On critical behaviour in gravitational collapse

Viqar Husain*, Erik A. Martinez†, and Darío Núñez‡

*Department of Mathematics and Statistics,
University of Calgary, Calgary, Alberta, Canada T2N 4N1.

†Theoretical Physics Institute,
University of Alberta, Edmonton, Alberta, Canada T6G 2J1.

‡Instituto de Ciencias Nucleares, UNAM,
Circuito Exterior CU, A.P. 70-543, México, D. F. 04510, México.

Abstract

We give an approach to studying the critical behaviour that has been observed in numerical studies of gravitational collapse. These studies suggest, among other things, that black holes initially form with infinitesimal mass. We show generally how a black hole mass formula can be extracted from a transcendental equation.

Using our approach, we give an explicit one parameter set of metrics that are asymptotically flat and describe the collapse of apriori unspecified but physical matter fields. The black hole mass formula obtained from this metric exhibits a mass gap - that is, at the onset of black hole formation, the mass is finite and non-zero.

PACS numbers: 04.20.Jb

Typeset using REVTeX

*email: vhusain@math.ucalgary.ca, martinez@phys.ualberta.ca, nunez@xochitl.nuclecu.unam.mx
A fundamental problem in general relativity is the investigation of the gravitational collapse of matter fields. The main motivation for studying this is the cosmic censorship conjecture, one form of which states that gravitational collapse always results in a black hole.

Recently the collapse problem has been studied numerically and the results are intriguing. For the spherically symmetric collapse of a scalar field, Choptuik [1] has found that when the initial matter field is an ingoing pulse, the collapsing matter forms a black hole with mass given by \( M = K(c - c_*)^\gamma \), where \( K \) is a constant, \( c \) is any one of the parameters in the initial data for the matter field, \( c_* \) is the critical value of this parameter that gives a zero mass black hole, and \( \gamma \sim .37 \). In particular, no black hole is formed when \( c < c_* \). An important feature of this result is that it appears to be independent of spherical symmetry and the type of matter fields: the same mass formula for the black hole has been found for the axisymmetric collapse of gravitational waves by Abraham and Evans [2] and for the spherically symmetric collapse of radiation by Coleman and Evans [3].

Thus, these results appear to reflect a universal property of the Einstein equations in strong field regions. So far there is no analytical understanding of this result, nor is there an explicit metric that exhibits this behaviour.

There have however been a number of attempts at explanations, most of which involve self-similar solutions [4,5]. However, the particular solutions discussed do not appear to be relevant for the collapse problem with asymptotically flat boundary conditions, because they are cosmological, and the collapse results in infinite mass black holes; the radius of the apparent horizon tends to infinity with time. Another (non-self-similar) time dependent solution, given by the authors, has the same shortcoming [6]. The problem has been discussed in two dimensions where an exponent of .5 was reported [7]. There have also been proposals for using perturbations of black holes to calculate the exponent analytically for the super-critical case \( c > c_* \), for which the black hole always forms [8–11]. Recently, further numerical perspectives on this problem have been given [12,13].

In this Letter we present an alternative analytical approach to this problem. The first
part of the paper gives a general approach that may be used to extract a black hole mass formula from a spherically symmetric time dependent metric. The mass formula arises from the solutions of a transcendental equation. The second part of the paper gives an explicit example of the procedure: We give a metric that satisfies the dominant energy condition and describes a realistic collapse. This specific example suggests that at the onset of black hole formation, there may be a mass gap. We comment on how our approach may be used to get a black hole mass formula without a mass gap.

We start with the general spherically symmetric line element

\[ ds^2 = -f e^{-2\psi} dv^2 + 2 e^{-\psi} dv dr + r^2 d\Omega^2 \]  

(1)

where \( f(r, v) \) and \( \psi(r, v) \) are functions of the radial coordinate \( r \), \((0 \leq r \leq \infty)\), and the advanced time coordinate \( v \), \((0 \leq v \leq \infty)\), and \( d\Omega^2 \) is the line element of the unit two-sphere. The parametrization (1) has been used to study the collapse of null shells \[14\]. The mass function \( m(r, v) \) is defined by

\[ f(r, v) = (1 - 2m(r, v)/r), \]

(2)

and is a measure of the mass contained within radius \( r \). The Schwarzschild solution results from setting \( m(r, v) = \text{constant} \) and \( \psi(r, v) = 0 \). The mass function will in general be parametrized by initial data parameters \( c_i \). For simplicity we will assume dependence on only one parameter, \( c \).

The apparent horizon is the three-surface that separates regions containing trapped surfaces from the normal regions of the spacetime. This surface is defined by the equation

\[ g^{ab} \partial_a r \partial_b r = 0. \]

For the metric (1) it is given by

\[ m(r, v; c) = \frac{r}{2}. \]

(3)

We are interested in the asymptotic \( v \to \infty \) solutions of this equation because this gives the large time behaviour of the apparent horizon, and this is what has been investigated numerically \[1,2\]. Taking this limit results in the one parameter family of transcendental equations
\[ M(r; c) := \lim_{v \to \infty} m(r, v; c) = \frac{r}{2}. \] (4)

The solutions of this equation give the radial coordinate of the asymptotic \((v \to \infty)\) apparent horizon as a function of \(c\). In general there will be ranges of \(c\) for which there are no solutions to (4). This will give the subcritical region where no black hole forms. Similarly there will be critical and supercritical solutions.

In the numerical work cited above, the black hole mass is defined by the radius of this asymptotic apparent horizon. This is a reasonable definition because it is in this limit that the apparent horizon approaches the event horizon [15]. In our discussion, this mass is given explicitly by the solutions \(r_{AH}(c)\) of these transcendental equations, namely

\[ M_{BH}(c) := \frac{r_{AH}(c)}{2}. \] (5)

A plot of the solution of the equations (4) as a function of \(c\) will give \(M_{BH}(c)\). We emphasize that the steps outlined above are general in the sense that if one is given an exact collapse solution, this procedure may be used to see if critical behaviour exists in the long time limit.

Our purpose in this paper is to study what specific mass formulas can be obtained from (4).

The mass function in the metric (3) must satisfy certain physical conditions in order to define a realistic collapse. These are: (i) The metric should be asymptotically flat, (ii) the mass function should satisfy \(m \geq 0\) and \(\partial m/\partial r \geq 0\), and should give, in the \(v \to \infty\) limit, a set of transcendental equations from which we can obtain \(M_{BH}(c)\), and (iii) the mass function should increase with advanced time corresponding to an implosion of matter, at least initially. Also, if the mass function leads to multiple apparent horizons, the outermost one will serve to extract a mass formula.

We are interested mainly in the supercritical case, where a black hole always forms at the end point of collapse, since the main goal of our work is to extract a black hole mass formula. However, as we will see, the subcritical case (no black hole) also arises naturally.

The Einstein equations \(G_{ab} = 8\pi T_{ab}\) for the metric (3) are

\[ m' = -4\pi r^2 T_v^v, \] (6)
\[
m = 4\pi r^2 T_v^r, \tag{7}
\]
\[
\psi' = 4\pi r T_{rr}, \tag{8}
\]
where the prime and dot denote \(r\) and \(v\) derivatives. We will not fix the stress-energy tensor by specifying any specific type of matter, but will instead determine this tensor in terms of \(m\) and \(\psi\). The goal then will be to see if these functions can be fixed so that the energy conditions for realistic matter are satisfied. (We note that \(\dot{\psi}\) is not fixed by the Einstein equations).

As an example of the above procedure, we now construct one metric that describes a realistic collapse. This requires a mass function satisfying the above three physical conditions, with \(\psi\) still arbitrary. A suitable choice is
\[
m(r, v; c) = c [\tanh(e^c \ln r) + 1] \tanh v. \tag{9}
\]
The Arnowit-Deser-Misner (ADM) mass for this choice is \(2c\).

The one parameter (\(c\)) set of transcendental equations (4) resulting from this in the \(v \to \infty\) limit are
\[
c [\tanh(e^c \ln r) + 1] = \frac{r}{2}, \tag{10}
\]
which give, using (5), the equation for the black hole mass \(M_{BH}\):
\[
c (\tanh[2M_{BH})] + 1) = M_{BH}. \tag{11}
\]

Equation (11) may be solved numerically, and there are solutions for \(c \geq c_* = .465727\). In the supercritical region \(c > c_*\), the solutions are fit approximately by the equation
\[
M_{BH} = 0.64 + 20.69(c - c_*) - 2205.15(c - c_*)^2 + 113620(c - c_*)^3. \tag{12}
\]
A plot of \(M_{BH}\) vs. \((c - c_*)\) is given in Fig. 1. The solutions of the transcendental equations (11) and the fit (12) were found using Mathematica, and also checked with Maple. (The Mathematica commands we used for Fig. 1 are: \(tt := Table[\{c - .465727, (M)/. FindRoot[ c (Tanh[ Exp[c] Log[2M]] +1) == M, \{M, 0\}] \}, \{c, .465728,.475000, .0001\}] Fit[ tt, \{1, x, x^2, x^3\}, x \].)
There are several comments in order for the fit ([12]): (i) The mass formula is not a pure power law as observed in the numerical integrations of the scalar field and null fluid [1,3]. (ii) This particular example exhibits a mass gap, because at criticality Eqn. (11) gives \( M_{BH} = 0.64 \), as Fig. 1 clearly shows. (iii) For \( c < c_\ast \) there are no solutions of (11), and hence no black hole formation. (iv) For \( c > c_\ast \) there are two intersections of the mass function with the \( r/2 \) curve, and hence an inner and outer apparent horizon. The above fit corresponds to the outer horizon, which is what is relevant for determining black hole masses.

In summary, although this mass function example does not give the mass formula obtained in the full numerical integrations [1–3], it does give the subcritical, critical and supercritical regions. Furthermore, as we will see below, this mass function corresponds to matter satisfying the dominant energy conditions, and thus gives a physical asymptotically flat collapse solution. This suggests that there are physical solutions that exhibit a mass gap.

So far the metric function \( \psi(r, v) \), is arbitrary. The task now is to see how this function is restricted so that the metric arises from realistic matter. We would like to give at least one example of such a \( \psi(r, v) \) in order to have an explicit metric. To do this we first turn to a discussion of the energy conditions.

The energy conditions may be imposed most simply by first diagonalizing the stress-energy tensor, (determined in terms of \( m \) and \( \psi \)), by solving \( \mathcal{T}_{ab}u_b = \lambda u_a \). Diagonalizing the \( (v, r) \) part of \( \mathcal{T}_{ab} \) gives the eigenvalues \( \lambda_\pm \):

\[
4\pi r^2 \lambda_\pm = -m' + \frac{rf\psi'}{2} \left[ 1 \pm \sqrt{1 + \frac{4\dot{m}e^{-\psi}}{rf^2\psi'}} \right].
\]

(13)

The corresponding \( (v, r) \) components of the eigenvectors are

\[
u_a^{(\pm)} = (1, \frac{4\pi r^2 \lambda_\pm + m'}{\dot{m}}),
\]

(14)

and their norms are

\[
|u^{(\pm)}| = \left( \frac{4\pi r^2 \lambda_\pm + m'}{\dot{m}} \right) \left[ 2e^{-\psi} + f\left( \frac{4\pi r^2 \lambda_\pm + m'}{\dot{m}} \right) \right].
\]

(15)
(The \((\theta, \phi)\) parts of the stress-energy tensor are determined by \(G_{\theta\theta}\) and \(G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}\), and are already diagonal (since \(G_{\theta\phi} \equiv 0\). The corresponding eigenvectors are spacelike.)

Stress-energy tensors are classified by their eigenvectors, and for physical fields this tensor must be either Type I, for which there is one timelike and three spacelike eigenvectors, or Type II, for which there are two null and two spacelike eigenvectors \([15]\). All physical fields are of type I, except for certain null fluid flows, which are of type II.

We will focus on the type I tensors. Let \(-\rho\) be the eigenvalue corresponding to the timelike eigenvector, and \(\pi_i\) \((i = 1, 2, 3)\) the eigenvalues corresponding to the three spacelike eigenvectors. Then the weak energy condition, which requires that the energy density be non-negative, is

\[
\rho \geq 0, \quad \text{and} \quad \pi_i \geq -\rho \quad \text{for} \quad i = 1, 2, 3.
\]  

The dominant energy condition, which requires that energy flows are never spacelike, imposes in addition to (16), the condition

\[
\pi_i \leq \rho \quad \text{for} \quad i = 1, 2, 3.
\]  

From our chosen mass function \((4)\), we see that \(\dot{m} > 0\), which requires that \(\psi' > 0\) for the square root in the eigenvalues \((13)\) to be real. We note from \((13)\) and \((15)\) that \(4\dot{m}e^{-\psi}/r f^2 \psi' > 0\), (which is always true), implies that the eigenvector \(u^\psi\) is timelike as required. Also, \(u^{(+)}\) is always spacelike. Therefore, for the weak energy condition we need a \(\psi\) that satisfies \(\psi' > 0\). In addition, to preserve the asymptotic flatness of the metric we require

\[
\lim_{r \to \infty} \psi(r, v = \infty) = \text{constant}.
\]  

Since we already have \(\lambda_+ > \lambda_-\), for the weak energy condition we require, in addition, a \(\psi\) such that

\[
\lambda_- \leq 0,
\]  

and
\[ G_{\theta\theta} = (r + m)\psi' - 3rm'\psi' + r^2 f(\psi')^2 - r^2 e^{-\psi}\psi' - rm'' + r^2 f\psi'' \geq \lambda_. \]  \hspace{1cm} (20)

For the dominant energy condition we also require

\[ -\lambda_\pm \geq \lambda_\pm \quad \text{and} \quad -\lambda_\pm \geq G_{\theta\theta}, \]  \hspace{1cm} (21)

that is, all the pressures must be bounded between \( \rho \) and \( -\rho \).

A \( \psi \) that satisfies both the weak and dominant energy conditions is of the form

\[ \psi(r, v) = -e^{-rf(v)} - g(v), \]  \hspace{1cm} (22)

where the functions \( f \) and \( g \) are everywhere positive and have a lower bound > 2. This last condition is necessary to enforce the dominant energy condition, which was checked to be true numerically for \( r \geq \) horizon radius, and for subcritical and supercritical ranges of \( c \).

Our main result is a general method for obtaining a mass formula for black holes via solutions to a transcendental equation: given any explicit collapse solution, a black hole mass formula must arise in this way, (and it may or may not be a pure power law). As an explicit example of the procedure, we have given a metric that describes a physically realistic spherically symmetric collapse, and that exhibits a mass gap at the onset of black hole formation.

It would be of interest to see if there are other choices for the mass function that describe a realistic collapse. In particular, it would be of much interest to find the mass functions that give gapless black hole mass formulas.

An important question concerning critical behaviour in our approach is what features of the mass function are responsible for giving criticality. The question actually has at least two parts: What features give the three subcritical, critical, and supercritical regimes, and what features give rise to a pure power law, and hence a critical exponent? The answer to the first question is that any mass function that grows with the parameter in approximately the same way as ours is sufficient. We speculate that the answer to the second question requires an approximately step shaped mass function (like ours) such that its knee remains just touching the \( r/2 \) curve (Fig. 2) for a wide range of parameter values \( c \), (instead of
crossing it, and thereby giving an inner horizon also, as ours does). In our approach the
former choice of mass function, (that just touches the $r/2$ curve), requires ‘fine tuning’
whereas the second choice appears to be more generic. A deeper question is what properties
of the Einstein equations in this context give rise to mass functions having these features.

A further point to note is that critical behaviour may occur even for unrealistic matter:
we simply leave $\psi$ arbitrary, up to requiring asymptotic flatness of the metric, and not worry
about imposing the energy conditions.

It seems possible to use this approach for axially-symmetric metrics as well; one could
for example use the mass function given here, and still have two remaining metric functions
at hand for satisfying the dominant energy condition.

We would like to thank Werner Israel, Ted Jacobson, Karel Kuchar, Don Page, Richard
Price, and Lee Smolin for very helpful comments. The work of V. H. and E. M. was
supported by the Natural Science and Engineering Research Council of Canada, and that
of D. N. partly by DGAPA, National A. University of México.
REFERENCES

[1] M. Choptuik, Phys. Rev. Lett. 70, 9 (1993).

[2] A. Abraham and C. R. Evans, Phys. Rev. Lett. 70, 2980 (1993).

[3] J. Coleman and C. R. Evans, Phys. Rev. Lett. 72, 1842 (1994).

[4] Y. Oshiro, K. Nakamura, and A. Tomimatsu, preprint gr-qc/9402017 (1994).

[5] P. R. Brady, Class. Quantum Grav. 11, 1255 (1994).

[6] V. Husain, E. A. Martinez and D. Nunez, Phys. Rev. D 50, 3783 (1994).

[7] A. Strominger and L. Thorlacius, Phys. Rev. Lett. 72, 1584 (1994).

[8] J. Traschen, preprint gr-qc/9403016 (1994).

[9] J. Pullin, preprint gr-qc/9409044 (1994).

[10] T. Koike, T. Hara, and S. Adachi, preprint gr-qc/9503007 (1995).

[11] D. Maison, preprint gr-qc/9504008 (1995).

[12] D. Garfinkle, preprint gr-qc/9412008 (1994).

[13] E. W. Hirschmann and D. M. Eardley, gr-qc/9412066 (1994).

[14] C. Barrabes and W. Israel, Phys. Rev. D 43, 1129 (1991).

[15] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, (Cambridge University Press, 1973).

[16] T. Jacobson, private communication.
FIGURE CAPTIONS

Figure 1. $M_{BH}$ vs. $(c - c_*)$ curve whose best fit is Eqn. (12).

Figure 2. The critical mass function ($c = .46573$) together with a typical subcritical case ($c = .3$). The other curve is the function $r/2$ which touches the critical mass function.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9505024v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9505024v1