INVERSE SEMIGROUPS AND THE CUNTZ-LI ALGEBRAS

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Abstract. In this paper, we apply the theory of inverse semigroups to the $C^*$-algebra $U[\mathbb{Z}]$ considered in [Cun08]. We show that the $C^*$-algebra $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries. We explicitly identify the groupoid $\mathcal{G}_{tight}$ associated to the inverse semigroup and show that $\mathcal{G}_{tight}$ is exactly the same groupoid obtained in [CL08].

1. Introduction

Ever since the appearance of the Cuntz algebras $O_n$ and the Cuntz-Krieger algebras $O_A$ there has been a great deal of interest in understanding the structure of $C^*$-algebras generated by partial isometries. The theory of graph $C^*$-algebras owes much to these examples. It has now been well known that these algebras admit a groupoid realisation and the groupoid turns out to be r-discrete. Another object that is closely related with an r-discrete groupoid is that of an inverse semigroup. The relationship between r-discrete groupoids and inverse semigroups was already clear from [Ren80].

An inverse semigroup $S$ is a semigroup together with an involution $*$ such that for every $s \in S$, $s^*ss^* = s^*$ . The universal example of an inverse semigroup is the semigroup of partial bijections on a set. Just like one can associate a $C^*$-algebra to a group, one can associate a universal $C^*$-algebra related with an inverse semigroup $S$ and is denoted $C^*(S)$. This universal $C^*$-algebra captures the representations of the inverse semigroup (as partial isometries on a Hilbert space). One can canonically associate an r-discrete groupoid $\mathcal{G}_S$ to an inverse semigroup $S$ such that the $C^*$-algebra of the groupoid $\mathcal{G}_S$ coincides with $C^*(S)$. For a more detailed account of inverse semigroups and r-discrete groupoids, we refer to [Pat99] and [Exe08].

Recently, Cuntz and Li in [CL08] has introduced a $C^*$-algebra associated to every integral domain with only finite quotients. Earlier in [Cun08], Cuntz considered the integral domain $\mathbb{Z}$. Let $R$ be an integral domain with only finite quotients. Then the universal algebra $U[R]$ is the universal $C^*$-algebra generated by a set of unitaries $\{u^n : n \in R\}$ and a set of partial isometries $\{s_m : m \in R^*\}$ satisfying certain relations. In [CL08], it was proved that $U[R]$ is simple and purely infinite. Moreover Cuntz and Li obtained a groupoid realisation of it which they later used it to compute the $K$-groups of these algebras for specific integral domains (See [CL11] and [CL09]). A concrete realisation of $U[R]$ can be obtained by representing $s_m$ and $u^n$

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on $\ell^2(R)$ by
\[ s_m \to S_m : \delta_r \to \delta_{rm} \]
\[ u^n \to U^n : \delta_r \to \delta_{r+n} \]
Then $U[R]$ is isomorphic to the $C^*$-algebra generated by $S_m$ and $U^n$ (by the simplicity of $U[R]$). The operator $S_m$ is implemented by the multiplication by $m$ (an injection) and $U^n$ is implemented by the addition by $n$ (a bijection). Thus it is immediately clear that $U[R]$ is generated by an inverse semigroup of partial isometries. Thus the theory of inverse semigroups should explain some of the results obtained by Cuntz and Li in [CL08]. The purpose of this paper is to obtain the groupoid realisation (obtained in [CL08]) by using the theory of inverse semigroups. We spell out the details only for the case $R = \mathbb{Z}$ as the analysis for general integral domains with finite quotients is similar. We should also remark that alternate approaches to the Cuntz-Li algebras were considered in [BE10] and in [KLQ10]. We should mention that this paper contains no new results. The point is if one uses the language of inverse semigroups one can obtain a groupoid realisation systematically without having to guess anything about the structure of the Cuntz-Li algebras.

Now we indicate the organisation of the paper. In Section 2, the definition of $U[\mathbb{Z}]$ is recalled and we show that $U[\mathbb{Z}]$ is generated by an inverse semigroup of partial isometries which we denote it by $T$. In Section 3, we recall the notion of tight representations of an inverse semigroup, a notion introduced by Exel in [Exe08]. We show that the identity representation of $T$ in $U[\mathbb{Z}]$ is in fact tight, and show that $U[\mathbb{Z}]$ is isomorphic to the $C^*$-algebra of the groupoid $G_{tight}$ (considered in [Exe08]) associated to $T$. In Sections 4 and 5, we explicitly identify the groupoid $G_{tight}$ which turns out to be exactly the groupoid considered in [CL08]. In Section 6, we show that $U[\mathbb{Z}]$ is simple. In Section 7, we digress a bit to explain the connection between Crisp and Laca’s boundary relations and Exel’s tight representations of Nica’s inverse semigroup. In the final Section, we give a few remarks of how to adapt the analysis carried out in Sections 1 – 6 for a general integral domain. A bit of notation: For non-zero integers $m$ and $n$, we let $[m,n]$ to denote the lcm of $m$ and $n$ and $(m,n)$ to denote the gcd of $m$ and $n$. For a ring $R$, $R^\times$ denotes the set of non-zero elements in $R$.

2. THE REGULAR $C^*$-ALGEBRA ASSOCIATED TO $\mathbb{Z}$

**Definition 2.1** ([Cun08]). Let $U[\mathbb{Z}]$ be the universal $C^*$-algebra generated by a set of unitaries $\{u^n : n \in \mathbb{Z}\}$ and a set of isometries $\{s_m : m \in \mathbb{Z}^\times\}$ satisfying the following relations.
\[ s_m s_n = s_{mn} \]
\[ u^n u^m = u^{n+m} \]
\[ s_m u^n = u^{mn} s_m \]
\[ \sum_{n \in \mathbb{Z}/(m)} u^n e_m u^{-n} = 1 \]
where $e_m$ denotes the final projection of $s_m$. 

Remark 2.2. Let \( \chi \) be a character of the discrete multiplicative group \( \mathbb{Q}^\times \). Then the universal property of the \( \mathcal{C}^* \)-algebra \( U[\mathbb{Z}] \) ensures that there exists an automorphism \( \alpha_\chi \) of the algebra \( U[\mathbb{Z}] \) such that \( \alpha_\chi(s_m) = \chi(m)s_m \) and \( \alpha_\chi(u^n) = u^n \). This action of the character group of the multiplicative group \( \mathbb{Q}^\times \) was considered in [CL08].

For \( m \neq 0 \) and \( n \in \mathbb{Z} \), consider the operators \( S_m \) and \( U^n \) defined on \( \ell^2(\mathbb{Z}) \) as follows:

\[
S_m(\delta_r) = \delta_{rm} \quad U^n(\delta_r) = \delta_{r+n}
\]

Then \( s_m \to S_m \) and \( u^n \to U^n \) gives a representation of the universal \( \mathcal{C}^* \)-algebra \( U[\mathbb{Z}] \) called the regular representation and its image is denoted by \( \mathcal{U}_r[\mathbb{Z}] \). We begin with a series of Lemmas (highly inspired and adapted from [Cun08] and from [CL08]) which will be helpful in proving that \( U[\mathbb{Z}] \) is generated by an inverse semigroup of partial isometries.

Lemma 2.3. For every \( m, n \neq 0 \), one has \( e_m = \sum_{k \in \mathbb{Z}/(n)} u^{km}e_{mn}u^{-km} \).

Proof. One has

\[
e_m = s_m s_m^* = s_m \left( \sum_{k \in \mathbb{Z}/(n)} u^k e_n u^{-k} s_m^* \right)
\]

\[
= \sum_{k \in \mathbb{Z}/(n)} s_m u^k s_n s_m^* u^{-k} s_m^*
\]

\[
= \sum_{k \in \mathbb{Z}/(n)} u^{km} s_m s_n s_m^* s_m u^{-km}
\]

\[
= \sum_{k \in \mathbb{Z}/(n)} u^{km} e_{mn} u^{-km}
\]

This completes the proof. \( \square \)

Lemma 2.4. For every \( m, n \neq 0 \), one has \( e_m e_n = e_{[m,n]} \) where \( [m,n] \) denotes the least common multiple of \( m \) and \( n \).

Proof. Let \( c := [m,n] \) be the lcm of \( m \) and \( n \). Then \( c = am = bn \) for some \( a, b \). Now from Lemma 2.3 it follows that

\[
e_m e_n = \sum_{r \in \mathbb{Z}/(a), s \in \mathbb{Z}/(b)} u^{mr} e_r u^{-mr} u^{ns} e_c u^{-ns}
\]

The product \( u^{mr} e_r u^{-mr} u^{ns} e_c u^{-ns} \) survives if and only if \( mr \equiv ns \mod c \). But the only choice for such an \( r \) and an \( s \) is when \( r \equiv 0 \mod a \) and \( s \equiv 0 \mod b \). [Reason: Suppose there exists \( r \) and \( s \) such that \( mr \equiv ns \mod c \). Then \( \frac{mr-ns}{c} \) is an integer. That is \( \frac{r}{a} - \frac{s}{b} \) is an integer.]
Let \( d = (m,n) \) and \( c = [m,n] \). Suppose \( r \equiv s \mod d \). Let \( k \) be such that \( k \equiv r \mod m \) and \( k \equiv s \mod n \). Then \( u^r e_m u^{-r} u^s e_n u^{-s} = u^k e_c u^{-k} \).

**Proof.** First note that \( u^r e_m u^{-r} u^s e_n u^{-s} = 0 \). Hence \( u^r e_m u^{-r} u^s e_d = 0 \). Now note that

\[
u^r e_m u^{-r} u^s e_n u^{-s} = u^r e_m (e_d u^{-r} u^s) e_d u^{-s} \quad \text{[by Lemma 2.4]}
\]

\[
= 0
\]

This completes the proof.

**Lemma 2.7.** For \( m,n \neq 0 \), one has \( s^*_m e_n s_m = e'_n \) where \( n' := \frac{n}{(n,m)} \).

**Proof.** First note that without loss of generality, we can assume that \( m \) and \( n \) are relatively prime. Otherwise write \( m := m_1 d \) and \( n := n_1 d \) where \( d \) is the gcd of \( m \) and \( n \). Then \( (m_1,n_1) = 1 \) and

\[
s^*_m e_n s_m = s^*_m s^*_d s^*_n s^*_n s^*_n s^*_d s^*_d s^*_m
\]

So now assume \( m \) and \( n \) are relatively prime. Observe that \( s^*_m e_n s_m \) is a selfadjoint projection. For \( s^*_m e_n s_m s^*_m e_n s_m = s^*_m e_n e_n s_m = s^*_m e_n e_m s_m = s^*_m e_n s_m \). Again,

\[
(s^*_m e_n s_m)^2 = s^*_m e_n e_m s_m
\]

\[
= s^*_m e_m s_m \quad \text{[by Lemma 2.4]}
\]

\[
= s^*_m s_m s_n s^*_n s^*_m s_m
\]

\[
= e_n
\]

This completes the proof.

**Lemma 2.8.** Let \( m,n \neq 0 \) and \( k \in \mathbb{Z} \) be given. If \( (m,n) \) does not divide \( k \) then one has \( s^*_m u^k e_n u^{-k} s_m = 0 \).

**Proof.** It is enough to show that \( x := e_n u^{-k} s_m \) vanishes. Thus it is enough to show that \( xx^* = e_n u^{-k} e_m u^k e_n \). Now Lemma 2.5 implies that \( xx^* = 0 \). This completes the proof.

**Lemma 2.9.** Let \( m,n \neq 0 \) and \( k \in \mathbb{Z} \) be given. Suppose that \( d := (m,n) \) divides \( k \). Choose an integer \( r \) such that \( mr \equiv k \mod n \). Then \( s^*_m u^k e_n u^{-k} s_m = u^r e_m u^{-r} \) where \( n_1 = \frac{n}{d} \).
Proof. Now observe that \( u^k e_n u^{-k} = u^m r e_n u^{-m} \). Hence one has

\[
\begin{align*}
  s_m^* u^k e_n u^{-k} s_m &= s_m^* u^m r e_n u^{-m} s_m \\
  &= u^r s_m^* e_n s_m u^{-r} \\
  &= u^r e_n u^{-r} \quad \text{[by Lemma 2.9]}
\end{align*}
\]

This completes the proof. \( \square \)

**Remark 2.10.** Let \( P := \{ u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z} \} \cup \{0\} \). Then the above observations show that \( P \) is a commutative semigroup of projections which is invariant under the map \( x \to s_m^* x s_m \).

The proof of the following proposition is adapted from [CL08].

**Proposition 2.11.** Let \( T := \{ s_m^* u^n e_k u^n s_m : m, m', k \neq 0, n, n' \in \mathbb{Z} \} \cup \{0\} \). Then \( T \) is an inverse semigroup of partial isometries. Let \( P := \{ u^n e_m u^{-n} : m \neq 0, n \in \mathbb{Z} \} \cup \{0\} \). Then the set of projections in \( T \) coincide with \( P \). Also the linear span of \( T \) is dense in \( U[\mathbb{Z}] \).

**Proof.** The fact that \( T \) is closed under multiplication follows from the following calculation.

\[
\begin{align*}
  s_m^* u^n e_r u^{-n'} s_m' s_k^* u^\ell e_s u^{-\ell'} s_k' &= s_m^* u^n e_r u^{-n'} s_m' s_m s_k^* s_m' u^\ell e_s u^{-\ell'} s_k' \\
  &= s_m^* u^n u^{-n'} e_r u^{-n'} e_s u^{-\ell'} u^\ell s_k' \\
  &= s_m^* u^n e_r u^{-n'} e_s u^{-\ell'} e_s u^\ell s_k' \quad \text{[where } e := u^n e_r u^{-n'} \text{ and } f = u^\ell e_s u^{-\ell}] \\
  &= s_m^* u^n s_k^* (s_k e s_k^*)(s_m f s_m')(s_m' f s_m') s_m' u^{-\ell'} s_k' \\
  &= s_m^* (s_k e s_k^*) s_m' f s_m' s_m' u^{-\ell'} s_k' \quad \text{[where } p := (s_k e s_k^*) (s_m f s_m')(s_m' f s_m') \in P \}
\end{align*}
\]

Thus we have shown that \( T \) is closed under multiplication. Clearly \( T \) is closed under the involution \( \ast \). Thus the linear span of \( T \) is a \( \ast \) algebra containing \( s_m \) and \( u^n \) for every \( m \neq 0 \) and \( n \in \mathbb{Z} \). Hence the linear span of \( T \) is dense in \( U[\mathbb{Z}] \).

Now we show that every element of \( T \) is in fact a partial isometry. Let \( v := s_m^* u^n e_k u^n s_m' \) be given. Now,

\[
vv^\ast = s_m^* u^n e_k u^n s_m' s_m^* u^{-n'} e_k u^{-n} s_m \\
= s_m^* u^n (e_k u^n e_m u^{-n'} e_k) u^{-n} s_m \\
= s_m^* u^n e u^{-n} s_m \quad \text{[where } e := (e_k u^n e_m u^{-n'} e_k) \in P \]
\]

Now it follows from Remark 2.10 that \( vv^\ast \in P \). It also shows that the set of projections in \( T \) coincide with \( P \). This completes the proof. \( \square \)

The following equality will be used later. Let us isolate it now.

\[
(2.1) \quad s_m^* u^k s_m u^k s_n = s_m^* u^{m_2 k_1} e_{m_2 n_1} u^{k_2 n_1} s_n
\]
3. Tight representations of an inverse semigroup

Let us recall the notion of tight characters and tight representations from [Exe08].

**Definition 3.1.** Let $S$ be an inverse semigroup with $0$. Denote the set of projections in $S$ by $E$. A character for $E$ is a map $x : E \rightarrow \{0, 1\}$ such that

1. the map $x$ is a semigroup homomorphism, and
2. $x(0) = 0$.

We denote the set of characters of $E$ by $\hat{E}_0$. We consider $\hat{E}_0$ as a locally compact Hausdorff topological space where the topology on $\hat{E}_0$ is the subspace topology induced from the product topology on $\{0, 1\}^E$.

For a character $x$ of $E$, let $A_x := \{e \in E : x(e) = 1\}$. Then $A_x$ is a nonempty set satisfying the following properties.

1. The element $0 \in A_x$.
2. If $e \in A_x$ and $f \geq e$ then $f \in A_x$.
3. If $e, f \in A_x$ then $ef \in A_x$.

Any nonempty subset $A$ of $E$ for which (1), (2) and (3) are satisfied is called a filter. Moreover if $A$ is a filter then the indicator function $1_A$ is a character. Thus there is a bijective correspondence between the set of characters and filters. We also call a character $x$ maximal or an ultrafilter if its support $A_x$ is maximal.

The set of maximal characters is denoted by $\hat{E}_\infty$ and its closure in $\hat{E}_0$ is denoted by $\hat{E}_{\text{tight}}$.

The following characterization of maximal characters is due to Exel and we refer to [Exe09] for a proof. Let $E$ be an inverse semigroup of projections. Let $e, f \in E$. We say that $f$ intersects $e$ if $fe \neq 0$.

**Lemma 3.2.** Let $E$ an inverse semigroup of projections with $0$ and $x$ be a character of $E$. Then the following are equivalent.

1. The character $x$ is maximal.
2. The support $A_x$ contains every element of $E$ which intersects every element of $A_x$.

**Corollary 3.3.** Let $A$ be a unital $C^*$-algebra and $E \subset A$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that $E$ contains a finite set $\{e_1, e_2, \cdots, e_n\}$ of mutually orthogonal projections such that $\sum_{i=1}^n e_i = 1$. Then for every maximal character $x$ of $E$, there exists a unique $e_i$ for which $x(e_i) = 1$.

**Proof.** The uniqueness of $e_i$ is clear as the projections $e_1, e_2, \cdots, e_n$ are orthogonal. Now to show the existence of an $e_i$ in $A_x$, we prove by contradiction. Assume that $e_i \notin A_x$ for every $i$. Then by Lemma 3.2 we have that for every $i$, there exists an $f_i \in A_x$ such that $e_i f_i = 0$. Let $f = \prod f_i$. Then $f \in A_x$ and thus nonzero and also $fe_i = 0$ for every $i$. As $\sum_i e_i = 1$, this forces $f = 0$. Thus we have a contradiction. □

Let us recall the notion of tight representations of semilattices from [Exe08] and from [Exe09]. The only semilattice we consider is that of an inverse semigroup of projections or in otherwords...
the idempotent semilattice of an inverse semigroup. Also our semilattice contains a maximal element 1. First let us recall the notion of a cover from [Exe08].

**Definition 3.4.** Let $E$ be an inverse semigroup of projections containing $\{0,1\}$ and $Z$ be a subset of $E$. A subset $F$ of $Z$ is called a cover for $Z$ if given a non-zero element $z \in Z$ there exists an $f \in F$ such that $fz \neq 0$. The set $F$ is called a finite cover if $F$ is finite.

The following definition is actually Proposition 11.8 in [Exe08].

**Definition 3.5.** Let $E$ be an inverse semigroup of projections containing $\{0,1\}$. A representation $\sigma : E \to \mathcal{B}$ of the semilattice $E$ in a Boolean algebra $\mathcal{B}$ is said to be tight if for every finite cover $Z$ of the interval $[0,x] := \{z \in E : z \leq x\}$, one has $\sup_{z \in Z} \sigma(z) = \sigma(x)$.

Let $A$ be a unital $C^*$ algebra and $S$ be an inverse semigroup containing $\{0,1\}$. Let $\sigma : S \to A$ be a unital representation of $S$ as partial isometries in $A$. Let $\sigma(C^*(E))$ be the $C^*$-subalgebra in $A$ generated by $\sigma(E)$. Then $\sigma(C^*(E))$ is a unital, commutative $C^*$-algebra and hence the set of projections in it is a Boolean algebra which we denote by $\mathcal{B}_{\sigma(C^*(E))}$. We say the representation $\sigma$ is **tight** if the representation $\sigma : E \to \mathcal{B}_{\sigma(C^*(E))}$ is tight.

**Lemma 3.6.** Let $X$ be a compact metric space and $E \subset C(X)$ be an inverse semigroup of projections containing $\{0,1\}$. Suppose that for every finite set of projections $\{f_1,f_2,\cdots,f_m\}$ in $E$, there exists a finite set of mutually orthogonal non-zero projections $\{e_1,e_2,\cdots,e_n\}$ in $E$ and a matrix $(a_{ij})$ such that

\[
\sum_{i=1}^{n} e_i = 1
\]

\[
f_i = \sum_{j} a_{ij} e_j.
\]

Then the identity representation of $E$ in $C(X)$ is tight.

**Proof.** Let $e \in E\{0\}$ be given and let $F$ be a finite cover for the interval $[0,e]$. Without loss of generality, we can assume that $e = 1$ (Just cut everything down by $e$). Let $F := \{f_1,f_2,\cdots,f_m\}$. Then by the hypothesis there exists a finite set of mutually orthogonal projections $\{e_1,e_2,\cdots,e_n\}$ and a matrix $(a_{ij})$ such that $f_i = \sum_{j} a_{ij} e_j$ and $\sum_{i} e_i = 1$. For a given $j$, let $A_j := \{i : a_{ij} \neq 0\}$. Since $F$ covers $C(X)$, it follows that for every $j$, $A_j$ is nonempty. In otherwords, given $j$, there exists an $i$ such that $f_i \geq e_j$. Thus $f := \sup_{i} f_i \geq e_j$ for every $j$. Hence $f \geq \sup_{j} e_j = 1$. This completes the proof.

In the next proposition, $T$ denotes the inverse semigroup associated to $U[\mathbb{Z}]$ in Proposition 2.11.

**Proposition 3.7.** The identity representation of $T$ in $U[\mathbb{Z}]$ is tight.

**Proof.** We apply Lemma 3.6. Let $\{u^{r_1}e_{m_1}u^{-r_1},u^{r_2}e_{m_2}u^{-r_2},\cdots,u^{r_k}e_{m_k}u^{-r_k}\}$ be a finite set of non-zero projections in $P$. By Lemma 2.3 it follows that each $f_i := u^{r_i}e_{m_i}u^{-r_i}$ is a linear
combination of \( \{ u^e u^{-s} : s \in \mathbb{Z}/(c) \} \) where \( c \) is the lcm of \( m_1, m_2, \ldots, m_k \). Then Lemma 3.6 implies that the identity representation of \( T \) in \( U[\mathbb{Z}] \) is tight. This completes the proof. \( \qed \)

Now we will show that the \( C^* \)-algebra of the groupoid \( G_{\text{tight}} \) of the inverse semigroup \( T \) is isomorphic to the algebra \( U[\mathbb{Z}] \). First let us recall the construction of the groupoid \( G_{\text{tight}} \) considered in \cite{Exe08}. Let \( S \) be an inverse semigroup with 0 and let \( E \) denote its set of projections. Note that \( S \) acts on \( E_0 \) partially. For \( x \in E_0 \) and \( s \in S \), define \( (x.s)(e) = x(ses^*) \). Then

- The map \( x.s \) is a semigroup homomorphism, and
- \( (x.s)(0) = 0 \).

But \( x.s \) is nonzero if and only if \( x(ss^*) = 1 \). For \( s \in S \), define the domain and range of \( s \) as

\[
D_s := \{ x \in E_0 : x(ss^*) = 1 \} \\
R_s := \{ x \in E_0 : x(s^*s) = 1 \}
\]

Note that both \( D_s \) and \( R_s \) are compact and open. Moreover \( s \) defines a homomorphism from \( D_s \) to \( R_s \) with \( s^* \) as its inverse. Also observe that \( E_{\text{tight}} \) is invariant under the action of \( S \).

Consider the transformation groupoid \( \Sigma := \{(x,s) : x \in D_s\} \) with the composition and the inversion being given by:

\[
(x,s)(y,t) := (x, st) \quad \text{if} \quad y = x.s \\
(x,s)^{-1} := (x.s, s^*)
\]

Define an equivalence relation \( \sim \) on \( \Sigma \) as \( (x,s) \sim (y,t) \) if \( x = y \) and if there exists an \( e \in E \) such that \( x \in D_e \) for which \( es = et \). Let \( \mathcal{G} = \Sigma/\sim \). Then \( \mathcal{G} \) is a groupoid as the product and the inversion respects the equivalence relation \( \sim \). Now we describe a topology on \( \mathcal{G} \) which makes \( \mathcal{G} \) into a topological groupoid.

For \( s \in S \) and \( U \) an open subset of \( D_s \), let \( \theta(s,U) := \{ [x,s] : x \in U \} \). We refer to \cite{Exe08} for the proof of the following two propositions. We denote \( \theta(s,D_s) \) by \( \theta_s \). Then \( \theta_s \) is homeomorphic to \( D_s \) and hence is compact, open and Hausdorff.

**Proposition 3.8.** The collection \( \{ \theta(s,U) : s \in S, U \text{ open in } D_s \} \) forms a basis for a topology on \( \mathcal{G} \). The groupoid \( \mathcal{G} \) with this topology is a topological groupoid whose unit space can be identified with \( E_0 \). Also one has the following.

1. For \( s, t \in S \), \( \theta_s \theta_t = \theta_{st} \).
2. For \( s \in S \), \( \theta_s^{-1} = \theta_{s^*} \), and
3. The set \( \{ 1_{\theta_s} : s \in T \} \) generates the \( C^* \) algebra \( C^*(\mathcal{G}) \).

We define the groupoid \( G_{\text{tight}} \) to be the reduction of the groupoid \( \mathcal{G} \) to \( E_{\text{tight}} \). In \cite{Exe08}, it is shown that the representation \( s \to 1_{\theta_s} \in C^*(G_{\text{tight}}) \) is tight and any tight representation factors through this universal one.

**Proposition 3.9.** Let \( T \) be the inverse semigroup associated to \( U[\mathbb{Z}] \) in Proposition 2.11. Let \( G_{\text{tight}} \) be the tight groupoid associated to \( T \). Then \( U[\mathbb{Z}] \) is isomorphic to \( C^*(G_{\text{tight}}) \).
Proof. Let $t_m, v^n$ denote the images of $s_m, u^n$ in $C^*(\mathcal{G}_{\text{tight}})$. The universality of the $C^*$-algebra $C^*(\mathcal{G}_{\text{tight}})$ together with Proposition 2.11 implies that there exists a homomorphism $\rho : C^*(\mathcal{G}_{\text{tight}}) \to U[\mathbb{Z}]$ such that $\rho(t_m) = s_m$ and $\rho(v^n) = u^n$.

Note that the mutually orthogonal set of projections $\{u^r e_m u^{-r} : r \in \mathbb{Z}/(m)\}$ cover $T$. Since the representation of $T$ in $C^*(\mathcal{G}_{\text{tight}})$ is tight, it follows that $\sum_r v^r t_m t_m^* v^{-r} = 1$. Now the universal property of $U[\mathbb{Z}]$ implies that there exists a homomorphism $\sigma : U[\mathbb{Z}] \to C^*(\mathcal{G}_{\text{tight}})$ such that $\sigma(s_m) = t_m$ and $\sigma(u^n) = v^n$. Now it is clear that $\rho$ and $\sigma$ are inverses of each other. This completes the proof.

In the next two sections, we identify the groupoid $\mathcal{G}_{\text{tight}}$ explicitly.

4. Tight characters of the inverse semigroup $T$

In this section, we determine the tight characters of the inverse semigroup $T$ defined in Proposition 2.11. Let us recall a few ring theoretical notions. We denote the set of strictly positive integers by $\mathbb{N}^+$. Consider the directed set $(\mathbb{N}^+, \leq)$ where we say $m \leq n$ if $m|n$. If $m|n$ then there exists a natural map from $\mathbb{Z}/(n)$ to $\mathbb{Z}/(m)$. The inverse limit of this system is called the profinite completion of $\mathbb{Z}$ and is denoted $\hat{\mathbb{Z}}$. In other words,

$$\hat{\mathbb{Z}} := \{ (r_m) \in \prod_{m \in \mathbb{N}^+] \mathbb{Z}/(m) : r_{mk} \equiv r_m \mod m \}$$

Also $\hat{\mathbb{Z}}$ is a compact ring with the subspace topology induced by the product topology on $\prod \mathbb{Z}/(m)$. Also $\mathbb{Z}$ embeds naturally in $\hat{\mathbb{Z}}$. We also need the easily verifiable fact that the kernel of the $m$th projection $r = (r_m) \to r_m$ is in fact $m\mathbb{Z}$.

For $r \in \hat{\mathbb{Z}}$, define a character $\xi_r : P \to \{0, 1\}$ by the following formula:

$$\xi_r(u^n e_m u^{-n}) := \delta_{r_m,n}$$
$$\xi_r(0) := 0$$

In the above formula, the Dirac-delta function is over the set $\mathbb{Z}/(m)$. Thus $\delta_{r_m,n} = 1$ if and only if $r_m \equiv n \mod m$.

**Proposition 4.1.** The map $r \to \xi_r$ is a topological isomorphism from $\hat{\mathbb{Z}}$ to $\hat{P}_{\text{tight}}$

**Proof.** First let us check that for $r \in \hat{\mathbb{Z}}$, $\xi_r$ is in fact a character and is maximal. Consider an element $r \in \hat{\mathbb{Z}}$. Let $e := u^{m_1} e_{m_1} u^{-m_1}$ and $f := u^{m_2} e_{m_2} u^{-m_2}$ be given. Let $d := (m_1, m_2)$ and $c := [m_1, m_2]$. Suppose $\xi_r(e) = \xi_r(f) = 1$. Then $r_{m_1} \equiv n_1 \mod m_1$ and $r_{m_2} \equiv n_2 \mod m_2$. Moreover, $r_c \equiv r_{m_i} \mod m_i$ for $i = 1, 2$. Thus $ef = u^{r_c} e_i u^{-r_c}$ by Lemma 2.6. Hence by definition $\xi_r(ef) = 1$. Now suppose $\xi_r(e) = 1$ and $e \leq f$. Then by Lemma 2.5 and Lemma 2.6 it follows that $m_2$ divides $m_1$ and $r_{m_2} \equiv r_{m_1} \equiv n_1 \equiv 0 \mod m_2$. Hence $\xi_r(f) = 1$. By definition 0 is not in the support of $\xi_r$. Thus we have shown that the support of $\xi_r$ is a filter or in other words $\xi_r$ is a character.

Now we claim $\xi_r$ is maximal. This follows from the observation that for every $m \in \mathbb{N}^+$, the set of projections $\{u^n e_m u^{-n} : n \in \mathbb{Z}/(m)\}$ are mutually orthogonal. Thus if $\xi$ is a character}
then for every $m$ there exists at most one $r_m$ for which $\xi(u^r m e_m u^{-r_m}) = 1$. This implies that if $\xi$ is a character which contains the support of $\xi_r$ then $\xi = \xi_r$.

Now let $\xi$ be a maximal character of $P$. Then by Corollary 3.3 and by the observation in the previous paragraph, it follows that for every $m$ there exists a unique $r_m$ such that $\xi(u^r m e_m u^{-r_m}) = 1$. Now let $k$ be given. Since both $u^r m e_m u^{-r_m}$ and $u^r m k e_m u^{-r_m}$ belong to the support of $\xi$, it follows that the product $u^r m e_m u^{-r_m} u^r m k e_m u^{-r_m}$ does not vanish. Then by Lemma 2.5, it follows that $r_m k = r_m \mod m$. Thus $r = (r_m) \in \hat{\mathbb{Z}}$ and the support of $\xi_r$ is contained in the support of $\xi$. Thus again by the observation in the preceding paragraph, it follows that $\xi = \xi_r$.

It is clear from the definition that the map $r \to \xi_r$ is one-one and continuous. As $\hat{\mathbb{Z}}$ is compact, it follows that the range of the map $r \to \xi_r$ which is $\hat{P}_\infty$ is also compact. Hence $\hat{P}_\infty = \hat{P}_{\text{tight}}$. Thus we have shown that $r \to \xi_r$ is a one-one and onto continuous map from $\hat{\mathbb{Z}}$ to $\hat{P}_{\text{tight}}$. Since $\hat{\mathbb{Z}}$ is compact, it follows that the above map is in fact a homeomorphism. This completes the proof.

From now on we will simply write $r(e)$ in place of $\xi_r(e)$ if $r \in \hat{\mathbb{Z}}$ and $e \in P$.

5. THE GROUPOID $G_{\text{tight}}$ OF THE INVERSE SEMIGROUP $T$

Let us recall a few ring theoretical constructions. Consider the directed set $(\mathbb{N}^+, \leq)$ where the partial order $\leq$ is defined by $m \leq n$ if $m$ divides $n$. For $m \in \mathbb{N}^+$, let $R_m := \hat{\mathbb{Z}}$. Let $\phi_{m \ell, m} : R_m \to R_{m \ell}$ be the map defined by multiplication by $\ell$. Then $\phi_{m \ell, m}$ is only an additive homomorphism and it does not preserve the multiplication. We let $R$ be the inductive limit of $(R_m, \phi_{m \ell, m})$. Then $R$ is an abelian group and $\hat{\mathbb{Z}}$ is a subgroup of $R$ via the inclusion $R_1 \subset R$.

Note that $R$ is a locally compact Hausdorff space. Moreover the group $P_Q := \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in \mathbb{Q}^\times, b \in \mathbb{Q} \right\}$ acts on $R$ by affine transformations. The action is described explicitly by the following formula. For $x \in R_p$

$$\begin{bmatrix} 1 & 0 \\ \frac{m}{m} & \frac{m}{m} \end{bmatrix} x = mx + np \in R_{m'p}$$

One can check that the above formula defines an action of $P_Q$ on $R$. We need the following lemma which has already appeared in [BE10]. We recall the proof for completeness.

**Lemma 5.1.** Let $a := \frac{n}{m}$ and $b := \frac{m}{m}$. Then $s^a u^n s_m$ depends only on $a$ and $b$. 
Theorem 5.3. Now observe that enough to consider the case $m = 1$. Thus $(r, s, s)$ is a topological groupoid isomorphism. We show that that $ss^* = e$. Hence if $r(ss^*) = 1$ then $r(tt^*) = 1$ and $r(e) = 1$. Moreover $es = et$. Thus $[(r, s, s)] = [(r, t, t)]$. This completes the proof. 

Proof. Suppose $\frac{m_1}{m_1} = \frac{m_2}{m_2}$ and $\frac{m_1}{m_1} = \frac{m_2}{m_2}$. Then $n_1m_1 = n_1n_2$ and $m_1m_2 = m_1m_2$. Now, we have

$$s^*, u_{m_1}n_1s_{m_1} = s^*, u_{m_2}n_2s_{m_2}u_{m_1}s_{m_1}$$

$$= s^*, u_{m_2}n_2s_{m_1}n_1s_{m_1}$$

$$= s^*, u_{m_2}n_2s_{m_1}n_1m_2s_{m_1}$$

$$= s^*, u_{m_2}n_2s_{m_1}n_1m_2s_{m_1}$$

$$= s^*, u_{m_2}s_{m_1}n_1m_2s_{m_1}$$

$$= s^*, u_{m_2}s_{m_1}n_2s_{m_1}$$

$$= s^*, u_{m_2}s_{m_1}n_2s_{m_1}$$

This completes the proof. 

Now we explicitly identify the groupoid $G_{tight}$ associated to the inverse semigroup $T$. When we consider transformation groupoids, we consider only right actions. Thus we let $P_\mathbb{Q}$ act on the right on the space $\mathcal{R}$ by defining $x.g = g^{-1}x$ for $x \in \mathcal{R}$ and $g \in P_\mathbb{Q}$. We show that that groupoid $G_{tight}$ of the inverse semigroup $T$ is isomorphic to the restriction of the transformation groupoid $\mathcal{R} \times P_\mathbb{Q}$ to the closed subset $\hat{\mathbb{Z}}$. (Here we consider $P_\mathbb{Q}$ as a discrete group.) Let us begin with a lemma which will be useful in the proof.

Lemma 5.2. In $G_{tight}$ one has $[(r, s_{m}, n_{m}e_{k}u_{n_{m}s_{m}})] = [(r, s_{m}, n_{m}e_{k}u_{n_{m}s_{m}})]$

Proof. First observe that $[(r, s_{m}, n_{m}e_{k}u_{n_{m}s_{m}})] = [(r, s_{m}, n_{m}e_{k}u_{n_{m}s_{m}})]$. Thus it is enough to consider the case $m = 1$. Now let $s := u_{k}e_{k}u_{n_{m}s_{m}}$, $t := u_{k}e_{k}u_{n_{m}s_{m}}$ and $e := u_{k}e_{k}u_{n_{m}s_{m}}$. Now observe that $ss^* = et$. Hence if $r(ss^*) = 1$ then $r(tt^*) = 1$ and $r(e) = 1$. Moreover $es = et$. Thus $[(r, s)] = [(r, t)]$. This completes the proof. 

Theorem 5.3. Let $\phi: \mathcal{R} \times P_\mathbb{Q}|_{\hat{\mathbb{Z}}} \rightarrow G_{tight}$ be the map defined by

$$\phi \left( r, \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} \right) = [(r, s_{m}, u_{k}s_{n})]$$

Then $\phi$ is a topological groupoid isomorphism.

Proof. The map $\phi$ is well defined.

Let $\left( r, \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} \right)$ be an element in $\mathcal{R} \times P_\mathbb{Q}|_{\hat{\mathbb{Z}}}$. Then we have $mr - k = ns$ for some $s \in \hat{\mathbb{Z}}$. Now we need to show that $r(s_{m}u_{k}s_{n}^{-1}s_{m}) = 1$. By Lemma 2.9 it follows that
implies that $mr$ of the form $e^{2.9}$ implies that $vv$ follows from Lemma 5.2. Now the surjectivity of $\phi$ follows from Lemma 5.2.

Injectivity of $\phi$:

First let us show that if $[s, s_m u^k s_n] \in \mathcal{G}_{tight}$ then $\phi([s, s_m u^k s_n]) = [r, s] = 1$. Since we are considering $P_Q$ as a discrete group, we can without loss of generality assume that $g_n = g$ for every $n$. Then, from Lemma 4.4, it follows that $\phi(r_n, g_n)$ converges to $\phi(r, g)$.

For an open subset $U$ of $\mathcal{Z}$ and $g := \left[ \begin{array}{cc} 1 & 0 \\ \frac{k}{m} & \frac{n}{m} \end{array} \right]$, consider the open set

$$\theta(U, g) := \{ (r, g) : r \in U \text{ and } r.g \in \mathcal{Z} \}.$$
Then the collection \( \{ \theta(U, g) : U \subseteq \hat{\mathbb{Z}}, g \in P\mathbb{Q}\} \) forms a basis for \( \mathcal{R} \times P\mathbb{Q}|_{\hat{\mathbb{Z}}} \). Moreover \( \phi(\theta(U, g)) = \theta(U, s_m u^k s_n) \). Hence \( \phi \) is an open map. Thus we have shown that \( \phi \) is a homeomorphism.

\( \phi \) is a groupoid morphism.

First we show that \( \phi \) preserves the source and range. By definition \( \phi \) preserves the range.

\[
\left( r, g := \begin{bmatrix} 1 & 0 \\ k & m \\ n & m \end{bmatrix} \right) \in \mathcal{R} \times P\mathbb{Q}|_{\hat{\mathbb{Z}}} \text{ be given. Let } v := s_m u^s s_n. \text{ Since } r, g \in \hat{\mathbb{Z}}, \text{ it follows that there exists } t \in \hat{\mathbb{Z}} \text{ such that } mr - k = nt. \text{ We need to show that } \xi_r \cdot v = \xi_t. \text{ (Just to keep things clear we write } \xi_r \text{ for the character determined by } r). \text{ It is enough to show that the support of } \xi_t \text{ and that of } \xi_r \cdot v \text{ coincide. But then both the characters are maximal and thus it is enough to show that the support of } \xi_t \text{ is contained in the support of } \xi_r \cdot v. \text{ Thus, suppose that } \xi_t(u^e s u^{-\ell}) = 1. \text{ Then } t_{ns} \equiv t_s \equiv \ell \mod s. \text{ This implies } mn_{ns} - k \equiv nt_{ns} \equiv n\ell \mod ns. \text{ Thus } mn_{ns} \equiv k + n\ell \mod ns. \text{ Let } n_1 := \frac{ns}{(ns, m)}. \text{ Now observe that }
\]

\[
(\xi_r \cdot v)(u^e s u^{-\ell}) = \xi_r(vu^e s u^{-\ell} v^*) \\
= \xi_r(s_m u^k s_n u^e s u^{-\ell} s_m u^{-k} s_n) \\
= \xi_r(s_m u^k n^\ell e_{ns} u^{(k + n\ell)} s_n) \\
= \xi_r(u^{n s} e_{n_1} u^{-r_{ns}}) \text{ [ By Lemma 2.9] } \\
= \delta_{r_{ns}, r_{n_1}} \\
= 1 \text{ [ Since } r_{ns} = r_{n_1} \text{ in } \mathbb{Z}/(n_1)]
\]

Thus we have shown that the support of \( \xi_t \) is contained in the support of \( \xi_r \cdot v \) which in turn implies that \( \xi_t = \xi_r \cdot v \). Hence \( \phi \) preserves the source.

Now we show that \( \phi \) preserves multiplication. Let \( \gamma_i := (r_i, \left[ \begin{smallmatrix} 1 & 0 \\ k_i & m_i \\ n_i & m_i \end{smallmatrix} \right] ) \) for \( i = 1, 2 \). Since \( \phi \) preserves the range and source, it follows that if \( \gamma_1 \) and \( \gamma_2 \) are composable, so do \( \phi(\gamma_1) \) and \( \phi(\gamma_2) \). Observe that

\[
\phi(\gamma_1) \phi(\gamma_2) = [r_1, s_{m_1} u^{k_1} s_{n_1} s_{m_2} u^{k_2} s_{n_2}] \\
= [r_1, s_{m_1} s_{m_2} u^{m_2 k_1} c_{m_2 n_1} u^{k_2 n_1} s_{n_1 n_2}] \text{ ( Eq. 2.1) } \\
= [r_1, s_{m_1} s_{m_2} u^{m_2 k_1 + n_1 k_2} s_{n_1 n_2}] \text{ (Lemma 5.2) } \\
= \phi(\gamma_1 \gamma_2)
\]

It is easily verifiable that \( \phi \) preserves inversion. This completes the proof. \( \square \)

**Remark 5.4.** Combining Proposition 3.2 and Theorem 8.3, we obtain that \( U[\mathbb{Z}] \) is isomorphic to \( C^*(\mathcal{R} \times P\mathbb{Q}|_{\hat{\mathbb{Z}}}) \) which is Remark 2 in page 17 of [CL08].
6. Simplicity of $U[\mathbb{Z}]$

First we recall a few definitions from [Ren09]. Let $\mathcal{G}$ be an r-discrete, Hausdorff and locally compact topological groupoid. Let $\mathcal{G}^0$ be its unit space. We denote the source and range maps by $s$ and $r$ respectively. The arrows of $\mathcal{G}$ define an equivalence relation on $\mathcal{G}^0$ as follows:

$$x \sim y \text{ if there exists } \gamma \in \mathcal{G} \text{ such that } s(\gamma) = x \text{ and } r(\gamma) = y.$$

A subset $E$ of $\mathcal{G}^0$ is said to be invariant if the orbit of $x$ is contained in $E$ whenever $x \in E$. For $x \in \mathcal{G}^0$, define the isotropy group at $x$ denoted $\mathcal{G}(x)$ by $\mathcal{G}(x) := \{ \gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x \}$.

A groupoid $\mathcal{G}$ is said to be

- topologically principal if the set of $x \in \mathcal{G}^0$ for which $\mathcal{G}(x) = \{ x \}$ is dense in $\mathcal{G}^0$.
- minimal if the only non-empty open invariant subset of $\mathcal{G}^0$ is $\mathcal{G}^0$.

We need the following theorem. We refer to [Ren09] for a proof.

**Theorem 6.1.** Let $\mathcal{G}$ be an r-discrete, Hausdorff and locally compact topological groupoid. If $\mathcal{G}$ is topologically principal and minimal then $C^*_\text{red}(\mathcal{G})$ is simple.

**Proposition 6.2.** The $C^*$-algebra $U[\mathbb{Z}]$ is simple.

**Proof.** Let $\mathcal{G}$ denote the groupoid $\mathcal{R} \times P_{\mathbb{Q}|\hat{\mathbb{Z}}}$. Since the group $P_{\mathbb{Q}}$ is solvable, it is amenable and thus by Proposition 2.15 of [MR82], it follows that the full groupoid $C^*$-algebra $C^*(\mathcal{G})$ is isomorphic to the reduced algebra $C^*_\text{red}(\mathcal{G})$. Now we apply Theorem 6.1 to complete the proof.

First let us show $\mathcal{G}$ is minimal. Let $U$ be a non-empty open invariant subset of $\mathcal{G}^0$. For $m = (m_1, m_2, \cdots, m_n) \in (\mathbb{Z}\setminus\{0\})^n$ and $k \in \mathbb{Z}$, let

$$U_{m,k} := \{ r \in \hat{\mathbb{Z}} : r_{m_i} \equiv k \mod m_i \}$$

Then, by the Chinese remainder theorem, it follows that the collection $\{U_{m,k}\}$ (where $m$ varies over $(\mathbb{Z}\setminus\{0\})^n$ (we let $n$ vary too) and $k \in \mathbb{Z}$) is a basis for the topology on $\hat{\mathbb{Z}}$. Also observe that for a given $m$, $\bigcup_{k \in \mathbb{Z}} U_{m,k} = \hat{\mathbb{Z}}$. Moreover the translation matrix $\begin{bmatrix} 1 & 0 \\ k_1 - k_2 & 1 \end{bmatrix}$ maps $U_{m,k_1}$ onto $U_{m,k_2}$. Now since $U$ is non-empty and open, there exists an $m$ and a $k_0$ such that $U_{m,k_0} \subseteq U$. But since $U$ is invariant, it follows that $U_{m,k} \subseteq U$ for every $k \in \mathbb{Z}$. Thus $\bigcup_{k \in \mathbb{Z}} U_{m,k} \subseteq U$. This forces $U = \hat{\mathbb{Z}}$. This completes the proof.

Now we show $\mathcal{G}$ is topologically principal. Let

$$E := \{ r \in \hat{\mathbb{Z}} : r \neq 0, r_p = 0 \ \forall i, \text{ except for finitely many primes } p \}$$

If one identifies $\hat{\mathbb{Z}}$ with $\prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$ then it is clear that $E$ is dense in $\hat{\mathbb{Z}}$. Now let $r \in E$ be given.

We claim that $\mathcal{G}(r) = \{ r \}$. Suppose $r. \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} = r$. Then $mr - k = nr$. But $r_p = 0$ except for finitely many primes. Thus it follows that $k$ is divisible by infinitely many primes which forces $k = 0$. Now $mr = nr$ and $r \neq 0$ implies $m = n$. Thus $\mathcal{G}(r) = \{ r \}$. This proves that $\mathcal{G}$ is topologically principal. This completes the proof. \[\Box\]
7. Nica-covariance, tightness and boundary relations

In this section, we digress a bit to understand some of the results in [Nic92], [CL07] and in [LR10] from the point of view of inverse semigroups. Let us recall the notion of quasi-lattice ordered groups considered by Nica in [Nic92]. Let \( G \) be a discrete group and \( P \) a subsemigroup of \( G \) containing the identity \( e \). Also assume that \( P \cap P^{-1} = \{e\} \). Then \( P \) induces a left-invariant partial order \( \leq \) on \( G \) defined by \( x \leq y \) if and only if \( x^{-1}y \in P \). The pair \( (G, P) \) is said to be quasi-lattice ordered if the following conditions are satisfied.

1. Any \( x \in PP^{-1} \) has a least upper bound in \( P \), and
2. If \( s, t \in P \) have a common upper bound in \( P \) then \( s, t \) have a least common upper bound.

If \( s, t \in P \) have a common upper bound in \( P \) then we denote the least upper bound in \( P \) by \( \sigma(s, t) \). It is easy to show that \( s, t \in P \) have a common upper bound if and only if \( s^{-1}t \in PP^{-1} \). Let us recall the Wiener-Hopf representation from [Nic92]. Consider the representation \( W : P \to B(\ell^2(P)) \) defined by
\[
W(p)(\delta_a) := \delta_{pa}
\]
where \( \{\delta_a : a \in P\} \) denotes the canonical orthonormal basis of \( \ell^2(P) \). Note that for \( s \in P \), \( W(s) \) is an isometry and \( W(s)W(t) = W(st) \) for \( s, t \in P \). For \( s \in P \), let \( M(s) = W(s)W(s)^* \) then
\[
(7.2) \quad M(s)M(t) = \begin{cases} 
M(\sigma(s, t)) & \text{if } s \text{ and } t \text{ have common upper bound in } P \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{N} := \{W(s)W(t)^* : s, t \in P\} \cup \{0\} \). Then Equation (5) of Proposition 3.2 in [Nic92] implies that \( \mathcal{N} \) is an inverse semigroup of partial isometries. The following definition is due to Nica.

**Definition 7.1** ([Nic92]). Let \( (G, P) \) be a quasi-lattice ordered group. An isometric representation \( V : P \to B(\mathcal{H}) \) on a Hilbert space \( \mathcal{H} \) (i.e. \( V(t)^*V(t) = 1 \) for \( t \in P \), \( V(e) = 1 \) and \( V(s)V(t) = V(st) \) for every \( s, t \in P \) is said to be Nica-covariant if the following holds
\[
(7.3) \quad L(s)L(t) = \begin{cases} 
L(\sigma(s, t)) & \text{if } s \text{ and } t \text{ have common upper bound in } P \\
0 & \text{otherwise}.
\end{cases}
\]

where we set \( L(t) = V(t)V(t)^* \). In other words a Nica-covariant representation of \( (G, P) \) is nothing but a unital representation of the inverse semigroup \( \mathcal{N} \) which sends \( 0 \) to \( 0 \).

Let us say a Nica-covariant representation is tight if the corresponding representation on \( \mathcal{N} \) is tight. Now one might ask what are the tight representations of the inverse semigroup \( \mathcal{N} \)? We prove that tight representations are nothing but Nica-covariant representations satisfying the boundary relations considered by Laca and Crisp in [CL07]. This fact is implicit in [CL07] and it is in fact explicit if one applies Theorem 13.2 of [Exc09]. The author believes that it is worth recording this connection and we do this in the next proposition.
First let us fix a few notations. A finite subset \( F \) of \( P \) is said to cover \( P \) if given \( x \in P \) there exists \( y \in F \) such that \( x \) and \( y \) have a common upper in \( P \). Let 
\[
\mathcal{F} := \{ F \subset P : F \text{ is finite and covers } P \}
\]

**Proposition 7.2.** Let \((G, P)\) be a quasi-lattice ordered group. Consider a Nica-covariant representation \( V : P \to B(\mathcal{H}) \). Then \( V \) is tight if and only if for every \( F \in \mathcal{F} \), one has 
\[
\prod_{t \in F}(1 - V(t)V(t)^*) = 0.
\]

**Proof.** Consider a Nica-covariant representation \( V : P \to B(\mathcal{H}) \). Suppose that \( V \) is tight. Let \( F \in \mathcal{F} \) be given. Note that \( F \) covers \( P \) if and only if \( \{ M(t) : t \in F \} \) covers the set of projections in \( \mathcal{N} \). Now the tightness of \( V \) implies that \( \sup_{t \in F} V(t)V(t)^* = 1 \). This is equivalent to saying that 
\[
\prod_{t \in F}(1 - V(t)V(t)^*) = 0.
\]
Thus we have the implication \( \Rightarrow \).

Let \( V \) be a Nica-covariant representation for which \( \prod_{t \in F}(1 - V(t)V(t)^*) = 0 \) for every \( F \in \mathcal{F} \). We denote the set of projections in \( \mathcal{N} \) by \( E \). Then \( E := \{ M(t) : t \in P \} \cup \{ 0 \} \). Let \( \{ M(t_1), M(t_2), \ldots, M(t_n) \} \subset [0, M(t)] \) be a finite cover. Then \( M(t_i) \leq M(t) \) for every \( i \). But this is equivalent to the fact that \( t \leq t_i \).

We claim that \( \{ t^{-1} t_i : i = 1, 2, \ldots, n \} \) covers \( P \). Let \( s \in P \) be given. Then \( t \leq s \) which implies \( M(ts) \leq M(t) \). Thus there exists a \( t_i \) such that \( M(ts)M(t_i) \neq 0 \). This implies that \( ts \) and \( t_i \) have a common upper bound in \( P \). In other words, \( (ts)^{-1} t_i = s^{-1} t^{-1} t_i \in PP^{-1} \). Thus \( s \) and \( t^{-1} t_i \) have a common upper bound in \( P \). This proves the claim.

By assumption it follows that 
\[
\prod_{i=1}^n (1 - L(t^{-1} t_i)) = 0
\]
where \( L(s) := V(s)V(s)^* \). Now multiplying this equality on the left by \( V(t) \) and on the right by \( V(t)^* \), we get 
\[
\prod_{i=1}^n (V(t)V(t)^* - V(t)V(t^{-1} t_i)V(t^{-1} t_i)^* V(t)^*) = 0
\]

\[
\prod_{i=1}^n (V(t)V(t)^* - V(t_i)V(t_i)^*) = 0
\]

But this is equivalent to \( \sup_i L(t_i) = L(t) \). This completes the proof. \( \square \)

**Remark 7.3.** The relations \( \prod_{x \in F}(1 - V(t)V(t)^*) = 0 \) for \( F \in \mathcal{F} \) are the boundary relations considered in [CL07].

Let \( Q_N \) be the C*-subalgebra of \( U[\mathbb{Z}] \) generated by \( u \) and \( \{ s_m : m > 0 \} \). In [Cun08], it was proved that \( Q_N \) is simple and purely infinite. Moreover in [Cun08], it was shown that \( U[\mathbb{Z}] \) is isomorphic to a crossed product of \( Q_N \) with \( \mathbb{Z}/2\mathbb{Z} \). Let 
\[
P_N := \left\{ \begin{bmatrix} 1 & 0 \\ k & m \end{bmatrix} : k \in \mathbb{N} \text{ and } m \in \mathbb{N}^\times \right\}
\]

Note that \( P_N \) is a semigroup of \( P_Q \).

**Remark 7.4.** In [LR10], it was proved that \((P_Q, P_N)\) is a quasi-lattice ordered group. Moreover it was shown in [LR10] that Nica-covariance together with boundary relations is equivalent to
Cuntz-Li relations and the universal C*-algebra made out of Nica-covariant representations satisfying the boundary relations is in fact $Q_N$.

8. The Cuntz-Li algebra for a general integral domain

We end this article by giving a few remarks of how to adapt the analysis in Section 1 – 6 for a general integral domain $R$. Now let $R$ be an integral domain such that $R/mR$ is finite for every non-zero $m \in R$. We also assume that $R$ is countable and $R$ is not a field.

**Definition 8.1** ([CL08]). Let $U[R]$ be the universal C*-algebra generated by a set of unitaries \( \{u^n : n \in R\} \) and a set of isometries \( \{s_m : m \in R^\times\} \) satisfying the following relations.

\[
\begin{align*}
    s_m s_n &= s_{mn} \\
    u^n u^m &= u^{n+m} \\
    s_m u^0 &= u^{mn}s_m \\
    \sum_{n \in R/mR} u^n e_m u^{-n} &= 1
\end{align*}
\]

where $e_m$ denotes the final projection of $s_m$.

Now the problem is the product $u^r e_m u^{-r} u^s e_n u^{-s}$ may not be of the form $u^k e_c u^{-k}$ for some $k$ and $c$. Nevertheless it will be in the linear span of \( \{u^k e_m u^{-k} : k \in R/(mn)\} \). Let $P$ denote the set of projections in $U[R]$ which is in the linear span of \( \{u^r e_m u^{-r} : r \in R/(m)\} \) for some $m$. Explicitly, a projection $e \in U[R]$ is in $P$ if and only if there exists an $m \in R^\times$ and $a_r \in \{0, 1\}$ such that $f = \sum_r a_r u^r e_m u^{-r}$.

Now it is easy to show that $P$ is a commutative semigroup of projections containing 0. Moreover $P$ is invariant under conjugation by $u^r$, $s_m$ and $s_m^*$. One can prove the following Proposition just as in the case when $R = \mathbb{Z}$.

**Proposition 8.2.** Let $T := \{s_m u^n e_m u^{-n} s_{m'} : e \in P, m, m' \neq 0, n, n' \in R\}$. Then $T$ is an inverse semigroup of partial isometries. Moreover the set of projections in $T$ coincide with $P$. Also the linear span of $T$ is dense in $U[R]$.

Let $\hat{R} := \{(r_m) \in \prod R/(m) : r_{mk} = r_m \text{ in } R/(m)\}$ be the profinite completion of the ring $R$. For $r \in \hat{R}$, define

\[ A_r := \{f \in P : f \geq u^m e_m u^{-m} \text{ for some } m\} \]

Then $A_r$ is an ultrafilter for every $r \in \hat{R}$ and the map $r \to A_r$ is a topological isomorphism from $\hat{R}$ to $\hat{P}_{\text{light}}$.

Let $Q(R)$ be the field of fractions of $R$. For $m \neq 0$, let $\mathcal{R}_m := \hat{R}$. For every $\ell \neq 0$, let $\phi_{m\ell,m} : \mathcal{R}_m \rightarrow \mathcal{R}_{\ell m}$ be the map defined by multiplication by $\ell$. Then $\phi_{m\ell,m}$ is only an additive homomorphism and it does not preserve the multiplication. We let $\mathcal{R}$ be the inductive limit of $(\mathcal{R}_m, \phi_{m\ell,m})$. Then $\mathcal{R}$ is an abelian group and $\hat{R}$ is a subgroup of $\mathcal{R}$ via the inclusion $\mathcal{R}_1 \subset \mathcal{R}$.
Note that $\mathcal{R}$ is a locally compact Hausdorff space. Moreover the group

$$P_{Q(R)} := \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a \in Q(R)^*, b \in Q(R) \right\}$$

acts on $\mathcal{R}$ by affine transformations. The action is described explicitly by the following formula. For $x \in \mathcal{R}_p$

$$\begin{bmatrix} 1 & 0 \\ \frac{m}{m'} & \frac{n}{m'} \end{bmatrix} x = mx + np \in \mathcal{R}_{m'p}$$

One can check that the above formula defines an action of $P_{Q(R)}$ on $\mathcal{R}$. Let $G_{\text{tight}}$ be the tight groupoid associated to the inverse semigroup $T$ defined in Proposition 8.2. Then as in the case when $R = \mathbb{Z}$, we have the following theorem.

**Theorem 8.3.** Let $\phi : \mathcal{R} \times P_{Q(R)}|_{\hat{R}} \to G_{\text{tight}}$ be the map defined by

$$\phi \left( (r, \begin{bmatrix} 1 & 0 \\ k & n \end{bmatrix}) \right) = [(r, s_m u^k s_n)]$$

Then $\phi$ is a topological groupoid isomorphism. Moreover the C*-algebra $U[R]$ is isomorphic to the full (and the reduced) C*-algebra of the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$.

We end this article by showing that $U[R]$ is simple.

**Proposition 8.4** ([CL08]). The C*-algebra $U[R]$ is simple.

**Proof.** Let us denote the groupoid $\mathcal{R} \times P_{Q(R)}|_{\hat{R}}$ by $\mathcal{G}$. As in Proposition 6.1 we need to show that $\mathcal{G}$ is minimal and topologically principal. The proof of the minimality of $\mathcal{G}$ is exactly similar to that in Proposition 6.1. We now show that $\mathcal{G}$ is topologically principal. For $g \in P_{Q(R)} \setminus \{1\}$, let us denote the set of fixed points of $g$ in $\hat{R}$ by $F_g$. It follows from Baire category theorem that $\mathcal{G}$ is topologically principal if and only if $F_g$ has empty interior for every $g \neq 1$.

Let $g = \begin{bmatrix} 1 & 0 \\ k & n \end{bmatrix}$ be a non-identity element in $P_{Q(R)}$. Suppose that $F_g$ contains a non-empty open set say $U$. Now note that $R$ is dense in $\hat{R}$. Thus $U \cap R$ is non-empty. Moreover $U \cap R$ is infinite. Let $r_1, r_2$ be two distinct points of $R$ in $U$. Since $r_1, r_2 \in F_g$, it follows that $mr_1 - k = nr_1$ and $mr_2 - k = nr_2$. Thus we have $(m - n)r_1 = k = (m - n)r_2$. This forces $m = n$ and $k = 0$. This is a contradiction to the fact that $g \neq 1$. Thus for every $g \neq 1$, $F_g$ has empty interior which in turn implies that $\mathcal{G}$ is topologically principal. This completes the proof. \[\square\]

**Remark 8.5.** In [KLQ10], Cuntz-Li type relations arising out of a semidirect product $N \rtimes H$ where $N$ is a normal subgroup and $H$ is an abelian group satisfying certain hypothesis were considered. It was shown in [KLQ10] that the universal C*-algebra generated by the Cuntz-Li type relations is isomorphic to a corner of a crossed product algebra. It is possible to apply inverse semigroups and tight representations to reconstruct this result. The details will be spelt out elsewhere.
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