Hopf Algebraic Structures in Proving Perturbative Unitarity

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Abstract

The coproduct of a Feynman diagram is set up through identifying the perturbative unitarity of the S-matrix with the cutting equation from the cutting rules. On the one hand, it includes all partitions of the vertex set of the Feynman diagram and leads to the circling rules for the largest time equation. Its antipode is the conjugation of the Feynman diagram. On the other hand, it is regarded as the integration of incoming and outgoing particles over the on-shell momentum space. This causes the cutting rules for the cutting equation. Its antipode is an advanced function vanishing in retarded regions. Both types of coproduct are well-defined for a renormalized Feynman diagram since they are compatible with the Connes–Kreimer Hopf algebra.

Key Words: Hopf algebra, Cutting rules
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1 Introduction

The present quantum field theory such as the standard model provides a theoretical description of particle physics which explains so far known high energy experiments [1]. But it has to appeal to regularization and renormalization schemes to extract finite physical quantities from divergent Feynman diagrams. The renormalizability has been one basic principle for physical interesting quantum field theories. It is satisfied in non-abelian gauge field theories by engaging the spontaneously symmetry breaking mechanism [2]. To uncover what is behind successful regularization and renormalization schemes, “some fundamental change in our ideas” is needed as Dirac argued [3].

Recent developments suggest that revisiting quantum field theory from Hopf algebraic points is one possible way out. The axioms of the Hopf algebra are listed in the appendix. The R-operation recipe in the BPH renormalization [4, 5] always yields local counter terms cancelling divergences of Feynman diagrams. It has been found to lead to the Hopf algebra of rooted trees, the Connes–Kreimer Hopf algebra [6, 7]. The Zimmermann forest formula in the BPHZ renormalization [4, 5, 8] represents the twisted antipode of the Connes–Kreimer Hopf algebra. The renormalized Feynman integral has the form of convolution between the twisted antipode and the Feynman rules, as relates the renormalization theory to the Birkhoff decomposition and the Riemann-Hilbert problem [7, 9]. Besides the above, the Wick normal ordering gives a coproduct with Laplace pairs so that the Wick theorem has a Hopf algebraic origin [10, 11].

The perturbative unitarity of the S-matrix is the perturbative realization of the unitarity of the S-matrix. It will be shown to have a Hopf algebraic structure. In terms of Feynman diagrams, it represents a diagrammatic equation, which is recognized as the cutting equation [12] derived from the cutting rules [13]. The point identifies the cutting propagator as the integration of incoming and outgoing particles over the on-shell momentum space. This shows that there exists a tensor product between Feynman diagrams and also specifies a coproduct of the Feynman propagator. The Hopf algebra of a Feynman diagram with oriented external lines is set up by solving the Hopf algebraic axioms.

The Hopf algebraic structures in the cutting rules has been introduced [14]. The cutting equation is an integral version of the largest time equation [12]. The latter one is derived by the circling rules [12, 15, 16] which assign an integral to a circled diagram with some vertexes encircled. The set of circled diagrams is obtained by partitioning the vertex set of a Feynman diagram. It is well known that all possible partitions of a set form a Hopf algebra. The largest time equation has a form of the convolution between the Feynman rules and the antipode representing the conjugation of a Feynman diagram.

With the constraint of the energy conservation at every vertex, the set of circled diagrams is reduced to the set of admissible cut diagrams for the cutting rules. This says that the Hopf algebra in the cutting rules is obtained by reducing the Hopf algebra of partitions but is explained in a different way. The coproduct
includes all admissible cuts of a Feynman diagram. The antipode is an advanced function vanishing in retarded region and so it does not contradict the causality principle.

The coproducts proposed above commute with the coproduct of the Connes–Kreimer Hopf algebra so they are well-defined for a renormalized Feynman diagram. The paper is organized as follows. In the second section, through comparing the diagrammatic equation for the perturbative unitarity with the cutting equation, the coproduct of a Feynman diagram is specified. In the third section, the Hopf algebraic structures in the circling rules and the cutting rules are constructed respectively. The compatibility to the Connes–Kreimer Hopf algebra is considered. In the last section, the universal coproduct of a Feynman diagram (a set of Feynman diagrams) is discussed and further research topics are suggested. In the appendix, the axioms of the Hopf algebra are listed.

2 Coproduct in proving the perturbative unitarity

The equation of Feynman diagrams realizing the perturbative unitarity is found to be the cutting equation derived by the cutting rules [12] in scalar field theory.

2.1 The perturbative unitarity of the S-matrix

In \( \varphi^4 \) model, the interaction Lagrangian density \( \mathcal{L}_{\text{int}} \) takes \( -\frac{1}{4!}\varphi^4 \), \( \lambda \) being coupling constant. Via the U-matrix approach, the S-matrix operator is given by the time-ordering product \( T \exp i \int d^4x \mathcal{L}_{\text{int}} \) [1]. Its hermitian \( S^\dagger \) satisfies the unitarity equation

\[
S S^\dagger = S^\dagger S = 1.
\]

In terms of T-matrix, \( S = 1 + iT \), the unitarity equation has a formalism of matrix entries,

\[
\langle f | iT | i \rangle + \langle f | (iT)^\dagger | i \rangle = -\int d^3m \langle f | T | m \rangle \langle m | T^\dagger | i \rangle,
\]

(1)

where \( | i \rangle \) and \( \langle f | \) respectively denote the incoming state \( | p_1, p_2, \ldots p_I \rangle \) and the outgoing state \( \langle q_1, q_2 \ldots q_F | \), the symbols \( p, q \) representing on-shell external momenta, and \( | m \rangle, \langle m | \) denote intermediate on-shell states.

In perturbative quantum field theory, the unitarity equation (1) leads to equations of Feynman diagrams. For the matrix entry \( \langle f | iT | i \rangle \), apply the Feynman rules, while for \( \langle f | (iT)^\dagger | i \rangle \), apply the conjugation rules to conjugation diagrams in order to avoid treating the conjugation operator \( T^\dagger \). A conjugation diagram is a Feynman diagram except that all vertexes are encircled. The conjugation rules assign the factor of \( \int d^4x (i\lambda) \) to the vertex \( x \) and the conjugation propagator \( \Delta_F^* \), the complex conjugation of \( \Delta_F \), to each internal line.

With connected Feynman diagrams, the unitarity equation (1) shows in a diagrammatic form
Figure 1. The unitarity equation of Feynman diagrams.

As a convention, incoming external lines are on the left hand side of the diagram and outgoing external lines on the right hand side so that the positive energy always flows from the left to the right. The dashed line distinguishes the Feynman diagram on its left hand side from the conjugation diagram on its right hand side.

2.2 Coproduct in proving the perturbative unitarity

The perturbative unitarity is proved by identifying the unitarity equation with the cutting equation [12, 15, 16]. For scalar field theories, the cutting equation [16] has a diagrammatic form,

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig2a}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig2b}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Here the dashed line represents the cut line through a diagram and the symbol “a.c.” means admissible cutting. The cutting rules and the cutting equation will be introduced in the next section.

The right part of Fig.2 has to be recognized to be the right part of Fig.1. Hence the cutting propagator \( \Delta_+ \) is required to be decomposed into an integration over the phase space of an incoming particle and outgoing particle,

\[
\Delta_+(x-y) = \int d^3k \frac{e^{-iky}}{\sqrt{(2\pi)^d 2\omega_k}} \frac{e^{ikx}}{\sqrt{(2\pi)^d 2\omega_k}}
\]

(2)

where \( \omega_k = \sqrt{k^2 + m^2} \) and \( m \) denotes mass of particle. The positive (negative) cutting propagators \( \Delta_+ (\Delta_-) \) takes the form

\[
\Delta_+(x-y) = \Delta_-(y-x) = \int \frac{d^4k}{(2\pi)^4} \theta(k_0) 2\pi \delta(k^2 + m^2) e^{ik(x-y)},
\]

(3)

which are combined into the Feynman propagator \( \Delta_F \),

\[
\Delta_F(x-y) = \theta(x^0 - y^0) \Delta_+(x-y) + \theta(y^0 - x^0) \Delta_-(x-y),
\]

(4)
θ function being the normal step function. The four propagators: $\Delta_F, \Delta^*_F, \Delta_+$ and $\Delta_-$ satisfy

$$(\Delta_F + \Delta^*_F - \Delta_+ - \Delta_-(x - y)) = 0$$

which is an example of the largest time equation [12, 15, 16].

Through comparing Fig.1 with Fig 2, we observe a tensor product structure in the proof for the perturbative unitarity. The equation (2) is regarded as a multiplication $m$ representing the integration over the tensor product of Feynman diagrams,

$$m(\quad \otimes \quad ) = \quad$$

Figure 3. Tensor product in proving the perturbative unitarity.

In terms of such tensor products, we construct a coproduct of the Feynman propagator $\Delta_F(x - y)$,

$$\Delta(\quad \otimes \quad ) = \quad \otimes e + e \otimes \quad + \quad \otimes \quad + \quad \otimes \quad$$

Figure 4. The coproduct of the Feynman propagator.

The symbol $e$ denotes the unit of Hopf algebra or the empty set $\emptyset$. With the character $\Phi$ representing the Feynman rules and the character $\Phi_c$ representing the conjugation rules, the largest time equation (5) has an algebraic formulation $m(\Phi \otimes \Phi_c)\Delta(\Delta_F) = 0$, namely the convolution $\Phi \ast \Phi_c(\Delta_F)$ between two characters $\Phi$ and $\Phi_c$.

## 3 Hopf algebraic structures in the cutting rules

With the above coproduct of the Feynman propagator, two types of Hopf algebras are set up. The first one reflects the circling rules [12] for the largest time equation. The second one is related to the cutting rules for the cutting equation [12]. Both survive renormalization at least the dimensional regularization and the minimal subtraction.

### 3.1 The Hopf algebra in the circling rules

Motivated by the simplest largest time equation (5) of the Feynman propagator, we are going to construct the largest time equation for an arbitrary $N$-vertex connected Feynman diagram $\Gamma$ in terms of $\Delta_\pm, \Delta_F$ and $\Delta^*_F$. With the observation that the coproduct in Fig.4 is regarded as a complete partition of the vertex set $\{x, y\}$, the circling rules [12] is devised and applied to a set of circled diagrams.

The Feynman diagram $\Gamma$ is specified by both the set $\mathcal{N}_N$ of its vertexes, $\mathcal{N}_N = \{x_1, x_2, \cdots, x_N\}$ and the set of all lines connecting vertexes $x_i$. It corresponds to $2^N$ circled diagrams with some vertexes encircled. The case of no vertexes encircled is $\Gamma$ itself and the case of all vertexes encircled is its conjugation diagram $\Gamma^c$. 
The circling rules map a circled diagram to an integral formalism. For a vertex \( x \), assign the factor of \( \int d^4 x (i\lambda) \); for an internal line connecting two circled vertexes, assign the conjugation propagator \( \Delta^* \); for an internal line connecting a circled vertex \( x_i \) to a uncircled vertex \( x_j \), assign the positive cutting propagator \( \Delta^+ (x_i - x_j) \), namely for an internal line connecting a uncircled vertex \( x_i \) to a circled vertex \( x_j \), assigns the negative cutting propagator \( \Delta^- (x_i - x_j) \); for other ingredients of a circled diagram, apply the Feynman rules. As an example of applying the circling rules, a two-loop four-point Feynman diagram \( \Gamma_{\text{Fig.5}} \) in \( -\frac{1}{4!} \phi^4 \) model has its largest time equation

\[
\begin{align*}
\begin{tikzpicture}[baseline=0, scale=0.7]

\draw[thick] (-0.5,0) -- (0.5,0);
\draw[thick] (-0.5,0.5) -- (0.5,0.5);
\draw[thick] (-0.5,0.5) arc (90:270:0.5);
\draw[thick] (0.5,0.5) arc (270:90:0.5);
\end{tikzpicture}
\end{align*}
\]

Figure 5. An example for the largest time equation.

In the following, a Hopf algebra in the circling rules is set up. \( H \) is a set generated by all connected Feynman diagrams with oriented external lines. \( F \) is a field representing the complex number \( \mathbb{C} \) with unit 1. The addition \( + \) is defined by the linear combination \( a \Gamma_1 + b \Gamma_2 \in H, a, b \in \mathbb{C}, \Gamma_1, \Gamma_2 \in H \) from which the triple \( (H, +; F) \) is a vector space. The multiplication \( m \) of two Feynman diagrams \( \Gamma_1 \) and \( \Gamma_2 \) is specified by their disjoint union \( \Gamma_1 \Gamma_2 = m(\Gamma_1 \otimes \Gamma_2) := \Gamma_1 \cup \Gamma_2 \). \( \eta \) is the unit map \( \eta(1) = e \) and specifies the empty set \( \emptyset \) as the unit \( e \). With these definitions, the associativity axiom (1) and the unit axiom (2) are satisfied.

For a Feynman diagram \( \Gamma \), the coproduct \( \Delta \) involves its subdiagrams and reduced subdiagrams. The subdiagram \( \gamma(\mathcal{V}_c) \) is specified by both the subset \( \mathcal{V}_c \) of \( \mathcal{V}_N \) and the set of all lines connecting vertexes in \( \mathcal{V}_c \). The reduced subdiagram \( \Gamma/\gamma \) is obtained by cutting the subdiagram \( \gamma \) out of \( \Gamma \). Hence the coproduct \( \Delta \) is defined by \( \Delta(\Gamma) = \sum_{\mathcal{P}} \gamma(\mathcal{V}_c) \otimes \Gamma/\gamma \), where \( \mathcal{P} \) denotes all possible partitions of the vertex set \( \mathcal{V}_N \). It expands as

\[
\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum_{1 \leq c < N} \gamma(\mathcal{V}_c) \otimes \Gamma/\gamma,
\]

where the summation is over all subdiagrams except \( \emptyset \) and \( \Gamma \). External lines of \( \gamma(\mathcal{V}_c) \) from cut internal lines are outgoing, while external lines of \( \Gamma/\gamma \) from cut internal lines are incoming. As an example, the coproduct of an one-loop four-point Feynman diagram \( \Gamma_{\text{Fig.6}} \) in the \( -\frac{1}{4!} \phi^4 \) model has the diagrammatic representation

\[
\begin{align*}
\begin{tikzpicture}[baseline=0, scale=0.7]

\draw[thick] (-0.5,0) -- (0.5,0);
\draw[thick] (-0.5,0.5) -- (0.5,0.5);
\draw[thick] (-0.5,0.5) arc (90:270:0.5);
\draw[thick] (0.5,0.5) arc (270:90:0.5);
\end{tikzpicture}
\end{align*}
\]

Figure 6. An example for the coproduct in the circling rules.

The counit \( \epsilon \) is chosen to satisfy \( \epsilon(e) = 1; \epsilon(\Gamma) = 0 \), if \( \Gamma \neq e \). With our choices of the coproduct and counit, the coassociativity axiom (5) and the counit axiom (6) are ensured. Obviously, the process of dividing the vertex set and further
dividing the vertex set of the subdiagram is equivalent to that of dividing the vertex set and further dividing the vertex set of the reduced subdiagram. At last, solving the equation representing the antipode axiom (8), the antipode $S$ is obtained to be $S(\Gamma) = -\Gamma - \sum_{1 \leq c < N} S(\gamma(V_c)) \Gamma / \gamma$ with $S(e) = e$.

In terms of the Feynman rules $\Phi$ and the conjugation rules $\Phi_c$, the largest time equation of $\Gamma$ has a form of convolution $\Phi * \Phi_c(\Gamma) = \Phi_c * \Phi(\Gamma) = 0$. As an example, the coproduct of the Feynman diagram $\Gamma_{\text{Fig.6}}$ leads to the largest time equation

$$\Phi * \Phi_c(\Gamma_{\text{Fig.6}}) = \Delta_2^+ + \Delta_2^+ - \Delta_2^- - \Delta_2^- = 0. \quad (7)$$

We go further to calculate the antipode $S(\Delta_F)$ of the Feynman propagator,

$$S(\frac{x}{y}) = -\frac{x}{y} + \frac{x}{y} + \frac{x}{y}$$

Figure 7. The antipode of the Feynman propagator.

With the simplest largest time equation (5), $S(\Delta_F)$ is the conjugation propagator $\Delta_F^*$. This suggests that the antipode $S(\Phi(\Gamma))$ of the Feynman diagram $\Gamma$ is obtained by applying the conjugation rules to its conjugation diagram $\Gamma^*$, namely, $S(\Phi(\Gamma)) = \Phi_c(\Gamma^*)$. Therefore the largest time equation has a typical Hopf algebraic form $S * \Phi(\Gamma) = \Phi * S(\Gamma) = 0$.

### 3.2 The Hopf algebra in the cutting rules

Circled diagrams with a vertex connecting other vertexes by the positive (negative) cutting propagator have vanishing Feynman integrals. At such a vertex, the conservation of energy is violated since $\Delta_+ (\Delta_-)$ only admits the positive (negative) energy flow. An admissible cut diagram is suitable to represent a non-vanishing circled diagram. For a connected Feynman diagram $\Gamma$, make a cut line through its internal or external lines to separate it into two parts. The left part denotes the admissible cut diagram $\gamma$ with at least one incoming external line, while the right part denotes the admissible reduced cut diagram $\Gamma / \gamma$ with at least one outgoing external line. The cases of only cutting incoming or outgoing lines are included.

The cutting rules [12, 15, 16] assign a Feynman integral to a cut diagram. Apply the Feynman rules to $\gamma$ and the conjugation rules to $\Gamma / \gamma$; assign $\Delta_+(x-y)$ to a cut internal line with the vertex $x$ in $\gamma$ and the vertex $y$ in $\Gamma / \gamma$. As an example, the cutting equation of the Feynman diagram $\Gamma_{\text{Fig.5}}$ at $x^0_1 < x^0_2, x^0_3$ has a diagrammatic form,

$$x_2 \quad x_2 \quad x_2 \quad x_2 \quad x_2 \quad x_2 = 0$$

Figure 8. An example for the cutting equation at $x^0_1 < x^0_2, x^0_3$.  

7
The Hopf algebra in the cutting rules is regarded as a reduced version of the Hopf algebra in the circling rules by imposing the conservation of energy at each vertex of a Feynman diagram. The set of admissible cut diagrams of the Feynman diagram $\Gamma$ specifies the coproduct

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum_{\text{a.c.}} \gamma \otimes \Gamma/\gamma,$$

where the symbol “a.c.” means admissible cutting and the summation excludes cases of only cutting external lines. As an example, the coproduct of the Feynman diagram $\Gamma_{\text{Fig.6}}$ at $y^0 < x^0$ has a diagrammatic form

$$\Delta\left( \begin{array}{c} x \\ y \end{array} \right) = x \otimes y \otimes e + e \otimes x \otimes y + y \otimes x$$

Figure 9. An example for the coproduct in the cutting rules.

In terms of the characters $\Phi$ and $\Phi_c$, the cutting equation has a coalgebraic form $m(\Phi \otimes \Phi_c)\Delta(\Gamma) = 0$. Such a coproduct satisfies the coassociativity axiom (3). Cutting $\Gamma$ twice leads to the set of all possible admissible cut diagrams $\gamma_1 \otimes \gamma_2 \otimes \gamma_3$ which is irrelevant to the order of two cuttings. For example, the Feynman diagram $\Gamma_{\text{Fig.5}}$ has a coassociative product, as is verified by

$$(\Delta \otimes \text{Id})\Delta\left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = (\text{Id} \otimes \Delta)\Delta\left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right)$$

Figure 10. An example for the coassociative coproduct in the cutting rules.

As the above, a coalgebraic structure has been set up. The bialgebraic axiom (5) and (6) have to be verified. To understand the axiom (5), study the Feynman diagram $\Gamma_{\text{Fig.5}}$ as an example

$$\Delta\left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \Delta\left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) \cdot \Delta\left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$$

Figure 11. An example for the bialgebraic axiom.

In the case that the multiplication between two Feynman diagrams is the disjoint union, the diagrammatic equation in Fig. 11 is regarded as a definition of the coproduct. In the other case that the multiplication represents the integration over the phase space of incoming and outgoing particles, it survives Feynman integrals. Considering Feynman integrands, its right hand side has more terms than its left hand side. With the choices of the coproduct and counit, the axiom (6) is easily checked to be satisfied.

Solving the antipode axiom (7), the antipode of the Feynman diagram $\Gamma$ in the cutting rules has the form $S(\Gamma) = -\Gamma - \sum_{\text{a.c.}} S(\gamma) \Gamma/\gamma$ with $S(e) = e$. The antipode of the Feynman propagator $\Delta_F$ takes the form
Figure 12. The antipode of the Feynman propagator at $x^0 > y^0$.

Its formalism of Feynman integrands is given by

$$S(\Delta_F(x)) = \theta(-x^0)(\Delta_+(x) - \Delta_-(x))$$

with $y = 0$, which vanishes in the retarded region. As a generalization, the antipode in the cutting rules is an advanced function. Calculating the antipode of $\Gamma$Fig.5 at $x_0^1 < x_0^2 < x_0^3$ to obtain

$$S(\Gamma) = (\Delta_+(x_{21})\Delta_+(x_{31}) - \Delta_F(x_{21})\Delta_F(x_{31}))\Delta^2_F(x_{23})$$

$$+ (\Delta_F(x_{21}) - \Delta_+(x_{21}))\Delta_+(x_{31})\Delta^2_-(x_{23})$$

where $x_{21} = x_2 - x_1, x_{31} = x_3 - x_1$ and $x_{23} = x_2 - x_3$. It vanishes at the retarded region $x_1^0 < x_2^0 < x_3^0$. Hence the Hopf algebraic structure in the cutting rules cooperates the perturbative unitarity of the S-matrix with its causality.

### 3.3 The Hopf algebraic structures under renormalization

For a divergent Feynman diagram $\Gamma$, the renormalized largest time equation and the renormalized cutting equation can be set up under the dimensional regularization and the minimal subtraction. They also have Hopf algebraic representations similar to the preceding constructions but involving the Connes–Kreimer Hopf algebra [6, 7, 9] denoting the BPH renormalization [4, 5].

The Connes–Kreimer Hopf algebra is defined in the space of 1PI Feynman diagrams. The product of two Feynman diagrams is their disjoint union. The coproduct of a divergent Feynman diagram $\Gamma$ expresses all possible disjoint unions of its divergent subdiagrams

$$\Delta_{CK}(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum \gamma \otimes \Gamma/\gamma.$$  \hspace{1cm} (11)

where $\gamma \subset \Gamma$. The antipode is obtained by solving the antipode axiom (7). The global counter term for $\Gamma$ is given by the twisted antipode

$$S_R(\Gamma) = -R(\Phi(\Gamma)) - R \sum S_R(\gamma) \Phi(\Gamma/\gamma)$$ \hspace{1cm} (12)

where $R$ denotes the minimal subtraction in the dimensional renormalization. The convolution $S_R \ast \Phi(\Gamma)$ leads a renormalized Feynman integral corresponding to the bare one $\Phi(\Gamma)$.

The tensor product in the circling rules reflects the partition of the vertex set of the Feynman diagram $\Gamma$ and in the cutting rules it shows that the cutting propagator decomposes into the integration over incoming and outgoing external lines, while in the Connes–Kreimer Hopf algebra it disentangles overlapping divergences or reduces nested divergences. That is to say that the coproducts in the
circling rules and in the cutting rules are compatible with the Connes–Kreimer Hopf algebra,
\[ \Delta(S_R \star \Phi(\Gamma)) = ((S_R \star \Phi) \otimes (S_R \star \Phi))\Delta(\Gamma). \] (13)

The term on the left hand side expands
\[ \Delta(S_R \star \Phi(\Gamma)) = \Delta(\Phi(\Gamma)) + \Delta(S_R(\Gamma)) + \sum S_R(\gamma)\Delta(\Phi(\Gamma/\gamma)) \] (14)
where the global counter term \( S_R(\Gamma) \) plays as a vertex and counter terms \( S_R(\gamma) \) for subdivergences act as coefficients. On the right hand side, the unit \( e \) or undivergent subdiagrams \( \gamma \) satisfy \( S_R \star \Phi(e) = e, S_R \star \Phi(\gamma) = \gamma \). As an example, the renormalized Feynman diagram \( \Gamma_{\text{Fig.6}} \) has a coproduct
\[ \Delta(x \bigotimes y + \bullet) = (x \bigotimes y + \bullet) \otimes e + e \otimes (x \bigotimes y + \bullet) + y \bigotimes x \]
Figure 13. An example for the coproduct of a renormalized Feynman diagram.

For the third term on the right hand side, the integration over the phase space of on-shell particles being finite ensures that the renormalized cutting equation is well-defined.

4 Concluding Remarks

For a Feynman diagram \( \Gamma \), there is a universal coproduct
\[ \Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum \gamma \otimes \Gamma/\gamma \] (15)
where \( \gamma \) denotes the subdiagram of \( \Gamma \) and the summation is over all nontrivial subdiagrams. When \( \gamma \) is specified by a subset of the vertex set of \( \Gamma \), the coproduct represents the circling rules for the largest time equation and the antipode denotes the conjugation diagram \( \Gamma^\ast \). When \( \gamma \) denotes an admissible cut diagram, the coproduct leads to the cutting rules for the cutting equation and the antipode is an advanced function vanishing in retarded regions. When \( \gamma \) is required to be the disjoint union of divergent subdiagrams, it represents the coproduct of the Connes–Kreimer Hopf algebra [6, 7, 9] and its twisted antipode is the Zimmermann’s forest formula [8].

The Hopf algebraic structures in proving the perturbative unitarity have been considered in scalar field theories. They are expected to be found in other field theories [16]. In fermionic field theories, the same Hopf algebraic structures will be obtained. But in gauge field theories, there are some subtle. The unphysical degrees of freedom such as ghosts and time-like component of gauge potential are involved in the cutting equation. They have to be removed by applying the Ward identities or the Slavnov–Taylor identities [15, 16, 17]. The meaningful coproduct has to be defined for the set of Feynman diagrams. Therefore in the universal coproduct, \( \Gamma \) may take the summation of \( \Gamma_i \), namely \( \Gamma = \sum_i \Gamma_i \).
Besides the renormalizability and unitarity, physically interesting quantum field theories have to satisfy the causality principle. With the admissible cut diagrams, the dispersion relation representing the causality takes the form of two-largest time equation [15, 16]. The Hopf algebraic structures similar to the case of the perturbative unitarity are expected to be found. In addition, the dispersion equation can be used to prove the power counting theorem or the locality of counter terms. The Hopf algebraic structures may play the important role since the Connes–Kreimer Hopf algebra solves the locality of counter terms.

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A The axioms of the Hopf algebra

Let \((H, +, m, \eta, \Delta, \epsilon, S; F)\) be a Hopf algebra over \(F\). We denote the set by \(H\), the field by \(F\), the addition by +, the product by \(m\), the unit map by \(\eta\), the coproduct by \(\Delta\), the counit by \(\epsilon\), the antipode by \(S\), the identity map by \(\text{Id}\), and the tensor product by \(\otimes\). The Hopf algebra has to satisfy the following seven axioms:

1. \(m(m \otimes \text{Id}) = m(\text{Id} \otimes m)\), \(m : H \otimes H \rightarrow H\), \(m(a \otimes b) = ab\), \(a, b \in H\);
2. \(m(\text{Id} \otimes \eta) = \text{Id} = m(\eta \otimes \text{Id})\), \(\eta : F \rightarrow H\) which denote the associative product \(m\) and the linear unit map \(\eta\) in the algebra \((H, +, m, \eta; F)\) over \(F\) respectively;
3. \((\Delta \otimes \text{Id}) \Delta = (\text{Id} \otimes \Delta) \Delta\), \(\Delta : H \rightarrow H \otimes H\), \(\Delta(e) = \text{Id}\) which denote the coassociative coproduct \(\Delta\) and the linear counit map \(\epsilon\) in the coalgebra \((H, +, \Delta, \epsilon; F)\) over \(F\) and where we have used \(F \otimes H = H\) or \(H \otimes F = H\);
4. \(\epsilon(ab) = \epsilon(a) \epsilon(b)\), \(\Delta(\epsilon) = \epsilon \otimes \epsilon\), \(\epsilon(e) = 1\), \(a, b, e \in H\) which are the compatibility conditions between the algebra and the coalgebra in the bialgebra \((H, +, m, \eta, \Delta, \epsilon; F)\) over \(F\), claiming that the coproduct \(\Delta\) and the counit \(\epsilon\) are homomorphisms of the algebra \((H, +, m, \eta; F)\) over \(F\) with the unit \(e\);
5. \(m(S \otimes \text{Id}) \Delta = \eta \circ \epsilon = m(\text{Id} \otimes S) \Delta\) which is the antipode axiom that can be used to define the antipode. The character \(\Phi\) is a nonzero linear functional over the algebra and is a homomorphism satisfying \(\Phi(ab) = \Phi(a) \Phi(b)\). The convolution between two characters \(\Phi\) and \(\Phi_c\) is defined by \(\Phi \ast \Phi_c := m(\Phi \otimes \Phi_c) \Delta\).

As an example, the coproduct for the matrix entry \(A_{ij}\) of the \(n \times n\) matrix \(A\) is defined as \(\Delta(A_{ij}) = \sum_{k=1}^{n} A_{ik} \otimes A_{kj}\), while the multiplication has \(\sum_{k=1}^{n} A_{ik} A_{kj} = A_{ij}\). The counit \(\epsilon\) is defined by \(\epsilon(A_{ij}) = \delta_{ij}\) and the antipode of
$A_{ij}$ is its inverse $A_{ij}^{-1}$. In our case, the following type of coproduct

$$\Delta(A_{ij}) = A_{ij} \otimes e + e \otimes A_{ij} + \sum_{k=1}^{n} A_{ik} \otimes A_{kj}$$

is considered.

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