Possible Finiteness of the Higgs-Boson Mass Renormalization

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Abstract

It is shown by explicit calculation that the one-loop mass renormalization of the Higgs boson in the standard model is gauge-independent. It could even be rendered finite if the following mass relationships were satisfied: $m_t^2 \simeq m_H^2 = (2M_W^2 + M_Z^2)/3$. Numerically, this would imply $m_t \simeq 84 \text{ GeV}$ which is below the current experimental lower bound of 91 GeV, but since higher-order corrections are yet to be calculated, the above hypothesis could still have a chance of being realized.
The standard $SU(2) \times U(1)$ gauge model\cite{1} has been firmly established in the past several years as the correct theory of fundamental electroweak interactions in all its aspects except for the role of its scalar sector. As is well-known, the spontaneous breaking of electroweak $SU(2) \times U(1)$ to electromagnetic $U(1)$ is most simply accomplished by the introduction of a fundamental scalar doublet $\Phi = (\phi^+, \phi^0)$, which acquires a nonzero vacuum expectation value $< \phi^0 > \equiv v$ and allows the $W$ and $Z$ gauge bosons to become massive\cite{2}. The one remaining physical degree of freedom appears as the scalar Higgs boson $H$, which is of course yet to be discovered. The existence of $H$ itself as a fundamental particle is considered by many people to be problematic because there must then be a quadratic divergence in the quantum field theory which appears as a correction to the square of the Higgs-boson mass $m_H$. Technically, one can always fine-tune the bare mass to compensate for this very large correction so that $m_H$ remains at the electroweak mass scale of order $10^2 \text{ GeV}$, but it is usually considered to be "unnatural". If one takes this as a hint that the standard model is either just an effective theory or one in which additional self-consistent constraints should be imposed, then the possibility exists that whereas $H$ may appear as a physical particle, the renormalization of $m_H$ itself may be actually finite. In the following, it will be shown that the standard model may indeed be consistent with such a hypothesis and two interesting mass relationships will be obtained.

In the standard model, let the Higgs potential be written as

$$V = \mu^2 (\Phi^\dagger \Phi) + \frac{1}{2} \lambda (\Phi^\dagger \Phi)^2, \quad (1)$$

then the vacuum expectation value of $\Phi$ is

$$v = (-\mu^2/\lambda)^{\frac{1}{2}}, \quad (2)$$

and the mass of the physical Higgs boson is given by

$$m_H^2 = 2\lambda v^2. \quad (3)$$

The masses of the vector gauge bosons are related to $v$ by

$$M_W^2 = \frac{1}{2} g_2^2 v^2, \quad (4)$$

and

$$M_Z^2 = \frac{1}{2} (g_1^2 + g_2^2) v^2, \quad (5)$$

with the weak mixing angle $\theta_W$ given by

$$\sin^2 \theta_W = \frac{g_1^2}{g_1^2 + g_2^2}. \quad (6)$$
The mass of a given fermion $f$ is then

$$m_f = g_f v. \quad (7)$$

Consider the renormalization of any physical mass found in the standard model such as $m_H$ or $m_f$. There are two kinds of contributions: the one-particle-irreducible (1PI) ones and the one-particle-reducible (1PR) ones. The latter are the result of tadpole diagrams with $H$ coupling to all massive particle-antiparticle pairs, as shown in Fig. 1. Using the $R_\xi$ gauge and a conventional cutoff procedure to regularize the divergent integrals,

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2} = \frac{-i}{16\pi^2} \left[ \Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right], \quad (8)$$

the sum of all one-loop tadpole contributions to the Higgs-boson two-point function $\Sigma(p^2)$ given by

$$-i\Sigma_R(p^2) = \frac{g^2}{M_W^2} \int \frac{d^4 k}{(2\pi)^4} \left[ -\frac{9m_H^2}{8(k^2 - m_H^2)} - \frac{9M_W^2}{2(k^2 - M_W^2)} - \frac{3m_H^2}{4(k^2 - \xi M_W^2)} \right. \left. - \frac{9M_Z^2}{4(k^2 - M_Z^2)} - \frac{3m_H^2}{8(k^2 - \xi M_Z^2)} + \sum_f \frac{3n_f m_f^2}{k^2 - m_f^2} \right] \quad (9)$$

is calculated to be

$$\Sigma_R(p^2) = -\frac{9g^2\Lambda^2}{64\pi^2M_W^2} \left[ m_H^2 + 2M_W^2 + M_Z^2 - 4 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \right]$$

$$+ \frac{9g^2}{64\pi^2M_W^2} \left[ \frac{1}{2} m_H^4 \ln \frac{\Lambda^2}{m_H^2} + 2M_W^4 \ln \frac{\Lambda^2}{M_W^2} + M_Z^4 \ln \frac{\Lambda^2}{M_Z^2} \right.$$ \left. - 4 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \ln \frac{\Lambda^2}{m_f^2} + \xi m_f^2 \left( \frac{1}{3} M_W^2 \ln \frac{\Lambda^2}{\xi M_W^2} + \frac{1}{6} M_Z^2 \ln \frac{\Lambda^2}{\xi M_Z^2} \right) \right], \quad (10)$$

where $n_f$ is the number of color degrees of freedom for the fermion $f$, i.e. 3 for quarks and 1 for leptons. In the above, if the coefficient of the quadratically divergent term is set equal to zero, the well-known condition first given by Veltman[3], namely

$$4m_i^2 \simeq 2M_W^2 + M_Z^2 + m_H^2, \quad (11)$$

is obtained, where all other fermion masses have been dropped because they are negligible. Recently, these tadpole contributions have been investigated by two groups. Osland and Wu[4] have obtained the quadratically divergent terms using point-splitting regularization; whereas Blumhofer and Stech[5] have obtained the logarithmically divergent terms as well.
Their results agree exactly with Eq. (10). Note that whereas $\Sigma_R(p^2)$ is actually momentum-independent, it is gauge-dependent, but that is not a problem because it is not a physically measurable quantity.

The one-loop 1PI contributions to $\Sigma(p^2)$ are shown in Fig. 2. They sum up to

$$-i\Sigma_f(p^2) = \frac{g^2}{M_W^2} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{3m_H^2}{8(k^2 - m_H^2)} + \frac{3M_W^2}{2(k^2 - M_W^2)} + \frac{m_H^2}{4(k^2 - \xi M_W^2)} \right.$$  
$$+ \frac{3M_Z^2}{4(k^2 - M_Z^2)} + \frac{m_H^2}{8(k^2 - \xi M_Z^2)} - \sum_f \frac{n_f m_f^2}{k^2 - m_f^2}$$  
$$+ \frac{9m_H^4}{8(k^2 - m_H^2)^2} + \frac{3M_W^4}{4(k^2 - M_W^2)^2} + \frac{m_H^4}{4(k^2 - \xi M_W^2)^2}$$  
$$+ \frac{3M_Z^4}{2(k^2 - M_Z^2)^2} + \frac{m_H^4}{8(k^2 - \xi M_Z^2)^2} - \sum_f \frac{2n_f m_f^4}{(k^2 - m_f^2)^2}$$  
$$-p^2 \left( \frac{(3 - \xi)M_W^2}{2(k^2 - \xi M_W^2)^2} + \frac{(3 - \xi)M_Z^2}{4(k^2 - \xi M_Z^2)^2} - \sum_f \frac{n_f m_f^2}{2(k^2 - m_f^2)^2} \right) \right]$$

+ finite terms. \hspace{1cm} (12)

The coefficient of the $p^2$ term is of course the wave-function (or field-operator) renormalization of $H$, which is both logarithmically divergent and gauge-dependent. To obtain the corresponding mass renormalization, let $p^2 = m_H^2$, then

$$\Sigma_f(m_H^2) = \frac{3g^2\Lambda^2}{64\pi^2 M_W^2} \left[ m_H^2 + 2M_W^2 + M_Z^2 - 4 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \right]$$

$$- \frac{3g^2}{64\pi^2 M_W^2} \left[ \frac{5}{2} m_H^4 \ln \frac{\Lambda^2}{m_H^2} + 6M_W^4 \ln \frac{\Lambda^2}{M_W^2} + 3M_Z^4 \ln \frac{\Lambda^2}{M_Z^2} \right]$$

$$- 12 \sum_f \left( \frac{n_f}{3} \right) m_f^4 \ln \frac{\Lambda^2}{m_f^2} + \xi m_H^2 \left( M_W^2 \ln \frac{\Lambda^2}{\xi M_W^2} + \frac{1}{2} M_Z^2 \ln \frac{\Lambda^2}{\xi M_Z^2} \right)$$

$$- m_H^2 \left( 2M_W^2 \ln \frac{\Lambda^2}{M_W^2} + M_Z^2 \ln \frac{\Lambda^2}{M_Z^2} - 2 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \ln \frac{\Lambda^2}{m_f^2} \right) \right]. \hspace{1cm} (13)$$

Adding the above to the tadpole contributions, we then have

$$\Sigma(m_H^2) = - \frac{3g^2\Lambda^2}{32\pi^2 M_W^2} \left[ m_H^2 + 2M_W^2 + M_Z^2 - 4 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \right] \hspace{1cm} (14)$$

$$- \frac{3g^2 m_H^2}{64\pi^2 M_W^2} \left[ m_H^2 \ln \frac{\Lambda^2}{m_H^2} - 2M_W^2 \ln \frac{\Lambda^2}{M_W^2} - M_Z^2 \ln \frac{\Lambda^2}{M_Z^2} + 2 \sum_f \left( \frac{n_f}{3} \right) m_f^2 \ln \frac{\Lambda^2}{m_f^2} \right].$$

Note first that $\Sigma(m_H^2)$ is gauge-independent as it should be, since the Higgs-boson mass in the standard model is a physical observable. Note also that the quadratically divergent
terms in both $\Sigma_R$ and $\Sigma_I$ are proportional to the same factor, which would vanish if Eq. (11) holds. Note finally that the $M_W^4$, $M_Z^4$, and $m_t^4$ logarithmically divergent terms in $\Sigma_R$ and $\Sigma_I$ exactly cancel in $\Sigma(m_H^2)$. Looking at the coefficient of the $\ln \Lambda^2$ term in Eq. (14), we cannot fail to notice that it would be zero if the condition

$$2m_t^2 \approx 2M_W^2 + M_Z^2 - m_H^2$$

is satisfied. Together with Eq. (11), this would imply

$$m_t^2 \approx m_H^2 = \frac{2}{3}M_W^2 + \frac{1}{3}M_Z^2.$$  

(16)

Using the experimental results $M_Z = 91.175 \pm 0.021 \text{ GeV}$[@6] and $M_W = 80.14 \pm 0.27 \text{ GeV}$[@7], we then obtain $m_t \approx m_H \approx 84 \text{ GeV}$. The current experimental lower bounds are 91 GeV[@8] and 59 GeV[@9] respectively. However, one must keep in mind that higher-order corrections have not yet been calculated, hence the above numerical result for $m_t$ could well increase by more than ten percent and be in agreement with data.

Since there have been a number of previous discussions on determining $m_t$ and $m_H$ by considering quadratic and logarithmic divergences in the standard model, it is important to note that whatever procedure one uses, it ought to be well-defined. In other words, one should deal only with physical (and thus gauge-independent) observables, namely masses and couplings. Now there is a problem with any given coupling in quantum field theory because it has to “run” with the energy. Even if one fine-tunes the values of certain parameters to get rid of the logarithmic divergence of a given coupling at a given energy[@4], it will not remain finite at a different energy because the $\beta$ functions of those certain parameters are in general not related in such a way for the next-order correction to vanish as well. Masses, on the other hand, are defined uniquely at their physical values, although quarks are somewhat different because they are permanently confined. In the above, once the conditions for $\Sigma(m_H^2)$ to be finite in one-loop order are found, i.e. Eqs. (11) and (15), the two-loop contributions are just perturbative corrections. The absence of both quadratic and logarithmic divergences in $\Sigma(m_H^2)$ is thus a well-defined and physically meaningful self-consistent constraint in the standard model.

Consider now some possible alternatives. Suppose one demands instead that the self-mass of the electron be finite[@10], then in addition to Eq. (11), one has

$$4m_t^4 \approx 2M_W^4 + M_Z^4 + \frac{1}{2}m_H^4 + 2\sin^2 \theta_W M_Z^2 m_H^2 - \frac{1}{2}m_e^2 m_H^2.$$  

(17)

Since $m_e$ is involved in the above, this would mean that $\mu$ and $\tau$ cannot have finite self-masses, or in other words, the electron must be fundamentally different from $\mu$ and $\tau$, yet there is
certainly no such indication \textit{a priori} in the standard model. Suppose one now considers
the self-mass of the electron neutrino\cite{11}, assuming that it has a right-handed component enabling it to have a Dirac mass. This is of course not a necessary feature of the standard model, but a possible minimal extension of it. One then obtains
\begin{equation}
4m_t^4 \simeq 2M_W^4 + M_Z^4 + \frac{1}{2}m_H^4 + \frac{1}{2}\left(m_e^2 - m_\nu^2\right)m_H^2,
\end{equation}
which again involves $m_e$ (and $m_\nu$) and is thus indicative of its arbitrariness. It is also a curiosity that if one sets the gauge parameter $\xi$ equal to zero in Eq. (10), then the condition for a vanishing logarithmic divergence in the sum of all tadpole contributions is the same as the above without the $m_H^2$ term. It was argued\cite{5} that this could be interpreted as a proper constraint on a putative gauge-invariant definition of the Higgs-boson vacuum expectation value. However, even if this were possible, such a quantity would still not be a physical observable.

Suppose one requires only that Eq. (11) be valid, then the absence of quadratic divergences for all self-masses is guaranteed at the common mass $m_H$. This is technically a well-defined constraint, but it is not very well motivated physically because all self-masses are then still divergent logarithmically. One may also assume that Eq. (11) has validity in the vicinity of $m_H$ so that its variation with mass scale should be set equal to zero\cite{12, 13}. This procedure is of course rather speculative; moreover, it does not allow a solution for $m_t$ and $m_H$ as it stands. If it is argued further that the gluon contribution should be dropped because it has nothing to do with masses, then a solution does exist\cite{12, 13}.

It is interesting to note that if dimensional regularization is used to extract the one-loop contribution of the quadratic divergence, there is a dependence on the space-time dimension $d$ in the residue of the pole at $d = 2$. If both this $d$ and the Dirac trace are set equal to 4, Eq. (11) is obtained; but if both are set equal to 2 as this procedure seems to require\cite{4, 14}, then
\begin{equation}
6m_t^2 \simeq 2M_W^2 + M_Z^2 + 3m_H^2
\end{equation}
is obtained instead. It has become a small controversy as to which is the right thing to do. As long as no physical significance is attached to the quadratic divergence, this question is really moot. However, a remarkable thing happens when Eq. (19) is used together with Eq. (15): the solution is again Eq. (16). In other words, \textit{the hypothesis of a finite $m_H$ renormalization is independent of regularization scheme}, hence Eqs. (11) and (19) are actually compatible with each other. This lends further support to Eq. (16) as a physically meaningful result.

In conclusion, if there are hints within the standard model for the existence of mass
relationships, the most physically meaningful quantity to consider is the Higgs-boson mass $m_H$. It has been demonstrated in the above that the one-loop renormalization of $m_H$ is gauge-independent and could even be rendered finite if the following mass relationships were satisfied: $m_t^2 \simeq m_H^2 = (2M_W^2 + M_Z^2)/3$. Numerically, this would imply $m_t \simeq m_H \simeq 84$ GeV. However, because of higher-order corrections to Eqs. (11) and (15) which have not yet been calculated, it cannot be established at this time that the above hypothesis is definitely inconsistent with the current experimental lower limit of 91 GeV on $m_t$. Clearly, the two-loop contributions to $m_H$ should be calculated but the work is by no means trivial and will take time.

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FIGURE CAPTIONS

Fig. 1. One-loop 1PR (tadpole) contributions to $m_H$. All massive particles are involved: $f$ refers to all the quarks and leptons; the $W$ and $Z$ contributions include their unphysical and ghost partners in the $R_\xi$ gauge.

Fig. 2. One-loop 1PI contributions to $m_H$. Labels are as in Fig. 1.