Abstract

It is demonstrated that, in the framework of the orbit method, a simple and damped harmonic oscillators are indistinguishable at the level of an abstract Lie algebra. This opens a possibility for treating the dissipative systems within the orbit method. In depth analysis of the coadjoint orbits of the $(1+1)$ dimensional Newton-Hooke group are presented. Further, it is argued that the physical interpretation is carried by a specific realisation of the Lie algebra of smooth functions on a phase space rather than by an abstract Lie algebra.

1 Introduction

Sidney Coleman famously said “The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction”. The accuracy of this dictum is striking when one considers the abundance of scientific papers devoted to this subject across many branches of physics. It has long been known that in the framework of the orbit method \[1,2,3\] the oscillator is described by the Newton–Hooke (NH) group. The NH type groups first appeared in the classification of the possible kinematical groups \[4\]. The thorough study of $(3+1)$ dimensional NH group was presented in \[5\]. The orbit method was employed in \[6\] to study the centrally extended NH group in $(2+1)$ dimensions. The coadjoint orbits and the irreducible representations were calculated therein. Besides the orbit method a planar system with exotic Newton–Hooke symmetry was constructed by the technique of nonlinear realisation \[7\], the analysis therein included the chiral decomposition. The idea of chiral decomposition was later applied to the non-commutative Landau problem \[8,9\] and to the rotation-less

\[1\] E-mail address: 800289@edu.p.lodz.pl
$NH$ symmetry of the 3D anisotropic oscillator [10]. Some work on the anisotropic (2+1) dimensional $NH$ group was presented in [11]. The extended conformal $NH$ type symmetries were also studied in connection to the Pais–Uhlenbeck oscillator [12–14]. This short overview is far from being complete but it shows, how telling a study of this simple system can be. It is worth to mention that orbit method was also successfully used to analyse systems with Galilei and Poincaré type symmetries both in the free case and with external electromagnetic fields [15–25].

In this paper an accessible yet illuminating example of harmonic oscillator is examined in the framework of the orbit method [1–3]. In the case of simple harmonic oscillator the Lie algebra of (1 + 1) $NH$ group is derived from the standard Hamiltonian description by a technique encouraged by [21]. Detailed analysis of the coadjoint action provides a full understanding of the physical interpretation. Clearly, analysis becomes more involved for the dissipative systems. However, a proper canonical transformation may allow for a significant simplification. For example, the damped harmonic oscillator can be described by the same Lie algebra as the undamped case. This simplification comes at a price of using rather elaborate coordinates. Consequently, at the level of the Lie algebra the damped harmonic oscillator is indistinguishable from the undamped one. This example illustrates a possible way of treating the dissipative systems within the framework of the orbit method. Apparently an abstract Lie algebra does not carry the physical interpretation of the system. The question arises how to use the orbit method so that the physical interpretation is not lost. The current paper is devoted to just this investigation.

This paper is structured as follows. In Section 2, starting with the Hamiltonian of the harmonic oscillator, the Lie algebra of the (1 + 1) dimensional Newton–Hooke group is derived to set the scene for the further analysis. Section 3 provides the coadjoint action of the group under investigation. Also, the symplectic structure on the coadjoint orbit of (1 + 1) Newton–Hooke group are given. The in depth analysis of the coadjoint orbits is presented in Section 4. Section 5 is devoted to the damped harmonic oscillator and shows that it may be described by the same abstract Lie algebra as the undamped case. The paper closes with conclusions in Section 6 where also some outlooks are provided.

## 2 The (1+1) Newton-Hooke group

To focus the attention take the Hamiltonian of the simple harmonic oscillator

\[ h(p,x) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \quad (2.1) \]

where $p$ is the kinematic momentum and $x$ is the displacement of the oscillator. Exploiting the canonical Poisson bracket

\[ \{F(p,x),G(p,x)\} = \frac{\partial F}{\partial x} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial x} \quad (2.2) \]

one arrives at the well known equations of motion

\[ \dot{p} = -m\omega^2 x, \quad \dot{x} = \frac{p}{m} \quad (2.3) \]
which when put together read $\ddot{x} = -\omega^2 x$. In order to use the orbit method to describe the simple harmonic oscillator an appropriate Lie algebra is needed. This algebra should be such that the equations of motion on its coadjoint orbits are equivalent to (2.3). Herein such a Lie algebra is constructed starting with the algebra of smooth functions on the phase space equipped with the Poisson bracket (2.2).

The method of constructing such a Lie algebra is based on Poisson’s theorem stating that the Poisson bracket of two quantities that are constants of motion is also a constant of motion. The Hamiltonian (2.1) i.e. the total energy of the system is the only integral of motion. This system also admits constants of motion e.g. $f(p, x, t) = t - \frac{1}{\omega} \arctan \frac{x}{p}$ which at $p = 0$ has to be understood in the sense of the limit. It is mentioned here for the sake of completeness, however will not be utilised in the present paper because there is no need to consider the time dependent generators. Therefore, $\hbar$ should be included in the set of generators. Inasmuch as some coordinates are needed, one just checks whether $p$ and $x$ could do the job. To this end calculate the Poisson bracket (2.2) for all the pairs selected from the set $\{h, p, x\}$ and find that the non-vanishing brackets are

$$\{h, p\} = m\omega^2 x, \quad \{h, x\} = -\frac{1}{m} p, \quad \{x, p\} = 1.$$ (2.4)

Quick conclusion is that, in order to have a closed algebra, the $\{h, p, x\}$ ought to be augmented by a constant function equal to 1. Even more elegantly one may replace $x$ with $k = mx$ and use a constant function equal $m$. In which case the Hamiltonian (2.1) becomes

$$h(p, k) = \frac{p^2}{2m} + \frac{\omega^2 k^2}{2m},$$ (2.5)

and the Poisson bracket (2.2), by the chain rule, reads now

$$\{F(p, k), G(p, k)\} = m \left( \frac{\partial F}{\partial k} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial k} \right).$$ (2.6)

The non-vanishing Poisson brackets are

$$\{h, p\} = \omega^2 k, \quad \{h, k\} = -p, \quad \{k, p\} = m.$$ (2.7)

Therefore the functions $J_1 = m, J_2 = h(p, k), J_3 = p, J_4 = k$ span the Poisson algebra under the Poisson bracket (2.6). Note that $J_1$ is a central generator. What was described above is known as the Lie algebra of the $(1 + 1)$ dimensional Newton-Hooke group. At the abstract level it is a four dimensional Lie algebra spanned by $J_1 = M, J_2 = H, J_3 = P, J_4 = K$ characterised by the following nonzero structure constants $c_{34}^1 = \omega^2, c_{24}^3 = -1, c_{34}^1 = -1$, in the above numbering of the basis, which will be kept throughout this paper.
3 Coadjoint action and dynamics

The matrices of the adjoint action \( m_{J_i}^{ad} \) corresponding to the generators \( J_1, \ldots, J_4 \) are given by \((m_{J_i}^{ad})_{jk} = c_{ik}^j \) where \( c_{ik}^j \) are the structure constants. Explicitly

\[
m_{J_2}^{Ad} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \omega^2 & 0 \end{bmatrix}, \quad m_{J_3}^{Ad} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}, \quad m_{J_4}^{Ad} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and \( m_{J_1}^{Ad} \) is the zero matrix since \( m \) is a central generator. The generic element \( g \) of our group can be written as \( g = (\eta, b, a, v) = e^{\eta M} e^{bH} e^{aP} e^{vK} \in G \). The coordinates for the dual to our Lie algebra \( g^* \) are realised by \( m, h, p, k \). The matrix of the coadjoint action of an element \( g \in G \) is then given by \( M_g^{coAd} = e^{-\tau m_{k}^{Ad}} e^{-\eta m_{P}^{Ad}} e^{-m_{H}^{Ad}} e^{-\eta m_{M}^{Ad}} \) which explicitly reads

\[
M_g^{coAd} = \begin{bmatrix} 1 & \frac{\omega^2 + a \omega^2}{2} & -v \cos b \omega - a \omega \sin b \omega & a \cos b \omega - \frac{v}{\omega} \sin b \omega \\ 0 & 1 & 0 & 0 \\ 0 & -v & \cos b \omega & \frac{\sin b \omega}{\omega} \\ 0 & a \omega^2 & -\omega \sin b \omega & \cos b \omega \end{bmatrix}.
\]

(3.1)

An element of \( g^* \) is represented as a row vector \( \xi = [m, h, p, k] \). Then the coadjoint action of \( g \in G \) is calculated by matrix multiplication of \( \xi \) by \( g^* \) on the right, which yields the following explicit form of the coadjoint action

\[
\begin{align*}
m' &= m, \\
h' &= h + \frac{1}{2}mv^2 + \frac{ma^2\omega^2}{2} - vp + a\omega^2k, \\
p' &= (p - mv) \cos b\omega - \omega(ma + k) \sin b\omega, \\
k' &= (ma + k) \cos b\omega + \frac{v - m}{\omega} \sin b\omega.
\end{align*}
\]

(3.2)

More detailed analysis of the action (3.2) will be presented in the following section. The next step is to calculate the invariants of the coadjoint action i.e. smooth functions \( C \) on \( g^* \), such that \( \forall g \in G \forall \xi \in g^* \quad C(\text{coAd}_g(\xi)) = C(\xi) \). They are solutions to the following set of differential equations \cite{26,30}

\[
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2k & -p \\ 0 & -\omega^2k & 0 & -m \\ 0 & p & m & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial C}{\partial m} \\ \frac{\partial C}{\partial h} \\ \frac{\partial C}{\partial p} \\ \frac{\partial C}{\partial k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

(3.3)

There are two solutions to (3.3), namely

\[
C_1 = m, \quad C_2 = k^2 - \frac{2mh}{\omega^2} + \frac{p^2}{\omega^2}.
\]

(3.4)

The first one is a trivial consequence of \( m \) being a central generator. Consider a map

\[
C : g^* \rightarrow \mathbb{R}^2, \\
\xi \mapsto (C_1(\xi), C_2(\xi)).
\]

(3.5)
At each point $\xi' = \text{co}Ad_g(\xi)$, $g \in G$ of the orbit through $\xi$ the value of $\text{(3.5)}$ is constant. Moreover, mapping $\text{(3.5)}$ is of a constant and maximal rank, therefore the preimage of a point is a submanifold in $g^*$. Each of its compact components is precisely a coadjoint orbit through $\xi$. In the present case, the orbit, denoted $O^{C_1,C_2}$, admits a single global parametrisation

$$\varphi : (p, k) \mapsto \left( m = C_1, h = \tilde{h} = \frac{p^2}{2m} + \frac{\omega^2 k^2}{2m} - \frac{\omega^2 C_2}{2m}, p, k \right)$$

(3.6)

so, in principle, it might be covered by a single map e.g. $\varphi^{-1}$. Note that, in the this example, the hamiltonian

$$\tilde{h}(p, k) = \frac{p^2}{2m} + \frac{\omega^2 k^2}{2m} - \frac{\omega^2 C_2}{2m}$$

(3.7)

which was derived from the invariants of the coadjoint action is, up to an additive constant, equivalent to the initial Hamiltonian (2.1). For the sake of completeness note that, the Jacobian of the map $(m, h, p, k) \mapsto (C_1, C_2, p, k)$ is $-\frac{2m}{\omega^2}$ so there is a singularity for $m = 0$. This case shall not be discussed in depth for it is of no physical interest. It suffices to say that, for the fixed $m = 0$, there is one invariant of the coadjoint action $C = k^2 + \frac{p^2}{\omega^2}$, and since $h$ is unrestricted, the orbits resemble the flatten cylinders. In what follows, $m \neq 0$ shall be assumed. The Poisson tensor $\Lambda$ on the orbit $O^{C_1,C_2}$, written in the chart $(O^{C_1,C_2}, \varphi^{-1})$ reads

$$\Lambda^{ij} = \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}$$

(3.8)

which is equivalent to (2.6). Since $m \neq 0$ on the orbit $O^{C_1,C_2}$ the Poisson structure is non-degenerate and one quickly finds, by the techniques presented in [31], that the corresponding symplectic two-form is

$$\omega = -\frac{1}{m}dp \wedge dk.$$ 

(3.9)

Therefore, employing the Hamiltonian (3.7), one finds the equations of motion to be

$$\dot{p} = -\omega^2 k, \quad \dot{k} = p$$

(3.10)

which, when combined with $k = mx$, are equivalent to (2.3).

### 4 Coadjoint orbits

Further insight into the structure of the coadjoint orbit can be gained by examining, one by one, the coadjoint actions of the group elements that correspond to the generators. Consider a test solution of (3.10) for example $p = -m \omega A \sin(\omega t), k = m \omega A \cos(\omega t)$ that is, at $t = 0$ the displacement is maximal and momentum is zero. The energy is constant and at any time is given by $\text{(3.7)}$ ($E = \tilde{h}(p, k)$), furthermore $m = C_1$ is also fixed. The trajectory of the system, as time flies, is then given by

$$\xi(t) = [m, E = \tilde{h}(p, k), p = -m \omega A \sin \omega t, k = m \omega A \cos \omega t].$$

(4.1)
The trajectory (4.1) lies on the coadjoint orbit characterised by $C_1 = m$ and a fixed $C_2$.

Coadjoint action of a group element $g = \exp(\tau H)$ generated by $H$ on (4.1) is: $m' = m$, $E' = E$ and

$$
\begin{align*}
p' &= -m\omega A \sin(\omega(t + \tau)), \\
k' &= mA \cos(\omega(t + \tau))
\end{align*}
$$

that is to say $m$ and $E$ remain constant and $p, k$ follow the elliptic trajectory. Clearly, $H$ generates temporal shifts. Moreover as the system evolves, it stays on the same orbit.

Next, let us consider a group element $g = \exp(lP)$ generated by $P$. Its coadjoint action on (4.1) is: $m' = m$, $p' = p$ and

$$
\begin{align*}
E' &= E + \frac{1}{2}m\omega^2 x'^2 + mx'\omega^2 A \cos \omega t, \\
k' &= k + ml.
\end{align*}
$$

The displacement ($x = \frac{k}{m}$) is increased by $l$ and energy is changed exactly in such a way, that the system stays on the same orbit which means that $P$ generates spatial shifts of the initial conditions.

Figure 1: Illustration of the coadjoint action of the elements of the group $NH(1 + 1)$ corresponding to the generators. The time evolution brings the system along the elliptical trajectory e.g. from point $A$ to $C$ or $B$ to $D$. Starting from the point $A$ the system can be moved to point $B$ by the spatial shift generated by $P$. Performing a boost generated by $K$ brings the system from the point $C$ to $D$. 

6
Finally, a group element \( g = \exp(uK) \) generated by \( K \) acts on (4.1) as:

\[
E' = E + \frac{1}{2}mu^2 + mu\omega A \sin \omega t, \quad p' = p - mu
\]
i.e. \( m \) and \( k \) are constant, momentum \( p \) is decreased by \( mu \) and energy is adjusted so that the system remains on the orbit. Clearly, \( K \) generates the momentum shifts. For the sake of completeness it is worth mentioning that the action of group elements generated by \( M \) is identity. The examples of above-described actions are presented in Figure 1 which also, by a small leap of imagination, allows us to visualise the coadjoint orbit.

5 Damped harmonic oscillator

A slightly more complex system which can be investigated by similar techniques is a damped harmonic oscillator. Take the following time dependent Hamiltonian

\[
h(P, q, t) = \frac{P^2}{2m}e^{-2\gamma t} + \frac{1}{2}m\omega_0^2 e^{2\gamma t}x^2
\]

where \( 2\gamma = \beta \) with \( \beta \) being the friction coefficient and \( \omega_0 \) is the undamped frequency of the oscillator. Note that \( P = m\dot{x}e^{2\gamma t} \) i.e. the canonical momentum does not coincide with the kinetic momentum \( p = m\dot{x} \). The Hamiltonian (5.1) with the canonical Poisson bracket yields the following equations of motion

\[
\dot{x} = \frac{P}{m}e^{-2\gamma t}, \quad \dot{P} = -m\omega_0^2 e^{2\gamma t}x
\]

or equivalently \( \ddot{x} + 2\gamma \dot{x} + \omega_0^2x = 0 \). It is an easy exercise to check that the procedure that was carried out in Section 2 for the undamped oscillator fails in the present case. Indeed, introducing the new coordinate \( k = mx \) one finds that the hamiltonian (5.1) becomes

\[
h(P, k, t) = \frac{P^2}{2m}e^{-2\gamma t} + \frac{1}{2}m\omega_0^2 e^{2\gamma t}k^2
\]

and, by the chain rule, new the Poisson bracket is just (2.6). Then, one quickly finds that

\[
\{h, P\} = \omega_0^2 e^{2\gamma t}k, \quad \{h, k\} = -e^{-2\gamma t}P, \quad \{k, P\} = m
\]

which fails to constitute a Lie algebra because there is an undesired time dependency of the structure constants. One way to deal with this problem is to use a generating function method to bring the Hamiltonian (5.1) to a more convenient form (see e.g. [32]). Consider the following generating function of the second kind

\[
F_2(k, P, t) = e^{\gamma t}kP - \frac{1}{2}m\gamma e^{2\gamma t}x^2.
\]

The transformation rules for the coordinates are

\[
P = \frac{\partial F_2}{\partial x} = e^{\gamma t}P - m\gamma e^{2\gamma t}x, \quad Q = \frac{\partial F_2}{\partial P} = e^{\gamma t}x
\]
Furthermore, the old $h$ and the new $H$ Hamiltonians obey

$$H - h = \frac{\partial F_2}{\partial t} = \gamma e^{\gamma t} x P - m \gamma^2 e^{2\gamma t} x^2 = \gamma Q P - m \gamma^2 Q^2. \quad (5.7)$$

The relations between old $(P, x)$ and new $(P, Q)$ coordinates can be written as

$$\begin{bmatrix} P \\ x \end{bmatrix} = \begin{bmatrix} e^{\gamma t} & -m \gamma e^{\gamma t} \\ 0 & e^{-\gamma t} \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}, \quad \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} e^{-\gamma t} & m \gamma e^{\gamma t} \\ 0 & e^{\gamma t} \end{bmatrix} \begin{bmatrix} P \\ x \end{bmatrix}. \quad (5.8)$$

By the chain rule $\frac{\partial}{\partial P} = \frac{\partial P}{\partial P} \frac{\partial}{\partial P} + \frac{\partial Q}{\partial P} \frac{\partial}{\partial Q} = e^{-\gamma t} \frac{\partial}{\partial P}$ and $\frac{\partial}{\partial x} = m \gamma e^{\gamma t} \frac{\partial}{\partial Q} + e^{\gamma t} \frac{\partial}{\partial Q}$ so one quickly finds that the Poisson bracket (2.2) becomes

$$\{F(P, Q), G(P, Q)\} = \frac{\partial F}{\partial Q} \frac{\partial G}{\partial P} - \frac{\partial F}{\partial P} \frac{\partial G}{\partial Q} \quad (5.9)$$

i.e. the transformation (5.7) is canonical but, since $H \neq h$ it is not a symmetry. Finally, the transformed Hamiltonian takes the following form

$$H(P, Q) = \frac{P^2}{2m} + \frac{1}{2} m(\omega_0^2 - \gamma^2)Q^2 \quad (5.10)$$

which, functionally is just (2.1) with $\omega^2 = \omega_0^2 - \gamma^2$. Therefore, in the new coordinates $P$ and $Q$ the procedure of constructing the Lie algebra as in Section 2 can be carried out. The resulting algebra is exactly $(1+1)$ Newton-Hooke algebra as it was for the undamped oscillator therefore, at the level of the abstract Lie algebra the two systems are indistinguishable. The difference lies in the realisation of the generators as smooth functions of the phase space coordinates. It is important to stress that the Hamiltonian derived from the invariants of the coadjoint action would be functionally equivalent (up to an additive constant) to (5.10) not to the initial Hamiltonian (5.1) as the interpretation of the coordinates has changed.

### 6 Concluding remarks

It was shown that, in the framework of the orbit method, a simple and damped harmonic oscillators can be described by the same abstract Lie algebra. The simple, yet striking example presented here shows that, when the dynamics on the coadjoint orbits are considered, a simple knowledge of an abstract Lie algebra does not suffice to provide the physical interpretation. What describes the system is rather a specific realisation of the Lie algebra in terms of the smooth functions on the classical phase space.

Particularly, the result presented in the current paper stresses the importance of keeping track of the physical interpretation when constructing the dynamics on the coadjoint orbits which might be crucial when the deformation quantisation on the coadjoint orbit is considered. One way to achieve that is to derive a relevant Lie algebra starting from the Hamiltonian formulation as was done in the current paper for the harmonic oscillator or in [21] for extended Galilei group also known in the literature as the Galilei-Maxwell group [24]. It is the intention of the author to follow with the application of the current results in case of the Poincaré-Maxwell group soon.
References

[1] A. A Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, Berlin Heidelberg 1976.

[2] A. A. Kirillov, *Bull. Amer. Math. Soc. (N.S.)* 34, 433 (1999).

[3] A. A. Kirillov, *Lectures on the Orbit Method*, American Mathematical Soc. 2004.

[4] H. Bacry and J. M. Lévy-Leblond, *J. Math. Phys.* 9, 1605 (1968).

[5] J. R. Derome and J. G. Dubois, *Il Nuovo Cim. B* 9, 351 (1972).

[6] O. Arratia, M. A. Martín, and M. A. del Olmo, *Int. J. Theor. Phys.* 50, 2035 (2011).

[7] P. D. Alvarez, J. Gomis, K. Kamimura, and M. S. Plyushchay, *Ann. Phys.* 322, 1556 (2007).

[8] P. D. Alvarez, J. Gomis, K. Kamimura and M. S. Plyushchay, *Phys. Lett. B* 659, 906 (2008).

[9] P. M. Zhang and P. A. Horváthy, *Ann. Phys.* 327, 1730 (2012).

[10] P. M. Zhang, P. A. Horváthy, K. Andrzejewski, J. Gonera and P. Kosiński, *Ann. Phys.* 333, 335 (2013).

[11] L. Todjihounde, A. Ngendakumana and J. Nzotungicimpaye, arXiv:math-ph/1102.0718.

[12] A. Galajinsky and I. Masterov, *Phys. Lett. B* 723, 190 (2013).

[13] K. Andrzejewski, A. Galajinsky, J. Gonera and I. Masterov, *Nucl. Phys. B* 885, 150 (2014).

[14] K. Andrzejewski, *Phys. Lett. B* 738, 405 (2014).

[15] J. Lukierski, P. C Stichel and W. J. Zakrzewski, *Ann. Phys. (N. Y.)* 260, 224 (1997).

[16] C. Duval and P. A. Horváthy, *Phys. Lett. B* 479, 284 (2000).

[17] R. Jackiw and V. P. Nair, *Phys. Lett. B* 480, 237 (2000).

[18] C. Duval and P. A. Horváthy, *J. Phys. A* 34, 10097 (2001).

[19] P. A. Horváthy, *Ann. Phys* 299, 128 (2002).

[20] P. A. Horváthy and M. S. Plyushchay, *J. High Energy Phys.* 2002, 033 (2002).

[21] P. A. Horváthy, L. Martina and P. C. Stichel, *Phys. Lett. B* 615, 87 (2005).

[22] S. Ghosh, *Phys. Lett. B* 638, 350 (2006).
[23] M. A. del Olmo and M. S. Plyushchay, *Ann. Phys. (N. Y.)* **321**, 2830 (2006).

[24] M. A. del Olmo, J. Negro and J. Tosiek, *J. Math. Phys.* **47**, 033508 (2006).

[25] P. A. Horváthy, L. Martina and P. C. Stichel, arXiv:hep-th/1002.4772.

[26] E. G Beltrametti and A, Blasi, *Phys. Lett.* **20**, 62 (1966).

[27] L. Abellanas and L. Martinez Alonso, *J. Math. Phys. (N.Y.)* **16**, 1580 (1975).

[28] J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus, *J. Math. Phys. (N.Y.)* **17**, 986 (1976).

[29] V. Boyko, J. Patera and R. Popovych, *J. Phys. A* **39**, 5749 (2006).

[30] L. Šnobl and P. Winternitz, *Classification and Identification of Lie Algebras*, CRM Monograph Series. American Mathematical Society, (2014)

[31] J. F. Cariñena, J. A. González, M. A. del Olmo and M. Santander, *Fortschr. Phys.* **38**, 681 (1990).

[32] W. Greiner, *Classical Mechanics: systems of particles and Hamiltonian dynamics*, Springer, Berlin Heidelberg 2009.

[33] M. Gadella, M. A. Martín, L. M. Nieto and M. A. del Olmo, *J. Math. Phys.*, **32**, 1182 (1991).

[34] A. Ballesteros, M. Gadella and M. A. del Olmo, *J. Math. Phys.* **33**, 3370 (1992).

[35] O. Arratia, M. A. Martín and M. A. del Olmo, arXiv:quant-ph/9611055.

[36] M. A Martín and M. A del Olmo, *J. Phys. A* **29**, 689 (1996).

[37] M. A. Lledó, *Int. J. Mod. Phys. B* **14**, 2397 (2000).

[38] R. Fioresi and M. A. Lledó, *Pacific J.Math.* **198**, 411 (2001).

[39] G. Dito and F. J. Turrubiates, *Physics Letters A*, **352** 309 (2006).