Appendix: Future Gradient Descent for Adapting the Temporal Shifting Data Distribution in Online Recommendation Systems

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Extra Notation. We introduce several new notations for the appendix. We use $\langle \cdot, \cdot \rangle$ to denote the inner product between two vectors and use $\circ$ to denote the entrywise product.

1 PROOF OF THEOREM

Proof. We start with a simple decomposition using the triangle inequality:

$$\|u_{w,t}(\theta_t)\| \leq \|u_{w,t}(\theta_t) - \bar{m}(\theta_t; t)\| + \|\bar{m}(\theta_t; t)\|.$$  

By the termination condition of Algorithm 2, we have $\|\bar{m}(\theta_t; t)\| \leq \delta$. Furthermore, it follows from (5) that

$$\|u_{w,t}(\theta_t) - \bar{m}(\theta_t; t)\| = \frac{1}{w} \|\nabla r_t(\theta_t) - m(\theta_t; t)\|.$$  

Hence, we obtain

$$\|u_{w,t}(\theta_t)\|^2 \leq \left( \delta + \frac{1}{w} \|\nabla r_t(\theta_t) - m(\theta_t; t)\| \right)^2 \leq 2\delta^2 + \frac{2}{w^2} \|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2.$$  

(1)

This further implies that

$$\mathfrak{R}_w(T) = \frac{1}{T} \sum_{t=1}^T \|u_{w,t}(\theta_t)\|^2 \leq \frac{2}{w^2T} \sum_{t=1}^T \|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 + 2\delta^2,$$  

(2)

and the main result follows from the fact that $\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 \leq \sup_{\theta} \|\nabla r_t(\theta) - m(\theta; t)\|^2$ for all $t \in [T]$. Furthermore, under the boundedness assumption, we have for all $t \in [T]$

$$\|\nabla r_t(\theta_t) - m(\theta_t; t)\|^2 \leq (\|\nabla r_t(\theta_t)\| + \|m(\theta_t; t)\|)^2 \leq 4M^2.$$  

(3)

Hence, (3) also implies $\mathfrak{R}_w(T) \leq 8M^2/w^2 + 2\delta^2$, which leads to $\mathfrak{R}_w(T) = O(1/w^2)$ when $\delta = 1/w$. \hfill \Box

2 DETAILS OF THE RESULT IN SECTION 4.4

Algorithm. Given $\theta_t$, define $h_t(\phi) = \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2$ as a function of $\phi$, where we view $\theta_t$ as a constant. Thus, if follows from that (3) that

$$\mathfrak{R}_w(T) \leq \frac{2}{w^2T} \sum_{t=1}^T h_t(\phi_t) + 2\delta^2.$$  

(4)
where we used Cauchy-Schwarz inequality in (6), the triangle inequality in (7) and the boundedness of the gradients in (8).

Algorithm 1 Generalized Future Gradient Descent for Smoothed Regret (simplified version for the theoretical study)

Input: The learning rate $\eta, \eta_\phi$ for updating the model parameter $\theta$ and $\phi$.
Initialize $\phi_1 = [1/b, \ldots, 1/b]$.
for $t \in [T]$ do
  Deploy the prediction model $f_{\theta_t}$ with the parameter $\theta_t$ and collect the new dataset $D_t$.
  Construct the function $h_t(\phi) = \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2$
  Initialize the model parameter $\theta$.
  while $\|m(\theta_{t+1}; \phi_{t+1}, t)\| \geq \delta$ do
    $\theta_{t+1} = \theta_t - \eta m(\theta_{t+1}; \phi_{t+1}, t)$.
  end while
end for

Thus, our goal is to minimize $\sum_{t=1}^{T} h_t(\phi_t)$ in an online manner, since we can only access $h_t(\phi_t)$ after $\phi_t$ is chosen. To achieve this, we use the classic exponentiated gradient method to update $\phi_t$. Specifically, for any $\phi = [a_1, \ldots, a_b] \in S_b$, define the negative potential function $\psi(\phi) = \sum_{i=1}^{b} a_i \log a_i$ and its Bregman divergence

$$B_\psi(\phi; \phi') = \psi(\phi) - \psi(\phi') - \langle \nabla \psi(\phi'), \phi - \phi' \rangle = \sum_{i=1}^{b} a_i \log \frac{a_i}{\phi'_i}.$$ 

Then $\phi_{t+1}$ is given by

$$\phi_{t+1} = \arg \min_{\phi \in S_b} \left( \nabla h_t, \phi + \frac{1}{\eta_\phi} B_\psi(\phi; \phi_t) \right) = \frac{\phi_t \exp(-\eta_\phi \nabla h_t(\phi_t))}{\|\phi_t \exp(-\eta_\phi \nabla h_t(\phi_t))\|_1},$$ 

where $\eta_\phi$ is the learning rate. See Section 6.6 in Orabona [2019] for the derivation of the last equality. Intuitively, $\frac{1}{\eta_\phi} B_\psi(\phi; \phi_t)$ stabilizes the algorithm by ensuring that $\phi_{t+1}$ remains close to $\phi_t$.

This simplified version of FGD is summarized in Algorithm 1. Note that when updating $\phi$, we only use the last recommendation model $\theta_t$.

Lemma 1. Suppose that we have $\|\nabla r_t(\theta)\| \leq M$ for all $\theta \in \Theta$ and $t$. Then $\|\nabla h_t(\phi)\|_\infty \leq 8M^2$ for all $\phi \in S_b$.

Proof. By definition, we have

$$h_t(\phi) = \|\nabla r_t(\theta_t) - \sum_{i=1}^{b} a_i \nabla r_{t-i}(\theta_t)\|^2 = \left\| \sum_{i=1}^{b} a_i (\nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t)) \right\|^2,$$

where we used the fact that $\sum_{i=1}^{b} a_i = 1$. Direct computation shows that

$$\left| \frac{\partial h_t}{\partial a_i}(\phi) \right| = 2 \left\langle \nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t), \sum_{j=1}^{b} a_j (\nabla r_t(\theta_t) - \nabla r_{t-j}(\theta_t)) \right\rangle \leq 2 \|\nabla r_t(\theta_t) - \nabla r_{t-i}(\theta_t)\| \left\| \sum_{j=1}^{b} a_j (\nabla r_t(\theta_t) - \nabla r_{t-j}(\theta_t)) \right\| \leq 2 \left( \|\nabla r_t(\theta_t)\| + \|\nabla r_{t-i}(\theta_t)\| \right) \left( \sum_{j=1}^{b} a_j \left( \|\nabla r_t(\theta_t)\| + \|\nabla r_{t-j}(\theta_t)\| \right) \right) \leq 8M^2,$$

where we used Cauchy-Schwarz inequality in (6), the triangle inequality in (7) and the boundedness of the gradients in (8). Hence, we conclude that $\|\nabla h_t(\phi)\|_\infty \leq 8M^2$. □
Proof of Theorem 2. Now we proceed to the proof of Theorem 2. This is a standard result in the online learning literature (see, e.g., [Orabona, 2019]). For completeness, we present the proof below.

Proof. As $\psi$ is $\lambda$-strongly convex with $\lambda = 1$, we have

$$B_\psi(\phi; \phi') \geq \frac{1}{2} \|\phi - \phi'\|^2_1. \tag{9}$$

Throughout the proof, we slightly abuse the notation by writing $\eta_\phi = \eta$ and $\nabla h_t = \nabla h_t(\phi_t)$ for simplicity. Notice that by our update rule $\phi_{t+1}$ is given by

$$\phi_{t+1} = \arg \min_{\phi \in S_b} (\eta(\nabla h_t, \phi) + B_\psi(\phi; \phi_t)).$$

From the first-order optimality condition, we get for any $\phi \in S_b$,

$$\langle \eta \nabla h_t + \nabla \psi(\phi_{t+1}) - \nabla \psi(\phi_t), \phi_{t+1} - \phi \rangle \leq 0$$

$$\Leftrightarrow \eta \langle \nabla h_t, \phi_t - \phi \rangle \leq \eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle + (\nabla \psi(\phi_{t+1}) - \nabla \psi(\phi_t), \phi - \phi_{t+1})$$

$$\Leftrightarrow \eta \langle \nabla h_t, \phi_t - \phi \rangle \leq \eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle - B_\psi(\phi; \phi_{t+1}) + B_\psi(\phi; \phi_t) - B_\psi(\phi_{t+1}; \phi_t),$$

where we used the three-point equality [Chen and Teboulle, 1993] in the last inequality. Furthermore,

$$\eta \langle \nabla h_t, \phi_t - \phi_{t+1} \rangle - B_\psi(\phi; \phi_{t+1}) \leq \eta \|\nabla h_t\|_\infty \|\phi_t - \phi_{t+1}\|_1 - \frac{1}{2} \|\phi_t - \phi_{t+1}\|^2_1$$

$$\leq \frac{\eta^2}{2} \|\nabla h_t\|^2_\infty + \frac{1}{2} \|\phi_t - \phi_{t+1}\|^2_1 - \frac{1}{2} \|\phi_t - \phi_{t+1}\|^2_1$$

$$= \frac{\eta^2}{2} \|\nabla h_t\|^2_\infty.$$

Combining these two bounds, we have

$$\eta \langle \nabla h_t, \phi_t - \phi \rangle \leq \frac{\eta}{2} \|\nabla h_t\|^2_\infty.$$

Since $h_t(\phi)$ is convex in $\phi$, we have $h_t(\phi_t) - h_t(\phi) \leq \langle \nabla h_t, \phi_t - \phi \rangle$ for any $\phi \in S_b$. By telescoping, we obtain

$$\sum_{t=1}^T (h_t(\phi_t) - h_t(\phi)) \leq \sum_{t=1}^T \langle \nabla h_t, \phi_t - \phi \rangle$$

$$\leq \frac{1}{\eta} \sum_{t=1}^T \left[ B_\psi(\phi; \phi_t) - B_\psi(\phi; \phi_{t+1}) + \frac{\eta^2}{2} \|\nabla h_t\|^2_\infty \right]$$

$$= \frac{1}{\eta} \left( B_\psi(\phi; \phi_1) - B_\psi(\phi; \phi_{T+1}) \right) + \frac{\eta}{2} \sum_{t=1}^T \|\nabla h_t\|^2_\infty$$

$$\leq \frac{1}{\eta} \log b + 32 \eta M^4 T.$$

where we used Lemma 8 $B_\psi(\phi; \phi_{T+1}) \geq 0$ and $B_\psi(\phi; \phi_1) = \psi(\phi) + \log b \leq \log b$ in the last inequality. Choosing $\eta = c \sqrt{(\log b) / (TM^4)}$ with some constant $c > 0$ leads to

$$\sum_{t=1}^T [h_t(\phi_t) - h_t(\phi)] \leq O(M^2 \sqrt{T \log b}). \tag{10}$$
Algorithm 2 Generalized Future Gradient Descent for Smoothed Loss

Input: The learning rate $\eta, \eta_\phi$ for updating the model parameter $\theta$ and $\phi$. The initial trajectory buffer $B$.

for $t \in [T]$ do

Deploy the prediction model $f_{\theta_t}$ with parameter $\theta_t$. Then collect the new dataset $D_t$.

Initialize the parameter of MFGG $\phi_{t+1}$.

for Inner loop iteration $k \in K$ do

$\phi_{t+1} \leftarrow \phi_{t+1} - \eta_\phi \sum_{\theta \in B} \nabla_\phi \|m(\theta; \phi_{t+1}, t) - \nabla r_t(\theta)\|^2$.

end for

Initialize the trajectory buffer $B = \emptyset$ and model parameter $\theta_{t+1}$.

while $\|\bar{m}(\theta_{t+1}; \phi_{t+1}, t + 1)\| \geq \delta$ do

$\theta_{t+1} \leftarrow \theta_{t+1} - \eta m(\theta_{t+1}; \phi_{t+1}, t + 1)$.

$B \leftarrow B \cup \{\theta_{t+1}\}$

end while

end for

Note that (10) holds for any $\phi \in S_b$. In particular, we can set $\phi = \phi^*$ defined by $\phi^* = \arg\min_{\phi \in S_b} \sum_{t=1}^{T} h_t(\phi)$. Therefore,

$$\sum_{t=1}^{T} h_t(\phi_t) \leq \sum_{t=1}^{T} h_t(\phi^*) + O(M^2 \sqrt{T \log b})$$

$$= \min_{\phi \in S_b} \sum_{t=1}^{T} \|\nabla r_t(\theta_t) - m(\theta_t; \phi, t)\|^2 + O(M^2 \sqrt{T \log b})$$

$$\leq \min_{\phi \in S_b} \sum_{t=1}^{T} \sup_{\theta} \|\nabla r_t(\theta) - m(\theta; \phi, t)\|^2 + O(M^2 \sqrt{T \log b}) = \min_{m \in M} Q[T; m] + O(M^2 \sqrt{T \log b}).$$

We thus conclude from (4) that

$$R_w(T) \leq \frac{2}{w^2 T} \left( \min_{m \in M} Q[T; m] + O(M^2 \sqrt{T \log b}) \right) + 2\delta^2.$$

3 A PRACTICAL GENERALIZED FGD ALGORITHM.

Compared with FGD in Algorithm 2, we use a smoothed version of MFGG $\bar{m}$ for training, which is due to the consideration of minimizing a smoothed loss in (2). For completeness, we also summarize the practical algorithm of the generalized version of FGD in Algorithm 2.

References

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Francesco Orabona. A modern introduction to online learning. arXiv:1912.13213, 2019.