ON FIBRATION STABILITY AFTER DERVAN-SEKTNAN AND SINGULARITIES

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Abstract. We introduce \( f \)-stability, a modification of fibration stability of Dervan-Sektnan [12], and show that \( f \)-semistable fibrations have only semi log canonical singularities. Moreover, \( f \)-stability puts restrictions on semi log canonical centers on Fano fibrations.

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1. Introduction

It is one of the most important problems in Kähler geometry that when a constant scalar curvature Kähler (cscK) metric exists on a polarized complex manifold \((X, L)\). The Yau-Tian-Donaldson (YTD) conjecture predicts that the existence of cscK metrics is equivalent to a certain algebro-geometric condition called K-polystability. Indeed, Berman-Darvas-Lu [4] proved that if \((X, L)\) admits a cscK metric, then \((X, L)\) is K-polystable and Chen-Donaldson-Sun [7] and Tian [39] proved YTD conjecture in Fano case independently. On the other hand, K-stability is the positivity of the leading term of Chow weight and is also an important notion in terms of the geometric invariant theory (GIT). Ross and Thomas [35] studied K-stability in an algebro-geometric way first, and Odaka [32] found out the relationship between K-stability and singularities by applying the minimal model program (MMP). He also proved that if a \(\mathbb{Q}\)-Gorenstein variety \(V\) is asymptotically Chow-semistable, then \(V\) has only slc singularities as a corollary.

On the other hand, the existence problem of cscK metrics on fibrations is well-studied in Kähler geometry. In this paper, fibrations mean algebraic fiber spaces \(f : X \to B\), where \(f\) are morphisms of varieties with connected general fiber. See the details in Definition 2.15. We call an algebraic fiber space \(f : X \to B\) is

- a Calabi-Yau fibration if there exists a line bundle \(L_0\) on \(B\) such that \(K_X \sim_{\mathbb{Q}} f^*L_0\).
- a Fano fibration (resp., a canonically polarized fibration) if \(K_X\) is \(f\)-antiample (resp., \(f\)-ample).
- a smooth (resp., flat) fibration if \(f\) is smooth (resp., flat).

Fine obtained a sufficiency condition [14, Theorem 1.1] for existence of a cscK metric on a smooth fibration whose fibers have cscK metrics first. Dervan and Sektnan [11] introduced
a differential geometric notion called optimal symplectic connection and proved the following generalization of the result of Fine:

**Theorem ([11], Theorem 1.2]).** Let \( f : (X, H) \to (B, L) \) be a polarized smooth fibration. If \( f \) admits an optimal symplectic connection and \((B, L)\) admits a twisted cscK metric with respect to the Weil-Petersson metric, then \((X, \delta H + f^*L)\) has cscK metrics for sufficiently small \( \delta > 0 \).

This is the definitive result for smooth fibrations. On the other hand, Jian-Shi-Song [24] proved that smooth good minimal models (i.e., \( K_X \) is semiample) have cscK metrics by the result of Chen-Cheng [6] and J-stability (cf., [38]). Here, we emphasize that they treated not only smooth fibrations but Calabi-Yau fibrations admitting a singular fiber. Sjöström Dyrefelt [36] and Song [37] generalized independently the result of Jian-Shi-Song to the case when \( X \) is a smooth minimal model (i.e., \( K_X \) is nef) later. On the other hand, the author proved K-stability of klt minimal models in [22].

Dervan and Sektnan [12] also conjectured that the existence of an optimal symplectic connection is equivalent to an algebro-geometric condition called fibration stability. Their theorem and conjecture predict that fibration stability and a certain stability of the base variety imply adiabatic K-stability (cf., Definition 2.16) of the total space as follows.

**Conjecture ([12], 1.3]).** Let \( f : (X, H) \to (B, L) \) be a smooth fibration. If \( f \) is fibration stable and \((B, L)\) has a twisted cscK metric with respect to the Weil-Petersson metric, then \((X, H)\) is adiabatically K-semistable i.e., \((X, \delta H + f^*L)\) is K-semistable for sufficiently small \( \delta > 0 \).

If it was true, we could obtain a criterion for K-stability of fibrations. The condition on the base in the conjecture would be necessary. Indeed, it is known that twisted K-stability of the base is necessary for adiabatic K-stability of the total space by the work of Dervan-Ross [9, Corollary 4.4].

In this paper, we introduce \( \mathcal{f} \)-stability (cf., Definition 4.4) as a modification of fibration stability of Dervan-Sektnan [12] and prove fundamental results on this. This is a stronger condition than fibration stability in numerical aspects. Dervan and Sektnan weakened the definition of fibration stability in [12]. On the other hand, the original one in [10] is the condition of the positivity of \( W_0 \) and \( W_1 \) we will explain in the next page. Note that the original fibration stability coincides with \( \mathcal{f} \)-stability for fibrations over curves such that each fiber is K-polystable. Since any polarized smooth curve is twisted K-stable, it is natural to conjecture that the original fibration stability implies existence of an optimal symplectic connection on fibrations over smooth curves rather than the new one. More generally, for a smooth fibration \( f : (X, H) \to (B, L) \), if \( B \) has a twisted cscK metric in the sense of the conjecture above and \( X \) has an optimal symplectic connection, it is easy to see that \( f \) is \( \mathcal{f} \)-semistable by [11, Theorem 1.2]. Taking these facts into account, it is worth studying \( \mathcal{f} \)-stability for K-stability of fibrations.

First, we introduce invariants \( W_i \) as the Donaldson-Futaki invariant for \( \mathcal{f} \)-stability. Similarly to results of [32] in K-stability, we establish the explicit formula to compute \( W_1 \) by taking general hyperplane sections of the base, and show that \( \mathcal{f} \)-stability puts some restrictions on singularities by applying MMP. We obtain the following.

**Theorem A** (Theorem 5.1). Let \( f : (X, \Delta, H) \to (B, L) \) be a polarized algebraic fiber space pair (cf., Definition 2.15). If \( f \) is \( \mathcal{f} \)-semistable, \((X, \Delta)\) has at most lc singularities.

We remark that \( \mathcal{f} \)-stability does not imply adiabatic K-stability in general. Indeed, a rational elliptic surface with a section and a \( II^* \) or \( IV^* \) fiber is \( \mathcal{f} \)-stable but adiabatic K-unstable over \( \mathbb{P}^1 \). We prove this fact in [23]. Thus, we can not apply Odaka’s result [32] to Theorem A directly.

We can also show that a \( \mathcal{f} \)-semistable flat Fano fibration is Kawamata log terminal (klt) if the base variety has only klt singularities. Moreover, as an application of \( \mathcal{f} \)-stability to K-stability, we show that adiabatically K-semistable flat Fano fibrations over klt varieties have only klt singularities.
Theorem B (Theorem 5.2). Let \( f : (X, H) \to (B, L) \) be a flat polarized algebraic fiber space. Suppose that there exist \( \lambda \in \mathbb{Q}_{>0} \) and a line bundle \( L_0 \) on \( B \) such that \( H + f^*L_0 \equiv -\lambda K_X \) and \( B \) has only klt singularities. If \( f \) is \( \mathfrak{f} \)-semistable, then \( X \) has only klt singularities. In particular, if \( f \) is adiabatically K-semistable (Definition 2.10), then \( X \) has only klt singularities.

On the other hand, we prove that klt Calabi-Yau fibrations are \( \mathfrak{f} \)-stable similarly to the well-known theorem [30, Theorem 2.10] in K-stability. Indeed, we show the following.

Theorem C (Theorem 4.15). Let \( f : (X, \Delta, H) \to (B, L) \) be a polarized algebraic fiber space pair with a line bundle \( L_0 \) on \( B \). Suppose that \( (X, \Delta, H) \) is a klt polarized pair and \( K_X + \Delta \equiv f^*L_0 \). Then \( f \) is \( \mathfrak{f} \)-stable.

We explain the definition of \( \mathfrak{f} \)-stability briefly as follows. Suppose that \( f : (X, H) \to (B, L) \) is a polarized algebraic fiber space. Let \( m = \text{rel.dim} f \) and \( d \text{im} B = n \). Roughly speaking, for any semiample test configuration \((X, \mathcal{H})\) for \((X, H)\), we define constants \( W_0(X, \mathcal{H}), \ldots, W_n(X, \mathcal{H}) \) and a rational function \( W_{n+1}(X, \mathcal{H})(j) \) in \( j \) so that

\[
V(H + jL)M^{NA}(X, \mathcal{H} + jL) = W_{n+1}(X, \mathcal{H})(j) + \sum_{i=0}^{n} j^i W_{n-i}(X, \mathcal{H})
\]

where \( \lim_{j \to \infty} W_{n+1}(X, \mathcal{H})(j) = 0 \) and \( M^{NA} \) is the non-Archimedean Mabuchi functional (cf., Definition 2.6 and Notation in §4). Then \( f : (X, H) \to (B, L) \) is \( \mathfrak{f} \)-semistable if

\[
\sum_{i=0}^{n} j^i W_{n-i}(X, \mathcal{H}) \geq 0
\]

for sufficiently large \( j > 0 \). We calculate \( W_i \) in Lemma 4.12 as follows,

\[
W_i(X, \mathcal{H}) = \binom{n+m}{n-i} \left( K_X^{\log} \right)_{\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{d-i}}} + S(X_b, H_b)_{m+i+1} \left( K_X^{\log} \right)_{\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots D_{d-i}}} + \sum_{k=n-i+1}^{n} C_k \mathcal{J}^{\mathcal{H}}_{\mathcal{X} \cap D_1 \cap D_2 \cdots \cap D_j}
\]

where \( C_k \) are constants and \( D_k \in |L| \) are general elements. Then we prove Theorems A, B by applying MMP results (cf., [34, Theorem 1.1], [20] and the adjunction formula [20, §16, §17]) and Lemma 4.12. On the other hand, we also show these theorems for deminormal pairs with boundaries in §§5.2 More precisely,

Theorem D (Theorem 5.13, Theorem 5.15). Suppose that \( f : (X, \Delta, H) \to (B, L) \) is a polarized deminormal algebraic fiber space pair (Definition 5.3).

1. If \( f \) is \( \mathfrak{f} \)-semistable, \( (X, \Delta) \) has at most slc singularities.

2. Suppose that there exist \( \lambda \in \mathbb{Q}_{>0} \) and a line bundle \( L_0 \) on \( B \) such that \( H + f^*L_0 \equiv -\lambda (K_X + \Delta) \), and \( f \) is \( \mathfrak{f} \)-semistable. Then any slc-center \( C \) of \((X, \Delta)\) is of fiber type, i.e., \( \text{codim}_X C \leq \text{codim}_B f(C) \) (Definition 5.3).

Note that if \( f \) is flat, \( C \) is of fiber type iff \( C \) contains an irreducible component of a fiber. Thus, Theorem A follows from Theorem D. On the other hand, \( \mathfrak{f} \)-semistability is a weaker condition than adiabatic K-semistability. Therefore, we also obtain the following.

Corollary E (Corollary 5.10). Let \( f : (X, \Delta, H) \to (B, L) \) be an adiabatically K-semistable polarized deminormal algebraic fiber space pair such that \( H + f^*L_0 \equiv -\lambda (K_X + \Delta) \) where \( L_0 \) is a line bundle on \( B \) and \( \lambda \in \mathbb{Q}_{>0} \). Then \((X, \Delta)\) is slc and any slc-center of \((X, \Delta)\) is of fiber type.
Outline of this paper. In §2, we prepare many terminology and facts on K-stability and algebraic fiber spaces. In §3, we recall results of Odaka [32] and Boucksom-Hisamoto-Jonsso n [5] on the relationship between singularities and K-stability. From §4, we state our original results. In §4, we calculate $W_i$ and deduce the Theorem C. In §§5.1, we apply results of MMP and the computations in §4 to obtain Theorem A and a generalization (Theorem 5.4) of Theorem [12]. In §§5.2, we extend theorems in §§5.1 to the deminormal case.

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2. Notation
In this paper, we work over $\mathbb{C}$. If $X$ is a scheme, we assume that $X$ is a scheme of finite type over $\mathbb{C}$ in this paper. If $X$ is a variety, we assume that $X$ is an irreducible, reduced and separated scheme. We follow the definitions of $\mathbb{Q}$-line bundles, $\mathbb{Q}$-Weil divisors and $\mathbb{Q}$-Cartier divisors, and the notations of the $\mathbb{Q}$-linearly equivalence $\sim_\mathbb{Q}$ and the numerical equivalence $\equiv$ from [21], [27] and [5]. A pair $(X, L)$ is a polarized scheme if $X$ is a proper, reduced and equidimensional scheme over $\mathbb{C}$ and $L$ is an ample $\mathbb{Q}$-line bundle over $X$. A $\mathbb{Q}$-Weil divisor $\Delta$ such that there exist integral divisors $F_i$ different from each other, $a_i \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $\Delta = \sum_{i=1}^{n} a_i F_i$ is a boundary if $K_{X,\Delta} := K_X + \Delta$ is $\mathbb{Q}$-Cartier on a normal variety $X$. Then, we call a pair $(X, \Delta)$ a (normal) log pair. Here, we do not assume that $[\Delta] = \sum [a_i] F_i$ is a reduced divisor. $A$ $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{r} a_i D_i$ on a smooth variety $X$ is simple normal crossing (snc) if each $\bigcap_{i,j \in J} D_i$ is smooth for any subset $J \subset \{ 1, 2, \cdots, r \}$.

First, recall the definitions of deminormal pairs and semi log canonical singularities,

Definition 2.1 (deminormal pair [25 Chapter 5]). Let $X$ be an equidimensional reduced scheme satisfying the Serre’s condition $S_2$. $X$ is a deminormal scheme if any codimension 1 point of $X$ is smooth or nodal. $A$ $\mathbb{Q}$-divisor $\Delta$ on $X$ is a boundary if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and any irreducible component of $\Delta$ is not contained in the singular locus $\text{Sing}(X)$. Then we call $(X, \Delta)$ a deminormal log pair.

Let $\nu : \tilde{X} \to X$ be the normalization. Then, $\mathfrak{d} = \text{Hom}_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$ is an ideal in both of $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$. We call the conductors of $X$ the closed subschemes defined by $\mathfrak{d}$ and we will denote these by $\text{cond}_X \subset X$ and $\text{cond}_{\tilde{X}} \subset \tilde{X}$ respectively. Note that $\text{cond}_X$ is a Weil divisor.

Further, let $L$ be a $\mathbb{Q}$-ample line bundle on $X$. Then, we call $(X, \Delta, L)$ a polarized deminormal (log) pair.

Definition 2.2 (Log discrepancy). Let $(X, \Delta)$ be a normal log pair and $v$ be a divisorial valuation on $X$. Suppose that $\sigma : Y \to X$ be a birational morphism such that there exist a positive constant $c > 0$ and a prime divisor $F$ on $Y$ such that $v = \text{cord}_F$. Then the log discrepancy of $v$ with respect to $(X, \Delta)$ is

$$A_{(X, \Delta)}(v) = v(K_Y - \sigma^* K_{(X, \Delta)}) - c.$$  

It is easy to see that the log discrepancy is independent of $\sigma$. We define $A_{(X, \Delta)}(v_{\text{triv}}) = 0$ if $v_{\text{triv}}$ is the trivial valuation.

We define $(X, \Delta)$ is

- sub Kawamata log terminal (subklt) if $A_{(X, \Delta)}(v) > 0$ for any non trivial divisorial valuation $v$.
- sub log canonical (sublc) if $A_{(X, \Delta)}(v) \geq 0$.

Let $c_X(v)$ be the center of $v$ on $X$. $c_X(v)$ is an lc center of $(X, \Delta)$ if $A_{(X, \Delta)}(v) = 0$. $(X, \Delta)$ is klt (resp., lc) if $(X, \Delta)$ is subklt (resp., sublc) and $\Delta$ is effective. We also say that $(X, \Delta)$ has only klt (resp., lc) singularities.
Let \((V, B)\) be a deminormal log pair and \(\nu : V' \to V\) be the normalization. Then, \((V, B)\) is semi log canonical (slc) if \((V', \nu_*^{-1}B + \text{cond})\) is lc.

Recall the notion of the minimal model program (MMP). We follow the fundamental notations of MMP in [27].

**Definition 2.3** (Minimal model). Let \((X, \Delta)\) be a projective normal log pair over a quasi projective normal variety \(S\) such that \([\Delta]\) is reduced. Let \(Y\) be a projective normal variety over \(S\). Then, a birational map \(\phi : X \to Y\) is a birational contraction if there is no \(\phi^{-1}\)-exceptional divisor. Suppose that \(K_Y + \phi_*\Delta\) is also \(\mathbb{Q}\)-Cartier. Then, \(\phi\) is \((K_X + \Delta)\)-negative (resp., \((K_X + \Delta)\)-non-positive) if we have \(p^*(K_Y + \Delta) = q^*(K_Y + \phi_*\Delta) + E\) where \(E\) is effective (resp., \(E\) is effective and \(\text{Supp} E\) contains all \(\phi^{-1}\)-exceptional divisors). Here, \(\Gamma\) is the resolution of singularities of the graph of \(\phi\) and \(p : \Gamma \to X\) and \(q : \Gamma \to Y\) are canonical projections. Now \(Y\) is

- a weak log canonical model (wlcm) of \((X, \Delta)\) if \(K_Y + \phi_*\Delta\) is nef over \(S\) and \(\phi\) is \((K_X + \Delta)\)-non-positive,
- a minimal model (resp., good minimal model) of \((X, \Delta)\) if \(K_Y + \phi_*\Delta\) is nef (resp., semiample) over \(S\) and \(\phi\) is \((K_X + \Delta)\)-negative,
- the log canonical model (lc model) if \(K_Y + \phi_*\Delta\) is ample over \(S\) and \(Y\) is a wlcm.

Next, recall K-stability. For example, see the detail in [5].

**Definition 2.4** (Test configuration). Let \(X\) be a proper scheme. Then, \(\pi : \mathcal{X} \to \mathbb{A}^1\) is a test configuration for \(X\) if \(\pi : \mathcal{X} \to \mathbb{A}^1\) satisfies the following properties:

1. \(\mathcal{X}\) has a \(\mathbb{G}_m\)-action.
2. \(\pi\) is a proper, flat and \(\mathbb{G}_m\)-equivariant morphism where \(\mathbb{A}^1\) admits a canonical \(\mathbb{G}_m\)-action.
3. \(\mathcal{X}_1 := \pi^{-1}(1) \cong X\).

If there is no fear of confusion, we will denote \(\pi : \mathcal{X} \to \mathbb{A}^1\) as \(\mathcal{X}\) simply. Let \((X, L)\) be a polarized deminormal scheme. Then, \((\mathcal{X}, \mathcal{L})\) is a (semi)ample test configuration for \((X, L)\) if

1. \(\mathcal{X}\) is a test configuration for \(X\).
2. \(\mathcal{L}\) is a \(\mathbb{A}^1\)-(semi)ample \(\mathbb{G}_m\)-equivariant \(\mathbb{Q}\)-line bundle.
3. \(\mathcal{L}|_{\mathcal{X}_1} = L\).

\((X_{\mathbb{A}^1}, L_{\mathbb{A}^1}) = (X \times \mathbb{A}^1, L \times \mathbb{A}^1)\) with a trivial \(\mathbb{G}_m\)-action is a semia ample test configuration. We call this the trivial test configuration. Note that any test configuration is birational. \((\mathcal{X}, \mathcal{L})\) is isomorphic to \((X_{\mathbb{A}^1}, L_{\mathbb{A}^1})\) in codimension 1, then we call \((\mathcal{X}, \mathcal{L})\) almost trivial.

\(\mathcal{X}\) dominates \(X_{\mathbb{A}^1}\) if there exists a birational morphism of test configurations \(\rho_X : \mathcal{X} \to X_{\mathbb{A}^1}\).

\((\mathcal{X}, \mathcal{L})\) is a product test configuration if \(\mathcal{X}\) is isomorphic to \(X \times \mathbb{A}^1\) as abstract varieties.

For simplicity, the central fiber of \((\mathcal{X}, \mathcal{L})\) is the fiber of \(\pi\) over \(0 \in \mathbb{A}^1\) and is denoted as \((\mathcal{X}_0, \mathcal{L}_0)\).

To define Donaldson-Futaki invariant, we need the following definition of the weight of \(\mathbb{G}_m\)-representation. Let \(W\) be a finite dimensional \(\mathbb{G}_m\)-representation space over \(\mathbb{C}\) and then \(W\) has the unique weight decomposition

\[ W = \bigoplus_{k = -\infty}^{\infty} W_k \]

where \(\lambda \in \mathbb{G}_m\) acts on \(v \in W_k\) in the way that \(v \mapsto \lambda^k v\). Then, the weight of \(W\) is

\[ \sum_k -k \dim W_k. \]

**Definition 2.5** (Donaldson-Futaki invariant, [33] Definition 3.2). Let \((X, \Delta, L)\) be an \(n\)-dimensional polarized (demi)normal pair and \((\mathcal{X}, \mathcal{L})\) be an ample test configuration. Let \(w(m)\) be the weight of \(H^0(X_0, m\mathcal{L}_0)\). It is well-known that \(w(m) = \sum_{i=0}^{n+1} b_i m^{n+1-i}\) is a polynomial function in \(m \gg 0\) with \(\deg w = n + 1\) (see [5], Theorem 3.1]). On the other hand,
let \( N_m = h^0(X, mL) = \sum_{i=0}^n a_i m^{n-i} \) be the Hilbert polynomial. Then, the Donaldson-Futaki invariant of \((\mathcal{X}, \mathcal{L})\) is

\[
\text{DF}(\mathcal{X}, \mathcal{L}) = 2b_1a_0 - a_1b_0.
\]

Moreover, let \( \Delta_X \) be the strict transformation of \( \Delta \times \mathbb{A}^1 \). Let \( \hat{w}(m) = \sum_{i=0}^n \hat{b}_i m^{n-i} \) be the weight polynomial of \( H^0(\Delta_{X,\theta}, mL)|_{\Delta_{X,\theta}} \) and \( \hat{N}_m = h^0(\Delta, mL|_{\Delta}) = \sum_{i=0}^{n-1} \hat{a}_i m^{n-1-i} \) be the Hilbert polynomial. Then, the log Donaldson-Futaki invariant of \((\mathcal{X}, \mathcal{L})\) is

\[
\text{DF}_\Delta(\mathcal{X}, \mathcal{L}) = \text{DF}(\mathcal{X}, \mathcal{L}) + \frac{\hat{b}_0a_0 - \hat{a}_0b_0}{a_0^2}.
\]

**Notation.** In this paper, we write line bundles and divisors interchangeably. For example,

\[
A \times \mathbb{A}^1.
\]

For simplicity, we will denote intersection products of line bundles or divisors as

\[
L \cdot (H + D) = L^m \cdot (H \otimes \mathcal{O}_X(D)).
\]

We define the non-Archimedean functionals as in [5].

**Definition 2.6** ([6], §6, 7]). Let \((X, \Delta, L)\) be an \( n \)-dimensional polarized normal pair and \( \pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1 \) be a normal semiample test configuration. Let also \((\mathcal{X}, \mathcal{E})\) be the \( \mathbb{G}_m \)-equivariant compactification over \( \mathbb{P}^1 \) such that the \( \infty \)-fiber \((\overline{\mathcal{X}}_\infty, \overline{\mathcal{E}}_\infty)\) is \( \mathbb{G}_m \)-equivariantly isomorphic to \((X, L)\) with a trivial \( \mathbb{G}_m \)-action (cf., [6] Definition 2.7). Suppose that there exists a \( \mathbb{G}_m \)-equivariant morphism \( \rho : \overline{\mathcal{X}} \to X_{\mathbb{P}^1} \) such that \( \rho \) is the identity on \( X \times \mathbb{P}^1 \setminus 0 \). Here, \((X_{\mathbb{P}^1}, L_{\mathbb{P}^1})\) is the \( \mathbb{G}_m \)-equivariant compactification of \((X_{\mathbb{A}^1}, L_{\mathbb{A}^1})\). We call \( \rho \) the canonical map. We also denote \( \rho^*L_{\mathbb{P}^1} \) by \( L_{\mathbb{P}^1} \). If

- \( V = V(L) := L^n \),
- \( S(X, \Delta, L) = \frac{n(K_{(X,\Delta)} L^{n-1})}{(L^n)} \) where \( K_{(X,\Delta)} = K_X + \Delta \),
- \( K^\log_{(\overline{X},\overline{\Delta}/\mathbb{P}^1)} = K_{(\overline{X},\overline{\Delta}/\mathbb{P}^1)} + (\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \) where \( \mathcal{X}_{0,\text{red}} \) is the reduced central fiber of \( \pi \) and \( \overline{\Delta} \) is the strict transformation of \( \Delta \times \mathbb{P}^1 \) in \( \overline{\mathcal{X}} \),
- For any irreducible component of \( \mathcal{X}_0 \), the divisorial valuation \( v_E \) on \( X \) is the restriction of \( b_{E,\mathbb{P}^1} \cdot \text{ord}_E \) to \( X \) where \( b_E = \text{ord}_E(\mathcal{X}_0) \) (cf., [6], §4),

then

- the non-Archimedean Monge-Ampère energy of \((\mathcal{X}, \mathcal{L})\) is
  \[
  E^{\text{NA}}(\mathcal{X}, \mathcal{L}) = \frac{\mathcal{E}^{n+1}}{(n+1)V(L)},
  \]
- the non-Archimedean I-functional of \((\mathcal{X}, \mathcal{L})\) is
  \[
  I^{\text{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1}(\mathcal{E} \cdot L^n_{\mathbb{P}^1}) - V^{-1}(\mathcal{E} - L_{\mathbb{P}^1}) \cdot \mathcal{E}^n,
  \]
- the non-Archimedean J-functional of \((\mathcal{X}, \mathcal{L})\) is
  \[
  J^{\text{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1}(\mathcal{E} \cdot L^n_{\mathbb{P}^1}) - E^{\text{NA}}(\mathcal{X}, \mathcal{L}),
  \]
- the \((\mathcal{J}^H)^{\text{NA}}\)-functional of \((\mathcal{X}, \mathcal{L})\) is
  \[
  (\mathcal{J}^H)^{\text{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1}(H_{\mathbb{P}^1} \cdot \mathcal{E}^n) - V^{-1}(nH \cdot L^{n-1}) E^{\text{NA}}(\mathcal{X}, \mathcal{L}),
  \]
  where \( H \) is a line bundle on \( X \),
- the non-Archimedean Ricci energy of \((\mathcal{X}, \mathcal{L})\) is
  \[
  R^{\text{NA}}(\mathcal{X}, \mathcal{L}) = (\mathcal{J}^{K_{(X,\Delta)}})^{\text{NA}}(\mathcal{X}, \mathcal{L}) - S(X, \Delta, L) E^{\text{NA}}(\mathcal{X}, \mathcal{L}),
  \]
• the non-Archimedean entropy of $(\mathcal{X}, \mathcal{L})$ is (see [3 Corollary 7.18])

$$H^\text{NA}_\Delta(\mathcal{X}, \mathcal{L}) = V^{-1}\sum_E A_{(X, \rho)}(v_E)(E \cdot \mathcal{L}^\rho) = V^{-1}(K^\log_{(\mathcal{X}, \Delta)/\mathbb{P}^1} - \rho^* K_{(X_1, \Delta_1)/\mathbb{P}^1}) \cdot \mathcal{L}^\rho$$

where $E$ runs over the irreducible components of $X_0$.

• the non-Archimedean Mabuchi functional of $(\mathcal{X}, \mathcal{L})$ is

$$M^\text{NA}_\Delta(\mathcal{X}, \mathcal{L}) = H^\text{NA}_\Delta(\mathcal{X}, \mathcal{L}) + (\mathcal{F}^K_{(X, \Delta)})^\text{NA}(\mathcal{X}, \mathcal{L}).$$

If there is no afraid of confusion, we denote $(\mathcal{X}, \mathcal{L})$ as $(\mathcal{X}, \mathcal{L})$. Note that those functional are pullback invariant, if there exists a $\mathbb{G}_m$-equivariant morphism $\mu : \tilde{X} \rightarrow X$ of normal test configurations for $X$ such that $\mu$ is the identity on $X$ and $\mu^* \mathcal{L}$ is the equivalence relation generated by dominations. See also [5, Proposition 2.17]. It is easy to see that $H^\text{NA}(L)$ is a set. Thus, the functional defined as above are well-defined on $H^\text{NA}(L)$. It is also easy to see that for any non-Archimedean positive metric $\phi \in H^\text{NA}(L)$ we can take a representative $(\mathcal{X}, \mathcal{L})$ of $\phi$ such that $\mathcal{X}$ is normal and there exists a $\mathbb{G}_m$-equivariant morphism $\rho : \mathcal{X} \rightarrow X_{p_1}$ such that $\rho$ is canonical.

**Proposition 2.7** ([33, Theorem 3.7], [5, Proposition 3.12]). Notations as in Definition 2.6.

Let $(X, \Delta, L)$ be a polarized normal pair and $(\mathcal{X}, \mathcal{L})$ be a normal semiample test configuration. Then,

$$\mathcal{D}^\Delta(\mathcal{X}, \mathcal{L}) = V^{-1}(K_{(\mathcal{X}, \Delta)/\mathbb{P}^1} - \rho^* K_{(X_1, \Delta_1)/\mathbb{P}^1}) \cdot \mathcal{L}^\rho + R^\text{NA}_{\Delta}(\mathcal{X}, \mathcal{L}) + S(X, \Delta, L)E^\text{NA}(\mathcal{X}, \mathcal{L}).$$

**Definition 2.8** (K-stability). Let $(X, \Delta, L)$ be a polarized deminormal pair. $(X, L)$ is

- **K-semistable if**

  $\mathcal{D}^\Delta(\mathcal{X}, \mathcal{L}) \geq 0$

  for any semiample test configuration,

- **K-polystable if** $(X, \Delta, L)$ is K-semistable and

  $\mathcal{D}^\Delta(\mathcal{X}, \mathcal{L}) = 0$

  if and only if $(\mathcal{X}, \mathcal{L})$ is isomorphic to a product test configuration in codimension 1,

- **K-stable if**

  $\mathcal{D}^\Delta(\mathcal{X}, \mathcal{L}) > 0$

  for any non-almost-trivial semiample test configuration,

- **uniformly K-stable if** there exists a positive constant $\epsilon > 0$ such that

  $\mathcal{D}^\Delta(\mathcal{X}, \mathcal{L}) \geq \epsilon I^\text{NA}(\mathcal{X}, \mathcal{L})$

  for any semiample test configuration.

We remark that the non-Archimedean $I$ and $J$-functionals are norm in the following sense. The non-Archimedean $I$ and $J$-functionals are nonnegative and

$$J^\text{NA}(\mathcal{X}, \mathcal{L}) = 0$$

if and only if the normalization of $(\mathcal{X}, \mathcal{L})$ is trivial by [3, Theorem 7.9]. They satisfy that

$$\frac{1}{n} J^\text{NA}(\mathcal{X}, \mathcal{L}) \leq I^\text{NA}(\mathcal{X}, \mathcal{L}) - J^\text{NA}(\mathcal{X}, \mathcal{L}) \leq n J^\text{NA}(\mathcal{X}, \mathcal{L})$$

by [3 Proposition 7.8].
Remark 2.9. Dervan \[8\] introduced the minimum norm of test configurations defined by 
\( J^{\text{NA}}(X, L) - J^{\text{NA}}(X, L) \) and also proved that this is a norm as the non-Archimedean I and 
J-functionals.

The following is well-known,

**Proposition 2.10** ([5 Proposition 8.2]). Let \((X, \Delta, L)\) be a polarized normal pair. Then \((X, L)\) is

- **K-semistable if and only if**
  \[ M^\text{NA}_{\Delta}(X, L) \geq 0 \]
  for any normal semiample test configuration,

- **K-stable if and only if**
  \[ M^\text{NA}_{\Delta}(X, L) > 0 \]
  for any non-trivial normal semiample test configuration,

- **uniformly K-stable if and only if**
  there exists a positive constant \( \epsilon > 0 \) such that
  \[ M^\text{NA}_{\Delta}(X, L) \geq \epsilon I^\text{NA}(X, L) \]
  for any normal semiample test configuration.

**Definition 2.11.** For \( \phi \in H^\text{NA}(L) \), we can define the new positive non-Archimedean metric \( \phi_d \)
for \( d \in \mathbb{Z}_{\geq 1} \) as follows. If \((X, L)\) is a representative of \( \phi \), then \( \phi_d \) is represented by \( (X^{(d)}, L^{(d)}) \)
that is the normalization of the base change \( X \times A^1 \) via the \( d \)-th power map \( A^1 \ni t \mapsto t^d \in A^1 \).

Note that \( d M^\text{NA}_{\Delta}(\phi) = M^\text{NA}_{\Delta}(\phi_d) \) and \( d \text{DF}_{\Delta}(\phi) \geq \text{DF}_{\Delta}(\phi_d) \) for any \( d \).
Moreover, for sufficiently divisible \( d \), \( M^\text{NA}_{\Delta}(\phi_d) = \text{DF}_{\Delta}(\phi_d) \) by [5 Proposition 7.16].

**Definition 2.12** (Slope stability, [35 Definition 4.17]). Let \((X, B, L)\) be a polarized log pair.
For any non-void closed subscheme \( Z \subset X \), the **deformation to the normal cone** \( \mathcal{X} \) of \( Z \) is a test configuration for \( X \) such that \( \mathcal{X} \) is the blow up along \( Z \times \{0\} \) (see also [19 Chapter 5]).
Then \((X, B, L)\) is slope K-(semi)stable if
\[ \text{DF}_{B}(\mathcal{X}, L) > 0, \text{ (resp.,} \geq 0) \]
for any non-almost-trivial semiample test configuration such that \( \mathcal{X} \) is a deformation to the normal cone of \( Z \).

We will use the following notation in §4

**Definition 2.13.** A non-Archimedean positive metric \( \phi \in H^\text{NA}(L) \) of \((X, L)\) is **normalized with respect to the central fiber** if we can take a representative \((\mathcal{X}, L)\) of \( \phi \) that satisfies the following,

1. \((\mathcal{X}, L)\) has a \( \mathbb{G}_m \)-equivariant dominant morphism \( \rho : (\mathcal{X}, L) \to (X_{A^1}, L_{A^1}) \).
2. The unique divisor \( D = \mathcal{L} - \rho^*L_{A^1} \) has the support \( \text{Supp} \mathcal{D} \not\ni \hat{X} \) where \( \hat{X} \) is the strict 

transformation of \( X \times \{0\} \).

Whenever there is no fear of confusion, a non-Archimedean metric \( \phi \) normalized with respect to the central fiber will be denoted by normalized non-Archimedean metric for short.

It follows immediately from the Definition 2.13 that

**Lemma 2.14.** Let \((\mathcal{X}, L)\) be a semiample test configuration over an \( n \)-dimensional polarized 
pair \((X, \Delta, L)\). Suppose that \((\mathcal{X}, L)\) is normalized and there exists the birational morphism \( \rho : \mathcal{X} \to X_{A^1} \).
Then the following hold,

1. \( J^{\text{NA}}(\mathcal{X}, L) = -E^{\text{NA}}(\mathcal{X}, L) \).
2. \( R^\text{NA}_{\Delta}(\mathcal{X}, L) = \frac{1}{t^N} \rho^* K_{(X_{d1}, \Delta_{d1})/P^1} \cdot L^n \).

We are interested in K-stability of the following object,
Definition 2.15. An algebraic fiber space \( f : X \to B \) is a surjective morphism between proper normal varieties with connected geometric fibers. If \( F \) is the generic geometric fiber, the relative dimension \( \text{rel.dim} f \) of \( f \) is \( \text{dim } F \). Note that \( F \) is normal and connected. If \( f : X \to B \) is an algebraic fiber space and flat, we call \( f \) a flat algebraic fiber space. If \( f : X \to B \) is an algebraic fiber space, \( H \) is an \( f \)-ample line bundle on \( X \) and \( L \) is an ample line bundle on \( B \), then we denote \( f : (X, H) \to (B, L) \) and call this a polarized algebraic fiber space. Moreover, if \( \Delta \) is a boundary on \( X \) and \( f \) is as above, then we denote \( f : (X, \Delta, H) \to (B, L) \) and call this a polarized algebraic fiber space pair.

Definition 2.16 (Adiabatic K-stability). Let \( f : X \to C \) be an algebraic fiber space, \( \Delta \) be the boundary of \( X \), \( H \) be a \( f \)-ample line bundle and \( L \) be an ample line bundle on \( C \). Then \( (X, \Delta, H) \) is adiabatically K-(poly, semi)stable over \( (C, L) \) if \( (X, \Delta, \epsilon H + f^*L) \) is K-(poly, semi)stable for sufficiently small \( \epsilon > 0 \).

Remark 2.17. In the above definition, we name the above stability after the adiabatic limit technique Fine [13], [14] and Dervan-Sektnan [12] used.

We need the following notation about algebraic fiber spaces.

Notation. Let \( f : X \to B \) be an algebraic fiber space and \( H \) and \( M \) are \( \mathbb{Q} \)-linearly equivalent over \( B \), \( H \sim_{\mathbb{Q}} M \) if there exists a \( \mathbb{Q} \)-line bundle \( L \) on \( B \) such that \( H \sim_{\mathbb{Q}} M + f^*L \). Furthermore, we will denote \( H + L = H + f^*L \) if there is no fear of confusion.

For K-stability of deminormal varieties, we can calculate the Donaldson-Futaki invariant by taking the normalization. We need the following,

Definition 2.18 (Partially normal test configuration, [31] Definition 3.7). Let \( X \) be a deminormal scheme and \( \mathcal{X} \) be a test configuration for \( X \). Let also \( \nu : \tilde{\mathcal{X}} \to \mathcal{X} \) be the normalization. The partially normalization \( \mathcal{X}' \) of \( \mathcal{X} \) is \( \text{Spec}_{\mathcal{X}}(\mathcal{O}_{X \times X'(-\nu(0))}) \cap \nu_{*}\mathcal{O}_{\tilde{\mathcal{X}}} \). Then, \( \mathcal{X} \) is partially normal if \( \mathcal{X}' \) is \( \mathcal{X} \). Note that \( \mathcal{X} \) is partially normal iff \( \mathcal{X} \) has no singular codimension 1 point over \( 0 \in \mathbb{A}^1 \) by [31] Proposition 3.8.

We recall the following fundamental result,

Proposition 2.19 (Proposition 5.1 of [35], Propositions 3.8 and 3.10 of [31]). Let \( (X, \Delta, L) \) be a polarized deminormal pair and \( (\mathcal{X}, \mathcal{L}) \) be a semiample test configuration for \( (X, L) \). Let also \( (\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \) be the normalization and \( (\mathcal{X}', \mathcal{L}') \) be the partially normalization. Here, \( (\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \) is a semiample test configuration for the polarized deminormal pair \( (\tilde{X}, \Delta_{\tilde{X}}, \nu^*L) \) where \( \nu : \tilde{X} \to X \) is the normalization and \( \Delta_{\tilde{X}} = \nu_{*}^{-1}\Delta + \text{cond}_{\tilde{X}} \). Then,

\[
\text{DF}_{\Delta_{\tilde{X}}} (\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) = \text{DF}_{\Delta} (\mathcal{X}', \mathcal{L}') \leq \text{DF}_{\Delta} (\mathcal{X}, \mathcal{L}).
\]

3. Review of Basic Results on K-stability and singularities

Let us recall results on the relationship between singularities and K-stability of [32] first. Next, we introduce the notion of \( \mathfrak{f} \)-stability, which is a modified version of fibration stability introduced by Dervan and Sektnan [12]. Then, we prove in §5 that \( \mathfrak{f} \)-stability puts restrictions on the singularities on \( (X, \Delta) \) as K-stability does. Moreover, we apply our results on \( \mathfrak{f} \)-stability to study singularities on adiabatic K-semistable log Fano fibrations.

Recall that Odaka proved that K-stability puts the restrictions on the singularities on varieties. In this section, recall those remarkable results of [32] and the logarithmic generalizations [5].

Definition 3.1. Let \( Z \) be a closed subscheme of a proper normal variety \( X \) and \( a_Z \subset \mathcal{O}_X \) be the coherent sheaf of ideals corresponding to \( Z \). Let also \( \pi : \tilde{X} \to X \) be the normalization of the blow up of \( X \) along \( Z \). We call \( \pi : \tilde{X} \to X \) normalized blow up and call the Cartier divisor \( D = \pi^{-1}(Z) \) corresponding to \( \pi^{-1}(a_Z) \) the inverse image of \( Z \). A divisorial valuation \( \nu \) on \( X \)
is Rees valuation of $Z$ if $v = \frac{\text{ord}_E}{\text{ord}_E(D)}$, where $E$ is an irreducible component of the exceptional divisor of $\pi$. We will denote the set of Rees valuations of $Z$ by Rees$(Z)$.

The following is the main theorem of [32],

**Theorem 3.2 ([32] Theorems 1.2, 1.3, [5] §9).** Let $(X, B, L)$ be a polarized demi-normal pair such that $L$ is effective. Then the following hold.

(i) If $(X, B, L)$ is K-semistable, then $(X, B)$ has only slc singularities.

(ii) If $(X, B, L)$ is K-semistable and $L = -K_X - B$, then $(X, B)$ has only klt singularities.

We will prove the generalization of this theorem in Theorems [5, 13] and [5, 13]. Let us recall the proof of this theorem briefly. First, we need the following result in the minimal model program, which is called the lc modification or slc modification,

**Theorem 3.3 ([33], [16]).** (i) Let $(X, B)$ be a normal log pair such that $B$ is effective. If $f : (Y, \Delta_Y) \to X$ is a birational morphism such that $K_Y + \Delta_Y$ is $f$-ample and $(Y, [\Delta_Y])$ is lc with $\Delta_Y = f_*^{-1}B + \sum E$ where $E$ runs over the irreducible and reduced $f$-exceptional divisors, then we call $f : (Y, [\Delta_Y]) \to X$ the lc modification of $(X, B)$ and the lc modification is unique up to isomorphism.

(ii) Let $(X, B)$ be a normal log pair such that $B$ is effective and $[B]$ is reduced. Then, there exists the lc modification $f : (Y, \Delta_Y) \to X$ of $(X, B)$.

(iii) Let $(X, B)$ be a demi-normal log pair such that $B$ is effective and $[B]$ is reduced. Then, there exists a morphism $f : (Y, \Delta_Y) \to X$ such that

- $f$ is projective and isomorphic in codimension 1,
- $(Y, \Delta_Y)$ is slc for $\Delta_Y = f_*^{-1}B + E$ where $E$ is the sum of all reduced $f$-exceptional divisors,
- $K_Y + \Delta_Y$ is $f$-ample.

We call such $f : (Y, \Delta_Y) \to X$ the slc modification of $(X, B)$ and the slc modification is unique up to isomorphism.

Let us recall the proof briefly. To prove (ii), we apply the results of [20] and [15] (see the detail in [34]). (i) follows from the argument of [16, §4]. Here, note that we do not assume that $[B]$ is reduced. To prove (iii), we take the normalization $\nu : (X', B' + D) \to (X, B)$ where $D$ is the conductor divisor on $X'$. Then, there exists the lc modification $f' : (Y', \Delta_Y') \to (X', B' + D)$ by (ii). Let $n : D^n \to D$ be the normalization and $D' = f'^{-1}D$. In general, $[\text{Diff}_D(B')]$ is not reduced. Here, Diff$_D(B')$ is the different and see the definition in [26, §16]. However, if $n' : D'^n \to D'$ is the normalization,

\[(D^n, \text{Diff}_D'((\Delta_Y - D'))) \to (D^n, \text{Diff}_D(B'))\]

is the lc modification in the sense of (i) (cf., [16, 4.4]). Therefore, the involution of $D^n$ lifts to $D'^n$ and (iii) follows from Kollár’s gluing theorem [25, Theorem 5.13] (see the detail in [34] and [16]).

By Theorem 3.3, we can conclude as follows.

**Corollary 3.4.** Let $(X, B)$ be a non-slc pair such that $B$ is effective and $[B]$ is reduced. Let also $\nu : (X', \nu_*^{-1}B + \text{cond}) \to (X, B)$ be the normalization. Then, there exists a closed subscheme $Z$ on $X$ such that $A(X, B)(v) := A(X', \nu_*^{-1}B + \text{cond})(v) < 0$ for $v \in \text{Rees}(\nu_*^{-1}Z)$.

We also remark the following theorem,

**Theorem 3.5 ([5, Theorem 4.8]).** Let $Z \subset X$ be a closed subscheme of a normal variety. Then, if $X$ is the normalization of the deformation to the normal cone of $Z$, the set of Rees valuations of $Z$ coincides with the set of valuations $v_E$, where $E$ runs over the irreducible components of $X_0$.

Therefore, Theorem 3.2 follows from the calculation as [5, Proposition 9.12]. We will prove in §5 that $f$-stability (we will define in §11 below) puts restrictions on the singularity as the above argument.
4. FIBRATION STABILITY AFTER DERVIN-SEKTNAN

In this section, we introduce the notion of $f$-stability that is a modified version of fibration
stability, which is introduced by Dervan and Sektnan [12]. Next, we show the fundamental
results of $f$-stability (e.g., Theorem 3) and compare $f$-stability with fibration stability.

First, recall the notion of fibration degeneration introduced by Dervan and Sektnan [12] as
test configuration in K-stability.

Definition 4.1. Let $\pi : (X, H) \to (B, L)$ be a polarized algebraic fiber space (cf., Definition
2.15). $\Pi : (\mathcal{X}, \mathcal{H}) \to B \times \mathbb{A}^1$ is a fibration degeneration for $f$ if the following hold,

- $(\mathcal{X}, \mathcal{H} + j\Pi^*(L \otimes O_{\mathbb{A}^1}))$ is an ample test configuration over $(X, H + j\pi^*L)$ for sufficiently
  large $j > 0$,
- $\Pi$ is $\mathbb{G}_m$-equivariant over $B \times \mathbb{A}^1$,
- Let $\Pi_1$ be the restriction of $\Pi$ to the fiber over $1 \in \mathbb{A}^1$. Then $\pi = \Pi_1$.

We will need a weaker concept which we call a semi fibration degeneration where $\mathcal{H}$ is relatively
semistable over $B \times \mathbb{A}^1$.

This notion coincides with [12] Definition 2.16 due to [loc.cit, Lemma 2.15 and Corollary
2.24] and next proposition when $f$ is flat.

Proposition 4.2. Let $\pi : (X, H) \to (B, L)$ be a polarized algebraic fiber space. If $(\mathcal{X}, \mathcal{H})$ is a
test configuration such that $(\mathcal{X}, \mathcal{H} + j\pi^*L_k)$ is a semiample test configuration for sufficiently
large $j$, where $\mathcal{X}$ dominates the trivial test configuration $X \times \mathbb{A}^1$ (hence also $B \times \mathbb{A}^1$), then
there exists a fibration degeneration for $f$ dominated by $(\mathcal{X}, \mathcal{H})$.

Proof. By assumption, $(\mathcal{X}, \mathcal{H} + j\pi^*L_k)$ is semiample for some large $j$ and hence $\mathcal{H}$ is semiample
over $B \times \mathbb{A}^1$. Therefore, if $\Pi$ is a canonical morphism from $\mathcal{X}$ to $B \times \mathbb{A}^1$, $\Pi^*\mathcal{H}^j \to \mathcal{H}^k$
defines a morphism $f_k : \mathcal{X} \to \mathcal{P}_{\mathbb{P}(\mathcal{H}^k)}(\mathcal{X})$ for sufficiently large $k$. If $(Y_k, \mathcal{X})$ is
the image of $f_k$, this is a fibration degeneration for $f$ dominated by $(\mathcal{X}, \mathcal{H})$. \hfill $\Box$

Then we define the almost triviality of fibration degenerations similarly to test configurations.

Definition 4.3. A semi fibration degeneration $(\mathcal{X}, \mathcal{H})$ is almost trivial if $(\mathcal{X}, \mathcal{H})$ is almost trivial
as a test configuration.

Now, we can define $f$-stability as follows,

Definition 4.4. Suppose that $f : (X, \Delta, H) \to (B, L)$ is a polarized algebraic fiber space with
a boundary $\Delta$. Let $m = \rel \dim f$ and $\dim B = n$. For any normal fibration degeneration
$(\mathcal{X}, \mathcal{H})$ for $f$, we define constants $W^\Delta_0(\mathcal{X}, \mathcal{H}), W^\Delta_1(\mathcal{X}, \mathcal{H}), \ldots, W^\Delta_n(\mathcal{X}, \mathcal{H})$ and a rational function
$W^\Delta_{n+1}(\mathcal{X}, \mathcal{H})_j$ so that the partial fraction decomposition of $M^\Delta_N(\mathcal{X}, \mathcal{H} + jL)$ in $j$ is as follows:

\[ V(H + jL)M^\Delta_N(\mathcal{X}, \mathcal{H} + jL) = W^\Delta_{n+1}(\mathcal{X}, \mathcal{H})_j + \sum_{i=0}^{n} j^i W^\Delta_{n-i}(\mathcal{X}, \mathcal{H}). \]

See also [12] Lemma 2.25. Then $f : (X, \Delta, H) \to (B, L)$ is called

- $f$-semistable if $W^\Delta_0 \geq 0$ and $W^\Delta_i(\mathcal{X}, \mathcal{H}) = W^\Delta_1(\mathcal{X}, \mathcal{H}) = \cdots = W^\Delta_i(\mathcal{X}, \mathcal{H}) = 0 \Rightarrow W^\Delta_{i+1}(\mathcal{X}, \mathcal{H}) \geq 0$
for $i = 0, 1, \ldots, n - 1$ for any fibration degeneration. Equivalently, $f$ is $f$-semistable if
$\sum_{i=0}^{n} j^i W^\Delta_{n-i}(\mathcal{X}, \mathcal{H}) \geq 0$ for sufficiently large $j > 0$.

- $f$-stable if $f$ is $f$-semistable and $W^\Delta_i(\mathcal{X}, \mathcal{H}) = 0, i = 0, 1, \ldots, n - 1 \Rightarrow W^\Delta_n(\mathcal{X}, \mathcal{H}) > 0$
for any non-almost-trivial fibration degeneration of $(X, H)$.

Note that $f$ is
Lemma 4.5. Let \( f \) be a morphism of varieties and \( D \) a Cartier divisor on \( B \) such that \( f(X) \not\subset D \), then we can define the Cartier divisor \( f^*D \). Let \( (X,\mathcal{H}) \) be a normal semi fibration degeneration for \( \pi \). Then, for sufficiently large \( k \in \mathbb{Z} \), there is \( D \in |kL| \) such that \( (X \cap D, \mathcal{H}_{|X\cap D}) \) is the test configuration of \( (X \cap D, H_{|X\cap D}) \) and \( k^{n+m-1}W_1(X,\mathcal{H}) = (n+m-1)W_1(X \cap D, \mathcal{H}_{|X\cap D}). \)

Proof. As in [12, p. 17], let

\[
C_1(\mathcal{X}, \mathcal{H}) = \frac{m}{m+2} \left( \frac{-K_{X/B} \cdot L^n \cdot H^{m-1}}{L^n \cdot H^m} \right) L^{n+1} \cdot H^{m+2},
\]

\[
C_2(\mathcal{X}, \mathcal{H}) = \frac{m}{m+1} \left( \frac{-K_{X/B} \cdot L^n \cdot H^{m-1}}{L^n \cdot H^m} \right)^2 L^n \cdot H^{m+1},
\]

\[
C_3(\mathcal{X}, \mathcal{H}) = \frac{-K_X \cdot L^{n-1} \cdot H^m}{L^n \cdot H^m} L^n \cdot H^{m+1},
\]

\[
C_4(\mathcal{X}, \mathcal{H}) = L^{n-1} \cdot H^{m+1} \cdot K_{X/B}^{\log}. \]

Then, we obtain

\[
W_1(X,\mathcal{H}) = \left( \frac{n+m}{n-1} \right) \sum_{j=1}^4 C_j(\mathcal{X}, \mathcal{H}).
\]

The set of \( D \in |kL| \) such that \( X \cap D, \mathcal{X} \cap D \) and \( D \) are normal is Zariski-dense open subset of \( |kL| \) for \( k \gg 0 \) by the Bertini theorem. We may assume that \( k = 1 \) by replacing \( L \). Moreover,
However, we can choose a general point \( b \in B \) so that \( X \cap D \) dominates \( \mathbb{P}^1 \), it is flat over \( \mathbb{P}^1 \). Thus, \( (\mathcal{X} \cap D, \mathcal{H}_{|\mathcal{X} \cap D}) \) is an ample test configuration over \( X \cap D \). We can apply the adjunction formula in [26, §16, §17] for \( D \) and the pullbacks of this to \( X \) and \( \mathcal{X} \) since the pullbacks of \( D \) are normal Cartier divisors. By these facts, we obtain

\[
(K_X - \pi^* K_B)|_{\pi^* D} = K_{\pi^* D} - \pi^* K_D + (\pi^* D - \pi^* D)|_{\pi^* D} = K_{\pi^* D} - \pi^* K_D.
\]

Note that \( K_X \) and \( \mathcal{X}_{0,\text{red}} \) are Cartier in codimension 1 but not \( \mathbb{Q} \)-Cartier entirely in general. However, we can choose \( D \) so general that \( K_X \) is also Cartier on any point of codimension 1 in \( D \) and then the adjunction formula holds for such \( D \) (cf., [26, 16.4.3]). Hence, we obtain

\[
\frac{C_1(\mathcal{X}, H) + C_2(\mathcal{X}, H) = C_1(\mathcal{X} \cap D, H_{|\mathcal{X} \cap D}) + C_2(\mathcal{X} \cap D, H_{|\mathcal{X} \cap D})}{\text{Lemma 4.9.}}.
\]

Notations as in the proof of Proposition 4.7. Then, it follows that

\[
\frac{\text{Lemma 4.8}}{\text{Lemma 4.9.}}.
\]

For the readers’ convenience, we prove this lemma here. First, we prove the following claim. Let \( \mathcal{X} \) be a fibration degeneration for \( f : X \to B \). Then a general fiber of \( \mathcal{X} \) over \( b \in B \) is a test configuration \( \mathcal{X}_b \) for \( X_b \).

It suffices to show that \( \mathcal{X}_b \) is flat over \( \mathcal{A}^1 \). Since \( B \) is normal, \( \mathcal{X} \) is flat over codimension 1 points of \( B \times \mathcal{A}^1 \). Therefore, \( \mathcal{X} \) is also flat over \( (b, 0) \) for general points \( b \in B \). On the other hand, by cutting by ample divisors, we may assume that \( \dim B = 1 \) and may also assume that \( X \) is flat over \( B \) due to the same argument of the proof of Proposition 4.7 and the Bertini theorem. Therefore, \( X \times (\mathcal{A}^1 \setminus \{0\}) \) is flat over \( B \times (\mathcal{A}^1 \setminus \{0\}) \). Hence, for general point \( b \in B \), \( \mathcal{X}_b \) is flat over \( \mathcal{A}_b^1 \). Thus, the assertion of the lemma follows from as in the proof of Proposition 4.7.

\[
\text{Lemma 4.8 (Lemma 2.33 of [12])}. \quad \text{Let} (\mathcal{X}, H) \text{ be a normal semi fibration degeneration for a polarized algebraic fiber space} f : (X, H) \to (B, L). \quad \text{Then}
\]

\[
W_0(\mathcal{X}, H) = \left( \frac{n + m}{n} \right) V(L) \cdot M^{\text{NA}}(\mathcal{X}_b, H_b)
\]

holds for general \( b \in B \).

\[
\text{Proof.} \quad \text{For the readers’ convenience, we prove this lemma here. First, we prove the following claim. Let} \ X \text{ be a fibration degeneration for} \ f : X \to B. \text{Then a general fiber of} \ \mathcal{X} \text{ over} \ B \text{ is a test configuration} \ \mathcal{X}_b \text{ for} \ X_b. \text{It suffices to show that} \ \mathcal{X}_b \text{ is flat over} \ \mathcal{A}_b^1. \text{Since} \ B \text{ is normal,} \ \mathcal{X} \text{ is flat over codimension 1 points of} \ B \times \mathcal{A}_b^1. \text{Therefore,} \ \mathcal{X} \text{ is also flat over} \ (b, 0) \text{ for general points} \ b \in B. \text{On the other hand, by cutting by ample divisors, we may assume that} \ \dim B = 1 \text{ and may also assume that} \ X \text{ is flat over} \ B \text{ due to the same argument of the proof of Proposition 4.7 and the Bertini theorem. Therefore,} \ X \times (\mathcal{A}_b^1 \setminus \{0\}) \text{ is flat over} \ B \times (\mathcal{A}_b^1 \setminus \{0\}). \text{Hence, for general point} \ b \in B, \ \mathcal{X}_b \text{ is flat over} \ \mathcal{A}_b^1. \text{Thus, the assertion of the lemma follows from as in the proof of Proposition 4.7.}
\]

To calculate \( W_1 \), it suffices to show the following by Proposition 4.7.

\[
\text{Lemma 4.9.} \quad \text{Let} \ \pi : (X, H) \to (B, L) \text{ be a polarized algebraic fiber space over a smooth curve} \ B. \text{Let} \ \deg L = 1. \text{If} (\mathcal{X}, H) \text{ is a normal semi fibration degeneration for} \ \pi, \text{then}
\]

\[
W_1(\mathcal{X}, H) = V(H)(M^{\text{NA}}(\mathcal{X}, H) + (S(X_b, H_b) - S(X, H))(E^{\text{NA}}(H) - E^{\text{NA}}(H_b))).
\]

\[
\text{Proof.} \quad \text{Notations as in the proof of Proposition 4.7. Then it follows that}
\]

\[
\begin{align*}
C_1(\mathcal{X}, H) &= V(H)E^{\text{NA}}(H)S(X_b, H_b) \\
C_2(\mathcal{X}, H) &= -V(H)E^{\text{NA}}(H_b)S(X_b, H_b) \\
C_3(\mathcal{X}, H) &= (V(H)E^{\text{NA}}(H_b)S(X, H) \\
C_4(\mathcal{X}, H) &= V(H)(M^{\text{NA}}(\mathcal{X}, H) - E^{\text{NA}}(H)S(X, H))
\end{align*}
\]
immediately from a direct computation.

For general cases, we can calculate $W_i$ by cutting $B$ by general divisors in $|L|_Q$.

Therefore, we can conclude that the original fibration stability defined by Dervan and Sektnan \[ is the property of codimension 1 points of the base $B$ as we see in the next remark.

**Remark 4.10.** In [12, Definitions 2.26, 2.28], the fibration stability is defined as the positivity of $W_0(\mathcal{X}, \mathcal{H})$ and $W_1(\mathcal{X}, \mathcal{H})$ for fibration degeneration $(\mathcal{X}, \mathcal{H})$ whose minimum norm

$$
\|((\mathcal{X}, \mathcal{H}))\|_m = \frac{L^n \cdot H^{m+1}}{m+1} + L^n \cdot H^m \cdot (H - H_{P1})
$$

(cf., [12, Definition 2.27]) does not vanish. Here, the notations as in Proposition 4.7. It is easy to see that a normal fibration degeneration that has the minimum norm $\|((\mathcal{X}, \mathcal{H}))\|_m = 0$ is not necessarily trivial but the general fiber is the trivial test configuration (see [12, Remark 2.30]). Indeed, let $f : X \to B$ be an algebraic fiber space over a surface and $(\mathcal{X}, \mathcal{H})$ be a deformation to the normal cone of a point $x \in X$. Then $\mathcal{X}$ has a general fiber that is a trivial test configuration and $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$ by Proposition 4.7. In this respect, $f$-stability is different from fibration stability even when the base is a curve. See also Example 4.17.

On the other hand, we obtain the following when $X$ is flat over $B \times \mathbb{A}^1$:

**Proposition 4.11.** Let $\pi : (X, H) \to (B, L)$ be a flat polarized algebraic fiber space. Let $(\mathcal{X}, \mathcal{H})$ is a normal fibration degeneration for $\pi$ such that $(\mathcal{X} \cap D, \mathcal{H}|_{\mathcal{X} \cap D})$ is a trivial test configuration for general $D \in |mL|$ and for sufficiently large $m \gg 0$. If the canonical morphism $\Pi : X \to B \times \mathbb{A}^1$ is flat, then $(\mathcal{X}, \mathcal{H})$ is the trivial test configuration.

**Proof.** We may assume that $\mathcal{X} \cap D$ is trivial for general member $D$ of $|mL|$, $m \gg 0$. By assumption, $(\mathcal{X}, \mathcal{H} + jL)$ is an ample test configuration for all $j \gg 0$, and hence it suffices to show that $X \times B \times \mathbb{A}^1$ is isomorphic in codimension 1 by the fact that if $(\mathcal{X}, \mathcal{H} + jL)$ is an almost trivial test configuration and normal then it is trivial. Let $a_i \in \mathbb{N}$, $D_i$ be the irreducible divisors and $X_0 = \sum a_i D_i$ be the central fiber over $\mathbb{A}^1$. We can check that $p_0 : X_0 \to B$ is a flat proper fibration and hence all the $D_i$ dominates $B$ via $p_0$. For general $D$, $X_0 \cap D = \sum a_i (D_i \cap D)$ is the cycle decomposition and coincides to the integral cycle $X \cap D$. Therefore, $X_0$ is integral.

By the valuative criterion for properness [21, II Theorem 4.7], we get a rational map $\psi : X_0 \dasharrow X$ from $\phi : \mathcal{X} \dasharrow X \times \mathbb{A}^1$ the canonical birational map. Let $\Gamma$ be the normalized graph of $\phi$ and $\pi : \Gamma \to X$ be the canonical projection respectively. Let $\Gamma_0$ be the central fiber of $\Gamma$ over $0 \in \mathbb{A}^1$ and $E_0$ be the irreducible component of $\Gamma_0$ corresponding to $X_0$. We must show that $q|_{E_0}$ is birational. Take the largest open subset $U \subset X$ such that $U$ has a morphism $\iota : U \to \Gamma$ that is the local inverse of $p$. Let $U_0$ be the central fiber of $U$ over $\mathbb{A}^1$. Note that $U_0 \neq \emptyset$ since codim$(\mathcal{X} \setminus U) \geq 2$.

Take $D \in |mL|$ such that $\mathcal{X} \cap D$, $X_0 \cap D$, $X \cap D$, $E_0 \cap D$ and $\Gamma \cap D$ are normal by the theorem of Bertini. We may assume that $\mathcal{X} \cap D$, $X_0 \cap D$, $X \cap D$, $E_0 \cap D$ and $\Gamma \cap D$ are also connected. Indeed, if dim $B \geq 2$, they are connected. Otherwise, dim $B = 1$. Then, we may assume that deg $D = 1$ and take a general fiber over $B$. We remark that $E_0 \cap D \neq \emptyset$ because $E_0$ dominates $B$. Since $\mathcal{X} \cap D$ and $X_{\mathbb{A}^1 \cap D}$ are isomorphic, $U \cap D$ and $\Gamma \cap D$ are birational equivalent. Assume that $q|_{E_0} : E_0 \to X$ is not a birational morphism. Assume also that $q|_{E_0}$ is not dominant, then $q|_{E_0 \cap D}$ is not birational for general $D$. This contradicts to $U_0 \cap D$ is embedded into $X$ via $q \circ \iota$. Hence $q|_{E_0}$ is dominant. Since we assume that $q|_{E_0}$ is not birational, deg$(q|_{E_0}) > 1$. This contradicts to deg$(q|_{E_0}) = \deg(q|_{E_0 \cap D}) = 1$. Therefore, $q|_{E_0} : E_0 \to X \times \{0\}$ is a birational morphism.

To prove Theorem 5.1 below, we need to calculate $W_i^A$ of an arbitrary normalized semi fibration degeneration $(\mathcal{X}, \mathcal{H})$ for $\pi$ and $i \geq 2$ as in Proposition 4.7. Indeed, we can prove the logarithmic version of Proposition 4.7 similarly. However, the coefficients are more complicated when $i \geq 2$ and we want to ignore the effects of higher codimensional points on $B$.

Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair. Suppose that $m = \text{rel.dim } f$ and $n = \text{dim } B$. First, we may assume that $H$ is ample by Lemma 4.15. Take general
ample divisors $D_1, D_2, \ldots, D_n \in |ML|$ for a fixed integer $M \gg 0$ in the sense of the Bertini theorem. Let $f(j)$ and $g(j)$ be $-K_{(X, \Delta)} \cdot (H + jL)^{n+m-1}$ and $(H + jL)^{n+m}$ respectively. Note that $f(j)$ and $g(j)$ depend only on $L$ and $H$. On the other hand,

$$S(X, \Delta, H + jL) = \frac{f(j)}{g(j)}.
$$

Note also that

$$(H + jL)^{n+m+1} = \sum_{i=0}^{n} j^{n-i} \binom{n + m + 1}{n - i} \mathcal{H}^{m+1} \cdot L^{n-i}.
$$

Then we can calculate $W_i^\Delta$ as follows:

**Lemma 4.12.** Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair with an ample line bundle $H$ and $(\mathcal{X}, \mathcal{H})$ be a normal semiample test configuration for $(X, H)$ dominating $X_{\Delta_1}$. Suppose also that $m = \text{rel.dim } f$, $n = \dim B$ and $(\mathcal{X}, \mathcal{H})$ is normalized with respect to the central fiber. Then, there exist constants $C_{n-i+1}, \ldots, C_n$ independent from $(\mathcal{X}, \mathcal{H})$ and $j$ such that

$$W_i^\Delta(\mathcal{X}, \mathcal{H}) = \binom{n + m}{n - i} \sum_{i=0}^{n} j^{n-i} \binom{n + m + 1}{n - i} \mathcal{H}^{m+1} \cdot L^{n-i} + \sum_{j=n-i+1}^{n} C_j \mathcal{J}^{\mathcal{N}A}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots D_n}),
$$

where $D_1, D_2, \ldots, D_n \in |ML|$ are general ample divisors for sufficiently large integer $M > 0$ that $ML$ is very ample. Here, $\mathcal{D}$ is the strict transformation of $\Delta \times \mathbb{P}^1$ on $X$. Furthermore,

$$W_i^\Delta(\mathcal{X}, \mathcal{H}) = M^{-(n-i)} \binom{n + m}{n - i} \mathcal{J}^{\mathcal{N}A}(\mathcal{H}_{|X \cap D_1 \cap D_2 \cap \cdots D_n}),
$$

**Notation.** If there is no fear of confusion, we will omit $\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j$ and denote as

$$\mathcal{J}^{\mathcal{N}A}(\mathcal{H}_{|X \cap D_1 \cap D_2 \cap \cdots D_j}) = \mathcal{J}^{\mathcal{N}A}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_j, \mathcal{H}_{|X \cap D_1 \cap D_2 \cap \cdots D_j}).
$$

**Proof of Lemma 4.12.** Note that $(\mathcal{X} \cap D_1, \mathcal{H}_{|\mathcal{X} \cap D_1})$ is also normalized with respect to the central fiber (cf., Definition 2.13). Replacing $L$ by $ML$, we may assume that $M = 1$. For the first assertion,

$$S(X, \Delta, H + jL) = -(n + m) \frac{K_{(X, \Delta)} \cdot (H + jL)^{n+m-1}}{(H + jL)^{n+m}} = S(X, \Delta, H) + O(j^{-1})
$$

where the coefficients of $O(j^{-1})$ depend only on $n, m, H$ and $L$. On the other hand, the $j^{n-i}$-term of $(H + jL)^{n+m+1}$ is as follows

$$\binom{n + m}{n - i} \binom{m + i + 1}{n + m + 1} \mathcal{H}^{m+1} \cdot L^{n-i},
$$

and the higher degree terms of $(H + jL)^{n+m+1}$ are the forms of $C_{n-i+1} \mathcal{J}^{\mathcal{N}A}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots D_n})$ where $l > 0$ and $C_{n-i+1}$ is a constant depending only on $l$ by the argument before this lemma. Here, we have

$$\mathcal{J}^{\mathcal{N}A}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots D_n}) = -\mathcal{E}^{\mathcal{N}A}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap D_2 \cap \cdots D_n}),
$$

for $k$ by Lemma 2.14. Thus, the first assertion holds. We can take $D_i \in |L|$ so general that $K\mathcal{X} \cap D_1 \cap \cdots \cap D_{l-1}$, $\mathcal{X}_{\text{reg}} \cap D_1 \cap \cdots \cap D_{l-1}$ and $\mathcal{D} \cap D_1 \cap \cdots \cap D_{l-1}$ are Cartier on any codimension 1 point of $\mathcal{X} \cap D_1 \cap \cdots \cap D_{l-1}$. By the adjunction formula [26] §16, for example, we have

$$K_{(X, \Delta)/\mathbb{P}^1}^{\log} |X \cap D_1 = K_{(X \cap D_1, D \cap D_1)/\mathbb{P}^1}^{\log} - \mathcal{X} \cap D_1, \mathcal{X} \cap D_1, \mathcal{X} \cap D_1;$$
as in the proof of Proposition 4.14 and hence we obtain the second assertion.

On the other hand, we calculate $W_i$ as Lemma 4.12 when there exists a line bundle $L_0$ on $B$ such that $K_{(X, \Delta)} = \lambda H + f^*L_0$ for $\lambda \in \mathbb{Q}$.

**Proposition 4.13.** Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair with $m = \text{rel.dim} f$ and $n = \dim B$. Furthermore, suppose that $H$ is ample and $L$ is very ample. Let $(X, \mathcal{H})$ be an arbitrary normal semistable test configuration for $(X, H)$ normalized with respect to the central fiber. Suppose also that $(X, \mathcal{H})$ dominates $X_{\Delta^1}$ and that there exist $\lambda \in \mathbb{Q}$ and a line bundle $L_0$ on $B$ such that $\lambda H \equiv K_X + \Delta + f^*L_0$. Then

$$
\left(\left(\frac{n + m}{n - k}\right)(H^{m+k} \cdot L^{-k})\right)^{-1} W^X_{\lambda} (\mathcal{H}_X, \mathcal{H}) = H^0_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}} (\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) + \lambda \left(f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right) - (k + 1) f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right)\right)
$$

if $f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cdots \cap D_{n-k+1}}\right), \ldots, f^NA (\mathcal{H}_b) = 0$ where $D_1, \ldots, D_{n-k} \in |L|$ and $b \in B$ are general.

**Proof.** It follows from Lemma 2.14 that

$$f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right) = -E^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right)$$

and if $f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k+1}}\right), \ldots, f^NA (\mathcal{H}_b) = 0$ for general $D_j$ and $b$, we have

$$\left(\left(\frac{n + m}{n - k}\right)(H^{m+k} \cdot L^{-k})\right)^{-1} R^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right) = p^* \left(\lambda H - f^*L_0\right) \cdot (\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}})^{m+k}$$

where $p : X \to X$ is the composition of canonical morphisms $X \to X_{\Delta^1}$ and $X_{\Delta^1} \to X$. The last equality follows from Lemma 4.14 below inductively. Therefore, we can see as [3 Lemma 7.25],

$$p^* H \cdot (\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}})^{m+k} = \left(\left(\frac{n + m}{n - k}\right)(H^{m+k} \cdot L^{-k})\right) \left(f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right) - (m + k + 1) f^NA \left(\mathcal{H}|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}\right)\right)$$

and $S(X_b, \Delta_b, H_b) = -m\lambda$. Therefore, the assertion follows from Lemma 4.12.

**Lemma 4.14.** Notations as in Proposition 4.13. If $f^NA (\mathcal{H}|_{X \cap D_1}) = 0$ for a general ample divisor $D_1$ that is $\mathbb{Q}$-linearly equivalent to $L$, then for any line bundle $M$ on $B$, $\mathcal{H}^{n+m} \cdot p^* f^*M = 0$.

**Proof.** Let $F = \mathcal{H} - H_{\Delta^1}$. Since $\mathcal{H}$ is normalized, $F$ is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and note that $\mathcal{H}|_F$ and $H_{\Delta^1}|_F$ are nef. By assumption, we have

$$0 = \mathcal{H}^{n+m} \cdot p^* f^*L = - \sum_{i=0}^{n+m-1} F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\Delta^1}^i \cdot p^* f^*L$$

since $H_{\Delta^1}^{n+m} \cdot p^* f^*L = 0$. On the other hand, $p^* f^*L|_F$ is also nef and hence $F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\Delta^1}^i \cdot p^* f^*L = 0$ for $0 \leq i \leq n + m - 1$. Take $k \gg 0$ such that $kL \pm M$ are very ample on $B$. Then $F \cdot H^{n+m-1-i} \cdot H_{\Delta^1}^i \cdot p^* f^*M = F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\Delta^1}^i \cdot p^* f^*(kL \pm M) \geq 0$. Therefore,

$$\sum_{i=0}^{n+m-1} F \cdot \mathcal{H}^{n+m-1-i} \cdot H_{\Delta^1}^i \cdot p^* f^*M = -\mathcal{H}^{n+m} \cdot p^* f^*M = 0.$$ 

Hence, we obtain Lemma 4.14 and Proposition 4.13.
Now, we can show that klt Calabi-Yau fibrations are $f$-stable as the well-known theorem in K-stability [33, Theorem 4.1].

**Theorem 4.15.** Let $f : (X, \Delta, H) \rightarrow (B, L)$ be a polarized algebraic fiber space pair with a line bundle $L_0$ on $B$. Suppose that $(X, \Delta, H)$ is a klt polarized pair and $K_X + \Delta \equiv f^*L_0$. Then $f$ is $f$-stable.

**Proof.** Let $(\mathcal{X}, \mathcal{H})$ be a normal semi-fibration degeneration for $f$ (cf., Definition 4.11), $m = \text{rel.dim } f$ and $n = \dim B$. By replacing $H$ by $H + jL$ for $j \gg 0$, we may assume that $H$ is ample and $(\mathcal{X}, \mathcal{H})$ is a normal semiample test configuration normalized with respect to the central fiber. We will prove that if $(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}, \mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k-1}})$ is trivial and $(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}, \mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k-1}})$ is trivial for general divisors $D_i \in |ML|$ for $M \gg 0$, then $W^\Delta_{k+1}(\mathcal{X}, \mathcal{H}) > 0$ by induction on $k$. Suppose that $(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k+1}, \mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k+1}})$ is trivial for general divisors $D_i \in |ML|$ for $M \gg 0$. By Proposition 4.13,

$$
(n + m - n - k) M^{-n-k} W^\Delta_k(\mathcal{X}, \mathcal{H}) = (H^{m+k} \cdot L^{n-k}) \mathcal{H}^{NA}_{\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}})
$$

for general $D_{n-k} \in |ML|$. It is easy to see by [3] Proposition 9.16 that if $(X, \Delta)$ is klt, then $X \cap D_1 \cap \cdots \cap D_{n-k}, \Delta \cap D_1 \cap \cdots \cap D_{n-k}$ is klt for general divisors $D_i \in |ML|$ and hence $H^{NA}_{\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}}(\mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}}) > 0$ unless $(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}, \mathcal{H}_{|\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}})$ is the trivial test configuration. 

Next, we discuss [12, Corollary 2.35]. Dervan and Sektnan conjectured that fibrations whose all fiber is K-stable are fiberation stable in the original sense [10]. They also conjectured the following,

**Conjecture 4.16** (Dervan-Sektnan [12]). Let $f : (X, H) \rightarrow (B, L)$ be a polarized smooth fibration such that any fiber is K-stable. If $\text{Aut}(X_b, H_b) = 0$ for any $b \in B$, $f$ is fiberation stable in the original sense [10].

In fact, they proved such fibrations are fiberation stable in the sense of [12]. However, there exists a following singular example for $f$-stability.

**Example 4.17.** Suppose that $B$ is a lc surface that has at least one isolated lc singularity (e.g., a projective cone of an elliptic curve) and $C$ is a curve whose genus $g(C) > 2$ with no automorphism other than the identity (cf., [11]). Let $X = C \times B$ be an algebraic fiber space over $B$. Note that $X$ has the relative canonical ample divisor over $B$. Let $L$ be an ample line bundle on $B$ and $H = K_{X/B} + rL$ for sufficiently large $r > 0$. Since $B$ has finite singularities, $X$ has lc centers of codimension 2. Let $b_0$ be one of lc singularities on $B$. Take the deformation to the normal cone $(\mathcal{X}, \mathcal{H})$ of $C \times b_0$. We can easily see that $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$ by Proposition 4.17. We can compute $W_2(\mathcal{X}, \mathcal{H})$ as follows. Note that $H^{NA}(\mathcal{X}, \mathcal{H}) = 0$ and $R^{NA}(\mathcal{X}, \mathcal{H}) + S(\mathcal{X}_b, K_{X_b})E^{NA}(\mathcal{X}, \mathcal{H}) = I^{NA}(\mathcal{X}, \mathcal{H}) - J^{NA}(\mathcal{X}, \mathcal{H})$ by Proposition 4.13. Let $E = H_{p_1} - \mathcal{H}$ be the exceptional divisor whose center has codimension 3 in $X \times \mathbb{P}^1$. We can see $H_{p_1} = H_{p_1}^2 \cdot E = H_{p_1}^2 \cdot E^2 = 0$. Then, we have

$$
V(H)I^{NA}(\mathcal{X}, \mathcal{H}) = E \cdot \mathcal{H}^3 = 3E^3 \cdot H_{p_1} - E^4,
$$

$$
-V(H)J^{NA}(\mathcal{X}, \mathcal{H}) = \frac{1}{4} \mathcal{H}^4 = -E^3 \cdot H_{p_1} + \frac{1}{4} E^4.
$$

Therefore,

$$
V(H)(I^{NA}(\mathcal{X}, \mathcal{H}) - 3J^{NA}(\mathcal{X}, \mathcal{H})) = -\frac{1}{4} E^4.
$$

Since $E$ is the pullback of the exceptional divisor on the blow-up of $B \times \mathbb{P}^1$, we have $E^4 = 0$. Hence, $(X, H) \rightarrow (B, L)$ is not $f$-stable. It also follows that Conjecture 4.16 does not hold in the original sense or for $f$-stability.
5. Singularities of semistable algebraic fiber spaces

In this section, we see that \( \mathcal{f} \)-semistability implies that \( X \) has only lc singularities similarly to [32]. Moreover, we obtain Theorem 5.2, which states that \( \mathcal{f} \)-semistable Fano fibrations have no horizontal non-klt singularities. In the latter part of this section, we consider the case when \( X \) is non-normal.

5.1. Normal case. We prove the following generalizations of [32] Theorems 1.2, 1.3 in terms of \( \mathcal{f} \)-stability respectively:

Theorem 5.1. Let \( f : (X, \Delta, H) \to (B, L) \) be a polarized algebraic fiber space pair. If \( f \) is \( \mathcal{f} \)-semistable, \( (X, \Delta) \) has at most lc singularities.

Theorem 5.2. Let \( f : (X, H) \to (B, L) \) be a flat polarized algebraic fiber space. Suppose that there exist \( \lambda \in \mathbb{Q}_{>0} \) and a line bundle \( L_0 \) on \( B \) such that \( H + f^*L_0 \equiv -\lambda K_X \) and \( B \) has only klt singularities. If \( f \) is \( \mathcal{f} \)-semistable, then \( X \) has only klt singularities. In particular, if \( f \) is adiabatically K-semistable (Definition 2.10), then \( X \) has only klt singularities.

Proof of Theorem 5.1. Suppose that \( m = \text{rel.dim} \, f \) and \( n = \text{dim} \, B \) and assume that \( (X, \Delta) \) is not lc. First, assume that \( [\Delta] \) is reduced. Then, by Theorem 1.1 of [34], there exists the lc modification \( \hat{X} \) of \( (X, \Delta) \). Thus, we can replace a closed subscheme \( Z \subset X \) whose Rees valuations \( v \) all satisfy \( A_{(X, \Delta)}(v) < 0 \) (cf. [5] Proposition 9.1) and \( \pi : \hat{X} \to X \) is the blow up along \( Z \). We can also easily check that the normalization of the deformation to the normal cone \( X \) of \( Z \) is a fiber degeneration for \( f \). Let \( \rho : \mathcal{X} \to X \times \mathbb{A}^1 \) be the canonical projection, \( E \) be the inverse image of \( Z \times \{0\} \) and \( \mathcal{H}_e = \rho^*H_K - \epsilon E \) for sufficiently small \( \epsilon > 0 \). We may assume that \( H \) is ample and there exists a positive constant \( \epsilon_0 > 0 \) such that \( (\mathcal{X}, \mathcal{H} = \mathcal{H}_0) \) is a normal ample test configuration by replacing \( H \) by \( H + jL \) for \( j \gg 0 \) (cf., Lemma 4.5). Hence, \( (\mathcal{X}, \mathcal{H}_e) \) is an ample test configuration for \( 0 < \epsilon < \epsilon_0 \). Note that \( (\mathcal{X}, \mathcal{H}_e) \) is normalized with respect to the central fiber. Now, we prove that for sufficiently small \( 0 < \epsilon < \epsilon_0 \), there exists \( k \) such that \( W_k^2(\mathcal{X}, \mathcal{H}_e) < 0 \) but \( W_k^2(\mathcal{X}, \mathcal{H}_e) = 0 \) for \( i < k \). Take \( D_1, D_2, \ldots, D_n \in |ML| \) general for sufficiently divisible integer \( M > 0 \). By replacing \( L \) by \( ML \), we may assume that \( M = 1 \) as in Lemma 4.12. Then there exists an integer \( k \) that is maximal among integers \( l \) such that \( (\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-l+1}, \mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-l+1}}) \) is trivial. Therefore, \( W_i^2(\mathcal{X}, \mathcal{H}_e) = 0 \) for \( i < k \) and \( J^{NA}(\mathcal{H}|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k+1}}) = \cdots = J^{NA}(\mathcal{H}_b) = 0 \). By Lemma 4.12,

\[
W_k^2(\mathcal{X}, \mathcal{H}_e) = \left( \frac{n + m}{n - k} \right) \left( \frac{L^{n-k}}{H^{k}} \right) \left( H_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) \right) + S(X_b, \Delta_b, H_b) \frac{m^{n-k+1}}{m + k + 1} \cdot n^{m-k}.
\]

If \( E \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} = \emptyset \), then \( (\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) \) is almost trivial and this contradicts to the maximality of \( k \). Thus we may assume that \( E \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \neq \emptyset \). As in the proof of Proposition 9.12 of [5], let \( \{F_i\}_{i=1}^r \) be the set of prime exceptional divisors of \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) to \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) and suppose that \( \min_{1 \leq i \leq r} \text{codim}_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}(p(F_i)) = r \). Then,

\[
R_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}^2(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = O(\epsilon^{r+1})
\]

\[
E_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}^2(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = O(\epsilon^{r+1})
\]

since the positive non-Archimedean metric \( (\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) \) is also normalized with respect to the central fiber. On the other hand, there exists a positive constant \( T > 0 \) such that

\[
H_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}^2(\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{\mathcal{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = -Te^r + O(\epsilon^{r+1})
\]
if \( A_{(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k})}(F_i) < 0 \) for all \( F_i \) such that \( \text{codim}(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}) \rho(F_i) = r \) by Theorem 3.3. Furthermore, we can take so general \( D_i \)'s that \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) are normal. Then, the theorem when \( [\Delta] \) is reduced holds by the following claim:

**Claim.** If a divisorial valuation \( v \) of \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) is one of the Rees valuations of \( Z \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \), then \( A_{(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k})}(v) < 0 \).

Now, we prove the claim. First, note that \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) is the (normalized) blow up along \( Z \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \). In fact, if \( 0 < k < n \), we can choose \( D_1, D_2, \ldots, D_{n-k} \) so general that \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) is normal and irreducible. Then we can see that \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) is isomorphic to the blow up of \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) along \( Z \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) by \([21]\) Corollary 7.15. On the other hand, suppose that \( k = 0 \). Then \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1} \) has the normal and connected generic geometric fiber over \( D_1 \cap D_2 \cap \cdots \cap D_{n-1} \). It is easy to see that \( g : X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1} \to D_1 \cap D_2 \cap \cdots \cap D_{n-1} \) satisfies that \( g_* \mathcal{O}_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1}} \cong \mathcal{O}_{D_1 \cap D_2 \cap \cdots \cap D_{n-1}} \). Since \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1} \) is birational to \( X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1} \), \( \hat{X} \cap D_1 \cap D_2 \cap \cdots \cap D_{n-1} \to D_1 \cap D_2 \cap \cdots \cap D_{n-1} \) has also connected fibers by the Zariski main theorem \([21] \) III Corollary 11.4. Therefore, \( \hat{X}_b \) is normal and connected for general \( b \in B \) and hence \( \hat{X}_b \to X_b \) is the blowing up along \( X_b \cap Z \). Thus we can conclude that the set of Rees valuations of \( Z \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) coincides with the set of irreducible components of \( E_i \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k} \) for any irreducible component \( E_i \) of \( \pi \)-exceptional divisors.

Next, for instance, we show the claim when \( k = 1 \). Since \( X \cap D_1 \) is a normal Cartier divisor and

\[
K_{\hat{X}} + \hat{X} \cap D_1 = \pi^*(K_X + \Delta + X \cap D_1) + \sum_i (A_{(X, \Delta)}(E_i) - 1)E_i
\]

where \( A_{(X, \Delta)}(E_i) < 0 \), we have by the adjunction formula (cf., \([26] \) 16.3, 16.4, 17.2),

\[
K_{\hat{X} \cap D_1} = \pi^*_{\hat{X} \cap D_1}(K_X \cap D_1 + \Delta \cap D_1) + \sum_i (A_{(X, \Delta)}(E_i) - 1)(E_i \cap D_1).
\]

Indeed, note that \( E_i \cap D_1 \)'s are reduced divisors (maybe reducible) and any \( \pi_{D_1} \)-exceptional divisor coincides with one of irreducible components of \( E_i \cap D_1 \) by Bertini’s theorem. We also remark that each \( E_i \) is not a \( \mathbb{Q} \)-Cartier but Weil divisor. In general, the above adjunction formula does not hold for any \( D_1 \). However, we can choose sufficiently general \( D_1 \) that each \( E_i \) is Cartier at codimension 1 points of \( \hat{X} \cap D_1 \). Then, \( E_i \cap D_1 = j^w E_i \) on \( \hat{X} \cap D_1 \) where \( j : \hat{X} \cap D_1 \to \hat{X} \). Here, \( j^w E_i \) is the extension of \( (j|_{D_1 \cap U})^*(E_i)|_U \to \hat{X} \cap D_1 \) where \( U \) is an open subset of \( \hat{X} \) such that \( \text{codim}(X \cap D_1 \setminus U) \geq 2 \) and \( (E_i)|_U \) is Cartier (see \([26] \) 16.3]). Therefore, the above adjunction formula holds and hence we have

\[
A_{(X \cap D_1, \Delta \cap D_1)}(E_i \cap D_1) = A_{(X, \Delta)}(E_i) < 0.
\]

We also check that \( E_i \cap D_1 \) is exceptional when we take so general \( D_1 \). Therefore, the claim holds when \( k = 1 \). The assertion when \( k > 1 \) follows similarly to the case when \( k = 1 \) inductively. Thus, we can conclude that the claim holds.

Finally, assume that \( \Delta = \sum a_i F_i \) for some \( a_i > 1 \) where \( F_i \) are distinct irreducible components. Fix \( F_j \) such that \( a_j > 1 \). Let \( (X, \mathcal{H}_T) \) be the deformation to the normal cone of \( F_j \) and \( \mathcal{H}_\epsilon = H_{\epsilon_1} - \epsilon E \) where \( E \) is the exceptional divisor for sufficiently small \( \epsilon > 0 \). Let \( k \) be maximal among integers \( l \) such that \( (X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-l+1}, \mathcal{H}_\epsilon|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-l+1}}) \) is trivial. Then by the argument in the previous paragraph, there exists a positive constant \( T > 0 \) such that

\[
H^N_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_\epsilon|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = -T \epsilon + O(\epsilon^2)
\]
since the Rees valuations $v \neq \text{ord}_{F_j}$ of $Z$ have the centers $C$ on $X$ such that $\text{codim}_X C \geq 2$ even if $F_j$ is not a $\mathbb{Q}$-Cartier divisor. On the other hand,

$$R_{\Delta \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}^{\text{NA}}(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = O(\epsilon^2)$$

$$E_{\text{NA}}^{\text{NA}}(X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}, \mathcal{H}_e|_{X \cap D_1 \cap D_2 \cap \cdots \cap D_{n-k}}) = O(\epsilon^2).$$

Thus, $W^\Delta_k(X, \mathcal{H}_e) < 0$ for sufficiently small $\epsilon > 0$. We complete the proof. \qed

To show Theorem 5.2, we need the following,

**Definition 5.3.** Let $f : X \to B$ be an algebraic fiber space. An irreducible closed subset $Z$ of $X$ is of fiber type if $\text{codim}_X Z \leq \text{codim}_B f(Z)$.

Moreover, we prove the following generalization of Theorem 5.2.

**Theorem 5.4.** Let $f : (X, \Delta, H) \to (B, L)$ be a polarized algebraic fiber space pair. Suppose that there exist $\lambda \in \mathbb{Q}_{>0}$ and a line bundle $L_0$ on $B$ such that $H + *f^*L_0 \equiv -\lambda(K_X + \Delta)$, and $f$ is $f$-semistable. Then $(X, \Delta)$ is lc and any lc-center of $(X, \Delta)$ is of fiber type.

**Claim.** Theorem 5.4 implies Theorem 5.2.

**Proof of Claim.** Note that subsets of non-fiber type of $X$ correspond to irreducible closed subsets that do not contain any irreducible component of any fiber of $f$ since $f$ is flat. Due to Theorem 5.4, it suffices to show that any non-klt center of $X$ does not contain any irreducible component of $X_b = f^{-1}(b)$ fiber of $f$ over $b \in B$ if $B$ is klt. This fact immediately follows from Proposition 5.6 and Lemma 5.5 below. The last assertion of Theorem 5.2 follows from the fact that adiabatic K-semistability implies $f$-semistability (see Remark 4.6). \qed

**Lemma 5.5.** Let $f : (X, H) \to (B, L)$ be a smooth surjective morphism. If $B$ has only klt singularities, so does $X$.

**Proof.** Let $\pi : B' \to B$ be a resolution of singularities of $B$ and $(X', \Delta') = (X \times_B B', X \times_B \Delta)$ where $\Delta = \text{Ex}(\pi)$ is snc. Note that $(X', \Delta')$ is log smooth since $f' : X' \to B'$ induced by $f$ is also a smooth morphism whose all fiber is connected. Note also that for any two prime divisors $D \neq D'$ on $B'$, $f'^*(D) = [f'^*(D)]$ and $f'^*(D)$ and $f'^*(D')$ have no common component due to the property of $f'$. Let $\mu : X' \to X$ be the canonical projection respectively. Since $f'$ is the base change of $f$, $\mu^*\Omega_{X/B} = \Omega_{X'/B'}$ and hence $\mu^*K_{X/B} = K_{X'/B'}$. On the other hand, $K_{B'} + F = \pi^*K_B + E$ where $E$ and $F$ are effective divisors that have no common components and $[F] = 0$. Therefore,

$$K_{X'} + f'^*(F) = \mu^*K_X + f'^*(E)$$

where $f'^*(F)$ and $f'^*(E)$ are effective divisors that have no common components and $[f'^*(F)] = 0$. Thus, $X$ has only klt singularities. \qed

**Proposition 5.6.** Let $f : (X, H) \to (B, L)$ be a flat polarized algebraic fiber space whose all fiber is connected and reduced. If $B$ has only klt singularities, $X$ has no lc center whose support contains any irreducible components of $X_b$ for $b \in B$.

**Proof.** Note that a general point of $X_b$ is smooth for $b \in B$. Since $f$ is faithfully flat, we conclude that there exists a closed subset $Z$ such that $f$ is smooth on $X \setminus Z$ and $Z$ contains no component of $X_b$ for any $b \in B$. Hence, the proposition follows from Lemma 5.5. \qed

In the proof of [32] Theorem 1.3, Odaka applied [3] 1.4.3. To prove Theorem 5.4, it is necessary to blow only up lc centers of non-fiber type and we cannot make use of the same argument directly. Hence, we need the slight modification of the technique developed for proving [32] Theorem 1.3 and [5] Proposition 9.9 as follows.
Proposition 5.7. Let \((X, \Delta)\) be an lc pair and \(E_j\) be prime divisors over \(X\) (maybe exceptional) such that \(A_{(X, \Delta)}(E_j) = 0\) for \(1 \leq j \leq r\). Then there exists a closed subscheme \(Z \subset \bigcup_{j=1}^r c_X(E_j)\) such that
\[
\emptyset \neq \text{Rees}(Z) \subset \left\{ \frac{\text{ord}_{E_j}}{\text{ord}_{E_j}(Z)} \mid \text{ord}_{E_j}(Z) \neq 0 \right\}_{j=1}^r.
\]
Furthermore, suppose that one of the following holds:

(1) \(r = 1\) and \(E_1\) is exceptional over \(X\),

(2) Any \(E_j\) is a divisor on \(X\).

Then, we have
\[
\text{Rees}(Z) = \left\{ \frac{\text{ord}_{E_j}}{\text{ord}_{E_j}(Z)} \right\}_{j=1}^r.
\]

Proof. Let \(f : \tilde{X} \to (X, \Delta)\) be a log resolution such that \(E_j\)'s are smooth divisors on \(\tilde{X}\). Suppose that
\[
K_{\tilde{X}} + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + \sum_i A_{(X, \Delta)}(F_i)F_i,
\]
where \(F_i\) are irreducible components of \(f\)-exceptional divisors and \(F = \sum_i F_i\). By assumption, \(A_{(X, \Delta)}(F_i) \geq 0\). Then by the proof of [15, 4.1], we can conclude that the \(K_{\tilde{X}} + f_*^{-1}\Delta + F\)-MMP with scaling over \(X\) terminates with a \(\mathbb{Q}\)-factorial dlt minimal model \((X_1, (f_1)^{-1}\Delta + F_1)\). Here, let \(f_1 : X_1 \to X\) be the structure morphism and \(D_1\) be the strict transform of \(F\). Note that any \(F_i\) such that \(A_{(X, \Delta)}(F_i) \neq 0\) is contracted but any \(E_j\) is not contracted on \(X_1\). Then
\[
K_{X_1} + (f_1)_*^{-1}\Delta + D_1 = f_1^*(K_X + \Delta)
\]
Let \(D'_1\) be the strict transformation of \(\sum_{j=1}^r E_j\) and \(D''_1 = (f_1)_*^{-1}\Delta + D_1 \sim_{X, Q} 0\). Then \(K_{X_1} + D''_1 = (1 - \delta)D'_1\)-MMP with scaling over \(X\) terminates with a good minimal model for \(0 < \delta < 1\) due to [2, Theorem 1.1] or [20, Theorem 1.6]. Therefore, there exists the log canonical model \(X_2\) of \((X_1, (f_1)^{-1}\Delta + (1 - \delta)D'_1)\) over \(X\). Let \(f_2 : X_2 \to X\) be the structure morphism and the strict transformations of \(D'_1\) and \(D''_1\) on \(X_2\) are \(D'_2\) and \(D''_2\) respectively. Then
\[
K_{X_2} + D''_2 + (1 - \delta)D'_2 = f_2^*(K_X + \Delta) - \delta D'_2
\]
and hence \(-D'_2\) is \(f_2\)-ample. Hence, the exceptional set \(\text{Exc}(f_2) \subset D'_2\) and any exceptional divisor other than \(E_1, \ldots, E_r\) is contracted. If the condition (2) holds, \(D'_2 = (f_2)_*^{-1} \left( \sum_{j=1}^r E_j \right) \neq 0\) and hence we have the second assertion. On the other hand, some \(E_j\) is not contracted on \(X_2\). Indeed, if any \(E_j\) is contracted, we have \(D'_2 = 0\). Then, \((X, \Delta) \cong (X_2, D''_2 + (1 - \delta)D'_2)\). This contradicts to that \((X_2, D''_2 + (1 - \delta)D'_2)\) is the log canonical model of \((X_1, (f_1)^{-1}\Delta + (1 - \delta)D'_1)\). Therefore, the first assertion holds. Furthermore, we have the second assertion by the first one if (1) holds.

Proof of Theorem 5.8 First, we may assume that \(\lambda = 1\). Let \(\dim X = N\) and \(\dim B = n\). Assume that the theorem failed. In other words, we assume that there exists an \(f\)-semistable polarized algebraic fiber space pair \(f : (X, \Delta, H) \to (B, L)\) such that there exists at least one lc center of non-fiber type. By Theorem 5.1, it follows that \((X, \Delta)\) is lc. As in the proof of Theorem 5.1, we may also assume that \(H\) is ample and \(L\) is very ample. Then there exists a closed subscheme \(Z\) whose Rees valuations \(v\) have log discrepancies \(A_{(X, \Delta)}(v) = 0\) and are not of fiber type due to Proposition 5.7. Note that \(Z\) is of non-fiber type. Let \((\mathcal{X}, \mathcal{H})\) be the normalization of the deformation to the normal cone of \(Z\), \(E = H_{k-\epsilon} - \mathcal{H}\) be the \(X \times \mathbb{A}^1\)-antiample exceptional divisor and \(\mathcal{H}_\epsilon = H_{k-\epsilon} - \epsilon E\) is a positive non-Archimedean metric normalized with respect to the central fiber. We may assume that \(\mathcal{H}\) is \(\mathbb{A}^1\)-ample. Decompose \(E = E_1 + E_2 + \cdots + E_N\) where each irreducible component \(E_i^{(s)}\) of \(E_i\) has center \(Z_i^{(s)} \subset X \times \{0\}\) such that \(\text{codim}_X Z_i^{(s)} = i\).

We will prove that there exists \(k\) such that \(W_k^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) < 0\) and \(W_i^\Delta(\mathcal{X}, \mathcal{H}_\epsilon) = 0\) for \(i < k\) for sufficiently small \(\epsilon > 0\) as in the proof of Theorem 5.1. First, we take general \(D_i\)'s. As in the proof, there exists \(k\) such that \(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k}\) is not trivial but \(\mathcal{X} \cap D_1 \cap \cdots \cap D_{n-k+1}\) is...
trivial. Similarly to the proof of Claim in Theorem 5.1, we can see all the \( Z \cap D_1 \cap \cdots \cap D_\ell \) also has Rees valuations \( v \) that have log discrepancies \( A_{(X\cap D_1\cap \cdots \cap D_\ell, \Delta \cap D_1(\cdots \cap D_\ell)}(v) = 0 \). Therefore, we may assume that \( k = n \) by cutting \( X \) and \( \mathcal{X} \) by general divisors in \( |L| \) by Proposition 4.13. Furthermore, we may also assume that the cycle \( L \cdot E = 0 \) and hence \( E_1 = E_2 = \cdots = E_{n-1} = E_n = \emptyset \) by the assumption. Let \( r = \min \{|i| E_i \neq \emptyset \} > n \). Note that \( W_{\Delta}^\lambda(\mathcal{X}, \mathcal{H}) \) is \( \Delta \mathcal{X}(\mathcal{X}, \mathcal{H}) - I_{\mathcal{X}}(\mathcal{X}, \mathcal{H}) + (n + 1) J_{\mathcal{X}}(\mathcal{X}, \mathcal{H}) \) and \( H_{\Delta}^\lambda(\mathcal{X}, \mathcal{H}) = 0 \) now by Proposition 4.13. Let \( \{ E_j \}_{j=1}^s \) be the set of irreducible components of \( E_j \). Here, we recall the notation in [5, §9.3]. Let also \( Z_{E_j}^{(i)} \) be the center of \( v_{E_j}^{(i)} \) on \( X \) and \( F_{E_j}^{(i)} \) be the generic fiber of the induced rational map \( E_j^{(i)} \to Z_{E_j}^{(i)} \times \{ 0 \} \). If \( E_j = \sum_{s=1}^{t_j} m_s E_j^{(s)} \) for \( m_s > 0 \), then for sufficiently small \( \epsilon > 0 \)

\[
a_i^{(j)}(\epsilon) := \sum_{s=1}^{t_j} m_s E_j^{(s)} \cdot \mathcal{H}_i \cdot H_{E_j}^{N-i} = \left\{ \begin{array}{ll} \\
\sum_{s=1}^{t_j} m_s \deg_{E_j^{(i)}(\mathcal{X}, \mathcal{H})} \left( \frac{i}{j} \right) (Z_{E_j}^{(i)} \cdot H^{n-j}) + O(\epsilon^{j+1}) & \text{for } i \geq j \\
0 & \text{for } i < j
\end{array} \right.
\]

where \( \deg_{E_j^{(i)}(\mathcal{X}, \mathcal{H})} = (F_{E_j}^{(i)} \cdot H^j) > 0 \). Note that \( \deg_{E_j^{(i)}(\mathcal{X}, \mathcal{H})}(Z_{E_j}^{(i)} \cdot H^{n-j}) > 0 \) by ampleness of \( \mathcal{H}_i \). Then, we have by [5, Lemma 7.4]

\[
(H)^N I_{\mathcal{X}}(\mathcal{H}) = \epsilon \sum_{j=r}^N a_N^{(j)}(\epsilon) = \epsilon a_N^{(r)}(\epsilon) + O(\epsilon^{r+2}),
\]

\[
(H)^N J_{\mathcal{X}}(\mathcal{H}) = \frac{1}{N+1} \epsilon \sum_{j=r}^N \sum_{i=j}^N a_i^{(j)}(\epsilon) = \frac{1}{N+1} \epsilon \sum_{i=r}^N a_i^{(r)}(\epsilon) + O(\epsilon^{r+2}).
\]

Therefore, we have

\[
(H)^N(I_{\mathcal{X}}(\mathcal{H}) - (n+1) J_{\mathcal{X}}(\mathcal{H})) = \epsilon a_N^{(r)}(\epsilon) - \frac{n+1}{N} \epsilon \sum_{i=r}^N a_i^{(r)}(\epsilon) + O(\epsilon^{r+2})
\]

\[
= \epsilon^r + \frac{1}{s} \sum_{s} m_s \deg_{E_j^{(i)}(\mathcal{X}, \mathcal{H})} (Z_{E_j}^{(i)} \cdot H^{n-r}) \left( \frac{N}{r} - \frac{n+1}{N+1} \sum_{i=r}^N \left( \frac{i}{r} \right) \right) + O(\epsilon^{r+2})
\]

\[
= \epsilon^{r+1} - \frac{1}{s} \sum_{s} m_s \deg_{E_j^{(i)}(\mathcal{X}, \mathcal{H})} (Z_{E_j}^{(i)} \cdot H^{n-r}) \left( \frac{N}{r} \right) \left( 1 - \frac{n+1}{r+1} \right) + O(\epsilon^{r+2}).
\]

Since \( r > n \), \( I_{\mathcal{X}}(\mathcal{H}) - (n+1) J_{\mathcal{X}}(\mathcal{H}) > 0 \) for sufficiently small \( \epsilon > 0 \). Therefore, \( W_{\Delta}^\lambda(\mathcal{X}, \mathcal{H}) < 0 \) for sufficiently small \( \epsilon > 0 \) and this is a contradiction. Thus, we complete the proof.

5.2. Non-normal case. Next, we consider the deminormal case. We will define f-stability of deminormal algebraic fiber spaces and prove generalizations (Theorems 5.13 and 5.15) of Theorem 5.1 and Theorem 5.2 in this subsection.

Definition 5.8. Let \( X \) be a deminormal scheme and \( \nu_i : X_i \to X \) be the normalization of irreducible components (compare this with Definition 2.1). A surjective morphism \( f : (X, \Delta) \to B \) of equidimensional reduced schemes is a deminormal algebraic fiber space pair if \( f_i : X_i \to B_i \) is an algebraic fiber space for any \( X_i \) where \( f_i \) is induced by \( f \circ \nu_i \) and \( B_i \) is the normalization of \( f \circ \nu_i(X_i) \), which is an irreducible component of \( B \).

Definition 5.9 (cf., Definition 4.4). Suppose that \( f : (X, \Delta, H) \to (B, L) \) is a deminormal polarized algebraic fiber space pair with a boundary \( \Delta \). Let \( \text{rel.dim } f = m \) and \( \text{dim } B = n \). We remark that we can define (semi) fibration degenerations for \( f \) similarly to Definition 4.1. For any fibration degeneration \( (\mathcal{X}, \mathcal{H}) \) for \( f \), we define constants \( W_0^\Delta(\mathcal{X}, \mathcal{H}), W_1^\Delta(\mathcal{X}, \mathcal{H}), \ldots, W_n^\Delta(\mathcal{X}, \mathcal{H}) \) and a rational function \( W_{n+1}^\Delta(\mathcal{X}, \mathcal{H})(j) \) so that the partial fraction decomposition of \( DF_{\Delta}(\mathcal{X}, \mathcal{H}+$$
Note that

where

This follows similarly to Definition 4.4. If \( \Delta = 0 \), we will denote

for \( k \)

Let

and the exceptional divisor

Then we have the following:

Then we will denote \( W_n^\Delta(X, \mathcal{H}) > 0 \) for any fibration degeneration not almost trivial as a test configuration for \((X, \mathcal{H})\).

Note that \( f \) is

- \( f \)-semistable if \( W_0^\Delta(X, \mathcal{H}) \geq 0 \) and \( W_i^\Delta(X, \mathcal{H}) = W_{i+1}^\Delta(X, \mathcal{H}) = \cdots = W_n^\Delta(X, \mathcal{H}) = 0 \)

- \( f \)-stable if \( f \) is \( f \)-semistable and \( W_i^\Delta(X, \mathcal{H}) = 0 \), \( i = 1, \ldots, \), \( n - 1 \) for any fibration degeneration not almost trivial as a test configuration for \((X, \mathcal{H})\).

This follows similarly to Definition \[4.4\] If \( \Delta = 0 \), we will denote \( W_n^\Delta = W_i \).

If \((X, \mathcal{H})\) is as partially normal and the normalization \((\widetilde{X}, \widetilde{\mathcal{H}})\) of \((X, \mathcal{H})\) has reduced central fiber, then

\[
\text{DF}_{\Delta}(X, \mathcal{H} + jL) = M_{\pi^*\Delta + \text{cond}_{\mathcal{X}}}^{\text{NA}}(\widetilde{X}, \widetilde{\mathcal{H}} + jL)
\]

where \( \pi : \widetilde{X} \to X \) is the normalization and hence it also holds that

for \( k \) (cf., \[31\]).

We prepare the following to calculate \( W_k^\Delta \) by taking the normalization,

**Definition 5.10.** Let \( X = \bigcup_{i=1}^r X_i \) be the irreducible decomposition. For \( 1 \leq i \leq r \), let \( b_i \) be general point of \( f(X_i) \). Let also \( \widetilde{X}_i \) be the normalization of \( X_i \). A deminormal algebraic fiber space pair \( f : (X, \Delta, H) \to (B, L) \) has the same scalar curvature with respect to the fiber of \( f \) if

\[
S \left( (X_i)_{b_i}, (\pi^*_s - \Delta + \text{cond}_{\mathcal{X}})|_{(X_i)_{b_i}}, H|_{(X_i)_{b_i}} \right) = S \left( (\widetilde{X}_i)_{b_i}, (\pi^*_s - \Delta + \text{cond}_{\mathcal{X}})|_{(\widetilde{X}_i)_{b_i}}, H|_{(\widetilde{X}_i)_{b_i}} \right)
\]

for \( 1 \leq i < j \leq r \).

Then we have the following:

**Lemma 5.11** (cf. \[22\] Theorem 6.6]). Notations as in 5.10. For \( 1 \leq i \leq r \), let \( b_i \) be general point of \( f(X_i) \). If \( f : (X, \Delta, H) \to (B, L) \) does not have the same scalar curvature with respect to the fiber of \( f \), then \( f \) is \( f \)-unstable.

**Proof.** Suppose that \( \dim B = n \) and \( \text{rel.dim } f = m \). Assume that

\[
S \left( (X_i)_{b_i}, (\pi^*_s - \Delta + \text{cond}_{\mathcal{X}})|_{(X_i)_{b_i}}, H|_{(X_i)_{b_i}} \right) > S \left( (\widetilde{X}_j)_{b_j}, (\pi^*_s - \Delta + \text{cond}_{\mathcal{X}})|_{(\widetilde{X}_j)_{b_j}}, H|_{(\widetilde{X}_j)_{b_j}} \right)
\]

for \( 2 \leq j \leq r \). Let \( B = \bigcup B_k \) be the irreducible decomposition and \( B_1 = f(X_1) \). Then we obtain as \[22\] Theorem 6.6]

\[
S \left( (X_1)_{b_1}, (\pi^*_s - \Delta + \text{cond}_{\mathcal{X}})|_{(X_1)_{b_1}}, H|_{(X_1)_{b_1}} \right) > \sum_k (L|_{B_k})^n S(X_{b_k}, \Delta_{b_k}, H_{b_k})
\]

Let \( Z = X_1 \cap \bigcup_{i \geq 2} X_i \) and \( \mathcal{X} \) be the partially normalization of the blow up of \( X_{\mathbb{A}^1} \) along \( Z \times \{0\} \) with the exceptional divisor \( E \). Let \( F \) be the strict transformation of \( X_1 \times \{0\} \). By taking finite base change via the \( d \)-th power map of \( \mathbb{A}^1 \), we may assume that the normalization \( \mathcal{X} \) of
\(X\) has the reduced central fiber as in the proof of [3, Proposition 7.16]. Choose \(\eta > 0\) such that 
\[-E + \eta F\] is \(X_{\Delta}\)-ample and let 
\[ \mathcal{H} = H_{\Delta} - \epsilon(E - \eta F) \]
be a polarization of \(X\) for sufficiently small \(\epsilon > 0\). Then we can prove that 
\[
\left(\frac{n + m}{n}\right)^{-1} W_0^\Delta(\mathcal{X}, \mathcal{H}) = \sum_k (L|B_k)^n \left( K(\Delta_{\Delta_{\Delta}})/\mathbb{P}^1 \cdot H_{\Delta_{\Delta_{\Delta}}} m + S(X_{\Delta_{\Delta}}, H_{\Delta_{\Delta}}) \frac{H_{\Delta_{\Delta}}}{m + 1} \right) 
\]
\[
= \epsilon \eta (H|_{X_{\Delta_1}})^n \left( \sum_k (L|B_k)^n S(X_{\Delta_{\Delta}}, H_{\Delta_{\Delta}}) \right) 
\]
\[
- (L)^n S \left( (\Delta_{\Delta_{\Delta}})_{X_{\Delta_1}}, (\pi_{\Delta_{\Delta_{\Delta}}})_1, H|_{(X_{\Delta_1})_{X_{\Delta_1}}} \right) + O(\epsilon^2) 
\]
for general \(b_k \in B_k\) similarly to the proof of [22, Theorem 6.6].

Therefore, if a reducible algebraic fiber space \(f : (X, \Delta, H) \to (B, L)\) is \(\mathfrak{f}\)-semistable, then we can decompose \(W_k^\Delta\) as follows,

**Lemma 5.12.** Let \(f : (X, \Delta, H) \to (B, L)\) be a deminormal polarized algebraic fiber space pair with the same scalar curvature with respect to the fiber of \(f\), \(0 < k \leq \dim B = n\) and \((X, \mathcal{H})\) be a partially normal semistable test configuration for \((X, H)\) dominating \(X_{\Delta}\). Suppose that \(H\) is ample, \(L\) is very ample and \(X\) has the reduced central fiber. Let \(\nu : \tilde{X} \to X\) be the normalization and \(\tilde{X} = \bigcup_{i=1}^r \tilde{X}_i\) be the irreducible decomposition. Let also \(\tilde{X}\) be the normalization of \(X\) and \(\tilde{X}_i\) be the irreducible decomposition where the indices corresponding to those of \(\tilde{X}\) are \(\tilde{X}_i = \bigcup_{i=1}^r \tilde{X}_i\).

If \((\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i})\) is normalized with respect to the central fiber and \((\tilde{X}_i \cap D_{1-j})_{X_{\Delta_1}} = \mathcal{H}|_{(\tilde{X}_i \cap D_{1-j})_{X_{\Delta_1}}} \) is trivial for \(0 \leq j < k\) and for general ample divisors \(D_1, \cdots, D_{n-k+1} \in |L|\), then

\[ W_k^\Delta(\mathcal{X}, \mathcal{H}) = \sum_{i=1}^r W_k^\Delta(\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i}). \]

**Proof.** By the assumption, we have

\[ DF_{\Delta}(\mathcal{X}, \mathcal{H} + jL) = M^{\mathfrak{f}}_{(\pi_{\Delta_{\Delta_{\Delta}}})_1, (\pi_{\Delta_{\Delta_{\Delta}}})_1}(\tilde{X}_i, \tilde{H} + jL) \]

for any \(j\). Moreover, since each \((\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i})\) is normalized, we have

\[ W_k(\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i}) = \sum_{i=1}^r W_k(\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i}) \]

by Lemmas 4.12 and 5.11.

Note that if \((\mathcal{X}, \mathcal{H})\) is a deformation to the normal cone of a closed subscheme \(Z\) of \(X\) with the exceptional divisor \(E\) such that \(\dim Z < \dim X\) and \(\mathcal{H} = H_{\Delta} - \epsilon E\) for sufficiently small \(\epsilon > 0\), then \((\tilde{X}_i, \mathcal{H}|_{\tilde{X}_i})\) is normalized with respect to the central fiber in Lemma 5.12. Then, we can prove the following,

**Theorem 5.13.** Let \(f : (X, \Delta, H) \to (B, L)\) be a polarized deminormal algebraic fiber space. If \(f\) is \(\mathfrak{f}\)-semistable, \((X, \Delta)\) has at most slc singularities.

**Proof.** Assume that \((X, \Delta)\) is not slc but \(f\) is \(\mathfrak{f}\)-semistable. If \([\Delta]\) is not reduced, let \(F\) be an irreducible component of \(\Delta\) whose coefficient is larger than 1. Then, as in the proof of Theorem 5.1 we can prove that the partially normalization of the deformation to the normal cone \((\mathcal{X}, \mathcal{H})\) of \(F\) with some polarization \(\mathcal{H}\) satisfies that there exists \(0 \leq k \leq \dim B\) such that

\[ W_k(\mathcal{X}, \mathcal{H}) < 0 \]
and
\[ W_i^\Delta(\mathcal{X}, \mathcal{H}) = 0 \]
for \( i < k \) by Lemma 5.11 and Lemma 5.12. Thus, we may assume that \([\Delta]\) is reduced. By [34 Corollary 1.2], there exists the slc modification \( \pi : Y \to X \). It is easy to see that there exists a closed subscheme \( Z \) such that \( \pi \) is the blow up along \( Z \). Let \( \nu : \tilde{X} \to X \) be the normalization and \( D = \text{cond}_X \) be the conductor. If \( \nu_\nu : \tilde{Y} \to Y \) is the normalization of \( Y \) and \( \tilde{\pi} : \tilde{Y} \to \tilde{X} \) is the induced morphism, then \( (\tilde{Y}, \Delta_{\tilde{Y}} + \tilde{\pi}^{-1}D) \) is the lc modification of \( (\tilde{X}, \Delta_{\tilde{X}} + D) \) where \( \Delta_{\tilde{Y}} = (\nu_\nu)_*^{-1}\Delta_Y \) and \( \Delta_{\tilde{X}} = \nu_\nu^{-1}\Delta_X \) (cf., [34 Lemma 3.1]). It is easy to see that \( \tilde{\pi} \) is the normalized blow up along \( \nu^{-1}Z \) and \( A((\tilde{X}, \Delta_{\tilde{X}} + D)(v) < 0 \) for \( v \in \text{Rees}(\nu^{-1}Z) \).

Let \((X, H) = H_{p_1} - \epsilon E\) be the partially normalized base change via \( A^1 \ni t \mapsto t^d \in A^1 \). Then
\[ M^{\text{NA}}_{(\Delta_{\tilde{X}} + D)}(\tilde{X}, \tilde{H}_\epsilon + jL) = DF((\Delta_{\tilde{X}} + D)) = DF_{\tilde{X}}(\tilde{X}, \tilde{H}_\epsilon + jL), \]
for \( j \in \mathbb{Q} \). Therefore,
\[ W^k((\Delta_{\tilde{X}} + D)) = W^k(\tilde{X}, \tilde{H}_\epsilon) \]
for \( 0 \leq k \leq \dim B \). Hence, by the proof of Theorem 5.7 and Lemmas 5.11 and 5.12, we can prove that there exists \( 0 \leq k \leq \dim B \) such that
\[ W^k(\tilde{X}, \tilde{H}_\epsilon) < 0 \]
and
\[ W^i(\tilde{X}, \tilde{H}_\epsilon) = 0 \]
for \( i < k \) and for sufficiently small \( \epsilon > 0 \).

As in Proposition 5.7 we need the following partial resolution.

**Lemma 5.14.** Let \((X, \Delta)\) be a projective slc pair and fix an slc center \( C \) such that \( \text{codim } C \geq 2 \) (i.e., if \( \nu : \tilde{X} \to X \) is the normalization, there exists an lc center \( C' \) such that \( \nu(C') = C \)). Then there exists a closed subscheme \( Z \) that satisfies the following conditions.

1. The reduced structure \( \text{red}(Z) \) is contained in \( C \),
2. \( A((X, \Delta))(v) = 0 \) for any \( v \in \text{Rees}(\nu^{-1}Z) \),
3. There exists at least one valuation \( v \in \text{Rees}(\nu^{-1}Z) \) such that the center of \( v \) dominates \( C \).

**Proof.** Fix an ample line bundle \( H \) and let \( \mathcal{I} \) be the ideal sheaf corresponding to the reduced structure of \( C \). Then the linear system \( \mathcal{I} = H^0(X, \mathcal{I} \otimes \mathcal{O}(mH)) \) is base point free outside from \( C \) for sufficiently large \( m > 0 \). We can choose \( D \in \mathcal{I} \) such that \( D \) contains no slc centers other than those contained in \( C \) and \( X_i \not\subset \text{supp}(D) \) for any irreducible component \( X_i \) of \( X \). Let \( f : Y \to \tilde{X} \) be a log resolution of \((\tilde{X}, \nu_*^{-1}\Delta + \nu^*D + \text{cond}_\tilde{X})\) and a resolution of the base locus of \( \mathcal{I} \). By replacing \( D \) by a general one, we may assume that \( f^*\nu^*D \) does not contain any prime divisor \( E \) on \( Y \) such that \( A((X, \Delta))(E) = 0 \) and \( \nu \circ f(E) \not\subset C \) due to the theorem of Bertini. Let \( \Delta_Y \) be the \( \mathbb{Q} \)-divisor satisfies that
\[ K_Y + \Delta_Y = f^*(K_{\tilde{X}} + \nu_*^{-1}\Delta + \text{cond}_\tilde{X}). \]

Then, since \((Y, \Delta_Y)\) is log smooth and suble, for non-lc centers \( C' \) on \((\tilde{X}, \nu_*^{-1}\Delta + \nu^*D + \text{cond}_\tilde{X})\), \( \nu(C') \subset C \) for sufficiently small rational \( \epsilon > 0 \). Note that \((X, \Delta + \nu^*D)\) is not slc along \( C \) but \([\nu^*D + \Delta]\) is reduced. Thanks to [34 Corollary 1.2], we take the slc modification \( g : W \to X \) of \((X, \Delta + \nu^*D)\) and there exists a closed subscheme \( Z \) such that \( g \) is the blow up along \( Z \). It is easy to see that red \((\nu^{-1}Z)\) is contained in \( C \), \( A((X, \Delta))(v) = 0 \) for any \( v \in \text{Rees}(\nu^{-1}Z) \) and there exists at least one valuation \( v \in \text{Rees}(Z) \) such that the center of \( v \) dominates \( C \). \( \square \)
Let \( f : (X, \Delta, H) \to (B, L) \) be a polarized demi-normal algebraic fiber space. Suppose that there exist \( \lambda \in \mathbb{Q}_{>0} \) and a line bundle \( L_0 \) on \( B \) such that \( H + f^*L_0 \equiv -\lambda (K_X + \Delta) \), and \( f \) is \( \mathcal{f} \)-semistable. Then, \( (X, \Delta) \) is slc and any slc-center of \( (X, \Delta) \) is of fiber type.

\[ \lambda \in \mathbb{Q}_{>0} \]

Proof. We may assume that \( H \) is ample and \( L \) is very ample. It follows from Theorem 5.13 that \( (X, \Delta) \) is slc. Assume that there exists at least one slc-center or singular locus of codimension 1 of \( (X, \Delta) \) that is of non-fiber type.

First, assume that there exists at least one irreducible component of non-fiber type of the conductor subscheme \( D = \text{cond} \mathcal{X} \) of \( X \). Let \( \nu : \mathcal{X} \to X \) be the normalization, \( \bar{D} = \text{cond} \mathcal{X} \) be the conductor divisor. Note that \( \tilde{\text{cond}} \mathcal{X} \) need not to be \( \mathbb{Q} \)-Cartier in general. Due to Proposition 5.7 there exists a coherent ideal sheaf \( \mathfrak{a} \subset \mathcal{O}_X \) such that \( \text{Rees}(\mathfrak{a}) \) is the set \( \{ \text{cond}_{\mathcal{D}} \} \) where \( D_i \) are all irreducible components of \( \bar{D} \). Since \( \mathfrak{a} \subset \mathcal{O}_X(-\bar{D}) \), we can consider \( \mathfrak{a} \) to be an ideal sheaf of \( X \) and let \( Z \) be the closed subscheme of \( X \) corresponding to \( \mathfrak{a} \). Then consider the partially normalization of the deformation to the normal cone \( (X, \mathcal{H}_\epsilon = H_{\lambda} - \epsilon E) \) of \( Z \) where \( E \) is the exceptional divisor for \( \epsilon > 0 \). We may assume that the central fiber of \( X \) is reduced by the same argument of the proof of Theorem 5.13. If \( \bar{X} \) is the normalization of \( X \), then \( H^\Delta_\Delta(\mathcal{X}_0, \mathcal{H}_\epsilon |_{\mathcal{X}_0}) = 0 \) for general \( b \in B \) and hence \( W^\Delta_0(\bar{X}, \mathcal{H}_\epsilon) < 0 \). Therefore, we may assume that any singular locus of codimension 1 of \( (X, \Delta) \) is of fiber type. In other words, we may assume that the general fiber is normal. Moreover, the general fiber is also lc since \( (X, \Delta) \) is slc.

Next, assume that there exists at least one irreducible component of non-fiber type \( F \) of \( |\Delta| \). Let \( X_1 \) be the irreducible component of \( X \) containing \( F \). There exists a closed subscheme \( Z \) of \( X_1 \) such that \( \text{Rees}(Z) = \{ F \} \) due to Proposition 5.7. For general point \( b \in f(X_1) \subset B \), \( ((X_1)_b, \Delta_{(X_1)_b}) \) is an lc pair and \( \text{Rees}(Z_b) = \{ F_b \} \). Therefore, it is easy to see that if \( (X, \mathcal{H}_\epsilon) \) is the partially normalization of the deformation of \( X \) to the normal cone of \( Z \) where \( E \) is the inverse image of \( Z \times \{ 0 \} \) and \( \mathcal{H}_\epsilon = H_{\lambda} - \epsilon E \), then we have \( W^\Delta_0(\bar{X}, \mathcal{H}_\epsilon) < 0 \) for sufficiently small \( \epsilon > 0 \). Thus, we conclude that \( |\Delta| \) is of fiber type.

Finally, let \( k = \min \{ \text{codim}_{F}(C) : \text{where } C \text{ is a slc center of non-fiber type} \} \) and fix a slc center \( C \) of non-fiber type such that \( \text{codim}_{F}(C) = k \). We may assume that \( \text{codim}_{F}(C) \geq 2 \). By Lemma 5.14, there exists a closed subscheme \( Z \) contained in \( C \) such that \( A_{\Delta}(v) = 0 \) for any \( v \in \text{Rees}(Z) \) and there exists at least one valuation \( v \in \text{Rees}(Z) \) such that the center of \( v \) dominates \( C \). Take general divisors \( D_1, \ldots, D_{n-k} \subset |L| \) and cut \( X \) by \( D_1, \ldots, D_{n-k} \). Now, let \( (X, \mathcal{H}_\epsilon) \) be the partially normalization of the deformation of \( X \) to the normal cone of \( Z \) where \( E \) is the exceptional divisor and \( \mathcal{H}_\epsilon = H_{\lambda} - \epsilon E \). Then, we will prove that \( W^\Delta_i(\bar{X}, \mathcal{H}_\epsilon) < 0 \) for sufficiently small \( \epsilon > 0 \). Here, we may assume that \( \text{dim} B = k \) by replacing \( X \) by \( X \cap D_1 \cap \cdots \cap D_{n-k} \). Since \( \text{dim} f(C) = 0 \), any center of \( v' \in \text{Rees}(Z) \) is of non-fiber type. Hence, we can prove that

\[ W^\Delta_k(\bar{X}, \mathcal{H}_\epsilon) < 0 \]

and

\[ W^\Delta_i(\bar{X}, \mathcal{H}_\epsilon) = 0 \]

for \( i < k \) and sufficiently small \( \epsilon > 0 \) similarly as Theorem 5.13 and Theorem 5.4. \( \square \)

Adiabatic K-semistability implies \( \mathcal{f} \)-semistability. Thus we conclude that the following holds by the previous theorem.

**Corollary 5.16.** Let \( f : (X, \Delta, H) \to (B, L) \) be an adiabatically K-semistable polarized demi-normal algebraic fiber space such that \( H + f^*L_0 \equiv -\lambda (K_X + \Delta) \) where \( L_0 \) is a line bundle on \( B \) and \( \lambda \in \mathbb{Q}_{>0} \). Then \( (X, \Delta) \) is slc and any slc-center of \( (X, \Delta) \) is of fiber type.

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