Power Corrections and KLN Cancellations.

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Abstract

We consider perturbative expansions in theories with an infrared cutoff $\lambda$. The infrared sensitive pieces are defined as terms nonanalytic in the infinitesimal $\lambda^2$ and powers of this cutoff characterize the strength of these infrared contributions. It is argued that the sum over the initial and final degenerate (as $\lambda \to 0$) states which is required by the Kinoshita - Lee - Nauenberg theorem eliminates terms of order $\lambda^0$ and $\lambda^1$. However, the quadratic and higher order terms in general do not cancel. This is investigated in simple examples of KLN cancellations, of relevance to the inclusive decay rate of a heavy particle, at the one loop level.
1 Introduction

Recently, there has been an upsurge of interest in the theoretical and phenomenological aspects of power-suppressed infrared contributions \([1, 5, 10, 4]\). In case of QCD, one analyzes terms of order \((\Lambda_{QCD}/M)^k\) where \(k\) is an integer, \(k \geq 0\), \(\Lambda_{QCD}\) is the infrared parameter of QCD, and \(M\) is a large mass scale which could be say, the total energy in \(e^+e^-\) annihilation or in the decays of heavy particles it is the mass of a heavy decaying quark. The case of \(k = 0\) corresponds to logarithmic divergences while positive \(k\) corresponds to what we call power-suppressed infrared contributions. Powers of \(\Lambda_{QCD}\) can emerge only through the so called dimensional transmutation. Perturbatively, it is more convenient to introduce an infrared cutoff parameter \(\lambda\) which could for example be an infinitesimal boson mass. Then the infrared sensitivity is studied by looking for the terms nonanalytical in \(\lambda^2\) in the perturbative expansion of the physical observable under consideration. This is because while terms like \(\lambda^2\) can come from propagators, those nonanalytic in \(\lambda^2\) can arise only as contributions from soft and collinear particles, thus signalling the importance of such contributions.

The Kinoshita-Lee-Nauenberg (KLN) theorem \([2, 3]\) is a fundamental theorem concerning infrared singularities in quantum mechanics and quantum field theory. The KLN theorem concerns infrared (IR) singularities which may arise in perturbative expansions, that is terms like \(\alpha \ln \lambda\). The theorem states that these terms disappear order by order in perturbation theory once summation over all initial and final states degenerate in the limit \(\lambda \to 0\) is performed. A natural question is whether this cancellation extends to the power-like suppressed IR terms of order \(\lambda, \lambda^2 \ln \lambda\), etc. In other words, whether the procedure of summing over the degenerate states brings about not only finiteness but analiticity as well. Although the question appears to be of fundamental importance in understanding the IR behaviour in quantum field theory, to our knowledge, it has never been addressed in a systematic way. Very recently \([4]\) it was conjectured that the KLN cancellation does extend to power-like terms. There, in particular, an attempt was made to use the theorem to understand the general reason...
behind the cancellation of the terms of order $\lambda/M$ in inclusive physical crossections.

In this paper we will explore the KLN cancellations at the level of IR power-like corrections. In section 2 we present arguments, based on the approach of Ref.[3] that a summation over degenerate initial and final states, in general, eliminates IR terms of order $\lambda^0$ (that is, $\log(\lambda)$) and $\lambda^1$. However, higher order nonanalytic terms like $\lambda^2\ln\lambda$ or $\lambda^3$ are in general, not cancelled. In the rest of the paper we consider some examples which verify this conclusion and elucidate the cancellation mechanism. Our examples have relevance to heavy particle inclusive weak decays in the one-loop approximation and we study the infrared behaviour by giving an infinitesimal mass, $\lambda$, to a boson. Then infrared sensitive pieces are identified as terms non-analytical in $\lambda^2$ [5]. In section 3 some simplifications relevant to the calculation of the nonanalytical terms in the correction to the mass and, of the bremsstrahlung diagrams are discussed. We also comment on the connection between the description of power suppressed infrared contributions in terms of the operator product expansion and that in terms of the nonanalytic terms obtained via Feynman graphs. In section 4 we discuss the KLN cancellations in an elementary example. Of particular technical interest is the implementation of the procedure of averaging over the initial states directly in terms of Feynman graphs (in contrast to the original techniques which make heavy use of disconnected graphs in the context of old fashioned perturbation theory). Drawing from the approach of Ref [4] we illustrate this in the context of the example. Our findings imply that the procedure of averaging over the initial states cannot be defined in a straightforward way beyond linear in $\lambda$ terms. We conclude with a brief summary.

2 Infrared Cancellations Following From the KLN Theorem

In this section we will outline an argument as to why in the KLN sum not only the terms singular in the infrared cutoff but those linearly dependant on it are cancelled as well. Our
discussion follows that of Ref. [3].

Consider the time evolution matrix, \( U(t, t_0) \) in terms of which the \( S \) matrix can be written as, \( U(0, \infty) \dagger U(0, -\infty) \). The Lee-Nauenberg probabilities \( P_{ab} \) for a transition from \( a \rightarrow b \) are given by:

\[
P_{ab} = \sum_{[a]} \sum_{[b]} |\langle b|S|a\rangle|^2
\]

Here the sum is over all degenerate initial and final states \([a]\) and \([b]\) respectively. This can be expressed in the form:

\[
P_{ab} = \sum_n \sum_m (T_{mn}^\dagger T_{mn}^\ast)
\]

\[
T_{mn}^\dagger = \sum_{[a]} \langle m|U(0, -\infty)|a\rangle^\ast \langle n|U(0, -\infty)|a\rangle.
\]

A similar expression holds for \( T_{mn}^\ast \) involving a sum over the final degenerate states \([b]\). The content of the KLN theorem is the statement that both \( T_{mn}^\dagger \) and \( T_{mn}^\ast \) are separately free of infrared divergences of the type \( \ln \lambda \) and hence so is \( P_{ab} \) (\( \lambda \) is an infrared cutoff). We will argue now that under certain conditions to be discussed, the absence of singularities of the type \( \ln \lambda \) in \( T_{mn}^\dagger \) separately implies that \( P_{ab} \) also does not contain any non analytical terms proportional to \( \lambda \).

To see this in the lowest order, consider the expansion of \( U(0, \pm \infty) \) in terms of \( gH_1 \) the interaction Hamiltonian. Then to order \( g \),

\[
T_{mn}^\dagger = \sum_{[a]} \left( \delta_{ma} \delta_{na} + g \frac{\delta_{ma}(1 - \delta_{na})}{E_a - E_n - i\epsilon} (H_1)^\ast_{na} + g \frac{\delta_{na}(1 - \delta_{ma})}{E_a - E_m - i\epsilon} (H_1)_{ma} + ... \right),
\]

with a similar expression for \( T_{mn}^\ast \). Infrared divergences arise from the vanishing of the energy denominators inside the brackets in (4). For example if \( |a\rangle \) represents the state of a quark, and \( |n\rangle \) represents that of a quark and a gluon, then the latter becomes degenerate with the former in the limit of a soft gluon. In this case the energy denominator \( E_a - E_n \) vanishes proportional to \( \omega \) the soft gluon energy, giving rise to the usual infrared divergence. A similar situation happens for the collinear divergence. Now, upon performing the degenerate...
sum in (4), it is easily seen [3] that there are no terms of order $1/\omega$ as $\omega \to 0$ in $T_{mn}^-$ and the same is true for $T_{mn}^+$ as well. Thus there are no terms of order $1/\omega^2$ in the summand of $P_{ab}$ (2), and since the phase space sum is proportional to $\omega d\omega$, there are no terms that diverge like $\ln \lambda$. (The same is true of any possible collinear divergence.) Now the point is that since the terms of order $1/\omega$ are separately cancelled in $T_{mn}^\pm$, then not only are there no terms of order $1/\omega^2$ in the summand of $P_{ab}$, but there are none of order $1/\omega$ as well. In fact, in general, the expansion of $(T_{mn}^+) T_{mn}^-$ for small $\omega$ starts off with the constant term. Thus non-analytical terms proportional to $\lambda$ are absent from the KLN probabilities. A similar argument relying on the proven finiteness of the $T_{mn}^\pm$ separately in the KLN sum and power counting generalizes to higher orders in $g$ as well. There is however a condition which must be satisfied and it makes its appearance only beyond the leading order brehmstrahlung diagrams.

To prove the finiteness of $T_{mn}^-$ to all orders [3] one must consider three cases: (1) Both $m$ and $n$ are in the degenerate set, $[a]$, (2) $m$ lies outside this set and $n$ may be contained in it, (3) the roles of $m$ and $n$ are reversed. Of these, case (1) follows from the unitarity of the $U$ matrices, and case (3) is related to (2) by hermiticity. For case (2) one uses induction. Since $U = U(0, \pm \infty)$ diagonalize the total Hamiltonian, then with $\hat{H}$ this diagonal quantity, we can write:

$$[U, \hat{H}] = (g H_1 + \Delta)U$$

where, the diagonal $\Delta = H_0 - \hat{H}$; $H_0$ is the free particle Hamiltonian with bare masses and $\hat{H}$ is the same with physical masses. Thus, for example for a massive quark, the matrix elements of $\Delta$ give the mass shift $\Delta m$. Let us denote by $O_\alpha$ the $\alpha^{th}$ term in the power series expansion of $O$, then by considering the appropriate matrix elements of (5) a recursion relation for $T_{\alpha mn}$ may be established expressing it in terms of $T_{\beta mn}^-$ and $\Delta_{\beta mn}$ for $\beta < \alpha$:

$$T_{\alpha mn}^- = \frac{1}{E_a - E_m} \left( \sum_p H_{1mp} T_{\alpha - 1 pn}^- \right. + \left. \sum_{\beta=1}^{\alpha-1} \Delta_{\beta mm} T_{\alpha - \beta mn}^- \right).$$

(6)
Together with the lowest order discussion given above, this establishes, not only the absence of infrared divergent terms proportional to $\ln \lambda$ but those linear in $\lambda$ as well, provided that $\Delta_\beta$ has this property for all $\beta \leq \alpha - 1$. This in turn implies that the renormalization procedure must be such as not to introduce any infrared sensitivity.

We conclude this general discussion with the following remarks: (1) In this general discussion we have assumed that a suitable infrared regulator $\lambda$ exists whose vanishing produces the degeneracy. Moreover, our discussion of infrared sensitivity is restricted to those arising due to this degeneracy alone. For explicit calculations, in field theories including the case of abelian gauge theories the regulator $\lambda$ can for example to be a non vanishing but small (gauge) boson mass.

(2) The statement on the absence of terms proportional to $\ln \lambda$ and $\lambda$ from the KLN sum is valid as we have just argued, in general in any field theory. Additional symmetries may require the vanishing of further terms nonanalytic in $\lambda^2$ like $\lambda^2 \ln \lambda$ and so on. An example of this is provided by gauge theories. Gauge invariance requires that if in the small $\omega$ expansion of $T_{mn}^\pm$ there are no terms of order $1/\omega$ then there are no constant terms (order $\omega^0$) as well and hence the leading term is proportional to $\omega$. This in turn implies that the leading non vanishing term in $P_{ab}$ due to soft gauge particles, that is non analytic in the infrared cutoff is proportional to $\lambda^4 \ln \lambda$.

(3) Let us consider the contribution of the collinear divergences in the lowest order expansion (4). Suppose that $|a\rangle$ represents a massless quark state and $|n\rangle$ that of the quark and a gluon. In the limit that the gluon is moving parallel to the quark the two states become degenerate. Then the energy denominator $E_a - E_n$ which goes like $(1 - \cos \theta)$ becomes of order $(\theta^2 + \theta^4)$. Now in most situations of physical interest the matrix element $(H_{1})_{na}$ itself for small $\theta$ becomes proportional to order $(\theta + \theta^3)$ due to helicity conservation at the vertex. Thus $T_{mn}^-$ becomes of order:

$$T_{mn}^- \sim \sum_{|a\rangle} \frac{1}{\theta} (1 + \theta^2)$$ (7)
and similarly for $T_{mn}^\pm$. Thus since the $1/\theta$ term in the degenerate sum for both $T_{mn}^\pm$ separately vanishes and since the phase space sum is proportional to $\sin\theta d\theta$, we see that collinear divergences in particular are not relevant for the terms linear in $\lambda$.

(4) The previous comment illustrates a feature that can be present even for the case of soft infrared sensitivity. Namely the power counting may be modified if there is kinematic suppression, as for instance, in the case of the anomalous magnetic moment interaction introduced as an example in section 4. Here the cancellation is of the term proportional to $\lambda^3$. The universal feature of the KLN theorem is the cancellation of the pole terms (like $1/\omega$) in the degenerate sum.

(5) It is known that in the calculation of $P_{ab}$ one must consider interference between graphs, some of which contain disconnected parts. In fact, in the covariant formulation of field theories an infinite number of such contributions may have to be considered in a given order [4]. Fortunately, at least for soft particles, it is possible to perform a rearrangement of the perturbation series so that the contribution of the purely disconnected pieces factor out. However, the disconnected pieces do leave behind a trace: Their effect is encoded in the rule that essentially every boson propagator in a graph is supplemented by its complex conjugate [4]. In addition of course, as dictated by the KLN sum, every emitted line is supplemented by an absorbed line. In this approach, we may forget about disconnected graphs and deal directly with Feynman graphs suitably modified in this manner. We apply this procedure to an example in section 4.

3 Computation of Terms Non-analytical in a Meson Mass$^2$.

In this section we develop considerations which are of relevance to the inclusive weak decays of a heavy fermion of mass $m$, ($m \gg \lambda$) which will be our principle example. Because the total energy available for the decay is of primary concern we study the analyticity of the
mass of the heavy particle in $\lambda^2$. For this purpose, since we are interested in matters of principle, we are free to choose a model which technically is most transparent. Thus we will introduce an interaction of the decaying fermion with various bosonic fields which couple to this fermion alone and not to the decay products. Of course, this scheme does not include the case of an interaction with a conserved charge but this is exactly the feature which makes the calculations most simple. Since it is only the initial particle that has new interactions the loop effects are encoded in the wave function and mass renormalizations of the initial particle. In particular, the renormalization of the mass affects directly the phase space factor in the decay probability and our problem is the evaluation of non-analytic contributions to a heavy particle mass.

We now derive some results involving the mass shift and the bremsstrahlung diagrams emphasizing the technical aspects and the calculational simplifications. In particular, we find that there is a simple prescription for finding the nonanalytic pieces in $\lambda^2$. These results are used in the following section to demonstrate the KLN cancellations and in section 5 some physical applications are mentioned.

We begin with an interaction of the form $G\bar{\psi}\Gamma\psi\varphi$, where $\varphi$ represents the "light" meson of mass $\lambda$ and $\psi$ the heavy fermion of mass $m$. Consider the well-known expression for the one-loop correction to the mass:

$$\Delta m \cdot \bar{w}w = -i \frac{G^2}{4\pi^3} \bar{w}(p)I(\hat{p})w(p)$$  \hspace{1cm} (8)

where $G$ is the coupling constant, $w(p)$ is the corresponding spinor (which in our case, describes the heavy particle at rest) and, finally, $I(\hat{p})$ is:

$$I(\hat{p}) = \int d^4k \frac{\Gamma(\hat{p} - \hat{k} + m)\Gamma}{(p - k)^2 - m^2 + i\epsilon (k^2 - \lambda^2 + i\epsilon)}$$  \hspace{1cm} (9)

Here $\lambda$ denotes the mass of the meson, $\Gamma$ is the vertex and we will concentrate for the moment on the case $\Gamma = \gamma_5, \gamma_5^2 = -1$. 

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As usual, we would integrate first over $k_0$. The meson propagator has poles at:

$$k_0 = \pm (k^2 + \lambda^2)^{1/2} \mp i\epsilon. \quad (10)$$

For the fermion propagator we expand in $1/m$ and keep only the nonrelativistic approximation for the kinetic energy. Then the poles are at:

$$k_0 = \frac{k^2}{2m} + i\epsilon, \quad k_0 = 2m + \frac{k^2}{2m} - i\epsilon. \quad (11)$$

Here the first pole corresponds to a static interaction if we neglect the $1/m$ terms altogether while the second pole corresponds to the production of a pair in the intermediate state.

Now, we close the contour in the lower half plane. Then we have two poles, at $k_0 = \omega = (k^2 + \lambda^2)^{1/2}$ and at $k_0 = 2m + \ldots$. A crucial point is that non-analytical terms can be associated only with the pole at $k_0 = \omega$. Because, if $k_0 \approx 2m$ then one can safely expand in $\lambda^2/m^2$ and no non-analytical terms can arise. (In fact, this is not entirely obvious since we are going to keep the relativistic corrections to terms non-analytical in $\lambda$.) We can check this considerable simplification by comparing with known results. In particular, for the correction to the mass in case of pseudoscalar interaction one has a closed form answer [3]:

$$\Delta m = \frac{G_P}{4\pi} \frac{m}{8\pi} \left( \log \frac{\Lambda_{UV}^2}{m^2} - \frac{1}{2} + \frac{\lambda^2}{m^2} + 2 \frac{\lambda^2}{m^2} (1 - \frac{\lambda^2}{m^2}) \log \frac{\lambda}{m} - 2 (\frac{\lambda}{m})^3 \sqrt{1 - \frac{\lambda^2}{2m^2}} \cos^{-1} \frac{\lambda}{2m} \right). \quad (12)$$

Let us check the terms of order $\sim \lambda^2 \log(\lambda/m)$ and $\sim (\lambda/m)^3$. We use:

$$\int d^4k \frac{1}{k^2 - \lambda^2 + i\epsilon} \Gamma(p - k + m)\Gamma \frac{\Gamma(p - k + m)\Gamma}{(p - k)^2 - m^2 + i\epsilon} w = (+2\pi i) \int \frac{d^3k}{2\omega} \bar{w} \Gamma(p - k + m)\Gamma \frac{\Gamma(p - k + m)\Gamma}{(-2m\omega + \lambda^2)\omega} w \quad (13)$$

where the equality is understood in the sense that we keep only one pole corresponding to $k_0 = \omega - i\epsilon$, as is explained above. Furthermore:

$$\bar{w} \frac{\Gamma(p - k + m)\Gamma}{(p - k)^2 - m^2} w = \bar{w} \frac{1}{2m(1 - \frac{\lambda^2}{2m\omega})} w \quad (14)$$
where we have accounted for \((\gamma_5)^2 = -1\), \(\bar{\omega}k\omega = \bar{\omega}w \cdot \omega\) (that is, \(\bar{\omega}\gamma_\lambda w \neq 0\) only if \(\lambda = 0\)). Moreover, as far as the non-analytical terms are concerned

\[
\int \frac{d^3k}{2(k^2 + \lambda^2)^{1/2}} = \pi \lambda^2 \log(\lambda).
\]  

(15)

Note that the log factor in the r.h.s. is in fact negative while the integral in the l.h.s. is apparently positive. As usual, this is due to the subtractions needed to extract the non-analytic term. Combining the factors we get for the leading non-analytic term

\[
\Delta m \approx -\frac{iG_P^2}{16\pi^4} (2\pi i)^{1/2} \frac{1}{2m} \pi \lambda^2 \log(\lambda) = \frac{G_P^2 \lambda^2 \log(\lambda)}{16\pi^2 m}.
\]  

(16)

which checks with (12).

For the next non-analytical term, we may expand the denominator above:

\[
\frac{1}{1 - \lambda^2/2m\omega} \approx 1 + \frac{\lambda^2}{2m\omega}.
\]  

(17)

Moreover, again keeping only the nonanalytic terms,

\[
\int \frac{d^3k \lambda^2}{\omega} = -2\pi^2 (\lambda^2)^{3/2}.
\]  

(18)

And in this way we reproduce the cubic term correctly:

\[
(\Delta m)_{\text{cubic}} = \frac{G_P^2}{16\pi^4} (2\pi) \frac{1}{8m^2} (-2\pi^2) (\lambda^2)^{3/2} = -\frac{G_P^2 (\lambda^2)^{3/2}}{32\pi m^2}.
\]  

(19)

Thus we see that the non-analytic terms (at least in the one-loop approximation we are considering) can be obtained in a very simple way, i.e., they arise only from the pole in the \(k_0\) plane corresponding to a real meson, in the intermediate state with a standard phase space factor. On the other hand, all the kinematics in the rest of the graph can be treated in a fully relativistic manner. That the infrared divergent terms are obtained by considering the massless boson to be on shell is well known, and what we see here is that the same is true for the other non-analytic terms as well. In general, it is a consequence of the Landau equations.
concerning the analytic structure of Feynman graphs which tell us that nonanalyticity in a mass results from putting the corresponding line on shell.

It should be emphasized that the above procedure for $\Delta m$ is different from just keeping those terms which are enhanced because of the small energy denominators. (i.e., keeping only the pole term $1/\omega$) This difference is an important point so let us discuss it further and examine the origin of the terms of order $\lambda^2 \ln \lambda$ and those of order $\lambda^3$ from the viewpoint of a non relativistic expansion.

The pseudoscalar interaction is ideal to illustrate the point because in this case the relativistic effects are not mere corrections but rather the leading terms. Indeed, in the non-relativistic limit:

$$\bar{w} \gamma_5 w \to \bar{u} \frac{(\sigma \cdot k)}{2m} u$$

where $u$ is a non-relativistic spinor, $k$ is the 3-momentum carried by the meson. Eq (20) expresses the equivalence (on the mass shell) of the pseudoscalar and pseudovector couplings. Therefore, in terms of the old-fashioned perturbation theory in the pole approximation the correction to the mass is:

$$\Delta m \cdot \bar{u} u = \frac{G_P^2}{4m^2} \int \frac{d^3k}{2\omega(2\pi)^3} \bar{u}(\sigma \cdot k) \frac{1}{-\omega} (\sigma \cdot k) u$$

where the essential factors are the vertices (see Eq. (20)), the energy denominator and the phase space of free mesons. As a result, we have

$$\Delta m = - \frac{G_P^2}{4m^2} \int \frac{d^3k |k|^2}{2\omega^2(2\pi)^3} = - \frac{G_P^2}{32\pi} \frac{(\lambda^2)^{3/2}}{m^2}.$$  

In other words, we reproduce the cubic term above (see Eq. (19)), which we see is dominated therefore by the pole.

On the other hand, the leading, or $\lambda^2 \ln \lambda$ term is not reproduced by old fashioned perturbation theory in the pole approximation. The reason is that the leading $\lambda^2 \log(\lambda)$ dependance is not due to the pole $1/\omega$ in the amplitude but rather is an entirely relativistic effect. If we
consider a heavy quark expansion then we should have first eliminated the lower components of the Dirac spinors and thus arrive at the effective term

\[ L_{\text{eff}} = \frac{G^2}{2m} \bar{\psi}(x)\psi(x)\varphi^2(x) \]  

(23)

where \( \psi \) and \( \varphi \) are the fermion and meson fields (see, e.g., Ref. [4], p. 411). The correction to the mass proportional to \( \lambda^2 \log(\lambda) \) arises clearly as a (perturbative) mesonic condensate-like contribution:

\[ \langle \varphi^2 \rangle_{\text{pert}} = \int \frac{d^3k}{2\omega(2\pi)^3} = \frac{\pi \lambda^2 \log(\lambda)}{(2\pi)^3} \]  

(24)

where we used the integral (15) above. Combining it with Eq. (23) we come again to (16). Thus we see that it is necessary to go beyond the most naive non-relativistic limit to reproduce the \( \lambda^2 \ln \lambda \) term.

We consider now the bremsstrahlung process, which is closely related to the self energy. In fact, previous calculations have found a Bloch Nordsieck [8] type cancellation in the total (weak decay) width to order \( \lambda \) [10, 5]. We can treat the bremsstrahlung graphs in a fully relativistic way, a la Feynman, without restricting ourselves to the \( 1/\omega \) terms in the amplitude. Then our "bremstrahlung" in this scalar case would not factorize any longer and the probability of radiation of a soft particle is no longer a product of an emission factor and of a non-radiative process. Indeed, the relativistic corrections are in fact not related to the pole, as emphasized above. The most convenient way that we have found to include the bremsstrahlung contribution for problems of relevance to the infrared sensitivity in inclusive weak decays is as follows. We will assume, (as is true in the realistic case), that integrating over the decay products gives \( \bar{w} p^5 w \) (for a review see for example [4]) where \( w \) is the spinor describing the heavy fermion but treated now as a field operator. Then we could write identically

\[ \bar{w} p^5 w = \bar{w}(\hat{p} - m + m)^5 w \approx 5m^4 \bar{w}(\hat{p} - m)w \]  

(25)

Moreover the factor \((\hat{p} - m)\) would cancel the propagator and we would reduce the graph describing bremsstrahlung to one describing \( \Delta m \). The corresponding correction to the total
width can be shown to be

\[ \delta \Gamma_{tot} |_{brem} = 5 \frac{\Delta m}{m} \Gamma^0 \]

(26)

where \( \Delta m \) now does include the leading \( \lambda^2 \log(\lambda) \) term, i.e. it corresponds to the fully relativistic evaluation of \( \Delta m \), the same as above. Thus contrary to the case of linear corrections to the mass of a heavy quark in a gauge theory \([10, 5]\) where a cancellation takes place, in this case, the corrections from the mass shift and the bremsstrahlung add instead of cancelling.

The above results have an interpretation in terms of a formalism that is close in spirit to the Operator Product Expansion. In fact we now apply it in a fully relativistic way, i.e., to Dirac fermions. We will see that the approach matches very nicely with the relativistic calculations discussed previously. Very briefly the idea is the following: Consider inclusive heavy quark decay and as mentioned in the previous paragraph, the result of integration over the decay products gives \( \bar{w} \gamma^5 w \) where the spinor \( w \) is treated now as an operator. The equations of motion read,

\[ (\not{p} - m - G_P \gamma_5 \varphi) w = 0 \]

(27)

where we have a pseudoscalar field not interacting with the decay products. We can conclude that the correction to the total rate due to this interaction is related to the matrix element:

\[ \langle \bar{w} G_P \gamma_5 \varphi w \rangle_{pert} \]

(28)

When evaluating this matrix element we note that the pseudoscalar interaction also enters through the exponential in:

\[ \langle \bar{w} G_P \gamma_5 \varphi w \rangle_{pert} = \langle T \bar{w} G_P \gamma_5 \varphi w \exp (i \int H_{int} dt) \rangle. \]

(29)

In this way we get:

\[ \langle \bar{w} G_P \gamma_5 \varphi w \rangle_{pert} = 2 \cdot \Delta m \]

(30)

where the factor of 2 is purely combinatorial. Moreover, as is explained above, \( \Delta m \) contains the following:

\[ \Delta m \sim G_P^2 \langle \varphi^2 \rangle \sim G_P^2 \lambda^2 \log(\lambda). \]

(31)
Thus the noncancellation of the $\lambda^2 \ln \lambda$ term can also be understood from an OPE like approach. In the language of the physical processes, the factor of 2 is due to the doubling of the effect of the mass shift through the radiation of mesons. The crucial point here is that the nonanalytic in $\lambda^2$ terms are entirely due to the pole corresponding to a ”real” meson in the intermediate state. That is why the doubling exhibited by the equation above is exact as far as the non-analytical terms are concerned, even if the relativistic corrections are included.

4 KLN Cancellations

In this section we first check that the KLN cancellation up to terms of order $\lambda$ found in section 2 on general grounds, actually takes place for single particle propagation at one loop. This example would be relevant to checking the KLN theorem for processes involving an incoming particle, such as in weak decays.

The KLN cancellations are better understood in the language of old-fashioned perturbation theory since the KLN theorem is rooted in Quantum Mechanics. As discussed in section 2, the KLN procedure deals with small energy denominators which enter the perturbative evaluation of wave functions:

$$\psi_{\text{pert}} \approx \psi_0 + \sum_n \frac{(\delta V)_n}{E_0 - E_n} \psi_n$$  \hspace{1cm} (32)

where $\psi_n$ is the unperturbed wave function and $E_n$ are unperturbed eigenvalues of the energy. Relevant to heavy particle decays are renormalization of the wave function and correction to the mass. Cancellation of the renormalization of the wave function by bremsstrahlung graphs is a very universal phenomenon which is to be expected and we concentrate, instead, on $\Delta m$. Let us start with the scalar interaction which has a non-vanishing non-relativistic limit:

$$\Delta m \cdot \bar{w}w = \frac{-i}{4\pi^3} \int d^4k \frac{\bar{w}(p - k + m)w}{(p - k)^2 - m^2} \frac{1}{k^2 - \lambda^2}.$$  \hspace{1cm} (33)

To examine the analyticity in $\lambda^2$ we keep (see section 3) only the contribution of the pole
associated with a "real" meson in the intermediate state:

\[ \Delta m = \frac{1}{2\pi^2} \frac{G_S^2}{4\pi} \int \frac{2m - \omega}{(-2m\omega + \lambda^2)} \frac{d^3k}{2\omega(2\pi)^3} \]  

(34)

Furthermore, let us expand (34) in \(1/m\):

\[ \frac{2m + \omega}{(-2\omega + \lambda^2)} \approx \frac{1}{-\omega} \left(1 - \frac{\omega}{2m} + \frac{\lambda^2}{\omega}\right) \approx \frac{1}{-\omega - \frac{k^2}{2m}}. \]  

(35)

We see that we have an energy denominator which corresponds to the transition from a heavy particle at rest to state of the same particle and a meson of 3-momentum \(k\) with the kinetic energy of the heavy particle taken into account to first order in \(1/m\).

To get a KLN cancellation we should add a process with the energy denominator of the opposite sign. The problem is how to interpret this. In the language of the Feynman graphs a natural step is to add a graph with a complex conjugated propagator of the meson. In fact this rule for the initial state summation was derived in [4] through an analysis of the relevant graphs. Thus one adds

\[ \frac{-i}{k^2 - \lambda^2 + i\epsilon} \rightarrow \left(\frac{-i}{k^2 - \lambda^2 + i\epsilon}\right)^* \]  

(36)

Where, the overall sign of the propagator is changed and \(+i\epsilon\) is replaced by \(-i\epsilon\). Because of this change the poles associated with the meson propagator are now located at:

\[ (k_0)_{pole} = \pm (k^2 + \lambda^2)^{1/2} \pm i\epsilon \]  

(37)

Closing the contour of integration over \(k_0\) in the lower half plane we get what we denote by \((\Delta m)_{KLN}\)

\[ (\Delta m)_{KLN} = \frac{1}{2\pi^2} \frac{G_S^2}{4\pi} \int \frac{d^3k}{2\omega(2\pi)^3} \frac{2m + \omega}{2m\omega + \lambda^2} \]  

(38)

In other words \((\Delta m)_{KLN}\) can be obtained from the standard \(\Delta m\) by changing \(\omega \rightarrow -\omega\) (in the amplitude but not in the phase space factor since we have also changed the overall sign of the propagator). A physical interpretation of (38) can be given in terms of a "KLN vacuum" which is populated with light mesons according to the phase space factor \(d^3k/2\omega(2\pi)^3\).
Then $(\Delta m)_{KLN}$ represents a forward scattering amplitude of the heavy particle at rest, off the mesons. We see that indeed $\Delta m$ and $(\Delta m)_{KLN}$ cancel each other as far as the $1/\omega$ terms are concerned. This then gives a cancellation including terms proportional to $\lambda$.

This is not true, however, for the relativistic corrections. Indeed, it is straightforward to see that adding the amplitudes with $+\omega$ and $-\omega$ does not eliminate quadratic terms:

$$\frac{2m + \omega}{2m\omega + \lambda^2} + \frac{2m - \omega}{-2m\omega + \lambda^2} = \frac{-k^2}{m(\omega^2 + \lambda^4/4m^2)}.$$  \hfill (39)

This implies, that because of the relativistic corrections

$$\Delta m + (\Delta m)_{KLN} \sim G_S^2 \lambda^2 \log(\lambda).$$  \hfill (40)

The above is in accordance with our discussion in section 2 in that while order $\lambda$ terms have cancelled, $\lambda^2 \ln \lambda$ terms have not. We could attempt to improve the cancellation by looking for an exact counterpart of $\Delta m$, e.g. some further averaging. In an obvious manner, the energy denominator with opposite sign arises if one considers the collision of a heavy particle and a light meson with opposite 3-momentum $k$. Thus, one may say that this state is indicated by the small energy denominator itself. Moreover, to ensure a coherent addition of the amplitudes one has to introduce then a coherent mixture of heavy particles in the initial state. Since we have already adopted the "KLN vacuum" with light particles this might seem a reasonable extention of the procedure. However, as far as heavy decays are concerned, a moving particle lives longer because of the time dilatation. Thus, at least at first sight we should amend for this factor which would then again destroy the complete cancellation of the two process (that is, with the meson in the initial state and the standard mass correction). In this way we see that, if at all, one may continue with the KLN cancellations with relativistic corrections included, only at the price of introducing artificial appearing procedures.

In any case, the KLN cancellation cannot be imposed on terms $\sim \lambda^2 \log(\lambda)$. Indeed, let us go back to the example of $\Delta m$ due to the pseudoscalar interaction considered in the previous subsection. If we add $(\Delta m)_{KLN}$, that is the same expression as for $\Delta m$ but with $\omega \rightarrow -\omega$,
the terms proportional to \( \lambda^2 \log(\lambda) \) are doubled, not cancelled:

\[
\Delta m + (\Delta m)_{KLN} \approx \frac{G_P^2}{32\pi^2} \frac{\lambda^2}{m} \log(\lambda) \approx \frac{G_P^2}{2m^2} \cdot \langle \varphi^2 \rangle_{\text{pert}}
\]

Thus, in a sense the, first KLN non-cancelling terms are of two types, i.e., a kinematical correction due to the \( 1/m \) terms and that due to a mesonic "condensate" which produces the \( \lambda^2 \log(\lambda) \) terms. It is worth noting that actually the difference between these two kinds of terms might depend on the specific procedure of introducing the infrared sensitive parameter. In particular, the KLN summation was reduced first to adding graphs with complex conjugated propagators in Ref [4]. In this reference the meson (photon in that case) is kept strictly massless and IR parameters are introduced through limits of integration. In that case adding amplitudes with \( \pm \omega \) eliminates poles in any case and the remaining terms look similar to local condensates.

It should be remarked here that when we classify IR terms by power counting we assume that there is no kinematical suppression. The counting of powers of \( \lambda \) may change when this further suppression is properly included. Introduce, for example an anomalous magnetic moment of the decaying fermion \( \kappa \). Then the corresponding leading contribution to \( \Delta m \) is:

\[
\Delta m = -\alpha_{\text{el}} \frac{\kappa^2}{2m^2} (\lambda^2)^{3/2}.
\]

This contribution arises from \( 1/\omega \) terms in the expression for \( \Delta m \) and is in fact subject to KLN cancellations.

In summary, the results of this section are in agreement with the general considerations of section 2 in the sense that that at least the nonanalytical terms including those linear in \( \lambda \) cancel from the KLN sum. For purely kinematical reasons it is difficult to define the KLN cancellation for terms of order \( \lambda^2 \). Moreover, the non-analytical terms of order \( \lambda^2 \log(\lambda) \) are not subject to the KLN cancellation in any case. Indeed, in a sense, these may be considered as arising, due to near degeneracy of the vacuum manifested in a non-vanishing perturbative condensate \( \langle \varphi^2 \rangle \sim \lambda^2 \log(\lambda) \). For QED we found in section 2 that the first nonanalytic terms
in the KLN sum were not $\lambda^2 \ln \lambda$ but actually $\lambda^4 \ln \lambda$. Since each term with a given power of $\lambda$ is separately gauge invariant, this is consistent with the idea that the first nonanalytic terms are of the condensate type ($\Pi$), because in QED the gauge invariant condensate term must be $\langle F_{\mu\nu}^2 \rangle$ which involves four powers of the momenta.

## 5 Summary

In summary, both by general arguments and by explicit examples we have shown that KLN cancellations can be only partly extended to the case of power suppressed non-analytical terms. Namely, what is cancelled are the pole contributions. Generally speaking, they are responsible for the two leading IR terms, that is, terms of order $\lambda^0, \lambda^1$. (Although the counting can be modified by an explicit suppression in the infrared region as in the example of the anomalous magnetic moment). It is worth pointing out here certain characteristics of the nonanalytic terms contributing to $\Delta m$. As discussed in section 3, $\Delta m$ among other nonanalytic pieces contains one proportional to $\lambda^2 \ln \lambda$ and another proportional to $\lambda^3$. The latter term was shown to arise from the pole approximation in distinction to the quadratic term which was relativistic in origin. The discussion at the end of section 3 was in the context of a Bloch Nordsieck type mechanism where the entire $\Delta m$ contributions which are of order $\lambda^2 \ln \lambda$ and $\lambda^3$ are uncancelled. However because of the origin of the $\lambda^3$ term and the fact that the KLN sum cancels terms with $1/\omega$ denominators, (see section 2), we see that it will be cancelled if both initial and final degenerate states are averaged. The same is not true of the term proportional to $\lambda^2 \ln \lambda$ since its origin is the scalar condensate. In this sense the cubic term is the analogue in the scalar case of the anomalous magnetic moment interaction.

At the technical level, in this paper we also found a simple way to extract non-analytical terms from Feynman graphs, and in addition given a straightforward way of averaging over initial states in the KLN sum, utilizing and extending the results of Ref [4]. Although for the
explicit examples, we concentrated on weak decays of heavy particles it is rather obvious that the results are of more general validity and apply to any process. Also, two-loop and higher loop graphs can be treated in a similar way. However, the introduction of a non-vanishing boson mass restricts the applicability of our methods for explicit calculations in higher loops to abelian theories only, for reasons of consistency.

6 Acknowledgements

We would like to thank M. Beneke, V. Braun, V. Khoze, G. Marchesini, A. Mueller, A. Sirlin, G. Sterman, and A. Vainshtein for discussions. This work was supported in part by the US Department of Energy.

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