CESÀRO AVERAGES FOR GOLDBACH REPRESENTATIONS
WITH SUMMANDS IN ARITHMETIC PROGRESSIONS

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Abstract. We consider weighted averages of the number of representations of an even integer as a sum of two prime numbers, where each summand lies in a given arithmetic progression modulo a common integer $q$. Our result is uniform in a suitable range for $q$.

1. Introduction

In recent years a great deal of papers treated various questions related to the average number of representations of an integer as a sum of prime numbers, or powers of primes. Unweighted averages have been considered by Languasco & Zaccagnini [11], and more recently Bhowmik, Halupczok, Matsumoto & Suzuki [2]. Weighted averages appear in a fairly large number of recent papers, as Cantarini [4], [5], [6], [7]; Languasco & Zaccagnini [12], [13], [14], [15], [16]; Brüdern, Kaczorowski & Perelli [3]; Goldston & Yang [10].

Here we deal with averages with a Cesàro weight, with the constraint that the summands in the Goldbach representations lie in fixed arithmetic progressions modulo a common integer $q$.

Let $q$ be a positive integer. For integers $a$ and $b$ coprime to $q$ we define

$$R_G(n; q, a, b) := \sum_{m_1, m_2 \equiv n \mod q} \Lambda(m_1)\Lambda(m_2),$$

where $\Lambda$ is the usual von Mangoldt-function. We could consider a more general definition with the summands lying in progressions with different $q_1$ and $q_2$, but then the conditions $m_1 \equiv a \mod q_1$ and $m_2 \equiv b \mod q_2$ would become $m_1 \mod q \in R_1$ and $m_2 \mod q \in R_2$, where $q$ is the least common multiple of $q_1$ and $q_2$ and $R_1$ and $R_2$ are suitable sets of residue classes mod $q$.

For $z = x + iy$ with $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$ we also set

$$\tilde{S}_{a, q}(z) := \sum_{m \equiv a \mod q} \Lambda(m) e^{-mqz}.$$

It is clear that $R_G$ is the generating function of $\tilde{S}_{a, q}(z)\tilde{S}_{b, q}(z)$, that is

$$\tilde{S}_{a, q}(z)\tilde{S}_{b, q}(z) = \sum_{m \geq 1} R_G(m; q, a, b) e^{-mqz}.$$

Our goal is to study averages of the quantities $R_G$, so we introduce a real parameter $k \geq 0$ and define

$$\Sigma_k(N; q, a, b) := \sum_{n \leq N} R_G(n; q, a, b) \frac{(N - n)^k}{k}$$

\begin{equation}
\frac{1}{k+1} \int_{(c)} e^{Nz} \tilde{S}_{a, q}(z)\tilde{S}_{b, q}(z) \frac{dz}{z^{k+1}} = \frac{1}{2\pi i} \int_\gamma e^{Nz} \tilde{S}_{a, q}(z)\tilde{S}_{b, q}(z) \frac{dz}{z^{k+1}},
\end{equation}

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where $\Gamma$ is the Euler Gamma-function. We will express $\Sigma_k$ as a sum of a main term, a secondary term and other smaller terms, depending explicitly on the zeros of the relevant Dirichlet $L$-functions. The inversion of infinite integral and series in (11) can be justified as in [13] for $k > 0$. See §§5-6.

This method was used in Cantarini [4], [5], [2]; Languasco & Zaccagnini [12], [13], [14], [15], [16] to deal with weights with $k > 1$, while Brüdern, Kaczorowski & Perelli [3] use a radically new approach in the spirit of the classical proof of the Prime Number Theorem.

For simplicity, we denote by $Z(\chi)$ the set of non-trivial zeros of the Dirichlet $L$-function $L(s, \chi)$. We can now define the terms in our decomposition:

$$M_k^{(1)}(N; q) = \frac{N^{k+2}}{\phi(q)^2 \Gamma(k + 3)}$$

$$M_k^{(2)}(N; q, a) = \frac{1}{\phi(q)^2} \sum_{\chi \mod q} \overline{\chi}(a) \sum_{\rho \in Z(\chi^*)} \frac{\Gamma(\rho)N^{k+1+\rho}}{\Gamma(k + 2 + \rho)},$$

$$M_k^{(3)}(N; q, a, b) = \frac{1}{\phi(q)^2} \sum_{\chi_1, \chi_2 \mod q} \overline{\chi_1(a)} \overline{\chi_2(b)} \sum_{\rho_1 \in Z(\chi_1^*), \rho_2 \in Z(\chi^*_2)} \frac{\Gamma(\rho_1)\Gamma(\rho_2)N^{k+\rho_1+\rho_2}}{\Gamma(k + 1 + \rho_1 + \rho_2)},$$

where here and throughout the paper $\chi^*$ denotes the primitive character that induces $\chi$. It is also convenient to write

$$M_k(N; q, a, b) = M_k^{(1)}(N; q) + M_k^{(2)}(N; q, a) + M_k^{(2)}(N; q, b) + M_k^{(3)}(N; q, a, b).$$

Throughout the paper, implicit constants may depend only on $k$. Our main result is the following theorem.

**Theorem 1.** For $k > 1$ and as $N \to +\infty$ we have

$$\Sigma_k(N; q, a, b) = M_k(N; q, a, b) + O((g(q) + \log N)N^{k+1}),$$

uniformly for $q \leq (\log N)^A$ and every $a, b$ with $(a, q) = (b, q) = 1$, for any fixed $A > 0$ and where

$$g(q) = \begin{cases} \log^2(q) & \text{if there is no exceptional zero modulo } q, \\ q^{1/2} \log^2(q) & \text{if there is an exceptional zero modulo } q. \end{cases} \quad (6)$$

We notice that we essentially detect the second main term in Bhowmik et al. [2] for $k = 0$, although we prove our Theorem 1 only for $k > 1$. The case $q = 1$ was treated in [13], and in fact we closely follow its proof. The weak uniformity bound in our Theorem and the value of $g$ are both due to the possible exceptional (or Siegel) zero of a Dirichlet $L$-function attached to a primitive real character modulo a “small” $q$. The turning point of the proof of Theorem 1 is Lemma 1 which we prove in §3. In fact, a weaker version would suffice in establishing the bounds which are necessary for the exchange of single and double series with the line integral, because in this case we may assume that $q$ is fixed.

We remark that (11) remains true also in the case $k = 0$, with a suitable interpretation. In fact, by partial summation we have

$$\tilde{S}_{a,q}(z)\tilde{S}_{b,q}(z) = \sum_{n \geq 1} R_G(m; q, a, b) e^{-mz} = z \int_0^{+\infty} f(t) e^{-zt} \, dt,$$

where

$$f(t) = \sum_{n \leq t} R_G(n; q, a, b) = \Sigma_0(t; q, a, b).$$
We remark that \( f \) is smooth in stretches and is bounded by an exponential. Hence, by standard inversion theorems (see, e.g., Theorem 8.4 of Folland [9]), we have

\[
\frac{1}{2}(f(t + 0) + f(t - 0)) = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{1/1-N-iT}^{1/1-N+iT} \tilde{S}_{a,q}(z)\tilde{S}_{b,q}(z)z^{-1}e^{iz} \, dz.
\]

2. OUTLINE OF THE PROOF

The four dominant terms in the statement, that is, in (5), arise multiplying formally the leading terms for \( \tilde{S}_{a,q} \) and \( \tilde{S}_{b,q} \) provided by Lemma [1] see (9). We have to show that we may exchange the summations with integration on the vertical line \( \Re(z) = x \), and also that the error term is small. We need the identity

\[
\frac{1}{2\pi i} \int_{(a)} u^{-s}e^{u} \, du = \frac{1}{\Gamma(s)}
\]

for \( a > 0 \) and its variant

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{itu}}{(a + iu)^s} \, du = \begin{cases} 
D^{s-1}e^{-aD}/\Gamma(s) & \text{if } D > 0, \\
0 & \text{if } D < 0,
\end{cases}
\]

in order to prove that (1) holds. Here \( \Re(a) > 0 \) and \( \Re(s) > 0 \). For the proof, see de Azevedo Pribitkin [11]. For brevity, we write

\[
\tilde{S}_{a,q}(z) = \frac{1}{\phi(q)} \left( \mathcal{M}(z; q, a) + \mathcal{E}(z; q, a) \right),
\]

where

\[
\mathcal{M}(z; q, a) = \frac{1}{z} - \sum_{\chi \bmod q} \chi(a) \sum_{\rho \in \mathbb{Z}\backslash \mathbb{R}} z^{-\rho} \Gamma(\rho).
\]

The bound for \( \mathcal{E}(z; q, a) \) is provided by Lemma [1]. We now substitute into (1) and find that

\[
\Sigma_k(N; q, a, b) = \frac{1}{2\pi i \phi^2(q)} \left( \int_{(s)} e^{Nz} \mathcal{M}(z; q, a)\mathcal{M}(z; q, b) \frac{dz}{z^{k+1}} \right)
\]

\[
+ \int_{(s)} e^{Nz} (\mathcal{M}(z; q, a)\mathcal{E}(z; q, b) + \mathcal{E}(z; q, a)\mathcal{M}(z; q, b)) \frac{dz}{z^{k+1}}
\]

\[
+ \int_{(s)} e^{Nz} \mathcal{E}(z; q, a)\mathcal{E}(z; q, b) \frac{dz}{z^{k+1}}.
\]

Expanding the first term in (10), exchanging summation with integration and using identity (7), we recover the terms in (2), (3) and (4), that is, the main term \( M_k(N; q, a, b) \) defined in (5). The proofs that the exchanges are legitimate are in §§5-6 while the proofs that single and double sums over zeros in (3) and (4) respectively converge follow closely the argument in [13] and [15], with minor modifications in the choices of the regions. We do not include details.

Now we deal with the error terms. We remark that

\[
|z|^{-1} \ll \begin{cases} 
x^{-1} & \text{if } |y| \leq x, \\
|y|^{-1} & \text{if } |y| > x.
\end{cases}
\]

We will eventually choose \( x = N^{-1} \). Hence, by Lemma [1] or simply by the Brun-Titchmarsh inequality we have

\[
\tilde{S}_{a,q}(z) \ll \frac{1}{\phi(q)x},
\]

provided that \( q \leq (\log N)^A \) for any fixed \( A > 0 \). Hence

\[
|M(z; q, a)| \leq \phi(q)\tilde{S}(x; q, a) + |\mathcal{E}(z; q, a)| \ll x^{-1} + |\mathcal{E}(z; q, a)|.
\]
Therefore we have
\[
\left| \int_{(x)} e^{Nz} M(z; q, a) \mathcal{E}(z; q, b) \frac{dz}{z^{k+1}} \right| \ll e^{Nz} \int_{(x)} |M(z; q, a)\mathcal{E}(z; q, b)| \frac{dy}{|z|^{k+1}} \ll e^{Nz} \int_{(x)} (x^{-1} + |\mathcal{E}(z; q, a)|) |\mathcal{E}(z; q, b)| \frac{dy}{|z|^{k+1}}.
\]

For brevity we set
\[
L(x; q, a) := \int_{(x)} |\mathcal{E}(z; q, a)|^2 \frac{dy}{|z|^{k+1}}.
\] (14)

By the Cauchy-Schwarz inequality, the total contribution of the terms in (11) and (12) is
\[
\ll e^{Nz} x^{-1} \int_{(x)} (|\mathcal{E}(z; q, a)| + |\mathcal{E}(z; q, b)|) \frac{dy}{|z|^{k+1}} + e^{Nz} \int_{(x)} |\mathcal{E}(z; q, a)| \cdot |\mathcal{E}(z; q, b)| \frac{dy}{|z|^{k+1}}
\ll e^{Nz} x^{-1} \sum_{h \in (a, b)} \left( \int_{(x)} \frac{dy}{|z|^{k+1}} L(x; q, h) \right)^{1/2} + e^{Nz} \left( L(x; q, a) L(x; q, b) \right)^{1/2}
\ll e^{Nz} \max_{(h, q) = 1} \left( x^{-1-k/2} L(x; q, h)^{1/2} + L(x; q, h) \right),
\]

by (13) again. We choose \( x = 1/N \) and use Lemma 2 to bound (14). We will complete the proof of Theorem 1 in §4.

3. Lemmas

For a Dirichlet character \( \chi \) mod \( q \) let
\[
\tilde{S}(z; \chi) := \sum_{m \geq 1} \chi(m) \Lambda(m) e^{-mz},
\]
so that, by orthogonality,
\[
\tilde{S}_{a,q}(z) = \frac{1}{\varphi(q)} \sum_{\chi \mod q} \overline{\chi}(a) \tilde{S}(z; \chi). \] (15)

We now express \( \tilde{S}(\chi^*) \) by means of \( \tilde{S}(\chi^*) \), where \( \chi^* \) mod \( q^* \) is the primitive character that induces \( \chi \). We have
\[
\left| \tilde{S}(z; \chi) - \tilde{S}(z; \chi^*) \right| \leq \sum_{m \geq 1} \Lambda(m) e^{-mx} \sum_{p|q} \log(p) \sum_{r \geq 1} e^{-p^rx} \leq e^{-x} \log(q). \] (16)

We recall some properties of the Dirichlet \( L \)-functions that we need in the proof of Lemma 1.

First, let \( \chi \) be a primitive odd character modulo \( q > 1 \) and let
\[
b(\chi) = \frac{L'}{L}(0, \chi) = -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \Gamma' \left( \frac{1}{2} \right) + B(\chi),
\]
where \( b(\chi) \) is defined in §19 of Davenport [8] and \( B(\chi) \) appears in the Weierstrass product for \( L(s, \chi) \). If \( \chi \) is even, we let
\[
b(\chi) = \lim_{s \to 0} \left( \frac{L'(s, \chi)}{L(s, \chi)} - \frac{1}{s} \right) = -\frac{1}{2} \log \frac{q}{\pi} + B(\chi) - \lim_{s \to 0} \left( \frac{1}{2} \Gamma' \left( \frac{s}{2} \right) + \frac{1}{s} \right).
\]

By the argument on pages 118–119 of Davenport [8] we have
\[
b(\chi) = O \left( \log(q) \right) - \sum_{|y| < 1} \frac{1}{\rho}.
\]
The Riemann-von Mangoldt formula for the number of zeros implies that the number of summands on the right is \( \ll \log(q) \). Each of the summands is \( \ll \log(q) \), unless \( \chi \) is a real character such that the relative \( L \)-function has an exceptional zero \( \beta \in [1 - c/\log(q), 1] \). Hence

\[
b(\chi) = O \left( \log^2(q) \right) - \frac{1}{\beta} - \frac{1}{1 - \beta} = O \left( \log^2(q) \right) - \frac{1}{1 - \beta}.
\]

The terms containing the exceptional zero are to be omitted if it does not exist. We now recall the upper bound \((1 - \beta)^{-1} \ll q^{1/2} \log^2(q)\) which is (12) on page 96 of Davenport. Hence we have \( b(\chi) = O(g(q)) \). Furthermore, for \( \sigma = \Re(w) \in [-1, 2] \), by (4) in §16 of [8] we have

\[
\frac{L'}{L}(w, \chi) = \sum_{\rho \neq 1} \frac{1}{w - \rho} + O \left( \log(q(|t| + 2)) \right).
\]

For \( w \) on the line \( \Re(s) = -\frac{1}{2} \) the summands are uniformly bounded, and their number is \( \ll \log(q(|t| + 2)) \) by the Riemann-von Mangoldt formula for the \( L \)-functions. Hence

\[
\frac{L'}{L} \left( -\frac{1}{2} + it, \chi \right) \ll \log(q(|t| + 2)).
\]

**Lemma 1.** Let \( M(z; q, a) \) and \( E(z; q, a) \) be defined by (8) and (9). Then

\[
E(z; q, a) \ll g(q) + 1 + \log(|z|) + |z|^{1/2} \log(q) + 1 \cdot \begin{cases}
1 & \text{if } |y| \leq x,
1 + \log^2(|y|/x) & \text{if } |y| > x.
\end{cases}
\]

The implicit constant is absolute.

**Proof.** Assume for the time being that \( \chi \) is a primitive character, and let \( \delta(\chi) = 1 \) if \( \chi \) is principal and \( \delta(\chi) = 0 \) otherwise. Following the proof in §4 of Linnik [17], or Lemma 4.1 of [13], we have

\[
\tilde{S}(z; \chi) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} z^{-w} \Gamma(w) \frac{L'}{L}(w, \chi) \, dw = \frac{\delta(\chi)}{z} - \sum_{\rho \in \mathbb{Z}(\chi)} z^{-\rho} \Gamma(\rho) + Q(z; \chi) + R(z; \chi),
\]

say, where

\[
R(z; \chi) := -\frac{1}{2\pi i} \int_{-1/2-\infty}^{-1/2+\infty} z^{-w} \Gamma(w) \frac{L'}{L}(w, \chi) \, dw
\]

and

\[
Q(z; \chi) := \begin{cases}
-(L'/L)(0, \chi) & \text{if } \chi \text{ is an odd character}
\end{cases}
\]

\[
-\gamma + b(\chi) + \log(z) & \text{if } \chi \text{ is an even character},
\]

(19)

taking into account the double pole of the integrand at \( s = 0 \). Hence, recalling (15) and (16), we have

\[
\tilde{S}_{aq}(z) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a) \tilde{S}(z; \chi^*) + O \left( \frac{1}{\phi(q)} \sum_{\chi \mod q} \left| \tilde{S}(z; \chi) - \tilde{S}(z; \chi^*) \right| \right)
\]

\[
= \frac{1}{\phi(q)z} - \frac{1}{\phi(q)} \sum_{\chi \mod q} \chi(a) \sum_{\rho \in \mathbb{Z}(\chi^*)} z^{-\rho} \Gamma(\rho)
\]

\[
+ O \left( \frac{1}{\phi(q)} \sum_{\chi \mod q} \left( |R(z; \chi^*)| + |Q(z; \chi^*)| \right) + e^{-x} \log(q) \right)
\]

5
In order to treat \( R(z; \chi) \) we need the bound \( \Gamma(w) \ll |t|^{-1} e^{-\pi|t|^2} \), (see Titchmarsh [20] §4.42) which is valid for \( |t| \to +\infty \). We split the line \( \sigma = -1/2 \) into the set \( L_c = \{ w = -1/2 + it : |t| > c \} \), where \( c > 0 \) is a suitable large absolute constant, and its complement. Arguing as in the proof of Lemma 4.1 of [13], and examining various cases we find

\[
\int_{L_c} \tau^w \Gamma(w) \frac{L'}{L}(w, \chi) \, dw \ll |z|^{1/2} \begin{cases} 
\log(q) + 1 & \text{if } |y| \leq x, \\
\log^2(|y|/x) + 1 + \log(q)(\log(|y|/x) + 1) & \text{if } |y| > x.
\end{cases}
\]

Finally, the integration over the complement of \( L_c \) yields a contribution \( O(1 + \log(q)) \).

We now turn to the estimation of \( |Q(z; \chi)| \). If \( \chi \) is an odd character, then we have

\[
\frac{L'}{L}(0, \chi) \ll \log(q),
\]

by formula 10.35 of §10 of Montgomery & Vaughan [18]. If \( \chi \) is even, we argue as above in (6), and by (19) we have

\[
Q(z; \chi) \ll g(q) + |\log(z)|.
\]

This implies that \( \mathcal{E} \) satisfies the bound in (18). \( \square \)

**Lemma 2.** For \( k > 1 \) we have

\[
\mathcal{L}(N^{-1}; q, a) := \int_{(1/N)} |E(z; q, a)|^2 \frac{dy}{|z|^{k+1}} \ll_k N^k (g(q) + \log(N))^2.
\]

**Proof.** We have

\[
\mathcal{L}(N^{-1}; q, a) \ll N^{k+1} \int_{-1/N}^{1/N} |E(z; q, a)|^2 \, dy + \int_{1/N}^{+\infty} \frac{|E(z; q, a)|^2 \, dy}{y^{k+1}}
\]

say. By Lemma 1 we easily get

\[
I_1 \ll \int_{-1/N}^{1/N} \left( g(q) + \log(N) + N^{-1/2} (\log(q) + 1) \right)^2 \, dy \leq N^{-1} (g(q) + \log(N))^2.
\]

Now we analyze \( I_2 \). Using again Lemma 1 we obtain

\[
I_2 \ll \int_{1/N}^{+\infty} \frac{(g(q) + 1 + |\log(y)| + y^{1/2} (\log(q) + 1) (1 + \log^2(Ny)))^2 \, dy}{y^{k+1}}
\]

\[
\ll \int_{1/N}^{+\infty} \frac{(g(q) + 1 + |\log(y)|)^2 \, dy}{y^{k+1}} + (\log(q) + 1)^2 \int_{1/N}^{+\infty} \frac{(1 + \log^2(Ny))^2 \, dy}{y^{k}}.
\]

The first summand can be easily estimated:

\[
\int_{1/N}^{+\infty} \frac{(g(q) + 1 + |\log(y)|)^2 \, dy}{y^{k+1}} \ll_k N^k (g(q) + 1)^2 + \int_{1/N}^{+\infty} \frac{\log^2(y) \, dy}{y^{k+1}}
\]

\[
\ll_k N^k \left( (g(q) + 1)^2 + \log^2(N) \right).
\]
We use the change of variables $v = Ny$ for the integral of the last summand and we get

$$
\int_{1/N}^{\infty} \left(1 + \log^2(Ny)\right)^2 \frac{dy}{y^k} \ll_k \left(N^{k-1} + \int_{1/N}^{\infty} \log^4(Ny) \frac{dy}{y^k}\right)
$$

which follows from the bound for $q$ since $k > 1$, and Lemma 2.

4. Completion of the proof of Theorem 1

In order to complete the proof it is enough to see that by Lemma 2 we have

$$
\max_{(h,q) = 1} \left(N^{1 + k/2} L(N^{-1}; q, h) \frac{1}{N} + L(N^{-1}; q, h)\right) \ll N^{k+1}(g(q) + \log(N)) + N^k(g(q) + \log(N))^2
$$

where the last estimation follows from the bound for $q$.

5. Interchange of the series over zeros with the line integral

It remains to prove that all the interchanges of the series and the integrals are legitimate.

We now state without proof two technical lemmas that can be proved by means of small variations on the arguments in Lemmas 2 and 3 in [13]. In both cases, we assume that $\chi \mod q$ is a primitive character and we let $\rho_\chi = \beta_\chi + i\gamma_\chi$ run over the non-trivial zeros of the associated Dirichlet $L$-function. Furthermore, we let $\alpha > 1$ be a parameter.

**Lemma 3.** We have

$$
\sum_{\gamma_\chi > 0} \gamma_\chi^{\beta_\chi - 1/2} \int_1^{\infty} \log^c(u) \exp(-\gamma_\chi \arctan(\frac{1}{u})) \frac{du}{u^{\alpha + \beta_\chi}} \ll_{\alpha, c} \sum_{\gamma_\chi > 0} \gamma_\chi^{1/2 - \alpha}
$$

for all $c \geq 0$. The series on the left converges for $\alpha > 3/2$ and diverges otherwise.

**Lemma 4.** Let $z = x + iy$ with $x \in (0, 1)$ and $y \in \mathbb{R}$. We have

$$
\sum_{\rho_\chi} |\gamma_\chi|^{\beta_\chi - 1/2} \int_{\mathbb{Y}} \log^c\left(\frac{|y|}{x}\right) \exp(\gamma_\chi \arctan(\frac{y}{x}) - \frac{\pi}{2}|\gamma_\chi|) \frac{dy}{|y|^{\alpha + \beta_\chi}}
$$

$$
\ll_{\alpha, c} x^{-\alpha} \sum_{\gamma_\chi > 0} \gamma_\chi^{\beta_\chi - 1/2} \exp\left(-\frac{\pi}{4} \gamma_\chi\right),
$$

for all $c \geq 0$, where $\mathbb{Y} = \mathbb{Y}_1 \cup \mathbb{Y}_2$, $\mathbb{Y}_1 = \{y \in \mathbb{R}: \gamma y \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-x, x]: \gamma y > 0\}$.

Let $z = x + iy$ and $w = u + iv$, where $u \geq 0$. We recall that

$$
|z^n| = |z|^u \exp\left(\nu \arctan\left(\frac{y}{x}\right)\right)
$$

and

$$
|\Gamma(w)| \leq (2\pi)^{1/2} |w|^{u-1/2} e^{-\pi|v|/2} \exp\left(\frac{1}{6|w|}\right).
$$
Using (20) and (21) we find
\[ \sum_{\chi \bmod q} \sum_{\rho \in \mathbb{Z}(\chi^*)} |\Gamma(\rho)| \int_{(1/N)} e^{Nz} \cdot |z^{-k-2} - \rho| \cdot |dz|. \]

Using (20) and (21) we find
\[ |\Gamma(\rho)| \int_{(1/N)} |z^{-k-2} - \rho| \cdot |dz| \ll |\gamma|^{\beta - 1/2} \exp \left( \frac{1}{6|\gamma|} \right) \int_{\mathbb{R}} \exp \left( \gamma \arctan(Ny) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^{k+2+\beta}}. \]

We sum over \( \rho \in \mathbb{Z}(\chi^*) \) and we can assume by symmetry that \( \gamma > 0 \). Recalling Lemma 4, we have
\[ \sum_{\gamma > 0} |\Gamma(\rho)| \int_{1/N}^{+\infty} e^{Nz} \cdot |z^{-k-2} - \rho| \cdot |dz| \ll_k N^{k+2} \sum_{\gamma > 0} \gamma^{\beta - 1/2} \exp \left( -\frac{\pi}{4} |\gamma| \right). \]

We still have to deal with the case \( \gamma > 0 \) and \( y > 1/N \). Using the identity \( \arctan(y) + \arctan(1/y) = \pi/2 \), we have
\[ \sum_{\gamma > 0} |\Gamma(\rho)| \int_{1/N}^{+\infty} e^{Nz} \cdot |z^{-k-2} - \rho| \cdot |dz| \ll_k N^{k+2} \sum_{\gamma > 0} \gamma^{\beta - 1/2} \int_{1}^{+\infty} \exp \left( -\gamma \arctan(1/|u|) \right) \frac{du}{u^{k+2+\beta}}. \]

Using Lemma 3, we see that this is
\[ \ll_k N^{k+2} \sum_{\gamma > 0} \gamma^{-k-3/2}, \]
which converges for \( k > -1/2 \), by standard zero-density estimates.

### 6. Interchange of the Double Series Over Zeros with the Line Integral

We start examining
\[ \sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)| \int_{(1/N)} e^{Nz} \cdot |z^{-k-1} - \rho_1| \cdot |\sum_{\chi_2 \bmod q} \chi_2(b) \sum_{\rho_2 \in \mathbb{Z}(\chi_2^*)} \Gamma(\rho_2) z^{-\rho_2}| \cdot |dz|. \]

Lemma 1 implies that
\[ \left| \sum_{\chi_2 \bmod q} \chi_2(b) \sum_{\rho_2 \in \mathbb{Z}(\chi_2^*)} \Gamma(\rho_2) z^{-\rho_2} \right| \leq \phi(q) |\bar{s}_{a,q}(z)| + \frac{1}{|z|} + \sum_{\chi_2 \bmod q} (|R(z; \chi_2^*)| + |Q(z; \chi_2^*)|) \ll q \frac{1}{|z|} + N + \log(|z|) + |z|^{1/2}, \]

hence we have to study
\[ \sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)|(A_1 + A_2 + A_3) \]
where
\[
A_1 = \int_{-1/N}^{1/N} N^{k+1+\beta_1} \exp(\gamma_1 \arctan(Ny)) \left( N + \log(N) + N^{-1/2} \right) dy,
\]
\[
A_2 = \int_{1/N}^{+\infty} y^{-k-1-\beta_1} \exp(\gamma_1 \arctan(Ny)) \left( \frac{1}{y} + N + |\log(y)| + y^{1/2}(1 + \log^2(Ny)) \right) dy,
\]
\[
A_3 = \int_{-\infty}^{-1/N} |y|^{-k-1-\beta_1} \exp(\gamma_1 \arctan(Ny)) \left( \frac{1}{|y|} + N + |\log(|y|)| + |y|^{1/2}(1 + \log^2(N|y|)) \right) dy.
\]

By symmetry, we can assume that \( \gamma_1 > 0 \). Clearly, by (21), we have
\[
\sum_{\chi_1 \mod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_1 \ll_q N^{k+2} \sum_{\chi_1 \mod q} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp \left( -\frac{\pi \gamma_1}{4} \right)
\]
and the series is obviously convergent, by standard density estimates.

Now, we consider \( A_2 \): we have, again by (21), that
\[
\sum_{\chi_1 \mod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_2 \ll_q \sum_{\chi_1 \mod q} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \times \int_{1/N}^{+\infty} y^{-k-1-\beta_1} \exp \left( \gamma_1 \left( \arctan(Ny) - \frac{\pi}{2} \right) \right) \left( \frac{1}{y} + N + |\log(y)| + y^{1/2}(1 + \log^2(Ny)) \right) dy.
\]

Then, taking \( v = Ny \), we obtain
\[
\sum_{\chi_1 \mod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_2 \ll_q N^k \sum_{\chi_1 \mod q} \sum_{\gamma_1 > 0} N^{\beta_1} \gamma_1^{\beta_1-1/2} \times \int_{1}^{\infty} v^{-k-1-\beta_1} \exp \left( -\gamma_1 \arctan \left( \frac{1}{v} \right) \right) \left( \frac{N}{v} + N + \log(vN) + v^{1/2}N^{-1/2} \left( 1 + \log^2(v) \right) \right) dv.
\]

and now, by Lemma[3] it is easy to see that the integral and the series are convergent for \( k > 1 \). If \( y \in (-\infty, -1/N) \) the result is trivial since \( \exp \left( \gamma_1 \left( \arctan(Ny) - \frac{\pi}{2} \right) \right) \leq \exp \left( -\gamma_1 \frac{\pi}{2} \right) \) and the condition \( k > 1 \) suffices to ensure the convergence of the integral.

Finally, if we consider
\[
\sum_{\chi_1 \mod q} \sum_{\chi_2 \mod q} \sum_{\rho_1 \in \mathbb{Z}(\chi_1^*)} |\Gamma(\rho_1)| \sum_{\rho_2 \in \mathbb{Z}(\chi_2^*)} |\Gamma(\rho_2)| \int_{1/N}^{1/N} |e^{Nz}| \left| z^{-k-1-\rho_1-\rho_2} \right| |dz|,
\]
we can simply argue as in [13], equation (21), getting the convergence for \( k > 1 \).

References
[1] W. de Azevedo Pribitkin, Laplace’s Integral, the Gamma Function, and Beyond, Amer. Math. Monthly 109 (2002), 235–245.
[2] G. Bhowmik, K. Halupczok, K. Matsumoto, and Y. Suzuki, Goldbach representations in arithmetic progressions and zeros of Dirichlet L-functions, Mathematika 65 (2019), no. 1, 57–97.
[3] J. Brüdern, J. Kaczorowski, and A. Perelli, Explicit formulae for averages of Goldbach representations, Trans. Amer. Math. Soc. 372 (2019), 6981–6999.
[4] M. Cantarini, On the Cesàro average of the “Linnik numbers”, Acta Arith. 180 (2017), no. 1, 45–62.
[5] M. Cantarini, On the Cesàro average of the numbers that can be written as a sum of a prime and two squares of primes, J. Number Theory 185 (2018), 194–217.
[6] M. Cantarini, Explicit formula for the average of Goldbach numbers, Indian Journal of Mathematics 61 (2019), 253–279.
[7] M. Cantarini, Some identities involving the Cesàro average of Goldbach numbers, Mathematical Notes (2019), To appear. Arxiv preprint 1711.08610.
[8] H. Davenport, *Multiplicative Number Theory*, third ed., Graduate Texts in Mathematics, vol. 74, Springer, 2000.
[9] G. B. Folland, *Fourier analysis and its applications*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.
[10] D. A. Goldston and L. Yang, *The average number of Goldbach representations*, Prime Numbers and Representation Theory, Lecture Series of Modern Number Theory, vol. 2, Science Press, Beijing, 2017, pp. 1–12.
[11] A. Languasco and A. Zaccagnini, *The number of Goldbach representations of an integer*, Proc. Amer. Math. Soc. 140 (2012), 795–804.
[12] , *A Cesàro average of Hardy-Littlewood numbers*, J. Math. Anal. Appl. 401 (2013), 568–577.
[13] , *A Cesàro average of Goldbach numbers*, Forum Mathematicum 27 (2015), no. 4, 1945–1960.
[14] , *Cesàro average in short intervals for Goldbach numbers*, Proc. Amer. Math. Soc. 145 (2017), no. 10, 4175–4186.
[15] , *A Cesàro average for an additive problem with prime powers*, Proceedings of the “Number Theory Week,” Poznań, September 4–8, 2017 (Warszawa) (Łukasz Pańkowski and Maciej Radziejewski, eds.), vol. 118, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences, 2019, pp. 137–152.
[16] , *A Cesàro average of generalised Hardy-Littlewood numbers*, Kodai Math. J. 42 (2019), no. 2, 358–375.
[17] Yu. V. Linnik, *A new proof of the Goldbach-Vinogradov theorem*, Rec. Math. [Mat. Sbornik] N.S. 19 (1946), 3–8.
[18] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge University Press, Cambridge, 2007.
[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, National Institute of Standards and Technology, Cambridge University Press, Cambridge, 2010.
[20] E. C. Titchmarsh, *The Theory of Functions*, second ed., Oxford University Press, Oxford, 1988.

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