Residual entanglement of accelerated fermions is not nonlocal

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We analyze the operational meaning of the residual entanglement in non-inertial fermionic systems in terms of the achievable violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality. We demonstrate that the quantum correlations of fermions, which were previously found to survive in the infinite acceleration limit, cannot be considered to be non-local. The entanglement shared by an inertial and an accelerated observer cannot be utilized for the violation of the CHSH inequality in case of high accelerations. Our results are shown to extend beyond the single mode approximation commonly used in the literature.

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I. INTRODUCTION

The extension of quantum information theory to relativistic settings has been a thriving area of research for some time\textsuperscript{1,3}. The effects of relativistic motion on entanglement and quantum information protocols have been studied extensively for inertial observers\textsuperscript{2,4,5}, as well as for accelerated systems\textsuperscript{1,6,7}. However, while the effects on entanglement have been analyzed in detail for both cases, the implications for tests of non-locality via the violation of Bell inequalities have received treatment only for the case of inertial motion\textsuperscript{2,4}. In the other situation, i.e., if (at least) one of the observers sharing a bipartite entangled state is moving with uniform acceleration, two scenarios naturally arise. The common quantum state can be entangled with respect to bosonic or, on the other hand, fermionic modes. The entanglement degradation of such a two mode state by accelerated motion, commonly attributed to the thermalization due to the Unruh effect\textsuperscript{10}, has been investigated for bosons\textsuperscript{8,9} and fermions\textsuperscript{1,6,8} respectively. The distinguishing feature of the fermionic from the bosonic case in these results is found to be a non-zero residual entanglement between the (anti-)fermionic modes in the infinite acceleration limit. The same result has been obtained for the Hawking effect in eternal black holes\textsuperscript{11}. Moreover, the remaining entanglement cannot be attributed to bound entanglement, since, even in the cases where bound entanglement can feature in principle, the entanglement persists even when measured by the negativity, a measure which only detects distillable entanglement\textsuperscript{12}.

It is the aim of this paper to shed light on this issue and assign operational meaning to the residual fermionic entanglement by applying the criterion introduced in Ref.\textsuperscript{13} to quantify the maximally possible violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality\textsuperscript{14}. We find that the quantum correlations remaining in the infinite acceleration limit cannot be used by the observers to demonstrate quantum non-locality, showing that acceleration effectively degrades correlations even in the fermionic case.

II. SETTING

We employ a scheme of two observers, Alice and (Anti)Rob, one of which, Alice, is inertial (we can assume without loss of generality that Alice is at rest), while the other one, (Anti)Rob, is uniformly accelerated, in complete analogy to the setup used in Ref.\textsuperscript{6} (see Fig. 1). To describe the point of view of an accelerated observer we introduce the so-called Rindler coordinates

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{rindler_coordinates.png}
\caption{The uniformly accelerated observers Rob and AntiRob are confined to the Rindler wedges I (|t| < x) and II (|t| < -x), respectively, which are causally disconnected from each other. Their worldlines are hyperbolas, which correspond to lines of constant $\chi = \frac{c^2}{a}$, where $a$ is their proper acceleration and $0 < \chi < \infty$.}
\end{figure}
\((\eta, \chi)\), which are the proper coordinates of an accelerated observer moving with fixed acceleration \(a\). These coordinates relate to Minkowskian coordinates \((t, x)\) by
\[
ct = \chi \sinh \left( \frac{a t}{c} \right), \quad x = \chi \cosh \left( \frac{a t}{c} \right).
\]
(1)

From (1) it can be seen, that for constant \(a\), \(t, x\) coordinates describe hyperbolic trajectories in space-time, which asymptotically approach the light cone.

Observers eternally in uniform acceleration are always restricted to either region I or II of the space-time, i.e., the light cone constitutes an effective event horizon for these observers. A quick inspection reveals that the Rindler coordinates defined in (1) do not cover the whole Minkowski space-time. Instead these coordinates only cover the right wedge (region I in Fig. 1). To cover the complete Minkowski space-time we need three more sets of coordinates, \(ct = -\chi \sinh \left( \frac{a t}{c} \right), x = -\chi \cosh \left( \frac{a t}{c} \right)\), for region II, corresponding to an observer accelerating leftward with regard to the Minkowskian origin, as well as \(ct = \pm \chi \cosh \left( \frac{a t}{c} \right), x = \pm \chi \sinh \left( \frac{a t}{c} \right)\) for regions F and P. For both relevant regions (I and II), the coordinates \((\chi, \eta)\) take values in the whole domain \((-\infty, +\infty)\), thus admitting completely independent procedures of canonical field quantization.

We can now expand the field in terms of a complete set of solutions of the Dirac equation in Minkowski coordinates \((i\gamma^\mu \partial_\mu + m)\phi = 0\), or, instead, in terms of a complete set of solutions of the Dirac equation in Rindler coordinates for regions (I and II), i.e., \([i\gamma^\mu (\partial_\mu - \Gamma_\mu) + m]\phi = 0\), where \(\gamma^\mu\) are the Dirac-Pauli matrices and \(\Gamma_\mu\) is the spinorial affine connection.

Hence, the field operator \(\phi\) can be expressed as
\[
\phi = N_M \sum_k \left( c_{k, M} u_{k, M}^+ + d_{k, M}^* u_{k, M} \right) = N_R \sum_{j} \left( c_{j, I} u_{j, I}^+ + d_{j, I}^* u_{j, I} + c_{j, II} u_{j, II}^+ + d_{j, II}^* u_{j, II} \right),
\]
where \(N_M, N_R\) are normalization constants. The label \(u_{k, M}^\pm\) denotes positive and negative energy solutions (particles and antiparticles) with regard to the Killing vector field \(\partial_\tau\), whereas \(u_{k, I}^\pm\) and \(u_{k, II}^\pm\) are the positive and negative frequency solutions of the Dirac equation in Rindler coordinates w.r.t. the corresponding timelike Killing vector field in regions I and II, respectively. By \(c_{j, \Sigma}, d_{j, \Sigma}\) with \(\Sigma = \text{I, II}\) we denote the Minkowski and Rindler particle/antiparticle operators, satisfying the usual anticommutation relations. The label \(k\) is a multilabel including frequency and spin \(k = (\Omega, s)\), where \(s\) is the component of the spin along the quantization axis.

The annihilation operators \(c_{k, M}, d_{k, M}\) define the Minkowski vacuum \(|0\rangle_M\) which must satisfy \(c_{k, M}|0\rangle_M = d_{k, M}|0\rangle_M = 0, \forall k\). In the same fashion \(c_{j, \Sigma}, d_{j, \Sigma}\) define the Rindler vacuum in regions \(\Sigma = \text{I, II}\).

From (2) we extract the transformation between the Minkowski and Rindler modes
\[
u_{j, M}^\pm = \sum_k \left[ a_{jk}^\pm u_{k, M}^+ + \beta_{jk}^\pm u_{k, M} + \alpha_{jk}^\pm u_{k, II} + \beta_{jk}^\pm u_{k, II} \right].
\]
(3)

The Bogoliubov coefficients that relate both sets of modes are given by the inner product \((u_{k, j}|u_{k, j}) = \int Dx u_{k, j}^\dagger u_{k, j}\) and are obtained after some elementary but lengthy algebra (see Refs. 11, 12, 13).

For fixed acceleration, combinations of Minkowski modes can be found, which transform into monochromatic Rindler modes 14, 15. These modes, which share the same vacuum state as the standard monochromatic Minkowski modes, are called Unruh modes, and their associated annihilation operators are
\[
C_{k, R} = \left( \cos r_k c_{k, I} - \sin r_k d_{k, II}^* \right), \quad C_{k, L} = \left( \cos r_k c_{k, II} - \sin r_k d_{k, I}^* \right),
\]
(4)

where \(tan r_k = e^{-\pi c t^2/a}\).

The associated single-particle state is obtained by applying the creation operator \(c_{k, R}^*\) to the vacuum. As is commonplace 14, 15, 16, we will work in the Grassmann scalar case, which is the simplest case that preserves the fundamental Dirac characteristics. With the shorthand notation \(|ijkl\rangle = |i\rangle_1 |j\rangle_2 |k\rangle_3 |l\rangle_4\), the Unruh vacuum state of mode \(\Omega\) can be expressed as
\[
|0\rangle_{\Omega} = \cos^2 r_0 |0000\rangle_{\Omega} - \sin r_0 \cos r_0 |0110\rangle_{\Omega} + \sin^2 r_0 |1111\rangle_{\Omega} - \cos r_0 \cos r_0 |1101\rangle_{\Omega} + \sin r_0 |1011\rangle_{\Omega} + \sin r_0 \cos r_0 |1001\rangle_{\Omega} + \cos r_0 |0101\rangle_{\Omega}.
\]
(6)

(see Ref. 17). Likewise, the particle and antiparticle states of Unruh mode \(\Omega\) in the Rindler basis are found to be
\[
|1\rangle_{\Omega}^+ = q_{\Omega} \left( \cos r_0 |0001\rangle_{\Omega} - \sin r_0 |1110\rangle_{\Omega} \right) + q_{\Omega} \left( \cos r_0 |0000\rangle_{\Omega} + \sin r_0 |1101\rangle_{\Omega} \right),
\]
\[
|1\rangle_{\Omega}^- = q_{\Omega} \left( \cos r_0 |0001\rangle_{\Omega} + \sin r_0 |1101\rangle_{\Omega} \right) + q_{\Omega} \left( \cos r_0 |0000\rangle_{\Omega} - \sin r_0 |1110\rangle_{\Omega} \right).
\]
(7)

III. ENTANGLEMENT AND NONLOCALITY

Following the same notation as in Ref. 18, we are endowing the fermionic Fock space with a particular tensor product structure. This can be problematic when working with entanglement measures, as shown in Ref. 19. However, our procedure is free from any ambiguity because we correctly treat the tensor product structure and anticommutation relations when we compute expectation values.

Let us then consider the following initial states
\[
|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_\Lambda |0\rangle_U + |1\rangle_\Lambda |1\rangle_U \right).
\]
(8)
where the tensor product structure refers to two distinct inertial observers, Alice and Bob, in Minkowski spacetime. We have explicitly assumed here that Bob’s mode is a positive (ψ+) or negative frequency (ψ−) Unruh mode, while Alice’s mode (labeled by the subscript “A”) can be either a Minkowski or Unruh mode of positive (ε = +") or negative frequency (ε = −") respectively. Let us now replace the second observer Bob by the previously discussed accelerated observers Rob or AntiRob. Due to their non-inertial motion, access to Minkowski space is limited. In particular, Rob is causally disconnected from region II, while AntiRob cannot interact with region I, which implies tracing over the unaccessible space-time regions. We additionally make the assumption, that each observer is concerned only with the (positive or negative frequency) mode, originally considered. For each of the initial states we thus obtain four different reduced two-qubit density matrices, computed in Ref. [8].

Tracing over region II and the antiparticle sector (i.e., the negative frequency mode) of the state ψ+ in Eq. (8), results in

$$\rho_{AR}^+ = \frac{1}{2} \begin{pmatrix} \cos^2 r_\Omega & 0 & 0 & q_R^* \cos r_\Omega \\ 0 & \sin^2 r_\Omega & 0 & 0 \\ 0 & 0 & 0 & q_R^* \cos r_\Omega \\ q_R \cos r_\Omega & 0 & 0 & 0 \end{pmatrix}$$

Likewise, the reduced state $\rho_{AR}^-$ for AntiRob, when he is only able to detect positive frequency modes is obtained by exchanging $q_R$ and $-q_R$. Similarly, the reduced state for Alice and Rob, when Rob’s detector registers only antiparticles is given by

$$\rho_{AR}^- = \frac{1}{2} \begin{pmatrix} \cos^2 r_\Omega & 0 & 0 & 0 \\ 0 & \sin^2 r_\Omega & q_L \sin r_\Omega & 0 \\ 0 & q_L \sin r_\Omega & 0 & 0 \\ 0 & 0 & 0 & q_R^* \cos r_\Omega \end{pmatrix}$$

and the corresponding state $\rho_{AR}^-$ for AntiRob is obtained from Eq. (10) by the interchange of $q_L$ and $-q_R$. An analogously, this procedure can be repeated for the initial state $\psi_-$ [see Eq. (3)], where the roles of positive and negative frequency modes are exchanged in the reduced density matrices, e.g., $\rho_{AR}^+$ is related to $\rho_{AR}^-$ by a simple sign change of $q_L$. For these states a tradeoff in entanglement (in terms of the negativity) between the particle and antiparticle sector has been demonstrated in Ref. [8].

We illustrate this behavior by studying another common entanglement measure, the concurrence [18]. The concurrence of a two-qubit density matrix $\rho$ is given by $C[\rho] = \max[0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}]$, where $\lambda_i$ are the eigenvalues of the matrix $\rho(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$ in decreasing order, $\sigma_2$ is the second Pauli matrix, and the asterisk denotes complex conjugation. As can be seen in Fig. 4, a change in entanglement in the particle sector of Alice and AntiRob for increasing $r_\Omega$ is accompanied by an opposite change in the corresponding antiparticle sector, and vice versa. To better understand the surviving correlations in the limit $r_\Omega \rightarrow \frac{\pi}{4}$ we want to infer how the entanglement of the reduced states can be utilized by the observers in an experiment to test Bell inequalities, in particular, the CHSH inequality [14], which is a suitable inequality for two qubits to test local-realistic theories. Any such theory must satisfy the bound

$$|\langle B_{\text{CHSH}} \rangle_\rho| \leq 2,$$

where $B_{\text{CHSH}} = a \cdot \sigma \cdot (b + b') \cdot \sigma + a' \cdot \sigma \cdot (b - b') \cdot \sigma$, a, a', b, and b' are unit vectors in $\mathbb{R}^3$, and $\sigma$ is the vector of Pauli matrices. It is known that for some choices of a, a', b, b' inequality (11) can be violated by certain states $\rho$ up to the value $2\sqrt{2}$. For a general two-qubit state Ref. [15] provides a criterion for the maximally possible violation of (11). The maximally possible value $\langle B_{\text{max}} \rangle_\rho$ of the Bell-CHSH expectation value [left-hand side of (11)] for a given state $\rho$ is determined by

$$\langle B_{\text{max}} \rangle_\rho = 2 \sqrt{\mu_1 + \mu_2},$$

where $\mu_1, \mu_2$ are the two largest eigenvalues of $U(\rho) = T_\rho \bar{T}_\rho$. The matrix $T = (t_{ij})$, where $t_{ij} = \text{Tr}[\rho(\sigma_i \otimes \sigma_j)]$, is the so-called correlation matrix of the generalized Bloch decomposition of the density operator $\rho$. For the reduced state $\rho$, we find

$$U(\rho_{AR}^\pm) = \begin{pmatrix} |q_R|^2 \cos^2 r_\Omega & 0 & 0 & 0 \\ 0 & |q_R|^2 \cos^2 r_\Omega & 0 & 0 \\ 0 & 0 & |q_R|^2 \cos^2 r_\Omega & 0 \\ 0 & 0 & 0 & |q_R|^2 \cos^2 r_\Omega \end{pmatrix}$$

Since $0 \leq r_\Omega < \frac{\pi}{4}$ and $|q_R| \leq 1$, the first two identical eigenvalues $|q_R|^2 \cos^2 r_\Omega$ are always larger than the third eigenvalue. From (12) we thus obtain

$$\langle B_{\text{max}} \rangle_{\rho_{AR}^\pm} = 2 \sqrt{2} |q_R| \cos r_\Omega.$$

Similarly, the maximally possible value of the Bell-CHSH parameter for the antiparticle sector of Alice and Rob is

$$\langle B_{\text{max}} \rangle_{\rho_{AR}^-} = 2 \sqrt{2} |q_L| \sin r_\Omega.$$
while the corresponding results for AntiRob are obtained by exchanging $q_R$ and $q_L$ in (13) and (15). For the initial state $\psi_-$, the results of the particle and antiparticle sectors are simply switched. As can be instantly seen from (13) and (15), the bound for local realism (right-hand side of (11)) cannot be surpassed in the infinite acceleration limit $r_\Omega \to \frac{\pi}{4}$. In fact, for the initial state $\psi_+$, another requirement for such a test of local realism is the ability to use different measurement bases. We assume this to be possible by swapping entanglement to another system, e.g., photons. This can be thought of as a device which produces photons of horizontal polarization when no fermion is incident, while it rotates the photon polarization by $\frac{\pi}{2}$, when a (anti)fermion interacts with the apparatus.

\section{Conclusions}

We have provided an operational explanation for the surviving fermionic entanglement in infinitely accelerated frames, bridging the gap to the bosonic case, where no entanglement remains in this limit. We have shown that in the fermionic case, the surviving entanglement cannot be used to violate the CHSH inequality, which is the optimal Bell inequality for this situation \cite{19}. Therefore, no quantum information tasks using these correlations can outperform states with appropriate classical correlations. This claim holds if the observers are not allowed to manipulate the final state by local operations or classical communication (LOCC). In particular, maximally entangled, nonlocal states could be distilled from the residual entanglement, if several copies of the state were supplied and the observers could freely communicate. However, this communication is severely limited in the infinite acceleration limit, such that any schemes of recurring local operations (see, e.g., Ref. \cite{21}) based on the classically transmitted measurement outcomes are excluded.

This is especially important not only for the results in the infinite acceleration limit but also if we identify this limit with a black hole situation where an observer is freely falling and another observer is resting arbitrarily close to the event horizon (see Ref. \cite{11}). Alice, when falling into a black hole, cannot communicate on a quantum information level with an observer who is resting near the horizon.

We have further found that the choice of Unruh mode (i.e., $q_R$ and $q_L$) crucially influences which of the accelerated observers, Rob or AntiRob, can in principle test the non-locality of the initial state.

Note added. Shortly after submission of this paper Ref. \cite{21} appeared, in which similar results, in accordance with this paper, are presented.

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[1] P. M. Alsing, I. Fuentes-Schuller, R. B. Mann and T. E. Tessier, Phys. Rev. A 74, 032326 (2006)
[2] D. Ahn, H. J. Lee, Y. H. Moon and S. W. Hwang, Phys. Rev. A 67, 012103 (2003)
[3] H. Terashima and M. Ueda, Int. J. Quantum Inform. 1, 93-114 (2003)
[4] N. Friis, R. A. Bertlmann, M. Huber and B. C. Hiesmayr, Phys. Rev. A 81, 042114 (2010)
[5] R. M. Gingrich and Ch. Adami, Phys. Rev. Lett. 89, 270402 (2002); A. Peres and D. R. Terno, Rev. Mod. Phys. 76, No. 1, 93-123 (2004); T. F. Jordan, A. Shaji and E. C. G. Sudarshan, Phys. Rev. A 75, 022101 (2007); M. Huber, N. Friis, A. Gabriel, Ch. Spengler and B. C. Hiesmayr, EPL 95, 20002 (2011)
[6] E. Martín-Martínez and I. Fuentes, Phys. Rev. A 83, 052306 (2011)
[7] P. M. Alsing and G. J. Milburn, Phys. Rev. Lett. 91, 180404 (2003)
[8] D. E. Bruschi, J. Louko, E. Martín-Martínez, A. Dragan and I. Fuentes, Phys. Rev. A 82, 042332 (2010)
[9] I. Fuentes-Schuller and R. B. Mann, Phys. Rev. Lett. 95, 120404 (2005)
[10] L. C. B. Crispino, A. Higuchi and G. E. A. Matsas, Rev. Mod. Phys. 80, No. 3, 787-838 (2008)
[11] E. Martín-Martínez, L. J. Garay and J. León, Phys. Rev. D 82, 064006 (2010)
[12] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002)
[13] R. Horodecki, P. Horodecki and M. Horodecki, Phys. Lett. A 200, 340-344 (1995)
[14] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23, 880-884 (1969)
[15] R. Jáuregui, M. Torres, and S. Hacyan, Phys. Rev. D 43, 3979 (1991).
[16] P. Langlois, Phys. Rev. D 70, 104008 (2004)
[17] M. Montero and E. Martín-Martínez, Phys. Rev. A 83, 062323 (2011)
[18] W. K. Wootters, Phys. Rev. Lett. 80, 2245-2248, (1998)
[19] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002)
[20] F. Verstraete, and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003)
[21] A. Smith, and R. B. Mann, e-print arXiv:1107.4633v1 [quant-ph] (2011)