All “static” spherically symmetric perfect fluid solutions of Einstein’s equations with equation of state \( p = w \rho \) and finite-polynomial “mass function”

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Abstract

We look for “static” spherically symmetric solutions of Einstein’s Equations for perfect fluid source with equation of state \( p = w \rho \). In order to include the possibilities of recently popularized dark energy and phantom energy possibly pervading the spacetime, we put no constraints on the constant \( w \). We consider all four cases compatible with the standard ansatz for the line element, discussed in previous work. For each case we derive the equation obeyed by the mass function or its analogs. For these equations, we find all finite-polynomial solutions, including possible negative powers.

For the standard case, we find no significantly new solutions, but show that one solution is a static phantom solution, another a black hole-like solution. For the dynamic and/or tachyonic cases we find, among others, dynamic and static tachyonic solutions, a Kantowski-Sachs (KS) class phantom solution, another KS-class solution for dark energy, and a second black hole-like solution.

The black hole-like solutions feature segregated normal and tachyonic matter, consistent with the assertion of previous work. In the first black hole-like solution, tachyonic matter is inside the horizon, in the second, outside.

The static phantom solution, a limit of an old one, is surprising at first, since phantom energy is usually associated with super-exponential expansion. The KS-phantom solution stands out since its “mass function” is a ninth order polynomial.

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1 Introduction and Motivation

Exact solutions of Einstein’s Field Equations

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \]  

are, of course, of interest for various purposes. Since the equations are very complicated, to find solutions one often makes simplifying assumptions about the left-hand-side and/or the right-hand-side. Popular simplifying assumptions about the left-hand-side include staticity and spherical symmetry. As is well known, the use of both assumptions together leads to the ansatz \[1\text{, Sect.23.2}\]

\[ ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\Omega^2 \]  

for the metric.

Most-often used simplifying assumptions about the right-hand-side of \[1\] are that \(T_{\mu\nu}\) represents vacuum (i.e. vanishes) or an electromagnetic field or a perfect fluid. For example, the vacuum assumption, together with the ansatz \[2\] gives uniquely the Schwarzschild metric, the simplest and best-known black hole solution.

The perfect fluid form of \(T_{\mu\nu}\), the stress-energy-momentum tensor, is

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \]  

where \(\rho\) and \(p\) are the energy density and pressure, respectively, as measured by an observer moving with the fluid, and \(u_\mu\) is its four-velocity. The use of this \(T_{\mu\nu}\) together with ansatz \[2\] describes the interiors of static spherically symmetric stars, for example. But the description \[3\] is not complete: \(\rho\) and \(p\) should also be specified as functions of particle number density, temperature, etc. One further simplifying assumption, justified under most circumstances, is that there is a relation, called an equation of state \(f(p, \rho) = 0\) between \(p\) and \(\rho\). In cosmology, one usually assumes that the equation of state is a proportionality,

\[ p = w\rho, \]  

with e.g. \(w = 0\) describing the matter-dominated (or "pressureless dust") case, \(w = 1/3\) the radiation-dominated case, \(w < -1/3\) dark energy, and \(w < -1\) phantom energy. The latter two concepts have been introduced into cosmology in the last decade \[2\text{, }3\], after the discovery of the acceleration of the expansion of the universe \[4\text{, }5\].

Now that a good case exists that the universe might be dominated by dark energy, even phantom energy, one should look for exact solutions with these sources. In particular, static spherically symmetric solutions would
be the easiest to find and might be relevant in the contexts of black holes or static stars. These solutions can be found starting from the ansatz (2), which for “static” perfect fluid source, (i.e. \( u^\mu = u^0 \delta^\mu_0 \)) leads to the well-known Oppenheimer-Volkoff (OV) equation [6]

\[
p' = -\frac{(\kappa pr^3 + F)}{2r(r-F)}(\rho + p) \tag{5}
\]

where

\[
F(r) = \kappa \int \rho r^2 dr, \tag{6}
\]

and prime denotes \( r \)-derivative. \( F(r) \) can be recognized as \( \kappa/4\pi \) times the ”mass function” defined in the literature. Into the OV equation (5) one must put \( p \) in terms of \( \rho \) via the equation of state, then \( \rho \) in terms of \( F' \), via (6), eventually getting a differential equation for \( F \). After solving for \( F \), the metric functions can be found via

\[
A(r) = \frac{r}{r - F(r)} \tag{7}
\]

\[
\frac{B'(r)}{B(r)} = \frac{\kappa pr^2 + 1}{r - F(r)} - \frac{1}{r}. \tag{8}
\]

The solutions can be interpreted as static only for positive \( A(r) \) and \( B(r) \), however. In general, the ansatz (2) admits four classes of solutions, called NS (the standard case), TD, ND (corresponding to Kantowski-Sachs [8, Sect.15.6.5], [7] case) and TS in [9]. The ND and TD solutions are not static, hence the quotes on “static” in the title and abstract. For each class, one gets a different OV-like equation.

The OV equation is valid in case NS. For equation of state (4), it becomes

\[
(w + 1)F'(wrF' + F) + 2w(rF'' - 2F')(r - F) = 0 \tag{9}
\]

where \( F(r) \) is written as \( F \) for brevity, and we put no constraint on \( w \) other than that it is a constant. This is a nonlinear equation whose general solution is difficult to find. One can attempt a series solution

\[
F(r) = \sum_{n=0}^{\infty} a_n r^n \tag{10}
\]

but the recursion expression one gets for \( a_n \) involves all of \( a_0 \ldots a_{n-1} \) and it seems not possible to even show that (10) converges, let alone find a closed expression for \( a_n \).

We can, however, find all of the finite-polynomial solutions of (9). This we do in the next section. In fact, we find all finite Laurent polynomials,
i.e. we consider also negative powers of $r$, but find none in case NS. Four of the found solutions are valid for particular values of $w$, and two for general $w$. While none of the solutions is totally original, the procedure shows that there are no other finite-polynomial solutions; and in Section 3 we discuss properties of the spacetimes.

In Section 4 we discuss similar solutions, derived in the appendix, for the TD, ND(KS) and TS cases. We also ask if we can find any solutions with finite-polynomial $A(r)$.

2 All finite-polynomial solutions for the mass function from the standard OV equation

In case NS, any power of $r$ less than 3 in $F(r)$ means a diverging density at the origin; in particular, a constant term corresponds to a point mass there, while negative powers mean diverging mass function, and therefore seem unnatural. On the other hand, the meaning of $F(r)$ is different in the TD, ND(KS) and TS cases, therefore negative powers are more acceptable.

The highest and lowest powers of $r$ in $F(r)$ we will call $m$ and $\tilde{m}$. The second-highest, third-highest, second-lowest and third-lowest powers of $r$ in $F(r)$ we will call $n$, $p$, $\tilde{n}$ and $\tilde{p}$ respectively, when they exist; and $A$, $B$, $C$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ will be the respective coefficients. We will substitute the polynomial into the left-hand-side of (9) and set coefficients of all powers of $r$ equal to zero.

For $m > 1$, the highest power of $r$ in eq. (9) is $2m - 1$, with coefficient

\[ (w + 1)m(wm + 1)A^2 - 2wm(m - 3)A^2 = 0 \]  

therefore in these cases $A$ is arbitrary and

\[ m = \frac{7w + 1}{w(1 - w)}. \]

1In the rest of this work, we will use “power” also when we really mean “order of the power”. It should be clear from the context which meaning is intended.
Table 1: Matrix of cases for solution of equations (9) and (44) by finite Laurent polynomials with highest power \( m \) and lowest power \( \tilde{m} \). In the first row and last column, an important equation valid for that row/column is indicated.

If this had been an integer, we would have found the order of the polynomial for arbitrary \( w \). Since it is not, we conclude that in these cases finite polynomial solutions exist for certain values of \( w \) only.

Of course, one can also solve for \( w \) in terms of \( m \):

\[
w = \frac{m - 7 \pm \sqrt{(m - 7)^2 - 4m}}{2m},
\]

or write

\[
w^2 = \frac{(m - 7)w - 1}{m}.
\]

Similarly, for \( \tilde{m} < 0 \) (that is, for cases 4,7,9 and 10), we can consider lowest power of \( r \) in eq.(8) and find

\[
\tilde{m} = \frac{7w + 1}{w(1 - w)}.
\]

Now, we can start the separate consideration of the cases in Table

Case 1. \( m > 1, \tilde{m} > 1 \)

For \( \tilde{m} > 1 \), the lowest power of \( r \) in eq.(9) is \( \tilde{m} \), the contribution coming from the right part. Its coefficient is

\[
2w[\tilde{m}(\tilde{m} - 1) - 2\tilde{m}]
\]

\( w = 0 \) is incompatible with eq.(11), therefore

\[
\tilde{m} = 3.
\]
Case 1.1. \( m > 3, \tilde{m} = 3 \).

In this case, \( n \) exists and the second-highest power of \( r \) in eq. (9) is \( m + n - 1 \), with coefficient

\[(w+1)[m(wn+1)+n(wm+1)]AB - 2w[m(m-3)+n(n-3)]AB = 0. \tag{18}\]

giving arbitrary \( B \) and, after elimination of \( w^2 \) by using (14), the equation

\[(4mn+7m-7n-2m^2-2n^2)w + m - n = 0 \tag{19}\]

which not only gives \( n \) in terms of \( m \) and \( w \), but also means that \( w \) is rational.

A careful inspection of (13) shows that there are only three values of \( m \) giving rational \( w \): 18, 15 and 3. For each \( m \), there are two \( w \) values, making a total of four subcases of subcase 1.1, since \( m \neq 3 \):

Case 1.1.1. \( m = 18, n > 1, w = \frac{1}{2} \).

In this case, solving (19) for \( n \) gives the values 18 and 27/2, both of which are unacceptable; the former because we should have \( m > n \), the latter because it is not an integer. Hence, this case fails.

One can also see this failure using a ‘brute force’ approach: If one puts a general 18th order polynomial (in effect, extending the argument down to \( \tilde{m} = 0 \)) into the left-hand-side of (9) and sets the coefficients of powers of \( r \) to zero, starting from the highest (35th) power, one gets \( a_{18} = A, \ a_{17} = a_{16} = ... = a_2 = 0, \ a_1 = \frac{108}{85}, \ a_0 = 0 \) by the time one arrives at the 17th power. But when this polynomial is put afresh into the left-hand-side of (9), one gets \( \frac{108}{25}r \) instead of zero, so lower powers don’t cancel entirely. This is not surprising, since there are 36 powers of \( r \) in (9), but 19 coefficients to be found.

Case 1.1.2. \( m = 18, n > 1, w = \frac{1}{5} \).

This time, for \( n \) we get 18 and 10, so we should take the latter. We could then continue, separating out the third highest power, but the ’brute force’ approach is more straightforward, especially since it can be executed with software. We find that this case also fails.

Similarly, we find that

Case 1.1.3. \( m = 15, n > 1, w = \frac{1}{3} \).
and

Case 1.1.4. $m = 15, n > 1, w = \frac{1}{5}$ fail too, finishing subcase 1.1.

Case 1.2. $m = \tilde{m} = 3$.

This subcase gives us two solutions,

\begin{align*}
\text{Solution 1 : } & w = -1, \quad F(r) = Ar^3 \\
\text{Solution 2 : } & w = -\frac{1}{3}, \quad F(r) = Ar^3,
\end{align*}

which finish case 1.

Case 2. $m > 1, \tilde{m} = 1$

The lowest power in (9) is now 1, the vanishing of whose coefficient gives

$$\tilde{A} = \frac{4w}{w^2 + 6w + 1}. \quad (22)$$

unless $w = -3 \pm 2\sqrt{2}$ (For these values, the coefficient cannot vanish at all). For more information, we consider the second-highest power in (9), $m + n - 1$. $n$ exists, but it may or may not be equal to $\tilde{m} = 1$. This necessitates consideration of two subcases:

Case 2.1. $m > 1, \tilde{m} = 1, n > 1$

In this subcase, eq. (18) is again valid, therefore the same chain of arguments can be followed ending with rational $w$ and allowed $m$ values of 18, 15 and 3. Subcases 2.1.1 - 2.1.4 ($m = 18, 15$) are covered by subcases 1.1.1 - 1.1.4 in the ‘brute force’ approach, giving no solutions. This leaves

Case 2.1.5. $m = 3, n > 1, w = -1$,

which fails because the solutions for $n$ are 3 and 0,

Case 2.1.6. $m = 3, n > 1, w = -\frac{1}{3}$,
which fails because solutions for \( n \) are 3 and 1; this finishes subcase 2.1.

Case 2.2. \( m > 1, \tilde{m} = 1 = n \)

Putting \( F(r) = Ar^m + \tilde{A}r \) into the OV eqn. (9), and using (12) and (22), we get

\[
(1 + 3w)(w^2 + 6w + 1) = 0. \tag{23}
\]

As pointed out after eq.(22), \((w^2 + 6w + 1)\) cannot vanish, so we get

**Solution 3:** \( w = -\frac{1}{3}, \ F(r) = Ar^3 + \frac{3}{2}r. \tag{24} \)

This finishes case 2. Solution 3 does not include Solution 2 as a special case.

Case 3. \( m > 1, \tilde{m} = 0 \)

This time, the lowest power in (9) is \( \tilde{n} - 1 \), with coefficient

\[
[(w + 1)\tilde{n} - 2w\tilde{n}(\tilde{n} - 3)]\tilde{A}\tilde{B} \tag{25}
\]

hence

\[
\tilde{n} = \frac{7w + 1}{2w} \tag{26}
\]

which again means that \( w \) is rational and leads to the same \( m - w \) pairs as in subcase 1.1. The subcases 3.1 - 3.4 \( (m = 18, 15) \) are again covered by subcases 1.1.1 - 1.1.4 in the 'brute force' approach with no solutions, leaving the \( m = 3 \) cases.

Case 3.5. \( m = 3, \tilde{m} = 0, w = -1 \).

This leads to \( \tilde{n} = 3 \), therefore to

**Solution 4:** \( w = -1, \ F(r) = Ar^3 + C \tag{27} \)

which includes solution 1 as a special case.

Case 3.6. \( m = 3, \tilde{m} = 0, w = -\frac{1}{3} \).

This gives \( \tilde{n} = 2 \), but also \( n = 1 \) (see subcase 2.1.6 or eq.(19)), an impossibility. This concludes case 3.
Case 4. $m > 1$, $\tilde{m} < 0$

In this case, eqs. (12) and (15) mean that $m = \tilde{m}$, hence this case fails.

Case 5. $m = 1 = \tilde{m}$

Very straightforwardly, one can derive

Solution 5: $w$ arbitrary, $F(r) = \frac{4w}{w^2 + 6w + 1} r$. \hfill (28)

Note that the coefficient $A$ was arbitrary in solutions 1-4, but it is determined in terms of $w$ in solution 5. Also, solutions 1-4 required certain values of $w$, while solution 5 is valid for arbitrary $w$.

Case 6. $m = 1$, $\tilde{m} = 0$

It turns out that for one value of $w$, one can add a constant term to the above solution:

Solution 6: $w = -\frac{1}{5}$, $F(r) = 5r + B$. \hfill (29)

Case 7. $m = 1$, $\tilde{m} < 0$

In a sense, this case is a mirror image of case 2. We have (15), and consideration of highest power of $r$ in (9) gives

$$A = \frac{4w}{w^2 + 6w + 1}. \hfill (30)$$

As in case 2, $(w^2 + 6w + 1)$ cannot vanish. For more information, we consider the second-lowest power $\tilde{m} + \tilde{n} - 1$, distinguishing if $\tilde{n}$ is equal to 1 or not.

Case 7.1. $m = 1$, $\tilde{m} < 0$, $\tilde{n} < 1$

The coefficient of $r^{\tilde{m} + \tilde{n} - 1}$ is given by the same expression as eq. (18) with $m \to \tilde{m}$, $n \to \tilde{n}$, $A \to \tilde{A}$ and $B \to \tilde{B}$. This makes again $w$ rational, but now $\tilde{m}$ must be 18 or 15 or 3, unacceptable because they are positive.

Case 7.2. $m = 1$, $\tilde{m} < 0$, $\tilde{n} = 1$

$F(r)$ consists of two terms, $F = Ar + \tilde{A}r^{\tilde{m}}$ now. The vanishing of the coefficient of $r^{\tilde{m}}$ in (9) reduces upon the substitutions (15) and (30) to (23)
again, giving the unacceptable (positive) \( \tilde{m} \) value 3.

\textit{Case 8.} \( m = 0, = \tilde{m} \)

This case is trivial:

\textbf{Solution 7:} \( w \) arbitrary, \( F(r) = A. \) \hfill (31)

\textit{Case 9.} \( m = 0, \tilde{m} < 0 \)

The highest power in \( (9) \) is \( n \) now, with coefficient \( 2wn(n - 3)B \), similar to lowest power in case 1. This cannot vanish; hence there is no solution in this case.

\textit{Case 10.} \( m, \tilde{m} < 0 \)

The same argument as above is valid here for \( n \rightarrow m \), so again there is no solution.

This completes all finite polynomial solutions of equation (9). Since Solution 1 is a special case of Solution 4, we will not consider it separately in the following section.

\section{Discussion of the solutions found from the standard (NS) OV equation}

To finalize the solutions, we calculate the metric functions \( A(r) \) and \( B(r) \) by using (7), (8), (4) and (6). The calculation of \( B(r) \) involves an arbitrary multiplicative constant at the last stage, the change of which is usually interpreted as a rescaling of \( t \), therefore physically irrelevant. But such rescaling cannot change the sign of that constant, so we consider the two choices of sign as two separate solutions, unless the requirement of correct signature forces a choice upon us. This happens for solutions 3, 4 and 7, whereas for solutions 2, 5 and 6 we have consider both signs. The results are shown in Table 2, where the well-known solutions are indicated in italics.

When the metric functions are negative, the spacetime cannot be supported by normal perfect fluid, the source fluid must be tachyonic. In other words, such a spacetime is of type TD in the terminology of [9]. In that case, the OV equation, (5), is not valid, but still, \( A(r)-B(r) \) pairs satisfy the same equation of pressure isotropy for cases NS and TD. Therefore negative metric functions found from NS-equations represent a valid TD solution, but
### Table 2: All finite-polynomial solutions of the equation (9) for the mass function in the standard (NS) OV case, together with the corresponding metric functions. Although we started with the NS OV equation, some of the solutions belong to class TD, as defined in [9]. In Solutions 2, 5 and 6, the upper signs in $B(r)$ apply to solutions a and lower signs to solutions b. The well-known solutions are indicated in *italics*.

| Sol.No. | $w$ | $F(r)$ | $B(r) = -g_{tt}$ | $A(r) = g_{rr}$ | Comments |
|---------|-----|--------|-----------------|-----------------|----------|
| 2 a,b   | $-\frac{1}{3}$ | $Ar^3$ | $\pm 1$ | $\frac{1}{1-Ar^2}$ | 2a, $A > 0$ : ESU; 2a, $A < 0$ : open, static; 2b : type TD |
| 3       | $-\frac{1}{3}$ | $Ar^3 + \frac{3}{2}r$ | $-\frac{1}{2}+Ar^2/r^2$ | $-\frac{1}{2}+Ar^2$ | $A > 0$ : type TD; $A < 0$ : BH-like |
| 4       | -1  | $Ar^3 + C$ | $1 - \frac{C}{r} - Ar^2$ | $\frac{1}{1-C/Ar^2}$ | Kőtler (*SdS*) |
| 5 a,b   | arbitrary, except $-1, -3 \pm 2\sqrt{2}$ | $\frac{4w}{w^2+6w+1}r$ | $\pm \left(\frac{r}{r_0}\right)^{\frac{3w}{w+1}}$ | $1 + \frac{4w}{(w+1)^2}$ | 5a: type NS, incl. static phantom; 5b: type TD |
| 6 a,b   | $-\frac{1}{5}$ | $5r + B$ | $\pm \frac{5a}{r}$ | $\frac{1}{r-4}$ | 6a: type NS; 6b: type TD |
| 7       | arbitrary | $A$ | $(1 - \frac{A}{r})$ | $\frac{1}{1-\frac{A}{r}}$ | Schwarzschild |

Table 2: All finite-polynomial solutions of the equation (9) for the mass function in the standard (NS) OV case, together with the corresponding metric functions. Although we started with the NS OV equation, some of the solutions belong to class TD, as defined in [9]. In Solutions 2, 5 and 6, the upper signs in $B(r)$ apply to solutions a and lower signs to solutions b. The well-known solutions are indicated in *italics*. 

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not with the equation of state that one has started with. If the NS equation of state is (4), the corresponding TD equation of state becomes \( p = -\frac{w}{1+2w} \rho \).

Solution 7 is the Schwarzschild solution. It may at first seem surprising that there is no restriction on \( w \). But since \( \rho \) vanishes, the value of \( w \) does not matter. In other words, it corresponds to a situation where all the fluid—whatever its equation of state parameter is—has already collapsed to the origin. Also, here we do not apply the usual restriction that \( A \) must be positive. If \( A \) is negative, the spacetime will give a naked singularity.

Solution 2a with positive \( A \) is also well-known: it is the Einstein static universe, with intimate historical connection to the cosmological constant \( \Lambda \), equivalent to \( w = -1 \). But this universe also contains matter \( (w = 0) \), whose attraction is precisely balanced by the repulsion of \( \Lambda \). So the matter density is proportional to \( \Lambda \) and the net effect is equivalent to a single fluid with \( w = -\frac{1}{3} \). Of course, “in the universe” \( Ar^2 < 1 \), so \( A(r) \) is positive and the signature correct. For negative \( A \), Solution 2a represents an open static universe, albeit with negative energy density, and no coordinate restriction.

Noting that the third well-known solution in Table 2 is Solution 4, the K"ottler (aka Schwarzschild-de Sitter) solution, the de Sitter part sometimes being called anti-de Sitter if \( A \) is negative, we turn to the discussion of less well-known solutions; changing the order slightly in the interest of presentation.

*Solution 2b* :

This solution has correct signature only for \( Ar^2 > 1 \), which means \( A \) must be positive. It is a dynamic spacetime, \( r \) being timelike, (it is solution TD1 of [9]) and describes a spacetime that first contracts, then expands in angular directions, while distances in the orthogonal spacelike direction stay fixed\(^2\). Even though we found this solution for \( w = -\frac{1}{3} \), the equation of state is actually \( p = \rho \). The solution can be identified with the \( n = 0 \) choice of Tolman V [13], if \( \text{const}=-1 \) is chosen\(^3\), with identification \( \frac{1}{R^2} \to A \).

*Solution 5a* :

Correct signature means positivity of \( A(r) \) in this solution, which in turn means that the solution is valid except for \(-3 - 2\sqrt{2} < w < -3 + 2\sqrt{2} \) (and it is of type NS). The cases \( w < -3 - 2\sqrt{2} \), for example, \( w = -6 \), represent static (ultra)phantom solutions. The \( w = \frac{1}{3} \) case is well-known [11].

\(^2\)The KS-like form of the metric is \( ds^2 = -d\tau^2 + dp^2 + \frac{1}{4} \cosh^2(\sqrt{A}\tau) \, d\Omega^2 \).

\(^3\)But Tolman chooses \( \text{const}=B^2 \) and later literature reports this form (e.g. [14]).
the \( w \to \infty \) limit, meaning zero density but nonzero pressure, is the metric called S1 in [10]; other valid cases with integer power of \( r \) in \( B(r) \) are \( w = 1 \) and \( w = 3 \).

The density is proportional to \( \frac{1}{r^2} \), but this is a mild singularity because the mass function goes to zero as \( r \to 0 \), i.e. there is no mass point at the origin. Of course, there is no event horizon, so the singularity is naked.

The origin is attractive to test particles for \( w < -3 - 2\sqrt{2} \) and for \( w > 0 \); repulsive for \( -3 + 2\sqrt{2} < w < 0 \). The sign of attraction correlates with the sign of \( \rho + 3p \), the so-called ”density of active gravitational mass” (e.g. [11]) for the fluid. The pressure is positive for all \( w \) ranges, and since \( p \propto \rho \propto \frac{1}{r^2} \), the pressure gradient is always towards the origin. So, we cannot understand the balance of a fluid element as in terms of \( \rho \) (both forces would accelerate the fluid element towards the origin in the ultraphantom case), but in terms of \( \rho + p \), the ”density of inertial mass” (e.g. [11]).

These static ultraphantom solutions constitute a counterexample to the impression in the literature (e.g. see [12]) that everywhere-phantom static spherically symmetric solutions cannot exist.

This solution can be identified with the \( n = \frac{2w}{w+1}, R \to \infty \) and \( B = \frac{1}{r_0} \) (or \( \text{const}=r_0^{-\frac{2w}{w+1}} \)) choice of Tolman V [13].

**Solution 5b:**

This is a TD solution (a subcase\(^4\) of TD2 of [9]) valid for \( -3 - 2\sqrt{2} < w < -3 + 2\sqrt{2} \), except \( w = -1 \). Assuming \( r \) is future-directed, this spacetime expands in the angular directions, and either expands (for \( w < -1 \)) or contracts (for \( w > -1 \)) in the orthogonal spacelike direction\(^5\). An infinite number of \( w \)-values, crowding -1, exist that give integer power of \( r \) in \( B(r) \).

The equation of state is \( p = -\frac{w}{1+2w} \rho \). The density is still proportional to \( \frac{1}{r^2} \), but because of the timelike nature of \( r \), \( F(r) \) cannot be interpreted as the mass function, and therefore we cannot make the same claim as to the mildness of the singularity as in solution 5a.

This solution can be identified with the almost same subcase of Tolman V [13] as solution 5a, except\(^3\) \( \text{const}=-r_0^{-\frac{2w}{w+1}} \).

**Solution 3:**

\(^4\)Which subcase it is depends on the sign of \( w + 1 \).

\(^5\)Metric in KS-like form: \( ds^2 = -d\tau^2 + \left( \frac{\tau^2}{|A| \rho_0} \right)^{\frac{2w}{w+1}} d\rho^2 + \frac{\tau^2}{|A|} d\Omega^2 \), where \( A = \frac{w^2 + 6w + 1}{(w + 1)^2} \).
For positive $A$, this solution is also of type TD, contracting in the angular directions and expanding in the orthogonal spacelike direction as $r \to 0$.

For negative $A$, both metric functions switch sign at $r = r_H = \sqrt{-\frac{1}{2A}}$, so that the spacetime is static (NS) for $r > r_H$ and dynamic (TD) for $r < r_H$. As far as test particle motion is concerned, this spacetime would be that of a black hole; but it must be supported by normal matter in the NS region, and tachyonic matter (with $p = \rho$) in the TD region. As unreasonable as this may seem, it is the only possible perfect fluid interpretation.

As the origin of $r$ is approached, the density again diverges like $\frac{1}{r^2}$, but again $r$ is timelike near the origin, and the same (non)conclusion applies to the singularity as in Solution 5b.

For positive $A$, this solution can be identified with the $a = 1$, $b = -1$, $m = 0$, $\frac{1}{R^2} = A$ (and the trivial $B = 1$ or const=1) choice of Tolman VIII [13].

Solution 6a:

This solution is type NS. $C$ must be positive and $r < \frac{C}{4}$. Interestingly, radially moving free particles oscillate between a minimum radius and $\frac{C}{4}$, which may be understood in terms of the repulsion of the negative mass point at the origin ($C = -B$ and $F(r)$ is the mass function) versus the attraction of the fluid, whose “enclosed active gravitational mass” (e.g. [11]) grows with $r$ (here, both $\rho$ and $\rho + 3p$ are positive).

The origin is a naked singularity, and not only due to the negative point mass there: The scalar curvature is $\frac{8}{r^2}$, that is, it diverges without containing $C$. But, after all, the scalar curvature does not contain $M$ in the Schwarzschild case, either (in fact, it vanishes). $r = \frac{C}{4}$ is a type of boundary, it is a turning point for all radial timelike geodesics.

This solution can be identified with the $n = -\frac{1}{2}$, $R \to -C$ and $B^2 = r_0$ choice of Tolman V [13].

Solution 6b:

This TD solution (with $p = \rho/3$) can be identified with the $n = -\frac{1}{2}$, $R \to -C$ and const=-$r_0$ choice of Tolman V [13]. There is no coordinate restriction for negative $C$, but $r$ must be larger than $\frac{C}{4}$ for positive $C$.

In the latter case, again $r = \frac{C}{4}$ is a turning point for timelike radial geodesics, but $r$ is timelike, so this spacetime first contracts in the angular directions while expanding in the orthogonal spacelike direction, then the evolution reverses.

The KS-like form of the metric is

$$ds^2 = -d\tau^2 + A \coth^2(\sqrt{A}\tau) d\rho^2 + \frac{1}{r^2} \sinh^2(\sqrt{A}\tau) d\Omega^2.$$
On the other hand, for negative $C$, the spacetime expands in the angular directions while contracting in the orthogonal spacelike direction, assuming $r$ is future-directed.

4 Discussion of solutions found from the OV-like equations in the other cases and another attempt

As discussed in the previous section, the TD solutions satisfy the same equation of pressure isotropy as the NS solutions, therefore a solution derived from the OV(like) equations of one class may in fact belong to the other class. Moreover, the same is valid for the ND and TS classes.

It turns out that for the equation of state $p = w\rho$, the OV-like equations of the TD and TS cases do not give any solutions not already covered by NS and ND cases, except for the special $w$ value $-\frac{1}{2}$. This should not be taken as an indication that the TD and TS solutions are trivial relabelings; for more complicated equations of state, there will be different solutions.

The proof of the above statement, the application of the procedure of Sect.2 to the ND case to find all ND and TS solutions with finite-polynomial $F(r)$ for $w \neq -\frac{1}{2}$, and the TD and TS solutions for $w = -\frac{1}{2}$ are given in the appendix.

In this section, we calculate the metric functions $A(r)$ and $B(r)$ for each solution from the appendix by using the relevant formulae, and discuss the solutions. We also show that no nontrivial solution with finite-polynomial $A(r)$ exists.

4.1 The TD case

For Solution 8, we get

$$A = \frac{1}{1 - C/r}$$

which for $r < C$ (only possible if $C > 0$) gives

$$B = -r_1^{-4} \left[(2r^2 + 5Cr - 15C^2) + \frac{C - r}{r} \left(C_1 - 15C^2 \tan^{-1} \sqrt{\frac{C - r}{r}} \right) \right]^2,$$

i.e. Solution TD3 of [9].

KS-like form of the metric is $ds^2 = -d\tau^2 + \frac{1}{r(r)} dr^2 + r^2(\tau) d\Omega^2$, where $\frac{dr}{d\tau} = \pm \sqrt{4 - \frac{C}{r}}$.
On the other hand, for \( r > C \) we find
\[
B = r_1^{-4} \left[ (2r^2 + 5Cr - 15C^2) + \sqrt{\frac{C - r}{r}} \left( C_1 + 15C^2 \ln\left( \frac{\sqrt{r - C} + \sqrt{r}}{|C|} \right) \right) \right]^2,
\]
the solution called NS1 in [9], found in [15] and named Kuch68 I in [14]. It describes a spacetime where pure pressure is in static equilibrium with its own gravitational attraction.

### 4.2 The ND(KS) and TS cases

The solutions found for the ND(KS) and TS cases, together with their metric functions, are shown in Table 3 (Solution 9 does not appear because it is a special case of Solution 11). As in Sect.3, the sign of \( B(r) \) is arbitrary, unless forced by the signature requirement.

The Schwarzschild and Köttrler (SdS) solutions, which appeared in Table 2, are found in this table as well, because they cannot really be classified in this scheme. Our classification is based upon the nature and direction of motion of the fluid, but for these solutions, the stress-energy-momentum tensor is independent of the fluid four-velocity: The \( u_\mu u_\nu \) term in \( T_{\mu\nu} \) is multiplied by \( p + \rho \); and \( p + \rho = 0 \) for the Köttrler solution, \( p = \rho = 0 \) for Schwarzschild. Hence, these solutions satisfy the equations for all four cases.

The other solutions in the table are less well-known:

**Solution 10:**

For positive \( A \), this solution is type ND (KS), representing a dynamic spacetime filled with a phantom perfect fluid. Assuming \( r \) is future-directed, the spacetime expands in the angular directions; in the perpendicular space-like direction, it first contracts, reaches a minimum, then expands.\(^8\) It is singular at both ends of the evolution, that is, at \( r = 0 \) and as \( r \to \infty \), the first singularity being in the finite past, the second in the infinite future. Of course, these attributes switch if \( r \) is past-directed.

For negative \( A \), Solution 10, like Solution 3, represents a black hole spacetime, as far as test particle motion is concerned; but it must be supported by two different fluids on the two sides of the horizon: tachyonic fluid in the outside, static region and normal fluid in the dynamic region inside/in the future.

---

\(^8\)The KS form of the metric is \( ds^2 = -d\tau^2 + \frac{1+8Av^8(\tau)}{\tau^8(\tau)} d\rho^2 + r^2(\tau) d\Omega^2 \), where \( \frac{dr}{d\tau} = \pm \sqrt{Av^8 + \frac{1}{\tau^8}} \).
| Sol.No. | $w$  | $F(r)$  | $B(r) = -g_{tt}$  | $A(r) = g_{rr}$  | Comments                                      |
|--------|------|---------|-------------------|------------------|-----------------------------------------------|
| 10     | $-3$ | $Ar^9 + \frac{9}{8}r$ | $\frac{1+8Ar^3}{r^6}$ | $\frac{8}{1+8Ar^8}$ | $A > 0$: type ND(KS), phantom-filled dynamic universe; $A < 0$: BH-like |
| 11     | $-1$ | $Ar^3 + B$ | $1 - \frac{B}{r} - Ar^2$ | $\frac{1}{1-B/r-Ar^2}$ | Köttler (SdS)                                  |
| 12 a,b | arbitrary, except $-\frac{1}{3}$ and 1 | $\frac{4w^2}{3w^2-2w-1}r$ | $\pm \left( \frac{r}{r_0} \right)^{-\frac{4w}{w+1}}$ | $-\frac{(w-1)(3w+1)}{(w+1)^2}$ | $-\frac{1}{3} < w < 1$: type TS; otherwise: type ND(KS), incl. DE, incl. phantom |
| 13 a,b | $\frac{1}{3}$ | $-\frac{1}{3}r + B$ | $\pm \frac{\rho_0}{r}$ | $\frac{3r}{4r-3B}$ | 13a: type TS; 13b: type ND(KS) |
| 14     | arbitrary | $A$ | $(1 - \frac{A}{r})$ | $\frac{1}{1-A/r}$ | Schwarzschild                                   |
| 15 a,b | $1$ | $\frac{C}{r}$ | $\pm 1$ | $\frac{1}{1-C/r^2}$ | 15a: type TS; 15b: type ND(KS) |
| 16     | $-\frac{1}{2}$ | $C$ | $(1 - \frac{C}{r})$ | $\frac{1}{1-C/r}$ | Schwarzschild                                   |
| 17     | $-\frac{1}{2}$ | $\frac{4}{3}r$ | $-(\frac{r}{r_0})^{-4}$ | $-3$ | type ND(KS)                                    |

Table 3: All finite-polynomial solutions for $F(r)$ in the ND(KS) and TS cases as defined in [9]; together with the corresponding metric functions. In Solutions 12, 13 and 15, the upper signs in $B(r)$ apply to solutions a and lower signs to solutions b. The well-known solutions are indicated in italic.
**Solutions 12a,b**:

Solution 12a is a TS solution, valid for $-\frac{1}{3} < w < 1$. For positive $w$, radially incoming test particles are reflected near the origin back to infinity, whereas for negative $w$, the origin constitutes a potential well from which they cannot escape.

Solution 12b, however, is valid for $w$ values other than $-\frac{1}{3} < w < 1$, and is identical to the $C_1 = 0$ special case of Solution ND2 of [9]. If $r$ is future-directed, it expands in the angular directions, and it expands or contracts in the perpendicular spacelike direction, if the sign of $\frac{w}{w+1}$ is negative or positive, respectively. Note that this means expansion for non-phantom dark energy ($-1 < w < -\frac{1}{3}$) and “radial” contraction for phantom energy.

**Solutions 13a,b**:

Solution 13a is a TS solution, where we must have $4r > 3B$, i.e. we have a restriction on $r$ if $B$ is positive. Either way, the equation of motion for test particles shows that tachyonic $w = \frac{1}{3}$ fluid is repulsive, consistent with the solution 12a.

Solution 13b is an ND (KS) solution, where $4r < 3B$. It represents a radiation-filled universe that expands and recollapses in angular directions while contracting and reexpanding in the perpendicular spacelike direction first found in [16].

**Solutions 15a,b**:

Solution 15a is solution TS1 of [9], where we must have $r^2 > C$. For positive $C$, $r_0 = \sqrt{C}$ is a turning point for radial geodesics; for negative $C$, there are no such turning points.

Solution 15b is solution ND1 of [9], apparently first found in [17], describing a finite-lifetime universe containing stiff matter, expanding and recollapsing in the angular directions.

**Solution 17**:

9The KS form of the metric is $ds^2 = -d\tau^2 + \left(\frac{x^2}{x^2+A}\right)^{-\frac{w}{w+1}} d\rho^2 + \frac{r^2}{A} d\Omega^2$, where $A = -\frac{(w-1)(3w+1)}{(w+1)^2}$.

10The KS form of the metric is $ds^2 = -d\tau^2 + \frac{r^2}{(\tau^2)} d\rho^2 + r^2(\tau) d\Omega^2$, where $\frac{d\tau}{d\tau} = \pm \sqrt{\frac{B}{r^2} - \frac{3}{r^2}}$.

The arbitrary $r_0$ must be chosen as $\frac{3}{2}B$ for agreement with [16], p.1684.

11The KS form of the metric is $ds^2 = -d\tau^2 + d\rho^2 + (C - \tau^2) d\Omega^2$. 

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This solution is the $C_1 = 0, A = -3$ special case of solution ND2 of [9], describing a spacetime containing pressure, but no density (because it is an ND (KS) solution found from the TS equations, its equation of state is \( p = -\frac{1}{2} \)); expanding in angular directions while contracting in the perpendicular spacelike direction\(^{12}\) if \( r \) is taken to be future-directed.

4.3 Finite-polynomial \( A(r) \)?

Another possible way to look for solutions is to work in terms of \( A(r) \) rather than \( F(r) \) by using equation (7). This leads to an equation with terms second to fourth order in \( A(r) \) and/or its derivatives. In trying to find a finite-polynomial solution for \( A(r) \), if the highest power of \( r \) in \( A(r) \) is \( m \), the highest power of \( r \) in the equation is \( 4m \); but it is multiplied by \( A^2(w+1)^2 \) in cases NS and TD, and \( -A^2(w+1)^2 \) in cases ND and TS. Setting the trivial \( w = -1 \) case aside, therefore, the highest possible value for \( m \) is zero. A similar argument shows that the lowest power in the \( A(r) \) polynomial must be zero or higher. Hence the only finite polynomial \( A(r) \) can be for equation of state \( p = w\rho \) is a constant.

5 Summary and Conclusions

We have considered spherically symmetric perfect fluid solutions in General Relativity and found all finite-polynomial solutions -including negative powers- of the equation satisfied by the so-called ”mass function” and its mathematical analogs for the equation of state \( p = w\rho \); and discussed the associated spacetimes.

The equation for the mass function follows from the Oppenheimer-Volkoff (OV) equation in the standard case where the fluid is static and normal (i.e. non-tachyonic, \( u^\mu u_\mu = -1 \)). However, the metric ansatz used in that analysis can also accommodate cases where the spacetime is dynamic in a certain way, or the fluid is tachyonic; as discussed in [9]. In these other cases analogous, but different functions exist, satisfying their own equations.

The solutions we found for the standard case, NS, are mathematically not very original; they are either some limiting cases of solutions found long ago by Tolman [13] or simple modifications thereof. Some aspects of the physical nature of these solutions can be seen in new light however, considering the classification in [9] and newly cosmologically relevant concepts of dark energy.

\(^{12}\)The KS form of the metric is \( ds^2 = -d\tau^2 + (\mathcal{A})^4 d\rho^2 + \frac{\tau^2}{\mathcal{A}} d\Omega^2 \).
and phantom energy. The solutions (Table 2) include dynamic spacetimes supported by tachyonic fluids (2b, 5b, 3 with $A > 0$, 6b) and a static spacetime containing a $w = -\frac{1}{5}$ fluid around a negative point mass (6a). The TD case gives two extra solutions, one describing a spacetime where pure pressure is in static equilibrium with its own gravitational attraction.

Some interesting solutions are also found from the ND(KS) and TS cases (Table 3): There are static solutions supported by tachyonic fluids (12a, 13a, 15a), the first two presumably original. Some solutions (10 for positive $A$, 12b, 13b, 15b, 17) are of the Kantowski-Sachs (KS) class: Solutions 13b, 15b and 17 describe dynamic KS-universes containing radiation, stiff matter and pure pressure, respectively.

We would like to particularly point out the following solutions:

- Solution 5a for $w < -3 - 2\sqrt{2}$ represents, perhaps unexpectedly, a family of static “ultraphantom” solutions.

- Solution 3 for negative $A$ is a black hole-like spacetime, which must be supported by normal matter outside the horizon and tachyonic fluid on the inside.

- Solution 10 for positive $A$ is a phantom KS solution, probably new.

- Solution 12b can also be valid for dark energy, including phantom, exhibiting anisotropic expansion for non-phantom dark energy.

- Solution 10 for negative $A$ is similar to Solution 3, a black hole-like spacetime, supported by segregated normal and tachyonic matter, except in this solution, the tachyonic fluid is outside and normal fluid is inside. It was concluded in [9] that black holes supported by perfect fluids cannot be “simple”.

There are no other solutions where $F(r)$ is a finite polynomial of $r$ for the assumed equation of state. One can also express the problem(s) in terms of $A(r)$, and then try to find finite polynomial solutions. The only such solution is $A(r)=$constant.
A All finite-polynomial solutions for $F(r)$ from the OV-like equations in the TD, ND(KS) and TS cases

A.1 The TD case

The TD OV equation \[9\] is

$$
p' = \frac{(\kappa pr^3 + F_{TD})}{2r(r - F_{TD})} (\rho + p)
$$

where

$$
F_{TD}(r) = -\kappa \int (\rho + 2p)r^2 dr,
$$

and the metric functions are found by

$$
A = \frac{r}{r - F_{TD}}
$$

$$
\frac{B'}{B} = \frac{\kappa pr^2 + 1}{r - F_{TD}} - \frac{1}{r}.
$$

The substitution $\tilde{\rho} = -(\rho + 2p)$ brings the TD OV equation (35) into the same form as the regular one (5), in terms of $\tilde{\rho}$ and $p$. When expressed in terms of $F_{TD}$, with equation of state (4), we get equation (9), but with the replacement $w \rightarrow -\frac{w}{1+2w}$. Since this is another constant equation of state parameter, we will not get any finite-polynomial solutions that are not already in Table 2, unless $1 + 2w = 0$. In that case, $F_{TD}$ becomes a constant,

\textbf{Solution 8 : } $w = -\frac{1}{2}, \ F_{TD}(r) = C. \tag{39}$

A.2 The ND case

The ND OV equation \[9\] is

$$
p' = \frac{3F_{ND} - 4r + \kappa pr^3}{2r(r - F_{ND})} (\rho + p)
$$

where

$$
F_{ND}(r) = -\kappa \int pr^2 dr
$$

and the metric functions can be found by

$$
A = \frac{r}{r - F_{ND}}
$$

$$
\frac{B'}{B} = \frac{1 - \kappa pr^2}{r - F_{ND}} - \frac{1}{r}.
$$

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In this case, $F_{ND}(r)$ obeys
\[(1 + w)F_{ND}'(3wF_{ND} - 4wr - rF_{ND}') + 2w(rF_{ND}' - 2F_{ND}'')(F_{ND} - r) = 0 \quad (44)\]

To find all finite-polynomial solutions of this equation (dropping label $ND$), we follow the same procedure as in Section 2, and for similar reasons, we use the same matrix of cases, shown in Table I.

For cases 1-4, the consideration of highest power of $r$ in eq. (44) gives
\[(w - 1)(3w + m) = 0 \quad (45)\]

Similarly, for cases 4, 7, 9 and 10, consideration of lowest power leads to
\[(w - 1)(3w + \tilde{m}) = 0. \quad (46)\]

**Case 1.** $m, \tilde{m} > 1$

Lowest power in eq. (44) gives
\[\tilde{m} = 1 - 2w. \quad (47)\]

Two subcases, $w = 1$ and $m = -3w$ follow from eq. (45).

**Case 1.1.** $m, \tilde{m} > 1; w = 1$

$w = 1$ gives $\tilde{m} = -1$ by eq. (47), which contradicts the definition of case 1.

**Case 1.2.** $m, \tilde{m} > 1; m = -3w$

$m = -3w$ gives
\[m = \frac{3}{2}(\tilde{m} - 1) \quad (48)\]

We may have $m = \tilde{m}$ or $m > \tilde{m}$:

**Case 1.2.1.** $m = -3w = \tilde{m} > 1$

gives

**Solution 9:** $w = -1, \quad F(r) = Ar^3 \quad (49)$

**Case 1.2.2.** $m = -3w > \tilde{m} > 1$

When $n$ exists and is larger than 1, as in this case, consideration of the second-highest power in eq. (44) gives
\[3(m + n)w^2 + [2(m^2 + n^2) - 2nm - 3(m + n)]w - 2nm = 0, \quad (50)\]
which together with \( m = -3w \) yields

\[
m = 2n + 3. \tag{51}
\]

But, this equation, together with eq.(48) gives

\[
n = \frac{3\tilde{m} - 9}{4}, \tag{52}
\]
impossible because it violates \( n \geq \tilde{m} \) for \( \tilde{m} > 1 \).

**Case 2.** \( m > 1, \tilde{m} = 1 \)

Lowest power in eq.(44) gives

\[
\tilde{A} = \frac{4w^2}{3w^2 - 2w - 1}. \tag{53}
\]

unless \( (3w^2 - 2w - 1) \) vanishes, that is, \( w = 1 \) or \( w = -\frac{1}{3} \). Highest power gives eq.(45), so this means \( m = -3w \). \( n \) exists, second-highest power gives eq.(50), if \( n > 1 \).

**Case 2.1.** \( m = -3w > 1, \tilde{m} = 1, n > 1 \)

Again, eq.(50), together with \( m = -3w \) gives eq.(51). Then the third-highest power in (44) is \( m + p - 1 = 2n + p + 2 \), as opposed to \( 2n - 1 \). We have to also distinguish if \( p > 1 \) or not.

**Case 2.1.1.** \( m = -3w > 1, \tilde{m} = 1, n > 1, p > 1 \)

Setting the relevant coefficient equal to zero gives

\[
(m - p)(2p + 3 - m) = 0, \tag{54}
\]
whose solutions, \( p = m \) and \( p = \frac{2n - 3}{2} = n \) are both unacceptable.

**Case 2.1.2.** \( m > 1, \tilde{m} = 1, m = -3w, n > 1, p = 1 \)

Setting the coefficient of the third-highest power in (44) equal to zero gives

\[
\tilde{A} = \frac{n(2n + 3)}{3(n^2 - 1)} = \frac{6w(w + 1)}{9w^2 + 18w + 5} \tag{55}
\]
which contradicts eq.(53).

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Case 2.2. $m > 1$, $\bar{m} = 1$, $m = -3w$, $n = 1$

Putting $F(r) = Ar^m + \tilde{A}r$ into eq. (44), and using (53), followed by $m = -3w$, we get

$$(m - 9)(m - 1) = 0. \quad (56)$$

This gives

**Solution 10:** $w = -3$, $F(r) = Ar^9 + \frac{9}{8}r. \quad (57)$

Case 3. $m > 1$, $\bar{m} = 0$

From consideration of the lowest power in (44), $\bar{n} - 1$, we get

$$3(w + 1) = 2(\bar{n} - 3), \quad (58)$$

$n$ exists and according to eq. (45) we have two subcases.

**Case 3.1.** $m > 1$, $\bar{m} = 0$, $w = 1$

$w = 1$ gives $\bar{n} = 6$ according to (58). Now, either $n \geq 6$, or $n = \bar{m} = 0$.

**Case 3.1.1.** $m > 1$, $\bar{m} = 0$, $w = 1$, $n \geq 6$

In this subcase, eq. (50) is applicable, but for $w = 1$, eq. (50) gives $2(m - n)^2$ on the left-hand-side, which cannot vanish.

**Case 3.1.2.** $m > 1$, $\bar{m} = 0 = n$, $w = 1$,

When $F(r) = Ar^6 + \tilde{A}$ is put into eq. (44) with $w = 1$, we get

$$72A\tilde{A}r^5 - 84Ar^6 = 0, \quad (59)$$

impossible to satisfy.

**Case 3.2.** $m = -3w > 1$, $\bar{m} = 0$

$m = -3w$, together with (58) gives

$$\bar{n} = \frac{3 - m}{2} + 3 \quad (60)$$

This makes $(m, \bar{n})$ pairs (3,3), (5,2), (7,1) possible.
Case 3.2.1. \( m = -3w = 3 = \tilde{n}, \tilde{m} = 0 \)

Solution 11: \( w = -1, \ F(r) = Ar^3 + B. \) \( (61) \)

This includes Solution 9 as a special case.

Case 3.2.2. \( m = -3w = 5, \tilde{n} = 2, \tilde{m} = 0 \)

This subcase gives no solution, which can be seen either by the ‘brute force’ method (Sect. 2, case 1.1.1) or by considering the second-highest power in eq. \( (44) \), which gives \( n = 1 \) or \( n = 5 \) (fails since we must have \( 5 > n \geq 2 \)).

Case 3.2.3. \( m = -3w = 7, \tilde{n} = 1, \tilde{m} = 0 \)

Again, the ‘brute force’ method shows that there is no solution.

Case 4. \( m > 1, \tilde{m} < 0 \)

Since both eq. \( (45) \) and eq. \( (46) \) apply, we must have \( w = 1. \ n \) exists. If \( n > 1 \), the argument in case 3.1.1 is applicable, so we must consider \( n \leq 1. \) A similar argument for second-lowest power in eq. \( (44) \) gives \( \tilde{n} \geq 1, \) so we have \( F(r) = Ar^m + Br + Ar^{\tilde{m}}. \) Now, consideration of the coefficients of the second-highest and second-lowest powers gives

\[
\frac{m(m+1)}{(m-1)^2} = B = \frac{\tilde{m}(\tilde{m}+1)}{(\tilde{m}-1)^2}
\]

with two solutions

\[
\tilde{m} = m, \quad \tilde{m} = \frac{m + 1}{3m - 1}
\]

neither of which is negative.

Case 5. \( m = 1 = \tilde{m} \)

Easily:

Solution 12: \( w \) arbitrary, \( F(r) = \frac{4w^2}{3w^2 - 2w - 1}r. \) \( (64) \)

As in case ND; this solution is valid for arbitrary \( w, \) with \( A \) is determined in terms of \( w; \) whereas in solutions 9-11 \( A \) was arbitrary, but \( w \) specified.
Case 6. \( m = 1, \tilde{m} = 0 \)

As in case ND, one can add a constant term to the above solution for one value of \( w \):

Solution 13: \( w = \frac{1}{3}, \quad F(r) = -\frac{1}{3}r + B. \) \hspace{1cm} (65)

Case 7. \( m = 1, \tilde{m} < 0 \)

We have eq.(46), and consideration of highest power in eq.(44) gives

\[ A = \frac{4w^2}{3w^2 - 2w - 1}; \] \hspace{1cm} (66)

as in case 2, \( w \) cannot be 1 or \(-\frac{1}{3}\), which means \( \tilde{m} = -3w \).

Case 7.1. \( m = 1, \tilde{m} = -3w < 0, \tilde{n} < 1 \)

The coefficient of the second-lowest power, \( r^{\tilde{m} + \tilde{n} - 1} \) is given by the same expression as in case 1.2.2, with \( m \rightarrow \tilde{m}, \quad n \rightarrow \tilde{n}, \quad A \rightarrow \tilde{A} \) and \( B \rightarrow \tilde{B} \); giving us

\[ \tilde{m} = 2\tilde{n} + 3, \] \hspace{1cm} (67)

which means that \( \tilde{m} < \tilde{n} < -3 \). We now consider the third-lowest power, whose value and coefficient depend on the value of \( \tilde{p} \).

Case 7.1.1. \( m = 1, \tilde{m} = -3w < 0, \tilde{n} < 3, \tilde{p} < 1 \)

The third-lowest power could be \( \tilde{m} + \tilde{p} - 1 \) or \( 2\tilde{n} - 1 \). Because of eq.(67), \( \tilde{p} = -3 \) is critical:

Case 7.1.1.1. \( m = 1, \tilde{m} = -3w < 0, \tilde{n} < -3, \tilde{p} < -3 \)

Power \( \tilde{m} + \tilde{p} - 1 \) is the third-lowest. Setting its coefficient equal to zero, using eq.(67) and \( \tilde{m} = -3w \), we get the two solutions \( \tilde{p} = \tilde{n} \) and \( \tilde{p} = 2\tilde{n} + 3 = \tilde{m} \), which are both unacceptable.

Case 7.1.1.2. \( m = 1, \tilde{m} = -3w < 0, \tilde{n} < -3, \tilde{p} = -3 \)

In this subcase, the coefficients \( \tilde{A}, \tilde{B} \) and \( \tilde{C} \) mix (unlike the previous subcase), so we resort to another ‘brute force’ approach: We put \( F(r) = Ar + B + Cr^{-1} + Dr^{-2} + \tilde{C}r^{-3} + \tilde{B}r^{\tilde{n}} + \tilde{A}r^{2\tilde{n}+3} \) into eq.(14), use \( w = -\frac{2\tilde{n}+3}{3} \), and sequentially set the coefficients of \( r^1 \cdots r^{-3} \) to zero, for each calculation.
using the result of the previous one, as well. We get $A = \frac{(2\tilde{n}+3)^2}{3(n^2+4n+3)}$ (compatible with eq.(69)), $B = 0$, $C = 0$, $D = 0$, $\tilde{C} = 0$ in sequence; this last result tells us that this subcase fails.

**Case 7.1.1.3.** $m = 1$, $\tilde{m} = -3w < \tilde{n} < -3$, $\tilde{p} > -3$

Power $2\tilde{n} - 1$ is the third-lowest. Setting its coefficient equal to zero, using eq.(67) and $\tilde{m} = -3w$, we get $\tilde{n} = \tilde{m}$, which is unacceptable.

**Case 7.1.2.** $m = 1$, $\tilde{m} = -3w < \tilde{n} < -3$, $\tilde{p} = 1$

Power $2\tilde{n} - 1$ is the third-lowest. Setting its coefficient equal to zero, using eq.(67) and $\tilde{m} = -3w$, we get $\tilde{n} = -\frac{3}{2}$ or $\tilde{n} = -1$ or $\tilde{n} = 3$ which are not acceptable.

**Case 7.2.** $m = 1$, $\tilde{m} < 0, \tilde{n} = 1$

$F(r)$ consists of two terms, $F = Ar + \tilde{A}r^{\tilde{m}}$ now. With this $F$, eq.(44) reduces, using eq.(67) and $\tilde{m} = -3w$ to

$$-2\tilde{A}\tilde{m}^2(\tilde{m} - 9)(\tilde{m} - 1)r^{\tilde{m}},$$

which cannot vanish.

**Case 8.** $m = 0, = \tilde{m}$

This case is trivial:

**Solution 14:** $w$ arbitrary, $F(r) = A.$  

(69)

**Case 9.** $m = 0$, $\tilde{m} < 0$

We have eq.(46), which leads to the subcases $w = 1$ and $\tilde{m} = -3w$. The highest power in eq.(44) is $n$, and consideration of its coefficient gives

$$n = 1 - 2w.$$  

(70)

**Case 9.1.** $m = 0$, $\tilde{m} < 0$, $w = 1$

Eq.(70) gives $n = -1$, which means that $\tilde{m}, \tilde{n} < 1$, so the second-lowest power in eq.(44) gives eq.(50), with $m \to \tilde{m}$, $n \to \tilde{n}$. However, for $w = 1$,  

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the left-hand-side of that equation cannot vanish.

Case 9.2. \( m = 0, -3w = \tilde{m} < 0 \)

Since \( \tilde{m}, \tilde{n} < 1 \), eq.\((67)\) is valid.

Case 9.2.1. \( m = 0, n = \tilde{m} = -3w < 0 \)

This means that \( F \) consists of two terms. We get \( w = -1 \) and \( n = 3 \), unacceptable for this case.

Case 9.2.2. \( m = 0, n > \tilde{m} = -3w < 0 \)

This means that a \( p < 1 \) exists. For \( p \neq -3 \), the subcases 7.1.1.1 and 7.1.1.3 carry over exactly; for \( p = -3 \), a calculation analogous to that of subcase 7.1.1.2 shows the nonexistence of solutions.

Case 10. \( m, \tilde{m} < 0 \)

Again, we have eq.\((46)\), therefore subcases \( w = 1 \) and \( \tilde{m} = -3w \). The highest power in eq.\((44)\) is \( m \), consideration of its coefficient giving

\[
m = 1 - 2w. \tag{71}
\]

Case 10.1. \( m, \tilde{m} < 0, w = 1 \)

Case 10.1.1. \( m = \tilde{m} < 0, w = 1 \)

Solution 15: \( w = 1, F(r) = C/r. \) \( \tag{72} \)

Case 10.1.2. \( \tilde{m} < m < 0, w = 1 \)

\( n, \tilde{n} \) exist; subcase 9.1 carries over exactly.

Case 10.2. \( m, \tilde{m} = -3w < 0, \)

Case 10.2.1. \( m = \tilde{m} = -3w < 0 \)

Gives \( w = -1, m = 3 \), unacceptable for this case.

Case 10.2.2. \( \tilde{m} = -3w < m < 0 \)
Since $\tilde{m}, \tilde{n} < 1$, eq. (67) is valid.

**Case 10.2.2.1.** $\tilde{m} = -3w = n < m < 0$

This means that $F$ consists of two terms. We get $w = 5$, $m = -9$ and $\tilde{m} = -15$, looking acceptable, but eq. (44) can still not be satisfied.

**Case 10.2.2.2.** $\tilde{m} = -3w < n < m < 0$

Subcase 9.2.2 applies.

This completes all finite polynomial solutions of equation (44).

### A.3 The TS case

The TS OV equation [9] is

$$\rho' + 2p' = \frac{3F_{TS} - 4r - \kappa(\rho + 2p)r^3}{2r(r - F_{TS})} (\rho + p)$$

where

$$F_{TS}(r) = -\kappa \int pr^2 dr,$$

and the metric functions are found by

$$A = \frac{r}{r - F_{TS}}$$

$$B' = \frac{1 + \kappa(\rho + 2p)r^2}{r - F_{TS}} - \frac{1}{r}$$

Again, for $F_{TS}$, with equation of state [14] we get equation (44), but with the replacement $w \to -\frac{w}{1+2w}$, leading to the only potentially new solutions for $1 + 2w = 0$. Then, we get two solutions:

**Solution 16:** $w = -\frac{1}{2}$, $F_{TS}(r) = C$,

**Solution 17:** $w = -\frac{1}{2}$, $F_{TS}(r) = \frac{4}{3}r$.

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