Extremal metrics and K-stability

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Declaration

The material presented in this thesis is the author's own, except where it appears with attribution to others.
Abstract

In this thesis we study the relationship between the existence of canonical metrics on a complex manifold and stability in the sense of geometric invariant theory. We introduce a modification of K-stability of a polarised variety which we conjecture to be equivalent to the existence of an extremal metric in the polarisation class. A variant for a complete extremal metric on the complement of a smooth divisor is also given. On toric surfaces we prove a Jordan-Hölder type theorem for decomposing semistable surfaces into stable pieces. On a ruled surface we compute the infimum of the Calabi functional for the unstable polarisations, exhibiting a decomposition analogous to the Harder-Narasimhan filtration of an unstable vector bundle.
## Contents

### Introduction

1 Finite dimensional GIT
   1.1 Stability ........................................ 11
   1.2 Kempf-Ness theorem ............................. 12
   1.3 Norm squared of the moment map ............... 14
   1.4 Modulus of stability ............................ 16
   1.5 Torus actions ..................................... 22

2 Extremal metrics
   2.1 Futaki invariant and Mabuchi functional .... 29
   2.2 Scalar curvature as a moment map ............. 31

3 Stability of varieties
   3.1 K-stability ........................................ 37
      3.1.1 Uniform K-stability .......................... 41
      3.1.2 K-stability of a pair $(V,D)$ .............. 42
   3.2 Relative K-polystability of a ruled surface ... 45
   3.3 Lower bound on the Calabi functional .......... 49

4 Toric varieties
   4.1 K-stability of toric varieties ................. 51
   4.2 Toric surfaces .................................... 57
      4.2.1 Uniform K-stability .......................... 58
      4.2.2 Measure majorisation ........................... 61
      4.2.3 Semistable surfaces ............................. 63

5 Ruled manifolds
   5.1 Summary of the momentum construction ......... 66
   5.2 A metric degeneration ............................ 70
   5.3 Extremal metrics on ruled surfaces ............. 77
   5.4 The infimum of the Calabi functional .......... 80

Bibliography ......................................... 83
Introduction

The subject of this thesis is finding canonical metrics on Kähler manifolds. The first result of this form is the classical uniformisation theorem, which states that every compact Riemann surface admits a metric of constant curvature, unique if we prescribe the total area. In higher dimensions a condition analogous to prescribing the total area is fixing the Kähler class of a metric. In [4] Calabi introduced the functional

\[ \int_M S(\omega) \frac{\omega^n}{n!} \]

for Kähler metrics \( \omega \) on \( M \) in a fixed cohomology class, where \( S(\omega) \) is the scalar curvature of \( \omega \). He proposed finding critical points of this functional (the Calabi functional) as candidates for a canonical metric in the Kähler class. Such a metric is called an extremal metric and the main problem is their uniqueness and existence. It has been shown in [7] that any two extremal metrics in a Kähler class are related by a holomorphic automorphism. The question of existence is still open.

Calabi showed that the Euler-Lagrange equation of the variational problem is that the gradient of the scalar curvature is a holomorphic vector field. A special case is that of Kähler-Einstein metrics since these have constant scalar curvature. In this case the first Chern class of the manifold is proportional to the Kähler class of the metric, and so it has to be negative definite, zero, or positive definite. In the case when the first Chern class is negative or zero, Yau [37] (also Aubin [3] in the negative case) showed that the variety admits a unique Kähler-Einstein metric, solving a conjecture of Calabi. The case of positive first Chern class proved to be more difficult and is still not completely resolved. Yau conjectured that in this case the existence of a Kähler-Einstein metric is related to the stability of the underlying variety in the sense of Mumford’s geometric invariant theory [26]. Tian made great progress towards understanding this (see [33]) giving an analytic “stability” condition which is equivalent to the existence of a Kähler-Einstein metric. This condition is the properness
of the Mabuchi functional, which is an energy functional on the Kähler class whose critical points are Kähler-Einstein metrics. In \cite{Tian97} Tian also defined the algebro-geometric notion of K-stability (not exactly the same as what we call K-stability), which is satisfied when the Mabuchi functional is proper.

In \cite{Donaldson90}, Donaldson showed that the scalar curvature arises as a moment map for a suitable infinite dimensional symplectic action (see also Fujiki \cite{Fujiki90}). This put earlier results into a new context, and explained on a formal level why the existence of a Kähler-Einstein metric, or more generally a metric of constant scalar curvature (cscK), is related to the stability of the variety. Moreover it made it possible to formulate precise conjectures. In particular in \cite{Donaldson90} Donaldson generalised Tian’s definition of K-stability by giving an algebro-geometric definition of the Futaki invariant, and conjectured that it is equivalent to the existence of a cscK metric.

The definition of K-stability is roughly the following (see Section \ref{sec:K-stability} for details). Given a polarised variety (this means we have chosen a Kähler class which is the first Chern class of an ample line bundle), we consider degenerations of it into possibly singular schemes. For each such test-configuration we define a number called the generalised Futaki invariant, and the variety is K-stable if this number is positive for all non-trivial test-configurations. The idea is that the Futaki invariant controls the asymptotic behaviour of the Mabuchi functional as we tend to a degenerate metric, so that properness of the Mabuchi functional corresponds to the Futaki invariant being positive for all nontrivial degenerations. This means that an important problem is to study the metric behaviour of such an algebro-geometric test-configuration. In \cite{Tian97} Tian studied the case where the central fibre is normal. In Section \ref{sec:deformation-normal-cone} we will study the case of deformation to the normal cone of the zero section of a ruled manifold, so the central fibre has a normal crossing singularity.

An interesting testing ground for these ideas is the case of toric varieties, which was developed by Donaldson. In \cite{Donaldson90} he showed that in the case of toric surfaces K-stability implies that the Mabuchi functional is bounded from below (this was later extended by Zhou-Zhu \cite{ZhouZhu03} to show properness of the Mabuchi functional) and a minimising sequence has a subsequence that converges in a weak sense. In \cite{Donaldson91} Donaldson proved interior estimates for the cscK equation for toric surfaces.

Unfortunately recent examples in \cite{Dolbeault06} show that positivity of the Futaki invariant for algebraic test-configurations may not be enough to ensure the existence of a cscK metric. One approach suggested in \cite{Sun11} is to allow more general
test-configurations with polarisations which are real linear combinations of line bundles, or with non-algebraic central fibres. In Section 3.1.1 we suggest an alternative way of strengthening the definition of K-stability to what we call uniform K-stability, and then in Section 4.2 we show that a K-polystable toric surface is uniformly K-polystable.

So far we have only considered cscK metrics, and it is natural to ask whether one can give a stability criterion for the existence of general extremal metrics. Such a criterion was proposed by the author in [31] (see also Mabuchi [22] for a different definition). Given the interpretation of the scalar curvature as a moment map, what one needs to do is to find a stability criterion satisfied by the orbit of a critical point of the norm squared of the moment map in general. The norm squared of the moment map was studied by Kirwan [19], but not exactly from this point of view, so we develop the finite dimensional theory in Chapter 1.

One advantage of extending the search for canonical metrics from cscK to general extremal metrics is that we have interesting explicit examples such as the ruled surfaces constructed in [36] and the more general constructions in [1]. In Section 5.3 we will use such explicit constructions to give complete extremal metrics on a ruled surface. These will then be used in Section 5.4 to determine the infimum of the Calabi functional for the unstable polarisations. The infimum is achieved by a degenerate metric, where the variety splits up into pieces which either admit a complete extremal metric, or collapse. In general if a variety is unstable one expects that there is such a decomposition into stable pieces in analogy with the Harder-Narasimhan filtration of an unstable vector bundle.

**Chapter summary**

In Chapter 1 we develop the finite dimensional theory of stability and the moment map. Apart from Section 1.4 the material in this chapter is contained in slightly different form in the work of Kirwan [19]. The main result is an extension of the Kempf-Ness theorem.

**Theorem 1.3.4** A point $x$ in $X$ is in the $G$-orbit of a critical point of $\|\mu\|^2$, if and only if it is polystable relative to a maximal torus which fixes it.

Here $G$ is a reductive group acting on a complex variety $X$, and $\mu$ is the moment map for the action of a maximal compact subgroup of $G$. In Section 1.4 we introduce the notion of the modulus of stability of a stable point and show that one can give a lower bound for the first eigenvalue of the derivative of the
moment map in terms of this modulus (see Theorem 1.4.2). In the final section we show how the theory works out in the case of a torus action.

Chapter 2 is a review of some well-known results about extremal metrics. In Section 2.2 we recall that the scalar curvature arises as a moment map for an infinite dimensional group action. Together with the results in the first chapter, this gives the motivation for the subsequent results, in particular the definition of K-stability in the next chapter.

In Chapter 3 we recall the definition of K-polystability from [10] and we introduce the notion of relative K-polystability which is the suitable generalisation to the case of extremal metrics with non-constant scalar curvature. In Section 3.1.1 we introduce the notion of uniform K-polystability. This addresses the problem mentioned above that K-polystability may not be enough to ensure the existence of a cscK metric, but it is still to be seen whether uniform K-polystability is the correct notion. We then consider the case of a pair \((X,D)\) where \(D \subset X\) is a divisor and define a variant of K-polystability for this situation. The aim is to find a condition for \(X \setminus D\) to admit a complete extremal metric, i.e. a complete metric such that the gradient of its scalar curvature is a holomorphic vector field, which is asymptotically hyperbolic near \(D\) (see Section 3.1.2 for the definition of the class of metrics we consider). In Section 3.2 we illustrate the definition of K-polystability on a ruled surface by finding destabilising test-configurations for certain polarisations. This will be complemented in Section 3.3 where we construct extremal metrics on this ruled surface (both compact metrics and complete metrics on the complement of a section) for the other polarisations.

In Section 3.3 we recall Donaldson’s theorem in [12] which gives a lower bound for the Calabi functional in terms of a destabilising test-configuration. This gives a fairly simple proof of the fact that a variety that admits a cscK metric must be K-semistable. We give the following refinement of the theorem.

**Theorem 3.3.2** Let \(T\) be a maximal torus of automorphisms of a polarised variety \((X,L)\) with corresponding extremal vector field \(\chi\). Suppose there is a test-configuration for \((X,L)\) compatible with \(T\) such that the modified Futaki invariant \(F_\chi(\alpha) < 0\) for the \(\mathbb{C}^*\)-action \(\alpha\) induced on the central fibre. Then for any metric \(\omega \in 2\pi c_1(L),\)

\[
\|S(\omega) - \hat{S}\|_{L^2}^2 \geq 2 \cdot (2\pi)^n \frac{F_\chi(\alpha)^2}{\|\alpha\|^2} + \|\chi\|_{L^2}^2.
\]

Here \(\hat{S}\) is the average scalar curvature. This theorem shows that a polarised
variety that admits an extremal metric is relatively K-semistable since if $\omega$ is an extremal metric then $\|S(\omega) - \hat{S}\|_{L^2}^2 = \|\chi\|_{L^2}^2$ (see Section 2.1).

In Chapter 4 we study toric varieties. First we generalise the toric test-configurations defined in [10] to bundles of toric varieties and compute their Futaki invariants (Theorem 4.1.2). We will use this in the next chapter to define test-configurations for a ruled manifold. In Section 4.2 we concentrate on toric surfaces and prove two results. The first is that a K-polystable toric surface is uniformly K-polystable, which relies on

**Proposition 4.2.2** Given a convex polygon $P$ there exists a constant $C$ such that for all non-negative continuous convex functions $f$ on $P$,

$$\|f\|_{L^2(P)} \leq C \int_{\partial P} f \, d\sigma.$$ 

We then use the notion of measure majorisation from convex geometry to study semistable surfaces, and prove

**Theorem 4.2.7** A K-semistable polygon $P$ has a canonical decomposition into subpolygons $Q_i$ each of which is either K-polystable, or a parallelogram with two opposite edges lying on edges of $P$.

A subpolygon $Q_i$ defines a pair $(X_i, D_i)$, with the divisor corresponding to the edges of $Q_i$ lying in the interior of $P$. K-polystability of $Q_i$ is interpreted as K-polystability of the pair $(X_i, D_i)$.

In the final chapter we study ruled manifolds using the explicit construction of metrics from momentum profiles due to Hwang-Singer [18]. In Section 5.2 we construct a sequence of metrics which model the deformation to the normal cone of a section, and show that the derivative of the Mabuchi functional along this degeneration tends to the Futaki invariant of the corresponding test-configuration. We then restrict attention to a ruled surface. First we use momentum profiles to construct extremal metrics on it, as well as complete extremal metrics on the complement of the zero or infinity section. We see that we obtain the same restrictions on the polarisation as in the stability computation in Section 3.2. In the last section we show that the infimum of the Calabi functional for the unstable polarisations is achieved by degenerate metrics assembled from the complete extremal metrics we have constructed.
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Chapter 1

Finite dimensional GIT

This chapter contains some background on the finite dimensional theory of geometric invariant theory and symplectic quotients. The basic references are Mumford-Fogarty-Kirwan [26] and Kirwan [19] (see also Thomas [32]). Essentially the only novelty is in Section 1.4; the results in the rest of the chapter can be obtained from the theory of Kirwan [19].

The aim of geometric invariant theory (GIT) is to define a quotient variety $X/G$ when an algebraic group $G$ acts on an algebraic variety $X$. It is natural to require functions over $X/G$ to be given by $G$ invariant functions over $X$, and this requirement gives a simple definition for the quotient. The difficulty is to understand what the quotient variety parametrises. In other words, we would like to understand the projection map from $X$ to $X/G$. There will be certain bad (unstable) orbits where this map is not defined, and also some semistable orbits which become identified with each other. This will be discussed in Section 1.4.

In symplectic geometry there is also a way of constructing quotients. Here we start with a symplectic manifold $M$ with symplectic form $\omega$, and a compact group $K$ acting on $M$, preserving $\omega$. In the case where $M$ is also an algebraic variety, then the Kempf-Ness theorem, discussed in Section 1.2, relates the symplectic quotient by $K$ to the GIT quotient by the complexification of $K$.

A central role is played by the norm squared of the moment map, which in the infinite dimensional setting is the Calabi functional. In Section 1.3 we show how one can characterise orbits of critical points of this functional using stability, generalising the Kempf-Ness theorem.

In Section 1.4 we introduce a notion we call the modulus of stability which measures how far a point is from being unstable, and we prove some simple
results about it. This is used in Chapter 3 to motivate the definition of uniform K-stability.

In Section 1.5 we illustrate the above theory in the case of a torus action, where everything can be seen quite explicitly. While a torus action may seem very special, the Cartan decomposition implies that many questions about general actions can be reduced to a torus action.

1.1 Stability

To give precise definitions let $(X, L)$ be a smooth complex projective variety with an ample line bundle, in other words a polarised variety. The graded ring of functions over $X$ is defined to be

$$R(X) = \bigoplus_{k=0}^{\infty} H^0(X, L^k).$$

Suppose a complex reductive group $G$ acts on $X$ by holomorphic automorphisms. Suppose we can lift this action of $G$ to a holomorphic action on $L$. A choice of such a lifting is called a linearisation of the action. This induces an action on $R(X)$, and we write $R(X)^G$ for the algebra of invariant functions. One can show that $G$ being reductive implies that this is a finitely generated algebra, so we can form the variety

$$X/G = \text{Proj} R(X)^G.$$

While this definition of the quotient space is very simple, what we need to understand is what its points represent. The inclusion $R(X)^G \to R(X)$ induces a rational map $X \dasharrow X/G$. The map is not defined at points $x \in X$ where every invariant section in $R(X)^G$ vanishes.

Definition 1.1.1. A point $x \in X$ is called unstable if every non-constant element of $R(X)^G$ vanishes at $x$. It is called semistable if it is not unstable.

If we denote the set of semistable points by $X^{ss}$, we now have a map $X^{ss} \to X/G$.

Definition 1.1.2. A point $x \in X$ is called polystable if there exists an element $f$ of $R(X)^G$ which does not vanish at $x$, the set $X_f$ where $f$ does not vanish is affine, and the action of $G$ on $X_f$ is closed (the orbit of each point is closed). If in addition $x$ has discrete isotropy group then it is called stable.
We call a $G$-orbit (poly/semi)-stable if a point in the orbit is. This does not depend on which point we choose. One can show that the closure of each semistable orbit contains a unique polystable orbit and the quotient $X/G$ parametrises the polystable orbits. The following alternative characterisation of polystable and semistable points is often useful.

**Proposition 1.1.3.** A point $x \in X$ is polystable if and only if for a choice of non-zero lift $\hat{x} \in L$, the orbit $G\hat{x}$ is closed in $L$. It is semistable if and only if the closure of the orbit $G\hat{x}$ does not intersect the zero section of $L$.

A central result in geometric invariant theory is the Hilbert-Mumford numerical criterion for stability. It says that the stability of a point can be determined by studying its orbits under one-parameter subgroups. Let $\lambda : \mathbb{C}^* \to G$ be a nontrivial one-parameter subgroup and $x \in X$. Since $X$ is projective, we can define

$$x_0 = \lim_{t \to 0} \lambda(t)x.$$  

We obtain an induced $\mathbb{C}^*$ action on the fibre $L_{x_0}$, which has a weight $-w(x, \lambda)$.

**Theorem 1.1.4 (Hilbert-Mumford criterion).** The point $x$ is

1. stable if and only if $w(x, \lambda) > 0$ for all $\lambda$,
2. semistable if and only if $w(x, \lambda) \geq 0$ for all $\lambda$,
3. polystable if and only if $w(x, \lambda) \geq 0$ for all $\lambda$ with equality only if $\lambda$ fixes $x$.

We will prove this theorem in the case of a torus action in Section 1.5.

**Example 1.1.5.** Let $X = S^n \mathbb{P}^1$, the space of unordered $n$-tuples of points on $\mathbb{P}^1$. We can identify such an $n$-tuple of points with a homogeneous polynomial of degree $n$, i.e. with a section of $\mathcal{O}(n)$, unique up to scaling. Thus $X = \mathbb{P}H^0(\mathcal{O}(n))$. Let $SL(2, \mathbb{C})$ act on $X$ via the natural action induced by the isomorphism $H^0(\mathcal{O}(n)) \cong S^n(\mathbb{C}^2)$.

Let us test whether a given $f \in H^0(\mathcal{O}(n))$ is stable for this action. Choose a $\mathbb{C}^*$ subgroup of $SL(2, \mathbb{C})$ and diagonalise it:

$$\lambda \mapsto \begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix},$$

in $[x : y]$ coordinates on $\mathbb{P}^1$, for some $k \geq 0$. In these coordinates we can write $f = \sum_{i=0}^n a_i x^i y^{n-i}$. As $\lambda \to 0$, the monomials $x^i y^{n-i}$ with $2i - n \leq 0$ do not
tend to zero. Thus the closure of the orbit as $\lambda \to 0$ does not contain the origin, unless $a_i = 0$ for all $i \leq n/2$; that is as long as $f$ does not vanish to order greater than $n/2$ at the point $[0 : 1]$. Changing the one-parameter subgroup corresponds to changing coordinates, so we can conclude that $f$ is semi-stable as long as it has no roots of multiplicity greater than $n/2$. Similarly, $f$ is polystable if it has no roots of multiplicity at least $n/2$ or if it has two roots of multiplicity $n/2$. Finally, $f$ is stable if it is polystable and has at least 3 distinct roots, since in this case the stabiliser is trivial.

1.2 Kempf-Ness theorem

Let us now consider taking quotients in the symplectic category. Let $(X, \omega)$ be a symplectic manifold with symplectic form $\omega$, and suppose that a compact group $K$ acts on $X$, preserving $\omega$. Write $\mathfrak{k}$ for the Lie algebra of $K$. To define the symplectic quotient we need a moment map for the action of $K$. This is a $K$-equivariant map $\mu : X \to \mathfrak{k}^*$, such that for each $\xi \in \mathfrak{k}$ the function $\langle \mu, \xi \rangle$ is a Hamiltonian for the vector field on $X$ induced by $\xi$. In other words,

$$d\langle \mu, \xi \rangle = \omega(\sigma(\xi), \cdot),$$

where $\sigma : \mathfrak{k} \to \text{Vect}(X)$ is the infinitesimal action. We will see shortly that a choice of moment map for the action is equivalent to a choice of linearisation of the action in GIT. Given a moment map $\mu : X \to \mathfrak{k}^*$, the symplectic quotient is defined to be $\mu^{-1}(0)/K$. This is a symplectic manifold if 0 is a regular value of $\mu$ and $K$ acts properly on $\mu^{-1}(0)$.

In order to relate this to the GIT quotient, we need some compatibility between the two setups. Suppose that $X$ is a Kähler variety, and let $L$ be an ample line bundle over $X$ endowed with a Hermitian metric with curvature form $-i\omega$. Suppose that the action of $K$ on $X$ preserves both the symplectic and holomorphic structures. Given an element $\xi \in \mathfrak{k}$ which induces a holomorphic vector field $v_\xi$ on $X$, we define a holomorphic vector field $\hat{v}_\xi$ on $L$ by

$$\hat{v}_\xi = \tilde{v}_\xi + i\langle \mu, \xi \rangle t,$$

where $\tilde{v}_\xi$ is the horizontal lift of $v_\xi$ and $t$ is the canonical vertical vector field on $L$. This gives an infinitesimal action of $\mathfrak{k}$ on $L$, which we can extend to the complexification $\mathfrak{g}$. Let us suppose that this infinitesimal action can be integrated to an action of $G$. We are now in the setup of GIT, with a complex
reductive group acting on a pair $(X, L)$.

**Example 1.2.1.** Suppose that the line bundle $L$ induces an embedding $X \hookrightarrow P^n$, and the Kähler metric on $X$ is the pullback of the Fubini-Study metric. Suppose the group $K$ acts on $X$ via a representation

$$\rho : K \rightarrow U(n + 1).$$

In this case we can write down a moment map for the action on $P^n$:

$$\mu(x).a = -i \overline{\rho_x(a) \hat{x}} \|\hat{x}\|^2,$$

for all $a \in \mathfrak{k}$, where $\hat{x} \in \mathbb{C}^{n+1} \setminus \{0\}$ is a lifting of $x \in P^n$. The moment map for the action on $X$ is just the restriction of this map to $X$.

The compatible linearisation is obtained by looking at the complexified representation $G \to GL(n + 1, \mathbb{C})$. The total space of the line bundle $\mathcal{O}_{P^n}(-1)$ is just the blowup of $\mathbb{C}^{n+1}$ in the origin, so we obtain an action on this line bundle. This induces an action on its dual, which when restricted to $X$ gives $L$.

**Theorem 1.2.2 (Kempf-Ness).** A $G$-orbit contains a zero of the moment map if and only if it is polystable. A $G$-orbit is semistable if and only if its closure contains a zero of the moment map.

The key idea in the proof of this theorem is to consider the following norm functional on the $G$-orbit of a point $x \in X$. Choose a non-zero lift $\hat{x} \in L_x$, and define

$$\phi : G/K \to \mathbb{R}$$

$$[g] \mapsto -\log \|g \cdot \hat{x}\|.$$

Let $\xi \in \mathfrak{k}$, and consider the restriction of $\phi$ to the geodesic $\exp(it\xi)$,

$$f(t) = -\log \|\exp(it\xi) \cdot \hat{x}\|.$$

Computing the derivative of $\phi$ in the direction $i\xi$, we find

$$f'(0) = \langle \mu(g \cdot x), \xi \rangle,$$

$$f''(0) = \|\sigma_x(\xi)\|^2,$$

where $\sigma_x : g \to T_xX$ is the infinitesimal action. This means that $\phi$ is convex along geodesics, and $g$ is a critical point of $\phi$ if and only if $\mu(g \cdot x) = 0$. 
Thinking of $\phi$ as a function on the $G$-orbit $G \cdot x$ now, we see that a critical point exists if and only if the $G$-orbit $G \cdot \hat{x}$ in $L_x$ is closed, i.e. $x$ is polystable.

Example 1.2.3. Let us consider Example 1.1.5 again, this time from the symplectic point of view. The symplectic form on $S^n \mathbb{P}^1 = \mathbb{P}^n$ induced by the standard symplectic form on $\mathbb{P}^1$ is just the standard symplectic form on $\mathbb{P}^n$. If we denote the moment map for the action of $SU(2)$ on $\mathbb{P}^1$ by $\mu$, then the moment map for the action of $SU(2)$ on $S^n \mathbb{P}^1$ is given by

$$\mu_n : S^n \mathbb{P}^n \to \mathfrak{su}(2)^*, \hspace{1cm} (x_1, \ldots, x_n) \mapsto \mu(x_1) + \ldots + \mu(x_n).$$

We can embed $\mathbb{P}^1$ as a coadjoint orbit in $\mathfrak{su}(2)^*$, and the moment map for the action of $SU(2)$ is just this embedding. Given an invariant inner product on $\mathfrak{su}(2)^*$, this orbit is a sphere, and we can see that the moment map $\mu_n$ simply gives the centre of mass of the $n$-tuple of points. Zeros of the moment map correspond to balanced configurations, which have centre of mass zero. The Kempf-Ness theorem in this case says that an $n$-tuple is polystable if and only if we can move the points to a balanced configuration by applying a transformation in $SL(2, \mathbb{C})$.

1.3 Norm squared of the moment map

We use the notation from the previous section. Let us now choose a rational invariant inner product on $\mathfrak{k}$. By rational we mean that for a maximal torus $T \subset K$ with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$, the inner product takes integral values on the kernel of the exponential map $\mathfrak{t} \to T$. Let us define the function

$$f : X \to \mathbb{R} \hspace{1cm} f(x) = \|\mu(x)\|^2.$$

The aim of this section is to study critical points of this function and generalise the Kempf-Ness theorem to characterise $G$-orbits of critical points of $f$ using a stability condition. In the following proposition we identify $\mathfrak{k}$ with its dual using the inner product.

**Proposition 1.3.1.** A point $x \in X$ is a critical point of $f$ if and only if the vector field on $X$ induced by $\mu(x)$ vanishes at $x$. Moreover when $\mu(x)$ is non-zero, it generates a circle subgroup of $K$. 

16
Proof. To prove the first statement we differentiate $f$. Write $v$ for the vector field on $X$ induced by $\mu(x)$. For a tangent vector $w$ at $x$,

$$df_x(w) = 2\langle d\mu(w), \mu(x) \rangle.$$ 

Since $\langle \mu, \mu(x) \rangle$ is a Hamiltonian for $v$, we have

$$df_x(w) = 2\omega(v, w).$$

Therefore $x$ is a critical point if and only if $\omega(v, w)$ evaluated at $x$ is zero for all $w$, i.e., if $v$ vanishes at $x$.

To prove the second statement let $\beta = \mu(x)$ and denote by $T$ the closure of the subgroup of $K$ generated by $\beta$. This is a compact connected Abelian Lie group, hence it is a torus. Letting $t$ be the Lie algebra of $T$, the moment map $\mu_T$ for the action of $T$ on $X$ is given by the composition of $\mu$ with the orthogonal projection from $\mathfrak{k}$ to $t$. Since by definition, $\beta \in t$, we have that $\mu(x) = \mu_T(x)$. Let $v_1, \ldots, v_k$ be an integral basis for the kernel of the exponential map from $t$ to $T$. Because of the rationality assumption on the inner product, what we need to show is that $\langle \mu_T(x), v_i \rangle$ is rational for all $i$, since then the orbit of $\mu(x)$ closes up to an $S^1$ orbit. Since $f_i = \langle \mu_T, v_i \rangle$ is the Hamiltonian function for the vector field induced by $v_i$, we know that $v_i$ acts on the fibre $L_x$ via $2\pi f_i(x)$. Since $\exp(v_i) = 1$, we find that $f_i(x)$ must be an integer.

Now we define the subgroups of $G$ which will feature in the stability condition. For a torus $T$ in $G$ with Lie algebra $t$, define two subalgebras of $\mathfrak{g}$:

$$\mathfrak{g}_T := \{ \alpha \in \mathfrak{g} | [\alpha, \beta] = 0 \text{ for all } \beta \in t \}$$

$$\mathfrak{g}_{T^\perp} := \{ \alpha \in \mathfrak{g}_T | \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in t \} \subset \mathfrak{g}_T. \quad (1.1)$$

Denote the corresponding connected subgroups by $G_T$ and $G_{T^\perp}$. Then $G_T$ is the identity component of the centraliser of $T$ and $G_{T^\perp}$ is a subgroup isomorphic to the quotient of $G_T$ by $T$. It is a closed subgroup of $G_T$ by the following Lemma and induction on the dimension of $T$.

**Lemma 1.3.2.** Let $H$ be a compact Lie group with Lie algebra $\mathfrak{h}$ endowed with a rational invariant inner product. Let $\beta \in \mathfrak{h}$ be in the centre of $\mathfrak{h}$, and suppose $\beta$ generates a circle subgroup of $H$. Write $H_{\beta^\perp}$ for the connected subgroup of
$H$ generated by the Lie algebra

$$h_{β⊥} := \{α ∈ h | \langle α, β \rangle = 0 \}.$$  

Then $H_{β⊥}$ is closed.

Proof. We will use the result of Malcev [24] stating that a subgroup of a Lie group corresponding to a Lie subalgebra is closed if and only if it contains the closure of all of its one-parameter subgroups.

Let $α ∈ h_{β⊥}$. We need to show that if $h ∈ H$ is in the closure of the one-parameter subgroup generated by $α$, then $h = \exp(γ)$ for some $γ ∈ h_{β⊥}$. Let us denote the closure of the subgroup generated by $α$ and $β$ by $T$. This is a compact connected Abelian group, so it is a torus and it contains $h$. Let $t$ be the Lie algebra of $T$, and let $t_{β⊥}$ be the subalgebra of elements orthogonal to $β$. Since $β$ is a rational element (it generates a circle) and the inner product is rational, we can choose a rational basis for $t_{β⊥}$, so the subgroup of $T$ generated by $t_{β⊥}$ is closed. In particular $h$ is in this subgroup, so $h = \exp(γ)$ for some $γ ∈ t_{β⊥} ⊂ h_{β⊥}$.

Working on the level of the compact subgroup $K$, if $t ⊂ k$, then the same formulae as in Equation 1.1 define Lie algebras $k_T, k_{T⊥}$ and subgroups $K_T, K_{T⊥}$ of $K$, such that

$$k_T = k ∩ g_T, \quad k_{T⊥} = k ∩ g_{T⊥}$$

$$K_T = K ∩ G_T, \quad K_{T⊥} = K ∩ G_{T⊥}.$$  

We can now write down the stability condition that we need.

**Definition 1.3.3.** Let $T$ be a torus in $G$ fixing $x$. We say that $x$ is **polystable relative to** $T$ if it is polystable for the action of $G_{T⊥}$ on $(X, L)$.

The main result of this section is the following.

**Theorem 1.3.4.** A point $x$ in $X$ is in the $G$-orbit of a critical point of $f$ if and only if it is polystable relative to a maximal torus in $G_x$, where $G_x$ is the stabiliser of $x$.

Before giving the proof, consider the effect of varying the maximal compact subgroup of $G$. If we replace $K$ by a conjugate $gKg^{-1}$ for some $g ∈ G$ and we replace $ω$ by $(g^{-1})^*ω$, then we obtain a new compact group acting by
symplectomorphisms. The associated moment map \( \mu_g \) is related to \( \mu \) by

\[
\mu_g(gx) = \text{ad}_g \mu(x) \in \text{ad}_g \mathfrak{k},
\]

where we identify the Lie algebra of \( gKg^{-1} \) with \( \text{ad}_g \mathfrak{k} \subset \mathfrak{g} \). Using the inner product on \( \text{ad}_g \mathfrak{k} \) induced by the bilinear form on \( \mathfrak{g} \), define the function \( f_g(x) = \|\mu_g\|^2 \). This satisfies \( f_g(gx) = f(x) \) by (1.2) and the \( \text{ad} \)-invariance of the bilinear form, so in particular the critical points of \( f_g \) are obtained by applying \( g \) to the critical points of \( f \).

**Proof of Theorem 1.3.4.** Suppose first that \( x \) is in the \( G \)-orbit of a critical point of \( f \). By replacing \( K \) with a conjugate if necessary, we can assume that \( x \) itself is a critical point, so \( \mu(x) \) fixes \( x \). If \( \mu(x) = 0 \) then Proposition 1.2.2 implies that \( x \) is polystable. If \( \mu(x) \neq 0 \) then by Lemma 1.3.1 we obtain a circle action fixing \( x \), generated by \( \beta = \mu(x) \). Choose a maximal torus \( T \) fixing \( x \), containing this circle. Since the moment map \( \mu_{T^⊥} \) for the action of \( K_{T^⊥} \) on \( X \) is the composition of \( \mu \) with the orthogonal projection from \( \mathfrak{k} \) to \( \mathfrak{k}_{T^⊥} \), we have that \( \mu_{T^⊥}(x) = 0 \). Using Proposition 1.2.2 this implies that \( x \) is polystable for the action of \( G_{T^⊥} \).

Conversely, suppose \( x \) is polystable for the action of \( G_{T^⊥} \) for a maximal torus \( T \) which fixes \( x \). Choose a maximal compact subgroup \( K \) of \( G \) containing \( T \). Then \( K_{T^⊥} \) is a maximal compact subgroup of \( G_{T^⊥} \) and using the assumption on \( x \), Proposition 1.2.2 implies that \( y = gx \) is in the kernel of the corresponding moment map \( \mu_{T^⊥} \) for some \( g \in G_{T^⊥} \). Then, for the moment map corresponding to \( K \), \( \mu(y) \) is contained in \( \mathfrak{t} \) (since \( T \) fixes \( y \) and \( \mu \) is equivariant, we have \( \mu(y) \in \mathfrak{t}_T \)), and therefore fixes \( y \). This means that \( y \) is a critical point of \( f \) by Proposition 1.3.1.

We will now reformulate this stability condition using the Hilbert-Mumford numerical criterion. Write \( G_x \) for the stabiliser of \( x \). Since \( G_x \) fixes \( x \), the action on the fibre \( L_x \) defines a map \( G_x \to \mathbb{C}^\ast \). The derivative at the identity gives a linear map \( \mathfrak{g}_x \to \mathbb{C} \) which we denote by \( -F_x \) in order to match with the sign of the Futaki invariant defined later. We say that \( -F_x(\alpha) \) is the weight of the action of \( \alpha \) on \( L_x \). According to the numerical criterion we have the following necessary and sufficient condition for a point \( x \) to be polystable: for all one-parameter subgroups \( t \to \exp(t\alpha) \) in \( G_{T^⊥} \), the weight on the central fibre \( L_{x_0} \) is negative, or equal to zero if \( \exp(t\alpha) \) fixes \( x \). Here \( x_0 \) is defined to
be \(\lim_{t \to 0} \exp(t\alpha)x\). In other words, the condition is that

\[ F_{x_0}(\alpha) \geq 0, \]

with equality if and only if \(x\) is fixed by the one-parameter subgroup.

It is inconvenient to restrict attention to one-parameter subgroups in \(G_{T^\perp}\) because the orthogonality condition is not a natural one for test-configurations which we will introduce later. We would therefore like to be able to consider one-parameter subgroups in \(G_T\) and adapt the numerical criterion. For a one-parameter subgroup in \(G_T\) generated by \(\alpha \in \mathfrak{k}_T\) we consider the one-parameter subgroup in \(G_{T^\perp}\) generated by the orthogonal projection\(^1\) of \(\alpha\) onto \(\mathfrak{k}_{T^\perp}\), which we denote by \(\overline{\alpha}\). We have

\[ \overline{\alpha} = \alpha - \sum_{i=1}^{k} \langle \alpha, \beta_i \rangle \beta_i, \]

where \(\beta_1, \ldots, \beta_k\) is an orthonormal basis for \(\mathfrak{t}\). Since \([\alpha, \mathfrak{t}] = 0\) and \(x\) is fixed by \(T\), the central fibre for the two one-parameter groups generated by \(\alpha\) and \(\overline{\alpha}\) is the same, the only difference is the weight of the action on this fibre. Since \(F_{x_0}\) is linear, we obtain

\[ F_{x_0}(\overline{\alpha}) = F_{x_0}(\alpha) - \sum_{i=1}^{k} \langle \alpha, \beta_i \rangle F_{x_0}(\beta_i). \]

The extremal vector field \(\chi\) is defined to be the element in \(\mathfrak{t}\) dual to the functional \(F_x\) restricted to \(\mathfrak{t}\) under the inner product. In other words, \(F_x(\alpha) = \langle \alpha, \chi \rangle\) for all \(\alpha \in \mathfrak{t}\). This generates a one-parameter subgroup by the same argument that was used in Proposition 1.3.1. If we now choose the orthonormal basis \(\beta_1\) such that \(\beta_1 = \chi/\|\chi\|\), then the previous formula reduces to

\[ F_{x_0}(\overline{\alpha}) = F_{x_0}(\alpha) - \langle \alpha, \chi \rangle. \]

If we define this expression to be \(F_{x_0,\chi}(\alpha)\), then the stability condition is equivalent to \(F_{x_0,\chi}(\alpha) \geq 0\) for all one-parameter subgroups generated by \(\alpha \in \mathfrak{k}_T\) with equality only if the one-parameter subgroup fixes \(x\). We therefore obtain the following

**Theorem 1.3.5.** A point \(x \in X\) is in the \(G\)-orbit of a critical point of \(f\),\(^1\) if this does not generate a one-parameter subgroup then we can approximate it with elements of \(\mathfrak{k}_T\) that do.

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\(^1\)If this does not generate a one-parameter subgroup then we can approximate it with elements of \(\mathfrak{k}_T\) that do.
if and only if for each one-parameter subgroup of $G$ generated by an element $\alpha \in \mathfrak{t}_T$ we have

$$F_{x_0,\chi}(\alpha) \geq 0,$$

with equality only if $\alpha$ fixes $x$. Here $T$ is a maximal torus fixing $x$ and $\chi$ is the corresponding extremal vector field.

We now ask what the infimum of the function $f = \|\mu\|^2$ is on a $G$-orbit.

**Theorem 1.3.6.** Let $x \in X$, and let $\alpha \in \mathfrak{t}_T$ generate a one-parameter subgroup such that the weight $F_{x_0,\chi}(\alpha) < 0$. Then

$$\inf_{g \in G} \|\mu(g \cdot x)\|^2 \geq \|\chi\|^2 + \frac{F_{x_0,\chi}(\alpha)^2}{\|\alpha\|^2}.$$

**Proof.** Suppose $F_{x_0}(\alpha) < 0$. We can arrange this by adding a multiple of $\chi$ to $\alpha$ if necessary. Consider the function

$$f(t) = \langle \mu(\exp(it\alpha) \cdot x), \alpha \rangle.$$

Computing the derivative of $f$, we find

$$f'(t) = \|\sigma_{\exp(it\alpha) \cdot x}(\alpha)\|^2 \geq 0,$$

so that $f$ is non-decreasing. Letting $t \to -\infty$ we get

$$f(t) \to \langle \mu(x_0), \alpha \rangle = -F_{x_0}(\alpha),$$

but $f(0) = \langle \mu(x), \alpha \rangle$, so that we must have $\langle \mu(x), \alpha \rangle \geq -F_{x_0}(\alpha)$. This implies

$$\|\mu(x)\|^2 \geq \frac{F_{x_0}(\alpha)^2}{\|\alpha\|^2}. \quad (1.3)$$

We now need to modify $\alpha$ carefully to get the result we want. Let $\overline{\alpha}$ be the component of $\alpha$ orthogonal to $\chi$ (the same remark as above applies if this does not generate a one-parameter subgroup). Since by our assumption $F(\overline{\alpha}) < 0$, we can choose a scalar $\lambda > 0$ such that $F_{x_0}(\lambda \overline{\alpha}) = -\|\lambda \overline{\alpha}\|^2$. Now define $\gamma = \lambda \overline{\alpha} - \chi$. As before, the central fibre for the one-parameter subgroup generated by $\gamma$ is $x_0$, just the weight is changed to

$$F_{x_0}(\lambda \overline{\alpha} - \chi) = -\|\lambda \overline{\alpha}\|^2 - \|\chi\|^2,$$
which is negative. Since
\[
\frac{F_{x_0}(\gamma)}{\|\gamma\|^2} = \|\lambda \gamma\|^2 + \|\chi\|^2 = \frac{F_{x_0}(\tau)}{\|\tau\|^2} + \|\chi\|^2,
\]
using Inequality 1.3 we obtain
\[
\|\mu(x)\|^2 \geq \frac{F_{x_0}(\tau)}{\|\tau\|^2} + \|\chi\|^2.
\]
Finally, since \( F_{x_0}(\tau) = F_{x_0,\chi}(\alpha) \), and \( \|\tau\| \leq \|\alpha\| \), we get the required inequality for \( \|\mu(x)\|^2 \). By replacing \( x \) by \( g \cdot x \) and \( \alpha \) by \( \text{ad}_g(\alpha) \) we obtain the same inequality for \( \|\mu(g \cdot x)\|^2 \).

Note that in this theorem we get the strongest inequality if we choose \( \alpha \) orthogonal to the chosen torus of automorphisms since that minimizes \( \|\alpha\| \). Note that if \( x \) is a critical point, then \( \|\mu(x)\| = \|\chi\| \), so this result implies a weak version of Theorem 1.3.4. This is the form in which it will be used in Chapter 3 to prove a necessary condition for a variety to admit an extremal metric.

### 1.4 Modulus of stability

Choose an invariant inner product on \( \mathfrak{k} \). Let \( x \in X \) be a polystable point, and write \( \pi_x : \mathfrak{k} \to \mathfrak{k}_x \) for the orthogonal projection onto the stabiliser of \( x \). Define the *modulus of stability* \( \lambda \) of \( x \) by
\[
\lambda = \inf_{\alpha} \frac{w(x, \alpha)}{\|\alpha - \pi_x(\alpha)\|},
\]
where the infimum is over all \( \alpha \in \mathfrak{k} \setminus \mathfrak{k}_x \) generating one-parameter subgroups. This is an invariant of the orbit of \( x \), and measures how far this orbit is from being unstable. Note that \( \lambda \) is strictly positive. To see this, note that we can restrict to \( \alpha \) in the orthogonal complement \( \mathfrak{k}_x^\perp \) and by continuity we can extend the function
\[
\psi(\alpha) = \frac{w(x, \alpha)}{\|\alpha\|}
\]
to all non-zero \( \alpha \in \mathfrak{k}_x^\perp \). The unit ball of \( \mathfrak{k}_x^\perp \) is compact so \( \psi \) achieves its infimum at some \( \beta \) which may or may not generate a one-parameter subgroup. To see that it does generate a one-parameter subgroup, restrict attention to the complex torus generated by \( \beta \) and use the arguments in Section 1.5. Since \( x \) is polystable, \( \psi(\beta) > 0 \) and so \( \lambda > 0 \).
Proposition 1.4.1. Let $x$ be polystable with modulus of stability $\lambda$. Let $x_0$ be the limit of $x$ under a one-parameter subgroup which does not fix $x$. Then $\|\mu(x_0)\| \geq \lambda$.

Proof. Let $\alpha \in \mathfrak{k}$ generate the one-parameter subgroup. We can assume that $\alpha$ is orthogonal to the stabiliser of $x$. By the definition of the weight, we have

$$\langle \mu(x_0), \alpha \rangle = -w(x, \alpha).$$

From this we obtain

$$\|\mu(x_0)\| \geq \frac{w(x, \alpha)}{\|\alpha\|} \geq \lambda.$$

\[\square\]

For each point $x \in X$, the infinitesimal action of $K$ induces a linear map $\sigma_x : \mathfrak{k} \to T_x X$. Using the metrics on $X$ and $\mathfrak{k}$ we form its adjoint $\sigma_x^*$. Suppose that the line bundle $L$ over $X$ is very ample and induces an embedding $X \subset \mathbb{P}^{n-1}$.

Theorem 1.4.2. Let $\mu(x) = 0$ and let the modulus of stability of $x$ be $\lambda$. Assume for simplicity that $x$ has trivial stabiliser. Then the smallest eigenvalue of $\sigma_x^* \sigma_x$ is bounded below by $2\lambda^2 / n$.

Proof. Consider the moment map restricted to a $G$-orbit,

$$\phi : G \to \mathfrak{k},$$

$$g \to \mu(g \cdot x).$$

We can compute

$$\langle d\phi_c(i\xi), \eta \rangle = \omega_x(\sigma_x(\eta), J \sigma_x(\xi)) = \langle \eta, \sigma_x^* \sigma_x(\xi) \rangle,$$

so the operator $\sigma_x^* \sigma_x$ is given by the derivative of $\phi$ in the $i\mathfrak{k}$ directions at the identity. To prove the result, we therefore need to show that for all $\xi \in \mathfrak{k}$

$$\langle d\phi_c(i\xi), \xi \rangle \geq \frac{2\lambda^2}{n} \|\xi\|^2,$$

and it is enough to restrict to the case when $\xi$ generates a $\mathbb{C}^*$ action.

Suppose the line bundle $L$ induces an embedding $X \subset \mathbb{P}(V)$ with $\dim V = n$, and $\xi$ generates a $\mathbb{C}^*$-action on $V$. Let $V = \bigoplus V_i$ be the weight decomposition of $V$, so that the action on $V_i$ has weight $w_i$, and $w_1 \leq w_2 \leq \ldots \leq w_n$. 

23
Choose an orthonormal basis \( \{ e_i \} \) with \( e_i \in V_i \). We can assume without loss of generality that \( x \) is in the orbit of \( x_0 = (1, 1, \ldots, 1) \) in these coordinates (if there were fewer non-zero coordinates, then we would get a sharper inequality in the end). The moment map is given by (see Example 1.2.1)

\[
\mu(t \cdot x_0) = \sum w_i e^{2w_i t} \sum e^{2w_i t}.
\]

We want to estimate the derivative of \( \mu(t \cdot x_0) \) with respect to \( t \) at the point \( t_0 \) for which \( x = t_0 \cdot x_0 \) (recall that \( \mu(x) = 0 \)). We have

\[
\frac{d}{dt} \mu(t_0 \cdot x_0) = \sum 2w_i^2 e^{2w_i t_0} \sum e^{2w_i t_0}.
\]

Let us suppose without loss of generality that \( t_0 \geq 0 \). Then \( \sum w_i^2 e^{2w_i t_0} \geq w_n^2 e^{2w_n t_0} \) and \( \sum e^{2w_i t_0} \leq n e^{2w_n t_0} \), so we obtain

\[
\frac{d}{dt} \mu(t_0 \cdot x_0) \geq \frac{2w_n^2}{n}.
\]

Since by the definition of the modulus of stability \( \lambda \) we have \( w_n \geq \lambda \| \xi \| \), the proof is complete. \( \square \)

### 1.5 Torus actions

In the case of a torus action, stability can be understood by analysing the weights of the action. Let \( T^c = (\mathbb{C}^*)^k \) act on \( \mathbb{P}(V) \) via a representation of \( T \) on \( V \). Choose a basis \( \{ e_1, \ldots, e_n \} \) for \( V \) such that the action is diagonal, given by weights \( \alpha_j \in t^* \). The action is given by

\[
\exp(\xi) e_j = \exp(i \langle \xi, \alpha_j \rangle) e_j, \quad \text{for all } \xi \in t.
\]

Let \( \{ X_1, \ldots, X_n \} \) be the dual basis for \( V^* \), on which the corresponding action is given by the same weights. Invariant monomials are given by \( \prod_i X_i^{a_i} \) such that \( \sum a_i \langle \xi, \alpha_i \rangle = 0 \) for all \( \xi \), ie.

\[
\sum_i a_i \alpha_i = 0.
\]

Invariant sections of \( O(m) \) over \( \mathbb{P}(V) \) are sums of these monomials with \( \sum a_i = m \).

Let \( x \in \mathbb{P}(V) \) and \( \hat{x} \in V \) a non-zero lifting. We define the weight polytope
$\Delta_x$ of $x$ to be the closed convex hull of the weights acting nontrivially on $x$:

$$\Delta_x = \overline{\text{co}\{\alpha_j | X_j(x) \neq 0\}} \subset t^*.$$ 

Note that $\Delta_x$ is contained in a proper affine subspace if and only if $x$ has non-discrete stabiliser. In the following theorem when referring to the interior of $\Delta_x$, we are considering $\Delta_x$ to be a subset of the minimal affine subspace containing it.

**Theorem 1.5.1.** Let $x \in P(V)$. We have

1. $x$ is semistable if and only if $\Delta_x$ contains the origin.
2. $x$ is polystable if and only if $\Delta_x$ contains the origin in its interior.
3. $x$ is stable if and only if $\Delta_x$ contains the origin in its interior and $\Delta_x$ is not contained in any proper subspace.

**Proof.**

1. By definition $x$ is semistable if and only if there is an invariant section of $O(m)$ for some $m$ which does not vanish at $x$. Invariant monomials which do not vanish at $x$ are products $\prod_i X_i^{a_i}$ with $\sum a_i \alpha_i = 0$ and $a_i = 0$ whenever $X_i(\hat{x}) = 0$. Such a section exists if and only if zero is contained in $\Delta_x$.

2. Let us first show that if $\Delta_x$ contains the origin in its interior then $x$ is polystable. For simplicity let us assume that $\hat{x} = (1, 1, \ldots, 1) \in V$. By the hypothesis we can choose non-zero $a_i$'s such that $s = \prod_i X_i^{a_i}$ is an invariant monomial. The set where $s$ does not vanish is the affine set $(C^*)^n$, and we need to show that the action of $T^c$ on this set is closed. It is enough to show that the action of the $R^k$ component of the torus is closed since $(S^1)^k$ is compact. Define the map

$$\psi : (C^*)^n \to R^n$$

$$(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|).$$

The images of the orbits under this map are subspaces of $R^n$, so since $\psi$ is continuous, the orbits are closed.

Conversely, suppose the origin is on the boundary of $\Delta_x$. Choose $\xi$ to be orthogonal to the face containing the origin, pointing inwards to $\Delta_x$. 
Then the lift of \( \lim_{t \to \infty} \exp(it\xi)x \) in \( V \) is given by

\[
\sum_{j: \langle \xi, \alpha_j \rangle = 0} e_j.
\]

Since \( x = (1, 1, \ldots, 1) \), this is not in the orbit of \( x \), so this orbit is not closed, and \( x \) is not polystable.

3. This follows from the remark before the theorem, since \( x \) is stable if and only if it is polystable with discrete stabiliser.

\[\square\]

Using the moment map we can give a different description of \( \Delta_x \). The compact torus \( T \) acts on the orbit \( T^c(x) \), and the interior of \( \Delta_x \) is the image of the moment map for this action. This follows from Atiyah’s convexity theorem (see [2]). Part 2 of the above theorem thus confirms the Kempf-Ness theorem in the case of a torus action.

Let us introduce a rational inner product on \( t \) so that we can identify \( t \) with \( t^* \). A one-parameter subgroup of \( T \) corresponds to an integral element \( \xi \in t \).

The limit \( \lim_{t \to -\infty} \exp(it\xi) \cdot x \) in \( P(V) \) is the sum of those \( e_i \) for which \( \langle \xi, \alpha_i \rangle \) is maximal and \( X_i(\hat{x}) \neq 0 \). The weight on the central fibre is therefore

\[
F(\xi) = \max_{i: X_i(\hat{x}) \neq 0} \{ \langle \xi, \alpha_i \rangle \}.
\]

The Hilbert-Mumford criterion says that \( x \) is stable if and only if \( F(\xi) > 0 \) for each integral \( \xi \). This means that for any rational hyperplane in \( t \) there are some \( \alpha_i \) on both sides of it. This is equivalent to the origin being contained in \( \Delta_x \), since the \( \alpha_i \) are rational. The argument with semistable and polystable points is similar, and this gives a proof of the Hilbert-Mumford criterion for torus actions.

It is easy to see that if \( x \) is polystable, then the modulus of stability of \( x \) is the distance of the boundary of \( \Delta_x \) from the origin. If on the other hand \( x \) is unstable, then the worst destabilising configuration (in the sense that \(-w(x, \alpha)/\|\alpha\| \) is maximal) is given by \(-\xi\), where \( \xi \) is the closest point of \( \Delta_x \) to the origin. The weight of this is \(-\|\xi\|^2\), so we see that in this case

\[
\inf_{t \in T^c} \|\mu(t \cdot x)\| = \sup_{\alpha} \frac{-w(x, \alpha)}{\|\alpha\|}.
\]

Note that this is a strengthening of Theorem 1.3.6 in the case of torus actions. In fact this stronger version is true in general, but we do not need it (see
Finally we describe relative polystability for a torus action. In Section 1.3 we saw that a point $x$ is a critical point of the norm squared of the moment map if $\mu(x)$ (as an element in $t$) fixes $x$. Recall that the image of the $T^c$-orbit of $x$ under the moment map is the interior of $\Delta_x$. Identifying $t$ with $t^*$ using the inner product we find that a vector $\xi \in t$ fixes $x$ if and only if $\xi$ is orthogonal to an affine subspace containing $\Delta_x$. We thus have the following

**Theorem 1.5.2.** The point $x$ is relatively polystable if and only if the orthogonal projection of the origin onto the minimal affine subspace containing $\Delta_x$ is in the interior of $\Delta_x$. 
Chapter 2

Extremal metrics

Extremal metrics were defined by Calabi [4] as an attempt to find canonical metrics in a given Kähler class on a Kähler manifold. For the definition let $(M,\omega_0)$ be a Kähler manifold. For any Kähler metric $\omega$ in the same cohomology class as $\omega_0$, define the Calabi functional

$$f(\omega) = \int_M (S(\omega) - \hat{S})^2 \frac{\omega^n}{n!},$$

where $S(\omega)$ is the scalar curvature, $\hat{S}$ is its average and $n$ is the dimension of $M$. We will see that $\hat{S}$ is independent of the choice of $\omega \in [\omega_0]$. The metric $\omega$ is called extremal if it is a critical point of this functional. Calabi showed that the Euler-Lagrange equation for this variational problem is that the gradient of the scalar curvature is a holomorphic vector field. The problem is the existence and uniqueness of extremal metrics. The uniqueness problem has been solved by Mabuchi [23] in the algebraic case and Chen-Tian [7] in general, in the sense that the extremal metric is unique up to the action of holomorphic automorphisms.

In Section 2.1 we will recall some of the more elementary theory of extremal metrics. Then in Section 2.2 we explain how the scalar curvature arises as a moment map for an infinite dimensional symplectic action. The Calabi functional then appears as the norm squared of the moment map, so the theory developed in the previous chapter becomes relevant to the study of extremal metrics. This point of view is used to motivate the definition of K-stability in Chapter 4. Also, all of the concepts introduced in Section 2.1 can be seen as special cases of the constructions in Chapter 1.
2.1 Futaki invariant and Mabuchi functional

As above, let \((M, \omega_0)\) be a Kähler manifold. For Kähler metrics \(\omega\) in the same cohomology class as \(\omega_0\) we define the following three functionals:

\[
\begin{align*}
    f(\omega) &= \int_M (S(\omega) - \hat{S})^2 \omega^n / n!, \\
    g(\omega) &= \int_M |\text{Ric}(\omega)|^2 \omega^n / n!, \\
    h(\omega) &= \int_M |\text{Riem}(\omega)|^2 \omega^n / n!,
\end{align*}
\]

where \(\text{Ric}\) is the Ricci curvature, and \(\text{Riem}\) is the full Riemannian curvature.

Calabi showed that these three functionals differ by constants depending only on the Kähler class, so their critical points are the same. Calabi showed (see [4])

**Proposition 2.1.1.** A metric \(\omega\) is a critical point of \(f\) if and only if the gradient of \(S(\omega)\) is a holomorphic vector field. Such a metric is called an extremal metric.

In particular a metric with constant scalar curvature (cscK) is an extremal metric, but there are also examples with non-constant scalar curvature (see eg. Section 5.3). In fact if we fix a maximal torus of holomorphic automorphisms of the manifold, then we can determine a priori what the gradient vector field of the scalar curvature of an extremal metric is if one exists. First of all the total scalar curvature is an invariant of the Kähler class, since

\[
\int_M S(\omega) \omega^n / n! = \int_M \rho \wedge \omega^{n-1} / (n-1)! = \frac{2\pi c_1(M) \cup [\omega]^{n-1}}{(n-1)!},
\]

where \(\rho\) is the Ricci form of \(\omega\). Since the volume is also an invariant of the Kähler class, we see that the average scalar curvature \(\hat{S}\) is fixed. In order to refine this, we need to define the Futaki invariant. This was introduced by Futaki in [14] as an obstruction to the existence of a Kähler-Einstein metric.

We first need some preliminaries about holomorphic vector fields.

Given a complex valued function \(f : M \to \mathbb{C}\) and a metric \(\omega\), we can define a vector field \(X_f\) of type \((1, 0)\) by

\[
X_f = \sum_\alpha g^{-\alpha} \partial f / \partial \bar{z}^\alpha \partial z^\alpha,
\]

where \(g\) is the metric corresponding to \(\omega\). This is the \((1, 0)\)-part of the gradient of \(f\).

**Definition 2.1.2.** We say \(f : M \to \mathbb{C}\) is a holomorphy potential if \(X_f\) is a
holomorphic vector field. Denote by $\mathfrak{h}$ the Lie algebra of holomorphic vector fields and by $\mathfrak{h}_1$ the subspace of holomorphic vector fields of the form $X_f$.

It is shown in Kobayashi [20] that when $M$ is a projective variety, then the space $\mathfrak{h}_1$ coincides with the space of holomorphic vector fields that can be lifted to an ample line bundle over $M$. We therefore write $\text{Aut}(M, L)$ for the group of automorphisms generated by $\mathfrak{h}_1$.

The Futaki invariant is defined as a functional $F_\omega : \mathfrak{h}_1 \to \mathbb{C}$

$$X_f \mapsto \int_M f(S(\omega) - \bar{S}) \frac{\omega^n}{n!},$$

The point is that this functional is independent of the choice of Kähler metric $\omega$ in the class $[\omega_0]$.

**Proposition 2.1.3 (cf. Calabi [5]).** The functional $F$ is independent of the choice of representative of the Kähler class.

Thus, if there is a metric $\omega \in [\omega_0]$ which has constant scalar curvature, then $F(X_f) = 0$ for all $X_f \in \mathfrak{h}_1$. This gives an obstruction to the existence of a cscK metric and was the original context in which the Futaki invariant was used.

Choose a maximal compact subgroup $K$ of $\text{Aut}(M, L)$, and a maximal torus inside $K$ with Lie algebra $\mathfrak{t} \subset \mathfrak{h}_1$. Let $\mathfrak{t}^\mathbb{C} \subset \mathfrak{h}_1$ be the complexification of $\mathfrak{t}$. We define an inner product on $\mathfrak{t}^\mathbb{C}$, following Futaki and Mabuchi [15] (they defined the inner product on a larger algebra, but we do not need that here). We choose a metric $\omega \in [\omega_0]$ which is invariant under $K$, and define

$$\langle X_f, X_g \rangle = \int_M f g \frac{\omega^n}{n!},$$

where we normalise $f, g$ to have integral zero on $M$. It is shown in [15] that this is invariant of the representative of the Kähler class chosen. This inner product is positive definite on $\mathfrak{t}$, and by duality the Futaki invariant defines a vector field $\chi \in \mathfrak{t}$. This is called the extremal vector field, and it only depends on the Kähler class and the choice of $K$ (it is in the centre of $\mathfrak{t}$). If we change $K$ to a conjugate, the new extremal vector field is a conjugate of $\chi$. In particular the norm $\langle \chi, \chi \rangle$ is an invariant of the Kähler class. From the definition of the Futaki invariant we see that it is given by $X_{\pi(S(\omega))}$ where $\pi(S(\omega))$ is the $L^2$-orthogonal projection of the scalar curvature $S(\omega)$ onto the space of holomorphy potentials. The fact that $X_{\pi(S(\omega))} \in \mathfrak{h}_1$ lies in $\mathfrak{t}$ is shown in [15]. This means
that the gradient of the scalar curvature of an extremal metric if it exists is given by $X_{\pi(S(\omega))}$ for any $K$-invariant $\omega$ in the Kähler class.

Note that if we normalise $\pi(S(\omega))$ to have zero mean, then we have

$$\int_M (S(\omega) - \hat{S})^2 \frac{\omega^n}{n!} = \int_M [S(\omega) - \hat{S} - \pi(S(\omega))]^2 \frac{\omega^n}{n!} + \|\pi(S(\omega))\|_{L^2}^2$$

(2.1)

This gives a lower bound on the Calabi functional for $K$-invariant metrics which is achieved by a metric $\omega$ if and only if $\omega$ is an extremal metric.

We now define the Mabuchi functional (see [21]) which is a functional on the set of Kähler metrics in a fixed Kähler class, whose critical points are constant scalar curvature metrics. Write $\mathcal{K}$ for the space of metrics in the Kähler class $[\omega_0]$. The tangent space to $\mathcal{K}$ at a metric $\omega$ can be identified as

$$T_\omega \mathcal{K} = \left\{ \phi \in C^\infty(M) \mid \int_M \phi \frac{\omega^n}{n!} = 0 \right\}.$$

We define the Mabuchi functional by its variation as follows:

$$d\mathcal{M}_\omega(\phi) = -\int_M \phi (S(\omega) - \hat{S}) \frac{\omega^n}{n!}.$$

This defines a closed 1-form on $\mathcal{K}$, and so it defines $\mathcal{M}$ up to a constant since $\mathcal{K}$ is contractible. From the definition it is clear that the critical points of the Mabuchi functional are metrics of constant scalar curvature. The space $\mathcal{K}$ can be thought of as an infinite dimensional symmetric space (analogous to $SL(n, \mathbb{C})/SU(n)$), and the Mabuchi functional is convex along geodesics. Therefore the existence of a constant scalar curvature metric in $\mathcal{K}$ is expected to be equivalent to the properness of $\mathcal{M}$. This has been shown in the case of Kähler-Einstein metrics by Tian [33] (see also [27]).

### 2.2 Scalar curvature as a moment map

In this section we show how the scalar curvature arises as the moment map in an infinite dimensional symplectic quotient problem. This was shown by Donaldson in [9]. We follow here the computation in local coordinates given by Tian [34].

Let $(M, \omega)$ be a symplectic manifold. An almost complex structure $J$ on $M$ is an endomorphism $J : TM \rightarrow TM$ such that $J^2 = -Id$. We say that the
almost complex structure is compatible with $\omega$ if the tensor $g_J$ defined by

$$g_J(u, v) = \omega(u, Jv)$$

is symmetric and positive definite, i.e., it defines a Riemannian metric. The almost complex structure $J$ is integrable if we can define local holomorphic coordinates on $(M, J)$. If $J$ is integrable and compatible with $\omega$, then together they define a Kähler structure on $M$.

Let us denote by $\mathcal{J}$ the space of (integrable) complex structures on $M$, compatible with the symplectic form. This is an infinite dimensional manifold with tangent space at $J$ given by

$$T_J \mathcal{J} = \{ A : TM \to TM : AJ + JA = 0, \omega(u, Av) = \omega(v, Au) = 0 \}.$$ 

For $A \in T_J \mathcal{J}$ define $\mu_A(u, v) = \omega(u, Av)$. We can check that $\mu_A(Ju, Jv) = -\mu_A(u, v)$ and $\mu_A(u, v) = \mu_A(v, u)$, and conversely these symmetric, anti $J$-invariant sections of $T^*M \otimes T^*M$ can be identified with $T_J \mathcal{J}$. This tangent space has a natural complex structure induced by $J$, namely

$$(J\mu)(u, v) = -\mu(Ju, v).$$

This complex structure is integrable and vector fields on $M$ acting on $\mathcal{J}$ preserve this complex structure. There is also an $L^2$ inner product on $T_J \mathcal{J}$ induced by $g_J$. Together these induce a Kähler structure on $\mathcal{J}$.

Assume for simplicity that $H^1(M) = 0$ and let $K = Symp^0(M, \omega)$ be the connected component of the identity in the group of symplectomorphisms of $M$. This acts on $\mathcal{J}$ preserving its symplectic structure, and we wish to identify a moment map for this action. The Lie algebra $\mathfrak{k}$ of $K$ can be identified with smooth functions on $M$ with zero mean, via the Hamiltonian construction. For an element $J \in \mathcal{J}$ we denote by $S(J)$ the scalar curvature of $g_J$, and by $\hat{S}$ the average scalar curvature, which is independent of $J$. We use the complex scalar curvature which is half of the usual Riemannian one. We can now state the result proved in [9].

**Proposition 2.2.1.** The map $J \mapsto 4(S(J) - \hat{S})$ is an equivariant moment map for the action of $K$ on $\mathcal{J}$, where we have identified $\mathfrak{k}$ with its dual via the $L^2$ pairing.
To prove this, we need to compute two maps:

\[ P : C^\infty_0(M) \to T_J \mathcal{J}, \]
\[ Q : T_J \mathcal{J} \to C^\infty_0(M), \]

where \( P \) is the infinitesimal action of \( \text{Symp}^0(M, \omega) \) on \( \mathcal{J} \) and \( Q \) is the infinitesimal change in the scalar curvature of \( g_J \) induced by an element in \( T_J \mathcal{J} \).

To do the computation we will choose local normal coordinates \( x_1, \ldots, x_{2n} \).

\[ g_J \text{ is Kähler, } dJ(0) = 0. \]

**Proposition 2.2.2.** Identifying \( T_J \mathcal{J} \) with symmetric 2-tensors as above, we have

\[ P(H)_{ij} = J^k_i H_{jk} + J^k_j H_{ik}. \]

**Proof.** Let us denote by \( X_H \) the Hamiltonian vector field corresponding to \( H \).

The components of \( X_H \) are given by \( X^i_H = -J^k_i g^{pq} H_{jk} + g^{ik} H_{kj} \).

Let us write

\[ \mathcal{L}_{X_H} J \left( \frac{\partial}{\partial x^j} \right) = A^i_j \frac{\partial}{\partial x^i}. \]

Using \( (\mathcal{L}_u J)(v) = \mathcal{L}_u (J(v)) - J \mathcal{L}_u v \) we can compute

\[ A^i_j = J^k_j J^p_i g^{pq} H_{jk} + g^{ik} H_{kj}. \]

Since \( A \) is related to \( P(H) \) by \( P(H)_{ij} = \omega_{ik} A^k_j \), we get

\[ P(H) = J^k_i H_{ik} + J^k_j H_{jk}. \]

\[ \square \]

**Proposition 2.2.3.** We have \( Q(\mu) = \frac{1}{2} \mu_{jk,kj} \).

**Proof.** Let us choose a path of complex structures \( J_t \) such that \( \frac{d}{dt} \big|_{t=0} J_t = \mu \).

Then the variation of the induced metrics \( g_t \) is also \( \mu \). Since the Christoffel symbols of \( g_t \) are of order \( t \), we have

\[ R^l_{ijkl} = \frac{\partial^2 g_{tl,ij}}{\partial x^l \partial x^i} - \frac{\partial^2 g_{tl,ik}}{\partial x^l \partial x^i} + O(t^2). \]

The scalar curvature is therefore

\[ S(g_t) = \frac{1}{2} g^k_t g^l_t \left( \frac{\partial^2 g_{tl,ij}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{tl,ik}}{\partial x^j \partial x^k} \right) + O(t^2). \]
Differentiating at \( t = 0 \), we obtain
\[
Q(\mu) = -\mu_{ik} \text{Ric}_{ik} + \frac{1}{2} (\mu_{jk,kj} - \mu_{jj,kk}).
\]

Since \( \text{Ric} \) is \( J \)-invariant and \( \mu \) is anti-\( J \)-invariant, the first term vanishes.
Since \( \mu \) is anti-\( J \)-invariant, the trace \( \mu_{jj} \) vanishes and so we get the result we wanted.

Putting the previous two results together, we can verify proposition \( 2.2.1 \).

We need to check that
\[
4(Q(\mu), H)_{L^2} = \Omega(P(H), \mu),
\]
where \( \Omega \) is the symplectic form on \( \mathcal{J} \) induced by the complex structure defined above and the \( L^2 \) product. We have
\[
(Q(\mu), H)_{L^2} = \frac{1}{2} \int_M \mu_{jk,kj} H \frac{\omega^n}{n!}
\]
\[
\Omega(P(H), \mu) = - (P(H), J\mu)_{L^2}.
\]

Since \( (J\mu)_{ij} = -J^k_i \mu_{kj} \), we have
\[
-(P(H), J\mu)_{L^2} = \int_M (J^k_i H_{jk} + J^k_j H_{ik}) J^j_p \mu_{pq} g^{ip} g^{jq} \frac{\omega^n}{n!}
\]
\[
= 2 \int_M H_{jk} \mu_{pq} g^{jk} g^{ip} \frac{\omega^n}{n!}.
\]

Integrating by parts we get the required result.

We now see that the norm squared of this moment map is the Calabi functional up to a scalar multiple. We can therefore hope to apply the results of Section \( 1.3 \) to characterise the complexified orbits of critical points of the functional in terms of stability. Unfortunately the complexification of \( \text{Symp}^0(M, \omega) \) does not exist, but we can think of the orbits of this complexification as follows.

The infinitesimal action of \( \text{Symp}^0(M, \omega) \) defines a distribution on \( \mathcal{J} \). It can be shown that the complexification of this distribution is integrable, so it defines a foliation of \( \mathcal{J} \). We think of the leaves of this foliation as the orbits of the complexified group.

At this point it is convenient to change our point of view. So far we have been looking at varying the complex structure on a symplectic manifold, but in the end we are interested in Kähler metrics on a complex manifold. If \( F : M \to M \) is a diffeomorphism then the metric defined by the pair \( (\omega, F^*(J)) \) is isometric...
to the one defined by \(((F^{-1})^* \omega, J)\). If \( \phi \in C_0^\infty (M) \), then the infinitesimal action of the vector field \(-JX_\phi\) on \(\omega\) is

\[
\mathcal{L}_{-JX_\phi} \omega = -dJd\phi = 2i \partial \bar{\partial} \phi.
\]

This shows that at least formally the orbits of the complexified group can be identified with the space Kähler metrics in a fixed Kähler class if we keep the complex structure fixed instead of the symplectic form by applying diffeomorphisms.

While we cannot directly use the results of the finite dimensional theory developed in the previous chapter to characterise Kähler classes which admit extremal metrics, we can use that theory to guide us to some extent. For example we can now reinterpret the Futaki invariant and the Mabuchi functional in this framework. Let us fix a complex structure \(J \in \mathcal{J}\). The stabiliser \(g_J\) of \(J\) in \(g\) is the space of holomorphic vector fields on \((M, J)\) which have holomorphy potentials (ie. the space \(h_1\) introduced in the previous section), and the inner product of Futaki and Mabuchi is just the restriction of the \(L^2\) product on \(g\).

We can now rewrite the definition of the Futaki invariant as a functional

\[
F : g_J \rightarrow \mathbb{C}
\]

\[\alpha \mapsto \langle \mu(J), \alpha \rangle,
\]

which we recognise to be the weight functional defined in Section 1.3. Similarly, the variation of the Mabuchi functional \(\mathcal{M}\) at the metric defined by \(J\) can be written as

\[
d\mathcal{M}_J : g \rightarrow \mathbb{C}
\]

\[\alpha \mapsto -\langle \mu(J), \alpha \rangle,
\]

which is the same as the variation of the norm functional defined in Section 1.2.

In analogy with the finite dimensional situation, to test whether a Kähler class contains a cscK metric, we need to look at the asymptotic rate of change of \(\mathcal{M}\) as we tend towards the boundary of the Kähler class. One problem is to identify what this boundary is, and another is to compute the asymptotics of the Mabuchi functional. In the next chapter we introduce the tools used to study this problem algebro-geometrically.
Chapter 3

Stability of varieties

In the previous chapter we have seen that the Calabi functional can be interpreted as the norm squared of a moment map, so if we apply the results of Chapter 1 at least on a formal level then we expect that a Kähler class admits an extremal metric if and only if it satisfies some kind of stability condition. In this chapter we make this more precise. In Section 3.1 we introduce the notion of K-polystability. A preliminary version was defined by Tian in [33] aiming to characterise Fano varieties which admit Kähler-Einstein metrics. The version given here is due to Donaldson [10] and was conjectured to be equivalent to the existence of a constant scalar curvature Kähler metric. We also define relative K-polystability due to the author (see [31]) which was conjectured to characterise Kähler classes containing extremal metrics. These conjectures are now likely to be false, since by an example in [1] one might need to consider test-configurations which are not algebraic. In Section 3.1.1 we suggest a way of strengthening the definition of K-stability to what we call uniform K-stability to address this problem.

It is desirable to generalise the notion of K-stability to pairs $\langle X, D \rangle$ where $X$ is a polarised variety, and $D$ a divisor. If $D$ is a smooth divisor then this would give a criterion to decide when $X \setminus D$ admits a complete extremal metric, and in general one expects that an unstable variety breaks up into such stable pairs. An example of this is given in Section 5.4. We propose a notion of K-stability for pairs in Section 3.1.2 and then we will consider an example computation on a ruled surface in Section 3.2. We will compare the results we obtain in this case with the explicit construction of extremal metrics in Section 5.3.1.

A fairly simple way to prove that a variety which admits a cscK metric is K-semistable was given in [12]. Here Donaldson showed that a destabilising
test-configuration gives a lower bound on the Calabi functional. We show that this can be extended to the case of extremal metrics in Section 3.3, which in particular proves that a polarised variety that admits an extremal metric is relatively K-semistable.

### 3.1 K-stability

We would like to motivate the definition of K-stability using the moment map picture we described in Section 2.2. In Section 1.1 we saw that the stability of a point \( x \) in geometric invariant theory can be verified by looking at the orbits of the point under one-parameter subgroups and evaluating a numerical weight on the limiting point. When this is positive for all one-parameter subgroups which do not fix \( x \), then \( x \) is polystable. The main problem in applying this directly to our infinite dimensional setting is understanding what the one-parameter subgroups are. Alternatively, we need to make sense of the boundary of the space of Kähler metrics in a fixed Kähler class. What we can do instead is to consider algebro-geometric degenerations of our complex manifold into possibly very singular schemes. These are the “test-configurations” that we will define below, and they are analogous to the orbits of one-parameter subgroups in the finite dimensional theory. In fact as was shown in [28], such a test-configuration can be used to define a weak geodesic ray of metrics in the Kähler class although we will not use this.

Given a test-configuration, we need to define the weight on the central fibre. We know that for trivial degenerations, which are induced by holomorphic vector fields on the manifold, this weight has to be the Futaki invariant we defined in Section 2.1.

We first recall the definition of the generalised Futaki invariant from Donaldson [10]. Let \( V \) be a polarised scheme of dimension \( n \) with a very ample line bundle \( L \). Let \( \alpha \) be a \( \mathbb{C}^* \)-action on \( V \) with a lifting to \( L \). This induces a \( \mathbb{C}^* \)-action on the vector space of sections \( H^0(V, L^k) \) for all integers \( k \geq 1 \). Let \( d_k \) be the dimension of \( H^0(V, L^k) \), and denote the infinitesimal generator of the action by \( A_k \). Denote by \( w_k(\alpha) \) the weight of the action on the top exterior power of \( H^0(V, L^k) \), which is the same as the trace \( \text{Tr}(A_k) \). Then \( d_k \) and \( w_k(\alpha) \) are polynomials in \( k \) of degree \( n \) and \( n + 1 \) respectively for \( k \).
sufficiently large, so we can write
\[ d_{k} = c_{0}k^{n} + c_{1}k^{n-1} + O(k^{n-2}), \]
\[ w_{k}(\alpha) = \text{Tr}(A_{k}) = a_{0}k^{n+1} + a_{1}k^{n} + O(k^{n-1}). \]

**Definition 3.1.1.** The Futaki invariant of the \( C^{*} \)-action \( \alpha \) on \((V,L)\) is defined to be
\[ F(\alpha) = \frac{c_{1}a_{0}}{c_{0}} - a_{1}. \]

The choice of lifting of \( \alpha \) to the line bundle is not unique, however \( A_{k} \) is defined up to addition of a scalar matrix. In fact if we embed \( V \) into \( \mathbb{P}^{d_{1}-1} \) using sections of \( L \), then lifting \( \alpha \) is equivalent to giving a \( C^{*} \)-action on \( \mathbb{C}^{d_{1}} \) which induces \( \alpha \) on \( V \) in \( \mathbb{P}^{d_{1}-1} \). Since the embedding by sections of a line bundle is not contained in any hyperplane, two such \( C^{*} \)-actions differ by an action that acts trivially on \( \mathbb{P}^{d_{1}-1} \) i.e. one with a constant weight, say \( \lambda \). We obtain that for another lifting, the sequence of matrices \( A'_{k} \) are related to the \( A_{k} \) by
\[ A'_{k} = A_{k} + k\lambda I, \]
where \( I \) is the identity matrix. A simple computation now shows that \( F(\alpha) \) is independent of the lifting of \( \alpha \) to \( L \). It is shown in [10] that when \( V \) is smooth and the \( C^{*} \)-action is induced by a holomorphic vector field then this generalised Futaki invariant coincides with the classical Futaki invariant we defined in Section 2.1 up to a scalar multiple. More precisely we have

**Proposition 3.1.2.** Suppose \( \omega \) is a Kähler metric in the class \( 2\pi c_{1}(L) \) and the \( C^{*} \)-action \( \alpha \) is generated by a vector field \( X \) with holomorphy potential \( f \).
Then
\[ 2 \cdot (2\pi)^{n} F(\alpha) = -\int_{V} f(S(\omega) - \hat{S}) \frac{\omega^{n}}{n!}. \]

We next recall the notion of a test-configuration from [10].

**Definition 3.1.3.** A test-configuration for \((V,L)\) of exponent \( r \) consists of a \( C^{*} \)-equivariant flat family of schemes \( \pi : \mathcal{V} \to \mathbb{C} \) (where \( C^{*} \) acts on \( \mathbb{C} \) by multiplication) and a \( C^{*} \)-equivariant ample line bundle \( \mathcal{L} \) over \( \mathcal{V} \). We require that the fibres \((\mathcal{V}_{t},\mathcal{L}|_{\mathcal{V}_{t}})\) are isomorphic to \((V,L_{t})\) for \( t \neq 0 \), where \( \mathcal{V}_{t} = \pi^{-1}(t) \). The test-configuration is called a product configuration if \( \mathcal{V} = V \times \mathbb{C} \). The Futaki invariant of the induced \( C^{*} \)-action on \((\mathcal{V}_{0},\mathcal{L}|_{\mathcal{V}_{0}})\) is called the Futaki invariant of the test-configuration.

With these definitions we can now define when a polarised variety is K-polystable.
Definition 3.1.4. A polarised variety \((V,L)\) is \(K\)-polystable if for all test-configurations the Futaki invariant is non-negative and is zero if and only if the test-configuration is a product configuration.

We would now like to define relative \(K\)-polystability, following the definitions in Section 1.3. This uses an inner product on the Lie algebra of the compact group in the moment map picture, which in our case is the \(L^2\) product on \(\mathcal{C}_0^\infty(V)\) when \(V\) is smooth. However we want to compute this algebro-geometrically on the central fibre of a test-configuration, so we define an inner product on \(\mathbb{C}^*\)-actions on a polarised variety \((V,L)\), which coincides with the \(L^2\) product when \(V\) is smooth. Note that \(\mathbb{C}^*\)-actions do not naturally form a vector space and we are really defining an inner product on a subspace of the Lie algebra of the automorphism group of \((V,L)\).

Let \(\alpha\) and \(\beta\) be two \(\mathbb{C}^*\)-actions on \(V\) with liftings to \(L\). If we denote the infinitesimal generators of the actions on \(H^0(V,L^k)\) by \(A_k, B_k\), then \(\text{Tr}(A_k B_k)\) is a polynomial of degree \(n+2\) in \(k\).

Definition 3.1.5. The inner product \(\langle \alpha, \beta \rangle\) is defined to be the leading coefficient in

\[
\text{Tr} \left[ \left( A_k - \frac{\text{Tr}(A_k)}{d_k} I \right) \left( B_k - \frac{\text{Tr}(B_k)}{d_k} I \right) \right] = \\
= \text{Tr}(A_k B_k) - \frac{w_k(\alpha) w_k(\beta)}{d_k} = \langle \alpha, \beta \rangle k^{n+2} + O(k^{n+1}) \quad \text{for } k \gg 1.
\]

Like the Futaki invariant, this does not depend on the particular liftings of \(\alpha\) and \(\beta\) to the line bundle since we are normalizing each \(A_k\) and \(B_k\) to have trace zero.

Let us see what this is when the variety is smooth. In this case we can consider the algebra of holomorphic vector fields on \(V\) which lift to \(L\). This is the Lie algebra of a group of holomorphic automorphisms of \(V\). Inside this group, let \(G\) be the complexification of a maximal compact subgroup \(K\). Let \(\mathfrak{g}, \mathfrak{k}\) be the Lie algebras of \(G, K\). Denoting by \(\mathfrak{t}_\mathbb{Q}\) the elements in \(\mathfrak{k}\) which generate circle subgroups, our inner product on \(\mathbb{C}^*\)-actions gives an inner product on \(\mathfrak{t}_\mathbb{Q}\). Since this is a dense subalgebra of \(\mathfrak{t}\), the inner product extends to \(\mathfrak{t}\) by continuity. We further extend this inner product to \(\mathfrak{g}\) by complexification and compute it differential geometrically. This is analogous to the proof of Proposition 3.1.2 in Donaldson [10]. Let us choose a \(K\)-invariant Kähler metric \(\omega\) in the class \(2\pi c_1(L)\). Note that \(\mathfrak{g}\) is a space of holomorphic vector fields on \(V\) which lift to \(L\). Let \(v, w\) be two holomorphic vector fields on \(V\), with liftings

39
\( \hat{v}, \hat{w} \) to \( L \). We can write

\[
\hat{v} = \nabla + if_L, \quad \hat{w} = \nabla + ig_L
\]

where \( \nabla \) (respectively \( \nabla \)) is the horizontal lift of \( v \) (respectively \( w \)), \( L \) is the canonical vector field on the total space of \( L \) defined by the action of scalar multiplication, and \( f, g \) are smooth functions on \( V \). As in [10] we have that

\[
\nabla f = - (i_v(\omega))^0,1, \quad \nabla g = - (i_w(\omega))^0,1,
\]

so in particular \( f \) and \( g \) are defined up to an additive constant, and we can normalise them to have zero integral over \( V \). We would like to show that

\[
\langle v, w \rangle = (2\pi)^{-n} \int_{V} f g \frac{\omega^n}{n!},
\]

where we have assumed that \( f, g \) have zero integral over \( V \). Making use of the identity \( \langle v, w \rangle = \frac{1}{2} \left( \langle v + w, v + w \rangle - \langle v, v \rangle - \langle w, w \rangle \right) \) it is enough to show this when \( v = w \). Furthermore, we can assume that \( v \) generates a circle action since \( \mathfrak{k}_Q \) is dense in \( \mathfrak{k} \).

We can find the leading coefficients of \( d_k, \text{Tr}(A^k), \text{Tr}(A^k A^k) \) for this circle action using the equivariant Riemann-Roch formula, in the same way as was done in [10]. We find that these leading coefficients are given by

\[
(2\pi)^{-n} \int_{V} \frac{\omega^n}{n!}, \quad (2\pi)^{-n} \int_{V} f \frac{\omega^n}{n!}, \quad (2\pi)^{-n} \int_{V} f^2 \frac{\omega^n}{n!},
\]

respectively. If we normalise \( f \) to have zero integral over \( V \), then we obtain the formula for the inner product that we were after.

To define relative K-polystability we also need to modify the definition of a test-configuration slightly.

**Definition 3.1.6.** We say that the test-configuration \((V, L)\) for \((V, L)\) is compatible with a torus \( T \) of automorphisms of \((V, L)\), if there is a torus action on \((V, L)\) which preserves the fibres of \( \pi : V \to \mathbb{C} \), commutes with the \( \mathbb{C}^* \)-action, and restricts to \( T \) on \((V_t, L|_{V_t})\) for \( t \neq 0 \).

Fix a maximal torus of automorphisms of \((V, L)\), and write \( \chi \) for the \( \mathbb{C}^* \)-action induced by the extremal vector field. This is defined as in Section 2.1 by requiring that \( F(\alpha) = \langle \chi, \alpha \rangle \) for all \( \mathbb{C}^* \)-actions \( \alpha \) in the torus. Because of the difference between the algebraic and differential-geometric definitions of the Futaki invariant and inner product, this is half of the differential geometric
extremal vector field. With these preliminaries we can state the definition of relative K-polystability.

**Definition 3.1.7.** A polarised variety $(V, L)$ is K-semistable relative to a maximal torus $T$ of automorphisms if

$$F_{\tilde{\chi}}(\tilde{\alpha}) := F(\tilde{\alpha}) - \langle \tilde{\chi}, \tilde{\alpha} \rangle \geq 0$$

for all test-configurations compatible with $T$. Here we denote by $\tilde{\alpha}$ and $\tilde{\chi}$ the $\mathbb{C}^*$-actions induced on the central fibre of the test-configuration. The variety is relatively K-polystable if in addition equality holds only if the test-configuration is a product configuration.

### 3.1.1 Uniform K-stability

In [31] we conjectured that a polarised variety is K-polystable relative to a maximal torus of automorphisms if and only if it admits an extremal metric, analogously to the conjecture in [10] in the cscK case. In Section 3.3 we will prove that a variety that admits an extremal metric is relatively K-semistable. As we mentioned in the introduction, an example of [10] shows that the converse statement is likely to be false and the conjectures need to be refined. Their example is a ruled manifold which is destabilised by a test-configuration with a non-algebraic polarisation. This is possible in the framework of slope-stability (see [30]).

A natural approach to remedy this situation is to strengthen the definition of K-polystability to uniform K-polystability as follows, extending the notion of the modulus of stability (see Section 1.4) to this setting. We choose a maximal torus of automorphisms $T$ of $(V, L)$, and define $(V, L)$ to be uniformly K-polystable if there exists a positive constant $\lambda > 0$ such that for all test-configurations compatible with $T$,

$$F(\alpha) \geq \lambda \|\alpha - \pi(\alpha)\|.$$  

Here $\alpha$ is the $\mathbb{C}^*$-action induced on the central fibre and $\pi(\alpha)$ is the orthogonal projection of $\alpha$ onto $T$. Then $\alpha - \pi(\alpha)$ might not generate a $\mathbb{C}^*$-action, but we can still define its norm.

In finite dimensional GIT such a $\lambda$ exists for all polystable points, but in the case of varieties this is no longer necessarily the case. In section 4.2 we will show that in the case of toric surfaces K-polystability implies uniform K-polystability, however for higher dimensional varieties we need to change the definition of the norm to an $L^{\frac{n}{n-1}}$ norm instead of the $L^2$ norm we used above if we want the
definition to make sense \((n\) is the complex dimension of the variety). We can define the \(L^p\)-norm of a \(\mathbb{C}^*\)-action as we have defined the \(L^2\)-norm, looking at the asymptotics of \(\text{Tr}(|A_k|^p)\) where the \(A_k\) are endomorphisms induced on \(H^0(V,L^k)\) by the \(\mathbb{C}^*\)-action as before. However unless \(p\) is an even integer, the equivariant Riemann-Roch formula can no longer be used to show that this coincides with the \(L^p\)-norm of a Hamiltonian. In any case it is tempting to conjecture that uniform \(K\)-polystability is the correct condition characterising the existence of \(cscK\) metrics. The stronger assumption of uniform \(K\)-polystability should make it easier to make analytic deductions. In particular we saw in Section 1.4 that control of the modulus of stability can be used to control the first eigenvalue of an operator \(\sigma^*\sigma\). In the infinite dimensional setting this is the Lichnerowicz operator and controlling its first eigenvalue is crucial to the analysis in trying to prove the existence of an extremal metric. The bound we gave in Section 1.4 is not good enough for this purpose, it is just an indication of what might be possible.

### 3.1.2 \(K\)-stability of a pair \((V, D)\)

In this subsection we propose a definition of \(K\)-stability for a pair \((V, D)\) consisting of a smooth polarised variety \((V, L)\) and a smooth divisor \(D \subset V\). The aim is to find a stability condition for the existence of a complete extremal metric on \(V \setminus D\) which is asymptotically hyperbolic near \(D\). A well-known case is the polarisation \(K_V + D\), where \(K_V\) is the canonical bundle of \(V\) and we assume that \(K_V + D\) is ample. In this case it was shown by Cheng-Yau [8] (see also Tian-Yau [35]) that a complete Kähler-Einstein metric exists on \(V \setminus D\), which is asymptotically hyperbolic near \(D\).

It is not yet clear what the precise class of metrics is that one should consider, but let us use the following as a preliminary definition:

**Definition 3.1.8.** A complete Kähler metric \(g\) on \(V \setminus D\) is asymptotically hyperbolic near \(D\) if near \(D\) it is asymptotic to a metric of the form

\[
g_0 = K \cdot \frac{|dz|^2}{(|z| \log |z|)^2} + h,
\]

where \(K\) is a smooth positive function on \(V\), the symmetric 2-tensor \(h\) is a smooth extension of a metric on \(D\), and \(z\) is a local defining holomorphic function for \(D\). By \(g\) being asymptotic to \(g_0\) near \(D\), we mean that for all \(i \in \mathbb{N}\),

\[
\lim_{z \to 0} \|\nabla^{(i)} (g - g_0)\|_{(g_0)} = 0.
\]
Note that $|dz|^2 / (|z| \log |z|)^2$ is the standard hyperbolic cusp metric on the punctured disk up to a scalar factor. The function $K$ is necessary because we do not want to prescribe the curvature near $D$ in the normal directions to $D$. For such a metric on $V \setminus D$ we define a Kähler class in $H^2(V)$ as follows. The Kähler form $\omega$ corresponding to $g$ defines an $L^2$-cohomology class in $H^2_L(V \setminus D, g)$. Since $g$ is quasi-isometric to the fibred cusp metrics of \cite{17}, according to Corollary 2 in that paper, this $L^2$-cohomology group is naturally isomorphic to the de Rham cohomology $H^2(V)$. The class in $H^2(V)$ defined in this way by $\omega$ is the Kähler class of our metric. For example for a metric on $\mathbb{P}^1$ which is asymptotically hyperbolic near a point, the Kähler class is simply given by the total area as an element in $H^2(\mathbb{P}^1) \cong \mathbb{R}$.

We incorporate the divisor $D$ into the definition of a test-configuration as follows.

**Definition 3.1.9.** A test-configuration for $(V, D, L)$ is a test-configuration $(V, L)$ for $(V, L)$ with a $\mathbb{C}^*$ invariant Cartier divisor $D \subset V$ which is flat over $\mathbb{C}$ and restricts to $D$ on the non-zero fibres.

The central fibre of such a test-configuration is a polarised scheme $(V_0, L_0)$ with a $\mathbb{C}^*$-action and a divisor $D_0 \subset V_0$ fixed by the $\mathbb{C}^*$-action. We define a modification of the Futaki invariant for this situation.

Let $(V, L)$ be a polarised scheme with a $\mathbb{C}^*$-action $\alpha$ and $D \subset V$ a divisor fixed by the $\mathbb{C}^*$-action. Let us write

$$H^0(V, L^k \otimes \mathcal{O}(-D)) \subset H^0(V, L^k)$$

for the sections which vanish along $D$. The inclusion is induced by a section of $\mathcal{O}(D)$ which vanishes along $D$. The assumption that $D$ is invariant under the $\mathbb{C}^*$-action means that this subspace is preserved by the action. As before, let us write $d_k, w_k$ for the dimension of $H^0(V, L^k)$ and for the total weight of the action on this space. Let us also write $\tilde{d}_k, \tilde{w}_k$ for the dimension and weight of the action on the space $H^0(V, L^k \otimes \mathcal{O}(-D))$. Define $c_0, c_1, a_0, a_1$ by

$$\frac{d_k + \tilde{d}_k}{2} = c_0 k^n + c_1 k^{n-1} + O(k^{n-2}),$$

$$\frac{w_k + \tilde{w}_k}{2} = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}).$$

(3.2)

The Futaki invariant of the $\mathbb{C}^*$-action $\alpha$ on the pair $(V, D)$ is then

$$F(\alpha) = \frac{c_1}{c_0} a_0 - a_1.$$
Let us also define $\alpha_1, \alpha_2$ by

$$\dim H^0(D, L^k|_D) = \alpha_1 k^{n-1} + \alpha_2 k^{n-2} + O(k^{n-3}).$$

**Definition 3.1.10.** We say that the triple $(V, D, L)$, where $L$ is an ample line bundle over $V$ and $D \subset V$ is a divisor, is K-polystable if the Futaki invariant of every test-configuration for $(V, D, L)$ is non-negative and is zero only for product configurations. In addition we require that $c_1 / c_0 < \alpha_2 / \alpha_1$.

Let us briefly explain the last condition. In Section 5.3 we will construct complete extremal metrics on the complement of a divisor on a ruled surface, and we will find that for a range of polarisations $m < k_2$ (where $m$ parametrises the polarisation and $k_2$ is a constant - see Section 5.3 for details) we obtain metrics which are asymptotically hyperbolic as in Definition 3.1.8. For $m = k_2$ we also obtain an extremal metric but it no longer has the asymptotic behaviour prescribed in Definition 3.1.8 but instead the fibre metrics behave like

$$\frac{|dz|^2}{|z|^2 (\log |z|)^{3/2}}$$

near the divisor. The non-degeneracy condition $c_1 / c_0 < \alpha_2 / \alpha_1$ is aimed to rule out this possibility. The condition arises when looking at deformation to the normal cone of the divisor $D$ (see Section 3.2 for the definition of deformation to the normal cone or [29] for more details). This gives a family of test-configurations parametrised by $c \in (0, \epsilon)$ where $\epsilon$ is a small positive number. It turns out that the Futaki invariant $F(c)$ of these test-configurations satisfies $F(0) = F'(0) = 0$ and our non-degeneracy condition is $F''(0) > 0$.

It is straightforward to extend the notion of relative K-polystability to pairs as well. We only consider automorphisms of $(V, L)$ which fix $D$ (but can induce a nontrivial automorphism of $D$), and define the extremal $C^*$-action in this group. The modified Futaki invariant is defined as before. The non-degeneracy condition is defined as above using deformation to the normal cone of $D$, but with the modified Futaki invariant.

We conjecture that if $D$ is a smooth divisor then $(V, D, L)$ is relatively K-polystable (with a positive modulus of stability) if and only if there exists a complete extremal metric on $V \setminus D$ in the cohomology class $c_1(L)$, which is asymptotically hyperbolic near $D$.

Our definition of a test-configuration was chosen because it seems natural, and it is satisfactory for the example that we compute in the next section. The definition of the Futaki invariant is motivated by the calculations involved in
the explicit construction of extremal metrics in Section 5.3. Those calculations also suggest that different combinations of $d_k$ and $\tilde{d}_k$ in Equation 3.2 could be used to characterise incomplete metrics with edge singularities along $D$ with various angles.

### 3.2 Relative K-polystability of a ruled surface

The aim of this section is to work out the stability criterion in a special case. Let $\Sigma$ be a genus two curve, and $\mathcal{M}$ a line bundle on it with degree one (the calculations also work for genus greater than two and a line bundle with degree greater than one). Define $X$ to be the ruled surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{M})$ over $\Sigma$. Tønnesen-Friedman [36] constructed a family of extremal metrics on $X$, which does not exhaust the entire Kähler cone (see also Section 5.3). We will show that $X$ is K-unstable (relative to a maximal torus of automorphisms) for the remaining polarisations (it was shown in [1] that $X$ does not admit an extremal metric for these unstable polarisations). We will also look at K-stability of the pairs $(X, S_0)$ and $(X, S_\infty)$ where $S_0$ and $S_\infty$ are the zero and infinity sections of $X$.

Since there are no non-zero holomorphic vector fields on $\Sigma$, a holomorphic vector field on $X$ must preserve the fibres. Thus, the holomorphic vector fields on $X$ are given by sections of $\text{End}_0(\mathcal{O} \oplus \mathcal{M})$. Here $\text{End}_0$ means endomorphisms with trace zero. The vector field given by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

generates a $\mathbb{C}^*$-action $\beta$, which is a maximal torus of automorphisms (see Maruyama [25] for proofs). Therefore this must be a multiple of the extremal vector field, which is then given by $\chi = \frac{F(\beta)}{\langle \beta, \beta \rangle} \beta$.

The destabilising test-configuration is an example of deformation to the normal cone of a subvariety, studied by Ross and Thomas [29] (see also Section 5.2, except we need to take into account the extremal $\mathbb{C}^*$-action as well. We consider the polarisation $L = C + mS_0$ where $C$ is the divisor given by a fibre, $S_0$ is the zero section (i.e. the image of $\mathcal{O} \oplus \{0\}$ in $X$, so that $S_0^2 = 1$) and $m$ is a positive constant. We denote by $S_\infty$ the infinity section, so that $S_\infty = S_0 - C$. Note that $\beta$ fixes $S_\infty$ and acts on the normal bundle of $S_\infty$ with weight 1.

We make no distinction between divisors and their associated line bundles, and use the multiplicative and additive notations interchangeably, so for example
$L^k = kC + mkS_0$ for an integer $k$.

The deformation to the normal cone of $S_\infty$ is given by the blowup

$$X := \widetilde{X} \times \mathbb{C} \xrightarrow{\pi} X \times \mathbb{C}$$

in the subvariety $S_\infty \times \{0\}$. Denoting the exceptional divisor by $E$, the line bundle $L_c = \pi^*L - cE$ is ample for $c \in (0, m)$. For these values of $x$ we therefore obtain a test-configuration $(X, L_c)$ with the $\mathbb{C}^*$ action induced by $\pi$ from the product of the trivial action on $X$ and the usual multiplication on $\mathbb{C}$. Denote the restriction of this $\mathbb{C}^*$-action to the central fibre $(X_0, L_0)$ by $\alpha$.

Since the $\mathbb{C}^*$-action $\beta$ fixes $S_\infty$ we obtain another action on the test-configuration, induced by $\pi$ from the product of the $\mathbb{C}^*$-action $\beta$ on $X$ and the trivial action on $\mathbb{C}$. Let us call the induced action on the central fibre $\beta$ as well. We wish to calculate $F^\chi(\alpha)$ where $\chi$ is a scalar multiple of $\beta$ as above.

For this we need the weight decomposition of the space $H^0(X_0, L^k_0)$. According to [29] we have

$$H^0(X_0, L^k_0) = H^0_X(kL - ckS_\infty) \oplus \bigoplus_{j=1}^{ck} \frac{H^0_X(kL - (ck - j)S_\infty)}{H^0_X(kL - (ck - j + 1)S_\infty)},$$

for $k$ large, with $t$ being the standard coordinate on $\mathbb{C}$. This gives the weight decomposition for the action of $\alpha$. For the action $\beta$, we need to further decompose $H^0_X(kL - ckS_\infty)$ into weight spaces as follows:

$$H^0_X(kL - ckS_\infty) = H^0_X(kL - mkS_\infty) \oplus \bigoplus_{i=1}^{mk-ck} \frac{H^0_X(kL - (mk - i)S_\infty)}{H^0_X(kL - (mk - i + 1)S_\infty)},$$

for $k$ large. This holds because of the following cohomology vanishing lemma.

**Lemma 3.2.1.** $H^1(X, kC + lS_0) = 0$ for $k \gg 0$ and $l \geq 0$.

**Proof.** Let $f : X \to \Sigma$ be the projection map. Since

$$\mathcal{O}_X(C) = f^*(\mathcal{O}_\Sigma(P))$$

where $P \in \Sigma$ is a point, we have

$$R^1f_*\mathcal{O}_X(kC + lS_0) = \mathcal{O}_\Sigma(kP) \otimes R^1f_*\mathcal{O}_X(lS_0).$$

The restriction of $\mathcal{O}_X(lS_0)$ to a fibre is $\mathcal{O}_{\mathbb{P}^1}(l)$ which for $l \geq 0$ has trivial $H^1$. This shows that $R^1f_*\mathcal{O}_X(kC + lS_0) = 0$ for $l \geq 0$. The Leray-Serre spectral
sequence now shows that \( H^1(X, kC + lS_0) = H^1(\Sigma, \mathcal{O}_\Sigma(kP) \otimes f_*\mathcal{O}_X(lS_0)) \).

Since \( \mathcal{M} \) has degree one, each summand in
\[
f_*\mathcal{O}_X(lS_0) = \bigoplus_{i=0}^l \mathcal{M}^\otimes i
\]
has non-negative degree, so for \( k \) large (in fact for \( k > 2 \)) we have
\[
H^1(\Sigma, \mathcal{O}_\Sigma(kP) \otimes f_*\mathcal{O}_X(lS_0)) = 0
\]
by Serre duality. This completes the proof. \( \square \)

In sum we obtain the decomposition
\[
H^0(X_0, L^k_0) = H^0_X(kL - mkS_\infty) \oplus \bigoplus_{i=1}^{mk-ck} \frac{H^0_X(kL - (mk - i)S_\infty)}{H^0_X(kL - (mk - i + 1)S_\infty)} \oplus \bigoplus_{j=1}^{ck} H^0_X(kL - (ck - j)S_\infty) / H^0_X(kL - (ck - j + 1)S_\infty).
\] (3.3)

According to [29] \( \alpha \) acts with weight \(-1\) on \( t \) that is, it acts with weight \(-j\) on the summand of index \( j \) above. Also, \( \beta \) acts on

\[
\frac{H^0_X(kL - lS_\infty)}{H^0_X(kL - (l + 1)S_\infty)}
\]

with weight \( l \), plus perhaps a constant independent of \( l \) which we can neglect, since the matrices are normalized to have trace zero in the formula for the modified Futaki invariant. The dimension of this space is \( k + l - 1 \) by the Riemann-Roch theorem. Writing \( A_k, B_k \) for the infinitesimal generators of the actions \( \alpha \) and \( \beta \) on \( H^0(X_0, L^k_0) \) and \( d_k \) for the dimension of this space, we can now compute
\[
d_k = \frac{m^2 + 2mk^2 + 2 - m}{2} + O(1),
\]
\[
\text{Tr}(A_k) = -\frac{c^3 + 3c^2}{6}k^3 + \frac{c^2 - c}{2}k^2 + O(k),
\]
\[
\text{Tr}(B_k) = \frac{2m^3 + 3m^2}{6}k^3 + \frac{m}{2}k^2 + O(k),
\] (3.4)
\[
\text{Tr}(A_kB_k) = -\frac{c^4 + 2c^3}{12}k^4 + O(k^3),
\]
\[
\text{Tr}(B_kB_k) = \frac{3m^4 + 4m^3}{12}k^4 + O(k^3).
\]
Using these, we can compute

\[ F_\chi(\alpha) = F(\alpha) - \langle \alpha, \chi \rangle = F(\alpha) - \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} F(\beta). \]

We obtain

\[ F_\chi(\alpha) = \frac{c(m - c)(m + 2)}{4(m^2 + 6m + 6)} \left[ (2m + 2)c^2 - (m^2 - 4m - 6)c + m^2 + 6m + 6 \right]. \]

If \( F_\chi(\alpha) \leq 0 \) for a rational \( c \) between 0 and \( m \), then the variety is K-unstable (relative to a maximal torus of automorphisms). We see that the variety is relatively K-unstable for \( m \geq k_1 \approx 18.889 \), where \( k_1 \) is the only positive real root of the quartic \( m^4 - 16m^3 - 52m^2 - 48m - 12 \).

Let us now look at relative K-stability of the pair \((X, S_\infty)\) and use the same test-configuration as above, i.e. deformation to the normal cone of \( S_\infty \) with parameter \( c \in (0, m) \). We have the same decomposition of \( H^0(X_0, L_0^k) \) as in Equation 3.3 and we can identify the quotient of \( H^0(X_0, L_0^k) \) by the space of sections which vanish along \( S_\infty \) with \( H^0(kL)/H^0(kL - S_\infty) \). We need to subtract half of the contribution of this space from the formulae in 3.3 to calculate the modified Futaki invariant for the pair. The dimension of this space is \( k - 1 \), the weight of \( \alpha \) on it is \( -ck \) and the weight of \( \beta \) is 0. The new formulae are therefore

\[ d'_k = \frac{m^2 + 2m}{2}k^2 + \frac{1 - m}{2}k + O(1), \]

\[ \text{Tr}(A'_k) = -\frac{c^3 + 3c^2}{6}k^3 + \frac{c^2}{2}k^2 + O(k), \]

while the other expansions remain unchanged. The new modified Futaki invariant is then

\[ F_\chi(\alpha) = \frac{c^2(m - c)}{2m^2(m^2 + 6m + 6)} \left[ c(2m^2 + 4m + 3) - m^3 + 3m^2 + 9m + 6 \right]. \]

The pair \((X, S_\infty)\) is relatively K-unstable if \( F_\chi(\alpha) \leq 0 \) for some rational \( c \in (0, m) \) or if the order of vanishing of \( F_\chi(\alpha) \) at \( c = 0 \) is greater than 2. This happens if \( m \geq k_2 \approx 5.0275 \), where \( k_2 \) is the only positive real root of the cubic \( m^3 - 3m^2 - 9m - 6 \).

The calculation for the pair \((X, S_0)\) is essentially the same, except we use deformation to the normal cone of \( S_0 \) in that case. These results should be compared to the results in Section 5.3 where we construct extremal metrics in the Kähler classes which are not destabilised by the test-configurations we
considered here.

### 3.3 Lower bound on the Calabi functional

In [12] Donaldson showed that a destabilising test-configuration gives a lower bound for the Calabi functional. The precise statement is the following.

**Theorem 3.3.1.** Let $\alpha$ be a destabilising test-configuration with Futaki invariant $F(\alpha) < 0$. Then for any metric $\omega$ in the class of our polarisation,

$$\|S(\omega) - \hat{S}\|_{L^2}^2 \geq 4 \cdot (2\pi)^n \frac{F(\alpha)^2}{\|\alpha\|^2}.$$  

The constant $4 \cdot (2\pi)^n$ arises from the difference between the differential-geometric and algebro-geometric Futaki invariants and inner products. This is analogous to the finite dimensional result, Theorem 1.3.6. In the same way as we did there, we can extend the result to the case of extremal metrics. We simply need to modify the test-configuration in such a way as to obtain the optimal inequality. This should be compared with Inequality 2.1.

**Theorem 3.3.2.** Let $T$ be a maximal torus of automorphisms of $(X, L)$ with corresponding extremal vector field $\chi$. Let $\mathcal{X}$ be a test-configuration compatible with $T$ such that $F_{\mathcal{X}}(\alpha) < 0$ for the $\mathbb{C}^*$-action $\alpha$ induced on the central fibre. Then for any metric $\omega \in 2\pi c_1(L)$,

$$\|S(\omega) - \hat{S}\|_{L^2}^2 \geq 2 \cdot (2\pi)^n \frac{F_{\chi}(\alpha)^2}{\|\alpha\|^2} + \|\chi\|_{L^2}^2.$$  

Here $\|\chi\|_{L^2}$ is the differential-geometric norm of the differential geometric extremal vector field, which is $2 \cdot (2\pi)^n/2$ times the algebraic norm of the algebraic extremal vector field. We will write $\|\chi\|$ without the $L^2$ subscript for the latter, hoping that it does not cause confusion.

**Proof.** Since the test-configuration is compatible with $T$, there is a $\mathbb{C}^*$-action $\tilde{\chi}$ on $\mathcal{X}$ fixing the base $\mathcal{C}$, which restricts to $\chi$ on the nonzero fibres. Write $\tilde{\alpha}$ for the $\mathbb{C}^*$-action on $\mathcal{X}$ induced by the test-configuration. We can modify the test-configuration by multiplying $\tilde{\alpha}$ by a multiple of $\tilde{\chi}$. The Futaki invariant of the new test-configuration will be $F(\alpha) + lF(\chi)$ for some integer $l$, where $F(\chi)$ is the Futaki invariant of the vector field $\chi$ on $X$. Note that $F(\chi) = \|\chi\|^2$ by the definition of the extremal vector field.
We can also pull back the test-configuration $X$ under a map

$$
\mathbb{C} \to \mathbb{C}
$$

$$
\lambda \mapsto \lambda^k,
$$

for positive integers $k$, which changes the Futaki invariant to $kF(\alpha)$. This means that we can construct a test-configuration with Futaki invariant equal to $kF(\alpha) + lF(\chi)$ for any integers $k, l$ with $k > 0$. Since we are only interested in the quotient of the Futaki invariant by the norm of the $\mathbb{C}^*$-action, we will assume that we can also have rational $k$ and $l$.

For irrational $k, l$ we can still define $\|k\alpha + l\chi\|$ and $F(k\alpha + l\chi)$ by continuity. We follow the proof of Theorem 1.3.6. Let $\overline{\alpha}$ be the component of $\alpha$ orthogonal to $\chi$, i.e. $\overline{\alpha} = \alpha - \lambda\chi$ for some $\lambda$ such that $\langle \overline{\alpha}, \chi \rangle = 0$. By our assumption $F(\overline{\alpha}) = F_\chi(\alpha)$ is negative, so we can choose a positive constant $\mu$ such that $F(\mu\overline{\alpha}) = -\|\mu\overline{\alpha}\|^2$. Now define $\gamma = \mu\overline{\alpha} - \chi$. We have

$$
F(\gamma) = -\|\mu\overline{\alpha}\|^2 - \|\chi\|^2 = -\|\gamma\|^2,
$$

which is negative, and

$$
\frac{F(\gamma)^2}{\|\gamma\|^2} = \frac{F_\chi(\alpha)^2}{\|\overline{\alpha}\|^2} + \|\chi\|^2.
$$

We can approximate $\gamma$ with a rational linear combination $k\alpha + l\chi$, and apply Theorem 3.3.1 to obtain

$$
\|S(\omega) - \hat{S}\|_{L^2} \geq 4 \cdot (2\pi)^n \frac{F_\chi(\alpha)^2}{\|\overline{\alpha}\|^2} + 4 \cdot (2\pi)^n \|\chi\|^2.
$$

Since $\|\overline{\alpha}\| \leq \|\alpha\|$, we get the required result. \square

Note that this proves that if $(X, L)$ admits an extremal metric then it is relatively K-semistable, since the extremal metric would satisfy $\|S(\omega) - \hat{S}\|_{L^2} = \|\chi\|_{L^2}$ (see Section 2.4). 

50
Chapter 4

Toric varieties

In [10] Donaldson developed the theory of K-stability for toric varieties. The main result is that on a toric surface the Mabuchi functional on torus invariant metrics is bounded from below if and only if the surface is K-polystable with respect to toric degenerations. It remains to be seen whether this implies the existence of a cscK metric on the variety, but much progress on this has been made in [11], giving interior a priori estimates for the PDE in the case of toric surfaces.

In Section 4.1 we recall the construction of toric test-configurations from [10], but we generalise it to test-configurations of bundles of toric varieties. We will use this generalisation in the next chapter.

In Section 4.2 we concentrate on toric surfaces. We first show that a K-polystable toric surface is uniformly K-polystable. Using the results of [10] this boils down to the statement that for a positive convex function on a polygon the integral on the boundary controls the $L^2$-norm on the interior. We will see that for this to hold in higher dimensions we need to use the $L^{\frac{n}{n+1}}$-norm instead of the $L^2$-norm, where $n$ is the dimension. We then give an alternative proof of a result in [10], using the notion of measure majorisation from convex geometry. Finally we use the same technique to prove that a semistable polygon can be decomposed into stable subpolygons. This is analogous to the Jordan-Hölder filtration of a semistable vector bundle.

4.1 K-stability of toric varieties

Let $\Delta \subset \mathbb{R}^n$ be the moment polytope of a smooth polarised toric variety. The polytope is defined by a finite number of linear inequalities $h_k(x) \geq c_k$, where
the $h_k$ are linear maps from $\mathbb{R}^n$ to $\mathbb{R}$ which induce primitive maps from the integer lattice $\mathbb{Z}^n$ to $\mathbb{Z}$. Let $d\mu$ be the standard Euclidean volume form on $\Delta$, and define a measure $d\sigma$ on the boundary $\partial \Delta$ as follows. On the face defined by the equation $h_r(x) = c_r$ we let $d\sigma$ be the constant $(n - 1)$-form such that $dh_r \wedge d\sigma$ is, up to sign, $d\mu$.

Recall the following result from [16]

**Theorem 4.1.1.** Let $Q : \Delta \to \mathbb{R}$ be a continuous function. We have for $k \gg 1$,

$$
\sum_{\alpha \in k\Delta \cap \mathbb{Z}^n} Q(\alpha) = k^n \int_{\Delta} Q \, d\mu + \frac{k^{n-1}}{2} \int_{\partial \Delta} Q \, d\sigma + O(k^n).$

**Test-configurations for toric bundles**

We now construct test-configurations for toric bundles, extending the construction for toric varieties in [10]. Let us first define toric bundles. The data is a principal $T = (\mathbb{C}^*)^n$-bundle $P \to M$ over a projective variety $M$ of dimension $m$, and an $n$-dimensional polarised toric variety $(V, O_V(1))$ with corresponding moment polytope $\Delta \subset t^*$. Define a bundle of toric varieties $\pi : X \to M$ by

$$
X = P \times_T V.
$$

Let $L_M \to M$ be an ample line bundle and define a line bundle $L$ over $X$ by

$$
L = \pi^* L_M \otimes (P \times_T O_V(1)).
$$

Let us assume that it is ample for our choice of data, so that the pair $(X, L)$ is a polarised variety.

For each $\alpha \in t^* \cap \mathbb{Z}^n$ we define a line bundle $F_\alpha$ over $M$ with transition functions induced by the map

$$
T \to \mathbb{C}^*
$$

$$
\exp(\xi) \mapsto \exp(i\alpha(\xi)), \quad \text{for } \xi \in t.
$$

The pushforward $\pi_* L^k$ is then given by

$$
\pi_* L^k = L^k_M \bigoplus_{\alpha \in k\Delta \cap \mathbb{Z}^n} F_\alpha.
$$
Let us define the functions $Q_1, Q_2 : \Delta \to \mathbb{R}$ by
\[
Q_1(\alpha) = c_1(L_M \otimes F_\alpha)^m, \\
Q_2(\alpha) = \frac{1}{2} c_1(L_M \otimes F_\alpha)^{m-1} \cup c_1(TM),
\]
first for rational $\alpha$ then extending by continuity. We then have
\[
\dim H^0(L^k_M \otimes F_{k\alpha}) = k^m Q_1(\alpha) + k^{m-1} Q_2(\alpha) + O(k^{m-2}),
\]
for rational $\alpha \in \Delta$ and $k \gg 1$. Since $H^0(X, L^k) = H^0(M, \pi_*(L^k))$ it follows using Theorem 4.1.1 that
\[
\dim H^0(L^k) = k^{m+n} \int_\Delta Q_1 d\mu + k^{m+n-1} \left( \frac{1}{2} \int_{\partial \Delta} Q_1 d\sigma + \int_\Delta Q_2 d\mu \right) + O(k^{m+n-2}).
\]
We can now state the main result.

**Theorem 4.1.2.** A rational piecewise-linear convex function $f$ on $\Delta$ defines a test-configuration for $(X, L)$ with Futaki invariant
\[
\frac{1}{2} \int_{\partial \Delta} f Q_1 d\sigma + \int_\Delta f Q_2 d\mu - \frac{a_1}{a_0} \int_\Delta f Q_1 d\mu,
\]
where
\[
a_0 = \int_\Delta Q_1 d\mu, \\
a_1 = \frac{1}{2} \int_{\partial \Delta} Q_1 d\mu + \int_\Delta Q_2 d\mu.
\]
The norm of the test-configuration (ie. the norm of the induced $\mathbb{C}^*$-action on the central fibre) is given by
\[
\left( \int_\Delta (f - \overline{f})^2 Q_1 d\mu \right)^{\frac{1}{2}},
\]
where $\overline{f}$ is the average of $f$ over $\Delta$ with respect to the measure $Q_1 d\mu$.

**Proof.** We define the test-configuration in the same way as was done in [10] for toric varieties. Suppose $f < R$ for some integer $R$, and let $\Delta'$ be the polytope
\[
\Delta' = \{ (x, t) : x \in \Delta, \ 0 < t < R - f(x) \} \subset t' \times \mathbb{R}.
\]
53
Let us assume that \( \Delta' \) is an integral polytope, otherwise we could replace it by \( k \Delta' \) for an integer \( k \). The polytope \( \Delta' \) defines a polarised toric variety \((W, \mathcal{O}_W(1))\). The face \( \overline{\Delta} \cap (t^* \times \{0\}) \) is a copy of \( \Delta \) so we get a natural embedding \( i : V \to W \) such that the restriction of \( \mathcal{O}_W(1) \) to \( V \) is isomorphic to \( \mathcal{O}_V(1) \). Write the \( n+1 \) torus action on \( W \) as \( T \times \mathbb{C}^* \), where the \( T \) action restricts in the obvious way to \( i(V) \). We now form the toric bundle

\[
Y = P \times_T W,
\]

with the line bundle

\[
\mathcal{L} = \pi^* L_M \otimes (P \times_T \mathcal{O}_W(1)),
\]

as we have done for \( X \). Note that we have only twisted \( W \) using the first \( n \) torus components (only \( T \), not \( T \times \mathbb{C}^* \)). This means that the corresponding \( Q_1, Q_2 : \Delta' \to \mathbb{R} \) are just the same as for \( \Delta \), composed with the projection \( \Delta' \to \Delta \).

In [10] Donaldson showed that there is a \( \mathbb{C}^* \)-equivariant map

\[
p : W \to \mathbb{P}^1,
\]

with \( p^{-1}(\infty) = i(V) \) such that the restriction of \( p \) to \( W \setminus i(V) \) is a test configuration for \((V, \mathcal{O}_V(1))\). Since the map \( p \) is \( T \)-invariant, we can define

\[
\tilde{p} : Y \to \mathbb{P}^1
\]

\[
(x, w) \mapsto p(w).
\]

and this defines a test configuration when restricted to \( Y \setminus \tilde{p}^{-1}(\infty) \). We can compute the Futaki invariant of this test configuration in the same way as was done in [10]. We have divisors \( X_0 = \tilde{p}^{-1}(0) \) and \( X \cong X_\infty = \tilde{p}^{-1}(\infty) \) defined by the vanishing of sections \( \sigma_0, \sigma_1 \) of the line bundle \( \tilde{p}^{-1}(\mathcal{O}(1)) \) over \( X \). When \( k \) is large we therefore get the following exact sequences:

\[
0 \to H^0(Y, \mathcal{L}^k(-1)) \xrightarrow{\sigma_0} H^0(Y, \mathcal{L}^k) \to H^0(X_0, \mathcal{L}^k|_{X_0}) \to 0
\]

\[
0 \to H^0(Y, \mathcal{L}^k(-1)) \xrightarrow{\sigma_1} H^0(Y, \mathcal{L}^k) \to H^0(X_\infty, \mathcal{L}^k|_{X_\infty}) \to 0
\]

The inclusion maps are multiplication by \( \sigma_0, \sigma_1 \). We first see that the dimension \( d_k \) of \( H^0(X_0, \mathcal{L}^k) \) is the same as that of \( H^0(X, \mathcal{L}^k) \). The \( \mathbb{C}^* \)-action acts with weight 0 on \( \sigma_0 \) and with weight 1 on \( \sigma_1 \), so the weight \( w_k \) of the action on \( \bigwedge^{d_k} H^0(X_0, \mathcal{L}^k) \) is given by the weight of the action on \( \bigwedge^{d_k} H^0(X_\infty, \mathcal{L}^k) \) plus

54
the dimension of $H^0(Y, L^k(-1))$. Since the action on $H^0(X, L^k)$ is trivial, we obtain

$$w_k = \dim H^0(Y, L^k(-1)) = \dim H^0(Y, L^k) - \dim H^0(X, L^k).$$

Using Theorem 4.1.1 we have

$$w_k = k^{m+n+1} \int_{\Delta} (R - f) Q_1 d\mu + k^{m+n} \left( \frac{1}{2} \int_{\partial \Delta} (R - f) Q_1 d\sigma + \int_{\Delta} (R - f) Q_2 d\mu \right) + O(k^{m+n-1}).$$

Recall that if

$$d_k = a_0 k^{m+n} + a_1 k^{m+n-1} + O(k^{m+n-2}),$$

$$w_k = b_0 k^{m+n+1} + b_1 k^{m+n} + O(k^{m+n-1}),$$

then the Futaki invariant is $\frac{d_k}{a_0} b_0 - b_1$. This gives the required formula for the Futaki invariant. The formula for the norm of the test configuration can be shown in the same way.

Let us assume now that $M$ is just a point, so that $(X, L)$ is a polarised toric variety with corresponding polytope $\Delta$. In this case $Q_1 = 1$ and $Q_2 = 0$, so we get back Donaldson's result in [10]: A rational piecewise linear convex function $f$ on $\Delta$ defines a test-configuration for $X$ whose Futaki invariant is given by $\frac{1}{2}\mathcal{L}(f)$, where

$$\mathcal{L}(f) = \int_{\partial \Delta} f d\sigma - a \int_{\Delta} f d\mu,$$

and $a = \frac{\text{Vol}(\partial \Delta, d\sigma)}{\text{Vol}(\Delta, \mu)}$. Note that this is slightly different from the formula in [10] because of our different convention for the definition of the Futaki invariant. Let us rescale the Futaki invariant to be $\mathcal{L}(f)$ to avoid unnecessary factors of two below.

Let us see what the modified Futaki invariant is (see Section 3.1). The maximal torus of automorphisms is the standard torus action on the toric variety. The toric test-configurations we have defined are compatible with this torus by definition. An affine linear function on $\Delta$ corresponds to a holomorphic vector field on $X$ by the Hamiltonian construction. The inner product of two vector fields is just the $L^2$ inner product of their Hamiltonians normalised to have mean zero (see Section 3.1). This means that the normalised Hamiltonian $B$ of
the extremal vector field satisfies
\[ \mathcal{L}(H) = -\int_{\Delta} BH \, d\mu \]
for all affine linear functions \( H \) (the minus sign appears because the test-configuration corresponding to \( H \) by the above construction is the product configuration corresponding to \( -H \)). There is a unique such \( B \) and it can be computed easily for specific toric varieties. Given a convex piecewise-linear function \( f \) and a vector field with Hamiltonian \( H \), the inner product of the induced \( \mathbb{C}^* \)-actions on the central fibre of the test-configuration induced by \( f \) is the \( L^2 \) product of \( -f \) and \( H \) normalised to have zero mean. This is because the weights of the \( \mathbb{C}^* \)-action on the central fibre induced by the test-configuration are just the values of \( -f \) plus some constant. This means that the modified Futaki invariant is
\[ \mathcal{L}(f) + \int_{\Delta} fB \, d\mu = \int_{\partial\Delta} f \, d\sigma - \int_{\Delta} (a - B)f \, d\mu. \]
By the definition of \( B \), this is zero for all affine linear \( f \). If we define \( A = a - B \), then we see that the modified Futaki invariant is given by
\[ \mathcal{L}_A(f) = \int_{\partial\Delta} f \, d\sigma - \int_{\Delta} Af \, d\mu. \]
Donaldson defines \( \mathcal{L}_A \) with this formula for all bounded \( A \) and conjectured that if \( \mathcal{L}_A(f) > 0 \) for all non-affine convex functions \( f \), then there is a Kähler metric on the toric variety with scalar curvature given by the function \( A \). There is a unique affine linear \( A \) such that \( \mathcal{L}_A(H) = 0 \) for all affine linear \( H \), and our discussion shows that for this \( A \) the condition \( \mathcal{L}_A(f) > 0 \) for all non-affine convex functions \( f \) means that the toric variety is relatively K-polystable with respect to toric test-configurations.

Let us now see what uniform K-polystability corresponds to. Define the projection map \( \pi : C(\Delta) \to C(\Delta) \) onto the \( L^2 \)-orthogonal complement of the space of affine linear functions. By definition \((X,L)\) is uniformly K-polystable with respect to toric degenerations, if there exists a \( \lambda > 0 \) for which
\[ \mathcal{L}(f) \geq \lambda \|\pi(f)\|_{L^{\frac{n}{n-1}}}, \]
for all convex \( f \). The choice of the \( L^{n/(n-1)} \)-norm will become clear at the end of Section 4.2.1. We can summarise all this as follows.
Proposition 4.1.3. Let $\Delta$ be a polytope corresponding to the polarised toric variety $(X, L)$ of dimension $n$.

- $(X, L)$ is K-semistable for toric test-configurations if $\mathcal{L}(f) \geq 0$ for all rational piecewise-linear convex functions $f$.

- If in addition $\mathcal{L}(f) = 0$ if and only if $f$ is affine linear, then $(X, L)$ is K-polystable.

- $(X, L)$ is uniformly K-polystable if there exists $\lambda > 0$ such that for all convex functions $f$,
  \[
  \mathcal{L}(f) \geq \lambda \| \pi(f) \|_{L^{\frac{n}{n-1}}}. \tag{4.1}
  \]

- Let $A$ be the unique affine linear function such that $\mathcal{L}_A(H) = 0$ for all affine linear $H$. Then $(X, L)$ is relatively K-polystable if $\mathcal{L}_A(f) \geq 0$ for all rational piecewise-linear convex functions $f$, with equality only if $f$ is affine linear.

4.2 Toric surfaces

We now restrict attention to toric surfaces. We first prove the following

Theorem 4.2.1. A K-polystable toric surface is uniformly K-polystable.

Let the toric surface correspond to the polygon $P$ containing the origin. Call a convex function $f$ normalised if $f(0) = 0$ and $f \geq 0$ on $P$. In [10] Donaldson proved that on a K-polystable toric surface there exists $\lambda > 0$, such that

\[
\mathcal{L}(f) \geq \lambda \int_{\partial P} f \, d\sigma
\]

for all normalised convex functions $f$. To prove our result, it is therefore enough to show the following, which we will prove in the next subsection.

Proposition 4.2.2. There exists a constant $C$ such that for all non-negative continuous convex functions $f$ on $P$,

\[
\| f \|_{L^2(P)} \leq C \int_{\partial P} f \, d\sigma.
\]

Together with Donaldson’s result, this shows that on a K-polystable toric surface there exists $\mu > 0$ such that

\[
\mathcal{L}(f) \geq \mu \| f \|_{L^2(P)},
\]

57
for all normalised convex functions $f$. This implies the inequality (4.1) for all convex functions $f$ since $\|\pi(f)\| \leq \|f\|$, and both sides of the inequality are invariant under adding affine linear functions to $f$.

In Section 4.2.2 we will reprove part of Donaldon’s result, namely the fact that on a toric surface if $\mathcal{L}(f) \geq 0$ for all convex $f$ and $\mathcal{L}(f) = 0$ for some convex $f$ which is not affine linear, then $\mathcal{L}(h) = 0$ for a simple piecewise linear convex function $h$. This is a piecewise linear function with one “crease”. Then in Section 4.2.3 we prove a decomposition theorem for K-semistable polygons.

### 4.2.1 Uniform K-stability

The aim of this section is to prove Proposition 4.2.2. Before giving the proof we need two lemmas. Define a simple piecewise linear function on $\mathbb{R}^2$ to be a function of the form

$$f(x) = \max(\lambda(x) + c, 0),$$

where $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is a linear function. We call the line $\lambda(x) = -c$ the crease of $f$.

**Lemma 4.2.3.** Let $h$ be a simple piecewise linear convex function on the triangle $\Delta = \{(x, y) | x, y \geq 0, x/a + y/b \leq 1\}$ with $h(a, 0) = h(0, b) = 0$. Then there is a constant $C$ independent of $a$ and $b$ such that

$$\left( \int_{\Delta} h^2 \, d\mu \right)^{1/2} \leq C \left( \int_0^a h(x, 0) \, dx + \int_0^b h(0, y) \, dy \right).$$

**Proof.** Suppose the crease of $h$ is the segment $(c, 0), (0, d)$. The inequality is invariant under multiplying $h$ by a constant so we can assume $h(0, 0) = 1$, so that on $\{(x, y) | x, y \geq 0, dx + cy \leq cd\}$ we have $h(x, y) = 1 - \frac{x}{c} - \frac{y}{d}$.

We have

$$\left( \int_{\Delta} h^2 \, d\mu \right)^{1/2} = \sqrt{cd} \left( \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx \right)^{1/2} = C\sqrt{cd},$$

for some constant $C$, and

$$\int_0^a h(x, 0) \, dx + \int_0^b h(0, y) \, dy = \frac{c + d}{2}.$$  

The result follows since $2\sqrt{cd} \leq c + d$.  

58
Lemma 4.2.4. Let $f$ be a non-negative convex function on the triangle $\Delta = \{(x,y) | x,y \geq 0, x/a + y/b \leq 1\}$, such that $f(x,0), f(0,y)$ are non-increasing. There exists a constant $C$, independent of $a,b$ such that

$$\left( \int_{\Delta} f^2 d\mu \right)^{1/2} \leq C \left( \int_0^a f(x,0) \, dx + \int_0^b f(0,y) \, dy + \sqrt{ab} \cdot \max\{f(a,0), f(0,b)\} \right).$$

Proof. It is enough to prove the assertion for piecewise linear convex functions. Let $f$ be a piecewise linear convex function on $\Delta$, and denote by $x_0 = a > x_1 > \ldots > x_k = 0$ and $y_0 = b > y_1 > \ldots > y_l = 0$ the points where the restriction of $f$ to the $x$ and $y$ axes is non-linear. Define the points $X_i = (x_i,0)$ and $Y_j = (0,y_j)$. We will define a new convex function $\tilde{f}$ which is equal to $f$ on the edges $X_0X_k$ and $Y_0Y_l$ and dominates it in the interior of $\Delta$. We will then prove the inequality for $\tilde{f}$.

We define $\tilde{f}$ by induction on rectangles $X_iY_jY_0X_0$. We start by defining $\tilde{f}$ on the “rectangle” $X_0Y_0Y_0X_0$ to be the linear interpolation between the values of $f$ at $X_0$, and $Y_0$. Supposing we have defined $\tilde{f}$ for some $i,j$, consider the triangles $X_iX_{i+1}Y_j$ and $X_iY_{j+1}Y_j$. Define linear functions $u$ and $v$ on these triangles, which are equal to $f$ on $\partial \Delta$. Extend $\tilde{f}$ by the function which is smaller on the intersection of the two triangles (or either function if they are equal). Note that $\tilde{f}$ is the greatest convex function equal to $f$ on the two orthogonal edges of $\Delta$. 

59
Define \( g = \tilde{f} - g_0 \), where \( g_0 \) is a linear function on \( \Delta \) such that \( g_0(a,0) = f(a,0), g_0(0,b) = f(0,b) \) and \( g_0(0,0) = \max\{f(a,0),f(0,b)\} \). We can write \( g \) as a sum of non-negative simple piecewise linear functions \( g_1, \ldots, g_k \) with creases inside \( \Delta \).

We have
\[
\|\tilde{f}\|_{L^2(\Delta)} \leq \sum_{i=1}^{k} \|g_i\|_{L^2(\Delta)} + \|g_0\|_{L^2(\Delta)}.
\]
The sum is handled by Lemma 4.2.3 and the last term is bounded above by \( \sqrt{ab} \cdot \max\{f(a,0),f(0,b)\} \) by definition.

**Proof of Proposition 4.2.2**

Denote by \( P_1, \ldots, P_k \) the vertices of the polytope (also let \( P_{k+1} = P_1 \)) and by \( e_i \) the edge joining \( P_i, P_{i+1} \) for \( i = 1, \ldots, k \). On each edge \( e_i \) choose a point \( Q_i \) where the restriction of \( f \) to \( e_i \) is minimal. Let us assume for simplicity that none of the \( Q_i \) coincide with a vertex (we can achieve this by perturbing \( f \) slightly). We first restrict attention to the triangles \( Q_i P_{i+1} Q_{i+1} \). The property of these triangles that we need is that \( f \) is non-decreasing on the edges \( Q_i P_{i+1} \) and \( Q_{i+1} P_{i+1} \).

We apply Lemma 4.2.4 to the triangle \( Q_i P_{i+1} Q_{i+1} \) to get that the \( L^2 \) norm of \( f \) on this triangle is bounded above by \( C \) times the integral on the segments \( Q_i P_{i+1} \) and \( P_{i+1} Q_{i+1} \) plus \( \sqrt{\text{Vol}(P)} \max\{f(Q_i),f(Q_{i+1})\} \). The \( L^2 \) norm of \( f \) on the interior of the polygon \( Q_1 Q_2 \ldots Q_k \) is bounded above by \( \sqrt{\text{Vol}(P)} \max\{f(Q_1),\ldots,f(Q_k)\} \). In sum we obtain that for some constant
\[ C_1, \quad \| f \|_{L^2(P)} \leq C_1 \left( \int_{\partial P} f \, d\sigma + \max \{ f(Q_1), \ldots, f(Q_k) \} \right). \]

Since \( f(Q_i) \) is the minimum of \( f \) on the edge \( e_i \), we have

\[ f(Q_i) \leq \frac{1}{\text{Vol}(e_i)} \int_{P_i} f \, d\sigma \leq C_2 \int_{\partial P} f \, d\sigma. \]

for some \( C_2 \). This completes the proof of the result. \( \square \)

This result does not hold in higher dimensions because of the way the \( L^p \) norms scale. For \( a < 1 \), consider the convex function

\[ h(x_1, \ldots, x_n) = \begin{cases} 1 - \frac{1}{a} (x_1 + \ldots + x_n), & \text{if } x_1 + \ldots + x_n < a, \\ 0, & \text{otherwise.} \end{cases} \]

on the set \( \{(x_1, \ldots, x_n) | \sum x_i \leq 1\} \). The \( L^p \)-norm on the interior of the set is \( a^{n/p} C_1 \) for some constant \( C_1 \) and the \( L^1 \)-norm on the boundary is \( a^{n-1} C_2 \) for another constant \( C_2 \). Therefore the natural inequality to consider for \( n > 2 \) is

\[ \| f \|_{L^p(P)} \leq C \int_{\partial P} f \, d\sigma, \]

with \( p = \frac{n}{n-1} \). This is the reason for our choice of norm in the definition of uniform \( K \)-stability in Section 3.1.1. It is an interesting question to see whether the inequality holds in this form for \( n > 2 \).

### 4.2.2 Measure majorisation

The aim of this section is to prove the following result. Recall that \( P \) is a polygon corresponding to a toric surface.

**Theorem 4.2.5.** Suppose that

\[ \mathcal{L}(f) = \int_{\partial P} f \, d\sigma - a \int_P f \, d\mu \geq 0 \]

for all continuous convex functions \( f \) on \( P \), but there is a continuous convex function \( u \) on \( P \) which is not affine linear and \( \mathcal{L}(u) = 0 \). Then there is a simple piecewise linear function \( f \) with crease passing through \( P \), such that \( \mathcal{L}(f) = 0 \).

This is proved in Donaldson [10], but we give a slightly different proof based on a result in [9]. The fact that \( \mathcal{L}(f) \geq 0 \) for all continuous convex functions
means that the measure \( d\sigma \) majorises \( a\,d\mu \). In this case (see [6]) there exists a family \( \{T_x\}_{x\in P} \) of probability measures on \( P \) such that the barycentre of \( T_x \) is \( x \), and

\[
\sigma = a \int_P T_x \,d\mu(x).
\]

(4.2)

Note that this implies that \( T_x \) is supported on \( \partial P \). Let us denote the convex hull of its support by \( l_x \) since we will normally think of them as line segments. For \( f \) convex, the Jensen inequality implies \( T_x(f) \geq f(x) \) with equality if and only if \( f \) is linear when restricted to \( l_x \). Hence

\[
\int_{\partial P} f \,d\sigma = a \int_P T_x(f) \,d\mu(x) \geq a \int_P f(x) \,d\mu(x)
\]

with equality if and only if \( f \) is linear when restricted to \( l_x \) for \( \mu \)-almost every \( x \). From this we can immediately see a case when there is a simple piecewise linear function giving equality. If there is a line \( L \) through \( P \) such that the set of \( x \) with \( l_x \) intersecting \( L \) transversally\(^1\) has measure zero (with respect to \( \mu \)), then any simple piecewise linear function with crease \( L \) will do. We wish to show that if there is no such \( L \), then the only convex functions giving equality are the linear ones. We need the following

**Lemma 4.2.6.** If a convex function \( f \) is linear when restricted to the convex sets \( l_x \) and \( l_y \) which intersect transversally, then \( f \) is linear when restricted to the convex hull of \( l_x \cup l_y \).

**Proof.** Suppose \( l_x \) and \( l_y \) are line segments. Let us denote the convex hull of \( l_x \cup l_y \) by \( S \). By subtracting a linear function from \( f \), we can assume that \( f \) restricted to \( l_x \cup l_y \) is zero. Since \( f \) is convex, it follows by definition, that \( f \) is non-positive on \( S \). Also, note that for any point \( p \) in \( S \), we can find a point \( q \) in \( l_x \cup l_y \) such that the segment \( pq \) intersects \( l_x \cup l_y \) in a point \( r \) with \( q \) lying between \( p \) and \( r \) (see Figure 4.1). Then, since \( f(q) = f(r) = 0 \) and \( f(p) \leq 0 \), we must have \( f(p) = 0 \) since \( f \) is convex when restricted to \( pr \). Thus, \( f \) is identically zero on \( S \).

In general if \( l_x \) and \( l_y \) are convex sets, then we can apply the previous argument to all pairs of line segments contained in \( l_x \) and \( l_y \) which intersect transversally.

Let \( \mathcal{L}(f) = 0 \) and let \( E \subset P^o \) (the interior of \( P \)) be the set of \( x \) such that \( f \) is linear when restricted to \( l_x \). The complement of \( E \) in \( P \) has measure zero.

\(^1\)By two convex sets intersecting transversally, we mean that their interiors (in the case of a line segment the complement of its endpoints) intersect.
Figure 4.1: Since \( f(q) = f(r) = 0 \), by convexity \( f(p) = 0 \).

For such an \( l_x \) if there is another \( l_y \) which intersects it transversally, then \( f \) is linear on the convex hull of \( l_x \cup l_y \) and thus linear on a neighbourhood of \( x \). We obtain that either there is a line \( L \) as above, or \( f \) is linear on \( P \). The proof is thus complete.

### 4.2.3 Semistable surfaces

Suppose we decompose \( P \) into subpolygons \( Q_i \). On each \( Q_i \) we have the Lebesgue measure \( d\mu \) and also a measure \( d\sigma_i \) on \( \partial Q_i \) which is the restriction of \( d\sigma \) (ie. it is equal to \( d\sigma \) on edges of \( Q_i \) which are subsets of edges of \( P \) and is zero on edges of \( Q_i \) which lie on the interior of \( P \)). For any bounded function \( A \) on \( Q_i \) we can define the functional

\[
\mathcal{L}_A(f) = \int_{\partial Q_i} f d\sigma_i - \int_{Q_i} Af d\mu.
\]

There is a unique affine linear function \( A \) for which \( \mathcal{L}_A(H) = 0 \) for all affine linear \( H \). Let us say that \( (Q_i, d\sigma_i) \) is relatively K-polystable if \( \mathcal{L}_A(f) > 0 \) for all non-affine convex functions \( f \). This is the same as relative K-polystability of the pair \( (V_{Q_i}, D_i) \) where \( V_{Q_i} \) is the variety corresponding to \( Q_i \) and \( D_i \) is the divisor corresponding to the edges of \( Q_i \) where \( d\sigma_i \) vanishes. In [10], Donaldson conjectured that if \( (Q_i, d\sigma_i) \) is relatively K-polystable then there exists a complete extremal metric on \( V_{Q_i} \setminus D_i \). Donaldson then suggested that if the toric variety corresponding to a polygon \( P \) is not K-polystable, then there should be a canonical decomposition of \( P \) into subpolygons \( Q_i \), with measures
$d\sigma_i$ induced by the measure on $\partial P$ as above and such that each $Q_i$ should either be relatively K-polystable, or be a parallelogram in which two opposite edges lie on edges of $P$. We prove this in the case of a K-semistable polygon.

**Theorem 4.2.7.** A K-semistable polygon has a canonical decomposition into rational subpolygons, each of which is either K-polystable or is a parallelogram in which two opposite edges lie on edges of $P$.

Suppose that $P$ is K-semistable, so that $L(f) \geq 0$ for all convex functions on $P$. Recall that we have associated to each point $x \in P$ a convex set $l_x$ containing $x$, which is the convex hull of a subset of $\partial P$. Let $F$ be the set of line segments $l$ joining points on the boundary of $P$ (and passing through the interior of $P$) such that

$$\mu(\{x \in P, l \text{ intersects } l_x \text{ transversally}\}) = 0.$$ 

From the discussion in the previous subsection we see that $F$ is the set of possible creases of a simple piecewise linear function $h$ such that $L(h) = 0$.

**Lemma 4.2.8.** If $l_1, l_2$ are line segments in $F$ then $l_1$ and $l_2$ cannot intersect.

**Proof.** Let $C$ be the convex hull of the union $l_1 \cup l_2$. Suppose $l_1$ and $l_2$ intersect transversally in the interior. For any $x$ in the interior of $C$ which doesn’t lie on $l_1$ or $l_2$ we have that $l_x$ intersects $l_1$ or $l_2$ transversally. Therefore we cannot have both $l_1, l_2 \in F$.

If $l_1$ and $l_2$ intersect on the boundary of $P$, in a point $y$, say, then for any $x$ in the interior of $C$ we have that $l_x$ either passes through $y$ or intersects $l_1$ or $l_2$ transversally. The set of $x$ with $l_x$ passing through $y$ but not intersecting $l_1$ or $l_2$ transversally must have measure zero, otherwise Equation 4.2 could not hold since the $\sigma$-measure of $y$ is zero. Therefore again, we cannot have both $l_1, l_2 \in F$. \(\square\)

The line segments $l \in F$ are therefore a set of disjoint line segments in $P$. Suppose now that $h$ is a simple piecewise linear function with crease $l$, so that $L(h) = 0$. In [10] (Section 6) Donaldson shows the following:

1. If one of the endpoints $x$ of $l$ is a vertex of $P$ then either the other endpoint is rational, or the other endpoint lies on an edge $J$ of $P$ such that every other line segment joining $x$ and $J$ is in $F$.

2. If $l$ joins two edges of $P$ which are not parallel, then $l$ is a rational line (its endpoints are rational).
3. If $l$ joins two parallel edges then either it is a rational line, or all other line segments parallel to $l$ joining the same two edges are in $F$.

The second possibility in case (1) is not possible because of Lemma 4.2.8. On the space of line segments joining two fixed edges of the polygon, the functional $\mathcal{L}$ is a polynomial. Therefore it can only have finitely many isolated zeroes. It follows that the set $F$ consists of finitely many rational line segments, and a finite number of families joining parallel edges of $P$ as in case (3). We therefore obtain the decomposition we were after into rational subpolygons $Q_i$. Note that the measure decomposition in Equation 4.2 can be restricted to the $Q_i$ because for almost every point $x \in Q_i$ the line segment $l_x$ lies inside $Q_i$ by our construction. This shows that each pair $(Q_i, d\sigma_i)$ is K-polystable. The argument also shows that the decomposition is canonical.
Chapter 5

Ruled manifolds

In this section we study extremal metrics on ruled manifolds. In Section 5.1 we summarise the momentum construction of Hwang-Singer [18] for writing down circle invariant metrics on a ruled manifold starting with a function of one variable (the momentum profile). Using this description in Section 5.2 we will write down a sequence of degenerating metrics which models the deformation to the normal cone of the zero section differential geometrically. To justify this we compute the asymptotics of the Mabuchi functional along this sequence of metrics and compare it to the Futaki invariant of the deformation to the normal cone.

We then concentrate on a ruled surface and construct explicit extremal metrics on it in Section 5.3 including complete extremal metrics on the complement of a divisor. The results fit in with the stability calculation in Section 3.2. We then use these extremal metrics to compute the infimum of the Calabi functional for the unstable polarisations by writing down a degenerate metric which achieves this infimum. The fact that it is the infimum follows because it gives equality in Donaldson’s Theorem 3.3.1.

5.1 Summary of the momentum construction

We briefly recall the momentum construction of circle invariant metrics on line bundles. The reference for this section is Hwang-Singer [18]. Let $(M, \omega_M)$ be a Kahler manifold of dimension $m$ and $(L, h)$ a Hermitian holomorphic line bundle over $M$. Let $\gamma = -\sqrt{-1} \partial \bar{\partial} \log h$ be the curvature form of $h$. Let $t = \log h$ be the logarithm of the fibrewise norm function. We want to consider
Kähler metrics on the total space of $L$ of the form

$$\omega = p^*\omega_M + 2\sqrt{-1}\partial\bar{\partial}f(t),$$  \hspace{1cm} (5.1)$$

where $p : L \to M$ is the projection map and $f$ is a suitably convex smooth function. Let us define $\tau = f'(t)$. For each $f$, the metric $\omega$ is invariant under the $S^1$ action rotating the fibres of $L$, and $\tau$ is the moment map for this action. Let $I \subset \mathbb{R}$ be the image of this moment map. Let $X$ be the generator of the $S^1$ action normalised so that $\exp(2\pi X) = 1$. The function $\|X\|_{\omega}$ is constant on level sets of $\tau$ so we can define a function $\phi : I \to \mathbb{R}$ such that $\phi(\tau) = \|X\|^2_{\omega}$. This function $\phi$ is called the momentum profile of the metric. We can reconstruct $f$ from $\phi$; in fact $t$ and $\tau$ are related by the Legendre transform with respect to $f$ and the Legendre transform $F$ of $f$ satisfies $F'' = 1/\phi$. This means that $t = F'(\tau)$, and

$$F(\tau) + f(t) = t\tau.$$  

We can also express the metric on the fibres using $\phi$, namely it is

$$\phi(\tau)\frac{|dz|^2}{|z|^2},$$

where $z$ is a coordinate on the fibre. In other words, $\phi$ gives the conformal factor relating the restriction of $\omega$ to the fibres, to the cylindrical metric. The advantage of this transformation is that in terms of $\phi$ the scalar curvature of $\omega$ is a second order linear differential expression, so we can compute with it conveniently.

We would now like to start with a momentum profile on an interval, and define a metric. For this we first need the following data.

**Definition 5.1.1.** *Horizontal data* $(p : (L,h) \to (M,\omega_M), I)$ consists of a Hermitian holomorphic line bundle over a Kähler manifold as above, together with a compatible momentum interval $I \subset \mathbb{R}$. The interval $I$ is compatible if for all $\tau \in I$ the form $\omega_M(\tau) = \omega_M - \tau\gamma$ is positive.

Given horizontal data, define a momentum profile to be a smooth function $\phi : I \to [0,\infty)$, which is positive on the interior of $I$. Define the ruled manifold $X = P(L \oplus O)$, and let $S_0, S_\infty$ be the zero and infinity sections. By the Legendre transform as above we obtain a function $f$ from $\phi$, and using we define a metric $\omega_\phi$ on a subset of $X$ with properties as follows. We assume for simplicity that the momentum profile is a rational function since that is all that
we need for applications.

**Theorem 5.1.2 (see [18]).** Let \( I = [a, b] \), and suppose \( \phi(a), \phi(b) = 0 \) and \( \phi \) is a rational function positive on \((a, b)\). The metric corresponding to the momentum profile \( \phi \) has the following properties depending on the boundary conditions of \( \phi \):

\[
\begin{align*}
\phi'(a) &= 2, \quad \phi'(b) = -2 \quad \text{smooth metric on } X, \\
\phi'(a) &= 0, \quad \phi'(b) = -2 \quad \text{complete metric on } X \setminus S_0, \\
\phi'(a) &= 2, \quad \phi'(b) = 0 \quad \text{complete metric on } X \setminus S_\infty, \\
\phi'(a) &= 0, \quad \phi'(b) = 0 \quad \text{complete metric on } X \setminus \{S_0 \cup S_\infty\}.
\end{align*}
\]

We now show that in the case of a complete metric if the order of vanishing of the momentum profile is precisely 2, then the metric is asymptotically hyperbolic as in Definition 3.1.8.

**Theorem 5.1.3.** Suppose \( I = [0, 1] \) and we have a momentum profile \( \phi \) such that \( \phi'(0) = 0 \) and \( \phi''(0) > 0 \). Then near the divisor \( \tau^{-1}(0) \) the metric is asymptotically hyperbolic.

**Proof.** Let us assume for simplicity that \( \phi''(0) = 2 \). We can then write

\[
\phi(\tau) = \tau^2 + a\tau^3 + O(\tau^4)
\]

for small \( \tau \) with some constant \( a \). In this proof by \( O(\tau^k) \) we do not just mean bounded by \( C\tau^k \) for some constant \( C \), but that the function has a convergent Taylor expansion with terms of order at least \( k \) for small \( \tau \). We have

\[
\frac{1}{\phi(\tau)} = \frac{1}{\tau^2(1 - a\tau)} + O(1).
\]

Integrating this, by the definition of the Legendre transform we get

\[
t = -\frac{1}{\tau} - a \log \tau + O(\tau). \tag{5.2}
\]

Integrating once more gives

\[
F(\tau) = -\log \tau - a\tau \log \tau + a\tau + O(\tau^2).
\]

The Kähler potential of the fibre metric is the Legendre transform of \( F \) so we obtain

\[
f(t) = t\tau - F(\tau) = \log \tau - 1 + O(\tau).
\]

68
From Equation 5.2 we obtain

\[ \log t = -\log \tau + \log(1 + a\tau \log \tau + O(\tau^2)). \]

With the previous formula for \( f(t) \) this implies

\[ f(t) + \log t = -1 + \log(1 + a\tau \log \tau + O(\tau^2)) + O(\tau) \quad (5.3) \]

with slight abuse of notation. Since \(-\log t\) is the Kähler potential of the hyperbolic cusp, to prove that our metric is asymptotically hyperbolic we need to show that the terms on the right hand side of Equation 5.3 have covariant derivatives whose norms tend to zero as \( z \to 0 \) (recall that \( t = \log |z| \)).

The norm squared \( \|dt\|^2 \) with respect to the hyperbolic metric is \( ct^2 \) for some constant \( c \).

The worst term on the right of Equation 5.3 is \( \tau \log \tau \), since the other terms are products of powers of \( \tau \log \tau \) with powers of \( \tau \). To prove the result it is therefore enough to show that for all \( k > 0 \)

\[ t^k \frac{\partial^k}{\partial \tau^k} (\tau \log \tau) \to 0, \quad \text{as } \tau \to 0. \]

By the chain rule \( \frac{\partial}{\partial \tau} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \) and since \( \frac{\partial}{\partial t} = \frac{1}{\phi(\tau)} \), we get \( \frac{\partial}{\partial \tau} = \phi(\tau) \frac{\partial}{\partial t} \).

By induction one can show that the term with smallest order of vanishing in \( \frac{\partial^k}{\partial \tau^k} (\tau \log \tau) \) is \( \tau^{k+1} \log \tau \). From Equation 5.2 we see that \( t\tau \) is bounded as \( \tau \to 0 \) and also \( \tau \log \tau \) tends to zero as \( \tau \to 0 \), hence

\[ t^k \tau^{k+1} \log \tau = (t\tau)^k \cdot \tau \log \tau \to 0, \quad \text{as } t \to 0. \]

This completes the proof. \( \square \)

Note that if the order of vanishing of \( \phi \) is greater than two, then the resulting metric is no longer asymptotically hyperbolic, but instead the fibre metrics are asymptotic to

\[ \frac{|dz|^2}{|z|^2 (\log |z|)^{k/(k-1)}} \]

for some \( k > 2 \).

Define \( Q : I \times M \to \mathbb{R} \) by \( Q(\tau) = \omega_M(\tau)^m / \omega_M^m \). The area of the fibres of \( X \) is \( 2\pi(b-a) \) and the volume of the zero section is \( Q(a)Vol(M, \omega_M) \). As for the scalar curvature of this metric \( \omega_\phi \), we have

**Theorem 5.1.4.** Let \( S_M(\tau) \) denote the scalar curvature of the metric \( \omega_M(\tau) \).
Then the scalar curvature of $\omega_\phi$ is given by

$$S(\omega_\phi) = S_M(\tau) - \frac{1}{2Q} \frac{\partial^2}{\partial \tau^2} (Q\phi)(\tau).$$

Since $\tau : X \to I$ is a moment map, we can consider the symplectic reductions $M_c = \tau^{-1}(c)/S^1$ for $c$ in the interior of $I$. From this point of view $\omega_M(c)$ and $S_M(c)$ give the induced Kähler form and its scalar curvature on $M_c$.

### 5.2 A metric degeneration

We will first construct a family of metrics on $\mathbb{P}^1$ using Kähler potentials, and then use their momentum profiles to define a sequence of metrics on a ruled manifold and we compute the asymptotic rate of change of the Mabuchi functional.

We define the sequence using Kähler potentials on $\mathbb{C} \setminus \{0\}$. Let $t = \log |z|$, and let $g(t)$ be the Kähler potential of a cusp metric on $\mathbb{P}^1$ minus a point. We can take $g(t)$ to be a strictly convex smooth real valued function such that $g(t) = -\log t$ for $t \gg 1$ and $g(t)$ is asymptotically $-ct$ as $t \to -\infty$ (meaning that $g(t) + ct$ converges to a constant as $t \to -\infty$), where $2\pi c$ is the area.

Let

$$g_s(t) = \begin{cases} g(t + s) - g(s + \frac{1}{s}), & t < 0 \\ 0, & t > 1/s \end{cases}$$

and let $g_s$ be smooth and strictly convex on $(0, 1/s)$. Define $h(t)$ to be smooth, $h(t) = 0$ for $t < 0$, and strictly convex for $t > 0$, such that $h(t)$ is asymptotically $dt$ for $t \gg 1$. Let

$$f_s(t) = h(t) + g_s(t) + ct.$$  

This is a strictly convex function on $\mathbb{R}$ and defines a metric of area $2\pi(c + d)$ on $\mathbb{P}^1$. As $s \to \infty$, the potential $f_s$ approaches the potential of a cusp metric of area $2\pi c$ on the interval $(-\infty, 0)$, so the $\mathbb{P}^1$ breaks up into two pieces.

Let us see what the corresponding momentum profiles look like. By definition $\tau_s = \frac{df_s}{dt}$, which changes with $s$, but we will normally drop the subscript $s$. The momentum profile $\phi$ is defined by $\phi_s(\tau) = 1/(F_s)'$, where $F_s$ is the Legendre transform of $f_s$. For each $s$, the momentum profile is a non-negative smooth function on the interval $[0, c + d]$, positive on the interior and $\phi_s'(0) = 2, \phi_s'(c + d) = -2$. As $s \to \infty$, we have $\phi_s(c), \phi_s'(c) \to 0$.

We can use these momentum profiles to define metrics on our ruled manifold.
We assume for simplicity that \( c + d \) is small enough so that the interval \([0, c + d]\) is a compatible momentum interval.

**Remark 5.2.1.** In general we expect that we could allow \( c + d \) to be as large as the Seshadri constant of the zero section by possibly changing our choice of \( \omega_M \) and \( h \), but we do not wish to discuss this in this thesis.

Let \( \omega_s \) be the metric corresponding to \( \phi_s \), and \( S(\omega_s) \) the scalar curvature given by Theorem 5.1.4 as

\[
S(\omega_s) = S_M(\tau) - \frac{1}{2Q} \frac{\partial^2}{\partial \tau^2}(Q\phi_s)(\tau).
\]

We observe that the scalar curvature is uniformly bounded as \( t \) varies since the \( \phi_s \) are bounded in \( C^2 \).

We would now like to compute the rate of change of the Mabuchi functional as \( s \to \infty \). The change in the Mabuchi functional is defined by

\[
\frac{d}{ds} M(\omega_s) = - \int_X \frac{df^*_s}{ds}(S(\omega_s) - \hat{S}) \frac{\omega^n_s}{n!},
\]

where \( \hat{S} \) is the average scalar curvature of \( X \), and \( n \) is the dimension of \( X \), ie. \( n = m + 1 \).

We have

\[
\frac{df_s(t)}{ds} = \begin{cases} 
g'(t + s) - (1 - s^{-2}) g'(s + s^{-1}), & t < 0, \\
0, & t > 1/s,
\end{cases}
\]

and also

\[
\tau_s = \frac{df_s(t)}{dt} = \begin{cases} 
g'(t + s) + c, & t < 0, \\
h'(t) + c, & t > 1/s.
\end{cases}
\]

We see that as \( s \to \infty \), the limit of the integrand is (writing \( \tau \) for \( \tau_\infty \))

\[
(\tau - c)(S(\omega_\infty) - \hat{S}), \quad \tau < c
\]

\[
0, \quad \tau > c.
\]

Here \( S(\omega_\infty) \) is defined by the formula (5.4) for \( \phi_\infty = \lim_{t \to \infty} \phi_t \), considered as functions on \([0, c + d]\) (note that \( \tau_t \) changes with \( t \)). Although \( \omega_\infty \) is a singular metric, \( S(\omega_\infty) \) is a continuous function on \( X \) and the volume form is given by

\[
\frac{\omega^n_\infty}{n!} = Q(\tau)d\tau \wedge d\theta \wedge \frac{\omega^m_M}{m!},
\]

71
which is bounded ($d\theta$ is the angular measure on the fibres). Since the convergence of the integrands is uniform (it is important here that the scalar curvature remains uniformly bounded as the metric degenerates), we can simply integrate the limit. We therefore find that

$$\lim_{s \to \infty} \frac{d}{ds} M(\omega_s) = \int_X (c - \tau)^+ (S(\omega_\infty) - \hat{S}) \omega_n^\pm / n!,$$

(5.5)

where $(c - \tau)^+ = \max\{c - \tau, 0\}$. We would like to show that up to a scalar multiple this is the Futaki invariant of an algebraic test-configuration.

### Deformation to the normal cone

Let $(X, \mathcal{L})$ be a polarised variety (the line bundle here is curly $\mathcal{L}$ to differentiate it from the line bundle $L$ over $M$ in the previous section). Deformation to the normal cone of a subscheme of $X$ was studied by Ross and Thomas [30]. Let us recall the construction. Suppose $Z \subset X$ is a subscheme, and define a test configuration $\mathcal{X} \to \mathbb{C}$ obtained by blowing up $X \times \mathbb{C}$ along $Z \times \{0\}$. Denote the exceptional divisor by $P$. The $\mathbb{C}^*$-action on $\mathcal{X}$ is induced by the product action on $X \times \mathbb{C}$ acting trivially on $X$ and by multiplication on $\mathbb{C}$. The central fibre of this test configuration can be written as $\hat{X} \cup E P$, where $\hat{X}$ is the blowup of $X$ along $Z$ with exceptional divisor $E$. If both $X$ and $Z$ are smooth, then $P = \mathbb{P}(\nu \oplus \mathcal{O})$, the projective completion of the normal bundle $\nu$ of $Z$ in $X$, and $E = \mathbb{P}(\nu)$.

There is a choice of line bundles on $\mathcal{X}$. Let $\pi$ denote the composition

$$\pi : \mathcal{X} \to X \times \mathbb{C} \to X.$$

For a positive rational number $c$ let $\mathcal{L}_c$ be the $\mathbb{Q}$-line bundle $\pi^*(\mathcal{L}) - cP$. The restriction of this to the general fibre of the test configuration is $\mathcal{L}$ and it is ample for sufficiently small $c$. In fact it is ample for $c < \epsilon(Z)$, where $\epsilon(Z)$ is the Seshadri constant of $Z$ (see [30]).

In [30] the Futaki invariant of this test-configuration is computed. Before stating the result we need some more definitions. Let $\mathcal{I}_Z$ be the ideal sheaf of $Z$, and for a fixed $x \in \mathbb{Q}_{>0}$ define $\alpha_i(x)$ by

$$\chi(\mathcal{L}^k \otimes \mathcal{I}_Z^{xk} / \mathcal{I}_Z^{x(k+1)}) = \alpha_1(x)k^{n-1} + \alpha_2(x)k^{n-2} + O(k^{n-3}), \quad k \gg 0, xk \in \mathbb{N}.$$
Define the slope of $X$ by

$$\mu(X) = -\frac{nK_X \cdot L^{n-1}}{2L^n}.$$  

The Futaki invariant is then

$$F(\mathscr{X}) = \int_0^c (c-x)\alpha_2(x)\,dx + \frac{c}{2} \alpha_1(0) - \left( \int_0^c (c-x)\alpha_1(x)\,dx \right) \mu(X).$$

When $Z$ is a divisor we can use the Riemann-Roch formula to compute

$$\alpha_1(x) = \frac{Z(L-xZ)^{n-1}}{(n-1)!}, \quad \alpha_2(x) = -\frac{Z(K_X+Z)(L-xZ)^{n-2}}{2(n-2)!}. $$

Recall now the ruled manifold $X = \mathbb{P}(L \oplus \mathcal{O})$ we defined before, where $p : L \to M$ is a line bundle over a Kähler manifold $(M, \omega_M)$. Let the $\mathbb{Q}$-polarisation $L$ over $X$ be given by a metric with momentum interval $[0, c+d]$ for rational $c, d > 0$. Consider the deformation to the normal cone $\mathscr{X}$ of the zero section $S_0$ with parameter $c$. We now show that its Futaki invariant is up to a positive multiple the asymptotic rate of change of the Mabuchi functional in Equation 5.5.

**Theorem 5.2.2.** Using the notation from Equation 5.5, we have

$$\lim_{s \to \infty} \frac{d}{ds} M(\omega_s) = \int_X (c-\tau)^+ (S(\omega_s) - \hat{S}) \frac{\omega^n}{n!} = 2(2\pi)^n F(\mathscr{X}).$$

**Proof.** Let us write

$$A = \int_X (c-\tau)^+ \frac{\omega^n}{n!}, \quad B = \int_X (c-\tau)^+ S_M(\tau) \frac{\omega^n}{n!},$$

$$C = \int_X (c-\tau)^+ \frac{1}{Q} \frac{\partial^2}{\partial\tau^2} (Q\phi_\infty)(\tau) \frac{\omega^n}{n!},$$

so we need to show $F(\mathscr{X}) = B - C/2 - \hat{S}A$.

First of all the average scalar curvature $\hat{S}$ is $2\mu(X)$. Let us compute $A$. The volume form is $\frac{\omega_M(\tau)^{n-1}}{(n-1)!} \wedge d\tau \wedge d\theta$, where $d\theta$ is the angular measure on the fibres. The integrand is constant on the $S^1$ fibres and also on the level sets of $\tau$, so

$$A = 2\pi \int_0^c (c-\tau) \text{Vol}(M, \omega_M(\tau)) \,d\tau.$$  

The volume $\text{Vol}(M, \omega_M(\tau))$ is the volume of the zero section with respect to the metric $\omega_M - \tau \gamma$, i.e. $(2\pi)^{n-1} S_0(L - \tau S_0)^{n-1}/(n-1)!$, which is just
\((2\pi)^{n-1}\alpha_1(\tau)\), since \(Z = S_0\). We therefore have

\[
A = (2\pi)^n \int_0^c (c - x)\alpha_1(x) \, dx. \tag{5.6}
\]

Now let us move on to B. We have

\[
B = 2\pi \int_0^c (c - \tau) \int_M S_M(\tau) \frac{\omega_M(\tau)^{n-1}}{(n-1)!} \, d\tau.
\]

The integral over \(M\) is the total scalar curvature of \(M\) with the polarisation \(\mathcal{L} - \tau S_0\), which using the adjunction formula is

\[
-(2\pi)^{n-1} \frac{K_{S_0}(\mathcal{L} - \tau S_0)^{n-2}}{(n-2)!} = -(2\pi)^{n-1} \frac{(K_X + S_0)S_0(\mathcal{L} - \tau S_0)^{n-2}}{(n-2)!}
\]

\[
= 2(2\pi)^{n-1}\alpha_2(\tau),
\]

so that we have

\[
B = 2(2\pi)^n \int_0^c (c - x)\alpha_2(x) \, dx. \tag{5.7}
\]

Finally let us compute C.

\[
C = 2\pi \int_M \int_0^c (c - \tau) \frac{\partial^2}{\partial \tau^2} (Q\phi_{\infty})(\tau) \, d\tau \frac{\omega_M^{n-1}}{(n-1)!},
\]

using that \(\omega_M(\tau)^{n-1} = Q(\tau)\omega_M^{n-1}\). We can integrate the inner integral by parts remembering that \(\phi_{\infty}(0) = \phi_{\infty}(c) = 0\), \(\phi'_{\infty}(0) = 2\) and \(\phi''_{\infty}(c) = 0\). We get

\[
C = -4\pi c \int_M Q(0) \frac{\omega_M^{n-1}}{(n-1)!} = -2(2\pi)^n c\alpha_1(0). \tag{5.8}
\]

Putting together equations \((5.6)\), \((5.7)\) and \((5.8)\) we obtain the required result. 

This result shows that for a ruled manifold if the deformation to the normal cone of the zero section destabilises for \(c\) sufficiently small (cf. Remark \(5.2.1\)), then the Mabuchi functional is not bounded from below. In particular the manifold cannot admit a cscK metric, although we knew this from the result in Section \(5.3\) already. With this approach however we have a direct relationship between a metric degeneration and the asymptotics of the Mabuchi functional, and a corresponding algebro-geometric test-configuration and its Futaki invariant. As we mentioned before, understanding this relationship in general is an important problem. It should be possible to extend the calculation here to deformation to the normal cone of a smooth divisor in a general manifold, transferring the metrics we have constructed here to a suitable tubular neighbourhood of the
divisor. At the time of writing this thesis I have not yet worked out how to do this.

We now perform a similar calculation but with the more general test-configurations for toric bundles we constructed in Section 4.1. For this, note that \( X \) is a toric bundle with base \( M \) and fibre \((\mathbb{P}^1, \mathcal{O}(l))\) with moment “polytope” \([0, l]\). Let the principal \( \mathbb{C}^* \)-bundle \( P \) on \( M \) be the complement of the zero section in \( L^{-1} \) so that the polarisation we defined in Section 4.1 coincides with the one obtained from the momentum construction for the same interval. The ample line bundle \( L_M \) over \( M \) is a holomorphic line bundle with first Chern class \( [\omega_M] \). The functions \( Q_1, Q_2 : [0, l] \to \mathbb{R} \) are given by

\[
Q_1(\tau) = \frac{1}{(2\pi)^m m!} \int_M (\omega_M - \tau \gamma)^m = (2\pi)^{-m} \int_M Q(\tau) \omega_M^m/m!
\]

\[
Q_2(\tau) = \frac{1}{2(2\pi)^{m-1}(m-1)!} \int_M (\omega_M - \tau \gamma)^{m-1} \wedge c_1(M)
= \frac{1}{2(2\pi)^m} \int_M S_M(\tau) Q(\tau) \omega_M^m/m!.
\]

Now according to Theorem 4.1.2 any rational piecewise linear convex function \( h \) on \([0, l]\) defines a test-configuration \( X \) for \( X \), with Futaki invariant equal to

\[
F(\mathcal{X}) = \frac{h(0)Q_1(0) + h(l)Q_1(l)}{2} + \int_0^l h(\tau)Q_2(\tau) d\tau - \frac{a_1}{a_0} \int_0^l h(\tau)Q_1(\tau) d\tau,
\]

where \( \hat{S} = \frac{2 a_1}{a_0} \) is the average scalar curvature of \( X \). Let us define \( F(h) \) with the same formula for any piecewise smooth function \( h \).

**Theorem 5.2.3.** Let \( h \) be any piecewise smooth convex function on \([0, l]\) which is smooth on the intervals \([l_i, l_{i+1}]\) for some \( 0 = l_0 < l_1 < \ldots < l_N = l \). Let \( \phi \in C^2([0, l]) \) be non-negative, satisfying

\[
\phi(0) = \phi(l_i) = \phi(l) = 0 \text{ for all } i,
\]

\[
\phi'(0) = 2, \phi'(l) = -2.
\]

Suppose in addition that \( h \) is linear on any interval \([l_i, l_{i+1}]\) on which \( \phi \) does not vanish identically. We then have

\[
F(h) = \frac{1}{2(2\pi)^n} \int_X h(\tau)(S(\omega) - \hat{S}) \omega^\phi_{\phi} / n!
\]

(5.9)

where \( \omega \phi \) is the singular metric corresponding to the “momentum profile” \( \phi \).
Note that $h$ does not define a test-configuration since it is not piecewise linear, and also $\phi$ does not define a metric since it vanishes on a subset of $(0, l)$. On the other hand $h$ can be uniformly approximated with piecewise linear functions, and $\phi$ can be approximated in $C^2$ with momentum profiles which do define metrics. Equation 5.9 will then hold in the limit for such approximating sequences of test-configurations and metrics. This is the setting in which the result will be used in Section 5.4.

Proof. As in the previous proof, we write

$$A = \int_X h(\tau) \frac{\omega^n}{n!}, \quad B = \int_X h(\tau) S_M(\tau) \frac{\omega^n}{n!},$$

$$C = \int_X h(\tau) \frac{1}{Q} \frac{\partial^2}{\partial \tau^2} (Q \phi_\infty) \frac{\omega^n}{n!},$$

and we need to show $F(\mathcal{X}) = B - C/2 - \hat{S} A$. In the same way as above, we get

$$A = (2\pi)^n \int_0^l h(\tau) Q_1(\tau) \ d\tau,$$

and also

$$B = 2(2\pi)^n \int_0^l h(\tau) Q_2(\tau) \ d\tau,$$

since $Q_2(\tau)$ is the total scalar curvature of $M$ with polarisation $c_1(L_M) - \tau c_1(L)$.

Also as before,

$$C = 2\pi \int_M \int_0^l h(\tau) \frac{\partial^2}{\partial \tau^2} (Q \phi_\infty)(\tau) \ d\tau \frac{\omega^{n-1}}{(n-1)!}.$$

Integrating by parts, using the assumption that $h$ is linear on $[l_i, l_{i+1}]$ if $\phi$ does not vanish there, we get that

$$\int_{l_i}^{l_{i+1}} h(\tau) \frac{\partial^2}{\partial \tau^2} (Q \phi_\infty) \ d\tau = \begin{cases} -2Q_1(0)h(0) & \text{if } i = 0, \\ 0 & \text{if } 0 < i < N - 1, \\ -2Q_1(l)h(l) & \text{if } i = N - 1. \end{cases}$$

Summing up, we obtain

$$C = -2(2\pi)^n (Q_1(0)h(0) + Q_1(l)h(l)).$$

Putting everything together, we obtain the result. \qed
5.3 Extremal metrics on ruled surfaces

We now specialise to a ruled surface. The base manifold $M$ is now a genus 2 curve equipped with a metric $\omega_M$ of constant scalar curvature and area $2\pi$, and $L$ is a degree $-1$ line bundle (this gives the same variety as a degree 1 line bundle that we used in Section 3.2 but enables us to use momentum profiles on $[0,m]$ instead of $[-m,0]$). The variety $X$ is the ruled surface $\mathbf{P}(L \oplus \mathcal{O})$. We pick a Hermitian metric on $L$ with curvature form $i\omega_M$. Using the description of circle-invariant metrics on ruled manifolds in Section 5.1 we now construct extremal metrics on $X$. This was done by Tønnessen-Friedman in [36], but we also construct complete metrics on $X \setminus S_0$ and $X \setminus S_\infty$. We choose the polarisation $\mathcal{L} = C + mS_\infty$ where $C$ is a fibre as in Section 5.2. This is equivalent to working on the momentum interval $[0,m]$. The function $Q$ needed in the formula for the scalar curvature (Theorem 5.1.4) is given by $Q(\tau) = 1 + \tau$, so the expression for the scalar curvature is

$$S(\omega_\phi) = \frac{1}{2(1 + \tau)}(-4 - [(1 + \tau)\phi]''),$$

where $\omega_\phi$ is the metric corresponding to a momentum profile $\phi : [0,m] \to \mathbb{R}$.

In order to find extremal metrics, we therefore need to find momentum profiles $\phi : [0,m] \to \mathbb{R}$ satisfying various boundary conditions, and solving the ODE

$$\frac{1}{2(1 + \tau)}(-4 - [(1 + \tau)\phi]'' = A\tau + B$$

for some constants $A, B$ since the gradient of a function $h(\tau)$ is holomorphic if and only if $h$ is linear. More explicitly,

$$[(1 + \tau)\phi]'' = -2A\tau^2 - 2(A + B)\tau - 2B - 4,$$

$$[(1 + \tau)\phi]' = -\frac{2A\tau^3}{3} - (A + B)\tau^2 - 2B\tau - 4\tau + C,$$

$$(1 + \tau)\phi = -\frac{A\tau^4}{6} - \frac{(A + B)\tau^3}{3} - B\tau^2 - 2\tau^2 + C\tau + D,$$

for some constants $C, D$. The resulting function defines a metric if it is positive on $(0,m)$.

Let us start with the case of a smooth metric on $X$. The boundary conditions are $\phi(0) = \phi(m) = 0$, $\phi'(0) = 2$, $\phi'(m) = -2$. Solving the resulting system of
linear equations for $A, B, C, D$, we obtain

$$\phi(\tau) = \frac{2\tau(m - \tau)}{m(m^2 + 6m + 6)(1 + \tau)} (\tau^2(2m + 2) + \tau(-m^2 + 4m + 6) + \tau^2 + m^2 + 6m + 6).$$

This will be positive on $(0, m)$, if and only if the quadratic expression (in $\tau$) in brackets is positive on this interval. This is the case for $m < k_1$, where $k_1$ is the only positive real root of the quartic $m^4 - 16m^3 - 52m^2 - 48m - 12$. Approximately $k_1 \approx 18.889$. This is also the result obtained by Tønessen-Friedman [36]. Figure 5.3 shows a plot of $\phi$ for $m = 17$. The scalar curvature is

$$S(\phi)(\tau) = \frac{24(m + 1)}{m(m^2 + 6m + 6)} \tau - \frac{6(3m^2 + 2m - 2)}{m(m^2 + 6m + 6)}.$$
the cubic $m^3 - 3m^2 - 9m - 6$. Approximately $k_2 \approx 5.0275$. Figure 5.3 shows a plot of $\phi$ for $m = 5$. For $m < k_2$ the order of vanishing of $\phi(\tau)$ at $\tau = m$ is precisely 2 so the metric is asymptotically hyperbolic near $S_\infty$. The scalar curvature is

$$S(\phi)(\tau) = \frac{12(m^2 - 2m - 3)}{m^2(m^2 + 6m + 6)} \tau - \frac{6(2m^2 - m - 4)}{m(m^2 + 6m + 6)}. \quad (5.10)$$

Figure 5.2: Momentum profile of an asymptotically hyperbolic extremal metric on $X \setminus S_\infty$, where $m = 5$

For a complete metric on $X \setminus S_0$ the boundary conditions are $\phi(0) = \phi(m) = 0$, $\phi'(0) = 0$, $\phi'(m) = -2$. We obtain

$$\phi(\tau) = \frac{2\tau^2(m - \tau)}{m^2(m^2 + 6m + 6)(1 + \tau)} \left( \tau(2m^2 + 4m + 3) - m^3 + 3m^2 + 9m + 6 \right).$$

This is positive on $(0, m)$ if the linear term in brackets is positive on this interval. This is the case for $m \leq k_2$, where $k_2$ is the same as above. Again, for $m < k_2$ the order of vanishing of $\phi(\tau)$ at $\tau = 0$ is exactly 2, so the resulting metric is asymptotically hyperbolic near $S_0$. The scalar curvature is

$$S(\phi)(\tau) = \frac{12(2m^2 + 4m + 3)}{m^2(m^2 + 6m + 6)} \tau - \frac{6(3m^2 + 5m + 2)}{m(m^2 + 6m + 6)}. \quad (5.11)$$
Note that if $\phi$ is a solution with these boundary conditions, then $\psi$ defined by

$$
\psi(\tau) = (a + 1) \phi \left( \frac{\tau - a}{a + 1} \right)
$$

is a solution with the same boundary conditions on the interval $[a, (a+1)m+a]$. This also gives a complete extremal metric on $X \setminus S_0$, just in a different Kähler class. The corresponding scalar curvature is given by

$$
S(\psi)(\tau) = \frac{1}{a+1} S(\phi) \left( \frac{\tau - a}{a + 1} \right).
$$

We now summarise these results in a proposition which we will use in the next section.

**Proposition 5.3.1.** There exists a complete extremal metric on $X \setminus S_{\infty}$ with momentum profile on $[0, m]$ for $m \leq k_2$ and scalar curvature given by (5.10). When $m < k_2$ the resulting metric is asymptotically hyperbolic. There exists a complete extremal metric on $X \setminus S_0$ with momentum profile on $[c, m]$ for any positive $c$ and $m > c$ such that

$$
m - c \leq k_2,
$$

with scalar curvature given by (5.12). If the inequality is sharp, the resulting metric is asymptotically hyperbolic.

Note that the polarisations for which we have not obtained asymptotically hyperbolic extremal metrics are K-unstable according to the calculations in Section 3.2.

### 5.4 The infimum of the Calabi functional

In this section we compute the infimum of the Calabi functional on the ruled surface considered above, for the unstable polarisations. According to Theorem 5.3.1, a test-configuration $\chi$ which has negative Futaki invariant $F(\chi) < 0$ gives a lower bound

$$
\| S(\omega) - \hat{S} \|_{L^2} \geq 4\pi \frac{-F(\chi)}{\| \chi \|}
$$

on the Calabi functional. Donaldson conjectured in [12] that the supremum of this lower bound over all test-configurations gives the infimum of the Calabi functional. We will show that for our ruled surface this is indeed the case.
Theorem 5.4.1. For the ruled surface $X$ we have

$$\inf_{\omega} \|S(\omega) - \hat{S}\|_{L^2} = 4\pi \sup_{\chi} \frac{-F(\chi)}{\|\chi\|},$$

where $\chi$ runs over all test-configuration for $X$.

Proof. Let the polarisation of $X$ be $L = C + mS_\infty$ working on the momentum interval $[0, m]$ as before, and $m \geq k_1$ so that $(X, L)$ is relatively K-unstable. We will define a sequence of momentum profiles $\phi_i$ and a sequence of test-configurations $\chi_i$ corresponding to rational piecewise linear convex functions $h_i$ such that

$$\lim_{i \to \infty} \|S(\omega_{\phi_i}) - \hat{S}\|_{L^2} = \lim_{i \to \infty} 4\pi \frac{-F(\chi_i)}{\|\chi_i\|}. \quad (5.13)$$

It will follow that this limit is the infimum of the Calabi functional. We will define these sequences by writing down their limits. In the case of $\phi_i$ this means a $C^2$ momentum profile $\phi$ which may be zero on a subset of $(0, m)$ and in the case of $\chi_i$ it means a continuous convex function $h$ which is not necessarily rational and piecewise linear.

There are two cases to consider. The first is when $k_1 \leq m \leq k_2(k_2 + 2) \cong 35.33$ (recall the constants $k_1, k_2$ defined in the previous section). In this case we define a constant $c = \sqrt{m + 1} - 1$, and define

$$\phi(\tau) = \begin{cases} 
\psi_1(\tau) & \text{for } \tau \in [0, c], \\
\psi_2(\tau) & \text{for } \tau \in [c, m].
\end{cases}$$

where $\psi_1, \psi_2$ are the momentum profiles of the complete extremal metrics given by Proposition 5.3.1 so that $\phi$ satisfies

$$\phi(0) = 0, \phi'(0) = 2, \quad \phi(c) = \phi'(c) = 0, \quad \phi(m) = 0, \phi'(m) = -2.$$ 

Note that the assumption on $m$ and $c$ ensures that the intervals $[0, c]$ and $[c, m]$ satisfy the conditions of Proposition 5.3.1 so that $\psi_1, \psi_2$ exist. We can check explicitly using the formulae in the previous section, that our choice of $c$ ensures that $\psi_1''(c) = \psi_2'(c)$, so $\phi$ is in $C^2$. The key point is that the scalar curvature $S(\omega_\phi)$ is concave (by construction it is linear on the intervals $[0, c]$ and $[c, m]$), which we can also verify explicitly (here the assumption $m \geq k_1$ is used). This allows us to define $h(\tau) = \hat{S} - S(\omega_\phi)$ which is a convex piecewise linear function. To verify Equation 5.13 we apply Theorem 5.2.3 which implies
that the normalised Futaki invariant is
\[
\frac{1}{2(2\pi)^2} \int_X (S(\omega) - \hat{S})^2 \frac{\omega^n}{n!} = \frac{1}{2\pi} \|S(\omega) - \hat{S}\|_{L^2}.
\]

The second case is when \( m > k_2(k_2 + 2) \). We now define a constant
\[
c = \frac{m + 1}{k_2 + 1} - 1,
\]
and we define
\[
\phi(\tau) = \begin{cases} 
\psi_1(\tau), & \tau \in [0, k_2], \\
0, & \tau \in [k_2, c], \\
\psi_2(\tau), & \tau \in [c, m],
\end{cases}
\]
where again, \( \psi_1 \) and \( \psi_2 \) are given by proposition 5.3.1. Once again our choice of the intervals guarantees that \( \phi \) is non-negative and in \( C^2 \), and also the scalar curvature \( S(\omega) \) is concave. On the interval \([k_2, c]\) it is given by \( -\frac{2}{1 + \tau} \), so it is not piecewise linear. We choose a sequence \( h_i \) of piecewise linear convex functions converging to \( h = \hat{S} - S(\omega) \). Theorem 5.2.3 implies Equation 5.13 once again.

In the proof we have exhibited the “worst destabilising test-configuration” for each unstable polarisation which breaks up the manifold into pieces. This is analogous to the Harder-Narasimhan filtration of an unstable vector bundle. Note that most of the time (except for when \( \sqrt{m+1} \) is rational in the first case) these are not algebraic test configurations. The first case is deformation to the normal cone with a real parameter, and the unstable manifold is split into two pieces which admit (complete) extremal metrics. The second case is more complicated, and the middle piece does not admit an extremal metric. Instead its cylindrical fibres become infinitely long and thin.
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