Berry phases for interacting spins in composite environments

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Abstract

Due to the potential application in quantum information process, geometric phase of interacting system arouse many interests. Some physicists concentrate on the system in pure classical environment, while others study the system in pure quantized environment. So a natural question is asked: how about an interacting system in composite environments made up of both classical and quantized field. In this letter, we analyze a quantum system composed of two interacting spins, of which one is in classical magnetic field and the other is in quantized field. First, classical magnetic field driven Berry phases for the whole system and subsystem are studied. The effect of couplings between particles and photon on these phases are analyzed. In comparison with the dynamical quantized field, We find that even a static quantized field in its vacuum state can also have an effect on Berry phase. Second, quantized field driven Berry phases for the whole system and subsystem are formulated, including both one and two mode of this field. The vacuum induced effects are elaborated, moreover compared with the constant vacuum induced phases in former papers, the counterpart in this letter varies according to classical magnetic field, couplings and other parameters. For the two mode quantized field, the rigorous relationship between the concurrence and Berry phase for subsystem are built up.

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I. INTRODUCTION

Berry phase had been discovered by Berry [2] in the context of adiabatic, unitary, cyclic evolution of time-dependent quantum system. He demonstrated that besides the usual dynamical phase, there exists an additional phase relating to the evolution of the state. Soon a geometrical interpretation of Berry phase was elaborated by Simon [15]. Berry’s result was extended to the nonadiabatic and cyclic case by Aharonov and Anandan [25]. In their definition, the dynamical phase was identified as a loop integral over the expectation value of the Hamiltonian. The Aharonov and Anandan phase (A-A phase) could be obtained by the difference between the total phase and the dynamical one. Likewise, Anandan [1] generalized the above one to the degenerate case. Depending on the Pancharatnam’s earlier work [13], Samuel and Bhandari [14] found a more general phase in the context of non-cyclic and non-unitary evolution of quantum mechanics. Subsequently, Munkunda and Simon established a quantum kinematic approach to geometric phases [12], which is the most general theory on geometric phases for pure quantum states.

Nevertheless, the above definitions of geometric phases can’t be applicable when the initial and final states are orthogonal. Manini and Pistolesi [10] first proposed the Abelian off-diagonal geometric phases to overcome the evident drawback of ordinary Berry phase during adiabatic evolution. One year later, the above definition was generalized to nonadiabatic cases by Mukunda et. al.[11]. Afterwards, Kult et al. [7] made a step forward in this direction by extending the concepts to non-Abelian cases.

The above definitions are all confined to pure states, what about the geometric phase for mixed state? This problem was first presented by Uhlmann [22] in mathematical context of purification. But, his definition depends on the chosen environment. Later, Sjöqvist et al. [17] redefine the non-degenerate mixed geometric phase in non-cyclic and unitary evolution under the background of quantum interference, which is independent of surroundings. Extensions of mixed-state geometric phases to the degenerate case [16] and the kinematic approach [20] had also been achieved. Likewise, the definition of mixed geometric phase was also extended to off-diagonal case by Filipp and Sjöqvist.

Besides its theoretical significance, Berry phase has many applications ranging from condensed matter physics [24] to quantum information and computation science [3, 4, 6, 28]. Most of the implementation of quantum information by geometric phase depends on com-
posite system. So this kind of systems is of concern. Sjöqvist [18] studied geometric phase for a pair of entangled spins in a time-independent uniform magnetic field, which was generalized by Tong, Kwek and Oh [21] to a rotating magnetic field. Yi, Wang and Zheng [27] investigated Berry phase two two coupled spin half system, one of which is driven by a slowly varying magnetic field. Sjöqvist et. al. [19] analyzed Berry phase for ground state of finite-size Lipkin-Meskov-Glick model including three spin half particles, which was extended to A-A geometric phase by Yang et. al. [26].

However, all the above researched system were in classical field. Recently, some scientists began to study spins in quantized field. Fuentes-Guridi et. al. [5] studied the Berry phase for spin half particle interacting with a quantized field and analyzed the vacuum induced effect, which was generalized by Liu et. al. [9] to $m$ photons process. Wang, Cui and Yi [23] considered about interacting spins with one driven by a quantized mode of field, while the counterpart of both of the two particles driven by the field was analyzed by Liang, Zhang and Yuan [8]. So a natural question is asked: how about an interacting system in composite environments made up of both classical and quantized field. In this letter, we analyze a quantum system composed of two interacting spins, of which one is in classical magnetic field and the other is in quantized field.

This paper is organised as follows. In the next section, Berry phases with classical magnetic field driving are discussed. The eigenstates of the whole system are worked out, moreover they are represented by triangular function of some introduced parameters, which made them elegant and easy to understand. Not only the Berry phase for the whole system but also the Berry phase for subsystem are formulated. Some special cases at utmost limit are discussed and the concurrence is used to explain the above phenomena. In Sec. III, the Berry phase with quantized field driving is calculated, including for both the whole system and subsystem. The vacuum induced Berry phase are placed emphasis on. The relationship between the concurrence and Berry phase for subsystem are built up. Furthermore, some special cases are elaborated. A conclusion is drawn in the last section.

II. BERRY PHASE WITH MAGNETIC FIELD DRIVING

Considering a system consisting of two interacting spin-1/2 particles in the presence of composite fields and supposing that particle 1 interacts with a single quantized mode of
an optic field in the rotating wave approximation and particle 2 is subject to a classical magnetic field, the Hamiltonian takes the form

\[
H = \frac{\omega_1}{2} \sigma^z_1 + \nu a^\dagger a + \lambda(\sigma^+_1 a + \sigma^-_1 a^\dagger) + J \sigma^z_1 \sigma^z_2 + \frac{1}{2} \mu B \cdot \vec{\sigma},
\]

(1)

where \(\omega_1\) is the transition frequency between the eigenstates of particle 1, \(\nu\) is the frequency of the field described in terms of the creation and annihilation operators \(a^\dagger\) and \(a\), \(\lambda\) is the coupling constant between the two particles, \(\mu\) represents the gyromagnetic ratio, \(\vec{B} = B \vec{n}\) stands for the magnetic field, \(\sigma_k^+ = (\sigma_k^x, \sigma_k^y, \sigma_k^z)\), \(\sigma_k^+ = (1/2)(\sigma_k^x + i\sigma_k^y)\) and \(\sigma_k^- = (1/2)(\sigma_k^x - i\sigma_k^y)\) are Pauli operators, the subscript denotes the particle.

In the invariant space spanned by \(\{|e_1e_2\rangle, |e_1g_2\rangle, |g_1e_2\rangle + 1\rangle, |g_1g_2\rangle + 1\rangle\},\) the Hamiltonian (1) can be expressed in a matrix form,

\[
\begin{pmatrix}
J + n \nu + \frac{\omega_1}{2} & \frac{1}{2} e^{-i\varphi} \omega_2 & \sqrt{n + 1} \lambda & 0 \\
\frac{1}{2} e^{i\varphi} \omega_2 & -J + n \nu + \frac{\omega_1}{2} & 0 & \sqrt{n + 1} \lambda \\
\sqrt{n + 1} \lambda & 0 & -J + (n + 1) \nu - \frac{\omega_1}{2} & \frac{1}{2} e^{-i\varphi} \omega_2 \\
0 & \sqrt{n + 1} \lambda & \frac{1}{2} e^{i\varphi} \omega_2 & J + (n + 1) \nu - \frac{\omega_1}{2}
\end{pmatrix},
\]

where \(\vec{n} = (\cos \varphi, \sin \varphi, 0)\) and \(\omega_2 = \mu B\). And the four nondegenerate eigenvectors are

\[
|\psi_j\rangle = (e^{-i\varphi} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} e^{-i\varphi} \sin \frac{\varphi}{2} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2})^T,
\]

(2)

where

\[
\begin{align*}
\cos \frac{\varphi}{2} \sin \frac{\varphi}{2} &= \frac{1}{\sqrt{N}} \{\omega_1[2J[\sqrt{F + 2A} + (-1)^j 2(J - \nu)] + (-1)^j (\omega^2_2 \mp A)]} \\
&\quad + (A \pm 2J \nu)[\mp \sqrt{F - 2A} + (-1)^j (\pm \nu \mp 2J)] + (-1)^j (2J \omega^2_2 - \nu \omega^2_2)\}
\end{align*}
\]

\[
\begin{align*}
\cos \frac{\varphi}{2} \cos \frac{\varphi}{2} &= \frac{1}{\sqrt{N}} \{\omega_1[\sqrt{F + 2A} + (-1)^{1+j} 2\nu] - \nu[\sqrt{F + 2A} + (-1)^{1+j} \nu] \\
&\quad + (-1)^j (4\lambda^2_n + \omega^2_1 \mp A)\}
\end{align*}
\]

\[
\begin{align*}
\sin \frac{\varphi}{2} \sin \frac{\varphi}{2} &= \frac{1}{\sqrt{N}} \{(-1)^j 2\lambda_n (2J \omega_1 + \omega^2_2 \mp A - 2J \nu)\}
\end{align*}
\]

\[
\begin{align*}
\sin \frac{\varphi}{2} \cos \frac{\varphi}{2} &= \frac{1}{\sqrt{N}} \{2\omega_2 \lambda_n \sqrt{F + 2A} + (-1)^j 2A\}
\end{align*}
\]

the normalized coefficients are

\[
N_j = \{\omega_1[2J[\sqrt{F + 2A} + 2(-1)^j (J - \nu)] + (-1)^j (\mp A + \omega^2_2)]} \\
\quad + (A \pm 2J \nu)[\mp \sqrt{F + 2A} + (-1)^j (\nu - 2J)] + (-1)^j (2J \omega^2_2 - \nu \omega^2_2)^2 \\
\quad + 4\omega^2_2 \lambda^2_n [\sqrt{F + 2A} + 2(-1)^j \nu] + \omega^2_2 (-1)^j \nu \sqrt{F + 2A} + (-1)^j \omega_1 \sqrt{F + 2A} \\
\quad \mp A + (\nu - \omega_1)^2 + 4\lambda^2_n + 4\lambda^2_n [\pm A \pm 2J (\nu - \omega_1) \mp \omega^2_2]^2
\]

5
\( j = 1, 2, 3, 4 \) (when \( j = 1, 2 \), the above sign is sensible; other cases, the bellow sign is sensible),

\[ A = \sqrt{4J^2 (\nu - \omega_1)^2 + \omega_2^2[\nu^2 + \omega_1 (\omega_1 - 2\nu) + 4\lambda^2(n + 1)]}, \quad F = 4J^2 + \nu^2 + \omega_1 (\omega_1 - 2\nu) + 4\lambda^2(n + 1) + \omega_2^2, \]

and \( \lambda_n = \lambda \sqrt{n + 1} \).

When \( \varphi \) is slowly changed from 0 to \( 2\pi \), the system undergoes an adiabatic and cyclic evolution. Then, the corresponding Berry phase can be calculated as the following formula,

\[ \gamma_j = \frac{i}{2} \int_0^{2\pi} d\varphi \langle \psi_j | \frac{d}{d\varphi} | \psi_j \rangle. \]  

(3)

Substituting the instantaneous eigenvector \( (2) \) into the above Eq. \( (3) \), one can obtain Berry phase,

\[ \frac{1}{2}[\sin^2(\chi_j) \Omega(\eta_j) + \cos^2(\chi_j) \Omega(\xi_j)], \]  

(4)

where

\[ \Omega(x) = 2\pi(1 - \cos x). \]  

(5)

The above Eq. \((5)\) hints that \( \Omega(\eta_j) \) and \( \Omega(\xi_j) \) can be regarded as the solid angle of a sphere of fix latitudes \( \eta_j \) and \( \xi_j \) respectively. So the geometric phase \((4)\) is a linear combination of solid angles. And \( \sin \frac{\chi_j}{2} \) and \( \cos \frac{\chi_j}{2} \) play the roles of weights. References \([5, 8, 9, 23]\) had investigated that when \( n = 0 \), the time-dependent quantized field induced a corresponding Berry phase. However, in this paper, we get another conclusion that even if the quantized field is static, when \( n = 0 \), the vacuum quantum field had an effect on Berry phase by the coupling constant \( \lambda \) or the frequency \( \nu \) of the quantized field. Moreover, if either \( \lambda \) or \( \nu \) is also zero, the corresponding Berry phase is also affected by \( \nu \) or \( \lambda \). The above discussion focus on the affection of quantized filed. Now, let’s turn our interests into the impaction of classical magnetic field. When \( \vec{B} = 0 \), i.e. \( \omega_2 = 0 \), the Berry phase \((4)\) disappears as we expect. Because when \( \vec{B} = 0 \), the instantaneous eigenstates become stationary. Therefore, no geometric phase can be generated. Next, let’s concentrate on some special cases. Under the condition that the coupling constant \( \lambda \rightarrow \infty \) or \( n \rightarrow \infty \), the geometric phase

\[ \gamma_j = \pi. \]  

(6)

In order to explain the above result, the concurrence of the instantaneous eigenstates is calculated as follows

\[ c_j = \sin(\chi_j) \sin \frac{\xi_j - \eta_j}{2}. \]  

(7)
It can be verified that under the condition that $\lambda \to \infty$ or $n \to \infty$, the related concurrence becomes zero. Hence, there is no relationship between particle 1 and particle 2. And the state of particle 1 is stationary. So the whole geometric phase is only generated by particle 2 precessing in the classical magnetic field. On the assumption of $J \to \infty$, the geometric phase

$$\gamma_j = 0.$$  

It may be explained as that when $J \to \infty$, the interaction between spins in Hamiltonian (1) becomes dominate part, hence the effect of classical magnetic field is negligible. Therefore, the Berry phase generated by the precession of the field is null. When $J = 0$, the concurrence vanishes. As the same circumstance above that $\lambda \to \infty$ or $n \to \infty$, the corresponding Berry phase is $\pi$.

Next, let us research mixed geometric phases for subsystems, which is proposed in [18]

$$\Gamma = \arg \sum_i p_i e^{i\beta_i},$$  

where $p_i$ and $\beta_i$ are the reduced density matrix’s eigenvalues and Berry phases generated by the eigenvectors respectively. The reduced density matrix of particle 2 is

$$\begin{pmatrix} \cos^2(\frac{\chi_j}{2})\sin^2(\frac{\xi_j}{2}) + \sin^2(\chi_j/2) \sin^2(\xi_j/2) & \frac{1}{2} e^{-i\varphi}[\sin(\xi_j) \cos^2(\chi_j/2) + \sin(\eta_j) \sin^2(\chi_j/2)] \\ \frac{1}{2} e^{i\varphi}[\sin(\xi_j) \cos^2(\chi_j/2) + \sin(\eta_j) \sin^2(\chi_j/2)] & \cos^2(\frac{\xi_j}{2})\cos^2(\frac{\chi_j}{2}) + \cos^2(\frac{\eta_j}{2}) \sin^2(\chi_j/2) \end{pmatrix}.$$  

Hence, by use of Eq. (8), we obtain the corresponding mixed state geometric phase for particle 2, which takes the form

$$\Gamma_j = -\tan^{-1}\left(\frac{1}{2} \sqrt{2 \sin^2(\chi_j) \cos(\eta_j - \xi_j) + \cos(2\chi_j)} + 3 \tan \frac{2\pi[\cos(\eta_j) \sin^2(\chi_j/2) + \cos(\xi_j) \cos^2(\chi_j/2)]}{\sqrt{2 \sin^2(\chi_j) \cos(\eta_j - \xi_j) + \cos(2\chi_j) + 3}}\right).$$  

As the geometric phase for the whole system, when the coupling constant $J = 0$, the geometric phase for particle 2 reduces to be

$$\Gamma_j = \pi$$  

as well.
III. BERRY PHASE WITH QUANTIZED FIELD DRIVING

In this section the phase shift operation \( U(\phi) = \exp(-i\phi a^\dagger a) \) is introduced to apply adiabatically to the Hamiltonian of the system (1). When \( \phi \) is slowly changed from 0 to \( 2\pi \), the corresponding Berry phase can be obtained as follows:

\[
\gamma_j^q = i \int_0^{2\pi} d\phi \langle \psi_j | U^\dagger(\phi) \frac{d}{d\phi} U(\phi) | \psi_j \rangle.
\]

(9)

Substituting the instantaneous state (2) into the above expression (9), we obtain

\[
\gamma_j^q = \pi (1 - \cos \chi_j) + 2\pi n.
\]

(10)

The static classical magnetic field \( \vec{B} \), the interaction between the two particles and quantum field all have impacts on the geometric phase of the system. Even when \( n = 0 \), the vacuum quantum field still have an effect on \( \gamma_j^q \) through \( \cos \chi_j \). However, when the classical magnetic field \( \vec{B} = 0 \), it gives no effect on the geometric phase. This may be the prominent feature of effect on geometric phase between the classical field and quantum field.

Moreover, in order to disclose the vacuum induced effect explicitly, the second mode of the field will introduced in this system [5, 23], whose creation and annihilation operators are labeled by \( b^\dagger \) and \( b \) respectively. The Hamiltonian of the whole system takes the form

\[
H_0^{2q} = \frac{\omega_1}{2}\sigma_1^z + \nu a^\dagger a + \nu b^\dagger b + \lambda (\sigma_1^+ a + \sigma_1^- a^\dagger) + J\sigma_1^z \sigma_2^z + \frac{1}{2} \mu \vec{B} \cdot \vec{\sigma}_2,
\]

(11)

which implies that the second mode of light has no interaction with the two spin 1/2 system or the first mode of light at the initial time. Hence, the eigenstates of the above Hamiltonian (11) read

\[
|\psi_j^{2q}\rangle = |\psi_j\rangle \otimes |n'\rangle,
\]

where \( |\psi_j\rangle \) is the eigenstates of the Hamiltonian (1) in the previous section. With the help of \( b^\dagger \) and \( b \) of the second mode field, the following transformed Hamiltonian is considered about

\[
U(\theta, \phi)H_0^{2q}U^\dagger(\theta, \phi),
\]

(12)

where \( U(\theta, \phi) = \exp(-\phi J_z) \exp(-\theta J_y) \) whose generators are \( J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \) and \( J_y = \frac{1}{2i}(a^\dagger b - ab^\dagger) \). Therefore, its eigenstates are

\[
U(\theta, \phi)|\psi_j^{2q}\rangle.
\]

(13)
If the parameters $\theta$ and $\phi$ are slowly changed so that the above eigenstates undertake adiabatic progress, there exist Berry phases

$$\gamma_{2q}^j = \oint \langle \psi_j^{2q} | U^{\dagger}(\theta, \phi) dU(\theta, \phi) | \psi_j^{2q} \rangle,$$  \hfill (14)

where $d$ denote exterior derivative. Substituting Eq. (13) into Eq. (14), one can obtain the corresponding Berry phases, which take the forms

$$\gamma_{2q}^j = -\frac{1}{2} \Omega \left[(n - n') + \sin^2 \frac{\chi_j}{2}\right],$$  \hfill (15)

where $\Omega = \int \sin \theta d\theta \wedge d\phi$ is the solid angle subtended by the closed loop in the Poincaré's sphere. From above equation, we can draw a conclusion that even though the field is in vacuum state, there is also an induced Berry phase, which is

$$\left(\gamma_{2q}^j\right)_{\text{zero}} = -\frac{1}{2} \Omega \left(\sin^2 \frac{\chi_j}{2}\right)_{n=0}.$$  \hfill (16)

Compared with the result obtained by Ref. [5] that the vacuum induced Berry phase is a definite value, our outcomes (16) varies according to $\sin^2 \frac{\chi_j}{2}$, whose explicit form is

$$\sin^2 \frac{\chi_j}{2} = \frac{4 \lambda^2}{N_j} \left\{ \left(2J\omega_1 + \omega_2^2 + A - 2J\nu\right)^2 + \omega_2^2 \sqrt{F + 2A + (-1)^j2J}\right\}.$$  

We can see that even when $n = n' = 0$, the quantum field $\nu$, the classical field $B$ and the coupling effect $\lambda$ can induce Berry phases. From the first mode and the second mode field, a comparison is shown that because the second mode field has no interaction with the particles, when $n' = 0$, it has no effect on Berry phase.

The above paragraphs discuss about Berry phase for the whole system. In the present section, let us concern on the Berry phase of subsystem. Using the same definition above (8), the Berry phase for subsystem composed of particle 1, the first mode quantum field and the second mode one takes the form

$$\Gamma_{2q}^j = -\frac{1}{2} \Omega \left[n - n' + \frac{1}{2}\right] + \tan^{-1} \left\{ \frac{\Omega \cos(\chi_j)}{4 \sqrt{\sin^2(\chi_j) \cos^2 \left(\frac{\xi_j - \eta_j}{2}\right) + \cos^2(\chi_j) \tan \left(\frac{\Omega \cos(\chi_j)}{4 \sqrt{\sin^2(\chi_j) \cos^2 \left(\frac{\xi_j - \eta_j}{2}\right) + \cos^2(\chi_j) \tan}\right)} \right\}.$$  \hfill (17)

By substituting Eq. (7) into above Eq. (17), we can obtain the relationship between Berry phase of subsystem and concurrence, which is

$$\Gamma_{2q}^j = -\frac{1}{2} \Omega \left[n - n' + \frac{1}{2}\right] + \tan^{-1} \left\{ \sqrt{1 - c_j^2} \tan \left(\frac{\Omega (1 - 2 \sin^2 \frac{\chi_j}{2})}{4 \sqrt{1 - c_j^2}}\right) \right\}. \hfill (18)$$
When the corresponding concurrence $c_j$ is zero (For example, when $J = 0$, $c_j = 0$.), the Berry phase takes the form

$$\left(\Gamma_j^{q_j}\right)_{c_j=0} = -\frac{1}{2} \Omega \left[n - n' + \sin^2 \frac{\chi_j}{2}\right],$$

which is very similar to the form (15).

IV. CONCLUSIONS AND ACKNOWLEDGEMENTS

This letter concentrates on a quantum system composed of two interacting spins, of which one is in classical magnetic field and the other is in quantized field. First, classical magnetic field driven Berry phases for the whole system and subsystem are studied. The effect of couplings between particles and photon on these phases are analyzed. In comparison with the dynamical quantized field, we find that even a static quantized field in its vacuum state can also have an effect on Berry phase. Second, quantized field driven Berry phases for the whole system and subsystem are formulated, including both one and two mode of this field. The vacuum induced effects are elaborated, moreover compared with the constant vacuum induced phases in former papers, the counterpart in this letter varies according to classical magnetic field, couplings and other parameters. For the two mode quantized field, the rigorous relationship between the concurrence and Berry phase for subsystem are built up.

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