Two Dimensional Quantum Gravity Coupled to Matter

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Abstract

A classical two dimensional theory of gravity which has a number of interesting features (including a Newtonian limit, black holes and gravitational collapse) is quantized using conformal field theoretic techniques. The critical dimension depends upon Newton’s constant, permitting models with $d = 4$. The constraint algebra and scaling properties of the model are computed.
1 Introduction

It has become clear in recent years that two dimensional quantum gravity has much to teach us, both in terms of obtaining a proper understanding of two dimensional systems and as a theoretical laboratory for the $(3 + 1)$ dimensional case. In the latter context it is hoped that the reduction in the number of degrees of freedom will yield a better understanding of short-distance problems, topology change, singularities and the cosmological constant problem. In the former context the problem of confronting two dimensional quantum gravity was avoided by considering massless matter systems. Classically, general coordinate invariance and Weyl invariance may be used to gauge fix all three degrees of freedom of the metric \cite{1}; upon quantization, Weyl invariance no longer holds except for particular values of the central charge of the matter (i.e. the critical dimensions). By restricting attention to these special cases the problem of quantizing the two dimensional gravitational field itself was avoided. This restriction was lifted when it was shown in light cone gauge that that such systems can be quantized in the presence of the Liouville mode of the metric for a genus zero surface \cite{2}. Shortly afterward it was shown how to obtain this result in conformal gauge for all genera \cite{3}. However, the central charge in such models is restricted to be less than or equal to one, and it is not clear how to recover the known critical cases.

A key problematic element in the above program is the choice of classical action for the gravitational field. The Einstein tensor vanishes in two dimensions for all metrics and the Einstein-Hilbert action simply yields the Euler number of the manifold. A number of suggestions have been made to address this problem \cite{4, 5, 6, 7, 8, 9, 10}. These typically involve taking the classical action to be trivial (either zero or the Einstein Hilbert action) and then gauge-fixing the symmetries of the metric to obtain a quantum action, or alternatively making use of topological interactions \cite{9}, a variant of which \cite{11} permits constant curvature solutions classically. In all cases the matter-gravity interaction is classically trivial, and quantum-mechanically quite limited.

However in the last few years it has been shown that classical gravity in two spacetime dimensions need not be so trivial \cite{12}. An interesting relativistic theory of gravitation in this context may be formed by setting the Ricci scalar $R$ equal to $T = T_{\mu}^{\mu}$, the trace of the conserved stress energy
where $G$ is Newton’s constant in two spacetime dimensions. In spite of its simplicity, this theory has a number of remarkable classical and semi-classical features, including a well-defined Newtonian limit [12], black holes [13, 14], a post-Newtonian expansion, gravitational waves, FRW cosmologies, gravitational collapse [15] and black hole radiation [16, 17, 18]. These features suggest that it is potentially a very useful tool in the study of quantum gravity: since its classical features are so similar to those of (3+1) dimensional general relativity, one might hope that its quantization would bear a similar resemblance to (3 + 1) dimensional quantum gravity. Indeed, it has been shown the system (1) along with the conservation of stress-energy

$$\nabla_\nu T^{\mu \nu} = 0$$

may be understood as the $D \rightarrow 1$ limit of the $(D + 1)$ dimensional Einstein equations [19]. The present paper is concerned with taking a first step towards quantization of this theory.

The system (1,2) may be derived from the action [17]

$$S = \frac{1}{8\pi G} \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} \partial_\mu \psi \partial_\nu \psi + \psi R - 8\pi G L_M \right)$$

which in turn yields the field equations

$$\frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} g^{\mu \nu} \partial_\nu \psi \right] - R = 0$$

$$\frac{1}{2} \left( \psi_\mu \psi_\nu - \frac{1}{2} g_{\mu \nu} \psi_\alpha \psi_\alpha \right) + g_{\mu \nu} g^{\alpha \beta} \psi_\alpha \psi_\beta - \psi_{;\mu ;\nu} = 8\pi G T_{\mu \nu}$$

plus the equations of motion for the matter fields. Insertion of the trace of (5) into (4) yields (1), and conservation of the stress energy tensor (eq. (2)) is guaranteed since the covariant divergence of the left hand side of (5) is identically zero when (4) holds. This also follows from considering the invariance of the action (3) under infinitesimal co-ordinate transformations, analogous to the (3 + 1)-dimensional case.

The action (3) with $T_{\mu \nu} = 0$ has been considered before [20, 21] in the context of finding a local action which produces the Polyakov theory. In
such cases the $\psi$-field is fully eliminated in terms of the metric degree of freedom by imposing boundary conditions which break reparametrization invariance, thereby inducing an anomaly into the Virasoro algebra. In the present case, $\psi$ is treated as an auxiliary field, since the classical evolution of the gravity/matter system is independent of the evolution of $\psi$. However the $\psi$ field obeys the equation

$$\frac{1}{2} \left( \psi_{;\mu} \psi_{;\nu} - \frac{1}{2} g_{\mu\nu} \psi_{;\alpha} \psi_{;\alpha} \right) + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \psi_{;\alpha;\beta} - \psi_{;\mu;\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \quad (6)$$

and so depends upon the evolution of the gravity/matter system (1,2).

Equations (4) and (5) form a system of 4 equations with 2 identities, which is equal to the number of unknowns (the metric degree of freedom and $\psi$). Reorganized in the form (1,2) (and (3)), one easily sees that upon setting $T_{\mu\nu} = g_{\mu\nu} \Lambda$ that they reduce to a proposal considered previously by Jackiw and Teitelboim for two dimensional gravity [5, 6], whose quantization was recently considered by Chamseddine [11]. Such a restriction permits only constant curvature solutions to the field equations. In contrast to this the theory based on (1) and (6) allows curvature and stress-energy to act upon one another in a manner which is very similar to the (3 + 1) dimensional situation.

Motivated by the above, the classical gravitational action is taken to be (3), and its quantum properties are considered herein. The Newtonian gravitational constant $G$ will be seen to significantly modify the relationship between the central charge and the conformal dimensions of the matter fields, permitting critical dimensions larger than 1.

## 2 Constraint Algebra

It is instructive to consider the constraint algebra which follows from (3). Working in conformal gauge, with $g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$ yields from (4) and (5)

$$\frac{1}{4} \left( \dot{\psi}^2 + (\psi')^2 \right) - \psi'' + \frac{1}{2} \dot{\phi} \dot{\psi} + \frac{1}{2} \dot{\phi'} \psi' = T_{00} \quad (7)$$

$$\frac{1}{2} \dot{\psi} \psi' + \frac{1}{2} \dot{\phi} \dot{\psi} + \frac{1}{2} \psi'' - \psi' = T_{01} \quad (8)$$
for the constraint equations and

\[
\frac{1}{4} \left( \dot{\psi}^2 + (\psi')^2 \right) - \ddot{\psi} + \frac{1}{2} \ddot{\phi} \psi + \frac{1}{2} \phi' \psi' = T_{11} \\
\ddot{\psi} - \psi'' + \ddot{\phi} - \phi'' = 0
\]

for the dynamical equations. Here \( \psi' \equiv \partial \psi / \partial x \) and \( \dot{\psi} \equiv \partial \psi / \partial t \). Subtracting (9) from (7) allows this latter set to be rewritten as

\[
\ddot{\phi} - \phi'' = - (T_{00} - T_{11})
\]

\[
\ddot{\psi} - \psi'' + \ddot{\phi} - \phi'' = 0.
\]

Following ref. [20], it is straightforward to show that

\[
P_\psi = - (\dot{\psi} + \dot{\phi})
\]

is the momentum conjugate to \( \psi \) and that the momenta conjugate to the metric is

\[
\Pi^{00} = 0 \quad \Pi^{01} = - \frac{1}{2} e^{-\phi} \psi' \quad \Pi^{11} = e^{-\phi} \dot{\psi}
\]

where \( \Pi^{\alpha \beta} \equiv \frac{\delta S}{\delta g_{\alpha \beta}} \). Clearly only \( P_\psi \) and \( \Pi^{11} \) are independent canonical variables in conformal gauge. It is useful to make a canonical transformation on these variables so that

\[
\chi = \psi + \phi \quad P_\chi = P_\psi = - \dot{\chi}
\]

and

\[
\Pi = e^\phi \Pi^{11} = \dot{\phi}.
\]

This guarantees that on a spatial slice

\[
\{ \chi(x), P_\chi(x') \} = \delta(x - x')
\]

\[
\{ \psi(x), \Pi(x') \} = \delta(x - x')
\]

are the canonical Poisson brackets.

Under this transformation the constraint equations (7) and (8) become respectively

\[
a_{\chi+}^2 - b_{\chi+} - (a_{\phi-}^2 + b_{\phi-}) + a_{\chi-}^2 + b_{\chi-} - (a_{\phi+}^2 + b_{\phi+}) = 2T_{00}
\]
\[-(a_{\chi+}^2 - b_{\chi+}) + (a_{\phi-}^2 + b_{\phi-}) + a_{\chi-}^2 + b_{\chi-} - (a_{\phi+}^2 - b_{\phi+}) = 2T_{01}\] (20)

where

\[a_{\chi\pm} \equiv \frac{1}{2} (P_{\chi \pm \chi'}) \quad a_{\phi\pm} \equiv \frac{1}{2} (P_{\phi \pm \phi'})\] (21)

and

\[b_{\chi\pm} \equiv 2a'_{\chi\pm} \quad b_{\phi\pm} \equiv 2a'_{\phi\pm}\] (22)

On spatial sections topologically equivalent to \(S^1\) these variables may be Fourier transformed so that

\[L_n^{\chi\pm} = \int_{-\pi}^{\pi} d\sigma e^{-in\sigma}(a_{\chi\pm}^2 \mp b_{\chi\pm})\] (23)

\[L_n^{\phi\pm} = \int_{-\pi}^{\pi} d\sigma e^{-in\sigma}(a_{\phi\pm}^2 \mp b_{\phi\pm})\] (24)

where the periodic spatial variable \(\sigma \in (-\pi, \pi)\). The Poisson brackets of these variables are

\[\{L_n^{\chi\pm}, L_m^{\chi\pm}\} = \pm i(n - m)\delta_{n+m,0}\] (25)

\[\{L_n^{\phi\pm}, L_m^{\phi\pm}\} = \pm i(n - m)\delta_{n+m,0}\] (26)

all other brackets being zero.

Adding the constraints (19) and (20) yields respectively

\[\mathcal{L}_n^{\chi\pm} = (\tilde{T}_{00}^{mn} \mp \tilde{T}_{01}^{mn})\] (27)

where \(\tilde{T}_{\mu\nu}^{mn}\) is the Fourier transform of the stress-energy tensor. The quantities

\[\mathcal{L}_n^{\chi\pm} \equiv L_n^{\chi\pm} - L_n^{\phi\mp}\] (28)

obey

\[\{\mathcal{L}_n^{\chi\pm}, \mathcal{L}_m^{\chi\pm}\} = i(n - m)\mathcal{L}_n^{\chi\pm\mp}\] (29)

for their Poisson bracket algebra.

Each of the \(\chi\), gravity sectors yields a classical Virasoro algebra with a non-zero central charge. However these cancel out in the combined \(\chi\)-gravity system, as is clear from (28,29). Classical coupling to conformally invariant matter will not affect this result, since the stress-energy tensor will in general have a Fourier decomposition in terms of operators which obey the Virasoro
algebra \( (29) \). For example, a massless scalar field \( \varphi \) will have stress-energy tensor components
\[
T_{00} \pm T_{01} = \frac{1}{2}(P_\varphi \pm \varphi')^2
\]
which upon Fourier decomposition will yield operators \( L^m_{\varphi \pm} \) whose Poisson brackets form a Virasoro algebra with vanishing central charge.

Other approaches to two-dimensional gravity typically impose additional constraints to the system described by the action \( (3) \) which introduce an anomaly at the classical level. For example, if the \( \psi \) degree of freedom is frozen out (i.e. if in \( (10) \) there are no homogenous solutions permitted for \( \psi \)) then \( \chi = 0 \) and consequently \( P_\chi = 0 \). The action \( (3) \) then becomes the non-local Polyakov action \( [1] \) and the constraint algebra develops an anomaly \( (29) \) due to this non-locality. Conformally coupling to two-dimensional matter and quantizing then permits one to cancel off this anomaly against the central charge of the matter \( [10] \). Alternatively, one could fix the two-dimensional metric \( g_{\mu \nu} = \hat{g}_{\mu \nu} \) which would freeze out the \( \phi \) degree of freedom. In this case the action \( (3) \) becomes that of the Liouville action \( [22] \). The field \( \psi \) is interpreted as the quantum-dynamical field whose exponential yields a conformal factor which provides the quantum corrections to \( \hat{g}_{\mu \nu} \) i.e. the full two-dimensional metric \( g_{\mu \nu} = e^\psi \hat{g}_{\mu \nu} \) \( [2, 3] \).

The approach taken here does not involve freezing out any of the degrees of freedom in the system described by the action \( (3) \). Classically the metric is locally \( g_{\mu \nu} = e^\phi \eta_{\mu \nu} \) and the \( \psi \) field is an auxiliary field whose classical evolution has no effect on the gravity/matter system.

3 Quantum Corrections

Following the remarks of the previous section, quantization of two dimensional gravity coupled to matter may be carried out by considering a functional integral over the field configurations of the metric, \( \psi \), a and matter fields. This may be done using the path integral
\[
Z = \int \frac{Dg}{V_{GC}} D\psi D\Phi e^{-S[\psi, g] - S_M[\Phi]} \tag{31}
\]
where \( S = S[\psi, g] + S_M[\Phi] \) is the Euclideanization of the action \( (3) \), with \( S_M \) being the matter part of the action and \( \Phi \) representing the matter fields. The volume of the diffeomorphism group, \( V_{GC} \), has been factored out.
Making the same scaling assumption as in refs. 3, 11 about the functional measure yields
\[ Z = \int [D\tau] D_g D_g b D_g c D_g \Phi e^{-S[\psi,\phi]-S_{gh}[b,c]-S_M[\Phi]} \]
where \([D\tau]\) represent the integration over the Teichmüller parameters and
\[ \hat{S}[\phi, \hat{g}] = \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}} \left( \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - Q \phi \hat{R} + \mu e^{\alpha \phi} \right) \]
is the Liouville action with arbitrary coefficients and \(g_{\mu\nu} = e^{\alpha \phi} \hat{g}_{\mu\nu}\). Note that \(R[e^{\alpha \phi} \hat{g}] = e^{-\alpha \phi} (\hat{R} - \alpha \hat{\nabla}^2 \phi)\) where \(\hat{R}\) and \(\hat{\nabla}^2\) are respectively the curvature scalar and Laplacian of the metric \(\hat{g}\). \(S_{gh}\) is the action for the ghost fields \(b\) and \(c\).

The approach is then to determine the parameters \(\alpha\) and \(Q\) from the requirement that the conformal anomaly vanish and that \(e^{\alpha \phi}\) is a conformal tensor of weight (1,1). The action \(S_{TOT} = \hat{S} + S[\psi,\hat{g}] + S_{gh} + S_M\) may be rewritten as
\[ S_{TOT} = S_{gh} + S_M + \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}} \left[ (1 + \frac{\alpha^2}{2G}) \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (Q - \frac{\alpha}{G}) \phi \hat{R} \right. \]
\[ - \frac{1}{G} \left( \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu (\psi + \alpha \phi) \partial_\nu (\psi + \alpha \phi) (\psi + \alpha \phi) \hat{R} \right) \]
\[ + (\mu + 8\pi \Lambda) e^{\alpha \phi} \right] \]
where \(\Lambda\) is the (possibly zero) classical cosmological constant. Upon rescaling the fields \(\psi\) and \(\phi\) so that
\[ \tilde{\phi} = \sqrt{1 + \frac{\alpha^2}{2G}} \phi \quad \tilde{\psi} = \frac{1}{\sqrt{2G}} (\psi + \alpha \phi) \]
(35) becomes
\[ S_{TOT} = \frac{1}{8\pi} \int d^2 x \sqrt{\hat{g}} \left\{ \hat{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{Q - \frac{\alpha}{G}}{\sqrt{1 + \frac{\alpha^2}{2G}}} \phi \hat{R} - \hat{g}^{\mu\nu} \partial_\mu \tilde{\psi} \partial_\nu \tilde{\psi} \right. \]
\[ - \sqrt{\frac{2}{G}} \tilde{\psi} \hat{R} + (\mu + 8\pi \Lambda) \exp \left[ \frac{\alpha}{\sqrt{1 + \frac{\alpha^2}{2G}}} \tilde{\phi} \right] \right\} + S_{gh} + S_M \]
The coefficients in front of the \( \tilde{\psi} \) terms are due to the sign of the kinetic energy term in (3). This may be dealt with by rescaling \( \tilde{\psi} \rightarrow i \tilde{\psi} \) so that the functional integral converges.

The action in (36) may now be analyzed using conformal field theoretic techniques. The fields \( \tilde{\phi}, \tilde{\psi} \) have propagators

\[
< \tilde{\phi}(z) \tilde{\phi}(w) > = -\ln(z - w) = < \tilde{\psi}(z) \tilde{\psi}(w) > \tag{37}
\]

and the contribution of the matter and ghost fields is as usual. Hence it is straightforward to compute the total central charge

\[
c_{\text{TOT}} = 1 + 3 \left( \frac{Q - \alpha}{G} \right)^2 + 1 - \frac{6}{G} - 26 + c_M
\]

\[
= c_M - 24 + 3 \frac{GQ^2 - 2(1 + \alpha Q)}{G + \alpha^2/2} \tag{38}
\]

using the operator product expansion of the stress-energy tensor which is associated with (36). The anomalous dimension of the (renormalized) cosmological constant term in (36) is

\[
\Delta = -\frac{1}{2} \frac{\alpha}{\sqrt{1 + \frac{\alpha^2}{2G}}} \left( \frac{Q - \alpha/G}{\sqrt{1 + \frac{\alpha^2}{2G}}} + \frac{\alpha}{\sqrt{1 + \frac{\alpha^2}{2G}}} \right) . \tag{39}
\]

In the \( G \rightarrow \infty \) limit these results yield what one would obtain from the Polyakov approach \[2, 3\], except that \( c_M \rightarrow c_M + 1 \) due to the presence of the additional scalar \( \psi \).

Setting \( \Delta = 1 \) so that \( e^{\alpha \phi} \) is a conformal tensor of weight (1,1) yields

\[
\alpha^2 + Q \alpha + 2 = 0 \tag{40}
\]

exactly as in ref. \[3\]. This yields the constraint \( Q^2 > 8 \) since \( \alpha \) must be real.

Setting \( c_{\text{TOT}} = 0 \) and using (36) yields

\[
\alpha^2 = \frac{(12 - c_M)G - 6 \pm \sqrt{(c_MG - 6)((c_M - 24)G - 6)}}{c_M - 12 + 6G} \tag{41}
\]

for \( \alpha \) in terms of \( c_M \).

As \( \alpha \) is real, \( \alpha^2 \) must be positive, imposing constraints on \( c_M \) and \( G \). There are three distinct cases to consider.
In this case the negative sign for the square root must be chosen in (41) and so
\[ \alpha^2 = \sqrt{(c_M G - 6)((c_M - 24)G - 6) + (c_M - 12)G + 6} \]
\[ 12 - 6G - c_M \]
is the only possible solution. Positivity of \( c_M \) implies \( G < 2 \).

(ii) \( c_M > 12 - 6G \)  
The discriminant in this case is not positive if \( 6/G < c_M < 24 + 6/G \). It is straightforward to show that if \( c_M \) is larger than \( 24 + 6/G \) then \( \alpha^2 \) is not positive, yielding \( c_M < 6/G \). There is no upper limit on \( G \), but positivity of \( \alpha^2 \) implies \( G > 1 \) regardless of which sign of the square root is taken. In the \( G \to \infty \) limit this is the familiar restriction \( 1 < c_M + 1 < 25 \) of string theory, since the additional scalar \( \psi \) adds one unit of central charge to \( c_M \).

(iii) \( c_M = 12 - 6G \)  
In this case
\[ \alpha^2 = \frac{2G}{G^2 - 1} \]
which implies \( 2 > G > 1 \) when \( c_M \) is positive.

Finally, one can compute the scaling dependence of the correlation functions. Inserting the factor
\[ 1 = \int D\delta \left( \int d^2 x \sqrt{g} e^{\alpha \phi - A} \right) \]
into the functional integral (32) implies that
\[ < \Pi_i e^{\alpha_i \Phi_i} >_A = A^{\xi - 1} < \Pi_i e^{\alpha_i \Phi_i} >_{A=1} \]
with
\[ \xi = \frac{QG - \alpha}{\alpha G} (h - 1) \]
where \( A \) is the area of the surface and \( h \) the number of handles.

4 Discussion

The model presented here is that of a two-dimensional quantum theory of gravity which has a non-trivial classical limit. As such it has a number of
interesting features which merit further study. Unlike virtually all other two-
dimensional theories of gravity, it has a well-defined Newtonian limit and
its classical features closely resemble those of $(3 + 1)$ dimensional general
relativity [19]. The restriction on the central charge now depends on the
dimensionless Newton constant $G$, and so for small enough $G$ one can avoid
the constraints imposed by the critical dimensions present in other models[8, 10]. For example, one could use this model as the basis of a non-critical
string theory in four dimensions provided $G < 4/3$.

The restriction to coupling to conformally invariant matter may be lifted
by extending the matter action so that

$$\tilde{S}_M = S_M + \sum_i m_i O_i$$

(45)

where $m_i$ are (dimensional) constants and $O_i$ are operators of conformal
dimension $\Delta_i$. Upon coupling to gravity, this term is modified to

$$\tilde{S}_M = S_M + \sum_i m_i e^{\alpha_i \phi} O_i$$

(46)

in (34). Expanding in the parameters $m_i$, one must as before require that
each term in (46) be a (1,1) operator and so (40) becomes

$$-\frac{1}{2} \frac{\alpha_i}{\sqrt{1 + \frac{\alpha_i^2}{2G}}} \left( \frac{Q - \alpha_i}{\sqrt{1 + \frac{\alpha_i^2}{2G}}} + \frac{\alpha_i}{\sqrt{1 + \frac{\alpha_i^2}{2G}}} \right) + \Delta_i = 1$$

(47)

which forms a constraint on the exponents $\alpha_i$. If $\mu + 8\pi \Lambda$ is set to zero in
(34), then the restriction that $Q^2 > 8$ no longer holds, but instead is modified
to

$$Q^2 > 8(1 - \Delta)(1 - \frac{\Delta}{G})$$

(48)

provided at least one of the constants ($m_1 = m$, say, with corresponding
operator of dimension $\Delta$) is non-zero.

It is of course possible to extend these results to include negative values
of $G$; although such models have antigravity, they have been of some field-
theoretic interest [13, 23]. From (34) it is clear that in this case a critical
value for $G$ exists ($G_{\text{crit}} = 2\alpha^2$) beyond which the model is no longer well-
defined. At this point the kinetic term for $\phi$ vanishes and the model becomes
singular, suggesting a phase transition to another model.

1Although the model in ref. [11] has no restriction on the critical dimension, it is also
limited to constant curvature solutions.
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