Arnold stable flow with conjugate point on 2D Riemannian manifold

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Abstract

Let $M$ be a compact 2-dimensional Riemannian manifold with smooth boundary and consider the incompressible Euler equation on $M$. In the case that $M$ is the straight periodic channel, the annulus or the disc with the Euclidean metric, it was proved by T.D. Drivas, G. Misiołek, B. Shi, and the second author that all Arnold stable solutions have no conjugate point on the volume-preserving diffeomorphism group $D^s_\mu(M)$. They also proposed a question which asks whether this is true or not for any $M$. In this article, we give a negative answer. More precisely, we construct an Arnold stable solution, which has a conjugate point, on an ellipsoid with the the top and bottom cut off.

Keywords: Euler equation, Arnold stable flow, diffeomorphism group, conjugate point.

MSC2020; Primary 35Q35; Secondary 35Q31.

1 Introduction

Let $(M,g)$ be a compact 2-dimensional Riemannian manifold with smooth boundary $\partial M$ and consider the incompressible Euler equation on $M$:

$$\begin{align*}
\frac{\partial u}{\partial t} + \nabla u &= -\text{grad } p \quad \text{on } M, \\
\text{div } u &= 0 \quad \text{on } M, \\
g(u,\nu) &= 0 \quad \text{on } \partial M,
\end{align*}$$

(1.1)

where $\nu$ is a unit normal vector field on $\partial M$. For the case that $M$ is the straight periodic channel, the annulus or the disc with the Euclidean metric, it was proved by T.D. Drivas, G. Misiołek, B. Shi, and the second author [4, Thm. 3] that all Arnold stable solutions (see Definition 2.5) has no conjugate point on the volume-preserving diffeomorphism group $D^s_\mu(M)$ on $M$. They also proposed a

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question [4] which asks whether this is true or not for any $M$. In this article, we give a negative answer of this question. For the precise statement, define 2-dimensional Riemannian manifolds by

$$M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\},$$

$$M^b_a := \{(x, y, z) \in M_a \mid -b \leq z \leq b\}$$

with the usual metric, where $a \leq 1$ and $0 < b < 1$. Note that $M_a$ is equal to $S^2$ if $a = 1$. Our main theorem of this article is the following.

**Theorem 1.1.** Let $a > 1$ and $0 < b < 1$. Then, there exists an Arnold stable solution of the incompressible Euler equation (1.1) on $M^b_a$ with a conjugate point on $\mathcal{D}_\mu(M)$.

By V. I. Arnold [1], geodesics on $\mathcal{D}_\mu(M)$ correspond to solutions of (1.1). Thus, the existence of a conjugate point is related to a Lagrangian stability of a corresponding solution.

This article is organized as follows. In Section 2, we recall the definition and properties of Arnold stability. In Section 3, we make some calculation related to Arnold stable solutions on $M_a$. In Section 4, we calculate the Misiolek curvature on $M^b_a$. Finally, we construct an Arnold stable flow on $M^b_a$ in Section 5.

## Acknowledgments

The authors are very grateful to G. Misiolek and T. D. Drivas for fruitful discussions. The research of TT was partially supported by Grant-in-Aid for JSPS Fellows (20J00101), Japan Society for the Promotion of Science (JSPS). The research of TY was partially supported by Grant-in-Aid for Scientific Research B (17H02860, 18H01136, 18H01135 and 20H01819), Japan Society for the Promotion of Science (JSPS).

### 2 Arnold stable flow

In this section, we recall that the definition of an Arnold stable flow and its basic property. Although almost all the materials in this section are well known, we prove some results for the convenience. Main references are [2 Sect. II.4.A], [3] and [4] Sect. 5.

Let $(M, g)$ be a compact 2-dimensional Riemannian manifold with smooth boundary $\partial M$ and consider the incompressible Euler equation (1.1) on $M$.

**Definition 2.1.** Let $u$ be a divergence-free vector field on $M$, which tangent to $\partial M$. A function $\psi$ on $M$ is called a stream function of $u$ if $\psi$ satisfies

$$\star \text{grad} \psi = u,$$

where $\star$ is the Hodge star. We write

$$\Delta := \text{div} \circ \text{grad}$$
Lemma 2.2. If \( u \) is a stationary solution of (1.1), then \( u = \star \text{grad} \psi \) and \( \text{grad} \omega \) are orthogonal. In particular, \( \text{grad} \psi \) and \( \text{grad} \omega \) are collinear.

Proof. Because \( u \) is a time independent solution of (1.1), we have

\[
\nabla_u u = -\text{grad} p.
\]

Applying the operator \( \text{div} \circ \star \) to this equation, we have the lemma.

Lemma 2.3. Let \( u \) be a stationary solution of (1.1), \( \psi \) its stream function, and \( \omega = \Delta \psi \). Then, at least locally, there exists a function \( F : \mathbb{R} \to \mathbb{R} \) satisfying

\[
\omega(x) = F(\psi(x)) \quad \text{for} \ x \in M.
\]

Proof. By Lemma 2.2, \( \psi \) and \( \omega \) have the same level set. Thus, there exists a function \( f : \mathbb{R} \to \mathbb{R} \) satisfying (locally on \( M \))

\[
\text{grad} \omega = f \text{grad} \psi.
\]

Take a primitive function \( F \) of \( f \) and define a function \( \omega_0 := F(\psi(x)) \). By the chain rule, we have

\[
\text{grad} \omega_0 = F'(\psi) \text{grad} \psi = f \text{grad} \psi = \text{grad} \omega.
\] (2.2)

Because the difference of functions which have the same gradient must be a constant function, this completes the proof.

Corollary 2.4. Let \( u \) be a stationary solution of (1.1), \( \psi \) its stream function and \( \omega = \Delta \psi \). Then, the function \( F \) in Lemma 2.3 satisfies

\[
F'(\psi) = \frac{\text{grad} \omega}{\text{grad} \psi} = \frac{\text{grad} \Delta \psi}{\text{grad} \psi}.
\] (2.3)

Proof. This is a consequence of (2.2). Note that by the collinearity of \( \text{grad} \omega \) and \( \text{grad} \psi \) (see Lemma 2.2), the fraction of (2.3) makes sense.

Write \( \lambda_1 > 0 \) for the first eigenvalue of \(-\Delta\).

Definition 2.5. We say that a stationary solution \( u \) of (1.1) is Arnol’d stable if the corresponding function \( F \) in Lemma 2.3 satisfies

\[
-\lambda_1 < F'(\psi) < 0, \quad \text{or} \quad 0 < F'(\psi) < \infty.
\] (2.4)

Lemma 2.6 ([3 Prop. 1.1]). Let \( X \) be a Killing vector field on \( M \), which tangents to \( \partial M \) and \( u = \star \text{grad} \psi \) an Arnol’d stable stationary solution of (1.1) with stream function \( \psi \). Then we have \( X \psi = 0 \).
Proof. Note that \( \Delta X = X \Delta \) because \( X \) is Killing. By the definition, we have
\[
\Delta \psi = F(\psi).
\]
The chain rule and \( X \Delta = \Delta X \) imply
\[
(\Delta - F'(\psi))X \psi = 0.
\]
Thus (2.4) implies the lemma.

3 \( M = M_a \) case

In this section, we apply the contents to the case that \( M \) is an ellipsoid.

Recall that we define a 2-dimensional Riemannian manifold \( M := M_a \) by
\[
M_a := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2(1 - z^2)\}
\]
for \( a \leq 1 \). Take a “spherical coordinate” of \( M_a \):
\[
\phi := \phi_a : (-d, d) \times (-\pi, \pi) \rightarrow M_a \quad \mapsto \quad (c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r))
\]
in such a way that \( c_2(0) = 0, c_1(r) > 0, \dot{c}_2(r) > 0 \), and that \( \dot{c}_1^2 + \dot{c}_2^2 = 1 \) (see [E] Sect. 4). Note that \( (c_1, c_2, d) = (\cos(r), \sin(r), \pi/2) \) in the case of \( a = 1 \) \((M_1 = S^2)\). Then, we obtain
\[
g(\partial_r, \partial_r) = 1, \quad g(\partial_r, \partial_\theta) = 0, \quad g(\partial_\theta, \partial_\theta) = c_1^2
\]
and
\[
\mu = c_1(r) dr \wedge d\theta.
\]
This implies
\[
\star \partial_r = \frac{-\partial_\theta}{c_1}, \quad \star dr = -c_1 d\theta, \quad \star \partial_\theta = c_1 \partial_r, \quad \star d\theta = \frac{dr}{c_1}. \tag{3.1}
\]
For a function \( f \) on \( M \) and \( u = u_1 \partial_r + u_2 \partial_\theta \), we have
\[
\begin{align*}
\text{grad } f &= \partial_r f \partial_r + c_1^{-2} \partial_\theta f \partial_\theta, \quad \text{grad } u = (\partial_r + c_1^{-1} \partial_r c_1) u_1 + \partial_\theta u_2, \tag{3.2} \\
\text{div } u &= (\partial_r + c_1^{-1} \partial_r c_1) u_1 + \partial_\theta u_2, \tag{3.3} \\
\Delta f &= \text{div } \circ \text{grad } f = (\partial_r^2 + \frac{\partial_\theta^2}{c_1^2} + \frac{c_1 \partial_r}{c_1}) f, \tag{3.4}
\end{align*}
\]
where \( \dot{c}_1 = \partial_r c_1 \). Hereafter, the dot always means the derivative by the variable \( r \). Set
\[
p_{\pm} := (0, 0, \pm 1) \quad \in M_a.
\]

Lemma 3.1. Let \( \psi \) be a stream function of an Arnol’d stable stationary solution \( u \) of (1.1). Then, \( \psi \) depends only on the variable \( r \), namely \( \psi = \psi(r) \).
Proof. Because $\partial \theta$ is Killing, this is a corollary of Lemma 2.6.

**Lemma 3.2.** Let $\psi = \psi(r)$ be a stream function of a solution $u$ of (1.1). Then, we have

$$F'(\psi) = \frac{\ddot{\psi}}{\psi} + \frac{\dot{\psi}}{c_1} \frac{\dot{\psi}}{c_1} + \frac{\dot{\psi}}{c_1} \frac{\dot{c}_1}{c_1} - \frac{(\dot{c}_1)^2}{c_1^2}$$.

Proof. By (3.2), (3.4) and $\omega = \Delta \psi$, we have

$$\text{grad } \psi = \dot{\psi} \partial_r, \quad \omega = \ddot{\psi} + \frac{\dot{\psi}}{c_1} \frac{\dot{\psi}}{c_1}, \quad \text{grad } \omega = \left( \ddot{\psi} + \frac{\dot{\psi}}{c_1} \frac{\dot{\psi}}{c_1} \right) \partial_r.$$

This completes the proof.

**Lemma 3.3.** Let $\psi = \psi(r)$ be a stream function of a solution $u$ of (1.1). Define a function $f = f(r)$ by $\psi = \frac{1}{c_1}$. Then, we have

$$F'(\psi) = \frac{Df}{f},$$

where $D$ is a differential operator defined by

$$D := \partial_r^2 - \frac{\dot{c}_1}{c_1} \partial_r.$$

Proof. By direct computations, we have

$$\ddot{\psi} = \frac{\ddot{f}}{c_1} - \frac{\dot{f}}{c_1} \frac{\dot{c}_1}{c_1}$$,

$$\dddot{\psi} = \frac{\dddot{f}}{c_1} - \frac{\ddot{f}}{c_1} \frac{\dot{c}_1}{c_1} - \frac{\dot{f}}{c_1} \frac{\dot{c}_1}{c_1} \frac{\ddot{c}_1}{c_1} + \frac{2f(\dot{c}_1)^2}{c_1^2},$$

$$\frac{\dddot{\psi}}{c_1} + \frac{\dot{c}_1}{c_1} \frac{\ddot{\psi}}{c_1} = \frac{\dddot{f}}{c_1} - \frac{\ddot{f}}{c_1} \frac{\dot{c}_1}{c_1} - \frac{\dot{f}}{c_1} \frac{\dot{c}_1}{c_1} \frac{\ddot{c}_1}{c_1} + \frac{f(\dot{c}_1)^2}{c_1^3} - \frac{f(\dot{c}_1)^2}{c_1^3}.$$

By Lemma 3.2 we have

$$F'(\psi) = \frac{\dddot{f}}{f} \frac{\dddot{f}}{f} \frac{\dot{c}_1}{c_1} + \frac{(\dot{c}_1)^2}{c_1^2} + \frac{\dot{c}_1}{c_1} \frac{\dot{c}_1}{c_1} - \frac{(\dot{c}_1)^2}{c_1^2}$$

$$= \frac{\dddot{f}}{f} \frac{\dddot{f}}{f} \frac{\dot{c}_1}{c_1}.$$

This completes the proof by (3.3).
Lemma 3.4 (cf. [8 Prop. 5.1]). We have

\[
\frac{\dot{c}_1}{c_1} = \frac{-a^2c_2}{c_1 \sqrt{c_1^4 + a^4c_2^2}}.
\]

Proof. Let \( E_a := \{(x, z) \in \mathbb{R}^2 \mid x^2 = a^2(1 - z^2)\} \) be an ellipse. We note that the gradient of the function \( x^2 - a^2(1 - z^2) \) on \( \mathbb{R}^2 \) is equal to \( 2x\partial_x + 2a^2z\partial_z \). Therefore \( x\partial_x + a^2z\partial_z \) is a normal vector field of \( E_a \).Thus \(-a^2z\partial_x + x\partial_z \) is tangent to \( E_a \). Note that \( M_a \) is a surface of revolution generated by \( E_a \). Thus, the assumption \( \dot{c}_1^2 + \dot{c}_2^2 = 1 \) implies

\[
(\dot{c}_1, \dot{c}_2) = \frac{1}{\sqrt{c_1^4 + a^4c_2^2}}(-a^2c_2, c_1).
\]

This completes the proof. \( \square \)

Lemma 3.5. Let \( f = f(r) \) be a function such that \( \dot{f} \) is a strictly monotonically increasing odd function. Then, we have

\[
Df > 0.
\]

Proof. Note that \( c_2(r) = z \) is an odd function and \( c_1 > 0 \) on \( M_a \setminus \{p_\pm\} \). Moreover, \( \dot{f} > 0 \) by the assumption of \( f \). Therefore,

\[
Df = \dot{f} + \frac{a^2c_2}{c_1 \sqrt{c_1^4 + a^4c_2^2}} \dot{f}
\]

is positive. \( \square \)

## 4 Misiolek curvature on \( M_a^b \)

In this section, we summarize the results of [8] in the case that a boundary exists. Define a manifold with boundary

\[
M_a^b := \{(x, y, z) \in M_a \mid -b \leq z \leq b\}
\]

for \( 0 < b < 1 \).

**Definition 4.1.** We call a vector field \( Z \) on \( M_a^b \) a zonal flow if it has the form

\[
Z = F(r)\partial_\theta
\]

for some function \( F \), which depends only on the variable \( r \).

**Remark 4.2.** Note that any zonal flow is a stationary solution of (1.1).

**Definition 4.3.** For a zonal flow \( Z \) and a divergence free vector field \( W \) on \( M_a^b \), define the Misiolek curvature \( MC_{Z,W} \) by

\[
MC_{Z,W} = \int_M g(\nabla_Z[Z,W] + \nabla_{[Z,W]}Z,W) \mu,
\]

where \( \mu \) is the volume form.
Lemma 4.4 ([8 Prop. 3.1]). Let \( Z = F(r) \partial_\theta \) be a zonal flow on \( M_a^b \). Then, for a divergence-free vector field \( W = W^1 \partial_r + W^2 \partial_\theta \) which tangents to \( \partial M \) (i.e., \( W^1(\pm b, \theta) = 0 \)), we have
\[
MC_{Z,W} = \int_{-b}^{b} \int_{-\pi}^\pi F^2 c_1 \left( - (\partial_\theta W_1)^2 - c_1^2 (\partial_r W_1)^2 + \left( (\partial_r c_1)^2 - c_1 \partial_\theta^2 c_1 \right) W_1^2 \right) d\theta dr.
\]

Proof. The assumption \( W^1(\pm b, \theta) = 0 \) guarantees to use the Stokes theorem in the same way of [8]. Thus, the same computation in [8] can be applied. \( \square \)

Definition 4.5. Let \( Z = F(r) \partial_\theta \) be a zonal flow. If \( F \) satisfies
(i) \( F^2 \) is an even function,
(ii) \( (F^2)' \) is a strictly monotonically decreasing function on \( r \geq 0 \),
then, we call \( Z = F(r) \partial_\theta \) a symmetric zonal flow.

Lemma 4.6 ([8 Prop. 3.2]). Let \( a > 1 \) and \( Z = F(r) \partial_\theta \) a symmetric zonal flow on \( M_a^b \) such that \( |F(\pm b)| \ll |F(0)| \). Then, there exists a divergence-free vector field \( W \) on \( M_a^b \) which tangents to \( \partial M_a^b \), satisfying \( M_{Z,W} > 0 \).

Proof. Let \( h(r) \) be a function on \( M_a^b \) with \( h(\pm b) = 0 \). Then,
\[
W_h := h(r) \sin \theta \partial_r + \left( \partial_r h(r) + \frac{h(r) \partial_\theta c_1(r)}{c_1(r)} \right) \cos \theta \partial_\theta
\]
defines a divergence-free vector field on \( M_a^b \), which tangents to \( \partial M_a^b \). By Lemma 4.4, we have
\[
MC_{V,W_h} = \pi \int_{-b}^{b} F^2 c_1 \left( h^2 c_1^2 - (\dot{h})^2 \right) dr, \tag{4.1}
\]
where \( \epsilon(r) = \sqrt{(\dot{c}_1)^2 - c_1 \dot{c}_1 - 1} \). Note that \( (\dot{c}_1)^2 - c_1 \dot{c}_1 - 1 \) is nonnegative by \( a \geq 1 \), see [8 Prop. 4.1]. We take \( h \) satisfying the following
(i) \( \partial_r h = 0 \) near \( r = 0 \),
(ii) \( 0 \ll \dot{h}^2 \) near \( r = 0 \).

For such \( h \), the value of the integrand in (4.1) at \( r = 0 \) is sufficiently large. Thus, the suitable choice of \( h \) makes \( MC_{Z,W_h} \) to be positive. \( \square \)

At the end of this section, we recall the Misiolek criterion for the existence of a conjugate point on the volume-preserving diffeomorphism group \( D^s_M(M) \) of a compact Riemannian manifold \( M \).

Fact 4.7 ([17 see also [23]]). Let \( s > 2 + \frac{2}{n} \) and \( M \) be a compact \( n \)-dimensional Riemannian manifold, possibly with smooth boundary. Suppose that \( V \in T_e D^s_M(M) \) is a stationary solution of the Euler equation (1.1) on \( M \) and take a geodesic \( \eta \) on \( D^s_M(M) \) satisfying \( V = \eta \circ \eta^{-1} \). Then if \( W \in T_e D^s_M(M) \) satisfies \( MC_{V,W} > 0 \) there exists a point conjugate to \( e \in D^s_M(M) \) along \( \eta(t) \) on \( 0 \leq t \leq t_0 \) for some \( t_0 > 0 \).
This was only proved for the case that $M$ has no boundary in [7] (and [8]). Thus, we explain how to apply the proof in [7] to the case $M$ has a boundary in the appendix.

5 Existence of Arnold stable flow on $M^b_a$.

In this section, we prove Theorem 1.1 by using the results obtained in the previous sections.

Proof of Theorem 1.1 We take a function $f(r)$ on $-b < r < b$ which satisfies the following conditions:

1. $f(r) > 0$ for $-b < r < b$;
2. $f$ is an even function;
3. $f(r)$ is a strictly monotonically decreasing function on $r \geq 0$;
4. $f(\pm b) \ll f(0)$;
5. $\frac{f^2}{c^2}$ is a strictly monotonically decreasing function on $r \geq 0$.

Note that it is clear that such a function actually exists. Moreover, Lemma 3.5 implies

\[ Df(r) > 0 \quad (5.1) \]

because $\dot{f}$ is a strictly monotonically increasing odd function by (2) and (3).

For such $f$, define a function $\psi_f(r)$ on $M^b_a$ by

\[ \dot{\psi}_f = \frac{\dot{f}}{c_1}. \]

Then,

\[ u_f := \ast \text{grad} \psi_f = \frac{\dot{f}}{c_1} \partial_\theta \quad (5.2) \]

is a solution of (1.1). Moreover, Lemma 3.3 implies

\[ F'(\psi_f) = \frac{Df}{f}, \]

which is positive on $M^b_a$ because $f > 0$ and $Df > 0$ by (1) and (5.1), respectively. Thus $u_f$ is Arnol’d stable. Moreover, (4), (5) and (5.2) implies that $u_f =: F(r) \partial_\theta$ is a symmetric zonal flow with $|F(\pm b)| \ll |F(0)|$. Thus, Lemma 4.6 implies that there exists a divergence-free vector field $W$ on $M^b_a$, which tangents to $\partial M^b_a$, satisfying $MC_{u_f,W} > 0$. This completes the proof by Fact 4.7.

Appendix: A sufficient criterion of Misiolek

In this appendix, we explain how to apply the proof of Fact 4.7 in [7] to the case $M$ has a boundary.
A.1 \( \mathcal{D}^s_\mu(M) \) in the case \( M \) has a boundary

In this subsection, we recall briefly the theory of volume-preserving diffeomorphism group \( \mathcal{D}^s_\mu(M) \) in the case that \( M \) has a boundary. Main reference is [5].

Let \( M \) be a compact \( n \)-dimensional Riemannian manifold with smooth boundary, \( \mathcal{D}^s_\mu(M) \) the group of all diffeomorphisms of Sobolev class \( H^s \) preserving the volume form on \( M \), and \( T_e \mathcal{D}^s_\mu(M) \) the tangent space of \( \mathcal{D}^s_\mu(M) \) at the identity element \( e \in \mathcal{D}^s_\mu(M) \), which is identified with the space of divergence-free vector fields on \( M \) tangent to \( \partial M \). If \( s > \frac{n}{2} + 1 \), \( \mathcal{D}^s_\mu(M) \) has an infinite dimensional Hilbert manifold structure with the right-invariant \( L^2 \) Riemannian metric given by

\[
\langle X, Y \rangle := \int_M g(X, Y) \mu,
\]

where \( X, Y \in T_e \mathcal{D}^s_\mu(M) \).

By V. I. Arnold [1], a solution \( u \) of the incompressible Euler equation (1.1) on \( M \) corresponds to a geodesic \( \eta \) on \( \mathcal{D}^s_\mu(M) \) starting at \( e \in \mathcal{D}^s_\mu(M) \) via \( u = \dot{\eta} \circ \eta^{-1} \). Thus, it is important to study of the geometry of \( \mathcal{D}^s_\mu(M) \). In particular, the existence of a conjugate point on a geodesic has attractive considerable attention because it is related to a Lagrangian stability of corresponding solution.

A.2 Sketch of proof of Fact 4.7

In this subsection, we explain how to apply the proof of Fact 4.7 in [7] to the case that \( M \) has a boundary. For the convenience, we rewrite Fact 4.7.

**Fact 4.7.** Let \( M \) be a compact \( n \)-dimensional Riemannian manifold with smooth boundary and \( s > 2 + \frac{n}{2} \). Suppose that \( V \in T_e \mathcal{D}^s_\mu(M) \) is a stationary solution of the Euler equation (1.1) on \( M \) and take a geodesic \( \eta \) on \( \mathcal{D}^s_\mu(M) \) satisfying \( V = \dot{\eta} \circ \eta^{-1} \). Then if \( W \in T_e \mathcal{D}^s_\mu(M) \) satisfies \( MC_{V,W} > 0 \) there exists a point conjugate to \( e \in \mathcal{D}^s_\mu(M) \) along \( \eta(t) \) on \( 0 \leq t \leq t_0 \) for some \( t_0 > 0 \).

**Sketch of a proof of Fact 4.7.** Because the Riemannian metric of \( \mathcal{D}^s_\mu(M) \) is right invariant, Theorem B.5 in [9] shows that there exist \( t_0 > 0 \) and a vector field \( \tilde{W} \) on \( \eta \) satisfying \( \tilde{W}(0) = \tilde{W}(t_0) = 0 \) and

\[
E''(\eta)_{t_0}^{t_0}(\tilde{W}, \tilde{W}) < 0 \quad \text{(A.1)}
\]

by the assumption \( MC_{V,W} > 0 \). Here \( E''(\eta)_{t_0}^{t_0}(\tilde{W}, \tilde{W}) \) is the second variation of the energy function \( E_{t_0}(\eta) \) of \( \eta \):

\[
E_{t_0}(\eta) := \frac{1}{2} \int_0^{t_0} \langle \dot{\eta}, \dot{\eta} \rangle dt.
\]

On the other hand, the same argument of [7] Lem. 3] gives

\[
E''(\eta)_{t_0}^{t_0}(Z, Z) \geq 0 \quad \text{(A.2)}
\]
for any vector field \( Z(t) \) on \( \eta \) with \( Z(0) = Z(t_0) = 0 \) if there exists no conjugate point on \( \eta(t) \) \((0 \leq t \leq t_0)\). The essential point of the argument of [7, Lem. 3] is that the differential of the exponential map is bounded operator, which is deduced by the boundedness of the curvature of \( \mathcal{D}_{\mu}(M) \) in [7, Lem. 3]. This boundedness is also guaranteed for the case that \( M \) has a boundary by [6, Prop. 3.6]. Thus, the same argument is valid in the case that \( M \) has a boundary and the contradiction of (A.1) to (A.2) gives the desired result. \( \square \)

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