t^1}.$$ 

Then we obtain

$$\psi + \psi_{q_i}\varepsilon H_{p_i} - \psi_{p_i}\varepsilon H_{q_i} + \varepsilon \psi_t = \psi + \frac{i\varepsilon}{\hbar}(pH_p - H)\psi.$$
The first order terms give

\[ i \psi_t = -i \sum H_{p_i} \partial_{q_i} + i \sum H_{q_i} \partial_{p_i} - \frac{1}{\hbar} \sum p_i H_{p_i} + \frac{1}{\hbar} H \psi. \]

In the case where

\[ H = \sum \frac{1}{2m_i} p_i^2 + V(x), \quad (x_i = q_i) \]

we have

\[ H_{p_i} = \frac{1}{m_i} p_i, \quad H_{q_i} = V_{x_i}. \]

Then we obtain

\[ i \psi_t = \left( -\frac{1}{2} \sum \frac{1}{\hbar m_i} p_i^2 - i \sum \frac{1}{m_i} p_i \partial_{x_i} + i \sum V_{x_i} \partial_{p_i} + \frac{1}{\hbar} V \right) \psi. \]

This is the Schroedinger equation in DetQM. This is a genuinely quantum equation, because there are terms depending on \( \hbar \). There is a principal difference between our equation and the 't Hooft’s equation of his deterministic QM [10]. Our equation gives non-dissipative quantum evolution.

The general Schroedinger equation (2.1) can be written on a general differential manifold. The analogical equation in QM on the general manifold cannot be written without an exact prescription of the order of operators. On the other hand, having given the Hamilton function \( H(q,p) \), we can directly write the “Schroedinger-like” equation (2.1).
3. The subquantum models SubQM and SubQM\textsubscript{RF}

Now we shall apply the deterministic QM in the situation of the SLO-vacuum, i.e. the medium composed of many space-like objects. We shall assume that there is a "sea" of space-like objects as it was already assumed above at the end of Section 1.

We shall suppose that for each particle (time-like object) and each space-like object there is a certain interaction. If $\psi(\vec{x}; t)$ is the state of a time-like object and $t = f_\alpha(\vec{x})$ is the trajectory of a space-like object, then the condition of the non-zero interaction between these two is the following

$$Spt\psi \cap \{(t, \vec{x}) | t = f_\alpha(\vec{x})\} \neq \emptyset.$$ 

This implies that a given particle (described possibly by DetQM) interacts with each space-like object.

At this moment, we are not able to specify the concrete form of the interaction term. In fact, it is not necessary in this paper.

We shall call the system of DetQM for particles + SLO-vacuum + interaction between them as SubQM – the general subquantum mechanical model.

We shall represent the total effect of the interaction particle – space-like object – as a random force acting on a particle. We shall denote this model SubQM\textsubscript{RF}. The random force depends on the point $(\vec{x}, t)$ and at this moment we suppose the random forces at two different space-time points are independent (components of the random force are also independent).

More concretely, we shall assume that the random force $F_i(t)$ acting on the $i$-th particle (in fact, on the $i$-th degree of freedom) at the time $t$ can be expressed as

$$F_i(t) = \phi(F(t, x_i(t)), i, t)$$

and that forces $F_i(t)$ and $F_j(t)$, $i \neq j$, are statistically independent, because mostly $x_i(t) \neq x_j(t)$.

So that the general (ideal) model is

$$\text{SubQM} \approx \text{DetQM} + \text{SLO-vacuum}$$
and the model with the random force representation of SLO-vacuum is

\[ \text{SubQM}_{RF} \approx \text{DetQM} + (\text{SLO-vacuum})_{RF}. \]

The last model consists of the following hypotheses.

(i) There exists a random force \( F_i(t) \) acting on the \( i \)-th degree of freedom, \( i = 1, \ldots, n \);
(ii) forces \( F_i(t) \) and \( F_j(t) \), \( i \neq j \), are statistically independent;
(iii) there is an amplitude distribution of the random force \( F_i \) given by

\[ A_{t_1, t_2}[F_i] = \exp \left\{ \frac{i}{\hbar} \frac{a^2}{2} \int_{t_1}^{t_2} F_i^2(t) \, dt \right\} \prod_{t \geq t_1} F_i(t) \]

for each degree of freedom \( i = 1, \ldots, n \);
(iv) the system with \( n \) degrees of freedom is described by DetQM – deterministic QM – with a given random force.

Assumption (iii) – the amplitude distribution of a random force – is chosen as a simplest possibility.

Note that we assume the probability amplitude distribution \( A_{t_1, t_2}[F_i] \) and not the classical probability distribution. This is a consequence of the first Feynman principle (PA), which we consider as a general basis of quantum (and subquantum) models.

From the independence assumption (ii) we obtain the joint amplitude distribution for forces \( F_1, \ldots, F_n \)

\[ A_{t_1, t_2}[F] = \exp \left\{ \frac{i}{\hbar} \frac{a^2}{2} \sum_{i=1}^{n} \int_{t_1}^{t_2} F_i^2(t) \, dt \right\} \prod_{t,i} dF_i(t), \]

where \( \prod_{t,i} dF_i(t) \) is "Feynman measure"

\[ \prod_{t,i} dF_i(t) = \prod_{i=1}^{n} \prod_{t \in [t_1, t_2]} dF_i(t). \]

The system with \( n \) degrees of freedom described by DetQM has the transition amplitude given by formulas of the preceding section.
but with a change corresponding to the existence of a random force. A half part of Hamilton equations is changed to

$$\dot{p}_i(t) = -H_{q_i}(q(t), p(t)) + F_i(t).$$

Then the transition amplitude is given by

$$K_{t^1,t^2}(q^1, p^1; q^2, p^2) = \int A_{t^1,t^2}[q,p] \prod_{t,i} \delta(q_i - H_{p_i}) \delta(p_i + H_{q_i} - F_i).$$

(3.1)

$$= \prod_{t,i} \delta(q_i(t^1) - q_i^1) \delta(p_i(t^1) - p_i^1) \delta(q_i(t^2) - q_i^2) \delta(p_i(t^2) - p_i^2).$$

$$\cdot \exp \left\{ i \frac{a}{2} \int_{t^1}^{t^2} \sum_i F_i^2(t^i) \, dt \right\} \prod_{t,i} dq_i(t) \prod_{t,i} dp_i(t) \prod_{t,i} dF_i(t).$$

For the special case when

$$H(x, p) = \sum_i \frac{1}{2m_i} p_i^2 + V(x)$$

we obtain after having done the integration on \( \prod dp_i(t) \)

$$K = \int A_{t^1,t^2}[x, \dot{x}] \prod_{t,i} \delta(m_i \dot{x}_i(t) + V_{x_i}(x(t)) - F_i(t)).$$

(3.2)

$$\cdot (BC\delta x p) \cdot \exp \left\{ i \frac{a}{2} \int \sum_i F_i^2(t^i) \, dt \right\} \prod_{t,i} dx_i(t) \prod_{t,i} dF_i(t),$$

where \((BC\delta x p)\) denotes the boundary conditions term

$$(BC\delta x p) := \prod_i \delta(x_i(t^1) - x_i^1) \delta(m_i \dot{x}_i(t^1) - p_i^1) \delta(x_i(t^2) - x_i^2) \delta(m_i \dot{x}_i(t^2) - p_i^2).$$

Similarly, we denote

$$(BC\delta q p) := \prod_i \delta(q_i(t^1) - q_i^1) \delta(p_i(t^1) - p_i^1) \delta(q_i(t^2) - q_i^2) \delta(p_i(t^2) - p_i^2).$$
and, as a definition of the domain

$$(BCxp) := \{ x_i(t^1) = x^1_i, m_i \dot{x}_i(t^1) = p^1_i, x_i(t^2) = x^2_i, m_i \dot{x}_i(t^2) = p^2_i, \quad i = 1, \ldots, n \} ,$$

$$(BCqp) := \{ q_i(t^1) = q^1_i, p_i(t^1) = p^1_i, q_i(t^2) = q^2_i, p_i(t^2) = p^2_i, \quad i = 1, \ldots, n \} .$$

Integrating (3.2) with respect to $\prod dF(t)$ we obtain

$$K = \int A_{t^1,t^2}[x, \dot{x}].(BCxp).$$

$$(3.3) \quad \exp \left\{ \frac{i}{\hbar} \int_{t^1}^{t^2} \sum_i \left( m_i \ddot{x}_i(t) + V_{x_i}(x(t)) \right)^2 dt \right\} \prod_{t,i} dx_i(t).$$

Then we obtain for the free evolution with $V \equiv 0$,

$$K_{t^1,t^2}(x^1, p^1; x^2, p^2) =$$

$$(3.4) \quad = \int_{(BCxp)} \exp \left\{ i \int_{t^1}^{t^2} \sum_i \left( \frac{m_i}{2} \dot{x}^2_i(t) + \frac{am_i^2}{2} \ddot{x}^2_i(t) \right) dt \right\} \prod_{t,i} dx_i(t).$$

The evolution of the wave function is then given by

$$\psi(x^2, p^2; t^2) = \int K_{t^1,t^2}(x^1, p^1; x^2, p^2) \psi(x^1, p^1; t^1) \prod_i dx_i dp_i.$$

Feynman integral in (3.4) is a Gaussian integral and it is separable for $i = 1, \ldots, n$. It is then sufficient to calculate it for each degree of freedom separately. Thus we can suppose $n = 1$ and we have

$$K_{t^1,t^2}(x^1, p^1; x^2, p^2) =$$

$$= \int_{(BCxp)} \exp \left\{ i \int_{t^1}^{t^2} \left( \frac{m}{2} \dot{x}^2(t) + \frac{am^2}{2} \ddot{x}^2(t) \right) dt \right\} \prod_t dx(t).$$
The result of the Gaussian integration has always the Gaussian form

$$K_{t^1, t^2} = N_T \exp \left\{ \frac{i}{\hbar} \tilde{S}_{t^1, t^2}(x^1, p^1; x^2, p^2) \right\},$$

where the normalization factor $N_T$ depends only on

$$T = t^2 - t^1$$

and $\tilde{S}$ denotes the action calculated along the classical trajectory (so-called classical action), i.e.

$$\tilde{S}_{t^1, t^2}(x^1, p^1; x^2, p^2) = \int_{t^1}^{t^2} \frac{1}{2}(m\ddot{x}^2 + am^2\dot{x}^2)dt$$

where $\ddot{x}(t)$ denotes the classical trajectory.

The classical action must satisfy the corresponding Euler’s equation

$$-m\partial_t^2 \ddot{x}(t) + am^2\partial_t^4 \ddot{x}(t) = 0$$

together with the boundary conditions

$$\text{(BC)xp} := \{ \ddot{x}(t^1) = x^1, m\dot{x}(t^1) = p^1, \dot{x}(t^2) = x^2, m\dot{x}(t^2) = p^2 \}.$$
then almost each trajectory contributes to the transition amplitude.

(ii) If the considered time interval is small with respect to $\tau$:

$$T = t^2 - t^1 \ll \tau,$$

then only perturbed classical trajectories contribute significantly to the transition amplitude.

As a consequence we can expect that

(i) for $T \gg \tau$ the transition amplitude $K_{t^1, t^2}$ is close to the standard quantum mechanical transition amplitude;
(ii) for $T \ll \tau$ the transition amplitude $K_{t^1, t^2}$ is close to the deterministic transition amplitude from DetQM.

The free behaviour of a "SubQM - particle" (more exactly a SubQM degree of freedom) is such that at short time intervals it is close to the DetQM, while on large intervals it is close to QM. The random force is the cause of the transition from the DetQM region to the QM region. This transition needs some time, which is of order of the relaxation time $\tau$. We shall study these limits – SubQM and QM limits: $T \ll \tau$ and $T \gg \tau$ – below in Sections 6, 7.

To calculate explicitly the transition amplitude we have to calculate at first the classical trajectory $\bar{x}$ and then the value of the action along this classical trajectory (classical action). Doing this calculation for one degree of freedom, $n = 1$, we set $t^1 = -T$, $t^2 = T$, $x^1 = -X$, $x^2 = X$ for $T > 0$, $x \in \mathbb{R}$. The classical trajectory satisfying boundary condition (3.5) is

$$\dot{x} = a_1 \sinh \beta t + a_2 \cosh \beta t + a_0,$$

where

$$a_1 = \frac{\Delta v}{\sinh \beta T},$$

$$a_2 = \frac{\bar{v} - V}{\cosh \beta T(1 - \tanh \beta T/\beta T)},$$

$$a_0 = \frac{V - \bar{v} \tanh \beta T/\beta T}{1 - (\beta T)^{-1} \tanh \beta T}.$$
and

\[ \Delta v = \frac{1}{2m} (p^2 - p^1), \]
\[ \bar{v} = \frac{1}{2m} (p^2 + p^1), \]
\[ V = \frac{x^2 - x^1}{t^2 - t^1}. \]

Then the classical action is

\[ \bar{S}_{t_1, t_2}(x^1, p^1; x^2, p^2) = \int_{t_1}^{t_2} \frac{m}{2} \left( \dot{\bar{x}}^2 + am\ddot{x}^2 \right) dt \]

and the propagator

\[ K_{t_1, t_2}(x^1, p^1; x^2, p^2) = N_T \exp \left\{ \frac{i}{\hbar} \bar{S}_{t_1, t_2}(x^1, p^1; x^2, p^2) \right\}. \]

By doing the integration we obtain

\[ \bar{S} = \frac{m}{\beta} \left[ (a_1^2 + a_2^2) \sinh \beta T \cosh \beta T + 2a_0a_2 \sinh \beta T + a_0^2 \beta T \right] \]

and then

\[ \bar{S} = \frac{m}{\beta} \left[ a_0^2 \beta T (1 - (\beta T)^{-1} \tanh \beta T) + \left( a_2 + \frac{a_0}{\cosh \beta T} \right)^2 + a_1^2 \right] \sinh \beta T \cosh \beta T \right]. \]

Using the formula

\[ a_2 + \frac{a_0}{\cosh \beta T} = \frac{\bar{v}}{\cosh \beta T} \]

we obtain

\[ S = \frac{m}{\beta} \left[ a_0^2 \beta T (1 - (\beta T)^{-1} \tanh \beta T) + \frac{\Delta v^2}{\tanh \beta T} + \bar{v}^2 \tanh \beta T \right] \]
\[ S = \frac{m(X - \frac{\bar{v}}{\beta} \tanh \beta T)^2}{T(1 - (\beta T)^{-1} \tanh \beta T)} + \frac{(p^1)^2 + (p^2)^2}{2m\beta \tanh 2\beta T} - \frac{p^1p^2}{m\beta \sinh 2\beta T}. \]

The first term in this formula can be written as

\[ m \left( x^2 - x^1 - \frac{p^1 + p^2}{m\beta} \tanh \beta T \right)^2 \frac{1}{4T(1 - (\beta T)^{-1} \tanh \beta T)}. \]

In the general case with \( n \geq 1 \) we have

\[ \bar{S}_{t_1,t_2}(x^1, p^1; x^2, p^2) = \sum_{i=1}^{n} \frac{1}{2\beta_i m_i} \left( \frac{(p_{i1}^1)^2 + (p_{i2}^2)^2}{\tanh \beta_i T} - \frac{2p_{i1}^1 p_{i2}^2}{\sinh \beta_i T} \right) + \]

\[ + \sum_{i} m_i \left[ x_{i2}^2 - x_{i1}^1 - (\beta_i m_i)^{-1} (p_{i1}^1 + p_{i2}^2) \tanh (\beta_i T/2) \right]^2 \frac{1}{2T(1 - (\beta_i T/2)^{-1} \tanh (\beta_i T/2))}, \]

where

\[ T = t^2 - t^1. \]

The normalization factor is given by

\[ N_T = (2\pi i\hbar)^{-n} \prod_i (\sinh \beta_i T)^{-1/2} (\beta_i T - 2 \tanh \beta_i T/2)^{-1/2}. \]

The simplest way to calculate \( N_T \) is using the well-known formula for quadratic Lagrangian \( L \) (where \( \vec{x} = (x, p) \)),

\[
\int_{\vec{x}(t_1) = \vec{x}_1}^{\vec{x}(t_2) = \vec{x}_2} \exp \left\{ \frac{1}{\hbar} \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}}) dt \right\} \prod_t dx(t) dp(t) =
\frac{1}{2\pi i\hbar} \left[ \det \left( -\frac{\partial^2 S_{CL}(\vec{x}_2, \vec{x}_1)}{\partial \vec{x}_2 \partial \vec{x}_1} \right) \right]^{1/2} \exp \left\{ \frac{i}{\hbar} S_{CL}(\vec{x}_2, \vec{x}_1) \right\}.
\]

Here \( S_{CL}(\vec{x}_2, \vec{x}_1) \) is the value of the action along the classical trajectory going from \( \vec{x}_1 \) to \( \vec{x}_2 \). The determinant inside is the well-known van Vleck-Pauli-Morette determinant (see [9], formula (55)).
Now we shall calculate the evolution equation. It is possible to do it by calculating all derivatives and then generalizing this to $n > 1$. A better way to prove this equation is presented in the next section.

The resulting equation for the wave function $\psi(x_1, p_1, \ldots, x_n, p_n; t)$ is

$$i\hbar \partial_t \psi = \left( -\frac{1}{2} \sum_{i=1}^{n} \frac{p_i^2}{m_i} - \frac{\hbar^2}{2} \sum_i m_i \beta_i^2 \partial_{p_i}^2 - i \hbar \sum_i \frac{p_i}{m_i} \partial_{x_i} \right) \psi. \tag{3.6}$$

Now we shall introduce the interaction term into this equation. We shall first consider its deterministic limit $\tau_i \to \infty$, i.e. $\beta_i \to 0$. We obtain (setting $\hbar = 1$)

$$i\partial_t \psi = \left( -\frac{1}{2} \sum \frac{p_i^2}{m_i} - i \sum \frac{p_i}{m_i} \partial_{x_i} \right) \psi$$

as a short-time limit of (3.5).

Comparison with the DetQM equation

$$i\partial_t \psi = \left( -\frac{1}{2} \sum \frac{p_i^2}{m_i} - i \sum \frac{p_i}{m_i} \partial_{x_i} + \sum V_{x_i} \partial_{p_i} + V \right) \psi$$

suggests the following evolution equation for the SubQM$_{RF}$ model

$$i\hbar \partial_t \psi = \hat{H} \psi, \tag{3.7}$$

$$\hat{H} = \left( -\frac{1}{2} \sum \frac{p_i^2}{m_i} - \frac{\hbar^2}{2} \sum_i m_i \beta_i^2 \partial_{p_i}^2 - i \hbar \sum_i \frac{p_i}{m_i} \partial_{x_i} + i \hbar \sum V_{x_i} \partial_{p_i} + V \right) \psi.$$

This is the basic equation of SubQM$_{RF}$. This is a "Schroedinger-like" equation for the wave function

$$\psi(x_1, p_1, \ldots, x_n, p_n; t),$$

considered for each $t$ as an element of the Hilbert space

$$H = L^2(\mathbb{R}^n_x \times \mathbb{R}^n_p) \cong L^2(\mathbb{R}^n).$$
Then $\hat{H}$ is an operator defined on (a dense part of) $H$ and we see that $\hat{H}$ is formally Hermitian. Thus the equation (3.6) generates the unitary evolution in SubQM$_{RF}$. Unitarity will be examined in more details in the next section.

Let us note that in the general case of a manifold $M$ the corresponding Hilbert space will be the $L^2$-space on the cotangent bundle $T^*M$ and the evolution equation (3.6) makes a good sense in this setting.
4. The unitary evolution in the model SubQM$_{RF}$

In this section we shall study the evolution equation

\[ i\hbar \partial_t \psi = \hat{H} \psi, \quad \hat{H} = \hat{H}_0 + \hat{H}_{int} \]

where

\[
\hat{H}_0 = -\frac{1}{2} \sum_{i=1}^n \frac{p_i^2}{m_i} - \frac{1}{2} \hbar^2 \sum m_i \beta_i^2 \partial_{p_i}^2 - i\hbar \sum \frac{p_i}{m_i} \partial_{x_i},
\]

\[
\hat{H}_{int} = i\hbar \sum V_x(x) \partial_{p_i} + V(x).
\]

The wave function

\[ \psi(x_1, p_1, \ldots, x_n, p_n; t) \]

is a function on \( \mathbb{R}^n(x) \times \mathbb{R}^n(p) \cong \mathbb{R}^{2n} \) for each \( t \in \mathbb{R} \).

From this equation we shall obtain the original Feynman integral. Then we shall show in more details unitary properties of this evolution.

By Trotter’s formula we have (for \( \varepsilon = T/m \))

\[
\langle x_0, p_0 | e^{-i\hat{H}_0 \varepsilon} e^{-i\hat{H}_{int} \varepsilon} \rangle \langle \xi_1, \eta_1 | x_1, p_1 \rangle \ldots \\
\langle x_{m-1}, p_{m-1} | e^{-i\hat{H}_0 \varepsilon} e^{-i\hat{H}_{int} \varepsilon} \rangle \langle \xi_m, \eta_m | x_m, p_m \rangle
\]

where \( \{|x_k, p_k\}\) is \( \delta \)-basis of states, \( x_k, p_k \in \mathbb{R}^n \),

\[ |x_k, p_k \rangle \sim \delta_{x_k}(x) \delta_{p_k}(p) \]

and \( \{ |\xi_k, \eta_k \rangle \}\) is the dual basis

\[ |\xi_k, \eta_k \rangle = \int e^{i\xi_k \cdot x_k} e^{i\eta_k \cdot p_k} |x_k, p_k\rangle d^n x_k d^n p_k \sim e^{i\xi_k \cdot x} e^{i\eta_k \cdot p}. \]

Here it is assumed the integration over all variables from the right hand side (RHS) of equation (4.1) which do not enter the LHS of this
equation. We shall obtain the standard product where we have written explicitly only the first term

\[
\int \exp \left\{ -i \varepsilon \left[ -\frac{1}{2} \sum \frac{p_{0i}^2}{m_i} + \frac{1}{2} \sum \beta_i^2 m_i \eta_i^2 + \sum \frac{p_{0i} \xi_{1i}}{m_i} - \sum V_{x_i}(x_0) \eta_i + V(x_0) \right] - i \sum \xi_{1i} x_{0i} - i \sum \eta_i p_{0i} + i \sum \xi_{1i} x_{1i} + i \sum \eta_i p_{1i} \right\} \ldots
\]

Then we arrive at the term

\[
\int \exp i \varepsilon \left\{ \sum \xi_{1i} x_{1i} - x_{0i} + \eta_i + \frac{p_{1i} - p_{0i}}{\varepsilon} + \frac{1}{2} \frac{p_{0i}^2}{m_i} - \frac{1}{2} \beta_i^2 m_i \eta_i^2 - \frac{p_{0i}}{m_i} \xi_{1i} + V_{x_i}(x_0) \eta_i + V(x_0) \right\} \ldots
\]

In the continuum limit \( m \to \infty \) we obtain

\[
\int_{(BC)} \exp i \left\{ \int_{t^1} \sum \xi_i(t) \dot{x}_i(t) + \eta_i(t) \dot{p}_i(t) + \frac{1}{2m_i} \frac{p_i^2(t)}{m_i} - \frac{1}{2} \beta_i m_i \eta_i^2(t) - \frac{p_i(t)}{m_i} \xi_i(t) + V_{x_i}(x(t)) \eta_i(t) - V(x(t)) \right\} dt \prod_{t,i} d\xi_i(t) d\eta_i(t) dp_i(t) dx_i(t).
\]

By integration with respect to \( \prod d\xi_i(t) d\eta_i(t) \) we obtain \( \delta \)-functions

\[
\int_{(BC)} \prod_{t,i} \delta \left( \dot{x}_i(t) - \frac{p_i(t)}{m_i} \right) \exp i \left\{ \int \frac{1}{2 \beta_i^2 m_i} \left( \dot{p}_i(t) + V_{x_i}(x(t)) \right)^2 dt + \int \left( \frac{1}{2} \frac{p_i^2(t)}{m_i} - V(x(t)) \right) dt \right\} \prod_{i,t} dp_i(t) dx_i(t).
\]

Using the preceding section we obtain here

\[
\int \left( \frac{1}{2} \sum \frac{p_i^2}{m_i} - V \right) dt = \bar{S},
\]

\[
\frac{1}{2 \beta_i^2 m_i} = \frac{a}{2}.
\]
So that we have arrived at the initial Feynman integral. Integrating by \( \prod dp_i(t) \) we obtain

\[
\int_{(BC)} \exp i \left\{ \int \frac{1}{2\beta^2 m_i} \left( m_i \ddot{x}_i(t) + V_{\chi_i}(x(t)) \right)^2 dt + \int \left( \frac{1}{2} m_i \dot{x}_i^2(t) - V(x(t)) \right) dt \right\} \prod_{t,i} dx_i(t).
\]

Now we shall study unitarity of the evolution in SubQM\(_{RF}\). We shall study namely the evolution in the \( p \)-space. We shall consider the case of one degree of freedom

\[
\varphi(p^2, t^2) = \int K_{t_1, t_2}(p^1; p^2) \varphi(p^1, t^1) dp^1.
\]

Here

\[
K_{t_1, t_2}(p^1; p^2) = N_T^{(p)} \exp \{ i S_{t_1, t_2}(p^1; p^2) \}, \quad T = t^2 - t^1
\]

and

\[
S_{t_1, t_2}(p^1; p^2) = \frac{1}{2\beta m} \left[ \frac{(p^1)^2 + (p^2)^2}{\tanh \beta T} - \frac{2p^1 p^2}{\sinh \beta T} \right].
\]

The reduced equation for \( p \) is an "imaginary harmonic oscillator"

\[
i \partial_t \varphi = \hat{H}_0(p), \quad \hat{H}_0(p) := -\frac{1}{2} \hbar^2 \frac{m^2 \beta^2}{\partial_p^2} - \frac{1}{2} \frac{p^2}{m}.
\]

The operator \(-\hat{H}_0(p)\) nor \(\hat{H}_0(p)\) is positive definite, so that the spectrum of \(\hat{H}_0(p)\) is not limited to a half-line. This is rather a non-standard case and we shall show explicitly that this operator creates a unitary group of operators. We shall also show that, in a certain sense, this unitary evolution has certain stability (relaxation) properties.

Let \( \{ T_t \}, \ T \in \mathbb{R}, \) be a group of unitary transformations defined for \( \varphi \in L^2(\mathbb{R}) \) by

\[
(T_t \varphi)(x) := e^{-\beta t/2} \varphi(e^{-\beta t} x), \ x \in \mathbb{R}.
\]
Let Fourier transforms be defined as
\[ (\mathcal{F}\varphi)(p) := \int e^{-i/hxp} \varphi(x) \, dx, \]
\[ (\mathcal{F}^{-1}\varphi)(x) := \int e^{i/hxp} \varphi(p) \frac{dp}{2\pi h}. \]

Let \( c_0 > 0 \) be a fixed constant and let \( \mathcal{U}_0 \) and \( \mathcal{U}_1 \) be two unitary transformations defined by
\[ (\mathcal{U}_0\varphi)(x) := \exp \left\{ \frac{i}{h} c_0 \beta m \frac{x^2}{4} \right\} \varphi(x), \]
\[ (\mathcal{U}_1\varphi)(p) := \exp \left\{ \frac{i}{h} \frac{1}{2 c_0 \beta m} p^2 \right\} \varphi(p). \]

Then the following theorem holds (its proof was suggested to the author by Dr. M. Šilhavý [8]).

Let us denote
\[ H_0^{(x)} = \hbar \beta \left( \frac{i}{2} + ix \partial_x \right). \]

**Theorem.**

(i) \( T_t = e^{i/hH_0^{(x)}t} \), i.e. \( T_t \) is generated by \( H_0^{(x)} \).

(ii) Let \( G = \mathcal{U}_1 \mathcal{F} \mathcal{U}_0 \). \( G \) is a unitary operator. If we denote
\[ H_1 = -\frac{1}{2} c_0 \hbar^2 \beta^2 m \partial_p^2 - \frac{1}{2 c_0 m} p^2 \]
then \( H_1 \) is a unitary transformation of \( H_0^{(x)} \) and
\[ H_1 = GH_0^{(x)} G^{-1}. \]

Thus \( H_1 \) is a generator of a unitary group
\[ e^{i/hH_1 t} = GT_t G^{-1}, \quad t \in \mathbb{R}. \]

(iii) We have an explicit form
\[ (e^{-i/hH_1 t} \varphi)(q) = (4\hbar c_0 \beta m)^{-1/2} (\sinh \beta t)^{-1/2} \int \exp \left\{ \frac{i}{h} \frac{i}{2 c_0 \beta m} \left[ p^2 + q^2 \tan \beta t - \frac{2pq}{\sinh \beta t} \right] \right\} \varphi(p) \, dp. \]

The proof uses the following lemma (due to M. Šilhavý, [8]).
Lemma. We have

(i) \( \partial_t T_t \big|_{t=0} = \frac{i}{\hbar} H_0^{(x)} \),

(ii) \( U_1 H_0^{(p)} U_1^{-1} = H_0^{(p)} + \frac{1}{c_0 m} p^2 \), where \( H_0^{(p)} := h\beta \left( \frac{i}{2} + ip\partial_p \right) \).

(iii) \( \mathcal{F} H_0^{(x)} \mathcal{F}^{-1} = -H_0^{(p)}, \mathcal{F}(x^2) \mathcal{F}^{-1} = -\hbar^2 \partial_p^2 \).

(iv) \( U_0 H_0^{(x)} U_0^{-1} = H_0^{(x)} + \frac{1}{2} c_0 \beta^2 m^2 x^2 \).

(v) \( \mathcal{F} U_0 H_0^{(x)} U_0^{-1} \mathcal{F}^{-1} = -H_0^{(p)} - \frac{1}{2} c_0 \hbar^2 \beta^2 m \partial_p^2 \).

(vi) \( U_1 \partial_p^2 U_1^{-1} = \partial_p^2 - \frac{p^2}{c_0 \hbar^2 \beta^2 m^2} - \frac{2}{c_0 \hbar^2 \beta^2 m} H_0^{(p)} \).

The proof of the lemma is done by an explicit calculation. Proof of the theorem follows as:

(i) and (ii) by calculation using lemma.

(iii) by calculation using the formula

\[
\int e^{ia/2p^2} e^{ipz} dp = \left( \frac{2\pi}{a} \right)^{1/2} e^{-\frac{a}{2\pi} z^2}. \]

For \( \exp(-iH_1 t) \) we have obtained a formula consistent with the propagator calculated before.

Now we shall study the transformed function

\[ \varphi(p; t) \]

for large times \( t \gg 1/\beta \). In this case we have

\[ \tanh \beta t \sim 1, \sinh \beta t \sim \frac{1}{2} e^{\beta t}. \]

Let

\[ \varphi_1(p) = \exp \left\{ i \frac{1}{2h\beta m} p^2 \right\} \varphi(p) \]

and let \( \psi_1(q) \) be Fourier transform

\[ \psi_1(q) = \int \exp \left\{ i \frac{1}{h\beta m} pq \right\} \varphi(p) dp. \]
If \( \varphi_1 \) is a compactly supported function then \( \psi_1 \) will be a \( C^\infty \)-function. By the Theorem (ii) we obtain that the evolution of \( \varphi \) can be represented by \( \psi_3 \) defined as

\[
\psi_2(q; t) := (\sinh \beta t)^{-1} \psi_1 \left( \frac{q}{\sinh \beta t} \right),
\]

\[
\psi_3(q; t) := \exp \left\{ -\frac{i}{\hbar \beta m} q^2 \right\} \psi_2(q; t).
\]

We shall show that \( \psi_2(\cdot; t) \) relaxes to the "constant" function for \( t \gg 1/\beta \). For \( t \gg 1/\beta \) we have

\[
\psi_2(q; t) \approx e^{-\beta t/2} \psi_1(q e^{-\beta t}).
\]

The transformation \( \psi_1 \mapsto \psi_2 \) is a unitary transformation.

\[
\int |\psi_2(q; t)|^2 dq = \int |\psi_1|^2 dq = \text{const}.
\]

But for derivatives we have

\[
\partial_q \psi_2 \approx e^{-\beta t/2} \psi_1 q (qe^{-\beta t}) e^{-\beta t}.
\]

Thus the \( L^2 \)-norm of \( D\psi_2 \) goes to 0:

\[
\int |D\psi_2(q; t)|^2 dq = e^{-2\beta t} \int |D\psi_1|^2 dq \approx \text{const}. e^{-2\beta t} \to 0.
\]

So that we have

\[
\psi_2(\cdot; t) \twoheadrightarrow \text{"const"}.
\]

In the same way we have

\[
\psi_3(\cdot; t) \twoheadrightarrow \text{"const}. \exp \left\{ -\frac{i}{\hbar \beta m} q^2 \right\}.
\]
5. The interpretation of SubQM-models

Up to now, we have defined the states and the evolution in sub-quantum models. The state of the system is defined by the wave function

$$\psi = \psi(x, p), \; x, p \in \mathbb{R}^n.$$ 

The wave function is considered as an amplitude distribution of position and momentum.

The evolution of the amplitude distribution in the momentum was studied in the preceding section. We have found that the evolution of the distribution $\varphi(p)$ is close to the "constant" distribution for times much greater than the relaxation time. We call this type of $\varphi(p)$ the relaxed distribution of momentum and shall use the name relaxed region for the situation when dependence on the momentum $p$ in the wave function $\psi(x, p)$ is the relaxed distribution.

In the relaxed region, a typical trajectory which contributes significantly to the transition amplitude is highly irregular and in the mathematical sense is non-differentiable. This means that the exact velocity does not exist and that the mean velocity approaches infinity on small time intervals. Such a behavior is typical for a Brownian particle. The trajectory of the particle in the mathematical model of Brownian motion is non-differentiable and the mean velocity is infinite.

By the analogy with the Brownian motion, we can suppose that a typical trajectory of the sub-quantum particle is similar to the typical Brownian trajectory, when we have the relaxed situation. On the other hand, this relaxed situation is achieved only approximately at finite times, so that the typical momentum is large but not infinite at finite times.

The conclusion is that it is very improbable that the dependence of $\psi(x, p)$ on the momentum $p$ could be observed. Thus we can assume the rather conservative point of view that only the dependence of the wave function $\psi(x, p)$ on the position variable $x$ can be observed. This leads to the following interpretation postulate.

We assume that in the subquantum models SubQM, SubQM$_{RF}$ and similar models, where the state is specified by the wave function $\psi(x, p)$ depending on the position and momentum variables, the only
observables are operators depending only on $x$

$$A(x, \partial_x).$$

The interpretation problem consists of two parts.
(i) If there is a 0-1 measurement (the corresponding observable is a projection) then one has to define what will be the state after the measurement,
(ii) one has to define what is a probability of a positive outcome of this 0-1 measurement.

The first part (i) creates no interpretation problem.

Let $x \in \mathbb{R}^n(x), p \in \mathbb{R}^n(p)$ so that the wave function $\psi(x, p)$ is defined on the space

$$\mathbb{R}^{2n}_{x,p} = \mathbb{R}^n_{x} \times \mathbb{R}^n_{p}.$$  

Then the state space is

$$L^2(\mathbb{R}^{2n}_{x,p}) \cong L^2(\mathbb{R}^n_{x}) \otimes L^2(\mathbb{R}^n_{p}).$$

$P$ is a projection in $L^2(\mathbb{R}^n_{x})$, i.e. the operator

$$P : L^2(\mathbb{R}^n_{x}) \to L^2(\mathbb{R}^n_{x})$$

satisfying

$$P^+ = P, \ P^2 = P, \ P \geq 0.$$  

Let $P \otimes id_{(p)}$ be the projection defined on $L^2(\mathbb{R}^n_{x}) \otimes L^2(\mathbb{R}^n_{p})$ by

$$\psi \otimes \varphi \mapsto P\psi \otimes \varphi$$

for $\psi \in \mathbb{R}^n_{x}, \ \varphi \in \mathbb{R}^n_{p}$, i.e. the projection $P$ operates only on the variable $x$.

The interpretation postulate says that possible outcomes of the measurement $P$ are 1 (answer = yes) or 0 (answer = no). After the measurement the system will be in the state

$$(P \otimes id_{(p)})\psi$$
if the result is 1, and in the state
\[(\text{id}_x - P) \otimes \text{id}_{(p)} \psi = (\text{id}_{(x,p)} - P) \otimes \text{id}_{(p)} \psi\]
if the result is 0.

The most typical situation is when
\[P \cong \chi_{a,b},\]
\[\chi_{a,b} = \begin{cases} 1 & \text{for } a < x < b, \\ 0 & \text{otherwise}, \end{cases}\]
i.e. it is a characteristic function of the interval \((a, b)\). This experiment is interpreted as passing through the "slit" \((a, b)\). Then the operation of \(P \otimes \text{id}_{(p)}\) is given by
\[\psi(x, p) \mapsto \chi_{a,b}(x) \psi(x, p).\]

On the other hand, we stated above that operators as \(\chi_{a,b}(p)\) operating on the momentum variable are not observable.

At this moment it is necessary to make a clear distinction between the particle's momentum \(p\) and the quantum-mechanical momentum (called shortly QM-momentum) defined as follows. The quantum-mechanical momentum is a property of the wave function, i.e. of the amplitude distribution \(\psi(x, p)\), and it is not a property of a particle. The QM-momentum is defined by the behavior of the wave function with respect to the translations in the physical space. The corresponding momentum operator \(-i\hbar \partial_x\) is the standard one. This does not depend on the \(p\)-distribution, since the particle's momentum \(p\) is conserved by translations. Thus the wave function with the QM-momentum \(\vec{\xi}\) is
\[\psi(x, p) = \text{const.} e^{i\vec{\xi} \cdot \vec{x}} \varphi(\vec{p}),\]
where the dependence of \(\varphi(\vec{p})\) on \(\vec{p}\) is a degeneracy factor.

Thus concepts of the particle's momentum and the QM-momentum are completely different. Particle's momentum is defined (not observable) at each point of the particle's trajectory while QM-momentum is a property of the function \(\psi(\cdot, p)\). Both quantities have their
amplitude distribution but the amplitude distribution of the QM-momentum is defined as a Fourier transform of $\psi(\cdot, p)$, the amplitude distribution of particle's momentum $\psi(x, \cdot)$ is independent on the $x$-variable of $\psi$. In fact, in the relaxed region the distribution in $p$ approaches a distribution of the type

$$\exp\left\{ i(2\hbar/bm)^{-1}p^2 \right\}.$$  

The second part of the interpretation – the formula for probabilities – is less clear. In general, one can construct the density operator corresponding to the pure state $\psi(x, p)$ as

$$\rho(x, p; x', p') := \psi(x, p)\psi^*(x', p').$$

The general density operator then will be

$$\rho(x, p; x', p') = \sum_{i=1}^{\infty} \alpha_i \psi_i(x, p)\psi_i^*(x', p'), \quad \alpha_i \geq 0, \quad \sum \alpha_i = 1.$$  

In the usual QM, the mean value of the observable $A(x, x')$ in the state $\rho(x, x')$ is given by the formula

$$Tr(A \rho) = \int A(x, x') \rho(x, x') dx dx'.$$

Now we have to treat variables $p$ and $p'$ in $\rho(x, p; x', p')$ in a certain way.

In general, let us consider the positive kernel $P(p, p')$. Let us define the relative mean value of $A$ as

$$Tr_P(A \rho) := \int A(x, x') \rho(x, p; x', p') P(p, p') dx dp dx' dp'$$

and then the absolute mean value is given by

$$Tr_P(A \rho)/Tr_P(\rho).$$

For the choice of $P$ we have two main possibilities.
We set \( P(p, p') = \delta(p - p') \).

This is formally the most standard (but wrong) choice. This means that the value \( p \) is principally observable; in Feynman’s language it says that the alternatives with \( p \neq p' \) are principally distinguishable.

For the projection \( P_{a, b} : \psi(x, p) \rightarrow \chi_{a, b}(x)\psi(x, p) \)

we obtain

\[
P(P_{a, b}\psi) = \int_a^b dx \int dp |\psi(x, p)|^2.
\]

In fact, this interpretation brings serious problems. There are concrete problems with the QM-limit, which we shall show later. This contradicts our point of view we explained above.

We set \( P(p, p') \equiv 1 \) for all \( p \) and \( p' \).

The physical meaning of this assumption is that situations with different \( p \neq p' \) are not principally distinguishable. The meaning is that for different \( p' \)'s we have to sum up amplitudes, not probabilities. This corresponds to the view that different \( p' \)'s mean something like different positions at the times \( t_2 \) and \( t_2 - \varepsilon/2 \). Here \( \varepsilon > 0 \) denotes the time step.

This means to sum up amplitudes for different \( p \) at first and then to do the square modulus. We have then

\[
P(P_{a, b}\psi) = \int_a^b dx \int dp dp' \psi(x, p)\psi^*(x, p').
\]

In general we have

\[
P(\text{pos. outcome}) = \int \rho(x, p; x, p') dx dp dp'.
\]

Of course, in this case it is necessary to make a renormalization of the wave function. The way we prefer is to consider the probability as a relative probability, which has to be renormalized for each time moment in such a way that the total probability be equal to 1.
6. THE LONG TIME APPROXIMATION
AND THE QUANTUM MECHANICAL LIMIT OF SubQMRF

The long time approximation is defined by time intervals satisfying

$$\Delta t \gg \frac{1}{\beta_i}, \ i = 1, \ldots, n.$$  

In this approximation all exponential functions are very close to their limits, i.e.

$$\tanh \beta T \sim 1, \ \sinh \beta T \sim \infty.$$  

This type of approximation can be called also the exponential approximation. The meaning is to set \(\exp(\beta T) = \infty\) and \(\exp(-\beta T) = 0\).

The long time (exponential) approximation to the propagator from Section 3 is the following.

(6.1) \(\bar{S}_{t^1,t^2}(x^1,p^1;x^2,p^2) = \)

\[
= \sum_{i=1}^{n} \frac{1}{2\beta_im_i} ((p^1_i)^2 + (p^2_i)^2) + \sum_{i=1}^{n} m_i \frac{[x^2_i - x^1_i - (\beta_im_i)^{-1}(p^1_i + p^2_i)]^2}{2T(1 - 2/(\beta_iT))}.
\]

The use of the iterated propagator needs another representation – separation of quantities into the groups:

(i) input quadratic form with the \(2 \times 2\) matrix \(Q_{inT}\),
(ii) output quadratic form with the \(2 \times 2\) matrix \(Q_{outT}\),
(iii) the in-out bilinear form with the \(2 \times 2\) matrix \(Q_{trT}\).

Here

\[
\bar{S} = at \frac{1}{2}(p^{12} + p^{22}) + bt^1p^2 + cT \frac{1}{2}(x^2 - x^1)^2 + dT(x^2 - x^1)(p^1 + p^2),
\]

(6.2)

\[
Q_{inT} = \begin{pmatrix} aT & -dT \\ dT & cT \end{pmatrix}, \quad Q_{outT} = \begin{pmatrix} aT & dT \\ dT & cT \end{pmatrix}, \quad Q_{trT} = \begin{pmatrix} bT & dT \\ -dT & -cT \end{pmatrix}
\]
and

\[
\begin{align*}
    a_T &= \frac{1}{\beta m} \left[ \frac{1}{\tanh \beta T} + \frac{\tanh^2(\beta T/2)}{\beta T - 2 \tanh(\beta T/2)} \right], \\
    b_T &= \frac{1}{\beta m} \left[ -\frac{1}{\sinh \beta T} + \frac{\tanh^2(\beta T/2)}{\beta T - 2 \tanh(\beta T/2)} \right], \\
    c_T &= \frac{\beta m}{\beta T - 2 \tanh(\beta T/2)}, \\
    d_T &= -\frac{\tanh(\beta T/2)}{\beta T - 2 \tanh(\beta T/2)}.
\end{align*}
\]

(6.3)

The propagator is equal to

\[
K_T = N_T \exp \frac{i}{\hbar} \left\{ \frac{1}{2} X_1^\top Q_{inT} X_1 + X_1^\top Q_{trT} X_2 + \frac{1}{2} X_2^\top Q_{outT} X_2 \right\},
\]

where

\[
X_1 = \left( \begin{array}{c} p_1 \\ x_1 \end{array} \right), \quad X_2 = \left( \begin{array}{c} p_2 \\ x_2 \end{array} \right).
\]

In the long time approximation we have

\[
\begin{align*}
    a_T &\approx \frac{1 + \omega_T}{\beta m}, \quad b_T \approx \frac{\omega_T}{\beta m}, \quad c_T \approx \beta m \omega_T, \quad d_T \approx -\omega_T
\end{align*}
\]

(6.5)

where

\[
\omega_T = \frac{1}{\beta T - 2}.
\]

This gives in the long time approximation

\[
\begin{align*}
    Q_{inT} &= \begin{pmatrix} \frac{1 + \omega_T}{\beta m} & \omega_T \\ \omega_T & \beta m \omega_T \end{pmatrix}, \quad Q_{outT} = \begin{pmatrix} \frac{1 + \omega_T}{\beta m} & -\omega_T \\ -\omega_T & \beta m \omega_T \end{pmatrix}, \\
    Q_{trT} &= \omega_T \begin{pmatrix} \frac{\beta m}{1} & -1 \\ 1 & -\beta m \end{pmatrix}.
\end{align*}
\]

(6.6)

It is important to study the evolution of Gaussian wave packets. Let us consider the wave function at the time \( t = t_1 \) in the form

\[
\psi_1(X^1) = \exp \frac{1}{\hbar} \left\{ -\frac{1}{2} X_1^\top A_1 X_1 + i B_1^\top X_1 \right\}
\]

(6.7)
where $A_1$ is a given $2 \times 2$ matrix and $B_1$ is a given 2-dimensional vector.

The evolution is given by

$$
\psi_2(p^2, x^2; t_2) = \int K_{t_2-t_1}(x^1, p^1; x^2, p^2)\psi_1(p^1, x^1; t_1)dp^1dx^1.
$$

As a result we obtain

$$
\psi_2(X^2; t_2) = \text{const}(T). \exp \frac{i}{\hbar} \left\{ \frac{1}{2} X^2^\top Q_{out}^T X^2 \right\}.
$$

The inverse matrix of a $2 \times 2$ matrix $A$ can be calculated by

$$
A^{-1} = \frac{A^{adj}}{\det A}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{adj} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

Let us start with the case where

$$
A_1 = \begin{pmatrix} -\frac{i}{\beta m} & 0 \\ 0 & 0 \end{pmatrix},
$$

i.e. the wave function $\psi_1$ depends on $p^1$ as $\exp i(p^1)^2/2\beta m$. For $R = iA_1 + Q_{in}T$ we obtain

$$
R^{-1} = \frac{1}{2\beta m} \begin{pmatrix} (\beta m)^2 & -\beta m \\ -\beta m & 2T + 1 \end{pmatrix}
$$

with $\det R = 2\omega_T$. Then we obtain

$$
Q_{out}^T - Q_{tr}^T R^{-1} Q_{tr} = \frac{1}{\beta m} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
R^{-1} Q_{tr} = \frac{1}{\beta m} \begin{pmatrix} 0 & 0 \\ 0 & 1 - \beta m \end{pmatrix}.
$$

Using this we obtain by the Gaussian integration

$$
\psi_2(X^2) = N_T \exp \left\{ \frac{i}{\hbar} \frac{1}{2} X^2^\top (Q_{out}^T - Q_{tr}^T R^{-1} Q_{tr}) X^2 \right\} \cdot \exp \left\{ -\frac{i}{\hbar} B_1^T R^{-1} Q_{tr} X^2 \right\} \cdot \exp \left\{ -\frac{i}{\hbar} B_1^T R^{-1} B_1 \right\}.
$$
In this way we obtain for $B_1 = \left( \frac{l^1}{k^1} \right)$,

$$
\psi_2(p^2, x^2) = N_T \exp \left\{ \frac{i}{\hbar} \frac{1}{2\beta m} (p^2)^2 \right\} \exp \left\{ \frac{i}{\hbar} k^1 \left( x^2 - \frac{p^2}{\beta m} \right) \right\} .
\right.

\exp \left\{ \frac{i}{4\hbar \beta m} \left( \beta ml^1 - k^1 \right)^2 - \frac{i(k^1)^2}{2\hbar \beta m \omega T} \right\}.

The absolute phase factor is equal to

$$
\exp \left\{ -\frac{i}{\hbar} \frac{(k^1)^2}{2m} T \right\} . \exp \left\{ \frac{i}{4\hbar \beta m} \left[ 4(k^1)^2 - (\beta ml^1 - k^1)^2 \right] \right\} .
\right.$$

The first factor is exactly the QM-factor $\exp\{-iEt\}$, while the second factor is a correction, which is a constant independent on $T$. In fact, in $\psi_1$ we have the QM-momentum

$$
\psi_1 \approx \exp \left\{ \frac{i}{\hbar} \left( l^1 p^1 + k^1 x^1 \right) \right\} .
\right.$$

The term

$$
\psi_2 \approx \exp \left\{ \frac{i}{\hbar} \frac{1}{\beta m} \frac{1}{2} (p^2)^2 \right\}
\right.$$

is the stabilized particle momentum distribution which is already in the propagator.

The term

$$
\psi_2 \approx \exp \left\{ \frac{i}{\hbar} k^1 x^2 \right\}
\right.$$

is the standard conservation of the QM-momentum $k^1$, i.e. $k^2 = k^1$. A completely new feature is the last term

$$
\psi_2 \approx \exp \left\{ -\frac{i}{\hbar} \frac{1}{\beta m} k^1 p^1 \right\} .
\right.$$

This means that in the long time approximation the dependence $\psi_1 \approx \exp\{il^1 p^1\}$ on $p^1$ is forgotten, the value $l^1$ enters only the time-independent absolute phase factor mentioned above. On the other
hand, the dependence of $\psi_2$ on $p^2$ is governed by $k^1/\beta m$, i.e. it depends on the QM-momentum.

Let us now suppose that $\psi_1$ is the superposition of waves with $l^1 \equiv 0$ and different $k^1$:

$$\psi_1(p^1, x^1) \approx \int dk^1 \exp \left\{ \frac{i(p^1)^2}{2\hbar \beta m} \right\} \exp \left\{ \frac{i}{\hbar} k^1 x^1 \right\} a(k^1).$$

Then by the superposition principle we have

$$\psi_2 \approx N_T \exp \left\{ \frac{i(p^2)^2}{2\hbar \beta m} \right\} \int \exp \left\{ -\frac{i(k^1)^2}{2\hbar m} \left( T - \frac{3}{2\beta} \right) \right\} \exp \left\{ \frac{i}{\hbar} k^1 \left( x^2 - \frac{p^2}{\beta m} \right) \right\} a(k^1)dk^1.$$

The difference between $t - 3/(2\beta)$ and $T$ is small for $\beta T \gg 1$. But dependence on $x^2 - p^2/(\beta m)$ is crucial.

Let us consider the simplest projector (the slit)

$$\chi_{a,b}(x^2).$$

Applying it we obtain $\varphi_2$:

$$\varphi_2(p^2, x^2) = \psi_2(p^2, x^2)\chi_{a,b}(x^2).$$

Using the interpretation postulate (IP$_1$) we obtain the probability

$$\text{Prob}(a, b) = \int \psi_2(p^2, x^2)\psi_2^*(p^2, x^2)\chi_{a,b}(x^2)dp^2dx^2 =$$

$$= \int a(k^1)a^*(k^{1'})\chi_{a,b}(x^2)\ldots dk^1dk^{1'}dx^2 \cdot \int \exp \left\{ -\frac{i}{\hbar \beta m}(k^1 - k^{1'})p^2 \right\} dp^2.$$

The last integral gives $\delta(k^1 - k^{1'})$ and then

$$\text{Prob}(a, b) = \int |a(k^1)|^2 \chi_{a,b}(x^2)dk^1dx^2$$
and this is clearly an incorrect result, in which the interference terms are neglected. We think that this excludes the probability interpretation (IP$_1$).

On the other hand, the probability formula in (IP$_2$)

\[ Prob(a, b) = \int \psi_2(p^2, x^2)\chi_a(x^2)dp^2dx^2 \]

is quite consistent.

Putting inside the formula for \( \psi_2(p^2, x^2) \) using the formula

\[
\int \exp \left\{ -\frac{i}{\hbar \beta m} \left( -\frac{(p^2)^2}{2} \right) \right\} \exp \left\{ -ip^2 \frac{k^1}{\hbar \beta m} \right\} dp^2 = \exp \left\{ -\frac{i(k^1)^2}{2\hbar \beta m} \right\}
\]

and calculating the integration with respect to \( p^2 \) and \( p^{2'} \) we obtain

\[
Prob(a, b) = N_T^2 \int \exp \left\{ \frac{i(p^2)^2 - (p^{2'})^2}{2\hbar \beta m} \right\} \cdot \exp \left\{ -i\frac{(k^1)^2 - (k^{1'})^2}{2hm} \left( T - \frac{3}{2\beta} \right) \right\} \exp \left\{ i\frac{x^2(k^1 - k^{1'})}{h} \right\} \cdot \chi_{a,b}(x^2) a(k^1) a^*(k^{1'}) dk^1 dk^{1'} dp^2 dp^{2'} dx^2 \approx \int \exp \left\{ -i\frac{(k^1)^2 - (k^{1'})^2}{2hm} \left( T - \frac{1}{2\beta} \right) \right\} \cdot \exp \left\{ i\frac{x^2(k^1 - k^{1'})}{h} \right\} \cdot \chi_{a,b}(x^2) a(k^1) a^*(k^{1'}) dk^1 dk^{1'} dx^2.
\]

Now the formula is quite close to the QM formula. If we change \( T - (2\beta)^{-1} \rightarrow T \) then we obtain exactly the QM evolution.

In fact, in QM we have

\[
\psi_2(x^2) = \int \exp \left\{ -\frac{i}{\hbar} \frac{(k^1)^2}{2m} T \right\} \exp \left\{ i\frac{k^1 x^2}{\hbar} \right\} a(k^1) dk^1
\]
and this implies the QM probability formula

$$\int |\psi|^2 \chi_{a,b} dx^2 = \int \exp \left\{ -i \frac{(k^1)^2 - (k^{1'})^2}{2\hbar m} T \right\} \cdot \exp \left\{ i \frac{(k^1 - k^{1'})x^2}{\hbar} \right\} a(k') a^\dagger(k^{1'}) dk^1 dk^{1'}.$$
7. The short-time approximation and the concentration effect

In this section we shall consider in details what evolution can happen during the short-time intervals satisfying

$$T \ll 1/\beta.$$ 

In these intervals the evolution is, in a certain sense, close to the DetQM evolution.

We shall proceed in the following steps:

(A) the short-time approximation to the propagator,
(B) the concept of the concentrated state,
(C) the preparation of concentrated states,
(D) the short-time evolution of concentrated states,
(E) some exact calculations.

In the preceding section we have found that after a (free) long-time evolution, the dependence of the wave function on the particle momentum $p$ is of the type

$$\psi(p, \cdot) \approx \exp \left\{ \frac{i}{\hbar} \left\{ \frac{1}{2m} p^2 \right\} \right\} \ldots,$$

i.e. all particle momenta $p$ contribute almost equally. We have called these states relaxed states and their behavior is closed to the QM-behavior.

The concentrated states are characterized as states where particle momenta are localized, i.e. close to a certain value $p_0$. These states are very far from the QM-states and at these states the main differences between QM and subquantum models are presented – and this is the main theme of the rest of the paper.

A typical concentrated state is of the form

$$\psi(p, \cdot) \approx \exp \left\{ -\frac{1}{2} \left( \frac{p - p_0}{\Delta p} \right)^2 \right\} \ldots$$

The first question is if it is possible to prepare concentrated states. It is possible by letting particles to pass through iterated slits.
The second question is to describe the non-QM behavior of the short-time evolution of concentrated states. The main feature is that the dispersion of the wave packets is slower than the QM-dispersion and that in short-time evolution the original particle momentum \( p_0 \) is partially remembered. After longer time, the memory of \( p_0 \) is almost forgotten and the relaxation happens.

(A) The short-time propagator is obtained by using the approximation for \( \beta T \ll 1 \).

\[
\begin{align*}
\tanh \beta T &\approx \beta T - \frac{1}{3}(\beta T)^3 \approx \beta T \\
\sinh \beta T &\approx \beta T + \frac{1}{6}(\beta T)^3 \approx \beta T \\
\cosh \beta T &\approx 1 + \frac{1}{2}(\beta T)^2 \\
1 - (\beta T/2)^{-1}\tanh(\beta T/2) &\approx \frac{1}{12}(\beta T)^2 \\
\tanh(\beta T/2) &\approx \frac{1}{2}\beta T - \frac{1}{24}(\beta T)^3 \approx \frac{1}{2}\beta T.
\end{align*}
\]

We obtain the short-time approximation

\[
(7.1) \quad \bar{S} = \frac{1}{2\beta m} \frac{(p^2 - p_1^2)}{\beta T} + \frac{\beta m}{2} \frac{12}{(\beta T)^3} \left[ x^2 - x^1 - \frac{1}{m} \frac{p^1 + p^2}{2} \right]^2.
\]

Then \( K = \exp\{ (i/\hbar)\bar{S} \} \).

We see that for short times, \( \beta T \ll 1 \), the dispersion of the wave packets, determined mainly by the second term, is similar to the QM but for \( T(\beta T)^2 \) instead of \( T \).

The first term is analogical to the \( x \)-propagator in QM, so that the short-time dispersion of the \( p \)-packet is similar to the QM-dispersion of the wave-packets. In the initial period, where particle momenta \( p^1 \) and \( p^2 \) are localized around \( p_0 \), the \( x \)-dispersion is very slow because of the term \( (\beta T)^3 \). Then the evolved distribution in \( x_2 \) is localized around

\[
x_0^1 + \frac{1}{m}p_0^1 T,
\]

where \( x_0^1 \) is the center of the initial wave packet. All these observations will be made more precise in what follows.
(B) The definition of the concentrated state is based on the idea of the localization such that there is a term

\[ \tilde{S} \approx -\frac{1}{2} \frac{(p - p_0)^2}{\Delta p^2} + \ldots, \quad \psi \approx \exp \left\{ \frac{i}{\hbar} \tilde{S} \right\} \]

in the wave function.

Thus we said that the wave function \( \psi(p, x) \) describes the concentrated state with the center \( p_0 \) and the dispersion \( \Delta p \) if there exists a constant \( c \) such that

\[ |\psi(p, x)| \leq c \exp \left\{ -\frac{1}{2} \frac{(p - p_0)^2}{\Delta p^2} \right\}. \tag{7.2} \]

If the wave function can be written in the form

\[
\psi(p, x) = e^{iS_0(p,x)} e^{-S_1(p,x)}, \quad S_0(p, x), S_1(p, x) \in \mathbb{R},
\]

then this condition means that

\[ c \cdot \frac{(p - p_0)^2}{2\Delta p^2} \leq S_1(p, x), \quad \forall p, x \in \mathbb{R}. \tag{7.3} \]

Let us note that the concentrated state has nothing in common with QM-states with the localized QM-momentum. The QM-state with the localized QM-momentum depends only on \( x \) and its SubQM-approximation is the state with the relaxed dependence on the particle momentum \( p \).

One can also consider the simultaneous localization in \( x \), so that the completely concentrated state satisfies

\[ |\psi(p, x)| \leq c \exp \left\{ -\frac{1}{2} \frac{(p - p_0)^2}{\Delta p^2} \right\} \exp \left\{ -\frac{1}{2} \frac{(x - x_0)^2}{\Delta x^2} \right\}. \tag{7.4} \]

We look for concentrated states with

\[ \Delta p \cdot \Delta x \ll \hbar / 2 \tag{7.5} \]

which, in a sense, break the Heisenberg principle.
We shall characterize such states by the "degree" of concentration $\kappa$ defined as

$$\kappa := \frac{2}{\hbar} \cdot \Delta x \cdot \Delta p.$$ 

Concentration states are states satisfying

$$\kappa < 1.$$ 

In this case we have still the Heisenberg uncertainty relation

$$\Delta p^{(QM)} \cdot \Delta x \geq \hbar / 2,$$

because this is a mathematical property of the Fourier transform. This relation does not depend on any physics, only on the definition of the QM-momentum. If we define the QM-momentum through Fourier transform, then the Heisenberg relation expresses the mathematical property of this object. The physics lies in defining the QM-momentum in this way.

With respect to the particle momentum $p$, the situation with Heisenberg relation is the following. In the relaxed state the particle momentum is completely dispersed:

$$\Delta p = \infty,$$

so that Heisenberg relation is obviously satisfied. In the concentrated state the localization in particle momentum can be small and the (analog of) Heisenberg relation can be not satisfied

$$\kappa = \frac{2}{\hbar} \Delta x \Delta p \ll 1.$$ 

This does not mean breakdown of Heisenberg relation, because it is a mathematical fact, but concerning $\Delta p^{(QM)}$.

On the other hand, physical consequences of the short-time evolution of concentrated states contradict Heisenberg principle. The evolution proceeds like an evolution with almost defined space and the (particle) momentum localization, i.e. close to the DetQM-evolution.

This type of behavior needs two assumptions to be satisfied: the starting state must be a concentrated state and the time interval
of the evolution must be short. Under these conditions Heisenberg principle is broken in subquantum models.

(C) The preparation of the concentrated states proceeds by applying (passing through) the iterated slits.

Passing through the slit \((a, b)\) of time \(t\) is represented by the projection
\[
\psi(p, x; t) \mapsto \psi'(p, x; t) = \psi(p, x; t) \chi_{a,b}(x).
\]

In the calculation below we approximate the projection by the multiplication by a "Gaussian" slit \(\tilde{\chi}_{a,b}\)

\[
\chi_{a,b}(x) \approx \tilde{\chi}(x_0, \Delta x; x) = c \exp \left\{-\frac{1}{2} \frac{(x - x_0)^2}{\Delta x^2} \right\},
\]

where the center \(x_0\) and the dispersion \(\Delta x\) are defined by
\[
x_0 = \frac{1}{2}(a + b), \quad \Delta x = \frac{1}{2}(b - a)
\]

and the normalization constant \(c\) is defined by the condition
\[
\int \tilde{\chi}(x_0, \Delta x; \cdot) dx = \int \chi_{a,b} dx = b - a = 2\Delta x.
\]

We shall denote the "passing the slit" projection by
\[
\psi(p, x; t) \mapsto \psi'(p, x; t) = \psi(p, x; t) \tilde{\chi}(x_0, \Delta x; x).
\]

The "iterated slit" process consists in the following

\[
(7.7) \quad \psi_1(p^1, x^1; 0) \mapsto \psi'_1(p^1, x^1; 0) = \psi_1(p^1, x^1; 0) \tilde{\chi}(x_0^1, \Delta x^1; x^1) \mapsto \\
\mapsto \psi'_2(p^2, x^2; T) = \int K_T(p^1, x^1; p^2, x^2) \psi'_1(p^1, x^1; 0) dp^1 dx^1 \mapsto \\
\mapsto \psi''_2(p^2, x^2; T) = \psi'_2(p^2, x^2; T) \tilde{\chi}(x_0^2, \Delta x^2; x^2).
\]

This means that we apply the slit \(\tilde{\chi}(x_0^1, \Delta x^1; \cdot)\) at time \(t = 0\) and then the second slit \(\tilde{\chi}(x_0^2, \Delta x^2; \cdot)\) at time \(t = T\). We shall show that if \(\Delta x^1\) and \(\Delta x^2\) are sufficiently small and if the evolution time \(T\)
is sufficiently small, $\beta T \ll 1$, then the resulting wave function $\psi_2''$ describes the concentrated state.

This will be done in two steps. At the first step we shall show that the short-time evolution $\psi_2'\left(\cdot, \cdot; T\right)$ contains the term bounding together $x^2$ and $p^2$,

$$\left|\psi_2(p^2, x^2; T)\right| \leq \text{const.} \exp\left\{ -\frac{(x^2 - p^2T/m)^2}{\text{const} \left(\Delta x^1\right)^2}\right\}.$$  

This comes from the fact that $p^1$ cannot differ much from $p^2$ and then the dispersion of $x^2 - p^2T/m$ is also small. We shall show this below.

The second (more simple) step combines together the term bounding $x^2$ and $p^2$ and the second slit term bounding $x^2$ to $x_0^2$. This creates simply the term bounding $p^2T/m$ to $x_0^2 - x_0^1$. This creates the completely concentrated state. The corresponding calculations will be described below.

In the case when $\delta = \Delta x_1 = \Delta x_2$ (both slits are of the same dimension) we obtain

$$\kappa^2 = \frac{4\hbar^2}{\delta^2} \Delta p_2^2 \approx 8\frac{m^2\delta^4}{T^2\hbar^2} + \frac{4}{9}(\beta T)^4.$$  

(D) The short-time evolution of the concentrated state.

Let us consider the concentrated state $\psi_1(p, x)$ satisfying the inequality (7.4) and let us assume that the evolution time $T$ satisfies the relation $\beta T \ll 1$. Then the evolved state

$$\psi_2(p^2, x^2; T) = \int K_T(p^1, x^1; p^2, x^2)\psi_1(p^1, x^1)dp^1dx^1$$

satisfies the relation

$$|\psi_2| \leq c. \exp\left\{ -\frac{1}{2} \frac{1}{\Delta x_2^2} \left(x^2 - x_0^1 - \frac{p_0}{m}T\right)^2\right\}$$

where the dispersion $\Delta x_2^2$ at the time $T$ is given by (the calculation can be found below)

$$\Delta x_2^2 = \Delta x_1^2 + \Delta p_1^2 \frac{T^2}{m^2} + \frac{1}{9}(\beta T)^6 \frac{\hbar^2}{\Delta p_1^2(\beta T)^2 + \Delta x_1^2 m^2 \beta^2}.$$
This has to be confronted with the QM-evolution (assuming the relaxed distribution of $\psi_1$ in $p^1$, or equivalently, that $\beta \to \infty$) which gives the standard result

$$\Delta x_2^2 (QM) = \Delta x_1^2 + \frac{\hbar^2}{4\Delta x_1^2} \cdot \frac{4T^2}{m^2}.$$  

This QM-result is completely consistent with Heisenberg principle, by which

$$\Delta p_1^2 (QM) \approx \frac{\hbar^2}{4\Delta x_1^2}.$$  

We see that in the both cases (7.10) and (7.11), the velocity of the growth of $\Delta x_2^2$ is proportional to $\Delta p_1^2$, but this quantity can be arbitrarily small in the subquantum models, while in the QM-case, the quantity $\Delta p_1^2 (QM)$ is strictly bounded from below by the Heisenberg principle. We have

$$\Delta p_1^2 (QM) \geq \frac{\hbar^2}{4\Delta x_1^2}$$  

in QM-case but it can happen

$$\Delta p_1^2 \ll \frac{\hbar^2}{4\Delta x_1^2}$$  

in the subquantum case. We shall call this situation the concentration effect in subquantum models. As a result we obtain that

$$(7.12) \quad \Delta x_2^2 \ll \Delta x_2^2 (QM)$$  

and this is the quantitative consequence of the concentration effect.

The corresponding gedanken experiment showing the observational difference between QM and SubQM$_{RF}$ is the following. We let pass the beam of particles through iterated slits (making, e.g. $x_0 = 0$, $p_0 = 0$) and we shall observe the particle on the screen behind the slits. Assuming that the slits are sufficiently narrow and that time intervals of the evolution between slits and between the last slit and the screen is sufficiently small, we obtain as an observational fact inequality
(7.12) expressing the clear difference between subquantum model and QM. More details on this type of experiments can be found in the last section.

It can be seen that this type of the effect is completely necessary in all subquantum models. The behavior on short time intervals is closer to the deterministic model DetQM and this implies both that the concentration states are created by passing through iterated slits and that the short-time evolution of the concentrated state will contradict Heisenberg principle. Thus this type of gedanken experiment is possible in all subquantum models. The order of quantities used in these gedanken experiments depends on $\beta$ and the order of $\beta$ is hypothetical.

(E) In this last part we shall describe detailed calculations giving formulas used above.

(i) Calculations will be done by using the following formulas. Let

$$X, E_k, B_0 \in \mathbb{R}^2, k = 1, \ldots, K,$$

be two-dimensional vectors,

$$a_k, b_k, a_0 \in \mathbb{C}, k = 1, \ldots, K,$$

be numbers satisfying $\text{Im}(a_k) \geq 0$. Let $\otimes$ denote the tensor product,

$$\left( \begin{array}{c} a \\ b \end{array} \right) \otimes \left( \begin{array}{c} c \\ d \end{array} \right) = \left( \begin{array}{cc} ac & ad \\ bc & bd \end{array} \right)$$

and let $\perp$ denote the 90-degree rotation,

$$\left( \begin{array}{c} a \\ b \end{array} \right) \perp = \left( \begin{array}{c} b \\ -a \end{array} \right).$$

Then we have

$$\int \exp \left\{ \frac{i}{2\hbar} \sum_{k=1}^{K} a_k \left( X^\top e_k + b_k \right)^2 + \frac{i}{\hbar} a_0 X^\top B_0 \right\} d^2 X =$$

$$= \text{const.} \exp \left\{ \frac{i}{2\hbar} \left\{ -\Delta^{-1} Y^\top \left( \sum_k a_k E_k^\perp \otimes E_k^\perp \right) Y + \sum_k a_k b_k \right\} \right\},$$
where
\[ Y = \sum_{k=1}^{K} a_k b_k E_k + a_0 B_0, \quad \Delta = \det \left( \sum_k a_k E_k \otimes E_k^\dagger \right). \]

In the one-dimensional situation we have the special case
\[
\int \exp \left\{ \frac{i}{2\hbar} \sum a_k (x + b_k)^2 + \frac{i}{\hbar} b_0 x \right\} d^1 x = \]
\[ = \text{const.} \exp \frac{i}{2\hbar} \left\{ - \left( \sum_k a_k \right)^{-1} \left( \sum_k a_k b_k + b_0 \right)^2 + \sum_k a_k b_k^2 \right\}. \]

(ii) We shall show that \( \psi_2(p^2; x^2; T) \) contains the concentration of \( (x^2 - Tm^{-1}p^2)^2 \). We shall use the short-time propagator described above and assume that \( \beta T \ll 1 \). Let the dimensionless quantity \( \sigma_1 \),
\[ \frac{\beta m}{\hbar} \sigma_1 = \frac{1}{\Delta x_1^2}, \]
describe the extension of the first and second slit. We have to calculate the following Gaussian integral
\[
\int \exp \left\{ \frac{i}{2\hbar} \left( \beta m \sigma_1 x_1^2 + \frac{1}{\beta m \beta T} (p_1 - p_2)^2 + \right. \right. \]
\[ \left. \left. + \beta m \omega_T \left( x_1 - x_2 + \frac{T}{2m} (p_1 + p_2) \right)^2 \right\} dp_1 dx_1, \]
where
\[ \omega_T = \frac{12}{(\beta T)^3}, \]
and we use the simpler notation with lower indices, \( x^1 \to x_1, p^1 \to p_1 \) etc. At first we shall change the notation with lower indices, \( x^1 \to x_1, p^1 \to p_1 \) etc. At first we shall change the third term using the formula for \( \omega_T \) to the form
\[ \frac{3}{\beta m \beta T} \left( \frac{2m}{T} x_1 - \frac{2m}{T} x_2 + p_1 + p_2 \right)^2 \]
and using the change of variables
\[ p_1 \to p_1 - p_2 + \frac{2m}{T} x_2, \]
we arrive at the integral
\[
\int \exp \left\{ \frac{i}{2\hbar} \left( \beta m \sigma_1 x_1^2 + \frac{1}{\beta m \beta T} \left( p_1 - 2p_2 + \frac{2m}{T} x_2 \right)^2 + \frac{3\beta m}{\beta T} \left( \frac{2m}{T} x_1 + p_1 \right)^2 \right) \right\} dp_1 dx_1.
\]

Now we shall use the formula from the preceding step (i) with \( K = 3 \). We have
\[
a_1 = \beta m \sigma_1, \quad a_2 = \frac{1}{\beta m \beta T}, \quad a_3 = \frac{3}{\beta m \beta T},
\]
\[
b_1 = 0, \quad b_2 = 2 \left( -p_2 + \frac{m}{T} x_2 \right), \quad b_3 = 0,
\]
\[
E_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 \\ 2m/T \end{pmatrix}.
\]

Then we have
\[
Y = \sum a_k b_k E_k = \frac{2}{(\beta T)^2} \left( x_2 - \frac{T p_2}{m} \right),
\]
\[
\sum a_k b_k^2 = \frac{4\beta m}{(\beta T)^3} \left( x_2 - \frac{T}{m} p_2 \right)^2,
\]
\[
\sum a_k E_k \otimes E_k^\top = \frac{1}{\beta m (\beta T)^3} \begin{pmatrix} 4(\beta T)^2 & 6\beta m \beta T \\ 6\beta m \beta T & 12(\beta m)^2 + i\sigma_1 (\beta m)^2 (\beta m)^3 \end{pmatrix},
\]
\[
\Delta^{-1} = \frac{3}{4(\beta m)^2 (\beta T)^2 (9 + \sigma_1^2 (\beta T)^6)} - \frac{i\sigma_1 \beta T}{4(\beta m)^2 (9 + \sigma_1^2 (\beta T)^6)}.
\]

For the inverse matrix we obtain the formula
\[
\left( \sum a_k E_k \otimes E_k^\top \right)^{-1} = \frac{6T}{4m(9 + \sigma_1^2 (\beta T)^6)} \begin{pmatrix} \sigma_1 (\beta T)^4 & 9(\beta m)^2 \\ -6\beta m \beta T & 4(\beta T)^2 \end{pmatrix} - \frac{i\sigma_1 (\beta T)^4}{4\beta m(9 + \sigma_1^2 (\beta T)^6)} \begin{pmatrix} 9(\beta m)^2 & -6\beta m \beta T \\ -6\beta m \beta T & 4(\beta T)^2 \end{pmatrix}.
\]
To obtain the resulting concentration we need only the imaginary part of this matrix:

$$\text{Im} \left( \sum a_k E_k \otimes E_k^\top \right)^{-1} = \frac{-i \sigma_1 (\beta T)^2}{4 \beta m (9 + \sigma_1^2 (\beta T)^6)} \left( \begin{array}{cc} 3 \beta m & -2 \beta T \\ -2 \beta T & 3 \beta m \end{array} \right) \otimes (3 \beta m, -2 \beta T).$$

Then we obtain

$$Y^\top \text{Im} \left( \sum a_k E_k \otimes E_k^\top \right)^{-1} Y = \frac{-i \sigma_1 \beta m}{1 + \frac{1}{9} \sigma_1^2 (\beta T)^6} \left( x_2 - \frac{T}{m} p_2 \right)^2$$

and the concentration term

$$\exp \left\{ -\frac{1}{2 \hbar} \cdot \frac{\sigma_1 \beta m}{1 + \frac{1}{9} \sigma_1^2 (\beta T)^6} \left( x_2 - \frac{T}{m} p_2 \right)^2 \right\}.$$

Applying the term $-i/2\hbar$ and the expression of $\sigma_1$ in terms of $\Delta x_1^2$ we obtain the concentration term

$$\exp \left\{ -\frac{1}{2} \frac{1}{\Delta x_2^2} \left( x_2 - \frac{T}{m} p_2 \right)^2 \right\}$$

with

$$\Delta x_2^2 = \Delta x_1^2 + \frac{\hbar^2}{9 \Delta x_1^2 (\beta m)^2 (\beta T)^6}.$$

For small $\beta T \ll 1$ we obtain the concentration of $(x_2 - T p_2/m)^2$ of the order of $\Delta x_1^2$.

(iii) At this moment, applying the second slit at the time $T$ to the wave function $\psi_2$, we obtain

$$|\psi'_2(p_2, x_2; T)| = |\psi_2(p_2, x_2; T)| \cdot \text{const.} \exp \left\{ -\frac{1}{2} \frac{1}{\Delta x_2^2} x_2^2 \right\}.$$

From part (ii) we know that

$$|\psi'_2(p_2, x_2; T)| \leq \text{const.} \exp -\frac{1}{2} \left[ \frac{1}{\Delta x_2^2} \left( x_2 - \frac{T}{m} p_2 \right)^2 + \frac{1}{\Delta x_2^2} x_2^2 \right].$$
with
\[ \tilde{\Delta}x_2^2 = \Delta x_1^2 + \frac{\hbar^2(\beta T)^6}{9\Delta x_1^2(\beta m)^2}. \]

From the inequality
\[ a(x_2 - \xi)^2 + bx_2^2 \geq \frac{ab}{a + b} \xi^2 \]
true for \( a + b > 0 \), we obtain
\[ \frac{1}{\tilde{\Delta}x_2^2} \left( x_2 - \frac{T}{m} p_2 \right)^2 + \frac{1}{\Delta x_2^2} x_2^2 \geq \frac{1}{\Delta p_2^2 p_2^2} \]
where
\[ \Delta p_2^2 = \left( \tilde{\Delta}x_2^2 + \Delta x_2^2 \right) \frac{m^2}{T^2}. \]

Using the formula for \( \tilde{\Delta}x_2^2 \) we arrive at
\[ \Delta p_2^2 \frac{T^2}{m^2} = \Delta x_1^2 + \Delta x_2^2 + \frac{\hbar^2(\beta T)^6}{9\Delta x_1^2(\beta m)^2} \]
and
\[ |\psi_2(p_2; x_2; T)| \leq \text{const.} \exp \left\{ -\frac{1}{2} \left\{ \frac{1}{\Delta x_2^2} x_2^2 + \frac{1}{\Delta p_2^2 p_2^2} \right\} \right\}. \]

For \( \Delta x_1 = \Delta x_2 \) we obtain the degree of concentration
\[ \kappa^2 = \frac{4}{\hbar^2} \Delta x_2^2 \Delta p_2^2 = 8 \frac{m^2}{T^2 \hbar^2} \cdot \Delta x_2^4 + \frac{4}{9}(\beta T)^4. \]

In the "equilibrated" situation, where first and second terms are of the same order, we have
\[ \Delta x_1 = \Delta x_1 \approx \frac{1}{2} \beta T \left( \frac{T \hbar}{m} \right)^{1/2} \]
and
\[ \kappa \approx (\beta T)^2. \]
In this way the concentrated states can be prepared.
(iv) Let us look for a concentration state which is centered around \( x_{20} \) and \( p_{20} \), so that the inequality

\[
|\psi'(p_2, x_2; T)| \leq \text{const.} \exp \left( -\frac{1}{2} \left\{ \frac{(x_2 - x_{20})^2}{\Delta x_2^2} + \frac{(p_2 - p_{20})^2}{\Delta p_2^2} \right\} \right)
\]

is fulfilled.

Of course, we suggest that the first slit has to be centered around \( x_{10} := x_{20} - \frac{p_{20}}{m} \).

Then the relevant integral is the following

\[
\int \exp \left\{ -\frac{i}{2\hbar} \left\{ \frac{1}{2} \frac{(x_1 - x_{10})^2}{\Delta x_1^2} + \frac{1}{2} \frac{(x_2 - x_{20})^2}{\Delta x_2^2} + \frac{1}{\beta m \beta T} (p_1 - p_2)^2 + \beta m \omega T \left( T \frac{p_1 + p_2}{2m} \right)^2 \right\} \right\} dp_1 dx_1.
\]

We shall make the substitution

\[
x_1 = \overline{x}_1 + x_{20} - \frac{p_{20}}{m} T, \quad x_2 = \overline{x}_2 + x_{20},
\]

\[
p_1 = \overline{p}_1 + p_{20}, \quad p_2 = \overline{p}_2 + p_{20}
\]

and obtain after the change \( dp_1 dx_1 = d\overline{p}_1 d\overline{x}_1 \)

\[
\int \exp \left\{ -\frac{i}{2\hbar} \left\{ \frac{1}{\Delta \overline{x}_1} + \frac{1}{\Delta \overline{x}_2} + \frac{1}{\beta m \beta T} (\overline{p}_1 - \overline{p}_2)^2 + \beta m \omega T \left( T \frac{\overline{p}_1 + \overline{p}_2}{2m} \right)^2 \right\} \right\} d\overline{p}_1 d\overline{x}_1.
\]

By the result of (iii) we obtain

\[
|\psi'(\overline{p}_2, \overline{x}_2; T)| \leq \text{const.} \exp \left( -\frac{1}{2} \left\{ \frac{\overline{x}_2^2}{\Delta \overline{x}_2^2} + \frac{\overline{p}_2^2}{\Delta \overline{p}_2^2} \right\} \right)
\]

and after the change \( \overline{x}_2 \rightarrow x_2 - x_{20}, \overline{p}_2 \rightarrow p_2 - p_{20} \) we arrive at the formula we were looking for.
(v) Now we shall start with the concentrated state

$$|\psi_1(p_1, x_1; 0)| \leq \text{const.} \exp - \frac{1}{2} \left\{ \frac{(x_1 - x_{10})^2}{\Delta x_1^2} + \frac{(p_1 - p_{10})^2}{\Delta p_1^2} \right\}$$

and it will be evaluated to

$$\psi_2(p_2, x_2; T) \leq \int K_T(p_1, x_1; p_2, x_2)\psi_1(p_1, x_1; 0)dp_1dx_1.$$  

We are interested mainly in the localization of $\psi_2$ in $x_2$ in the form of the interpretation from Section 5.

$$\left| \int \psi_2(p_2, x_2; T)dp_2 \right| \leq \text{const.} \exp \left\{ -\frac{1}{2} \frac{(x_2 - x_{20})^2}{\Delta x_2^2} \right\}.$$  

It is sufficient to make an integration on $dp_2$ inside the propagator and to obtain the reduced propagator

$$\tilde{K}_T(p_1, x_1; x_2) := \int K_T(p_1, x_1; p_2, x_2)dp.$$  

We shall obtain

$$\tilde{K}_T(p_1, x_1; x_2) = \exp \left\{ \frac{i}{2\hbar} \beta m \tilde{\omega}_T \left( x_2 - x_1 - \frac{T}{m}p_1 \right)^2 \right\},$$

where

$$\tilde{\omega}_T = \frac{3}{(\beta T)^3}.$$  

We have to calculate the integral

$$\int \exp \frac{i}{2\hbar} \left\{ \frac{1}{\beta m \beta T} (p_2 - p_1)^2 + \beta m \frac{12}{(\beta T)^3} \left( x_2 - x_1 - \frac{T}{2m} (p_1 + p_2) \right)^2 \right\}dp_2.$$  

The second term may be rewritten as

$$\frac{3}{\beta m \beta T} \left( p_2 + p_1 - (x_2 - x_1) \frac{2m}{T} \right)^2.$$
Using the last formula from (i) for $K = 2$ we obtain the formula for $\tilde{K}_T$.

(vi) In the calculation of the short-time evolution of the concentrated state we shall use the reduced propagator $\tilde{K}_T$. We have to calculate the following integral (where $\tilde{\omega}_T = 3(\beta T)^{-3}$)

$$
\int \exp \frac{i}{\hbar} \left\{ i \beta m \sigma_1 (x_1 - x_{10})^2 + i \frac{\rho_1}{\beta m} (p_1 - p_{10})^2 + \beta m \tilde{\omega}_T \left( x_1 - x_2 + \frac{T}{m} p_1 \right)^2 \right\} \, dx_1 dp_1.
$$

We make substitutions

\[
\begin{align*}
p_1 & \rightarrow p_1 + p_{10}, \\
x_1 & \rightarrow x_1 + x_{10}, \\
\bar{x}_2 & = x_2 - x_{10} - \frac{T}{m} p_{10}
\end{align*}
\]

and then we obtain the integral

$$
\int \exp \frac{i}{\hbar} \left\{ i \beta m \sigma_1 x_1^2 + i \frac{\rho_1}{\beta m} p_1^2 + \beta m \tilde{\omega}_T \left( x_1 + \frac{T}{m} p_1 - \bar{x}_2 \right)^2 \right\} \, dx_1 dp_1.
$$

Now we shall apply the formula from (i) with $K = 3$ and

\[
\begin{align*}
a_1 &= i \beta m \sigma_1, \\
a_2 &= i \frac{\rho_1}{\beta m}, \\
a_3 &= \beta m \tilde{\omega}_T, \\
b_1 &= b_2 = 0, \\
b_3 &= -\bar{x}_2, \\
E_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
E_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
E_3 &= \begin{pmatrix} T/m \\ 1 \end{pmatrix}.
\end{align*}
\]

Then we obtain

\[
A^{-1} := \left( \sum a_k E_k \otimes E_k^\top \right)^{-1}
= \Delta^{-1} \frac{1}{\beta m} \left[ \tilde{\omega}_T \left( \begin{pmatrix} \beta m & -\beta T \end{pmatrix} \otimes (\beta m - \beta T) + i \begin{pmatrix} \sigma_1 \beta m^2 & 0 \\ 0 & p_1 \end{pmatrix} \right) \right],
\]
and where
\[ \Delta^{-1} = -\rho_1 \sigma_1 - \rho_1 \sigma_1 \beta^2 T^2 + \rho_1 \overline{\omega_T} (\sigma_1 \beta^2 T^2 + \rho_1) \overline{\omega_T} \left( \sigma_1 \beta^2 T^2 + \rho_1 \right)^2. \]

Using
\[ Y = \sum a_k b_k E_k = -\overline{\omega_T} x_2 \left( \frac{\beta T}{\beta m} \right) \]
we obtain
\[ Y^\top A^{-1} Y = \Delta^{-1} i \beta m \overline{\omega_T} (\sigma_1 \beta^2 T^2 + \rho_1) \overline{x_2}^2. \]

We are interested only in the localization term so that only the imaginary part contributes (the term \( \sum a_k b_k^2 \) contributes to the real part),
\[ \text{Im}(Y^\top A^{-1} Y) = -x_2^2 \beta m \overline{\omega_T}^2 (\sigma_1 \beta^2 T^2 + \rho_1) \sigma_1 \rho_1 \overline{x_2}^2. \]

From the equality
\[ \exp \left\{ -\frac{1}{2} \Delta x_2^2 \right\} = \exp \left\{ \frac{1}{2\hbar} \text{Im}(Y^\top A^{-1} Y) \right\} \]
and from equations
\[ \frac{1}{\sigma_1} = \frac{\beta m}{\hbar} \Delta x_1^2, \]
\[ \frac{1}{\rho_1} = \frac{1}{\hbar \beta m} \Delta p_1^2 \]
we obtain the final formula for the dispersion \( \Delta x_2^2 \) of the wave packet at the time \( T \)
\[ \Delta x_2^2 = \Delta x_1^2 + \frac{T^2}{m^2} \Delta p_1^2 + \frac{1}{9} \frac{T^2}{m^2} (\beta T)^4 \frac{h^2}{\Delta x_1^2} + \frac{T^2}{m^2} \Delta p_1^2. \]

(vii) The analogical formula for the dispersion \( \Delta x_2^2 \) in QM is standard. The integral to be calculated is
\[ \int \exp \left\{ \frac{i}{\hbar} \left( \frac{\hbar (x_1 - x_{10})^2}{\Delta x_1^2} + \frac{m}{T} (x_1 - x_2)^2 \right) \right\} dx_1. \]
Making the substitution $x_1 \rightarrow x_1 + x_{10}$, $\overline{x}_{2} = x_2 - x_{10}$ we obtain the integral

$$\int \exp \left\{ \frac{i}{2\hbar} \left\{ \frac{i\hbar}{\Delta x_1^2} x_1^2 + \frac{m}{T} (x_1 - \overline{x}_{2})^2 \right\} \right\} dx_1.$$ 

Using the last formula from (i) we obtain that the real part of the resulting Gaussian is

$$\exp \left\{ -\frac{1}{2} \frac{x_2^2}{\Delta x_2^2} \right\}$$

where

$$\Delta x_2^2 = \Delta x_1^2 + \frac{\hbar^2}{\Delta x_1^2} \cdot \frac{T^2}{m^2}.$$ 

If we introduce the conjugated quantity

$$\Delta p_1^{(QM)} = \frac{\hbar^2}{4\Delta x_1^2}$$

then we have an analogical formula

$$\Delta x_2^2 = \Delta x_1^2 + \frac{4T^2}{m^2} \Delta p_1^{(QM)}.$$ 

This quantity satisfies, of course, the Heisenberg relation.
8. **The Correlated Random Force Model SubQM\(_{\text{CRF}}\)** and the Correlation Effect

The starting point of subquantum model was the idea of the SLO-vacuum – the medium composed from space-like objects. The random force served as a model of the interaction of the system with such a medium. We have supposed that the random forces \( F(t, \vec{x}_1) \) and \( F(t, \vec{x}_2), \vec{x}_1 \neq \vec{x}_2 \), representing interaction with the SLO-vacuum, are stochastically independent.

The opposite hypothesis, that these forces are not completely independent, is also possible. Let us consider the model of the space-like objects with the zero (space-like) velocity

\[
t = f^\alpha(\vec{x}) \equiv t_0^\alpha, \quad \alpha \in \mathbb{Z}.
\]

One can then think on idea that the random force is the same at different places in the space and that it depends only on the time

\[
F(t, \vec{x}) \equiv F_0(t)
\]

and then the forces \( F_i(t) = F(t, \vec{x}_i) \) and \( F_j(t) = F(t, \vec{x}_j), \ i \neq j, \) are equal.

The completely opposite assumption that the random forces \( F_i(t) \) and \( F_j(t), \forall i, j, \) are the same is too strong. We shall suppose that the random forces will contain the part \( G_0 \) which is the same for all particles and the part \( G_i \) which is different for different particles. The hypotheses will be the following; they substitute hypotheses (i)-(iv) from Section 3:

(i) There exists a random force \( F_i(t) \) acting on the \( i \)-th degree of freedom, \( i = 1, \ldots, n. \)

(ii) Forces \( F_i(t) \) can be expressed as

\[
F_i(t) = G_0(t) + G_i(t), \quad i = 1, \ldots, n,
\]

where forces \( G_0(t), G_1(t), \ldots, G_n(t) \) are statistically independent.

(iii) There is an amplitude distribution of the random forces given by

\[
\mathcal{A}_{t^1, t^2}[G_0] = \exp \left\{ \frac{i}{\hbar} \frac{a_0}{2} \int_{t^1}^{t^2} G_0^2(t) dt \right\} \prod_t dG_0(t),
\]

\[
\mathcal{A}_{t^1, t^2}[G_j] = \exp \left\{ \frac{i}{\hbar} \frac{a_1}{2} \int_{t^1}^{t^2} G_j^2(t) dt \right\} \prod_t dG_j(t), \quad j = 1, \ldots, n.
\]
(iv) The system with \( n \) degrees of freedom is described by DetQM with a given random force. Forces \( F_i(t) \) and \( F_j(t) \), \( i \neq j \), are correlated, because they both contain the common part \( G_0(t) \), while other parts \( G_i(t) \) and \( G_j(t) \) are independent. We shall proceed in the following steps.

(A) The Feynman integral,
(B) the propagator,
(C) definition and preparation of the correlated states,
(D) evolution of the correlated states.

(A) The Feynman integral for the transition amplitude in the SubQM\(_{\text{CRF}}\) model is

\[
A = \int_{(BC)} \exp \left\{ \frac{i}{\hbar} \mathcal{A}_{t_1,t_2}[x_1, \ldots, x_n] \right\} \prod_{i,t} \delta(m_i \ddot{x}_i(t) - G_0(t) - G_i(t)) \cdot \exp \left\{ \int_{t_1}^{t_2} \left( a_0 G_0^2(t) + a_1 \sum_{i=1}^{n} G_i^2(t) \right) dt \right\} \cdot \prod_t dG_0(t) \prod_{t,i} dx_i(t).
\]

The boundary conditions are standard: \( x_i(x^s) = x_i^s \), \( m_i \ddot{x}_i(t^s) = p_i^s \), \( s = 1, 2, i = 1, \ldots, n \). For simplicity we shall suppose that

\[
m_i = m \quad (\forall i),
\]

\[
A \left[ x_i \right] = \int_{t_1}^{t_2} \left( \frac{m}{2} \sum_{i=1}^{n} \dot{x}_i^2(t) - V(x_i(t)) \right) dt,
\]

\[
n = 3n_0,
\]

i.e. that we have an interacting system of \( n_0 \) particles subjected to correlated random forces and all particles have the same mass \( m \).

At first we shall make the integration with respect to \( \prod dG_i(t) \). The \( \delta \)-functions imply that \( G_i = m\ddot{x}_i - G_0 \), so that we shall obtain the integral

\[
\int_{(BC)} \exp \left\{ \frac{i}{\hbar} \int \frac{m}{2} \sum_i \dot{x}_i^2 - V(x_i) dt \right\} \cdot \exp \left\{ \int \left( a_0 G_0^2 + a_1 \sum (m\ddot{x}_i - G_0)^2 \right) dt \right\} \cdot \prod_t dG_0(t) \prod_{t,i} dx_i(t)
\]
and this gives
\[ \int_{(BC)} \exp \left( \frac{i}{2\hbar} \left\{ \int \left( m \sum x_i^2 - 2V(x_i) + a_1m^2 \sum \dddot{x}_i^2 + (a_0 + na_1)G_0^2 \right) dt \right\} \right) \cdot \exp \left\{ -\frac{i}{\hbar} \int a_1mG_0 \sum \dddot{x}_i dt \right\} \prod_t dG_0(t) \prod_t dx_i(t). \]

Integrating with respect to \( \prod dG_0(t) \) we obtain finally
\[ A = \int_{(BC)} \exp \left( \frac{i}{2\hbar} \left\{ \int \left( m \sum x_i^2 - 2V(x_i) + a_1m^2 \sum \dddot{x}_i^2 + a_2m^2 (\sum \dddot{x}_i)^2 \right) dt \right\} \prod_t dx_i(t), \]

where
\[ a_2 = -\frac{a_1^2}{a_0 + na_1}. \]

We see that the collective term with \( a_2 \) is a new feature of this model. If \( a_2 = 0 \) and \( a_1 = a \), this model is the same as SubQM_{RF}. The term \( (\sum \dddot{x}_i)^2 \) creates certain interaction among particles.

(B) In the calculation of the propagator we shall assume that the interaction term is zero,
\[ V \equiv 0. \]

Our way to diagonalize the new term requires to do the orthogonal transformation
\[ x_i(t) = \sum_{j=1}^{n} R_{ij} y_j(t) \]

such that
\[ R_{in} = (n)^{-1/2} \text{ for } i = 1, \ldots, n. \]

Columns of the orthogonal matrix \( R \) compose a basis of \( \mathbb{R}^n \) and they are, for example,
\[ R_{ij} = (i^2 + i)^{-1/2} \text{ for } j \leq i \leq n - 1, \]
\[ R_{nj} = n^{-1/2} \text{ for } j \leq n, \]
\[ R_{j,j+1} = -j(j^2 + j)^{-1/2} \text{ for } j \leq n - 1, \]
\[ R_{ij} = 0 \text{ for } j \geq i + 2. \]
The inverse transformation is

\[ y_k(t) = \sum_i R_{ik} x_i(t). \]

So that \( R^\top R = 1 \). We shall introduce the corresponding boundary conditions for \( y_j \)'s

\[
(BC)_Y : \quad y_j(t^s) = y^s_j, \quad m\dot{y}_j(t^s) = q^s_j, \quad Y_j^s = \left( \begin{array}{c} q^s_j \\ y^s_j \end{array} \right), \quad s = 1, 2,
\]

where

\[
y^s_j = \sum R_{ij} x^s_i, \quad q^s_j = \sum R_{ij} p^s_i, \quad Y_j^s = \sum R_{ij} X^s_i, \quad s = 1, 2.
\]

From orthogonality of \( R \) we obtain

\[
\sum \dot{x}_i^2(t) = \sum \dot{y}_j^2(t),
\]

\[
\sum \ddot{x}_i^2(t) = \sum \ddot{y}_j^2(t),
\]

\[
(\sum \dddot{x}_i)^2 = n\dddot{y}_n^2.
\]

We shall make the orthogonal change of variables in the Feynman integral and we obtain

\[
\mathcal{A} = \left\{ \prod_{i=1}^{n-1} \int_{(BC)_Y} \exp \left[ \frac{i}{2\hbar} \int m\dot{y}_i^2 + a_1 m^2 \ddot{y}_i^2 dt \right] \prod_t dy_i(t) \right\} \cdot \int_{(BC)_Y} \exp \left[ \frac{i}{2\hbar} \int m\dddot{y}_n^2 + a_3 m^2 \dddot{y}_n^2 dt \right] \prod_t dy_n(t),
\]

where

\[
a_3 = a_1 + a_2 n = \frac{a_0}{n + a_0/a_1} \approx \frac{a_0}{n}.
\]

Here we assume that

\[
\tau_0 \ll \tau_1, \quad \tau_0 = (a_0 m)^{1/2}, \quad \tau_1 = (a_1 m)^{1/2},
\]
i.e. that the relaxation time of the $G_0$ is shorter than the relaxation time of the $G_i$-forces. This means that for $T$, $\tau_0 \ll T \ll \tau_1$, the $G_0$-process is already relaxed but the $G_i$-processes are not relaxed. Equivalently, we have $a_0 \ll a_1$ and thus also

$$a_3 \approx n^{-1} a_0 \ll a_1.$$ 

We shall introduce

$$\tau_3 = (a_3m)^{1/2}, \quad \beta_3 = 1/\tau_3, \quad \beta_1 = 1/\tau_1.$$ 

The last Feynman integrals may be simply calculated if we shall use formula (6.4) using matrices $Q_{inT}$, $Q_{trT}$ and $Q_{outT}$, where also their parametrical dependence on $\beta$ is denoted by $Q_{\beta_i}^{\beta_i}$ etc. The resulting propagator is $K_T = N_T \exp i\hbar^{-1} S_T$, where

$$S_T = \sum_{i=1}^{n-1} \left( \frac{1}{2} Y_i^1 \top Q_{inT}^{\beta_1} Y_i^1 + Y_i^1 \top Q_{trT}^{\beta_1} Y_i^2 + \frac{1}{2} Y_i^2 \top Q_{outT} Y_i^2 \right) +$$

$$+ \frac{1}{2} Y_n^1 \top Q_{inT}^{\beta_3} Y_n^1 + Y_n^1 \top Q_{trT}^{\beta_3} Y_n^2 + \frac{1}{2} Y_n^2 \top Q_{outT} Y_n^2.$$ 

To transform this quantity into $X$-variables we introduce

$$\overline{X}^s = \frac{1}{n} \sum_{i=1}^{n} X_i^s, \quad s = 1, 2,$$

$$\Delta X_i^s = X_i^s - \overline{X}^s, \quad s = 1, 2, \quad i = 1, \ldots, n.$$ 

In this way we obtain

$$Y_n^s = n^{-1/2} \sum_j x_j^s = n^{1/2} \overline{X}^s.$$ 

Using the formula of the type ($Q$ is any $2 \times 2$ matrix)

$$\sum_{i=1}^{n} X_i^s \top Q X_i^r = \Delta X_i^s \top Q \Delta X_i^r + n \overline{X}^s \top Q \overline{X}^r, \quad s, r = 1, 2,$$
and
\[ \sum_{i=1}^{n} Y_{i}^{s} Q Y_{i}^{r} = \sum_{j=1}^{n} X_{j}^{s} Q X_{j}^{r}, \quad s, r = 1, 2. \]

We obtain that
\[ \sum_{i=1}^{n} Y_{i}^{s} Q Y_{i}^{r} = \sum_{j=1}^{n} \Delta X_{j}^{s} Q \Delta X_{j}^{r}, \quad s, r = 1, 2. \]

As a result we obtain the formula
\[ S_{T} = \sum_{i=1}^{n} \left( \frac{1}{2} \Delta X_{i}^{1 T} Q_{inT}^{\beta_{1}} \Delta X_{i}^{1} + \Delta X_{i}^{1 T} Q_{trT}^{\beta_{1}} X_{i}^{2} + \frac{1}{2} \Delta X_{i}^{2 T} Q_{outT}^{\beta_{1}} X_{i}^{2} \right) + \]
\[ + n \left( \frac{1}{2} X_{i}^{1 T} Q_{inT}^{\beta_{3}} X_{i}^{1} + X_{i}^{1 T} Q_{trT}^{\beta_{3}} X_{i}^{2} + \frac{1}{2} X_{i}^{2 T} Q_{outT}^{\beta_{3}} X_{i}^{2} \right). \]

To obtain the final form of the propagator we shall use formulas of the type
\[ \sum_{i=1}^{n} (y_{i}^{s})^{2} = \sum_{i=1}^{n} (\Delta x_{i}^{s})^{2}, \quad \sum_{i=1}^{n} (p_{i}^{s})^{2} = \sum_{i=1}^{n} (\Delta p_{i}^{s})^{2}, \quad (y_{n}^{s})^{2} = n(\bar{x}^{s})^{2}, \ldots \]

In this way we obtain
\[ S_{T} = \sum_{i=1}^{n} \frac{1}{2\beta_{1} m} \left( \frac{(\Delta p_{i}^{1})^{2} + (\Delta p_{i}^{2})^{2}}{\tanh \beta_{1} T} - \frac{2\Delta p_{i}^{1} \Delta p_{i}^{2}}{\sinh \beta_{1} T} \right) + \]
\[ + \frac{n}{2\beta_{3} m} \left( \frac{(\bar{p}^{1})^{2} + (\bar{p}^{2})^{2}}{\tanh \beta_{3} T} - \frac{2\bar{p}^{1} \bar{p}^{2}}{\sinh \beta_{3} T} \right) + \]
\[ + \frac{m}{2T} \left\{ \sum_{i=1}^{n} \left( \frac{\Delta x_{i}^{2} - \Delta x_{i}^{1} - (\beta_{1} m)^{-1}(\Delta p_{i}^{1} + \Delta p_{i}^{2}) \tanh(\beta_{1} T/2)}{1 - (\beta_{1} T/2)^{-1} \tanh(\beta_{1} T/2)} \right)^{2} - \right. \]
\[ \left. \frac{\Delta x_{i}^{2} - \bar{x}^{1} - (\beta_{3} m)^{-1}(\bar{p}^{1} + \bar{p}^{2}) \tanh(\beta_{3} T/2)}{1 - (\beta_{3} T/2)^{-1} \tanh(\beta_{3} T/2)} \right)^{2} \right\}. \]

The result is that the relative positions and relative momenta evolve with the relaxation constant \( \beta_{1} \) while the mean value of positions and the mean value of momenta evolve with relaxation.
constant $\beta_3 \gg \beta_1$. This phenomenon creates certain time interval during which mean values are already relaxed (i.e. long-time case) and relative values are still not relaxed (i.e. short-time case) – such $\mathcal{T}$ that $\beta_3 \mathcal{T} \gg 1$ and $\beta_1 \mathcal{T} \ll 1$. For such times the correlation effect happens.

(C) The correlated state is the $n$-particle state satisfying the inequality

$$|\psi(p_i, x_i)| \leq c \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n}(x_i^1)^2 \frac{\Delta x^2}{\Delta x^2} - \frac{1}{2} \sum_{i=1}^{n}(p_i^1)^2 \frac{\Delta p^2}{\Delta p^2} \right\}$$

for some positive constants $\Delta x$, $\Delta p$, $c$, where

$$\Delta p \cdot \Delta x \ll \hbar.$$

This means that in the correlated state the relative positions and relative momenta are concentrated in the sense of the preceding section.

Preparation of the correlated state is similar to preparation of concentrated states – particles pass through repeated slits. Assuming two slits preparation, one has to introduce into the original Feynman integral two Gaussians representing the process of passing through slits,

$$\exp \left\{ -\frac{1}{2} \sum_{i=1}^{n}(x_i^1)^2 \Delta x^2 - \sum_{i=1}^{n}(y_i^2)^2 \Delta x^2 \right\}.$$

Now we shall make transformation of variables to new variables $y$. From the term describing the effect of slits we obtain

$$\exp \left\{ -\frac{1}{2} \sum_{i=1}^{n}(y_i^1)^2 \Delta x^2 - \sum_{i=1}^{n}(q_i^2)^2 \Delta x^2 \right\}.$$

Now, using results of the preceding section (part (C)) we obtain, assuming $\beta_1 \mathcal{T} \ll 1$, that variables $y_i^2$ and $q_i^2$, $i = 1, \ldots, n - 1$, will be concentrated, because they evolve with the relaxation constant $\beta_1$. Making then the inverse transformation $y, q \rightarrow x, p$, we obtain from

$$|\psi'_{2}| \leq \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n-1}(y_i^2)^2 \frac{\Delta x^2}{\Delta x^2} + (q_i^2)^2 \frac{\Delta p^2}{\Delta p^2} \right\}$$
the inequality

\[ |\psi'_2| \leq \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \frac{(\Delta x_i^2)^2}{\Delta x^2} + \frac{(\Delta p_i^2)^2}{\Delta p^2} \right\}. \]

Dependence of \( \psi'_2 \) on variables \( y_n, q_n \) or \( \bar{x}, \bar{p} \) need not be concentrated and in fact, assuming \( \beta_3 T \gg 1 \), it will be relaxed.

(D) The evolution of the correlated state constructed above can be analyzed in terms of variables \( y_i, q_i \). The correlated state \( \psi_1 \) expressed in variables \( \hat{y}_i \) and \( \hat{q}_i \) satisfies \( (\Delta x \cdot \Delta p \ll \hbar) \)

\[ |\psi_1(y_1, \ldots, y_n, q_1, \ldots, q_n)| \leq c. \exp \left\{ -\frac{1}{2\Delta x_1^2} \sum_{i=1}^{n-1} (y_i^1)^2 - \frac{1}{2\Delta p_1^2} \sum_{i=1}^{n-1} (q_i^1)^2 \right\}. \]

If the evolution time \( T = t^2 - t^1 \) satisfies

\[ \beta_1 T \ll 1 \ll \beta_3 T, \]

then the evolution in variables \( y_i, q_i, i \leq n-1 \), has the same properties as the evolution of the concentrated state in the preceding section (part (D)). We obtain that the evolution of \( \psi_2 \) is much slower than the QM-evolution – details are in the preceding section. We obtain that the dispersion of \( \psi_2 \) in \( y_i, i = 1, \ldots, n - 1 \), is of order

\[ \Delta x^2_2 \leq \Delta x^2_1 + \Delta p^2_1 \frac{T^2}{m^2} (1 + o(\beta_1 T)). \]

On the other hand, there is no control on \( y_n, q_n \). After transforming this back to variables \( \Delta x_i, \bar{x} \) etc. we obtain that the dispersion of

\[ \sum_{i=1}^{n} (\Delta x_i^2)^2 \]

is of order \( \Delta x^2_2 \), while \( \bar{x}^2 \) is relaxed.

This implies the following behavior of the \( n \)-particle system in \( \text{SubQM}_{CRF} \) (assuming \( \beta_1 T \ll 1 \ll \beta_3 T \)):

(i) The evolution of the correlated state during time \( T \) gives the state in which the relative positions are small,
(ii) the mean position behaves quantum-mechanically, because $\beta_3 T \gg 1$, so that the long-time approximation applies to $y_n^2 \sim x^2$.

(iii) the resulting picture contradicts QM in this, that the group of particles behaves as a correlated system, i.e. as a whole, as a certain "superparticle" in the QM-law, but the inner dispersion inside the group is much smaller than in QM.
9. Proposed experiments

Tests which can differ between QM and SubQM are based on the existence of the concentration, resp. correlation effects in SubQM.

The possible subquantum effects depend on the value of the parameter $a$, resp. $\tau_0 = 1/\beta$ characterizing the subquantum model. The values of parameters $L, \delta, T_0, m, V$ characterizing preparation of the particles are related to the value of $\tau_0$ in part (B) below.

The goal of tests is to find a result implying the existence of some subquantum effect. The result would be then the lower estimate for the relaxation time $\tau_0$. The upper estimate of $\tau_0$ is another problem not discussed in this paper.

In the description of each test we have to specify:

(i) preparation of the state of particles,
(ii) preparation of the beam of particles,
(iii) type of searched effect,
(iv) configuration of the screen and of the other measuring devices,
(v) description of results indicating presence of a subquantum effect.

Description of tests will be given in three parts in which the first part (A) will be common for all tests.

(A) Preparation of the state of particles (parameters $L, \delta$) and preparation of the beam of particles (parameters $T_0, m, V$) – i.e. (i) + (ii).

(B) Description of parts (iii)-(v) for each particular test.

(C) The discussion of possible physical values of parameters $L, \delta, T_0, m, V$ in the relation to possible values of $\tau_0 = 1/\beta$ (including the parameter $L_{sc}$ and other parameters describing geometry of the screen).

The idea is to look for concentration and correlation effects, which are typical for any subquantum model and which are excluded by QM. The first step is to create, by using iterated slits, the concentrated or correlated state of particles. In the concentrated short-time pulse we have concentration of three quantities: space position, momentum and time position.

(A) The preparation part of any test consists in passing through iterated slits. Here we shall describe the standard form of iterated
slits and the possible variants will be described below. Let us assume that the particles are moving in the direction of the axis $x_3$. Let $\delta$ denote the radius of the hole and $L$ denote the distance between slits/holes. Then the first hole $H_{-1}$ means the solid screen with the hole at the center

$$H_{-1} = \{ x \in \mathbb{R}^3 | x_3 = -L, \ x_1^2 + x_2^2 \geq \delta^2 \}$$

and the second hole will be

$$H_0 = \{ x \in \mathbb{R}^3 | x_3 = 0, \ x_1^2 + x_2^2 \geq \delta^2 \}.$$ 

The true slits will be one-dimensional objects

$$S_{-1} = \{ x \in \mathbb{R}^3 | x_3 = -L, \ |x_1| \geq \delta \},$$
$$S_0 = \{ x \in \mathbb{R}^3 | x_3 = 0, \ |x_1| \geq \delta \}.$$ 

We shall consider in details only the first situation – the holes, since the case with slits is similar. The screen will be at the distance $L_{sc}$,

$$S_{sc} = \{ x \in \mathbb{R}^3 | x_3 = L_{sc} \}.$$ 

One can also consider the case with three or more iterated holes (resp. slits) with the other hole

$$H_{-2} = \{ x \in \mathbb{R}^3 | x_3 = -2L, \ x_1^2 + x_2^2 \geq \delta^2 \},$$ etc.

We shall suppose that particles in the beam move with the velocity

$$V > 0$$

in the direction of the axis $x_3$.

We shall suppose also that the beam has a form of a pulse with the duration

$$T_0 > 0.$$ 

We shall consider two types of pulses:

- the short-time pulse satisfying $T_0 \ll \tau_0 = 1/\beta$,
- the long-time pulse satisfying $T_0 \gg \tau_0$. 

In this way we are able to prepare particles in the concentrated state assuming that $L$ and $\delta$ are sufficiently small (with respect to $V\tau$).

Preparation of the correlated state requires that

$$\tau_0 < T_0 \ll \tau_1.$$ 

Thus the concentrated state may exist in the form of both long-time and short-time pulses, while the correlated state requires the short-time pulse beam.

The last parameter of a particle will be its mass $m > 0$.

The simplest form of the screen will be the plain $S_{sc}$ with distance $L_{sc} = L$. If $L_{sc}/V \gg \tau_0 \gg \tau_1$, respectively, or $L/V \gg \tau_0 \gg \tau_1$, resp.) then probably all subquantum effects disappear, in particular, the concentration and correlation effects disappear.

The main step in the preparation of the beam is to let it pass through iterated holes (resp. slits): $H_{-1}, H_0$. To obtain the concentrated state after passing the last hole, we need

$$T := L/V \ll \tau_0, \text{ i.e. } \beta T \ll 1.$$ 

We have

$$\kappa_0^2 = \left(\frac{2}{\hbar} \Delta x_0 \Delta p_0 \right)^2 \approx 8 \frac{m^2}{T^2 \hbar^2} \cdot \Delta x_0^4 + \frac{4}{9} (\beta T)^4$$

so that the second term is already small. The concentrated state then needs (together with $\tau \ll \tau_0$) that

$$\delta^2 = \Delta x_0^2 \ll \frac{T \hbar}{3m} \ll \frac{\tau_0 \hbar}{3m}.$$ 

These two conditions are sufficient for the creation of the concentrated state by passing through two iterated holes (or slits).

The degree of the concentration $\kappa$ depends crucially on $T = L/V$.

E.g., for $T \gg \tau_0$ (i.e. $L \gg V\tau_0$) $\kappa \gg 1$ and all subquantum effects disappear.

(B) The distance between the screen and the last hole $H_0$ (resp. slit $S_0$) will be denoted $L_{sc}$. We shall consider two types of measurements:

(i) screen,
(ii) detectors.
In the first case – screen – we measure the density of observed particles. The measurement is a measurement of the position observable. Usually the observed density can be decomposed into a slowly varying amplitude part and a rapidly oscillating part. We have

\[ \rho(x) \approx \bar{\rho}(x) \sin^2 \phi(x), \]

where \( \phi \) describes the ”rapidly oscillating” part and \( \bar{\rho} \) is the ”slowly varying” component. The main part density \( \bar{\rho} \) can be obtained as a mean value of \( \rho \) over oscillations.

This decomposition into \( \bar{\rho} \) and \( \phi \) is standard for interference pictures. If \( \lambda_0 \) is the typical wave-length, then \( \bar{\rho} \) is slowly varying on distances of order \( \lambda_0 \).

In the second case, we shall consider certain number of detectors placed on the screen. Typically we shall consider the detector as a hole (or a slit) such that particles passing through this hole (or slit) are registered. We shall consider the following types of detectors

\[ H_{sc}(x_1^0, x_2^0, r) := \{ x \in \mathbb{R}^3 | x_3 = L_{sc}, (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 < r^2 \} \]
\[ S_{sc}(x_1^0, r) := \{ x \in \mathbb{R}^3 | x_3 = L_{sc}, |x_1 - x_1^0| < r \}. \]

The observed number of particles can be related to the particle density by

\[ \text{Num} \left[ H_{sc}(x_1^0, x_2^0, r) \right] = \int_{H_{sc}(x_1^0, x_2^0, r)} \rho \, dx, \]

and similarly for \( S_{sc}(x_1^0, r) \).

Let the probability that the particle approaches the screen at \( H_{sc}(x_1^0, x_2^0, r) \) be \( p, 0 < p < 1 \). Let \( N_0 \) be the total number of particles having passed through the preparation part of the system. Then the mean value of the number of particles arriving at \( H_{sc}(x_1^0, x_2^0, r) \) is

\[ \bar{N} = N_{all} \cdot p, \quad N_{all} = \text{total number of particles} \]

and the mean quadratic deviation is

\[ \sigma_0 = \sqrt{N_{all} \sqrt{p(1 - p)}} = \sqrt{\bar{N}} \cdot \sqrt{1 - p} \leq \sqrt{\bar{N}}. \]
If we assume that $p$ is rather small, we obtain that the fluctuation of the observed number $N$ of particles arriving at $H_{sc}(x_1^0, x_2^0, r)$ is of order

$$\sigma_0 \approx \sqrt{N}.$$  

Typical situation is the following – there are three detectors

$$D_0 = H_{sc}(0, 0, r_0), \quad D_\pm = H_{sc}(\pm x_1^0, 0, r_1),$$

where $x_1^0 > r_0 + r_1$.

Analogously with slits,

$$D_0 = S_{sc}(0, r_0), \quad D_\pm = S_{sc}(\pm x_1^0, r_1), \quad x_1^0 > r_0 + r_1.$$  

The evolution after the last hole (slit), i.e. on $0 < x_3 < L_{sc}$, is the following. Let $\Delta x_0^2$, $\Delta p_0^2$ and $\kappa_0 = \Delta x_0 \Delta p_0^2/\hbar$ be parameters at $x_3 = 0$, i.e. after passing the last preparation hole (slit). Then denoting by

$$\tilde{\Delta} x_{sc}^2 := \Delta x_0^2 + \Delta p_0^2 \frac{T_{sc}^2}{m^2}, \quad T_{sc} = L_{sc}/V,$$

the ”pure” dispersion and by

$$\tilde{\Delta} p_{sc} := \frac{\hbar^2}{4\tilde{\Delta} x_{sc}^2}$$

the corresponding Heisenberg’s dual quantity, we obtain the ”real” dispersion

$$\Delta x_{sc}^2 = \Delta x_0^2 + \Delta p_0^2 \frac{T_{sc}^2}{m^2} + \frac{4}{9}\tilde{\Delta} p_{sc}^2 \cdot m^2 \cdot (\beta T_{sc})^4.$$  

This is rewriting of formula (7.10)

$$\Delta x_1^2 = \Delta x_0^2 + \Delta p_0^2 \frac{T_{sc}^2}{m^2} + \frac{1}{9}(\beta T)^6 \frac{\hbar^2}{\Delta p_0^2 (\beta T)^2 + \Delta x_0^2 m^2 \beta^2}.$$  

This shows that the conditions for subquantum effects are the following

(9.3) \hspace{1cm} T_{sc} := L_{sc}/V \ll \tau_0, \text{ i.e. } \beta T_{sc} \ll 1,
and

\[ \kappa_0 = \frac{2}{\hbar} \Delta x_0 \Delta p_0 \ll 1. \]  

The last condition expresses the fact that the prepared state must be concentrated.

Now we can describe proposed experiments. In the situation of detectors, it is reasonable (for obtaining stable results), but not necessary, to assume that the dimensions of detectors are larger than the wave-length of oscillations

\[ r_0, r_1 \gg \lambda_0. \]  

(Exp. 1). In the situation described above assuming that conditions (9.1)-(9.4) are satisfied, we obtain in the "screen-like" situation the following relation

\[ \Delta x_{sc}^2 \ll \Delta x_{sc}^{2(QM)}, \]  

where \( \Delta x_{sc} \) is the observed dispersion of the position on the screen and \( \Delta x_{sc}^{(QM)} \) is the dispersion expected by QM.

This is a direct manifestation of the concentration effect. In this case:

(iii) Type of the searched effect: the slower dispersion than in QM,
(iv) configuration: the density on the screen,
(v) indication of the subquantum effect: inequality (9.6) – a small dispersion.

(Exp. 2). Here the situation is the same as in Exp. 1, but instead of observing the density on the screen we use three detectors \( D_0, D_\pm \) (in the hole or slit variations).

Let us denote by \( N_0, N_\pm \) the number of particles observed at detectors \( D_0, D_\pm \) and let \( N_0^{(QM)}, N_\pm^{(QM)} \) be corresponding numbers predicted by QM. Then, assuming (9.1)-(9.5) we have

\[ \frac{N_\pm}{N_0} \ll \frac{N_\pm^{(QM)}}{N_0^{(QM)}}, \]
or more precisely (but similarly)

\[
\frac{N_{\pm}}{N_{\text{all}}} \ll \frac{N_{\pm}^{(QM)}}{N_{\text{all}}}.
\]

This expresses the fact that the dispersion in the subquantum situation is slower than QM-dispersion.

A still more stable indication is expressed by the inequality

\begin{equation}
N_{\pm} + N_{-} \ll N_{0}^{(QM)} + N_{\pm}^{(QM)} + N_{-}^{(QM)}.
\end{equation}

In this case

(iii) searched effect: small dispersion,
(iv) configuration: three detectors \(D_{0}, D_{\pm}\),
(v) indication: inequality (9.7) – small dispersion.

(Exp. 3). This experiment is proposed for testing possible fluctuation of the basic parameter of subquantum models – the quantity \(\tau_{0}\) – the relaxation time. This quantity is expressed as a function of the basic parameter \(a\) and the mass \(m\) of a particle by

\[
\tau_{0}^{2} = am.
\]

The parameter \(a\) describes, in a sense, the ”density” of space-like objects in SLO-vacuum.

We have assumed that this ”density” (and hence also parameter \(a\)) is constant with respect to the time.

But, by the proper physical idea of SLO-vacuum, this (and the ”density”) is a dynamical property and it is reasonable to assume that there may be fluctuation of this quantity with respect to time.

Of course, these fluctuations are significant only on short time intervals. Thus it is necessary to use short-time pulses. Let us suppose that there are \(i = 1, \ldots, I\) pulses of particles, each of the duration \(T_{0}\), where

\begin{equation}
\beta T_{0} \ll 1.
\end{equation}
In these pulses we have corresponding quantities

\[ N_{all}^i, \ N_0^i, \ N_{\pm}^i \] for \( i = 1, \ldots, I \).

Let the mean values be denoted by

\[ \bar{N}_0 = I^{-1} \sum N_0^i, \ \bar{N}_\pm = I^{-1} \sum N_{\pm}^i. \]

In the case of negligible fluctuations, we have the mean square deviation satisfying the inequality mentioned above

\[ \sigma_0 := \left[ I^{-1} \sum_{i=1}^I \left( N_+^i + N_-^i - \bar{N}_+ - \bar{N}_- \right)^2 \right]^{1/2} \leq \left[ \bar{N}_+ + \bar{N}_- \right]^{1/2}. \]

This corresponds to the inequality \( \sigma(N) \leq \sqrt{N} \) for \( N = N_+ + N_- \).

In the case of fluctuating value of \( \beta \) we can assume that the mean square deviation of the quantity \( N_+^i + N_-^i \) will be larger due to the change of \( \beta \) and not only due to the standard statistical deviation. So that we look for satisfaction of the opposite inequality

\[ I^{-1} \sum_{i=1}^I (N_+^i + N_-^i - \bar{N}_+ - \bar{N}_-)^2 > \bar{N}_+ + \bar{N}_-. \] (9.9)

In this experiment it is necessary to consider the random quantity

\[ N = N_+ + N_- \]

with the sample values

\[ N_i = N_+^i + N_-^i, \ i = 1, \ldots, I, \]

because there may exist another subquantum effect – the correlation effect – which typically gives large fluctuations of \( N_+ \) and \( N_- \), but smaller fluctuation of \( N_+ + N_- \) (see below). Of course, realization that (9.9) is satisfied needs also a reasonably large number \( I \) of pulses. This is a standard statistical argument relating \( I \) to the gap in inequality (9.9).

We have

(iii) searched effect: fluctuation of the parameter \( a \) (resp. the density of SLO’s)

(iv) configuration: three detectors \( D_0, D_\pm \),

(v) indication: inequality (9.9) – large quadratic deviation of \( N_+ + N_- \).
(Exp. 4). The basic correlation effect. The correlation effect uses the (hypothetical) correlation between particles in the same pulse.

If there are $I$ pulses, $i = 1, \ldots, I$, in each there is a resulting density $\rho_i(x)$, $x = (x_1, x_2)$, which is decomposed as

$$\rho_i(x) \cong \bar{\rho}_i(x) \sin^2 \phi_i(x), \quad i = 1, \ldots, I,$$

into a slowly varying part $\bar{\rho}_i$ and a rapidly oscillating part $\sin^2 \phi_i(x)$.

The total observed density is

$$\rho(x) \cong \bar{\rho}(x). \Phi(x),$$

where $\bar{\rho} = \sum \bar{\rho}_i$ and $0 \leq \Phi \leq 1$ ($\Phi$ is the rapidly oscillating part of $\rho$).

There are three time intervals

$$T_0 = \text{duration of the pulse},$$
$$T = \frac{L}{V},$$
$$T_{sc} = \frac{L_{sc}}{V},$$

and they should satisfy

$$T_0 \ll \tau_0,$$
$$\tau_0 \ll T \ll \tau_1,$$
$$\tau_0 \ll T_{sc} \ll \tau_1.$$

Thus the length $L_0 = VT_0$ has to satisfy

$$L_0 \ll V \tau_0 \ll L, \quad L_{sc} \ll V \tau_1.$$

The correlation effect consists in the following behavior: particles in a pulse behave collectively like one group. This is true in a certain approximation where (9.10) is satisfied.

In consequence of this group-like behavior we can assume (see Sect. 8) that

$$\bar{\rho}_i(x) \approx \exp \left\{ -\frac{1}{2} \frac{|x - x_{i0}|^2}{\tau_0^2} \right\}, \quad x = (x_1, x_2),$$
where

\[ x^i_0 = (x^i_{01}, x^i_{02}) \]

is a center of this "Gaussian" wave packet and \( r_0 \) is its approximate radius. The correlation effect says that centers \( x^i_0 \) are far from each other (relatively to \( r_0 \)), i.e. that

\[ |x^i_0 - x^j_0| \gtrsim r_0 \]

at least for many couples \( i \neq j \).

In QM any particle in any pulse is independent on other particles and by the standard statistics of fluctuations we have for the dispersion of centers

\[ I^{-1} \sum |x^i_0 - \bar{x}_0|^2 \ll r_0^2, \]

where \( \bar{x}_0 = I^{-1} \sum x^i_0 \).

The inequality indicating the correlation effect is the following

\[ (9.11) \quad I^{-1} \sum |x^i_0 - \bar{x}_0|^2 \gtrsim r_0^2. \]

In fact, it is not simple to identify centers \( x^i_0 \) (for example by registration of the density after each pulse) – if it is possible to decompose reasonably \( \bar{\rho} \) into \( \sum \bar{\rho}_i \), it means that the correlation effect takes place. In the QM situation we should have

\[ \bar{\rho} \approx \exp \left\{ -\frac{1}{2} \frac{|x - \bar{x}_0|^2}{r_0'^2} \right\}, \]

where \( r_0 \lesssim r_0' \).

Thus the effect consists in the fact that \( \bar{\rho} \) looks like a sum of Gaussians rather that a certain one Gaussian. Hence experiment requires a small number of pulses:

\[ 2 \leq I \leq 8. \]

We have

(iii) searched effect: the group-like behavior of short pulses,
(iv) configuration: the screen, the short-time pulses,
(v) indication: inequality (9.11) or a decomposition of \( \bar{\rho} \) into more than one Gaussian.
(Exp. 5). This is also an experiment looking for correlation effect using short-time pulses. But instead of a screen as in Exp. 4 we shall use detectors $D_0$, $D_\pm$.

Let us denote by $N^i_0$, $N^i_\pm$ the number of particles of the $i$-th pulse arrived at the detector $D_0$, resp. $D_\pm$. Then we can calculate the mean values

$$\bar{N}_0 = I^{-1} \sum N^i_0, \quad \bar{N}_\pm = I^{-1} \sum N^i_\pm.$$ 

At first we shall assume that the total number of particles in each pulse is the same,

$$N^1_{\text{all}} = N^2_{\text{all}} = \cdots = N^I_{\text{all}}.$$ 

Let us denote by $\sigma(N_\pm)$ the mean quadratic deviation

$$\sigma(N_\pm) := \left[ I^{-1} \sum_{i=1}^{I} \left( N^i_\pm - \bar{N}_\pm \right)^2 \right]^{1/2}$$

of the (sample) quantities $N^i_\pm$. Then by the statistical argument mentioned above and using the basic QM fact stating that all particles are mutually independent one has the estimate

$$\sigma(N_\pm) \lesssim \sqrt{\bar{N}_\pm}.$$ 

This estimate depends on the assumption that $I$ (the number of trials-pulses) is sufficiently large. But this is a standard property of all statistical assertions.

Thus the indication of the subquantum correlation effect is fulfillment of the opposite inequality (inequalities)

$$\sigma(N_\pm) \sqrt{N_\pm} > 1.$$ 

(9.12)

The gap in this inequality has to be considered in relation with the number $I$ of trials-pulses.

Let us assume that quantum mechanically

$$\bar{N}_+ \approx \bar{N}_-,$$
which happens in the situation when detectors $D_+$ and $D_-$ are placed symmetrically. Then, following QM, we have the standard deviations of the random quantities $N_\pm$

$$N_\pm^i \cong \bar{N}_\pm \pm \sqrt{\bar{N}_\pm}, \ i = 1, \ldots, I,$$

where $\bar{N}_\pm = \frac{1}{I} \sum N_\pm^i$ are the mean values. Let us define

$$\bar{N} := \sqrt{\bar{N}_+ \bar{N}_-}.$$

From

$$\left| \frac{N_\pm^i}{\bar{N}_\pm} - 1 \right| \approx \frac{1}{\sqrt{\bar{N}_\pm}}$$

we obtain

$$\left| \frac{N_+^i}{N_+} - \frac{N_-^i}{N_-} \right| \approx \frac{1}{\sqrt{N_+}} + \frac{1}{\sqrt{N_-}} = \frac{\sqrt{\bar{N}_+} + \sqrt{\bar{N}_-}}{\bar{N}}$$

and then

$$\left| N_+^i \bar{N}_- - N_-^i \bar{N}_+ \right| \lesssim \bar{N} \left( \sqrt{\bar{N}_+} + \sqrt{\bar{N}_-} \right).$$

Using $\sqrt{N_+} + \sqrt{N_-} \approx 2\sqrt{N}$ we obtain the inequality

$$\left| N_+^i \frac{\bar{N}_-}{N} - N_-^i \frac{\bar{N}_+}{N} \right| \lesssim 2\sqrt{N}.$$

Here $\frac{\bar{N}_-}{N}$ and $\frac{\bar{N}_+}{N}$ are correction factors related to the possibly non-equilibrated situation $\bar{N}_+ \neq \bar{N}_-$.

In the correlation effect it often happens that $N_+^i$ and $N_-^i$ are substantially different. This gives the effect that fluctuations of $N_+^i - N_-^i$ are larger than in QM. Thus the indicating inequality is

$$I^{-1} \sum_i \left| N_+^i \frac{\bar{N}_-}{N} - N_-^i \frac{\bar{N}_+}{N} \right| > 2\sqrt{N}, \ \bar{N} := \sqrt{\bar{N}_+ \bar{N}_-}.$$

In this case we have

(iii) searched effect: correlation inside the group-pulse,

(iv) configuration: detectors $D_\pm$,

(v) indication: inequality (9.12) or (9.13) – large deviation.
(Exp. 6). This is a variant of Exp. 5.
There are two detectors in the form of half-plains

\[ D_+ = \{ (x_1, x_2) | x_1 > 0 \}, \]
\[ D_- = \{ (x_1, x_2) | x_1 < 0 \}. \]

Let \( N^i_\pm \) be numbers of particles observed at the \( i \)-th pulse in detectors \( D_\pm \). Let mean values be

\[ \bar{N}^i_\pm := I^{-1} \sum_{i=1}^{I} N^i_\pm. \]

It is possible to define the corrected numbers

\[ \tilde{N}^i_\pm := N^i_\pm \cdot \frac{\bar{N}_+ + \bar{N}_-}{N^+_i + N^-_i}. \]

Then we can consider inequalities (9.12) or (9.13) written for \( \tilde{N}^i_\pm \) as indicating presence of subquantum effects.

We have

(iii) searched effect: correlation inside a pulse,
(iv) configuration: two detectors – half-plains,
(v) indication: inequalities (9.12) or (9.13) for corrected numbers \( \tilde{N}^i_\pm \).

This type of an experiment was proposed in [6] under the name "sub-quantum coherence effect". It is possible to consider experiments that are variants of those already proposed. For example

(Exp. 2'). The indicating inequality may be also

\[ \frac{N_0}{N_{all}} \gg \frac{N_0^{(QM)}}{N_{all}}. \]

(C) In part C we have to consider concrete forms of proposed experiments. There are two possibilities:

(C1) massive particles like electrons or protons or neutrons,
(C2) mass-less particles – photons.
Our theory is purely non-relativistic, so that formulas used in this section cannot be directly applied to photons. But we shall consider experiments Exp. 1-Exp. 6 also for photons by analogy and by using \( V = c \), i.e. \( T = L/c \), \( T_{sc} = L_{sc}/c \) etc.

Of course, all possible results indicating presence of a certain sub-quantum effect depend on the parameters of considered subquantum model. Namely on parameter \( a \), or, equivalently, \( \tau_0 \) or \( \tau_i \) (resp. \( \beta \) or \( \beta_i \)). There is no indication how large or small this parameter can be.

(C1). Massive particles. In this part we shall propose possible physical values of parameters. There are different cases corresponding to a possible value of \( \tau_0 \).

We need to satisfy the following inequalities

\[
T \ll \tau_0, \quad \delta^2 \ll \frac{T\hbar}{3m}.
\]

We shall calculate values for \( m = m_e \), the mass of the electron.

(i) If we assume that

\[
\tau_0 \sim 10^{-9} s
\]

and using the approximate values \( \hbar \sim 10^{-34} Js \), \( m_e \sim 10^{-30} kg \) we obtain the possible values

\[
T, T_{sc} \sim 10^{-10} s, \\
\delta \sim 10^{-7} m
\]

and then

\[
L, L_{sc} \lesssim V.10^{-10} m,
\]

where \( V \) is the velocity of particles. If \( V = 0.3c \) then

\[
L, L_{sc} \lesssim V.10^{-2} m.
\]

(ii) If we assume

\[
\tau_0 \sim 10^{-11} s,
\]
then
\[ T, T_{sc} \sim 10^{-12}s, \]
\[ \delta \sim 10^{-8}m \]
and then for \( V = 0.3c \)
\[ L, L_{sc} \lesssim 10^{-4}m. \]

The appropriate value of \( T_0 \) will be in both cases
\[ T_0 \lesssim 0.1T. \]

The appropriate value of \( I \) will be \( I \gtrsim 100 \) but also \( I \gtrsim 10 \) may be already significant.

(C2). Photons. All experiments can be considered as optical experiments with photons. All formulas using the mass \( m \) are more or less meaningless.

We can consider all proposed experiments with photons, but
(i) without formulas containing the mass \( m \), i.e. the velocity
\[ V = c, \]
the times \( T = L/c, T_{sc} = L_{sc}/c. \)
(ii) We cannot calculate \( \beta \) (nor \( \tau_0 \)) from the parameter \( a \), but we can suppose that there is certain time \( \tau_0 \), possibly \( \tau_0 = \tau_0(\nu) \), where \( \nu \) is frequency of the light, which defines the relaxation time.
(iii) We can look for subquantum effects trying different combinations of parameters \( L, L_{sc}, \delta \). There are two inequalities which have to be satisfied
\[ T \ll \tau_0, \quad T_{sc} \ll \tau_0, \text{ i.e. } L, L_{sc} \leq \tau_0c. \]

In experiments Exp. 4-Exp. 6 with pulses we have to assume that the ”length” of the pulse is sufficiently short,
\[ cT_0 \ll L, L_{sc}, \]
where $T_0$ is the duration of the pulse.

The correlation effect is more clear, if the "length" of the pulse is sufficiently short, so that the pulse should be as short as possible.

The values of parameters:

(i) if, say, $\tau_0 = 10^{-9}s$, then $\tau_0 c = 0.3 \, m$ and this seems to be too long;

(ii) if, say, $\tau_0 = 10^{-11}s$, then $\tau_0 c = 3 \, mm$ and we can assume $L, L_{sc} \lesssim 1 \, mm$ and for the pulse we can suppose that $T_0 = 10^{-12}s$, i.e. $cT_0 \sim 0.3 \, mm$;

(iii) the best way is to look for the pulse as short as possible, say, $T_0 \lesssim 10^{-13}s$, i.e. $T_0 c \lesssim 0.03 \, mm$ and then consider the length $L, L_{sc} \gg T_0 c$.

The value of $\delta$ should be reasonable with respect to $L$ and $L_{sc}$.

The advantages of optical experiments are

(i) the possibility of extremally short pulses,

(ii) large range of wave phenomena,

(iii) good standard detectors.

The disadvantage

(i) there is not an explicit subquantum model.

A typical optical frequency is

$$\nu \sim 5.10^{14} Hz.$$ 

Corresponding energy is

$$E_{opt} = \hbar \nu \sim 3.10^{-19} J$$

and the "effective" mass is

$$m_{opt} = E_{opt} c^{-2} \sim 3.10^{-36} kg.$$ 

For $T \sim 10^{-10}s$ we obtain the "effective"

$$\delta \sim 3.10^{-4}.$$
We have presented subquantum models that can be useful in at least two directions:

(i) new phenomena in subquantum models,
(ii) new quantization procedure.

The new subquantum effects considered in this paper are:

(a) the concentration effect which says that under certain conditions particles move almost deterministically during short-time intervals,
(b) short pulses of particles move as a group.

The general subquantum models are based on the hypotheses of

(a) deterministic quantum model,
(b) the subquantum medium composed of space-like objects.

The influence of this subquantum medium on deterministic quantum particles is modelled by (quantum) random forces. There are two models: independent random forces (the model SubQM\(_{RF}\)) and correlated random forces (the model SubQM\(_{CRF}\)).

The short-time behavior of the subquantum models contain concentration and correlation effects, which are basis of proposed experiments distinguishing between QM and subquantum models.

We have proposed a new quantization procedure which divides quantization into two steps.

(i) The first step is to postulate the corresponding deterministic quantum model DetQM. We shall call this step the 0-th quantization. Construction of the deterministic quantum model is defined uniquely by the classical system, this model is completely local. The corresponding theory is the quantum model containing \(\hbar\). This is something like a ”bare” quantum theory. There are all quantum probability rules (Feynman’s rules), but the evolution is deterministic, without spreading out of wave functions. Heisenberg’s uncertainty principle is completely violated in this deterministic quantum model. There are states with an exact localization in the position and momentum variables. These states do not spread out and remain localized in the deterministic quantum model.
The second step consists in an introduction of the appropriate subquantum medium and, moreover, in representation of this medium as a quantum random force acting on the deterministic quantum system. The second step is a dynamical mechanism which assures (approximate) satisfaction of Heisenberg’s uncertainty principle.

There are at least two areas of application of this quantization schema:

(a) Quantum gravity,
(b) renormalization in QFT.

0-th quantization of a gravity. With respect to (i), an important step will be construction of deterministic quantum gravity (even construction of deterministic Newtonian quantum gravity would be a big step forward). As noted by many theoreticians (e.g. R. Penrose), there is the main conflict between locality of the General Relativity and non-locality of QT. We think that the construction of deterministic quantum gravity would be possible, since both theories, General Relativity and Deterministic QT are local. Thus there will be no conflict on locality. On the other hand, introduction of the subquantum medium may reflect more closely conditions at the Big Bang (or in the early period of the Universe).

Renormalization of QFT. The behavior of subquantum models on small distances is milder than in the standard QFT. There is a hope that the QFT will not require (infinite) renormalization and that the subquantum models of QFT will be finite. (The finite renormalization will be, of course, as useful as before.)
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