ON A CALABI-TYPE ESTIMATE FOR PLURICLOSED FLOW

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ABSTRACT. The regularity theory for pluriclosed flow hinges on obtaining $C^\alpha$ regularity for the metric assuming uniform equivalence to a background metric. This estimate was established in [14] by an adaptation of ideas from Evans-Krylov, the key input being a sharp differential inequality satisfied by the associated ‘generalized metric’ defined on $T \oplus T^*$. In this work we give a sharpened form of this estimate with a simplified proof. To begin we show that the generalized metric itself evolves by a natural curvature quantity, which leads quickly to an estimate on the associated Chern connections analogous to, and generalizing, Calabi-Yau’s $C^3$ estimate for the complex Monge Ampere equation.

1. INTRODUCTION

Pluriclosed flow [17] is a geometric flow extending Kähler-Ricci flow to more general complex manifolds while preserving the pluriclosed condition for a Hermitian metric, $\sqrt{-1} \partial \bar{\partial} \omega = 0$. Associated to a pluriclosed metric we have the Bismut connection, the unique Hermitian connection with skew symmetric torsion, defined by

$$\nabla^B = D + \frac{1}{2} g^{-1} H, \quad H = d^c \omega,$$

where $D$ denotes the Levi-Civita connection. Let $\Omega^B$ denote the curvature of $\nabla^B$, and furthermore let

$$\rho_B = \text{tr} \Omega^B \in \Lambda^2.$$

This is a closed form representing $\pi c_1$, but is not in general of type $(1,1)$. The pluriclosed flow can be expressed using the Bismut connection as

$$\frac{\partial}{\partial t} \omega = -\rho^1_{B}, \quad \frac{\partial}{\partial t} \beta = -\rho^2_{B}, \quad (1.1)$$

where $\beta \in \Lambda^{2,0}$ is the ‘torsion potential’ along the solution (i.e. $\overline{\partial} \beta = \partial \omega$). First introduced in [14], $\beta$ is not strictly necessary to describe associated the Hermitian metric, but plays a central role in obtaining a priori estimates.

By now there are many global existence and convergence results for pluriclosed flow ([1] [10] [14] [15] [16]). All of these results exploit some underlying structure of the given background to obtain a priori $L^\infty$ estimates for the metric tensor along the flow. In the setting of Kähler-Ricci flow, reduced to a parabolic Monge-Ampere equation, this corresponds to having a $C^{1,1}$ estimate for the potential, at which point one applies either the Evans-Krylov method [5] [7] to obtain a $C^{2,\alpha}$ estimate, or Calabi’s $C^3$ estimate [2] [18] [13], after which Schauder estimates can be applied to obtain $C^\infty$ estimates. As pluriclosed metrics cannot be described locally by a single function, the pluriclosed flow does not admit a scalar reduction, so the method of Evans-Krylov cannot be applied. As pluriclosed flow is a parabolic system of equations for the Hermitian metric $g$, obtaining a $C^\alpha$ estimate for the metric is similar to
the DeGiorgi-Nash-Moser/Krylov-Safonov \cite{DeGiorgi, Nash, Moser, Krylov, Safonov} estimate for uniformly parabolic equations. However, these results are known to be false in general for systems of equations \cite{4}. Thus, a key step in obtaining regularity of pluriclosed flow is to turn \( L^\infty \) control over the metric into a \( C^{\alpha} \) estimate.

This regularity barrier was overcome by the second author in \cite{14}, with a key role played by the generalized metric associated to \( g \) and \( \beta \). Specifically, given \( g \) a Hermitian metric and \( \beta \in \Lambda^{2,0} \), the associated generalized metric is expressed in local coordinates,

\[
G = \begin{pmatrix}
g_{ij} + \beta_{ik} \beta_{jk} g^{kl} \\
\frac{1}{g} \beta_{jk} g^{lp} \\
\beta_{jp} g^{lp} & g^{jk}
\end{pmatrix}.
\]

This is a Hermitian metric on \( T^{1,0} \oplus \Lambda^{1,0} \), which has unit determinant. In local complex coordinates, it turns out that \( G \) is a matrix subsolution of the linear heat equation. This together with the fact that \( G \) has unit determinant allows one to adapt the strategy behind Evans-Krylov regularity to obtain a \( C^{\alpha} \) estimate for \( G \), after which one obtains a \( C^{\alpha} \) estimate for \( g \) and \( \beta \).

Our purpose here is to give a simplified and sharpened version of this estimate which reveals further structure of pluriclosed flow related to the generalized metric and its associated Chern connection. As a Hermitian metric on a holomorphic vector bundle, \( G \) has a canonically associated Chern connection, and curvature tensor \( \Omega \in \Lambda^{1,1} \otimes \text{End}(T \oplus T^*) \). Taking its trace with respect to the Hermitian metric \( g \) and lowering the final index with \( G \) yields a natural curvature operator

\[
S_{AB} = g_{ij} \Omega_{ijAB}.
\]

As we show in \S 3, under the pluriclosed flow equations \cite{11}, the associated generalized metric evolves by

\[
\frac{\partial}{\partial t} G = - S.
\]

This remarkably simple formula leads to a clean evolution equation for the Chern connection associated to \( G \), to which the maximum principle can be applied to obtain a \( C^1 \) estimate for the metric assuming uniform equivalence of \( G \) with a background metric. This computation is similar in style, and in fact generalizes, the classic \( C^3 \) estimate of Calabi-Yau for real/complex Monge-Ampere equations and Kähler-Ricci flow \cite{2, 13, 18}. Given this, a blowup argument adapted from \cite{14} leads to sharp scale-invariant estimates on all derivatives of \( G \), yielding our main result. Before stating it we record some notation.

**Definition 1.1.** Given a complex manifold \((M^{2n}, J)\) and Hermitian metrics \( g \) and \( \tilde{g} \), let

\[
\Upsilon(g, \tilde{g}) := \nabla^g - \nabla^{\tilde{g}}
\]

denote the difference of the associated Chern connections. Similarly, given \( G \) and \( \tilde{G} \) Hermitian metrics on \( T^{1,0} \oplus \Lambda^{1,0} \) let

\[
\Upsilon(G, \tilde{G}) := \nabla^G - \nabla^{\tilde{G}}
\]

\(^1\)In the terminology of generalized geometry, the generalized metric is an endomorphism of \( T \oplus T^* \) as opposed to a symmetric inner product. Our metric is related to such an endomorphism by raising one index using the associated neutral inner product.
be the difference of the associated Chern connections. Furthermore, let
\[ f_k = f_k(G, \tilde{G}) := \sum_{j=0}^{k} |\nabla^j \Upsilon|^{\frac{2}{1+j}}. \]
This is a fixed-scale measure of the \((k+1)\)-st derivatives of \(G\).

**Theorem 1.2.** Given \((M^{2n}, J)\), fix \((\omega, \beta)\) a solution to pluriclosed flow \((1.1)\) on \([0, \tau)\), \(\tau \leq 1\), with associated generalized metric \(G\). Fix a background generalized metric \(\tilde{G}(\tilde{g}, \tilde{\beta})\) such that \(\Lambda^{-1} \tilde{G} \leq G \leq \Lambda \tilde{G}\). There exists \(\rho > 0\) depending on \(\tilde{g}\) such that for all \(0 < R < \rho\), and \(k \in \mathbb{N}\), there exists a constant \(K = K(k, \Lambda, \tilde{G})\) such that
\[ \sup_{B_R(p) \times \{t\}} f_k(x, t) \leq K \left( \frac{1}{t} + R^{-4} \right). \]

As the quantity \(f_k\) dominates the corresponding \(C^{k+1}\) norm of the metric \(g\) (cf. Lemma 3.1), Theorem 1.2 recovers the estimates of ([14] Theorem 1.7). In fact, it has strengthened that estimate in several ways. First, we have explicitly localized the estimate in terms of the geometry of a background metric \(\tilde{g}\). Furthermore, the quantity \(\rho\) is equivalent to the curvature radius of \(\tilde{g}\), allowing us to obtain regularity for \(G\) in settings where the metric is collapsing.

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### 2. Curvature of Generalized Metric

All connections below are Chern connections unless otherwise stated; those that are decorated with a \(g\) are of the classical metric on \(T^{1,0}\) and those that are undecorated are of the generalized metric \(G\) on \(T^{1,0} \oplus \Lambda^{1,0}\). The same convention holds for curvatures and traces thereof. Furthermore, we use tildes to denote quantities associated to a fixed background metric \(\tilde{g}\) or generalized metric \(\tilde{G}(\tilde{g}, \tilde{\beta})\). We use Einstein summation convention wherein lower case indices are summed over elements of either the tangent or cotangent spaces and capital letters are summed over the entire generalized tangent space \(T^{1,0} \oplus \Lambda^{1,0}\). Also, choosing complex coordinates \(z^i\), we let \(Z^i = \frac{\partial}{\partial z^i}\) and \(W^i = dz^i\). Thus in particular the generalized metric \(G\) in (1.2) can be expressed in these components as follows:

\[
G_{Z^i \overline{Z}^j} = g_{i\overline{j}} + \beta_{ik} \overline{\beta_{j\overline{k}}} g^k \\
G_{Z^i W^j} = g_{i\overline{j}} \beta_{j\overline{p}} g^p \\
G_{W^i \overline{Z}^j} = g_{i\overline{j}} \beta_{j\overline{p}} g^p \\
G_{W^i W^j} = g_{i\overline{j}}
\]

Using this, further elementary computations yield the inverse matrix
\[
G^{-1} = \begin{pmatrix}
g_{i\overline{j}} & g_{i\overline{p}j\overline{p}} \beta_{j\overline{p}} g^p \\
\beta_{j\overline{p}} g^p & g_{i\overline{k}} + \beta_{kp} \overline{\beta_{j\overline{p}}} g^p
g_{i\overline{k}} & \beta_{j\overline{p}} g^p & g_{i\overline{k}} + \beta_{kp} \overline{\beta_{j\overline{p}}} g^p
\end{pmatrix}
\]
Or, in components,

\[
G_{Z^i} = g_{\tilde{j}i},
\]
\[
G_{\bar{Z}^i} = \beta_{p} g_{\bar{j}p},
\]
\[
G_{W^i} = g_{\bar{j}i},
\]
\[
G_{\bar{W}^i} = \beta_{\bar{j}p} g_{\bar{p}i},
\]
\[
G_{\bar{W}^i} = g_{\bar{j}i} + \beta_{p} g_{\bar{j}p} g_{\bar{p}i}.
\]

Before computing the curvature of \(G\) below, we observe an important convention used throughout the paper. We assume that the metric \(\omega\) and torsion potential \(\beta\) satisfy

\[
\partial \omega = \partial \beta.
\]

Given \(\omega\), such \(\beta\) can always be found at least locally. More generally one can expect to solve globally for

\[
\partial \omega = \tilde{T} + \partial \beta,
\]

where \(\tilde{T}\) denotes the Chern torsion with respect to some background metric. All of the results below generalize to this case.

**Lemma 2.1.** Given \((M^{2n}, J)\) a complex manifold and generalized metric \(G\), the Chern curvature associated to \(G\) is

\[
\Omega_{ijW^aW^b} = -g^{b\sigma} (\Omega_{ijm}^g - g^{\sigma\nu} T_{m\sigma\nu}),
\]
\[
\Omega_{ijZ^aW^b} = -g^{b\sigma} \nabla_i \nabla_j \beta_{an},
\]
\[
\Omega_{ijZ^aZ^b} = (\Omega^g)_{ijab} - T_{akj} T_{mli} g^{kl}.
\]

**Proof.** First recall the general formula for Chern curvature of a general Hermitian metric \(h\) in complex coordinates,

\[
\Omega_{ij} = -\partial_{j} \Gamma_{i\alpha}^{\beta} = -\partial_{j} (\partial_{i} h_{\alpha\gamma} h^{\gamma\beta}) = -\partial_{j} \partial_{i} h_{\alpha\gamma} h^{\gamma\beta} + \partial_{i} h_{\alpha\gamma} h^{\gamma\delta} \partial_{j} h_{\delta\gamma} h^{\beta\gamma}.
\]

Lowering the index yields

\[
\Omega_{ij\sigma} = -\Omega_{j\alpha} h_{\alpha\sigma} = -h_{\alpha\sigma,ij} + h_{\alpha\gamma,ij} h^{\gamma\delta} h_{\delta\gamma}.
\]

We note that the claimed formulas are invariant under transformations \(\beta \mapsto \beta - \gamma\) where \(\gamma\) is a holomorphic local section of \(\Lambda^{2,0} T^*\). To take advantage of this, let \(p \in M\) and choose \(\gamma = \beta_{ij}(p) d z^i \wedge d z^j\). Clearly \(\gamma\) is holomorphic, and after subtracting it we have forced \(\beta(p) = 0\). Thus, we can without loss of generality compute at a single point and suppose \(\beta\) vanishes at that point. In particular notice that \(G_{ZW} = 0\) at \(p\).

With that in mind, let us begin by expanding

\[
\Omega_{jW^aW^b} = -\partial_{j} \partial_{i} G_{W^aW^b} + \partial_{i} G_{W^a\gamma} G^{\gamma\beta} \partial_{j} G_{\beta W^b}
\]
\[
= -\partial_{i} \partial_{j} G_{W^aW^b} + \left[ \partial_{i} G_{W^a\gamma} G^{\gamma\beta} Z^{i} \partial_{j} G_{Z^bW^b} + \partial_{j} G_{W^a\gamma} G^{\gamma\beta} \partial_{j} Z^{i} \partial_{i} G_{Z^bW^b} \right]
\]
\[
= -\partial_{i} \partial_{j} G_{W^aW^b} + A_1 + A_2.
\]
Then, we can compute
\[
\partial_i \partial_j G_{\tilde{W}W} = g_{\tilde{J}i}^\tilde{m} \partial_i (g_{\tilde{m}\tilde{n}} g_{\tilde{m}j}) \\
= - \partial_i (g_{\tilde{m}\tilde{n}} g_{\tilde{m}j} + g_{\tilde{m}\tilde{i}} g_{\tilde{m}i} g_{\tilde{m}j} + g_{\tilde{m}\tilde{j}} g_{\tilde{m}i} g_{\tilde{m}j}) \\
= g_{\tilde{m}\tilde{n}} (\Omega_{ijm}^\tilde{n} + g_{\tilde{m}i} g_{\tilde{m}j}).
\]

Further,
\[
A_1 = g_{\tilde{m}\tilde{n}} \partial_i (g_{\tilde{m}\tilde{n}} g_{\beta_{i\tilde{n}}} - g_{\tilde{m}\tilde{n}} g_{\beta_{m\beta}}), \\
A_2 = g_{\tilde{m}\tilde{n}} \partial_j (g_{\tilde{m}\tilde{n}} g_{\beta_{j\tilde{n}}} - g_{\tilde{m}\tilde{n}} g_{\beta_{m\beta}}).
\]

Gathering these all up gives
\[
\Omega_{ijW\tilde{W}^b} = - g_{\tilde{m}\tilde{n}} (\Omega_{ijm}^\tilde{n} + g_{\tilde{m}i} g_{\tilde{m}j}) + g_{\tilde{m}\tilde{n}} g_{\beta_{m\beta}},
\]

Next, we expand
\[
\Omega_{ijZ\tilde{W}^b} = - \partial_i \partial_j G_{Z\tilde{W}^b} + \partial_i G_{Z\tilde{W}^b} \partial_j G_{\tilde{W}^b} \\
= - \partial_i \partial_j G_{Z\tilde{W}^b} + \left[ \partial_i G_{Z\tilde{W}^b} G_{Z\tilde{W}^b} \partial_j G_{Z\tilde{W}^b} + \partial_i G_{Z\tilde{W}^b} G_{Z\tilde{W}^b} \partial_j G_{Z\tilde{W}^b} \right] \\
= - \partial_i \partial_j G_{Z\tilde{W}^b} + A_1 + A_2.
\]

Then we can compute
\[
\partial_i \partial_j G_{Z\tilde{W}^b} = \partial_i (g_{pq} \beta_{ap}) \\
= \partial_i (g_{pq} \beta_{ap} + g_{pq} \beta_{aq}) \\
= g_{pq} \beta_{ap} + g_{pq} \beta_{aq}.
\]

And also,
\[
A_1 = g_{pq} \partial_i (g_{pq} \beta_{ap} + g_{pq} \beta_{aq}), \\
A_2 = g_{pq} \partial_j (g_{pq} \beta_{ap} + g_{pq} \beta_{aq}).
\]

Assembling these gives
\[
\Omega_{ijZ\tilde{W}^b} = - g_{pq} \beta_{ap} + g_{pq} \beta_{aq} + g_{pq} \beta_{ap} + g_{pq} \beta_{aq} + (g_{pq} g_{pq} \beta_{ap} + g_{pq} \beta_{aq}) \\
= - g_{pq} \beta_{ap} + g_{pq} \beta_{aq} + g_{pq} \beta_{ap} + g_{pq} \beta_{aq}.
\]

as claimed. Lastly, we expand
\[
\Omega_{ijZ\tilde{W}^b} = - \partial_i \partial_j G_{Z\tilde{W}^b} + \partial_i G_{Z\tilde{W}^b} \partial_j G_{\tilde{W}^b} \\
= - \partial_i \partial_j G_{Z\tilde{W}^b} + \left[ \partial_i G_{Z\tilde{W}^b} G_{Z\tilde{W}^b} \partial_j G_{Z\tilde{W}^b} + \partial_i G_{Z\tilde{W}^b} G_{Z\tilde{W}^b} \partial_j G_{Z\tilde{W}^b} \right] \\
= - \partial_i \partial_j G_{Z\tilde{W}^b} + A_1 + A_2.
\]
Computing gives
\[ \partial_t \partial_j G^{Z+ \bar{Z}'} = g_{\bar{a} j \bar{j}} + (\beta_{ak, j} \overline{\beta_{ik}} + \beta_{ak, i} \overline{\beta_{jk}}) g^{\bar{k}}. \]
And also,
\begin{align*}
A_1 &= g^{\bar{p}} g_{\bar{a} i} g_{\bar{p} j}, \\
A_2 &= g^{\bar{m}} (\beta_{ap, i} g_{\bar{p} k}) (\beta_{bn, j} g^{\bar{n}}) = g^{\bar{m}} \beta_{ap, i} \beta_{bn, j}.
\end{align*}
Putting this together gives:
\begin{align*}
\Omega_{\bar{a} b} &\equiv 0 + \partial_k (\beta_{ap, i} g_{\bar{p} k}) (\beta_{bn, j} g^{\bar{n}}) = g^{\bar{p}} \beta_{ap, i} \beta_{bn, j},
\end{align*}
as claimed. 

\[ \square \]

Lemma 2.2. Given \((M^{2n}, J)\) a complex manifold and generalized metric \(G\), the tensor \(S\) associated to \(G\) satisfies
\begin{align*}
S_{W^a W^b} &= -g^{m n} g^{m a} (S_{m n} - T_{m n}^2), \\
S_{Z^a Z^b} &= -g^{m n} \Delta g^{m a}, \\
S_{Z^a Z^b} &= S_{Z^a Z^b} = S_{a b} - T_{a b}^2,
\end{align*}
where here and below we refer to \(T_{l m} \overline{T}_{j k m} g^{l k} g^{m n}\) as \(T_{ij}^2\). 

Proof. Taking the trace in Lemma 2.1 proves the result. 

\[ \square \]

3. Evolution Equations

In this section we establish formula (1.3), the evolution equation for \(G\) under pluriclosed flow. Before we begin computing, it will be useful to establish some B-field transformations of various quantities.

Lemma 3.1. Given \((M^{2n}, J)\) a complex manifold, \(\gamma \in \Lambda^{2,0}\) and a generalized Hermitian metric \(G\), the Christoffel symbols of the generalized metric \(\tilde{G} = e^{-\gamma} Ge^\gamma\) are given by
\[ \tilde{\Gamma} = e^{-\gamma} G^{-1} \begin{pmatrix} 0 & 0 \\ -\partial_{\gamma} & 0 \end{pmatrix} Ge^\gamma + e^{-\gamma} \Gamma e^\gamma + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]
In particular, if \(\gamma\) is constant, then
\[ \tilde{\Omega} = e^{-\gamma} \Omega e^\gamma \]
\[ S\tilde{G} = e^{-\gamma} S G e^\gamma. \]
Proof.
\[ \tilde{\Gamma} = \tilde{G}^{-1} \partial \tilde{G} \]
\[ = (e^{-\gamma} G e^{\gamma})^{-1} \partial (e^{-\gamma} G e^{\gamma}) \]
\[ = e^{-\gamma} G^{-1} (e^{\gamma} \partial G) e^{\gamma} \]
\[ = e^{-\gamma} G^{-1} e^{\gamma} \Gamma e^{\gamma} + e^{-\gamma} \partial e^{\gamma} \]
\[ = e^{-\gamma} G^{-1} \begin{pmatrix}
0 & 0 \\
-\partial \gamma & 0
\end{pmatrix} Ge^{\gamma} + e^{-\gamma} \Gamma e^{\gamma} + \begin{pmatrix}
0 & 0 \\
0 & \partial \gamma
\end{pmatrix} \]

Given this, in the case \( \gamma \) is constant we have \( \tilde{\Gamma} = e^{-\gamma} \Gamma e^{\gamma} \), and differentiating again we have
\[ \tilde{\Omega} = -\partial \tilde{\Gamma} = -\partial e^{-\gamma} \Gamma e^{\gamma} = -e^{-\gamma} \partial \Gamma e^{\gamma} = e^{-\gamma} \Omega e^{\gamma}. \]

\[ \square \]

**Proposition 3.2.** Given \((M^{2n}, J)\) and \((\omega, \beta)\) a solution to pluriclosed flow (1.1), the associated generalized metric \( G \) satisfies
\[ \frac{\partial}{\partial t} G = -S. \]

**Proof.** We will use the following equations
\[ \rho_B^{11} = S^g - T^2 \quad \rho_B^{20} = \partial \overline{\partial} \omega \]
These can be derived in using the Bianchi identity for the Chern curvature and type decomposing the identity \( \rho_B - \rho_C = dd^* \omega \) (cf. [6]).

Furthermore, arguing as above by applying Lemma 3.1, we can without loss of generality compute at a space-time point where \( \beta \) vanishes. Using this we can compute
\[ \frac{\partial}{\partial t} G_{Z^j \bar{Z}^k} = \frac{\partial}{\partial t} g_{Z^j \bar{Z}^k} + \frac{\partial}{\partial t} (\beta_{ik}\overline{\partial}^{ij} g^{jk}) \]
\[ = - (\rho_B^{11})_{ij} - (\rho_B^{20})_{ik} g_{j l} + \beta_{ik} (\rho_B^{20})_{ij} g_{j k} + \beta_{ik} (\rho_B^{11})_{ik} g_{j k} \]
\[ = - (S^g_{ij} - T^2_{ij}) \]
\[ = - S_{Z^j \bar{Z}^k}. \]

Also,
\[ \frac{\partial}{\partial t} G_{Z^j \bar{Z}^i} = \frac{\partial}{\partial t} (\beta_{ip} g_{j p}) \]
\[ = g_{j p} \frac{\partial}{\partial t} \beta_{ip} - \beta_{ip} g^{jm} g_{m \bar{p}} \frac{\partial}{\partial t} g_{n \bar{m}} \]
\[ = - g_{j p} (\rho_B^{20})_{ij} \]
\[ = - g_{j p} (\partial \overline{\partial} \omega_{ij}) \]
\[ = g_{j p} \Delta_g \beta_{ip} \]
\[ = - S_{Z^j \bar{Z}^i}. \]
Further,

\[
\frac{\partial}{\partial t} G_{\bar{W}W'} = \frac{\partial}{\partial t} \bar{g}^\lambda_{\bar{\lambda}}^{\bar{\alpha} \beta} = - \bar{g}^{\bar{\beta}} \bar{g}^{\lambda} \left( \bar{\rho}_B \right)_{\bar{\lambda} \bar{\mu}}^{1,1} - \bar{g}^{\bar{\beta}} \bar{g}^{\lambda} \left( \bar{S}^{\bar{\gamma}}_{\bar{\mu} \bar{\nu}} - \bar{T}^{2}_{\bar{\mu} \bar{\nu}} \right) = - \bar{S}^{\bar{G}}_{\bar{W} \bar{W'}}.
\]

\[\square\]

**Proposition 3.3.** Given \((M^{2n}, J)\) and \((\omega, \beta)\) a solution to pluriclosed flow (1.1), with associated generalized metric \(G\) and background generalized metric \(\tilde{G}\), one has

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\Upsilon|^2 = - |\nabla \Upsilon|^2 - |\nabla \Upsilon + T \cdot \Upsilon|^2 + T \ast \Upsilon \ast \tilde{\Omega} + \Upsilon \ast \Upsilon \ast \tilde{\Omega} + \Upsilon \ast \tilde{\nabla} \tilde{\Omega}.
\]

**Proof.** A general calculation for the variation of the Chern connection associated to a Hermitian metric yields

\[
\frac{\partial}{\partial t} \Upsilon^B_{iA} = \nabla_i \frac{\partial}{\partial t} G^B_A.
\]

Specializing this result using Proposition 3.2 we thus have

(3.1)

\[
\frac{\partial}{\partial t} \Upsilon^B_{iA} = - \nabla_i S^B_A.
\]

Using this, we compute

\[
\Delta \Upsilon^B_{iA} = \bar{g}^{\bar{\beta}} \nabla_i \nabla_k \Upsilon^B_{iA} = \bar{g}^{\bar{\beta}} \nabla_i \left( \Omega^B_{\bar{\kappa}\bar{i}A} - \tilde{\Omega}^B_{\bar{\kappa}\bar{i}A} \right) = \bar{g}^{\bar{\beta}} \nabla_i \Omega^B_{\bar{\kappa}\bar{i}A} + \bar{g}^{\bar{\beta}} \nabla_i \Omega^B_{\bar{\kappa}\bar{p}A} + \bar{g}^{\bar{\beta}} \nabla_i \tilde{\Omega}^B_{\bar{\kappa}\bar{i}A}
\]

\[
= \partial_i \Upsilon^B_{iA} - \bar{g}^{\bar{\beta}} T^p_{\bar{\mu}^A} \Omega^B_{\bar{\mu} \bar{p} \bar{i}A} + \bar{g}^{\bar{\beta}} \nabla_i \tilde{\Omega}^B_{\bar{\kappa} \bar{i}A} + \bar{g}^{\bar{\beta}} \left( \Upsilon^q_{\bar{i}A} \tilde{\Omega}^B_{\bar{q} \bar{i}A} + \Upsilon^B_{\bar{i}A} \tilde{\Omega}^B_{\bar{q} \bar{i}A} - \Upsilon^B_{\bar{i}A} \tilde{\Omega}^B_{\bar{q} \bar{i}A} \right).
\]

Next we observe the commutation formula

(3.2)

\[
\Delta \bar{Y}^B_{\bar{jA}} = \bar{g}^\lambda_{\bar{\kappa} \bar{l}A} \nabla_{\bar{k} \bar{l}} \bar{Y}^B_{\bar{jA}} = \bar{g}^\lambda_{\bar{\kappa} \bar{l}A} \left[ \nabla_{\bar{j} \bar{k}} \bar{Y}^B_{\bar{jA}} - (\Omega^q)^{\bar{q} \bar{k}}_{\bar{j} \bar{l}A} \bar{Y}^B_{\bar{q} \bar{j}A} - \Omega^B_{\bar{q} \bar{k} \bar{l}A} \bar{Y}^B_{\bar{q} \bar{j}A} + \Omega^B_{\bar{k} \bar{q} \bar{l}A} \bar{Y}^B_{\bar{q} \bar{j}A} \right]
\]

\[
= \Delta \bar{Y}^B_{\bar{jA}} - (S^q)^{\bar{q} \bar{jA}} + S^B_{\bar{q} \bar{jA}} \bar{Y}^B_{\bar{q} \bar{j}A} + S_{\bar{q} \bar{jA}} \bar{Y}^B_{\bar{q} \bar{j}A}.
\]
Combining (3.1)-(3.3), the evolution equations of Proposition 3.2 and the pluriclosed flow equation we obtain

\[
\frac{\partial}{\partial t} |\gamma|_{g^{-1}, G}^{2} = \frac{\partial}{\partial t} \left[ g^{i\alpha} G^{\alpha} g B \gamma_{\alpha} \overline{\gamma}_{\beta} \right] \\
= -g^{\bar{n}} \left( -S_{\bar{n}}^{B} + T_{\bar{n}}^{B} \right) g^{i\alpha} G^{\alpha} g B \gamma_{\alpha} \overline{\gamma}_{\beta} - g^{i\alpha} G^{\alpha} \left( -S_{\alpha \beta}^{G} \right) g^{i\alpha} \gamma_{\alpha} \overline{\gamma}_{\beta} \\
+ g^{i\alpha} G^{\alpha} \left( -S_{\alpha \beta}^{G} \right) \gamma_{\alpha} \overline{\gamma}_{\beta} + g^{i\alpha} G^{\alpha} g B \left( \Delta \gamma_{\alpha} + g(T)_{\alpha}^{B} \Omega_{\alpha}^{B} \right) \\
- g^{i\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} - g^{i\alpha} \left[ \gamma_{\alpha}^{A} \tilde{\Omega}_{\alpha}^{B} + \gamma_{\alpha}^{B} \tilde{\Omega}_{\alpha}^{M} - \gamma_{\alpha}^{B} \tilde{\Omega}_{\alpha}^{M} \right] \overline{\gamma}_{\beta} \\
+ g^{i\alpha} G^{\alpha} g B \gamma_{\alpha} \overline{\gamma}_{\beta} \left( \Delta \gamma_{\alpha}^{B} - g \left( T^{\beta} \right)_{\alpha}^{B} \Omega_{\alpha}^{B} \right) + g^{i\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} \\
- g^{i\alpha} \left[ \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} + \nabla_{\alpha} \overline{\gamma}_{\beta} \right].
\]

We furthermore compute

\[
\Delta |\gamma|_{g^{-1}, G}^{2} = g^{i\alpha} G^{\alpha} g B \nabla_{i} \nabla_{\alpha} (\gamma_{\alpha} \overline{\gamma}_{\beta}) \\
= (\Delta \gamma, \overline{\gamma}) + (\gamma, \Delta \overline{\gamma}) + |\nabla \gamma|^{2} + |\nabla \overline{\gamma}|^{2} \\
= (S_{\alpha \beta}^{G})_{\alpha}^{B} \gamma_{\alpha} \overline{\gamma}_{\beta} + S_{\alpha \beta}^{G} \gamma_{\alpha} \overline{\gamma}_{\beta} + (\Delta \gamma, \overline{\gamma}) + (\gamma, \Delta \overline{\gamma}) + |\nabla \gamma|^{2} + |\nabla \overline{\gamma}|^{2}.
\]

Subtracting the two equations above yields

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\gamma|^{2} = - |\nabla \gamma|^{2} - |\nabla \overline{\gamma}|^{2} \\
+ g^{i\alpha} G^{\alpha} g B \left( - (T)_{\alpha}^{B} \right) \gamma_{\alpha} \overline{\gamma}_{\beta} + T_{\alpha}^{B} \Omega_{\alpha}^{B} \overline{\gamma}_{\beta} - T_{\alpha}^{B} \Omega_{\alpha}^{B} \gamma_{\alpha} \\
+ \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} - \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} \\
- \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} - \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} + \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B} \\
- \gamma_{\alpha} \nabla_{\alpha} \tilde{\Omega}_{\beta}^{B}.
\]

Then, observing that the second through fifth terms above form a perfect square we arrive at

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\gamma|^{2} = - |\nabla \gamma|^{2} - |\nabla \overline{\gamma} + T \cdot \gamma|^{2} + T \cdot \gamma \cdot \tilde{\Omega} + T \cdot T \cdot \tilde{\Omega} + \gamma \cdot \gamma \cdot \tilde{\Omega} + \gamma \cdot \nabla \tilde{\Omega},
\]

as claimed. \( \square \)

**Corollary 3.4.** Supposing that the initial generalized metric is uniformly equivalent to a background, i.e. \( \Lambda^{-1} G \leq G \leq \Lambda G \), then along a solution to the generalized Ricci flow,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\gamma|^{2} \leq - |\nabla \gamma|^{2} - |\nabla \overline{\gamma} + T \cdot \gamma|^{2} + C(\Lambda, \bar{g}) |\gamma| \left( |T| + |\gamma| + 1 \right).
\]
Proof. This follows by applying the Cauchy-Schwartz inequality to the result of the previous proposition. \hfill \Box

**Lemma 3.5.** Given \((M^{2n}, J)\) and \((\omega, \beta)\) a solution to pluriclosed flow \((1.1)\), with associated generalized metric \(G\) and background generalized metric \(\tilde{G}\), one has

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_G \tilde{G} = -|\mathcal{Y}|_{g^{-1}, G^{-1}, \tilde{G}}^2 + G^{\overline{B}A} g^{mn} \tilde{\Omega}^{mn}_{\overline{m}nB}.
\]

**Proof.** Using Proposition 3.2 we compute

\[
(3.4) \quad \frac{\partial}{\partial t} \text{tr}_G \tilde{G} = \frac{\partial}{\partial t} (G^{\overline{B}A} \tilde{G}_{\overline{AB}}) = -G^{\overline{B}A} G^{\overline{D}B} S_{DC} \tilde{G}_{\overline{AB}}.
\]

Furthermore,

\[
\Delta \text{tr}_G \tilde{G} = g^{mn} \nabla_m \nabla_{\overline{n}} (G^{\overline{B}A} \tilde{G}_{\overline{AB}}) = g^{mn} G^{\overline{B}A} \nabla_m \nabla_{\overline{n}} \tilde{G}_{\overline{AB}}.
\]

However,

\[
\nabla_m \nabla_{\overline{n}} \tilde{G}_{\overline{AB}} = (\nabla_m - \nabla_{\overline{n}}) \tilde{G}_{\overline{AB}} = -\tilde{G}_{\overline{AC}} \tilde{\Omega}^{\overline{C}}_{\overline{m}B}.
\]

The second covariant derivative then yields

\[
\nabla_m \nabla_{\overline{n}} \tilde{G}_{\overline{AB}} = -\nabla_m (\tilde{G}_{\overline{AC}} \tilde{\Omega}^{\overline{C}}_{\overline{m}B}) = \tilde{G}_{\overline{DC}} \tilde{\Omega}^{\overline{D}}_{\overline{m}A} \tilde{\Omega}^{\overline{C}}_{\overline{n}B} + \tilde{G}_{\overline{AC}} (\tilde{\Omega}^{\overline{D}}_{\overline{m}B} - \tilde{\Omega}^{\overline{D}}_{\overline{n}B}).
\]

Contracting this with \(g\) yields

\[
(3.5) \quad \Delta \text{tr}_G \tilde{G} = |\mathcal{Y}|_{g^{-1}, G^{-1}, \tilde{G}}^2 + G^{\overline{B}A} \tilde{G}_{\overline{AC}} (\tilde{\Omega}^{\overline{D}}_{\overline{m}B} - \tilde{\Omega}^{\overline{D}}_{\overline{n}B}).
\]

Combining (3.4) and (3.5) gives the result. \hfill \Box

**Corollary 3.6.** Given \((M^{2n}, J)\) and \((\omega, \beta)\) a solution to pluriclosed flow \((1.1)\), with associated generalized metric \(G\) and background generalized metric \(\tilde{G}\), such that \(\Lambda^{-1} G \leq \Lambda \tilde{G}\), one has

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_G \tilde{G} \leq -\Lambda^{-1} |\mathcal{Y}|_{g^{-1}, G^{-1}, \tilde{G}}^2 + K(\Lambda, \tilde{\vartheta}).
\]

**Proof.** This follows in a straightforward way from Lemma 3.5 using the uniform equivalence of \(G\) and \(\tilde{G}\). \hfill \Box

**Lemma 3.7.** On a Hermitian manifold \((M, J)\) with metrics \(g\) and \(\tilde{g}\) having torsion potentials \(\beta\) and \(\tilde{\beta}\) respectively satisfying \(\Lambda^{-1} \tilde{g} \leq g \leq \Lambda \tilde{g}\) and \(|\beta|^2 \leq \Lambda\), Chern connections \(\nabla\) and \(\tilde{\nabla}\)

\[
|\mathcal{Y}_g|^2 \leq \Lambda^{-2} |\mathcal{Y}_{\tilde{g}}|^2_{g^{-1}, G^{-1}, \tilde{G}} + K(\Lambda, \tilde{\vartheta}).
\]

**Proof.** We will compute \(\mathcal{Y}\) using Lemma 3.1. Set \(\gamma_{ij} \equiv \beta_{ij}(p)\) and \(\tilde{\gamma} \equiv \tilde{\beta}_{ij}(p)\), and compute at \(p\) as follows

\[
\mathcal{Y}(p) = \Gamma - \tilde{\Gamma} = e^{-\gamma} \left( \begin{array}{cc} g^{ij} & 0 \\ 0 & g^{ij} g^{-1} \end{array} \right) e^{\gamma} - e^{-\tilde{\gamma}} \left( \begin{array}{cc} \tilde{g}^{ij} & 0 \\ 0 & \tilde{g}^{ij} \tilde{g}^{-1} \end{array} \right) e^{\tilde{\gamma}}.
\]

But \(g \partial g^{-1}\) is the natural connection action on \((T^*)^{1,0}\) since

\[
g \partial_i g^{-1}(dz^j) = g \partial_i (g^{ij} \partial_j) = g (g^{ij} \partial_j) = -g^{kl} g^{ij} g_{klm} dz^l = -(\Gamma^j_l)^d dz^l.
\]
Thus, the above can naturally be written as

$$\Upsilon(p) = e^{-\gamma} \begin{pmatrix} \Gamma^g & 0 \\ 0 & \Gamma^g \end{pmatrix} e^\gamma - e^{-\gamma} \begin{pmatrix} \Gamma^g & 0 \\ 0 & \Gamma^g \end{pmatrix} e^\gamma$$

Then since $\gamma(p) = \beta(p)$ and $\tilde{\gamma}(p) = \tilde{\beta}(p)$, after some simplification, we get

$$\Upsilon = \left( -\tilde{\beta} \Gamma^\tilde{\beta} - \Gamma^\gamma \beta + \beta \Gamma^\gamma + \Gamma^\gamma \beta \right) G^g = \left( \beta \Upsilon^g + \Upsilon^g \beta + (\beta - \tilde{\beta}) \Gamma^\gamma + \Gamma^\gamma (\beta - \tilde{\beta}) \right) G^g.$$

Computing in Hermitian coordinates about $p$ simplifies this to

$$\Upsilon = e^\beta \begin{pmatrix} \Upsilon^g & 0 \\ 0 & \Upsilon^g \end{pmatrix} e^\beta + \left( (\beta - \tilde{\beta}) \ast T \right) 0.$$

By tensoriality, this is equality is global. But then, letting $\tilde{G}$ be the diagonal metric with $G = e^{-\gamma} \tilde{G} e^\beta$, it is clear that

$$|\Upsilon|_{g, G, G}^2 \geq \left| \begin{pmatrix} \Upsilon^g & 0 \\ 0 & \Upsilon^g \end{pmatrix} \right|^2_{g^{-1}, G^{-1}, G} - K(\Lambda, \tilde{g}) \geq \Lambda^{-2} |\Upsilon|^2_{g} - K(\Lambda, \tilde{g}),$$

as claimed. \qed

In the proposition below we will use a cutoff function to get a localized estimate. In particular we will say that $\eta$ is a cutoff function for a ball of radius $R$ at $p \in M$ with respect to a background metric $\tilde{g}$ if

$$|\eta|_{B(p, R)} \equiv 1, \supp \eta \subset B(p, R)$$

$$|\nabla \eta| \leq \frac{C}{R}, \quad |\nabla^2 \eta| \leq \frac{C}{R^2},$$

where the constant $C$ depends on the background metric $\tilde{g}$.

**Proposition 3.8.** Given $(M^{2n}, J)$, fix $(\omega, \beta)$ a solution to pluriclosed flow \([1, 1]\), with associated generalized metric $G$. Fix a background metric generalized metric $\tilde{G}(\tilde{g}, \tilde{\beta})$ such that $\Lambda^{-1} \tilde{G} \leq G \leq \Lambda \tilde{G}$, a cutoff function $\eta$ for a ball of radius $R$ with respect to $\tilde{g}$, constants $\Lambda, p \geq 1$, and define

$$\Phi = t \eta^p |\Upsilon|^2 + A \eta^{p-2} \Phi_G \tilde{G}.$$

There exists a constant $K = K(\Lambda, \tilde{G}, p) > 0$ such that

$$\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq -2 \Re \left\langle \nabla \Phi, p \eta^{-1} \nabla \eta \right\rangle$$

$$+ |\Upsilon|^2 \eta^{p-2} \left( 1 + tK \left( 1 + R^{-2} \right) - A \Lambda^{-1} \right) + KA \left( 1 + R^{-2} \right),$$

**Proof.** We first compute using Corollary 3.4

$$\left( \frac{\partial}{\partial t} - \Delta \right) (t \eta^p |\Upsilon|^2) = \eta^p |\Upsilon|^2 + t \eta^p \left( \frac{\partial}{\partial t} - \Delta \right) (|\Upsilon|^2) - 2pt \eta^{p-1} \Re \left\langle \nabla |\Upsilon|^2, \nabla \eta \right\rangle$$

$$- p(p-1) \eta^{p-2} |\nabla \eta|^2 |\Upsilon|^2 - pt \eta^{p-1} |\Upsilon|^2 \Delta \eta$$

$$\leq \eta^p |\Upsilon|^2 + K(\Lambda) t \eta^p |\Upsilon| (|T| + |\Upsilon| + 1) - 2pt \eta^{p-1} \Re \left\langle \nabla |\Upsilon|^2, \nabla \eta \right\rangle$$

$$- p(p-1) \eta^{p-2} |\nabla \eta|^2 |\Upsilon|^2 - pt \eta^{p-1} |\Upsilon|^2 \Delta \eta.$$
For ease of computing we set \( q = p - 2 \). Using Corollary 3.6 we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \eta^q \text{tr}_G \tilde{G} = \eta^q \left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_G \tilde{G} - 2q \eta^{q-1} \Re \left\langle \nabla \text{tr}_G \tilde{G}, \nabla \eta \right\rangle
\]
\[
- q(q - 1) \eta^{q-2} \text{tr}_G \tilde{G} |\nabla \eta|^2 - q \eta^{q-1} \text{tr}_G \tilde{G} \Delta \eta
\]
\[
\leq - \eta^q \Lambda^{-1} |\mathcal{Y}|^2 + K(\Lambda, \tilde{g}) - 2q \eta^{q-1} \Re \left\langle \nabla \text{tr}_G \tilde{G}, \nabla \eta \right\rangle
\]
\[
- q(q - 1) \eta^{q-2} \text{tr}_G \tilde{G} |\nabla \eta|^2 - q \eta^{q-1} \text{tr}_G \tilde{G} \Delta \eta.
\]
Combining the two inequalities above, Lemma 3.7 and (3.6) yields
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq \eta^q |\mathcal{Y}|^2 + t K \eta^q |\mathcal{Y}| (|T| + |\mathcal{Y}| + 1) - 2pt \eta^{p-1} \Re \left\langle \nabla |\mathcal{Y}|^2, \nabla \eta \right\rangle
\]
\[
- p(p - 1) tr \eta^{p-2} |\nabla \eta|^2 |\mathcal{Y}|^2 - pt \eta^{p-1} |\mathcal{Y}|^2 \Delta \eta
\]
\[
+ A \left\{ - \eta^q \Lambda^{-1} |\mathcal{Y}|^2 + K - 2q \eta^{q-1} \Re \left\langle \nabla \text{tr}_G \tilde{G}, \nabla \eta \right\rangle \right.
\]
\[
- Aq(q - 1) \eta^{q-2} \text{tr}_G \tilde{G} |\nabla \eta|^2 - q \eta^{q-1} \text{tr}_G \tilde{G} \Delta \eta \right\}
\]
\[
\leq - 2pt \eta^{p-1} \Re \left\langle \nabla |\mathcal{Y}|^2, \nabla \eta \right\rangle - 2A \eta^{q-1} \Re \left\langle \nabla \text{tr}_G \tilde{G}, \nabla \eta \right\rangle
\]
\[
+ |\mathcal{Y}|^2 \eta^q (1 + t (K + CR^{-2}) - A \Lambda^{-1}) + K(\Lambda, p) A (1 + R^{-2})
\]
We also directly compute
\[
\nabla \Phi = p t \eta^{p-1} \nabla \eta |\mathcal{Y}|^2 + t K \eta^p \nabla |\mathcal{Y}| (|T| + |\mathcal{Y}| + 1) + A \left\{ q \eta^{q-1} \nabla \text{tr}_G \tilde{G} + \eta^q \nabla \text{tr}_G \tilde{G} \right\}. \tag{3.7}
\]
Taking the inner product with \( \eta^{-1} \nabla \eta \), multiplying by \( p \), and rearranging we find
\[
-2 \Re \left\langle p t \eta^{p-1} \nabla |\mathcal{Y}|^2, \nabla \eta \right\rangle
\]
\[
= - 2 \Re \left\langle \nabla \Phi, p \eta^{-1} \nabla \eta \right\rangle + 2p(pt \eta^{p-2} |\mathcal{Y}|^2 + A \eta^{q-2} \text{tr}_G \tilde{G}) |\nabla \eta|^2
\]
\[
+ 2A \eta^{q-1} \Re \left\langle \nabla \text{tr}_G \tilde{G}, \nabla \eta \right\rangle. \tag{3.8}
\]
We also will exploit the inequality
\[
|\nabla \text{tr}_G \tilde{G}|^2 = g^{jk}(\nabla_k \text{tr}_G \tilde{G})(\nabla_j \text{tr}_G \tilde{G})
\]
\[
= g^{jk} G^{\alpha \beta} \nabla_k \tilde{G}_{\alpha \beta} \nabla_j \tilde{G}_{\alpha \beta}
\]
\[
= g^{jk} G^{\alpha \beta} \nabla_k (\tilde{G}_{\alpha \beta} - \tilde{\nabla}_k) \tilde{G}_{\alpha \beta} \tilde{\nabla}_j
\]
\[
= g^{jk} G^{\alpha \beta} (\mathcal{Y} \tilde{G}_a) \tilde{\nabla}_j \tilde{G}_{\alpha \beta} \tag{3.9}
\]
\[
\leq C(\Lambda) |\mathcal{Y}|^2.
\]
Using (3.8) and (3.9) in (3.7) yields
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq - 2 \Re \left\langle \nabla \Phi, p \eta^{-1} \nabla \eta \right\rangle
\]
\[
+ |\mathcal{Y}|^2 \eta^{p-2} (1 + t K (1 + R^{-2}) - A \Lambda^{-1}) + K A (1 + R^{-2}),
\]
as claimed. \[\square\]
4. Main Results

**Proposition 4.1.** Given \((M^{2n}, J)\), fix \((\omega, \beta)\) a solution to pluriclosed flow \((1.1)\) on \([0, \tau]\), \(\tau \leq 1\), with associated generalized metric \(G\). Fix a background metric generalized metric \(\tilde{G}(\tilde{g}, \tilde{\beta})\) such that \(\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}\), and some \(0 < R < 1\). There exists a constant \(K = K(p, \Lambda, \tilde{G})\) such that

\[
\sup_{B_{\tilde{g}}(p) \times \{t\}} \left| \Upsilon(G, \tilde{G}) \right|_{2} \leq K \left( \frac{1}{t} + R^{-4} \right).
\]

**Proof.** Let \(\eta\) denote a cutoff function for \(B_{R}\), fix \(p = 2\), and define \(\Phi\) as in Proposition 3.8. We can choose \(A \geq C\Lambda KR^{-2}\) so that for times \(t \leq 1\), Proposition 3.8 yields

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Phi \leq -2\Re(\nabla \Phi, p\eta^{-1}\nabla \eta) + KA\left( 1 + R^{-2} \right)
\]

\[
\leq -2\Re(\nabla \Phi, p\eta^{-1}\nabla \eta) + KR^{-4}.
\]

Since \(\Phi \leq K\) at time 0, it follows from the maximum principle that

\[
\sup_{M \times \{t\}} \Phi \leq K \left( 1 + tR^{-4} \right)
\]

But then for any \(x \in B_{\tilde{g}}\),

\[
|\Upsilon|^{2}(x, t) = \eta^{p}(x) |\Upsilon|^{2}(x, t) \leq \frac{1}{t} \Phi(x, t) \leq K \left( \frac{1}{t} + R^{-4} \right),
\]

as claimed.

\(\square\)

**Theorem 4.2.** Given \((M^{2n}, J)\), fix \((\omega, \beta)\) a solution to pluriclosed flow \((1.1)\) on \([0, \tau]\), \(\tau \leq 1\), with associated generalized metric \(G\). Fix a background metric generalized metric \(\tilde{G}(\tilde{g}, \tilde{\beta})\) such that \(\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}\). There exists \(\rho > 0\) depending on \(\tilde{g}\) such that for all \(0 < R < \rho\), and \(k \in \mathbb{N}\), there exists a constant \(K = K(\rho, \Lambda, \tilde{G})\) such that

\[
\sup_{B_{\tilde{g}}(p) \times \{t\}} f_{k}(x, t) \leq K \left( \frac{1}{t} + R^{-4} \right).
\]

**Proof.** The case \(k = 0\) is established in Proposition 4.1. For \(k > 0\), suppose otherwise. We choose \(\rho > 0\) so that, for all \(x \in B_{R}(p)\), the exponential map associated to \(\tilde{g}\) is a local diffeomorphism on a ball of radius \(\rho\) with uniform estimates on the pullback metric \(\exp_{x}^{*}\tilde{g}\). Given \(G_{t}\) a solution to pluriclosed flow as described and \(0 < R < \rho\), suppose there exist points \((x_{i}, t_{i}) \in B_{R}(p) \times [0, \tau)\) s.t.

\[
\frac{t_{i}f_{k,i}(x_{i}, t_{i})}{1 + t_{i}d(x_{i}, \partial B_{R})^{-4}} \not\to \infty.
\]

We refine to points \((\tilde{x}_{i}, \tilde{t}_{i})\) such that

\[
(4.1) \quad \frac{\tilde{t}_{i}f_{k,i}(\tilde{x}_{i}, \tilde{t}_{i})}{1 + \tilde{t}_{i}d(\tilde{x}_{i}, \partial B_{R})^{-4}} = \sup_{B_{\tilde{g}}(p) \times [0, t_{i}]} \frac{t_{i}f_{k,i}(x_{i}, t_{i})}{1 + t_{i}d(x_{i}, \partial B_{R})^{-4}}.
\]

From here on, we will drop these decorations and refer only to the refined points. By construction, for each \(x_{i}\) we can use the exponential map of \(\tilde{g}\) on a ball of radius \(d(x_{i}, \partial B_{R})\).
to pullback $G$ and so work in complex coordinate charts. Set $\sigma_i = f_k(x_i, t_i)$, and define a family of metrics on $B(0, \sqrt{\sigma_i} d(x_i, \partial B_R))) \times [-t_i \sigma_i, (\tau - t_i) \sigma_i)$ by

$$G'_i(x, t) = G_i(x_i + \frac{x}{\sqrt{\sigma_i}}; t_i + \frac{t}{\sigma_i})$$

where $\tilde{G}'_i$ is defined similarly. Each of these primed metrics is actually a solution to 3.2 on their domains as $\frac{\partial}{\partial t} G'_i(x, t) = \frac{1}{\sigma_i} \frac{\partial}{\partial t} G_i$ and $S_i'(x, t) = \frac{1}{\sigma_i} S_i$. Associated to these metrics, the quantities $f'_{k,i}$ scale as given by

$$f'_{k,i}(x, t) = \frac{1}{\sigma_i} f_{k,i}(x_i + \frac{x}{\sqrt{\sigma_i}}; t_i + \frac{t}{\sigma_i})$$

and thus by construction

$$f'_{k,i}(0, 0) = 1.$$

Notice that since $t_i \sigma_i \to \infty$ and $d(x_i, \partial B_R)^2 \sigma_i \to \infty$, eventually all of these solutions and associated quantities exist on $B_1(0) \times [-1, 0] \subset \mathbb{C}^n \times \mathbb{R}$.

Also note that since for sufficiently large $i$, $t_i \sigma_i \geq 2$, one has

$$t_i + \frac{t}{\sigma_i} \geq \frac{t_i}{2}$$

for any $t \in [-1, 0]$. Similarly, for sufficiently large $i$ it will hold for $x \in B_1(0)$ that

$$d(x_i + \frac{x}{\sqrt{\sigma_i}}, \partial B_r) \geq \frac{1}{2} d(x_i, \partial B_r)$$

This implies using (4.1) that for any $x, t \in B_1(0) \times [-1, 0]$, one has

$$\frac{t_i}{64(1 + t_i d(x_i, \partial B_R)^{-4})} f_{k,i}(x_i + \frac{x}{\sqrt{\sigma_i}}; t_i + \frac{t}{\sigma_i})$$

$$\leq \frac{t_i}{1 + t_i d(x_i, \partial B_R)^{-4}} f_{k,i}(x_i + \frac{x}{\sqrt{\sigma_i}}; t_i + \frac{t}{\sigma_i})$$

Thus we obtain

$$f'_{k,i}(x, t) = \frac{1}{\sigma_i} f_{k,i}(x_i + \frac{x}{\sqrt{\sigma_i}}; t_i + \frac{t}{\sigma_i}) \leq 64$$

on $B_1(0) \times [-1, 0]$ for sufficiently large $i$. This is a uniform $C^{k+1}$-estimate. As our equation is of the form

$$\frac{\partial}{\partial t}(G'_i)_{\eta \overline{\eta}} = (G'_i)^{\eta \overline{\eta}}(G'_i)_{\eta \overline{\eta}} \partial G'_i + \partial G'_i \ast \partial G'_i$$

and a uniform $C^{k+1}$-estimate for $G'_i$ implies a uniform $C^k$-estimate for $\partial G'_i \ast \partial G'_i$, we are exactly in the case of the Schauder estimates. So, on $B(0, \frac{1}{2}) \times [-\frac{1}{2}, 0]$, we have a uniform $C^{k+2}$-estimate. Applying Ascoli-Arzelá then gives subsequential $C^{k+1}$-convergence of the metrics to some limit $G'_\infty$. This convergence in particular implies that

$$f'_{k,\infty}(0, 0) = 1.$$
We note that the rescaled background metrics converge in $C^\infty$ to a metric $\tilde{G}'_\infty$ which must be Euclidean. Furthermore, using the estimate on $f_0$ it follows that for points in $B(0, \frac{1}{2}) \times [-\frac{1}{2}, 0]$, $$\lim_{i \to \infty} f'_{1,i}(x, t) = \frac{1}{\sigma_i} \left( \frac{x}{\sqrt{\sigma_i}} + t \sqrt{\sigma_i} - d(x, B_R) \right) \to 0.$$ Hence the metric $G'_\infty$ is constant in space and time after the blow-up, and thus $f'_{k,\infty} \equiv 0$, a contradiction. \[ \square \]

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