Numerical simulation of bottom topography transformation taking into account the coastal shore protection structures

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Abstract. The article presents the results of studies and modeling the dynamics of nonlinear processes of sediment transport and the diagnostics of changes in shallow water relief, taking into account underwater structures that are widely used for coastal protection. The study of these processes was carried out on the basis of 2D model of sediment transport and 3D model of wave hydrodynamics, taking into account the geometry of the bottom topography and the function of elevating the level of the aquatic environment. A discrete model of sediment transport is obtained by approximating the corresponding linearized continuous model. Particular attention is paid to the creation of effective software for conducting hydrodynamic computational experiments. The developed software package allows predicting the dynamics of bottom surface behavior, the appearance of marine braids and ridges, their growth and transformation.

1. Introduction
One of the manifestations of global climate change is the increase in the number of adverse and dangerous phenomena in the coastal zones of water bodies, which include changes in the bottom topography, including siltation of shipping and access channels, changes in the coastline due to accumulation of bottom sediments, as well as opposite processes shore abrasion. In recent years, the results of comprehensive studies of a number of Russian scientists have been published, and environmental management issues in coastal zones: N.A. Aybulatov, G.G. Matishov, S.V. Berdnikov, E.A. Schwartz, V.A. Ivanov, etc. At the same time, it should be noted that unified approaches to system analysis and modeling of coastal zones have not yet been created, which is partially explained by their natural-climatic and physical-geographical differences for different seas of Russia, such as the Azov, Baltic, Barents, White, Okhotsk and others.

The current ecological state of coastal systems is largely determined by the variety of particulate matter of both mineral and organic origin. Under the influence of a complex of intra-water processes, transformation and sedimentation of particles of suspended matter and, as a result, the formation of bottom sediments occurs. Sediment of coastal systems is a complex heterogeneous physicochemical system, the study of the processes inside which under modern conditions is an important and urgent task [1], [2]. In particular, it seems relevant to assess the flows of pollutants and determine the amount of sediment deposited associated with the development of industrial and recreational activities in coastal areas [3]. As a rule, research in this area requires the construction of mathematical models that
are as close as possible to real processes and allow predicting the distribution of suspended matter in an aqueous medium [4]-[6].

Non-stationary 2D model of sediment transport is used to predict changes in the bottom topography. It took into account the physically significant factors and parameters as soil porosity; the critical value of the shear stress at which sediment movement begins; turbulent exchange; dynamically changing bottom geometry, wind currents and bottom friction [7]-[10]. A discrete model of sediment transport, proposed as a result of approximation of the corresponding linearized continuous model, is proposed and investigated. The sediment transport model is complemented by a model of the movement of the aquatic environment and turbulence. Since the mathematical problems corresponding to the constructed models must be solved in real or accelerated time scales, on grids including $10^6$–$10^{12}$ nodes, it is necessary to develop parallel algorithms for computer systems with mass parallelism. For the proposed models of hydrodynamic processes, parallel algorithms are developed that are implemented as a complex of programs. Numerical experiments are performed for model problems of bottom sediment transport and bottom topography transformation taking into account underwater structures, the results of which are consistent with real physical experiments.

2. Statement of the problem

Water flows carry large amount of solid particles, such as sediment, moving particles of clay, mud, gravel, pebbles, sand, loess, carbonate compounds, emulsions of mineral oils, petroleum products and other components. The surfaces of sediment particles are capable of absorbing various pollutants, including heavy metals and pesticides that have a negative impact on the ecological situation of a water body [11], [12]. Sediment can be carried by the flow in a suspended state (suspended sediment), and can be moved in the bottom layer of the stream by rolling, sliding, saltation (entrained sediment). Particles of sediment carried by the flow can transform from a suspended state to an entrained state and vice versa, entrained particles can stop moving, and motionless particles can start to move. The nature of the movement of suspended and entrained sediments is determined by the flow velocity, depth, and other hydraulic elements of the water flow.

2.1. Mathematical description of the sediment transport problem

According to [13]-[15], the sediment transport equation is considered:

$$
(1-\varepsilon) \frac{\partial H}{\partial t} = \text{div} \left( k \frac{\tau_b}{\sin \varphi_0} \text{grad} H \right) - \text{div} (k \tau_b),
$$

(1)

where $H = H(x,y,t)$ is the depth of the reservoir, $\varepsilon$ is the porosity of bottom materials; $\tau_b$ is the vector of tangential stress at the bottom of the reservoir; $\tau_{bc}$ is the critical value of the tangential stress; $\varphi_0 = \alpha \sin \varphi_0$, $\alpha$ is the angle of soil repose in the reservoir; $k = k(H,x,y,t)$ is non-linear coefficient determined by the ratio

$$
k = \frac{A d}{(\rho - \rho_b) g d} \left( \frac{\tau_b}{\sin \varphi_0} \text{grad} H \right)^{-\lambda - 1},
$$

(2)

($\rho, \rho_b$ are particle density of the bottom material and the aquatic environment, respectively; $g$ is the acceleration of gravity; $\varphi$ is the wave frequency; $A$ and $\beta$ are dimensionless constants; $d$ are the characteristic sizes of soil particles).

The first term on the right-hand side of equation (1) describes the change in the bottom topography due to the random movement of particles due to the combined action of gravity and the wave motion of the aqueous medium, which has a stochastic nature. The second term is the direction of motion of sediment particles under the action of shear stress near the bottom caused by the movement of the aqueous medium.
Let $D \subset \mathbb{R}^2$ is the region where the process takes place, and $\partial$ is its boundary, which is a piecewise-smooth line. The domain of equation (1) is a three-dimensional cylinder $\mathcal{U}_\tau = D \times (0, T)$ with height $T$ with a base $D$. Its border consists of a lateral surface $S = \{0, T\}$ and two bases: the lower $D \times \{0\}$ and the upper $D \times \{T\}$. Further, for simplicity, equation (1) is considered in a rectangular region $D(x,y) = \{0 < x < L_x, 0 < y < L_y\}$.

Let supplement equation (1) with the initial condition, assuming that the function of the initial conditions belongs to the corresponding smoothness class:

$$H(x,y,0) = H_0(x,y), H_0(x,y) \in C^3(D) \cap C(\overline{D}), \quad \text{grad}_{(x,y)} H_0 \in C(\overline{D}), (x,y) \in \overline{D}$$

Let formulate the conditions on the boundary of the region:

$$\left. H \right|_{\partial D} = 0,$$  \hspace{1cm} (4)

$$H(L_x,y,t) = H(L_x,y,0), 0 \leq y \leq L_y,$$  \hspace{1cm} (5)

$$H(0,y,t) = H(0,y,0), 0 \leq y \leq L_y,$$  \hspace{1cm} (6)

$$H(x,0,t) = H(x,0,0), 0 \leq x \leq L_x,$$  \hspace{1cm} (7)

$$H(x,L_y,t) = 0, 0 \leq x \leq L_x.$$  \hspace{1cm} (8)

In addition to the boundary conditions (5)-(8), we assume that their smoothness conditions are satisfied, the existence of continuous derivatives on the boundary of the region $D$:

$$\text{grad}_{(x,y)} H \in C(\overline{D}) \cap C(\mathcal{U}_\tau).$$  \hspace{1cm} (9)

The non-degeneracy condition for the operator of the problem has the form:

$$k \geq k_0 = \text{const} > 0, \forall (x,y) \in \overline{D}, 0 \leq t \leq T.$$  \hspace{1cm} (10)

The vector of tangential stress at the bottom is expressed using unit vectors of the coordinate system in a natural way:

$$\tau_b = \hat{b}_x \tau_{b_x} + \hat{b}_y \tau_{b_y}, \quad \tau_{b_x} = \tau_{b_x}(x,y,t), \quad \tau_{b_y} = \tau_{b_y}(x,y,t).$$  \hspace{1cm} (11)

### 2.2. Linearization of the sediment transport problem

Using the methods described in [14], [15], we create a linearized model on a time interval $0 \leq t \leq T$, by constructing a uniform grid $\omega$ with a step $\tau$, i.e. set of points and linearize the initial-boundary-value problem (1)-(8), approximating with an error $O(\tau)$ in the space $L_1$ the initial initial-boundary-value problem (1)-(8).

Let introduce the following notation:

$$k^{(n-1)} = \frac{\Delta x \Delta y}{(\rho - \rho_b) g d}, \quad \tau_{b_x} = \frac{\tau_{b_x}}{\sin \Phi_b}, \quad \text{grad} H^{(n-1)}(x,y,t_{n-1})^\tau.$$  \hspace{1cm} (12)

Then write equation (1) after linearization in the form:

$$(1 - \delta) \frac{\partial H^{(n)}}{\partial t} = \text{div} \left( k^{(n-1)} \frac{\tau_{b_x}}{\sin \Phi_b} \text{grad} H^{(n)} \right) - \text{div} \left( k^{(n-1)} \tau_{b_x} \right), 0 \leq t \leq t_n, n = 1, 2, ..., N$$  \hspace{1cm} (13)

and supplement it with the initial conditions:

$$H^{(0)}(x,y,0) = H_0(x,y), H^{(n)}(x,y, t_{n-1}) = H^{(n-1)}(x,y, t_{n-1}), (x,y) \in \overline{D}, n = 2, ..., N.$$  \hspace{1cm} (14)
2.3. Discrete sediment transport problem

Let construct a finite-difference scheme approximating problem (13), (14), (4) - (8). Cover the region $D$ with a uniform rectangular computational grid $\omega_i = \omega_x \times \omega_y$, assuming that the grid in time $\omega_t$ is defined earlier (Section 2.2)

$$\omega_x = \{x = ih_i, 0 \leq i \leq N_x - 1, h_x = h(N_x - 1)\},$$

$$\omega_y = \{y = jh_j, 0 \leq j \leq N_y - 1, h_y = h(N_y - 1)\},$$

where $n, i, j$ are the indices of the nodes of the grids constructed in the temporal $Ot$ and spatial $Ox, Oy$ directions, respectively, $\tau h_x, \tau h_y$ are the steps of the grids in the temporal and spatial directions, respectively, $N_x, N_y$ is the number of nodes in the temporal and spatial directions, respectively.

To obtain a difference scheme, we use the balance method. Integrate both sides of equation (13) over the region: $D_{xy}$:

$$D_{xy} \in \left\{ t \in t_{n+1}, x \in [x_{i-1/2}, x_{i+1/2}], y \in [y_{j-1/2}, y_{j+1/2}] \right\},$$

as a result, obtain the following equality:

$$\iint_{D_{xy}} (1-\varepsilon) H^{(n)} \frac{\partial t}{\partial x} dtdxdy + \iint_{D_{xy}} (k^{(n-1)} \tau_{b, x}) y dt dxdy + \iint_{D_{xy}} (k^{(n-1)} \tau_{b, y}) x dt dxdy =$$

$$= -\iint_{D_{xy}} \left( k^{(n-1)} \frac{\tau_{bc}}{\sin 0} H^{(0)}\right) y dtdxdy + \iint_{D_{xy}} \left( k^{(n-1)} \frac{\tau_{bc}}{\sin 0} H^{(0)}\right) x dtdxdy. \tag{15}$$

In equality (15), we calculate approximately the integrals by the formulas of the rectangles, divide the resulting equation into a product of factors, $\tau h_x, \tau h_y$, and, replacing the approximate equality by the exact one, we obtain a difference scheme approximating a linearized continuous problem:

$$\frac{(1-\varepsilon)}{\tau} H^{(n)}_{i,j} - H^{(n)}_{i,j} = \frac{k^{(n)}}{h_x} \left( \tau_{b, x}\right)_{i-1/2,j} - \frac{k^{(n)}}{h_x} \left( \tau_{b, x}\right)_{i+1/2,j} + \frac{k^{(n)}}{h_y} \left( \tau_{b, y}\right)_{i,j-1/2} - \frac{k^{(n)}}{h_y} \left( \tau_{b, y}\right)_{i,j+1/2} =$$

$$= \frac{\tau_{bc}}{\sin 0} \left( k^{(n)} \frac{H^{(n)} - H^{(0)}}{h_x} \right)_{i,j-1/2} + \frac{\tau_{bc}}{\sin 0} \left( k^{(n)} \frac{H^{(n)} - H^{(0)}}{h_y} \right)_{i,j+1/2} + \frac{\tau_{bc}}{\sin 0} \left( k^{(n)} \frac{H^{(n)} - H^{(0)}}{h_x} \right)_{i-1/2,j} - \frac{\tau_{bc}}{\sin 0} \left( k^{(n)} \frac{H^{(n)} - H^{(0)}}{h_y} \right)_{i,j-1/2},$$

where

$$\left( \tau_{b, x}\right)_{i-1/2,j} = \frac{\left( \tau_{b, x}\right)_{i,j} + \left( \tau_{b, x}\right)_{i,j-1}}{2}, \quad \left( \tau_{b, y}\right)_{i,j} = \frac{\left( \tau_{b, y}\right)_{i,j+1} + \left( \tau_{b, y}\right)_{i,j-1}}{2}.$$
\[
A_{cr} \left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} - \frac{\tau_{bc}}{\rho_0} \left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} \left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} - \tau_{bc} \left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} \right).
\]

The value \( \left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} \) is written in the form

\[
\left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} = \frac{H_{r+i/2,j} - H_{r+i/2,j-1}}{h_y} + \frac{H_{r+i/2,j} - H_{r+i/2,j-1}}{2h_y}
\]

Similarly, obtain the following approximation:

\[
\left( \frac{\partial H}{\partial r} \right)_{r+i/2,j} = \frac{H_{r+i/2,j} - H_{r+i/2,j-1}}{2h_y} + \frac{H_{r+i/2,j} - H_{r+i/2,j-1}}{h_y}
\]

3. Description of the parallel algorithm of the developed software package

A software package implemented in C++ is designed to build turbulent flows of an incompressible velocity field of the aquatic environment on high-resolution grids for predicting sediment transport and possible scenarios of changing the geometry of the bottom region of shallow water bodies.

The adaptive modified alternating-triangular method of minimum corrections was used to solve this problem. In parallel implementation, decomposition methods of grid regions were used for computationally time-consuming diffusion-convection problems, taking into account the architecture and parameters of a multiprocessor computing system. The decomposition of the calculated two-dimensional region is performed according to two spatial variables \( x, y \). The parallel algorithm is described taking into account the peculiarities of the HPC cluster, which is a computing system with distributed memory of 128 nodes connected by an InfiniBand network with a throughput of 20 Gb/s, each node in turn is an SMP system with 16 computers and volume RAM 32 GB. Parallel computing using MPI technology was performed on a distributed computing cluster using 128 processors. To evaluate the characteristics of a parallel algorithm, we need some actual data on the performance of a computing system. The data was obtained as a result of testing with Linpack and Pallas MPI benchmark packages. Peak MVS performance is 18.8 TFlops. MVS includes 8 computer racks. The computing field of a multiprocessor computing system is based on the HP BladeSystem c-class infrastructure with integrated communication modules, power supply and cooling systems. As computing nodes, 128 identical 16-core HP ProLiant BL685c Blade servers are used, each of which is equipped with four 4-core AMD Opteron 8356 2.3GHz processors and 32GB RAM. The total number of computing cores in the complex is 2048, the total amount of RAM is 4 TB. To control the MVS, 3 HP ProLiant DL385G5 control servers are used. For backup tasks, the MSL4048 library is used.

4. Results of numerical experiments

A series of numerical experiments was performed to simulate the dynamics of changes in the topography of the bottom of a complex configuration in the coastal zone of a reservoir. In model problems, the presence of obstacles (groyne, underwater breakwaters, breakwaters, dumps, etc.) and various irregularities underlying its surface was assumed to be on the bottom surface.

The simulation section under consideration has dimensions of 55 m \( \times \) 55 m horizontally and 2 m vertically (in depth), the peak point rises above sea level to 1 m. Suppose that the liquid is at rest at the initial time. The size of the computational grid is 110 \( \times \) 110, the step in spatial variables is 0.05 m, the time step is 0.01 s, the wind speed is 5 m/s and is directed from left to right.

Figure 1 presents the results of modeling the dynamics of bottom changes for the case when there are obstacles in the form of pointed structures, such as intermittent groyne on its surface. Due to sediment retention, the groyne not only stop the movement of the material carried by the waves along the coast, but also contribute to its deposition. These structures are one of the best means to protect the coast and prevent the invasion of the sea in the mainland.
Figure 1. The geometry of the computational domain with structures in the form of groynes (a - the initial moment, b, c - after 5, 15 minutes of processor time from the moment the simulation began, respectively).

Figure 1a shows the initial position of the contour lines of the function of the depths and topography of the bottom, a feature of which is the presence of three bottom groynes. Fluctuations in the isolines of the depth function are observed in the central part of the calculated one. These structures have a length of up to 15 m and are located at a distance not exceeding 10 m from each other. The structures are completely submerged in the water body, and their maximum height is 1.25 m. Modeling of the sediment transport process has shown that over time there is a smoothing of surface irregularities, sediment formation, a decrease in the depth of the slope of the bottom of the coastal zone and, as a consequence, a gradual shallowing of the considered zones of a reservoir. So, after 5 min. the contours of the depth function in the center of the computational domain have a wave-like shape, and in the bun location zone, they have a meandering shape. Sediments are deposited between groynes, and the pointed peaks of the groynes are deformed and take the form of gentle slides. As a result of these processes, a decrease in the depth of the coastal zone and an increase in the beach area are observed (Figure 1b). Figure 1c shows the results more clearly, the simulation time was 15 min. The contours of the depth function take on a soft wave-like shape over the entire computational domain, including the region of peak values. An active process of coastal movement of sediments and a decrease in the level of depth was observed. So, during the indicated estimated time in the inter-compartment compartments the decrease in depth was about 0.5 m. The height of the slides decreases, and the slides themselves take on a more smoothed appearance.

Figure 2 shows the case when the initial position of the computational domain contains four gentle structures of underwater breakwaters form. As well as groynes, coastal underwater breakwaters can significantly change the nature of the water flow in the coastal zone and contribute to the reformation of the coast due to sediment accumulation.

Underwater breakwaters are parallel to the coastline. The structures have a length of 30 m, a width of 15 m and are located at the same distance from each other - 5 m. The ridges of underwater breakwaters are located 1.6 m below sea level and rise towards the coast to 0.4 m below sea level. Figure 2b, c shows that after 5 and 15 min. after the start of modeling, gradually the contours of the
depth function in the center of the computational domain take on a wavy shape, and in the area adjacent to the coast, oval-shaped lines appear. The buns are deformed and acquire a hill-like appearance, the height of which becomes 2 m less. Due to sediment transport, the depth of the reservoir near the coast decreases. The distance between the underwater breakwaters is filled with sediments by 1.3 m.

Figure 2. The geometry of the computational domain with structures in the form of underwater breakwaters (a - the initial moment, b, c - after 5, 15 minutes of processor time from the moment the simulation began, respectively).

Figure 3. The geometry of the computational domain with spiky structures in the form of underwater banquets (a - the initial moment, b, c - after 5, 15 minutes of processor time from the moment the simulation began, respectively).
Figure 3 shows a feature of the computational domain at the initial moment with the presence of three structures of the type of underwater banquets from a stone sketch, two of which are located on the sea side at a distance of about 7.5 m, and the third is behind them at a distance of 2.5 m.

Underwater banquets have different heights, which range from 1.2 to 0.2 m and are below 0.45 m from the level of the free surface. Figure 3b, c shows an active process along the coastal movement of sediment and their accumulation on the shore. After 5 minutes since the beginning of the simulation, underwater banquets acquire a hilly appearance, the height of which does not exceed 0.65 m from the level of the free surface, and after 15 min - 0.7 m. An underwater banquet located on the coast side gradually disappears during the simulation due to the coastal movement of a large volume of coastal-marine sediments. In general, the depth of the reservoir near the coast is significantly reduced. The contours of the depth function within the computational domain within 15 min take the form of almost horizontal lines, lead to a restructuring of the spatial distribution of sediment flow parameters, which largely determine the main trends in modern coastline reformation in the study area.

5. Conclusion
In the article, the dynamics of the bottom topography of coastal systems and the associated process of moving alongshore sediment flows was explored. In order to describe the dynamics of the relief of the bottom surface, multicomponent mathematical and numerical models are constructed that take into account many factors. A non-stationary 2D model of transport of bottom sediments in the coastal zone of shallow water bodies is proposed, supplemented by the equations: Navier-Stokes, continuity, state of the aquatic environment. A discrete model of sediment transport is obtained by approximating the corresponding linearized continuous model. Particular attention is paid to the creation of effective software for conducting hydrodynamic computational experiments. The developed software package allows for numerical modeling of bottom deformation in the coastal zone of a reservoir. The results of numerical experiments are presented.

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