Multipliers for continuous frames in Hilbert spaces

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Abstract

In this paper, we examine the general theory of continuous frame multipliers in Hilbert space. These operators are a generalization of the widely used notion of (discrete) frame multipliers. Well-known examples include anti-Wick operators, STFT multipliers or Calderón–Toeplitz operators. Due to the possible peculiarities of the underlying measure spaces, continuous frames do not behave quite as their discrete counterparts. Nonetheless, many results similar to the discrete case are proven for continuous frame multipliers as well, for instance compactness and Schatten-class properties. Furthermore, the concepts of controlled and weighted frames are transferred to the continuous setting.

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1. Introduction

A discrete frame is a countable family of elements in a separable Hilbert space, which allows stable but not necessarily unique decomposition of arbitrary elements into expansion of the frame elements. The concept of generalization of frames was proposed by Kaiser [31] and independently by Ali et al [2] to a family indexed by some locally compact space endowed with a Radon measure. These frames are known as continuous frames. Gabardo and Han in [29] called these frames frames associated with measurable spaces and Askari-Hemmat et al in [4] called these frames generalized frames and they are linked to coherent states in mathematical physics [2]. For more studies, the interested reader can also refer to [1, 3, 16, 28].

Bessel and frame multipliers were introduced by one of the authors [5–7] for Hilbert spaces. For Bessel sequences, the investigation of the operator \( M = \sum m_k (f, \psi_k) \psi_k \), where the analysis coefficients \( (f, \psi_k) \) are multiplied by a fixed symbol \( m_k \) before resynthesis (with \( \varphi_k \)), is very natural. There are numerous applications of this kind of operator. As a particular
way to implement time-variant filters, Gabor frame multipliers [26] are used, also known as Gabor filters [35]. Such operators find application in psychoacoustics [9], denoising [33], computational auditory scene analysis [49], virtual acoustics [32] and seismic data analysis [34]. On a more theoretical level, Bessel multipliers of $p$-Bessel sequences in Banach spaces are introduced in [42].

Wavelet and Gabor frames are used very often in a signal-processing algorithm. Both systems are derived from a continuous frame transform. For these two special systems, continuous frame multipliers have been investigated as STFT multipliers [26], or anti-Wick operators [21] and Calderón–Toeplitz operators [37, 44], respectively. In this paper, we investigate multipliers for continuous frames in the general setting, with some comments on the mentioned special cases in section 3.4.

This paper is organized as follows. In section 2, we collect a number of notions and preliminaries on continuous Bessel mappings and frames and their most basic properties and present some well-known examples. In section 3, we define continuous Bessel and frame multipliers as generalizations of discrete Bessel and frame multipliers, develop their theory and prove a number of statements on the compactness of multipliers as well as on mapping properties with respect to Schatten classes. We also investigate perturbation results and the continuous dependence of the multiplier on the symbol and on the analysis and synthesis frames. We also look at the particular instances of STFT and wavelet multipliers, the latter are known as Calderón–Toeplitz operators, and compare our results to existing ones. In section 4, we generalize the concepts of controlled and weighted frames to the continuous setting.

2. Preliminaries

2.1. Operator theory and functional analysis

Throughout this paper, $\mathcal{H} (\mathcal{H}_1, \mathcal{H}_2)$ will be complex Hilbert spaces, with the inner product $\langle x, y \rangle$, linear in the first coordinate and conjugate linear in the second coordinate and the norm $\|x\| = \sqrt{\langle x, x \rangle}$ for $x, y \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. This set is a Banach space with the norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$. We define $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with bounded inverse. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we simply write $\mathcal{B}(\mathcal{H})$ and $GL(\mathcal{H})$. By $(e_n)$, we always denote an orthonormal basis for a Hilbert space. A map $\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a sesquilinear form if it is linear in the first variable and conjugate linear in the second variable. For such a map, we have the following assertion.

**Theorem 2.1** [40]. Let $\Psi$ be a bounded sesquilinear form on a Hilbert space $\mathcal{H}$. Then, there is a unique operator $u$ on $\mathcal{H}$ such that

$$\Psi(x, y) = \langle u(x), y \rangle \quad (x, y \in \mathcal{H}).$$

Moreover, $\|u\| = \|\Psi\|$. A bounded operator $T$ is called positive (respectively non-negative), if $\langle Tf, f \rangle > 0$ for all $f \neq 0$ (respectively $\langle Tf, f \rangle \geq 0$ for all $f \in \mathcal{H}$). We say $S > T$ if $S - T > 0$ (respectively $S \geq T$, if $S - T \geq 0$). For a non-negative operator $T$, there exists a unique non-negative operator $S$ on $\mathcal{H}$, such that $S^2 = T$ and $S$ commutes with every operator that commutes with $T$. See, e.g., [17] or [30] for good accounts of elementary operator theory.

A linear operator $T$ from the Banach space $X$ into the Banach space $Y$ is called compact if the image under $T$ of the closed unit ball in $X$ is a relatively compact subset of $Y$, or, equivalently, if the image of any bounded sequence contains a convergent subsequence. A well-known characterization of compact operators is the following.
Lemma 2.2 [19]. Let $X$ and $Y$ be Banach spaces. A bounded operator $T : X \to Y$ is compact if and only if $\|Tx_n\| \to 0$, whenever $x_n \to 0$ weakly in $X$.

For any compact operator $T : \mathcal{H} \to \mathcal{K}$, the operator $T^*T : \mathcal{H} \to \mathcal{H}$ is compact and non-negative. The unique non-negative operator $S$, such that $S^2 = T^*T$, is also compact. The eigenvalues of $S$ are called the singular values of $T$. They form a non-increasing sequence of non-negative numbers that either consists of only finitely many nonzero terms or converges to zero. If the sequence of singular values $(s_n)$ is in $\ell^p$, $1 \leq p < \infty$, then $T$ belongs to the Schatten $p$-class $S_p(\mathcal{H})$. In particular, if $\sum |s_n| < \infty$, then $T$ is a trace-class operator; if $\sum |s_n|^2 < \infty$, then $T$ is a Hilbert–Schmidt operator. A good source for information on Schatten-class operators is, e.g., [52].

We recall the definition of a discrete frame.

**Definition 2.3.** A family $(f_n) \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A > 0$ and $B < \infty$, such that

$$A \|f\|^2 \leq \sum_n |\langle f, f_n \rangle|^2 \leq B \|f\|^2$$

for all $f \in \mathcal{H}$. If $A = B$, then it is called a tight frame.

### 2.2. Continuous frames

**Definition 2.4.** Let $\mathcal{H}$ be a complex Hilbert space and $(\Omega, \mu)$ be a measure space with positive measure $\mu$. The mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to $(\Omega, \mu)$, if

(i) $F$ is weakly measurable, i.e. for all $f \in \mathcal{H}$, $\omega \to \langle f, F(\omega) \rangle$ is a measurable function on $\Omega$;

(ii) there exist constants $A, B > 0$, such that

$$A \|f\|^2 \leq \int_{\Omega} |\langle f, F(\omega) \rangle|^2 \, d\mu(\omega) \leq B \|f\|^2 \quad (f \in \mathcal{H}).$$

The constants $A$ and $B$ are called continuous frame bounds. If $A = B$, then $F$ is called a tight continuous frame, if $A = B = 1$ a Parseval frame. The mapping $F$ is called the Bessel mapping or in short Bessel if only the inequality on the right in (ii) holds. In this case, $B$ is called the Bessel constant or Bessel bound.

If $\Omega = \mathbb{N}$ and $\mu$ is a counting measure, then $F$ is a discrete frame. In this sense, continuous frames are the more general setting.

The first inequality in (ii) shows that $F$ is complete, i.e.

$$\text{span}\{F(\omega)\}_{\omega \in \Omega} = \mathcal{H}.$$  

It is well known that discrete Bessel sequences in a Hilbert space are norm bounded above:

$$\sum_n |\langle f, f_n \rangle|^2 \leq B \|f\|^2$$

for all $f \in \mathcal{H}$, then

$$\|f_n\| \leq \sqrt{B}$$

for all. For continuous Bessel mappings, however, this is not necessary. Consider the following example.
Example 2.5. Take an (essentially) unbounded (Lebesgue) measurable function \( a : \mathbb{R} \to \mathbb{C} \), such that \( a \in L^2(\mathbb{R}) \setminus L^\infty(\mathbb{R}) \). It is easy to see that such functions indeed exist; consider for example the function

\[
b(x) := \begin{cases} 
\frac{1}{\sqrt{|x|}}, & \text{if } 0 < |x| < 1, \\
\frac{1}{|x|^2}, & \text{if } |x| \geq 1, \\
0, & \text{if } x = 0.
\end{cases}
\]

This function is clearly in \( L^1(\mathbb{R}) \setminus L^\infty(\mathbb{R}) \), and furthermore, \( b(x) \geq 0 \) for all \( x \in \mathbb{R} \).

Now take \( a(x) := \sqrt{b(x)} \). Choose a fixed vector \( h \in \mathcal{H}, h \neq 0 \). Then, the mapping

\[
F : \mathbb{R} \to \mathcal{H}, \quad \omega \mapsto F(\omega) := a(\omega) \cdot h
\]

is weakly (Lebesgue) measurable and a continuous Bessel mapping, since

\[
\int_{\mathbb{R}} |\langle f, F(\omega) \rangle|^2 d\mu(\omega) = \int_{\mathbb{R}} |a(\omega)|^2 |\langle f, h \rangle|^2 d\mu(\omega)
= |\langle f, h \rangle|^2 \int_{\mathbb{R}} |a(\omega)|^2 d\mu(\omega)
\leq \|h\|^2 \|a\|^2_{L^2(\mathbb{R})} \|f\|^2
\]

for all \( f \in \mathcal{H} \), but

\[
\|F(\omega)\| = \|a(\omega)h\| = |a(\omega)||h|
\]

is unbounded, since \( a \) is unbounded.

Even continuous frames need not necessarily be norm bounded.

Example 2.6. Let \( F : \mathbb{R} \to \mathcal{H} \) be a norm-unbounded continuous Bessel mapping with the Bessel constant \( B_F \), as in the previous example. Let \( G : \mathbb{R} \to \mathcal{H} \) be a norm-bounded continuous frame (for example, a continuous wavelet or Gabor frame, cf section 2.3) with continuous frame bounds \( 0 < A_G \leq B_G \) and norm bound \( M > 0 \), i.e. \( \|G(\omega)\| \leq M \) for a.e. \( \omega \in \mathbb{R} \).

Then, \( G + \epsilon F \) is a norm-unbounded continuous frame, for all sufficiently small \( \epsilon > 0 \).

To see this, first note that it is obvious that the mapping \( G + \epsilon F : \mathbb{R} \to \mathcal{H} \) is weakly measurable for any choice of \( \epsilon > 0 \). It satisfies the upper frame bound, since

\[
\int_{\mathbb{R}} |\langle f, G(\omega) + \epsilon F(\omega) \rangle|^2 d\mu(\omega) \leq \int_{\mathbb{R}} (|\langle f, G(\omega) \rangle|^2 + \epsilon |\langle f, F(\omega) \rangle|^2) d\mu(\omega)
\leq 2 \cdot \int_{\mathbb{R}} (|\langle f, G(\omega) \rangle|^2 + \epsilon^2 |\langle f, F(\omega) \rangle|^2) d\mu(\omega)
\leq 2 \cdot (B_G + \epsilon^2 B_F) \cdot \|f\|^2.
\]

For the lower frame bound, observe that

\[
\left( \int_{\mathbb{R}} |\langle f, G(\omega) + \epsilon F(\omega) \rangle|^2 d\mu(\omega) \right)^{1/2}
\geq \left( \int_{\mathbb{R}} |\langle f, G(\omega) \rangle|^2 d\mu(\omega) \right)^{1/2} - \left( \int_{\mathbb{R}} \epsilon^2 |\langle f, F(\omega) \rangle|^2 d\mu(\omega) \right)^{1/2}
\geq \sqrt{A_G} \|f\| - \epsilon \sqrt{B_F} \|f\|
= (\sqrt{A_G} - \epsilon \sqrt{B_F}) \|f\|.
\]

Now choose \( \epsilon < \sqrt{\frac{A_G}{B_F}} \), and then \( \sqrt{A_G} - \epsilon \sqrt{B_F} > 0 \), and the lower frame bound is established.
This continuous frame is, however, not norm bounded, since
\[ \|G(\omega) + \varepsilon F(\omega)\| \geq \varepsilon \|F(\omega)\| - \|G(\omega)\| \geq \varepsilon \|F(\omega)\| - M; \]
with \(F\) being unbounded, this is unbounded as well.

The construction in the last two examples depends crucially on the existence of an unbounded square-integrable function, or equivalently, on the existence of an unbounded integrable function. It can be generalized to the following theorem.

**Theorem 2.7.** Let \((\Omega, \mu)\) be a measure space, such that \(L^1(\Omega, \mu) \nsubseteq L^\infty(\Omega, \mu)\), i.e. there exist unbounded integrable functions. Then, the following holds.

If there exist any continuous frames at all with respect to \((\Omega, \mu)\), then there are also norm-unbounded ones.

**Proof.** Fix a vector \(h \in \mathcal{H}, h \neq 0\). Pick a function \(b : \Omega \to \mathbb{C}, b \in L^1(\Omega, \mu) \setminus L^\infty(\Omega, \mu)\). Then, \(a := \sqrt{|b|}\) is a function in \(L^2(\Omega, \mu) \setminus L^\infty(\Omega, \mu)\), i.e. an unbounded square-integrable function. As in example 2.5, the mapping \(F : \Omega \to \mathcal{H}, F(\omega) = a(\omega) \cdot h\) is a norm-unbounded continuous Bessel mapping. If there exists a norm-bounded continuous frame \(G : \Omega \to \mathcal{H}\), then one can show as in example 2.6 that the mapping \(G + \varepsilon F\) is a norm-unbounded continuous frame, for sufficiently small \(\varepsilon\). \(\square\)

Concerning the existence of continuous frames, we have the following result.

**Theorem 2.8.** Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space. Then, there exists a continuous tight frame \(F : \Omega \to \mathcal{H}\) with respect to \((\Omega, \mu)\).

**Proof.** Since \(\Omega\) is \(\sigma\)-finite, it can be written as a disjoint union \(\Omega = \bigcup \Omega_k\) of countably many subsets \(\Omega_k \subseteq \Omega\), such that \(\mu(\Omega_k) < \infty\) for all \(k\). Without loss of generality, assume that \(\mu(\Omega_k) > 0\) for all \(k\). If there are infinitely many such subsets \(\Omega_k, k \in \mathbb{N}\), then let \((e_k)_{k \in \mathbb{N}}\) be an orthonormal basis of an infinite-dimensional separable Hilbert space \(\mathcal{H}\). Define the function \(F : \Omega \to \mathcal{H}\) by
\[ \omega \mapsto F(\omega) := \frac{1}{\sqrt{\mu(\Omega_k)}} e_k, \quad \text{for } \omega \in \Omega_k. \]
Then, for all \(f \in \mathcal{H}\),
\[ \int_\Omega |\langle f, F(\omega) \rangle|^2 \, d\mu(\omega) = \sum_k \int_{\Omega_k} |\langle f, F(\omega) \rangle|^2 \, d\mu(\omega) \]
\[ = \sum_k |\langle f, e_k \rangle|^2 \frac{1}{\mu(\Omega_k)} \mu(\Omega_k) \]
\[ = \|f\|^2. \]
Thus, \(F\) is a continuous tight frame with frame bound 1. If there are only finitely many \(\Omega_k, k = 1, \ldots, N\), then take for \(\mathcal{H}\) an \(N\)-dimensional Hilbert space instead and proceed analogously. \(\square\)

For the convenience of the reader, we briefly repeat some basic facts and notions on continuous frames. Details may be found, e.g., in [2] or [41].

Let \(F\) be a continuous frame with respect to \((\Omega, \mu)\); then, the mapping
\[ \Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \]
defined by
\[ \Psi(f, g) = \int_{\Omega} \langle f, F(\omega) \rangle \langle F(\omega), g \rangle \, d\mu(\omega) \]
is well defined, sesquilinear and bounded. By the Cauchy–Schwarz inequality, we obtain
\begin{align*}
|\Psi(f, g)| & \leq \int_{\Omega} |\langle f, F(\omega) \rangle | \, d\mu(\omega) \\
& \leq \left( \int_{\Omega} |\langle f, F(\omega) \rangle |^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} |\langle F(\omega), g \rangle |^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \\
& \leq B \|f\| \|g\|.
\end{align*}
Hence, \( \|\Psi\| \leq B \). By theorem 2.1, there exists a unique operator \( S_F : \mathcal{H} \to \mathcal{H} \) such that
\[ \Psi(f, g) = \langle S_F f, g \rangle, \quad (f, g \in \mathcal{H}) \]
and moreover \( \|\Psi\| = \|S\| \).

Since \( \langle S_F f, f \rangle = \int_{\Omega} |\langle f, F(\omega) \rangle |^2 \, d\mu(\omega), S_F \) is positive and \( A I \leq S_F \leq B I \). Hence, \( S_F \) is invertible, positive and \( \frac{1}{2} I \leq S_F^{-1} \leq \frac{1}{2} I \). We call \( S_F \) the continuous frame operator of \( F \) and we use the notation \( S_F f = \int_{\Omega} \langle f, F(\omega) \rangle F(\omega) \, d\mu(\omega) \), which is valid in the weak sense. Thus, every \( f \in \mathcal{H} \) has the (weak) representations
\[ f = S_F^{-1} S_F f = \int_{\Omega} \langle f, F(\omega) \rangle S_F^{-1} F(\omega) \, d\mu(\omega), \]
\[ f = S_F S_F^{-1} f = \int_{\Omega} \langle f, S_F^{-1} F(\omega) \rangle F(\omega) \, d\mu(\omega). \]

**Theorem 2.9** [45]. Let \((\Omega, \mu)\) be a measure space and let \( F \) be a Bessel mapping from \( \Omega \) to \( \mathcal{H} \). Then, the operator \( T_F : L^2(\Omega, \mu) \to \mathcal{H} \) weakly defined by
\[ \langle T_F \psi, h \rangle = \int_{\Omega} \psi(\omega) \langle F(\omega), h \rangle \, d\mu(\omega) \quad (h \in \mathcal{H}) \]
is well defined, linear and bounded, and its adjoint is given by
\[ T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega) \rangle \quad (\omega \in \Omega). \]
The operator \( T_F \) is called the synthesis operator and \( T_F^* \) is called the analysis operator of \( F \).

Such as in the discrete case we have the next proposition.

**Proposition 2.10** [45]. Let \( F : \Omega \to \mathcal{H} \) be a Bessel function with respect to \((\Omega, \mu)\). By the above notations, \( S_F = T_F T_F^* \).

Using an analogous statement as in [41] for the synthesis operator, it is easy to prove a characterization of continuous frames in terms of the frame operator.

**Theorem 2.11**. Let \((\Omega, \mu)\) be a \( \sigma \)-finite measure space.

The mapping \( F : \Omega \to \mathcal{H} \) is a continuous frame with respect to \((\Omega, \mu)\) for \( \mathcal{H} \) if and only if the operator \( S_F \) is a bounded and invertible operator.

**Definition 2.12**. Let \( F \) and \( G \) be continuous frames with respect to \((\Omega, \mu)\) for \( \mathcal{H} \). We call \( G \) a dual of \( F \) if the following holds true:
\[ \langle f, g \rangle = \int_{\Omega} \langle f, F(\omega) \rangle \langle G(\omega), g \rangle \, d\mu \quad (f, g \in \mathcal{H}). \]
In this case, \( (F, G) \) is called a dual pair. It is clear that (2.12) is equivalent with \( T_G T_F^* = I \).

It is certainly possible for a continuous frame \( F \) to have only one dual. In this case, we call \( F \) a Riesz-type frame.

**Proposition 2.13** [33]. Let \( F \) be a continuous frame with respect to \((\Omega, \mu)\) for \( \mathcal{H} \). Then, \( F \) is a Riesz-type frame if and only if \( R(T_F^*) = L^2(\Omega, \mu) \).
2.3. Gabor and wavelet systems

Well-known examples for frames are wavelet and Gabor systems. The corresponding continuous wavelet and STFT transforms give rise to continuous frames. We make use of the following unitary operators on $L^2(\mathbb{R})$:

- Translation: $T_x f(t) := f(t - x)$, for $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$.
- Modulation: $M_y f(t) := e^{2 \pi i y t} f(t)$, for $f \in L^2(\mathbb{R})$ and $y \in \mathbb{R}$.
- Dilation: $D_z f(t) := \frac{1}{|z|} f\left(\frac{t}{z}\right)$, for $f \in L^2(\mathbb{R})$ and $z > 0$.

Definition 2.14. Let $\psi \in L^2(\mathbb{R})$, and let

$$C_\psi := \int_{-\infty}^{+\infty} |\hat{\psi}(\gamma)|^2 d\gamma,$$

where $\hat{\psi}$ denotes the Fourier transform of $\psi$. The function $\psi$ is called admissible if $0 < C_\psi < +\infty$. For $a, b \in \mathbb{R}$ with $a \neq 0$, let

$$\psi^{a, b}(x) := (T_b D_a \psi)(x) = \frac{1}{|a|^{\frac{1}{2}}} \psi\left(\frac{x - b}{a}\right), \quad (x \in \mathbb{R}).$$

Then, the continuous wavelet transform $W_\psi$ is defined by

$$W_{\psi}(f)(a, b) := (f, \psi^{a, b}) = \int_{-\infty}^{+\infty} f(x) \frac{1}{|a|^{\frac{1}{2}}} \psi\left(\frac{x - b}{a}\right) dx, \quad f \in L^2(\mathbb{R}).$$

For an admissible function $\psi$ in $L^2$, the system $\{\psi^{a, b}\}_{a \neq 0, b \in \mathbb{R}}$ is a continuous tight frame for $L^2(\mathbb{R})$ with respect to $\Omega = \mathbb{R} \setminus \{0\} \times \mathbb{R}$ equipped with the measure $\frac{da}{a^2}$ and for all $f \in L^2(\mathbb{R})$

$$f = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_\psi(f)(a, b) \psi^{a, b} \frac{da}{a^2} db,$$

where the integral is understood in weak sense (this formula is known as the Calderón reproducing formula, cf [23]). This system constitutes a continuous tight frame with frame bound $\frac{1}{C_\psi}$. If $\psi$ is suitably normed so that $C_\psi = 1$, then the frame bound is 1, i.e. we have a continuous Parseval frame. For details, see proposition 11.1.1 and corollary 11.1.2 of [15].

Definition 2.15. Fix a function $g \in L^2(\mathbb{R}) \setminus \{0\}$. The short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to the window function $g$ is given by

$$\Psi_g(f)(y, \gamma) = \int_{-\infty}^{+\infty} f(x) g(x - y) e^{-2 \pi i x \gamma} dx, \quad (y, \gamma \in \mathbb{R}).$$

Note that in terms of modulation operators and translation operators, $\Psi_g(f)(y, \gamma) = \langle f, M_y T_{-\gamma} g \rangle$.

Let $g \in L^2(\mathbb{R}) \setminus \{0\}$, then $\{M_y T_{-\gamma} g\}_{a, b \in \mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $\Omega = \mathbb{R}^2$ equipped with the Lebesgue measure. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi_{g_1}(f_1)(a, b) \Psi_{g_2}(f_2)(a, b) db da = \langle f_1, f_2 \rangle \langle g_2, g_1 \rangle.$$

So, this system represents a continuous tight frame with the bound $\|g\|^2$. For details, see proposition 8.1.2 of [15].
3. Continuous frame multipliers

Gabor multipliers [26] led to the introduction of Bessel and frame multipliers for abstract Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators [5–7].

**Definition 3.1.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k) \subseteq \mathcal{H}_2$ be Bessel sequences. Fix $m = (m_k) \in l^\infty$. The operator $M_{(m), (\psi), (\phi)} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by

$$M_{(m), (\psi), (\phi)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k, \quad (f \in \mathcal{H}_1)$$

is called the Bessel multiplier for the Bessel sequences $(\psi_k)$ and $(\phi_k)$. The sequence $m$ is called the symbol of $M$. For frames, the resulting Bessel multiplier is called a frame multiplier, and for the Riesz sequence, a Riesz multiplier.

This motivates the following definition in the continuous case.

**Definition 3.2.** Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ with respect to $(\Omega, \mu)$ and $m : \Omega \to \mathbb{C}$ be a measurable function. The operator $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\langle M_{m,F,G} f, g \rangle = \int_\Omega m(\omega) \langle f, F(\omega) \rangle \langle G(\omega), g \rangle \, d\mu(\omega)$$

for all $f, g \in \mathcal{H}$, is called the continuous Bessel multiplier of $F$ and $G$ with respect to the mapping $m$, called the symbol.

We use the following notation to be understood in weak sense as above:

$$M_{m,F,G} f := \int_\Omega m(\omega) \langle f, F(\omega) \rangle \langle G(\omega), g \rangle \, d\mu(\omega).$$

**Lemma 3.3.** Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ with respect to $(\Omega, \mu)$ with bounds $B_F$ and $B_G$. Let $m \in L^\infty(\Omega, \mu)$. The operator $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\langle M_{m,F,G} f, g \rangle = \int_\Omega m(\omega) \langle f, F(\omega) \rangle \langle G(\omega), g \rangle \, d\mu(\omega)$$

for all $f, g \in \mathcal{H}$, is well defined and bounded with

$$\|M_{m,F,G}\| \leq \|m\|_\infty \sqrt{B_F B_G}.$$

**Proof.** It is clear that for each $f, g \in \mathcal{H}$,

$$|\langle M_{m,F,G} f, g \rangle| \leq \|m\|_\infty \int_\Omega |\langle f, F(\omega) \rangle \langle G(\omega), g \rangle| \, d\mu(\omega)$$

$$\leq \|m\|_\infty \left( \int_\Omega |\langle f, F(\omega) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left( \int_\Omega |\langle G(\omega), g \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}$$

$$\leq \|m\|_\infty \sqrt{B_F B_G} \|f\| \|g\|.$$

Thus, $M_{m,F,G}$ is well defined and bounded. $\square$

It is easy to prove that if $m(\omega) > 0$ a.e., then for any Bessel function $F$ the multiplier $M_{m,F,F}$ is a positive operator, and if $m(\omega) \geq \delta > 0$ for some positive constant $\delta$, then $M_{m,F,F}$ is just the frame operator of $\sqrt{m}F$ and thus is positive, self-adjoint and invertible.

By using synthesis and analysis operators, one easily shows that

$$M_{m,F,G} = T_0 D_m T_F^*,$$

where $D_m : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ and $(D_m \psi)(\omega) = m(\omega) \psi(\omega)$. It is proved that if $m \in L^\infty(\Omega, \mu)$, then $D_m$ is bounded and $\|D_m\| = \|m\|_\infty$ [17].
Proposition 3.4. Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ with respect to $(\Omega, \mu)$ and $m : \Omega \to \mathbb{C}$ be a measurable function; then, $(M_{m,F,G})^* = M_{m,F,G}^\ast$.

Proof. For $f, g \in \mathcal{H}$,
\[
\langle f, M_{m,F,G}^\ast g \rangle = \langle M_{m,F,G}^\ast f, g \rangle = \int_{\Omega} m(\omega) \langle f, F(\omega) \rangle \langle g, F(\omega) \rangle \, d\mu(\omega) = \int_{\Omega} (f, \langle m(\omega) \rangle G(\omega)) \, d\mu(\omega) = \langle f, M_{m,G,F} g \rangle.
\]

3.1. Multiplication operators on $L^2$

Motivated by the discrete case one might expect that $m \in L^p$ implies $D_m \in S_p$, where $S_p(\mathcal{H})$ denotes the family of Schatten $p$-class operators on $\mathcal{H}$. For $p = 1$, we have trace-class operators, and for $p = 2$, we have Hilbert–Schmidt operators. If this were true, we could easily, using representation (3.1), obtain results like in [6], since $S_p(\mathcal{H}_1, \mathcal{H}_2)$ is a two-sided $*$-ideal of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Unfortunately, the following proposition shows that at least for multiplication operators on $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, dx)$ (with $dx$ denoting the Lebesgue measure) the above considerations are never true, which constitutes a major difference between the discrete and the continuous cases. The result seems to be mathematical folklore; we give a full proof for completeness. We will use the following lemma, which is of independent interest.

Lemma 3.5. Let $A \subseteq \mathbb{R}^d$ be a measurable set of positive Lebesgue measures, $\lambda(A) > 0$. Then, there exists a partition of $A$ into countably infinitely many mutually disjoint measurable sets $A_n$, $n \in \mathbb{N}$, of positive measures, i.e. such that

1. $A = \bigcup_{n=1}^{\infty} A_n$,
2. $A_n \cap A_m = \emptyset$ for $n \neq m$,
3. $\lambda(A_n) > 0$ for all $n \in \mathbb{N}$.

Proof. It suffices to show that any $A \subseteq \mathbb{R}^d$ with $\lambda(A) > 0$ can be decomposed into two disjoint measurable sets $B$ and $C$, such that $A = B \cup C$, $B \cap C = \emptyset$ and $\lambda(B) > 0, \lambda(C) > 0$, since the claim follows from this by induction. Without loss of generality, assume further that $\lambda(A) =: L < \infty$. The whole space $\mathbb{R}^d$ can be covered by mutually disjoint $d$-dimensional half-open cubes $I_n, n \in \mathbb{N}$, sufficiently small such that $\lambda(I_n) < \frac{L}{2}$ for all $n \in \mathbb{N}$. Then, $A = \bigcup_{n \in \mathbb{N}} (A \cap I_n)$. Since $(A \cap I_n) \cap (A \cap I_m) = \emptyset$, we have $L = \lambda(A) = \sum_{n \in \mathbb{N}} \lambda(A \cap I_n)$. Now set

\[ N := \{ n \in \mathbb{N} : \lambda(A \cap I_n) > 0 \}. \]

Since $\lambda(A \cap I_n) \leq \lambda(I_n) \leq \frac{L}{2}$ for all $n \in \mathbb{N}, N$ must clearly contain at least two elements, say $n_1$ and $n_2$. Now set

\[ B := A \cap (I_{n_1}) \]

and

\[ C := (A \cap I_{n_1}) \cup \bigcup_{n \in N}(A \cap I_n). \]

Then, $B$ and $C$ have the stated properties. \qed

Now we can prove the following.
Proposition 3.6. Let $a \in L^\infty(\mathbb{R}^d)$. Denote by $D_a : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $f \mapsto a \cdot f$, the bounded multiplication operator with the symbol $a$.

Then, $D_a$ is a compact operator if and only if $a = 0$.

Proof. Assume $a \neq 0$. Let $\|a\|_{L^\infty} = c > 0$. Define

$$A := \left\{ x \in \mathbb{R}^d : |a(x)| > \frac{c}{2} \right\}.$$ 

Then, $A$ is a set of positive Lebesgue measure, $\lambda(A) > 0$. Find a partition of $A$ as in the preceding lemma, i.e. into countably infinitely many measurable subsets $A_n$, $n \in \mathbb{N}$, such that

1. $A = \bigcup_{n \in \mathbb{N}} A_n$, (2) $A_n \cap A_m = \emptyset$ for $n \neq m$, i.e. the sets $A_n$ are mutually disjoint, and (3) $\lambda(A_n) > 0$ for all $n \in \mathbb{N}$, i.e. all the sets $A_n$ have a strictly positive Lebesgue measure. Then, set

$$f_n := \chi_{A_n} \cdot \frac{1}{\sqrt{\lambda(A_n)}},$$

with $\chi_{A_n}$ being the characteristic function of $A_n$. Since

$$\langle f_n, f_m \rangle = \int_{\mathbb{R}^d} \chi_{A_n}(x) \cdot \frac{1}{\sqrt{\lambda(A_n)}} \cdot \chi_{A_m}(x) \cdot \frac{1}{\sqrt{\lambda(A_m)}} \, dx$$

$$= \int_{A_n \cap A_m} \frac{1}{\sqrt{\lambda(A_n)} \lambda(A_m)} \, dx$$

$$= \begin{cases} f_n \cdot \frac{1}{\sqrt{\lambda(A_n)}} \, dx = 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

the sequence of functions $(f_n)$ constitutes an orthonormal system in $L^2(\mathbb{R}^d)$. As such, it satisfies $f_n \to 0$ weakly by Bessel’s inequality. But $D_a f_n(x) = a(x) \cdot \chi_{A_n}(x) \frac{1}{\sqrt{\lambda(A_n)}}$, and

$$\|D_a f_n\|^2 = \int_{\mathbb{R}^d} |a(x)|^2 \cdot |\chi_{A_n}(x)| \frac{1}{\sqrt{\lambda(A_n)}}^2 \, dx$$

$$= \int_{A_n} |a(x)|^2 \frac{1}{\lambda(A_n)} \, dx$$

$$\geq \int_{A_n} \left( \frac{c}{2} \right)^2 \frac{1}{\lambda(A_n)} \, dx$$

Thus, $\|D_a f_n\| \to 0$. Hence, $D_a$ cannot be compact by lemma 2.2. \qed

In order to prove sufficient conditions for compactness of continuous frame multipliers, we thus have to choose a different approach than in the discrete setting. This will be addressed in the next section.

3.2. Compact multipliers

Theorem 3.7. Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ with respect to $(\Omega, \mu)$ and let either $F$ or $G$ be norm bounded, i.e. there is a constant $M > 0$ such that $\|F(\omega)\| \leq M$ resp. $\|G(\omega)\| \leq M$ for almost every $\omega \in \Omega$. Let $m : \Omega \to \mathbb{C}$ be a (essentially) bounded measurable function with support of a finite measure, i.e. there exists a subset $K \subseteq \Omega$ with $\mu(K) < \infty$ such that $m(\omega) = 0$ for almost every $\omega \in \Omega \setminus K$.

Then, $M_{m,F,G}$ is a compact operator.
Proof. We have
\[ M_{m,F,G} = T_G \circ D_m \circ T_F^* \]
with \( T_F^* \) being the analysis operator for \( F \), \( D_m \) the multiplication operator with the symbol \( m \) and \( T_G \) the synthesis operator for \( G \). Assume first that \( F \) is bounded, \( \|F(\omega)\| \leq M \) for almost all \( \omega \in \Omega \). We will show that \( D_m \circ T_F^* : \mathcal{H} \to L^2(\Omega, \mu) \) is compact. To this end, let \( f_n \to 0 \) weakly. Then,
\[
\|D_m T_F^* f_n\|^2 = \int_{\Omega} |m(\omega)|^2 \cdot |\langle f_n, F(\omega) \rangle|^2 \, d\mu(\omega)
= \int_K |m(\omega)|^2 \cdot |\langle f_n, F(\omega) \rangle|^2 \, d\mu(\omega).
\]
For the integrand,
\[
|m(\omega)|^2 \cdot |\langle f_n, F(\omega) \rangle|^2 \to 0
\]
for \( n \to \infty \) pointwise for every fixed \( \omega \in K \), since the weak convergence of \( \langle f_n, F(\omega) \rangle \) implies \( \langle f_n, F(\omega) \rangle \to 0 \) for every \( \omega \in \Omega \) fixed. Furthermore, weakly convergent sequences are bounded; thus, there is a constant \( C > 0 \), such that \( \|f_n\| \leq C \) for all \( n \in \mathbb{N} \), and
\[
|m(\omega)|^2 \cdot |\langle f_n, F(\omega) \rangle|^2 \leq ||m||^2_{\infty} \cdot ||f_n||^2 \cdot ||F(\omega)||^2 \\
\leq ||m||^2_{\infty} \cdot C^2 \cdot M^2
\]
for all \( n \in \mathbb{N} \). This constant is an integrable majorant on \( K \), so by Lebesgue’s dominated convergence theorem
\[
\int_K |m(\omega)|^2 \cdot |\langle f_n, F(\omega) \rangle|^2 \, d\mu(\omega) \to 0
\]
for \( n \to \infty \). Hence, the operator \( D_m \circ T_F^* \) maps weakly convergent sequences to norm convergent ones and is compact by lemma 2.2. So, \( M_{m,F,G} = T_G \circ (D_m \circ T_F^*) \) is compact as well.

If \( G \) is bounded instead of \( F \), consider the adjoint operator
\[ M_{m,F,G}^* = M_{m,G,F} = T_F \circ D_m \circ T_G^* ; \]
by what we have already shown, \( M_{m,G,F} \) is compact, and hence also \( M_{m,F,G} \). \( \square \)

Corollary 3.8. Let \( F \) and \( G \) be the Bessel mappings for \( \mathcal{H} \) with respect to \( (\Omega, \mu) \) and let either \( F \) or \( G \) be norm bounded. Let \( m : \Omega \to \mathbb{C} \) be a (essentially) bounded measurable function that vanishes at infinity, i.e. for every \( \varepsilon > 0 \) there is a set of finite measure \( K = K(\varepsilon) \subseteq \Omega \), \( \mu(K) < \infty \), such that \( m(\omega) \leq \varepsilon \) for almost every \( \omega \in \Omega \setminus K \). Then, \( M_{m,F,G} \) is compact.

Proof. For every \( n \in \mathbb{N} \), choose a set \( K_n \subseteq \Omega \), such that \( \mu(K_n) < \infty \) and \( |m(\omega)| \leq \frac{1}{n} \) for all \( \omega \not\in K_n \). Set
\[ m_n(\omega) := m(\omega) \cdot \chi_{K_n}(\omega), \]
where \( \chi_{K_n} \) denotes the characteristic function of the set \( K_n \). Then, obviously
\[ \|m_n - m\|_{\infty} \leq \frac{1}{n} \to 0 \]
for \( n \to \infty \), and thus
\[
\|M_{m,F,G} - M_{m,F,G}\| \leq \|m_n - m\|_{\infty} \sqrt{B_F B_G} \to 0
\]
by lemma 3.3. The functions \( m_n \) are bounded and of finite support, so \( M_{m,F,G} \) is compact for every \( n \in \mathbb{N} \) by the preceding theorem, and hence, \( M_{m,F,G} \) is also compact. \( \square \)

Now assume that both \( F \) and \( G \) are norm bounded. Then, we can prove a trace-class result. We use the following criterion.
Lemma 3.9 [43]. Let $\mathcal{H}$ be a Hilbert space. A bounded operator $T : \mathcal{H} \to \mathcal{H}$ is trace class if and only if $\sum_n |\langle Te_n, e_n \rangle| < \infty$ for every orthonormal basis $(e_n)$ of $\mathcal{H}$. Moreover,

$$\|T\|_{S^1} = \sup \left\{ \sum_n |\langle Te_n, e_n \rangle| : (e_n) \text{ orthonormal basis} \right\}.$$ 

Theorem 3.10. Let $F$ and $G$ be the norm-bounded Bessel mappings with norm bounds $L_F$ and $L_G$, respectively. Let $m \in L^1(\Omega, \mu)$. Then, $M_{m,F,G}$ is a well-defined bounded operator and a trace-class operator with

$$\|M_{m,F,G}\|_{S^1} \leq \|m\|_1 L_F L_G.$$ 

Proof. For arbitrary $f, g \in \mathcal{H}$, we have

$$\int_{\Omega} |m(\omega)||\langle f, F(\omega) \rangle||\langle G(\omega), g \rangle| \, d\mu(\omega)$$

$$\leq \int_{\Omega} |m(\omega)||f||\|F(\omega)||\|g||\|G(\omega)|| \, d\mu(\omega)$$

$$\leq \|f\|\|g\|L_F L_G \int_{\Omega} |m(\omega)| \, d\mu(\omega)$$

$$= \|f\|\|g\|L_F L_G \|m\|_1.$$ 

Thus, $M_{m,F,G}$ is a well-defined bounded linear operator by theorem 2.1. 

Take an arbitrary orthonormal basis $(e_n)$ of $\mathcal{H}$. Then,

$$\sum_n |\langle M_{m,F,G} e_n, e_n \rangle|$$

$$= \sum_n \left| \int_{\Omega} m(\omega) \langle e_n, F(\omega) \rangle \langle G(\omega), e_n \rangle \, d\mu(\omega) \right|$$

$$\leq \sum_n \left| \int_{\Omega} |m(\omega)| \cdot |\langle e_n, F(\omega) \rangle| \cdot |\langle G(\omega), e_n \rangle| \, d\mu(\omega) \right|$$

$$\leq \int_{\Omega} |m(\omega)| \sum_n |\langle e_n, F(\omega) \rangle| \cdot |\langle G(\omega), e_n \rangle| \, d\mu(\omega)$$

$$\leq \int_{\Omega} \left| m(\omega) \right| \left( \sum_n |\langle e_n, F(\omega) \rangle|^2 \right)^{1/2} \left( \sum_n |\langle G(\omega), e_n \rangle|^2 \right)^{1/2} \, d\mu(\omega)$$

$$\leq \|m\|_1 L_F L_G.$$ 

where we have used Fubini’s theorem and Cauchy–Schwarz’s inequality at the indicated places. Hence, $M_{m,F,G}$ is a trace class with the norm estimate $\|M_{m,F,G}\|_{S^1} \leq \|m\|_1 L_F L_G$, by the previous lemma 3.9. 

Having established the trace class case, we are now able to extend the result to the whole family of Schatten $p$-classes by complex interpolation; see, e.g., [12].

Theorem 3.11. Let $F$ and $G$ be the norm-bounded Bessel mappings with norm bounds $L_F$ and $L_G$, respectively. Let $m \in L^p(\Omega, \mu)$, $1 < p < \infty$. Then, $M_{m,F,G}$ is a well-defined bounded operator that belongs to the Schatten $p$-class $S_p(\mathcal{H})$, with a norm estimate

$$\|M_{m,F,G}\|_{S_p} \leq \|m\|_p (L_F L_G)^{1/p} (B_F B_G)^{1/2q}.$$
Proof. We first show that the operator is well defined by the weak definition 3.2. To this end, let $f, g \in \mathcal{H}$ be fixed. Observe that the functions $\omega \mapsto \langle f, F(\omega) \rangle$ resp. $\omega \mapsto \langle G(\omega), g \rangle$ are bounded (by $L_F \| f \|$ resp. $L_G \| g \|$) and belong to $L^2(\Omega, \mu)$ (because $F$ and $G$ are the Bessel mappings); hence, their product $\omega \mapsto \langle f, F(\omega) \rangle \langle G(\omega), g \rangle$ is in $L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu)$. But $L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu) \subseteq L^q(\Omega, \mu)$ for all $1 \leq q < \infty$.

Thus, for all $f, g \in \mathcal{H}$,

$$|\langle M_{\theta,F,G} f, g \rangle| \leq \int_\Omega |m(\omega)| |\langle f, F(\omega) \rangle |G(\omega), g\rangle| \, d\mu(\omega) \leq \|m\|_p \|\langle f, F(\omega) \rangle \rangle \|G(\omega), g\|_q$$

by Hölder’s inequality, with $\frac{1}{p} + \frac{1}{q} = 1$. The second term can be estimated as

$$\| \langle f, F(\omega) \rangle \rangle \|G(\omega), g\|_q \leq \| \langle f, F(\omega) \rangle \rangle \|G(\omega), g\|_q^{-1} \| \langle f, F(\omega) \rangle \rangle \|G(\omega), g\|_q$$

$$\leq L_F^{-1} \| f \|^{q-1} L_G^{-1} \| g \|^{q-1} \int_\Omega |\langle f, F(\omega) \rangle \rangle \|G(\omega), g\| \, d\mu(\omega)$$

$$\leq L_F^{-1} L_G^{-1} \| f \|^{q-1} \| g \|^{q-1} \| F \| \| B_G \| \| f \|^{q} \| g \|^{q}.$$ 

Now assume that $\| f \|, \| g \| \leq 1$. Then,

$$|\langle M_{\theta,F,G} f, g \rangle| \leq \|m\|_p \|F \| \| B_G \| \| f \| \| g \|.$$ 

This proves that $M_{\theta,F,G}$ is a well-defined bounded operator.

Now lemma 3.3 shows that the mapping $L^\infty(\Omega, \mu) \rightarrow B(\mathcal{H}), m \mapsto M_{\theta,F,G}$, is a bounded linear operator. The same is true for the mapping $L^1(\Omega, \mu) \rightarrow S_1(\mathcal{H}), m \mapsto M_{\theta,F,G}$, by theorem 3.10. Now let $\theta = 1 - \frac{1}{p} = \frac{1}{q}$ (i.e. such that $\frac{1}{p} = \frac{1}{\theta} + \frac{1}{\infty}$). A standard complex interpolation ([12]), between the Banach spaces $[L^1(\Omega, \mu), L^\infty(\Omega, \mu)]_\theta = L^p(\Omega, \mu)$ on the one hand and $[S_1(\mathcal{H}), B(\mathcal{H})] = [S_1(\mathcal{H}), S_{\infty}(\mathcal{H})]_\theta = S_p(\mathcal{H})$ on the other, proves that the mapping $m \mapsto M_{\theta,F,G}$ gives also a bounded linear operator from $L^p(\Omega, \mu)$ to the Schatten $p$-class $S_p(\mathcal{H})$ with the norm estimate

$$\| M_{\theta,F,G} \|_{S_p} \leq \|m\|_p^{\frac{1}{p}} (\| F \| \| B_G \|)^{\frac{1}{p}} = \|m\|_p^{\frac{1}{p}} (\| F \| \| B_G \|)^{\frac{1}{2p}}.$$ 

□

3.3. Changing the ingredients

Like discrete Bessel multipliers [5], a continuous Bessel multiplier clearly depends on the chosen symbol, analysis and synthesis functions. The following natural questions arise. What happens if these items are changed? Are the frame multipliers similar to each other if the symbol or the frames are similar to each other (in the right similarity sense)? Do the multipliers depend continuously on the input data?

Let $m, m' \in L^\infty$ and $F, F', G, G'$ be the Bessel functions. Representation (3.1) and linearity of the operators $T_F, T_\mathcal{F}, T_G, T_\mathcal{G}, D_m$ and $D_m$ result

$$M_{\theta,F,G} - M_{\theta',F,G} = T_G D_{m} T_F^{\mathcal{F}} = M_{m,m',F,G},$$

$$M_{\theta,F,G} - M_{\theta,F',G} = T_G D_{m} T_{F-F'}^{\mathcal{F}} = M_{m,F-F',G},$$

$$M_{\theta,F,G} - M_{\theta,F,G'} = T_{G,G'} D_{m} T_F^{\mathcal{G}} = M_{m,F,G-G'}.$$
By adapting the methods in [5] and using the above identities, we can prove the following theorem about continuous Bessel multipliers.

**Theorem 3.12.** Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ with respect to $(\Omega, \mu)$ and $m : \Omega \to \mathbb{C}$ be a measurable function. Let $m^{(n)}$ be functions indexed by $n \in \mathbb{N}$ such that $m^{(n)} \to m$ in $L^p(\Omega, \mu)$. Then, $M_{m^{(n)}, F, G}$ converges to $M_{m, F, G}$ in the Schatten-$p$-norm, i.e. $\|M_{m^{(n)}, F, G} - M_{m, F, G}\|_{S_p} \to 0$, as $n \to \infty$.

**Proof.** The proof follows immediately from (3.3) and the norm estimate in lemma 3.3 and theorem 3.11.

In particular, this is also valid for trace class ($p = 1$) operators and bounded operators ($p = \infty$).

**Theorem 3.13.** Let $m \in L^2(\Omega, \mu)$. Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$. Let $F^{(n)}$ be a sequence of Bessel mappings such that $F^{(n)}(\omega) \to F(\omega)$ in a uniform strong sense. Then, $M_{m^{(n)}, F, G}$ converges to $M_{m, F, G}$ in the operator norm.

**Proof.** Let $f, g \in \mathcal{H}$. For given $\epsilon > 0$, choose $N$ such that $\|F^{(n)}(\omega) - F(\omega)\| \leq \epsilon$ for all $n \geq N$, for all $\omega \in \Omega$. Then,

\[
\|(M_{m^{(n)}, F, G} - M_{m, F, G})f, g\| \leq \int_{\Omega} |m(\omega)| |(f, F^{(n)}(\omega))| |(g, F(\omega))| d\mu(\omega)
\]

\[
\leq \left( \int_{\Omega} |m(\omega)|^2 |(f, F^{(n)}(\omega))|^2 d\mu(\omega) \right)^{1/2} (B_G)^{1/2} \|g\|
\]

\[
\leq \epsilon \|m\|_2 \|f\| (B_G)^{1/2} \|g\|.
\]

Thus, by theorem 2.1

\[
\|M_{m^{(n)}, F, G} - M_{m, F, G}\| \leq \epsilon \|m\|_2 (B_G)^{1/2},
\]

so $M_{m^{(n)}, F, G}$ converges to $M_{m, F, G}$ in the operator norm. \qed

For symbols $m \in L^1(\Omega, \mu)$, we can find the following theorem.

**Theorem 3.14.** Let $m \in L^1(\Omega, \mu)$. Let $F$ and $G$ be the Bessel mappings for $\mathcal{H}$ and $G$ be norm bounded. Let $F^{(n)}$ be a sequence of Bessel mappings such that $F^{(n)}(\omega) \to F(\omega)$ in a uniform strong sense. Then, $M_{m^{(n)}, F, G}$ converges to $M_{m, F, G}$ in the operator norm.

**Proof.** For given $\epsilon > 0$, choose $N$ such that $\|F^{(n)}(\omega) - F(\omega)\| \leq \epsilon$ for all $n \geq N$, for all $\omega \in \Omega$. Then,

\[
\|(M_{m^{(n)}, F, G} - M_{m, F, G})f\| \leq \int_{\Omega} |m(\omega)| |(f, F^{(n)}(\omega)) - (f, F(\omega))| \|G(\omega)\| d\mu(\omega)
\]

\[
\leq \epsilon \|m\|_{L^1} \|f\|. \quad \text{□}
\]

In the last two results, the roles of $F$ and $G$ can be switched.

### 3.4. Examples: continuous STFT and wavelet multipliers

Particular cases of continuous frame multipliers, which means multipliers for certain continuous frames, have already been studied and used before. In this section, we briefly summarize some earlier results on STFT multipliers and Calderón–Toeplitz operators.
3.4.1. STFT multipliers. Continuous frame multipliers have been discussed and extensively used earlier for the continuous frame of definition 2.15, i.e. the STFT. An operator of the form

\[ M_{m,\phi,\psi} f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} m(a, b)\Psi_\phi(f)(a, b)M_{\psi, T_a} da \, db \]

is called an STFT multiplier. In this context, the associated continuous frame multipliers are also known as time–frequency localization operators. They were first introduced and studied by Daubechies and Paul [24, 25], where they are used as a mathematical tool to extract specific features of interest of a signal on phase space from its time–frequency representation. The Wigner distribution constitutes a continuous frame that is essentially identical to the STFT, cf [50] or [27]. It is closely related to the so-called Weyl calculus of quantum mechanics. In physics, multipliers for the Wigner distribution have been around for quite a long time in connection with questions of quantization, under the name ‘anti-Wick operators’ in the work of Berezin [11]. They also appeared earlier in the theory of pseudodifferential operators, cf [22]. In these early works, the symbol is usually taken to be the characteristic function of some portion of the time–frequency plane. In [43], results on decay properties of the eigenvalues as well as smoothness of the eigenfunctions of Wigner multipliers with characteristic functions as symbols are derived. A first result on Schatten-class properties is contained in [40], where it is shown for the Weyl correspondence that symbols in \( L^p \) lead to Hilbert–Schmidt operators. Boundedness and mapping properties with respect to other Schatten classes of the correspondence between the symbol of a multiplier and the resulting operator are considered extensively in [13, 14] (for anti-Wick operators) and in [18, 19] (for STFT multipliers). In these works, the operators are often interpreted as pseudodifferential operators. In [13], it is shown that symbols in \( L^p \) generate anti-Wick operators in the Schatten \( p \)-class. This result is a special case of our theorem 3.11. In [14], the theory of anti-Wick operators is extended to symbols in distributional Sobolev spaces. The paper [18] can very well serve as a comprehensive first survey on localization operators, i.e. STFT multipliers. The theory is developed in the framework of time–frequency analysis, see also [21]. As symbol classes, the so-called modulation spaces are considered. This requires that the window functions for the STFTs that form the continuous analysis and synthesis frames also belong to modulation spaces, usually to the Feichtinger algebra \( S_0 \). In this case, it is shown that symbols in the modulation space \( M^{p, \infty} \) are sufficient for localization operators in the Schatten \( p \)-class, \( 1 \leq p < \infty \). Since \( L^p \) spaces are continuously embedded in the modulation spaces \( M^{p, \infty} \), this extends our results in the considered special case. In [19], the authors also present necessary conditions for Schatten classes. Symbolic calculus and Fredholm properties for localization operators are discussed in [20]. The PhD thesis in [10] is concerned with questions of approximation of operators by localization operators and density properties of the set of all localization operators with symbols in certain symbol classes in spaces of operators, equipped with different topologies.

3.4.2. Calderón–Toeplitz operators. An operator defined by

\[ \mathbf{M}_{m,\psi} f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} m(a, b)W_\psi(f)(a, b)\psi_{a, b} \frac{da \, db}{a^2} \]

(in the notation of definition 2.14) is called a Calderón–Toeplitz operator. This is a multiplier for the continuous frame given by the continuous wavelet transform. In this case, the function \( m \) is referred to as the upper symbol of the operator, whereas the so-called lower symbol is given by \( m(a, b) = \langle \mathbf{M}_{m,\psi} \psi_{a, b}, \psi_{a, b} \rangle \). The concept was first introduced in [44] in 1990 as an analogue in terms of the wavelet transform to Toeplitz operators on spaces of analytic functions, for example Bergman spaces. The lower symbol corresponds analogously to the...
Berezin transform of a Toeplitz operator. Some interesting results on the spectral theory of these operators are shown in [45], for example the so-called correspondence principle—a statement on the dimensions of the spectral projections for certain bounded symbols. A number of mapping properties for Calderón–Toeplitz operators (with a sufficiently smooth window function $\psi$) depending on the lower symbol are contained in [37], for example the boundedness of the operator if and only if the lower symbol is bounded, or the compactness of the operator if and only if the lower symbol vanishes at infinity. These are stronger versions of theorems 3.3 and 3.8 in this specialized setting. Some Schatten-class properties for lower symbols in $L^p$, see theorem 3.11 for the general case, as well as for positive upper symbols are proven. Eigenvalue estimates are given in [46] and [38]. Calderón–Toeplitz operators are (along with STFT multipliers) proposed as a tool for the time–frequency localization in [23]. A unified treatment of the elementary theory of STFT multipliers and wavelet transform multipliers (based on the underlying group structures) is given in the textbook [51].

4. Controlled and weighted continuous frames

The notion of controlled and weighted frames as introduced in [8] for discrete frames is closely linked to multipliers. So here we look at the corresponding properties for continuous frames.

4.1. Controlled continuous frames

Definition 4.1. Let $C \in GL(\mathcal{H})$. A $C$-controlled continuous frame is a map $F : \Omega \to \mathcal{H}$ such that there exist $m_{CL} > 0$ and $M_{CL} < \infty$ such that

$$m_{CL}\|f\|^2 \leq \int_{\Omega} \langle f, F(\omega)(CF(\omega), f) \rangle d\mu \leq M_{CL}\|f\|^2 \quad (f \in \mathcal{H}).$$

We call $L_C f = \int_{\Omega} \langle f, F(\omega) \rangle CF(\omega) d\mu$ (in weak sense) the controlled continuous frame operator. Analogous to proposition 2.4 of [8] one can show that $L_C \in GL(\mathcal{H})$.

Proposition 4.2. Let $F : \Omega \to \mathcal{H}$ be a $C$-controlled continuous frame for some $C \in GL(\mathcal{H})$. Then, $F$ is a continuous frame for $\mathcal{H}$.

Proof. Since $C$ is linear, we have

$$S_F f = \int_{\Omega} \langle f, F(\omega) \rangle F(\omega) d\mu = C^{-1} \int_{\Omega} \langle f, F(\omega) \rangle CF(\omega) d\mu = C^{-1} L_C f.$$  

Therefore, $S_F$ is a bounded, positive and invertible operator and so $F$ is a continuous frame. □

By definition, $L_C$ is positive and $L_C = CS_F = S_F C^*$. Therefore, it is easy to show that given $C \in GL(\mathcal{H})$ is a self-adjoint operator, then the mapping $F$ is a $C$-controlled frame if and only if it is a continuous frame for $\mathcal{H}$, and $C$ is positive and commutes with $S_F$.

The following proposition shows that we can retrieve a continuous frame multiplier from a multiplier of controlled frames. Actually, the role played by controlled operators is that of a precondition matrix.

Proposition 4.3. Let $C, D \in GL(\mathcal{H})$ be self-adjoint operators. If $F$ and $G$ are $C$- and $D$-controlled frames, respectively, and $M$ is their multiplier operator with respect to $m$, then $D^{-1}M C^{-1} = M_{m,F,G}$.

Proof. It is easy to see that for the $C$- and $D$-controlled frames $F$ and $G$, we have $T_C = CT$ and $T_D = T^* D$. Now, representation (3.1) results $D^{-1}M C^{-1} = M_{m,F,G}$. □
4.2. Weighted continuous frames

**Definition 4.4.** Let $\mathcal{H}$ be a complex Hilbert space and $(\Omega, \mu)$ be a measure space with a positive measure $\mu$ and $m : \Omega \rightarrow \mathbb{R}^+$. The mapping $F : \Omega \rightarrow \mathcal{H}$ is called a weighted continuous frame with respect to $(\Omega, \mu)$ and $m$, if

1. $F$ is weakly measurable and $m$ is measurable;
2. there exist constants $A, B > 0$ such that

   $$A \|f\|^2 \leq \int_{\Omega} m(\omega) |\langle f, F(\omega) \rangle|^2 \, d\mu(\omega) \leq B \|f\|^2 \quad (f \in \mathcal{H}). \quad (4.1)$$

The mapping $F$ is called weighted Bessel if the second inequality in $(4.1)$ holds.

By using some ideas of [47], we have the following result.

**Theorem 4.5.** Let $M_{m,F,G}$ be invertible. Then, we have the following.

1. If $G$ is a Bessel map, then $mF$ satisfies the lower frame condition.
2. If $F$ is a Bessel map, then $mG$ satisfies the lower frame condition.
3. If $F$ and $mG$ ($G$ and $mF$ respectively) are the Bessel maps, then they are continuous frames.
4. If $G$ is a Bessel map and $m \in L^\infty$, $m \neq 0$, then $F$ has a lower bound.
5. If $F$ and $G$ are the Bessel maps and $m \in L^\infty$, $m \neq 0$, then both of $F$ and $G$ are continuous frames.

**Proof.**

1. For $f, g \in \mathcal{H}$, we have

   $$|\langle M_{m,F,G} f, g \rangle| \leq \left( \int_{\Omega} |\langle f, (mF)(\omega) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left( \int_{\Omega} |\langle (G(\omega), g) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}$$

   without loss of generality, we can assume $f \neq 0$ and

   $$\int_{\Omega} |\langle f, (mF)(\omega) \rangle|^2 \, d\mu(\omega) < \infty.$$

   So

   $$|\langle M_{m,F,G} f, g \rangle| \leq \sqrt{B_G} \|g\| \left( \int_{\Omega} |\langle f, (mF)(\omega) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}.$$

   By letting $g = (M_{m,F,G}^*)^{-1} f$, we have

   $$\|f\|^2 \leq \sqrt{B_G} \|M_{m,F,G}^*\| \|f\| \left( \int_{\Omega} |\langle f, (mF)(\omega) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}.$$

   So

   $$\|f\| \leq \frac{1}{\sqrt{B_G} \|M_{m,F,G}^*\|} \left( \int_{\Omega} |\langle f, (mF)(\omega) \rangle|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}.$$

2. Similar to (1).

3. Let $F$ be a Bessel map, then by part (1), $mG$ has a lower bound and so it is a frame. If $mG$ is a Bessel map, then by (2) $1 \cdot F = F$ satisfies the lower frame inequality and therefore is a frame.
(4) By (1), $mF$ satisfies the lower frame inequality. Therefore

$$A \left\| f \right\|^2 \leq \int_{\Omega} \left| \langle f, (mF)(\omega) \rangle \right|^2 d\mu(\omega) \leq \left\| m \right\|_\infty^2 \int_{\Omega} \left| \langle f, \mathbf{F}(\omega) \rangle \right|^2 d\mu(\omega).$$

And so

$$A \left\| m \right\|^2_\infty \leq \int_{\Omega} \left| \langle f, \mathbf{F}(\omega) \rangle \right|^2 d\mu(\omega).$$

(5) Follows from (1), (2) and (3). □

The following theorem finds a dual of a continuous frame in the case that the multiplier operator is invertible. (Analogous to the discrete results in [48].)

**Theorem 4.6.** Let $\mathbf{M}_{m,F,G}$ be invertible and $\mathbf{G}$ be a continuous frame. Then, $(\mathbf{M}_{m,F,G}^{-1})^* \mathbf{mF}$ is a dual.

**Proof.** By replacing $f$ with $\mathbf{M}_{m,F,G}^{-1}f$ in

$$\langle \mathbf{M}_{m,F,G}f, g \rangle = \int_{\Omega} m(\omega) \langle f, \mathbf{F}(\omega) \rangle \langle \mathbf{G}(\omega), g \rangle d\mu,$$

we obtain

$$\langle f, g \rangle = \int_{\Omega} m(\omega) \langle \mathbf{M}_{m,F,G}^{-1}f, \mathbf{F}(\omega) \rangle \langle \mathbf{G}(\omega), g \rangle d\mu$$

$$= \int_{\Omega} \langle f, (\mathbf{M}_{m,F,G}^{-1})^* m(\omega)F(\omega) \rangle \langle \mathbf{G}(\omega), g \rangle d\mu.$$ □

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