Leading-order hadronic contribution to the anomalous magnetic moment of the muon from $N_f = 2 + 1 + 1$ twisted mass fermions

Grit Hotzel$^1$

in collaboration with Florian Burger$^1$, Xu Feng$^2$, Karl Jansen$^3$, Marcus Petschlies$^4$, Dru B. Renner$^5$

$^1$Humboldt University Berlin, Germany
$^2$KEK, Japan
$^3$NIC, DESY Zeuthen, Germany
$^4$The Cyprus institute, Cyprus
$^5$Jefferson Lab, USA

Lattice 2013, Mainz, Germany
The anomalous magnetic moment of the muon, $a_\mu$, can be measured very precisely: [B. Lee Roberts, Chinese Phys. C 34, 2010]

$$a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$$

$$a_\mu^{\text{SM}} = 116591828(49) \times 10^{-11}$$

[Hagiwara et al., J. Phys. G38, 2011]

There is a $\approx 3\sigma$ discrepancy between $a_\mu^{\text{exp}}$ and $a_\mu^{\text{SM}}$:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 261(80) \times 10^{-11}$$
Charm quark necessary to reach required precision

Current discrepancy \[ [Hagiwara et al., J. Phys. G38, 2011] \]

\[
a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = 261(80) \times 10^{-11}
\]

- charm quark contribution computed in perturbation theory \[ [Bodenstein, Dominguez, Schilcher, Phys.Rev. D85, 2012] \]
  \[
a_{\mu}^{\text{hvp,c}} = 144(1) \times 10^{-11}
\]

- comparable to hadronic light-by-light scattering contribution \[ [Prades, de Rafael, Vainshtein, arXiv:0901.0306 [hep-ph], 2009] \]
  \[
a_{\mu}^{\text{hlbl}} = 105(26) \times 10^{-11}
\]

- and also to electroweak contribution \[ [Jegerlehner, Nyffeler, Phys. Rept. 477, 2009] \]
  \[
a_{\mu}^{\text{EW}} = 153(2) \times 10^{-11}
\]
Leading hadronic contribution $a_{\mu}^{hvp}$

$$a_{\mu}^{QCD} = a_{\mu}^{lo,hvp} + a_{\mu}^{ho,hvp} + a_{\mu}^{lbl}$$

- can be computed directly in Euclidean space-time [T. Blum, PRL 91, 2003]

$$a_{\mu}^{hvp} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{m_{\mu}^2} \right) \Pi_R(Q^2)$$

where $\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$

- main ingredient: hadronic vacuum polarisation tensor

$$\Pi_{\mu\nu}(Q) = \int d^4 x \ e^{iQ \cdot (x-y)} \langle J_{\mu}^{em}(x) J_{\nu}^{em}(y) \rangle = (Q_{\mu} Q_{\nu} - Q^2 g_{\mu\nu}) \Pi(Q^2)$$

with

$$J_{\mu}^{em}(x) = \frac{2}{3} \bar{u}(x) \gamma_\mu u(x) - \frac{1}{3} \bar{d}(x) \gamma_\mu d(x) + \frac{2}{3} \bar{c}(x) \gamma_\mu c(x) - \frac{1}{3} \bar{s}(x) \gamma_\mu s(x)$$
Mixed-action set-up

- configurations generated by ETMC [Baron et al., JHEP 1006, 2010]
- light quarks: twisted mass action for mass-degenerate fermion doublet [Frezzotti, Rossi, JHEP 0408, 2004]
  \[
  S_F[\chi, \bar{\chi}, U] = \sum_x \bar{\chi}(x) \left[ D_W + m_0 + i\mu_q \gamma_5 \tau^3 \right] \chi(x)
  \]
- heavy sea quarks: twisted mass action for non-degenerate fermion doublet [Frezzotti, Rossi, Nucl. Phys. Proc. Suppl. 128, 2004]
  \[
  S_F[\chi_h, \bar{\chi}_h, U] = \sum_x \bar{\chi}_h(x) \left[ D_W + m_0 + i\mu_\sigma \gamma_5 \tau^1 + \mu_\delta \tau^3 \right] \chi_h(x)
  \]
- heavy valence quarks: Osterwalder-Seiler action [Frezzotti, Rossi, JHEP 0410, 2004]
  \[
  S_F[\chi_h, \bar{\chi}_h, U] = \sum_x \bar{\chi}_h(x) \left[ D_W + m_0 + i \begin{pmatrix} \mu_c & 0 \\ 0 & -\mu_s \end{pmatrix} \gamma_5 \right] \chi_h(x)
  \]
- tune bare mass parameters $\mu_{c/s}$ such that physical kaon and D-meson masses are reproduced
The $N_f = 2 + 1 + 1$ ensembles

| Ensemble    | $\beta$ | $a[\text{fm}]$ | $L^3 \times T$ | $m_{PS}[\text{MeV}]$ | $L[\text{fm}]$ |
|-------------|---------|----------------|----------------|---------------------|----------------|
| D15.48      | 2.10    | 0.061          | $48^3 \times 96$ | 227                 | 2.9            |
| D30.48      | 2.10    | 0.061          | $48^3 \times 96$ | 318                 | 2.9            |
| D45.32sc    | 2.10    | 0.061          | $32^3 \times 64$ | 387                 | 1.9            |
| B25.32t     | 1.95    | 0.078          | $32^3 \times 64$ | 274                 | 2.5            |
| B35.32      | 1.95    | 0.078          | $32^3 \times 64$ | 319                 | 2.5            |
| B35.48      | 1.95    | 0.078          | $48^3 \times 96$ | 314                 | 3.7            |
| B55.32      | 1.95    | 0.078          | $32^3 \times 64$ | 393                 | 2.5            |
| B75.32      | 1.95    | 0.078          | $32^3 \times 64$ | 456                 | 2.5            |
| B85.24      | 1.95    | 0.078          | $24^3 \times 48$ | 491                 | 1.9            |
| A30.32      | 1.90    | 0.086          | $32^3 \times 64$ | 283                 | 2.8            |
| A40.32      | 1.90    | 0.086          | $32^3 \times 64$ | 323                 | 2.8            |
| A50.32      | 1.90    | 0.086          | $32^3 \times 64$ | 361                 | 2.8            |
First $N_f = 2$ configurations at the physical point

More details: Talk by Bartosz Kostrzewa, Monday, 16:50

- again use Iwasaki action in gauge sector
- add clover-term to twisted mass action for non-degenerate fermion doublet

$$S_F[\chi, \bar{\chi}, U] = \sum_x \bar{\chi}(x) \left[ D_W + m_0 + i\mu q\gamma_5\tau^3 \right] \chi(x)$$

$$+ c_{SW} \sum_x \bar{\chi}(x) \left[ \frac{i}{4} \sigma_{\mu\nu} F_{\mu\nu} \right] \chi(x)$$

- very preliminary parameters of first ensemble:

| $\beta$ | $c_{SW}$ | $a[fm]$ | $L^3 \times T$ | $m_{PS}[MeV]$ | $L[fm]$ |
|--------|---------|---------|----------------|----------------|---------|
| 2.10   | 1.57551 | 0.096   | $48^3 \times 96$ | 128            | 4.6     |
How the observables are determined

- use conserved (point-split) vector current
  \[
  J^C_{\mu}(x) = \frac{1}{2} \left( \bar{\chi}(x + \hat{\mu})(1 + \gamma_{\mu}) U_{\mu}^\dagger(x) Q_{\text{el}} \chi(x) 
  - \bar{\chi}(x)(1 - \gamma_{\mu}) U_{\mu}(x) Q_{\text{el}} \chi(x + \hat{\mu}) \right)
  \]
  where \( Q_{\text{el}} = \text{diag}(\frac{2}{3}, -\frac{1}{3}) \)

- use redefinition \cite{Feng, Jansen, Petschlies, Renner, PRL 107, 2011}
  \[
  a^{\text{hvp}}_{\mu} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{H^2} \frac{H^2_{\text{phys}}}{m_{\mu}^2} \right) \Pi_R(Q^2)
  \]
  which goes to \( a^{\text{hvp}}_{\mu} \) for \( m_{PS} \rightarrow m_{\pi} \), i.e. when \( H \rightarrow H_{\text{phys}} \)

- effectively, redefinition of muon mass
  \[
  m_{\mu} = m_{\mu} \cdot \frac{H}{H_{\text{phys}}}
  \]

- in the following will always use \( H = m_{\nu} - \rho\)-meson mass
Fitting the hadronic vacuum polarisation function

- have $\Pi(\hat{Q}^2)$ depending on discrete momenta
- to obtain smooth function fit this for each flavour to

$$\Pi(Q^2) = (1 - \theta(Q^2 - Q_{\text{match}}^2))\Pi_{\text{low}}(Q^2) + \theta(Q^2 - Q_{\text{match}}^2)\Pi_{\text{high}}(Q^2)$$

with

$$\Pi_{\text{low}}(Q^2) = \sum_{i=1}^{M} g_i^2 \frac{m_i^2}{Q^2 + m_i^2} + \sum_{j=0}^{N-1} a_j(Q^2)^j$$

and

$$\Pi_{\text{high}}(Q^2) = \sum_{k=0}^{C-1} c_k(Q^2)^k + \left(\sum_{l=0}^{B-1} b_l(Q^2)^l\right) \cdot \log(Q^2)$$

- different matching conditions and functions possible
- Padé approximants [Aubin, Blum, Golterman, Peris, Phys.Rev. D86, 2012] under investigation
Light quark contribution on $N_f = 2 + 1 + 1$ sea

$H = m_V$

- $N_f = 2 + 1 + 1$ result: $a_{\mu, ud}^{hvp} = 5.67(11) \cdot 10^{-8}$
- $N_f = 2$ result: $a_{\mu, ud}^{hvp} = 5.72(16) \cdot 10^{-8}$

[Feng, Jansen, Petschlies, Renner, PRL 107, 2011]
Light quark contribution on $N_f = 2 + 1 + 1$ sea

Comparing to preliminary $N_f = 2$ result at physical pion mass

- $N_f = 2 + 1 + 1$ result: $a_{\mu, ud}^{\text{hvp}} = 5.67(11) \cdot 10^{-8}$
- new $N_f = 2$ result at physical point: $a_{\mu, ud}^{\text{hvp}} = 5.55(70) \cdot 10^{-8}$
Adding the strange quark in the valence sector

- for strange quark lattice artefacts have to be taken into account

\[
\alpha_{\mu,uds}^{hvp}(m_{PS}, a) = A + B m_{PS}^2 + C a^2
\]

with fit parameters \(A, B, C\)

light sector: cannot discriminate \(a^2\) effects

Grit Hotzel (HU Berlin)

The muon \(g - 2\)

02/08/13 12 / 18
Three-flavour contribution on $N_f = 2 + 1 + 1$ sea

$H = m_V$

- $N_f = 2 + 1 + 1$ result: $a_{\mu,uds}^{hvp} = 6.55(21) \cdot 10^{-8}$
- Three-flavour result extracted from dispersive analysis of [Jegerlehner, Szafron, Eur. Phys. J C71, 2011]: $a_{\mu,uds}^{hvp} = 6.79(05) \cdot 10^{-8}$
The four-flavour contribution on $N_f = 2 + 1 + 1$ sea

$H = m_V$

- $N_f = 2 + 1 + 1$ result: $a_{hvp}^\mu = 6.74(21) \cdot 10^{-8}$
- Result from dispersive analysis: [Jegerlehner, Szafron, Eur. Phys. J C71, 2011] $a_{hvp}^\mu = 6.91(05) \cdot 10^{-8}$
Systematic uncertainties

- from choosing fit ranges for vector mesons:
  \[ \Delta_V = 0.13 \cdot 10^{-8} \]

- from choosing different number of terms in fit function:
  \[ \Delta_{MNBC} = 0.12 \cdot 10^{-8} \]

- found to be negligible: \( m_{PSL} > 3.8, m_{PS} < 400 \text{ MeV} \), varying matching momentum between \([1 \text{ GeV}^2, 3 \text{ GeV}^2]\)

- not quantified yet: disconnected contributions, wrong sea quark masses

- preliminary final result:
  \[ a_{\mu}^{hvp} = 6.74(21)(18) \cdot 10^{-8} \]
Comparison of results for $a_{\mu}^{\text{hvp}}$ in $10^{-8}$ for different numbers of valence quarks:

|                       | $u,d$  | $u,d,s$ | $u,d,s,c$ |
|-----------------------|--------|---------|-----------|
| this work             | 5.67(11) | 6.55(21) | 6.74(27)  |
| ETMC 2011             | 5.72(16) | -       | -         |
| Mainz 2011            | 5.46(66) | 6.18(64) | -         |
| RBC-UKQCD 2011        | -      | 6.41(33) | -         |
| HLS estimate 2011     | 5.59(04) | 6.71(05) | 6.83(05)  |
| (dispersive analysis + flavour weighting) |         |         |           |

ETMC 2011: [Feng, Jansen, Petschlies, Renner, PRL 107, 2011]
Mainz 2011: [Della Morte, Jäger, Jüttner, Wittig, JHEP 1203, 2012]
RBC-UKQCD 2011: [Boyle, Del Debbio, Kerrane, Zanotti, Phys. Rev. D85, 2012]
HLS estimate 2011: [Benayoun, David, DelBuono, Jegerlehner, Eur. Phys. J. C72, 2012]
First $N_f = 2 + 1 + 1$ lattice calculation of $a^\text{hvp}_\mu$ gives compatible result with dispersive analyses.

Modified method [Feng, Jansen, Petschlies, Renner, PRL 107, 2011] works for $N_f = 2 + 1 + 1$ computation.

Chiral extrapolation in light sector to be checked with computation at physical pion mass.
Outlook

- improve data by more statistics, especially in heavy sector, and all-mode-averaging
- use Padé approximants
- disconnected contributions
- more $N_f = 2$ configurations at physical point
- $N_f = 2 + 1 + 1$ configurations at physical point, probably several lattice spacings needed
- include isospin breaking effects
- different observables: $\Delta \alpha_{\text{QED}}^{\text{hvp}}$, Adler function, weak mixing angle, S-parameter
- light-by-light scattering
Why it works

- redefinition [Feng, Jansen, Petschlies, Renner, PRL 107, 2011]

\[ a_{\mu}^{\text{hvp}} = \alpha_0^2 \int_0^\infty \frac{dQ^2}{Q^2} w \left( \frac{Q^2}{H^2} \frac{H_{\text{phys}}^2}{m_{\mu}^2} \right) \Pi_R(Q^2) \]

which goes to \( a_{\mu}^{\text{hvp}} \) for \( m_{\text{PS}} \rightarrow m_{\pi} \), i.e. when \( H \rightarrow H_{\text{phys}} \)

- effectively, redefinition of muon mass

\[ m_{\mu} = m_{\mu} \cdot \frac{H}{H_{\text{phys}}} \]

- leading vector meson contribution

\[ a_{\mu}^{\text{hvp}} \propto \alpha^2 g_V^2 \frac{m_{\mu}^2}{m_V^2} \]

⇒ strong dependence on \( m_{\text{PS}} \) via \( m_V \)
Comparison with Padé approximants

- Padé approximants model-independent, systematically improvable way of fitting vacuum polarisation [Aubin et al., Phys.Rev. D86, 2012]
- vacuum polarisation integrated up to \( Q_{\text{max}}^2 = 1.5 \text{ GeV}^2 \):

\[
H = m_V
\]

\[
H = 1
\]

- will use Padé approximants when chiral extrapolation no longer needed
Example for standard fit

\[ \beta = 1.95, \frac{L}{a} = 32, \mu_{\text{light}} = 0.0025 \]

-\( \Pi_{\text{light}} \)

The muon \( g - 2 \)

Grit Hotzel (HU Berlin)