GRÖBNER BASES FOR (ALL) GRASSMANN MANIFOLDS

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Abstract. Grassmann manifolds $G_{k,n}$ are among the central objects in geometry and topology. The Borel picture of the mod 2 cohomology of $G_{k,n}$ is given as a polynomial algebra modulo a certain ideal $I_{k,n}$. The purpose of this paper is to understand this cohomology via Gröbner bases. Reduced Gröbner bases for the ideals $I_{k,n}$ are determined. An application of these bases is given by proving an immersion theorem for Grassmann manifolds $G_{5,n}$, which establishes new immersions for an infinite family of these manifolds.

1. Introduction

Mod 2 cohomology of Grassmann manifolds $G_{k,n} = O(n + k)/O(k) \times O(n)$ is the polynomial algebra in Stiefel-Whitney classes $w_1, \ldots, w_k$ of the canonical bundle over $G_{k,n}$ modulo the ideal $I_{k,n}$ generated by dual classes $\overline{w}_{n+1}, \ldots, \overline{w}_{n+k}$. Although the description of this ideal is simple enough, concrete calculations in cohomology of Grassmann manifolds may be rather difficult to perform. The question of whether a certain cohomology class is zero is rather important in various applications — for example, in determining the span of Grassmannians, in discussing immersions and embeddings in Euclidean spaces, in the determination of cup-length (which is related to the Lusternik-Schnirelmann category), in some geometrical problems which may be reduced to the question of the existence of a non-zero section of a bundle over a Grassmann manifold, etc. It is known that Gröbner bases are useful when one works with polynomial algebras modulo certain ideal. The first use of Gröbner bases in this context appears in [9] where the Gröbner bases for $I_{2,n}$ were established for $n$ of the form $n = 2^s - 3$ and $n = 2^s - 4$. These bases were used to prove an immersion result for corresponding Grassmann manifolds. Another application of Gröbner bases in the similar context may be found in [11].

In [12] and [13], reduced Gröbner bases for $I_{2,n}$ and $I_{3,n}$ (for all $n$) were established and used to obtain some new immersion results for Grassmann manifolds. At the time of writing of these papers, the authors were not aware of the paper [8], where additive bases for mod 2 cohomology of Grassmann manifolds were established. These additive bases together with information about Gröbner bases

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obtained directly in \cite{12} and \cite{13} allowed us to obtain reduced Gröbner bases for all $I_{k,n}$.

The plan of the presentation is as follows. In Section 2 some necessary facts about cohomology algebra $H^*(G_{k,n}; \mathbb{Z}_2)$ are reviewed. Section 3 contains main results, namely, the determination of reduced Gröbner bases for all $I_{k,n}$. Section 4 is devoted to an application of the obtained results to the immersion problem for $G_{5,n}$ for $n$ divisible by 8.

2. The cohomology algebra $H^*(G_{k,n}; \mathbb{Z}_2)$

In this section $n$ and $k$ are fixed integers such that $n \geq k \geq 2$. Let $G_{k,n} = G_k(\mathbb{R}^{n+k})$ be the Grassmann manifold of $k$-dimensional subspaces in $\mathbb{R}^{n+k}$. Let $\gamma_k$ be the canonical vector bundle over $G_{k,n}$ and $w_1, w_2, \ldots, w_k$ its Stiefel-Whitney classes. It is a direct consequence of Borel’s result (\cite{2}) that the mod 2 cohomology algebra of $G_{k,n}$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ modulo the ideal $I_{k,n}$ generated by the dual classes $\overline{w}_{n+1}, \overline{w}_{n+2}, \ldots, \overline{w}_{n+k}$. The following equality holds for these dual classes:

$$(1 + w_1 + w_2 + \cdots + w_k)(1 + \overline{w}_1 + \overline{w}_2 + \cdots) = 1,$$

and therefore, they satisfy the recurrence relation

$$(2.1) \quad \overline{w}_{r+k} = \sum_{i=1}^{k} w_i \overline{w}_{r+k-i}, \quad r \geq 1.$$ 

Also, it is not hard to verify that the explicit formula for $\overline{w}_r$ $(r \geq 1)$ is the following (see \cite{13} p. 3):

$$(2.2) \quad \overline{w}_r = \sum_{a_1+2a_2+\cdots+ka_k = r} [a_1, a_2, \ldots, a_k] w_1^{a_1} w_2^{a_2} \cdots w_k^{a_k},$$

where $a_1, a_2, \ldots, a_k$ are understood to be nonnegative integers and $[a_1, a_2, \ldots, a_k]$ denotes the multinomial coefficient

$$[a_1, a_2, \ldots, a_k] = \begin{pmatrix} a_1 + a_2 + \cdots + a_k \\ a_1 \end{pmatrix} \begin{pmatrix} a_2 + \cdots + a_k \\ a_2 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} + a_k \\ a_{k-1} \end{pmatrix} = \prod_{t=2}^{k} \binom{\sum_{j=t-1}^{k} a_j}{a_{t-1}}.$$ 

On the other hand, in \cite{8} Jaworowski detected an additive basis for $H^*(G_{k,n}; \mathbb{Z}_2)$. Let us conclude this brief opening section by stating his result.

**Theorem 2.1** (\cite{8}). The set $B = \{ w_1^{a_1} w_2^{a_2} \cdots w_k^{a_k} \mid a_1 + a_2 + \cdots + a_k \leq n \}$ is a vector space basis for $H^*(G_{k,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{k,n}$.

3. Gröbner bases

As usual, $\mathbb{Z}$ denotes the set of all integers. Recall that for $\alpha, \beta \in \mathbb{Z}$ the binomial coefficient $\binom{\alpha}{\beta}$ is defined by

$$\binom{\alpha}{\beta} := \left\{ \begin{array}{ll} \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}, & \beta > 0 \\ 1, & \beta = 0 \\ 0, & \beta < 0 \end{array} \right.,$$

and therefore, the following lemma is straightforward.
Lemma 3.1. If \((\binom{\alpha}{\beta}) \neq 0\), then \(\alpha \geq \beta\) or \(\alpha \leq -1\).

Recall also the well-known formula (which holds for all \(\alpha, \beta \in \mathbb{Z}\))
\[
\binom{\alpha}{\beta} = \left(\frac{\alpha - 1}{\beta} \right) + \left(\frac{\alpha - 1}{\beta - 1} \right).
\]

Let us now introduce some notations that we are going to use throughout this
section. For an integer \(m \geq 2\) and an \(m\)-tuple \(N\) of integers we define the following
\(m\)-tuples obtained from \(N\) (for \(i \leq m\) and \(i < j \leq m\):
\[
\begin{align*}
N^i & \text{ by adding 1 to the } i\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N^i := N); \\
N_i & \text{ by subtracting 1 from the } i\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N_i := N); \\
N^{i,j} & \text{ by adding 1 to the } i\text{-th and } j\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N^{i,j} := N^{j,i}); \\
N^{i,i} & \text{ by adding 2 to the } i\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N^{i,i} := N); \\
N_{i,j} & \text{ by subtracting 1 from the } i\text{-th and } j\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N_{i,j} := N_{j,i}); \\
N_{i,i} & \text{ by subtracting 2 from the } i\text{-th coordinate of } N \text{ (if } i < 1\text{, then } N_{i,i} := N).
\end{align*}
\]

For an integer \(k \geq 2\), a \(k\)-tuple \(A = (a_1, a_2, \ldots, a_k)\) and a \((k-1)\)-tuple \(M = (m_2, m_3, \ldots, m_k)\) of integers, let:
\[
\begin{align*}
S_A & := \sum_{j=1}^k a_j, S'_A := \sum_{j=1}^k j a_j, \text{ and } S_M := \sum_{j=2}^k m_j, S'_M := \sum_{j=2}^k (j-1)m_j; \\
P_t(A, M) & := \left(\sum_{j=t}^k a_j - \sum_{j=t}^k t m_j\right), \text{ for } t = 2, k; \\
P(A, M) & := \prod_{t=2}^k P_t(A, M).
\end{align*}
\]

For example, \(P_2(A, M) = \left(S_A - S_M \atop a_1\right)\). Also, \(P(A, 0) = [a_1, a_2, \ldots, a_k]\), where \(0 = (0, 0, \ldots, 0)\).

Henceforth, the integers \(k\) and \(n\) with the property \(n \geq k \geq 2\) are fixed. Observe
the polynomial algebra \(\mathbb{Z}_2[w_1, w_2, \ldots, w_k]\). Let us now define certain polynomials
in \(\mathbb{Z}_2[w_1, w_2, \ldots, w_k]\) which will be important in our considerations.

Definition 3.2. For a \((k-1)\)-tuple of nonnegative integers \(M = (m_2, \ldots, m_k)\), let
\[
g_M := \sum_{S_A' = n+1 + S'_M} P(A, M) \cdot W^A,
\]
where the sum is taken over all \(k\)-tuples of nonnegative integers \(A = (a_1, a_2, \ldots, a_k)\)
such that \(S'_A = n + 1 + S'_M\), and \(W^A = w_1^{a_1} w_2^{a_2} \cdots w_k^{a_k}\).

Moreover, let
\[
G := \{g_M \mid S_M \leq n + 1\}.
\]

Note that, by \([3, 2]\), \(\omega_{n+1} = g_0 \in G\).

Our aim is to prove that \(G\) is a Gröbner basis for \(I_{k,n} = (\omega_{n+1}, \ldots, \omega_{n+k})\)
which determines the cohomology algebra \(H^*(G_{k,n}; \mathbb{Z}_2)\). In order to do so, first
we need to specify a term ordering in \(\mathbb{Z}_2[w_1, w_2, \ldots, w_k]\). We shall use the grelex
ordering (which will be denoted by \(\preceq\)) on terms (monomials) in \(\mathbb{Z}_2[w_1, w_2, \ldots, w_k]\).
with $w_1 > w_2 > \cdots > w_k$. It is defined as follows. The terms are compared by
the sum of the exponents and if these are equal for two terms, they are compared
lexicographically from the left. That is, for $k$-tuples $A$ and $B$ of nonnegative integers
we shall write $W^A < W^B$ if either $S_A < S_B$ or else $S_A = S_B$ and $a_s < b_s$ where
$s = \min\{i \mid a_i \neq b_i\}$. Of course, $W^A \preceq W^B$ means that either $W^A < W^B$ or
$W^A = W^B$.

In fact, we are going to prove that $G$ is the reduced Gröbner basis for $I_{k,n}$ with
respect to the grlex ordering $\preceq$. We start with a lemma.

**Lemma 3.3.** If a $k$-tuple $A = (a_1, \ldots, a_k)$ and a $(k-1)$-tuple $M = (m_2, \ldots, m_k)$ of
nonnegative integers are such that $P(A, M) \neq 0$, then $S_A < S_M$ or else $\sum_{j=1}^{k} a_j \geq
\sum_{j=t}^{k} m_j$ for all $t = 2, k$.

**Proof.** Let us assume that $S_A \geq S_M$. We will prove by induction on $t$ that
$\sum_{j=1}^{k} a_j \geq \sum_{j=t}^{k} m_j$, for $t = 2, k$. Since $(S_A - S_M) = P_2(A, M) \neq 0$ and $S_A - S_M \geq 0$, by
Lemma 3.1 we have that $S_A - S_M \geq a_1$, and therefore $\sum_{j=2}^{k} a_j \geq \sum_{j=2}^{k} m_j$.

Suppose now that $\sum_{j=1}^{k} a_j \geq \sum_{j=t}^{k} m_j$ for some $t$ such that $2 \leq t \leq k - 1$. Since
$P_{t+1}(A, M) \neq 0$ and $\sum_{j=1}^{k} a_j \geq \sum_{j=t}^{k} m_j \geq \sum_{j=t+1}^{k} m_j$, again by Lemma 3.1 we
conclude that $\sum_{j=1}^{k} a_j - \sum_{j=t+1}^{k} m_j \geq a_t$. Hence, $\sum_{j=t+1}^{k} a_j \geq \sum_{j=t+1}^{k} m_j$. $\Box$

For a nonzero polynomial $f = \sum_{i=1}^{r} t_i \in \mathbb{Z}[w_1, w_2, \ldots, w_k]$, where $t_i$ are pairwise
different terms, let $T(f) := \{t_1, t_2, \ldots, t_r\}$ ($T(0) := \emptyset$). The leading term of $f \neq 0$, denoted by $LT(f)$, is defined as $\max T(f)$ with respect to $\preceq$.

**Proposition 3.4.** Let $M = (m_2, \ldots, m_k)$ be a $(k-1)$-tuple of nonnegative integers
such that $S_M \leq n + 1$ (i.e., such that $g_M \in G$). Then $g_M \neq 0$ and $LT(g_M) = W^M$, where
$W^M = (n + 1 - S_M, m_2, \ldots, m_k)$. Moreover, if $W^A \in T(g_M) \setminus \{W^M\}$ for some
$k$-tuple $A$ of nonnegative integers, then $S_A < n + 1$.

**Proof.** If we define $m_1 := n + 1 - S_M$, then obviously $P_t(M, M) = \binom{m_t - 1}{m_t - 1} = 1$, for
$t = 2, k$, and therefore $P(M, M) = 1$. Furthermore,

$$S_M^t = \sum_{j=1}^{k} jm_j = n + 1 - S_M + \sum_{j=2}^{k} jm_j = n + 1 + \sum_{j=2}^{k} (j - 1)m_j = n + 1 + S_M^t,$$

and hence $W^M \in T(g_M)$. So, $g_M \neq 0$.

Now take a $k$-tuple $A = (a_1, \ldots, a_k)$ of nonnegative integers such that $S_A' = n + 1 + S_M^t$ and $P(A, M) \equiv 1 \pmod{2}$, i.e., $W^A \in T(g_M)$. Since $S_M^t = n + 1$, in order to finish the proof of the proposition, it suffices to show that if $S_A \geq n + 1$, then $A = M$.

Since $S_M \leq n + 1 \leq S_A$, by Lemma 3.3 we have the following $k - 1$ inequalities:

$$a_k \geq m_k,$$

$$a_{k-1} + a_k \geq m_{k-1} + m_k,$$

$$\vdots$$

$$a_2 + \cdots + a_k \geq m_2 + \cdots + m_k.$$

(3.2)
Summing up these inequalities, we get
\[ \sum_{j=2}^{k} (j-1)a_j \geq \sum_{j=2}^{k} (j-1)m_j. \]

On the other hand, since \( S_A \geq n + 1 \) and \( S_A' = n + 1 + S_M' \),
\[ \sum_{j=2}^{k} (j-1)a_j = \sum_{j=2}^{k} (j-1)a_j = S_A' - S_A \leq S_A' - (n + 1) = S_M' = \sum_{j=2}^{k} (j-1)m_j, \]
so all the inequalities in (3.2) are in fact equalities and \( S_A = n + 1 \). Hence, \( a_t = m_t \)
for \( t = 2, k \), and \( a_1 = S_A - \sum_{j=2}^{k} a_j = n + 1 - S_M \), i.e., \( A = M \).

Prior to the formulation of the following lemma, we would like to emphasize that
for a \( (k-1) \)-tuple \( M = (m_2, \ldots, m_k) \), by our definition, \( M^i = (m_2, \ldots, m_{i+1} + 1, \ldots, m_k) \), \( i = 1, k-1 \), and likewise for \( M^{ij}, M^{ij}, M_i \), etc. For example, the
\( (k-1) \)-tuple \( M^2 \) is defined as \( (m_2, m_3+1, \ldots, m_k) \), and not as \( (m_2+1, m_3, \ldots, m_k) \).

**Lemma 3.5.** Let \( A = (a_1, a_2, \ldots, a_k) \) be a \( k \)-tuple and \( M = (m_2, \ldots, m_k) \) a \( (k-1) \)-
tuple of integers.

(a) For \( 1 \leq i \leq j \leq k-2 \),
\[ P(A, M^{ij}) \equiv P(A, M^j) + P(A, M^{i-1,j+1}) + P(A_{j+1}, M^{i-1}) \pmod{2}. \]

(b) For \( 1 \leq i \leq k-1 \),
\[ P(A, M^{ik}) \equiv P(A_i, M^{i-1}) + P(A_k, M^{i-1}) \pmod{2}. \]

**Proof.** Let \( 1 \leq i \leq j \leq k-1 \). It is immediate from the definition that for all \( t = \frac{2}{k} \),
\[ P_t(A, M^{ij}) = \begin{pmatrix} a_{t-1} + a_t + \cdots + a_k - m_t - \cdots - m_k - \delta_t \\ a_{t-1} \end{pmatrix}, \]
where \( \delta_t = \begin{cases} 2, & t \leq i + 1 \\ 1, & i + 2 \leq t \leq j + 1 \\ 0, & t > j + 1 \end{cases} \).

Also, if \( t \neq i + 1 \), then
\[ P_t(A_i, M^j) = \begin{pmatrix} a_{t-1} + a_t + \cdots + a_k - m_t - \cdots - m_k - \delta_t \\ a_{t-1} \end{pmatrix}, \]
and so,
\[ (3.3) \quad P_t(A, M^{ij}) = P_t(A_i, M^j), \quad \text{for } t \neq i + 1. \]

Likewise, using formula (5.1) we get
\[ (3.4) \quad P_{i+1}(A, M^{ij}) + P_{i+1}(A_i, M^j) = P_{i+1}(A, M^j), \]
since the left-hand side is equal to
\[ \frac{a_i + \cdots + a_k - m_{i+1} - \cdots - m_k - 2}{a_i} + \left( \frac{a_i + \cdots + a_k - m_{i+1} - \cdots - m_k - 2}{a_i - 1} \right) \]
and the right-hand side to
\[ \frac{a_i + \cdots + a_k - m_{i+1} - \cdots - m_k - 1}{a_i}. \]
(a) In this case, similarly as for (3.3) and (3.4), one obtains the following equalities:

\[(3.5) \quad P_t(A_i, M^j) = P_t(A, M^{i-1,j+1}), \quad \text{for } t \not\in \{i + 1, j + 2\}\]

\[(3.6) \quad P_t(A, M^{i-1,j+1}) = P_t(A_{j+1}, M^{i-1}), \quad \text{for } t \neq j + 2\]

\[(3.7) \quad P_{i+1}(A, M^j) = P_{i+1}(A, M^{i-1,j+1})\]

\[(3.8) \quad P_{j+2}(A_i, M^j) = P_{j+2}(A, M^{i-1,j+1}) + P_{j+2}(A_{j+1}, M^{i-1}).\]

So, using identities (3.3)–(3.8), we have

\[P(A, M^{i-j}) = \prod_{t=2}^{k} P_t(A, M^{i-j}) = P_{i+1}(A, M^{i-j}) \cdot \prod_{t=2}^{k} P_t(A_i, M^j)\]

\[\equiv \left( P_{i+1}(A_i, M^j) + P_{i+1}(A, M^j) \right) \cdot \prod_{t=2}^{k} P_t(A_i, M^j)\]

\[= \prod_{t=2}^{k} P_t(A_i, M^j) + P_{i+1}(A, M^{i-1,j+1}) \cdot \prod_{t=2}^{k} P_t(A_i, M^j)\]

\[= P(A_i, M^j) + P_{j+2}(A_i, M^j) \cdot \prod_{t=2}^{k} P_t(A, M^{i-1,j+1})\]

\[= P(A_i, M^j) + \left( P_{j+2}(A, M^{i-1,j+1}) + P_{j+2}(A_{j+1}, M^{i-1}) \right) \prod_{t=2}^{k} P_t(A, M^{i-1,j+1})\]

\[= P(A_i, M^j) + P(A, M^{i-1,j+1}) + P(A_{j+1}, M^{i-1}) \pmod{2}.\]

(b) In a similar manner as before, for \(1 \leq i \leq k-1\) one can obtain two additional equalities:

\[(3.9) \quad P_t(A_i, M^{k^1}) = P_t(A_k, M^{i-1}), \quad \text{for } t \neq i + 1,\]

\[(3.10) \quad P_{i+1}(A, M^{k-1}) = P_{i+1}(A_k, M^{i-1}).\]

Now, using identities (3.3)–(3.4) and (3.9)–(3.10), we have

\[P(A, M^{i,k-1}) = \prod_{t=2}^{k} P_t(A, M^{i,k-1}) = P_{i+1}(A, M^{i,k-1}) \cdot \prod_{t=2}^{k} P_t(A_i, M^{k-1})\]

\[\equiv \left( P_{i+1}(A_i, M^{k-1}) + P_{i+1}(A, M^{k-1}) \right) \cdot \prod_{t=2}^{k} P_t(A_i, M^{k-1})\]

\[= P(A_i, M^{k-1}) + P(A_k, M^{i-1}) \pmod{2},\]

and we are done. \(\square\)

Note that we could unify parts (a) and (b) of the previous lemma by stating that

\[P(A, M^{i-j}) \equiv P(A_i, M^j) + P(A_{j+1}, M^{i-1}) + P(A, M^{i-1,j+1}) \pmod{2},\]

for \(1 \leq i \leq j \leq k-1\), with convention that \(P(A, M^{i-1,j+1}) = 0\) if \(j = k - 1\).
Proposition 3.6. Let \( M = (m_2, \ldots, m_k) \) be a \((k-1)\)-tuple of nonnegative integers and \( 1 \leq i \leq j \leq k - 1 \). Then in the polynomial algebra \( \mathbb{Z}[w_1, w_2, \ldots, w_k] \), we have the identity

\[
g_{M^{i,j}} = w_i g_{M^j} + w_{j+1} g_{M^{i-1}} + g_{M^{i-1,j+1}},
\]

where the polynomial \( g_{M^{i-1,j+1}} \) is understood to be zero if \( j = k - 1 \).

Proof. By Lemma 3.5 we have

\[
g_{M^{i,j}} = \sum_{S'_{M^{i,j}} = n + 1 + S'_{M^{i,j}}} P(A, M^{i,j}) \cdot W^A = \sum_{S'_{M^{i,j}} = n + 1 + S'_{M^{i,j}}} (P(A_i, M^j) + P(A_{j+1}, M^{i-1}) + P(A, M^{i-1,j+1})) \cdot W^A
\]

\[
= \sum_{S'_{M^{i,j}} = n + 1 + S'_{M^{i,j}}} P(A_i, M^j) \cdot W^A + \sum_{S'_{M^{i,j}} = n + 1 + S'_{M^{i,j}}} P(A_{j+1}, M^{i-1}) \cdot W^A + g_{M^{i-1,j+1}},
\]

since \( S'_{M^{i,j}} = S'_M + i + j = S'_M + i - 1 + j + 1 = S'_{M^{i-1,j+1}} \) (for \( j \leq k - 2 \)).

Observe also that the equality \( S'_{A_i} = S'_A - i = n + 1 + S'_{M^{i,j}} - i = n + 1 + S'_M + j = n + 1 + S'_{M^{i,j}} \), and likewise, it is equivalent to \( S'_{A_{j+1}} = n + 1 + S'_{M^{i-1}}, \)

Now, consider the first sum in the upper expression. Since the sum is taken over the \( k \)-tuples \( A = (a_1, a_2, \ldots, a_k) \) of nonnegative integers (such that \( S'_{A_i} = n + 1 + S'_{M^{i,j}} \)), the coordinates of \( A_i \) are also nonnegative with exception that its \( i \)-th coordinate might be \(-1\) (if \( a_i = 0 \)). But, in that case, \( P_{i+1}(A_i, M^j) = (a_i + \cdots + a_k - m_{i+1} - \cdots - m_k - 2) = 0 \), and so \( P(A_i, M^j) = 0 \). Therefore, we may assume that \( a_i \geq 1 \), and consequently, that \( A_i \) runs through the set of \( k \)-tuples of nonnegative integers (such that \( S'_{A_i} = n + 1 + S'_{M^{i,j}} \)). Hence,

\[
\sum_{S'_{A_i} = n + 1 + S'_{M^{i,j}}} P(A_i, M^j) \cdot W^A = w_i \sum_{S'_{A_i} = n + 1 + S'_{M^{i,j}}} P(A_i, M^j) \cdot W^{A_i} = w_i g_{M^j}.
\]

So, we are left to prove that the second sum in the upper expression for \( g_{M^{i,j}} \) is equal to \( w_{j+1} g_{M^{i-1}} \). Let \( A = (a_1, a_2, \ldots, a_k) \) be a \( k \)-tuple of nonnegative integers such that \( S'_{A_i} = n + 1 + S'_{M^{i,j}}, \) i.e., \( S'_{A_{j+1}} = n + 1 + S'_{M^{i-1}} \). It suffices to show that \( a_{j+1} = 0 \) implies \( P(A_{j+1}, M^{i-1}) = 0 \), since then the proof follows as for the first sum.

If \( j + 1 < k \), then \( a_{j+1} = 0 \) implies \( P_{j+2}(A_{j+1}, M^{i-1}) = 0 \), and so, \( P(A_{j+1}, M^{i-1}) = 0 \).

For \( j = k - 1 \), let us assume to the contrary that \( a_k = 0 \) and \( P(A_k, M^{i-1}) \neq 0 \). First we shall prove that

\[
(3.11) \quad a_{t-1} + a_t + \cdots + a_{k-1} \leq m_t + \cdots + m_k + \varepsilon_t, \quad \text{for all } t = \frac{2}{2}, k,
\]

where \( \varepsilon_t = \begin{cases} 1, & 2 \leq t \leq i \\ 0, & i + 1 \leq t \leq k \end{cases} \). The proof is by reverse induction on \( t \). For the induction base we prove \((3.11)\) for \( t = k \). Since \( a_{k-1} - 1 - m_k \neq 0 \) and \( a_{k-2} - 1 - m_k < a_{k-1} - 1 - m_k \leq -1 \), by Lemma 3.1 we conclude that \( a_{k-1} - 1 - m_k \leq -1 \), so \( a_{k-1} \leq m_k = m_k + \varepsilon_k \). For the inductive step, let \( 2 \leq t \leq k - 1 \), and suppose that
\[ a_t + \cdots + a_{k-1} \leq m_{t+1} + \cdots + m_k + \varepsilon_{t+1}. \] Since obviously \( \varepsilon_{t+1} \leq \varepsilon_t \), we actually have that \( a_t + \cdots + a_{k-1} \leq m_{t+1} + \cdots + m_k + \varepsilon_t \). Since

\[
P_t(A_k, M^{i-1}) = \left( a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t \right) a_{t-1} \neq 0,
\]

and \( a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t \leq a_{t-1} - 1 - m_t < a_{t-1} \), according to Lemma 3.1 we have that \( a_{t-1} + a_t + \cdots + a_{k-1} - 1 - m_t - m_{t+1} - \cdots - m_k - \varepsilon_t \leq -1 \), i.e., \( a_{t-1} + a_t + \cdots + a_{k-1} \leq m_t + m_{t+1} + \cdots + m_k + \varepsilon_t \).

Now, summing up inequalities (3.11), we get

\[
S'_A \leq S'_M + \sum_{i=2}^{k} \varepsilon_i < S'_M + k - 1 < S'_M + n + 1 \leq S'_M + n + 1 = S'_{A_k},
\]

which is a contradiction since \( S'_A > S'_A - k = S'_{A_k} \). \qed

In the following lemma we establish a connection between polynomials \( g_M \) and polynomials (dual classes) \( \overline{w}_r \in \mathbb{Z}_2[w_1, w_2, \ldots, w_k] \) from the previous section.

**Lemma 3.7.** For \( m \geq 0 \) and \( (k-1) \)-tuple \( M = (m, 0, \ldots, 0) \) we have that

\[
g_M = \sum_{i=0}^{m} \binom{m}{i} \overline{w}_{1}^{m-i} \overline{w}_{n+1+i}.
\]

**Proof.** The polynomials \( g_M \) were introduced in Definition 3.2 and they depend on the (previously fixed) integer \( n \). In this proof (and only in this proof) we allow \( n \) to vary through the set \( \{k, k+1, \ldots\} \), while the integer \( k \geq 2 \) is still fixed (we are working in the polynomial algebra \( \mathbb{Z}_2[w_1, w_2, \ldots, w_k] \)). Note that the polynomials \( \overline{w}_r, r \geq 1 \), are defined independently of \( n \). We emphasize the dependence of \( g_M \) on \( n \) by using an appropriate superscript, and we actually prove the following claim:

\[
g_M^{(n)} = \sum_{i=0}^{m} \binom{m}{i} \overline{w}_{1}^{m-i} \overline{w}_{n+1+i}, \quad \text{for all } m \geq 0 \text{ and all } n \geq k.
\]

The proof is by induction on \( m \). We have already noticed that \( g_0^{(n)} = \overline{w}_{n+1} \), and therefore, the claim is true for \( m = 0 \) (and all \( n \geq k \)). So, let \( m \geq 1 \) and assume that the claim is true for the integer \( m-1 \) and all \( n \geq k \). Let \( M = (m, 0, \ldots, 0) \) and \( n \geq k \). Then \( M_1 = (m-1, 0, \ldots, 0) \) and since, for all \( k \)-tuples \( A \) of integers, \( P_2(A, M) \equiv P_2(A_1, M_1) + P_2(A, M_1) \) (mod 2) by (3.1) and \( P_t(A, M) = P_t(A_1, M_1) = P_t(A, M_1) \)
for \( t = 3, k \), we have that
\[
\begin{align*}
g_M^{(n)} &= \sum_{S' = n+1+S'} P(A, M) \cdot W^A \\
&= \sum_{S' = n+1+S'} \left( (P_2(A_1, M_1) + P_2(A, M_1)) \cdot \prod_{i=3}^k P_i(A, M) \right) \cdot W^A \\
&= w_1 \sum_{S' = n+1+S'} P(A_1, M_1) \cdot W^{A_1} + \sum_{S' = (n+1)+1+S'} P(A, M_1) \cdot W^A \\
&= w_1 g_{M_1}^{(n)} + g_{M_1}^{(n+1)} \\
&= w_1 \sum_{i=0}^{m-1} \binom{m-1}{i} w_1^{m-1-i} w_{n+1+i} + \sum_{i=0}^{m-1} \binom{m-1}{i} w_1^{m-1-i} w_{n+2+i} \\
&= \sum_{i=0}^{m} \binom{m-1}{i} w_1^{m-i} w_{n+1+i} \\
&= \sum_{i=0}^{m} \binom{m}{i} w_1^{m-i} w_{n+1+i},
\end{align*}
\]
and the proof is completed.

Proposition 3.8. \( G \subseteq I_{k,n} \).

Proof. Since the ideal \( I_{k,n} \) is generated by the polynomials \( \overline{w}_{n+1}, \overline{w}_{n+2}, \ldots, \overline{w}_{n+k} \), note that, by the recurrence relation (2.1), not only these \( k \) polynomials, but all \( \overline{w}_r \) for \( r \geq n+1 \) belong to \( I_{k,n} \). Likewise, we shall prove that \( g_M \in I_{k,n} \) for all \((k-1)\)-tuples \( M \) of nonnegative integers, and not only for those with the property \( S_M \leq n+1 \) (i.e., \( g_M \in G \)).

We define the relation \( \prec_{\text{lexr}} \) on the set of all \((k-1)\)-tuples of nonnegative integers by
\[
(a_1, a_2, \ldots, a_{k-1}) \prec_{\text{lexr}} (b_1, b_2, \ldots, b_{k-1}) \iff a_s < b_s, \text{ where } s = \max \{ i \mid a_i \neq b_i \},
\]
which is exactly the strict part of the lexicographical right ordering. This is a well ordering and our proof is by induction on \( \prec_{\text{lexr}} \).

For the \((k-1)\)-tuple \( M = (m, 0, \ldots, 0) \), where \( m \geq 0 \) is arbitrary integer, from Lemma 3.7 and our remark at the beginning of this proof, we immediately get that \( g_M \in I_{k,n} \). So, let us now take a \((k-1)\)-tuple \( M = (m_2, m_3, \ldots, m_k) \) such that the greatest integer \( s \) with the property \( m_{s+1} > 0 \) is at least 2. Hence, \( 2 \leq s \leq k-1 \) and \( M = (m_2, \ldots, m_{s+1}, 0, \ldots, 0) \). Let us also assume that \( g_{M'} \in I_{k,n} \) for all \( M' \) such that \( M' \prec_{\text{lexr}} M \). We wish to prove that \( g_M \in I_{k,n} \). By Proposition 3.8 applied to the \((k-1)\)-tuple \( M_s, i = 1 \) and \( j = s - 1 \),
\[
g_M = g_{M', s} + w_1 g_{M', s-1} + w_s g_{M'},
\]
Since \( M_s \prec_{\text{lexr}} M_{s-1} \prec_{\text{lexr}} M_{s-2} \prec_{\text{lexr}} M \), we conclude that \( g_M \in I_{k,n} \). \( \square \)

In the following proposition we formulate a characterization of Gröbner bases which we shall use for the proof that the set \( G \) is a Gröbner basis for the ideal \( I_{k,n} \). The proof of the proposition can be found in [1 Proposition 5.38(vi)].
Proposition 3.9. Let $\mathbb{F}$ be a field, $\mathbb{F}[x_1, x_2, \ldots, x_k]$ the polynomial algebra, and $I$ an ideal in $\mathbb{F}[x_1, x_2, \ldots, x_k]$. Suppose that a term ordering $\preceq$ in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ is fixed. Let $G$ be a finite subset of $I$ such that $0 \notin G$, and let $B \subseteq \mathbb{F}[x_1, x_2, \ldots, x_k]/I$ be the set of cosets of all terms which are not divisible by any of the leading terms $LT(g)$, $g \in G$. Then $G$ is a Gröbner basis for $I$ with respect to $\preceq$ if and only if $B$ is a vector space basis for $\mathbb{F}[x_1, x_2, \ldots, x_k]/I$.

We are now finally in position to prove our main result.

**Theorem 3.10.** The set $G$ (see Definition 3.2) is the reduced Gröbner basis for the ideal $I_{k,n}$ with respect to the grex ordering $\preceq$.

**Proof.** By Propositions 3.4 and 3.8 $0 \notin G \subseteq I_{k,n}$, and it is obvious from the definition that $G$ is finite. Also, according to Proposition 3.4 again, the set $\{LT(g) \mid g \in G\}$ is exactly the set of all terms in $\mathbb{Z}[w_1, w_2, \ldots, w_k]$ with the sum of the exponents equal to $n+1$, that is $\{LT(g) \mid g \in G\} = \{W^A \mid S_A = n+1\}$. Therefore, the set of all terms which are not divisible by any of the terms in $\{LT(g) \mid g \in G\}$ is just the set $\{W^A \mid S_A \leq n\}$. By Proposition 3.9 again, and Theorem 3.10, $G$ is a Gröbner basis for $I_{k,n}$.

Since $\{LT(g) \mid g \in G\} = \{W^A \mid S_A = n+1\}$ and all terms of $g \in G$ except the leading one have the sum of the exponents at most $n$ (Proposition 3.4), no term of $g$ is divisible by any other leading term in $G$. This means that Gröbner basis $G$ is the reduced one.

Propositions 3.4 and 3.6 enable us to explicitly determine the polynomials $g_M \in G$ for the $(k-1)$-tuples $M = (m_2, m_3, \ldots, m_k)$ such that $m_k$ is close to $n$. Namely, if $g_M \in G$ and $W^A \in T(g_M) \setminus \{W^M\}$ (where $M = (n-1-S_M, m_2, \ldots, m_k)$), then $S_A \leq n$ by Proposition 3.4. Consequently, $S_A' = \sum_{j=1}^{k} j a_j \leq k \sum_{j=1}^{k} a_j = kS_A \leq kn$. On the other hand, $S_A' = n+1+S_M'$, and so, we conclude that $g_M = W^M$ whenever $S_A' > (k-1)n-1$.

Let $N$ be the $(k-1)$-tuple $(0, \ldots, 0, n)$. Since $S_N > S_N = (k-1)n$ (for $s = \frac{1}{k-1}$, by the previous remark we have that

$$g_N = w_1 w_k^n \quad \text{and} \quad g_N = w_{s+1} w_k^n, \quad 1 \leq s \leq k-1.$$  

If we apply Proposition 3.6 to the $(k-1)$-tuple $N_{k-1} = (0, \ldots, 0, n-1)$, $i = 1$ and $j = k-1$, we obtain the relation $w_k g_{N_{k-1}} = g_N + w_1 g_N$. Both summands on the right-hand side contain $w_k$ as a factor, so $w_k$ cancels out and using (3.12) we get

$$g_{N_{k-1}} = w_1 w_k^{n-1} + w_2 w_k^{n-1}.$$  

Likewise, by applying Proposition 3.6 to $N_{k-1}$, $i = s+1$ and $j = k-1$, one obtains that $w_k g_{N_{k-1}} = w_{s+1} g_N + g_{N_{s+1}}$, and so

$$g_{N_{k-1}} = w_1 w_{s+1} w_k^{n-1} + w_{s+2} w_k^{n-1}, \quad 1 \leq s \leq k-2.$$  

Identities (3.13) and (3.14) determine $g_M \in G$ when $m_k = n-1$ and $S_M \leq n$. For computing $g_M \in G$ when $m_k = n-1$ and $S_M = n+1$ for a concrete integer $k$, one can use Proposition 3.6 and apply it first to $N_{k-1}$, $i = 1$ and all $j = 1, k-2$, then to $N_{k-1}$, $i = 2$ and all $j = 2, k-2$ and so on. After that, the polynomials $g_M \in G$ for $m_k = n-2$ can be obtained in the same manner – by suitable applications of Proposition 3.6. Actually, in the cases $k = 2$ and $k = 3$, all the members $g_M$ of Gröbner basis $G$ for $m_k \geq n-5$ were listed in [12] (for $k = 2$) and [13] (for $k = 3$).
Since our application of Gröbner bases (given in the next section) treats the case \( k = 5 \), let us write down the relations (3.13) and (3.14) in this case:

\[
g_{(0,0,0,n-1)} = w_1^2 w_5^{n-1} + w_2 w_5^{n-1},
\]

\[
g_{(1,0,0,n-1)} = w_1 w_2 w_5^{n-1} + w_3 w_5^{n-1},
\]

\[
g_{(0,1,0,n-1)} = w_1 w_3 w_5^{n-1} + w_4 w_5^{n-1},
\]

\[
g_{(0,0,1,n-1)} = w_1 w_4 w_5^{n-1} + w_n^5.
\]

(3.15)

Remark 3.11. Since the description of the mod 2 cohomology of the complex Grassmann manifolds \( G_k(\mathbb{C}^{n+k}) \) is essentially the same as the one in the real case (the only difference being in the fact that dimensions of the Stiefel-Whitney classes are multiplied by 2), it is immediate that the reduced Gröbner basis for the corresponding ideal in this case can be obtained from the Gröbner basis \( G \) (Definition 3.2) by substituting \( w_{2i} \) for \( w_i \) (\( i = 1, k \)) in all polynomials \( g_M \in G \).

4. Application to immersions

In this section we consider the (real) Grassmannians \( G_{5,n} \), where \( n \) is divisible by 8. As before, \( w_i \in H^i(G_{5,n}; \mathbb{Z}_2) \), \( i = 1, 5 \), is the \( i \)-th Stiefel-Whitney class of the canonical bundle \( \gamma_5 \) over \( G_{5,n} \).

Lemma 4.1. Let \( n \equiv 0 \pmod{8} \) and let \( \nu \) be the stable normal bundle over Grassmann manifold \( G_{5,n} \). Then for the Stiefel-Whitney classes of this bundle, the following equalities hold: \( w_2(\nu) = w_1^2 + w_2 \) and \( w_i(\nu) = 0 \) when \( i \geq 5n - 4 \).

Proof. Let \( r \geq 3 \) be the integer such that \( 2^r < n + 5 \leq 2^{r+1} \). Note that this implies \( n \geq 2^r \) since \( n \equiv 0 \pmod{8} \). In [10] p. 365 Hiller and Stong proved that

\[
(4.1) \quad w(\nu) = w(\gamma_5 \otimes \gamma_5) \cdot (1 + w_1 + w_2 + w_3 + w_4 + w_5) 2^{r+1} - n - 5,
\]

and that the top nonzero class in this expression is in dimension \( 20 + 5(2^{r+1} - n - 5) \). Since \( n \geq 2^r \), we have that \( 20 + 5(2^{r+1} - n - 5) \leq 20 + 5(2^r - 5) = 5 \cdot 2^r - 5 \leq 5n - 5 \). This proves the second equality in the statement of the lemma.

For the first one, we need the fact \( w_1(\gamma_5 \otimes \gamma_5) = w_2(\gamma_5 \otimes \gamma_5) = 0 \), which is not hard to check by the method described in [10] Problem 7-C]. Using this fact and (4.1), one obtains that

\[
w_2(\nu) = \left( \frac{2^{r+1} - n - 5}{2} \right) w_1^2 + (2^{r+1} - n - 5) w_2 = w_1^2 + w_2,
\]

since \( 2^{r+1} - n - 5 \equiv 3 \pmod{8} \). \( \square \)

Theorem 4.2. If \( n \equiv 0 \pmod{8} \), then \( G_{5,n} \) immerses into \( \mathbb{R}^{10n-3} \).

Proof. Since \( \dim G_{5,n} = 5n \), in order to prove that there is an immersion of \( G_{5,n} \) into \( \mathbb{R}^{10n-3} \), by Hirsch’s theorem (7 Theorem 6.4) it suffices to show that the classifying map \( f_\nu: G_{5,n} \to BO \) of the stable normal bundle \( \nu \) over \( G_{5,n} \) lifts up to \( BO(5n - 3) \).

\[ BO(5n - 3) \]

\[ G_{5,n} \]

\[ f_\nu \]

\[ BO \]
We shall use the method of modified Postnikov towers (MPT) introduced by Gitler and Mahowald in [5] and extended to fibrations \( p : BO(m) \to BO \) for \( m \) odd by Nussbaum in [11]. So, we factor out the map \( p : BO(5n - 3) \to BO \) as indicated in the following diagram and then we lift the map \( f_\nu \) one level at the time. The diagram presents the 5n-MPT for the fibration \( p (K_m \text{ stands for the Eilenberg-MacLane space } K(\mathbb{Z}_2, m)) \). Also, the relations that determine the \( k \)-invariants of the tower are listed in the table.

\[
\begin{array}{c}
BO(5n-3) \\
G_{5,n} \xrightarrow{f_\nu} BO \xrightarrow{k_0^1 \times k_2^1} K_{5n-2} \times K_{5n-1} \xrightarrow{k_2^2} K_{5n} \\
E_1 \xrightarrow{k_1^2} K_{5n} \\
E_2 \xrightarrow{k_0^2} K_{5n} \\
\end{array}
\]

\[
\begin{array}{c}
k_0^1 = w_{5n-2} \\
k_0^2 = w_{5n} \\
k_1^1 : (Sq^2 + w_2)k_0^1 = 0 \\
k_2^1 : (Sq^2 + w_2)k_1^1 + Sq^1 k_0^2 = 0 \\
k_2^2 : (Sq^2 + w_2)k_1^1 + Sq^1 k_2^1 = 0
\end{array}
\]

According to Lemma 4.1.1, \( w_{5n-2} = w_{5n} = 0 \), so, we can lift \( f_\nu \) up to \( E_1 \). Also, since

\[
Sq^1(w_4 w_5^{n-1}) = (w_1 w_4 + w_5)w_5^{n-1} + w_4(n - 1)w_1 w_5^{n-1} = nw_1 w_4 w_5^{n-1} + w_5^n = w_5^n,
\]

and since \( w_5^n \neq 0 \) in \( H^{5n}(G_{5,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) (Theorem 2.1), by looking at the relations in the table for \( k_1^1 \) and \( k_2^1 \), we see that it is easy to overcome these two \( k \)-invariants. Therefore, the only obstruction left to deal with comes from the \( k \)-invariant \( k_1^2 \). Since \( H^{5n-1}(G_{5,n}; \mathbb{Z}_2) \cong \mathbb{Z}_2 \), it suffices to show that the map \( (Sq^2 + w_2(\nu)) : H^{5n-3}(G_{5,n}; \mathbb{Z}_2) \to H^{5n-1}(G_{5,n}; \mathbb{Z}_2) \) is nontrivial. We use Lemma 4.1.1 formulas of Wu and Cartan and the polynomials from [3,13] (which are trivial in \( H^*(G_{5,n}; \mathbb{Z}_2) \) since they are members of the Gröbner basis \( G \) for the ideal \( I_{5,n} \), and hence, they...
belong to $I_{5,n}$ to calculate
\[
(Sq^2 + w_2(\nu))(w_2w_5^{n-1}) = (Sq^2 + w_1^2 + w_2)(w_2w_5^{n-1})
\]
\[
= w_2^2w_5^{n-1} + (w_1w_2 + w_3)(n-1)w_1w_5^{n-1}
+ w_2\left((n-1)w_2w_5^{n-1} + \binom{n-1}{2}w_1^2w_5^{n-1}\right) + w_1^2w_2w_5^{n-1} + w_2^2w_5^{n-1}
= w_2^2w_5^{n-1} + w_1w_3w_5^{n-1} + w_2^2w_5^{n-1}
= w_2g(0,0,0,n-1) + g(0,1,0,n-1) + w_4w_5^{n-1}
= w_4w_5^{n-1},
\]
and this class is nonzero by Theorem 2.1.

By the famous result of Cohen [3], Grassmanian $G_{5,n}$ can be immersed into $\mathbb{R}^{10n-\alpha(5n)}$, where $\alpha(5n)$ denotes the number of ones in the binary expansion of $5n$. This means that Theorem 4.2 improves this result whenever $\alpha(5n) = 2$ (and $n \equiv 0 \mod 8$). Such a case occurs when $n$ is a power of two, and it is known that then $G_{5,n}$ cannot be immersed into $\mathbb{R}^{10n-6}$ ([3, p. 365]). So, if $n$ is a power of two, then for $\text{imm}(G_{5,n}) = \min\{d \mid G_{5,n} \text{ immerses into } \mathbb{R}^d\}$ the following inequalities hold
\[
10n - 5 \leq \text{imm}(G_{5,n}) \leq 10n - 3.
\]
Actually, a sufficient and necessary condition for $\alpha(5n) = 2$ and $n \equiv 0 \mod 8$ is that $n$ is of the form $2^r + \sum_{i=0}^{s} (2^{r+2i+4i} + 2^{r+3i+4i})$, $r \geq 3$, $s \geq -1$ (where the case $s = -1$ corresponds to the case $n = 2^r$).

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