Stationary growth and unique invariant harmonic measure of cylindrical DLA

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We prove that the harmonic measure is stationary, unique and invariant on the interface of DLA growing on a cylinder surface. We provide a detailed theoretical analysis puzzling together multiscaling, multifractality and conformal invariance, supported by extensive numerical simulations of clusters built using conformal mappings and on lattice. The growth properties of the active and frozen zones are clearly elucidated. We show that the unique scaling exponent characterizing the stationary growth is the DLA fractal dimension.

Introduction.– Albeit the diffusion limited aggregation model (DLA) was introduced 30 years ago [1], it still embodies the perfect puzzle for theorists. In spite of the enormous amount of numerical works, only few rigorous results are proven, and these all concern the generalized dimensions of the aggregate. The notion of varying dimensionality characterizes the DLA growth in two ways: multiscaling and multifractality. Multiscaling suggests that the aggregate’s fractal dimension attains different local values \( d(r/R) \) on each circular ring of radius \( r \), while the overall gyration radius \( R(N) \sim N^{1/D} \) (\( N \) and \( D \) cluster’s particles number and fractal dimension) [2,3]. Multifractality [4] entails the introduction of generalized dimensions \( D(q) \), which correspond to the fractal dimension of the \( q \) points correlation function. The aforementioned rigorous results are: \( D(0) \geq 3/2 \) [5], \( D(1) = 1 \) [6] and \( D(3) = D(0)/2 \) [7]. A simpler definition of \( D(q) \) is provided by the connection with the harmonic measure, i.e. the growth probability at the interface. In general, the determination of the harmonic measure within the fjords of the fractal, is a quite impossible task if resorting to the usual numerical techniques [8]. Iterated conformal mapping [10] provides a solution to this problem [8]: indeed it is based on the representation of the growth dynamics via the convolution of elementary complex functions \( \phi_{\lambda_n, \theta_n}(w) \), hereafter named \( \phi_n(w) \), which map the unitary circle onto a unitary circle with a bump of linear size \( \sqrt{\lambda_n} \) placed at position \( w = e^{i\theta_n} \) [11]. The ensuing function \( z = \Phi^{(N)}(e^{i\theta}) = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_N(e^{i\theta}) \) transforms the unitary circle \( e^{i\theta} \) onto the cluster’s interface \( z \). Furthermore conformal mapping has been extended to DLA growing on a cylinder surface [12]: in this case the mapping function is

\[
\Xi^{(N)}(e^{i\theta}) = -i \frac{L}{2\pi} \ln \left[ \Phi^{(N)}(e^{i\theta}) \right],
\]

where \( L \) is the cylinder circumference and \( \Phi^{(N)} \) represents the radial deformation of the cylindrical aggregate (see section I in suppl. mat.). The DLA overall height grows as \( h(N) \sim L \sum_{n=1}^{N} \phi_n \) where \( F_{1}(N) = \Pi_{n=1}^{N} \sqrt{1 + \lambda_n} \) is the first Laurent coefficient of \( \Phi^{(N)} \) [10,11].

In this letter we provide another rigorous result concerning the harmonic measure of a DLA growing in a cylindrical geometry. We show that the growth probability at the interface is stationary, unique and invariant, leading to the conformal invariance of the complex mapping function. Our framework provides the natural connection between multiscaling and multifractality, and offers a clearcut definition of the frozen and active zone of the aggregate. Moreover, we show that the stationarity leads to the appeareance of the fractal dimension \( D \) as the unique exponent governing the growth dynamics, in striking contrast to DLA in radial geometry.

Height’s scaling and growth velocity.– The average height of a cylindrical aggregate grows according to the following law [12]:

\[
\langle h(N) \rangle = L \left( \frac{\lambda}{\langle n \rangle} \right)^{\beta} \quad \text{if} \quad N \ll \langle n \rangle
\]

and

\[
\langle h(N) \rangle = L \left( \frac{N}{\langle n \rangle} \right)^{\beta} \quad \text{if} \quad N \gg \langle n \rangle \quad (\beta \approx 4/3, \quad \langle n \rangle = 1 + \lambda / (\langle n \rangle))
\]

The first regime accounts for a self-affine initial growth, while the DLA attains the linear self-similar regime after an average time \( \langle n \rangle \), namely the transient on which the cluster forgets about its initial condition, i.e. the cylinder’s baseline. From the scaling of \( \langle h(N) \rangle \) and the expression of \( F_1^{(N)} \), one obtains

\[
\langle h(N) \rangle \approx \frac{L}{4\pi} \sum_{n=1}^{N} \langle \lambda_n \rangle
\]

which furnishes the expression of the average height’s growth velocity:

\[
d\langle h(N) \rangle/dN \simeq \frac{L}{4\pi} \langle \lambda_N \rangle
\]

Thus the average elementary increment \( \langle \lambda_n \rangle \) grows as \( 4\pi \beta^{\frac{n-1}{3}} \) for \( n \ll \langle n \rangle \), and it gets to the stationary value \( \frac{4\pi}{\langle n \rangle} \) for \( n \gg \langle n \rangle \). This suggests that self-similarity is intimately connected to the height’s velocity stationary growth, in contrast with DLA in radial geometry [11]: in this case the radius growth is clearly non-stationary and \( \langle \lambda_n \rangle = 2/(nDR) \), with \( DR \approx 1.71 \) radial cluster’s dimension.

Harmonic measure.– We now proceed to the evaluation of the harmonic measure [1]. The DLA interface is the union of different arcs, which are the (remaining) boundaries of the \( N \) particles composing the cluster. Each arc may be labeled by \( \Delta \lambda_n \) where \( n \) is the gener-
ation when the n-th bump was added to the structure; besides, we identify the middle point \( z_n \) as the representative point of the entire arc (Fig. 2, inset (b)), and we calculate the probability \( P(N)(z_n) \) as function of generation \( n \): dotted line corresponds to the self-affine regime, i.e. \( \frac{n^3}{n_P^2} e^{-\frac{n^2}{n_P}} (\beta = 4/3) \). (b) Rescaling of \( P(N)(z_n) \): probabilities at different \( N \) collapse on top of each other exhibiting the \( N \sim n \) dependence. Dashed lines show the self-affine region \( (n_0) \) and the active zone \( (n_{AZ}) \). (c) Ratio between \( P(N_{max})(z_n) \) \( (N_{max} = 6 \times 10^4) \) and the curves in panel (a), showing the stationarity in the frozen zone.

The angle \( \vartheta^N_n \) represents the counterimage of \( z_n \) on the unitary circle, and must be subjected to a reparametrization whenever a new particle is added \[ 9, 12 \]. Indeed consider a DLA at two different generations \( N \) and \( N - k \), with \( k \in [1, N - n] \): the position \( z_n \) remains unchanged whether the cluster has \( N \) or \( N - k \) particles, i.e. \( z_n = \Xi(N) \left( e^{i\vartheta^N_n} \right) = \Xi(N - k) \left( e^{i\vartheta^{N-k}_n} \right) \), yielding

\[
e^{i\vartheta^N_n} = \phi_1^{N-1} \circ \phi_{N-1}^{-1} \circ \cdots \circ \phi_{N-k+1}^{-1} \left( e^{i\vartheta^{N-k}_n} \right),
\]

where \( \phi_n^{-1} \) is the inverse of \( \phi_n \) \[ 9, 12 \]. Thus any \( \vartheta^N_n \) can be determined from its initial value \( \vartheta_n \sim \vartheta_n \). The electric field at \( z_n \) can be calculated as \( |E(N)(z_n)| = \frac{|E^{(n)}(z_n)|}{W_N^{n_{k+1}} \left[ \phi'_n(e^{i\vartheta^N_n}) \right]} \) where \( \phi' \) is the derivative of \( \phi_n \). Fig. 1 shows the probability \( P(N)(z_n) \) as a function of \( n \), for a single DLA realization at 6 different values of \( N \). The harmonic measure \( \mu_{N,n} \) is the ensemble average of \( P(N)(z_n) \), i.e. \( \mu_{N,n} \left( \frac{\Xi(N)}{L} \right) = \langle P(N)(z_n) \rangle \) from the numerics its behavior is stationary, depending solely on the difference \( N - n \):

\[
\mu_{N,n} \sim \frac{1}{\langle n_P \rangle} \times \begin{cases} 
\left( \frac{n}{\langle n_P \rangle} \right)^{\beta - \frac{n}{n_P}} e^{-\frac{n^2}{\langle n_P \rangle}} & n \ll \langle n_0 \rangle \\
\left( \frac{n}{\langle n_P \rangle} \right)^{\beta - \frac{n}{n_P}} \frac{n}{\langle n_P \rangle} & \langle n_0 \rangle \ll n \ll N - \langle n_{AZ} \rangle \\
\left( \frac{n}{\langle n_P \rangle} \right)^{\beta - \frac{n}{n_P}} & N - \langle n_{AZ} \rangle \ll n \ll N
\end{cases}
\]

\[ 3 \] \( n_{AZ} \) represents the number of particles that compose the active zone of the DLA \[ 14 \]. Firstly, we focus on the frozen zone for which \( n \ll N - \langle n_{AZ} \rangle \): a quantitative analysis of the measure’s scaling in the active region, will be given in the following. The probability of a point belonging to the frozen interface exhibits an apparent exponential decay, with a characteristic time \( n_P \). This arises from the electric field expression, indeed for any point in the frozen zone \( |\phi_k (e^{i\vartheta^N_n})| \simeq e^{1/n_P} \), if \( k \gg n + n_{AZ} \). This suggests that a point in the frozen zone cannot influence the growth dynamics, since the probability in \( z_n \) does not depend on the specific choice of the elementary function \( \phi_k \) when \( k \gg n + n_{AZ} \). On the other side, it points to the notion of conformal invariance \[ 10 \], since two conformal transformations \( \phi_n \) and \( \phi_k \) commute whenever \( k \gg n + n_{AZ} \). Indeed, given a mapping function \( \Xi(N)(w) \) with \( N - n_{AZ} \gg k \gg n + n_{AZ} \), the size of the new bump \( \sqrt{\lambda_{N+1}} \) and the ensuing growth dynamics will remain unchanged whether \( \phi_k \) is swapped with \( \phi_n \). \( n_P \) can be accurately measured for any \( z_n \) lying in the frozen zone, indeed from \[ 3 \] one has \( n_P = (N_2 - N_1) / [\ln P(N_{1})(z_n) - \ln P(N_{2})(z_n)] \), with \( N_2 \) and \( N_1 \) two arbitrary generations. The scaling of \( \langle n_P \rangle \) is dis-
Numerical simulations show that \( \alpha \) fulfills the scaling law
\[
\langle n \rangle \sim \langle n \rangle^{\alpha}
\]
(Fig.[1] in suppl. mat.), we get \( \alpha \) (see section II in suppl. mat.).

In conformal mapping, the stochastic process is defined by the angles \( \theta_1, \ldots, \theta_N \). Hence, we consider as independent initial conditions \( \theta^{(1)}_1, \ldots, \theta^{(1)}_{N_{init}} \) and \( \theta^{(2)}_1, \ldots, \theta^{(2)}_{N_{init}} \) (\( N_{init} \gg \langle n_0 \rangle \)), and we extract subsequent angles according to \( \theta^{(1)}_N = \theta^{(2)}_N (N > N_{init}) \): collapse arises if and only if \( \langle \Delta \theta^{(1)}_N \rangle = \langle \Delta \theta^{(2)}_N \rangle \) for \( n \in [N - \langle n_{AZ} \rangle, N] \) (Fig.[5] in suppl. mat.). However, this procedure is strongly affected by sistemical errors induced by \( \phi_n \) (see section III of suppl. mat.): if a growth attempt is made close to the frozen regions, unphysical particles tend to fill the fjords of the aggregates, leading to clusters’ divergence rather than collapse. Thus, we apply the collapsing procedure to DLA on lattice.

For DLA on lattice, randomness is given by the Brownian nature of the upcoming particle’s path: a collapsing protocol may consist on taking the same diffusive trajectories, for particles released from the upper cylinders’ boundaries in both DLA. Therefore, after two initial conditions have been built (Fig.[4] panel (a)), protocol is started and it is stopped only when both DLA have collapsed (Fig.[4] panel (b)), i.e. when they are identical within a window of height 3L. The average collapsing time \( \langle n_C \rangle \) shows a stretched exponential behaviour,
decay is robust to the change of lattice geometry and results strongly indicate that the stretched exponential behavior is explained within the conformal mapping framework. Most important, the uniqueness and invariance of the harmonic measure paves the way for the notion of ergodicity in fractal growth phenomena.

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Conclusions—We have shown that the harmonic measure is stationary, unique and invariant on the DLA interface. As a matter of fact, within this comprehensive framework, the system’s stationarity entails that multiscaling, multifractality and conformal invariance appear as a unique emergent property of the growth dynamics. Moreover the stationarity allows the precise definition of characteristic times, whose scaling exhibit a sole critical exponent: the aggregate’s fractal dimension. This is at odds with radial DLAs, for which a stationary phase and an ensuing single scaling exponent cannot be identified, casting very fundamental doubts on the possible existence and definition of a fractal dimension in this geometry. The observed stretched exponential behavior is given by

\[ e^{\frac{1}{2} L^{-0.5}} \]

valid in square and in hexagonal lattices (Fig. 4 panel (e)). The outlined collapsing protocol requires that both DLA are overlapping in a spatial window: we now want that collapse occurs when they are identical within a temporal window \( n_{AZ} \). In this case, DLA collapse is the physical distance between the positions of the last \( n_{AZ} \) homologous particles is 0. Our results strongly indicate that the stretched exponential decay is robust to the change of lattice geometry and definition of collapsing criteria (Fig. 4 panel (e)), but is definitely different from what has been observed for \( n_0 \), \( n_P \), \( n_{AZ} \) and \( n_{OLD} \) is that the average number of particles composing a DLA in a box \( L \times L \).

The observed stretched exponential behavior is explained within the conformal mapping framework. The collapsing time \( n_C \) can be expressed as

\[ n_C^{-1} \sim \frac{1}{p \left( \left\{ (1) \Delta \vartheta_k^N \right\} \right) p \left( \left\{ (2) \Delta \vartheta_k^N \right\} \right) \Pi_{k=0}^{n_{AZ}} \delta \left( (1) \Delta \vartheta_k^N - (2) \Delta \vartheta_k^N \right) \}

where the joint probability distribution \( p \left( \left\{ (1) \Delta \vartheta_k^N \right\} \right) \) is \( p \left( (1) \Delta \vartheta_k^N - n_{AZ}, \ldots, (1) \Delta \vartheta_N^N \right) \), and \( \delta(x) \) is the Dirac delta function. Assuming \( \Delta \vartheta_k^N \) uncorrelated, we have that

\[ n_C^{-1} \sim \frac{1}{\Pi_{k=0}^{n_{AZ}} p \left( (1) \Delta \vartheta_k^N \right)^2} \].

Now, taking \( p \left( (1) \Delta \vartheta_k^N \right) \sim e^{-\frac{1}{(1) \Delta \vartheta_k^N}} \), thanks to we finally obtain

\[ n_C \sim \Pi_{k=0}^{n_{AZ}} \left( e^{2(p(N)(1)_{N-k})} \right)^{-\alpha \left( \frac{N-k}{N-k} \right)} \sim \]

\[ \Pi_{k=0}^{n_{AZ}} \left( e^{2P(N)(N-k)} \right)^{-\alpha \left( \frac{N-k}{N-k} \right)} \]

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