IN VARIANTS OF FANO VARIETIES IN FAMILIES

FRANK GOUNELAS AND ARIYAN JAVANPEYKAR

Abstract. We show that the Picard rank is constant in families of Fano varieties (in arbitrary characteristic) and we moreover investigate the constancy of the index.

1. Introduction

The aim of this paper is to extend basic properties of invariants of complex algebraic (smooth) Fano varieties to positive characteristic.

For a smooth projective variety over a field $k$, the Néron-Severi group $\text{NS}(X)$ is finitely generated. We let $\rho(X)$ denote the rank of $\text{NS}(X_{\bar{k}})$, where $\bar{k}$ is an algebraic closure of $k$. We refer to $\rho(X)$ as the (geometric) Picard rank of $X$. Recall that a smooth projective geometrically connected variety $X$ over a field $k$ is a (smooth projective) Fano variety if the dual $\omega_X^\vee$ of the canonical invertible sheaf is ample. It is not hard to show that the Picard rank $\rho(X)$ of a Fano variety $X$ over $\mathbb{C}$ equals the second Betti number of $X$ [15, Prop. 2.1.2]. In particular, as Betti numbers are constant in smooth complex algebraic families, the Picard rank is constant in any complex algebraic family of Fano varieties parametrized by a connected variety.

The constancy of the Picard rank in families plays an important role in the classification of Fano varieties. Motivated by the classification problem for Fano varieties in positive characteristic, we show that the Picard rank is constant in families of Fano varieties.

Note that Fano varieties are rationally chain connected [3, 14]. A proof of the following two results, using the decomposition of the diagonal, is well-known to experts; we include this proof in the appendix.

Theorem 1.1. Let $X \to S$ be a smooth proper morphism of schemes whose geometric fibres are rationally chain connected smooth projective varieties. If $S$ is connected, then the geometric Picard rank is constant on the fibres, i.e., for all $s$ and $t$ in $S$, we have $\rho(X_s) = \rho(X_t)$.

We were first led to investigate this problem when studying integral points on the complex algebraic stack of Fano varieties; see [16, §2]. The results obtained in loc. cit. use deformation-theoretic techniques and only deal with characteristic zero or mixed characteristic. In fact, in loc. cit. Kodaira vanishing plays a crucial role. However, there are (smooth) Fano varieties in characteristic two which violate Kodaira vanishing [20].

It is not difficult to see that the Picard rank can jump in families of non-Fano varieties (e.g. K3 surfaces). However, note that the Picard rank is constant in families of Enriques surfaces [22]. (The constancy of the Picard rank in families of Enriques surfaces also follows from the results in [21].)

By the smooth proper base change theorem for étale cohomology, to prove Theorem 1.1 it suffices to show that the Picard rank equals the second $\ell$-adic Betti number for all prime numbers $\ell$ invertible in the base field. As Fano varieties are rationally chain connected, the latter follows from the next result.

Theorem 1.2. Let $X$ be a smooth projective rationally chain connected variety over an algebraically closed field $k$. For all prime numbers $\ell$, the homomorphism

$$\text{NS}(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1))$$



2010 Mathematics Subject Classification. 14J45, (14K30, 14D23).

Key words and phrases. Fano varieties, Picard rank, index, crystalline cohomology, rational connectedness.
is an isomorphism of \( \mathbb{Z}_p \)-modules.

For \( X \) a scheme, we let \( \text{Br}(X) \) be the cohomological Brauer group \( H^2_{\text{et}}(X, \mathbb{G}_m) \). Note that, if \( X \) is a regular scheme, then \( \text{Br}(X) \) is a torsion abelian group [3 Prop. 1.4]. To prove Theorem 1.2 we use geometric arguments following a suggestion of Jason Starr. Namely, we use simple facts about Brauer groups of compact type curves (see Section 2) to prove that the Brauer group of a rationally chain connected variety is killed by some integer \( d \geq 1 \) (see Proposition 4.2). A well-known argument involving the Kummer sequence then concludes the proof of Theorem 1.2 (and thus Theorem 1.1).

Another natural invariant of a Fano variety is its index. Let \( X \) be a Fano variety over a field \( k \). We define its (geometric) index \( r(X) \) to be the largest \( r \in \mathbb{N} \) such that \(-K_X^r\) is divisible by \( r \) in Pic \( X \). It is not hard to show that the index is constant in families of Fano varieties over the complex numbers. It seems reasonable to suspect that the index is constant in families of Fano varieties in mixed or positive characteristic; see Proposition 6.3 for some partial results. In general, we are only able to establish the constancy of the index up to multiplication by powers of the characteristic of the residue field.

**Theorem 1.3.** Let \( S \) be a trait with generic point \( \eta \). Assume that the characteristic \( p \) of the closed point \( s \) of \( S \) is positive. Let \( X \to S \) be a smooth proper morphism of schemes whose geometric fibres are Fano varieties. Then, there is an integer \( i \geq 0 \) such that \( r(X_s) = p^i r(X_\eta) \).

In the hope of establishing the constancy of the index for families of Fano varieties, we investigate \( p \)-adic “crystalline” analogues of Theorem 1.2. For a perfect field \( k \), we let \( W = W(k) \) be the Witt ring of \( k \).

**Theorem 1.4.** Let \( X \) be a smooth projective connected scheme over an algebraically closed field \( k \) of characteristic \( p > 0 \). If \( X \) is separably rationally connected or \( X \) is a Fano variety with \( H^0(X, \Omega_X^1) = 0 \), then \( H^2_{\text{crys}}(X/W) \) is torsion-free, \( H^1(X, \mathcal{O}_X) = 0 \), and the morphisms of \( \mathbb{Z}_p \)-modules

\[
\text{NS}(X) \otimes \mathbb{Z}_p \to H^2_{\text{ppd}}(X, \mathbb{Z}_p(1)) \to H^2_{\text{crys}}(X/W)^{F^p}
\]

are isomorphisms.

It is currently not known whether \( H^1(X, \mathcal{O}_X) = 0 \), for \( X \) a Fano variety over an algebraically closed field \( k \). In [24] Shepherd-Barron proves that \( H^1(X, \mathcal{O}_X) = 0 \) for all Fano varieties \( X \) of dimension at most three (see also [24 Corollary 2]).

**Acknowledgements.** The authors would like to thank Jason Starr for sketching how an argument for proving Proposition 1.2 should work. We are most grateful to Jean-Louis Colliot-Thélène and Charles Vial for their comments and suggestions, and for pointing out a mistake in our proof of Theorem A.1 in an earlier version. We also thank Ben Bakker, François Charles, Cristian González-Avilés, Ofer Gabber, Raju Krishnamoorthy, Christian Liedtke, Daniel Loughran, and Kay Rülling for helpful discussions. The second named author gratefully acknowledges support from SFB/Transregio 45.

2. **Brauer groups of compact type curves**

In this section, we let \( k \) be a field (of arbitrary characteristic). The aim of this section is to investigate Brauer groups of mildly singular curves over \( k \). Similar statements (assuming \( k \) is of characteristic zero) are obtained by Harpaz-Skorobogatov [11].

A proper connected reduced one-dimensional scheme over \( k \) with only ordinary double points is a curve of compact type if its dual graph is a tree. In other words, a nodal proper connected reduced curve over \( k \) is of compact type if and only if every node is disconnecting. Note that the normalization of a compact type curve is the disjoint union of its irreducible components.

We will frequently use “partial normalizations”. More precisely, let \( s \) in \( C(k) \) be a singular point of \( C \). Since \( s \) is a disconnecting node, there are precisely two closed subschemes \( C_1 \) and \( C_2 \) such that \( s \in C_1(k), s \notin C_2(k), C = C_1 \cup C_2, \) and \( C_1 \cap C_2 = \{s\} \). Note that \( C_1 \) and \( C_2 \) are unique (up to renumbering). We define the partial normalization of \( C \) with respect to \( s \) to be the morphism \( C' \to C \), where \( C' = C_1 \cup C_2 \). It is clear that the normalization \( \tilde{C} \to C \) of \( C \) factors as \( \tilde{C} \to C' \to C \).
Lemma 2.1. Let $C$ be a compact type curve over $k$ and $Z$ an integral scheme of finite type over $k$. Let $0 \in C(k)$ be a singular point and let $C' \rightarrow C$ be the partial normalization with respect to 0. Then the pullback morphism $\text{Br}(Z \times C) \rightarrow \text{Br}(Z \times C')$ is injective.

Proof. Note that $C'$ is the disjoint union of two compact type curves, $C_1$ and $C_2$ say. Also, note that $0 \in C(k)$ induces a point $0_1 \in C_1(k)$ and a point $0_2 \in C_2(k)$. Let $Y = Z \times C'$ and let $\nu : Y \rightarrow Z \times C$ be the morphism induced by the partial normalization $C' \rightarrow C$. Note that $Y$ is the disjoint union of $Y_1 := Z \times C_1$ and $Y_2 := Z \times C_2$. Let $j : Z \cong Z \times \{0\} \rightarrow Z \times C$ be the closed immersion induced by $0 \in C(k)$. We have a short exact sequence of étale sheaves on $Z \times C$

$$0 \rightarrow G_m \rightarrow \nu_* G_m \rightarrow j_* G_m \rightarrow 0.$$ 

Since $\nu$ is a finite morphism, this induces an exact sequence

$$\text{Pic}(Z \times C') \xrightarrow{j} \text{Pic} Z \rightarrow \text{Br}(Z \times C) \rightarrow \text{Br}(Z \times C').$$

Note that, to prove the lemma, it suffices to show that $f$ is surjective. To do so, let $L$ be (the isomorphism class of) a line bundle on $Y$. Let $L_1$ be the induced line bundle on $Y_1$ and $L_2$ be the induced line bundle on $Y_2$. The homomorphism $f$ maps $L$ to the line bundle $p^* L_1 \otimes q^* L_2^{-1}$ on $Z$, where $p : Z \cong Z \times \{0\} \rightarrow Y_1$ is the section induced by $0_1$ and $q : Z \cong Z \times \{0_2\} \rightarrow Y_2$ is the section induced by $0_2$. It is clear that this map is surjective.

Corollary 2.2. Let $C$ be a compact type curve over $k$ and let $Z$ be a smooth integral finite type scheme over $k$ with function field $L = K(Z)$. Then, the natural morphism $\text{Br}(Z \times C) \rightarrow \text{Br}(\text{Spec } L \times \times C)$ is injective.

Proof. Let $N$ be the number of irreducible components of $C$. We argue by induction on $N$. If $N = 1$, then $Z \times_k C$ is integral over $k$ and the natural pull-back morphism $\text{Br}(Z \times_k C) \rightarrow \text{Br}(K(Z \times_k C))$ is injective [25 Cor IV.2.6]. In particular, as the natural morphism $\text{Spec } K(Z \times_k C) \rightarrow Z \times_k C$ factors as $\text{Spec } K(Z \times_k C) \rightarrow \text{Spec } L \times_k C \rightarrow Z \times_k C$, it follows that $\text{Br}(Z \times_k C) \rightarrow \text{Br}(\text{Spec } L \times_k C)$ is injective.

Thus, to prove the corollary, we may and do assume that $N \geq 2$. The natural morphism $\text{Spec } L \rightarrow Z$ induces the natural morphism $i : \text{Spec } L \times C \rightarrow Z \times C$. Let $C' \rightarrow C$ be the partial normalization with respect to the choice of a singular point $0$ in $C(k)$. Note that there is a natural morphism $j : \text{Spec } L \times C' \rightarrow Z \times_k C'$, and that $C' = C_1 \sqcup C_2$, where $C_1$ and $C_2$ are compact type curves. Note that the number of irreducible components of $C_1$ (resp. $C_2$) is less than $N$.

There are natural morphisms $f : Z \times_k C' \rightarrow Z \times_k C$ and $g : \text{Spec } (L) \times_k C' \rightarrow \text{Spec } (L) \times_k C$ induced by the morphism $C' \rightarrow C$. Note that $f^*$ is injective by Lemma 2.1. Since the number of irreducible components of $C_1$ (resp. $C_2$) is less than $N$, by the induction hypothesis, the natural pull-back morphisms $j_1^* : \text{Br}(Z \times C_1) \rightarrow \text{Br}(\text{Spec } L \times C_1)$ and $j_2^* : \text{Br}(Z \times C_2) \rightarrow \text{Br}(\text{Spec } L \times C_2)$ are injective. Note that

$$\text{Br}(Z \times C') = \text{Br}(Z \times C_1) \oplus \text{Br}(Z \times C_2)$$

and likewise

$$\text{Br}(\text{Spec } L \times C') = \text{Br}(\text{Spec } L \times C_1) \oplus \text{Br}(\text{Spec } L \times C_2).$$

Thus, we see that $j^* = j_1^* \oplus j_2^*$ is injective. It follows that $j^* \circ f^*$ is injective. As $g^* \circ i^* = j^* \circ f^*$, we conclude that $i^*$ is injective.

A compact type curve $C$ over $k$ is of genus zero if all irreducible components of $C_k$ have genus zero. Note that, in this case, all irreducible components of $C_k$ are isomorphic to $\mathbb{P}^1_k$ over $k$.

Lemma 2.3. Let $C$ be a compact type curve of genus zero over $k$. Assume that all irreducible components of $C$ are geometrically irreducible, and that $C(k) \neq \emptyset$. Then the natural pull-back morphism $\text{Br}(k) \rightarrow \text{Br}(C)$ is an isomorphism.

Proof. Let $N$ be the number of irreducible components of $C$. We argue by induction on $N$. Firstly, if $N = 1$, then the result is well-known. Indeed, if $N = 1$, then $C \cong \mathbb{P}^1_k$. Let $k \rightarrow k^s$ be a separable closure of $k$. It is not hard to show that $\text{Br}(k) = \ker[\text{Br}(\mathbb{P}^1_k) \rightarrow \text{Br}(\mathbb{P}^1_{k^s})]$; see [23] Thm. 42.8]. By a
theorem of Grothendieck [39 Corollaire 5.8] (see also [29]), we have that \( \text{Br}(\mathbb{P}^1) = 0 \). We conclude that the result holds for \( N = 1 \). Thus, to prove the lemma, we may and do assume that \( N \geq 2 \).

Let \( s : \text{Spec} \ k \to C \) be a singular point. Let \( \nu : C' \to C \) be the partial normalization of \( C \) with respect to \( s \). Note that \( C' = C_1 \sqcup C_2 \), where \( C_1 \) and \( C_2 \) are compact type curves of genus zero over \( k \). Note that the irreducible components of \( C_1 \) and \( C_2 \) are geometrically irreducible, and that \( C_1(k) \neq \emptyset \) and \( C_2(k) \neq \emptyset \). Let \( s_1 : \text{Spec} \ k \to C_1 \) and \( s_2 : \text{Spec} \ k \to C_2 \) be the sections corresponding to \( s : \text{Spec} \ k \to C \). Note that the following diagram of groups

\[
\begin{array}{ccc}
\text{Br}(C) & \xrightarrow{\nu^*} & \text{Br}(C_1) \oplus \text{Br}(C_2) \\
\downarrow{s^*} & & \downarrow{s_1^* \oplus s_2^*} \\
\text{Br}(k) & \xrightarrow{\Delta} & \text{Br}(k) \oplus \text{Br}(k)
\end{array}
\]

is commutative. It follows from Lemma 2.1 that the morphism \( \nu^* \) is injective. By the induction hypothesis, since the number of irreducible components of \( C_1 \) and \( C_2 \) is less than \( N \), the natural pull-back morphisms \( \text{Br}(k) \to \text{Br}(C_1) \) and \( \text{Br}(k) \to \text{Br}(C_2) \) are isomorphisms. In particular, \( s_1^* \oplus s_2^* : \text{Br}(C_1) \oplus \text{Br}(C_2) \to \text{Br}(k) \oplus \text{Br}(k) \) is an isomorphism. We conclude that \( (s_1^* \oplus s_2^*) \circ \nu^* \) is injective. However, since \( \Delta^* \circ s^* = (s_1^* \oplus s_2^*) \circ \nu^* \), we conclude that \( s^* \) is injective. Since \( s^* : \text{Br}(C) \to \text{Br}(k) \) is surjective, we conclude that \( s^* : \text{Br}(C) \to \text{Br}(k) \) is an isomorphism. The result follows.

**Remark 2.4.** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) with \( H^1_{\text{et}}(\text{Spec} \ \mathbb{Q}, E) \neq 0 \). As is shown in [21, §2], the Leray spectral sequence induces a short exact sequence

\[
\text{Br}(\mathbb{Q}) \to \text{Br}(E) \to H^1_{\text{et}}(\text{Spec} \ \mathbb{Q}, E) \to 0.
\]

Since \( E(\mathbb{Q}) \neq \emptyset \), the natural pull-back morphism \( \text{Br}(\mathbb{Q}) \to \text{Br}(E) \) is injective. As \( H^1_{\text{et}}(\text{Spec} \ \mathbb{Q}, E) \) is non-trivial, the natural morphism \( \text{Br}(\mathbb{Q}) \to \text{Br}(E) \) is not an isomorphism.

**Corollary 2.5.** Let \( Z \) be a smooth integral scheme of finite type over \( k \). Let \( C \) be a compact type curve of genus zero over \( k \) with \( C(k) \neq \emptyset \) and whose irreducible components are geometrically irreducible. Then the natural pull-back morphism \( \text{Br}(Z) \to \text{Br}(Z \times C) \) is an isomorphism.

**Proof.** Let \( 0 \in C(k) \) be a rational point and \( s : Z \to Z \times C \) the corresponding section to the projection morphism \( p : Z \times C \to Z \). Consider the induced maps on Brauer groups \( p^* : \text{Br}(Z) \to \text{Br}(Z \times C) \) and \( s^* : \text{Br}(Z \times C) \to \text{Br}(Z) \). Since the composition \( s^* \circ p^* \) is the identity, it follows that \( s^* \) is surjective.

Let \( L = k(Z) \) be the function field of \( Z \). Let \( i : \text{Spec} \ L \to Z \) be the natural morphism. Let \( j : \text{Spec} \ L \times C \to Z \times C \) be the corresponding natural morphism. Note that \( s \circ i = j \circ t \), where \( t : \text{Spec} \ L \to \text{Spec} \ L \times C \) is induced by the point \( 0 \in C(k) \). In particular, since \( s \circ i = j \circ t \), we have \( i^* \circ s^* = t^* \circ j^* \). By Corollary 2.2, the morphism \( j^* \) is injective. Moreover, by Lemma 2.3, the morphism \( t^* \) is an isomorphism (hence injective). In particular, the composition \( t^* \circ j^* \) is injective. However, since \( t^* \circ s^* = t^* \circ j^* \), it follows that \( s^* \) is injective. We conclude that \( s^* \) is an isomorphism, and that \( p^* \) is its inverse.

**Remark 2.6.** Corollary 2.5 fails without the assumption that \( C(k) \neq \emptyset \). Indeed, if \( C \) is a smooth proper curve of genus zero over a field \( k \), then \( \text{Br}(k) \to \text{Br}(C) \) is injective if and only if \( C \) has a rational point.

3. The cycle class map

Let \( X \) be a smooth projective connected scheme over an algebraically closed field \( k \). For any prime \( \ell \) we have the Kummer sequence of fpff sheaves

\[
0 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0.
\]

Using Grothendieck's theorem that for smooth sheaves the fpff and étale cohomology agree, we see that the long exact sequence in cohomology induces

\[
0 \to \text{Pic}(X)/\ell^n \to H^2_{\text{fpff}}(X, \mu_{\ell^n}) \to \text{Br}(X)[\ell^n] \to 0.
\]
where the last term is the $\ell^n$-torsion in the cohomological Brauer group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Since $\text{Pic}^0(X)$ is the group of closed points on a abelian variety over an algebraically closed field, it is a divisible group. In particular, the equality $\text{Pic}(X) / \text{Pic}^0(X) = \text{NS}(X)$ induces $\text{Pic}(X) / \ell^n = \text{NS}(X) / \ell^n$. Passing to the projective limit we obtain

$$0 \to \text{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ppf}}(X, \mathbb{Z}_\ell(1)) \to T_\ell \text{Br}(X) \to 0;$$

see [13 (5.8.5)].

**Lemma 3.1.** If there is an integer $d \geq 1$ such that $\text{Br}(X)$ is killed by $d$, then for all prime numbers $\ell$, the homomorphism $\text{NS}(X) \otimes \mathbb{Z}_\ell \to H^2_{\text{ppf}}(X, \mathbb{Z}_\ell(1))$ is an isomorphism of $\mathbb{Z}_\ell$-modules.

**Proof.** Our assumption on the Brauer group of $X$ implies that, for all prime numbers $\ell$, the $\ell$-adic Tate module $T_\ell \text{Br}(X)$ is zero. The statement therefore follows immediately from the discussion above. □

Note that, for a prime $\ell \in k^*$, we have that $H^2_{\text{ppf}}(X, \mathbb{Z}_\ell(1)) = H^2_{\text{et}}(X, \mathbb{Z}_\ell(1))$. We will often use this.

**Remark 3.2.** Let $X$ be a smooth projective geometrically connected scheme over a finitely generated field $K$, let $\overline{K}$ be an algebraic closure of $K$, and let $\ell$ be a prime number with $\ell \in K^*$. Assume that $\text{Br}(X)$ is killed by some positive integer $d$. Then, by Lemma 3.1, the natural map

$$\text{NS}(X_{\overline{K}}) \otimes \mathbb{Z}_\ell \to H^2_{\text{et}}(X_{\overline{K}}, \mathbb{Z}_\ell)$$

is an isomorphism of $\mathbb{Z}_\ell$-modules compatible with the action of the absolute Galois group $G = \text{Gal}({\overline{K}}/K)$. If we take $G$-invariants on both sides of this isomorphism we obtain an integral version of the Tate conjecture for divisors on $X$. (Note that $\text{NS}(X)$ is not necessarily isomorphic to the subgroup of $G$-invariants in $\text{NS}(X_{\overline{K}})$. In particular, the “naïve integral” analogue of Tate’s conjecture for divisors fails for such $X$ over $K$; see see for instance [30].)

**Corollary 3.3.** Assume that the characteristic $p$ of $k$ is positive. Let $W$ be the Witt ring of $k$ and let $\mathbb{K}$ be the fraction field of $W$. If there is an integer $d \geq 1$ such that $\text{Br}(X)$ is killed by $d$, then the $K$-linear morphism

$$\text{NS}(X) \otimes K \to H^2_{\text{crys}}(X/W) \otimes K$$

is an isomorphism of $K$-vector spaces.

**Proof.** Note that the homomorphism $\text{NS}(X) \otimes K \to H^2_{\text{crys}}(X/W) \otimes K$ is injective [13 Remarque 6.8.5]. Let $\ell$ be a prime number with $\ell \in k^*$. By Lemma 3.1 we have

$$\dim_K \text{NS}(X) \otimes \mathbb{Z}_\ell = \dim_K H^2_{\text{ppf}}(X, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell = \dim_K H^2_{\text{et}}(X, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell.$$ 

Moreover, by [17] (see also [13 1.3.1]), we have that

$$\dim_K H^2_{\text{et}}(X, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell = \dim_K H^2_{\text{crys}}(X/W) \otimes K.$$ 

This concludes the proof (as any injective $K$-linear map of finite-dimensional $K$-vector spaces of equal dimension is an isomorphism).

4. BRAUER GROUPS OF RATIONALLY CHAIN CONNECTED VARIETIES

In this section we prove Theorems 1.1 and 1.2. We let $k$ denote an algebraically closed field.

**Lemma 4.1.** Let $X$ and $Y$ be quasi-projective connected reduced schemes over $k$. Let $f : X \to Y$ be a generically finite dominant morphism. If $Y$ is smooth, then there is an integer $d \geq 1$ such that the kernel of the induced map $f^* : \text{Br}(Y) \to \text{Br}(X)$ is killed by $d$.

**Proof.** Let $X' \subset X$ be an open subscheme such that $X'$ is a smooth quasi-projective geometrically integral scheme over $k$ and $X' \to Y$ is dominant. In particular, $X' \to Y$ is a generically finite dominant morphism of smooth geometrically integral quasi-projective schemes over $k$. Therefore, by [12 Prop. 1.1] (which holds over any field), the kernel of the natural pull-back morphism $\text{Br}(Y) \to \text{Br}(X')$ is killed by the generic degree of $X' \to Y$. Since the kernel of the morphism $\text{Br}(Y) \to \text{Br}(X)$ is contained in the kernel of $\text{Br}(Y) \to \text{Br}(X')$, the result follows. □
Proposition 4.2. Let $X$ be a smooth projective geometrically connected scheme over $k$. If $X$ is rationally chain connected, then there is an integer $d \geq 1$ such that $\text{Br}(X)$ of $X$ is killed by $d$.

Proof. Fix a general point $x$ in $X$. Since $X$ is rationally chain connected, there exist a smooth geometrically integral variety $Z$ with $\dim(Z) = \dim(X) - 1$, a compact type curve $T$ of genus zero over $k$, and a dominant generically finite morphism $F : Z \times T \to X$ such that, for all $z$ in $Z$, the image of $F_z : T \cong \{z\} \times T \to X$ contains $x$. Note that, replacing $Z$ by a dense open if necessary, there is a section $s : Z \to Z \times T$ such that the composition

$$Z \xrightarrow{s} Z \times T \xrightarrow{\pi} X$$

contracts $Z$ to $x$. In particular, as the pullback morphism $(F \circ s)^* : \text{Br}(X) \to \text{Br}(Z)$ factors through $\text{Br}(k) = 0$, we see that $(F \circ s)^*$ is the zero map. Now, since the natural pull-back morphism $\text{Br}(Z) \to \text{Br}(Z \times T)$ is an isomorphism (Corollary 2.5), we see that $\text{Br}(Z) \to \text{Br}(Z \times T)$ is the zero map. However, by Lemma 3.1 there is an integer $d \geq 1$ such that the kernel of $\text{Br}(X) \to \text{Br}(Z \times T)$ is killed by $d$. As the kernel of $\text{Br}(X) \to \text{Br}(Z \times T)$ equals $\text{Br}(X)$, we conclude that $\text{Br}(X)$ is killed by $d$ as required. □

Proof of Theorem 1.2. By Proposition 4.2 the Brauer group of a rationally chain connected variety over an algebraically closed field is killed by some integer $d \geq 1$. In particular, the theorem follows from Lemma 3.1. □

Remark 4.3. Let $X$ be a smooth projective rationally chain connected variety over the algebraic closure of a finite field. Then, it follows from [27, Theorem 0.4.(b)] and the fact that $\text{Br}(X)$ is killed by some integer $d \geq 1$ (Proposition 4.2) that the Brauer group of $X$ is finite.

Remark 4.4. Combining Remark 3.2 with Theorem 1.2, we obtain an integral analogue of the Tate conjecture for divisors on rationally chain connected smooth projective varieties.

Proof of Theorem 1.3. Let $s$ and $t$ be points in $S$. Let $\ell$ be a prime number such that $\ell \neq \text{char}(k(s))$ and $\ell \neq \text{char}(k(t))$. By the smooth proper base-change theorem, $\dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_s} H^2_{\text{ét}}(X, \mathbb{Q}_s)$. In particular, since étale and fppf cohomology with $\mathbb{Q}_s$-coefficients agree, Theorem 1.2 implies that the Picard ranks of $X_s$ and $X_t$ are equal. □

5. Crystalline cohomology of separably rationally connected varieties

We now prove a $p$-adic analogue of Theorem 1.2 for separably rationally connected varieties; we refer the reader to [18, IV.3] for the basic definitions.

Proposition 5.1. Let $X$ be a smooth projective connected scheme over an algebraically closed field $k$ of characteristic $p > 0$.

1. If $X$ is separably rationally connected, then $H^1(X, \Omega^1_X) = H^2(X, \mathcal{O}_X) = 0$. If, in addition, $p > 0$, then $H^i_{\text{crys}}(X/W)$ is torsion-free for $i \leq 2$.

2. If $X$ is a Fano variety with $H^0(X, \Omega^1_X) = 0$, then $H^1(X, \mathcal{O}_X) = 0$. If, in addition, $p > 0$, then $H^i_{\text{crys}}(X/W)$ is torsion-free for $i \leq 2$.

Proof. We may and do assume that $p > 0$. If $X$ is separably rationally connected, then it follows from [18, IV.3.8] that $H^0(X, \Omega^1_X) = 0$ which we may now assume to prove both statements of the proposition. Now, in the separably rationally connected case, the result about $H^1(X, \mathcal{O}_X)$ is contained in [27]; we extend the arguments to the Fano case as well. Since $\pi^1_1(X)$ is finite [4] and the $p$-power torsion in $\pi^1_1(X)$ is trivial by a result of Esnault [5, Proposition 8.4], we see that $H^1_{\text{crys}}(X, \mathbb{F}_p) = \text{Hom}(\pi^1_1(X), \mathbb{F}_p) = 0$. Note that $H^i(X, \mathcal{F}_p) \otimes k$ is the semi-simple part of the action of Frobenius on $H^i(X, \mathcal{O}_X)$. Thus, Frobenius is nilpotent on $H^i(X, \mathcal{O}_X)$. However, Frobenius is also injective on $H^i(X, \mathcal{O}_X)$, as its kernel is $H^0(X, \mathcal{O}_X/F \mathcal{O}_X) \subset H^0(X, \Omega^1_X) = 0$. Since $F$ is both injective and nilpotent on $H^1(X, \mathcal{O}_X)$, the group $H^1(X, \mathcal{O}_X)$ has to be zero.

To conclude the proof, one can now apply [13, Thm. II.5.16] or argue more directly as follows. By [14, 1.3.7], there is a universal coefficient exact sequence

$$0 \to H^1_{\text{crys}}(X/W) \otimes k \to H^1_{\text{DR}}(X/k) \to \text{Tor}^1_{\mathbb{Z}}(H^2_{\text{crys}}(X/W), k) \to 0.$$
The existence of the Fröhlicher spectral sequence shows that \( h^{0,1}(X) + h^{1,0}(X) \geq h^{1}\text{dr}(X/k) \). As \( h^{0,1}(X) = h^{1,0}(X) = 0 \), it follows that \( h^{1}\text{dr}(X/k) = 0 \). Thus,
\[
\text{Tor}_1^W(\mathcal{H}^2_{\text{cris}}(X/W), k) = 0,
\]
thereby showing that \( \mathcal{H}^2_{\text{cris}}(X/W) \) is torsion-free. That \( \mathcal{H}^1_{\text{cris}}(X/W) \) is torsion free for any smooth projective variety is a standard result.

**Proof of Theorem 1.2.** By Proposition 5.1, \( \mathcal{H}^2_{\text{cris}}(X/W) \) is torsion-free and \( H^0(X, \Omega^1_X) = H^0(Z_1 \Omega^1_Z) = 0 \), so that the result follows from [13 Thm. II.5.14] and Theorem 1.2. \( \square \)

6. The index in families of Fano varieties

We first show that the (geometric) index \( r(X) \) of a Fano variety \( X \) is bounded by \( \dim(X) + 1 \).

**Lemma 6.1.** Let \( X \) be a Fano variety over a field \( k \). Then \( r(X) \leq \dim(X) + 1 \).

**Proof.** We may and do assume that \( k \) is algebraically closed. Let \( n := \dim X \). By [18 V.1.6.1], there is a rational curve \( C \) so that \( -K_X.C \leq \dim X + 1 \). Writing \( -K_X = r(X)H \), we get \( r(X) \leq r(X)H.C = -K_X.C \leq \dim X + 1 \). \( \square \)

**Proposition 6.2.** Let \( S \) be a trait with generic point \( \eta \) and closed point \( s \). Let \( p = \text{char}(k(s)) \). Let \( f : X \to S \) be a smooth proper morphism of schemes whose geometric fibres are Fano varieties.

1. If \( p = 0 \), then \( r(X_s) = r(X_\eta) \).
2. If \( p > 0 \), then there is an integer \( i \geq 0 \) such that \( r(X_s) = p^i r(X_\eta) \).
3. If \( r(X_s) > r(X_\eta) \), then \( 0 < p \leq (n+1)/r(X_\eta) \).

**Proof.** We may and do assume that \( k(s) \) is algebraically closed. Let \( X_\eta \) denote the geometric generic fibre of \( X \to S \). We may and do assume that \( \text{NS}(X_\eta) = \text{NS}(X_\eta) \).

Firstly, since \( S \) is regular integral and noetherian, the homomorphism \( \text{Pic}(X) \to \text{Pic}(X_\eta) \) is an isomorphism [10 Lem. 3.1.1].

For all \( \ell \neq p \), the natural morphism \( \text{NS}(X_\eta) \otimes \mathbb{Z}_\ell = \text{NS}(X_\eta) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X_\eta, \mathbb{Z}_\ell) \) is an isomorphism (Theorem 1.2). Similarly, for all \( \ell \neq p \), the natural morphism \( \text{NS}(X_s) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(X_s, \mathbb{Z}_\ell) \) is an isomorphism (Theorem 1.2). By smooth proper base change for \( \ell \)-adic cohomology, for all \( \ell \neq p \), there is a natural \( \mathbb{Z}_\ell \)-linear isomorphism \( H^2_{\text{ét}}(X_\eta, \mathbb{Z}_\ell) = H^2_{\text{ét}}(X_s, \mathbb{Z}_\ell) \). This proves (1) and (2).

Finally, to prove (3), by (1), we may and do assume that \( p > 0 \). Also, by our assumption and (2), we have that \( r(X_s) = p^i \cdot r(X_\eta) \) with \( i \geq 1 \). Therefore, by Lemma 6.1, we immediately get \( p \leq p^i = r(X_s)/r(X_\eta) \leq (n+1)/r(X_\eta) \). \( \square \)

**Proposition 6.3.** Let \( S \) be a trait with generic point \( \eta \) and closed point \( s \). Let \( p = \text{char}(k(s)) \). Let \( f : X \to S \) be a smooth proper morphism whose geometric fibres are Fano varieties. Assume any one of the following conditions:

1. \( p > \dim X_s + 1 \), or
2. \( H^2(X_s, \mathcal{O}_{X_s}) = 0 \), or
3. There exists an integral 1-cycle \( T \) on \( X_\eta \) such that \( -K_{X_\eta}.T = r(X_\eta) \).

Then, the index is constant on the fibres of \( f : X \to S \), i.e. \( r(X_s) = r(X_\eta) \).

**Remark 6.4.** Kollár asked in [13 V.1.13] whether there exists a rational curve \( T \) (instead of an integral sum of 1-cycles) which satisfies the third condition of the proposition.

**Proof of Proposition 6.3.** We may assume that \( k(s) \) is algebraically closed. Assume that \( p > \dim X_s + 1 \). Then, the result follows from (3) in Proposition 6.2.

Under the second assumption, assume that \( r(X_s) = p^i r(X_\eta) \) with \( i > 0 \). Then there exists an ample divisor \( H_0 \in \text{Pic}(X_s) \) so that \( -K_{X_s} = p^i r(X_\eta) H_0 \). But the assumption \( H^2(X_s, \mathcal{O}_{X_s}) = 0 \) implies that all the obstructions to lifting a line bundle to the formal completion vanish, so \( H_0 \) lifts to a divisor \( H \in \text{Pic}(X_\eta) \). This gives a contradiction to the definition of the index \( r(X_\eta) \).
Proposition 6.5. Then, for an algebraically closed field $k$ of characteristic zero and a Fano variety $X$ over $k$, there exists an integral 1-cycle $T$ on $X$ such that $-K_X.T = r(X)$.

Proof. First, assume $k = \mathbb{C}$. Let $n := \dim X$. Since $X$ is Fano, the Hodge decomposition and the fact that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ give $H^{2n-2}(X, \mathbb{C}) = H^{n-1,n-1}(X)$. From Poincaré duality and the fact that the Picard group is torsion free, we have a unimodular pairing $H^2(X, \mathbb{Z}) \times H^{2n-2}(X, \mathbb{Z}) \to \mathbb{Z}$ given by cup product. Write now $-K_X = rH$ for $r = r(X)$ the index and define a homomorphism $H^2(X, \mathbb{Z}) \to \mathbb{Z}$ sending $H \mapsto 1$ and extending by zero elsewhere. Unimodularity of the cup product implies that there is a class $T \in H^{2n-2}(X, \mathbb{Z})$ so that $(H, T) = 1$. By the surjectivity of the cycle class map $CH_1(X) \to H^{n-1,n-1}(X) \cap H^{2n-2}(X, \mathbb{Z})$ (which follows from our assumptions), the cohomology class $T$ can be represented by a cycle $\sum m_iC_i$, where $m_i \in \mathbb{Z}$ and $C_i$ irreducible curves on $X$. This concludes the proof of the proposition for $k = \mathbb{C}$.

To conclude the proof of the proposition, let $X$ be a Fano variety over an algebraically closed field $k$ of characteristic zero. Let $K \subset k$ be an algebraically closed subfield with an embedding $K \to \mathbb{C}$ and let $Y \to \text{Spec } K$ be a Fano variety over $K$ such that $X_K \cong X$. Then, by what we have just shown, there is an integral 1-cycle $T'$ on $Y_{\mathbb{C}}$ such that $-K_{Y_{\mathbb{C}}}.T' = r(Y_{\mathbb{C}}) = r(Y) = r(X)$. We now use a standard specialization argument to conclude the proof. Let $K \subset L \subset \mathbb{C}$ be a subfield of $\mathbb{C}$ which is finitely generated over $K$ such that the integral 1-cycle $T'$ on $Y_{\mathbb{C}}$ descends to an integral 1-cycle $T''$ on $Y_L$. Moreover, let $U$ be a quasi-projective integral scheme over $K$ whose function field $K(U)$ is $L$, and let $Y \to U$ be a smooth proper morphism whose geometric fibres are Fano varieties such that $Y \times_U \text{Spec } L \cong Y_L$. Let $T$ be the closure of $T''$ in $Y$, and note that $T$ extends the integral 1-cycle $T''$ on $Y_L$. Now, note that the generic fibres of $Y \times_K U \to U$ and $Y \to U$ are isomorphic over $L$. Therefore, replacing $U$ by a dense open if necessary, we have that $Y \times_K U$ is isomorphic to $Y$ over $U$ (by “spreading out” of isomorphisms). Let $u$ be a closed point of $U$. The integral 1-cycle $T$ on $Y \times_K U$ restricts to an integral 1-cycle $T_u$ on $Y_u = Y \times_K \text{Spec } k(u)$ such that $-K_Y.T_u = r(Y)$. Define $T$ to be the $T_u \times_K \text{Spec } k$. Note that $T$ is an integral 1-cycle on $X$ with the sought property. 

Corollary 6.6. Let $S$ be a trait with generic point $\eta$ and closed point $s$. Let $X \to S$ be a smooth proper morphism of schemes whose geometric fibres are Fano varieties. If $\text{char}(k(\eta)) = 0$ and the integral Hodge conjecture holds for 1-cycles on Fano varieties over $\mathbb{C}$, then $r(X_s) = r(X_\eta)$.

Proof. This follows from (3) of Proposition 6.3 and Proposition 6.5.

Appendix A. Varieties with $\text{CH}_0(X) = \mathbb{Z}$

To show that the Brauer group of a rationally chain connected variety is killed by some positive integer (Proposition 6.2), one can also argue using the decomposition of the diagonal of Bloch-Srinivas, as we show now.
Theorem A.1. Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p \geq 0$. Assume that there is an algebraically closed field $\Omega$ of infinite transcendence degree over $k$ such that that $\text{CH}_0(X_\Omega) = \mathbb{Z}$. Then there exists an integer $m \geq 1$ such that $\text{Br}(X)$ is $m$-torsion.

Proof. Let $n = \dim X$. To prove the result, by a standard specialization argument, it suffices to show that $\text{Br}(X_\Omega)$ is killed by some integer $m \geq 1$. Thus, to prove the theorem, we may and do assume that $X = X_\Omega$.

By the decomposition of the diagonal of Bloch and Srinivas [2] there is an integer $m \geq 1$ such that in $\text{CH}^n(X \times X)$ we have

$$m\Delta = X \times x + Z,$$

where $x \in X$ is any point and $Z \subset X \times X$ is an $n$-cycle whose projection under $\text{pr}_1$ does not dominate $X$ [32 Thm. 3.10]. In particular, there is a divisor $D$ on $X$ such that $Z$ is supported on $D \times X$. Now, start with a class $\beta \in \text{Br}(X) = \text{H}_1^{\text{et}}(X, \mathbb{G}_m)$. Under our assumptions, it will be a torsion cohomology class. Assume it is $d$-torsion, so that it is an element of $\text{Br}(X)[d]$. From the Kummer sequence in the fppf topology (note here that if $(d, p) = 1$ then we can work in the étale topology), we have a short exact sequence

$$0 \to \text{Pic}(X)/d \text{Pic}(X) \to H_2^{\text{fppf}}(X, \mu_d) \to \text{Br}(X)[d] \to 0$$

where we have used that $H_2^{\text{fppf}}(X, \mathbb{G}_m) = H_1^{\text{et}}(X, \mathbb{G}_m) = \text{Pic } X$ and likewise for second cohomology of $\mathbb{G}_m$ by a theorem of Grothendieck saying that fppf and étale cohomology agree for smooth groups. Let $\alpha \in H_2^{\text{fppf}}(X, \mu_d)$ be any class in the preimage of $\beta$. (If $(d, p) = 1$, then the group $H_2^{\text{et}}(X, \mu_d)$ is finite, and hence so is $\text{Br}(X)[d]$). However, $H_2^{\text{fppf}}(X, \mu_d)$ does not have to be finite.) Each element $\gamma$ in $\text{CH}^n(X \times X)$ is a correspondence on $X$, and therefore induces a morphism $\gamma^* : H_2^{\text{fppf}}(X, \mu_d) \to H_2^{\text{fppf}}(X, \mu_d)$ given by $\gamma^* \alpha := \text{pr}_{2*}(\text{pr}_1^* \alpha \cup [\gamma])$ where $[\gamma] \in H_2^{\text{et}}(X \times X, \mu_d)$ is the image of $\gamma$ under the cycle class map $\text{cl} : \text{CH}^n(X \times X) \to H_2^{\text{et}}(X \times X, \mu_d)$; see Remark A.3 below. In particular, the diagonal induces the identity morphism on $H_2^{\text{fppf}}(X, \mu_d)$, whereas the class $X \times x$ is the zero map. We thus have

$$m\alpha = [Z]^*\alpha.$$ 

Let now $p, q$ be the projections of $Z$ onto $D$ and $X$ respectively. Hence, if $i : D \to X$ denotes the inclusion, then we have proven that multiplication by $m$ on $H_2^{\text{fppf}}(X, \mu_d)$ factors as follows:

$$
\begin{array}{c}
H_2^{\text{fppf}}(D, \mu_d) \\
p_+q^* \downarrow i_*
\end{array}
\quad
\begin{array}{c}
H_2^{\text{fppf}}(X, \mu_d) \\
m \downarrow
\end{array}
\quad
\begin{array}{c}
H_2^{\text{fppf}}(X, \mu_d) \\
m \downarrow
\end{array}
\quad
\begin{array}{c}
\text{Br}(X)[d] \\
m \downarrow
\end{array}
\quad
\begin{array}{c}
\text{Br}(X)[d].
\end{array}
$$

The map $i_*$ maps a multiple of the fundamental class of $D$ (a divisor) to its Chern class, so in particular the image of $i_*$ is contained in the image of $\text{Pic}(X)/d \text{Pic}(X)$. Thus, when projected down to $\text{Br}(X)[d]$, the image of $i_*$ must be zero. In other words, every $d$-torsion class in the Brauer group of $X$ is killed by $m$.

Remark A.2. If $X$ is a non-supersingular K3 surface over an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, then $\text{CH}_0(X) = \mathbb{Z}$. (Let $A_0(X)$ be the Chow group of zero cycles of degree zero on $X$. By 26, 28, the natural map $A_0(X) \to \text{Alb}(X)(\overline{\mathbb{F}}_p)$ induces an isomorphism on torsion subgroups. Thus, as the Albanese of $X$ is trivial, the group $A_0(X)$ is torsion-free. However, as $A_0(X)$ is a torsion abelian group, we conclude that $A_0(X) = 0$ and thus $\text{CH}_0(X) = \mathbb{Z}$.) Now, since $X$ is non-supersingular, there is no integer $d \geq 1$ such that the Brauer group of $X$ is killed by $d$. \qed
Remark A.3. Let $d$ be a positive integer. Note that the Kummer sequence $0 \rightarrow \mu_d \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$ is exact in the fppf topology. In particular, every class in $\text{Br}(X)[d]$ comes from a class in $H^2_{\text{fppf}}(X, \mu_d)$. Now, even though the latter group does not have to be finite, we do have cycle class maps in the fppf topology. We could not find an explicit reference for this, but taking a locally free resolution of the ideal sheaf of a subvariety, using the existence of a $c_1$ map for line bundles in the fppf topology and extending by linearity and cup product gives Chern classes and hence a cycle class map into $H^2_{\text{fppf}}(X, \mu_d)$.

Remark A.4. Note that the assumption in Theorem A.1 holds for a rationally chain connected smooth projective variety $X$ over $k$.

Remark A.5. The fact that the prime-to-$p$ part of the Brauer group of a rationally chain connected variety $X$ is killed by some integer $d \geq 1$ can also be deduced from work of Colliot-Thélène [8] (see also [1] Theorem 1.4 and Lemma 1.7)).

Remark A.6 (Colliot-Thélène). Salberger has proven (unpublished) the following generalization of Theorem A.1. Let $X$ be a smooth projective variety over an algebraically closed field $k$. Assume that there is a curve $C$ over $k$ and a morphism $C \rightarrow X$ such that, for any algebraically closed field $\Omega$ containing $k$, the induced morphism $\text{CH}_0(C_\Omega) \rightarrow \text{CH}_0(X_\Omega)$ is surjective. Then, there exists an integer $m \geq 1$ such that $\text{Br}(X)$ is $m$-torsion.

References

[1] A. Auel, Colliot-Thélène J.-L., and Parimala R. Universal unramified cohomology of cubic fourfolds containing a plane. In Brauer groups and obstruction problems: moduli spaces and arithmetic (Palo Alto, 2013), pages 29–56. Birkhäuser Basel, 2017.

[2] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. Amer. J. Math., 105(5):1235–1253, 1983.

[3] F. Campana. Connexité rationnelle des variétés de Fano. Ann. Sci. École Norm. Sup. (4), 25(5):539–545, 1992.

[4] A. Chambert-Loir. A propos du groupe fondamental des variétés rationnellement connexes. 2003.

[5] A. Chambert-Loir. Points rationnels et groupes fondamentaux: applications de la cohomologie $p$-adique (d’après P. Berthelot, T. Ekedahl, H. Esnault, etc.). Astérisque, (294):viii, 125–146, 2004.

[6] Jean-Louis Colliot-Thélène. Un théorème de finitude pour le groupe de Chow des zéro-cycles d’un groupe algébrique linéaire sur un corps $p$-adique. Invent. Math., 159(3):589–606, 2005.

[7] F. Gounelas. The first cohomology of separably rationally connected varieties. C. R. Math. Acad. Sci. Paris, 352(11):871–873, 2014.

[8] A. Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In Dix exposés sur la cohomologie des schémas, volume 3 of Adv. Stud. Pure Math., pages 67–87. North-Holland, Amsterdam, 1968.

[9] A. Grothendieck. Le groupe de Brauer. III. Exemples et compléments. In Dix exposés sur la cohomologie des schémas, volume 3 of Adv. Stud. Pure Math., pages 88–188. North-Holland, Amsterdam, 1968.

[10] D. Harari. Méthode des fibrations et obstruction de Manin. Duke Math. J., 75(1):221–260, 1994.

[11] Yonatan Harpaz and Alexei N. Skorobogatov. Singular curves and the étale Brauer-Manin obstruction for surfaces. Ann. Sci. Éc. Norm. Supér. (4), 47(4):765–778, 2014.

[12] E. Ieronymou, A. N. Skorobogatov, and Y. G Zarhin. On the Brauer group of diagonal quartic surfaces. arXiv.org, (3):659–672, December 2009.

[13] L. Illusie. Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. (4), 12(4):501–661, 1979.

[14] L. Illusie. Crystalline cohomology. In Motives (Seattle, WA, 1991), volume 55 of Proc. Sympos. Pure Math., pages 43–70. Amer. Math. Soc., Providence, RI, 1994.

[15] V. A. Iskovskikh and Yu. G. Prokhorov. Fano varieties. In Motives (Seattle, WA, 1991), volume 55 of Proc. Sympos. Pure Math., pages 43–70. Amer. Math. Soc., Providence, RI, 1994.

[16] A. Javanpeykar and D. Loughran. Good reduction of Fano threefolds and sextic surfaces. Annali di Scuola Normale Superiore Pisa, to appear.

[17] N. M. Katz and W. Messing. Some consequences of the Riemann hypothesis for varieties over finite fields. Invent. Math., 23:73–77, 1974.

[18] J. Kollár. Rational curves on algebraic varieties, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996.

[19] J. Kollár, Y. Miyaoka, and S. Mori. Rational connectedness and boundedness of Fano manifolds. J. Differential Geom., 36(3):765–779, 1992.
[20] N. Lauritzen and A. P. Rao. Elementary counterexamples to Kodaira vanishing in prime characteristic. *Proc. Indian Acad. Sci. Math. Sci.*, 107(1):21–25, 1997.

[21] S. Lichtenbaum. Zeta functions of varieties over finite fields at \( s=1 \). In *Arithmetic and geometry, Vol. I*, pages 173–194. Birkhäuser Boston, Boston, MA, 1983.

[22] C. Liedtke. The Picard rank of an Enriques surface. *Math. Res. Letters*, to appear.

[23] Y. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[24] G. Megyesi. Fano threefolds in positive characteristic. *J. Algebraic Geom.*, 7(2):207–218, 1998.

[25] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[26] J. S. Milne. Zero cycles on algebraic varieties in nonzero characteristic: Rojtman’s theorem. *Compositio Math.*, 47(3):271–287, 1982.

[27] J. S. Milne. Values of zeta functions of varieties over finite fields. *American Journal of Mathematics*, 108(2):297–360, 1986.

[28] A. A. Rojtman. The torsion of the group of \( 0 \)-cycles modulo rational equivalence. *Ann. of Math. (2)*, 111(3):553–569, 1980.

[29] D. J. Saltman. The Brauer group and the center of generic matrices. *J. Algebra*, 97(1):53–67, 1985.

[30] C. Schoen. An integral analog of the Tate conjecture for one-dimensional cycles on varieties over finite fields. *Math. Ann.*, 311(3):493–500, 1998.

[31] N. I. Shepherd-Barron. Fano threefolds in positive characteristic. *Compositio Math.*, 105(3):237–265, 1997.

[32] C. Voisin. *Chow rings, decomposition of the diagonal, and the topology of families*, volume 187 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2014.

Frank Gounelas, Humboldt Universität Berlin, Berlin, Germany.

E-mail address: gounelas@mathematik.hu-berlin.de

Ariyan Javanpeykar, Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany.

E-mail address: peykar@uni-mainz.de