COHOMOLOGIES OF 3-LIE ALGEBRAS WITH DERIVATIONS

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Abstract. In this paper, we consider a 3-Lie algebra with a derivation (called a 3-LieDer pair). We define cohomology for a 3-LieDer pair with coefficients in a representation. We use this cohomology to study deformations and abelian extensions of 3-LieDer pairs. We give the notion of a 3-Lie2Der pair, which can be viewed as the categorification of a 3-LieDer pair. We show that skeletal 3-Lie2Der pairs are classified by triples given by 3-LieDer pairs, representations and 3-cocycles. We define crossed modules of 3-LieDer pairs and show that there exists a one-to-one correspondence between strict 3-Lie2Der pairs and crossed modules of 3-LieDer pairs.

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1. Introduction

Generalizations of Lie algebras to higher arities, including 3-Lie algebras and more generally, n-Lie algebras (also called Filippov algebras) ([17]), have attracted attention from both mathematics and physics. n-Lie algebras are the algebraic structure corresponding to Nambu mechanics ([25]). In particular, the structure of 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the word-volume theory of multiple M2-branes ([5, 6]) and the Bagger-Lambert theory has a novel local gauge symmetry which is based on the metric 3-Lie algebras ([12] [13]). Many extensive literatures are related to this pioneering work. See [10] [19] [26] and the review article [11] for more details. Structure theory of n-Lie algebras are widely studied. Representation theory of n-Lie algebras was studied by Kasymov in [20] and cohomology theory of n-Lie algebras was studied by Takhajan and Gautheron in [18] [29]. Deformations of n-Lie algebras were studied in [1] [24] [29]. See [11] [22] [32] for more details on extensions of n-Lie algebras.
Derivations are useful tool to study various algebraic structures. Derivations can be used to construct homotopy Lie algebras ([31]), deformation formulas ([8]) and differential Galois theory ([21]). They also play an essential role in mathematical logic ([27]), functional analysis ([7, 9]) and control theory ([3, 4]). In [16, 23], the authors study algebras with derivations from the operadic point of view. Recently, the authors in [30] study the cohomology, extensions and deformations of Lie algebras with derivations (called LieDer pairs). The results of [30] have been extended to associative algebras with derivations (called AssDer pairs) and Leibniz algebras with derivations (called LeibDer pairs) in [14, 15].

In this paper, we consider 3-Lie algebras with derivations (called 3-LieDer pairs). We develop a cohomology theory of 3-LieDer pairs that controls the extensions and deformations of 3-LieDer pairs.

The paper is organized as follows. In Section 2, we introduce representations and cohomologies of 3-LieDer pairs. We construct a degree −1 graded Lie algebra structure on the underlying graded vector space of the cochain complex such that the Maurer-Cartan elements of this graded Lie algebra are precisely 3-LieDer pairs. In Section 3, we consider deformations of 3-LieDer pairs. We define one-parameter formal deformations of 3-LieDer pairs and show that $n$-infinitesimals of equivalent one-parameter formal deformations of a 3-LieDer pair are in the same cohomology class. We study deformations of order $n$ of 3-LieDer pairs and show that the obstruction of a deformation of order $n$ extending to a deformation of order $n + 1$ can be controlled by the third cohomology groups. In Section 4, we study abelian extensions of 3-LieDer pairs and show that abelian extensions of 3-LieDer pairs are classified by the second cohomology groups. In Section 5, we introduce the concept of 3-Lie2Der pairs, which is the categorification of 3-LieDer pairs. We show that there is a one-to-one correspondence between skeletal 3-Lie2Der pairs and the triples consisted of 3-LieDer pairs, representations and 3-cocycles. We define crossed modules of 3-LieDer pairs and show that there exists a one-to-one correspondence between strict 3-Lie2Der pairs and crossed modules of 3-LieDer pairs. Furthermore, we show that there exists a one-to-one correspondence between strict isomorphism classes of strict 3-Lie2Der pairs and equivalent classes of crossed modules of 3-LieDer pairs.

In this paper, we work over an algebraically closed field $F$ of characteristic 0 and all the vector spaces are over $F$ and finite-dimensional.

2. REPRESENTATIONS AND COHOMOLOGIES OF 3-LIE Der PAIRS

In this section, we consider 3-LieDer pairs. We give representations and cohomologies of 3-LieDer pairs. Finally, we construct a degree −1 graded Lie algebra whose Maurer-Cartan elements are precisely 3-LieDer pairs.

2.1. 3-LieDer pairs and their representations.

**Definition 2.1.** A 3-Lie algebra is a vector space $L$ together with a skew-symmetric linear map $[[\cdot, \cdot, \cdot]]_L : \otimes^3 L \rightarrow L$ such that the following Fundamental Identity holds:

\[
\begin{align*}
[x_1, x_2, [x_3, x_4, x_5]]_L &= [([x_1, x_2, x_3]_L, x_4, x_5)_L + [x_3, [x_1, x_2, x_4]_L, x_5]_L + [x_3, x_4, [x_1, x_2, x_3]_L]_L, \\
\text{where } x_1, x_2, x_3, x_4, x_5 &\in L.
\end{align*}
\]

Elements in $\wedge^2 L$ are called fundamental objects of the 3-Lie algebra $(L, [[\cdot, \cdot, \cdot]]_L)$. There is a bilinear operation $[\cdot, \cdot, \cdot]_F$ on $\wedge^2 L$, which is given by

\[
\begin{align*}
[X, Y]_F &= [x_1, x_2, y_1]_L \wedge y_2 + y_1 \wedge [x_1, x_2, y_2]_L, \quad \forall X = x_1 \wedge x_2, \; Y = y_1 \wedge y_2 \in \wedge^2 L.
\end{align*}
\]
It is well known that $(\wedge^2 L, [\cdot, \cdot]_F)$ is a Leibniz algebra, which plays an important role in the theory of 3-Lie algebras. The Leibniz rule can be written as

\begin{equation}
[X, [Y, Z]_F]_F = [[X, Y]_F, Z]_F + [Y, [X, Z]_F]_F, \quad \forall X, Y, Z \in \wedge^2 L.
\end{equation}

Moreover, the fundamental identity (2.1) is equivalent to

\begin{equation}
[X, [Y, z]] - [Y, [X, z]] = [[X, Y]_F, z], \quad \forall X, Y \in \wedge^2 L, \ z \in L.
\end{equation}

**Definition 2.2.** ([20]). A representation of a 3-Lie algebra $L$ on a vector space $V$ is a linear map $\rho: \wedge^2 L \to \text{End}(V)$ such that for all $x_1, x_2, x_3, x_4 \in L$,

\[
\rho(x_1, x_2)\rho(x_3, x_4) = \rho([x_1, x_2, x_3]_L, x_4) + \rho(x_3, [x_1, x_2, x_4]_L) + \rho(x_1, [x_3, x_2, x_4]_L) + \rho(x_1, x_2)\rho(x_3, x_4),
\]

\[
\rho(x_1, [x_2, x_3, x_4]_L) = \rho(x_1, x_2)\rho(x_3, x_4) - \rho(x_2, x_3)\rho(x_1, x_4) + \rho(x_3, x_4)\rho(x_1, x_2).
\]

In the sequel, a representation is denoted by $(V; \rho)$.

**Proposition 2.1.** Let $L$ be a 3-Lie algebra, $V$ a vector space and $\rho: \wedge^2 L \to \mathfrak{gl}(V)$ a bilinear map. Then $(V; \rho)$ is a representation of $L$ on $V$ if and only if there is a 3-Lie algebra structure on the direct sum $L \oplus V$ of vector spaces (called the semidirect product) defined by

\[
[x_1 + v_1, x_2 + v_2, x_3 + v_3]_\rho = [x_1, x_2, x_3]_L + \rho(x_1, x_2)v_3 + \rho(x_3, x_1)v_2 + \rho(x_2, x_3)v_1, \quad \forall x_i \in L, v_i \in V.
\]

A semidirect product 3-Lie algebra is denoted by $L \ltimes V$ or simply $L \ltimes V$.

The cohomology theory for a 3-Lie algebra $L$ with coefficients in a representation $(V; \rho)$ is given as follows. Denote by $C^n(L; V) := \text{Hom}(\wedge^n \wedge^2 L \wedge L, V)$, the space of $n$-cochains. The corresponding coboundary operator $d : C^n(L; V) \to C^{n+1}(L; V)$ is defined by

\[
(df_n)(\overline{x}_1, \ldots, \overline{x}_n, z) = \sum_{1 \leq j < k \leq n} (-1)^j f_n(\overline{x}_1, \ldots, \hat{x}_j, \ldots, \overline{x}_k, [\overline{x}_j, \overline{x}_k]_F, \ldots, \overline{x}_n, z) + \sum_{j=1}^n (-1)^j f_n(\overline{x}_1, \ldots, \hat{x}_j, \ldots, \overline{x}_n, [x_j, y_j, z]_L) + \sum_{j=1}^n (-1)^{i+1} \rho(\overline{x}_j) f_n(\overline{x}_1, \ldots, \hat{x}_j, \ldots, \overline{x}_n, z) + (-1)^n \rho(y_n, z) f_n(\overline{x}_1, \ldots, \overline{x}_{n-1}, x_n) + (-1)^n \rho(x_n, z) f_n(\overline{x}_1, \ldots, \overline{x}_{n-1}, y_n)
\]

for all $f_n \in C^n(L; V)$ and $\overline{x}_i = x_i \wedge y_i \in \wedge^2 L$, $z \in L$, $i = 1, 2, \ldots, n$, where $\cdot, \cdot)_F$ is given by (2.2). We denote the corresponding $n$-th cohomology group by $H^n(L; V)$.

Recall that a derivation on a 3-Lie algebra $L$ is a linear map $\theta_L : L \to L$ satisfying

\begin{equation}
\theta_L[x, y, z]_L = [\theta_L(x), y, z]_L + [x, \theta_L(y), z]_L + [x, y, \theta_L(z)]_L, \quad \forall x, y, z \in L.
\end{equation}

We denote the set of derivations on the 3-Lie algebra $L$ by $\text{Der}(L)$.

**Definition 2.3.** A 3-LieDer pair consists of a 3-Lie algebra $(L, [\cdot, \cdot, \cdot]_L)$ and a derivation $\theta_L \in \text{Der}(L)$. We denote a 3-LieDer pair by $(L, [\cdot, \cdot, \cdot]_L, \theta_L)$ or simply by $(L, \theta_L)$.

**Example 2.1.** Let $L$ be the 4-dimensional 3-Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and bracket

\[
[e_1, e_2, e_3]_L = e_4, [e_1, e_2, e_4]_L = e_3, [e_1, e_3, e_4]_L = e_2, [e_2, e_3, e_4]_L = e_1.
\]
Then any derivation $D = (d_{ij}) \in \text{Der}(L)$ has the form as follows

$$
\begin{bmatrix}
ed_1 \\
ed_2 \\
ed_3 \\
ed_4
\end{bmatrix} =
\begin{bmatrix}
0 & d_{12} & d_{13} & d_{14} \\
d_{12} & 0 & d_{23} & d_{24} \\
-d_{13} & d_{23} & 0 & d_{34} \\
d_{14} & -d_{24} & d_{34} & 0
\end{bmatrix}
\begin{bmatrix}
ed_1 \\
ed_2 \\
ed_3 \\
ed_4
\end{bmatrix}.
$$

See [2] for more examples and applications of derivations on 3-Lie algebras.

**Definition 2.4.** Let $(L, [\cdot, \cdot, \cdot]_L, \theta_L)$ and $(K, [\cdot, \cdot, \cdot]_K, \theta_K)$ be two 3-LieDer pairs. A 3-LieDer pair morphism from $(L, [\cdot, \cdot, \cdot]_L, \theta_L)$ to $(K, [\cdot, \cdot, \cdot]_K, \theta_K)$ is a 3-Lie algebra morphism $\eta : L \rightarrow K$ such that $\eta \circ \theta_L = \theta_K \circ \eta$.

Next we give the definition of representations of 3-LieDer pairs.

**Definition 2.5.** A representation of a 3-LieDer pair $(L, \theta_L)$ on a vector space $V$ is a pair $(\rho, \theta_V)$, where $\rho : \wedge^2 L \rightarrow \text{End}(V)$ and $\theta_V \in \text{End}(V)$, such that $\rho$ is a representation of the 3-Lie algebra $L$ on $V$ and satisfies

$$\tag{2.7} \theta_V \circ \rho(x, y) - \rho(x, y) \circ \theta_V = \rho(\theta_L(x), y) + \rho(x, \theta_L(y)).$$

We denote a representation of a 3-LieDer pair by $(V; \rho, \theta_V)$.

**Example 2.2.** Let $(L, [\cdot, \cdot, \cdot]_L, \theta_L)$ be a 3-LieDer pair. Then $(L; \text{ad}, \theta_L)$ is a representation of the 3-LieDer pair $(L; \theta_L)$, which is called the adjoint representation of $(L, \theta_L)$.

**Lemma 2.1.** Let $L$ be a 3-Lie algebra and $(V; \rho)$ a representation of $L$. Then $(V; \rho, \theta_V)$ is a representation of a 3-LieDer pair $(L, \theta_L)$ if and only if $(L \ltimes \rho V, \theta_L + \theta_V)$ is a 3-LieDer pair, which we call the semi-direct product of the 3-LieDer pair $(L, \theta_L)$ by the representation $(V; \rho, \theta_V)$.

**Proof.** Assume that $(L \ltimes \rho V, \theta_L + \theta_V)$ is a 3-LieDer pair. Since $L \ltimes \rho V$ is a semi-direct product 3-Lie algebra, $\rho$ is a representation of the 3-Lie algebra $L$ on $V$. Since $\theta_L + \theta_V$ is a derivation on the semi-direct product 3-Lie algebra $L \ltimes \rho V$, for $x, y \in L, v \in V$, we have

$$0 = (\theta_L + \theta_V)((x, y, v)_\rho) - ([\theta_L + \theta_V](x, y, v)_\rho + [x, (\theta_L + \theta_V)(y), v])_\rho + [x, y, (\theta_L + \theta_V)(v)]_\rho
= (\theta_V \circ \rho(x, y))(v) - (\rho(\theta_L(x), y) + \rho(x, \theta_L(y))) + \rho(x, \theta_V(v)),$$

which implies that

$$\theta_V \circ \rho(x, y) = \rho(\theta_L(x), y) + \rho(x, \theta_L(y)) + \rho(x, y) \circ \theta_V.$$

Thus $(V; \rho, \theta_V)$ is a representation of a 3-LieDer pair $(L, \theta_L)$.

The converse can be proved similarly. We omit the details. \(\square\)

Let $(V; \rho, \theta_V)$ be a representation of a 3-LieDer pair $(L, \theta_L)$. With a slight abuse of notation, we still use $\theta_L$ to denote the endomorphism on $\wedge^2 L$ given by

$$\theta_L(X) = \theta_L(x) \wedge y + x \wedge \theta_L(y), \ \forall X = x \wedge y \in \wedge^2 L.$$

By a direct calculation, we have

$$\theta_L([X, Y]_F) = [\theta_L(X), Y]_F + [X, \theta_L(Y)]_F, \tag{2.8}$$

where $[\cdot, \cdot]_F$ is given by (2.2) and $X, Y \in \wedge^2 L$. Thus $\theta_L$ is a derivation on the Leibniz algebra $(L, [\cdot, \cdot]_F)$. 
2.2. **Cohomologies of 3-LieDer pairs.** Let \((V; \rho, \theta_V)\) be a representation of a 3-LieDer pair \((L, \theta_L)\). We define cochain groups by \(C^1_{3-LieDer}(L; V) := \text{Hom}(L, V)\) and

\[
C^p_{3-LieDer}(L; V) := C^p(L; V) \times C^{p-1}(L; V), \quad p \geq 2.
\]

For \(p \geq 1\), we define an operator \(\delta : C^p(L; V) \to C^p(L; V)\) by

\[
(\delta \alpha)(x_1 \otimes \cdots \otimes x_p) = \sum_{i=1}^{p-1} \alpha(x_1 \otimes \cdots \otimes x_i \theta_L(x_i) \otimes \cdots \otimes x_p) + \alpha(x_1 \otimes \cdots \otimes x_p, \theta_L(z)) - \theta_V(\alpha(x_1 \otimes \cdots \otimes x_p \otimes z)),
\]

where \(x_i \in \wedge^2 L\) and \(z \in L\).

Define \(\partial : C^1_{3-LieDer}(L; V) \to C^2_{3-LieDer}(L; V)\) by

\[
\partial \alpha_1 = (d\alpha_1, -\delta \alpha_1), \quad \alpha_1 \in \text{Hom}(L, V).
\]

For \(p \geq 2\), define \(\partial : C^p_{3-LieDer}(L; V) \to C^{p+1}_{3-LieDer}(L; V)\) by

\[
\partial(\alpha_p, \beta_{p-1}) = (d\alpha_p, d\beta_{p-1} + (-1)^p \delta \alpha_p).
\]

**Lemma 2.2.** The operator \(\delta\) and \(d\) are commutative, i.e., \(d \circ \delta = \delta \circ d\).

**Proof.** It follows by a straightforward tedious calculations. \(\square\)

**Proposition 2.2.** The map \(\partial\) is a coboundary operator, i.e. \(\partial \circ \partial = 0\).

**Proof.** Let \((\alpha_p, \beta_{p-1}) \in C^p_{3-LieDer}(L; V)\). Then, by Lemma 2.2 we get

\[
(\partial \circ \partial)(\alpha_p, \beta_{p-1}) = \partial(d\alpha_p, d\beta_{p-1} + (-1)^p \delta \alpha_p)
= (d(d\alpha_p), d(d\beta_{p-1} + (-1)^p \delta \alpha_p) + (-1)^p \delta(\delta \alpha_p))
= (0, 0 + (-1)^p \delta(d\alpha_p) - (-1)^p \delta(\delta \alpha_p))
= (0, 0).
\]

\(\square\)

**Definition 2.6.** Let \((V; \rho, \theta_V)\) be a representation of a 3-LieDer pair \((L, \theta_L)\). The cohomology of the cochain complex \((C^*_{3-LieDer}(L; V), \partial)\) is taken to be the **cohomology of the 3-LieDer pair** \((L; \theta_L)\). The corresponding \(p\)-th cohomology group is defined by

\[
\mathcal{H}^p_{3-LieDer}(L; V) = Z^p_{3-LieDer}(L; V)/B^p_{3-LieDer}(L; V),
\]

where \(Z^p_{3-LieDer}(L; V)\) is the space of \(p\)-cocycles and \(B^p_{3-LieDer}(L; V)\) is the space of \(p\)-coboundaries.

In particular, when \(p = 1\), we have the following

**Corollary 2.1.** Let \((V; \rho, \theta_V)\) be a representation of a 3-LieDer pair \((L, \theta_L)\). Then

\[
\mathcal{H}^1_{3-LieDer}(L; V) = \{\alpha \in \text{Hom}(L, V) \mid \alpha[x, y, z]_L = \rho(x, y)\alpha(z) + \rho(y, z)\alpha(x) - \rho(x, z)\alpha(y), \alpha \circ \theta_L = \theta_V \circ \alpha, \forall x, y, z \in L\}.
\]

**Proof.** For any \(\alpha \in C^1_{3-LieDer}(L; V)\), \(\alpha\) is closed if and only if

\[
\partial \alpha = (d\alpha, -\delta \alpha) = (0, 0),
\]

which is equivalent to that \(\alpha[x, y, z]_L = \rho(x, y)\alpha(z) + \rho(y, z)\alpha(x) - \rho(x, z)\alpha(y)\) and \(\alpha(\theta_L(x)) = \theta_V(\alpha(x))\) for \(x, y, z \in L\). Moreover, since there is no exact 1-cochain, we get the conclusion. \(\square\)
2.3. Maurer-Cartan characterizations of 3-LieDer pairs. A permutation $\sigma \in S_n$ is called an $(i,n-i)$-shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. If $i = 0$ or $i = n$, we assume $\sigma = \text{Id}$. The set of all $(i,n-i)$-shuffles will be denoted by $S_{(i,n-i)}$.

Let $L$ be a vector space. Denote by $C^*(L; L) = \mathcal{C}^0_{n=1}C^0(L; L)$, where $C^0(L; L) = \text{Hom}(\otimes^{n-1}\wedge^2 L) \wedge L, L)$. It was shown in [28] that the graded vector space $C^*(L; L)$ equipped with the following bracket

$$[P, Q]_{3\text{Lie}} = P \circ Q - (-1)^{pq} Q \circ P, \quad \forall P \in C^{p+1}(L; L), Q \in C^{q+1}(L; L),$$

is a degree $-1$ graded Lie algebra, where $P \circ Q \in C^{p+q+1}(L; L)$ is defined by

$$(P \circ Q)(\mathcal{x}_1, ..., \mathcal{x}_{p+q}, z)
= \sum_{k=1}^{p} (-1)^{(k-1)q} \sum_{\sigma \in S_{(k-1,q)}} (-1)^{\sigma} P(\mathcal{x}_{\sigma(1)}, ..., \mathcal{x}_{\sigma(k-1)}, Q(\mathcal{x}_{\sigma(k)}, ..., \mathcal{x}_{\sigma(k+q-1)}, \mathcal{x}_{k+q}) \wedge y_{k+q}, \mathcal{x}_{k+q+1}, ..., \mathcal{x}_{p+q}, z)
+ \sum_{k=1}^{q} (-1)^{(k-1)p} \sum_{\sigma \in S_{(k-1,q)}} (-1)^{\sigma} P(\mathcal{x}_{\sigma(1)}, ..., \mathcal{x}_{\sigma(k-1)}, Q(\mathcal{x}_{\sigma(k)}, ..., \mathcal{x}_{\sigma(k+q-1)}, y_{k+q}), \mathcal{x}_{k+q+1}, ..., \mathcal{x}_{p+q}, z)
$$

(2.15) $+ \sum_{\sigma \in S_{(p,q)}} (-1)^{pq} (-1)^{\sigma} P(\mathcal{x}_{\sigma(1)}, ..., \mathcal{x}_{\sigma(p)}, Q(\mathcal{x}_{\sigma(p+1)}, ..., \mathcal{x}_{\sigma(p+q)}, z))$

for all $\mathcal{x}_i = x_j \wedge y_j \in \wedge^2 L$, $1 \leq i \leq p + q$, $z \in L$. In particular, $\pi : \wedge^3 L \to L$ defines a 3-Lie algebra structure on $L$ if and only if $[\pi, \pi]_{3\text{Lie}} = 0$, i.e. $\pi$ is a Maurer-Cartan element of graded Lie algebra $(C^*(L; L), [, , ]_{3\text{Lie}})$. Moreover, the coboundary map $d$ of the 3-Lie algebra with coefficients in the adjoint representation can be given by

$$df_p = (-1)^{p-1}[\pi, f_p]_{3\text{Lie}}, \quad \forall f_p \in C^p(L; L).$$

Proposition 2.3. Let $L$ be a vector space. The bracket $[,]_{3\text{Lie}} : C^0_{3\text{Lie}}(L; L) \times C^0_{3\text{Lie}}(L; L) \to C^0_{3\text{Lie}}(L; L)$ given by

$$[(\alpha_p, \beta_{p-1}), (\alpha_q, \beta_{q-1})]_{3\text{Lie}} = ((\alpha_p, \alpha_q)_{3\text{Lie}}, (-1)^{p+1}[\alpha_p, \beta_{q-1}]_{3\text{Lie}} + [\beta_{p-1}, \alpha_q]_{3\text{Lie}})$$

(2.16) defines a degree $-1$ graded Lie bracket on the graded vector space $C^*_{3\text{Lie}}(L; L)$. Moreover, its Maurer-Cartan elements are precisely the 3-LieDer pairs on $L$.

Proof. The proof of $C^*_{3\text{Lie}}(L; L)$ equipped with the bracket $[,]_{3\text{Lie}}$ is a degree $-1$ graded Lie algebra following as that $(C^*(L; L), [, , ]_{3\text{Lie}})$ is a degree $-1$ graded Lie algebra. For $\omega \in \text{Hom}(\wedge^3 L, L)$ and $\varphi \in \text{Hom}(L, L)$, then we have

$$[(\omega, \varphi), (\omega, \varphi)]_{3\text{Lie}} = ([\omega, \omega]_{3\text{Lie}}, -2[\omega, \varphi]_{3\text{Lie}}).$$

It is obvious that $(\omega, \varphi)$ is a Maurer-Cartan element if and only if $[\omega, \omega]_{3\text{Lie}} = 0$ and $[\omega, \varphi]_{3\text{Lie}} = 0$. For $x, y, z \in L$, we have

$$[\omega, \varphi]_{3\text{Lie}}(x, y, z) = \omega(\varphi(x), y, z) + \omega(x, \varphi(y), z) + \omega(x, y, \varphi(z)) - \varphi(\omega(x, y, z)).$$

Thus $(\omega, \varphi)$ is a Maurer-Cartan element if and only if $(L, \omega, \varphi)$ is a 3-LieDer pair.

The following proposition shows that the coboundary operator $\partial$ with coefficient in the adjoint representation of a 3-LieDer pair $(L, \omega, \varphi)$ can be presented by the degree $-1$ graded Lie algebra $(C^*_{3\text{Lie}}(L; L), [, , ]_{3\text{Lie}})$.

Proposition 2.4. Let $(L, \omega, \varphi)$ be a 3-LieDer pair. Then we have

$$\partial(\alpha_p, \beta_{p-1}) = (-1)^{p-1}[(\omega, \varphi), (\alpha_p, \beta_{p-1})]_{3\text{Lie}}, \quad \forall (\alpha_p, \beta_{p-1}) \in C^p_{3\text{Lie}}(L; L).$$

Proof. It follows by a direct calculation. We omit the details.
3. Deformations of 3-LieDer pairs

In this section, we study one-parameter formal deformations of 3-LieDer pairs, in which the 3-Lie algebra and the distinguished derivation are simultaneous deformations.

**Definition 3.1.** A one-parameter formal deformation of a 3-LieDer pair \((L, [\cdot, \cdot, \cdot]_L, \theta_L)\) is a pair of formal power series \((f_i, g_i)\) of the form

\[
f_i = \sum_{i \geq 0} f_i t^i, \quad g_i = \sum_{i \geq 0} g_i t^i,
\]

where \(f_0 = [\cdot, \cdot, \cdot]_L\) and \(g_0 = \theta_L\) such that \((L[[t]], f, g)\) is a 3-LieDer pair over \(\mathbb{F}[[t]]\).

By a direct calculation, we have

**Lemma 3.1.** A pair \((f_i, g_i)\) is a one-parameter formal deformation of \((L, \theta_L)\) if and only if the following equations hold:

\[
0 = \sum_{i + j = n \geq 0} \left(f_i(f_j(x_1, x_2, x_3), x_4, x_5) + f_i(x_3, f_j(x_1, x_2, x_4), x_5) + f_i(x_3, f_j(x_1, x_2, x_4), x_5)\right)
\]

\[
0 = \sum_{i + j = n \geq 0} \left(f_i(x_3, x_4, f_j(x_1, x_2, x_5)) - f_i(x_1, x_2, f_j(x_3, x_4, x_5))\right),
\]

\[
0 = \sum_{i + j = n \geq 0} \left(g_i(f_j(x_1, x_2, x_3)) - f_j(g_i(x_1), x_2, x_3)\right)
\]

In particular, for \(n = 1\), (3.2) and (3.3) imply that

\[
0 = f_1(\theta_0(x_1, x_2, x_3), x_4, x_5) + f_1(x_3, \theta_0(x_1, x_2, x_4), x_5) + f_1(x_3, \theta_0(x_1, x_2, x_4), x_5)
\]

\[
+ f_0(x_3, x_4, f_1(x_1, x_2, x_3)) - f_0(x_1, x_2, f_1(x_3, x_4, x_5)),
\]

\[
0 = g_1(\theta_0(x_1, x_2, x_3)) - f_0(g_1(x_1), x_2, x_3) - f_0(x_1, g_1(x_2), x_3)
\]

\[
- f_0(x_1, x_2, g_1(x_3)) + g_0(f_1(x_1, x_2, x_3)) - f_1(g_0(x_1), x_2, x_3)
\]

Note that (3.4) implies that \(df_1 = 0\) and (3.5) implies that \(dg_1 + \delta f_1 = 0\). Thus we have

**Lemma 3.2.** \((f_1, g_1)\) is a 2-cocycle, i.e. \((f_1, g_1) \in \mathcal{Z}^2_{-\text{LieDer}}(L; L)\).

**Proof.** Since \(\delta(f_1, g_1) = (df_1, dg_1 + \delta f_1) = (0, 0)\), the conclusion follows. \(\square\)

Similar to the proof of Lemma 3.2, we can show that

**Proposition 3.1.** If \((f_i, g_i) = 0, 1 \leq i < n\), then \((f_n, g_n) \in \mathcal{Z}^2_{-\text{LieDer}}(L; L)\).

A 2-cochain \((f_n, g_n)\) is called the \(n\)-infinitesimal of \((f_i, g_i)\) if \((f_i, g_i) = 0\) for all \(1 \leq i < n\). In particular, the 2-cocycle \((f_1, g_1)\) is called the infinitesimal (or 1-infinitesimal) of \((f_i, g_i)\).

Therefore, by Proposition 3.1, we see that the \(n\)-infinitesimal \((f_n, g_n)\) of \((f_i, g_i)\) is a 2-cocycle whenever \(n \geq 1\).
Definition 3.2. Suppose that \((f_i, g_i)\) and \((f'_i, g'_i)\) are one-parameter formal deformations of a 3-LieDer pair \((L, \theta_L)\). \((f_i, g_i)\) and \((f'_i, g'_i)\) are called equivalent (i.e., \((f_i, g_i) \sim (f'_i, g'_i)\)), if there exists a linear isomorphism \(\Phi_t = \sum_{i \geq 0} \phi_i t^i : (L[[t]], f_i, g_i) \to (L[[t]], f_i, g_i)\), where \(\phi_i \in C^1(L; V)\) and \(\phi_0 = id_L\) satisfying that

\[
\Phi_t \circ f'_i = f_i \circ (\Phi_t \times \Phi_t \times \Phi_t), \quad \Phi_t \circ g'_i = g_i \circ \Phi_t. \tag{3.6}
\]

Motivated by Proposition 3.1, we have the following

Proposition 3.2. The \(n\)-infinitesimals of two equivalent one-parameter formal deformations of a 3-LieDer pair \((L, \theta_L)\) belong to the same cohomology class.

Proof. Let \((f_i, g_i)\) and \((f'_i, g'_i)\) be one-parameter formal deformations of \((L, \theta_L)\). It suffices to show that \([(f_n, g_n)] = [(f'_n, g'_n)] \in H^2_{3-LieDer}(L; L)\). By exacting the coefficients of \(t^n\), we see that (3.6) is equivalent to

\[
\sum_{i+j=k \geq 0} \phi_i(f_j(x, y, z)) = \sum_{i+j+k+l=n \geq 0} f'_i(\phi_i(y), \phi_j(y), \phi_k(z)),
\]

\[
\sum_{i+j=k \geq 0} \phi_i(g_j(x)) = \sum_{i+j=k \geq 0} g'_i(\phi_j(x))
\]

for all \(x, y, z \in L\).

Since \(\phi_0 = id_L\), we have

\[
f_n(x, y, z) - f'_n(x, y, z) = -\phi_1(f_0(x, y, z)) + f_0(\phi_1(x), y, z) + f_0(x, \phi_1(y), z)
\]

\[
+ f_0(x, y, \phi_1(z)),
\]

\[
g_n(x) - g'_n(x) = -\phi_1(\theta_L(x)) + \theta_L(\phi_1(x)),
\]

which implies that \(f_n - f'_n = d\phi_1\) and \(g_n - g'_n = -\delta\phi_1\). Thus we have

\[
(f_n, g_n) - (f'_n, g'_n) = (d\phi_1, -\delta\phi_1) = \partial \phi_1.
\]

This means that \([(f_n, g_n)] = [(f'_n, g'_n)] \in H^2_{3-LieDer}(L; L)\). \(\square\)

Next we shall investigate the rigidity of 3-LieDer pairs.

Definition 3.3. Let \((f_i, g_i)\) be a one-parameter formal deformation of a 3-LieDer pair \((L, \theta_L)\). If \((f_i, g_i) \sim (f_0, g_0)\), then \((f_i, g_i)\) is said to be trivial. Moreover, \((L, \theta_L)\) is called rigid if every one-parameter formal deformation of \((L, \theta_L)\) is trivial.

Theorem 3.1. Let \((L, \theta_L)\) be a 3-LieDer pair. If \(H^2_{3-LieDer}(L; L) = 0\), then \((L, \theta_L)\) is rigid.

Proof. Let \((f_i, g_i)\) be a one-parameter formal deformation of \((L, \theta_L)\). Set \(f_i = f_0 + \sum_{i \geq r} f_i t^i, \quad g_i = g_0 + \sum_{i \geq r} g_i t^i\). By Proposition 3.1, \((f_i, g_i) \in Z^2_{3-LieDer}(L; L)\). Since \(H^2_{3-LieDer}(L; L) = 0\), there is a map \(\phi_r \in C^1(L; L)\) such that \((f_r, g_r) = -\delta \phi_r = (-d\phi_r, \delta \phi_r)\). Then \(\Phi_t := id_L + \phi_r t^r\) is a linear isomorphism of \(L\). Define \(f'_i, g'_i\) as

\[
f'_i = \Phi_t^{-1} \circ f_i \circ (\Phi_t \times \Phi_t \times \Phi_t), \quad g'_i = \Phi_t^{-1} \circ g_i \circ \Phi_t. \tag{3.7}
\]

It is straightforward to check that \((f'_i, g'_i)\) is a one-parameter formal deformation of \((L, \theta_L)\). Thus, by (3.7) and Definition 3.2, one has \((f'_i, g'_i) \sim (f_i, g_i)\).
By a direct calculation, we have
\[
\begin{aligned}
f'_t(x, y, z) &= f_0(x, y, z) + \left( f_t(x, y, z) - \phi_t(f_0(x, y, z)) + f_t(\phi_t(x), y, z) \right) t' + \cdots, \\
g'_t(x) &= g_0(x) + \left( g_t(x) + g_0(\phi_t(x)) - \phi_t(g_0(x)) \right) t' + \cdots.
\end{aligned}
\] (3.8) (3.9)

Since \( f_t = -\partial \phi_t \) and \( g_t = \delta \phi_t \), (3.8) and (3.9) reduce to
\[
f'_t(x, y, z) = f_0(x, y, z) + \sum_{i \geq r+1} f'_i(x, y, z)t^i, \quad g'_t(x) = g_0(x) + \sum_{i \geq r+1} g'_i(x)t^i.
\]

Then by repeating the argument, we obtain that \((f_t, g_t) \sim (f_0, g_0)\). \( \square \)

**Definition 3.4.** A deformation of order \( n \) of a 3-LieDer pair \((L, \theta_L)\) is a pair \((f_t, g_t)\) such that \( f_t = \sum_{i=0}^n f_i t^i \) and \( g_t = \sum_{i=0}^n g_i t^i \) endow the \( \mathbb{F}[t]/(t^{n+1}) \)-module \( L[t]/(t^{n+1}) \) the 3-LieDer pair structure with \((f_0, g_0) = \langle L, \theta, \cdot, \cdot \rangle \) \( L, \theta_L \). Furthermore, if there exists a 2-cochain \((f_{n+1}, g_{n+1})\) belongs to \( C^2_{3-\text{LieDer}}(L; L) \) such that \((f'_t, g'_t)\) defined by
\[
(f'_t, g'_t) = (f_t + f_{n+1}, g_t + g_{n+1})
\]
is a deformation of order \( n + 1 \) of \((L, \theta_L)\), we say that \((f_t, g_t)\) is extensible.

Let \((f_t, g_t)\) be a deformation of order \( n \) of a 3-LieDer pair \((L, \theta_L)\). For any \( X_1 = x_1 \wedge x_2, X_2 = x_3 \wedge x_4 \in \wedge^2 L \) and \( x_5 \in L \), define a 3-cochain \((\Omega^3_{n+1}, \Omega^2_{n+1}) \in C^3_{3-\text{LieDer}}(L; L) \) as
\[
\begin{aligned}
\Omega^3_{n+1}(X_1, X_2, x_5) &= \sum_{i + j = n + 1 \atop i, j > 0} \left( f_i(f_j(x_1, x_2, x_3), x_4, x_5) + f_i(x_3, f_j(x_1, x_2, x_4), x_5) \\
&\quad \quad + f_i(x_3, x_4, f_j(x_1, x_2, x_5)) - f_i(x_1, x_2, f_j(x_3, x_4, x_5)) \right), \\
\Omega^2_{n+1}(X_1, x_3) &= \sum_{i + j = n + 1 \atop i, j > 0} \left( g_i(f_j(x_1, x_2, x_3)) - f_j(g_i(x_1), x_2, x_3) \\
&\quad \quad - f_j(x_1, g_i(x_2), x_3) - f_j(x_1, x_2, g_i(x_3)) \right).
\end{aligned}
\]

**Proposition 3.3.** The 3-cochain \((\Omega^3_{n+1}, \Omega^2_{n+1})\) is closed, i.e. \( \partial(\Omega^3_{n+1}, \Omega^2_{n+1}) = 0 \).

**Proof.** It is straightforward to check that \( \Omega^3_{n+1} = df_{n+1} \) and \( \Omega^2_{n+1} = dg_{n+1} + \delta f_{n+1} \). It is obvious that \( d\Omega^3_{n+1} = d(df_{n+1}) = 0 \). By Lemma 2.2, we also have
\[
d\Omega^2_{n+1} - \delta \Omega^3_{n+1} = d(dg_{n+1} + \delta f_{n+1}) - \delta(df_{n+1}) = (d \circ \delta - \delta \circ d)f_{n+1} = 0.
\]

Thus \( \partial(\Omega^3_{n+1}, \Omega^2_{n+1}) = (d\Omega^3_{n+1}, d\Omega^2_{n+1} - \delta \Omega^3_{n+1}) = (0, 0) \). The proof is completed. \( \square \)

The main result of this section is given in the following

**Theorem 3.2.** Let \((f_t, g_t)\) be a deformation of order \( n \) of a 3-LieDer pair \((L, \theta_L)\). Then \((f_t, g_t)\) is extensible if and only if the cohomology class \([\Omega^3_{n+1}, \Omega^2_{n+1}]\) in \( \mathcal{H}^3_{3-\text{LieDer}}(L; V) \) is trivial.

**Proof.** Suppose that \((f_t, g_t)\) is extensible, i.e., \((f'_t, g'_t)\) is a deformation of order \( n + 1 \), where
\[
f'_t = f_t + f_{n+1}, \quad g'_t = g_t + g_{n+1}.
\]
It is straightforward to check that \( \Omega^3_{n+1} = df_{n+1}, \Omega^2_{n+1} = dg_{n+1} + \delta f_{n+1} \), which imply that
\[
(\Omega^3_{n+1}, \Omega^2_{n+1}) = (df_{n+1}, dg_{n+1} + \delta f_{n+1}) = \partial(f_{n+1}, g_{n+1}).
\]

Thus the cohomology class \([\Omega^3_{n+1}, \Omega^2_{n+1}]\) is trivial.

Conversely, suppose that \((\Omega^3_{n+1}, \Omega^2_{n+1})\) is trivial. Then there exists a 2-cochain \((f_{n+1}, g_{n+1}) \in C^2_{3-LieDer}(L; L)\) such that \((\Omega^3_{n+1}, \Omega^2_{n+1}) = \partial(f_{n+1}, g_{n+1})\). Set \(f'_i = f_i + f_{n+1}, g'_i = g_i + g_{n+1}\), then by a direct check we see that \((f'_i, g'_i)\) is a 3-LieDer pair of \((L, \theta)\) and hence \((f'_i, g'_i)\) is a deformation of order \(n + 1\) due to Definition 3.4. Namely, \((f_i, g_i)\) is extensible. The proof is finished. \(\Box\)

4. Abelian extensions of 3-LieDer pairs

In this section, we study abelian extensions of 3-LieDer pairs and show that equivalent abelian extensions of 3-LieDer pairs can be classified by the second cohomology groups.

**Definition 4.1.** Let \((A, \theta_A)\) and \((B, \theta_B)\) be two 3-LieDer pairs. An abelian extension of \((B, \theta_B)\) by \((A, \theta_A)\) is an exact sequence of 3-LieDer pair morphisms

\[
0 \longrightarrow A \xrightarrow{i} L \xrightarrow{\pi} B \longrightarrow 0
\]

such that \(A\) is an abelian ideal of \(L\), i.e.,
\[
\pi L = B, \quad \pi A = 0.
\]

A section of an abelian extension of \((B, \theta_B)\) by \((A, \theta_A)\) is a linear map \(s : B \rightarrow L\) such that \(\pi \circ s = i d_B\).

Let \((B, \theta_B)\) be a 3-LieDer pair and \((A; \rho, \theta_A)\) a representation of \((B, \theta_B)\). Suppose that \((\psi, \lambda) \in C^2_{3-LieDer}(B; A)\). Define \([\cdot, \cdot, \cdot]_{\rho, \psi} : \wedge^3(B \oplus A) \rightarrow B \oplus A\) and \(\theta_1 : B \oplus A \rightarrow B \oplus A\) as

\[
[x_1 + a_1, x_2 + a_2, x_3 + a_3]_{\rho, \psi} = [x_1 + a_1, x_2, x_3]_B + \psi(x_1, x_2, x_3) + \rho(x_1, x_2)(a_3)
\]

\[
+ \rho(x_2, x_3)(a_1) + \rho(x_3, x_1)(a_2),
\]

\[
\theta_1(x + a) = \theta_B(x) + \lambda(x) + \theta_A(a)
\]

for all \(x, a, a_i \in B, a, a_i \in A\). Denote \(B \oplus A\) with the bracket operation (4.2) by \(B \ltimes_{\rho, \psi} A\).

**Proposition 4.1.** With the above notation. Then \((B \ltimes_{\rho, \psi} A, \theta_1)\) is a 3-LieDer pair if and only if \((\psi, \lambda) \in Z^2_{3-LieDer}(B; A)\). In this case, \(0 \rightarrow (A, \theta_A) \leftrightarrow (B \ltimes_{\rho, \psi} A, \theta_1) \overset{p}{\rightarrow} (B, \theta_B) \rightarrow 0\) is an abelian extension, where \(p\) is the canonical projection.

**Proof.** Suppose that \((B \oplus A, \theta_1)\) is a 3-LieDer pair. It is straightforward to check that \(B \oplus A\) equipped with the bracket \([\cdot, \cdot, \cdot]_{\rho, \psi}\) is a 3-Lie algebra if and only if \(d\psi = 0\). Moreover, by the fact that \(\theta_1\) is a derivation on \(B \ltimes_{\rho, \psi} A\), \(\theta_B\) is a derivation on \(B\) and \((A; \rho, \theta_A)\) is a representation of \((B, \theta_B)\), for any \(x_i \in B, a_i \in A\), we have

\[
[\theta_1(x_1 + a_1), x_2 + a_2, x_3 + a_3]_{\rho, \psi} = [x_1 + a_1, \theta_1(x_2 + a_2), x_3 + a_3]_{\rho, \psi} + [x_1 + a_1, \theta_1(x_2 + a_2), x_3 + a_3]_{\rho, \psi} - \theta_1([x_1 + a_1, x_2 + a_2, x_3 + a_3]_{\rho, \psi})
\]

\[
= \bigcup_{x_1, x_2, x_3} [\theta_B(x_1), x_2, x_3]_B - \theta_B([x_1, x_2, x_3]_B) - \theta_A([x_1, x_2, x_3])
\]

\[
+ \bigcup_{x_1, x_2, x_3} \rho(x_1, x_2)\lambda(x_1) + \bigcup_{x_1, x_2, x_3} \psi(\theta_B(x_1), x_2, x_3) - \theta_A(\psi(x_1, x_2, x_3))
\]

\[
+ \rho(\theta_B(x_1), x_2)(a_3) + \rho(x_1, \theta_B(x_2))(a_3) + \rho(x_1, x_2)\theta_A(a_3) - \theta_A(\rho(x_1, x_2)(a_3))
\]
+ρ(θ_B(x_2), x_3)(a_1) + ρ(x_2, θ_B(x_3))(a_1) + ρ(x_2, x_3)θ_A(a_1) - θ_A(ρ(x_2, x_3)(a_1))
+ρ(θ_B(x_3), x_1)(a_2) + ρ(x_3, θ_B(x_1))(a_2) + ρ(x_3, x_1)θ_A(a_2) - θ_A(ρ(x_3, x_1)(a_2))

= -λ([x_1, x_2, x_3]_B) + \bigcup_{x_1, x_2, x_3} ψ(x_1, x_2, x_3) + \bigcup_{x_1, x_2, x_3} ψ(θ_B(x_1), x_2, x_3) - θ_A(ψ(x_1, x_2, x_3))

= (dλ + δψ)(x_1, x_2, x_3) = 0,

where \bigcup_{x_1, x_2, x_3} denotes the summation over the cyclic permutations of \(x_1, x_2, x_3\). Furthermore, we have

\[ \partial(ψ, λ) = (dψ, dλ + δψ) = (0, 0), \]

which implies that \((ψ, λ) \in Z^2_{3-LieDer}(B; A)\).

The converse can be proved similarly. We omit the details. \(\square\)

Let \(ε_{(L, θ)}\) be an abelian extension of \((B, θ_B)\) by \((A, θ_A)\) and \(s : B \rightarrow L\) a section. Define \(\rho : \Lambda^2 B \rightarrow \text{End}(A), \omega : \Lambda^3 B \rightarrow A\) and \(μ : B \rightarrow A\) by

\[ ρ(x, y)(a) = [s(x), s(y), a]_L, \]
\[ ω(x, y, z) = [s(x), s(y), s(z)]_L - s([x, y, z]_B), \]
\[ μ(x) = θ_L(s(x)) - s(θ_B(x)), \forall x, y, z ∈ B, a ∈ A. \]

Note that \(ρ\) does not depend on the choice of sections.

**Proposition 4.2.** Let \(ε_{(L, θ)}\) be an abelian extension of \((B, θ_B)\) by \((A, θ_A)\). Then \((A; ρ, θ_A)\) is a representation of the 3-LieDer pair \((B, θ_B)\) and \((ω, μ) \in Z^2_{3-LieDer}(B; A)\). Moreover, the cohomology class \([ω, μ] \in H^2_{3-LieDer}(B; A)\) does not depend on the choice of sections.

**Proof.** It is routine to check that \(ρ\) is a representation of the 3-Lie algebra \(B\) on \(A\). We only need to show the following equality

\[ θ_A(ρ(x, y)(a)) = ρ(θ_B(x), y)(a) + ρ(x, θ_B(y))(a) + ρ(x, y)(θ_A(a)), \forall x, y ∈ B, a ∈ A. \]

By a direct calculation, we have

\[ ρ(θ_B(x), y)(a) + ρ(x, θ_B(y))(a) + ρ(x, y)(θ_A(a)) \]
\[ = [s(θ_B(x)), s(y), a]_L + [s(x), s(θ_B(y)), a]_L + [s(x), s(y), θ_L(a)]_L \]
\[ = [θ_L(s(x)) - μ(x), s(y), a]_L + [s(x), θ_L(s(y)) - μ(y), a]_L + [s(x), s(y), θ_L(a)]_L \]
\[ = [θ_L(s(x)), s(y), a]_L + [s(x), θ_L(s(y)), a]_L + [s(x), s(y), θ_L(a)]_L \]
\[ = θ_L([s(x), s(y), a]_L) \]
\[ = θ_A(ρ(x, y)(a)). \]

Thus \((A; ρ, θ_A)\) is a representation of the 3-LieDer pair \((B, θ_B)\).

Since \(ε_{(L, θ)}\) is an abelian extension of \((B, θ_B)\) by \((A, θ_A)\), by Proposition 4.1, \((ω, μ) \in Z^2_{3-LieDer}(B; A)\). Let \(s_1\) and \(s_2\) be two sections of \(π\). Set \(λ(x) = s_1(x) - s_2(x), x ∈ B\). Note that \(λ \in \text{Hom}(B, A)\). By a direct calculation, we obtain that \(ω_1 - ω_2 = dλ\). Moreover,

\[ μ_1(x) - μ_2(x) = θ_L(s_1(x)) - s_1(θ_B(x)) - (θ_L(s_2(x)) - s_2(θ_B(x))) \]
\[ = θ_L(s_1(x) - s_2(x)) - (s_1(θ_B(x)) - s_2(θ_B(x))) \]
\[ = θ_A(λ(x)) - λ(θ_B(x)) \]
\[ = - (δλ)(x). \]
Then we have \((\omega_1, \mu_1) - (\omega_2, \mu_2) = (d\lambda, -\delta\lambda) = \partial\lambda\). Thus the cohomology class \((\omega_1, \mu_1)\) and \((\omega_2, \mu_2)\) are in the same cohomological class.

**Definition 4.2.** Two abelian extensions \(e_{(L_1, \theta_{L_1})}, e_{(L_2, \theta_{L_2})}\) of \((B, \theta_B)\) by \((A, \theta_A)\) are called **equivalent**, if there exists a 3-LieDer pair morphism \(\eta : (L_1, \theta_{L_1}) \rightarrow (L_2, \theta_{L_2})\) such that we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & (A, \theta_A) \\
& \| \quad \eta \downarrow \| \\
0 & \longrightarrow & (A, \theta_A) \\
& \| \quad \eta \downarrow \| \\
& \quad \| \longrightarrow \| \\
& \quad \| \longrightarrow \| \\
& \quad \| \longrightarrow \| \\
& \quad \| \longrightarrow \| \\
\end{array}
\]

\[
(\omega_1, \mu_1) = (\omega_2, \mu_2)
\]

Assume that \(e_{(L_1, \theta_{L_1})}\) and \(e_{(L_2, \theta_{L_2})}\) are equivalent abelian extensions of \((B, \theta_B)\) by \((A, \theta_A)\). It is not hard to see that these equivalent abelian extensions of 3-LieDer pairs give the same representation \(\rho\) defined by \((4.4)\) of the 3-Lie algebra \(B\) on \(A\). Thus \((A; \rho, \theta_A)\) is also a representation of the 3-LieDer pair \((B, \theta_B)\), which does not depend on equivalence classes of abelian extensions of \((B, \theta_B)\) by \((A, \theta_A)\).

**Theorem 4.1.** There is a one-to-one correspondence between equivalence classes of abelian extensions of \((B, \theta_B)\) by \((A, \theta_A)\) and the second cohomology group \(\mathcal{H}_{3-LieDer}^2(B; A)\).

**Proof.** Let \(e_{(L_1, \theta_{L_1})}, e_{(L_2, \theta_{L_2})}\) be two equivalent abelian extensions of \((B, \theta_B)\) by \((A, \theta_A)\). Choose a section \(s_1 : B \rightarrow L_1\) of \(\pi_1\). Since

\[
\pi_2 \circ (\eta \circ s_1) = (\pi_2 \circ \eta) \circ s_1 = \pi_1 \circ s_1 = id_B,
\]

we obtain that \(s_2 := \eta \circ s_1\) is a section of \(\pi_2\). Let \(\omega_i, \mu_i\) respectively given by \((4.5), (4.6)\) corresponding to \(s_i, i = 1, 2\). As \(\eta\) is a 3-LieDer pair morphism and \(\eta|_A = id_A\), for any \(x, y, z \in B\), we have

\[
\omega_2(x, y, z) = [s_2(x), s_2(y), s_2(z)]_L_2 - s_2([x, y, z]_B)
\]

\[
= [\eta(s_1(x)), \eta(s_1(y)), \eta(s_1(z))]_L_2 - \eta(s_1([x, y, z]_B))
\]

\[
= \eta([s_1(x), s_1(y), s_1(z)]_L_1 - s_1([x, y, z]_B))
\]

\[
= \eta(\omega_1(x, y, z)) = \omega_1(x, y, z).
\]

Similarly, by \(\eta \circ \theta_{L_1} = \theta_{L_2} \circ \eta\), we have

\[
\mu_2(x) = \theta_{L_2}(s_2(x)) - s_2(\theta_{L_1}(x))
\]

\[
= \theta_{L_2}(\eta(s_1(x))) - \eta(s_1(\theta_{L_1}(x)))
\]

\[
= \eta(\theta_{L_2}(s_1(x))) - \eta(s_1(\theta_{L_1}(x)))
\]

\[
= \eta(\theta_{L_2}(s_1(x)) - s_1(\theta_{L_1}(x)))
\]

\[
= \eta(\mu_1(x)) = \mu_1(x).
\]

Hence we have \((\omega_1, \mu_1) = (\omega_2, \mu_2)\). By Proposition 4.2, the cohomology class \([(\omega_1, \mu_1)]\) of the abelian extension \(e_{(L_1, \theta_{L_1})}\) and the cohomology class \([(\omega_2, \mu_2)]\) of the abelian extension \(e_{(L_2, \theta_{L_2})}\) do not depend on the choice of sections. Thus equivalence abelian extensions of \((B, \theta_B)\) by \((A, \theta_A)\) give the same element in \(\mathcal{H}_{3-LieDer}^2(B; A)\).

Conversely, let \([(\omega_1, \mu_1)] = [(\omega_2, \mu_2)] \in \mathcal{H}_{3-LieDer}^2(B; A)\). Then there exists a map \(\lambda \in \text{Hom}(B, A)\) such that

\[
(\omega_1, \mu_1) - (\omega_2, \mu_2) = \partial\lambda = (d\lambda, -\delta\lambda).
\]
By Proposition 4.1, $0 \to (A, \theta_A) \hookrightarrow (B \bowtie_{\rho, \omega_1} A, \theta_{\mu_1}) \xrightarrow{\rho_1} (B, \theta_B) \to 0$ and $0 \to (A, \theta_A) \hookrightarrow (B \bowtie_{\rho, \omega_2} A, \theta_{\mu_2}) \xrightarrow{\rho_2} (B, \theta_B) \to 0$ are abelian extensions. Define $\eta : B \bowtie_{\rho, \omega_1} A \to B \bowtie_{\rho, \omega_2} A$ by
\begin{equation}
\eta(x + a) = x + \lambda(x) + a, \forall x \in B, a \in A.
\end{equation}
By a direct computation, we can show that $\eta$ is a 3-LieDer pair morphism such that (4.8) commutes. Thus these two abelian extensions are equivalent.

5. Classification of skeletal and strict 3-Lie2Der pairs

In this section, first we recall the concept of 2-term 3-Lie$_{\omega_1}$-algebra, which is equivalent to 3-Lie 2-algebra. Then we introduce the definition of a 3-Lie2Der pair, which consists of a 3-Lie 2-algebra and a 2-derivation. We show that skeletal 3-Lie2Der pairs can be classified by the triple which consists of a 3-LieDer pair, a representation and a 3-cocycle. Analogous to crossed modules of 3-Lie algebras ([33]), we introduce the definition of crossed modules of 3-LieDer pairs. We show that there exists a one-to-one correspondence between strict 3-Lie2Der pairs and crossed modules of 3-LieDer pairs.

**Definition 5.1.** ([33]) A 3-Lie 2-algebra $V = (V_1, V_0, d, l_3, l_5)$ consists of the following data:
- a complex of vector spaces $V_1 \overset{d}{\to} V_0$,
- a completely skew-symmetric trilinear maps $l_3 : V_i \times V_j \times V_k \to V_{i+j+k}$, where $0 \leq i+j+k \leq 1$.
- a multilinear map $l_5 : (\wedge^3 V_0) \otimes (\wedge^3 V_0) \to V_1$,

such that for any $x, y, z \in V_0$ and $u, v, w \in V_1$, the following equalities are satisfied:
\begin{enumerate}
  \item $d(l_3(x, y, u)) = l_3(x, y, du)$,
  \item $l_3(u, v, w) = 0$; $l_3(u, v, x) = 0$,
  \item $l_3(u, v, x) = l_3(u, dv, x)$,
  \item $d(l_3(x_1, x_2, x_3, x_4, x_5)) = l_3(l_3(x_1, x_2, x_3), x_4, x_5) + l_3(x_3, l_3(x_1, x_2, x_4), x_5) + l_3(x_3, x_4, l_3(x_1, x_2, x_5)) - l_3(x_1, x_2, l_3(x_3, x_4, x_5))$,
  \item $l_3(u, v, x) = l_3(u, x, l_3(x_2, x_3, x_4, x_5))$,
  \item $l_3(u, l_3(x_1, x_2, x_3, x_4, x_5)) - l_3(x_1, x_2, l_3(x_3, x_4, x_5))$,
  \item $l_3(l_3(x_1, x_2, x_3, x_4, x_5), x_6, x_7) + l_3(x_1, l_3(x_2, x_3, x_4, x_5), x_6) + l_3(x_1, x_2, l_3(x_3, x_4, x_5, x_6, x_7))$
  \item $l_3(x_3, x_4, l_3(x_1, x_2, x_5, x_6, x_7)) + l_3(l_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7))$,
  \item $l_3(x_3, x_4, l_3(x_1, x_2, x_5, x_6, x_7)) + l_3(l_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7))$.
\end{enumerate}

A 3-Lie 2-algebra $(V_1, V_0, d, l_3, l_5)$ is called **skeletal (strict)** if $d = 0$ ($l_5 = 0$).

**Definition 5.2.** ([33]) Let $V = (V_1, V_0, d, l_3, l_5)$ and $V' = (V'_1, V'_0, d', l'_3, l'_5)$ be two 3-Lie 2-algebras. A morphism $\varphi : V \to V'$ consists of
- a chain map $f : V \to V'$, which consists of linear maps $f_0 : V_0 \to V'_0$ and $f_1 : V_1 \to V'_1$ satisfying $f_1 \circ d = d' \circ f_0$,
- a completely skew-symmetric trilinear map $f_2 : V_0 \wedge V_0 \wedge V_0 \to V'_1$, such that for any $x, y \in V_0$ and $v \in V_1$, we have
\begin{enumerate}
  \item $d'(f_2(x_1, x_2, x_3)) = f_0(l_3(x_1, x_2, x_3)) - l'_3(f_0(x_1), f_0(x_2), f_0(x_3))$,
  \item $f_2(x_1, x_2, dv) = f_1(l_3(x_1, x_2, v)) - l'_3(f_0(x_1), f_0(x_2), f_1(v))$.
\end{enumerate}
(c) \( l'_2(f_0(x_1), f_0(x_2), f(x_3), f_0(x_4), f_0(x_5), f_0(x_6)) = l'_2(f_0(x_1), f_0(x_2), f(x_3), f_0(x_4), f_0(x_5), f_0(x_6)) \)
\(-l'_2(f_0(x_1), f_0(x_2), f(x_3), f_0(x_4), f_0(x_5)) = l'_2(f_0(x_1), f_0(x_2), f(x_3), f_0(x_4), f_0(x_5)), \)
\(-l'_2(f_0(x_1), f_0(x_2), f(x_3), f_0(x_4), f_0(x_5), f_0(x_6)) = f_2(l_3(x_1, x_2, x_3), x_4, x_5) \)
\(+ f_2(x_3, l_3(x_1, x_2, x_5), x_5) + f_2(x_3, x_4, l_3(x_1, x_2, x_3)) \)
\(- f(x_1, x_2, l_3(x_3, x_4, x_5)) - f_1(l_5(x_1, x_2, x_3, x_4, x_5)). \)

If \( f_0 \) and \( f_1 \) are invertible, then we call \( f \) is an isomorphism.

In the following, we give the definition of 2-derivations on 3-Lie 2-algebras.

**Definition 5.3.** A 2-derivation of a 3-Lie 2-algebra \( \mathcal{V} \) is a triple \( (X_0, X_1, l_X) \), where \( X = (X_0, X_1) \in \text{End}(V_0) \oplus \text{End}(V_1) \) and \( l_X : V_0 \wedge V_0 \to V_1 \), such that for any \( x, y, z, x_i \in V_0 \), \( v \in V_1 \), the following equalities hold:

(a) \( X_0 \circ d = d \circ X_1, \)
(b) \( d l_X(x, y, z) = X_0 l_5(x, y, z) = l_3(x_0, x, y, z) = l_3(x, x_0, y_0, z), \)
(c) \( l_3(x, y, v) = X_1 l_5(x, y, v) = l_3(x, x_0, v) = l_5(x, x_0, x_1, y_1), \)
(d) \( X_1 l_5(x_1, x_2, x_3, x_4, x_5) = l_3(l_3(x_1, x_2, x_3), x_4, x_5) + l_3(l_3(x_1, x_2, x_3), x_4, x_5) \)
\(+ l_3(l_3(x_1, x_2, x_3), x_4, x_5) + l_3(l_3(x_1, x_2, x_3), x_4, x_5) \)
\(+ \sum_{i=1}^5 l_5(x_1, \cdots, X_0(x_5)), \cdots, x_5). \)

If \( l_X = 0 \), then we call the triple \( (X_0, X_1, 0) \) is a strict 2-derivation of the 3-Lie 2-algebra \( \mathcal{V} \).

**Example 5.1.** Let \( \mathcal{V} = (V_1, V_0, d, l_3, l_5) \) be a 3-Lie 2-algebra. Fix any \( x \wedge y \in \wedge^2 V_0 \). Define \( ad_{x \wedge y} \in \text{End}(V_0 \oplus V_1) \) by
\[ ad_{x \wedge y}(z + v) = l_3(x, x_0, z + v), \quad z \in V_0, v \in V_1. \]
Set \( X_0 = ad_{x \wedge y}|_{V_0}, X_1 = ad_{x \wedge y}|_{V_1}. \) Define \( l_X : V_0 \wedge V_0 \to V_1 \) by
\[ l_X(x_1, x_2, x_3) = -l_5(x, x_1, x_2, x_3), \quad x_1, x_2, x_3 \in V_0. \]
Then \( (X_0, X_1, l_X) \) is a 2-derivation of \( \mathcal{V} \).

A 3-Lie 2-algebra \( \mathcal{V} \) with a 2-derivation will be denoted by \( \langle \mathcal{V}; (X_0, X_1, l_X) \rangle \), which we call it a \textbf{3-Lie2Der pair}. In particular, a skeletal 3-Lie 2-algebra (resp. a strict 3-Lie 2-algebra) with a 2-derivation (resp. a strict 2-derivation) will be called a \textbf{skeletal 3-Lie2Der pair} (resp. a \textbf{strict 3-Lie2Der pair}).

**Definition 5.4.** Let \( \langle \mathcal{V}; (X_0, X_1, l_X) \rangle \) and \( \langle \mathcal{V}'; (X'_0, X'_1, l'_X) \rangle \) be two 3-Lie2Der pairs. An isomorphism from \( \langle \mathcal{V}; (X_0, X_1, l_X) \rangle \) to \( \langle \mathcal{V}'; (X'_0, X'_1, l'_X) \rangle \) is a quadruple \( (f_0, f_1, f_2, g) \), where \( f_0 : V_0 \to V'_0 \), \( f_1 : V_1 \to V'_1 \), \( f_2 : V_0 \wedge V_0 \to V'_1 \) and \( g : V_0 \to V'_1 \) are linear maps, such that \( (f_0, f_1, f_2) \) is a 3-Lie 2-algebra isomorphism and the following identities hold for all \( x, y, z, x_i \in V_0, v \in V_1: \)

(a) \( X'_0(f_0(x)) - f_0(X_0(x)) = d(g(x)), \)
(b) \( X'_1(f_1(v)) = g(d(v)), \)
(c) \( f_1(l_X(x, y, z)) + f_2(X_0(x, y, z)) + f_3(x, X_0(y, z)) + f_2(x, x_0, y, z) \)
\(- l'_3(f_0(x), f_0(y), f_0(z)) = l'_3(g(x), f_0(y), f_0(z)) \)
\(+ l'_5(f_0(x), g(y), f_0(z)) + l'_5(f_0(x), f_0(y), g(z)) - g(l_3(x, y, z)). \)

If \( f_2 = g = 0 \), then the quadruple \( (f_0, f_1, 0, 0) \) is called a \textbf{strict isomorphism}.

**Proposition 5.1.** There exists a one-to-one correspondence between skeletal 3-Lie2Der pairs and triples \((L, \theta), (V; \rho, \theta_V), (\alpha_1, \alpha_2)\), where \((L, \theta_L)\) is a 3-LieDer pair, \((V; \rho, \theta_V)\) is a representation of \((L, \theta_L)\), and \((\alpha_1, \alpha_2)\) is a 3-cocycle associated to the representation \((V; \rho, \theta_V)\).
Proof. Suppose that \((V; (X_0, X_1, l_X))\) is a skeletal 3-Lie2Der pair, where \(V = (V_1, V_0, 0, l_3, l_5)\). By condition (d) in Definition 5.1 we deduce that \(V_0\) is a 3-Lie algebra with the 3-Lie bracket given by \(l_3\). By condition (b) in Definition 5.3 we obtain that \(X_0\) is a derivation of \(V_0\) and hence \((V_0, X_0)\) is a 3-LieDer pair.

Define \(\rho : \wedge^2 V_0 \rightarrow \text{End}(V_1)\) by

\[
\rho(x, y)(v) = l_3(x, y, v), \quad \forall \, x, y \in V_0, v \in V_1.
\]

By conditions (e) and (f) in Definition 5.1 \(\rho\) is a representation of 3-Lie algebra \(V_0\). Then by Definition 2.5 and condition (c) in Definition 5.3 \((V_1; \rho, X_1)\) is a representation of 3-LieDer pair \((V_0, X_0)\). Moreover, by condition (g) in Definition 5.1 we get \(dl_5 = 0\) and by condition (d) in Definition 5.3 we get \(dl_X - \delta l_5 = 0\). Thus we have

\[
\partial(l_5, l_X) = (dl_5, d(l_X) + (-1)^3 \delta l_5) = (0, 0),
\]

which implies that \((l_5, l_X)\) is a 3-cocycle.

Conversely, let \((V; \rho, \theta, \sigma)\) be a representation of 3-LieDer pair \((L, \theta_L)\) and \((\alpha_3, \alpha_2) \in \mathbb{Z}_3^\text{LieDer}(L; V)\).

Set \(V_1 = V, V_0 = L\). For \(x, y, z \in V_0, u, v \in V_1\), define \(l_5 : V_1 \times V_1 \times V_1 \rightarrow V_1\) by

\[
\begin{align*}
    l_5(x, y, z) &= [x, y, z]_L, \\
    l_5(x, u, v) &= l_3(u, x, v) = l_3(u, v, x) = 0, \\
    l_5(u, x, y) &= l_3(x, u, y) = l_3(x, y, u) = \rho(x, y)(u)
\end{align*}
\]

and for any \(x_i, 1 \leq i \leq 5\), define \(l_5^i : (\wedge^2 V_0) \otimes (\wedge^3 V_0) \rightarrow V_1\) by

\[
l_5^i(x_1, x_2, x_3, x_4, x_5) = \alpha_3(x_1, x_2, x_3, x_4, x_5).
\]

Then \(V := (V_1, V_0, 0, l_3, l_5)\) is a skeletal 3-Lie 2-algebra.

Define \(l_X : V_0 \wedge V_0 \wedge V_0 \rightarrow V_1\) by

\[
l_X(x, y, z) = -l_X(y, x, z) = -l_X(x, z, y) = \alpha_2(x, y, z).
\]

Then \((\theta_V, \theta_L, l_X)\) is a 2-derivation of the skeletal 3-Lie 2-algebra \(V\). Therefore, \((V; (\theta_V, \theta_L, l_X))\) is a skeletal 3-Lie2Der pair.

In the following, we shall define the equivalence relation between triples \(((L, \theta_L), (V; \rho, \theta_V), (\alpha_3, \alpha_2))\) and show that there is a one-to-one correspondence between equivalent classes of such triples and isomorphism classes of skeletal 3-Lie2Der pairs.

Definition 5.5. Let \(((L, \theta_L), (V; \rho, \theta_V), (\alpha_3, \alpha_2))\) and \(((L', \theta_{L'}), (V'; \rho', \theta_{V'}), (\alpha'_3, \alpha'_2))\) be two triples as described in Proposition 5.7. They are called equivalent if there exist 3-LieDer pair isomorphism \(\sigma : (L, \theta_L) \rightarrow (L', \theta_{L'})\), linear isomorphism \(\tau : V \rightarrow V'\) and two linear maps \(\lambda : L \wedge L \wedge L \rightarrow V'\), \(\mu : L \rightarrow V'\) such that for any \(x, y, z \in L\), the following equalities hold:

(a) \(\theta_{V'} \circ \tau = \tau \circ \theta_V\),
(b) \(\tau \circ \rho(x, y) = \rho'(\sigma(x), \sigma(y))\),
(c) \(\lambda([x_1, x_2, x_3]_L, x_4, x_5) + \lambda(x_3, [x_1, x_2, x_4]_L, x_5) + \lambda(x_3, x_4, [x_1, x_2, x_5]_L) - \lambda(x_1, x_2, [x_3, x_4, x_5]_L)_L - \lambda(x_3, x_4, [x_1, x_2, x_5]_L) = \rho'([\sigma(x_1), \sigma(x_2), \sigma(x_3)]_L, \sigma(x_4), \sigma(x_5)) + \rho'([\sigma(x_1), \sigma(x_2), \sigma(x_3)], \sigma(x_4), \sigma(x_5)) + \rho'([\sigma(x_1), \sigma(x_2), \sigma(x_3)], \sigma(x_4), \sigma(x_5)) - \alpha'_3(\sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4), \sigma(x_5))\).


\( \tau(\alpha_2(x, y, z)) + \lambda(\theta_L(x), y, z) + \lambda(\alpha_2, \theta_T(y), z) + \lambda(x, \theta_L(z)) \)

\(-\theta_V(\lambda(x, y, z)) - \alpha'_2(\sigma(x), \sigma(y), \sigma(z)) = \rho'(\sigma(x), \sigma(y), \mu(z)) \\
\quad + \rho'(\sigma(y), \sigma(z))\mu(x) + \rho'(\sigma(z), \sigma(x))\mu(y) - \mu([x, y, z]_L). \)

**Theorem 5.1.** There exists a one-to-one correspondence between isomorphism classes of skeletal 3-Lie2Der pairs and equivalent classes of triples \((L, \theta_L), (V; \rho, \theta_V), (\alpha_3, \alpha_2)\), where \((L, \theta_L)\) is a 3-LieDer pair; \((V; \rho, \theta_V)\) is a representation of \((L, \theta_L)\), and \((\alpha_3, \alpha_2)\) is a 3-cocycle associated to the representation \((V; \rho, \theta_V)\).

**Proof.** Let \((f_0, f_1, f_2, g)\) be an isomorphism between skeletal 3-Lie2Der pairs \((V; (X_0, X_1, l_3))\) and \((V'; (X'_0, X'_1, l'_3))\). Let \(((V_0, X_0), (V_1; \rho, X_1), (l_5, l_3))\) and \(((V'_0, X'_0), (V'_1; \rho', X'_1), (l'_5, l'_3))\) be the corresponding triples given by Proposition 5.1. Set \(\sigma = f_0, \tau = f_1, \lambda = f_2, \mu = g\). It is obvious that \(\tau : V_1 \to V'_1\) is a linear isomorphism. By condition (a) in Definition 5.5 and condition (a) in Definition 5.4, \(\sigma : (V_0, \theta_{V_0}) \to (V'_0, \theta_{V'_0})\) is a 3-LieDer pair isomorphism.

By conditions (b) and (c) in Definition 5.4 we can show that conditions (a) and (d) in Definition 5.5 hold. By conditions (b) and (c) in Definition 5.2 we obtain that conditions (b) and (c) in Definition 5.5 hold. Thus the triples \(((V_0, X_0), (V_1; \rho, X_1), (l_5, l_3))\) and \(((V'_0, X'_0), (V'_1; \rho', X'_1), (l'_5, l'_3))\) are equivalent.

The converse can be proved similarly. We omit the details. \(\square\)

In the following, we give the definition of crossed modules of 3-LieDer pairs.

**Definition 5.6.** A crossed module of 3-LieDer pairs is a quadruple \((A, \theta_A, B, \theta_B, (A; \rho, \theta_A, \eta))\), where \((A, \theta_A)\) and \((B, \theta_B)\) are 3-LieDer pairs, \((A; \rho, \theta_A)\) is a representation of 3-LieDer pair \((B, \theta_B)\) and \(\eta : (A, \theta_A) \to (B, \theta_B)\) is a morphism of 3-LieDer pairs, such that for any \(u, v, w \in A, x, y \in B,\)

\[
\begin{align*}
\eta(\rho(x, y)(u)) &= [x, y, \eta(u)]_B, \\
\rho(\eta(u), \eta(v))(w) &= [u, v, w]_A, \\
\rho(x, \eta(u), v)(w) &= -\rho(x, \eta(v))(u), \\
\rho(x, y)[u, v, w]_A &= [\rho(x, y)(u), v, w]_A + [u, \rho(x, y)(v), w]_A + [u, v, \rho(x, y)(w)]_A.
\end{align*}
\]

**Proposition 5.2.** There exists a one-to-one correspondence between strict 3-Lie2Der pairs and crossed modules of 3-LieDer pairs.

**Proof.** Suppose that \((V; (X_0, X_1, 0))\) is a strict 3-Lie2Der pair, where \(V = (V_1, V_0, d, l_3, 0)\). Let \(A = V_1\) and \(B = V_0\). Define the 3-ary bracket operations \([\cdot, \cdot, \cdot]_A\) and \([\cdot, \cdot, \cdot]_B\) by

\[
\begin{align*}
[u, v, w]_A &= l_3(du, dv, w), \\
[x, y, z]_B &= l_3(x, y, z).
\end{align*}
\]

Then it is straightforward to check that \((A, [\cdot, \cdot, \cdot]_A)\) and \((B, [\cdot, \cdot, \cdot]_B)\) are 3-Lie algebras. Moreover, by conditions (a) and (c) in Definition 5.3 we have

\[
\begin{align*}
X_1([u, v, w]_A) &= X_1l_3(du, dv, w) \\
&= l_3(X_0(du), dv, w) + l_3(du, X_0(dv), w) + l_3(du, dv, X_1(w)) \\
&= l_3(d(X_1(u)), dv, w) + l_3(du, d(X_1(v)), w) + l_3(du, dv, X_1(w)) \\
&= [X_1(u), v, w]_A + [u, X_1(v), w]_A + [u, v, X_1(w)]_A,
\end{align*}
\]

which implies that \(X_1\) is a derivation of \((A, [\cdot, \cdot, \cdot]_A)\). Similarly, by conditions (b) in Definition 5.3 we deduce that \(X_0\) is a derivation of \((B, [\cdot, \cdot, \cdot]_B)\). Therefore, \((A, X_1)\) and \((B, X_0)\) are 3-LieDer pairs.
Define \( \rho : \wedge^2 B \to \text{End}(A) \) by

\[
(5.7) \quad \rho(x, y)(u) = l_3(x, y, u).
\]

By conditions (e) and (f) in Definition \([5.1]\), \( \rho \) is a representation of 3-Lie algebra \( B \). By Definition \([2.5]\) and condition (c) in Definition \([5.3]\), we obtain that \( (A; \rho, X_1) \) is a representation of 3-LieDer pair \( (B, X_0) \).

Let \( \eta = d : A \to B \). Then by condition (a) in Definition \([5.3]\) and the fact that

\[
\eta([u, v, w]) = dl_3(du, dv, dw) = l_3(du, dv, dw) = [du, dv, dw]_B = [\eta(u), \eta(v), \eta(w)]_B.
\]

We deduce that \( \eta \) is a 3-LieDer pair morphism.

By condition (f) in Definition \([5.1]\) we have

\[
\rho(x, y)([u, v, w]) = l_3(x, y, l_3(u, dv, dw))
\]

\[
= l_3(l_3(x, y, u), dv, dw) + l_3(u, l_3(x, y, dv), dw) + l_3(u, dv, l_3(x, y, dw))
\]

\[
= l_3(\rho(x, y)(u), dv, dw) + l_3(u, dl_3(x, y, v), dw) + l_3(u, dv, dl_3(x, y, w))
\]

\[
= [\rho(x, y)(u), v, w]_A + [u, \rho(x, y)(v), w]_A + [u, v, \rho(x, y)(w)]_A,
\]

which implies that \([5.4]\) holds.

By a direct calculation, we have

\[
\eta(\rho(x, y)(u)) = d\rho(x, y)(u) = dl_3(x, y, u) = l_3(x, y, du) = [x, y, du]_B,
\]

\[
\rho(\eta(u), \eta(v))(w) = l_3(\eta(u), \eta(v), w) = l_3(du, dv, w) = [u, v, w]_A,
\]

\[
\rho(x, \eta(u))(v) = l_3(x, du, v) = -l_3(x, dv, u) = -\rho(x, dv)(u),
\]

which imply that \([5.1] - [5.3]\) hold. Therefore, the quadruple \( ((A, X_1), (B, X_0), (A; \rho, X_1), \eta) \) is a crossed module of 3-LieDer pairs.

The converse can be proved similarly. We omit the details. \( \square \)

**Definition 5.7.** Let \( ((A, \theta_A), (B, \theta_B), (A; \rho, \theta_A), \eta) \) and \( ((A', \theta_{A'}), (B', \theta_{B'}), (A; \rho', \theta_{A'}), \eta') \) be crossed modules of 3-LieDer pairs. They are called **equivalent** if there exist 3-LieDer pair isomorphisms \( \sigma : (B, \theta_B) \to (B', \theta_{B'}) \) and \( \tau : (A, \theta_A) \to (A', \theta_{A'}) \) such that for any \( x, y \in B \), the following equalities hold:

(a) \( \tau \circ \eta = \eta' \circ \sigma \),

(b) \( \tau \circ \rho(x, y) \circ \tau^{-1} = \rho'(\sigma(x), \sigma(y)) \).

**Theorem 5.2.** There exists a one-to-one correspondence between strict isomorphism classes of strict 3-Lie2Der pairs and equivalent classes of crossed modules of 3-LieDer pairs.

**Proof.** Let \( (f_0, f_1, 0, 0) \) be the strict isomorphism between the strict 3-Lie2Der pairs \( (V; (X_0, X_1, 0)) \) and \( (V'; (X'_0, X'_1, 0)) \). Let \( ((V, X_1), (V_0, X_0), (\rho, V_1, X_1)), \eta) \) and \( ((V', X'_1), (V'_0, X'_0), (\rho', V'_1, X'_1)), \eta') \) be crossed modules of 3-LieDer pairs, corresponding to \( (V; (X_0, X_1, 0)) \) and \( (V'; (X'_0, X'_1, 0)) \) given by Proposition \([5.2]\) respectively. By conditions (a) and (b) in Definition \([5.4]\), we get that \( \sigma : (V_0, X_0) \to (V'_0, X'_0) \) and \( \tau : (V_1, X_1) \to (V'_1, X'_1) \) are 3-LieDer pair isomorphisms.

Set \( \sigma = f_0, \tau = f_1, \eta = d \) and \( \eta' = d' \). Define \( \rho : \wedge^2 V_0 \to \text{End}(V_1) \) and \( \rho' : \wedge^2 V'_0 \to \text{End}(V'_1) \) by

\[
\rho(x, y)(u) = l_3(x, y, u), \quad x, y \in V_0, \quad u \in V_1,
\]

\[
\rho(x', y')(u') = l_3'(x', y', u'), \quad x', y' \in V'_0, \quad u' \in V'_1.
\]
By the condition $f_1 \circ d = d' \circ f_0$ in Definition [5.2], we have $\tau \circ \eta = \eta' \circ \sigma$. By condition (b) in Definition [5.2], we deduce that $\tau \circ \rho(x, y) \circ \tau^{-1} = \rho' (\sigma(x), \sigma(y))$. Therefore, the above two crossed modules of 3-LieDer pairs are equivalent.

The converse can be proved similarly. We omit the details. □

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