A conservative difference scheme with optimal pointwise error estimates for two-dimensional space fractional nonlinear Schrödinger equations

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Abstract
In this paper, a linearized semi-implicit finite difference scheme is proposed for solving the two-dimensional (2D) space fractional nonlinear Schrödinger equation (SFNSE). The scheme has the property of mass and energy conservation at the discrete level, with an unconditional stability and a second-order accuracy for both time and spatial variables. The main contribution of this paper is an optimal pointwise error estimate for the 2D SFNSE, which is rigorously established for the first time. Moreover, a novel technique is proposed for dealing with the nonlinear term in the equation, which plays an essential role in the error estimation. Finally, the numerical results confirm well with the theoretical findings.

KEYWORDS
conservative difference scheme, pointwise error estimate, Riesz fractional derivative, second-order convergence, unconditional stability
1 | INTRODUCTION

Schrödinger equation is one of the most important equations in mathematical physics, which describes non-relativistic quantum mechanical behavior, including modeling the hydrodynamics of Bose–Einstein condensate [1]. In the 1960s, Feynman and Hibbs [10] used path integrals over Brownian paths to derive the standard (non-fractional) Schrödinger equation. Fractional quantum mechanics is a theory used to discuss quantum phenomena in fractal environments. Laskin et al. [20, 21] first successfully attempt to apply the fractal concept to reformulate the standard Schrödinger equation over Lévy-like quantum mechanical paths and develop the fractional Schrödinger equation, in which the fractional space derivative replaces the second-order Laplacian in the standard Schrödinger equation. Laskin [22] established the parity conservation law for fractional Schrödinger equation. Guo et al. [11] studied the existence and uniqueness of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation by using the energy method. Dong et al. [7] investigated the analytic solution of space fractional Schrödinger equation with linear potential, delta-function potential, and Coulomb potential, respectively. There are also numerous papers devoted to the theoretical property and practical application of fractional Schrödinger equations [2, 6, 8, 31, 41]. In the literature, there are some researchers concentrated on finding the solution of fractional Schrödinger equation by using the analytical approach. For instance, Herzallah et al. [16] successfully applied the adomian decomposition method to find the approximated analytical solution of the Schrödinger equation with time and space fractional derivatives. However, exact solutions of the fractional Schrödinger equation are generally not available. Numerical computation is extremely important for fractional Schrödinger equations. The time fractional Schrödinger equations have been numerically studied by many researchers [3, 4, 17], we will not describe these literature in detail.

Due to the wide application of the space fractional Schrödinger equation, performing efficient and accurate numerical simulations for the space fractional Schrödinger equation plays an essential role in many real applications. For example, Wang et al. [43, 44] studied the Crank–Nicolson finite difference scheme and implicit conservative finite difference scheme for the coupled space fractional one-dimensional (1D) nonlinear Schrödinger equations with the Riesz space fractional derivative, where the convergent results in $L^2$-norm were obtained. Afterwards, they investigated the maximum-norm error analysis for the 1D coupled space fractional nonlinear Schrödinger equations (SFNSEs) [45]. Wang and Huang [46] proposed an energy conservative difference scheme for the nonlinear fractional Schrödinger equations with convergence results in $L^2$-norm. Huang et al. [48] obtained the pointwise error estimate of a conservative difference scheme for the 1D fractional Schrödinger equation. Li et al. [30] investigated the time–space-fractional Schrödinger equation using implicit finite difference scheme. Zhang et al. [9] proposed three Fourier spectral methods for 1D fractional Schrödinger equation. Yang [50] proposed a class of linearized energy-conserved difference schemes for SFNSEs with second-order convergence in the $L^2$-norm. Ran et al. [40] established a conservative finite difference scheme for strongly coupled 1D fractional Schrödinger equation, where the solvability, stability, and convergence in the maximum norm were established. Khaliq et al. [19] proposed a fourth-order implicit–explicit scheme for 1D fractional Schrödinger equation, where the stability and convergence in $L^2$-norm were obtained. Li et al. [24] constructed a conservative linearized Crank–Nicolson Galerkin FEMs for the nonlinear fractional Schrödinger equation, where unconditional $L^2$-norm error estimates are derived by using a new error splitting technique. Closely followed by the previous work, Li et al. [25] further studied a fast linearized conservative finite element method for the strongly coupled 1D nonlinear fractional Schrödinger equations.

To the best of authors’ knowledge, there are very few works in the literature for the numerical methods and numerical analysis for the high dimensional space fractional Schrödinger equation.
In the small amount of researches, Zhao et al. [54] and Wang et al. [47] have studied the numerical two-dimensional (2D) SFNSE and obtained an error estimates in $L^2$-norms. By using Strang’s splitting technique, Owolabi et al. [35] simulated the SFNSE with the Riesz fractional derivative from 1D to 3D with the exponential time-difference method in time and the Fourier pseudo-spectral method in space. Khaliq et al. [26] solved the multi-dimensional SFNSEs, where the empirical convergence analysis and calculation of the local truncation error were exhibited. After carefully studying the research papers listed above, we found that the pointwise error estimation for high dimensional space-fractional Schrödinger equation has never been investigated. Furthermore, the optimal pointwise error estimate especially the one in the sense of $L^\infty$ norm, which is much more difficult to obtain and has more significant impacts, are still unavailable.

In this paper, we consider the following 2D SFNSE

$$i u_t + L_\alpha u + |u|^2 u = 0, \quad (x, y) \in \mathbb{R}^2, \quad 0 < t < T,$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2, \quad 0 < t < T,$$

where $i = \sqrt{-1}$ is the complex unit, $u(x, y, t)$ is a complex-valued function of the time variable $t$ and space variable $x, y$, $u_0(x, y)$ is the complex-valued function satisfying certain regularity, and $1 < \alpha \leq 2$. The 2D Riesz fractional derivative $L_\alpha u (\alpha \in (1, 2])$ is defined in $\mathbb{R}^2$ as

$$L_\alpha u(x, y, t) := L_N^\alpha u(x, y, t) + L_S^\alpha u(x, y, t).$$

The fractional differential operator $L_N^\alpha u$ is defined on $\mathbb{R}^2$ as follows

$$L_N^\alpha u(x, y, t) := \frac{1}{2 \cos \left( \frac{\alpha \pi}{2} \right)} (-\infty D_N^\alpha u(x, y, t) + x D_S^\alpha u(x, y, t)),$$

in which the left and right Riemann–Liouville fractional derivatives are defined in (3) respectively,

$$-\infty D_N^\alpha u(x, y, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{u(\xi, y, t)}{(x - \xi)^{\alpha-1}} d\xi,$$

$$x D_S^\alpha u(x, y, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_{x}^{\infty} \frac{u(\xi, y, t)}{(\xi - x)^{\alpha-1}} d\xi.$$

$L_S^\alpha u(x, y, t)$ can be defined similarly. When $\alpha = 2$, the 2D Riesz fractional derivative operator $L_\alpha$ reduces into the standard Laplacian operator and the system reduces into the classical 2D nonlinear Schrödinger’s system [13, 39].

For problems (1) and (2), by using the similar method in [11], we can obtain the following mass and energy conservation

$$Q(t) = Q(0), \quad E(t) = E(0),$$

where

$$Q(t) = \| u(\cdot, \cdot, t) \|_2, \quad E(t) = \left\| (-\Delta)^{\frac{\alpha}{4}} u(\cdot, \cdot, t) \right\|_2^2 - \frac{1}{2} \| u(\cdot, \cdot, t) \|_4^4.$$

$\| \cdot \|_2$ denotes the $L^2$-norm and $\| \cdot \|_4$ denotes the $L^4$-norm. The detailed interpretation is provided in Appendix A.

In this paper, we develop and analyze a linearized three-level implicit finite difference scheme for solving the 2D SFNSE. The main contribution of the paper is that the optimal pointwise error estimate is obtained based on the discrete fractional Sobolev embedding theorem in $H^\alpha (1 < \alpha \leq 2)$ norm. And the technique for dealing with the nonlinear term in the error estimate process is completely new. As
far as we know, this is the first result to show the uniform convergence of the numerical solution for 2D SFNSE.

The rest of this paper is organized as follows. Section 2 summarizes the fractional Sobolev space and provides the 2D discrete fractional Sobolev inequality. The linearized implicit finite difference method is established in Section 3. Section 4 analyzes the theoretical results including the unique solvability, convergence, and stability. The numerical results are presented in Section 5. Concluding remarks are provided in the last section.

2 | PRELIMINARIES

2.1 | Spatial discretization

In the past few years, numerous researchers have focused on the approximation of the Riesz fractional derivative, including Meerschaert et al. [32, 33], Yang et al. [49], Deng et al. [42, 55], Ortigueira [34], Celik and Duman [5], Zhao et al. [54], and many others. In the current paper, Riesz fractional centered difference discretization for the fractional derivative will be adopted.

**Lemma 1** (see [5]). Let $f \in C^5(\mathbb{R}) \cap L^1(\mathbb{R})$ and all spatial derivatives of $f$ up to fifth order are $L^1(\mathbb{R})$. Then

$$L_2^\alpha f(x) = -\frac{1}{h^2} \sum_{j=-\infty}^{\infty} c_j^\alpha f(x - jh_x) + O(h^2),$$

for $1 < \alpha \leq 2$, where

$$c_0^\alpha = \frac{\Gamma(\alpha + 1)}{\left(\Gamma\left(\frac{\alpha}{2} + 1\right)\right)^2}, \quad c_s^\alpha = \frac{(-1)^s \Gamma(\alpha + 1)}{\Gamma\left(\frac{\alpha}{2} - s + 1\right) \Gamma\left(\frac{\alpha}{2} + s + 1\right)} = \left(1 - \frac{\alpha + 1}{\frac{\alpha}{2} + s}\right) c_{s-1}^\alpha, \quad \text{for } s \in \mathbb{Z},$$

and

$$c_j^\alpha = c_{-j}^\alpha \leq 0, \quad \text{for } j = \pm 1, \pm 2, \ldots \quad (5)$$

2.2 | Fractional Sobolev norm

An infinite 2-D mesh grid can be denoted by $Z_h = \{(x_j = jh_x, y_k = kh_y), j, k \in \mathbb{Z}\}$. For any two grid functions $U = \{U_{j,k}\}$, $V = \{V_{j,k}\}$ defined on $Z_h$, the discrete inner product and the norms are defined as

$$(U, V) = h_x h_y \sum_{j,k \in \mathbb{Z}} U_{j,k}^* V_{j,k}, \quad \|U\|_2 = \sqrt{(U, U)}, \quad \|U\|_\infty = \max_{j,k \in \mathbb{Z}} |U_{j,k}|,$$

where $U^*$ is the complex conjugate of $U$. Let $L_h^2 = \{V | V \in Z_h, \|V\|_2 < +\infty\}$.

The semi-discrete Fourier transformation $\hat{V}(k_1, k_2) \in L^2\left(\left[\frac{-\pi}{h_x}, \frac{\pi}{h_x}\right] \times \left[\frac{-\pi}{h_y}, \frac{\pi}{h_y}\right]\right)$ is defined as

$$\hat{V}(k_1, k_2) = \frac{h_x h_y}{2\pi} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} V_{j,k} e^{-ik_1 x_j - ik_2 y_k}$$

and the inversion formula is given by

$$V_{j,k} = \frac{1}{2\pi} \int_{-\frac{\pi}{h_x}}^{\frac{\pi}{h_x}} \int_{-\frac{\pi}{h_y}}^{\frac{\pi}{h_y}} \hat{V}(k_1, k_2) e^{ik_1 x_j + ik_2 y_k} dk_1 dk_2.$$
Let $H$ Using these definitions, we have a discrete version of the summation property of Sobolev's norms,

$$\int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \hat{U}(k_1, k_2) \hat{V}^*(k_1, k_2) dk_1 dk_2. \quad (9)$$

Therefore, we can also use a definition $\| U \|_H^2 = \left( \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} | \hat{U}(k_1, k_2)|^2 dk_1 dk_2 \right)^{\frac{1}{2}}$.

For a given constant $\alpha \in (1, 2]$, the fractional Sobolev norm $\| V \|_{H^\alpha}$ and semi-norms $\| V \|_{H^\alpha}$, $\| V \|_{H^\alpha}$ can be defined as follows

$$\| V \|_{H^\alpha}^2 = \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \left( |k_1|^{\alpha} + |k_2|^{\alpha} + 2 |k_1 k_2|^{\alpha} + |k_1|^{2\alpha} + |k_2|^{2\alpha} \right) \hat{V}(k_1, k_2)^2 dk_1 dk_2. \quad (10)$$

$$\| V \|_{H^\alpha}^\infty = \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \left( |k_1|^{\alpha} + |k_2|^{\alpha} \right) \hat{V}(k_1, k_2)^2 dk_1 dk_2. \quad (11)$$

$$\| V \|_{H^\alpha}^2 = \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \left( 2 |k_1 k_2|^{\alpha} + |k_1|^{2\alpha} + |k_2|^{2\alpha} \right) \hat{V}(k_1, k_2)^2 dk_1 dk_2. \quad (12)$$

Using these definitions, we have a discrete version of the summation property of Sobolev's norms,

$$\| V \|_{H^\alpha}^2 = \| V \|^2 + \| V \|_{H^\alpha}^\infty + \| V \|_{H^\alpha}^2. \quad (13)$$

Let $H^\alpha_h : = \{ V | V \in Z_h, \| V \|_{H^\alpha} < +\infty \}$.

**Lemma 2** (Discrete fractional Sobolev's inequality). For any $1 < \alpha \leq 2$ and $U \in H^\alpha_h$, it holds that

$$\| U \|_{\infty} \leq C_\alpha \| U \|_{H^\alpha}, \quad (13)$$

where $C_\alpha$ is a positive constant that depends on $\alpha$ but does not depend on $h_x$ or $h_y$.

**Proof.** From the inversion of the Fourier transformation (8), we have

$$\| U \|_{\infty} = \max_{j, k \in \mathbb{Z}} \left| \frac{1}{2\pi} \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \hat{U}(k_1, k_2) e^{i k_1 j + i k_2 k} dk_1 dk_2 \right| \leq \frac{1}{2\pi} \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} | \hat{U}(k_1, k_2) | dk_1 dk_2$$

$$\leq \frac{1}{2\pi} \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \left( |k_1|^{\alpha} + |k_2|^{\alpha} \right) \hat{U}(k_1, k_2)^2 (1 + |k_1|^{\alpha})^{\frac{1}{2}} (1 + |k_2|^{\alpha})^{\frac{1}{2}} dk_1 dk_2$$

$$\leq \frac{1}{2\pi} \left( \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \frac{1}{(1 + |k_1|^{\alpha})(1 + |k_2|^{\alpha})} dk_1 dk_2 \right)^{\frac{1}{2}}$$

$$\times \left( \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} (1 + |k_1|^{\alpha})(1 + |k_2|^{\alpha}) \hat{U}(k_1, k_2)^2 dk_1 dk_2 \right)^{\frac{1}{2}}$$

$$\leq C_\alpha \| U \|_{H^\alpha},$$

where $C_\alpha = \frac{1}{2\pi} \left( \int \frac{\xi}{\eta} \int \frac{\xi}{\eta} \frac{1}{(1 + |k_1|^{\alpha})(1 + |k_2|^{\alpha})} dk_1 dk_2 \right)^{\frac{1}{2}} > 0$. \hfill \blacksquare
Lemma 3 (Fractional seminorm equivalence). For every $1 < \alpha \leq 2$ and $U \in H_h^a$, the second-order difference scheme for the space fractional derivatives is given by [5, 34]

$$
\mathcal{L}_x^\alpha U_{j,k} \triangleq -\frac{1}{h_x^\alpha} \sum_{l=-\infty}^{+\infty} c_l^\alpha U_{j-l,k}, \quad \mathcal{L}_y^\alpha U_{j,k} \triangleq -\frac{1}{h_y^\alpha} \sum_{m=-\infty}^{+\infty} c_m^\alpha U_{j,k-m},
$$

(14)

then

$$
\left(\frac{2}{\pi}\right)^\alpha |U|^2_{H_x^2} \leq (-)(\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U, U) \leq |U|^2_{H_x^2}, \quad \forall \ U \in H_h^a,
$$

and

$$
\left(\frac{2}{\pi}\right)^{2\alpha} |U|^2_{H^2} \leq ((\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U, (\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U) \leq |U|^2_{H^2}, \quad \forall \ U \in H_h^a.
$$

Proof. Applying the semi-discrete Fourier transform on $\mathcal{L}_x^\alpha U_{j,k} + \mathcal{L}_y^\alpha U_{j,k}$, and using the generating functions

$$
\sum_{l=-\infty}^{+\infty} c_l^\alpha e^{ik_1x_l} = \sin \left(\frac{k_1 h_x}{2}\right)^\alpha, \quad \sum_{m=-\infty}^{+\infty} c_m^\alpha e^{ik_2y_m} = \sin \left(\frac{k_2 h_y}{2}\right)^\alpha,
$$

we conclude that

$$
(\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U_{j,k} = -\left(\frac{1}{h_x^\alpha} \sum_{l=-\infty}^{+\infty} c_l^\alpha e^{ik_1x_l} + \frac{1}{h_y^\alpha} \sum_{m=-\infty}^{+\infty} c_m^\alpha e^{ik_2x_m}\right) \hat{U}(k_1, k_2)
$$

$$
= -\left(\frac{1}{h_x^\alpha} 2 \sin \left(\frac{k_1 h_x}{2}\right)^\alpha + \frac{1}{h_y^\alpha} 2 \sin \left(\frac{k_2 h_y}{2}\right)^\alpha\right) \hat{U}(k_1, k_2).
$$

The Parseval’s identity leads to that

$$
(-)(\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U, U)
$$

$$
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{h_x^\alpha} \left[2 \sin \left(\frac{k_1 h_x}{2}\right)^\alpha + \frac{1}{h_y^\alpha} \left[2 \sin \left(\frac{k_2 h_y}{2}\right)^\alpha\right]\right]|\hat{U}(k_1, k_2)|^2\,dk_1dk_2.
$$

Since $k_1 \in [-\frac{\pi}{h_x}, \frac{\pi}{h_x}]$ and $k_2 \in [-\frac{\pi}{h_y}, \frac{\pi}{h_y}]$, then $\frac{k_1 h_x}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\frac{k_2 h_y}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. It hold a few estimates:

$$
\frac{2}{\pi} \left|\frac{k_1 h_x}{2}\right| \leq \sin \left(\frac{k_1 h_x}{2}\right) \leq \left|\frac{k_1 h_x}{2}\right|, \quad \frac{2}{\pi} \left|\frac{k_2 h_y}{2}\right| \leq \sin \left(\frac{k_2 h_y}{2}\right) \leq \left|\frac{k_2 h_y}{2}\right|,
$$

$$
\left(\frac{2}{\pi}\right)^\alpha |k_1|^\alpha \leq \frac{1}{h_x^\alpha} \left[2 \sin \left(\frac{k_1 h_x}{2}\right)^\alpha\right] \leq |k_1|^\alpha, \quad \left(\frac{2}{\pi}\right)^\alpha |k_2|^\alpha \leq \frac{1}{h_y^\alpha} \left[2 \sin \left(\frac{k_2 h_y}{2}\right)^\alpha\right] \leq |k_2|^\alpha.
$$

Therefore,

$$
\left(\frac{2}{\pi}\right)^\alpha \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (|k_1|^\alpha + |k_2|^\alpha)|\hat{U}(k_1, k_2)|^2\,dk_1dk_2
$$

$$
\leq (-)(\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U, U)
$$

$$
\leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (|k_1|^\alpha + |k_2|^\alpha)|\hat{U}(k_1, k_2)|^2\,dk_1dk_2,
$$

which can be understood as

$$
\left(\frac{2}{\pi}\right)^\alpha |U|^2_{H_x^2} \leq (-)(\mathcal{L}_x^\alpha + \mathcal{L}_y^\alpha)U, U) \leq |U|^2_{H_x^2}, \quad \forall \ U \in H_h^a.
$$
Again, using the Parseval’s identity, we have
\[(\mathcal{L}_x^a + \mathcal{L}_y^a)U, (\mathcal{L}_x^a + \mathcal{L}_y^a)U)\]
\[= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\frac{h_1 h_2}{2}\right)^2 \left(\frac{h_1 h_2}{2}\right)^2 |\hat{U}(k_1, k_2)|^2 dk_1 dk_2.\]
Therefore,
\[\left(\frac{2}{\pi}\right)^2 \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |k_1|^a |k_2|^a |\hat{U}(k_1, k_2)|^2 dk_1 dk_2 \leq\]
\[\|(\mathcal{L}_x^a + \mathcal{L}_y^a)U, (\mathcal{L}_x^a + \mathcal{L}_y^a)U\|_{H^a} \leq |U|^2_{H^a}, \quad \forall \ U \in H^a_h. \]

3 | A THREE-LEVEL LINEARIZED IMPLICIT DIFFERENCE SCHEME

Since the solution of problem (1) and (2) vanishes as \((x, y) \to \infty\), then in practical calculations, the original problem can be regarded as that defined on a sufficiently large but bounded domain with homogeneous Dirichlet boundary condition. Namely,
\[u(x, y, t) = 0, \quad (x, y) \text{ on } \partial \Omega, \quad t \in [0, T], \tag{15}\]
where \(
\Omega = [a, b] \times [c, d], \ a, \ c \text{ are sufficiently large negative numbers and } b, \ d \text{ are sufficiently large positive numbers such that the truncation error is negligible. In what follows, we will make no distinction between } u(x, y, t) \text{ defined in } \mathbb{R}^2 \text{ and } u(x, y, t) \text{ defined in } \Omega = [a, b] \times [c, d], \text{ and always suppose that the equation is augmented by the zero-boundary constraint.}
\]

Let \(N, M_x, \) and \(M_y\) be positive integers, and define \(\Delta t = T/N, \ h_x = (b - a)/M_x, \ h_y = (d - c)/M_y\) to be the time step and spatial mesh-size, respectively. Now \(\Omega \times [0, T]\) can therefore be covered by
\[\Omega^0_h = \{(x_j, y_k, t_n) | x_j = jh_x, \ y_k = kh_y, \ t_n = n\Delta t, \ j = 0, \ldots, M_x, \ k = 0, \ldots, M_y, \ n = 0, \ldots, N\}. \]
In addition, let \(U^n = (U^n_{j,k})_{(M_x+1)(M_y+1)}\) be the numerical solution at time level \(t = t_n\). Following this definition, we have the discrete Dirichlet boundary condition (15)
\[U^n_{0,k} = U^n_{M_x,k} = U^n_{j,0} = U^n_{j,M_y} = 0, \quad n = 0, \ldots, N. \tag{16}\]

For every grid function \(U^n\), we define four difference operators:
\[U^n_{j,k} = \frac{U^{n+1}_{j,k} + U^{n-1}_{j,k}}{2}, \quad \delta_t U^n_{j,k} = \frac{U^{n+1}_{j,k} - U^{n-1}_{j,k}}{2\Delta t}, \tag{17}\]
\[\delta_t U^{n+\frac{1}{2}}_{j,k} = \frac{1}{\Delta t} (U^{n+1}_{j,k} - U^n_{j,k}), \quad U^n_{j,k} = \frac{U^{n+2}_{j,k} + 2U^n_{j,k} + U^{n-2}_{j,k}}{4}.\]

We also need the discrete inner product and norms for each two grid functions \(U, V\). Define
\[(U, V) = h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} U_{j,k} V^*_j, \quad \|U\|_2 = \sqrt{(U, U)}, \quad \|U\|_\infty = \max_{1 \leq j \leq M_x-1, 1 \leq k \leq M_y-1} |U^n_{j,k}|. \tag{17}\]
We should notice that for any \( U \in \Omega_h^0 \), we can extend the grid function into an infinite-grid-defined function by letting \( U_{j,k} = 0 \) when \((x_j = j\Delta x, y_k = k\Delta y)\) is outside the domain \( \Omega \). Thus the norms and inner product defined in (6) can also be defined for grid functions in \( \Omega_h^0 \). Moreover, Lemma 1 and Lemma 2 are also valid for every \( U \in \Omega_h^0 \). As a result, we will not make distinctions between the inner products and norms defined in (6) and (17) in all the later sections.

Note that Equation (1) can be rewritten as

\[
\mathbf{u}_t = \mathbf{i}L_{x}^\alpha u + \mathbf{i}L_{y}^\alpha u + |\mathbf{u}|^2u.
\]  

For time discretization, we use a three-level linear implicit scheme for (1) around \( t = t_n \), which causes a second-order error of time step:

\[
\frac{u^{n+1} - u^{n-1}}{2\tau} = \mathbf{i}L_{x}^\alpha \frac{u^{n+1} + u^{n-1}}{2} + \mathbf{i}L_{y}^\alpha \frac{u^{n+1} + u^{n-1}}{2} + |u|^2 \frac{u^{n+1} + u^{n-1}}{2} + O(\tau^2).
\]  

After collecting the terms of \( u^{n+1} \) and \( u^{n-1} \) in (19), the above relation is changed into

\[
\left( \frac{1}{\tau} - \mathbf{i} |u|^2 - \mathbf{i}L_{x}^\alpha - \mathbf{i}L_{y}^\alpha \right) u^{n+1} = \left( \frac{1}{\tau} + |u|^2 + \mathbf{i}L_{x}^\alpha + \mathbf{i}L_{y}^\alpha \right) u^{n-1} + O(\tau^2).
\]  

Under the assumption of the homogenous boundary condition (15), the second-order difference scheme for space fractional derivatives (14) is given below

\[
(L_{x}^\alpha U)_{j,k} = -\frac{1}{h_x^\alpha} \sum_{s=j-M_x+1}^{j-1} c_s^\alpha U_{j-s,k} = -\frac{1}{h_y^\alpha} \sum_{s=1}^{M_y-1} c_s^\alpha U_{j,k-1},
\]  

\[
(L_{y}^\alpha U)_{j,k} = -\frac{1}{h_y^\alpha} \sum_{s=k-M_y+1}^{k-1} c_s^\alpha U_{j,k-s} = -\frac{1}{h_x^\alpha} \sum_{s=1}^{M_x-1} c_s^\alpha U_{j,k}.
\]  

At last, our linear finite difference scheme is summarized as follows:

\[
\left( \frac{1}{\tau} - \mathbf{i} |U^{n}_{j,k}|^2 - \mathbf{i}L_{x}^\alpha - \mathbf{i}L_{y}^\alpha \right) U^{n+1}_{j,k} = \left( \frac{1}{\tau} + |U^{n}_{j,k}|^2 + \mathbf{i}L_{x}^\alpha + \mathbf{i}L_{y}^\alpha \right) U^{n-1}_{j,k},
\]  

for \( 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1 \). For such a time-iterating scheme, an initialization is always needed for starting the loop. Considering the given initial data, the solution at the initial step is then given by

\[
U^{0}_{j,k} = (u_0)_{j,k} = u_0(x_j, y_k) \quad 0 < j < M_x, \quad 0 < k < M_y.
\]  

Moreover, since the linear scheme (23) involves three time levels, the first step values \( \{ U^{1}_{j,k} \} \) are also indispensable to begin the loop. We accomplish this by using Taylor’s expansion:

\[
u^1 = u_0 + \tau u_t(x, 0) + \frac{\tau^2}{2} u_{tt}(x, \tilde{\tau})
\]

\[
= u_0 + \tau (\mathbf{i}L_{x}^\alpha u_0 + \mathbf{i}L_{y}^\alpha u_0) + |u_0|^2 u_0 + \frac{\tau^2}{2} u_{tt}(x, \tilde{\tau}),
\]  

for some \( \tilde{\tau} \in (0, \tau) \), and Equation (18) is used here to substitute the time derivative. Hence in the numerical approximation, \( \{ U^{1}_{j,k} \} \) can be obtained by the following second-order scheme

\[
U^{1}_{j,k} = U^{0}_{j,k} + \tau (\mathbf{i}L_{x}^\alpha u_0)_{j,k} + \mathbf{i}L_{y}^\alpha u_0 + \mathbf{i} |u_0|^2 (u_0)_{j,k},
\]  

for \( 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1 \).
4 | THEORETICAL ANALYSIS

The following conclusions are essential for the theoretical analysis of the numerical solution.

Lemma 4  For any $U, V \in \Omega_0^h$, the following product can be taken apart:

$$\delta_t(U^n V^m) = U^n \delta_t V^m + V^m \delta_t U^n. \quad (27)$$

The multiplication in the above formula and the proof below is an element-wise operation of grid functions in $\Omega_0^h$.

Proof. Notice that

$$\delta_t(U^n V^m) = \frac{U^{n+1} V^{n+1} - U^{n-1} V^{n-1}}{2\tau}$$

$$= \frac{(U^{n+1} - U^{n-1}) V^{n+1} + U^{n-1} (V^{n+1} - V^{n-1})}{2\tau}$$

$$= \delta_t U^n V^{n+1} + U^{n-1} \delta_t V^n.$$ 

On the other hand, we can compute symmetrically

$$\delta_t(U^n V^m) = \frac{U^{n+1} V^{n+1} - U^{n-1} V^{n-1}}{2\tau}$$

$$= \frac{(V^{n+1} - V^{n-1}) U^{n+1} + V^{n-1} (U^{n+1} - U^{n-1})}{2\tau}$$

$$= \delta_t V^n U^{n+1} + V^{n-1} \delta_t U^n.$$ 

Taking the average of the two equations, we conclude that

$$\delta_t(U^n V^m) = \frac{U^{n+1} + U^{n-1}}{2} \delta_t V^n + \frac{V^{n+1} + V^{n-1}}{2} \delta_t U^n = U^\theta \delta_t V^n + V^\theta \delta_t U^n.$$ 

\[ \blacksquare \]

Lemma 5  For any $U, V \in \Omega_0^h$, there exists a linear operator $\Lambda^a$ such that

$$(-\mathcal{L}_x^a + \mathcal{L}_y^a)U, V) = (\Lambda^a U, \Lambda^a V). \quad (28)$$

Proof. Let

$$C_x = \frac{1}{h_x^a} \begin{pmatrix} c_0^a & c_{a-1}^a & \cdots & c_{a-M+2}^a \\ c_1^a & c_0^a & \cdots & c_{a-M+3}^a \\ \vdots & \vdots & \ddots & \vdots \\ c_{M-2}^a & c_{M-3}^a & \cdots & c_0^a \end{pmatrix}, \quad C_y = \frac{1}{h_y^a} \begin{pmatrix} c_0^a & c_{a-1}^a & \cdots & c_{a-M+2}^a \\ c_1^a & c_0^a & \cdots & c_{a-M+3}^a \\ \vdots & \vdots & \ddots & \vdots \\ c_{M-2}^a & c_{M-3}^a & \cdots & c_0^a \end{pmatrix}.$$ 

$$D = I_{M-1} \otimes C_x + C_y \otimes I_{M-1},$$

$$\mathcal{U} = (U_{1,1}, \ldots, U_{M-1,1}, U_{1,2}, \ldots, U_{M-1,2}, \ldots, U_{1,M-1}, \ldots, U_{M-1,M-1})^T,$$

$$\mathcal{V} = (V_{1,1}, \ldots, V_{M-1,1}, V_{1,2}, \ldots, V_{M-1,2}, \ldots, V_{1,M-1}, \ldots, V_{M-1,M-1})^T.$$ 

Since $C_x$ and $C_y$ are both real symmetric positive definite matrices [46], $D$ is also a real symmetric positive definite matrix. Form the spectral theorem, there exist a real orthogonal matrix $P$ and a diagonal matrix $A$ with positive diagonal entries such that

$$D = P A P^T = (P A^{1/2} (P^T)^T) (P A^{1/2} (P^T)^T)^T = L^T L,$$

where $L = P A^{1/2} (P^T)^T$ is also a real symmetric positive definite matrix.
Now that
\[-(L_x^a + L_y^a)U = D \mathcal{U},\]  
(29)
hence
\[-(L_x^a + L_y^a)U, V = h_x h_y \mathcal{V}^H D \mathcal{U} = h_x h_y \mathcal{V}^H L^T L \mathcal{U} = h_x h_y (L \mathcal{V})^H L \mathcal{U} = (L U, L V),\]  
(30)
where $\mathcal{V}^H$ is the Hermitian transpose of $\mathcal{V}$. If we define the linear operator $\Lambda^a U = L U$ for any $U \in \Omega_0^n$, then we can obtain (28). ▪

By using Lemma 5, the following lemma is easy to verify.

**Lemma 6**  
For any $U^n \in \Omega_0^n$, we have
\[\text{Im}((L_x^a + L_y^a)U, U) = 0,\]  
(31)
\[\text{Re}(-(L_x^a + L_y^a)U^n, \delta_t U^n) = \frac{1}{4\tau} (\|\Lambda^a U^{n+1}\|_2^2 - \|\Lambda^a U^{n-1}\|_2^2).\]  
(32)

**Lemma 7**  (Discrete Gronwall’s inequality [12, 18]). Let $\{u_k\}$ and $\{w_k\}$ be nonnegative sequences and $\alpha$ be a nonnegative constant. They together satisfy
\[u_n \leq \alpha + \sum_{0 \leq k < n} w_k u_k \quad \text{for} \quad n \geq 0.\]

Then for all $n \geq 0$, it holds that
\[u_n \leq \alpha \exp \left( \sum_{0 \leq k < n} w_k \right).\]

### 4.1 Energy conservation

It can be verified that (23) on the mesh grids is equivalent to
\[
\frac{U_{j,k}^{n+1} - U_{j,k}^{n-1}}{2\tau} + \left( L_x^a \frac{U_{j,k}^{n+1} + U_{j,k}^{n-1}}{2} + L_y^a \frac{U_{j,k}^{n+1} + U_{j,k}^{n-1}}{2} \right) + |U_{j,k}^n|^2 \frac{U_{j,k}^{n+1} + U_{j,k}^{n-1}}{2} = 0. 
\]

(33)

Now we prove the following discrete energy conservative law, which is similar to the continuous case.

**Theorem 1**  The schemes (33) with the initialization (26) are conservative in the sense of discrete energy:
\[E^n = E^0, \quad 1 \leq n \leq N - 1,\]
(34)
where
\[E^n = \frac{1}{2} (\|\Lambda^a U^{n+1}\|_2^2 + \|\Lambda^a U^n\|_2^2) - \frac{1}{2} h_x h_y \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} |U_{j,k}^n|^2 |U_{j,k}^{n+1}|^2.\]
(35)

**Proof.** Taking inner product of (33) with $-\delta_t U^n$, we have
\[-i \|\delta_t U^n\|_2^2 - (L_x^a U^n + L_y^a U^n, \delta_t U^n) - (|U^n|^2 U^n, \delta_t U^n) = 0.\]
(36)
Now consider the real part of (36). It follows from Lemma 5 and Lemma 6 that
\[
\frac{1}{4\tau}(||A^aU^{n+1}||^2_2 - ||A^aU^{n-1}||^2_2) - h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U_{j,k}^n|^2 \cdot \frac{1}{4\tau}(|U_{j,k}^{n+1}|^2 - |U_{j,k}^{n-1}|^2) = 0.
\]
Therefore,
\[
\frac{1}{2}(||A^aU^{n+1}||^2_2 + ||A^aU^{n-1}||^2_2) - \frac{1}{2} h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U_{j,k}^n|^2 \cdot |U_{j,k}^{n+1}|^2 = \frac{1}{2}(||A^aU^{n}||^2_2 + ||A^aU^{n-2}||^2_2) - \frac{1}{2} h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |U_{j,k}^{n-1}|^2 \cdot |U_{j,k}^n|^2.
\]
This completes the proof.

\[\blacksquare\]

4.2 | Solvability of difference scheme

**Theorem 2**  The finite difference scheme (16), (23), (24), and (26) is uniquely solvable.

**Proof.** Since the schemes (16), (23), (24), and (26) are derived form a linear system of equations, it is sufficient to show that the homogeneous linear system has only zero solution. The data of initial level and the first level have been uniquely determined by (24) and (26). Now we suppose that \(U^1, \ldots, U^n\) have already been uniquely obtained, then the iteration yields a new linear system for \(U^{n+1}\)

\[
\frac{i}{\tau}U_{j,k}^{n+1} + (L_x^a U_{j,k}^{n+1} + L_y^a U_{j,k}^{n+1}) + |U_{j,k}^n|^2 U_{j,k}^{n+1} = 0.
\]

Computing the inner product of \(U^{n+1}\) with both sides of the above equation, and following the result from Lemma 5, we can show that \(||U^n||_2 = 0, n = 2, 3, \ldots\) This implies, by a closed induction, that the finite difference schemes (16), (23), (24), and (26) are uniquely solvable.

\[\blacksquare\]

4.3 | \(L^\infty\) convergence and stability

Let \(u(x, y, t)\) be the exact solution of problem (1) and (2) and \(U_{j,k}^n\) be the solution of the numerical schemes (16), (23), (24), and (26). Define \(u_{j,k}^n = u(x_j, y_k, t_n)\). Then the error function is the difference between \(u\) and \(U\):

\[e_{j,k}^n = u_{j,k}^n - U_{j,k}^n, \quad j = 1, 2, \ldots, M_x - 1, \quad k = 1, 2, \ldots, M_y - 1, \quad n = 1, 2, \ldots, N.\]

By substituting \(U\) with \(u\) in the schemes (16), (23), and (24), we can define the truncation errors as follows:

\[
r_{j,k}^n = \frac{1}{2} \left( \frac{1}{\tau} - i |u_{j,k}^n|^2 - iL_x^a - iL_y^a \right) u_{j,k}^{n+1} - \frac{1}{2} \left( \frac{1}{\tau} + i |u_{j,k}^n|^2 + iL_x^a + iL_y^a \right) u_{j,k}^{n-1},
\]

for \(1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1\).

Through simple calculations, we see that

\[
r_{j,k}^n = \delta_j u_{j,k}^{n-1} - i(L_x^a + L_y^a) u_{j,k}^n - i|u_{j,k}^n|^2 u_{j,k}^n,
\]

for \(1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1\).
However, it follows from the difference scheme (23) that
\[ 0 = \delta_i U_{j,k}^n - i(\mathcal{L}_x^a + \mathcal{L}_y^a)U_{j,k}^n - i|U_{j,k}^n|^2 U_{j,k}^n, \]  
for \( 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1. \)

This implies that we obtain a relation between the error and the truncation error, by subtracting (39) from (40):
\[ E_{j,k}^n = \delta_i e_{j,k}^n - i(\mathcal{L}_x^a + \mathcal{L}_y^a)e_{j,k}^n - iP_{j,k}^n, \]
where
\[ P_{j,k}^n = (|u_{j,k}^n|^2 u_{j,k}^n - |U_{j,k}^n|^2 U_{j,k}^n), \]
for \( 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1. \)

**Lemma 8** Suppose that the solution of problem (1) and (2) is sufficiently smooth and satisfies the homogeneous Dirichlet’s boundary condition. Then it holds that
\[ |E_{j,k}^n| \leq C_R(\tau^2 + h_x^2 + h_y^2), \]
\[ |\delta_i E_{j,k}^n| \leq C_R(\tau^2 + h_x^2 + h_y^2), \]
\[ 1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1, 1 \leq n \leq N - 1, \]
where \( C_R \) is a positive constant independent of \( \tau, h_x, \) and \( h_y. \)

**Proof.** Using the Taylor’s expansion of the exact solution \( u(x, y, t) \) at \( (x_j, y_k, t_n) \), we have
\[ \frac{u(x_j, y_k, t_{n+1}) - u(x_j, y_k, t_{n-1})}{2\tau} = \frac{\partial^2 u(x_j, y_k, t_n) + s\tau}{\partial t^2} \]
\[ = \frac{\tau^2}{4} \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1 - s)^2 ds. \]

From the proof in [5], we have
\[ \mathcal{L}_x^a u(x_j, y_k, t_{n+1}) + u(x_j, y_k, t_n) = \frac{\tau^2}{2} \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1 - s) ds + O(h_x^2). \]

Similarly,
\[ \mathcal{L}_y^a u(x_j, y_k, t_{n+1}) + u(x_j, y_k, t_n) = \frac{\tau^2}{2} \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1 - s) ds + O(h_y^2). \]

In addition, the nonlinear term can also be expanded
\[ |u(x_j, y_k, t_n)|^2 \frac{u(x_j, y_k, t_{n+1}) + u(x_j, y_k, t_{n-1})}{2} - |u(x_j, y_k, t_n)|^2 u(x_j, y_k, t_n) \]
\[ = |u(x_j, y_k, t_n)|^2 \left( \frac{u(x_j, y_k, t_{n+1}) + u(x_j, y_k, t_{n-1})}{2} - u(x_j, y_k, t_n) \right) \]
\[ = |u(x_j, y_k, t_n)|^2 \frac{\tau^2}{2} \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1 - s) ds. \]
Summarizing all the computations, it holds

\[
\begin{align*}
 r_{j,k}^n &= \frac{\tau^2}{4} \int_0^1 \left( \frac{\partial^3 u(x_j, y_k, t_n + s\tau)}{\partial t^3} + \frac{\partial^3 u(x_j, y_k, t_n - s\tau)}{\partial t^3} \right) (1-s)^2 ds \\
&- \frac{\tau^2}{2} (L_x^n + L_y^n) \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1-s) ds \\
&- \frac{\tau^2}{2} \left| u(x_j, y_k, t_n) \right|^2 \int_0^1 \left( \frac{\partial^2 u(x_j, y_k, t_n + s\tau)}{\partial t^2} + \frac{\partial^2 u(x_j, y_k, t_n - s\tau)}{\partial t^2} \right) (1-s) ds \\
&+ O(h_\tau^2 + O(h_y^2)). 
\end{align*}
\]

Therefore,

\[
\delta_r^n r_{j,k} = O(\tau^2 + h_\tau^2 + h_y^2),
\]

By using (43), it will be easy to see that

\[
\delta_r^n r_{j,k} = O(\tau^2 + h_\tau^2 + h_y^2), \quad 1 \leq j \leq M_x - 1, \quad 1 \leq k \leq M_y - 1, \quad 1 \leq n \leq N - 1,
\]

completing the proof.

Following a similar proof of Lemma 9 in [14], we can obtain the lemma below.

**Lemma 9** Suppose that the solution of problem (1) and (2) is sufficiently smooth and vanishes as \((x, y) \to \infty\). Then for the difference scheme (26), we have

\[
\begin{align*}
\left| e_{j,k}^n \right| &\leq C_e (\tau^2 + \tau h_\tau^2 + \tau h_y^2), \\
\left| (L_x^a + L_y^a) e_{j,k}^n \right| &\leq C_e (\tau^2 + \tau h_\tau^2 + \tau h_y^2), \\
1 &\leq j \leq M_x - 1, \quad 1 \leq k \leq M_y - 1, \quad 1 \leq n \leq N - 1,
\end{align*}
\]

where \(C_e\) is a positive constant independent of \(\tau\) and \(h\).

**Theorem 3** Let us denote \(e^n = u^n - U^n\). Suppose that the solution of problem (1) and (2) is smooth enough and vanishes as \((x, y) \to \infty\). Then there exist two small positive constants \(\tau_0\) and \(h_0\), such that, when \(\tau < \tau_0\) and \(h_\tau < h_0\), \(h_y < h_0\), the numerical solution \(U^n\) of the difference scheme (16), (23), (24), and (26) converges to the exact solution \(u^n\) in the sense of \(L^\infty\)-norm, with an optimal convergence order \(O(\tau^2 + h_\tau^2 + h_y^2)\), that is,

\[
\| e^n \|_{\infty} \leq C_0 (\tau^2 + h_\tau^2 + h_y^2), \quad 1 \leq n \leq N,
\]

where \(C_0\) is a positive constant independent of \(\tau, h_\tau,\) and \(h_y\).

**Proof.** We use mathematical induction to prove the result. Instead of directly showing (44), we will prove a stronger conclusion as follow:

\[
\| e^n \|_2 + \| \Lambda^a e^n \|_2 + \| (L_x^a + L_y^a) e^n \|_2 \leq C^0 (\tau^2 + h_\tau^2 + h_y^2), \quad 1 \leq n \leq N,
\]

which, once verified, can easily derive the estimate (44). Because it follows directly from Lemma 2, Lemma 3 and some basic inequalities that

\[
\| e^n \|_{\infty} \leq C_a \| e^n \|_{H^a} \leq C_a \left( \| e^n \|_2 + \| e^n \|_{H^a} + \| e^n \|_{H^a}^2 \right) \\
\leq C_a \left( \frac{\pi}{2} \right)^{2a} \left( \| e^n \|_2 + \| \Lambda^a e^n \|_2 + \| (L_x^a + L_y^a) e^n \|_2 \right)
\]

(47)
\[ \leq C_a \left( \frac{\pi}{2} \right)^{2a} C^0(\tau^2 + h_x^2 + h_y^2) \] (48)

\[ := C_0(\tau^2 + h_x^2 + h_y^2). \] (49)

First, it can be proved that (45) holds for \( n = 1 \). As a matter of fact, from Lemma 9, it holds

\[ \|e^1\|_2 = (e^1, e^1) \leq h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |e_{j,k}^1| \cdot |e_{j,k}^1| \]

\[ \leq (b - a)(d - c) \max_{1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1} |e_{j,k}^1| \cdot |e_{j,k}^1| \]

\[ \leq (b - a)(d - c) C_e^2(\tau^2 + h_x^2 + h_y^2)^2, \]

and similarly,

\[ \|L^a e^1\|_2 = ((L_x^a + L_y^a)e^1, e^1) \leq h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |(L_x^a + L_y^a)e_{j,k}^1| \cdot |e_{j,k}^1| \]

\[ \leq (b - a)(d - c) \max_{1 \leq j \leq M_x - 1, 1 \leq k \leq M_y - 1} |(L_x^a + L_y^a)e_{j,k}^1| \cdot |e_{j,k}^1| \]

\[ \leq (b - a)(d - c) C_e^2(\tau^2 + h_x^2 + h_y^2)^2, \]

Taking \( C^0 = 3 \sqrt{(b - a)(d - c)} C_e \), then the estimate for \( n = 1 \) is valid.

Now the induction is started. Assume that (45) is true for \( m \leq n \). We want to show that (45) is also valid for \( n + 1 \). By the assumption, we have

\[ \|e^m\|_\infty \leq C_0(\tau^2 + h_x^2 + h_y^2), \quad 1 \leq m \leq n, \]

and

\[ \|U^m\|_\infty \leq \|u^m\|_\infty + \|e^m\|_\infty \]

\[ \leq C_m + C_0(\tau^2 + h_x^2 + h_y^2) \]

\[ \leq C_m + C_0(\tau^2 + 2b^2) \]

\[ \leq C_m + 1, \quad 1 \leq m \leq n, \] (50)
Step 1. Computing the discrete inner product of (41) with $e^h$ and taking the real part of the resulting equation, we have

$$\frac{\|e^{n+1}\|_2^2 - \|e^n\|_2^2}{4\tau} = \text{Re}[(r^n, e^h)] - \text{Im}[(P^n, e^h)]$$

$$\leq \|r^n\|_2 \|e^h\|_2 + \|P^n\|_2 \|e^h\|_2$$

$$\leq \frac{1}{2} \|r^n\|_2^2 + \frac{1}{2} \|P^n\|_2^2 + \|e^h\|_2^2$$

$$\leq \frac{1}{2} \|r^n\|_2^2 + \frac{1}{2} \|P^n\|_2^2 + \frac{1}{2} \|e^{n+1}\|_2^2 + \frac{1}{2} \|e^{n-1}\|_2^2,$$

(51)

where

$$P^n_{j,k} = |u^n_{j,k}|^2 u^n_{i,j,k} - \hat{U}^n_{j,k} \hat{U}^n_{j,k}$$

$$= (|u^n_{j,k}|^2 - |U^n_{j,k}|^2) u^n_{i,j,k} + |U^n_{j,k}|^2 e^h_{j,k}$$

$$= (|u^n_{j,k}|^2 - |U^n_{j,k}|^2)(|u^n_{j,k}| + |U^n_{j,k}|) u^n_{i,j,k} + |U^n_{j,k}|^2 e^h_{j,k}.$$  

(52)

It follows from the assumption (50) that

$$\|P^n\|_2 \leq (\|u^n\|_\infty + \|U^n\|_\infty)\|u^n\|_\infty \|e^n\|_2 + \|U^n\|_2 \|e^h\|_2$$

$$\leq (C_m + C_m + 1) C_m \|e^n\|_2 + (C_m + 1)^2 \|e^h\|_2$$

$$\leq C_1 (\|e^n\|_2 + \|e^h\|_2)$$

$$\leq C_1 (\|e^{n-1}\|_2 + \|e^n\|_2 + \|e^{n+1}\|_2),$$

(53)

for $\tau < \tau_1$ and $h < h_1$, where

$$C_1 = 2(C_m + 1)^2.$$

Therefore,

$$\|P^n\|_2^2 \leq 3C_1^2 (\|e^{n-1}\|_2^2 + \|e^n\|_2^2 + \|e^{n+1}\|_2^2),$$

(54)

for $\tau < \tau_1$ and $h < h_1$. Following (51) and (54),

$$\|e^{n+1}\|_2^2 - \|e^{n-1}\|_2^2 \leq 2\tau \|r^n\|_2^2 + (12C_1^2 + 2)\tau (\|e^{n-1}\|_2^2 + 2\|e^n\|_2^2 + \|e^{n+1}\|_2^2).$$

(55)

Now define $E_{n+1} := \|e^{n+1}\|_2^2 + \|e^n\|_2^2$, then (55) can be simplified into

$$E_{n+1} - E_n \leq (12C_1^2 + 2)\tau (E_{n+1} + E_n) + 2\tau \|r^n\|_2^2.$$

(56)

This is equivalent to

$$(1 - (12C_1^2 + 2)\tau) (E_{n+1} - E_n) \leq (24C_1^2 + 4)\tau E_n + 2\tau \|r^n\|_2^2.$$

(57)

Let $\tau_1 = \frac{1}{24C_1^2 + 4}$, then as $\tau < \tau_1$, we have $1 - (12C_1^2 + 2) \geq \frac{1}{2}$, and

$$E_{n+1} - E_n \leq (48C_1^2 + 8)\tau E_n + 4\tau \|r^n\|_2^2.$$

(58)

Replacing $n$ by $k$ above and summing $k$ over $1$ by $n$, then we obtain that

$$E_{n+1} \leq (48C_1^2 + 8)\tau \sum_{k=1}^n E_k + 4\tau \sum_{k=1}^n \|r^k\|_2^2 + E_1.$$

(59)
Since Lemma 9 shows that $|e_k^j| \leq C_e(\tau^2 + \tau h_\nu^2 + \tau^2 h_\nu^2)$ for some positive constant $C_e$ and $1 \leq j \leq M_x - 1$, $1 \leq k \leq M_y - 1$, then it apparently holds that

$$E_1 = \|e^1\|_2^2 = h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} |e_k^j|^2 \leq h_x h_y \sum_{j=1}^{M_x-1} \sum_{k=1}^{M_y-1} C_e^2(\tau^2 + h_\nu^2 + h_\nu^2)^2$$

$$\leq (b - a)(d - c)C_e^2(\tau^2 + h_\nu^2 + h_\nu^2)^2,$$

where $\tau < 1$ is used again. In addition, Lemma 8 provides that

$$4\tau \sum_{k=1}^{n} \|r^k\|_2^2 \leq 4TC_e^2(\tau^2 + h_\nu^2 + h_\nu^2)^2.$$ (60)

This prior estimate can be applied to set a bound:

$$E_{n+1} \leq (48C_e^2 + 8)\sum_{k=1}^{n} E_k + (4TC_e^2 + (b - a)(d - c)C_e^2)(\tau^2 + h_\nu^2 + h_\nu^2)^2$$

$$= (48C_e^2 + 8)\sum_{k=1}^{n} E_k + C_2(\tau^2 + h_\nu^2 + h_\nu^2)^2,$$ (62)

Now the discrete Gronwall’s inequality shows us that

$$\frac{1}{4\tau} \|e^{n+1}\|_2^2 + \|e^n\|_2^2 = E_{n+1} \leq C_2 \exp\left(\sum_{k=1}^{n} (48C_e^2 + 8)\tau\right) \left(\tau^2 + h_\nu^2 + h_\nu^2\right)^2$$

$$\leq C_2 e^{(48C_e^2+8)\tau} \left(\tau^2 + h_\nu^2 + h_\nu^2\right)^2$$

$$=: C_3(\tau^2 + h_\nu^2 + h_\nu^2)^2,$$ (63)

**Step 2.** To carry out further estimations, we compute the discrete inner product of (41) with $(\mathcal{L}_x^a + \mathcal{L}_y^a)e^n$ and consider the real part of the resulting equation. Now we have a new relation

$$\frac{1}{4\tau} \|\Lambda^a e^{n+1}\|_2^2 - \|\Lambda^a e^{n-1}\|_2^2$$

$$= Re[\langle r^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^n \rangle] - Im[\langle P^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^n \rangle]$$

$$\leq \|r^n\|_2\|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2 + \|r^n\|_2\|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2 + \|P^n\|_2\|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2$$

$$+ \|P^n\|_2\|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2$$

$$\leq \|r^n\|_2^2 + \|P^n\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2^2.$$ (64)

We should notice what we have concluded at the end of Step 1,

$$\|P^n\|_2^2 \leq C_1^2(\|e^{n+1}\|_2^2 + 2\|e^n\|_2^2 + \|e^{n-1}\|_2^2) = 3C_1^2(E_{n+1} + E_n) \leq 6C_1^2C_3(\tau^2 + h_\nu^2 + h_\nu^2)^2.$$ (65)

Again, we compute the discrete inner product of (41) with $(\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^n$ and analyze the imaginary part of the resulting equation, namely,

$$\frac{1}{4\tau} \|\Lambda^a e^{n+1}\|_2^2 - \|\Lambda^a e^{n-1}\|_2^2$$

$$= -Im[\langle r^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^n \rangle] - Re[\langle P^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^n \rangle].$$ (66)

Now combining (64) and (66), we have

$$\|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2$$

$$+ \|\Lambda^a e^{n+1}\|_2^2$$

$$- \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2^2$$

$$- \|\Lambda^a e^{n-1}\|_2^2$$

$$= \|P^n\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2$$

$$+ \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2^2$$

$$- \|\Lambda^a e^{n+1}\|_2^2$$

$$- \|\Lambda^a e^{n-1}\|_2^2.$$
Using some basic estimates, we also have that
\[ \|r^n\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n-1}\|_2^2 \]

Using the same method as what appears in Step 1, we denote that
\[ F_{n+1} = \|((\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^n\|_2^2 + \|\Lambda e^{n+1}\|_2^2 + \|\Lambda e^n\|_2^2, \]
by which the inequality (67) is reduced to
\[ F_{n+1} - F_n \leq 4\tau(F_{n+1} + F_n) - 4\tau Im[(r^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^n)] - 4\tau Re[(P^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^n)]. \]

This can be further estimated as \( \tau < \frac{1}{8} \), that
\[ F_{n+1} - F_n \leq 16\tau F_n - 8\tau Im[(r^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^n)] - 8\tau Re[(P^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^n)] + 8\tau(\|r^n\|_2^2 + \|P^n\|_2^2). \]

We replace the index \( n \) by \( k \) in (69) and sum over \( k \) from 1 to \( n \). It yields
\[ F_{n+1} - F_1 \leq 16\tau \sum_{k=1}^n F_k - 8\tau \sum_{k=1}^n Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^k)] - 8\tau \sum_{k=1}^n Re[(P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^k)] + 8\tau(\|r^k\|_2^2 + \|P^k\|_2^2). \]

**Step 3.** In order to use the discrete Gronwall’s inequality, we need to make some further estimations. First, following from Lemma 8 and our previous consequences, we obtain
\[ 8\tau \sum_{k=1}^n(\|r^k\|_2^2 + \|P^k\|_2^2) \leq 8T(C_k^2 + 6C_k^2C_3)(\tau^2 + h_k^2 + h_k^2)^2 = C_4(\tau^2 + h_k^2 + h_k^2)^2. \]

Using some basic estimates, we also have that
\[ 8\tau \sum_{k=1}^n Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^k)] = 16\tau \sum_{k=1}^n \frac{1}{2\tau} Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)(e^{k+1} - e^{k-1})]\]
\[ = 8\tau \sum_{k=1}^n \frac{1}{2\tau} Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1})] - 8\tau \sum_{k=1}^n \frac{1}{2\tau} Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1})]\]
\[ = 8\tau \sum_{k=2}^{n+1} \frac{1}{2\tau} Im[(r^{k-1}, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k)] - 8\tau \sum_{k=0}^{n-1} \frac{1}{2\tau} Im[(r^{k+1}, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k)]\]
\[ = 4Im[(r^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}) + (r^{n-1}, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^n) - (r^1, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^0)\]
\[ - (r^0, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^1)] - 8\tau \sum_{k=2}^{n-1} Im[(\delta_t r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k)]. \]

It follows from Lemma 8 and Lemma 9, after some simple calculations,
\[ 8\tau \sum_{k=1}^n Im[(r^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_t e^k)] \leq 4[\|r^n\|_2 \cdot \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2 + \|r^{n-1}\|_2 \cdot \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^n\|_2] \]
\[ + \| r^2 \|_2 \cdot \| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2 \| + 8\tau \sum_{k=2}^{n-1} \| \delta_i r^k \|_2 \cdot \| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2 \]

\[ \leq 4 \left[ 4\| r^0 \|_2^2 + \frac{1}{16}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1} \|_2^2 + 4\| r^{k-1} \|_2^2 + \frac{1}{16}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 + \frac{1}{2}\| r^2 \|_2^2 \right] \]

\[ + \frac{1}{2}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 + 4\| \sum_{k=2}^{n-1} \| \delta_i r^k \|_2^2 + 4\| \sum_{k=2}^{n-1} \| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 \]

\[ = 16\| r^0 \|_2^2 + 16\| r^{k-1} \|_2^2 + 2\| r^2 \|_2^2 + 2\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 + \frac{1}{4}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1} \|_2^2 \]

\[ + \frac{1}{4}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 + 4\| \sum_{k=2}^{n-1} \| \delta_i r^k \|_2^2 + 4\| \sum_{k=2}^{n-1} \| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 \]

\[ \leq 4\| \sum_{k=2}^{n-1} \| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 + \frac{1}{4}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1} \|_2^2 + \frac{1}{4}\| (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k \|_2^2 \]

\[ + C_5(\tau^2 + h_x^2 + h_y^2) \]

(72)

Since \( \| r^0 \|_2^2 \) and \( \| \delta_i r^k \|_2^2 \) are all bounded by a constant multiple of \((\tau^2 + h_x^2 + h_y^2)^2\), we can use \( C_5 \) as a general positive constant to represent the sum of all coefficients of \((\tau^2 + h_x^2 + h_y^2)^2\) in the above estimation.

Next we need to estimate \( 8\tau \sum_{k=1}^{n} \text{Re}\{(P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^k)\} \). We notice that

\[ 8\tau \sum_{k=1}^{n} \text{Re}\{(P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^k)\} \]

\[ = 4\text{Re}(P^0, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1}) + \text{Re}(P^{k-1}, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k) - \text{Re}(P^1, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^0) \]

\[ - \text{Re}(P^2, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^1) - 8\tau \sum_{k=2}^{n-1} \text{Re}(\delta_i P^k, e^k), \]

(73)

where

\[ P^k = |u^k|^2 u^k - |U^k|^2 U^k \]

\[ = u^k(u^k)^* u^k - (u^k - e^k)((u^k)^* - (e^k)^*)(u^k - e^k) \]

\[ = (u^k)^* u^k e^k + u^k u^k(e^k)^* + u^k(u^k)^* e^k - (e^k)^* e^k u^k - e^k e^k(u^k)^* \]

\[ - e^k(e^k)^* u^k + e^k(e^k)^* e^k \]

(74)

and

\[ \delta_i P^k = \delta_i((u^k)^* u^k e^k) + \delta_i(u^k u^k(e^k)^*) + \delta_i(u^k(e^k)^* e^k) - \delta_i((e^k)^* e^k u^k) \]

\[ - \delta_i(e^k e^k(u^k)^*) - \delta_i(e^k(e^k)^* u^k) + \delta_i(e^k(e^k)^* e^k) = \sum_{j=1}^{n} \delta_i \]

(75)

In order to estimate \( \delta_i e^k \) and similar terms which shows in the expansion of \( \delta_i P^k \), we first use the truncation error (41) to find

\[ ||\delta_i e^k||_2 \leq ||e^k||_2 + ||P^k||_2 + ||\mathcal{L}_x^a + \mathcal{L}_y^a||_2 \]

\[ \leq ||e^k||_2 + ||P^k||_2 + ||(\mathcal{L}_x^a + \mathcal{L}_y^a)e^k||_2 + ||(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}||_2 \]

(76)

Next, we observe that the exact solution \( u \) is sufficiently smooth, so that \( ||u||_\infty \) and \( ||u_t||_\infty \) must be bounded in the domain \([0, T] \times [a, b] \times [c, d]\). In addition, we may as well
take use of the result in Step 1, that is, \( \| e^{n+1} \|_2 + \| e^n \|_2 \leq \sqrt{3C_3(\tau^2 + h_x^2 + h_y^2)} < C \) as \( \tau, h_x, h_y < 1 \). Therefore, combining with Lemma 4, the seven items can be estimated as follows:

\[
\| I_1 \|_2 = \| \delta_t(u^k) e^k u^k + \delta_t e^k(u^k)^* e^k + \delta_t u^k (u^* e)^k \|_2 \\
\leq \| u_t \|_\infty \cdot \| u \|_\infty \cdot \| e^k \|_2 + \| u \|_\infty^2 \cdot \| \delta_t e^k \|_2 + \| u_t \|_\infty \cdot \| u \|_\infty (\| e^{k+1} \|_2 + \| e^{k-1} \|_2) \\
\leq C(\| e^{k+1} \|_2 + \| e^{k-1} \|_2 + \| \delta_t e^k \|_2) \\
\leq C(\| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+1} \|_2 + \| (L_x^a + L_y^a)e^{k-1} \|_2),
\]

\[
\| I_2 \|_2 = \| \delta_t u^k (e^k)^* u^k + \delta_t (e^k)^* u^k e^k + \delta_t u^k (u^* e)^k \|_2 \\
\leq \| u_t \|_\infty \cdot \| u \|_\infty \cdot \| e^k \|_2 + \| u \|_\infty^2 \cdot \| \delta_t e^k \|_2 + \| u_t \|_\infty \cdot \| u \|_\infty (\| e^{k+1} \|_2 + \| e^{k-1} \|_2) \\
\leq C(\| e^{k+1} \|_2 + \| e^{k-1} \|_2 + \| \delta_t e^k \|_2) \\
\leq C(\| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+1} \|_2 + \| (L_x^a + L_y^a)e^{k-1} \|_2),
\]

\[
\| I_3 \|_2 = \| \delta_t u^k (e^k)^* e^k + \delta_t (e^k)^* u^k e^k + (uu^*)^k \delta_t e^k \|_2 \\
\leq 2\| u_t \|_\infty \cdot \| u \|_\infty \cdot \| e^k \|_2 + \| u \|_\infty^2 \cdot \| \delta_t e^k \|_2 \\
\leq C(\| e^{k+2} \|_2 + \| e^{k-2} \|_2 + \| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2).
\]

Especially for \( 2 \leq k \leq n-1 \), the estimation (76) only involves values from the first \( n \)th time layers, thus by the induction assumption, we have \( \| \delta_t e^k \|_2 \leq C \) when \( \tau, h_x, h_y < 1 \). By using the estimate (46) under the induction assumption and Lemma 2, we have

\[
\| I_4 \|_2 = \| \delta_t u^k (e^k)^* e^k + \delta_t (e^k)^* u^k e^k + \delta_t u^k (u^* e)^k \|_2 \\
\leq \| u_t \|_\infty \cdot \| e^k \|_\infty \cdot \| e^k \|_2 + \| u \|_\infty \cdot \| \delta_t e^k \|_2 \cdot \| e^k \|_2 + \| u \|_\infty (\| e^{k+1} \|_\infty + \| e^{k-1} \|_\infty) \cdot \| \delta_t e^k \|_2 \\
\leq C(\| e^{k+2} \|_2 + \| e^{k-2} \|_2 + \| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2).
\]

\[
\| I_5 \|_2 = \| \delta_t u^k (e^k)^* e^k + \delta_t (e^k)^* u^k e^k + \delta_t u^k (u^* e)^k \|_2 \\
\leq \| u_t \|_\infty \cdot \| e^k \|_\infty \cdot \| e^k \|_2 + \| u \|_\infty \cdot \| \delta_t e^k \|_2 \cdot \| e^k \|_2 + \| u \|_\infty (\| e^{k+1} \|_\infty + \| e^{k-1} \|_\infty) \cdot \| \delta_t e^k \|_2 \\
\leq C(\| e^{k+2} \|_2 + \| e^{k-2} \|_2 + \| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2).
\]

\[
\| I_6 \|_2 = \| \delta_t u^k (e^k)^* u^k + \delta_t (e^k)^* u^k e^k + \delta_t u^k (u^* e)^k \|_2 \\
\leq 2\| u_t \|_\infty \cdot \| e^k \|_\infty \cdot \| \delta_t e^k \|_2 + \| u_t \|_\infty (\| e^{k+1} \|_\infty + \| e^{k-1} \|_\infty) \cdot \| \delta_t e^k \|_2 \\
\leq C(\| e^{k+2} \|_2 + \| e^{k-2} \|_2 + \| e^{k+1} \|_2 + \| e^{k-1} \|_2 + ||P_k||_2 + \| (L_x^a + L_y^a)e^{k+2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2 + \| (L_x^a + L_y^a)e^{k-2} \|_2).
\]
\[ \leq C(\|e^{k+1}\|_2 + \|e^{k-1}\|_2 + \|\delta_i e^k\|_2) \]
\[ \leq C(\|e^{k+1}\|_2 + \|e^{k-1}\|_2 + \|P^k\|_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1}\|_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}\|_2), \]

\[ \|I_7\|_2 = \|\delta_i e^k(e^k) + \delta_i(e^k) e^k + \delta_i e^k(e^k)\|_2 \]
\[ \leq 2\|\delta_i e^k\|_2\|e^k\|_\infty\|e^k\|_\infty + \|\delta_i e^k\|_2(\|e^{k+1}\|_\infty^2 + \|e^{k-1}\|_2^2) \]
\[ \leq C(\|e^k\|_\infty + \|\delta_i e^k\|_2) \]
\[ \leq C(\|e^{k+2}\|_{H^0} + \|\delta_i e^k\|_2) \]
\[ \leq C(\|e^{k+2}\|_{H^0} + \|e^k\|_{H^0} + \|e^{k-2}\|_{H^0} + \|\delta_i e^{k+1}\|_2 + \|\delta_i e^{k-1}\|_2) \]
\[ \leq C(\|e^{k+2}\|_2 + \|e^k\|_2 + \|e^{k-2}\|_2 + \|\Lambda^a e^{k+1}\|_2 + \|\Lambda^a e^{k-1}\|_2 + \|\Lambda^a e^{k-2}\|_2) \]
\[ + C(\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_e^{k+2})_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^k\|_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}\|_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-2}\|_2 \]
\[ + \|r^{k+1}\|_2 + \|r^{k-1}\|_2 + \|P^{k+1}\|_2 + \|P^{k-1}\|_2). \] (77)

When \( \tau, \eta_x, \eta_y < 1 \).

Now we have for \( 2 \leq k \leq n - 1, \)

\[ \|\delta_i P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k\| \]
\[ \leq \sum_{i=1}^7 \|I_i\|_2(\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_e^k) \]
\[ \leq C(\|e^{k+2}\|_2^2 + \|e^{k+1}\|_2^2 + \|e^k\|_2^2 + \|e^{k-1}\|_2^2 + \|e^{k-2}\|_2^2) \]
\[ + C(\|\Lambda^a e^{k+2}\|_2^2 + \|\Lambda^a e^{k+1}\|_2^2 + \|\Lambda^a e^k\|_2^2 + \|\Lambda^a e^{k-1}\|_2^2 + \|\Lambda^a e^{k-2}\|_2^2) \]
\[ + C(\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_e^{k+2})_2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^k\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-2}\|_2^2 \]
\[ + C(\|r^{k+1}\|_2^2 + \|r^{k-1}\|_2^2 + \|P^{k+1}\|_2^2 + \|P^{k-1}\|_2^2). \]

Therefore, it follows from Lemma 9 and the estimation (65), that

\[ 8\tau \sum_{k=1}^n Re[(P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)\delta_i e^k)] \]
\[ = 4[Re(P^n, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}) + Re(P^{n-1}, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^n) - Re(P^1, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^0) \]
\[ - Re(P^2, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^1)] - 8\tau Re \sum_{k=2}^{n-1} (\delta_i P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k) \]
\[ \leq 8\tau \sum_{k=2}^{n-1} (\delta_i P^k, (\mathcal{L}_x^a + \mathcal{L}_y^a)e^k) + 4 \left( |P^n|^2 + 4 |P^{n-1}|^2 + \frac{1}{16} \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{n+1}\|_2^2 \right) \]
\[ + 4 \left( \frac{1}{16} \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^n\|_2^2 + (6C^2 C^2 + C^2)(\tau^2 + h_x^2 + h_y^2)^2 \right) \]
\[ \leq 8\tau \sum_{k=2}^{n-1} \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^k\|_2^2 \]
\[ + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-2}\|_2^2 \]
\[ + 8\tau \sum_{k=2}^{n-1} \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+2}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k+1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^k\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-1}\|_2^2 + \|(\mathcal{L}_x^a + \mathcal{L}_y^a)e^{k-2}\|_2^2 \].
\[ + 8Ct \sum_{k=2}^{n-1} (\|e^{k+2}\|_2^2 + \|e^{k+1}\|_2^2 + \|e^k\|_2^2 + \|e^{k-1}\|_2^2 + \|e^{k-2}\|_2^2) \]

\[ + 8Ct \sum_{k=2}^{n-1} (\|r^{k+1}\|_2^2 + \|r^k\|_2^2 + \|r^{k-1}\|_2^2 + \|r^{k-2}\|_2^2 + \|r^{k-3}\|_2^2) \]

\[ + 8\tau C (\|e^{k+1}\|_2^2 + \|e^{k+1}\|_2^2 + \|e^k\|_2^2 + \|e^{k-1}\|_2^2 + \|e^{k-2}\|_2^2) \]

\[ + 16\|P^\alpha\|_2^2 + 16\|P^{\alpha-1}\|_2^2 + \frac{1}{4}((\mathcal{L}_x^a + \mathcal{L}_y^a)e^{\alpha+1}\|_2^2 + \frac{1}{4}((\mathcal{L}_x^a + \mathcal{L}_y^a)e^{\alpha}\|_2^2 \]

\[ + 4(6C_1C_3 + C_2^2)(\tau^2 + h_\alpha^2 + h_\gamma^2)^2 \]

\[ \leq 40Ct \sum_{k=1}^{n} (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\Lambda^\alpha e^k\|_2^2) + (8Ct + \frac{1}{4}) (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\Lambda^\alpha e^\alpha\|_2^2) \]

\[ + C_6(\tau^2 + h_\alpha^2 + h_\gamma^2)^2. \]  

(78)

Since \(\|e^{\alpha+1}\|_2^2 + \|e^\alpha\|_2^2 + \|P^\alpha\|_2^2\), and \(\|r^\alpha\|_2^2\) are all bounded by constant multiples of \((\tau^2 + h_\alpha^2 + h_\gamma^2)^2\), we can use \(C_6\) as a general positive constant to represent the sum of all coefficients of \((\tau^2 + h_\alpha^2 + h_\gamma^2)^2\) in the above estimation.

Now we put these estimations (72)–(78) backward into the inequality (70), to obtain

\[ F_{n+1} - F_1 \]

\[ \leq 16\tau \sum_{k=1}^{n} F_k + (40Ct + 4\tau) \sum_{k=1}^{n} (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\Lambda^\alpha e^k\|_2^2) \]

\[ + (8Ct + \frac{1}{2}) (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2) \]

\[ + 8\tau (\|\Lambda^\alpha e^{\alpha+1}\|_2^2 + \|\Lambda^\alpha e^{\alpha}\|_2^2) + (C_4 + C_5)(\tau^2 + h_\alpha^2 + h_\gamma^2)^2 \]

\[ \leq (40Ct + 20\tau) \sum_{k=1}^{n} F_k + (8Ct + \frac{1}{2}) (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2) \]

\[ + 8\tau (\|\Lambda^\alpha e^{\alpha+1}\|_2^2 + \|\Lambda^\alpha e^{\alpha}\|_2^2) + (C_4 + C_5 + C_6)(\tau^2 + h_\alpha^2 + h_\gamma^2)^2. \]  

(79)

Furthermore, let \(\tau_3 = \frac{1}{32C}\). If we take \(\tau < \tau_3\), then it holds that \(1/2 - 8Ct \geq 1/4\). And since

\[ \left( \frac{1}{2} - 8Ct \right) F_{n+1} - F_1 \]

\[ \leq F_{n+1} - F_1 - \left( \frac{1}{2} + 8Ct \right) (\|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2 + \|\mathcal{L}_x^a + \mathcal{L}_y^a\|_2^2) \]

\[ - 8\tau (\|\Lambda^\alpha e^{\alpha+1}\|_2^2 + \|\Lambda^\alpha e^{\alpha}\|_2^2) \]

\[ \leq (40Ct + 20\tau) \sum_{k=1}^{n} F_k + (C_4 + C_5 + C_6)(\tau^2 + h_\alpha^2 + h_\gamma^2)^2, \]  

(80)

then we have an estimation for \(F_{n+1}\):

\[ F_{n+1} \leq 4F_1 + 4(40C + 20)\tau \sum_{k=1}^{n} F_k + 4(C_4 + C_5 + C_6)(\tau^2 + h_\alpha^2 + h_\gamma^2)^2. \]  

(81)
As for the rest terms, from the results of Lemma 9,
\[ F_1 = \| (L_x^a + L_y^a)e_1 \|^2_2 + \| \Lambda e_1 \|^2_2 \leq 2(b-a)(d-c)C_\varepsilon (\tau^2 + h_x^2 + h_y^2)^2. \] (82)
Substituting (82) into (81) gives
\[ F_{n+1} \leq (160C + 80) \tau \sum_{k=1}^{n} F_k + C_7 (\tau^2 + h_x^2 + h_y^2)^2, \]
where \( C_7 \) is the sum of all coefficients of \((\tau^2 + h_x^2 + h_y^2)^2\) in the resulting estimation. Thus by the discrete Gronwall’s equation, it yields
\[ F_{n+1} \leq C_7 \exp \left( \sum_{k=1}^{n} (160C + 80) \tau \right) (\tau^2 + h_x^2 + h_y^2)^2 \]
\[ \leq C_7 \exp \{(160C + 80)T\}(\tau^2 + h_x^2 + h_y^2)^2 := C(\tau^2 + h_x^2 + h_y^2)^2. \] (83)
Now we combine the consequences (63) and (83). Then it holds that
\[
\| e^{n+1} \|^2_2 + \| \Lambda e^{n+1} \|^2_2 + \| (L_x^a + L_y^a)e^{n+1} \|^2_2 \\
\leq \| e^{n+1} \|^2_2 + \| \Lambda e^{n+1} \|^2_2 + \| (L_x^a + L_y^a)e^{n+1} \|^2_2 \\
= E_{n+1} + F_{n+1} \\
\leq (C_3 + C_8)(\tau^2 + h_x^2 + h_y^2)^2.
\]
From the mean value inequality, we see
\[
\| e^{n+1} \|^2_2 + \| \Lambda e^{n+1} \|^2_2 + \| (L_x^a + L_y^a)e^{n+1} \|^2_2 \leq \sqrt{3} \sqrt{\| e^{n+1} \|^2_2 + \| \Lambda e^{n+1} \|^2_2 + \| (L_x^a + L_y^a)e^{n+1} \|^2_2}.
\]
Thus we need to take \( C^0 = \max\{3(C_3 + C_8), 3\sqrt{(b-a)(d-c)C_\varepsilon} \} \). Since \( C^0 = C_a(\pi/2)^{2a} \cdot C^0 \), once \( C^0 \) is fixed, the condition for \( \tau_1, h_1 \), that is, \( \tau_1^2 + 2h_1^2 < 1/C_0 \) can be used to determine \( \tau_1, h_1 \). Therefore, let \( \tau_0 = \min\{\tau_1, \tau_2, \tau_3, 1\}, h_0 = \min\{h_1, 1\}, \) and \( h_0^2 = \min\{h_1, 1\} \), then (45) is valid for \( n + 1 \). This completes the mathematical induction, i.e., it holds that
\[
\| e^{n+1} \|^2_2 + \| \Lambda e^{n+1} \|^2_2 + \| (L_x^a + L_y^a)e^{n+1} \|^2_2 \leq C(\tau^2 + h_x^2 + h_y^2)^2, \quad 1 \leq n \leq N, \] (84)
which also shows that
\[
\| e^n \| \leq C_a \left( \frac{\pi}{2} \right)^{2a} C^0(\tau^2 + h_x^2 + h_y^2) =: C(\tau^2 + h_x^2 + h_y^2)^2, \quad 1 \leq n \leq N. \] (85)

**Theorem 4** Suppose that the solution of problem (1) and (2) is smooth enough, then the solution \( U_n \) of the difference scheme (23)–(26) is bounded in the \( L^\infty \)-norm for \( \tau \leq \tau_0 \) and \( h \leq h_0 \), that is,
\[
\| U_n \| \leq C_\varepsilon, \quad 1 \leq n \leq N, \] (86)
where \( \tau_0 \) and \( h_0 \) are the same positive constants in Theorem 3.

**Proof:** From Theorem 3, we have
\[
\| U^n \| \leq \| u^n \| + \| e^n \| \leq C_m + C_0(\tau^2 + 2h^2), \quad 1 \leq n \leq N,
\]
for \( \tau < \tau_0, h < h_0. \)
| $\tau = h$ | $\alpha = 1.2$ | $\alpha = 1.5$ | $\alpha = 1.9$ | $\alpha = 2.0$ |
| --- | --- | --- | --- | --- |
| | $\|u^n - U^n\|_\infty$ | Order | $\|u^n - U^n\|_\infty$ | Order | $\|u^n - U^n\|_\infty$ | Order | $\|u^n - U^n\|_\infty$ | Order |
| $1/16$ | $4.62e-02$ | — | $3.01e-02$ | — | $4.70e-02$ | — | $2.08e-02$ | — |
| $1/32$ | $9.67e-03$ | 2.26 | $1.05e-02$ | 1.52 | $7.60e-03$ | 2.63 | $5.55e-03$ | 1.91 |
| $1/64$ | $1.81e-03$ | 2.42 | $2.29e-03$ | 2.19 | $1.55e-03$ | 2.30 | $1.27e-03$ | 2.13 |
| $1/128$ | $5.48e-04$ | 2.17 | $5.24e-04$ | 2.13 | $3.89e-04$ | 2.19 | $3.07e-04$ | 2.05 |
| $1/256$ | $1.21e-04$ | 2.18 | $1.26e-04$ | 2.06 | $1.08e-04$ | 2.15 | $7.65e-05$ | 2.00 |

Since $C_0(\tau^2 + 2h^2) < 1$ for $\tau < \tau_0, h < h_0$, we have

$$\|U^n\|_\infty \leq C_s, \quad 1 \leq n \leq N,$$

where $C_s = C_m + 1$. This completes the proof.

## 5 | NUMERICAL RESULTS

In this section, we present some numerical results of the proposed difference scheme (16), (23), (24) and (26) to support our main theoretical findings. In the computation, uniform spatial meshsizes $h_x = h_y = h$ are used.

### Example 1

In order to test the accuracy and verify the unconditional stability of the proposed scheme, we consider the following equation with source terms:

$$u_t(x, y, t) = iL_\alpha u(x, y, t) + i|u(x, y, t)|^2 u(x, y, t) + g(x, y, t).$$

The initial conditions, and the source term $g(x, y, t)$ are determined by the exact solution

$$u(x, y, t) = i \sin(t)x^4(2 - x)^4y^4(2 - y)^4.$$

We note that the solution satisfies homogeneous Dirichlet condition

$$u|_{\partial \Omega} = 0, \quad \text{where} \quad \Omega = [0, 2] \times [0, 2].$$

Table 1 shows the maximum norm errors and corresponding convergence rates for various choices of $\alpha$. It can be easily seen that the convergence order of the proposed scheme approaches to 2, which is consistent with Theorem 3 in the above section.

To show the unconditional stability of the method, we fix $h$ and vary $\tau$, results for $\alpha = 1.2$ and $\alpha = 1.5$ are plotted in Figure 1. As we can see that the results clearly show that the time step is not related to the spatial meshsize, and as the time steps go to zero, the dominant errors come from the spatial parts.

### Example 2

In this example, we compute the practical problem with the initial conditions as follows:

$$u(x, y, 0) = \frac{2}{\sqrt{\pi}} \exp\{-(x^2 + y^2)\}, \quad (x, y) \in \Omega \triangleq [-5, 5] \times [-5, 5].$$
Since the initial value $u(x, y, 0)$ exponentially decays to zero when $(x, y)$ is away from the origin, we impose the homogeneous boundary condition in the computation

$$u(x, y, 0) = 0, \quad (x, y) \text{ on } \partial \Omega.$$  

To show the discrete energy conservative property, we choose $\tau = 1/20$, $h = 1/20$ and present the discrete energy $E^n$ at different times for different $\alpha$ ($\alpha = 1.2, \alpha = 1.5, \alpha = 1.8$) in Table 2. It is found that the proposed scheme preserves the energy conservation property very well. Therefore, the scheme is suitable for long-time simulation.

Taking $\tau = 1/16$ and $h = 1/16$, we plot the numerical solutions that derived from the proposed scheme to investigate the propagation of solution profile by using different $\alpha$. The results are shown in Figures 2–4. One can see that the value of $\alpha$ significantly affects the shape of the solution. The larger $\alpha$ is, the more heterogeneity of the solution at late time stages is.

### CONCLUDING REMARKS

In this paper, we developed a linearized semi-implicit finite difference scheme for 2D SFNSE. A rigorous analysis of the proposed finite difference scheme is carried out, which includes the conservation,
FIGURE 2 The propagation of solutions with $\alpha = 1.3$, Example 2

(a) $\alpha = 1.3, \ T = 0.5$

(b) $\alpha = 1.3, \ T = 1.0$

(c) $\alpha = 1.3, \ T = 2.0$

(d) $\alpha = 1.3, \ T = 4.0$

FIGURE 3 The propagation of solutions with $\alpha = 1.7$, Example 2

(a) $\alpha = 1.7, \ T = 0.5$

(b) $\alpha = 1.7, \ T = 1.0$

(c) $\alpha = 1.7, \ T = 2.0$

(d) $\alpha = 1.7, \ T = 4.0$
the unique solvability, the unconditional stability and the second-order convergence in the $L^\infty$-norm. What is worth mentioning is that the theoretical conclusions such as the optimal pointwise error estimate are presented for the first time in this field. In the end, we implemented the difference scheme through two numerical tests, which showed a perfect consistency with our theoretical findings.

In addition, our numerical method and the analysis technique of the optimal pointwise error estimates can be easily extended to the cases with spatial fourth-order accuracy [54], the 2D coupled SFNSE [40], 2D space fractional Ginzburg–Landau equation [37, 53] and some other space fractional diffusion equation in 2D and 3D [15, 51, 52, 56], which will be our future work. In addition, for the resulting systems of algebraic equations, the coefficient matrices have Toeplitz structure, which can be solved by adopting a super-fast solver with preconditioner [23, 27–29] or multigrid methods [36, 38] to reduce the CPU time and storage requirement in future.

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CONFLICT OF INTEREST
The authors declare that they have no conflict of interest.
DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Taking real part of (A.2), we have
\[ \Re \{ u_t(x, y, t) u^* (x, y, t) \} = \frac{1}{2} \| u(x, y, t) \|^2, \]
and integrating in \( \mathbb{R}^2 \), we have
\[ \frac{d}{dt} \int_{\mathbb{R}^2} \frac{1}{2} |u(x, y, t)|^2 dx dy = 0. \]

It implies that the first equation holds in (4).

Multiplying (A.1) by \(-u_t^* (x, y, t)\) and integrating in \( \mathbb{R}^2 \), we have
\[ -i \int_{\mathbb{R}^2} |u_t(x, y, t)|^2 dx dy - \int_{\mathbb{R}^2} L_{u_t} u_t(x, y, t) u^* (x, y, t) dx dy - \int_{\mathbb{R}^2} \| u(x, y, t) \|^2 u_t(x, y, t) u^* (x, y, t) dx dy = 0. \]

Once notices that
\[ - \int_{\mathbb{R}^2} L_{u_t} u_t(x, y, t) u^* (x, y, t) dx dy = \int_{\mathbb{R}^2} (\Delta)^{\frac{\alpha}{2}} u(x, y, t) \frac{\partial}{\partial t} \left( (\Delta)^{\frac{\alpha}{2}} u(x, y, t) \right) dx dy. \]

Taking real part of (A.3), we have
\[ - \Re \{ L_{u_t} u_t(x, y, t) u^* (x, y, t) dx dy \} = \frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \| (\Delta)^{\frac{\alpha}{2}} u(x, y, t) \|^2 \right) dx dy \]
\[ = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \| (\Delta)^{\frac{\alpha}{2}} u(x, y, t) \|^2 dx dy. \]

Taking real part of (A.2), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \| (\Delta)^{\frac{\alpha}{2}} u(x, y, t) \|^2 dx dy - \frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^2} \| u(x, y, t) \|^4 dx dy = 0, \]
which implies that the second equation holds in (4).