The rate of convergence of some Riemann-Stieltjes sums

Adrian Holhoș

March 14, 2014

Abstract

We give the rate of convergence of some optimal lower Riemann-Stieltjes sums toward the integral.

1 Introduction

Let $[a, b]$ be a bounded closed interval. Let $f, g$ be two functions defined on $[a, b]$. Consider an $n$-division $\Delta$ of $[a, b]$ defined by

$$\Delta: a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

and consider $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ such that $\xi_i \in [t_{i-1}, t_i]$ for every $1 \leq i \leq n$. The Riemann-Stieltjes sum is defined by

$$RS(f, g, \Delta, \xi) = \sum_{i=1}^{n} f(\xi_i) \cdot [g(t_i) - g(t_{i-1})].$$

The function $f$ is said to be Riemann-Stieltjes integrable with respect to $g$ if there is an $I \in \mathbb{R}$ with the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every division $\Delta$ of $[a, b]$ with mesh $\|\Delta\| = \max_{1 \leq i \leq n} (t_i - t_{i-1})$ less than $\delta$ and every choice of the points $\xi_i$ in $[t_{i-1}, t_i]$ we have

$$|RS(f, g, \Delta, \xi) - I| < \varepsilon.$$ 

The number $I$ is denoted $\int_{a}^{b} f(t) \, dg(t)$ and is called the Riemann-Stieltjes integral of $f$ with respect to $g$. When $g(x) = x$ we obtain the Riemann integrability.

Consider the lower Riemann-Stieltjes sum of a continuous function $f$ on $[a, b]$

$$RS(f, g, \Delta, \text{min}) = RS(f, g, \Delta, \xi),$$

where the points $\xi_i$ are chosen such that $f(\xi_i) = \min_{t \in [a,b]} f(t)$. The set of all $n$-divisions of $[a, b]$ is compact and $\Delta \mapsto RS(f, g, \Delta, \text{min})$ is continuous, so there is an optimal $n$-division $\Delta_{\text{opt}}$ at which the lower Riemann-Stieltjes
sum is maximum. This optimal $n$-division may not be unique, but the sum $\text{RS}(f, g, \Delta_{\text{opt}}, \text{min})$ is unique.

In Theorem 6 we give the rate of approximation of the Riemann-Stieltjes integral by the optimal lower Riemann-Stieltjes sums, a result which generalizes Theorem 1.2 of [2].

2 Main results

We give next a generalization of a Lemma found in [1].

**Lemma 1.** Let $g : [a, b] \rightarrow \mathbb{R}$ be a non-negative continuous function which is not identically zero on any open subinterval of $[a, b]$ and let $h : [a, b] \rightarrow \mathbb{R}$ be a strictly positive and continuous function on $[a, b]$. For any positive integer $n$ there exists a division of $[a, b]$: $t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ such that the quantities

$$(t_i - t_{i-1}) \cdot \max_{t \in [t_{i-1}, t_i]} g(t) \cdot \max_{t \in [t_{i-1}, t_i]} h(t), \quad 1 \leq i \leq n$$

are all equal to each other. Moreover, if $J_n$ is the common value of all these quantities, then

$$\lim_{n \to \infty} nJ_n = \int_a^b g(t)h(t) \, dt.$$

**Proof.** Parametrize the $(n-1)$ simplex $\sigma$ by $n$-tuples $(u_1, u_2, \ldots, u_n)$, where $u_i \geq 0$ and $\sum_{i=1}^n u_i = 1$. Let this $n$-tuple correspond to the partition of $[a, b]$ given by

$$t_i = a + (b - a) (u_1 + u_2 + \cdots + u_i), \quad 1 \leq i \leq n, \text{ and } t_0 = a.$$

Let us define the function

$$\psi(u_1, u_2, \ldots, u_n) = (w_1, w_2, \ldots, w_n), \quad w_i = \frac{v_i}{\sum_{i=1}^{n} v_i},$$

where $v_i = (b - a) u_i \cdot \max_{t \in [t_{i-1}, t_i]} g(t) \cdot \max_{t \in [t_{i-1}, t_i]} h(t)$. We have

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} (t_i - t_{i-1}) \max_{t \in [t_{i-1}, t_i]} g(t) \cdot \max_{t \in [t_{i-1}, t_i]} h(t)$$

is an upper Riemann sum for $\int_a^b g(t)h(t) \, dt$ and $\sum_{i=1}^{n} v_i > 0$.

Since the maximum value of a continuous function over a closed interval depends continuously on the endpoints of that interval, $\psi$ is a continuous function. Because $w_i = 0$ implies $u_i = 0$, $\psi$ maps every face of $\sigma$ into itself.
All this prove that $\psi$ is surjective. So there exists $(u_1, u_2, \ldots, u_n)$ such that $\psi(u_1, u_2, \ldots, u_n) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. This proves the first part of our Lemma.

We have $nJ_n \leq (b - a)\|g\| \cdot \|h\|$. Let $\varepsilon > 0$ be given. From the continuity of $g$ and $h$ on $[a, b]$ there is a $\delta > 0$ so that $|t - t'| < \delta$ implies $|g(t) - g(t')| < \varepsilon$ and $|h(t) - h(t')| < \varepsilon$. We choose $n > \frac{(b - a)\|g\| \cdot \|h\|}{\varepsilon^2}$.

If $\max_{t \in [t_{i-1}, t_i]} g(t) \geq \varepsilon$ and $\max_{t \in [t_{i-1}, t_i]} h(t) \geq \varepsilon$ we have

$$J_n = (t_i - t_{i-1}) \cdot \max_{t \in [t_{i-1}, t_i]} g(t) \cdot \max_{t \in [t_{i-1}, t_i]} h(t) \geq \varepsilon^2(t_i - t_{i-1}),$$

which proves that

$$t_i - t_{i-1} \leq \frac{J_n}{\varepsilon^2} = \frac{nJ_n}{n^2 \varepsilon^2} \leq \frac{(b - a)\|g\| \cdot \|h\|}{n^2 \varepsilon^2} < \delta.$$ 

This implies that the oscillations of $g$ and of $h$ over $[t_{i-1}, t_i]$ are at most $\varepsilon$. Considering $\eta_i$ and $\xi_i$ the points of maximum for $g$ and $h$ over the interval $[t_{i-1}, t_i]$ and applying the Mean Value Theorem for integrals we obtain

$$\left| J_n - \int_a^b g(t)h(t) \, dt \right| = \left| \sum_{i=1}^n \left[ (g(\eta_i)h(\xi_i) - g(c_i)h(c_i)) (t_i - t_{i-1}) \right] \right|$$

$$\leq \sum_{i=1}^n \left( |g(\eta_i) - g(c_i)| \cdot |h(\xi_i)| + |g(c_i) - h(c_i)| \cdot |h(\xi_i)| \right) (t_i - t_{i-1})$$

$$\leq \varepsilon(||h|| + ||g||)(b - a).$$

This proves that $nJ_n$ tends to $\int_a^b g(t)h(t) \, dt$.

Consider now the case when $\max_{t \in [t_{i-1}, t_i]} g(t) < \varepsilon$ or $\max_{t \in [t_{i-1}, t_i]} h(t) < \varepsilon$. Suppose $g(t) < \varepsilon$ for every $t \in [t_{i-1}, t_i]$. The case when $\max_{t \in [t_{i-1}, t_i]} h(t) < \varepsilon$ can be analysed similarly. Because $g$ is nonnegative we deduce also that the oscillation of $g$ over the interval $[t_{i-1}, t_i]$ is at most $\varepsilon$. As we have done before $nJ_n$ differs from the integral $\int_a^b g(t)h(t) \, dt$ by less than $\varepsilon(b - a)3(||h||)$. \[\square\]

Lemma 2. For every function $f \in C^1[a, b]$ and every $g \in C^1[a, b]$ with $g'(t) > 0$, for every $t \in [a, b]$, we have

$$\int_a^b f(t) \, g(t) - |g(b) - g(a)| \min_{t \in [a, b]} f(t) \leq \frac{1}{2} (b - a)^2 \|f'\| \cdot \|g'\|.$$

Proof. Let $c \in [a, b]$ be the minimum point of $f$ over $[a, b]$. We have

$$\int_a^b f(t) \, g(t) - |g(b) - g(a)| \min_{t \in [a, b]} f(t) = \int_a^b |f(t) - f(c)|g'(t) \, dt$$

$$\leq \|f'\| \cdot \|g'\| \cdot \int_a^b |t - c| \, dt.$$
The proof is completed by using the inequality:

$$\int_a^b |t - c| \, dt = \frac{(c - a)^2}{2} + \frac{(b - c)^2}{2} \leq \frac{(b - a)^2}{2}.$$  \(\square\)

**Lemma 3.** Consider \(f\) a function of class \(C^1\) defined on \([a, b]\) with the derivative \(f'\) having a finite number of zeros. Let \(g \in C^1[a, b]\) be a function with \(g'(t) > 0\) for every \(t\) in \([a, b]\). Then

$$\limsup_{n \to \infty} n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{\text{opt}}, \min) \right) \leq \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt \right)^2.$$

**Proof.** We apply Lemma 1 to the functions \(|f'(t)|^{\frac{1}{2}}\) and \(|g'(t)|^{\frac{1}{2}}\) and obtain a division \(\Delta'\): \(a = t_0 < t_1 < t_2 < \cdots < t_n = b\) such that

$$J_n = (t_i - t_{i-1}) \cdot \max_{t \in [t_{i-1}, t_i]} |f'(t)|^{\frac{1}{2}} \cdot \max_{t \in [t_{i-1}, t_i]} |g'(t)|^{\frac{1}{2}},$$

has the same value for all values of \(i \in \{1, 2, \ldots, n\}\) and

$$\lim_{n \to \infty} nJ_n = \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt.$$

Using Lemma 2 we obtain

$$\int_a^b f(t) \, dg(t) - RS(f, g, \Delta', \min) = \sum_{i=1}^n \left( \int_{t_{i-1}}^{t_i} f(t) \, dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right)$$

$$\leq \frac{1}{2} \sum_{i=1}^n (t_i - t_{i-1})^2 \cdot \max_{t \in [t_{i-1}, t_i]} |f'(t)| \cdot \max_{t \in [t_{i-1}, t_i]} |g'(t)|$$

and finally

$$\limsup_{n \to \infty} n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{\text{opt}}, \min) \right)$$

$$\leq \limsup_{n \to \infty} n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta', \min) \right)$$

$$\leq \limsup_{n \to \infty} n \frac{1}{2} nJ_n^2 = \frac{1}{2} \lim_{n \to \infty} (nJ_n)^2$$

$$= \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt \right)^2.$$

\(\square\)
Lemma 4. Consider $f$ a function of class $C^1$ defined on $[a, b]$. Let $g \in C^1[a, b]$ be a function with $g'(t) > 0$ for every $t$ in $[a, b]$. If $f'(t) \neq 0$ in a subinterval $[p, q]$ of $[a, b]$, then for every $\xi \in [p, q]$ we have

$$\left| \int_p^q f(t) \, dg(t) - [g(q) - g(p)] \min_{t \in [p, q]} f(t) - \frac{1}{2}(q - p)^2 |f'(\xi)| g'(|\xi|) \right|$$

$$\leq \frac{1}{2}(q - p)^2 \cdot \|g'\| \cdot \omega(f', q - p) + \|f'\| \cdot \omega(g', q - p)$$

where $\omega(h, \delta)$ is the usual modulus of continuity of the function $h$.

Proof. Suppose $f' > 0$ on $[p, q]$. The case when the derivative of $f$ is strictly negative on $[p, q]$ can be treated similarly. Because $f$ is strictly increasing the minimum of $f$ is attained in $p$. We have

$$\int_p^q f(t) \, dg(t) - [g(q) - g(p)] \min_{t \in [p, q]} f(t) = \int_p^q [f(t) - f(p)] g'(t) \, dt.$$ 

Applying the Mean Value Theorem for integrals twice we obtain

$$\int_p^q [f(t) - f(p)] g'(t) \, dt = g'(c) \cdot \int_p^q [f(t) - f(p)] \, dt$$

$$= g'(c) \cdot \int_p^q \int_p^t f'(u) \, du \, dt$$

$$= g'(c) \cdot \int_p^q f'(u)(q - u) \, du$$

$$= g'(c) \cdot f'(d) \cdot \frac{(q - p)^2}{2},$$

for some $c, d \in (p, q)$. Because

$$|g'(c) \cdot f'(d) - f'(\xi) g'(|\xi|)| \leq \|g'\| \cdot \omega(f', q - p) + \|f'\| \cdot \omega(g', q - p)$$

the proof is complete. \qed

Lemma 5. Consider $f$ a function of class $C^1$ defined on $[a, b]$ with the derivative $f'$ having a finite number of zeros. Let $g \in C^1[a, b]$ be a function with $g'(t) > 0$ for every $t$ in $[a, b]$. Then

$$\liminf_{n \to \infty} n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) \geq \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt \right)^2 .$$

Proof. We first prove that for any $\delta > 0$ there exists a positive integer $r$ such that for any $n$-division $\Delta$ of $[a, b]$ the following inequality is true:

$$(n + r)^{\frac{1}{2}} \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta, \min) \right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{2}} \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt - \delta(b - a).$$
Since the function \( x \mapsto x^{1/2} \) is uniformly continuous on \([0, \infty)\), there exists \( \delta_1 > 0 \) such that for any \( x \) and \( y \) in \([0, \infty)\) if \(|x - y| < \delta_1\) then \(|x^{1/2} - y^{1/2}| < \delta\).

We take a subinterval \([p, q]\) of \([a, b]\) and suppose \( f'(t) \neq 0 \) in \([p, q]\). Because of the continuity of the derivatives of \( f \) and \( g \) there exists \( \eta > 0 \) such that if \( q - p < \eta \) then \( \frac{1}{2}[\|g'\| \cdot \omega(f', q - p) + \|f'\| \cdot \omega(g', q - p)] < \delta_1\). Using Lemma 4 we obtain

\[
\left| \frac{\int_p^q f(t) \, dg(t) - [g(q) - g(p)] \min_{t \in [p, q]} f(t)}{(q - p)^2} - \frac{1}{2} |f'(\xi)|g'(\xi)| \right| \leq \delta_1,
\]

for any \( \xi \in [p, q] \). Therefore, we have

\[
\left| \left( \frac{\int_p^q f(t) \, dg(t) - [g(q) - g(p)] \min_{t \in [p, q]} f(t)}{q - p} \right) - \frac{1}{\sqrt{2}} \sqrt{\|f'(\xi)|g'(\xi)(q - p)|} \right| \leq \delta,
\]

which is equivalent with

\[
\left| \left( \int_p^q f(t) \, dg(t) - [g(q) - g(p)] \min_{t \in [p, q]} f(t) \right) - \frac{1}{\sqrt{2}} \sqrt{\|f'(\xi)|g'(\xi)(q - p)|} \right| \leq \delta(q - p).
\]

Since \( f' \) is uniformly continuous on \([a, b]\), for the above \( \delta > 0 \) there exists \( \zeta > 0 \) such that \(|x - y| < \zeta\) implies \(|f'(x) - f'(y)| < \delta^2/\|g'\|\). We denote by \( Z \) the zero set of \( f' \):

\[ Z = \{ t \in [a, b] \mid f'(t) = 0 \} \]

and define the \( \zeta \)-neighborhood \( Z_{\zeta} \) of \( Z \) by

\[ Z_{\zeta} = \{ u \in [a, b] \mid \exists t \in Z : |t - u| < \zeta \} \].

Then for any \( t \in Z_{\zeta} \) we have \( g'(t)|f'(t)| < \delta^2 \) and \( f' \) is not equal to 0 on the complement of \( Z_{\zeta} \). By the definition of \( Z_{\zeta} \) and the properties of \( f' \) we can see that \( Z_{\zeta} \) is a disjoint union of finitely many intervals (by choosing \( \zeta \) small enough). We denote by \( r_1 \) the number of all endpoints of the intervals of \( Z_{\zeta} \).

For \( \eta > 0 \) obtained above we take a positive integer \( r_2 \) satisfying \( r_2 \geq (b - a)/\eta \) and set \( r = r_1 + r_2 \). For any \( n \)-division \( \Delta \) of \([a, b]\) we can add at most \( r_2 \) points to \( \Delta \) such that the mesh of the new division is less than or equal to \( \eta \). Moreover we add the endpoints of all the intervals of \( Z_{\zeta} \) and denote the new division by

\[ \Delta': t_0 = a < t_1 < \ldots < t_m = b. \]

By the definition of \( \Delta' \) we have \( m \leq n + r \) and \( t_i - t_{i-1} \leq \eta \). Each interval \([t_{i-1}, t_i]\) satisfies \([t_{i-1}, t_i] \subset \overline{Z_{\zeta}} \) or \([t_{i-1}, t_i] \subset [a, b] \setminus Z_{\zeta} \). In both cases we can take \( c_i \in [t_{i-1}, t_i] \) satisfying

\[ \int_{t_{i-1}}^{t_i} \sqrt{|f'(t)| \cdot g'(t)} \, dt = \sqrt{|f'(c_i)| \cdot g'(c_i)(t_i - t_{i-1})}. \]
In the case $[t_{i-1}, t_i] \subset Z_\zeta$ we have

$$\frac{1}{\sqrt{2}} \sqrt{|f'(c_i)| \cdot g'(c_i)(t_i - t_{i-1})} \leq \frac{1}{\sqrt{2}} \delta(t_i - t_{i-1})$$

$$\leq \left( \int_{t_i}^{t_{i-1}} f(t) dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right)^{\frac{1}{2}} + \delta(t_i - t_{i-1}).$$

In the case $[t_{i-1}, t_i] \subset [a, b] \setminus Z_\zeta$, $f'$ is not equal to 0 in $[t_{i-1}, t_i]$, so

$$\frac{1}{\sqrt{2}} \sqrt{|f'(c_i)| \cdot g'(c_i)(t_i - t_{i-1})}$$

$$\leq \left( \int_{t_i}^{t_{i-1}} f(t) dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right)^{\frac{1}{2}} + \delta(t_i - t_{i-1}).$$

Adding all these inequalities for $i = 1, 2, \ldots, m$ we get

$$\frac{1}{\sqrt{2}} \int_a^b \sqrt{|f'(t)| \cdot g'(t)} dt$$

$$\leq \sum_{i=1}^m \left( \int_{t_i}^{t_{i-1}} f(t) dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right)^{\frac{1}{2}} + \delta(b - a).$$

Applying the Cauchy-Schwarz inequality to the first term of the right-hand side of the above inequality we obtain

$$\sum_{i=1}^m \left( \int_{t_i}^{t_{i-1}} f(t) dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right)^{\frac{1}{2}}$$

$$\leq m^{\frac{1}{2}} \left( \sum_{i=1}^m \left( \int_{t_i}^{t_{i-1}} f(t) dg(t) - [g(t_i) - g(t_{i-1})] \min_{t \in [t_{i-1}, t_i]} f(t) \right) \right)^{\frac{1}{2}}$$

$$= m^{\frac{1}{2}} \left( \int_a^b f(t) dg(t) - RS(f, g, \Delta') \min f(t) \right)^{\frac{1}{2}}.$$

From these we have

$$\frac{1}{\sqrt{2}} \int_a^b \sqrt{|f'(t)| \cdot g'(t)} dt \leq m^{\frac{1}{2}} \left( \int_a^b f(t) dg(t) - RS(f, g, \Delta') \min f(t) \right)^{\frac{1}{2}} + \delta(b - a).$$

Because

$$[g(c) - g(b)] \min_{t \in [b, c]} f(t) + [g(b) - g(a)] \min_{t \in [a, b]} f(t) \geq [g(c) - g(a)] \min_{t \in [a, c]} f(t),$$

for every $a < b < c$, we obtain

$$\int_a^b f(t) dg(t) - RS(f, g, \Delta') \min f(t) \leq \int_a^b f(t) dg(t) - RS(f, g, \Delta, \min).$$
This estimate and \( m \leq n + r \) imply
\[
\frac{1}{\sqrt{2}} \int_a^b \sqrt{|f'(t)|g'(t)} \, dt \leq (n + r)^{\frac{1}{2}} \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta, \min) \right)^{\frac{1}{2}} + \delta(b - a).
\]

Now, let us prove Lemma 5. From the continuity of the function \( x \mapsto x^2 \) in \( x_0 = \frac{1}{2} \int_a^b \sqrt{|f'(t)|g'(t)} \, dt \), for any \( \varepsilon > 0 \) there exists \( \xi > 0 \) such that if \( x_0 - x \leq \xi \) we have \( x_0^2 - x^2 \leq \frac{\varepsilon}{2} \). We take \( \delta > 0 \) which satisfies \( \delta(b - a) < \xi \).

We can apply the result obtained above and get a positive integer \( r \) such that for any \( n \)-division \( \Delta \) of \([a, b]\)
\[
\xi \geq \delta(b - a)
\]
\[
\geq \frac{1}{\sqrt{2}} \int_a^b \sqrt{|f'(t)|g'(t)} \, dt - (n + r)^{\frac{1}{2}} \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta, \min) \right)^{\frac{1}{2}},
\]
which implies
\[
\frac{\varepsilon}{2} \geq \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)|g'(t)} \, dt \right)^2 - (n + r) \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta, \min) \right).
\]

We can substitute the optimal division \( \Delta_{opt} \) for \( \Delta \) in the above inequality and get
\[
(n + r) \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) \geq \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)|g'(t)} \, dt \right)^2 - \frac{\varepsilon}{2}.
\]

Since
\[
\lim_{n \to \infty} \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) = 0,
\]
we can choose a positive integer \( N \) such that for \( n \geq N \) the inequality
\[
0 \leq r \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) \leq \frac{\varepsilon}{2}
\]
holds. Thus
\[
n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) \geq \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)|g'(t)} \, dt \right)^2 - \varepsilon,
\]
for every \( n \geq N \). This completes the proof. \( \square \)

**Theorem 6.** Consider \( f \) a function of class \( C^1 \) defined on \([a, b]\) with the derivative \( f' \) having a finite number of zeros. Let \( g \in C^1[a, b] \) be a function with \( g'(t) > 0 \) for every \( t \) in \([a, b]\). Then
\[
\lim_{n \to \infty} n \left( \int_a^b f(t) \, dg(t) - RS(f, g, \Delta_{opt}, \min) \right) = \frac{1}{2} \left( \int_a^b \sqrt{|f'(t)| \cdot g'(t)} \, dt \right)^2.
\]

**Proof.** The result follows from the inequalities of Lemma 8 and 5. \( \square \)
References

[1] A. M. Gleason, *A curvature formula*, Amer. J. Math., 101 (1979), 86–93.

[2] H. Tasaki, *Convergence rates of approximate sums of Riemann integrals*, J. Approx. Theory, 161, 2 (2009), 477–490.