A COSMOLOGICAL THEORY WITHOUT SINGULARITIES

R. Brandenberger\textsuperscript{1)}, V. Mukhanov\textsuperscript{2,∗}) and A. Sornborger\textsuperscript{1)}

\textsuperscript{1)} Physics Department
Brown University
Providence, RI 02912, USA

\textsuperscript{2)} Institut für Theoretische Physik
ETH Zürich, Hönggerberg
CH–8093 Zürich, Switzerland

Abstract

A theory of gravitation is constructed in which all homogeneous and isotropic solutions are nonsingular, and in which all curvature invariants are bounded. All solutions for which curvature invariants approach their limiting values approach de Sitter space. The action for this theory is obtained by a higher derivative modification of Einstein’s theory. We expect that our model can easily be generalized to solve the singularity problem also for anisotropic cosmologies.

∗ On leave of absence from Institute for Nuclear Research, Academy of Sciences, 117 312 Moscow, Russia.
1. Introduction

One of the outstanding problems in the theory of gravitation (and more generally in the quest for a unified theory of all interactions) is the singularity problem. According to the Penrose–Hawking theorems, general relativity (GR) manifolds are, in general, geodesically incomplete, which is a sign that singularities in space–time occur.

Singularities are undesirable for a theory which claims to be complete, since their existence implies that space–time cannot be continued past them. The space–time structure becomes unpredictable already at the classical level.

Two important examples of singularities in GR are the initial and final singularities in a closed Universe and the singularity in the center of the black hole. In the former case, the singularity implies we cannot answer the question what will happen after the “big crunch”, or (in the case of an expanding Universe) what was before the “big bang”.

The presence of singularities is an indication that GR is an incomplete theory. Wheeler even talks about a “crisis in physics”. It is a widespread opinion that either quantum gravity or a more fundamental theory such as string theory will provide a cure for the “sickness” of GR. However, quantum gravity does not yet exist as a self–consistent non–perturbative theory. Neither does string theory exist as a unique theory capable of addressing the singularity problem of gravity in a definitive way, although interesting string–specific ideas have recently been put forwards.

Because of the absence of a completely developed fundamental theory on the basis of which we could address the singularity problem, we will use a rather different approach. Any fundamental theory will, in the region of low curvature, give an effective action for a four–dimensional space–time metric $g_{\mu\nu}$ which to lowest order must agree with the Einstein action. We will try to construct (guess) an effective action for $g_{\mu\nu}$ which solves the singularity problem and which in the low curvature limit reduces to the Einstein action. It is possible that in such a manner we will be able to discover important features of the future fundamental theory. We might also gain information which will help in finding this fundamental theory.

Before discussing the ideas behind our construction of the effective action for gravity, we return to the Penrose–Hawking theorems. They do not give us any detailed information about the nature of the singularity. However, in the two examples discussed above –
collapsing Universe and black hole – we know that at the singularity some of the physically measurable curvature invariants like $R$, $R_{\mu\nu} R^{\mu\nu}$, $C^2 = C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$ diverge (here, $R$ is the Ricci scalar, $R_{\mu\nu}$ the Ricci tensor, and $C_{\alpha\beta\gamma\delta}$ the Weyl tensor). It is reasonable to assume that the divergence of some curvature invariants at the singularity is a fairly general phenomenon. In fact, for singularities reached on timelike curves in a globally hyperbolic space–time it can be proved\(^1\) that the Riemann tensor becomes infinite. Hence, as a first step we will find a mechanism to bound all the curvature invariants.

Limitation principles play a very important role in physics. Special relativity includes as one of its fundamentals the principle that no particle velocity can exceed the speed of light. The cornerstone of quantum mechanics is the uncertainty principle which states that the second fundamental constant, Planck’s constant $\hbar$, gives the minimal phase space volume a particle can be localized in. The third fundamental constant, Newton’s gravitational constant $G$, has not yet been used in any limitation principle.

Thus it is natural to assume that there exists a fundamental length $\ell_p \sim (G\hbar c^{-3})^{1/2} \simeq 10^{-33}$ cm in nature (determined by $G$) such that there is no curvature corresponding to scales $\ell < \ell_p$. There are strong indications that this will in fact arise in quantum gravity\(^5\) or string theory\(^3\). From the existence of a fundamental length it follows by simple dimensional considerations that all curvature invariants are limited:

$$|R| \leq \ell_p^{-2}, \quad |R_{\mu\nu} R^{\mu\nu}| \leq \ell_p^{-4}, \quad |C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}| \leq \ell_p^{-8}, \ldots$$

(1.1)

To realize the idea of a fundamental length it is necessary to construct a theory in which all curvature invariants are bounded. Since there are an infinite number of curvature invariants and since bounds on low order invariants do not necessarily imply bounds on higher order invariants, it is a rather formidable task to construct such a theory.

Fortunately we can simplify the problem drastically by making use of the “limiting curvature hypothesis” (LCH) construction\(^6\), according to which one looks for a theory in which:

i) A finite number of invariants are bounded by an explicit construction (e.g. $|R| \leq \ell_p^{-2}$ and $|R_{\mu\nu} R^{\mu\nu}| \leq \ell_p^{-4}$).

ii) When these invariants take on their limiting values, any solution of the field equations reduces to a definite nonsingular solution (e.g. de Sitter space).
In this case it follows automatically that all curvature invariants are limited.

The LCH incorporates in a natural manner Penrose’s hypothesis\(^7\) that the Weyl tensor \(C\) should vanish at the beginning (and end) of the Universe. This follows since by the LCH the Universe near the big bang and big crunch is de Sitter and that \(C = 0\) for a de Sitter Universe.

A theory in which the LCH is realized has some attractive features, both for cosmology\(^6\) and for black holes\(^8\). In cosmology, the present homogeneous expanding Universe would have started out with a de Sitter phase. In this case, we would have some (maybe unusual) realization of the oscillating Universe scenario. Entropy considerations tell us that only for a perfectly homogeneous and isotropic Universe could we have perfect periodicity. In general, we must have a nontrivial realization. Including inhomogeneities, we might obtain a multi–Universe model in which one collapsing Universe splits into several de Sitter bounces.

For black holes, the LCH gives the attractive picture that inside of the horizon instead of a singularity at the center we would have a piece of a de Sitter Universe which could be the source of other Friedmann (baby) Universes (see Fig. 1). In this case, the difficult question\(^9\) concerning information loss when matter falls into a black hole have a natural answer: the information which is lost to the observer external to the Schwarzschild horizon is stored in the baby Universe. In addition, using this picture provides a good starting point to attack the issue of the final stage of an evaporating black hole, a problem which has recently been of high interest in the context of two–dimensional quantum gravity\(^10\).

In this paper we construct an effective action for gravity in which all homogeneous and isotropic solutions are nonsingular and at high curvature approach de Sitter space. (A brief summary of our work was published in Ref. 11.) In order to implement the LCH we proceed in analogy to a technique by which point particle velocities can be limited, thus achieving the transition between Newtonian mechanics and point particle motion in special relativity (SR) (see also Ref. 12). An extension of our construction to inhomogeneous cosmologies and to black hole metrics will be presented separately.\(^{13}\)

In the following section we present the general theory of how to implement the LCH. We obtain a fairly general effective action for gravity as a higher derivative modification of the Einstein action, specialize to the case of an isotropic, homogeneous Universe and derive the resulting equations of motion.
In Section 3 we analyze a simple model which yields a nonsingular Universe without limiting curvature. We discuss the effects of including spatial curvature (i.e. $k \neq 0$) and hydrodynamical matter. The analysis of the more complicated model with limiting curvature is given in Section 4. Section 5 contains conclusions and further discussion.

2. Theory

In order to realize the LCH and hence to avoid singularities it is necessary to abandon at least one of the key assumptions on which the Penrose–Hawking theorems are based. The two most important assumptions are:

i) The energy dominance condition, a simplified version of which appropriate for cosmology is $\epsilon > 0$ and $\epsilon + 3p \geq 0$, where $\epsilon$ and $p$ are matter energy density and pressure respectively.

ii) The Einstein equations are universally true.

There is no reason to believe that these assumptions will be valid at very high energies and curvatures. First of all, already in matter theories routinely studied by particle physicists, the energy dominance condition is not always true. For example, the effective equation of state for a homogeneous, slowly varying scalar field configuration with potential energy is $p \simeq -\epsilon$, thus violating the energy dominance condition. This matter evolution scenario is in fact the basis for the inflationary Universe$^{14}$.

Note, however, that inflationary Universe models do not cure the problem of the final singularity. There may be nonsingular solutions for a collapsing Universe filled with scalar field matter, but they are of measure zero. Rather, in this case the typical solutions have an effective equation of state $p = \epsilon$ (the kinetic term for the scalar field dominates), not $p = -\epsilon$, and hence have a final singularity. Our goal is to construct a theory in which all solutions are nonsingular.

Concerning the second key assumption of the Penrose–Hawking theorems, it is well known that the Einstein theory can only be the effective theory of gravity at low curvatures. Perturbative quantum gravity calculations$^{15}$, vacuum polarization effects of quantum matter fields in an external gravitational background$^{16}$, and also considerations based on string theory$^{17}$ all show that the effective equations for the gravitational field should be modified
at higher curvatures. In a perturbative analysis, the modifications take the form of higher
derivative terms which are usually important only at very high (Planck) curvatures. Hence,
provided the effective action approach is valid at all at high curvatures, this effective action
will certainly not be of pure Einstein form.

To summarize, there are two ways to modify the theory at high curvatures in order to
avoid singularities:

i) Modify the matter action by including terms which violate the energy dominance
   condition.

ii) Modify the gravitational field equations.

The first approach was explored in Ref. 18. However, the weakness of this approach
is the absence of a good physical motivation for the modification. In addition, it seems
impossible to avoid singularities associated with purely gravitational modes which do not
couple to matter.

The second approach is much better motivated since higher derivative correction terms
to the Einstein action are predicted by many theories\textsuperscript{15–17}. Hence, our starting point will
be to look for an effective action for gravity of the form

\[ S_g = -\frac{1}{16\pi G} \int F \left( R, R_{\mu\nu} R^{\mu\nu}, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \ldots \right) \sqrt{-g} \, d^4 x + \text{nonlocal terms} , \quad (2.1) \]

where the dots denote the dependence of \( F \) on other curvature invariants. At low curvatures,
the leading term in \( F \) is simply \( R \).

The action (2.1) can be viewed as the effective action of some fundamental theory such
as quantum gravity or string theory. In these theories we are at present unable to calculate
the nonperturbative effective action. Hence, as mentioned in the introduction, our approach
will be to construct (guess) an effective action of the form (2.1) to obtain a theory in which
all solutions are nonsingular.

To simplify the considerations, we shall neglect nonlocal terms. In our approch, this
is justifiable since if we are able to solve the singularity problem in a purely local theory,
we expect that the nonlocal terms (which are inevitable for example because of particle
production) will not drastically change the asymptotic behavior of our theory because of its
special properties (see Section 5).
Key to the analysis is the assumption about validity of the background field approximation for the gravitational field up to high curvatures. Such an approximation will only be justified if the quantum fluctuations around this metric are sufficiently small. If the gravitational field is asymptotically free at high curvatures (see Section 3), we can hope that this approach will be valid. As we shall see, there are features in our theory which indicate that this will really be the case.

For the moment we shall ignore matter (later we will show that the presence of matter does not change the solutions at high curvatures in an important way). Thus, our starting point is the effective action

\[
S_g = -\frac{1}{16\pi G} \int F \left( R, R_{\mu\nu} R^{\mu\nu}, C^2, \ldots \right) \sqrt{-g} \, d^4x. \tag{2.2}
\]

The usual Einstein theory in the absence of matter has only one solution – Minkowski space – for a homogeneous and isotropic Universe. Any non–Einstein theory of gravity gives rise to fourth (or higher) order equations of motion and hence to a large number of cosmological solutions. In general, the singularity problems of such a theory are much worse than in Einstein gravity. A simple example is \(R^2\) gravity

\[
F(R) = R + \alpha R^2 \tag{2.3}
\]

which is conformally equivalent\(^{19}\) to Einstein gravity plus scalar field matter and which hence has many isotropic singular solutions (even without matter). Thus, the theory we are looking for must be a very special higher derivative gravity model.

We wish to construct an effective action for gravity in which all homogeneous and isotropic solutions are nonsingular and in which all curvature invariants are limited (in Section 5 we will indicate how to extend our analysis to anisotropic models\(^{13}\)). To motivate our construction, it is useful to keep in mind ways of writing the action for two well-known physical theories in which certain physical quantities are bounded: special relativity and the Born–Infeld theory of electromagnetism\(^{20}\).

To impose bounds on physical quantities in an explicit manner, it is convenient to employ a Lagrange multiplier technique proposed by Altshuler\(^{12}\). To explain how this technique works, we first consider the simple example of point particle motion. We start with the
action for a nonrelativistic particle of mass \( m \) and world line \( x(t) \). We demonstrate how to explicitly implement the limitation on the particle velocity, and in particular how to obtain the action for point particle motion in special relativity. The nonrelativistic action with which we start is

\[
S_{\text{old}} = m \int dt \frac{1}{2} \dot{x}^2. \tag{2.4}
\]

In order to construct a new theory with bounded velocity, we introduce a “Lagrange multiplier field” \( \phi(t) \), which couples to some function of the quantity whose value we want to limit, and a potential \( V(\phi) \) for this field:

\[
S_{\text{new}} = m \int dt \left[ \frac{1}{2} x^2 + \phi \ddot{x}^2 - V(\phi) \right] \tag{2.5}
\]

Let us stress that \( \phi \) is not a dynamical field. Provided that \( \partial V/\partial \phi \) is bounded, the constraint equation (i.e. the variational equation with respect to \( \phi \)) ensures that \( \dot{x} \) is bounded. In order to obtain the correct Newtonian limit for small \( \dot{x} \) and small \( \phi \), \( V(\phi) \) must be proportional to \( \phi^2 \) as \( |\phi| \to 0 \). One of the simplest potentials which satisfies the above asymptotic conditions,

\[
V(\phi) = \frac{2\phi^2}{1 + 2\phi}, \tag{2.6}
\]

leads to special relativity. In fact, eliminating the Lagrange multiplier using the constraint equation and substituting the result into (2.5) yields (up to a constant term which does not effect the equations of motion) the relativistic point particle action

\[
S_{\text{new}} = m \int dt \sqrt{1 - \dot{x}^2}. \tag{2.7}
\]

Let us return to the theory of gravitation. In the notation of the above example, the “old” theory will be given by the Einstein action. In order to implement the LCH, we wish to impose restrictions on some curvature invariants \( I_1, I_2, \ldots, I_n \) in an explicit manner. The general form of a higher derivative local modification of the Einstein action involving the invariants \( I_1, \ldots, I_n \) is

\[
S_g = -\frac{1}{16\pi G} \int [R + F(I_1, I_2, \ldots, I_n)] \sqrt{-g} d^4x, \tag{2.8}
\]

where \( F \) is some function of the invariants \( I_1, \ldots, I_n \).
By introducing Lagrange multiplier fields $\phi_1(t), \ldots, \phi_n(t)$, the above action can be rewritten as
\[
S_g = -\frac{1}{16\pi G} \int [R + \phi_1 f_1(I_1) + \ldots + \phi_n f_n(I_n) + V(\phi_1 \ldots \phi_n)] \sqrt{-g} d^4x, \tag{2.9}
\]
where $f_i(I_i)$ are functions we can choose as we want. The actions (2.8) and (2.9) are equivalent provided that the potential $V(\phi_1, \ldots, \phi_n)$ satisfies the following partial differential equation
\[
-\sum_{i=1}^n \phi_i \frac{\partial V}{\partial \phi_i} + V(\phi_1 \ldots \phi_n) = F\left( f_1^{-1}\left( \frac{\partial V}{\partial \phi_1} \right), \ldots, f_n^{-1}\left( \frac{\partial V}{\partial \phi_n} \right) \right). \tag{2.10}
\]
This follows immediately by using the constraint equations for (2.9)
\[
f_i(I_i) = \frac{\partial V}{\partial \phi_i} \quad i = 1, \ldots, n. \tag{2.11}
\]

We see from the constraint equations (2.11) that by appropriate choice of the functions $f_i$ and $V$ we can implement bounds on the invariants $I_1, \ldots, I_n$. Variation of the action (2.9) with respect to $g_{\mu\nu}$ yields the field equations.

First, we try to construct the simplest theory in which the LCH is realized. At least for simple models (like the isotropic Universe) it is natural to choose as one of the invariants
\[
I_1 = R - \sqrt{3} (4R_{\mu\nu} R^{\mu\nu} - R^2)^{1/2} \tag{2.12}
\]
since for a homogeneous, spatially flat Universe it is equal to $12H^2$. This invariant will be used to limit the curvature by some (e.g. Planckian) value. The second invariant $I_2$ we will take on such a form as to implement in the theory the condition that in the asymptotic regions all of the solutions evolve to de Sitter. The simplest way to do this is to pick $I_2$ such that $I_2 = 0$ only for de Sitter space (Minkowski space is included as a special case), and to make sure that
\[
I_2 \to 0 \quad \text{as} \quad |\phi_2| \to \infty. \tag{2.13}
\]

For homogeneous and isotropic space–times, it can be shown that
\[
I_2 = 4R_{\mu\nu} R^{\mu\nu} - R^2 \tag{2.14}
\]
is a good choice, since $I_2 = 0$ only for de Sitter space. Note that in general, $I_2$ is positive.
semidefinite. However, for inhomogeneous and anisotropic space–times (e.g. when $C^2 \neq 0$), the above form of $I_2$ is insufficient to single out de Sitter space as an asymptotic solution. This is obvious from considering the Schwarzschild metric for which $I_2 = 0$. Hence, in the general case we should add to (2.14) terms which depend on $C^2$ and vanish for conformally flat space–times.

However, for a homogeneous and isotropic Universe it is (as we will show) sufficient to consider the action in the following general form:

$$S_g = -\frac{1}{16\pi G} \int \left[ R + \phi_1 f_1 (I_1) + \phi_2 f_2 (I_2) + V(\phi_1, \phi_2) \right] \sqrt{-g} \, d^4 x.$$  \hspace{1cm} (2.15)

The variational field equations which follow from (2.15) are

$$R^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta R - \frac{1}{2} \delta^\alpha_\beta V = \left( \phi_1 \frac{\partial f_1}{\partial I_1} \right)_\beta^\alpha - \left( \phi_1 \frac{\partial f_1}{\partial I_1} \right)_\sigma^\sigma - \phi_1 \frac{\partial f_1}{\partial I_1} R^\alpha_\beta - \frac{1}{2} \phi_1 f_1 \delta^\alpha_\beta - 4 \left( \phi_2 \frac{\partial f_2}{\partial I_2} R^\sigma_\tau \right)_\tau^\sigma \delta^\alpha_\beta - 4 \left( \phi_2 \frac{\partial f_2}{\partial I_2} R^\alpha_\beta \right)_\tau^\tau - 8 \phi_2 \frac{\partial f_2}{\partial I_2} R^\alpha_\beta R^\gamma_\gamma \delta^\alpha_\beta$$

\hspace{1cm} + 2 \phi_2 \frac{\partial f_2}{\partial I_2} R R^\alpha_\beta + \frac{1}{2} \delta^\alpha_\beta \phi_2 f_2,

and the constraint equations are

$$f_1(I_1) = -\frac{\partial V}{\partial \phi_1}$$

\hspace{1cm} (2.16)

$$f_2(I_2) = -\frac{\partial V}{\partial \phi_2}.$$  \hspace{1cm} (2.17)

We will simplify the theory further by assuming a factorizable potential

$$V(\phi_1, \phi_2) = V_1(\phi_1) + V_2(\phi_2).$$  \hspace{1cm} (2.18)

The asymptotic conditions on the potentials $V_1$ and $V_2$ follow from demanding that the theory reduces to the Einstein theory at small curvatures, and that the LCH is realized. The
first condition yields
\[ V_i(\phi_i) \sim \phi_i^2 \quad |\phi_i| \ll 1; \quad i = 1, 2. \]  
(2.19)

In order to limit \( R \) explicitly we can try a potential which to leading order takes the form
\[ V_1(\phi_1) \sim \phi_1 \quad |\phi_1| \gg 1, \]  
(2.20)

and to obtain de Sitter solutions in the asymptotic regions we need a potential which at large \( \phi_2 \) increases less quickly than \( \phi_2 \). We assume an asymptotic form:
\[ V_2(\phi_2) \sim \text{const} \quad |\phi_2| \gg 1. \]  
(2.21)

In this case, provided \( f_2(I_2) \to 0 \) as \( I_2 \to 0 \), the constraint equation (2.17) implies that \( I_2 \to 0 \) as \( |\phi_2| \to \infty \), and we have a chance of realizing the LCH, provided that the evolution of the scalar fields \( \phi_1 \) and \( \phi_2 \) is appropriate, a question which needs detailed investigation.

To conclude this section we will write down equations (2.16) and (2.17) explicitly for a homogeneous and isotropic metric with scale factor \( a(t) \) in the contracting phase (i.e. \( H < 0 \)):
\[ ds^2 = dt^2 - a^2(t) \left( \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \]  
(2.22)

We choose simple functions \( f_1 \) and \( f_2 \):
\[ f_1(I_1) = I_1 \]
\[ f_2(I_2) = -\sqrt{I_2} \]  
(2.23)

Thus, our final action takes the form
\[ S_g = -\frac{1}{16\pi G} \int \left[ (1 + \phi_1)R - (\phi_2 + \sqrt{3} \phi_1) \sqrt{4R_{\mu\nu}R^{\mu\nu} - R^2} + V_1(\phi_1) + V_2(\phi_2) \right] \sqrt{-g} d^4x. \]  
(2.24)

As is well known from the derivation of the Friedmann–Robertson–Walker equations in Einstein gravity, the only independent equation of motion is the \( 0 - 0 \) equation. In our case, we have in addition the constraint equations (2.17). The full set of equations can be obtained by inserting the metric (2.22) into (2.16) and (2.17).
The resulting $\phi_1, \phi_2$ and $0-0$ equations are

\[ H^2 + \frac{k}{a^2} = \frac{1}{12} V_1' \]  

(2.25)

\[ \dot{H} - \frac{k}{a^2} = -\frac{1}{\sqrt{12}} V_2' \]  

(2.26)

\[- \frac{1}{2}(V_1 + V_2) + 3H^2(1 - 2\phi_1) + 3 \frac{k}{a^2}(4\phi_1 + 1) = \sqrt{3} H \left( \dot{\phi}_2 + 3H\phi_2 - \frac{k}{Ha^2}\phi_2 \right)\]  

(2.27)

Another way to obtain the same equations is to substitute the ansatz (2.22) with $g_{00} = N(t)^2$ into the action (2.24) and to vary with respect to $N, \phi_1$ and $\phi_2$ (see e.g. Ref. 21). Adding to the system matter with action

\[ S_m = \int L_m \sqrt{-g} \, d^4x \]  

(2.28)

where $L_m$ is the matter Lagrangian, only leads to an additional term

\[ \frac{8\pi}{3} G \rho_m \]  

(2.29)

on the left hand side of the $0-0$ equation.

In the following sections, we shall show that all solutions of the above equations are free of singularities.

3. Nonsingular Universe without Limiting Curvature

Since our goal is primarily to construct a nonsingular Universe model and only secondarily to limit the curvature, we first consider a simple model in which the $\phi_1$ field is absent. In this case, it is easier to discuss our techniques of analysis.

We will show that for this model all solutions for a collapsing Universe are nonsingular and asymptotically approach de Sitter solutions. However, there is no general (i.e. solution independent) bound on the effective cosmological constant of the de Sitter period.
In this section we set $\phi_2 \equiv \phi$ and $V_2 \equiv V$. The equations of motion are given by (2.26) and (2.27). Let us first consider a spatially flat ($k = 0$) collapsing model without matter. In this case, the equations of motion are

$$\dot{H} = -\frac{1}{2\sqrt{3}} V', \quad (3.1)$$

$$\dot{\phi} = -3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V. \quad (3.2)$$

The phase space of this model is the two dimensional $(\phi, H)$ plane. The phase space trajectories can be understood by considering $dH/d\phi$ (determined immediately from (3.1) and (3.2))

$$\frac{dH}{d\phi} = -\frac{V'}{\sqrt{12}} \left(-3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V\right)^{-1}. \quad (3.3)$$

From (3.3), it follows that provided that $V(\phi)$ is bounded at large $\phi$ (as postulated in (2.21)), then as $\phi$ tends to infinity, $H$ approaches a finite value, i.e. for any solution, the effective cosmological constant in the large $\phi$ region is bounded. In this case, it follows from (3.2) that in a collapsing Universe, for large $\phi$

$$\phi(t) \sim e^{3|H|t}. \quad (3.4)$$

Our choice of invariant $I_2$ has led to the conclusion that the asymptotic de Sitter solutions are attractor solutions. This conclusion holds independent of the specific choice of the potential $V(\phi)$, as long as the asymptotic condition (2.21) is satisfied.

From (3.4) it follows that all solutions for a contracting Universe are free of singularities. It takes infinite time to reach $\phi = \infty$.

To concretize the consideration, we consider a simple potential which satisfies the asymptotic conditions (2.19) and (2.21):

$$V = \sqrt{12} H_0^2 \frac{\phi^2}{1 + \phi^2}, \quad (3.5)$$

where $H_0$ is a constant (in the model with limiting curvature discussed in Section 4, $H_0$ sets the scale of this limiting curvature).
The phase space trajectories \((\phi(t), H(t))\) in a collapsing Universe are shown in Fig. 2. The numerical results were obtained using the specific potential (3.5). However, as discussed above, the main features of the diagram depend only on the asymptotic properties.

First we note that there is only one singular point \((\dot{\phi} = \dot{H} = 0)\) in the phase plane. This point is

\[
(\phi, H) = (0, 0)
\]

and corresponds to Minkowski space–time.

There are two classes of trajectories which are asymptotically de Sitter. Those starting at large positive values of \(\phi\) go off to \(\phi = \infty\), reaching their asymptotic value of \(H\) from above (i.e. \(\dot{H} < 0\)). Those starting with large negative values of \(\phi\) tend to \(\phi = -\infty\) with \(\dot{H} > 0\).

For small values of \(H\) and \(\phi\) we can use the asymptotic condition (2.19) on \(V(\phi)\) to conclude that there are periodic solutions about Minkowski space. In this limit, the basic equations (3.1) and (3.2) become

\[
\dot{H} = -\frac{1}{2\sqrt{3}} \frac{\partial V}{\partial \phi} \simeq -2H_0^2\phi
\]

\[
\dot{\phi} \simeq \frac{1}{\sqrt{3}} \frac{3H^2 - \frac{1}{2} V}{H} \simeq H_0 \frac{\sqrt{3}(H/H_0)^2 - \phi^2}{H/H_0},
\]

where for \(V(\phi)\) we have inserted the general asymptotic form

\[
V(\phi) \simeq 2\sqrt{3} H_0^2 \phi^2,
\]

valid for small \(\phi\). The numerical factor \(2\sqrt{3}\) has been inserted to eliminate numerical constants in the following equations.

It is convenient to introduce a rescaled time

\[
\tau \equiv H_0 t
\]

and a dimensionless measure of \(H\):

\[
y \equiv H/H_0
\]
With \(d/d\tau\) denoted by a prime, equations (3.7) and (3.8) become

\[
y' = -2\phi \\
\phi' = \frac{\sqrt{3} y^2 - \phi^2}{y}
\] (3.12)

To see the oscillatory nature of the solutions, we introduce radial and angular coordinates \(r\) and \(\psi\)

\[
\phi = r \sin \psi \\
y = -3^{-1/4} r (1 - \cos \psi).
\] (3.13)

The resulting equations for \(r\) and \(\psi\) are

\[
\psi' = \omega \\
r' = 0.
\] (3.14)

where the frequency is \(\omega = 2 \cdot 3^{1/4}\). The corresponding solutions oscillate with frequency given by \(H_0\) (which we expect to be Planck scale) about Minkowski space.

Based on the preceding discussion of asymptotic solutions we see that there is a separatrix in phase space dividing solutions which tend to \(\phi = \infty\) from those which oscillate or tend to \(\phi = -\infty\). We observe that for large \(|H|\), the separatrix will asymptotically (and from the right hand side on Fig. 2) approach the line of turning points given by \(d\phi/dH = 0\). From (3.1), it follows that for large \(|H|\) the turning points lie at

\[
\phi \simeq \frac{1}{\sqrt{3}}.
\] (3.15)

For small values of \(\phi\) and \(H\), the separatrix is well to the right of the line of turning points given by

\[
\phi \simeq 3^{1/4} \frac{|H|}{H_0}.
\] (3.16)

The above analysis of the phase space trajectories is an indication that in our theory, Minkowski space is stable towards homogeneous perturbations. As long as the initial values of \(|\dot{H}|\), \(|\dot{\phi}|\) and \(\dot{\phi}/|\dot{H}|\) are small, a solution starting close to Minkowski space will remain close for all times. The issue of stability of Minkowski space towards inhomogeneous perturbations is an important unsolved problem.
We stress again that all the general features of the phase space analysis are true for any potential \( V(\phi) \) which satisfies the required asymptotic conditions (2.19) and (2.21). However, the results depend crucially on the choice of the invariant \( I_2 \).

Next, we include hydrodynamical matter with energy density

\[
\rho_m(t) = c a(t)^{-n}, \tag{3.17}
\]

where \( n = 3 \) for dust and \( n = 4 \) for radiation. For the moment, we keep to a collapsing spatially flat model. In this case, equation (3.1) is unchanged while equation (3.2) becomes

\[
\dot{\phi} = -3H\phi + \sqrt{3H} - \frac{1}{2\sqrt{3H}} V - \frac{8\pi G}{\sqrt{3H}} c a(t)^{-n}. \tag{3.18}
\]

With matter, phase space is three dimensional – the third dimension being \( a(t) \). In Fig. 3, we show the projection of some of the trajectories onto the \((\phi(t), H(t))\) plane, for potential \( V(\phi) \) given by (3.5). All trajectories have \( 8\pi G c = 1 \) and \( a(t_0) = 10, t_0 \) being the initial time. The main impression is that the trajectories look very similar to those without matter in the asymptotic region. We shall now explain why this is the case.

First, we note that as \( |\phi| \to \infty \), the solutions approach de Sitter space since \( \dot{H} \to 0 \). Hence,

\[
a(t) \simeq e^{-|H|(t-t_0)} a(t_0). \tag{3.19}
\]

Next, we combine (3.1) and (3.18) to obtain for \(|\phi| >> 1\)

\[
\frac{dH}{d\phi} \simeq \frac{V'}{2\sqrt{3}} \left( 3H\phi + \frac{8\pi G c}{\sqrt{3H}} a(t)^{-n} \right)^{-1}. \tag{3.20}
\]

Our model incorporates a very important feature: in the asymptotic de Sitter region matter does not have an important effect on the geometry. The effective gravitational constant which describes the influence of matter on the geometry goes to zero as space-time approaches de Sitter space. In this sense the model is asymptotically free.
Some understanding of asymptotic freedom can be obtained by solving the $\phi$ and $H$ equations of motion (3.1) and (3.18) in the asymptotic region $|\phi| >> 1$. Equation (3.18) becomes

$$\dot{\phi} \simeq 3|H|\phi + \frac{c}{|H|}a(t_0)^{-n}e^{n|H|(t-t_0)}$$

(3.21)

(where we have incorporated the factor $8\pi G/\sqrt{3}$ into the definition of $c$). From (3.21) it follows that $\phi(t)$ is a linear combination of the homogeneous solution (3.4) and (assuming that $H \simeq$ const) the inhomogeneous contribution $\phi_I(t)$

$$\phi_I(t) = \frac{c}{n|H|^2}a(t_0)^{-n}\left(e^{n|H|(t-t_0)} - 1\right)$$

(3.22)

For dust ($n = 3$), both the homogeneous and inhomogeneous terms grow at the same rate, and the coefficient of the inhomogeneous term is smaller. Hence, matter does not effect even the time dependence of the phase space trajectories. For radiation ($n = 4$), $\phi_I(t)$ grows faster than (3.4). At sufficiently late times, therefore, it will dominate. In this period, however, we can (for potential (3.5)) solve the $H$ equation (3.6) to obtain

$$H(t) \simeq H(t_1) - \frac{2H_0^2}{3n|H|} \left(\frac{\sqrt{3n|H|^2}}{c}\right)^3 a(t_0)^{3n}e^{-3n|H|(t_1-t_0)} ,$$

(3.23)

(where $t_1$ is some time $\gg t_0$ well into the asymptotic region) which shows that the presence of matter does not effect the final value of the curvature when starting the evolution in the asymptotic region.

For small $|\phi|$, the presence of matter does have a significant effect on the phase space trajectories. As $a(t_0)$ decreases (or, equivalently, $c$ and thus the matter energy density increase), the distortions of the trajectories increase, as can be seen by comparing Figs. 3 and 4. Figure 4 corresponds to a matter energy density which is ten times larger.

Finally, we consider the effects of spatial curvature. In this case, equations (3.1) and (3.2) generalize to (see (2.26) and (2.27))

$$\dot{H} = -\frac{1}{2\sqrt{3}}V' + \frac{k}{a^2}$$

(3.24)

$$\dot{\phi} = -3H\phi + \frac{k}{Ha^2}\phi - \frac{1}{2\sqrt{3H}}V + \sqrt{3H} + \sqrt{3}\frac{k}{Ha^2} - \frac{c}{Ha^n} ,$$

(3.25)

where the constant $c$ is as in Eq. (3.21).
In the case of the potential (3.5) and for \( c = 0 \), some resulting phase space trajectories projected onto the \((\phi/H)\) plane are shown in Figs. 5 and 6. For the trajectories of Fig. 6, the initial value of \( a(t) \) was chosen to be ten times smaller than in Fig. 5. Hence, the effects of curvature are more pronounced.

Consider a sample trajectory of Fig. 5. It starts out with large initial value of \( a \). The trajectory tends towards \(|\phi| >> 1\) and \( \dot{H} \to 0 \), as in the case \( k = 0 \). Since \( a(t) \) is now decreasing almost exponentially, the role of curvature increases. At a critical value of \( \phi \), the value of \( \dot{H} \) becomes 0. This will occur when

\[
\frac{1}{2\sqrt{3}} V'(\phi(t)) = \frac{k}{a^2(t)} . \tag{3.26}
\]

Hence, the smaller the initial value of \( a(t) \), the earlier (3.25) will be satisfied (compare Figs. 5 and 6). At a similar time, the curvature terms also start to dominate in Eq. (3.24). Therefore, as is obvious from the \( k \) dependent terms in (3.24), \( \phi(t) \) will rapidly decrease, as will \(|H(t)|\). At some finite and negative value of \( \phi \), \( H(t) \) vanishes. Thereafter, the Universe reexpands. The evolution of this model for small \( a(t) \) resembles a de Sitter bounce.

Note that all solutions are nonsingular. In particular, the solutions can be integrated through the point when \( H = 0 \) (when terms on the right hand side of (3.29) become infinite).

In conclusion, we have constructed a higher derivative modification of Einstein’s theory in which all homogeneous and isotropic solutions are nonsingular. Without curvature (i.e. for \( k = 0 \)), the solutions either are periodic about Minkowski space or else converge to a \( k = 0 \) de Sitter solution. For \( k \neq 0 \) the solutions which do not remain close to Minkowski space go through a de Sitter bounce and are future extendible to \( t = \infty \). In addition, we have shown that our model is asymptotically free in the sense that the effective coupling of matter to gravity goes to zero as \( \alpha \) increases.
4. Nonsingular Universe with Limiting Curvature

Now we turn to the discussion of the full model in which the LCH is implemented, the model given by the action (2.24), in which for a homogeneous and isotropic metric the equations of motion reduce to (2.25 – 2.27). We include hydrodynamical matter with energy density given by (3.17).

In the general case \((k \neq 0 \text{ and } c \neq 0)\), the phase space of the model is three dimensional: \(\phi_1(t), \phi_2(t)\) and \(a(t)\). For \(k = 0\) and \(c = 0\), the dependence on \(a(t)\) drops out and the phase space can be reduced to the two dimensional \(\phi_1/\phi_2\) diagram. The first order equations of motion in phase space are found by combining equations (2.25 – 2.27). To derive the equation for \(\phi_1(t)\), we differentiate (2.25) with respect to \(t\) and use (2.26) to substitute for \(\dot{H}\) to obtain

\[
\dot{\phi}_1 = -4\sqrt{3} \frac{H V'_1}{V''_1}. \tag{4.1}
\]

The equation of motion for \(\phi_2\) is (2.27):

\[
\dot{\phi}_2 = -3H\phi_2 + \frac{k}{Ha^2}\phi_2 + \frac{1}{\sqrt{3}H} \left(3H^2(1 - 2\phi_1) + 3\frac{k}{a^2}(4\phi_1 - 1) - \frac{c}{2} (V_1 + V_2) - \frac{c}{a^n}\right) \tag{4.2}
\]

where \(H\) can be expressed in terms of \(\phi_1\) and \(a\) via (2.25). From (4.1), (4.2) and (2.25) it is obvious that for \(k = c = 0\) the \(a(t)\) dependence disappears.

In the case \(k = c = 0\) we may use (2.25) to get

\[
\frac{d\phi_2}{d\phi_1} = -\frac{V''_1}{4V'_1} \left[-\sqrt{3}\phi_2 + (1 - 2\phi_1) - \frac{2}{V'_1}(V_1 + V_2)\right], \tag{4.3}
\]

the key equation for the following phase space analysis.

For all potentials \(V_1(\phi_1)\) and \(V_2(\phi_2)\) with asymptotical behavior

\[
V_i \propto \phi_i^2 \quad \phi_i \ll 1 \tag{4.4}
\]

and

\[
V_1 \propto \phi_1 - \ln \phi_1 + O\left(\frac{1}{\phi_1}\right) \quad \phi_1 \gg 1 \tag{4.5}
\]

\[
V_2 \propto \text{const} + O\left(\frac{1}{\phi_2}\right) \quad \phi_2 \gg 1 \tag{4.6}
\]

the phase diagrams have the same features as depicted schematically in Fig. 8 for spatially
collapsing Universes without matter. The numerical solutions depicted in Fig. 7 were obtained for the particular choice of potentials (which satisfy the asymptotic conditions of (4.4 - 4.6))

\[ V_1(\phi_1) = 12H_0^2 \frac{\phi_1^2}{1 + \phi_1} \left( 1 - \frac{\ell n(1 + \phi_1)}{1 + \phi_1} \right) \]  

(4.7)

\[ V_2(\phi_2) = 2\sqrt{3}H_0^2 \frac{\phi_2^2}{1 + \phi_2^2}. \]  

(4.8)

The presence of the logarithmic term in (4.4) will be justified shortly.

We can identify four classes of trajectories. Note that by (2.26), \(|\phi_2| \to \infty\) implies that the evolution approaches de Sitter space. The first class of trajectories starts in the de Sitter phase at \(\phi_2 \to -\infty\) and evolves to de Sitter at \(\phi_2 \to \infty\). For small initial values of \(\phi_1\), trajectories starting at \(\phi_2 = -\infty\) reach a turning point and return to \(\phi_2 = -\infty\). The third class of trajectories are periodic solutions about Minkowski space–time \((\phi_1 = \phi_2 = 0)\). Finally, trajectories starting with small \(\phi_1\) and \(\phi_1/\phi_2\) with \(\phi_2\) positive evolve towards de Sitter solutions at \(\phi_2 = \infty\). There are two separatrices dividing phase space into regions corresponding to the four above classes (see Fig. 8).

Note that in order to prevent solutions starting with \(\phi_1 >> 1\) and \(\phi_2 \simeq 0\) from escaping to \(\phi_1 = \infty\) at \(\phi_2 < 1\) in finite time - such solutions would violate the LCH and would lead to singularities in higher order curvature invariants - it was necessary to add the logarithmic correction term to \(V_1(\phi_1)\).

Phase space is the half plane \(\phi_1 \geq 0\). Negative values of \(\phi_1\) are unphysical since by (2.25), and using the small \(\phi_1\) asymptotic form of \(V_1(\phi_1)\), they would correspond to imaginary values for \(H(t)\). This half plane can be divided into four regions: in Region A, \(\phi_1 >> 1\) and \(|\phi_2| >> 1\), in Region B, \(\phi_1 >> 1\) and \(|\phi_2| << 1\), in Region C, \(\phi_1 << 1\) and \(|\phi_2| << 1\) and in Region D, \(\phi_1 << 1\) and \(|\phi_2| >> 1\). We will analyze the phase space trajectories in each of the above regions, focusing on three features: the asymptotic expressions for \(d\phi_2/d\phi_1\) (which give the tangent vectors to the trajectories), the separatrices, and the equations for the trajectories. To concretize the discussion, we use the potentials (4.7) and (4.8). However, except in Region B, the asymptotical solutions are independent of the specific choice of potentials.
In Region A, Equation (4.3) becomes

\[
\frac{d\phi_2}{d\phi_1} \simeq \frac{\sqrt{3}}{4} \frac{\phi_1^3}{\phi_2^4} \left[ \sqrt{3 + 4 \phi_1} + 4 \phi_2 \right].
\] (4.9)

The direction of the tangent vectors is sketched in Fig. 8. Arrows indicate the direction of increasing time and are obtained by inspecting (4.1) and (4.2) directly. By inspecting the tangent vectors it is clear that all solutions in the upper region $\phi_2 > 1$ quickly approach de Sitter space ($|\phi_2| \to \infty$ implies de Sitter space). In the lower region $\phi_2 < -1$ there are two domains separated by a separatrix which for $\phi_1 \gg 1$ and $|\phi_2| \gg 1$ is close to the line of turning points where $d\phi_2/d\phi_1 = 0$, its equation being given by

\[
\frac{\phi_2}{\phi_1} = -\frac{4}{\sqrt{3}}.
\] (4.10)

(see Fig. 8). To the right of the separatrix, trajectories correspond to solutions starting out in a de Sitter phase. To the left of the line given by (4.10), we have $d\phi_2/d\phi_1 > 0$ and trajectories go off to de Sitter space at $\phi_2 \to -\infty$. In all cases, de Sitter space is reached at finite $\phi_1$ values. This is seen by explicitly integrating (4.9). In the region where the first term on the right hand side of (4.9) dominates we have

\[
\phi_1 \simeq c - \frac{2}{9} \phi_2^{-3},
\] (4.11)

while in the domain where the second term dominates the approximate solution is

\[
\phi_1 \simeq \frac{1}{\frac{9}{4} \phi_2^3 + c},
\] (4.12)

($c$ is a constant of integration). Note that all of the solutions starting in Region A start in de Sitter space and end up in de Sitter space.

In Region B, the tangents in phase space are given by

\[
\frac{d\phi_2}{d\phi_1} \simeq \sqrt{3} \frac{1}{\phi_1 \phi_2},
\] (4.13)

which integrates to

\[
\phi_1 = c \exp \left\{ \frac{1}{2 \sqrt{3}} \phi_2^2 \right\}.
\] (4.14)

The tangent vectors are again sketched in Fig. 8. From (4.14) it follows that trajectories leave Region B at a finite value of $\phi_1$. They enter Region A and hence asymptotically approach de Sitter space.
In Region C, Equation (4.3) becomes
\[
\frac{d\phi_2}{d\phi_1} \simeq -\frac{\sqrt{3}}{2\phi_2} \left[ 1 - 3\phi_1 - \frac{1}{\sqrt{12}} \phi_2^2 \right].
\] (4.15)

The separatrix in the upper half planes is close to the line of turning points \(d\phi_2/d\phi_1 = 0\) for large values of \(\phi_i\):
\[
\phi_2 \simeq \pm 12^{1/4} \phi_1^{1/2} (1 - 3\phi_1)^{1/2}.
\] (4.16)

Where the first term dominates, the trajectories obey
\[
\phi_1 \simeq -\frac{1}{\sqrt{3}} \phi_2^2 + c.
\] (4.17)

From the sketch of Fig. 8 it is clear that the trajectories which pass through \(\phi_1 = \phi_2 = 0\) with \(\dot{\phi}_2/\dot{\phi}_1(\phi_1 = \phi_2 = 0)\) not too large correspond to periodic motion about Minkowski space. This – as in the model of Section 3 – is an indication that Minkowski space is stable in our theory towards homogeneous perturbations.

Finally, in Region D the equation for the tangent vector is
\[
\frac{d\phi_2}{d\phi_1} \simeq +\frac{\sqrt{3}}{2} \phi_2^3 \left[ \sqrt{3}\phi_2 + \frac{1}{2\sqrt{3}\phi_1} \right].
\] (4.18)

There is a separatrix which is (for large \(\phi_i\)) approximately described by
\[
\phi_2 \simeq -\frac{1}{6\phi_1}.
\] (4.19)

To the right of this line, the trajectories are given by
\[
\phi_1 \simeq c - \frac{2}{9} \phi_2^{-3},
\] (4.20)

to the left by
\[
\phi_1 \simeq ce^{-2\phi_2^2}.
\] (4.21)

In conclusion, all solutions are either periodic about Minkowski space or are asymptotically de Sitter. All solutions can be extended to \(t = \pm \infty\), and hence there are no singularities.
As in the model of Section 3, including matter does not effect the asymptotic solutions. The coupling between matter and gravity is asymptotically free also in the theory with action (2.24). However, including matter changes the nature of solutions starting near Minkowski space. These solutions now approach de Sitter space (see Fig. 9). This result is not surprising, since also in Einstein gravity, Minkowski space is not a solution of the field equations in the presence of matter.

The projection of some phase space trajectories onto the \((\phi_1, \phi_2)\) plane in a model with \(k \neq 0\) but \(c = 0\) are shown in Fig. 10. Like in the single field model of Section 3, the trajectories initially evolve as for \(k = 0\) towards de Sitter space. Hence, for finite \(\phi_1, \phi_2\) becomes very large. Eventually, however, the curvature terms become important, \(\phi_2\) reaches a turning point and rapidly (within time period \(H_0^{-1}\)) relaxes to zero (for finite value of \(\phi_1\)). As is obvious from Fig. 11, the rapid decrease in \(\phi_2\) corresponds to the de Sitter bounce during which \(H\) changes sign.

5. Conclusions and Discussion

We have constructed a theory of gravity in which all homogeneous and isotropic solutions (not only special solutions as in some other models\(^{23}\)) are nonsingular, regardless of the matter content of the Universe. Our effective action for gravity contains higher derivative terms which modify the Einstein action at high curvatures. Such terms are expected to be important near the Planck curvature in any fundamental theory such as quantum gravity or string theory.

Most higher derivative gravity theories have much worse singularity properties than Einstein gravity. We use a particular construction based on implementing the “Limiting Curvature Hypothesis” to obtain a class of models without singularities. We discussed two models, one in which all curvature invariants are bounded and all solutions except those periodic about Minkowski space asymptotically approach de Sitter space (Section 4), and a simpler model without limiting curvature (Section 3).

The theory presented in this paper is “asymptotically free” in the sense that the coupling of matter to gravity goes to zero as the curvature approaches its limiting value (similar features have been discussed by Linde\(^{24}\) under the name “gravitational confinement”).
When applied to an expanding Universe, our theory implies that it started out in a de Sitter phase with scale factor \( a(t) = e^{Ht} \) (for \( k = 0 \)) or else (for \( k = 1 \)) it emerged from a de Sitter bounce. In particular, there was a period of inflation driven by gravity. This is no surprise as it is well known\(^{25}\) that higher derivative gravity theories often produce inflation.

Note that the property of asymptotic freedom might also justify using the effective action approach to gravity until Planck curvatures. Asymptotic freedom will also play an important role in controlling nonlocal terms. For example, nonlocal terms due to particle production may be expected to vanish in the asymptotic regions of phase space.

Our action is constructed by adding two Lagrange multiplier terms (and their corresponding potentials) to the Einstein action. Each Lagrange multiplier is coupled to a curvature invariant. The role of the first Lagrange multiplier is to limit the curvature, the role of the second one (\( \phi_2 \)) is to force space–time to be de Sitter at large curvature. For a homogeneous and isotropic model, it was sufficient to couple \( \phi_2 \) to the invariant \( I_2 = 4R_{\mu\nu}R^{\mu\nu} - R^2 \), since in this case \( I_2 = 0 \) singles out de Sitter space.

However, for an anisotropic cosmology, we must extend the invariant \( I_2 \) by including a term which effects the anisotropy. In a subsequent paper\(^{13}\) (see also Ref. 26) we show that \( I_2 = +R_{\mu\nu}R^{\mu\nu} - R^2 + C^2 \) is an appropriate invariant. This invariant also works for a spherically symmetric metric. Thus, in a model like the one presented here, but with the new \( I_2 \), we are able to show that there will be no singularities inside the black hole horizon\(^{13}\).

Open questions include the generalization of our model to general inhomogeneous metrics, and a full stability analysis.

**Acknowledgement**

For useful discussions we are grateful to B. Altshuler, A. Linde, M. Markov, M. Mohazzab and M. Trodden. This work has been supported in part by the U.S. Department of Energy under grant No. DE-AC02-76-ER03130 (Task A), by an Alfred P. Sloan Foundation fellowship to R.B., and by the Swiss National Science Foundation (V.M.).
REFERENCES

1. R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965);
   S. Hawking, *Proc. R. Soc. London*, **A300**, 182 (1967).

2. C. Misner, K. Thorne and J. Wheeler, *Gravitation* (Freeman, New York, 1972).

3. R. Brandenberger and C. Vafa, *Nucl. Phys.* **B316**, 391 (1989);
   A. Tseytlin and C. Vafa, *Nucl. Phys.* **B372**, 443 (1992);
   G. Veneziano, *Phys. Lett.* **B265**, 287 (1991);
   M. Gasperini and G. Veneziano, *Phys. Lett.* **B277**, 256 (1992);
   M. Gasperini and G. Veneziano, “Pre–Big-Bang in String Cosmology”, CERN preprint TH6572/92 (1992).

4. C. Clarke, *Comm. Math. Phys.* **41**, 65 (1975).

5. J. Narlikar and T. Padmanabhan, *Ann. Phys. (N.Y.)* **150**, 289 (1983).

6. M. Markov, *Pis’ma Zh. Eksp. Teor. Fiz.* **36**, 214 (1982);
   M. Markov, *Pis’ma Zh. Eksp. Teor. Fiz* **46**, 431 (1987);
   V. Ginsburg, V. Mukhanov and V. Frolov, *Zh. Eksp. Teor. Fiz.* **94**, 1 (1988).

7. R. Penrose, in “General Relativity: An Einstein Centenary”, ed. by S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).

8. V. Frolov, M. Markov and V. Mukhanov, *Phys. Lett.* **B216**, 272 (1989);
   V. Frolov, M. Markov and V. Mukhanov, *Phys. Rev.* **D41**, 383 (1990);
   D. Morgan, *Phys. Rev.* **D43**, 3144 (1991).

9. S. Hawking, *Phys. Rev.* **D14**, 2460 (1976).

10. C. Callen, S. Giddings, J. Harvey and A. Strominger, *Phys. Rev.* **D45**, R1005 (1992).

11. V. Mukhanov and R. Brandenberger, *Phys. Rev. Lett.* **68**, 1969 (1992).

12. B. Altshuler, *Class. Quant. Grav.* **7**, 189 (1990).

13. R. Brandenberger, M. Mohazzab, V. Mukhanov, A. Sornborger and M. Trodden, in preparation (1993).

14. A. Guth, *Phys. Rev.* **D23**, 347 (1981).
15. G. ’t Hooft and M. Veltman, *Ann. Inst. H. Poincaré* **20**, 69 (1974);  
   A. Barvinsky and G. Vilkovisky, *Phys. Rep.* **119**, 1 (1985).

16. N. Birrell and P. Davies, “Quantum Fields in Curved Space”, (Cambridge Univ. Press, Cambridge, 1992).

17. J. Scherk and J. Schwarz, *Nucl. Phys.* **B81**, 118 (1974);  
   T. Yoneya, *Prog. Theor. Phys.* **51** 1907 (1974);  
   C. Lovelace, *Phys. Lett.* **135B**, 75 (1984);  
   E. Fradkin and A. Tseytlin, *Nucl. Phys.* **B261**, 1 (1985).

18. M. Markov and V. Mukhanov, *Nuovo Cimento* **86B**, 97 (1985).

19. B. Whitt, *Phys. Lett.* **B145** 176 (1984).

20. M. Born and L. Infeld, *Proc. R. Soc. London* **A144**, 425 (1934).

21. R. Brandenberger, R. Kahn and W. Press, *Phys. Rev.* **D28**, 1809 (1983).

22. see e.g. V. Arnold, *Ordinary Differential Equations* (MIT Press, Cambridge, 1973).

23. T. Ruzmaikina and A. Ruzmaikin, *JETP Lett.* **30**, 372 (1970);  
   H. Nariai and K. Tomita, *Prog. Theor. Phys.* **46**, 776 (1971);  
   M. Giesswein and E. Steernwitz, *Acta Phys. Austriaca* **41**, 41 (1975);  
   V. Gurovich, *Zh. Eksp. Teor. Fiz.* **73**, 369 (1977);  
   J. Barrow and A. Ottewill, *J. Phys.* **A16**, 2757 (1983).

24. A. Linde, *Rep. Prog. Phys.* **47**, 925 (1984).

25. A. Starobinski, *Phys. Lett.* **B91**, 99 (1980).

26. B. Altshuler, in “Proc. of lst Int. A.D. Sakharov Conference”, Moscow, May 1991 (NOVA Science Publishing, Inc., New York, 1992).
Figure Captions

Fig. 1: Penrose diagram of an eternal black hole in Einstein gravity (a) and in the nonsingular Universe theory (b). The singularities (S) are replaced by de Sitter phases (dS) which couple to Friedmann Universes (FRW). The horizons (H) are not affected.

Fig. 2: Phase-space diagram \((\phi, H)\) - arrows indicating the direction of time evolution - for a spatially flat Universe without limiting curvature and with no matter \((k = c = 0)\). Generated using the potential (3.5).

Fig. 3: A projection onto the \((\phi, H)\) plane of the three dimensional phase-space diagram \((\phi, H, a)\) - arrows again indicating direction of time evolution - for a spatially flat Universe without limiting curvature but with matter \((k = 0, c \neq 0)\). Generated using the potential (3.5) with initial condition \(a(t_0) = 10\).

Fig. 4: Phase-space diagram as in Fig. 3, but with \(a(t_0) = 1\). Therefore, the initial matter energy density is larger than for the trajectories of Fig. 3.

Fig. 5: A projection onto the \((\phi, H)\) plane of the three dimensional phase-space diagram \((\phi, H, a)\) in a closed \((k = 1)\) Universe without limiting curvature and in the absence of matter \((c = 0)\). Potential (3.5) was used, and \(a(t_0) = 10\) was chosen as initial condition.

Fig. 6: Same as in Fig. 5, but with initial condition \(a(t_0) = 1\). Notice the different scales on the axes.

Fig. 7: Phase-space diagram for the spatially flat \((k = 0)\) Universe with limiting curvature based on the potentials (4.7) and (4.8). There is no matter \((c = 0)\).

Fig. 8: Sketch of the generic phase-space diagram for a two field model with \(k = c = 0\) and potentials satisfying the asymptotic conditions (4.4) - (4.6). Lines with arrows indicate phase space trajectories, arrows pointing in the direction of increasing time. Separatrices are shown as dashed lines. With \(A, B, C\) and \(D\) we denote the asymptotic regions of phase space discussed in the text.

Fig. 9: A projection onto the \((\phi_1, \phi_2)\) plane of the three dimensional phase-space diagram \((\phi_1, \phi_2, a)\), for a two field model which is spatially flat but contains matter. The potentials used are (4.7) and (4.8).

Fig. 10: The same for a two field model without matter but including spatial curvature \((k \neq 0)\).
Fig. 11: Trajectories in the $(I_2, H)$ plane for the same model as in Fig. 10.