Mathematical models of the development of industrial enterprises, with the effect of lagging internal and external investments

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Abstract. The article proposes new models of the dynamic development of enterprises that restore their production through internal and external delayed investments. Differential balance equations for such enterprises are established, which describe changes in factors of production and output. Proportional, progressive and digressive depreciation deductions are considered and their interaction with internal and external investments is investigated. The conditions of the limiting state of work of enterprises are formulated and the corresponding values of the factors of production are calculated. Various options for the development of enterprises, including their stable operation, temporary suspension of their activities for the period of technical re-equipment, and temporary partial curtailment of their production, have been investigated. The regularities of the influence of the volumes of depreciation deductions, internal and external investments on the dynamics of enterprise development are revealed. Numerical analysis of the developed models showed good agreement with the known statistical data on the work of industrial enterprises.

1. Introduction
The improvement of mathematical methods of calculation and modeling of indicators of the dynamics of economic development of industrial enterprises is one of the urgent modern economic theory problems, the successful solution of which allows to analyze the activities of enterprises in efficient way, calculate the limiting values for their production factors, predict production output, costs and profits quite accurately, evaluate the effects of substitution of production factors, etc.

The development and growth of economic systems in general and manufacturing enterprises in particular is a long-term trend towards an increase in the indicators of the national economy. A significant contribution to the development of theoretical principles of economic growth is presented in works [1–7].

On the basis of these provisions, a number of researchers have developed growth models that take into account the influence of technological progress and the role of information processes [8–18].

The nature of the dynamic development of a manufacturing enterprise is determined by the interaction of the investment in production, the amount of depreciation of production factors, and the cost of modernizing the means of production.
The introduction of internal and external investments into the production of an enterprise is a process distributed over time, therefore, the development model of an enterprise should take into account not only the investments coming in at a given period of time, but also the entire history of their step by step introduction.

For model construction of economic development of enterprises, considering the lag of internal and external investments, differential equations are widely used, containing the distribution functions of gradual and continuous input of capital investments [19–30].

The aim of our work is to develop new economic and mathematical models of the dynamics of the development of industrial enterprises through internal and external delayed investments.

The scientific novelty and features of these models lie in the fact that they describe the interaction of proportional, progressive and digressive depreciation deductions with internal and external lagging investments and allow calculating the limiting values of factors of production and output.

Variants of stable progressive development of the enterprise, suspension of its work during the re-equipment of production and temporary crisis curtailment of production when replacing equipment are considered.

2. Statement of the problem
Let the volume of output of the enterprise be provided by a set of resources in the form of production factors \( (Q_1, Q_2, \ldots, Q_n) \), which are fixed capital, working capital, financial capital, labor resources, materials, technologies and innovations involved in production, etc.

The quantities of \( Q_s \) change over time and represent some continuous and continuously differentiable functions \( Q_s(t) \). The units of measurement of a variable, depending on the economic situation under consideration, can be one month, a quarter or one year.

Functions \( Q_s = Q_s(t) \) are limited, they have their own upper and lower limits \( Q_0^s \leq Q_s(t) \leq Q_\infty^s \), where \( Q_0^s = Q_s(0) \) is the initial value of the factor of production, and \( Q_\infty^s = \lim_{t \to \infty} Q_s(t) \) is its limiting value.

The values of the components \( Q_0^s \) at the initial moment of time of the process under consideration \( t = 0 \) are considered known. Limiting values of quantities \( Q_\infty^s \) follow from the developing economic situation and are subject to calculation.

The production output of an enterprise is described by the Cobb–Douglas multifactor production function

\[
V = P \prod_{s=1}^{n} Q_s^{a_s},
\]

where \( P \) is the cost of products produced per unit volume of resources, \( a_s \) are the elasticities of output for the corresponding resources \( Q_s \), \( 0 < a_s < 1 \).

Over a certain short period of time \( \Delta t \), the increment in the volumes of production factors \( Q_s = Q_s(t) \) will amount to values \( \Delta Q_s \), that can be represented as three terms

\[
\Delta Q_s(t) = \Delta Q_s^A(t) + \Delta Q_s^I(t) + \Delta Q_s^G(t),
\]

where \( \Delta Q_s^A(t) \) are the increment in depreciation of production factors \( Q_s \), \( \Delta Q_s^I(t) \) are the partial recovery of production factors \( Q_s \) due to internal investment, and \( \Delta Q_s^G(t) \) are the partial recovery of factors of production \( Q_s \) due to external investment in the enterprise.

The increments in partial depreciation \( \Delta Q_s^A(t) \) over a period of time \( \Delta t \) can be represented as

\[
\Delta Q_s^A(t) = -A_s \theta(t) Q_s^{\nu_s}(t) \Delta t,
\]

where \( A_s \) are the amortization coefficients, the shares of the volumes of production factors \( Q_s \) retired per unit of time, \( \nu_s \) are the indicators of the amortization intensity.
When \( u_s = 1 \) there is the usual proportional depreciation, when \( u_s > 1 \) the depreciation deductions increase and become progressive, while \( u_s < 1 \), on the contrary, the depreciation charges decrease and become regressive.

The function \( \theta = \theta(t) \) in equations (3) determines the options for the development of the considered enterprise. For a constant and single function \( \theta(t) \equiv 1 \), the development of the enterprise will be stable. Different sizes of the deviation of the value of the function \( \theta(t) \) from one in the direction of decreasing will correspond to a slowdown in the development of an enterprise, its temporary halt during a change in production technologies, and a partial curtailment of production [25].

The increase in internal investment over a period of time is determined by the relations

\[
\Delta Q_s^I(t) = \theta(t)W_s(t)\Delta t, \tag{4}
\]

where

\[
W_s(t) = \int_{-\infty}^{t} R_s(t, \tau)I_s(\tau)d\tau \tag{5}
\]

– the volume of internal investments accumulated by the enterprise due to the factors of production \( Q_s \) at the moment of time \( t \); \( R_s(t, \tau) \) – the distribution functions of the gradual and continuous input of investments, corresponding to the factors of production \( Q_s \) for the entire period of operation of the enterprise; \( I_s(\tau) \) – volume investments, corresponding to the factor of production \( Q_s \) and made at the moment of time \( \tau \).

With a lag in capital investment, the flows of internal investment are redistributed over time, but the amount of investment for the entire period remains constant. Thus, the investment input distribution functions \( R_s(t, \tau) \) satisfy the normalization conditions

\[
\int_{\tau}^{\infty} R_s(t, \tau)d\tau = 1. \tag{6}
\]

The investment input process is considered stationary, therefore, formula (6) takes the form

\[
W_s(t) = \int_{-\infty}^{t} R_s(t - \tau)I_s(\tau)d\tau. \tag{7}
\]

For the exponential distribution of investment input \( R_s(t - \tau) = \lambda_s e^{-\lambda_s(t-\tau)} \), relation (7) takes the form

\[
W_s(t) = \lambda_s \int_{-\infty}^{t} e^{-\lambda_s(t-\tau)}I_s(\tau)d\tau, \tag{8}
\]

where \( \lambda_s \) are the distribution parameters describing the degree of influence of previously made internal investments on capital investments of the current moment.

Obviously, the larger the values of the quantities \( \lambda_s \), the less this influence and vice versa. Differentiation of both sides of the integral equation (8) in time shows that it is equivalent to the differential equation

\[
\frac{dW_s(t)}{dt} = \lambda_s B_s V(t) - \lambda_s W_s(t). \tag{9}
\]

Internal investments \( I_s(t) \) is related to the production function \( V(t) \) by the relations

\( I_s(t) = B_s V(t) \), where \( B_s \) are the rates of accumulation of internal investments \( (0 \leq B_s \leq 1) \).
Increases in external investments $\Delta Q_s^G(t)$ over a period of time $\Delta t$ are determined by the relations

$$\Delta Q_s^G(t) = \theta(t) U_s(t) \Delta t,$$  \hspace{1cm} (10)

where

$$U_s(t) = \int_{-\infty}^{t} S_s(t, \tau) G_s(\tau) d\tau,$$  \hspace{1cm} (11)

– the volumes of external investments accumulated by the enterprise at a point in time $t$, $S_s(t, \tau)$ are the distribution functions of the gradual and continuous input of external investments for the entire period of the enterprise’s operation, $G_s = \eta_s G(\tau)$ are the volumes of external investments made at a point in time $\tau$, $G(t)$ is the total volume of external investment attributable to all volumes of production factors $Q_s(t)$, $\eta_s$ – distribution coefficients of the volumes of external investments between the volumes of production factors.

Obviously, the coefficients $\eta_s$ are not independent, but satisfy the relation $\sum_{s=1}^{n} \eta_s = 1$.

With a lag in capital investment, the flows of external investment are redistributed over time, but the amount of investment for the entire period remains constant. Thus, the investment input distribution functions $S_s(t, \tau)$ satisfy the normalization conditions

$$\int_{\tau}^{\infty} S_s(t, \tau) d\tau = 1.$$  \hspace{1cm} (12)

The external investment input processes are considered stationary; therefore, formulas (11) take the form

$$U_s(t) = \int_{-\infty}^{t} S_s(t - \tau) G_s(\tau) d\tau.$$  \hspace{1cm} (13)

For exponential distributions of investment inputs $S_s(t - \tau) = \mu_s e^{-\mu_s(t-\tau)}$ relations (13) take the form

$$U_s(t) = \mu_s \int_{-\infty}^{t} e^{-\mu_s(t-\tau)} G_s(\tau) d\tau,$$  \hspace{1cm} (14)

where $\mu_s$ are the distribution parameters describing the degree of influence of previously made external investments on capital investments of the current moment.

Obviously, the larger the values of the quantities $\mu_s$, the less this influence and vice versa.

Successive differentiation of both sides of integral equations (14) in time $t$ shows that they are equivalent to the system of differential equations

$$\frac{dU_s(t)}{dt} = \mu_s G_s(t) - \mu_s U_s(t).$$  \hspace{1cm} (15)

Substituting formulas (3), (4), and (10) into balance equations (2), we obtain

$$\Delta Q_s(t) = \theta(t) \left( -A_s Q_s^{Us}(t) + W_s(t) + U_s(t) \right) \Delta t.$$  \hspace{1cm} (16)

The passage to the limit in relation (16) at $\Delta t \to 0$ leads to the system of nonlinear differential equations

$$\frac{dQ_s(t)}{dt} = \theta(t) \left( -A_s Q_s^{Us}(t) + W_s(t) + U_s(t) \right).$$  \hspace{1cm} (17)
Equations (9), (15), and (17) form a system of normal nonlinear coupled first-order equations

\[
\begin{aligned}
\frac{dQ_s(t)}{dt} &= \theta(t) \left( -A_sQ^u_s(t) + W_s(t) + U_s(t) \right), \\
\frac{dW_s(t)}{dt} &= \lambda_sB_s(t)V(t) - \lambda_sW_s(t), \\
\frac{dU_s(t)}{dt} &= \mu_sG_s(t) - \mu_sU_s(t).
\end{aligned}
\]  

(18)

Substitution of formula (2) for the production function into the system of equations (18) gives

\[
\begin{aligned}
\frac{dQ_s(t)}{dt} &= \theta(t) \left( -A_sQ^u_s(t) + W_s(t) + U_s(t) \right), \\
\frac{dW_s(t)}{dt} &= \lambda_sB_sP \prod_{k=1}^{n} Q_k^a(t) - \lambda_sW_s(t), \\
\frac{dU_s(t)}{dt} &= \mu_sG_s(t) - \mu_sU_s(t).
\end{aligned}
\]  

(19)

The initial conditions for system (19) have the form

\[
\begin{aligned}
Q_s|_{t=0} &= Q_s(0) = Q_s^0, \\
W_s|_{t=0} &= W_s(0) = W_s^0, \\
U_s|_{t=0} &= U_s(0) = U_s^0.
\end{aligned}
\]  

(20)

In the general case, the nonlinear Cauchy problem (19), (20) has no analytical solution and can be solved only numerically.

If Dirac functions \(R_s(t, \tau) = S_s(t, \tau) = \delta(t - \tau)\) are chosen as the distribution functions of the gradual and continuous input of internal and external investments \(R_s(t, \tau)\) and \(S_s(t, \tau)\), then the Cauchy problem (19), (20) takes the form

\[
\frac{dQ_s(t)}{dt} = \theta(t) \left( -A_sQ^u_s(t) + B_sP \prod_{k=1}^{n} Q_k^a(t) + \eta_sG(t) \right),
\]

(21)

and

\[
Q_s|_{t=0} = Q_s(0) = Q_s^0,
\]

(22)

and coincides with the results of [30], in which the effect of the lagging of internal and external investments is not taken into account.

The structure of the system of equations (19) shows that the development of the enterprise will continue as long as the sum of the volumes of internal and external investments exceeds the volumes of depreciation deductions. If the sum of the volumes of internal and external investments becomes equal to the volume of depreciation, the development of the enterprise will stop. The limiting values \(Q^\infty_s\) of the volumes of production factors \(Q_s(t)\) are found from the conditions

\[
\begin{aligned}
\frac{dQ_s(t)}{dt} &= \theta(t) \left( -A_sQ^u_s(t) + W_s(t) + U_s(t) \right) = 0, \\
\frac{dW_s(t)}{dt} &= \lambda_sB_s(t)V(t) - \lambda_sW_s(t) = 0, \\
\frac{dU_s(t)}{dt} &= \mu_sG_s(t) - \mu_sU_s(t) = 0.
\end{aligned}
\]  

(23)
Eliminating the quantities $W_s(t)$ and $U_s(t)$ from equations (23), we obtain a system of equations for calculating the limiting values of the volumes of production factors $Q_s^\infty$

$$-A_s Q_s^{u_s}(t) + B_s P \prod_{k=1}^{n} Q_k^{a_k}(t) + G_s(t) = 0. \quad (24)$$

The forms of the integral curves of equations (12) and (14) significantly depend on the type of function $\theta(t)$ that determines the center of the time interval, its length and the amount of deviation from the unit value at which the enterprise operates stably.

If in the time interval $(t^* - \sigma, t^* + \sigma)$ an enterprise makes a complete or partial replacement of technological equipment, then the function $\theta(t)$ can be written in the form [25,26]

$$\theta(t) = 1 - \omega e^{-(t^* - t)^2 / 2\sigma^2}. \quad (25)$$

where $\omega$ is the maximum size of the deviation of the function $\theta(t)$ from unity, $t^*$ is the center of the time interval, and $\sigma$ is the radius of the time interval.

If $\omega = 0$, then the enterprise will work stably, if $0 < \omega < 1$, then in the vicinity of the point $t = t^*$ the growth of functions $Q_s(t)$ slows down, if $\omega = 1$, then at the moment of time $t = t^*$ the growth of functions $Q_s(t)$ stops, and in the time interval $(t^* - \sigma, t^* + \sigma)$ there is a re-equipment of production, if $\omega > 1$, then in the interval of time $(t^* - \sigma, t^* + \sigma)$ there is a re-equipment of production, accompanied by some reduction.

3. Mathematical model of the development of a one-factor manufacturing enterprise, with the effect of lagging internal investment

Consider a manufacturing enterprise, the output of which is provided by only one resource in the form of a production factor $Q = Q(t)$, there are no external investments $G(t) = 0$, and the development of the enterprise is carried out only at the expense of its own internal investments.

A continuous and continuously differentiable function $Q = Q(t)$ is bounded on the number semiaxis $(0 \leq t < \infty)$, $Q_0 \leq t \leq Q_\infty$, by its limiting values $Q_0 = Q(0)$, $Q_\infty = \lim_{t \to \infty} Q(t)$. In this case, the general system of equations (19) and initial conditions (20) take the form

$$\begin{aligned}
\frac{dQ(t)}{dt} &= \theta(t) (-AQ^u(t) + W(t)), \\
\frac{dW(t)}{dt} &= \lambda BPQ^a(t) - \lambda W(t),
\end{aligned} \quad (26)$$

and

$$\begin{aligned}
Q|_{t=0} &= Q(0) = Q_0, \\
W|_{t=0} &= W(0) = W_0.
\end{aligned} \quad (27)$$

The limiting value $Q_\infty$ of the volume of the production factor is found from the equation

$$-AQ^u + BPQ^a = 0, \quad (28)$$

whose solution has the form

$$Q_\infty = \left(\frac{BP}{A}\right)^{\frac{1}{u-a}}. \quad (29)$$

The general system of equations (21), (22), which does not take into account the effect of delayed internal investment, takes the form

$$\frac{dQ(t)}{dt} = \theta(t) (-AQ^u(t) + BPQ^a(t)), \quad (30)$$
and

\[ Q_{t=0} = Q(0) = Q_0. \]  

(31)

Figure 1 shows a comparison of the curves of the dynamic development of the enterprise, obtained as a result of the numerical solution of the Cauchy problem (26), (27) and the numerical solution of the Cauchy problem (30), (31) for cases of stable operation, suspension of work and some curtailment of production.

\[ Q \]

\[ Q_\infty = 111.005. \]

It should be noted that the enterprise model that takes into account the effect of lagging internal investment gives lower numerical values than the enterprise model that does not take into account this effect.

Let us now apply the constructed models (26), (27) and (30), (31) to calculate the indicators of development of the enterprise PJSC “Chelyabinsk Pipe-Rolling Plant”, the statistical data on the output of which are given in table 1 [31].

**Table 1.** Statistical data on the output of the enterprise PJSC “Chelyabinsk Pipe-Rolling Plant”.

| Year | \(Q\) (mill.rub.) | \(V\) (mill.rub.) |
|------|------------------|------------------|
| 2008 | 72.698296        | 63.721902        |
| 2009 | 77.103839        | 66.246199        |
| 2010 | 87.578960        | 72.564189        |
| 2011 | 92.435837        | 76.645429        |
| 2012 | 97.656699        | 84.048139        |
| 2013 | 100.399083       | 82.721179        |
| 2014 | 115.118761       | 97.184656        |
| 2015 | 123.270175       | 112.285286       |
| 2016 | 128.353653       | 109.806604       |
| 2017 | 142.265642       | 116.090570       |

In accordance with the statistical data of table 1, the parameters of the system of equations (26), (27) take the calculated values: \(P = 4.35; \ a = 0.65; \ u = 0.905; \ A = 0.12; \ B = 0.2; \ \lambda = 0.5; \ t^* = 5; \ \sigma = 0.45; \ \omega = 1; \ Q_0 = 72.398. \)

Figure 2 shows a comparison of the curves of the dynamic development of the enterprise PJSC “Chelyabinsk Pipe-Rolling Plant” obtained as a result of the numerical solution of the Cauchy problem (26), (27) and the numerical solution of the Cauchy problem (30), (31) for the case of its suspension at the time (2013) associated with the re-equipment of production.
4. Mathematical model of the development of a one-factor manufacturing enterprise, with the effect of lagging internal and external investments

Let now, in addition to its own internal investments, external investments be made in a one-factor manufacturing enterprise.

In this case, the general system of equations (19) and the initial conditions (20) will be written in the form

\[
\begin{align*}
\frac{dQ(t)}{dt} &= \theta(t) \left(-AQ^u(t) + W(t) + U(t)\right), \\
\frac{dW(t)}{dt} &= \lambda BPQ^a(t) - \lambda W(t), \\
\frac{dU(t)}{dt} &= \mu G(t) - \mu U(t),
\end{align*}
\] (32)

and

\[
\begin{align*}
Q|_{t=0} &= Q(0) = Q_0, \\
W|_{t=0} &= W(0) = W_0, \\
U|_{t=0} &= U(0) = U_0.
\end{align*}
\] (33)

The general system of equations (21), (22), which does not take into account the effect of lagging internal and external investments, takes the form

\[
\frac{dQ(t)}{dt} = \theta(t) \left(-AQ^u(t) + BPQ^a(t) + G(t)\right),
\] (34)

\[
Q|_{t=0} = Q(0) = Q_0.
\] (35)

The type of function of the volume of external investments \(G(t)\) will significantly depend on the investment conditions. If the largest amount of external investments is invested in production at the initial moment of time, and then their level gradually decreases to a certain limit, then the function \(G(t)\) is conveniently represented in the form [32]

\[
G_T(t) = G_{\text{min}} + (G_{\text{max}} - G_{\text{min}})e^{-vt},
\] (36)

where constants \(G_{\text{max}}\) and \(G_{\text{min}}\) represent the maximum and minimum values of attracted external investments, \(v\) is a parameter characterizing the rate of decrease in the volume of external investments.

In this case, the limiting value \(Q_\infty\) of the volume of the production factor \(Q(t)\) is found from the equation

\[
-AQ^u + BPQ^a + G_{\text{min}} = 0.
\] (37)
which, in contrast to equation (28), can only be solved numerically.

If external investments are made in production over a relatively short time interval, increasing first from some initial value to its maximum value, and then decreasing to zero, then the function $G_T(t)$ is conveniently represented as [32]

$$G_T(t) = G_{\text{max}} e^{-\frac{(t-t_G)^2}{2\sigma_G^2}},$$

(38)

where $G_{\text{max}}$ is the maximum value of attracted external investments, $t_G$ is the time point corresponding to the maximum value of external investments, $\sigma_G$ is the radius of the time interval for external investments.

Figure 3 shows the curves of stable development of the enterprise obtained as a result of the numerical solution of the Cauchy problem (32), (33) and the numerical solution of the Cauchy problem (34), (35) with external investments given by formula (36).

Figure 4 shows the curves of stable development of the enterprise obtained as a result of the numerical solution of the Cauchy problem (32), (33) and the numerical solution of the Cauchy problem (34), (35) with external investments given by formula (38).

Let us now compare the constructed design models (32), (33) and (34), (35) with the statistical data for the development indicators of the enterprise LLC LADA Izhevsk Automobile Plant, given in table 2 [33].
Table 2. Statistical data on the production of the enterprise LLC "LADA Izhevsk Automobile Plant".

| Year | Q (mill.rub.) | V (mill.rub.) |
|------|---------------|---------------|
| 2008 | 13.217574     | 8.456392      |
| 2009 | 14.207309     | 8.931730      |
| 2010 | 14.95581      | 9.372198      |
| 2011 | 17.083839     | 10.683700     |
| 2012 | 17.801323     | 11.779749     |
| 2013 | 21.029907     | 13.996108     |
| 2014 | 40.349890     | 22.646566     |
| 2015 | 47.985065     | 29.901182     |
| 2016 | 59.579269     | 45.173967     |
| 2017 | 72.758050     | 52.328860     |

Statistical data $Q_S$ in table 2 show that until 2013 the enterprise developed monotonously due to internal investments, and after 2013 its development was significantly influenced by external investments. The statistical data $Q_S$ in table 2 for the period from 2008 to 2013 are well approximated by the theoretical function

$$Q_T(t) = Q_0 + 0.26t^{2.1}. \quad (39)$$

In the period from 2013 to 2018, function (39) describes the possible development of an enterprise in the absence of external investment.

The differences between the statistical data $Q_S$ in table 2 and the theoretical values of function (39) $Q_T$ are statistical data $G_S$ on the volume of external investment, which can be well approximated by the theoretical function (38).

Figure 5 shows a comparison of the statistical data $Q_S$ of table 2 with the theoretical values of the function (39) $Q_T$ and a comparison of the statistical data $G_S$ of the volume of external investments with the theoretical values of the function (38) $G_T$.

Figure 6 shows a comparison of the curves of the dynamic development of the enterprise LLC LADA Izhevsk Automobile Plant, obtained as a result of the numerical solution of the Cauchy problem (32), (33) and the numerical solution of the Cauchy problem (34), (35) with the statistical data $Q_S$ of table 2.
5. Conclusion

New models of dynamic development of a multifactorial manufacturing enterprise have been developed, through gradual distributed input of internal and external investments.

The interaction of proportional, progressive and digressive depreciation deductions with internal and external lagging investments was investigated.

The conditions of the equilibrium state of the enterprise were formulated and equations were obtained to determine the limiting value of the factor of production, upon reaching which the further growth of output by the enterprise stops.

Three options for the development of enterprises were considered. In the first case, the enterprise is developing steadily and progressively. In the second case, the enterprise temporarily suspends the growth of production, re-equipping production and replacing technological equipment. In the third case, the enterprise is forced to temporarily curtail production when the technological order changes.

Numerical analysis of the presented models showed good agreement with the known statistical data on the work of industrial enterprises.

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