Poincaré gauge theory with coupled even and odd parity spin-0 modes: Cosmological normal modes

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We are investigating the dynamics of a new Poincaré gauge theory of gravity model, which has cross coupling between the spin-0 + and spin-0 − modes. To this end we are considering a very appropriate situation – homogeneous-isotropic cosmologies – which is relatively simple, and yet all the modes have non-trivial dynamics which reveals physically interesting and possibly observable results. More specifically we consider manifestly isotropic Bianchi class A cosmologies. Here the first order equations obtained from an effective Lagrangian are linearized and the normal modes are found. These turn out to control the asymptotic late time cosmological normal modes. Numerical evolution confirms the late time asymptotic approximation and shows the expected effects of the cross parity pseudoscalar coupling.

1 Introduction

Physically (and geometrically) it is reasonable to consider gravity as a gauge theory of the local Poincaré symmetry of Minkowski spacetime. A theory of gravity based on local spacetime geometry gauge symmetry, the quadratic Poincaré gauge theory of gravity (PG, aka PGT) was worked out some time ago [1–7]. We briefly sketch this theory, and how the search for good dynamical propagating modes led to focusing on the two scalar modes.

There is no known fundamental reason why the gravitational coupling should respect parity. With this in mind, the general quadratic PG theory has recently seen renewed interest in including all possible couplings between even and odd parity quantities. The appropriate cross parity pseudoscalar coupling constants have been incorporated, in particular, into the special case of the dynamically favored two scalar mode model to give the BHN model [8], which is the most general PG model that we expect to have problem free dynamics. Here, in a cosmological context, we investigate certain aspects of the dynamics of this extended model.

We are especially interested in investigating the dynamics of the PG BHN model. This can be expected to be very clearly revealed in purely time dependent solutions, hence we considered homogeneous cosmologies. The two dynamical connection modes that we wish to study carry spin 0 + and spin 0 − (and are thus referred to as scalar modes or, more specifically, as the scalar and pseudoscalar mode). Consequently in a homogeneous situation they cannot pick out any spatial direction, and thus they have no interaction with spatial anisotropy, so for a study of their dynamics it is most simple and appropriate to look to isotropic models. Very recently [9,10] for the PG BHN model, following the technique used in [11], we constructed an effective Lagrangian and Hamiltonian as well as a system of first order dynamical equations for Bianchi class A isotropic homogeneous cosmological models, and presented some sample evolution which showed the effect of the cross parity coupling.

Here we wish to further analyze the aforementioned first order equations obtained from the effective Lagrangian. Extending the analysis technique which was successful in the more restricted model considered in [11], the normal modes are identified, and it is shown analytically how they control the late time asymptotics. A numerical evolution example is presented which shows that the asymptotic late time normal mode evolution is a good approximation.

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2 The Poincaré gauge theory

In the Poincaré gauge theory of gravity [1–7], the two sets of local gauge potentials are, for “translations”, the orthonormal co-frame \( \theta^a = e^a_i dx^i \), where the metric is \( g = -\theta^a \otimes \theta^a + \theta_{ab} \theta^b \otimes \theta^a \), and, for “rotations”, the metric-compatible (Lorentz Lie-algebra valued) connection 1-forms \( \Gamma^a_{\mu} \). The associated field strengths are the torsion and curvature 2-forms:

\[
T^a := d\theta^a + \Gamma^a_{\mu} \wedge \theta^\mu = \frac{1}{2} R^a_{\mu \nu \rho} \theta^\mu \wedge \theta^\nu,
\]

\[
R^{a \beta} := d\Gamma^{a \beta} + \Gamma^a_{\gamma} \Gamma^{\gamma \beta} = \frac{1}{2} R^{a \beta}_{\mu \nu} \theta^\mu \wedge \theta^\nu,
\]

which satisfy the respective Bianchi identities:

\[
DT^a = R^{a \beta} \wedge \theta^\beta, \quad DR^{a \beta} = 0.
\]

The PG Lagrangian density is generally taken to have the standard quadratic Yang-Mills form, which leads to quasi-linear second order equations for the gauge potentials. Qualitatively,

\[
\mathcal{L}[\theta, \Gamma] \sim \kappa^{-1}[\Lambda + \text{curvature} + \text{torsion}^2] + \varrho^{-1} \text{curvature}^2,
\]

where \( \Lambda \) is the cosmological constant, \( \kappa = 8\pi G/c^4 \) is the usual gravitational constant, and \( \varrho^{-1} \) has the dimensions of action. The field equations, including source terms, obtained by variation w.r.t. the two gauge potentials have the respective general forms

\[
\Lambda + \text{curvature} + D \text{torsion} + \text{torsion}^2 + \text{curvature}^2
\]

\sim \text{energy-momentum density},

\[
\text{torsion} + D \text{curvature} \sim \text{spin density}.
\]

From these two equations, with the aid of the Bianchi identities (3), one can obtain, respectively, the conservation of source energy-momentum and angular momentum statements.

Earlier investigations generally considered models with even parity terms; the models had 10 dimensionless scalar coupling constants. The recent BHN investigation [8] systematically considered all the possible odd-parity Lagrangian terms, introducing 7 new pseudoscalar coupling constants.

Not all of these coupling constants are physically independent, since there are 3 topological invariants: the (odd parity) Nieh-Yan [12] identity \( d(\theta^a \wedge T_a) = T^a \wedge T_a + R_{a \beta} \wedge \theta^{a \beta} \), the (even parity) Euler 4-form \( R^{a \beta}_{\mu \nu} \wedge \theta^\mu \wedge \theta^\nu \), and the (odd parity) Pontryagin 4-form \( R^{a \beta}_{\mu \nu} \wedge R^{\beta a}_{\mu \nu} \). For detailed discussions of the BHN Lagrangian and the topological boundary terms see [8] and the new work [13].

Investigations (especially [3, 14]) of the linearized PG theory identified six possible dynamic connection modes; they carry spin-2\(^\pm \), spin-1\(^\pm \), spin-0\(^\pm \). A good dynamic mode should transport positive energy and should not propagate outside the forward null cone. The linearized investigations found that at most three modes can be simultaneously dynamic; all the acceptable cases were tabulated; many combinations of three modes are satisfactory to linear order. Complementing this, the Hamiltonian analysis revealed the related constraints [15]. Then detailed investigations of the Hamiltonian and propagation [16–19] concluded that effects due to nonlinearities in the constraints could be expected to render all of these cases physically unacceptable except for the two “scalar modes”, carrying spin-0\(^+\) and spin-0\(^-\).

In order to further investigate the dynamical possibilities of these PG scalar modes, Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models were considered. Using a \( k = 0 \) model it was found that the 0\(^+\) mode naturally couples to the acceleration of the universe and could account for the present day observations [20, 21]; this model was then extended to include the 0\(^-\) mode [11].

After systematically developing the general odd parity PG theory, in BHN [8] the two scalar torsion mode PG Lagrangian was extended to include the appropriate pseudoscalar coupling constants that provide cross parity coupling, such terms are often referred to as “parity violating” terms.

The BHN Lagrangian [8] has the specific form

\[
\mathcal{L}_{BHN}[\theta, \Gamma] = \frac{1}{2\kappa} \left[ -2\Lambda + g_0 R + b_0 X - \frac{1}{2} \sum_{n=1}^{3} a_n^{(n)} T^2 + 3 a_2 V_{\mu} A^\mu \right] - \frac{1}{2\rho} \left[ \frac{w_0}{12} R^2 - \frac{w_3}{12} X^2 + \frac{\mu_3}{12} RX \right],
\]

where \( R \) is the scalar curvature and \( X \) is the pseudoscalar curvature (specifically \( X/6 = R_{(123)}^0 \) is the magnitude of the one independent component of the totally antisymmetric curvature), and \( V_{\mu} := T_\alpha^{a \mu} = (2)T_\alpha^{a \mu}, A_{\mu} := \frac{1}{4} \epsilon_{\mu \nu}^{a \beta} T_\nu^{a \beta} \) are the torsion trace and axial vectors. The parameters \( a_0, a_1, a_2, w_3 \) and \( w_0 \) are scalars, whereas \( b_0, \sigma_2 \) and \( \mu_3 \) are pseudoscalars. For an extensive discussion of the mathematics and physics of the PG theory and this model as well as further references see BHN [8] and the new work [13].

In BHN the general field equations were worked out, and then specialized to find the most general 2-scalar mode PG FLRW cosmological model. We have taken an alternative approach, considering manifestly isotropic Bianchi models.
3 The PG scalar mode Bianchi I and IX cosmological model

PG cosmological investigations have a long history. For earlier PG cosmological investigations see Minkevich and coworkers, e.g., [22–26] and Goenner and Müller-Hoissen [27]; for recent work see [8, 11, 20, 21, 28–31].

For the usual FLRW models, although they are actually homogeneous and isotropic, the representation is not manifestly so. Indeed they are merely manifestly isotropic-about-a-chosen-point. In contrast, the representation we have used for the isotropic Bianchi I and IX models has the virtue of being manifestly homogeneous and manifestly isotropic. (Indeed these are the only two Bianchi models which admit such a representation.)

For homogeneous, isotropic Bianchi type I and IX (respectively equivalent to FLRW $k = 0$ and $k = \pm 1$) cosmological models the isotropic orthonormal coframe has the form

$$\theta^0 := dt, \quad \theta^a := a \sigma^a,$$

where $a = a(t)$ is the scale factor and $\sigma^j$ depends on the (not needed here) spatial coordinates in such a way that

$$d\sigma^j = \zeta^e_j k \sigma^e \wedge \sigma^k,$$

where $\zeta = 0$ for Bianchi I and $\zeta = 1$ for Bianchi IX, thus $\zeta^2 = k$, the sign of the FLRW Riemannian spatial curvature.

Because of isotropy, the only non-vanishing connection one-form coefficients are necessarily of the form

$$\Gamma^a_0 = \psi(t) \sigma^a, \quad \Gamma^a_b = \chi(t) e^a_{bc} \sigma^c.$$

Here $e_{abc} := e_{[abc]}$ is the usual 3 dimensional Levi-Civita anti-symmetric symbol.

From the definition of the curvature (2), all the independent nonvanishing curvature 2-form components are found to be

$$R^a_b = \chi dt \wedge e^a_{bc} \sigma^c$$

$$+ (\psi^2 - \chi^2) \sigma^a \wedge \sigma^b + \chi \zeta e^a_{bc} e^c_{ij} \sigma^i \wedge \sigma^j,$$

$$R^a_0 = \psi dt \wedge \sigma^a - \chi \psi a^b \wedge e^a_{bc} \sigma^c + \psi \zeta e^a_{bc} \sigma^b \wedge \sigma^c.$$

Consequently, the scalar and pseudoscalar curvatures are, respectively,

$$R = 6(\dot{a}/a - 1) \psi + a^{-2}(\psi^2 - |\chi - \zeta|^2 + \zeta^2),$$

$$X = 6(\dot{a}/a - 1) \chi + 2a^{-2} \psi (\chi - \zeta).$$

Because of isotropy, the only nonvanishing torsion tensor components are of the form

$$T^a_{bo} = u(t) \delta^a_b, \quad T^a_{bc} = -2x(t) e^a_{bc},$$

where $u$ and $x$ are referred to as the scalar and pseudoscalar torsion, respectively. From the definition of the torsion (1), the relations between these torsion components and the gauge variables are found to be

$$u = a^{-1}(\dot{a} - \psi), \quad x = a^{-1}(\chi - \zeta).$$

Note that

$$a^{-1} \psi = H - u,$$

where $H = a^{-1} \dot{a}$ is the Hubble function. From this relation it can clearly be seen that kinematically scalar torsion directly couples to the rate of expansion of the universe. On the other hand pseudoscalar torsion, from the second relation in (16), is directly connected with the totally antisymmetric part of the connection, which dynamically couples to all of the fundamental spin-$\frac{1}{2}$ fermions.

For the material source of gravity, because of the symmetry assumptions of our model, the source material energy-momentum tensor is necessarily of the fluid form described by an energy density $\rho$, pressure $p$ and flow vector along the cosmological time axis. Although we expect the source spin density to play an important role in the very early universe, it is reasonable to assume, as we do here, that the material spin density at later times is negligible.

4 Effective Lagrangian and first order equations

The dynamical equations for these homogeneous cosmologies can be obtained by imposing the Bianchi symmetry on the general field equations found by BHN from their Lagrangian density. Using an alternative approach, we have obtained the dynamical equations directly (generalizing the procedure used in [11] to include Bianchi IX, fluid pressure and the new pseudoscalar coupling constants) from a classical mechanics type effective Lagrangian, which in this case is simply obtained by restricting the BHN Lagrangian density to the Bianchi symmetry (this step is where the manifestly homogeneous
representation plays an essential role). This procedure is known to successfully give the correct dynamical equations for all Bianchi class A models (which includes our cases) in GR [32], and it is conjectured to also work equally as well for the PG theory. Our calculations (for the details see [9,10]) explicitly verify this property for the isotropic Bianchi I and IX models. Indeed the equations we obtained in this way are equivalent to those found (at a later date) by BHN for their FLRW models (which for $k = 0,1$ are equivalent to our isotropic Bianchi models) by restricting to FLRW symmetry their general dynamical PG equations. This has proved to be a useful cross-check.

Our effective Lagrangian $L_{\text{eff}} = L_G + L_{\text{int}}$ includes the interaction Lagrangian: $L_{\text{int}} = p a^3$, where $p = p(t)$ is the pressure, and the gravitational Lagrangian:

$$L_G = \frac{1}{2\kappa} (a_0 R + b_0 X - 2\Lambda) a^3 + \frac{3}{2\kappa} (-a_2 u^2 + 4a_3 x^2 + 4\sigma_2 ux) a^3 - \frac{1}{24\rho} (w_6 R^2 - w_3 X^2 + \mu_3 RX) a^3.$$  \hspace{1cm} (18)

It should be noted that the parameter restrictions

$$a_2 < 0, \quad w_6 < 0, \quad w_3 > 0, \quad \mu_3^2 + 4w_3 w_6 < 0$$  \hspace{1cm} (19)

are, in the light of Eqs. (13), (14), (16), necessary for the least action principle, which requires positive quadratic-kinetic-terms. (The last of these, which restricts the magnitude of $\mu_3$ relative to $w_3, w_6$, follows from completing the square in the quadratic curvature terms.)

In the following we often take for simplicity units such that $\kappa = 1 = \rho$. These factors can be easily restored in the final results by noting that in the Lagrangian they occur in conjunction with certain PG parameters. Hence in the final results one need merely make the replacements $(a_0, a_2, a_3, b_0, \Lambda, \sigma_2) \rightarrow \kappa^{-1}(a_0, a_2, a_3, b_0, \Lambda, \sigma_2)$, $(w_3, w_6, \mu_3) \rightarrow \kappa^{3}(w_3, w_6, \mu_3)$.

The gravitational Lagrangian has the usual form of a sum of terms homogeneous in “velocities” $L_G = L_0 + L_1 + L_2$; the associated energy function is thus

$$\mathcal{E}_G := \frac{\partial L_G}{\partial \dot{\psi}} \dot{\psi} + \frac{\partial L_G}{\partial \dot{\chi}} \dot{\chi} + \frac{\partial L_G}{\partial \dot{a}} \dot{a} - L_G = L_2 - L_0$$

$$= a^3 \left\{ -3 \left( a_0 - \frac{1}{2} a_2 \right) u^2 - 3a_0 H^2 + 3x^2 (a_0 - 2a_3) + 6u H \left( a_0 - \frac{1}{2} a_2 \right) + 6(b_0 + \sigma_2) x (H - u) - 3a_0 \frac{\zeta^2}{a^2} + \Lambda \right\}$$

$$+ \frac{w_6}{24} \left[ R^2 - 12R \left\{ (H - u)^2 - x^2 + \frac{\zeta^2}{a^2} \right\} \right]$$

$$+ \frac{\mu_3}{24} \left\{ X^2 + 24X x (H - u) \right\}$$

$$- \frac{\mu_3}{24} \left\{ R X - 6X \left\{ (H - u)^2 - x^2 + \frac{\zeta^2}{a^2} \right\} + 12Rx (H - u) \right\}.$$  \hspace{1cm} (20)

The energy value (20) has the form $-\frac{1}{2} \rho \dot{a}$ where, from a comparison with Eq. (167) in BHN [8], it can be seen that $\rho$ is the value of the material energy density.

Making use of the formulas for the torsion and curvature components in terms of the gauge variables (13), (14), (16), from $L_{\text{eff}}$ we obtained the Euler-Lagrange equations for $\psi$, $\chi$, and $a$ and found [9,10], respectively,

$$-\frac{w_6}{2} \ddot{R} - \frac{\mu_3}{4} \ddot{X} = - \left( 3(2a_0 - a_2) - w_6 R - \frac{\mu_3}{2} X \right) u$$

$$- \left( 6(b_0 + \sigma_2) - \frac{\mu_3}{2} R + w_3 X \right) x;$$  \hspace{1cm} (21)

$$\frac{\mu_3}{4} \ddot{R} + \frac{w_3}{2} \ddot{X} = - \left( 6(b_0 + \sigma_2) - \frac{\mu_3}{2} R + w_3 X \right) u$$

$$+ \left( 6(a_0 - 2a_3) - w_6 R - \frac{\mu_3}{2} X \right) x;$$  \hspace{1cm} (22)

$$-3(a_2 \ddot{u} - 2\sigma_2 \ddot{x}) = (a_0 R + b_0 X - 3\Lambda) + 6H (a_2 u - 2\sigma_2 x)$$

$$+ \frac{3}{2} (-a_2 u^2 + 4a_3 x^2 + 4\sigma_2 ux) - \frac{1}{24} (w_6 R^2 - w_3 X^2 + \mu_3 RX)$$

$$+ \frac{a_0}{2} \left\{ -6(H - u)^2 + 6x^2 - 6\frac{\zeta^2}{a^2} \right\} + 6b_0 x (H - u)$$

$$- \frac{1}{24} \left[ 2w_6 R + \mu_3 X \right] \left\{ -6(H - u)^2 + 6x^2 - 6\frac{\zeta^2}{a^2} \right\}$$

$$+ \frac{1}{24} \left[ 2w_3 X - \mu_3 R \right] [12x(H - u)] + 3p.$$  \hspace{1cm} (23)

Since $L_G$ is time independent, the energy function (20) satisfies an energy conservation relation:

$$\dot{\mathcal{E}} = \frac{\partial L_G}{\partial a^3} \dot{a}^3 = - \frac{\partial L_G}{\partial \psi} \dot{\psi} - \frac{\partial L_G}{\partial \chi} \dot{\chi} - \frac{\partial L_G}{\partial a} \dot{a}$$

$$= \frac{\delta L_{\text{int}}}{\delta a} \dot{a} = 3pa^2 \dot{a},$$  \hspace{1cm} (24)

which, with the interpretation of $\rho$ mentioned above, is just the perfect fluid relation

$$- \frac{d(pa^3)}{dt} = \rho \frac{d a^3}{dt}.$$  \hspace{1cm} (25)

The above equations (21,22,23) are 3 second order equations for the gauge potentials $a, \psi, \chi$. However they can in an alternative way be used as part of a set of 6 first order equations along with the Hubble relation $\dot{a} = aH$.
and the following two relations, obtained by taking the time derivatives of the torsion (16) and using the curvature definitions (13), (14):

\[
\dot{x} = -H x - \frac{X}{6} - 2x(H - u),
\]

(26)

\[
\dot{H} - \dot{u} = \frac{R}{6} - H(H - u) - (H - u)^2 + x^2 - \frac{\zeta^2}{a^2} - \frac{(\rho - 3p)}{3\alpha_2} + \frac{4\Lambda}{3\alpha_2},
\]

(27)

One advantage of such a reformulation is that the variables are now all observables.

Our 6 first order dynamical equations and the energy equation have the form

\[
\dot{a} = aH,
\]

(28)

\[
\dot{\alpha} = \frac{1}{6\alpha_2}(\tilde{a}_2 R - 2\tilde{a}_2 X) - 2H^2 - \frac{\tilde{a}_2 - 4\tilde{a}_3}{a_2} x^2 - \frac{\zeta^2}{a^2} + \frac{(\rho - 3p)}{3\alpha_2} + \frac{4\Lambda}{3\alpha_2},
\]

(29)

\[
\dot{u} = -\frac{1}{3\alpha_2}(a_0 R + \tilde{a}_2 X) - 3Hu + u^2 - \frac{4a_3}{a_2} x^2 + \frac{(\rho - 3p)}{3\alpha_2} + \frac{4\Lambda}{3\alpha_2},
\]

(30)

\[
\dot{\tilde{x}} = -\frac{X}{6} - (3H - 2u)x,
\]

(31)

\[
-\frac{w_6}{2} \dot{\tilde{X}} - \frac{\mu_3}{4} \tilde{X} = \left[3\tilde{a}_2 + w_6 R + \frac{\mu_3}{2} X\right] u + \left[-6\tilde{a}_2 + \frac{\mu_3}{2} R - w_3 X\right] x,
\]

(32)

\[
\frac{w_3}{2} \dot{\tilde{X}} - \frac{\mu_3}{4} \tilde{R} = \left[-6\tilde{a}_2 + \frac{\mu_3}{2} R - w_3 X\right] u - \left[12\tilde{a}_3 + w_6 R + \frac{\mu_3}{2} X\right] x,
\]

(33)

\[
\rho = 3\left(-\frac{1}{2} \tilde{a}_2 + 2\tilde{a}_3\right) x^2 + \frac{3a_2}{2} \left[H^2 + \frac{\zeta^2}{a^2}\right] - \Lambda + \left(-6\tilde{a}_2 + \frac{\mu_3}{2} R - w_3 X\right) x(H - u)
\]

\[
+ \frac{1}{24}(w_6 R^2 - w_3 X^2 + \mu_3 RX)
\]

\[
- \frac{1}{2}\left(3\tilde{a}_2 + w_6 R + \frac{\mu_3}{2} X\right) \left[(H - u)^2 - x^2 + \frac{\zeta^2}{a^2}\right],
\]

(34)

where we have introduced certain modified parameters \(\tilde{a}_2, \tilde{a}_3\) and \(\tilde{\sigma}_2\) with the definitions

\[
\tilde{a}_2 := a_2 - 2a_0, \quad \tilde{a}_3 := a_3 - \frac{a_0}{2}, \quad \tilde{\sigma}_2 := \sigma_2 + b_0.
\]

(35)

For some purposes, e.g., numerical evolution, it is more convenient to have the last two dynamical equations (32,33) in a form which is resolved for \(\tilde{R}\) and \(\tilde{X}\):

\[
\tilde{R} = \frac{6}{\alpha} \left[(w_3 \tilde{a}_2 - \mu_3 \tilde{\sigma}_2) u - (2w_3 \tilde{a}_2 + 2\mu_3 \tilde{a}_3)x\right]
\]

\[
-2Ru - \frac{4w_3^2 + \mu_3^2}{2\alpha} Xx + \frac{(w_3 - w_6)\mu_3}{\alpha} Rx,
\]

(36)

\[
\tilde{X} = \frac{6}{\alpha} \left[2w_6 \tilde{a}_2 + \frac{1}{2} \mu_3 \tilde{a}_2\right] u + (4w_6 \tilde{a}_3 - 3\mu_3 \tilde{\sigma}_2)x
\]

\[
-2Xu + \frac{4w_3^2 + \mu_3^2}{2\alpha} Rx - \frac{(w_3 - w_6)\mu_3}{\alpha} Xx,
\]

(37)

where

\[
\alpha := -w_3 w_6 - \frac{\mu_3^2}{4}.
\]

(38)

It should be noted that for the range of parameters of physical interest, from (19), we should have \(\alpha > 0\).

We should mention that an alternative set of 6 canonical Hamiltonian dynamical equations for this system was also presented in [10]. Here we will further analyze the 6 equations given above.

We have recast our system of second order equations obtained from our effective Lagrangian into six first order equations for (3D) tensorial quantities, equations which are suitable for numerical evolution and comparison with observations. This system of equations (28)–(33), (34), and (36), (37) – aside from some small changes of notation – are the same as the corresponding FLRW equations found in BHN [8].

In order to calculate the time evolution, one needs to have some relation which determines the pressure \(p\). A popular choice is some equation of state, \(p = p(\rho)\). Except in the early universe, for all known forms of matter, a reasonable approximation is vanishing pressure, i.e., the cosmic fluid is treated as noninteracting dust. We will use that assumption for our numerical evolutions discussed here. In the case of \(p = 0\), the above equations are self contained, the energy relation (34) can be regarded as defining a material density whose value can be calculated entirely in terms of the geometric variables. With this assumption the above system of 6 dynamic equations (28)–(31), (36), (37) is a closed deterministic system.

### 5 Linearized equations and normal modes

Following the procedure used in [11], for vanishing \(\Lambda\) and \(\zeta\), by dropping higher than linear order terms in \(H, u, x,\)
$R$, $X$, we can linearize our model. This leads to the first order *linearized* versions of Eqs. (28)–(33):

$$\dot{a} = aH,$$

$$3a_2\dot{H} = \frac{1}{2}a\dot{a}_2R - \dot{\sigma}_2X,$$

$$3a_2\dot{u} = -a_0\dot{R} - \dot{\sigma}_2X,$$

$$\dot{x} = \frac{X}{6},$$

$$-\frac{\sigma_6}{2}R + \frac{\mu_3}{4} \dot{\chi} = 3\dot{a}_2u - 6\dot{\sigma}_2x,$$

$$-\frac{\mu_3}{4}R + \frac{w_3}{2} \dot{X} = -6\dot{\sigma}_2u - 12\dot{a}_3x,$$

with the associated (to lowest, i.e., quadratic, order) “energy”:

$$\mathcal{E} = a^3 \left\{ \frac{3}{2} \dot{a}_2u^2 - 3a_0H^2 - 6\dot{a}_3x^2 - 3uH\dot{a}_2 + 6\dot{\sigma}_2x(H - u) - \frac{w_6}{24}R^2 + \frac{w_3}{24}X^2 - \frac{\mu_3}{24}RX \right\}.$$  

(45)

Note (as expected): (i) The odd parity coupling terms lead to mixing of the even ($R,u$) and odd ($X,x$) dynamical variables; this is especially apparent in (43,44). (ii) In contrast to the model of [11] without cross parity pseudoscalar coupling constants, from (40) the acceleration is now also driven by the odd pseudoscalar curvature.

To analyze this system we first introduce a new variable combination:

$$z := a_0H + \frac{\dot{a}_2}{2}u - \dot{\sigma}_2x,$$

(46)

which to linear order from (40)–(42) is constant:

$$\dot{z} = a_0H + \frac{\dot{a}_2}{2}\dot{u} - \dot{\sigma}_2\dot{x} = 0.$$  

(47)

This is, to linear order, a *zero frequency normal mode*.

To analyze the system further we rewrite (43) and (44) in matrix form:

$$\mathbb{T} \begin{pmatrix} \dot{R} \\ \dot{X} \end{pmatrix} = -6\mathbb{M} \begin{pmatrix} u \\ x \end{pmatrix},$$

(48)

with the two *symmetric* matrices

$$\mathbb{T} := \begin{pmatrix} -\frac{\sigma_6}{2} & -\frac{\mu_3}{4} \\ -\frac{\mu_3}{4} & \frac{w_3}{2} \end{pmatrix}, \quad \mathbb{M} := \begin{pmatrix} -\frac{\dot{a}_2}{2} & \ddot{\sigma}_2 \\ \ddot{\sigma}_2 & 2\dot{a}_3 \end{pmatrix}.$$  

From (41) and (42), we get

$$\begin{pmatrix} \dot{u} \\ \dot{x} \end{pmatrix} = -\mathbb{N} \begin{pmatrix} R \\ X \end{pmatrix}, \quad \mathbb{N} := \begin{pmatrix} a_0/3a_2 & \ddot{\sigma}_2/3a_2 \\ 0 & 1/6 \end{pmatrix}.$$  

(49)

Differentiating (43) and (44) using (49) gives

$$\mathbb{V} \begin{pmatrix} \dot{R} \\ \dot{X} \end{pmatrix} = -6\mathbb{M} \begin{pmatrix} \dot{u} \\ \dot{x} \end{pmatrix} = -\mathbb{V} \begin{pmatrix} R \\ X \end{pmatrix}.$$  

(50)

where

$$\mathbb{V} := -6\mathbb{M}\mathbb{N} = -2\begin{pmatrix} -\frac{1}{2}a_0\ddot{\sigma}_2 & a_0\ddot{\sigma}_2 \\ a_0\ddot{\sigma}_2 & \ddot{\sigma}_2^2 + a_0\dot{a}_3 \end{pmatrix}.$$  

(51)

is also a symmetric matrix (even though $\mathbb{N}$ is not).

Now one can find the normal modes and the normal frequencies. Assuming

$$\begin{pmatrix} R(t) \\ X(t) \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} R_0 \\ X_0 \end{pmatrix},$$  

(52)

from (50) we get

$$\mathbb{V} \begin{pmatrix} R_0 \\ X_0 \end{pmatrix} = \omega^2 \mathbb{T} \begin{pmatrix} R_0 \\ X_0 \end{pmatrix}.$$  

(53)

This is a standard eigenvalue/eigenvector problem; the eigenvalues are obtained from

$$0 = \det[\mathbb{V} - \omega^2 \mathbb{T}] = \det \mathbb{V} - \beta \omega^2 + \omega^4 \det \mathbb{T},$$  

(54)

where $\beta := V_{11}T_{22} + T_{11}V_{22} - 2T_{12}V_{12}$. The two roots

$$\omega^2_k = \beta \pm \sqrt{\beta^2 - 4\det \mathbb{T} \det \mathbb{V}}$$  

(55)

give the two normal frequencies $\omega_k$. The parameters should be chosen such that these are real values. The physically appropriate range of parameters is such that the matrices $\mathbb{T}$ and $\mathbb{V}$ have *positive* eigenvalues, which is equivalent to them having positive traces and determinants. The requirement that both $\omega^2_+$ and $\omega^2_-$ be positive may impose some additional restriction on the magnitude of the pseudoscalar parameters.

It should be noted that our formulation is such that both $\mathbb{T}$ and $\mathbb{V}$ are symmetric. Recall that two such symmetric matrices can be simultaneously diagonalized by using a matrix whose columns are the eigenvectors, which are guaranteed to be *orthogonal* (see, e.g., Chap. 6 in [33]). Let $\mathbb{A}$ be the matrix of eigenvectors which are normalized by $\mathbb{T}$ and diagonalize $\mathbb{V}$:

$$\mathbb{A}^\top \mathbb{T} \mathbb{A} = \mathbb{I}, \quad \mathbb{A}^\top \mathbb{V} \mathbb{A} = \mathbb{D} = \begin{pmatrix} \omega^2_+ & 0 \\ 0 & \omega^2_- \end{pmatrix}.$$  

(56)

---

$^2$ It should be noted that under these assumptions $\rho$ and hence $p$ are non-linear.
Then the solution of (50) has the form

\[
\begin{pmatrix}
R \\
X
\end{pmatrix} = A \begin{pmatrix}
q_+ \\
q_-
\end{pmatrix},
\]  
(57)

where \(q_+, q_-\) satisfy

\[
\dot{q}_\pm + \omega_\pm^2 q_\pm = 0,
\]  
(58)

which has solutions of the form

\[
q_\pm(t) = A_\pm \cos \omega \pm t + B_\pm \sin \omega \pm t,
\]  
(59)

where the 4 constants \(A_\pm, B_\pm\) are determined by the initial conditions. It follows from (48) that

\[
\begin{pmatrix}
u \\
x
\end{pmatrix} = -\frac{1}{6} (\Lambda M)^{-1} \begin{pmatrix}
\dot{R} \\
\dot{X}
\end{pmatrix}
= -\frac{1}{6} (\Lambda M)^{-1} \begin{pmatrix}
\dot{q}_+ \\
\dot{q}_-
\end{pmatrix} = -\frac{1}{6} (\Lambda M)^{-1} \begin{pmatrix}
\dot{q}_+ \\
\dot{q}_-
\end{pmatrix}.
\]  
(60)

From these results we find the solution to (40) to be

\[
H = H_0 - \frac{1}{3 a_2}(a_2 / 2, -a_2) \Upsilon^{-1} \bar{A}^{-1} \begin{pmatrix}
\dot{q}_+ \\
\dot{q}_-
\end{pmatrix}.
\]  
(61)

It should be kept in mind that the physical signs should be positive for \(a_0, w_3\) and negative for \(w_6, a_2\). This means negative for \(a_2\). Requiring that there be a good limit under the condition of vanishing odd parameters to the case previously investigated in [11] gives also a negative value for \(a_3\).

### 6 Late time asymptotic expansion

At late times in an expanding universe as the scale factor \(a\) becomes larger the field amplitudes should be decreasing. We can then expect the quadratic terms in (34) to be dominant; hence if the cosmological constant vanishes the late time asymptotic behavior of \(H, u, x, R, \) and \(X\) should have an \(a^{-3/2}\) fall off. So, following [11], we parameterize them according to

\[
\begin{align*}
H &= H a^{-3/2}, \\
u &= \mu a^{-3/2}, \\
x &= \lambda a^{-3/2}, \\
R &= \xi a^{-3/2}, \\
X &= \chi a^{-3/2}.
\end{align*}
\]  
(62)

The current Universe corresponds to \(a^{3/2} \gg 1\). Substituting (62) into (28)–(31), (36), (37), (34) gives

\[
\dot{a} = a^{-1/2} H,
\]  
(63)

\[
\dot{H} = a^{-3/2} \left[ \frac{\dot{a}_2 - 4 \ddot{a}_2}{a_2} X^2 - \frac{H^2}{2} + \frac{\kappa}{3 a_2^2} \rho_0 a^3(0) \right],
\]  
(64)

\[
\dot{u} = a^{-3/2} \left[ -\frac{4 a_3^2}{a_2} + \frac{\kappa}{3 a_2^2} \rho_0 a^3(0) \right],
\]  
(65)

\[
\dot{x} = a^{-3/2} \left[ \frac{2 H u - 3 H^2}{2} \right] - \frac{X}{6},
\]  
(66)

\[
\dot{\bar{R}} = a^{-3/2} X
\]  
(67)

\[
\begin{align*}
&= a^{3/2} \Lambda + \frac{3 a_2}{2} [(H - u)^2 - \frac{a^2}{4} + \zeta^2 a] + \frac{3}{2} \left( \frac{1}{2} \ddot{a}_2 - 2 \dddot{a}_3 \right) \frac{x^2}{a^2} \\
&\quad + \frac{w_3}{24} X^2 - \frac{w_6}{24} R^2 - \frac{w_3}{24} R X - \frac{3}{2} a_2 \left( H^2 + \zeta^2 a \right) + 6 \dot{a}_2 \dddot{X} (H - u) \\
&\quad + \left( \frac{w_6}{2} R + \frac{w_3}{4} \frac{X}{H - u} \right) \left( (H - u)^2 - \frac{x^2}{a^2} + \zeta^2 a \right),
\end{align*}
\]  
(68)

Now let us restrict our further considerations to \(a^3 \gg 1\), dropping the higher order terms, with vanishing \(\Lambda\) and \(\zeta\), the energy expression becomes

\[
\begin{align*}
-a^3 \kappa \rho &= -\kappa \rho_0 a^3(0) \\
&= \frac{3 a_2}{2} \left( (H - u)^2 - \frac{a^2}{4} + \zeta^2 a \right) + \frac{w_3}{24} X^2 - \frac{w_6}{24} R^2 - \frac{3}{2} a_2 \left( H^2 + \zeta^2 a \right) + 6 \dot{a}_2 \dddot{X} (H - u) \\
&\quad + \left( \frac{w_6}{2} R + \frac{w_3}{4} \frac{X}{H - u} \right) \left( (H - u)^2 - \frac{x^2}{a^2} + \zeta^2 a \right).
\end{align*}
\]  
(69)
and the rescaled variables satisfy the six linear equations:

\[ \dot{a} = a^{-1/2} \frac{\dot{H}}{H}, \]  
\[ \dot{H} = \frac{1}{6a^2} [ \dot{a} R - \dot{2}\sigma X ], \]  
\[ \dot{u} = -\frac{1}{3a^2} [a_0 R + \sigma_2 X ], \]  
\[ \dot{\sigma} = \frac{-X}{6}, \]  
\[ \dot{R} = \frac{6}{a} \left[ (w_3 \sigma_2 - \mu_3 \sigma_2) u - 2(w_3 \sigma_2 + \mu_3 \sigma_2) X \right], \]  
\[ \dot{X} = \frac{6}{a} \left[ (2w_3 \sigma_2 + \frac{1}{2} \mu_3 \sigma_2) u + (4w_3 \sigma_2 + \frac{1}{2} \mu_3 \sigma_2) X \right]. \]  

These 6 equations turn out to be equivalent to the linearized system (39–44) which was considered earlier. Consequently there are 3 late time normal modes, \( z \), which is a constant, and the two normal modes of frequency \( \omega_{\pm} \) involving \( R, X, u, \sigma \).

The linearized mode \( z \) is an asymptotic rescaling of a certain combination of the frame/metric scale expansion factor with the scalar and pseudoscalar torsion:

\[ z := a_0 H + \frac{\ddot{a}}{2} u - \dot{\sigma} \sigma, \]  

which to linear order is constant, but in general evolves according to (here including \( \xi^2 \) and \( \Lambda \))

\[ \dot{z} = -2\dot{a} \dot{\sigma} x^2 - \frac{\ddot{a}}{2} u (3H - u), \]  
\[ -2a_0 H^2 + \dot{\sigma} x (3H - u) + \frac{k \rho}{6} - a_0 \frac{\dot{\sigma}^2}{a^2} + \frac{2\Lambda}{3}. \]  

From the above one can see that at late times (with vanishing cosmological constant) the Hubble expansion rate has the form

\[ H = a^{-3/2} \times \text{const.}, \]  
\[ \frac{\ddot{a}}{2d0} u + \dot{\sigma} \sigma x, \]  

with the \( 0^+ \) torsion amplitude \( u \) and the \( 0^- \) torsion amplitude \( x \) having damped oscillations at a superposition of the frequencies \( \omega_{\pm} \). With the pseudoscalar cross parity coupling constants, not only does the even parity \( 0^+ \) mode effect the expansion rate at late times, but so also does the \( 0^- \) odd parity mode. The late time acceleration from (72) is

\[ \ddot{a} = a^{-1/2} \frac{1}{6a^2} (\ddot{a} R - 2\dot{\sigma} X), \]  

which has oscillations at superpositions of the two periodic rates \( \omega_{\pm} \). In this model sometimes the expansion rate is accelerating and sometimes it is slowing down.

Figure 1 (online color at: www.ann-phys.org) Hubble function \( H \), “constant mode” \( z \), scalar curvature \( R \), pseudoscalar curvature \( X \), scalar torsion \( \sigma \) and pseudoscalar torsion, \( \dot{X} \). The blue (solid) lines represent the rescaled late time evolution and the red (dashed) lines represent the linear approximation evolution.
7 Numerical demonstration

The validity of our late time asymptotic analytic results has been tested numerically. Taking the parameters as

\[ a_0 = 1, \quad a_2 = -0.83, \quad a_3 = -0.35, \quad w_6 = -1.1, \]

\[ w_3 = 0.091, \quad \sigma_2 = 0.4 \quad \text{and} \quad \mu_3 = -0.07, \]

(81)

which allows for significant contributions from the pseudoscalar coupling between even and odd parity, we considered several sets of initial values. We plot a typical case in Fig. 1 showing the Hubble function \( \mathcal{H} \), the “constant” function \( z \), the scalar curvature \( R \), the pseudoscalar curvature \( X \), the torsion \( \tau \) and the axial torsion \( x \). Here we compare our model for the late time asymptotic evolution with the linear result. The solid (blue) lines represent the rescaled evolution at late time and the dashed (red) lines represent the linear approximation evolution.

8 Concluding discussion

We have been investigating the dynamics of the Poincaré gauge theory of gravity. Recently, the model with two good propagating modes carrying spin \( 0^+ \), \( 0^- \) (referred to as the scalar and pseudoscalar modes) has been extended to include pseudoscalar constants that couple the two different parity modes.

Here we have considered the dynamics of this BHN model in the context of manifestly homogeneous and isotropic cosmological models. A system of first order equations obtained from an effective Lagrangian was linearized, the normal modes were identified, and it was shown analytically how they control the late time asymptotics. A numerical evolution example was presented which shows that the late time linear mode approximation is good.

The analysis of the equations confirms certain expected effects of the pseudoscalar coupling constants – which provide a direct interaction between the even and odd parity modes. In these models, at late times the acceleration oscillates. It can be positive at the present time. As far as we know the scalar torsion mode does not directly couple to any known form of matter, but we noted that it does couple directly to the Hubble expansion, and thus it can directly influence the acceleration of the universe. On the other hand, the pseudoscalar torsion couples directly to fundamental fermions; with the newly introduced pseudoscalar coupling constants it too can directly influence the cosmic acceleration.

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