Equations of motion for a (non-linear) scalar field model as derived from the field equations

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July 11, 2018

Abstract

The problem of derivation of the equations of motion from the field equations is considered. Einstein’s field equations have a specific analytical form: They are linear in the second order derivatives and quadratic in the first order derivatives of the field variables. We utilize this particular form and propose a novel algorithm for the derivation of the equations of motion from the field equations. It is based on the condition of the balance between the singular terms of the field equation. We apply the algorithm to a nonlinear Lorentz invariant scalar field model. We show that it results in the Newton law of attraction between the singularities of the field moved on approximately geodesic curves. The algorithm is applicable to the \(N\)-body problem of the Lorentz invariant field equations.

PACS: 04.25.-g, 45.50.Dd, 95.10.Ce

1 Introduction

General Relativity (GR) is unique among the class of field theories in the treatment of the equations of motion. When the particles are modeled by the singularities of the field, their motion is completely determined by the field equation. By comparison, in classical electrodynamics, the equations of motion of matter sources are postulated independently from the field equations. A precise analysis of the basis assumptions of classical electrodynamics was given recently in the premetric axiomatic framework of Hehl and Obukhov \([1],[2]\). This way, two independent assumptions have to be postulated separately from the Maxwell field equations: (i) The agent of interaction — the Lorentz force expression, and (ii) The agent of inertia — the mass-times-acceleration term.

In GR, the field equations are nonlinear so a superposition of solutions does not satisfy the field equations. Consequently, the motion of sources cannot be completely independent of the field equations. Moreover, it was shown already in the early days of GR that the motion of a point-like particles is determined by the field equations and should not be postulated separately. This result was achieved in two essentially different ways:

(i) Due to Einstein and his collaborators \([3] — [9]\) (see also \([10] — [15]\)), the equations of motion of the point-like particles were derived from the vacuum field equations

\[
R_{ij} - \frac{1}{2}g_{ij}R = 0 .
\]  

(1)

Point-like particles are represented by the singularities of the gravitational field, i.e, of the solutions of \([11]\).

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(ii) Fock \[16\], see also \[17\], considered particles as represented by a suitable energy-momentum
tensor $T_{ij}$. Correspondingly, the equation of motion was derived from the full Einstein field equation
\[ R_{ij} - \frac{1}{2} g_{ij} R = T_{ij}. \]
\[ (2) \]
Papapetrou \[18\], does not assume a specific form of the energy-momentum. Thus his approximation
method is fairly general and, in principle, can be carried out to arbitrary high orders.

In both models, a suitable approximation scheme was used so that the equations of motion of
massive particles, were approximated by geodesic curves.

The Einstein-Infeld-Hoffmann (EIH) equations were used for high order PPN approximations of
the $N$-body problem \[19\] — \[22\], for calculation of the gravitational radiation reaction force \[23\],
and for describing the motion of a gyroscope \[24\], \[25\]. For the geodesic line approximation of the
motion of bodies of a sufficiently small size and mass, see \[26\] and the reference given therein.

The EIH-approach was used to establish arguments for the plausibility of the geodesic postulate.
In particular, Sternberg \[27\] proved the geodesic postulate in a general form for a system that
incorporates also Yang-Mills fields.

The aim of our paper is different. In this article, we adopt Einstein’s idea of describing point-like
particles by the singularities of a field. Then, some natural questions can be considered:

(i) What analytical properties of the Einstein field equations are used in order to derive the proper
equations of motion for singularities?

(ii) What parts of the field equation have to be identified with the agent of inertia and the agent
of the force?

(iii) What other nonlinear field equations lend themselves to derivability of the equations of motion?

In this article we submit partial answers. First we define a suitable class of the field equations.
These are linear in the second order derivatives and quadratic in the first order derivatives of the
field variables. We also require the equations to be Lorentz invariant. We show that the linear part
serves as the agent of inertia while the quadratic part is the agent of the force.

We suggest a nonlinear scalar Lorentz invariant field model. There is an exact static solution
to this equation, which is singular at $N$ points. This solution does not imply interaction between
singularities, it is non physical. Indeed, for the Newtonian gravity as formulated by a field theory,
$\triangle \phi = 0$, a similar exact solution\[^1\] $\phi = \sum_i m_i/r_i$ exists. It is rejected by an additional axiom,
which is Newton’s law of attraction. Also for Maxwell electrodynamics, the potentials satisfy (in
the Lorentz gauge) the wave equation $\Box A = 0$. An exact static solution is $A = \sum_i q_i/r_i$. Also this
solution is non physical (it does not describe the field of free non-constrained sources) because it
contradicts Coulomb law. In both examples, one cannot distinguish between the quasi-static and
dynamic solutions by the field equations.

We do not pre-postulate the law of interaction. It is derived from the field equation. For our
model, the non-physical exact static solution is rejected, the same way the quasi-static solution was
rejected in Newton’s and Maxwell’s theories. Instead, an approximate solution is constructed. When
our algorithm is applied to this solution, it yields the Newton-type law of attraction.

Our algorithm can be briefly described as follows.

(i) Start with a nonlinear Lorentz invariant field equation. Let its static spherical symmetric
solution (exact or approximate) be singular at a point. Due to Einstein’s description, this is
the field of one particle.

(ii) The field of a particle moving with a constant velocity is constructed by a Lorentz transforma-
tion of the static field.

(iii) The field of a particle moving on a curved trajectory\[^2\] $\psi(t)$ is constructed by a Lorentz transforma-
tion with an instantaneous velocity $\dot{\psi}(t)$.

\[^1\] We use a system of units with $c = 1, G = 1$.

\[^2\] The bold symbols denote 3-dimensional vectors.
(iv) The field of $N$ particles is taken to be, approximately, a superposition of the fields of the single particles.

(v) When this field of $N$ particles is inserted into the field equation, two singular expressions emerge: one from the linear part and the other from the quadratic part. By equating the highest order terms of these expressions the strength of the singularity of the solution is reduced. The corresponding balance equation yields Newton's law of attraction between the singularities.

We apply this algorithm to a nonlinear Lorentz invariant scalar field model. In this context, the Newton law of attraction is derived.

The paper is organized as followed: In section 2, we construct a nonlinear scalar Lorentz invariant field model. It shares some analytical properties of the Einstein field equation. In section 3, exact static solutions are presented. The solution with $N$ static singularities does not allow to describe the interaction. We treat it as non-physical. In section 4, we construct a non-static solution with $N$ singularities. In section 5, we show that the balance between the singular contributions yields, in the first order, the Newton law of attraction. In section 6, we present a qualitative technical summary of our algorithm. In section 7, we give a brief outlook of the proposed alternative algorithm and its possible extensions.

2 A nonlinear scalar model

2.1 Pointwise singularities in GR

In vacuum, the Einstein field equation can be written as

$$R_{\mu\nu} = 0.$$  \hspace{1cm} (3)

Here and in the sequel, the Greek indices refer to the coordinates, $\mu, \nu, \cdots = 0, 1, 2, 3$, while the Roman indices denote the specific singular point of the solutions, $i, j, \cdots = 1, \cdots, N$. From an analytical point of view, (3) is a system of ten equations for ten components of the metric tensor $g_{\mu\nu}$. All the equations are:

(i) Hyperbolic nonlinear PDE of the second order;

(ii) Linear in the second order derivatives of the metric tensor;

(iii) Quadratic in its first order derivatives.

The field of a pointwise source is described by the Schwarzschild solution of (3), which nonzero components are:

(i) Smooth for $r \neq 0$;

(ii) Singular at the origin $r = 0$;

(iii) Have the Taylor expansion of the form

$$g_{\mu\nu} \sim 1 + \frac{m}{r} + C_1 \left(\frac{m}{r}\right)^2 + C_2 \left(\frac{m}{r}\right)^3 + \cdots.$$  \hspace{1cm} (4)

The positive parameter $m$ in (4) is interpreted as a mass of a pointwise source located at the origin of the coordinates $r = 0$, while the constants $C_1, C_2, \cdots$ are dimensionless.

Due to the translation invariance of (3), the mass $m$ can be located at an arbitrary point, $r = r_0$. Moreover, it was proved, see [23], that the system (3) has a smooth solution with $N$ singularities located at the points $r = r_i(t)$. Although, even for $N = 2$, an exact form of such a solution is not available, the leading terms of its Taylor expansion can be calculated. The main output of the EIH analysis, is the fact that the functions $r_i(t)$ describe the proper motion of small bodies. Particularly, in the first order approximation, the functions $r_i(t)$ satisfy the Newton equation of motion with the Newton force of attraction at the right hand side. The higher order approximations yield the geodesic equation.
2.2 Nonlinear scalar model

In order to clarify how the field equation itself can determine an interaction between singularities, we will study a scalar field model which shares the special analytical properties of the Einstein equation mentioned above.

Let a scalar field \( \varphi = \varphi(x) \) be given on a flat Minkowskian 4D-space with a metric \( \eta_{\mu\nu} = \eta^{\mu\nu} = diag(1,-1,-1,-1) \). We consider a nonlinear scalar field equation

\[
\eta^{\mu\nu} (\varphi_{,\mu\nu} - k\varphi_{,\mu}\varphi_{,\nu}) = 0 ,
\]

which is equivalent to

\[
\Box \varphi - k (\dot{\varphi}^2 - \|\nabla \varphi\|^2) = 0 .
\]

Here \( k \) is a dimensionless constant, its value will be discussed in the sequel.

The commas denote partial derivatives \( \varphi_{,\mu} = \partial \varphi / \partial x^\mu \), the wave operator is \( \Box = \partial^2 / \partial t^2 - \partial^2 / \partial x^2 - \partial^2 / \partial y^2 - \partial^2 / \partial z^2 \), \( \nabla \) is the spatial gradient operator, the scalar product \( \langle \cdots \rangle \) and the corresponding norm \( \| \cdots \| \) are Euclidean.

In our approach, the scalar model (5) is used only as a simplified illustration to the way an interaction between singularities can emerge in GR. The model, however, is interesting by itself. In particular, the numerical evolution of a scalar field model similar to the one under consideration was recently studied in [29].

3 Exact static solution

3.1 Linear equation

In order to derive a suitable exact solution to (7), we start with a special case \( k = 0 \). For this value, the equation (5) is linear

\[
\Box \varphi = \eta^{\mu\nu} \varphi_{,\mu\nu} = 0 .
\]

Its unique static, spherically symmetric, asymptotically zero solution is

\[
\varphi = \frac{m}{r} .
\]

Note that this function is smooth for \( r \neq 0 \) and singular at the origin of the coordinate system. It is natural to interpret such a solution as a field of a pointwise particle with a mass (or a charge) equal to \( m \) located at the origin. The translational invariance of (7), yields that the singularity can be located at an arbitrary point \( \mathbf{r}_0 \)

\[
\varphi = \frac{m}{R} ,
\]

where

\[
R = |\mathbf{r} - \mathbf{r}_0| , \quad R = ||\mathbf{R}|| .
\]

The linearity of the field equation (7) yields the existence of a static solution which is singular at \( N \) distinct points

\[
\varphi = \sum_{i=1}^{N} \frac{m_i}{R_i} ,
\]

where

\[
R_i = |\mathbf{r} - \mathbf{r}_i| , \quad R_i = ||\mathbf{R}_i|| .
\]

This solution approaches zero at infinity and is approximately spherically symmetric in small neighborhoods of every singularity. For long distance from the set of the singular points, the full spherical symmetry is reinstated

\[
\varphi = \frac{M}{r} , \quad M = \sum_{i=1}^{N} m_i .
\]

We interpret (11) as a field generated by a static configuration of \( N \) particles of masses \( m_i \), which are located at the points \( \mathbf{r}_i \). Since the solution (11) is static, an interaction between the particles is absent.

3.2 A nonlinear equation

Let us derive now a corresponding static solutions of the nonlinear equation (5) with arbitrary values of the parameter \( k \). We redefine the scalar field as

\[
\varphi = -\frac{1}{k} \ln \phi, \quad \phi = e^{-k\varphi}.
\]  

(14)

It follows that

\[
\eta^{\mu\nu} (\varphi_{,\mu} - k\varphi_{,\mu}) = -\frac{1}{k\phi} \eta^{\mu\nu} \varphi_{,\mu\nu} = 0.
\]  

(15)

Thus, under the transformation (14), the nonlinear equation (5) is transformed to the linear one (7). Consequently, the field \( \varphi \) satisfies the equation (5) if and only if the new field \( \phi \) satisfies (7). We use this transformational property to derive certain exact static solutions of (5).

Start with a solution of (7) of the form

\[
\phi = 1 - k \frac{m}{R}, \quad R = ||r - r_0||.
\]  

(16)

Under (14), it transforms to a 1-singular solution of (5)

\[
\varphi = -\frac{1}{k} \ln \left( 1 - k \frac{m}{R} \right).
\]  

(17)

This solution vanishes at infinity as \( \varphi \to m/R \). Also, in the limiting case \( k \to 0 \), the solution (17) approaches the Newtonian potential. Although these limits hold independently of the sign of \( k \), for \( k < 0 \), (17) is singular only at a point \( r = r_0 \) and smooth at other points. We will see in the sequel that precisely the negative values of the parameter \( k \) yield the attraction between the singularities.

Due to the transformation (14), the solution (11) with \( N \) singular points also can be transformed to a static solution of (5). We take it in the following form

\[
\varphi = -\frac{1}{k} \ln \left( 1 - k \sum_{i=1}^{N} \frac{m_i}{R_i} \right),
\]  

(18)

where \( R_i = r - r_i \) while \( r_i \) is a radius vector to the \( i \)-th point. Some analytical properties of this solution can be motivated from the physical point of view. It is singular at the points \( r = r_i \) and smooth otherwise. In the small neighborhoods of the singular points, the solution is approximately spherically symmetric. This full spherical symmetry is reinstated on a long distance from the set of \( N \) singular points, where the solution is approximated by

\[
\varphi = -\frac{1}{k} \ln \left( 1 - k \frac{M}{r} \right), \quad M = \sum_{i=1}^{N} m_i.
\]  

(19)

In other words, the masses of the sources are approximately additive. Since the solution (18) is static its singularities do not interact.

4 Non-static solution generated

4.1 Instantaneous dynamical Lorentz transformations

The solutions exhibited above are static. In order to describe an interaction between singularities we are going to derive a certain set of dynamical solutions with the same type of singularities. For this we have to move the singular points on curved trajectories. The corresponding field \( \varphi \) will also move. For a Lorentz invariant equation the dynamical solution can be derived by a corresponding Lorentz transformation.
Let us start with a unique singular point which moves on a straight trajectory with a constant velocity \( v \). This motion can be represented as a Lorentz transformation of a stationary point \( r \) with the velocity \(-v\). Using the Lorentz transformation with the velocity \(-v\) we have (see Appendix A)

\[
r \rightarrow r - \alpha v <v, r> - \beta vt,
\]

where the Lorentz parameters are denoted by

\[
\beta = \frac{1}{\sqrt{1 - v^2}}, \quad \alpha = \frac{1}{v^2} \left(1 - \frac{1}{\sqrt{1 - v^2}}\right).
\]

To represent a motion of \( N \) singular points on distinct straight trajectories, we apply, in their neighborhoods, \( N \) independent Lorentz transformations

\[
r_i \rightarrow r_i - \alpha_i v_i <v_i, r_i> - \beta_i v_i t,
\]

with the Lorentz parameters

\[
\beta_i = \frac{1}{\sqrt{1 - v_i^2}}, \quad \alpha_i = \frac{1}{v_i^2} \left(1 - \frac{1}{\sqrt{1 - v_i^2}}\right).
\]

In order to construct a time dependent solution, we apply, for the static solutions, an instantaneous Lorentz transformations with time dependent velocities. Every such transformation acts only in a neighborhood of the corresponding singular point of the mass \( m_i \). In particular, for a curved trajectory \( \psi(t) \), we substitute in \( v \) by \( \dot{\psi} \) and \( vt \) by \( \psi \). Consequently, a dynamical Lorentz transformation which depends on the trajectory is

\[
r \rightarrow r - \alpha \dot{\psi} <\dot{\psi}, r> - \beta \psi.
\]

In order to deal with a motion of of \( N \) distinct singularities, we define a set of instantaneous dynamical Lorentz transformations \((i = 1, \cdots, N)\)

\[
(r - r_i) \rightarrow (r - r_i) - \alpha \dot{\psi}_i <\dot{\psi}_i, (r - r_i)> - \beta \psi_i,
\]

where the Lorentz parameters \( \alpha_i \) and \( \beta_i \) are functions only of the velocity \( ||\dot{\psi}_i||^2 \)

\[
\beta_i = \frac{1}{\sqrt{1 - ||\dot{\psi}_i||^2}}, \quad \alpha_i = \frac{1}{||\dot{\psi}_i||^2} \left(1 - \frac{1}{\sqrt{1 - ||\dot{\psi}_i||^2}}\right).
\]

4.2 Non-static solution of the linear field equation

Now, consider the linear field equation

\[
\eta^{\mu\nu} \varphi_{,\mu\nu} = 0.
\]

Recall that this equation is a special case, \( k = 0 \), of our non-linear model \( \phi \).

We start with an exact static solution of \( \varphi \)

\[
\varphi = \sum_{i=1}^{N} \frac{m_i}{R_i}, \quad R_i = r - r_i.
\]

The transformation \( R_i \) leads to a time depended ansatz

\[
\varphi = \sum_{i=1}^{N} \frac{m_i}{R_i}, \quad R_i = (r - r_i) - \alpha_i \dot{\psi}_i <\dot{\psi}_i, (r - r_i)> - \beta_i \psi_i.
\]
Clearly, for an arbitrary function $\psi(t)$, (29) does not satisfy (27). Let us look for functions $\psi_i(t)$ so that (29) is a solution of (27), at least approximately. Calculate, approximately, the left hand side of (27) for $\varphi$ given in (29). In particular, we omit the time derivatives of the Lorentz functions $\alpha_i$ and $\beta_i$, which bring only terms $O(\dot{\psi}\ddot{\psi})$. In this lowest order approximation, cf. Appendix B,

\[
\Box \varphi = \sum_{i=1}^{N} \frac{m_i \beta_i}{R_i^3} < \ddot{\psi}_i, R_i > + O(\dot{\psi}\dot{\psi}). \tag{30}
\]

Consequently $\varphi$ is an approximate solution of the field equation (7) if and only if $\ddot{\psi}_i = 0$. Thus, the linear field equation (7) can serve as a field model for a free inertial motion of a system of $N$ singularities.

In fact, for $\ddot{\psi}_i = 0$, i.e. for $\psi_i = \dot{\psi}_i t$, this result is exact. Indeed, at the point $R_i = 0$

\[
r - r_i = \dot{\psi}_i \left( \alpha_i < \dot{\psi}_i, (r - r_i) > + \beta_i t \right), \tag{31}
\]

which results in the equation of motion of a free particle

\[
r - r_i = \dot{\psi}_i t. \tag{32}
\]

### 4.3 Non-static solution of the non-linear field equation

Let us now turn to the full non-linear scalar field equation

\[
\eta^{\mu\nu} (\varphi, \mu\nu - k\varphi, \mu\varphi) = 0. \tag{33}
\]

If the transformation (25) is applied to its static solution

\[
\varphi = -\frac{1}{k} \ln \left( 1 - k \sum_{i=1}^{N} \frac{m_i}{R_i} \right), \quad R_i = r - r_i. \tag{34}
\]

we will obtain no more than the inertial motion $\ddot{\psi} = 0$. This is because the equation (33) is Lorentz invariant. Instead of that, we start with a corresponding approximate solution,

\[
\varphi = -\frac{1}{k} \sum_{i=1}^{N} \ln \left( 1 - k \frac{m_i}{R_i} \right), \quad R_i = r - r_i. \tag{35}
\]

which is a superposition of $N$ independent exact solutions.

In order to generate a non-static (interaction) solution, we apply, in a neighborhood of every singular point, Lorentz transformations with variable velocities. Thus we come to

\[
\varphi = -\frac{1}{k} \sum_{i=1}^{N} \ln \left( 1 - k \frac{m_i}{R_i} \right), \quad R_i = (r - r_i) - \alpha_i \dot{\psi}_i < \dot{\psi}_i, (r - r_i) > - \beta_i \psi_i. \tag{36}
\]

As in the linear case, for an arbitrary function $\psi(t)$, this ansatz is not a solution of (33). Let us now search for conditions under which (36) turns into an approximate solution of (33) in the lowest order.

Calculating for (36) the linear second order derivative part of (33), we get (see Appendices C and D)

\[
\sum_{i=1}^{N} \frac{m_i \beta_i}{R_i^3} < \frac{R_i}{1 - k \frac{m_i}{R_i}}, \ddot{\psi}_i > = -k \sum_{i \neq j}^{N} \frac{\frac{m_i}{R_i}}{1 - k \frac{m_i}{R_i}} \frac{\frac{m_j}{R_j}}{1 - k \frac{m_j}{R_j}} \left[ < R_i, R_j > + < \dot{\psi}_i, R_i > < \dot{\psi}_j, R_j > \right] + O(\dot{\psi}\dot{\psi}). \tag{37}
\]
The expressions are calculated up to $O(\dot{\psi}\ddot{\psi})$. To this accuracy, the derivatives of the Lorentz parameters $\alpha_i, \beta_i$ are zero.

The second line of (37) is of the order $O(\dot{\psi}_i\dot{\psi}_j)$. For the lowest approximation, we neglect this term. Moreover, to the same accuracy, we can take $\beta_i = 1$. Thus, from this stage, our results will be valid only for slow motion of the singularities.

To sum up, we derived that the ansatz (36) is an approximate solution of the field equation (33) only if the functions $\psi_i(t)$ satisfy

\begin{equation}
\sum_{i=1}^{N} \frac{m_i}{R_i^3} < R_i, \ddot{\psi}_i > = -k \sum_{i \neq j}^{N} \frac{m_i}{R_i^3} \frac{m_j}{R_j^3} < R_i, R_j > ,
\end{equation}

where

\begin{equation}
R_i = (r - r_i) - \alpha_i \dot{\psi}_i < \dot{\psi}_i, (r - r_i) > - \beta_i \psi_i .
\end{equation}

5 An interaction is generated

Both sides of the equation (38) are functions of a point $r$. It is singular at $N$ points $R_i = 0$ and regular at other points. Let us examine how this relation can be satisfied in a neighborhood of a singular point. Consider the $p$-th singularity. We will examine the field at an arbitrary point $r$ close to this singularity, i.e., we have $R_p \to 0$. For the $i$-th singularity

\begin{equation}
R_i \to 0 \text{ yields } R_i \to R_{ip} \text{ for } i \neq p ,
\end{equation}

where $R_{ip}$ is a vector directed from the point $i$ to the point $p$.

On the left hand side of the equation (38), there is only one singular term

\begin{equation}
\frac{m_p}{R_p^3} < R_p, \ddot{\psi}_p > .
\end{equation}

The singular term on the right hand side of (38) is

\begin{equation}
-k \frac{m_p}{R_p^3} \sum_{j \neq p} \frac{m_j}{R_j^3} < R_p, R_j > .
\end{equation}

The terms (41) and (42) are of the order $O(R^{-2})$ near the $p$-th singularity.

They cancel each other if and only if

\begin{equation}
< R_p, \ddot{\psi}_p > = -k \sum_{j \neq p} \frac{m_j}{R_j} < R_p, R_j > .
\end{equation}

Take into account that the point $r$ (in $R_p$) is arbitrary. Hence (38) is valid only if

\begin{equation}
\ddot{\psi}_p = -k \sum_{j \neq p} \frac{m_j}{R_j} R_j .
\end{equation}

For the limiting values, we have in (10)

\begin{equation}
\ddot{\psi}_p = -k \sum_{j \neq p} \frac{m_j}{R_{jp}^3} R_{jp} .
\end{equation}

For distances greater than the Schwarzschild radius $r = km$, (45) remains in the form

\begin{equation}
\ddot{\psi}_p = -k \sum_{j \neq p} \frac{m_j}{R_{jp}^3} R_{jp} .
\end{equation}
For a system of two singular points, it yields
\[ m_1 \ddot{\psi}_1 = k \frac{m_1 m_2}{R_{12}^3} R_{12}. \]  
(47)

For \( k < 0 \), (46) and (47) result in attraction between the particles. The absolute value of \( k \) is unimportant, since it amounts to the rescaling of the mass.

This way the Newton-type law of attraction is derived from the scalar field equation.

6 Qualitative Description of the Algorithm

In this section we give a qualitative description of the proposed algorithm for deriving the equations of motion from the field equation. Consider a field equation of a general type
\[ a \Phi_{,\mu\nu} - b \Phi_{,\mu} \Phi_{,\nu} = 0 \]  
(48)
without specification of the tensorial nature of the field variable \( \Phi \). Here, the coefficients \( a \) and \( b \) are the dimensionless functions of the field variable (or constants)
\[ a = a(\Phi), \quad b = b(\Phi). \]

We do require, however, that (48) is Lorentz invariant.

Let the field equation (48) has a static spherical symmetric solution \( \Phi(r - r_0) \) with a singularity located at \( r = r_0 \). Denote the Lorentz transformation based on the velocity \( v \) by \( L(v) \). Consequently, if \( \Phi = \Phi(r - r_0) \) is a time independent solution of (48) then \( L(v) \Phi \) is also a solution of the same equation for an arbitrary Lorentz transformation \( L(v) \). Note that if \( \Phi \) is a multi-component quantity like a tensor then \( L(v) \) involves not only a coordinate change but, also, a transformation of components of \( \Phi \). The solution \( L(v) \Phi \) describes the field of a pointwise singularity moving with a constant velocity \( v \) on the trajectory \( \psi = vt \).

Let us try to construct a generalization of the Lorentz transformation used above such that the origin moves on a curved trajectory \( \psi = \psi(t) \). Denote such a transformation by \( N(\psi) \). The choice
\[ N(\psi) = L(\dot{\psi}) \]
is a plausible candidate. Correspondingly, \( N(\psi) \Phi \) is a rigid motion of the field.

Now substitute \( N(\psi) \Phi \) in (48). If \( \dot{\psi} = \text{const} \) then \( N(\psi) \Phi \) is also a solution of (48). If not, then the linear part produces extra terms. These extra terms come from two sources:

(i) The derivatives of the Lorentz root \( \sqrt{1 - |\dot{\psi}|^2} \). In our consideration the velocity of the particle \( \dot{\psi} \) as well as its time and spatial derivatives are assumed to be small. It follows that the derivatives of the root are of the form \( O(|\dot{\psi}|) \). Thus they may be rejected.

(ii) The linear part. Since \( \Phi \) is time independent the first order derivatives of \( \Phi \) are only the spatial ones. Thus, the first order derivatives of \( L(\dot{\psi}) \Phi(x) \) involve spatial derivatives of \( \Phi \) multiplied by \( \dot{\psi} \). The second order derivatives of \( L(\dot{\psi}) \Phi(x) \) cancel each other exactly in the same way as in a Lorentz transformed solution \( L(v) \Phi(x) \). One exception: The second order derivative \( \ddot{\psi} \) multiplied by spatial first order derivatives of \( \Phi \) do remain. This extra term that comes from the linear part of the field equation is the agent of inertia.

(iii) The quadratic part. It involves only first order derivatives of the field. Consequently the fact that \( \dot{\psi} \) is variable does not affect it’s form.

Construct now a solution that describes a field of \( N \) particles, i.e. a field with \( N \) singular points. It can be approximated by a superposition of 1-singular solutions moving on arbitrary trajectories \( \psi_j = \psi_j(t) \).
\[ \Phi = \sum_j L_j(\dot{\psi}_j) \Phi_j(x). \]
Substituting this approximate solution in (48) we obtain, in the linear part, only the second derivatives $\ddot{\psi}_j$ multiplied by the spatial first derivatives of $\Phi$.

Consider the quadratic part. It is composed of the first order derivatives of $L_j(\dot{\psi}_j)\Phi_j(x)$ multiplied by the first order derivatives of $L_k(\dot{\psi}_k)\Phi_k(x)$. If $j = k$, since $\Phi_k(x)$ is a solution of (18), these products are cancelled by the linear part operating on $\Phi_k(x)$. If $j \neq k$ the products will be declared to be an approximation to the interaction between the $j$-th and the $k$-th particles.

Near the $k$-th singularity, for the linear part, only the terms coming from $\Phi_k(x)$, similar to (11), will be dominant. Likewise, for the quadratic part, only the terms involving the derivatives of $\Phi_k(x)$, similar to (12), will be dominant. Equating, near the singularity, of the two terms above (again to the leading order) should, hopefully, result in the Newton-type law of attraction.

7 Conclusions

The motivation for our paper is to learn what properties of the Einstein field equations lead to derivability the proper equations of motion for the singularities. We conjecture that it pertains to the specific structure of the field equations: As partial differential equations, they are linear in the second order derivatives and quadratic in the first order derivatives of the field variables. Furthermore, we conjecture that the quadratic terms are the agent of interaction, they have to generate the force expression. The second order derivatives play a role of the agent of inertia, it has to generate the mass-times-acceleration term.

We carry out the above plan for a simple non-linear scalar model. It is remarkable that only with the slow motion assumption, one is ultimately led to the Newton-type law of attraction. It should be emphasized that our scheme does not pretend to be more precise than the alternative methods cited above. Instead, we are looking for a proper identification of the terms of the field equation with the inertial and interaction terms of the equations of motion.

In order to apply the proposed scheme to Einstein’s equation, one has to solve the problem of consistency [30]. Every one of ten independent equations $R_{\mu\nu} = 0$ has to give the same equation of motion. Also the Bianchi identities have to be taken into account and their role in our scheme has to be revealed. It is well known that the Bianchi identities play a crucial role in EIH-procedure.

An other extension is in the direction of embedding [30]. The trajectories obtained are approximate. At this point, two avenues are open. The first one, which is adopted by EIH is to get higher order approximations to the trajectories. This procedure is also used in the PPN approach. By these methods, the successive approximations become highly singular near the particle trajectories. The second avenue is to embed the singularities in a field satisfying the field equations. For that purpose, the successive approximations should add regular terms (and, possible, low order singular terms) near the trajectories.

The method proposed in this paper can hopefully be useful for a rigorous proof of the geodesic postulate.

Acknowledgment

We are grateful to the anonymous referee for most careful reading of our manuscript. His valuable comments and remarks as well as his guide to the relevant literature were very helpful.

A Lorentz transformations

Lorentz transformations are usually written in a very special form when the axes of two reference systems are parallel one to the other. In particular, when a reference system $\{\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\}$ moves relative to another reference system $\{t, x, y, z\}$ with a velocity parallel to the $x$-axis, the corresponding Lorentz transformation is

$$\tilde{x} = \beta(x + vt), \quad \tilde{y} = y, \quad \tilde{z} = z, \quad \tilde{t} = \beta(t + vx),$$

(49)
with the Lorentz parameter
\[ \beta = \frac{1}{\sqrt{1 - v^2}}. \] (50)

Recall that we use a system of units with \( c = 1 \).

Since the different directed Lorentz transformations do not commute, a general transformation (with an arbitrary directed vector of velocity) cannot be generated by a successive application of three orthogonal transformations (relative to three axes). Although, a formula for the Lorentz transformation with a general velocity vector is known from the literature, we present here a brief derivation from (49). This derivation is instructive because the proof conforms with the method used in this article.

Consider a reference system which axes are parallel to the corresponding axes of a rest reference system. Let the origin of the reference system move with an arbitrary directed velocity \( v \). Consider a radius vector \( r \) directed to an arbitrary point in space. Its projection on the direction of \( v \) is
\[ P_v r = v < v, r > \frac{v}{v^2}. \] (51)

Exhibit the vector \( r \) as a sum of its tangential and normal parts
\[ r = P_v r + N_v r, \quad N_v r = r - v < v, r > \frac{v}{v^2}. \] (52)

Due to (49), for a Lorentz transformation directed by \( v \), these two parts transform as
\[ P_v \tilde{r} = \beta (P_v r + vt), \]
\[ N_v \tilde{r} = N_v r. \] (53) (54)

Consequently, the transform of the spatial coordinates is
\[ \tilde{r} = P_v \tilde{r} + N_v \tilde{r} = \beta (P_v r + vt) + N_v r, \] (55)

or, explicitly,
\[ \tilde{r} = r + v (\beta t - \alpha < r, v >), \] (56)

where
\[ \alpha = (1 - \beta) = \frac{1}{v^2} \left( 1 - \frac{1}{\sqrt{1 - v^2}} \right). \] (57)

The change of the time coordinate is also governed only by the tangential part of the vector \( r \)
\[ \tilde{t} = \beta (t + ||P_v r|| v) = \beta (t + < P_v r, v >), \] (58)

or, explicitly,
\[ \tilde{t} = \beta (t + < r, v >). \] (59)

In the special case of a motion parallel to the axis \( x \), the relations (56), (59) reduce to the ordinary form of the Lorentz transformation (49). Therefore an arbitrary Lorentz transformation takes the form
\[ \begin{cases} \tilde{t} = \beta (t + < r, v >), \\ \tilde{r} = r + v (\beta t - \alpha < r, v >) \end{cases}. \] (60)

B The linear equation: The 1-point solution moved

Calculate the d’Alembertian of the field \( \varphi \) of a single particle moving on a trajectory \( \psi(t) \)
\[ \varphi = \frac{m}{R}, \quad R = ||R||, \] (61)

where
\[ R = (r - r_0) - \alpha \dot{\psi} < \dot{\psi}, (r - r_0) > - \beta \psi. \] (62)
Consequently, the second order derivative is
\[
\beta = \frac{1}{\sqrt{1 - ||\dot{\psi}||^2}} \approx 1 + \frac{1}{2}||\dot{\psi}||^2 + \frac{3}{8}||\dot{\psi}||^4 + \cdots
\] (63)
and
\[
\alpha = \frac{1}{||\dot{\psi}||^2} \left( 1 - \frac{1}{\sqrt{1 - ||\dot{\psi}||^2}} \right) \approx -\frac{1}{2} \left( 1 - \frac{3}{2}||\dot{\psi}||^2 - \frac{5}{8}||\dot{\psi}||^4 + \cdots \right).
\] (64)
The time derivatives of these functions are correspondingly
\[
d\beta = 2\beta' \langle \dot{\psi}, \ddot{\psi} \rangle = \langle \dot{\psi}, \ddot{\psi} \rangle \left( 1 + \frac{3}{2}||\dot{\psi}||^2 + \cdots \right),
\] (65)
and
\[
d\alpha = 2\alpha' \langle \dot{\psi}, \ddot{\psi} \rangle = -\langle \dot{\psi}, \ddot{\psi} \rangle \left( \frac{3}{2} + \frac{5}{4}||\dot{\psi}||^2 + \cdots \right),
\] (66)
where \(\alpha', \beta'\) denote the derivatives relative to the variable \(\dot{\psi}\).
The first order time derivative of the vector (62) is
\[
\dot{R} = -(2\alpha' \dot{\psi} < \dot{\psi}, \ddot{\psi} > + \alpha \ddot{\psi}) < \dot{\psi}, r - r_0 > -\alpha \ddot{\psi} < \dot{\psi}, r - r_0 > -2\beta' \dot{\psi} < \dot{\psi}, \ddot{\psi} > -\beta \ddot{\psi}.
\] (67)
Due to (68–69), up to the order \(O(\dot{\psi} \ddot{\psi})\), the second order time derivative of the field is
\[
\frac{\partial^2 \varphi}{\partial t^2} = \frac{m\beta}{R^3} \left( < \dot{\psi}, R > + < \dot{\psi}, R_t > \right) - 3\frac{m\beta}{R^5} < \dot{\psi}, R gardening \dot{R}, \dot{R}>.
\] (68)
Thus we get
\[
\frac{\partial^2 \varphi}{\partial t^2} = \frac{m\beta}{R^3} \left( < \dot{\psi}, R > - \beta ||\dot{\psi}||^2 \right) + 3\frac{m\beta^2}{R^5} < \dot{\psi}, R >
\]
\[
= \frac{m\beta}{R^3} < \dot{\psi}, R > + \frac{m\beta^2}{R^5} \left( 3 < \dot{\psi}, R > - R^2 ||\ddot{\psi}||^2 \right).
\] (69)
Let us calculate now the spatial derivatives of the field (61). Introduce a set of unit vectors \(e_1, e_2, e_3\) directed along the axes \(x, y, z\) correspondingly. Thus the \(x\)-component of the vector (62) can be written as
\[
R_x = e_1 - \alpha \dot{\psi} < \dot{\psi}, e_1 >.
\] (70)
Consequently, the second order derivative is
\[
\frac{\partial^2 \varphi}{\partial x^2} = -\frac{m}{R^3} ||R_x||^2 + 3\frac{m\beta}{R^5} < R_x, R >
\] (71)
Due to the spherical symmetry of the field \(\varphi\), its Laplacian takes the form
\[
\Delta \varphi = 3\frac{m}{R^5} (||R_x||^2 + ||R_y||^2 + ||R_z||^2) - \frac{m}{R^3} (||R_x||^2 + ||R_y||^2 + ||R_z||^2).
\] (72)
Since
\[
< R_x, R > = < R, e_1 > - \alpha < \dot{\psi}, e_1 > < \dot{\psi}, R >
\] (73)
we get
\[
< R_x, R >^2 + < R_y, R >^2 + < R_z, R >^2 = R^2 - 2\alpha < \dot{\psi}, R >^2 + \alpha^2 ||\ddot{\psi}||^2 < \dot{\psi}, R >^2,
\] (74)
Consequently, the d'Alembertian of the 1-singular ansatz (61) takes the form

\[ \alpha, \beta \]

Using the expressions (63) for the functions \( R \)

Thus the Laplacian of the field \( \varphi \) takes the form

\[ \Delta \varphi = \frac{m \beta}{R^3} (\alpha^2 ||\dot{\psi}||^2 - 2 \alpha)(3 < \dot{\psi}, R > - 2 ||\dot{\psi}||^2 R^2). \]  

(76)

Correspondingly, the d'Alembertian of the field is

\[ \Box \varphi = \frac{m \beta}{R^3} < \ddot{\psi}, R > + \frac{m^2}{R^0} (\beta^2 + 2 \alpha - \alpha^2 ||\dot{\psi}||^2) \left( 3 < \dot{\psi}, R > - R^2 ||\dot{\psi}||^2 \right). \]  

(77)

Using the expressions (63) for the functions \( \alpha, \beta \), we derive

\[ \beta^2 + 2 \alpha - \alpha^2 ||\dot{\psi}||^2 = 0. \]  

(78)

Consequently, the d'Alembertian of the 1-singular ansatz (61) takes the form

\[ \Box \varphi = \frac{m \beta}{R^3} < \ddot{\psi}, R >. \]  

(79)

It is clear that a singularity that moves on a straight trajectory with a constant velocity \( \ddot{\psi} = 0 \) is described by a solution of the linear equation \( \Box \varphi = 0 \).

C The non-linear equation: The motion of a 1-point singularity

In this section we substitute the ansatz with one singular point —

\[ \varphi = - \frac{1}{k} \ln \left( 1 - k \frac{m}{R} \right), \quad R = (r - r) - \alpha \dot{\psi} < \dot{\psi}, (r - r) > - \beta \psi. \]  

(80)

into the left hand side of the non-linear field equation (33). From the calculations above, the time derivatives of the vector \( R \) are

\[ R_t = - \beta \dot{\psi}, \quad R_{tt} = - \beta \ddot{\psi}. \]  

(81)

Thus the second order time derivative is

\[ \varphi_{tt} = \frac{m \beta}{R^3} \left( 3 \beta < R, \dot{R} >^2 + R^2 < R, \ddot{R} > - \beta R^2 ||\dot{\psi}||^2 \right) + \frac{m^2 \beta^2 k}{R^6} \left( 3 < \dot{\psi}, R > - R^2 ||\dot{\psi}||^2 \right). \]  

(82)

As for the spatial derivatives we use

\[ R_x = e_1 - \alpha \dot{\psi} < \dot{\psi}, e_1 >, \quad R_{xx} = 0, \]  

(83)

where \( e_1 \) is a unit vector along the \( x \) axis. Consequently,

\[ \Delta \varphi = 3 \frac{m}{R^6} \frac{< R, R_x >^2 + < R, R_y >^2 + < R, R_z >^2}{1 - k \frac{m}{R}} - \frac{m}{R^3} \frac{< R_x, R_x > + < R_y, R_y > + < R_z, R_z >}{1 - k \frac{m}{R}} + \frac{km^2}{R^6} \frac{< R, R_x >^2 + < R, R_y >^2 + < R, R_z >^2}{(1 - k \frac{m}{R})^2}. \]  

(84)

Substituting here the value of \( R_x \) we get

\[ \Delta \varphi = 3 \frac{m}{R^6} \frac{R^2 - 2 \alpha < \dot{\psi}, R >^2 + \alpha^2 ||\dot{\psi}||^2 < \dot{\psi}, R >^2}{1 - k \frac{m}{R}} - \frac{m}{R^3} \frac{3 - 2 \alpha ||\dot{\psi}||^2 + \alpha^2 ||\dot{\psi}||^4}{1 - k \frac{m}{R}} + \frac{km^2}{R^6} \frac{R^2 - 2 \alpha < \dot{\psi}, R >^2 + \alpha^2 ||\dot{\psi}||^2 < \dot{\psi}, R >^2}{(1 - k \frac{m}{R})^2}. \]  

(85)
Using once more the relation (87) we get

\[ \square \varphi = \frac{m \beta}{R^3} \left( \frac{3 \beta < R, \dot{\psi} > + R^2 < R, \ddot{\psi} > - \beta R^2 \| \dot{\psi} \|^2}{1 - k \frac{m}{\rho}} \right) + \frac{m^2 \beta k}{R^6} \frac{< R, \ddot{\psi} >}{(1 - k \frac{m}{\rho})^2} + \frac{3 m}{R^6} \left( \frac{R^2 - 2 \alpha < \dot{\psi}, R > + \alpha^2 \| \dot{\psi} \|^2}{1 - k \frac{m}{\rho}} \right) < \dot{\psi}, R >^2 - m \left( \frac{3 - 2 \alpha \| \dot{\psi} \|^2 + \alpha^2 \| \ddot{\psi} \|^2}{1 - k \frac{m}{\rho}} \right) < \dot{\psi}, R >^2 \]

Thus

\[ \square \varphi = \frac{m \beta}{R^3} \left( \frac{3 \beta < R, \dot{\psi} > + R^2 < R, \ddot{\psi} > - \beta R^2 \| \dot{\psi} \|^2}{1 - k \frac{m}{\rho}} \right) + \frac{m^2 \beta k}{R^6} \frac{< R, \ddot{\psi} >}{(1 - k \frac{m}{\rho})^2} \]

we obtain

\[ \square \varphi = \frac{m \beta}{R^3} \frac{< R, \dot{\psi} >}{1 - k \frac{m}{\rho}} - \frac{m^2 k}{R^6} \frac{1}{(1 - k \frac{m}{\rho})^2}. \]  

(88)

As for the quadratic part of the field equation, to the same accuracy,

\[ \eta^{ab, \varphi, \alpha \varphi, b} = \frac{m^2 \beta^2}{R^6} \frac{< R, \dot{\psi} >^2}{(1 - k \frac{m}{\rho})^2} - \frac{m^2}{R^6} \frac{< R, R_x >^2 + < R, R_y >^2 + < R, R_z >^2}{(1 - k \frac{m}{\rho})^2} \]

\[ = \frac{m^2 \beta^2}{R^6} \frac{< R, \dot{\psi} >^2}{(1 - k \frac{m}{\rho})^2} - \frac{m^2}{R^6} \frac{R^2 - 2 \alpha < \dot{\psi}, R >^2 + \alpha^2 \| \dot{\psi} \|^2}{(1 - k \frac{m}{\rho})^2} < \dot{\psi}, R >^2 \]

\[ = - \frac{m^2}{R^4} \frac{1}{(1 - k \frac{m}{\rho})^2} + \frac{m^2}{R^6} (\beta^2 + 2 \alpha - \alpha^2 \| \ddot{\psi} \|^2) < R, \dot{\psi} >^2 \]  

(89)

Using once more the relation (87) we get

\[ \eta^{ab, \varphi, \alpha \varphi, b} = \frac{m^2}{R^4} \frac{1}{(1 - k \frac{m}{\rho})^2}. \]  

(90)

When (88) and (90) are substituted into the left hand side of the field equation (88) we obtain

\[ \square \varphi - k \eta^{ab, \varphi, \alpha \varphi, b} = \frac{m \beta}{R^3} \frac{< R, \dot{\psi} >}{1 - k \frac{m}{\rho}}. \]  

(91)

Consequently, the field equation (88) is satisfied only for a singularity that moves on a straight trajectory with a constant velocity.

**D  The non-linear equation. The N-point solution moved**

Take the field of N singular points as a superposition

\[ \varphi = - \frac{1}{k} \sum_{i=1}^{N} \ln \left( 1 - k \frac{m_i}{R_i} \right), \quad R_i = (r - r_i) - \alpha_i \dot{\psi}_i, \quad (r - r_i) > - \beta_i \psi_i. \]  

(92)

Substitute it in the left hand side of the non-linear field equation (88). Since this ansatz a superposition of N independent solutions, the linear part of the field equation is a sum of the expressions given in (88)

\[ \square \varphi = \sum_{i=1}^{N} \frac{m_i \beta_i}{R_i^3} \frac{< R_i, \dot{\psi}_i >}{1 - k \frac{m_i}{\rho_i}} - \frac{m_i^2 k}{R_i^6} \frac{1}{(1 - k \frac{m_i}{\rho_i})^2}. \]  

(93)
Calculate now the non-linear part
\[ \varphi_t = - \sum_{i=1}^{N} \left( \frac{\langle \dot{R}_i, R_i \rangle}{1 - k \frac{m_i}{R_i}} \right) = \sum_{i=1}^{N} \left( \frac{\langle \dot{\psi}_i, R_i \rangle}{1 - k \frac{m_i}{R_i}} \frac{m_i \beta_i}{R_i^3} \right). \] (94)

Thus
\[ (\varphi_t)^2 = \sum_{i,j=1}^{N} \frac{m_i \beta_i}{R_i^3} \frac{m_j \beta_j}{R_j^3} \left( \langle \dot{R}_{ix}, R_i \rangle \langle \dot{R}_{ix}, R_j \rangle + \langle \dot{R}_{iy}, R_i \rangle \langle \dot{R}_{iy}, R_j \rangle + \langle \dot{R}_{iz}, R_i \rangle \langle \dot{R}_{iz}, R_j \rangle \right). \] (95)

As for the spatial derivatives
\[ \langle \nabla \varphi, \nabla \varphi \rangle = \sum_{i,j=1}^{N} \frac{m_i \beta_i}{R_i^3} \frac{m_j \beta_j}{R_j^3} \left( \langle \dot{R}_{ix}, R_i \rangle \langle \dot{R}_{ix}, R_j \rangle + \langle \dot{R}_{iy}, R_i \rangle \langle \dot{R}_{iy}, R_j \rangle + \langle \dot{R}_{iz}, R_i \rangle \langle \dot{R}_{iz}, R_j \rangle \right). \] (96)

Applying the relation
\[ R_{ix} = e_1 - \alpha_i \dot{\psi}_i < \dot{\psi}_i, e_1 >, \] (97)

the expression in the brackets of (96) takes the form
\[ \left( \cdots \right) = \langle \dot{R}_i, R_j \rangle - \alpha_j \left( \langle \dot{\psi}_j, R_i \rangle \langle \dot{\psi}_j, R_j \rangle \right) - \alpha_i \left( \langle \dot{\psi}_i, R_j \rangle \langle \dot{\psi}_i, R_i \rangle \right) + \alpha_i \alpha_j \left( \langle \dot{\psi}_i, R_j \rangle \langle \dot{\psi}_i, \dot{\psi}_j \rangle \right). \] (98)

Consequently, the quadratic part of the field equation takes the form
\[ k\eta^{ab} \varphi_{,a} \varphi_{,b} = -k \sum_{i,j=1}^{N} \frac{m_i}{R_i^3} \frac{m_j}{R_j^3} \left( \langle \dot{R}_i, R_j \rangle + \langle \dot{\psi}_i, R_j \rangle \langle \dot{\psi}_i, R_i \rangle \right), \] (99)

Observe that, for \( i = j \), this expression coincides with (90). For \( i \neq j \), the expressions in the second line of (99) are not canceled, they can, however, be neglected in the lowest approximation.

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