ALL POISSON STRUCTURES IN $\mathbb{R}^3$

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Abstract

Hamiltonian formulation of $N = 3$ systems is considered in general. The most general solution of the Jacobi equation in $\mathbb{R}^3$ is proposed. Compatible Poisson structures and the corresponding bi-Hamiltonian $N = 3$ systems are also discussed.
1. Introduction.

Hamiltonian formulation of $N = 3$ systems has been intensively considered in the last two decades. Recent works [1], [2] on this subject give a very large class of the solutions of the Jacobi equation for the Poisson matrix $J$. Very recently generalizing the solutions given in [1] we proposed new classes of solutions of the Jacobi equation and gave also some compatible Poisson structures. In the same work we have considered also bi-Hamiltonian systems corresponding to compatible Poisson structures [3].

In this work we give the most general solution of the Jacobi equation in $\mathbb{R}^3$, and give the condition for the existence of the compatible pairs of Poisson structures. Finally we give the conditions for the existence of the bi-Hamiltonian systems.

Matrix $J = (J^{ij})$, $i, j = 1, 2, 3$, defines a Poisson structure in $\mathbb{R}^3$ if it is skew-symmetric, $J_{ij} = -J_{ji}$, and its entries satisfy the Jacobi equation

$$J_{li} \partial_l J_{jk} + J_{lj} \partial_l J_{ki} + J_{lk} \partial_l J_{ij} = 0, \quad (1)$$

where $i, j, k = 1, 2, 3$. Here we use the summation convention, meaning that repeated indices are summed up. We show that the general solution of the above equation (1) has the following form

$$J^{ij} = \mu \epsilon^{ijk} \partial_k \Psi, \quad (2)$$

where $\mu$ and $\Psi$ are arbitrary differentiable functions of $x^i$, $i = 1, 2, 3$ and $\epsilon^{ijk}$ is the Levi-Civita symbol. This has a very natural geometrical explanation. Let $\Psi = c_1$ and $H = c_2$ define two surfaces $S_1$ and $S_2$ respectively in $\mathbb{R}^3$, where $c_1$ and $c_2$ are some constants. Then the intersection of these surfaces define a curve $C$ in $\mathbb{R}^3$. The velocity vector $\frac{dx}{dt}$ of this curve is parallel to the vector product of the normal vectors $\nabla \Psi$ and $\nabla H$ of the surfaces $S_1$ and $S_2$, respectively, i.e.,

$$\frac{dx}{dt} = -\mu \nabla \Psi \times \nabla H, \quad (3)$$

where $\mu$ is any arbitrary function in $\mathbb{R}^2$. This equation defines a Hamiltonian system in $\mathbb{R}^3$. We shall prove that all Hamiltonian systems in $\mathbb{R}^3$ are of the form given in (3).

As we shall see in Section 3 and 4, having the Poisson structure in the form (2) allows us to construct compatible Poisson structures $J$ and $\tilde{J}$ and...
hence the corresponding bi-Hamiltonian systems. For further details please see [3].

2. General Solution.

Let $J^{12} = u$, $J^{13} = -v$, $J^{23} = w$, where $u, v$ and $w$ are some differentiable functions of $x_1, x_2$ and $x_3$. Then Jacobi equation (1) takes the form

$$u\partial_1 v - v\partial_1 u + w\partial_2 u - w\partial_2 u + v\partial_3 w - w\partial_3 v = 0.$$  

(4)

Assuming that $u \neq 0$, let $\rho = \frac{v}{u}$ and $\chi = \frac{w}{u}$ then equation (4) can be written as

$$\partial_1 \rho - \partial_2 \chi + \rho \partial_3 \chi - \chi \partial_3 \rho = 0.$$  

(5)

This equation can be put in a more suitable form by writing it

$$(\partial_1 - \chi \partial_3)\rho - (\partial_2 - \rho \partial_3)\chi = 0.$$  

(6)

Introducing differential operators $D_1$ and $D_2$ defined by

$$D_1 = \partial_1 - \chi \partial_3 \quad D_2 = \partial_2 - \rho \partial_3,$$  

(7)

one can write equation (6) as

$$D_1 \rho - D_2 \chi = 0.$$  

(8)

**Lemma 1.** Let equation (8) be satisfied. Then there are new coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ such that

$$D_1 = \partial_{\bar{x}_1}, \quad \text{and} \quad D_2 = \partial_{\bar{x}_2}.$$  

(9)

**Proof.** If equation (8) is satisfied, it is easy to show that the operators $D_1$ and $D_2$ commute, i.e.,

$$D_1 \circ D_2 - D_2 \circ D_1 = 0.$$  

Hence, by the Frobenius theorem (see [4] p. 40) there exist coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ such that the equalities (9) hold. ☐

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The coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are described by the following lemma.

**Lemma 2.** Let $\zeta$ be a common invariant function of $D_1$ and $D_2$, i.e.

$$D_1 \zeta = D_2 \zeta = 0,$$

then the coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$ of Lemma 1 are given by

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = \zeta.$$  \hspace{1cm} (11)

Moreover from (10) we get ($\partial_3 \zeta \neq 0$),

$$\rho = \frac{\partial_1 \zeta}{\partial_3 \zeta}, \quad \chi = \frac{\partial_2 \zeta}{\partial_3 \zeta}.$$  \hspace{1cm} (12)

**Theorem 1.** All Poisson structures in $\mathbb{R}^3$ take the form (2), i.e., $J^{ij} = \mu \epsilon^{ijk} \partial_k \zeta$. Here $\mu$ and $\zeta$ are some differentiable functions in $\mathbb{R}^3$

**Proof.** Using (12), the entries of matrix $J$, in the coordinates $\bar{x}_1, \bar{x}_2, \bar{x}_3$, can be written as

$$u = \mu \partial_3 \zeta, \quad v = \mu \partial_2 \zeta, \quad w = \mu \partial_1 \zeta.$$  \hspace{1cm} (13)

Thus matrix $J$ has the form (2) ($\Psi = \zeta$). \hspace{1cm} $\square$

**Remark 1.** So far we assumed that $u \neq 0$. If $u = 0$ then the Jacobi equation becomes quite simpler $v \partial_3 w - w \partial_3 v = 0$ which has the simple solution $w = v \xi(x_1, x_2)$, where $\xi$ is an arbitrary differentiable of $x_1$ and $x_2$. This class is also covered in our general solution (2) by letting $\Psi$ independent of $x_3$.

A well known example of a dynamical system with Poisson structure of the form (2) is the Euler equations.

**Example 1.** Consider the Euler equations ([4], pp.397–398,)

$$\dot{x}_1 = \frac{I_2 - I_3}{I_2 I_3} x_2 x_3,$$

$$\dot{x}_2 = \frac{I_3 - I_1}{I_3 I_1} x_3 x_1,$$

$$\dot{x}_3 = \frac{I_1 - I_2}{I_1 I_2} x_1 x_2.$$  \hspace{1cm} (14)
where $I_1, I_2, I_3 \in \mathbb{R}$ are some (non-vanishing) real constants. This system admits Hamiltonian representation of the form (2). The matrix $J$ can be defined in terms of function $\Psi = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ and $\mu = 1$, so

$$
\begin{align*}
    u &= -x_3, \\
    v &= -x_2, \\
    w &= -x_1,
\end{align*}
$$

and $H = \frac{x_1^2}{2I_1} + \frac{x_2^2}{2I_2} + \frac{x_3^2}{2I_3}$.

**Remark 2.** All of the Poisson structures described in [1] and [3] have the form (2). The Poisson structure $J$, described in these references are given by

$$
\begin{align*}
    u(x) &= \eta(x_1, x_2, x_3)\psi_1(x_1)\psi_2(x_2)\phi_3(x_3), \\
    v(x) &= \eta(x_1, x_2, x_3)\psi_1(x_1)\phi_2(x_2)\psi_3(x_3), \\
    w(x) &= \eta(x_1, x_2, x_3)\phi_1(x_1)\psi_2(x_2)\psi_3(x_3),
\end{align*}
$$

where $\eta(x_1, x_2, x_3)$. $\psi_i(x_i)$. $\phi_i(x_i)$. $i = 1, 2, 3$, are arbitrary non-vanishing differentiable functions. Defining $\mu = \eta(x_1, x_2, x_3)\psi_1(x_1)\psi_2(x_2)\psi_3(x_3)$ and $\Psi = \int_{x_1}^{x_2} \frac{\phi_1}{\psi_1} dx_1 - \int_{x_2}^{x_3} \frac{\phi_2}{\psi_2} dx_2 + \int_{x_3}^{x_1} \frac{\phi_3}{\psi_3} dx_3$ one obtains that $J$ has form (2).

**Remark 3.** In general the Darboux theorem states that (see [4]) , locally, all Poisson structures can be reduced to the standard one (Poisson structure with constant entries). The above theorem , Theorem 1 , resembles the Darboux theorem for $N = 3$. All Poisson structures, at least locally, can be cast into the form (2). This result is important because the Dorboux theorem is not suitable for obtaining multi-Hamiltonian systems in $\mathbb{R}^3$, but we will show that our theorem is effective for this purpose. Writing the Poisson structure in the form (2) allows us to construct bi-Hamiltonian representations of a given Hamiltonian system.

**Definition 1.** Two Hamiltonian matrices $J$ and $\bar{J}$ are compatible, if the sum $J + \bar{J}$ defines also a Poisson structure.

**Lemma 3.** Let Poisson structures $J$ and $\bar{J}$ have form (2), so $J^{ij} = \mu \epsilon^{ijk} \partial_k \Psi$ and $\bar{J}^{ij} = \bar{\mu} \epsilon^{ijk} \partial_k \bar{\Psi}$. Then $J$ and $\bar{J}$ are compatible if and only if there exist
a differentiable function $\Phi(\Psi, \tilde{\Psi})$ such that
\[
\tilde{\mu} = \mu \frac{\partial \Phi}{\partial \Psi},
\] (17)
provided that $\partial \Phi \equiv \frac{\partial \phi}{\partial \Psi} \neq 0$ and $\partial \Phi \equiv \frac{\partial \psi}{\partial \Psi} \neq 0$

**Remark 4.** In [2] a method is given to find new solutions of the Jacobi equation (1) from the known ones. Construction of compatible structures, Lemma 3, covers all such cases. Letting, for instance, $\tilde{\Psi} = x_1 + x_2 + x_3$ and $\tilde{\mu} = \xi$ (to have the same notation as [2]) we obtain the most general solution of the Jacobi equation with $u = u_0 + \xi, v = v_0 + \xi, w = w_0 + \xi$ where $u_0 = \mu \Psi_3, v_0 = \mu \Psi_2, w_0 = \mu \Psi_1$ as the known solution.

The above Lemma 4 suggests that all Poisson structures in $\mathbb{R}^3$ have compatible pairs, because the condition (17) is not so restrictive on the Poisson matrices $J$ and $\tilde{J}$. Such compatible Poisson structures can be used to construct bi-Hamiltonian systems.

**Definition 2.** A Hamiltonian equation is said to be bi-Hamiltonian if it admits two Hamiltonian representations $H$ and $\tilde{H}$, with compatible Poisson structures $J$ and $\tilde{J}$ such that
\[
\frac{dx}{dt} = J \nabla H = \tilde{J} \nabla \tilde{H},
\] (18)

**Lemma 4.** Let $J$ be given by (2) and $H(x_1, x_2, x_3)$ is any differentiable function then the Hamiltonian equation
\[
\frac{dx}{dt} = J \nabla H = -\mu \nabla \Psi \times \nabla H,
\] (19)
is bi-Hamiltonian with the second structure given by $\tilde{J}$ with entries
\[
\begin{align*}
\tilde{u}(x) &= \tilde{\mu} \partial_1 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \\
\tilde{v}(x) &= \tilde{\mu} \partial_2 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \\
\tilde{w}(x) &= \tilde{\mu} \partial_3 g(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)),
\end{align*}
\] (20)
and $\tilde{H} = h(\Psi(x_1 x_2 x_3), H(x_1, x_2, x_3)), \tilde{\Psi} = g(\Psi(x_1, x_2, x_3), H(x_1, x_2, x_3))$
\[
\tilde{\mu} = \mu \frac{\partial \Phi}{\partial \Psi},
\] Provided that there exist differentiable functions $\Phi(\Psi, \tilde{\Psi}), h(\Psi, H)$,
and \( g(\Psi, H) \) satisfying the following equation
\[
\frac{\partial g}{\partial \Psi} \frac{\partial h}{\partial H} - \frac{\partial g}{\partial H} \frac{\partial h}{\partial \Psi} = \frac{\Phi_1(\Psi, g)}{\Phi_2(\Psi, g)},
\]
where \( \Phi_1 = \partial_\Psi \Phi \big|_{(\Psi, g)} \), \( \Phi_2 = \partial_\Psi \Phi \big|_{(\Psi, g)} \).

**Proof.** By Lemma 3, \( J \) and \( \bar{J} \) are compatible and it can be shown by a straightforward calculation that the equality (being a bi-Hamiltonian system)
\[
\bar{J} \nabla \bar{H} = J \nabla H
\]
(22)
or
\[
\bar{\mu} \nabla \bar{\Psi} \times \nabla \bar{H} = \mu \nabla \Psi \times \nabla H
\]
(23)
is guaranteed by (21). Hence the system
\[
\begin{align*}
\frac{dx_1}{dt} &= \mu \partial_3 \Psi \partial_2 H - \partial_2 \Psi \partial_3 H \\
\frac{dx_2}{dt} &= -\mu \partial_3 \Psi \partial_1 H + \partial_1 \Psi \partial_3 H \\
\frac{dx_3}{dt} &= \mu \partial_2 \Psi \partial_1 H - \partial_1 \Psi \partial_2 H
\end{align*}
\]
(24)
is bi-Hamiltonian. \( \square \)

**Remark 5.** The Hamiltonian function \( H \) is a conserved quantity of the system. It is clear from the expression (24) that the function \( \Psi \) is another conserved quantity of the system. Hence for a given Hamiltonian system there is a duality between \( \bar{H} \) and \( \Psi \). Such a duality arises naturally because a simple solution of the equation (21) is \( \bar{\Psi} = H \), \( \bar{H} = \Psi \) and \( \bar{\mu} = -\mu \).

Using the Lemma 4 we can construct infinitely many compatible Hamiltonian representations by choosing functions \( \Phi \), \( g \), \( h \) satisfying (21). If we fix functions \( \Phi \), \( g \) then equation (21) became linear first order partial differential equations for \( h \). For instance, taking \( g = \Psi H \) and \( \bar{\mu} = -\mu \), which fixes \( \Phi \), we obtain \( h = \ln H \). Thus we obtain second representation of equation (3) with \( \bar{J} \) given by \( \bar{\Psi} = \Psi H \) and \( \bar{H} = \ln H \). Let us give some examples of Hamiltonian systems. All the autonomous examples in [1] with bi-Hamiltonian representations can be listed below but we prefer time dependent ones.
Example 2. Consider systems that are obtained from RTW system [8]

\[
\begin{align*}
\dot{x} &= \gamma x + \delta y + z - 2y^2 \\
\dot{y} &= \gamma y - \delta x + 2xy \\
\dot{z} &= -2z(x + 1).
\end{align*}
\]

(25)

for appropriate subset of parameters by recalling. Following [6] we have:

RTW(1) system

\[
\begin{align*}
\dot{x}_1 &= \delta x_2 + x_3e^{-2t} - 2x_2^2 \\
\dot{x}_2 &= -\delta x_1 + 2x_1x_2 \\
\dot{x}_3 &= -x_1x_3,
\end{align*}
\]

(26)

where \( \delta \) is an arbitrary constant. The matrix \( J \) is given by (2) with \( \mu = 1 \) and \( \Psi = \frac{1}{2}(x_1^2 - x_2^2 + x_3e^{-t}) \) and Hamiltonian \( H = x_3(2x_2 - \delta) \).

The matrix \( \tilde{J} \) is given by (2) with \( \tilde{\mu} = 1 \) and \( \tilde{\Psi} = \frac{x_3}{2}(x_1^2 - x_2^2 + x_3e^{-t})(2x_2 - \delta) \) and Hamiltonian \( \tilde{H} = \ln(x_3(2x_2 - \delta)) \).

RTW(3) system

\[
\begin{align*}
\dot{x}_1 &= (x_3 - 2x_2)e^{-t} \\
\dot{x}_2 &= 2x_1x_2e^{-t} \\
\dot{x}_3 &= -2x_1x_3e^{-t}.
\end{align*}
\]

(27)

The matrix \( J \) is given by with \( \mu = 1 \) and \( \Psi = (x_1^2 - x_2^2 + x_3)e^{-t} \) and Hamiltonian \( H = x_2x_3 \).

The matrix \( \tilde{J} \) is given by (2) with \( \tilde{\mu} = 1 \) and \( \tilde{\Psi} = x_2x_3(x_1^2 - x_2^2 + x_3)e^{-t} \) and Hamiltonian \( \tilde{H} = \ln(x_3x_2) \).

RTW(4) system

\[
\begin{align*}
\dot{x}_1 &= x_3e^{-(\gamma+2)t} - 2x_2^2e^{\gamma t} \\
\dot{x}_2 &= 2x_1x_2e^{\gamma t} \\
\dot{x}_3 &= -2x_1x_3e^{\gamma t},
\end{align*}
\]

(28)

where \( \gamma \) is an arbitrary constant. The matrix \( J \) is given by with \( \mu = 1 \) and \( \Psi = (x_1^2 - x_2^2) + x_3)e^{\gamma t} + x_3e^{-(\gamma+2)t} \) and Hamiltonian \( H = x_2x_3 \).

The matrix \( \tilde{J} \) is given by (2) with \( \tilde{\mu} = 1 \) and \( \tilde{\Psi} = x_2x_3[(x_1^2 - x_2^2) + x_3)e^{\gamma t} + \)
\[ x_3 e^{-(\gamma+2)t} \] and Hamiltonian \( \tilde{H} = \ln(x_3 x_2) \).

**RTW(5) system**

\[
\begin{align*}
\dot{x}_1 &= \delta x_2 + x_3 - 2x_2^2 e^{-2t} \\
\dot{x}_2 &= -\delta x_1 + 2x_1 x_2 e^{-2t} \\
\dot{x}_3 &= -2x_1 x_3 e^{-2t},
\end{align*}
\]

(29)

where \( \delta \) is a non-vanishing constant. The matrix \( J \) is given by with \( \mu = 1 \) and \( \Psi = \delta e^{-2t} (x_1^2 - x_2^2) + \frac{\delta}{2} x_3 \) and Hamiltonian \( H = x_1^2 + x_2^2 + \frac{2}{\delta} x_2 x_3 \).

The matrix \( \tilde{J} \) is given by (2) with \( \tilde{\mu} = 1 \) and \( \tilde{\Psi} = [x_1^2 + x_2^2 + \frac{2}{\delta} x_2 x_3] [\frac{2}{\delta} e^{-2t} (x_1^2 - x_2^2) + \frac{\delta}{2} x_3] \) and Hamiltonian \( \tilde{H} = \ln(x_1^2 + x_2^2 + \frac{2}{\delta} x_2 x_3) \).

**Remark 6.** If a system in \( \mathbb{R}^3 \) is Hamiltonian then \( \Psi \), and \( H \) are constants of motion. This reduces the system of three coupled ODE to a single ODE. As an example let us consider Euler top equation given in Example 4. The constants of motion are \( \Psi = -\frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \), and \( H = \frac{x_1^2}{2I_1} + \frac{x_2^2}{2I_2} + \frac{x_3^2}{2I_3} \).

So, the equalities \( \Psi = c_1 \) and \( H = c_2 \) give (assuming \( I_1 \neq I_2 \neq I_3 \))

\[
\begin{align*}
x_1 &= \sqrt{\tilde{c}_1 + \frac{I_1(I_3-I_2)}{I_3(I_2-I_1)} x_3^2} \\
x_2 &= \sqrt{\tilde{c}_2 + \frac{I_2(I_3-I_1)}{I_3(I_1-I_2)} x_3^2}
\end{align*}
\]

(30)

where \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are new constants and \( x_3 \) is given by

\[
t - t_0 = \int \frac{dx_3}{\sqrt{\left( \tilde{c}_1 + \frac{I_1(I_3-I_2)}{I_3(I_2-I_1)} x_3^2 \right) \left( \tilde{c}_2 + \frac{I_2(I_3-I_1)}{I_3(I_1-I_2)} x_3^2 \right)}}
\]

(31)

where \( t_0 \) is an integration constant. Hence one has the general solution with the required number of integration constants.

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