SYMMETRY IN MAXIMAL \((s-1, s+1)\) CORES

RISHI NATH

Abstract. We explain a “curious symmetry” for maximal \((s-1, s+1)\)-core partitions first observed by T. Amdeberhan and E. Leven. Specifically, using the \(s\)-abacus, we show such partitions have empty \(s\)-core and that their \(s\)-quotient is comprised of 2-cores. This imposes strong conditions on the partition structure, and implies both the Amdeberhan-Leven result and additional symmetry. We also find a more general family of partitions that exhibits these symmetries.

1. Introduction

The study of simultaneous core partitions, which began only fifteen years ago, has seen a recent spike of interest. Much of the attention has focused around either a conjecture of Armstrong on the average size of an \((s, t)\)-core or generalizing known results on the \((s, s+1)\) (Catalan) case. Results in a recent paper of Amdeberhan and Leven deviate from this slightly to examine \((s-1, s+1)\)-cores in the case where \(s\) is even and greater than 2; they note a symmetry in the set of first column hook numbers of \(\kappa_{s\pm1}\), the \((s-1, s+1)\)-core of maximal size. [Theorem 2.2 in this paper states their result.]

Hidden by their proof (which involves involves the integral and fractional parts of a real number) is a connection with the \(s\)-core and \(s\)-quotient structure viewed on the \(s\)-abacus. From this vantage point, the Amdeberhan-Leven theorem is a result on the symmetry of runners (columns) of the \(s\)-abacus of maximal \((s-1, s+1)\)-cores.

Given a partition \(\lambda\), let \(\lambda^0\) be the \(s\)-core of \(\lambda\) and \((\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(s-1)})\) be the \(s\)-quotient of \(\lambda\). Let \(\kappa_{s\pm1}\) be the unique maximal simultaneous \((s-1, s+1)\)-core partition and \(\tau_\ell = (\ell, \ell-1, \ell-2, \cdots, 1)\) be the \(\ell\)-th 2-core partition. We state our main theorem.

**Theorem 1.1.** Let \(s = 2k > 2\). Then \(\kappa_{s-1,s+1}\) has the following \(s\)-core and \(s\)-quotient structure:

1. \((\kappa_{s-1,s+1})^0 = \emptyset\).

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In Section 3.1 we describe the $s$-abacus of $\kappa_{s\pm1}$, which we use to prove Theorem 1.1. We provide an alternate proof of the Amdeberhan-Leven result in Section 3.2. In Section 4.1 we demonstrate an additional symmetry in the rows of the $s$-abacus of $\kappa_{s-1,s+1}$. We formalize both the runner and row symmetries exhibited by $\kappa_{s\pm1}$ in Section 4.2, and describe the most general family of partitions which satisfy them.

**Example 1.2.** The 8-abacus of $\kappa_{7,9}$ and the associated 8-quotient are shown in Figure 1 and Figure 2 respectively. [Note: the 8-quotient consists of a sequence of 2-core partitions, arising from the structure of the 8-abacus.]

2. PRELIMINARIES

2.1. Basic definitions. Let $\mathbb{N} = \{0, 1, \cdots \}$ and $n \in \mathbb{N}$. A partition $\lambda$ of $n$ is defined as a finite, non increasing sequence of positive integers $(\lambda_1, \lambda_2, \cdots)$ that sums to $n$. Each $\lambda_\gamma$ is known as a component of $\lambda$. Then $\sum_\gamma \lambda_\gamma = n$, and $\lambda$ is said to have size $n$, denoted $|\lambda| = n$. We also use the notation $\lambda_i^m$ to indicate that $\lambda_i$ occurs $m$-times as a component of $\lambda$.

The Young diagram $[\lambda]$ is a graphic representation of $\lambda$ in which rows of boxes corresponding to the integer values in the partition sequence are left-aligned. Then $\lambda^*$ is the conjugate partition of $\lambda$ obtained by
exchanging rows and columns of the Young diagram of $\lambda$. Then $\lambda$ is self-conjugate if $\lambda = \lambda^*$. Using matrix notation, a hook $h_{\iota\gamma}$ of $[\lambda]$ with corner $(\iota, \gamma)$ is the set of boxes to the right of $(\iota, \gamma)$ in the same row, below $(\iota, \gamma)$ in the same column, and $(\iota, \gamma)$ itself. Given $h_{\iota\gamma}$, its length $|h_{\iota\gamma}|$ is the number of boxes in the hook. The set $\{h_1\gamma\}$ are the first-column hooks of $\lambda$.

One can remove a hook $h$ of $\lambda$ by deleting boxes in $[\lambda]$ which comprise $h$ and migrating any remaining detached boxes up-and-to-the-left. In this way a new partition $\lambda'$ of size $n - |h_{\iota\gamma}|$ is obtained. An $s$-hook is a hook of length $s$. An $s$-core partition $\lambda$ is one in which no hook of length $s$ appears in the Young diagram.

### 2.2. Simultaneous $(s,t)$-core partitions.

Let $r, s, t$ be positive integers. A simultaneous $(s,t)$-core partition is one in which no hook of length $s$ or $t$ appears. In 1999, J. Anderson [5] proved when $(s, t) = 1$, there are exactly $(s+t)/(s + t)$ simultaneous $(s,t)$-cores. Subsequent work by B. Kane [10], J. Olsson and D. Stanton [13], J. Vandehey [16] confirmed the existence of a unique maximal $(s,t)$-core of size $\frac{(s^2-1)(t^2-1)}{24}$ which contains all other $(s,t)$-cores. This maximal simultaneous $(s,t)$-core partition is denoted by $\kappa_{s,t}$. [A. Tripathi [15] and M. Fayers [8] obtained some of the results above using different methods.]

When it is convenient we will denote $\kappa_{s-1,s+1}$ by $\kappa_{s\pm 1}$.

**Theorem 2.1.** [Olsson-Stanton, Theorem 4.1, [13]] Suppose $(s, t) = 1$. There is a unique maximal simultaneous $(s,t)$-core $\kappa_{s,t}$ of size $\frac{(s^2-1)(t^2-1)}{24}$. In particular, $\kappa_{s,t}$ is self-conjugate.

A recent paper of D. Armstrong, C. Hanusa and B. Jones [6] includes a conjecture (the Armstrong conjecture) that the average size of a $(s,t)$-core is $\frac{(s+t+1)(s-1)(t-1)}{24}$. R. Stanley and F. Zenello [14] subsequently resolved the Catalan ($t = s + 1$) case of the Armstrong conjecture; they employ a bijection between lower ideals in the poset $P_{s,t}$ and simultaneous $(s,t)$-cores. [Here $P_{s,t}$ is the partially ordered set whose
elements are all positive integers not contained in the numerical semigroup generated by \( s, t \). The partial order requires \( z_1 \in P_{s,t} \) to cover \( z_2 \in P_{s,t} \) if \( z_1 - z_2 \) is either \( s \) or \( t \). Under this map a lower ideal \( I \) of \( P_{s,t} \) corresponds to an \((s, t)\)-core partition whose first-column hook lengths are exactly the values in \( I \). Then \( P_{s,t} \) corresponds to \( \kappa_{s,t} \).

These two papers have led to renewed interest in simultaneous core partitions. The Armstrong conjecture has been verified for self-conjugate partitions by W. Chen, H. Huang, and L. Wang [7] and for \((s, ms + 1)\) by A. Aggarwal [1]. T. Amdeberhan and E. Leven [4] extended Stanley and Zanello’s bijection to lower poset ideals and simultaneous \((s_1, s_2, \cdots, s_k)\)-cores. Several conjectures of T. Ambederhan [5] on the maximal and average size simultaneous \((s, s + 1, s + 2)\)-cores have been proved first by J. Yang, M. Zhong and R. Zhou [18] and later by H. Xiong [17]. A. Aggarwal has also proved a partial converse to a theorem of Vandehey on the containment of simultaneous \((r, s, t)\)-cores [2].

2.3. A “curious symmetry”. Amdeberhan and Leven also examine \( P_{r,r+2} \) for \( r \) odd. They first construct a \((r - 1) \times (r + 1)\) rectangle \( R \) as follows: the bottom-left corner is labelled by 1, the numbers increase from left-to-right and bottom-to-top, and the largest position, in the upper-right corner, is labeled by \((r - 1)(r + 1)\). If \( x \in P_{r,r+2} \) then \( x \) is entered into this rectangle, otherwise the position is left blank. Using a runner-row index, counting runners (or columns) \( a \) from left-to-right in the \( x \)-coordinate \((1 \leq a \leq r + 1)\), and rows \( b \) from bottom-to-top in the \( y \)-coordinate \((1 \leq b \leq r - 1)\), they prove the following result, which they call a “curious symmetry.”

**Theorem 2.2.** [Amdeberhan-Leven, Theorem 2.2, [4]] For \( r \geq 3 \) the \((a, b)\) entry of \( R \) is an element of \( P_{r,r+2} \) if and only if \( \{a, r - 1 - b\} \) is not. Equivalently, for \( 1 \leq a \leq r + 1 \) and \( 1 \leq b \leq r - 1 \), \((r + 1)(b - 1) + a \in P_{r,r+2} \) if and only if \((r + 1)(r - 1 - b) + a \notin P_{r,r+2} \).
[There is a precedent for the case Amdeberhan-Leven consider. For 
\( r = 2k+1 > 1 \), the maximal simultaneous \((r, r+2)\)-core is self-conjugate
by Theorem 2.1. The author and C. Hanusa showed in [10] that it
is more natural to think about simultaneous \((r, r+2)\)-core partitions
than simultaneous \((s, s+1)\)-core partitions, which behave better in the
non-self-conjugate case.] For the remainder of this paper we will let
\( s = r + 1 \), and will consider maximal \((s - 1, s + 1)\)-core, where \( s \) is
even and greater than 2. We now review the \( s \)-abacus, \( s \)-core, and
\( s \)-quotient constructions.

2.4. bead-sets. A bead-set \( X \) corresponding to a partition \( \lambda \) is gen-
eralization of the set of first column hooks in the following sense:
\( X = \{0, 1, \cdots, k, |h_{i1}|+k, |h_{i2}|+k, |h_{i3}|+k, \cdots \} \) for some non-negative
integer \( k \). It can also be seen as a finite set of non-negative integers,
represented by beads at integral points of the \( x \)-axis, i.e. a bead at posi-
tion \( x \) for each \( x \) in \( X \) and spacers at positions not in \( X \). A minimal
bead-set \( X \) is one where the first space is counted as 0. Then \(|X|\) is
the number of beads that occur after the zero position, where ever that
may fall. We say \( X = \{0, 1, \cdots, k, |h_{i1}|+k, |h_{i2}|+k, |h_{i3}|+k, \cdots \} \)
is normalized with respect to \( s \) if \( k \) is the minimal integer such that
\(|X| \equiv 0 (\mod s)\).

2.5. 2-cores and staircase partitions. [The results in this section
are stated without proof; for more details see Section 2 in [12].] The set
of hooks \( \{h_{i\gamma}\} \) of \( \lambda \) correspond bijectively to pairs \((x, y)\) where \( x \in X, \)
\( y \not\in X \) and \( x > y \); that is, a bead in the bead-set \( X \) of \( \lambda \) and a spacer
to the left of it. Hooks of length \( s \) are those such that \( x - y = s \).

Each first-column hook length, or bead \( x_i \) in the minimal bead-set
\( X \), also corresponds to a row, or component \( \lambda_i \) of \( \lambda \) The following result
allows us to recover the size of the components from \( X \).

Lemma 2.3. The size of the component \( \lambda_i \) corresponding to the bead
\( x_i \in X \) is the number of spacers to the left of the bead; that is, \( \lambda_i = \)
\( |y \notin X : y < x_i| \).

Let \( \tau_k = (k, k - 1, \cdots, 1) \) be the \( k \)-th staircase partition. Then
\(|\tau_k| = t_k \) where \( t_k = \binom{k+1}{2} \) (the \( k \)-th triangular number). The following
lemmas are well-known.

Lemma 2.4. The 2-core partitions are exactly the staircase partitions.

Lemma 2.5. The minimal \( X \) for the 2-core \( \tau_k \) is \( \{1, 3, 5, \cdots, 2k - 3, 2k - 1\} \). In other words, the 2-core partitions are sequence of alternating spacers-and-beads of length \( 2k - 1 \).
2.6. The s-abacus. Given a fixed integer $s$, we can arrange the non-negative integers in an array of columns and consider the columns as runners.

\[
\begin{array}{ccc}
ms & (m + 1)s - 1 \\
\vdots & \ddots \\
n & s + 1 & 2s - 1 \\
0 & 1 & \cdots & s - 1
\end{array}
\]

The column containing $i$ for $0 \leq i \leq s - 1$ will be called runner $i$. The positions $0, 1, 2, \cdots$ on the $i$th runner corresponding to $i, i + s, i + 2s, \cdots$ will be called $i$-positions. Placing a bead at position $x_j$ for each $x_j \in X$ gives the $s$-abacus diagram of $X$. A normalized abacus will be one whose bead-set $X$ is normalized, a minimal abacus is one in which $X$ is minimal (or, the first spacer is counted as the zero position).

2.7. The $s$-core and $s$-quotient. By removing a sequence of $s$-hooks from $\lambda$ until no $s$-hooks remain, one obtains its $s$-core $\lambda^0$. The $s$-abacus of $\lambda^0$ can be found from the $s$-abacus of $\lambda$ by pushing beads in each runner down as low as they can go (Theorem 2.7.16, [9]: we have changed the orientation). This implies $\lambda^0$ is unique since it is independent of the way the $s$-hooks are removed. For $0 \leq i \leq p - 1$ let $X_i = \{ j : i + js \in X \}$ and let $\lambda_{(i)}$ be the partition represented by the bead-set $X_i$. The $s$-quotient of $\lambda$ is the sequence $(\lambda_{(0)}, \cdots, \lambda_{(s-1)})$ obtained from $X$. The next lemma is Proposition 3.5 in [12].

**Lemma 2.6.** Let $\lambda$ be a partition with $s$-core $\lambda^0$ and $s$-quotient $(\lambda_{(i)})$, $0 \leq i \leq s - 1$. Then

1. Every 1-hook in $\lambda_{(i)}$ corresponds to a $s$-hook in $\lambda$ for $0 \leq i \leq s - 1$.
2. $n = |\lambda^0| + \sum_i |\lambda_{(i)}|$.

**Lemma 2.6** implies that there exists a bijection between a partition $\lambda$ and its $s$-core and $s$-quotient, such that each node in some $\lambda_i$ corresponds to an $s$-hook in $\lambda$. The situation is strengthened when $\lambda$ is self-conjugate.

**Lemma 2.7.** Suppose $|X| = 0 \pmod{s}$. Let $\lambda^*$ be the conjugate of $\lambda$, $(\lambda^*)^0$ its $s$-core and let $(\lambda_{(i)}^*)$ be the $s$-quotient of $\lambda^*$, $0 \leq i \leq s - 1$. Then

1. $(\lambda^*)^0 = (\lambda^0)^*$
2. $(\lambda_{(i)}^*)^* = \lambda_{(s-1-i)}$.

In particular, $\lambda = \lambda^*$ if and only if $\lambda^0 = (\lambda^0)^*$ and $(\lambda_{(i)}^*)^* = (\lambda^*)_{(i)}$. 

2.8. **The axis of symmetry.** The following results and their proofs can be found in Section 4, [11].

**Proposition 2.8.** Suppose $\lambda$ is a partition of $n$ and let $X$ be a bead-set for $\lambda$. Then there exists a half-integer $\theta(\lambda)$ such that the number of beads to the right of $\theta(\lambda)$ equals the number of spaces to its left. Conversely, given a bead-spacer sequence and a half-integer $\theta(\lambda)$ such that the number of beads to the right equals the number of spaces to the left, one can recover the unique partition $\lambda$.

**Lemma 2.9.** Let $X$ be a minimal bead-set for $\lambda$. If $x' \in X$ is the entry with maximum value, $\theta(\lambda) = \frac{x'}{2}$. We call $\theta(\lambda)$ the axis of $\lambda$. If $\lambda$ is self-conjugate we say $X$ has a axis of symmetry.

**Corollary 2.10.** Let $X$ be a bead-set for $\lambda$. Then $\lambda$ is a self-conjugate partition if and only if there exists a half-integer $\theta(\lambda)$ such that beads and spaces in $X$ to the right of $\theta(\lambda)$ are reflected respectively to spaces and beads in $X$ to its left.

When $\lambda^0 = \emptyset$, each $\lambda_i$ has an axis of symmetry $\theta(\lambda_i)$ induced by $X$.

**Lemma 2.11.** Suppose $X$ is normalized. Then $|X| = ms, \lambda^0 = \emptyset$, and each runner has exactly $m$ beads if and only if $\theta(\lambda_{(i)}) = \theta(\lambda_{(i')}) = m - \frac{1}{2}$ for all $0 \leq i, i' \leq s - 1$.

**Example 2.12.** The maximum $(5,7)$-core $\kappa_{5,7}$ has empty 8-core. In the normalized (minimal) 8-abacus in Figure 1, each $\lambda_{(i)}$ has axis $\theta(\lambda_{(i)}) = \frac{5}{2}$.

### 3. The $s$-Quotient of $\kappa_{s \pm 1}$

#### 3.1. The $s$-abacus of $\kappa_{s \pm 1}$

We begin with a classical result of Sylvester.

**Lemma 3.1.** The largest integer in $P_{s,t}$ is $st - s - t$.

The Amdeberhan-Leven rectangle $R$ is constructed to begin at 0; the $(r + 1)$-abacus of $\kappa_{r,r+2}$ starts at 0. However $0 \not\in P_{r,r+2}$ and by Lemma 3.1 neither is $(r + 1)(r - 1)$. Hence $R$ and the minimal $(r + 1)$-abacus of $\kappa_{r,r+2}$ include the same values.

Recall $s = r + 1$. We now interpret the Amdeberhan-Leven result in terms of the $s$-abacus $\kappa_{s-1,s+1}$. We use a runner-row index. We start with a definition.

**Definition 3.2.** Let $s = 2k > 2$. Then $\alpha(s)$ is an $s$-abacus with $s$ runners, indexed from left-to-right by $0 \leq i \leq s - 1$ and $s - 2$ rows,
indexed from bottom-to-top by $0 \leq i \leq s - 3$, which is constructed as follows: For each $i \in [0, k-2]$, the runners $i$ and $2k-i-1$ are composed firstly of beads in rows $j$ where $0 \leq j \leq i$. Then rows $j > i$ consist of alternating spacers-and-beads, until the total number of beads in each runner is $(k-1)$. Spacers fill the remainder of the rows.

**Example 3.3.** $\alpha(8)$ has three beads in each runner. Runners $i=3$ and $4$ consist of three beads below three spacers; $i=2$ and $5$ have two beads followed by a spacer-and-bead, then two spacers; $i=1$ and $6$ have one bead followed by spacer-beadspacer-bead; and runners $i=0$ and $7$ have an alternating sequence of spacers-and-beads. [See Figure 1.]

**Lemma 3.4.** The $s$-abacus $\alpha(s)$ is normalized with respect to $s$.

*Proof.* The total number of beads in $\alpha(s)$ is $2k(k-1) = \frac{s^2 - 2s}{2}$, a multiple of $s$. □

**Lemma 3.5.** Fix $1 < j < 2k - 3$ and $0 \leq i < k - 1$.

1. There is a bead in row $j$ of runner 0 if and only if there is a bead in row $j - 1$ of runner 1.
2. There is a bead in row $j$ of runner $2k - 1$ if and only if there is a bead in row $j - 1$ of runner $2k - 2$.
3. There is a spacer in row $j$ of runner 0 if and only if there is a spacer in row $j + 1$ of runner 1.
4. There is a spacer in row $j$ of runner $2k - 1$ if and only if there is a spacer in row $j$ of runner $2k - 2$.

*Proof.* By Definition 3.2, runner $i = 0$ begins in row $j = 0$ with a spacer, and continues upwards with alternating beads-and-spacers. Runner $i = 1$ begins with a bead in row 1, and continues upwards, alternating spacers-and-beads. Since both columns have $2k - 2$ rows, (1) and (3) follow. For (2) and (4), a similar argument holds. □

**Lemma 3.6.** The $s$-abacus $\alpha(s+2)$ can be obtained from the $s$-abacus $\alpha(s)$ using the following procedure:

1. Append a new row of $2k$ beads below $\alpha(s)$.
2. Append a new row of $2k$ spacers above $\alpha(s)$.
3. Append a new runner of length $2k - 2$ consisting of alternating beads-and-spacers to the left (and an identical column to the right) of $\alpha(s)$.
4. Append a single spacer to the bottom, and a single bead at the top of, both new runners in step (3). [The total number of beads in all runners, both the two new runners, as well as the $s = 2k$ previous runners, will now be $k$.]
(5) Renumber the runners with $i'$ so $0 \leq i' \leq 2k + 1$ and the rows with $j'$ so that $0 \leq j' \leq 2k - 1$. Renumber the abacus positions, with 0 in the bottom left-most corner, increasing from left-to-right and bottom-to-top, with final position $(2k + 1)(2k - 1)$ in the upper-right-hand corner.

**Proof.** It is enough to see that the outcome satisfies Definition 3.2 for $\alpha(s + 2)$.

**Example 3.7.** To see how Lemma 3.6 is used to obtain $\alpha(10)$ from $\alpha(8)$, see Appendix A, Figure 9 and Figure 8.

Recall $\lambda^0$ is the $s$-core partition of $\lambda$, $(\lambda_{i+1})$ is the $s$-quotient (where $0 \leq i \leq s - 1$), and that $\tau_{\ell}$ the the $\ell$-th 2-core partition. For the following two lemmas we abuse notation and let $\alpha(s)$ refer not only to the $s$-abacus but also to its corresponding partition.

**Lemma 3.8.** Suppose $s = 2k > 2$. Then

1. $\alpha(s)^0 = \emptyset$
2. $\alpha(s)_{i+1} = \alpha(s)(s - i - 1) = \tau_{2k - i + 1}$.

**Proof.** We proof each condition separately.

1. Since each runner $\alpha(s)_{i}$ has $k - 1$ beads and $(k - 1)$ spacers, the removal of all $s$-hooks will result in an $s$-abacus with each runners having $k - 1$ beads beneath $k - 1$ spacers. This corresponds to the empty partition.

2. We use induction on $k$. For $k = 2$ it is true. Assume it is for $k$. We obtain the $\alpha(s + 2)$ from $\alpha(s)$ by Lemma 3.2. By construction, for $1 \leq i' \leq 2k$ we have $|\alpha(s)(i' - 1)| = |\alpha(s + 2)(i')|$; hence, by the inductive hypothesis and since $i + 1 = i'$, $|\alpha(s + 2)(i')| = \tau_{2k - i'}$. It only remains to check $i' = 0, 2k + 1$. The proof is finished using (3) and (4) of Lemma 3.6 and Lemma 2.5.

**Example 3.9.** $\alpha(8)$ has 8-quotient $(\lambda_0, \cdots, \lambda_{s-1})$

$$(3, 2, 1), (2, 1), (1), \emptyset, \emptyset, (1), (2, 1), (3, 2, 1))$$

[See Appendix A, Figure 8 and Appendix B, Figure 12]

**Lemma 3.10.** Let $s = 2k > 2$. Then $\alpha(s)$ is the minimal $s$-abacus for $\kappa_{s-1, s+1}$.

**Proof.** By construction, $\alpha(s)$ is minimal, since the first spacer labels zero. We must show:

1. $|\alpha(s)| = \frac{(2k - 1)^2 - 1}{24}$;
(2) \( \alpha(s) \) contains no \((2k - 1)\)-hooks or \((2k + 1)\)-hooks.

Then by the uniqueness implied by Theorem 2.1, \( \alpha(s) = \kappa_{s \pm 1} \). We use the structure of \( \alpha(s) \) and induction on \( k \).

By Theorem 2.6 each 1-hook in the \( s \)-quotient corresponds to a \( s \)-hook in \( \lambda \). Hence, to prove (1), since \( \alpha(s)^0 = \emptyset \), it is enough to calculate \( \sum t_i \lambda(i) \) and multiply by \( s = 2k \). This equals \( 2k \cdot 2 \sum_{i=1}^{k-1} t_i = (4k)(k(k^2-1)/6 \) = 16k^4 - 16k^2 = (4k^2 - 4k)\((4k^2 + 4k)\), which, after completing-the-square, equals to \( 2^{k-1}(2k-1)\) \((2k+1)^2 - 1)\). We are done.

To prove (2), we use induction on \( k > 2 \). For the basic case, \( s=4 \), it holds: \( \alpha(4) \) has no 3-hooks or 5-hooks. [See Appendix A, Figure 6.]

By the inductive hypothesis we know the \( 2k \)-abacus of \( \kappa_{2k \pm 1} \) contains no \((2k - 1)\)-hooks or \((2k + 1)\)-hooks. More specifically, no bead in \( \alpha(s) \) has a spacer either \( 2k + 1 \) or \( 2k - 1 \) positions below it. Apply Lemma 3.6 to obtain \( \alpha(s + 2) \); this adds two additional positions between the beads and spacers arising from \( \alpha(s) \). Hence there are no \((2k + 1)\)-hooks or \((2k + 3)\)-hooks arising from bead-spacer pairs \( (x, y) \) where both \( x \) and \( y \) are in runners \( 1 < i' < 2k - 2 \). It remains to examine the beads and spacers introduced by runners \( i' = 0, 2k + 1 \).

If a bead in row \( j' \) of runner \( i' = 0 \) were to add a new \((2k + 3)\)-hook, a spacer would have to appear in row \( j' - 2 \) of the runner \( i' = 2k + 1 \). By construction, such positions are occupied by beads, since runners \( 0 \) and \( 2k + 1 \) are identical. If a bead in row \( j' \) of \( i' = 0 \) were to add a new \((2k + 1)\)-hook, a spacer would have to appear in row \( j' - 1 \) of runner \( i' = 1 \). But by the Lemma 3.5(1), this position is always occupied by a bead.

If a bead in row \( j' \) on runner \( i' = 2k + 1 \) were to add a new \((2k + 3)\)-hook, a spacer would appear in row \( j' - 1 \) of runner \( i' = 2k \). But by Lemma 3.5(2) this position is always occupied by a bead. If a bead in row \( j' \) of \( i' = 2k + 1 \) were to add a new \((2k + 1)\)-hook, a spacer would have to appear in the same row in the runner \( i' = 0 \). By construction, the two runners are identical, so a bead in one implies a bead in the other.

If a spacer in row \( j' \) of runner \( i' = 0 \) were to add a new \((2k + 3)\)-hook, a bead would have to appear in row \( j' + 1 \) of runner \( i' = 1 \). But by Lemma 3.5(3), this position is always occupied by a spacer. If a spacer in row \( j' \) of \( i' = 0 \) were to add \((2k + 1)\)-hook, a bead would have to appear in the same row of runner \( i' = 2k + 1 \). By construction, the two runners are identical, so a spacer in one implies a spacer in the other.
If a spacer in row $j'$ of runner $i' = 2k + 1$ were to add a new $(2k + 3)$-hook, a bead would have to appear in row $j' + 2$ in runner $i' = 0$; by construction, since both runners are identical alternating sequences of spacer-and-beads, such positions are occupied by spacers. If a spacer in row $j'$ of runner $i' = 2k + 1$ were to add a new $(2k + 1)$-hook, a bead would have to appear in row $j' + 1$ of runner $i' = 2k$. But by Lemma 3.5(4) this position is occupied by a spacer.

\[ \square \]

3.2. An alternative proof of Amdeberhan-Leven. Using the results of this section we offer an alternative proof to Theorem 2.2.

**Proof of Theorem 2.2** By Lemma 3.10, the $s$-core of $\kappa_{s \pm 1} = \emptyset$, and $X$ is normalized. Again by Lemma 3.10, each $\lambda_i$ is self-conjugate, so each runner obeys Lemma 2.10. By Lemma 2.11 all $(\kappa_{s-1,s+1})$ have the same axis of symmetry, which is at $i$-position $\frac{s-3}{2}$. Our runner-row index is $0 \leq j \leq s - 3$ with $s = r - 1$, which finishes the proof.

\[ \square \]

4. Generalizations

4.1. Additional symmetry. Using Theorem 1.1 we can strengthen Amdeberhan-Leven to include additional symmetry.

**Theorem 4.1.** Let $s = 2k > 2$ and let $\alpha(s)$ be the $s$-abacus of $\kappa_{s \pm 1}$. Then the following are equivalent:

1. $(i, j) \in \alpha(s)$
2. $(i, s - 3 - j) \not\in \alpha(s)$
3. $(s - 1 - i, j) \in \alpha(s)$.

**Proof.** By Theorem 2.2 is sufficient to prove $(1) \iff (3)$. This follows from Lemma 3.10 and Lemma 3.8.

\[ \square \]

4.2. Horizontal anti-symmetry and vertical symmetry. The symmetries exhibited by the $s$-abacus of $\kappa_{s \pm 1}$ can be formalized and generalized to a larger family of partitions. For the remainder of this section we assume that the bead-set $X$ of $\lambda$ is normalized with respect to $s$. Suppose that the $s$-abacus of $\lambda$ has maximum value $i + (q - 1)s$. In particular, the $s$-abacus of $\lambda$ has $s$ columns and $q$ rows.

**Definition 4.2.** We say the $s$-abacus of $\lambda$ exhibits *horizontal anti-symmetry* if a there is a bead in the $(i, j)$th-position if and only if there is a spacer in the $(i, q - j - 1)$ position.

**Definition 4.3.** We say the $s$-abacus of $\lambda$ exhibits *vertical symmetry* if there is a bead in the $(i, j)$th-position if and only if there is a bead in the $(s - i - 1, j)$th-position.
Lemma 4.4. The $s$-abacus of $\lambda$ exhibits horizontal anti-symmetry if and only if $q$ is even, $\lambda(i) = \lambda^*(i)$, and each runner has $q$ beads.

Proof. Suppose the $s$-abacus of $\lambda$ exhibits horizontal anti-symmetry. Clearly $q$ must be even, otherwise there would exist a bead or spacer in a row that would not have a spacer or bead to pair with. Let $q = 2m$. Horizontal symmetry also implies each runner $i$ must have the same axis of symmetry, $\theta(\lambda_i) = \frac{q-1}{2} = m - \frac{1}{2}$. Lemma 2.10 implies $\lambda = \lambda^*$. By Lemma 2.11 each runner has exactly $q$ beads. The proof in the other direction is clear. □

Lemma 4.5. The $s$-abacus of $\lambda$ exhibits vertical symmetry if and only if $s$ is even, runner $i$ and runner $s - i - 1$ have the same number of beads, and $\lambda_i = \lambda_{s-i-1}$ for $0 \leq i \leq s - 1$.

Proof. Suppose the $s$-abacus of $\lambda$ exhibits horizontal symmetry. Then $s$ must be even, otherwise there would be a bead or spacer in a runner that would not have a bead or spacer to pair with. Vertical symmetry also implies that each runner $i$ and $s - i - 1$ must be identical. This means runners $i$ and $s - i - 1$ have the same number of beads and $\lambda_i = \lambda_{s-i-1}$ for each $0 \leq i \leq s - 1$. The proof in the other direction is clear. □

Theorem 4.6. $\lambda$ exhibits both horizontal anti-symmetry and vertical symmetry with respect to $s$ if and only if $s$ and $q$ are both even and the following three conditions hold for all $0 \leq i \leq s - 1$

1. $\lambda^0 = \emptyset$
2. $\lambda(i) = \lambda^*(i)$
3. $\lambda(i) = \lambda(s-i-1)$.

Proof. This follows from Lemma 4.4 and Lemma 4.5. □

Example 4.7. $\lambda = (8, 6^4, 1^2)$ exhibits horizontal anti-symmetry and vertical symmetry with respect to $s = 4$, but is neither a 3-core nor a 5-core. See Figure 5.

The following corollary is immediate.

Corollary 4.8. Let $s = 2k > 1$. The $s$-abacus of $\kappa_{s\pm 1}$ exhibits horizontal anti-symmetry and vertical symmetry.

Corollary 4.9. If the $s$-abacus of $\lambda$ exhibits both horizontal anti-symmetry and vertical symmetry then $\lambda$ is self-conjugate.

Proof. By Theorem 4.6 since $\lambda(i) = \lambda(s-i-1)$ and $\lambda(i) = \lambda^*(i)$, we have $\lambda(i) = \lambda^*(s-i-1)$. Since $\lambda^0 = \emptyset$, and by assumption $|X| = 0 \pmod{s}$, we have $\lambda = \lambda^*$ by Lemma 2.7. □
Figure 5. The minimal 4-abacus of $\lambda = (8, 6^4, 1^2)$ (see Example 4.7)

5. Further Study

5.1. Simultaneous $(s - 1, s, s + 1)$-cores. The following theorem is a recently-proven conjecture of Amdeberhan [5].

Theorem 5.1. (Yang-Zhong-Zhou, [18]; H. Xiong, [17]) The size of the largest $(s - 1, s, s + 1)$-core is

1. $k \binom{k+1}{3}$ if $s = 2k > 2$
2. $(k + 1) \binom{k+1}{3} + \binom{k+2}{3}$ if $s = 2k + 1 > 2$.

Let $\kappa_{s-1,s,s+1}$ is a (not necessarily unique) simultaneous $(s - 1, s, s + 1)$-core of maximal size. Theorem 5.1 allows us to compare $|\kappa_{s \pm 1}|$ with $|\kappa_{s-1,s,s+1}|$.

Proposition 5.2. Let $s = 2k > 2$. Then $|\kappa_{s \pm 1}| > |\kappa_{(s-1,s,s+1)}|$. In particular, $|\kappa_{s \pm 1}| = 4|\kappa_{(s-1,s,s+1)}|$

Proof. Since $s$ is even, by Theorem 5.1(1) above $|\kappa_{s-1,s,s+1}| = \frac{k^4 - k^2}{6}$. However by Theorem 2.1 $|\kappa_{s \pm 1}| = \frac{(s-1)^2-1((s+1)^2-1)}{24}$. This simplifies to $\frac{4(k^4-k^2)}{6}$. The result follows. □

Corollary 5.3. $\kappa_{s,s+2}$ is never an $s$-core.

Corollary 5.3 also follows from Theorem 1.1 which says $\kappa_{s,s+2}$ is comprised completely of $s$-hooks. Is there interpretation (either in the geometry of the $s$-abacus or in the manipulation of Young diagrams) of the factor of 4 that appears above? A cursory examination of $\kappa_{(3,5)}$ and $\kappa_{(3,4,5)}$ does not suggest an obvious one.

5.2. Other proofs using the $s$-abacus. In their proof of Theorem 2.2 Amdeberhan and Leven use the following result (Corollary 2.1 (ii), [4]).

Lemma 5.4. Exactly half of the integers in $\{1, 2, \cdots, (s-1)(t-1)\}$ belong to $P_{s,t}$. 
They cite a result of T. Popoviciu on the integral and fractional parts of an integer of which this is a consequence. We provide an alternative proof using only the geometry of the \(s\)-abacus.

**Proof of Lemma 5.4.** Since by Lemma 3.1 neither \((s-1)(t-1)\) nor 0 are in \(P_{s,t}\), it is equivalent to prove half of the integers in \(\{0, 1, 2, \ldots, st - s - t\}\) are in the minimal bead-set of \(\kappa_{s,t}\). By Lemma 2.8, the axis is \(\theta(\kappa_{s,t}) = \frac{st - s - t}{2}\). This implies the result. \(\Box\)

Perhaps there are other results on simultaneous core partitions that can be understood using bead-sets and the geometry of the \(s\)-abacus.

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APPENDIX A

The $s$-abaci $\alpha(s)$ of $\kappa_{s\pm1}$

**Figure 6.** $s = 4$

```
| 4 | 5 | 6 | 7 |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
```

**Figure 7.** $s = 6$

```
| 18 | 19 | 20 | 21 | 22 | 23 |
|----|----|----|----|----|----|
| 12 | 13 | 14 | 15 | 16 | 17 |
| 6  | 7  | 8  | 9  | 10 | 11 |
| 0  | 1  | 2  | 3  | 4  | 5  |
```

**Figure 8.** $s = 8$

```
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 |
|----|----|----|----|----|----|----|----|
| 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
| 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
| 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
```

**Figure 9.** $s = 10$

```
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
|----|----|----|----|----|----|----|----|----|----|
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
```
APPENDIX B

The $s$-quotients of $\kappa_{s\pm 1}$

Figure 10. 4-quotient of $\kappa_{3,5}$

\[ \square, \emptyset, \emptyset, \square \]

Figure 11. 6-quotient of $\kappa_{5,7}$

\[ \square, \square, \emptyset, \emptyset, \square, \square \]

Figure 12. 8-quotient of $\kappa_{7,9}$

\[ \square, \square, \square, \emptyset, \emptyset, \square, \square, \square, \square \]

Figure 13. 10-quotient of $\kappa_{9,11}$

\[ \square, \square, \square, \square, \emptyset, \emptyset, \square, \square, \square, \square, \square, \square \]
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REFERENCES

[1] A. Aggarwal, *Armstrong’s conjecture for $(k, mk + 1)$-core partitions*, http://arxiv.org/abs/1407.5134

[2] A. Aggarwal, *A Converse to Vandehey’s Theorem on Simultaneous Core Containment*, http://arxiv.org/abs/1408.0550

[3] T. Amdeberhan, *Theorems, Problems, conjectures*, http://129.81.170.14/∼tamdeberhan/conjectures.pdf

[4] T. Amdeberhan and E. Leven, *Multi-cores, posets, and lattice paths*, arXiv:1406.2250.

[5] J. Anderson, *Partitions which are simultaneously $t_1$- and $t_2$-core*, Discrete Math. 248 (2002), 237243.

[6] D. Armstrong, C. R. H. Hanusa, B. C. Jones, *Results and conjectures on simultaneous core partitions*, http://arxiv.org/abs/1308.0572

[7] William Y.C. Chen, Harry H.Y. Huang, Larry X.W. Wang, *The Average Size of a Self-conjugate $(s, t)$-core Partition*, http://arxiv.org/abs/1406.2583

[8] M. Fayers *The $t$-core of an $s$-core*, J. Combin. Theory Ser. A 118 (2011) 15251539.

[9] G. James and A. Kerber, *The Representation Theory of the Symmetric Groups*. Encyclopedia of Mathematics, 16.

[10] C.R.H. Hanusa and R. Nath *The number of self-conjugate partitions*, J.Number Theory, 133:751768, 2013.

[10] B. Kane, Masters Thesis, Unpublished.

[11] R. Nath, *On diagonal hooks of self-conjugate partitions*, http://arxiv.org/abs/0903.2494

[12] J. Olsson, *Combinatorics and Representation Theory of Finite Groups*, Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1993.

[13] J. Olsson and D. Stanton, *Block inclusions and cores of partitions*, Aequationes Math. 74 (2007) 90-110.

[14] Richard P. Stanley, Fabrizio Zanello, *The Catalan case of Armstrong’s conjecture on core partitions*, http://arxiv.org/abs/1312.4352

[15] A. Tripathi, *On the largest size of a partition that is both an $s$ and $t$ core*, Journal of Number Theory Volume 129, Issue 7, July 2009, Pages 18051811.

[16] J. Vandehey, *Containment in $(s, t)$-core partitions*, http://arxiv.org/abs/0809.2134.

[17] H. Xiong, *On the largest size of $(t, t+1, \ldots, t+p)$-core partitions*, http://arxiv.org/abs/1410.2061.

[18] Jane Y.X. Yang, Micheal X.X. Zhong, Robin D.P. Zhou, *On the Enumeration of $(s, s+1, s+2)$-core Partitions*, http://arxiv.org/pdf/1406.2583.pdf

York College/City University of New York
E-mail address: rnath@york.cuny.edu