Parity and Relative Parity in Knot Theory

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Abstract

In the present paper we give a simple proof of the fact that the set of virtual links with orientable atoms is closed. More precisely, the theorem states that if two virtual diagrams $K$ and $K'$ have orientable atoms and they are equivalent by Reidemeister moves, then there is a sequence of diagrams $K = K_1 \rightarrow \cdots \rightarrow K_n = K'$ all having orientable atoms where $K_i$ is obtained from $K_{i-1}$ by a Reidemeister move. The initial proof heavily relies on the topology of virtual links and was published in [IM]. Our proof is based on the notion of parity which was introduced by the second named author in 2009.

We split the set of crossings of a virtual link diagram into sets of odd and even in accordance with a fixed rule. The rule must only satisfy several conditions of Reidemeister’s type. Then one can construct functorial mappings of link diagrams by using parity. The concept of parity allows one to introduce new invariants and strengthen well-known ones [Ma1].

1 Parity and Relative Parity

In the present section we introduce the concept of parity. Section 2 contains a short description of the theory of atoms. The main theorem stated in the Abstract is proved in the Section 3.

Definition 1.1 A 4-valent graph $\Gamma$ is called framed if for each vertex of $\Gamma$ the four emanating half-edges are split into two pairs of (formally) opposite.

One can define Reidemeister moves for framed 4-graphs. The first move is an addition/removal of a loop, see Fig. 1, left. The second Reidemeister move adds/removes a bigon formed by a pair of edges which are adjacent in
Definition 1.2 An equivalence class of framed 4-graphs modulo Reidemeister moves is called a free link.

Definition 1.3 Two edges $a, a'$ belong to the same unicursal component of a graph $\Gamma$ if there exist sequences of edges $a = a_1, \ldots, a_{n+1} = a'$ and vertices $c_1, \ldots, c_n$ such that for $i = 1, \ldots, n$ the edges $a_i, a_{i+1}$ are opposite in $c_i$.

The number of unicursal components is an invariant under Reidemeister moves.

Definition 1.4 By a parity we mean a rule for endowing all vertices of all 4-valent framed graphs (or all vertices of all graphs from a subset closed under Reidemeister moves) by elements of $\mathbb{Z}_2$ such that the following axioms hold:

1. Each crossing taking part in the first Reidemeister move is endowed with 0.
2. Each two crossings taking part in the second Reidemeister move are either both odd (i.e. are endowed with 1) or both even (i.e. are endowed with 0).
3. The sum (modulo 2) of parities of three vertices participating in the third Reidemeister move equals 0. Moreover, parities of corresponding vertices ($a$ and $a'$, $b$ and $b'$, $c$ and $c'$ in Fig. 2) are the same.

Also we require that vertices which do not take part in a move preserve their parities.

Example 1.1 Free knots (i.e., one-component free links) are encoded by Gauss diagrams without signs and arrows in such a way that there is a natural
correspondence between vertices of a knot and chords of its diagram. Two chords are called *linked* if their ends are alternated on the circle of the Gauss diagram. We say that a chord of a Gauss diagram is even if the number of chords it is linked with, is *even*, and *odd* otherwise. It can be easily checked that this yields a parity. We call it *Gaussian parity*.

**Example 1.2** This example deals with graphs having 2 unicursal components. A vertex is even if all four incident half-edges belong to the same component; otherwise a vertex is odd.

Therefore a parity for the given graph $\Gamma$ is described by a pair $(\Gamma, \mathcal{P})$. Here $\mathcal{P}$ is a function on the set of vertices of $\Gamma$ to $\mathbb{Z}_2$ satisfying axioms 1)-3). In the case of more complicated knot-like objects by a *parity* we mean also a rule satisfying the same axioms under Reidemeister moves.

**Definition 1.5** A *virtual knot diagram* is an image of a smooth immersion $f : S^1 \rightarrow \mathbb{R}^2$ such that each intersection point is double, transverse and endowed with classical (with a choice for underpass and overpass specified) or virtual crossing structure. A *virtual knot* is an equivalence class of virtual knot diagrams modulo generalized Reidemeister moves. The notion of a virtual knot was introduced by L. H. Kauffman in [K].

In the case of virtual link diagrams (where chords of the Gauss diagram are endowed with signs and arrows) the parity axioms must hold only for those vertices which can participate in a Reidemeister move. Note that each parity for free knots induces a parity for virtual knots, but not vice versa.

Let $\Gamma$ be a framed 4-graph with one unicursal component. The group $H_1(\Gamma, \mathbb{Z}_2)$ is generated by "halves" which correspond to vertices: for every vertex $v$ there exist two halves $\Gamma_{1,v}$ and $\Gamma_{2,v}$ (see Fig. 3) obtained by smoothing of the graph $\Gamma$ at the vertex $v$. If the set of framed 4-graphs is endowed with a parity, one can consider a cohomology class $h$ defined by equalities.
Collecting the properties of this cohomology class we see that:

1. For every framed 4-graph $\Gamma$ we have $h(\Gamma) = 0$.

2. If $\Gamma'$ is obtained from $\Gamma$ by a first Reidemeister move adding a loop then for every basis $\{\alpha_i\}$ of $H_1(\Gamma, \mathbb{Z}_2)$ there exists a basis of the group $H_1(\Gamma', \mathbb{Z}_2)$ consisting of one element $\beta$ corresponding to the loop and a set of elements $\alpha'_i$ naturally corresponding to $\alpha_i$. Then we have $h(\beta) = 0$ and $h(\alpha_i) = h(\alpha'_i)$.

3. Let $\Gamma'$ be obtained from $\Gamma$ by a third increasing Reidemeister move. Then for every basis $\{\alpha_i\}$ of $H_1(\Gamma, \mathbb{Z}_2)$ there exists a basis in $H_1(\Gamma', \mathbb{Z}_2)$ consisting of one "bigon" $\gamma$, the elements $\alpha'_i$ naturally corresponding to $\alpha_i$, and one additional element $\delta$, see Fig. 4, left. Then the following holds: $h(\alpha_i) = h(\alpha'_i), h(\gamma) = 0$.

4. Let $\Gamma'$ be obtained from $\Gamma$ by a third Reidemeister move. Then there exists a graph $\Gamma''$ with one vertex of valency 6 and the other vertices of valency 4 which is obtained from either of $\Gamma$ or $\Gamma'$ by contracting the small triangle to the point. This generates the mappings $i: H_1(\Gamma, \mathbb{Z}_2) \to H_1(\Gamma'', \mathbb{Z}_2)$ and $i': H_1(\Gamma', \mathbb{Z}_2) \to H_1(\Gamma'', \mathbb{Z}_2)$, see Fig. 4, right. Then the following holds: the cocycle $h$ is equal to zero for small triangles, besides that if for $a \in H_1(\Gamma, \mathbb{Z}_2), a' \in H_1(\Gamma', \mathbb{Z}_2)$ we have $i(a) = i'(a')$, then $h(a) = h(a')$.

Let us consider now graphs with $n > 1$ unicursal components.

**Theorem 1.1** In this case $H_1(\Gamma, \mathbb{Z}_2)$ is generated not only by halves but by the following set of cycles:

1. All halves,
2). All bigons,  
3). (for \( n \geq 3 \)) Cycles on \( \Gamma \) corresponding to cycles on the intersection graph of unicursal components.

**Proof.** Consider an arbitrary cycle \( \gamma \) on \( \Gamma \). Denote by \( M_\Gamma \) the set of cycles described in the statement of the theorem and by \( Y \) the set of vertices where \( \gamma \) rotates.

Let \( \gamma \) rotates in a vertex \( y \in Y \) which is incident to edges \( l_1, l_2 \in \Gamma \) and the edges belong to the same unicursal component. If \( l_1 \) and \( l_2 \) belong to the same half (say, \( \Gamma_{1,y} \)), then by adding the half \( \Gamma_{1,y} \) to the graph \( \Gamma \) (note than both halves are the elements of \( M_\Gamma \)) we can obtain a cycle \( \gamma' \) which rotates only in vertices from \( Y \setminus \{ y \} \). If \( l_1 \in \Gamma_{1,y} \) and \( l_2 \in \Gamma_{2,y} \) we can add any of the two halves to \( \Gamma \) to decrease the number of vertices in \( Y \).

After applying this procedure several times obtained cycle \( \tilde{\gamma} \) will rotate only in those vertices where two pairs of incident half-edges belong to other unicursal components. Thus \( \tilde{\gamma} \) belong to \( M_\Gamma \).

We use properties 2)-4) of \( h \) to define it in the case of \( n > 1 \).

**Definition 1.6** An element \( h \in H^1(\Gamma, \mathbb{Z}_2) \) is called a homological parity if the following conditions hold:

1). \( h(l_i) = 0 \) for each unicursal component \( l_i \) of any framed 4-graph \( \Gamma \).
2)-4). This properties coincide with ones listed above for \( n = 1 \).

**Definition 1.7** We say that homological parity \( h \in H^1(\Gamma, \mathbb{Z}_2) \) agrees with a parity \( \mathcal{P} \) if for any cycle \( \gamma \in \Gamma \) holds \( h(\gamma) = \sum_v \mathcal{P}(\gamma) \). Here the sum is over all vertices \( v \) such that \( \gamma \) rotates in \( v \).

**Example 1.3** Suppose that for each unicursal component \( l_i \) (\( i = 1, \ldots, n \)) of a graph \( \Gamma \) the total number of its intersection points with other components is even. Denote the set of all framed 4-graphs with this property by \( \mathcal{G} \). We say that homological parity is Gaussian if its value on any cycle \( \gamma \) is equal to the number of vertices where \( \gamma \) go transversally. Gaussian homological parity is well defined on the set \( \mathcal{G} \).
**Definition 1.8** A parity is called *Gaussian* if it agrees with the Gaussian homological parity (in a sense of Definition 1.7).

Note that this definition of Gaussian parity coincides with one given in Example 1.1.

The parity axioms listed at the beginning of this section lead to a filtration on the set of 4-valent graphs and on the set of virtual link diagrams.

Consider a virtual link diagram \( L \). Let \( f(L) \) be the diagram obtained from \( L \) by making all odd crossings virtual. By analogy one can state the graph version of the action of \( f \).

The next fact follows from definition.

**Theorem 1.2** The map \( f \) is a well-defined map on the set of all virtual links.

*Proof.* We must prove that if link diagrams \( L_1 \) and \( L_2 \) are equivalent by a Reidemeister move then the diagrams \( f(L_1) \) and \( f(L_2) \) are also equivalent. We check this only for the third Reidemeister move (other cases are similar).

Recall that if vertices \( a, b, c \) take part in the third move then there is either one or three even vertices among them (according to the third parity axiom). If all the three vertices are even, then \( f(L_1) = L_1 \) and \( f(L_2) = L_2 \), so that \( f(L_1) \) and \( f(L_2) \) also differ by the same move. In the other case virtual diagrams are equivalent by the detour move (this is one of the generalized Reidemeister moves). \( \square \)

Note that analogous theorem holds for *any* knot-like objects (e.g., for free links) and for *any* parity on these objects.

It turns out that one can construct a filtration on the set of all virtual link by means of the map \( f \). Let us introduce the following notation: denote the set of link diagrams having all even crossings by \( \mathcal{A}^0 \), the set of link diagrams \( L \) such that \( f(L) \in \mathcal{A}^0 \) by \( \mathcal{A}^1 \), etc. Then there is a natural filtration \( \mathcal{A}^0 \subset \mathcal{A}^1 \subset \ldots \subset \mathcal{A}^n \subset \ldots \)

2 Atoms and their orientability

The notion of atom was introduced by A.T.Fomenko in [F] for the study of bifurcations of integrable Hamiltonian systems. In particular, atoms describe the structure of Morse functions on 2-manifolds in the neighbourhood of the
critical level with several critical points belonging to it. First connections
between atoms and knots were discovered by V. O. Manturov in papers on
graphs embedded into surfaces and his construction of Khovanov homology
for virtual links (see [Ma2],[Ma3]).

Let $P$ be a smooth closed compact 2-manifold. Let $\Gamma$ be a graph of
valency 4 embedded into $P$ such that the graph splits $P$ into cells.

**Definition 2.1** A pair $(P, \Gamma)$ is called an atom if the set of connected
components in $P \setminus \Gamma$ can be divided into two subsets (black and white cells)
so that every edge of $\Gamma$ is incident to one black and to one white cell.

**Definition 2.2** The graph $\Gamma$ is called a frame of the atom $(P, \Gamma)$.
Atoms will be considered up to a natural isomorphism.

**Definition 2.3** Two atoms $(P_1, \Gamma_1)$ and $(P_2, \Gamma_2)$ are called isomorphic if
there exists a homeomorphism $\phi: P_1 \to P_2$, taking frame into frame, black
cells to black cells, and white cells to white ones.

**Definition 2.4** An atom $(P, \Gamma)$ is called orientable if the surface $P$ is
orientable, otherwise it is called non-orientable.

The framing of $\Gamma$ is induced by its embedding into $P$.

**Definition 2.5** One says that an orientation of edges of $\Gamma$ determines
the source-sink structure if they can be oriented in such a way that in every
vertex two opposite edges are incoming, and the other two are emanating.

Note that if it is possible to determine the source-sink structure on $\Gamma$ then
it is determined completely by choosing orientation of any single edge of $\Gamma$.

The following three theorems were proved in [Ma4]:

**Theorem 2.1** An atom $(P, \Gamma)$ is orientable if and only if it is possible to
determine the source-sink structure on its frame.

**Theorem 2.2** An atom $(P, \Gamma)$ is orientable if and only if any cycle of $\Gamma$
is orientable.

**Theorem 2.3** A cycle of an atom $(P, \Gamma)$ is orientable if and only if this
cycle goes through opposite half-edges in even number of vertices of $\Gamma$.

We point out that starting from a frame with $n$ vertices it is possible to
construct $2^n$ atoms (some of them might turn out to be isomorphic) choosing
a pair of non-opposite half-edges in every vertex, which form the border of
black cells. It follows from Theorem 2.1 that all atoms obtained in this way
are either simultaneously orientable or non-orientable.
3 Proof of the main theorem

Theorem 3.1 If two virtual diagrams \( K \) and \( K' \) have orientable atoms and they are equivalent by Reidemeister moves, then there is a sequence of diagrams \( K = K_1 \to \ldots \to K_n = K' \) all having orientable atoms where \( K_i \) is obtained from \( K_{i-1} \) by a Reidemeister move.

Proof. By standard applying of the second Reidemeister move one can obtain another sequence \( K_1 \to \ldots \to K'_1 \to K'_2 \to \ldots \to K'_n \to \ldots \to K_n \) where every two neighbouring diagrams differ by a Reidemeister move and the intersection graph of unicursal components is connected.

Since the diagram \( K'_1 \) has an orientable atom there exists the source-sink structure on \( K'_1 \). Therefore \( K'_1 \in \mathcal{G} \) (here \( \mathcal{G} \) is the set of framed 4-graphs defined in Example 1.3). For any unicursal component of \( K'_1 \) the parity of the total number of intersection points with other components is an invariant under Reidemeister moves. So we have \( K'_2, \ldots, K'_n \in \mathcal{G} \).

Consider the Gaussian homological parity \( h \) on \( \mathcal{G} \). Choose a Gaussian parity \( \mathcal{P}_1 \) such that \( \mathcal{P}_1(v) = 0 \) for any vertex \( v \in K'_1 \).

It follows from orientability of atoms for \( K'_n \) that for each vertex \( w \in K'_n \) holds \( \mathcal{P}_1(w) = 0 \).

Apply the map \( f \) to the sequence of diagrams \( K'_1 \to \ldots \to K'_n \) as many times as necessary to make all intermediate link diagrams having orientable atom. Since \( f(K'_1) = K'_1 \) and \( f(K'_n) = K'_n \) we will obtain a sequence of diagrams with orientable atoms between \( K'_1 \) and \( K'_n \).

Then we perform the same method to sequences \( K_1 \to \ldots \to K'_1 \) and \( K'_n \to \ldots \to K_n \). In both cases we obtain sequences with all intermediate diagrams having orientable atoms.

This completes the proof of the Theorem. \( \square \)

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