LOGARITHMIC UNCERTAINTY PRINCIPLES FOR THE HANKEL WAVELET TRANSFORM

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ABSTRACT. The aim of this paper is to prove a logarithmic and a Hirschman-Beckner entropic uncertainty principles for the Hankel wavelet transform. Then we derive a general form of Heisenberg-type uncertainty inequality for this transformation.

1. Introduction

Let $d \geq 1$ be the dimension, and let us denote by $\langle \cdot, \cdot \rangle$ the scalar product and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$. Define also the classical translation and dilation operators on $L^2(\mathbb{R}^d)$ by

$$\tau_{\alpha} g(t) = g(t - x) \quad \text{and} \quad \delta_{\alpha} g(t) = a^{d/2} f(at), \quad a \in (0, \infty), \quad x, t \in \mathbb{R}^d. \quad (1.1)$$

Then the Fourier transform is defined for a function $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by:

$$\mathcal{F}(g)(\xi) = \hat{g}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(x)e^{-i\xi \cdot x} \, dx, \quad (1.2)$$

and it is extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way. Moreover, if we take $\phi \in L^2(\mathbb{R}^d)$ an admissible wavelet, that satisfies the following admissibility condition,

$$0 < A_\phi := \int_0^\infty |\hat{\phi}(a\xi)|^2 \frac{da}{a} < \infty, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad (1.3)$$

then the continuous wavelet transform of a signal $f \in L^2(\mathbb{R}^d)$ is defined by (see e.g. [10]),

$$W_\phi g(a, x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{g}(t)\overline{\phi_{a,x}(t)} \, dt, \quad (a, x) \in U, \quad (1.4)$$

where $U = (0, \infty) \times \mathbb{R}^d$ is the affine group, and $\phi_{a,x}$ is the wavelet atom defined, by

$$\phi_{a,x} = A^{-1/2}_{\psi} \tau_{\alpha} \delta_{\alpha} \phi. \quad (1.5)$$

Wavelet atoms (which are the dilation with a scale parameter $a$ and the translation by the position (or time) parameter $x$ of admissible wavelets) has their energy well localized in position, while their Fourier transform is mostly concentrated in a limited frequency band.

Now, if $g(x) = f(|x|)$ is a radial function on $\mathbb{R}^d$, then $\hat{g}(\xi) = \mathcal{H}_{d/2-1}(f)(\xi|)$, where for $\alpha \geq -1/2$, $\mathcal{H}_\alpha$ is the Hankel transform (also known as the Fourier-Bessel transform) defined by (see e.g. [24]):

$$\mathcal{H}_\alpha(f)(\xi) = \int_{\mathbb{R}^+} f(x)j_\alpha(x\xi) \, d\mu_\alpha(x), \quad \xi \in \mathbb{R}_+ = [0, +\infty). \quad (1.6)$$

In particular $\mu_\alpha$ is the weight measure defined by $d\mu_\alpha(x) = \frac{x^{2\alpha+1}}{2\Gamma(\alpha+1)} \, dx$ and $j_\alpha$ (see e.g. [24] [27]) is the spherical Bessel function given by:

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha + 1)J_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n}, \quad (1.7)$$

where $J_\alpha$ is the Bessel function of the first kind and $\Gamma$ is the gamma function.
Throughout this paper, \( \alpha \) will be a real number such that \( \alpha > -1/2 \), since for \( \alpha = -1/2 \), we have \( \mu_{-1/2} \) is the Lebesgue measure and \( \mathcal{H}_{-1/2} \) is the Fourier-cosine transform, which is the Fourier transform \( \mathcal{F} \) restricted to even functions on \( \mathbb{R} \).

For \( \alpha > -1/2 \), let us recall the Poisson representation formula (see e.g. [25] (1.71.6), p. 15):

\[
j_\alpha(x) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1/2) \Gamma(\frac{1}{2})} \int_{-1}^{1} (1 - s^2)^{\alpha - 1/2} \cos(sx) \, ds.
\]

(1.8)

Therefore, \( j_\alpha \) is bounded with \( |j_\alpha(x)| \leq j_\alpha(0) = 1 \). As a consequence,

\[
\| \mathcal{H}_\alpha(f) \|_\infty \leq \| f \|_{1, \mu_\alpha},
\]

(1.9)

where \( \| \cdot \|_\infty \) is the usual essential supremum norm on the space of essentially bounded functions \( L^\infty_\alpha(\mathbb{R}_+) \), and for \( 1 \leq p < \infty \), we denote by \( L^p_\alpha(\mathbb{R}_+, \mu_\alpha) \) the Banach space consisting of measurable functions \( f \) on \( \mathbb{R}_+ \) equipped with the norms:

\[
\| f \|_{p, \mu_\alpha} = \left( \int_0^\infty |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p}.
\]

(1.10)

It is also well-known (see [24]) that the Hankel transform extends to an isometry on \( L^2_\alpha(\mathbb{R}_+) \):

\[
\| \mathcal{H}_\alpha(f) \|_{2, \mu_\alpha} = \| f \|_{2, \mu_\alpha}.
\]

(1.11)

Uncertainty principles in Fourier analysis set a limit to the possible concentration of a function and its Hankel transform in the time-frequency domain. The most familiar form is the Heisenberg-type inequalities, in which concentration is measured by dispersions (see [4 19 8]): For all \( f \in L^2_\alpha(\mathbb{R}_+) \),

\[
\|xf\|_{2, \mu_\alpha} \| \xi \mathcal{H}_\alpha(f) \|_{2, \mu_\alpha} \geq (\alpha + 1) \| f \|_{2, \mu_\alpha}^2,
\]

(1.12)

with equality, if and only if \( f \) is a multiple of a suitable Gaussian function.

A little less known principles consist of logarithmic uncertainty principles. In particular, for any even function \( f \) in the Schwartz space \( \mathcal{S}_c(\mathbb{R}) \) we have (see [18] Theorem 3.10)),

\[
\int_0^{\infty} \ln(x) |f(x)|^2 \, d\mu_\alpha(x) + \int_0^{\infty} \ln(\xi) |\mathcal{H}_\alpha(f)(\xi)|^2 \, d\mu_\alpha(\xi) \geq \left( \ln(2) + \frac{\Gamma'(\alpha + 1)}{\Gamma(\alpha + 1/2)} \right) \| f \|_{2, \mu_\alpha}^2.
\]

(1.13)

The proof of the last inequality is based on a Pitt-type inequality for the Hankel transform (see [18] Theorem 3.9), and from which we derive the following Heisenberg-type uncertainty inequality for the Hankel transform: For all even function \( f \) in the Schwartz space \( \mathcal{S}_c(\mathbb{R}) \),

\[
\|xf\|_{2, \mu_\alpha} \| \xi \mathcal{H}_\alpha(f) \|_{2, \mu_\alpha} \geq 2 \exp \left( \frac{\Gamma'(\alpha + 1)}{\Gamma(\alpha + 1/2)} \right) \| f \|_{2, \mu_\alpha}^2,
\]

(1.14)

where

\[
2 \exp \left( \frac{\Gamma'(\alpha + 1)}{\Gamma(\alpha + 1/2)} \right) \approx (\alpha + 1), \quad \text{for} \quad \alpha \gg 1,
\]

(1.15)

which is the optimal constant in the sharp Heisenberg's Inequality (1.12).

Moreover, by using the sharp Hausdorff-Young inequality for the Hankel transform, the author in [5, Theorem 3.3], proved the following entropic uncertainty inequality for the Hankel transform (in which concentration is measured by entropy, and which implies also Inequality (1.12)), that is, for all nonzero \( f \in L^2_\alpha(\mathbb{R}_+) \),

\[
- \int_0^{\infty} |f(x)|^2 \ln \left( |f(x)|^2 \right) \, d\mu_\alpha(x) - \int_0^{\infty} |\mathcal{H}_\alpha(f)(\xi)|^2 \ln \left( |\mathcal{H}_\alpha(f)(\xi)|^2 \right) \, d\mu_\alpha(\xi) \\
\geq \left( (2\alpha + 2) \ln \left( \frac{e}{2} \right) - 2 \ln \left( \| f \|_{2, \mu_\alpha}^2 \right) \right) \| f \|_{2, \mu_\alpha}^2.
\]

(1.16)

As for the Fourier transform, if we take a radial admissible wavelet \( \phi(t) = \psi(|t|) \), then for all radial function \( g(t) = f(|t|) \), its continuous wavelet transform \( W_\psi g \) coincides with the Hankel
wavelet transform $\mathbf{W}_\psi f$ (with $\alpha = d/2 - 1$), defined by

$$
\mathbf{W}_\psi f(a, x) = c_\psi^{-1/2} \int_0^\infty f(t) \tau_2^\alpha(D_{a, \psi}^\alpha)(t) \, d\mu_\alpha(t), \quad (a, x) \in \mathbb{R}_+^2 = (0, \infty) \times [0, \infty),
$$

(1.17)

where $c_\psi$ is the admissibility condition given in (2.19), and $\tau_2^\alpha$, $D_{a, \psi}^\alpha$ are the Hankel translation and the dilation operators given in (2.14), (2.15) respectively.

Wavelet theory is often seen as a relatively new concept for time-frequency analysis, rather than the Fourier, the Hankel and the windowed Fourier transforms, because of its advantage on better locality in the time-scale variations of a signal. However, the uncertainty principles set a limit to the maximal time-frequency or time-scale resolutions. These uncertainty principles for the continuous wavelet transform, can be found in [11, 16, 17, 23, 28], and those for the Hankel wavelet transform can be found in [1, 12].

The aim of this paper is to extend the uncertainty principles (1.13) and (1.16) for the Hankel wavelet transform. In particular we prove the following Pitt-type inequality: For all $0 \leq \beta < \alpha + 1$, and all function $f \in \mathcal{S}_c(\mathbb{R})$ such that $\mathbf{W}_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R})$, we have

$$
\int_0^\infty \int_0^\infty a^{-2\beta} |\mathbf{W}_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) \leq C_{\alpha, \beta}(\psi) \int_0^\infty \int_0^\infty x^{2\beta} |\mathbf{W}_\psi f(a, x)|^2 \, d\nu_\alpha(a, x),
$$

(1.18)

where $d\nu_\alpha(a, x) = a^{2\alpha+1} \, da \, d\mu_\alpha(x)$, and the constant $C_{\alpha, \beta}(\psi)$ is given in (4.14). This inequality allows to prove the following logarithmic uncertainty principle for the Hankel wavelet transform: For all function $f \in \mathcal{S}_c(\mathbb{R})$ such that $\mathbf{W}_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R})$, we have

$$
\int_{\mathbb{R}_+^2} \ln(a) \, |\mathbf{W}_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) + \int_{\mathbb{R}_+^2} \ln(x) \, |\mathbf{W}_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) \geq C_\alpha(\psi) \|f\|_{2, \mu_\alpha}^2,
$$

(1.19)

where $\mathbb{R}_+^2 = (0, \infty) \times \mathbb{R}_+$, and the constant $C_\alpha(\psi)$ is given in (4.16). Consequently, we derive the following Heisenberg-type uncertainty inequality: For all $f \in \mathcal{S}_c(\mathbb{R})$ such that $\mathbf{W}_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R})$, we have

$$
\|ax\mathbf{W}_\psi f\|_{2, \mu_\alpha} \|x\mathbf{W}_\psi f\|_{2, \mu_\alpha} \geq e^{C_\alpha(\psi)} \|f\|_{2, \mu_\alpha}^2,
$$

(1.20)

where $\|\cdot\|_{p, \nu_\alpha}; \ (p \geq 1)$ are the usual norms in the Banach spaces $L_p^\alpha(\mathbb{R}_+^2) = L^p(\mathbb{R}_+^2, \nu_\alpha)$ given by,

$$
\|F\|_{p, \nu_\alpha}^p = \int_0^\infty \int_0^\infty |F(a, x)|^p \, d\nu_\alpha(a, x), \quad F \in L_p^\alpha(\mathbb{R}_+^2).
$$

(1.21)

Finally, we prove the following Hirschman-Beckner entropic uncertainty inequality for the Hankel wavelet transform, that is, for any admissible wavelet $\psi$ such that $\|\psi\|_{2, \mu_\alpha}^2 \leq c_\psi$, we have for all nonzero function $f \in L_\alpha^2(\mathbb{R}_+)$,

$$
- \int_{\mathbb{R}_+^2} |\mathbf{W}_{\psi} f(a, x)|^2 \ln \left( |\mathbf{W}_{\psi} f(a, x)|^2 \right) \, d\nu_\alpha(a, x) \geq \|f\|_{2, \mu_\alpha}^2 \ln \left( \frac{c_\psi}{\|\psi\|_{2, \mu_\alpha}^2 \|f\|_{2, \mu_\alpha}^2} \right).
$$

(1.22)

The last inequality implies a general form of Heisenberg-type uncertainty inequality for functions in $L_\alpha^2(\mathbb{R}_+)$, that is, for all $s, \beta > 0$, and any admissible wavelet $\psi$ such that $\|\psi\|_{2, \mu_\alpha}^2 \leq c_\psi$, there exists a positive constant $C(s, \alpha, \beta)$ such that, for all nonzero function $f \in L_\alpha^2(\mathbb{R}_+)$,

$$
\|a^s \mathbf{W}_{\psi} f\|_{2, \mu_\alpha}^p \|ax^\beta \mathbf{W}_{\psi} f\|_{2, \mu_\alpha}^\beta \geq C(s, \alpha, \beta) \|f\|_{2, \mu_\alpha}^{p + \beta},
$$

(1.23)

where the constant $C(s, \alpha, \beta)$ is given in (1.39). This inequality improve a result proved in [1], in which the constant involves the Mellin transform.

As a side result, we prove that for any admissible wavelet $\psi$, and any function $f \in L_\alpha^2(\mathbb{R}_+)$, its Hankel wavelet transform $\mathbf{W}_{\psi} f$ belongs to $L_p^\alpha(\mathbb{R}_+^2), \ p \geq 2$, satisfying the following Lieb-type inequality

$$
\|\mathbf{W}_{\psi} f\|_{p, \nu_\alpha} \leq \left( \frac{\|\psi\|_{2, \mu_\alpha}^2}{c_\psi} \right)^{\frac{1}{2} - \frac{1}{p}} \|f\|_{2, \mu_\alpha}.
$$

(1.24)
In particular if a nonzero function \( f \in L^2_\alpha(\mathbb{R}_+) \) is \((\varepsilon, \psi)\)-time-scale-concentrated in a subset \( \Sigma \subset \mathbb{R}^2_+ \) of finite measure, i.e.

\[
\int_{\mathbb{R}^2_+ \setminus \Sigma} \|W_\psi f(a, x)^2\, dv_\alpha(a, x) \leq \varepsilon \|f\|_{L^2_\alpha}^2,
\]

where \( 0 < \varepsilon < 1 \), then \( \Sigma \) satisfies for every \( p > 2 \),

\[
\nu_\alpha(\Sigma) \geq \frac{c_\psi}{\|\psi\|_{L^2_\alpha}} (1 - \varepsilon)^{\frac{p}{p-2}}.
\]

Consequently, if \( \varepsilon = 0 \), then this implies that the Hankel wavelet transform is concentrated in \( \Sigma \) and its support \( \Sigma \) satisfies

\[
\nu_\alpha(\text{supp} \, W_\psi f) \geq \frac{c_\psi}{\|\psi\|_{L^2_\alpha}} \tag{1.27}
\]

which means that, the support of the Hankel wavelet transform of a nonzero function \( f \in L^2_\alpha(\mathbb{R}_+) \) cannot be too small.

The remainder of this paper is arranged as follows, in the second section we recall some useful harmonic analysis results associated with the Hankel and the Hankel wavelet transforms. Section 3 is devoted to the study of an uncertainty principle for the Hankel wavelet transform in subsets of small measures. In Section 4 we prove some logarithmic uncertainty principles for the Hankel wavelet transform, and derive some Heisenberg-type uncertainty inequalities.

2. Preliminaries

2.1. Notation. For a measurable subset \( E \), we will write \( E^c \) for its complement, and we will denote by \( S_\alpha(\mathbb{R}) \) the Schwartz space, constituted by the even infinitely differentiable functions on the real line, rapidly decreasing together with all their derivatives.

2.2. The Hankel transform. In this subsection, we will recall some harmonic analysis results related to the Hankel transform that we shall use later (see e.g. [24, 27]). We denote by \( \ell_\alpha \) the Bessel operator defined on \((0, \infty)\) by

\[
\ell_\alpha(u) = u'' + \frac{2\alpha + 1}{r} u'.
\]

For all \( \lambda \in \mathbb{C} \), the following system

\[
\begin{cases}
\ell_\alpha(u) = -\lambda^2 u, \\
u(0) = 1, \, u'(0) = 0;
\end{cases}
\]

admits a unique solution given by the modified Bessel function \( x \mapsto j_\alpha(x, \lambda) \), where

\[
j_\alpha(z) = 2^\alpha \frac{\Gamma(\alpha + 1)}{\pi^\alpha} J_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(\alpha + n + 1)} \left(\frac{z}{2}\right)^{2n} \tag{2.3}
\]

The Hankel translation operator \( \tau^\alpha_x \), \( x \in \mathbb{R}_+ \) is defined by

\[
\tau^\alpha_x(f)(y) = \frac{\Gamma(\alpha + 1)}{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha + 1}{2}\right)} \int_0^{\infty} f \left( \sqrt{x^2 + y^2 + 2xy \cos(\theta)} \right) \sin(\theta)^{2\alpha} \, d\theta,
\]

whenever the integral in the right-hand side is well defined. Then

\[
\tau^\alpha_x(j_\alpha(\lambda \cdot))(y) = j_\alpha(\lambda x)j_\alpha(\lambda y), \quad x, \, y, \, \lambda \geq 0.
\]

Moreover, for all \( x, \, y > 0 \),

\[
\tau^\alpha_x(f)(y) = \int_0^{\infty} f(t) K_\alpha(t, x, y) \, d\mu_\alpha(t),
\]

where \( K_\alpha \) is the kernel given by

\[
K_\alpha(t, x, y) = \begin{cases}
\frac{\Gamma^2(\alpha + 1)}{\sqrt{\pi} 2^{\alpha-1} \Gamma(\alpha + \frac{1}{2})} \left(\frac{(x+y)^{1/2} t^2 - (x-y)^{1/2}}{(xy)^{\alpha/2}}\right)^{\alpha - 1/2}, & \text{if } |x - y| < t < x + y, \\
0, & \text{otherwise.}
\end{cases}
\]
The kernel $K_\alpha$ is symmetric in the variables $t, x, y$, and satisfies
\[ \int_0^\infty K_\alpha(t, x, y) \, d\mu_\alpha(t) = 1. \] (2.8)

From Hölder’s inequality, Relations (2.6) and (2.8) we have, for every $f \in L_p^\alpha(\mathbb{R}_+)$, $p \in [1, \infty]$, the function $\tau_\alpha^\alpha(f) \in L_p^\alpha(\mathbb{R}_+)$ and
\[ \|\tau_\alpha^\alpha(f)\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}. \] (2.9)

Moreover, for $f \in L_1^\alpha(\mathbb{R}_+)$, we have for all $x \in \mathbb{R}_+$,
\[ \int_0^\infty \tau_\alpha^\alpha(f)(y) \, d\mu_\alpha(y) = \int_0^\infty f(y) \, d\mu_\alpha(y), \] (2.10)
and for every $f \in L_p^\alpha(\mathbb{R}_+)$, $p = 1, 2$,
\[ \mathcal{H}_\alpha(\tau_\alpha^\alpha(f))(\lambda) = j_\alpha(x\lambda)\mathcal{H}_\alpha(f)(\lambda), \quad \lambda \in \mathbb{R}_+. \] (2.11)

The convolution product of $f, g \in L_1^\alpha(\mathbb{R}_+)$ is defined by
\[ f \ast_\alpha g(x) = \int_0^\infty \tau_\alpha^\alpha(f)(y)g(y) \, d\mu_\alpha(y), \] (2.12)
which is commutative and associative in $L_1^\alpha(\mathbb{R}_+)$. Then, for every $1 \leq p, q, r \leq \infty$, such that $1/p + 1/q = 1 + 1/r$, we have $f \ast_\alpha g \in L_r^\alpha(\mathbb{R}_+)$, with
\[ \|f \ast_\alpha g\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha} \|g\|_{q,\mu_\alpha}. \]

In particular, for every $f \in L_1^\alpha(\mathbb{R}_+)$ and $g \in L_2^\alpha(\mathbb{R}_+)$, $p = 1, 2$, the function $f \ast_\alpha g$ is in $L_2^\alpha(\mathbb{R}_+)$, and then
\[ \mathcal{H}_\alpha(f \ast_\alpha g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g). \] (2.13)

Moreover, for every $f, g \in L_2^\alpha(\mathbb{R}_+)$, the function $f \ast_\alpha g \in L_2^\alpha(\mathbb{R}_+)$ if and only if $\mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g) \in L_2^\alpha(\mathbb{R}_+)$, and then
\[ \mathcal{H}_\alpha(f \ast_\alpha g) = \mathcal{H}_\alpha(f)\mathcal{H}_\alpha(g). \] (2.14)

2.3. The Hankel wavelet transform. Now, we will recall some preliminaries results of the Hankel wavelet transform (HWT for short), which is first introduced in [26], (see also [1][12][16][29]).

The dilation operator $\mathcal{D}_a^\alpha$ of a function $f \in L_2^\alpha(\mathbb{R}_+)$ is defined by
\[ \mathcal{D}_a^\alpha f(x) = a^{\alpha+1}f(ax), \quad a > 0. \] (2.15)

It satisfies the following properties.

**Properties 2.1.**

1. For all $f \in L_2^\alpha(\mathbb{R}_+)$ we have
\[ \|\mathcal{D}_a^\alpha f\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}, \quad \mathcal{H}_\alpha(\mathcal{D}_a^\alpha f) = \mathcal{D}_a^\alpha \mathcal{H}_\alpha(f). \] (2.16)

2. For all $f, g \in L_2^\alpha(\mathbb{R}_+)$, we have
\[ \langle \mathcal{D}_a^\alpha f, g \rangle_{\mu_\alpha} = \langle f, \mathcal{D}_a^{-\alpha} g \rangle_{\mu_\alpha}, \] (2.17)
where $\langle \cdot, \cdot \rangle_{\mu_\alpha}$ is the usual inner product on the Hilbert space $L_2^\alpha(\mathbb{R}_+)$. (3) For all $x \geq 0$, we have
\[ \mathcal{D}_a^\alpha \tau_\alpha^\alpha = \tau_\alpha^\alpha \mathcal{D}_a^\alpha. \] (2.18)

**Definition 2.2.** A nonzero function $\psi \in L_1^\alpha(\mathbb{R}_+) \cap L_\infty^\alpha(\mathbb{R}_+)$ is said to be an admissible wavelet, if it satisfies,
\[ 0 < c_\psi = \int_0^\infty |\mathcal{H}_\alpha(\psi)(a)|^2 \, da < \infty. \] (2.19)

Moreover, if $\psi$ is an admissible wavelet, then $\psi \in L_p^\alpha(\mathbb{R}_+)$, for all $1 \leq p \leq \infty$, since for $p \in (1, \infty)$, we have
\[ \|\psi\|_{p,\mu_\alpha}^p = \int_{\mathbb{R}_+} |\psi(x)||\psi(x)|^{p-1} \, d\mu_\alpha(x) \leq \|\psi\|_{\infty}^{p-1}\|\psi\|_{1,\mu_\alpha}. \] (2.20)
Definition 2.3. Let ψ be an admissible wavelet. Then the Hankel wavelet transform $W^\alpha_\psi$ is defined on $L^2_\alpha(\mathbb{R}^+)$ by

$$W^\alpha_\psi f(a, x) = \int_0^\infty f(t)\overline{\psi^\alpha_{a,x}(t)}\,d\mu_\alpha(t), \quad (a, x) \in \mathbb{R}^2_+,$$

(2.21)

where, $\psi^\alpha_{a,x} = c^{-1/2}_\psi r^{\alpha}_x (D^\alpha_\psi)^{-1}(a, x)$.

Equality (2.21) can be also written as

$$W^\alpha_\psi f(a, x) = c^{-1/2}_\psi \ast_\alpha D^\alpha_\psi (\overline{\psi})(x) = \langle f, \psi^\alpha_{a,x}\rangle_{\mu_\alpha}.$$

(2.22)

Moreover, the Hankel wavelet transform satisfies the following properties.

Properties 2.4. Let ψ be an admissible wavelet. Then for every $f \in L^2_\alpha(\mathbb{R}^+)$, we have:

1. The function $W^\alpha_\psi f$ is bounded, with

$$\|W^\alpha_\psi f\|_\infty \leq c^{-1/2}_\psi \|f\|_{2,\mu_\alpha} \|\psi\|_{2,\mu_\alpha},$$

(2.23)

where $\|\cdot\|_\infty$ is the essential supremum norm in the space $L^\infty_\alpha(\mathbb{R}^2)$.

2. The function $W^\alpha_\psi f$ belongs to $L^2_\alpha(\mathbb{R}^2)$, and satisfies the following Plancherel formula,

$$\|W^\alpha_\psi f\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}.$$

(2.24)

3. For every $(a, x) \in \mathbb{R}^2_+$,

$$W^\alpha_\psi (D^\alpha_\psi f)(a, x) = W^\alpha_\psi f\left(\frac{a}{\lambda}, \lambda x\right), \quad \lambda > 0.$$

(2.25)

4. For every $(a, x) \in \mathbb{R}^2_+$, we have

$$H^\alpha_{\lambda} (W^\alpha_\psi f(a, \cdot)) (\xi) = c^{-1/2}_\psi \mu^\alpha_{a+1} H^\alpha_{\lambda} (f) \overline{H^\alpha_{\lambda} (\psi)} \left(\frac{\xi}{a}\right).$$

(2.26)

3. Uncertainty principles for the HWT in subsets of small measures

In this section Σ will be a subset in $\mathbb{R}^2_+$ of finite measure $0 < \nu_\alpha(\Sigma) < \infty$, and ψ will be an admissible wavelet. The aim of this section is to prove an uncertainty results limiting the concentration of the HWT in subsets of small measures (see e.g. [7] and [10] Theorem 3.3.3) for related results for the Fourier and the windowed Fourier transforms).

Definition 3.1. Let $0 \leq \varepsilon < 1$. Then, a nonzero function $f \in L^2_\alpha(\mathbb{R}^+)$ is $(\varepsilon, \psi)$-time-scale-concentrated in Σ if,

$$\int_{\Sigma} |W^\alpha_\psi f(a, x)|^2 d\nu_\alpha(a, x) \leq \varepsilon \|f\|_{2,\mu_\alpha}^2.$$  

(3.1)

Notice also that, if $\varepsilon = 0$ in (3.1), then Σ will be the exact support of $W^\alpha_\psi f$ (denoted by $\text{supp}W^\alpha_\psi f$), so that Inequality (3.1) with $0 < \varepsilon < 1$, means that $W^\alpha_\psi f$ is “practically zero” outside Σ, and then Σ may be considered as the “essential” support of $W^\alpha_\psi f$.

We denote by $L^2_\alpha(\varepsilon, \Sigma, \psi)$ the set of all functions in $L^2_\alpha(\mathbb{R}^+)$ that are $(\varepsilon, \psi)$-time-scale-concentrated in Σ, where $\varepsilon \in (0, 1)$. Then we have the following Donoho-Stark type uncertainty relation. (see [11] Proposition 3.1).

Proposition 3.2. If $f$ is in $L^2_\alpha(\varepsilon, \Sigma, \psi)$, then the essentiol support of its HWT satisfies

$$\nu_\alpha(\Sigma) \geq \frac{c^{-1}_\psi}{\|\psi\|_{2,\mu_\alpha}^2} \left(1 - \varepsilon\right).$$

(3.2)

Proof. Let $f \in L^2_\alpha(\varepsilon, \Sigma, \psi)$ such that $\|f\|_{2,\mu_\alpha} = 1$. By (2.23), we have for all $(a, x) \in \mathbb{R}^2_+$,

$$|W^\alpha_\psi f(a, x)| \leq \frac{\|\psi\|_{2,\mu_\alpha}}{c^{-1}_\psi},$$

then

$$\int_{\Sigma} |W^\alpha_\psi f(a, x)|^2 d\nu_\alpha(a, b) \leq \|W^\alpha_\psi f\|_{2,\mu_\alpha}^2 \nu_\alpha(\Sigma) \leq \frac{\|\psi\|_{2,\mu_\alpha}^2}{c^{-1}_\psi} \nu_\alpha(\Sigma).$$
by Plancherel formula (2.24),
\[
\int_{\Sigma} |W_{\psi}^\alpha f(a,x)|^2 dv_{\alpha}(a,x) = 1 - \int_{\Sigma^c} |W_{\psi}^\alpha f(a,x)|^2 dv_{\alpha}(a,x) \geq 1 - \varepsilon. \tag{3.3}
\]
This allows to conclude.

In particular, when \(\varepsilon = 0\), then (3.3) implies that the Hankel wavelet transform is concentrated in \(\Sigma\) and its support satisfies
\[
\nu_{\alpha}(\text{supp } W_{\psi}^\alpha f) \geq \frac{c_{\psi}}{\|\psi\|^2_{2,\mu_\alpha}}. \tag{3.4}
\]
This means that, the support or the essential support of the Hankel wavelet transform cannot be too small. On the other hand if we take a small subset \(\Sigma \subset \mathbb{R}_+^2\) such that \(\nu_{\alpha}(\Sigma) < \frac{c_{\psi}}{\|\psi\|^2_{2,\mu_\alpha}}\), then (see e.g. [12, Proposition 3.1]), for all \(f \in L^2_{\alpha}(\mathbb{R}_+)\),
\[
\int_{\Sigma} |W_{\psi}^\alpha f(a,b)|^2 dv_{\alpha}(a,b) \geq \left(1 - \frac{\|\psi\|_{2,\mu_\alpha}^2}{c_{\psi}} \right) \|f\|_{2,\mu_\alpha}^2. \tag{3.5}
\]
In particular if the Hankel wavelet transform is supported in \(\Sigma\) (that is supp \(W_{\psi}^\alpha f \subset \Sigma\)), such that \(\nu_{\alpha}(\Sigma) < \frac{c_{\psi}}{\|\psi\|^2_{2,\mu_\alpha}}\), then \(f\) is the zero function.

**Theorem 3.3.** Let \(p \geq 1\) and let \(\psi, \phi\) two admissible wavelets. Then for all \(f, g \in L^2_{\alpha}(\mathbb{R}_+)\), the function \(W_{\psi}^\alpha f W_{\phi}^\alpha g\) belongs to \(L^p_{\alpha}(\mathbb{R}_+^2)\), and
\[
\|W_{\psi}^\alpha f W_{\phi}^\alpha g\|_{p,\nu_{\alpha}}^p \leq \left(\frac{\|\psi\|_{2,\mu_\alpha}^2 \|\phi\|_{2,\mu_\alpha}^2}{c_{\psi} c_{\phi}}\right)^{p-1} \|f\|_{2,\mu_\alpha}^{p} \|g\|_{2,\mu_\alpha}^{p}. \tag{3.6}
\]

**Proof.** By Cauchy-Schwartz’s inequality and the Plancherel formula (2.24), we have
\[
\int_{\mathbb{R}_+^2} |W_{\psi}^\alpha f(a,x)W_{\phi}^\alpha g(a,x)| dv_{\alpha}(a,x) \leq \|W_{\psi}^\alpha f\|_{2,\nu_{\alpha}} \|W_{\phi}^\alpha g\|_{2,\nu_{\alpha}} = \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha}.
\]
This implies that the function \(W_{\psi}^\alpha f W_{\phi}^\alpha g\) belongs to \(L^1_{\alpha}(\mathbb{R}_+^2)\), and
\[
\|W_{\psi}^\alpha f W_{\phi}^\alpha g\|_{1,\nu_{\alpha}} \leq \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha}. \tag{3.7}
\]
On the other hand, from (2.23), we have for \((a,x) \in \mathbb{R}_+^2\),
\[
|W_{\psi}^\alpha f(a,x)W_{\phi}^\alpha g(a,x)| \leq \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha} \frac{\|\psi\|_{2,\mu_\alpha} \|\phi\|_{2,\mu_\alpha}}{c_{\psi} c_{\phi}}.
\]
Therefore, the function \(W_{\psi}^\alpha f W_{\phi}^\alpha g\) belongs to \(L^\infty_{\alpha}(\mathbb{R}_+^2)\), and
\[
\|W_{\psi}^\alpha f W_{\phi}^\alpha g\|_{\infty} \leq \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha} \frac{\|\psi\|_{2,\mu_\alpha} \|\phi\|_{2,\mu_\alpha}}{c_{\psi} c_{\phi}}. \tag{3.8}
\]
Thus by (3.7), (3.8) and by an interpolation theorem we obtain the desired result.

**Remark 3.4.** Theorem 3.3 implies in particular that \(W_{\psi}^\alpha f \in L^p_{\alpha}(\mathbb{R}_+^2)\), \(p \geq 2\), with
\[
\|W_{\psi}^\alpha f\|_{p,\nu_{\alpha}}^p \leq \left(\frac{\|\psi\|_{2,\mu_\alpha}^2}{c_{\psi}}\right)^{p/2-1} \|f\|_{2,\mu_\alpha}^{2p}. \tag{3.9}
\]
This result can be considered as a Lieb-type inequality [15].

**Corollary 3.5.** Let \(f \in L^2_{\alpha}(\varepsilon, \Sigma, \psi)\). Then for every \(p > 2\),
\[
\nu_{\alpha}(\Sigma) \geq \frac{c_{\psi}}{\|\psi\|^2_{2,\mu_\alpha}} (1 - \varepsilon)^{\frac{2p}{p-2}}. \tag{3.10}
\]
Proof. Let \( f \in L^2_\alpha(\mathbb{R}^+), \) such that \( \|f\|_{2,\mu_\alpha} = 1. \) Then by Hölder’s inequality,

\[
\int_{\Sigma} \left| W_\psi^\alpha f(a,x) \right|^2 \, d\nu_\alpha(a,x) \leq \nu_\alpha(\Sigma)^{1-2/p} \left( \int_{\mathbb{R}^2} \left| W_\psi^\alpha f(a,x) \right|^p \, d\nu_\alpha(a,x) \right)^{2/p}.
\]

Thus by \( 3.3 \) and \( 3.9, \) we have

\[
1 - \varepsilon \leq \int_{\Sigma} \left| W_\psi^\alpha f(a,x) \right|^2 \, d\nu_\alpha(a,x) \leq \nu_\alpha(\Sigma)^{1-2/p} \left( \frac{\|\psi\|_{2,\mu_\alpha}^2}{c_\psi} \right)^{1-2/p}.
\]

The proof is complete. \( \square \)

4. Logarithmic uncertainty principles for the HWT

4.1. On Heisenberg-type uncertainty inequalities for the HWT. The Heisenberg-type uncertainty principle for the Hankel transform states [8, Theorem 2.1] (see also [4, 19]): For all \( f \in L^2_\alpha(\mathbb{R}^+), \)

\[
\|x f\|_{2,\mu_\alpha}^2 + \|\xi \mathcal{H}_\alpha(f)\|_{2,\mu_\alpha}^2 \geq (2\alpha + 2)\|f\|_{2,\mu_\alpha}^2,
\]

and by a well-known dilation argument (see [8, Corollary 2.2]), we obtain

\[
\|x f\|_{2,\mu_\alpha}^2 \|\xi \mathcal{H}_\alpha(f)\|_{2,\mu_\alpha} \geq (\alpha + 1)\|f\|_{2,\mu_\alpha}^2.
\]

Some Heisenberg-type uncertainty inequalities for the HWT were proved in [11]. In this subsection we will prove the following Heisenberg-type uncertainty inequality, comparing the concentration of \( W_\psi^\alpha f \) in position (with respect to the \( x \)-variable) and the concentration of \( \mathcal{H}_\alpha(f) \) in frequency (with respect to the \( \xi \)-variable).

Theorem 4.1. For all \( f \in L^2_\alpha(\mathbb{R}^+), \) we have

\[
\|x W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \|\xi \mathcal{H}_\alpha(f)\|_{2,\nu_\alpha}^2 \geq (2\alpha + 2)\|f\|_{2,\mu_\alpha}^2,
\]

or equivalently,

\[
\|x W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \|\xi \mathcal{H}_\alpha(f)\|_{2,\nu_\alpha} \geq (\alpha + 1)\|f\|_{2,\mu_\alpha}^2.
\]

Proof. It is clear that, if Inequality \( 4.4 \) holds then Inequality \( 4.3 \) is trivial, since \( s^2 + t^2 \geq 2st. \) Therefore it is enough to prove Inequality \( 4.3, \) and that \( 4.3 \) implies \( 4.4. \)

Heisenberg’s inequality \( 4.1 \) for the function \( W_\psi^\alpha f(a, \cdot) \in L^2_\alpha(\mathbb{R}^+) \) leads to:

\[
\int_0^\infty x^2 |W_\psi^\alpha f(a,x)|^2 \, d\mu_\alpha(x) + \int_0^\infty \xi^2 |\mathcal{H}_\alpha(W_\psi^\alpha f(a, \cdot))|^2 \, d\mu_\alpha(\xi) \geq (2\alpha + 2)\|W_\psi^\alpha f(a, \cdot)\|_{2,\mu_\alpha}^2.
\]

Then by \( 2.26, \)

\[
\int_0^\infty x^2 |W_\psi^\alpha f(a,x)|^2 \, d\mu_\alpha(x) + \int_0^\infty |\xi|^2 |\mathcal{H}_\alpha(f)(\xi)|^2 \frac{|\mathcal{H}_\alpha(\psi)(\xi/a)|^2}{c_\psi a^{2\alpha+2}} \, d\mu_\alpha(\xi) \geq (2\alpha + 2)\|W_\psi^\alpha f(a, \cdot)\|_{2,\mu_\alpha}^2.
\]

Integrating with respect to \( a^{2\alpha+1} \, da, \) we obtain from the admissibility condition \( 2.19 \) and Plancherel formula \( 2.24, \)

\[
\int_0^\infty \int_0^\infty x^2 |W_\psi^\alpha f(a,x)|^2 \, d\mu_\alpha(a,x) + \int_0^\infty \xi^2 |\mathcal{H}_\alpha(f)(\xi)|^2 \, d\mu_\alpha(\xi) \geq 2(\alpha + 1)\|f\|_{2,\mu_\alpha}^2.
\]

This proves \( 4.3. \) Now by replacing \( f \) by \( D_\lambda^\alpha f \) in the previous inequality, we obtain by \( 2.16 \) and \( 2.25, \)

\[
\int_0^\infty \int_0^\infty x^2 |W_\psi^\alpha f\left(\frac{a}{\lambda}, \lambda x\right)|^2 \, d\mu_\alpha(a,x) + \lambda^{-2\alpha-2} \int_0^\infty \xi^2 |\mathcal{H}_\alpha(f)(\frac{\xi}{\lambda})|^2 \, d\mu_\alpha(\xi) \geq 2(\alpha + 1)\|f\|_{2,\mu_\alpha}^2.
\]

Thus, by a suitable change of variables, we get

\[
\lambda^{-2} \int_0^\infty \int_0^\infty x^2 |W_\psi^\alpha f(a,x)|^2 \, d\mu_\alpha(a,x) + \lambda^2 \int_0^\infty \xi^2 |\mathcal{H}_\alpha(f)(\xi)|^2 \, d\mu_\alpha(\xi) \geq (2\alpha + 2)\|f\|_{2,\mu_\alpha}^2.
\]
Minimizing the left-hand side of that inequality over $\lambda > 0$, we obtain
\[
\left( \int_0^\infty \int_0^\infty x^2 |W_\psi^\alpha f(a, x)|^2 \, da \, dx \right)^{1/2} \leq \sqrt{\frac{c_\psi}{M(|H_\alpha(\psi)|^2)(2s)}} \| \alpha^s W_\psi^\alpha f \|_{2, \mu_\alpha},
\]
where $M$ is the Mellin transform defined by
\[
M(f)(z) = \int_0^\infty x^{-z} f(x) \, dx.
\]
Then by (4.2), (4.5), we have the following Heisenberg-type uncertainty inequality, comparing the concentration of $f$ in position (with respect to the $x$-variable) and the concentration of $W_\psi^\alpha f$ in scale (with respect to the $a$-variable),
\[
\| f \|_{2, \mu_\alpha} \| a \, W_\psi^\alpha f \|_{2, \mu_\alpha} \geq \sqrt{c_\psi^{-1} M(|H_\alpha(\psi)|^2)(2s)} \| f \|_{2, \mu_\alpha}^2,
\]
and by (4.2), (4.5), we obtain the following Heisenberg-type uncertainty inequality which seems the most natural one, comparing the concentration of $W_\psi^\alpha f$ in position (with respect to the $x$-variable) and the concentration of $W_\psi^\alpha f$ in scale (with respect to the $a$-variable),
\[
\| a \, W_\psi^\alpha f \|_{2, \mu_\alpha} \| x \, W_\psi^\alpha f \|_{2, \mu_\alpha} \geq \sqrt{c_\psi^{-1} M(|H_\alpha(\psi)|^2)(2s)} \| f \|_{2, \mu_\alpha}^2.
\]

More generally, by following the same way of Theorem 4.1 and by using (4.5), and the following general form of Heisenberg-type uncertainty principle for the Hankel transform,
\[
\| x^s f \|_{2, \mu_\alpha} \| x^s H_\alpha(f) \|_{2, \mu_\alpha}^s \geq c(s, \alpha, \beta) \| f \|_{2, \mu_\alpha}^{s+\beta},
\]
the author in [18, Theorem 4.2] has deduced that, for any $s, \beta > 0$ there exists a constant $c(s, \alpha, \beta)$ such that for all $f \in L_2^2(\mathbb{R}^+)$,
\[
\| a^s W_\psi^\alpha f \|_{2, \mu_\alpha} \| x^s W_\psi^\alpha f \|_{2, \mu_\alpha}^s \geq c(s, \alpha, \beta) \left( c_\psi^{-1} M(|H_\alpha(\psi)|^2)(2s) \right)^{s/2} \| f \|_{2, \mu_\alpha}^{s+\beta}.
\]

In this paper we are interested in Heisenberg-type uncertainty inequalities like (4.8) and (4.11) (but without using the Mellin transform), limiting the concentration of $W_\psi^\alpha f$ in position and scale. A related results for the windowed Fourier and windowed Hankel transforms were proved in [3, 7], limiting the concentration in position and frequency.

### 4.2. Pitt-type inequality for the HWT

The aim of this subsection is to prove an analogue of Pitt’s inequality for the Hankel wavelet transform. Recall that, the Pitt-type inequality for the Hankel transform (see [18, Theorem 3.9]) states: For every $f$ in the Schwartz class $\mathcal{S}(\mathbb{R})$ and $0 < \beta < \alpha + 1$,
\[
\| \xi^{-\beta} H_\alpha(f) \|_{2, \mu_\alpha} \leq C_{\alpha, \beta} \| x^\beta f \|_{2, \mu_\alpha},
\]
where
\[
C_{\alpha, \beta} = 2^{-\beta} \frac{\Gamma \left( \frac{2-\beta+1}{2} \right)}{\Gamma \left( \frac{2+\beta+1}{2} \right)}.
\]

As noted in [13], the constant $C_{\alpha, \beta}$ is the best (smallest) constant for Pitt’s inequality (4.11). Then we derive the following Pitt-type inequality for the Hankel wavelet transform.

**Theorem 4.3.** For all function $f \in \mathcal{S}(\mathbb{R})$ such that $W_\psi^\alpha f(a, \cdot) \in \mathcal{S}(\mathbb{R})$, we have
\[
\| a^{-\beta} W_\psi^\alpha f \|_{2, \mu_\alpha} \leq C_{\alpha, \beta}(\psi) \| x^\beta W_\psi^\alpha f \|_{2, \mu_\alpha},
\]
where $0 < \beta < \alpha + 1$. 

\[
\| a^{-\beta} W_\psi^\alpha f \|_{2, \mu_\alpha} \| x^\beta W_\psi^\alpha f \|_{2, \mu_\alpha} \geq \sqrt{c_\psi^{-1} M(|H_\alpha(\psi)|^2)(2s)} \| f \|_{2, \mu_\alpha}^2,
\]
and by (4.2), (4.5), we have
\[
\| a \, W_\psi^\alpha f \|_{2, \mu_\alpha} \| x \, W_\psi^\alpha f \|_{2, \mu_\alpha} \geq \sqrt{c_\psi^{-1} M(|H_\alpha(\psi)|^2)(2s)} \| f \|_{2, \mu_\alpha}^2.
\]
where
\[
C_{\alpha,\beta}(\psi) = \sqrt{c_{\psi}^{-1} \mathcal{M} (|H_\alpha(\psi)|^2) (-2\beta)} 2^{-\beta} \frac{\Gamma \left( \frac{\alpha-\beta+1}{2} \right)}{\Gamma \left( \frac{\alpha+\beta+1}{2} \right)},
\] (4.14)

Proof. By applying the Pitt-type inequality (4.11) for the function \( W^\alpha_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R}) \), we obtain
\[
\int_{0}^{\infty} \xi^{-2\beta} |H_\alpha(W^\alpha_\psi f(a, \cdot))(\xi)|^2 \, d\mu_\alpha(\xi) \leq C_{\alpha,\beta}^2 \int_{0}^{\infty} x^{2\beta} |W^\alpha_\psi f(a, x)|^2 \, d\mu_\alpha(x).
\]

Then, by (2.26),
\[
\frac{1}{c_{\psi}} \int_{0}^{\infty} \xi^{-2\beta} |H_\alpha(f)(\xi)|^2 |H_\alpha(\psi)(\xi/a)|^2 \, d\mu_\alpha(\xi) \leq C_{\alpha,\beta}^2 \int_{\mathbb{R}_+^2} x^{2\beta} |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x).
\]

Integrating the last inequality with respect to the measure \( a^{2\alpha+1} \, da \), we obtain
\[
\frac{1}{c_{\psi}} \int_{0}^{\infty} \xi^{-2\beta} |H_\alpha(f)(\xi)|^2 \left( \int_{0}^{\infty} |H_\alpha(\psi)(\xi/a)|^2 \, \frac{da}{a} \right) \, d\mu_\alpha(\xi) \leq C_{\alpha,\beta}^2 \int_{\mathbb{R}_+^2} x^{2\beta} |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x).
\]

Therefore, since \( \psi \) is an admissible wavelet, then by (2.19), we conclude that
\[
\int_{0}^{\infty} \xi^{-2\beta} |H_\alpha(f)(\xi)|^2 \, d\mu_\alpha(\xi) \leq C_{\alpha,\beta}^2 \int_{\mathbb{R}_+^2} x^{2\beta} |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x).
\]

Thus by (4.5), we obtain the desired result. \( \square \)

Remark 4.4. Notice that, if \( \beta = 0 \) then \( C_{\alpha,0} = 1 \) and \( \mathcal{M}(|H_\alpha(\psi)|^2)(0) = c_{\psi} \), then we obtain an equality in Pitt-type inequality (4.13) for any function \( f \in \mathcal{S}_c(\mathbb{R}) \) such that \( W^\alpha_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R}) \).

4.3. Benkner-type uncertainty principle for the HWT. Now will use the Pitt-type inequality (4.13) to obtain the following logarithmic uncertainty principle (also known as Beckner-type uncertainty principle) for the HWT.

Theorem 4.5. For all function \( f \in \mathcal{S}_c(\mathbb{R}) \) such that \( W^\alpha_\psi f(a, \cdot) \in \mathcal{S}_c(\mathbb{R}) \), we have
\[
\int_{\mathbb{R}_+^2} \ln(x) |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) + \int_{\mathbb{R}_+^2} \ln(x) |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) \geq C_\alpha(\psi) \| f \|_{2,\mu_\alpha}^2, \quad (4.15)
\]
where
\[
C_\alpha(\psi) = \ln(2) + \frac{\Gamma \left( \frac{\alpha+1}{2} \right)}{\Gamma \left( \frac{\alpha+3}{2} \right)} - c_{\psi}^{-1} \left( \int_{0}^{\infty} \ln(\alpha) a |H_\alpha(\psi)(a)|^2 \, da \right).
\] (4.16)

Proof. Since \( W^\alpha_\psi f(a, \cdot) \) is in the Schwartz class \( \mathcal{S}_c(\mathbb{R}) \), then from (1.13),
\[
\int_{0}^{\infty} \ln(x) |W^\alpha_\psi f(a, x)|^2 \, d\mu_\alpha(x) + \int_{0}^{\infty} \ln(|\alpha|) |H_\alpha(W^\alpha_\psi f(a, \cdot))(\xi)|^2 \, d\mu_\alpha(\xi) \geq C_\alpha \| W^\alpha_\psi f(a, \cdot) \|^2_{2,\mu_\alpha},
\]
and by (2.26), we have
\[
\int_{0}^{\infty} \ln(x) |W^\alpha_\psi f(a, x)|^2 \, d\mu_\alpha(x) + \int_{0}^{\infty} \ln(|\alpha|) |H_\alpha(f)(\xi)|^2 \frac{|H_\alpha(\psi)(\xi/a)|^2}{c_{\psi} a^{2\alpha+2}} \, d\mu_\alpha(\xi) \geq C_\alpha \| W^\alpha_\psi f(a, \cdot) \|^2_{2,\mu_\alpha}.
\]

Then by integrating the previous inequality with respect to the measure \( a^{2\alpha+1} \, da \), and by (2.19), (2.21), we obtain
\[
\int_{\mathbb{R}_+^2} \ln(x) |W^\alpha_\psi f(a, x)|^2 \, d\nu_\alpha(a, x) + \int_{0}^{\infty} \ln(|\alpha|) |H_\alpha(f)(\xi)|^2 \, d\mu_\alpha(\xi) \geq C_\alpha \| f \|^2_{2,\mu_\alpha}. \quad (4.17)
\]
The logarithmic uncertainty principle for the HWT (4.15) can be written as:
\[ \int f(x)^2 \nu_\alpha(x) dx = \int_0^\infty \ln(1 + a^2) \left( \int_0^\infty |W^\alpha_x f(a,x)|^2 d\mu_\alpha(x) \right) a^{2n+1} da 
= c_\psi^{-1} \int_0^\infty |H_\alpha(f)(\xi)|^2 \left( \int_0^\infty \ln(1 + a^2) |H_\alpha(\psi)(\xi/a)|^2 d\mu_\alpha(\xi) \right) a^{2n+1} da 
= c_\psi^{-1} \int_0^\infty |H_\alpha(f)(\xi)|^2 \left( \int_0^\infty (\ln(1 + a^2) - (\ln(a)|H_\alpha(\psi)(\xi/a)|^2 d\mu_\alpha(\xi) \right) 
= \int_0^\infty \ln(1 + a^2) |H_\alpha(\psi)(a)|^2 d\mu_\alpha(a) - C_\psi \|f\|_{2,\mu_\alpha}^2, \]
where
\[ C_\psi = c_\psi^{-1} \left( \int_0^\infty \frac{\ln(1 + a^2)}{a^2} |H_\alpha(\psi)(a)|^2 da \right). \] (4.18)

This completes the proof. \[ \square \]

**Remark 4.6** (Another proof of Theorem 4.5). If we define the function \( \Phi \) on \([0, \alpha + 1]\) by:
\[ \Phi(\beta) = \int_{R^2_+} a^{-2\beta} |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x) - C_{\alpha,\beta}(\psi)^2 \int_{R^2_+} x^{2\beta} |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x), \]
then
\[ \Phi'(\beta) = -2 \int_{R^2_+} a^{-2\beta} \ln(1 + a^2) |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x) - 2C_{\alpha,\beta}(\psi)^2 \int_{R^2_+} x^{2\beta} \ln(x) |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x) 
- \frac{d}{d\beta} \left( C_{\alpha,\beta}(\psi)^2 \right) \int_{R^2_+} x^{2\beta} |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x). \] (4.19)

Moreover, the Pitt-type inequality implies that \( \Phi(\beta) \leq 0 \) for every \( \beta \in (0, \alpha + 1) \), and by Remark 4.4 we have \( \Phi(0) = 0 \). Therefore \( \Phi'(0^+) \leq 0 \). Thus by (2.24),
\[ \int_{R^2_+} \ln(1 + a^2) |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x) + \int_{R^2_+} \ln(x) |W^\alpha_x f(a,x)|^2 d\nu_\alpha(a,x) \geq \frac{C}{2} \|f\|_{2,\mu_\alpha}^2, \]
where
\[ C = \left. \frac{d}{d\beta} \left( C_{\alpha,\beta}(\psi)^2 \right) \right|_{\beta=0} = -2 \left( \ln 2 + \frac{\Gamma'(\alpha/2)}{\Gamma(\alpha/2)} \right) - c_\psi^{-1} \left( \int_0^\infty \frac{\ln(1 + a^2)}{a^2} |H_\alpha(\psi)(a)|^2 da \right). \]

From the logarithmic uncertainty principle for the HWT (4.15), we can derive a Heisenberg-type uncertainty inequality for functions in \( \mathcal{S}_c(\mathbb{R}) \).

**Corollary 4.7.** For all \( f \in \mathcal{S}_c(\mathbb{R}) \) such that \( W^\alpha_x f(a, \cdot) \in \mathcal{S}_c(\mathbb{R}) \), we have
\[ \|a W^\alpha_x f\|_{2,\mu_\alpha} \geq c_{\alpha,\psi} \|f\|_{2,\mu_\alpha}^2. \] (4.20)

**Proof.** The logarithmic uncertainty principle for the HWT (4.15) can be written as follow:
\[ \frac{1}{2} \int_{R^2_+} \ln(a^2) \left( \frac{|W^\alpha_x f(a,x)|^2}{\|f\|_{2,\mu_\alpha}^2} d\nu_\alpha(a,x) \right) + \frac{1}{2} \int_{R^2_+} \ln(a^2) \left( \frac{|W^\alpha_x f(a,x)|^2}{\|f\|_{2,\mu_\alpha}^2} d\nu_\alpha(a,x) \right) \geq C_{\alpha,\psi}, \] (4.21)
where by (2.24), \( \left( \frac{|W^\alpha_x f(a,x)|^2}{\|f\|_{2,\mu_\alpha}^2} d\nu_\alpha(a,x) \right) \) is a probability measure on \((0, \infty)^2\).
Now since the logarithm is a concave function, then by Jensen’s inequality, we have for all nonzero function \( f \in S_\alpha (\mathbb{R}) \),

\[
\ln \left( \int_{\mathbb{R}^2_+} a^2 \frac{|W_\alpha f (a, x)|^2}{\|f\|_{2, \mu_\alpha}^2} \, d\nu_\alpha (a, x) \right)^{1/2} + \ln \left( \int_{\mathbb{R}^2_+} x^2 \frac{|W_\alpha f (a, x)|^2}{\|f\|_{2, \mu_\alpha}^2} \, d\nu_\alpha (a, x) \right)^{1/2} \geq C_\alpha (\psi).
\]

Then

\[
\ln \left( \frac{\|a W_\alpha f\|_{2, \nu_\alpha} \|x W_\alpha f\|_{2, \nu_\alpha}}{\|f\|_{2, \mu_\alpha}^2} \right) \geq C_\alpha (\psi).
\]

Therefore

\[
\|a W_\alpha f\|_{2, \nu_\alpha} \|x W_\alpha f\|_{2, \nu_\alpha} \geq e^{C_\alpha (\psi)} \|f\|_{2, \mu_\alpha}^2.
\]

Thus the result follows. \( \square \)

**Remark 4.8.** By adapting the proof of the last corollary to the logarithmic uncertainty principle \( \text{(1.13)} \), we will derive a new Heisenberg-type uncertainty inequality for the Hankel transform, that is, for all nonzero function \( f \) in the Schwartz space \( S_\alpha (\mathbb{R}) \), we have

\[
\|xf\|_{2, \mu_\alpha} \|\mathcal{H}_\alpha (f)\|_{2, \mu_\alpha} \geq 2 \exp \left( \frac{\Gamma' (\frac{\alpha + 1}{2})}{\Gamma (\frac{\alpha + 1}{2})} \right) \|f\|_{2, \mu_\alpha}^2.
\]

Now since

\[
\frac{\Gamma' (z)}{\Gamma (z)} = \ln z - \frac{1}{2z} - 2 \int_0^\infty \frac{t}{(t^2 + z^2)(e^{2\pi t} - 1)} \, dt,
\]

then

\[
2 \exp \left( \frac{\Gamma' (\frac{\alpha + 1}{2})}{\Gamma (\frac{\alpha + 1}{2})} \right) \approx (\alpha + 1), \quad \text{for} \quad \alpha \gg 1,
\]

which is the optimal constant in Heisenberg’s Inequality \( \text{(4.24)} \).

### 4.4. Hirschman-Beckner entropic uncertainty inequality for the HWT.

Following a conjecture by Hirschman \( \text{(13)} \), Beckner \( \text{(2)} \) proved that for all \( f \in L^2 (\mathbb{R}^d) \), such that \( \|f\|_2 = 1 \),

\[
- \int_{\mathbb{R}^d} |f(x)|^2 \ln (|f(x)|^2) \, dx - \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \ln (|\hat{f}(\xi)|^2) \, d\xi \geq d \ln (e/2).
\]

Now we define the Shannon’s differential entropy by (see \( \text{(22)} \)),

\[
E(\rho) = - \int_{\mathbb{R}^d_+} \rho(a, x) \ln \left( \rho(a, x) \right) \, d\nu_\alpha (a, x),
\]

where \( \rho \) is probability density function on \( \mathbb{R}^d_+ \) satisfying \( \|\rho\|_{1, \nu_\alpha} = 1 \). The aim of this section is to prove a Hirschman-Beckner entropic uncertainty inequality, and a Heisenberg-type uncertainty relations for the HWT. As a first result, we state the following preliminary lemma.

**Lemma 4.9.** For every \( x \in [0, 1] \) and every \( p \in (2, 3] \), we have

\[
0 \leq \frac{x^2 - x^p}{p - 2} \leq -x^2 \ln x.
\]

**Proof.** For \( x \in (0, 1) \), we define the function

\[
p \mapsto S_x (p) = \frac{x^p - x^2}{p - 2}, \quad p \in (2, 3].
\]

Then it derivative satisfies

\[
S'_x (p) = \frac{(p - 2)x^p \ln x + x^2 - x^p}{(p - 2)^2}, \quad p \in (2, 3].
\]
For every $x \in (0, 1)$, the function $p \mapsto D_x(p) = (p - 2)x^p \ln x + x^2 - x^p$ is differentiable on $(2, 3]$, and
$$D'_x(p) = (p - 2)x^p \ln^2 x \geq 0.$$ 
Then for every $x \in (0, 1)$, the function $D_x$ is increasing on $(2, 3]$ and $D_x(2^+) = 0$. Thus $S_x$ is also an increasing function on $(2, 3]$. In particular, for every $x \in (0, 1)$,
$$x^2 \ln x = \lim_{p \to 2^+} S_x(p) \leq S_x(p), \quad p \in (2, 3]. \tag{4.31}$$ 
The last inequality remains true for $x \to 0^+$, and then
$$0 \leq \frac{x^2 - x^p}{p - 2} \leq -x^2 \ln x, \quad x \in (0, 1), \quad p \in (2, 3]. \tag{4.32}$$ 
This completes the proof of the lemma. \qed

Now we will prove the main result of this section.

**Theorem 4.10.** Let $\psi$ an admissible wavelet such that $\|\psi\|_{2,\mu_\alpha}^2 \leq c_\psi$. Then for all nonzero function $f \in L^2_\alpha(\mathbb{R}_+)$,
$$- \int_{\mathbb{R}^2_+} |W_\psi^\alpha f(a, x)|^2 \ln (|W_\psi^\alpha f(a, x)|^2) \, dv_\alpha(a, x) \geq \|f\|_{2,\mu_\alpha}^2 \ln \left( \frac{c_\psi}{\|\psi\|_{2,\mu_\alpha}^2 \|f\|_{2,\mu_\alpha}^2} \right). \tag{4.33}$$ 
Proof. Let $f$ be a nonzero function in $L^2_\alpha(\mathbb{R}_+)$, such that $\|f\|_{2,\mu_\alpha} = 1$, and suppose that
$$E(|W_\psi^\alpha f|^2) = - \int_{\mathbb{R}^2_+} |W_\psi^\alpha f(a, x)|^2 \ln (|W_\psi^\alpha f(a, x)|^2) \, dv_\alpha(a, x) < \infty.$$ 
Then by (4.28), we have for all $(a, x) \in \mathbb{R}^2_+$,
$$|W_\psi^\alpha f(a, x)| \leq 1.$$ 
Therefore
$$E(|W_\psi^\alpha f|^2) \geq 0,$$ 
and from Lemma 4.10, we have for every $p \in (2, 3)$,
$$0 \leq \frac{|W_\psi^\alpha f(a, x)|^2 - |W_\psi^\alpha f(a, x)|^p}{p - 2} \leq -|W_\psi^\alpha f(a, x)|^2 \ln (|W_\psi^\alpha f(a, x)|). \tag{4.34}$$ 
Now, from Theorem 4.33, we know that $W_\psi^\alpha f \in L^p_\alpha(\mathbb{R}_+^2)$, $p \geq 2$, with
$$\|W_\psi^\alpha f\|_{p,\mu_\alpha}^p \leq \left( \frac{\|\psi\|_{2,\mu_\alpha}^2}{c_\psi} \right)^{p/2 - 1}.$$ 
Then the function $p \mapsto \varphi(p)$ defined on $[2, \infty)$ by
$$\varphi(p) = \|W_\psi^\alpha f\|_{p,\mu_\alpha}^p - \left( \frac{\|\psi\|_{2,\mu_\alpha}^2}{c_\psi} \right)^{p/2 - 1}$$ 
is negative and by Plancherel formula (4.24), it satisfies $\varphi'(2) = 0$. Thus $\varphi'(2) \leq 0$.

On the other hand by (4.34), we have for every $p \in (2, 3)$, and all $(a, x) \in \mathbb{R}^2_+$,
$$\int_{\mathbb{R}^2_+} \left| \frac{|W_\psi^\alpha f(a, x)|^2 - |W_\psi^\alpha f(a, x)|^p}{p - 2} \right| \, dv_\alpha(a, x) \leq - \int_{\mathbb{R}^2_+} |W_\psi^\alpha f(a, x)|^2 \ln (|W_\psi^\alpha f(a, x)|) \, dv_\alpha(a, x),$$ 
$$= - \frac{1}{2} E(|W_\psi^\alpha f|^2) < \infty.$$
Moreover by Plancherel formula \( L^2(\mathbb{R}^2) \), we have for every \( p > 3 \), and all \((a, b) \in \mathbb{R}^2\),
\[
\int_{\mathbb{R}^2} \left| \frac{W^\alpha_{\psi} f(a, x)}{p - 2} - \frac{W^\alpha_{\psi} f(a, x)}{p - 2} \right|^p \, dv_\alpha(a, x)
\]
\[
\leq 2 \int_{\mathbb{R}^2} |W^\alpha_{\psi} f(a, x)|^2 \, dv_\alpha(a, x) = 2 < \infty.
\]
Consequently, by the Lebesgue dominated convergence theorem,
\[
\left( \frac{d}{dp} \|W^\alpha_{\psi} f\|^p_{p, \mu_\alpha} \right)_{p=2^+} = \int_{\mathbb{R}^2} \lim_{p \to 2^+} \left| W^\alpha_{\psi} f(a, x) \right|^p - \|W^\alpha_{\psi} f(a, x)\|^2 \, dv_\alpha(a, x)
\]
\[
= \int_{\mathbb{R}^2} \|W^\alpha_{\psi} f(a, x)\|^2 \ln (\|W^\alpha_{\psi} f(a, x)\|) \, dv_\alpha(a, x)
\]
\[
= -\frac{1}{2} E (\|W^\alpha_{\psi} f\|^2).
\]

Therefore
\[
\varphi'(2^+) = \frac{d}{dp} \varphi(p)_{p=2^+}
\]
\[
= -\frac{1}{2} E (\|W^\alpha_{\psi} f\|^2) + \frac{1}{2} \ln \left( \frac{c_{\psi}}{\|\psi\|^2_{2, \mu_\alpha}} \right) \leq 0.
\]

Thus
\[
E (\|W^\alpha_{\psi} f\|^2) \geq \ln \left( \frac{c_{\psi}}{\|\psi\|^2_{2, \mu_\alpha}} \right). \tag{4.35}
\]

Finally, if \( f \in L^2_\alpha(\mathbb{R}^+) \) is any nonzero function, then if we replace \( f \) by \( \frac{f}{\|f\|_{2, \mu_\alpha}} \) in (4.35), we obtain the desired result. \( \square \)

Consequently we derive the following Heisenberg-type uncertainty inequality for the HWT.

**Corollary 4.11.** Let \( s, \beta > 0 \), and let \( \psi \) be an admissible wavelet such that \( \|\psi\|^2_{2, \mu_\alpha} \leq c_\psi \). Then there exists a positive constant \( C_{s, \alpha, \beta} \) such that, for all nonzero function \( f \in L^2_\alpha(\mathbb{R}^+) \),
\[
\|a^s W^\alpha_{\psi} f\|^2_{2, \mu_\alpha} + \|x^\beta W^\alpha_{\psi} f\|^2_{2, \mu_\alpha} \geq C_{s, \alpha, \beta} \|f\|^2_{2, \mu_\alpha}, \tag{4.36}
\]
where
\[
C_{s, \alpha, \beta} = \frac{(s + 1)(s + \beta)}{s\beta} \exp \left[ -1 + \frac{s\beta}{(s + 1)(s + \beta)} \ln \left( \frac{2^{s+2} \Gamma(s+1)}{\Gamma(\frac{s+1}{s}) \Gamma(\frac{s+1}{\beta})} \right) \right]. \tag{4.37}
\]

**Proof.** Let \( f \in L^2_\alpha(\mathbb{R}^+) \), such that \( \|f\|_{2, \mu_\alpha} = 1 \), and let \( t, s, \beta > 0 \). Then a straightforward computation gives,
\[
\int_0^\infty \int_0^\infty e^{-2^{s+2} \frac{t^{s+1} + a^s}{t}} \, dv_\alpha(a, x) = c(s, \alpha, \beta) t^{\frac{s+1}{s} + \frac{\alpha}{\beta}}.
\]
where
\[
c(s, \alpha, \beta) = \frac{\Gamma(s+1) \Gamma(\frac{s+1}{\beta})}{2^{s+2} s^s \Gamma(\alpha+1)}.
\]
Now let \( Q_{t, s, \alpha, \beta} \) the function defined on \( \mathbb{R}^2_+ \) by
\[
Q_{t, s, \alpha, \beta}(a, x) = \frac{e^{-2^{s+2} \frac{t^{s+1} + a^s}{t}}}{c(s, \alpha, \beta) t^{\frac{s+1}{s} + \frac{\alpha}{\beta}}}.
\]
Then \( Q_{t,s,\alpha,\beta}(a,x) \, d\nu_\alpha(a,x) \) is a probability measure on \( \mathbb{R}_+^2 \), and since the function \( t \mapsto t \ln t \) is convex, we obtain by Jensen’s inequality,

\[
\int_{\mathbb{R}_+^2} |W_\psi^\alpha f(a,x)|^2 \ln \left( \frac{|W_\psi^\alpha f(a,x)|^2}{Q_{t,s,\alpha,\beta}(a,x)} \right) \, d\nu_\alpha(a,x) \geq 0.
\]

Therefore

\[
\frac{1}{t} \left( \|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \right) \geq E \left( |W_\psi^\alpha f|^2 \right) - \ln(t^{\frac{s}{s+\beta}} + t^{\frac{\beta}{s+\beta}}) - \ln(c(s,\alpha,\beta)).
\]

It follows that by 11.37,

\[
\|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \geq t \left[ \ln \left( \frac{c_\nu}{\|\psi\|_{2,\nu_\alpha}^2} \right) - \ln(t^{\frac{s}{s+\beta}} + t^{\frac{\beta}{s+\beta}}) - \ln(c(s,\alpha,\beta)) \right].
\]

Minimizing the right hand side with \( t_0 = e^{\ln(c_\nu/\|\psi\|_{2,\nu_\alpha}^2) - 1} \), we obtain

\[
\|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \geq C_{s,\alpha,\beta}
\]

where \( C_{s,\alpha,\beta} = \frac{(s+\beta)(s+\beta-1)}{s^2\beta^2} t_0 \), and \( c = \ln \left( \frac{c_\nu}{\|\psi\|_{2,\nu_\alpha}^2} \right) \). Finally replacing \( f \) by \( f_{\|f\|_{2,\nu_\alpha}} \), we conclude the desired result.

Moreover we have the following Heisenberg-type uncertainty like 11.10, without involving the Mellin transform.

**Corollary 4.12.** Let \( s, \beta > 0 \), and let \( \psi \) an be an admissible wavelet such that \( \|\psi\|_{2,\nu_\alpha}^2 \leq c_\psi \). Then there exists a positive constant \( C(s,\alpha,\beta) \) such that, for all nonzero function \( f \in L_1^s(\mathbb{R}_+) \),

\[
\|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \geq C(s,\alpha,\beta) \|f\|_{2,\nu_\alpha}^{s+\beta},
\]

where

\[
C(s,\alpha,\beta) = \left( \frac{s}{\beta} \right)^{\frac{s}{s+\beta}} \left( \frac{\beta}{s+\beta} C_{s,\alpha,\beta} \right)^{\frac{s+\beta}{s}}.
\]

**Proof.** Replacing \( f \) by \( D_\lambda^s f \) in Inequality (4.38), we obtain by (2.25),

\[
\int_{\mathbb{R}_+^2} a^{2s} |W_\psi^\alpha f \left( \frac{a}{\lambda}, \lambda x \right)|^2 \, d\nu_\alpha(a,x) + \int_{\mathbb{R}_+^2} x^{2\beta} |W_\psi^\alpha f \left( \frac{a}{\lambda}, \lambda x \right)|^2 \, d\nu_\alpha(a,x) \geq C_{s,\alpha,\beta} \|f\|_{2,\nu_\alpha}^2.
\]

Then a suitable change of variables gives

\[
\lambda^{2s} \|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2 + \lambda^{-2\beta} \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2 \geq C_{s,\alpha,\beta} \|f\|_{2,\nu_\alpha}^2.
\]

Minimizing the left hand side with \( \lambda = \left( \frac{\beta \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^2}{s \|a^s W_\psi^\alpha f\|_{2,\nu_\alpha}^2} \right)^{\frac{1}{s+\beta}} \), we obtain

\[
\|a^{-s} W_\psi^\alpha f\|_{2,\nu_\alpha}^{2s} \|x^\beta W_\psi^\alpha f\|_{2,\nu_\alpha}^{2\beta} \geq C_{s,\alpha,\beta} \|f\|_{2,\nu_\alpha}^{2s+\beta},
\]

which allows to conclude.

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