Periodicity in the cumulative hierarchy

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Abstract

We investigate the structure of rank-into-rank elementary embeddings incompatible with the Axiom of Choice. Assuming ZF set theory and the existence of a (non-trivial) elementary embedding

\[ j : V_{\alpha+1} \rightarrow V_{\alpha+1} \]

of a rank initial segment of \( V \) into itself, we show that the structure of \( V_\alpha \) is fundamentally different to that of \( V_{\alpha+1} \). (Here \( \alpha \) may be either a limit or a successor ordinal.) We show that \( j \) is definable from parameters over \( V_{\alpha+1} \) if and only if \( \alpha + 1 \) is an odd ordinal. Moreover, if \( \alpha + 1 \) is odd then \( j \) is definable over \( V_{\alpha+1} \) from its restriction \( j \upharpoonright V_\alpha \), and uniformly so. This parameter is optimal in that \( j \) is not definable from any parameter in \( V_\alpha \).

Further, we also show that \( \Sigma_1 \)-elementary embeddings \( j : V_\lambda \rightarrow V_\lambda \) are non-definable for all limit \( \lambda \).

It is moreover known that if there is a Reinhardt cardinal, then for all sufficiently large ordinals \( \alpha \), there is an elementary \( j : V_\alpha \rightarrow V_\alpha \), and therefore the cumulative hierarchy is eventually periodic (with period 2).

1 Introduction

The universe \( V \) of all sets is the union of the cumulative hierarchy \( \langle V_\alpha \rangle_{\alpha \in \text{OR}} \). Here \( \text{OR} \) denotes the class of all ordinals, and \( V_\alpha \) is the set obtained by iterating the power set operation \( X \mapsto \mathcal{P}(X) \) transfinitely, starting with \( V_0 = \emptyset \), setting \( V_{\alpha+1} = \mathcal{P}(V_\alpha) \), and \( V_\eta = \bigcup_{\alpha < \eta} V_\alpha \) for limit ordinals \( \eta \).

Before Cantor’s discovery of the transfinite ordinals, mathematicians typically only considered sets lying quite low in the infinite levels of the cumulative hierarchy (below \( V_{\omega+5} \)). This paper, however, investigates the farthest reaches of the universe of sets. At this level, the cumulative hierarchy appears to become quite uniform: for large infinite limit ordinals \( \eta \) and large numbers \( n \) and \( m \), it is hard to differentiate \( V_{\eta+n} \) and \( V_{\eta+m} \) in terms of natural set theoretic properties. One might therefore expect that \( V_{\eta+813} \), for example, is somehow structurally indistinguishable from \( V_{\eta+814} \). But the key result of this paper shows that assuming \( \eta \) is very large — so large, in fact, as to violate the Axiom of Choice — \( V_{\eta+813} \) and \( V_{\eta+814} \) display fundamental structural differences. More generally, the properties of \( V_{\eta+n} \) depend the parity of \( n \).

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Exactly how large must $\eta$ be for these differences to arise? To answer this question requires introducing some basic concepts from the theory of large cardinals, one of the main areas of research in modern set theory. The simplest example of a large cardinal is the inaccessible cardinal. An uncountable ordinal $\kappa$ is inaccessible if any function from $V_\alpha$ to $\kappa$ with $\alpha < \kappa$ is bounded strictly below $\kappa$.\(^1\) So inaccessible cardinals are “unreachable from below”, and form a natural kind of closure point of the set theoretic universe. By Gödel’s Incompleteness Theorem, inaccessible cardinals cannot be proven to exist in ZF, because if $\kappa$ is inaccessible then $V_\kappa$ models all of the ZF axioms, as does $V_\alpha$ for unboundedly many ordinals $\alpha < \kappa$. (The Zermelo-Fränkel axioms, denoted ZF, are the usual axioms of set theory, without the Axiom of Choice AC. And ZFC denotes ZF augmented with AC.)

But inaccessibles are just the beginning. Further up in the hierarchy, large cardinals are typically exhibited by some form of elementary embedding

$$j : V \rightarrow M$$

from the universe $V$ of all sets to some transitive\(^2\) class $M \subseteq V$. Elementary means that $j$ preserves the truth of all first-order statements in parameters between $V$ and $M$ (see §1.1 for details). One can show that if $j$ is not the identity, then there is an ordinal $\kappa$ such that $j(\kappa) > \kappa$, and the least such ordinal is called the critical point $\text{crit}(j)$ of $j$; if $\text{ZFC}^3$ holds then such a critical point is called a measurable cardinal. The critical point of an elementary embedding is inaccessible, and in fact there are unboundedly many inaccessible cardinals $\eta < \kappa$. So such critical points transcend inaccessible cardinals. Critical points are transcended by still larger large cardinals.

Large cardinal axioms are by far the most widely accepted and well-studied principles extending the standard axioms of set theory.\(^4\) One of the main reasons for this is the empirical fact that large cardinal axioms are arranged in an essentially linear hierarchy of strength, with each large cardinal notion transcending all the preceding ones.\(^5\) There is no known example of a pair of incompatible large cardinal axioms, and the linearity phenomenon suggests that none will ever arise.

The strength of a large cardinal notion $j : V \rightarrow M$ depends in large part on the extent to which $M$ resembles $V$ and contains fragments of $j$. So taking the notion to its logical extreme, William Reinhardt suggested in his dissertation taking $M = V$; that is, a (non-identity) elementary embedding

$$j : V \rightarrow V.$$  

The critical point of such an embedding became known as a Reinhardt cardinal. But Kunen proved in [10] (also see [6]) that, assuming $\text{ZFC}$, that Reinhardt

\(^1\)Assuming the Axiom of Choice AC, inaccessibility is usually defined slightly differently, but under AC, the definitions are equivalent. The definition we give here is the appropriate one when one does not assume AC.

\(^2\)That is, for all $x \in M$, we have $x \subseteq M$.

\(^3\)Under $\text{ZFC}$, this notion is equivalent to measurability, but the notions are not equivalent in general under $\text{ZF}$ alone.

\(^4\)An example of a large cardinal axiom is the assertion that there is an inaccessible cardinal or the assertion that there is a critical cardinal. While there is no formal definition of the term "large cardinal axiom," there is little controversy over which principles qualify as large cardinal axioms.

\(^5\)This is a bit of an oversimplification.
cardinals do not exist. Actually, suppose $j : V \to M$ is elementary where $M \subseteq V$ is a transitive class and $j$ is not the identity. Letting $\kappa_0 = \text{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$, then because $j$ is order-preserving (an easy consequence of elementarity),

$$\kappa_0 < \kappa_1 < \ldots < \kappa_n < \ldots.$$ 

Let their supremum be $\lambda = \sup_{n<\omega} \kappa_n$. We write $\kappa_n(j) = \kappa_n$ and $\kappa_\omega(j) = \lambda$. Kunen proved in [10] (assuming ZFC) that $V_{\lambda+1} \not\subseteq M$. He also proved that there is no ordinal $\lambda'$ and elementary embedding $j : V_{\lambda'+2} \to V_{\lambda'+2}$.

So ZFC enforces a rather abrupt upper limit to the large cardinal hierarchy.

Following Kunen’s result, set theorists understandably focused their attention on large cardinals below this level. But it has remained a mystery whether AC is actually needed to prove there can be no elementary $j : V \to V$. Suzuki [16] showed in ZF alone that such a $j$ cannot be definable from parameters over $V$. This leads to a metamathematical question: what exactly is a class? In the most restrictive formulation, classes are all definable from parameters, so in this setting, Suzuki’s result rules out an elementary $j : V \to V$ from ZF alone, and the matter is settled – though not the $j : V_{\lambda+2} \to V_{\lambda+2}$ matter, which is completely immune to Suzuki’s argument. But one can also formulate classes more generally, and appropriately formulated, there is no known way to disprove the existence of $j : V \to V$ without AC. For the most part in this paper, we focus anyway on embeddings of set size, so the precise definition of classes is not so important for us here.\footnote{In §6 we will deal with actual Reinhardt cardinals, and will mention an appropriate formulation of classes there.} However, it is well known that Kunen’s theorem is also really about sets, hence is independent of the formulation of classes used, so ZFC rules out $j : V \to V$ in general.

In the last few years, there has been growing interest in investigating large cardinal notions like $j : V \to V$ and, in fact, beyond, assuming ZF (often augmented with fragments of AC). This paper sits within that line of investigation, and in fact just beyond the level which violates choice, with (set-sized) elementary, or partially elementary, embeddings of the form

$$j : V_\alpha \to V_\alpha.$$ 

These are known as rank-into-rank embeddings, because $V_\alpha$ is a rank initial segment of $V$. If there is a Reinhardt cardinal then there are many rank-into-rank embeddings; see Theorem 6.1.

We primarily consider the following question, with ZF as background theory. Let $\alpha$ be an ordinal and $j : V_\alpha \to V_\alpha$ be elementary. Is $j$ definable from parameters over $V_\alpha$? That is, we investigate whether there is $p \in V_\alpha$ and some formula $\varphi$ in the language of set theory (with binary predicate symbol $\in$ for membership) such that for all $x,y \in V_\alpha$, we have

$$j(x) = y \iff V_\alpha \models \varphi(p,x,y),$$ 

where $\models$ is the usual model theoretic truth satisfaction relation.

It turns out that there is a very simple answer to this question, generalizing Suzuki’s theorem, but with a twist. We say that an ordinal $\alpha$ is even iff $\alpha =$
\[ \eta + 2n \text{ for some } n < \omega, \text{ with } \eta = 0 \text{ or } \eta \text{ a limit ordinal. Naturally, odd means not even.} \]

**Theorem 1.1.** Let \( j : V_\alpha \rightarrow V_\alpha \) be fully elementary, \( j \neq \text{id} \). Then \( j \) is definable from parameters over \( V_\alpha \) iff \( \alpha \) is odd.

So if there is an elementary \( j : V_{\eta+184} \rightarrow V_{\eta+184} \) (and hence an elementary embedding from \( V_{\eta+183} \) to \( V_{\eta+183} \), namely \( j \upharpoonright V_{\eta+183} \)), then \( V_{\eta+183} \) and \( V_{\eta+184} \) are indeed different (but \( V_{\eta+182} \) analogous to \( V_{\eta+184} \), etc). The proof will also yield much more information about such \( j \)'s, and in the successor case, give a characterization of them, and reveal strong structural differences between the odd and even levels which admit such embeddings. A consequence of Theorem 6.1 will also be that if there is a Reinhardt cardinal, and \( j : V \rightarrow V \), then all ordinals \( \eta \geq \kappa_\omega(j) \) are indeed large enough for this periodicity phenomenon to take hold.

Periodicity phenomena (with period 2) are of course a familiar feature of logical quantifiers: \( \forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \). They are pervasive in descriptive set theory (in particular in the Periodicity Theorems, see [12]). But in these cases, when analyzing complexity classes and so forth arising from quantifier alternation, the periodicity is built into the definitions in the first place. This is in contrast with the periodicity present in Theorem 1.1, where the definitions of the cumulative hierarchy and elementary embeddings do not seem to have any obvious periodicity built into them. In the cases of both the Periodicity Theorems just mentioned and Theorem 1.1, there are stark differences between the even and odd sides. Also, the periodicity in the \( V_\alpha \)'s seems to manifest certain “\( \forall/\exists \)” features, even if these are not explicit in its definition.

For more on recent developments on large cardinals beyond/without AC, the reader should refer to [3], [5], [4], [1], [2], [15], [14], [13], [7], [17], [8].

We record some history on the development of the work. The results on the limit case in §3.1 and §5 are due to the second author, and were first distributed in the notes [14, v1], which also contain various other related work. The analysis of embeddings \( j : V_\lambda+n \rightarrow V_\lambda+n \) for limit \( \lambda \) and \( n = 2 \) in terms of Reinhardt filters, in §4, was discovered in some form by the first author in 2017, and he communicated this to the second author shortly after the release of [14]. The first author then discovered Theorem 1.1, and used this to generalize Woodin’s \( I_0 \)-theory to higher levels; for this work see [7]. A few months later, also attempting to generalize the first author’s analysis of embeddings to \( n > 2 \), the second author rediscovered Theorem 1.1. Our two proofs of non-definability in the even successor case (Theorem 3.19) were somewhat different; the one we give here is that due to the second author, and the original one, due to the first author, can be seen in [7]. Motivated by these ideas, the second author has found results on the consistency of \( ZF \) with embeddings \( j : V_{\lambda+2} \rightarrow V_{\lambda+2} \); for this see [13].

### 1.1 Terminology, notation, basic facts

We will assume the reader is familiar with basic first-order logic and set theory. But much of the material, particularly in the earlier parts of the paper, does not require extensive background in set theory, so we aim to make the paper reasonably broadly accessible. Therefore we do explain some points in the paper.
which are standard set theory, and summarize in this section some basic facts for convenience; the reader should refer to texts like [11] for more details.

The language of set theory is the first-order language with the membership relation $\in$. The Zermelo-Fr"ankel axioms are denoted by $\text{ZF}$, and $\text{ZFC}$ denotes $\text{ZF} + \text{AC}$, where $\text{AC}$ is the Axiom of Choice. We sometimes discuss $\text{ZF}(\bar{A})$, where $\bar{A}$ is an extra predicate symbol; this is just like $\text{ZF}$, but in the expanded language with both $\in$ and $\bar{A}$, and incorporates the Collection and Separation schemata for all formulas in the expanded language. A model of $\text{ZF}(\bar{A})$ has the form $(V, \in, A)$, abbreviated $(V, A)$, where $V$ is the universe of sets and $A \subseteq V$. Thus, $A$ is automatically a class of this model (and in the interesting case, $A$ is not already definable from parameters over $V$).

We write $\Sigma_0 = \Pi_0 = \Delta_0$ for the class of formulas (in the language of set theory) in which all quantifiers are bounded, meaning of the form "$\forall x \in y$" or "$\exists x \in y$". Then $\Sigma_{n+1}$ formulas are those of the form "$\exists x_1, \ldots, x_n \psi(x_1, \ldots, x_n, y)$", where $\psi$ is $\Pi_n$, and $\Pi_{n+1}$ formulas are negations of $\Sigma_{n+1}$. A relation is $\Delta_{n+1}$ if expressed by both $\Sigma_{n+1}$ and $\Pi_{n+1}$ formulas.

Given structures $M = ([M], R_1, R_2, \ldots, R_n)$ and $N = ([N], S_1, S_2, \ldots, S_n)$ for the same first order language $\mathcal{L}$, with universes $[M]$ and $[N]$ respectively, a map $\pi : M \to N$ (literally, $\pi : [M] \to [N]$) is elementary, just in case

$$M \models \varphi(\bar{x}) \iff N \models \varphi(\pi(\bar{x}))$$

(1)

for all first order formulas $\varphi$ of $\mathcal{L}$ and all finite tuples $\bar{x} \in M^{<\omega}$. We can refine this notion by considering formulas of only a certain complexity: We say $\pi$ is $\Sigma_n$-elementary iff line (1) holds for all $\bar{x} \in M^{<\omega}$ and $\Sigma_n$ formulas $\varphi$.

An elementary substructure is of course the special case of this in which $\pi$ is just the inclusion map. We write $M \preceq N$ for a fully elementary substructure, and $M \preceq_\pi N$ for $\Sigma_n$-elementary.

Given $X \subseteq M$ and $p \in M$, $X$ is definable over $M$ from the parameter $p$ iff there is a formula $\varphi \in \mathcal{L}$ such that for all $x \in M$ (literally $x \in [M]$), we have

$$x \in X \iff M \models \varphi(x, p).$$

This can also be refined to $\Sigma_n$-definable from $p$, if we demand $\varphi$ be a $\Sigma_n$ formula, and likewise for $\Pi_n$. We say that $X$ is definable over $M$ without parameters if we can take $p = \emptyset$. We say $X$ is definable over $M$ from parameters if $X$ is definable over $M$ from some $p \in M$.

Recall that a set $M$ is

- transitive iff $\forall x \in M \forall y \in x [y \in M]$,
- extensional iff $\forall x, y \in M [x \neq y \implies \exists z \in M [z \in x \iff z \notin y]]$;

note these notions are $\Delta_0$. The Mostowski collapsing Theorem asserts that if $M$ is a set and $E$ a binary relation on $M$ which is wellfounded and $(M, E)$ satisfies $E$-extensionality (that is, $\forall x, y \in M [x \neq y \implies \exists z \in M [zEx \iff zEy]]$), then there is a unique transitive set $\bar{M}$, and unique map $\pi : \bar{M} \to M$, such that $\pi$ is an isomorphism

$$\pi : (\bar{M}, \in) \to (M, E);$$

here $\bar{M}$ is called the Mostowski or transitive collapse of $(M, E)$, and $\pi$ the Mostowski unicollapse map. The most important example of transitive sets in this paper are the segments $V_\alpha$ of the cumulative hierarchy.
A key fact for transitive sets is that of absoluteness with respect to $\Delta_0$ truth: Let $M$ be transitive. Then $\Delta_0$ formulas are absolute to $M$, meaning that if $\psi$ is $\Delta_0$ and $\vec{x} \in M^{<\omega}$, then

$$\psi(\vec{x}) \iff [M \models \psi(\vec{x})].$$

Here the blanket assertion “$\psi(\vec{x})$” on the left implicitly means “$V \models \psi(\vec{x})$” where $V$ is the ambient universe in which we are working. This equivalence is proven by an induction on the formula length. It follows that if $\psi$ is $\Delta_0$ then

$$[M \models \exists y \psi(y, \vec{x})] \implies [M \models \exists y \psi(y, \vec{x})],$$
(in fact any witness $y \in M$ also works in $V$), so conversely,

$$[\forall y \psi(y, \vec{x})] \implies [M \models \forall y \psi(y, \vec{x})].$$

We write $\text{OR}$ for the class of all ordinals. Ordinals $\alpha, \beta$ are represented as sets in the standard form: $0 = \emptyset$, $\alpha + 1 = \alpha \cup \{\alpha\}$, and we take unions at limit ordinals $\eta$. The standard ordering on the ordinals is then $\alpha < \beta \iff \alpha \in \beta$, and this ordering is wellfounded. Being an ordinal is a $\Delta_0$-definable property, because $x$ is an ordinal iff $x$ is transitive and (the elements of) $x$ are linearly ordered by $\in$. Therefore being an ordinal is absolute for transitive sets, and preserved by $\Sigma_0$-elementary embeddings between transitive sets. That is, if $M, N$ are transitive and $x \in M$ then

$$x \text{ is an ordinal } \iff M \models x \text{ is an ordinal},$$
and if $j : M \rightarrow N$ is also $\Sigma_0$ elementary then

$$M \models x \text{ is an ordinal } \iff N \models j(x) \text{ is an ordinal}.$$

So this will hold in particular for the elementary embeddings $j : V_\alpha \rightarrow V_\beta$ that we consider. Note that transitivity of sets is also a $\Delta_0$-definable property, so absolute. Note that if $M$ is a transitive set then $\text{OR} \cap M$ is also an ordinal, in fact the least ordinal not in $M$.

If $N$ is a model of $ZF$ (possibly non-transitive), we write

$$\text{OR}^N = \{ \alpha \in N : N \models \text{“}\alpha \in \text{OR}\text{”}\}.$$ 

Similarly, if $\alpha \in \text{OR}^N$ we write

$$V_\alpha^N = \text{the unique } v \in N \text{ such that } N \models \text{“}v = V_\alpha\text{”}.$$ 

We use analogous superscript-$N$ notation whenever we have a notion defined using some theory $T$ and $N \models T$. So superscript-$N$ means “as computed/defined in/over $N$”.

Given a set $x$, the rank of $x$, denoted $\text{rank}(x)$, denotes the least ordinal $\alpha$ such that $x \subseteq V_\alpha$. (The Axiom of Foundation ensures that this is well-defined.)

Given a function $f : X \rightarrow Y$, $\text{dom}(f)$ denotes the domain of $f$, $\text{rg}(f)$ the range, and given $A \subseteq X$, $f[A]$ or $f^*A$ denotes the pointwise image of $A$.

Let $j : V \rightarrow M$ be elementary, where $M \subseteq V$ and $j$ is non-identity. An argument by contradiction can be used to show that there is an ordinal $\kappa$ such that $j(\kappa) > \kappa$, and the least such is called the critical point of $j$, denoted $\text{crit}(j)$. 

6
The same holds more generally, for example if \( j : M \to M \) is elementary where \( M \) is a transitive set or class. It follows that, in particular, such a \( j \) cannot be surjective, so there is no non-trivial \( \in \)-isomorphism of transitive \( M \).

If \( j : M \to N \) is elementary between transitive \( M, N \), then \( M \cong \text{rg}(j) \preceq N \), and \( \text{rg}(j) \) is a wellfounded extensional set, and therefore the Mostowski collapsing theorem applies to it. The transitive collapse is just \( M \), and \( j \) is the uncollapse map. So from \( j \) we can compute \( \text{rg}(j) \) (and \( M = \text{dom}(j) \)), and from \( \text{rg}(j) \) we can recover \( M, j \).

2 Suzuki’s Fact: Non-definability of \( j : V \to V \)

Suzuki [16] proved the following basic fact. We will use variants of its proof later, and the proof is short, so we include it as a warm-up. Everything in this section is well known.

**Fact 2.1 (Suzuki).** Assume ZF.\(^7\) Then no class \( k \) which is definable from parameters is an elementary \( k : V \to V \).

Here when we say simply “definable from parameters”, we mean over \( V \). Of course, the theorem is really a theorem scheme, giving one statement for each possible formula \( \varphi \) being used to define \( k \) (from a parameter). We need a couple of lemmas. The first is a little easier to consider in the case that \( \alpha \) in the proof is a limit ordinal, but the proof goes through in general.

**Lemma 2.2.** Let \( j : V_\delta \to V_\lambda \) be \( \Sigma_1 \)-elementary. Then \( j(V_\alpha) = V_\beta \) for all \( \alpha < \delta \).

**Proof.** Fix \( \alpha < \delta \). Note that \( V_\delta \) satisfies the following statements about the parameters \( \alpha \) and \( V_\alpha \):\(^8\)

- “\( V_\alpha \) is transitive”
- “For every \( X \in V_\alpha \) and every \( Y \subseteq X \), we have \( Y \in V_\alpha \)”,
- “\( V_\alpha \) satisfies ‘For every ordinal \( \beta \), \( V_\beta \) exists’.” \(^9\)

The first statement here is \( \Sigma_0 \) (in parameter \( V_\alpha \)), the second is \( \Pi_1 \), and the third \( \Delta_1 \), so \( V_\lambda \) satisfies the same assertions of the parameter \( j(V_\alpha) \). It follows that \( j(V_\alpha) = V_\beta \) for some \( \beta < \lambda \). But also \( \alpha = V_\alpha \cap \text{OR} \), another fact preserved by \( j \) (again by \( \Sigma_1 \)-elementarity), so \( j(\alpha) = j(V_\alpha) \cap \text{OR} \), so \( \beta = j(\alpha) \). \( \square \)

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\(^7\)That is, we are assuming that the universe \( V \models \text{ZF} \). We often use this language and then make statements which are to be interpreted in/over \( V \).

\(^8\)When we write “\( V_\alpha \)” in the 3 statements, we refer to the object \( x = V_\alpha \) as a parameter, as opposed to the object defined as the \( \alpha \)th stage of the cumulative hierarchy. But note that the “\( \beta \)” and “\( V_\beta \)” are quantified variables, and here \( V_\beta \) does refer to the \( \beta \)th stage of the cumulative hierarchy.

\(^9\)The reader might notice that this needs to be formulated appropriately, because if \( \alpha = \beta + 1 \), then the standard definition of \( (V_\gamma)_{\gamma \leq \beta} \) is the function \( f : \beta + 1 \to V \) where \( f(\gamma) = V_\gamma \), and \( f \notin V_\alpha \). But it is straightforward to reformulate things appropriately. For the case in which \( j : V_\delta \to V_\delta \) and \( \delta \) is a limit, one can also get around these things in other ways, since we can just talk about elements of \( V_\delta \), instead of literally talking about something that \( V_\alpha \) satisfies.
The following fact is [9, Proposition 5.1] (though it is stated under the assumption that $M,N$ are transitive proper class inner models there). We will need to prove analogues later (and we will actually appeal to a relativization of it to models of $\text{ZF}(\bar{A})$, the proof of which we leave it to the reader to fill in). So let us look at the proof, also as a warm-up:

**Fact 2.3.** Let $M, N$ be models of $\text{ZF}$. Let $j : M \to N$ be $\Sigma_1$-elementary and $\in$-cofinal. Then $j$ is fully elementary.

**Proof.** We prove by induction on $n < \omega$, that $j$ is $\Sigma_n$-elementary.

Because $j$ is $\Sigma_1$-elementary, we have $j(V_\alpha) = V_{j(\alpha)}$ for each $\alpha$.

Suppose $j$ is $\Sigma_n$-elementary where $n \geq 1$. Let $C_n \subseteq \text{OR}^M$ be the $M$-class of all $\alpha$ such that $V^M_\alpha \equiv_\alpha V^M_{\lambda}$ (note that $C_n$ is as defined over $M$, without parameters). $\text{ZF}$ proves (via standard model theoretic methods) that $C_n$ is unbounded in OR.

Let $\alpha \in C_n$. We claim that $j(\alpha) \in C^N_n$ (with $C^N_n$ defined analogously over $N$; see §1.1). For suppose $N \models \varphi(x)$ where $x \in V^N_{j(\alpha)}$ and $\varphi$ is $\Sigma_n$, but that $V^N_{j(\alpha)} \models \neg \varphi(x)$. The existence of such an $x$ is a $\Sigma_n$ assertion about the parameter $V^N_{j(\alpha)}$, satisfied by $N$, so $M$ satisfies the same about $V^M_\alpha$ (by $\Sigma_n$-elementarity of $j$). But $\alpha \in C_n$, contradiction.

Now suppose that $N \models \varphi(j(x))$, where $\varphi$ is $\Sigma_{n+1}$. Then by the $\in$-cofinality of $j$ and the previous remarks, we may pick $\alpha \in C_n$ such that $x \in V^M_\alpha$ and $V^N_{j(\alpha)} \models \varphi(j(x))$. But then $V^M_\alpha \models \varphi(x)$, and since $\alpha \in C_n$, it follows that $M \models \varphi(x)$, as desired. \qed

**Proof of 2.1.** Suppose that $k : V \to V$ is elementary and there is a $\Sigma_n$ formula $\varphi$ and $p \in V$ such that for all $x,y$, we have

$$k(x) = y \iff \varphi(p,x,y).$$

Given any parameter $q$, attempt to define a function $j_q$ by:

$$j_q(x) = y \iff \varphi(q,x,y).$$

Say that $q$ is bad iff $j_q : V \to V$ is a $\Sigma_1$-elementary, non-identity map. Because $j_q$ is defined using the fixed formula $\varphi$ and we only demand $\Sigma_1$-elementarity, badness is a definable notion (without parameters). And $p$ above is bad.

Now by Fact 2.3 above, if $q$ is bad then $j_q$ is in fact fully elementary.

Now let $\kappa_0$ be the least critical point $\text{crit}(j_q)$ of all $j_q$ for bad parameters $q$. Note then that the singleton $\{\kappa_0\}$ is definable over $V$, from no parameters. (That is, there is a formula $\psi$ such that $\psi(x) \iff x = \kappa_0$, for all sets $x$.)

Let $q_0$ witness the choice of $\kappa_0$. As mentioned above, $j_{q_0}$ is in fact fully elementary, and we have $\text{crit}(j_{q_0}) = \kappa_0$. So $j_{q_0}(\kappa_0) > \kappa_0$, whereas $j_{q_0}(\alpha) = \alpha$ for all $\alpha < \kappa_0$. Note then that it follows that $\kappa_0 \notin \text{rg}(j_{q_0})$. But by the (full) elementarity of $j_{q_0} : V \to V$ and definability of $\{\kappa_0\}$, we must have $j_{q_0}(\kappa_0) = \kappa_0 \in \text{rg}(j_{q_0})$, a contradiction. \qed

We remark that Suzuki actually proved a more general theorem, considering elementary embeddings of the form $j : M \to V$ where $M \subseteq V$. 

8
3 Definability of rank-to-rank embeddings

3.1 The limit case

Most investigations of rank-to-rank cardinals to date have focused on elementary embeddings \( j : V_\alpha \rightarrow V_\alpha \) where \( \alpha = \kappa_\omega(j) \) or \( \alpha = \kappa_\omega(j) + 1 \), for the obvious reason that assuming the Axiom of Choice, these are the only rank-to-rank embeddings there are. The following very simple fact turns out to play a central role in these investigations: if \( \lambda \) is a limit ordinal, an elementary embedding from \( V_\lambda \) to \( V_\lambda \) extends in at most one way to an elementary embedding from \( V_{\lambda+1} \) to \( V_{\lambda+1} \).

**Definition 3.1.** Suppose \( \lambda \) is a limit ordinal and \( j : V_\lambda \rightarrow V_\lambda \) is an elementary embedding. The **canonical extension** of \( j \) is the function \( j^+ : V_{\lambda+1} \rightarrow V_{\lambda+1} \) defined by

\[
j^+(X) = \bigcup_{\alpha<\lambda} j(X \cap V_\alpha).
\]

Note that the canonical extension \( j^+ \) is indeed a function \( V_{\lambda+1} \rightarrow V_{\lambda+1} \). However, it is not clear that it is elementary with respect to \( V_{\lambda+1} \), and in fact, it is well known that there are examples which fail to be so. (For example, let \( \kappa \) be least such that there is an elementary \( j : V_\lambda \rightarrow V_\lambda \) with \( \text{crit}(j) = \kappa \), and show that \( j^+ \) is not elementary.) But if \( j \) does extend to an elementary embedding \( i : V_{\lambda+1} \rightarrow V_{\lambda+1} \), in fact even just \( \Sigma_1 \)-elementary, then clearly \( i(V_\lambda) = V_\lambda \) and \( i = j^+ \).

**Definition 3.2.** An ordinal \( \lambda \) has the **unique extension property** if any elementary embedding \( V_\lambda \rightarrow V_\lambda \) admits at most one extension to an elementary embedding from \( V_{\lambda+1} \) to \( V_{\lambda+1} \).

As a consequence of the preceding discussion, we have the following fact:

**Proposition 3.3.** Every limit ordinal has the unique extension property. In fact, if \( \lambda \) is a limit and \( i : V_{\lambda+1} \rightarrow V_{\lambda+1} \) is elementary then \( i = (i \upharpoonright V_\lambda)^+ \).

It follows that for limit ordinals \( \lambda \), every elementary \( j : V_{\lambda+1} \rightarrow V_{\lambda+1} \) is definable over \( V_{\lambda+1} \) from parameters, in fact, from its own restriction \( j \upharpoonright V_\lambda : V_\lambda \rightarrow V_\lambda \). (Since \( V_\lambda \) is closed under ordered pairs, \( j \upharpoonright V_\lambda \) is literally an element of \( V_{\lambda+1} \).) On the other hand, \( j \) is not definable over \( V_{\lambda+1} \) from any element of \( V_\lambda \), and also, no elementary embedding from \( V_\lambda \) to \( V_\lambda \) is definable from parameters over \( V_\lambda \):

**Theorem 3.4.** Let \( j : V_\delta \rightarrow V_\delta \) be elementary and \( p \in V_\delta \) and suppose that \( j \) is definable over \( V_\delta \) from the parameter \( p \). Then \( \delta = \beta + 1 \) is a successor and \( \text{rank}(p) = \beta \).

**Proof.** We adapt the proof of Suzuki’s Fact. Suppose the theorem fails. Fix \( k < \omega \) and a \( \Sigma_k \) formula \( \varphi \) such that for some \( \beta < \delta \) and \( p \in V_\beta \), the set

\[
\tilde{j}_p = \{(x, y) \in V_\delta : V_\delta \models \varphi(p, x, y)\}
\]

In other words, \( j^+ : \mathcal{P}(V_\delta) \rightarrow \mathcal{P}(V_\delta) \)
is such that
\[ j_p : V_\delta \to V_\delta \] and is fully elementary.

Let \( \mu_0 \) be the least possible critical point of all such (fully elementary) embeddings \( j_p \) (minimizing over all \( p \in V_\beta \)). Let \( p_0 \in V_\beta \) and \( j_{p_0} \) witness this, so \( \text{crit}(j_{p_0}) = \mu_0 \).

For \( n < \omega \), say that \( q \in V_\beta \) is \( n \)-bad iff
\[ j_q : V_\delta \to V_\delta \] and is \( \Sigma_n \)-elementary.

Let \( A_n = \{ q \in V_\beta : q \text{ is } n \text{-bad} \} \). So \( A_n \in V_\beta \) and note that \( A_n \) is definable over \( V_\delta \) from the parameter \( \beta \).

Since \( j = j_{p_0} \) is fully elementary, \( j(A_n) \cap V_\beta = A_n \). (It is no problem if \( j(\beta) > \beta \).) Let \( A = \bigcap_{n<\omega} A_n \), so also \( A \in V_\delta \). Note that for every \( q \in A \), \( j_q : V_\delta \to V_\delta \) is fully elementary.

The sequence \( \langle A_n \rangle_{n<\omega} \) can easily be coded by a set in \( V_\delta \) (or is literally an element of \( V_\delta \) if \( \delta \) is a limit), and therefore
\[ j(A) = \bigcap_{n<\omega} j(A_n), \]
so \( p_0 \in j(A) \). Therefore \( V_\delta \models \exists q \in j(A) \text{ such that } \text{crit}(j_q) < j(\mu_0) \)" (as witnessed by \( p_0 \)). Pulling this back with the elementarity of \( j \), \( V_\delta \models \exists q \in A \text{ such that } \text{crit}(j_q) < \mu_0 \). But this contradicts the minimality of \( \mu_0 \). □

**Corollary 3.5.** Let \( \xi \) be a limit ordinal, \( n < \omega \) and \( j : V_{\xi+n+2} \to V_{\xi+n+2} \) be elementary. Then \( j \) is not definable over \( V_{\xi+n+2} \) from \( j \upharpoonright V_{\xi+1} \).\(^{11}\)

**Proof.** If \( n > 0 \) then this is immediate by Theorem 3.4. Suppose \( n = 0 \) and \( j \) is definable over \( V_{\xi+2} \) from \( j \upharpoonright V_{\xi+1} \). Then \( j \upharpoonright V_{\xi+1} = (j \upharpoonright V_{\xi})^+ \), so in fact, \( j \) is definable over \( V_{\xi+2} \) from \( j \upharpoonright V_{\xi} \), and \( j \upharpoonright V_{\xi} \in V_{\xi+1} \). This contradicts Theorem 3.4. □

These observations lead to the following natural questions, which lead to the topic of investigation in the next section. Let \( j : V_{\xi+n+3} \to V_{\xi+n+3} \) be elementary, where \( n < \omega \). Can \( j \) be definable over \( V_{\xi+n+3} \) from \( j \upharpoonright V_{\xi+n+2} \)? More generally, let \( j : V_{\xi+n+2} \to V_{\xi+n+2} \) be elementary, where \( n < \omega \). Can \( j \) be definable over \( V_{\xi+n+2} \) from some parameter (of rank \( \xi + n + 1 \))?

### 3.2 The successor case

In this section, we consider the definability of elementary embeddings from \( V_{\alpha+1} \) to \( V_{\alpha+1} \) without the assumption that \( \alpha \) is a limit ordinal. Most of the results of the previous section turn out to generalize to the case that \( \lambda \) is an arbitrary even ordinal. We start with the following theorem:

**Theorem 3.6.** Every even ordinal has the unique extension property.

\(^{11}\)With the standard representation of functions as a set of ordered pairs \( \{x, y\} = \{\{x\}, \{x, y\}\} \), then the elements of \( j \upharpoonright V_{\xi+1} \) are of the form \( (x, y) \) where \( x, y \in V_{\xi+1} \), which means that, at worst, we get \( (x, y) \in V_{\xi+3} \), so \( j \upharpoonright V_{\xi+1} \in V_{\xi+4} \). Thus, if \( n < 2 \) then it does not literally make sense to talk about \( j \upharpoonright V_{\xi+1} \) as an element of \( V_{\xi+n+2} \). But one can actually represent \( j \upharpoonright V_{\xi+1} \) as a set in \( V_{\xi+2} \), and that is what we mean here. This sort of representation is discussed further in the proof of Lemma 3.7.

\(^{12}\)Literally, since \( \xi \) is a limit, so \( V_{\xi} \) is closed under ordered pairs.
In fact we will prove a somewhat stronger theorem, Theorem 3.16.

At first glance, it seems that the proof of the unique extension property for limit ordinals $\lambda'$ required structural features specific to $V_{\lambda'}$ that are not available at the other even levels. In particular, Definition 3.1 uses the existence of the hierarchy $\langle V_\alpha \rangle_{\alpha<\lambda'}$ for $V_{\lambda'}$; this hierarchy seems to have no analog at the level of $V_{\lambda'+2n}$ when $n>0$. But suppose $\lambda$ is an arbitrary ordinal (which will end up being even). On closer inspection, all Definition 3.1 really requires is a family of sets $I \subseteq V_{\lambda'+1}$ (so $\Gamma \subseteq V_{\lambda}$ for each $\Gamma \in I$) closed downwards under $\subseteq$ such that for any elementary embedding
\[ j : V_{\lambda} \rightarrow V_{\lambda} \]
there is a map
\[ \tilde{j} : I \rightarrow V_{\lambda+1} \]
such that
\begin{enumerate}
\item for all elementary $i : V_{\lambda+1} \rightarrow V_{\lambda+1}$ extending $j$, we have $i \upharpoonright I = \tilde{j}$,
\item $V_{\lambda} = \bigcup_{\Gamma \in I} \tilde{j}(\Gamma)$.
\end{enumerate}

Given $j$ and $\tilde{j}$ satisfying (1) and (2), if $i : V_{\lambda+1} \rightarrow V_{\lambda+1}$ is an elementary embedding extending $\tilde{j}$, then we must have
\[ i(X) = \bigcup_{\Gamma \in I} \tilde{j}(X \cap \Gamma) \]
for all $X \in V_{\lambda+1}$. (Note that the right-hand side makes sense since $I$ is closed downwards under $\subseteq$.) For by (1) we have $i \upharpoonright I = \tilde{j}$. So the right-to-left inclusion $\supseteq$ is an immediate consequence of elementarity, and the left-to-right inclusion $\subseteq$ follows from (2): for any $a \in i(X)$, fixing $\Gamma \in I$ such that $a \in \tilde{j}(\Gamma)$, we then have
\[ a \in i(X) \cap \tilde{j}(\Gamma) = i(X \cap \Gamma) = i(X \cap \Gamma) = \tilde{j}(X \cap \Gamma). \]

Denote by $[X]^Y$ the collection of subsets of $X$ that are the surjective image of a subset of $Y$.\[^{13}\] We aim to verify that when $\lambda$ is an even successor ordinal, $I = [V_\lambda]^{V_{\lambda-1}}$ is a family satisfying (1) and (2). (Trivially, $I$ is downwards closed under $\subseteq$.)

Towards (1), we prove a preliminary extension lemma, which is a standard coding argument in set theory. Note that for any successor ordinal $\lambda$,
\[ V_{\lambda} \subseteq [V_\lambda]^{V_{\lambda-1}} \subseteq V_{\lambda} \]

**Lemma 3.7.** For any ordinal $\lambda$, any elementary $j : V_{\lambda} \rightarrow V_{\lambda}$ extends uniquely to an elementary $j^* : [V_\lambda]^{V_{\lambda-1}} \rightarrow [V_\lambda]^{V_{\lambda-1}}$.

**Sketch.** Let $p : V_{\lambda-1} \rightarrow V_{\lambda-1} \times V_{\lambda-1}$ be a surjection that is $\Sigma_1$-definable without parameters over $V_{\lambda-1}$. We leave the construction of this pairing function $p$ to the reader, but one way to proceed is to use Quine-Rosser pairs. For every $A \in V_{\lambda}$, let $R_A = p[A]$. In this way, every binary relation on $V_{\lambda-1}$ is coded by an element of $V_{\lambda}$. For every binary relation $R$ on $V_{\lambda-1}$, let
\[ \Gamma_R = \{ R_a : a \in V_{\lambda-1} \} \in [V_\lambda]^{V_{\lambda-1}} \]
\[^{13}\]We allow subsets of $Y$, instead of just $Y$ itself, so that $\emptyset$ is included in $[X]^Y$. 11
where \( R_0 = \{ b \in V_{\lambda-1} : (a, b) \in R \} \in V_\lambda \). In this way, every element of \([V_\lambda]^{V_\lambda-1}\) is coded by a binary relation on \( V_{\lambda-1} \).

Suppose \( i : [V_\lambda]^{V_\lambda-1} \to [V_\lambda]^{V_\lambda-1} \) is an elementary embedding. Then by the elementarity of \( i \),

\[
i(\Gamma_{R_a}) = \Gamma_{R_{i(a)}}
\]

Thus \( i \) is uniquely determined by \( i \upharpoonright V_\lambda \). Moreover, if \( j : V_\lambda \to V_\lambda \) is elementary, then one can check that the embedding \( j^* : [V_\lambda]^{V_\lambda-1} \to [V_\lambda]^{V_\lambda-1} \) defined by \( j^*(\Gamma_{R_a}) = \Gamma_{R_{j(a)}} \) is elementary.

If \( \lambda \) is a successor ordinal and \( j : V_\lambda \to V_\lambda \) is an elementary embedding, let us denote its unique elementary extension to \([V_\lambda]^{V_\lambda-1}\) by \( j^* : [V_\lambda]^{V_\lambda-1} \to [V_\lambda]^{V_\lambda-1} \).

Now let us return to our attempt to extend an elementary embedding

\[
j : V_\lambda \to V_\lambda
\]

to \( V_{\lambda+1} \) in the case that \( \lambda \) is a successor ordinal. Taking \( I = [V_\lambda]^{V_\lambda-1} \) and \( j = j^* \), the uniqueness of \( j^* \) secures (1).

The problem is now to secure (2), so let us give this property a name.

**Definition 3.8.** An ordinal \( \lambda \) has the cofinal embedding property if either \( \lambda \) is a limit ordinal or \( \lambda \) is a successor ordinal and for every elementary embedding \( j : V_\lambda \to V_\lambda \), we have

\[
V_\lambda = \bigcup_{\Gamma \in [V_\lambda]^{V_\lambda-1}} j^*(\Gamma).
\]

The cofinal embedding property derives its name from its equivalence to the assertion that for every \( j : V_\lambda \to V_\lambda \), the range of \( j^* : [V_\lambda]^{V_\lambda-1} \to [V_\lambda]^{V_\lambda-1} \) is cofinal in \([V_\lambda]^{V_\lambda-1}\) with respect to inclusion.

**Definition 3.9.** For any structure \( M \), let \( \mathcal{E}(M) \) denote the set of elementary embeddings from \( M \) to \( M \).

The argument sketched above shows:

**Lemma 3.10.** Suppose \( \lambda \) is an ordinal with the cofinal embedding property. Then \( \lambda \) has the unique extension property.

**Proof.** By 3.3, we may assume that \( \lambda \) is a successor ordinal.

Suppose \( i \in \mathcal{E}(V_{\lambda+1}) \) extends \( j \in \mathcal{E}(V_\lambda) \). Fix \( X \in V_{\lambda+1} \). We claim that

\[
i(X) = \bigcup_{\Gamma \in [V_\lambda]^{V_\lambda-1}} j^*(X \cap \Gamma)
\]

This implies that \( i \) is uniquely determined by \( j \).

The right-to-left inclusion \( \supseteq \) does not require the cofinal embedding property: for all \( \Gamma \in [V_\lambda]^{V_\lambda-1} \), \( j^*(X \cap \Gamma) = i(X \cap \Gamma) \subseteq i(X) \). For the inclusion \( \subseteq \), fix \( A \in i(X) \), and we will show \( A \in j^*(X \cap \Gamma) \) for some \( \Gamma \in [V_\lambda]^{V_\lambda-1} \). Indeed, by the cofinal embedding property, we can find \( \Gamma \in [V_\lambda]^{V_\lambda-1} \) such that \( A \in j^*(\Gamma) \).

Now \( A \in i(X) \cap j^*(\Gamma) = i(X) \cap i(\Gamma) = i(X \cap \Gamma) \), as desired.

The proof of Lemma 3.10 suggests a generalization of the canonical extension operation (Definition 3.1) assuming the cofinal embedding property:

12
Definition 3.11. Suppose \( \lambda \) is a successor ordinal with the cofinal embedding property. For any \( j \in \mathcal{E}(V_{\lambda}) \), the canonical extension of \( j \) is the function

\[
j^+: V_{\lambda+1} \to V_{\lambda+1}
\]
defined by

\[
j^+(X) = \bigcup_{\Gamma \in [V_{\lambda}]^{V_{\lambda+1}-1}} j^*(X \cap \Gamma)
\]

Having made this definition, one can formulate a property stronger than the unique extension property which follows from the cofinal embedding property by the proof of Lemma 3.10.

Definition 3.12. An ordinal \( \lambda \) has the canonical extension property if for any \( i \in \mathcal{E}(V_{\lambda+1}) \),

\[
i = (i | V_{\lambda})^+.
\]

In the case that \( \lambda \) is a limit ordinal, \((i | V_{\lambda})^+\) denotes the canonical extension operation defined in Definition 3.1. As a consequence of this definition:

Proposition 3.13. Every limit ordinal has the canonical extension property.

Also, by the proof of Lemma 3.10:

Lemma 3.14. Suppose \( \lambda \) is an ordinal with the cofinal embedding property. Then \( \lambda \) has the canonical extension property.

The key observation, which leads to the periodicity phenomenon, is that the cofinal embedding property follows from the canonical extension property:

Lemma 3.15. Suppose \( \lambda \) is an ordinal with the canonical extension property. Then \( \lambda + 2 \) has the cofinal embedding property.

Proof. Fix \( j \in \mathcal{E}(V_{\lambda+2}) \) and \( A \in V_{\lambda+2} \). We must find a set \( \Gamma \in [V_{\lambda+2}]^{V_{\lambda+1}} \) such that \( A \in j^*(\Gamma) \). Let

\[
\Gamma = \{(k^+)^{-1}[A] : k \in \mathcal{E}(V_{\lambda})\}.
\]

Since \( \mathcal{E}(V_{\lambda}) \) is the surjective image of \( V_{\lambda+1} \), \( \Gamma \in [V_{\lambda+2}]^{V_{\lambda+1}} \). Note that the canonical extension operation is definable without parameters over \([V_{\lambda+2}]^{V_{\lambda+1}}\) in the sense that

\[
\{(k, X, Y) \in \mathcal{E}(V_{\lambda}) \times V_{\lambda+1} \times V_{\lambda+1} : k^+(X) = Y\}
\]

is definable over the structure \([V_{\lambda+2}]^{V_{\lambda+1}}, 14\) Since \( j^*: [V_{\lambda+2}]^{V_{\lambda+1}} \to [V_{\lambda+2}]^{V_{\lambda+1}} \) is elementary,

\[
j^*(\Gamma) = \{(k^+)^{-1}[j(A)] : k \in \mathcal{E}(V_{\lambda})\}
\]

Since \( A = (j^+)^{-1}[j(A)] \), \( A \in j^*(\Gamma) \). This proves the lemma.

Now a simple induction proves the cofinal embedding property and the canonical extension property:

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14 This relation is “morally” definable over \( V_{\lambda+1} \), but unfortunately \( \mathcal{E}(V_{\lambda}) \) may not even be a subset of \( V_{\lambda+2} \) since a Kuratowski pair of elements of \( V_{\lambda} \) can have rank as large as \( \lambda + 1 \). On the other hand, \( \mathcal{E}(V_{\lambda}) \) is literally a subset of \([V_{\lambda+2}]^{V_{\lambda}}\).
Theorem 3.16. Every even ordinal has the canonical extension property.

Proof. Suppose $\lambda$ is a limit ordinal. We prove by induction on $n$ that $\lambda + 2n$ has the canonical extension property for all $n < \omega$. The base case $n = 0$ follows from Proposition 3.13. Therefore assume by induction that $\lambda + 2n$ has the canonical extension property. By Lemma 3.15, $\lambda + 2n + 2$ has the cofinal embedding property. Therefore by Lemma 3.14, $\lambda + 2n + 2$ has the canonical extension property. This completes the induction step.

Of course, the unique extension property (Theorem 3.6) follows as an immediate corollary, and, applying Lemma 3.15, so does the cofinal embedding property.

Corollary 3.17. Every even ordinal has the cofinal embedding property.

Using these results, we can analyze the definability of rank-to-rank embeddings:

Theorem 3.18. If $\lambda$ is an even ordinal and $i : V_{\lambda + 1} \to V_{\lambda + 1}$ is an elementary embedding, then $i$ is definable over $V_{\lambda + 1}$ from $i \upharpoonright V_{\lambda}$.\footnote{Again $i$ is “morally” definable over $V_{\lambda + 1}$ from the parameter $i \upharpoonright V_{\lambda}$, but unfortunately $i \upharpoonright V_{\lambda} \notin V_{\lambda + 1}$ if $\lambda$ is a successor ordinal.}

Proof. For any elementary embedding $k \in \mathcal{E}(V_{\lambda})$, $k$ is the inverse of the transitive collapse map $\pi : k^{-}V_{\lambda} \to V_{\lambda}$. Therefore the graph of $k$ is definable over $V_{\lambda + 1}$ from the parameter $k^{-}V_{\lambda}$. It is easy to see that $k^{+}$ is definable over $V_{\lambda + 1}$ using the graph of $k$ as a predicate, and it therefore follows that $k^{+}$ is definable over $V_{\lambda + 1}$ from $k^{-}V_{\lambda}$. But $i = (i \upharpoonright V_{\lambda})^{+}$ since $\lambda$ has the canonical extension property. Hence $i$ is definable over $V_{\lambda + 1}$ from $i \upharpoonright V_{\lambda}$.

The cofinal embedding property also implies the non-definability of elementary embeddings of even ranks. The proof given here is an elaboration on that of Theorem 3.4.

Theorem 3.19. Suppose $j : V_{\alpha + 1} \to V_{\alpha + 1}$ is an elementary embedding. Then $j$ is not definable over $V_{\alpha + 1}$ from parameters in $\bigcup \{j^{*}(\Gamma) : \Gamma \in [V_{\alpha + 1}]^{V_{\alpha}}\}$.

Proof. Suppose the theorem fails. Fix a formula $\varphi(v_{0}, v_{1}, v_{2})$ and a parameter $p \in \bigcup \{j^{*}(\Gamma) : \Gamma \in [V_{\alpha + 1}]^{V_{\alpha}}\}$ such that for all $x, y \in V_{\alpha + 1}$,

$$
j(x) = y \iff V_{\alpha + 1} \models \varphi(x, y, p).
$$

For each $q \in V_{\alpha + 1}$, let $j_{q} : V_{\alpha + 1} \to V_{\alpha + 1}$ be defined by setting

$$
j_{q}(x) = y \iff V_{\alpha + 1} \models \varphi(x, y, q)
$$

Let $A$ be the set of $q \in V_{\alpha + 1}$ such that $j_{q}$ is a well-defined function. For $k : V_{\alpha + 1} \to V_{\alpha + 1}$ a $\Sigma_{1}$-elementary embedding, let

$$
k^{*} : [V_{\alpha + 1}]^{V_{\alpha}} \to [V_{\alpha + 1}]^{V_{\alpha}}
$$
be as defined in Lemma 3.7. For $1 \leq n \leq \omega$, we say that $q \in A$ is $n$-bad if

$$j_q : V_{\alpha + 1} \rightarrow V_{\alpha + 1}$$

is a $\Sigma_n$-elementary embedding and $q \in \bigcup\{ j_q^*(\Gamma) : \Gamma \in [V_{\alpha + 1}]^{V_\alpha}\}$. Thus $p$ is $\omega$-bad.

Let

$$\kappa = \min\{\text{crit}(j_q) : q \text{ is } \omega\text{-bad}\}$$

Fix an $\omega$-bad parameter $r$ such that $\text{crit}(j_r) = \kappa$. Fix $\Gamma \in [V_{\alpha + 1}]^{V_\alpha}$ such that $r \in j_r(\Gamma)$. For each $1 \leq n \leq \omega$, let

$$A_n = \{ q \in \Gamma : q \text{ is } n\text{-bad}\}$$

Since $j_r$ is fully elementary, for each $1 \leq n < \omega$,

$$j_r(A_n) = \{ q \in j_r(\Gamma) : q \text{ is } n\text{-bad}\}$$

Since $(A_n)_{1 \leq n < \omega}$ is coded by an element of $V_{\alpha + 1}$,

$$j_r(A_\omega) = j_r\left( \bigcap_{1 \leq n < \omega} A_n \right) = \bigcap_{1 \leq n < \omega} j_r(A_n)$$

In other words, $j_r(A_\omega) = \{ q \in j_r(\Gamma) : q \text{ is } \omega\text{-bad}\}$. Therefore $r \in j_r(A_\omega)$ and every $q \in j_r(A_\omega)$ is $\omega$-bad. Hence

$$\min\{\text{crit}(j_q) : q \in j_r(A_\omega)\} = \kappa$$

Therefore $\kappa = j_r(\min\{\text{crit}(j_q) : q \in A_\omega\})$, so $\kappa$ is in the range of $j_r$. This contradicts that $\kappa = \text{crit}(j_r)$.

Theorem 3.20. Suppose $\lambda$ is an even ordinal and $j : V_\lambda \rightarrow V_\lambda$ is an elementary embedding. Then $j$ is not definable from parameters over $V_\lambda$.

Proof. If $\lambda$ is a limit ordinal, this follows from Theorem 3.4. Otherwise it follows from Theorem 3.19, which implies that $\lambda$ has the cofinal embedding property.

4 Reinhardt ultrafilters

Solovay’s discovery of supercompactness in the late 1960’s marked the beginning of the modern era of large cardinal theory. In the context of ZFC, supercompactness has both a combinatorial characterization in terms of normal ultrafilters and a “model theoretic” characterization in terms of elementary embeddings $j : V \rightarrow M$ where $M$ is an inner model. In the choiceless context, however, the equivalence between the two characterizations is no longer provable, and instead supercompactness splinters into a number of inequivalent but interrelated concepts.

The rank-into-rank embeddings $j : V_\delta \rightarrow V_\delta$ studied here represent one manifestation of supercompactness, and in this section we aim to establish a characterization in terms of normal ultrafilters for these embeddings in the case that $\delta = \alpha + 2$. Since this forces us into the choiceless regime, there is no way forward but face these subtleties head on. 16

16 In this section we assume familiarity with ultrapowers as used in set theory; the reader familiar with supercompactness measures should be fine.
4.1 Preliminaries: normal ultrafilters in ZF

Definition 4.1. Suppose $I$ is a set and $\mathcal{P}$ is a family of subsets of $I$. For any $I$-indexed sequence $(S_i)_{i \in I}$ of subsets of $\mathcal{P}$, the diagonal intersection of $(S_i)_{i \in I}$, denoted $\triangle_{i \in I} S_i$, is the set $\{ \sigma \in \mathcal{P} : \sigma \in \bigcap_{i \in \sigma} S_i \}$. 

We suppress the dependence of the diagonal intersection on the family $\mathcal{P}$.

Definition 4.2. Suppose $\mathcal{P}$ is a family of sets and $F$ is a filter over $\mathcal{P}$.

- $F$ is fine if for all $x \in \bigcup \mathcal{P}$, $x \in \sigma$ for $F$-almost all $\sigma \in \mathcal{P}$.
- $F$ is normal if $F$ is closed under diagonal intersections indexed by $\bigcup \mathcal{P}$.

The most familiar example of a normal filter is the closed unbounded filter over a regular uncountable cardinal $\kappa$. In fact, every normal fine filter over a regular cardinal extends the closed unbounded filter.

We will really only be interested in normal fine ultrafilters here. Although one cannot prove Lö"{o}s's Theorem without the Axiom of Choice, one can still attempt to establish equivalents of these properties in terms of ultrapowers.

Given an ultrafilter $\mu$ over $\mathcal{P}$, we write $\text{Ult}(M, \mu)$ for the ultrapower of $M$ by $\mu$ (formed using all functions $f : \mathcal{P} \to M$). Here $M$ will always be a transitive set in our applications. If the ultrapower is isomorphic to a transitive set $U$, then we identify it with $U$. We drop the superscript “$M$” and/or subscript “$\mu$” if these are suppressed or clear from context. We write $i^\mu_M$ for the associated ultrapower embedding $i^\mu_M : M \to \text{Ult}(M, \mu)$; that is, $i^\mu_M(x) = [c_x]$ where $c_x$ is the constant function with constant value $x$.

Lemma 4.3. Suppose $\mathcal{P}$ is a family of sets, $X = \bigcup \mathcal{P}$, and $\mu$ is an ultrafilter over $\mathcal{P}$.

- $\mu$ is fine if and only if $j_\mu X \subseteq [id]_\mu$.
- If $\mu$ is normal, then $[id]_\mu \subseteq j_\mu X$. Moreover, if $X$ is wellorderable, then the converse holds.

In the context of ZFC, normality is equivalent to a version of Fodor’s Lemma. Recall that a function $f$ is regressive if $f(\sigma) \in \sigma$ for all $\sigma \in \text{dom}(f)$.

Proposition 4.4 (ZFC). Suppose $F$ is a filter over $\mathcal{P}$. Then $F$ is normal if and only if every regressive function defined on an $F$-positive set assumes a constant value on an $F$-positive set.

Dropping Choice, however, this equivalence may fail depending on whether $\bigcup \mathcal{P}$ is wellorderable. We now explore this in a bit more detail. ***? The ZF generalization We say a binary relation $R$ is regressive if for all $A \in \text{dom}(R)$ and $x \in \text{ran}(R)$, $A R x$ implies $x \in A$.

Proposition 4.5. A filter $F$ over $\mathcal{P}$ is normal if and only if for every regressive relation $R$ whose domain is an $F$-positive subset of $\mathcal{P}$, there is some $x \in \text{ran}(R)$ such that $R x$ for an $F$-positive set of $\sigma \in \mathcal{P}$.
Proof. Let \( X = \bigcup \mathcal{P} \). For the forwards direction, assume that \( F \) is normal, and let \( R \) be a regressive relation whose domain is an \( F \)-positive subset of \( \mathcal{P} \). Assume towards a contradiction that for each \( x \in X \), the set of \( \sigma \in \mathcal{P} \) such that \( \sigma R x \) is \( F \)-null. For each \( x \in X \), let \( A_x \) be the set of \( \sigma \in \mathcal{P} \) such that \( \sigma R x \), so that by our assumption, \( A_x \in F \). By normality, the diagonal intersection \( A \) of the sequence \( \langle A_x \rangle_{x \in X} \) belongs to \( F \). Since \( \text{dom}(R) \) is \( F \)-positive, there is some \( \sigma \in \text{dom}(R) \cap A \). Since \( \sigma \in A \), for every \( x \in \sigma \), \( \sigma \in A_x \) and hence \( \sigma R x \) fails. Since \( R \) is regressive, it follows that \( \sigma \) is not in the domain of \( R \). This is a contradiction.

For the converse, fix \( \langle A_x \rangle_x \subseteq F \). Consider the relation \( R \) defined by setting \( \sigma R x \) if and only if \( \sigma \in \mathcal{P} \), \( x \in \sigma \), and \( \sigma \notin A_x \). Assume towards a contradiction that the diagonal intersection of \( \langle A_x \rangle_x \) is not in \( F \). Then its complement in \( \mathcal{P} \), which is equal to the domain of \( R \), is \( F \)-positive, so by our hypothesis on \( F \), there is then an \( x \) such that \( \sigma R x \) for an \( F \)-positive set of \( \sigma \in \mathcal{P} \). In other words, the set of \( \sigma \in \mathcal{P} \setminus A_x \) is \( F \)-positive, which contradicts that \( A_x \in F \). \( \square \)

As a corollary, one obtains an ultraproduct characterization of normality and fineness of ultrafilters:

**Corollary 4.6.** Suppose \( \mathcal{P} \) is a family of sets with union \( X \) and \( \mu \) is an ultrafilter over \( \mathcal{P} \). Define \( r : \mathcal{P}(X) \to \prod_{\sigma \in \mathcal{P}} \mathcal{P}(\sigma) / \mu \) by \( r(S) = [\langle S \cap \sigma \rangle_{\sigma \in \mathcal{P}}]_\mu \). Then \( \mu \) is fine if and only if \( r \) is injective, and \( \mu \) is normal if and only if \( r \) is surjective.

**Proof.** We omit the part involving fineness, which is easy, and turn to normality.

For the forward direction, assume \( \mu \) is normal. Fix \( f \in \prod_{\sigma \in \mathcal{P}} \mathcal{P}(\sigma) \), and we will show that \( [f]_\mu \in \text{ran}(r) \). Let \( S \) be the set of \( x \in X \) such that \( x \in f(\sigma) \) for \( \mu \)-almost all \( \sigma \in \mathcal{P} \). We will show that \( r(S) = [f]_\mu \).

Consider the regressive relation \( R \) defined by setting \( \sigma R x \) if and only if \( x \in f(\sigma) \setminus S \). Clearly there can be no \( x \) such that \( \sigma R x \) for \( \mu \)-almost all \( \sigma \in \mathcal{P} \). It follows from Proposition 4.5 that \( \text{dom}(R) \) is \( \mu \)-null, which implies that \( f(\sigma) \subseteq S \) for \( \mu \)-almost all \( \sigma \in \mathcal{P} \). Hence \( f(\sigma) \subseteq S \cap \sigma \) for \( \mu \)-almost all \( \sigma \in \mathcal{P} \).

Turning to the reverse inclusion, for \( x \in S \), let \( A_x = \{ \sigma : x \in f(\sigma) \} \). We have \( A_x \in F \) for all \( x \in S \), so

\[
\Delta_{x \in S} A_x = \{ \sigma \in \mathcal{P} : x \cap S \subseteq f(\sigma) \}
\]

belongs to \( \mu \) by normality. It follows that \( S \cap \sigma \subseteq f(\sigma) \) for \( \mu \)-almost all \( \sigma \in \mathcal{P} \). Combining this with the previous paragraph, \( r(S) = [\langle S \cap \sigma \rangle_{\sigma \in \mathcal{P}}]_\mu = [f]_\mu \), as desired.

The reverse direction is essentially immediate from Proposition 4.5, so we omit the proof. \( \square \)

### 4.2 Successor rank-into-rank embeddings as ultrapowers

In this section we sketch an alternate proof of Theorem 1.1, one which is equivalent to that presented already, but superficially different, and maybe more standard for set theory.

We will again define for all even ordinals \( \delta \) a canonical extension function

\[
k \mapsto k^+,
\]

17
with domain $\mathcal{E}(V_\delta)$, such that $k^+ : V_{\delta+1} \rightarrow V_{\delta+1}$ (but $k^+$ is not claimed to be elementary in general), and such that if $k$ extends to an elementary map $V_{\delta+1} \rightarrow V_{\delta+1}$, then it is $k^+$. The function $k \mapsto k^+$, with domain $\mathcal{E}(V_\delta)$, will be definable over $V_{\delta+1}$ without parameters (meaning the set of “tuples” $(k, x, y)$ such that $k \in \mathcal{E}(V_\delta)$, $x, y \in V_{\delta+1}$ and $k^+(x) = y$ will be so definable). The definition is by induction on $\delta$.

If $\delta$ is a limit, then $k^+$ is defined as in Definition 3.1.

Suppose now that $\delta = \eta + 2$ where $\eta$ is even. Let $j : V_{\eta+2} \rightarrow V_{\eta+2}$ be elementary; we want to define $j^+$ and prove some facts. Let $\mu$ be the ultrafilter over $\mathcal{E}(V_\eta)$ derived from $j$ with seed $j \upharpoonright V_\eta$:

$$\mu = \{ \sigma \subseteq \mathcal{E}(V_\eta) \mid j \upharpoonright V_\eta \in j(\sigma) \}.$$ 

Note here that $\sigma$ above is coded by an element of $V_{\eta+2}$, so $j(\sigma)$ makes sense. Let:

- $U = \text{Ult}(V_{\eta+2}, \mu)$ and $i_\mu : V_{\eta+2} \rightarrow U$ be the ultrapower map,
- $\tilde{U} = \text{Ult}(V_{\eta+3}, \mu)$ and $\tilde{i}_\mu : V_{\eta+3} \rightarrow \tilde{U}$ be the ultrapower map.

We will eventually show that $i_\mu = j$ and $j \subseteq \tilde{i}_\mu$, and define $j^+ = \tilde{i}_\mu$.

We don’t yet know that $U, \tilde{U}$ are extensional/wellfounded, so formally consider these ultrapowers at the “representation” level; that is, their elements are equivalence classes $[f]_\mu$ of functions $f$.

Now consider the hull

$$H = \text{Hull}^{V_{\eta+2}}(\text{rg}(j) \cup \{ j \upharpoonright V_\eta \}),$$

where $\text{Hull}^M(X)$, for $X \subseteq M$, denotes the set of all $x \in M$ such that $x$ is definable over $M$ from parameters in $X$. The following claim is a typical feature of ultrapowers via a measure derived from an embedding, although part 2 only holds because $j \upharpoonright V_\eta$ encodes enough information, and for this it is crucial that the canonical extension $(j \upharpoonright V_\eta)^+ = j \upharpoonright V_{\eta+1}$, and that this operation is definable over $V_{\eta+2}$ (in fact it is over $V_{\eta+1}$), a fact we know by induction.

**Claim 1.** Recall $U = \text{Ult}(V_{\eta+2}, \mu)$. We have:

1. $U$ is extensional and wellfounded; moreover, $U \cong H$.
2. $H = V_{\eta+2}$.
3. $i_\mu = j$, after we identify $U$ with its transitive collapse $V_{\eta+2}$.

**Proof.** Part 2: As noted above, from the parameter $j \upharpoonright V_\eta$, $V_{\eta+2}$ can definably recover

$$k = (j \upharpoonright V_\eta)^+ = j \upharpoonright V_{\eta+1}.$$ 

Now let $x \in V_{\eta+2}$, so $x \subseteq V_{\eta+1}$. Then note that

$$x = k^{-1} \cdot j(x),$$

and since $j(x) \in \text{rg}(j)$, this suffices.\(^1\)

\(^1\)Note that the proof actually shows that $V_{\eta+2} = \text{Hull}_\Sigma(V_{\eta+2}^{\eta+2})(\text{rg}(j) \cup \{ j \upharpoonright V_\eta \})$. 

18
Part 1: Define the map \( \pi : U \to V_{\eta+2} \) by
\[
\pi([f]_{\mu}^{V_{\eta+2}}) = j(f)(j \restriction V_{\eta}) .
\]
We claim this is an isomorphism (with respect to both = and \( \in \)), and in particular, the ultrapower is extensional and wellfounded.

First note that if \( \pi \) is well-defined then \( \pi \) is surjective, because given \( x \in V_{\eta+2} \), just let \( f_x : \mathcal{E}(V_{\eta}) \to V_{\eta+2} \) be the function
\[
f_x(k) = (k+1)^{-1}x,
\]
and then note that \( j(f_x)(j \restriction V_{\eta}) = x \).

To see that \( \pi \) is well-defined and injective, we just unravel the definitions to observe that the following are equivalent:
- \([f]_{\mu}^{V_{\eta+2}} = [g]_{\mu}^{V_{\eta+2}}\)
- \(j(V_{\eta}) \in j(A)\) where \( A = \{k \in \mathcal{E}(V_{\eta}) \mid f(k) = g(k)\}\),
- \(j(f)(j \restriction V_{\eta}) = j(g)(j \restriction V_{\eta})\).

Replacing “=” with “\( \in \)” above shows that \( \pi \) is also an \( \in \)-isomorphism.

Part 3: We have \( i_{\mu}(x) = [c_{\mu}^{V_{\eta+2}}] \) where \( c_{\mu} \) is the constant function taking value \( x \). But note that \( \pi \circ i_{\mu} = j \), because
\[
\pi([c_{\mu}^{V_{\eta+2}}]) = j(c_{\mu})(j \restriction V_{\eta}) = c_{j(x)}(j \restriction V_{\eta}) = j(x),
\]
since \( j \) is elementary. But after identifying the ultrapower with \( V_{\eta+2} \), \( \pi \) is actually just the identity, so we are done.

Claim 2. We have:

1. \( \bar{U} \) is wellfounded.
2. The following are equivalent:
   - (a) \( \bar{U} \) is extensional,
   - (b) \( j \) extends to a \( \Sigma_0 \)-elementary \( \ell : V_{\eta+3} \to V_{\eta+3} \),
   - (c) for all \( R \subseteq \mathcal{E}(V_{\eta}) \times V_{\eta+2} \), there is a \( \mu \)-uniformization of \( R \).
3. If \( \ell : V_{\eta+3} \to V_{\eta+3} \) is a \( \Sigma_0 \)-elementary extension of \( j \) then (identifying \( \bar{U} \) with its transitive collapse):
   - (a) \( V_{\eta+2} \subseteq \bar{U} \subseteq V_{\eta+3} \), and in fact \( \mu \notin \bar{U} \),
   - (b) \( \ell = \tilde{i}_{\mu} \).

Definition 4.7. Let \( R \subseteq \mathcal{E}(V_{\eta}) \times V \) be a relation. A \( \mu \)-uniformization of \( R \) is a function \( f : \mathcal{E}(V_{\eta}) \to V \) such that for \( \mu \)-measure one many \( k \in \mathcal{E}(V_{\eta}) \), we have
\[
\text{if there is } x \text{ such that } (k, x) \in R \text{ then } (k, f(k)) \in R.
\]

Having analyzed \( j \) as an ultrapower map, we now consider extending \( j \) to \( V_{\eta+3} \). Recall \( \bar{U} = \text{Ult}(V_{\eta+3}, \mu) \) and \( \tilde{i}_{\mu} = i_{\mu}^{V_{\eta+3}} \).
Proof. Part 1: By Claim 1, the part of the ultrapower formed by functions with codomain $V_{\eta+2}$ is isomorphic to $V_{\eta+2}$. It follows that $\hat{U}$ is wellfounded.

Part 2: Suppose $\ell : V_{\eta+3} \to V_{\eta+3}$ is $\Sigma_0$-elementary and $j = \ell \restriction V_{\eta+2}$. Let us first observe that $\ell(V_{\eta+2}) = V_{\eta+2}$. Since $\ell(V_{\eta+2}) \subseteq V_{\eta+3}$, we have $\ell(V_{\eta+2}) \subseteq V_{\eta+2}$, so we just need $V_{\eta+2} \subseteq \ell(V_{\eta+2})$. So let $x \in V_{\eta+2}$. Then with $f_x$ defined as before, we have $x = j(f_x)(j \restriction V_\eta)$. But

\[ V_{\eta+3} \models \text{"}f_x(k) \in V_{\eta+2}\text{ for all } k \in \mathcal{E}(V_\eta)\text{"}, \]

which is a $\Sigma_0$ statement of the parameters $f_x, V_{\eta+2}, \mathcal{E}(V_\eta)$, and therefore

\[ V_{\eta+3} \models \text{"}\ell(f_x)(k) \in \ell(V_{\eta+2})\text{ for all } k \in \ell(\mathcal{E}(V_\eta))\text{"}, \]

but $\ell(f_x) = j(f_x)$ and $\ell(\mathcal{E}(V_\eta)) = j(\mathcal{E}(V_\eta)) = \mathcal{E}(V_\eta)$, so

\[ x = j(f_x)(j \restriction V_\eta) \in \ell(V_{\eta+2}), \]

as desired.

We next show that $i_\mu = \ell$. For we know $i_\mu = j$ already, so consider $X \in V_{\eta+3}\setminus V_{\eta+2}$, so $X \subseteq V_{\eta+2}$. Let $x \in V_{\eta+2}$. Let

\[ D = \{ k \in \mathcal{E}(V_\eta) \mid f_x(k) \in X \}. \]

Then $x \in i_\mu(X)$ iff $D \in \mu$ iff $j \restriction V_\eta \in j(D) = \ell(D)$ iff (by $\Sigma_0$-elementarity) $\ell(f_x)(j \restriction V_\eta) \in \ell(X)$ iff $x = j(f_x)(j \restriction V_\eta) \in \ell(X)$.

Now let us deduce that (c) holds. So let $R \subseteq \mathcal{E}(V_\eta) \times V_{\eta+2}$ and let $D$ be the domain of $R$; that is,

\[ D = \{ k \in \mathcal{E}(V_\eta) \mid \exists x \ [(k, x) \in R] \}. \]

We may assume $D \in \mu$, so $j \restriction V_\eta \in j(D)$. Now $R \in V_{\eta+3}$ and

\[ V_{\eta+3} \models \forall k \in D \exists x \in V_{\eta+2} \ [(k, x) \in R]. \]

So by $\Sigma_0$-elementarity and since $\ell(V_{\eta+2}) = V_{\eta+2}$,

\[ V_{\eta+3} \models \forall k \in \ell(D) \exists x \in V_{\eta+2} \ [(k, x) \in \ell(R)], \]

and since $D \in \mu$, therefore we can fix $x \in V_{\eta+2}$ such that $(j \restriction V_\eta, x) \in R$. We have the function $f_x : \mathcal{E}(V_\eta) \to V_{\eta+2}$, with $f_x(k) = (k')^{-1}x$. We claim that $f_x$ a $\mu$-uniformization of $R$. For the complement $\bar{R} = \mathcal{E}(V_\eta) \times V_{\eta+2} \setminus \ell(R)$ is also in $V_{\eta+3}$, so let

\[ C = \{ k \in \mathcal{E}(V_\eta) \mid (k, f_x(k)) \in \bar{R} \}. \]

Suppose $C \in \mu$. Then $j \restriction V_\eta \in j(C) = \ell(C)$, and by $\Sigma_0$-elementarity of $\ell$, therefore $(j \restriction V_\eta, \ell(f_x)(j \restriction V_\eta)) \in \ell(R)$, and $\ell(R) = V_{\eta+2} \setminus \ell(R)$. So $(j \restriction V_\eta, x) \notin R$, contradiction.

Now assume (c) holds (the $\mu$-uniformization); we will show that $\hat{U}$ is extensional and Los’ theorem holds for $\Sigma_0$ formulas, which implies that

\[ \tilde{i}_\mu : V_{\eta+3} \to \hat{U} \subseteq V_{\eta+3} \]

More precisely, $f_x$ is coded by some $f' \in V_{\eta+2}$, and $\mathcal{E}(V_\eta)$ coded by some $\mathcal{E}' \in V_{\eta+1}$, and it is a $\Sigma_0$ statement of parameters $f', V_{\eta+2}, \mathcal{E}'$ to assert the displayed statement about $f_x, V_{\eta+2}, \mathcal{E}(V_\eta)$.
is $\Sigma_0$-elementary, and therefore in fact $\tilde{i}_\mu : V_{\eta+3} \to V_{\eta+3}$ is $\Sigma_0$-elementary.

For extensionality, let $f, g : {\mathcal{E}}(V_\eta) \to V_{\eta+3}$ be such that $[f] \neq [g]$; that is,

$$D = \{ k \in {\mathcal{E}}(V_\eta) \mid f(k) \neq g(k) \} \in \mu.$$  

Then define the relation

$$R = \{(k,x) \in {\mathcal{E}}(V_\eta) \times V_{\eta+2} \mid x \in f(k) \text{ iff } x \notin g(k)\}.$$  

Note that for all $k \in D$, since $f(k), g(k) \subseteq V_{\eta+2}$ and $f(k) \neq g(k)$, there is $x$ with $(k,x) \in R$. So we can $\mu$-uniformize $R$ with some $h : {\mathcal{E}}(V_\eta) \to V_{\eta+2}$. Since $\mu$ is an ultrafilter, either (i) for $\mu$-measure one many $k$, we have $h(k) \in f(k) \setminus g(k)$, or (ii) vice versa. Suppose (i) holds. Then $[h] \in [f]$ and $[h] \notin [g]$, verifying extensionality for $[f],[g]$.

It follows now that $\tilde{U}$ is isomorphic to some subset of $V_{\eta+3}$ (and we already know $V_{\eta+2} \subseteq \tilde{U}$). Now observe that the assumed $\mu$-uniformization is enough for essentially the usual proof of Los’ theorem to go through, but just with respect to $\Sigma_0$ formulas (with parameters given by functions $f : {\mathcal{E}}(V_\eta) \to V_{\eta+3}$): that is, for example with one such function $f$ and $\varphi$ a $\Sigma_0$ formula, we have

$$\tilde{U} \models \varphi([f]) \iff \{ k \in {\mathcal{E}}(V_\eta) \mid V_{\eta+3} \models \varphi(f(k)) \} \in \mu.$$  

It follows as usual that $\tilde{i}_\eta$ is $\Sigma_0$-elementary as a map $V_{\eta+3} \to \tilde{U}$, and hence as a map $V_{\eta+3} \to V_{\eta+3}$, as desired.

Finally suppose that $\mu$-uniformization as in (c) fails; we will show that $\tilde{U}$ is not extensional. Let $R \subseteq {\mathcal{E}}(V_\eta) \times V_{\eta+2}$ be a counterexample to $\mu$-uniformization. We have the constant function $c_\eta$. Define $f : {\mathcal{E}}(V_\eta) \to V_{\eta+3}$ by

$$f(k) = \{ x \mid (k,x) \in R \}.$$  

Note that $f(k) \neq \emptyset$ for almost all $k$ ($R$ is trivially uniformizable otherwise). So there is no $g$ such that $[g] \in [f]$, and therefore $\tilde{U}$ is non-extensional with respect to $[f],[c_\eta]$.

Part 3: We already saw these things in the proof of part 2.

**Definition 4.8** (Canonical extension via ultrapowers). Let $j : V_{\eta+2} \to V_{\eta+2}$ be elementary. Then $j^+ : V_{\eta+3} \to V_{\eta+3}$ is defined $j^+ = \tilde{i}_\mu$, defined as above.

We can now reprove Theorem 1.1, by induction, using the extension $j^+$ just defined. The argument is essentially as before. Let $\lambda$ be a limit and $j : V_{\lambda+2} \to V_{\lambda+2}$ be elementary. Let $\mu$ be the measure derived from $j$ with seed $j \upharpoonright V_\lambda$. By Claim 1, $V_{\lambda+2} = \text{Ult}(V_{\lambda+2}, \mu)$. Supposing there is $p \in V_{\lambda+2}$ such that $j$ is definable from $p$ over $V_{\lambda+2}$, we have $j = j^+\mid V_{\lambda+2}$ and

$$p = [f_p]^{V_{\lambda+2}} = j(f_p)(j \upharpoonright V_\lambda),$$  

so $p \in \text{rg}(j(f_p))$. One now argues as in the proof of Theorem 3.19 to reach a contradiction.

Next, if $\ell : V_{\lambda+3} \to V_{\lambda+3}$ is elementary and $j = \ell \upharpoonright V_{\lambda+2}$, we know that $\ell = j^+$ by the claims above. But $\mu \in V_{\lambda+3}$, and it is straightforward to see that the ultrapower map $j^+ = \tilde{i}_\mu^{V_{\lambda+3}}$ is definable over $V_{\lambda+3}$ from the parameter $\mu$, or equivalently, from the parameter $j$. So $\ell$ is definable as desired.

Now suppose $j : V_{\lambda+4} \to V_{\lambda+4}$ is elementary. Let $\mu$ be the measure derived from $k$ with seed $j \upharpoonright V_{\lambda+2}$. Then since $j \upharpoonright V_{\lambda+3} = (j \upharpoonright V_{\lambda+2})^+$, the claims give that $\text{Ult}(V_{\lambda+4}, \mu) = V_{\lambda+4}$ and $j$ is the ultrapower map, so like before, we get that $j$ is not definable from parameters. Etc.
4.3 Reinhardt ultrafilters

One can abstract out a notion of filter which corresponds precisely to elementary embeddings \( j : V_{\eta+2} \rightarrow V_{\eta+2} \) for even \( \eta \). In the previous section, the ultrafilters measured sets of embeddings \( k : V_\eta \rightarrow V_\eta \). But such a \( k \) is equivalent to its range, and in the following definition we measure sets of ranges of embeddings instead of sets of embeddings themselves.

**Definition 4.9 (Reinhardt ultrafilter).** Let \( \eta \) be even and \( \mu \) be an ultrafilter over \( V_{\eta+1} \), so the measure one sets are families of subsets of \( V_\lambda \). We say that \( \mu \) is:

1. **rank-Jónsson** iff \( \sigma = \{ A : A \subseteq V_\eta \text{ and the transitive collapse of } A \text{ is } V_\eta \} \in \mu \).
2. **fine** iff for each \( x \in V_\eta \), \( \sigma = \{ A \subseteq V_\eta : x \in A \} \in \mu \),
3. **normal** iff for each \( \langle \sigma_x \rangle_x \subseteq \mu \), the diagonal intersection \( \Delta_{x \in V_\eta} \sigma_x = \{ A \subseteq V_\eta : A \in \sigma_x \text{ for each } x \in A \} \in \mu \),
4. **pre-Reinhardt** iff rank-Jónsson, fine and normal,
5. **Reinhardt** iff pre-Reinhardt and every relation \( R \subseteq V_{\eta+1} \times V_{\eta+2} \) such that \( R \) is definable over \( V_{\eta+2} \) from parameters, can be \( \mu \)-uniformized.

**Theorem 4.10.** We have:

1. Let \( \eta \) be even and \( j : V_{\eta+2} \rightarrow V_{\eta+2} \) be elementary. Let \( \mu \) be the measure over \( V_{\eta+1} \) derived from \( j \) with seed \( j^* V_\eta \). Then \( \mu \) is Reinhardt.
2. Let \( \mu \) be a Reinhardt ultrafilter over \( V_{\eta+1} \). Let \( U = \text{Ult}(V_{\eta+2}, \mu) \). Then the ultrapower is extensional and wellfounded, and identifying it with its transitive collapse, we have \( U = V_{\eta+2} \) and \( j_{\mu}^{V_{\eta+2}} : V_{\eta+2} \rightarrow V_{\eta+2} \) is fully elementary.

**Proof.** To be added.

\[ \square \]

5 Σ₁-elementarity at limit rank-into-rank

It is natural to ask whether we can prove a version of Theorem 1.1 when we assume less than full elementarity of the maps. In this case, we can answer the question completely for the limit case, but not in the successor case. It is easy to see that if we only demand Σ₀-elementarity, then the embedding can easily be definable from parameters:
Example 5.1. Assume ZFC, let $\mu$ be a normal measure and $j : V \rightarrow \text{Ult}(V, \mu)$ be the ultrapower map, and identify $\text{Ult}(V, \mu)$ with transitive $M \subseteq V$. Then note that in fact, $j : V \rightarrow V$ is $\Sigma_0$-elementary and definable from the parameter $\mu$. (Therefore, if $\mu$ is the unique normal measure on $\text{crit}(j)$ then $j$ is actually definable without parameters.)

We will now consider the case that $\delta$ is a limit and $j : V_\delta \rightarrow V_\delta$ is just $\Sigma_1$-elementary. Here we need some more standard set theoretic notions, but expressed appropriately for the ZF context.

Definition 5.2. An ordinal $\kappa$ is inaccessible if whenever $\alpha < \kappa$ and $\pi : V_\alpha \rightarrow \kappa$, then $\text{rg}(\pi)$ is bounded in $\kappa$.

The cofinality $\text{cof}(\kappa)$ of $\kappa$ is the least $\eta \in \text{OR}$ such that there is a map $\pi : \eta \rightarrow \kappa$ with $\text{rg}(\pi)$ unbounded in $\kappa$.

We say $\kappa$ is regular iff $\text{cof}(\kappa) = \kappa$.

A norm on a set $X$ is a function $\pi : X \rightarrow \text{OR}$. The associated prewellorder on $X$ is the relation $R$ on $X$ given by

$$xRy \iff \pi(x) \leq \pi(y).$$

One can also axiomatize prewellorders on $X$ as those relations $R$ on $X$ which are linear, total, reflexive, with wellfounded strict part (the strict part $\prec R$ is the relation defined $[xRy \wedge \neg yRx]$).

If $\kappa$ is regular but not inaccessible, and $\alpha$ is least such that there is a cofinal map $\pi : V_\alpha \rightarrow \kappa$, then the Scott ordertype of $\kappa$, denoted $\text{scot}(\kappa)$, is the set of all prewellorders of $V_\alpha$ whose ordertype is $\kappa$.

Remark 5.3. Suppose $\kappa$ is regular but not inaccessible, and let $\alpha$ be as above and $\pi : V_\alpha \rightarrow \kappa$ be cofinal. Then $\text{rg}(\pi)$ has ordertype $\kappa$, as otherwise $\kappa$ is singular. Moreover, $\alpha$ is a successor ordinal. For otherwise, by the minimality of $\alpha$, we get a cofinal function $f : \alpha \rightarrow \kappa$ by defining $f(\beta) = \text{sup}(\pi^{-1}V_\delta)$ for $\beta < \alpha$, again contradicting regularity.

Definition 5.4. Let $\delta$ be a limit and $j : V_\delta \rightarrow V_\delta$ be $\Sigma_1$-elementary. For $A \subseteq V_\delta$, define $j^+(A)$ just as in Definition 3.1. Define $j^1 = j$ and for $n \geq 1$ define $j^{n+1} = j^+(j^n)$. For $x \in V_\delta$, we say that $x$ is $(j, n)$-stable iff $j^m(x) = x$ for all $m \in [n, \omega)$.

Before we state the next theorem, we state an immediate corollary:

Corollary 5.5. Let $\delta \in \text{OR}$ be a limit and $j : V_\delta \rightarrow V_\delta$ be $\Sigma_1$-elementary. Suppose that either $V_\delta$ is inaccessible, or $\delta$ is singular and $j(\text{cof}(\delta)) = \text{cof}(\delta)$, or $\delta$ is regular non-inaccessible and $j(\text{scot}(\delta)) = \text{scot}(\delta)$. Then $j$ is fully elementary. In fact, for every $A \subseteq V_\delta$,

$$j : (V_\delta, A) \rightarrow (V_\delta, j^+(A))$$

is fully elementary (with respect to the language $\mathcal{L}_A$, with $\in$ and a predicate symbol $A$ interpreting $A, j^+(A)$).

19This is an abbreviation of Scott ordertype. The second author thank Asaf Karagila for suggesting this terminology.

23
Theorem 5.6 (An iterate is elementary). Let $\delta \in \text{OR}$ be a limit and $j : V_\delta \to V_\delta$ be $\Sigma_1$-elementary.²⁰ Then:

1. Every $j^n : V_\delta \to V_\delta$ is $\Sigma_1$-elementary; in fact, for each $A \subseteq V_\delta$,
   \[ j^n : (V_\delta, A) \to (V_\delta, j^{n+}(A)) \]

   is $\Sigma_1$-elementary in the language $\mathcal{L}_A$.

2. $j^{n+1} = j^{n+}(j^n)$.

3. If $x \in V_\delta$ and $j^n(x) = x$ then $x$ is $(j, n)$-stable.

4. For each $\alpha < \delta$ there is $n < \omega$ such that $\alpha$ is $(j, n)$-stable.

5. For each $\alpha < \delta$ and $\xi \in \text{OR}$, letting $P$ be the set of all preworders of $V_\alpha$ of length $\xi$, there is $n < \omega$ such that $P$ is $(j, n)$-stable. In particular, if $\delta$ is regular non-inaccessible then there is $n$ with $\text{scot}(\delta)$ being $(j, n)$-stable.

6. Suppose that either:
   - $\delta$ is inaccessible, or
   - $\delta$ is regular non-inaccessible and $\text{scot}(\delta)$ is $(j, n)$-stable, or
   - $\delta$ is singular and $\text{cof}(\delta)$ is $(j, n)$-stable.

Then $j^n : V_\delta \to V_\delta$ is fully elementary, and in fact, for each, $A \subseteq V_\delta$,
\[ j^n : (V_\delta, A) \to (V_\delta, j^{n+}(A)) \]

is fully elementary.

Proof. Part 1: Let $\alpha < \delta$ and $\alpha' = j(\alpha)$ and $j' = j \upharpoonright V_\alpha$. So $j' : V_\alpha \to V_{\alpha'}$ is fully elementary. This fact is preserved by $j$, by $\Sigma_1$-elementarity. Clearly also $j^+(j) : V_\delta \to V_\delta$, and is therefore $\Sigma_0$-elementary with respect to these models. But $j^+(j)$ is also $\mathcal{E}$-cofinal, hence $\Sigma_1$-elementary (with respect to $\mathcal{E}$).

For the $\Sigma_1$-elementarity of $j^n : (V_\delta, A) \to (V_\delta, j^{n+}(A))$, let $x \in V_\delta$ and $\varphi$ be $\Sigma_0$ (in the expanded language), and suppose
\[ (V_\delta, j^{n+}(A)) \models \exists y \varphi(j^n(x), y). \]

Let $\alpha < \delta$ be sufficiently large that $x \in V_\alpha$ and
\[ (V_{j^n(\alpha)}, j^{n+}(A) \cap V_{j^n(\alpha)}) \models \exists y \varphi(j^n(x), y). \]

Then by the $\Sigma_1$-elementarity of $j^n$ (just in the language with $\mathcal{E}$),
\[ (V_\alpha, A \cap V_\alpha) \models \exists y \varphi(x, y), \]

so $(V_\delta, A) \models \exists y \varphi(x, y)$ as desired.

Part 2: For $n = 1$ this is just the definition. For $n = 2$ note that:
\[ j^3 = j^+(j^2) = j^+(j^+(j)) = (j^+(j))^+(j^+(j)) = (j^2)^+(j^2). \]

The rest is similar.

²⁰Recall that by Lemma 2.2, $j(V_\alpha) = V_{j(\alpha)}$ for each $\alpha < \delta$. 
Part 3: If \( x = j(x) \) then \( j(x) = j(j(x)) = j^+(j)(j(x)) = j^+(j)(x) \).

Part 4: Suppose not and let \( \alpha < \delta \) be least otherwise. We use the argument in [15] (which is just a slight variant on the standard proof of linear iterability). For \( n \in [1, \omega) \) let
\[
A_n = \{ \beta < \alpha : j^n(\beta) = \beta \}.
\]
So \( \alpha = \bigcup_{n < \omega} A_n \) and \((A_n)_{n < \omega} \in V_\delta \). By \( \Sigma_1 \)-elementarity, note
\[
j(A_n) = \{ \beta < j(\alpha) : j^{n+1}(\beta) = \beta \}.
\]
and
\[
j(\alpha) = j \left( \bigcup_{n < \omega} A_n \right) = \bigcup_{n < \omega} j(A_n).
\]
But \( \alpha < j(\alpha) \) by choice of \( \alpha \) and part 3, so there is \( n \) such that \( \alpha \in j(A_n) \). But then \( j^{n+1}(\alpha) = \alpha \), contradiction.

Part 5: By the above, there is \( n_0 \) such that \( \alpha \) is \((j, n_0)\)-stable. Now argue as in the previous part from \( n_0 \) onward, and using the parameter \( \alpha \), define the collection \( P \) of prewellorders of \( V_\alpha \) of the form \( P = P_\xi \) for some ordinal \( \xi \), with \( \xi \) least such that for no \( n \in [n_0, \omega) \) is \( j_n(P) = \bar{P} \). Here \( \xi \geq \delta \) is possible. Note that the notion of prewellorder (regarding relations \( R \in V_\delta \)) is simple enough that it is preserved by our \( \Sigma_1 \)-elementary maps. Likewise, the lengths of 2 prewellorders can be compared in a simple enough fashion, and hence we always have \( j_n(P_\xi) = P_\xi' \) with some \( \xi'_n \). In fact, we get \( \xi'_n > \xi \), since \( j_n(P_\xi) = P_\xi' \) for \( \xi < \xi' \). One can now argue for a contradiction much as before.

Part 6: For this proof we just write \( j(A) \) instead of \( j^+(A) \) for \( A \subseteq V_\delta \); note this is unambiguous in the case that \( A \in V_\delta \).

If \( \delta \) is inaccessible then for every \( A \subseteq V_\delta \), \( (V_\delta, A) \models \text{ZF(A)} \).21 By part 1, \( j : (V_\delta, A) \to (V_\delta, j(A)) \) is \( \Sigma_1 \)-elementary. Therefore a direct relativization of Fact 2.3 shows that \( j \) is fully elementary in the expanded language.

Now consider the case that \( \delta \) is singular and let \( \gamma = \text{cof}(\delta) \). By renaming, we may assume \( j(\gamma) = \gamma \). Let \( A \subseteq V_\delta \). We know \( j : (V_\delta, A) \to (V_\delta, j(A)) \) is \( \Sigma_1 \)-elementary, and must show it is fully elementary.

We begin with \( \Sigma_2 \)-elementarity. Let \( x \in V_\delta \) and \( \varphi \) be \( \Pi_1 \) and suppose that
\[
(V_\delta, j(A)) \models \exists y \varphi(j(x), y),
\]
and let \( \beta < \delta \) be such that some \( y \in V_{j(\beta)} \) witnesses this.

Suppose first that \( \gamma < \delta \); so we are assuming \( j(\gamma) = \gamma \). Let \( f : \gamma \to \delta \) be cofinal and increasing. For \( \xi < \gamma \) let
\[
B_\xi = \{ z \in V_\beta : (V_{f(\xi)}, A \cap V_{f(\xi)}) \models \varphi(x, z) \}.
\]
Then note that
\[
j(B_\xi) = \{ z \in V_{j(\beta)} : (V_{j(f(\xi))}, j(A) \cap V_{j(f(\xi))}) \models \varphi(j(x), z) \}.
\]
Therefore \( y \in j(B_\xi) \), so in fact
\[
y \in \left( \bigcap_{\xi < \gamma} j(B_\xi) \right) \neq \emptyset.
\]

21That is, \( \text{ZF} \) augmented with Collection and Separation for formulas in the language with \( \in \) and \( A \), and \( A \) interprets \( A \).
As \( \gamma < \delta \), we have \( (B_\xi)_{\xi<\gamma} \in V_\delta \). Also,
\[
\xi_0 < \xi_1 \implies B_{\xi_1} \subseteq B_{\xi_0}.
\]
So the same holds of \( j((B_\xi)_{\xi<\gamma}) \), and since \( j(\gamma) = \gamma \), we have \( j^{-1}(\gamma) \) cofinal in \( j(\gamma) \), and so letting \( j((B_\xi)_{\xi<\gamma}) = \bigcup B_\xi' \) where
\[
\bigcup B_\xi' = \bigcap \bigcup B_j(\xi) = \bigcap j(B_\xi) \neq \emptyset.
\]
So \( \bigcap_{\xi<\gamma} B_\xi \neq \emptyset \). But letting \( z \in \bigcap_{\xi<\gamma} B_\xi \), note that
\[
(V_\delta, A) \models \varphi(x, z),
\]
as desired.

Now suppose instead that \( \delta \) is regular non-inaccessible. Define \( (B_\xi)_{\xi<\delta} \) as before, except that now \( f(\xi) = \xi \) for \( \xi < \delta \). If there is \( \xi_0 < \delta \) such that \( B_\xi = B_{\xi_0} \) for all \( \xi \in (\xi_0, \delta) \), then we easily have that \( B_{\xi_0} \neq \emptyset \), and any \( z \in B_{\xi_0} \) witnesses \( \exists y \varphi(x, y) \) as before. Now suppose there is no such \( \xi_0 \). Given \( z_0, z_1 \in B = \bigcup_{\xi<\delta} B_\xi \), say that \( z_0 \prec^* z_1 \) iff there is \( \xi < \delta \) such that \( z_1 \in B_\xi \) but \( z_0 \notin B_\xi \). Then \( \prec^* \) is a prewellorder on \( B \), and \( \prec^* \) is in \( V_\delta \), and because \( \gamma = \delta \) and there is no \( \xi_0 \) as above, \( \delta \) is the the ordertype of \( \prec^* \). So let \( P = \text{scot}(\delta) \), so by assumption \( j(P) = P \), which easily gives that \( j(\prec^*) \) also has ordertype \( \delta \). The function \( z \mapsto B_{\text{rank}^*(z)} \), with domain \( B \), and where \( \text{rank}^*(z) \) is the \( \prec^* \)-rank of \( z \), is also in \( V_\delta \). But then we can argue as before to show \( \bigcap_{\xi<\delta} B_\xi \neq \emptyset \), which suffices, also as before.

So we have \( \Sigma_2 \)-elementarity (with respect to an arbitrary \( A \subseteq V_\delta \)). Now suppose we have \( \Sigma_k \)-elementarity where \( k \geq 2 \). Define the theory
\[
T = T_{k-1}^A = \text{Th}_{\Sigma_{k-1}}^{(V_\delta, A)}(V_\delta);
\]
this denotes the theory consisting of all pairs \( (\varphi, x) \) such that \( \varphi \) is a \( \Sigma_{k-1} \) formula and \( (V_\delta, A) \models \varphi(x) \). The \( \Sigma_k \)-elementarity of \( j \) gives:

**Claim 3.** \( j(T) = \text{Th}_{\Sigma_{k-1}}^{(V_\delta, j(A))}(V_\delta) \).

**Proof.** Given \( \alpha < \delta \), we have
\[
(V_\delta, A) \models \forall x \in V_\alpha \left( \forall \Sigma_{k-1} \text{ formulas } \varphi \text{ of } L_A \left[ \varphi(x) \iff (\varphi, x) \in T \cap V_\alpha \right] \right).
\]
which is a \( \Pi_k \) assertion of parameter \( (V_\alpha, T \cap V_\alpha) \), which therefore lifts to \( (V_\delta, j(A)) \) regarding the parameter \( (V_j(\alpha), j(T) \cap V_j(\alpha)) \).

So by what we have proved above, but with \( (A, T) \) replacing \( A \), we have that \( j \) is \( \Sigma_2 \)-elementary as a map
\[
j : (V_\delta, (A, T)) \to (V_\delta, (j(A), j(T))).
\]
Now let \( \varphi \) be \( \Sigma_{k-1} \) and suppose that
\[
(V_\delta, j(A)) \models \exists y \forall z \left[ \varphi(j(x), y, z) \right];
\]

equivalently,

\[(V_\delta, (j(A), j(T))) \models \exists y \forall z \ [(\varphi, (j(x), y, z)) \in j(T)].\]

By the $\Sigma_2$-elementarity of $j$ with respect to the structures in line (2) above, therefore

\[(V_\delta, (A, T)) \models \exists y \forall z \ [(\varphi, (x, y, z)) \in T];\]
equivalently, $(V_\delta, A) \models \exists y \forall z \ [\varphi(x, y, z)]$, as desired. \qed

Using the preceding theorem, we can easily improve on Theorem 3.4:

**Theorem 5.7.** Let $\delta$ be a limit ordinal. Then there is no $j : V_\delta \to V_\delta$ which is $\Sigma_1$-elementary and definable from parameters over $V_\delta$.

**Proof.** Suppose otherwise. Then by Theorem 5.6, there is $n < \omega$ such that $j^n : V_\delta \to V_\delta$ is fully elementary, and since $j$ is definable from parameters over $V_\delta$, so is $j^n$. This contradicts Theorem 3.4. \qed

The following theorem, due to Andreas Lietz and the second author, shows that if a Reinhardt cardinal exists then it is at times necessary to pass from $j$ to $j^n$ to secure full elementarity:

**Theorem 5.8** (Lietz, S.). Suppose $j : V_{\lambda^+} \to V_{\lambda^+}$ is elementary where $\lambda = \kappa \omega(j)$. Then for each $n < \omega$ there is a limit $\delta < \lambda^+$ such that $j^n \delta \subseteq \delta$,

\[j \upharpoonright V_\delta : V_\delta \to V_\delta\]
is $\Sigma_1$-elementary, but $j = j^1, j^2, \ldots, j^n$ are not $\Sigma_2$-elementary.

**Proof.** First consider $n = 1$. Let $\kappa = \text{crit}(j)$ and $\delta = \lambda + \kappa$ and $k = j \upharpoonright V_\delta$. Since $j(\lambda) = \lambda$ and $j \upharpoonright \kappa = \text{id}$, we have $k : V_\delta \to V_\delta$, and clearly $k$ is $\in$-cofinal and $\Sigma_0$-elementary, hence $\Sigma_1$-elementary. But consider the $\Pi_2$ formula

\[\varphi(\kappa, \lambda) = \exists \alpha < \kappa \ \exists \xi \in \text{OR} \ [\xi = \lambda + \alpha].\]

Then $V_{\lambda^+} \models \varphi(\kappa, \lambda)$, but $V_{\lambda^+} \models \neg \varphi(j(\kappa), j(\lambda))$; that is, $V_{\lambda^+} \models \neg \varphi(j(\kappa), \lambda)$, since $\alpha \equiv \kappa < j(\kappa)$, but $\lambda + \kappa \not\in V_{\lambda^+}$. For this example, $k^2(\kappa) = \kappa = \text{cof}(\lambda + \kappa)$, so $k^2$ is fully elementary, by Theorem 5.6.

Now let $n$ be arbitrary.

**Claim.** $j$ has $\lambda^+$-many fixed points $< \lambda^+$.

**Proof.** By Theorem 5.6, for each $\alpha < \lambda^+$, there is $n$ such that $j^n(\alpha) = \alpha$. Let $F_n$ be the set of $\alpha < \lambda^+$ such that $j^n(\alpha) = \alpha$. Then the ordertypes of the $F_n$ are unbounded in $\lambda^+$ (or one of them has ordertype $\lambda^+$), since otherwise one easily constructs a surjection $\pi : \lambda \to \lambda^+$ (consider the uncollapse maps $\pi_n : \alpha_n \to F_n$ where $\alpha_n \in \text{OR}$ is the ordertype of $F_n$). Now $F_1$ is unbounded in $\lambda^+$. For suppose not, and let $\alpha_1 < \lambda^+$ be strictly higher than some bound $\beta$. Let $\pi_1 : \lambda \to \alpha_1$ be a surjection. Let $\pi_{n+1} = j(\pi_n)$ and $\alpha_{n+1} = \text{rg}(\pi_{n+1})$, so $j(\alpha_n) = \alpha_{n+1}$. Then note that from $\langle \pi_n \rangle_{n < \omega}$ we get a surjection $\lambda \to \beta = \sup_{n < \omega} \alpha_n$.\footnote{The second author initially noticed the $n = 1$ example, then Lietz generalized this to $n > 1$ via basically the method at the end of the proof, but from a stronger assumption to secure fixed points, and then the second author observed the claim on fixed points, leading to the version here.}
Therefore $\beta < \lambda^+$, but note $\text{cof}(\beta) = \omega$, so $j(\beta) = \beta$, a contradiction to the choice of $\alpha_1 \leq \beta$. Now we claim that $F_1$ has ordertype $\lambda^+$. For suppose not and let $\beta_1$ be its ordertype, and $\beta_{n+1}$ be the ordertype of $F_{n+1}$. Then note that $\beta_{n+1} = \sup j^* \beta_n = \sup j^n \beta_n$ (using that $F_n$ is cofinal in $\lambda^+$ now). Then letting $\eta < \lambda^+$ be a fixed point with $\beta_1 < \eta$, note $\beta_n < \eta$ for all $n$, a contradiction.

Now let $\delta$ be the supremum of the first $\kappa_n(j)$ fixed points of $j$ which are $> \lambda$. Then $j^* \delta \subseteq \delta$, so $k = j \upharpoonright \eta$ is again $\Sigma_1$-elementary. Let $P$ be the set of all prewellorders of $\lambda$ in ordertype $\delta$ (note $P \neq \emptyset$ as $\lambda < \delta < \lambda^+$). Then $V_\delta \models \forall W \in P$ for all proper segments $w$ of $W$.

there is $\alpha \in \text{OR}$ which is isomorphic to $w$.

But for $m < n$, $j^m(W)$ is the set of all prewellorders of $j^m(\lambda) = \lambda$ in ordertype some $\delta'_m$, and $\delta < \delta'_m$, because $\text{cof}(W) = \kappa_n(j)$ for each $W \in P$, a fact seen by $V_\delta$, so $\text{cof}(W) = \kappa_{n+1}(j)$ for each $W \in j^m(P)$. (And we can’t have $\delta > \delta'_m$, because $j$ embeds the prewellorder $w$ into the prewellorder $j(w)$. Therefore $V_\delta$ does not satisfy the statement above about $j^m(P)$, so $j^m$ is not $\Sigma_2$-elementary.

\section{Which ordinals are large enough?}

We said in the introduction that if an ordinal $\eta$ is large enough, then $V_{\eta+183}$ and $V_{\eta+184}$ are very different from each other. Of course, we have seen that there are such differences assuming there is an elementary $j : V_{\eta+184} \rightarrow V_{\eta+184}$. So we could take this as the definition of “large enough”, but then the term is not very natural, because then it needn’t be that $\eta + 1$ is also “large enough”. To get a good notion of “large enough”, we assume that there is a Reinhardt cardinal. Let then $j : V \rightarrow V$ be elementary with $\kappa_\omega(j)$ minimal. Then we say that $\eta$ is “large enough” iff $\eta \geq \kappa_\omega(j)$. Below, $\text{ZF}(j)$ denotes the Zermelo Fräenkel axioms in the language $L_j$ with symbols $\in, j$, augmented with Collection and Separation for all formulas in $L_j$.

Under this theory, we can assert that “$j : V \rightarrow V$ is elementary” with the single formula “$j : V \rightarrow V$ is $\Sigma_1$-elementary”, by Fact 2.3. The following theorem was mentioned to the first author by Koellner a few years ago, but may be folklore. There are some further related things in [14]:

\textbf{Theorem 6.1} (Folklore?). \textit{Assume ZF(j) and $j : V \rightarrow V$ is elementary (non-identity). Let $\lambda = \kappa_\omega(j)$. Then for all $\alpha \geq \lambda$ and all $\eta < \lambda$, there is an elementary $k : V_\alpha \rightarrow V_\alpha$ such that $\text{crit}(k) > \eta$ and $\kappa_\omega(k) = \lambda$.}

\textbf{Proof.} Suppose not and let $(\eta, \alpha)$ be the lexicographically least counterexample. Then $(\eta, \alpha)$ is definable from the parameter $\lambda$, and hence fixed by $j$. But then $j(\alpha) = \alpha$, so $j \upharpoonright V_\alpha : V_\alpha \rightarrow V_\alpha$, and $j(\eta) = \eta < \lambda$, so $\eta < \text{crit}(j) = \text{crit}(j \upharpoonright V_\alpha)$, so $j \upharpoonright V_\alpha$ contradicts the choice of $(\eta, \alpha)$.

So above $\lambda = \kappa_\omega(j)$, the cumulative hierarchy is periodic the whole way up.

\textbf{Remark 6.2.} For the reader familiar with [3], note that the property stated of $\lambda = \kappa_\omega(j)$ in the theorem above is just that of a Berkeley cardinal (see [3]) with
\footnote{We don’t know that $\lambda^+$ is regular; the first author has results in regard to this.}
respect to rank segments of $V$ (except that we have also stated it for $V_\lambda$ itself, although $\lambda \notin V_\lambda$). One could call such a $\lambda$ a rank-Berkeley cardinal. Note that unlike Reinhardtness, rank-Berkeleyness is first-order. If there is a Reinhardt, then which is less, the least Reinhardt or the least rank-Berkeley? If $j : V \rightarrow V$ and $\lambda = \kappa_j(j)$ is the least rank-Berkeley, then note that for every $k : V \rightarrow V$ with $\text{crit}(k) < \lambda$, we have $\lambda_{\omega,k} = \lambda$. In particular, if $\kappa$ is super Reinhardt then the least rank-Berkeley is $< \kappa$. We show next that the least rank-Berkeley being below the least Reinhardt, has consistency strength beyond that of a Reinhardt.

We remark that arguing further as above shows that every rank-Berkeley is HOD-Berkeley. Can be/is the least HOD-Berkeley $< $ the least rank-Berkeley?

**Theorem 6.3.** Suppose $(V,j) \models \text{ZF}(j)$ and $j : V \rightarrow V$, and let $\kappa = \text{crit}(j)$ and $\lambda = \kappa_{\omega,j}$, and suppose the least rank-Berkeley is $\delta < \lambda$. Let $\mu_j$ be the normal measure over $\kappa$ derived from $j$. Then $\delta < \kappa$ and there is $\kappa' < \delta$ such that for $\mu_j$-measure one many $\gamma < \kappa$, $(V_\gamma, V_{\gamma+1}) \models \kappa' \text{ is a Reinhardt cardinal}.$

**Proof.** Suppose $\delta < \lambda$ is rank-Berkeley, so $\delta < \kappa$. Then there is $k : V_\kappa \rightarrow V_\kappa$ which is elementary and non-identity. Let $\kappa' = \text{crit}(k)$. Then $\kappa$ is inaccessible and $(V_\kappa, V_{\kappa+1}) \models \text{ZF}_2 + \kappa' \text{ is Reinhardt, as witnessed by } k$. Since $\kappa = \text{crit}(j)$, the theorem follows routinely.

**Corollary 6.4.** Suppose ZF$(j) + \text{ "} j : V \rightarrow V \text{"}$ is consistent. Then so is $\text{ZF}(j) + \text{ "} j : V \rightarrow V \text{"} + \text{ "} \kappa_{\omega}(j) \text{ is the least rank-Berkeley} \text{"}.$

This also gives that $\lambda = \kappa_{\omega}(j)$ can be definable over $V$ without parameters. But there is anyway another way to see that $j : V \rightarrow V$ with $\lambda$ non-definable is stronger than just $j : V \rightarrow V$. For since $\lambda$ is a limit of inaccessibles, if $\lambda$ is non-definable, then $V$ has inaccessibles $\delta > \lambda$, and taking the least such, $j(\delta) = \delta$, so we get $(V_\delta, V_{\delta+1}) \models \text{ZF}_2 + \text{ "} \text{There is a Reinhardt} \text{"}$ (actually the latter holds for every inaccessible $\delta > \lambda$, since $j^n(\delta) = \delta$ for some $n$).

## 7 Questions and related work

In §5 we ruled out the definability of $\Sigma_1$-elementary embeddings $j : V_\delta \rightarrow V_\delta$ for $\delta$ a limit. Note that we also observed that for $\delta$ even, $\Sigma_1$-elementary maps $j : V_{\delta+1} \rightarrow V_{\delta+1}$ are always definable from the parameter $j \upharpoonright V_\delta$. But what about partially elementary maps $V_{\delta+2} \rightarrow V_{\delta+2}$? Can they be definable from parameters over $V_{\delta+2}$? If so, what can one say about the complexity of the definition in relation to the degree of elementarity?

One can also generalize the notion of “definable from parameters” to allow higher order definitions, such as looking in $L(V_\delta)$. If $\delta$ is a limit and $L(V_\delta) \models \text{ "} \text{cof}(\delta) > \omega \text{"}$ then $L(V_\delta)$ has no elementary $j : V_\delta \rightarrow V_\delta$ (see [14]; the case that $\delta$ is inaccessible was established earlier by the first author). There is a little on the cofinality $\omega$ case in [14], but this case is much more subtle, and not much is known.

The existence of the canonical extension $j^+$ of an embedding $j : V_\lambda \rightarrow V_\lambda$ for limit $\lambda$ is of fundamental importance to the analysis of $I_0$; see for example [18]. But this is now naturally generalized to all even $\lambda$. It turns out that much of the $I_0$ theory generalizes in turn, and this is one of the topics of [7].
Of course a significant question looming over this work is whether embeddings of the form we are considering can even exist. Some recent progress in this regard, establishing the consistency of $\mathbf{ZF} + j : V_{\lambda+2} \rightarrow V_{\lambda+2}$ relative to ZFC + $I_0$, is the topic of [13].

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