How to compute the rank of a Delaunay polytope

Mathieu DUTOUR SIKIRIĆ  Viatcheslav GRISHUKHIN
Institut Rudjer Bosković, Zagreb  CEMI RAN, Russia

Abstract

Roughly speaking, the rank of a Delaunay polytope (first introduced in [2]) is its number of degrees of freedom. In [3], a method for computing the rank of a Delaunay polytope $P$ using the hypermetrics related to $P$ is given. Here a simpler more efficient method, which uses affine dependencies instead of hypermetrics is given. This method is applied to classical Delaunay polytopes.

Then, we give an example of a Delaunay polytope, which does not have any affine basis.

1 Introduction

A lattice $L$ is a set of the form $v_1\mathbb{Z} + \cdots + v_n\mathbb{Z} \subset \mathbb{R}^n$. A Delaunay polytope $P$ is inscribed into an empty sphere $S$ such that no point of $L$ is inside $S$ and the vertex-set of $P$ is $L \cap S$. The Delaunay polytopes of $L$ form a partition of $\mathbb{R}^n$.

The vertex-set $V = V(P)$ of a Delaunay polytope $P$ is the support of a distance space $(V, d_P)$. For $u, v \in V(P)$, the distance $d_P(u, v) = \|u - v\|^2$ is the Euclidean norm of the vector $u - v$. A distance vector $(d(v, v'))$ with $v, v' \in V$ is called a hypermetric on the set $V$ if it satisfies $d(v, v') = d(v', v)$, $d(v, v) = 0$ and the following hypermetric inequalities:

$$H(b)d = \sum_{v, v' \in V} b_v b_v' d(v, v') \leq 0 \quad \text{for any } b = (b_v) \in \mathbb{Z}^V \quad \text{with} \quad \sum_{v \in V} b_v = 1. \quad (1)$$

The set of distance vectors, satisfying (1) is called the hypermetric cone and denoted by $HYP(V)$.

The distance $d_P$ is a hypermetric, i.e., it belongs to the hypermetric cone $HYP(V)$. The rank of $P$ is the dimension of the minimal by inclusion face $F_P$ of $HYP(V)$ which contains $d_P$.

It is shown in [3] that $d_P$ determines uniquely the Delaunay polytope $P$. When we move $d_P$ inside $F_P$, the Delaunay polytope $P$ changes, while its affine type remain the same. In other words, like the rank of $P$, the affine type of $P$ is an invariant of the face $F_P$.

The above movement of $d_P$ inside $F_P$ corresponds to a perturbation of each basis of $L$, and, therefore, of each Gram matrix (i.e., each quadratic form) related to $L$. In this paper, we show that there is a one-to-one correspondence between the space spanned by $F_P$ and the space $B(P)$ spanned by the set of perturbed quadratic forms. Hence, those two spaces have the same
dimension. It is shown here, that if one knows the coordinates of vertices of \( P \) in a basis, then it is simpler to compute \( \dim(\mathcal{B}(P)) \) than \( \dim(F_P) \). This fact is illustrated by computations of ranks of cross polytope and half-cube.

In the last section, we describe a non-basic repartitioning Delaunay polytope recently discovered by the first author.

\section{Equalities of negative type and hypermetric}

A sphere \( S = S(c, r) \) of radius \( r \) and center \( c \) in an \( n \)-dimensional lattice \( L \) is said to be an \textit{empty sphere} if the following two conditions hold:

(i) \( \|a - c\|^2 \geq r^2 \) for all \( a \in L \),

(ii) the set \( S \cap L \) contains \( n + 1 \) affinely independent points.

A Delaunay polytope \( P \) in a lattice \( L \) is a polytope, whose vertex-set is \( L \cap S(c, r) \) with \( S(c, r) \) an empty sphere.

Denote by \( L(P) \) the lattice generated by \( P \). In this paper, we can suppose that \( P \) is \textit{generating} in \( L \), i.e., that \( L = L(P) \). A subset \( V \subseteq V(P) \) is said to be \textit{\( \mathbb{K} \)-generating}, with \( \mathbb{K} \) being a ring, if every vertex \( w \in V(P) \) has a representation \( w = \sum_{v \in V} z(v)v \) with \( 1 = \sum_{v \in V} z(v) \) and \( z(v) \in \mathbb{K} \). If \( |V| = n + 1 \), then \( V \) is called an \textit{\( \mathbb{K} \)-affine basis}; the Delaunay polytope \( P \) is called \textit{\( \mathbb{K} \)-basic} if it admits at least one \( \mathbb{K} \)-affine basis. In this work \( \mathbb{K} \) will be \( \mathbb{Z} \), \( \mathbb{Q} \) or \( \mathbb{R} \) and if the ring is not precised, it is \( \mathbb{Z} \). Furthermore, let

\[
Y(P) = \{ y \in \mathbb{Z}^{|V(P)|} : \sum_{v \in V(P)} y(v)v = 0, \quad \sum_{v \in V(P)} y(v) = 0 \} \tag{2}
\]

be the \( \mathbb{Z} \)-module of all integral dependencies on \( V(P) \). If the Delaunay polytope \( P \) is a simplex, then \( Y(P) = \{0\} \).

A dependency on \( V(P) \) implies some dependencies between distances \( d_P(u, v) \) as follows. Let \( c \) be the center of the empty sphere \( S \) circumscribing \( P \). Then all vectors \( v - c, v \in V(P) \), have the same norm \( \|v - c\|^2 = r^2 \), where \( r \) is the radius of the sphere \( S \). Hence,

\[
d_P(u, v) = \|u - v\|^2 = \|u - c - (v - c)\|^2 = 2(r^2 - \langle u - c, v - c \rangle) \tag{3}
\]

Multiplying this equality by \( y(v) \) and summing over \( v \in V(P) \), we get

\[
\sum_{v \in V(P)} y(v)d_P(u, v) = 2r^2 \sum_{v \in V(P)} y(v) - 2\langle u - c, \sum_{v \in V(P)} y(v)(v - c) \rangle.
\]

Since \( y \in Y(P) \), we obtain the following important equality

\[
\sum_{v \in V(P)} y(v)d_P(u, v) = 0, \text{ for any } u \in V(P) \text{ and } y \in Y(P). \tag{4}
\]
Denote by $\mathcal{S}_{\text{dist}}(P)$ the system of equations (1) for all integral dependencies $y \in Y(P)$ and all $u \in V(P)$, where the distances $d_P(u,v)$ are considered as unknowns.

Multiplying the equality (1) by $y(u)$ and summing over all $u \in V(P)$, we obtain

$$
\sum_{u,v \in V(P)} y(u) y(v) d_P(u,v) = 0.
$$

(5)

This equality is called an equality of negative type and the system of such equality is denoted $\mathcal{S}_{\text{neg}}(P)$. Hence, the equalities of $\mathcal{S}_{\text{neg}}(P)$ are implied by the one of $\mathcal{S}_{\text{dist}}(P)$.

Each integral dependency $y \in Y(P)$ determines the following representation of a vertex $w \in V(P)$ as an integer combination of vertices from $V(P)$:

$$
w = w + \sum_{v \in V(P)} y(v)v = \sum_{v \in V(P)} y^w(v)v,
$$

where

$$
y^w(v) = \begin{cases} y(v) & \text{if } v \neq w, \\ y(w) + 1 & \text{if } v = w \end{cases} \quad \text{and} \quad \sum_{v \in V(P)} y^w(v) = 1.
$$

Let $\delta_w$ be the indicator function of $V(P)$: $\delta_w(v) = 0$ if $v \neq w$, and $\delta_w(w) = 1$. Obviously, $\delta_w$ is $y^w$ for the trivial representation $w = w$. We have $y^w = y + \delta_w$. Conversely, every representation $w = \sum_{v \in V(P)} y^w(v)v$ provides the dependency $y = y^w - \delta_w \in Y(P)$. Substituting $y = y^w - \delta_w$ in (5), we obtain the following equality

$$
\sum_{u,v \in V(P)} y(u) y(v) d_P(u,v) = \sum_{u,v \in V(P)} y^w(u) y^w(v) d_P(u,v) - 2 \sum_{v \in V(P)} y^w(v) d_P(w,v).
$$

Since $d_P(w,w) = 0$, we can set $y^w = y$ in the last sum. For any $w \in V(P)$, we use this equality in the following form using equations (1) and (5)

$$
\sum_{u,v \in V(P)} y^w(u) y^w(v) d_P(u,v) = \sum_{u,v \in V(P)} y(u) y(v) d_P(u,v) + 2 \sum_{v \in V(P)} y(v) d_P(w,v) = 0.
$$

(6)

The equality

$$
\sum_{u,v \in V(P)} z(u) z(v) d_P(u,v) = 0, \quad \text{where} \quad \sum_{v \in V(P)} z(v) = 1, \quad z(v) \in \mathbb{Z},
$$

is the hypermetric equality. Denote by $\mathcal{S}_{\text{hyp}}(P)$ the system of all hypermetric equalities which hold for $d_P(u,v)$, considering the distances $d_P(u,v)$ as unknowns.

In [3], the following lemma is proved. For the sake of completeness, we give its short proof.

**Lemma 1** Let $P$ be a Delaunay polytope with vertex-set $V(P)$. Let $y^w \in \mathbb{Z}^{V(P)}$, such that $\sum_{v \in V(P)} y^w(v) = 1$. Then the following assertions are equivalent

(i) a vertex $w \in V(P)$ has the representation $w = \sum_{v \in V(P)} y^w(v)v$;
(ii) the distance \( d_P \) satisfies the hypermetric equality 
\[
\sum_{u,v \in V(P)} y^w(u)y^w(v)d_P(u,v) = 0.
\]

**Proof.** (i)⇒(ii) Obviously, \( y = y^w - \delta_w \) is a dependency, i.e. \( y \in Y(P) \). Hence, this implication follows from the equalities (i)\textsuperscript{3}, (i)\textsuperscript{1} and (i)\textsuperscript{5}.

(ii)⇒(i) Substituting the expression (i)\textsuperscript{6} for \( d_P \) in the hypermetric equality of (ii) we obtain the equality
\[
2r^2 - 2\| \sum_{v \in V(P)} y^w(v)(v - c) \|^2 = 0.
\]

Obviously, \( \sum_{v \in V(P)} y^w(v)c = c \) and \( \sum_{v \in V(P)} y^w(v)v \) is a point of \( L(P) \). Denote this point by \( w \). Then the above equality takes the form \( \| w - c \|^2 = r^2 \). Hence, \( w \) lies on the empty sphere circumscribing \( P \). Therefore, \( w \in V(P) \) and (i) follows. \( \square \)

According to Lemma 1\textsuperscript{1} each hypermetric equality of the system \( S_{hyp}(P) \) corresponds to a representation \( y^w \) of a vertex \( w \in V(P) \). Since the relation \( y = y^w - \delta_w \) gives a one-to-one correspondence between dependencies on \( V(P) \) and non-trivial representations \( y^w \) of vertices \( w \in V(P) \), we can prove the following assertion:

**Lemma 2** The systems of equations \( S_{dist}(P) \) and \( S_{hyp}(P) \) are equivalent, i.e., their solution sets coincide.

**Proof.** The equality (i)\textsuperscript{3} shows that each equation of the system \( S_{hyp}(P) \) is implied by equations of the system \( S_{dist}(P) \).

Now, we show the converse implication. Suppose the unknowns \( d(u,v) \) satisfy all hypermetric equalities of the system \( S_{hyp}(P) \). The equality (i)\textsuperscript{6} implies the equality
\[
2 \sum_{v \in V(P)} y(v)d(w,v) = - \sum_{u,v \in V(P)} y(u)y(v)d(u,v),
\]
where \( y = y^w - \delta_w \). This shows that, for the dependency \( y \) on \( V(P) \), \( \sum_{v \in V(P)} y(v)d(w,v) \) does not depend on \( w \); denote it by \( A(y) \). Hence, we have
\[
-2A(y) = \sum_{u,v \in V(P)} y(u)y(v)d(u,v) = A(y) \sum_{u \in V(P)} y(u).
\]

According to equation (i)\textsuperscript{4}, the last sum equals zero. This implies the equalities (i)\textsuperscript{5} and hence the equalities of the system \( S_{dist}(P) \). \( \square \)

Obviously, the space determined by the system \( S_{hyp}(P) \) (and also of the system \( S_{dist}(P) \)) is a subspace \( X(P) \) of the space spanned by all distances \( d(u,v) \), \( u,v \in V(P) \). The dimension of \( X(P) \) is the rank of \( P \). According to Lemma 2\textsuperscript{2} in order to compute the rank of \( P \), we can use only equations of the system \( S_{dist}(P) \).

Let \( V_0 = \{ v_0, v_1, \ldots, v_n \} \) be an \( \mathbb{R} \)-affine basis of \( P \). Then each vertex \( w \in V(P) \) has a unique representation through vertices of \( V_0 \) as follows
\[
w = \sum_{v \in V_0} x(v)v, \quad \sum_{v \in V_0} x(v) = 1, \quad x(v) \in \mathbb{R}.
\]
Since the vertices of \( P \) are points of a lattice, in fact, \( x(v) \in \mathbb{Q} \). Hence, the above equation can be rewritten as an integer dependency

\[
y(w)w + \sum_{v \in V_0} y(v)v = 0, \quad y(w) + \sum_{v \in V_0} y(v) = 0, \text{ with } y(w) \in \mathbb{Z}.
\]

One sets \( y_u = 0 \) for \( u \in V(P) - (V_0 \cup \{w\}) \) and gets \( y_u \in Y(P) \). Any dependency \( y \in Y(P) \) is a rational combination of dependencies \( y_w, \ w \in V(P) - V_0 \). Hence, the following equality holds:

\[
\beta y = \sum_{w \in V(P) - V_0} \beta_w y_w, \text{ with } \beta_w \in \mathbb{Z} \text{ and } 0 < \beta \in \mathbb{Z}
\]

Since the equalities \( \beta \) are linear over \( y \in Y(P) \), the dependencies \( y_w, \ w \in V(P) - V_0 \) provide the following system, which is equivalent to \( S_{\text{dist}}(P) \)

\[
y(w)d_P(u, w) + \sum_{v \in V_0} y(v)d_P(u, v) = 0, \text{ with } u \in V(P) \text{ and } w \in V(P) - V_0. \tag{7}
\]

We see that, for \( u \in V(P) - V_0 \), the distance \( d_P(u, w), \ w \in V(P) - V_0 \), is also expressed through distances between \( u \) and \( v \in V_0 \). But for \( u \in V_0 \), the distance \( d_P(u, w) \) is expressed through distances between \( u, v \in V_0 \). This implies that the distance \( d_P(u, w) \) for \( u, w \in V(P) - V_0 \) can be also represented through distances \( d_P(u, v) \) for \( u, v \in V_0 \). Hence, the dimension of \( X(P) \) does not exceed \( \frac{n(n+1)}{2} \), where \( n + 1 = |V_0| \), which is the dimension of the space of distances between the vertices of \( V_0 \).

In order to obtain dependencies between \( d_P(u, v) \) for \( u, v \in V_0 \), we use equation (7) for \( u = w \). Since \( d_P(w, w) = 0 \), we obtain the equations

\[
\sum_{v \in V_0} y(w)(v)d_P(v, w) = 0, \ w \in V(P) - V_0.
\]

Multiplying the above equation by \( y(w) \) and using equation (7), we obtain

\[
0 = \sum_{u \in V_0} y(w)(u)(y(w)(d_P(u, w))) = -\sum_{u \in V_0} y(w)(u) \sum_{v \in V_0} y(v)d_P(u, v).
\]

So, we obtain the following main equations for dependencies between \( d_P(u, v) \) for \( u, v \in V_0 \)

\[
\sum_{u, v \in V_0} y(u)y(v)d_P(u, v) = 0, \ w \in V(P) - V_0. \tag{8}
\]

Note, that if \( V_0 \) is an affine basis of \( L(P) \), then one can set \( y(w) = -1 \). In this case, the equation \( y(w) + \sum_{v \in V_0} y(v) = 0 \) takes the form \( \sum_{v \in V_0} y(v) = 1 \). This implies that the above equations are hypermetric equalities for a \( \mathbb{Z} \)-basic Delaunay polytope \( P \). If \( P \) is \( \mathbb{Z} \)-basic, then the distance \( d_P \) restricted to the set \( V_0 \) lies on the face of the cone \( HY P(V_0) \) determined by the hypermetric equalities (8). But if \( P \) is not \( \mathbb{Z} \)-basic, then the equations (8) are not hypermetric, and the distance \( d_P \) restricted on the set \( V_0 \) lies inside the cone \( HY P(V_0) \). On the other hand,
the distance $d_P$ on the whole set $V(P)$ lies on the boundary of the cone $HYP(V(P))$. This implies that, in this case, the rank of $d_P$ restricted to $V_0$ is greater than the rank of $d_P$ on $V(P)$.

This can be explained as follows. We can consider the cone $HYP(V_0)$ as a projection of $HYP(V(P))$ on a face of the positive orthant $\mathbb{R}^N_+$, where $N = |V(P)|$. This face is determined by the equations $d(u, v) = 0$ for $v \in V(P) - V_0$ or/and $u \in V(P) - V_0$. By this projection, the distance $d_P$, lying on the boundary of the cone $HYP(V(P))$, is projected into the interior of the cone $HYP(V_0)$. This hypermetric space corresponds to a wall of an $L$-type domain, which lies inside the cone $HYP(V_0)$.

But, in order to compute the rank of $P$, it is sufficient to find the dimension of the space determined by the system (8).

3 Dependencies between lattice vectors

Now we go from affine realizations to linear realizations. Take $v_0 \in V_0$ as origin of the lattice $L(P)$ and choose the lattice vectors $a_i = a(v_i) = v_i - v_0$, $1 \leq i \leq n$ such that $\{a_i : 1 \leq i \leq n\}$ forms a $\mathbb{Q}$-basis of $L(P)$. If $P$ is basic, we can choose $v_i$ such that $\{a_i : 1 \leq i \leq n\}$ is a $\mathbb{Z}$-basis of $L(P)$. Using the expressions $d_P(v_i, v_j) = ||a_i - a_j||^2$, it is easy to verify that there is the following relation between distances $d_P(v_i, v_j)$, $u, v \in V_0$, and inner products $\langle a_i, a_j \rangle$:

$$d_P(v_i, v_0) = ||a_i||^2, \quad d_P(v_i, v_j) = ||a_i||^2 - 2\langle a_i, a_j \rangle + ||a_j||^2.$$

And conversely,

$$||a_i||^2 = d_P(v_i, v_0), \quad \langle a_i, a_j \rangle = \frac{1}{2}(d_P(v_i, v_0) + d_P(v_j, v_0) - d_P(v_i, v_j)).$$

This shows that there is a one-to-one correspondence between the set of distances $d_P(v_i, v_j)$, $0 \leq i < j \leq n$, and the set of inner products $\langle a_i, a_j \rangle$, $1 \leq i \leq j \leq n$.

We substitute the above expressions for $d_P(v_i, v_j)$, $0 \leq i, j \leq n$, into the equations (8), where we set $y_w(i) = y_w(v_i)$, and use the equality $\sum_{i=0}^n y_w(i) = -y_w(w)$. We obtain the following important equations

$$-y_w(w) \sum_{i=1}^n y_w(i)||a_i||^2 = ||\sum_{i=1}^n y_w(i)a_i||^2, \quad w \in V(P) - V_0. \quad (9)$$

We can obtain the equation (8) directly, as follows. For $v \in V(P)$, the vector $a(v) = v - v_0$ is a lattice vector of $L(P)$. For $y \in Y(P)$, we have obviously $\sum_{v \in V(P)} y(v)a(v) = 0$. In particular, for $y = y_w$, this equation has the form

$$y_w(w)a(w) + \sum_{i=1}^n y_w(i)a_i = 0$$

and allows to represent the vectors $a(w)$ in the $\mathbb{Q}$-basis $\{a_i : 1 \leq i \leq n\}$.

Recall that the lattice vector $a(w)$ of each vertex $w \in V(P)$ of a Delaunay polytope $P$ satisfies the equation $||a(w) - c||^2 = r^2$. Since $v_0 \in V$, $a(v_0) = 0$, which implies $||c||^2 =
\[ \|0 - c\|^2 = r^2. \] The vertex-set of \( P \) provides the following system of equations \( \|a(w) - c\|^2 = \|c\|^2, \ w \in V(P), \) i.e.,
\[
2\langle c, a(w) \rangle = \|a(w)\|^2, \ w \in V(P).
\]
Since \( y_w(w)a(w) = -\sum_{i=1}^{n} y_{i_1}(i)a_i, \) the above equations take the form
\[
-y_w(w)\sum_{i=1}^{n} y_{i_1}(i)2\langle c, a_i \rangle = \|\sum_{i=1}^{n} y_{i_1}(i)a_i\|^2. \tag{11}
\]
Recall that \( a_i = a(v_i). \) Hence, the vertices \( v_i \) give \( 2\langle c, a_i \rangle = \|a_i\|^2, \) and the above equation takes the form of equation (10).

We will use the equations (10) mainly for basic Delaunay polytopes. In this case, we can set \( y_w(w) = -1, \) and \( a_i = b_i, \ 1 \leq i \leq n, \) where \( B = \{b_i : 1 \leq i \leq n\} \) is the basis of \( L(P) \) consisting of lattice vectors of the basic Delaunay polytope \( P. \)

For a given \( \mathbb{Q} \)-affine basis \( V_0 \subseteq V(P) \) of a Delaunay polytope \( P, \) the set of affine dependencies \( \{y_w \in Y(P) : w \in V(P) - V_0\} \) is uniquely determined up to integral multipliers and form a \( \mathbb{Q} \)-basis of the \( \mathbb{Z} \)-module \( Y(P). \) This implies that the equations (10) determine a subspace
\[
\mathcal{A}(P) = \{a_{ij} : -y_w(w)\sum_{i=1}^{n} y_{i_1}(i)a_i = \sum_{1 \leq i, j \leq n} y_w(i)y_w(j)a_{ij}, \ w \in Y(P), w \in V(P) - V_0\}
\]
in the \( \frac{n(n+1)}{2} \)-dimensional space of all symmetric \( n \times n \)-matrices \( a_{ij} = a_{ji}, \ 1 \leq i < j \leq n, \)

Since there is a one-to-one correspondence between distances \( d(v_i, v_j), \ 0 \leq i < j \leq n, \) and inner products \( a_{ij} = \langle a_i, a_j \rangle, \ 1 \leq i \leq j \leq n, \) the dimension of the subspace \( \mathcal{A}(P) \) is equal to the rank of \( P. \) So, in order to compute the rank of \( P, \) we have to find the dimension of \( \mathcal{A}(P). \)

### 4 The space \( \mathcal{B}(P) \) and our computational method

Fix a basis \( B = \{b_i : 1 \leq i \leq n\} \) of the lattice \( L. \) Every lattice vector \( a(v), \ v \in V(P), \) has a unique representation \( a(v) = \sum_{i=1}^{n} z_i(v)b_i. \) Define \( Z_B(P) = \{z_i(v) : 1 \leq i \leq n, v \in V(P)\}. \)

Recall that the cone \( \mathcal{P}_n \) of all positive semi-definite forms on \( n \) variables is partitioned into \( L \)-type domains. Each \( L \)-type domain is an open polyhedral cone of dimension \( k, \) where \( 1 \leq k \leq \frac{n(n+1)}{2}. \) It consists of form having affinely equivalent partitions into Delaunay polytopes, i.e. Delaunay partitions. More exactly, an \( L \)-type domain is the set of quadratic forms \( f(x) = \|\sum_{i=1}^{n} x_ib_i\|^2 \) having the same set of matrices \( Z_B(P) \) for all non-isomorphic Delaunay polytopes \( P \) of its Delaunay partition. So, this set is not changed when the basis \( B \) changes such that the form \( f(x) \) belongs to the same \( L \)-type domain. In other words, \( Z_B(P) \) is an invariant of this \( L \)-type domain.

We set \( z_{ij} = z_i(v_j) \) for \( v_j \in V_0 - \{v_0\}, \ 1 \leq j \leq n. \) The matrix \( Z_B = (z_{ij})_i^l \) is non-degenerate and gives a correspondence between the linear bases of \( P \) and bases of \( L(P). \) In particular, this correspondence maps the space \( \mathcal{A}(P) \) in the space \( \mathcal{B}(P) \) of matrices \( b_{ij} = \langle b_i, b_j \rangle \) of the quadratic form \( f(x). \) If \( P \) is basic and \( b_i = a_i, \ 1 \leq i \leq n, \) then \( Z_B \) is the identity matrix \( I, \) and \( \mathcal{A}(P) = \mathcal{B}(P). \)
Substituting in the equations (10) the above representations of the vectors \(a(v), v \in V(P)\), in the basis \(B\), we obtain explicit equations, determining the space \(B(P)\). In fact, we have

\[
2 \sum_{i=1}^{n} z_i(v)\langle c, b_i \rangle = \sum_{1 \leq i, j \leq n} z_i(v)z_j(v)b_{ij}, \; v \in V(P). \tag{12}
\]

We have the following \(\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}\) parameters in the equations (12):

\[
b_{ij} = \langle b_i, b_j \rangle, \; 1 \leq i \leq j \leq n, \quad \text{and} \quad \langle c, b_i \rangle, \; 1 \leq i \leq n,
\]

Hence, all these parameters can be represented through a number of independent parameters. This number is just the rank of \(P\). Recall that a Delaunay polytope is called extreme if \(\text{rk}(P) = 1\). Hence, in order to be extreme, a Delaunay polytope should have at least \(\frac{n(n+3)}{2}\) vertices.

Note that, for \(v = v_0\), the equation (12) is an identity, since \(a(v_0) = 0\) and therefore \(z_i(v_0) = 0\) for all \(i\). So, we have \(|V(P)| - 1\) equations (12). For \(v = v_i, 1 \leq i \leq n\), one gets \(n\) equations that give a representation of the parameters \(\langle c, b_i \rangle, 1 \leq i \leq n\) in terms of the parameters \(\langle b_i, b_j \rangle, 1 \leq i \leq j \leq n\). Hence, the equations (12), for \(v \in V(P) - V_0\), allow to find dependencies between the main parameters \(\langle b_i, b_j \rangle\). Now, we write out explicitly dependencies between \(\langle b_i, b_j \rangle\).

Since the basic vectors \(b_i \in B\) are mutually independent, a dependency \(\sum_{v \in V} y(v)a(v) = 0\) implies the dependencies \(\sum_{v \in V} y(v)z_i(v) = 0\) between the coordinates \(z_i(v), 1 \leq i \leq n\).

Multiplying equation (12) by \(y(v)\), and summing over all \(v \in V(P)\), we obtain that the \(\mathbb{Z}\)-module \(Y(P)\) determines the following subspace of the space of parameters \(b_{ij} = \langle b_i, b_j \rangle\):

\[
B(P) = \{b_{ij} : \sum_{i,j=1}^{n} (\sum_{v \in V} y(v)z_i(v)z_j(v))b_{ij} = 0, \; y \in Y(P)\}.
\]

In the Delaunay partition of the lattice \(L(P)\), there are infinitely many Delaunay polytopes equivalent to \(P\). Each of them has the form \(a \pm P\), where \(a = \sum_{i=1}^{n} z_i^a b_i\) is an arbitrary lattice vector of \(L(P)\). Now, we show that the space \(B(P)\) is independent on a representative of \(P\) in \(L(P)\), i.e., that \(B(P) = B(a \pm P)\).

Let \(v_a = a \pm v\) be the vertex of the polytope \(a \pm P\) corresponding to a vertex \(v\) of \(P\). Obviously, \(z_i(v_a) = z_i^a \pm z_i(v)\). Substituting these values of \(z_i(v_a)\) into the equations determining \(B(a \pm P)\), we obtain

\[
\sum_{v_a} y(v_a)z_i(v_a)z_j(v_a) = \sum_{v \in V(P)} y(v)(z_i^a z_j^a \pm z_i^a z_j(v) + z_i(v)z_j(v)).
\]

Since \(y\) is a dependency between vertices of \(P\), the sums with \(z_i^a\) equal zero. This shows that \(B(P)\) does not depend on a representative of \(P\).

Since the equalities determining the space \(B(P)\) are linear in \(y\), we can consider these equalities only for basic dependencies \(y_w, w \in V(P) - V_0\). We obtain the following main system of equations describing dependencies between the parameters \(b_{ij}\):

\[
\sum_{i,j=1}^{n} (\sum_{v \in V} y_w(v)z_i(v)z_j(v))b_{ij} = 0, \; w \in V(P) - V_0. \tag{13}
\]
A unimodular transformation maps a basis of $L(P)$ into another basis. This transformation generates a transformation which maps the space $B(P)$ into another space related to $P$. The dimension of the space $B(P)$ is an invariant of the lattice $L(P)$ generated by $P$.

In [1], a non-rigidity degree of a lattice was defined. In terms of this paper, the non-rigidity degree of a lattice $L$ is the dimension of the intersection of spaces $B(P)$ related to all non-isomorphic Delaunay polytopes of a star of Delaunay polytopes of $L$. Hence,

$$\text{nrd}(L) = \dim(\cap_P B(P)).$$

In fact, the space $\cap_P B(P)$ is the supporting space of the $L$-type domain of the lattice $L$.

## 5 Centrally symmetric construction

In many cases, the computation of the rank of a Delaunay polytope $P$ using the equations [12] is easier than by using the hypermetric equalities generated by $P$. We demonstrate this by giving a simpler proof of Lemma 15.3.7 of [3]. Recall that a Delaunay polytope is either centrally symmetric or asymmetric. Let $c$ be the center of the empty sphere circumscribing $P$. For any $v \in V(P)$, the point $v^* = 2c - v$ is centrally symmetric to $v$. If $P$ is centrally symmetric, then $v^* \in V(P)$ for all $v \in V(P)$. If $P$ is asymmetric, then $v^* \not\in V(P)$ for all $v \in V(P)$.

**Lemma 3** Let $P$ be an $n$-dimensional basic centrally symmetric Delaunay polytope of a lattice $L$ with the following properties:

1. The origin $0 \in V(P)$ and the vectors $e_i$, $1 \leq i \leq n$, are basic vectors of $L$, whose endpoints are vertices of $P$.

2. The intersection $P_1 = P \cap H$ of $P$ with the hyperplane $H$ generated by the vectors $e_i$, $1 \leq i \leq n - 1$, is an asymmetric Delaunay polytope of the lattice $L_1 = L \cap H$.

3. If the endpoint $v_n$ of the basic vector $e_n$ is $v^*$ for some $v \in V(P_1)$, then there is a vertex $u \in V(P)$ such that $u \neq v, v^*$ for all $v \in V(P_1)$.

Then $\text{rk}(P) \leq \text{rk}(P_1)$.

**Proof.** It is sufficient to prove that the $n$ parameters $\langle e_i, e_n \rangle$, $1 \leq i \leq n$, can be expressed through the parameters $a_{ij} = \langle e_i, e_j \rangle$, $1 \leq i \leq j \leq n - 1$.

Let $c$ be the center of $P$. Obviously, $2c = 0^* \in V(P)$. Since $P_1$ is asymmetric, $2c \not\in L_1$. It is easy to see that $2c = a_0 + ze_n$, with $a_0 = \sum_{i=1}^{n-1} y_i e_i \in L_1$ and $0 \neq z \in \mathbb{Z}$. Hence, the equation $2\langle c, e_i \rangle = \|e_i\|^2$ takes the form $\langle a_0 + ze_n, e_i \rangle = \|e_i\|^2$, and the parameters $\langle e_i, e_n \rangle$ are represented through the parameters $\langle e_i, e_j \rangle$ as follows

$$\langle e_i, e_n \rangle = \frac{1}{z}(\|e_i\|^2 - \langle a_0, e_i \rangle), \quad 1 \leq i \leq n - 1.$$

Now, using the equation $2\langle c, e_n \rangle = \|e_n\|^2$, we obtain $\langle a_0 + ze_n, e_n \rangle = \|e_n\|^2$, i.e., $\|e_n\|^2(1 - z) = \langle a_0, e_n \rangle = \sum_{i=1}^{n-1} y_i \langle e_i, e_n \rangle$. Hence, if $z \neq 1$, we can represent $\|e_n\|^2$ through $\langle e_i, e_j \rangle$. 


1 \leq i \leq j \leq n - 1$, too. But if $z = 1$, then the endpoint $v_n$ of $e_n$ belongs to $(V(P_1))^*$. In this case there is a vertex $u$ such that $u = \sum_{i=1}^n z_i e_i = u_0 + z_n e_n$, where $u_0 \in L_1$ and $z_n \neq 0, 1$. Using the equation $2\langle c, u \rangle = \|u\|^2$, where now $2c = a_0 + e_n$, we have $\langle a_0 + e_n, u_0 + z_n e_n \rangle = \|u_0 + z_n e_n\|^2$. This equation gives

$$\|e_n\|^2 = \frac{1}{z_n(z_n-1)}[(a_0 - u_0, u_0) + \langle z_n a_0 + (1 - 2z_n)u_0, e_n \rangle].$$

The strict inequality $\text{rk}(P) < \text{rk}(P_1)$ is possible if some vertices of the set $V(P) - V(P_1)$ provide additional relations between the parameters $\langle e_i, e_j \rangle$, $1 \leq i \leq j \leq n - 1$.

**Examples**, where $\text{rk}(P) < \text{rk}(P_1)$, can be given by some extreme Delaunay polytopes.

**Corollary 1** Let $P$ be a basic centrally symmetric Delaunay polytope satisfying the conditions of Lemma 3. $P$ is extreme if $P_1$ is extreme.

6 Computing the rank of simplexes, cross-polytopes and half-cubes

**Simplexes**. Let $\Sigma$ be an $n$-dimensional simplex with vertices $0, v_i$, $1 \leq i \leq n$. The vertex $v_i$ is the end-point of the basic vector $e_i$, $1 \leq i \leq n$. We have only $n$ equations $2\langle c, e_i \rangle = \|e_i\|^2$ determining only the coordinates of the center $c$ of $\Sigma$ in the basis $\{e_i : 1 \leq i \leq n\}$. Since there is no relation between the $\frac{n(n+1)}{2}$ parameters $\langle e_i, e_j \rangle = a_{ij}$, all these parameters are independent. Hence,

$$\dim(B(\Sigma)) = \frac{n(n+1)}{2}, \text{ i.e., \text{rk}(\Sigma) = \frac{n(n+1)}{2}}.$$ 

**Cross-polytopes**. An $n$-dimensional cross-polytope $\beta_n$ is a basic centrally symmetric Delaunay polytope. It is the convex hull of $2n$ end-points of $n$ linearly independent segments intersecting in the center $c$ of the circumscribing sphere. The set $V(\beta_n)$ is partitioned into two mutually centrally symmetric $n$-subsets each of which is the vertex-set of an $(n-1)$-dimensional simplex $\Sigma$. So, $V(\beta_n) = V(\Sigma) \cup V(\Sigma^*)$. Let $V(\Sigma) = \{0, v_i : 1 \leq i \leq n - 1\}$. All $\mathbb{Z}$-affine bases of $\beta_n$ are of the same type: $n - 1$ basic vectors $e_i$, $1 \leq i \leq n - 1$, with end vertices $v_i$, are basic vectors of the simplex $\Sigma$, and $e_n = 2c$, which is the segment which connects the vertex 0 with its opposite vertex $0^*$. Let $a_i$ be the lattice vector endpoint of which is the vertex $v_i^* \in \Sigma^*$. Obviously, $a_i = 2c - e_i = e_n - e_i$. The equality $2\langle c, a_i \rangle = \|a_i\|^2$ gives $\langle e_i, e_n \rangle = \|e_i\|^2$, $1 \leq i \leq n - 1$. So, we obtain $n - 1$ independent relations between the parameters $\langle e_i, e_j \rangle$, and they are the only relations. Hence,

$$\text{rk}(\beta_n) = \frac{n(n+1)}{2} - (n-1).$$

(Cf., the first formula on p.232 of [3].)

**Half-cubes**. Take $N = \{1, 2, \ldots, n\}$, a basis $(e_{i})_{i \in N}$ and defines $e(T) = \sum_{i \in T} e_i$ for any $T \subseteq N$. Call a set $T \subseteq N$ even if its cardinality $|T|$ is even. A half-cube $h\gamma_n$ is the convex hull
of endpoints of all vectors $e(T)$ for all even $T \subseteq N$. Note that $h\gamma_3$ is a simplex, and $h\gamma_4$ is the cross-polytope $\beta_4$. Hence,

$$\text{rk}(h\gamma_3) = \frac{3(3+1)}{2} = 6, \quad \text{and} \quad \text{rk}(h\gamma_4) = \frac{4(4+1)}{2} - 3 = 7.$$ 

The rank of $h\gamma_n$ is computed from the following system of equations:

$$2\langle c, e(T) \rangle = \|e(T)\|^2, \quad T \subseteq N, \quad T \text{ is even.} \tag{14}$$

Let $T_1$ and $T_2$ be two disjoint even subsets of $N$. Since the set $T = T_1 \cup T_2$ is even, we have

$$2\langle c, e(T_1 \cup T_2) \rangle = 2\langle c, e(T_1) + e(T_2) \rangle = \|e(T_1) + e(T_2)\|^2 = \|e(T_1)\|^2 + \|e(T_2)\|^2 + 2\langle e(T_1), e(T_2) \rangle.$$ 

Comparing this equation with the equations (14) for $T = T_1$ and $T = T_2$, we obtain that for any two disjoint even subsets the following orthogonality conditions hold:

$$\langle e(T_1), e(T_2) \rangle = 0, \quad \text{if} \quad T_1 \cap T_2 = \emptyset, \quad T_i \subset N, \quad \text{and} \quad T_i \text{ is even,} \quad i = 1, 2.$$ 

Note that, for $n = 3$, we have no orthogonality condition. If $n \geq 4$, take 4 elements $i, j, k$ and $l$ and write three equalities corresponding to three partitions:

$$\langle e_i + e_j, e_k + e_l \rangle = 0, \quad \langle e_i + e_k, e_j + e_l \rangle = 0, \quad \langle e_i + e_l, e_j + e_k \rangle = 0.$$ 

It is easy to verify that these equalities are equivalent to the following three equalities

$$\langle e_i, e_j \rangle + \langle e_k, e_l \rangle = 0, \quad \langle e_i, e_k \rangle + \langle e_j, e_l \rangle = 0, \quad \langle e_i, e_l \rangle + \langle e_j, e_k \rangle = 0. \tag{15}$$

In the particular case $n = 4$, we conclude again that $\text{rk}(h\gamma_4) = \frac{4(4+1)}{2} - 3 = 7$.

We show that, for $n \geq 5$, the orthogonality conditions are equivalent to mutual orthogonality of all vectors $e_i, i \in N$. To this end, it is sufficient to consider even subsets of cardinality two and use equation (15) for each quadruple $\{i, j, k, l\} \subseteq N$. Considering arbitrary subsets of $N$ of cardinality 4, we obtain that, for $n \geq 5$, the system of equalities (15) for all quadruples has the following unique solution

$$\langle e_i, e_j \rangle = 0, \quad 1 \leq i < j \leq n, \quad \text{for} \quad n \geq 5.$$ 

So, all the basic vectors are mutually orthogonal. Obviously, the orthogonality of basic vectors implies the orthogonality conditions. Hence, the only independent parameters are the $n$ parameters $\|e_i\|^2, i \in N$. This implies that

$$\text{rk}(h\gamma_n) = n \quad \text{if} \quad n \geq 5.$$ 

Note that we use a basis of $h\gamma_n$, which is not a basis of the lattice generated by $h\gamma_n$. But the spaces $B(P)$ have the same dimension for all bases. See another proof in [4].
7 A non-basic repartitioning Delaunay polytope

The example $P_0$ given in this section is 12 dimensional; its 14 vertices belong to two disjoint sets of vertices of regular simplexes $\Sigma_i^2$, $i = 1, 2$, of dimension 2, and two disjoint sets of vertices of regular simplexes $\Sigma_i^3$, $i = 1, 2$, of dimension 3.

Let $V(\Sigma_i^q)$ be the vertex-set of the four simplex $\Sigma_i^q$, $i = 1, 2$, $q = 2, 3$. Then $V = \cup V(\Sigma_i^q)$ is the vertex-set of $P_0$. The distances between the vertices of $P_0$ are as follows

$$d(u, v) = \begin{cases} 
7 & \text{if } u, v \in \Sigma_i^q, \quad i = 1, 2, \quad q = 2, 3; \\
6 & \text{if } u \in \Sigma_i^2, \quad v \in \Sigma_i^3, \quad i = 1, 2; \\
10 & \text{if } u \in \Sigma_i^2, \quad v \in \Sigma_i^3; \\
12 & \text{if } u \in \Sigma_i^3, v \in \Sigma_i^3 \text{ or } u \in \Sigma_i^2, v \in \Sigma_i^3, \text{ or } u \in \Sigma_i^2, v \in \Sigma_i^3.
\end{cases}$$

We show that, for every $u \in V$, the set $V - \{u\}$ is an $\mathbb{R}$-affine basis of $P_0$. In fact, let $V - \{u\} = \{v_i : 0 \leq i \leq 12\}$ and let $a_i = v_i - v_0$, $1 \leq i \leq 12$. For the Gram matrix $a_{ij} = \langle a_i, a_j \rangle$, we have $a_{ii} = \|a_i\|^2 = \|v_i - v_0\|^2 = d(v_i, v_0)^2$. The relations between $a_{ij}$ and $d(v_i, v_j)$ are $a_{ij} = \frac{1}{2}(d(v_i, v_0) + d(v_j, v_0) - d(v_i, v_j))$. Now, one can verify that the Gram matrix $(a_{ij})$ is not singular. Hence, $\{a_i : 1 \leq i \leq n\}$ is a basis, i.e. the dimension of $P_0$ is, in fact, 12.

The space $Y(P_0)$ of affine dependencies of vertices of $P_0$ is one-dimensional. For $v \in V$, let

$$y(v) = \begin{cases} 
3 & \text{if } v \in \Sigma_1^2, \\
-3 & \text{if } v \in \Sigma_2^2, \\
2 & \text{if } v \in \Sigma_2^3, \\
-2 & \text{if } v \in \Sigma_3^3.
\end{cases}$$

Obviously, $\sum_{v \in V} y(v) = 0$. It is easy to verify that for any $u \in V$ the following equality holds

$$\sum_{v \in V} y(v)d(u, v) = 0. \tag{16}$$

Let $S(c, r)$ be the sphere circumscribing $P_0$. Then $\|v - c\|^2 = r^2$ for all $v \in V$. We have $d(u, v) = \|u - v\|^2 = \|(u - c) - (v - c)\|^2 = 2(r^2 - (u - c, v - c))$. Since $\sum_{v \in V} y(v) = 0$, the equality (16) takes the form

$$\langle u - c, \sum_{v \in V} y(v)(v - c) \rangle = 0,$$

i.e., $\langle u - c, \sum_{v \in V} y(v)(v - c) \rangle = \langle u - c, \sum_{v \in V} y(v)v \rangle = 0.$

Since this equality holds for all $u \in V$, and $\{u - c \mid u \in V\}$ span $\mathbb{R}^{12}$, we obtain $\sum_{v \in V} y(v)v = 0$, i.e., $y \in Y(P_0)$. Since $Y(P_0)$ is one dimensional and the coefficient of $y$ have greatest common divisor 1, one has $Y(P_0) = y\mathbb{Z}$.

Using the basis $\{a_i : 1 \leq i \leq 12\}$, for non-basic $a(w)$, we obtain $a(w) = -\frac{1}{y(w)} \sum_{i=1}^{12} y(v_i)a_i$. Since there exist an $i$ such that $\frac{y(v_i)}{y(w)} \notin \mathbb{Z}$ for any choice of $w \in V$, the polytope $P_0$ is not basic and the $\mathbb{Q}$-basis $\{a_i : 1 \leq i \leq n\}$ is not a $\mathbb{Z}$-basis of any lattice $L$ having $P_0$ as a Delaunay polytope.
Remark that we can put the vector $y$ in equation (9), we obtain the following equation
\[-y(w) \sum_{i=1}^{12} y(v_i) ||a_i||^2 = || \sum_{i=1}^{12} y(v_i) a_i ||^2.\]

which implies that $rk(P_0) = rk(V,d) = \frac{12 \times 13}{2} - 1 = 77$.

It is useful to compare the above computation of $rk(P)$ with the following computations using distances. Recall that $rk(V(P_0),d)$ is equal to the dimension of the face of the hypermetric cone $HYP(V(P_0)) = HYP(V)$, where the distance $d$ lies. The dimension of $HYP(V)$ is $N = \frac{|V||V|-1}{2} = \frac{14 \times 13}{2} = 91$.

As in Section 2, we obtain that, for every $w \in V = V(P_0)$, the equality (16) implies the following hypermetric equality
\[
\sum_{v,v' \in V} y^w(v)y^w(v')d(v,v') = 0,
\]
where $y^w(v) = y(v) + \delta_w$. It is easy to see that the 14 equalities (17) for 14 vertices $w \in V$ are mutually independent. In fact, these 14 equalities are equivalent to the 14 equalities (16) for the 14 vertices $u \in V$. The two equations (16) corresponding to two vertices $u,w \in V$ have only one common distance $d(u,w)$. The intersection of the corresponding 14 facets is a face of dimension $91 - 14 = 77$.

But, for every $u \in V$, the hypermetric space $(V - \{u\},d)$ has rank $\frac{(|V| - |\{u\}|)(|V| - |\{u\}|-1)}{2} = 78$, which is greater than $rk(V,d) = 77$.

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