Research Article

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Degree bounds for modular covariants

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Abstract: Let $V, W$ be representations of a cyclic group $G$ of prime order $p$ over a field $k$ of characteristic $p$. The module of covariants $k[V, W]^G$ is the set of $G$-equivariant polynomial maps $V \to W$, and is a module over $k[V]^G$. We give a formula for the Noether bound $\beta(k[V, W]^G, k[V]^G)$, i.e. the minimal degree $d$ such that $k[V, W]^G$ is generated over $k[V]^G$ by elements of degree at most $d$.

Keywords: Invariant theory, modular representation, cyclic group, module of covariants, Noether bound

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1 Introduction

Let $G$ be a finite group, $k$ a field and $V$, $W$ a pair of finite-dimensional $kG$-modules. Let $k[V]$ denote the symmetric algebra on the dual $V^*$ of $V$ and let $k[V, W] = k[V] \otimes_k W$. Elements of $k[V]$ represent polynomial functions $V \to k$ and elements of $k[V, W]$ represent polynomial functions $V \to W$; for $f \otimes w \in k[V, W]$ the corresponding function takes $v$ to $f(v)w$. The group $G$ acts by algebra automorphisms on $k[V]$ and hence diagonally on $k[V, W]$. The fixed points $k[V, W]^G$ of this action are called covariants and represent $G$-equivariant polynomial functions $V \to W$. The the fixed points $k[V]^G$ are called invariants. For $f \in k[V]^G$ and $\phi \in k[V, W]^G$ we define the product

$$f\phi(v) = f(v)\phi(v).$$

Then $k[V]^G$ is a $k$-algebra and $k[V, W]^G$ is a finite $k[V]^G$-module. Modules of covariants in the non-modular case ($|G| \neq 0 \in k$) were studied by Chevalley [3], Shephard–Todd [10], Eagon–Hochster [7]. In the modular case far less is known, but recent work of Broer and Chuai [1] has shed some light on the subject. A systematic attempt to construct generating sets for modules of covariants when $G$ is a cyclic group of order $p$ was begun by the first author in [5].

Let $A = \bigoplus_{d \geq 0} A_d$ be any graded $k$-algebra and $M = \sum_{d \geq 0} M_d$ any graded $A$-module, where $A_d$ and $M_d$ denote the $d$-th homogeneous components of $A$ and $M$, respectively. Then the Noether bound $\beta(A)$ is defined to be the minimum degree $d > 0$ such that $A$ is generated by the set $\{a : a \in A_k, k \leq d \}$. Similarly, $\beta(M, A)$ is defined to be the minimum degree $d > 0$ such that $M$ is generated over $A$ by the set $\{m : m \in M_k, k \leq d \}$, and we write $\beta(M) = \beta(M, A)$ when the context is clear.

Noether famously showed that $\beta(C[V]^G) \leq |G|$ for arbitrary finite $G$, but computing Noether bounds in the modular case is highly nontivial. When $G$ is cyclic of prime order, the second author along with Fleischmann, Shank and Woodcock [6] determined the Noether bound for any $kG$-module. The purpose of this article is to find results similar to those in [6] for covariants. Our main result can be stated concisely as follows.

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Theorem 1. Let $G$ be a cyclic group of order $p$, $k$ a field of characteristic $p$, $V$ a reduced $kG$-module and $W$ a nontrivial indecomposable $kG$-module. Then

$$\beta(k[V, W]^G) = \beta(k[V]^G)$$

unless $V$ is indecomposable of dimension $2$.

Here by reduced we mean that the direct sum decomposition of $V$ contains no summands on which $G$ acts trivially; see also remarks following Proposition 4.

2 Preliminaries

For the rest of this article, $G$ denotes a cyclic group of order $p > 0$, and we let $k$ be a field of characteristic $p$. We choose a generator $\sigma$ for $G$. Over $k$, there are $p$ indecomposable representations $V_1, \ldots, V_p$ and each indecomposable representation $V_i$ is afforded by a Jordan block of size $i$. Note that $V_p$ is isomorphic to the free module $k\sigma$, and this is the unique free indecomposable $kG$-module.

Let $\Delta = \sigma - 1 \in kG$. We define the transfer map $\text{Tr} : k[V] \to k[V]$ by $\sum_{1 \leq i \leq p} \sigma^i$. Notice that we also have $\text{Tr} = \Delta^{p-1}$. Invariants that are in the image of $\text{Tr}$ are called transfers.

Remark 2. Let $e_1, \ldots, e_t$ be an upper triangular basis for the $i$-dimensional indecomposable representation $V_i$. Then $\Delta(e_j) = e_{j-1}$ for $2 \leq j \leq i$ and $\Delta(e_1) = 0$. Therefore $\Delta(V_i) = 0$ for all $j \geq 1$. Note that for any indecomposable module $V_i$ we have $\Delta(V_i) = V_{i-1}$ for $2 \leq i \leq p$ and $\Delta(V_1) = 0$. It follows that an invariant $f$ is in the image of the linear map $\Delta' : k[V] \to k[V]$ if and only if it is a linear combination of fixed points in indecomposable modules of dimension at least $j + 1$. In particular, an invariant is in the image of the transfer map ($= \Delta^{p-1}$) if and only if it is a linear combination of fixed points of free $kG$-modules.

We assume that $V$ and $W$ are $kG$-modules with $W$ indecomposable and we choose a basis $w_1, \ldots, w_n$ for $W$ so that we have

$$\sigma w_i = \sum_{1 \leq j \leq i} (-1)^{i-j} w_j$$

for $1 \leq i \leq n$. For $f \in k[V]$ we define the weight of $f$ to be the smallest positive integer $d$ with $\Delta^d(f) = 0$. Note that $\Delta^p = (\sigma - 1)^p = 0$, so the weight of a polynomial is at most $p$.

A useful description of covariants is given in [5]. We include this description here for completeness.

Proposition 3 ([5, Proposition 3]). Let $f \in k[V]$ with weight $d \leq n$. Then

$$\sum_{1 \leq j \leq d} \Delta^{d-1}(f) w_j \in k[V, W]^G.$$

Conversely, if

$$f_1 w_1 + f_2 w_2 + \cdots + f_n w_n \in k[V, W]^G,$$

then there exists $f \in k[V]$ with weight $\leq n$ such that $f_j = \Delta^{d-1}(f)$ for $1 \leq j \leq n$.

For a non-zero covariant $h = f_1 w_1 + f_2 w_2 + \cdots + f_n w_n$, we define the support of $h$ to be the largest integer $j$ such that $f_j \neq 0$. We denote the support of $h$ by $s(h)$. We shall say $h$ is a transfer covariant if there exists a non-negative integer $k$ and $f \in k[V]$ such that $f_1 = \Delta^k(f), f_2 = \Delta^{k+1}(f), \ldots, f_{s(h)} = \Delta^{p-1}(f)$ for some $f \in k[V]$.

We call a homogeneous invariant in $k[V]^G$ indecomposable if it is not in the subalgebra of $k[V]^G$ generated by invariants of strictly smaller degree. Similarly, a homogeneous covariant in $k[V, W]^G$ is indecomposable if it does not lie in the submodule of $k[V, W]^G$ generated by covariants of strictly smaller degree.

3 Upper bounds

We first prove a result on decomposability of a transfer covariant. In the proof below we set $\gamma = \beta(k[V], k[V]^G)$. 
Proposition 4. Let \( f \in \mathbb{k}[V] \) be homogeneous and let \( h = \Delta^k(f)w_1 + \Delta^{k+1}(f)w_2 + \cdots + \Delta^{p-1}(f)w_{s(h)} \) be a transfer covariant of degree \( > \gamma \). Then \( h \) is decomposable.

Proof. Let \( g_1, \ldots, g_t \) be a set of homogeneous polynomials of degree at most \( \gamma \) generating \( \mathbb{k}[V] \) as a module over \( \mathbb{k}[V]^G \). So we can write \( f = \sum_{1 \leq i \leq t} q_i g_i \), where each \( q_i \in \mathbb{k}[V]^G \) is a positive degree invariant. Since \( \Delta^j \) is \( \mathbb{k}[V]^G \)-linear, we have \( \Delta^j(f) = \sum_{1 \leq i \leq t} q_i \Delta^j(g_i) \) for \( k \leq j \leq p-1 \). It follows that

\[
  h = \sum_{1 \leq i \leq t} q_i (\Delta^j(g_i)w_1 + \cdots + \Delta^{p-1}(g_i)w_{s(h)}).
\]

Note that \( \Delta^j(g_i)w_1 + \cdots + \Delta^{p-1}(g_i)w_{s(h)} \) is a covariant for each \( 1 \leq i \leq t \) by Proposition 3. We also have \( q_i \in \mathbb{k}[V]^G \) so it follows that \( h \) is decomposable.

Write \( V = \bigoplus_{j=1}^m V_n \) as a sum of indecomposable modules. Note that

\[
  \mathbb{k}[\bigoplus_{j=1}^m V_n] = \mathbb{k}[V]^G = (S(V^*) \otimes S(V_1^*)) \otimes W^G = \mathbb{k}[V] \otimes \mathbb{k}[V_1].
\]

Therefore we will assume that \( n_j > 1 \) for all \( j \); such representations are called reduced. Choose a basis \( \{x_{i,j} : 1 \leq i \leq n_j, 1 \leq j \leq m\} \) for \( V^* \), with respect to which we have

\[
  \sigma(x_{i,j}) = \begin{cases} x_{i,j} + x_{i+1,j}, & i < n_j, \\ x_{i,j}, & i = n_j. \end{cases}
\]

This induces a multidegree on \( \mathbb{k}[V] = \bigoplus_{d \in \mathbb{N}^m} \mathbb{k}[V]_d \) which is compatible with the action of \( G \). For \( 1 \leq j \leq m \) we define \( N_j = \prod_{i=0}^{n_j-1} \sigma^j x_{1,j} \), and note that the coefficient of \( x_{1,j}^p \) in \( N_j \) is 1. Given any \( f \in \mathbb{k}[V_n] \), we can therefore perform long division, writing

\[
  f = q_j N_j + r,
\]

where \( q_j \in \mathbb{k}[V_n] \) for all \( j \) and \( r \in \mathbb{k}[V_n] \) has degree \( < p \) in the variable \( x_{1,j} \). This induces a vector space decomposition

\[
  \mathbb{k}[V_n] = N_j \mathbb{k}[V_n] \oplus B_j,
\]

where \( B_j \) is the subspace of \( \mathbb{k}[V_n] \) spanned by monomials with \( x_{1,j} \)-degree \( < p \), but the form of the action implies that \( B_j \) and its complement are \( \mathbb{k}G \)-modules, so we obtain a \( \mathbb{k}G \)-module decomposition. Since \( \mathbb{k}[V] = \bigotimes_{j=1}^m \mathbb{k}[V_n] \), it follows that

\[
  \mathbb{k}[V] = N_j \mathbb{k}[V] \oplus (B_j \otimes \mathbb{k}[V^*]),
\]

where \( V^* = V_{n_1} \oplus \cdots \oplus V_{n_{s_2}} \oplus \cdots \oplus V_{n_m} \). From this decomposition it follows that if \( M \) is a \( \mathbb{k}G \) direct summand of \( \mathbb{k}[V]_d \), then \( N_j M \) is a \( \mathbb{k}G \) direct summand of \( \mathbb{k}[V]_{d+p} \) with the same isomorphism type. Further, any \( f \in \mathbb{k}[V]^G \) can be written as

\[
  f = q N_j + r,
\]

with \( q \in \mathbb{k}[V]^G \) and \( r \in (B_j \otimes \mathbb{k}[V^*])^G \). If in addition \( \deg(f) = (d_1, d_2, \ldots, d_m) \) with \( d_j > p - n_j \), then the degree \( d_j \) homogeneous component of \( B_j \) is free by [8, 2.10] and since tensoring a module with a free (projective) module gives a free (projective) module we may further assume, by Remark 2, that \( r \) is in the image of the transfer map.

If \( h = \sum_{i=1}^{s(h)} \Delta^{i-1}(f)w_i \in \mathbb{k}[V, W]^G \), we define the multidegree of \( h \) to be that of \( f \). Since \( G \) preserves the multidegree, this is the same as the multidegree of \( \Delta^{i-1}(f) \) for all \( i \leq s(h) \). Then the analogue of this result for covariants is the following:

Proposition 5. Let \( h \) be a covariant of multidegree \( d_1, d_2, \ldots, d_m \) with \( d_j > p - n_j \) for some \( j \). Then there exist a covariant \( h_1 \) and a transfer covariant \( h_2 \) such that \( h = N_j h_1 + h_2 \).

Proof. We proceed by induction on the support \( s(h) \) of \( h \). If \( s(h) = 1 \), then by Proposition 3, we have that \( h = f w_1 \) with \( f \in \mathbb{k}[V]^G \). Then we can write \( f = q N_j + \Delta^{p-1}(t) \) for some \( q \in \mathbb{k}[V]^G \) and \( t \in \mathbb{k}[V] \). Then both \( q w_1 \) and \( \Delta^{p-1}(t) w_1 \) are covariants by Proposition 3 and therefore \( h = q N_j w_1 + \Delta^{p-1}(t) w_1 \) gives us the desired decomposition.
Lemma 7. Let \( s(h) = k \). Then by Proposition 3 there exists \( f \in \mathbb{k}[V] \) such that
\[
h = f w_1 + \Delta(f) w_2 + \cdots + \Delta^{k-1}(f) w_k,
\]
with \( \Delta^k(f) = 0 \). Since \( \Delta^k(f) \in \mathbb{k}[V]^G \) and \( d_j > p - n_j \), we can write \( \Delta^k(f) = q N_j + \Delta^{p-1}(t) \) for some \( q \in \mathbb{k}[V]^G \) and \( t \in \mathbb{k}[V] \). It follows that \( q N_j \) is in the image of \( \Delta^k \). But since multiplication by \( N_j \) preserves the isomorphism type of a module, it follows that \( q \) is in the image of \( \Delta^k \). Write \( q = \Delta^k(f') \) with \( f' \in \mathbb{k}[V] \). Set
\[
h_1 = f' w_1 + \Delta(f') w_2 + \cdots + \Delta^{k-1}(f') w_k \quad \text{and} \quad h_2 = \Delta^{p-k}(t) w_1 + \cdots + \Delta^{p-1}(t) w_k.
\]
Since \( \Delta^{k-1}(f') \in \mathbb{k}[V]^G \), it follows that \( h_1 \) is a covariant by Proposition 3. Consider the covariant
\[
h' = h - N_j h_1 - h_2.
\]
Since \( \Delta^{k-1}(f) = \Delta^{p-1}(t) + \Delta^{k-1}(f') N_j \), the support of \( h' \) is strictly smaller than the support of \( h \). Moreover, \( h_2 \) is a transfer covariant and so the assertion of the proposition follows by induction.

We obtain the following upper bound for the Noether number of covariants:

Proposition 6. We have \( \beta(\mathbb{k}[V, W]^G) \leq \max(\beta(\mathbb{k}[V]), \mathbb{k}[V]^G), mp - \dim(V) \).

Proof. Let \( h \in \mathbb{k}[V, W]^G \) with degree \( d > \max(\beta(\mathbb{k}[V]), \mathbb{k}[V]^G), mp - \dim(V) \). Let \( (d_1, d_2, \ldots, d_m) \) be the multidegree of \( h \). Then we must have \( d_j > p - n_j \) for some \( j \). Consequently, we may apply Proposition 5, writing
\[
h = N_j h_1 + h_2,
\]
where \( h_2 \) is a transfer covariant. Since \( \deg(h_2) > \beta(\mathbb{k}[V], \mathbb{k}[V]^G) \), it follows that \( h_2 \) is decomposable by Proposition 4, and so we have shown that \( h \) is decomposable.

4 Lower bounds

Indecomposable transfers are one method of obtaining lower bounds for \( \beta(\mathbb{k}[V]^G) \). Recall that we have written \( V = \bigoplus_{j=1}^m V_{n_j} \) as a sum of indecomposable modules. The analogous result for covariants is:

Lemma 7. Let \( n \geq 2 \) and let \( \Delta^{p-1}(f) \in \mathbb{k}[V]^G \) be an indecomposable homogeneous transfer. Then the transfer covariant
\[
h = \Delta^{p-n}(f) w_1 + \cdots + \Delta^{p-1}(f) w_n
\]
is indecomposable.

Proof. Assume on the contrary that \( h \) is decomposable. Then there exist homogeneous \( q_i \in \mathbb{k}[V]^G \) and \( h_i \in \mathbb{k}[V, W]^G \) such that \( h = \sum_{1 \leq i \leq t} q_i h_i \). Write \( h_i = h_{i,1} w_1 + \cdots + h_{i,n} w_n \) for \( 1 \leq i \leq t \). Then we have
\[
\Delta^{p-1}(f) = \sum_{1 \leq i \leq t} q_i h_{i,n}.
\]
By Proposition 3 we have \( \Delta(h_{i,n-1}) = h_{i,n} \) and so \( h_{i,n} \in \mathbb{k}[V]^G \) because \( n \geq 2 \). It follows that \( \sum_{1 \leq i \leq t} q_i h_{i,n} \) is a decomposition of \( \Delta^{p-1}(f) \) in terms of invariants of strictly smaller degree, contradicting the indecomposability of \( \Delta^{p-1}(f) \).

Corollary 8. Suppose \( n \geq 2 \) and \( \beta(\mathbb{k}[V]^G) > \max(p, mp - \dim(V)) \). Then \( \beta(\mathbb{k}[V]^G) \leq \beta(\mathbb{k}[V, W]^G) \).

Proof. By [8, Lemma 2.12], \( \mathbb{k}[V]^G \) is generated by the norms \( N_1, N_2, \ldots, N_m \), invariants of degree at most \( mp - \dim(V) \), and transfers. Since there exists an indecomposable invariant of degree \( \beta(\mathbb{k}[V]^G) \), if the hypotheses of the corollary above hold, then \( \mathbb{k}[V]^G \) contains an indecomposable transfer with this degree. By Lemma 7, \( \mathbb{k}[V, W]^G \) contains a transfer covariant of degree \( \beta(\mathbb{k}[V]^G) \) which is indecomposable, from which the conclusion follows.
5 Main results

We are now ready to prove Theorem 1. Note that $k[V, V_1]^G$ is generated over $k[V]^G$ by $w_1$ alone, which has degree zero, and therefore $\beta(k[V, V_1]^G) = 0$. For this reason we assume $n \geq 2$ throughout.

Proof. Suppose first that $n_j > 3$ for some $j$. Then by [6, Proposition 1.1 (a)], we have
\[ \beta(k[V]^G) = m(p - 1) + (p - 2). \]
Since $V$ is reduced, we have dim($V$) $\geq 2m$ and hence
\[ \beta(k[V]^G) > mp - \dim(V). \]
Also, $\beta(k[V]^G) \geq 2p - 3 > p$ since $n_j \leq p$ for all $j$. Therefore Corollary 8 implies that $\beta(k[V]^G) \leq \beta(k[V, W]^G)$. On the other hand, [6, Lemma 3.3] shows that the top degree of $k[V]/k[V]^Gk[V]$ is bounded above by $m(p - 1) + (p - 2)$. By the graded Nakayama Lemma it follows that $\beta(k[V], k[V]^G) \leq m(p - 1) + (p - 2)$. We have already shown that this number is at least $mp - \dim(V) + 1$, so by Proposition 6 we get that
\[ \beta(k[V, W]^G) \leq m(p - 1) + (p - 2) = \beta(k[V]^G) \]
as required.

Now suppose that $n_i \leq 3$ for all $i$ and $n_j = 3$ for some $j$. Then by [6, Proposition 1.1 (b)], we have
\[ \beta(k[V]^G) = m(p - 1) + 1. \]
Since $V$ is reduced, we have dim($V$) $\geq 2m$ and hence
\[ \beta(k[V]^G) > mp - \dim(V). \]
Also $\beta(k[V]^G) \geq 2p - 1 > p$ provided $m \geq 2$. In that case Corollary 8 applies. If $m = 1$, then Dickson [4] has shown that $k[V]^G = k[x_1, x_2, x_3]^G$ is minimally generated by the invariants $x_3, x_3^2 - 2x_1x_3 - x_2x_3$, $N$, $\Delta^p - (x_1^{p-1} - x_2)$. It follows that $\Delta^p - (x_1^{p-1} - x_2)$ is an indecomposable transfer, so by Lemma 7, $k[V, W]^G$ contains an indecomposable transfer covariant of degree $p = \beta(k[V]^G)$. In either case we obtain
\[ \beta(k[V, W]^G) \geq \beta(k[V]^G). \]

On the other hand, by [9, Corollary 2.8], $m(p - 1) + 1$ is an upper bound for the top degree of $k[V]/k[V]^G$. By the same argument as before we get $\beta(k[V]^G, k[V]) \leq m(p - 1) + 1$. We have already shown that this number is at least $mp - \dim(V) + 1$, so by Proposition 6 we get that
\[ \beta(k[V, W]^G) \leq m(p - 1) + 1 = \beta(k[V]^G) \]
as required.

It remains to deal with the case $n_i = 2$ for all $i$, i.e. $V = MV$. We assume $m \geq 2$. In this case Campbell and Hughes [2] showed that $\beta(k[V]^G) = (p - 1)m$. As dim($V$) $= 2m$, we have $\beta(k[V]^G) > mp - \dim(V)$. If $m \geq 3$ or $m = 2$ and $p > 2$, then we have
\[ \beta(k[V]^G) > p \]
and Corollary 8 applies. In case $m = 2 = p$, $k[V]^G = k[x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}]^G$ is a hypersurface, minimally generated by $(x_{2,2}, N_1, x_{2,2}, N_2, \Delta^{p-1}(x_{1,1}, x_{1,2}))$. In particular, $\Delta^{p-1}(x_{1,1}, x_{1,2})$ is an indecomposable transfer, so by Lemma 7, $k[V, W]^G$ contains an indecomposable transfer covariant of degree 2. In both cases we get
\[ \beta(k[V, W]^G) \geq \beta(k[V]^G). \]
On the other hand, by [9, Theorem 2.1], the top degree of $k[V]/k[V]^Gk[V]$ is bounded above by $m(p - 1)$. We have already shown this number is at least $mp - \dim(V) + 1$. Therefore, by Proposition 6, we get
\[ \beta(k[V, W]^G) \leq \beta(k[V]^G) \]
as required. $\square$
Remark 9. The only reduced representation not covered by Theorem 1 is \( V = V_2 \). An explicit minimal set of generators of \( k[V_2, W]^G \) as a module over \( k[V_2]^G \) is given in [5], the result is

\[
\beta(k[V_2, W]) = n - 1.
\]

This is the only situation in which the Noether number is seen to depend on \( W \).

Remark 10. Suppose \( V \) is any reduced \( kG \)-module and \( W = \bigoplus_{i=1}^r W_i \) is a decomposable \( kG \)-module. Then

\[
k[V, W]^G = (S(V^*) \otimes \bigoplus_{i=1}^r W_i)^G = \bigoplus_{i=1}^r (S(V^*) \otimes W_i)^G.
\]

So \( \beta(k[V, W]^G) = \max_i \beta(k[V, W_i]^G) : i = 1, \ldots, r \) unless \( V \) is indecomposable of dimension 2, in which case we have

\[
\beta(k[V_2, W]^G) = \max_i \beta(k[V_2, W_i]^G) : i = 1, \ldots, r = \max_i (\dim(W_i) - 1 : i = 1, \ldots, r).
\]

Thus, the results of this paper can be used to compute \( \beta(k[V, W]^G) \) for arbitrary \( kG \)-modules \( V \) and \( W \).

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