THE VISUAL BOUNDARY OF HYPERBOLIC FREE-BY-CYCLIC GROUPS

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* The first and third authors were supported by ISF grant 1941/14.
** The second author has been supported by the ANR grant DAGGER ANR-16-CE40-0006-01.
† The third author was supported by the Azrieli Foundation and was partially supported at the Technion by a Zuckerman STEM Leadership Postdoctoral Fellowship.

Received May 2, 2019 and in revised form January 5, 2020
Let $\phi$ be an atoroidal outer automorphism of the free group $F_n$. We study the Gromov boundary of the hyperbolic group $G_\phi = F_n \rtimes_\phi \mathbb{Z}$. Using the Cannon–Thurston map, we explicitly describe a family of embeddings of the complete bipartite graph $K_{3,3}$ into the boundary of the free-by-cyclic group. To do so, we define the directional Whitehead graph and use it to relate the topology of the boundary to the structure of the Rips Machine associated to a fully irreducible outer automorphism of the free group. In particular, we prove that an indecomposable $F_n$-tree is Levitt type if and only if one of its directional Whitehead graphs contains more than one edge. As an application, we obtain a new proof of Kapovich–Kleiner’s theorem [KK00] that $\partial G_\phi$ is homeomorphic to the Menger curve if the automorphism is atoroidal and fully irreducible.

1. Introduction

The family of free-by-cyclic groups is an intriguing example of a collection of groups that is simply defined, yet has a rich geometric structure theory. Denote by $F_n = \langle x_1, \ldots, x_n \rangle$ the free group on $n \geq 2$ generators. Let $\text{Aut}(F_n)$ be the group of automorphisms of $F_n$, and let $\text{Out}(F_n)$ be the quotient of $\text{Aut}(F_n)$ by the normal subgroup of inner automorphisms. Given an outer automorphism $\phi \in \text{Out}(F_n)$, the group $G_\phi$ is defined by the HNN presentation

\begin{equation}
G_\phi := F_n \rtimes_\phi \mathbb{Z} = \langle x_1, \ldots, x_n, t \mid t^{-1}x_i t = \Phi(x_i), 1 \leq i \leq n \rangle
\end{equation}

where $\Phi \in \text{Aut}(F_n)$ is any automorphism of $F_n$ in the outer class $\phi$. (Different choices of $\Phi$ yield Tietze-equivalent presentations of $G_\phi$.) There is a satisfying correspondence between geometric properties of the group $G_\phi$ and algebraic properties of the outer automorphism $\phi$, which is summarized in Table 1.

The following theorems imply that if $G_\phi$ is hyperbolic and does not split over a virtually cyclic subgroup, then its boundary is homeomorphic to the Menger curve. We begin by noting that the cohomological dimension of a free-by-cyclic group is two, hence, by a result of Bestvina–Mess [BM91], the topological dimension of $\partial G_\phi$ is one. Kapovich–Kleiner [KK00] proved that if $G$ is a hyperbolic group that does not split over a virtually cyclic subgroup and whose boundary is one dimensional, then the boundary of $G$ is homeomorphic to either the unit circle, the Sierpinski carpet, or the Menger curve. Euler characteristic
Table 1. The first relationship is due to Brinkmann [Bri00] and the second to Kapovich–Kleiner [KK00, Proof of Corollary 15].

| Properties of $G_\phi$ | Dynamics of $\phi$ |
|------------------------|---------------------|
| $G_\phi$ is a hyperbolic group | $\phi$ is atoroidal, i.e., no conjugacy class of $F_n$ is invariant under a power of $\phi$ |
| The visual boundary $\partial G_\phi$ contains a local cut point | A power of $\phi$ preserves a free splitting of $F_n$ |
| $\iff$ [Bow98] |
| $G_\phi$ splits over $\mathbb{Z}$ |

arguments rule out the possibility that the boundary is a circle or Sierpinski carpet. Indeed, Tukia–Gabai–Casson–Jungreis [Tuk88, Gab92, CJ94] proved hyperbolic groups with circle boundary act geometrically on the hyperbolic plane, and hence have negative Euler characteristic; Kapovich–Kleiner [KK00] proved hyperbolic groups with Sierpinski carpet boundary have negative Euler characteristic. Since the Euler characteristic of a free-by-cyclic group is zero, $\partial G_\phi$ is homeomorphic to the Menger curve.

In this paper we use different tools—the theory of outer automorphisms of free groups and the theory of Cannon–Thurston maps—to explicitly find a non-planar subset of $\partial G_\phi$.

**Theorem 1.1:** If $\phi$ is an atoroidal automorphism in $\text{Out}(F_n)$, then $\partial G_\phi$ contains a copy of the complete bipartite graph $K_{3,3}$.

**Remark 1.2:** Theorem 1.1 is easily deduced from the case that $\phi$ is fully irreducible and atoroidal (proof of Theorem 9.1). We henceforth concentrate on the fully irreducible case.

To find a non-planar set in the boundary, we use the existence and structure of the Cannon–Thurston map from the boundary of the free group to the boundary of the free-by-cyclic group. Mitra (Mj) [Mit98] proved there exists a continuous surjection $\hat{\iota} : \partial F_n \to \partial G_\phi$ that extends the injection $\iota : F_n \to G_\phi$ in the presentation (1.1). The map $\hat{\iota}$ is called the **Cannon–Thurston map for the subgroup** $F_n \leq G_\phi$. 


The Cannon–Thurston map factors through well-studied $\mathbb{R}$-trees in the following way. Levitt–Lustig [LL03] proved that if $\phi$ is fully irreducible, then $\phi$ acts with North-South dynamics on compactified Outer Space (see Definition 2.5), fixing an attracting tree $T_+$ and a repelling tree $T_-$. They define surjections $Q_+ : \partial F_n \to \widehat{T}_+$ and $Q_- : \partial F_n \to \widehat{T}_-$, where $\widehat{T}_\pm$ is the compactification of the metric completion of $T_\pm$. Kapovich–Lustig [KL15] proved that the following diagram commutes, where $\mathcal{R}_\pm$ is a surjection (see Section 7).

\[
\begin{array}{ccc}
\partial F_n & \xrightarrow{\iota} & \partial G_\phi \\
Q_+ & \downarrow & \downarrow \mathcal{R}_+ \\
\widehat{T}_+ & \xrightarrow{\mathcal{R}_-} & \widehat{T}_-
\end{array}
\]

Our contribution is to describe how these $\mathbb{R}$-trees $T_+$ and $T_-$ interact in the boundary $\partial G_\phi$ to form non-planar subsets. That is, to prove Theorem 1.1, we realize the embedded graph $K_{3,3}$ in $\partial G_\phi$ as a union of a subtree that lies in the tree $T_+$ and a subtree that lies in the tree $T_-$. If $\phi$ admits the existence of these subtrees in a particular way, then we say that $\phi$ satisfies the $T_\pm$-pattern; see Definition 8.2. The $T_\pm$-pattern maps into $\partial G_\phi$ to form an embedded $K_{3,3}$, as shown in Proposition 8.4.

To prove that every atoroidal fully irreducible outer automorphism admits a $T_\pm$-pattern we introduce the directional Whitehead graph of a dense $F_n$-tree $T$ in compactified Outer Space; see Definition 6.1. (In particular, these graphs are defined for any fully irreducible outer automorphism and the corresponding attracting and repelling laminations.) We prove the following theorems that relate the structure of the directional Whitehead graph to both the Rips Machine and to the structure of the visual boundary.

**Theorem 1.3 (Theorem 8.5):** If $\phi \in \text{Out}(F_n)$ is fully irreducible and $T_-$ its repelling tree, then:

A directional Whitehead graph of $T_-$ contains more than one edge $\iff$ $\phi$ satisfies the $T_\pm$-pattern.
Theorem 1.4 (Theorem 8.6): If $T$ is an indecomposable $F_n$-tree, then:

A directional Whitehead graph of $T$ contains more than one edge. $\iff$ $T$ has Levitt type; in particular, there exists a system of partial isometries associated to $T$ for which the Rips Machine never halts. See Definition 5.5.

In the spirit of Table 1, this analysis gives a new correspondence between dynamical properties of $\phi$, i.e., its action on $\hat{T}_-$ and its dual lamination, and topological properties of $\partial G_\phi$, i.e., that it contains a copy of $K_{3,3}$. Paulin [Pau96] proved that the quasi-conformal type of the visual boundary of a hyperbolic group is a complete quasi-isometry invariant; background is given by Haissinsky [Hai09]. A first step towards analyzing metric and conformal properties of the boundary is to understand its topology in a more direct fashion, and this was our main motivation when writing this paper.

Another possible application is to further understand how the decomposition in Diagram 1.2 changes as a function of the splitting of $G_\phi$ arising from a rational point in the connected component of the BNS-invariant of the cohomology class inducing $\phi$. The work of Dowdall–Kapovich–Leininger [DKL15, DKL17b, DKL17a] shows the splittings in the same component share many of the same properties. Our paper suggests that while the free group $F_{n,\alpha}$ changes as we vary $\alpha$ in the component of the BNS invariant, the image of the interior of the tree $\hat{T}_\alpha$ in $\partial G_\phi$ does not. In light of Theorems 1.3 and 1.4 this could imply that all automorphisms in a component are parageometric or they are all ageometric.

Additionally, our techniques demonstrate how a difference in dynamical properties of geometric and non-geometric outer automorphisms yields a difference in the topology of the boundary of the corresponding free-by-cyclic groups. Recall, $\phi \in \text{Out}(F_n)$ is geometric if $\phi$ is induced by a homeomorphism of a punctured surface. In this case, $\phi$ is toroidal and can be fully irreducible. If $\phi$ is fully irreducible and geometric, then iterating the Rips Machine on the system of partial isometries associated to $\phi$ halts, producing an interval exchange transformation. Hence, Theorem 1.4 shows that every directional Whitehead graph for $\phi$ contains no more than one edge. Indeed, in this case, the CAT(0) boundary of $G_\phi$ [HK05, Rua05] and the Bowditch boundary of $G_\phi$ relative to the flat stabilizers [Bow12] embed in the 2-sphere.
In the process of proving the theorems above, we establish new results which may be of independent interest. For instance, we further investigate systems of partial isometries on compact trees in continuation of the work of Coulbois, Hilion and Lustig [CHL09, CH14, CH12]. Precise definitions of the objects considered here can be a bit technical and will be given in Sections 2 and 5. Combining our results with previous results of Coulbois–Hilion [CH12], we obtain the following characterization of the type (Levitt or surface) of a free mixing $F_n$-tree.

**Theorem 1.5 (Corollary 5.16):** Let $T$ be a free mixing $F_n$-tree, and let $A$ be a basis of $F_n$. Let $S_0 = (K_0, A_0)$ be the associated system of partial isometries, and let $S_i = (K_i, A_i)$ denote the output after the $i$th iteration of the Rips Machine. Let $K_{i}^{\geq 3}$ be the subset of the subtree $K_i$ on which at least three distinct partial isometries in $A_i$ are defined, and let $\text{vol}(K_{i}^{\geq 3})$ denote the volume of $K_{i}^{\geq 3}$. Then:

- $T$ is pseudo-surface if and only if $\text{vol}(K_{i}^{\geq 3}) = 0$ for some $i \in \mathbb{N}$;
- $T$ is Levitt type if and only if $\text{vol}(K_{i}^{\geq 3}) > 0$ for all $i \in \mathbb{N}$.

We also unify two possible definitions of the **ideal Whitehead graph** of a fully irreducible outer automorphism $\phi \in \text{Out}(F_n)$. Handel–Mosher [HM11, Chapter 3] introduced the ideal Whitehead graph to study the asymptotic relations between singular leaves of the attracting lamination of $\phi$; see Section 2.3.

**Definition 1.6:** The graph $\text{Wh}_\phi$ is the $F_n$-quotient of the graph whose vertex set is the union of nonrepelling fixed points in $\partial F_n$ of principal automorphisms (see Definition 2.20) representing $\phi$. An edge of $\text{Wh}_\phi$ corresponds to a leaf of the attracting lamination $\Lambda^\phi_+$. The following definition will be more useful for us.

**Definition 1.7:** A **singular leaf** of $\Lambda^\phi_+$ is a leaf whose asymptotic class contains more than one element. The graph $\text{Wh}_{\Lambda^\phi_+}$ is the $F_n$-quotient of the graph whose vertex set is the union of endpoints of singular leaves of $\Lambda^\phi_+$. An edge of $\text{Wh}_{\Lambda^\phi_+}$ corresponds to a (equivalence class of a) singular leaf of $\Lambda^\phi_+$.

We prove that the two definitions are equivalent in Section 3.1.

**Theorem 1.8:** The ideal Whitehead graphs $\text{Wh}_\phi$ and $\text{Wh}_{\Lambda^\phi_+}$ of a fully irreducible outer automorphism $\phi \in \text{Out}(F_n)$ are isomorphic.
Acknowledgements. We wish to thank the Mathematical Science and Research Institute for their hospitality in the fall of 2016. We thank the referees for helpful comments.

2. Preliminaries

2.1. Trees, directions, and the observers’ topology. A metric space \((T,d)\) is an \(\mathbb{R}\)-tree if for any two points \(x, y \in T\), there is a unique topological arc \(p_{x,y} : [0, 1] \to T\) connecting \(x\) to \(y\) and so that the image of \(p_{x,y}\) is isometric to the segment \([0, d(x,y)]\). We denote the image of \(p_{x,y}\) by \([x,y]\), and we refer to this arc as the segment in \(T\) from \(x\) to \(y\). An arc is non-trivial if it contains at least 2 distinct points.

If \(x \in T\) is a point, a direction at \(x\) in \(T\) is a component of \(T \setminus \{x\}\). A point \(x \in T\) is a branch point if there are at least 3 directions at \(x\); it is an extremal point if there is only one direction at \(x\). Let \(\text{Int}(T)\) denote the interior of a tree \(T\), i.e., the set of points of \(T\) which are not extremal. Two arcs \([x,y]\), \([x,y']\) are germ-equivalent if \([x,y] \cap [x,y'] \neq \{x\}\); an equivalence class is a germ at \(x\). The map that associates to the arc \([x,y]\) the direction containing \(y\) induces a bijection from the set of germs at \(x\) to the set of directions at \(x\).

A ray in \(T\) is the image of an immersion \(\mathbb{R}_+ \to T\). Two rays \(\rho\) and \(\rho'\) are asymptotic if \(\rho \cap \rho'\) has infinite diameter. Let \(\partial T\) denote the Gromov boundary of \(T\); as a set, \(\partial T\) consists of asymptotic classes of rays in \(T\). Let \(\overline{T}\) denote the metric completion of \(T\). The points in \(\overline{T} \setminus T\) are extremal in \(\overline{T}\).

As explained by Coulbois–Hilion–Lustig [CHL07], the set \(\hat{T}\) may be equipped with the observers’ topology, which makes the space homeomorphic to a dendrite. The observers’ topology on \(\hat{T}\) is the topology generated (in the sense of a subbasis) by the set of directions in \(\hat{T}\), where the notion of direction in \(T\) extends to \(\hat{T}\), and points of \(\partial T\) are extremal in \(\hat{T}\). This topology is weaker than the topology induced by the metric on an \(\mathbb{R}\)-tree. On any interval of \(\overline{T}\), the metric topology and the observers’ topology agree.

Definition 2.1 (Actions on trees): An \(F_n\)-tree is an \(\mathbb{R}\)-tree \(T\) together with a homomorphism \(\rho : F_n \to \text{Isom}(T)\); the homomorphism is often repressed. The action of \(F_n\) on \(T\) is minimal if \(F_n\) does not leave invariant any non-trivial subtree of \(T\). The action is very small if (1) all edge stabilizers are cyclic (i.e., \(\{1\}\)
or $\mathbb{Z}$; (2) for every non-trivial $g \in F_n$, the fixed subtree $\text{Fix}(g)$ is isometric to a subset of $\mathbb{R}$; and (3) $\text{Fix}(g)$ is equal to $\text{Fix}(g^p)$ for all $p \geq 2$. The action is **mixing** if for any non-degenerate segments $I$ and $J$ in $T$ the segment $I$ is covered by finitely many translates of $J$: there exist finitely many elements $u_1, \ldots, u_k \in F_n$ so that $I \subset u_1 J \cup u_2 J \cup \cdots \cup u_k J$. The action is **indecomposable** if the action is mixing and, in addition, the elements $u_1, \ldots, u_k \in F_n$ may be chosen so that $u_i J \cap u_{i+1} J$ is a non-degenerate segment for any $i = 1, \ldots, k - 1$. In particular, if $T$ is mixing then orbits are dense.

### 2.2. The $Q$-map and dual lamination.

**Theorem 2.2** ([LL03, Section 3] [CHL07, Proposition 2.3]): Let $T$ be a minimal, very small, $F_n$-tree with dense orbits. There exists an $F_n$-equivariant surjective map

$$Q : \partial F_n \to \hat{T}$$

which is continuous with respect to the observers’ topology. In addition, points in $\partial T$ have exactly one pre-image by $Q$. ■

The $Q$-map given in the previous theorem may be used to define a lamination of $F_n$ as follows.

**Definition 2.3:** The **double boundary** of $F_n$ is

$$\partial^2 F_n := (\partial F_n \times \partial F_n) \setminus \Delta,$$

where $\Delta$ is the diagonal. Let $i : \partial^2 F_n \to \partial^2 F_n$ denote the involution that exchanges the factors. The double boundary $\partial^2 F_n$ is endowed with the topology induced by the product topology, and $F_n$ acts diagonally on $\partial^2 F_n$. A **lamination** of $F_n$ is a non-empty, closed, $F_n$-invariant, $i$-invariant subset of $\partial^2 F_n$. A lamination of $F_n$ is **minimal** if it does not contain any proper sublamination other than $\emptyset$. Two leaves $(X,Y)$ and $(X',Y')$ of a lamination of $F_n$ are **asymptotic** if either $X = X'$ or $Y = Y'$.

**Definition 2.4:** Let $F_n$ act by isometries on an $\mathbb{R}$-tree $T$ so that the action is minimal, very small, and has dense orbits. Let $Q : \partial F_n \to \hat{T}$ be the $Q$-map given in Theorem 2.2. The **dual lamination** of $T$ is

$$\mathcal{L}(T) = \{(X,Y) \in \partial^2 F_n \mid Q(X) = Q(Y)\}.$$

A **leaf** of the dual lamination $\mathcal{L}(T)$ is a pair $(X,Y) \in \mathcal{L}(T)$. 
2.3. Attracting and repelling trees and laminations of an outer automorphism.

Definition 2.5 (Outer space): For \( n \geq 2 \), **Culler–Vogtmann outer space**, denoted \( CV_n \), is the projectivized space of minimal, free, discrete actions by isometries of the free group \( F_n \) on \( \mathbb{R} \)-trees. Its topology is induced by embedding \( CV_n \) into the space of length functions [CM87]. Let \( \overline{CV}_n \) denote the compactification of \( CV_n \), which is the set of projective classes of minimal, very small, \( F_n \)-trees [CM87, CL95, BF94]. Let \( \partial CV_n = \overline{CV}_n \setminus CV_n \). These spaces admit an action of \( \text{Out}(F_n) \); an element \( \phi \in \text{Out}(F_n) \) sends an \( F_n \)-tree \((T, \rho)\) to \((T, \rho \circ \Phi)\), where \( \Phi \in \text{Aut}(F_n) \) is in the class \( \phi \). For background, see [CV86], [Vog02].

Definition 2.6: An outer automorphism \( \phi \in \text{Out}(F_n) \) is **fully irreducible** (or, **iwip**) if no conjugacy class of a proper free factor of \( F_n \) is fixed by a positive power of \( \phi \).

Theorem 2.7 ([LL03, Theorem 1.1]): If \( \phi \in \text{Out}(F_n) \) is fully irreducible, then \( \phi \) acts on \( \overline{CV}_n \) with North-South dynamics and projectively fixes two trees \( T_+, T_- \in \partial CV_n \).

Definition 2.8: We refer to the tree \( T_+ = T_+^\phi \) as the **attracting tree** of \( \phi \) and to the tree \( T_- = T_-^\phi \) as the **repelling tree** of \( \phi \). We omit the \( \phi \)-notation when the outer automorphism is clear from context. Notice that

\[
T_+^\phi = T_-^{\phi^{-1}} \quad \text{and} \quad T_-^\phi = T_+^{\phi^{-1}}.
\]

If \( \phi \) is a fully irreducible outer automorphism, then the group \( F_n \) acts on the trees \( T_+ \) and \( T_- \) with dense orbits. In fact, these actions exhibit a much stronger dynamical property.

Theorem 2.9 ([CH12, Theorem 2.1]): The actions of \( F_n \) on the attracting and repelling trees of a fully irreducible outer automorphism of \( F_n \) are indecomposable.

Definition 2.10: Let \( T_+ \) and \( T_- \) be the attracting and repelling trees of a fully irreducible free group automorphism. By Theorem 2.2, there are \( F_n \)-equivariant, surjective maps

\[
Q_+: \partial F_n \to \hat{T}_+ \quad \text{and} \quad Q_-: \partial F_n \to \hat{T}_-
\]

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which are continuous with respect to the observers’ topology. In addition, points in $\partial T_+$ and $\partial T_-$ have exactly one pre-image by $Q_+$ and $Q_-$, respectively. Define the dual laminations $L(T_+)$ and $L(T_-)$ as in Definition 2.4.

**Proposition 2.11 ([KL15, Proposition 3.22]):** $L(T_+) \cap L(T_-) = \emptyset$.

**Remark 2.12:** Let $T$ be an $\mathbb{R}$-tree with a very small, minimal action of $F_n$ by isometries and dense orbits, let $Q : \partial F_n \to \hat{T}$ be the $Q$-map, and let $L(T)$ be the dual lamination. The map $Q$ induces a map

$$Q : L(T) \to \hat{T} \quad \text{by} \quad Q(X,Y) = Q(X) = Q(Y).$$

Thus, there are maps $Q_+ : L(T_+) \to \hat{T}_+$ and $Q_- : L(T_-) \to \hat{T}_-$.

**Notation 2.13:** Bestvina–Feighn–Handel [BFH97] define **attracting** and **repelling laminations** $\Lambda_+ = \Lambda_{\phi}^+$ and $\Lambda_- = \Lambda_{\phi}^-$, respectively, associated to a fully irreducible $\phi \in \text{Out}(F_n)$. As we do not use the definition of these laminations explicitly, we refer the reader to [BFH97] for details and briefly recall the facts relevant to this paper. Similar to the structure of the attracting and repelling trees $T_+$ and $T_-$, the attracting and repelling laminations satisfy $\Lambda_{\phi}^+ = \Lambda_{\phi}^- 1$ and $\Lambda_{\phi}^- = \Lambda_{\phi}^+ 1$.

We will make use of the following relationships between the laminations $\Lambda_+$ and $L(T_-)$.

**Definition 2.14:** Let $S \subset \partial^2 F_n$. The **diagonal extension** of $S$, denoted $\text{diag}(S)$, is the set of all elements $(X,Y) \in \partial^2 F_n$ such that, for some integer $m \geq 1$, there exist elements $X = Z_0, Z_1, \ldots, Z_m = Y$ in $\partial F_n$ such that $(Z_{i-1}, Z_i) \in S$ for $i \in \{1, \ldots, m\}$.

**Theorem 2.15 ([KL14, Theorem 1.2] [CHR15, Corollary 1.3]):** $L(T_-) = \text{diag}(\Lambda_+)$ and $L(T_+) = \text{diag}(\Lambda_-)$.

**Definition 2.16:** Let $\mathcal{L}$ be a lamination in $\partial^2 F_n$. The **derived lamination** of $\mathcal{L}$, denoted $\mathcal{L}'$, is the set of limit points in $\mathcal{L}$. That is,

$$\ell \in \mathcal{L}' \iff \exists \{\ell_i\} \subset \mathcal{L} \text{ such that } \ell_i \neq \ell \forall i, \text{ and } \lim_{i \to \infty} \ell_i = \ell.$$

The next theorem can be derived from work of Coulbois–Hilion–Reynolds [CHR15, Theorem A]; as the statement below does not appear in their paper, we include a brief proof.
Theorem 2.17 ([CHR15]): Let $\phi$ be a fully irreducible outer automorphism, let $\Lambda_+$ denote its attracting lamination, and let $L(T_-)$ denote the dual lamination of its repelling tree. Then $\Lambda_+ = L(T_-)'$. 

Proof. Let $\phi$ be a fully irreducible outer automorphism. By Theorem 2.9, the action of $F_n$ on the tree $T_-$ is indecomposable. Thus, by [CHR15, Theorem A], the lamination $L(T_-)$ contains a unique minimal sublamination which equals its derived lamination $L(T_-)'$. On the other hand, the attracting lamination $\Lambda_+$ is minimal by the construction of Bestvina–Feighn–Handel [BFH97]. Moreover, $\Lambda_+$ is contained in $L(T_-)$ by Theorem 2.15. Therefore, $L(T_-)' = \Lambda_+$. □

2.4. Automorphisms, topological representatives, and homotheties.

We will use the following facts regarding the correspondence between a fully irreducible outer automorphism $\phi$ and an automorphism in $\text{Aut}(F_n)$ representing $\phi$ that were proven by Gaboriau–Jaeger–Levitt–Lustig [GJLL98] and Handel–Mosher [HM11]. We begin with the relevant definitions.

Definition 2.18: Let $R_n$ denote the graph with one vertex $v$ and $n$ edges. Choosing a basis $A$ of $F_n$ and identifying each oriented edge with a distinct element of $A$ identifies $\pi_1(R_n,v)$ with $F_n$. Moreover, each automorphism $\Phi \in \text{Aut}(F_n)$ is represented by a map $f: R_n \to R_n$ sending the vertex $v$ to itself and an edge of $R_n$ to an immersed edge path so that $f$ is a homotopy equivalence. The correspondence between such self-maps of $R_n$ and elements of $\text{Aut}(F_n)$ is a bijection.

Let $G$ be a graph and $\mu: R_n \to G$ a homotopy equivalence called a marking. A homotopy equivalence $f: G \to G$ gives rise to an outer class $[f_*] \in \text{Out}(F_n)$. If $f(e)$ is an immersed edge path for every edge $e \in E(G)$, then we say that $f$ is a topological representative of $[f_*]$. The map $f$ is a train track map if, in addition, $f^k|_{G-E(G)}$ is locally injective for all $k > 0$.

Let $\phi \in \text{Out}(F_n)$, let $f: G \to G$ be a topological representative of $\phi$, and let $\widetilde{G}$ be the universal cover of $G$. The identification, via the homotopy equivalence $\mu$, of $\pi_1(G,\ast)$ with the free group $F_n$ (up to conjugation) gives rise to an action of $F_n$ on $\widetilde{G}$ by deck transformations $\rho: F_n \to \text{Aut}(\widetilde{G})$. The map $\rho$ is well defined up to precomposing it with $i_g$, where $i_g \in \text{Aut}(F_n)$ denotes conjugation by $g$.

The following lemma is elementary and its proof is left to the reader.
Lemma 2.19: Let $\phi \in \text{Out}(F_n)$ and $g: G \to G$ be a topological representative of $\phi$. Let $\tilde{G}$ denote the universal cover of $G$ and $\rho: F_n \to \text{Aut}(\tilde{G})$ the action of $F_n$ by deck transformations. There is a bijection between automorphisms $\Phi \in \phi$ and lifts $\tilde{g}: \tilde{G} \to \tilde{G}$ of $g$ given by the equation
\begin{equation}
\tilde{g} \circ \rho(\gamma) = \rho(\Phi(\gamma)) \circ \tilde{g}.
\end{equation}

Definition 2.20: Let $\phi \in \text{Out}(F_n)$ be fully irreducible, and let $\Phi \in \text{Aut}(F_n)$ so that $\Phi \in \phi$. We say that $\Phi$ is a principal automorphism if the extension of $\Phi$ to $\partial F_n$, which we denote by $\partial \Phi$, fixes at least three nonrepelling points.

Lemma 2.21 ([GJLL98] [HM11, Theorems 2.15, 2.16]): Let $\phi \in \text{Out}(F_n)$ be fully irreducible, and let $T_+$ be its attracting tree.

1. If $\Phi$ is a principal automorphism representing $\phi$, then there exists a unique homothety $h_+: T_+ \to T_+$ so that $h_+(\gamma x) = \Phi(\gamma)h_+(x)$ for all $\gamma \in F_n$ and $x \in T_+$. This homothety stretches the distances in $T_+$ by the dilatation of $\phi$.
2. The homothety $h_+$ from the previous item fixes a branch point $b \in T_+$.
3. The correspondence $\Phi \mapsto b$ defines a bijection between the set of principal automorphisms representing $\phi$ and the set of branch points of $T_+$.
4. For each automorphism $\Phi$ representing $\phi$ there exists a unique homothety $h_-: T_- \to T_-$ so that $h_-(\gamma x) = \Phi(\gamma)h_-(x)$ for all $\gamma \in F_n$ and for all $x \in T_-$.

Proof. Item (4) is the only one that does not appear in [HM11]. Since orbits are dense in $T_-$ uniqueness follows from satisfying the equation in (4). Since $\phi$ acts on $\overline{CV}_n$ by North-South dynamics, $T_-$ is equivalent to $T_- \cdot \phi$. This means that there exists a homothety $h_-: T_- \to T_-$ which is $F_n$-equivariant. That is, if $\rho: F_n \to \text{Aut}(T_-)$ is the $F_n$-action on $T_-$ then
\begin{equation}
h_-(\rho(\gamma)x) = \rho(\Phi(\gamma))h_-(x).
\end{equation}
Suppressing $\rho$ yields the statement in the lemma.

Remark 2.22: Let $f : G \to G$ be a train-track map. A path $\rho$ in $G$ is a Nielsen path if $f_\#(\rho) = \rho$, where $f_\#(\rho)$ denotes the tightening of $\rho$. An indivisible Nielsen path (INP) is a Nielsen path that is not the concatenation of two non-trivial Nielsen paths. Let $\phi$ be a fully irreducible outer automorphism, and let $f : G \to G$ be a train-track map representing $\phi$ with minimal number of INPs
(among all the train-track maps representing $\phi$). As explained in [BH92], $f$ has either zero INPs or one INP. Moreover, $\phi$ is atoroidal if and only if $f$ has either no INPs or the INP of $f$ is not a loop. This fact has some implications for the repelling tree $T_-$ and the attracting lamination $\Lambda^\phi_+$:

- The INP $\rho$ does not occur as a subpath of any (realization in $G$ of) leaf of $\Lambda^\phi_+$, the attracting lamination of $\phi$. Indeed, by construction, subpaths of $\Lambda^\phi_+$ are legal, contrary to $\rho$.
- In the atoroidal case, the action of $F_n$ on $T_-$ is free. In the other case (when $\rho$ is a loop), there is one orbit of points with non-trivial stabilizer: these stabilizers are all conjugated, each one is infinite cyclic and generated by an element in the conjugacy class defined by the loop $\rho$.

Let $\phi \in \text{Out}(F_n)$ be fully irreducible. We call a triplet $(X,Y,Z)$ of distinct points in $\partial F_n$ special if $(X,Y), (Y,Z) \in \Lambda^\phi_+$. The set of special triplets is $F_n$-invariant.

**Lemma 2.23:** There exists a number $M := M(n)$ so that for any fully irreducible outer automorphism $\phi \in \text{Out}(F_n)$, there are at most $M F_n$-orbits of special triplets.

**Proof.** The proof is based on [CH14, Theorem 5.3]. Consider the map

$$Q_- : \partial F_n \to T_-.$$

If $a \in T_-$, then the group $\text{Stab}(a) \leq F_n$ acts on the set $Q_-(a)$. Coulbois–Hilion define the $Q$-index of $a$, denoted $\text{ind}(a)$, as

$$\text{ind}(a) = |Q^{-1}_-(a)/\text{Stab}(a)| + 2 \text{rank}(\text{Stab}(a)) - 2. \tag{2.2}$$

If $(X,Y,Z)$ is a special triplet, then by Theorem 2.15,

$$Q_-(X) = Q_-(Y) = Q_-(Z).$$

Hence, either $\text{Stab}(Q_-(X)) \neq \{1\}$ or $|Q^{-1}_-(Q_-(X))| > 1$, and in both cases $\text{ind}(Q_-(X)) > 0$. Let $A$ be a set of representatives of orbits of the $F_n$ action on $T_-$. By [CH14, Theorem 5.3],

$$\sum_{a \in A} \max\{0, \text{ind}(a)\} \leq 2n - 2. \tag{2.3}$$
Consider the map $Q_- : \{\text{special triplets}\} \to T_-$ defined by

$$Q_-(X, Y, Z) = Q_-(X).$$

This map is $F_n$-equivariant so descends to a map on the sets of orbits. Moreover, the map $Q_-$ maps onto the set of orbit points that contribute to the sum in equation (2.3). Thus the cardinality of the orbits of special triplets is bounded above by $2n - 2$ times the number of orbits of special triplets which are contained in $Q_-$ point preimages.

If $\phi$ is non-geometric, then all point stabilizers are trivial. So, equation (2.2) becomes $\text{ind}(a) = |Q_-(a)| - 2$ which is bounded by $2n - 2$ by inequality (2.3) so the cardinality of special triples is bounded by $2n(2n - 2)$. If $\phi$ is geometric, an extra argument is needed. There is one orbit of points $\mathcal{A}$ in $T_-$ whose stabilizer is non-trivial and is, in fact, infinite cyclic. Let $g$ be a generator of $\text{Stab}(a_0)$ for this $a_0 \in A$. Inequality (2.3) implies

$$|Q_-(a_0)/\text{Stab}(a_0)| \leq 2n - 2.$$ 

Let $[\cdot]_g$ denote a $g$-orbit and consider the map sending the orbit of the special triple $[(X, Y, Z)]_g$ to $([X]_g, [Y]_g, [Z]_g) \in (Q_-(a_0)/\langle g \rangle)^3$. We will show that it is at most 25 to 1. Let $(X, Y, Z), (X', Y', Z')$ be two special triplets that map to $a_0$ and so that $X, X'$ are in the same $\langle g \rangle$-orbit, as well as $Y, Y'$ and $Z, Z'$. By applying a power of $g$ we may assume that $X' = X$. Now $Y' = g^iY$ and if $|i| \geq 2$ then one of the leaves $X^{-1}Y, X^{-1}Y'$ of $\Lambda_+$ would contain a copy of $g$ which is impossible by Remark 2.22. The same is true for $Z$ and $Z'$. So the number of $F_n$-orbits of triplets that map to $a_0$ is bounded by $25(2n - 2)^3$ and the total number of orbits is bounded by $25(2n - 2)^3 + 2n(2n - 2)$.

**Lemma 2.24:** For each $n$ there exists $p \geq 1$ so that for each fully irreducible $\phi \in \text{Out}(F_n)$ and for any distinct asymptotic leaves $\ell_1, \ell_2 \in \Lambda^\phi_+$, there exists an automorphism $\Phi \in \text{Aut}(F_n)$ representing $\phi^p$ so that the endpoints of $\ell_1$ and $\ell_2$ are non-repelling fixed points of $\partial \Phi$.

**Proof.** Let $\phi \in \text{Out}(F_n)$. By Lemma 2.23, there are finitely many $F_n$-orbits of special triplets. The outer automorphism $\phi$ permutes these orbits. Thus, there exists $r \geq 1$ so that $\phi^r$ fixes the orbits of special triplets. Let $W$ be the special triplet defined by $\ell_1$ and $\ell_2$. Let $\Phi \in \text{Aut}(F_n)$ be an automorphism representing $\phi^r$. Then, there exists an element $h \in F_n$ so that $\Phi(W) = h \cdot W$. 

Let
\[ \Phi' = i_{h^{-1}} \circ \Phi, \]
where \( i_{h^{-1}} \) denotes the inner automorphism given by conjugation by \( h^{-1} \); that is, \( i_{h^{-1}}(g) = h^{-1}gh \). Then, \( \Phi'(W) = W \). So, \( \Phi'^{3!} \) fixes each endpoint of \( W \). Since \( \ell_1, \ell_2 \in \Lambda^\phi_+ \), these endpoints are non-repelling fixed points of \( \partial \Phi^6 \).

**Lemma 2.25 ([CH12, Equation 4.2]):** Let \( \Phi \) be an automorphism and
\[ h_\pm : T_\pm \to T_\pm \]
the corresponding homotheties guaranteed by Lemma 2.21. Then, for each \( \xi \in \partial F_n \),
\[ Q_\pm(\partial \Phi(\xi)) = h_\pm(Q_\pm(\xi)). \]

3. Realization of leaves

**Definition 3.1:** Let \( \mathcal{L} \) be a lamination of \( F_n \). The **ends** of the lamination \( \mathcal{L} \) is the set
\[ \mathcal{E} \mathcal{L} = \{ X \in \partial F_n \mid \exists Y \in \partial F_n \text{ such that } (X, Y) \in \mathcal{L} \}. \]

**Lemma 3.2 ([HM11]):** Let \( \phi \in \text{Out}(F_n) \) be a fully irreducible outer automorphism, let \( T_\pm \) be the attracting and repelling trees of \( \phi \), and let \( Q_+ : \partial F_n \to \hat{T}_+ \) be the \( Q \)-map. For every \( X \in \mathcal{E} \mathcal{L}(T_-) \), the point \( Q_+(X) \in \partial T_+ \).

**Proof.** This argument follows from work of Handel–Mosher [HM11], though the exact statement does not appear in their paper. So, we include a proof for clarity, keeping the notation used by the reference.

Let \( g : \Gamma \to \Gamma \) be an affine train track representative of \( \phi \), and let \( \widetilde{\Gamma} \) denote the universal cover of the graph \( \Gamma \). (That is, \( g \) is a train track map as in Definition 2.18 with the additional property that \( \Gamma \) is a metric graph and there exists \( \lambda > 1 \) so that the restriction of each iterate \( g^i \) to the interior of each edge locally stretches the metric by the factor \( \lambda^i \).) Via the marking \( \mu : R_n \to \Gamma \), the boundary \( \partial \widetilde{\Gamma} \) may be identified with \( \partial F_n \), and the attracting lamination \( \Lambda_+ \) may be identified with a set of geodesic lines in the graph \( \widetilde{\Gamma} \).
There exists an $F_n$-equivariant edge-isometry $f_g : \widetilde{\Gamma} \to T_+$ which is a $\Lambda_+$-isometry; that is, for each leaf $\ell \in \Lambda_+$ viewed as an isometric embedding $\ell : \mathbb{R} \to \widetilde{\Gamma}$, the map $f_g \circ \ell : \mathbb{R} \to T_+$ is an isometric embedding [HM11, Corollary 2.14]. Therefore, the limit $\lim_{t \to \pm \infty} f_g \circ \ell(t)$ exists and lies in $\partial T_+$. Let

$$f_g(\ell(\pm \infty)) := \lim_{t \to \pm \infty} f_g \circ \ell(t).$$

By [LL03, Section 3] $Q_+(\ell(\pm \infty)) = f_g(\ell(\pm \infty))$. Hence, $Q_+$ maps the ends of $\Lambda_+$ to $\partial T_+$. By Theorem 2.15, $\mathcal{E}L(T_-) = \mathcal{E}\Lambda_+$, so $Q_+(X) \in \partial T_+$. 

**Definition 3.3:** Let $\ell = (X,Y) \in L(T_-)$. By Lemma 3.2, $Q_+(X), Q_+(Y) \in \partial T_+$, and by Proposition 2.11, $Q_+(X) \neq Q_+(Y)$. The realization of $\ell$ in $T_+$ is the bi-infinite geodesic, denoted $\ell^+$, that connects $Q_+(X)$ and $Q_+(Y)$ in $T_+$.

Proposition 3.7 below compares the convergence of leaves of the lamination $L(T_-)$ in the topology on $\partial^2 F_n$ to the convergence of the realization of leaves in the attracting tree $T_+$ in the Hausdorff topology.

**Definition 3.4:** If $\{\ell^+_i \mid i \in \mathbb{N}\}$ is a sequence of bi-infinite geodesics in $T_+$ and $\ell^+$ is a bi-infinite geodesic in $T_+$, then $\lim_{i \to \infty} \ell^+_i = \ell^+$ in the Hausdorff topology on $T_+$ if for any subarc $I \subset \ell^+$, there exists $N \in \mathbb{N}$ so that $I \subset \ell^+_i$ for all $i > N$.

**Definition 3.5:** Let $A$ be a basis for $F_n$. The set $\partial^2 F_n$ may be identified with the space of pairs $(X,Y)$ of infinite reduced words in $A \cup A^{-1}$ with $X \neq Y$. For an infinite reduced word $X$, let $X_1$ denote the first letter of $X$. Let

$$C_A := \{(X,Y) \in \partial^2 F_n \mid X_1 \neq Y_1\}$$

be the unit cylinder associated to $A$.

The proof of the next lemma is elementary and left to the reader.

**Lemma 3.6:** Let $A$ be a basis of $F_n$. The unit cylinder associated to $A$ is open and compact in $\partial^2 F_n$. 

**Proposition 3.7:** Let $A$ be a basis of $F_n$. Let $C_A \subset \partial^2 F_n$ be the unit cylinder associated to $A$ and let $\{\ell_i \mid i \in \mathbb{N}\} \cup \{\ell\} \subset L(T_-) \cap C_A$. Then, $\lim_{i \to \infty} \ell_i = \ell \in \partial^2 F_n$ if and only if $\lim_{i \to \infty} \ell^+_i = \ell^+ \subset T_+$ in the Hausdorff topology, where $\ell^+_i$ and $\ell^+$ are the realizations of these leaves in $T_+$ as in Definition 3.3.
Proof. Let $\ell = (\xi, \eta) \in L(T_-) \cap C_A$ and $\ell_i = (\xi_i, \eta_i) \in L(T_-) \cap C_A$ so that $\lim_{i \to \infty} \ell_i = \ell$. Let $I = [a, b] \subset \ell^+$, and assume that $a$ is between $Q_+^{\xi}(\xi)$ and $b$. Let $d_1$ be the direction at $a$ containing $Q_+^{\xi}(\xi)$. Since $Q_+^{\xi}$ is continuous with respect to the observers’ topology, there exists $N_1 \in \mathbb{N}$ so that for all $i > N_1$, $Q_+^{\xi}(\xi_i) \in d_1$. Similarly, if $d_2$ is the direction at $b$ containing $Q_+^{\eta}(\eta)$, then there exists $N_2 \in \mathbb{N}$ so that for all $i > N_2$, $Q_+^{\eta}(\eta_i) \in d_2$. Let $i > \max\{N_1, N_2\}$. Then $\ell_i^+$ contains $[a, b]$ as desired.

Suppose now that $\lim_{i \to \infty} \ell_i^+ = \ell^+$ in the Hausdorff topology. Since $L(T_-) \cap C_A$ is compact by Lemma 3.6, the sequence $\{\ell_i\}$ has a convergent subsequence. Let $\tau$ be a partial limit; that is, $\lim_{j \to \infty} \ell_{ij} = \tau \in \partial^2 F_n$. By the arguments in the previous paragraph, $\lim_{j \to \infty} \ell_{ij}^+ = \tau^+$ in the Hausdorff topology on $T_+$. Thus, $\ell^+ = \tau^+$. By Theorem 2.2, the map $Q_+$ is injective on $\mathcal{E}L(T_-)$, so $\tau = \ell$. Therefore, the sequence $\{\ell_i\}$ has a unique partial limit; thus, $\{\ell_i\}$ converges to $\ell$. ■

In Section 8 we will use the following property of the realization of leaves of the lamination $L(T_-)$ in the tree $T_+$.

Definition 3.8: A star is a wedge of intervals or a wedge of rays. The wedge point is called the middle of the star.

Proposition 3.9: Let $\ell_1, \ell_2, \ldots, \ell_k$ be leaves of the lamination $L(T_-)$ such that $\ell_i$ is asymptotic to $\ell_{i+1}$ for each $i = 1, \ldots, k - 1$. Then $\bigcup_{i=1}^k \ell_i^+$ is a star in $T_+$.

Proof. Since $L(T_-) = \text{Diag}(\Lambda_+^\phi)$ we may find a list $\ell_1', \ell_2', \ldots, \ell_s'$ so that $\ell_i' \in \Lambda_+^\phi$, $\ell_i'$ is asymptotic to $\ell_{i+1}'$ and the set of ends of $\{\ell_1, \ldots, \ell_k\}$ is contained in the ends of $\{\ell_1', \ldots, \ell_s'\}$. It is clear that if we prove the theorem for the second list of leaves then it will follow for the first so we abuse notation and assume that the original list of leaves is in $\Lambda_+^\phi$. By Lemma 2.24 there exists a principal automorphism $\Phi$ representing $\phi^p$ for some $p \in \mathbb{N}$ so that the endpoints of $\ell_1$ and $\ell_2$ are attracting fixed points of $\partial \Phi$. Similarly, since $\ell_2$ and $\ell_3$ are asymptotic, there exists an automorphism $\Phi'$ representing $\phi^p$ so that the endpoints of $\ell_2$ and $\ell_3$ are attracting fixed points of $\partial \Phi'$. Attracting fixed points of two automorphisms in the same outer class are disjoint [Hil07, Theorem 1.1] (see alternatively [HM11, Corollary 2.9] for two principal automorphisms), so $\Phi = \Phi'$. Continuing in this fashion proves that the endpoints of $\ell_1, \ldots, \ell_k$ are attracting fixed points of $\partial \Phi$. By Lemma 2.21(1), there exists a homothety $h : T_+ \to T_+$.
representing $\Phi$. Let $S \subset T_+$ be the union of the realizations of $\{\ell_i\}_{i=1}^k$. Then, $S$ is invariant by $h$ by Lemma 2.25. Moreover, $S$ has finitely many vertices, and $h$ permutes the finitely many vertices of $S$. Since $h$ is a homothety, $S$ contains only one (internal) vertex. \[\square\]

3.1. PROOF OF THEOREM 1.8. Let $Wh_\phi$ be the $F_n$-quotient of the graph whose vertex set is the union of nonrepelling fixed points in $\partial F_n$ of principal automorphisms representing $\phi$; see Definition 2.20. An edge of $Wh_\phi$ corresponds to a leaf of the attracting lamination $\Lambda^\phi_+$ of $\phi$. Let $Wh_{\Lambda^\phi_+}$ be the $F_n$-quotient of the graph whose vertex set is the union of endpoints of singular leaves of $\Lambda^\phi_+$, where a singular leaf has an asymptotic class containing more than one element. An edge of $Wh_{\Lambda^\phi_+}$ corresponds to a singular leaf of $\Lambda^\phi_+$. By Lemma 2.24, $Wh_\phi \cong Wh_{\Lambda^\phi_+}$, proving Theorem 1.8.

4. The band complex

4.1. SYSTEMS OF PARTIAL ISOMETRIES AND THE BAND COMPLEX.

Definition 4.1: A **compact forest** is a finite union of compact $\mathbb{R}$-trees. A **partial isometry** of a compact forest $K$ is an isometry $a : J \to J'$, where $J$ and $J'$ are compact subtrees of $K$. The **domain** of $a$, denoted $\text{dom}(a)$, is $J$, and the **range** of $a$ is $J'$. A partial isometry is **non-empty** if its domain is non-empty. A **system of partial isometries** is a pair $S = (K, A)$, where $K$ is a compact forest and $A$ is a finite collection of non-empty partial isometries of $K$.

Definition 4.2: Let $S = (K, A)$ be a system of partial isometries, and let $I = [0, 1]$ denote the unit interval. For each $a_i \in A$, let $b_i \subset K$ be the domain of $a_i$. Let $B_i := b_i \times I$ be called a **band**. The **band complex** $B$ is the quotient of $K \sqcup_i B_i$, where $b_i \times \{0\}$ is identified to the domain of $a_i$, and $b_i \times \{1\}$ is identified to the range of $a_i$ by isometries.

Definition 4.3: Each band $B = b \times I \subset B$ is foliated by leaves of the form $x \times I$ for $x \in b$. The foliation of the bands yields a foliation of $B$. A finite, infinite, or bi-infinite path $\gamma$ in $B$ is an **admissible leaf path** if $\gamma$ is a locally isometric, immersed path contained in a leaf. A **half-leaf based at** $x \in K$ is an admissible leaf path $\rho : [0, \infty) \to B$ with $\rho(0) = x$. A finite admissible path $\gamma$ travels through a finite sequence of bands $B_1, \ldots, B_k$ and, in turn,
corresponds to the sequence of partial isometries $a_1,\ldots,a_k$ that give rise to the bands. The domain of $\gamma$ is then $\text{dom}(a_k \circ \cdots \circ a_1)$. Note that the domain contains $\gamma(0)$. This definition extends to admissible rays and bi-infinite lines. The limit set $\Omega$ of $S = (K,A)$ is the set of elements of $K$ which are in the domain of a bi-infinite admissible reduced path. The lamination $L(B)$ is the set of bi-infinite admissible leaf paths in $B$. For an $\mathbb{R}$-tree $J$, let $\mu_J$ denote the Lebesgue measure on $J$, which consists of the Lebesgue measures on the segments of $J$. The foliated space $B$ has a transverse measure associated to the Lebesgue measure on the components of $K$.

**Notation 4.4:** Let $x \in K$, and let $\ell = \cdots z_{-2}z_{-1}z_0z_1z_2\cdots \subset L(B)$ be a bi-infinite admissible leaf path through $x$; so, $z_i \in A \cup A^{-1}$ for $i \in \mathbb{Z}$. Suppose $\ell_1 = z_0z_1z_2\cdots$ and $\ell_2 = z_{-1}^{-1}z_{-2}^{-1}\cdots$ denote the half-leaves based at $x$. Then, $\ell = \ell_1 \cup \ell_2$, and to record the point $x$ in the leaf $\ell$, we use the notation

$$\ell = \cdots z_{-2}z_{-1}z_0z_1z_2\cdots.$$ 

**Definition 4.5:** Let $S = (K,A)$ be a system of partial isometries, and let $B$ be the associated band complex. Let $\tilde{B}$ denote the universal cover of $B$. The foliation and transverse measure on $B$ lift to a foliation and transverse measure on $\tilde{B}$. Collapsing each leaf in $\tilde{B}$ to a point yields an $\mathbb{R}$-tree $T_B$, called the dual tree to $B$.

### 4.2. System of partial isometries associated to a free group automorphism.

Coulbois–Hilion–Lustig in [CHL09] use the $Q$-map to construct a system of partial isometries for an $\mathbb{R}$-tree $T$ with a very small, minimal action of $F_n$ by isometries and dense orbits.

**Construction 4.6 ()**: [CHL09]. Let $T$ be a very small, minimal $F_n$-tree with dense orbits, let $A$ be a basis of $F_n$, and let $C_A \subset \partial^2 F_n$ be its unit cylinder. Let

$$\Omega_A := Q(L(T) \cap C_A) \subset \overline{T}.$$ 

The compact heart $K_A$ of $T$ relative to $A$ is the convex hull of $\Omega_A$ in $\overline{T}$. By [CHL09, Theorem 1.1], $K_A$ is indeed a compact subtree of $\overline{T}$. Let $S = (K_A,A)$ be the system of partial isometries so that for each $a \in A$, the partial isometry associated to $a$ is the maximal restriction of $a^{-1}$ to $K_A$. As defined in Section 4.1, let $B$ be the associated band complex of $S$, and let $T_B$ denote the $F_n$-tree dual to $B$. 
Remark 4.7: If \( K_A \) is the compact heart of \( T \) relative to the basis \( A \) of \( F_n \), then \( gK_A \) is the compact heart of \( T \) relative to the basis \( gA^{-1} \) of \( F_n \). Moreover, the unit cylinder satisfies \( C_{gA^{-1}} = gC_A \) and \( \Omega_{gA^{-1}} = g\Omega_A \). We leave the verification of these facts to the reader.

**Theorem 4.8** ([CHL09, Theorem 5.4]): Let \( T \) be a minimal, very small, \( F_n \)-tree with dense orbits, and let \( A, C_A, B, \) and \( T_B \) be defined as in Construction 4.6.

1. The tree \( T \) is equal to the minimal subtree of the tree \( T_B \).
2. \( L(T) \cap C_A = L(B) \), where \( L(T) \cap C_A \) and \( L(B) \) are identified with bi-infinite reduced words in \( A \cup A^{-1} \) as in the above construction.

**Notation 4.9:** When we apply Construction 4.6 and Theorem 4.8 to the attracting or repelling trees \( T_+ \) or \( T_- \) of a fully irreducible outer automorphism, we denote each object with the appropriate subscript; we write \( S_+, \Omega_+, \ldots \), and so on.

5. Rips Induction and overlapping bands

5.1. Rips Induction.

**Definition 5.1:** Let \( S = (K, A) \) be a system of partial isometries. The ** output of the Rips Machine applied to \( S \)** is a new system of partial isometries \( S' = (K', A') \) defined as follows.

\[
K' := \{ x \in K \mid x \in \text{dom}(a) \cap \text{dom}(a') \text{ for some } a \neq a' \in A \cup A^{-1} \}.
\]

Since \( A \) is finite and the intersection of two domains is a compact \( \mathbb{R} \)-tree, \( K' \) is a compact forest. Let \( A' \) be the set of all maximal restrictions of the elements of \( A \) to pairs of components of \( K' \). Then \( S' = (K', A') \) is a system of partial isometries.

**Definition 5.2** ([CH14, Definition 3.11]): Let \( S_0 = (K_0, A_0) \) be a system of partial isometries, and let \( S_1 = (K_1, A_1) \) denote the output of the Rips Machine. The system of partial isometries \( S_0 \) is **reduced** if for any partial isometry \( a \in A_0^\pm \), the set of extremal points of the domain of \( a \) is contained in \( K_1 \).

**Proposition 5.3** ([CH14, Propositions 5.6, 3.14]): Let \( \phi \in \text{Out}(F_n) \) be a fully irreducible outer automorphism, and let \( S_+ = (K_+^\lambda, A) \) and \( S_- = (K_-^\lambda, A) \) be the systems of partial isometries defined in Construction 4.6. Then \( S_+ \) and \( S_- \) are reduced, as is any output of \( S_+ \) or \( S_- \) under the Rips Machine.
Definition 5.4: Let $S_0 = (K_0, A_0)$ be a system of partial isometries. Let $S_i = (K_i, A_i)$ denote the output of the $i^{th}$ iteration of the Rips Machine. If for some $i$, $K_i = K_{i+1}$, then the Rips Machine halts on $S_i$, and the Rips Machine eventually halts on $S_0$.

Definition 5.5: Let $S_0$ be a system of partial isometries. If the Rips Machine eventually halts on $S_0$, then $S_0$ is called surface type. If the Rips Machine does not eventually halt on $S_0$ and

$$\lim_{i \to \infty} \max_{a \in A_i} \text{diam } a = 0,$$

then $S_0$ is called of Levitt type.

The definition of ‘Levitt type’ given here is equivalent to the original one in [CH12, Section 5], which states that the limit set $\Omega$, see Definition 4.3, is totally disconnected.

Notation 5.6: Let $\phi \in \text{Out}(F_n)$ be a fully irreducible outer automorphism, and let $S_+$ and $S_-$ be the systems of partial isometries defined in Construction 4.6. If $S_+$, respectively $S_-$, is of Levitt type, we say $T_+$, respectively $T_-$, is of Levitt type.

The following proposition is a direct consequence of a result of Coulbois–Hilion [CH12].

Proposition 5.7 ([CH12, Theorem 5.2]): Let $\phi \in \text{Out}(F_n)$ be a fully irreducible atoroidal outer automorphism. Then either both $T_+$ and $T_-$ are of Levitt type, or one of the trees $T_+$ and $T_-$ is of Levitt type and the other one is of surface type.

The proof of Theorem 1.1 uses the fact that at least one of $T_+$ and $T_-$ is of Levitt type.

5.2. Volume of an $\mathbb{R}$-tree and overlapping domains. The main aim of this section is to prove Theorem 5.15, which is a converse to [CH14, Proposition 4.3], and states that if $T$ is an $F_n$-tree with dense orbits and of Levitt type, then the associated Rips Machine (beginning from the compact heart) has the property that at each step there are three overlapping bands.

Definition 5.8: A compact $\mathbb{R}$-tree $K$ is finite if $K$ has a finite number of extremal points. In this case, $K$ has also a finite number of branch points. Removing
these branch points from \( K \) yields to a finite set of arcs; the \textbf{volume} \( \text{vol}(K) \) of \( K \) is the sum of the lengths of these arcs. The volume \( \text{vol}(K) \) of a compact \( \mathbb{R} \)-tree \( K \) is the supremum of the volume of the finite subtrees contained in \( K \).

A \textbf{compact forest} \( K \) is a finite disjoint union of compact trees \( K_1, \ldots, K_p \); its \textbf{volume} is

\[
\text{vol}(K) = \sum_{1 \leq i \leq p} \text{vol}(K_i).
\]

This volume of an \( \mathbb{R} \)-tree may be finite or infinite. In either case, the following proposition applies.

**Proposition 5.9:** Let \( T \) be a compact \( \mathbb{R} \)-tree. Let \( \epsilon > 0 \) and let \( \mathcal{E} \) be a set of disjoint arcs in \( T \) of length \( \epsilon \). Then \( \mathcal{E} \) is finite.

For the proof of Proposition 5.9, we will use the following combinatorial lemma (that we state in the context of \( \mathbb{R} \)-trees even if it is valid for more general notions of trees).

**Lemma 5.10:** Let \( T \) be an \( \mathbb{R} \)-tree. Let \( \mathcal{E} \) be an infinite set of pairwise disjoint subtrees of \( T \). There exists a sequence \( (T_n)_{n \in \mathbb{N}} \) of pairwise distinct elements of \( \mathcal{E} \) with the property that, for all \( n \), \( T_{n+1} \) does not intersect the convex hull \( K_n \) of \( \bigcup_{i=1}^{n} T_i \).

**Proof.** First, suppose there exists an arc \( \gamma \) in \( T \) intersecting an infinite number of elements of \( \mathcal{E} \). Choose an orientation on \( \gamma \), which induces a linear ordering \( \prec \) on the set of elements of \( \mathcal{E} \) that intersect \( \gamma \). Any countable subset of \( \mathcal{E} \) gives rise to a sequence \( (T_n)_{n \in \mathbb{N}} \) of pairwise disjoint subtrees of \( T \) such that for all \( n \in \mathbb{N} \), \( T_n \prec T_{n+1} \). In particular, \( T_{n+1} \) does not intersect the convex hull of \( \bigcup_{i=1}^{n} T_i \).

In the remaining case, every arc of \( T \) intersects a finite number of elements of \( \mathcal{E} \). We build the sequence \( (T_n)_{n \in \mathbb{N}} \) by induction. First, pick an element \( T_1 \) of \( \mathcal{E} \). Suppose \( T_1, \ldots, T_n \) have been chosen so that no element of \( \mathcal{E} - \bigcup_{i=1}^{n} T_i \) intersects the convex hull \( K_n \) of \( \bigcup_{i=1}^{n} T_i \). Let \( S \) be an element of \( \mathcal{E} - \bigcup_{i=1}^{n} T_i \). By assumption, \( S \) does not intersect \( K_n \). Let \( \gamma \) be the arc in \( T \) joining \( S \) and \( K_n \). Set \( T_{n+1} \) to be the element of \( \mathcal{E} \) closest to \( K_n \) that intersects \( \gamma \).

**Proof of Proposition 5.9.** Assume towards a contradiction there is a sequence \( (\gamma_n)_{n \in \mathbb{N}} \) of disjoint arcs in \( T \) of length \( \epsilon > 0 \). By Lemma 5.10, we can assume that up to reordering indices, \( \gamma_{n+1} \) does not lie in the convex hull \( K_n \) of \( \bigcup_{i=1}^{n} \gamma_i \). Since \( \gamma_{n+1} \) has length \( \epsilon \), at least one of its extremal points, denoted by \( x_{n+1} \),
is at distance more than $\epsilon/2$ from $K_n$. Thus, for all $i \neq j \in \mathbb{N}$, $d(x_i, x_j) > \epsilon/2$. Hence, no subsequence of $(x_k)_{k \in \mathbb{N}}$ is convergent (since such a subsequence is not a Cauchy sequence), which contradicts the compactness of $T$. \qed

Remark 5.11: Let $K$ be a (non-empty) compact tree. Then $\text{vol}(K) = 0$ if and only if $K$ is point. Indeed, if $K$ contains two distinct points, then $K$ contains the arc joining these two points, and hence the volume of $K$ is as least as big as the length of this interval.

Notation 5.12: Let $S = (K, A)$ be a system of partial isometries. Let

$$A^{-1} = \{a^{-1} \mid a \in A\},$$

and let

$$A^\pm = A \cup A^{-1}.$$  

We suppose from now on that $A \cap A^{-1} = \emptyset$. Let $v(x)$ be the band valence of $x$; that is,

$$v(x) = \#\{a \in A^\pm \mid x \in \text{dom}(a)\}.$$ 

Let

$$K^{=i} = \{x \in K \mid v(x) = i\}.$$ 

Define $K^{\geq i}$ and $K^{\leq i}$ similarly.

Lemma 5.13: Let $S = (K, A)$ be a system of partial isometries, such that $\text{vol}(K^{\geq 3}) = 0$. Then:

(i) $K^{\geq 3}$ is a finite set of points in $K$.

(ii) If, moreover, the valence of every point in $K$ is finite, then the set

$$\{\delta \text{ direction at } x \mid x \in K^{\geq 3}\}$$

of directions in $K$ at a point in $K^{\geq 3}$ is finite.

Proof. Property (ii) follows immediately from Property (i). According to Remark 5.11, since $\text{vol}(K^{\geq 3}) = 0$, the components of $K^{\geq 3}$ are points. It remains to show there are only finitely many components. Since the domains of the partial isometries in $A^{\pm 1}$ are subtrees of the tree $K$, an intersection of domains is also a tree (and possibly the empty set). Thus, the set of components of $K^{\geq 3}$ injects in the set of subsets of $A^{\pm 1}$ of cardinality at least 3, which is a finite set since $A^{\pm 1}$ is itself a finite set. \qed
Let $T$ be a free $\mathbb{R}$-tree with dense orbits and of Levitt type, and let $A$ be a basis of $F_n$. Let $K_0$ be the compact heart of $T$ relative to $A$, and let $S_0 = (K_0, A_0)$ be the associated system of partial isometries as in Construction 4.6. Since $T$ is of Levitt type, the Rips Machine does not halt. Let $S_i = (K_i, A_i)$ denote the output after the $i^{th}$ iteration of the Rips Machine. Then, $\text{vol}(K_i^{\geq 3}) > 0$ for all $i \in \mathbb{N}$.

Remark 5.14:  
(i) All orbits of $S_0 = (K_0, A_0)$ are infinite; that is, every leaf in the band complex $B$ is infinite. Indeed, [CH14, Proposition 5.6] ensures the system $S_0$ is reduced, which, by definition, implies every orbit is infinite.

(ii) The valence of the points in $K_0$ is bounded. Indeed, $K_0$ is a subtree of $T$, and the valence of the points of $T$ is bounded (by $2n$ [GJLL98]).

Theorem 5.15: Let $T$ be an $F_n$-tree with dense orbits and of Levitt type, and let $A$ be a basis of $F_n$. Let $S_0 = (K_0, A_0)$ be the associated system of partial isometries, and let $S_i = (K_i, A_i)$ denote the output after the $i^{th}$ iteration of the Rips Machine. Then $\text{vol}(K_i^{\geq 3}) > 0$ for all $i \in \mathbb{N}$.

Proof. By contradiction, suppose $\text{vol}(K_{i_0}^{\geq 3}) = 0$ for some $i_0 \in \mathbb{N}$. By definition of the Rips Machine, $K_{i_0+1}^{\geq 3} \subseteq K_{i_0}^{\geq 3}$. In particular, the sequence $(\text{vol}(K_i^{\geq 3}))_i$ is decreasing, and thus $\text{vol}(K_i^{\geq 3}) = 0$ for all $i \geq i_0$. We can suppose that $i_0 = 0$.

We define by induction a sequence $(\gamma_n)$ of subarcs of $K_0$. Let $\gamma_0$ be an arc contained in $K_0^{\geq 1}$. There is only one partial isometry $a$ defined on $\gamma_0$; let $\gamma_1 = \gamma_0 \cdot a$, the image of $\gamma_0$ by $a$. Since $S_0$ has no finite orbit, $\gamma_1 \subseteq K_0^{\geq 2}$. In particular, $\gamma_0$ and $\gamma_1$ are disjoint.

Case 1: $\gamma_1 \subseteq K_0^{\geq 2}$. There are exactly two partial isometries defined on $\gamma_1$: one is $a^{-1}$; let $b$ denote the other. Set $\gamma_2 = \gamma_1 \cdot b$. Again, $\gamma_2 \subseteq K_0^{\geq 2}$ since $S_0$ has no finite orbits.

Case 2: $\gamma_1 \cap K_0^{\geq 3} \neq \emptyset$. Replace $\gamma_0$ by a subarc so $\gamma_1 \subseteq K_0^{\geq 2}$; this procedure is possible since $K_0^{\geq 3}$ is a finite subset of $K_0$ by Lemma 5.13. Define $\gamma_2$ as in Case 1.

Iterate this process. Since the valence of the points of $K$ is finite (see Remark 5.14), statement (ii) of Lemma 5.13 ensures that after a finite number of iterations, Case 2 ceases to occur.

Finally, the arcs $\gamma_n$ are pairwise disjoint. Indeed, $\gamma_0 \subseteq K_0^{\geq 1}$ and $\gamma_i \subseteq K_0^{\geq 2}$ for all $i \in \mathbb{N}$. (In other words, one can perform a sequence of “Rips moves” successively based on $\gamma_0$, $\gamma_1$, $\gamma_2$, and so on.) This collection contradicts Proposition 5.9. \[\blacksquare\]
By [CH14, Proposition 4.3], if \( S_0 = (K_0, A_0) \) is a pseudo-surface system of partial isometries, then \( \text{vol}(K_0^{\geq 3}) = 0 \). As shown in [CH14, Proposition 5.14], a free mixing \( F_n \)-tree \( T \) is either pseudo-surface—in which case \( \text{vol}(K_0^{\geq 3}) = 0 \) by [CH14, Proposition 4.3]—or Levitt type—in which case \( \text{vol}(K_i^{\geq 3}) > 0 \) for all \( i \in \mathbb{N} \) by Theorem 5.15. Combining these two propositions yields the following corollary.

**Corollary 5.16:** Let \( T \) be a free mixing \( F_n \)-tree and let \( A \) be a basis of \( F_n \). Let \( S_0 = (K_0, A_0) \) be the associated system of partial isometries, and let \( S_i = (K_i, A_i) \) denote the output after the \( i^{th} \) iteration of the Rips Machine. Then:

- \( T \) is a pseudo-surface if and only if \( \text{vol}(K_i^{\geq 3}) = 0 \) for some \( i \in \mathbb{N} \),
- \( T \) is Levitt type if and only if \( \text{vol}(K_i^{\geq 3}) > 0 \) for all \( i \in \mathbb{N} \).

**6. The directional Whitehead graph**

In this section, we introduce a tool called the directional Whitehead graph to study asymptotic relations between singular leaves. We define the graph in two ways: first, using the \( Q \)-map and its dual lamination, and second, using the band complex associated to a system of partial isometries and its dual lamination. For fully irreducible outer automorphisms, there is a well-studied correspondence between these objects, and we prove in Lemma 6.5 that these two definitions agree in this setting. Theorem 1.1 hinges on a certain property of the directional Whitehead graphs of \( T_\phi \).

**6.1. Directional Whitehead graph.**

**Definition 6.1:** Let \( T \) be an \( \mathbb{R} \)-tree with a very small, minimal action of \( F_n \) by isometries and dense orbits, and let \( Q : \partial F_n \to T \) be the \( Q \)-map given in Theorem 2.2. Let \( L(T) \) be the lamination dual to \( T \). Let \( x \in T \) be a branch point, and let \( d \) be a component of \( T \setminus \{x\} \). A **\( d \)-leaf** is a leaf \( \ell = (Y, Y') \in L(T) \) such that \( Q(\ell) = x \) and for which there exists a sequence \( \{\ell_i\}_{i=1}^{\infty} \subset L(T) \) limiting to \( \ell \) and so that \( Q(\ell_i) \in d \) for all \( i \). A vertex in the **directional Whitehead graph of** \( T \) at \( d \), denoted \( \text{Wh}_T(x,d) \), is an end of a \( d \)-leaf. The edges of \( \text{Wh}_T(x,d) \) correspond to the \( d \)-leaves.

Some questions immediately arise.
Question 6.2:  
(1) Is the directional Whitehead graph computable?  
(2) Can a directional Whitehead graph be empty?  
(3) Can a directional Whitehead graph be disconnected?  
(4) Can a leaf $\ell \in L(T)$ be a $d$-leaf and a $d'$-leaf for distinct directions $d$ and $d'$ at $x$? That is, suppose $\phi$ is a fully irreducible outer automorphism and let $T_-$ be its repelling tree. Let $x \in T_-$ be a branch-point. Then $Q_-(x)$ is a component, $C(x)$, of the ideal Whitehead graph of $\phi$. Definition 6.1 implies that for each direction $d$ at $x$, there is an injection  
$$h_d: \text{Wh}_T(x,d) \rightarrow C(x).$$  
Let $\ell \in C(x)$ be some leaf. By Theorem 2.17 there exists a direction $d$ such that $\ell \in \text{Im}(h_d)$. Moreover, a priori, it seems possible that $\ell$ is in the image of $h_d$ for more than one $d$. Is this in fact possible?  

The second definition of the directional Whitehead graph is given in terms of systems of partial isometries. The second definition is used in Theorem 8.6.  

**Definition 6.3:** Let $(K,A)$ be a system of partial isometries, and let $\mathcal{B}$ be the corresponding band complex. Let $x \in K$, and let $d$ be a direction of $K$ at $x$. The **directional Whitehead graph of $d$ with respect to $(K,A)$**, denoted $\text{Wh}_{(K,A)}(x,d)$, is defined as follows. The vertex set is a subset of the ends of the lamination (Definition 3.1) as follows. There is an edge $(\xi,\eta) \in \text{Wh}_{(K,A)}(x,d)$ if there exists a leaf $(\xi,\eta) \in L(\mathcal{B})$ based at $x$ and sequences of leaves $\{(\xi_i,\eta_i)\}_{i \in \mathbb{N}}$ based at $x_i \in d$ for all $i$ and so that  
$$\xi_i \rightarrow \xi, \quad \eta_i \rightarrow \eta.$$  
See Figure 6.1 for an illustration.  

**Remark 6.4:** In the above definition, $\lim_{i \to \infty} x_i = x$ since $Q$ is continuous with respect to the observers’ topology by Theorem 2.2.  

**Lemma 6.5:** Let $T$ be a very small, minimal $F_n$-tree with dense orbits. For every $x \in T$ and direction $d$ at $x$ there exists a compact subtree $K \subset T$ and a reduced system of partial isometries $S = (K,A)$ so that $x \in K$, the subtree $K$ contains a germ in the direction $d$, and  
$$\text{Wh}_T(x,d) = \text{Wh}_{(K,A)}(x,d).$$
Figure 6.1. On the left are local pictures of band complexes $\mathcal{B}$ and $\mathcal{B}'$; the colored lines represent leaves of the lamination. To the right of each complex is drawn $\text{Wh}(x)$, the ideal Whitehead graph at $x$ (see Theorem 1.8), and $\text{Wh}(x, d_i)$, the directional Whitehead graph at $x$ in direction $d_i$. Note that if the bands overlap, then a directional Whitehead graph may contain more than one edge.

Proof. Let $e$ be an edge of $\text{Wh}_T(x,d)$. By Definition 6.1, there exists a leaf $\ell \in L(T)$ corresponding to $e$ so that $Q(\ell) = x$, and there exists a sequence $\{\ell_i\}_{i=1}^{\infty} \subset L(T)$ so that $Q(\ell_i) = x_i \in d$ and $\lim_{i \to \infty} \ell_i = \ell$. Let $A$ be a basis of $F_n$ so that $\ell \in C_A$ (a translation of the original basis will do the trick). Since $C_A$ is an open neighborhood of $\ell \in \partial^2 F_n$ there exists an $N \in \mathbb{N}$ so that for all $i > N$, $\ell_i \in C_A$. We truncate the first $N$ elements of the sequence $\{\ell_i\}$. Thus, $x = Q_-(\ell) \in \Omega_A$ and $x_i = Q_-(\ell_i) \in \Omega_A$. The space $K_A$ is the convex hull of $\Omega_A$ in $\mathcal{T}$, so $\{x_i\}_{i=1}^{\infty} \cup \{x\} \subset K_A$. Hence, $K_A$ contains $x$ and the germ corresponding to the direction $d$. By Theorem 4.8, $L(T) \cap C_A = L(\mathcal{B})$. Therefore, $\ell_i \in L(\mathcal{B})$ for all $i$, and converges to $\ell \in L(\mathcal{B})$. Hence, there is a corresponding edge in $\text{Wh}_{(K,A)}(x,d)$. 
As for the other direction, an edge $e$ in $\text{Wh}_{(K,A)}(x,d)$ corresponds to a leaf $\ell \in L(\mathcal{B}) = C_A \cap L(T)$ based at $x$ such that there exist leaves $\ell_i \in L(\mathcal{B})$ based at $x_i \in d$ so that $\lim_{i \to \infty} \ell_i = \ell$. Thus, $Q(\ell_i) = x_i$, $Q(\ell) = x$, and hence $\ell$ corresponds to an edge of $\text{Wh}_T(x,d)$.

7. Cannon–Thurston maps

Generalizing work of Cannon and Thurston [CT07], Mitra (Mj) [Mit98] proved the existence of a Cannon–Thurston map for any hyperbolic normal subgroup of a hyperbolic group. The structure of Cannon–Thurston maps for hyperbolic free-by-cyclic groups which do not virtually split over $\mathbb{Z}$ was investigated by Kapovich–Lustig [KL15]. We will use the following results.

**Theorem 7.1** ([Mit98]): Let $G_{\phi} = F_n \rtimes_{\phi} \mathbb{Z}$ be a hyperbolic group. There exists a continuous surjection $\hat{\iota} : \partial F_n \to \partial G_{\phi}$. The map $\hat{\iota}$ is the Cannon–Thurston map.

**Proposition 7.2** ([KL15, Proposition 4.8, Lemma 4.9]): The Cannon–Thurston map $\hat{\iota} : \partial F_n \to \partial G_{\phi}$ factors through the maps $Q_+ : \partial F_n \to \hat{T}_+$ and $Q_- : \partial F_n \to \hat{T}_-$, and thus induces well-defined maps $R_+ : \hat{T}_+ \to \partial G_{\phi}$ and $R_- : \hat{T}_- \to \partial G_{\phi}$ which are surjective and $F_n$-equivariant. Furthermore:

(a) $R_+(\overline{T}_+) \cap R_+(\hat{T}_+ \setminus \overline{T}_+) = \emptyset$. Likewise, $R_-((\hat{T}_- \setminus \overline{T}_-)) = \emptyset$.

(b) $R_+(\overline{T}_+) \cap R_-((\hat{T}_- \setminus \overline{T}_-)) = \emptyset$.

(c) The restriction $R_+|_{\overline{T}_+}$ of $R_+$ to the metric completion of $T_+$ is injective. Likewise, the restriction $R_-|_{\overline{T}_-}$ of $R_-$ to the metric completion of $T_-$ is injective.
8. The $T_{\pm}$-pattern and $K_{3,3}$ subcomplexes

Notation 8.1: Throughout this section, suppose that $\phi \in \text{Out}(F_n)$ is fully irreducible, and let $T_+$ and $T_-$ denote the attracting and repelling trees for $\phi$, respectively.

An illustration of the following definition appears in Figure 8.1.

![Figure 8.1](image)

**Figure 8.1.** The $T_{\pm}$-pattern. The right side of the figure is contained in $\overline{T_-}$. The configuration in $\hat{T}_+$ is either like the left side or the middle of the figure depending on whether $(\ell_1^+ \cap \ell_b^+) \cup (\ell_2^+ \cap \ell_c^+) \subset T_+$ in Definition 8.2 has 3 or 4 prongs.

**Definition 8.2:** The outer automorphism $\phi$ satisfies the $T_{\pm}$-pattern if the following holds. There exists a point $a \in T_-$ such that $|Q_-^1(a)| \geq 3$, and there exists a direction $d$ at $a$ containing two points $b, c \in d$ with

$$|Q_-^1(b)| = |Q_-^1(c)| = 2,$$

which have the following properties. There exist leaves $\ell_1, \ell_2, \ell_b, \ell_c \in L(T_-)$ such that

$$Q_-(\ell_1) = Q_-(\ell_2) = a, \quad Q_-(\ell_b) = b, \quad Q_-(\ell_c) = c,$$

and $(\ell_1^+ \cap \ell_b^+) \cup (\ell_2^+ \cap \ell_c^+) \subset T_+$ is a star with at least three prongs and with midpoint $y \in \text{Int}(\ell_1^+ \cap \ell_b^+) \cap \text{Int}(\ell_2^+ \cap \ell_c^+)$. 

**Remark 8.3:** Suppose $\phi \in \text{Out}(F_n)$ can be represented by an automorphism of $F_n$ which is induced by a pseudo-Anosov homeomorphism of a surface with negative Euler characteristic and non-empty boundary. Then, $\phi$ does not satisfy the $T_{\pm}$-pattern. Indeed, the leaves of the attracting and repelling laminations can be realized as (non-crossing) embedded lines in $\mathbb{H}^2$ with dual trees $\hat{T}_+$ and $\hat{T}_-$, respectively. In this case, a direction at a point $a \in T_-$ corresponds to a half-space in $\mathbb{H}^2$. We leave the details to the reader.
**Proposition 8.4:** If $\phi$ satisfies the $T_{\pm}$-pattern (or the $T_\mp$-pattern), then there exists an embedding of $K_{3,3}$ in $\partial G_\phi$, where $K_{3,3}$ denotes the complete bipartite graph with two vertex sets of size three.

Figure 8.2. An embedding of $K_{3,3}$ in $\partial G_\phi$ built using the $T_{\pm}$-pattern. The points $\pi_i$ and $\Upsilon$ are the images of $p_i$ and $y$ under the map $\mathcal{R}_+$, respectively. In addition, $\alpha = \mathcal{R}_+(A_i) = \mathcal{R}_-(a)$, $\beta = \mathcal{R}_+(B_i) = \mathcal{R}_-(b)$, and $\gamma = \mathcal{R}_+(C_i) = \mathcal{R}_-(c)$.

**Proof.** Suppose $\phi$ satisfies the $T_{\pm}$-pattern. An embedding of $K_{3,3}$ in $\partial G_\phi$ is illustrated in Figure 8.2 and is described as follows. Recall from Section 7 that the Cannon–Thurston map $\hat{\iota} : \partial F_n \to \partial G_\phi$ factors through the maps

$$ Q_+ : \partial F_n \to \hat{T}_+ \quad \text{and} \quad Q_- : \partial F_n \to \partial \hat{T}_-. $$

The induced maps $\mathcal{R}_+ : \hat{T}_+ \to \partial G_\phi$ and $\mathcal{R}_- : \hat{T}_- \to \partial G_\phi$ are embeddings restricted to $\text{Int}(T_+)$ and $\text{Int}(T_-)$, and the images of $\text{Int}(T_+)$ and $\text{Int}(T_-)$ in $\partial G_\phi$ are disjoint. Thus, the images of the interiors of the colored paths in $\hat{T}_+$ drawn in Figure 8.1 do not intersect in $\partial G_\phi$. The endpoints of these paths on $\partial T_+$ are identified, via $\mathcal{R}_+$ and $\mathcal{R}_-$, to points in the interior of $T_-$ to form a $K_{3,3}$. ■
Theorem 8.5: An outer automorphism $\phi$ satisfies the $T_{\pm}$-pattern if and only if there exists a point $a \in T_-$ and a direction $d$ at $a$ such that $\text{Wh}_{T_-}(a, d)$ contains at least three vertices.

Proof. Suppose there exist a point $a \in T_-$ and a direction $d$ at $a$ so that $\text{Wh}_{T_-}(a, d)$ contains three vertices, corresponding to points $\alpha_1, \alpha_2, \alpha_3 \in \partial F_n$. By definition, $Q_-(\alpha_i) = a$ and each $\alpha_i$ is an end of a leaf of the lamination $L(T_-)$. Thus, there exist two leaves $\ell_1 \neq \ell_2 \in L(T_-)$ so that

$$\{\alpha_1, \alpha_2, \alpha_3\} \subset (\ell_1 \cup \ell_2).$$

By the definition of the directional Whitehead graph, there exist sequences $\{\sigma_k\}_{k=1}^\infty, \{\rho_k\}_{k=1}^\infty \subset L(T_-)$ such that $\lim_{k \to \infty} \sigma_k = \ell_1, \lim_{k \to \infty} \rho_k = \ell_2$, and $Q_-(\sigma_k), Q_-(\rho_k) \in d$ for all $k$. By Proposition 3.9, the realization in $T_+$ of the leaves corresponding to edges of a Whitehead graph is a star. Therefore, $\ell_1^+ \cap \ell_2^+$ is either a point $y$ or a ray initiating at a point $y$. Since $\lim_{k \to \infty} \sigma_k = \ell_1$, by Proposition 3.7 $\sigma_k^+ \cap \ell_1^+$ is a non-trivial segment containing $y$ in its interior for a large enough $k$. Likewise, for large enough $k$, $\rho_k^+ \cap \ell_2^+$ is a non-trivial segment containing $y$ in its interior. Set $\ell_a = \sigma_k$ and $\ell_b = \rho_k$ for large enough $k$ so that $(\ell_1^+ \cap \ell_b^+) \cup (\ell_2^+ \cap \ell_c^+)$ is a star with $y$ its midpoint. Therefore, $\phi$ satisfies the $T_{\pm}$-pattern, concluding one direction of the proof.

Conversely, suppose $\phi$ satisfies the $T_{\pm}$-pattern. We will show $\ell_1$ and $\ell_2$ yield distinct edges of $\text{Wh}_{T_-}(a, d)$. By Lemma 2.21, there exists a homothety

$$h_+: T_+ \to T_+$$

that fixes the middle $y$ of the star $(\ell_1^+ \cap \ell_b^+) \cup (\ell_2^+ \cap \ell_c^+)$ in $T_+$. The homothety $h_+$ corresponds to a principal automorphism $\Phi$ representing $\phi$. We replace $\phi$ by a power so that $h_+$ fixes all directions at $y$ and $\partial \Phi(\ell_j) = \ell_j$ for $j = 1, 2$. Let $h_-$ be the corresponding homothety of $T_-$ guaranteed by Lemma 2.21. By Lemma 2.25,

$$h_-(a) = h_-(Q_-(\ell_1)) = Q_-(\Phi(\ell_1)) = Q_-(\ell_1) = a,$$

where $a \in T_-$ is the point given in the definition of the $T_{\pm}$-pattern. Thus, $h_-$ permutes the directions at $a$. We replace $\Phi, h_+, \text{and } h_-$ by powers so that $h_-$ fixes the directions at $a$. Since $h_+$ is a homothety fixing the directions at $y$,

$$\lim_{k \to \infty} h_+^k(\ell_1^+) = \ell_1^+.$$
By Proposition 3.7, $\lim_{k \to \infty} \Phi^k(\ell_b) = \ell_1$. Likewise, $\lim_{k \to \infty} \Phi^k(\ell_c) = \ell_2$. Recall that $b = Q_-(\ell_b)$ and $c = Q_-(\ell_c)$ belong to the same component $d$ of $T_- \setminus \{a\}$ by assumption. Since $h_-$ fixes the directions at $a$,

$$Q_-(\Phi^k(\ell_b)) = h_-(Q_-(\ell_b)) \in d$$

and likewise, $Q_-(\Phi^k(\ell_c)) \in d$. Therefore, $\ell_1$ and $\ell_2$ yield distinct edges in $\text{Wh}_{T_-(a,d)}$, concluding the proof.

**Theorem 8.6:** There exists a directional Whitehead graph of a point in $T_-$ with at least three vertices if and only if the Rips Machine for $T_-$ never halts.

**Proof.** Suppose the Rips Machine never halts, and let $(K_m, A_m)$ denote the output after the $m^{th}$ iteration of the Rips Machine. This means that the tree $T_-$ is not of surface type, and thus, Proposition 5.7 ensures that $T_-$ is of Levitt type. In particular, if $a_m \in A_m$ (for each $m \in \mathbb{N}$), then

$$\lim_{m \to \infty} \text{diam(dom}(a_m)) = 0.$$

By Theorem 5.15, for each $m$ there exists a compact non-trivial interval $I_m \subset K_m$ and three distinct elements $a_m, b_m, c_m \in A_m$ such that

$$I_m \subset \text{dom}(a_m) \cap \text{dom}(b_m) \cap \text{dom}(c_m).$$

We may choose $\{I_m\}$ to be nested because for each $m$, if an interval is contained in the domain of three partial isometries in $A_m$, then it is contained in the domain of three partial isometries in $A_{m-1}$. We may also choose the sequences of partial isometries so that for each $f = a, b, c$, $f_m|_{K_{m+1}} = f_{m+1}$. Thus, there exists a point $x$ with $x \in \bigcap_{m=1}^{\infty} I_m$. Moreover, because there are only finitely many directions at $x$, there exists at least one germ $d$ based at $x$, so that for each $m$, the interval $I_m$ contains a subsegment in the germ $d$.

To find an infinite sequence of leaves, let $x_m \neq x$ be an extremal point of $\text{dom}(a_{m+1})$ so that the segment $[x, x_m]$ is contained in the germ $d$. By Proposition 5.3, $x_m \in \Omega$, where $\Omega$ is the limit set of the system (see Definition 4.3). So, there exists a leaf $\sigma_m \in L(\mathcal{B}_-)$ with $Q_-(\sigma_m) = x_m$. In the alphabet $A_0, \sigma_m = \cdots a_0 \cdots$ for all $m \in \mathbb{N}$; see Notation 4.4. Since $L(\mathcal{B}_-)$ is compact, there is a subsequence of $\{\sigma_m\}_{m \in \mathbb{N}}$ that converges in $\partial^2 F_n$ to a leaf $\ell_1$. Moreover, $\text{diam(dom}(a_m)) \to 0$, so $\lim_{m \to \infty} x_m = x$. The map $Q_-$ is continuous with respect to the observers’ topology, so

$$Q_-(\ell_1) = \lim_{m \to \infty} Q_-(\sigma_m) = \lim_{m \to \infty} x_m = x.$$
Thus, there exists an edge $e$ in $\text{Wh}_K(x,d)$ corresponding to $\ell_1$. Since $x_m \in \text{dom}(a_m)$, in the alphabet $A_m$, $\sigma_m = \cdots a_m \cdots$. Thus, one half-leaf of $\ell_1$ is the limit of half-leaves which begin with $\{a_m\}_{m \in \mathbb{N}}$. Without loss of generality, assume that the other end of $\ell_1$ is the limit of half-leaves which begin with $\{b_m\}_{m \in \mathbb{N}}$. Perform the same construction with $\{c_m\}_{m \in \mathbb{N}}$ instead of $\{a_m\}_{m \in \mathbb{N}}$ to obtain a sequence $\{y_m\}_{m \in \mathbb{N}} \subset d$ and a sequence $\{\rho_m\}_{m \in \mathbb{N}} \subset \mathcal{L}(\mathcal{B}_-)$ so that $\rho_m = \cdots c_m \cdots$ in the alphabet $A_m$ and $\lim_{m \to \infty} \rho_m = \ell_2$. Then

$$Q_- (\ell_2) = x.$$ 

The leaf $\ell_2$ is the limit of half-leaves which begin with $\{c_m\}_{m \in \mathbb{N}}$, so $\ell_1 \neq \ell_2$. Therefore, there exists another edge $e' \neq e$ in $\text{Wh}_{(K,A)}(x,d)$ where $e'$ corresponds to $\ell_2$, and $e$ corresponds to $\ell_1$. By Lemma 6.5, $\text{Wh}_{T_-}(x,d)$ contains two edges as well.

Conversely, suppose $\text{Wh}_{T_-}(x,d)$ contains three vertices. By Lemma 6.5, there exists a system of partial isometries $(K,A)$ so that $\text{Wh}_{(K,A)}(x,d)$ contains three vertices. So, $\text{Wh}_{(K,A)}(x,d)$ contains two edges corresponding to leaves $\ell_1, \ell_2$ in $L(\mathcal{B}_-)$. By definition, there exist sequences $\{\sigma_k\}_{k \in \mathbb{N}}, \{\rho_k\}_{k \in \mathbb{N}} \subset \mathcal{L}(\mathcal{B}_-)$ so that $\lim_{k \to \infty} \sigma_k = \ell_1, \lim_{k \to \infty} \rho_k = \ell_2$, and $Q_- (\sigma_k), Q_- (\rho_k) \in d$. For each $m \in \mathbb{N}$, there are two distinct pairs $(a, b), (c, f)$ of partial isometries $a, b, c, f \in A_m$ so that in the alphabet of $A_m$, $\ell_1 = \cdots b^{-1}a \cdots$ and $\ell_2 = \cdots f^{-1}c \cdots$. (Note that the pairs may share one element.) Therefore,

$$x \in \text{dom}(a) \cap \text{dom}(b) \cap \text{dom}(c) \cap \text{dom}(f).$$

Let $k$ be large enough so that in the alphabet $A_m$,

$$\sigma_k = \cdots b^{-1}a \cdots \text{ and } \rho_k = \cdots f^{-1}c \cdots.$$ 

Thus,

$$Q_- (\sigma_k) \in \text{dom}(a) \cap \text{dom}(b) \text{ and } Q(\rho_k) \in \text{dom}(c) \cap \text{dom}(f).$$

Since the domain of a partial isometry is convex, the arc $[Q_- (\sigma_k), x]$ is contained in $\text{dom}(a) \cap \text{dom}(b)$, and the arc $[Q_- (\rho_k), x]$ is contained in $\text{dom}(c) \cap \text{dom}(f)$. Hence, the domains of $a, b, c, f$ intersect in a non-trivial arc (in the direction $d$). By [CH14, Proposition 4.3], there exists a free band, that is, an interval $J \in K$ and a unique partial isometry $g \in A_m$ so that $J \subset \text{dom}(g)$. Thus, for all $m \in \mathbb{N}$ the Rips Machine does not stop at the $m$-th step. □
9. The boundary $\partial G_\phi$

**Theorem 9.1:** If $G_\phi$ is a hyperbolic group, then $\partial G_\phi$ contains a copy of the complete bipartite graph $K_{3,3}$.

**Proof.** If $\phi$ is fully irreducible, let $T_+$ and $T_-$ be the attracting and repelling trees of $\phi$ in the boundary of $CV_n$. By Proposition 5.7, either $T_-$ or $T_+$ has Levitt type. Without loss of generality, suppose $T_-$ has Levitt type. By Theorem 8.6, some directional Whitehead graph of $T_-$ contains more than one edge. By Theorem 8.5, $\phi$ satisfies the $T_{\pm}$ pattern; hence, by Proposition 8.4, the boundary $\partial G_\phi$ contains an embedded copy of $K_{3,3}$.

If $\phi$ is not fully irreducible, then there exists a free factor $A_0 < F_n$ and a power $\phi^k$ so that $[A_0]$, the conjugacy class of $A_0$ in $F_n$, is $\phi^k$-invariant. Let $A$ be a minimal such factor. Since $G_\phi$ is hyperbolic, $\phi$ is atoroidal, hence $\text{rank}(A) \geq 2$. Moreover, $\phi$ induces a well-defined outer automorphism $\phi'$ on $A$ [HM13, Fact 1.4]. Since $\phi$ is atoroidal, so is $\phi'$, and hence $\text{rank}(A) \geq 3$. By the minimality of $A$, the outer automorphism $\phi'$ is fully irreducible. Therefore, the boundary of

$$G' = A \rtimes_{\phi'} \mathbb{Z}$$

contains an embedded copy of $K_{3,3}$ by the previous paragraph. The subgroup $G'$ quasi-isometrically embeds in $G_{\phi^k}$.

Thus, $\partial G'$ embeds into $\partial G_{\phi^k}$. The group $G_{\phi^k}$ is a finite-index subgroup of $G_\phi$, so the boundaries of $G_\phi$ and $G_{\phi^k}$ are homeomorphic. □

**Remark 9.2:** Since the visual boundary of a hyperbolic group is locally connected [Bow99, Swa96], the existence of a non-planar graph in the boundary is equivalent to non-planarity of the boundary by a theorem of Claytor [Cla34]. Kuratowski [Kur30] proved that a graph is non-planar if and only if it admits an embedding of the complete bipartite graph $K_{3,3}$ or the complete graph $K_5$.

In what follows, we describe the work of Kapovich–Kleiner [KK00] relevant to this paper.

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1 This follows from the arguments in [Mit98, Section 3]. Indeed, $G'$ is a retract of $G$. If one takes a “ladder” over all of $A$, Mitra’s arguments show that there is a Lipshitz retraction of $G$ to this ladder. Therefore the ladder is quasi-convex in $G$, and it is also the universal cover of $G'$.
Theorem 9.3 ([KK00, Theorem 4]): Let $G$ be a hyperbolic group which does not split over a finite or virtually cyclic subgroup, and suppose $\partial G$ is one-dimensional. Then one of the following holds:

1. $\partial G$ is a Menger curve,
2. $\partial G$ is a Sierpinski carpet,
3. $\partial G$ is homeomorphic to $S^1$, and $G$ maps onto a Schwartz triangle group with finite kernel.

Theorem 9.3 quickly follows from a compilation of results: the characterization of the Menger curve [And58a, And58b], the characterization of the Sierpinski carpet [Why58], and the structure of local and global cut points in the boundary of a hyperbolic group [BM91, Bow98, Bow99, Swa96].

Corollary 9.4 ([KK00, Corollary 15]): Let $F$ be a finitely generated free group, and $\phi: F \to F$ an atoroidal automorphism. If no power of $\phi$ preserves a free splitting of $F$, then the Gromov boundary of $G_\phi := F \rtimes_\phi \mathbb{Z}$ is the Menger curve. In particular, if $\phi$ is fully irreducible, then $\partial G_\phi$ is the Menger curve.

Remark 9.5: The hypothesis of the above corollary in [KK00] states that $\phi$ is irreducible, not fully irreducible. However, in the proof they obtain a contradiction by producing a $\phi_j$-invariant free decomposition of $F$ for some positive $j > 0$. This property only contradicts that $\phi$ is fully irreducible.

To prove the above corollary, Kapovich–Kleiner [KK00] first show that if $\phi \in \text{Out}(F_n)$ is fully irreducible (or no power of $\phi$ preserves a free splitting of $F$), then $G_\phi$ does not split over a trivial or cyclic subgroup. (See also Brinkmann [Bri02].) Then, Theorem 9.3 implies $\partial G_\phi$ is either the circle, the Sierpinski carpet, or the Menger curve. At this point, using Theorem 1.1, one can directly rule out the circle and Sierpinski carpet, since these spaces are planar. Alternatively, Kapovich–Kleiner rule out the circle by applying the work of Tukia–Gabai–Casson–Jungreis [Tuk88, Gab92, CJ94], which classifies the hyperbolic groups with boundary homeomorphic to $S^1$ as precisely the groups that act discretely and cocompactly by isometries on the hyperbolic plane. Finally, Kapovich–Kleiner prove a rather deep result: if $G$ is a hyperbolic group with Sierpinski carpet boundary, then $G$ together with the stabilizers of the peripheral circles of the Sierpinski carpet forms a Poincaré duality pair (see [KK00, Corollary 12]). In particular, they conclude that in this case $\chi(G) < 0$. Since $\chi(G_\phi) = 0$, the boundary $\partial G_\phi$ is the Menger curve.
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