Scalar and Vector Perturbations in Quantum Cosmological Backgrounds

Emanuel J. C. Pinho and Nelson Pinto-Neto
ICRA - Centro Brasileiro de Pesquisas Físicas – CBPF,
rua Xavier Sigaud, 150, Urca, CEP22290-180, Rio de Janeiro, Brazil
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Generalizing a previous work concerning cosmological linear tensor perturbations, we show that the lagrangians and hamiltonians of cosmological linear scalar and vector perturbations can be put in simple form through the implementation of canonical transformations and redefinitions of the lapse function, without ever using the background classical equations of motion. In particular, if the matter content of the Universe is a perfect fluid, the hamiltonian of scalar perturbations can be reduced, as usual, to a hamiltonian of a scalar field with variable mass depending on background functions, independently of the fact that these functions satisfy the background Einstein’s classical equations. These simple lagrangians and hamiltonians can then be used in situations where the background metric is also quantized, hence providing a substantial simplification over the direct approach originally developed by Halliwell and Hawking.

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I. INTRODUCTION

In the theory of linear cosmological perturbations, simple evolution equations for the perturbations have been obtained [1]. Lagrangians and hamiltonians describing the dynamics of scalar, vector, and tensor perturbations coming from the Einstein-Hilbert lagrangian have been greatly simplified in different cosmological scenarios under the assumption that the background metric satisfies Einstein classical field equations, and after taking out space and time total derivatives [1]. Once these simple lagrangians and hamiltonians are obtained, the quantization of linear cosmological perturbations becomes easy, with a quite simple interpretation: they can be seen as quantum fields which behave essentially as scalar fields with a time dependent effective mass. The time varying background scale factor which is responsible for this “mass” acts as a pump field [2], creating or destroying modes of the perturbations. In this framework, one can assume an initial vacuum state for the perturbations, yielding primordial perturbation spectra which can be compared with observations. In the cosmological inflationary scenario [3], the resulting spectrum for scalar perturbations is in good agreement with the data [4].

However, this state of affairs is rather incomplete: the overwhelming majority of classical backgrounds possess an initial singularity at which the classical theory is expected to break down, and one needs to justify the initial conditions for inflation and quantum perturbations. Hence, a full quantum treatment including the background must be constructed. The first approach in this direction was made in Ref. [5], where the canonical quantization of the perturbations and background was implemented through the derivation of the super-hamiltonian constraint of the whole system and its consequent Wheeler-DeWitt equation \( H(A, P_A, X, P_X)\psi = 0 \), where \( A \) and \( P_A \) represent the phase space background variables, and \( X \) and \( P_X \) the perturbation phase space variables. They claim that the no boundary proposal can set the initial conditions for inflation and the vacuum initial state for the perturbations. Then, through the imposition of the ansatz on the wave functional \( \Psi(A, X, t) = \varphi(A(t))\psi(A, X, t) \), they could manage to separate the quantum effects in the background from the quantum perturbations, where the wave function for the background \( \varphi(A, t) \) obeys an independent quantum min-superspace description where back reactions terms from the quantum perturbations are negligible. The singularity is bypassed through an euclidianization of spacetime near it, and a consequent beginning of time when (or where) the geometry passes from the euclidian signature to the lorentzian one. The quantum perturbations are described in the oscillatory part of the background wave function, where a WKB approximation can be used. Then, the evolution of the scale factor in time may be obtained through the equation \( \dot{a} \propto \partial S/\partial a \), where \( S \) is a solution of the classical Hamilton-Jacobi equation. Hence, this evolution is the classical one, and we are back to a semiclassical description of the perturbations.

In parallel to that, the possibility that the singularity could be avoided through a bounce connecting the present expanding phase with a preceding contracting phase has been explored. In this case, the Universe is eternal: there is no beginning of time, nor horizons. Many frameworks where bounces may occur have been proposed [6, 7, 8]. These new features of the background introduce a new picture for the evolution of cosmological perturbations: vacuum initial conditions may now be imposed when the Universe was very big and almost flat, and effects due to the contracting and bouncing phases, which are not present in models with a beginning of time, may change the subsequent evolution of perturbations in the expanding phase. Because of that, the evolution of
cosmological perturbations in bouncing models has been cause of intense debate \cite{9}.

In the framework of quantum cosmology in minisuperspace models, bouncing models had also been proposed where the bounce occurs due to quantum effects in the background \cite{10, 11, 12, 13, 14, 15, 16}. Some approaches have used an ontological interpretation of quantum mechanics, the Bohm-de Broglie \cite{17} one, to interpret the results \cite{13, 14, 15}. In this interpretation, quantum Bohmian trajectories, the quantum evolution of the scale factor $a_q(t)$ at zeroth order, can be defined through the relation $\dot{a} \propto \partial S/\partial a$, where $S$ is now the phase of the background wave function $\varphi(A,t)$ without any approximation: it is not a solution of the classical Hamilton-Jacobi equation. In fact it satisfies a modified Hamilton-Jacobi equation derived from the Wheeler-DeWitt equation for $\varphi(A,t)$, and hence $a_q(t)$ is not the classical trajectory: in the regions where the quantum effects cannot be neglected, the quantum trajectory $a_q(t)$ performs a bounce which connect two asymptotic classical regions where the quantum effects are negligible. One than has in hands a definite function of time for the background, even at the quantum level, which realizes a soft transition from the contracting phase to the expanding one. Due to the results of Ref. \cite{5}, where the background minisuperspace Wheeler-DeWitt equation for $\varphi(A,t)$ continue to hold when quantum perturbations are present because back reaction terms are negligible (which can also be justified through other ansatz for the wave function or verified ‘a posteriori’), this background quantum function $a_q(t)$ is sufficient to describe the whole quantum features of the background. The natural question to ask is what happens with the perturbations when it passes through this well defined and regular quantum bounce. One could then use the hamiltonian $H$ of Ref. \cite{6} to investigate the evolution of quantum perturbations in this quantum background. However, the structure of $H$ is rather complicated, turning it difficult to obtain any detailed result about the spectra of perturbations, specially the scalar ones. Also, a simplification of $H$ using the zeroth order classical equations, as done in Ref. \cite{1} and described in the beginning of this section, is not possible because the background is also quantized and it does not satisfy the classical Einstein’s equations. This state of affairs motivated us to find a way to simplify the hamiltonian of Ref. \cite{6}, without ever recurring to the background classical equations, and apply it to these quantum systems.

Recently, we have managed to put the hamiltonian of tensor perturbations into a very simple form through the implementation of canonical transformations and redefinitions of the lapse function only, without recurring to any classical equations of motion \cite{18, 19}. Its consequences were explored in Ref. \cite{18}. However, tensor perturbations are very special (they are automatically gauge invariant, their equations do not depend on the matter background) and it remained to investigate if it would be possible to do the same procedures to simplify the hamiltonian of scalar and vector perturbations. Note that such perturbations are not gauge invariant from the beginning, and they have contributions from the matter perturbations, which renders the calculations much more intricate.

The aim of this paper is to show that it is indeed possible to put the complicated hamiltonians of scalar and vector perturbations of Ref. \cite{6} into the very simple form of Ref. \cite{1} without using any classical background equations. We will exhibit the canonical transformations and lapse functions redefinitions which make the job. The simplified constraints obtained have direct physical interpretations. The quantization of the theory yields a very simple Wheeler-DeWitt equation for the perturbations and background, which can be used with whatever interpretation and choice of time one makes.

When the matter content is a perfect fluid, the Wheeler-DeWitt equation assumes a Schrödinger form, and a further simplification can be achieved provided one uses the ontological interpretation of Bohm and de Broglie \cite{17}. As in this case a quantum Bohmian trajectory $a_q(t)$) at zeroth order can be defined, a time dependent unitary transformation can be implemented in the scalar perturbation sector using this $a_q(t)$, and, like in Ref. \cite{1}, the hamiltonian can be further simplified rendering equations governing the scalar perturbations which are formally equivalent to simple equations for a scalar field with an effective mass depending on the quantum solution for the scale factor $a_q(t)$, the quantum "pump field".

The paper is organized as follows. In the following section, we specify the action and hamiltonian by restricting attention to the particular case of a Friedmann-Lemaître-Robertson-Walker (FLRW) background and perturbations around it, without yet making any separation in scalar, vector and tensor perturbations. In sections III and IV we analyze the cases of vector and scalar perturbations, respectively. We concentrate in the hydrodynamical fluid case letting the scalar field case to Appendix A. In section IV we present all the steps to simplify the scalar part of the hamiltonian. In section V we quantize this system. After separating the background Schrödinger equation from the perturbed one, we show how to use the Bohm-de Broglie interpretation in order to perform the last canonical transformations which yields quantum equations for the perturbations with the same form as those presented in Ref. \cite{1}. Finally, Sec. VI ends this paper with some general conclusions. Appendix B presents the explicit canonical transformations used in section IV.

II. LINEAR COSMOLOGICAL PERTURBATIONS

Let the geometry of spacetime be given by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$$  \hspace{1cm} (1)
where \( g^{(0)}_{\mu\nu} \) represents a homogeneous and isotropic cosmological background,
\[
ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = N^2(t) dt^2 - a^2(t) \gamma_{ij} dx^i dx^j,
\]
where \( \gamma_{ij} \) is the spatial metric of the spacelike hypersurfaces with constant curvature \( K = 0, \pm 1 \), and the \( h_{\mu\nu} \) are the linear perturbations, which we decompose into
\[
\begin{align*}
h_{00} &= 2N^2 \phi, \\
h_{0i} &= -NaA_i, \\
h_{ij} &= a^2 \epsilon_{ij}.
\end{align*}
\]
Substituting Eq. (3) into the Einstein-Hilbert action
\[
S_{gr} = - \frac{1}{6l^2} \int d^4 x \sqrt{-g} R,
\]
where \( l^2 = 8\pi G/3 \), yields the zeroth and second order actions
\[
S^{(0)}_{gr} = \frac{1}{6l^2} \int d^4 x N \gamma^{\frac{1}{2}} a^3 \left( - \frac{6a^2}{a^2 N^2} + \frac{6K}{a^2} \right)
\]
identified with the pressure. The particle number density \( \rho \) is given by
\[
\rho = \frac{F(a^4)}{\sqrt{-g}J},
\]
where \( F \) is an arbitrary function of lagrangian variables, \( \sigma \) is a time parameter along the particle world lines, and \( J \) is the jacobian of the transformation from lagrangian variable to eulerian ones.

The energy-momentum tensor of the fluid reads
\[
T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g}L)}{\delta g_{\mu\nu}} = \varepsilon V^{\mu}V^\nu - p(g^{\mu\nu} - V^{\mu}V^\nu),
\]
where it is clear that \( \varepsilon \) e \( p \) corresponds to the energy density and pressure, respectively. The sound velocity \( c_s \) is defined by
\[
c_s^2 = \frac{\partial \varepsilon}{\partial p}.
\]
Perturbations displace the particles from their background positions \( x_0^\mu \) to the \( x^\mu \) positions given by
\[
x_0^\alpha \rightarrow x^\alpha = x_0^\alpha + \xi^\alpha (x_0),
\]
meaning a change into their eulerian position, which implies modifications in the jacobian,
\[
J = J_0 \left( 1 + \xi^0 + \xi^i_x, + \frac{1}{2} \xi^i \cdot \xi^j + \xi^0 \xi^i_x + \frac{1}{2} \xi^i \cdot \xi^j - \xi^0 \cdot \xi^i, \right),
\]
in the determinant,
\[ \sqrt{-g(x_0 + \xi)} = \sqrt{-g^{(0)}(x_0)} \left( 1 + \phi - \frac{1}{2} \epsilon + \frac{N}{N - \xi^0} \right) - \frac{1}{2} \frac{\dot{N}}{N} \xi^0 + \frac{3\dot{a}}{a} \xi^0 \phi - \frac{3}{2} \frac{\dot{a}}{a} \xi^0 \phi \xi^0 + \phi \xi^i - \frac{1}{2} \dot{\xi}^0 + \frac{3\dot{\dot{a}}}{a^2} \xi^0 \phi + \frac{\dot{N}}{2N} \xi^0 \phi \xi^0 + \frac{3\ddot{a}}{2a} \xi^0 - \frac{1}{2} \dot{\gamma} \dot{\xi}^i, \right) \]

and in

\[ \sqrt{g_{\mu\nu}} \frac{\partial x^\mu}{\partial x_0} \frac{\partial x^\nu}{\partial x_0} = \sqrt{g^{(0)}_{\mu\nu}} \frac{\partial x^\mu}{\partial x_0} \frac{\partial x^\nu}{\partial x_0} \left( 1 + \phi + \frac{N}{N - \xi^0} \right) - \frac{1}{2} \frac{\dot{N}}{N} \xi^0 + \frac{3\dot{\dot{a}}}{a^2} \xi^0 \phi \xi^0 + \frac{\dot{N}}{2N} \xi^0 \phi \xi^0 + \frac{3\ddot{a}}{2a} \xi^0 - \frac{1}{2} \dot{\gamma} \dot{\xi}^i. \right) \]

The particle number density at point \( x_0 \) is then given by

\[ \rho(x_0) = \rho_0 \left( 1 + \frac{1}{2} \epsilon - \xi^i | i + \dot{\xi}^i | i \xi^0 + \xi^i | i \xi^i - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \dot{\xi}^i \dot{\xi}^j - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j \right) - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^i + \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j \left[ \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^i + \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j - \frac{1}{2} \frac{a^2}{N^2} \gamma_{ij} \xi^i \xi^j \right] \]

The total lagrangian including the gravitational sector then reads

\[ L = \frac{\dot{\gamma}^2}{a^2} + Na^3 \epsilon_0 V + \frac{Na}{6l^2} \int d^3x \gamma^2 \left( A^{(4)} A_{(4)} - \frac{1}{2} \epsilon \xi^i | i \right) + \rho_0 \left( \frac{1}{2} \epsilon \phi - \frac{1}{4} \epsilon^i | i \xi^i \right) - \frac{1}{2} \frac{\dot{\gamma}^2}{a^2} + \frac{Na^3 \epsilon_0}{6l^2} \int d^3x \gamma^2 \left( A^{(4)} A_{(4)} - \frac{1}{2} \epsilon \xi^i | i \xi^i \right) - \frac{1}{2} \frac{\dot{\gamma}^2}{a^2} + \frac{Na^3 \epsilon_0}{6l^2} \int d^3x \gamma^2 \left( A^{(4)} A_{(4)} - \frac{1}{2} \epsilon \xi^i | i \xi^i \right) - \frac{1}{2} \frac{\dot{\gamma}^2}{a^2} + \frac{Na^3 \epsilon_0}{6l^2} \int d^3x \gamma^2 \left( A^{(4)} A_{(4)} - \frac{1}{2} \epsilon \xi^i | i \xi^i \right) - \frac{1}{2} \frac{\dot{\gamma}^2}{a^2} + \frac{Na^3 \epsilon_0}{6l^2} \int d^3x \gamma^2 \left( A^{(4)} A_{(4)} - \frac{1}{2} \epsilon \xi^i | i \xi^i \right). \]

The procedure of Ref. 11 to simplify Eq. 20 begins as follows: using the background equation of motion

\[ \ddot{a} = \frac{\dot{a}}{a} \left( \dot{\gamma} + \frac{a^3}{N} - \frac{3l^2 N^3 a}{2} \right), \]

and discarding a total time derivative

\[ \left[ \frac{a^2 \dot{a}}{6l^2 N} \int d^3x \gamma^2 \left( \epsilon^i | i \xi^i \right) - \frac{1}{2} \dot{\gamma}^2 \right] \]

we obtain
The hamiltonian from Eq. (20) reads

\[ H = - \frac{\delta^2 a V}{t^2 N} - Na^3 \xi_0 V + Na \int d^3 x \gamma^\frac{4}{3} \left( A^{ij} A_{[ij]} - \frac{1}{4} \epsilon^{ijk} \epsilon_{ijk} + \frac{a}{N} \dot{A}_i \epsilon^{ij} + \frac{1}{2} \epsilon^{ij} | \epsilon^{k} | + \phi_i \epsilon^{ij} \right) \\
- \frac{1}{2} \epsilon^{ij} | \epsilon^{ij} | - \phi_i \epsilon^{ij} + \frac{1}{4} \epsilon^{ij} | \epsilon^{ij} | \right) + \frac{a^3}{24t^2 N} \int d^3 x \gamma^\frac{4}{3} (\dot{A}_i + \frac{1}{2} A_i \epsilon^{ij} | |) - \frac{a^2 \dot{a}^2}{3t^2 N} \int d^3 x \gamma^\frac{4}{3} \dot{\phi} - \frac{a^2 \dot{a}}{6t^2} \int d^3 x \gamma^\frac{4}{3} \dot{A}_i \\
- Na^3 \xi_0 \int d^3 x \gamma^\frac{4}{3} \left( - \frac{1}{2} \phi^2 + \frac{1}{2} A^2 A_i - \phi | \epsilon^{ij} | | \right) - Na^3 p_0 \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{2} \dot{\phi} - \phi | \epsilon^{ij} | \right) \\
+ \frac{1}{2} Na^3 (\xi_0 + p_0) \int d^3 x \gamma^\frac{4}{3} \left( \frac{a^2}{N^2} \xi^i \dot{\epsilon} + 2 \frac{a}{N} A_i \dot{\epsilon} + A_i A^i \right) - \frac{1}{2} \epsilon_{ik} A^3 (\xi_0 + p_0) \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{4} \dot{\phi}^2 + \xi^i \dot{\epsilon}^i | - \epsilon | \epsilon^i | \right). \tag{23} \]

Now using the other background equation

\[ \frac{\dot{a}^2 a}{6t^2 N} - Na^3 \xi_0 \tag{24} \]

we arrive at the simplified lagrangian

\[ L = - \frac{\delta^2 a V}{t^2 N} - Na^3 \xi_0 V + Na \int d^3 x \gamma^\frac{4}{3} \left( A^{ij} A_{[ij]} - \frac{1}{4} \epsilon^{ijk} \epsilon_{ijk} + \frac{a}{N} \dot{A}_i \epsilon^{ij} + \frac{1}{2} \epsilon^{ij} | \epsilon^{k} | + \phi_i \epsilon^{ij} \right) \\
- \frac{1}{2} \epsilon^{ij} | \epsilon^{ij} | - \phi_i \epsilon^{ij} + \frac{1}{4} \epsilon^{ij} | \epsilon^{ij} | \right) + \frac{a^3}{24t^2 N} \int d^3 x \gamma^\frac{4}{3} (\dot{A}_i + \frac{1}{2} A_i \epsilon^{ij} | |) - \frac{a^2 \dot{a}^2}{3t^2 N} \int d^3 x \gamma^\frac{4}{3} \dot{\phi} - \frac{a^2 \dot{a}}{6t^2} \int d^3 x \gamma^\frac{4}{3} \dot{A}_i \\
- Na^3 \xi_0 \int d^3 x \gamma^\frac{4}{3} \left( - \frac{1}{2} \phi^2 + \frac{1}{2} A^2 A_i - \phi | \epsilon^{ij} | | \right) - Na^3 p_0 \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{2} \dot{\phi} - \phi | \epsilon^{ij} | \right) \\
+ \frac{1}{2} Na^3 (\xi_0 + p_0) \int d^3 x \gamma^\frac{4}{3} \left( \frac{a^2}{N^2} \xi^i \dot{\epsilon} + 2 \frac{a}{N} A_i \dot{\epsilon} + A_i A^i \right) - \frac{1}{2} \epsilon_{ik} A^3 (\xi_0 + p_0) \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{4} \dot{\phi}^2 + \xi^i \dot{\epsilon}^i | - \epsilon | \epsilon^i | \right). \tag{25} \]

Note that it is not necessary to use Eq. (24) in order to pass from Eq. (23) to Eq. (25): the redefinition of the lapse function

\[ N = : \dot{N} \left[ 1 + \frac{1}{2t^2} \int d^3 x \gamma^\frac{4}{3} (\dot{\epsilon} + \phi^2 - A^2 A_i) \right] \tag{26} \]

takes Eq. (24) into Eq. (25). Note that these two lapse functions related by Eq. (25) are equivalent at first order. Hence, this procedure does not modify the equations of motion at first order when we make a time gauge choice.

Let us now calculate the hamiltonians of these lagrangians for perfect fluids with equation of state

\[ p_0 = \lambda \xi_0 \tag{27} \]

The hamiltonian from Eq. (25) reads

\[ H_T = - \frac{N t^2 P^2 a}{4a V} + N \frac{P_T}{a^3} + \frac{N t^2 P^2 a}{a V^2} \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{8} \phi^2 + \frac{1}{2} \epsilon^{ij} | \epsilon^{ij} | + \frac{1}{4} \epsilon^{ij} | \epsilon^{ij} | \right) + \frac{N P_a}{6 V} \int d^3 x \gamma^\frac{4}{3} \left( \phi A^i + \epsilon^{ij} | A_i \right) \\
+ \frac{2N t^2 P a}{a V} \int d^3 x \gamma^\frac{4}{3} \left( \phi A^i + \epsilon^{ij} | A_i \right) - \frac{N t^2 P a}{2a^2 V^2} \int d^3 x \pi^\frac{2}{3} A_{,i} A_{,i} + \frac{N P_a}{12 V} \int d^3 x \gamma^\frac{4}{3} \epsilon^{ij} A^i | | + \frac{N t^2 P a}{a^2 V} \int d^3 x \phi + \frac{6N t^2}{a^3} \int d^3 x \pi^{ij} \pi^{ij} \gamma^\frac{4}{3} \right) \\
- \frac{3t^2 N}{a^3} \int d^3 x \pi^\frac{2}{3} - \frac{N a}{2t^2} \int d^3 x \gamma^\frac{4}{3} A^i | A^i | - \frac{N}{a} \int d^3 x \pi^{ij} A^i | + \frac{N}{2a^2 (\lambda + 1) \xi_0} \int d^3 x \pi^{ij} \xi^{ij} | + \frac{N}{a} \int d^3 x \pi^{ij} A^i \pi^{ij} \\
- \frac{Na}{6t^2} \int d^3 x \gamma^\frac{4}{3} \left( A^{ij} A_{[ij]} - \frac{1}{4} \epsilon^{ijk} \epsilon_{ijk} + \frac{1}{2} \epsilon^{ij} | \epsilon^{k} | k + \phi_i \epsilon^{ij} | - \frac{1}{2} \epsilon^{ij} | \epsilon^{i} | + \frac{1}{4} \epsilon^{ij} | \epsilon^{i} | \right) \\
+ \frac{Na^3 \xi_0}{4} \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{2} \phi^2 + \frac{1}{2} A^2 A_i - \phi | \epsilon^{ij} | \right) + Na^3 \lambda \xi_0 \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{4} \phi + \frac{1}{4} \epsilon^{ij} | \epsilon^{i} | - \frac{1}{8} \phi | \epsilon^{ij} | \right) \\
+ \frac{1}{2} Na^3 \lambda (\lambda + 1) \xi_0 \int d^3 x \gamma^\frac{4}{3} \left( \frac{1}{4} \phi + \frac{1}{4} \epsilon^{ij} | \epsilon^{i} | - \frac{1}{8} \phi | \epsilon^{ij} | \right), \tag{28} \]
while that from Eq. (23) is given by

\[
H_T = -\frac{N^2 P^2}{4a^2} + N^2 P\frac{a}{a^3} + \int d^3x \gamma^2 \left( \frac{1}{8} \phi^2 + \frac{1}{8} \phi - \frac{1}{8} A^i A_i \right) + \int d^3x \gamma^2 \left( \phi A^i_{|i} + e^j_{|j} A_i \right) + \frac{N^2 P}{a^2} \int d^3x \pi \phi
\]

\[
+ 6N^2 \frac{\pi^i}{a^2} \int d^3x \gamma^2 \left( \frac{2\pi^j}{a^2} - \frac{N}{a^2} \int d^3x \gamma^2 \theta^i_\lambda \right) + \frac{N}{a} \int d^3x \gamma^2 \left( A^i_{|i} \right) - \frac{1}{4} \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} + \frac{1}{2} \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{4} \epsilon_{ij}^{(1)} \right)
\]

\[
+ \frac{1}{2} \epsilon_{ij}^{(1)} \left( 1 - \frac{1}{2} \phi^2 + \frac{1}{2} A^i_{|i} - \phi \xi_{ij}^{(1)} \right) + Na^3 e_0 \int d^3x \gamma^2 \left( \frac{1}{4} \epsilon_{ij}^{(1)} - \epsilon_{ij}^{(1)} \right)
\]

(29)

The quantity \( P_T \) appearing in the second term of the zeroth order term of both hamiltonians is just the kine-

matical constant \( P_T \equiv e_0 a^{3+3} V \). We have introduced it

as a canonical momentum to a variable \( T \) which is cyclic,

implying indeed that \( P_T \) is a constant. We have made an

inverse Routh procedure. The variable \( T \) plays the role

of time when the system is quantized. This form of the

zeroth order hamiltonian appears in other approaches to

a lagrangian formulation of fluids; see e.g. Ref. [20]

for details.

One can now use the total time derivative \( \mathcal{F} \) to

construct the generator of canonical transformations

\[
\mathcal{F} = a \tilde{P}_a - \frac{a}{12V} \int d^3x \pi^i \epsilon_{ij}^{(1)} - \frac{\pi^a}{12V} \int d^3x \gamma^2 \left( \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{2} \epsilon_{ij}^{(1)} \right)
\]

yielding

\[
a = \hat{a} \left[ 1 + \frac{1}{12V} \int d^3x \gamma^2 \left( \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{2} \epsilon_{ij}^{(1)} \right) \right]
\]

\[
P_a = \hat{P}_a \left[ 1 - \frac{1}{12V} \int d^3x \gamma^2 \left( \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{2} \epsilon_{ij}^{(1)} \right) \right]
\]

\[
\pi^i = \pi^i + \frac{a}{6V} \left[ \frac{1}{2} \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{2} \epsilon_{ij}^{(1)} \right]
\]

\[
\tilde{\epsilon}^{ij} = \tilde{\epsilon}^{ij}.
\]

(30)

(31)

Using the fact that \( \rho \propto a^{-3} \), the particle number density

transforms to

\[
\rho = \hat{\rho} \left[ 1 - \frac{1}{4V} \int d^3x \gamma^2 \left( \epsilon_{ij}^{(1)} \epsilon_{ij}^{(1)} - \frac{1}{2} \epsilon_{ij}^{(1)} \right) \right] =: \hat{\rho} - \delta \rho
\]

(32)

Substituting this last equation into Eqs. (10) and (9) we obtain

\[
\epsilon_0 = \hat{\epsilon}_0 - \frac{\left( \epsilon_0 + \hat{\rho}_0 \right)}{\hat{\rho}} \delta \rho.
\]

(33)

Inserting Eqs. (31) and (33) into Eq. (28), we obtain (29).

Hence, in the lagrangian point of view, one can pass from

Eq. (20) to Eq. (23) without using any background equa-
tions of motion. As we have shown that we can pass

from Eq. (24) to Eq. (26) just through a redefinition of

\( N \), then it is proven that lagrangian (20) is equivalent to

lagrangian (24) at this order of approximation irrepro-
dent of the classical background equations of motion.

In order to proceed from this point\(^1\), we will now sep-

arate the perturbations into scalar, vector, and tensor

perturbations. We make the decomposition:

\[
A_i = B_{||i} + S_i
\]

\[
e_{ij} = 2 \psi \gamma_{ij} - 2E_{||ij} - F_{||ij} - F_{||ij} + w_{ij}
\]

(34)

in the gravitational sector, while the quantities \( w_{ij}, F_i \) e

\( S_i \) satisfy

\[
S^{||i} = F^{||i} = 0
\]

\[
w_{||ij} = 0
\]

\[
w_{||i} = 0,
\]

(35)

and

\[
\xi^{||i} = \eta^{||i}
\]

(36)

with

\[
\eta^{||i} = \xi^{||i}
\]

(37)

in the matter sector. Substituting the above decomposi-
tions into eq. (25) leads to a separation of this lagrangian,

with some total derivatives discarded, into three indepen-
dent sectors: scalar, vector and tensor sectors. We will

focus our attention on the vector and scalar sectors be-

cause the case of tensor perturbations has already been

treated in Ref. (18).

\(^1\) Equation (25) corresponds to Eq. (10.37) of Ref. (1) if one is

restricted to scalar perturbations, and if one reads \( \beta \) in the latter

as \( \beta = 3a^2 \epsilon_0 (\epsilon_0 + P_0)/2 \).
III. VECTOR PERTURBATIONS

Combining the contributions of gravitational and matter sectors and defining the gauge invariant quantities

\[ V^i = S^i - \frac{a}{N} \dot{F}^i, \quad (38) \]

we obtain

\[ L^V = \frac{Na}{12l^2} \int d^3 x \gamma^i V^{ij} V_{ij} + \frac{1}{2} Na^2 (\lambda + 1) \varepsilon_0 \int d^3 x \gamma^i \left( \frac{a}{N} \dot{\eta}^{i (gi)} + V^i \right) \left( \frac{a}{N} \eta^{i (gi)} + V_i \right). \quad (40) \]

When constructing the hamiltonian, the primary constraint \( \Pi_i \approx 0 \) appears, where \( \approx \) means a weak equality in the sense of Dirac \([21]\), and \( \Pi_i \) is the momentum canonically conjugate to \( V^i \). The hamiltonian then reads

\[ H^V = N \left[ - \frac{l^2 P_a^2}{4aV} + \frac{P_x}{a^3} + \int d^3 x \left( \frac{P_i P_i}{P_T a^2 - 3l^2 \gamma V^i} + \frac{V^i P_i}{a} - \frac{a}{12l^2} \gamma^2 V^{ij} V_{ij} \right) \right] + \int d^3 x \Lambda^i \pi_i, \quad (41) \]

where \( P_i \) is the momentum canonically conjugate to \( \eta^{i (gi)} \), and the \( \Lambda^i \) are Lagrange multipliers.

The conservation of \( \Pi_i \) in time imposes a secondary constraint

\[ \Pi^i = \{ \Pi^i, H^V \} = \phi^i \frac{1}{a} \pi_i - \frac{a}{6l^2} \gamma^2 V^{ij} \phi_{ij}. \quad (42) \]

The conservation in time of \( \phi^i_0 \) fixes the lagrange multipliers \( \Lambda^i \) to the value

\[ \Lambda^i = \frac{l^2 P_a}{a^2 V} V^i. \quad (43) \]

Then the equations of motion for \( V^i \) and \( \eta^{i (gi)} \) imply that

\[ V^i = \frac{V_0^i}{a^2}, \quad (44) \]

and

\[ \varphi^i \equiv \left( \frac{a}{N} \dot{\eta}^{i (gi)} + V^i \right) = \frac{\nabla^2 V^i_0}{(\lambda + 1) P_T a^{-1 - \lambda}} \quad (45) \]

These solutions correspond to the classical result, which was obtained without recurring to the classical background equations.

\[ L^E = \frac{Na}{3l^2} \int d^3 x \gamma^i \left( \psi^i \psi_{,i} - 2 \phi^i \psi_{,i} \right) - \frac{2a^2}{3l^2} \int d^3 x \gamma^i \left( \dot{\psi} + \frac{\dot{\phi} a}{a} \phi \right) F^{,i}_{,i} - \frac{a^3}{Nl^2} \int d^3 x \gamma^i \left( \dot{\psi} + \frac{\dot{\phi} a}{a} \phi \right)^2 \]

\[ - \frac{Na^3 (\lambda + 1) \varepsilon_0}{2} \int d^3 x \gamma^i \left[ \lambda (3 \psi - \zeta^{i (gi)})^2 + 2 \phi (3 \psi - \zeta^{i (gi)}) \right] + \frac{Na}{12l^2 a} \int d^3 x \gamma^i \varphi^i \varphi_{,i}. \quad (49) \]

The two constraints obtained are second class. After defining the corresponding Dirac brackets \([21]\), they become strong equalities which can be used to obtain some variables from others.

IV. SCALAR PERTURBATIONS

Defining the quantities

\[ F = B - \frac{a}{N} E, \quad (46) \]

\[ \zeta^{(gi)} = \zeta + E, \quad (47) \]

which is a gauge invariant quantity, and

\[ \varphi = \frac{\sqrt{6l^2 a^2 \sqrt{(\lambda + 1) \varepsilon_0}} \cdot \frac{a}{N} \zeta^{(gi)} + F}{\sqrt{\lambda}}, \quad (48) \]

which can be identified with the perturbed velocity potential of the fluid particles, the scalar lagrangian reads
As in the vector sector, some constraints appear, and, because of definition \(13\) which involves a time derivative, we have to use the Ostrogradsky method \[22\] through the definition \(\pi_\zeta = \partial L / \partial \dot{\zeta} - \dot{\pi}_\zeta\). The constraints are

\[
\phi_1 = P_N; \quad \phi_2 = \pi_F; \quad \phi_3 = \pi_\varphi; \quad \phi_7 = \pi_\varphi; \quad \phi_9 = P_\mu,
\]

and the Hamiltonian is

\[
H = N \left[ H_0 + \int d^3 x \Lambda_\phi \pi_\phi + \int d^3 x \Lambda_F \pi_F + \int d^3 x \Lambda_\varphi \pi_\varphi \right] + \Lambda_N P_N
\]

where \(H_0\) reads

\[
H_0 = - \frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^{a/3}} + \frac{(\lambda + 1) P_T}{2a^{a/3} V} \int d^3 x \gamma^{\lambda/2} \left[ \Lambda(3\psi - \zeta^{ij}(\mathfrak{g} i)^{\lambda})^2 + 2\phi(3\psi - \zeta^{ij}(\mathfrak{g} i)^{\lambda}) \right] + \frac{l^2 P_a}{2a^2 V} \int d^3 x \phi \pi_\psi
\]

\[
+ \frac{1}{a} \int d^3 x \pi_\zeta \left( \sqrt{V} \sqrt{\lambda} \right) \left( \alpha^{-\lambda/3} \pi - F \right) - \frac{a}{l^2} \int d^3 x \gamma^{\lambda/2} \left( \frac{l^2}{2a^2} \sqrt{\pi} + \frac{1}{3} F^{ij} \right)^2
\]

\[
+ \frac{\lambda}{12l^2 a} \int d^3 x \gamma^{\lambda/2} \varphi \psi^{ij} , \lambda \int d^3 x \gamma^{\lambda/2} (\psi - 2\phi) \psi^{ij} , i.
\]

Conservation in time of the primary constraints \[60\] leads to the secondary constraints

\[
\phi_4 = \frac{H}{N}
\]

\[
\phi_5 = \frac{1}{a} \pi_\zeta + \frac{2a^2}{3} \gamma^{\lambda/2} \left( \frac{l^2}{2a^2} \pi_\psi + \frac{1}{3} F^{ij} , i \right)^{\lambda},
\]

\[
\phi_6 = - \left( \frac{\lambda + 1}{a^{a/3} V} \right) \left( 3\psi - \zeta^{ij} \right) - \frac{l^2 P_a}{2a^2 V} \pi_\psi + \frac{2a^2}{3} \gamma^{\lambda/2} \psi^{ij} , i,
\]

\[
\phi_8 = \frac{1}{a} \sqrt{\frac{V}{6l^2 a}} \left( \frac{1}{3} \lambda \right) \left( \frac{\sqrt{\lambda}}{6l^2 a} \right) \gamma^{\lambda/2} \varphi^{ij} , i.
\]

Neglecting third order terms, conservations in time of \(\phi_4\) and \(\phi_6\) are identically satisfied, whereas conservation of \(\phi_5\) and \(\phi_8\) determines the Lagrange multipliers \(\Lambda_F\) and \(\Lambda_\varphi\). The Lagrange multiplier \(\Lambda_F\) reads

\[
a \Lambda_F = \psi - \phi + \frac{l^2 P_a}{aV} F.
\]

As \(\Lambda_F = \hat{F}/N\), then

\[
\frac{a}{N} \hat{F} = \psi - \phi - \frac{2a}{N} F,
\]

which, when expressed in terms of the gauge invariant Bardeen potentials, yields

\[
\Phi = \Psi,
\]

a well known result.

\[\text{The Poisson brackets among the constraints read}\]

\[
\{\phi_2(x), \phi_5(x')\} = - \frac{2a}{9l^2} \delta(x - x')^{ij} , i
\]

\[
\{\phi_7(x), \phi_8(x')\} = - \frac{\sqrt{\lambda}}{6l^2 a} \delta(x - x')^{ij} , i
\]

\[
\{\phi_6(x), \phi_5(x')\} = \frac{2}{9l^2} \delta(x - x')^{ij} , i - \frac{l^2}{aV} \pi_\psi(x) \pi_\zeta(x')
\]

\[
\{\phi_6(x), \phi_8(x')\} = \frac{\sqrt{(\lambda + 1) P_T}}{\sqrt{6l^2 a}} a^{-2(1+\lambda) \gamma^{\lambda/2} \delta(x - x')^{ij} , i}
\]

\[
- \frac{l}{4\sqrt{6l^2 a}} (1 - 3\lambda) a^{-2(1+3\lambda) \pi_\psi(x) \pi_\zeta(x')}
\]

Defining

\[
\tilde{\phi}_6 = \phi_6 + \frac{1}{a} \phi_2 + \frac{\sqrt{(\lambda + 1) P_T} \sqrt{\lambda} a^{-2(1+3\lambda) \phi_7}}{\sqrt{\lambda^2 V}}
\]

one can prove that \(\tilde{\phi}_6\) is a first class constraint; it has zero Poisson brackets with all others constraints up to third order. We are then left with four second class constraints, \(\phi_2, \phi_5, \phi_7, \text{ and } \phi_8\). Hence, from the 10 degrees of freedom of phase space corresponding to the variables \(\phi, \psi, F, \varphi, \text{ and } \zeta\), we have to extract 4 from the second class constraints and \(2 \times 2 = 4\) from the two first class constraints \(\phi_6\) and \(\phi_3\), remaining 2 degrees of freedom in phase space, as expected for this problem.

In order to eliminate the second class constraints, we have to define the Dirac brackets associated with them. The Dirac brackets among the variables of phase space which are not canonical are (excepting the ones involving \(F\) and \(\pi_F\), which are not relevant)

\[
\{\zeta^{ij}(\mathfrak{g})x, \varphi(x')\}^D = - \frac{\sqrt{\lambda} \sqrt{V} a^{-2(1+3\lambda)}}{\sqrt{\lambda} (\lambda + 1) P_T \gamma^{\lambda/2} \delta(x - x')}
\]

\[\text{2 The explicit value of } \Lambda_\varphi \text{ is not important for what follows.}\]
\[
\{P_a, \varphi(x)\}^D = \frac{1}{2a}(1-3\lambda)\varphi(x) \tag{59}
\]

Defining the quantities

\[
\varphi(c) = \frac{a^3(1-3\lambda)}{\sqrt{\lambda}} \varphi
\]

\[
\pi_{\varphi}(c) = -\sqrt{\lambda} \sqrt{(\lambda + 1) P_T} \frac{\gamma^\frac{1}{2}}{\sqrt{6lV}} (3\psi - \zeta^i \delta^i_{,i}),
\]

\[
\pi_{\psi}(c) = \pi_{\psi} - \frac{3\sqrt{\lambda} \sqrt{(\lambda + 1) P_T}}{\sqrt{6lV}} \gamma^\frac{1}{2} \varphi(c), \tag{60}
\]

we obtain that the Dirac brackets for these quantities are canonical. The Hamiltonian in terms of these new variables then reads

\[
H = N H_0 - N \int d^3 x \phi \dot{\phi}_b + \int d^3 x \Lambda_{\phi} \pi_{\phi} \tag{61}
\]

where \( H_0 \) is given by

\[
H_0 = \frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{3l^2}{a^{3\lambda}} \int d^3 x \frac{\pi_{\varphi}(c)}{\gamma^\frac{1}{2}} - \frac{\lambda}{12l^2 a^{2-3\lambda}} \int d^3 x \gamma^\frac{1}{2} \varphi(c) \varphi(c)_{,i} - \frac{3\lambda(\lambda + 1) P_T}{8a^2 V} \int d^3 x \gamma^\frac{1}{2} \varphi(c)^2
\]

\[-\frac{\sqrt{\lambda}}{2} \frac{3l \sqrt{(\lambda + 1) P_T}}{a^{3\lambda} V} \int d^3 x \varphi(c) \pi_{\psi}(c) + \frac{a}{3l^2} \int d^3 x \gamma^\frac{1}{2} \psi \psi_{,i}, \tag{62}
\]

and \( \phi_b \) by (we omitted the bars)

\[
\phi_b = \frac{\sqrt{6l \sqrt{(\lambda + 1) P_T}}}{\sqrt{\lambda V}} a^{-3\lambda} \pi_{\varphi}(c) - \frac{l^2 P_a}{2a^2 V} \pi_{\psi}(c) - \frac{3\sqrt{\lambda} l P_a \sqrt{(\lambda + 1) P_T}}{2 \sqrt{6} a^2 V^\frac{3}{2}} \gamma^\frac{1}{2} \varphi(c) + \frac{2a}{3l^2} \gamma^\frac{1}{2} \psi_{,i} \tag{63}
\]

From the second class constraints we obtain the identity,

\[
\frac{l^2}{2a^2 \gamma^\frac{1}{2}} \pi_{\psi} + \frac{1}{3} F_{i},i = \frac{3l^2 \sqrt{\lambda} \sqrt{(\lambda + 1) P_T}}{2 \sqrt{6} a^2 V^\frac{3}{2}} a^{-\frac{3}{2}(1+\lambda)} \varphi, \tag{64}
\]

which in terms of the new canonical variables reads

\[
\frac{l^2}{2a^2 \gamma^\frac{1}{2}} \pi_{\psi}(c) + \frac{1}{3} F_{i},i = 0. \tag{65}
\]

We will need this equation later.

If we now perform a canonical transformation generated by

\[
F_1 = T \dot{P}_a + a \ddot{P}_a + \int d^3 x \left[ \frac{1}{\sqrt{6l}} a^{-\frac{3}{2}(1-3\lambda)} \varphi(c)_{,i} + \psi \pi_{\psi} + \frac{2\sqrt{\lambda} \sqrt{(\lambda + 1) P_T}}{l^2 P_a \sqrt{\lambda}} a^{-\frac{3}{2}(1-\lambda)} \psi \pi_{\phi} - \frac{\gamma^\frac{1}{2}}{2} \alpha \varphi^2(c) \right] \tag{66}
\]

where \( \alpha \) is given by

\[
\alpha = \frac{(\lambda + 1) \dot{P}_a}{2l^2 P_a a} + \frac{(1 - 3\lambda) \ddot{P}_a}{24V} a^{-2(3\lambda)}, \tag{67}
\]

constructed in order to introduce the \( \psi \) variable of Ref. \[3\] (\( \pi \) is its canonical momentum), the new \( H_0 \) reads (the explicit canonical transformations are given in Appendix B)

\[
H_0 = \frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2a} \int d^3 x \frac{\pi^2}{\gamma^\frac{1}{2}} + \frac{\lambda}{2a} \int d^3 x \frac{1}{2} \gamma^\frac{1}{2} v_{,i} v_{,i} + \sqrt{\lambda} \frac{(\lambda + 1) P_T}{2 P_a} a^{-6\lambda} - \frac{9\lambda(\lambda + 1) P_T}{2 P_a} a^{-2(1+\lambda)}
\]

\[+ \frac{12(18a^2 - 18\lambda^2)}{64a^2 V^2} \int d^3 x \frac{1}{2} \gamma^\frac{1}{2} v_{,i} v_{,i} + \int d^3 x \frac{1}{2} \gamma^\frac{1}{2} \psi_{,i} \tag{68}
\]

\[
\int d^3 x \gamma^\frac{1}{2} \psi_{,i}, \tag{69}
\]

\[\text{\textsuperscript{3}}\text{The term proportional to } \alpha \text{ in Eq. 66}, \tag{66}
\]

as well as the specific form given by Eq. 67 are made in order to eliminate a term proportional to \( \psi \psi \) in the Hamiltonian.
\[-\frac{\sqrt{3}t^2\sqrt{(\lambda+1)P_T}}{2}a^{-\frac{3}{2}(5+3\lambda)} \int d^3x v \pi + \int d^3x \phi \left\{ \frac{18[(\lambda+1)P_T]^2 V_a}{l^4 P_a^3 \sqrt{\lambda}} a^{(2-\lambda)} - \frac{18[(\lambda+1)P_T]^2 P_T V_a}{l^4 P_a^3} a^{(2-\lambda)} \right\}
\]
\[+ \frac{-29 + 9\lambda - 9\lambda^2}{8\lambda V} [(\lambda + 1)P_T] a^{-3\lambda} - \frac{3[(\lambda + 1)P_T]^2}{l^2 P_a^2 \sqrt{\lambda}} a^{(1-6\lambda)} + \frac{3(3 + 2\lambda + 3\lambda^2)(\lambda + 1)P_T}{2l^2 P_a^2 \sqrt{\lambda}} a^{(1-6\lambda)} \gamma^2 \psi \]
\[+ \left[ \frac{-2 + 3\lambda}{8\lambda V} \sqrt{(\lambda + 1)P_T} a^{-\frac{1}{2}(1+3\lambda)} - \frac{6\lambda \sqrt{\sqrt{(\lambda + 1)P_T} P_T a^{\frac{1}{2}(1-9\lambda)}}}{2l^2 P_a^2 \sqrt{\lambda}} \right] \pi \]
\[+ \left[ \frac{-18[(\lambda + 1)P_T]^2 \sqrt{\sqrt{(\lambda + 1)P_T} P_T a^{\frac{1}{2}(1-15\lambda)}}}{l^2 P_a^2 \sqrt{\lambda}} a^{\frac{1}{2}(1-15\lambda)} + \frac{18\sqrt{(\lambda + 1)P_T} \sqrt{\sqrt{(\lambda + 1)P_T} P_T a^{\frac{1}{2}(1-15\lambda)}}}{2l^2 P_a^2 \sqrt{\lambda}} \right] \gamma^2 \psi \]
\[+ \left[ \frac{3(3 + 2\lambda + 3\lambda^2)(\lambda + 1)P_T}{2l^2 P_a^2 \sqrt{\lambda}} a^{\frac{1}{2}(1+9\lambda)} \right] \gamma^2 \psi \]
\[+ \left[ \frac{a}{3l^2} - \frac{2(\lambda + 1)P_T}{l^4 P_a^2} a^{-3\lambda} \right] \gamma^2 \psi \]
\[\hfill (68) \]

This canonical transformation applied to \( \phi_6 \) shows that \( v \) is a gauge invariant quantity. The same is not true for its momentum \( \pi \), which has a non-zero Poisson Bracket with the first class constraint \( \phi_6 \). In order to obtain a gauge invariant momentum \( \pi \) we now make the canonical transformation generated by

\[ F_2 = aP_a + \int d^3x \{ v \pi - \frac{P_a}{4aV} \int d^3x \gamma^2 \psi \psi \}. \]
\[ F_3 = aP_a + \frac{1}{a} \int d^3x v \pi - \frac{P_a}{4aV} \int d^3x \gamma^2 \psi \psi^2. \]
\[ \hfill (70) \]

The constraint \( \phi_6 \) reads then

\[ \phi_6 = - \frac{P_a}{2a^2V} \pi \psi, \]
\[ \hfill (71) \]

and aiming at eliminating a term in \( v^2 \) proportional to \( P_T \) in the final form of the Hamiltonian, we perform the last canonical transformation

\[ H_0 = \frac{-l^2 P_a^2}{4aV} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2a^3} \int d^3x \frac{\pi^2}{\gamma^2} + \frac{a}{2} \int d^3x \gamma^2 \psi \psi^2 \]
\[ + \frac{a}{2} \int d^3x \gamma^2 \psi \psi^2 \]
\[ + H_0^{(0)} \frac{1}{l^2 P_a^2} \int d^3x \left\{ - \frac{9(\lambda + 1)P_T}{l^2 P_a^2} a^{2-3\lambda} \psi^2 + \frac{6\sqrt{\sqrt{(\lambda + 1)P_T} P_T a^{\frac{1}{2}(1-3\lambda)}}}{l^2 P_a^2 \sqrt{\lambda}} \right\} \psi \]
\[+ \left[ \frac{18[(\lambda + 1)P_T]^2}{l^2 P_a^2 \sqrt{\lambda}} a^{\frac{1}{2}(1-3\lambda)} + \frac{9(\lambda + 1)P_T}{l^2 P_a^2 \sqrt{\lambda}} a^{\frac{1}{2}(1-3\lambda)} \right] \psi^2 \]
\[+ \frac{2a^2V}{l^4 P_a^2} \psi \psi^2 \\] 
\[ = H_0^{(0)} + H_0^{(2)} \frac{1}{l^2 P_a^2} \int d^3x \left[ F^{(1)} \phi_6 + F^{(2)} H_0^{(0)} \right]. \]
\[ \hfill (73) \]

where \( H_0^{(0)} \) and \( H_0^{(2)} \) are the zeroth and second order Hamiltonian constraints, and \( F^{(1)} \) and \( F^{(2)} \) are first and second order functions which can be read from Eq. (73).
We now make the redefinitions of $N$ and $\phi$ as $\tilde{N} = N(1 + \int d^3 x F^{(2)})$, which would again just imply a different irrelevant time gauge choice with terms beyond first order, and $\tilde{\phi} = N\left(-\frac{l^2 P_a}{2a^2 V} \phi + F^{(1)}\right)$. From the inverse of the transformations (70) and (69), and definition (48), $\tilde{\phi}$ is given by

$$\frac{l^2 P_a}{2a^2 V} \phi - \frac{3l^2}{2} N a (\lambda + 1) \epsilon_\alpha \left( \frac{a^2}{N} \gamma + B \right)$$

(74)

As $\tilde{\phi}$ is, through the equations of motion, equal to $\tilde{\psi}$, we obtain, imposing $N = a$, the constraint equation (10.39) of reference [1].

Inserting expression (73) into Eq. (61), and the above redefinitions, we obtain, omitting the tilda,

$$H = N (H_0^{(0)} + H_0^{(2)}) + \Lambda N P_N + \int d^3 x \dot{\phi} \phi + \int d^3 x \Lambda \dot{\phi} \phi,$$

with

$$H_0^{(0)} = -\frac{l^2 P_a^2}{4aV} + \frac{P_T}{a^3 a},$$

(75)

(76)

and

$$H_0^{(2)} = \frac{1}{2a^3} \int d^3 x \frac{\pi^2}{\gamma^2} + \frac{a^2}{2} \int d^3 x \gamma^2 \dot{\psi}^2 v_i v_j.$$  

(77)

Now we are left with two first class constraints (in fact one plus $\infty^3$ constraints): one with the homogeneous lapse function $N$ as its associated Lagrange multiplier, which in the quantization procedure will lead to the Wheeler-DeWitt equation, and the other $\infty^3$ constraints with $\phi(x^i)$ as their Lagrange multiplier, which is nothing but the inhomogeneous lapse function (see definition [4]), which, as anticipated, has been tremendously simplified to imply a simple consequence when quantized: the wave functional does not depend on $\psi$. The supermomentum constraint is automatically satisfied because the $v$ variable is gauge invariant.

The connection between $\bar{v} = av$ (the Mukhanov-Sasaki variable) and $\Phi$ can be obtained from Eq. (69) which, after implementing the canonical transformations (49), reads (the bars are omitted)

$$\frac{l^2}{2a^2 \gamma^2} \pi \psi + \frac{2aV}{3l^2 P_a} \left( \psi + \frac{l^2 P_a}{2aV} F \right) \gamma^i + \left[ \frac{3\sqrt{a} (\lambda + 1) P_T}{P_a \sqrt{a}} a^{-\frac{1}{2}(1+\lambda)} + \frac{(1+3\lambda) \sqrt{(\lambda + 1) P_T} a^{-3/2(1+\lambda)}}{4\sqrt{a}\sqrt{V}} \right] v = 0$$

(78)

Using that $P_a = -\frac{2aV \dot{a}}{N^2}$, we can identify the quantity $\psi + \frac{l^2 P_a}{2aV} F$ with the Bardeen potential $\Psi$ which, from Eq. (60), is equal to $\Phi$. After some algebraic manipulations, we obtain:

$$\frac{3l^4 P_a}{4a^3 V \gamma^2} \pi \psi + \Phi \gamma^i + \frac{3l^2 \sqrt{(\lambda + 1) P_T}}{2\sqrt{a} V} a^{-\frac{1}{2}(1+\lambda)} \left\{ \frac{\pi}{\gamma^2} + \left[ \frac{(l^2 P_a}{2aV} + \frac{3(\lambda + 1) H_0}{P_a} v \right] \right\} + O(3) = 0$$

(79)

Using again that $P_a = -\frac{2aV \dot{a}}{N^2}$, and that $\pi = \gamma^2 \dot{v}$, $H_0 \approx 0$, $\pi \approx 0$, and choosing the gauge $N = a$ (conformal time), we obtain

$$\Phi \gamma^i = -\frac{3l^2 \sqrt{(\lambda + 1) P_T}}{2\sqrt{a} V} a^{-\frac{1}{2}(1+3\lambda)} \left( \frac{v}{a} \right)$$

(80)

Equation (80) coincides with equation (12.8) of Ref. [10] relating $v$ and $\Phi$ when the classical equations of motion are used.

V. DIRAC QUANTIZATION

In this section we will focus only in the quantization of scalar perturbations. Vector perturbations are trivial and the quantization of tensor perturbations was done in Ref. [18].

A. The functional Schrödinger equation

In the Dirac quantization procedure, the first class constraints must annihilate the wave functional $\chi[N, a, \phi(x^i), \psi(x^i), v(x^i), T]$, yielding

$$\frac{\partial}{\partial N} \chi = 0,$$

$$\frac{\delta}{\delta \phi} \chi = 0,$$

$$\frac{\delta}{\delta \psi} \chi = 0,$$

$$H \chi = 0.$$  

(81)
The first three equations impose that the wave functional does not depend on \( N \), \( \phi \) and \( \psi \): as mentioned above, \( N \) and \( \phi \) are, respectively, the homogeneous and inhomogeneous parts of the total lapse function, which are just Lagrange multipliers of constraints, and \( \psi \) has been substituted by \( v(x^i) \), the unique degree of freedom of scalar perturbations, as expected.

\[
\frac{i}{\tilde{\eta}} \chi = \frac{1}{4} \left\{ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \left[ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \right] \chi - \left[ \frac{a^{3\lambda-1}}{2} \int d^3x \frac{\delta^2}{\delta v^2} - \frac{a^{3\lambda+1}}{2} \int d^3x v^i v_i \right] \chi, \right.
\]

where we have chosen the factor ordering in \( a \) in order to yield a covariant Schrödinger equation under field redefinitions, and \( V \) and \( l \) have been absorbed in redefinitions of the fields.

**B. Further developments using the Bohm-de Broglie interpretation**

If one makes the ansatz

\[
\chi[a,v,T] = \chi_0(a,T) \chi_2[v,T]
\]

then we obtain for \( \chi_2(a,v,T) \) the equation

\[
\frac{\partial}{\partial T} \chi_2(a,v,T) = \frac{-a^{(3\lambda-1)}}{2} \int d^3x \frac{\delta^2}{\delta v^2} \chi_2(a,v,T) + \frac{\lambda a^{(3\lambda+1)}}{2} \int d^3x v^i v_i \chi_2(a,v,T)
\]

**Solutions of the zeroth order equation** are known \([13,15]\). If one uses the ontological Bohm-de Broglie interpretation of quantum mechanics in order to obtain the Bohm trajectories \( a(T) \) from Eq. \([85]\), this \( a(T) \) can be viewed as a given function of time in the second equation \([86]\). Going to conformal time \( d\tilde{\eta} = a^{3\lambda-1}dT \), and performing the unitary transformation

\[
U = e^{i[f d^3x \gamma^i \frac{\delta}{\delta v^i}]} e^{i[f d^3x (\frac{\lambda}{2} \chi_2 + \ln(\frac{1}{a}))]},
\]

the Schrödinger functional equation for the perturbations is transformed to

\[
\frac{i}{\tilde{\eta}} \chi_2[v,\eta] = \int d^3x \left( \frac{1}{2} \frac{\delta^2}{\delta \eta^2} + \frac{\lambda}{2} v^i v^i - \frac{a''}{2a} v^2 \right) \chi_2[v,\eta],
\]

where we have gone to the new quantum variable \( \tilde{v} = av \), the Mukhanov-Sasaki variable defined in Ref. \([1]\), after performing transformation \([87]\), and we have omitted the bars.

As \( P_T \) appears linearly in \( H \), and making the gauge choice \( N = a^{3\lambda} \), one can interpret the \( T \) variable as a time parameter. Hence, the equation

\[
H \chi = 0
\]

assumes the Schrödinger form

\[
\frac{i}{\tilde{\eta}} \chi_0(a,T) = 1 \left\{ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \left[ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \right] \right\} \chi_0(a,T),
\]

where \( \chi_0(a,T) \) satisfies the equation,

\[
\frac{i}{\tilde{\eta}} \chi_0(a,T) = \frac{1}{4} \left\{ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \left[ a^{(3\lambda-1)/2} \frac{\partial}{\partial a} \right] \right\} \chi_0(a,T),
\]

the corresponding time evolution equation for the operator \( v \) in the Heisenberg picture is given by

\[
v'' + \lambda v^i v_i - \frac{a''}{a} v = 0,
\]

where a prime means derivative with respect to conformal time. In terms of the normal modes \( v_k \), the above equation reads

\[
v_k'' + \left( \frac{k^2}{a} - \frac{a''}{a} \right) v_k = 0.
\]

These equations have the same form as the equations for scalar perturbations obtained in Ref. \([1]\) (for one single fluid, the pump function \( z''/z \) obtained in \([1]\) is exactly equal to \( a''/a \) obtained here, if we make use of the background equations). The difference is that the function \( a(\eta) \) is no longer a classical solution of the background equations but a quantum Bohmian trajectory of the quantized background, which may lead to different power spectra.
VI. CONCLUSION

In this paper we have managed to obtain simple hamiltonians for scalar perturbations when the matter content is described either by a perfect fluid or by a scalar fluid, without recurring to the background classical equations. Performing canonical transformations and redefining the homogeneous lapse functions with terms which do not alter the linear perturbation equations, the constraint connected to the inhomogeneous part of the lapse function is greatly simplified implying that the momentum canonically conjugate to the scalar perturbation $\psi$ is weakly zero. The hamiltonian constraint is also greatly simplified when written in terms of a new variable which is exactly equal to the usual Mukhanov-Sasaki's variable $\tilde{\eta}$.

This simplified hamiltonian can now be used in the Dirac quantization procedure not only to quantize the perturbations but also the background, yielding a Wheeler-DeWitt equation much simpler to handle than the one of Ref. \cite{WheelerDeWitt}. In the case of perfect fluids, where a preferred time variable appears and the Wheeler-DeWitt equation can be put in a Schrödinger form, and using the Bohm-de Broglie interpretation of quantum mechanics to perform a last unitary transformation, one obtains an equation for the modes which has the same form as in Ref. \cite{MukhanovSasaki}, where the pump field is obtained from a scale factor which now takes into account the quantum effects, the quantum Bohmian trajectory of the background.

In future publications, we will apply these results to specific models, and evaluate the power spectrum of scalar perturbations which arise on them in order to compare, when taken together with the results of Ref. \cite{MukhanovSasaki}, with observations.

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APPENDIX A: THE SCALAR FIELD

In this paper we have managed to obtain simple hamiltonians for scalar perturbations when the matter content is described either by a perfect fluid or by a scalar fluid, without recurring to the background classical equations. Performing canonical transformations and redefining the homogeneous lapse functions with terms which do not alter the linear perturbation equations, the constraint connected to the inhomogeneous part of the lapse function is greatly simplified implying that the momentum canonically conjugate to the scalar perturbation $\psi$ is weakly zero. The hamiltonian constraint is also greatly simplified when written in terms of a new variable which is exactly equal to the usual Mukhanov-Sasaki's variable $\tilde{\eta}$.

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In future publications, we will apply these results to specific models, and evaluate the power spectrum of scalar perturbations which arise on them in order to compare, when taken together with the results of Ref. \cite{MukhanovSasaki}, with observations.
Its hamiltonian is given by

\[
H = N \left\{ -\frac{t^2 P_a^2}{4aV} - \frac{K a V}{I^2} + \frac{P^2}{2a^3 V} + \frac{a^3 V U}{2} + \frac{t^2 P_a^2}{8 a V^2} + \frac{t^2 P_a^2}{24 a V^2} + \frac{a^3 V U}{2} + \frac{\frac{t^2 P_a^2}{8 a V^2}}{2} \right\} \int d^3 x \gamma \hat{\phi}^2 + \frac{t^2 P_a^2}{24 a V^2} \int d^3 x \gamma \hat{\phi}^2 - \frac{t^2 P_a^2}{8 a V^2} \int d^3 x \gamma \hat{\phi}^2 A^i A_i \\
+ \frac{5t^2 P_a^2}{48 a V^2} \int d^3 x \gamma \hat{\phi}^2 e_i^j e_{ij} - \frac{t^2 P_a^2}{32 a V^2} \int d^3 x \gamma \hat{\phi}^2 + \frac{P_a}{6V} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} \phi + \frac{P_a}{6V} \int d^3 x \gamma \hat{\phi}^2 A^i_{|j} e_{ij} \\
- \frac{P_a^2}{4a^3 V} \int d^3 x \gamma \hat{\phi}^2 (\phi^2 - \epsilon \phi - A^i A_i - \frac{1}{2} \epsilon_{ij} e_{ij} - \frac{1}{4} \epsilon^2 + \frac{P_a}{a^3 V} \int d^3 x (\phi + \frac{1}{2} \epsilon) \pi \phi - \frac{P_a}{a^3 V} \int d^3 x \gamma \hat{\phi}^2 \delta \pi A^i_{|i} \\
+ \frac{6t^2}{a^3} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} - \frac{3t^2}{a^3} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} + \frac{3t^2}{a^3} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} - \frac{a}{4t^2} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} A^i_{|j} - \frac{1}{a} \int d^3 x \pi A^i_{|i} \\
+ \frac{t^2 P_a}{a^2 V} \int d^3 x \pi \phi + \frac{2t^2 P_a}{a^2 V} \int d^3 x \pi \epsilon_{ij} e_{ij} + \frac{1}{2a} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} - \frac{a^3}{6t^2} \int d^3 x \gamma \hat{\phi}^2 A^{ij} A_{ij} + \frac{1}{2} \left( e_{ij} e_{ij} - \epsilon \phi - A^i A_i - 3 \phi^2 \right) \right\} \int d^3 x \gamma \hat{\phi}^2 \\
+ \frac{3t^2 P_a}{a^2 V} \int d^3 x \pi \phi + \frac{2t^2 P_a}{a^2 V} \int d^3 x \pi \epsilon_{ij} e_{ij} + \frac{1}{2a} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} - \frac{a^3}{6t^2} \int d^3 x \gamma \hat{\phi}^2 A^{ij} A_{ij} + \frac{1}{2} \left( e_{ij} e_{ij} - \epsilon \phi - A^i A_i - 3 \phi^2 \right) \\
+ \frac{3t^2 P_a}{a^2 V} \int d^3 x \pi \phi + \frac{2t^2 P_a}{a^2 V} \int d^3 x \pi \epsilon_{ij} e_{ij} + \frac{1}{2a} \int d^3 x \frac{\pi \epsilon_{ij} e_{ij}}{\gamma} - \frac{a^3}{6t^2} \int d^3 x \gamma \hat{\phi}^2 A^{ij} A_{ij} + \frac{1}{2} \left( e_{ij} e_{ij} - \epsilon \phi - A^i A_i - 3 \phi^2 \right)
\]

Performing the canonical transformation generated by

\[
F = a \tilde{P}_\alpha + \phi_0 \tilde{\phi}_\alpha - \int d^3 x (\tilde{\phi}_\pi \phi_\alpha + \tilde{\phi}_\pi \phi_\alpha + \epsilon_\pi e_{\pi} + \delta \pi \phi_\pi) - \frac{a \tilde{P}_a}{12V} \int d^3 x \gamma \hat{\phi}^2 (\epsilon_\pi e_{\pi} - \frac{1}{2} \epsilon^2) - \tilde{\phi}_\alpha \int d^3 x \gamma \hat{\phi}^2 (\tilde{\phi} + \frac{1}{2} \epsilon) \delta \phi
\]

which are

\[
a = \tilde{a} + \frac{\tilde{a}}{12V} \int d^3 x \gamma \hat{\phi}^2 (\epsilon_\pi e_{\pi} - \frac{1}{2} \epsilon^2)
\]

\[
P_a = \tilde{P}_a - \tilde{P}_a \int d^3 x \gamma \hat{\phi}^2 (\epsilon_\pi e_{\pi} - \frac{1}{2} \epsilon^2)
\]

\[
\phi_0 = \varphi_0 + \frac{1}{\sqrt{V}} \int d^3 x \gamma \hat{\phi}^2 (\tilde{\phi} + \frac{1}{2} \epsilon) \delta \phi
\]

\[
\pi_{\phi} = \pi_{\phi} - \frac{P_a}{V} \gamma \hat{\phi}^2 \delta \phi
\]

\[
\pi_{ij} = \pi_{ij} - \tilde{P}_a \frac{\gamma \hat{\phi}^2}{6V} (\epsilon_\pi e_{\pi} - \frac{1}{2} \epsilon^2) - \tilde{P}_a \frac{\gamma \hat{\phi}^2}{2V} \delta \phi \gamma_{ij}
\]

\[
\pi_{\phi} = \pi_{\phi} - \tilde{P}_a \frac{\gamma \hat{\phi}^2}{V} (\tilde{\phi} + \frac{1}{2} \epsilon)
\]

yields the new hamiltonian

\[
H = N \left\{ -\frac{t^2 P_a^2}{4aV} - \frac{K a V}{I^2} + \frac{P^2}{2a^3 V} + \frac{a^3 V U}{2} + \frac{t^2 P_a^2}{8 a V^2} + \frac{t^2 P_a^2}{24 a V^2} + \frac{a^3 V U}{2} + \frac{\frac{t^2 P_a^2}{8 a V^2}}{2} \right\} \int d^3 x \gamma \hat{\phi}^2 + \frac{t^2 P_a^2}{24 a V^2} \int d^3 x \gamma \hat{\phi}^2 - \frac{t^2 P_a^2}{8 a V^2} \int d^3 x \gamma \hat{\phi}^2 A^i A_i \\
+ \frac{P_a}{6V} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} \phi + \frac{P_a}{6V} \int d^3 x \gamma \hat{\phi}^2 A^i_{|j} e_{ij} - \frac{P_a^2}{4a^3 V} \int d^3 x \gamma \hat{\phi}^2 (3 \phi^2 + \epsilon \phi - A^i A_i) + \frac{t^2 P_a}{a^2 V} \int d^3 x \pi \phi \\
+ \frac{3t^2 P_a}{a^2 V} \int d^3 x \pi \delta \phi + \frac{P_a}{a^2 V} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} + \frac{a^3}{a^3} \int d^3 x \frac{\pi \delta \phi}{\gamma} - \frac{3t^2}{a^3} \int d^3 x \frac{\pi \delta \phi}{\gamma} - \frac{a}{4t^2} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} A^i_{|j} \\
- \frac{1}{a} \int d^3 x \pi A^i_{|i} - \frac{9t^2 P_a^2}{4a^3 V^2} \int d^3 x \gamma \hat{\phi}^2 \delta \phi^2 - \frac{3t^2 P_a P_a}{2a^2 V} \int d^3 x \gamma \hat{\phi}^2 \delta \phi + \frac{1}{2a} \int d^3 x \frac{\pi \delta \phi}{\gamma} - \frac{a^3}{6t^2} \int d^3 x \gamma \hat{\phi}^2 A^i_{|i} A^i_{|j} + \frac{1}{2} \frac{\epsilon_{ij} e_{ij}}{a} \epsilon_{ij} e_{ij} - \epsilon \phi - A^i A_i - 3 \phi^2 \\
- \frac{1}{4} \epsilon_{ij} e_{ij} e_{ij} - \epsilon \phi - A^i A_i - 3 \phi^2 \\
+ a^3 U_{\phi} \int d^3 x \gamma \hat{\phi}^2 \delta \phi^2 - \frac{a^3 U_{\phi}}{4} \int d^3 x \gamma \hat{\phi}^2 (\phi^2 + \epsilon \phi - A^i A_i) + \frac{a^3}{2} \int d^3 x \gamma \hat{\phi}^2 (\frac{1}{a^2} \delta \phi^2 e_{ij} e_{ij} + \frac{a}{2} U_{\phi} \delta \phi^2)
\]
Going back to its corresponding lagrangian, and redefining $N$ as

$$N = \tilde{N} \left[ 1 + \frac{1}{2V} \int d^3x \gamma^\frac{1}{2}(e\phi + \phi^2 - A^i A_i) \right]$$ (A9)

we obtain

$$L = -\frac{\dot{a}^2 a V}{l^2 N} + \frac{N K a V}{l^2} + \frac{\dot{\phi}^2 a^3 V}{2N} - \frac{Na^3 V U}{2} + \frac{Na}{6l^2} \int d^3x \gamma^\frac{1}{2} \left[ A_{ij} A_{ij} - \frac{1}{4} \epsilon_{ijk} \epsilon_{ij} \right] + \frac{a}{N} A_i \epsilon_{ij} + \frac{1}{2} \epsilon_{ijk} \epsilon_i \right]$$

$$+ \frac{a}{N} \int d^3x \gamma^\frac{1}{2} \phi^2 - \frac{2a}{3l^2} \int d^3x \gamma^\frac{3}{2} (\phi A_i - \frac{1}{2} A_i \epsilon_{ij} | j) - \frac{a}{4l^2} \int d^3x \gamma^\frac{3}{2} \phi \dot{\epsilon} - \frac{a}{2l^2} \int d^3x \gamma^\frac{3}{2} \dot{A}_i | i$$

$$+ \frac{a^3 \phi_0}{N} \int d^3x \gamma^\frac{3}{2} (\phi + \frac{1}{2} \epsilon) \delta \phi + a^2 \phi_0 \int d^3x \gamma^\frac{3}{2} \delta \phi A_i | i - Na^3 U_\phi \int d^3x \gamma^\frac{3}{2} \phi \delta \phi - \frac{\phi_0 a^3}{2N} \int d^3x \gamma^\frac{3}{2} \phi^2$$

$$+ \frac{Na}{2} \int d^3x \gamma^\frac{3}{2} \left( \frac{\delta \phi}{N^2} - \frac{\delta \phi^3}{a^2} \int d^3x \gamma^\frac{3}{2} S_i | i = 0 \right)$$ (A10)

Splitting as before the perturbations into their tensorial, vector and scalar parts

$$A_i = B_{ij} + S_i$$

$$\epsilon_{ij} = 2\psi \gamma_{ij} - 2E_{ij} - F_{ij} + w_{ij}$$, (A11)

with

$$S^i | i = F^i | i = 0$$

Using the gauge invariant quantity

$$V_i = S_i - \frac{a}{N} \dot{F}_i$$, (A14)

this lagrangian simplify to

$$\frac{Na}{6l^2} \int d^3x \gamma^\frac{1}{2} (V^{ij} V_{ij} - 2K V^i V_i)$$ (A15)

its associated hamiltonian reads

$$H^{(V)} = \frac{Na}{6l^2} \int d^3x \gamma^\frac{1}{2} V_i (\frac{1}{2} V^{ij} V_{ij} + K V_i) + \int d^3x \lambda_i \pi_i$$, (A16)

where we have the constraint

$$\pi_i^V \approx 0$$ (A17)

Consortion of the constraint $\pi_i^V \approx 0$ leads to the secondary constraint

$$\frac{1}{2} V^{ij} V_{ij} + K V \approx 0$$ (A18)

whose conservation fixes the Lagrange multiplier $\lambda_i$, which means that both constraints are second class. Defining the associated Dirac brackets, they become strong equalities, yielding the well known result for a universe filled only with a scalar field:

$$V^i = 0$$ (A19)

In the scalar sector we have
The hamiltonian reads

\[ H = \mathcal{H}_0 + \Lambda N P_N + \int d^3x \Delta_F \pi_F + \int d^3x \Lambda_\phi (\pi_\phi - \frac{P_\phi}{V} \gamma^\frac{2}{3} \delta \phi) \]  

where \( \mathcal{H}_0 \) is given by

\[
\mathcal{H}_0 = -\frac{l^2 P_\phi^2}{4aV} + \frac{P_\phi^2}{2a^3V} - \frac{KaV}{l^2} + \frac{a^3VU}{2} + \frac{l^2 P_\phi}{2a^3V} \int d^3x \phi \pi_\psi - \frac{P_\phi^2}{2a^3V} \int d^3x \gamma^\frac{2}{3} \phi^2 + \frac{3l^2 P_\phi}{2a^3V} \int d^3x \phi \delta \phi
\]

Performing the canonical transformation

\[
\varphi_0 = \varphi - \frac{1}{V} \int d^3x \gamma^\frac{2}{3} \phi \delta \phi
\]

\[
\pi_\phi = \pi_\phi + \frac{P_\phi}{V} \gamma^\frac{2}{3} \delta \phi
\]

\[
\pi_\phi = \pi_\phi + \frac{P_\phi}{V} \gamma^\frac{2}{3} \phi
\]

generated by

\[
\mathcal{F} = \mathcal{I} - \frac{P_\phi}{V} \int d^3x \gamma^\frac{2}{3} \phi \delta \phi,
\]

where \( \mathcal{I} \) represents the identity transformation, the new \( \mathcal{H} \) reads

\[
\mathcal{H} = \mathcal{H}_0 + \Lambda N P_N + \int d^3x \Delta_F \pi_F + \int d^3x \Lambda_\phi \pi_\phi, \quad \text{(A26)}
\]

\[
\mathcal{H}_0 = -\frac{l^2 P_\phi^2}{4aV} + \frac{P_\phi^2}{2a^3V} - \frac{KaV}{l^2} + \frac{a^3VU}{2} + \frac{l^2 P_\phi}{2a^3V} \int d^3x \phi \pi_\psi - \frac{3l^2 P_\phi}{2a^3V} \int d^3x \gamma^\frac{2}{3} \phi^2
\]

\[
+ \frac{a}{3l^2} \int d^3x \gamma^\frac{2}{3} \delta \phi \pi_\phi + \frac{a}{2} \int d^3x \gamma^\frac{2}{3} \delta \phi \delta \phi - \frac{a^3 U_\phi}{4a^3V} + \frac{a^3 U_\pi_\psi}{4} \int d^3x \gamma^\frac{2}{3} \phi^2
\]
\( + \left(-\frac{3l^2 P_a P_\varphi}{2a^2 V^2} + \frac{a^3 U_\varphi}{2}\right) \int d^3 x \gamma^2 \phi \delta \varphi - \frac{a}{3l^2} \int d^3 x \gamma^2 \left[ \psi^i \psi_i - 2\phi^i \psi_i + K (-3\psi^2 + 6\phi) \right] - \frac{aK}{3l^2} \int d^3 x \gamma^2 F F^i, i \)

\( + \left(P_\varphi \right)_{a^3 V} \int d^3 x \phi \pi_\varphi + \frac{1}{2a^3} \int d^3 x \pi_\varphi^2 + \frac{a}{2} \int d^3 x \gamma^2 \phi \delta \varphi^2 + (-\frac{9l^2 P_\varphi^2}{4a^3 V^2} + \frac{a^3 U_\varphi}{4}) \int d^3 x \gamma^2 \phi \delta \varphi^2 \)

\( \text{(A27)} \)

Conservation of the primary constraints \( \text{(A21)} \) leads to the secondary constraints

\[ H_0 \approx 0 \] \( \text{(A28)} \)

\[ \phi_5 \equiv \frac{1}{3a} \pi_\psi + \frac{2a}{9l^2} \gamma^2 F^i, i + \frac{2aK}{3l^2} \gamma^2 F \approx 0 \] \( \text{(A29)} \)

\[ \phi_6 \equiv - \frac{l^2 P_a}{2a^2 V} \pi_\psi + \left(\frac{3l^2 P_a P_\varphi}{2a^2 V^2} - \frac{a^3 U_\varphi}{2}\right) \gamma^2 \phi \delta \varphi + \frac{2a}{3l^2} \gamma^2 \phi \psi^i, i + \frac{6aK}{3l^2} \gamma^2 \psi - \frac{P_\varphi}{a^3 V} \pi_\varphi \approx 0 \] \( \text{(A30)} \)

Conservation of \( H_0 \) is identically satisfied. Conservation of \( \phi_6 \) leads to a term proportional to \( H_0 \) up to second order terms. Finally, \( \phi_5 \) fixes the Lagrange multiplier \( \Lambda_F \):

\[ a \Lambda_F = - \phi + \frac{l^2 P_a}{aV} F. \] \( \text{(A31)} \)

Substituting \( \tilde{F} = \{F, H\} = \Lambda_F \) into the above equation, we get for the gauge invariant Bardeen potentials \( \Phi \) and \( \Psi \)

\[ \Phi = \Psi \] \( \text{(A32)} \)

Calculating the non null Poisson brackets among the constraints yields

\[ \{\phi_3, \phi_5\} = -\frac{2a}{9l^2} \gamma^2 \delta^3 (x - x')^i, i - \frac{2aK}{3l^2} \gamma^2 \delta^3 (x - x') \]

\[ \{\phi_5, \phi_6\} = -\frac{2}{9l^2} \gamma^2 \delta^3 (x - x')^i, i - \frac{2K}{3l^2} \gamma^2 \delta^3 (x - x'). \] \( \text{(A33)} \)

The \( \phi_3 \) and \( \phi_5 \) constraints are second class, while

\[ \tilde{\phi}_6 = : \phi_6 + \frac{1}{a} \phi_3 \] \( \text{(A34)} \)

is a first class constraint. Defining the Dirac brackets, the second class constraints can be substituted in the Hamiltonian.

Making \( K = 0 \), and performing the canonical transformations generated by

\[ \mathcal{F}_1 = a \tilde{P}_a + \varphi \tilde{P}_\varphi + \int d^3 x \left[ a \pi \delta \varphi + \psi \pi_\psi - \frac{2 \tilde{P}_\varphi}{l^2 P_a} \psi \pi \right. \]

\[ + \left. \frac{\alpha}{2} \gamma^2 \phi \delta \varphi^2 \right], \] \( \text{(A35)} \)

where

\[ \alpha = \frac{3P_\varphi^2}{aP_a V} + \frac{a^2 P_a}{2V}, \] \( \text{(A36)} \)

and

\[ \mathcal{F}_2 = a \tilde{P}_a + \varphi \tilde{P}_a + \int d^3 x \left\{ \psi \pi_\psi + \nu \pi + \frac{2 \tilde{P}_\varphi}{l^2 P_a} - \frac{a^4 U_\varphi}{l^2 a^3 P_a V} \right\} \gamma^2 \psi \pi_\psi \]

\[ + \frac{2a^3 V}{3l^4 P_a} \gamma^2 \phi \psi^i, i \right\}, \] \( \text{(A37)} \)

and making a redefinition of \( N \), we finally obtain

\[ \tilde{\phi}_6 = \pi_\psi, \] \( \text{(A38)} \)

and

\[ H = NH_0 + \int d^3 x \left( \frac{l^2 P_a}{2a^2 V} \phi + \frac{3P_\varphi^2}{a^4 V} \psi + \frac{3l^2 P_a}{2a^4 V^2} \right) \phi \tilde{\phi}_6 + a N_\phi \phi_6 + \int d^3 x \Lambda_\phi \pi_\phi, \] \( \text{(A39)} \)
where

\[ H_0 = \frac{l^2 P^2}{4aV} + \frac{P^2_\varphi}{2a^3 V} + \frac{a^3 VU}{2} + \frac{1}{2a} \int d^3 x \frac{\pi^2}{\gamma^2} + \frac{1}{2a} \int d^3 x \gamma^\frac{1}{2} \hat{v}^* \hat{v}, \]

\[ + \left( \frac{15 l^2 P^2}{4a^3 V^2} + au_\varphi \varphi \right) - \frac{3l^2 U a}{8} + \frac{9U P_\varphi^2}{4aP_a^2} - \frac{l^4 P^2_\varphi}{16a^3 V^2} - \frac{27P_\varphi^1}{4a^2 V^2 P_\varphi^2} - \frac{3P_\varphi U_\varphi}{P_a} \right) \int d^3 x \gamma^\frac{1}{2} \hat{v}^2. \]  

(A40)

Using the background classical equations one can show that the coefficient of \( \hat{v}^2 \) can be written as \( z''/z \) as in [1]. Without their use, this is the simplest form the Hamiltonian of scalar perturbations can have in scalar field models.

**APPENDIX B: THE EXPLICIT CANONICAL TRANSFORMATIONS**

The explicit canonical transformations obtained from the generators \( \mathcal{F}_1, \mathcal{F}_2 \) and \( \mathcal{F}_3 \) of section III are, respectively

\[ a = \tilde{a} \left[ 1 + \frac{2(\lambda + 1)P_T}{l^2 P_a^3} a^2 \hat{\phi} \right] \int d^3 x \hat{\psi} + \frac{1}{2a} \frac{\partial \alpha}{\partial P_a} \int d^3 x \gamma^\frac{1}{2} \left( \sqrt{6} \gamma^\frac{1}{2} a^2 \hat{\phi} \right)^2 \]

\[ P_a = \tilde{P}_a - \frac{(1 + 3 \lambda)}{2\sqrt{6}l} \tilde{a}^2 \hat{\phi} \int d^3 x \sqrt{6} \tilde{a} \hat{\phi} (1 - 3 \lambda) \int \frac{d^3 x \gamma^\frac{1}{2} \left( \sqrt{6} \gamma^\frac{1}{2} \tilde{a} \hat{\phi} \right)^2}{l P_a \sqrt{\lambda}} \]

\[ \pi_{\varphi} = \frac{\tilde{a}^2 \hat{\phi} (1 - 3 \lambda)}{\gamma^\frac{1}{2} \tilde{a}} - \alpha \sqrt{6} \tilde{a} \hat{\phi} (1 - 3 \lambda) \gamma^\frac{1}{2} \hat{\psi} + \frac{2\alpha \sqrt{6} (\lambda + 1)P_T}{l P_a \sqrt{\lambda}} \tilde{a}^2 \hat{\phi} \gamma^\frac{1}{2} \hat{\psi} \]

\[ \varphi(c) = \sqrt{6} \tilde{a} \hat{\phi} (1 - 3 \lambda) \hat{v} - \frac{2\sqrt{6} \gamma^\frac{1}{2} (\lambda + 1)P_T}{l P_a \sqrt{\lambda}} \tilde{a}^2 \hat{\phi} \gamma^\frac{1}{2} \hat{\psi} \]

\[ \pi_{\psi} = \tilde{\pi}_{\psi} + \frac{2\sqrt{6} (\lambda + 1)P_T}{l^2 P_a \sqrt{\lambda}} \tilde{a} \hat{\phi} (1 - \lambda) \pi \]

(B1)

\[ a = \tilde{a} \left[ 1 + \frac{12 \sqrt{V}((\lambda + 1)P_T)^2}{l^2 P_a^3} a^2 (9\lambda - 1) \right] \int d^3 x \gamma^\frac{1}{2} \hat{\psi} + \left[ \frac{18V((\lambda + 1)P_T)^2}{l^4 P_a^3} a^2 - 6\lambda + \frac{(1 - 3\lambda)(\lambda 1)P_T}{2l^2 P_a^3} a^2 - 3\lambda \right] \int d^3 x \gamma^\frac{1}{2} \hat{\psi}^2 \]

\[ + \frac{2\tilde{a}^2 V}{3l^4 P_a^3} \int d^3 x \gamma^\frac{1}{2} \hat{\psi} \hat{\psi}^* d^i \hat{\psi} \]

\[ P_a = \tilde{P}_a + \left[ \frac{9(1 - 3 \lambda) \sqrt{V}((\lambda + 1)P_T)^2}{l^2 P_a^3} a^2 (9\lambda - 1) - \frac{(1 - 9\lambda^2) \sqrt{V}((\lambda + 1)P_T)}{4l \sqrt{\lambda} \sqrt{V}} \right] \int d^3 x \gamma^\frac{1}{2} \hat{\psi} \]

\[ + \left[ \frac{18V(1 - 2\lambda)(\lambda + 1)P_T^2}{l^4 P_a^3} a^2 - 6\lambda + \frac{2 + 3\lambda - 9\lambda^2(\lambda + 1)P_T}{2l^2 P_a^3} a^2 - 3\lambda \right] \int d^3 x \gamma^\frac{1}{2} \hat{\psi}^2 + \frac{2\tilde{a}^2 V}{l^4 P_a} \int d^3 x \gamma^\frac{1}{2} \hat{\psi} \hat{\psi}^* d^i \hat{\psi} \]

\[ \pi = \tilde{\pi} + \left[ \frac{6 \sqrt{V}((\lambda + 1)P_T)^2}{l^2 P_a^3} a^2 (1 - 3\lambda) - \frac{(1 + 3\lambda) \sqrt{V}((\lambda + 1)P_T)}{2l \sqrt{\lambda} \sqrt{V}} \right] \gamma^\frac{1}{2} \hat{\psi} \]

\[ \pi_{\psi} = \tilde{\pi}_{\psi} + \left[ \frac{6 \sqrt{V}((\lambda + 1)P_T)^2}{l^2 P_a^3} a^2 (1 - 3\lambda) - \frac{(1 + 3\lambda) \sqrt{V}((\lambda + 1)P_T)}{2l \sqrt{\lambda} \sqrt{V}} \right] \gamma^\frac{1}{2} \hat{\psi} \]

\[ + \left[ \frac{-12V((\lambda + 1)P_T)^2}{l^4 P_a^3} a^2 - 2\lambda + \frac{(1 + 3\lambda)(\lambda + 1)P_T}{l^2 P_a^3} a^2 - 3\lambda \right] \gamma^\frac{1}{2} \hat{\psi} + \frac{4\tilde{a}^3 V}{3l^4 P_a} \gamma^\frac{1}{2} \hat{\psi} \]

(B2)

\[ a = \tilde{a} + \frac{l^3 P_a^2}{4a} \int d^3 x \gamma^\frac{1}{2} \hat{v} \]

\[ P_a = \tilde{P}_a - \frac{1}{\tilde{a}} \int d^3 x \hat{v} + \frac{l^2 P_a}{4V} \int d^3 x \gamma^\frac{1}{2} \hat{v}^2 \]
\[ \pi = \frac{1}{a^2} \tilde{P}_a \gamma^\frac{1}{2} \tilde{v} \]

\[ v = a \dot{v}. \]  

The intermediary hamiltonian between \( F_2 \) and \( F_3 \) reads,

\[
H_0 = - \frac{l^2 P_a}{4 a V} + \frac{P_T}{a^{3\lambda}} + \frac{1}{2 a} \int d^3 x \frac{\pi^2}{\gamma^\frac{1}{2}} + \frac{\lambda}{2 a} \int d^3 x \gamma^{\frac{1}{2}} v^2 + \frac{3(\lambda + 1) P_T}{P_a} a^{-(1 + \lambda)} \psi \right] \pi_\psi + H_0^{(0)} \int d^3 x \left\{ \frac{9(\lambda + 1) P_T}{l^2 P_a^{\frac{3}{2}} a^\lambda} \frac{1}{2 \sqrt{V} \sqrt{V}} a^{-(1 + \lambda)} - \frac{3(1 - 3\lambda)(\lambda + 1) P_T}{l^2 P_a^{\frac{3}{2}} a^\lambda} \frac{1}{2 \sqrt{V} \sqrt{V}} a^{-(1 + \lambda)} \right\} \gamma^\frac{1}{2} v^2.
\]

\[ \text{(B3)} \]

\[ \text{(B4)} \]