Pseudo-Hermitian Hamiltonians Generating Waveguide Mode Evolution

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We study the properties of Hamiltonians defined as the generators of transfer matrices in quasione-dimensional waveguides. For single- or multi-mode waveguides obeying flux conservation and time-reversal invariance, the Hamiltonians defined in this way are non-Hermitian, but satisfy symmetry properties that have previously been identified in the literature as “pseudo Hermiticity” and “spectral anti-PT symmetry”. We show how simple one-channel and two-channel models exhibit symmetry-breaking transitions between real, imaginary, and complex eigenvalue pairs.

I. INTRODUCTION

In 1998, Bender and co-workers\textsuperscript{[1–3],} pointed out that Hamiltonians which are symmetric under a combination of parity and time reversal (PT) possess an interesting feature: despite being non-Hermitian, they can have eigenvalues that are strictly real. Moreover, tuning the Hamiltonian parameters can induce a non-Hermitian symmetry-breaking transition, between PT-symmetric eigenstates with real eigenvalues and PT-broken eigenstates with complex and conjugate-paired eigenvalues. The original intention of Bender et al. was to use PT symmetry to extend fundamental quantum mechanics, but in 2008 Christodoulides and co-workers showed that PT symmetry could be realized in optical structures with balanced gain and loss\textsuperscript{[4–7].} Since then, research into PT-symmetric optics has progressed rapidly, and the idea of “gain and loss engineering” in photonics, based on PT symmetry, has led to devices with highly promising applications, such as low-power optical isolation\textsuperscript{[8, 9]} and laser mode selection\textsuperscript{[10, 11].}

In this paper, we look at a class of physically-motivated non-Hermitian Hamiltonians that are not PT-symmetric in the original sense of Bender et al.\textsuperscript{[1–3],} but nonetheless exhibit symmetry-breaking transitions between real and complex eigenvalues. These Hamiltonians are the generators of transfer matrices in single- or multi-channel waveguides without gain or loss\textsuperscript{[12].} They form a subgroup of the “pseudo-Hermitian” matrices, a generalization of PT symmetric matrices identified by Mostafazadeh and co-workers\textsuperscript{[13–15].} This subgroup is restricted by a further symmetry relation that has previously been identified by Sukhorukov et al. as “spectral anti-PT symmetry”, in the context of beam evolution in parametric amplifiers\textsuperscript{[16].}

For an $N$-channel waveguide (either an optical waveguide, or a quantum electronic waveguide\textsuperscript{[12]) that obeys flux conservation as well as time-reversal invariance, it is known\textsuperscript{[17]} that the group of transfer matrices, at each energy (or frequency), has a one-to-one mapping to the symplectic group $\text{Sp}(2N, \mathbb{R})$. Each transfer matrix is generated by a $2N \times 2N$ matrix, $H$, which is typically non-Hermitian; these matrices map onto the group of real Hamiltonian matrices, $\text{sp}(2N, \mathbb{R})$, which are the generators of $\text{Sp}(2N, \mathbb{R})$. We can regard each $H$ as a “Hamiltonian” depending parametrically on the operating energy. The eigenvalues of $H$ are not energies, but rather the modal wavenumbers of a translationally invariant waveguide, with real eigenvalues corresponding to propagating modes and complex eigenvalues to evanescent (in-gap) modes. As shown below, $H$ supports real eigenvalues despite being non-Hermitian because it satisfies a certain pair of symmetries, one equivalent to pseudo-Hermiticity\textsuperscript{[13–15]} and the other to spectral anti-PT symmetry\textsuperscript{[16].} These symmetries are tied to the physical conditions of flux conservation and time-reversal invariance in the underlying waveguide.

To motivate the interpretation of transfer matrix generators as “Hamiltonians”\textsuperscript{[12],} consider a segment of an $N$-channel waveguide with negligible back-reflection. The position along the waveguide axis, $z$, can be thought of as playing the role of “time”. At a given energy $E$, the transfer matrix is a $2N \times 2N$ block-diagonal matrix of the form

$$M = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$

where $U_1$ and $U_2$ describe the mode-mixing of the right- and left-moving modes, respectively. In the absence of gain or loss, $U_1$ and $U_2$ are unitary, and $M$ is generated by a $2N \times 2N$ Hermitian matrix

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}.$$ 

The eigenvalues of $H_{1,2}$ are the wavenumbers of the right- and left-moving modes. For such a reflection-free waveguide, we could focus on the Hermitian submatrix $H_1$ as the Hamiltonian of the $N$ forward modes; this leads to the well-known mapping between beam propagation and the Schrödinger wave equation\textsuperscript{[18].}

When back-reflection is non-negligible, $M$ is no longer block-diagonal; its generator $H$ is neither block-diagonal nor Hermitian\textsuperscript{[12].} However, the eigenvalues of $H$ can...
still be regarded as modal wavenumbers, which now consist of a mix of right- and left-moving components. Band extrema correspond to exceptional points of $H$, where its eigenvectors coalesce and the matrix becomes defective. At these points, the modal wavenumbers exhibit “symmetry-breaking” transitions between real pairs (propagating modes), and either purely imaginary pairs (purely evanescent gap modes) or complex pairs (quasi-evanescent gap modes). This is reminiscent of PT symmetry breaking \[11, 2\], but, as mentioned above, it is not PT symmetry that is responsible for these eigenvalue transitions \[3, 13–16\].

### II. HAMILTONIAN SYMMETRIES

For an $N$-channel waveguide, the wavefunction at position $z$ can be expressed by $2N$ complex wave amplitudes:

$$|\Psi(z)\rangle \equiv \begin{pmatrix} \Psi^+(z) \\ \Psi^-(z) \end{pmatrix}, \quad \text{where} \quad \Psi^\pm(z) \equiv \begin{pmatrix} \psi^\pm_1(z) \\ \vdots \\ \psi^\pm_N(z) \end{pmatrix}. \quad (3)$$

Here, $\pm$ denotes wave components moving in the $\pm \hat{z}$ direction, and the wave components are normalized so that $|\psi^\pm_1|^2$ is an energy flux. The wavefunctions at any two points, $z_1$ and $z_2$, are related by a transfer matrix:

$$M(z_1, z_2) |\Psi(z_2)\rangle = |\Psi(z_1)\rangle. \quad (4)$$

Let us assume that the waveguide is flux-conserving and time-reversal invariant \[17\]. This imposes two symmetry constraints on $M$. Firstly, flux conservation states that the incoming flux into the segment between $z_1$ and $z_2$ must equal the outgoing flux, which implies that

$$\Sigma_z = M^\dagger \Sigma_z M, \quad \text{where} \quad \Sigma_z \equiv \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix}. \quad (5)$$

Secondly, time-reversal invariance states that for each solution, there is also a solution obtained by taking the complex conjugate of the wavefunctions. Hence,

$$M^* = \Sigma_x M \Sigma_x, \quad \text{where} \quad \Sigma_x \equiv \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}, \quad (6)$$

with $\mathcal{I}$ denoting the $N \times N$ identity matrix. Both symmetry relations are preserved under composition of transfer matrices. Moreover, the combination of Eqs. \[5\] and \[6\] implies that waveguide propagation is reciprocal, i.e. the scattering matrix is symmetric \[19\]. Details of these symmetries are given in Appendix A.

From the transfer matrix, we define a Hamiltonian $H$ using a Schrödinger-like equation:

$$-i \frac{\partial}{\partial z} M(z, z_0) = H(z) M(z, z_0). \quad (7)$$

$H$ is Hermitian if and only if $M$ is unitary (i.e., the reflection-free waveguide discussed in the Introduction).

In the more general case where $M$ is non-unitary, $H$ is non-Hermitian. To determine the symmetry constraints on $H$, we use the well-known fact that the exponential map commutes with the adjoint action:

$$e^{J^{-1} (i H \Delta z) J} = J^{-1} e^{i H \Delta z} J, \quad (8)$$

where

$$J \equiv \Sigma_z \Sigma_z = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}. \quad (9)$$

Applying this to Eqs. \[5\]–\[6\] gives a pair of symmetries:

$$\Sigma_z H \Sigma_z = H^\dagger, \quad (10)$$

$$\Sigma_x H \Sigma_x = -H^*. \quad (11)$$

The $H$ matrices are closely connected to the symplectic structure of the transfer matrices. As noted in Ref. \[17\], the transfer matrices can be mapped to the real symplectic group $\text{Sp}(2N, \mathbb{R})$. In a similar way, the $H$ matrices defined by Eq. \[7\] are isomorphic (in the vector space sense) to the real-valued Hamiltonian matrices, $\text{sp}(2N, \mathbb{R})$, which are the Lie algebra generators of $\text{Sp}(2N, \mathbb{R})$. For details, see Appendix A.

Any $2N \times 2N$ Hamiltonian that satisfies Eqs. \[10\]–\[11\] must have the form

$$H = \begin{pmatrix} \mathcal{H} & \mathcal{A} \\ -\mathcal{A}^* & -\mathcal{H}^* \end{pmatrix}, \quad (12)$$

where $\mathcal{H}$ and $\mathcal{A}$ are $N \times N$ matrices satisfying

$$\mathcal{H} = \mathcal{H}^\dagger, \quad \mathcal{A} = \mathcal{A}^T. \quad (13)$$

### III. EIGENVALUE PROPERTIES

The original PT symmetry criterion identified by Bender and co-workers \[11, 13\] is not the only way for non-Hermitian Hamiltonians to support real eigenvalues. Mostafazadeh and co-workers have developed the concept of “pseudo-Hermiticity” as a necessary condition for real eigenvalues \[13, 14\]. On a given Hilbert space, a linear operator $H$ is pseudo-Hermitian if there is a linear, invertible, Hermitian operator $\eta$ such that

$$H^\dagger = \eta H \eta^{-1}. \quad (14)$$

This defines a possibly indefinite inner product on $\mathcal{H}$, with $\eta$ as the metric operator \[14\]. In our case, the matrix $\Sigma_z$ in Eqs. \[10\] plays the role of $\eta$. Thus, the Hamiltonians discussed in this paper form a subgroup of the pseudo-Hermitian group. The subgroup is further restricted by the time-reversal symmetry \[11\], which has previously been noticed and named “spectral anti-PT symmetry” by Sukhorukov et al., in the context of beam evolution in a parametric amplifier \[16\].

To understand what the eigenvalues of $H$ mean in physical terms, consider a waveguide that is invariant
under a translation by $L$. A waveguide mode at fixed energy $E$ satisfies
\[ M(z + L, z)|\Psi\rangle = \exp(ikL)|\Psi\rangle, \] (15)
for some wavenumber $\kappa$. If the waveguide is invariant under continuous translations, the Hamiltonian defined via Eq. (7) is $z$-independent, and the waveguide modes satisfy
\[ H|\Psi\rangle = \kappa|\Psi\rangle. \] (16)
Note that “continuous translational invariance”, in this discussion, does not imply the absence of back-reflection. Back-reflection can be induced by inhomogeneities at length scales much smaller than what we are interested in, such as a sub-wavelength Bragg grating. In such cases, we assume that Eq. (15) holds for $L$ larger than the inhomogeneity length scale, and define $H$ in the Floquet-Bloch sense [20], restricting our attention to situations where $|\text{Re}(\kappa)|$ is far from the Brillouin zone edge.

The Hamiltonian $H$ satisfies Eqs. (10)–(11), which implies that its eigenvalues come in pairs. If $\kappa$ is an eigenvalue, then both $\kappa^*$ and $-\kappa$ are also eigenvalues. To show this, suppose that
\[ H \begin{pmatrix} v \\ w \end{pmatrix} = \kappa \begin{pmatrix} v \\ w \end{pmatrix}, \] (17)
for some $v, w \in \mathbb{C}^N$ and $\kappa \in \mathbb{C}$. By Eq. (10),
\[ H^\dagger \begin{pmatrix} v \\ -w \end{pmatrix} = \kappa \begin{pmatrix} v \\ -w \end{pmatrix}. \] (18)
Since $H$ and $H^T$ share the same eigenvalues, $\kappa^*$ is an eigenvalue of $H$. Similarly, from the spectral anti-PT symmetry [11], we obtain
\[ H^* \begin{pmatrix} w \\ v \end{pmatrix} = -\kappa \begin{pmatrix} w \\ v \end{pmatrix}, \] (19)
which implies that $-\kappa^*$ is an eigenvalue of $H$.

Thus, for each eigenvalue $\kappa$, either (i) $\kappa$ is purely real, (ii) $\kappa$ is purely imaginary, or (iii) there are four distinct complex eigenvalues \{ $\kappa, -\kappa, \kappa^*, -\kappa^*$\}. Note that case (iii) can only occur for $N \geq 2$. We can induce transitions between these cases by tuning various parameters in $H$, such as the operating energy $E$. In the following sections, we examine how these transitions occur in specific waveguide models.

**IV. ONE-CHANNEL WAVEGUIDES**

For $N = 1$, Eqs. (12)–(13) reduce to
\[ H = \begin{pmatrix} \mathcal{E} & a \\ -a^* & -\mathcal{E} \end{pmatrix}, \] (20)
where $\mathcal{E} \in \mathbb{R}$ and $a \in \mathbb{C}$.

In the context of quantum electronic waveguides, this type of non-Hermitian $2 \times 2$ Hamiltonian was previously investigated by Mathur [12], starting from a model of coupled chiral edge states in a quasi-one-dimensional quantum Hall gas. The chiral edge states satisfy the time-independent Schrödinger equation
\[ \begin{pmatrix} -i\partial_z -a(z) \\ -a(z)^* \partial_z \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \mathcal{E} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \] (21)
where $\psi^\pm$ are the wave amplitudes for the chiral edge states on opposite edges, $\mathcal{E}$ is the edge state energy, and $a(z)$ is the coupling between the edge states (which could be induced by strong disorder in the bulk). Re-arranging Eq. (21), and using the definition (7), yields Eq. (20).

When $a$ is a non-zero constant, $H$ has eigenvalues
\[ \kappa = \pm \sqrt{\mathcal{E}^2 - |a|^2}. \] (22)
These are either both real or both imaginary, with the transition occurring at $\kappa = 0$. If this transition is induced by tuning the operating frequency, it corresponds to the well-known phenomenon of a band extremum. As there are just two eigenvalues, we cannot have a transition from real eigenvalues to complex eigenvalues with non-vanishing real parts.

By direct calculation, we can derive the most general matrix $U$ satisfying (i) unitarity and (ii) $[UK, H] = 0$, where $K$ is the complex conjugation operator. This matrix has the form
\[ U = e^{i\theta} \begin{pmatrix} e^{i\text{arg}(a)} & 0 \\ 0 & e^{-i\text{arg}(a)} \end{pmatrix}, \] (23)
where $\theta \in \mathbb{R}$ is a free parameter. Any such matrix always satisfies $(UK)^2 = 1$, and thus falls into the class of generalized PT symmetric matrices discussed in Ref. [3]. These authors showed that a Hamiltonian obeying generalized PT symmetry can support real eigenvalues because there is a basis in which the $H$ matrix (and hence the secular equation) is real [3].

There are two special cases to consider. Firstly, if $a$ is real, then $U$ has the trivial form $U = \exp(i\theta)\mathbb{1}$. Secondly, if $a$ is imaginary, then we can take $U = \sigma_z$, which is indeed a valid and non-trivial parity operator, which means that $H$ is PT symmetric [1]. But for all other cases, where $a$ is non-real and non-imaginary, we cannot satisfy $U^2 = 1$, so $U$ does not represent a parity operation.

**V. TWO-CHANNEL WAVEGUIDES**

Next, we consider the $N = 2$ case. In a similar spirit to the $N = 1$ example developed by Mathur [12], based on the quantum Hall effect, we can identify a physically-motivated quantum electronic model for a $N = 2$ waveguide of the desired form. It is based on the following
Eq. (24), we obtain the spin and the direction of motion. By re-arranging which conjugates the wave amplitudes as well as reversing equation is invariant under the time-reversal operation ping between edges without spin-flip. This Schrödinger ping between edges with spin flip; and

Here, \( m \) induced by spin-orbit edge impurities; \( m \) represents hop- ping between edges with spin flip; and \( m \) represents hopping between edges without spin-flip. This Schrödinger equation is invariant under the time-reversal operation

\[
\psi^+ \leftrightarrow (\psi^-)^*, \quad \psi^- \leftrightarrow (\psi^+)^*.
\]

which conjugates the wave amplitudes as well as reversing the spin and the direction of motion. By re-arranging Eq. (24), we obtain

\[
H = \begin{pmatrix}
\mathcal{E} & -a & -m_1 & -b \\
-a^* & \mathcal{E} & -b & -m_2 \\
m_1^* & b^* & -\mathcal{E} & a^* \\
b^* & m_2^* & a & -\mathcal{E}
\end{pmatrix},
\]

where \( \mathcal{E} \in \mathbb{R} \) and \( a, b, m_1, m_2 \in \mathbb{C} \). We can see that this is the most general \( 4 \times 4 \) matrix satisfying Eqs. (12)–(13).

There are two special cases. Firstly, if \( m_1 = m_2 = 0 \), then the eigenvalues of \( H \) are

\[
\kappa = \pm \sqrt{(|\mathcal{E}|^2 - |b|^2},
\]

where the two \( \pm \) signs are independent. Secondly, if \( a = 0 \) and \( m_1 = m_2 = m \), then

\[
\kappa = \pm \sqrt{E^2 - |b|^2}.
\]

Like the \( N = 1 \) case discussed in Section [V], these eigenvalues are either real or purely imaginary, with transitions occurring at \( \kappa = 0 \).

When \( a \neq 0 \) and \( m_1, m_2 \neq 0 \), there can occur transitions between purely real or purely imaginary eigenvalues and complex eigenvalues. We can demonstrate such transitions by taking case \( b = 0 \) and \( m_1 = m_2 = m \), and varying \( \mathcal{E} \). The results are shown in Fig. 1. It can be shown that the bifurcations occur when

\[
\mathcal{E} = \pm |m| \sin[\arg(a)].
\]

Moreover, the bifurcation occurs along the real-\( \kappa \) axis if

\[
|a| > |m\cos[\arg(a)]|,
\]

and along the imaginary-\( \kappa \) axis for the opposite case. The bifurcation along the real-\( \kappa \) axis is caused by pseudo-Hermiticity, Eq. (11). The bifurcation along the imaginary-\( \kappa \) axis is caused by the spectral anti-PT symmetry, Eq. (10).

VI. CONCLUSION

We have shown that the generator of the transfer matrix, when regarded as a non-Hermitian Hamiltonian, exhibits the features of “pseudo-Hermiticity” [13][15] and “spectral anti-PT symmetry” [16]. In the literature on non-Hermitian systems, these symmetries have previously been put forward as generalizations and variants of the original PT symmetry concept [11][12]. In this context, however, they arise from the simple physical requirements of flux conservation and time-reversal symmetry respectively.

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Appendix A: Transfer matrix properties

This Appendix summarizes the properties of transfer matrices in flux-conserving and time-reversal invariant
waveguides. As discussed in the main text, such transfer matrices satisfy [17]

\[ \Sigma_z = M^\dagger \Sigma_x M, \quad (A1) \]
\[ M^* = \Sigma_x M \Sigma_x \]

Note that Eq. (A1) follows directly from flux conservation if we let \(|\psi_n|^2\) represent the magnitude of the flux corresponding to wave amplitude \(\psi_n^\pm\). It can also be written, equivalently, as \(M^\dagger \Sigma_x M^\dagger = \Sigma_z\).

We can combine Eqs. (A1)–(A2) to obtain

\[ M^T J M = J. \quad (A3) \]

This symmetry relation states that the waveguide is reciprocal [19]; to show this, consider two arbitrary independent sets of wave amplitudes \(\psi_A^\pm\) and \(\psi_B^\pm\), such that

\[ M \begin{pmatrix} \psi_A^+ \\ \Phi_A^+ \end{pmatrix} = \begin{pmatrix} \Phi_A^+ \\ \Phi_B^+ \end{pmatrix}, \quad M \begin{pmatrix} \psi_B^+ \\ \Phi_B^+ \end{pmatrix} = \begin{pmatrix} \Phi_B^+ \\ \Phi_B^- \end{pmatrix}. \quad (A4) \]

We can also define the scattering matrix \(S\), which relates incoming to outgoing waves:

\[ S \begin{pmatrix} \psi^+ \\ \phi^+ \end{pmatrix} = \begin{pmatrix} \psi^- \\ \phi^- \end{pmatrix}, \quad (A5) \]

for both \(A\) and \(B\) subscripts. Using Eqs. (A3)–(A4),

\[ \begin{pmatrix} \psi_A^+ \\ \Phi_A^+ \end{pmatrix}^T M^T J M \begin{pmatrix} \psi_B^+ \\ \Phi_B^+ \end{pmatrix} = \begin{pmatrix} \psi_A^+ \\ \Phi_A^+ \end{pmatrix}^T J \begin{pmatrix} \psi_B^+ \\ \Phi_B^+ \end{pmatrix}. \quad (A6) \]

Using Eq. (A5), we can simplify this to

\[ \begin{pmatrix} \psi_A^+ \\ \Phi_A^+ \end{pmatrix}^T (S - S^T) \begin{pmatrix} \psi_B^+ \\ \Phi_B^+ \end{pmatrix} = 0. \quad (A7) \]

Since this holds for independent sets of wave amplitudes, \(S\) must be symmetric [19].

It is important to note that Eqs. (A1)–(A2), together, form a stronger set of constraints than Eq. (A3). In optical waveguides with gain and/or loss, Eqs. (A1)–(A2) are violated, but the reciprocity relation (A3) still holds.

Returning to transfer matrices that satisfy Eqs. (A1)–(A2), we can show by direct substitution into Eq. (A2) that such matrices must take the form

\[ M = \begin{pmatrix} C & B \\ B^* & C^* \end{pmatrix}, \quad (A8) \]

where \(B\) and \(C\) are complex \(N \times N\) matrices. Using Eq. (A1), we then find that

\[ CC^\dagger - BB^\dagger = 1 \quad \text{and} \quad CB^T = B^T C. \quad (A9) \]

We can define \(C = \mathcal{X} + i\mathcal{Y} \quad \text{and} \quad B = \mathcal{F} + i\mathcal{G}\), where \(\{\mathcal{X}, \mathcal{Y}, \mathcal{F}, \mathcal{G}\}\) are real \(N \times N\) matrices. This allows us to map \(M\) to a real \(2N \times 2N\) matrix as follows [17]:

\[ f(M) = W = \begin{pmatrix} \mathcal{X} - \mathcal{G} & \mathcal{F} + \mathcal{Y} \\ \mathcal{F} - \mathcal{Y} & \mathcal{X} + \mathcal{G} \end{pmatrix}. \quad (A10) \]

The \(f\) map is one-to-one and onto, and one can show that \(W\) is symplectic (i.e., \(W JW^T = J\)) if and only if \(M\) satisfies Eqs. (A8)–(A9). Note, however, that the group operation of \(\text{Sp}(2N, \mathbb{R})\) (i.e., multiplication of the \(W\) matrices) does not correspond to the composition operation (matrix multiplication) of the transfer matrices.

In Section [1], we introduced the set of matrices \(H\) that are the infinitesimal generators of \(M\). These satisfy Eqs. (A12)–(A13). Using the same map \(f\) defined in Eq. (A10), we can show that

\[ if(H) = \begin{pmatrix} -\text{Im}(\mathcal{H}) - \text{Re}(\mathcal{A}) & -\text{Re}(\mathcal{H}) - \text{Im}(\mathcal{A}) \\ \text{Re}(\mathcal{H}) - \text{Im}(\mathcal{A}) & -\text{Im}(\mathcal{H}) + \text{Re}(\mathcal{A}) \end{pmatrix}. \quad (A11) \]

This is a real \(2N \times 2N\) matrix of the “Hamiltonian” form, meaning that it satisfies

\[ [Jf(H)]^T = Jf(H), \quad (A12) \]

where \(J\) is the skew-symmetric matrix defined in Eq. [9]. The Hamiltonian matrices are so-called because they occur naturally in systems of equations formed by Hamilton’s equations of classical mechanics [22].

Note that Eq. (A11) is one-to-one, and thus constitutes an isomorphism to the group of Hamiltonian matrices, where the group operations are addition operations on the \(H\) matrices as well as the Hamiltonian matrices. In turn, the group of Hamiltonian matrices is the Lie algebra \(\text{sp}(2N, \mathbb{R})\) that generates the Lie group \(\text{Sp}(2N, \mathbb{R})\).

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