EMBEDDING ALMOST-COMPLEX MANIFOLDS IN ALMOST-COMPLEX EUCLIDEAN SPACES

ANTONIO J. DI SCALA AND DANIELE ZUDDAS

Abstract. We show that any compact almost-complex manifold \((M, J)\) of complex dimension \(m\) can be pseudo-holomorphically embedded in \(\mathbb{R}^{6m}\) equipped with a suitable almost-complex structure \(\tilde{J}\).

Keywords: embedding, almost-complex structure, manifold, pseudo-holomorphic embedding.

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1. Introduction

An almost-complex structure on a \(2n\)-dimensional smooth manifold \(M\) is a tensor \(J \in \text{End}(TM)\) such that \(J^2 = -\text{id}\). If \(M\) is oriented we say that \(J\) is positive if the orientation induced by \(J\) on \(M\) agrees with the given one. An almost-complex structure is called integrable if it is induced by a holomorphic atlas. In dimension two any almost-complex structure is integrable, while in higher dimension this is far from true. A smooth map \(f: N \rightarrow M\) between two almost-complex manifolds \((N, J')\), \((M, J)\) is called pseudo-holomorphic if \(J \circ T f = T f \circ J'\), where \(T f: TN \rightarrow TM\) is the tangent map of \(f\). When the map \(f\) is an embedding, \((N, J')\) is said to be an almost-complex submanifold of \((M, J)\). In this case we can identify \(N\) with its image \(f(N) \subset M\) and the almost-complex structure \(J'\) with the restriction of \(J\) to \(TN \cong T(f(N)) \subset TM\).

If we equip \(\mathbb{R}^{2n}\) with the canonical complex structure, that is to say \(\mathbb{R}^{2n} \cong \mathbb{C}^n\), then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds pseudo-holomorphically embedded in \(\mathbb{R}^{2n}\) equipped with an integrable or non-integrable almost-complex structure.

In [2] Calabi and Eckmann constructed the first examples of compact, simply connected complex manifolds \(M_{p,q}\) which are not algebraic. Topologically \(M_{p,q}\) is the product \(S^{2p+1} \times S^{2q+1}\). Then by deleting a point on each factor one obtains a complex structure \(J\) on \(\mathbb{R}^{2p+2q+2}\). In section 5 of [2] it was shown that when \(p, q > 1\) there exists a complex torus as a complex submanifold of \((\mathbb{R}^{2p+2q+2}, J)\) [2, p. 499]. It follows that the Calabi-Eckmann complex structure \(J\) on \(\mathbb{R}^{2n}\) cannot be tamed by any symplectic form and in particular cannot be Kähler. Calabi and Eckmann also observed that the only holomorphic functions on \((\mathbb{R}^{2p+2q+2}, J)\) are the constants answering negatively to a question raised by Bochner about the uniformization of complex structures on \(\mathbb{R}^{2n}\).

In [1] Bryant constructed pseudo-holomorphic non-constant maps \(\varphi: M^2 \rightarrow S^6\) for any compact Riemann surface \(M^2\), where \(S^6\) is equipped with the almost-complex structure induced by

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the octonion multiplication. These maps realize compact Riemann surfaces as pseudo-holomorphic singular curves in \( S^6 \).

In [3] it was shown that any almost-complex torus \( \mathbb{T}^n = \mathbb{R}^{2n}/\Lambda \) can be pseudo-holomorphically embedded into \((\mathbb{R}^{4n}, J_\Lambda)\) for a suitable almost-complex structure \( J_\Lambda \). It follows that any compact Riemann surface can be realized as a pseudo-holomorphic curve of some \((\mathbb{R}^{2n}, J)\), where \( J \) is a suitable almost-complex structure.

In this paper we prove the following general theorem.

**Theorem 1.** Any compact almost-complex manifold \((M, J)\) of real dimension \(2m\) can be pseudo-holomorphically embedded in \((\mathbb{R}^{6m}, \tilde{J})\) for a suitable positive almost-complex structure \( \tilde{J} \).

In particular, any compact Riemann surface can be realized as a pseudo-holomorphic curve in \((\mathbb{R}^4, \tilde{J})\). In [3] was shown that the torus is the only compact Riemann surface that can be pseudo-holomorphically embedded in \((\mathbb{R}^4, \tilde{J})\) for some \( \tilde{J} \).

2. Preliminaries

The space of positive linear complex structures on \( \mathbb{R}^{2n} \) is diffeomorphic to the homogeneous space \( \tilde{\mathfrak{J}}(n) = GL^+(2n, \mathbb{R})/GL(n, \mathbb{C}) \) and is homotopy equivalent to \( \mathfrak{J}(n) = SO(2n)/U(n) \). So, an almost-complex structure \( J \) on \( \mathbb{R}^{2n} \) can be regarded as a smooth map \( J : \mathbb{R}^{2n} \to \tilde{\mathfrak{J}}(n) \).

**Lemma 2.** Let \( M \subset \mathbb{R}^{2n} \) be a closed submanifold and let \( J : M \to \tilde{\mathfrak{J}}(n) \) be a smooth map. Then there exists a smooth extension \( \tilde{J} : \mathbb{R}^{2n} \to \tilde{\mathfrak{J}}(n) \) if and only if \( J \) is homotopic to a constant.

**Proof.** The ‘only if’ part follows immediately from the fact that \( \mathbb{R}^{2n} \) is contractible.

Let us prove the ‘if’ part. Consider a smooth homotopy \( H : M \times [0, 1] \to \tilde{\mathfrak{J}}(2n) \) such that \( H_0(x) = J_0 \) for all \( x \in M \), and \( H_1 = J \) where \( H_t(x) = H(x, t) \) and \( J_0 \in \tilde{\mathfrak{J}}(n) \). We can extend \( H \) to \( \mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n} \times [0, 1] \) by setting \( H(x, 0) = J_0 \) for any \( x \in \mathbb{R}^{2n} \). By the homotopy extension property [4, Chapter 0] there exists \( \tilde{H} : \mathbb{R}^{2n} \times [0, 1] \to \tilde{\mathfrak{J}}(n) \) which extends \( H \). We conclude the proof by setting \( \tilde{J} = \tilde{H}_1 \).

Let \((M, J)\) be an almost-complex manifold. The strategy to prove Theorem 1 will be to choose an arbitrary embedding \( f : M \hookrightarrow \mathbb{R}^{6m} \), which exists for the weak Whitney embedding theorem, and to show that \( J \) extends to the pullback \( f^*(T\mathbb{R}^{6m}) \) and this extension is null-homotopic.

Consider the standard filtration \( SO(1) \subset SO(2) \subset \cdots \). Since \( SO(n-1) \) contains the \((n-2)\)-skeleton of \( SO(n) \) (because the standard fibration \( SO(n) \to S^{n-1} \)) it follows that the \( k \)-skeleton of \( SO(n) \) is contained on \( SO(k+1) \) for \( 0 \leq k \leq n-2 \).

Since \( SO(n) \subset U(n) \) it follows that \( U(n) \) contains the \((n-1)\)-skeleton of \( SO(2n) \) for \( n \geq 1 \). Then the homomorphism induced by the inclusion \( i_* : \pi_j(U(n)) \to \pi_j(SO(2n)) \) is an isomorphism for \( j \leq n-2 \) and is an epimorphism for \( j = n-1 \).

From the homotopy exact sequence of the fibre bundle \( SO(2n) \to \mathfrak{J}(n) \) given by the projection map it follows that \( \pi_j(\tilde{\mathfrak{J}}(n)) \cong \pi_j(\mathfrak{J}(n)) \cong 0 \) for \( j \leq n-1 \).

**Definition 3.** A space \( X \) is said to be \( n \)-connected if \( \pi_j(X) \cong 0 \) for all \( j \leq n \).

In particular, 0-connected means path-connected.

From the above considerations we have that \( \tilde{\mathfrak{J}}(n) \) is \((n-1)\)-connected. The following proposition is well-known in the theory of CW-complexes.
Proposition 4. If $X$ is $n$-connected then any map $Y \to X$ defined on a CW-complex $Y$ of dimension $\leq n$ is homotopic to a constant.

Also the following proposition is standard, and we give only the idea of the proof.

Proposition 5. Let $\xi : E \to M$ be an oriented real vector bundle of rank $2k$ over an $m$-manifold $M$. If $k \geq m$ then $\xi$ admits a positive complex structure.

Proof. Consider the bundle $\xi^3 : \tilde{\mathfrak{J}}(E) \to M$ with fibre $\tilde{\mathfrak{J}}(k)$ induced by $\xi$. Namely, for any $p \in M$ the fibre of $\xi^3$ over $p$ is the space of positive linear complex structures on $\xi^{-1}(p)$. Since $\tilde{\mathfrak{J}}(k)$ is $(k - 1)$-connected, it follows that $\xi^3$ admits a section if $k \geq m$, see [7, Part III]. This section is a positive complex structure on $\xi$.

Let $f : M \to \mathbb{R}^N$ be an immersion. The normal bundle $\nu_f(M)$ is, as usual, the orthogonal complement of $TM$ in $f^*(T\mathbb{R}^N)$, that is to say:

$$f^*(T\mathbb{R}^N) = TM \oplus \nu_f(M).$$

If $M$ is oriented then the normal bundle can be equipped with a canonical orientation, namely that which makes the splitting of $f^*(T\mathbb{R}^N)$ into a Whitney sum of oriented fibre bundles, where $\mathbb{R}^N$ is considered with the standard orientation.

3. Proof of the main results

Theorem 6. Let $M \subset \mathbb{R}^{2n}$ be a submanifold of even dimension endowed with an almost-complex structure $J$. If the normal bundle of $M$ in $\mathbb{R}^{2n}$ admits a positive complex structure with respect to the canonical orientation, then for any $k \geq \max(0, \dim_{\mathbb{R}} M - n + 1)$ there exists an almost-complex structure $\tilde{J}$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ such that $M \times \{0\} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2k}$ is an almost-complex submanifold.

Proof. Let us choose a positive complex structure on the normal bundle of $M$. Then by taking the Whitney sum with the almost-complex structure on $M$ we get a complex structure on $(T\mathbb{R}^{2n})|_M$. So we obtain a smooth map $J : M \to \tilde{\mathfrak{J}}(n)$.

In view of Lemma 2 our target is to get a $J$ null-homotopic. This is so if $\dim_{\mathbb{R}} M \leq n - 1$ because $\tilde{\mathfrak{J}}(n)$ is $(n - 1)$-connected and Proposition 4.

If $\dim_{\mathbb{R}} M > n - 1$ we take the product $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$, where $\mathbb{R}^{2k}$ is endowed with the standard complex structure, and we embed $M$ as $M \times \{0\}$. We get a complex structure on the normal bundle of $M$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ in the obvious way. So we obtain a map $J_k : M \to \tilde{\mathfrak{J}}(n + k)$. It follows that $J_k$ is homotopic to a constant if $k \geq \dim_{\mathbb{R}} M - n + 1$. In this case $J_k$ extends on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ by Lemma 2.

It follows that if $(M, J)$ is contained in $\mathbb{C}^n$ with a complex normal bundle and if $n \geq 2 \dim_{\mathbb{C}} M + 1$, then there is a positive almost-complex structure $\tilde{J}$ on $\mathbb{C}^n$ which makes $(M, J)$ an almost-complex submanifold of $(\mathbb{C}^n, \tilde{J})$.

Proof of Theorem 1. Let $f : M \hookrightarrow \mathbb{R}^{6m}$ be any embedding. The normal bundle $\nu_f(M)$ has rank $4m$ and is orientable. By Proposition 5 there is a complex structure on the normal bundle and then we conclude by an application of Theorem 6 with $k = 0$.

In some cases we can construct an embedding in an euclidean space of lower dimension. Recall that an $s$-inverse of the tangent bundle $TM$ is a vector bundle $\xi$ such that $TM \oplus \xi$ is a trivial vector bundle. Observe that if $f : M \to \mathbb{R}^N$ is an immersion then the normal bundle $\nu_f(M)$ is a
real s-inverse of the tangent bundle $TM$. The converse also holds and is a Theorem of Hirsch [5], and is given as follows.

**Theorem 7.** (Hirsch [5]) Any s-inverse of $TM$ is the normal bundle of some immersion $f : M \to \mathbb{R}^N$.

Let $\xi$ be a complex s-inverse of $(TM, J)$ of complex rank $k$, namely $TM \oplus \xi$ is trivial as a real vector bundle. Now Hirsch’s Theorem 7 implies that there exists an immersion $f : M \to \mathbb{R}^{2(m+k)}$ such that $\xi$ is isomorphic to $\nu_f(M)$ as real vector bundles. So $\nu_f(M)$ carries a complex structure.

Up to a product with some $\mathbb{R}^{2\lambda}$, we can assume that $k \geq m + 1$, and then $f$ is regularly homotopic, namely homotopic through immersions, to an embedding $f_1 : M \to \mathbb{R}^{2(m+k)}$. It follows that $\nu_{f_1}(M) \cong \nu_f(M)$ carries a complex structure. Now apply Theorem 6 to get $\tilde{J}$.

If the rank of $\xi$ satisfies $m + 1 \leq k \leq 2m - 1$ we get a pseudo-holomorphic embedding in an euclidean space of complex dimension $m + k < 3m$.

Let $(S^6, J)$ be the six-dimensional sphere equipped with the standard almost-complex structure $J$ obtained from the octonion multiplication. Theorem 1 implies that $(S^6, J)$ can be pseudo-holomorphically embedded in $(\mathbb{R}^{18}, \tilde{J})$ for a suitable positive almost-complex structure $\tilde{J}$. Using the existence of a low-dimensional s-inverse of $(TS^6, J)$ we have the following result.

**Corollary 8.** The almost-complex sphere $(S^6, J)$ can be pseudo-holomorphically embedded in $(\mathbb{R}^{14}, \tilde{J})$ for a suitable positive almost-complex structure $\tilde{J}$.

**Proof.** Since $S^6$ is embedded in $\mathbb{R}^8$ with trivial normal bundle we conclude by an application of Theorem 6 with $k = 3$.

Notice that $(S^6, J)$ can not be pseudo-holomorphically embedded in $(\mathbb{R}^{12}, \tilde{J})$. In fact, the Euler class of the normal bundle of any embedding of $S^6$ in $\mathbb{R}^{12}$ is zero by a theorem of Whitney, see [6, p. 138]. On the other hand, if $S^6$ is contained pseudo-holomorphically in $(\mathbb{R}^{12}, \tilde{J})$, by a straightforward computation with the Chern class, we obtain for the Euler class $e(\nu(S^6)) = c_3(\nu(S^6)) = -2\lambda \neq 0$, which is a contradiction, where $\lambda \in H^6(S^6)$ is the standard generator.

We conclude with a question. Since our construction is essentially homotopy-theoretic, we are unable to control the integrability of the almost-complex structure $\tilde{J}$ of Theorem 1. So the following question is very natural.

**Question 9.** Let $(M, J)$ be an integrable complex manifold. Is there an embedding of $(M, J)$ into an integrable $(\mathbb{R}^{2n}, \tilde{J})$?

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Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail address: antonio.discala@polito.it

Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: d.zuddas@gmail.com