An approach to the time-dependent Jaynes-Cummings model without the rotating wave approximation

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This report presents an approach to the exact solutions of the time-dependent Jaynes-Cummings (J-C) model without the rotating wave approximation (RWA). It is shown that there is a squeezing-operator unitary transformation for relating the J-C model without RWA and the one with RWA. Thus by using an appropriate squeezing unitary transformation, the time-dependent J-C model without RWA can be transformed into the one with RWA that has been exactly solved previously, and based on this one can readily obtain the exact solutions of the time-dependent Jaynes-Cummings (J-C) model without RWA. The approach presented here also shows that we can treat these two kinds of time-dependent J-C models (both with and without RWA) in a unified way.

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The interaction between a two-level atom and a quantized single-mode electromagnetic field is described by the Jaynes-Cummings (J-C) model [1], which can be applied to the study of many quantum effects such as the quantum collapses and revivals of the atomic inversion, photon antibunching, squeezing of the radiation inversionless light amplification [2–4] as well as electromagnetically induced transparency [5–7]. Recently, problems of time-dependent quantum systems attract attention of many researchers in various areas such as quantum optics [8,9], condensed matter physics [10,11], atomic and molecular physics (including molecular reaction) [12–14] and gravity theory [15,16] as well. More recently, Guedes et al. solved the one-dimensional Schrödinger equation with a time-dependent linear potential [17,18]. As far as we are concerned, by making use of the Lewis-Riesenfeld invariant theory [19], we have obtained the exact solutions of the time-dependent supersymmetric multiphoton J-C model with the rotating wave approximation [20,21].

It is well known that the J-C model without the rotating wave approximation (RWA) has close relation to the effects of virtual photon fields. However, to the best of our knowledge, in the literature, only the time-dependent J-C model with RWA and the time-independent J-C model without RWA have been exactly solved. It may be believed that the time-dependent J-C model without RWA may get less attention than deserves. The Hamiltonian of the J-C model without RWA has the non-rotating-wave term, which makes the Hamiltonian without RWA more complicated than that with RWA. Historically, maybe for this reason, it was not easy for us to solve such time-dependent model. In the present report, we will show that there is a relationship (unitary transformation) between the J-C model without RWA and the one with RWA: specifically, a squeezing-operator unitary transformation can relate these two kinds of time-dependent J-C model (both with and without RWA) in a unified way.

Thus by using an appropriate squeezing unitary transformation, the time-dependent J-C model without RWA can be transformed into the one with RWA that has been exactly solved previously, and based on this one can readily obtain the exact solutions of the time-dependent Jaynes-Cummings (J-C) model without RWA. The approach presented here also shows that we can treat these two kinds of time-dependent J-C models (both with and without RWA) in a unified way.

The Hamiltonian of the time-dependent two-level J-C model without RWA is written in the form (in the unit $\hbar = 1$)

$$H(t) = \omega(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + \gamma(t)a^\dagger\sigma_- + \gamma^*(t)a\sigma_+ + \lambda(t)a\sigma_- + \lambda^*(t)a^\dagger\sigma_+ ,$$  \hspace{1cm} (1)

where $a^\dagger$ and $a$ denote the creation and annihilation operators of the monomode photon fields interacting with the two-level atoms characterized by the Pauli’s third-component matrix $\sigma_z$; the Pauli matrices $\sigma_\pm$ represent the atomic transition operators satisfying the commutation relations $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$ and $[\sigma_+, \sigma_-] = \sigma_z$; the expression

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\(\lambda(t)a\sigma_- + \lambda^*(t)a^\dagger \sigma_+\) in the Hamiltonian (1) stands for the non-rotating-wave term, which can describe the effects of virtual photon fields. Note that here all the coefficients \((\omega, \omega_0, \gamma, \gamma^*, \lambda, \lambda^*)\) in the expression (1) are time-dependent parameters (functions). The time-dependent Schrödinger equation governing this time-dependent J-C model without RWA is

\[
H(t)|\Psi(t)\rangle = i \frac{\partial}{\partial t} |\Psi(t)\rangle,
\]

which will be exactly solved in this report.

Unfortunately, due to the presence of the non-rotating-wave term in the Hamiltonian (1), it is not easy to obtain the analytical solutions of the time-dependent Schrödinger equation (2). In order to overcome this difficulty, in what follows we will employ a time-dependent squeezing-operator unitary transformation, which may transform the Hamiltonian (1) without RWA into the one with RWA (in which the non-rotating-wave term is absent). Under this squeezing-operator transformation \(S(\zeta)\) (which will be referred to as “squeezing unitary transformation”), the time-dependent Schrödinger equation (2) may be rewritten as follows

\[
H_s(t)|\Psi_s(t)\rangle = i \frac{\partial}{\partial t} |\Psi_s(t)\rangle
\]

with \(|\Psi_s(t)\rangle = S^\dagger(\zeta)|\Psi(t)\rangle, \ H_s(t) = S^\dagger(\zeta)H(t)S(\zeta) - S^\dagger(\zeta)i \frac{\partial}{\partial t} S(\zeta), \) where the applied squeezing operator takes the form \(S(\zeta) = \exp\left(\frac{\zeta}{2}a^\dagger \sigma_+ - \frac{\zeta^*}{2}a \sigma_-\right)\). Note that \(\zeta = \zeta(t)\) is a time-dependent function which will be determined in the following. With the help of the Baker-Campbell-Hausdorff formula [22] and the Glauber formula, one can arrive at

\[
-S^\dagger(\zeta) \frac{\partial}{\partial t} S(\zeta) = \frac{-i \left(\zeta^* - \zeta\right)}{8\zeta^* \zeta} \left(\cosh \sqrt{4\zeta^* \zeta} - 1\right) (a^\dagger a + aa^\dagger)
\]

\[
+ \frac{i \left(\zeta^* - \zeta\right)}{\sqrt{4\zeta^* \zeta}} \left(\sinh \sqrt{4\zeta^* \zeta} - \sqrt{4\zeta^* \zeta}\right) \left(\zeta^* a^2 + a^2 \zeta + i \left(\frac{\zeta}{2}a^2 - \frac{\zeta^*}{2}a^{12}\right)\right)
\]

(4)

and

\[
S^\dagger(\zeta) H(t) S(\zeta) = \omega(t) \left[ -\frac{1}{2} + \frac{\cosh \sqrt{4\zeta^* \zeta}}{2} (a^\dagger a + aa^\dagger) - \frac{\sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} (\zeta^* a^2 + a^2 \zeta) \right] + \frac{\omega_0(t)}{2} \sigma_z
\]

\[
+ \left[ \gamma \cosh \sqrt{4\zeta^* \zeta} - \frac{\lambda \zeta \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} \right] a^\dagger \sigma_- + \left[ \gamma^* \cosh \sqrt{4\zeta^* \zeta} - \frac{\lambda^* \zeta^* \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} \right] a \sigma_+
\]

\[
+ \left[ \lambda \cosh \sqrt{4\zeta^* \zeta} - \frac{\gamma \zeta \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} \right] a \sigma_- + \left[ \lambda^* \cosh \sqrt{4\zeta^* \zeta} - \frac{\gamma^* \zeta^* \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} \right] a^\dagger \sigma_+,
\]

(5)

where dot denotes the derivative of \(\zeta\) and \(\zeta^*\) with respect to time \(t\). If we set

\[
\Omega = \left[ \omega \cosh \sqrt{4\zeta^* \zeta} - \frac{i \left(\zeta^* - \zeta\right)}{4\zeta^* \zeta} \left(\cosh \sqrt{4\zeta^* \zeta} - 1\right) \right], \quad C = \left[ \omega - \frac{i \left(\zeta^* - \zeta\right)}{8\zeta^* \zeta} \right] \left(\cosh \sqrt{4\zeta^* \zeta} - 1\right),
\]

\[
A = \frac{\zeta \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}} \left[ \frac{i \left(\zeta^* - \zeta\right)}{4\zeta^* \zeta} - \omega \right] - \frac{i \left(\zeta^* - \zeta\right)}{4\zeta^* \zeta} - \frac{i}{2} \zeta^*,
\]

\[
g = \gamma \cosh \sqrt{4\zeta^* \zeta} - \frac{\lambda \zeta \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}}, \quad \Lambda = \lambda \cosh \sqrt{4\zeta^* \zeta} - \frac{\gamma \zeta \sinh \sqrt{4\zeta^* \zeta}}{\sqrt{4\zeta^* \zeta}},
\]

(6)

then the obtained Hamiltonian via the squeezing unitary transformation may be simplified into the following one

\[
H_s(t) = \Omega(t)a^\dagger a + \frac{\omega_0(t)}{2} \sigma_z + A(t)a^2 + A^*(t)a^\dagger a^2 + g(t)a^\dagger \sigma_- + g^*(t)a \sigma_+ + \Lambda(t)a^2 \sigma_- + \Lambda^*(t)a^\dagger \sigma_+ + C(t).
\]

(7)

Evidently, if the two constraints, \(A(t) = 0\) and \(\Lambda(t) = 0\), are satisfied, then the quadratic-form terms \(a^2\) and \(a^\dagger a^2\), and the non-rotating-wave term in the Hamiltonian (7) will be absent. It should be further pointed out that the two equations \(A(t) = 0\) and \(\Lambda(t) = 0\) are just employed to determine the time-dependent functions \(\zeta\) and \(\zeta^*\) in the
squeezing operator $S(\zeta)$. Namely, only when the time-dependent functions $\zeta$ and $\zeta^*$ agree with these two equations will the squeezing operator $S(\zeta)$ change $H(t)$ without RWA into the form with RWA (see in the following). Thus we obtain the time-dependent Hamiltonian without the non-rotating-wave term, i.e., $H_s(t) = \Omega(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + g(t)a^\dagger\sigma_- + g^*(t)a\sigma_+ + C(t)$. It is readily verified that if $|\Psi_s(t)\rangle$ is rewritten as $|\Psi_s(t)\rangle = \exp\left[\frac{1}{i}\int_0^t C(t')dt'\right]|\Psi_{s\text{RWA}}(t)\rangle$, then the time-dependent Schrödinger equation (3) may be rewritten in the following form

$$H_s^{\text{RWA}}(t)|\Psi_{s\text{RWA}}(t)\rangle = \frac{\partial}{\partial t}|\Psi_{s\text{RWA}}(t)\rangle$$

(8)

with $H_s^{\text{RWA}}(t) = H_s(t) - C(t)$, i.e.,

$$H_s^{\text{RWA}}(t) = \Omega(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + g(t)a^\dagger\sigma_- + g^*(t)a\sigma_+,$$

(9)

which is a standard form of the time-dependent Hamiltonian of the J-C model with RWA. Thus we have shown that there is a connection (unitary transformation) between the two kinds of time-dependent J-C model (without RWA and with RWA), and that under the so-called squeezing unitary transformation, the complicated time-dependent J-C model without RWA can be reduced to the simple J-C model with RWA.

To close this report, we will briefly discuss the exact solutions of the time-dependent J-C model with RWA characterized by Eq.(9). It is well known that the Lewis-Riesenfeld invariant theory [19] can be applied to the exact solutions of the time-dependent quantum models, the algebraic generators in the Hamiltonians of which form a certain closed Lie algebra. Unfortunately, the commutation relations among the generators in the Hamiltonian (9) are $[a\sigma_+, a^\dagger\sigma_-] = N'\sigma_z$, $[a\sigma_+, \sigma_z] = -2a\sigma_+$, $[a^\dagger\sigma_-, \sigma_z] = 2a^\dagger\sigma_-$, which shows that $\{a\sigma_+, a^\dagger\sigma_- , \sigma_z , N'\}$ do not form a closed Lie algebra. But note that here $N'$ is of the form

$$N' = \begin{pmatrix} aa^\dagger & 0 \\ 0 & a^\dagger a \end{pmatrix},$$

(10)

which commutates with all the generators in the Hamiltonian (9), namely, $[N', H_s^{\text{RWA}}(t)] = 0$. This, therefore, means that the operator $N'$ is just a time-independent Lewis-Riesenfeld invariant that agrees with the Liouville-Von Neumann equation [19] and that one can obtain a closed quasi-algebra $\{a\sigma_+, a^\dagger\sigma_- , \sigma_z \}$ in the sub-Hilbert-space of $N'$ so long as $N'$ in the commutation relation $[a\sigma_+, a^\dagger\sigma_-] = N'\sigma_z$ is replaced with its certain eigenvalue [20,21]. Further analysis shows that $N'$ possesses the eigenvalue $m + 1$ and that the corresponding eigenvalue equation can be written in the form

$$N'( |m\rangle_{m+1} ) = (m + 1)( |m\rangle_{m+1} ),$$

(11)

where $|m\rangle$ and $|m + 1\rangle$ denote the number states of photon field with the corresponding occupation numbers being $m$ and $m + 1$, respectively.

Hence, by working in the sub-Hilbert-space of $N'$ corresponding to the eigenvalue $m + 1$, we can exactly solve Eq.(8) of the time-dependent J-C model with RWA by using the Lewis-Riesenfeld invariant theory [19] and the invariant-related unitary transformation formation [23]. In the following we will present only the final results without providing the mathematical procedure. For the detailed calculation, readers may be referred to Ref. [21].

In the Lewis-Riesenfeld invariant theory [19,23], it is readily verified that the time-dependent unitary transformation $V(t) = \exp\left[\beta(t)a^\dagger\sigma_- - \beta^*(t)a\sigma_+\right]$ with $\beta(t) = -\frac{\theta(t)\exp[-i\phi(t)]}{2\sqrt{m + 1}}$, $\beta^*(t) = -\frac{\theta(t)\exp[i\phi(t)]}{2\sqrt{m + 1}}$ will transform the time-dependent Lewis-Riesenfeld invariant [19,21]

$$\mathcal{I}(t) = -\frac{\sin \theta(t)}{\sqrt{m + 1}} \left[ \exp[-i\phi(t)]a^\dagger\sigma_- + \exp[i\phi(t)]a\sigma_+ \right] + \cos \theta(t)\sigma_z$$

(12)

into a time-independent one, i.e., $V^+(t)\mathcal{I}(t)V(t) = \sigma_z$ [21]. The eigenvalue of $\sigma_z$ is $\sigma = \pm 1$. So, in accordance with Eq.(11), in the sub-Hilbert-space of $N'$ the two eigenstates of the Lewis-Riesenfeld invariant (12) corresponding to the eigenvalues $\sigma = \pm 1$ are $V(t)|m\rangle_{m+1}$, $V(t)|0\rangle_{m+1}$, respectively. According to the Lewis-Riesenfeld theory [19], the solutions of the time-dependent Schrödinger equation (8) can be constructed in terms of the eigenstates of the invariant (12), one can therefore obtain the following two explicit solutions (i.e., without chronological product)
is defined to be \[ [21] \]

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Schrödinger equation (2) without RWA, which are as follows

Hence, by the aid of the solutions of Eq.(3) and (8), one can finally obtain the exact solutions of the time-dependent Schrödinger equation (2) without RWA, which are as follows

\[
|\Psi_{s}^{\text{RWA}}(\sigma = +1, N' = m + 1; t)\rangle = \exp \left\{ \frac{1}{i} \int_{0}^{t} [C(t') + \phi_{\sigma=+1}(t')] dt' \right\} S(\zeta)V(t)\left( \begin{array}{c} m \\ 0 \end{array} \right),
\]

\[
|\Psi_{s}^{\text{RWA}}(\sigma = -1, N' = m + 1; t)\rangle = \exp \left\{ \frac{1}{i} \int_{0}^{t} [C(t') + \phi_{\sigma=-1}(t')] dt' \right\} S(\zeta)V(t)\left( \begin{array}{c} 0 \\ m + 1 \end{array} \right).
\]

(13)

To summarize, we have shown that the time-dependent J-C model without RWA can be transformed into the one with RWA under a squeezing unitary transformation, which enables us to easily obtain the exact solutions of the time-dependent J-C model without RWA. Moreover, in this report the fact that by using the squeezing unitary transformation one can treat the two time-dependent J-C models (both with and without RWA) in a unified way is demonstrated, which also implies that we can consider the quantum effects of virtual photons in the J-C model from the point of view of the squeezing transformation. In addition, we think that the method presented here may seem to be somewhat ingenious and therefore deserves further consideration in its application to other quantum-mechanical models without RWA.

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