FRACTIONAL INPUT STABILITY AND ITS APPLICATION TO NEURAL NETWORK

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Abstract. This paper deals with fractional input stability, and contributes to introducing a new stability notion in the stability analysis of fractional differential equations (FDEs) with exogenous inputs using the Caputo fractional derivative. In particular, we study the fractional input stability of FDEs with exogenous inputs. A Lyapunov characterization of this notion is proposed and several examples are provided to explain the fractional input stability of FDEs with exogenous inputs. The applicability and simulation of this method are illustrated by studying the particular class of fractional neutral networks.

1. Introduction. Numerous works on stability notions exist in the literature, and deal with the stability of fractional differential equations (FDEs) without inputs. The stability notions in fractional calculus are: Mittag-Leffler [20], asymptotic [13, 20], fractional exponential [30], practical, and conditional (with respect to small input) [29] stability, among others. Numerous works [4, 12, 14, 33, 34, 35] exist on the stability notions of differential equations with exogenous inputs, and various results have been provided, see also [31]. All of these results were obtained with ordinary derivative. In the context of FDEs with exogenous inputs, many works have investigated the problem of stabilization or synchronization. The conditional asymptotic stability of FDEs with exogenous inputs has recently been introduced in the literature (see [29]). Conditional stability simply refers to the stability with respect to small input. In this paper, we focus on studying the input-to-state stability (ISS) in the context of the non-integer order. Traditionally, the ISS property of nonlinear systems requires that the norm of solutions be upper-bounded by a vanishing transient term, depending on the initial state, plus a term that is somewhat proportional to the magnitude of the input signal applied to the given system [8]. The ISS property was introduced by Sontag in 1989 [33]. In our opinion, this property can be studied in the context of FDEs with exogenous inputs, which can be demonstrated by obtaining the solution of the FDE defined by

\[ D_\alpha^\varepsilon x = Ax + Bu. \]
ISS in the context of the non-integer order is conventionally referred to as fractional input stability (FIS), see definition 2.11. We demonstrate that FIS exhibits two important properties: the converging input converging state (CICS) and bounded input bounded state (BIBS), as explained in section 2.

The fractional derivative has a long history, starting in 1695, when l’Hospital asked a remarkable question to Leibniz and the community of mathematicians: what does it mean when \( n = \frac{1}{2} \)? Numerous limiting definitions of a fractional derivative have been introduced to answer to this question, such as the Riemann-Liouville derivative [26, 28], Caputo derivative [26, 28], Gronwald-Letnikov derivative [13], conformable derivative [1, 16, 18], Atangana-Baleanu derivatives [5], and Atangana-Koca-Caputo fractional derivative [6]. Looking for these various types of fractional derivatives encouraged Machado and Ortigueira to ask the question: which derivative? [23]. Fractional calculus is a generalization of ordinary differentials and integration of an arbitrary non-integer order.

In this paper, we use the Caputo concept of the fractional derivative, which is defined later. This paper deals with the FIS of FDEs with exogenous inputs. The stability of nonlinear systems has received increasing attention, owing to its important role in the science and engineering fields. A large number of monographs and papers have been devoted to fractional nonlinear systems [11, 21, 37]; for variations in fractional calculus, see [3]. This paper deals with FIS and Lyapunov characterization of FDEs with exogenous inputs. In numerous cases, providing FIS using trajectories is very difficult; therefore, we propose an alternative using Lyapunov characterization to address this issue. Two particular properties will be studied, namely the CICS and BIBS (see subsection 2.2).

The remainder of this paper is organized as follows. In section 2, we introduce certain necessary definitions, describe the class of FDEs, and provide the main results. In section 3, we provide several examples and illustrate our main results. Our proofs, conclusions, and remarks are summarized in section 4.

2. Preliminary definitions and main results.

2.1. Preliminary definitions. In this section, we introduce certain notations and preliminary definitions that we will use to establish the main results. We begin by introducing the comparison functions that are fundamental to stability analysis, and easy to manipulate. \( \mathcal{PD} \) denotes the set of all continuous functions \( \alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) satisfying \( \alpha(0) = 0 \), and \( \alpha(s) > 0 \) for all \( s > 0 \). A class \( \mathcal{K} \) function is an increasing \( \mathcal{PD} \) function. The class \( \mathcal{K}_\infty \) represents the set of all unbounded \( \mathcal{K} \) functions. A continuous function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is said to be of class \( \mathcal{KL} \) if \( \beta(., t) \in \mathcal{K} \) for any \( t \geq 0 \), and \( \beta(s,.) \) is non-increasing and tends to zero as its arguments tend to infinity. Given \( x \in \mathbb{R}^n \), \( \|x\| \) stands for its Euclidean norm: 
\[ \|x\| := \sqrt{x_1^2 + \ldots + x_n^2}. \]

We begin with the fractional derivatives and fractional integral used in this paper.

Definition 2.1. [20, 28, 24] Given a function \( f : [0, +\infty[ \rightarrow \mathbb{R} \), the Riemann-Liouville fractional derivative of \( f \) of order \( \alpha \) is defined by
\[ D_\alpha^{RL} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds \tag{1} \]
for all \( t > 0 \), \( \alpha \in (0, 1) \), where \( \Gamma(.) \) is gamma function.
Definition 2.2. [20, 28] Given a function $f : [0, +\infty[ \rightarrow \mathbb{R}$, the Caputo fractional derivative of $f$ of order $\alpha$ is defined by

$$D^\alpha_c f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds$$

(2)

for all $t > 0$, $\alpha \in (0, 1)$, where $\Gamma(\cdot)$ is gamma function.

Definition 2.3. [9, 10] Given a function $f : [0, +\infty[ \rightarrow \mathbb{R}$, the Riemann-Liouville integral of $f$ of order $\alpha$ is defined by

$$I^\alpha_{RL} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$$

(3)

for all $t > 0$, $\alpha \in (0, 1)$, where $\Gamma(\cdot)$ is gamma function.

One can refer to Udita, whose works were devoted to generalizing the fractional integral [9, 15]. It is well known that the Riemann-Liouville and Caputo derivatives verify the Leibniz property. We define the Mittag-Leffler function, which is a generalization of the first-order exponential function. This function plays an important role in the stability analysis of FDEs. We have the following definition.

Definition 2.4. [2, 22, 25] Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$. The Mittag-Leffler function is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$  

(4)

The classical exponential $E_{1,1}(z) = \exp(z)$ is obtained by taking $\alpha = 1$ and $\beta = 1$. The following lemma will contribute to simplifying the calculation when we use the quadratic Lyapunov function.

Lemma 2.5. [7] Let $x \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any time $t \geq 0$, the following relationship holds

$$D^\alpha_c (x^T Px) \leq 2x^T PD^\alpha_c x \quad \alpha \in (0, 1)$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, square, symmetric and positive definite matrix.

For those who are interested in the right Caputo fractional derivative of the function $f(x) = |x|$, we provide the following result.

Lemma 2.6. [27] Let $\alpha \in (0, 1)$. Then, the Caputo derivative of $f$ is given by

$$D^\alpha_c |x| = \text{sign}(x) D^\alpha_c x$$

The explicit proof can be found in [27], but the following definition should be noted:

$$\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0 
\end{cases}$$

This right Caputo fractional derivative can play an important role in proving the fractional input stability of the fractional neural network.

We obtain the solution to the following FDE, which is fundamental to our main results. The form of the solution to the equation described in the following lemma is well known in the literature.
**Definition 2.11.** The FDE defined by (7) is said to be fractional input stable if,

\[ b, K > 0 \text{ and } \lambda \leq 0. \]

The FIS imposes an asymptotic gain (\( \gamma \)) decay of the state norm up to a function of the signal \( \|u\|_\infty \). In particular, the FIS offers two main properties, the motives for which are introduced in the literature.

We can observe from (9) that, if the input converges to zero, all solutions to the FDEs with exogenous inputs (7) converge asymptotically to zero, which is known.
as the CICS property. A further remark is that if all inputs are bounded, with (9), all of the solutions to the FDEs with exogenous inputs (7) are also well bounded, which is known as the BIBS property.

From the above definitions we obtain in particular the definition of the Mittag-Leffler input stability.

**Definition 2.12.** The FDE defined by (7) is said to be Mittag-Leffler input stable (MLIS) if, for any input $u \in \mathbb{R}^m$, there exist a class $\mathcal{K}_\infty$ function $\gamma$ such that for any initial condition $\|x_0\|$, its solution satisfies

$$\|x(t, x_0, u)\| \leq \{K \|x_0\| E_\alpha (\lambda(t - t_0)^{\alpha})\}^{\frac{1}{\beta}} + \gamma(\|u\|_\infty),$$

where $b, K > 0$ and $\lambda \leq 0$.

If the FDE with exogenous input is FIS (respectively, MLIS), the trivial equilibrium point ($x = 0$) of the unforced FDE $D^\alpha_0 x = f(t, x, 0)$ is uniformly globally asymptotically stable (respectively, Mittag-Leffler stable). The proof is straightforward to see, because $\gamma(0) = 0$ in (10) we recover (8). That is, the first fundamental consequence of the FIS is related to the trivial equilibrium point. This justifies the current study on the fractional neutral network, for which in many cases the external input is taken as zero.

**Definition 2.13.** [17] The origin of the FDEs defined by (7) without input ($u = 0$) is said to be globally uniformly asymptotically stable if there exists a class $\mathcal{KL}$ function $\beta$ such that, for any bounded initial condition $\|x_0\|$, its solution satisfies

$$\|x(t, x_0)\| \leq \beta(\|x_0\|, t - t_0).$$

For certain solutions that we will need to use later, we provide the right Caputo fractional derivatives [19] of B-splines with $t \geq 0$, $n \geq \lceil \alpha \rceil$, and $\alpha \geq 0$:

$$D^\alpha_0 t^n = \frac{\Gamma(n + 1)}{\Gamma(n - \alpha + 1)} t^{n - \alpha}.$$

2.2. **Basic properties of FIS.** In this section, we study the CICS and BIBS properties of the FDEs with exogenous inputs. We wish to explain explicitly what the FIS (respectively, MLIS) really means, by providing certain basic examples.

- We begin with the fractional linear differential equation, which is mathematically represented by the following form:

$$D^\alpha_0 x = Ax + Bu,$$

where $x \in \mathbb{R}^n$ is a state variable, $A$ is a matrix in $\mathbb{R}^{n \times n}$, $B$ is a matrix in $\mathbb{R}^{n \times m}$, and $u \in \mathbb{R}^m$ is the exogenous input. The solution to the FDE (12) using Lemma 2.7 is given by

$$x(t) = x_0 E_\alpha (A(t - t_0)^{\alpha}) + \int_{t_0}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^{\alpha}) Bu(s)ds$$

from which it follows that

$$\|x(t)\| \leq \|x_0\| \|E_\alpha (A(t - t_0)^{\alpha})\| + \|B\| \|u\| \int_{t_0}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^{\alpha}) ds.$$

If the state matrix $A$ satisfies the condition $|\arg(\lambda(A))| > \frac{\alpha \pi}{2}$, according to [28], there exist a positive constant $M$ such that

$$\int_{t_0}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^{\alpha}) ds \leq M.$$
Then, we have the following inequality:

$$\|x(t)\| \leq \|x_0\|\|E_\alpha (A(t-t_0)^\alpha)\| + \|B\|\|u\|_{\infty} M.$$  \hspace{1cm} (13)

We can observe that, if we take a converging input $\|u\| \to 0$, the trivial solution to the FDE satisfies $\|x(t)\| \leq \|x_0\|\|E_\alpha (A(t-t_0)^\alpha)\|$, and all solutions converge to zero asymptotically, which is the CICS property. According to the condition of (13), we can observe that a bounded input generates a bounded state, which is the BIBS property.

It is important to note that we have any information on the state if the state matrix $A$ does not satisfy the classical condition $|\arg(\lambda(A))| > \frac{\alpha \pi}{2}$. This is fundamental if we are aiming for the FIS or MLIS property of the fractional linear differential equations. We will also prove later, using the Lyapunov characterization, that the condition is necessary and sufficient for the MLIS of the fractional linear differential equations.

- We now investigate what occurs with the fractional bilinear differential equation, mathematically represented in the following form:

$$D^c_\alpha x = Ax + Bu$$  \hspace{1cm} (14)

where $x \in \mathbb{R}^n$ is a state variable, $A$ is a matrix in $\mathbb{R}^{n \times n}$, $B$ is a matrix in $\mathbb{R}^{n \times m}$, and $u \in \mathbb{R}$ is the exogenous input (can depend on time), satisfying the CICS and BIBS properties. Note that, without lost generality, we suppose that the state matrix $A$ satisfies the classical condition $|\arg(\lambda(A))| > \frac{\alpha \pi}{2}$. For example, we take the following particular fractional bilinear differential equation, defined by

$$D^c_\alpha x = -x + xu.$$  \hspace{1cm} (15)

If we take the exogenous input $u = 0$, we obtain $D^c_\alpha x = -x$, the solution for which is given by

$$x(t) = x_0 E_\alpha (-\gamma) .$$

In other words, the fractional bilinear differential equation without input, defined by (15), is Mittag-Leffler stable.

It is demonstrated in [29] that these categories of FDEs are asymptotically stable with respect to the small inputs. In other words, they are conditionally asymptotically stable if the condition $\|u\| < 1$, as can be observed here. This condition is important because, for example, if the exogenous input takes a constant value $u(t) = 3$, the fractional bilinear differential equation defined by (15) becomes $D^c_\alpha x = 2x$, the solution for which is given by

$$x(t) = x_0 E_\alpha (2\gamma) .$$

We can observe that, if there exists a trajectory starting at $x_0 = 1$, the above solution diverges (see Figure 1). Perhaps the input is bounded, but the generated solution is not. In this case, we do not have the BIBS property, because the fractional bilinear differential equation defined by (15) is not MLIS.

Regarding this example, it is now certain that the fractional bilinear differential equations are not FIS or MLIS in general.

- We investigate the following example. The considered scalar FDE is mathematically represented by the following form:

$$D^c_\alpha x = -x + \sqrt{2t + 2(x + \sqrt{\gamma})u}.$$  \hspace{1cm} (16)
Let $x_0 = 1$, $\alpha = 1/2$ and $u = \frac{1}{\sqrt{2t}+2}$; we can observe that the input converges to zero and is bounded. Under these conditions, the FDE becomes

$$D^{\alpha/2}_c x = \sqrt{x}.$$ 

Its solution is given by $x(t) = \frac{2}{\sqrt{\pi}}t$, which clearly diverges. We observe that the applied input is bounded and converges to zero, but the generated solution is not bounded and diverges. Thus, the BIBS and CICS properties are not satisfied.

Observing all of these examples in order to guarantee acceptable behavior of the stable FDEs with exogenous inputs is necessary for the required CICS and BIBS properties.

2.3. Results and Lyapunov characterization. At this point, we can state the main results of this paper, which are provided in Section 4. In general, the considered FDEs are mathematically represented in the following form:

$$D^\alpha_c x = f(t, x, u),$$

where $x \in \mathbb{R}^n$ is a state variable, $u \in \mathbb{R}^m$ represents the exogenous input, and $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous locally Lipschitz function satisfying Caratheodory conditions and $f(t, 0, 0) = 0$. Note that, with FIS, we must prove the existence of the trivial equilibrium point of the FDE without input. This necessity makes FIS an interesting stability notion of the FDEs in the form of (17). This is also significant in the application of this notion in fractional neural network work.

For the fractional linear differential equations, we provide the following result to prove the fractional Mittag-Leffler input stability.

**Theorem 2.14.** If the state matrix $A$ with $\alpha \in (0, 1)$ satisfies the condition $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, the fractional differential linear equation (12) is fractional MLIS.

The proof for this theorem using the solution is provided in Section 2. Furthermore, we prove this theorem using Lyapunov characterization in Section 3.

As determining the FDE solutions can become highly complex, we propose an alternative for studying the FIS of FDEs with exogenous inputs.

**Theorem 2.15.** Let there exist a positive function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ that is continuous and differentiable, and a class $K_\infty$ of functions $\chi_1, \chi_2$, and class $K$ functions $\chi_3, \chi_4$, satisfying the following conditions:
where \( \theta \) and \( \chi \) have been selected.

1. \( \chi_1(||x||) \leq V(t,x) \leq \chi_2(||x||) \).
2. \( V(t,x) \) has a Caputo fractional derivative of order \( \alpha \) for all \( t > t_0 \geq 0 \).
3. If for any \( ||x|| \geq \chi_4(||u||) \) implies \( D_\alpha^\alpha V(t,x) \leq -\chi_3(||x||) \).

Then, the FDE (17) with exogenous input is fractional input stable.

This theorem was first introduced and proven in a first integer order differential equation by Sontag in [33]. An explicit proof of this theorem in a first-order differential equation was provided in [36]. However, a similar proof cannot be adopted in the non-integer order context, because we can not achieve the monotonicity of the Lyapunov function using the Caputo derivative. Therefore, the most appropriate option is the use of the fractional differential comparison lemma.

3. Fractional input stability of particular class of fractional neural network. In this section, two examples are provided in order to illustrate the Lyapunov characterization of FIS provided in the previous section. We begin with an introductory example. We firstly study the FIS of the fractional bilinear differential equation.

- For an illustration of our results, let the following fractional linear system be defined as

\[
D_\alpha^\alpha x = Ax + Bu, \tag{18}
\]

where \( x \in \mathbb{R}^n \) is a state variable, \( A \) is a matrix in \( \mathbb{R}^{n \times n} \) satisfying the condition \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \), \( B \) is a matrix in \( \mathbb{R}^{n \times n} \), and the exogenous input is \( u \in \mathbb{R}^n \). We select a Lyapunov candidate function \( V(t,x) = x^T P x \), where \( A^T P + PA = -Q \) and \( P = I_n \). The \( \alpha \)-derivative of \( V \) along the trajectories of (18) is as follows:

\[
D_\alpha^\alpha V(t,x) \leq 2x^T PD_\alpha^\alpha x = (Ax + Bu)^T P x + x^T P (Ax + Bu) = x^T A^T P x + (Bu)^T P x + x^T P (Bu) = x^T (A^T P + PA) x + (Bu)^T P x + x^T P (Bu) \leq -\lambda_{\min}(Q) ||x||^2 + 2\lambda_{\max}(P) ||B|| ||u|| ||x|| \leq -(1-\theta) \lambda_{\min}(Q) ||x||^2 - \theta \lambda_{\min}(Q) ||x||^2 + 2\lambda_{\max}(P) ||B|| ||u|| ||x||,
\]

where \( \theta \in (0, 1) \). We can see that, if \( ||x|| \geq \frac{2\lambda_{\max}(P)||B||||u||}{\lambda_{\min}(Q)} \),

\[
D_\alpha^\alpha V(t,x) \leq 2x^T PD_\alpha^\alpha x \leq -(1-\theta) \lambda_{\min}(Q) ||x||^2,
\]

where \( \chi_4(r) = \frac{2\lambda_{\max}(P)||B||r}{\lambda_{\min}(Q)} \). Using Theorem 2.15, we conclude that the fractional linear differential equation is MLIS if the state matrix satisfies the condition \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \). Note that the condition \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \) consequently implies that the matrix \( Q \) is positive definite. We can also adopt the following method:

\[
D_\alpha^\alpha V(t,x) \leq 2x^T PD_\alpha^\alpha x \leq -\lambda_{\min}(Q) ||x||^2 + 2\lambda_{\max}(P) ||B|| ||u|| ||x||.
\]

Select any \( \theta \in (0, 1) \) and set

\[
k = \frac{2 ||P|| ||B||}{\lambda_{\min}(Q) - \theta} \chi_4(r) = kr.
\]

Thus, if \( ||x|| \geq \chi_4(||u||) \) it is implied that \( D_\alpha^\alpha V(t,x) \leq -\theta ||x||^2 \). Then, we obtain the Mittag-Leffler input stability of the given FDE.
Consider the bi-dimensional FDE defined by
\[
\begin{align*}
D_\alpha^c x_1 &= -x_1 + x_2^2 \\
D_\alpha^c x_2 &= -x_1 x_2 - x_2 + u
\end{align*}
\] (19)
where \( x = [x_1, x_2]^T \in \mathbb{R}^2 \). We select a Lyapunov candidate function defined by the function
\[ V(t, x) = \frac{1}{2} \| x \|^2 \].

The \( \alpha \)-derivative of \( V \) along the trajectories of (19) is determined using Lemma 2.5
\[
D_\alpha^c V(t, x) \leq x^T D_\alpha^c x = -x_1^2 + x_2^2 x_1 - x_2^2 x_1 - x_2^2 + x_2 u
\]
\[
= -\| x \|^2 + x_2 u
\]
\[
\leq -(1 - \theta) \| x \|^2 - \theta \| x \|^2 + | x_2 | \| u \| ,
\]
where \( \theta \in (0, 1) \). Applying the same reasoning as in the above example and based on the fact that \( | x_2 | \leq \| x \| \), we can observe that, if \( \| x \| \geq \frac{| u |}{\gamma} \), we have that
\[
D_\alpha^c V(t, x) \leq x^T D_\alpha^c x \leq -(1 - \theta) \| x \|^2 .
\]
Using Theorem 2.15, we conclude that the fractional linear differential equation (19) is FIS.

Consider a particular class of a fractional neutral network described by
\[
D_\alpha^c x = -Ax + Tg(t, x) + I,
\] (20)
where \( A = \text{diag}(a_1, a_2, ..., a_n) \in \mathbb{R}^{n \times n} \) is the matrix of self-regular parameters of the neurons, satisfying the condition \( \| \text{arg}(\lambda(-A)) \| > \frac{\alpha\pi}{2} \), \( T \in \mathbb{R}^{n \times n} \) is the interconnection matrix, the function defined by
\[
g(t, x) = (g_1(t, x_1), g_2(t, x_2), ..., g_n(t, x_n))^T
\]
is a neuron input-output activation function and satisfies in the paper \( g(0) = 0 \), and \( I = (I_1, I_2, ..., I_n)^T \) designs an exogenous constant input. Explicit statements on the model are not detailed in this paper. We furthermore assume no delay, else the result may change considerably. The objective here is to present the FIS. In a forthcoming paper by us or another author it will be specified whether or not there is delay. One can refer to neutral network systems for further details. The above model is that after proving the existence of the equilibrium point.

Let a Lyapunov candidate function be defined by the function \( V(t, x) = \| x \| \). The \( \alpha \)-derivative of \( V \) along the trajectories of (20) is determined using lemma (2.6):
\[
D_\alpha^c V(t, x) = D_\alpha^c | x | = \text{sign}(x) D_\alpha^c x
\]
\[
= \text{sign}(x) [-Ax + Tg(t, x) + I]
\]
\[
= -A \text{sign}(x) x + T \text{sign}(x) g(t, x) + \text{sign}(x) I
\]
\[
\leq -k | x | + \| T \| \| g(t, x) \| + | I | .
\]
We assume that \( | g(t, x) | \leq \gamma | x | \), and note that the existence of the constant \( k \) stems from the fact that the matrix \( A \) is diagonal. Thus, we obtain
\[
D_\alpha^c V(t, x) = D_\alpha^c | x | \leq -k | x | + \| T \| \| g(t, x) \| + | I |
\]
\[
\leq -k | x | + \| T \| \gamma | x | + | I |
\]
\[
\leq -(1 - \theta) k | x | - \theta k | x | + \| T \| \gamma | x | + | I | ,
\]
where \( \theta \in (0, 1) \). Then, if \( | x | \geq \frac{\| I \|}{\theta k + \| T \| \gamma} \), it is implied that \( D_\alpha^c V(t, x) \leq -(1 - \theta) k | x | \). Using theorem 2, we conclude that the fractional neutral network is fractional.
input stable with respect to the external input $I$. The consequence is interesting: the FDE (20) without exogenous input ($I = 0$) admits an equilibrium point; particularly, $x = 0$, which is asymptotically stable (obligatory the origin if it exist). For simulation, one can replace $|.|$ with the norm $||.||$. For numerical illustration (see similar model in [38]), we consider a particular class of the fractional differential neutral network with

$$
A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0.6 & -0.3 \\ -0.37 & -0.1 \end{pmatrix}
$$

$I = (-1.7, 1.2)^T$ and $g(t, x) = (\sin x_1, \sin x_2)^T$. The fractional differential neural network is expressed by

$$
\begin{align*}
D_c^\alpha x_1 &= -2x_1 + 0.6 \sin x_1 - 0.3 \sin x_2 - 1.7 \\
D_c^\alpha x_2 &= -x_2 - 0.37 \sin x_1 - 0.1 \sin x_2 + 1.2.
\end{align*}
$$

(21)

The fractional neutral network (21) is fractional input stable if the following condition is satisfied: $\|x\| \geq \frac{\|I\|}{\theta k_1 + \gamma}$, then $D_c^\alpha V(t, x) \leq - (1 - \theta) \|x\|$, where $k_1 = 1 = \lambda_{\text{min}}(A)$. Finally, the solution satisfies with $k_1 = 1$, $\theta = 0.5$, and $x_0 = 1$:

$$
\|x(t, x_0, u)\| \leq E_\alpha (-0.5t^\theta) + \gamma(\|u\|),
$$

where $\gamma(s) = cs$ with $c = \frac{1}{\theta k_1 + \gamma}$. One can refer to the simulation described in Figure 2 for the solution behavior with $\alpha = 0.75$. It can explicitly be observed that, if the state norm exceeds 1.89, all of the states converge, which is significant behavior offering FIS. Thus, in particular, if we take $I = 0$, the perturbed systems admit $x = 0$ (another value may be possible but with FIS, only the trivial solution interests us) as an equilibrium point, which is automatically asymptotically stable. In the simulation described in Figure 3, the state behavior with time can be observed. A further conclusion is that, under all bounded exogenous input, the generated state is also bounded.

Fractional input stability exhibits three important properties: the converging input converging state (CICS), bounded input bounded state (BIBS), and the trivial solution of FDE obtained when no controls are applied is uniformly globally asymptotically stable (0-GAS).

![Figure 2. FIS of fractional neural network](image-url)

$x = 0$ (another value may be possible but with FIS, only the trivial solution interests us) as an equilibrium point, which is automatically asymptotically stable. In the simulation described in Figure 3, the state behavior with time can be observed. A further conclusion is that, under all bounded exogenous input, the generated state is also bounded.

Fractional input stability exhibits three important properties: the converging input converging state (CICS), bounded input bounded state (BIBS), and the trivial solution of FDE obtained when no controls are applied is uniformly globally asymptotically stable (0-GAS).
Claim 1. If there exists $t_0 \geq 0$ such that $x_0 \in S$, $x(t) \in S$ for all $t \geq t_0$.

**Proof.** Suppose this is not the case. Then, there exists $\epsilon > 0$ such that $V(t) > \epsilon + c$. Let the set $\tau = \inf \{t \geq t_0 : V(t) \geq \epsilon + c\}$. It then follows that $\|x(\tau)\| \geq \chi_4(\|u\|)$, from which we obtain under the assumption (3) $D^\alpha_0 V(t)_{t=\tau} \leq -\chi_3 \circ \chi_2^{-1}(V(t))_{t=\tau} < 0$ for all $t \in [t_0, \tau]$. Using the fractional differential comparison lemma, we have $V(t) \geq V(\tau)$. We notice that this contradicts the minimality of the set $\tau$. Thus, $x(t) \in S$ for all $t \geq t_0$. Let $t_1 = \inf \{t \geq 0 : x(t) \in S\}$, and according to the above reasoning, it follows that $V(t) \leq \chi_2 \circ \chi_4(\|u\|)$ for all $t \geq t_1$. Under the first assumption of the theorem, we obtain
\[
\|x(t)\| \leq \chi_1^{-1} \circ \chi_2 \circ \chi_4(\|u\|).
\] (22)

Let $\gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4$. According to the comparison function, it is clearly a $\mathcal{K}_\infty$ function.

**Claim 2.** There exists a $\mathcal{KL}$ function $\beta$ such that, for each $\|x_0\|$ and each bounded control $u$, there exists a time instant $T > 0$ (necessarily $T = t_1$) such that $\|x(t)\| \leq \beta(\|x_0\|, t-t_0)$ for all $t \leq t_1$, and $x(t) \in S$ for all $t \geq t_1$.

**Proof.** For all $t \leq t_1$, $x(t) \notin S$, which implies $\|x(t)\| \geq \chi_4(\|u\|)$, and under assumption (3), we have $D^\alpha_0 V(t) \leq -\chi_3(\chi_2^{-1}(V(t)) = -g(V(t))$. Let $y(t)$ be the solution to the equation defined by $D^\alpha_0 y = -g(y)$, whenever $y(t_0) \geq V(t_0)$. According to Sene in [32], there exists a class $\mathcal{KL}$ function $\sigma$ such that
\[
V(t) \leq \sigma(V(t_0), t-t_0).
\]

Now, from the inequality in assumption (1), we obtain
\[
\|x(t)\| \leq \chi_1^{-1} \circ \sigma(V(t_0), t-t_0).
\]

We let the function $\beta(\|x_0\|, t-t_0) = \chi_1^{-1} \circ \sigma(V(t_0), t-t_0)$, which is clearly also a class $\mathcal{KL}$ function. Then, we conclude that
\[
\|x(t)\| \leq \beta(\|x_0\|, t-t_0) \quad \text{for all } t \in [t_0, t_1].
\] (23)
Furthermore, superposing the solutions in (22) and (23), we obtain
\[ \|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|u\|_\infty) \]
for all \( t \geq t_0 \), where the asymptotic gain is given by \( \gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4 \in \mathcal{K}_\infty \).

5. **Conclusion.** In this paper, we have discussed the FIS of FDEs with exogenous input using a Caputo derivative. We have introduced a Lyapunov characterization for FIS when obtaining the explicit solution to the FDE becomes complicated. Several examples have been provided to explain the FIS concept. As a result, all applications using FDEs with exogenous inputs are open to investigation. The author encourages, in particular, the application of FIS to fractional neural networks with delay. Owing to space limitations, numerous examples and the fractional Mittag-Leffler input stability characterization have been omitted. Several examples will be provided in the forthcoming note.

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