ON HIGHER DIMENSIONAL POISSONIAN PAIR CORRELATION

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Abstract. In this article we study the pair correlation statistic for higher dimensional sequences. We show that for any $d \geq 2$, strictly increasing sequences $(a^{(1)}_n), \ldots, (a^{(d)}_n)$ of natural numbers have metric Poissonian pair correlation with respect to sup-norm if their joint additive energy is $O(N^{3-\delta})$ for any $\delta > 0$. Further, in dimension two, we establish an analogous result with respect to the 2-norm.

As a consequence, it follows that $(\{n\alpha\}, \{n^2\beta\})$ and $(\{n\alpha\}, \{[n \log n]^{\beta}\})$ $(A \in [1,2])$ have Poissonian pair correlation for almost all $(\alpha, \beta) \in \mathbb{R}^2$ with respect to sup-norm and 2-norm. This gives a negative answer to the question raised by Hofe r and Kaltenb"ock [15]. The proof uses estimates for ‘Generalized’ GCD-sums.

1. Introduction

Let $(x_n) \in [0,1)^d$ be a sequence, $s > 0$ be a real number and $N$ be a natural number. The pair correlation statistic of $(x_n)$ is defined as follows:

$$R^d_2(s, (x_n), N) := \frac{1}{N} \# \left\{ 1 \leq n \neq m \leq N : \|x_n - x_m\| \leq \frac{s}{N} \right\},$$

where for any real $x$, $\|x\| := \inf_{m \in \mathbb{Z}} |x + m|$, the nearest integer distance. The sequence $(x_n)$ is said to have Poissonian pair correlation (PPC) if for all $s > 0$,

$$\lim_{N \to \infty} R^d_2(s, (x_n), N) = 2s.$$

This concept originated from theoretical physics and it plays a crucial role in the Berry–Tabor conjecture. Rudnick and Sarnak [21] first studied this notion from a mathematical point of view. Since then this topic has received wide attention [22, 23], and several generalizations are known (see [4, 13, 14, 20, 25]).

The theory of uniform distribution (or equidistribution) of a sequence has a long history. It is known that Poissonian pair correlated sequences are necessarily uniformly distributed (see [2, 23, 12]) but the converse is not true.

Only recently, a concept of pair correlation for the higher dimensional sequences have been introduced in [14] with respect to sup-norm and in [25] with respect to 2-norm. Throughout the article, we assume that $d \geq 2$. For $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d$, we denote $\|x\|_{\infty} = \max\{\|x^{(1)}\|, \ldots, \|x^{(d)}\|\}$ and $\|x\|_2 = (\|x^{(1)}\|^2 + \cdots + \|x^{(d)}\|^2)^{1/2}$. Let $(x_n)_{n \geq 1}$ be a sequence in $[0,1)^d$. For $s > 0$ we write

$$R^{(d)}_{2,\infty}(s, (x_n), N) := \frac{1}{N} \# \left\{ 1 \leq m \neq n \leq N : \|x_n - x_m\|_{\infty} \leq \frac{s}{N^{1/d}} \right\}.$$
and

\[ R_{2,2}^{(d)}(s,(x_n),N) := \frac{1}{N} \# \{ 1 \leq n \neq m \leq N : \|x_n - x_m\|_2 \leq \frac{s}{N^{1/d}} \} \].

**Definition.** A sequence \((x_n)_{n \geq 1}\) in \(\mathbb{R}^d\) is said to have \(\infty\)-PPC if for all \(s > 0\),

\[ R_{2,\infty}^{(d)}(s,(x_n),N) \rightarrow (2s)^d \text{ as } N \rightarrow \infty, \]

and 2-PPC if for all \(s > 0\),

\[ R_{2,2}^{(d)}(s,(x_n),N) \rightarrow w_d s^d \text{ as } N \rightarrow \infty, \]

where \(w_d\) is the volume of the unit ball of \(\mathbb{R}^d\) in 2-norm.

Similar to the one-dimensional case, it has been proved in [14] that \(\infty\)-PPC implies uniform distribution and in [25] that 2-PPC implies uniform distribution for higher dimensional sequences.

**Remark 1.** Instead of defining the counting function \(R_2\) for balls centred at the origin, we could have defined it for any balls and consequently defined a stronger version of PPC by demanding convergence for every ball. In that case, the notion of 2-PPC and \(\infty\)-PPC would coincide; furthermore, it is going to be independent of the norm. But there is no clear way to show this equivalence for our present definitions although the norms are topologically equivalent. On the contrary, we believe that such an equivalence does not hold, though we do not have any example to demonstrate so.

The purpose of this article is to show \(\infty\)-PPC and 2-PPC for some higher dimensional sequences of the form \(\{a_n^{(i)}\}_{i=1}^d\), where \((a_n^{(i)})\), for \(i = 1, 2, \ldots, d\) are sequences of natural numbers and \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}_d^d\). For simplicity, in this case we denote the respective pair correlation statistics by \(R_{2,\infty}^{(d)}(s,\alpha,N)\) and \(R_{2,2}^{(d)}(s,\alpha,N)\).

**Definition.** We say that a \(d\)-dimensional sequence \((a_n^{(1)}, \ldots, a_n^{(d)})\) has metric Poissonian pair correlation with respect to sup-norm \((\infty\text{-MPPC})\) if

\[ R_{2,\infty}^{(d)}(s,\alpha,N) \rightarrow (2s)^d \text{ as } N \rightarrow \infty, \]

for almost all \(\alpha \in \mathbb{R}^d\). Moreover, 2-MPPC is defined analogously.

Our results depend on the notion of additive energy of integer sequences. For a finite subset \(A\) of integers, the additive energy \(E(A)\) of \(A\) is defined by

\[ E(A) := \# \{ a, b, c, d \in A : a + b = c + d \}. \]

Recently, Aistleitner et al. [1] proved that for a strictly increasing sequence \((a_n)\) of natural numbers the sequence \((\alpha a_n)\) has Poissonian pair correlation for almost all \(\alpha\), provided \(E(A_N) \ll N^{3-\epsilon}\) for some \(\epsilon > 0\), where \(A_N\) denotes the set of first \(N\) elements of \((a_n)\). In [3], Bloom and Walker improved their result by relaxing the condition on the upper bound of additive energy. Analogous results for some special higher dimensional sequences with respect to sup-norm were established in [14], where the following theorem was specifically proved:

**Theorem A.** Let \((a_n)\) be a strictly increasing sequence of natural numbers, \(A_N\) denote the first \(N\) elements of \((a_n)\) and suppose that

\[ E(A_N) = O \left( \frac{N^3}{(\log N)^{1+\epsilon}} \right), \text{ for some } \epsilon > 0, \]
then for almost all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, $(\{a_n\alpha\}) = (\{a_n\alpha_1\}, \ldots, \{a_n\alpha_d\})$ has $\infty$-PPC.

We consider more general sequences, namely, $(\{a_n^{(1)}\alpha_1\}, \ldots, \{a_n^{(d)}\alpha_d\})$ and study their pair correlation property. To state our results, we introduce the notion of joint additive energy for several increasing sequences of natural numbers.

**Definition (Joint additive energy).** Let $(a_n^{(1)}), (a_n^{(2)}), \ldots, (a_n^{(d)})$ be strictly increasing sequences of natural numbers and $A_N^{(i)}$ denote the first $N$ elements of $(a_n^{(i)})$, for $1 \leq i \leq d$. The joint additive energy $E(A_N^{(1)}; \ldots; A_N^{(d)})$ is given by

$$ E\left(A_N^{(1)}; \ldots; A_N^{(d)}\right) = \#\left\{1 \leq n, m, k, l \leq N : a_n^{(i)} + a_m^{(i)} = a_k^{(i)} + a_l^{(i)}, i = 1, \ldots, d \right\}. $$

Note that joint additive energy is additive energy in higher dimensions. In section 2, we study joint additive energy in detail and obtain an upper bound of it.

Now, we state our results.

**Theorem 1.1.** Let $(a_n^{(1)}), (a_n^{(2)}), \ldots, (a_n^{(d)})$ be strictly increasing sequences of natural numbers and $A_N^{(i)}$ denote the first $N$ elements of $(a_n^{(i)})$, for $1 \leq i \leq d$. Suppose that for some $\delta > 0$,

$$ E\left(A_N^{(1)}; \ldots; A_N^{(d)}\right) = O\left(N^{3-\delta}\right). $$

Then $(a_n^{(1)}, \ldots, a_n^{(d)})$ has $\infty$-MPPC.

An immediate consequence is the following.

**Corollary 1.2.** Suppose that for some $\delta > 0$,

$$ \min_{1 \leq i \leq d} E(A_N^{(i)}) = O\left(N^{3-\delta}\right). $$

Then $(a_n^{(1)}, \ldots, a_n^{(d)})$ has $\infty$-MPPC.

In [25], Steinerberger introduced the notation of 2-PPC but did not indicate any sequences which satisfy 2-PPC. In the following theorems we study 2-PPC for certain sequences.

**Theorem 1.3.** Let $(a_n)$ be a strictly increasing sequence of natural numbers and $A_N$ denote the first $N$ elements of $(a_n)$. Assume that

$$ E(A_N) = O\left(\frac{N^3}{(\log N)^{1+\epsilon}}\right), \text{ for some } \epsilon > 0. $$

Then for almost all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, the sequence $(\{a_n\alpha\}) = (\{a_n\alpha_1\}, \ldots, \{a_n\alpha_d\})$ has 2-PPC.

**Theorem 1.4.** Let $(a_n), (b_n)$ be strictly increasing sequences of natural numbers and $A_N$, $B_N$ denote their first $N$ elements respectively. Suppose that

$$ E\left(A_N; B_N\right) = O\left(N^3 \exp\left(- (\log N)^{2+\delta}\right)\right), \text{ for some } \delta > 0. $$

Then $(a_n, b_n)$ has 2-MPPC.

**Remark 2.** Due to technical complexity, we prove Theorem 1.4 only for dimension two, though the arguments can be extended to higher dimensions.

Recently, Hofer and Kaltenböck [15] asked the following question:
Theorem 1.5. Let $A \in [1, 2]$ be any real number. Then the sequence $(n, [n(\log n)^A])$ has $\infty$-MPPC and 2-MPPC.

Currently, we do not know any result showing that $(\{n(\log n)^A\alpha\})$ has PPC for any real $\alpha$. However, from [9, Corollary 1] we obtain the associated additive energy $\ll N^3(\log N)^{1-A}$.

Notation: Let $h \geq 2$ be an integer, $a$ be a real number and $x = (x_1, \ldots, x_h), y = (y_1, \ldots, y_h) \in \mathbb{R}^h$.

- Coordinate-wise product $xy = (x_1y_1, \ldots, x_hy_h)$.
- Inner product $x \cdot y = \sum_{1 \leq i \leq h} x_iy_i$.
- Fractional part $\{x\} = (\{x_1\}, \ldots, \{x_h\})$.
- Scalar product $ax = (ax_1, \ldots, ax_h)$.
- $\Sigma_{x \in \mathbb{Z}^h}$ will mean sum over $x$ with $x_i \neq 0$ for all $i$.

Further, for $a, b \in \mathbb{Z}$ their greatest common divisor (GCD) is denoted by $(a, b)$. Define $e(x) := e^{2\pi ix}$ for $x \in \mathbb{R}$.

Let $h \geq 1$. If $a_n^{(i)} = a_n$ for all $i$ then we denote $A_N^{(i)}$’s by $A_N$ and $E(A_N^{(1)}; \ldots; A_N^{(h)})$ by $E(A_N)$. For $v = (v_1, \ldots, v_h), v_i \in \mathbb{Z}$ is non-zero for all $i$, we define the representation function $R_N(v)$ by

\[
R_N(v) := \#\{1 \leq m \neq n \leq N : a_n^{(i)} - a_m^{(i)} = v_i, 1 \leq i \leq h\}.
\]

We use $R_N(v)$ and $R_N(v_1, \ldots, v_h)$ interchangeably. One can see that,

\[
\sum_{v \in \mathbb{Z}^h} R_N(v)^2 \leq E(A_N^{(1)}; \ldots; A_N^{(h)}).
\]

2. Joint additive energy

Let $A_N^{(i)}$ denote the first $N$ elements of $(a_n^{(i)})$, for $1 \leq i \leq d$. It is easy to see that the joint additive energy satisfies the following trivial estimate

\[
N^2 \leq E(A_N^{(1)}; \ldots; A_N^{(d)}) \leq \min_{1 \leq i \leq d} E(A_N^{(i)}) \leq N^3.
\]

From this observation we conclude that for any strictly increasing sequence $A := (a_n)_{1 \leq n \leq N}$ of $N$ natural numbers and any fixed integer $l \geq 2$, $B_l := \{n^l : 1 \leq n \leq N\}$ we have $E(A; B_l) \ll N^{2+\epsilon}$ for any $\epsilon > 0$ (since $E(B_l) \ll N^{2+\epsilon}$, see [1]).

For $1 \leq s \leq d$, Vinogradov’s mean value $J_{s,d}(N)$ is the number of solutions in $\mathbb{N}$ of the system:

\[
x_1^i + \cdots + x_s^i = y_1^i + \cdots + y_s^i, \quad 1 \leq i \leq d.
\]
Proposition 2.1. We have the following bounds for the joint additive energy:

\[ E(B_1; B_2; \ldots; B_d) = J_{2,d}(N) \ll N^{2+\epsilon}, \]

for any \( \epsilon > 0 \). A natural question is how small the joint additive energy can be when all the components have considerably large additive energy. For example, [10, Theorem 1] says that the sequence \( \lfloor n \log n \rfloor \) has additive energy \( \gg N^3 \log N \), but we show that the joint additive energy of \( (n) \) and \( \lfloor n \log n \rfloor \) is much smaller.

Let \( (f(n)), (g(n)) \) be two strictly increasing sequences of natural numbers, and let \( F_N \) and \( G_N \) be the sets of their first \( N \) elements, respectively. For a given positive integer \( l \) with \( 1 \leq l < N \), denote by \( Q_l := Q_l(N) \) the number of solutions of the system of equations

\[
\begin{aligned}
&f(x) + f(y) = f(x + l) + f(z), \\
g(x) + g(y) = g(x + l) + g(z),
\end{aligned}
\]

where \( 1 \leq x < x + l \leq z < y \leq N \).

We have the following bounds for the joint additive energy:

**Proposition 2.1.** For any \( 0 < \epsilon < 1 \) we have

\[
\frac{N^4}{(f(N) + 1)(g(N) + 1)} \ll E(F_N; G_N) \ll N^{2+\epsilon} + N^\epsilon \sum_{1 \leq l \leq N^{1-\epsilon}} Q_l.
\]

**Proof.** We adopt the idea of the proof of [9, Theorem 1] and [10, Theorem 1] for the upper and lower bounds, respectively, to obtain these estimates.

For any real \( \alpha_1, \alpha_2 \) we set

\[
S(\alpha_1, \alpha_2) = \sum_{1 \leq n \leq N} e(f(n)\alpha_1 + g(n)\alpha_2).
\]

Then, by orthogonality of the exponential function, we obtain

\[
E(F_N; G_N) = \int_{[0,1]^2} |S(\alpha_1, \alpha_2)|^4 d\alpha.
\]

For integers \( 1 \leq s \leq [N^\epsilon] \), set \( I_s := \{ n \in \mathbb{Z} : (s - 1)N^{1-\epsilon} < n \leq sN^{1-\epsilon} \} \) and \( I_{[N^\epsilon]+1} := \{ n \in \mathbb{Z} : [N^\epsilon]N^{1-\epsilon} < n \leq N \} \). Then, by separating diagonal and off-diagonal terms, applying triangular inequality and the partition of \([1, N]\) into \( I_s \)'s we deduce

\[
|S(\alpha_1, \alpha_2)|^4 \ll N^2 \left( 1 + \sum_{1 \leq s \leq [N^\epsilon]+1} \sum_{n \in I_s} \sum_{1 \leq n < m \leq N} e((f(n) + f(m))\alpha_1 + (g(n) + g(m))\alpha_2) \right)^2.
\]

Now, by applying Cauchy-Schwarz inequality on the sum over \( s \), we get

\[
|S(\alpha_1, \alpha_2)|^4 \ll N^2 + N^\epsilon \sum_{1 \leq s \leq [N^\epsilon]+1} \left( \sum_{n \in I_s} \sum_{1 \leq n < m \leq N} e((f(n) + f(m))\alpha_1 + (g(n) + g(m))\alpha_2) \right)^2.
\]

We apply the above bound of \( |S(\alpha_1, \alpha_2)|^4 \) in (2.2), expand the squares, and integrate to obtain

\[
E(F_N; G_N) \ll N^2 + N^\epsilon \sum_{1 \leq s \leq [N^\epsilon]+1} \sum_{n \in I_s} \sum_{1 \leq n < m \leq N} 1.
\]
Observe that \( n, n_1 \in I_s \) imply that \( |n - n_1| \leq N^{1-\epsilon} \). If \( n_1 = n \), then \( m = m_1 \), and in this case the contribution of the sums in the right-hand side of (2.3) is \( N^2 \). Otherwise, by writing \( |n - n_1| = l \), the contribution of such sums is

\[
\ll \sum_{l \leq N^{1-\epsilon}} Q_l.
\]

Combining these estimates with (2.3), we obtain the upper bound.

To estimate the lower bound, we start with the identity

\[
N^2 = \int_{[0,1]^2} S^2(\alpha_1, \alpha_2) \sum_{m=0}^{2f(N)} e(-m \alpha_1) \sum_{n=0}^{2g(N)} e(-n \alpha_2) d\alpha.
\]

The lower bound follows from an application of the Cauchy-Schwarz inequality to the right-hand side.

**Theorem 2.2.** Let \( f(x) = x \) and \( g(x) = [h(x)] \), where \( h \) is a real valued function which is three times continuously differentiable on the segment \([1, N]\) with \( h'(x) > 0 \), \( h''(x) > 0 \) and \( h'''(x) < 0 \). For any real \( 0 < \epsilon < 1 \),

\[
E(F_N; G_N) \ll N^{2+\epsilon} + \frac{N^{1+\epsilon} \log N}{h''(N)}.
\]

The following corollary is immediate.

**Corollary 2.3.** Let \( 1 \leq A \leq 2 \) and \( 0 < \epsilon < 1 \). If \( h(n) = n(\log n)^A \), then

\[
E(F_N; G_N) \ll N^{2+\epsilon} (\log N)^{2-A}.
\]

**Proof of Theorem 2.2.** Here we take \( f(n) = n \) and \( g(n) = [h(n)] \), so \( Q_l \) in (2.1) reduces to the number of solutions of the equation

\[
[h(n)] + [h(m + l)] = [h(n + l)] + [h(m)], \text{ with } 1 \leq n < n + l \leq m < m + l \leq N.
\]

Now, following the proof of [10, Theorem 1] we get

\[
Q_l \ll N \left( \frac{2}{lh''(N)} + 1 \right).
\]

Combining this bound with Proposition 2.1 we obtain the required upper bound.

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### 3. “Generalized” GCD Sums

Given a function \( f : \mathbb{N} \to \mathbb{C} \) with finite support and \( \alpha \in (0, 1] \), the so-called GCD sum (also known as Gál sum) is defined as

\[
S_f(\alpha) := \sum_{a, b} f(a) f(b) \frac{(a, b)^{2\alpha}}{(ab)^\alpha}.
\]

This sum plays a key role in finding large values of the Riemann zeta function (see [3, 6]). Moreover, it is connected to the theory of equidistribution and pair correlation (for instance, see [11, 13]). We introduce here a new type of GCD sum in higher dimensions. Let \( f : \mathbb{N}^d \to \mathbb{C} \) be a function with finite support and \( \alpha \in (0, 1] \). Define \( d \)-dimensional GCD sum

\[
S_f(d, \alpha) := \sum_{a, b \in \mathbb{N}^d} f(a) f(b) \prod_{i=1}^d \frac{(a_i, b_i)^{2\alpha}}{(a_i b_i)^\alpha}.
\]
The next result provides an upper bound for $S_f(d; \alpha)$.

**Proposition 3.1.** Let $f : \mathbb{N}^d \to \mathbb{C}$ be an arithmetic function with finite support of cardinality $K$. Then we have the following estimates:

$$S_f(d; \alpha) \ll \begin{cases} (\log \log K)^{O(1)} \|f\|_2^2, & \text{if } \alpha = 1, \\ \exp(C(\alpha)(\log \log K)^{-\alpha}(\log \log K)^{-\alpha}) \|f\|_2^2, & \text{if } \frac{1}{2} < \alpha < 1, \end{cases}$$

where $C(\alpha)$ is an absolute positive constant depending on $\alpha$.

Furthermore, if $f$ is a positive real-valued satisfying $\|f\|_1 \geq 3$ and $K \geq \log \|f\|_1$, then

$$S_f(d; 1/2) \ll \frac{\exp(C(\log K \log \log ||f||_1)^{1/2})}{(\log ||f||_1 + O(1))^d} \|f\|_2^2,$$

for some positive absolute constant $C$.

**Example 1.** Let $\alpha = 1$, $d = 2$ and $f = 1_{[1,N]^2}$. Then,

$$S_f(2; 1) = \sum_{a,b \in [1,N]^2} \prod_{i=1}^2 \frac{(a_i b_i)^2}{a_i b_i} = \left( \sum_{a,b \in [1,N]} \frac{(a,b)^2}{ab} \right)^2.$$

Then by applying a result of Gál [8], we have

$$S_f(2; 1) \ll N^2(\log \log N)^4.$$

For simplicity, we prove Proposition 3.1 for the case $d = 2$ by extending an idea of a random model given by Lewko and Radziwill in [19]. For $d \geq 3$, the proof is a straightforward extension of the method.

Let $\{X(p) : p \text{ prime}\}$ and $\{Y(p) : p \text{ prime}\}$ be two collections of independent random variables uniformly distributed on $\mathbb{S}^1$. Also, assume that $\{X(p), Y(p) : p \text{ prime}\}$ is an independent collection. For every $n \in \mathbb{N}$, we define $X(n) := \prod_{p|n} X(p)$ and similarly $Y(n) := \prod_{p|n} Y(p)$. The random zeta function associated with $X(n)$ is defined by

$$\zeta_X(\alpha) := \sum_{n \geq 1} \frac{X(n)}{n^\alpha}, \text{ where } \alpha > 1/2,$$

and $\zeta_Y(\alpha)$ is defined similarly. For fixed $\alpha > 1/2$, these series converge a.e. by Kolmogorov three series theorem (see [16, Theorem 15.51]).

We recall the moment estimates of $\zeta_X(\alpha)$ from Lemma 7 of [3] and use them to prove Proposition 3.1.

**Lemma 3.2.** For a real number $l$,

$$\log \mathbb{E}[|\zeta_X(\alpha)|^{2l}] \ll \begin{cases} l \log \log l, & \text{if } \alpha = 1, \\ C'(\alpha)l^{1/\alpha}(\log l)^{-1}, & \text{if } 1/2 < \alpha < 1, \\ l^2 \log((\alpha - 1/2)^{-1}), & \text{if } \frac{1}{2} < \alpha, \end{cases}$$

where $C'(\alpha)$ is a positive constant, $l \geq 3$ for first two cases and $l \geq 1$ for the final case.

**Proof of Proposition 3.1.** Let us start by defining the double sum

$$D(X, Y) := \sum_{a,b} f(a,b)X(a)Y(b).$$
Then, we look at the expectation of \(|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2\), expressed as
\[
E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] = \sum_{m_1, n_1, m_2, n_2, a, b, c, d} \frac{f(a, b)\overline{f(c, d)}}{m_1^\alpha n_1^\alpha m_2^\alpha n_2^\alpha} \mathbb{1}_{n_1a=n_2c \quad m_1b=m_2d}.
\]
Now, indicator functions allow us to write
\[
n_1 = \frac{h_1c}{(a, c)}, \quad n_2 = \frac{h_2a}{(a, c)} \quad \text{and} \quad m_1 = \frac{h_2d}{(b, d)}, \quad m_2 = \frac{h_2b}{(b, d)}
\]
for positive integers \(h_1, h_2\). Thus,
\[
(3.1) \quad E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] = \sum_{h_1, h_2, a, b, c, d} f(a, b)\overline{f(c, d)} \frac{(a, c)^{2\alpha}(b, d)^{2\alpha}}{(abcd)^\alpha} \frac{1}{h_1^2 h_2^2} = \zeta(2\alpha)^2 S_f(2, \alpha).
\]
Also, we notice that \(E[|D(X, Y)|^2] = \|f\|_2^2\). Let \(V\) and \(l\) be positive real parameters to be chosen later. Consider the event \(\mathcal{A} = (|\zeta_X(\alpha)| < V, |\zeta_Y(\alpha)| < V)\). Now by splitting the expectation over \(\mathcal{A}\) and its complement, we get
\[
(3.2) \quad E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] \leq V^4 \|f\|_2^2 + V^{2-2l} E[|\zeta_Y(\alpha)|^{2+2l}|D(X, Y)|^2] \\
+ V^{2-2l} E[|\zeta_X(\alpha)|^{2+2l}|D(X, Y)|^2] \\
+ V^{-4l} E[|\zeta_X(\alpha)|^{2+2l}|\zeta_Y(\alpha)|^{2+2l}].
\]
By Cauchy-Schwarz inequality, \(|D(X, Y)|^2 \leq \|f\|_2^2 K\). Using the inequality above, we get:
\[
(3.3) \quad E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] \leq \|f\|_2^2 (V^4 + V^{2-2l} K \mathbb{E}[|\zeta_Y(\alpha)|^{2+2l}] \\
+ V^{2-2l} K \mathbb{E}[|\zeta_X(\alpha)|^{2+2l}] + V^{-4l} K \mathbb{E}[|\zeta_X(\alpha)|^{2+2l}|\zeta_Y(\alpha)|^{2+2l}].
\]
Case 1. \(\alpha = 1\). Using Lemma \([3.2]\) in \([3.3]\) we get the upper bound
\[
\|f\|_2^2 (V^4 + V^{2-2l} K \exp(Cl \log \log l)).
\]
We choose \(l = \log K + 3\) and \(V = (\log l)^C\), where \(C\) is a large positive constant, and obtain
\[
E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] \leq \|f\|_2^2 (\log \log K)^O(1).
\]
Case 2. \(\frac{1}{2} < \alpha < 1\). Applying Lemma \([3.2]\) in \([3.3]\) to get
\[
E[|\zeta_X(\alpha)\zeta_Y(\alpha)D(X, Y)|^2] \leq \|f\|_2^2 (V^4 + V^{2-2l} K \exp(C'(\alpha) l^{1/\alpha} (\log l)^{-1})).
\]
Choosing \(l = (\log K)^{\alpha} (\log \log K)^{\alpha} + 3\) and \(V = \exp(C(\alpha) l^{-1+1/\alpha} (\log l)^{-1})\) we get the upper bound
\[
\|f\|_2^2 \exp(C(\alpha) (\log K)^{1-\alpha} (\log \log K)^{-\alpha}).
\]
Case 3. \(\alpha = \frac{1}{2}\). Let \(\beta = \frac{1}{2} + \frac{1}{\log \|f\|_1}\). Then, \([3.3]\) and Lemma \([3.2]\) give us
\[
(3.4) \quad E[|\zeta_X(\beta)\zeta_Y(\beta)D(X, Y)|^2] \leq \|f\|_2^2 (V^4 + V^{2-2l} K \exp(C' l^2 \log \log \|f\|_1)).
\]
By Hölder’s inequality we get
\[
(3.5) \quad S_f(2, 1/2) \leq \left( S_f(2, \beta) \right)^{\frac{1}{2\beta}} \|f\|_1^{2(1-\frac{1}{2\beta})} \ll \left( S_f(2, \beta) \right)^{\frac{1}{2\beta}}.
\]
For $s \to 1$ we have

$$\zeta(s) = \frac{1}{s-1} + O(1).$$

Using this in (3.1) and combining with (3.4) and (3.5), we obtain the upper bound

$$(3.6) \quad S_f(2, 1/2) \ll (\log \|f\|_1 + O(1))^{-2} \|f\|_2^2 (V^4 + V^2 - 2l \exp(C'^2 \log \log \|f\|_1)).$$

Hence we choose $l = (\log K)^{1/2}(\log \log \|f\|_1)^{-1/2}$ and $V = \exp(C''(l \log \log \|f\|_1))$ in (3.6) to get the upper bound

$$S_f(2, 1/2) \ll (\log \|f\|_1 + O(1))^{-2} \|f\|_2^2 \exp(C(\log K \log \log \|f\|_1)^{1/2}).$$

This completes the proof. \[\square\]

4. Proof of Theorem 1.1

Proof. Let $s > 0$ be fixed and $N \geq (2s)^d$ be an integer. For $\alpha \in \mathbb{R}^d$ and $a_n \in \mathbb{N}^d$, consider the sequence $(x_n) = \{a_n \alpha\}$. Then,

$$R_{2,\infty}^{(d)}(s, \alpha, N) = \frac{1}{N} \sum_{1 \leq m \neq n \leq N} \chi_{s,N}(\alpha(a_m - a_n)),$$

where $\chi_{s,N}$ is a characteristic function defined for $x \in \mathbb{R}^d$ as follows:

$$\chi_{s,N}(x) = \begin{cases} 1 & \text{if } \|x\|_\infty \leq s/N^{1/d}, \\ 0 & \text{otherwise}. \end{cases}$$

The Fourier series expansion of $\chi_{s,N}$ is given by

$$(4.1) \quad \chi_{s,N}(\alpha) \sim \sum_{r \in \mathbb{Z}^d} c_r e(r.\alpha),$$

where

$$c_r = c_{r,s} = \int_{-s/N^{1/d}}^{s/N^{1/d}} \cdots \int_{-s/N^{1/d}}^{s/N^{1/d}} e\left(-\sum_{i \leq d} r_i \alpha_i\right) d\alpha_1 \cdots d\alpha_d$$

$$= c_{r_1} \cdots c_{r_d},$$

and

$$c_{r_j} = \int_{-s/N^{1/d}}^{s/N^{1/d}} e(-r_j \alpha_j) d\alpha_j, \quad j = 1, 2, \ldots, d.$$

Further, one gets the following upper bound:

$$|c_{r_j}| \leq \min\left(2sN^{-\frac{d}{2}}, |r_j|^{-1}\right).$$

A straightforward calculation gives the expectation:

$$\mathbb{E}[R_{2,\infty}^{(d)}(s, \alpha, N)] = \int_{[0,1]^d} R_{2,\infty}^{(d)}(s, \alpha, N) d\alpha = (2s)^d \frac{N - 1}{N}.$$

Now, the variance of $R_{2,\infty}^{(d)}(s, \alpha, N)$ is defined as

$$\text{Var}(R_{2,\infty}^{(d)}(s, \alpha, N)) : = \int_{[0,1]^d} \left( R_{2,\infty}^{(d)}(s, \alpha, N) - \frac{(2s)^d(N-1)}{N} \right)^2 d\alpha.$$
By using Fourier series expansion of $\chi_{s,N}$ from (4.1), we write

$$\var{R_{2,\infty}^{(d)}(s, N)} = \frac{1}{N^2} \int_{[0,1]^d} \left( \sum_{1 \leq m \neq n \leq N} \sum_{r \in \mathbb{Z}^d \setminus \{0\}} c_r e\left(r \cdot (\alpha(a_m - a_n))\right) \right)^2 d\alpha.$$ 

After squaring the integrand, we want to interchange the summations and integrations. Such rearrangement can be justified by using the fact that the partial sums of a Fourier series of an indicator function are uniformly bounded. Hence, the dominated convergence theorem is applicable (for example, see [21], Chapter 3, Exercise 18). Thus, $\var{R_{2,\infty}^{(d)}(s, N)}$ equals

$$\frac{1}{N^2} \sum_{1 \leq m \neq n \leq N} \sum_{r, t \in \mathbb{Z}^d \setminus \{0\}} c_r c_t \int_{[0,1]^d} e\left(r \cdot (\alpha(a_m - a_n)) - t \cdot (\alpha(a_k - a_l))\right) d\alpha$$

$$= \frac{1}{N^2} \sum_{1 \leq m \neq n \leq N} \sum_{r, t \in \mathbb{Z}^d \setminus \{0\}} c_r c_t \prod_{i \leq d} \int_{[0,1]} e\left(r_i(a_m^i - a_n^i) - t_i(a_k^i - a_l^i)\right) d\alpha_i.$$ 

But, the innermost integral is equal to 1 whenever $r_i = t_i = 0$ or $r_i(a_m^i - a_n^i) = t_i(a_k^i - a_l^i)$, otherwise the integral is zero. Note that when all components of $r, t$ are nonzero, the associated sums will produce the main contribution. Otherwise, when some of them are zero, we save powers of $N$. Precisely, the variance is equal to

$$\frac{1}{N^2} \sum_{1 \leq m \neq n \leq N} \sum_{v, w} \mathcal{R}_N(v_{j_1}, \ldots, v_{j_q}) \mathcal{R}_N(w_{j_1}, \ldots, w_{j_q}) \left(\frac{2s}{N^{1/d}}\right)^{2(q-d)} \sum_{r, t \in \mathbb{Z}^q} c_r c_t,$$

where $\mathcal{R}_N$ is as defined in (1.1).

The relation $r_i v_i = t_i w_i$ allows us to write $r_i, t_i$ as

$$r_i = \frac{h_i v_i}{\gcd(v_i, w_i)} \quad \text{and} \quad t_i = \frac{h_i v_i}{\gcd(v_i, w_i)},$$

where $h_i$ is a nonzero integer. Now we separate the sum over $h_i$ into three sub-intervals of the real line given by

$$|h_i| \leq \frac{N^{1/d} \gcd(v_i, w_i)}{s \max(|v_i|, |w_i|)},$$

$$\frac{N^{1/d} \gcd(v_i, w_i)}{s \max(|v_i|, |w_i|)} < |h_i| \leq \frac{N^{1/d} \gcd(v_i, w_i)}{s \min(|v_i|, |w_i|)},$$

$$|h_i| \geq \frac{N^{1/d} \gcd(v_i, w_i)}{s \min(|v_i|, |w_i|)}.$$ 

For any fixed $j_1 < j_2 < \cdots < j_q$, we follow the arguments in page 344 of [14] and get

$$\sum_{r, t \in \mathbb{Z}^q} c_r c_t \ll \frac{s^q (\log N)^q}{N^{q/d}} \prod_{1 \leq \beta \leq q} \frac{\gcd(v_{j_\beta}, w_{j_\beta})}{\sqrt{v_{j_\beta} w_{j_\beta}}}.$$
Replace the above bound in the right hand side of (4.2) to get
\[ \text{Var}(R_2^{(d)}(s, ., N)) \ll \frac{s^d (\log N)^d}{N^3} \sum_{v, w \in \mathbb{Z}^d} \mathcal{R}_N(v) \mathcal{R}_N(w) \prod_{1 \leq i \leq d} \frac{\gcd(v_i, w_i)}{\sqrt{v_i w_i}} \]
\[ + \sum_{j_1 < \cdots < j_q \leq d, 1 \leq q < d} \frac{s^{2d-q} (\log N)^q}{N^{4-q/d}} \sum_{v, w \in \mathbb{Z}^q} \mathcal{R}_N(v) \mathcal{R}_N(w) \prod_{1 \leq \beta \leq q} \frac{\gcd(v_{j_{\beta}}, w_{j_{\beta}})}{\sqrt{v_{j_{\beta}} w_{j_{\beta}}}}. \]

Now applying Proposition 3.1 to the above GCD sums, we get for any \( \gamma > 0 \),
\[ \text{Var}(R_2^{(d)}(s, ., N)) \ll \frac{s^d (\log N)^d}{N^3} E\left(A_N^{(1)}; \ldots; A_N^{(d)}\right) \exp\left(C \sqrt{\log N \log \log N}\right) \frac{1}{(\log N + O(1))^d} + \frac{1}{N^{1/d - \gamma}}. \]

Thus, under the hypothesis of the theorem, it follows that the variance is \( O(N^{-\delta/2}) \). This is the main part of the proof. The rest of the arguments follow a standard method of applying Chebyshev’s inequality and Borel-cantelli lemma (see the proof of Theorem 1 in [1] and also [14, 23].) \( \square \)

5. Proof of Theorems 1.3 and 1.4

5.1. Properties of Bessel functions. Here we state some properties of the Bessel function, which are important tools in the proof of Theorem 1.3 and Theorem 1.4. For \( \nu \) complex order with \( \Re(\nu) > -1/2 \) and \( t \geq 0 \), the Bessel function \( J_\nu \) is defined by
\[ J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^{1} e^{itx}(1-x^2)^{\nu} \frac{dx}{\sqrt{1-x^2}}. \]
This definition is also valid for \( t \in \mathbb{C} \). The function \( J_\nu \) has many interesting properties. For our application, we mention a few of them below and these are essentially given in [11] Appendix B1. For \( \Re(\nu) > -1/2 \), \( t \geq 1 \), we have a nice approximation that gives an asymptotic formula as \( t \to \infty \),
\[ J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos \left( t - \frac{\pi \nu}{2} - \frac{\pi}{4} \right) + O(t^{-3/2}). \]
For \( 0 < t \leq 1 \) and \( \Re(\nu) > -1/2 \), we have the following upper bound
\[ J_\nu(t) \ll \nu \exp \left( \max\{ (\Re(\nu) + 1/2)^{-2}, (\Re(\nu) + 1/2)^{-1}\} |\Im(\nu)| \right) t^{\Re(\nu)}. \]
Whenever \( t > 0 \) and \( \nu \in \mathbb{N} \), the Bessel function reduces to a simple form:
\[ J_\nu(t) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos(t \sin \theta - \nu \theta) d\theta. \]

**Lemma 5.1.** Let \( N, \nu \geq 2 \) be two integers and \( s, r > 0 \) be real numbers. Then
1. \( J_{\nu/2}(r) \ll_{\nu} 1 \) if \( r \leq 1 \),
2. \( J_{\nu/2}(r) \ll_{\nu} \frac{1}{\sqrt{r}} \) if \( r > 1 \),
3. \( J_{\nu/2}(2\pi r \sqrt{N}^{-1/\nu}) \ll_{\nu} \frac{\nu^{3/2}}{\sqrt{N}} \), for any \( r > 0 \),
\[(4) \ |J_\mu(r)| \leq 1, \text{ for any } r > 0, \text{ and } \mu \in \mathbb{N}.\]

Properties 1, 2, 3 and 4 follows from (5.3), (5.2), (5.1) and (5.4), respectively.

5.2. **Preparation of the proofs of Theorem 1.3 and Theorem 1.4.** For the sequence \((x_n) = \{\{a_n, \alpha\}\}\), the \(d\)-dimensional pair correlation statistic in 2-norm is

\[
R^{(d)}_{2,2}(s, \alpha, N) = \frac{1}{N} \sum_{1 \leq m \neq n \leq N} I_{s,N}(\alpha(a_m - a_n)),
\]

where \(I_{s,N}\) is the indicator function for all \(x \in \mathbb{R}^d\) which satisfy \(\|x\|_2 \leq s/N^{1/d}\). Then, the Fourier series expansion of \(I_{s,N}\) is given by

\[
I_{s,N}(\alpha) \sim \sum_{r \in \mathbb{Z}^d} c_r e(r.\alpha),
\]

where

\[
(5.5) \quad c_r = c_{r,s} = \begin{cases} \omega_d \frac{s^{d/2}}{N} J_d/2 \left( \frac{2\pi s}{N^{1/d}} \|r\|_2 \right), & \text{if } r = 0, \\ \frac{1}{\sqrt{N}} \frac{\omega_d^{d/2}}{s^{d/2}} J_d/2 \left( \frac{2\pi s}{N^{1/d}} \|r\|_2 \right), & \text{if } r \in \mathbb{Z}^d \setminus \{0\}. \end{cases}
\]

For the details of such formulas, one can see [11, Appendix B].

It is not hard to verify that

\[
\mathbb{E}(R^{(d)}_{2,2}(s, .., N)) = \omega_d s \frac{N - 1}{N}.
\]

We now proceed to calculate the variance of \(R^{(d)}_{2,2}(s, .., N)\), that is,

\[
\text{Var}(R^{(d)}_{2,2}(s, .., N)) = \int_{[0,1]^d} \left( R^{(d)}_{2,2}(s, \alpha, N) - \omega_d \frac{(N - 1)}{N} \right)^2 d\alpha
\]

\[
= \frac{1}{N^2} \int_{[0,1]^d} \left( \sum_{1 \leq m \neq n \leq N} \sum_{r \in \mathbb{Z}^d \setminus \{0\}} c_r e(r.\alpha(a_m - a_n)) \right)^2 d\alpha.
\]

By interchanging summation and integration \(\text{Var}(R^{(d)}_{2,2}(s, .., N))\) is seen to be bounded above by

\[
(5.6) \quad \frac{1}{N^2} \sum_{1 \leq m \neq n \leq N} \sum_{1 \leq k \neq l \leq N} c_r c_t \prod_{i \leq d} \int_{[0,1]} e \left( (r_i(a_m^{(i)} - a_n^{(i)}) - t_i(d_k^{(i)} - d_l^{(i)})) \alpha_i \right) d\alpha_i.
\]

Such rearrangement can be justified as follows. We split the Fourier series of the indicator function at some parameter \(M\) and consider the partial sum up to \(M\) and the tail part. We apply the Cauchy-Schwarz inequality to the tail part to interchange the sums and the integration, which leaves us with the square-integral of the tail. However, the 2-norm of the tail goes to zero as \(M \to \infty\). Thus, for arbitrarily large \(M\) the partial sum up to \(M\) remains, and the rearrangement is obvious.

As in Theorem 1.1, it is enough to show that the variance is arbitrarily small.
Proof of Theorem 1.3. In this theorem, we consider \( a_n^{(i)} = a_n \) for all \( 1 \leq i \leq d \), and the orthogonality of exponential function allow us to write

\[
\text{Var}(R_{2d}(s, N)) \leq \frac{1}{N^d} \sum_{v, w \in \mathbb{Z} \setminus \{0\}} R_N(v) R_N(w) \sum_{r, t \in \mathbb{Z}^d \setminus \{0\}} |c_r c_t|.
\]

Let us fix \( j_1 < j_2 < \cdots < j_q \leq d \) where \( 1 \leq q \leq d \). We claim that

\[
\sum_{r, t \in \mathbb{Z}^d \setminus \{0\}} |c_r c_t| \ll_d s^{2d-q} (v, w)^q \sqrt{v w^q}
\]

where the sum \( \Sigma'' \) is over \( r, t \) with all components zero other than \( r_{j_\beta}, t_{j_\beta} \) for \( 1 \leq \beta \leq q \). This implies that for \( q < d \), we will always save a power of \( N \) and the main contribution to the variance comes from the case \( q = d \).

Now, from the relations \( r_{j_\beta} v = r_{j_\beta} w \) we can write

\[
r_{j_\beta} = \frac{w h_{j_\beta}}{(v, w)} \quad \text{and} \quad t_{j_\beta} = \frac{v h_{j_\beta}}{(v, w)} \quad \text{for} \quad 1 \leq \beta \leq q,
\]

where \( h_{j_\beta} \) takes integer values. Let us denote the \( q \)-tuple \( (h_{j_1}, \ldots, h_{j_q}) \) by \( h \) and from now onwards, we use the notation \( \|h\|_2 \) for the usual 2-norm.

Using the definition of \( c_r \)'s from (5.5) the left hand side of (5.8) is bounded by

\[
\ll \frac{s^d}{N} \sum_{h \in \mathbb{Z}^d} (v, w)^d \left| J_d/2 \left( \frac{2\pi s|v|}{N^{1/d}(v, w)} \|h\|_2 \right) \right| \left| J_d/2 \left( \frac{2\pi s|w|}{N^{1/d}(v, w)} \|h\|_2 \right) \right|.
\]

Now we divide this sum into three parts \( \Sigma' = \Sigma'_1 + \Sigma'_2 + \Sigma'_3 \) depending on the following range of values of \( \|h\|_2 \):

\[
1 \leq \|h\|_2 \leq \frac{N^{1/d}(v, w)}{2\pi s \max(|v|, |w|)} : = \max(h),
\]

\[
\frac{N^{1/d}(v, w)}{2\pi s \max(|v|, |w|)} \leq \|h\|_2 \leq \frac{N^{1/d}(v, w)}{2\pi s \min(|v|, |w|)} : = \min(h),
\]

\[
\|h\|_2 \geq \frac{N^{1/d}(v, w)}{2\pi s \min(|v|, |w|)}.
\]

We are using (3) of Lemma 5.1 to get

\[
\Sigma'_1 \ll \frac{s^d}{N} (\max(h))^q \ll \frac{s^{d-q}(v, w)^q}{N^{1-q/d} \sqrt{|vw|^q}}.
\]

By using (1) and (2) of Lemma 5.1 we obtain

\[
\Sigma'_2 \ll \frac{(v, w)^d}{(|vw|)^{d/2}} \sum_{\max(h) \leq \|h\|_2 \leq \min(h)} \frac{\sqrt{\max(h)} \|h\|^{d+1/2}}{N^{1-q/d} |vw|^{d/2}} \ll \frac{s^{d-q}(v, w)^q}{N^{1-q/d} \sqrt{|vw|^q}}.
\]
Again, using case (2) of Lemma 5.1, we deduce

\[
\sum' \ll \frac{(v, w)^{d+1} N^{1/d}}{s([vw])^{d/2+1/2}} \sum_{\|h\|_{2} > \min_{h} \|h\|_{2}} \frac{1}{\|h\|_{2}^{d+1}}
\]

\[
\ll \frac{(v, w)^{d+1} N^{1/d}}{s([vw])^{d/2+1/2}} \frac{1}{(\min_{h})^{d-q+1}} \ll \frac{s^{d-q}(v, w)^{q}}{N^{1-q/d} \sqrt{|vw|^{q}}}
\]

Combining all three bounds with (5.9) we obtain the claim (5.8). By inserting (5.8) into (5.7), we get

\[
\Var(R_{2,2}^{(d)}(s, N)) \ll \sum_{1 \leq q \leq d} \frac{s^{2d-q}}{N^{4-q/d}} \sum_{v,w \in \mathbb{Z} \setminus \{0\}} \mathcal{R}_{N}(v) \mathcal{R}_{N}(w) \frac{(v, w)^{q}}{\sqrt{|vw|^{q}}}
\]

Now, using the result on GCD sums by Gál [8], we obtain

\[
\Var(R_{2,2}^{(d)}(s, N)) \ll \begin{cases} \frac{s^{2}}{N^{2}} E_{N}(A_{N})(\log \log N)^{2} + \frac{s^{2}}{N^{d/2-1}}, & \text{if } d = 2, \\ \frac{s^{d}}{N^{d}} E_{N}(A_{N}) + \frac{s^{d}}{N^{1-d}}, & \text{if } d \geq 3, \end{cases}
\]

for some sufficiently small \( \gamma > 0 \). Thus, under the hypothesis of this theorem, the variance is \( O((\log N)^{1-\varepsilon/2}) \). The rest of the arguments follow from the proof of [3, Theorem 5]. \( \square \)

**Proof of Theorem 1.4.** For \( d = 2, \) (5.6) gives us

\[
(5.10) \quad \Var(R_{2,2}^{(2)}(s, N)) \leq \frac{1}{N^{2}} \sum_{v, w \in \mathbb{Z} \setminus \{0\}} \mathcal{R}_{N}(v) \mathcal{R}_{N}(w) \sum_{r, t \in \mathbb{Z}^{2}} |c_{r} c_{t}|
\]

\[
+ \frac{1}{N^{2}} \sum_{v, w \in \mathbb{Z} \setminus \{0\}} \mathcal{R}_{N}(v) \mathcal{R}_{N}(w) \sum_{r, t \in \mathbb{Z}^{2}} |c_{r(0,0)} c_{t(0,0)}|
\]

\[
+ \frac{1}{N^{2}} \sum_{v, w \in \mathbb{Z} \setminus \{0\}} \mathcal{R}_{N}(v) \mathcal{R}_{N}(w) \sum_{r, t \in \mathbb{Z}^{2}} |c_{r(0,0)} c_{t(0,0)}|
\]

Note that the second and third terms on the right hand side above are \( O(N^{-1/2+\gamma}) \) for some \( \gamma > 0 \), as it follows from (5.8) with \( d = 2 \) and \( q = 1 \).

Let us call the inner sum in the first term \( I \). From the relation \( r_{i} v_{i} = t_{i} w_{i} \), we have \( r_{i} = \frac{w_{i} h_{i}}{v_{i} w_{i}} \) and \( t_{i} = \frac{v_{i} h_{i}}{v_{i} w_{i}} \) for \( i = 1, 2 \) where \( h_{i}'s \) are nonzero integers. For simplicity we write

\[
A = \frac{v_{1}}{(v_{1}, w_{1})}, \quad B = \frac{w_{2}}{(v_{2}, w_{2})}, \quad C = \frac{v_{1}}{(v_{1}, w_{1})}, \quad D = \frac{w_{2}}{(v_{2}, w_{2})}.
\]

By using (5.3) we obtain

\[
I \ll \frac{s^{2}}{N} \sum_{h_{1}, h_{2} \neq 0} \left| J_{1} \left( \frac{2\pi s}{\sqrt{N}} (A^{2} h_{1}^{2} + B^{2} h_{2}^{2})^{1/2} \right) \right| \left| J_{1} \left( \frac{2\pi s}{\sqrt{N}} (C^{2} h_{1}^{2} + D^{2} h_{2}^{2})^{1/2} \right) \right|
\]

\[
= \frac{s^{2}}{N |AC|} \sum_{h_{1} \neq 0} \frac{1}{h_{1}^{2}} \sum_{h_{2} \neq 0} \left| J_{1} \left( \frac{2\pi s |Ah_{1}|}{\sqrt{N}} (1 + \frac{B^{2}}{A^{2} h_{1}^{2}}) \right) \right| \left| J_{1} \left( \frac{2\pi s |Ch_{1}|}{\sqrt{N}} (1 + \frac{D^{2}}{C^{2} h_{1}^{2}}) \right) \right|
\]

\[
\times \left( 1 + \frac{B^{2}}{A^{2} h_{1}^{2}} \right)^{1/2} \left( 1 + \frac{D^{2}}{C^{2} h_{1}^{2}} \right)^{1/2}.
\]
Now we divide the sum over $h_1$ into two parts,

$$|h_1| \leq \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)} \quad \text{and} \quad |h_1| > \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)}.$$

Applying (4) and (2) of Lemma 5.1 respectively, for the small and large arguments of the Bessel function in the above inequality, we get

$$I \ll \frac{s^2}{N|AC|} \left( \sum_{|h_1| \leq \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)}} \frac{1}{h_1^2} \sum_{h_2 \neq 0} \left( 1 + \frac{B^2}{A^2h_1^2}h_2^2 \right)^{-\frac{1}{4}} \left( 1 + \frac{D^2}{C^2h_1^2}h_2^2 \right)^{-\frac{1}{4}} \right) + \sum_{|h_1| > \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)}} \frac{\sqrt{N}}{s|AC|^{1/2}|h_1|^3} \sum_{h_2 \neq 0} \left( 1 + \frac{B^2}{A^2h_2^2}h_1^2 \right)^{-\frac{1}{4}} \left( 1 + \frac{D^2}{C^2h_1^2}h_2^2 \right)^{-\frac{1}{4}} \right).$$

Now observe that for $a_1, a_2 \in \mathbb{R}$, $(1 + a_1^2)(1 + a_2^2) \geq (1 + a_1a_2)^2$. Then, the sum on $h_2$ in the above two terms can be bounded by

$$\int_0^\infty \left( 1 + \frac{|BD|}{|AC|} \frac{x^2}{h_1^2} \right)^{-1} dx \quad \text{and} \quad \int_0^\infty \left( 1 + \frac{|BD|}{|AC|} \frac{x^2}{h_1^2} \right)^{-3/2} dx,$$

respectively, and both are bounded by $|h_1||\frac{AC}{BD}|^{1/2}$. Thus, we have

$$I \ll \frac{s^2}{N|AC|} \left( \left| \frac{AC}{BD} \right|^{1/2} \sum_{|h_1| \leq \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)}} \frac{1}{h_1^2} \right) + \sum_{|h_1| > \frac{\sqrt{N}}{2\pi s \min(|A|, |C|)}} \frac{s^2 \log N}{N\min(|A|, |C|)} \ll \frac{s^2}{N|AC|} \left( \left| \frac{AC}{BD} \right|^{1/2} \log N + \min(|A|, |C|) \right) \ll \frac{s^2}{N|AC|} \left( \log N \right).$$

Hence, we deduce the variance estimate

$$(5.11) \quad \text{Var}(R_{2,2}(s, N)) \ll \frac{s^2 \log N}{N^3} \sum_{v_1, v_2} \mathcal{R}_N(v_1, v_2) \mathcal{R}_N(w_1, w_2) \frac{(v_1, w_1)(v_2, w_2)}{\sqrt{|v_1w_1v_2w_2|}}.$$

Now applying Proposition 3.1 in (5.11), we conclude that

$$\text{Var}(R_{2,2}(s, N)) \ll \frac{s^2}{N^3 \log N} E(A^{(1)} N, A^{(2)} N) \exp(C \sqrt{\log N \log \log N}).$$

This completes the proof. \(\square\)

6. PROOF OF THEOREM 1.5

This follows easily by combining Theorem 1.1 with Corollary 2.3 and Theorem 1.4 with Corollary 2.3.

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