GLOBAL MILD SOLUTIONS TO THREE-DIMENSIONAL MAGNETOHYDRODYNAMIC SYSTEM IN MORREY SPACES

FENG LIU, SHUAI XI, AND SHENGGUO ZHU

ABSTRACT. In this article, the Cauchy problem of three-dimensional (3-D) incompressible magnetohydrodynamic system was investigated. If the initial $M^{1,1}$ norms of the vorticity $\omega$ and the current density $j$ are both sufficiently small, then some uniform estimates with respect to time for the coupling terms between the fluid and the magnetic field can be established, which lead to a global-in-time well-posedness of mild solutions in Morrey spaces via some effective arguments.

1. Introduction

Magnetohydrodynamics (MHD) is part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. The motion of an electrically conducting, viscous incompressible fluid in $\mathbb{R}^3$ can be described by the Magnetohydrodynamic equations (see [7, 22]):

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \left( P + \frac{1}{2} |b|^2 \right) = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty),$$

$$\partial_t b - \eta \Delta b + \nabla \times (b \times u) = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty),$$

$$\text{div} u = \text{div} b = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty),$$

and the initial data

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x) \quad \text{for} \quad x \in \mathbb{R}^3,$$

where $(x = (x_1, x_2, x_3))^T$, $t \in \mathbb{R}^3 \times \mathbb{R}_+$, the unknowns $u(x, t) = (u^1, u^2, u^3)^T(x, t)$, $P(x, t)$ and $b(x, t) = (b^1, b^2, b^3)^T(x, t)$ denote the fluid velocity, pressure and magnetic field, respectively. Here, $P_T = P + \frac{1}{2} |b|^2$ is the total kinetic pressure, $\nu$ is the kinematic viscosity and $\eta$ is the resistivity. For simplicity, we assume throughout that both $\nu$ and $\eta$ are equal to 1.

The global existence of MHD equations with finite energy initial data, i.e., $(u_0, b_0) \in L^2(\mathbb{R}^3)$ was independently proved by Duvaut-Lions [8] and Sermange-Temam [26]. Mild solutions, for the Navier-Stokes (NS) equations, were first constructed by Kato-Fujita [10] in the spaces $H^s(\mathbb{R}^d)$ for $s \geq \frac{d}{2} - 1$, and then in $L^p(\mathbb{R}^d)$ ($p \geq d$) spaces (classical admissible spaces) by Kato [17]. The other global well-posedness of mild solutions for small initial data is due to Cannone [3] and Planchon [25] in the Besov space $B_{p,\infty}^{\frac{d}{2}}(\mathbb{R}^d)$, with $1 < p < \infty$, Koch-Tataru [20] in the space $BMO^{-1}$, and Lei-Lin [23] in the space $\chi^{-1} = \{ f \in \mathcal{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{-1} |\hat{f}(\xi)| d\xi < \infty \}$.

Our purpose, in this paper, is to establish well-posedness results of MHD equations with the initial data highly concentrated in “small sets”(“rough data”), such as the initial vortex profiles are vortex rings and filaments. Such kind of rough initial data is of infinite energy and thus outside the scope of the standard energy method and the classical Leary’s theory ([8], [26]).

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For the case of the multi-dimensional incompressible Navier-Stokes (NS) equations, the well-posedness with such kind of rough initial data has been studied by many mathematicians in this field. We attribute the corresponding development into two stages:

**Stage I.** (Two-dimensional (2-D) NS equation). The global existence of solutions to 2-D NS equations with large initial data in the space of finite measures $M(\mathbb{R}^2)$ was first shown by Cottet [5], and independently by Giga-Miyakawa-Osada [16], and the proof in [16] was simplified by Kato [19]. Under the assumption that the atomic part of the initial vorticity is sufficiently small, uniqueness was also proved in [16, 19]. Later, Gallagher-Gallay [11] extended the uniqueness theory to the general initial data in $M(\mathbb{R}^2)$ via some arguments developed in [12], which allows to handle large Dirac masses. In summary, NS equations are well-posed for arbitrary data in the space of $M(\mathbb{R}^2)$.

**Stage II.** (3-D NS equation). For 3-D NS equations, to the best of our knowledge, the progress is not as good as that of 2-D case, of which the reason is mainly due to the different forms of vorticity stretching term in the equation of vorticity, which can be shown as follows,

$$(u \cdot \nabla)\omega \quad \text{in} \quad \mathbb{R}^2 \quad \text{and} \quad (u \cdot \nabla)\omega - (\omega \cdot \nabla)u \quad \text{in} \quad \mathbb{R}^3.$$  

As a result, compared with the 2-D case, more estimates on $u$ are required to control the $L^1(\mathbb{R}^3)$ norm of the vorticity. The first attempt in the measure direction, for the well-posedness of the 3-D case, is not as good as that of 2-D case, of which the reason is mainly due to the different forms of $\omega$ and $u$. As a result, compared with the 2-D case, more estimates on $u$ are required to control the $L^1(\mathbb{R}^3)$ norm of the vorticity. The first attempt in the measure direction, for the well-posedness of the 3-D case, is not as good as that of 2-D case, of which the reason is mainly due to the different forms of $\omega$ and $u$.

In order to get rid of the finite total variation assumption, Giga and Miyakawa introduced the Morrey-type space $M^p(\mathbb{R}^3)$ of measure (see Definition 2.1) in [15], and established the existence of global solutions whose initial vorticity is a small Radon measure belonging to $M^2(\mathbb{R}^3)$. In order to solve NS equations for the velocity $u(x,t)$ instead of the vorticity $\omega$, Kato [18] introduced a more general Morrey space $M^{m,k}(\mathbb{R}^3)$ (see Definition 2.1) and established a more general result: if $u_0$ is small enough in $M^{m,3-m}(\mathbb{R}^3)$ for $1 < m \leq 3$, then there is a global solution $u \in M^{m,3-m}(\mathbb{R}^3)$, which is unique subject to certain restrictions. This partially generalizes the result of [15] by eliminating the differentiability of $u_0$ except the case for $m = 1$. The analysis on the meaningful case, $m = 1$, which admits certain measures, encounters some essential mathematical difficulties, since the Calderon-Zygmund type singular operator is not bound in $L^1(\mathbb{R}^3)$. Some other related progress, for 3-D NS equations in Morrey space, can also be found in [4, 9, 21, 27].

For the MHD system, the situation is more complicated due to the strong coupling effect between the velocity vector field $u(x,t)$ and the magnetic field $b(x,t)$. In order to clearly analyse the coupling effect of this system, we give the equations on the time evolution of the vorticity $\omega = (\omega^1, \omega^2, \omega^3)^T$ and the current density $j = (j^1, j^2, j^3)^T$:

$$\begin{align*}
\partial_t \omega & - \Delta \omega + \partial_i (u^i \omega - \omega^i u - b^i j + j^i b) = 0, \\
\partial_t j & - \Delta j + \nabla \times ((u \cdot \nabla) b - (b \cdot \nabla) u) = 0, \\
u &= K \ast \omega, \\
b &= K \ast j, \\
\text{div} \omega &= \text{div} j = 0, \\
\omega(x,0) &= \omega_0, \\
j(x,0) &= j_0,
\end{align*}$$

where $K(x) = -\frac{1}{4\pi} \frac{(x_1 x_2 x_3)}{|x|^3}$ is the Biot-Savart kernel. Based some elaborate analysis of the above system, in the current paper, we will present the global mild solutions of (1.5). If the $W^{1,1}$ norms of $\omega$ and $j$ are sufficiently small initially, then some uniform estimates with respect to time for the coupling terms between the fluid and the magnetic field can be obtained, which lead to a global-in-time well-posedness of mild solutions in Morrey spaces via some effective arguments.
2. Preliminary definitions and main results

In this section, we will give some necessary definitions and then state our main result. Moreover, the main strategy of our following proof will be also discussed.

2.1. Preliminary definitions

We start with the definition of Morrey spaces.

**Definition 2.1** ([15, 18]). Let \( \mu \) be a measurable function. For \( 1 \leq p < \infty \) and \( 0 \leq \lambda < 3 \), the Morrey space \( M^{p, \lambda}(\mathbb{R}^3) \) is defined as

\[
\|\mu\|_{M^{p, \lambda}(\mathbb{R}^3)} = \begin{cases} 
\sup_{x \in \mathbb{R}^3} r^{-\lambda} \|\mu\|_{TV(B(x, r))}, & p = 1, \\
\sup_{x \in \mathbb{R}^3, r > 0} r^{-\lambda/p} \left( \int_{B(x, r)} |\mu|^p(y)dy \right)^{1/p}, & p > 1.
\end{cases}
\]

Here \( B(x, r) \) is the open ball in \( \mathbb{R}^3 \) with radius \( r \) centered at \( x \) and \( \|\mu\|_{TV(B(x, r))} \) is the total variation of \( \mu \). Particularly, we denote \( M^{1, \lambda}(\mathbb{R}^3) = M^{q}(\mathbb{R}^3) \), for all \( 1 \leq q < \infty \) (see [15]).

**Definition 2.2** ([2]). Let \( G(t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t}) \), \( \omega = \nabla \times u \), \( j = \nabla \times b \), \( u = (u^1, u^2, u^3) \), \( \omega = (\omega^1, \omega^2, \omega^3) \), \( j = (j^1, j^2, j^3) \) and \( b = (b^1, b^2, b^3) \). We say that a pair \( (\omega, j) \) constitutes the mild solution to the IVP (1.5) if they satisfy: for all \( 0 < t < \infty \), \( x \in \mathbb{R}^3 \) and \( i \in \{1, 2, 3\} \),

\[
\omega(x, t) = G(t) \ast \omega_0(x) \\
+ \int_0^t \int_{\mathbb{R}^3} G(x-y, t-s) \partial_{y_i} (u^i \omega - u \omega^i - b^i j + b^i j^i)(y, s) dy ds,
\]

\( j(x, t) = G(t) \ast j_0(x) \\
+ \int_0^t \int_{\mathbb{R}^3} G(x-y, t-s) \nabla_y \times ((u \cdot \nabla) b - (b \cdot \nabla) u)(y, s) dy ds
\]

and

\[
\omega(x, t) \xrightarrow{\text{weak}*} \omega_0(x) \quad \text{and} \quad j(x, t) \xrightarrow{\text{weak}*} j_0(x) \quad \text{as} \quad t \to 0.
\]

Here, formula (2.1) is understood in the sense of \( L^1(\mathbb{R}^3) \) and formula (2.2) means that for any given \( \phi(x) \in C_0(\mathbb{R}^3) \), the functions

\[
g(t) = \begin{cases} 
\int_{\mathbb{R}^3} \omega(x, t) \phi(x) dx, & t > 0, \\
\int_{\mathbb{R}^3} \phi(x) d\omega_0, & t = 0,
\end{cases}
\]

and

\[
h(t) = \begin{cases} 
\int_{\mathbb{R}^3} j(x, t) \phi(x) dx, & t > 0, \\
\int_{\mathbb{R}^3} \phi(x) d j_0, & t = 0,
\end{cases}
\]

are both continuous at \( t = 0 \).

Here and throughout this paper, the letter \( C \) or \( c \), sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. We set \( \mathbb{R}^+ := (0, \infty) \). For \( 1 \leq p \leq \infty \), we denote \( p' \) by the conjugate index of \( p \), that is \( 1/p + 1/p' = 1 \) (here we set \( 1'/2 = \infty \) and \( \infty' = 1 \)). We also adopt the following simplified notations:

\[
\|f\|_p = \|f\|_{M^p(\mathbb{R}^3)}, \quad \|f\|_{p, \lambda} = \|f\|_{M^{p, \lambda}(\mathbb{R}^3)}, \quad \|f\|_{L^\infty} = \|f\|_{L^{\infty}(\mathbb{R}^3)}.
\]

We assume that all the functions in this paper are real-valued.

2.2. Main results
We now state our main results as follows. The Global-in-time well-posedness of (1.5) can be formulated as follows:

**Theorem 1. (Global-in-time well-posedness).** We define
\[
A_1 := \{ (p,q) | p \in (1,2), \quad q \in [1,3-p), \quad (p-2)(q-4) \leq 2, \quad 3 < 2p + q < 6 \},
\]
\[
A_2 := \{ (p,q) | q \in [1,2), \quad p \in \left( \frac{3(q-3)}{4-q}, 3-q \right) \}.
\]
Assume that
\[
\|\omega_0\|_{1,1} \text{ and } \|j_0\|_{1,1}
\]
are small enough,

then
\[(i) \text{ if } (p,q) \in A_1, \text{ there exists a unique mild solution } (\omega, j) \text{ in } [0,\infty) \times \mathbb{R}^3 \text{ to the problem (1.5) such that}
\]
\[
1-\frac{3}{2q}\omega, \quad 1-\frac{3}{2q}j \in L^\infty(\mathbb{R}^+; M^{1,q}(\mathbb{R}^3)),
\]
\[
\omega, \quad j \in L^{\frac{2p}{p+2}}(\mathbb{R}^+; M^{p,q}(\mathbb{R}^3)),
\]
\[
t^{\frac{3}{2q}}\nabla \omega, \quad t^{\frac{3}{2q}}\nabla j \in L^\infty(\mathbb{R}^+; M^{p,q}(\mathbb{R}^3)),
\]
\[
\nabla \omega, \quad \nabla j \in L^{\frac{2p}{p+2}}(\mathbb{R}^+; M^{p,q}(\mathbb{R}^3)),
\]
and the solution \((\omega, j)\) solves (1.5) in the classical sense for \(t > 0\);

\[(ii) \text{ if } (p,q) \in A_1 \cap A_2, \text{ there exists a unique mild solution } (\omega, j) \text{ in } [0,\infty) \times \mathbb{R}^3 \text{ to the problem (1.5) such that}
\]
\[
t^{\frac{3}{4q}}\omega, \quad t^{\frac{3}{4q}}j \in L^\infty(\mathbb{R}^+; M^{1,q}(\mathbb{R}^3)),
\]
\[
\omega, \quad j \in L^{\frac{2p}{p+2}}(\mathbb{R}^+; M^{1,q}(\mathbb{R}^3)),
\]
\[
t^{\frac{3}{4q}}\nabla \omega, \quad t^{\frac{3}{4q}}\nabla j \in L^\infty(\mathbb{R}^+; M^{1,q}(\mathbb{R}^3)),
\]
\[
\nabla \omega, \quad \nabla j \in L^{\frac{2p}{p+2}}(\mathbb{R}^+; M^{1,q}(\mathbb{R}^3)),
\]
and the solution \((\omega, j)\) solves (1.5) in the classical sense for \(t > 0\).

**Remark 2.** By (ii) of Theorem 1, taking \(q = 1\), it is easy to see that there exists a global unique mild solution \((\omega, j)\) in \([0,\infty) \times \mathbb{R}^3\) to the problem (1.5) such that
\[
\omega, \quad j \in L^\infty(\mathbb{R}^+; M^{1,1}(\mathbb{R}^3)),
\]
\[
t^\frac{3}{4q} \nabla \omega, \quad t^\frac{3}{4q} \nabla j \in L^\infty(\mathbb{R}^+; M^{1,1}(\mathbb{R}^3)),
\]
\[
\nabla \omega, \quad \nabla j \in L^2(\mathbb{R}^+; M^{1,1}(\mathbb{R}^3)),
\]
when \(\|\omega_0\|_{1,1} \text{ and } \|j_0\|_{1,1}\) are small enough. For this case, the result obtained above can be regarded as an extension of the theory for NS equations in [15] to MHD. However, our proof indeed needs some new ideas due to the strong coupling between the fluid and the magnetic field.

### 2.3. Main strategies

The main purpose of the current work is to prove the global-in-time well-posedness of mild solutions to the Cauchy problem (1.5), which generated a sequence from the iterative scheme of the vorticity equations and converged to a root by the Fixed-point theorem. Actually, in order to eliminate the total kinetic pressure term \(\nabla P_T\) in (1.1), one usual method is to apply the Leray projector \(P\) in [18] (\(P\) is a matrix \(3 \times 3\) with elements \((P)_{i,j} = \delta_{ij} + R_k R_j\), where \(R_k = \delta_k(\Delta)^{1/2}\) \((k = 1, 2, 3)\) are the Riesz transforms) or the curl operator in [15]. Considering the unboundedness of the Leray projector \(P\) in \(M^{p}(\mathbb{R}^3)\), we have to use the curl operator in this paper to handle \(\nabla P_T\) along the spirit of [15].

However, the approaches used in [15] for establishing the existence of the NS system fail to apply to the corresponding problem of magnetohydrodynamic system directly due to some new mathematical difficulties:
• the strong coupling between the magnetic filed $b$ and the fluid velocity $u$;

• one can not control $\|\nabla u\|_p$ and $\|\nabla b\|_p$ by $\|\nabla \times u\|_p$ and $\|\nabla \times b\|_p$ in $\mathcal{M}^p(\mathbb{R}^3)$, which are mainly reflected in estimating the cross term $(u \cdot \nabla) b$ and $(b \cdot \nabla) u$.

Hence, compared with the theory for pure NS system, in order to overcome the difficulties mentioned above, some new observations are indeed required. Actually, we found that the following estimate plays a key role in establishing the global-in-time estimates for the desired mild solutions:

$$\|(b \cdot \nabla) u\|_{1, \lambda} \leq \|\nabla u\|_{\theta, \tau} \|b\|_{r, s},$$

for $1 = \frac{1}{\theta} + \frac{1}{r}$ and $\lambda = \frac{\tau}{\theta} + \frac{s}{r}$,

which means that we need to estimate $\|\omega\|_{\theta, \tau}$ and $\|b\|_{r, s}$. For this purpose,

• first, we estimate $\|\omega\|_{\theta, \tau}$ when the initial norms $\|\omega_0\|_{1, 1}$ and $\|j_0\|_{1, 1}$ are both sufficiently small. Lemma 3 in the Appendix B is introduced for this step. Particularly, we established the bounds for the map

$$T : \mathcal{M}^{q_1, \lambda_1}(\mathbb{R}^3) \to \mathcal{M}^{q_2, \lambda_2}(\mathbb{R}^3)$$

with $0 \leq \lambda_1 \leq \lambda_2 < 3$, which is new for the case $0 \leq \lambda_1 < \lambda_2 < 3$, if $T$ is convolution operator with heat kernel (see Proposition 4 in Appendix A);

• second, under the same initial data, we can control $\|b\|_{r, s}$ by $\|\omega\|_{0, \tau}$ and $\|\omega\|_{1, \lambda}$.

At last, we used a inequality that is similar to Lemma 3 in the Appendix B to get the well-posedness of the desired global solution.

2.4. Outline

The rest of the paper is organized as follows. In Section 3, we give the proof of Theorem 3, which is divided into the following two steps:

• Step (i): in Subsection 3.1, we establish the global-in-time well-posedness of mild solutions in $\mathcal{M}^{p,q}(\mathbb{R}^3)$ for some $p > 1$ and $q \in [0, 3)$;

• Step (ii): in Subsection 3.2, we establish the global-in-time well-posedness of mild solutions in $\mathcal{M}^{1,q}(\mathbb{R}^3)$ for some $q \in [0, 4)$.

The proof of Theorem 1, which can be regarded as the special case of Theorem 3, is listed at the end of the Section 3.

In Appendix, we give some basic properties of Morrey spaces and some technical lemmas. Among those, Appendix A is devoted to presenting some basic properties of Morrey spaces and some inequalities for Riesz potential and convolution operators with heat kernel and Biot-Savart kernel on Morrey spaces, and Appendix B is established to give some preliminary lemmas, which play an important role in our proof.

3. Proof of Theorem 1

This section will give the proof of Theorem 1. Actually, Theorem 1 can be deduced by the following result, which is of interest in its own right.
Theorem 3. We set
\[ E_1 := \{(p,q,p_0,q_0) : p_0 \in [1, \infty), \ p_0 \leq p, \ q_0 \in [0,3), \ 2p_0 + q_0 = 3, \]
\[ 1 < p, \ q_0 \leq q < 3, \ p + q < 3, \ 3 < 2p + q < 6, \ (q - 4)(p - 2) \leq 2 \}; \]
\[ E_2 := \{(p,q,p_0,q_0,q_1) : (p,q,p_0,q_0,q_1) \in E_1, \there \exists q_2, q_3 \in [0,3) \]
\[ \text{and } \bar{p} \in (1, \min\{p, p'\}) \text{ such that } 0 \leq q_1 - \bar{q}_0 < 1, \ 0 \leq q_1 - q_2 < 1, \]
\[ q_2 = \frac{\bar{p}q_0 + \frac{q}{p}}{2}, \ \frac{1}{p} = \frac{1}{p} + \frac{3 - q_1}{q_1} \]
\[ q_3 = q_1 \left( \frac{p'}{p} - \frac{p'}{p} \right) + \frac{2}{p'}(1 + p' - \frac{p'}{p}) \\]
\[ q_2 - \bar{q}_0 + \frac{1}{2} = q_1 - \bar{q}_0 + \frac{2p - 3 + q}{2p}(1 + p' - \frac{p'}{p}) \} \]
(i) Let \((p,q,p_0,q_0) \in E_1. \) Assume that
\[ \|\omega_0\|_{p_0,q_0} \text{ and } \|j_0\|_{p_0,q_0} \text{ are small enough.} \]
Then there exists a global unique mild solution \((\omega, j)\) on \(\mathbb{R}^+\) of (1.5) such that
\[ t^{\frac{1 - \frac{1}{p}q}{p}} \omega, \ t^{\frac{1 - \frac{1}{p}q}{p}} j \in L^\infty(\mathbb{R}^+; \mathcal{M}^{p,q}(\mathbb{R}^3)), \]
\[ \omega, \ j \in L^{\frac{2p}{2p - 1 - q}}(\mathbb{R}^+; \mathcal{M}^{p,q}(\mathbb{R}^3)), \]
\[ t^{\frac{3}{2} - \frac{3}{2p} q^2} \nabla \omega, \ t^{\frac{3}{2} - \frac{3}{2p} q^2} \nabla j \in L^\infty(\mathbb{R}^+; \mathcal{M}^{p,q}(\mathbb{R}^3)), \]
\[ \nabla \omega, \ \nabla j \in L^{\frac{2p}{2p - 1 - q}}(\mathbb{R}^+; \mathcal{M}^{p,q}(\mathbb{R}^3)). \]
(ii) Let \((p,q,p_0,q_0,q_1) \in E_2. \) Assume that
\[ \|\omega_0\|_{p_0,q_0} \text{ and } \|j_0\|_{p_0,q_0} \text{ are small enough,} \]
and
\[ \omega_0, j_0 \in \mathcal{M}^{1,q_1}(\mathbb{R}^3). \]
Then there exists a global unique mild solution \((\omega, j)\) on \(\mathbb{R}^+\) of (1.5) such that
\[ t^{\frac{q_1 - \bar{q}_0}{q_1}} \omega, \ t^{\frac{q_1 - \bar{q}_0}{q_1}} j \in L^\infty(\mathbb{R}^+; \mathcal{M}^{1,q_1}(\mathbb{R}^3)), \]
\[ \omega, \ j \in L^{\frac{2}{q_1 - \bar{q}_0}}(\mathbb{R}^+; \mathcal{M}^{1,q_1}(\mathbb{R}^3)), \]
\[ t^{\frac{1 + q_1 - \bar{q}_0}{q_1}} \nabla \omega, \ t^{\frac{1 + q_1 - \bar{q}_0}{q_1}} \nabla j \in L^\infty(\mathbb{R}^+; \mathcal{M}^{1,q_1}(\mathbb{R}^3)), \]
\[ \nabla \omega, \ \nabla j \in L^{\frac{2}{q_1 - \bar{q}_0}}(\mathbb{R}^+; \mathcal{M}^{1,q_1}(\mathbb{R}^3)). \]

In order to prove Theorem 3, we need to use the standard successive approximation scheme:
\[ \left\{ \begin{array}{l}
\omega^{(0)}(x,t) = G(\cdot,t) \ast \omega_0(x), \ j^{(0)}(x,t) = G(\cdot,t) \ast j_0(x), \\
\omega^{(k+1)}(x,t) = \omega^{(0)}(x,t) + \int_0^t \int_{\mathbb{R}^3} G(x - y,t - s) \\
\quad \times \partial_y(u^{(k)}(\omega^{(k)} - u^{(k)} \omega^{(k)} - b^{(k)} j^{(k)}) + b^{(k)} j^{(k)})(y,s)dyds, \\
(j^{(k+1)}(x,t) = j^{(0)}(x,t) + \int_0^t \int_{\mathbb{R}^3} G(x - y,t - s) \\
\quad \times \nabla_y ((u^{(k)}(\cdot) \cdot b^{(k)}(x)) - (b^{(k)} \cdot \nabla)u^{(k)})(y,s)dyds, \\
(u^{(k)}(x,t) = K \ast \omega^{(k)}(x,t), \ b^{(k)}(x,t) = K \ast j^{(k)}(x,t). \end{array} \right. \]

We now prove Theorem 3 by the following two parts:

3.1. Proof for part (i) of Theorem 3
Let \((p, q, p_0, q_0) \in E_1\). There exist two real numbers \(r, \theta\) such that
\[
1 \leq \theta < p, \quad \frac{1}{\theta} = \frac{1}{r} + \frac{1}{p}, \quad \frac{1}{p} - \frac{1}{r} = \frac{1}{3 - q}.
\]

For convenience, we set
\[
W_{k,p,q}^0 = \sup_{t \in \mathbb{R}^+} t^{\frac{3 - q}{p}} \|\omega^{(k)}(\cdot, t)\|_{p,q}, \quad \tilde{W}_{k,p,q}^0 = \|\|\omega^{(k)}(\cdot, t)\|_{p,q}\|_{L_t^{\frac{2p}{3 + q}}(\mathbb{R}^+)},
\]
\[
W_{k,p,q}^1 = \sup_{t \in \mathbb{R}^+} t^{\frac{3 - q}{p}} \|\nabla \omega^{(k)}(\cdot, t)\|_{p,q}, \quad \tilde{W}_{k,p,q}^1(u) = \|\|\nabla \omega^{(k)}(\cdot, t)\|_{p,q}\|_{L_t^{\frac{2p}{3 + q}}(\mathbb{R}^+)},
\]
\[
J_{k,p,q}^0 = \sup_{t \in \mathbb{R}^+} t^{\frac{3 - q}{p}} \|j^{(k)}(\cdot, t)\|_{p,q}, \quad J_{k,p,q}^0 = \|\|j^{(k)}(\cdot, t)\|_{p,q}\|_{L_t^{\frac{2p}{3 + q}}(\mathbb{R}^+)},
\]
\[
J_{k,p,q}^1 = \sup_{t \in \mathbb{R}^+} t^{\frac{3 - q}{p}} \|\nabla j^{(k)}(\cdot, t)\|_{p,q}, \quad J_{k,p,q}^1(u) = \|\|\nabla j^{(k)}(\cdot, t)\|_{p,q}\|_{L_t^{\frac{2p}{3 + q}}(\mathbb{R}^+)},
\]

The proof for part (i) of Theorem 3 will be divided into five steps:

**Step 1: Estimates for the terms** \(W_{k,p,q}^0, J_{k,p,q}^0, W_{k,p,q}^0\), and \(J_{k,p,q}^0\).

By (3.3) and Proposition 4, there exists a constant \(C_1 > 0\) such that
\[
\|\omega^{(k+1)}(\cdot, t)\|_{p,q} \leq \|G(\cdot, t) * \omega_0\|_{p,q} + C_1 \int_0^t (t - s)^{-\frac{1}{2} - \frac{1}{4} \left(\frac{3 - q}{p}\right)} \times \|\|\omega^{(k)}(\cdot, t) - u^{(k)}(\cdot, t)\|_{p,q}\|_{\theta_0} ds,
\]
and
\[
\|j^{(k+1)}(\cdot, t)\|_{p,q} \leq \|G(\cdot, t) * j_0\|_{p,q} + C_1 \int_0^t (t - s)^{-\frac{1}{2} - \frac{1}{4} \left(\frac{3 - q}{p}\right)} \times \|\|j^{(k)}(\cdot, t)\|_{p,q}\|_{\theta_0} ds.
\]
Inequality (3.8) together with (3.9) yields that
\[
\|u_{k+1}(\cdot,t)\|_{p,q} \leq 2\|G(\cdot, t) * (|\omega_0| + |j_0|)\|_{p,q} + 4C_1C_2 \int_0^t (t-s)^{-\frac{3-p}{2}} \|u_k(\cdot,s)\|_{p,q} ds.
\] (3.10)

On the other hand, by Proposition 4, there exists a constant \(A_1 > 0\) such that
\[
\|u_1(\cdot, t)\|_{p,q} \leq A_1 t^{-\frac{3}{2} - \frac{p}{2}} \|u_0\|_{p_0,q_0} \leq A_1 t^{-\frac{3}{2p^2} - 1} \|u_0\|_{p_0,q_0}
\] (3.11)
for all \(u_0 \in \mathcal{M}^{p_0,q_0}(\mathbb{R}^3)\). By (3.10), (3.11) and invoking Lemma 3, we can get
\[
W_{k,p,q}^0 + J_{k,p,q}^0 + W_{k,p,q}^1 + J_{k,p,q}^1 \leq 16A_1 \|\omega_0\|_{p_0,q_0} + |j_0|_{p_0,q_0} \leq 16A_1 \left(\|\omega_0\|_{p_0,q_0} + |j_0|_{p_0,q_0}\right)
\] (3.12)
for all \(k \geq 1\), whenever
\[
\|\omega_0\|_{p_0,q_0} + |j_0|_{p_0,q_0} \leq \frac{1}{32A_1C_1C_2} \min \left\{ \frac{1}{C(1 - \frac{3}{2p^2}, \frac{3}{p} - 1)} \right\} =: G_1.
\] (3.13)

**Step 2: Estimates for the terms** \(W_{k,p,q}^1\) and \(J_{k,p,q}^1\).

By (3.3) and Proposition 4 again, there exists a constant \(C_3 > 0\) such that
\[
\|\nabla \omega^{k+1}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot, t) * \omega_0\|_{p,q} + C_3 \int_{t/2}^t (t-s)^{-\frac{1}{2} - \frac{1}{2} \left(\frac{3}{p} - \frac{1}{2}\right)} \times \left( \|\left((u_0^{(k)} \cdot \nabla) \omega^{(k)} + (\omega_0^{(k)} \cdot \nabla) u^{(k)} - (b^{(k)} \cdot \nabla) j^{(k)} + (j^{(k)} \cdot \nabla) b^{(k)}(\cdot,s)\|_{q_0,q_0} ds
\right)
\] (3.14)
\[
\|\nabla j^{(k+1)}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot, t) * j_0\|_{p,q} + C_3 \int_{t/2}^t (t-s)^{-\frac{1}{2} - \frac{1}{2} \left(\frac{3}{p} - \frac{1}{2}\right)} \times \left( \|\left((u_0^{(k)} \cdot \nabla) j^{(k)} + \nabla u^{(k)} \times b^{(k)}_u - (b^{(k)} \cdot \nabla) \omega^{(k)} - \nabla b^{(k)}(\cdot,s)\|_{q_0,q_0} ds
\right)
\] (3.15)
By Propositions 2 (i) and 3, one finds that
\[
\|\left((u_0^{(k)} \cdot \nabla) \omega^{(k)} - (\omega_0^{(k)} \cdot \nabla) u^{(k)} - (b^{(k)} \cdot \nabla) j^{(k)} + (j^{(k)} \cdot \nabla) b^{(k)}(\cdot,s)\|_{q_0,q_0} \leq \|u_0^{(k)}(\cdot,s)\|_{p,q} \|\nabla \omega^{(k)}(\cdot,s)\|_{p,q} + \|\omega_0^{(k)}(\cdot,s)\|_{p,q} \|\nabla u^{(k)}(\cdot,s)\|_{p,q}
\] (3.16)
\[
\|\left((u_0^{(k)} \cdot \nabla) j^{(k)} + \nabla u^{(k)} \times b^{(k)}_u - (b^{(k)} \cdot \nabla) \omega^{(k)} - \nabla b^{(k)}(\cdot,s)\|_{q_0,q_0} \leq \|u_0^{(k)}(\cdot,s)\|_{p,q} \|\nabla j^{(k)}(\cdot,s)\|_{p,q} + \|\omega_0^{(k)}(\cdot,s)\|_{p,q} \|\nabla u^{(k)}(\cdot,s)\|_{p,q}
\] (3.17)
Applying Proposition 4, there exists a constant \(A_2 > 0\) such that
\[
\|\nabla G(\cdot,t) * u_0\|_{p,q} \leq A_2 t^{-\frac{3}{2} - \frac{p}{2}} \|u_0\|_{p_0,q_0} = A_2 t^{-\frac{3}{2p^2} - \frac{2}{p}} \|u_0\|_{p_0,q_0}.
\] (3.18)
It now follows from (3.14)-(3.18), (3.6) and (3.7) that
\[
\| \nabla \omega^{k+1}(\cdot, t) \|_{p,q} 
\leq A_2 \left( \frac{1}{2} \right)^{\frac{3-q}{p}} \| \omega_0 \|_{p_0,q_0} + 2C_2 C_3 \int_{1/2}^{t} (t-s)^{-\frac{1}{2} - \frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q} \| \nabla \omega^{k}(\cdot, s) \|_{p,q} 
+ \| j^{(k)}(\cdot, s) \|_{p,q} \| \nabla j^{(k)}(\cdot, s) \|_{p,q} ds \n+ C_2 C_3 \int_{0}^{t/2} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 + \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds \n+ C_2 C_3 \int_{0}^{t/2} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 + \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds,
\]
(3.19)

Some direct computations may yield that
\[
\int_{1/2}^{t} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q} \| \nabla \omega^{k}(\cdot, s) \|_{p,q} + \| j^{(k)}(\cdot, s) \|_{p,q} \| \nabla j^{(k)}(\cdot, s) \|_{p,q} ds 
\leq (W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1) \int_{1/2}^{t} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 ds \n+ \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds \n+ 2C_2 C_3 \int_{0}^{t/2} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 + \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds \n+ \int_{0}^{t/2} (t-s)^{-\frac{3-q}{p}} (3\| \omega^{k}(\cdot, s) \|_{p,q} \| \nabla \omega^{k}(\cdot, s) \|_{p,q} + \| j^{(k)}(\cdot, s) \|_{p,q} \| \nabla j^{(k)}(\cdot, s) \|_{p,q}) ds \n\leq \left( \frac{2}{2p+q-5} \right)^{2} \left( \frac{3}{2p+q-5} \right)^{2} (3W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1(3W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1), \n\int_{0}^{t/2} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 + \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds \n\leq \left( W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1 \right) \int_{1/2}^{t} (t-s)^{-\frac{3-q}{p}} \| \omega^{k}(\cdot, s) \|_{p,q}^2 ds \n+ \| j^{(k)}(\cdot, s) \|_{p,q}^2 ds \n\leq \left( \frac{2}{2p+q-5} \right)^{2} \left( \frac{3}{2p+q-5} \right)^{2} (3W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1).
\]
(3.21)

Inequality (3.19) together with (3.20)-(3.23) implies that there exists a constant $C_4 > 0$ such that
\[
W_{k+1,p,q}^1 \leq A_2 \| \omega_0 \|_{p_0,q_0} + C_4 (W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1) \right) + C_4 (W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1),
\]
(3.24)
\[
J_{k+1,p,q}^1 \leq A_2 \| j_0 \|_{p_0,q_0} + C_4 (W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1) \right) + C_4 (W_{k,p,q}^0 W_{k,p,q}^1 + J_{k,p,q}^0 J_{k,p,q}^1).
\]
(3.25)

Therefore, by (3.12), (3.24) and (3.25), we have that, there exists $C_5 > 0$ such that
\[
W_{k+1,p,q}^1 + J_{k+1,p,q}^1 \leq C_5 (\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0}) (1 + W_{k,p,q}^1 + J_{k,p,q}^1) \text{ for all } k \geq 1
\]
(3.26)
whenever
\[
\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0} \leq \min \{ G_1, 1 \}.
\]
(3.27)

By (3.18), it holds that
\[
W_{1,p,q}^1 + J_{1,p,q}^1 \leq A_2 (\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0}).
\]
(3.28)
Combining (3.26) with (3.28) implies that
\[
W_{k,p,q}^1 + J_{k,p,q}^1 \leq \left(1 + \frac{A_2}{C_5}\right) C_6 \left(\|\omega_0\|_{p_0,q_0} + \|f_0\|_{p_0,q_0}\right) \left(1 - \left(C_5 \left(\|\omega_0\|_{p_0,q_0} + \|f_0\|_{p_0,q_0}\right)\right)^k\right).
\]
(3.29)
whenever
\[
\|\omega_0\|_{p_0,q_0} + \|f_0\|_{p_0,q_0} \leq \min\{G_1, 1, (2C_5)^{-1}\}.
\]
(3.30)

**Step 3: Estimates for the terms \(W_{k,p,q}^1\) and \(J_{k,p,q}^1\)**

By (3.3) and Proposition 4, there exists a constant \(C_6 > 0\) such that
\[
\|\nabla \omega^{(k+1)}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot,t) \ast \omega_0\|_{p,q} + C_6 \int_0^t (t-s)^{-\frac{\alpha}{2} - \frac{3-\alpha}{2}}\|\nabla \omega^{(k)} - (\omega^{(k)} \ast \nabla) u^{(k)} - (b^{(k)} \ast \nabla) j^{(k)} + (j^{(k)} \ast \nabla) b^{(k)}\|_{p,q}(\cdot,s)\|\theta_q\|_{\theta_q} ds,
\]
(3.31)
\[
\|\nabla \omega^{(k+1)}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot,t) \ast f_0\|_{p,q} + C_6 \int_0^t (t-s)^{-\frac{\alpha}{2} - \frac{3-\alpha}{2}}\|\nabla \omega^{(k)} - (\omega^{(k)} \ast \nabla) u^{(k)} - (b^{(k)} \ast \nabla) j^{(k)} + (j^{(k)} \ast \nabla) b^{(k)}\|_{p,q}(\cdot,s)\|\theta_q\|_{\theta_q} ds.
\]
(3.32)

It follows from (3.16), (3.17), (3.31) and (3.32) that
\[
\|\nabla \omega^{(k+1)}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot,t) \ast \omega_0\|_{p,q} + 2C_2C_6 \int_0^t (t-s)^{-\frac{\alpha}{2}}\|\omega^{(k)}\|_{p,q} \|\nabla \omega^{(k)}\|_{p,q} + \|j^{(k)}\|_{p,q} \|\nabla j^{(k)}\|_{p,q}\|\theta_q\|_{\theta_q} ds,
\]
(3.33)
\[
\|\nabla j^{(k+1)}(\cdot,t)\|_{p,q} \leq \|\nabla G(\cdot,t) \ast f_0\|_{p,q} + 2C_2C_6 \int_0^t (t-s)^{-\frac{\alpha}{2}}\|\omega^{(k)}\|_{p,q} \|\nabla \omega^{(k)}\|_{p,q} + \|j^{(k)}\|_{p,q} \|\nabla j^{(k)}\|_{p,q}\|\theta_q\|_{\theta_q} ds.
\]
(3.34)

We set
\[
a = \left(\frac{3}{2} - \frac{3-q}{2p}\right)^{-1}, \quad b = \left(\frac{5}{2} - \frac{3-q}{p}\right)^{-1}.
\]

Note that \(a > 1, b > 1\) and
\[
\frac{1}{a} + 1 = \frac{1}{b} + \frac{3-q}{2p}, \quad \frac{1}{b} = \left(\frac{3}{2} - \frac{3-q}{2p}\right) + \left(1 - \frac{3-q}{2p}\right).
\]

By (3.33), (3.34), Proposition 4, Lemma 2, Hölder’s inequality and the following Young’s inequality
\[
\|f \ast g\|_{L^\infty(\mathbb{R}^+)} \leq \|f\|_{L^\frac{2p}{p+q}(\mathbb{R}^+)} \|g\|_{L^p(\mathbb{R}^+)},
\]
it holds that
\[
\tilde{W}_{k+1,p,q}^1 \leq A_3 \|\omega_0\|_{p_0,q_0} + 2C_2C_6 \int_0^t \|\nabla \omega^{(k)}\|_{p,q} \|\nabla j^{(k)}\|_{p,q},
\]
(3.35)
\[
\begin{aligned}
\bar{J}_{k+1}^{1,p,q} & \leq A_3 \| j_{0} \|_{p_0,q_0} + 3C_2C_6 \| f \|_{L^\frac{2p}{p+q}(\mathbb{R}^+)} \\
& \quad \times \left( \left\| (\omega^{(k)} (\cdot, \cdot), \nabla j^{(k)} (\cdot, \cdot), j_{p,q} \right\|_{p,q} + \left\| j^{(k)} (\cdot, \cdot) \right\|_{p,q} \| \nabla \omega^{(k)} (\cdot, \cdot) \|_{p,q} \right)_{L^0(\mathbb{R}^+)} \\
& \leq A_3 \| j_{0} \|_{p_0,q_0} + 3C_2C_6 (3\bar{W}_{k,q}^{1,p,q} \bar{J}_{k,p,q}^{1} + \bar{J}_{k,p,q}^{0}).
\end{aligned}
\] (3.36)

It follows from (3.12), (3.35) and (3.36) that there exists a global mild solution whenever the condition (3.27) holds.

Applying Proposition 4 and Lemma 2, there exists a constant \( A_4 \) such that
\[
\bar{W}_{1,p,q}^{1} + \bar{J}_{1,p,q}^{1} \leq A_4 (\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0}).
\] (3.38)

Combining (4.38) with (4.37) implies that
\[
\bar{W}_{k,p,q}^{1} + \bar{J}_{k,p,q}^{1} \leq 2(A_4 + C_7) (\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0})
\] (4.39)
whenever
\[
\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0} \leq \min \{ G_1, 1, (2C_7)^{-1} \}.
\] (4.40)

**Step 4: Conclusions.**

Define the following mappings:
\[
T_{1} \omega^{(k)} = \omega^{(k+1)}; \quad T_{2} j^{(k)} = j^{(k+1)}.
\]

Let \( \omega_{0} \), \( j_{0} \) satisfy
\[
\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0} \leq G_1.
\] (3.41)

We set \( R = 16A_1 (\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0}) \) and
\[
B_R = \left\{ f \in L^{\frac{2p}{p+q}}(\mathbb{R}^+; M^{p,q}(\mathbb{R}^3)) : \left\| f \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+) \leq R \right\}.
\]

By Step 1 we see that \( \{ \omega^{(k)} \} \subset B_R \) and \( \{ j^{(k)} \} \subset B_R \). Moreover, \( L^{\frac{2p}{p+q}}(\mathbb{R}^+; M^{p,q}(\mathbb{R}^3)) \) is a Banach space. Applying Banach’s fixed point Theorem, there exists a global mild solution \( (\omega, j) \) on \( \mathbb{R}^+ \) such that the estimates (3.1) hold under the condition (3.41) holds. Moreover, it holds that
\[
\left\| \omega^{(k)} - \omega \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+) \quad \text{and} \quad \left\| j^{(k)} - j \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\] (3.42)

**Step 5: Proof of the uniqueness.**

Assume that there exist two global mild solution \( (\omega, j) \) and \( (\tilde{\omega}, \tilde{j}) \) on \( \mathbb{R}^+ \) of (1.2) such that the estimates (3.1) hold under the condition (3.41) holds. Moreover, it holds that
\[
\left\| \omega^{(k)} - \tilde{\omega} \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+) \quad \text{and} \quad \left\| j^{(k)} - \tilde{j} \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\] (3.43)

\[
\bar{W}_{k,p,q}^{0} + \bar{J}_{k,p,q}^{0} \leq 16A_1 (\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0}) \quad \text{for all} \quad k \geq 1.
\] (4.44)

whenever (3.13) holds.

For convenience, given a function \( f : \mathbb{R}^3 \times (0, \infty) \), we set
\[
I_{p,q}(f) := \left\| f (\cdot, \cdot) \right\|_{p,q} L^{\frac{2p}{p+q}}(\mathbb{R}^+).
\]

Then (3.42)-(4.44) together with the Minkowski’s inequality imply that
\[
I_{p,q}(\omega) + I_{p,q}(j) + I_{p,q}(\tilde{\omega}) + I_{p,q}(\tilde{j}) \leq 20A_1 (\| \omega_{0} \|_{p_0,q_0} + \| j_{0} \|_{p_0,q_0}),
\] (4.45)
whenever (3.13) holds.
For convenience, we set
\[ u = K \ast \omega, \quad \tilde{u} = K \ast \tilde{\omega}, \quad b = K \ast j, \quad \tilde{b} = K \ast \tilde{j}, \]
\[ \omega(0,x) = \tilde{\omega}(0,x) = \omega_0(x), \quad j(0,x) = \tilde{j}(0,x) = j_0(x), \]
\[ [\omega] = \omega - \tilde{\omega}, \quad [j] = j - \tilde{j}, \quad [u] = u - \tilde{u}, \quad [b] = b - \tilde{b}. \]

Clearly,
\[ [u] = K \ast [\omega], \quad [b] = K \ast [j]. \]

By (2.1), it holds that
\[
[\omega](x,t) = \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) \partial_y (|u|^i \omega - |u|^i j + [b]^i j^i)(y,s)dyds
+ \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) \partial_y (\tilde{u}^i [\omega] - \tilde{u}^i [\omega] - \tilde{b}^i [j] + \tilde{b}^i [j])^i(y,s)dyds,
\]
\[
[j](x,t) = \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) \nabla_y \times ((|u| \cdot \nabla) [b] - ([b] \cdot \nabla) [u]) \gamma(x,s)dyds
+ \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) \nabla_y \times ((\tilde{u} \cdot \nabla) [b] - ([b] \cdot \nabla) \tilde{u}) \gamma(x,s)dyds
+ \int_0^t \int_{\mathbb{R}^3} G(x - y, t - s) \nabla_y \times ((\tilde{u} \cdot \nabla) [b] - ([b] \cdot \nabla) \tilde{u}) \gamma(x,s)dyds.
\]

By Proposition 4, (3.46) and (3.47), there exists a constant \( C_8 > 0 \) such that
\[
\| [\omega] \|_{p,q} \leq C_8 \int_0^t (t - s)^{-\frac{3q}{2p}} \| ([u]^i \omega - |u|^i j + [b]^i j^i) + \tilde{u}^i [\omega] - \tilde{u}^i [\omega] - \tilde{b}^i [j] + \tilde{b}^i [j]^i \gamma(x,s)dyds,
\]
\[
\| [j] \|_{p,q} \leq C_8 \int_0^t (t - s)^{-\frac{3q}{2p}} \| \nabla_y \times ((|u| \cdot \nabla) [b] - ([b] \cdot \nabla) [u]) \gamma(x,s) + \nabla_y \times ((\tilde{u} \cdot \nabla) [b] - ([b] \cdot \nabla) \tilde{u}) \gamma(x,s) \|_{\partial_d q}.
\]

By Propositions 2 (i) and 3, there exists a constant \( C_9 > 0 \) such that
\[
\| (|u|^i \omega - |u|^i j + [b]^i j^i) \|_{p,q} \leq C_9 \| (|u|^i \omega) \|_{p,q} + \| [u] \|_{p,q} + \| [\tilde{u}] \|_{p,q} + \| [\tilde{b}] \|_{p,q} + \| [j] \|_{p,q} + \| [b] \|_{p,q}.
\]

It follows from (3.49)-(3.51) that
\[
\| [\omega] \|_{p,q} \leq C_8 C_9 \int_0^t (t - s)^{-\frac{3q}{2p}} \| [\omega] \|_{p,q} + \| [j] \|_{p,q} + \| [\tilde{u}] \|_{p,q} + \| [\tilde{b}] \|_{p,q} \| _{\partial_d q} ds,
\]
\[
\| [j] \|_{p,q} \leq C_8 C_9 \int_0^t (t - s)^{-\frac{3q}{2p}} \| [\omega] \|_{p,q} + \| [j] \|_{p,q} + \| [\tilde{u}] \|_{p,q} + \| [\tilde{b}] \|_{p,q} + \| [\tilde{u}] \|_{p,q} + \| [\tilde{b}] \|_{p,q} \| _{\partial_d q} ds.
\]

By the Hölder’s inequality and the following Young’s inequality
\[
\| f \ast g \|_{L^p(\mathbb{R}^+)} \leq \| f \|_{L^{2p/(2p-1)}(\mathbb{R}^+)} \| g \|_{L^p(\mathbb{R}^+)}
\]
with \( a = \frac{2p}{2p-3+q} \) and \( b = \frac{p}{2p-3+q} \), we get from (3.52)-(3.53) that

\[
I_{p,q}(\|\omega\|) \leq C_8 C_9 \| \frac{1}{\beta(2) - \frac{2p}{2p-3+q}} \| L_{2p}^{2p-3+q}(\mathbb{R}^+)^2 \| \|\omega\|_{p,q} \|\omega\|_{p,q} + \| f \|_{p,q} \| f \|_{p,q}
\]

\[
+ \|\bar{\omega}\|_{p,q} \|\omega\|_{p,q} + \| f \|_{p,q} \| f \|_{p,q} \| L_{2p}^{2p-3+q}(\mathbb{R}^+) \|
\]

\[
\leq C_8 C_9 (I_{p,q}(\|\omega\|)I_{p,q}(\|\omega\|) + I_{p,q}(\|f\|)I_{p,q}(\|f\|)
\]

\[
+ I_{p,q}(\|\omega\|)I_{p,q}(\|f\|) + I_{p,q}(\|f\|)I_{p,q}(\|f\|)),
\]

(3.54)

\[
I_{p,q}(\|f\|) \leq C_8 C_9 (I_{p,q}(\|\omega\|)I_{p,q}(\|f\|) + I_{p,q}(\|\omega\|)I_{p,q}(\|f\|) + I_{p,q}(\|\omega\|)I_{p,q}(\|f\|)).
\]

(3.55)

Hence, (3.53)-(3.55) and (3.45) yield that

\[
I_{p,q}(\|\omega\|) + I_{p,q}(\|f\|) \leq C_{10}(\|\omega_0\|_{p_0,q_0} + \| f_0 \|_{p_0,q_0})(I_{p,q}(\|\omega\|) + I_{p,q}(\|f\|))
\]

(3.56)

whenever (3.13) holds.

Letting

\[
\|\omega_0\|_{p_0,q_0} + \| f_0 \|_{p_0,q_0} < \min \{ G_1, (2C_9)^{-1} \},
\]

inequality (3.56) gives \( I_{p,q}(\|\omega\|) + I_{p,q}(\|f\|) = 0 \), which implies that \( \|\omega\| = 0 \) and \( \| f \| = 0 \) for almost every \((x, t) \in \mathbb{R}^3 \times \mathbb{R}^+ \). This proves the uniqueness. \( \square \)

3.2. Proof for part (ii) of Theorem 3

Let \((p, q, p_0, q_0, q_1, q_1) \in E_2 \). Then there exist \( q_2, q_3 \in [0, 3] \), \( \bar{p} \in (1, \min \{ p, p' \}) \) and \( \theta \in (0, 1) \) such that

\[
0 \leq q_1 - q_0 < 1, \quad 0 \leq q_1 - q_2 < 1,
\]

\[
q_2 = \frac{q_1}{p'} + \frac{\bar{p}}{p}, \quad \frac{1}{p} = \frac{1}{p} + \frac{1}{q_3}, \quad \frac{1}{p} = \theta + \frac{1-\theta}{2},
\]

\[
\frac{q_2}{p} = q_1 \theta + \frac{q_1}{p}(1-\theta), \quad \frac{q_2 - q_1}{p} = \frac{q_1 - q_0}{2} \theta + \frac{q_1 - q_0}{2p}(2-\theta).
\]

For convenience, we set

\[
W_{k,1,q_1}^0 = \sup_{t \in \mathbb{R}^+} \frac{1}{t^{1-q_0/q_3}} \| \omega^{(k)}(\cdot, t) \|_{1,q_1}, \quad W_{k,1,q_1}^0 = \| \| \omega^{(k)}(\cdot, t) \|_{1,q_1} \|_{L_t^{q_1} - \omega_0(\mathbb{R}^+)},
\]

\[
W_{k,1,q_1}^1 = \sup_{t \in \mathbb{R}^+} \frac{1}{t^{1-q_1/q_3}} \| \nabla \omega^{(k)}(\cdot, t) \|_{1,q_1}, \quad W_{k,1,q_1}^1 = \| \| \nabla \omega^{(k)}(\cdot, t) \|_{1,q_1} \|_{L_t^{q_1} - \omega_0(\mathbb{R}^+)},
\]

\[
J_{k,1,q_1}^0 = \sup_{t \in \mathbb{R}^+} \frac{1}{t^{1-q_1/q_3}} \| j^{(k)}(\cdot, t) \|_{1,q_1}, \quad J_{k,1,q_1}^0 = \| \| j^{(k)}(\cdot, t) \|_{1,q_1} \|_{L_t^{q_1} - \omega_0(\mathbb{R}^+)},
\]

\[
J_{k,1,q_1}^1 = \sup_{t \in \mathbb{R}^+} \frac{1}{t^{1-q_1/q_3}} \| \nabla j^{(k)}(\cdot, t) \|_{1,q_1}, \quad J_{k,1,q_1}^1 = \| \| \nabla j^{(k)}(\cdot, t) \|_{1,q_1} \|_{L_t^{q_1} - \omega_0(\mathbb{R}^+)}.
\]

Step 1: Estimates for the terms \( W_{k,1,q_1}^0, J_{k,1,q_1}^0, J_{k,1,q_1}^1 \).

By (3.3) and Proposition 4, one finds that

\[
\| \omega^{(k+1)}(\cdot, t) \|_{1,q_1} \leq \| G(\cdot, t) \ast \omega_0 \|_{1,q_1} + B_1 \int_0^t \int_{\mathbb{R}^3} (t-s)^{-\frac{1-2q_1}{q_1}} \| (u^{(k)} - u^{(k)}) \omega^{(k)}(\cdot, s) - b^{(k)} j^{(k)}(\cdot, s) \|_{1,q_2} \| f \|_{1,q_2} \| f \|_{1,q_2},
\]

(3.57)

\[
\| j^{(k+1)}(\cdot, t) \|_{1,q_1} \leq \| G(\cdot, t) \ast f_0 \|_{1,q_1} + B_1 \int_0^t \int_{\mathbb{R}^3} (t-s)^{-\frac{1-2q_1}{q_1}} \| (u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) u^{(k)}(\cdot, s) \|_{1,q_2} \| f \|_{1,q_2} \| f \|_{1,q_2},
\]

(3.58)

Note that \( q_2 = \frac{q_1}{p'} + \frac{q_1}{p} \). By Proposition 2 (i), one has

\[
\| (u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) u^{(k)}(\cdot, s) \|_{1,q_2} + B_1 \int_0^t \int_{\mathbb{R}^3} (t-s)^{-\frac{1-2q_1}{q_1}} \| (u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) u^{(k)}(\cdot, s) \|_{1,q_2} \| f \|_{1,q_2} \| f \|_{1,q_2},
\]

(3.59)
Using (3.58) and (3.66), one has
\[\| (u^{(k)}(t) \cdot \nabla) b^{(k)}(t) - (b^{(k)}(t) \cdot \nabla) u^{(k)}(t) \|_{1,q_2} \leq \| (u^{(k)}(t), s) \|_{p', q_3} \| f^{(k)}(s) \|_{p, q} + \| b^{(k)}(t, s) \|_{p', q_3} \| \omega^{(k)}(s) \|_{p, q}. \tag{3.60} \]

Note that \( \tilde{p} < p' \) and \( \frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\theta}{q} \). This together with Proposition 3 implies that
\[\| u^{(k)}(t, s) \|_{p', q_3} \leq B_2 \| \omega^{(k)}(s) \|_{\tilde{p}, 1}, \tag{3.61} \]
\[\| b^{(k)}(t, s) \|_{p', q_3} \leq B_2 \| f^{(k)}(s) \|_{\tilde{p}, 1}. \tag{3.62} \]

Since
\[1 < \tilde{p} < \min\{p, p'\}, \quad \frac{1}{\tilde{p}} = \theta + \frac{(1 - \theta)}{p}, \quad \frac{q_3}{\tilde{p}} = q_1 \theta + \frac{q}{p} (1 - \theta), \]
then by invoking (vi) of Proposition 1, we can get
\[\| \omega^{(k)}(s) \|_{p, q} \leq B_3 \| \omega^{(k)}(s) \|_{1, q_1}^{\theta} \| \omega^{(k)}(s) \|_{p, q}^{1 - \theta}, \tag{3.63} \]
\[\| f^{(k)}(s) \|_{p, q} \leq B_3 \| f^{(k)}(s) \|_{1, q_1}^{\theta} \| f^{(k)}(s) \|_{p, q}^{1 - \theta}. \tag{3.64} \]

Combining (3.59) with (3.61)-(3.64) implies that
\[\| (u^{(k)}(t) \cdot \nabla) \omega^{(k)}(s) - (b^{(k)}(t) \cdot \nabla) b^{(k)}(s) - b^{(k)}(t) \cdot \hat{J}(s, t) + b^{(k)}(s) \cdot \hat{J}(t, s) \|_{1,q_2} \leq 2B_2 \| \omega^{(k)}(s) \|_{1, q_1} \| \omega^{(k)}(s) \|_{p, q} + 2B_2 \| f^{(k)}(s) \|_{1, q_1} \| f^{(k)}(s) \|_{p, q} \tag{3.65} \]
\[\leq 2B_2 B_3 \| \omega^{(k)}(s) \|_{1, q_1}^{\theta} \| \omega^{(k)}(s) \|_{p, q}^{1 - \theta} + \| f^{(k)}(s) \|_{1, q_1}^{\theta} \| f^{(k)}(s) \|_{p, q}^{1 - \theta}. \]

It follows from (3.60)-(3.64) that
\[\| ((u^{(k)} \cdot \nabla) b^{(k)}(s) - (b^{(k)} \cdot \nabla) u^{(k)}(s)) \|_{1,q_2} \leq B_2 \| \omega^{(k)}(s) \|_{1, q_1} \| \omega^{(k)}(s) \|_{p, q} + B_2 \| f^{(k)}(s) \|_{1, q_1} \| f^{(k)}(s) \|_{p, q} \tag{3.66} \]
\[\leq B_2 B_3 \| \omega^{(k)}(s) \|_{1, q_1}^{\theta} \| \omega^{(k)}(s) \|_{p, q}^{1 - \theta} + \| f^{(k)}(s) \|_{1, q_1}^{\theta} \| f^{(k)}(s) \|_{p, q}^{1 - \theta}. \]

Inequalities (3.57) and (3.65) imply that
\[\| \omega^{(k+1)}(s) \|_{1, q_1} \leq \| G(\cdot, t) * \omega_0 \|_{1, q_1} + 2B_1 B_2 B_3 \int_0^1 (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} \| \omega^{(k)}(s) \|_{1, q_1}^{\theta} \| \omega^{(k)}(s) \|_{p, q}^{1 - \theta} ds \tag{3.67} \]
\[+ 2B_1 B_2 B_3 \int_0^1 (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} \| f^{(k)}(s) \|_{1, q_1}^{\theta} \| f^{(k)}(s) \|_{p, q}^{1 - \theta} ds. \]

Using (3.58) and (3.66), one has
\[\| f^{(k+1)}(s) \|_{1, q_1} \leq \| G(\cdot, t) * f_0 \|_{1, q_1} + B_1 B_2 B_3 \int_0^1 (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} \| \omega^{(k)}(s) \|_{1, q_1}^{\theta} \| \omega^{(k)}(s) \|_{p, q}^{1 - \theta} ds \tag{3.68} \]
\[+ B_1 B_2 B_3 \int_0^1 (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} \| f^{(k)}(s) \|_{1, q_1}^{\theta} \| f^{(k)}(s) \|_{p, q}^{1 - \theta} ds. \]

Invoking Proposition 4, (3.67) and (3.68) yield that
\[W_{k+1,1,q_1}^{0} \leq A_1 \| \omega_0 \|_{1, q_0} + 2B_1 B_2 B_3 (W_{k,1,q_1}^{0})^{\theta} (W_{k,p,q}^{0})^{2 - \theta} \times \int_0^t (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} s^{-\frac{q_3}{2}} \delta^{\frac{1}{2} - \frac{q_3}{2}} (s^{\frac{1}{2} - \frac{q_3}{2}} - 1) (2 - \theta) ds \tag{3.69} \]
\[+ 2B_1 B_2 B_3 (J_{k,1,q_{1}}^{0})^{\theta} (J_{k,p,q}^{0})^{2 - \theta} \]
\[\times \int_0^t (t - s)^{-\frac{1}{2} + \frac{q_3}{2}} s^{-\frac{q_3}{2}} \delta^{\frac{1}{2} - \frac{q_3}{2}} (s^{\frac{1}{2} - \frac{q_3}{2}} - 1) (2 - \theta) ds, \]
It follows from (3.69)-(3.71) that
\[
J_{k+1,q_1}^0 \leq A_5 \| j_0 \|_{1,q_0} + B_4 ((W_{k+1,q_1}^0)^{\theta} (W_{k,q}^0)^{2-\theta}) + B_4 ((J_{k+1,q_1}^0)^{\theta} (J_{k,q}^0)^{2-\theta}),
\]
where
\[
W_{k+1,q_1}^0 \leq A_5 \| \omega_0 \|_{1,q_0} + B_4 ((W_{k+1,q_1}^0)^{\theta} (W_{k,q}^0)^{2-\theta}) + B_4 ((J_{k+1,q_1}^0)^{\theta} (J_{k,q}^0)^{2-\theta}),
\]
and
\[
J_{k+1,q_1}^0 \leq A_5 \| j_0 \|_{1,q_0} + B_4 ((W_{k+1,q_1}^0)^{\theta} (W_{k,q}^0)^{1-\theta} J_{k,q}^0) + B_4 ((J_{k+1,q_1}^0)^{\theta} (J_{k,q}^0)^{1-\theta} W_{k,q}^0).
\]
Inequality (3.12) together with (3.72) and (3.73) may lead to
\[
W_{k+1,q_1}^0 \leq A_5 \| \omega_0 \|_{1,q_0} + B_5 ((\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0})^{2-\theta} (W_{k+1,q_1}^0 + J_{k+1,q_1}^0)^{\theta}),
\]
whenever (3.13) holds.

On the other hand, by Proposition 4, it holds that
\[
W_{k+1,q_1}^0 \leq A_5 \| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0}.
\]
Assume that (3.13) holds and
\[
\max \{ A_5 (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0}), B_5 (\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0})^{2-\theta} \} \leq 1/2.
\]
Then (3.74) and (3.75) may yield that
\[
W_{k+1,q_1}^0 + J_{k+1,q_1}^0 \leq 2 \quad \text{for all} \quad k \geq 1,
\]
whenever (3.13) and (3.75) hold. (3.77) together with (3.74) implies that
\[
W_{k+1,q_1}^0 + J_{k+1,q_1}^0 \leq C_{10} (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0} + \| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0}) \quad \text{for all} \quad k \geq 1
\]
whenever (3.13) and (3.76) hold.

**Step 2: Estimates for the terms \( \bar{W}_{k+1,q_1}^0 \) and \( J_{k+1,q_1}^0 \).**

Note that
\[
\frac{2}{q_1 - \hat{q}_0} > 1, \quad \frac{2}{1 + q_1 - q_2} > 1, \quad \frac{2}{q_2 - \hat{q}_0 + 1} > 1, \quad \frac{q_1 - \hat{q}_0 + 1}{2} = 1 + \frac{q_1 - q_2}{2} + \frac{q_2 - \hat{q}_0 + 1}{2}.
\]

By (3.67), (3.68), Proposition 4, Lemma 2 and the following Young’s inequality
\[
\| f * g \|_{L^{\frac{2}{q_1 - \hat{q}_0}}(\mathbb{R}^+)} \leq \| f \|_{L^{\frac{2}{1 + q_1 - q_2}}(\mathbb{R}^+)} \| g \|_{L^{\frac{2}{q_2 - \hat{q}_0 + 1}}(\mathbb{R}^+)},
\]
one can obtain that
\[ W_{k+1,1,q_1}^0 \leq A_5 \| \omega_0 \|_{1, \phi_0} + 2B_1B_2B_3 \left\| t^{\frac{1}{1+q_1-q_0}} \right\|_{L^{\frac{1}{1+q_1-q_0}}(\mathbb{R}^+)} \]
\[ \times \left\| \left( \left( \omega^{(k)}(\cdot,t) \right)^0_{1,q_1} \right) \right\|_{1,q_1} \| \omega^{(k)}(\cdot,t) \|_{p,q}^{2-\theta} \]
\[ + \left\| j^{(k)}(\cdot,t) \right\|_{1,q_1} \| j^{(k)}(\cdot,t) \|_{p,q}^{2-\theta} \right\|_{L^{\frac{1}{1+q_1-q_0}}(\mathbb{R}^+)}. \]  
(3.79)

\[ j_{k+1,1,q_1}^0 \leq A_5 \| j_0 \|_{1, \phi_0} + B_6 \left( (\bar{W}_{k,1,q_1}^0)^0_{k,p,q} \right)^{2-\theta} + \left( (j_{k,1,q_1}^0)^0_{k,p,q} \right)^{2-\theta}, \]  
(3.80)

By (3.3), (3.81) and (3.12), we have
\[ W_{k+1,1,q_1}^0 + j_{k+1,1,q_1}^0 \leq A_5 \left( \| \omega_0 \|_{1, \phi_0} + \| j_0 \|_{1, \phi_0} \right) \]
\[ + 16A_1B_6 \left( (\bar{W}_{k,1,q_1}^0)^0_{k,1,q_1} + (j_{k,1,q_1}^0)^0_{k,1,q_1} \right)^{2-\theta} \]  
(3.82)

whenever (3.13) holds.

On the other hand, by Proposition 4 and Lemma 2, it holds that
\[ W_{1,1,q_1}^0 + j_{1,1,q_1}^0 \leq A_5 \left( \| \omega_0 \|_{1, \phi_0} + \| j_0 \|_{1, \phi_0} \right) \]  
(3.84)

By (3.83), (3.84) and the argument similar to those used in deriving (3.78), it holds that
\[ W_{k,1,q_1}^0 + j_{k,1,q_1}^0 \leq 2 \quad \text{for all } k \geq 1 \]  
(3.85)

whenever (3.13) holds and
\[ \max \left\{ A_5 \left( \| \omega_0 \|_{1, \phi_0} + \| j_0 \|_{1, \phi_0} \right), 16A_1B_6 \left( \| \omega_0 \|_{p_0, \phi_0} + \| j_0 \|_{p_0, \phi_0} \right)^{2-\theta} \right\} \leq 1/2. \]  
(3.86)

Inequality (3.85) together with (3.83) also yields that
\[ W_{k,1,q_1}^0 + j_{k,1,q_1}^0 \leq C_{11} \left( \| \omega_0 \|_{1, \phi_0} + \| j_0 \|_{1, \phi_0} + \| \omega_0 \|_{p_0, \phi_0} + \| j_0 \|_{p_0, \phi_0} \right) \quad \text{for all } k \geq 1 \]  
(3.87)

whenever (3.13) and (3.86) hold.

Step 3: Estimates for the terms \( W_{k,1,q_1}^1, j_{k,1,q_1}^1 \).

By (3.3) and Proposition 4, we get
\[ \left\| \nabla \omega^{(k+1)} \right\|_{1,q_1} \]
\[ \leq \left\| \nabla G(\cdot,t) * \omega^{(k)} \right\|_{1,q_1} + B_3 \int_{t/2}^t (t-s)^{-\frac{1}{2} - \frac{d}{2(n-d)}} \left\| \left( u^{(k)} \cdot \nabla \right) \omega^{(k)} - (\omega^{(k)} \cdot \nabla) u^{(k)} \right\| \right|_{1,q_1} ds \]
\[ - (b^{(k)} \cdot \nabla) j^{(k)} + (j^{(k)} \cdot \nabla) b^{(k)}) (\cdot, s) \right\|_{1,q_1} ds \]
\[ + B_3 \int_0^{t/2} (t-s)^{-\frac{1}{2} - \frac{d}{2(n-d)}} \left\| (u^{(k)} \cdot \nabla) \omega^{(k)} - u^{(k)} \omega^{(k)} \right\| \right|_{1,q_1} ds \]
\[ - b^{(k)} j^{(k)} + b^{(k)} j^{(k)} (\cdot, s) \right\|_{1,q_2} ds, \]  
(3.88)
\[ \| \nabla j^{(k+1)} \|_{1,q_1} \]
\[ \leq \| \nabla G(\cdot,t) \ast j_0 \|_{1,q_1} + B_3 \int_{t/2}^t (t-s)^{-\frac{3}{4}} \| ((u^{(k)} \cdot \nabla) j^{(k)} + \nabla u^{(k)} \times b^{(k)}_i) - (b^{(k)} \cdot \nabla) \omega^{(k)} - \nabla b^{(k)} \times u^{(k)}_i ) (\cdot,s) \|_{1,q_2} ds \]
\[ + B_3 \int_{t/2}^t (t-s)^{-1-\frac{2q_2}{q_1}} \| ((u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) u^{(k)}) (\cdot,s) \|_{1,q_2} ds. \] (3.89)

By the arguments similar to those used in deriving (3.65) and (3.66), one can get
\[ \| ((u^{(k)} \cdot \nabla) \omega^{(k)} - (\omega^{(k)} \cdot \nabla) u^{(k)} - (b^{(k)} \cdot \nabla) j^{(k)} + (j^{(k)} \cdot \nabla) b^{(k)} ) (\cdot,s) \|_{1,q_2} \]
\[ \leq B_4 \left( \| \omega^{(k)} (\cdot,s) \|_{p,q} + \| \nabla \omega^{(k)} (\cdot,s) \|_{p,q} \right) \]
\[ + B_4 \left( \| \nabla j^{(k)} (\cdot,s) \|_{p,q} + \| \nabla b^{(k)} (\cdot,s) \|_{p,q} \right) \]
\[ \leq B_4 \left( \| \omega^{(k)} (\cdot,s) \|_{1,q_1} \| \omega^{(k)} (\cdot,s) \|_{1,q_2} \| \nabla \omega^{(k)} (\cdot,s) \|_{p,q} \right) \]
\[ + B_4 \left( \| \nabla j^{(k)} (\cdot,s) \|_{1,q_1} \| \nabla j^{(k)} (\cdot,s) \|_{1,q_2} \| \nabla b^{(k)} (\cdot,s) \|_{p,q} \right). \] (3.90)

It follows from (3.90) that
\[ \int_{t/2}^t (t-s)^{-\frac{3}{4}} \| ((u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) \omega^{(k)} - \nabla b^{(k)} \times u^{(k)}_i ) (\cdot,s) \|_{1,q_2} ds \]
\[ \leq B_4 \int_{t/2}^t (t-s)^{-\frac{3}{4}} \| \omega^{(k)} (\cdot,s) \|_{1,q_1} \| \omega^{(k)} (\cdot,s) \|_{1,q_2} \| \nabla \omega^{(k)} (\cdot,s) \|_{p,q} ds \]
\[ + B_4 \int_{t/2}^t (t-s)^{-\frac{3}{4}} \| \nabla j^{(k)} (\cdot,s) \|_{1,q_1} \| \nabla j^{(k)} (\cdot,s) \|_{1,q_2} \| \nabla b^{(k)} (\cdot,s) \|_{p,q} ds \] (3.92)

Noting that
\[ \frac{1}{2} - \frac{q_1 - q_2}{2} > 0, \quad \frac{1}{2} - \frac{q_1 - \tilde{q}_1 \theta}{2} \frac{2p - 3 + q}{2p} (1 - \theta) - \frac{3p - 3 + q}{2p} + 1 > 0, \]
\[ \frac{1}{2} - \frac{q_1 - q_2}{2} - \frac{q_1 - \tilde{q}_1 \theta}{2} \frac{2p - 3 + q}{2p} (1 - \theta) - \frac{3p - 3 + q}{2p} = \tilde{q}_1 - q_1 - 1. \]
It follows that
\[
\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| \omega^{(k)}(\cdot,s) \|_{1,q_t} + \| \omega^{(k)}(\cdot,s) \|_{p,q} \right) ds \\
\leq (W_{k.p,q}^0)^{\theta}(W_{k.p,q}^0)^{1-\theta}W_{k.p,q}^1 \\
	imes \int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} s^{-\frac{q_1 - q_2}{2}} \frac{\theta}{p} s^{-\frac{2p-3(q_1 - q_2)}{2}} (1-s)^{-\frac{3q_1 - q_2}{2p}} ds \\
\leq B_5(W_{k.p,q}^0)^{\theta}(W_{k.p,q}^0)^{1-\theta}W_{k.p,q}^1 t^{\frac{q_1 - q_2}{2}} \\
(3.93)
\]

Similarly, we can get
\[
\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| \nabla \omega^{(k)}(\cdot,s) \|_{1,q_t} + \| \nabla \omega^{(k)}(\cdot,s) \|_{p,q} \right) ds \\
\leq B_5(W_{k.p.q}^1)^{\theta}(W_{k.p.q}^1)^{1-\theta}W_{k.p,q}^0 \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}} \\
\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| \nabla j^{(k)}(\cdot,s) \|_{1,q_t} + \| \nabla j^{(k)}(\cdot,s) \|_{p,q} \right) ds \\
\leq B_5(J_{k.p,q}^1)^{\theta}(J_{k.p,q}^1)^{1-\theta}W_{k.p,q}^0 \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}} \\
(3.94)
\]

\[
\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| j^{(k)}(\cdot,s) \|_{1,q_t} + \| j^{(k)}(\cdot,s) \|_{p,q} \right) ds \\
\leq B_5(J_{k.p,q}^0)^{\theta}(J_{k.p,q}^0)^{1-\theta}J_{k.p,q}^1 \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}} \\
(3.95)
\]

\[
\int_{t/2}^{t} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| (u^{(k)} \cdot \nabla) \omega(\cdot,s) - (\omega(\cdot,s) \cdot \nabla) u^{(k)} - (b^{(k)} \cdot \nabla) j^{(k)}(\cdot,s) + (j^{(k)} \cdot \nabla) b^{(k)}(\cdot,s) \right) ds \\
\leq B_4 B_5 (W_{k.p,q}^0)^{\theta}(W_{k.p,q}^0)^{1-\theta}W_{k.p,q}^1 + (W_{k.p,q}^1)^{\theta}(W_{k.p,q}^1)^{1-\theta}W_{k.p,q}^0 \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}} \\
+ B_4 B_5 (J_{k.p,q}^1)^{\theta}(J_{k.p,q}^1)^{1-\theta}W_{k.p,q}^0 + (J_{k.p,q}^0)^{\theta}(J_{k.p,q}^0)^{1-\theta}J_{k.p,q}^1 \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}} \\
(3.96)
\]

On the other hand, by (3.65) we get
\[
\int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| u^{(k)}(\cdot,s) - u^{(k)} \|_{1,q_t} + \| j^{(k)}(\cdot,s) \|_{1,q_t} + \| \omega(\cdot,s) \|_{p,q} \right)^{2-\theta} ds \\
\leq 2B_2 B_3 \int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| \omega(\cdot,s) \|_{1,q_t} + \| \omega(\cdot,s) \|_{p,q} \right)^{2-\theta} ds \\
+ 2B_2 B_3 \int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| j^{(k)}(\cdot,s) \|_{1,q_t} + \| j^{(k)}(\cdot,s) \|_{p,q} \right)^{2-\theta} ds \\
(3.97)
\]

Note that
\[
-1 + \frac{q_1 - q_2}{2} - \frac{q_1 - q_0 + 2p - 3 + q + 2}{2p} (2-\theta) + 1 = \frac{q_0 - q_1 - 1}{2} \\
\]

It follows that
\[
\int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| \omega(\cdot,s) \|_{1,q_t} + \| \omega(\cdot,s) \|_{p,q} \right)^{2-\theta} ds \\
\leq (W_{k.p,q}^0)^{\theta}(W_{k.p,q}^0)^{2-\theta} \int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} s^{-\frac{q_1 - q_2}{2}} \frac{\theta}{p} s^{-\frac{2p-3(q_1 - q_2)}{2}} (2-\theta) ds \\
(3.98)
\]

Similarly, we can get
\[
\int_{0}^{t/2} (t-s)^{-\frac{1}{2}} \frac{q_1 - q_2}{2} \left( \| j^{(k)}(\cdot,s) \|_{1,q_t} + \| j^{(k)}(\cdot,s) \|_{p,q} \right)^{2-\theta} ds \leq B_6(J_{k.p,q}^0)^{\theta}(J_{k.p,q}^0)^{2-\theta} \frac{q_1 - q_2}{2} t^{\frac{q_1 - q_2}{2}}. \\
(3.99)
\]
It follows from (3.88), Proposition 4 and (3.92)-(3.100) that
\[
W_{k+1,q_1}^1 \leq B_{71} ||\omega||_{1,q_0} + B_{72}(W_{k+1,q_1}^0 ||W_{k+1}^0||_{1,q_1}^0 - \theta W_{k+1,q_1}^0 + B_7(J_{k+1,q_1})^0 ||J_{k+1,q_1}||_{1,q_1}^0 - \theta J_{k+1,q_1}^1)
\]
\[
+ B_7(W_{k+1,q_1}^0 ||W_{k+1}^0||_{1,q_1}^0 - \theta W_{k+1,q_1}^0 + B_7(J_{k+1,q_1})^0 ||J_{k+1,q_1}||_{1,q_1}^0 - \theta J_{k+1,q_1}^1)
\]
\[
+ B_7(W_{k+1,q_1}^0 ||W_{k+1}^0||_{1,q_1}^0 - \theta W_{k+1,q_1}^0 + B_7(J_{k+1,q_1})^0 ||J_{k+1,q_1}||_{1,q_1}^0 - \theta J_{k+1,q_1}^1)
\]
\[
(3.101)
\]

We get from (3.91) that
\[
\int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(k) \cdot \nabla) J_s^{(k)} + \nabla u^{(k)} \times b_s^{(k)} - (b^{(k)}(k) \cdot \nabla) \omega_s^{(k)} - \nabla b^{(k)} \times u_s^{(k)}((s) ||u_s^{(k)}(s) ||_{1,q_2} \text{ds)}
\]
\[
\leq B_4 \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
+ B_4 \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
+ 2B_4 \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
(3.102)
\]

Similar arguments to those in deriving (3.93) may yield that
\[
\int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
\leq B_8(W_{k+1,q_1}^0 ||W_{k+1,q_1}^0||_{1,q_1}^0 - \theta W_{k+1,q_1}^0 + B_7(J_{k+1,q_1})^0 ||J_{k+1,q_1}||_{1,q_1}^0 - \theta J_{k+1,q_1}^1)
\]
\[
(3.103)
\]

Note that
\[
-\frac{1}{2} - \frac{q_1 - q_2}{2} + \frac{3 - q}{2p} - 1 + \frac{2q_1 - q_2}{2} - \frac{3 - q}{2p} - 1 = \frac{2q_1 - q_2}{2}.
\]

It follows that
\[
\int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
\leq W_{k+1,q_1}^0 ||J_{k+1,q_1}^1 ||_{1,q_1}^0 - \theta \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
(3.105)
\]

It follows from (3.102)-(3.105) that
\[
\int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
- (b^{(k)}(s) \cdot \nabla) \omega_s^{(k)} - \nabla b^{(k)} \times u_s^{(k)}((s) ||u_s^{(k)}(s) ||_{1,q_2} \text{ds)}
\]
\[
\leq B_10 \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
+ B_10 \int_{t/2}^t (t-s)^{-\frac{1}{2}} ||(u^{(k)}(s), s)||_{1,q_1}^0 ||(u^{(k)}(s), s)||_{1,q_1}^0 \text{ds}
\]
\[
(3.106)
\]
Note that
\[-1 - \frac{q_1 - q_2}{2} - \frac{q_1 - q_0}{2} \theta + \frac{3 - q - 2p}{2p} (1 - \theta) + \frac{3 - q - 2p}{2p} + 1 = \frac{q_0 - q_1 - 1}{2}.
\]
By (3.66), it holds that
\[
\int_0^{t/2} (t - s)^{-1 - \frac{q_1 - q_2}{2}} \|((u^{(k)} \cdot \nabla) b^{(k)} - (b^{(k)} \cdot \nabla) u^{(k)})\|_{1,q} ds
\leq B_2 B_3 \int_0^{t/2} (t - s)^{-1 - \frac{q_1 - q_2}{2}} \|\omega^{(k)}\|_{1,q} \|\omega^{(k)}\|_{1,q}^{1 - \theta} (s) \|\omega^{(k)}\|_{p,q} ds
\]
\[+ B_2 B_3 \int_0^{t/2} (t - s)^{-1 - \frac{q_1 - q_2}{2}} \|j^{(k)}\|_{1,q} \|j^{(k)}\|_{1,q}^{1 - \theta} \|\omega^{(k)}\|_{p,q} ds \quad \tag{3.107}
\]
\[\leq B_1 t^{\frac{q_1 - q_2}{2}} (W_{l,1,q_1}^0)^{(1 - \theta)} (W_{l,1,q_1}^0)^{-1 - \theta} + (J_{l,1,q_1}^0)^{(1 - \theta)} (J_{l,1,q_1}^0)^{-1 - \theta} W_{l,1,q_1}^0.
\]
Using (3.89), (3.106) and Proposition 4, we have
\[
J_{k+1,1,q_1}^1 \leq B_{12} \|J_0\|_{1,q_0} + B_{12} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} + B_{12} \|J_0\|_{1,q_0} + B_{12} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0
\]
\[+ B_{12} (W_{k,1,q_1}^0)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 + B_{12} (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0
\]
\[+ B_{12} (W_{k,1,q_1}^0)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 + B_{12} (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 \tag{3.108}
\]
Therefore, we get from (3.101), (3.108), (3.12), (3.29) and (3.77) that
\[
W_{k+1,1,q_1}^1 + J_{k+1,1,q_1}^1 \leq (B_7 + B_{12}) \|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} + B_{12} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0
\]
\[+ B_{12} (W_{k,1,q_1}^0)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 + B_{12} (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0
\]
\[+ B_{12} (W_{k,1,q_1}^0)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 + B_{12} (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 (J_{k,1,q_1}^1)^{(1 - \theta)} W_{k,1,q_1}^0 \tag{3.109}
\]
\[\leq B_{12} \|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} + B_{12} \|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} + B_{12} \|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} \quad \tag{3.110}
\]
when (13.1), (3.30) and (3.76) hold. On the other hand, by Proposition 4, it holds that
\[
W_{1,1,q_1}^1 + J_{1,1,q_1}^1 \leq \sup_{\tau \in \mathbb{R}^+} \int^{\tau + \frac{q_1 - q_0}{2}} \|\nabla G(\cdot, t) * \omega_0\|_{1,q_1} + \sup_{\tau \in \mathbb{R}^+} \int^{\tau + \frac{q_1 - q_0}{2}} \|\nabla G(\cdot, t) * J_0\|_{1,q_1}
\]
\[\leq B_{14} \|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} \tag{3.110}
\]
By (3.109), (3.110) and the arguments similar to those used to derive (3.77) and (3.78), it holds that
\[
W_{k,1,q_1}^1 + J_{k,1,q_1}^1 \leq C_{12} (\|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} + \|\omega_0\|_{p_0,q_0} + \|J_0\|_{p_0,q_0}) \quad \text{for all } k \geq 1,
\]
when (13.1), (3.30) and (3.76) hold and
\[
(B_{13} + B_{14}) (\|\omega_0\|_{1,q_0} + \|J_0\|_{1,q_0} + (\|\omega_0\|_{p_0,q_0} + \|J_0\|_{p_0,q_0})^{2 - \theta} \leq 1/2 \tag{3.113}
\]
**Step 4:** Estimates for the terms \(\tilde{W}_{k,1,q_1}^1, J_{k,1,q_1}^1\).
By (3.3), Proposition 4 and (3.90), it holds that
\[
\|\nabla \omega^{(k+1)}\|_{1,q_1} \\
\leq \|\nabla G(\cdot,t) * \omega_0\|_{1,q_1} \\
+ B_4 \int_0^t (t-s)^{-\frac{1}{2} - \frac{q_1-q_2}{2}} \|\omega^{(k)}(\cdot,s)\|_{1,q_1}^{\theta} \|\omega^{(k)}(\cdot,s)\|_{p,q}^{\theta(1-\theta)} \|\nabla \omega^{(k)}(\cdot,s)\|_{p,q} \|\omega^{(k)}(\cdot,s)\|_{p,q} \|\nabla \omega^{(k)}(\cdot,s)\|_{p,q} \|\omega^{(k)}(\cdot,s)\|_{p,q} ds
\]
(3.114)

It was observed that
\[
\frac{1+q_1-q_0}{2} + 1 = \frac{1+q_1-q_2}{2} + \frac{(q_2-q_0}{2} + 1),
\]
\[
\frac{2}{q_1-q_0} > \theta, \quad \frac{2p}{2p-3+q} > 1 - \theta, \quad \frac{2p}{3p-3+q} > 1,
\]
\[
\frac{q_2-q_0}{2} + 1 = \frac{q_1-q_0}{2} + \frac{2p-3+q}{2p} (1 - \theta) + \frac{3p-3+q}{2p},
\]
\[
\frac{2}{1+q_1-q_0} > \theta, \quad \frac{2p}{2p-3+q} > 1 - \theta, \quad \frac{2p}{3p-3+q} > 1,
\]
\[
\frac{q_2-q_0}{2} + 1 = \frac{q_1-q_0}{2} + \frac{3p-3+q}{2p} (1 - \theta) + \frac{2p-3+q}{2p}.
\]

These facts together with (3.114), Proposition 4, Lemma 2, Young’s inequality and Hölder’s inequality imply that
\[
\bar{W}_{k+1,1,q_1}^1 \leq B_{15} \|\omega_0\|_{1,q_0} + B_{15} \|t^{-\frac{1}{2} - \frac{q_1-q_2}{2}} \|_{L^{\frac{2p}{2p-3+q}}(R^+)}
\]
\[
\times \left( \|\|\omega^{(k)}(\cdot,t)\|_{1,q_1}^{\theta} \|\omega^{(k)}(\cdot,t)\|_{p,q}^{\theta(1-\theta)} \|\nabla \omega^{(k)}(\cdot,t)\|_{p,q} \|\nabla \omega^{(k)}(\cdot,t)\|_{p,q} \right)_{L^{\frac{2p}{2p-3+q}}(R^+)}
\]
\[
+ \|\|\nabla \omega^{(k)}(\cdot,t)\|_{1,q_1}^{\theta} \|\nabla \omega^{(k)}(\cdot,t)\|_{p,q}^{\theta(1-\theta)} \|\omega^{(k)}(\cdot,t)\|_{p,q} \|\nabla \omega^{(k)}(\cdot,t)\|_{p,q} \|\omega^{(k)}(\cdot,t)\|_{p,q} \right)_{L^{\frac{2p}{2p-3+q}}(R^+)}
\]
\[
+ \|\|\nabla j^{(k)}(\cdot,t)\|_{1,q_1}^{\theta} \|\nabla j^{(k)}(\cdot,t)\|_{p,q}^{\theta(1-\theta)} \|j^{(k)}(\cdot,t)\|_{p,q} \|\nabla j^{(k)}(\cdot,t)\|_{p,q} \right)_{L^{\frac{2p}{2p-3+q}}(R^+)}
\]
\[
\leq B_{15} \|\omega_0\|_{1,q_0} + B_{15} \|W_{k,1,q_1}^0\|_{1,q_1} (1-\theta) + B_{15} \|J_{k,1,q_1}^0\|_{1,q_1} (1-\theta) + B_{15} \|J_{k,1,q_1}^0\|_{1,q_1} (1-\theta) + B_{15} \|J_{k,1,q_1}^0\|_{1,q_1} (1-\theta).
\]
(3.115)

By (3.3), Proposition 4 and (3.91), it holds that
\[
\|\nabla j^{(k+1)}\|_{1,q_1} \\
\leq \|\nabla G(\cdot,t) * j_0\|_{1,q_1} \\
+ B_4 \int_0^t (t-s)^{-\frac{1}{2} - \frac{q_1-q_2}{2}} \|\omega^{(k)}(\cdot,s)\|_{1,q_1}^{\theta} \|\omega^{(k)}(\cdot,s)\|_{p,q}^{\theta(1-\theta)} \|\nabla j^{(k)}(\cdot,s)\|_{p,q} \|\nabla j^{(k)}(\cdot,s)\|_{p,q} ds
\]
(3.116)
Inequality (3.116) together with the arguments similar to those used to derive (3.115) yields that
\[
\tilde{J}_{k+1,q_1}^1 \leq B_{16} \| \tilde{j}_0 \|_{1,q_0} + B_{16} \tilde{W}_{k,p,q}^0 (\tilde{J}_{k+1,q_1}^1)^\theta (\tilde{J}_{k,p,q}^1)^{1-\theta} + B_{16} (\tilde{W}_{k+1,q_1}^1)^\theta (\tilde{W}_{k,p,q}^0)^{1-\theta} \tilde{J}_{k,p,q}^1 + (\tilde{J}_{k+1,q_1}^1)^\theta (\tilde{J}_{k,p,q}^0)^{1-\theta} \tilde{W}_{k,p,q}^0).
\]
(3.117)

Hence, by (3.12), (3.29), (3.39), (3.85), (3.115) and (3.117), we have
\[
\tilde{W}_{k+1,q_1} + \tilde{J}_{k+1,q_1} \leq B_{17} (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0}) + B_{17} (\| \omega_0 \|_{p_0,q_0} + \| j_0 \|_{p_0,q_0})^{2-\theta} + B_{17} (\tilde{W}_{k+1,q_1} + \tilde{J}_{k+1,q_1})^\theta
\]
whenever (3.13), (3.30), (3.40) and (3.86) hold. On the other hand, applying Proposition 4 and Lemma 2, we can get
\[
\tilde{W}_{k+1,q_1} + \tilde{J}_{k+1,q_1} \leq \| \nabla G(\cdot,t) * \omega_0 \|_{1,q_1} \|_{L^{\frac{2}{\theta-q_0}}(\mathbb{R}^+)} + \| \nabla G(\cdot,t) * j_0 \|_{1,q_1} \|_{L^{\frac{2}{\theta-q_0}}(\mathbb{R}^+)}
\]
\[
\leq B_{18} (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0}).
\]
(3.119)

Inequality (3.118) together with (3.119) and similar arguments to those in getting (3.77) and (3.78) leads to
\[
\tilde{W}_{k+1,q_1} + \tilde{J}_{k+1,q_1} \leq 2 \quad \text{for all } k \geq 1,
\]
(3.120)
\[
\tilde{W}_{k,q_1} + \tilde{J}_{k,q_1} \leq C_{19} (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0} + \| \omega_0 \|_{p_{0,q_0}} + \| j_0 \|_{p_{0,q_0}})
\]
when (3.13), (3.30), (3.40) and (3.86) hold and
\[
(B_{17} + B_{18}) (\| \omega_0 \|_{1,q_0} + \| j_0 \|_{1,q_0}) + B_{17} (\| \omega_0 \|_{p_{0,q_0}} + \| j_0 \|_{p_{0,q_0}})^{2-\theta} \leq 1/2.
\]

The rest of the proof is essentially analogous to Steps 4 and 5 in the proof of Theorem 3 (i). We omit the details.

We now turn to prove Theorem 1.

Proof of Theorem 1. Let \( p_0 = q_0 = 1 \), then \( E_1 = A_1 \). This proves (i) of Theorem 1. Taking \( q_3 = q_1 \) and \( \bar{q}_0 = 1 \) in \( E_2 \). One can easily get that
\[
q_1 = q_2 = q_3 = q, \quad \bar{p} = \frac{p(3-q)}{3-q+p'}.
\]
By the fact that \( 1 < \bar{p} < \min\{p,p'\} \) and \( 0 \leq q_1 - \bar{q}_0 < 1 \), it holds that
\[
q \in [1,2), \quad \frac{2(3-q)}{4-q} < p < 3-q.
\]
This proves (ii) of Theorem 1.

\[\square\]

Appendix

This appendix will be devoted to presenting some notations, lemmas and propositions, which are frequently used in our proof.

Appendix A

Appendix A contains some propositions. We start with some basic properties of Morrey spaces:

**Proposition 1 (Basic Properties of Morrey space):**

(i) \( \mathcal{M}^{1,0}(\mathbb{R}^3) = \mathcal{M}^1(\mathbb{R}^3) \) is the set of finite measures \( \mathcal{M} \) and \( \| \mu \|_1 = |\mu| \);

(ii) \( \mathcal{M}^{p,0}(\mathbb{R}^3) = L^p(\mathbb{R}^3) \) for \( p > 1 \);

(iii) \( L^p(\mathbb{R}^3) \subset L^{p,\infty}(\mathbb{R}^3) \subset \mathcal{M}^p(\mathbb{R}^3) \) for \( 1 < p < \infty \), where \( L^{p,\infty}(\mathbb{R}^3) \) denotes the Lorentz space;
(iv) Inclusion relations: for $1 \leq r, s$, $\tau, \lambda < \infty$ satisfying $s \leq r$, $\tau \leq \lambda$ and $\frac{3-\lambda}{r} = \frac{3-\tau}{s}$, 
\[
\mathcal{M}^{r\lambda}(\mathbb{R}^3) \subset \mathcal{M}^{s\tau}(\mathbb{R}^3);
\]

(v) Interpolation inequality: if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $0 < \theta < 1$, then 
\[
\mathcal{M}^{p_0}(\mathbb{R}^3) \cap \mathcal{M}^{p_1}(\mathbb{R}^3) \subset \mathcal{M}^{P}(\mathbb{R}^3)
\]
and 
\[
\|\mu\|_P \leq \|\mu\|_{p_0}^{1-\theta} \|\mu\|_{p_1}^\theta \quad \text{for} \quad \mu \in \mathcal{M}^{p_0}(\mathbb{R}^3) \cap \mathcal{M}^{p_1}(\mathbb{R}^3).
\]

(vi) Let $1 \leq p_1 < p_2 < p_3$, $0 \leq \mu_1, \mu_2, \mu_3 < 3$ and $k \in (0, 1)$ be such that 
\[
\frac{1}{p_3} = \frac{k}{p_1} + \frac{1-k}{p_2}, \quad \frac{\mu_3}{p_3} = \frac{\mu_1}{p_1} + \frac{\mu_2}{p_2}(1-k).
\]
Then 
\[
\|f\|_{p_1, \mu_1} \leq C\|f\|_{p_1, \mu_1}^{1-k} \|f\|_{p_2, \mu_2}.
\]

Proof. It should be pointed out that (i)-(iii) and (v) of Proposition 1 follows from [15] and (iv) of Proposition 1 follows from [18]. The proof of (vi) of Proposition 1 follows easily from the arguments same to those used to derive [1, Lemma 2.1 (iv)]. Here we omit the details. □

Remark 1. By Proposition 1 (iii), some works of [15, 18] can be regarded as the generalizations of the $L^p$ theory on NS problem in [13, 14, 17, 28].

Proposition 2. Let $\mu = (\mu_1, \mu_2, \mu_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$ and $\omega = \nabla \times u$, then one has

(i) (The Hölder inequality in Morrey space): for $1 \leq l, s, m, r \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{m} + \frac{1}{s}$ and $\frac{1}{r} = \frac{1}{m} + \frac{1}{s}$, then
\[
\|\mu\|_{r,0} \leq \|\mu\|_{m,\lambda} \|\nu\|_{s,\lambda}.
\]
Particularly, if $1 \leq p \leq \infty$, $\frac{1}{r} = \frac{1}{q} + \frac{1}{s}$, and $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$, then
\[
\|\mu\|_{r} \leq \|\mu\|_{p} \|\nu\|_{q},
\]
\[
\|\nabla u\|_{r} \leq \|\omega\|_{s,\tau} \|b\|_{r,\tau}.
\]

(ii) (Inequalities about the Biot-Savart kernel):
(a) If $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$ and $\mu \in \mathcal{M}^{p}(\mathbb{R}^3)$, then $K \ast \mu \in \mathcal{M}^{q}(\mathbb{R}^3)$ and
\[
\|K \ast \mu\|_{q} \leq \frac{1}{4\pi} \|\mu\|_{p}.
\]
(b) If $0 \neq p < 3 < q$ and $\mu \in \mathcal{M}^{p}(\mathbb{R}^3) \cap \mathcal{M}^{q}(\mathbb{R}^3)$, then $K \ast \mu \in \mathcal{M}^{\theta}(\mathbb{R}^3)$ and
\[
\|K \ast \mu\|_{\infty} \leq \frac{1}{4\pi} \|\mu\|_{p} \left(\frac{1}{\theta} - \frac{1}{p}\right)^{-1} \left(\frac{1}{\theta} - \frac{1}{q}\right)^{-1} \|\mu\|_{q} \left(\frac{1}{\theta} - \frac{1}{p}\right)^{-1} \left(\frac{1}{\theta} - \frac{1}{q}\right)^{-1}.
\]
Particularly, if we choose $q = 2p$ and $\theta = \frac{2}{p}$ with $p \in (\frac{3}{2}, 3)$, then
\[
\|K \ast \mu\|_{\infty} \leq \|\mu\|_{p} \left(\frac{1}{\theta} - \frac{1}{p}\right)^{-1} \|\mu\|_{\infty}^{2\theta - \theta}.
\]

The following proposition focuses the mapping properties for the Riesz potential on the Morrey spaces (see [27, Proposition 3.7]).

Proposition 3. Let $S(x) = |x|^\delta - 3$ for some $\delta \in (0, 3)$. Then, for $1 < p < q < \infty$, $0 \leq \theta < 3$, $\frac{1}{p} - \frac{1}{q} = \frac{\delta}{3 - \delta}$ and $f \in \mathcal{M}^{\theta}(\mathbb{R}^3)$, there exists a positive constant $C$ independent of $f$ such that
\[
\|S \ast f\|_{q, \theta} \leq C\|f\|_{p, \theta}.
\]

By motivated by the idea in [18, 24], we have the following inequalities for the heat kernel and Biot-Savart kernel:
Proposition 4. Let \( 1 \leq q_1 \leq q_2 \leq \infty, \ 0 \leq \lambda_1 \leq \lambda_2 < 3, \) and for \( t > 0, \)
\[
G(x,t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t}).
\]
We define the following operators as follows:
\[
T_{1,t} f = G(\cdot,t) * f(x), \quad T_{2,t} f = \nabla G(\cdot,t) * f(x), \quad T_{3,t} f = \partial_t G(\cdot,t) * f(x).
\]
Then the operators \( T_{i,t} (i=1,2,3) \) are bounded from \( \mathcal{M}^{q_1,\lambda_1}(\mathbb{R}^3) \) to \( \mathcal{M}^{q_2,\lambda_2}(\mathbb{R}^3) \) and depend on \( t \) continuously. Furthermore, one has for \( f \in \mathcal{M}^{q_1,\lambda_1}(\mathbb{R}^3), \)
\[
t^{\frac{1}{2}(\alpha_1 - \alpha_2)} ||T_{1,t} f||_{q_2,\lambda_2} \leq C ||f||_{q_1,\lambda_1}, \tag{A.1}
\]
\[
t^{\frac{1}{2}(\alpha_1 - \alpha_2)} ||T_{2,t} f||_{q_2,\lambda_2} \leq C ||f||_{q_1,\lambda_1}, \tag{A.2}
\]
\[
t^{1 + \frac{1}{2}(\alpha_1 - \alpha_2)} ||T_{3,t} f||_{q_2,\lambda_2} \leq C ||f||_{q_1,\lambda_1}, \tag{A.3}
\]
where \( \alpha_i = \frac{3 - \lambda_i}{q_i} (i=1,2) \) and constants \( C \) depends on \( q_1, q_2, \lambda_1, \lambda_2. \)

Proof. The case \( \lambda_1 = \lambda_2 \) was proved in [29]. We shall adopt the ideas used in the proof of Proposition 2.4 in [29] to prove the case \( \lambda_1 < \lambda_2. \) It was shown in [29] that
\[
\partial_t G(x,t) = ct^{-\frac{1}{2}} g_1(x), \quad \partial_t G(x,t) = ct^{-1} g_2(x).
\]
Moreover, the function \( g_i \) is another radial function enjoying the same properties as \( G(x,t) \) does. Hence, we only prove (A.1) since (A.2) and (A.3) can be proved similarly.

We now prove (A.1). Let \( \beta = \frac{3 - \lambda_2}{q_2}. \) It is clear that \( \beta > 1 \) since \( \lambda_1 < \lambda_2 < n. \) Fix \( t > 0 \) and \( R > 0. \) It is clear that
\[
\int_{|x-y| < R} |T_{1,t} f(y)|^{q_2} \, dy \leq ||T_{1,t} f||_{L^{q_2}}^{q_2 - \frac{q_1}{q_2}} \int_{|x-y| < R} |T_{1,t} f(y)|^{q_1} \, dy. \tag{A.4}
\]
By Hölder’s inequality with exponents \( p = \beta \) and \( p' = \beta', \) we see that
\[
\int_{|x-y| < R} |T_{1,t} f(y)|^{\frac{q_2}{\beta}} \, dy \leq CR^\alpha (1 - \beta') \left( \int_{|x-y| < R} |T_{1,t} f(y)|^{q_1} \, dy \right)^{1/\beta}. \tag{A.5}
\]
(A.5) together with (A.4) yields that
\[
R^{-\lambda_2} \int_{|x-y| < R} |T_{1,t} f(y)|^{q_2} \, dy \leq ||T_{1,t} f||_{L^{q_2}}^{q_2 - \frac{q_1}{q_2}} ||T_{1,t} f||_{q_1,\lambda_1}^{q_1}. \tag{A.6}
\]
Combining (A.6) with the known estimates (A.2) and (A.3) in [29] implies
\[
R^{-\lambda_2} \int_{|x-y| < R} |T_{1,t} f(y)|^{q_2} \, dy \leq C ||f||_{q_1,\lambda_1}^{q_2 - \frac{q_1}{q_2}} (q_2 - \frac{q_1}{q_2}) = C ||f||_{q_1,\lambda_1} T^{\frac{\lambda_2}{2}(q_2 - \frac{q_1}{q_2})}, \tag{A.7}
\]
where \( \alpha_i = \frac{3 - \lambda_i}{q_i} (i=1,2). \) Then (A.1) follows easily from (A.7). \qed

Appendix B

Appendix B is devote to presenting some technique lemmas, which are useful in the proof of of Theorem 3. Let \( \mathcal{B}(a,b) \) be the beta function defined by
\[
\mathcal{B}(a,b) = \int_0^t (t-s)^{a-1}s^{b-1} \, ds.
\]
It is well-known that if \( a,b > 0, \) then
\[
\mathcal{B}(a,b) = C(a,b) t^{a+b-1} \quad \text{with} \quad C(a,b) = \int_0^1 (1-s)^{a-1}s^{b-1} \, ds > 0.
\]

Lemma 1. Let \( \{s_k\}_{k \geq 0} \) be a sequence of nonnegative real numbers and \( f : [0,\infty) \to \mathbb{R} \) be a function. Suppose that \( f \) satisfies the following conditions:
Notice that

where

which yields

Proof. It is clear that $x_0 \leq x^*$. Assume that $x_{l-1} \leq x^*$ for all $l \in \{0, 1, 2, \ldots, k\}$ with some $k \geq 1$.

This assumption yields that

$$x_k - x^* \leq f(x_{k-1}) - x^* \leq f(x^*) - x^* = 0,$$

which yields $x_k \leq x^*$. This concludes the desired conclusion by induction.

As a direct application of Lemma 1, we can get the following result.

**Corollary 1.** Let $a_1 > 0$, $b_1 > 0$ and $1 - 4a_1b_1 > 0$. Let $\{X_k\}_{k \geq 0}$ be a sequence of nonnegative real numbers such that

(i) $X_0 \leq \frac{1 - \sqrt{1 - 4a_1b_1}}{2b_1}$;

(ii) $X_{k+1} \leq a_1 + b_1X_k^2$ for all $k \geq 0$.

Then one has

$$X_k \leq \frac{2a_1}{1 + \sqrt{1 - 4a_1b_1}} < 2a_1.$$

The following results play key roles in the proof of Theorem 3.

**Lemma 2.** Let $G(x, t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t})$ and $k = 0$. We denote $\nabla^0 u = u$ and $\nabla^1 u = \nabla u$.

Suppose that for fixed $p_0 \in [1, \infty)$, $q_0 \in (0, 3)$ and $u \in \mathcal{M}^{p_0, q_0} (\mathbb{R}^n)$, there exists a constant $A > 0$ independent of $u, t$ such that

$$|||\nabla^k G(\cdot, t) * u|||_{l, p, q} \leq A t^{-\frac{k}{2}} \left( \frac{1-q_0}{p_0} - \frac{1-q}{p} + \frac{3}{k} \right)^{-\frac{3}{2}} ||u||_{p_0, q_0},$$

for all $p \in (p_0, \infty)$ and $q \in [q_0, 3)$. Fix $p_0 \in [1, \infty)$, $q_0 \in (0, 3)$ and $u_0 \in \mathcal{M}^{p_0, q_0} (\mathbb{R}^n)$, then for any $p \in (p_0, \infty)$ and $q \in [q_0, 3)$, there exists a constant $C > 0$ independent of $u_0$ such that

$$|||\nabla^k G(\cdot, t) * u_0|||_{l, p, q} \leq C ||u_0||_{p_0, q_0},$$

(A.8)

where

$$a = 2 \left( \frac{3 - q_0}{p_0} - \frac{3 - q}{p} + \frac{k}{3} \right)^{-1}.$$

Proof. Fix $p > p_0$, $q \geq q_0$ and $q \in (0, 3)$. By our assumption we known that there exist $p_1, p_2$ with $p_1 < p$, $p_2 < p$, $p_1 < p_0 < p_2$ such that

$$|||\nabla^k G(\cdot, t) * u_0|||_{l, p, q} \leq A t^{-\frac{k}{2}} \left( \frac{1-q_0}{p_1} - \frac{1-q}{p} + \frac{3}{k} \right)^{-\frac{3}{2}} ||u_0||_{p_1, q_0},$$

(A.9)

$$|||\nabla^k G(\cdot, t) * u_0|||_{l, p, q} \leq A t^{-\frac{k}{2}} \left( \frac{1-q_0}{p_2} - \frac{1-q}{p} + \frac{3}{k} \right)^{-\frac{3}{2}} ||u_0||_{p_2, q_0}.$$  

(A.10)

There exists a constant $\Theta \in [0, 1]$ such that $\frac{\theta}{p_1} + \frac{1 - \theta}{p_2} = \frac{1}{p_0}$. Let

$$a_1 = \frac{2}{\frac{1-q_0}{p_1} - \frac{3 - q}{p} + \frac{k}{3}}, \quad a_2 = \frac{2}{\frac{1-q_0}{p_2} - \frac{3 - q}{p} + \frac{k}{3}}.$$

(A.9)-(A.10) together with the fact that $||t^{-\frac{k}{2}}||_{L^{p_1/q_1, p_2/q_2}} = 1$ yield that

$$|||G(\cdot, t) * u_0|||_{l, p_1, q_1} \leq C ||u_0||_{p_1, q_0},$$

(A.11)

$$|||G(\cdot, t) * u_0|||_{l, p_2, q_2} \leq C ||u_0||_{p_2, q_0}.$$  

(A.12)

Notice that $\frac{\theta}{a_1} + \frac{1 - \theta}{a_2} = \frac{1}{a}$, $a = \frac{a_1 a_2}{a_1 + a_2}$. An interpolation between (A.11) and (A.12) may yields (A.8).
Lemma 3. Let $G(x, t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t})$. Assume that given $\rho_0 \in [1, \infty)$, $q_0 \in (0, 3)$ and $u \in M^{\rho_0, -q_0}(\mathbb{R}^n)$, there exists a constant $A > 0$ independent of $u, t$ such that

$$
\|G(\cdot, t) * u\|_{p, q} \leq A t^{-\frac{3}{2}} \|u(\cdot, t)\|_{\rho_0, q_0}
$$

for all $p \in (p_0, \infty)$ and $q \in [q_0, 3]$. Fix $p_0 \in [1, \infty)$, $q_0 \in (0, 3)$, $p \in (p_0, \infty)$ and $q \in [q_0, 3)$ such that

$$
a = \frac{1}{2} \left( \frac{3 - q_0}{p_0} - \frac{3 - q}{p} \right) < \frac{1}{2}.
$$

Let $u_0$ be a function such that

$$
\|u_0\|_{p_0, q_0} \leq \frac{1}{4AB_1B_2} \min \left\{ \frac{1}{C(a, 1 - 2a)}, 1 \right\},
$$

and \{u_k\}_{k \geq 1} be a sequence of functions satisfying the following

$$
\begin{align*}
\|u_k(\cdot, t)\|_{p, q} &\leq B_1 \|G(\cdot, t) * u_0\|_{p, q} + B_2 \int_0^t (t - s)^{a - 1} \|u_k(s)\|_{p, q}^2 \, ds \\
\|u_k\|_{L^{1/a}_q} &\leq 2B_1^2 \|u_0\|_{L^{1/a}_q}
\end{align*}
$$

for some $B_1, B_2 > 0$. Then, for all $k \geq 1$, it holds that

(i) $\sup_{t > 0} \|u_k\|_{p, q} \leq 2AB_1 \|u_0\|_{p_0, q_0}$,

(ii) $\|u_k\|_{L^{1/a}_q} \leq 2AB_1 \|u_0\|_{p_0, q_0}$.

Proof. Letting $A_k = \sup_{t > 0} \|u_k\|_{p, q}$, it follows from Lemma 2 and Young’s inequality that

$$
A_{k+1} \leq AB_1 \|u_0\|_{p_0, q_0} + B_2 t^a \int_0^t (t - s)^{a - 1} \|u_k(s)\|_{p, q}^2 \, ds
$$

$$
\leq AB_1 \|u_0\|_{p_0, q_0} + B_2 C(a, 1 - 2a) \sup_{t > 0} t^{2a} \|u_k\|_{p, q}^2
$$

$$
\leq AB_1 \|u_0\|_{p_0, q_0} + B_2 C(a, 1 - 2a) A_k^2.
$$

Notice that $1 - 4AB_1B_2 C(a, 1 - 2a) \|u_0\|_{p_0, q_0} > 1$. Invoking Corollary 1, we can get

$$
A_k < 2AB_1 \|u_0\|_{p_0, q_0}.
$$

This proves (i). Let $\tilde{A}_k = \|u_k\|_{p, q} \|u_k\|_{L^{1/a}_q}$. It follows from Lemma 2 and Young’s inequality that

$$
\|f * g\|_{L^{1/a}_q} \leq \|f\|_{L^{1/(1-a)}_q} \|g\|_{L^{1/a}_q},
$$

$$
\tilde{A}_{k+1} \leq B_1 \|G(\cdot, t) * u_0\|_{p, q} \|u_k\|_{L^{1/a}_q} + B_2 \int_0^t (t - s)^{a - 1} \|u_k(s)\|_{p, q}^2 \, ds
$$

$$
\leq AB_1 \|u_0\|_{p_0, q_0} + \|u_k\|_{L^{1/(1-a)}_{1/a}}^2 - B_2 \|u_k\|_{p, q}^2
$$

$$
\leq AB_1 \|u_0\|_{p_0, q_0} + B_2 \tilde{A}_k^2,
$$

which, along with Corollary 1, yields (ii).

Finally, we would like to remark that the following general formulas are useful for calculations with vector fields in $\mathbb{R}^3$.

$$
\nabla (F \cdot G) = (F \cdot \nabla) G + (G \cdot \nabla) F + F \times (\nabla \times G) + G \times (\nabla \times F),
$$

$$
div (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G),
$$

$$
\nabla \times (F \times G) = F div G - G div F + (G \cdot \nabla) F - (F \cdot \nabla) G.
$$

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REFERENCES

[1] M.F. de Almeida and L.C.F. Ferreira, On the Navier-Stokes equations in the half-space with initial and boundary rough data in Morrey spaces, J. Differential Equations **254**: 1548–1570, 2013.
[2] M. Ben-Artzi, J. Croisille and D. Fishelov, Navier-Stokes equations in planar domains, World Scientific, 2013.
[3] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoam. **13**: 515–541, 1997.
[4] M. Cannone, Ondelettes, paraproduits et Navier-Stokes, Diderot Editeur, Paris, 1995.
[5] G. Cottet, Équations de Navier-Stokes dans le plan avec tourbillon initial mesuere, C. R. Acad. Sci. Paris Sér. I Math. **303**: 105–108, 1986.
[6] G. Cottet and J. Soler, Three-dimensional Navier-Stokes equations for singular filament initial data, J. Differential Equations **74**: 234–253, 1988.
[7] T. G. Cowling, Magnetohydrodynamics, Interscience Tracts on Physics and Astronomy, No. 4. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1957.
[8] G. Duvaut and J.-L. Lions, Inéquations en thermoélasticité et magnétoturbulent, Arch. Rational Mech. Anal. **46**: 241–279, 1972.
[9] P. Federbush, Navier and Stokes meet the wavelet, Commun. Math. Phys. **155**: 219–248, 1993.
[10] H. Fujita and T. Kato, On the Navier-Stokes initial value problem, I, Arch. Rational Mech. Anal. **16**: 269–315, 1964.
[11] I. Gallagher and T. Gallay, Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity, Math. Ann. **332**: 287–327, 2005.
[12] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations **62**: 186–212, 1986.
[13] Y. Giga and T. Miyakawa, Solutions in $L_2$ of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal. **89**: 267–281, 1985.
[14] Y. Giga and T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, Commun. Partial Differ. Eq. **14**: 577–618, 1989.
[15] Y. Giga, T. Miyakawa and H. Osada, Two-dimensional Navier-Stokes flow with measures as initial vorticity, Arch. Ration. Mech. Anal. **104**: 223–250, 1988.
[16] T. Kato, Strong $L^p$ solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. **187**: 471–480, 1984.
[17] T. Kato, Strong solutions of the Navier-Stokes equation in Morrey spaces, Bol. Soc. Brasil. Mat. (N.S.) **22**: 127–155, 1992.
[18] T. Kato, The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity, Differ. Integ. Equ. **7**: 949–966, 1994.
[19] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math., **157**: 22–35, 2001.
[20] H. Kozono and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. Commun. Partial Differ. Eq. **19**: 959–1014, 1994.
[21] L. Landau and E. Lifchitz, Physique théorique (“Landau-Lifshits”), Tome 8. Traduit du Russe. “Mir”, Moscow, 1990. Electrodynamik des milieus continus. [Electrodynamics of continuous media], Second Russian edition revised by Lifshitz [Lifshiç] and L. Pitaevsky [L. P. Pittåvskjii], Translated from the second Russian edition by Anne Sokova.
[22] Z. Lei, F. Lin, Global Mild Solutions of Navier-Stokes Equations Commun. Pure Appl. Math., **LXIV**: 1297–1304, 2011.
[23] J. Peetre, On the theory of $L_{p,\lambda}$ spaces, J. Funct. Anal. **4**: 71–87, 1969.
[24] F. Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$, Ann. Inst. Henri Poincare, Anal. Non Lineaire **13**: 319–336, 1996.
[25] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, Commun. Pure Appl. Math. **36**: 635–664, 1983.
[26] M. E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Commun. Partial Differ. Eq. **17**: 1407–1456, 1992.
[27] W. Wahl, The Equations of Navier-Stokes and Abstract Parabolic Equations, Aspects of Mathematics, E8. Friedr. Vieweg & Sohn, Braunschweig, 1985.
[28] J. Wu, Quasi-geostrophic-type equations with initial data in Morrey spaces, Nonlinearity **10**: 1409–1420, 1997.
FENG LIU: COLLEGE OF MATHEMATICS AND SYSTEM SCIENCE, SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, QINGDAO, SHANDONG 266590, PEOPLE’S REPUBLIC OF CHINA
E-mail address: FLiu@sdu.edu.cn

SHUAI XI: COLLEGE OF MATHEMATICS AND SYSTEM SCIENCE, SHANDONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, QINGDAO, SHANDONG 266590, PEOPLE’S REPUBLIC OF CHINA
E-mail address: shuaixi@sdu.edu.cn

SHENGGUO ZHU: MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD OX2 6GG, UK.
E-mail address: zhus@maths.ox.ac.uk