A WEIGHTED PRÉKOPA–LEINDLER INEQUALITY
AND SUMSETS WITH QUASICUBES

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Abstract. We give a short, self-contained proof of two key results
from a paper of four of the authors. The first is a kind of weighted
discrete Prékopa-Leindler inequality. This is then applied to show
that if $A, B \subseteq \mathbb{Z}^d$ are finite sets and $U$ is a subset of a “quasicube”
then $|A + B + U| \geq |A|^{1/2} |B|^{1/2} |U|$. This result is a key ingredient
in forthcoming work of the fifth author and Pálvölgyi on the sum-
product phenomenon.

1. Introduction

Quasicubes. The notion of a quasicube $\Sigma \subseteq \mathbb{Z}^d$ is defined inductively.
When $d = 1$, a quasicube is simply a set of size two. For larger $d$, $\Sigma$ is
a quasicube if

(1) $\pi(\Sigma) = \{x_0, x_1\}$ is a set of size two, where $\pi : \mathbb{Z}^d \to \mathbb{Z}$ is the
coordinate projection onto the final coordinate, and

(2) The fibre $\Sigma_i := \Sigma \cap \pi^{-1}(x_i)$ (considered as a subset of $\mathbb{Z}^{d-1}$) is
a quasicube.

Thus, for instance, the usual cube $\{(0, 1)^d$ is a quasicube. Another ex-
ample of a quasicube with $d = 2$ is the set $\Sigma = \{(0, 0), (1, 0), (0, 1), (1, 2)\}$.

The following result is established in [5].

Theorem 1.1. Let $A, B \subseteq \mathbb{Z}^d$ be finite sets and suppose that $U \subseteq \mathbb{Z}^d$
is contained in a quasicube. Then $|A + B + U| \geq |A|^{1/2} |B|^{1/2} |U|$. 

Our aim in this note is to give a short, self-contained proof of this
result.

2. A Weighted Discrete Prékopa–Leindler Inequality

As in [5], we deduce Theorem 1.1 from a weighted discrete Prékopa–
Leindler inequality. Let $a, b : \mathbb{Z} \to [0, \infty)$ be compactly supported

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functions. We define the *max-convolution*
\[ a \ast b(n) := \sup_{m \in \mathbb{Z}} a(n - m)b(m), \]
and we write
\[ \|a\|_2 := \left( \sum_n a(n)^2 \right)^{1/2}, \quad \|b\|_2 := \left( \sum_n b(n)^2 \right)^{1/2}. \]

The following result is equivalent to [5, Theorem 11.1].

**Proposition 2.1.** Let \( a, b : \mathbb{Z} \to [0, \infty) \) be compactly supported functions and let \( p \in [0, 1] \). Then we have
\[
\sum_n \max(pa \ast b(n), (1 - p)a \ast b(n - 1)) \geq \|a\|_2 \|b\|_2.
\]

In the case \( p = \frac{1}{2} \), this is (2.4) in the paper of Prékopa [6], where it is used to establish the 1-dimensional case of what is now known as the Prékopa–Leindler inequality (we will recall the statement of this below). We will proceed in the opposite direction, deducing Proposition 2.1 from Prékopa–Leindler.

Suppose that \( f, g : \mathbb{R} \to [0, \infty) \) are compactly supported, piecewise continuous functions. Then the (1-dimensional) Prékopa–Leindler inequality states that
\[
\int f \ast g \geq 2\|f\|_2 \|g\|_2,
\]
where the max-convolution is defined by
\[ f \ast g(x) := \sup_{y \in \mathbb{R}} f(x - y)g(y), \]
and the norms are the usual Lebesgue norms
\[ \|f\|_2 := \left( \int f^2 \right)^{1/2}, \quad \|g\|_2 := \left( \int g^2 \right)^{1/2}. \]

(It should always be clear from context whether we are applying \( \ast \) or \( \| \cdot \|_2 \) with functions on \( \mathbb{Z} \) or functions on \( \mathbb{R} \)). We note that Brascamp and Lieb [2] found a much shorter proof of (2.1) than the original (see also this survey of Gardner [3]).

**Proof of Proposition 2.1.** By continuity we may assume that \( p \in (0, 1) \). Set \( \lambda := \log(\frac{1}{p} - 1) \). Apply (2.1) with functions \( f, g \) defined by
\[
f(x) := e^{\lambda x}a([x]), \quad g(y) := e^{\lambda y}b([y]).
\]
Let \( n \in \mathbb{Z} \) and \( 0 < t < 1 \). Suppose that \( x + y = n + t \). Then, since \( x - 1 < \lfloor x \rfloor \leq x \), we have \( n - 2 < \lfloor x \rfloor + \lfloor y \rfloor < n + 1 \), or in other words \( \lfloor x \rfloor + \lfloor y \rfloor = n - 1 \) or \( n \). If \( \lfloor x \rfloor + \lfloor y \rfloor = n - 1 \) then
\[
f(x)g(y) \leq e^{\lambda(t+1)}a\bar{a}b(n-1),
\]
whilst if \( \lfloor x \rfloor + \lfloor y \rfloor = n \) then
\[
f(x)g(y) \leq e^{\lambda}a\bar{a}b(n).
\]
Therefore
\[
\int f \bar{g}(n + t) \leq e^{\lambda} \max(a\bar{a}b(n), e^{\lambda}a\bar{a}b(n-1)).
\]
Integrating over \( t \in [0, 1) \) and then summing over \( n \in \mathbb{Z} \) yields
\[
\int f \bar{g} \leq e^{\lambda} \frac{1}{\lambda} \sum_{n} \max(a\bar{a}b(n), e^{\lambda}a\bar{a}b(n-1)). \tag{2.2}
\]
On the other hand,
\[
\|f\|^2 = e^{2\lambda} - 1 \|a\|^2, \quad \|g\|^2 = e^{2\lambda} - 1 \|b\|^2.
\]
Substituting into (2.1) gives
\[
\sum_{n} \max(a\bar{a}b(n), e^{\lambda}a\bar{a}b(n-1)) \geq (e^{\lambda} + 1) \|a\| \|b\|.
\]
Recalling the choice of \( \lambda \) (thus \( p = \frac{1}{e^{\lambda+1}} \)), the proposition follows. \( \square \)

3. Proof of the main theorem

The arguments of this section are all in [5], but there they form part of a more general framework. Here we provide a self-contained account tailored to the specific purpose of proving Theorem 1.1.

Proof of Theorem 1.1. We proceed by induction on \( d \). The proof of the inductive step also proves the base case \( d = 1 \).

Suppose that \( U \) is contained in a quasicube \( \Sigma \subset \mathbb{Z}^d \). Suppose that \( \pi(\Sigma) = \{x_0, x_1\} \), where \( \pi : \mathbb{Z}^d \to \mathbb{Z} \) is projection onto the last coordinate. Since the inequality is translation-invariant, we may assume that \( x_0 = 0 \) and \( x_1 = q > 0 \). Suppose first that \( q = 1 \).

Let \( A_i := A \cap \pi^{-1}(n) \) be the fibre of \( A \) above \( n \), and similarly for \( B \). The set \( U \) has just two fibres \( U_0, U_1 \) and, by the definition of quasicubes, they are both contained in quasicubes of dimension \( d - 1 \).
Observe that the fibre of $A + B + U$ above $n$ contains $A_x + B_y + U_0$ whenever $x + y = n$, and $A_x + B_y + U_1$ whenever $x + y = n - 1$. By induction,

$$|A_x + B_y + U_0| \geq |A_x|^{1/2} |B_y|^{1/2} |U_0|,$$

$$|A_x + B_y + U_1| \geq |A_x|^{1/2} |B_y|^{1/2} |U_1|,$$

and so the fibre $(A + B + U)_n$ of $A + B + U$ above $n$ has size at least

$$\max \left( |U_0| \frac{a(3.1)}{\max |A_x|^{1/2} |B_y|^{1/2} }, |U_1| \frac{a(3.2)}{\max |x+y=n-1 |A_x|^{1/2} |B_y|^{1/2} } \right).$$

This is equal to

$$|A + B + U| = \sum_n (|A + B + U)_n| \geq |U| \sum_n \max \left( p a(3.1) + (1 - p) a(3.2), n \right) \geq |U| \sum_n \max \left( a(3.1) + (1 - p) a(3.2), n \right),$$

This proves the result when $q = 1$. Suppose now that $q$ is arbitrary, and foliate $A = \bigcup_{r \in \mathbb{Z}/q\mathbb{Z}} A_r$, $B = \bigcup_{s \in \mathbb{Z}/q\mathbb{Z}} B_s$, where $A_r := \{ a \in A : \pi(a) \equiv r(\text{mod } q) \}$ and similarly for $B_s$. Let $r_*$ be such that $|A_r| \leq |A_{r_*}|$ for all $r$, and $s_*$ be such that $|B_s| \leq |B_{s_*}|$ for all $s$.

The sets $A_{r_*} + B_{s_*} + U$ are disjoint as $s$ varies, and so by the case $q = 1$ (rescaled) we have

$$|A + B + U| \geq \sum_s |A_{r_*} + B_{s_*} + U| \geq |U| |A_{r_*}|^{1/2} \sum_s |B_{s_*}|^{1/2}. \quad (3.1)$$

Similarly,

$$|A + B + U| \geq |U| |B_{s_*}|^{1/2} \sum_r |A_r|^{1/2}. \quad (3.2)$$

Taking products of (3.1), (3.2) and using

$$|A_{r_*}|^{1/2} \sum_r |A_r|^{1/2} \geq \sum_r |A_r| = |A|,$$

$$|B_{s_*}|^{1/2} \sum_s |B_s|^{1/2} \geq \sum_s |B_s| = |B|,$$

the result follows. \qed
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