ON SOME PROPERTIES AND INEQUALITIES FOR THE
NIELSEN’S $\beta$-FUNCTION

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Abstract. In this study, we obtain some convexity, monotonicity and additivity properties as well as some inequalities involving the Nielsen’s $\beta$-function which was introduced in 1906.

1. Introduction and Preliminaries

The Nielsen’s $\beta$-function, $\beta(x)$ which was introduced in [9] is defined as

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} \, dt, \quad x > 0$$

(1)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0$$

(2)

and by change of variables, the representation (1) can be written as

$$\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} \, dt, \quad x > 0.$$ 

(3)

The function $\beta(x)$ is also defined as [9]

$$\beta(x) = \frac{1}{2} \left\{ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right\}$$

(4)

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\Gamma(x)$ is the Euler’s Gamma function. See also [1], [3], [5] and [7].

It is known that function $\beta(x)$ satisfies the following properties [1], [9].

$$\beta(x+1) = \frac{1}{x} - \beta(x), \quad (5)$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \quad (6)$$

In particular, $\beta(1) = \ln 2$, $\beta \left( \frac{1}{2} \right) = \frac{\pi}{2}$, $\beta \left( \frac{3}{2} \right) = 2 - \frac{\pi}{2}$ and $\beta(2) = 1 - \ln 2$. 

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Proposition 1.1. The function $\beta(x)$ is related to the classical Euler’s beta function, $B(x, y)$ in the following ways.

\[
\beta(x) = -\frac{d}{dx} \left\{ \ln B \left( \frac{x}{2}, \frac{1}{2} \right) \right\}, \tag{7}
\]

\[
\beta(x) + \beta(1 - x) = B(x, 1 - x). \tag{8}
\]

Proof. By the Euler’s beta function $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we obtain

\[
B \left( \frac{x}{2}, \frac{1}{2} \right) = \frac{\Gamma \left( \frac{x}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{x+1}{2} \right)}. \tag{9}
\]

Then by taking the logarithmic derivative of (9) and using (4), we obtain

\[
\frac{d}{dx} \left\{ \ln B \left( \frac{x}{2}, \frac{1}{2} \right) \right\} = \frac{1}{2} \frac{B' \left( \frac{x}{2}, \frac{1}{2} \right)}{B \left( \frac{x}{2}, \frac{1}{2} \right)} = \frac{1}{2} \left\{ \frac{\Gamma' \left( \frac{x}{2} \right)}{\Gamma \left( \frac{x}{2} \right)} - \frac{\Gamma' \left( \frac{x+1}{2} \right)}{\Gamma \left( \frac{x+1}{2} \right)} \right\}
\]

\[
= \frac{1}{2} \left\{ \psi \left( \frac{x}{2} \right) - \psi \left( \frac{x+1}{2} \right) \right\}
\]

\[
= -\beta(x)
\]

yielding the result (7). The result (8) follows easily from the relation (6).

Remark 1.2. The function $\beta(x)$ is referred to as the incomplete beta function in [1] and [7]. However, this should not be confused with the incomplete beta function which is usually defined as

\[
B(a; x, y) = \int_0^a t^{x-1}(1 - t)^{y-1} dt \quad x > 0, y > 0
\]

or the regularized incomplete beta function which is defined as

\[
I_a(x, y) = \frac{B(a; x, y)}{B(x, y)} \quad x > 0, y > 0.
\]

Also, the function should not be confused with Dirichlet’s beta function which is defined as [4]

\[
\beta^*(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x} = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}} dt, \quad x > 0.
\]

We shall use the notations $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in the rest of the paper.
By differentiating $m$ times of (1), (2) and (3), we obtain

\[ \beta^{(m)}(x) = \int_0^1 \frac{(\ln t)^m t^{x-1}}{1 + t} \, dt, \quad x > 0 \]

(10)

\[ = (-1)^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + x)^{m+1}}, \quad x > 0 \]

(11)

\[ = (-1)^m \int_0^\infty \frac{m e^{-xt}}{1 + e^{-t}} \, dt, \quad x > 0 \]

(12)

for $m \in \mathbb{N}_0$. It is clear that $\beta^{(0)}(x) = \beta(x)$. In particular, we have

\[ \beta^{(m)}(1) = (-1)^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)^{m+1}} = (-1)^m m! \eta(m + 1), \quad m \in \mathbb{N}_0 \]

(13)

\[ = (-1)^m m! \left( 1 - \frac{1}{2^m} \right) \zeta(m + 1), \quad m \in \mathbb{N} \]

(14)

where $\eta(x)$ is the Dirichlet’s eta function and $\zeta(x)$ is the Riemann zeta function defined as

\[ \eta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)^x}, \quad x > 0 \quad \text{and} \quad \zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}, \quad x > 1. \]

Then by differentiating $m$ times of (4) and (5), we obtain respectively

\[ \beta^{(m)}(x + 1) = \frac{(-1)^m m!}{x^{m+1}} - \beta^{(m)}(x) \]

(15)

and

\[ \beta^{(m)}(x) = \frac{1}{2^{m+1}} \left\{ \psi^{(m)} \left( \frac{x + 1}{2} \right) - \psi^{(m)} \left( \frac{x}{2} \right) \right\}. \]

(16)

For rational arguments $x = \frac{p}{q}$, the function $\psi^{(m)}(x)$ takes the form

\[ \psi^{(m)} \left( \frac{p}{q} \right) = (-1)^{m+1} m! q^{m+1} \sum_{k=0}^{\infty} \frac{1}{(qk + p)^{m+1}}, \quad m \geq 1 \]

(17)

which implies

\[ \psi^{(m)} \left( \frac{3}{4} \right) - \psi^{(m)} \left( \frac{1}{4} \right) = (-1)^{m+1} m! 4^{m+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^{m+1}} - \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^{m+1}} \right\}. \]

(18)

Let $x = \frac{1}{2}$ in (16). Then we obtain

\[ \beta^{(m)} \left( \frac{1}{2} \right) = \frac{1}{2^{m+1}} \left\{ \psi^{(m)} \left( \frac{3}{4} \right) - \psi^{(m)} \left( \frac{1}{4} \right) \right\} \]

(19)

which by (18) can be written as

\[ \beta^{(m)} \left( \frac{1}{2} \right) = (-1)^{m+1} m! 2^{m+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^{m+1}} - \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^{m+1}} \right\}. \]

(20)
Now let $m = 1$ in (20). Then we obtain
\[
\beta'\left(\frac{1}{2}\right) = 4 \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^2} - \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^2} \right\} = -4G \tag{21}
\]
where $G = 0.915965594177...$ is the Catalan’s constant.

**Remark 1.3.** The Catalan’s constant has several interesting representations [2], and amongst them are:
\[
G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2},
\]
\[
G = -\frac{\pi^2}{8} + 2 \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^2}, \tag{22}
\]
\[
G = \frac{\pi^2}{8} - 2 \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^2}. \tag{23}
\]
Thus, (21) is a consequence (22) and (23).

Equivalently, by letting $m = 1$ in (19) we obtain
\[
\beta'\left(\frac{1}{2}\right) = \frac{1}{4} \left\{ \psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right) \right\} = -4G
\]
since $\psi'\left(\frac{1}{4}\right) = \pi^2 + 8G$ and $\psi'\left(\frac{3}{4}\right) = \pi^2 - 8G$. See [1] and [6].

By using (13), (14), (15) and (21), we derive the following special values.
\[
\beta'(1) = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12},
\]
\[
\beta'(2) = -1 + \frac{\pi^2}{12},
\]
\[
\beta'(3) = \frac{3}{4} - \frac{\pi^2}{12},
\]
\[
\beta'\left(\frac{3}{2}\right) = 4(G - 1),
\]
\[
\beta'\left(\frac{5}{2}\right) = \frac{40}{9} - 4G.
\]

More special values may be derived by using similar procedures. As shown in [1] and [5], the Nielsen’s $\beta$-function is very useful in evaluating certain integrals.

2. Main Results

To start with, we recall the following well-known definitions.

**Definition 2.1.** A function $f : I \rightarrow \mathbb{R}$ is said to be logarithmically convex if
\[
\log f(ux + vy) \leq u \log f(x) + v \log f(y)
\]
or equivalently
\[
f(ux + vy) \leq (f(x))^u(f(y))^v
\]
for each $x, y \in I$ and $u, v > 0$ such that $u + v = 1$. 
Definition 2.2. A function \( f : (0, \infty) \to \mathbb{R} \) is said to be completely monotonic if \( f \) has derivatives of all order and
\[
(-1)^k f^{(k)}(x) \geq 0 \quad \text{for} \quad x \in (0, \infty), \quad k \in \mathbb{N}_0.
\]

Lemma 2.3. For \( x > 0 \), the following statements hold.

(i) \( \beta(x) \) is decreasing.
(ii) \( \beta^{(m)}(x) \) is positive and decreasing if \( m \) is even.
(iii) \( \beta^{(m)}(x) \) is negative and increasing if \( m \) is odd.
(iv) \( |\beta^{(m)}(x)| \) is decreasing for all \( m \in \mathbb{N} \).

Proof. These follow easily from (3) and (12).

Proposition 2.4. The function \( \beta(x) \) is completely monotonic.

Proof. Let \( x > 0 \) and \( k \in \mathbb{N}_0 \). Then by (12) obtain
\[
(-1)^k \beta^{(k)}(x) = (-1)^{2k} \int_0^\infty \frac{t^k e^{-xt}}{1 + e^{-t}} \, dt \geq 0
\]
which completes the proof.

Remark 2.5. More generally, \( \beta^{(m)}(x) \) is completely monotonic if \( m \) is even and \( -\beta^{(m)}(x) \) is completely monotonic if \( m \) is odd. To see this, note that for \( x > 0 \) and \( k, m \in \mathbb{N}_0 \), we obtain
\[
(-1)^k \beta^{(m+k)}(x) = (-1)^{m+2k} \int_0^\infty \frac{t^{m+k} e^{-xt}}{1 + e^{-t}} \, dt \geq (\leq) 0
\]
respectively for even(odd) \( m \).

Theorem 2.6. The double-inequality
\[
\frac{\beta'(a) + \beta'(b)}{2} \leq \frac{\beta(b) - \beta(a)}{b - a} \leq \beta' \left( \frac{a + b}{2} \right)
\]
holds for \( a, b > 0 \).

Proof. We employ the classical Hermite-Hadamard inequality which states that if \( f : [a, b] \to \mathbb{R} \) is convex, then
\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]
Without loss of generality, let \( b \geq a > 0 \) and \( f(x) = -\beta'(x) \) for \( x \in [a, b] \). Then \( f(x) \) is convex and consequently, we obtain
\[
-\beta' \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b \beta'(x) \, dx \leq -\frac{\beta'(a) + \beta'(b)}{2}
\]
which gives the result (24). Alternatively, since \( \beta'(x) \) is continuous and concave (i.e. \( \beta''(x) < 0 \)) on \( (0, \infty) \), then by Theorem 1 of [8], we obtain the desired result.
Theorem 2.7. Let \( m, n \in \mathbb{N}_0, a > 1, \frac{1}{a} + \frac{1}{b} = 1 \) such that \( \frac{m}{a} + \frac{n}{b} \in \mathbb{N}_0 \). Then, the inequality
\[
\left| \beta\left(\frac{m}{a} + \frac{n}{b}\right) \left(\frac{x}{a} + \frac{y}{b}\right) \right| \leq \left| \beta^{(m)}(x) \right|^{\frac{1}{a}} \left| \beta^{(n)}(y) \right|^{\frac{1}{b}}
\] (25)
holds for \( x, y > 0 \).

Proof. By the relation (12) and the Hölder’s inequality, we obtain
\[
\left| \beta\left(\frac{m}{a} + \frac{n}{b}\right) \left(\frac{x}{a} + \frac{y}{b}\right) \right| = \int_0^\infty \frac{t^{\left(\frac{m}{a} + \frac{n}{b}\right)} e^{-\left(\frac{x}{a} + \frac{y}{b}\right)t}}{1 + e^{-t}} \, dt
\]
\[
= \int_0^\infty \frac{t^m e^{-\frac{xt}{a}}}{(1 + e^{-t})^\frac{1}{a}} \left(\frac{1}{1 + e^{-t}} \right)^\frac{1}{b} e^{-\frac{yt}{b}} \, dt
\]
\[
\leq \left( \int_0^\infty \frac{e^{-xt}}{1 + e^{-t}} \, dt \right)^{\frac{1}{a}} \left( \int_0^\infty \frac{e^{-yt}}{1 + e^{-t}} \, dt \right)^{\frac{1}{b}}
\]
\[
= \left| \beta^{(m)}(x) \right|^{\frac{1}{a}} \left| \beta^{(n)}(y) \right|^{\frac{1}{b}}
\]
which completes the proof.

Remark 2.8. Note that the absolute signs in (25) are not required if \( m \) and \( n \) are even.

Remark 2.9. If \( m = n \) is even in Theorem 2.7, then the inequality (25) becomes
\[
\beta^{(m)} \left(\frac{x}{a} + \frac{y}{b}\right) \leq \left( \beta^{(m)}(x) \right)^{\frac{1}{a}} \left( \beta^{(m)}(y) \right)^{\frac{1}{b}}
\] (26)
which implies that the function \( \beta^{(m)}(x) \) is logarithmically convex for even \( m \). Moreover, if \( m = 0 \) in (26), then we obtain
\[
\beta \left(\frac{x}{a} + \frac{y}{b}\right) \leq \left( \beta(x) \right)^{\frac{1}{a}} \left( \beta(y) \right)^{\frac{1}{b}}
\] (27)
implies that \( \beta(x) \) is logarithmically convex.

Remark 2.10. Let \( a = b = 2, x = y \) and \( m = n + 2 \) in Theorem 2.7. Then we obtain the Turan-type inequality
\[
\left| \beta^{(n+1)}(x) \right|^2 \leq \left| \beta^{(n+2)}(x) \right| \left| \beta^{(n)}(x) \right|.
\] (28)
Furthermore, if \( n = 0 \) in (28) then we get
\[
(\beta'(x))^2 \leq \beta''(x) \beta(x).
\] (29)

Theorem 2.11. Let \( m \in \mathbb{N}_0 \) be even. Then the function
\[
Q(x) = e^{ax} \beta^{(m)}(x)
\] (30)
is convex for \( x > 0 \) and any real number \( a \).

Proof. Let \( m \) be even and \( a \) be any real number. Then for \( x > 0 \),
\[
Q'(x) = a e^{ax} \beta^{(m)}(x) + e^{ax} \beta^{(m+1)}(x),
\]
\[
Q''(x) = a^2 e^{ax} \beta^{(m)}(x) + 2ae^{ax} \beta^{(m+1)}(x) + e^{ax} \beta^{(m+2)}(x)
\]
\[
= e^{ax} \left[ a^2 \beta^{(m)}(x) + 2a \beta^{(m+1)}(x) + \beta^{(m+2)}(x) \right].
\]
The quadratic function $f(a) = a^2\beta^{(m)}(x) + 2a\beta^{(m+1)}(x) + \beta^{(m+2)}(x)$ has a discriminant $\Delta = 4 \left[ \left(\beta^{(m+1)}(x)\right)^2 - \beta^{(m)}(x)\beta^{(m+2)}(x) \right] \leq 0$ as a result of (28). Then, since $\beta^{(m)}(x) > 0$, it follows that $f(a) \geq 0$. Thus, $Q''(x) \geq 0$ and this completes the proof.

**Theorem 2.12.** Let $m \in \mathbb{N}_0$ be even. Then the function

$$P(x) = \left[ \beta^{(m)}(x) \right]^\alpha$$

is convex for $x > 0$ and $\alpha > 0$.

**Proof.** Let $m$ be even, $x > 0$ and $\alpha > 0$. Then

$$\ln P(x) = \alpha \ln \beta^{(m)}(x) \quad \text{implies} \quad \frac{P'(x)}{P(x)} = \alpha \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}.$$

That is,

$$P'(x) = \alpha P(x) \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}$$

and then

$$P''(x) = P(x) \left\{ \left( \frac{P'(x)}{P(x)} \right)^2 + \alpha \frac{\beta^{(m+2)}(x)\beta^{(m)}(x) - (\beta^{(m+1)}(x))^2}{[\beta^{(m)}(x)]^2} \right\}$$

$$\geq 0$$

as a result of (28).

**Theorem 2.13.** Let $m \in \mathbb{N}_0$ be even. Then the function

$$U(x) = \frac{\beta^{(m)}(kx)}{\beta^{(m)}(x)^k}$$

is increasing if $k > 1$ and decreasing if $0 < k \leq 1$.

**Proof.** For $x > 0$ and $m$ even, define a function $S$ by

$$S(x) = \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}.$$

Then direct differentiation yields

$$S'(x) = \frac{\beta^{(m+2)}(x)\beta^{(m)}(x) - (\beta^{(m+1)}(x))^2}{[\beta^{(m)}(x)]^2}$$

and by (28), we conclude that $S'(x) \geq 0$. Hence $S(x)$ is increasing. Next, let $u(x) = \ln U(x)$. Then we obtain

$$u'(x) = k \left[ \frac{\beta^{(m+1)}(kx)}{\beta^{(m)}(kx)} - \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)} \right].$$

Since $S(x)$ is increasing, it follows that $u'(x) > 0$ if $k > 1$ and $u'(x) \leq 0$ if $0 < k \leq 1$. This completes the proof.
Corollary 2.14. Let \( m \in \mathbb{N}_0 \) be even and \( 0 < x \leq y \). Then the inequality
\[
\left( \frac{\beta^{(m)}(y)}{\beta^{(m)}(x)} \right)^k \leq \frac{\beta^{(m)}(ky)}{\beta^{(m)}(kx)}
\]
is satisfied if \( k > 1 \). It reverses if \( 0 < k \leq 1 \).

Proof. This follows from the monotonicity property of \( U(x) \) as defined in (32).

Theorem 2.15. Let \( m \in \mathbb{N}_0 \) be even and \( a > 0 \). Then for \( x > 0 \), the function
\[
\Omega(x) = \frac{\beta^{(m)}(a)}{\beta^{(m)}(x + a)}
\]
is increasing and logarithmically concave, and the inequality
\[
1 < \frac{\beta^{(m)}(a)}{\beta^{(m)}(x + a)}
\]
is satisfied.

Proof. Define \( \mu \) for \( m \in \mathbb{N}_0 \) even, \( a > 0 \) and \( x > 0 \) by
\[
\mu(x) = \ln \Omega(x) = \ln \beta^{(m)}(a) - \ln \beta^{(m)}(x + a).
\]
Then
\[
\mu'(x) = -\frac{\beta^{(m+1)}(x + a)}{\beta^{(m)}(x + a)} > 0
\]
which implies that \( \mu(x) \) is increasing. Consequently, \( \Omega(x) = e^{\mu(x)} \) is increasing. Next, we have
\[
(\ln \Omega(x))'' = -\left[ \frac{\beta^{(m+2)}(x + a)\beta^{(m)}(x + a) - (\beta^{(m+1)}(x + a))^2}{[\beta^{(m)}(x + a)]^2} \right] \leq 0
\]
which implies that \( \Omega(x) \) is logarithmically concave. Furthermore,
\[
\lim_{x \to 0^+} \Omega(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} \Omega(x) = \infty.
\]
Then since \( \Omega(x) \) is increasing, we obtain the result (34).

Theorem 2.16. Let \( m \in \mathbb{N}_0 \). Then the following inequalities hold for \( x, y > 0 \).
\[
\beta^{(m)}(x + y) \leq \beta^{(m)}(x) + \beta^{(m)}(y)
\]
if \( m \) is even, and
\[
\beta^{(m)}(x + y) \geq \beta^{(m)}(x) + \beta^{(m)}(y)
\]
if \( m \) is odd.
Proof. Let $m$ be even and $H(x) = \beta^{(m)}(x + y) - \beta^{(m)}(x) - \beta^{(m)}(y)$. Then for a fixed $y$, we obtain
\[
H'(x) = \beta^{(m+1)}(x + y) - \beta^{(m+1)}(x) \\
= (-1)^{(m+1)} \int_{0}^{\infty} \frac{t^m (e^{-(x+y)t} - e^{-xt})}{1 + e^{-t}} \, dt \\
= - \int_{0}^{\infty} \frac{t^m}{1 + e^{-t}} (e^{-yt} - 1) \, dt \\
\geq 0.
\]
Hence, $H(x)$ is increasing. Moreover,
\[
\lim_{x \to \infty} H(x) = \lim_{x \to \infty} \left[ \beta^{(m)}(x + y) - \beta^{(m)}(x) - \beta^{(m)}(y) \right] \\
= (-1)^{m} \lim_{x \to \infty} \left[ \int_{0}^{\infty} \frac{t^m}{1 + e^{-t}} \left( e^{-(x+y)t} - e^{-xt} - e^{-yt} \right) \, dt \right] \\
= - \int_{0}^{\infty} \frac{t^m e^{-yt}}{1 + e^{-t}} \, dt \\
\leq 0.
\]
Therefore, $H(x) \leq 0$ which gives the result (35). Similarly, for $m$ odd, we obtain $H'(x) \leq 0$ and $\lim_{x \to \infty} H(x) \geq 0$ which implies that $H(x) > 0$ and this gives the result (36).

Remark 2.17. Theorem 2.16 is another way of saying that the function $\beta^{(m)}(x)$ is subadditive if $m$ is even, and superadditive if $m$ is odd.

Theorem 2.18. Let $m \in \mathbb{N}_0$. Then for $m$ odd, the function $\beta^{(m)}(x)$ is star-shaped on $(0, \infty)$. That is,
\[
\beta^{(m)}(\alpha x) \leq \alpha \beta^{(m)}(x)
\]
for all $x \in (0, \infty)$ and $\alpha \in (0, 1]$.

Proof. Let $m$ be odd and $T(x) = \beta^{(m)}(\alpha x) - \alpha \beta^{(m)}(x)$. Then for $x \in (0, \infty)$ and $\alpha \in (0, 1]$, we have
\[
T'(x) = \alpha \left[ \beta^{(m+1)}(\alpha x) - \beta^{(m+1)}(x) \right] \\
\geq 0.
\]
Thus, $T(x)$ is increasing. Recall that $\beta^{(n)}(x)$ is decreasing for even $n$. Then since $0 < \alpha x \leq x$, we have $\beta^{(m+1)}(\alpha x) \geq \beta^{(m+1)}(x)$. Furthermore,
\[
\lim_{x \to \infty} T(x) = \lim_{x \to \infty} \left[ \beta^{(m)}(\alpha x) - \alpha \beta^{(m)}(x) \right] \\
= \lim_{x \to \infty} \left[ \int_{0}^{\infty} \frac{t^m e^{-\alpha xt}}{1 + e^{-t}} \, dt - \alpha \int_{0}^{\infty} \frac{t^m e^{-xt}}{1 + e^{-t}} \, dt \right] \\
= 0.
\]
Therefore, $T(x) \leq 0$ which completes the proof.
Theorem 2.19. Let $m \in \mathbb{N}_0$. Then the inequality
\[ [\beta^{(m)}(xy)]^2 \leq \beta^{(m)}(x)\beta^{(m)}(y) \] (38)
holds for $x \geq 1$ and $y \geq 1$.

Proof. We have $xy \geq x$ and $xy \geq y$ since $x \geq 1$ and $y \geq 1$. If $m$ is even, then we obtain
\[ 0 < \beta^{(m)}(xy) \leq \beta^{(m)}(x) \]
and
\[ 0 < \beta^{(m)}(xy) \leq \beta^{(m)}(y) \]
since $\beta^{(m)}(x)$ is decreasing for even $m$ (see Lemma 2.3). That implies
\[ [\beta^{(m)}(xy)]^2 \leq \beta^{(m)}(x)\beta^{(m)}(y). \]
Also, if $m$ is odd, then we have
\[ 0 > \beta^{(m)}(xy) \geq \beta^{(m)}(x) \]
and
\[ 0 > \beta^{(m)}(xy) \geq \beta^{(m)}(y) \]
since $\beta^{(m)}(x)$ is increasing for odd $m$, and that also implies
\[ [\beta^{(m)}(xy)]^2 \leq \beta^{(m)}(x)\beta^{(m)}(y) \]
which completes the proof.

A generalization of Theorem 2.19 is given as follows.

Theorem 2.20. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ such that $m$ is even. Then the inequality
\[ \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) \leq \left(\prod_{i=1}^{n} \beta^{(m)}(x_i)\right)^{\frac{1}{n}} \] (39)
holds for $x_i \geq 1$, $i = 1, 2, 3 \ldots, n$.

Proof. Since $x_i \geq 1$ for $i = 1, 2, 3 \ldots, n$, we have $\prod_{i=1}^{n} x_i \geq x_j$ for $j = 1, 2, 3 \ldots, n$.
For $m$ even, we have
\[ 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) \leq \beta^{(m)}(x_1), \]
\[ 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) \leq \beta^{(m)}(x_2), \]
\[ \vdots \]
\[ 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) \leq \beta^{(m)}(x_n). \]

Then by taking products of these inequalities, we obtain
\[ \left[\beta^{(m)}\left(\prod_{i=1}^{n} x_i\right)\right]^n \leq \prod_{i=1}^{n} \beta^{(m)}(x_i) \]
which completes the proof.

**Theorem 2.21.** Let $m, n \in \mathbb{N}_0$ and $s \geq 1$. Then, the inequality

$$(|\beta^{(m)}(x)| + |\beta^{(n)}(y)|) \leq |\beta^{(m)}(x)| + |\beta^{(n)}(y)|$$

holds for $x, y > 0$.

**Proof.** Note that $u^s + v^s \leq (u + v)^s$, for $u, v \geq 0$ and $s \geq 1$. Then by the Minkowski’s inequality, we obtain

$$(|\beta^{(m)}(x)| + |\beta^{(n)}(y)|)^\frac{1}{s} = \left(\int_0^\infty \frac{t^m e^{-xt}}{1 + e^{-t}} dt + \int_0^\infty \frac{t^n e^{-yt}}{1 + e^{-t}} dt\right) \frac{1}{s}$$

$$= \left(\int_0^\infty \left[\left(\frac{t^m e^{-xt}}{1 + e^{-t}}\right)^s + \left(\frac{t^n e^{-yt}}{1 + e^{-t}}\right)^s\right] dt\right) \frac{1}{s}$$

$$\leq \left(\int_0^\infty \frac{t^m e^{-xt}}{1 + e^{-t}} dt\right) \frac{1}{s} + \left(\int_0^\infty \frac{t^n e^{-yt}}{1 + e^{-t}} dt\right) \frac{1}{s}$$

$$= |\beta^{(m)}(x)| \frac{1}{s} + |\beta^{(n)}(y)| \frac{1}{s}$$

which yields the desired result.

**Remark 2.22.** Notice that $|\beta^{(m)}(x)| = (-1)^m \beta^{(m)}(x)$ for $m \in \mathbb{N}_0$ and $x > 0$. Then by the recurrence relation (41), we obtain

$$|\beta^{(m)}(x + 1)| = \frac{m!}{x^{m+1}} - |\beta^{(m)}(x)|$$

which implies

$$|\beta^{(m)}(x)| \leq \frac{m!}{x^{m+1}}.$$

**Theorem 2.23.** Let $m \in \mathbb{N}_0$ and $0 < a < b$. Then, there exists a $\lambda \in (a, b)$ such that

$$|\beta^{(m)}(b) - \beta^{(m)}(a)| \leq (b - a) \frac{(m + 1)!}{\lambda^{m+2}}$$

**Proof.** By the classical mean value theorem, there exist a $\lambda \in (a, b)$ such that

$$\frac{\beta^{(m)}(b) - \beta^{(m)}(a)}{b - a} = \beta^{(m+1)}(\lambda).$$

Thus, $\frac{|\beta^{(m)}(b) - \beta^{(m)}(a)|}{(b - a)} = |\beta^{(m+1)}(\lambda)|$ and by (42), we obtain the result (43).

3. Conclusion

In this study, we obtained some convexity, monotonicity and additivity properties as well as some inequalities involving the Nielsen’s $\beta$-function. The established results may be useful in evaluating or estimating certain integrals. Furthermore, the findings could provide useful information for further study of the function.
CONFICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publication of this paper.

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