Functional Inequalities for Metric-Preserving Functions with Respect to Intrinsic Metrics of Hyperbolic Type

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Abstract: We obtain functional inequalities for functions which are metric-preserving with respect to one of the following intrinsic metrics in a canonical plane domain: hyperbolic metric or some restrictions of the triangular ratio metric, respectively, of a Barrlund metric. The subadditivity turns out to be an essential property, being possessed by every function that is metric-preserving with respect to the hyperbolic metric and also by the composition with some specific function of every function that is metric-preserving with respect to some restriction of the triangular ratio metric or of a Barrlund metric. We partially answer an open question, proving that the hyperbolic arctangent is metric-preserving with respect to the restrictions of the triangular ratio metric on the unit disk to radial segments and to circles centered at origin.

Keywords: metric-preserving function; subadditive function; intrinsic metric

1. Introduction

Metric-preserving functions have been studied in general topology from a theoretical point of view and have applications in fixed point theory [1,2], as well as in metric geometry to construct new metrics from known metrics, as the metrics $d + t$ and the $α$—snowlake $d^α$, $α ∈ (0, 1)$ associated to each metric $d$ [3–5]. The theory of metric-preserving functions, that can be traced back to Wilson and Blumenthal, has been developed by Borsík, Dobos, Piotrowski, Vallin [6–9] and others, being recently generalized to semimetric spaces and quasimetric spaces [10] (see also [11–13]). As we will show below, there is a strong connection between metric-preserving functions and subadditive functions. The theory of subadditive function is well-developed [14,15], the functional inequality corresponding to subadditivity being viewed as a natural counterpart of Cauchy functional equation [16,17].

Given a function $f : [0, ∞) → [0, ∞)$, it is said that $f$ is metric-preserving if for every metric space $(X, d)$ the function $f ∘ d$ is also a metric on $X$, i.e., $f$ transfers every metric to a metric. The function $f : [0, ∞) → [0, ∞)$ is called amenable if $f^{-1}(\{0\}) = \{0\}$. If there exists some metric space $(X, d)$ such that the function $f ∘ d$ is also a metric on $X$, then $f : [0, ∞) → [0, ∞)$ is amenable. The symmetry axiom of a metric is obviously satisfied by $f ∘ d$ whenever $d$ is a metric. Given a amenable, $f$ is metric-preserving if and only if $f ∘ d$ satisfies triangle inequality whenever $d$ is a metric.

Each of the following properties is known to be a sufficient condition for an amenable function to be metric-preserving [10,11]:

1. $f$ is concave;
2. $f$ is nondecreasing and subadditive;
3. $f$ is tightly bounded (that is, there exists $a > 0$ such that $f(x) ∈ [a, 2a]$ for every $x > 0$).

For instance, every function $f : [0, ∞) → [0, ∞)$ with $f(0) = 0$ for which $\frac{f(t)}{t}$ is nonincreasing on $(0, ∞)$ with $f(0) = 0$ is concave, then $f$ is nondecreasing [18] and Jensen inequality shows that $\frac{f(t)}{t}$ is nonincreasing on $(0, ∞)$; hence $f$ is nondecreasing and subadditive.
One proves that every metric-preserving function \( f : [0, \infty) \to [0, \infty) \) is subadditive, using a particular choice of the metric \( d \), e. g. the usual metric on \( \mathbb{R} \). However, a subadditive amenable function \( f : [0, \infty) \to [0, \infty) \) need not be metric-preserving, as in the case of \( f(t) = \frac{t}{1 + t^2} \) \([11]\). Recall that a function \( f : [0, \infty) \to [0, \infty) \) which is convex and vanishes at the origin is subadditive if and only if \( f \) is linear (\([11]\) Theorem 3.5).

We are interested in the following problem: given a particular metric \( d \) on a subset \( A \) of the complex plane, find necessary conditions satisfied by amenable functions \( f : [0, \infty) \to [0, \infty) \) for which \( f \circ d \) is a metric. In other terms, we look for solutions of the functional inequality

\[
 f(d(x, z)) \leq f(d(x, y)) + f(d(y, z)) \quad \text{for all } x, y, z \in A.
\]

If we can find for every \( a, b \in [0, \infty) \) some points \( x, y, z \in A \) such that \( d(x, y) = a \), \( d(y, z) = b \) and \( d(x, z) = a + b \), then \( f \) is subadditive on \( [0, \infty) \). For some metrics \( d \) it could be difficult or impossible to find such points.

We will consider the cases where \( d \) is a hyperbolic metric, a triangular ratio metric or some other Barrlund metric. Recall that all these metrics belong to the class of intrinsic metrics, which is recurrent in the study of quasiconformal mappings \([4]\).

The hyperbolic metric \( \rho_D \) on the unit disk \( \mathbb{D} \) is given by

\[
 \tanh \frac{\rho_D(x, y)}{2} = \frac{|x - y|}{|1 - \overline{xy}|},
\]

that is, \( \rho_D(x, y) = 2\arctanh p_D(x, y) \), where \( p_D(x, y) = \frac{|x - y|}{|1 - \overline{xy}|} \) is the pseudo-hyperbolic distance and we denoted by \( \arctanh \) the inverse of the hyperbolic tangent \( \tanh \) \([19]\). The hyperbolic metric \( \rho_{\mathbb{H}} \) on the upper half plane \( \mathbb{H} \) is given by

\[
 \tanh \frac{\rho_{\mathbb{H}}(x, y)}{2} = \frac{|x - y|}{|x - \overline{y}|}.
\]

For every simply-connected proper subdomain \( \Omega \) of \( \mathbb{C} \) one defines, via Riemann mapping theorem, the hyperbolic metric \( \rho_\Omega \) on \( \Omega \). We prove that, given \( f : [0, \infty) \to [0, \infty) \), if \( f \circ \rho_\Omega \) is a metric on \( \Omega \), then \( f \) is subadditive. In the other direction, if \( f : [0, \infty) \to [0, \infty) \) is amenable, nondecreasing and subadditive, then \( f \circ \rho_\Omega \) is a metric on \( \Omega \).

The triangular ratio metric \( s_G \) of a given proper subdomain \( G \subset \mathbb{C} \) is defined as follows for \( x, y \in G \) \([20]\)

\[
 s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|}. \tag{1}
\]

For the triangular ratio metric \( s_{\mathbb{H}} \) on the half-plane, it is known that \( s_{\mathbb{H}}(x, y) = \tanh \frac{p_{\mathbb{H}}(x, y)}{2} \) for all \( x, y \in \mathbb{H} \). If \( F : [0, 1) \to [0, \infty) \) and \( f \circ s_{\mathbb{H}} \) is a metric on the upper half-plane \( \mathbb{H} \), we show that \( F \circ \tanh \) is subadditive on \([0, \infty)\).

The triangular ratio metric \( s_{\mathbb{D}}(x, y) \) on the unit disk can be computed analytically as

\[
 s_{\mathbb{D}}(x, y) = \frac{|x - y|}{|x - \overline{z_0}| + |y - \overline{z_0}|}, \quad \text{where } z_0 \in \partial \mathbb{D} \text{ is the root of the algebraic equation}
\]

\[
 x^4 z^4 - (x + y) z^3 + (x + y) z - xy = 0
\]

for which \(|x - z| + |z - y|\) has the least value \([21]\). However, a simple explicit formula for \( s_{\mathbb{D}}(x, y) \) is not available in general.

As \( \arctanh_{\mathbb{H}} \) is a metric on the upper half-plane \( \mathbb{H} \), it is natural to ask if \( \arctanh_{\mathbb{D}} \) is a metric on the unit disk \( \mathbb{D} \). The answer is unknown, but we prove that some restrictions of \( \arctanh_{\mathbb{D}} \) are metrics, namely the restriction to each radial segment of the unit disk and the restriction to each circle \(|z| = \rho < 1\). Given \( f : [0, 1) \to [0, \infty) \) such that the restriction of \( f \circ s_{\mathbb{D}} \) to some radial segment of the unit disk \( \mathbb{D} \) is a metric, we prove that
symmetry if and only if the unit disk, it suffices to consider the case 

\[
F\left(\frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}}\right) \leq F(\tanh(a)) + F(\tanh(b)).
\]

For a proper subdomain \(G \subset \mathbb{R}^n\), for a number \(p \geq 1\), and for points \(x, y \in G\), let 

\[
b_{G,p}(x,y) = \sup_{z \in \partial G} \frac{|x - y|}{\sqrt{|x - z|^p + |z - y|^p}}.
\]

The above formula defines a metric, as shown by A. Barrlund [22] for \(G = \mathbb{R}^n \setminus \{0\}\) and by P. Hästö [5] in the general case. This metric is called a Barrlund metric and is studied in [23]. In addition, the limit case \(p = \infty\) is considered and it is shown that the formula 

\[
b_{G,\infty}(x,y) = \sup_{z \in \partial G} \frac{|x - y|}{\max(|x - z|, |z - y|)}, x, y \in G
\]
defines a metric. Note that \(b_{G,p}\) is invariant to similarities for every \(p \in [1, \infty]\) and that for \(p = 1\) the Barrlund metric coincides with the triangular ratio metric. We will consider Barrlund metrics with \(p = 2\) on canonical domains in plane, the upper half plane and the unit disk, that have explicit formulas [23]. For \(G \in \{\mathbb{H}, \mathbb{D}\}\), assuming that \(F \circ b_{G,2}\) is a metric on some subset \(A \subset G\) which is a ray, a line or a circle, we obtain a functional inequality satisfied by \(F\), under the form of the subadditivity of a composition \(F \circ \varphi\), where the function \(\varphi\) depends only on \(A\).

2. The Case of Hyperbolic Metrics

Let \(\mathbb{D}\) be the unit disk with the hyperbolic metric \(\rho_{\mathbb{D}}\).

Let \(f : [0, \infty) \to [0, \infty)\) be amenable. If \(f\) is subadditive and nondecreasing, then \(f \circ \rho_{\mathbb{D}}\) is a metric on the unit disk \(\mathbb{D}\).

**Proposition 1.** If \(f : [0, \infty) \to [0, \infty)\) and \(f \circ \rho_{\mathbb{D}}\) is a metric on the unit disk \(\mathbb{D}\), then \(f\) is subadditive.

**Proof.** Let \(f : [0, \infty) \to [0, \infty)\) such that \(f \circ \rho_{\mathbb{D}}\) is a metric on the unit disk \(\mathbb{D}\).

Denote \(g(t) = f(2\arctanh(t)), t \in [0, \infty)\). Then \(g : [0, \infty) \to [0, \infty)\) satisfies

\[
g(p_{\mathbb{D}}(x,y)) \leq g(p_{\mathbb{D}}(x,z)) + g(p_{\mathbb{D}}(z,y))\] for all \(x, y, z \in \mathbb{D}\).

Since the pseudo-hyperbolic metric is invariant to the Möbius automorphisms of the unit disk, it suffices to consider the case \(z = 0\). In conclusion, \(g \circ p_{\mathbb{D}}\) satisfies the triangle inequality if and only if

\[
g(p_{\mathbb{D}}(x,y)) \leq g(p_{\mathbb{D}}(x,0)) + g(p_{\mathbb{D}}(0,y))\] for all \(x, y, 0 \in \mathbb{D}\). (3)

However, \(p_{\mathbb{D}}(x,0) = |x|, p_{\mathbb{D}}(0,y) = |y|\); hence, \(g(p_{\mathbb{D}}(x,0)) + g(p_{\mathbb{D}}(0,y)) = g(|x|) + g(|y|)\).

On the other hand, \(p_{\mathbb{D}}(x,y)^2 = \frac{|x - y|^2}{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)} = \frac{|x|^2 + |y|^2 - 2|x| \cdot |y| \cos \alpha}{1 + |x|^2 \cdot |y|^2 - 2|x| \cdot |y| \cos \alpha},\) where \(\alpha := \angle(x0y)\).

Let \(x, y \in \mathbb{D}\). Since

\[
d \left(\frac{|x|^2 + |y|^2 - 2|x| \cdot |y| \cos \alpha}{1 + |x|^2 \cdot |y|^2 - 2|x| \cdot |y| \cos \alpha}\right) = \frac{2|x| \cdot |y| (1 - |x|^2) (1 - |y|^2)}{(1 + |x|^2 \cdot |y|^2 - 2|x| \cdot |y| \cos \alpha)^2} \sin \alpha,
\]
the function $\alpha \mapsto \frac{|x|^2 + |y|^2 - 2|x| |y| \cos \alpha}{1 + |x|^2 - |y|^2 - 2|x| |y| \cos \alpha}$ is nonincreasing on $[-\pi, 0]$ and nondecreasing on $[0, \pi]$.

Then $\alpha \mapsto \frac{|x|^2 + |y|^2 - 2|x| |y| \cos \alpha}{1 + |x|^2 - |y|^2 - 2|x| |y| \cos \alpha}$ attains its maximum if and only if $\cos \alpha = -1$, in which case $p_{\mathbb{D}}(x, y) = \frac{|x| + |y|}{1 + |x||y|}$.

If (3) holds, then

$$g \left( \frac{r + s}{1 + rs} \right) \leq g(r) + g(s) \text{ for all } r, s \in [0, 1).$$

Conversely, if (4) holds and $g$ is nondecreasing, then (3) is satisfied.

Let $\varphi : \mathbb{C} \setminus \{-1\} \rightarrow \mathbb{C} \setminus \{-1\}$, $\varphi(w) = \frac{1-w}{1+w}$. Then $\varphi^{-1} = \varphi$ and $\varphi \left( \frac{z_1 + z_2}{1 + z_1 z_2} \right) = \varphi(z_1) \varphi(z_2)$ for every $z_1, z_2 \in \mathbb{C} \setminus \{-1\}$ with $z_1 z_2 \in \mathbb{C} \setminus \{-1\}$. Note that $\varphi([0,1)) = (0,1)$.

Denoting $\rho = \varphi(r)$ and $\sigma = \varphi(s)$, (4) is equivalent to

$$(g \circ \varphi)(\rho \cdot \sigma) \leq (g \circ \varphi)(\rho) + (g \circ \varphi)(\sigma) \text{ for all } \rho, \sigma \in (0,1].$$

Now, denoting $\ln \rho = -u$ and $\ln \sigma = -v$, we see that the above condition reduces to

$$(g \circ \varphi \circ \exp)(-u - v) \leq (g \circ \varphi \circ \exp)(-u) + (g \circ \varphi \circ \exp)(-v) \text{ for all } u, v \in [0, \infty).$$

Denote $h(t) := (g \circ \varphi \circ \exp)(-t)$, $t \in [0, \infty)$. Condition (4) holds if and only if $h$ is subadditive on $[0, \infty)$.

We note that $(g \circ \varphi \circ \exp)(-t) = \tanh \frac{t}{2}$ for $t \in [0, \infty)$. Then for all $t \in [0, \infty)$ we have $h(t) = g(\tanh \frac{t}{2}) = f(t).

We proved that $f$ is subadditive. 

**Remark 1.** The functional equation associated to the inequality (4) $g \left( \frac{r + s}{1 + rs} \right) = g(r) + g(s)$, $r, s \in [0, 1)$ reduces via the substitution $h(t) = g(\tanh \frac{t}{2})$ to the Cauchy equation $h(u + v) = h(u) + h(v)$, $u, v \in [0, \infty)$. Extending $h$ to an odd function, we may assume that $h$ is additive on $[0, \infty)$. If $g$ is bounded on one side on a set of positive Lebesgue measure, then $h$ is linear [16]; hence, there exists some positive constant $c$ such that $g(t) = c \text{arctanh}(t)$, $t \in [0, \infty)$.

Let $\mathbb{H}$ be the upper half-plane with the hyperbolic metric $\rho_{\mathbb{H}}$. We are interested in the amenable functions $f : [0, \infty) \rightarrow [0, \infty)$ for which $f \circ \rho_{\mathbb{H}}$ is a metric on $\mathbb{H}$. Consider the Cayley transform $T : \mathbb{H} \rightarrow \mathbb{D}$, $T(z) = \frac{z - 1}{z + 1}$, which is a bijective conformal map. Noting that for all $x, y \in \mathbb{H}$ we have $\rho_{\mathbb{H}}(x, y) = \rho_\mathbb{D}(T(x), T(y))$, it follows that $f \circ \rho_{\mathbb{H}}$ is a metric on $\mathbb{H}$ if and only if $f \circ \rho_{\mathbb{D}}$ is a metric on $\mathbb{D}$. From Proposition 1 we get the following

**Corollary 1.** If $f : [0, \infty) \rightarrow [0, \infty)$ is amenable and $f \circ \rho_{\mathbb{H}}$ is a metric on upper half-plane $\mathbb{H}$, then $f$ is subadditive.

More generally, for every proper simply-connected subdomain $\Omega$ of $\mathbb{C}$ there exists, by Riemann mapping theorem, a conformal mapping $T_\Omega : \Omega \rightarrow \mathbb{D}$. The hyperbolic metric $\rho_\Omega$ on $\Omega$ is defined by $\rho_\Omega(x, y) = \rho_\mathbb{D}(T_\Omega(x), T_\Omega(y))$.

Clearly, $f \circ \rho_\Omega$ is a metric on $\Omega$ if and only if $f \circ \rho_\mathbb{D}$ is a metric on $\mathbb{D}$. Now Proposition 1 leads to following generalization of itself.

**Theorem 1.** Let $\Omega$ be a proper simply-connected subdomain of $\mathbb{C}$ and $\rho_\Omega$ be the hyperbolic metric on $\Omega$. If $f : [0, \infty) \rightarrow [0, \infty)$ and $f \circ \rho_\Omega$ is a metric on $\Omega$, then $f$ is subadditive.

**Corollary 2.** Let $\Omega$ be a proper simply-connected subdomain of $\mathbb{C}$ and $\rho_\Omega$ be the hyperbolic metric on $\Omega$. Let $f : [0, \infty) \rightarrow [0, \infty)$ amenable and nondecreasing. Then $f \circ \rho_\Omega$ is a metric on $\Omega$ if and only if $f$ is subadditive on $[0, \infty)$. 

3. The Case of Unbounded Geodesic Metric Spaces

We can give another proof of Theorem 1, based on geometric arguments in geodesic metric spaces. The main idea is that in a geodesic metric space the distance is additive along geodesics.

A topological curve \( \gamma : I \to X \) in a metric space \((X, d)\), where \( I \subset \mathbb{R} \) is an interval, is called a geodesic if \( L(\gamma_{|t_1,t_2}) = d(\gamma(t_1), \gamma(t_2)) \) for every subinterval \([t_1, t_2] \subset I\), i.e., the length of every arc of the geodesic is equal to the distance between the endpoints of the arc. A metric space is called a geodesic metric space if every pair of its points can be joined by a geodesic path.

**Lemma 1.** In a geodesic metric space \((X, d)\) that is unbounded, for every positive numbers \(a\) and \(b\) there exists some points \(x, y, z \in X\) such that \(d(x, y) = a\), \(d(y, z) = b\) and \(d(x, z) = a + b\).

**Proof.** Let \(a, b\) be positive numbers. Fix an arbitrary point \(x \in X\). As \((X, d)\) is unbounded, there exists a point \(w \in X\) such that \(d(x, w) > a + b\). As \((X, d)\) is a geodesic metric space, there exists a geodesic path joining \(x\) and \(w\) in \(X\). We may assume that this path is parameterized by arc-length, let us denote it by \(\gamma : [0, L] \to X\), where \(L = L(\gamma) = d(x, w)\).

Then the length of the restriction of \(\gamma\) to \([0, t]\) is \(L(\gamma_{|0,t}) = t\), where \(t \in [0, L]\). Since \(L = d(x, w) > a + b\), we may consider \(\gamma(a) =: y\) and \(\gamma(a + b) =: z\). By the definition of a geodesic curve, \(d(x, y) = L(\gamma_{|[0,a]}) = a\), \(d(x, z) = L(\gamma_{|[0,a+b]}) = a + b\) and \(d(x, y) = L(\gamma_{|[a,a+b]}) = b\). \(\square\)

**Proposition 2.** If the geodesic metric space \((X, d)\) is unbounded, then every function \(f : [0, \infty) \to [0, \infty)\) which is metric-preserving with respect to \(d\) must be subadditive on \([0, \infty)\).

**Proof.** Let \((X, d)\) be a geodesic metric space that is unbounded. Assume that \(f : [0, \infty) \to [0, \infty)\) is metric-preserving with respect to \(d\).

We have to prove that \(f(a + b) - f(a) - f(b) \leq 0\) for all nonnegative numbers \(a\) and \(b\). If \(a = 0\) and \(b = 0\) this inequality is obvious. Let \(a\) and \(b\) be positive numbers.

By Lemma 1, there exists some points \(x, y, z \in X\) such that \(d(x, y) = a, d(y, z) = b\) and \(d(x, z) = a + b\). Then \(f(a + b) - f(a) - f(b) = (f \circ d)(x, z) - [(f \circ d)(x, y) + (f \circ d)(y, z)] \leq 0\), since \(f \circ d\) satisfies triangle inequality. \(\square\)

**Remark 2.** The metric space \((\Omega, \rho_{\Omega})\), where \(\Omega\) is a proper simply-connected subdomain of \(\mathbb{C}\) and \(\rho_{\Omega}\) is the hyperbolic metric on \(\Omega\), is a geodesic metric space and is unbounded. By Proposition 2 we get another proof for Theorem 1. Note that the pseudo-hyperbolic distance on \(\Omega\) is not additive along geodesics of \((\Omega, \rho_{\Omega})\) [19].

4. The Case of Triangular Ratio Metric on a Canonical Plane Domain

The triangular ratio metric \(s_G\) of a given proper subdomain \(G \subset \mathbb{R}^n\) is defined as follows for \(z_1, z_2 \in G\) [20]

\[
s_G(z_1, z_2) = \sup_{z \in \partial G} \frac{|z_1 - z_2|}{|z_1 - z| + |z - z_2|}.
\]

Note that \(s_G(z_1, z_2) \leq 1\) for all \(z_1, z_2 \in G\). If no segment joining two points in \(G\) intersects the boundary \(\partial G\), as it is the case if \(G\) is convex, then \(s_G(z_1, z_2) < 1\) for all \(z_1, z_2 \in G\).

We have \(s_G(z_1, z_2) = \frac{|z_1 - z_2|}{\inf_{z \in \partial G} (|z_1 - z| + |z - z_2|)}\) and the infimum \(\inf_{z \in \partial G} (|z_1 - z| + |z - z_2|)\) is always attained. A point \(u \in \partial G\) is called a Ptolemy–Alhazen point for \(z_1, z_2 \in G\) if a light ray from \(z_1\) is reflected at \(u\) on the circle, such that the reflected ray goes through the point...
z_2. Every point at which \( \inf_{z \in \partial G} (|z_1 - z| + |z - z_2|) \) is attained is a Ptolemy–Alhazen point for \( z_1, z_2 \in G \).

For a subset \( A \) of a convex domain \( G \) we may study the functional inequality

\[
F(s_G(x, y)) \leq F(s_G(x, z)) + F(s_G(z, y)) \quad \text{for all } x, y, z \in A,
\]

where \( F : [0, 1) \to [0, \infty) \).

4.1. The Triangular Ratio Metric on the Upper Half-Plane

Recall that \( s_H(x, y) = \tanh \frac{\rho_H(x, y)}{2} = \frac{|x-y|}{|1-xy|} \) for all \( x, y \in \mathbb{H} \), that is \( s_H \) coincides with the pseudo-hyperbolic distance \( \rho_H \).

Using the subadditivity of functions preserving the hyperbolic metric of the upper half-plane, we get the following

**Proposition 3.** If \( F : [0, 1) \to [0, \infty) \) and \( F \) is metric-preserving with respect to \( s_H \) on the upper half-plane \( \mathbb{H} \), then \( F \circ \tanh \) is subadditive on \( [0, \infty) \).

**Proof.** \( F(s_H(x, y)) = F \left( \tanh \frac{\rho_H(x, y)}{2} \right) = G(\rho_H(x, y)) \) for all \( x, y \in \mathbb{H} \), where we denoted \( G(t) := F(\tanh \frac{t}{2}) \), \( t \in [0, \infty) \).

Note that \( F \circ s_H \) is a metric on \( \mathbb{H} \) if and only if \( G \circ \rho_H \) is a metric on \( \mathbb{H} \).

If \( F \circ s_H \) is a metric on \( \mathbb{H} \), by Corollary 1, it follows that \( G \) is subadditive on \( [0, \infty) \). Then \( H = F \circ \tanh \) is also subadditive on \( [0, \infty) \), as \( H(t) = G(2t) \) for all \( t \in [0, \infty) \). \( \square \)

**Remark 3.** Note that \( \tanh \) maps \( [0, \infty) \) onto \( [0, 1) \) and is subadditive on \( [0, \infty) \). If \( F : [0, 1) \to [0, \infty) \) is subadditive on \( [0, 1) \), then \( F \circ \tanh \) is subadditive on \( [0, \infty) \). The converse is not true, as it is shown in the case \( F(t) = \arctanh(t) \), \( t \in [0, 1) \). Note that

\[
\arctanh(a) + \arctanh(b) = \arctanh \left( \frac{a + b}{1 + ab} \right) \leq \arctanh(a + b) \quad \text{for all } a, b \in [0, 1].
\]

4.2. The Triangular Ratio Metric on the Unit Disk

There is an open conjecture stating that \( \arctanhs_D \) is a metric on the unit disk [23]. Note that \( \arctanhs_D \) is a metric on \( A \subseteq D \) if and only if

\[
s_D(x, z) \leq \frac{s_D(x, y) + s_D(y, z)}{1 + s_D(x, y)s_D(y, z)} \quad \text{for all } x, y, z \in A.
\]

We will prove that \( \arctanh \) is metric-preserving with respect to the restrictions of the triangular ratio metric on the unit disk to radial segments, respectively, to circles centered at the origin.

**Lemma 2 ([23] Theorem 2.2).** For \( x, y \in D \), \( s_D(x, y) \leq \frac{|x-y|}{2\sqrt{|x|\cdot|y|}} \). Equality holds if and only if \( 0, x \) and \( y \) are collinear.

**Lemma 3.** The following addition formula holds: if \(-1 < r \leq s \leq t < 1\), then

\[
\arctanhs_D(r, t) = \arctanhs_D(r, s) + \arctanhs_D(s, t).
\]

The restriction of \( \arctanhs_D \) to each diameter of the unit disk is a metric.

**Proof.** Recall that \( \arctanh(u) = \frac{1}{2} \log \frac{1+u}{1-u} \) for all \( u \in (-1, 1) \). In particular, if \(-1 < u \leq v \leq 1\), then

\[
\arctanhs_D(u, v) = \frac{1}{2} \log \frac{1 + \frac{(v-u)}{2(v+u)}}{1 - \frac{(v-u)}{2(v+u)}} = \frac{1}{2} \log \frac{1 - u}{1 - v}.
\]
Let \(-1 < r \leq s \leq t < 1\). Then

\[
\arctanhs_D(r,s) + \arctanhs_D(s,t) = \frac{1}{2} \log \frac{1-r}{1-s} + \frac{1}{2} \log \frac{1-s}{1-t} \\
= \frac{1}{2} \log \frac{1-r}{1-t} = \arctanhs_D(r,t).
\]

Since \(s_D\) is invariant to rotations around the origin (more generally, \(s_D\) is invariant to similarities), it suffices to prove that the restriction of \(\arctanhs_D\) to the intersection of the unit disk with the real axis is a metric. This follows from the above addition formula, observing that \(0 \leq \max(\arctanhs_D(r,s), \arctanhs_D(s,t)) \leq \arctanhs_D(r,t). \)

We prove that the restriction of the triangular ratio metric of the unit disk \(s_D\) to each radial segment of the unit disk takes all values between 0 and 1.

**Lemma 4.** For every \(\lambda \in (0,1)\) and \(r \in [0, \frac{1}{2})\), there exists \(s \in (r, \frac{1}{2})\) such that \(s_D(r,s) = \lambda\).

**Proof.** Let \(\lambda \in (0,1)\) and \(r \in [0, \frac{1}{2})\).

Assume that \(s \in (r, 1)\). Then \(s_D(r,s) = \lambda\) if and only if \(\frac{s-r}{1-(s+r)} = \lambda\), i.e., \(s = \frac{(1-\lambda)r+\lambda}{1+\lambda}\).

Note that \(s = \frac{(1-\lambda)r+\lambda}{1+\lambda}\) implies \(s-r = \frac{\lambda(1-2r)}{1+\lambda} > 0\) and \(s - \frac{1}{2} = \frac{(1-\lambda)(2r-1)}{2(1+\lambda)} < 0\). \(\square\)

**Proposition 4.** Let \(f : [0,1) \to [0,\infty)\).

1. If the restriction of \(f \circ s_D\) to some radial segment of the unit disk \(D\) is a metric, then \(f \circ \tanh\) is subadditive on \([0,\infty)\).
2. If \(f\) is amenable and \(f \circ \tanh\) is subadditive and nondecreasing on \([0,\infty)\), then the restriction of \(f \circ s_D\) to every diameter of the unit disk \(D\) is a metric.

**Proof.** (1) Let \(F := f \circ \tanh : [0,\infty) \to [0,\infty)\).

As \(s_D\) is invariant to rotations around the origin, we may assume that the restriction of \(f \circ s_D\) to \(F \circ \arctanhs_D\) to \(\{ x + iy : x \in [0,1), y = 0 \}\) is a metric.

Since \(F(0) = 0\), it suffices to prove that \(F\) is subadditive on \([0,\infty)\).

Let \(a, b \in (0,\infty)\). We prove that \(F(a + b) \leq F(a) + F(b)\).

Denote \(\tanh a = \lambda\) and \(\tanh b = \mu\).

Fix \(r \in [0, \frac{1}{2})\). By Lemma 4, there exists \(s \in \left(r, \frac{1}{2}\right)\) such that \(s_D(r,s) = \lambda\). Applying again Lemma 4, we get \(t \in \left(s, \frac{1}{2}\right)\) such that \(s_D(s,t) = \mu\).

The addition formula \(\arctanhs_D(r,t) = \arctanhs_D(r,s) + \arctanhs_D(s,t)\) shows that \(\arctanhs_D(r,t) = a + b\).

Since the restriction of \(F \circ \arctanhs_D\) to \(\{ x + iy : x \in [0,1), y = 0 \}\) is a metric,

\[
F(\arctanhs_D(r,t)) \leq F(\arctanhs_D(r,s)) + F(\arctanhs_D(s,t)),
\]

i.e., \(F(a + b) \leq F(a) + F(b)\), q.e.d.

(2) The function \(F := f \circ \tanh : [0,\infty) \to [0,\infty)\) is amenable, subadditive and nondecreasing self-mapping of \([0,\infty)\); therefore, \(F\) is metric-preserving. By Lemma 3, the restriction of \(\arctanhs_D\) to every diameter of the unit disk \(D\) is a metric. Then the restriction of \(f \circ s_D = F \circ \arctanhs_D\) to every diameter of the unit disk \(D\) is a metric. \(\square\)

**Remark 4.** Let \(f : [0,1) \to [0,\infty)\). If \(f\) is amenable, subadditive and nondecreasing on \([0,1)\), then \(f \circ s_D\) is a metric on the entire unit disk \(D\) and obviously \(f \circ \tanh\) is amenable, subadditive and nondecreasing on \([0,\infty)\).

We prove that each restriction of \(\arctanhs_D\) to a circle centered at origin and contained in the unit disk is a metric.
If $x, y \in \mathbb{D}$ with $|x| = |y| = \rho < 1$ it is known from ([24] Remark 3.14) that, denoting $\omega_0 := 2 \arccos \rho$ and $2 \alpha := \angle(x, 0, y)$, we have

$$s_{\mathbb{D}}(x, y) = \left\{ \begin{array}{ll} \frac{\rho}{\sin \alpha}, & \text{if } 2 \alpha > \omega_0 \\ \frac{\rho \sin \alpha}{\sqrt{1 + \rho^2 - 2 \rho \cos \alpha}}, & \text{if } 2 \alpha \leq \omega_0 \end{array} \right..$$

(8)

If $2 \alpha = \omega_0$, then $\frac{\rho \sin \alpha}{\sqrt{1 + \rho^2 - 2 \rho \cos \alpha}} = \frac{\rho \sqrt{1 - \rho^2}}{\sqrt{1 + \rho^2 - 2 \rho \cos \alpha}} = \rho$.

We will see that the restriction of the triangular ratio metric of the unit disk $s_{\mathbb{D}}$ to any circle $|z| = \rho < 1$ takes all values from 0 and $\rho$.

**Lemma 5.** Let $\rho \in (0, 1)$. Denote $\omega_0 := 2 \arccos \rho$ and let $f : [0, \frac{\omega_0}{2}] \to \mathbb{R}$ be defined by

$$f(\theta) = \frac{\rho \sin \theta}{\sqrt{1 + \rho^2 - 2 \rho \cos \theta}}.$$ Then:

1. $f$ is increasing on $[0, \frac{\omega_0}{2}]$ and $f''(\theta) < 0$ for all $\theta \in [0, \frac{\omega_0}{2}]$. In particular, $f(\theta) \leq f\left(\frac{\omega_0}{2}\right) = \rho$ for all $\theta \in [0, \frac{\omega_0}{2}]$ and for every $\lambda \in [0, \rho]$ there exists an unique $\theta \in [0, \frac{\omega_0}{2}]$ such that $f(\theta) = \lambda$.

2. The function $h : [0, \frac{\omega_0}{2}] \to \mathbb{R}$ defined by $h(\theta) := \frac{f'(\theta)}{f(\theta)}$ is decreasing on $[0, \frac{\omega_0}{2}]$.

3. For every $k \in (0, \omega_0)$ the function $R_k(\theta) := \frac{f(\theta) + f(\theta + k)}{f(\theta)} + \frac{\rho}{f(\theta)}$, with $\max(0, k - \frac{\omega_0}{2}) \leq \theta \leq \min(\frac{\omega_0}{2}, k)$, is increasing on $\left[\max(0, k - \frac{\omega_0}{2}), \frac{\omega_0}{2}\right]$ and decreasing on $\left[\frac{\omega_0}{2}, \min(k, \frac{\omega_0}{2})\right]$.

**Proof.**

1. We have

$$f'(\theta) = \frac{\rho (\cos \theta - \rho) (1 - \rho \cos \theta)}{(1 + \rho^2 - 2 \rho \cos \theta)^2}, \quad \theta \in \left[0, \frac{\omega_0}{2}\right].$$

We note that $\theta \in \left[0, \frac{\omega_0}{2}\right]$ implies $\cos \theta - \rho \geq 0$; hence, $f'(\theta) \geq 0$, with equality if and only if $\theta = \frac{\omega_0}{2}$. Then $f$ is increasing on $\left[0, \frac{\omega_0}{2}\right]$; therefore, $f(\theta) \leq f\left(\frac{\omega_0}{2}\right)$ for all $\theta \in [0, \frac{\omega_0}{2}]$. Note that $f\left(\frac{\omega_0}{2}\right) = \rho$.

By the intermediate value property and the monotonicity of $f$, for every $\lambda \in [0, \rho] = \left[f(0), f\left(\frac{\omega_0}{2}\right)\right]$ there exists an unique $\theta \in [0, \frac{\omega_0}{2}]$ such that $f(\theta) = \lambda$. Moreover, in this case $\cos \theta = t \in [\rho, 1]$ is the unique root from $[\rho, 1]$ of the quadratic function $Q(t) = \rho^2 t^2 - 2 \rho^2 t + \lambda^2 - \rho^2 + \lambda^2 \rho^2$.

Furthermore,

$$f''(\theta) = \frac{-\rho \sin \theta}{(\rho^2 - 2 \cos \theta \rho + 1)^2} \left(\rho^2 (\rho - \cos \theta)^2 + (1 - \rho^2)(1 - \rho \cos \theta)\right) \leq 0$$

for all $\theta \in [0, \frac{\omega_0}{2}]$, with equality if and only if $\theta = 0$.

2. By computation we get

$$h(\theta) = \frac{\rho (\cos \theta - \rho)}{(1 - \rho \cos \theta) \sqrt{1 + \rho^2 - 2 \rho \cos \theta}} \geq 0,$$

for all $\theta \in [0, \frac{\omega_0}{2}]$. Then $\rho^{-2}(h(\theta))^2 = H(\cos \theta)$, where

$$H(t) = \frac{(t - \rho)^2}{(1 - \rho t)^2 (1 + \rho^2 - 2 \rho t)}, \quad t \in [\rho, 1].$$

We have

$$H'(t) = \frac{2(t - \rho)}{(1 - \rho t)^3 (1 + \rho^2 - 2 \rho t)^2} \left[(1 - \rho^2)(1 + \rho^2 - 2 \rho t) + \rho (\rho - t) (\rho t - 1)\right].$$
Then, \( H'(t) \geq 0 \) for all \( t \in \{\rho, 1\} \), with \( H'(t) = 0 \) only for \( t = \rho \), therefore, \( H \) is increasing on \([\rho, 1]\). This shows that \( h \) is decreasing on \([0, \omega_0]\).

(3) Let \( k \in (0, \omega_0) \). Denote \( a = \max(0, k - \omega_0) \) and \( b = \min(k, \omega_0) \). Note that \( a < b \) and \( R_k(\theta) \) is well-defined on \([a, b] \). Moreover, \( x, y \in (a, b) \). If \( k \in (0, \omega_0) \), then \( a = 0 \) and \( b = k \). If \( k \in (\omega_0, \omega_0) \), then \( a = k - \omega_0 \) and \( b = \omega_0 \).

The derivative
\[
R'_k(\theta) = \frac{\left(1 - (f(\theta))^2\right) \left(1 - (f(k - \theta))^2\right)}{(1 + f(\theta)f(k - \theta))^2} \left(\frac{f'(\theta)}{1 - (f(\theta))^2} - \frac{f'(k - \theta)}{1 - (f(k - \theta))^2}\right)
\]

vanishes if and only if \( \theta = k - \theta \), i.e., \( \theta = \frac{k}{2} \). Since \( h \) is decreasing on \([0, \omega_0]\) \( \supseteq [a, b] \), we have \( R'_k(\theta) > 0 \) for \( \theta \in \left[a, \frac{k}{2}\right] \) and \( R'_k(\theta) < 0 \) for \( \theta \in \left(\frac{k}{2}, b\right) \); hence, \( R_k \) is increasing on \([a, \frac{k}{2}] \) and decreasing on \([\frac{k}{2}, b]\). \( \Box \)

**Remark 5.** The maximum of \( R_k \) on \([a, b]\) is \( \frac{2f(\frac{k}{2})}{1 + f(\frac{k}{2})} \). The minimum of \( R_k \) on \([a, b]\) is
\[
\min(R_k(a), R_k(b)) = \begin{cases} f(k), & \text{if } k \in (0, \omega_0) \\ f(\omega_0) + f(k - \omega_0), & \text{if } k \in (\omega_0, \omega_0) \end{cases}
\]

**Theorem 2.** The restriction of \( \text{arctanh}_{S_D} \) to each circle \( |z| = \rho < 1 \) is a metric.

**Proof.** Let \( x, y, z \in \mathbb{D} \) with \( |x| = |y| = |z| = \rho < 1 \).

Denote \( 2a := \angle(x, 0, y) \), \( 2\beta := \angle(y, 0, z) \) and \( 2\gamma := \angle(z, 0, x) \), where \( a, \beta, \gamma \in (0, \pi/2) \). Set \( \tilde{\rho}_a := s_D(x, y), \tilde{\rho}_\beta := s_D(y, z) \) and \( \tilde{\rho}_\gamma := s_D(z, x) \). Without loss of generality, we may assume that the triangle \( \Delta xyz \) is positively oriented.

Note that \( \max(\tilde{\rho}_a, \tilde{\rho}_\beta, \tilde{\rho}_\gamma) \leq \rho < 1 \).

The expression
\[
\text{arctanh}_{S_D}(x, y) + \text{arctanh}_{S_D}(y, z) - \text{arctanh}_{S_D}(x, z) = \frac{1}{2} \ln \left(1 + \frac{2(1 + \tilde{\rho}_a \tilde{\rho}_\beta)}{(1 - \tilde{\rho}_a)(1 - \tilde{\rho}_\beta)(1 + \tilde{\rho}_\gamma)} \left(\tilde{\rho}_a + \tilde{\rho}_\beta - \tilde{\rho}_\gamma + \tilde{\rho}_a \tilde{\rho}_\beta \tilde{\rho}_\gamma\right)\right)
\]

has the same sign as \( E(\alpha, \beta, \gamma) := \tilde{\rho}_a + \tilde{\rho}_\beta - \tilde{\rho}_\gamma - \tilde{\rho}_a \tilde{\rho}_\beta \tilde{\rho}_\gamma \).

By symmetry, \( E(\alpha, \beta, \gamma) = E(\beta, \alpha, \gamma), E(\alpha, \gamma, \beta) = E(\gamma, \alpha, \beta) \) and \( E(\beta, \gamma, \alpha) = E(\gamma, \beta, \alpha) \).

We have to prove that \( E(\alpha, \beta, \gamma), E(\beta, \gamma, \alpha) \) and \( E(\alpha, \gamma, \beta) \) are always nonnegative.

We will consider \( f : \left[0, \frac{\omega_0}{2}\right] \to \mathbb{R} \) be defined by \( f(\theta) = \frac{\rho \sin \theta}{\sqrt{1 + \rho^2 - 2 \rho \cos \theta}} \) as in the previous lemma.

**Case 1.** Assume that \( 2(\alpha + \beta) \leq \omega_0 \).

Obviously, \( 2a \leq \omega_0, 2\beta \leq \omega_0 \) and \( 2\gamma = 2(a + \beta) \leq \omega_0 \); hence, \( \tilde{\rho}_\theta = f(\varphi) \) for \( \varphi \in \{\alpha, \beta, \gamma\} \).

We have \( 0 < \alpha < \gamma \leq \frac{\omega_0}{2} \) and \( 0 < \beta < \gamma \leq \frac{\omega_0}{2} \); therefore, \( \tilde{\rho}_a < \tilde{\rho}_\gamma \) and \( \tilde{\rho}_\beta < \tilde{\rho}_\gamma \), since \( f \) is increasing on \([0, \omega_0] \).

Then \( E(\alpha, \gamma, \beta) = \tilde{\rho}_a (1 - \tilde{\rho}_\beta \tilde{\rho}_\gamma) + \tilde{\rho}_\gamma - \tilde{\rho}_\beta > 0 \) and \( E(\beta, \gamma, \alpha) = \tilde{\rho}_\beta (1 - \tilde{\rho}_a \tilde{\rho}_\gamma) + \tilde{\rho}_\gamma - \tilde{\rho}_a > 0 \).

We may write
\[
E(\alpha, \beta, \gamma) = (1 + f(\alpha)f(\gamma - \alpha)) \left(\frac{f(\alpha) + f(\gamma - \alpha)}{1 + f(\alpha)f(\gamma - \alpha)} - f(\gamma)\right),
\]
where $0 < \alpha < \gamma$.

Fix $\gamma \in \left[0, \frac{\omega_0}{2}\right]$. With the notation from Lemma 5 (3), we have

$$E(\alpha, \beta, \gamma) = (1 + f(\alpha)f(\gamma - \alpha))(R_{\gamma}(\alpha) - f(\gamma)).$$

Since $R_{\gamma}$ is increasing on $[0, \frac{\pi}{2}]$ and decreasing on $[\frac{\pi}{2}, \gamma]$, we have $R_{\gamma}(\theta) > R_{\gamma}(0) = R_{\gamma}(\gamma) = f(\gamma)$ for all $\theta \in (0, \gamma)$.

Then $E(\alpha, \beta, \gamma) > 0$.

**Case 2.** Assume that $2\alpha \leq \omega_0$ and $2\beta \leq \omega_0$ and $2(\alpha + \beta) > \omega_0$.

**Subcase 2.1.** Assume that $2(\alpha + \beta) \leq \pi$.

Then $2\gamma = 2(\alpha + \beta) > \omega_0$; hence, $\bar{\rho}_{\gamma} = \rho = \max\{\bar{\rho}_{\alpha}, \bar{\rho}_{\beta}\}$.

We see that $E(\alpha, \gamma, \beta) = \bar{\rho}_{\alpha}(1 - \bar{\rho}_{\alpha}\rho_{\gamma}) + \bar{\rho}_{\gamma} - \rho_{\beta} > 0$ and $E(\beta, \gamma, \alpha) = \bar{\rho}_{\beta}(1 - \bar{\rho}_{\beta}\rho_{\gamma}) + \rho_{\gamma} - \bar{\rho}_{\alpha} > 0$.

Now

$$E(\alpha, \beta, \gamma) = E\left(\alpha, \beta, \frac{\omega_0}{2}\right) = f(\alpha + f(\beta) - \rho - \rho f(\alpha)f(\beta) = f(\beta)(1 - \rho f(\alpha)) + f(\alpha) - \rho.$$

Since $f$ is increasing on $[0, \frac{\omega_0}{2}]$ and $\frac{\omega_0}{2} - \alpha < \beta < \frac{\omega_0}{2}$, we have $E(\alpha, \beta, \frac{\omega_0}{2}) > E(\alpha, \frac{\omega_0}{2} - \alpha, \frac{\omega_0}{2})$.

Let $\alpha' = \alpha, \beta' = \frac{\omega_0}{2} - \alpha$ and $\gamma' = \frac{\omega_0}{2}$. As $2(\alpha' + \beta') = \omega_0$, according to Case 1 we have $E(\alpha', \beta', \gamma') > 0$, i.e., $E(\alpha, \frac{\omega_0}{2} - \alpha, \frac{\omega_0}{2}) > 0$.

The latter two inequalities show that $E(\alpha, \beta, \gamma) > 0$.

**Subcase 2.2.** Assume that $2(\alpha + \beta) > \pi$.

Then $2\gamma = 2\pi - 2(\alpha + \beta)$. We compare $2\gamma$ to $\omega_0$.

**Subcase 2.2.1.** Assume that $2\gamma < \omega_0$.

Note that $\omega_0 \geq \frac{\pi}{2}$, that is, $0 < \rho \leq \frac{\pi}{2}$.

For each $\varphi \in \{\alpha, \beta, \gamma\}$ we have $2\varphi \leq \omega_0$; hence, $\bar{\rho}_{\varphi} = f(\varphi)$.

Since $\beta = \pi - \gamma - \alpha$, we have $E(\alpha, \beta, \gamma) = (1 + f(\alpha)f(\gamma - \alpha))(R_{\pi - \gamma}(\alpha) - f(\gamma))$, where $0 < \pi - \gamma - \alpha \leq \frac{\omega_0}{2} \leq \frac{\omega_0}{2} - \gamma$. We have $\frac{\pi - \gamma - \alpha}{2} = \frac{\pi}{2} - \frac{\omega_0}{2}$ with equality only if $2\alpha = 2\beta = \omega_0$. In addition, $\frac{\pi - \gamma}{2} > \pi - \gamma - \frac{\omega_0}{2}$, with equality only if $2\alpha = 2\beta = \omega_0$.

If $2\alpha = 2\beta = \omega_0$, then $\bar{\rho}_{\alpha} = \bar{\rho}_{\beta} = \rho$; hence, $E(\alpha, \beta, \gamma) = 2\rho - \bar{\rho}_{\gamma} - \rho^2\bar{\rho}_{\gamma} = \rho(1 - \rho^2) > 0$.

Letting aside the case $2\alpha = 2\beta = \omega_0$, we get

$$0 < \pi - \gamma - \frac{\omega_0}{2} < \pi - \frac{\omega_0}{2} < \frac{\omega_0}{2}.$$

Fix $\gamma \in \left[0, \frac{\omega_0}{2}\right]$. From Lemma 5 (3), $R_{\pi - \gamma}$ is increasing on $\left[\pi - \gamma - \frac{\omega_0}{2}, \pi - \frac{\omega_0}{2}\right]$ and decreasing on $\left[\frac{\pi}{2}, \pi - \frac{\omega_0}{2}\right]$.

The minimum of $R_{\pi - \gamma}$ on $\left[\pi - \gamma - \frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$ is $\frac{f(\omega_0)}{1+f(\omega_0)}$, attained at both endpoints of the interval.

It follows that $R_{\pi - \gamma}(\theta) - f(\gamma) \geq \frac{f(\omega_0)}{1+f(\omega_0)} - f(\gamma) > 0$ for every $\theta \in \left[\pi - \gamma - \frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$; hence, $E(\alpha, \beta, \gamma) > 0$.

Due to the symmetry of the assumptions $2\varphi \leq \omega_0$ for $\varphi \in \{\alpha, \beta, \gamma\}$ and $\alpha + \beta + \gamma = \pi$, it follows similarly that $E(\alpha, \gamma, \beta) > 0$ and $E(\beta, \gamma, \alpha) > 0$.

**Subcase 2.2.2.** Assume that $2\gamma > \omega_0$. Then $\bar{\rho}_{\gamma} = \rho$.

We have $E(\alpha, \gamma, \beta) = \bar{\rho}_{\alpha} + \bar{\rho}_{\gamma} - \rho_{\beta} - \rho_{\alpha}\rho_{\beta}\rho_{\gamma} = \bar{\rho}_{\alpha}(1 - \rho_{\alpha}\rho_{\beta}) + \rho_{\gamma} - \rho_{\beta} > 0$ and $E(\beta, \gamma, \alpha) = \bar{\rho}_{\beta}(1 - \rho_{\beta}\rho_{\gamma}) + \rho_{\alpha} - \rho_{\beta} > 0$.

Note that $2\alpha \leq \omega_0, 2\beta \leq \omega_0$ and $2(\alpha + \beta) > \pi$ imply $\omega_0 > \frac{\pi}{2}$.

We have $2\alpha \leq \omega_0, 2\beta \leq \omega_0$ and $\frac{\pi}{2} < \alpha + \beta = \pi - \gamma \leq \min\left\{\omega_0, \pi - \frac{\omega_0}{2}\right\}$.

Here $\min\left\{\omega_0, \pi - \frac{\omega_0}{2}\right\} = \omega_0$ if $\frac{\pi}{2} < \omega_0 < 2\pi$ and $\min\left\{\omega_0, \pi - \frac{\omega_0}{2}\right\} = \pi - \frac{\omega_0}{2}$ if $2\pi \leq \omega_0 < \pi$. 

We write
\[ E(\alpha, \beta, \gamma) = (1 + f(\alpha)f(\pi - \gamma - \alpha))(R_{\pi - \gamma}(\alpha) - \rho), \]
where \( \pi - \gamma - \frac{\omega_0}{\pi} \leq \alpha \leq \frac{\omega_0}{\pi} < \pi - \gamma \). In addition, \( \pi - \gamma - \frac{\omega_0}{\pi} \leq \frac{\pi - \gamma}{2} \leq \frac{\omega_0}{\pi} \).

Note that \( \pi - \gamma - \frac{\omega_0}{\pi} = \frac{\pi - \gamma}{2} \) if and only if \( 2\alpha = 2\beta = \omega_0 \) in which case \( \tilde{\rho}_\alpha = \tilde{\rho}_\beta = \rho \); hence,
\[ E(\alpha, \beta, \gamma) = 2\rho - \rho - \rho^3 = \rho \left( 1 - \rho^2 \right) > 0. \]

Similarly, \( \frac{\pi - \gamma}{2} = \frac{\omega_0}{\pi} \) if and only if \( 2\alpha = 2\beta = \omega_0 \).

We may assume that \( \pi - \gamma - \frac{\omega_0}{\pi} < \frac{\pi - \gamma}{2} < \frac{\omega_0}{\pi} \). Fix \( \gamma \in \left[ \frac{\omega_0}{\pi}, \frac{\pi}{2} \right] \).

As in Subcase 2.2.1. \( R_{\pi - \gamma} \) is increasing on \( \left[ \pi - \gamma - \frac{\omega_0}{\pi}, \frac{\pi - \gamma}{2} \right] \) and decreasing on \( \left[ \frac{\pi - \gamma}{2}, \frac{\omega_0}{\pi} \right] \).

The minimum of \( R_{\pi - \gamma} \) on \( \left[ \pi - \gamma - \frac{\omega_0}{\pi}, \frac{\omega_0}{\pi} \right] \) is \( \frac{f(\omega_0)}{1+f(\omega_0)}f(\pi - \gamma - \frac{\omega_0}{\pi}) \).

It follows that \( R_{\pi - \gamma}(\theta) - \rho \geq f(\omega_0) + f(\pi - \gamma - \frac{\omega_0}{\pi}) - \rho > f(\omega_0) - \rho = 0 \)
for every \( \theta \in \left[ \pi - \gamma - \frac{\omega_0}{\pi}, \frac{\omega_0}{\pi} \right] \), in particular \( E(\alpha, \beta, \gamma) > 0 \).

Case 3. Assume that \( 2\alpha \geq \omega_0 \) or \( 2\beta \geq \omega_0 \).

We may consider that \( 2\alpha \geq \omega_0 \), the other case being analogous. Then \( \tilde{\rho}_\alpha = \rho \geq \max \{ \tilde{\rho}_\beta, \tilde{\rho}_\gamma \} \).

We have \( E(\alpha, \beta, \gamma) = \tilde{\rho}_\beta(1 - \rho \tilde{\rho}_\gamma) + \rho - \tilde{\rho}_\gamma > 0 \) and \( E(\alpha, \gamma, \beta) = \tilde{\rho}_\gamma(1 - \rho \tilde{\rho}_\beta) + \rho - \tilde{\rho}_\beta > 0 \).

It remains to analyze the sign of \( E(\beta, \gamma, \alpha) = \tilde{\rho}_\gamma + \tilde{\rho}_\gamma - \rho - \rho \tilde{\rho}_\gamma \).

Case 3.1. Assume that \( 2\beta \geq \omega_0 \) or \( 2\gamma \geq \omega_0 \).

If \( 2\beta \geq \omega_0 \), then \( \tilde{\rho}_\gamma = \rho \) and \( E(\beta, \gamma, \alpha) = \rho + \tilde{\rho}_\gamma - \rho - \rho^2 \tilde{\rho}_\gamma = \tilde{\rho}_\gamma(1 - \rho^2) > 0 \).

Similarly, if \( 2\gamma \geq \omega_0 \), then \( \tilde{\rho}_\gamma = \rho \) and \( E(\beta, \gamma, \alpha) = \tilde{\rho}_\gamma + \rho - \rho^2 \tilde{\rho}_\beta = \tilde{\rho}_\gamma(1 - \rho^2) > 0 \).

Case 3.2. Assume that \( 2\beta \leq \omega_0 \) and \( 2\gamma \leq \omega_0 \).

We cannot have \( 2(\alpha + \beta) \leq \pi \), since this implies \( 2\gamma = 2(\alpha + \beta) > 2\alpha \geq \omega_0 \), a contradiction. Then \( 2(\alpha + \beta) > \pi \) and \( 2\gamma = 2\pi - 2(\alpha + \beta) \).

In Subcase 2.2.2. we proved that \( E(\alpha, \beta, \gamma) > 0 \) under the following assumptions:
\( 2(\alpha + \beta + \gamma) = 2\pi, 2\alpha \leq \omega_0, 2\beta \leq \omega_0 \) and \( 2\gamma \geq \omega_0 \).

In the present case, \( 2(\alpha + \beta + \gamma) = 2\pi, 2\beta \leq \omega_0, 2\gamma \leq \omega_0 \) and \( 2\alpha \geq \omega_0 \); hence \( E(\beta, \gamma, \alpha) > 0 \). \( \square \)

**Corollary 3.** Let \( F : [0,1) \to [0,\infty) \) with \( F^{-1}(\{0\}) = \{0\} \). If \( F \circ \tanh \) is subadditive and nondecreasing on \( [0, \infty) \), then the restriction of \( F \circ s_D \) to every circle \( |z| = r < 1 \) is a metric. Moreover, if \( F \) is subadditive and nondecreasing on \( [0,1) \), then the restriction of \( F \circ s_D \) to the entire unit disk is a metric.

**Proof.** Let \( r \in (0,1) \). Theorem 2 shows that the restriction of \( \text{arctanh}_D \) to the circle \( |z| = r \) is a metric.

The function \( G := F \circ \tanh \) is metric-preserving. Therefore, \( F \circ s_D = G \circ \text{arctanh}_D \) is a metric on the circle \( |z| = r \). \( \square \)

**Remark 6.** Triangle inequality for the restriction of \( \text{arctanh}_D \) to any circle \( |z| = r \) with \( r \in (0,1) \) is always strict, as we see from the proof of Theorem 2. Therefore, given \( a,b \in (0,\infty) \) we cannot find \( x,y,z \) on a circle \( |z| = r \) such that \( \text{arctanh}_D(x,y) = a, \text{arctanh}_D(y,z) = b \) and \( \text{arctanh}_D(x,z) = a + b \). This prevents us from obtaining the subadditivity of \( F \circ \tanh \) under the assumption that \( F : [0,1) \to [0,\infty) \) with \( F^{-1}(\{0\}) = \{0\} \) is metric-preserving with respect to the restriction of the triangular ratio metric \( s_D \) to every circle \( |z| = r < 1 \).
We prove a functional inequality similar to (4) satisfied by continuous functions \( F : [0, 1) \rightarrow [0, \infty) \) with \( F^{-1}([0]) = \{0\} \) which are metric-preserving with respect to the restriction of the triangular ratio metric \( s_D \) to every circle \( |z| = r < 1 \).

**Theorem 3.** Assume that the continuous amenable function \( F : [0, 1) \rightarrow [0, \infty) \) is metric-preserving with respect to the restriction of the triangular ratio metric \( s_D \) to every circle \( |z| = r < 1, r \in (0, 1) \). Then, for every \( \lambda, \mu \in (0, 1) \), the following inequality holds:

\[
F\left( \frac{\lambda \sqrt{1 - \mu^2} + \mu \sqrt{1 - \lambda^2}}{\sqrt{\left(\lambda \sqrt{1 - \mu^2} + \mu \sqrt{1 - \lambda^2}\right)^2 + (1 - \lambda^2)(1 - \mu^2)}} \right) \leq F(\lambda) + F(\mu). \tag{9}
\]

Equivalently, for every \( a, b \in [0, \infty) \) we have

\[
F\left( \frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}} \right) \leq F(\tanh(a)) + F(\tanh(b)). \tag{10}
\]

**Proof.** For \( \lambda = 0 \) or \( \mu = 0 \) the inequality is trivial. Fix \( \lambda, \mu \in (0, 1) \).

Denote by \( C_r \) the circle \( |z| = r < 1 \).

For every \( r \) with \( \max(\lambda, \mu) < r < 1 \) there exist \( x_r, y_r, z_r \in C_r \) such that, denoting \( 2\alpha := \angle(x_r, 0, y_r) \) and \( 2\beta := \angle(y_r, 0, z_r) \), the following conditions are satisfied:

(i) \( \alpha, \beta \in \left(0, \frac{1}{2} \right) \arccos r \);

(ii) \( d(x_r, y_r) = \lambda \) and \( d(y_r, z_r) = \mu \).

Using the Formula (8) we look for \( \alpha \in \left(0, \frac{1}{2} \arccos r \right) \), i.e., with \( \cos \alpha \in \left(\sqrt{\frac{1}{1 + r^2}}, 1 \right) \), such that

\[
\frac{r \sin \alpha}{\sqrt{1 + r^2 - 2r \cos \alpha}} = \lambda.
\]

The above requirements are satisfied if and only if \( \cos \alpha = \frac{1}{2} \left(\lambda^2 + \sqrt{(1 - \lambda^2)(r^2 - \lambda^2)}\right) \). Then we compute \( \sin \alpha = \frac{\lambda}{\sqrt{1 - \lambda^2 + r^2 - \lambda^2}} \). Taking \( x_r \in C_r \) arbitrary and \( y_r = x_r e^{i\alpha} \), it follows that \( d(x_r, y_r) = \lambda \).

Similarly, we find an unique \( \beta \in \left(0, \frac{1}{2} \arccos r \right) \) such that \( \frac{r \sin \beta}{\sqrt{1 - r^2 - 2r \cos \beta}} = \mu \) and obtain

\[
\cos \beta = \frac{1}{2} \left(\mu^2 + \sqrt{(1 - \mu^2)(r^2 - \mu^2)}\right) \quad \text{and} \quad \sin \beta = \frac{\mu}{\sqrt{1 - \mu^2 + \sqrt{r^2 - \mu^2}}}.
\]

Now, taking \( z_r = y_r e^{i\beta} \), it follows that \( d(y_r, z_r) = \mu \).

Denote \( 2\gamma := \angle(z_r, 0, x_r) \). Since \( \alpha, \beta \in \left(0, \frac{1}{2} \arccos r \right) \), it follows that \( 2\gamma = 2(\alpha + \beta) \in (0, 2 \arccos r) \subset (0, \pi) \); therefore, \( 2\gamma = 2(\alpha + \beta) < 2 \arccos r \). Using (8), it follows that

\[
d(x_r, z_r) = \frac{r \sin(\alpha + \beta)}{\sqrt{1 + r^2 - 2r \cos(\alpha + \beta)}}.
\]

Computing \( \sin(\alpha + \beta) \) and \( \cos(\alpha + \beta) \), and using the notations \( H(r, \tau) = \sqrt{1 - \tau^2} + \sqrt{r^2 - \tau^2} \) and \( K(r, \tau) = \tau^2 + \sqrt{(1 - \tau^2)(r^2 - \tau^2)} \), we obtain

\[
d(x_r, z_r) = \frac{1 - r^2}{r} \frac{A(r, \lambda, \mu)}{B(r, \lambda, \mu)}
\]

where

\[
A(r, \lambda, \mu) = \lambda \frac{K(r, \mu)}{H(r, \mu)} + \mu \frac{K(r, \lambda)}{H(r, \mu)}
\]
and
\[ B(r, \lambda, \mu) = \left( 1 + r^2 - \frac{2}{r} K(r, \lambda) K(r, \mu) - \lambda \mu (1 - r^2)^2 \frac{1}{H(r, \lambda) H(r, \mu)} \right)^{1/2}. \]

Let the function \( F : [0, 1) \to [0, \infty) \) be amenable and metric-preserving with respect to the restriction of the triangular ratio metric \( \tilde{s}_D \) to every circle \( |z| = r < 1. \)

Let \( r \) satisfying \( \max(\lambda, \mu) < r < 1. \) The triangle inequality \( F(d(x_r, z_r)) \leq F(d(x_r, y_r)) + F(d(y_r, z_r)) \) may be written as
\[
F \left( \frac{1 - r^2}{r} A(r, \lambda, \mu) \right) \leq F(\lambda) + F(\mu). \tag{11}
\]

We compute
\[
\lim_{r \to 1^+} \frac{1 - r^2}{r} A(r, \lambda, \mu) = \frac{\lambda \sqrt{(1 - \mu_2^2) + \mu \sqrt{(1 - \lambda_2^2)}}}{2 \lambda \mu \sqrt{(1 - \lambda^2)(1 - \mu^2) + 1 - \lambda^2 \mu^2}}.
\]

We have
\[
\frac{\lambda \sqrt{1 - \mu^2 + \mu \sqrt{1 - \lambda^2}}}{\sqrt{2 \lambda \mu \sqrt{(1 - \lambda^2)(1 - \mu^2) + 1 - \lambda^2 \mu^2}}} = \frac{\lambda \sqrt{1 - \mu^2 + \mu \sqrt{1 - \lambda^2}}}{\sqrt{(\lambda \sqrt{1 - \mu^2 + \mu \sqrt{1 - \lambda^2}})^2 + (1 - \lambda^2)(1 - \mu^2)}} \in [0, 1) \text{ whenever } \lambda, \mu \in (0, 1).
\]

If \( F \) is continuous on \([0, 1)\), then letting \( r \) tend to 1 from below in inequality (11) we get (9).

Let \( a, b \in [0, \infty) \). Denote \( \lambda = \tanh(a) \) and \( \mu = \tanh(b) \). Then
\[
\frac{\lambda \sqrt{1 - \mu_2^2 + \mu \sqrt{1 - \lambda_2^2}}}{\sqrt{2 \lambda \mu \sqrt{(1 - \lambda_2^2)(1 - \mu_2^2) + 1 - \lambda_2 \mu^2}}} = \frac{\tanh(a) + \tanh(b)}{\sqrt{1 + (\tanh(a) + \tanh(b))^2}}. \]

Since the function \( \tanh : [0, \infty) \to (0, 1) \) is surjective, the inequalities (9) and (10) are equivalent. \( \square \)

**Remark 7.** Since \( \sinh \) is supradditive and the function \( x \mapsto \frac{x}{\sqrt{1 + x}} \) is increasing on \( \mathbb{R} \), we have
\[
\frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}} \leq \frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}} = \tanh(a + b).
\]

If \( F \) is nonincreasing, then inequality (10) implies the subadditivity of the function \( F \circ \tanh \) on \([0, \infty)\). If \( F \) is nondecreasing, then inequality (10) is implied by the subadditivity of the function \( F \circ \tanh \) on \([0, \infty)\).

Numerical experiments show that \( \frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}} \) and \( \tanh(a + b) \) are close to each other for all \( a, b \in [0, \infty) \).

**Lemma 6.** For every \( a, b \in [0, \infty) \) we have
\[
0 \leq \tanh(a + b) - \frac{\sinh(a) + \sinh(b)}{\sqrt{1 + (\sinh(a) + \sinh(b))^2}} \leq \frac{2\sqrt{t_0(t_0 - 1)}}{2t_0 - 1} - \frac{2\sqrt{t_0 - 1}}{4t_0 - 3},
\]
where \( t_0 \) is the unique real positive root of the polynomial \( P(t) = 16t^4 - 16t^3 - 56t^2 + 80t - 27 \). Equivalently, for all \( \lambda, \mu \in (0, 1) \)
\[
0 \leq \frac{\lambda + \mu}{1 + \lambda \mu} - \frac{\lambda \sqrt{1 - \mu^2} + \mu \sqrt{1 - \lambda^2}}{\sqrt{(\lambda \sqrt{1 - \mu^2} + \mu \sqrt{1 - \lambda^2})^2 + (1 - \lambda^2)(1 - \mu^2)}} \leq \frac{2\sqrt{t_0(t_0 - 1)}}{2t_0 - 1} - \frac{2\sqrt{t_0 - 1}}{4t_0 - 3}.
\]
Proof. Let \( E(x, y) = \tanh(x + y) - \frac{\sinh(x) + \sinh(y)}{\sqrt{1 + (\sinh(x) + \sinh(y))^2}} \), where \( x, y \in [0, \infty) \). We observe that \( E(x, y) \) tends to zero as \( x \to 0 \) or \( y \to 0 \), respectively, as \( x \to \infty \) or \( y \to \infty \). Then there exists the maximum of \( E \) on \([0, \infty) \times [0, \infty)\), attained at some point \((x_0, y_0) \in [0, \infty) \times [0, \infty)\).

The partial derivatives \( \frac{\partial}{\partial x}(x, y) = \frac{1}{(\cosh(x + y))^2} - \frac{\cosh(x)}{(1 + (\sinh(x) + \sinh(y))^2)^{3/2}} \) and \( \frac{\partial}{\partial y}(x, y) = \frac{1}{(\cosh(x + y))^2} - \frac{\cosh(y)}{(1 + (\sinh(x) + \sinh(y))^2)^{3/2}} \) vanish at \((x_0, y_0)\); hence, \( x_0 = y_0 \) and \( x_0 > 0 \) is a solution of the equation

\[
\frac{1}{(\cosh(2x))^2} = \frac{\cosh(x)}{\left(1 + 4(\sinh(x))^2\right)^{3/2}}.
\]

Using the change of variable \((\cosh(x))^2 = t\), the above equation transforms into \(t(2t - 1)^4 = (4t - 3)^3\). However, \( t(2t - 1)^4 - (4t - 3)^3 = (t - 1)P(t) \) and \((\cosh(x_0))^2 > 1\) is a root of \(P\). It turns out that \(P\) has one positive root \(t_0\), one negative root and two complex nonreal roots. Then \(\cosh(x_0) = \sqrt{t_0}\) and

\[
\max\{E(x, y) : (x, y) \in [0, \infty) \times [0, \infty)\} = E(x_0, x_0) = \tanh(2x_0) - \frac{2 \sinh(x_0)}{\sqrt{1 + 4(\sinh(x_0))^2}}
\]

\[
= \frac{2\sqrt{t_0(t_0 - 1)}}{2t_0 - 1} - \frac{2\sqrt{t_0 - 1}}{\sqrt{4t_0^2 - 3}}.
\]

\[\square\]

Remark 8. Using the approximate value \( t_0 \cong 1.6638 \) we get \( \frac{2\sqrt{t_0(t_0 - 1)}}{2t_0 - 1} - \frac{2\sqrt{t_0 - 1}}{\sqrt{4t_0^2 - 3}} \cong 0.050705. \)

5. The Case of Barrlund Metric with \( p = 2 \) on a Canonical Plane Domain

We will consider Barrlund metrics on canonical domains in plane: the upper half plane and the unit disk. For \( p = 2 \) and \( G \in \{\mathbb{H}, \mathbb{D}\} \) explicit formulas for \( b_{G,p} \) have been proved in [23], as follows:

\[
b_{\mathbb{H},2}(z_1, z_2) = \frac{\sqrt{2}|z_1 - z_2|}{\sqrt{|z_1 - z_2|^2 + (\text{Im}(z_1 + z_2))^2}} \quad \text{for all } z_1, z_2 \in \mathbb{H}
\]

and

\[
b_{\mathbb{D},2}(w_1, w_2) = \frac{|w_1 - w_2|}{\sqrt{2 + |w_1|^2 + |w_2|^2 - 2|w_1 + w_2|}} \quad \text{for all } w_1, w_2 \in \mathbb{D}.
\]

Using parallelogram’s rule, we can write

\[
b_{\mathbb{D},2}(w_1, w_2) = \frac{\sqrt{2}|w_1 - w_2|}{\sqrt{|w_1 - w_2|^2 + (2 - |w_1 + w_2|)^2}}.
\]

We can see that \( b_{\mathbb{H},2}(\mathbb{H} \times \mathbb{H}) = [0, \sqrt{2}] \) and \( b_{\mathbb{D},2}(\mathbb{D} \times \mathbb{D}) = [0, \sqrt{2}]. \)

Next, we study the restrictions of \( b_{\mathbb{H},2} \) to vertical rays \( V_{x_0} : (\text{Re}(z) = x_0 \text{ and } \text{Im}(z) > 0), x_0 \in \mathbb{R}, \) to rays through origin \( O_m : (\text{Im}(z) = m\text{Re}(z) \text{ and } \text{Im}(z) > 0) \), \( m \in \mathbb{R}^+ \) and to horizontal lines \( L_c : (\text{Im}(z) = c), c \in (0, \infty) \).

Proposition 5. Let \( F : [0, 1] \to [0, \infty) \) be an amenable function and \( \psi : \mathbb{R} \to (-1,1), \psi(t) = \frac{t^{p-1}}{\sqrt{2}^{p-1}}. \) The following are equivalent:

1. \( F \) is metric-preserving with respect to the restriction of \( b_{\mathbb{H},2} \) to every ray \( V_{x_0}, x_0 \in \mathbb{R}; \)
(2) $F$ is metric-preserving with respect to the restriction of $b_{\Xi,2}$ to some ray $V_{x_0}$, $x_0 \in \mathbb{R}$; 
(3) $F \circ |\psi|$ is subadditive on $\mathbb{R}$.

**Proof.** Consider the ray $V_{x_0} : (\text{Re}(z) = x_0$ and $\text{Im}(z) > 0), x_0 \in \mathbb{R}$. For $z_1 = x_0 + iy_1, z_2 = x_0 + iy_2 \in V_{x_0}$, denoting $\frac{y_1}{y_2} = e^u, u \in \mathbb{R}$ we have

$$b_{\Xi,2}(z_1, z_2) = \frac{|y_1 - y_2|}{\sqrt{y_1^2 + y_2^2}} = |e^u - 1| \sqrt{2e^{2u} + 1} = \psi(u).$$

The functions $F \circ \left( b_{\Xi,2} |_{V_{x_0}} \right)$ and $F \circ \psi$ are well-defined, since $\psi(u) \in [0,1)$ for every $u \in \mathbb{R}$.

For $z_k = x_0 + iy_k \in V_{x_0}$, $k = 1, 2, 3$ denote $\frac{y_1}{y_2} = e^u$ and $\frac{y_2}{y_3} = e^v$, where $u, v \in \mathbb{R}$.

With these notations, the triangle inequality

$$\text{(F \circ b_{\Xi,2})(z_1, z_3) \leq (F \circ b_{\Xi,2})(z_1, z_2) + (F \circ b_{\Xi,2})(z_2, z_3)} \quad (12)$$

is equivalent to

$$\text{(F \circ \psi)(u + v) \leq (F \circ \psi)(u) + (F \circ \psi)(v)}. \quad (13)$$

(1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Assume that $F$ is metric-preserving with respect to the restriction of $b_{\Xi,2}$ to some fixed ray $V_{x_0}$.

For every $u, v \in \mathbb{R}$ there exist $z_k = x_0 + iy_k \in V_{x_0}$, $k = 1, 2, 3$ such that $\frac{y_2}{y_1} = e^u$ and $\frac{y_3}{y_2} = e^v$. Since $z_1, z_2, z_3$ satisfy (12), it follows that (13) holds. Therefore, $F \circ \psi$ is subadditive on $\mathbb{R}$.

(3) $\Rightarrow$ (1) Assume that $F \circ \psi$ is subadditive on $\mathbb{R}$. Fix $x_0 \in \mathbb{R}$. For every $z_k = x_0 + iy_k \in V_{x_0}$, $k = 1, 2, 3$ there exist $u, v \in \mathbb{R}$ such that $\frac{y_2}{y_1} = e^u$ and $\frac{y_3}{y_2} = e^v$. Since $u$ and $v$ satisfy (13), we obtain the triangle inequality (12). It follows that the restriction of $F \circ b_{\Xi,2}$ to $V_{x_0}$ is a metric, q.e.d. \[\Box\]

**Proposition 6.** Let $O_m : (\text{Im}(z) = m \text{Re}(z)$ and $\text{Im}(z) > 0), m \in \mathbb{R} \setminus \{0\}$. Let $F : \left[0, \sqrt{2}\right) \rightarrow [0, \infty)$ be an amenable function and $\zeta_m : \mathbb{R} \rightarrow (-\sqrt{2}, \sqrt{2}), \zeta_m(u) = \frac{\tanh \left( \frac{u}{\sqrt{2}} \right) \sqrt{2} \left( \left( \frac{\tanh \left( \frac{u}{\sqrt{2}} \right) \sqrt{2} \right)^2 + \frac{m^2}{m^2 + 1} \right)} \sqrt{\left( \frac{\tanh \left( \frac{u}{\sqrt{2}} \right) \sqrt{2} \right)^2 + \frac{m^2}{m^2 + 1}}$. Then $F$ is metric-preserving with respect to the restriction of $b_{\Xi,2}$ to the ray $S_m$ if and only if $F \circ |\zeta_m|$ is subadditive on $\mathbb{R}$.

**Proof.** For $z_1 = x_1 + imx_1, z_2 = x_2 + imx_2 \in O_m$, denoting $\frac{x_2}{x_1} = e^u, u \in \mathbb{R}$ we have

$$b_{\Xi,2}(z_1, z_2) = \frac{|x_1 - x_2| \sqrt{2(m^2 + 1)}}{\sqrt{(1 + m^2)(x_1 - x_2)^2 + m^2(x_1 + x_2)^2}}$$

$$= \frac{|1 - \frac{x_2}{x_1}| \sqrt{2(m^2 + 1)}}{\sqrt{(1 + m^2)(1 - \frac{x_2}{x_1})^2 + m^2(1 + \frac{x_2}{x_1})^2}}$$

$$= \frac{|e^u - 1| \sqrt{2(m^2 + 1)}}{\sqrt{(1 + m^2)(1 - e^u)^2 + m^2(1 + e^u)^2}}$$

$$= \frac{\tanh \left( \frac{u}{\sqrt{2}} \right) \sqrt{2}}{\sqrt{\left( \frac{\tanh \left( \frac{u}{\sqrt{2}} \right) \sqrt{2} \right)^2 + \frac{m^2}{m^2 + 1}}} = \zeta_m(u).$$
The functions $F \circ \left( b_{\mathbb{H},2} |_{S_m} \right)$ and $F \circ |_{\kappa_m}$ are well-defined, since $|\kappa_m(u)| \in \left[0, \sqrt{2}\right]$ for every $u \in \mathbb{R}$.

For $z_k = (1 + im)x_k \in O_m$, $k = 1, 2, 3$ denote $\frac{z_k}{x_k} = e^u$ and $\frac{z_k}{x_k} = e^v$, where $u, v \in \mathbb{R}$.

With these notations, the triangle inequality

$$ (F \circ b_{\mathbb{H},2})(z_1, z_3) \leq (F \circ b_{\mathbb{H},2})(z_1, z_2) + (F \circ b_{\mathbb{H},2})(z_2, z_3) $$

is equivalent to

$$ (F \circ |_{\kappa_m})(u + v) \leq (F \circ |_{\kappa_m})(u) + (F \circ |_{\kappa_m})(v).$$

If $F$ is metric-preserving with respect to the restriction of $b_{\mathbb{H},2}$ to the ray $O_m$, then for every $u, v \in \mathbb{R}$ we find $z_k = (1 + im)x_k \in O_m$, $k = 1, 2, 3$ such that $\frac{z_k}{x_k} = e^u$ and $\frac{z_k}{x_k} = e^v$ and applying (14) we get (15). Conversely, if $F \circ \kappa_m$ is subadditive on $\mathbb{R}$, then for every $z_k = (1 + im)x_k \in O_m$, $k = 1, 2, 3$ we find $u, v \in \mathbb{R}$ such that $\frac{z_k}{x_k} = e^u$ and $\frac{z_k}{x_k} = e^v$ and applying (14) we get (15). 

\textbf{Proposition 7.} Let $F : \left[0, \sqrt{2}\right] \to [0, \infty)$ and $c > 0$. Denote $\varphi_c(t) = \frac{\sqrt{2}t}{\sqrt{t^2 + 4c^2}}$, $t \in \mathbb{R}$. The restriction of $F \circ b_{\mathbb{H},2}$ to the line $\Im(z) = c$ is a metric if and only if $F \circ |\varphi_c|$ is subadditive on $\mathbb{R}$.

\textbf{Proof.} By our assumption, for all $x_1, x_2, x_3 \in \mathbb{R}$ we have $F(b_{\mathbb{H},2}(x_1 + ic, x_2 + ic)) \leq F(b_{\mathbb{H},2}(x_1 + ic, x_3 + ic)) + F(b_{\mathbb{H},2}(x_2 + ic, x_3 + ic))$, i.e.,

$$ F\left(\frac{\sqrt{2}|x_1 - x_3|}{\sqrt{|x_1 - x_3|^2 + 4c^2}}\right) \leq F\left(\frac{\sqrt{2}|x_1 - x_2|}{\sqrt{|x_1 - x_2|^2 + 4c^2}}\right) + F\left(\frac{\sqrt{2}|x_2 - x_3|}{\sqrt{|x_2 - x_3|^2 + 4c^2}}\right).$$

Let $\varphi_c(t) = \frac{\sqrt{2}t}{\sqrt{t^2 + 4c^2}}$, $t \in [0, \infty)$. The above inequality is equivalent to

$$ (F \circ |\varphi_c|)(x_1 - x_3) \leq (F \circ |\varphi_c|)(x_1 - x_2) + (F \circ |\varphi_c|)(x_2 - x_3) \quad \text{for all } x_1, x_2, x_3 \in \mathbb{R}. \quad (16)$$

Note that $|\varphi_c(t)| \in \left[0, \sqrt{2}\right)$ for every $t \in [0, \infty)$; therefore, the functions $F \circ \left( b_{\mathbb{H},2} |_{\Im(z) = c} \right)$ and $F \circ |\varphi_c|$ are well-defined.

We see that $F \circ |\varphi_c|$ satisfies (16) if and only if $F \circ |\varphi_c|$ is subadditive on $\mathbb{R}$. 

Next, we study metric-preserving functions with respect to the restriction of the Barrlund distance on the unit disk $b_{\mathbb{D},2}$ to some one-dimensional manifolds, such as radial segments, diameters or circles centered at origin.

\textbf{Proposition 8.} Let $F : [0, 1) \to [0, \infty)$ be an amenable function and $\psi : \mathbb{R} \to (-1, 1)$, $\psi(t) = \frac{t^{\frac{1}{\sqrt{e^t+1}}}}{\sqrt{e^t+1}}$. The following are equivalent:

1. $F$ is metric-preserving with respect to the restriction of $b_{\mathbb{D},2}$ to every radial segment in the unit disk;
2. $F$ is metric-preserving with respect to the restriction of $b_{\mathbb{D},2}$ to some radial segment in the unit disk;
3. $F \circ |\psi|$ is subadditive on $\mathbb{R}$.

\textbf{Proof.} Obviously, (1) $\Rightarrow$ (2). Using the invariance of the Barrlund distance on the unit disk with respect to rotations around the origin, it follows that (2) $\Rightarrow$ (1), since (1) holds if and only if $F$ is metric-preserving with respect to the restriction of $b_{\mathbb{D},2}$ to the intersection $I = [0, 1)$ between the unit disk and the non-negative semiaxis.
In order to prove that (2) and (3) are equivalent, we may assume without loss of generality that the radial segment in (2) is \( I = [0, 1) \). For \( z_1 = x_1, z_2 = x_2 \in I \), denoting \( \frac{1-x_1}{x_1} = e^u \), \( u \in \mathbb{R} \) we have

\[
\begin{align*}
b_{D,2}(z_1, z_2) &= \frac{|x_1 - x_2|}{\sqrt{2 + x_1^2 + x_2^2 - 2(x_1 + x_2)}} = \frac{|(1-x_2) - (1-x_1)|}{\sqrt{(1-x_2)^2 + (1-x_1)^2}} \\
&= \frac{|e^u - 1|}{\sqrt{e^{2u} + 1}} = \psi(u).
\end{align*}
\]

For \( z_k = x_k \in I, k = 1, 2, 3 \) denote \( \frac{1-x_1}{x_1} = e^u \) and \( \frac{1-x_3}{x_2} = e^v \), where \( u, v \in \mathbb{R} \).

With these notations, the triangle inequality

\[
(F \circ b_{D,2})(z_1, z_3) \leq (F \circ b_{D,2})(z_1, z_2) + (F \circ b_{D,2})(z_2, z_3) \tag{17}
\]

is equivalent to

\[
(F \circ |\psi|)(u + v) \leq (F \circ |\psi|)(u) + (F \circ |\psi|)(v). \tag{18}
\]

Assume that \( F \) is metric-preserving with respect to the restriction of \( b_{D,2} \) to the radial segment \( I \). For every \( u, v \in \mathbb{R} \) we find \( z_k = x_k \in I, k = 1, 2, 3 \) such that \( \frac{1-x_k}{x_k} = e^u \) and \( \frac{1-x_k}{x_k} = e^v \). Indeed, we may choose any \( x_k \) between \( \max\{0, 1 - e^{-u}, 1 - e^{-u-v}\} \) and 1. Then \( \frac{1-x_1}{x_1} = e^u \) if and only if \( x_2 = 1 - e^u(1-x_1) \), but \( 0 < 1 - x_1 < e^{-u} \); therefore, \( 0 < x_2 < 1 \). Moreover, \( 0 < 1 - x_1 < e^{-u-v} \) implies \( x_2 > 1 - e^{-v} \). Since \( \frac{1-x_3}{x_2} = e^v \) if and only if \( x_3 = 1 - e^v(1-x_2) \), where \( 0 < 1 - x_2 < e^{-v} \), it follows that \( 0 < x_3 < 1 \). Now applying (17) we get (18).

Conversely, if \( F \circ |\psi| \) is subadditive on \( \mathbb{R} \), then for every \( z_k = x_k \in I, k = 1, 2, 3 \) we find \( u, v \in \mathbb{R} \) such that \( \frac{1-x_k}{x_k} = e^u \) and \( \frac{1-x_k}{x_k} = e^v \) and applying (18) we get (17). \( \Box \)

We give a sufficient condition for a function to be metric-preserving with respect to the restriction of the Barrlund metric \( b_{D,2} \) to some diameter of the unit disk, under the form of a functional inequality.

**Proposition 9.** Let \( F : \left( 0, \sqrt{2} \right] \to [0, \infty) \). Assume that the restriction of \( F \circ b_{D,2} \) to some diameter of the unit disk is a metric. Then

\[
F \left( \frac{r \sqrt{2}}{\sqrt{r^2 + 1}} \right) \leq 2F \left( \frac{r}{\sqrt{r^2 - 2r + 2}} \right) \text{ for all } r \in [0, 1). \tag{19}
\]

**Proof.** Since \( b_{D,2} \) is invariant to rotations around the origin, if a function is metric-preserving with respect to the restriction of the Barrlund metric \( b_{D,2} \) to some diameter of the unit disk, then that function is metric-preserving with respect to the restriction of the Barrlund metric \( b_{D,2} \) to every diameter of the unit disk. We may assume that the given diameter is on the real axis.

Note that \( b_{D,2}(0, w) = b_{D,2}(0, -w) = \frac{|w|}{\sqrt{2 + |w|^2 - 2|w|}} = \frac{2|w|}{\sqrt{2 + 2|w|^2}} = \frac{|w| \sqrt{2}}{\sqrt{1 + |w|^2}} \).

The above inequality writes as

\[
(F \circ b_{D,2})(r, -r) \leq (F \circ b_{D,2})(0, r) + (F \circ b_{D,2})(0, -r) \text{ for all } r \in [0, 1),
\]

which is true, due to the assumption that the restriction of \( F \circ b_{D,2} \) to the diameter \( \{ x + i0 : -1 < x < 1 \} \) of the unit circle is a metric. \( \Box \)
Remark 9. Let $F : [0, \sqrt{2}] \rightarrow [0, \infty)$. Assume that the restriction of $F \circ b_{\mathbb{D},2}$ to some radial segment of the unit disk is a metric. By Proposition 8, $F \circ \psi$ is subadditive on $\mathbb{R}$, where $\psi : \mathbb{R} \rightarrow [0,1)$, $\psi(t) = \frac{e^{2t} - 1}{\sqrt{e^{2t} - 1}}$. In particular, $F \left( \frac{|e^{a} - 1|}{\sqrt{e^{a} - 1}} \right) \leq 2F \left( \frac{|e^{a} - 1|}{\sqrt{e^{a} - 1}} \right)$ for every $a \in \mathbb{R}$. If $r \in [0,1)$, denoting $e^{a} = 1 - r$, the previous inequality becomes

$$F \left( \frac{r(2-r)}{\sqrt{(r-1)^4 + 1}} \right) \leq 2F \left( \frac{r}{\sqrt{r^2 - 2r + 2}} \right) \quad (20)$$

Note that $\frac{r\sqrt{2}}{\sqrt{r^2 + 1}} \leq \frac{r(2-r)}{(r-1)^4 + 1}$ for every $r \in [0,1)$. If $F$ is nondecreasing on $[0, \sqrt{2}]$, then the above inequality (20) is stronger than (19).

Propositions 5 and 8 are very similar and, together with Propositions 6 and 7, have a common pattern.

Lemma 7. Each of the functions $\psi : \mathbb{R} \rightarrow (-1,1)$ with $\psi(t) = \frac{e^{2t} - 1}{\sqrt{e^{2t} - 1}}$, $\varphi_\eta : \mathbb{R} \rightarrow (-\sqrt{2}, \sqrt{2})$ with $\varphi_\eta(u) = \frac{\tanh \left( \frac{u}{\sqrt{2}} \right)}{\sqrt{\left( \tanh \left( \frac{1}{\sqrt{2}} \right) \right)^2 + \frac{1}{r^2-1}}}$ and $\varphi_c : \mathbb{R} \rightarrow (-\sqrt{2}, \sqrt{2})$ with $\varphi_c(t) = \frac{\sqrt{2}t}{\sqrt{2t^2 + 4\pi^2}}$, generically denoted by $\varphi$, has the following properties: it is odd on $\mathbb{R}$, nonnegative, increasing and concave on $[0, \infty)$; hence, it is subadditive on $[0, \infty)$.

Lemma 8. If the restriction to $[0, \infty)$ of a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is nonnegative, nondecreasing and subadditive and if $|\varphi|$ is even on $\mathbb{R}$, then $|\varphi|$ is subadditive on $[0, \infty)$.

Proof. We prove that $|\varphi(a+b)| \leq |\varphi(a)| + |\varphi(b)|$ for every $a, b \in \mathbb{R}$. This is clear for $a, b \in [0, \infty)$, taking into account that the restriction to $[0, \infty)$ of $\psi$ is nonnegative and subadditive. If $a, b \in (-\infty, 0]$, using the fact that $|\varphi|$ is even on $\mathbb{R}$ and the previous case we get $|\varphi(a+b)| = |\varphi((-a) - (-b))| \leq |\varphi(-a)| + |\varphi(-b)| = |\varphi(a)| + |\varphi(b)|$. It remains to study the case where $a \cdot b < 0$. By symmetry, it suffice to assume that $a < 0 \leq b$ and $a + b > 0$. Then $|\varphi(a+b)| = |\varphi(a - b)| = |\varphi(a) - |b|) + |\varphi(a)| \leq |\varphi(a)| + |\varphi(b)|$, since $\varphi$ is nondecreasing and nonnegative on $[0, \infty)$. \qed

Corollary 4. (a) If $F : [0,1) \rightarrow [0, \infty)$ is subadditive, then $F$ is metric-preserving with respect to the restriction of $b_{\mathbb{D},2}$ to each vertical ray $V_{\lambda_0}$ and with respect to the restriction of $b_{\mathbb{D},2}$ to each radial segment of the unit disk.

(b) If $F : [0, \sqrt{2}) \rightarrow [0, \infty)$ is subadditive, then $F$ is metric-preserving with respect to the restrictions of $b_{\mathbb{D},2}$ to each oblique ray $S_{\lambda}$ and to each horizontal line $L_c$.

Proof. Using Lemmas 7 and 8, we see that the modulus of each of the functions $\psi$, $\varphi_\eta$ and $\varphi_c$ is a subadditive function on $\mathbb{R}$. The composition of two subadditive functions is subadditive. For (a) Then we apply Propositions 5 and 8 for (a), respectively, Propositions 6 and 7 for (b). \qed

Finally, we obtain a characterization of functions $F$ which are metric-preserving with respect to the restriction of $b_{\mathbb{D},2}$ to each circle centered at origin.

Proposition 10. Let $F : [0, \sqrt{2}) \rightarrow [0, \infty)$ be an amenable function, $r \in [0,1)$ and $\theta_r : \mathbb{R} \rightarrow \mathbb{R}$, $\theta_r(t) = \frac{r|\sin t|}{\sqrt{r^2 - 2r|\cos t| + 1}}$. Then the restriction of $F \circ b_{\mathbb{D},2}$ to the circle $|z| = r$ is a metric if and only if $F \circ \theta_r$ is subadditive on $\mathbb{R}$. 

we compute is equivalent to the functional inequality (10) related to the subadditivity of \( f \), namely Theorem 1 and Propositions 3, 4, 5, 6, 7, 8 and 10. Within the table we use the Barrlund metric on \( R \) the class of all nondecreasing subadditive self-maps on \( MP \) geometry and functional inequalities. For a metric space, transferring some special metrics to metrics, establishing connections between metric

\[ d(x, y) = \frac{|z_1 - z_2|}{\sqrt{z_1 + z_2}} = \frac{2r|\sin \frac{\alpha_1 - \alpha_2}{2}|}{\sqrt{2 + 2r^2 - 4r|\cos \frac{\alpha_1 - \alpha_2}{2}|}} = \frac{r|\sin \frac{\alpha_1 - \alpha_2}{2}|\sqrt{2}}{\sqrt{r^2 - 2r|\cos \frac{\alpha_1 - \alpha_2}{2}| + 1}} = \frac{r|\sin u|\sqrt{2}}{\sqrt{r^2 - 2r|\cos u| + 1}} = \theta_i(u). \]

For \( z_k = re^{ia_k} \in C \) with \( a_k \in \mathbb{R} \) for \( k = 1, 2, 3 \), we denote \( \frac{\alpha_1 - \alpha_2}{2} = u \) and \( \frac{\alpha_2 - \alpha_3}{2} = v \). Then

\[ u + v = \frac{\alpha_1 - \alpha_3}{2}. \]

With the above notations, the triangle inequality

\[ (F \circ b_{D,2})(z_1, z_3) \leq (F \circ b_{D,2})(z_1, z_2) + (F \circ b_{D,2})(z_2, z_3) \]

(21)

is equivalent to

\[ (F \circ \theta_i)(u + v) \leq (F \circ \theta_i)(u) + (F \circ \theta_i)(v). \]

(22)

First we assume that \( F \circ \theta_i \) is subadditive on \( \mathbb{R} \). As above, for every \( z_k = re^{ia_k} \in C \) with \( a_k \in \mathbb{R} \) for \( k = 1, 2, 3 \), we denote \( \frac{\alpha_1 - \alpha_3}{2} = u \) and \( \frac{\alpha_2 - \alpha_3}{2} = v \). It suffices to take \( a_3 = 0, a_2 = 2\alpha \) and \( a_1 = 2(u + v) \). Finally, applying (21) with \( z_k = re^{ia_k} \in C \) for \( k = 1, 2, 3 \) we get (22).

6. Conclusions

In this paper, we investigated properties related to subadditivity of the functions transferring some special metrics to metrics, establishing connections between metric geometry and functional inequalities. For a metric space, \( (X, d) \) such that \( d(x, y) \in [0, T) \) for all \( x, y \in X \), where \( 0 < T \leq \infty \), let us denote by \( MP(X, d) \) the class of functions \( f : [0, T) \to \mathbb{R} = [0, \infty) \) with the property that \( f \circ d \) is a metric on \( d \). It is known from the theory of metric-preserving functions that the intersection of all classes \( MP(X, d) \) includes the class of all nondecreasing subadditive self-maps on \( \mathbb{R}_+ \) and is included in the class of all subadditive self-maps on \( \mathbb{R}_+ \). We obtained functional inequalities satisfied by functions in \( MP(X, d) \) in several cases, where \( X \) is some subset of \( G \in \{ H, D \} \) and \( d \) is the restriction to \( X \) of an intrinsic metric on \( G \), namely the hyperbolic metric, the triangular ratio metric \( s_G \) or the Barrlund metric \( b_{G,2} \). We will denote by \( Sa((0, T)) \) the class of functions \( f : [0, T) \to \mathbb{R}_+ \) that are subadditive. In addition, denote by \( X_r \) the circle of radius \( r \in (0, 1) \) centered at origin.

We summarize in Table 1 most of our results, excepting Proposition 2 and Theorem 3, namely Theorem 1 and Propositions 3, 4, 5, 6, 7, 8 and 10. Within the table we use the abbreviation \( F = MP(X, d) \).

We determined the functions \( \psi \) (that is fixed), \( \alpha_m, \varphi_c \) and \( \theta_i \) (each depending only on the respective parameter).

Moreover, denoting \( d_r = s_p|_X \), we proved that every \( f \in \mathbb{R} \in (0, 1) \) \( MP(X, d_r) \) satisfies the functional inequality (10) related to the subadditivity of \( f \circ \tanh \), as follows. If \( f \) is nonincreasing on \( [0, 1) \) and satisfies (10), then \( f \circ \tanh \) is subadditive on \( \mathbb{R}_+ \). If \( f \) is nondecreasing on \( [0, 1) \) and \( f \circ \tanh \) is subadditive on \( \mathbb{R}_+ \), then \( f \) satisfies (10).
Since \( \text{arctanh}_D \) is a metric on \( D \), it would be interesting to know if \( \text{arctanh}_D \) is a metric on \( \mathbb{H} \) ([23] Conjecture 2.1). We proved that \( \text{arctanh}_D \) induces a metric on each diameter of \( D \) and on each circle of radius \( r \in (0, 1) \) centered at origin. The above conjecture remains open.

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### References

1. Petrusel, A.; Rus, I.A.; Şerban, M.A. The role of equivalent metrics in fixed point theory. *Topol. Methods Nonlinear Anal.* **2013**, *41*, 85–112.
2. Pongsriiam, P.; Termwuttipong, I. On metric-preserving functions and fixed point theorems. *Fixed Point Theory Appl.* **2014**, *179*, 14. [CrossRef]
3. Fujimura, M.; Mocanu, M.; Vuorinen, M. A new intrinsic metric and quasiregular maps. *Complex Anal. Its Synerg.* **2021**, *7*, 6. [CrossRef]
4. Hariri, P.; Klén, R.; Vuorinen, M. *Conformally Invariant Metrics and Quasiconformal Mappings*; Springer Monograph in Mathematics; Springer: Berlin, Germany, 2020.
5. Hästö, P. A new weighted metric: The relative metric I. *J. Math. Anal. Appl.* **2002**, *274*, 38–58. [CrossRef]
6. Borsík, J.; Doboš, J. On metric preserving functions. *Real Anal. Exch.* **1988**, *13*, 285–293. [CrossRef]
7. Doboš, J.; Piotrowski, Z. Some remarks on metric preserving functions. *Real Anal. Exch.* **1994**, *19*, 317–320. [CrossRef]
8. Doboš, J.; Piotrowski, Z. When distance means money. *Internat. J. Math. Ed. Sci. Technol.* **1997**, *28*, 513–518. [CrossRef]
9. Vallin, R.W. Continuity and differentiability aspects of metric-preserving functions. *Real Anal. Exch.* **1999**, *25*, 849–868. [CrossRef]
10. Jachymski, J.; Turobos, F. On functions preserving regular semimetrics and quasimetrics satisfying the relaxed polygonal inequality. *RACSAM* **2020**, *114*, 1. [CrossRef]
11. Corazza P. Introduction to metric-preserving functions. *Amer. Math. Monthly* **1999**, *104*, 309–323. [CrossRef]
12. Doboš, J. Metric Preserving Functions. Online Lecture Notes. Available online: [http://web.science.upjs.sk/jozefdobos/wp-content/uploads/2012/03/mpf1.pdf](http://web.science.upjs.sk/jozefdobos/wp-content/uploads/2012/03/mpf1.pdf) (accessed on 25 August 2021).
13. Dovgoshey, O.; Martio, O. Functions transferring metrics to metrics. *Beitr. Algebra Geom.* **2013**, *54*, 237–261. [CrossRef]
14. Bingham, N.H.; Ostaszewski, A.J. Generic subadditive functions. *Proc. Am. Math. Soc.* **2008**, *136*, 4257–4266. [CrossRef]
15. Matkowski, J.; Świątkowski, T. On subadditive functions. *Proc. Am. Math. Soc.* **1993**, *119*, 187–197. [CrossRef]
16. Bingham, N.H.; Ostaszewski, A.J. Additivity, subadditivity and linearity: automatic continuity and quantifier weakening. *Indag. Math.* 2017, 29, 687–713. [CrossRef]
17. Kuczma, M. *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy’s Equation and Jensen’s Inequality*, 2nd ed.; Springer Science & Business Media: Basel, Switzerland, 2009.
18. Ger, R.; Kuczma, M. On the boundedness and continuity of convex functions and additive functions. *Aequationes Math.* 1970, 4, 157–162. [CrossRef]
19. Beardon, A.F.; Minda, D. The Hyperbolic Metric and Geometric Function Theory. In *Quasiconformal Mappings and Their Applications*; Ponnusamy, S., Sugawa, T., Vuorinen, M., Eds.; Narosa Publishing House: New Delhi, India, 2007; pp. 9–56.
20. Hariri, P.; Vuorinen, M; Zhang, X. Inequalities and bilipschitz conditions for triangular ratio metric. *Rocky Mountain J. Math.* 2017, 47, 1121–1148. [CrossRef]
21. Fujimura, M.; Hariri, P.; Mocanu, M.; Vuorinen, M. The Ptolemy–Alhazen problem and spherical mirror reflection. *Comput. Methods Funct. Theory* 2019, 19, 135–155. [CrossRef]
22. Barrlund, A. The p-relative distance is a metric. *SIAM J. Matrix Anal. Appl.* 1999, 21, 699–702. [CrossRef]
23. Fujimura, M.; Mocanu, M.; Vuorinen, M. Barrlund’s distance function and quasiconformal maps. *Complex Var. Elliptic Equ.* 2021, 66, 1225–1255. [CrossRef]
24. Hariri, P.; Klén, R.; Vuorinen, M.; Zhang, X. Some remarks on the Cassinian metric. *Publ. Math. Debrecen* 2017, 90, 269–285. [CrossRef]