Study of control problems for the stationary MHD equations

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Abstract. The optimal control problems for the stationary magnetohydrodynamic equations under inhomogeneous mixed boundary conditions for a magnetic field are considered. The role of control in control $x$s under study is played by normal component of the magnetic field on the part of the boundary. In the capacity of cost functionals quadratic tracking–type functionals for a velocity, magnetic field or pressure are taken.

1. Introduction. Statement of the boundary value problem
Great attention has been paid recently to the optimal control problems for the models of magnetohydrodynamics (MHD) for a viscous conducting incompressible fluid (see [1–6]).

In this paper we study control problems for the new MHD model in which stationary equations of magnetohydrodynamic are considered under mixed boundary conditions for a magnetic field and under Dirichlet boundary condition for a velocity with tracking–type functionals. The role of controls in control problems under study is played by a normal component of the magnetic field on the part of the boundary.

One of the features of this work consist in proving of the solvability of optimal control problem under minimum requirements on a normal component of the magnetic field given on the part of boundary and playing the role of control. This result, in contrast to [3], allows to use a simple $L^2$-norm for the normal component of the magnetic field instead of the $H^{1/2}$-norm on the part of the boundary by Tikhonov regularization of considered incorrect control problem. This regularization is needed to prove stability and uniqueness of the control problem’s solution.

Let $\Omega$ be a bounded domain of space $\mathbb{R}^3$ with boundary $\partial \Omega$ consisting of two parts $\Sigma_\nu$ and $\Sigma_\tau$. In this paper we study control problems for the stationary magnetohydrodynamic equations of viscous incompressible fluid

$$-\nu \Delta u + (u \cdot \nabla)u + \nabla p - \omega \text{rot } H \times H = f, \quad \text{div } u = 0,$$

$$\nu_1 \text{rot } H - \rho_0^{-1} E + \omega H \times u = \nu_1 j, \quad \text{div } H = 0, \quad \text{rot } E = 0,$$

considering in domain $\Omega$ under following inhomogeneous boundary conditions:

$$u|_{\partial \Omega} = g, \quad H \cdot n|_{\Sigma_\nu} = q, \quad H \times n|_{\Sigma_\nu} = 0, \quad E \times n|_{\Sigma_\tau} = k.$$

Here $u$ is the velocity vector, $H$ and $E$ are magnetic and electric fields, respectively, $p = P/\rho_0$, where $P$ is the pressure, $\rho_0 = \text{const}$ is a fluid density, $\omega = \mu/\rho_0$, $\nu_1 = 1/\rho_0 \sigma = \omega \nu_m$, $\nu$ and
\( \nu_m \) are constant kinematic and magnetic viscosity coefficients, \( \sigma \) is a constant conductivity, \( \mu \) is a constant magnetic permeability, \( n \) is the outer normal to \( \partial \Omega \), \( j \) is the current density. In the remainder of the paper we will refer to problem (1)–(3) for given functions \( f, j, g, k \) and \( q \) as Problem 1. We note that all the quantities in (1)–(3) are dimensional and their physical dimensions are defined in terms of SI units. Physically the boundary conditions for the electromagnetic field in (3) correspond to the situation when the part \( \Sigma_v \) of the boundary \( \partial \Omega \) is a perfect insulator.

For the first time a global solvability of Problem 1 under homogeneous boundary conditions for a velocity and electromagnetic field was proved in [11] without the requirement \( \Sigma_r \cap \Sigma_v = \emptyset \). It is essential to use mathematical tools from [12–14] A global solvability of the inhomogeneous mixed Problem 1 was proved in [15], see also [16, 17]. The mixed boundary condition for a magnetic field in (3) generalizes previously used two types of boundary conditions see [2, 9, 18, 19].

2. Function spaces. The preliminary results

Below we will use the Sobolev spaces \( H^s(D), s \in \mathbb{R} \), \( H^0(D) \equiv L^2(D) \), where \( D \) denotes \( \Omega \) or the boundary \( \partial \Omega \). Corresponding spaces of vector–functions are denoted by \( H^s(D)^3 \) and \( L^2(D)^3 \).

The inner products and norms in the spaces \( H^s(\Omega) \) and \( H^s(\Omega)^3 \) are denoted by \( (\cdot, \cdot)_{s, \Omega} \) and \( \| \cdot \|_{s, \Omega} \). The inner products and norms in \( L^2(\Omega) \) and \( L^2(\Omega)^3 \) are denoted by \( (\cdot, \cdot) \) and \( \| \cdot \| \). By \( \| \cdot \|_{1, \Omega} \) and \( | \cdot |_{1, \Omega} \) we denote norm and seminorm in \( H^1(\Omega) \) or \( H^1(\Omega)^3 \). For arbitrary Hilbert space \( H \) by \( H^* \) we denote the dual space of \( H \). As in [13, 14] we assume that the following conditions to \( \Omega \) are satisfied:

(i) \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) and the boundary \( \partial \Omega \) is the union of a finite number of disjoint closed \( C^2 \) surfaces, each surface having finite surface area;

(ii) \( \Sigma_r \) is nonempty open subset of \( \partial \Omega \) with \( M + 1 \) disjoint nonempty open components \( \{ \sigma_0, \sigma_1, \ldots, \sigma_M \} \) and there is a positive \( d_0 \) such that dist \( d(\sigma_i, \sigma_j) \geq d_0 > 0 \) when \( i \neq j \) and \( M \geq 1 \). The boundary of each \( \sigma_i \) is either empty or \( C^{1,1} \) curve. We set \( \Sigma_v = \partial \Omega \setminus \Sigma_r \).

Let \( \mathcal{D}(\Omega) \) be the space of infinitely differentiable compactly supported functions in \( \Omega \), \( H^1_0(\Omega) \) be the closure of \( \mathcal{D}(\Omega) \) in \( H^1(\Omega) \), \( V = \{ \mathbf{v} \in H^1_0(\Omega)^3 : \text{div} \mathbf{v} = 0 \} \), \( H^{-1}(\Omega)^3 = (H^1_0(\Omega)^3)^* \), \( L^2_0(\Omega) = \{ \mathbf{p} \in L^2(\Omega) : (p, 1) = 0 \} \), \( H^1(\Omega, \Sigma_r) = \{ \varphi \in H^1(\Omega) : \varphi|_{\Sigma_r} = 0 \} \), \( C_{\Sigma_0,0}(\Omega)^3 := \{ \mathbf{v} \in C^0(\Omega)^3 : \mathbf{v} \cdot n|_{\Sigma_r} = 0, \mathbf{v} \times n|_{\Sigma_r} = \mathbf{0} \} \). In addition to the spaces introduced above we will use the spaces \( H(\text{div}, \Omega) = \{ \mathbf{v} \in L^2(\Omega)^3 : \text{div} \mathbf{v} \in L^2(\Omega) \}, H(\text{curl}, \Omega) = \{ \mathbf{v} \in L^2(\Omega)^3 : \text{curl} \mathbf{v} \in L^2(\Omega)^3 \}, H^0(\text{curl}, \Omega) = \{ \mathbf{v} \in H(\text{curl}, \Omega) : \text{curl} \mathbf{v} = \mathbf{0} \} \) and the space \( H_{DC}(\Omega) = H(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \), equipped with the Hilbert norm \( \| \mathbf{u} \|_{DC}^2 := \| \mathbf{u} \|_H^2 + \| \text{div} \mathbf{u} \|_L^2 + \| \text{curl} \mathbf{u} \|_L^2 \).

We will use the following Green’s formulae [7]

\[
\int_\Omega \mathbf{v} \cdot \text{grad} \varphi \, dx + \int_\Omega \text{div} \mathbf{v} \varphi \, dx = \int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \varphi \, d\sigma \quad \forall \mathbf{v} \in H^1(\Omega)^3, \varphi \in H^1(\Omega),
\]

\[
\int_\Omega (\mathbf{v} \cdot \text{curl} \mathbf{w} - \mathbf{w} \cdot \text{curl} \mathbf{v}) \, dx = \int_{\partial \Omega} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{w} \times \mathbf{T} \, d\sigma \quad \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3.
\]

Let \( H^{1}_{\Sigma_{0},0}(\Omega)^3 \) be the closure of the space \( C_{\Sigma_{0},0}(\Omega)^3 \cap H^{1}(\Omega)^3 \) with respect to the norm \( \| \cdot \|_{1, \Omega} \) and \( H_{DC}^{1}(\Omega) \) be the closure of \( C_{\Sigma_{0},0}(\Omega)^3 \cap H^{1}(\Omega)^3 \) with respect to the norm \( \| \cdot \|_{DC} \). Let

\[
\mathcal{H}_{\Sigma_r}(\Omega) = \{ \mathbf{h} \in L^2(\Omega)^3 : \text{div} \mathbf{h} = 0, \mathbf{h} \cdot n|_{\Sigma_r} = 0, \mathbf{h} \times n|_{\Sigma_r} = 0 \},
\]

\[
\mathcal{H}_{\Sigma_v}(\Omega) = \{ \mathbf{h} \in L^2(\Omega)^3 : \text{div} \mathbf{h} = \mathbf{0} \text{ in } \Omega, \mathbf{h} \cdot n|_{\Sigma_v} = 0, \mathbf{h} \times n|_{\Sigma_v} = 0 \},
\]

\[
\mathcal{V}_{\Sigma_{0}}(\Omega) = \{ \mathbf{v} \in H_{DC}^{1}(\Omega) : \text{div} \mathbf{v} = 0 \text{ in } \Omega \} \cap \mathcal{H}_{\Sigma_r}(\Omega)^{\perp}.
\]

A number of properties of the function spaces introduced above has been proved in [13, 14]. We formulate these properties as the following theorem.
Theorem 2.1. We assume that conditions (i), (ii) hold. Then:
1) the spaces $H_{\text{div}}(\Omega)$ and $H_{\Sigma_r}(\Omega)$ are finite dimensional;
2) $H_{DC, \Sigma_r}(\Omega) = H_{1, \Sigma_r}(\Omega)^3$ and the norm $\| \cdot \|_{DC}$ is equivalent to the norm $\| \cdot \|_{1, \Omega}$;
3) there is a constant $\delta_1$ dependent on $\Omega$ and $\Sigma_r$ such that the coercitivity inequality holds

$$\|\text{curl } h\|^2 \geq \delta_1 \|h\|^2_{1,\Omega} \quad \forall h \in V_{\Sigma_r}(\Omega);$$

(7)

4) the following orthogonal decomposition holds:

$$L^2(\Omega)^3 = \nabla H^1(\Omega, \Sigma_r) \oplus \text{curl } H_{DC, \Sigma_r}(\Omega) \oplus H_{\Sigma_r}(\Omega)$$

(8)

and $\text{curl } H_{DC, \Sigma_r}(\Omega) \equiv \text{curl } V_{\Sigma_r}(\Omega)$.

Along with spaces $H_{DC}(\Omega)$ and $H^0(\text{curl}, \Omega)$ we will use their subspaces

$$H^{1/2}_{\text{div}}(\Omega) := H^{1/2}(\Omega)^3 \cap \{ h \in H_{DC}(\Omega) : \text{div } h = 0, \ h \times n = 0 \text{ on } \Sigma_r \} \cap H_{\Sigma_r}(\Omega)^1,$$

$$H^0_{\Sigma_r}(\text{curl}, \Omega) := \{ e \in H^0(\text{curl}, \Omega) : e \times n|_{\Sigma_r} \in L^2(\Sigma_r) \}$$

equipped natural norms $\|h\|_{H^{1/2}_{\text{div}}(\Omega)} = \|h\|_{1/2,\Omega} + \|\text{curl } h\|_{\Omega}$, $\|e\|_{H^0_{\Sigma_r}(\text{curl}, \Omega)} := \|e\|_\Omega + \|e \times n\|_{\Sigma_r}$.

Here $l$ is a dimensional factor of the dimension $|l| = L_0$ and its value is equal to 1, $L_0$ denote the SI dimensions of the length. The spaces $H^{1/2}_{\text{div}}(\Omega)$ and $H^0_{\Sigma_r}(\text{curl}, \Omega)$ will be used below for describing the magnetic and electric field, respectively.

Let $H^1_{\text{div}}(\Omega) := \{ v \in H^1_{\text{div}}(\Omega) : \text{div } v = 0 \}$. Along with Theorem 2.1 we will use also a number of properties of bilinear and trilinear forms related with linear and nonlinear terms in equations (1), (2). We formulate them as the following Lemma (see for details [13, 14]).

Lemma 2.1. Under condition (i) there exist constants $\delta_i = \delta_i(\Omega) > 0$ and $\gamma_i = \gamma_i(\Omega) > 0$, $i = 0, 1$, depending on $\Omega$ such that

$$(\nabla v, \nabla v) \geq \delta_0 \|v\|^2_{1,\Omega} \quad \forall v \in H^1_{0}(\Omega)^3, \ (\text{rot } \Psi, \text{rot } \Psi) \geq \delta_1 \|\Psi\|^2_{1,\Omega} \quad \forall \Psi \in V_{\Sigma_r}(\Omega),$$

$$\| (\mathbf{u} \cdot \nabla) w \| \leq \gamma_0 \|\mathbf{u}\|_{1,\Omega} \|w\|_{1,\Omega}, \ \| (\text{rot } \mathbf{u} \times w) \| \leq \gamma_1 \|\mathbf{u}\|_{1,\Omega} \|w\|_{1,\Omega} \quad \forall \mathbf{u}, w \in H^1(\Omega)^3,$$

$$\| (\text{rot } \mathbf{u} \times w) \| \leq \gamma_1 \|\mathbf{u}\|_{H^{1/2}_{\text{div}}(\Omega)} \|w\|_{H^{1/2}_{\text{div}}(\Omega)} \quad \forall \mathbf{u}, w \in H^{1/2}_{\text{div}}(\Omega), \ v \in H^1(\Omega)^3.$$ 

(9)

Let the following conditions hold in addition to (i), (ii):

(iii) $f \in H^{-1}(\Omega)^3, \ j \in L^2(\Omega)^3; (iv) g \in H^{1/2}_{\text{div}}(\partial \Omega), \ q \in L^2(\Sigma_r), \ k \in (\gamma_r \Sigma_r) H^0_{\Sigma_r}(\text{curl}, \Omega)$.

As usual, while studying control problems for MHD system we will deal with weak form of Problem 1. It consists of finding a triple $(\mathbf{u}, \mathbf{H}, p) \in H^1_{\text{div}}(\Omega) \times H^{1/2}_{\text{div}}(\Omega) \times L^2(\Omega)$ satisfying

$$\nu (\nabla \mathbf{u}, \nabla v) + \nu_1 (\text{rot } \mathbf{H}, \text{rot } \Psi) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, v) + \alpha (\text{rot } \Psi \times \mathbf{H}, \mathbf{u}) - (\text{rot } \mathbf{H} \times \mathbf{H}, v) -$$

$$-(\text{div } v, p) = \langle f, v \rangle + (\nu_1 \mathbf{j} \cdot \text{rot } \Psi + \rho_0^{-1}(\mathbf{k} \cdot \text{rot } \Psi)_{\Sigma_r}, \Psi \rangle \quad \forall v \in H^1_{0}(\Omega)^3 \times V_{\Sigma_r}(\Omega), \ p \in L^2(\Omega)$$

(10)

$$\text{div } \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = g \text{ on } \Sigma, \ \mathbf{H} \cdot n = q \text{ on } \Sigma_r.$$ 

(11)

In order to obtain (10) one should multiply the first equation in (1) by a function $v \in H^1_0(\Omega)^3$, the first equation in (2) by rot $\Psi$ where $\Psi \in V_{\Sigma_r}(\Omega)$, integrate over $\Omega$, apply Green’s formulas, add the obtained results and make use of the identity

$$(\mathbf{E}, \text{curl } \Psi) = \int_{\Sigma_r} (\mathbf{E} \times n|_{\Sigma_r}) \cdot \Psi_T d\sigma = (\mathbf{k}, \Psi_T)_{\Sigma_r} = (\mathbf{E}_0, \text{curl } \Psi) \quad \forall \mathbf{E} \in V_{\Sigma_r}(\Omega).$$

(12)

The identity (10) does not contain electric field $\mathbf{E} \in H^1_0(\text{curl}, \Omega)$ which was eliminated with the help of (12). However, using a condition on a boundary vector $\mathbf{k}$ in (iii) vector $\mathbf{E}$ can be
reconstructed uniquely from triple \((u, H, p) \in H^1_{\text{div}}(\Omega) \times H^{1/2}_{\text{div}}(\Omega) \times L^2(\Omega)\) satisfying (10) so that the first equation in (2) holds a.e. in \(\Omega\) (see details in [15]). This allows us to refer below to mentioned triple \((u, H, p)\) as a weak solution to Problem 1.

Let in addition to (i)–(iv) the following condition holds

\(|\nabla u| |\nabla H| |\nabla p| \leq M_u, \|H\|_{H^{1/2}_{\text{div}}(\Omega)} \leq M_H, \|p\|_{\Omega} \leq M_p. \) (13)

Here \(M_u, M_H, M_p\) are continuous nondecreasing functions of \(\|f\|_{-1, \Omega}, \|j\|_{\Omega}, \|k\|_{\Sigma_r}, \|g\|_{1/2, \Sigma} , \|q\|_{\Sigma_r}. \)

To prove the Problem 1’s local uniqueness we introduce the functional space

\[ \tilde{V}_\Sigma(\Omega) = \{v \in H(\text{rot}, \text{div}; \Omega) : \text{div } v = 0, v \cdot n|_{\Sigma_r} = 0, v \times n|_{\Sigma_r} = 0\} \cap H_\Sigma(\Omega). \]

It’s obvious that in a common case \(V_\Sigma(\Omega) \subset \tilde{V}_\Sigma(\Omega). \)

**Lemma 2.2.** Let conditions (i), (ii), (v) hold and function \(f \in L^2(\Omega)^3\) satisfies equality \((f, \text{rot } v) = 0\) for all \(v \in V_\Sigma(\Omega). \) Then \((f, \text{rot } h) = 0\) for all \(h \in \tilde{V}_\Sigma(\Omega). \)

**Proof.** From relation \((f, \text{rot } v) = 0\) for all \(v \in V_\Sigma(\Omega)\) on the strength (8) results \(f = \nabla \phi + e\) a.e. in \(\Omega, \) where \(\phi \in \nabla H^1(\Omega, \Sigma_r), e \in H_\Sigma(\Omega). \)

Under condition (v) the embedding \(V_\Sigma(\Omega) \subset H^1(\Omega)^3\) is true (see [20, p. 174]) which gives an opportunity to use Green’s formula (5) to the inner product \((e, \text{rot } \Psi)\):

\[ (e, \text{rot } \Psi) = (e \times n, \Psi_T)_{\Sigma} = (e \times n, \Psi_T)_{n} + (e \times n, \Psi_T)_{\Sigma_r} = 0, \forall \Psi \in \tilde{V}_\Sigma(\Omega). \]

Let us introduce the space \(C^\infty_{0, \Sigma_r}(\Omega) = \{h \in C^\infty(\Omega) : \|h\|_{\Sigma_r} = 0\}. \) It’s clear that \(C^\infty_{0, \Sigma_r}(\Omega)\)

embded densely in \(H^1(\Omega, \Sigma_r)\) by norm \(\|\cdot\|_{1, \Omega}\) at condition (v).

Let \(\varphi_m\) be a sequence of functions from the space \(C^\infty_{0, \Sigma_r}(\Omega),\) converging to \(\varphi \in H^1(\Omega, \Sigma_r)\) by the norm \(\|\cdot\|_{1, \Omega}. \)

Let us apply the Green’s formula (5) to the inner product \((\nabla \varphi_m, \text{rot } \Psi)\):

\[ (\nabla \varphi_m, \text{rot } \Psi) = (\nabla \varphi_m \times n, \Psi_T)_{\Sigma} = (\nabla \varphi_m \times n, \Psi_T)_{\Sigma_r} + (\nabla \varphi_m \times n, \Psi_T)_{\Sigma_r} = 0, \forall \Psi \in \tilde{V}_\Sigma(\Omega). \]

It’s not complicated to show further that

\[ \|((\nabla \varphi_m, \text{rot } \Psi) | - (\nabla \varphi, \text{rot } \Psi))\| \leq \| \varphi_m - \varphi\|_{1, \Omega} \| \Psi\|_{1, \Omega} \to 0 \text{ at } m \to \infty \forall \Psi \in \tilde{V}_\Sigma(\Omega). \]

From lemma 2.2 follows that the Problem 1’s weak formulation is also true for all functions \((v, \Psi) \in V \times \tilde{V}_\Sigma(\Omega),\) but not only for functions \((v, \Psi) \in V \times V_\Sigma(\Omega). \)

From the corollary 7.4 in [12, p. 981] the coercive inequality follows \(\|\text{rot } h\|_{\Omega} \geq \alpha \|h\|_{\Omega}\) for all \(h \in \tilde{V}_\Sigma(\Omega),\) according to which \(\|\cdot\|_{\tilde{V}_\Sigma(\Omega)} \equiv \||\cdot||_{\Omega}. \)

As at (v) the embedding takes place \(\tilde{V}_\Sigma(\Omega) \subset H^1(\Omega)^3,\) then the inequality, illustrating the equivalence of two norms on the space \(\tilde{V}_\Sigma(\Omega),\) is met:

\[ \|\text{rot } h\|_{\Omega} \geq \delta_1' \|h\|_{1, \Omega}, \forall h \in \tilde{V}_\Sigma(\Omega). \] (14)

Then from Theorem 2.2 and Lemma 2.2 follows

**Theorem 2.3.** Under assumptions (i)–(v) there exists a weak solution \((u, H, p)\) of Problem 1 and a priori estimate (13) takes place. If, besides, functions \(f, j, k, g, q\) and \(q\) are small (or “viscosity coefficients” \(\nu, \nu_m\) are large) in the sense

\[ \gamma_0 M_u + \gamma_1 (\sqrt{\alpha}/2) M_H < \delta_0, \gamma_1 M_u + \gamma_1 (\sqrt{\alpha}/2) M_H < \delta_1 M_m, \] (15)

where constants \(\delta_0, \gamma_0, \gamma_1\) are introduced in Lemma 2.1, constant \(\delta_1\) is from (14), then the weak solution is unique.
3. Statement and solvability of control problem

The control problem consists in minimization of certain functionals depending on the state \((u, H, p)\) and other unknown functions (controls) satisfying the state equations (1)–(3).

As the cost functional we choose one of the following:

\[
I_1(v) = \|v - v_d\|_Q^2, \quad I_2(H) = \|H - H_d\|_{\mathcal{H}_1}^2, \quad I_3(H) = \|H - H_d\|_{\mathcal{H}_1^{1/2}}^2, \quad I_4(p) = \|p - p_d\|_Q^2. \tag{16}
\]

Here the function \(v_d \in L^2(Q)^3\) denotes some desired velocity field given in a subdomain \(Q \subset \Omega\). Functions \(H_d \in L^2(Q)^3\) (or \(H_d \in \mathcal{H}_1^{1/2}(Q)\)) and \(p_d \in L^2(Q)\) have similar sense for the magnetic field or pressure.

As controls we choose the boundary function \(q\) and assume that \(q\) is changed over set \(K\) satisfying the condition

\(\text{(j) } K \subset L^2(\Sigma_r)\) is nonempty convex closed set.

Setting \(x = (u, H, p) \in X = H^1_\text{div}(\Omega) \times \mathcal{H}^{1/2}_\text{div}(\Omega) \times L^2_0(\Omega), Y = H^{-1}(\Omega)^3 \times V_{\Sigma_1}(\Omega)^* \times L^2_0(\Omega) \times H^1_\text{div}(\Sigma) \times L^2(\Sigma_r)\), we introduce an operator \(F \equiv (F_1,F_2,F_3,F_4) : X \times K \times H^{-1}_\text{div}(\Sigma) \to Y\) by

\[
\langle F_1(x),(v,\Psi) \rangle = \nu(\nabla u,\nabla v) + \nu_1(\text{rot } H,\text{rot } \Psi) + ((u \cdot \nabla) u, v) - (\text{div } v, p) +
\]

\[
+ \omega[(\text{rot } \Psi \times H, u) - (\text{rot } H \times H, v)] - (f,v) - (\nu_1,\text{rot } \Psi) - p_0^{-1}(k,\text{rot } \Psi_{\Sigma_3}) (\forall v,\Psi \in H^1_0(\Omega) \times V_{\Sigma_3}(\Omega), \langle F_2(x,r) \rangle = -(\text{div } u, r) \forall r \in L^2_0(\Omega), F_3(x,g) = u|_{\Gamma} - g \in H^{1/2}_\text{div}(\Sigma), F_4(x,q) = H \cdot n - q \in L^2(\Sigma_r), \text{ and rewrite the weak form (10), (11) of Problem 1 in the form of the operator equation}
\]

\[
F(x,q) = F(u,H,p,q) = 0. \tag{17}
\]

Let \(I : X \to \mathbb{R}\) be a weakly lower semicontinuous cost functional. Consider the following optimal control problem:

\[
J(x,q) = (\mu_0/2)I(x) + (\mu_1/2)\|q\|^2_{\Sigma_3} \to \inf, \quad F(x,q) = 0, \quad (x,q) \in X \times K. \tag{18}
\]

Here \(\mu_0 > 0\) and \(\mu_1 \geq 0\) are nonnegative parameters.

**Theorem 3.1.** Let under assumptions (i), (ii) and (j), \(\mu_0 > 0\), \(\mu_1 > 0\) or \(\mu_0 > 0\), \(\mu_1 \geq 0\) and \(K\) be the bounded set. Then problem (18) has at least one solution for \(I = I_k, k = 1,4\).

**Proof.** Let us denote by \((x_m,q_m) \in Z_{ad}, m \in N\) a minimizing sequence, for which \(\lim_{m \to \infty} J(x_m,q_m) = \inf_{(x,q) \in Z_{ad}} J(x,q) \equiv J^*\).

On the strength of Theorem 3.1, the following estimates hold for the control \(q_m\) and corresponding components of Problem 1’s solution: \(\|q_m\|_{\Sigma_3} \leq c_1, \|u_m\|_1,\Omega \leq c_2, \|H_m\|_{\mathcal{H}^{1/2}_\text{div}(\Omega)} \leq c_3, \|p_m\|_0 \leq c_4\), where \(c_1, c_2, \ldots\) are some constants, not depending on \(m\). From given estimates follows that there are weak limits \(q^* \in L^2(\Sigma_r), u^* \in H^1_\text{div}(\Omega), H^* \in \mathcal{H}^{1/2}_\text{div}(\Omega), p^* \in L^2_0(\Omega)\) of some subsequences of sequences \(\{q_m\}, \{u_m\}, \{H_m\}, \{p_m\}\), it can be considered that at \(m \to \infty\)

\[
q_m \to q^* \text{ weakly in } L^2(\Sigma_r), \quad u_m \to u^* \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^4(\Omega)^3,
\]

\[
p_m \to p^* \text{ weakly in } L^2(\Omega), \quad H_m \to H^* \text{ weakly in } \mathcal{H}^{1/2}_\text{div}(\Omega) \text{ and strongly in } L^2(\Omega)^3. \tag{19}
\]

It is clear that \(q^* \in K, F_2(x^*,q^*) = 0, F_3(x^*,q^*) = 0, F_4(x^*,q^*) = 0\). Let us show that \(F_1(x^*,q^*) = 0\), i.e. that

\[
\nu(\nabla u^*,\nabla v) + \nu_1(\text{rot } H^*,\text{rot } \Psi) + ((u^* \cdot \nabla) u^*, v) - (\text{div } v, p^*) + \omega[(\text{rot } \Psi \times H^*, u^*) - (\text{rot } H^* \times H^*, v)] =
\]

\[
= (f,v) + (j,\text{rot } \Psi) + (k,\text{rot } \Psi_{\Sigma_3}) \quad (\forall v,\Psi \in H^1_0(\Omega) \times V_{\Sigma_3}(\Omega)). \tag{20}
\]
For this purpose let us note that \( u_m, H_m, p_m \) satisfy the relation

\[
\nu(\nabla u_m, \nabla v) + \nu_1(\text{rot} H_m, \text{rot} \Psi) + \left( (u_m \cdot \nabla) u_m, v \right) + \nu[(\text{rot} \Psi \times H_m, u_m) - (\text{rot} H_m \times H_m, v)] =
\]

\[-(\text{div} v, p_m) = (f, v) + (j, \text{rot} \Psi) + (k, \Psi_T)_{\Sigma_r} \forall (v, \Psi) \in H^1_0(\Omega)^3 \times V_{\Sigma_r}(\Omega)
\]

(21)

Let us pass to a limit in (21) at \( m \to \infty \). From (19) it follows that all linear summands in (21) pass to corresponding linear summands in (20) at \( m \to \infty \). Now the nonlinear summands starting with \( (\text{rot} H_m \times H_m, v) \) can be considered. It is clear that

\[
(\text{rot} H_m \times H_m, v) - (\text{rot} H^* \times H^*, v) = (\text{rot} H_m \times (H_m - H^*), v) + (\text{rot} (H_m - H^*) \times H^*, v).
\]

Let us choose a random number \( \varepsilon > 0 \) and fix a test function \( v \in H^1_0(\Omega)^3 \). It can be proved that such number \( M = M(\varepsilon, v) \) exists:

\[
|(\text{rot} H_m \times (H_m - H^*), v)| \leq \varepsilon \forall m \geq M.
\]

(22)

As \( D(\Omega)^3 \) embeded densely in \( H^1_0(\Omega)^3 \) by norm \( \| \cdot \|_{1, \Omega} \) that for any function \( v \in H^1_0(\Omega)^3 \) there is a sequence \( v_n \in D(\Omega)^3 \), converging to \( v \) by norm \( \| \cdot \|_{1, \Omega} \) at \( n \to \infty \). Using \( v_n \), we have

\[
|(\text{rot} H_m \times (H_m - H^*), v)| \leq |(\text{rot} H_m \times (H_m - H^*), v_n)| + |(\text{rot} H_m \times (H_m - H^*), v_n - v)| \forall m, n \in \mathbb{N}
\]

(23)

On the strength of uniform boundedness by \( m \) of values \( \|\text{rot} H_m\|_\Omega \) and \( \|H_m - H^*\|_{L^3(\Omega)} \) there is such number \( N = N(\varepsilon, v) \) that for the second summand in (23) the estimate takes place

\[
|(\text{rot} H_m \times (H_m - H^*), v_n - v)| \leq \varepsilon/2 \forall n \geq N, \forall m \in \mathbb{N}.
\]

(24)

Let us consider the first summand in (23). By Holder inequality for any function \( v_n \in D(\Omega)^3 \) the inequality is true \( |(\text{rot} H_m \times (H_m - H^*), v_n)| \leq \|\text{rot} H_m\|_\Omega \|H_m - H^*\|_\Omega \|v_n\|_{L^\infty(\Omega)^3} \). As \( \|\text{rot} H_m\|_\Omega \leq \|H_m\|_{H^{1/2}(\Omega)} \leq c_4 \) and \( H_m \to H^* \) in \( L^2(\Omega)^3 \) at \( m \to \infty \) by (19), then, starting with some number \( M = M(\varepsilon, v, N) \), the estimate \( |(\text{rot} H_m \times (H_m - H^*), v_n)| \leq \varepsilon/2 \) holds for all \( m \geq M, n = N \). From this and from (23), (24) the estimate (22) follows.

On the strength of randomness of \( \varepsilon \) it means that

\[
|(\text{rot} H_m \times (H_m - H^*), v)| \to 0 \text{ at } m \to \infty \forall v \in H^1_0(\Omega)^3.
\]

(25)

From the weak convergence of the sequence \( H_m \) in the space \( H^{1/2}(\Omega) \) the weak convergence of \( \text{rot} H_m \) in \( L^2(\Omega)^3 \) takes place. In such case

\[
(\text{rot} (H_m - H^*) \times H^*, v) = (\text{rot} (H_m - H^*), H^* \times v) \to 0 \text{ at } m \to 0,
\]

(26)

as \( H^* \times v \in L^2(\Omega)^3 \). From (22)–(26) it follows that \( (\text{rot} H_m \times H_m, v) \to (\text{rot} H^* \times H^*, v) \) at \( m \to \infty \) for all \( v \in H^1_0(\Omega)^3 \).

Applying similar methods we show that \( (\text{rot} \Psi \times H_m, u_m) \to (\text{rot} \Psi \times H^*, u^*) \) at \( m \to \infty \) for all \( \Psi \in V_{\Sigma_r}(\Omega) \). (From the definition of the space \( V_{\Sigma_r}(\Omega) \) follows that the subspace of the space of smooth function \( C^\infty(\Omega) = \{ h \in C^\infty(\Omega) : h \cdot n|_{\Sigma_r} = 0, h \times n|_{\Sigma_r} = 0 \} \) is dense in \( V_{\Sigma_r}(\Omega) \) by norm \( \| \cdot \|_{1, \Omega} \).)

As \( u \in H^1(\Omega)^3 \) without any problems we can conclude that \( ((u_m \cdot \nabla) u_m, v) \to ((u^* \cdot \nabla) u^*, v) \) at \( m \to \infty \) (see [7]).

Passing to the limit in (20) at \( m \to \infty \), coming to (21), and from continuity of the functional \( J \) on \( X \times L^2(\Sigma_r) \) we have that \( J(x^*, q^*) = J^* \).
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