Numerical study on bending response of auxetic 2D-lattice plates

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Abstract
In this study, the bending response of 2D-lattice plates having auxetic unit cells is determined by using the homogenization method based on equivalent strain energy. The auxetic unit cells in this study are different topologies of re-entrant hexagonal unit cells. In the homogenization method, the effective out-of-plane elastic properties of an auxetic 2D-lattice plate are obtained from the strain energy values of its unit cell under different curvature modes, determined by the finite element method. In the analysis, the auxetic unit cells are considered as frames whose struts are modeled as Euler beams. The effective elastic properties, i.e., the effective bending moduli, Poisson’s ratios, and shear modulus, are described as the bending response of a 2D-lattice plate subjected to out-of-plane bending. In the validation, the effective elastic properties of some auxetic 2D-lattice plates obtained from their unit cells by the homogenization method are numerically compared with those obtained from direct structural analysis of the plates. Besides, the obtained results show how the bending response of the auxetic 2D-lattice plates can be adjusted by varying their unit-cell geometries, especially the internal cell angle.

Keywords: Auxetic structure, Re-entrant hexagon, Periodic plate, Bending modulus, Negative Poisson’s ratio.

1. Introduction
The two-dimensional lattices are periodic structures that have a wide range of applications. The 2D lattices are characterized by having a unit-cell pattern repeated on a plane in two orthogonal directions. Since mechanical properties can be designed to yield the desired properties, directly depending on their unit-cell patterns, 2D lattices have been explored by many researchers [1-9]. Two-dimensional lattices can be modeled as frames and used as plane or plate structures. If there are enough unit cells in a 2D lattice, it can be considered as an equivalent homogeneous material. The material properties of the equivalent homogeneous material of a 2D lattice can be represented by effective elastic properties that can be determined by homogenization methods [3-6, 10-13]. One of the most widely used homogenization methods is the homogenization method based on equivalent strain energy. In the homogenization method, the strain energy in an equivalent homogeneous material, under uniform far-field boundary conditions that yield uniform stress and strain fields in the equivalent homogeneous material, is considered to be the same as the strain energy in its original inhomogeneous material, under
the same boundary conditions [14]. The effective material law of the inhomogeneous material can be obtained by considering the relationship between the average stresses and strains that occur in the material under these boundary conditions. Since 2D lattices are modeled as frames, their struts can be considered as beams. The effective material law of a 2D lattice can be determined from frame analysis of a selected unit cell of the lattice using the homogenization method.

One of the popular types of 2D lattices is honeycombs whose unit-cell patterns are typically in hexagonal geometries. Among various hexagonal unit-cell patterns, re-entrant hexagonal unit cells are featured because they can exhibit negative Poisson’s ratio for 2D lattices or the so-called auxetic 2D lattices. An auxeticity of a material can be described by its specific geometries [15,16]. For an auxetic 2D lattice, the auxeticity can be designed by its unit-cell topologies. Mechanical properties of auxetic 2D lattices have been expansively investigated or designed by many researchers [1,4,10,15,17-20]. Most of the researchers focused on the elastic properties of auxetic 2D lattices subjected to in-plane loading. For example, auxetic cellular structures were suggested as 2D lattices with re-entrant unit cells by Gibson et al. [1]. In their work, the auxetic effect of the 2D lattices was analytically determined. In the work by Scarpa et al. [19], the effective in-plane elastic properties, including the effective Young’s moduli and Poisson’s ratio, of 2D lattices with re-entrant hexagonal unit cells were investigated using finite element simulation and experiment. Effects of various geometries of the unit cells (aspect ratio, thickness, and internal cell angles) on the effective elastic properties were also investigated. Closed-form expressions of the effective in-plane elastic constants of conventional and re-entrant hexagonal lattices, considering variable dimensional parameters, e.g., aspect ratio and internal cell angle, were derived by Zhang et al. [10]. In the work by Alderson et al. [20], the finite element method was used to simulate the curvatures of various auxetic 2D lattices under out-of-plane bending.

A 2D lattice used as a plate structure can be termed as a 2D-lattice plate. When a common 2D-lattice plate is subjected to an out-of-plane bending moment, it is expected that the resulting curvature on the perpendicular edges occurs in the opposite direction of those on the edges that the moment applied. On the other hand, an auxetic 2D-lattice plate yields the resulting curvature in the same direction as those happening on the applied moment edges. This is a notable property resulting from negative Poisson’s ratio of auxetic 2D-lattice plates. For this reason, in this study, the attractive properties of auxetic 2D-lattice plates are numerically investigated. The effective out-of-plane elastic properties of 2D-lattice plates with auxetic unit cells are determined using the homogenization method based on equivalent strain energy. The considered auxetic unit cells are different re-entrant hexagonal unit cells. These unit cells are obtained by adapting internal cell angles and the aspect ratio of hexagonal geometry. The effective elastic properties proposed in this study include the effective bending moduli, Poisson’s ratios, and shear modulus. In the validation, the effective elastic properties of some auxetic 2D-lattice plates obtained from their unit cells by the homogenization method are compared with those obtained from direct structural analysis of the plates. All analyses are done by the finite element method. The obtained results show how the effective out-of-plane elastic properties of 2D-lattice plates with auxetic unit cells can be adjusted by varying the geometries of the unit cells.

2. Effective Out-of-Plane Properties of 2D-Lattice Plates
The effective out-of-plane properties in this study are used to represent the bending response of 2D-lattice plates and can be determined by using the homogenization method based on equivalent strain energy. In the homogenization method, the strain energy in an equivalent homogeneous plate, under periodic far-field boundary conditions, is equal to the strain energy in its original periodic plate, under the same boundary conditions [6,14]. Since under periodic far-field boundary conditions, the displacement, stress, and strain fields of a periodic plate are periodic, a unit cell of the plate under periodic boundary conditions is sufficient for creating these periodic response fields. The detailed information of the method can be found in [6]. Only important equations for the determination of the effective properties of 2D-lattice plates are briefly shown.

Consider a thin periodic plate, composed of a significantly large number of unit cells, in a coordinate system \(x_1 - x_2 - x_3\) that its origin lies on a midplane \(A\). The periodicity of the plate is only in the \(x_1\) –
\(x_2\) plane. A set of periodic kinematic boundary conditions is applied to the plate in a way that results in the following deflection \(w\) that is a displacement in \(x_3\) direction, i.e.,

\[
w(x_1, x_2) = w^o(x_1, x_2) + w^p(x_1, x_2) = \left[ \frac{\kappa_{11}^o}{2} x_1^2 + \frac{\kappa_{22}^o}{2} x_2^2 + \kappa_{12}^o x_1 x_2 \right] + w^p(x_1, x_2)
\]

(1)

where \(\kappa_{ij}^o\) is a constant symmetric tensor, and \(w^p\) is a periodic function of \(x_1\) and \(x_2\). In addition, \(w^o\) represents the deflection of a homogeneous plate subjected to uniform kinematic boundary conditions that yield a constant curvature of \(\kappa_{ij}^o\). The deflection \(w\) in equation (1) is extended everywhere in the midplane of the plate, including the inside of any voids, which can be considered as the limit cases of infinitely soft inclusions [14]. The rotation \(\phi_i\) and curvature \(\kappa_{ij}\) of the periodic plate are respectively defined as \(\phi_i = w_i\) and \(\kappa_{ij} = w_{ij}\) [6].

The average of \(\kappa_{ij}\) over the midplane area \(A\) of the plate can be represented by \(\kappa_{ij}^o\) [6], i.e.,

\[
\langle \kappa_{ij} \rangle = \frac{1}{A} \int_A \kappa_{ij} dA = \kappa_{ij}^o.
\]

(2)

The equation of the effective stiffness tensor \(D_{ijkl}^*\) of the periodic plate is defined as

\[\langle \kappa_{ijkl} \rangle = \frac{1}{A} \int_A \kappa_{ijkl} dA = -D_{ijkl}^* (\kappa_{kl}) = -D_{ijkl}^* \kappa_{kl}^o
\]

(3)

where \(\langle \kappa_{ijkl} \rangle = \kappa_{ij}^o\) is the average of the bending moments \(\kappa_{ijkl}\) over \(A\).

The average strain energy of the plate \(\bar{U}\) over \(A\), and the strain energy of the unit cell \(U_C\) are respectively given by [6]

\[
\bar{U} = -\frac{1}{2A} \int_A \kappa_{ijkl} dA = -\frac{1}{2} \langle \kappa_{ijkl} \rangle \langle \kappa_{ij} \rangle = \frac{1}{2} D_{ijkl}^* \kappa_{kl}^o \kappa_{ij}^o;
\]

(4)

\[U_C = \bar{U} A_C = \frac{1}{2} D_{ijkl}^* \kappa_{kl}^o \kappa_{ij}^o A_C
\]

(5)

where, \(A_C\) denotes the midplane area of the unit cell. Equation (5) can be used to determine \(D_{ijkl}^*\). In the determination, different modes of \(\kappa_{ij}^o\) are applied to the unit cell and the values of \(U_C\) are determined from structural analysis. Here, the curvature modes of \(\kappa_{ij}^o\) are created via periodic boundary conditions obtained from equation (1).

For an orthotropic periodic plate, equation (3) can be written in matrix form as

\[
\begin{pmatrix}
D_{11}^* \\
D_{22}^*
\end{pmatrix} = \begin{pmatrix}
D_{11}^* & D_{12}^* & 0 \\
D_{22}^* & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\kappa_{11}^o \\
\kappa_{22}^o
\end{pmatrix}
\]

(6)

Since there are four independent effective constants \(D_{ij}^*\) in the above equation, i.e., \(D_{11}^*, D_{22}^*, D_{33}^*,\) and \(D_{12}^*\), four different modes of \(\kappa_{ij}^o\) in equation (5) are required [6]. Denote the strain energy of the unit cell to which \(\kappa_{ij}^o = \kappa_{ij}^{o(i)}\), where \(i\) denotes the modes of \(\kappa_{ij}^o\), is applied as \(U_C^{(i)}\). The relationship between
$U_C^{(i)}$ and $\kappa_{ij}^{(i)}$ are used to compute the corresponding values of $D_{ij}^*$. The effective out-of-plane properties of 2D-lattice plates can be written in terms of $D_{ij}^*$ as

$$E_1^* = \frac{12 D_{11}^*}{t^3} \left[ 1 - \frac{D_{12}^*}{D_{11}^* D_{22}^*} \right]; \quad E_2^* = \frac{D_{22}^*}{D_{11}^*} E_1^*;$$

$$\nu_{12}^* = \frac{D_{12}^*}{D_{22}^*}; \quad \nu_{21}^* = \frac{D_{12}^*}{D_{11}^*}; \quad G_{12}^* = \frac{12 D_{33}^*}{t^3}. \quad (7)$$

Here, $t$ is the thickness of the plate. Since $D_{ij}^*$ is the function of $U_C^{(i)}$, the effective properties are also the function of $U_C^{(i)}$ and can be obtained directly in terms of $U_C^{(i)}$.

Besides, $E_i^*$, $\nu_{ij}^*$, and $G_{12}^*$ can be computed by direct structural analysis of a whole 2D-lattice plate. The effective stiffness equations for unidirectional bending of a thin periodic are given by [6]

$$E_1^* = \frac{12 M_{11}^0}{t^3 \kappa_{11}^0}; \quad E_2^* = \frac{12 M_{22}^0}{t^3 \kappa_{22}^0}; \quad \nu_{12}^* = \frac{\kappa_{22}^0}{\kappa_{11}^0}; \quad \nu_{21}^* = - \frac{\kappa_{11}^0}{\kappa_{22}^0}; \quad G_{12}^* = \frac{6 M_{12}^0}{t^3 \kappa_{12}^0}. \quad (8)$$

### 3. Re-Entrant Hexagonal Unit-Cell Geometry

The auxetic unit cells considered in this study are re-entrant hexagonal unit cells as shown in figure 1. Two sets of unit-cell struts are defined as vertical struts of length $h$ and inclined struts of length $l$. All struts are the same cross section area. A hexagonal unit-cell geometry is defined by some dimensionless parameters, e.g., internal cell angle $\theta$ and aspect ratio $h/l$ of the unit cell. Different hexagonal unit-cell geometries can be obtained by varying $\theta$ and $h/l$. Figure 1, in which $\theta$ are all negative, shows different re-entrant hexagonal unit cells. For the unit cells with $h = l$, to keep an appropriate hexagonal shape, the negative internal cell angle should be limited when the angle is close to $-30^\circ$. If the negative internal cell angle is required over the angle limit, the re-entrant hexagonal unit cells with $h > l$ can be used to avoid collision of vertical and inclined struts as shown in figure 1(b). The coordinate system in figure 1(a) is used for all unit cells. The boundaries of the unit cells are presented as dashed lines. The areas inside the boundary edges can be defined as the midplane areas $A_C$ of the unit cells.

![Figure 1. Different geometries of re-entrant hexagonal unit cells](image-url)
4. Results and Discussion

The effective out-of-plane elastic properties of 2D-lattice plates with re-entrant hexagonal unit cells shown in figure 1 are determined by using the homogenization method. In this investigation, all struts of the unit cells are modeled as Euler beams. The inclined strut’s length \( l \), defined in figure 1, is set to 1. The cross-sections of all struts in the 2D-lattice plates are set to be square sections of 0.1×0.1. The struts are placed such that the thickness of the 2D-lattice plates is also equal to 0.1. Young’s modulus and Poisson’s ratio of the base material that is isotropic are set to 1.0 and 0.2, respectively. Note that, to investigate the trend of bending response of the plates, mathematically, it is not necessary to use physically real values. In the homogenization method, numerical results of strain energy, \( U_c^{(1)} \), \( U_c^{(2)} \), \( U_c^{(3)} \), and \( U_c^{(4)} \), for the four modes of \( \kappa_i^j \), can be obtained from finite element analysis. After that, the effective elastic properties in equations (7) are numerically determined using the obtained strain energy values. Figure 2 shows plotting results of the effective elastic properties, \( E_i^* \), \( v_{ij}^* \), and \( G_{12}^* \) against \( \theta \), plotted by every 5\(^\circ\), from 0\(^\circ\) to −30\(^\circ\). It can be seen from the results that, when \( \theta \) is different from 0\(^\circ\) to negative, the obtained 2D-lattice plates are more auxetic. Reducing \( \theta \) from 0\(^\circ\) to −30\(^\circ\) obviously increases the value of \( E_1^* \) but slightly reduces the value of \( E_2^* \). Note that, \( E_1^* \) is always greater than \( E_2^* \) and, at \( \theta = −30\(^\circ\) \), \( E_1^* \) is significantly greater than \( E_2^* \). For \( v_{12}^* \) and \( v_{21}^* \), the negative values of both results increase when \( \theta \) is reduced to more negative. The absolute values of \( v_{12}^* \) are always greater than those of \( v_{21}^* \). Reducing \( \theta \) continuously from 0\(^\circ\) to −30\(^\circ\) slightly increases the values of \( G_{12}^* \).

![Figure 2](image-url)  
*Figure 2. Plots of the effective elastic properties, \( E_i^* \), \( v_{ij}^* \), and \( G_{12}^* \) against \( \theta \).*

To validate the results obtained from the homogenization method, direct structural analysis of the 2D-lattice plates with auxetic unit cells is performed. Finite element models of 2D-lattice plates with the considered re-entrant hexagonal unit cells are created in a commercial finite element software, MSC Marc. In the direct structural analysis, the lattices, each of which consists of 64×64 unit cells, are subjected to pure bending and torsional loading for determining \( E_i^* \), \( v_{ij}^* \), and \( G_{12}^* \). For example, pure bending is applied by using the prescribed rotations on the two opposite edges, schematically shown in figure 3. The reactions at the non-free edges and the resulting rotations at the free edges are then used to compute the effective bending moduli and Poisson’s ratios directly from equation (8).

Table 1 shows the effective bending moduli \( E_i^* \), Poisson’s ratios \( v_{ij}^* \), and shear modulus \( G_{12}^* \) of some auxetic 2D-lattice plates obtained from the homogenization method and the direct structural analysis. For the plates, four different topologies of re-entrant hexagonal unit cells, detailed in table 1, are selected to perform the validation in this study. Particularly good agreement is observed. For the values of \( G_{12}^* \), there is a small fluctuation of the results between the homogenization method and the direct structural analysis since the effect of heterogeneity of the plate. The differences in table 1 are expected to reduce and converge to zero difference when there are enough unit cells in a 2D-lattice plate. Besides, when comparing the effective elastic properties of between the plates with re-entrant hexagonal unit cells.
having \( h/l = 1 \) and those having \( h/l = 2 \) at the same \( \theta = -30^\circ \), it is found that the values of \( E'_1 \) and \( G_{12}' \) as well as the negative value of \( \nu_{12}' \) reduce while the values of \( E'_2 \) and the negative value of \( \nu_{21}' \) increase.

**Figure 3.** Example loading scheme of direct structural analysis of a lattice applied pure bending

**Table 1.** Comparison of the results from the homogenization method and direct structural analysis

| Re-entrant hexagonal unit cell | Homogenization method, \( a \) | Direct analysis of the lattices, \( b \) | Difference, \( [(b - a)/a] \times 100\% \) |
|------------------------------|-------------------------------|----------------------------------|-----------------------------------------------|
| Re-entrant hexagon with \( \theta = -5^\circ \) and \( h = l \) | \( E'_1 \) \( 1.088 \times 10^{-1} \) | \( E'_1 \) \( 1.088 \times 10^{-1} \) | 0.000% |
|                              | \( E'_2 \) \( 2.680 \times 10^{-2} \) | \( E'_2 \) \( 2.680 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{12}' \) \( -3.992 \times 10^{-2} \) | \( \nu_{12}' \) \( -3.992 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{21}' \) \( -9.835 \times 10^{-3} \) | \( \nu_{21}' \) \( -9.835 \times 10^{-3} \) | 0.000% |
|                              | \( G_{12}' \) \( 2.484 \times 10^{-2} \) | \( G_{12}' \) \( 2.468 \times 10^{-2} \) | -0.644% |
| Re-entrant hexagon with \( \theta = -15^\circ \) and \( h = l \) | \( E'_1 \) \( 1.267 \times 10^{-1} \) | \( E'_1 \) \( 1.267 \times 10^{-1} \) | 0.000% |
|                              | \( E'_2 \) \( 2.261 \times 10^{-2} \) | \( E'_2 \) \( 2.261 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{12}' \) \( -1.339 \times 10^{-1} \) | \( \nu_{12}' \) \( -1.339 \times 10^{-1} \) | 0.000% |
|                              | \( \nu_{21}' \) \( -2.389 \times 10^{-2} \) | \( \nu_{21}' \) \( -2.389 \times 10^{-2} \) | 0.000% |
|                              | \( G_{12}' \) \( 2.740 \times 10^{-2} \) | \( G_{12}' \) \( 2.709 \times 10^{-2} \) | -1.131% |
| Re-entrant hexagon with \( \theta = -30^\circ \) and \( h = l \) | \( E'_1 \) \( 1.567 \times 10^{-1} \) | \( E'_1 \) \( 1.567 \times 10^{-1} \) | 0.000% |
|                              | \( E'_2 \) \( 1.741 \times 10^{-2} \) | \( E'_2 \) \( 1.741 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{12}' \) \( -2.867 \times 10^{-1} \) | \( \nu_{12}' \) \( -2.867 \times 10^{-1} \) | 0.000% |
|                              | \( \nu_{21}' \) \( -3.186 \times 10^{-2} \) | \( \nu_{21}' \) \( -3.186 \times 10^{-2} \) | 0.000% |
|                              | \( G_{12}' \) \( 3.500 \times 10^{-2} \) | \( G_{12}' \) \( 3.469 \times 10^{-2} \) | -0.914% |
| Re-entrant hexagon with \( \theta = -30^\circ \) and \( h = 2l \) | \( E'_1 \) \( 5.222 \times 10^{-2} \) | \( E'_1 \) \( 5.222 \times 10^{-2} \) | 0.000% |
|                              | \( E'_2 \) \( 3.258 \times 10^{-2} \) | \( E'_2 \) \( 3.258 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{12}' \) \( -9.558 \times 10^{-2} \) | \( \nu_{12}' \) \( -9.558 \times 10^{-2} \) | 0.000% |
|                              | \( \nu_{21}' \) \( -5.963 \times 10^{-2} \) | \( \nu_{21}' \) \( -5.963 \times 10^{-2} \) | 0.000% |
|                              | \( G_{12}' \) \( 1.637 \times 10^{-2} \) | \( G_{12}' \) \( 1.627 \times 10^{-2} \) | -0.611% |
5. Conclusion
In this study, the effective out-of-plane elastic properties of auxetic 2D-lattice plates are determined using the homogenization method based on equivalent strain energy. The considered unit cells, including different re-entrant hexagonal unit cells, are created by varying the internal cell angles and aspect ratios of hexagonal geometry. In the employed homogenization method, the strain energy values of unit cells under different curvature modes are computed by the finite element method and used to determine the effective elastic properties. The effective elastic properties of the considered 2D-lattice plates obtained from the homogenization method compared satisfactorily with those obtained from direct structural analysis of the plates. It is found that varying internal cell angles of hexagonal geometry can significantly change the values of the effective elastic properties of the resulting 2D-lattice plates, especially effective negative Poisson’s ratio. The effective negative Poisson’s ratio can significantly increase the degrees of auxeticity of these plates. Reducing the internal cell angles to negative also allows $E_1^*$ of the plates under out-of-plane bending to be significantly increased. Moreover, increasing the aspect ratio of the re-entrant hexagonal unit cells causes reductions of $E_1^*$ and $G_{12}^*$, while negative Poisson’s ratios still appear whenever $\theta$ is negative.

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