On some hypergeometric supercongruence conjectures of Long

Michael Allen

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Abstract

In 2003, Rodriguez Villegas conjectured 14 supercongruences between hypergeometric functions arising as periods of certain families of rigid Calabi–Yau threefolds and the Fourier coefficients of weight 4 modular forms. Uniform proofs of these supercongruences were given in 2019 by Long, Tu, Yui, and Zudilin. Using p-adic techniques of Dwork, they reduce the original supercongruences to related congruences which involve only the hypergeometric series. We generalize their techniques to consider six further supercongruences recently conjectured by Long. In particular we prove an analogous version of Long, Tu, Yui, and Zudilin’s reduced congruences for each of these six cases. We also conjecture a generalization of Dwork’s work which has been observed computationally and which would, together with a proof of modularity for Galois representations associated to our hypergeometric data, yield a full proof of Long’s conjectures.

Keywords Hypergeometric series · Modular forms · Supercongruences · p-adic Gamma functions · P-adic hypergeometric series · Hypergeometric motives

Mathematics Subject Classification 33C20 · 11F33

1 Introduction and statement of results

Hypergeometric functions have long been studied throughout many different branches of mathematics. In recent years, there has been significant progress in studying hypergeometric functions in relation to automorphic forms. Finite field analogs of hypergeometric functions were introduced by Greene in 1987 [1], and several authors have since related point counts on abelian varieties to these finite field hypergeometric
functions [1–4]. Values of truncated hypergeometric series are often congruent modulo primes $p$ to various objects of mathematical interests. In rarer instances, these congruences hold modulo larger than expected powers of $p$, in which case we refer to them as supercongruences. Supercongruences have been established relating hypergeometric functions to Ramanujan–Sato series [5–10], to Apéry numbers and similar sequences [11, 12], and to modular forms [13–16]. In this paper, we consider the latter type of supercongruence. In particular, we investigate a number of supercongruences between truncated hypergeometric series and Fourier coefficients of modular forms which were recently conjectured by Long [17].

We consider the hypergeometric datum given as the tuple $(\vec{\alpha}, \vec{\beta}, \lambda)$, where $\lambda$ is a non-zero element of an abelian extension of $\mathbb{Q}$ which were recently conjectured by Long [17]. We define the classical (generalized) hypergeometric series appearing as periods of certain families of rigid Calabi–Yau threefolds. These supercongruences have recently been proven by Long, Tu, Yui, and Zudilin [15].

Rodriguez Villegas conjectured the following fourteen supercongruences for hypergeometric series appearing as periods of certain families of rigid Calabi–Yau threefolds. These supercongruences have recently been proven by Long, Tu, Yui, and Zudilin [15].

**Theorem 1.1** (Long, Tu, Yui, Zudilin [15]) Let $r_1, r_2 \in \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \right\}$ or $(r_1, r_2) \in \left\{ \left( \frac{1}{2}, \frac{3}{5} \right), \left( \frac{1}{8}, \frac{3}{8} \right), \left( \frac{1}{10}, \frac{3}{10} \right), \left( \frac{1}{12}, \frac{5}{12} \right) \right\}$. Then for each prime $p > 5$, the congruence

$$4F_3 \left[ \begin{array}{c} r_1 & 1 - r_1 & r_2 & 1 - r_2 \\ 1 & 1 & 1 & 1 \end{array} ; 1 \right]_{p-1} \equiv a_p(f_{[r_1, 1-r_1, r_2, 1-r_2]}) \quad (\text{mod } p^3)$$

holds for some explicit modular form $f_{[r_1, 1-r_1, r_2, 1-r_2]}$ of weight 4.
Proofs of individual cases of Theorem 1.1 had previously been given by Kilbourn [14], McCarthy [16], and Fuselier and McCarthy [13]. Long, Tu, Yui, and Zudilin [15] give the first unified proof of all cases of this theorem, and in fact give two proofs. The first uses $p$-adic techniques and holds for all ordinary primes—primes at which $a_p(f) \not\equiv 0 \pmod p$. The second proof uses character sum arguments and establishes the supercongruence for all primes greater than 7. Recently, Long [17] has conjectured further supercongruences for hypergeometric functions following a similar shape to those in Theorem 1.1, but with more complicated parameters appearing in $\tilde{\beta}$.

**Conjecture 1.2 (Long [17])**

Let $(r_1, r_2, q) \in \left\{ (1/2, 1/2, 3), (1/2, 1/2, 6), (1/2, 1/3, 7), (1/2, 1/3, 3/4), (1/2, 1/4, 7/6), (1/2, 1/5, 3/4) \right\}$. Set $\tilde{\alpha}_{(r_1, r_2)} := (1 - r_1, 1 - r_2, 1 - r_2)$ and $\tilde{\beta}_q := (1, 1, q, 2 - q)$. Let $\text{HD}_{(r_1, r_2, q)}$ denote the hypergeometric datum $(\tilde{\alpha}_{(r_1, r_2)}, \tilde{\beta}_q, 1)$. For each of these hypergeometric data, there exists an explicit weight 4 modular form $f_{(r_1, r_2, q)}$ and Dirichlet character $\chi_{(r_1, r_2, q)}$ such that, for all primes $p \geq 7$,

$$p \cdot _4F_3 \left[ \begin{array}{c} r_1 - 1 - r_1 \ r_2 - 1 - r_2 \ 1 \ q \ 2 - q \end{array} ; 1 \right]_{p-1} \equiv \chi_{(r_1, r_2, q)}(p)a_p(f_{(r_1, r_2, q)}) \pmod{p^3}.$$

Multiplication by $p$ is necessary for the left-hand side to be $p$-adically integral, as the truncated hypergeometric series can only be guaranteed to have $p$-adic valuation greater than or equal to $-1$, which we will show in Proposition 3.5.

Long, Tu, Yui, and Zudilin’s [15] $p$-adic approach consists of two major components. First, they use $p$-adic methods of Dwork [18] and known modularity of the particular Calabi–Yau threefolds from which these hypergeometric series arise to reduce Theorem 1.1 to the following congruence, which involves only the hypergeometric series.

**Theorem 1.3 (Long–Tu–Yui–Zudilin [15])** Let $\tilde{\alpha} = (1 - r_1, 1 - r_2, 1 - r_2)$, and $\tilde{\beta} = (1, 1, 1, 1)$. For each choice of $(r_1, r_2)$ appearing in Theorem 1.1, we have

$$F_{s+1}(\tilde{\alpha}, \tilde{\beta}) \equiv F_s(\tilde{\alpha}, \tilde{\beta})F_1(\tilde{\alpha}, \tilde{\beta}) \pmod{p^3}.$$ 

They then establish this reduced congruence using $p$-adic techniques. The main goal of this paper is to generalize the approach taken by Long, Tu, Yui, and Zudilin to prove an analogous version of Theorem 1.3 for the hypergeometric series appearing in Conjecture 1.2.

**Theorem 1.4** Fix a prime $p \geq 7$. For each of the six hypergeometric data $\text{HD}_{(r_1, r_2, q)}$ appearing in Conjecture 1.2 and all $s \geq 0$ we have

$$p^{s+1}F_{s+1}(\tilde{\alpha}, \tilde{\beta}) \equiv p^sF_s(\tilde{\alpha}, \tilde{\beta}) \cdot pF_1(\tilde{\alpha}, \tilde{\beta}) \pmod{p^3}.$$ 

Once again, these powers of $p$ are necessary to ensure that each term is $p$-adically integral. This will again be proven fully by Proposition 3.5. To fully prove Conjecture 1.2 from Theorem 1.4 it would be necessary to generalize Dwork’s work for these cases
Fig. 1 The values of $\chi$ and $f$ in Conjecture 1.2 for the hypergeometric data considered by Li, Long, and Tu

| $(r_1, r_2, q)$         | $(c, f)$         | $\chi(r_1, r_2, q)$ | $f(r_1, r_2, q)$ |
|-------------------------|------------------|----------------------|------------------|
| $(1/2, 1/2, 4/3)$       | $(1/6, 1/2)$     | $\epsilon$           | $f_{24.4.a.a}$   |
| $(1/2, 1/2, 7/6)$       | $(1/3, 1/2)$     | $\epsilon$           | $f_{12.4.a.a}$   |
| $(1/2, 1/3, 7/6)$       | $(1/3, 1/3)$     | $(3/7)$               | $f_{48.4.a.c}$   |

and to establish the expected modularity of the Galois representations associated to our hypergeometric data by Katz [19, 20] and Beukers, Cohen, and Mellit [2]. Li, Long, and Tu [21] have recently proven this modularity for the hypergeometric data corresponding to the first three tuples $(r_1, r_2, q)$ listed in Conjecture 1.2. The Dirichlet characters and modular forms corresponding to these hypergeometric data are listed below in Figure 1. In the table, we use the LMFDB [22] label for the particular modular forms, and use $\epsilon$ to denote the trivial character.

Watkins’ [23] implementation of hypergeometric motives in Magma can be used to check a generalization of Dwork’s results to these hypergeometric series. The following conjecture holds computationally for all primes up to 1000 when $s = 1$, all primes up to 100 when $s = 2$, and all primes up to 50 when $s = 3$ or 4.

**Conjecture 1.5** Assume $p \geq 7$ is a prime, and let $s \geq 1$. For each $\text{HD}_{(r_1, r_2, q)}$ appearing in Conjecture 1.2, $p \text{F}_s(\vec{\alpha}, \vec{\beta})/\text{F}_{s-1}(\vec{\alpha}, \vec{\beta}) \in \mathbb{Z}_p$ and there exists $\gamma_p = \gamma_p(\vec{\alpha}, \vec{\beta}) \in \mathbb{Z}_p^\times$ such that

$$p \frac{\text{F}_s(\vec{\alpha}, \vec{\beta})}{\text{F}_{s-1}(\vec{\alpha}, \vec{\beta})} \equiv \gamma_p \mod p^s.$$ 

Moreover, letting $\rho(\vec{\alpha}, \vec{\beta})$ denote the Galois representation of the absolute Galois group of $G_\mathbb{Q}$ associated to the hypergeometric data $(\vec{\alpha}, \vec{\beta}, 1)$ by Beukers, Cohen, and Mellit [2], the limit $\gamma_p$ is a unit root of the characteristic polynomial of $\rho(\vec{\alpha}, \vec{\beta})(\text{Frob}_p)$.

A proof of Conjecture 1.5, along with the modularity of the given Galois representations, would together imply Conjecture 1.2 from Theorem 1.4. However, as our hypergeometric series do not always have $p$-adically integral coefficients this conjecture does not immediately generalize from the approach taken by Dwork.

The rest of the paper is organized as follows. First, we record some necessary background about $p$-adic interpretations of hypergeometric functions in Sect. 2. In Sect. 3, we prove some preliminary results on the $p$-adic valuations of the given hypergeometric coefficients. Section 4 is dedicated to the proof of Theorem 1.4.

## 2 $p$-adic background

Throughout the paper, we work over the $p$-adic integers $\mathbb{Z}_p$. The first of the two proofs of Theorem 1.1 appearing in [15] utilizes $p$-adic perturbation techniques originating in work of Long and Chan, Long, and Zudilin [5, 24] and further developed by Long and
As the Pochhammer symbol \((a_k)\) can be written as \(\Gamma(x+n)/\Gamma(x)\), we can translate into the \(p\)-adic setting using Morita’s \(p\)-adic \(\Gamma\)-function [26], which is defined on integers \(n\) by setting

\[
\Gamma_p(x) = (-1)^x \prod_{0 < i < x \atop p \nmid i} i, \tag{1}
\]

and then extended continuously to the \(p\)-adic integers \(\mathbb{Z}_p\). By comparing (1) to the definition of the classical \(\Gamma\) function, we obtain the following identity for all integers \(n\):

\[
\Gamma(n) = (-1)^n \Gamma_p(n) \left[ \frac{n - 1}{p} \right]! p^{\lfloor (n-1)/p \rfloor}. \tag{2}
\]

For an overview of the function \(\Gamma_p\) and its applications see for example Diamond [27]. The function \(\Gamma_p\) satisfies the following identities, which are analogous to the functional equation and Euler’s reflection formula for the classical \(\Gamma\) function.

**Lemma 2.1** Let \(x \in \mathbb{Z}_p\) and \(\Gamma_p\) be defined as in (1). We have

\[
\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} 
-x, & \text{if } x \in \mathbb{Z}_p^\times \\
-1, & \text{if } x \in p\mathbb{Z}_p,
\end{cases}
\]

and

\[
\Gamma_p(x)\Gamma_p(1-x) = (-1)^{x_0},
\]

where \(x_0 \in \{1, 2, \cdots, p\}\) satisfies \(x - x_0 \equiv 0 \pmod{p}\).

For each \(k \geq 0\), define the function \(G_k(a) := \Gamma_p^{(k)}(a)/\Gamma_p(a)\). These functions are considered by Long and Ramakrishna [25], and satisfy many nice analytic properties. For example, logarithmically differentiating the second identity in Lemma (2.1) yields the reflection formula

\[
G_1(a) = G_1(1-a). \tag{3}
\]

Rewriting in terms of \(\Gamma_p\) and differentiating again produces

\[
G_2(a) + G_2(1-a) = 2G_1^2(a). \tag{4}
\]

Long and Ramakrishna additionally show that \(G_2(0) = G_1^2(0)\). Combining this with (3) and (4) when \(a = 0\) gives us the related identity

\[
G_1^2(1) = G_2(1). \tag{5}
\]
The following theorem of Long and Ramakrishna makes the $p$-adic approach particularly appealing for our supercongruences as it allows us to rewrite quotients of $\Gamma_p$ functions, which arise naturally from hypergeometric functions, as a $p$-adic series involving the $G_i$ functions.

**Theorem 2.2** (Long–Ramakrishna, [25]) For $p \geq 5$, $r \in \mathbb{N}$, $a \in \mathbb{Z}_p$, $m \in \mathbb{C}_p$ satisfying $v_p(m) \geq 0$ and $t \in \{0, 1, 2\}$ we have

$$\frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^{t} \frac{G_k(a)(mp^r)^k}{k!} \mod p^{(t+1)r}.$$ 

The above result also holds for $t = 4$ if $p \geq 11$.

Once again we can take logarithmic derivatives to obtain identities for the $G_i$ functions. The only cases of these related identities we will need are when $t = 0$ and $r = 1$, in which case we find that, for $k \in \{1, 2\}$,

$$G_k(a + mp) - G_k(a) = O(p). \quad (6)$$

Also of use to us is Dwork’s framework for $p$-adic hypergeometric functions [18]. Throughout the paper, we use $\lfloor x \rfloor$ to denote the floor function and for any $a \in \mathbb{Z}_p$ we use $[a]_0$ to denote the first $p$-adic digit of $a$. Dwork’s dash operation is the map $\check{\cdot} : \mathbb{Q} \cap \mathbb{Z}_p \to \mathbb{Q} \cap \mathbb{Z}_p$ defined by

$$a' = \frac{a + [-a]_0}{p}.$$ 

Each of the six possible choices of $\tilde{\alpha} = \tilde{\alpha}(r_1, r_2)$ appearing in Theorem 1.4 are closed under this operation for all primes $p \geq 7$. That is, as multisets,

$$\tilde{\alpha} = \{r_1, 1 - r_1, r_2, 1 - r_2\} = \{r'_1, (1 - r_1)', r'_2, (1 - r_2)\}' = \tilde{\alpha}'. \quad (8)$$

Our choices of $\tilde{\beta} = \tilde{\beta}_q$ are not closed under the Dwork dash operation, instead we have

$$\tilde{\beta}' = \{1', 1', q', (2 - q)'\} = \{1, 1, q - 1, 2 - q\}. \quad (7)$$

The exact relation between $\tilde{\beta}$ and $\tilde{\beta}'$, in particular whether $q - 1 = q'$ or $(2 - q)'$, depends on the congruence of $p$ modulo the denominator of $q$.

For the remainder of the paper, with our multisets $\tilde{\alpha}$ and $\tilde{\beta}$ and our prime $p \geq 7$ fixed, we relabel $\tilde{\alpha} = \{r_1, r_2, r_3, r_4\}$ and $\tilde{\beta} = \{q_1, q_2, q_3, q_4\}$ so that

$$r'_1 \leq r'_2 \leq r'_3 \leq r'_4 \quad \text{and} \quad q'_1 \leq q'_2 \leq q'_3 \leq q'_4. \quad (8)$$
For each HD\(_{(r_1, r_2, q)}\), this choice ensures that \(r_2 = r_3 = \frac{1}{2}\) and \(q_3 = q_4 = 1\). For each \(a \in \mathbb{Q} \cap \mathbb{Z}_p\), we observe

\[
1 - a' = \frac{p - a - [-a]_0}{p} = \frac{1 - a + p - 1 - [-a]_0}{p} = \frac{1 - a + [a - 1]_0}{p} = (1 - a)'.
\]

(9)

By definition of \(\tilde{a}\), \(1 - r_1\) must belong to \(\tilde{a}\), so one of \(r_2, r_3,\) or \(r_4\) must equal \(1 - r_1\). If \(r_i = 1 - r_1, (9)\) implies that \(r_i' = 1 - r_i'\). Our choice of ordering in (8) thus guarantees that \(r_i' = 1 - r'_i\) and hence that \(r_i = 1 - r_1\). A similar relationship holds between \(r_2\) and \(r_3\). Therefore

\[
r_i' + r_i' = r_2' + r_3' = 1 \quad \text{and} \quad r_1 + r_4 = r_2 + r_3 = 1.
\]

(10)

By definition of \(\tilde{\beta}\), we have \(\{q_1', q_2'\} = \{q_i - 1, 2 - q_i\}\), where \(i \in \{1, 2\}\) is chosen so that \(q_i > 1\). Thus,

\[
q_1' + q_2' = 1 \quad \text{and} \quad q_1 + q_2 = 2.
\]

(11)

For each \(1 \leq j \leq 4\) we define

\[
t_j := [-r_j]_0 = pr_j' - r_j \quad \text{and} \quad u_j := [-q_j]_0 = pq_j' - q_j.
\]

(12)

As \(r_i' \leq r_i'\), the definition of the dash operation implies that \(r_1 - r_2 \leq t_2 - t_1\). But \(r_1 - r_2 > -1\) and \(t_2 - t_1 \in \mathbb{Z}\), and so \(t_1 \leq t_2\). Additionally, \(r_1 + r_4 + t_1 + t_4\) is divisible by \(p\) by definition of \(t_i\) and (10), and so \(t_1 + t_4\) is congruent to \([- (r_1 + r_4)]_0\) modulo \(p\). We also know that \(r_1 + r_4 = 1\) so \([- (r_1 + r_4)]_0 = p - 1\). As \(0 \leq t_1, t_4 \leq p - 1\) it must therefore be the case that \(t_1 + t_4 = p - 1\). Similar arguments using (10) and (11) hold for the remaining terms, and so we have

\[
t_1 \leq t_2 \leq t_3 \leq t_4 \quad \text{and} \quad t_1 + t_4 = t_2 + t_3 = p - 1
\]

(13)

and also

\[
u_1 \leq u_2 \leq u_3 \leq u_4 \quad \text{and} \quad u_1 + u_2 = p - 2, \quad u_3 = u_4 = p - 1.
\]

(14)

For each HD\(_{(r_1, r_2, q)}\) we have \(q_i' < r_i' < q_2'\) for each \(i \in \{1, 2, 3, 4\}\), and so weaving these inequalities together we have

\[
u_1 < t_1 \leq t_2 = t_3 \leq t_4 < u_2 < u_3 = u_4 = p - 1.
\]

(15)

Finally, we observe that \(\tilde{\beta}\) can be written as

\[
\tilde{\beta} = \{1, 1, q_1' + 1, q_2'\}.
\]

(16)
3 \( p \)-adic valuations of hypergeometric coefficients.

The \( p \)-adic valuation of a rational number \( r \), which we denote by \( v_p(r) \), is equal to the exponent \( k \) on \( p \) when \( r \) is written in the form \( (a/b)p^k \) with \( a \) and \( b \) both relatively prime to \( p \). As we wish to reduce our hypergeometric series modulo \( p^3 \), it will be useful to know the exact \( p \)-adic valuations of each of the hypergeometric coefficients

\[
H(k) := H_{\vec{\alpha}, \vec{\beta}}(k) := \frac{(r_1)_k(r_2)_k(r_3)_k(r_4)_k}{(q_1)_k(q_2)_k(1)_k},
\]

where \( \vec{\alpha}, \vec{\beta} \) are as in Conjecture 1.2. Given \( a = \sum_{n \geq 0} a_n p^n \in \mathbb{Z}_p \), set

\[
[a]_i = \sum_{n=0}^i a_n p^n. \tag{17}
\]

We first consider the valuations of the rising factorials.

**Lemma 3.1** Let \( p \) be a prime, let \( k \in \mathbb{N} \), and let \( a \in \mathbb{Q} \cap \mathbb{Z}_p \). For each integer \( i \geq 0 \), let \( [a]_i \) be defined as in (17). The \( p \)-adic valuation of \( (a)_k \) is given by the formula

\[
v_p((a)_k) = \sum_{i=1}^{\infty} \left\lfloor \frac{k + p^i - [-a]_{i-1} - 1}{p^i} \right\rfloor.
\]

**Proof** Expanding the rising factorial we have

\[
v_p(a)_k = v_p [a(a+1) \cdots (a+k-1)] = \sum_{j=0}^{k-1} v_p(a+j)).
\]

For each \( i \), the smallest non-negative integer \( j \) such that \( a + j \) is divisible by \( p^i \) is \( j = [-a]_{i-1} \). As the indexing of \( j \) begins at zero, this occurs at the \( ([-a]_{i-1} + 1)^{st} \) term of the rising factorial. Multiples of \( p^i \) will further appear at each term which differs from \( a + [-a]_{i-1} \) by a multiple of \( p^i \). There are \( (k - [-a]_{i-1} - 1)/p^i \) such terms after the first which are divisible by \( p^i \). Thus, the total number of terms which are divisible by \( p^i \) is equal to

\[
1 + \left\lfloor \frac{k - [-a]_{i-1} - 1}{p^i} \right\rfloor = \left\lfloor \frac{k + p^i - [-a]_{i-1} - 1}{p^i} \right\rfloor.
\]

So, the sum

\[
\sum_{i=1}^{\infty} \left\lfloor \frac{k + p^i - [-a]_{i-1} - 1}{p^i} \right\rfloor
\]

counts every term in \( (a)_k \) which is divisible by \( p \) once, every term that is divisible by \( p^2 \) twice, and so on. Therefore it is equal to \( v_p((a)_k) \), as was to be shown. \( \square \)
Remark 3.2 In the case where \( a = 1 \), we have \((1)_k = k! \) and Lemma 3.1 reduces to Legendre’s formula

\[ v_p(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{p^i} \right\rfloor. \]

The next lemma allows us to compute the truncation \([a]_i\) as defined in (17) for any \( a \in \mathbb{Q} \cap \mathbb{Z}_p \).

Lemma 3.3 Let \( p \) be a prime and \( a/b \in \mathbb{Q} \) be written in reduced terms such that \( b \geq 1 \) and \( p \nmid b \). Let \( \lambda_i \) denote the least positive residue of \( ap^{-i} \) modulo \( b \). Then for each \( i \in \mathbb{N} \),

\[ [-a/b]_{i-1} = \frac{\lambda_i p^i - a}{b}. \]

Proof By definition, \([-a/b]_{i-1}\) is the smallest positive integer such that

\[ p^i \mid (a + b [-a/b]_{i-1}). \tag{18} \]

Thus, there exists \( n \in \mathbb{N} \) such that \( a + b [-a/b]_{i-1} = np^i \). Solving this equation for \([-a/b]_{i-1}\) yields

\[ [-a/b]_{i-1} = \frac{np^i - a}{b} \tag{19} \]

The fact that \([-a/b]_{i-1}\) is the least integer satisfying (18) implies that it is the smallest integer of the form \((np^i - a)/b\). Thus, finding \([-a/b]_{i-1}\) is equivalent to finding the smallest \( n \in \mathbb{N} \) such that \((np^i - a)/b \in \mathbb{Z}\). For this to be an integer, we must have \(np^i - a \equiv 0 \pmod{b}\). Thus,

\[ n \equiv ap^{-i} \pmod{b}. \]

As \( n \) is the least such integer, it follows that \( n = \lambda_i \). This, along with (19), completes the proof. \( \square \)

For \( \vec{\alpha} = \{r_1, r_2, r_3, r_4\} \), define \( \vec{\alpha}_i \) as the multiset \([-r_1]_i, [-r_2]_i, [-r_3]_i, [-r_4]_i\) for each \( i \geq 0 \). For \( \vec{\beta} = \{1, 1, q_1, q_2\} \) we define \( \vec{\beta}_i \) similarly. Using Lemma 3.3 we are able to compute \( \vec{\alpha}_i \) and \( \vec{\beta}_i \) for each HD\((r_1, r_2, q)\) and for each \( i \). These values are recorded in Figure 2.

From these calculations, we conclude the following generalization of (15).

Corollary 3.4 Let \( p \geq 7 \) be prime and \( i \geq 1 \). With notation as above, label \( \vec{\alpha}_i = \left\{t_j^{(i)}\right\}_{j=1}^{4} \) and \( \vec{\beta}_i = \left\{u_j^{(i)}\right\}_{j=1}^{4} \) such that

\[ t_1^{(i)} \leq t_2^{(i)} \leq t_3^{(i)} \leq t_4^{(i)} \quad \text{and} \quad u_1^{(i)} \leq u_2^{(i)} \leq u_3^{(i)} \leq u_4^{(i)}. \]
Then
\[ p^i \leq u_1^{(i)} < t_1^{(i)} \leq t_2^{(i)} \leq t_3^{(i)} < u_2^{(i)} < u_3^{(i)} = u_4^{(i)} = p^{i+1} - 1. \]

In particular, for each \( 0 \leq k \leq p^i - 1 \) and each of our hypergeometric data, none of the Pochhammer symbols appearing in \( H_{\vec{\alpha}, \vec{\beta}}(k) \) are divisible by \( p^{i+1} \).

The following proposition inductively finds the valuations of all \( H_{\vec{\alpha}, \vec{\beta}}(k) \) for each \( \text{HD}(r_1, r_2, q) \). In particular, it shows that both sides of the congruence in Theorem 1.4 are \( p \)-adically integral and hence the congruence is well-defined.

**Proposition 3.5** Let \( p \geq 7 \) be prime and \( s \in \mathbb{N} \). For each \( \text{HD}(r_1, r_2, q) \) appearing in Conjecture 1.2, we have \( p^s H_{\vec{\alpha}, \vec{\beta}}(k) \in \mathbb{Z}_p \) for all \( 0 \leq k \leq p^s - 1 \). In particular, \( p^s F_s(\vec{\alpha}, \vec{\beta}) \in \mathbb{Z}_p \).

**Proof** By the strong triangle inequality of the \( p \)-adic valuation, we have
\[
 v_p(F_s(\vec{\alpha}, \vec{\beta})) \geq \min_{0 \leq k \leq p^s - 1} \left\{ v_p(H_{\vec{\alpha}, \vec{\beta}}(k)) \right\}.
\]

Therefore it suffices to show that, for all \( 0 \leq k \leq p^s - 1 \), we have \( v_p(H(k)) \geq -s \).

We induct on \( s \). For the base case \( s = 1 \),
\[
 v_p(H(k)) = \sum_{i=1}^{4} v_p((r_i)_k) - v_p((q_i)_k)
\]
and so by (15) and Lemma 3.1, we have

$$v_p(H(k)) = \begin{cases} 
0 & 0 \leq k \leq u_1 \\
-1 & u_1 < k \leq t_1 \\
0 & u_1 < k \leq t_2 \\
2 & t_3 < k \leq t_4 \\
3 & t_4 < k \leq u_2 \\
2 & u_2 < k \leq p - 1.
\end{cases}$$

This completes the base case.

For the inductive step, we fix $s \geq 1$ and assume that $v_p(H(k)) \geq -s$ for all $0 \leq k \leq p^s - 1$. Define a function $f : \mathbb{N} \to \mathbb{Z}$ by setting $f(n) = v_p(H(r))$, where $r$ is the remainder of $n$ when dividing by $p^s$. The function $f$ relates closely to the $p$-adic valuations of $H(k)$ for $k \geq p^s$. In particular, it only differs from the actual valuations by failing to account fully for multiples of $p^s+1$ which appear in the rising factorials. We correct for this using Corollary 3.4 and Lemma 3.1. For example, when $k = u_1^{(s)} + 1$, we introduce a multiple of $p^s$ to $(q_1)_k$ by Lemma 3.1. However, $f$ will only decrease by $s - 1$ at this index. So, for $u_1^{(s)} < k \leq t_1^{(s)}$, we have $v_p(H(k)) = f(k) - 1$. Continuing this reasoning, we obtain

$$v_p(H(k)) = \begin{cases} 
f(k) & 0 \leq k \leq u_1^{(s)} \\
f(k) - 1 & u_1^{(s)} < k \leq t_1^{(s)} \\
f(k) & t_1^{(s)} < k \leq t_2^{(s)} \\
f(k) + 2 & t_3^{(s)} < k \leq t_4^{(s)} \\
f(k) + 3 & t_4^{(s)} < k \leq u_2^{(s)} \\
f(k) + 2 & u_2^{(s)} < k \leq p^{s+1}.
\end{cases}$$

As $f(k) \geq -s$ for all $k$ by the inductive hypothesis, $v_p(H(k)) \geq -(s + 1)$ for all $0 \leq k \leq p^{s+1} - 1$. \qed

The valuations of $pH(k)$ established for $0 \leq k \leq p - 1$ in the base case will be of particular use, and so we record them separately in the following corollary for future reference.

**Corollary 3.6** For each $HD_{(r_1,r_2,q)}$, prime $p \geq 7$, and $0 \leq k \leq p - 1$, we have

$$v_p(pH_{\vec{a},\vec{b}}(k)) = \begin{cases} 
1 & 0 \leq k \leq u_1 \\
0 & u_1 < k \leq t_1 \\
1 & u_1 < k \leq t_2 \\
3 & t_3 < k \leq t_4 \\
4 & t_4 < k \leq u_2 \\
3 & u_2 < k \leq p - 1.
\end{cases}$$
In particular, $pH(k) \equiv 0 \pmod{p^3}$ for all $t_3 < k \leq p - 1$.

4 Proof of Theorem 1.4

We now turn our attention to proving Theorem 1.4, beginning by reinterpreting the supercongruences $p$-adically. Using the Dwork dash operation, Long, Tu, Yui, and Zudilin [15] give a formula for rewriting a quotient of shifted factorials in terms of $\Gamma_p(a)$, which we extend from $\mathbb{Z}_p^\times$ to all of $\mathbb{Z}_p$ in the following Lemma.

**Lemma 4.1** Let $k \in \mathbb{Z}_{\geq 0}$, $a = [k]_0$, and $b = (k - a)/p$, or equivalently $k = a + bp$. Then for any $r \in \mathbb{Z}_p$,

$$
\frac{(r)_k}{(1)_k} = -\frac{\Gamma_p(r + k)}{\Gamma_p(1 + k) \Gamma_p(r)} \frac{(r')_b}{(1)_b} ((r' + b)p)^{\nu(a, [-r]_0)},
$$

where

$$
\nu(a, x) = -\left\lfloor \frac{x - a}{p - 1} \right\rfloor = \begin{cases} 
0 & \text{if } a \leq x, \\
1 & \text{if } x < a < p.
\end{cases}
$$

**Proof** By density of the positive integers in $\mathbb{Z}_p$, it suffices to prove the identity for $r \in \mathbb{Z}_{\geq 0}$ and extend continuously. The case where $r \in \mathbb{Z}_p^\times$ is proved by Long, Tu, Yui, and Zudilin [15], so we assume $r \in \mathbb{Z}_{\geq 0} \cap p\mathbb{Z}_p$. Let $h \in \mathbb{Z}_p$ be such that $r = ph$. We note that $[-r]_0 = 0$ and hence $r' = h$. Then, using (2), we compute

$$
\frac{(r)_k}{(1)_k} = \frac{\Gamma(r + k)}{\Gamma(1 + k) \Gamma(r)}
$$
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By applying Lemma 4.1 twice to

\[ \frac{(r)_k}{(1)_k} \left( \frac{-\Gamma_p(r + k)}{\Gamma_p(r + k)} (r)_b \right) \left( \frac{(r')_b}{(1)_b} (r' + b)^v(a,[-r]_0) \right) \]

we obtain the following:

**Corollary 4.2** Let \( k, a, b, \) and \( v \) be defined as in Lemma 4.1. Then for any \( r, q \in \mathbb{Z}_p \),

\[ \frac{(r)_k}{(q)_k} = \frac{(r)_k (1)_k}{(1)_k (q)_k}, \]

As \( r = ph \) and \( k = a + bp \), it follows that \( r + k = a + p(b + h) \) and

\[ \left\lfloor \frac{r + k - 1}{p} \right\rfloor = h + b + \begin{cases} 
-1 & a = 0 \\
0 & a > 0.
\end{cases} \]

Additionally, \( [k/p] = b \) and \( [(r - 1)/p] = h - 1 \). Thus, if \( a > 0 \),

\[ \frac{(r)_k}{(1)_k} \left( \frac{-\Gamma_p(1 + k)\Gamma_p(r)}{\Gamma_p(r + k)} (r)_b \right) \left( \frac{(r')_b}{(1)_b} (r' + b)^p(h+1)(h-1) \right) = \frac{(r')_b}{(1)_b}. \]

And if \( a = 0 \),

\[ \frac{(r)_k}{(1)_k} = \frac{-\Gamma_p(r + k)}{\Gamma_p(r + k)} \frac{(r')_b}{(1)_b} ((r' + b)^p)^v(a,[-r]_0), \]

as was to be shown. \( \square \)

By applying Lemma 4.1 twice to

\[ \frac{(r)_k}{(q)_k} = \frac{(r)_k (1)_k}{(1)_k (q)_k}, \]

we obtain the following:
And so for general $k = a + bp$,

$$\frac{(r)_{a+bp}}{(q)_{a+bp}} = \frac{\Gamma_p(r + a) \Gamma_p(q)}{\Gamma_p(q + a) \Gamma_p(r + k)} \frac{(r')_b (q')_b (q' + b)^{v(a, [-r]_0)}}{(q'+b)^{v(a, [-q]_0)}} \times p^{v(a, [-r]_0) - v(a, [-q]_0)} \frac{\Gamma_p(q + a) \Gamma_p(r + k)}{\Gamma_p(r + a) \Gamma_p((q + a) + bp)}$$

(20)

We collect terms in (20) by defining

$$\Lambda_{\mathbf{a}, \mathbf{b}}(a + bp) := \prod_{j=1}^{4} \left( 1 + \frac{b}{r'_j} \right)^{v(a, t_j)} \left( 1 + \frac{b}{q'_j} \right)^{-v(a, u_j)},$$

where $t_j$ and $u_j$ are defined as in (12). Using Theorem 2.2, we compute

$$\prod_{j=1}^{4} \Gamma_p((r_j + a) + bp) \equiv \prod_{j=1}^{4} \Gamma_p(r_j + a) \left( 1 + bp \left( \sum_{i=1}^{4} G_1(r_i + a) \right) + (bp)^2 \left[ \frac{1}{2} \sum_{i=1}^{4} G_2(r_i + a) + \sum_{1 \leq i < j \leq 4} G_1(r_i + a) G_1(r_j + a) \right] \right) \pmod{p^3}. $$

(21)

Similarly,

$$\prod_{j=1}^{4} \Gamma_p((q_j + a) + bp) \equiv \prod_{j=1}^{4} \Gamma_p(q_j + a) \left( 1 + bp \left( \sum_{i=1}^{4} G_1(q_i + a) \right) + (bp)^2 \left[ \frac{1}{2} \sum_{i=1}^{4} G_2(q_i + a) + \sum_{1 \leq i < j \leq 4} G_1(q_i + a) G_1(q_j + a) \right] \right) \pmod{p^3}. $$

(22)

Taking the quotient of (21) and (22), we have

$$\prod_{j=1}^{4} \frac{\Gamma_p((r_j + a) + bp)}{\Gamma_p((q_j + a) + bp)} \equiv \prod_{j=1}^{4} \frac{\Gamma_p(r_j + a) \Gamma_p((q_j + a) + bp)}{\Gamma_p(q_j + a) \Gamma_p(r_j + a)} (1 + J_1(a) bp + J_2(a)(bp)^2) \pmod{p^3}. $$(23)
where $J_1 : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ and $J_2 : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ are defined by

$$J_1(a) := J_1(a; \vec{\alpha}, \vec{\beta}) := \sum_{j=1}^{4} (G_1(r_j + a) - G_1(q_j + a))$$

and

$$J_2(a) := J_2(a; \vec{\alpha}, \vec{\beta}) := \frac{1}{2} \sum_{j=1}^{4} [G_2(r_j + a) - G_2(q_j + a)]$$

$$+ \sum_{i=1}^{4} G_1(q_i + a) \sum_{j=1}^{4} [G_1(q_j + a) - G_1(r_j + a)]$$

$$+ \sum_{1 \leq i < j \leq 4} [G_1(r_i + a)G_1(r_j + a) - G_1(q_i + a)G_1(q_j + a)].$$

When $\vec{\beta} = \{1, 1, 1, 1\}$, these definitions reduce to the corresponding functions $J_1(a; \vec{\alpha})$ and $J_2(a; \vec{\alpha})$ considered by Long, Tu, Yui, and Zudilin [15].

Applying Corollary 4.2 and (23) to $p^{s+1}F_{s+1}(\vec{\alpha}, \vec{\beta})$, we find

$$p^{s+1}F_{s+1}(\vec{\alpha}, \vec{\beta}) = p^{s+1} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \prod_{j=1}^{4} \frac{(r_j)_a + bp}{(q_j)_a + bp}$$

$$= p^{s+1} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \prod_{j=1}^{4} \left[ \frac{(r_j)_a}{(q_j)_a} \frac{(r'_j)_b}{(q'_j)_b} \left( 1 + \frac{b}{r'_j} \right)^{v(a, r_j)} \right]$$

$$\times \left( 1 + \frac{b}{q'_j} \right)^{-v(a, b_j)} \frac{\Gamma_p(q_j + a) \Gamma_p((r_j + a) + bp)}{\Gamma_p(r_j + a) \Gamma_p((q_j + a) + bp)}$$

$$= p^{s+1} \sum_{b=0}^{p-1} \prod_{j=1}^{4} \frac{(r'_j)_b}{(q'_j)_b} \sum_{a=0}^{p-1} \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a}$$

$$\times \Lambda_{\vec{\alpha}, \vec{\beta}}(a + bp) \left( 1 + J_1(a)bp + J_2(a)(bp)^2 \right) \pmod{p^3}. \ (24)$$

Recall that, for each of our hypergeometric data $HD_{(r_1, r_2, q)}$, $\vec{\alpha}$ is closed under the Dwork dash operation, so $\prod_{j=1}^{4} (r'_j)_b = \prod_{j=1}^{4} (r_j)_b$. However, $\vec{\beta}$ is not closed under the dash operation. Instead, by (16)

$$\prod_{j=1}^{4} (q'_j)_b = (1)_b(1)_b(q'_1)_b(q'_2)_b = (1)_b(1)_b \left[ (q'_1)(q'_1 + 1)(q'_1 + b) \right] (q'_2)_b$$
\[
\frac{1}{b/q_1^r + 1} (1)_b (1)_b (q_1)_b (q_2)_b = \frac{1}{b/q_1^r + 1} \prod_{j=1}^{4} (q_j)_b. \tag{25}
\]

Hence,
\[
p^{s+1} F_{s+1}(\alpha, \beta) \equiv p^{s+1} \sum_{b=0}^{p^s-1} \prod_{j=1}^{4} \frac{(b)_b}{(q_j)_b} \prod_{a=0}^{p-1} \frac{(r_j)_a}{(q_j)_a} \times \left[ \Lambda_{\alpha, \beta} (a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) \right] \pmod{p^3}. \tag{26}
\]

Pulling the \(b/q_1^r + 1\) term inside the inner sum along with one multiple of \(p\), we have
\[
p^{s+1} F_{s+1}(\alpha, \beta) \equiv p^{s} \sum_{b=0}^{p^s-1} \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} \prod_{a=0}^{p-1} \frac{(r_j)_a}{(q_j)_a} \left( \frac{b}{q_1^r + 1} \right) \times \left[ \Lambda_{\alpha, \beta} (a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) \right] \pmod{p^3}. \tag{27}
\]

Therefore, Theorem 1.4 reduces to the following congruence
\[
\sum_{b=0}^{p^s-1} p^s \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} \prod_{a=0}^{p-1} \frac{(r_j)_a}{(q_j)_a} \left( \frac{b}{q_1^r + 1} \right) \Lambda_{\alpha, \beta} (a + bp) \times \left( 1 + J_1(a)bp + J_2(a)(bp)^2 \right) - 1 \equiv 0 \pmod{p^3}. \tag{28}
\]

In fact, we prove the stronger condition that each term of the sum indexed by \(b\) in (28) vanishes modulo \(p^3\). Define the inner sum \(I(b)\) at \(b \in \{0, 1, \cdots, p^s - 1\}\) by
\[
I(b) := \sum_{a=0}^{p-1} p^s \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left( \frac{b}{q_1^r + 1} \right) \Lambda(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1. \tag{29}
\]

Thus, to prove Theorem 1.4 it suffices to show that, for each \(s \geq 0\) and \(0 \leq b \leq p^s - 1\),
\[
p^s \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} I(b) \equiv 0 \pmod{p^3}. \tag{30}
\]
We define $C_1$ and $C_2$ as

$$C_1 = p \left[ \sum_{a=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left( pJ_1(a) + \frac{1}{q'_1} \right) + \sum_{a=u_1+1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} J_1(a) \right] + \sum_{a=t_1+1}^{t_2} \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left( J_1(a) p + \frac{1}{r'_1} \right)$$

$$C_2 = p^2 \left[ \sum_{a=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \frac{J_1(a)}{q'_1} + \sum_{a=u_1+1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} J_2(a) \right] + \sum_{a=t_1+1}^{t_2} \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \frac{J_1(a)}{r'_1}.$$  \hspace{1cm} (31)

We aim to show that both of these quantities vanish modulo $p^3$. By way of the following proposition, this will then prove (30) and hence Theorem 1.4.

**Proposition 4.3** Let $C_1, C_2$ be as in (31) and $I(b)$ as in (29). For each $b$ with $0 \leq b \leq p^s - 1$, if

$$C_1 b + C_2 b^2 \equiv 0 \pmod{p^3},$$

then

$$p^s \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} I(b) \equiv 0 \pmod{p^3}.$$  

Before proving Proposition 4.3, we prove a few useful Lemmas concerning the $p$-adic valuations of particular parts of $I(b)$. In particular, we consider cases of $b$, depending on whether or not the terms involving $b$ in $I(b)$ are $p$-adically integral. We first determine exactly when this $p$-integrality occurs.

**Lemma 4.4** Fix a prime $p \geq 7$, and let $0 \leq a \leq p - 1$ and $b \in \mathbb{N}$. Additionally define $u_i$ and $t_i$ as in (12). For each $\text{HD}_{(r_1, r_2, q)}$ appearing in Conjecture 1.2, if either $a \leq u_2$ or $b \not\equiv -q'_2 \pmod{p}$, then $(1 + b/q'_1) \Lambda_{\vec{a}, \vec{b}}(a + bp) \in \mathbb{Z}_p$.  


Lemma 4.6 Let \( b \) be such that \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \in \mathbb{Z}_p \), and let \( C_1 \) and \( C_2 \) be as in (31). We have

\[
I(b) \equiv C_1 b + C_2 b^2 \pmod{p^3}.
\]

**Proof** Assume \( b \geq 0 \) is such that \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \in \mathbb{Z}_p \). Then the entire term

\[
\left[ \left( \frac{b}{q_1'} + 1 \right) \Lambda_{\vec{a}, \vec{\beta}}(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right]
\]

By choice of \( \vec{a}, \vec{\beta}, \) and \( p \), each \( r_j' \) and \( q_j' \) all belong to \( \mathbb{Z}_p \), and so the numerator of \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \) belongs to \( \mathbb{Z}_p \). Thus, the only way that this term could fail to be \( p \)-adically integral is if \( u_2 < a \leq p - 1 \) and \( p \mid 1 + b/q_2' \). As \( q_2' \in \mathbb{Z}_p \), this second condition is equivalent to \( b \equiv -q_2' \pmod{p} \), completing the proof. \( \square \)

Remark 4.5 The converse of the above lemma does not hold. It can be the case that \( u_2 < a \leq p - 1 \) and \( b \equiv -q_2' \pmod{p} \), but \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \) is \( p \)-adically integral if the numerator is more highly divisible by \( p \) then \( 1 + b/q_2' \).

In the next Lemma, we consider the case where \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \in \mathbb{Z}_p \). We show that in this case \( I(b) \equiv C_1 b + C_2 b^2 \pmod{p^3} \). In Proposition 3.5 we showed that, for any \( 0 \leq b \leq p^s - 1 \),

\[
p^s \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} \in \mathbb{Z}_p.
\]

Thus, establishing this congruence between \( I(b) \) and \( C_1 b + C_2 b^2 \) implies Proposition 4.3 for such \( b \).

Lemma 4.6 Let \( b \) be such that \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \in \mathbb{Z}_p \), and let \( C_1 \) and \( C_2 \) be as in (31). We have

\[
I(b) \equiv C_1 b + C_2 b^2 \pmod{p^3}.
\]

**Proof** Assume \( b \geq 0 \) is such that \( (1 + b/q_1') \Lambda_{\vec{a}, \vec{\beta}}(a + bp) \in \mathbb{Z}_p \). Then the entire term

\[
\left[ \left( \frac{b}{q_1'} + 1 \right) \Lambda_{\vec{a}, \vec{\beta}}(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right]
\]
is $p$-adically integral. Thus, for all $a$ such that the $p$-adic valuation of $H(k)$ is at least 2, the corresponding term of the sum in $I(b)$ will vanish mod $p^3$. By Corollary 3.6,

$$I(b) \equiv \sum_{a=0}^{t_2} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a}$$

$$\times \left[ \left( \frac{b}{q_1^2} + 1 \right) \Lambda(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right] \text{ (mod $p^3$).} \quad (33)$$

Using the explicit values of $\Lambda(a + bp)$ in (32), we have

$$I(b) \equiv \sum_{a=0}^{u_1} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left[ \left( pJ_1(a) + \frac{1}{q_1^2} \right) b + \left( p^2 J_2(a) + \frac{p^2 J_1(a)}{q_1^2} b^2 + \frac{p^2 J_2(a)}{q_1^2} b^3 \right) \right]$$

$$+ \sum_{a=u_1+1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left[ pJ_1(a) b + p^2 J_2(a) b^2 \right]$$

$$+ \sum_{a=t_1+1}^{t_2} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left[ \left( pJ_1(a) + \frac{1}{r_1^2} \right) b + \left( p^2 J_2(a) + \frac{p J_1(a)}{r_1^2} b^2 + \frac{p^2 J_2(a)}{r_1^2} b^3 \right) \right].$$

Regrouping the above terms by powers of $b$, we have $I(b) \equiv C_1 b + C_2 b^2 + C_3 b^3$ (mod $p^3$), where

$$C_1 = p \cdot \left[ \sum_{a=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left( pJ_1(a) + \frac{1}{q_1^2} \right) + \sum_{a=u_1+1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} J_1(a) \right.$$\n
$$+ \left. \sum_{a=t_1+1}^{t_2} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left( J_1(a) + \frac{1}{r_1^2} \right) \right].$$

$$C_2 = p^2 \cdot \left[ \sum_{a=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left( \frac{J_1(a)}{q_1^2} + pJ_2(a) \right) + \sum_{a=u_1+1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} J_2(a) \right.$$\n
$$+ \left. \sum_{a=t_1+1}^{t_2} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \left( pJ_2(a) + \frac{J_1(a)}{r_1^2} \right) \right].$$

$$C_3 = p^3 \cdot \left[ \sum_{a=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \frac{J_2(a)}{q_1^2} + \sum_{a=t_1+1}^{t_2} \prod_{j=1}^{4} \frac{(r_j) a}{(q_j) a} \frac{J_2(a)}{r_1^2} \right).$$

The term inside of the parentheses is $p$-integral in each case, as the only piece which is not is $H_{a,b^2}(a)$ when $u_1 < a < t_1$, but this is made up for in each $C_i$ by an extra factor of $p$ appearing in that portion of the sum. In particular, we see that $C_3 \equiv 0$.
Proof of Proposition 4.3) Lemmas 4.4 and 4.6 give the result in the case where \( b \not\equiv -q_2' \pmod{p} \). Thus, we consider the case where \( b \equiv -q_2' \pmod{p} \). Define 

\[
\tilde{\Lambda}_{\alpha, \beta}(a + bp) \quad \text{as follows:}
\]

\[
\tilde{\Lambda}_{\alpha, \beta}(a + bp) = \Lambda_{\alpha, \beta}(a + bp)(1 + b/q_2')^{v(a, u_2)}
\]

\[
= \begin{cases}
\Lambda_{\alpha, \beta}(a + bp) & 0 \leq a < u_2 \\
(1 + b/q_2')\Lambda_{\alpha, \beta}(a + bp) & u_2 \leq a \leq p - 1.
\end{cases}
\]

(35)

From the discussion in the proof of Lemma 4.4, \((1 + b/q_2')\tilde{\Lambda}_{\alpha, \beta}(a + bp)\) belongs to \(\mathbb{Z}_p\). We define \(\tilde{I}(b)\) similarly to \(I(b)\), but with \(\Lambda\) replaced by \(\tilde{\Lambda}\). That is,

\[
\tilde{I}(b) = \sum_{a=0}^{p-1} p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left[ \left( \frac{b}{q_1} + 1 \right) \tilde{\Lambda}(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right].
\]

(36)

As \( J_1, J_2 \in \mathbb{Z}_p^2 \), it follows that the expression inside of the brackets is \(p\)-integral. This means that the \(a^{th}\) term of the sum \(\tilde{I}(b)\), which we will denote by \(\tilde{I}_a(b)\), satisfies

\[
v_p(\tilde{I}_a(b)) \geq v_p \left( p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \right).
\]

Thus, Corollary 3.6 implies \(\tilde{I}_a(b)\) vanishes modulo \(p^3\) whenever \(a > t_2\). Therefore, modulo \(p^3\) we may write \(\tilde{I}(b)\) as

\[
\sum_{a=0}^{t_2} p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left[ \left( \frac{b}{q_1} + 1 \right) \tilde{\Lambda}(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right].
\]

But, we know that \(\tilde{\Lambda}_{\alpha, \beta}(a + bp) = \Lambda_{\alpha, \beta}(a + bp)\) for all \(0 \leq a \leq t_2\). Thus, \(\tilde{I}(b)\) simplifies further modulo \(p^3\) as

\[
\sum_{a=0}^{t_2} p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \left[ \left( \frac{b}{q_1} + 1 \right) \Lambda_{\alpha, \beta}(a + bp)(1 + J_1(a)bp + J_2(a)(bp)^2) - 1 \right].
\]
This is exactly how we rewrote $I(b)$ in (33), and so by the same calculations we have that

$$\tilde{I}(b) \equiv C_1 b + C_2 b^2 \pmod{p^3}.$$ 

To finish the proof of the proposition, it now suffices to show for $b \equiv -q_2' \pmod{p}$ that, if $\tilde{I}(b)$ vanishes modulo $p^3$ then so does $p^s \prod_{j=1}^{4} \frac{(r_j)_{b}}{(q_j)_{b}} I(b)$. To do so, we show

$$p^s \prod_{j=1}^{4} \frac{(r_j)_{b}}{(q_j)_{b}} I(b) \equiv p^s \prod_{j=1}^{4} \frac{(r_j)_{b}}{(q_j)_{b}} \tilde{I}(b) \pmod{p^3}. \quad (37)$$ 

From the definition of $I(b)$ and $\tilde{I}(b)$, the difference of the expressions in the above congruence is

$$p^s \prod_{j=1}^{4} \frac{(r_j)_{b}}{(q_j)_{b}} \sum_{a=0}^{p-1} p \prod_{j=1}^{4} \frac{(r_j)_{a}}{(q_j)_{a}} \left[ \left( \frac{b}{q_1'} + 1 \right)(\Lambda(a + bp) - \tilde{\Lambda}(a + bp)) (1 + J_1(a)bp + J_2(a)(bp^2)) \right] \quad (38)$$

By (35), the term $\Lambda - \tilde{\Lambda}$ vanishes for all $a \leq u_2$. Thus, we need only consider indices $a$ satisfying $u_2 < a \leq p - 1$. For such $a$, we have

$$\Lambda(a + bp) - \tilde{\Lambda}(a + bp) = \left( \frac{1}{1 + b/q_2'} \right) \tilde{\Lambda}(a + bp) - \tilde{\Lambda}(a + bp)$$

$$= \tilde{\Lambda}(a + bp) \left( \frac{-b}{q_2'(1 + b/q_2')} \right).$$

So, (38) simplifies to

$$p^s \frac{-b(1 + b/q_2')}{q_2'(1 + b/q_2')} \prod_{j=1}^{4} \frac{(r_j)_{b}}{(q_j)_{b}} \sum_{a=2+1}^{p-1} p \prod_{j=1}^{4} \frac{(r_j)_{a}}{(q_j)_{a}} \tilde{\Lambda}(a + bp) (1 + J_1(a)bp + J_2(a)b^2 p^2).$$

If we show that this expression vanishes mod $p^3$, we will be done. To do this, we first consider the sum

$$\sum_{a=u_2+1}^{p-1} p \prod_{j=1}^{4} \frac{(r_j)_{a}}{(q_j)_{a}} \tilde{\Lambda}(a + bp) \left( 1 + J_1(a)bp + J_2(a)b^2 p^2 \right).$$

Each of $\tilde{\Lambda}(a+bp)$ and $(1 + J_1(a)bp + J_2(a)b^2 p^2)$ are $p$-adically integral. By Corollary 3.4, the hypergeometric coefficient has $p$-adic valuation equal to 2. Hence, for each $a$
in this range,

\[
v_p \left( p \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \tilde{\Lambda}(a + bp) \left( 1 + J_1(a)bp + J_2(a)b^2p^2 \right) \right)
\]

\[
= v_p(p) + v_p \left( \prod_{j=1}^{4} \frac{(r_j)_a}{(q_j)_a} \right)
\]

\[
+ v_p \left( \tilde{\Lambda}(a + bp)(1 + J_1(a)bp + J_2(a)b^2p^2) \right)
\]

\[
\geq 1 + 2 + 0 = 3.
\]

Thus, we need only show that the expression

\[
p^s \frac{-b(1 + b/q'_1)}{q'_2(1 + b/q'_2)} \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b}
\]

is \(p\)-adically integral. To do so, we first note that, as \(p \nmid b\),

\[
v_p \left( \frac{-b(1 + b/q'_1)}{q'_2} \right) = 0,
\]

and so we need only consider

\[
p^s \left( \frac{1}{1 + b/q'_2} \right) \prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b}.
\]

Let \(i\) be the largest integer such that \(b \equiv -q'_2 \pmod{p^i}\). Then we have

\[
v_p \left( \frac{1}{1 + b/q'_2} \right) = -i.
\]

The condition \(b \equiv -q'_2 \pmod{p^i}\) is equivalent to saying that \(b = [-q'_2]_{i-1} + np^i\). Additionally, our labeling of \(\tilde{\beta}\) guarantees us that \(q'_2 \in \tilde{\beta}\), and so therefore \([-q'_2]_{i-1} \in \tilde{\beta}_{i-1}\), where \(\tilde{\beta}_{i-1}\) is defined as in Corollary 3.4. Thus, there exists \(j \in \{1, 2\}\) such that

\[
b = u_j^{(i-1)} + np^i.
\]

From the inductive procedure to find the \(p\)-adic valuations of \(H(k)\) described in Proposition 3.5, there must be a decrease in the \(p\)-adic valuations of \(H(k)\) that occurs
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at \( u^{(i-1)} + np^i + 1 \), as at this index a multiple of \( p^i \) is added to the denominator. In particular,

\[
v_p(H(b)) = v_p(H(b + 1)) + i.
\]

From Proposition 3.5 we also know that \( v_p(H(b + 1)) \geq -s \). Therefore,

\[
\frac{p^s}{1 + b/q_2}
\prod_{j=1}^{4} \frac{(r_j)_b}{(q_j)_b} \geq s - i + (i - s) = 0.
\]

We have thus shown that this term is \( p \)-adically integral, completing the proof.

The discussion at the beginning of the section and Proposition 4.3 imply that Theorem 1.4 follows from establishing that \( C_1 b + C_2 b^2 \equiv 0 \pmod{p^3} \) for all \( b \). Before showing this, we introduce a number of necessary identities. First, from (2) and the definition of the Pochhammer symbol, for all \( 0 \leq a \leq p - 1 \) we have

\[
(t)_a = (-1)^a \frac{\Gamma_p(t + a)}{\Gamma_p(t)}(t + [-t]_0)^{\nu(a, [-t]_0)}.
\]

Taking logarithmic derivatives of both sides we obtain

\[
\frac{d}{dt}(t)_a = G_1(t + a) - G_1(t) + \frac{\nu(a, [-t]_0)}{t + [-t]_0}.
\]

And differentiating again we obtain

\[
\frac{d^2}{dt^2}(t)_a = \left( G_1(t + a) - G_1(t) + \frac{\nu(a, [-t]_0)}{t + [-t]_0} \right)^2 + G_2(t + a) - G_2(t) - G^2_1(t + a) - G^2_1(t) - \frac{\nu(a, [-t]_0)}{(t + [-t]_0)^2}.
\]

Now, to show that \( C_1 \equiv C_2 \equiv 0 \pmod{p^3} \), we introduce particular rational functions whose residues we relate to the \( C_i \). This is similar to the approach taken by Long, Tu, Yui, and Zudilin, as well as other authors to establish similar hypergeometric identities \([10, 12, 15, 28]\). We define, for \( i \in \{1, 2\} \)

\[
R_i(t) = \frac{\prod_{j=1}^{4} (-t + 1 - pr_j)_{t_j}(-t + q_1)_{u_1+1}}{(-t + 1 - pq_1')_{u_1+1}t_p^{(i+1)}}.
\]

By (13) and (14), the degree of the numerator of \( R_i(t) \) is \( 2p - 1 + u_1 \), whereas the denominator has degree \( (i + 1)p + 1 + u_1 \). For both choices of \( i \) the degree of the denominator is at least 2 greater than that of the numerator, so the residue sum theorem
implies that the sum of the residues of $R_i$ is equal to zero. These rational functions have partial fraction decompositions

$$R_1(t) = \sum_{k=0}^{p-1} \frac{A_k^{(1)}}{(t+k)^2} + \sum_{k=1}^{p-1} \frac{B_k^{(1)}}{t+k} + \sum_{k=1}^{u_1+1} \frac{D_k^{(1)}}{(-t+k-pq_1')},$$

and

$$R_2(t) = \sum_{k=0}^{p-1} \left( \frac{A_k^{(2)}}{(t+k)^3} + \frac{E_k^{(2)}}{(t+k)^2} + \frac{B_k^{(2)}}{t+k} \right) + \sum_{k=1}^{u_1+1} \frac{D_k^{(2)}}{(-t+k-pq_1')}.$$

By the residue theorem, it follows that

$$0 = \sum_{k=0}^{p-1} B_k^{(1)} + \sum_{k=1}^{u_1+1} D_k^{(1)}.$$

and

$$0 = \sum_{k=0}^{p-1} B_k^{(2)} + \sum_{k=1}^{u_1+1} D_k^{(2)}.$$

Our goal is to relate these residue sums to the $C_i$ in order to show that $p^3$ divides $C_i$ for each $i$. We begin with a lemma which shows that each $D_k^{(i)}$ is small $p$-adically.

**Lemma 4.7** Set notation as above, and let $1 \leq k \leq u_1+1$. For $i \in \{1, 2\}$, there exists $\delta_k^{(i)} \in \mathbb{Z}_p$ such that $D_k^{(i)} = p^{3-i}\delta_k^{(i)}$.

**Proof** The statement of the lemma is equivalent to saying $v_p(D_k^{(i)}) \geq 3 - i$ for all $p$ and $k$. We compute

$$D_k^{(i)} = (-t+k-pq_1')R_i(t)\bigg|_{t=k-pq_1'} = \prod_{j=1}^{4}(-k+1+p(q_1'-r_j'))_{tj}(-k+q_1+pq_1')_{u_1+1} \frac{(1-k)_{u_1+1-k}(k-pq_1'^{(3-i)})}{(1-k)_{u_1+1-k}(k-pq_1'^{(3-i)})}. $$

Using (39), we have first that, for each $j \in \{1, 2, 3, 4\}$,

$$(-k+1+p(q_1'-r_j'))_{tj} = (-1)^{t_j} \frac{\Gamma_p(1-k+t_j-pr_j'+pq_1')}{\Gamma_p(1-k+p(q_1'-r_j'))} \left( p(q_1'-r_j') \right)^{v(t_j,k-1)}.$$

As $\Gamma_p$ takes $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$, the $p$-adic valuation of the above expression depends only on the term $(p(q_1'-r_j'))^{v(t_j,k-1)}$. From (15), $t_j > u_1$ for all $j$. Therefore, for $1 \leq k \leq u_1+1,$
$t_j > k - 1$, and so $v(t_j, k - 1) = 1$. For each of our hypergeometric data except $\text{HD}_{(1/2, 1/4, 7/6)}$ we have $q'_{1} - r^j_{1} \in \mathbb{Z}^\times_p$ for all $p \geq 7$, and in this one exceptional case the same holds for all $p \geq 11$. For each of these primes, $p(q'_{1} - r^j_{1})$ has $p$-adic valuation exactly one, and so $(-k + 1 + p(q'_{1} - r^j_{1}))_{t_j}$ does as well. In the exceptional case $\alpha = \{1/2, 1/2, 3/4, 5/4\}$ and $\beta = \{1, 1, 7/6, 5/6\}$ and with $p = 7$ and $j$ chosen so that $r^j_{1} = 3/4$ we have $q'_{1} - r^j_{1} = 1/6 - 3/4 = 7/12$, and so the $p$-adic valuation of $(-k + 1 + p(q'_{1} - r^j_{1}))_{t_j}$ is in fact equal to 2. In all cases, we conclude

$$v_p \left( \prod_{j=1}^{4} (-k + 1 + p(q'_{1} - r^j_{1}))_{t_j} \right) \geq 4$$

for all $1 \leq k \leq u_1 + 1$.

Next we consider $(-k + q_{1} + pq'_{1})_{u_1+1}$. As before, the $p$-adic valuation depends only on the final term in (39), and in this case the exponent is $v(u_1 + 1, [k - q_{1} - pq'_{1}]_0)$, which reduces to $v(u_1 + 1, [k - q_{1}]_0)$. By definition, the leading $p$-adic digit of $-q_{1}$ is $u_1$, and so as long as $k < [-u_1]_0$, we will have $[k - q_{1}]_0 = k + u_1$. This is the case, as by (14) $[-u_1]_0 = u_2 + 2$, and $k \leq u_1 + 1 < u_2 + 2$. Therefore,

$$u_1 + 1 \leq [k - q_{1}]_0 \leq 2u_1 + 1 < p,$$

and so $v(u_1 + 1, [k - q_{1}]_0) = 0$ for all $1 \leq k \leq u_1 + 1$. In particular, we see that $v_p((-k + q_{1} + pq'_{1})_{u_1+1}) = 0$.

For the terms in the denominator, we observe that $(1 - k)_{k-1}$ and $(1)_{u_1+1-k}$ together contain all $u_1$ integers from $1 - k$ to $u_1 + 1 - k$ excluding zero. As $u_1 + 1 < p$, there is thus no multiple of $p$ appearing in these two terms, as the only such multiple in this range is the zero we have removed. Thus, $v_p((1 - k)_{k-1}(1)_{u_1+1-k}) = 0$. Finally, the shifted factorial $(k - pq'_{1})_{p}$ will necessarily have exactly one multiple of $p$ appearing within it, namely at the term $k + p - k - pq'_{1} = p(1 - q'_{1})$. As $1 - q'_{1} \in \mathbb{Z}^\times_p$ for all $p \geq 7$ and $\tilde{\beta}$ within our consideration, we conclude that $v_p((k - pq'_{1})_{p}^{i+1} = i + 1$.

Putting these calculations together, we conclude that

$$v_p(D^{(i)}_k) \geq 4 - (i + 1) = 3 - i,$$

completing the argument. \[\square\]

We are now prepared to prove Theorem 1.4.

**Proof of Theorem 1.4** We have seen from Proposition 4.3 and (30) that it suffices to show that each $C_i \equiv 0 \pmod{p^3}$, which we will do by showing that a certain multiple of each $C_i$ is equivalent modulo $p^3$ to the residue sums of the rational functions $R_i$ defined in (42). We first consider $i = 1$. To compute the residues $B_k^{(1)}$, we first must
compute \( A_k^{(1)} \). This computation can be done directly, as

\[
A_k^{(1)} = (t + k)^2 R(t) \bigg|_{t = -k} = \frac{\prod_{j=1}^{4} (k + 1 - pr_j') t_j (k + q_1)_{u_1 + 1}}{(k + 1 - pq_1')_{u_1 + 1}(k)^2(k - 1)^2 p^{-k-1}}.
\]

We will interpolate this expression \( p \)-adically using (39), but first we make a few observations we will need for our reductions. By the definitions of the function \( v \) and of \( t_j \), we have

\[
v(t_j, p - k - 1) = v(k + 1, p - t_j) = v(k + 1, [r_j]_0).
\]

For the term \((k + q_1)_{u_1 + 1}\), we can use (39) to see that

\[
(k + q_1)_{u_1 + 1} = (-1)^{u_1 + 1} \frac{\Gamma_p(k + 1 + pq_1')}{\Gamma_p(k + q_1)} (k + q_1 + [-k - q_1]_0)^{v(u_1 + 1, [-k - q_1]_0)}.
\]

As \( q_1 = -u_1 + pq_1' \), we have that \(-k - q_1 = u_1 - k - pq_1'\) and so

\[
[-k - q_1]_0 = [u_1 - k - pq_1']_0 = \begin{cases} u_1 - k & 0 \leq k \leq u_1 \\ p + u_1 - k & u_1 < k \leq p - 1. \end{cases}
\]

Additionally, we have

\[
v(u_1 + 1, [-k - q_1]_0) = \begin{cases} v(u_1 + 1, u_1 - k) & 0 \leq k \leq u_1 \\ v(u_1 + 1, p + u_1 - k) & u_1 < k \leq p - 1 \\ 1 & 0 \leq k \leq u_1 \\ 0 & u_1 < k \leq p - 1 \end{cases} = v(u_1 + 1, k).
\]

Combining these two calculations yields

\[
(k + q_1 + [-k - q_1]_0)^{v(u_1 + 1, [-k - q_1]_0)} = (pq_1')^{v(u_1 + 1, k)}.
\]

We choose \( i \in \{1, 2\} \) such that \( u_i \neq 1 \), which is possible by (14). This choice allows us to use Lemma 2.1 to conclude that

\[
\Gamma_p(q_1) \Gamma_p(q_2) = \Gamma_p(q_i) \Gamma_p(1 - q_i) \frac{\Gamma_p(2 - q_i)}{\Gamma_p(1 - q_i)} = (-1)^{p-u_i+1}(1 - q_i) = (-1)^{u_i+1+\epsilon} q_1',
\]
where $\epsilon_t \in \{0, 1\}$ depends only on $\vec{w}$ and $p$. To finish simplifying $A_k^{(1)}$ we use Lemma 2.1, (39), (39), (43), as well as Theorem 2.2 and its logarithmic derivative. We find

$$A_k^{(1)} = \frac{\prod_{j=1}^{4} \Gamma_p(k+1-r_j)}{\prod_{j=1}^{4} \Gamma_p(k+1-pr_j)} \left( p(1-r'_j) \right)^{v(k+1;j_0)} \left( -1 \right)^{u_1+1} \frac{\Gamma_p(k+1+pq'_1)}{\Gamma_p(k+1+pq'_2)} \left( pq'_1 \right)^{v(u_1+1,k)}$$

$$= \frac{\prod_{j=1}^{4} \Gamma_p(k-r_j) \Gamma_p(k+1-pq'_1) \Gamma_p(k+1-pq'_2) \Gamma_p(k+1+p-pk)}{\prod_{j=1}^{4} \Gamma_p(k+r_j) \Gamma_p(k+1+pr'_j) \Gamma_p(k+1+pq'_1)}$$

$$= (-1)^{t_1+t_2+p} \prod_{j=1}^{4} \frac{\Gamma_p(k+1+pq'_1)}{\Gamma_p(k+1+pq'_2)} \frac{\Gamma_p(k+1-pq'_1)}{\Gamma_p(k+1+p-pk)} \prod_{j=1}^{4} \Gamma_p(k+1+pr'_j)$$

$$= (-1)^{t_1+t_2+u_1+1} \prod_{j=1}^{4} \frac{\Gamma_p(k+1+pq'_1)}{\Gamma_p(k+1+pq'_2)} \frac{\Gamma_p(k+1-pq'_1)}{\Gamma_p(k+1+p-pk)} \prod_{j=1}^{4} \Gamma_p(k+1+pr'_j)$$

$$= (-1)^{t_1+t_2+u_1+\epsilon_1+1} \prod_{j=1}^{4} \frac{\Gamma_p(k+1+pq'_1)}{\Gamma_p(k+1+pq'_2)} \frac{\Gamma_p(k+1-pq'_1)}{\Gamma_p(k+1+p-pk)} \prod_{j=1}^{4} \Gamma_p(k+1+pr'_j)$$

$$= (-1)^{t_1+t_2+u_1+\epsilon_1+1} \prod_{j=1}^{4} \frac{\Gamma_p(k+1+pq'_1)}{\Gamma_p(k+1+pq'_2)} \frac{\Gamma_p(k+1-pq'_1)}{\Gamma_p(k+1+p-pk)} \prod_{j=1}^{4} \Gamma_p(k+1+pr'_j)$$

The residues $B_k^{(1)}$ can be computed explicitly as well. For each $0 \leq k \leq p - 1$, we have

$$B_k^{(1)} = \lim_{t \to -k} \frac{d}{dt} \left( (t+k)^2 R(t) \right)$$

$$= A(k) \left( \sum_{j=1}^{4} \frac{d}{dt} \left( -t+1-pr'_j \right) \right) \frac{d}{dt} \left( -t+q_1 \right)_{u_1+1} \frac{d}{dt} \left( -t+1-pq'_1 \right)_{u_1+1} \frac{d}{dt} \left( -t+1-pq'_2 \right)_{u_2+1}$$

$$- 2 \frac{d}{dt} \left( t \right)_k - 2 \frac{d}{dt} \left( t+k+1 \right)_{p-k-1} \frac{d}{dt} \left( t+k+1 \right)_{p-k-1}$$

Each of these logarithmic derivatives can be written in terms of the $G_j$ functions using (40), and subsequently we use (6) to simplify these expressions. After a large amount of simplification, one finds that

$$B_k^{(1)} = -A_k^{(1)} \left( J_1(k; \vec{w}, \vec{w}) + \sum_{j=1}^{4} \frac{v(k, t_j)}{pr'_j} + \frac{v(u_1+1, k)}{pq'_1} - \frac{v(k, u_2)}{pq'_2} \right) + O(p^2).$$
Therefore, if we consider the full residue sum multiplied by \((-1)^{t_1+t_2+u_i+？e_1} p\) and use the \(p\)-adic valuations from Corollary 3.6 and Lemma 4.7, we have

\[
0 = (-1)^{t_1+t_2+u_i+？e_1} p \left( \sum_{k=0}^{p-1} B_k^{(1)}(1) + \sum_{k=1}^{u_1+1} D_k^{(1)} \right) \\
= \sum_{k=0}^{p-1} p^2 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \left( J_1(k) + \sum_{j=1}^{4} \frac{v(k, t_j)}{pr_j} + \frac{v(u_1+1, k)}{pq'_1} - \frac{v(k, u_2)}{pq'_2} \right) + O(p^3) \\
= \sum_{k=0}^{u_1} p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \left( pJ_1(k) + \frac{1}{q'_1} \right) + \sum_{k=u_1+1}^{t_2} p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \left( pJ_1(k) + \frac{1}{q'_1} \right) + O(p^3) \\
= C_1 + O(p^3). 
\]

Therefore, \(C_1 \equiv 0 \pmod{p^3}\).

Showing that \(C_2 \equiv 0 \pmod{p^3}\) is very similar, although the particular details are significantly more involved as we now must compute residues at poles of order 3. As such, we again omit many of the details. To calculate \(A_k^{(2)}\), we choose \(i \in \{1, 2\}\) so that \(u_i \neq 1\) which allows for the simplifications

\[
A_k^{(2)} = \frac{p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k}}{(k+1-pr_j)_{t_j}(k+q_1)_{u_1+1}} \frac{q'_1 \Gamma_p(k+1+pq'_1) \Gamma_p(k+1-pq'_1)}{\Gamma_p(q_1) \Gamma_p(q_2) \Gamma_p(k+1) \Gamma_p(k+1-pr_j)} \\
= (-1)^{t_1+t_2+u_i+？e_1} p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \frac{\Gamma_p(k+1+pq'_1) \Gamma_p(k+1-pq'_1)}{\Gamma_p(k+1) \prod_{j=1}^{4} \Gamma_p(k+1-pr_j)} \\
= (-1)^{t_1+t_2+u_i+？e_1} p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \frac{\Gamma_p(k+1+pq'_1) \Gamma_p(k+1-pq'_1)}{\prod_{j=1}^{4} \Gamma_p(k+1-pr_j)} \\
= (-1)^{t_1+t_2+u_i+？e_1} p^4 \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} (1 + O(p)).
\]

We can compute each

\[
B_k^{(2)} = \frac{1}{2} \lim_{t \to -k} \frac{d^2}{dt^2} \left( (t+k)^3 R_2(t) \right)_{t=-k}
\]

explicitly using the residue formula. The derivative can be written purely in terms of logarithmic derivatives of Pochhammer symbols, which we can then apply (40) and (41) to in order to obtain an expression in terms of \(\Gamma_p, G_1\), and \(G_2\). We omit the exact
expression for $B_k^{(2)}$, as it is a sum over 36 distinct terms. Although the computation is necessarily much larger than that needed to determine $B_k^{(1)}$, the underlying ideas and techniques are much the same. Once expressed in terms of $\Gamma_p, G_1, \text{and } G_2$, we are able to utilize Lemma 2.1 and Theorem 2.2, as well as the identities (3), (4), and (5) that we have established for $\Gamma_p, G_1, \text{and } G_2$ to eventually reduce $B_k^{(2)}$ to

$$B_k^{(2)} = A_k^{(2)} \left( O(p) + J_2(k; \bar{\alpha}, \bar{\beta}) \right)$$

$$+ \frac{1}{p} \left( \sum_{j=1}^{4} \frac{v(k, t_j)}{r'_j} + \frac{v(u_1 + 1, k)}{q'_1} - \frac{v(k, u_2)}{q'_1} \right) \left( J_1(k) + O(p) \right)$$

$$+ \frac{1}{p^2} \left( 2 \frac{v(k, u_2)}{(q'_2)^2} + 2 \sum_{1 \leq i < j \leq 4} \frac{v(k, t_j)}{r'_i r'_j} - 2 \sum_{j=1}^{4} \frac{v(k, u_2)}{r'_j q'_2} \right).$$

Note that the term being multiplied by $1/p^2$ is equal to zero for all $0 \leq k \leq t_2$. We now multiply our residue sum by $(-1)^{t_1 + t_2 + u_i + \epsilon_i}$ which yields

$$0 = (-1)^{t_1 + t_2 + u_i + \epsilon_i} p^2 \left( \sum_{k=0}^{p-1} B_k^{(2)} + \sum_{k=1}^{u_1 + 1} D_k^{(2)} \right).$$

By Lemma 4.7 we have $p^2 D_k^{(2)} \equiv 0 \pmod{p^3}$ for all $1 \leq k \leq u_1 + 1$. Additionally, by Corollary 3.6 the $p$-adic valuation of the hypergeometric coefficient $H(k)$ is at least 2 for all $t_2 \leq k < p - 1$. It follows that $p^2 B_k^{(2)} \equiv 0 \pmod{p^3}$ for all such $k$. Therefore,

$$0 \equiv (-1)^{t_1 + t_2 + u_i + \epsilon_i} p^3 \sum_{k=0}^{t_2} B_k^{(2)}$$

$$\equiv p^2 \left[ \sum_{k=0}^{u_1} \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \frac{J_1(k)}{q'_1} + \sum_{k=u_1 + 1}^{t_1} p \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} J_2(k) \right. \left. \quad + \sum_{k=t_1 + 1}^{t_2} \prod_{j=1}^{4} \frac{(r_j)_k}{(q_j)_k} \frac{J_1(k)}{r'_1} \right] \pmod{p^3}.$$

This final expression is exactly $C_2$, and so $C_2 \equiv 0 \pmod{p^3}$. This completes the proof of Theorem 1.4.
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Declarations

Conflicts of interest  The author has no relevant financial or non-financial interests to disclose.

Data availability  Data sharing is not applicable to this manuscript.

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