HÖLDER REGULARITY FOR THE GRADIENT OF THE INHOMOGENEOUS PARABOLIC NORMALIZED $p$-LAPLACIAN

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Abstract. In this paper we study an evolution equation involving the normalized $p$-Laplacian and a bounded continuous source term. The normalized $p$-Laplacian is in non divergence form and arises for example from stochastic tug-of-war games with noise. We prove local $C^{\alpha,\frac{\alpha}{2}}$ regularity for the spatial gradient of the viscosity solutions. The proof is based on an improvement of flatness and proceeds by iteration.

1. Introduction

In this paper we are interested in local regularity properties for viscosity solutions of the normalized $p$-Laplace equation

$$\partial_t u(x, t) - \Delta_N^p u(x, t) = f(x, t),$$

where $p \in (1, \infty)$ and $f$ is a continuous and bounded function.

The normalized $p$-Laplacian can be seen as the one-homogeneous version of the standard $p$-Laplacian and also as a combination of the Laplacian and the normalized infinity Laplacian,

$$\Delta_N^p u := |Du|^{2-p} \Delta_p u = \Delta u + (p-2) \Delta_N^\infty u = \Delta u + (p-2)|Du|^{-2} \sum_{ij} u_{ij} u_i u_j,$$

where $\Delta_p u = \text{div}(|Du|^{p-2}Du)$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and $u_i = \frac{\partial u}{\partial x_i}$.

Recently, a connection between the theory of stochastic tug-of-war games and non-linear equations of $p$-Laplacian type has been investigated. In the elliptic case, this connection started with the seminal
work of Peres, Schramm, Sheffield and Wilson \[PS08, PSSW09\]. In the parabolic case, Manfredi, Parviainen and Rossi \[MPR10\] showed that solutions to (1.1) can be obtained as limits of values of tug-of-war games with noise when the parameter that controls the length of the possible movements goes to zero.

Equations of type (1.1) have been suggested in connection to economics \[NP14\] and image processing \[Doe11, ETT15\]. Existence of viscosity solutions of (1.1) has been proved using different techniques, including game-theoretic arguments \[MPR10\] and approximation methods (see \[BG13, BG15, Doe11\]). Regularity issues for this problem were analyzed in \[BG15, Doe11\] where the authors proved Lipschitz estimates and bounds for the modulus of continuity of (1.1) using PDE techniques. In \[PR16\] the authors obtained local Hölder and Lipschitz estimates using a game theoretic method for the case \(p = p(x,t) > 2\). The asymptotic behavior for (1.1) has been investigated in \[Juu14, BG13, Doe11\].

The Hölder continuity of the solutions follows from the general regularity theory developed by Krylov and Safonov for equations in non-divergence form \[Kry87, Wa92I, IS12\]. Recently, for the homogeneous case \(f = 0\), Jin and Silvestre \[JS15\] proved Hölder gradient estimates for solutions of (1.1). They extended their method to prove Hölder gradient estimates for a class of singular or degenerate parabolic equations \[IJS16\].

The normalized \(p\)-Laplacian enjoys the good properties of being uniformly parabolic and 1-homogeneous, the main difficulty in proving regularity results comes from the discontinuity at \(\{D u = 0\}\).

Let \(\Omega\) be a bounded domain of \(\mathbb{R}^n\) and \(T > 0\). Denote \(\Omega_T = \Omega \times (0,T)\). Our main contribution is the following.

**Theorem 1.1.** Assume that \(p > 1\) and \(f \in L^\infty(\Omega_T) \cap C(\Omega_T)\). There exists \(\alpha = \alpha(p, n) > 0\) such that any viscosity solution \(u\) of (1.1) is in \(C^{1+\alpha, \frac{1+\alpha}{2}}(\Omega_T)\). Moreover, for any \(\Omega' \subset \subset \Omega\) and \(\varepsilon > 0\), we have

\[
|Du| \leq C \left(||u||_{L^\infty(\Omega_T)} + ||f||_{L^\infty(\Omega_T)}\right)
\]

and

\[
sup_{(x,t),(x,s) \in \Omega' \times (\varepsilon, t-\varepsilon)} \frac{|u(x,t) - u(x,s)|}{|t-s|^{\frac{1+\alpha}{2}}} \leq C \left(||u||_{L^\infty(\Omega_T)} + ||f||_{L^\infty(\Omega_T)}\right),
\]

where \(C = C(p, n, d, \varepsilon, d')\), \(d' = \text{dist}(\Omega', \partial \Omega)\) and \(d = \text{diam}(\Omega)\).
The proof is inspired by [IS13] and involves an improvement of flatness: if $u$ can be approximated by an affine function in a cylinder $Q$, then we can find a better approximation in a smaller cylinder and we can iterate the process. In fact, we prove by induction that for some $\rho, \alpha \in (0, 1)$ and $C = C(p,n)$, there exists a sequence $q_k$ such that $\text{osc}_{Q_{\rho}^k}(u(x,t) - q_k \cdot x) \leq C\rho^{k(1+\alpha)}$. The inductive step is based on proving improvement of flatness for the rescaled function $w_k = \rho^{-k(1+\alpha)}(u(\rho^k x) - q_k \cdot (\rho^k x))$. The main task is then to study the equation satisfied by the deviation of $u$ from a linear approximation that we denote $w(x,t) = u(x,t) - q \cdot x$ and to obtain a local $C^{\beta,\beta/2}$ estimate for $w$ independent of $q$. This will be the purpose of Lemma 3.1. By the Arzelà-Ascoli theorem we get compactness which, along with the regularity estimates for the homogeneous equation, we use to improve our approximation of $u$ in a smaller cylinder $Q_r$ by finding a linear approximation for $w$ (see Lemma 3.3). Providing a linear approximation for $w$ independently of the value of $q$ is based on a contradiction argument which requires a uniform $C^{1+\beta,\frac{\beta+1}{2}}$ estimate with respect to $q$ for the associated homogeneous equation. For this purpose, we adapt the strategy of [JS15] once we have obtained a Lipschitz control on $w$ independent of $q$ when $|q|$ is large enough. Finally, we prove (1.2) by using Lemma 3.3 and an iteration in Lemma 3.4. Once the regularity with respect to the space variable is obtained, we use a standard barrier argument in order to control the oscillation of $u$ and derive $C^{1+\alpha/2}$ regularity with respect to the time variable in Lemma 3.5.

Let us also mention that the extremal cases $p = 1$ and $p = \infty$ have also received attention. The case $p = 1$ is known as the mean curvature flow equation, we refer the reader to the works of Evans and Spruck [ES91]. The normalized infinity Laplacian is related to certain geometric problems and was studied by [JK06, LY15]. We refer to [Eva07, KS06] for game theoretic interpretations of these equations for the elliptic case.

We can adapt some tools from the regularity theory of fully nonlinear parabolic equations (ABP estimates, Harnack inequality, Hölder and Lipschitz regularity...). However the $C^{1+\alpha,(1+\alpha)/2}$ regularity results do not seem to be straightforwardly applicable to quasilinear equations of the type (1.1) due to the discontinuity of the operator with respect to $Du$. For classical results on regularity theory for equations in non-divergence form, we refer the reader to Krylov and Safonov [Kry87, KSS80], where they used perturbation techniques in order
to get $C^{1+\alpha,\frac{1+\alpha}{2}}$ regularity for fully nonlinear parabolic PDEs under a smallness condition on the solution. We also refer to the works of L. Wang [Wa92I, Wa92II] where he used compactness arguments in the case where the oscillation of the diffusion term is small enough. For a nice introduction to fully nonlinear parabolic equations, we refer to the lecture notes of Imbert and Silvestre [IS12].

The paper is organized as follows. In Section 2 we present some preliminary tools, in Section 3 we prove Theorem 1.1 and in the last section we provide estimates for the homogeneous equation satisfied by the deviation of $u$ from a linear function.

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2. Preliminaries

For $r > 0$, denote $Q_r := B_r(0) \times (-r^2, 0]$ and $Q_r(x, t) := Q_r + (x, t)$ the parabolic cylinders. The normalized $p$-Laplacian can be seen as a uniformly parabolic operator in trace form on the set $\{Du \neq 0\}$. Indeed it can be written in the form

$$\Delta_p^N u = \text{tr}(A(Du)D^2u)$$

where

$$A(Du) = I + (p - 2)\frac{Du \otimes Du}{|Du|^2},$$

satisfies

$$\lambda|\xi|^2 \leq \langle A(Du)\xi, \xi \rangle \leq \Lambda|\xi|^2.$$

For $p > 1$, we denote by $\Lambda$ and $\lambda$ the ellipticity constants of the normalized $p$-Laplacian $\Delta_p^N$. It is easy to see that $\Lambda = \max(p - 1, 1)$ and $\lambda = \min(p - 1, 1)$.

We denote by $S^n$ the set of symmetric $n \times n$ matrices. For $a, b \in \mathbb{R}^n$, we denote by $a \otimes b$ the $n \times n$-matrix for which $(a \otimes b)_{ij} = a_i b_j$.

We will use the Pucci operators

$$P^+(X) := \sup_{A \in A_{\lambda, \Lambda}} -\text{tr}(AX)$$

and

$$P^-(X) := \inf_{A \in A_{\lambda, \Lambda}} -\text{tr}(AX),$$

where $A_{\lambda, \Lambda} \subset S^n$ is the set of symmetric $n \times n$ matrices whose eigenvalues belong to $[\lambda, \Lambda]$. 
Since we study Hölder and $C^{1+\alpha,(1+\alpha)/2}$ regularity in parabolic cylinders $Q_r$, for $\alpha \in (0, 1)$, we will use the notation

$$[u]_{C^{\alpha,\alpha/2}(Q_r)} := \sup_{(x,t),(y,s) \in Q_r, (x,t) \neq (y,s)} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{1/2}}.$$ 

$$\|u\|_{C^{\alpha,\alpha/2}(Q_r)} := \|u\|_{L^\infty(Q_r)} + [u]_{C^{\alpha,\alpha/2}(Q_r)}$$

for Hölder continuous functions.

We define the space $C^{1+\alpha,(1+\alpha)/2}(Q_r)$ as the space of all functions with finite norm

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(Q_r)} := \|u\|_{L^\infty(Q_r)} + \|Du\|_{L^\infty(Q_r)} + [u]_{C^{1+\alpha,(1+\alpha)/2}(Q_r)},$$

where

$$[u]_{C^{1+\alpha,(1+\alpha)/2}(Q_r)} := \sup_{(x,t),(y,s) \in Q_r, (x,t) \neq (y,s)} \frac{|Du(x,t) - Du(y,s)|}{|x-y|^\alpha + |t-s|^{1/2}}$$

$$+ \sup_{(x,t),(x,s) \in Q_r, t \neq s} \frac{|u(x,t) - u(x,s)|}{|t-s|^1}.$$

In this paper $C$ and $\tilde{C}$ will denote generic constants which may change from line to line.

The normalized $p$-Laplacian is undefined when $Du = 0$, where it has a bounded discontinuity. This difficulty can be resolved by adapting the notion of viscosity solution using the upper and lower semicontinuous envelopes (relaxations) of the operator, see [CIL92].

**Definition 2.1.** Let $\Omega$ be a bounded domain, $T > 0$, $1 < p < \infty$ and $f \in C(\Omega_T)$. An upper semicontinuous function $u$ is a viscosity subsolution of (1.1) if for all $(x_0,t_0) \in \Omega_T$ and $\varphi \in C^2(\Omega_T)$ such that $u - \varphi$ attains a local maximum at $(x_0,t_0)$, one has

$$\begin{cases} 
\varphi_t(x_0,t_0) - \Delta_p^N \varphi(x_0,t_0) \leq f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) \neq 0, \\
\varphi_t(x_0,t_0) - \Delta \varphi(x_0,t_0) - (p-2)\lambda_{\max}(D^2\varphi(x_0,t_0)) \leq f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) = 0 \text{ and } p \geq 2, \\
\varphi_t(x_0,t_0) - \Delta \varphi(x_0,t_0) - (p-2)\lambda_{\min}(D^2\varphi(x_0,t_0)) \leq f(x_0,t_0), & \text{if } D\varphi(x_0,t_0) = 0, 1 < p < 2. 
\end{cases}$$

A lower semicontinuous function $u$ is a viscosity supersolution of (1.1) if for all $(x_0,t_0) \in \Omega_T$ and $\varphi \in C^2(\Omega_T)$ such that $u - \varphi$ attains a local minimum at $(x_0,t_0)$, one has
Proof. If $D\varphi(x_0, t_0) \neq 0$, then $\varphi_t(x_0, t_0) - \Delta^N\varphi(x_0, t_0) \geq f(x_0, t_0)$, where $\varphi_t(x_0, t_0) \geq f(x_0, t_0)$, $\varphi_t(x_0, t_0) - \Delta\varphi(x_0, t_0) - (p - 2)\lambda_{\text{min}}(D^2\varphi(x_0, t_0)) \geq f(x_0, t_0)$, if $D\varphi(x_0, t_0) = 0$ and $p \geq 2$.

We say that $u$ is a viscosity solution of (1.1) in $\Omega_T$ if it is both a viscosity sub- and supersolution.

In Sections 3 and 4 we will need the following lemma which allows to control the oscillation in space-time for a solution of a uniformly parabolic equation with bounded discontinuities, once its oscillation in space is controlled in every time slice (see [JS15, Lemma 4.3] and [BBL02, Lemma 9.1]). The viscosity solution to (2.1) is defined analogously to (1.1).

**Lemma 2.2.** Let $v \in C(\overline{Q}_r)$ be a viscosity solution to

$$v_t - \Delta v - (p - 2)\left< D^2v, \frac{Dv + a}{|Dv + a|} \right> = f \quad \text{in} \quad Q_r,$$

where $a \in \mathbb{R}^n$ and $f \in L^\infty(Q_r) \cap C(Q_r)$. Suppose that for all $t \in [-r^2, 0]$

$$\text{osc}_{B_r} v(\cdot, t) \leq A,$$

then

$$\text{osc}_{Q_r} v \leq C(n, p)A + 4r^2||f||_{L^\infty(Q_r)}.$$

**Proof.** Denote $\Lambda = \max(1, p - 1)$ and define

$$v(x, t) := \bar{v} + \left(2||f||_{L^\infty(Q_r)} + \frac{5A\Lambda}{r^2}\right)t + 2Ar^{-2}|x|^2$$

where $\bar{v}$ is chosen so that $\bar{v}(\cdot, -r^2) \geq v(\cdot, -r^2)$ and $\bar{v}(\bar{x}, -r^2) = v(\bar{x}, -r^2)$ for some $\bar{x} \in B_r$. Notice that $\bar{x}$ must belong to $B_r$ since otherwise we would have

$$2A = \bar{v}(\bar{x}, -r^2) - \bar{v}(0, -r^2) \leq v(\bar{x}, -r^2) = v(0, -r^2)$$

$$\leq \text{osc}_{B_r} v(\cdot, -r^2) \leq A,$$

which is impossible. Now we claim that $\bar{v} \geq v$ in $Q_r$. We argue by contradiction. If this is not true, set $m = -\inf_{Q_r}(\bar{v} - v) > 0$. Let $(\hat{x}, \hat{t}) \in \overline{Q}_r$ be such that $m = v(\hat{x}, \hat{t}) - \bar{v}(\hat{x}, \hat{t})$. By assumption, we have
that osc $v(\cdot, \hat{t}) \leq A$. Since by construction, $\bar{v}(\bar{x}, -r^2) = v(\bar{x}, -r^2)$, we have $\hat{t} > -r^2$. Noticing that $\bar{v} + m \geq v$ and $\bar{v}(\hat{x}, \hat{t}) + m = u(\hat{x}, \hat{t})$, then arguing as above, we see that $\hat{x} \in B_r$. Since $v$ is a viscosity solution of (2.1) and $\bar{v}$ is a smooth test function, it follows by definition that if $D\bar{v}(\hat{x}, \hat{t}) + a \neq 0$, we have

$$||f||_{L^\infty(Q_r)} + \frac{5An\Lambda}{r^2} \leq \partial_t\bar{v}(\hat{x}, \hat{t}) - f(\hat{x}, \hat{t})$$

$$\leq \Delta\bar{v}(\hat{x}, \hat{t}) + (p-2) \left(D^2\bar{v}(\hat{x}, \hat{t}), \frac{D\bar{v}(\hat{x}, \hat{t}) + a}{|D\bar{v}(\hat{x}, \hat{t}) + a|}, \frac{D\bar{v}(\hat{x}, \hat{t}) + a}{|D\bar{v}(\hat{x}, \hat{t}) + a|}\right)$$

$$\leq \frac{4n\Lambda A}{r^2},$$

and if $D\bar{v}(\hat{x}, \hat{t}) + a = 0$, then

$$||f||_{L^\infty(Q_r)} + \frac{5An\Lambda}{r^2} \leq \partial_t\bar{v}(\hat{x}, \hat{t}) - f(\hat{x}, \hat{t})$$

$$\leq \Delta\bar{v}(\hat{x}, \hat{t}) + (p-2)\lambda_{\max}(D^2\bar{v}(\hat{x}, \hat{t}))$$

$$\leq \frac{4n\Lambda A}{r^2},$$

which is impossible. Using similar arguments, we can show that for some suitable $\eta$ the function

$$\nu(x, t) := \eta - \left(2||f||_{L^\infty(Q_r)} + \frac{5An\Lambda}{r^2}\right)t - 2Ar^{-2}|x|^2$$

satisfies $\nu(\cdot, -r^2) \leq \nu(\cdot, -r^2)$ and $\nu(x, -r^2) = \nu(x, -r^2)$ for some $x \in B_r$ and hence

$$\nu \leq v \in Q_r.$$

Moreover, since $\bar{v}(x, -r^2) - \nu(x, -r^2) \leq osc_{B_r} v(\cdot, -r^2) \leq A$, we have

$$\bar{\eta} - \eta \leq (10n\Lambda + 1)A + 4r^2 ||f||_{L^\infty(Q_r)}$$

and it follows that

$$osc \nu \leq sup_{Q_r} \bar{v} - inf_{Q_r} v \leq \bar{\eta} - \eta + 4A \leq (10n\Lambda + 5)A + 4r^2 ||f||_{L^\infty(Q_r)}.$$  

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We assume that $p > 1$ and $f \in L^\infty(\Omega_T) \cap C(\Omega_T)$ and want to show that there exists $\alpha = \alpha(p, n) > 0$ such that any viscosity solution $u$ to equation (1.1) is of class $C^{1+\alpha, \frac{\alpha+1}{2}}_{loc}(\Omega_T)$. 
3.1. Reduction of the problem and preliminary lemmas. First we reduce the problem by rescaling. For $\varepsilon_0 > 0$, let
\[
\kappa = (2 \|u\|_{L^\infty(\Omega_T)} + \varepsilon_0^{-1} \|f\|_{L^\infty(\Omega_T)})^{-1}.
\]
Setting $\tilde{u} = \kappa u$, then $\tilde{u}$ satisfies
\[
\partial_t \tilde{u} - \Delta^N_p (\tilde{u}) = \tilde{f},
\]
where $\tilde{f} := \kappa f$ and $\|\tilde{u}\|_{L^\infty(\Omega_T)} \leq \frac{1}{2}$ and $\|\tilde{f}\|_{L^\infty(\Omega_T)} \leq \varepsilon_0$. Hence, without loss of generality we may assume in Theorem 1.1 that $\|u\|_{L^\infty(\Omega_T)} \leq 1/2$ and $\|f\|_{L^\infty(\Omega_T)} \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(p, n)$ is chosen later. Next, we notice that it is sufficient to prove that there exists some constant $C = C(p, n) > 0$ such that for fixed $(x, t) \in \Omega \times (0, T)$ and small enough $r \in (0, 1)$, there exists $q = q(r, x, t) \in \mathbb{R}^n$ for which
\[
\text{osc}_{Q_r(x,t)} (u(y, s) - u(x, t) - q \cdot (x - y)) \leq Cr^{1+\alpha}.
\]
It suffices to choose a suitable $\rho \in (0, 1)$ such that the previous inequality holds true for $r = r_k = \rho^k$, $q = q_k$ and $C = 1$ by proceeding by induction on $k \in \mathbb{N}$. Using a covering of $\Omega_T$ by cubes $Q_r(x, t)$ where $x \in \Omega$, $t \in (0, T)$ and $r < \text{dist}((x, t), \partial\Omega_T)$, we may work on cubes $Q_r(x, t)$. By translation, it is enough to show that the solution is $C^{1+\alpha, \frac{\alpha+1}{2}}$ at 0, and by considering
\[
\tilde{u}_r(y, s) = r^{-2}u(x + ry, t + r^2s),
\]
we may work on the unit parabolic cylinder $Q_1(0, 0)$. Finally, considering $u(x, t) - u(0, 0)$ if necessary, we may suppose that $u(0, 0) = 0$.

Notice that if $w(x, t) = u(x, t) - q \cdot x$, then $w$ satisfies
\[
\partial_t w - \Delta w - (p - 2) \left\langle D^2 w, \frac{Dw + q}{|Dw + q|} \frac{Dw + q}{|Dw + q|} \right\rangle = f \quad \text{in} \quad \Omega_T. \tag{3.1}
\]
In order to prove Theorem 1.1, we will first need the following equi-continuity lemma.

**Lemma 3.1.** For all $r \in (0, 1)$, there exist $\beta = \beta(p, n) \in (0, 1)$ and $C = C(p, n, r) > 0$ such that any viscosity solution $w$ of (3.1) with $\text{osc}_{Q_1} w \leq 1$ and $\|f\|_{L^\infty(\Omega_1)} \leq 1$ satisfies
\[
\|w\|_{C^{3,\beta/2}(Q_r)} \leq C. \tag{3.2}
\]

**Proof.** Equation (3.1) can be rewritten as
\[
\partial_t w - \text{tr} \left( \left( I + (p - 2) \frac{Dw + q}{|Dw + q|} \right) \frac{Dw + q}{|Dw + q|} \right) D^2 w = f.
\]
Recalling the definitions for Pucci operators $P^+$ and $P^-$ respectively, the equation takes the form

\[
\begin{cases}
\partial_t w + P^+(D^2 w) + |f| \geq 0 \\
\partial_t w + P^-(D^2 w) - |f| \leq 0
\end{cases}
\]

By the classical results of [KS80, Wa92I] (see also [IS12]), there exist $\beta = \beta(p, n) \in (0, 1)$ and $C_\beta = C(r, p, n)$ such that

\[
||w||_{C^{\beta, \beta/2}(Q_r)} \leq C_\beta \left( \text{osc}_{Q_1} w + ||f||_{L^{n+1}(Q_1)} \right) \leq C_\beta.
\]

Our next step consists in showing a linear approximation result for solutions to equation (3.1). We proceed by contradiction, using the previous proposition together with known regularity results for uniformly parabolic linear PDEs and the following result for the homogeneous equation associated to (3.1). For convenience, we postpone the technical proof of Proposition 3.2 and present it in the last section.

**Proposition 3.2.** Assume that $f \equiv 0$ and let $w$ be a viscosity solution to equation (3.1) with $\text{osc}_{Q_1} w \leq 1$. For all $r \in (0, \frac{3}{4})$, there exist constants $C_0 = C_0(p, n) > 0$ and $\beta_1 = \beta_1(p, n) > 0$ such that

\[
||w||_{C^{1+\beta_1,(1+\beta_1)/2}(Q_r)} \leq C_0.
\] (3.3)

Now we are in position to prove the following improvement of flatness lemma.

**Lemma 3.3.** There exist $\varepsilon_0 = \varepsilon_0(p, n) \in (0, 1)$ and $\rho = \rho_0(p, n) \in (0, 1)$, such that for any $q \in \mathbb{R}^n$ and for any viscosity solution $w$ of (3.1) with $\text{osc}_{Q_1} w \leq 1$ and $||f||_{L^\infty(Q_1)} \leq \varepsilon_0$, there exists $q' \in \mathbb{R}^n$ with $|q'| \leq \bar{C}(p, n)$ such that

\[
\text{osc}_{Q_\rho} (w(x, t) - q' \cdot x) \leq \frac{1}{2} \rho.
\]

_Proof._ Suppose by contradiction that there exist a sequence of functions $(f_j)$ with $||f_j||_{L^\infty(Q_1)} \to 0$, a sequence of vectors $(q_j)$ and a sequence of viscosity solutions $(w_j)$ with $\text{osc}_{Q_1} w_j \leq 1$ to

\[
\partial_t w_j - \Delta w_j - (p-2) \left\langle D^2 w_j \frac{Dw_j + q_j}{|Dw_j + q_j|}, \frac{Dw_j + q_j}{|Dw_j + q_j|} \right\rangle = f_j \quad \text{in} \quad Q_1,
\]

such that, for all $q' \in \mathbb{R}^n$ and any $\rho \in (0, 1)$

\[
\text{osc}_{Q_\rho} (w_j(x, t) - q' \cdot x) > \frac{\rho}{2}.
\] (3.4)
Using the Arzelà-Ascoli compactness result with Lemma 3.1, there exists a continuous function $w_\infty$ such that $w_j \to w_\infty$ uniformly in $Q_\rho$ for any $\rho \in (0, 1)$. Passing to the limit in (3.4), we have that for any vector $q'$,

$$\text{osc}_{Q_\rho}(w_\infty(x, t) - q' \cdot x) > \frac{\rho}{2}. \tag{3.5}$$

Let us first suppose that the sequence $(q_j)$ is bounded. Then, up to a subsequence, it converges to $q_\infty$. Using the relaxed limit, we get that $w_\infty$ satisfies

$$\partial_t w_\infty - \Delta w_\infty - (p - 2) \left\langle D^2 w_\infty \frac{Dw_j + q_j}{|Dw_\infty + q_\infty|}, \frac{Dw_j + q_\infty}{|Dw_\infty + q_\infty|} \right\rangle = 0,$$

applying Proposition 3.2 we have that, there exists $C_0 = C_0(p, n) > 0$ such that

$$||w_\infty||_{C^{1+\beta_1, (1+\beta_1)/2}(Q_1)} \leq C_0.$$

If the sequence $(q_j)$ is unbounded, take a subsequence, still denoted by $(q_j)$, for which $|q_j| \to \infty$, and then a converging subsequence from $e_j = \frac{q_j}{|q_j|}$, $e_j \to e_\infty$. We have that in $Q_1$

$$\partial_t w_j - \Delta w_j - (p - 2) \left\langle D^2 w_j \frac{|q_j|^{-1} + e_j}{|Dw_j|q_j^{-1} + e_j|}, \frac{|q_j|^{-1} + e_j}{|Dw_j|q_j^{-1} + e_j|} \right\rangle = f_j.$$ 

Passing to the limit we obtain

$$\partial_t w_\infty - \Delta w_\infty - (p - 2) \left\langle D^2 w_\infty e_\infty, e_\infty \right\rangle = 0 \quad \text{in} \quad Q_1, \tag{3.6}$$

with $|e_\infty| = 1$. Noticing that equation (3.6) can be written as

$$\partial_t w_\infty - \text{tr} \left((I + (p - 2)e_\infty \otimes e_\infty)D^2 w_\infty\right) = 0,$$

we see that equation (3.6) is linear, uniformly parabolic and depends only on $\partial_t w_\infty$ and $D^2 w_\infty$. By the regularity result of [Lie96, Lemma 12.13], there is $\beta_2 > 0$ so that $w_\infty \in C^{1+\beta_2, (1+\beta_2)/2}_{\text{loc}}$ and the Hölder norm of $Dw_\infty$ is bounded by a constant depending only on $p, n, ||w_\infty||_{L^\infty(Q_1)}$ that we still denote by $C_0$.

We have thus shown that $w_\infty \in C^{1+\beta, (1+\beta)/2}_{\text{loc}}$ for $\beta = \min(\beta_1, \beta_2) > 0$ with a Hölder norm independent of the sequence $(q_j)$. Choose $\rho \in (0, 1/2)$ such that

$$C_0 \rho^\beta \leq \frac{1}{4}.$$

By $C^{1+\beta, (1+\beta)/2}_{\text{loc}}$ regularity there exists a vector $k_\rho$ with $||k_\rho|| \leq \tilde{C}(p, n)$ (see Proposition 3.2) such that

$$\text{osc}_{Q_\rho}(w_\infty(x, t) - k_\rho \cdot x) \leq C_0 \rho^{1+\beta} \leq \frac{1}{4} \rho. \tag{3.7}$$
This contradicts (3.5), and the proof is complete. The boundedness of \(|q'|\) can be obtained as follows. Since \(w_j\) converges to \(w_\infty\), we get that for all \(\varepsilon_1 > 0\) there exists \(\varepsilon_0 = \varepsilon_0(p, n)\) sufficiently small such that if \(\|f\|_{L^\infty(Q_1)} \leq \varepsilon_0\) then
\[
\text{osc}_{Q_\rho}(w - w_\infty) \leq \varepsilon_1. \tag{3.8}
\]
Taking \(\varepsilon_1 = \frac{\rho}{4}\), we conclude from (3.7) and (3.8) that for \(\varepsilon_0\) sufficiently small, if \(\|f\|_{L^\infty(Q_1)} \leq \varepsilon_0\) then
\[
\text{osc}_{Q_\rho}(w(x, t) - k_\rho \cdot x) \leq \frac{\rho}{2}. \quad \square
\]

3.2. Iteration and proof of the main theorem.

**Hölder estimate for the gradient in the space variable**

In order to control the Hölder continuity of the gradient of a function with respect to the space variable, it is standard to make sure that, around each point, the function can be deviated from a plane so that its oscillation in a ball of radius \(r > 0\) is of order \(r^{1+\alpha}\) (see [Lie96, Lemma 12.12] and Appendix A).

The Hölder regularity with respect to the space variable stated in Theorem 1.1 is a direct consequence of the following lemma and Lemma A.1 after scaling back from \(\tilde{u}\) to \(u\).

**Lemma 3.4.** Assume that \(\rho, \varepsilon_0 \in (0, 1)\) are as in Lemma 3.3 and let \(u\) be a viscosity solution of (1.1) with \(\text{osc}_{Q_1} u \leq 1\) and \(\|f\|_{L^\infty(Q_1)} \leq \varepsilon_0\). Then there exists \(\alpha \in (0, 1)\) such that, for all \(k \in \mathbb{N}\), there exists \(q_k \in \mathbb{R}^n\) such that
\[
\text{osc}_{Q_{r_k}}(u(y, t) - q_k \cdot y) \leq r_k^{1+\alpha}, \tag{3.9}
\]
where \(r_k := \rho^k\).

**Proof.** For \(k = 0\), the estimate (3.9) follows from the assumption \(\text{osc}_{Q_1} u \leq 1\) and \(q_0 = 0\). Next we take \(\alpha \in (0, 1)\) such that \(\rho^\alpha > 1/2\). We assume that \(k \geq 0\) and that we constructed already \(q_k \in \mathbb{R}^n\) such that (3.9) holds true. To prove the inductive step \(k \rightarrow k + 1\), we denote \(r_k := \rho^k\) and we rescale the solution considering for \(x \in Q_1\)
\[
w_k(x, t) = r_k^{-1-\alpha}\left(u(r_k x, r_k^2 t) - q_k \cdot (r_k x)\right).
\]

We have by induction assumption, \(\text{osc}_{Q_1}(w_k) \leq 1\) and \(w_k\) satisfies
\[
\partial_t w_k - \Delta w_k - (p - 2) \left\langle D^2 w_k \frac{Dw_k + (q_k/r_k^\alpha)}{|Dw_k + (q_k/r_k^\alpha)|}, \frac{Dw_k + (q_k/r_k^\alpha)}{|Dw_k + (q_k/r_k^\alpha)|} \right\rangle = f_k,
\]
where \( f_k(x, t) = r_k^{1-\alpha}f(r_kx, r_k^2t) \) with \( ||f_k||_{L^\infty(Q_1)} \leq \varepsilon_0 \) since \( \alpha < 1 \). Using the result of Lemma 3.3, there exists \( l_k \in \mathbb{R}^n \) with \( ||l_k|| \leq \tilde{C}(p, n) \) such that

\[
\text{osc}_{B_\rho}(w_k(x, t) - l_k \cdot x) \leq \frac{1}{2}\rho.
\]

Setting

\[ q_{k+1} = q_k + l_k r_k^\alpha, \quad (3.10) \]

we get

\[
\text{osc}_{Q_{r_{k+1}}}(u(x, t) - q_{k+1} \cdot x) \leq \frac{\rho}{2} r_k^{1+\alpha} \leq r_{k+1}^{1+\alpha}. \quad \square
\]

**Proof of estimate (1.2) in Theorem 1.1:** We show that \( q_k \) converges to a vector \( q_\infty \). Indeed from (3.10), we have that for \( m \geq k \), \( |q_m - q_k| \leq \sum_{j=k}^{m-1} r_j^\alpha \leq C \rho^{\kappa_\alpha} \), where \( C = C(p, n) > 0 \). It follows that \( q_k \) converges and

\[
\sup_{(x, t) \in Q_{r_k}} (q_k \cdot x - q_\infty \cdot x) \leq C \rho^{k(1+\alpha)}, \quad \text{osc}_{(x, t) \in Q_{r_k}} (u(x, t) - q_k \cdot x) \leq \rho^{k(1+\alpha)}.
\]

Consequently, we have that

\[
\sup_{(x, t) \in Q_{r_k}} |u(x, t) - q_\infty \cdot x - u(0, 0)| \leq C r_k^{1+\alpha}, \quad (3.11)
\]

where \( C = C(p, n) \). It follows from Lemma A.1 that \( Du \) is of class \( C^{\alpha, \alpha/2} \) at \((0, 0)\) and we will denote \( q_\infty \) by \( Du(0, 0) \) in the sequel. The estimate (1.2) in Theorem 1.1 follows from translation arguments, estimate (3.11) and Lemma A.1. \( \square \)

**Hölder estimate for the solution in the time variable**

Here we show the regularity of \( u \) with respect to the time variable to finish the proof of Theorem 1.1 by constructing suitable barriers in order to control the oscillation of \( u \) (see [JS15, Lemma 4.3]).

**Lemma 3.5.** Under the hypothesis of Lemma 3.4, there exists \( C = C(p, n) > 0 \) such that for all \( t \in (-r^2, 0) \), we have

\[
|u(0, t) - u(0, 0)| \leq C|t|^{\frac{1+\alpha}{2}}. \quad (3.12)
\]

**Proof.** For \((x, t) \in Q_r\), set

\[
v(x, t) := u(x, t) - u(0, 0) - Du(0, 0) \cdot x.
\]

It follows from (3.9) that for \( x_1, x_2 \in B_r \) and \( t \in [-r^2, 0] \), we have

\[
|v(x_1, t) - v(x_2, t)| \leq \text{osc}_{(y,s) \in Q_r} (u(y, s) - Du(0, 0) \cdot y) \leq C r^{1+\alpha}.
\]
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We conclude that, for all $t \in [-r^2, 0]$, we have
$$\overset{\overset{\text{osc}}{B_r}}{v}(\cdot, t) \leq Cr^{1+\alpha} =: A.$$ 

We claim that there exists a constant $C = C(p, n) > 0$ such that
$$\overset{\overset{\text{osc}}{Q_r}}{v} \leq CA + 4r^2 ||f||_{L^\infty(Q_1)}.$$ 

Indeed, denoting $b = Du(0, 0)$, we observe that $v$ satisfies in $Q_r$ the following equation in the viscosity sense
$$\partial_t v - \Delta v - (p - 2) \left< D^2 v, \frac{Dv + b}{|Dv + b|}, \frac{Dv + b}{|Dv + b|} \right> = f.$$ 

We conclude from Lemma 2.2 that
$$\overset{\overset{\text{osc}}{Q_r}}{v} \leq C(p, n)r^{1+\alpha} + 4r^2 ||f||_{L^\infty(Q_1)} \leq C(p, n)r^{1+\alpha}.$$ 

In particular for $t \in (-r^2, 0)$, we have
$$|u(0, t) - u(0, 0)| = |v(0, t)| \leq C(p, n)|t|^\frac{1+\alpha}{2}.$$ 

Since the equation (1.1) is invariant under translation, we get the desired result and have thus proven the second estimate (1.3) in Theorem 1.1. □

4. LOCAL HÖLDER ESTIMATE FOR THE GRADIENT OF THE LIMITING EQUATION

In this section we derive estimates for bounded viscosity solutions to the following equation
$$w_t - \Delta w - (p - 2) \left< D^2 w, \frac{Dw + q}{|Dw + q|}, \frac{Dw + q}{|Dw + q|} \right> = 0 \quad \text{in } Q_1, \quad (4.1)$$

and in particular prove Proposition 3.2.

Introducing the function $v(x, t) := w(x, t) + q \cdot x$, it is easy to check that $v$ is a viscosity solution to
$$v_t - \Delta_p^N v = 0 \quad \text{in } Q_1.$$ 

By the regularity result of Jin and Silvestre [JS15, Theorem 1.1], there is $\beta_1 = \beta_1(p, n) > 0$ so that $v \in C^{1+\beta_1, (1+\beta_1)/2}_{\text{loc}}(Q_1)$ and hence also $w \in C^{1+\beta_1, (1+\beta_1)/2}_{\text{loc}}(Q_1)$. The main difficulty is to provide $C^{1+\beta_1, (1+\beta_1)/2}$ estimates which are uniform with respect to $q$.

The main idea is to divide the study to the cases $|q|$ small and $|q|$ large (depending on $p$ and $n$). For $|q|$ large enough, that is $|q| >$
\( L_0(p, n) \) where \( L_0(p, n) \) will be chosen later, our strategy is to prove that the Lipschitz norm of \( w \) is controlled independently from \( q \), hence the equation is no longer discontinuous and we can adapt the proof of \([JS15]\) to derive \( C^{1+\beta_1, (1+\beta_1)/2} \) estimates uniform with respect to \( q \). When \( |q| \leq L_0 \) the result follows immediately from \([JS15, Theorem 1.1]\). Indeed, since 

\[
\text{osc } v \leq \text{osc } w + 2|q| \leq 1 + 2L_0,
\]

we get that 

\[
\|w\|_{C^{1+\beta_1, (1+\beta_1)/2}(Q_{1/2})} \leq \|v\|_{C^{1+\beta_1, (1+\beta_1)/2}(Q_{1/2})} + 2|q| \\
\leq C(p, n)\text{osc } v + 2L_0 \leq C_0(p, n).
\]

Hence, from now on we will focus on the case \( |q| \) large.

### 4.1. Lipschitz through the Ishii-Lions method

In order to prove Proposition \([32]\) we first need the following technical lemma concerning Lipschitz regularity of solutions of equation \((4.1)\). For \( n \times n \) matrices we use the matrix norm 

\[
\|A\| := \sup_{|x| \leq 1} \{|Ax|\}.
\]

**Lemma 4.1.** Let \( w \) be a bounded viscosity solution to equation \((4.1)\) with \( \text{osc } w \leq 1 \). For all \( r \in (0, \frac{3}{4}) \), there exists a constant \( \nu_0 = \nu_0(p, n) > 0 \) such that, if \( |q| > \nu_0 \), then for all \( x, y \in \overline{B_r} \) and \( t \in [-r^2, 0] \),

\[
|w(x, t) - w(y, t)| \leq \tilde{C} |x - y|,
\]

where \( \tilde{C} = \tilde{C}(p, n) > 0 \).

**Proof.** We use the viscosity method introduced by Ishii and Lions in \([IL90]\) (see also \([IS12, JJS16]\) for further applications).

**Step 1.** Notice that it suffices to show that \( w \) is Lipschitz in \( Q_{3/4} \), this will imply that \( w \) is Lipschitz in any smaller cube \( Q_r \) for \( r \in (0, \frac{3}{4}) \) with the same Lipschitz constant. In the sequel we take \( r = 3/4 \). First we fix \( x_0, y_0 \in B_r \), \( t_0 \in (-r^2, 0) \) and introduce the auxiliary function

\[
\Phi(x, y, t) := w(x, t) - w(y, t) - L\varphi(|x - y|)
\]

\[
- \frac{M}{2} |x - x_0|^2 - \frac{M}{2} |y - y_0|^2 - \frac{M}{2} (t - t_0)^2,
\]
where $\varphi$ is defined below. Our aim is to show that $\Phi(x, y, t) \leq 0$ for $(x, y) \in \overline{B}_r \times \overline{B}_r$ and $t \in [-r^2, 0]$. For a proper choice of $\varphi$, this yields the desired regularity result. We take

$$
\varphi(s) = \begin{cases} 
    s - s^\gamma \kappa_0 & 0 \leq t \leq s_1 := (\frac{1}{\gamma \kappa_0})^{1/(\gamma - 1)} \\
    \varphi(s_1) & \text{otherwise},
\end{cases}
$$

where $2 > \gamma > 1$ and $\kappa_0 > 0$ is such that $s_1 \geq 2$ and $\gamma \kappa_0 2^{\gamma - 1} \leq 1/4$.

Then

$$
\varphi'(s) = \begin{cases} 
    1 - \gamma s^{\gamma - 1} \kappa_0 & 0 \leq s \leq s_1 \\
    0 & \text{otherwise},
\end{cases}
$$

$$
\varphi''(s) = \begin{cases} 
    -\gamma (\gamma - 1) s^{\gamma - 2} \kappa_0 & 0 < s \leq s_1 \\
    0 & \text{otherwise}.
\end{cases}
$$

In particular, $\varphi'(s) \in \left[\frac{3}{4}, 1\right]$ and $\varphi''(s) < 0$ when $s \in [0, 2]$.

**Step 2.** We argue by contradiction and assume that $\Phi$ has a positive maximum at some point $(x_1, y_1, t_1) \in \overline{B}_r \times \overline{B}_r \times [-r^2, 0]$. Notice that $x_1 \neq y_1$, otherwise the maximum of $\Phi$ would be non positive. Since $w$ is continuous and its oscillation is bounded, choosing

$$
M \geq \frac{32 \text{osc}_{Q_1} w}{\max(d((x_0, t_0), \partial Q_r), d((y_0, t_0), \partial Q_r))^2},
$$

we get

$$
|x_1 - x_0| + |t_1 - t_0| \leq 2 \sqrt{\frac{2|w(x_1, t_1) - w(y_1, t_1)|}{M}} \leq 2 \sqrt{\frac{2 \text{osc}_{Q_1} w}{M}}
$$

$$
\leq \frac{d((x_0, t_0), \partial Q_r)}{2},
$$

$$
|y_1 - y_0| + |t_1 - t_0| \leq 2 \sqrt{\frac{2|w(x_1, t_1) - w(y_1, t_1)|}{M}} \leq 2 \sqrt{\frac{2 \text{osc}_{Q_1} w}{M}}
$$

$$
\leq \frac{d((y_0, t_0), \partial Q_r)}{2},
$$

so that $x_1$ and $y_1$ are in $B_r$ and $t_1 \in (-r^2, 0)$.

We know by Lemma 3.1 that, $w$ is locally Hölder continuous and that there exists a constant $C_\beta > 0$ depending only on $p, n$ and $\text{osc}_w$ such that

$$
|w(x, t) - w(y, t)| \leq C_\beta |x - y|^\beta \quad \text{for } x, y \in B_r, t \in (-r^2, 0).
$$
Using that \( w \) is Hölder continuous, it follows, adjusting the constants (by choosing \( 2M \leq C_{\beta} \)), that
\[
M |x_1 - x_0| \leq C_{\beta} |x_1 - y_1|^{\beta/2} ,
\]
\[
M |y_1 - y_0| \leq C_{\beta} |x_1 - y_1|^{\beta/2} .
\]

By Jensen-Ishii’s lemma (also known as theorem of sums, see [CIL92, Theorem 8.3] and [IS12], there exist
\[
(\sigma, \tilde{\zeta}_x, X) \in \mathcal{P}^{2,+}(w(x_1, t_1) - \frac{M}{2} |x_1 - x_0|^2 - \frac{M}{2} (t_1 - t_0)^2),
\]
\[
(\sigma, \tilde{\zeta}_y, Y) \in \mathcal{P}^{2,-}(w(y_1, t_1) + \frac{M}{2} |y_1 - y_0|^2),
\]
that is
\[
(\sigma + M(t_1 - t_0), a, X + MI) \in \mathcal{P}^{2,+}w(x_1, t_1),
\]
\[
(\sigma, b, Y - MI) \in \mathcal{P}^{2,-}w(y_1, t_1),
\]
where \( \tilde{\zeta}_x = \tilde{\zeta}_y \)
\[
a = L\varphi'(|x_1 - y_1|) \frac{x_1 - y_1}{|x_1 - y_1|} + M(x_1 - x_0) = \tilde{\zeta}_x + M(x_1 - x_0),
\]
\[
b = L\varphi'(|x_1 - y_1|) \frac{x_1 - y_1}{|x_1 - y_1|} - M(y_1 - y_0) = \tilde{\zeta}_y - M(y_1 - y_0).
\]
If \( L \) is large enough (depending on the Hölder constant \( C_{\beta} \)), we have
\[
|a|, |b| \geq L\varphi'(|x_1 - y_1|) - C_{\beta} |x_1 - y_1|^{\beta/2} \geq \frac{L}{2}.
\]
Moreover, by Jensen-Ishii’s lemma, for any \( \tau > 0 \), we can take \( X, Y \in \mathcal{S}^a \) such that
\[
- \left[ \tau + 2 ||B|| \right] \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \tag{4.3}
\]
and
\[
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \frac{2}{\tau} \begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}, \tag{4.4}
\]
where
\[
B = L\varphi''(|x_1 - y_1|) \frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|} + \frac{L\varphi'(|x_1 - y_1|)}{|x_1 - y_1|} \left( I - \frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|} \right).
\]
and
\[
B^2 = \frac{L^2(\varphi'(|x_1 - y_1|))^2}{|x_1 - y_1|^2} \left( I - \frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|} \right) \right] \\
\quad \quad \quad \quad \quad + L^2(\varphi''(|x_1 - y_1|))^2 \frac{x_1 - y_1}{|x_1 - y_1|} \otimes \frac{x_1 - y_1}{|x_1 - y_1|}.
\]

Notice that
\[
||B|| \leq L \varphi'(|x_1 - y_1|), \quad (4.5)
\]
\[
||B^2|| \leq L^2 \left( |\varphi''(|x_1 - y_1|)| + \frac{\varphi'(|x_1 - y_1|)}{|x_1 - y_1|} \right)^2, \quad (4.6)
\]
and for \( \xi = \frac{x_1 - y_1}{|x_1 - y_1|} \), we have
\[
\langle B\xi, \xi \rangle = L \varphi''(|x_1 - y_1|) < 0, \quad \langle B^2\xi, \xi \rangle = L^2(\varphi''(|x_1 - y_1|))^2.
\]
Choosing \( \tau = 4L \left( |\varphi''(|x_1 - y_1|)| + \frac{\varphi'(|x_1 - y_1|)}{|x_1 - y_1|} \right) \), we have that for \( \xi = \frac{x_1 - y_1}{|x_1 - y_1|} \):
\[
\langle B\xi, \xi \rangle + \frac{2}{\tau} \langle B^2\xi, \xi \rangle = L \left( \varphi''(|x_1 - y_1|) + \frac{2}{\tau} L(\varphi''(|x_1 - y_1|))^2 \right) \leq \frac{L}{2} \varphi''(|x_1 - y_1|) < 0. \quad (4.7)
\]
In particular applying inequalities (4.3) and (4.4) to any vector \((\xi, \xi)\) with \( |\xi| = 1 \), we have that \( X - Y \leq 0 \) and \( ||X||, ||Y|| \leq 2 ||B|| + \tau \). We refer the reader to [IL90, CIL92] for details.

Thus, setting \( \eta_1 = a + q, \eta_2 = b + q \), we have for \( |q| \) large enough (depending only on \( L \))
\[
|\eta_1| \geq |q| - |a| \geq \frac{|a|}{2} \geq \frac{L}{4},
\]
\[
|\eta_2| \geq |q| - |b| \geq \frac{|b|}{2} \geq \frac{L}{4}, \quad (4.8)
\]
where \( L \) will be chosen later on and \( L \) will depend only on \( p, n, C_\beta \).

Writing the viscosity inequalities
\[
M(t_1 - t_0) + \sigma \leq \text{tr}(X + MI) + (p - 2) \frac{\langle (X + MI)(a + q), (a + q) \rangle}{|a + q|^2},
\]
\[
\sigma \geq \text{tr}(Y - MI) + (p - 2) \frac{\langle (Y - MI)(b + q), (b + q) \rangle}{|b + q|^2},
\]

We refer the reader to [IL90, CIL92] for details.
we get
\[
M(t_1 - t_0) + \sigma \leq \text{tr}(A(\eta_1)(X + MI))
\]
\[
-\sigma \leq -\text{tr}(A(\eta_2)(Y - MI))
\]
where for \( \eta \neq 0 \bar{\eta} = \frac{\eta}{|\eta|} \) and
\[
A(\eta) := I + (p - 2)\bar{\eta} \otimes \bar{\eta}.
\]
Adding the two inequalities, we get
\[
0 \leq \text{tr}(A(\eta_1)(X + MI)) - \text{tr}(A(\eta_2)(Y - MI)) + M|t_1 - t_0|.
\]
It follows that
\[
0 \leq \text{tr}(A(\eta_1)(X - Y)) + \text{tr}((A(\eta_1) - A(\eta_2))Y)
\]
\[
+ M\left[\text{tr}(A(\eta_1)) + \text{tr}(A(\eta_2))\right] + 2Mr^2.
\] (4.9)
Notice that all the eigenvalues of \( X - Y \) are non positive. Moreover, applying the previous matrix inequality (4.4) to the vector \((\xi, -\xi)\) where \( \xi := \frac{x_1 - y_1}{|x_1 - y_1|} \) and using (4.7), we obtain
\[
\langle (X - Y)\xi, \xi \rangle \leq 4 \left( \langle B\xi, \xi \rangle + \frac{2}{r} \langle B^2\xi, \xi \rangle \right)
\]
\[
\leq 2L\varphi''(|x_1 - y_1|) < 0. \quad (4.10)
\]
Hence at least one of the eigenvalue of \( X - Y \) that we denote by \( \lambda_{i_0} \) is negative and smaller than \( 2L\varphi''(|x_1 - y_1|) \). The eigenvalues of \( A(\eta_1) \) belong to \([\min(1, p - 1), \max(1, p - 1)]\). Using (4.10), it follows by [Theo75] that
\[
\text{tr}(A(\eta_1)(X - Y)) \leq \sum_i \lambda_i(A(\eta_1))\lambda_i(X - Y)
\]
\[
\leq \min(1, p - 1)\lambda_{i_0}(X - Y)
\]
\[
\leq 2\min(1, p - 1)L\varphi''(|x_1 - y_1|).
\]
It is easy to see that
\[
A(\eta_1) - A(\eta_2) = \bar{\eta}_1 \otimes \bar{\eta}_1 - \bar{\eta}_2 \otimes \bar{\eta}_2
\]
\[
= (\bar{\eta}_1 - \bar{\eta}_2) \otimes \bar{\eta}_1 - \bar{\eta}_2 \otimes (\bar{\eta}_2 - \bar{\eta}_1)
\]
\[
= (\bar{\eta}_1 - \bar{\eta}_2) \otimes \bar{\eta}_1 + \bar{\eta}_2 \otimes \bar{\eta}_1 - \bar{\eta}_2 \otimes (\bar{\eta}_2 - \bar{\eta}_1) - \bar{\eta}_2 \otimes \bar{\eta}_1
\]
\[
= (\bar{\eta}_1 - \bar{\eta}_2) \otimes \bar{\eta}_1 - \bar{\eta}_2 \otimes (\bar{\eta}_2 - \bar{\eta}_1)
\]
and hence
\[
\text{tr}((A(\eta_1) - A(\eta_2))Y) \leq n\|Y\|\|A(\eta_1) - A(\eta_2)\|
\]
\[
\leq n|p - 2|\|Y\|\|
\]
\[
\bar{\eta}_1 - \bar{\eta}_2 |(\bar{\eta}_1 + |\bar{\eta}_2|)
\]
\[
|\eta_1 - \eta_2| = \left| \frac{\eta_1}{p} - \frac{\eta_2}{2\eta_2} \right| \leq \max \left( \left| \frac{\eta_2 - \eta_1}{2\eta_2} \right|, \left| \frac{\eta_2 - \eta_1}{\eta_1} \right| \right) \\
\leq \frac{8C_\beta}{L} |x_1 - y_1|^\beta/2,
\]
where we used (4.8).

On the other hand, by (4.3)–(4.6),
\[
\|Y\| = \max_\xi |\langle Y \xi, \xi \rangle| \leq 2\|B \xi, \xi \rangle| + \frac{4}{7} \|B^2 \xi, \xi \rangle| \\
\leq 4L \left( \frac{\varphi'(|x_1 - y_1|)}{|x_1 - y_1|} + |\varphi''(|x_1 - y_1|)| \right).
\]
Hence, remembering that \(|x_1 - y_1| \leq 2\), we end up with
\[
\text{tr}((A(\eta_1) - A(\eta_2))Y) \leq 128n \|p - 2\| C_\beta \varphi'(|x_1 - y_1|) |x_1 - y_1|^{1+\beta/2} \\
+ 128n \|p - 2\| C_\beta |\varphi''(|x_1 - y_1|)|.
\]

Finally, we have
\[
M(\text{tr}(A(\eta_1)) + \text{tr}(A(\eta_2))) \leq 2Mn \max(1, p - 1).
\]

**Step 3.** Gathering the previous estimates with (4.9) and recalling the definition of \(\varphi\), we get
\[
0 \leq 128n \|p - 2\| C_\beta \left( \frac{\varphi'(|x_1 - y_1|)}{|x_1 - y_1|^{\beta/2 - 1}} + \frac{\varphi''(|x_1 - y_1|)}{|x_1 - y_1|} \right) \\
+ 2 \min(1, p - 1) L \varphi''(|x_1 - y_1|) + 2M r^2 + 2Mn \max(1, p - 1) \\
\leq 128n \|p - 2\| C_\beta |x_1 - y_1|^{\beta/2 - 1} + 2nM \max(1, p - 1) \\
+ 128n \|p - 2\| C_\beta \gamma(\gamma - 1) \kappa_0 |x_1 - y_1|^{\gamma - 2} \\
- 2 \min(1, p - 1) \gamma(\gamma - 1) \kappa_0 L |x_1 - y_1|^{\gamma - 2} + 2M r^2.
\]
Taking \(\gamma = 1 + \beta/2 > 1\) and choosing \(L\) large enough depending on \(p, n, C_\beta\), we get that
\[
0 \leq \frac{-\min(1, p - 1) \gamma(\gamma - 1) \kappa_0}{200} L |x_1 - y_1|^{\gamma - 2} < 0,
\]
which is a contradiction. Hence choosing first \(L\) such that
\[
0 > 128n \|p - 2\| C_\beta \left( \frac{\varphi'(|x_1 - y_1|)}{|x_1 - y_1|^{\beta/2 - 1}} + \frac{\varphi''(|x_1 - y_1|)}{|x_1 - y_1|} \right) \\
+ \min(1, p - 1) L \varphi''(|x_1 - y_1|) + 2Mn \max(1, p - 1) + 2M r^2
\]
and then taking $|q|$ large enough (depending on $L$, it suffices that $|q| > 6L > \frac{3}{2}|a|$, see (1.8)), we reach a contradiction and hence $\Phi(x, y, t) \leq 0$ for $(x, y, t) \in B_r \times B_r \times (-r^2, 0)$. The desired result follows since for $x_0, y_0 \in B_r, t_0 \in (-r^2, 0)$, we have $\Phi(x_0, y_0, t_0) \leq 0$, we get

$$|w(x_0, t_0) - w(y_0, t_0)| \leq L \varphi(|x_0 - y_0|) \leq L|x_0 - y_0|.$$ 

\[\square\]

4.2. Improvement of oscillation and small perturbation result.

Notice that if $|q|$ is large enough then the equation (4.1) satisfied by $w$ is uniformly parabolic and the operator can no longer be discontinuous. Indeed, taking $\nu_0$ from Lemma 4.1 and assuming that $|q| > \nu_0$, we know from Lemma 4.1 that for $(x, t) \in Q_r$, the gradient $|Dw(x, t)|$ is controlled by some constant $\tilde{C}$ depending only on $p, n, ||w||_{L^\infty(Q_1)}$ and independent of $|q|$. It follows that, if $q$ satisfies

$$|q| \geq L_0 := \max(\nu_0, 4\tilde{C}) \geq 4 ||w||_{L^\infty(Q_r)},$$

we have

$$3\tilde{C} \leq |q| - |Dw| \leq |Dw + q|.$$ 

Equation (4.1) can be rewritten as

$$w_t - \text{tr}(A(x, t)D^2w(x, t)) = 0,$$

where

$$A(x, t) = I + (p - 2) \frac{Dw(x, t) + q}{Dw(x, t) + q} \otimes \frac{Dw(x, t) + q}{Dw(x, t) + q}.$$ 

Since we already know that $w$ is locally of class $C^{1+\beta_1,(1+\beta_1)/2}$ (see the beginning of this section where we introduced $v(x, t) := w(x, t) + q \cdot x$ and used the result of [JS15, Theorem 1.1]), we can see that $w$ solves a uniformly parabolic equation with Hölder coefficients. By standard regularity results (see [Lie96, Theorem 14.10] and [LU68 Theorem 5.1]), we conclude that $w \in C^{2+\gamma,1+\gamma/2}_{\text{loc}}(Q_1)$.

For $r \in (\frac{1}{2}, \frac{3}{4})$, considering $\bar{w}(x, t) = \frac{w(rx, r^2t)}{\tilde{C}}$ where $\tilde{C}$ is given in Lemma 4.1, we have that $\bar{w}$ is a smooth solution to

$$\partial_t \bar{w} - \Delta \bar{w} - (p - 2) \left( D^2\bar{w} \frac{D\bar{w} + \bar{q}}{|D\bar{w} + \bar{q}|} \frac{Dw + q}{|Dw + q|} \right) = 0 \quad \text{in} \quad Q_1,$$

with $|D\bar{w}| \leq 1$ in $Q_1$ and $|\bar{q}| = \frac{|rq|}{\tilde{C}} > 4r > 2$.

Hence without loss of generality, when $|q|$ is large enough, we can work with smooth solutions to (4.1) satisfying $|Dw| \leq 1$ and $|q| > 2$. 
In order to derive a Hölder gradient estimate independent of \( q \) for solutions of (4.1), we need to consider two alternatives: either we can use an improvement of oscillation or we can show that the solution is close to a linear function and then use a small perturbation result (see [Wa92II]). Proofs of these auxiliary results for (4.1) are quite similar to those in [JS15], but for the reader’s convenience we decided to give the details.

First we start with an improvement of oscillation for the projection of \( Dw \) on an arbitrary direction.

**Lemma 4.2.** Let \( w \) be a smooth solution of (4.1) with \( |Dw| \leq 1 \) in \( Q_1 \), and \( |q| > 2 \). For every \( \ell \in (0, 1), \mu > 0 \), there exist \( \tau(\mu, n) > 0, \delta(n, p, \mu) > 0 \), such that for any \( e \in S^{n-1} \), if

\[
\left|\{(x, t) \in Q_1 : Dw(x, t) \cdot e \leq 1 - \ell}\right| > \mu |Q_1|,
\]

then

\[
Dw \cdot e \leq 1 - \delta \text{ in } Q_{\tau} = B_r(0) \times (-\tau^2, 0].
\]

**Proof.** Notice that the equation (4.1) can be rewritten as

\[
w_t - \sum_{ij} A_{ij}(Dw) \cdot w_{ij} = 0,
\]

where

\[
A_{ij}(s) = \delta_{ij} + (p - 2) \frac{(s_i + q_i)(s_j + q_j)}{|s + q|} \frac{|s| + |q|}{|s + q|}
\]

satisfies

\[
\min(1, p - 1) I \leq A \leq \max(1, p - 1) I,
\]

that is \( w \) is a solution of a uniformly parabolic equation with bounded coefficients.

Moreover, notice that

\[
A_{ijm}(s) := \frac{\partial A_{ij}(s)}{\partial s_m} = (p - 2) \frac{\delta_{jm}(s_j + q_j) + \delta_{mj}(s_i + q_i)}{|s + q|^2}
\]

\[
- \frac{2(p - 2)(s_m + q_m)(s_i + q_i)(s_j + q_j)}{|s + q|^4}.
\]

Since by assumption \( |q| > 2 \) and \( |Dw| \leq 1 \), that is \( |Dw + q| \geq |q| - |Dw| \geq 1 \), we observe that

\[
|A_{ijm}(Dw)| \leq \frac{4|p - 2|}{|Dw + q|} \leq 4|p - 2|. \tag{4.11}
\]
The proof proceeds as in \cite[Lemma 4.1]{JS15} by deriving the equation, constructing good barriers and using a weak Harnack inequality \cite[Proposition 2.3]{JS15} for nonnegative supersolutions of uniformly parabolic equations with bounded coefficients. Indeed, for $\rho > 0$, defining $h(x,t) := Dw(x,t) \cdot e - l + \rho |Dw(x,t)|^2$, differentiating the equation and using (4.11), we get that, for $c_1$ appropriately chosen, the function $\bar{h}(x,t) := 1 - e^{-c_1((1-l+\rho)-h(x,t))}$ is a nonnegative supersolution of a uniformly parabolic equation with bounded coefficients.

If we can iterate the previous lemma to all directions, then proceeding by iteration, we derive an improvement of oscillation for $|Dw|$. 

**Corollary 4.3.** Assume that $|q| > 2$ and let $w$ be a smooth solution to (4.1) with $|Dw| \leq 1$. For every $l \in (0,1)$ and $\mu > 0$, there exist $\tau = \tau(\mu,n) \in (0,1/4)$ and $\delta = \delta(\mu,n,p) > 0$, such that for every $k \in \mathbb{N}$, if

$$a) \quad \left| \left\{ (x,t) \in Q_{\tau^i} : Dw(x,t) \cdot e \leq l(1-\delta)^i \right\} \right| > \mu |Q_{\tau^i}|$$

for all $e \in S^{n-1}$ and $i = 0, ..., k$

then

$$b) \quad |Dw| \leq (1-\delta)^{i+1} \text{ in } Q_{\tau^{i+1}} \text{ for } i = 0, ..., k.$$

**Proof.** We proceed by induction on $k$. The case $k = 0$ follows from the previous lemma. Assume that if the assumption $a)$ holds true for $i = 0, \ldots, k-1$ then the conclusion $b)$ follows for $i = 0, \ldots, k-1$. Then it is enough to show that if assumption $a)$ is satisfied for $i = 0, \ldots, k$ then claim $b)$ follows for $i = 0, \ldots, k$. Hence, knowing that the assumption $a)$ is satisfied for $i = 0, \ldots, k$ and claim $b)$ holds for $i = 0, \ldots, k-1$, it remains to show that claim $b)$ holds for $i = k$.

For this purpose, we define the rescaled function $v(x,t) = \frac{w(\tau^k x, \tau^{2k} t)}{\tau^k(1-\delta)^k}$. From the induction hypothesis, we have that

$$|Dv| \leq 1 \text{ in } Q_1,$$

$$| \left\{ (x,t) \in Q_1 : Dw(x,t) \cdot e \leq l \right\} | > \mu |Q_1|.$$

Moreover $v$ solves in $Q_1$

$$v_t - \Delta v - (p-2) \left\langle D^2 v, \frac{Dv(1-\delta)^k + q}{|Dv(1-\delta)^k + q|}, \frac{Dv(1-\delta)^k + q}{|Dv(1-\delta)^k + q|} \right\rangle = 0.$$

That is $v$ solves

$$v_t - \Delta v - (p-2) \left\langle D^2 v, \frac{Dv + \tilde{q}}{|Dv + \tilde{q}|}, \frac{Dv + \tilde{q}}{|Dv + \tilde{q}|} \right\rangle = 0$$

where $|\bar{q}| = \left| \frac{q}{(1 - \delta)^k} \right| > |q| > 2$.

It follows from Lemma (4.2), that

$$Dv \cdot e \leq (1 - \delta) \quad \text{in} \quad Q_\tau \quad \text{for all} \quad e \in S^{n-1}.$$ 

This implies that $|Dv| \leq 1 - \delta$ in $Q_\tau$ and consequently

$$|Dw| \leq (1 - \delta)^{k+1} \quad \text{in} \quad Q_{\tau^{k+1}}. \quad \Box$$

Next, we show that under some conditions, $w$ can be arbitrarily close to a linear function.

**Lemma 4.4.** Fix a constant $\eta > 0$ and let $w$ be a smooth solution to the uniformly parabolic equation (4.1) with $|Dw| \leq 1$, $|q| > 2$ and $w(0,0) = 0$. Suppose that

$$|\{(x,t) \in Q_1 : |Dw(x,t) - e| > \varepsilon_0\}| \leq \varepsilon_1 \quad \text{for some} \quad e \in S^{n-1}.$$ 

Then if $\varepsilon_0, \varepsilon_1 \geq 0$ are small enough, it follows that

$$\text{osc}_{(x,t) \in Q_{1/2}} (w(x,t) - e \cdot x) \leq \eta.$$ 

The smallness of $\varepsilon_0, \varepsilon_1$ depends on $n, \eta$ and $p$.

**Proof.** The proof follows as in [JS15, Lemma 4.4] by first controlling the oscillation of $w(x,t) - e \cdot x$ on $B_{1/2}$ and using Lemma 2.2. \quad \Box

Next we state a simple calculus fact, which links the assumptions of Lemma 4.3 and Lemma 4.4. The proof is provided in [JS15, pages 14-15].

**Lemma 4.5.** Let $\ell \in (0,1)$ and $\mu > 0$. Let $v : Q_1 \to \mathbb{R}$ be a smooth function satisfying $|Dv| \leq 1$ in $Q_1$ and

$$|\{(x,t) \in Q_1 : Dv(x,t) \cdot e \leq \ell\}| \leq \mu |Q_1|$$

for some $e \in S^{n-1}$, then for $\varepsilon_0 := \sqrt{2(1 - \ell)}$ and $\varepsilon_1 := \mu |Q_1|$,

$$|\{(x,t) \in Q_1 : |Dv(x,t) - e| > \varepsilon_0\}| \leq \varepsilon_1.$$ 

We will also use the following regularity estimate for small perturbation of solutions of fully nonlinear parabolic equations, which was proved by Wang [Wan13] and is a parabolic analogue to the result of Sav07.
Theorem 4.6. Let $w$ be a smooth solution to (4.1) in $Q_1$ with $|q| > 2$ and $|Dw| \leq 1$. Then for each $\gamma > 0$, there are $\eta(n, p, \gamma), C(n, p, \gamma) > 0$ such that if for some linear function $L(x)$ with $\frac{1}{2} \leq |DL| \leq 2$ it holds
\[
||w - L||_{L^\infty(Q_1)} \leq \eta
\]
then
\[
||w - L||_{C^{2+\gamma, 1+\gamma/2}(Q_{1/2})} \leq C.
\]

4.3. Proof of Proposition 3.2. We start with the case $|q|$ large enough.

Theorem 4.7. Let $w$ be a bounded viscosity solution of (4.1) with \(\text{osc}_{Q_1} w \leq 1\). Then there exist $L_0 = L_0(p, n, ||w||_{L^\infty(Q_1)}) > 0$, $\alpha = \alpha(p, n) \in (0, 1)$ and $C = C(p, n) > 0$ such that if $|q| > L_0$, then
\[
||w||_{C^{1+\alpha, (1+\alpha)/2}(Q_{1/2})} \leq C ||w||_{L^\infty(Q_1)}.
\]

Proof. Let $L_0 = \max(\nu_0, 4\tilde{C})$ where $\nu_0$ and $\tilde{C}$ are given in Lemma 4.1. If $|q| \geq L_0$, then viscosity solutions $w$ to (4.1) are smooth solutions. Without loss of generality we can assume that $w(0, 0) = 0$, $|Dw| \leq 1$ and $|q| > 2$. Let $\tau$ and $\delta$ be the constants given by Corollary 4.3. Let $\eta > 0$ be as in Theorem 4.6 for a fixed $\gamma \in (0, 1)$. Let $\ell \in (0, 1)$ be sufficiently close to 1 and $\mu > 0$ small so that $\varepsilon_0, \varepsilon_1 > 0$ defined by
\[
\varepsilon_0 := \sqrt{2(1 - \ell)}, \quad \varepsilon_1 := \mu |Q_1|
\]
are sufficiently small in order that the result of Lemma 4.5 holds.

Let $k$ be the minimum of the nonnegative integers such that the condition $a)$ in Corollary 4.3 does not hold. The proof splits into two cases.

Case 1) $k = \infty$:
In this case we have $Dw(0, 0) = 0$. There exists $i \in \mathbb{N}$ large enough so that, we have $\tau^{i+1} \leq |x| + \sqrt{|t|} \leq \tau^i$, and
\[
i \leq \frac{\log(|x| + \sqrt{|t|})}{\log(\tau)} \leq (i + 1).
\]
Hence, by Corollary 4.3 it follows that
\[
|Dw(x, t)| \leq \frac{1}{1 - \delta} (1 - \delta)^i \leq C(1 - \delta)^{\frac{\log(|x| + \sqrt{|t|})}{\log(\tau)}}
\]
\[
= C(1 - \delta)^{\frac{\log(|x| + \sqrt{|t|})}{\log(1 - \delta)}} \leq C(1 - \delta)^{\alpha},
\]
where \( \alpha = \frac{\log(1-\delta)}{\log(\tau)} \) and \( C = (1-\delta) \). This proves the claim in this case.

**Case 2) \( k < \infty \):**

In this case we have

\[
\left| \{(x,t) \in Q_{\tau^k} : Dw(x,t) \cdot e \leq \ell(1-\delta)^k \} \right| \leq \mu |Q_{\tau^k}|
\]

for some \( e \in \mathbb{S}^{n-1} \). Set

\[
v(x,t) = \frac{1}{\tau^k(1-\delta)^k} w(\tau^k x, \tau^{2k} t)
\]

which satisfies

\[
v_t - \Delta v - (p-2) \left< D^2 v, \frac{Dv + \bar{q}}{|Dv + \bar{q}|}, \frac{Dv + \bar{q}}{|Dv + \bar{q}|} \right> = 0 \quad \text{in } Q_1
\]

where \(|\bar{q}| = \left| \frac{q}{(1-\delta)^k} \right| > |q| > 2\).

Moreover

\[
| \{(x,t) : Dv(x,t) \cdot e \leq \ell \}| \leq \mu |Q_1| \quad \text{for some } e \in \mathbb{S}^{n-1},
\]

and since condition \( a) \) holds for \( k-1 \), we have \(|Dv| \leq 1 \) in \( Q_1 \).

It follows from Lemma 4.5 that for \( \varepsilon_0 := \sqrt{2(1-\ell)} \) and \( \varepsilon_1 := \mu |Q_1| \), we have

\[
| \{(x,t) \in Q_1 : |Dv(x,t) - e| > \varepsilon_0 \}| \leq \varepsilon_1,
\]

and hence by Lemma 4.3, we conclude that

\[
|v(x,t) - e \cdot x| \leq \eta \quad \text{for all } (x,t) \in Q_{1/2},
\]

where \( \eta > 0 \) can be made arbitrarily small by choosing \( \mu > 0 \) small and \( \ell \in (0,1) \) close to one. By Theorem 4.6 there exists \( C = C(p,n) > 0 \) such that

\[
|Dv(x,t) - e| \leq C(|x| + \sqrt{|t|}) \gamma \quad \text{in } Q_{1/4}
\]

and since \(|Dv| \leq 1 \), this estimate also holds in \( Q_1 \).

Rescaling back and assuming that \((1-\delta)/\tau \leq 1\), we have

\[
|Dw(x,t) - (1-\delta)^k e| \leq C(1-\delta)^k (|x| \tau^{-k} + \sqrt{|t|} \tau^{-k}) \gamma
\]

\[
\leq C(|x| + \sqrt{|t|}) \gamma
\]

for \( (x,t) \in Q_{\tau^k} \).

This implies the result near zero. Standard translation and scaling arguments imply that

\[
||Dw||_{C^{\gamma/2}(Q_{1/2})} \leq C(p,n) ||w||_{L^\infty(Q_1)}.
\]
The $C^{1+\gamma}$ Hölder continuity in time for $w$ follows as in the proof of Lemma 3.5. The proof of Proposition 3.2 is now complete since for $|q| \leq L_0$, the result follows from the regularity estimate from [JS15] for $v(x,t) = w(x,t) + q \cdot x$ (see the beginning of this section).

**Appendix A. Characterizations of functions with Hölder continuous gradient**

**Lemma A.1.** Let $u \in C(Q_{2R}(0,0))$ for some $R > 0$ and assume that there are some positive constants $c_0$ and $\alpha$ with $\alpha \leq 1$ such that for any $(x_0, t_0)$ in $Q_R(0,0)$ there is a vector $q(x_0, t_0)$ satisfying for any $r \in (0, R]$,

$$
\sup_{Q_r(x_0, t_0)} |u(x, t) - u(x_0, t_0) - q(x_0, t_0) \cdot (x - x_0)| \\
\leq \text{osc}_{Q_r(x_0, t_0)} (u(x, t) - q(x_0, t_0) \cdot x) \\
\leq c_0 r^{1+\alpha}.
$$

(A.1)

Then $u$ is differentiable with respect to the space variable in $Q_R(0,0)$, $Du \in C^{\alpha, \frac{\alpha}{2}}(Q_{\frac{R}{2}}(0,0))$ and

$$
[Du]_{C^{\alpha, \frac{\alpha}{2}}(Q_{\frac{R}{2}}(0,0))} \leq C(n)c_0.
$$

(A.2)

We give details of the proof for the reader’s convenience, the result is well known. There are two ways to prove Lemma A.1. A first idea of the proof can be found in [Lie96, Lemma 12.12]. In this section we decided to adapt to the parabolic setting the arguments used in [Kry96, AZ15].

**Proof.** First notice that the estimate (A.1) implies that $u$ is differentiable with respect to the space variable in $Q_R(0,0)$ and for $(x_0, t_0)$ in $Q_R(0,0)$, we have

$$
Du(x_0, t_0) = q(x_0, t_0).
$$

Let $(x, t), (y, s) \in Q_{\frac{R}{2}}(0,0)$ and define $r$ as $r^2 = |x - y|^2 + |t - s|$. Without loss of generality we can assume that $x = -\frac{r}{2}e_1$ and $y = \frac{r}{2}e_1$ where $e_i$ are the vector of the canonical base of $\mathbb{R}^n$.

Using (A.1), we have

$$
|u(y, s) - Du(x, t) \cdot (y - x) - u(x, t)| \leq c_0 r^{1+\alpha},
$$

(A.3)

$$
|u(x, t) - Du(y, s) \cdot (x - y) - u(y, s)| \leq c_0 r^{1+\alpha}.
$$

(A.3)
Adding these two inequalities we get
\[ |(Du(x, t) - Du(y, s)) \cdot (x - y)| \leq 2c_0r^{1+\alpha}, \]
that is
\[ |\partial_1 u(y, s) - \partial_1 u(x, t)| \leq 4c_0r^\alpha. \quad (A.4) \]
For \( i = 2, \ldots, n \), we fix \( z = \frac{r}{4}e_i \). Using (A.1), we have
\[ |u(z, t) - Du(x, t) \cdot (z - x) - u(x, t)| \leq c_0r^{1+\alpha}, \]
\[ |u(y, s) + Du(y, s) \cdot (z - y) - u(z, t)| \leq c_0r^{1+\alpha}. \]
Adding the two inequalities and using the triangle inequality, we get
\[ |u(y, s) - u(x, t) - Du(x, t) \cdot (z - x) + Du(y, s) \cdot (z - y)| \leq 2c_0r^{1+\alpha}, \]
that is
\[ |u(y, s) - u(x, t) - Du(x, t) \cdot (y - x) + (Du(y, s) - Du(x, t)) \cdot (z - y)| \leq 2c_0r^{1+\alpha}. \quad (A.5) \]
By the definition of \( z \), we have
\[ (Du(y, s) - Du(x, t)) \cdot (z - y) = \frac{r}{4}(\partial_i u(y, s) - \partial_i u(x, t)) \]
\[ - \frac{r}{4}(\partial_1 u(y, s) - \partial_1 u(x, t)). \]
Using (A.3)–(A.5), it follows that
\[ \frac{r}{4}|\partial_i u(y, s) - \partial_i u(x, t)| \]
\[ \leq \frac{r}{4}|\partial_1 u(y, s) - \partial_1 u(x, t)| \]
\[ + |u(y, s) - u(x, t) - Du(x, t) \cdot (y - x) + (Du(y, s) - Du(x, t)) \cdot (z - y)| \]
\[ + |u(y, s) - u(x, t) - Du(x, t) \cdot (y - x)| \]
\[ \leq 6c_0r^{1+\alpha}. \]

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