Clique Separator Decomposition of Hole- and Diamond-Free Graphs and Algorithmic Consequences

Andreas Brandstädter* Vassilis Giakoumakis†

January 21, 2013

Abstract

Clique separator decomposition introduced by Tarjan and Whiteides is one of the most important graph decompositions. A graph is an atom if it has no clique separator. A hole is a chordless cycle with at least five vertices, and an antihole is the complement graph of a hole. A graph is weakly chordal if it is hole- and antihole-free. $K_4 - e$ is also called diamond. Paraglider has five vertices four of which induce a diamond, and the fifth vertex sees exactly the two vertices of degree two in the diamond. In this paper we show that atoms of hole- and diamond-free graphs (of hole- and paraglider-free graphs, respectively) are either weakly chordal or of a very specific structure. Hole- and paraglider-free graphs are perfect graphs. The structure of their atoms leads to efficient algorithms for various problems.

Keywords: Clique separator decomposition; hole- and diamond-free graphs; hole- and paraglider-free graphs; perfect graphs; efficient algorithms.

1 Introduction, Motivation and Related Work

A clique separator (or clique cutset) of a graph $G$ is a clique $K$ such that $G[V \setminus K]$ has more connected components than $G$. An atom is a graph without clique separator. In [32, 34], it is shown that a clique separator decomposition tree of a graph can be determined in polynomial time, and in [32], this decomposition is applied to various problems such as Minimum Fill-in, Maximum Weight Independent Set (MWIS), Maximum Weight Clique and Coloring; if the problem is solvable in polynomial time on the atoms of a hereditary graph class $C$, it is solvable in polynomial time on class $C$. In this paper, we are going to analyze the structure of atoms in two subclasses of hole-free graphs.

A hole is a chordless cycle with at least five vertices, and an antihole is the complement graph of a hole. A graph is hole-free (antihole-free, respectively) if it contains no induced subgraph which is isomorphic to a hole (an antihole, respectively). $K_4 - e$ (i.e., a clique of four vertices minus one edge) is called diamond. A paraglider has five vertices four of which induce a diamond, and the fifth vertex sees exactly the two vertices of degree two in the diamond (see Figure 1). Note that paraglider is the

*Fachbereich Informatik, Universität Rostock, A.-Einstein-Str. 21, D-18051 Rostock, Germany, ab@informatik.uni-rostock.de
†MIS (Modélisation, Information & Systèmes), Université de Picardie Jules Verne, Amiens, France. e-mail: vassilis.giakoumakis@u-picardie.fr
complement graph of the disjoint union $P_2 \cup P_3$ (where $P_n$ denotes a chordless path with $n$ vertices and $n - 1$ edges).

Cycle properties of graphs and their algorithmic aspects play a fundamental role in combinatorial optimization, discrete mathematics and computer science. Various graph classes are characterized in terms of cycle properties - among them are the classes of chordal graphs, weakly chordal graphs and perfect graphs which are of fundamental importance for algorithmic graph theory and various applications. A graph is \textit{chordal} (also called \textit{triangulated}) if it is hole- and $C_4$-free (where $C_4$ denotes the chordless cycle of four vertices). See e.g. \cite{13, 22, 30} for the many facets of chordal graphs. A graph is completely decomposable by clique separator decomposition if and only if it is chordal. A graph is \textit{weakly chordal} (also called \textit{weakly triangulated}) if it is hole- and anti-hole-free. These graphs have been extensively studied in \cite{25, 26, 28, 31}; they are perfect. In \cite{2, 27}, recognition of weakly chordal graphs is solved in time $O(m^2)$, and the MWIS problem on weakly chordal graphs is solved in time $O(n^4)$. Chordal graphs are weakly chordal.

The celebrated \textit{Strong Perfect Graph Theorem} (SPGT) by Chudnovsky et al. says:

\textbf{Theorem 1 (SPGT \cite{19}).} A graph is perfect if and only if it is odd-hole-free and odd-anti-hole-free.

It is also well known that a graph is the line graph of a bipartite graph if and only if it is (claw,diamond,odd-hole)-free (see e.g. \cite{13}). These graphs play a fundamental role in the proof of the SPGT.

Since every hole $C_k$, $k \geq 7$, contains the disjoint union of $P_2$ and $P_3$ (and the paraglider is the complement graph of $P_2 \cup P_3$), it follows that HP-free graphs are $C_k$-free for every $k \geq 7$. Thus, by the SPGT, HP-free graphs are perfect. Our structural results for atoms of HP-free graphs, however, give a more direct way to show perfection of HP-free graphs.

Hole- and diamond-free graphs generalize the important class of chordal bipartite graphs (which are exactly the hole- and triangle-free graphs), and diamond-free chordal graphs are the well-known block graphs - see \cite{13} for various characterizations and the importance of chordal bipartite graphs as well as of block graphs. In \cite{10, 17}, various characterizations of (dart,gem)-free chordal graphs are given; among others, it is shown that a graph is (dart,gem)-free chordal if and only if it results from substituting cliques into the vertices of a block graph.

Recently there has been much work on related classes such as even-hole-free (forbidding also $C_4$) and diamond-free graphs \cite{29} (see also \cite{33}) and \cite{21} dealing with the structure and recognition of $C_4$- and diamond-free graphs.

Hole- and paraglider-free graphs obviously generalize chordal graphs. The classes of weakly chordal graphs and HP-free graphs are incomparable as the examples of paraglider (which is weakly chordal but not HP-free) and $C_6$ (which is HP-free but not weakly chordal) show but HP-free graphs are closely related to weakly chordal graphs:

Our main result in this paper shows that atoms of hole- and paraglider-free graphs (HP-free graphs for short) are either weakly chordal or of a very simple structure close to matched co-bipartite graphs. By \cite{32}, this has various algorithmic consequences; in section \cite{5} we describe these and others.
2 Further Basic Notions

Let $G$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. Adjacency of vertices $x, y \in V$ is denoted by $xy \in E$, or $x \sim y$, or we simply say that $x$ and $y$ see each other. Nonadjacency is denoted by $xy \notin E$, or $x \not\sim y$, or $x$ and $y$ miss each other.

The open neighborhood $N(v)$ of a vertex $v$ in $G$ is $N(v) = \{u \mid uv \in E\}$, the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$, and the antineighborhood $A[v]$ of $v$ is $A(v) = \{u \mid u \not\sim v\}$. The neighborhood $N(X)$ of a subset $X \subseteq V$ is the set of all neighbors of $x \in X$ outside $X$. For a subgraph $H$ of $G$, let $N_H(x)$ denote the set $N(x) \cap V(H)$ and let $N_H(X)$ denote the set $N(X) \cap V(H)$.

For graph $G$, let $\overline{G}$ (or co-$G$) denote the complement graph of $G$, i.e., $\overline{G} = (V(G), \{xy \mid x \not\sim y\})$. For $H \subseteq V$, let $G[H]$ denote the induced subgraph of $H$ in $G$.

Let $P_k$ denote a chordless path with $k$ vertices $x_1, \ldots, x_k$ and edges $x_ix_{i+1}, 1 \leq i \leq k - 1$, and let $C_k$ denote a chordless cycle with the same $k$ vertices and edges $x_ix_{i+1}, 1 \leq i \leq k - 1$, and $x_kx_1$.

A vertex set $U \subseteq V$ is independent if the vertices of $U$ are pairwise nonadjacent. $U$ is a clique if the vertices of $U$ are pairwise adjacent. Let $S_r$ ($K_r$, respectively) denote an independent vertex set (a clique, respectively) with $r$ vertices.

For vertex $x$ of graph $G$ and $H \subseteq V(G)$, $x \oplus H$ means that $x$ is adjacent to all vertices of $H$. In this case, we also say that $x$ is total or universal with respect to $H$. Correspondingly, $x \ominus H$ means that $x$ is adjacent to no vertex of $H$.

For $H \subseteq V(G)$ and $Q \subseteq V(G)$ with $H \cap Q = \emptyset$, $H \odot Q$ means that every vertex of $H$ is adjacent to every vertex of $Q$ (we also say that $H$ and $Q$ form a join) and $H \oslash Q$ means that no vertex of $H$ is adjacent to any vertex of $Q$ ($H$ and $Q$ form a co-join).

Let $G$ be a graph. $G \setminus H$ or $G - H$ denotes the graph $G[V(G) - V(H)]$ induced by the set of vertices $V(G) - V(H)$.

Let $\mathcal{F}$ be a set of graphs. $G$ is $\mathcal{F}$-free if no induced subgraph of $G$ is an element of $\mathcal{F}$. As already mentioned, $G$ is hole-free (is antihole-free, respectively) if no induced subgraph of $G$ is isomorphic to a hole (an antihole, respectively).

A co-matched bipartite graph results from a complete bipartite graph $K_{k,k}$ by deleting a perfect matching. A matched co-bipartite graph is the complement of a co-matched bipartite graph, i.e., it consists of two disjoint cliques of the same size $k$, and the edges between them form a matching with $k$ edges.

Note that $\overline{C_6}$ is a matched co-bipartite graph with six vertices. Let $A$ be a matched co-bipartite graph. Then left$(A)$ denotes one of the maximal cliques of $A$ and right$(A)$ denotes the other maximal clique of $A$. Clearly left$(A)$ and right$(A)$ form a bipartition of the co-matched bipartite graph $\overline{A}$ (and thus a corresponding partition of the vertex set of $A$). Subsequently, the edges between left$(A)$ and right$(A)$ are called matching edges.

3 Adjacency Properties for (Hole,Paraglider)-Free Graphs Containing $\overline{C_6}$

In this section we describe some adjacency properties of HP-free graphs containing $\overline{C_6}$ which will be useful in the structural description of atoms of hole- and paraglider-free
graphs.

3.1 Neighbors of $C_6$ in HP-Free Graphs

Throughout this section, let $G$ be an HP-free graph. As mentioned already in the introduction, the only possible antihole in an HP-free graph is $C_6$; if $G$ is $C_6$-free, it is weakly chordal. The following propositions are dealing with HP-free graphs containing $C_6$. Obviously, the following holds:

**Proposition 1.** Pairs $x, y$ with $x \not\sim y$ in a $C_6$ $A$ are endpoints of a $P_4$ $(x, a, b, y)$ and two $P_3$'s $(x, c, y)$, $(x, d, y)$ such that $(c, a, b, d)$ is another $P_4$ in $A$.

Let $A$ be a graph isomorphic to a $C_6$. The set of vertices outside $A$ having distance $i \geq 1$ from $A$ will be denoted by $D_i(A)$. Moreover, $D_1 = D_1(A) = A_1 \cup \ldots \cup A_6$, where $A_i$, $i \in \{1, \ldots, 6\}$, denotes the set of vertices outside $A$ with distance one from $A$ and having exactly $i$ neighbors in $A$ (note that $A_i$ contain only vertices which are not in $A$).

Obviously, the next property holds:

**Proposition 2.** If $x, y \in A_1$ with $x \sim y$, and $N_A(x) = \{t\}$, $N_A(y) = \{z\}$ with $t \neq z$ then $t \sim z$.

For neighbors outside $A$ which see more than one vertex in $A$, the situation is as follows:

**Proposition 3.**

(i) The two $A$-neighbors of any vertex in $A_2$ form an edge in $A$.

(ii) The three $A$-neighbors of any vertex in $A_3$ form a triangle in $A$.

(iii) $A_4 = A_5 = \emptyset$.

(iv) $A_6$ is a clique. Moreover, in a hole- and diamond-free graph, $A_6 = \emptyset$.

(v) If $x$ sees $A$ and $N_A(x)$ is not a clique then $x \in A_6$.

**Proof.** (i): If $x \in A_2$ sees $y$ and $z$ in $A$ with $y \not\sim z$ then by Proposition 1, there is a $P_4$ $P$ in $A$ with endpoints $y$ and $z$. It follows that $x$ together with $P$ induce a $C_5$ in $G$, a contradiction.

(ii): If the neighborhood of $x \in A_3$ in $A$ is not a triangle then without loss of generality, $x$ sees two vertices in $\text{left}(A)$, say $a$ and $b$, and one in $\text{right}(A)$, say $c$. If $c$ misses $a$ and $b$ then
$x, a, b, c$ together with the neighbor of $c$ in left$(A)$ induce a paraglider, and if $c$ sees $a$ then $x, a, b, c$ together with the neighbor of $b$ in right$(A)$ induce a paraglider - contradiction.

$(iii)$: If $x \in A_1$ sees all three vertices in left$(A)$, say $a, b, c$, and one in right$(A)$, say $d$, then if $a$ sees $d$, $x, a, b, d$ together with the neighbor of $b$ in right$(A)$ induce a paraglider. If $x$ sees two vertices in left$(A)$, say $a, b$, and two vertices in right$(A)$, say $c, d$ then if $a$ sees $c$ and $b$ sees $d$, $x, a, d$ and the matching edge which $x$ is missing induce a $C_5$. If $a$ misses $d$ and $b$ sees $c$ then $x, a, b, c$ and the neighbor of $a$ in right$(A)$ induce a paraglider.

If $x \in A_5$ sees all three vertices in left$(A)$ and two in right$(A)$, say $d, e$, then $x, d, e$ together with the vertex $f$ which $x$ misses in right$(A)$ and the neighbor of $f$ in left$(A)$ induce a paraglider.

$(iv)$: If there are $x, y \in A_6$ with $x \not\sim y$ then $x$ and $y$ together with any $P_1 \cup P_2$ from $A$ form a paraglider. Moreover, the vertices of any $P_3$ in $A$ together with any vertex of $A_6$ induce a diamond.

$(v)$: This property easily follows from the preceding ones. 

\begin{proposition}
Let $x \sim y$. If $x \in A_1$ and $y \in A_2 \cup A_3$ or $x \in A_2$ and $y \in A_3$ then $N_A(x)$ and $N_A(y)$ are comparable with respect to set inclusion.
\end{proposition}

\begin{proof}
As before, let $A$ be a $C_6$, say with cliques $\text{left}(A) = \{v_1, v_2, v_3\}$, $\text{right}(A) = \{v_4, v_5, v_6\}$ and matching edges $v_1v_4, v_2v_5$ and $v_3v_6$.

First let $x \in A_1$; without loss of generality, let $N_A(x) = \{v_1\}$ and assume that $y \not\sim v_1$.

Recall that $y \in A_2$ or $y \in A_3$. If $\{v_2, v_3\} \subseteq N_A(y)$ then $x, y, v_1, v_2, v_3$ induce a paraglider. Thus $y$ must see at least one vertex from right$(A)$. If $y$ sees $v_5$ then either $x, v_1, v_4, v_5, y$ or $x, v_1, v_2, v_5, y$ is a $C_5$ since by Proposition 3 $N_A(y)$ is a clique, and similarly if $y$ sees $v_6$. Thus $y$ misses $v_5$ and $v_6$ which implies that $y$ sees $v_4$. Since by assumption, $y$ misses $v_1, y$ sees $v_4$ and $v_2$ or $v_3$ but this contradicts Proposition 3.

Now let $x \in A_2$ and $y \in A_3$; by Proposition 3 $N_A(y) = \text{left}(A)$ or $N_A(y) = \text{right}(A)$ and $N_A(x)$ is an edge in $A$. If $N_A(x) = \{v_1, v_2\}$ and $N_A(x)$ and $N_A(y)$ are not comparable then $N_A(y) = \text{right}(A)$ but now $x, v_2, v_3, v_6, y$ is a $C_5$ - contradiction. If however $N_A(x) = \{v_1, v_3\}$ and without loss of generality, $N_A(y) = \text{left}(A)$ then $x, y, v_2, v_5, v_4$ is a $C_5$ which shows Proposition 3.

\begin{proposition}
For all $x, y \in A_2$ with $x \sim y$, $N_A(x) \cup N_A(y)$ is a clique.
\end{proposition}

\begin{proof}
By Proposition 3 $N_A(x)$ and $N_A(y)$ are edges. Assume to the contrary that there are $z \in N_A(x)$ and $t \in N_A(y)$ with $z \not\sim t$. Thus $z \notin N_A(y)$ and $t \notin N_A(x)$. By Proposition 1 there is a $P_4 (z, u, v, t)$ in $A$. Since $N_A(x)$ is an edge, $x$ misses $v$, and likewise $y$ misses $u$. To avoid a hole in the subgraph induced by $\{x, z, u, v, t, y\}$, we obtain $x \sim u$ and $y \sim v$ which implies that $N_A(x) \cup N_A(y) = \{z, u, v, t\}$. Then by Proposition 1 there is a $P_3 (z, w, t)$ in $A$ such that $x$ and $y$ miss $w$ and consequently $x, z, w, t, y$ induce a $C_5$ in $G$, a contradiction.

Now it is easy to see that by Propositions 2, 3, 4, and 5 we obtain:

\begin{corollary}
For all $x, y \in A_3$ with $x \sim y$ and at least one of $x, y$ does not belong to $A_3$, $N_A(x) \cup N_A(y)$ is a clique.
\end{corollary}
Proposition 6. Let \( x, y \in D_1 \) with \( x \not\sim y \) be the endpoints of a chordless path \( P \) whose internal vertices do not belong to \( D_1 \cup A \). Then

(i) \( P \) contains exactly three vertices \( x, w, y \) and

(ii) \( N_A(x) \) and \( N_A(y) \) are comparable.

Proof. (i): Assume to the contrary that \( P \) contains at least four vertices. Let \( u \) and \( v \) be two vertices of \( A \) such that \( u \in N_A(x) \) and \( v \in N_A(y) \) and let \( Q \) be a chordless path in \( A \) joining \( u \) and \( v \) (possibly \( \text{length}(Q) = 0 \), i.e., \( u = v \)). Now it is easy to verify that the graph induced by the vertices of \( P \cup Q \) contains a hole, a contradiction.

(ii): Assume to the contrary that \( N_A(x) \) and \( N_A(y) \) are not comparable. Let \( z \) and \( t \) be two vertices of \( A \) such that \( z \in N_A(x) \) \( - \) \( N_A(y) \) and \( t \in N_A(y) \) \( - \) \( N_A(x) \). If \( z \) is adjacent to \( t \) then \( x, z, t, y, w \) (where \( w \) is the vertex from condition (ii)) induce a \( C_5 \). Hence \( z \not\sim t \), and by Proposition 1 there is a \( P_4 \) \((z, a, b, t)\) in \( A \). Since by Proposition 3 \( N_A(x) \) and \( N_A(y) \) are cliques, neither \( x \) nor \( y \) can be adjacent to both vertices \( a \) and \( b \). It follows that the subgraph induced by \( x, z, a, b, t, y, w \) contains a hole, a contradiction. \( \square \)

Proposition 7. Let \( A^* \) be a maximal matched co-bipartite subgraph of \( G \) containing \( A \). Then the following hold:

(i) Every vertex of \( A_6 \) is total with respect to \( V(A^*) \).

(ii) If \( x \) and \( y \) are vertices of \( G \setminus A^* \) with \( x, y \in A_3 \), \( N_A(x) = \text{left}(A) \) and \( N_A(y) = \text{right}(A) \) then \( x \not\sim y \).

Proof. (i): Assume to the contrary that for some \( x \in A_6 \) and \( y \in V(A^*) \setminus V(A) \), \( x \not\sim y \) holds. Assume without loss of generality that \( y \in \text{left}(A^*) \) and let \( z \) be the neighbor of \( y \) in \( \text{right}(A^*) \). Consider the subgraph \( H \) of \( G \) induced by \( a, b, c, d, y, z \) where \( a, b, c, d \) are four vertices of \( A \) forming a \( C_4 \). Clearly, \( H \) is isomorphic to \( \overline{C_6} \). Since \( x \) is total with respect to \( \{a, b, c, d\} \), \( x \) will be adjacent to four or five vertices of \( H \) and we obtain a contradiction to Proposition 3.

(ii): First observe that if \( A^* = A \) then \( x \not\sim y \) for otherwise the graph induced by \( V(A) \cup \{x, y\} \) is a matched co-bipartite graph and this contradicts the maximality of \( A^* \). Thus, we can suppose that \( V(A^*) \setminus V(A) \neq \emptyset \).

Assume to the contrary that \( x \sim y \) and consider any edge \( zt \) of \( A^* \setminus A \) such that \( z \in \text{left}(A^*) \) and \( t \in \text{right}(A^*) \). Let \( Q \) be the graph induced by \( z, t \) and four vertices \( a, b, c, d \) forming a \( C_4 \) in \( A \) such that \( \{a, b\} \subset \text{left}(A) \) and \( \{c, d\} \subset \text{right}(A) \). Clearly \( Q \) is isomorphic to \( C_6 \).

We shall prove that \( x \sim z, y \sim t, x \not\sim t \) and \( y \not\sim z \). Observe first that since \( x \) misses \( c, d \) and \( y \) misses \( a, b \), we must have that \( x \not\sim t \) and \( y \not\sim z \) for otherwise \( N_Q(x) \) or \( N_Q(y) \) would not be a clique which contradicts Proposition 3.

Let \( Q_2 \) (\( Q_3 \), respectively) denote the vertices outside \( Q \) having exactly two neighbors (three neighbors, respectively) in \( Q \). Now \( x \sim z \) and \( y \sim t \) for otherwise since \( x \) sees \( a \) and \( b \), and \( y \) sees \( c \) and \( d \), we would have \( x \in Q_2 \) and \( y \in Q_2 \cup Q_3 \) or \( x \in Q_2 \cup Q_3 \) and \( y \in Q_2 \), and we obtain a contradiction to Proposition 4 or Proposition 5. Hence \( x \) \( \text{left}(A^*) \), \( x \) \( \text{right}(A^*) \), \( y \) \( \text{left}(A^*) \) and \( y \) \( \text{right}(A^*) \) and consequently \( V(A^*) \cup \{x, y\} \) induces a graph isomorphic to a matched co-bipartite graph which contradicts to the assumed maximality of \( A^* \). \( \square \)
3.2 A Lemma for Atoms of HP-Free Graphs

The subsequent Lemma describes an essential property of HP-free atoms which will lead to a structural description of HP-free graphs.

Let \( G \) be an HP-free graph, let \( A \) be an induced \( C_6 \) in \( G \) and let \( xy \) be a matching edge of \( A \) with \( x \in \text{left}(A) \) and \( y \in \text{right}(A) \). We use the following notation:

- \( A_2[xy] := \{ u \mid u \in A_2, N_A(u) = \{ x, y \} \} \)
- \( A_1[xy] := \{ uv \in E \mid u, v \in A_1, N_A(u) = \{ x \}, N_A(v) = \{ y \} \}. \)

By \( V(A_1[xy]) \), we denote the set of vertices in \( A_1[xy] \).

**Lemma 1.** In an HP-free atom, \( A_1[xy] = A_2[xy] = \emptyset \).

**Proof.** Assume to the contrary that at least one of the two sets is nonempty. Recall that by Proposition \( \{ iv \} \), \( A_6 \) is a clique which implies that \( \{ x, y \} \cup A_6 \) is a clique. Let \( G' := G \setminus \{(x, y) \cup A_6\} \) and \( A' := A \setminus \{ x, y \} \). Clearly the vertices of \( A' \) form a \( C_4 \), say \( C = \{ a, b, c, d \} \) with \( \text{left}(A) = \{ x, a, d \} \) and \( \text{right}(A) = \{ y, b, c \} \). Since \( G \) is an atom, \( \{ x, y \} \cup A_6 \) cannot be a clique cutset and consequently, \( G' \) contains a path between some vertex \( x_0 \in A_2[xy] \cup V(A_1[xy]) \) and \( x_k \in A' \). Let \( L = (x_0, x_1, \ldots, x_k) \) be such a path of minimum length in \( G' \). If \( x_0y_0 \in A_1[xy] \) then we assume without loss of generality that \( x_0 \sim x \) and \( y_0 \sim y \).

**Claim 1.** length(\( L \)) \= 2.

**Proof of Claim** Assume not - then \( L = (x_0, x_1, x_2) \) with \( x_2 \in A' \).

Assume first that \( x_0 \in A_2[xy] \). Since by Proposition \( \{ iv \} \), \( N_A(x_1) \) is a clique (recall that \( x_1 \notin A_6 \) and \( N_A(x_1) \cap \{ a, b, c, d \} \neq \emptyset \), if \( x_1 \in A_1 \cup A_3 \) then \( N_A(x_0) \) is not comparable with \( N_A(x_1) \) which contradicts Proposition \( \{ iv \} \) and if \( x_1 \in A_2 \), \( N_A(x_0) \cup N_A(x_1) \) is not a clique which contradicts Proposition \( \{ iv \} \).

Assume now that \( x_0 \in V(A_1[xy]) \) (recall that we assumed \( x_0 \sim x \)). By Proposition \( \{ 2 \} \) and Proposition \( \{ 4 \} \) we deduce that \( N_A(x_1) \subseteq \{ x, a, d \} \) and that \( y_0 \not\sim x_1 \). Let \( u \) be a neighbor of \( x_1 \) in \( \{ a, d \} \) and \( v \) the vertex of \( \{ b, c \} \) adjacent to \( u \). Then \( x_0, x_1, u, v, y, y_0 \) induce a \( C_6 \), a contradiction which shows Claim 1.

Since length(\( L \)) is assumed to be minimum, none of \( x_1, \ldots, x_{k-2} \) can be in \( A_2 \cup A_3 \cup V(A_1[xy]) \cup A_2[xy] \). It follows that if a vertex \( x_i \in \{ x_1, \ldots, x_{k-2} \} \) belongs to \( D_1 \) then \( x_i \in A_1 \setminus V(A_1[xy]) \). Let

\[ Q := \{ x_1, \ldots, x_{k-2} \} \cap (A_1 \setminus V(A_1[xy])). \]

**Claim 2.** If \( x_0 \in A_2[xy] \) then \( Q \neq \emptyset \).

**Proof of Claim** Assume \( Q = \emptyset \); then none of \( x_1, \ldots, x_{k-2} \) belongs to \( D_1 \) and consequently by Proposition \( \{ 3 \} \), \( N_A(x_{k-1}) \) and \( N_A(x_0) = \{ x, y \} \) must be comparable. By Proposition \( \{ 3 \} \), \( N_A(x_{k-1}) \) must be a clique (recall that \( x_k \in \{ a, b, c, d \} \), and since the path in \( G' \) contains no vertex from \( A_6 \), we have \( x_{k-1} \notin A_6 \)). Thus we obtain a contradiction which shows Claim 2.
Claim 3. If $Q \neq \emptyset$ then either $N_A(Q) = \{x\}$ or $N_A(Q) = \{y\}$.

Proof of Claim 3. Assume not; then there are two vertices $x_i$ and $x_j$ in $Q$, $1 \leq i < j \leq k-2$, such that $N_A(x_i) \neq N_A(x_j)$ and for all $k$, $i < k < j$, $x_k \notin D_1$. Observe that $j > i + 1$ for otherwise $x_i$ would be adjacent to $x_j$ and consequently $x_i$ and $x_j$ would belong to $V(A_1[x,y])$, a contradiction. Now $N_A(x_i)$ and $N_A(x_j)$ are not comparable - a contradiction to Proposition 6 which shows Claim 3. \hfill \Box

Claim 4. If $Q \neq \emptyset$ then $N_A(Q) = \{x\}$ implies that $N(x_{k-1}) \subseteq \text{left}(A)$ and $N_A(Q) = \{y\}$ implies that $N(x_{k-1}) \subseteq \text{right}(A)$.

Proof of Claim 4. Let $x_s$, $1 \leq s \leq k-2$, be a vertex of path $L$ with $x_s \in Q$ such that $s$ is maximum with respect to these properties.

Assume first that $x_{k-1} \in A_1$. Then $x_s \sim x_{k-1}$ for otherwise, by Proposition 6, $N_A(x_{k-1})$ must be comparable with $N_A(x_s)$ and we obtain a contradiction to the fact that $x_{k-1}$ has a neighbor in $\{a, b, c, d\}$. Proposition 2 implies that $N_A(x_{k-1}) \sim N_A(x_s)$ and consequently $N_A(x_{k-1})$ is contained either in $\{a, d\} \subseteq \text{left}(A)$ if $N_A(x_s) = \{x\}$ or in $\{b, c\} \subset \text{right}(A)$ if $N_A(x_s) = \{y\}$.

Now assume that $x_{k-1} \in A_2 \cup A_3$. Then Proposition 4 and Proposition 6 imply that $N_A(x_{k-1})$ and $N_A(x_s)$ must be comparable. Claim 4 follows from the fact that $N_A(x_{k-1})$ is a clique and at least one of the vertices of $\{a, b, c, d\}$ belongs to $N_A(x_{k-1})$. \hfill \Box

Claim 5. For $x_0 \in V(A_1[x,y])$, the following hold:

(i) If $Q \neq \emptyset$ then $N_A(Q) = \{x\}$.

(ii) $N_A(x_{k-1}) \subseteq \text{left}(A)$.

Proof of Claim 5

(i): Recall that for $x_0 \in V(A_1[x,y])$, we assumed that $N_A(x_0) = \{x\}$. Let $x_i$ be a vertex such that $x_i \in Q$ and $i$ is as small as possible. Recall that by Claim 3 either $N_A(Q) = \{x\}$ or $N_A(Q) = \{y\}$ holds.

If $i = 1$ and $N_A(Q) = \{y\}$ then $x_1 \in V(A_1[x,y])$ since $x_1 \sim x_0$ - a contradiction to the fact that every vertex of $Q$ belongs to $A_1 - V(A_1[x,y])$. Thus, $N_A(x_1) = \{x\}$ and also $N_A(Q) = \{x\}$. If $i > 1$ then $x_1 \in D_2$ and by Proposition 6 we obtain that $i = 2$ and $N_A(x_2) = \{x\}$. Then by Claim 2 we obtain that $N_A(Q) = \{x\}$ as claimed.

(ii): If $Q \neq \emptyset$ then $N_A(x_{k-1}) \subseteq \text{left}(A)$ follows by the fact that $N_A(Q) = \{x\}$ and Claim 4.

In the other case, if $Q = \emptyset$ then no vertex of $\{x_1, \ldots, x_{k-2}\}$ is in $D_1$. Proposition 4 implies that $N_A(x_{k-1})$ and $N_A(x_0)$ must be comparable, and since by assumption $N_A(x_0) = \{x\}$ and $N_A(x_{k-1})$ is a clique, we obtain Claim 5. \hfill \Box

Let $u \in \{a, d\}$ be a neighbor of $x_{k-1}$ and let $v$ be the neighbor of $u$ in $\text{right}(A)$ which clearly is different from the vertex $y$. If $x_0 \in A_2[x,y]$ then by Claim 2 $Q \neq \emptyset$ and by Claim 3 $N_A(Q) = \{x\}$ or $N_A(Q) = \{y\}$. Assume without loss of generality that $N_A(Q) = \{x\}$; then by Claim 3 we have $N(x_{k-1}) \subseteq \text{left}(A)$. Then the subgraph induced by $x_0, \ldots, x_{k-1}, u, v, y$ is a hole, a contradiction. Hence $x_0 \in V(A_1[x,y])$. By Claim 5 if $Q \neq \emptyset$ then $N_A(Q) = \{x\}$. It follows that the subgraph induced by $x_0, \ldots, x_{k-1}, u, v, y, y_0$ is a hole, a contradiction which shows Lemma 11. \hfill \Box
4 Structure of (Hole,Paraglider)-Free and (Hole,Diamond)-Free Atoms

Recall that HP-free (HD-free, respectively) denotes hole- and paraglider-free (hole- and diamond-free, respectively).

**Theorem 2.** If $G$ is an HP-free atom containing an induced $\overline{C_6}$, $A$, and $A_6$ denotes the set of vertices which are universal for $A$ then $G \setminus A_6$ is a matched co-bipartite graph.

**Proof.** Assume the contrary; let $G' := G \setminus A_6$ and let $A^*$ be a maximal matched co-bipartite subgraph in $G'$ containing $A$. Let $W := V(G') - V(A^*)$; by assumption, $W \neq \emptyset$. We define a partition $\pi(W)$ of the vertices of $W$ according to their distance from $A^*$: $W = W_1 \cup \ldots \cup W_k$ where $W_i := \{x \in W \mid d(x, A^*) = i\}, i = 1, \ldots, k$. Thus, $W_1 = (W \cap (A_1 \cup A_2 \cup A_3)) \cup (W \cap D_2)$ where $D_2$ denotes the set of vertices which are in distance two from $A$ and which see a vertex in $A^*$. The vertices in $W_2$ have distance at least two from $A$.

**Claim 6.** No vertex in $W_1$ has neighbors in both left($A^*$) and right($A^*$).

**Proof of Claim 6.** Assume to the contrary that for some $x \in W_1$, there are $y$ and $z$ with $y \in$ left($A^*$) and $z \in$ right($A^*$) such that $x \sim y$ and $x \sim z$. Suppose first that $y \sim z$. Consider the graph $Q$ induced by $y, z$ and four vertices $a, b, c, d$ of $A$ forming a $C_4$ such that $\{y, z\} \cap \{a, b, c, d\} = \emptyset$. Clearly $Q$ is isomorphic to a $\overline{C_6}$. Then since by Lemma 1, $Q_2[yz] = \emptyset$ (where as before, $Q_2[yz]$ denotes the vertices outside $Q$ seeing exactly $y$ and $z$ in $Q$), $x$ can not belong to $D_2(A)$ and consequently $N(x) \cap \{a, b, c, d\} = \emptyset$, that is, $x \in A_1 \cup A_2 \cup A_3$. Since by Proposition 3, $N_Q(x)$ is a clique and by assumption $x$ sees both $y$ and $z$, we obtain a contradiction.

Now suppose that $y \not\sim z$ and consider the graph $H$ induced by $y, z, y_1, z_1, a, b$ where $y_1$ is the neighbor of $y$ in right($A^*$), $z_1$ is the neighbor of $z$ in left($A^*$), $ab$ is any edge of $A$ such that $a \in$ left($A$), $b \in$ right($A$) and $\{a, b\} \cap \{y, y_1, z, z_1\} = \emptyset$. Clearly $H$ is isomorphic to a $\overline{C_6}$. Since by assumption $x$ sees both $y$ and $z$, $N_H(x)$ is not a clique which by Proposition 3 implies that $x$ sees all vertices of $H$ and thus also $x \sim a$ and $x \sim b$ with $a \in$ left($A$) and $b \in$ right($A$). Since by Proposition 3, $x \not\in A_3$, by Lemma 1, $x \not\in A_2[a, b]$ and by assumption, $x \not\in A_6$, we obtain a contradiction. ◊

We define now the following sets:

$$\text{left}(W_1) := \{x \in W_1 \mid N_{A^*}(x) \subseteq \text{left}(A^*)\}$$

$$\text{right}(W_1) := \{x \in W_1 \mid N_{A^*}(x) \subseteq \text{right}(A^*)\}.$$

By Claim 6, $\text{left}(W_1) \cap \text{right}(W_1) = \emptyset$. Thus $W_1 = \text{left}(W_1) \cup \text{right}(W_1)$ is a partition of $W_1$.

**Claim 7.** There is no edge between left($W_1$) and right($W_1$).

**Proof of Claim 7.** Assume to the contrary that $x \sim y$ for some $x \in$ left($W_1$) and $y \in$ right($W_1$). Recall that $D_1$ denotes the vertices in distance one to $A$. We first show:

$$x \text{ and } y \text{ cannot be both in } D_1. \tag{1}$$
Assume to the contrary that $x, y \in D_1$. Then by Proposition 3 (ii), $x, y \notin A_3$ is impossible. Suppose without loss of generality that $x \notin A_2$, i.e., $x \in A_1 \cup A_2$ and $y \in A_1 \cup A_2 \cup A_3$. If $x \in A_1$ and $y \in A_2 \cup A_3$ or $x \in A_2$ and $y \in A_1 \cup A_3$, Proposition 4 implies that $N_A(x)$ and $N_A(y)$ are incomparable, and if $x, y \in A_2$, Proposition 5 implies that $N_A(x) \cup N_A(y)$ is a clique. But since $x \in \left(W_1\right)$ and $y \in \left(W_1\right)$, none of these cases can occur. It follows that $x, y \in A_1$. However, by Lemma 1 such a pair of adjacent vertices cannot exist, a contradiction. $\diamond$

It follows that at least one of $x$ or $y$ is in $D_2$. Assume that $x \in D_2$ and let $u$ be a neighbor of $x$ in $D_1$. Suppose first that also $y \in D_2$ and let $v$ be a neighbor of $y$ in $D_1$. Obviously $u \in \left(W_1\right)$ and $v \in \left(W_1\right)$. Since by assumption $x, y \in D_2$, Proposition 6 (i) implies that $u \sim v$ and we obtain a contradiction with (f). Consequently, $y \in D_1$. Since $N_A(u)$ and $N_A(y)$ are not comparable, Proposition 6 (ii) implies that $u \sim y$ and again we obtain a contradiction with (f). This shows Claim 9. $\diamond$

For the partition $\pi(W) = \{W_1, \ldots, W_k\}$, $k \geq 1$, define the following sets for every $i \in \{2, \ldots, k\}$:

$$
\text{left}(W_i) := \{x \in W_i \mid \exists y \in \text{left}(W_{i-1}) \text{ such that } x \sim y\}
$$

$$
\text{right}(W_i) := \{x \in W_i \mid \exists y \in \text{right}(W_{i-1}) \text{ such that } x \sim y\}.
$$

Claim 8. ($\text{left}(W_1) \cup \ldots \cup \text{left}(W_k)) \cap (\text{right}(W_1) \cup \ldots \cup \text{right}(W_k)) = \emptyset$ and $(\text{left}(W_1) \cup \ldots \cup \text{left}(W_k)) \ominus (\text{right}(W_1) \cup \ldots \cup \text{right}(W_k))$.

Proof of Claim 8. We shall prove the claim by induction on $k$. By Claims 4 and 7 the result is true for $k = 1$. By the induction hypothesis the result is true for $k < s$, $s > 1$. Assume to the contrary that the result is false for $W_s \in \pi(W)$. Then there must be a chordless path $L_1 = (x_1, \ldots, x_{s-1}, x, y_{s-1}, \ldots, y_1)$ or a chordless path $L_2 = (x_1, \ldots, x_{s-1}, x, y, y_{s-1}, \ldots, y_1)$ such that $x_i \in \text{left}(W_i)$, $y_i \in \text{right}(W_i)$, $i \in \{1, \ldots, s-1\}$ and $x, y \in W_s$. By the induction hypothesis there is no edge between $\{x_1, \ldots, x_{s-1}\}$ and $\{y_1, \ldots, y_{s-1}\}$. Let $L = (x_1, z_1, \ldots, z_r, y_1)$, $r \geq 2$, be a chordless path joining $x_1$ and $y_1$ such that $z_i \in A^*$, $i \in \{1, \ldots, r\}$, which clearly exists. It is easy to see that the graph induced by the vertices of $L_1$ and $L$ or by the vertices of $L_2$ and $L$ is isomorphic to a hole - a contradiction. This shows Claim 8. $\diamond$

Let

$$
\text{left}(W) := (\text{left}(W_1) \cup \ldots \cup \text{left}(W_k))
$$

$$
\text{right}(W) := (\text{right}(W_1) \cup \ldots \cup \text{right}(W_k)).
$$

By Claim 8 left($W$) and right($W$) form a partition of $W$.

Claim 9. left($W$)$\ominus$right($W$) $\cup$ right($A^*$) and right($W$)$\ominus$left($W$) $\cup$ left($A^*$).

Proof of Claim 9. Indeed, by Claim 8 we have that left($W$)$\ominus$right($W$). By Claim 6 we have that left($W_1$)$\ominus$right($A^*$) and right($W_1$)$\ominus$left($A^*$), and by the construction of $W_2, \ldots, W_k$ we have that $(W_2 \cup \ldots \cup W_k)\ominus V(A^*)$. This shows Claim 9. $\diamond$

Since by assumption $G' = G \setminus A_6$ is not isomorphic to a matched co-bipartite graph, we must have that $W \neq \emptyset$. Assume without loss of generality that left($W$) $\neq \emptyset$. Then
since by Proposition 7 (i), $A_6 \cup \text{left}(A^*)$ is a clique and since by Claim 9 there is no edge between left($W$) and right($W') \cup \text{right}(A^*)$, $A_6 \cup \text{left}(A^*)$ would be a clique cutset in $G$ which contradicts our assumption that $G$ is an atom. This finishes the proof of Theorem 2.

**Corollary 2.** Let $G$ be a (hole,paraglider)-free graph.

(i) If $G$ is $C_6$-free then $G$ is weakly chordal.

(ii) If $G$ is an atom containing an induced $C_6$ then $G$ is the join of a matched co-bipartite graph and a clique.

**Proof.** (i): Recall that HP-free graphs are $C_k$-free for $k \geq 7$.

(ii): Indeed by Theorem 2 for a $C_6$ $A$ in $G$, $G' = G \setminus A_6$ is a matched co-bipartite graph. By Proposition 7 $A_6 \cup V(G')$, and by Proposition 3 $A_6$ is a clique.

Since by Proposition 3 (iv), in (hole,diamond)-free graphs $A_6 = \emptyset$, we have:

**Corollary 3.** Let $G$ be a (hole,diamond)-free graph.

(i) If $G$ is $C_6$-free then $G$ is weakly chordal.

(ii) If $G$ is an atom containing an induced $C_6$ then $G$ is a matched co-bipartite graph.

### 5 Algorithmic Consequences

In [32], for various problems such as Minimum Fill-in, Maximum Independent Set, Maximum Clique and Coloring, it is shown that whenever these problems are efficiently solvable on the atoms of a graph class, they are efficiently solvable on the graphs of the class. For perfect graphs, Maximum Independent Set, Maximum Clique and Coloring are known to be solvable in polynomial time [23, 24] using the ellipsoid method (but from a practical point of view, this is not an efficient solution of the problems).

(Hole,paraglider)-free graphs are perfect as the Strong Perfect Graph Theorem implies (a more direct way can use Theorem 2 and Corollary 2 and the fact that a graph is perfect if its atoms are perfect).

The clique separator approach gives direct combinatorial algorithms for the problems mentioned above:

Recognition of weakly chordal graphs can be done in $O(m^2)$ [21, 27], and recognition of matched co-bipartite graphs can be easily done in linear time. Thus, given an input graph, determine its atoms and check whether they are either weakly chordal or are the join of a clique and a matched co-bipartite graph. If not then the input graph is not (hole,paraglider)-free. Otherwise solve the problems on the atoms and finally combine the solutions as described in [32].

For matched co-bipartite graphs, MWIS is trivial. A first polynomial time algorithm for weakly chordal graphs is given in [26], and in [31], MWIS is solved in time $O(n^4)$ for weakly chordal graphs. Thus, the time bound for MWIS on HP-free graphs is roughly $O(n^6)$: Determine whether the input graph is weakly chordal. If yes, use the algorithm for weakly chordal graphs. If not, check whether all prime atoms are matched co-bipartite,
and if yes, then use the trivial algorithm for these graphs. If not, the input graph is not HP-free.

For Maximum Clique and Coloring one can proceed in a similar way. For Maximum Clique on diamond-free graphs, however, there is a more direct way to solve the problem efficiently by switching to the complement graph and the complement problem MWIS: If $G$ is gem-free (see Figure 1 for gem) then $\overline{G}$ has the property that for every vertex, its antineighborhood is $P_4$-free, i.e., a cograph. This means that one can solve the MWIS problem for such graphs in time $O(nm)$ in the obvious way. In [5], a $O(n^6)$ algorithm is given for Minimum Fill-In on weakly chordal graphs. Minimum Fill-In on matched co-bipartite graphs is efficiently solvable in the obvious way.

The Maximum Weight Induced Matching (MWIM) problem is another example of a problem which can be added to the list of problems above: A set $M$ of edges is an induced matching in $G$ if the pairwise distance of the edges in $M$ is at least two in $G$. The MWIM problem asks for an induced matching of maximum weight. In [16], it is shown that for a hereditary class $C$ of graphs, MWIM is solvable in polynomial time if MWIM is solvable in polynomial time on the atoms of $C$. This can be applied to (hole,paraglider)-free graphs since for weakly chordal graphs, a polynomial time solution is given in [18], and obviously, matched co-bipartite graphs are $3K_2$-free, which means that in such graphs (and in the join of a matched co-bipartite graph and a clique) one has to check only pairs of edges.

6 Conclusion

In this paper we have described the structure of (hole, paraglider)-free atoms (of (hole, diamond)-free atoms, respectively) and some algorithmic consequences. In a forthcoming paper [3], we will analyze the structure of (hole,diamond)-free graphs and its algorithmic consequences in more detail; in particular, we show that weakly chordal diamond-free atoms are either cliques or chordal bipartite.

There are various other aspects and papers which are related to our work as described subsequently:

6.1 Related results for subclasses of $P_5$-free graphs

In [1], Alekseev showed that $P_5$- and paraglider-free atoms are $3K_2$-free which leads to a polynomial time algorithm for the MWIS problem since $3K_2$-free graphs contain at most $O(n^4)$ inclusion-maximal independent sets. In [11], we improved this result by generalizing the forbidden paraglider subgraph. In [8], we give a more detailed structural analysis of $P_5$- and paraglider-free atoms. In [15], we describe the structure of prime $P_5$- and co-chair-free graphs and give algorithmic applications. The complexity of the MWIS problem for $P_5$-free graphs is an open problem. It is also open for $(P_5, C_5)$-free graphs; such graphs are hole-free. Thus, it is interesting to study subclasses of $P_5$-free graphs (subclasses of $(P_5, C_5)$-free graphs, respectively).

6.2 Clique-width

In [6], we describe the simple structure of $(P_5,diamond)$-free graphs; such graphs can contain $C_5$ and thus, $P_5$- and diamond-free graphs are in general not perfect and incomparable.
with (hole,diamond)-free graphs. \((P_5,\text{diamond})\)-free graphs have bounded clique-width - see e.g. [20] for the notion and algorithmic implications of bounded clique-width which has tremendous consequences for efficiently solving hard problems on such graph classes. For the more general class of \((P_5,\text{gem})\)-free graphs, the situation is similar: By the Strong Perfect Graph Theorem, \((\text{hole,gem})\)-free graphs are perfect since antiholes with at least seven vertices contain gem. The structure of \((P_5,\text{gem})\)-free graphs and some algorithmic applications were described in [4, 9]. In [12], it was shown that \((P_5,\text{gem})\)-free graphs have bounded clique-width.

The clique-width of \((\text{hole,diamond})\)-free graphs, however, is unbounded since e.g. the subclass of chordal bipartite graphs (which are the (hole, triangle)-free graphs), has unbounded clique-width [14]. This illustrates that corresponding subclasses of hole-free graphs are more interesting than those of \(P_5\)-free graphs.

6.3 Open problems

It would be interesting to describe the structure of \((\text{hole,gem})\)-free graphs. In particular, how can one avoid to use the Strong Perfect Graph Theorem for showing that \((\text{hole,gem})\)-free graphs are perfect?

In [7], we give a polynomial time algorithm for the MWIS problem on hole- and co-chair-free graphs. It would be interesting to obtain better structural results on these graphs.

References

[1] V.E. Alekseev, On easy and hard hereditary classes of graphs with respect to the independent set problem, Discrete Applied Math. 132 (2004) 17-26.
[2] A. Berry, J.-P. Bordat, P. Heggernes, Recognizing weakly triangulated graphs by edge separability, Nordic Journal of Computing 7 (2005) 164-177.
[3] A. Berry, A. Brandstäd, V. Giakoumakis, Recognizing diamond-free hole-free graphs in \(O(n^2)\) time, manuscript 2011.
[4] H. Bodlaender, A. Brandstädt, D. Kratsch, M. Rao, and J.P. Spinrad, On algorithms for \((P_5,\text{gem})\)-free graphs, Theor. Computer Science 349 (2005) 2-21.
[5] V. Bouchitté, I. Todinca, Treewidth and Minimum Fill-In: Grouping the Minimal Separators, SIAM J. on Computing 31 (2001) 212-232.
[6] A. Brandstädt, \((P_5,\text{diamond})\)-free graphs revisited: structure and linear time optimization, Discrete Applied Math. 138 (2004) 13-27.
[7] A. Brandstädt, V. Giakoumakis, Maximum Weight Independent Sets in Hole- and Co-Chair-Free Graphs, manuscript 2011, submitted.
[8] A. Brandstädt and Chinh T. Hoang, On clique separators, nearly chordal graphs and the Maximum Weight Stable Set problem, Extended abstract in: M. Jünger and V. Kaibel (Eds.), IPCO 2005, LNCS 3509, pp. 265-275, 2005. Theoretical Computer Science 389 (2007) 295-306
[9] A. Brandstädt and D. Kratsch, On the structure of \((P_5,\text{gem})\)-free graphs; Discrete Applied Math. 145 (2005) 155-166.
[10] A. Brandstädt and V.B. Le, Simplicial powers of graphs, Theoretical Computer Science 410 (2009) 5443-5454.
[11] A. Brandstädt, V.B. Le, and S. Mahfud, New applications of clique separator decomposition for the Maximum Weight Stable Set problem, Theor. Computer Science 370 (2007) 229-239.
[12] A. Brandstädt, Hoang-Oanh Le, and R. Mosca, Chordal co-gem-free graphs and \((P_5,\text{gem})\)-free graphs have bounded clique-width, Discrete Applied Math. 145 (2005) 232-241.
[13] A. Brandstädt, V.B. Le, and J.P. Spinrad, Graph Classes: A Survey, SIAM Monographs on Discrete Math. Appl., Vol. 3, Philadelphia, 1999.

[14] A. Brandstädt, V.V. Lozin, On the linear structure and clique width of bipartite permutation graphs, Ars Combinatoria Vol. LXVII (2003) 273-281.

[15] A. Brandstädt, R. Mosca, On the Structure and Stability Number of $P_5$- and Co-Chair-Free Graphs, Discrete Applied Math. 132 (2004) 47-65.

[16] A. Brandstädt, R. Mosca, On Distance-3 Matchings and Induced Matchings, available online in Discrete Applied Math. 2010.

[17] A. Brandstädt and P. Wagner, Characterising $(k,l)$-leaf powers, Discrete Applied Math. 158 (2010) 110-122.

[18] K. Cameron, R. Sritharan, Y. Tang, Finding a maximum induced matching in weakly chordal graphs, Discrete Math. 266 (2003) 133-142.

[19] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Ann. of Math. 164 (2006) 51-229.

[20] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique width, Theory of Computing Systems 33 (2000) 125-150.

[21] E.M. Eschen, C.T. Hoång, J.P. Spinrad, R. Sritharan, On graphs without a $C_4$ or a diamond, CoRR abstract 0909.4719 (2009), electronically available in Discrete Applied Math. 2011.

[22] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs. Academic Press, 1980.

[23] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 169-197, Corrigendum: Combinatorica 4 (1984) 291-295.

[24] M. Grötschel, L. Lovász, A. Schrijver, Polynomial algorithms for perfect graphs, Annals of Discrete Math. 21 (1984) 325-356.

[25] R.B. Hayward, Weakly triangulated graphs, J. Combin. Theory Series B 39 (1985) 200-208.

[26] R.B. Hayward, C.T. Hoång, F. Maffray, Optimizing weakly triangulated graphs, Graphs and Combinatorics 5 (1989) 339-349; erratum in 6 (1990) 33-35.

[27] R.B. Hayward, J.P. Spinrad, R. Sritharan, Weakly chordal graph algorithms via handles, Proceedings of the 11th Symposium on Discrete Algorithms 42-49, 2000.

[28] R.B. Hayward, J.P. Spinrad, R. Sritharan, Improved algorithms for weakly chordal graphs, Graphs and Combinatorics 3 (2007) no. 2, Art. 14.

[29] T. Kloks, H. Müller, K. Vušković, Even-hole-free graphs that do not contain diamonds: A structure theorem and its consequences, Journal of Combinatorial Theory, Series B 99 (2009) 733-800.

[30] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, SIAM Monographs on Discrete Math. and Appl. Vol. 2, Philadelphia, 1999.

[31] J.P. Spinrad, R. Sritharan, Algorithms for weakly triangulated graphs, Discrete Applied Math. 59 (1995) 181-191.

[32] R.E. Tarjan, Decomposition by clique separators, Discrete Math. 55 (1985) 221-232.

[33] K. Vušković, Even-hole-free graphs: a survey, electronically available in Appl. Anal. and Discrete Math., 2010.

[34] S.H. Whitesides, A method for solving certain graph recognition and optimization problems, with applications to perfect graphs, in: Berge, C. and V. Chvátal (eds), Topics on perfect graphs, North-Holland, Amsterdam, 1984.