Weyl Cohomology and the Effective Action for Conformal Anomalies

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We present a general method of deriving the effective action for conformal anomalies in any even dimension, which satisfies the Wess-Zumino consistency condition by construction. The method relies on defining the coboundary operator of the local Weyl group, \( g_{ab} \rightarrow \exp(2\sigma)g_{ab} \), and giving a cohomological interpretation to counterterms in the effective action in dimensional regularization with respect to this group. Non-trivial cocycles of the Weyl group arise from local functionals that are Weyl invariant in and only in the physical even integer dimension \( d = 2k \). In the physical dimension the non-trivial cocycles generate covariant non-local action functionals characterized by sensitivity to global Weyl rescalings. The non-local action so obtained is unique up to the addition of trivial cocycles and Weyl invariant terms, both of which are insensitive to global Weyl rescalings. These distinct behaviors under rigid dilations can be used to distinguish between infrared relevant and irrelevant operators in a generally covariant manner. Variation of the \( d = 4 \) non-local effective action yields two new conserved geometric stress tensors with local traces equal to the square of the Weyl tensor and the Gauss-Bonnet-Euler density respectively. The second of these conserved tensors becomes \( (3)H_{ab} \) in conformally flat spaces, exposing the previously unsuspected origin of this tensor. The method may be extended to any even dimension by making use of the general construction of conformal invariants given by Fefferman and Graham. As a corollary, conformal field theory behavior of correlators at the asymptotic infinity of either anti-de Sitter or de Sitter spacetimes follows, i.e. AdS\(_{d+1}\) or deS\(_{d+1}/\text{CFT}_d\) correspondence. The same construction naturally selects all infrared relevant terms (and only those terms) in the low energy effective action of gravity in any even integer dimension. The infrared relevant terms arising from the known anomalies in \( d = 4 \) imply that the classical Einstein theory is modified at large distances.

I. INTRODUCTION

The existence of conformal or trace anomalies in quantum field theories has been known for several decades [1–4]. Nevertheless, their full mathematical structure as well as their physical implications have remained a subject of discussion [5]. Accordingly, our purpose in this paper will be twofold. First, we wish to clarify the mathematical structure of conformal anomalies as non-trivial cocycles of the cohomology of the Weyl group, by defining the Weyl coboundary operator acting on functionals of the spacetime metric, and its corresponding cochain. This definition of the cohomology of the Weyl group will enable us to show that the trace anomaly terms are in one-to-one correspondence to local counterterms in the dimensionally regulated effective action in \( d \) dimensions, which become Weyl invariant in \textit{and only in} the physical even dimension \( d = 2k \). The limit, \( d \rightarrow 2k \) gives rise to a well defined non-local action for each non-trivial cocycle in the physical dimension, which is unique up to Weyl invariant and cohomologically trivial terms. In addition to supplying a well-defined and efficient algorithm for calculating the effective action of trace anomalies in any even dimension (provided that the local Weyl invariants in that dimension are known), the use of dimensional regularization will serve our second purpose, which is to expose the physical implications of the resulting anomalous effective action. In fact, exactly the same property of the effective action that identifies it as a non-trivial cocycle of the Weyl group, its multi-valuedness, also implies that it is sensitive to global Weyl rescalings, and therefore infrared relevant in the Wilson renormalization group sense. That is, the elements of the non-trivial cohomology of the Weyl group necessarily give rise to terms in the low energy effective action of gravity which do not decouple in limit of low energies or large spacetime volumes. Specializing to \( d = 4 \) spacetime dimensions, this means that the known trace anomalies necessarily imply modifications of classical general relativity in the effective low energy, long distance theory of gravity.
The relationship of anomalies to conformal invariants in even integer spacetime dimensions allows for a broader connection between pure mathematics and the physics of the renormalization group and low energy effective actions, implicit in the recent literature on AdS/CFT correspondence \[1\]. Fefferman and Graham (hereafter referred to as FG) provided an algorithm for constructing conformal invariants in \(d\) dimensions by embedding the physical space of interest in a \(d + 2\) dimensional Ricci flat ambient space, in such a way that coordinate invariant scalars in the embedding space (which are easy to construct) give rise to conformal Weyl invariants (which are generally more difficult to find) in the physical space \([2]\). The construction makes use of the conformal properties of the light cone in the \(d + 2\) dimensional ambient space and the foliation of this space by \(d + 1\) dimensional surfaces of asymptotically constant Ricci curvature which approach this light cone in a specific coordinatization. The FG algorithm is exactly what is needed to generate local conformal invariants and hence non-trivial cocycles of the Weyl group in any even dimension in accordance with our general dimensional regularization method.

Furthermore, because the physical space of interest lies at a conformal boundary of the \(d + 1\) dimensional bulk or embedding space, conformal transformations of the boundary form a special class of coordinate transformations in the bulk space of asymptotically constant curvature. The geometric radial coordinate of the bulk space geometry \(\rho\) corresponds to the length scale of the finite size rescalings in the effective theory on the boundary in a precise way. The set of local diffeomorphism invariants in the bulk action which diverge as the conformal boundary is approached generate exactly the infrared relevant operators of the Wilson effective action on the boundary, local or not, and a shift in the \(\rho\) coordinate of the bulk geometry is precisely a finite size scale transformation in the \(d\) dimensional boundary theory. The infrared divergent terms as \(\rho \to 0\) may also be regularized by dimensional continuation. Thus the FG embedding together with dimensional regularization provide just the mathematical tools necessary to define the infrared relevant operators of the Wilson effective action in a theory with general coordinate invariance in a precise way.

A by-product of this study of conformal infinity in the FG coordinates is the conclusion that the structure of conformal field theories in the AdS/CFT correspondence is actually representative of a more general feature of the FG embedding, which requires neither AdS (since it works equally well for asymptotically de Sitter spaces), nor a supersymmetric CFT on the conformal boundary. The conformal behavior at infinity (as opposed to specific values of coefficients in the effective action or correlation functions) and the generation of infrared relevant terms in the effective gravitational theory asymptotically as \(\rho \to 0\) is a purely kinematic property of the FG embedding of physical spacetime in a space of one higher dimension with constant positive or negative scalar curvature. Hence, there is a deS/CFT correspondence as well as an AdS/CFT correspondence, although the former has no evident connection with string theory backgrounds, as in the AdS/CFT case \([3]\).

Before presenting the general method and exploring these relationships in detail, let us review some well known features of conformal anomalies and previous treatments of their cohomology. The standard route to exhibiting the algebraic structure of the trace anomaly is to treat the parameters of infinitesimal local conformal transformations as anti-commuting Grassmann variables \([4]\). This guarantees nilpotence of the generator of the infinitesimal Weyl transformations, and distinguishes trivial from non-trivial cocycles in a clear manner. However, an important aspect of the anomaly is left unexplored in this abstract algebraic approach. Non-trivial cohomology implies that although the conformal anomaly itself is local, it cannot be written as the Weyl variation of some local coordinate invariant action. Instead the action whose Weyl variation is the local anomaly must itself be non-local in any even integer dimension. The Wess-Zumino (WZ) consistency condition is just the integrability condition that this coordinate invariant non-local action exists \([5, 6]\). Consistency is automatically satisfied in the algebraic approach by the nilpotence of two successive anti-commuting Weyl transformations, but the algebraic method furnishes no means of constructing the WZ consistent effective actions associated with the non-trivial cocycles, and sheds little light on their physical meaning.

A complementary approach to conformal anomalies is to construct the WZ effective action directly by ‘integrating the trace anomaly’ with respect to the local conformal factor variation \(\delta \sigma(x)\). In two dimensions this procedure is almost immediate and can be performed by inspection \([7]\), while in four and higher even integer dimensions it requires adding a admixture of the trivial cocycles (i.e. those terms which can be written as the Weyl variation of a local coordinate invariant action) in order to bring the anomaly into a form linear in \(\sigma(x)\), which can be integrated easily \([8]\). Although this approach certainly produces an action satisfying WZ consistency \([9]\), it leaves the deeper cohomological aspects of conformal anomalies unexamined. Moreover, since it produces a WZ action that involves an \(d^{th}\) order differential operator on \(\sigma\) in \(d = 2k\) even dimensions, it seems to imply strong ultraviolet (UV) behavior, and raises concerns about ghosts similar to local higher derivative theories of gravity in dimensions greater than two. Conversely and perhaps paradoxically, the non-local form of the WZ action obtained in this approach seems to imply more severe IR behavior than expected with unphysical \(p^{-4}\) poles in anomalous Ward identities \([10]\). Yet the anomaly is the effect of integrating out massless fields in all the standard one-loop calculations and the fully covariant non-local form of the WZ anomalous action should be associated with long distance or low energy physics, in which there can be neither unphysical UV ghosts nor higher order IR poles. Hence this approach raises a number of questions which should be resolved before it can be deemed completely satisfactory.
In this paper we present a general, unified treatment of conformal anomalies in any even dimension by using the same method both to define the non-trivial cohomology of the Weyl group and to derive the local WZ effective action, as well as its fully covariant non-local form. The key observation which will make this possible is that the non-trivial cocycles of the Weyl cohomology in any even integer dimensions are in one-to-one correspondence with those local counterterms in the dimensionally regulated effective action which become Weyl invariant in and only in exactly the physical dimension $d = 2k$.

The technical reason for this one-to-one correspondence is that such local invariants have Weyl variations which are proportional to $d - 2k$ near the physical dimension and hence cancel the $(d - 2k)^{-1}$ pole of dimensional regularization, yielding a finite action $\Gamma_{WZ}$ for $d = 2k$, which is local in terms of $\sigma$, and which automatically satisfies the WZ consistency condition. This becomes clear when $\sigma$ is eliminated from the action by solving for the unique conformal factor that Weyl translates between two different metrics $g_{ab}$ and $\bar{g}_{ab} \equiv e^{2\sigma}g_{ab}$ in the same conformal equivalence class. Then the local WZ action becomes the difference of two fully covariant but non-local actions,

$$\Gamma_{WZ}[g; \sigma] = S_{\text{anom}}[ge^{2\sigma}] - S_{\text{anom}}[\bar{g}] .$$

This difference may be viewed as a finite Weyl coboundary operation, $\Delta_{\sigma} \circ S_{\text{anom}}[\bar{g}]$, which produces a one-form from a scalar functional $S[\bar{g}]$. The anti-symmetrized coboundary operator on $k$-forms may be defined by a straightforward generalization from standard Riemannian geometry, and the nilpotency condition of $\Delta_{\sigma}$ on the cochain easily checked (cf. Sec. 2). Hence the WZ consistency of $\Gamma_{WZ}$ follows immediately. Its cohomology is nevertheless non-trivial since $S_{\text{anom}}[\bar{g}]$ is a non-local functional of $g_{ab}$ when written entirely in the physical dimension $d = 2k$, and $\Gamma_{WZ}[\bar{g}; \sigma]$ is not a single-valued functional of the original metric, $g_{ab}$. This multi-valuedness of the configuration space of metrics is due to the exclusion of the singular metrics $g_{ab} = 0$ and its inverse, $g^{ab} = 0$, corresponding to vanishing or diverging conformal factor $\Omega = e^{\sigma}$. These singular metrics correspond to punctures that allow the topological Euler number of the manifold to change. Because of this topological obstruction in the configuration space of smooth metrics, $\Gamma_{WZ}[\bar{g}; \sigma]$ is not invariant under the shift $\sigma(x) \to \sigma(x) + i\pi q$, corresponding to integer $q$ winding around the obstruction, and for the same reason $\Gamma_{WZ}[\bar{g}; \sigma]$ is necessarily non-invariant under the constant global scale transformation, $\sigma(x) \to \sigma(x) + \sigma_0$. It is this sensitivity to finite volume scaling of the effective WZ action which reveals its physical interpretation in terms of standard Wilson renormalization group principles.

The use of dimensional regularization to define the cohomology of the Weyl group has a number of advantages. From a purely mathematical point of view, it reduces the general classification of the non-trivial cocycles, usually associated with quantum anomalies, to the construction of conformal invariants, a problem of classical differential geometry that has been (nearly) rigorously solved in any even dimension by Fefferman and Graham. Since the non-trivial cocycles and the corresponding non-local effective action $S_{\text{anom}}$ are unique up to addition of trivial cocycles and completely Weyl invariant terms, the explicit construction of the WZ effective action eliminates doubts about the correctness of the procedure of integrating the anomaly. As a consequence, in $d = 4$ we can conclude definitively that there are no non-local terms in the effective action, of the form $C_{abcd} \log(-\square)C^{abcd}$, generated by the anomaly, as has long been conjectured. Moreover, dimensional regularization is the natural way to regularize the solutions of the Einstein equations on embedding spaces of asymptotically constant curvature, as required by the FG algorithm for constructing conformal invariants. Hence continuation in the number of dimensions removes the ‘FG ambiguity,’ regulates the infrared divergences at large volumes in the FG construction, and also illuminates why the trace anomaly of conformal field theories is given by the AdS/CFT correspondence conjecture.

From a more physical point of view, dimensional regularization is closely connected to renormalization and the renormalization group (RG) flow of a quantum theory, and allows a treatment of UV and IR renormalization effects in a unified way. It is well known in statistical physics and renormalization theory that UV behavior is enhanced by increasing the number of spacetime dimensions in loop integrations while conversely, the IR behavior is enhanced by decreasing $d$ in a given graph. Since $d$ is not required a priori to be greater than or less than the physical dimension $2k$ in the dimensional regularization procedure, the limit $d \to 2k$ of counterterms in the quantum effective action can (and does) contain both UV and IR logarithmic effects in the physical dimension, treated on an equal footing. In the quantum effective action poles replace the UV cutoff and IR effects appear as finite terms, with no singular behavior as $d$ approaches the physical dimension. Conversely, from the Wilson infrared effective action point of view, the UV cutoff is fixed and terms in the effective action are classified according to their behavior under rigid scale transformations with a sliding IR cutoff, which we may take to be the total $d$-volume of the system. Dimensional continuation may be used again, this time to regulate the IR behavior, preserving generally coordinate invariance as the volume is taken to infinity. In this case, poles signal large volume divergences or IR relevant operators in the Wilson RG approach, while terms which remain finite as the volume goes to infinity are IR irrelevant and contain no poles at the physical dimension.

These considerations and the non-single valuedness of $\Gamma_{WZ}$ under global scale transformations imply that the non-trivial Weyl cocycles are best understood as infrared marginally relevant operators of the Wilson effective action of
low energy gravity under the RG, which control the critical exponents and scaling behavior of the theory near its IR conformal fixed point(s). These fixed point(s) are Gaussian because of the quadratic form of $\Gamma_{WZ}$, provided that the coefficients of the trivial cocycles (i.e. the UV relevant counterterms) flow to zero in the extreme infrared $[20]$. This physical RG interpretation of conformal anomalies is considerably more general than existing anomaly calculations in classically conformally invariant free field theories, and implies that the appearance of $S_{\text{anom}}$ in the effective action of low energy gravity with some coefficient is quite generic within the standard framework of low energy effective field theories. The RG scaling interpretation is completely consistent with the scaling behavior of $d = 2$ CFT’s coupled to gravity both in the continuum and in lattice simulations, where the KPZ critical exponents are determined by the Polyakov action $[22,23]$. In $d = 4$ it is consistent with the IR stability of the Gaussian fixed point $[20]$, with the recent consideration of the scaling behavior of the effective action of a massless, but not conformally coupled scalar field in de Sitter space $[23]$, and with the fact there are no higher order poles or unphysical propagating ghost states in the $d = 4$ effective quantum theory determined by $\Gamma_{WZ}$ $[25]$.

A final bonus of the complete classification of the Weyl cohomology by dimensional regularization in any even dimension is that it automatically yields the conserved energy-momentum tensor corresponding to the non-local action $S_{\text{anom}}[g]$. Although these tensors are also non-local in general, in four dimensions one of them becomes the local geometric tensor $^{(3)}H_{ab}$, defined by $\left(\frac{1}{2}\Gamma_{WZ}\right)$ in $d = 4$ conformally flat spacetimes, which had been found some time ago $[20]$. Thus the existence of the covariant non-local action functional $S_{\text{anom}}$ turns out to be the underlying reason for the existence of this local tensor, which has been called ‘accidentally conserved’ $[1]$. Its non-local generalization to non-conformally flat spacetimes agrees with the direct calculation from $\Gamma_{WZ}$ given in ref. $[25]$, and its trace is proportional to the Gauss-Bonnet-Euler density in $d = 4$.

The paper is organized as follows. In the next section we define the anti-symmetric coboundary operator of finite Weyl shifts, generalizing the ideas of differential forms and de Rham cohomology to the functional space of metrics, without the use of infinitesimal anticommuting Grassmann numbers. In Section 3 we illustrate our general technique for constructing $\Gamma_{WZ}$ in $d = 2$ dimensions, recovering both the Polyakov action and the energy-momentum tensor corresponding to it. In Section 4 we follow the same procedure in $d = 4$ dimensions, deriving the form of $\Gamma_{WZ}$ previously found by integrating the anomaly, as well as its fully covariant non-local form, $S_{\text{anom}}$. The conserved geometrical stress tensors corresponding to this action are derived in Section 5, where the connection to the tensor $^{(3)}H_{ab}$ is also demonstrated. Section 6 contains a brief mathematical interlude, recapitulating the FG embedding in spaces of asymptotically constant curvature, the conformal infinity of both AdS and de Sitter space in suitable coordinates, and the subset of diffeomorphisms in the bulk or embedding space which become local Weyl transformations on the conformal boundary. In Section 7 the FG construction is applied to finite volume scale transformations, which serves to select exactly the infrared relevant operators of the effective Wilson gravitational boundary action. Section 8 summarizes our conclusions and contains the main results of the paper in a concise form.

Wherever feasible detailed formulae used in the main text are relegated to the three Appendices. The first catalogs the conformal variations of various tensors needed in the text, the second contains a proof of an interesting identity of the Weyl tensor in four dimensions, and the third is a computation of the new non-local tensor $C_{ab}$ that appears in the variation of the $d = 4$ effective action in the general case.

II. COHOMOLOGY OF THE WEYL GROUP AND DIMENSIONAL REGULARIZATION

We consider the abelian Weyl group of local conformal transformations on the metric,

$$\bar{g}_{ab}(x) \rightarrow \tilde{g}_{ab}(x) \equiv e^{2\sigma(x)} \bar{g}_{ab}(x), \quad (2.1)$$

where $\sigma(x)$ is any smooth function of the coordinates. Let $S[g]$ denote any scalar functional (local or not) of the metric $\bar{g}_{ab}$. The action of the Weyl group (2.1) suggests that a finite difference or coboundary operator $\Delta_{\sigma}$ on scalar functionals $S[g]$ be defined by

$$\Delta_{\sigma} \circ S[\tilde{g}] \equiv (\Delta S)_{\sigma} \equiv S[e^{2\sigma} \tilde{g}] - S[\bar{g}] \equiv \Delta S[\tilde{g}] - S[\bar{g}]. \quad (2.2)$$

By definition, functionals $S_{\text{inv}}$ which are invariant under the Weyl transformation (2.1) satisfy $\Delta_{\sigma} \circ S_{\text{inv}} = 0$.

In general, $\Delta_{\sigma} \circ S \neq 0$, and this quantity depends on both the initial or fiducial metric, $\bar{g}_{ab}(x)$ and the finite Weyl shift $\sigma(x)$. It may be regarded as a one-form, $\Gamma^{(1)}[\tilde{g}; \sigma]$ with respect to the finite shift Weyl group $[1]$. This is a natural generalization of the concept of a one-form from the differential geometry of finite dimensional Riemannian manifolds to the present case of the infinite dimensional functional space of metrics. The definition of the coboundary operator $\Delta_{\sigma}$ above acting on scalar functionals can be generalized to $\Delta_{\sigma_{k+1}}$ acting on $k$-forms $\Gamma^{(k)}[\tilde{g}; \sigma_1, \ldots, \sigma_k]$ by the anti-symmetrized chain rule.
\[
\Delta_{\sigma_{k+1}} \circ \Gamma^{(k)}[\bar{g}; \sigma_1, \ldots, \sigma_k] \equiv \sum_{i=1}^{k+1} (-1)^{i+1} (\Delta \Gamma^{(k)})_{\sigma_i}[\bar{g}; \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{k+1}],
\]

(2.3)

where

\[
(\Delta \Gamma^{(k)})_{\sigma_i}[\bar{g}; \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{k+1}] \equiv \Gamma^{(k)}[\bar{g}c^{2\sigma_i}; \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{k+1}] - \Gamma^{(k)}[\bar{g}; \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{k+1}],
\]

(2.4)

is the Weyl finite difference operator on the functional, $\Gamma^{(k)}[\bar{g}; \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{k+1}]$ with $\sigma_i$ omitted, treated as a scalar functional of the metric $\bar{g}$ with all the other $\sigma$ dependences unaffected. This anti-symmetrized coboundary operation (2.3) produces a $(k+1)$-form from a $k$-form, which is totally anti-symmetric under interchange of any two of its arguments, $\sigma_i$ and $\sigma_j$, and defines a finite derivation operation in the space of metrics with the same algebraic properties as the exterior derivative in Riemannian geometry. If the Weyl transformation by $\sigma$, (2.1), is replaced by an infinitesimal anticommuting Grassmann variable, then the definition (2.3) becomes equivalent to the standard definition \[10\]. The anti-commuting Grassmann variables are simply a device to keep track of anti-symmetrization, and are not essential, provided the coboundary operator $\Delta_{\sigma}$ is defined in an explicitly anti-symmetric manner for a finite $\sigma$ Weyl shift, as in (2.3). The previous definition (2.2) is a special case of the general definition (2.3) with $k=0$, or in other words, zero-forms $\Gamma^{(0)}[\bar{g}]$ are just coordinate invariant scalar functionals of the metric. The Einstein-Hilbert action of classical general relativity or the general coordinate invariant effective action of a quantum field theory $S_{eff}[\bar{g}]$ in a general curved background metric $\bar{g}$ are examples of such coordinate invariant scalar functionals of the metric.

The definition of the cochain (2.3) allows us to give a precise meaning to the cohomology and WZ consistency for the Weyl group. Applying the general definition for $k=1$, the operation of the coboundary operator $\Delta_{\sigma}$ on one-forms produces the two-form,

\[
\Gamma^{(2)}[\bar{g}; \sigma_1, \sigma_2] = \Delta_{\sigma_2} \circ \Gamma^{(1)}[\bar{g}; \sigma_1] = (\Delta \Gamma^{(1)})_{\sigma_1}[\bar{g}; \sigma_2] - (\Delta \Gamma^{(1)})_{\sigma_2}[\bar{g}; \sigma_1]
\]

\[
= \Gamma^{(1)}[\bar{g}c^{2\sigma_1}; \sigma_2] - \Gamma^{(1)}[\bar{g}; \sigma_2] - \Gamma^{(1)}[\bar{g}c^{2\sigma_2}; \sigma_1] + \Gamma^{(1)}[\bar{g}; \sigma_1].
\]

(2.5)

If $\Gamma^{(1)}[\bar{g}; \sigma_1]$ is the one-form (2.2), i.e. if we apply (2.3) to the case, $\Gamma^{(1)}[\bar{g}; \sigma_1] = \Delta_{\sigma_1} \circ S[\bar{g}]$, then we obtain the algebraic identity,

\[
(\Delta_{\sigma_2} \circ \Delta_{\sigma_1}) \circ S[\bar{g}] \equiv \Delta_{\sigma_2} \circ (\Delta_{\sigma_1} \circ S[\bar{g}]) = 0.
\]

(2.6)

This nilpotency property of $\Delta_{\sigma}$ is easily seen to be generally true on any $k$-form, owing to the anti-symmetry of the general definition (2.3). That is, quite generally we have the nilpotency property,

\[
(\Delta^2)_{\sigma_1, \sigma_2} = \Delta_{\sigma_1} \circ \Delta_{\sigma_2} = 0,
\]

(2.7)

an essential property of a coboundary operator, which justifies the use of this term from ordinary differential geometry. To define the cohomology, let us call any one-form $\Gamma^{(1)}[\bar{g}; \sigma_1]$ which satisfies

\[
\Delta_{\sigma_2} \circ \Gamma^{(1)}[\bar{g}; \sigma_1] = 0 \quad \text{closed},
\]

(2.8)

and any one-form $\Gamma^{(1)}[\bar{g}; \sigma_1]$ that can be written,

\[
\Gamma^{(1)}[\bar{g}; \sigma_1] = \Delta_{\sigma_1} \circ S_{local}[\bar{g}] \quad \text{exact},
\]

(2.9)

if $S_{local}[\bar{g}]$ is some local, single-valued scalar functional of the metric. Eq. (2.6) shows that all exact one-forms are closed, because of the nilpotency property (2.7) of the coboundary shift operator $\Delta_{\sigma}$. This is the trivial cohomology of the Weyl group. However, all closed forms are not necessarily exact, and the non-trivial cohomology of the Weyl group is defined precisely as the set of these closed but non-exact one-forms. That is, elements of the non-trivial cohomology of the Weyl group are one-forms obeying (2.8) that cannot be written as Weyl transforms of any local, single-valued functional of the metric, as in (2.9).

The insistence on local single-valued functionals of the metric, $S_{local}[\bar{g}]$ for the exactness property (2.9) is essential. The underlying reason is that the measure of integration over the space of metrics is single-valued and local, so that any $S_{local}[\bar{g}]$ can be absorbed into the definition of the functional measure, whereas non-local functionals in the effective action cannot be so absorbed, and constitute the genuine anomalies. The Weyl group involves the exponential map (2.1), which is certainly single-valued and local. However, the inverse of the exponential map is a logarithm, which is not single-valued, and as we shall see, non-locality results from solving for $\sigma$ in terms of the original metric $\bar{g}_{ab}$ and its
Weyl translate $g_{ab}$, which involves inverting a differential operator, a non-local operation. Hence, arbitrary functionals $\Gamma^{(1)}[\bar{g}; \sigma_1]$ of $\bar{g}$ and $\sigma$ separately (even if restricted to purely local ones) need not be single-valued, local functions of the full Weyl transformed metric $g = e^{2\sigma} \bar{g}$. These are the closed but non-exact functionals which determine the non-trivial cohomology of the Weyl group, and for which we seek an algorithm to compute explicitly.

As the non-exactness of one-forms defines the non-trivial cohomology of the Weyl group and corresponds to genuine conformal anomalies, the condition of closure (2.8) corresponds precisely to the WZ consistency condition on the anomalous effective action. Indeed let $\Gamma_{WZ}[\bar{g}; \sigma]$ be the one-form effective action whose variation generates the Weyl anomaly $A$ in the full (Weyl-transformed) metric $g_{ab} = g_{ab} e^{2\sigma}$, i.e.

$$
\frac{\delta}{\delta \sigma} \Gamma_{WZ}[\bar{g}; \sigma] = A[ge^{2\sigma}].
$$

(2.10)

Then the statement that $\Gamma_{WZ}[\bar{g}; \sigma]$ is closed,

$$
\Delta_{\sigma_2} \circ \Gamma_{WZ}[\bar{g}; \sigma_1] = \Gamma_{WZ}[\bar{g} e^{2\sigma_1}; \sigma_2] - \Gamma_{WZ}[\bar{g}; \sigma_2] - \Gamma_{WZ}[\bar{g} e^{2\sigma_2}; \sigma_1] + \Gamma_{WZ}[\bar{g}; \sigma_1] = 0,
$$

(2.11)

is precisely the statement that $\Gamma_{WZ}[\bar{g}; \sigma]$ satisfies the WZ consistent anomaly condition \[11,12\]. It is the finite shift generalization of the infinitesimal form of WZ integrability condition for the abelian Weyl group, i.e.

$$
\frac{\delta^2 S}{\delta \sigma_2 \delta \sigma_1} = \frac{\delta^2 S}{\delta \sigma_1 \delta \sigma_2}.
$$

(2.12)

Although this is guaranteed algebraically by the use of anti-commuting infinitesimal Weyl parameters, it is important to recognize that the physical content of the condition (2.12) is that some scalar effective action functional $S[\bar{g}]$ must exist, whose first variation is the anomaly, and whose second variation is necessarily hermitian with respect to interchange of the order of successive infinitesimal Weyl variations \[11,12\].

With these preliminaries we come now to the essential observation which will enable us to give a general technique for constructing the WZ consistent anomalous effective actions in any even $d = 2k$ spacetime dimension, by using well-known properties of dimensional regularization. In dimensional regularization we are instructed to write down

$$
\Gamma[\bar{g}; \sigma] = \lim_{d \to 2k} \frac{S_d[\bar{g} e^{2\sigma}]}{d - 2k}
$$

$$
\Gamma[\bar{g}; \sigma] = \lim_{d \to 2k} \frac{S_d[\bar{g} e^{2\sigma}]}{d - 2k} = \lim_{d \to 2k} \frac{\Delta_{\sigma} \circ S_d[\bar{g}]}{d - 2k},
$$

(2.13)

exists in the physical dimension $d = 2k$. Whenever this limit exists the resulting functionally satisfies the WZ consistency condition (2.11), since it is constructed explicitly as an exact one-form of the Weyl group in $d$ dimensions, and the nilpotency property (2.7) is purely algebraic, independent of $d$, commuting with the limit in (2.11). However, as we shall see, after the indicated limit has been taken, the effective action $\Gamma[\bar{g}; \sigma]$ can no longer be written as the Weyl variation of a local action in the physical dimension, and hence each non-zero counterterm $S_d$ for which the limit (2.13) exists will generate an element of the WZ consistent non-trivial cohomology of the Weyl group in precisely $d = 2k$ dimensions. Conversely, since the anomaly $A$, is generated by taking the infinitesimal Weyl variation of the effective action $\Gamma[\bar{g}; \sigma]$ via (2.10), and it is composed of local dimension $2k$ scalar invariants with finite coefficients, by taking the full set of these invariants near the physical dimension and finding all conformal invariant combinations, all non-trivial cohomologies of the Weyl group must be generated in dimensional regularization by counterterms in the effective action whose UV pole singularity at the physical dimension is cancelled by a $d - 2k$ factor in the numerator. Only in this way can a local anomaly in the trace of the renormalized energy-momentum tensor with a finite UV cutoff independent coefficient be generated. Hence dimensional regularization naturally classifies the dimension $2k$ scalar invariants into those corresponding to non-trivial and trivial first cocycles of the Weyl group, depending on whether their Weyl variations do or do not vanish linearly as $d \to 2k$. There do not appear to be invariants which vanish faster than linearly as $d \to 2k$, which would correspond to higher non-trivial cohomological structures in the space of metrics.
These general observations reduce the problem of finding all the non-trivial elements of the cohomology of the Weyl group in $2k$ even dimensions to finding all the local Weyl invariants in that dimension. This latter problem of classical differential geometry has been solved by the construction of Fefferman and Graham which we review in Section 6. Since the counterterms in dimensional regularization are integrals over local invariants, we must allow also for the possibility of non-trivial cocycles arising from local densities that are Weyl invariant only up to total derivatives. Such counterterms correspond to topological invariants, of which there is exactly one, the Gauss-Bonnet-Euler invariant $E_{2k}$ in any even dimension [27]. This one topological invariant gives rise to the type A anomalies while the FG local Weyl invariants give rise to the type B anomalies [18]. Both have Weyl variations that vanish linearly as $d \to 2k$ and hence the continuation of each of these two kinds of invariants away from $d = 2k$ determines a finite WZ action $\Gamma[\bar{g}; \sigma]$ independent of the UV regulator pole for which the limit in (2.13) exists.

In this way the explicit construction of the non-trivial cohomology of the Weyl group can be carried out, determining the general form of the conformal anomaly and the non-local WZ consistent effective action corresponding to it in any even dimension. In the following two sections we show that for $d = 2$ and $d = 4$ dimensions this construction generates precisely the Polyakov action and the four dimensional anomalous action analogous to it discussed in previous work [13,14,20]. Generalizations to higher even dimensions are also straightforward. Odd dimensional spacetimes have no conformal anomalies and must be treated differently, e.g. as boundaries of spacetimes of one higher dimension.

### III. WZ ACTION AND ENERGY-MOMENTUM IN TWO DIMENSIONS

Let us illustrate our general approach first in $d = 2$ dimensions. The unique dimension-two local scalar function of the metric is the Ricci scalar, and hence its spacetime integral is the only possible counterterm near $d = 2$ dimensions. Thus we consider

$$
\Gamma[\bar{g}; \sigma] = \lim_{d \to 2} \int \frac{d^d x \sqrt{-g} \mathcal{R} - \int d^d x \sqrt{-g} \mathcal{R}}{d - 2},
$$

(3.1)

where $\mathcal{R} = R[g]$ is the Ricci scalar in $d$ dimensions evaluated on the $d$-dimensional metric $g_{ab} = e^{2\sigma} \bar{g}_{ab}$, and $\mathcal{R} = R[g]$, evaluated on the $d$-dimensional fiducial metric $g_{ab}$. Now in $d$ dimensions,

$$
\sqrt{-g} \mathcal{R} = \sqrt{-g} e^{(d-2)\sigma} \left[ \mathcal{R} - 2(d-1) \Box \sigma - (d-1)(d-2) \sigma^a \sigma_a \right],
$$

(3.2)

where all covariant derivatives and contractions are performed with the metric $g_{ab}$ and we have introduced the shorthand notations, $\nabla_a \sigma \equiv \sigma_a$, $\Box \sigma \equiv \nabla_a \nabla^a \equiv \sigma^a \sigma_a$. Expanding (3.2) to first order in $d - 2$, subtracting $\sqrt{-g} \mathcal{R}$ and taking the limit indicated in (3.1), we obtain

$$
\Gamma[\bar{g}; \sigma] = \int d^2 x \sqrt{-\bar{g}} \left[ -\sigma \Box \sigma + \sigma \mathcal{R} \right],
$$

(3.3)

after an integration by parts and ignoring total derivatives which give possible surface contributions. Up to a multiplicative normalization this is exactly the Polyakov action found by functionally integrating with respect to $\sigma$ the form of the trace anomaly for the metric $g_{ab} = \bar{g}_{ab} e^{2\sigma}$ directly in $d = 2$ dimensions [13], i.e.

$$
\frac{\delta \Gamma[\bar{g}; \sigma]}{\delta \sigma} = \sqrt{-\bar{g}} (\mathcal{R} - 2 \Box \sigma) = \sqrt{-\bar{g}} \mathcal{R}.
$$

(3.4)

When the conformal property of the self-adjoint hermitian differential operator, $\Box$ in two dimensions,

$$
\sqrt{-\bar{g}} \Box = \sqrt{\bar{g}} \Box,
$$

(3.5)

is used, together with (3.2) for $d = 2$, it is easily checked that the effective action $\Gamma[\bar{g}; \sigma]$ in (3.3) obeys the WZ consistency condition (2.11). Indeed, this is automatic from its construction (3.1) as an exact one-form in $d \neq 2$ dimensions, followed by a limiting process which commutes with the algebraic nilpotency property of $\Delta_\sigma$.

However, since (3.3) is a simple polynomial in $\sigma$ it cannot be written as a single-valued local functional of the full metric $\bar{g} e^{2\sigma}$. This is clear from the fact that the complex transformation,

$$
\sigma \to \sigma + i \pi q
$$

(3.6)

for any integer $q$ leaves the full metric $g = \bar{g} e^{2\sigma}$ invariant but under this transformation,
\[ \Gamma[\tilde{g}; \sigma] \rightarrow \Gamma[\tilde{g}; \sigma] + iq\pi \int d^2 x \sqrt{-g} \ R = \Gamma[\tilde{g}; \sigma] + 4\pi^2 i q \tilde{\chi}_e \] (3.7)

where \( \tilde{\chi}_e \) is the Euler number of the metric \( \tilde{g} \). Hence the action \( \Gamma[\tilde{g}; \sigma] \) is not a single-valued local functional of \( g = \tilde{g} e^{2\sigma} \), but rather an explicit representation of the non-trivial cohomology of the Weyl group in \( d = 2 \) physical dimensions. Evidently, the expansion of the exponential conformal \( \sigma \) dependence of (3.3) required by the limit (3.1) is responsible for \( \Gamma \) being a simple polynomial in \( \sigma \) without the periodicity under (3.6), and this is associated in turn with the fact that the integral of the Ricci scalar is invariant under the Weyl group only in exactly two dimensions. Since the integral of \( R \) is proportional to the Euler number in two dimensions, the existence of a single element of the non-trivial cohomology and a single anomaly coefficient is a consequence of the existence of one and only one (type A) topological invariant. There are no (type B) local Weyl invariants in two dimensions.

Notice also that if instead of \( q \) being an integer we take \( i\pi q \) to be a real constant \( \sigma_0 \), then the transformation (3.6) is a global rescaling of the metric,

\[ g_{ab} \rightarrow g_{ab} e^{2\sigma_0}. \] (3.8)

Then (3.7) informs us that the multi-valuedness of the action \( \Gamma[\tilde{g}; \sigma] \) necessarily implies that it transforms non-trivially under such a global rescaling. Hence, non-trivial behavior under global rescaling is a necessary feature of a representative of the non-trivial cohomology of the Weyl group.

Associated with the one-form \( \Gamma \), which is local in \( \sigma \), obeying WZ consistency there is a non-local zero form, \textit{i.e.} a non-local quantum effective action of the full metric \( g = \tilde{g} e^{2\sigma} \). Since the action (3.3) is quadratic in \( \sigma \) this non-local action is easily constructed by adding to \( \Gamma \) a \( \sigma \) independent piece which ‘completes the square’ and leaves the variation (3.4) and the WZ condition unaffected. To find this \( \sigma \) independent term explicitly, we need only solve the linear differential equation (3.4) for \( \sigma \) by introducing the Green’s function inverse of the second order Weyl covariant differential operator (3.5), namely

\[ -\sqrt{-g} \, D_2(x, x') = \delta^2(x, x'), \] (3.9)

where \( \delta^2(x, x') \) is a density of weight two, \textit{i.e.} its integral \( \int d^2 x' \delta^2(x, x') = 1 \). If the metric has a Lorentzian signature then the boundary conditions needed to specify the particular Green’s function solution of (3.9) will depend on the application. If \( \Box \) has normalizable zero modes, such as on the Euclidean two sphere \( S^2 \), then \( D_2(x, x') \) will have to be defined by the inverse of \( \Box \) on the orthogonal complement to its kernel, the \( \delta^2 \) function on the right side of (3.9) being appropriately modified. Using the Green’s function, \( D_2 \) we obtain from (3.4),

\[ \sigma(x) = \frac{1}{2} \int d^2 x' D_2(x, x') (\sqrt{-g} R - \sqrt{-g} \bar{R})_{x'}. \] (d = 2).

(3.10)

This is a formal solution of the Poincare uniformization problem in two dimensions, since the metric \( \tilde{g}_{ab} \) may be chosen as that of constant scalar curvature \( \bar{R} \), and then (3.10) specifies the conformal factor needed to bring an arbitrary metric \( g_{ab} \) (with the same fixed Euler number as \( \tilde{g}_{ab} \)) to the metric with uniform scalar curvature [25].

Substituting the solution for \( \sigma \) (3.10) into (3.3) and using the fact that \( D_2(x, x') = \bar{D}_2(x, x') \) is Weyl invariant (owing to the Weyl invariance of \( \sqrt{-g} \bar{R} \)) gives

\[ \Gamma[\tilde{g}; \sigma] = \frac{1}{4} \int d^2 x \sqrt{-g} \int d^2 x' \sqrt{-g'} R(x) D_2(x, x') R(x') - \frac{1}{4} \int d^2 x \sqrt{-g} \int d^2 x' \sqrt{-g'} \bar{R}(x) \bar{D}_2(x, x') \bar{R}(x'). \] (3.11)

This difference form shows again that \( \Gamma[\tilde{g}; \sigma] \) is the Weyl variation \( \Delta_\sigma \) of a scalar action functional (although now non-local) in the physical dimension \( d = 2 \), which accounts for it automatically satisfying the WZ condition. Clearly, the multi-valuedness of this functional (3.3) is related to its non-locality by the fact that it depends on \( \sigma \) directly (rather than \( e^{2\sigma} \)) and eliminating \( \sigma \) by (3.11) introduces the non-local Green’s function \( D_2(x, x') \). This example illustrates how the non-trivial cohomology of the Weyl group is associated with non-local action functionals in the physical dimension, despite the fact that we began with a local functional in (3.1) defined in \( d \) dimensions.

The last term in (3.11) is precisely the \( \sigma \) independent term required to bring \( \Gamma \) into the fully covariant but non-local form. Defining \( \Gamma_{WZ} \) by multiplying \( \Gamma \) by the correct normalization factor of \( -Q^2/4\pi \equiv c/24\pi \), corresponding to the standard two-dimensional trace anomaly coefficient \( c_m = N_s + N_f \) for \( N_s(N_f) \) free massless scalar (fermion) matter fields, we obtain

\[ \Gamma_{WZ}[\tilde{g}; \sigma] = \Delta_\sigma \circ S_{anom}[\tilde{g}] = S_{anom}[g] - S_{anom}[\tilde{g}], \] (3.12)

8
with

\[ S_{anom}[\sigma] = -\frac{Q^2}{16\pi} \int d^2 x \sqrt{-g} \int d^2 x' \sqrt{-g'} R(x) D_2(x, x') R(x') , \tag{3.13} \]

which is the fully covariant non-local form of the Polyakov action in two dimensions. The total central charge is \( c = c_m - 26 + 1 \) when the effects of ghosts and the \( \sigma \) field itself are included in the trace anomaly coefficient and \( Q^2 = (25 - c_m)/6 \).

In this way the non-local WZ consistent effective action \([3.13]\) corresponding to the trace anomaly in two dimensional spacetime can be constructed from the counterterm in dimensional regularization near \( d = 2 \) dimensions. Since

\[ \int d^2 x \sqrt{-g} R = 4\pi \chi \]

is the unique integral Weyl invariant in two dimensions, there are no local UV counterterms of dimension two which can be added to the effective action and \([3.13]\) cannot be removed or altered by any UV regulator. On the contrary, the anomaly calculation shows that its coefficient is determined by the number of massless excitations in the far infrared, which is a genuine feature of the low energy theory of 2D gravity, independent of any UV regulator. This physical meaning of the non-trivial cohomology of the Weyl group in two dimensions through the non-local IR effective action \([3.13]\) and its properties under global rescalings has been verified by simplicial lattice simulations \([23, 29]\).

Corresponding to the covariant non-local effective action \([3.13]\) there is a conserved energy-momentum tensor. The most rapid route to deriving this tensor is to vary the local form of the WZ action with respect to the background metric \( \bar{g} \).

Finally we note that since the solution for \( \sigma \), \((3.10)\) can be written as the difference of two terms evaluated on the metrics \( g_{ab} \) and \( \bar{g}_{ab} \), respectively, we can introduce this solution into \((3.14)\). Using the identities of Appendix A relating the covariant derivatives with respect to the two metrics, we find that the mixed terms cancel and the stress tensor may be written as the difference of two tensors, \( T_{ab}^{WZ} = T_{ab}^{(anom)}[\bar{g}] - T_{ab}^{(anom)}[g] \), each of which is a non-local function of a single metric. Indeed,

\[ T_{ab}^{(anom)}[g] \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{anom}[\sigma]}{\delta g^{ab}} = \frac{Q^2}{4\pi} \left[ -\nabla_a \nabla_b \varphi + g_{ab} \Box \varphi - \frac{1}{2} (\nabla_a \varphi)(\nabla_b \varphi) + \frac{1}{4} g_{ab} (\nabla_c \varphi)(\nabla^c \varphi) \right] , \tag{3.16} \]

where

\[ \varphi(x) \equiv \int d^2 x' D_2(x, x') \sqrt{-g'} R' \quad (d = 2) . \tag{3.17} \]

It is easily checked that the non-local tensor \( T_{ab}^{anom}[g] \) is conserved by virtue of the vanishing of the Einstein tensor in \( d = 2 \) dimensions, and that it has the trace,

\[ g^{ab} T_{ab}^{anom} = \frac{Q^2}{4\pi} \Box \varphi = -\frac{Q^2}{4\pi} R = \frac{c}{24\pi} R , \tag{3.18} \]

which is the local trace anomaly in two dimensions.
IV. THE WZ ACTION IN FOUR DIMENSIONS

In $d = 4$ spacetime dimensions there are four local scalar functions of the metric with dimension four, viz. $R^2$, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$, and $\Box R$. The last of these is a total derivative in any number of dimensions, so it gives no volume contribution when integrated. Thus there are just three possible counterterms for the dimensionally continued effective action for gravity near four dimensions. However, the existence of the total derivative $\Box R$ indicates a new feature absent in $d = 2$, since $\Box R$ can appear in the trace anomaly but disappears from the volume effective action. This is associated with the existence of a trivial cocycle in four dimensions and leads to one constraint between the four possible terms in the trace anomaly $\Box R$.

Let us define the two linear combinations in $d$ near equal to four dimensions,

$$E_d = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 + \frac{(d-4)}{18}R^2; \quad \text{and}$$

$$F_d = [C_{abcd}C^{abcd}]_d = R_{abcd}R^{abcd} - \frac{4}{d-2}R_{ab}R^{ab} + \frac{2}{(d-1)(d-2)}R^2$$

which together with $R^2$ form a basis for the three remaining independent scalar invariants in the effective action. At $d = 4$, $E_4$ is the integrand of the Gauss-Bonnet-Euler topological invariant, analogous to $R$ at $d = 2$, while $F_4$ is the Weyl tensor squared, a local Weyl invariant. Each of these is Weyl invariant in $d = 4$ when integrated over all space. Thus, each of these two terms will generate a non-trivial cocycle of the Weyl group in $d = 4$. The addition of the $R^2$ term with the particular coefficient $(d-4)/18$ in (4.1a) adds a particular admixture of the trivial cocycle in defining the $E_d$ invariant away from $d = 4$, chosen with a view ahead to simplify the $d \to 4$ limit. We do not add any such term to $F_d$ since it already transforms as a local density of weight 4 under the Weyl group, and

$$\sqrt{-g}F_d = \sqrt{-g}e^{(d-4)\sigma} \mathcal{T}_d = \sqrt{-g} \mathcal{T}_d + (d-4)\sigma\sqrt{-g}\mathcal{T}_d + \mathcal{O}(d-4)^2;$$

becomes a local Weyl invariant in exactly $d = 4$ dimensions.

Our general algorithm for the construction of the WZ consistent effective action over the two non-trivial cocycles now requires that we evaluate

$$\Gamma_{WZ}[\bar{g}; \sigma] = b \lim_{d \to 4} \left\{ \int \frac{d^d x}{d-4} \sqrt{-g} F_d - \int \frac{d^d x}{d-4} \sqrt{-g} \mathcal{T}_d \right\} + b' \lim_{d \to 4} \left\{ \int \frac{d^d x}{d-4} \sqrt{-g} E_d - \int \frac{d^d x}{d-4} \sqrt{-g} \mathcal{E}_d \right\},$$

with arbitrary coefficients $b$ and $b'$. Expanding the simple transformation law (4.2) to linear order in $d-4$ immediately gives the form of the first non-trivial cocycle, namely

$$b \lim_{d \to 4} \left\{ \int \frac{d^d x}{d-4} \sqrt{-g} F_d - \int \frac{d^d x}{d-4} \sqrt{-g} \mathcal{T}_d \right\} = b \int d^d x \sqrt{-g} F_4 \sigma,$$

which is linear in $\sigma$.

The algebra required to compute the second cocycle in (4.3) is somewhat more tedious. The necessary relations are cataloged in Appendix A. To simplify the task one may note first that

$$E_d = F_d - [2 + (d-4)] \left( R_{ab}R^{ab} - \frac{1}{3}R^2 \right) + \mathcal{O}(d-4)^2,$$

so that the only non-trivial quantity whose $\sigma$ dependence we need near four dimensions is $R_{ab}R^{ab} - \frac{1}{3}R^2$. Using

$$R_{ab} = \mathcal{R}_{ab} - (d-2)(\sigma_{ab} - \sigma_a\sigma_b + \bar{g}_{ab}\sigma^c\sigma_c) - \bar{g}_{ab}\sigma^c_{\cdot c},$$

and (3.2) we show in Appendix A that

$$\int d^d x \sqrt{-g} \left( R_{ab}R^{ab} - \frac{1}{3}R^2 \right) = \int d^d x \sqrt{-g} \left( \mathcal{R}_{ab} \mathcal{R}^{ab} - \frac{1}{3} \mathcal{R}^2 \right)$$

$$+ (d-4) \int d^d x \sqrt{-g} \sigma \left( \mathcal{R}_{ab} \mathcal{R}^{ab} - \frac{1}{3} \mathcal{R}^2 + \frac{1}{3} \Box \mathcal{R} \right) - (d-4) \int d^d x \sqrt{-g} \sigma \Delta_4 \sigma + \mathcal{O}(d-4)^2,$$
up to surface terms which we systematically neglect. In this expression

\[ \Delta_4 = \Box^2 + 2R^{ab}\nabla_a \nabla_b + \frac{1}{3}(\nabla^a R)\nabla_a - \frac{2}{3}R\Box \]  

(4.8)
is the fourth order scalar operator satisfying the Weyl invariance property in four dimensions,

\[ \sqrt{-g}\Delta_4 = \sqrt{-g}\Delta_4 \quad (d = 4) , \]  

(4.9)

analogous to (3.8) in two dimensions. The cancellation of all terms cubic and quartic in \( \sigma \) which \( a \ priori \) could appear in the last line of (4.7) is noteworthy. Using (4.7) we have

\[
\int d^d x \sqrt{-g} E_d = \int d^d x \sqrt{-g} F_d - [2 + (d - 4)] \int d^d x \sqrt{-g} \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) + \mathcal{O}(d - 4)^2
\]

\[
= \int d^d x \sqrt{-g} F_d + (d - 4) \int d^d x \sqrt{-g} F_d \sigma - [2 + (d - 4)] \int d^d x \sqrt{-g} \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right)
\]

\[-2(d - 4) \int d^d x \sqrt{-g} \left( R_{ab} R^{ab} - \frac{1}{3} R^2 + \frac{1}{3} \Box R \right) \sigma + 2(d - 4) \int d^d x \sqrt{-g} \sigma \Delta_4 \sigma + \mathcal{O}(d - 4)^2
\]

\[
= \int d^d x \sqrt{-g} E_d + (d - 4) \int d^d x \sqrt{-g} \left\{ E_d - \frac{2}{3} \Box R \right\} \sigma + 2 \sigma \Delta_4 \sigma \right\} + \mathcal{O}(d - 4)^2.
\]  

(4.10)

Therefore, neglecting possible surface terms we find that the second term in (4.3) becomes

\[
b' \lim_{d \to 4} \left\{ \int d^d x \sqrt{-g} E_d - \int d^d x \sqrt{-g} E_d \right\} = b' \int d^d x \sqrt{-g} \left\{ \left( E_4 - \frac{2}{3} \Box R \right) \sigma + 2 \sigma \Delta_4 \sigma \right\} ,
\]  

(4.11)

and the general element of the non-trivial cohomology of the Weyl group in four dimensions is given by

\[
\Gamma_{WZ}[\tilde{g}; \sigma] = b \int d^d x \sqrt{-g} F_4 \sigma + b' \int d^d x \sqrt{-g} \left\{ E_4 - \frac{2}{3} \Box R \right\} \sigma + 2 \sigma \Delta_4 \sigma \right\} .
\]  

(4.12)

Notice that this construction of \( \Gamma_{WZ} \) contains only terms up to quadratic order in \( \sigma \), which was arranged by the addition of the local \( R^2 \) term with the particular coefficient \( (d - 4)/18 \) in (4.14).

In exactly \( d = 4 \) dimensions by using the invariance property (4.13) of the self-adjoint hermitian differential operator \( \Delta_4 \), it is easily checked that the Weyl one-form \( \Gamma_{WZ}[\tilde{g}; \sigma] \) satisfies the WZ consistency condition, as was first shown in ref. [12]. In the present treatment this follows immediately from the construction of (4.3) as the limit of a \( d \) dimensional exact form. The \( \sigma \) variation of \( \Gamma_{WZ}[\tilde{g}; \sigma] \),

\[
\frac{\delta \Gamma_{WZ}[\tilde{g}; \sigma]}{\delta \sigma} = b \sqrt{-g} F_4 + b' \sqrt{-g} \left\{ \left( E_4 - \frac{2}{3} \Box R \right) \sigma + 4 \sigma \Delta_4 \sigma \right\}
\]

\[
= b \sqrt{-g} F_4 + b' \sqrt{-g} \left( E_4 - \frac{2}{3} \Box R \right),
\]  

(4.13)

is the non-trivial conformal trace anomaly of massless quantum matter fields in the full metric, \( g_{ab} = \tilde{g}_{ab} \exp(2\sigma) \), as is also easily checked using the conformal variation formulae derived in Appendix A, since

\[
\sqrt{-g} \left( E_4 - \frac{2}{3} \Box R \right) = \sqrt{-g} \left( E_4 - \frac{2}{3} \Box R \right) + 4 \sqrt{-g} \Delta_4 \sigma .
\]  

(4.14)

Using this relation for \( g_{ab} \) and \( \tilde{g}_{ab} \) interchanged and \( \sigma \to -\sigma \) is the most immediate way of proving the invariance property (4.9). The independent construction of \( \Gamma_{WZ}[\tilde{g}; \sigma] \) by the dimensional continuation limiting process (4.13) explains why the action obtained by integration of the trace anomaly by undoing the variation (4.13) gives a WZ consistent effective action for the local conformal factor of the metric in four dimensions [12].

As in two dimensions, the fact that \( \Gamma_{WZ}[\tilde{g}; \sigma] \) is a simple second order polynomial in \( \sigma \) means that it cannot be written as a single-valued local functional of the full metric \( g_{ab} \), although its \( \sigma \) variation can be through (4.13). One can again consider the complex transformation (4.6) and observe that

\[
\Gamma_{WZ}[\tilde{g}; \sigma] \to \Gamma_{WZ}[\tilde{g}; \sigma] + i\pi q b \int d^d x \sqrt{-g} F_4 + 32\pi^3 i q b' \tilde{\chi}_E ,
\]  

(4.15)
so that neither term is single valued. Replacing $i\pi q$ by a constant real shift $\sigma_0$ shows that this multi-valuedness of the action $\Gamma_{WZ}[g; \sigma]$ associated with its non-trivial cohomology necessarily implies its sensitivity to global Weyl rescalings of the metric. Neither term of the non-trivial cocycle is invariant under global dilations, contrary to what is claimed in ref. [13] for the type A anomaly, and both terms in the WZ consistent effective action arise from the same mechanism through a cancellation of the pole in dimensional regularization. It is also noteworthy that the admixture of the $\Box R$ term in the second cocycle drops out of the global dilation (4.15), since it is a total derivative. The sensitivity to finite volume scaling comes only from the Weyl invariants $E_4$ and $E_4$ in four dimensions.

Further, we can exhibit the non-local but fully covariant form of the WZ effective action by introducing the Green’s function of the fourth order Weyl covariant differential operator (4.8). Defining this Green’s function, $D_4$ by

$$\sqrt{-g} \Delta_4 D_4(x, x') = \delta^4(x, x'),$$

with the same qualifying remarks as follow (3.9), allows us formally to invert the relation (4.14) for $\sigma$ to obtain

$$\sigma(x) = \frac{1}{4} \int d^4x' D_4(x, x') \left[ \sqrt{-g} \left( E_4 - \frac{2}{3} \Box R \right) - \sqrt{-g} \left( E_4 - \frac{2}{3} \Box R \right) \right]_{x'}.$$

This is a formal solution to the four dimensional uniformization problem of bringing an arbitrary metric to one with constant $E_4 - \frac{4}{3} \Box R$ by a local Weyl conformal transformation. We remark that although the Poincare-Yamabe conjecture has been proven in two dimensions [12], this four dimensional uniformization conjecture (namely, that the $\sigma$ of our formal inversion of $\Delta_4$ exists and is unique) has not been proven. However, the conformal property of $\Delta_4$ suggests that it should be possible to generalize the two dimensional case in this way.

Substituting (4.17) into (4.12) and using $D_4(x, x') = D_4(x, x')$ shows that $\Gamma_{WZ}[g; \sigma]$ can be written explicitly as a difference of non-local actions,

$$\Gamma_{WZ}[\bar{g}; \sigma] = \Delta_\sigma \circ S_{\text{anom}}[\bar{g}] = S_{\text{anom}}[\bar{g}] - S_{\text{anom}}[\bar{g}],$$

with

$$S_{\text{anom}}[g] = \frac{b}{4} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g'} F_4 D_4(x, x') \left( E_4 - \frac{2}{3} \Box R \right)' + \frac{b'}{8} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g''} \left( E_4 - \frac{2}{3} \Box R \right) D_4(x, x') \left( E_4 - \frac{2}{3} \Box R \right)'.$$

If we introduce the auxiliary field $\varphi$ defined in $d = 4$ by

$$\varphi(x) = \frac{1}{2} \int d^4x' D_4(x, x') \sqrt{-g'} \left( E_4 - \frac{2}{3} \Box R \right)' \quad (d = 4),$$

then the non-local action may be written in the form,

$$S_{\text{anom}}[g] = \frac{b}{2} \int d^4x \sqrt{-g} F_4 \varphi - \frac{b'}{2} \int d^4x \sqrt{-g} \left[ \varphi \Delta_4 \varphi - \left( E_4 - \frac{2}{3} \Box R \right) \varphi \right] \equiv S_{\text{anom}}^{(F)} + S_{\text{anom}}^{(E)}.$$  

The result (4.19) corrects a misprint in eq. (2.14) of ref. [20].

In this way the non-local WZ consistent covariant effective action corresponding to the non-trivial cohomology of the Weyl group in four dimensions may be constructed directly from Weyl invariant counterterms in dimensional regularization. The trivial cocycle corresponds to the Weyl non-invariant local action,

$$S_{\text{local}}^{(4)}[g] = - \frac{(2b + 2b' + 3b'')}{18} \int d^4x \sqrt{-g} R^2,$$

which conforms to Duff’s parameterization of the three independent trace anomaly coefficients [3], namely

$$T_a^a = A_4 = \frac{1}{\sqrt{-g} \delta \sigma} \left( S_{\text{anom}} + S_{\text{local}}^{(4)} \right) = b \left( F_4 + \frac{2}{3} \Box R \right) + b' E_4 + b'' \Box R.$$  

Whereas the coefficients $b$ and $b'$ are determined by the number of massless fields of each spin, the $b''$ coefficient is scheme dependent, as we expect for the true UV counterterm $\int d^d x \sqrt{-g} R^2$ that has non-vanishing local Weyl variation in the physical dimension $d = 4$ and no cancellation of the $(d - 4)^{-1}$ pole multiplying it.
The local $R^2$ invariant which is a trivial cocycle of the Weyl group differs from the non-trivial $F$ and $E$ cocycles in another way. As we have seen, these are sensitive to global Weyl rescalings $\sigma \to \sigma + \sigma_0$ in the physical dimension, due to their multi-valuedness under the complex periodic transformation of the metric $\sigma \to \sigma + i\pi q$. This means that the non-trivial cocycles scale non-trivially as the finite volume of the system is scaled. However, the trivial $R^2$ cocycle is single valued under the same global transformation. Indeed,

$$\sqrt{-g} \ R^2 = \sqrt{-g} \ (\mathbf{R} - 6 \sigma - 6 \sigma_0 \sigma^0)^2 \quad (d = 4),$$

so its $\sigma$ dependence enters purely through derivatives, and its integral is invariant under rigid global rescalings in the physical dimension. Hence it is insensitive to finite volume rescalings, as one would expect for a term relevant in the UV but irrelevant in the IR. The rigid dilation invariance has associated with it a Noether current $J^a$, whose divergence $\nabla_a J^a$ is proportional to the total derivative $\mathbf{R}$ in the anomaly. Since the integral of the anomaly $\int d^4 x \sqrt{-g} A_4$ is nothing but the global Weyl variation of the WZ effective action, in order to have vanishing contribution to this global anomaly and therefore insensitivity to global Weyl rescalings, trivial cocycles must yield total divergences in $A_4$, in distinction to both the non-trivial $F$ and $E$ cocycles. This clear separation of behavior under global dilations shows that the trivial and non-trivial cocycles of the Weyl group are associated with UV and IR physics respectively.

We can proceed further to deduce a general property of the effective action for gravity by classifying the behavior under the Weyl group. Since $\Gamma_{WZ}$ satisfying WZ consistency is unique up to an arbitrary admixture of local trivial cocycles $\Delta_\sigma \circ S_{local}$ and $\Gamma_{WZ}$ itself can be written as a finite Weyl shift on an anomalous action, as in (4.18), the only possible additions to $S_{anom}$ are local terms or arbitrary (generally non-local) but Weyl invariant terms, $S_{inv}$ which drop out of difference (4.18). That is, the full effective action of any covariant theory must be of the form,

$$S_{eff}[g] = S_{local}[g] + S_{inv}[g] + S_{anom}[g],$$

where

$$\Delta_\sigma \circ S_{inv} = 0.$$

In addition to (4.22) the $S_{local}$ in this expression can contain local terms of both higher and low dimension than four, multiplied by coefficients with negative or positive mass dimensions, respectively. The higher dimension terms are strictly irrelevant in the IR, since they scale to zero with negative powers of $e^{\sigma_0}$ and may be neglected for physics far below the Planck scale, while the lower dimension local terms are nothing but the terms of the usual Einstein-Hilbert classical action, i.e.

$$S_{local}[g] = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} (R - 2\Lambda) + S^{(4)}_{local} + \sum_{n=3}^{\infty} S^{(2n)}_{local}. \quad (4.27)$$

The classical terms grow as positive powers of $e^{\sigma_0}$ under global dilations and are clearly IR relevant terms. The term (4.22) is the only allowed dimension four, Weyl non-invariant local term. The local dimension four term involving the Weyl tensor squared is Weyl invariant and among the many terms that can appear in $S_{inv}$. Because both of these are neutral under global dilations we expect them to be marginally irrelevant in the IR (conversely, marginally relevant in the UV). All the higher dimension local terms in the sum in (4.27) for $n \geq 3$ scale to zero as $\sigma_0 \to \infty$ and are clearly strictly IR irrelevant. Any non-local, Weyl non-invariant terms generate non-trivial cocycles of the Weyl group. If there are only two non-trivial cocycles in $d = 4$, then the most general non-local, Weyl non-invariant action is given by $\mathcal{L}$. These scale linearly with $\sigma_0$ or equivalently, logarithmically with length or volume rescalings. Although the precise matter content of the massless fields which are integrated out to obtain the effective gravitational action influence the values of the $b$ and $b'$ coefficients, the form of $S_{anom}$ is the completely general solution to the non-trivial cocycle action in four dimensions and its response under global scale transformations cannot be changed by local terms or Weyl invariant terms.

Since the complete classification of terms in the effective action according to their response under the Weyl group allows only local or completely Weyl invariant terms to be added to $S_{anom}$, according to (4.23), we can conclude definitively that there is no non-local term in $S_{eff}$ of the form,

$$\int d^4 x \sqrt{-g} C_{abcd} \log (H_{\mu^2}) C^{abcd},$$

as has long been conjectured (1.13, 1.18). Indeed, such a term has no simple transformation properties under the local Weyl group and its local Weyl variation does not generate either the $b$ or $b'$ term in the local anomaly (4.23). Although a plethora of complicated non-local terms are generated in the full effective action of a quantum field theory in curved
spacetime, their Weyl non-invariant contributions which are not either absorbable into redefinitions of the coupling constants in the relevant parts of $S_{\text{local}}, n = 0, 1$ or strictly irrelevant, $(n \geq 3$ in the expansion (4.27) are completely determined by the trace anomaly of the renormalized energy-momentum tensor [31]. Therefore the effective action must always contain as one piece, $S_{\text{anom}}[g]$, if its variation is to yield the correct local $F_4$ and $E_4$ trace anomalies in its conformal limit, and these non-trivial Weyl cocycle terms cannot be removed or altered by the addition of local terms. Since the term (4.28) is neither local nor Weyl invariant, and it does not reduce to the terms in the non-trivial cocycle that produce the correct local Weyl anomaly (by construction), it cannot appear in the effective action in a general background metric (although it may mimic some effects of the correct $S_{\text{anom}}$ in the weak field expansion around flat space). Although this conclusion and the decomposition of the general form of the full low energy effective action of gravity in four dimensions (4.25) has been reached after lengthy calculations from the form of the heat kernel expansion of the quantum effective action [31], in fact (4.25) follows only from general covariance and the classification of terms in the effective action according to their behavior under the Weyl group (2.1).

As in $d = 2$ the appearance of a locally Weyl covariant scalar differential operator $\Delta_4$ is a necessary feature of the non-trivial cohomology. It cannot be removed by the addition of local terms. The propagator of a conformal differential operator $D_2$ or $D_3$ is a logarithm in coordinate space in any number of dimensions. Its IR properties in position space do not become any worse in higher dimensions than in $d = 2$. In momentum space there is nothing unphysical about either $\Gamma_{WZ}$ or $S_{\text{anom}}$, despite the appearance of the fourth order differential operator. Although the apparent $p^{-4}$ pole produced some premature concern [3], more careful attention to all the powers of momentum in the numerator of flat space amplitudes shows that this concern is baseless and that $S_{\text{anom}}$ is fully consistent with conformal Ward identities in flat space [32]. This certainly has to be the case without detailed calculation, since the WZ consistent effective action is nothing but the generating functional of precisely these conformal Ward identities. Any new non-local terms of the kind suggested recently in [32] are both not needed and inconsistent with the general form (4.25).

To conclude this section we remark also that canonical quantization of the quadratic WZ action $\Gamma_{WZ}$ on the cylindrical background Einstein space, $R \times S^3$ shows that there are no propagating ghost states which survive imposition of the constraints of diffeomorphism invariance. Instead the physical Hilbert space of the pure WZ theory (i.e. with the local terms $S_{\text{local}}$ set to zero) consists of a particular global mode of the $S^3$ with a discrete spectrum labelled by a single integer [25]. This is exactly what one would expect of a theory with a single IR degree of freedom, but no local UV degrees of freedom (ghost or otherwise) as occur in local higher derivative gravity theories with $C_{abcd} C_{abcd}$ or $R^2$ actions. Those local higher derivative actions certainly give rise corrections to Einstein gravity at the Planck scale, where the entire low energy effective action approach breaks down. The non-trivial cocycle behavior very differently, showing its IR character which is quite distinguishable from ultra-short distance behavior in the canonical quantization approach. Finally, in the far IR the WZ effective theory is stable to marginal deformations by the local $R^2$ term, i.e. the a priori arbitrary coefficient,

$$[b'' + \frac{2}{3}(b + b')]_{1,\kappa} = 0,$$

(4.29)

is a stable IR fixed point under RG flow induced by the non-trivial $\Delta_4$ operator in the Gaussian effective action $\Gamma_{WZ}$ [20]. Hence, the coefficient of $S_{\text{local}}^{(4)}$ in (4.22) flows to zero logarithmically at large distances, again just as one would expect for a UV marginally relevant but IR marginally irrelevant operator at a Gaussian fixed point.

V. THE ENERGY-MOMENTUM TENSOR OF $\Gamma_{WZ}$ IN FOUR DIMENSIONS

We consider next the conserved energy-momentum tensors corresponding to the non-local effective actions of the two non-trivial cocycles in $d = 4$. Using the identities in Appendix A, the tensor obtained by varying to the first (b term) in $\Gamma_{WZ}$ is

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^4 x \sqrt{-g} F_4 = 2 \partial_a \partial_b F_{cd} \sigma + 4 \nabla_c \nabla_d \left( \partial_{(a} C_{b) d} \right).$$

(5.1)

It can be verified that the same stress tensor is obtained by first varying the $b$ term in the $d$ dimensional anomalous action and then taking the limit $d \to 4$. Indeed,

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int d^d x \sqrt{-g} F_{ab} \equiv F_{ab} = H_{ab} - \frac{4}{d-2} (^{(2)} H_{ab} + \frac{2}{(d-1)(d-2)} (^{(1)} H_{ab}) = 2 C_{cde} C_b^{cde} - \frac{g_{ab}}{2} C_{cdef} C^{cdef} + \frac{4}{d-2} C_{a}^e R_{bde} + 4 \nabla_c \nabla_d C_{(a} C_{b)}. \quad (5.2)$$
where the three tensors $H_{ab}$, (1) $H_{ab}$ and (2) $H_{ab}$ are defined in \[[A11]\] of Appendix A. The first two terms in the last expression vanish in $d = 4$ dimensions (cf. Appendix B). However, in $d \neq 4$ dimensions this tensor is non-zero. Let us define

$$C_{ab} \equiv \lim_{d \to 4} \left\{ \frac{g_{ab} C_{cdef} C^{cdef} - 4 C_{ab} C_{cdef} C_{b}^{cdef}}{d - 4} \right\} .$$

(5.3)

Evidently this tensor is non-trivial since its trace,

$$g^{ab} C_{ab} = C_{cdef} C^{cdef}$$

(5.4)

is non-vanishing. Its $\sigma$ dependence in any dimension is very simple, due to the conformal transformation property of the Weyl tensor,

$$C^{\alpha}_{\beta c d} = C^{\beta a}_{\alpha c d} .$$

(5.5)

Using \[[A18]\] of the Appendix for the $\sigma$ dependence of the remaining terms in \[[5.2]\], we find

$$\begin{align*}
&\ e^{(d-2)\sigma} F_{ab} - \mathcal{T}_{ab} = \\
&= 4(d - 4) \left[ C_{ab} \right] + (d - 4) C_{ab} \sigma d d + \frac{1}{2} \sigma C_{ab} \right] \mathcal{R}_{bd} + \sigma \mathcal{D} C_{ab} \mathcal{D} C_{ab} + O(d - 4)^2 .
\end{align*}$$

(5.6)

The factor of $e^{(d-2)\sigma}$ in the first term is a consequence of the fact that \( \sqrt{-g} \delta g^{ab} = e^{(d-2)\sigma} \sqrt{-g} \delta g^{ab} \) in the variation holding $\sigma$ fixed, and the variation of the $C_{ab}$ term vanishes to linear order in $d - 4$. Dividing the difference of tensors \[[5.8]\] by $d - 4$ and taking the limit $d \to 4$ reproduces \[[5.3]\].

Since \[[5.1]\] is linear in $\sigma$, substituting the solution for $\sigma$ \[[1.17]\],

$$\sigma = \frac{1}{2} (\varphi - \bar{\varphi}) ,$$

(5.7)

with the previous definition of the auxiliary field $\varphi$, \[[1.20]\] and $\bar{\varphi}$ defined similarly with all terms evaluated at the metric $g_{ab}$ into \[[2.1]\] yields the difference,

$$\left[ C^{a b}_{c d} R_{c d} \varphi + 2 \mathcal{D} C_{a b} \mathcal{D} \varphi \right] - \left[ C^{a b}_{c d} R_{c d} \bar{\varphi} + 2 \mathcal{D} C_{a b} \mathcal{D} \bar{\varphi} \right] .$$

(5.8)

Taking into account the $\sigma$-independent contribution from the first two terms in \[[5.2]\], we see that the energy-momentum tensor obtained by varying the first cocycle (b term) in the non-local anomalous action \[[4.19]\] is

$$\left[ F(T) \right]_{ab} = - \frac{2}{\sqrt{-g}} \delta g^{ab} S_{\text{anom}} \left[ g \right] = b C_{ab} - 2 b \left[ C^{a b}_{c d} R_{c d} \varphi + 2 \mathcal{D} C_{a b} \mathcal{D} \varphi \right] ,$$

(5.9)

in $d = 4$ dimensions, where now all indices are raised and lowered with the single metric $g_{ab}$, and the correct normalization has been restored. An explicit form for the non-local tensor $C_{ab}$ appearing in this expression and defined by \[[5.3]\] cannot be obtained from the $\sigma$ dependence of the WZ action, but requires varying the non-local action $S_{\text{anom}} \left[ g \right]$ directly. Comparison of \[[5.9]\] with \[[3.21]\] shows that $C_{ab}$ comes from the variation of the auxiliary $\varphi$ field itself. Since the second term in \[[5.9]\] is traceless, the trace $g^{ab} (F) T_{ab} [g] = b C_{cdef} C^{cdef}$ comes from the $C_{ab}$ term alone. An explicit form for $C_{ab}$ is given in Appendix C.

For the second cocycle the direct variation of the four dimensional WZ action was calculated in ref. \[[25]\]. To check this result we may evaluate

$$\begin{align*}
& \frac{1}{\sqrt{-g}} \delta g^{ab} \int d^4 x \sqrt{-g} E_d \equiv E_{ab} = H_{ab} - 4 (2) H_{ab} + (1) H_{ab} \\
& = 2 C_{acde} C_{b}^{cdef} - \frac{g_{ab}}{2} C_{cdef} C^{cdef} + (d - 4) \left[ (3) H_{ab} + \frac{1}{18} (1) H_{ab} \right] ,
\end{align*}$$

(5.10)

where

$$\begin{align*}
(3) H_{ab} & \equiv - \frac{4}{d - 2} C_{acde} R^{ced} + \frac{2(d - 3)}{(d - 2)^2} R_{ced} R^{ced} g_{ab} - \frac{4(d - 3)}{(d - 2)^2} R_{a}^{c} R_{b}^{d} \\
& + \frac{2d(d - 3)}{(d - 1)(d - 2)^2} RR_{ab} - \frac{(d + 2)(d - 3)}{2(d - 1)(d - 2)^2} R^2 g_{ab} .
\end{align*}$$

(5.11)
is a generalization of the tensor $^{(3)} H_{ab}$ in four dimensional conformally flat spacetimes \cite{30}. Because $E_4$ is the Euler density, the tensor $E_{ab}$ vanishes identically in $d = 4$, just as the Einstein tensor $G_{ab}$ does in $d = 2$.

The same combination of Weyl squared tensors appears in \cite{5.10} as in \cite{6.2}, with its simple $e^{-2\sigma}$ dependence on $\sigma$. Considering the explicit factor of $d - 4$ in the remaining terms of \cite{5.10}, the stress tensor corresponding to the second cocycle of the WZ action in $d = 4$ dimensions is

$$
(E) T_{ab}[g] = -\frac{2\delta}{\sqrt{-g} g^{ab}} S_{anom}^{(E)}[g] = b' C_{ab} - 2b' \left[^{(3)} H_{ab} + \frac{1}{18}(1) H_{ab} \right]_{d=4}
$$

$$
= b' \left[ C_{ab} + 4C_{abcd} R^{cd} + \frac{2}{9}(\nabla_a \nabla_b R - g_{ab} \Box R - 7R R_{ab}) + \frac{5}{9}g_{ab}R^2 + 2R_{a}^{\;e} R_{bc} - g_{ab} R^{cd} R_{cd} \right]. \tag{5.12}
$$

The conformal variation of this tensor is obtained by computing

$$
e^{2\sigma} \left[^{(3)} H_{ab} + \frac{1}{18}(1) H_{ab} \right] - \left[^{(3)} \ddot{H}_{ab} + \frac{1}{18}(1) \ddot{H}_{ab} \right] = 4C^{\alpha \beta}_{\quad cd} (\sigma_{cd} - \sigma_c \sigma_d) + 4R^{(a}_{\quad (a} \sigma_{b)c} - 4R^{(a}_{\quad (a} \sigma_{b)c} + 8R^{ab} \Box \sigma
$$

$$
- \frac{2}{3} R_{ab} \sigma_c \sigma^c + \frac{2}{3} R^{(a}_{\quad (a} \sigma_{b)c} - \frac{4}{3} R \sigma_{ab} + \frac{2}{3} R \sigma_{ab} - 4\sigma_{a}^{\sigma \epsilon} \sigma_{b \epsilon} + 4\sigma_{ab} \Box \sigma + \frac{2}{3} (\Box \sigma)_{ab} + \frac{2}{3} (\sigma \sigma^{\epsilon})_{ab} - 4\sigma_{(a}(\Box \sigma)_{b)}
$$

$$+ \dot{g}_{ab} \left[ -2R^{cd} \sigma_{cd} + \frac{2}{3} R^{d} \sigma_{c} \sigma_{d} - \frac{1}{3} R \sigma^{2} \sigma + \frac{2}{3} \Box \sigma - \frac{2}{3} \Box^{2} \sigma + \frac{2}{3} \sigma \sigma^{\epsilon} \sigma_{cd} - (\Box \sigma)^{2} + \frac{2}{3} (\Box \sigma)_{x} \sigma^{x} \right]. \tag{5.13}
$$

A straightforward exercise in commuting covariant derivatives and use of the Bianchi identities shows that this tensor is precisely equal to the negative of that given by the right hand side of eq. \cite{2.9} of ref. \cite{25} (where the notation for the background metric $\bar{g}_{ab}$ omitted the overbar). In ref. \cite{25} this energy-momentum tensor was used to study the physical state Hilbert space of the quantized $\sigma$ field in the Einstein space background, $R \times S^3$.

Because of \cite{1.5} and \cite{3.4} the trace of the tensor $(E) T_{ab}[g]$ is

$$
g^{ab} (E) T_{ab}[g] = b' \left[ C_{cdef} C^{cdef} - 2R_{cd} R^{cd} + \frac{2}{3} R^2 - \frac{2}{3} \Box R \right] = b' \left( E_4 - \frac{2}{3} \Box R \right) \tag{5.14}
$$

in $d = 4$ dimensions.

Both the tensors $(F) T_{ab}[g]$ and $(E) T_{ab}[g]$ obtained by varying the non-local anomalous action in 4 dimensions are themselves non-local in general. However, if the spacetime is conformally flat then the Weyl tensor, $C_{ab}$ and $(F) T_{ab}[g]$ both vanish, while $(E) T_{ab}[g]$ becomes proportional to the local tensor $(3) H_{ab} + \frac{1}{18}(1) H_{ab}$. Since $(1) H_{ab}$ is the variation of the local $R^2$ action, it is conserved on its own, and we conclude that $(3) H_{ab}$ with $C_{abcd} = 0$ must be conserved in $d = 4$ conformally flat spacetimes. The conservation of this tensor in this case had been noted some time ago, \cite{1.20} and it has been called accidentally conserved.’ Evidently this local tensor which is conserved in conformally flat spacetimes owes its existence to the non-local anomalous action corresponding to the second cocycle of the Weyl group in four dimensions. The necessary appearance of $(3) H_{ab}$ in the renormalized stress tensor of a matter field in curved space after renormalization is explained by the contribution of $S_{anom}$ to the effective action with a finite coefficient, which as we have seen appears automatically in dimensional regularization \cite{37}. Because the action $S_{anom}$ is non-invariant under global Weyl rescalings, $(3) H_{ab}$ carries information about the global scaling behavior of a quantum theory in a state having this geometrical term as the expectation value of its energy-momentum tensor. This fact has been used to extract the IR $b'$ coefficient of a massless scalar field for any curvature coupling $\xi$ by examining the attractor behavior of its energy-momentum tensor at late times in de Sitter space, showing that $S_{anom}$ appears also in the effective action of theories which are not classically Weyl invariant \cite{24}.

VI. CONFORMAL INFINITY IN THE FG CONSTRUCTION

Since conformal invariants are central to obtaining the WZ effective action of the non-trivial cocycles of the Weyl group by the dimensional continuation method, we review in this section the FG procedure for constructing such conformal invariants in any dimension. Dimensional continuation is also the natural technique to handle a certain obstruction in the FG expansion at even integer dimensions \cite{6}. As discussed in the introduction, the FG embedding may be used to classify all the IR relevant terms in the effective action on the physical boundary space as well. Thus the treatment and applications of the FG construction in this paper will be somewhat different than that in the recent literature \cite{33,34}. 

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The FG approach is based upon generalization of the following elementary example of embedding of the round sphere $S^d$ in a flat Minkowski space of two dimensions higher. In this example the $d + 2$ dimensional space (called the ambient space) has signature $(d + 1, 1)$ and flat metric $\eta_{AB}$, i.e. its line element is

$$ds^2 = \eta_{AB} dX^A dX^B = -(dX^0)^2 + (dX^1)^2 + \ldots + (dX^{d+1})^2 \equiv -(dX^0)^2 + \vec{d}X \cdot \vec{d}X,$$

with the proper Lorentz isometry group $SO(d + 1, 1)$,

$$X^A = \Lambda^A_B X^B.$$

This group has three classes of orbits in the ambient space, namely

1. Degenerate: The future (or past) light cone, $\eta_{AB} X^A X^B = 0$, with $X^0 > 0$ (or $X^0 < 0$);
2. Lobachevski (Euclidean AdS): The upper (or lower) sheet of the two-sheeted hyperboloid, $\eta_{AB} X^A X^B = -\ell^2$, with $X^0 > 0$ (or $X^0 < 0$) and negative intrinsic scalar curvature, $R = -d(d + 1)/\ell^2$;
3. de Sitter: The single sheeted hyperboloid, $\eta_{AB} X^A X^B = +\ell^2$, with positive intrinsic scalar curvature, $R = d(d + 1)/\ell^2$.

In the first class of orbits the Lorentz group acts projectively, and the space of null directions on the future light cone is isomorphic to the sphere $S^d$. Indeed let

$$N^A \equiv \frac{X^A}{X^0} = (1, n^i), \quad i = 1, \ldots d + 1. \quad (6.3)$$

Then $\eta_{AB} N^A N^B = -1 + n^i n^i$ implies the vector $\vec{n}$ lies on the unit sphere $S^d$. Under the Lorentz transformation

$$n^i \to \tilde{n}^i = \frac{X^i}{X^0} = \frac{\Lambda^i_0 + \Lambda^i_j n^j}{\Lambda^0_0 + \Lambda^0_j n^j}. \quad (6.4)$$

Using $dX^i = X^0 dn^i + n^i dX^0$, the ambient metric $[6.1]$ restricted to the light cone is

$$ds^2 = (X^0)^2 d\vec{n} \cdot d\vec{n}. \quad (6.5)$$

Since the light cone is invariant under $SO(d + 1, 1)$ proper Lorentz transformations in the ambient spacetime, the projective transformation $[6.4]$ may be viewed as a local conformal transformation,

$$X^0 \to \tilde{X}^0 = \Omega(\vec{n}) X^0, \quad \Omega = e^\sigma = \Lambda^0_0 + \Lambda^0_i n^i \quad (6.6)$$

of $S^d$. Hence the conformal structure on the sphere $S^d$ may be investigated by studying the simpler Lorentz group in the ambient space.

On the second class of orbits, Lobachevski space, one may introduce the standard hyperbolic projective coordinates,

$$X^0 = \ell \left[ \frac{1 + y^2}{1 - y^2} \right], \quad X^i = \ell \frac{2y^i}{1 - y^2}, \quad y^2 \equiv \vec{y} \cdot \vec{y}, \quad (6.7)$$

so that the metric takes the standard Lobachevski form,

$$ds_L^2 = \frac{4\ell^2 d\vec{y} \cdot d\vec{y}}{(1 - y^2)^2}. \quad (6.8)$$

with the boundary at $y^2 = 1$. Alternately, one can introduce the coordinates

$$\rho = \frac{4}{\ell^2} \frac{(1 - y)^2}{(1 + y)^2}, \quad \vec{n} = \frac{\vec{y}}{y} \quad (6.9)$$

to bring the Lobachevski line element into the FG form,

$$ds_L^2 = \frac{\ell^2}{4} \left( \frac{d\rho}{\rho} \right)^2 \frac{1}{\rho} \left[ \left( 1 - \frac{\ell^2 \rho^2}{4} \right)^2 d\vec{n} \cdot d\vec{n} \right] = \frac{\ell^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} g_{ab}(x, \rho) dx^a dx^b, \quad (6.10)$$

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with $g_{ab}(x, 0)$ the round sphere metric on $S^d$, and $x^a, a = 1, \ldots, d$ coordinates on $S^d$. Thus the boundary of Lobachewski space at $\rho = 0$ is isomorphic to the light cone of the first orbit which it approaches asymptotically. This is the conformal boundary of $d + 1$ dimensional Lobachewski space, and in this elementary example of embedding the conformally flat $S^d$, the Lobachewski metric (called the bulk metric) $g_{ab}(x, \rho)$ possesses a regular Taylor expansion in $\rho$ which terminates at the second order, after the factor of $\rho^{-1}$ has been extracted as in (6.10). The usefulness of this coordinatization is that the conformal group of the boundary $S^d$ metric, $SO(d + 1, 1)$ at $\rho = 0$ has been represented as the isometry group of the bulk (and ambient) space, which makes possible the study of the conformal structure on the boundary by ordinary geometric properties both in the $d + 1$ bulk Lobachewski space, as well as the $d + 2$ ambient flat space. The Lorentzian signature AdS metric with isometry group $SO(d, 2)$ is obtained by changing the signature of one of the $X^i$ in the ambient Minkowski space metric (6.1).

The analogous construction works equally well in the case of the third class of orbits, namely de Sitter space. Indeed introducing the change of variables,

$$\rho = \frac{4}{\ell^2} \exp \left( -\frac{2\tau}{\ell} \right)$$

(6.11)

into a standard form of the de Sitter line element,

$$ds^2_{deS} = -d\tau^2 + \ell^2 \cosh^2 \left( \frac{\tau}{\ell} \right) d\vec{n} \cdot d\vec{n},$$

(6.12)

with closed $S^d$ spatial sections brings it into the FG form,

$$ds^2_{deS} = -\frac{\ell^2}{4} \left( \frac{d\rho}{\rho} \right)^2 + \frac{1}{\rho} \left[ \frac{1}{2} \left( 1 + \ell^2 \rho \right)^2 d\vec{n} \cdot d\vec{n} \right] = -\frac{\ell^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} g_{ab}(x, \rho) dx^a dx^b,$$

(6.13)

which is exactly the same as the Euclidean AdS case (6.11) with $\ell^2 \to -\ell^2$. The conformal infinity at $\rho = 0, \tau = \infty$ is also asymptotic to the same future light cone in the $d + 2$ dimensional ambient space and therefore it enjoys all the same conformal properties as the $\rho = 0$ boundary of Euclidean AdS. Thus, the FG method of extracting conformal invariants and conformal field theory behavior will work equally well in asymptotically de Sitter spacetime at its spacelike future infinity at $\rho = 0$, and there is CFT behavior at the conformal infinity of bulk de Sitter space as well.

To show the conformal behavior in de Sitter space arises more explicitly, let us use the flat spatial sections of de Sitter space,

$$ds^2_{deS} = -dt^2 + \ell^2 \exp \left( \frac{2t}{\ell} \right) d\vec{x} \cdot d\vec{x}$$

(6.14)

and make the change of variables $\rho = \ell^{-2} \exp \left( -\frac{2t}{\ell} \right)$ to bring the de Sitter line element into the alternative form,

$$ds^2_{deS} = -\frac{\ell^2}{4} \left( \frac{d\rho}{\rho} \right)^2 + \frac{1}{\rho} d\vec{x} \cdot d\vec{x} = -\frac{\ell^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} \delta_{ab} dx^a dx^b.$$

(6.15)

In this case the metric takes the FG form with $g_{ab}(x, \rho) = g_{ab}(x, 0) = \delta_{ab}$ independent of $\rho$. Then consider the behavior of the two-point function of a scalar field with mass $m$ in $d + 1$ dimensional de Sitter spacetime, viz. [20]

$$\langle T\Phi(t, \vec{x})\Phi(t', \vec{x}') \rangle = \frac{\ell^{1-d}}{(4\pi)^{d/2} \Gamma(d / 2 + \nu)} \frac{\Gamma(d - \nu) \Gamma(d / 2 + \nu)}{\Gamma(d + 1)} 2F_1 \left( \frac{d}{2} ; \nu, \frac{d + 1}{2} ; 1 - s^2(t, \vec{x}; t', \vec{x}'); \right),$$

(6.16)

where

$$\nu \equiv \sqrt{\frac{d^2}{4} - m^2 \ell^2} \quad \text{and}$$

$$s^2(t, \vec{x}; t', \vec{x}') = \frac{1}{4} \exp \left( \frac{t + t'}{\ell} \right) \left[ - \left( e^{-\frac{t}{\ell}} - e^{-\frac{t'}{\ell}} \right)^2 + (\vec{x} - \vec{x}')^2 \right]$$

(6.17)

depends only on the invariant distance between the two points $(t, \vec{x})$ and $(t', \vec{x}')$. Going to the conformal boundary at $t \sim t' \to \infty, \rho \sim \rho' \to 0$, $s^2$ becomes proportional to the flat section invariant distance $(\vec{x} - \vec{x}')^2$, and the asymptotic behavior of the hypergeometric function $2F_1$ implies that the two-point function of the scalar field behaves as [37]
\[ \langle \Phi(t, \vec{x}) \Phi(t', \vec{x}') \rangle \to C_- s^{-2\Delta_-} + C_+ s^{-2\Delta_+}, \]  

(6.18)

characteristic of a conformal theory in flat space with conformal weights,

\[ \Delta_{\pm} = \frac{d}{2} \pm \nu = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2 \ell^2}. \]  

(6.19)

We note that the de Sitter case propagator comes from a massive unitary field theory and gives complementary information about the conformal representation corresponding to \( \Delta_- \) which cannot be obtained in the AdS case without going to tachyonic \( m^2 \to -m^2 \). General de Sitter correlation functions which depend on the invariant distance between two points in the de Sitter bulk will have conformal behavior as \( \rho \to 0 \) as well, for purely geometric reasons of the embedding. The asymptotic behavior of quantum field theory on de Sitter spacetime, \( deS_{d+1} \) induces conformal field theory behavior, \( CFT_d \) on its asymptotic conformal infinity. This is so because of the pure kinematic fact of group isomorphism: the de Sitter isometry group \( SO(d + 1, 1) \) is the same as the conformal group \( C(S^d) \) of the asymptotic conformal infinity, which consists of the asymptotic past and the asymptotic future \( S^d \) of \( deS_{d+1} \). In other words, there exists \( deS_{d+1}/CFT_d \) correspondence \([35, 36]\). There is another aspect of the \( deS_{d+1}/CFT_d \) correspondence, namely, for even integer \( d = 2k \), conformal anomalies can be computed from the bulk gravitational theory on a spacetime of constant positive curvature of one higher dimension. This connection with conformal anomalies is described in the next section.

To proceed now consider the FG generalization of the above simple example of embedding \( S^d \) to an arbitrary \( d \) dimensional metric. FG showed that any \( d \) dimensional metric (with the appropriate signature) may be embedded at the conformal boundary of a space of asymptotically constant curvature, with \( g_{ab}(x, 0) = g_{ab}^{(0)} \) the boundary metric and the bulk metric in the vicinity of the boundary given by expanding \( g_{ab}(x, \rho) \) in a Taylor series in integer power of \( \rho \), i.e.

\[ g_{ab}(x, \rho) = \sum_n g_{ab}^{(n)}(x) \rho^n. \]  

(6.20)

The coefficients \( g_{ab}^{(n)} \) are determined order by order in \( \rho \) by solving Einstein’s equations for the \( d + 1 \) dimensional bulk metric. However as noted by FG themselves \([7]\) if \( d = 2k \) is an even integer this Taylor series breaks down at order \( n = k \), and logarithmic terms appear for the general embedded metric. In that case only the trace and the covariant divergence of \( g_{ab}^{(k)} \) is determined by Einstein’s equations in the bulk. The remaining part of \( g_{ab}^{(k)} \) cannot be determined and has been called ‘the FG ambiguity’ \([35]\).

The breakdown in the expansion \( (6.20) \) at \( n = k \) for even integer dimensions \( d = 2k \) has to do with the existence of certain traceless conformal tensors on the boundary metric \([7]\). However, as is already apparent from the power series solution of Einstein’s equations for general \( d \) in ref. \([19]\), the series \( (6.20) \) is well-defined for all \( n \) if \( d \neq 2k \) and there are no logarithmic terms, their place being taken instead by simple poles at \( d = 2n \) in the coefficients, \( g_{ab}^{(n)} \). Thus dimensional regularization supplies just the means of realizing the FG idea of obtaining a unique \( d + 1 \) dimensional bulk metric corresponding to an arbitrary \( d \) dimensional boundary metric by solving the Einstein equations order by order in a power series in \( \rho \), with the general \( g_{ab}(x, \rho) \) given by \( (6.20) \), replacing the simple terminating series we found in the exact Lobachewski or de Sitter cases. This is the first point of contact between the FG construction and dimensional regularization.

Furthermore, the residues of the poles in \( g_{ab}^{(n)} \) at \( d = 2n \) are linear combinations of precisely the conserved tensors \( G_{ab} \) and \( F_{ab} \), \( E_{ab} \) encountered in the previous sections for \( n = 1 \) and \( n = 2 \) respectively, showing the close connection of these terms in the dimensionally continued FG expansion with the non-trivial cocycles in two and four dimensions. Indeed, the explicit form of the first few expansion coefficients in arbitrary dimension \( d \) was reported in ref. \([12]\), which after taking account of the different conventions for the Riemann tensor reads

\[ g_{ab}^{(1)} = \frac{-\ell^2}{d-2} \left( R_{ab} - \frac{1}{2(d-1)} R g_{ab}^{(0)} \right), \]  

(6.21a)

\[ g_{ab}^{(2)} = \frac{\ell^4}{d-4} \left( \frac{1}{8(d-1)} \nabla_a \nabla_b R + \frac{1}{4(d-2)} \Box R_{ab} - \frac{1}{8(d-1)(d-2)} \Box R g_{ab}^{(0)} - \frac{1}{2(d-2)} R_{cd} R_{abcd} \right) + \frac{d-4}{2(d-2)^2} R_a^c R_{bc} + \frac{1}{(d-1)(d-2)} R R_{ab} + \frac{3d}{16(d-1)^2(d-2)^2} R_{cd} R_{abcd} g_{ab}^{(0)} \]  

(6.21b)

The expansion coefficients are completely well-defined and there is no FG ambiguity in the dimensionally continued coefficients of the expansion \( (6.20) \). The coefficients of the conformal anomaly determined by this construction are
the same on the de Sitter side as on the AdS side modulo the $\ell^2 \rightarrow -\ell^2$ changes in the basic formulae. The residue of the pole at $d = 2$ in $g^{(1)}_{ab}$ is proportional to the Einstein tensor, $G_{ab}$, while the residue of the pole at $d = 4$ in $g^{(2)}_{ab}$ is easily verified by means of the identities in Appendix A to be proportional to a linear combination of $E_{ab}$ and $F_{ab}$. In fact, $G_{ab}$ and $E_{ab}$ vanish in two and four dimensions respectively, and do not contribute to the pole terms, while $F_{ab}$ is the traceless Bach tensor which is the explicit obstruction to the $\rho$ expansion in four dimensions. The reason for this connection is that the residues of the pole terms, which correspond to the logarithms in even integer dimensions, are necessarily conserved due to the Einstein equations in the bulk geometry, and they transform with a definite conformal weight $2(1 - n)$ under conformal transformations of the boundary metric. It follows from this that the residue of $g^{(n)}_{ab}$ at $d = 2n$ must be a linear combination of the conserved traceless tensors obtained by varying the $2n$ dimensional action composed of conformal invariants in that dimension. This is the second point of contact between the dimensionally continued FG expansion and the development of the previous sections.

To demonstrate the transformation properties of the coefficients $g^{(n)}_{ab}$ in detail requires the form of the coordinate transformation of the bulk metric which leaves the FG form,

$$ds^2 = \pm \frac{\ell^2}{\rho^2} + g_{ab}(x, \rho) dx^a dx^b,$$  \hspace{1cm} (6.22)

invariant \cite{39-41}. These special (PBH) transformations take the infinitesimal form,

$$\rho = \rho' e^{-2\sigma(x')} \simeq \rho' (1 - 2\sigma(x')),$$

$$x^a = x'^a + \xi^a(x', \rho').$$ \hspace{1cm} (6.23)

Requiring the FG bulk line element have no mixed $dx^a d\rho'$ terms gives

$$\xi^a(x, \rho) = \pm \frac{\ell^2}{2} \int_0^\rho d\rho' g^{ab}(x, \rho') \partial_b \sigma(x) + \xi^{(0)}(x).$$ \hspace{1cm} (6.24)

Then $\xi^a$ may be developed as a power series in $\rho$, i.e.

$$\xi^a(x, \rho) = \sum_{n=0}^{\infty} \xi^{(n)}(x) \rho^n,$$ \hspace{1cm} (6.25)

which is completely determined by the expansion (6.20) when we impose the Einstein equations in the bulk geometry. The diffeomorphism (6.23) generates the transformation,

$$\delta g_{ab}(x, \rho) = 2\sigma(x)(1 - \rho \partial_\rho) g_{ab}(x, \rho) + \nabla_a \xi_b(x, \rho) + \nabla_b \xi_a(x, \rho)$$ \hspace{1cm} or

$$\delta g^{(n)}_{ab}(x) = 2(1 - n)\sigma(x) g^{(n)}_{ab}(x) + \nabla_a \xi^{(n)}_b(x) + \nabla_b \xi^{(n)}_a(x)$$ \hspace{1cm} (6.26, 6.27)

on the metric, where $\nabla_a$ is the covariant derivative with respect to the zeroth order boundary metric $g^{(0)}$. At $\rho = 0$, or for $n = 0$ this transformation reduces to an infinitesimal Weyl transformation \cite{21} of the boundary metric $g_{ab}(x, \rho = 0)$, up to the diffeomorphism $\xi^{(0)}(x)$. The transformation of the coefficient $g^{(n)}_{ab}$ under global Weyl transformations, $\sigma(x) = \sigma_0 = \text{const.}$ is that of a tensor of weight $2(1 - n)$, as required. Hence a similar relationship of the higher order terms in the expansion possessing poles at larger even integer dimensions, with the variations of the Weyl invariant terms that give rise to traceless conserved tensors in those dimensions is to be expected. Particular linear combinations of these tensors enter the expansion (6.20) because the solution of the bulk metric is determined by the second order Einstein equations on the $d + 1$ dimensional embedding space, and this solution contains no arbitrary coefficients in non-integer dimensions.

It is important to recognize that the use of the Einstein equations to determine the coefficients of the power series in $\rho$ for the bulk metric is the simplest route to generalizing the example of the embedding of the sphere in Lobachevsky or de Sitter space. The Einstein equations have no dynamical content as equations of motion following from some variational principle in this purely mathematical construction of the bulk embedding geometry. Rather, once the bulk metric embedding has been determined by solving the Einstein equations for fixed constant Ricci scalar as a power series in $\rho$, we are free to evaluate any coordinate invariant scalar action functional on this bulk metric, and indeed this was the method FG proposed to construct conformal Weyl invariants of the boundary metric. Since a particular subset of bulk coordinate transformations, the PBH transformations \cite{23} are local conformal transformations on the boundary, coordinate invariant scalars in the bulk give rise to conformal invariant scalars on the boundary. In the original paper \cite{8}, the construction of conformal invariants was proposed by considering
coordinate invariant scalars in the \( d + 2 \) dimensional (now non-flat) ambient space, of dimension less than \( 2n \) to avoid the logarithmic obstruction to the series expansion (6.20). However, as the considerations of this section show, dimensional continuation allows one to lift this restriction, and to use invariant scalars in the \( d + 1 \) dimensional bulk embedding space, which contain the same geometric information as the ambient space embedding. Thus, dimensional continuation furnishes exactly the missing ingredient in the FG construction, which repays the favor by furnishing the Weyl invariants needed to construct the non-trivial cocycles in the general procedure of the previous sections.

Moreover, as we have seen, for the general embedded geometry the region near \( \rho = 0 \) is similar to the light cone of the simple prototype example of embedding \( S^d \), and the expansion in powers of \( \rho \) works equally well for the metric of the \( d + 1 \) dimensional asymptotic AdS or deS space in the vicinity of \( \rho = 0 \), since both spaces of positive or negative scalar curvature asymptotically approach the same light cone, which is the conformal infinity of the \( d + 2 \) dimensional ambient spacetime.

\[
\int \sqrt{g} \, d^d x \, \rho^{-d} \, B(x, \rho) = \int \sqrt{g} \, d^{d+1} x \, \rho^{-d-1} \, \sqrt{g^{(0)}} \, b(x, \rho),
\]

where we have chosen to place the \( \rho \) dependence of the volume element of the metric \( g_{ab}(x, \rho) \) into the density \( b(x, \rho) \). From the fact that \( B = \sqrt{g^{(0)}} b / \sqrt{g} \) in (7.1) transforms as a scalar under the PBH diffeomorphisms (6.23), one can find the transformation rule for \( b \), namely

\[
\delta b(x, \rho) = -2 \sigma(x) \rho \partial_\rho b(x, \rho) + \nabla_a (b(x, \rho) \xi^a(x, \rho)),
\]

From this the transformation for the coefficients in the \( \rho \) expansion,

\[
b(x, \rho) = \sum_{n=0} b_n \rho^n
\]

may be found, \( i.e. \)

\[
\delta b_n = -2n \sigma b_n + \nabla_a \left( \sum_{j=0}^{n-1} b_j \xi^{(n-j)} \right),
\]

where the \( \xi^{(j)} \) are the expansion coefficients of \( \xi^a \) determined by (5.20) and (5.24). The form of the transformation (7.4) shows that

\[
\int \sqrt{g^{(0)}} \, d^{2k} x \, b_{2k}(x)
\]

is invariant under local Weyl transformations in \( d = 2k \) dimensions, for \textit{any} diffeomorphism invariant action function \( S_{\text{bulk}} \) in the bulk. Hence substituting explicit forms for scalar functions in the bulk and carrying out the expansion of \( b \) in \( \rho \) to order \( n = k \) in dimensionally continued \( d \) dimensions by solving the Einstein equations (with either positive or negative cosmological constant) generates precisely the conformally invariant scalars one needs to construct the non-trivial cocycles in \( d = 2k \) even dimensions. This is the explicit proof of the construction of conformal invariants in the coordinates (6.23). Notice also that both the strictly local Weyl invariants (type B) for which the inhomogeneous

\[
\text{VII. FINITE VOLUME SCALING AND THE IR EFFECTIVE ACTION FOR GRAVITY}
\]

The discussion in the previous section of the embedding procedure and dimensional regularization to realize the original FG idea for constructing the conformal invariants of the boundary metric is essentially mathematical and kinematic in nature. The FG construction of Weyl invariants is just what is needed to generalize the method of obtaining the anomalous WZ effective action corresponding to non-trivial cocycles of the Weyl group in any even dimension, as proposed in Section 2. In the physics literature to date the FG embedding has been used almost entirely to check features of the AdS/CFT correspondence, which specifies that the classical supergravity bulk action be used \[6\]. However, the embedding of an arbitrary metric in a space of one dimension higher, in such a way that conformal transformations on the boundary become coordinate transformations in the bulk has broader consequences for anomalies and infrared scaling behavior than perhaps is evident at first. It is this physical application of the FG embedding to the construction of the low energy effective action for gravity that we explore in this section.

Let us consider an arbitrary coordinate invariant local action in the \( d + 1 \) dimensional bulk,

\[
S_{\text{bulk}} = \int d\rho d^d x \, \rho^{-d-1} \sqrt{g} \, B(x, \rho) = \int d^d x, \rho \, \rho^{-d-1} \sqrt{g^{(0)}} \, b(x, \rho),
\]

where we have chosen to place the \( \rho \) dependence of the volume element of the metric \( g_{ab}(x, \rho) \) into the density \( b(x, \rho) \). From the fact that \( B = \sqrt{g^{(0)}} b / \sqrt{g} \) in (7.1) transforms as a scalar under the PBH diffeomorphisms (6.23), one can find the transformation rule for \( b \), namely

\[
\delta b(x, \rho) = -2 \sigma(x) \rho \partial_\rho b(x, \rho) + \nabla_a (b(x, \rho) \xi^a(x, \rho)),
\]

From this the transformation for the coefficients in the \( \rho \) expansion,

\[
b(x, \rho) = \sum_{n=0} b_n \rho^n
\]

may be found, \( i.e. \)

\[
\delta b_n = -2n \sigma b_n + \nabla_a \left( \sum_{j=0}^{n-1} b_j \xi^{(n-j)} \right),
\]

where the \( \xi^{(j)} \) are the expansion coefficients of \( \xi^a \) determined by (5.20) and (5.24). The form of the transformation (7.4) shows that

\[
\int \sqrt{g^{(0)}} \, d^{2k} x \, b_{2k}(x)
\]
total derivative term in (7.4) is absent, and the topological invariants (type A) for which this term is present are both contained in the solutions of (7.4).

It is instructive to write down the infinitesimal PBH conditions which the first few $b_n$ must satisfy:

\[
\begin{align*}
\delta b_0 &= 0, \\
\delta b_1 &= -2\sigma b_1 \pm \frac{\ell^2}{2} b_0 \Box \sigma, \\
\delta b_2 &= -4\sigma b_2 \pm \frac{b_0 \ell^2}{4(d-2)} \left[ R^{ab} \nabla_a \nabla_b \sigma - \frac{1}{2} R \Box \sigma \right],
\end{align*}
\]

(7.6)

which are solved by

\[
\begin{align*}
b_0 &= \text{const.}, \\
b_1 &= \pm b_0 \frac{\ell^2}{4(d-1)} R, \\
b_2 &= b_0 \frac{\ell^4}{32(d-2)(d-3)} E_4 + c_2 \ell^4 C_{abcd}C^{abcd}.
\end{align*}
\]

(7.7)

The de Sitter case is again recovered from the AdS case by a simple change of $\ell^2 \rightarrow -\ell^2$ in all AdS formulae. The coefficients $b_n$ are the Euler (type A) and Weyl (type B) invariants which we have used in our previous construction of the non-trivial cocyles in 2 and 4 dimensions. Note that the Euler invariant in (7.7) is associated with the inhomogeneous solution to (7.4) induced from the lower order $b_n$ in the second total derivative term of (7.4) or (7.6), corresponding to the fact that the Euler density is Weyl invariant only up to a total derivative. On the other hand the local Weyl invariant $C_{abcd}C^{abcd}$ is a solution to the homogeneous eq. $\delta b_2 = -4\sigma b_2$. Notice also that only the non-trivial cocyles are generated by this procedure. The trivial $R^2$ cocyle is not a solution of the PBH equations (7.4). Clearly this is because the trivial cocyles are not locally Weyl invariant, and only local Weyl invariants (up to surface terms) can be generated by bulk coordinate invariants due to the PBH symmetry. Certainly no term like (4.28) is generated.

Because of the PBH symmetry $b$ necessarily satisfies the infinitesimal form of the WZ consistency condition,

\[
\int d^{2k} x \sqrt{g^{(0)}} (\sigma_1 \delta \sigma_2 b - \sigma_2 \delta \sigma_1 b) = 0,
\]

(7.8)

since this is just a subset of diffeomorphisms of the bulk action. Since the $b_n$ coefficients appear both in the action and in the trace anomaly after variation with respect to $\sigma$, this excludes $\Box R$ from $b_2$ in (7.7) as well. Indeed comparison of the third variation in (7.6) with eq. (A17) of Appendix A shows that $\Box R$ does not satisfy this condition. Thus the FG construction is precisely what is required to construct the Weyl invariants (and only those invariants) which give rise to the non-trivial cocyles in any even dimension. Taking arbitrary linear combinations of coordinate invariant scalars in the bulk action will give rise to arbitrary linear combinations of the conformal invariants on the boundary, in contrast to the AdS/CFT conjecture which applies to a specific bulk action and specific set of anomaly coefficients in the boundary theory.

The pole term in the expansion of the metric (6.20) at order $n = k$ is cancelled and does not appear in the expansion of the action density $b(x, \rho)$. However when the expansion (7.3) is substituted in (7.1) the $\rho$ integral diverges at small $\rho$ in even integer dimensions, or equivalently the $n^{th}$ order term in the expansion of the classical bulk action possesses a pole at $d = 2n$, similar to the dimensional regularization counterterms in a quantum field theory at the boundary. However, unlike the UV regulator of quantum theory, dimensional regularization now appears as an IR regulator of the large volume divergences of the classical bulk action.

The fact that the limit $\rho \rightarrow 0$ is an infrared limit is clear also from the PBH transformation (6.23), which shows that $\rho \rightarrow 0$ if $\sigma = \sigma_0 \rightarrow +\infty$, with $\rho'$ fixed. The explicit form of the change of variables (6.11) to bring the de Sitter metric into the FG form also shows conformal infinity is reached by stretching all length scales to the extreme IR limit, where the conformal behavior of the de Sitter correlation functions becomes apparent, as in (6.18). This stretching of physical length scales (with some UV cutoff imposed if necessary to regulate short distance behavior) is precisely what is contemplated in the Wilson description of the renormalization group [17]. Rescaling of $\rho$ in the bulk metric is completely equivalent to a finite volume scaling in the conformal boundary metric, and the expansion of the general bulk action in powers of $\rho$ is just an expansion of the general boundary action in scale dimensions of the volume.

From the general PBH transformation and the explicit examples we see that $\rho$ scales like $\lambda^{-2}$ if $\lambda$ is a physical length scale in the boundary metric. Substituting the expansion (7.3) in (7.1) and cutting off the lower limit of the $\rho$ integral at $\rho = \lambda^{-2}$ shows that the dimensionally regulated series is of the form,
where we neglect the $\lambda$ independent terms coming from the (fixed) upper limit of the $\rho$ integral. In the $d$ dimensional boundary theory the $\lambda$ dependent terms in this series have precisely the form of terms in the effective action with mass dimension $2n$, $b_{2n}(\lambda) = \lambda^{-2n}b_{2n}(x)$, the power of $\lambda$ classifying their behavior under global Weyl transformations, i.e. finite volume scaling. The terms with $n < \frac{d}{2}$ are strictly relevant terms at large volumes. Conversely those with $n > \frac{d}{2}$ are strictly irrelevant in the infrared limit of large volumes. The marginally relevant terms at $d = 2n$ are obtained by taking the logarithmic variation, $\delta b_{2n}$ which cancels the pole when $d \to 2k$, yielding the finite result (7.3), which are the non-trivial cocycles of Weyl anomaly in $2k$ even dimensions. As we have seen explicitly in $d = 4$ the trivial $R^2$ cocycle is not included in $b_1$. Thus the FG construction selects precisely the infrared relevant terms and only those terms in the limit $\rho \to 0$ (or $\lambda \to \infty$).

The absence of the trivial cocycle terms which are required for UV renormalization is a manifestation of the fact that the poles in the expansion of the dimensionally regulated bulk action (7.3) are infrared poles, their formal similarity to the UV counterparts of dimensionally regulated quantum theories notwithstanding. Taking the logarithmic variation of (7.4), the physical limit $d \to 2k$ and integrating again with respect to $d\lambda/\lambda$ gives

$$
\Gamma_{eff}^{IR}[g^{(0)}; \lambda] = \sum_{n=0}^{k-1} \frac{\lambda^{2k-2n}}{2k-2n} \int d^{2k}x \sqrt{-g^{(0)}} b_n(x) + \log \lambda \int d^{2k}x \sqrt{-g^{(0)}} b_{2k}(x) + O(\lambda^0)
$$

which are all the infrared relevant terms in the boundary effective action which grow with either a positive power of $\lambda$ or $\log \lambda$ as $\lambda$ grows. We see that the non-trivial cocycles of the Weyl anomaly are the latter which are marginally relevant under finite volume rescaling. If $\lambda$ is replaced by $e^{\rho}$, then (7.10) is exactly of the form of the WZ consistent effective action $\Gamma_{WZ}[g^{(0)}; \sigma_0]$, augmented by the relevant local terms for $n < k$ in (4.27) to give the total effective action of (4.23). The Weyl invariant terms $S_{inv}$ are order $\lambda^3$ and cannot be calculated by either method, but neither are they relevant in the low energy, long distance limit compared to the terms kept in (7.11) or (4.23).

Thus the FG embedding does much more than generate just the local Weyl invariants, which was its original purpose, and which are needed to construct the non-trivial cocycles of the anomaly in any even integer dimension. The terms explicitly displayed in (7.10) which diverge as $\lambda$ and the physical volume are taken to infinity are just the IR relevant terms of the Wilson effective action of the generally covariant boundary theory. This provides an unambiguous definition and extension of the Wilson RG scale transformation to low energy effective theories of gravity. From this Wilson effective action point of view the integral of $R^2$ is absent in four dimensions because it is a marginally irrelevant operator in the IR which is independent of global Weyl rescalings, remaining volume independent in the infinite volume $\lambda \to \infty$ limit. Arbitrary $\lambda$ independent non-local terms in (7.10), corresponding to $S_{inv}$ in (4.23) are also neutral under finite volume scale transformations and therefore are marginally irrelevant operators in the infrared as well. Thus, we arrive at a precise formulation of low energy effective field theory for gravity in $d$ dimensions from the classical FG construction in $d+1$ dimensions. Its direct proof in a full quantum field theory setting is more difficult than this simple classical construction and would require that one systematically integrate out all the fluctuations between two scales, say $\lambda$ and $2\lambda$, to show that the new effective action is of the same form as the previous one, (7.10) with renormalized coefficients, provided that this effective action is used to calculate only soft processes with momenta $p \ll \lambda$. In principle, this Wilson-Kadanoff exact renormalization group blocking procedure can be applied either in the continuum or on a lattice. Although this direct analysis would be welcome, as it would give detailed information about the RG flow for different matter or gravitational field representations and couplings, the general classification of terms in the effective action according to their properties under Weyl rescaling and the FG implementation of this rescaling as a coordinate transformation in one dimension higher is sufficient to fix the general form of the IR effective action (4.23) or (7.10).

The low energy effective action for physical four dimensional spacetime includes in addition to the familiar cosmological term $b_0$ which scales as the volume, $\lambda^4$ and the Einstein-Hilbert term $b_1$ which scales as $\lambda^2$, also the non-local $S_{anom}$ corresponding to the two non-trivial cocycles in $d = 4$ which scale as $\log \lambda$. Hence these anomalous terms from the non-trivial cocycles are not irrelevant in the infrared, and in principle modify the Einstein theory even at low energies and large distances. This conclusion is perhaps less surprising if one recalls the origin of the anomaly as the effect of massless excitations which do not decouple at arbitrarily large distances. As in the example of the $U(1)$ chiral anomaly, the most general low energy effective Lagrangian consistent with symmetry contains a marginally relevant WZ term which dominates the decay, $\pi^0 \to 2\gamma$ in the low energy limit of QCD [12].
VIII. CONCLUSIONS

Since several different aspects of conformal anomalies have been presented in this paper, and some parts of these overlap with earlier work, we summarize here the main conclusions for the benefit of the reader:

- The finite shift coboundary operator of the Weyl group may be defined without the use of anti-commuting Grassmann variables by Eqs. (2.3) and (2.4).
- The first cohomology of the Weyl group is defined as one-forms of the cochain which are closed but non-exact in the sense of Eqs. (2.8) and (2.9), and are represented by non-local functionals of the metric in the physical even dimension $d = 2k$.
- The non-local cocycles of the Weyl group may be constructed in dimensional regularization by considering all the local counterterms of dimension $2k$ near $d = 2k$, and selecting those which are conformal invariant in and only in the physical dimension.
- The non-trivial cocycles are of two kinds, corresponding to two kinds of conformal invariants, those invariant up to surface terms, of which there is only one (type A, Euler density) and those which are locally conformal invariant (type B), of which there are an increasing number in higher even dimensions $d = 2k$.
- Both kinds of non-trivial cocycles have integrals whose local Weyl variation vanishes linearly as $d \to 2k$, and lead to UV finite effective actions, local in terms of $\sigma$, which automatically satisfy the WZ consistency condition. The action obtained this way is identical to that obtained by integrating the local anomaly with respect to $\sigma$.
- The effective actions constructed by this method are nevertheless non-trivial due to the multi-valuedness under the global shift (3.6), which signal sensitivity to winding about the obstruction in the space of metrics at the singular metrics $g_{ab} = 0$ and $g^{\bar{a}b} = 0$.
- Exactly this same multi-valuedness property indicates sensitivity of the non-trivial cocycle actions to global Weyl rescalings (3.8), which imply that they correspond to marginally relevant operators in the IR.
- The non-local but fully covariant action corresponding to each non-trivial cocycle may be constructed explicitly by solving a linear differential equation for the conformal transformation $\sigma$ between two members of the conformal equivalency class $g_{ab}$ and $\bar{g}_{ab}$.
- The conformal differential operator appearing in this equation for $\sigma$ is $\Box$ in $d = 2$ dimensions and $\Delta_4$ defined in (4.8) in $d = 4$ dimensions. In the latter case a new uniformization conjecture analogous to the Poincare-Yamabe conjecture for two dimensional Riemannian manifolds suggests itself.
- There are analogous $d^{th}$ order conformal differential operators on scalar functions in higher even dimensions, and although this has not been proven in all generality, it appears to be possible by simple counting of invariants to bring the anomalous action in higher even dimensions to a Gaussian form in $\sigma$ by a suitable admixture of the trivial cocycle terms. This has been verified explicitly in $d = 2, 4, 6$ dimensions, with the $d = 6$ case treated first in ref. [10] and also in [43]. In all even dimensions the propagator of the conformal invariant differential operator is logarithmic.
- Corresponding to each non-trivial cocycle there is a conserved energy momentum tensor which is generally non-local also in its fully covariant form. One of these tensors becomes the local geometric tensor $^{(3)}\mathcal{H}_{ab}$ in $d = 4$ conformally flat spacetimes, showing the true origin of this tensor.
- In $d = 2$ there are no local Weyl invariants and the non-local effective action found by Polyakov is the single infrared relevant term in the effective action for 2D gravity in addition to the volume cosmological term, all other possible terms in $S_{eff}$ being strictly irrelevant in the IR.
- In higher dimensions the non-trivial cocycles determine the effective action up to local terms and strictly Weyl invariant terms, as in (1.23). Although this is considerably less information than in two dimensions, it already determines all the IR relevant terms in the effective action in $d = 4$ and is sufficient to preclude any term of the form (1.28) for the Weyl squared anomaly.
- The Fefferman-Graham embedding of an arbitrary space at the conformal infinity of a higher dimensional space is exactly the construction needed to generate the local conformal invariants, and therefore non-trivial cocycles and WZ effective actions in all higher even dimensions by the dimensional regularization method.
Conversely, dimensional regularization of the FG series (6.20) eliminates the obstruction or ambiguity in the original FG construction, providing a well-defined bulk metric of either positive or negative scalar curvature (deS or AdS) in the form (6.22).

Although it makes use of the classical Einstein equations in the bulk, the FG construction is essentially kinematic, embedding the local Weyl group of the boundary metric in the diffeomorphisms of the embedding space, and the asymptotically de Sitter embedding space has the same conformal behavior at infinity in the coordinates (1.13) as the asymptotically AdS embedding. Both give rise to CFT behavior at conformal infinity $\rho \to 0$. In other words, there is $deS_{d+1}/CFT_d$ correspondence as well.

By cataloging all scalar invariants in the $d + 1$ dimensional bulk geometry of the dimensionally continued FG construction, whose volume integrals diverge as the conformal limit $\rho \to 0$ is approached, all IR relevant terms (and only those terms) in the low energy effective action of gravity in any integer dimension may be obtained. In this case dimensional regularization may be used as well, with poles appearing to regulate now the IR (instead of UV) divergences at infinite volume.

Hence, the FG embedding of the local Weyl group into diffeomorphisms of one higher dimension gives a precise meaning to finite volume rescaling in the Wilson RG sense to theories possessing low energy general coordinate invariance.

The non-trivial cocycles of the Weyl group and the non-local effective actions they generate are marginally relevant in the IR, while the trivial cocycles and Weyl invariant terms in (4.25) or (7.10) are either marginally or strictly irrelevant ($\lambda^p, p \leq 0$) and may be neglected in the low energy effective action of gravity.

The known anomalies in $d = 4$ lead to effective actions which are marginally relevant, implying modification of the classical Einstein theory at low energies or large distances.

Finally the higher order differential operator in $d = 4$ leads to no problems with ghosts or unphysical poles [25], and indicates instead an additional global scalar degree of freedom in the low energy effective theory of $d = 4$ gravity over and above the Einstein theory. Since the conformal mode is completely frozen in the classical Einstein theory, the new mode can lead to qualitatively new effects in the modified theory. In fact, the fluctuations of this new degree of freedom in the quadratic action (4.21) generate an infrared stable Gaussian fixed point characterized by restoration of conformal symmetry and anomalous scaling of both the Einstein-Hilbert and cosmological terms [20]. At this fixed point the scaling dimensions of the Einstein and cosmological terms are different from their classical dimension in the classification of relevant, marginal and irrelevant terms in the low energy effective action of the previous section. These refer to the perturbative (therefore also Gaussian) fixed point of flat spacetime where scaling dimensions of terms under the RG are given by their canonical dimensions. The new non-perturbative fixed point found in ref. [20] describes a conformal invariant phase of strong gravity in $d = 4$ where the effective cosmological $\Lambda$ and inverse Newtonian $G^{-1}$ terms flow to zero and it is the $S_{anom}$ term of (4.21) which controls the physics [44]. Possible consequences of this conformal invariant phase of gravity for the Cosmic Microwave Background have been investigated in [15].

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APPENDIX A: CONFORMAL VARIATIONS

For the reader’s convenience we catalog in this Appendix the various tensors and their conformal variations needed to derive the detailed formulae of the text. The conformal variations may be derived from the relation between the covariant derivative with respect to the metric $\bar{g}_{ab}$ and

$$g_{ab} = e^{2\sigma} \bar{g}_{ab}.$$  \hfill (A1)

By definition,

$$\nabla_a V^b_c \equiv \partial_a V^b_c + \Gamma^b_{ac} V^d_c - \Gamma^d_{ac} V^b_d; \quad \nabla_a V^b_c \equiv \partial_b V^b_c + \Gamma^b_{ad} V^d_c - \Gamma^d_{ad} V^b_d.$$  \hfill (A2a, A2b)

where the Christoffel connections are related by

$$\Gamma^b_{ac} = \frac{g^{bd}}{2} (-\partial_d g_{ac} + \partial_a g_{cd} + \partial_c g_{da})$$

$$= \frac{\bar{g}^{bd}}{2} (-\partial_d \bar{g}_{ac} + \partial_a \bar{g}_{cd} + \partial_c \bar{g}_{da}) + \bar{g}^{bd} (-g_{ac}\partial_d \sigma + \bar{g}_{cd}\partial_a \sigma + \bar{g}_{ad}\partial_c \sigma)$$

$$= \Gamma^b_{ac} + \Delta \Gamma^b_{ac} ,$$  \hfill (A3)

with

$$\Delta \Gamma^b_{ac} = -\bar{g}_{ac} \sigma^b + \delta^b_a \sigma_a + \delta^b_a \sigma_c.$$  \hfill (A4)

This is the basic relation from which all conformal variations may be derived. From these relations and the definition of the Riemann tensor,

$$[\nabla_c, \nabla_d] v^a = R^a_{bcd} v^b,$$  \hfill (A5)

we obtain its conformal variation,

$$R^a_{bcd} = \bar{R}^a_{bcd} + 2\delta^a_{[d|\sigma|c]} + 2\bar{g}_{[d|\sigma|c]} + 2\bar{g}_{[d|\sigma|c]} + 2\delta^a_{[d|\sigma|c]} + 2\delta^a_{[d|\sigma|c]} + 2\delta^a_{[d|\sigma|c]} + 2\delta^a_{[d|\sigma|c]} ,$$  \hfill (A6)

where $\sigma_a = \nabla_a \sigma$ and $\sigma_{ab} = \nabla_a \nabla_b \sigma = \nabla_a \nabla_b \sigma$, all barred covariant derivatives taken with respect to the metric $\bar{g}_{ab}$, and $2V_{[ab]} = V_{ab} - V_{ba}$ denotes anti-symmetrization of the bracketed indices. The contractions of this formula in $d$ dimensions,

$$R_{cd} = \bar{R}_{cd} - (d-2)(\sigma_{c;cd} - \sigma_c \sigma_d) - \bar{g}_{cd} \left[ (\nabla \sigma + (d-2)\sigma_a \sigma^a) \right] ;$$  \hfill (A7a)

$$R = e^{-2\sigma} \left\{ \bar{R} - 2(d-1)(\nabla \sigma + (d-1)(d-2)\sigma_a \sigma^a) \right\} ,$$  \hfill (A7b)

follow immediately.

The conformal factor dependence of various tensors quadratic in the curvature may be worked out next:

$$R^c_{ab} R_{bc} = e^{-2\sigma} \left\{ \bar{R}^c_{ab} \bar{R}_{bc} - 2(d-2)\bar{R}_{[a|\sigma|bc]} + 2(d-2)\bar{R}_{(a|\sigma|b)c} - 2\bar{R}_{ab} \left[ (\nabla \sigma + (d-2)\sigma_a \sigma^a) \right] \right\} ;$$  \hfill (A8a)

$$R^{cd} R_{cd} = e^{-4\sigma} \left\{ \bar{R}^{cd} \bar{R}_{cd} - 2(d-2)\bar{R}^{cd} (\sigma_{c;cd} - \sigma_c \sigma_d) - 2\bar{R}_{cd} \sigma_a \sigma^a - 2\bar{R} \left[ (\nabla \sigma + (d-2)\sigma_a \sigma^a) \right] \right\} ;$$  \hfill (A8b)

$$RR_{ab} = e^{-2\sigma} \left\{ \bar{R} R_{ab} - 2(d-1)d_{ab} \nabla \sigma - (d-1)(d-2)\bar{R}_{ab} \sigma_a \sigma^a - (d-2)\bar{R} (\sigma_{c;ab} - \sigma_a \sigma_b) \right\} ;$$  \hfill (A8c)

$$R^2 = e^{-4\sigma} \left\{ \bar{R}^2 - 4(d-1)\bar{R} \left[ (\nabla \sigma - (d-1)(d-2)\bar{R} \sigma_a \sigma^a + 4(d-1)^2(\nabla \sigma)^2 \right] \right\} ;$$  \hfill (A8d)
From these conformal dependences we derive

\[
\sqrt{-g} \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) = \sqrt{-g} \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) + \sqrt{-g} \left\{ -4T^{ab} (\sigma_{ab} - \sigma_a \sigma_b) + 2T^{a\alpha} - 4(\sigma_a^2 - 4\sigma_a \sigma_b + 4\sigma_{ab}(\sigma^{ab} - 2\sigma^a \sigma^b)) \right\} \\
+ (d-4) \left\{ R_{ab} R^{ab} - \frac{1}{3} R^2 - 4T^{ab} (\sigma_{ab} - \sigma_a \sigma_b) + 2T^{a\alpha} - 4(\sigma_a^2 - 4\sigma_a \sigma_b \sigma_b^2 + 4\sigma_{ab}(\sigma^{ab} - 2\sigma^a \sigma^b)) \right\} \\
+ (d-4) \left\{ -2T^{b} (\sigma_{ab} - \sigma_a \sigma_b) + \frac{4}{3} R^{a} - 4T_\alpha - 5(\sigma_a^2) + 4\sigma_{ab}(\sigma^{ab} - 2\sigma^a \sigma^b) \\
- 10\sigma_a \sigma_b \sigma_a \sigma_b^2 - 4\sigma_a \sigma^a \sigma_b \sigma_b^2 \right\} + O(d-4)^2. \tag{A9}
\]

The terms involving \( \sigma \) but no factor of \( d-4 \) in the second line of this expression can be written as a total derivative, and hence give only a surface term when integrated. Ignoring any such surface contributions we have then

\[
\int \sqrt{-g} \, d^d x \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) - \int \sqrt{-g} \, d^d x \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) = (d-4) \int \sqrt{-g} \, \sigma \left( R_{ab} R^{ab} - \frac{1}{3} R^2 \right) \\
- 4(d-4) \int \sqrt{-g} \, d^d x \, \sigma \left\{ -(\sqrt{-g} - \frac{1}{2} R^{ab} R_{ab} \sigma_{ab} - \sigma^a \Box \sigma - \sigma^a \sigma_b \sigma_b^2 \right\} \\
+ (d-4) \int d^d x \sqrt{-g} \left\{ -2T^{b} (\sigma_{ab} - \sigma_a \sigma_b) + \frac{4}{3} R^{a} - 4T_\alpha - 5(\sigma_a^2) + 4\sigma_{ab}(\sigma^{ab} - 2\sigma^a \sigma^b) \\
- 10\sigma_a \sigma_b \sigma_a \sigma_b^2 - 4\sigma_a \sigma^a \sigma_b \sigma_b^2 \right\} + O(d-4)^2 \\
= (d-4) \int \sqrt{-g} \, \sigma \left( R_{ab} R^{ab} - \frac{1}{3} R^2 + \frac{1}{3} \Box R \right) - (d-4) \int d^d x \sqrt{-g} \, \Delta \sigma + O(d-4)^2, \tag{A10}
\]

up to terms of linear order in \( d-4 \) in the expansion around \( d=4 \). This is the formula used in [17] of the text.

Following standard notation we define the three tensors,

\[
H_{ab} = \frac{1}{\sqrt{-g}} \delta^{cd} \int d^d x \sqrt{-g} R_{cd} R_{cd} = 2R_a^{cde} R_{bcde} - \frac{g_{ab}}{2} R^{cde} R_{cd} + 4Q_{ab} \tag{A11a}
\]

\[
-2\nabla_a \nabla_b R - 4R_{abc} + 4R^{cd} R_{acbd} \tag{A11b}
\]

\[
(2) H_{ab} = \frac{1}{\sqrt{-g}} \delta^{cd} \int d^d x \sqrt{-g} R_{cd} R_{cd} = 2R_a^{cd} R_{cd} - \frac{g_{ab}}{2} R^{cd} R_{cd} - \nabla_a \nabla_b R + 4Q_{ab} \tag{A11c}
\]

\[
(1) H_{ab} = \frac{1}{\sqrt{-g}} \delta^{cd} \int d^d x \sqrt{-g} R^2 = 2g_{ab} R - 2\nabla_a \nabla_b R + 2R_{ab} - \frac{g_{ab}}{2} R^2. \tag{A11d}
\]

These differ from ref. [30] by an overall minus sign due to neglect of the sign change between \( \delta g_{ab} \) and \( \delta g_{ab} = -g_{ac} g_{bd} \delta g_{cd} \) in ref. [30]. The definition of the Weyl tensor in \( d \) dimensions is

\[
C^a_{bcd} \equiv R^a_{bcd} + \frac{2}{d-2} \left( \delta^a_{[d} R_{eb]} + g_{bc} R^a_{d} \right) + \frac{2}{(d-1)(d-2)} \delta^a_{[c} g_{d]b} R, \tag{A12}
\]

which is conformally invariant in \( d \) dimensions, i.e.

\[
C^a_{bcd} = C^a_{bcd}. \tag{A13}
\]

The quadratic contractions,

\[
C^a_{cde} C_{bcd} = R_a^{cde} R_{bcde} - \frac{4}{d-2} R_a^{c e} R_{c d} + \frac{2}{(d-2)^2} (2R R_{ab} - n R^e_a R_{eb}) \\
+ \frac{2}{(d-2)^2} g_{ab} R_{cd} R_{cd} - \frac{2}{(d-1)(d-2)} g_{ab} R^2, \tag{A14}
\]

and
\[
C_{cdef}C^{cdef} = R_{cdef}R^{cdef} - \frac{4}{d-2}R_{cd}R^{cd} + \frac{2}{(d-1)(d-2)}R^2,
\]
(A15)

also have simple transformation properties under the local Weyl group.

From the transformation rule for covariant derivatives (A1) and (A2) one may derive also

\[
\nabla_a \nabla_b R = e^{-2\sigma} \left\{ \square_{ab} - 6\Gamma_{i(a}\sigma_{b)} + 2\Gamma_{i(a}\sigma_{b)} - 2(d-1)(\square_{ab} - (d-1)(d-2)(\sigma_{c}\sigma_{c})_{ab} + 4(d-1)\square_{c}\sigma_{ab} + 12(d-1)(\square_{i(a}\sigma_{b)} + (d-1)(d-2)(\sigma_{c}\sigma_{c})_{ab}
+ 12(d-1)(d-2)\sigma_{c}\sigma_{b})\sigma_{c} - 16(d-1)\sigma_{c}\sigma_{b} \square_{i(a}\sigma_{b)} - 8(d-1)(d-2)\sigma_{c}\sigma_{c}\sigma_{c} \\
+ g_{ab}[\Gamma_{i(a}\sigma_{b)} - 2\Gamma_{i(a}\sigma_{b)} - 2(d-1)(\square_{c\sigma_{c} - 2(d-1)(d-2))\sigma_{c\sigma_{c}} - (d-1)(d-2))\sigma_{c\sigma_{c}})
+ 4(d-1)(\square_{c\sigma_{c} + 2(d-1)(d-2))\sigma_{c\sigma_{c}}}
\right\},
\]
(A16)

and finally,

\[
\nabla_c \nabla_d C_{(a\ b)} = e^{-2\sigma} \left\{ \nabla_c \nabla_d C_{(a\ b)} + (d-3)C_{(a\ b)\sigma_{c\sigma_{c}} + 2(d-4)\sigma_{d}\nabla_c C_{(a\ b)} + (d-3)(d-5)C_{(a\ b)\sigma_{c\sigma_{c}}}
\right\}.
\]
(A18)

**APPENDIX B: AN IDENTITY OF THE WEYL TENSOR IN FOUR DIMENSIONS**

Using the van der Waerden method the Weyl tensor may be decomposed into its irreducible components, (2, 0) and (0, 2) of $SL(2, C) \oplus SL(2, C)$ corresponding to the self-dual and the anti-self-dual parts $C_{abcd} = C_{+ab} + C_{-ab}$. The Weyl tensor itself corresponds to the representation $(2, 0) \oplus (0, 2)$. The finite-dimensional irreducible representations of $SL(2, C)$ are given by the space of complex completely symmetric spinors with the number of indices equal to the twice the total spin, 2s. The completely symmetric spinor with $N(s)$ indices has $N(s) = 2s + 1$ independent components, which is exactly the dimension of the angular momentum s representation of $SU(2) \subset SL(2, C)$. The explicit mapping between $d = 4$ spacetime or tangent space indices and two-component spinorial indices is given by the Pauli-van der Waerden matrices $\sigma_{AB}$. The decomposition of the Weyl tensor into its self-dual and anti-self-dual components correspond in the spinorial description to

\[
C_{abcd} = C_{+ab} + C_{-ab} \Leftrightarrow \Psi_{ABCD} \epsilon_{\ A\ B\ C\ D} + \Psi_{ABCD} \epsilon_{\ A\ B\ C\ D}.
\]
(B1)

The anti-symmetric spinors $\epsilon_{AB}$ and $\epsilon_{AB}^c$ correspond to the ‘metric’ on the space of spinors; they are used to lower upper spinor indices and their ‘contravariant’ form raises spinor indices. Care must be taken as far as the order of spinor indices is concerned because of the anti-symmetric nature of $\epsilon$’s. Each of these representations in (B1) have a definite helicity and in physical terms the linearized Weyl tensor decomposition may be identified with the spin-2 helicity $\pm 2$ wave functions of the graviton field.

With this short introduction to $SL(2, C)$ spinors, we are ready to demonstrate that the symmetric tensor,

\[
C_{cdef}C^{cdef} - \frac{1}{4}\delta_{\ a}^{\ b}C_{cdef}C^{cdef} \equiv 0,
\]
(B2)

vanishes identically in $d = 4$ dimensions. Let us compute the first term in the above identity using the spinorial notation, (B1)

\[
C_{cdef}C_{bcde} \Leftrightarrow \left( \Psi^{ACDE} \epsilon^{ACDE} + \Psi^{ACDE} \epsilon^{ACDE} \right) \left( \Psi_{BCDE} \epsilon_{BCDE} + \Psi_{BCDE} \epsilon_{BCDE} \right)
= \Psi^{ACDE} \Psi_{BCDE} \epsilon^{ACDE} \epsilon_{BCDE} + c.c.
= 2\Psi^{ACDE} \Psi_{BCDE} \epsilon^{ACDE} \epsilon_{BCDE} + c.c.
= \Psi^{CDEF} \Psi_{CDEF} \delta^{A\ B} + c.c.
= \left( \Psi^{CDEF} \Psi_{CDEF} + \Psi^{CDEF} \Psi_{CDEF} \right) \delta^{A\ B}.
\]
(B3)
The variation of the Green’s function $\delta S$.

We shall evaluate all four terms in the variation following from the symmetry of $\Psi$ and the antisymmetry of $\epsilon$, namely

$$\Psi^{ACDE}\Psi_{BCDE} = \frac{1}{2}\Psi^{CDEF}\Psi_{CDEF}\delta_A^B. \quad (B4)$$

On the other hand the square of the Weyl tensor is given by the contraction over $A, B$ and $A, B$ of $[B3]$, i.e.

$$C^{bcde}C_{bcde} = 4\left(\Psi^{BCDE}\Psi_{BCDE} + \overline{\Psi}^{BCDE}\overline{\Psi}_{BCDE}\right). \quad (B5)$$

Converting $\delta_A^B\delta_A^B$ back to spacetime indices by multiplying $[B3]$ by the Pauli matrices $\sigma_u^{AB}$ and taking account of $[B5]$ then yields $[B2]$.

**APPENDIX C: THE TENSOR $C_{AB}$**

In this Appendix we compute the nonlocal tensor $C_{ab}$ defined by $[5.8]$ of the text. If we use the definition of the auxiliary field $\varphi$ given by $[4.21]$ and vary the action corresponding to the $F$ cocycle of the $d = 4$ anomaly given by the first term of $[4.21]$ we obtain

$$\delta S_{anom}^{(F)}[g] = \frac{b}{2} \int d^4x \left(\delta(\sqrt{-g} F_4)\varphi + \sqrt{-g} F_4\delta \varphi\right). \quad (C1)$$

The first term in the variation of $S_{anom}^{(F)}$ is already known and it is presented in the text. The second term leads to $C_{ab}$, which will be computed below. We need to compute the variation of $\varphi$ which is

$$\delta \varphi(x) = \frac{1}{2} \int d^4x' \left[\delta \left(\sqrt{-g}(E_4 - \frac{2}{3} \Box R)\right) D_4(x, x') + \sqrt{-g'}(E_4 - \frac{2}{3} \Box R')\delta D_4(x, x')\right]. \quad (C2)$$

The variation of the Green’s function $D_4(x, x')$ is

$$\delta D_4(x, x') = -\int d^4x'' D_4(x, x'')\delta \left(\sqrt{-g} \Delta_4\right)'' D_4(x'', x'). \quad (C3)$$

Defining a new non-local scalar $\psi$ by the formula,

$$\psi(x) = \frac{1}{2} \int d^4x'' (F_4)'' D_4(x'', x), \quad (C4)$$

we obtain the following formula for the total variation of $S_{anom}^{(F)}$:

$$\delta S_{anom}^{(F)}[g] = \frac{b}{2} \int d^4x \left[\delta(\sqrt{-g} F_4)\varphi + \sqrt{-g} \psi\delta \left(E_4 - \frac{2}{3} \Box R\right) - 2\sqrt{-g} \psi\delta \left(\Delta_4\right)\varphi\right], \quad (C5)$$

after taking advantage of the cancellation of terms obtained by varying the two $\sqrt{-g}$ factors in $[C2]$ and $[C3]$. Using the relation $[4.3]$ for $d = 4$, we can rewrite the variation of $S_{anom}^{(F)}$ in the form,

$$\delta S_{anom}^{(F)}[g] = \frac{b}{2} \int d^4x \left[\delta(\sqrt{-g} F_4)\varphi + \sqrt{-g} \psi\delta F_4 - 2\psi\delta(\Delta_4)\varphi - 2\psi\delta \left(R^{ab} R_{ab} - \frac{1}{3} R^2 + \frac{1}{3} \Box R\right)\right]. \quad (C6)$$

We shall evaluate all four terms in the variation $\delta S_{anom}^{(F)} = \delta S_1^{(F)} + \delta S_2^{(F)} + \delta S_3^{(F)} + \delta S_4^{(F)}$, where

$$\delta S_1^{(F)} = \frac{b}{2} \int d^4x \delta(\sqrt{-g} F_4)\varphi, \quad (C7)$$
\[ \delta S_{2}^{(F)} = \frac{b}{2} \int d^4 x \sqrt{-g} \psi \delta F_4, \]  

\[ \delta S_3^{(F)} = -b \int d^4 x \sqrt{-g} \psi \delta \Delta_4 \varphi \]  

\[ \delta S_4^{(F)} = -b \int d^4 x \sqrt{-g} \psi \delta \left( R^{ab} R_{ab} - \frac{1}{3} R^2 + \frac{1}{3} \Box R \right). \]  

Let \( \delta g^{ab} = h^{ab} \), then we have

\[ \delta S_1^{(F)} = \frac{b}{2} \int d^4 x \sqrt{-g} h^{ab} \left( 2 R^{cd} C_{acbd} \varphi + 4 \nabla^c \nabla^d (C_{acbd} \varphi) \right), \]  

\[ \delta S_2^{(F)} = \frac{b}{2} \int d^4 x \sqrt{-g} h^{ab} \left( 2 R^{cd} C_{acbd} \psi + 4 \nabla^c \nabla^d (C_{acbd} \psi) + \frac{1}{2} g_{ab} F_4 \psi \right), \]  

\[ \delta S_3^{(F)} = \frac{b}{2} \int d^4 x \sqrt{-g} h^{ab} \left[ -g_{ab} \left( 2 \nabla^2 \varphi - \Box \nabla \varphi + \frac{1}{3} \Box (\nabla^c \nabla_c \varphi) - \frac{2}{3} \nabla^c \nabla_d \varphi - \frac{2}{3} R \Box \varphi - \frac{2}{3} R \nabla^c \nabla_d \varphi + \frac{1}{3} \nabla^c R \nabla_d \varphi \right) 
+ 2 R^{cd} \nabla_c \varphi \nabla_d \varphi + 2 \psi R^{cd} \nabla_c \varphi \nabla_d \varphi 
+ 2 \nabla_a \Box \nabla_b \varphi + 2 \nabla_a \Box \nabla_b \varphi - 2 \nabla^c \nabla^d \psi \nabla_a \nabla_b \varphi - 2 \nabla^c \nabla^d \psi \nabla_a \nabla_b \varphi \right) \]  

\[ + \frac{4}{3} \nabla_a \nabla_b (\nabla^c \nabla_d \varphi) - 2 \Box \nabla_a \nabla_b \varphi - 2 \varphi \nabla_a \nabla_b \varphi \varphi + 4 R^c \nabla_a \nabla_b \varphi \nabla_c \varphi \nabla_b \varphi - \frac{4}{3} R_{ab} \nabla^c \nabla_d \varphi - \frac{4}{3} R \nabla_a \nabla_b \varphi \]  

\[ \delta S_4^{(F)} = \frac{b}{2} \int d^4 x \sqrt{-g} h^{ab} \left\{ -2 g_{ab} \left[ \frac{1}{3} \nabla^2 \psi + R^{cd} \nabla_c \nabla_d \psi - \frac{2}{3} R \Box \psi - \frac{1}{6} \nabla^c R \nabla_c \psi \right] 
- 2 \left[ \psi \left( 2 R^{cd} R_{acbd} - \frac{2}{3} R R_{ab} + \Box R_{ab} - \frac{1}{3} \nabla_a \nabla_b R \right) + \frac{4}{3} R_{ab} \Box \psi + \frac{2}{3} R \nabla_a \nabla_b \psi \right] 
- 2 R_a \nabla_b \nabla_c \psi + 2 \nabla_c \psi (\nabla_a R_{ab} - \nabla_a R_{bc}) - \frac{1}{3} \nabla_a \nabla_b \Box \psi \right\}. \]  

From the last three equations one can read off the formula for the \( C_{ab} \) tensor, \( \text{i.e.} \)

\[ C_{ab} = -2 R^{cd} C_{abcd} + 4 \nabla^c \nabla^d (C_{(ab)cd} \psi) - \frac{1}{2} g_{ab} F_4 \psi + 2 g_{ab} \left( \frac{1}{3} \nabla^2 \psi + R^{cd} \nabla_c \nabla_d \psi - \frac{2}{3} R \Box \psi - \frac{1}{3} \nabla^c R \nabla_c \psi \right) 
+ \psi \left( 4 R^{cd} R_{acbd} - \frac{4}{3} R R_{ab} + 2 \nabla_a \nabla_b R \right) - \frac{2}{3} \nabla_a \nabla_b \Box \psi + \frac{8}{3} R_{ab} \Box \psi + \frac{4}{3} R \nabla_a \nabla_b \psi - 4 R^c (\nabla_a R_{bc}) \nabla_c \psi 
+ 4 \nabla^c \psi (\nabla_a R_{ab} - \nabla (a R_{bc})) + g_{ab} \left( \nabla^2 \varphi - \Box \nabla^2 \varphi + \frac{1}{3} \Box (\nabla^c \nabla_c \varphi) - \frac{2}{3} \nabla^c \nabla_d \varphi - \frac{2}{3} R \nabla^c \nabla_d \varphi + \frac{1}{3} \nabla^c R \nabla_d \varphi \right) 
+ 2 R^{cd} \nabla_c \psi (\nabla_a \nabla_b \varphi + 2 \nabla_a \nabla_b \psi \nabla^d \varphi) + 2 \nabla^d \nabla (\nabla^c \nabla_c \varphi) - 2 \nabla (\nabla^c \nabla^d \varphi) - 2 \nabla (\nabla^c \nabla^d \varphi) + 2 \nabla^c \nabla^d \psi \nabla_c \nabla_d \varphi + 2 \nabla^c \nabla^d \psi \nabla_c \nabla_d \psi \right) \]  

\[ + 2 \Box \nabla_a \nabla_b \varphi + 2 \varphi \nabla_a \nabla_b \varphi + \frac{4}{3} \nabla_a \nabla_b (\nabla^d \nabla^c \nabla_c \varphi) - 4 R^c (\nabla_a \nabla_b \psi + \nabla_c \nabla_d \varphi) + \frac{4}{3} R_{ab} \nabla^c \nabla_d \psi + \frac{4}{3} R \nabla (\nabla_a \nabla_b \varphi). \]  

Using the definitions of \( \varphi \) and \( \psi \) one can verify that the trace of the \( C_{ab} \) tensor as given above is local and equal to

\[ g^{ab} C_{ab} = C^{abcd} C_{abcd} = F_4, \]  

in agreement with (5.4) of the text.