On the Extrapolation of Magnetohydrostatic Equilibria on the Sun

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Abstract

Modeling the interface region between the solar photosphere and corona is challenging because the relative importance of magnetic and plasma forces change by several orders of magnitude. While the solar corona can be modeled by the force-free assumption, we need to take plasma forces into account (pressure gradient and gravity) in photosphere and chromosphere, here within the magnetohydrostatic (MHS) model. We solve the MHS equations with the help of an optimization principle and use vector magnetogram as the boundary condition. Positive pressure and density are ensured by replacing them with two new basic variables. The Lorentz force during optimization is used to update the plasma pressure on the bottom boundary, which makes the new extrapolation work even without pressure measurements on the photosphere. Our code is tested using a linear MHS model as reference. From the detailed analyses, we find that the newly developed MHS extrapolation recovers the reference model at high accuracy. The MHS extrapolation is, however, numerically more expensive than the nonlinear force-free field extrapolation and consequently one should limit their application to regions where plasma forces become important, e.g., in a layer of about 2 Mm above the photosphere.

Key words: Sun: magnetic fields

1. Introduction

It is a challenging problem to reconstruct the magnetic field and plasma together in the solar atmosphere. Usually in the corona, the magnetic field is expected to dominate over plasma because of the low plasma \( \beta \) (Gary 2001). The magnetic field is then modeled by the so-called force-free assumption (Wiegelmann & Sakurai 2012). However, in the photosphere and lower chromosphere, there always exists high \( \beta \) regions where the pressure gradient and gravity are also important. Still under the assumption of a stationary state, the more general extrapolation that takes into account the nonmagnetic force is called the magnetohydrostatic (MHS) extrapolation.

While sophisticated approaches of force-free extrapolation have been developed in the past few decades: Schmidt (1964), Semel (1967) for potential field; Chiu & Hilton (1977), Seehafer (1978) for linear force-free field (LFFF); and Sakurai (1981), Wu et al. (1990), Wheatland et al. (2000), Yan & Sakurai (2000), Régnier et al. (2002), Wiegelmann & Neukirch (2003), Wiegelmann (2004), Wheatland (2004), Valori et al. (2005), Amari et al. (2006), Wiegelmann et al. (2006), He & Wang (2008), Jiang & Feng (2012), Inoue et al. (2014), and Guo et al. (2016) for nonlinear force-free field (NLFFF), much less papers addressed the MHS extrapolation.

In the generic case, the MHS equations are not soluble analytically. However, a special class of MHS equilibria can be obtained by the ansatz

\[
\nabla \times \mathbf{B} = \alpha_0 \mathbf{B} + f(z) \nabla B_z \times \mathbf{e}_z, \tag{1}
\]

where the first term is a field line parallel current and the second term defines the current perpendicular to the gravity (Low 1985). For this special form of the current, the MHS equations can be solved by the separation of variables (Low 1985, 1991, 1992; Neukirch & Rastätter 1999) or a fast Fourier transform (Alissandrakis 1981). This is the so-called linear MHS model, which reduces to an LFFF for \( f(z) = 0 \). Aulanier et al. (1998) modeled the magnetic field using MHS equations derived by Low (1992), taking into account the pressure and gravity. The parameters in the linear MHS model, \( \alpha \) and \( a \), are constant in the entire computational region and a scale-height of 2 Mm was used. The authors pointed out main properties of magnetostatic configurations computed with this model, namely that the field aligned part of the current density contains two parts, the \( \alpha \mathbf{B} \) term and the horizontal currents. Different from linear force-free fields, where the current density is strictly parallel to the magnetic field, this property adds some nonlinearity regarding the field aligned currents. Another interesting property noted by Aulanier et al. (1998) is that (using their Equation (4)) the changes in plasma pressure (compared to the background atmosphere model) are stronger the more vertical the field is. This property is consistent with the observation of a reduced plasma pressure in strong field regions like sunspots. We would like to point out that the linear MHS model requires global constants \( a \) and \( \alpha \) and this excludes strong localized concentration of electric current and Lorentz forces. While Aulanier et al. (1998) used the linear MHS configuration to model solar structures, the main emphasis of our paper is to develop and test a nonlinear MHS code, which does not have such limitations. As we are not aware of exact nonlinear MHS solutions in 3D, we, however, test the code by comparison with a linear MHS model.

For general MHS equations, the computationally expensive numerical codes are required. Different numerical methods have been developed for this aim, e.g., Grad & Rubin (1958) solved a system of linear equations iteratively to approach the solution of nonlinear MHS equations. An advantage of the Grad-Rubin approach is that the underlying mathematical problem is well posed. A disadvantage of providing certain boundary conditions (currents or \( \alpha \) in the NLFFF, additional pressure in MHS) is that in reality the boundary data on both footpoints are not consistent due to measurement errors. This can lead to large differences between the solutions computed from positive and negative footpoints as shown in
Schröder et al. (2008) for the force-free approach. The Grad-Rubin method has been extended to solve the MHS equations with gravity by Gilchrist & Wheatland (2013) and Gilchrist et al. (2016). Wiegelmann & Neukirch (2006) developed an optimization principle for computing the magnetic field and plasma pressure consistently without considering gravity. The method was tested by application to a semi-analytic MHS solution that is axisymmetric. Zhu et al. (2013) modeled the MHS equilibria through a magnetohydrodynamic (MHD) relaxation method. The method was tested by a Sun-like numerical model and Hα fibril observation in the chromosphere (Zhu et al. 2016).

On the other hand, high spatial resolution data make MHS extrapolations necessary to extrapolate photospheric vector magnetograms upward and thereby resolve the physics of the upper photosphere and chromosphere. Compared with the height, say about 2 Mm, of the non-force-free layer, the common spatial resolution of vector magnetograms (e.g., 700 km for SDO/HMI) was too low to resolve it. The modeling of this thin layer, however, becomes possible with the unprecedented small pixel size of 40 km from the SUNRISE/IMaX observation. Wiegelmann et al. (2015)/Wiegelmann et al. (2017) applied a linear MHS model to a quiet/an active region using the line-of-sight (LOS)/vector magnetogram from IMaX observation during its first/second flight in 2009/2013.

In this paper we present a more general optimization model, where the magnetic field, plasma pressure, and density are computed self-consistently. The photospheric boundary (vector magnetogram, typical pressure, and density in the quiet region) is the only input of this model, which makes it applicable to real data. The basic equations are described in Section 2, a method used to update the pressure at the bottom boundary during optimization is presented in Section 3, then the algorithm is presented in Section 4. An analytic linear MHS solution for testing the code is described in Section 5 and the results are presented in Section 6. In Section 7 we present our conclusions and discuss some questions in the future application.

2. Basic Equations

The MHS equations are given by

\[
\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p - \rho g \hat{z} = 0, \tag{2}
\]

\[
\nabla \cdot \mathbf{B} = 0, \tag{3}
\]

where \(\mathbf{B}, p, \rho, g,\) and \(\mu_0\) are magnetic field, plasma pressure, plasma density, gravitational acceleration, and vacuum permeability, respectively. As the gravitational acceleration changes only 0.57\% (from 272.407 to 273.975 m s\(^{-2}\)) in the 2 Mm non-force-free layer, \(g\) is treated as a constant. We define the functional

\[
\mathcal{L}(\mathbf{B}, p, \rho) = \int_V \omega_a B^2 \Omega_a^2 + \omega_b B^2 \Omega_b^2 dV, \tag{4}
\]

with

\[
\Omega_a = B^{-2} \left[ \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p - \rho g \hat{z} \right], \tag{5}
\]

\[
\Omega_b = B^{-2} (\nabla \cdot \mathbf{B}) \mathbf{B},
\]

where \(\omega_a\) and \(\omega_b\) are the weighting functions with cos-profile.

The problem of solving Equation (2)–(3) is replaced with following minimization problem:

\[
\text{minimize } \mathcal{L}(\mathbf{B}, p, \rho) \tag{6}
\]

subject to: \(p > 0\) \tag{7}

\(\rho > 0. \tag{8}\)

The constraints can be eliminated using the variable transformation to \(p\) and \(\rho\)

\[
p = Q^2, \tag{9}
\]

\[
\rho = \frac{R^2}{g H_s}, \tag{10}
\]

where pressure scale-height \(H_s\) is a constant. \(H_s\) and \(g\) in Equation (10) are used to make \(R\) as the dimension of \(B\) and \(Q\). Then the above constrained optimization problem is changed to an unconstrained one,

\[
\text{minimize } \mathcal{L}(\mathbf{B}, Q, R). \tag{11}
\]

According to Wheatland et al. (2000), Wiegelmann (2004), and Wiegelmann & Neukirch (2006), the optimization can be simply extended to solve the MHS equations with gravity. Taking the functional derivative of the functional (4) with respect to an iteration parameter \(t\) leads to

\[
\frac{1}{2} \frac{dL}{dt} = -\int_V \left( \frac{\partial B}{\partial t} \cdot \tilde{F} + \frac{\partial Q}{\partial t} F_1 + \frac{\partial R}{\partial t} F_2 \right) dV
\]

\[
- \int_S \left( \frac{\partial B}{\partial t} \cdot \tilde{G} + \frac{\partial Q}{\partial t} G_1 \right) dS, \tag{12}
\]

where \(\tilde{F}, F_1, F_2, \tilde{G}\) and \(G_1\) are defined in the Appendix.

If \(\mathbf{B}, p, \rho\) are fixed on the boundary of the computation box, \(L\) can be minimized by solving the equations

\[
\frac{\partial B}{\partial t} = \mu_1 \tilde{F}, \quad \frac{\partial Q}{\partial t} = \mu_2 F_1, \quad \frac{\partial R}{\partial t} = \mu_3 F_2 \tag{13}
\]

iteratively. In the paper, \(\mu_1 = \mu_2 = \mu_3 = 1\).

3. Consistent Evolution of Pressure on the Boundary

Because of the observational limitation, only the vector magnetogram on the photosphere can be used as a boundary input. The weighting functions diminish the effect of the unknown top and lateral boundaries (Wiegelmann 2004), but different from the NLFFF extrapolation we need additional information regarding the plasma pressure and density on the bottom boundary.

Because the gravitational force is only in the vertical direction, we derive the following simplified MHS equations on the 2D photospheric layer:

\[
\nabla_{ph} p = f_{ph}, \tag{14}
\]

where \(f_{ph}\) is the 2D Lorentz force on the photosphere and \(\nabla_{ph} = \hat{x} \partial_x + \hat{y} \partial_y\). Taken another divergence operation on both sides of Equation (14) results in the Poisson’s equation

\[
\Delta_{ph} p = \nabla \cdot f_{ph}, \tag{15}
\]

where \(\Delta_{ph} = \partial_x^2 + \partial_y^2\) is the 2D Laplacian. If we know the Lorentz force, the pressure is determined when the pressure on the four edges of the bottom plane (xy-plane) is assigned; the typical pressure of the quiet region can be used as the pressure on the edges if the computation box is much larger than the
active region. Although we do not know the Lorentz force of the MHS equilibria to be extrapolated, we can compute it at any step during the optimization. Then an iterative approach can be designed to update the pressure on the photosphere consistently with the magnetic field (detailed description of the algorithm in Section 4).

From another perspective, any vector field can be decomposed into curl-free and divergence-free components (Helmholtz decomposition). For the Lorentz force on the photosphere, however, it is curl-free if the stationary state is maintained. But the Lorentz force has a divergence-free component during the optimization. Taking additional divergence operations to Equation (14) extracts the curl-free component of the Lorentz force. The curl-free component of the Lorentz force determines the pressure.

So far, we used information regarding the Lorentz force during optimization to update the bottom pressure. It appears that the density $\rho$ can be easily computed from force balance in the $z$-direction, $\rho = \left( \frac{1}{\mu_0} (\nabla \times B) \times B - \nabla p \right)_z / g$. However, the test shows no improvement of the results. We will further study this issue in the future. In this paper, the bottom density is uniform and fixed during optimization.

4. Numerical Implementation

We have developed a code to compute 3D-MHS equilibria, based on the previous optimization code (Wiegelmann 2004; Wiegelmann & Neukirch 2006).

1. Calculate an NLFFF using the vector magnetogram.
2. Insert an isothermal gravity stratified atmosphere. The $p$ and $\rho$ on the photosphere are uniformly distributed.
3. Iterate for $B$, $p$, and $\rho$ by Equation (13). This step is repeated until $L$ reaches its minimum.
4. Update $p$ on the photosphere by solving Poisson’s Equation (15) with Lorentz force computed from $J \times B$, and repeat from step 3 if $p$ is not changed for the given tolerance, iteration stops and outputs $B$, $p$, and $\rho$. 

Figure 1. Magnetic field for test 1 with different models. The field lines start from the same seeds which are uniformly distributed in the bottom plane.
5. Reference MHS Solution

Low (1985, 1991) presented a class of analytic solutions of the 3D static, magnetized atmospheres. The solutions are characterized by two parts of electric currents as described in Equation (1), namely the component parallel to the magnetic field and the component perpendicular to the gravitational field. Assume that \( f(z) \) has the form

\[
f(z) = a \exp^{-\kappa z},
\]

where \( a \) and \( \kappa \) control the magnitude and effective height of Lorentz force. Using Fourier transforming \( B \) with respect to \( x \) and \( y \), Equation (1) can be solved by the separation of variables with the LOS magnetogram as the bottom boundary. With this magnetic structure, the pressure and density have the following distribution:

\[
p = p_0(z) - \frac{1}{2\mu_0} f(z) B_z^2,
\]

\[
\rho = \frac{1}{g} \frac{dp_0}{dz} + \frac{1}{\mu_0 g} \frac{df}{dz} B_z^2 + f(B \cdot \nabla) B_z.
\]

It is apparent from the two equations above that the plane-parallel hydrostatic atmosphere \( (\rho_0 = -\frac{1}{g} \frac{dp_0}{dz}; p_0) \) is disturbed by the magnetic field. The pressure is weak in the strong \( B_z \) region with \( f > 0 \).

To determine all of the variables in the computation box, we use the following parameter set: Low & Lou (1990) LOS magnetogram labeled \( n = m = 1 \), \( \Phi = \pi/4 \), and \( l = 0.3 \) in their notation; field line parallel linear current with \( \alpha = -3.0 \) and nonmagnetic force with \( a = 0.5; \kappa = 0.02 \) means that the effective height of Lorentz force is 50 grids; the background atmosphere with \( \rho_0(z = 0) = 9.0 \times 10^{-4} \text{kg m}^{-3} \) and temperature of \( T_0(z) = 6000/5500/7840 \) K at the height 0/0.5/1.28 Mm (use linear interpolation to derive inter point temperature). For more sophisticated modeling of the vertical temperature profile see Vernazza et al. (1981). Then a linear MHS solution is generated in the Cartesian box (unit: Mm)

\[
V = \{(x, y, z)| -1.6 \leq x \leq 1.6, \quad -1.6 \leq y \leq 1.6, \quad 0 \leq z \leq 1.28\}.
\]

All above parameters are chosen to mimic a small magnetic pole on the Sun. The grid points \( 80 \times 80 \times 32 \) are used to resolve this reference model. The grid size is 40 km, which is the same with the SUNRISE/IMaX data.

6. Results

We use the figures of merit introduced by Schrijver et al. (2006) to quantify the difference between the reconstructed magnetic field \( B \) and the reference one \( b \), and supplement these with a \( C \)-value between the field lines, linear Pearson correlation coefficients both for the 3D and LOS integration (along the \( z \)-axis) of plasma pressure \( \text{corr}3D,p \), \( \text{corr}2D,p \), and density \( \text{corr}3D,\rho, \text{corr}2D,\rho \). They are defined as:

1. Vector correlation

\[
C_{\text{vec}} = \sum_{i} \frac{B_i \cdot b_i}{\left( \sum_{i} |B_i|^2 \sum_{i} |b_i|^2 \right)^{1/2}}.
\]
better field lines than the NLFFF extrapolation. See also Table 1 of the C-values of the individual field line. The mean C-values of NLFFF and MHS extrapolated lines are 0.162 and 0.016, with corresponding standard deviations of 0.128 and 0.026, respectively. The above comparisons show how the Lorentz force affects the field line patterns.

**Figure 3.** LOS integration of the plasma pressure (top) and density (bottom) in the central field of view \( x, y \in [-1.2, 1.2] \) (unit: Mm). Left/right panels correspond to the reference/reconstructed solution.

**Table 1**

| No. | \( C_{\text{nlfff}} \) | \( C_{\text{mhs}} \) | No. | \( C_{\text{nlfff}} \) | \( C_{\text{mhs}} \) | \( (C_{\text{nlfff}}) \pm \sigma \) | \( (C_{\text{mhs}}) \pm \sigma \) |
|-----|----------------|----------------|-----|----------------|----------------|----------------------|----------------------|
| 1   | 0.059          | 0.007          | 11  | 0.277          | 0.009          | 21                   | 0.003                | 0.005                |
| 2   | 0.105          | 0.011          | 12  | 0.103          | 0.012          | 22                   | 0.025                | 0.005                |
| 3   | 0.131          | 0.016          | 13  | 0.191          | 0.009          | 23                   | 0.189                | 0.024                |
| 4   | 0.104          | 0.017          | 14  | 0.164          | 0.006          | 24                   | 0.393                | 0.076                |
| 5   | 0.055          | 0.019          | 15  | 0.248          | 0.010          | 25                   | 0.084                | 0.005                |
| 6   | 0.013          | 0.001          | 16  | 0.234          | 0.013          | 26                   | 0.035                | 0.005                |
| 7   | 0.214          | 0.006          | 17  | 0.133          | 0.012          | 27                   | 0.026                | 0.006                |
| 8   | 0.255          | 0.012          | 18  | 0.520          | 0.014          | 28                   | 0.031                | 0.007                |
| 9   | 0.250          | 0.012          | 19  | 0.221          | 0.009          | 29                   | 0.392                | 0.138                |
| 10  | 0.262          | 0.012          | 20  | 0.025          | 0.003          | 30                   | 0.118                | 0.015                |

**Note.** The footpoints of 30 lines are randomly distributed in the negative region \( (B_z < 0) \).
The order of the figures of merit (see Table 2) agrees with the conclusion from the above visual quality. Figure 3 shows LOS integration of plasma pressure and density along the $z$-axis. We also notice that the MHS extrapolation needs five times more steps and six times more CPU time than the NLFFF extrapolation.

6.2. Test II: Bottom Vector Magnetogram with a Weighted Boundary Layer

In test II, we only use the bottom vector magnetogram as the boundary input, which mimics the real situation. In this test, the total $80 \times 80 \times 32$ grids of the box consist of the inner region ($64 \times 64 \times 24$) and layer ($nd = 8$ grids) at the lateral and top boundaries with cos-profile weighting functions (Wiegelmann 2004). To see if the pressure update on the photosphere improves the result, we perform two test runs for MHS extrapolation. The difference between them is that in one of the runs, the pressure is uniform and fixed on the photosphere during optimization, while in the other run, we update the pressure using the method mentioned in Section 3.

Figure 4 shows the overall magnetic field line patterns from different models for test II using the bottom magnetogram.

![Magnetic field in the inner region](image)

**Figure 4.** Magnetic field in the inner region (smaller box) for test II with different models.

| Model   | $C_{vec}$ | $C_{cs}$ | $1 - E_w$ | $1 - E_n$ | $corr_{2D, \rho}$ | $corr_{2D, \rho}$ | $corr_{3D, \rho}$ | $corr_{3D, \rho}$ | Step ($\times 10^3$) |
|---------|-----------|----------|-----------|-----------|-------------------|-------------------|-------------------|-------------------|------------------|
| Potential | 0.8911    | 0.7841   | 0.4952    | 0.4080    | /                 | /                 | /                 | /                 | /                |
| NLFFF    | 0.9875    | 0.9747   | 0.8531    | 0.8405    | /                 | /                 | /                 | /                 | 110               |
| MHS      | 0.9979    | 0.9911   | 0.9492    | 0.9237    | 0.9988            | 0.9993            | 1.0000            | 0.9999            | 590               |

The order of the figures of merit (see Table 2) agrees with the conclusion from the above visual quality. Figure 3 shows LOS integration of plasma pressure and density along the $z$-axis. We also notice that the MHS extrapolation needs five times more steps and six times more CPU time than the NLFFF extrapolation.
From Figure 5, we can see that the MHS extrapolation produces better field lines than the NLFFF extrapolation. (See also Table 3 of the C-values of the individual field line.) The mean C-values of NLFFF and MHS extrapolated lines are 0.103 and 0.059, with corresponding standard deviations of 0.061 and 0.049, respectively. Although the field line geometry difference between the two MHS extrapolations is not large, the integration of plasma pressure and density along z-axis (see Figure 6) shows rather large differences. Updating the bottom pressure significantly improves the pressure and density results.

The process of MHS extrapolation is optimizing the magnetic field and plasma. Figure 7 shows how far the final
plasma deviates from the initially gravity stratified atmosphere. We can see the final solution is close to the gravity stratified atmosphere at the low height. When $z$ increases, the difference becomes larger. To check if the MHS equations are fulfilled in the extrapolated solution of test II, the field line components of $-\nabla p$ and $\rho g$ are calculated. Defining

$$\text{Ratio} = \frac{\hat{B} \cdot (-\nabla p + \rho g)}{|\nabla p| + |\rho g|},$$

where $\hat{B} = B/B$ is the unit vector along the magnetic field line. For an MHS equilibrium, the ratio = 0 anywhere. Here we compute the ratio along four field lines (same lines in Figure 5). For all 272 points, the mean ratio is 0.86% with a standard deviation of 0.96% (see Figure 8). The extremely small ratio means that the recovered plasma satisfies the field line component of the MHS equation at high accuracy.

6.2.1. Influence of Initial Conditions

Here we investigate in the dependence of the result on the choice of the initial condition. The two initial magnetic fields we use are (1) the potential field (Seehafer 1978) and (2) the NLFFF produced by the optimization code (Wiegelmann 2004), while the two initial atmospheres are the (1) isothermal atmosphere and the (2) more realistic 1D model described in Section 5. This results in four combinations (see Table 5). The choice of the initial magnetic field configuration has a significant influence on the resulting magnetic field and plasma equilibrium. Similar conclusions were found in previous studies for the NLFFF (Wiegelmann 2004; Schrijver et al. 2006): a starting state that is near the true solution leads to a better result. That means we better use a multigrid approach to give a better starting state, similar to that used as a standard in the NLFFF extrapolation. However, notice that the initial potential field results in a somewhat more accurate density solution.
Unlike the magnetic field, the choice of the initial atmosphere has negligible influence on the results. Either the isothermal atmosphere or the Sun-like atmosphere give almost the same solution in this test.

### 6.2.2. Influence of Noise

Until now, we input the magnetic field on the bottom boundary as it is known exactly. However, this is not the case when the real vector magnetogram is used. In this subsection, we study the influence of the noise of the bottom magnetic field by adding some random noise (2% in \( B_z \), \( n_l \) in \( B_x \) and \( B_y \)) to the magnetogram. \( n_l = 5\%, \, 10\%, \, 15\%, \, 20\% \) are the noise levels of the transverse field for the different test runs. The same cross-profile weighting functions and boundary layer \( n_d = 8 \) are used in these test runs.

Table 6 shows the results. The random noise of the magnetic field is independent of the neighboring grids. This leads to high frequent noise of the current and Lorentz force on the photosphere, which makes the extrapolation inaccurate. As a result, all metrics are getting worse with increasing noise.

### 7. Discussion and Conclusions

In this work, we have generalized the optimization method to apply to the MHS equilibria. Compared with the NLFFF approach, the MHS optimization confronts two new challenges: (1) how to ensure positive pressure and density and (2) how to deal with boundary pressure and density. The first problem is actually how to deal with the positivity constraint in optimization. This constraint can be eliminated by the variable transformation of Equations (9) and (10). The second problem is more complex because no measurements of the plasma pressure and density are available. Some information, however, is included in the data of the vector magnetogram. Based on the assumption of the force balance in the bottom plane, we obtain Poisson’s Equation (15) for computing pressure on the photosphere. Then we design an algorithm to update the bottom pressure consistently within the optimization procedure.

In test II, we need to update the bottom pressure 18 times, and most steps (153 K in total 191 K) are in the first round of \( L \) minimization.

We conclude the following from the above tests. (1) The MHS equilibria are reconstructed at relatively high accuracy by...
the generalized optimization principle for iterating the magnetic field, plasma pressure, and mass density simultaneously. (2) Updating the bottom pressure using the Lorentz force significantly improves the results of the MHS extrapolation. (3) The initial choice of the magnetic field influences the final results significantly, whereas, MHS extrapolation using an NLFFF model as the initial condition produces much better results than using a potential field.

We also test our code with vanishing $f_z$ by setting $a = 0.0$. The model with $a = 0.0$ is an LFFF. As supposed, our code can recover the LFFF at almost the same accuracy with the results obtained by the NLFFF approach.

Notice that the bottom density is still uniformly distributed in the current extrapolation. We would like to address this issue in a future article. In test II, the MHS extrapolation takes about $6.5$ CPU hours on a $2.1$ GHz processor. An application to IMaX vector magnetogram embedded by HMI data (about $2000 \times 2000$ grids; see Wiegelmann et al. 2017) needs a large amount of computational resources. For practical application, we can limit the calculation to the SUNRISE-FOV ($936 \times 936$) to reduce the computation. A multigrid approach is likely to enable faster convergence with high resolution magnetograms. Furthermore, the MHS model should be restricted to the non-force-free layer (about $2$ Mm above the photosphere) to reduce the computation time. In the force-free corona above, computational, less expensive NLFFF extrapolations can be used.

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Appendix

Variable Definitions

The variables in Equation (12) are defined as

$$F_1 = -2Q \nabla \cdot (\omega_a \Omega_a),$$

where $\hat{n}$ is the inward unit vector on the surface $S$.

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