Two-photon exchange in elastic $e\pi$ scattering

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We calculate two-photon exchange amplitude for the elastic electron-pion scattering in the dispersion-relation inspired approach, including both elastic and inelastic contributions. The latter was modelled as a sum of $\rho$ and $b_1(1235)$ meson contributions. We find that at $Q^2 \lesssim 2 \text{ GeV}^2$ the elastic contribution is dominant, similarly to electron-proton scattering case. At higher $Q^2$ the inelastic contribution is not negligible, but still smaller than the elastic one. We also explain observed rapid amplitude growth at backward angles.

I. INTRODUCTION

In last years, two-photon exchange (TPE) is widely discussed in the literature. The role of TPE in the elastic electron-proton scattering was studied most thoroughly. In particular, it was shown that TPE can be responsible for the discrepancy between Rosenbluth and polarization transfer measurements of the proton form factors (FFs) \cite{1,2}. These intriguing results have triggered a study of TPE effects in other processes, such as elastic $ed$ \cite{3}, deep inelastic scattering \cite{4}, and so on.

Two recent papers \cite{5,6} discuss TPE in the elastic $e\pi$ scattering. Both are limited to so-called elastic contribution, and report significant increase of TPE amplitude at backward angles. These papers use different approaches to loop integral calculation: in Ref. \cite{5} it was calculated approximately, under the assumption that both photons carry about a half of the transferred momentum, whereas in Ref. \cite{6} the loop integral was expressed through 't Hooft-Veltman $n$-point functions. However, the starting expression for the TPE diagram in Refs. \cite{5,6} is written somewhat heuristically. In particular the (virtual) Compton scattering amplitude, constructed in the same manner, might be not gauge-invariant\footnote{For example, in Ref. \cite{6} TPE amplitude results from contracting leptonic and hadronic tensors, $L_{\mu\nu}$ and $H_{\mu\nu}$. It is tempting to identify the hadronic tensor, after proper crossing symmetrization, with the virtual Compton scattering amplitude: $T_{\mu\nu} \sim H_{\mu\nu} + H_{\nu\mu}(k_1 \leftrightarrow k_2)$. However with $H_{\mu\nu}$ from Eq.(8) of Ref. \cite{6} we have $T_{\mu\nu} k_2^\nu = 2k_1^\mu$ instead of 0, as required by gauge invariance.}

In the present paper we evaluate TPE amplitude for the elastic $e\pi$ scattering using dispersion-relation inspired approach, developed in Ref. \cite{7}. The positive features of this approach are (for more detail see discussion in the Introduction of Ref. \cite{7}):

- no need for off-shell FFs,
- clear and unambiguous definition of elastic and inelastic contributions,
- correct analytic structure of the resulting TPE amplitude (i.e. it is an analytic function with only branch cuts dictated by unitarity).

We evaluate both elastic and inelastic contributions. Following common practice \cite{2}, the latter is modelled as a sum of resonance contributions, namely, $\rho$ and $b_1(1235)$ meson contributions. Certainly, contributions of other meson resonances can be easily included.

II. TPE AMPLITUDE

Throughout the paper we use the notation, similar to Ref. \cite{7}. The initial and final pion (electron) momenta are denoted $p$ and $p'$ ($k$ and $k'$), respectively. The transferred momentum is $q = p' - p$, the pion mass is $M$, the electron mass $m$ is assumed to be infinitely small.

In one-photon exchange approximation, the elastic electron-pion scattering amplitude is

$$M_{fi} = -\frac{4\pi\alpha}{q^2} u'\gamma^\mu u (p + p')_\mu F(q^2)$$  \hspace{1cm} (1)

where $\alpha$ is fine structure constant and the real-valued function $F(q^2)$ is called pion electromagnetic FF. Since the pion has zero spin, it is easy to see that even in general case the amplitude keeps the same structure \footnote{For example, in Ref. \cite{6} TPE amplitude results from contracting leptonic and hadronic tensors, $L_{\mu\nu}$ and $H_{\mu\nu}$. It is tempting to identify the hadronic tensor, after proper crossing symmetrization, with the virtual Compton scattering amplitude: $T_{\mu\nu} \sim H_{\mu\nu} + H_{\nu\mu}(k_1 \leftrightarrow k_2)$. However with $H_{\mu\nu}$ from Eq.(8) of Ref. \cite{6} we have $T_{\mu\nu} k_2^\nu = 2k_1^\mu$ instead of 0, as required by gauge invariance.}, if we neglect...
the electron mass. The only difference with one-photon exchange case is that the function $F$ becomes complex and depends on both $t \equiv q^2$ and $\nu = (p + p')(k + k')$. We may write

$$F(t, \nu) = F(t) + \delta F(t, \nu) + O(\alpha^2),$$

where $\delta F$ is TPE contribution. Because of charge conjugation and crossing symmetry, $\delta F$ should be an odd function of $\nu$.

To calculate the TPE amplitude, we start with its absorptive part, for which we have the unitarity condition

$$M_{f \bar{f}} - M_{if}^* = \frac{i}{4\pi^2} \sum_h \int M_{fi, n} M_{if, n}^* \theta(p_0^t) \delta(p''^{t'} - M_0^2) \theta(\nu_0^t) \delta(k''^{t'}) d^4 k''$$

where subscript $n$ denotes intermediate state, which consists of the electron with momentum $k''$ and some hadronic state $h$. We restrict ourselves to two-particle intermediate states, thus $h$ can only be a single meson with positive charge and negative $C$-parity ($\pi$, $\rho$ an so on). Its mass is denoted $M_h$ and momentum is $p'' = p + k - k''$. Retaining only the term corresponding to $h = \pi$, we obtain so-called elastic contribution $\delta F^{(el)}$. Other contributions are referred to as inelastic ones.

### A. Elastic contribution

Substituting the elastic amplitude (1) in the unitarity condition (3), we obtain

$$\bar{u} \gamma^\mu u (p + p') \mu \text{ Im } \delta F^{(el)}(t, \nu) = -\frac{i \alpha t}{2\pi} \int \bar{u} \gamma^\mu \bar{p}^{'''} \gamma^\nu u (p' + p'') \mu \nu \bar{F}(t_1) \bar{F}(t_2) \theta(p_0^t) \delta(p''^{t'} - M^2) \theta(\nu_0^t) \delta(k''^{t'}) d^4 k''$$

where $t_1 = (p'' - p)^2$, $t_2 = (p'' - p')^2$ and $\bar{F}(t) = F(t)/t$, and, after some transformations

$$\text{Im } \delta F^{(el)}(t, \nu) = -\frac{i \alpha t}{2\pi} (\nu - t) \int \left\{ 1 + \frac{2M^2 + \nu - t}{\nu^2 + t(4M^2 - t)} \right\} \bar{F}(t_1) \bar{F}(t_2) \theta(p_0^t) \delta(p''^{t'} - M^2) \theta(\nu_0^t) \delta(k''^{t'}) d^4 k''$$

where $t_p = t_1 + t_2 - t$.

Now, to obtain box-type amplitude $\delta F^{(el)}_{box}$ we should first change, according to Ref. [7],

$$\theta(p_0^t) \delta(p''^{t'} - M^2) \theta(\nu_0^t) \delta(k''^{t'}) \to \frac{1}{2\pi^2 i k''^{t'}(p''^{t'} - M^2)}$$

which gives

$$\delta F^{(el)}_{box}(t, \nu) = \frac{i \alpha t}{4\pi^3} (\nu - t) \int \left\{ 1 + \frac{2M^2 + \nu - t}{\nu^2 + t(4M^2 - t)} \right\} \bar{F}(t_1) \bar{F}(t_2) \frac{d^4 k''}{k''^{t'}(p''^{t'} - M^2)}$$

(7)

This quantity is marked with a tilde, because, due to the denominator of the second term in curly braces, it has unphysical poles at $\nu = \pm \nu_0 = \pm \sqrt{-t(4M^2 - t)}$. As described in Ref. [7], we should subtract appropriate rational function of $\nu$ to obtain correct analytic behaviour of the amplitude. This is easily achieved with the help of Eq. (B4) of Ref. [7], yielding

$$\delta F^{(el)}_{box}(t, \nu) = \frac{i \alpha t}{4\pi^3} (\nu - t) \int \left\{ 1 + \frac{(2M^2 + \nu - t)p_p - (\nu - t)(p''^{t'} - M^2) - (4M^2 + \nu - t)k''^{t'}}{\nu^2 + t(4M^2 - t)} \right\} \bar{F}(t_1) \bar{F}(t_2)\frac{d^4 k''}{k''^{t'}(p''^{t'} - M^2)}$$

(8)

The subtracted quantity is a rational function of $\nu$, since the integrals $\int \bar{F}(t_1) \bar{F}(t_2) \frac{d^4 k''}{k''^{t'}(p''^{t'} - M^2)}$ and $\int \bar{F}(t_1) \bar{F}(t_2) \frac{d^4 k''}{k''^{t'}(p''^{t'} - M^2)}$ are independent of $\nu$; thus the subtraction does not introduce new cuts or violate unitarity.

The full TPE amplitude is the sum of box and crossed-box amplitudes,

$$\delta F^{(el)}(t, \nu) = \delta F^{(el)}_{box}(t, \nu) - \delta F^{(el)}_{box}(t, -\nu)$$

(9)

The crossed box amplitude $-\delta F^{(el)}_{box}(t, -\nu)$ has no imaginary part in the s-channel ($\nu > 0$), but provides correct imaginary part in the u-channel ($\nu < 0$).
The elastic part of the TPE amplitude is infra-red divergent. To obtain physically meaningful finite result, standard Mo&Tsai contribution is usually subtracted, and we also follow this way. It can be shown that for \( m \approx 0 \) Mo&Tsai contribution is equal to

\[
\delta F_{\text{box}}^{(MT)}(t, \nu) = \frac{\alpha}{\pi} F(t) \left\{ 2 \ln \frac{\lambda}{M} + \ln \frac{\nu - t}{2mM} - \ln^2 \frac{\nu - t}{2M^2} - \text{Li}_2 \left( 1 - \frac{\nu - t}{2M^2} \right) \right\}
\]

(10)

where \( \lambda \) is infinitely small photon mass, introduced to regulate the divergence. The logarithmic dependence on \( m \) disappears when adding box and crossed box amplitudes.

### B. \( \rho \)-meson inelastic contribution

This contribution arises from the \( e \rho \) intermediate state in the r.h.s. of Eq. (3). To calculate it we need \( \rho \pi \gamma^* \) vertex, which can be written in general form as

\[
\mathcal{M} = \sqrt{4\pi\alpha} \frac{2M_\rho}{M_\rho^2 - M^2} g(q^2) e^{\mu\nu\sigma\tau} \epsilon_\mu q_\nu p_\sigma v_\tau
\]

(11)

where \( M_\rho \) is \( \rho \)-meson mass, \( q \) and \( p \) are photon and pion momenta, \( \epsilon \) and \( v \) are photon and \( \rho \)-meson polarizations, respectively, and \( g(q^2) \) is dimensionless form factor. Its normalization is established from the \( \rho \rightarrow \pi \gamma \) decay width

\[
\Gamma_{\rho \rightarrow \pi \gamma} = \alpha |g(0)|^2 \frac{M_\rho^2 - M^2}{6M_\rho}
\]

(12)

Using the latest value \( \Gamma_{\rho \rightarrow \pi \gamma} = 68 \text{ keV} \), we obtain \( g(0) = 0.272 \). For \( q^2 \) dependence we use simple vector-dominance-inspired form

\[
g(q^2) = g(0) \frac{M_\omega^2}{M_\omega^2 - q^2}
\]

(13)

where \( M_\omega = 0.872 \text{ GeV} \). With \( \rho \pi \gamma^* \) vertex we obtain the contribution to the imaginary part of the amplitude

\[
\bar{u}' \gamma^\mu (p + p')_\mu \text{Im} \delta F^{(\rho)}(t, \nu) = \frac{\alpha t}{2\pi} \left( \frac{2M_\rho}{M_\rho^2 - M^2} \right)^2 \int \bar{u}' \gamma^\mu \hat{k}''^\nu \gamma_\nu u \times

\times e^{\mu'\nu'\sigma'\tau'} p'_\mu p'_\nu e^{\mu\nu\sigma\tau} p_\rho p_\tau \frac{g(t_1)g(t_2)}{t_1 t_2} \theta(y_0') \delta(p''^2 - M_\rho^2) \theta(k''^2) \delta(k''^2) d^4k''
\]

(14)

The real part reconstruction procedure is the same as for the elastic contribution, except that the unphysical poles are subtracted with the help of the identity

\[
\int d^4p'' f(p'') \left\{ \frac{s t_p - (s - M^2) \Delta M^2}{k'^2(p''^2 - M_\rho^2)} - \frac{s - M^2}{k'^2} \frac{s + M^2}{p''^2 - M_\rho^2} \right\} \bigg|_{\nu = \nu_0} = 0
\]

(15)

(\( \Delta M^2 = M_\rho^2 - M^2 \)), which can be obtained from Eq. (B4) of Ref. putting \( f(p'') \rightarrow f(p'')(p''^2 - M^2)/(p''^2 - M_\rho^2) \).

The final result is

\[
\delta F_{\text{box}}^{(\rho)}(t, \nu) = \frac{-i\alpha t}{32\pi^2} \left( \frac{2M_\rho}{M_\rho^2 - M^2} \right)^2 \int \left\{ \frac{2M^2 t_p - (\nu + t)(p''^2 - M^2) - (4M^2 - \nu - t)k''^2}{\nu^2 + t(4M^2 - t)} \right\} \left[ t^2 + 2t \Delta M^2 \right] + \frac{(\nu - t)k''^2}{\nu^2 + t(4M^2 - t)} \left[ t^2 + t_1 t_2 - (t_1 + 2t_2)M^2 + (t - t_1)\Delta M^2 + \Delta M^2 \right] +

2t_1 t_2 + t \Delta M^2 + t_1 (\nu/2 - t - \Delta M^2) \right\} \frac{g(t_1)g(t_2) d^4k''}{t_1 t_2 k''^2(p''^2 - M_\rho^2)}
\]

(16)
Only $\Lambda = 0$ particles were taken from Ref. [9]. In what follows $\epsilon$ where further procedure is analogous to previous cases, and the result is

$$\Gamma_{b_1 \rightarrow \pi\gamma} = \alpha|g_1(0)|^2 \frac{M^2_b - M^2}{6M_b}$$

Thus we neglect second form factor and assume

$$g_1(q^2) = g_1(0) \frac{M^2_\omega}{M^2_\omega - q^2}, \quad g_2(q^2) = 0$$

where $g_1(0) = 0.40$ is obtained from Eq. (18) and PDG value $\Gamma_{b_1 \rightarrow \pi\gamma} = 230$ keV.

C. $b_1$ meson inelastic contribution

Next allowed intermediate state with sufficiently large $\pi\gamma$ branching ratio is $b_1(1235)$. In general case, $b_1 \rightarrow \pi\gamma^*$ transition amplitude depends on two form factors:

$$M = \sqrt{4\pi\alpha} \frac{2M_b}{M_b^2 - M^2} \left[ g_1(q^2)(q^\mu p^\nu - g^{\mu\nu} p q) + g_2(q^2)(q^2 g_{\mu\nu} - q_{\mu} q_{\nu}) \right] v_{\mu} e_{\nu}$$

Only $g_1$ contributes to $b_1 \rightarrow \pi\gamma$ decay width:

$$\Gamma_{b_1 \rightarrow \pi\gamma} = \alpha|g_1(0)|^2 \frac{M^2_b - M^2}{6M_b}$$

Thus we neglect second form factor and assume

$$g_1(q^2) = g_1(0) \frac{M^2_\omega}{M^2_\omega - q^2}, \quad g_2(q^2) = 0$$

where $g_1(0) = 0.40$ is obtained from Eq. (18) and PDG value $\Gamma_{b_1 \rightarrow \pi\gamma} = 230$ keV.

Further procedure is analogous to previous cases, and the result is

$$\delta F^{(b_1)}(t, \nu) = - \frac{i \alpha t}{32\pi^2} \left( \frac{2M_b}{M^2_b - M^2} \right)^2 \left\{ \frac{2M^2 t_p - (\nu + t)(p^{\prime 2} - M^2) - (4M^2 - \nu - t)k^{\prime 2}}{\nu^2 + t(4M^2 - t)} \left[ t^2 + 2t\Delta M^2 \right] + \frac{(\nu - t)t_p + 2t(p^{\prime 2} - M^2) - 2\nu k^{\prime 2}}{\nu^2 + t(4M^2 - t)} \left[ t^2 + t_1 t_2 + t_p M^2 + (3t + t_p)\Delta M^2 + \Delta M^4 \right] + 2t_1 t_2 + t\Delta M^2 + t_p(\nu/2 - t - \Delta M^2) \right\} \frac{g_1(t_1)g_1(t_2)d^4k^{\prime \prime}}{t_1 t_2 k^{\prime 2}(p^{\prime 2} - M^2)}$$

III. RESULTS

In the numerical calculation we use monopole parameterization of pion form factor: $F(Q^2) = \Lambda^2/(Q^2 + \Lambda^2)$, where $\Lambda = 0.719$ GeV is chosen so as to reproduce measured charge radius of the pion. Its value as well as masses of all particles were taken from Ref. [9]. In what follows $\epsilon = \frac{\nu^2 - Q^2(4M^2 + Q^2)}{\nu^2 + Q^2(4M^2 + Q^2)}$ is virtual photon polarization parameter.

The calculated elastic part of TPE amplitude (Fig. 1 blue curves) agrees well with the results presented in Ref. [6]. The coincidence is, most likely, accidental: we know that for the electron-proton scattering these two approaches give different analytical results [7].
The inelastic part of TPE amplitude, calculated with the inclusion of $\rho$ and $b_1$ meson contributions, is shown in Fig. 2. More detailed numerical study reveal that it diverges logarithmically at the thresholds $s = M_\rho^2$ and $s = M_{b_1}^2$. This seems to be a consequence of neglecting respective meson widths. If one takes finite width into account, the curves become "smeared" and the divergence should disappear. In comparison with the elastic one, the inelastic contribution is almost negligible, except at very high $Q^2$ (Fig. 1, right). However at high $Q^2$ our scheme for inelastic contribution calculation becomes doubtful. Indeed, we begin to exploit such an ambiguous thing as "the contribution of a resonance away from the resonance", and trust that it is main contribution. The inclusion of heavier resonances is likely needed. The contribution of non-resonant multi-particle states also may be significant, and needs to be estimated somehow. Therefore we think that at present the magnitude of the inelastic contribution at high $Q^2$ is not reliably known.

Both Refs. [5, 6] and our work find that the elastic part of TPE amplitude at high $Q^2$ sharply grows at backward angles (i.e. near $\varepsilon = 0$). The inelastic contribution has similar tendency, as one can infer from Fig. 2 (b). Note that the amplitude does not diverge, it remains finite at $\varepsilon = 0$. In our approach this holds automatically, since $\varepsilon = 0$ is the physical region boundary, corresponding to $\sqrt{\frac{4M^2 + Q^2}{2M^2}}$, whereas the amplitude is constructed to be finite at this point ([7], Sec. II A).

The explanation of this phenomenon is quite simple. The full amplitude is the sum of box and x-box amplitudes, and each of them has a singularity (a branching point) at $s = M^2$ or $u = M^2$, respectively (neglecting the electron mass). Though both singularities lie in the unphysical region, the $u$-channel singularity $u = M^2$ corresponds to $\varepsilon = -\left(1 + Q^2/2M^2\right)^{-1}$; for $Q^2 \gg M^2$ this is very close to $\varepsilon = 0$, explaining the rapid amplitude growth near this point. A good illustration is Fig. 3 where box and x-box amplitudes are plotted separately (a proper constant

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**FIG. 2:** Inelastic contribution ($\rho + b_1$) as a function of $\varepsilon$ at fixed $Q^2$ (a) and as a function of $s$ at fixed c.m. scattering angle (b).

**FIG. 3:** Box (blue, lower curves) and x-box (red, upper curves) parts of the elastic contribution.
was subtracted to make the amplitudes vanish at \( \varepsilon \to 1 \). Looking at the TPE amplitudes for the electron-proton scattering \([10]\), one can see a similar effect, which is just less pronounced, because the proton mass is much higher.

IV. CONCLUSION

We have calculated TPE amplitude for the elastic electron-pion scattering in the dispersion approach, including both elastic and inelastic contributions. For the latter we take into account \( \rho \) and \( b_1(1235) \) mesons as intermediate states.

We find that at not-so-high \( Q^2 \) (up to \( 2 \text{ GeV}^2 \)) the elastic contribution is dominant, as in electron-proton scattering. At higher \( Q^2 \) the inelastic contribution is not negligible, but still smaller than the elastic one. However we believe that the former should be estimated more carefully, and no conclusion can be drawn at the moment.

We also explain the behaviour of the amplitude at backward angles and \( Q^2 \gg M^2 \). As \( Q^2 \) increases, the \( u \)-channel threshold singularity approaches physical region boundary \( \varepsilon = 0 (\theta = 180^\circ) \), resulting in sharp amplitude growth.

[1] J.Arrington, W.Melnitchouk, and J.A.Tjon, Phys. Rev. C 76, 035205 (2007).
[2] S.Kondratyuk, P.G.Blunden, W.Melnitchouk, and J.A.Tjon, Phys. Rev. Lett. 95, 172503 (2005); S.Kondratyuk and P.G.Blunden, Phys. Rev. C 75, 038201 (2007).
[3] Y.B.Dong and D.Y.Chen, Phys. Lett. B 675, 426 (2009); A.P.Kobushkin, Ya.D.Krivenko-Emetov, and S.Dubnička, Phys. Rev. C 81, 054001 (2010).
[4] A.Afanasev, M.Strikman, and C.Weiss, Phys. Rev. D 77, 014028 (2008).
[5] Y.-B.Dong and S.D.Wang, Phys. Lett. B 684, 123 (2010).
[6] P.G.Blunden, W.Melnitchouk, and J.A.Tjon, Phys. Rev. C 81, 018202 (2010).
[7] D.Borisyuk and A.Kobushkin, Phys. Rev. C 78, 025208 (2008).
[8] Y.S.Tsai, Phys. Rev. 122, 1898 (1961).
[9] C.Amsler et al. [Particle Data Group], Phys. Lett. B 667, 1 (2008).
[10] D.Borisyuk and A.Kobushkin, Phys. Rev. C 74, 065203 (2006).