Flow equations in the light-front perturbation theory

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Abstract

The method of flow equations is applied to QED in the light-front dynamics. To second order in the coupling the particle number conserving part of the effective QED Hamiltonian has two terms of different structure. The first term gives the Coulomb interaction and the correct spin splittings of positronium; the contribution of the second term to mass spectrum depends on the explicit form of unitary transformation and may influence the spin-orbit.

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1 Introduction

In the previous work [4] we have outlined the strategy to construct an effective renormalized Hamiltonian by means of flow equations. Requiring that the particle number (Fock state) conserving terms in the Hamiltonian were considered to be diagonal and the other terms off-diagonal the block-diagonal effective Hamiltonian was obtained. The main advantage of this procedure is, that finally states of different particle number (Fock sectors) are completely decoupled in the effective Hamiltonian, and thus the bound state problem, which is in general a many-body one, reduces to a few-body problem. For the positronium this means, that one is left with a two-particle problem, since the particle number violating contributions are eliminated.

We perform the unitary transformation to bring the bare cutoff Hamiltonian of the gauge field theory to a block-diagonal form. We distinguish therefore the 'diagonal'-particle number (Fock state) conserving and the 'rest'-particle number (Fock state) changing sectors. The elimination of the "rest" part of Hamiltonian generates the new interactions in the "diagonal" sectors.

We study QED on the light-front in this approach. We use one of the most convenient prescription of light-front perturbation theory as formulated by Brodsky and Lepage [5]. In second order in coupling there is a new interaction between electron and positron that arise from the eliminating matrix elements of electron-photon vertex.

Making use of Brodsky, Lepage light-front perturbation theory it is possible to separate two different structures in the effective electron-positron interaction. The contribution of both of them to the mass spectrum is considered.

2 Flow equations and the effective Hamiltonian

Flow equations perform the unitary transformation, which brings the Hamiltonian to a block-diagonal form with the number of particles (or Fock state) conserving in each block. In what follows we distinguish between the 'diagonal' (here Fock state conserving) and 'rest' (Fock state changing) sectors of the Hamiltonian. We break the Hamiltonian as

\[ H = H_{0d} + H_d + H_r \]  

where \( H_{0d} \) is the free Hamiltonian; and the indices 'd','r' correspond to 'diagonal','rest' parts of the Hamiltonian, respectively. The flow equation for the Hamiltonian and the generator of unitary transformation are written [4]

\[
\frac{dH}{dl} = [\eta, H_d + H_r] + [[H_d, H_r], H_{0d}] + [[H_{0d}, H_r], H_{0d}] \\
\eta = [H_{0d}, H_r] + [H_d, H_r] 
\]  

In the basis of the eigenfunctions of the free Hamiltonian

\[ H_{0d}|i> = E_i|i> \]  

one obtains for the matrix-elements between the many-particle states

\[
\frac{dH_{ij}}{dl} = [\eta, H_d + H_r]|_{ij} - (E_i - E_j)[H_d, H_r]|_{ij} - (E_i - E_j)^2 H_{rij} \\
\eta_{ij} = (E_i - E_j)H_{rij} + [H_d, H_r]|_{ij} 
\]
The energy differences are given by
\[ E_i - E_j = \sum_{k=1}^{n_2} E_{i,k} - \sum_{k=1}^{n_1} E_{j,k} \]  
where \( E_{i,k} \) and \( E_{j,k} \) are the energies of the created and annihilated particles, respectively. The generator belongs to the 'rest' sector, i.e. \( \eta_{ij} = \eta_{rij}, \eta_{dij} = 0 \). In what follows we use
\[
[\hat{O}_r, \hat{H}_d] = 0 \\
[\hat{O}_r, \hat{H}_d] \neq 0
\]  
where \( \hat{O}_r \) is the operator from the 'rest' sector (for example \( \hat{H}_r \) or \( \hat{\eta}_r \)) and \( \hat{H}_d \) is the diagonal part of Hamiltonian.

For the 'diagonal' \( (n_1 = n_2) \) and 'rest' \( (n_1 \neq n_2) \) sectors of the Hamiltonian one has correspondingly
\[
\frac{dH_{dij}}{dl} = [\eta, H_{dij}] \\
\frac{dH_{rij}}{dl} = [\eta, H_d + H_r]_{rij} - (E_i - E_j)[H_d, H_r]_{rij} + \frac{du_{ij}}{dl} H_{rij}
\]  
where we have introduced the cutoff function \( u_{ij}(l) \)
\[ u_{ij}(l) = e^{-(E_i - E_j)^2l} \]  
For simplicity we neglect the dependence of the energy \( E_i \) on the flow parameter \( l \). The main difference between these two sectors is the presence of the third term in the 'rest' sector \( \frac{du_{ij}}{dl} H_{rij} \), which insures the band-diagonal structure for the 'rest' part
\[ H_{rij} = u_{ij} \tilde{H}_{rij} \]  
i.e. in the 'rest' sector only the matrix elements with the energy difference \( |E_i - E_j| < 1/\sqrt{l} \) are not zero. In the similarity renormalization scheme \[2\] the width of the band corresponds to the UV cutoff \( \lambda \). The connection between the two quantities is given
\[ l = \frac{1}{\lambda^2} \]  
The matrix elements of the interactions, which change the Fock state, are strongly suppressed, if the energy difference exceeds \( \lambda \), while for the Fock state conserving part of the Hamiltonian the matrix elements with all energy differences are present. As the flow parameter \( l \rightarrow \infty \) (or \( \lambda \rightarrow 0 \)) the 'rest' part is completely eliminated, except maybe for the matrix elements with \( i = j \). One is left with the block-diagonal effective Hamiltonian.

Generally, the flow equations are written
\[
\frac{dH_{ij}}{dl} = [\eta, H_d + H_r]_{ij} - (E_i - E_j)[H_d, H_r]_{ij} + \frac{du_{ij}}{dl} H_{ij} \\
\eta_{ij} = [H_d, H_r]_{ij} + \frac{1}{E_i - E_j} \left( \frac{du_{ij}}{dl} H_{ij} \right)
\]  
where the following condition on the cutoff function in 'diagonal' and 'rest' sectors, respectively, is imposed
\[
u_{dij} = 1 \\
u_{rij} = u_{ij}
\]
One recovers with this condition the flow equations eq. (7) for both sectors. Other unitary transformations, which bring the Hamiltonian to the block-diagonal form, with the Fock state conserving in each block, are used \[2\]

\[
\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_{d} + H_{r}]_{ij} + r_{ij} \frac{du_{ij} H_{ij}}{d\lambda} u_{ij}
\]

\[
\eta_{ij} = \frac{r_{ij}}{E_{i} - E_{j}} \left( [\eta, H_{d} + H_{r}]_{ij} - \frac{du_{ij} H_{ij}}{d\lambda} u_{ij} \right)
\]

(13)

and \[3\]

\[
\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_{d} + H_{r}]_{ij} + \frac{du_{ij} H_{ij}}{d\lambda} u_{ij}
\]

\[
\eta_{ij} = \frac{1}{E_{i} - E_{j}} \left( r_{ij} [\eta, H_{d} + H_{r}]_{ij} - \frac{du_{ij} H_{ij}}{d\lambda} u_{ij} \right)
\]

(14)

where \(u_{ij} + r_{ij} = 1\); and the constrain eq. (12) on the cutoff function in both sectors is implied. One can choose the sharp cutoff function

\[
u_{ij} = \theta(\lambda - |\Delta_{ij}|).
\]

We consider the flow equations in the perturbative frame, therefore we break the Hamiltonian

\[H = H_{0d} + \sum_{n}(H_{d}^{(n)} + H_{r}^{(n)})\]

(15)

where \(H^{(n)} \sim e^{n}\), \(e\) is the bare coupling constant (here we do not refer to the definite field theory). To the order of \(n\) in coupling constant the flow equations in both sectors are given

\[
\frac{dH_{dij}^{(n)}}{dl} = \sum_{k}[\eta^{(k)}, H_{r}^{(n-k)}]_{dij}
\]

\[
\eta_{dij}^{(n)} = 0
\]

\[
\frac{dH_{rij}^{(n)}}{dl} = \sum_{k}[\eta^{(k)}, H_{d}^{(n-k)} + H_{r}^{(n-k)}]_{rij} - (E_{i} - E_{j}) \sum_{k}[H_{d}^{(k)}, H_{r}^{(n-k)}]_{rij} + \frac{du_{ij} H_{rij}^{(n)}}{dl} u_{ij}
\]

\[
\eta_{rij}^{(n)} = \frac{1}{E_{i} - E_{j}} \left( r_{ij} [\eta, H_{d} + H_{r}]_{rij} - \frac{du_{ij} H_{rij}^{(n)}}{dl} u_{ij} \right)
\]

\[
\eta^{(n)} = \eta_{d}^{(n)} + \eta_{r}^{(n)}
\]

(16)

We solve these equations in the leading order for the Fock state conserving sector.

\[
\frac{dH_{dij}^{(2)}}{dl} = [\eta_{r}^{(1)}, H_{r}^{(1)}]_{dij}
\]

\[
\eta_{dij}^{(1)} = -\frac{1}{E_{i} - E_{j}} \frac{dH_{dij}^{(1)}}{dl}
\]

\[
\frac{dH_{rij}^{(1)}}{dl} = \frac{du_{ij} H_{rij}^{(1)}}{dl} \frac{dH_{rij}^{(1)}}{dl} u_{ij}
\]

(17)

and \(H_{rij}^{(1)} = H_{rij}^{(1)}\). Explicitly one has

\[
\frac{dH_{dij}^{(2)}}{dl} = -\sum_{k} \left( \frac{1}{E_{i} - E_{k}} \frac{dH_{dik}^{(1)}}{dl} H_{rjk}^{(1)} + \frac{1}{E_{j} - E_{k}} \frac{dH_{rjk}^{(1)}}{dl} H_{dik}^{(1)} \right) d
\]

\[
H_{rij}^{(1)}(l) = H_{rij}^{(1)}(l = 0) \frac{f_{ij}(l)}{f_{ij}(l = 0)}
\]

(18)
where we have introduced the function $f_{ij}$ defining the leading order solution for the 'rest' part. Further we refer to it as similarity function. Here

$$f_{ij}(l) = u_{ij}(l) = e^{-(E_i - E_j)^2l}$$  \hspace{1cm} (19)

The similarity function $f_{\lambda}(\Delta)$ has the same behavior (when $\lambda \rightarrow \infty \quad f_{\lambda}(\Delta) = 1$, and when $\lambda \rightarrow 0 \quad f_{\lambda}(\Delta) = 0$ as the cutoff function $u_{\lambda}(\Delta)$; in the leading order the similarity function shows how fast the 'rest' sector is eliminated.

Making use of the connection $l = 1/\lambda^2$, we get

$$\frac{dH^{(2)}_{dij}}{d\lambda} = - \sum_k \left( \frac{1}{E_i - E_k} \frac{dH^{(1)}_{r_{ik}}}{d\lambda} H^{(1)}_{r_{jk}} + \frac{1}{E_j - E_k} \frac{dH^{(1)}_{r_{jk}}}{d\lambda} H^{(1)}_{r_{ik}} \right) \delta_{ij}$$

$$H^{(1)}_{r_{ij}}(\lambda) = H^{(1)}_{r_{ij}}(\Lambda \rightarrow \infty) \frac{f_{ij}(\lambda)}{f_{ij}(\Lambda \rightarrow \infty)}$$  \hspace{1cm} (20)

The equations eq. (20) are the same also for the unitary transformations given above eqs. (13) and (14) up to the choice of the function $f_{ij}(\lambda)$. Neglecting the dependence of the energy $E_i$ on the cutoff, one has

$$H^{(2)}_{d_{ij}}(\lambda) = H^{(2)}_{d_{ij}}(\Lambda \rightarrow \infty) + \left( \sum_k H^{(1)}_{r_{ik}}(\Lambda \rightarrow \infty) H^{(1)}_{r_{jk}}(\Lambda \rightarrow \infty) \right) \delta_{ij}$$

$$\times \left( \frac{1}{E_i - E_k} \int_{\lambda}^{\infty} \frac{df_{ik}(\lambda')}{d\lambda'} f_{jk}(\lambda') d\lambda' + \frac{1}{E_j - E_k} \int_{\lambda}^{\infty} \frac{df_{jk}(\lambda')}{d\lambda'} f_{ik}(\lambda') d\lambda' \right) \delta_{ij}$$  \hspace{1cm} (21)

where $\Lambda$ is the bare cutoff; the sum $\sum_k$ is over all intermediate states; and the label 'd' denotes the 'diagonal' sector. Specifying the function $f_{ij}$ we get the interaction $H^{(2)}_{d}$, generated by different unitary transformations. One has corresponding for the flow equations eq. (11)

$$f_{ij}(\lambda) = u_{ij}(\lambda) = e^{-\Delta_{ij}^2/\lambda^2}$$  \hspace{1cm} (22)

and for the unitary transformations eqs. (13) and (14)

$$f_{ij}(\lambda) = u_{ij}(\lambda)e^{r_{ij}(\lambda)}$$

$$f_{ij}(\lambda) = u_{ij}(\lambda)$$

$$u_{ij} = \theta(\lambda - |\Delta_{ij}|), \quad u_{ij} + r_{ij} = 1$$  \hspace{1cm} (23)

where $\Delta_{ij} = \sum_{k=1}^{n_2} E_{i,k} - \sum_{k=1}^{n_1} E_{j,k}$.

To illustrate the method we consider QED on the light-front in the next section. We calculate the generated interaction eq. (21) in the electron-positron sector to solve for the positronium mass spectrum. In this case $H^{(1)}_{r}(\Lambda \rightarrow \infty)$ is the electron-photon vertex with the bare coupling constant $e$; and the initial value of generated interaction is $H^{(2)}_{d}(\Lambda \rightarrow \infty) = 0$.

3 Renormalized effective electron-positron interaction

In this section we give the effective Hamiltonian in the light front dynamics for the positronium system, generated by the unitary transformation $\mathbb{H}$.

The light front Schrödinger equation for the positronium model reads

$$H_{LC}|\psi_n> = M_n^2|\psi_n>$$  \hspace{1cm} (24)
where $H_{LC} = P^\mu P_\mu$ is the invariant mass (squared) operator, referred for convenience as the light front Hamiltonian of positronium and $|\psi_n>$ is the corresponding eigenfunction.

The canonical Hamiltonian of the system $H_{LC}$ is in general an infinite dimensional matrix, namely contains infinite many Fock sectors (i.e. one has for the positronium wave function $|\psi_n> = ce\bar{e}[(e\bar{e})_n] + e\bar{e}\gamma[(e\bar{e}\gamma)_n] + ...$) and the states with infinite large energies. We obtain a finite dimensional Hamiltonian by:

1. introducing the bare cutoff (regularization) with the result $H_{LC}^B(\Lambda)$ - the bare Hamiltonian;
2. performing the unitary transformation with the result $H_{LC}^{eff}$ - the effective renormalized Hamiltonian;
3. truncating the Fock space to the lowest Fock sector ($|e\bar{e}>$) with the result $\tilde{H}_{LC}^{eff}$ - the effective renormalized Hamiltonian acting in the electron-positron sector.

The eigenvalue equation is written then

$$\tilde{H}_{LC}^{eff} |(e\bar{e})_n> = M_n^2 |(e\bar{e})_n>$$

(25)

where $n$ labels all quantum numbers. The effective light front Hamiltonian is splitted into the free (noninteracting) part and effective electron-positron interaction

$$\tilde{H}_{LC}^{eff} = H_{LC}^{(0)} + V_{LC}^{eff}$$

(26)

The continuum version of the light front equation eq. (25) is then expressed by the integral equation (Jacobi momenta are introduced on fig. 1)

$$\left( M_n^2 - \frac{m^2 + \vec{k}_\perp^2}{x(1 - x)} \right) \psi_n(x, \vec{k}_\perp; \lambda_1, \lambda_2) = \sum_{\lambda_1', \lambda_2'} \int_D \frac{dx'd^2\vec{k}_\perp'}{2(2\pi)^3} <x, \vec{k}_\perp; \lambda_1, \lambda_2|V_{LC}^{eff} |x', \vec{k}'_\perp; \lambda_1', \lambda_2' > \psi_n(x', \vec{k}'_\perp; \lambda_1', \lambda_2')$$

(27)

where the wave function is normalized

$$\sum_{\lambda_1, \lambda_2} \frac{dx^2d^2\vec{k}_\perp}{2(2\pi)^3} \psi^*(x, \vec{k}_\perp; \lambda_1, \lambda_2)\psi_n(x, \vec{k}_\perp; \lambda_1, \lambda_2) = \delta_{nn'}$$

(28)

The integration domain $D$ is restricted by the covariant cutoff condition of Brodsky and Lepage

$$\frac{m^2 + \vec{k}_\perp^2}{x(1 - x)} \leq \Lambda^2 + 4m^2$$

(29)

which allows for states to have the kinetic energy below the bare cutoff $\Lambda$.

For the effective electron-positron interaction one has in the exchange and annihilation channels

$$V_{LC}^{eff} = V_{exch} + V_{ann} = \sum_{\text{channel}} \lim_{\lambda \to 0} (V_{\lambda}^{gen} + V_{\lambda}^{PT} + V_{\lambda}^{inst})$$

(30)

where the terms in eq. (30) correspond to the interaction generated by the unitary transformation eq. (21), the perturbative photon exchange with the energy below the cutoff and the instantaneous exchange term arising in the light-front gauge, respectively. Here $\lambda$ is the ‘running’ cutoff, that defines the ‘continuous’ step of the unitary transformation $U(\lambda, \Lambda)$ and related to the flow parameter $l$ as $l = 1/\lambda^2$. 

6
Following the rules of light-front perturbative theory \[5\] one has for the interaction eq. (21), generated in the second order \(O(e^2)\) in the electron-positron sector

\[
V^{\text{gen}}_\lambda = -e^2 < \gamma^\mu \gamma^\nu > \\
\times \left\{ \theta(q^+) D_{\mu\nu}(q) \left( \frac{1}{D_1} \int_\lambda^\infty \frac{df^{\lambda'}(D_1)}{d\lambda'} f^{\lambda'}(D_2) d\lambda' + \frac{1}{D_2} \int_\lambda^\infty \frac{df^{\lambda'}(D_2)}{d\lambda'} f^{\lambda'}(D_1) d\lambda' \right) \\
+ \theta(-q^+) D_{\mu\nu}(-q) \left( \frac{1}{-D_1} \int_\lambda^\infty \frac{df^{\lambda'}(-D_1)}{d\lambda'} f^{\lambda'}(-D_2) d\lambda' + \frac{1}{-D_2} \int_\lambda^\infty \frac{df^{\lambda'}(-D_2)}{d\lambda'} f^{\lambda'}(-D_1) d\lambda' \right) \right\}
\]

(31)

where we sum \((\Sigma_k \text{ in eq. (21)})\) the two terms corresponding to the two time-ordered diagrams; \(f_\lambda(\Delta)\) is the similarity function, arising from the unitary transformation and specified below; \(D_{\mu\nu}(q) = \frac{q^{\mu} q^{\nu}}{q^2} \eta_{\mu\nu} + \frac{i}{q^2} (\eta_{\mu} q^\nu - \eta_{\nu} q^\mu) - g^\mu_{\mu\nu}\) is the photon propagator in light-front gauge \[6\], \(\eta_{\mu} = (0, \eta^+ = 1, 0, 0)\); \(D_1, D_2\) and \(D\) are energy denominators given below; \(q\) is the exchanged momentum. The notation \(< \gamma^\mu \gamma^\nu >\) is introduced for the matrix element given in exchange channel as fig. \[3\]

\[
< \gamma^\mu \gamma^\nu > |_{\text{exch}} = \frac{\bar{u}(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1) \bar{v}(p_2, \lambda_2) \gamma^\nu v(p_2, \lambda_2)}{\sqrt{p_1^+} \sqrt{p_1^+} \sqrt{p_2^+} \sqrt{p_2^+}} P^{2+}
\]

(32)

where \(p_i, p'_i\) are light-front three-momenta carried by the constituents, \(\lambda_i, \lambda'_i\) are their light-front helicities, \(u(p_1, \lambda_1), v(p_2, \lambda_2)\) are their spinors \[3\]; index \(i = 1, 2\) refers to electron and positron, respectively; \(P = (P^+, P^\perp)\) is light-front positronium momentum. (For the annihilation channel see Appendix B).

Making use of the symmetry

\[
f_\lambda(-D) = f_\lambda(D) \\
D_{\mu\nu}(-q) = D_{\mu\nu}(q)
\]

(33)

we have the following electron-positron interaction generated by the unitary transformation

\[
V^{\text{gen}}_\lambda = -e^2 < \gamma^\mu \gamma^\nu > \frac{1}{q^+} D_{\mu\nu}(q) \left( \frac{1}{D_1} \int_\lambda^\infty \frac{df^{\lambda'}(D_1)}{d\lambda'} f^{\lambda'}(D_2) d\lambda' + \frac{1}{D_2} \int_\lambda^\infty \frac{df^{\lambda'}(D_2)}{d\lambda'} f^{\lambda'}(D_1) d\lambda' \right)
\]

(34)

We combine all the interactions eq. (30) together \[3\]

\[
V^{\text{gen}}_\lambda = -e^2 < \gamma^\mu \gamma^\nu > \frac{1}{q^+} D_{\mu\nu}(q) \left( \frac{1}{D_1} \int_\lambda^\infty \frac{df^{\lambda'}(D_1)}{d\lambda'} f^{\lambda'}(D_2) d\lambda' + \frac{1}{D_2} \int_\lambda^\infty \frac{df^{\lambda'}(D_2)}{d\lambda'} f^{\lambda'}(D_1) d\lambda' \right) \\
V^{PT}_\lambda = -e^2 < \gamma^\mu \gamma^\nu > \frac{1}{q^+} D_{\mu\nu}(q) \frac{1}{D} f_\lambda(D_1) f_\lambda(D_2) \\
V^{\text{inst}} = -e^2 < \gamma^\mu \gamma^\nu > \frac{1}{q^+} \eta_{\mu\nu}
\]

(35)

The energy denominators in the generated interaction \(V^{\text{gen}}_\lambda\) and the exchanged momentum are in the exchange channel

\[
D_1^{\text{exch}} = \Delta_{p_1, p_1}; \quad D_2^{\text{exch}} = \Delta_{p_2, p'_2}; \quad q^{\text{exch}} = p'_1 - p_1 = q
\]

(36)

in the annihilation channel

\[
D_1^{\text{ann}} = \Delta_{p'_1, -p'_2}; \quad D_2^{\text{ann}} = \Delta_{p_2, -p_1}; \quad q^{\text{ann}} = p_1 + p_2 = P
\]

(37)
where the notation is introduced $\Delta_{p_1 p_2} = p_1^- - p_2^- - (p_1 - p_2)^-$. The energy denominator in the
perturbative term $V^{PT}_\lambda$ is given
\[ D = \sum_{\text{inc}} p^- - \sum_{\text{interm}} p^- \] (38)
where the sums are over the light-front energies, $p^-$, of the incident (inc) and intermediate (interm) particles.

In what follows we use Jacobi momenta fig. (1)
\[ p_1(xP^+, x\vec{P}_\perp + \vec{k}_\perp) \]
\[ p_2((1 - x)P^+, (1 - x)\vec{P}_\perp - \vec{k}_\perp) \] (39)
and corresponding for the momenta $p_1', p_2'$; here $x$ is the light-front fraction of electron momentum and $P(P^+, \vec{P}_\perp)$ is the total momentum of positronium. For convenience introduce
\[ D^{\text{exch}}_1 = \frac{\Delta_1}{P^+} = -\frac{\tilde{\Delta}_1}{q^+}, \quad D^{\text{exch}}_2 = \frac{\Delta_2}{P^+} = -\frac{\tilde{\Delta}_2}{q^+}, \quad D^{\text{exch}} = -\frac{\tilde{\Delta}_3}{q^+} \]
\[ D^{\text{ann}}_1 = \frac{M^2}{P^+}, \quad D^{\text{ann}}_2 = \frac{M'_2}{P^+}, \quad D^{\text{ann}} = \frac{M^2}{P^+} \] (40)
and from now on we use the rescaled value of the cutoff $\lambda \rightarrow \lambda^2/P^+$. Then the following terms contribute to the effective electron-positron interaction fig. (1)
in the exchange channel
\[ V^{\text{gen}}_\lambda = -e^2 N_1 \left( \frac{1}{\Delta_1} \int_\lambda^\infty \frac{d\lambda'}{d\lambda} f_{\lambda'}(\Delta_1) d\lambda' + \frac{1}{\Delta_2} \int_\lambda^\infty \frac{d\lambda'}{d\lambda} f_{\lambda'}(\Delta_2) d\lambda' \right) \]
\[ V^{\text{PT}}_\lambda = -e^2 N_1 \frac{1}{\Delta_3} f_{\lambda}(\Delta_1) f_{\lambda}(\Delta_2) \]
\[ V^{\text{inst}}_\lambda = -e^2 <\gamma^\mu \gamma^\nu>_{\text{exch}} \eta_\mu \eta_\nu \frac{1}{q^{+2}} \] (41)
in the annihilation channel
\[ V^{\text{gen}}_\lambda = e^2 N_2 \left( \frac{1}{M^2_0} \int_\lambda^\infty \frac{d\lambda'}{d\lambda} f_{\lambda'}(M^2_0) d\lambda' + \frac{1}{M'_2} \int_\lambda^\infty \frac{d\lambda'}{d\lambda} f_{\lambda'}(M'_2) d\lambda' \right) \]
\[ V^{\text{PT}}_\lambda = e^2 N_2 \frac{1}{M^2_0} f_{\lambda}(M^2_0) f_{\lambda}(M'_2) \]
\[ V^{\text{inst}}_\lambda = -e^2 <\gamma^\mu \gamma^\nu>_{\text{ann}} \eta_\mu \eta_\nu \frac{1}{P^{+2}} \] (42)
where
\[ N_1 = -<\gamma^\mu \gamma^\nu>_{\text{exch}} D_{\mu\nu}(q) \]
\[ N_2 = -<\gamma^\mu \gamma^\nu>_{\text{ann}} D_{\mu\nu}(P) \] (43)
$q = p_1' - p_1$ is the exchanged photon momentum, with $q^- = \frac{q^{+2}}{q^+}$; and $P = p_1 + p_2$ is the total momentum. The matrix elements of the effective interaction: current-current terms $N_1, N_2$ and $<\gamma^\mu \gamma^\nu> \eta_\mu \eta_\nu$ in both channels, are calculated according to the rules of light-front perturbation.
theory as formulated by Brodsky, Lepage [5]. (see Appendices A and B for the exchange and annihilation channels, respectively). The energy denominators in eq. (41) read
\[ \tilde{\Delta}_1 = \frac{(xk_1 - x'k_1')^2 + m^2(x - x')^2}{xx'}; \quad \tilde{\Delta}_2 = \tilde{\Delta}_1 \big|_{x \rightarrow (1 - x), x' \rightarrow (1 - x')} \]
\[ \tilde{\Delta}_3 = (k_\perp - k'_\perp)^2 + \frac{(x - x')^2}{2} \left( \frac{1}{1 - x} - \frac{1}{x} \right) - k_\perp^2 \left( \frac{1}{1 - x'} - \frac{1}{x'} \right) \]
\[ + m^2 \left( \frac{1}{xx'} + \frac{1}{(1 - x)(1 - x')} \right) + |x - x'| \left( \frac{1}{2}(M_0^2 + M'_0^2) - M_n^2 \right) \]
\[ M_0^2 = \frac{k_\perp^2 + m^2}{x(1 - x)}; \quad M'_0 = \frac{k'_\perp^2 + m^2}{x'(1 - x')} \]
\[ P^- = \frac{(P^\perp)^2 + M_n^2}{P^+}; \quad P = (P^+, P^\perp) \] (44)

Note \( \tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3 \) are positive defined.

The exchange channel brings the dominant contribution to the mass spectrum. In what follows we focus on the effective interaction in the exchange channel.

One can simplify the current-current term in the exchange channel, \( N_1 \). Note, that for the vector \( [q_\mu - (p'_{1\mu} - p_{1\mu})] \)' and \( ' \perp \)' components vanish by momentum conservation, i.e. it is proportional to the null vector \( \eta_\mu \). Therefore one can represent
\[ q_\mu = p'_{1\mu} - p_{1\mu} - \eta_\mu \frac{D_1}{2} \]
\[ q_\nu = p_{2\nu} - p'_{2\nu} - \eta_\nu \frac{D_2}{2} \] (45)

Making use of the Dirac equation, one has
\[ \bar{\pi}(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1)\bar{\pi}(p'_2, \lambda'_2) \gamma^\nu v(p_2, \lambda_2) \times (\eta_\mu q_\nu + \eta_\nu q_\mu) \]
\[ = -\bar{\pi}(p_1, \lambda_1) \gamma^\mu u(p'_1, \lambda'_1)\bar{\pi}(p'_2, \lambda'_2) \gamma^\nu v(p_2, \lambda_2) \eta_\mu \eta_\nu \left( \frac{D_1 + D_2}{2} \right) \] (46)

Then the current-current term in the exchange channel can be written
\[ N_1 = -<\gamma^\mu \gamma^\nu > D_{\mu\nu}(q) \rightarrow <\gamma^\mu \gamma^\nu > g_{\mu\nu} - <\gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{2q^+} \] (47)

we omit the label 'exch'- exchange channel.

Below we use the approximation
\[ M_n^2 = \frac{1}{2}(M_0^2 + M'_0^2) \] (48)

that simplifies the calculations. In this approximation the following holds
\[ \tilde{\Delta} = \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{2} \] (49)
where now the energy denominator for the perturbative photon exchange reads

\[ \Delta = \tilde{\Delta}_3(M_n^2 = \frac{1}{2}(M_0^2 + M_n^2)) \]

\[ = (\vec{k}_\perp - \vec{k}'_\perp)^2 + \frac{(x - x')}{2} \left( \frac{1}{1-x} - \frac{1}{1-x'} \right) \left( \frac{1}{1-x} - \frac{1}{1-x'} \right) \]

\[ + m^2 \frac{(x - x')^2}{2} \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) \]

(50)

Combining all the terms eq. (11) together one has for the effective interaction in the exchange channel

\[ V_{LC}^{\text{eff}}(\lambda) = V_{\Lambda}^{\text{gen}} + V_{\Lambda}^{\text{PT}} + V_{\lambda}^{\text{inst}} \]

\[ = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\Theta_{1\lambda}}{\Delta_1} + \frac{\Theta_{2\lambda}}{\Delta_2} + \frac{f_{1\lambda} f_{2\lambda}}{\Delta} \right) \]

\[ - e^2 < \gamma^\mu \gamma^\nu > \eta_{\mu\nu} \frac{1}{2q^2} (\Delta_1 - \Delta_2) \left( \frac{\Theta_{1\lambda}}{\Delta_1} - \frac{\Theta_{2\lambda}}{\Delta_2} \right) \]

(51)

where we have introduced

\[ \Theta_{1\lambda} = \int_{\lambda}^{\Delta_1} \frac{df_{\lambda'}(\Delta_1)}{d\lambda'} f_{\lambda'}(\Delta_1) d\lambda' \]

\[ \Theta_{2\lambda} = \int_{\lambda}^{\Delta_2} \frac{df_{\lambda'}(\Delta_2)}{d\lambda'} f_{\lambda'}(\Delta_1) d\lambda' \]

\[ f_{1\lambda} = f_{\lambda}(\Delta_1); \ f_{2\lambda} = f_{\lambda}(\Delta_2) \]

(52)

and one has

\[ \Theta_{1\lambda} + \Theta_{2\lambda} = 1 - f_{1\lambda} f_{2\lambda} \]

(53)

Remind, that \( f_{\lambda}(\Delta) \) is the similarity function specified below.

Instantaneous exchange with the energy below the cutoff \( e^2 < \gamma^\mu \gamma^\nu > \eta_{\mu\nu} \frac{1}{q^2} f_{1\lambda} f_{2\lambda} \) is canceled in the effective interaction \( V_{LC}^{\text{eff}}(\lambda) \) eq. (51).

In the eq. (51) the ”\( \eta_{\mu\nu} \)” term is spin-independent, and it is at least one power of momenta higher than the leading spin-independent piece in the ”\( g_{\mu\nu} \)” term. This form of the effective interaction eq. (51) is convenient to separate the structures: ”\( \eta_{\mu\nu} \)” term, containing the collinear singularity when \( x \sim x' \), from the ”\( g_{\mu\nu} \)” term.

The initial value of the effective interaction is given at the bare cutoff \( \Lambda \)

\[ V_{LC}^{\text{eff}}(\lambda = \Lambda \to \infty) = V_{\Lambda=\Lambda \to \infty}^{\text{PT}} + V_{\lambda}^{\text{inst}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta} \]

(54)

This is the result of light-front perturbative theory [3]. Note, that though \( V_{\lambda}^{\text{inst}} \) is singular as \( x \sim x' \), the interaction \( V_{\Lambda=\Lambda \to \infty}^{\text{PT}} + V_{\lambda}^{\text{inst}} \) is free of collinear singularity.

The resulting value of the effective interaction \( V_{LC}^{\text{eff}} \) is defined at \( \lambda \to 0 \) (except maybe for the point of Coulomb singularity \( q(q_z, \vec{q}_\perp) = 0 \), i.e. \( (x = x', \vec{k}_\perp = \vec{k}'_\perp) \))

\[ V_{LC}^{\text{eff}} = V_{LC}^{\text{eff}}(\lambda \to 0) = V_{\Lambda=0}^{\text{gen}} + V_{\lambda}^{\text{inst}} \]

\[ = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\Theta_1}{\Delta_1} + \frac{\Theta_2}{\Delta_2} \right) \]

\[ - e^2 < \gamma^\mu \gamma^\nu > \eta_{\mu\nu} \frac{1}{2q^2} (\Delta_1 - \Delta_2) \left( \frac{\Theta_1}{\Delta_1} - \frac{\Theta_2}{\Delta_2} \right) \]

(55)
where
\[
\Theta_1 = \Theta_{1\lambda}|_{\lambda=0} \\
\Theta_2 = \Theta_{2\lambda}|_{\lambda=0}
\] (56)

Estimate the difference between the initial and final values of the effective interaction. Introduce \( \tilde{\Delta}_1 - \tilde{\Delta}_2 = 2\delta \), then together with eq. (49) one has
\[
\tilde{\Delta}_1 = \tilde{\Delta} + \delta; \quad \tilde{\Delta}_2 = \tilde{\Delta} - \delta
\]
(57)

where usually (in the nonrelativistic case and in the collinear limit \( x \sim x' \)) holds \( |\delta| << \tilde{\Delta} \). For the effective interaction one has
\[
V_{\text{eff, LC}}^{(i)} = -e^2 <\gamma^\mu \gamma^\nu> g_{\mu\nu} \frac{1}{\Delta} \left( 1 - (\Theta_1 - \Theta_2) \frac{\delta}{\Delta} + O\left(\frac{\delta^2}{\Delta^2}\right) \right)
\]
(58)

index above in \( \Delta V^{(i)} \) shows the order of expansion with respect to \( (\delta/\Delta) \). The leading order in eq. (58) is given by the result of perturbation theory
\[
V^{(0)} = -e^2 <\gamma^\mu \gamma^\nu> \eta_{\mu} \eta_{\nu} \frac{1}{q^2} \left( (\Theta_1 - \Theta_2) \frac{\delta}{\Delta} + O\left(\frac{\delta^2}{\Delta^2}\right) \right)
\] (59)

We change the variables, which defines \( p_z : \ (x, k_{\perp}) \rightarrow \tilde{p}(p_z, k_{\perp}) \), \( \tilde{p} \) is the three momentum in the center of mass frame
\[
x = \frac{1}{2} \left( 1 + \frac{p_z}{\sqrt{p_z^2 + m^2}} \right)
\] (60)

The term eq. (59) gives in the leading order of nonrelativistic approximation \( |\tilde{p}|/m << 1 \) the 3-dimensional Coulomb interaction
\[
V^{(0)} \approx -\frac{16e^2m^2}{q^2}
\] (61)

where \( q_z \) is the exchanged momentum.

It was shown in [7], [8], that the nonrelativistic expansion of the term eq. (59) to the second order \( O\left((\frac{q_z^2}{m^2})^2\right) \) gives rise to the correct spin splittings of positronium and the rotational invariance (due to the degeneracy of triplet states) is restored.

Corrections \( \Delta V_{g_{\mu\nu}}^{(1)}; \Delta V_{\eta_{\mu}\eta_{\nu}}^{(1)} \) arise due to the unitary transformation, i.e. the corrections due to the energy denominators in the "\( g_{\mu\nu} \)" term and the "\( \eta_{\mu}\eta_{\nu} \)" term. Estimate the first order corrections \( \Delta V_{g_{\mu\nu}}^{(1)}; \Delta V_{\eta_{\mu}\eta_{\nu}}^{(1)} \ O\left(\frac{\delta}{\Delta}\right) \) in the nonrelativistic case. We choose for this purpose the explicit form of the similarity function
\[
f_\lambda(\Delta) = e^{-\frac{\Delta^2}{\lambda^2}}
\] (62)
Then one has
\[ \Theta_1 = \frac{\Delta_1^2}{\Delta_1^2 + \Delta_2^2} \] (63)
and
\[ V_{LC}^{eff} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{\Delta_1 + \Delta_2}{\Delta_1^2 + \Delta_2^2} - e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{2q_{\nu}^2} \frac{(\Delta_1 - \Delta_2)^2}{\Delta_1^2 + \Delta_2^2} \] (64)

The series eq. (58) for the effective interaction then reads
\[ V_{LC}^{eff} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta} \left( 1 - \frac{\delta^2}{\Delta^2} + O\left(\frac{\delta^3}{\Delta^3}\right) \right) \]
\[ -e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q_{\nu}^2} \left( \frac{\delta^2}{\Delta^2} + O\left(\frac{\delta^3}{\Delta^3}\right) \right) \] (65)
where \( \delta \approx -\frac{2\pi}{m}(\vec{q}, \vec{p} + \vec{p}') \).

The corrections are given correspondingly
\[ \Delta V_{g_{\mu\nu}}^{(1)} = e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta} \frac{\delta^2}{\Delta^2} \approx \frac{16e^2m^2}{q^2} \left( \frac{q_z(\vec{q}, \vec{p} + \vec{p}')}{{mq}^2} \right)^2 \]
\[ \Delta V_{\eta_\mu \eta_\nu}^{(1)} = e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q_{\nu}^2} \frac{\delta^2}{\Delta^2} \approx \frac{16e^2m^2}{q_z^2} \left( \frac{q_z(\vec{q}, \vec{p} + \vec{p}')}{{mq}^2} \right)^2 \] (66)
where \( x - x' \approx \frac{q_z}{2m} \).

The corrections due to energy denominators in the "\( g_{\mu\nu} \)" term are spin-independent and do not affect the spin-dependent interactions.

The correction from "\( \eta_\mu \eta_\nu \)" term is of the order \( e^2q^0 \). Quite generally the interaction given by the light-front perturbation theory \( V_{\lambda \rightarrow \infty}^{PT} + V_{\lambda \rightarrow 0}^{inst} \) and the effective interaction generated by the unitary transformation \( V_{\lambda \rightarrow \infty}^{gen} + V_{\lambda \rightarrow 0}^{inst} \) have both a leading Coulomb behavior eq. (61), but they differ by spin-independent "\( \eta_\mu \eta_\nu \)" term in the order \( e^2q^0 \), which contributes in the first order bound state perturbation theory to the mass in the order \( \alpha^4 \). In the order of fine structure splitting \( \alpha^4 \) also terms of order \( e^4q^{-1} \) and \( e^6q^{-2} \) are important [1]. We expect that the structure "\( \eta_\mu \eta_\nu \)" in order \( e^2q^0 \) will be compensated in the mass spectrum (in its spin-independent part) by the corresponding terms in the order \( (e^4) \) and \( (e^6) \). [1]

\[ ^1 \text{For the similarity function } f_\lambda(\Delta) = \theta(\lambda^2 - |\Delta|) \text{ the corrections are } \]
\[ \Delta V_{g_{\mu\nu}}^{(1)} = e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta} \frac{\delta |\delta|}{\Delta^2} \approx \frac{16e^2m^2}{q^2} \frac{|q_z||\vec{q}(\vec{p} + \vec{p}')|}{mq^2} \]
\[ \Delta V_{\eta_\mu \eta_\nu}^{(1)} = e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q_{\nu}^2} \frac{\delta |\delta|}{\Delta^2} \approx \frac{16e^2m^2}{q_z^2} \frac{|q_z||\vec{q}(\vec{p} + \vec{p}')|}{mq^2} \] (67)

In this case the "\( \eta_\mu \eta_\nu \)" term is of the order \( e^2q^{-1} \) and contributes to the mass in the order \( \alpha^3 \). One needs to calculate the effective interaction up to the term \( e^4q^{-2} \) to cancel this contribution from "\( \eta_\mu \eta_\nu \)" term in the mass.

In the case of \( f_\lambda(\Delta) = e^{-\frac{\Delta^2}{\alpha^2}} \), (see the main text)
\[ \Delta V_{g_{\mu\nu}} = 0, \quad \Delta V_{\eta_\mu \eta_\nu} = 0. \] (68)
It is argued in [8], that though "$\eta_\mu \eta_\nu$" term does not affect spin-spin and tensor interactions, but it may influence in the second order bound state perturbation theory the spin-orbit.

We remind also that "$\eta_\mu \eta_\nu$" term in the effective interaction can have the singular behavior in the collinear limit as $x \sim x'$ (Appendix C).

One can generate the result of perturbation theory by the flow equations [1], if one chooses for the similarity function

$$f_\lambda(\Delta) = e^{-\frac{|\Delta|}{\lambda^2}}$$

(69)

Then $\Theta$-factor eq. (70) is $\Theta_1 = \frac{\Delta_1}{\Delta_1 + \Delta_2}$; "$\eta_\mu \eta_\nu$" term in eq. (70) vanishes and one has for the effective interaction

$$V_{eff}^{LC} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta}$$

(70)

This is true with the approximation done before eqs. (48) and (49).

Other choices of the similarity functions $f_\lambda(\Delta)$ are possible (eqs. (22) and (23), see also Appendix C). We expect that the mass spectrum stay intact with respect to different similarity functions.

In order to introduce the spectroscopic notation for positronium mass spectrum we integrate out the angular degree of freedom ($\varphi$) by substituting it with the discrete quantum number $J_z = n$, $n \in \mathbb{Z}$ (actually for the annihilation channel only $|J_z| \leq 1$ is possible)

$$< x, k_\perp ; J_z, \lambda_1, \lambda_2 | V_{eff}^{LC} | x', k'_\perp, J'_z, \lambda'_1, \lambda'_2 >$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} d\varphi e^{-iL_\varphi \varphi} \int_{0}^{2\pi} d\varphi' e^{iL_\varphi' \varphi'} (-\frac{1}{2(2\pi)^2}) < x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{eff}^{LC} | x', k'_\perp, \varphi'; \lambda'_1, \lambda'_2 >$$

(71)

where $L_z = J_z - S_z$; $S_z = \frac{\lambda_1}{2} + \frac{\lambda_2}{2}$ and the states can be classified (strictly speaking only for rotationally invariant systems) according to their quantum numbers of total angular momentum $J$, orbit angular momentum $L$, and total spin $S$. It should be noted that the definition of angular momentum operators in light-front dynamics is problematic because they include the interaction.

The general matrix elements for the effective interaction depending on the angles $\varphi, \varphi'$ (actually on the difference $(\varphi - \varphi')$) $< x, k_\perp, \varphi; \lambda_1, \lambda_2 | V_{eff}^{LC} | x', k'_\perp, \varphi'; \lambda'_1, \lambda'_2 >$ and also the matrix elements of the effective interaction after the angular integration for the total momentum $< x, k_\perp; J_z, \lambda_1, \lambda_2 | \tilde{V}_{eff}^{LC} | x', k'_\perp, J'_z, \lambda'_1, \lambda'_2 >$ in the exchange and annihilation channels are given in Appendices A and B, respectively.

To perform the angular integration eq. (71) analytically we choose for the similarity function eq. (14)

$$f_\lambda(\Delta) = u_\lambda(\Delta) = \theta(\lambda^2 - |\Delta|)$$

(72)

where for simplicity we use the sharp cutoff function $u_\lambda(\Delta)$. Then the effective interaction reads

$$V_{eff}^{LC} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\theta(\tilde{\Delta}_1 - \tilde{\Delta}_2)}{\Delta_1} + \frac{\theta(\tilde{\Delta}_2 - \tilde{\Delta}_1)}{\Delta_2} \right)$$

$$-e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{2q^2} |\tilde{\Delta}_1 - \tilde{\Delta}_2| \left( \frac{\theta(\tilde{\Delta}_1 - \tilde{\Delta}_2)}{\Delta_1} + \frac{\theta(\tilde{\Delta}_2 - \tilde{\Delta}_1)}{\Delta_2} \right)$$

(73)
where \( \theta(x) - \theta(-x) = \text{sign}(x) \).

We proceed now to solve for the positronium spectrum in all sectors of \( J_z \). For this purpose we formulate the light-front integral equation eq. (27) in the form where the integral kernel is given by the effective interaction for the total momentum \( J_z \) eq. (71). After the change of variables eq. (60) (\( \vec{k}_\perp; x \) = \( (k_\perp, \varphi; x) \) → \( \vec{p} = (\vec{k}_\perp, p_z) \) = \( (\mu \sin \theta \cos \varphi, \mu \sin \theta \sin \varphi, \mu \cos \theta) \))

\[
x = \frac{1}{2} \left( 1 + \frac{\mu \cos \theta}{\sqrt{\mu^2 + m^2}} \right)
\]

(74)

where the Jacobian reads

\[
J = \frac{\mu^2 m^2 \mu^2 (1 - \cos^2 \theta)}{2 (m^2 + \mu^2)^{3/2}} \sin \theta
\]

(75)

one has the following integral equation

\[
(M_n^2 - 4(m^2 + \mu^2))\tilde{\psi}_n(\mu, \cos \theta; J_z, \lambda_1, \lambda_2) + \sum_{J'_z, \lambda'_1, \lambda'_2} \int_{D} d\mu' \int_{-1}^{+1} d\cos \theta' \frac{\mu^2 m^2 + \mu'^2 (1 - \cos^2 \theta')}{2 (m^2 + \mu'^2)^{3/2}}
\times <\mu, \cos \theta; J_z, \lambda_1, \lambda_2|\tilde{V}_{eff}^{LC}\tilde{V}_{eff}^{LC}|\mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2> \tilde{\psi}_{n'}(\mu', \cos \theta'; J'_z, \lambda'_1, \lambda'_2) = 0
\]

(76)

The integration domain \( D \) eq. (29) is given now by \( \mu \in [0; \frac{\Lambda}{2}] \). Neither \( L_z \) nor \( S_z \) are good quantum numbers; therefore we set \( L_z = J_z - S_z \).

The wave function is normalized

\[
\sum_{J_z, \lambda_1, \lambda_2} \int d\mu \ d\cos \theta \tilde{\psi}_n^*(\mu, \cos \theta; J_z, \lambda_1, \lambda_2)\tilde{\psi}_{n'}(\mu, \cos \theta; J_z, \lambda_1, \lambda_2) = \delta_{nn'}
\]

(77)

where \( n \) labels all quantum numbers.

The integral equation eq. (76) may be used for the calculations of positronium mass spectrum numerically.

4 Conclusions

We have considered the particle number conserving part of the effective QED Hamiltonian, generated in second order in the coupling by unitary transformation. The new generated interaction between electron and positron has two terms of different structure.

The first term has spin-independent as well as spin-dependent parts. The spin-independent part, combined with the instantaneous exchange interaction, leads to the Coulomb interaction. The light-front spin-dependent part after a simple unitary transformation to rotate the spins gives the familiar Breit-Fermi spin-spin and tensor interactions [8], that lead to the correct spin splittings of positronium and rotational invariance is restored [7].

These properties of the first term do not depend on the explicit form of the unitary transformation performed.

The second term is spin-independent and do depend on the unitary transformation. In the first order bound state perturbation theory it contributes the spin-independent part to the mass in order of fine structure. We expect, that this contribution in mass is cancelled by the terms from higher orders in the unitary transformation. The second term may also influence the spin-orbit in the second order bound state perturbation theory [8].

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A Matrix elements of the effective interaction. Exchange channel

In the Appendices A and B we follow the scheme of the work [9] to calculate the matrix elements of the effective interaction in the exchange and annihilation channels, respectively. Here, we list the general, angle-dependent matrix elements defining the effective interaction in the exchange channel eqs. (51) and (55) (part I) and the corresponding matrix elements of the effective interaction for arbitrary \( J_z \), after integrating out the angles eq. (71) (part II).

The whole is given for the similarity function \( f_\lambda(\Delta) = u_\lambda(\Delta) = \theta(\lambda^2 - |\Delta|) \) eq. (14) with the sharp cutoff. The effective interaction generated by the unitary transformation in the exchange channel reads eq. (73)

\[
V_{LC}^{\text{eff}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\theta(a_1 - a_2)}{\Delta_1} + \frac{\theta(a_2 - a_1)}{\Delta_2} \right)
\]

\[
- e^2 < \gamma^\mu \gamma^\nu > \eta_{\mu\nu} \frac{1}{2q^2} \left( \frac{(a_1 - a_2)\theta(a_1 - a_2)}{\Delta_1} + \frac{(a_2 - a_1)\theta(a_2 - a_1)}{\Delta_2} \right)
\]

\[
= - e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\theta(a_1 - a_2)}{\Delta_1} + \frac{\theta(a_2 - a_1)}{\Delta_2} \right)
\]

\[
- e^2 < \gamma^\mu \gamma^\nu > \eta_{\mu\nu} \frac{1}{2q^2} |a_1 - a_2| \left( \frac{(a_1 - a_2)}{\Delta_1} + \frac{(a_2 - a_1)}{\Delta_2} \right)
\]

where fig. (1)

\[
< \gamma^\mu \gamma^\nu > |_{\text{exch}} = \frac{\bar{u}(p_1, \lambda_1) \gamma^\mu u(p_2, \lambda_2)}{\sqrt{p_1^+}} \frac{\bar{v}(p_2', \lambda_2') \gamma^\nu v(p_2, \lambda_2)}{\sqrt{p_2'^+}}
\]

\[
q = p_2' - p_1 \text{ is the momentum transfer. One has in eq. (78)}
\]

\[
\tilde{\Delta}_1 = a_1 - 2k_\perp k'_\perp \cos(\varphi - \varphi')
\]

\[
\tilde{\Delta}_2 = a_2 - 2k_\perp k'_\perp \cos(\varphi - \varphi')
\]

\[
\tilde{\Delta} = a - 2k_\perp k'_\perp \cos(\varphi - \varphi')
\]

\[
\tilde{k}_\perp = k_\perp (\cos \varphi, \sin \varphi)
\]

\[
a_1 = \frac{x'}{x} \frac{k^2_\perp + x' k'^2_\perp + m^2 (x - x')^2}{x x'}
\]

\[
= k^2_\perp + k'^2_\perp + (x - x') \left( k^2_\perp \left( \frac{1}{x} \right) - k'^2_\perp \left( \frac{1}{x'} \right) \right) + m^2 \frac{(x - x')^2}{x x'}
\]

\[
a_2 = \frac{1 - x'}{1 - x} k^2_\perp + \frac{1 - x}{1 - x'} k'^2_\perp + m^2 \frac{(x - x')^2}{(1 - x)(1 - x')}
\]

\[
= k^2_\perp + k'^2_\perp + (x - x') \left( k^2_\perp \left( \frac{1}{1 - x} \right) - k'^2_\perp \left( \frac{1}{1 - x'} \right) \right) + m^2 \frac{(x - x')^2}{(1 - x)(1 - x')}
\]

\[
a = \frac{k^2_\perp + k'^2_\perp + (x - x')}{2} \left( k^2_\perp \left( \frac{1}{1 - x} \right) - k'^2_\perp \left( \frac{1}{1 - x'} \right) \right)
\]

\[
+ m^2 \frac{(x - x')^2}{2} \left( \frac{1}{x x'} + \frac{1}{(1 - x)(1 - x')} \right)
\]

\[
a = \frac{1}{2} (a_1 + a_2)
\]
The energy denominator ($\tilde{\Delta}$ and $a$ corresponding) in the case of perturbative theory is given for completeness.

It is useful to display the matrix elements of the effective interaction in the form of tables. The matrix elements depend on the one hand on the momenta of the electron and positron, respectively, and on the other hand on their helicities before and after the interaction. The dependence on the helicities occur during the calculation of these functions $E(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2)$ in part I and $G(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2)$ in part II as different Kronecker deltas \[5\]. These functions are displayed in the form of helicity tables. We use the following notation for the elements of the tables

$$F_i(1, 2) \rightarrow E_i(x, k_\perp; x', k'_\perp); \quad G_i(x, k_\perp; x', k'_\perp)$$

(82)

Also we have used in both cases for the permutation of particle and anti-particle

$$F_3^*(x, k_\perp; x', k'_\perp) = F_3(1-x, -k_\perp; 1-x', -k'_\perp)$$

(83)

one has the corresponding for the elements of arbitrary $J_z$; in the case when the function additionally depends on the component of the total angular momentum $J_z = n$ we have introduced

$$\tilde{F}_i(n) = F_i(-n)$$

(84)

### A.1 The general helicity table.

To calculate the matrix elements of the effective interaction in the exchange channel we use the matrix elements of the Dirac spinors listed in Table 1 [5]. Also the following holds

$$\bar{v}_{\lambda'}(p)\gamma^\mu u_\lambda(q) = \bar{u}_\lambda(q)\gamma^\alpha u_{\lambda'}(p).$$

| $\mathcal{M}$ | $\frac{1}{\sqrt{k + k'}} \bar{u}(k', \lambda') \mathcal{M} u(k, \lambda)$ |
|--------------|--------------------------------------------------|
| $\gamma^+$  | $2\delta^\lambda_{\lambda'}$                      |
| $\gamma^-$  | $\frac{2}{k^+ k'} \left[ (m^2 + k_\perp k'_\perp e^{+i\lambda \varphi - \varphi'}) \delta^\lambda_{\lambda'} - m\lambda \left( k'_\perp e^{+i\lambda \varphi'} - k_\perp e^{+i\lambda \varphi} \right) \delta^\lambda_{-\lambda'} \right]$ |
| $\gamma^\bot_1$ | $\left( k'_\perp \frac{e^{-i\lambda \varphi'} + k_\perp}{k^+ e^{+i\lambda \varphi}} \right) \delta^\lambda_{\lambda'} + m\lambda \left( \frac{1}{k^+} - \frac{1}{k^+} \right) \delta^\lambda_{-\lambda'}$ |
| $\gamma^\bot_2$ | $i\lambda \left( k'_\perp \frac{e^{-i\lambda \varphi'} - k_\perp}{k^+ e^{+i\lambda \varphi}} \right) \delta^\lambda_{\lambda'} + im \left( \frac{1}{k^+} - \frac{1}{k^+} \right) \delta^\lambda_{-\lambda'}$ |

Table 1: Matrix elements of the Dirac spinors.

We introduce for the matrix elements entering in the effective interaction eq. (78)

$$2E^{(1)}(x, k_\perp; \lambda_1, \lambda_2|x', k'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^{\mu}\gamma^{\nu} \rangle g_{\mu\nu} =$$
These functions are displayed in the Table 2.

| final : initial | $(\lambda_1', \lambda_2') = \uparrow\uparrow$ | $(\lambda_1', \lambda_2') = \uparrow\downarrow$ | $(\lambda_1', \lambda_2') = \downarrow\uparrow$ | $(\lambda_1', \lambda_2') = \downarrow\downarrow$ |
|-----------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $(\lambda_1, \lambda_2) = \uparrow\uparrow$ | $E_1(1, 2)$ | $E_3^*(1, 2)$ | $E_3(1, 2)$ | 0 |
| $(\lambda_1, \lambda_2) = \uparrow\downarrow$ | $E_3^*(2, 1)$ | $E_2(1, 2)$ | $E_4(1, 2)$ | $-E_3(2, 1)$ |
| $(\lambda_1, \lambda_2) = \downarrow\uparrow$ | $E_3(2, 1)$ | $E_4(1, 2)$ | $E_2(1, 2)$ | $-E_3^*(2, 1)$ |
| $(\lambda_1, \lambda_2) = \downarrow\downarrow$ | 0 | $-E_3(1, 2)$ | $-E_3^*(1, 2)$ | $E_1(1, 2)$ |

Table 2: General helicity table defining the effective interaction in the exchange channel.

The matrix elements $E_i^{(n)}(1, 2) = E_i^{(n)}(x, \vec{k}_\perp; x', \vec{k}'_\perp)$ $(n = 1, 2)$ are the following

\[
E_1^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + \frac{k_\perp k'_\perp}{xx'(1-x)(1-x')} e^{-i(\varphi-\varphi')}
\]

\[
E_2^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + k_\perp^2 \frac{1}{x(1-x)} + k'_\perp^2 \frac{1}{x'(1-x')}
\]

\[
+ k_\perp k'_\perp \left( \frac{e^{i(\varphi-\varphi')}}{xx'} + \frac{e^{-i(\varphi-\varphi')}}{(1-x)(1-x')} \right)
\]

\[
E_3^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m \frac{k'_\perp e^{i\varphi'} - k_\perp \frac{1-x'}{1-x} e^{i\varphi}}{xx'}
\]

\[
E_4^{(1)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')}
\]

and

\[
E_1^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_2^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = \frac{2}{(x-x')^2}
\]

\[
E_3^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = E_4^{(2)}(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 0
\]

A.2 The helicity table of the exchange channel for arbitrary $J_z$.

Following the description given in the main text eq. (71) we integrate out the angles in the effective interaction in the exchange channel eqs. (73) and (78). For the matrix elements of the effective interaction for an arbitrary $J_z = n$ with $n \in \mathbb{Z}$

\[G(x, k_\perp; \lambda_1, \lambda_2 | x', k'_\perp; \lambda'_1, \lambda'_2) = \langle x, k_\perp; J_z, \lambda_1, \lambda_2 | \bar{V}_{LC}^{eff} | x', k'_\perp; J'_z, \lambda'_1, \lambda'_2 \rangle \]

in the exchange channel one obtains the helicity Table 3.
| (λ₁, λ₂) = ↑↑ | (λ₁, λ₂) = ↑↓ | (λ₁, λ₂) = ↓↑ | (λ₁, λ₂) = ↓↓ |
|---|---|---|---|
| G₁(1, 2) | G₃*(1, 2) | G₃(1, 2) | 0 |
| G₃*(2, 1) | G₂(1, 2) | G₄(1, 2) | −G₃(2, 1) |
| G₃(2, 1) | G₄(1, 2) | G₂(1, 2) | −G₃*(2, 1) |
| 0 | −G₃(1, 2) | −G₃(1, 2) | G₁(1, 2) |

Table 3: Helicity table of the effective interaction for J_z = ±n, x > x'

Here, the functions G_i(1, 2) = G_i(x, k⊥; x', k'⊥) are given

\[
G₁(x, k⊥; x', k'⊥) = m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) \text{Int}_{a₁a₂}(|1-n|)
+ \frac{k₁k'₁}{xx'(1-x)(1-x')} \text{Int}_{a₁a₂}(|n|) + \frac{|a₁ - a₂|}{(x-x')^2} \text{Int}_{a₁a₂}(|1-n|)
\]

\[
G₂(x, k⊥; x', k'⊥) = (m^2 \left( \frac{1}{xx'} + \frac{1}{(1-x)(1-x')} \right) + k₁^{-2} \frac{1}{x(1-x)} + k'₁^{-2} \frac{1}{x'(1-x')} \text{Int}_{a₁a₂}(|n|)
+ k₁k'₁ \left( \frac{1}{xx'} \text{Int}_{a₁a₂}(|1-n|) + \frac{1}{(1-x)(1-x')} \text{Int}_{a₁a₂}(|1+n|) \right)
+ \frac{|a₁ - a₂|}{(x-x')^2} \text{Int}_{a₁a₂}(|n|)
\]

\[
G₃(x, k⊥; x', k'⊥) = -\frac{m}{xx'} \left( k'₁ \text{Int}_{a₁a₂}(|1+n|) - k₁ \frac{1-x'}{1-x} \text{Int}_{a₁a₂}(|n|) \right)
\]

\[
G₄(x, k⊥; x', k'⊥) = -m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} \text{Int}_{a₁a₂}(|n|)
\]

(89)

where we have introduced the functions

\[
\text{Int}_{a₁a₂}(n) = \theta(a₁ - a₂)\text{Int}_{a₁}(n) + \theta(a₂ - a₁)\text{Int}_{a₂}(n)
\]

\[
\text{Int}_{a₁}(n) = \frac{\alpha}{\pi} (-A(a₁))^{-n+1} \left( \frac{B(a₁)}{k₁k'₁} \right)^n
\]

\[
A(a_i) = \frac{1}{\sqrt{a_i^2 - 4k_i^2 k'₁^2}}
\]

\[
B(a_i) = \frac{1}{2} (1 - a_i A(a_i))
\]

(90)

and the functions a_i, i = 1, 2 are given in eq. [81].

The following integrals were used by the calculation of the matrix elements [2]

\[
\frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \frac{\cos(n(\varphi - \varphi'))}{a_i - 2k₁k'₁ \cos(\varphi - \varphi')} = 2\pi (-A(a_i))^{-n+1} \left( \frac{B(a_i)}{k₁k'₁} \right)^n
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \frac{\sin(n(\varphi - \varphi'))}{a_i - 2k₁k'₁ \cos(\varphi - \varphi')} = 0
\]

(91)

The last terms in G₁, G₂ arise from the "ημη'" structure of the effective interaction eq. [78], and are defined only for L_z = L'z = 0, i.e. for the total angular momentum J_z = S_z = λ₁/2 + λ₂/2.
Due to the corresponding $\delta$-functions they are equal to

$$
\Delta G = \frac{|a_1 - a_2|}{(x - x')^2} \int n_{a_1 a_2}(0) = \left( -\frac{\alpha}{\pi} \right) \frac{(a_1 - a_2)}{(x - x')^2} \left( \frac{\theta(a_1 - a_2)}{\sqrt{a_1^2 - 4k_{2\perp}^2 k_{1\perp}^2}} - \frac{\theta(a_2 - a_1)}{\sqrt{a_2^2 - 4k_{2\perp}^2 k_{1\perp}^2}} \right)
$$

(92)

In the collinear limit this term is singular \[ (\text{see also the first point of eq. (111)}) \]

$$
\Delta G |_{x \sim x'} = \left( -\frac{\alpha}{\pi} \right) \frac{1}{x(1 - x)} \frac{1}{|x - x'|} + \text{const}
$$

(94)

we have used

$$
A(a_1)\theta(a_1 - a_2) - A(a_2)\theta(a_2 - a_1)
= \frac{1}{2} (A(a_1) - A(a_2)) (\theta(a_1 - a_2) + \theta(a_2 - a_1)) + \frac{1}{2} (A(a_1) + A(a_2)) (\theta(a_1 - a_2) - \theta(a_2 - a_1))
= \frac{1}{2} (A(a_1) - A(a_2)) + \frac{1}{2} (A(a_1) + A(a_2)) \text{sign}(a_1 - a_2)
$$

(95)

The condition on the parameter space $(x, k_{\perp})$ due to the $\theta$-function, namely $\theta(a_1 - a_2)$ (i.e. when $a_1 > a_2$) reads

$$
(x - x') (x(1 - x)(k_{1\perp}^2 + m^2) - x'(1 - x')(k_{2\perp}^2 + m^2)) > 0
$$

(96)

Making use of the coordinate change eq. (60) this is equivalent to

$$
\left( \frac{\mu \cos \theta}{\sqrt{\mu^2 + m^2}} - \frac{\mu' \cos \theta'}{\sqrt{\mu'^2 + m^2}} \right) (\mu - \mu') < 0
$$

(97)

---

2 In fact the IR singularities in the effective interaction arise from the singular behavior of the generator of unitary transformation $\eta(l)$ in the collinear limit $q^+ \to 0$. The following condition must be imposed on the generator of transformation

$$
l \lim_{q^+ \to 0} \eta(l) = 0
$$

(93)

to insure the effective interaction to be finite in the collinear limit. This is true, for example, for the second and the third choice of the generator in Appendix C.
Matrix elements of the effective interaction. Annihilation channel

We repeat the same calculations for the matrix elements of the effective interaction in the annihilation channel. In this case the effective interaction generated by the unitary transformation reads eqs. (30) and (42)

\[ V_{\text{eff}}^{\text{LC}} = e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\theta(M_0^2 - M_0'^2)}{M_0^2} + \frac{\theta(M_0'^2 - M_0^2)}{M_0'^2} \right) \]

\[-e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{P^{+2}} \quad (98)\]

where

\[ < \gamma^\mu \gamma^\nu > = \frac{\bar{v}(p'_2, \lambda'_2) \gamma^\mu u(p'_1, \lambda'_1)}{\sqrt{p'_2^+}} \frac{\bar{u}(p_1, \lambda_1)}{\sqrt{p_1^+}} \frac{\gamma^\nu v(p_2, \lambda_2)}{\sqrt{p_2^+}} P^{+2} \quad (99)\]

\[ P^{+} = p^{+_1} + p^{+_2} \text{ is the total momentum; and} \]

\[ M_0^2 = \frac{k^2 + m^2}{x(1 - x)} \]

\[ M_0'^2 = \frac{k'^2 + m^2}{x'(1 - x')} \quad (100)\]

All particle momenta are depicted on fig. (1). Note that the energy denominators of the effective interaction in the annihilation channel do not depend on the angles \( \varphi, \varphi' \).

| \( \mathcal{M} \) | \[ \frac{1}{\sqrt{k + k'}} \bar{v}(k', \lambda') \mathcal{M} u(k, \lambda) \] |
|------------------|----------------------------------|
| \( \gamma^+ \)   | \[ 2\delta^\lambda_{\lambda'} \] |
| \( \gamma^- \)   | \[ \frac{2}{k^+ k'^+} \left[ -(m^2 - k^+ k'^+ e^{+i\lambda(\varphi - \varphi')}) \delta^\lambda_{\lambda'} - m \lambda \left(k'^{e^{+i\lambda\varphi'}} + k^{e^{+i\lambda\varphi}}\right) \delta^\lambda_{\lambda'} \right] \] |
| \( \gamma^1_{\perp} \) | \[ \left( \frac{k'^+ e^{-i\lambda\varphi'}}{k^+ e^{+i\lambda\varphi}} + \frac{k^+ e^{+i\lambda\varphi'}}{k'^+ e^{-i\lambda\varphi}} \right) \delta^\lambda_{\lambda'} - m \lambda \left( \frac{1}{k^+} + \frac{1}{k'^+} \right) \delta^\lambda_{\lambda'} \] |
| \( \gamma^2_{\perp} \) | \[ i\lambda \left( \frac{k'^+ e^{-i\lambda\varphi'}}{k^+ e^{+i\lambda\varphi}} - \frac{k^+ e^{+i\lambda\varphi'}}{k'^+ e^{-i\lambda\varphi}} \right) \delta^\lambda_{\lambda'} - im \left( \frac{1}{k^+} + \frac{1}{k'^+} \right) \delta^\lambda_{\lambda'} \] |

Table 4: Matrix elements of the Dirac spinors.
B.1 The general helicity table

For the calculation of matrix elements of effective interaction in the annihilation channel we use the matrix elements of the Dirac spinors listed in Table 4 [5]. Also the following holds

\((\bar{v}_\lambda(p)\gamma^\alpha u_\lambda(q))^+ = -u_\lambda(q)\gamma^\alpha v_\lambda(p)\).

We introduce

\[2H^{(1)}(x, \vec{k}_\perp; \lambda_1, \lambda_2|x', \vec{k}'_\perp; \lambda'_1, \lambda'_2) = \langle \gamma^\mu \gamma^\nu \rangle \gamma^\mu u(x', \vec{k}'_\perp; \lambda'_1) \bar{u}(x, \vec{k}_\perp; \lambda_1) \gamma^\nu v(1 - x, -\vec{k}_\perp; \lambda_2)\]

\[= \langle \gamma^\mu \gamma^\nu \rangle \frac{\bar{v}(1 - x', -\vec{k}_\perp; \lambda_2) \gamma^\mu u(x', \vec{k}'_\perp; \lambda'_1) \bar{u}(x, \vec{k}_\perp; \lambda_1) \gamma^\nu v(1 - x, -\vec{k}_\perp; \lambda_2)}{\sqrt{xx'(1 - x)(1 - x')}}\]

These functions are displayed in the Table 5.

| final:initial | (\lambda'_1, \lambda'_2) = \uparrow\uparrow | (\lambda'_1, \lambda'_2) = \uparrow\downarrow | (\lambda'_1, \lambda'_2) = \downarrow\uparrow | (\lambda'_1, \lambda'_2) = \downarrow\downarrow |
|---------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| (\lambda_1, \lambda_2) = \uparrow\uparrow | H_1(1, 2) | H_3(2, 1) | H_3^*(2, 1) | 0 |
| (\lambda_1, \lambda_2) = \uparrow\downarrow | H_3(1, 2) | H_3^*(1, 2) | H_4(2, 1) | 0 |
| (\lambda_1, \lambda_2) = \downarrow\uparrow | H_3^*(1, 2) | H_4(1, 2) | H_2(1, 2) | 0 |
| (\lambda_1, \lambda_2) = \downarrow\downarrow | 0 | 0 | 0 | 0 |

Table 5: General helicity table defining the effective interaction in the annihilation channel.

Here, the matrix elements \(H^{(n)}_i(x, \vec{k}_\perp; x', \vec{k}'_\perp)\) are the following

\[H^{(1)}_1(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m^2 \left( \frac{1}{x} + \frac{1}{1 - x} \right) \left( \frac{1}{x'} + \frac{1}{1 - x'} \right)\]

\[H^{(1)}_2(x, \vec{k}_\perp; x', \vec{k}'_\perp) = k_\perp k'_\perp \left( \frac{e^{i(\varphi - \varphi')}}{xx'} + \frac{e^{-i(\varphi - \varphi')}}{(1 - x)(1 - x')} \right)\]

\[H^{(1)}_3(x, \vec{k}_\perp; x', \vec{k}'_\perp) = m\lambda_1 \left( \frac{1}{x'} + \frac{1}{1 - x'} \right) \frac{k_\perp}{1 - x} e^{i\varphi}\]

\[H^{(1)}_4(x, \vec{k}_\perp; x', \vec{k}'_\perp) = -k_\perp k'_\perp \left( \frac{e^{i(\varphi - \varphi')}}{x(1 - x')} + \frac{e^{-i(\varphi - \varphi')}}{x'(1 - x)} \right)\]

and

\[H^{(2)}_1(x, \vec{k}_\perp; x', \vec{k}'_\perp) = H^{(2)}_3(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 0\]

\[H^{(2)}_2(x, \vec{k}_\perp; x', \vec{k}'_\perp) = H^{(2)}_4(x, \vec{k}_\perp; x', \vec{k}'_\perp) = 2\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
final:initial & $(\lambda_1', \lambda_2') = \uparrow \uparrow$ & $(\lambda_1', \lambda_2') = \uparrow \downarrow$ & $(\lambda_1', \lambda_2') = \downarrow \uparrow$ & $(\lambda_1', \lambda_2') = \downarrow \downarrow$ \\
\hline
$(\lambda_1, \lambda_2) = \uparrow \uparrow$ & $F_1(1, 2)$ & $F_3(2, 1)$ & $F_3^*(2, 1)$ & 0 \\
$\quad = \uparrow \downarrow$ & $F_3(1, 2)$ & $F_4^*(1, 2)$ & $F_4(1, 2)$ & 0 \\
$\quad = \downarrow \uparrow$ & $F_3^*(1, 2)$ & $F_4(1, 2)$ & $F_2(1, 2)$ & 0 \\
$\quad = \downarrow \downarrow$ & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{Helicity table of the effective interaction in the annihilation channel for $J_z \geq 0$.}
\end{table}

**B.2 The helicity table of the annihilation channel for $|J_z| \leq 1$**

The matrix elements of the effective interaction for $J_z \geq 0$ in the annihilation channel (the sum of the generated interaction for $J_z = +1$ and instantaneous graph for $J_z = 0$) are given in Table 6.

The function $F_i(1, 2) = F_i(x, k_\perp; x', k'_\perp)$ are the following

- $F_1(x, k_\perp; x', k'_\perp) = -\frac{\alpha}{\pi} \frac{1}{\Omega} \frac{m^2}{xx'(1-x)(1-x')} \delta_{|J_z|,1}$
- $F_2(x, k_\perp; x', k'_\perp) = -\frac{\alpha}{\pi} \left( \frac{1}{\Omega} \frac{k_\perp k'_\perp}{xx'} \delta_{|J_z|,1} + 2\delta_{J_z,0} \right)$
- $F_3(x, k_\perp; x', k'_\perp) = -\frac{\alpha}{\pi} \frac{1}{\Omega} \frac{1}{\lambda_1} \frac{m}{x(1-x')} \frac{k_\perp}{1-x} \delta_{|J_z|,1}$
- $F_4(x, k_\perp; x', k'_\perp) = -\frac{\alpha}{\pi} \left( -\frac{1}{\Omega} \frac{k_\perp k'_\perp}{x(1-x')} \delta_{|J_z|,1} + 2\delta_{J_z,0} \right)$ (105)

where we have introduced

\[
\frac{1}{\Omega} = \frac{\theta(M_0^2 - M_0'^2)}{M_0^2} + \frac{\theta(M_0'^2 - M_0^2)}{M_0'^2} \tag{106}
\]

The table for $J_z = -1$ is obtained by inverting all helicities, i.e.

$F(J_z = +1; \lambda_1, \lambda_2) = -\lambda_1 F(J_z = -1; -\lambda_1, -\lambda_2)$ (107)

The matrix elements of the effective interaction in the annihilation channel are nonzero only for $|J_z| \leq 1$ due to the restriction on the angular momentum of the photon.
C Collinear limit

We estimate in this appendix the correction of \( \eta_{\mu} \eta_{\nu} \) term in the collinear limit. With respect to different choices of the similarity function one has either singular or finite corrections. The expansion of the effective interaction in the exchange channel eq. (65) with respect to \( \frac{|\delta|}{\Delta} \ll 1 \) reads

\[
V^{\text{eff}}_{\text{LC}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta} \left( 1 - (\Theta_1 - \Theta_2) \frac{\delta}{\Delta} + \left( \frac{\delta}{\Delta} \right)^2 \right)
- e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q^2+2} \left( (\Theta_1 - \Theta_2) \frac{\delta}{\Delta} + \left( \frac{\delta}{\Delta} \right)^2 \right) + O \left( \frac{\delta^3}{\Delta^3} \right)
\]

\[
= V^{(0)} + \Delta V_{\eta_\mu \eta_\nu} + \Delta V_{\eta_\mu \eta_\nu}
\]

where the leading order electron-positron interaction \( V^{(0)} \) is given by eq. (59) in the main text, the corrections from next to leading orders of \( g_{\mu\nu} \) and \( \eta_\mu \eta_\nu \) terms are called \( \Delta V_{\eta_\mu \eta_\nu} \) and \( \Delta V_{\eta_\mu \eta_\nu} \), respectively; "\( \Theta \)-factors" are given

\[
\Theta_1 = \int_0^\infty \frac{df(\Delta_1)}{d\lambda'} f(\Delta_2) d\lambda'
\]
\[
\Theta_2 = \int_0^\infty \frac{df(\Delta_2)}{d\lambda'} f(\Delta_1) d\lambda'
\]

The effective interactions with the corresponding similarity functions and "\( \Theta \)-factors" are written below; also the leading order corrections as \( x \sim x' \Delta \tilde{V}_{\eta_\mu \eta_\nu} \) are given

1. \( f_\lambda(\Delta) = \theta(\lambda^2 - |\Delta|); \quad \Theta_1 = \theta(\tilde{\Delta}_1 - \tilde{\Delta}_2) = \theta(\delta) \)

\[
V^{\text{eff}}_{\text{LC}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \left( \frac{\theta(\tilde{\Delta}_1 - \tilde{\Delta}_2)}{\Delta_1} + \frac{\theta(\tilde{\Delta}_2 - \tilde{\Delta}_1)}{\Delta_2} \right)
- e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q^2+2} |\tilde{\Delta}_1 - \tilde{\Delta}_2| \left( \frac{\theta(\tilde{\Delta}_1 - \tilde{\Delta}_2)}{\Delta_1} + \frac{\theta(\tilde{\Delta}_2 - \tilde{\Delta}_1)}{\Delta_2} \right)
\]

\[
\Delta V_{\eta_\mu \eta_\nu} \approx -e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q^2+2} \frac{|\delta|}{\Delta} \approx -\frac{1}{x-x'} \frac{2e^2}{x(1-x)} \frac{|k^2_\perp - k'^2_\perp|}{x^2(1-x)^2 (k^2_\perp - k'^2_\perp)^2} + \frac{e^2}{x^2(1-x)^2} \frac{(k^2_\perp - k'^2_\perp)^2}{(k^2_\perp - k'^2_\perp)^4}
\]

2. \( f_\lambda(\Delta) = e^{-\frac{\Delta^2}{\lambda^2}}; \quad \Theta_1 = \frac{\tilde{\Delta}_1^2}{\Delta_1^2 + \Delta_2^2} \)

\[
V^{\text{eff}}_{\text{LC}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\Delta_1^2 + \Delta_2^2} - e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q^2+2} \frac{(\tilde{\Delta}_1 - \tilde{\Delta}_2)^2}{\Delta_1^2 + \Delta_2^2}
\]

\[
\Delta V_{\eta_\mu \eta_\nu} \approx -e^2 < \gamma^\mu \gamma^\nu > \eta_\mu \eta_\nu \frac{1}{q^2+2} \frac{\delta^2}{\Delta^2} \approx -\frac{e^2}{x^2(1-x)^2} \frac{(k^2_\perp - k'^2_\perp)^2}{(k^2_\perp - k'^2_\perp)^4}
\]

3. \( f_\lambda(\Delta) = e^{-\frac{\Delta}{\lambda^2}}; \quad \Theta_1 = \frac{\tilde{\Delta}_1}{\Delta_1 + \Delta_2} \)

\[
V^{\text{eff}}_{\text{LC}} = V^{\text{PT}}_{\lambda=\Delta \to \infty} + V^{\text{inst}} = -e^2 < \gamma^\mu \gamma^\nu > g_{\mu\nu} \frac{1}{\Delta}
\]

\[
\Delta V_{\eta_\mu \eta_\nu} = 0
\]
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Figure 1: The effective electron-positron interaction in the exchange channel; the diagrams correspond to the generated and instantaneous interactions. The perturbative photon exchange with two different time orderings is also depicted.