PARABOLA-INSPIRED PONCELET POLYGONS
DERIVED FROM THE BICENTRIC FAMILY

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Abstract. We study loci and properties of a Parabola-inscribed family of
Poncelet polygons whose caustic is a focus-centered circle. This family is
the polar image of a special case of the bicentric family with respect to its
circumcircle. We describe closure conditions, curious loci, and new conserved
quantities.

1. Introduction

This is a continuation of our investigation of Euclidean phenomena of Poncelet
families [12, 11, 14, 20]. Recall Poncelet’s porism: specially-chosen pairs of conics
$C, C'$ admit a one-parameter family of polygons inscribed in $C$ while simultaneously
circumscribed about $C'$ [5, 7, 8].

Here we consider a certain family such that $C$ is a parabola $P$ while $C'$ is a
circle centered on the focus of $P$. As shown in Figure 1, this is simply the polar
image of the bicentric family (interscribed between two circles) with respect to its
circumcircle, see Appendices A and B for construction details. We derive closure
conditions for this new family for $N = 3, 4, 5, 6$ cases ($N$ is the number of sides) and
describe some of its properties and loci of associated points. Also considered is its
polar image with respect to $P$.

Main results.

- The loci of vertex, perimeter, and area centroids are parabolas. Recall that
  in general, the locus of the perimeter centroid is not a conic [22].
- The loci of vertex and area centroids of polar polygons are straight lines,
  whereas that of the perimeter centroid is a non-conic.
- In the $N = 3$ case, the locus of the orthocenter is a straight line as are those
  of many triangle centers of the polar family. The Euler line of the polar
  family always passes through the parabola’s focus.
- Several centers of the $N = 3$ polar family are stationary and/or sweep circles.
  In the latter case, they all belong to a single parabolic pencil.
- We prove that the quantity $\sum \sin \theta_i/2$ is conserved, where $\theta_i$ are the interior
  angles of parabola-inscribed polygons. In fact, this quantity is conserved by
  any conic-inscribed polar image of the bicentric family.

Most of the above properties were first noticed via simulation [26], and later proved
with a computer-algebra system (CAS) [17], using the explicit parametrizations
given in Appendix A. For brevity, we omit any CAS-based proofs.

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Figure 1. Several configurations of the parabola-inscribed Poncelet family (green), obtainable as the polar image of the bicentric family (blue) with respect to the outer circle (black), provided the bicentric incircle passes through the circumcenter, see Appendix B. Also shown is the polar family (red) of the parabola-inscribed one with respect to the parabola itself. This family is inscribed in a hyperbola (dark dashed red). So you can think of this trio (blue, green, red) as successive polar images with respect to the outer conic of each preceding family.

Related work. We can roughly divide it into three groups: (i) the study of point loci over certain triangle families [18, 19, 28], (ii) proving that loci of certain Poncelet triangle families are of a given curve type [9, 13, 21, 24], and (iii) proving properties and invariants over $N \geq 3$ Poncelet families [2, 4, 6, 22]. Also related is the Steiner-Soddy Poncelet family which are the polar image of the so-called Brocard porism with respect to the circumcircle [10].

Article organization. In Sections 2 and 3 we examine parabola-inscribed Poncelet triangles (as well as its polar polygons with respect to the parabola), deriving closure conditions and expressions for many of its their triangle center loci. In Sections 4 to 6 we derive geometric closure conditions for $N = 4, 5, 6$ families, respectively, detecting the abovementioned pattern for the loci of their centroids (as well as in the polar family), followed by conjectured generalizations in Section 7. In Section 8 we describe a new quantity conserved by the parabola-inscribed family (and variations thereof).

In Appendix A we provide explicit parametrizations for the vertices of both the $N = 3$ and $N = 4$ families, as well as their respective polar families. In Appendix B
we explore the relation of parabola-inscribed families with the traditional bicentric family.

2. LOCI OF PARABOLA-INSERBED TRIANGLES

Referring to Figure 2, consider a Poncelet family \( T \) of triangles inscribed in a parabola \( \mathcal{P} \), and circumscribed about a focus-centered circle. Let \( F = [-f,0] \) and \( V = [0,0] \) denote focus and vertex, respectively, where \( f \) is the focal distance. Consider a circle \( \mathcal{C} \) centered at \( F \) with radius \( r \).

**Proposition 1.** \( \mathcal{P} \) and \( \mathcal{C} \) will admit a Poncelet family of triangles iff \( r/f = 2(\sqrt{2} - 1) \).

**Proof.** Consider the Poncelet triangle with two parallel sides shown in Figure 3, inscribed in the parabola \( y = x^2/(4f) \), where \( f \) is the focal length. At \( x = r \) the parabola must be at \( y = f - r \), i.e., \( f - r = r^2/(4f) \), and the result follows. \( \square \)

2.1. **Straight-line orthocenter locus.** Let \( \mathcal{T} \) denote our parabola-inscribed triangle family and \( \mathcal{D} \) the directrix of \( \mathcal{P} \). Henceforth we shall adopt Kimberling’s notation \( X_k \) to refer to triangle centers [16].

**Proposition 2.** Over \( \mathcal{T} \), the locus of the orthocenter \( X_4 \) is the line parallel to \( \mathcal{D} \) given by \( x = (5 - 2\sqrt{2})f \), with \( y \in \mathbb{R} \setminus \{\pm2f/(\sqrt{2} + 1)\} \).
The proof below was kindly contributed by Alexey Zaslavsky [27]:

**Proof.** Let $C$ be the unit circle in the complex plane and $A, B, C$ the touching points with the sides of the parabola-inscribed triangles. The polar transformation with center $F$ maps the parabola to a circle with center $I$ passing through $F$ and touching $AB, BC, CA$. Using Euler’s formula $|FI|^2 = r^2 = R(R - 2r)$ [25], with $R = 1$ its radius, and $r = \sqrt{2} - 1$. Consider the line $FI$ as the real axis. Since $I$ is self-conjugated with respect to $ABC$, we have

$$a + b + c = 2\sqrt{2} - 2 + (3 - 2\sqrt{2})abc,$$
$$ab + bc + ca = 3 - \sqrt{2} + (2\sqrt{2} - 2)abc.$$  

The polar images of the altitudes of the original triangle are the common points of $BC, CA, AB$ with the lines passing through $F$, and perpendicular to $FA, FB, FC$ respectively. We have to calculate the common point of the line passing through these three points and the real axis. The coordinate functions of this point are symmetric functions in $a, b, c$, so we can express them as elementary symmetric functions on said variables, and verify that they are constant.

□

In Appendix B we describe how the parabola-inscribed family is the polar image of the bicentric family with respect to its circumcircle. Referring to Figure 4, Proposition 2 is actually a special case of:

**Proposition 3.** The locus of $X_4$ of an family which is the polar image of $N = 3$ bicentrics with respect to its outer circle is an ellipse, straight line, or hyperbola if the circumcenter of the bicentric triangle lies in the interior, on top, or outside its incircle.

2.2. Three parabolic loci. Referring to Figure 2, we show below that over $\mathcal{T}$, the loci of the barycenter, circumcenter, and Spieker centers are all parabolas. The first and last correspond to the vertex and perimeter centroids of a triangle. This is curious since, in general, the locus of the perimeter centroid of a Poncelet family is not a conic [23].

**Proposition 4.** Over $\mathcal{T}$, the locus of the barycenter $X_2$ is a parabola coaxial with $\mathcal{P}$, with focus $F_2 = [-f/3, 0]$, and vertex $V_2 = [2f(1 - 2\sqrt{2})/3, 0]$.

**Proposition 5.** Over $\mathcal{T}$, the locus of the circumcenter $X_3$ of $\mathcal{T}$ is a parabola coaxial with $\mathcal{P}$, with focus $F_3 = [-f(2\sqrt{2} - 3)/2, 0]$, and vertex $V_3 = [-f(2\sqrt{2} + 3)/2, 0]$.

**Proposition 6.** Over $\mathcal{T}$, the locus of the Spieker center $X_{10}$ is a parabola coaxial with $\mathcal{P}$, with focus $F_{10} = [(1 - 2\sqrt{2})f, 0]$ and vertex $V_{10} = [f(3/2 - 2\sqrt{2}), 0]$. In particular, $F_{10} = [-f - r, 0]$, i.e., it lies on the left extreme of $C$. 

*Figure 3. Construction used to derive $r/f$ in Proposition 1. Side $P_1P_2$ of the Poncelet triangle is perpendicular to the axis while the other two sides are parallel to it, i.e., vertex $P_3$ lies on the line at infinity.*
Figure 4. Consider perturbing to the bicentric family (blue) such that the circumcenter $O$ is interior (resp. exterior) to the incircle, as shown on the left (resp. right). The tangential family (green) becomes ellipse- (resp. hyperbola-) inscribed (gold curve). In the former (resp. latter) case, the locus of the orthocenter $X_4$ is an ellipse (resp. hyperbola).

3. THE POLAR $N = 3$ FAMILY

Referring to Figure 5, let $T'$ denote the polar triangle of a triangle $T$ in $\mathcal{T}$, i.e., whose sidelines are the polars of $T$ with respect to $\mathcal{P}$. Since $T$ is inscribed in $\mathcal{P}$ these are simply the tangents.

Recall some known properties of the polar triangle with respect to any parabola $[3]$: (i) the circumcircle of $T'$ passes through the focus $F$; (ii) the orthocenter of $T'$ is on the directrix; (iii) its area is half that of the reference triangle.

**Proposition 7.** The $T'$ family is Ponceletian. It is circumscribed about $\mathcal{P}$ and is inscribed in a hyperbola $\mathcal{H}$ with center $[f,0]$. Its axes are the axis and directrix of $\mathcal{P}$. Its implicit equation reads

$$\mathcal{H} : \left(\sqrt{2} + \frac{3}{2}\right)(x - f)^2 - \frac{y^2}{2} - 2f^2 = 0.$$

3.1. **Straight and nearly-straight loci.** An enduring conjecture has been that the locus of the incenter $X_1$ of a Poncelet triangle family can only be a conic if the pair is confocal $[15]$.

As shown in Figure 5, over the polars, the locus of the incenter is, to the naked eye, a straight line. However, upon an algebraic investigation:

**Proposition 8.** The locus of the incenter $X_1'$ of $T'$ is one of four branches of the following quartic:

$$X_1' : (-5\sqrt{2} - 6)x^2y^2 + (4\sqrt{2} + 2)f^2x^2 + (10\sqrt{2} + 12)fxy^2 + (8\sqrt{2} + 4)f^3x + (3\sqrt{2} - 16)f^2y^2 - 14f^4 = 0.$$

Specifically, the branch

$$X_1' = \left[\frac{\sqrt{2}y^2 + 2 + 2y^2 - \sqrt{-4y^2 + 4y^4 + 8\sqrt{2} + 8\sqrt{2}y^2 - 2\sqrt{2}y^4}}{\sqrt{2}y^2 - 2 + 2y^2}, y\right],$$
Figure 5. The polar triangle $T'$ (red) with respect to the parabola $P$ (gold) to which our Poncelet family (green) is inscribed. It is Ponceletian as it is inscribed in a hyperbola (dashed dark red). Well-know properties include (i) the circumcircle (dashed red) passes through the focus $F$, and (ii) the orthocenter $X'_4$ lies (and therefore sweeps) the directrix of $P$ [3]. Also shown are the visually-straight, though quartic loci of the polar incenter $X'_1$ and Spieker center $X'_{10}$ (red and olive, respectively). The loci of $X'_k$, $k = 2, 3, 4$ are straight lines parallel to the directrix (red, red, and dashed gold), the latter the directrix itself.

where $y \neq \pm \sqrt{2 - \sqrt{2}}$.

The locus of $X'_1$ is bounded by two lines parallel to the directrix and approximately $f/850$ apart, see Figure 6.

Still referring to Figure 5:

**Proposition 9.** The locus of the barycenter $X'_2$ of $T'$ is a line parallel to $D$ and parametrized by

$$X'_2 = \frac{1}{3} \left[ (2\sqrt{2} - 1)f, \frac{(4 - 8\sqrt{2})f^2 y + y^3}{(8\sqrt{2} - 12)f^2 + y^2} \right].$$

**Proposition 10.** The locus of the circumcenter $X'_3$ of $T'$ is a line parallel to $D$ and parametrized by

$$X'_3 = \left[ (\sqrt{2} - 1)f, \frac{(3\sqrt{2} + 2)(2\sqrt{2}y^2 - 28f^2 + y^2)y}{14(y\sqrt{2} - 2f + y)(y\sqrt{2} + 2f + y)} \right].$$

Referring to Figure 7, the above expressions for $X'_2$ and $X'_3$ yield:
Corollary 1. The (varying) Euler line $X'_2X'_4$ of the polar family passes through the focus $F = [-f, 0]$ of $\mathcal{P}$.

Still referring to Figure 7, the next 4 propositions were obtained from experimental evidence and verification by CAS:

Proposition 11. The locus of the symmedian point $X'_6$ of $T'$ is a line parallel to $D$ and parametrized by

$$X'_6 = \left[ (5 - 3\sqrt{2})f, \frac{(3\sqrt{2} + 4)(2\sqrt{2}y^2 - 28f^2 + y^2)y}{14(y\sqrt{2} - 2f + y)(y\sqrt{2} + 2f + y)} \right].$$

Proposition 12. The locus of $X'_{10}$ of the polar family is an algebraic curve of degree four given by

$$X'_{10} : 4 \left( 11\sqrt{2} + 16 \right) x^4 - 4 \left( 3\sqrt{2} + 5 \right) x^2y^2 - 4 \left( 37\sqrt{2} + 50 \right) f x^3 + 8 \left( 2\sqrt{2} + 1 \right) f xy^2 + 21 \left( 5\sqrt{2} + 8 \right) f^2x^2 - 4 \left( 9\sqrt{2} + 8 \right) f^3x - \left( \sqrt{2} + 4 \right) f^2y^2 + 7 f^4 = 0$$

This locus is tightly bound by the following two lines parallel to the directrix:

$$x = \left( \sqrt{2} - 1 + \frac{10 - 7\sqrt{2}}{2} \right) f \quad \text{and} \quad x = \left( \sqrt{2} - 2^{-1/4} \right) f.$$

The distance between these lines is approx. $f/1700$.

3.2. Stationary points. The circumcenter of the tangential triangle appears as $X_{26}$ on [16].

Proposition 13. Point $X'_{26}$ of $T'$ is stationary at the focus $F$ of $\mathcal{P}$.
Figure 7. Over the polar family (red), the Euler line (dashed magenta) will always pass through the focus $F$ of the parabola-inscribed family (green). $X'_{26}$ (resp. $X'_{68}$ and $X'_{110}$) remain stationary at the focus $F$ (resp. the two vertices of the hyperbola to which the polar family is inscribed). Experimentally, $X'_{161}$ is stationary at the intersection of the caustic with the parabola axis farthest from the latter’s vertex. Also shown is the Kiepert inparabola (magenta), whose focus is $X'_{110}$ and directrix is the Euler line. Thus the polar family simultaneously inscribes the original parabola (gold) and the Kiepert (magenta). Finally, the figure depicts the circular locus of Steiner point $X'_{99}$ of the polar family.

Note $X_{26}$ does not lie in general on the circumcircle of a reference triangle. In our case it does since, as mentioned above, the circumcircle of the polar contains the focus.

The Kiepert parabola of a triangle is an inscribed parabola whose focus is labeled $X_{110}$ on [16]. Its directrix is the Euler line [25]. Referring to Figure 7:

**Proposition 14.** The focus $X'_{110}$ of the Kiepert parabola (resp. the Prasolov point $X'_{68}$) of the polar family is stationary at the vertex of $H$ farthest (resp. closest) to the focus of $P$. Furthermore, $X'_{161}$ is stationary at the intersection of the incircle with the parabola axis farthest from the parabola vertex, i.e., at $[1 - 2\sqrt{2})f, 0]$.

**Observation 1.** Over the polar family, the vertex of its Kiepert parabola sweeps a circle.

3.3. Linear loci galore. Referring to Figure 8:

**Observation 2.** Over the first 1000 triangle centers in [16], the following triangle centers of $T'$ sweep linear loci parallel to $D$: $X'_{k}, k = 2, 3, 4, 5, 6, 20, 22, 23, 24, 25,$
Figure 8. Many triangle centers of the polar family sweep lines parallel to the directrix. The following are shown: \( X_k \), \( k = 2, 3, 4, 5, 6, 20, 22, 24, 25, 49, 51, 54, 64, 66, 67, 69, 74 \).

49, 51, 52, 54, 64, 66, 67, 69, 74, 113, 125, 140, 141, 143, 146, 154, 155, 156, 159, 182, 184, 185, 186, 193, 195, 206, 235, 265, 323, 343, 368, 370, 376, 378, 381, 382, 389, 394, 399, 403, 427, 428, 468, 546, 547, 548, 549, 550, 567, 568, 569, 575, 576, 578, 597, 599, 631, 632, 858, 895, 973, 974.

3.4. A pencil of circular loci. Referring to Figure 7:

**Proposition 15.** The locus of the Steiner point \( X'_{99} \) is a circle whose center \( O'_{99} \) lies on the axis of \( P \) of radius \( R'_{99} \) such that at its right endpoint it touches \( X'_{110} \).

Explicitly,

\[
O'_{99} = \left[ (6\sqrt{2} - 7)f, 0 \right], \quad R'_{99} = 2f \sqrt{17 - 12\sqrt{2}}.
\]

Referring to Figure 9:

**Observation 3.** Over the first 1000 triangle centers in [16], the following triangle centers of \( T' \) sweep circular loci with centers on the axis of \( P \) and passing through \( X'_{110} \): \( X_k', k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935 \).

This gives credence to:
Figure 9. Over the polar family we find that if a certain triangle center sweeps a circular locus, said locus will be an element of a parabolic pencil with $X_{110}$ as their common point (not labeled). In the figure the circular loci of $X_k, k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935$ are shown. Notice all lie on the dynamically-moving circumcircle (dashed red) except for $X_{249}$.

**Conjecture 1.** If the locus of $X_k'$ is a circle with nonzero radius, it is in the parabolic pencil with $X_{110}$ as a common point.

### 4. Parabola-Inscribed Quadrilaterals

Referring to Figure 10, consider a Poncelet family $Q$ of quadrilaterals inscribed in a parabola $P$, and circumscribed about a focus-centered circle $C$ of radius $r$. As before, let $f$ denote the parabola’s focal distance, and $V = [0, 0], F = [-f, 0]$, its vertex and focus, respectively.

**Proposition 16.** $P$ and $C$ will admit a Poncelet family of convex quadrilaterals iff $r/f = 2\sqrt{5} - 2$.

**Proof.** Referring to Figure 11, consider the symmetric Poncelet quadrilateral $P_i = [x_i, y_i], i = 1, \ldots, 4$, inscribed in the parabola $y = x^2/(4f)$, i.e., $x = 2\sqrt{fy}$. Clearly, $y_1 = f - r$, and $y_2 = f + r$. Requiring that $P_1P_2$ be tangent to $C$ yields the quartic $r^2 + 4f\sqrt{f^2 - r^2} = 0$. The claim is the one positive root of this quartic.

Note: more generally, Cayley’s conditions may be used to include the non-convex case, see [8].
Figure 10. A Poncelet quadrilateral (green) is shown inscribed in a parabola \( P \) (gold) and circumscribed about a focus-centered circle (brown). Over the family, (i) the intersection \( W \) of its diagonals (dashed green) is stationary; (ii) the loci of vertex \( C_0 \), perimeter \( C_1 \), and area \( C_2 \) centroids sweep 3 distinct parabolas (blue) coaxial with \( P \) with foci on \( F_0, F_1 \), and \( F_2 \). Notice the vertex of \( C_0 \) is \( F \) and that of \( C_1 \) is \( F_0 \). (iii) \( C_0, C_2, W \) are collinear (dashed blue). Also shown is the polar quadrilateral \( Q' \) (red) with respect to \( P \), inscribed in a hyperbola (dashed, red) centered at \( f, 0 \). One observes that: (i) its diagonals (dashed red) also intersect at \( W \); (ii) the loci of its vertex \( C'_0 \) and area \( C'_2 \) centroids are lines (dashed orange) perpendicular to the axis of \( P \); (iii) \( C'_0, C'_2, W \) are collinear (dashed red); (iv) the locus of the polar perimeter centroid \( C'_1 \) is algebraic and of degree 10.

Figure 11. Construction used to derive \( r/f \) in for parabola-inscribed convex quadrilaterals in Proposition 16.

The next 3 propositions, first identified experimentally, were then confirmed via CAS.

Proposition 17. Over \( Q \), the two major diagonals \( P_1P_3 \) and \( P_2P_4 \) intersect at a stationary point \( W = [(2 - \sqrt{5})f, 0] \).

4.1. The three centroids. Referring to Figure 10, let \( C_0, C_1, \) and \( C_2 \) denote the vertex, perimeter, and area centroids of the quadrilaterals in \( Q \), respectively.

Proposition 18. Over the family, \( C_0, C_2, \) and \( W \) are collinear.

Proposition 19. Over the Poncelet family, the loci of \( C_0, C_1, C_2 \) are parabolas coaxial with \( P \), whose foci and vertices locations are listed in Table 1.
Table 1. Location of centroids $C_0, C_1, C_2$ in the convex $N = 4$ family.

| Centroid (N=4) | Focal Dist. | Vertex $x/f$ | Vtx. $x/f$ (num.) |
|----------------|-------------|--------------|-------------------|
| $C_0$          | $f/4$       | $-1$         | $-1$              |
| $C_1$          | $f/2$       | $(\sqrt{5} - 5)/2$ | $-1.381966$      |
| $C_2$          | $f/3$       | $\sqrt{5}/3 - 2$ | $-1.25464$       |

4.2. The polar quadrilateral. Referring to Figure 10, consider the polar quadrilateral whose sides are the tangents to $P$ at the vertices of the original family. Let $P_i', i = 1, \ldots, 4$ denote its vertices and $C_0', C_1', C_2'$ denote its vertex, perimeter, and area centroids.

Proposition 20. The locus of the polar quadrilateral’s vertices is the hyperbola $\mathcal{H}$ given by

$$\mathcal{H} : \frac{(x - f)^2}{4(\sqrt{5} - 2)f^2} - \frac{y^2}{4f^2} - 1 = 0,$$

with center at $[f, 0]$ and foci $[f(1 \pm 2\sqrt{\sqrt{5} - 1}), 0]$.

Let $W$ be defined as in Proposition 17. The next two propositions result from visual (and numerical) detection, followed by verification by CAS.

Proposition 21. The two diagonals of the polar quadrilateral intersect at $W$.

Proposition 22. Over the polar quadrilateral family, $C_0', C_1', C_2'$, and $W$ are collinear.

Proposition 23. Over $Q$, the loci of $C_0'$ and $C_2'$ are lines parallel to the parabola’s directrix and given by $C_0' : x = (3 - \sqrt{5})f/2$, and $C_2' : x = (4 - \sqrt{5})f/3$.

Rather laborious CAS manipulation yields:

Proposition 24. Over $Q$, the locus of $C_1'$ is one connected component of an algebraic curve of degree ten, given by the following equation:

$$C_1' = -\left(1457008 \sqrt{5} + 3257968\right) f x^7 y^2 + \left(122156 \sqrt{5} + 273148\right) f^2 x^4 y^4 + \left(465164 \sqrt{5} + 1040132\right) f^2 x^6 y^2$$

$$- (96506 \sqrt{5} + 215698) f^4 x^2 y^2 - \left(119256 \sqrt{5} + 266664\right) f^3 x^2 y^4 + \left(505052 \sqrt{5} + 1129268\right) f^3 x^4 y^2$$

$$+ (8564 \sqrt{5} + 19204) f^6 x y^2 - \left(881712 \sqrt{5} + 1971568\right) f^8 x^2 y^4 - \left(43955 \sqrt{5} + 98268\right) f^6 x^2 y^4$$

$$+ (24568 \sqrt{5} + 54936) f^8 x^2 y^4 - \left(7250 \sqrt{5} + 16210\right) f^9 x y^4 - \left(1274930 \sqrt{5} + 2850838\right) f^9 x^4 y^2$$

$$+ (1235568 \sqrt{5} + 2762832) f^{10} x^5 y^2 + \left(4457696 \sqrt{5} + 9967712\right) f^{12} x^2 y^6 - \left(7787152 \sqrt{5} + 17412608\right) f^{10} x^5 y^2$$

$$+ \left(5470456 \sqrt{5} + 12232344\right) f^{12} x^2 y^6 - \left(1690535 + 755997 \sqrt{5}\right) f^{12} x^2 y^6 - \left(812098 \sqrt{5} + 1815898\right) f^{12} x^5 y^2$$

$$+ (330222 \sqrt{5} + 738968) f^{14} x^3 y^4 + \left(1002448 \sqrt{5}\right) f^{16} y^4 - \left(228 \sqrt{5} + 672\right) f^{16} y^2$$

$$- (7300 \sqrt{5} + 16956) f^{18} x^3 + \left(2750 \sqrt{5} + 7150\right) f^{18} y^6 - \left(16145 \sqrt{5} + 36103\right) f^{18} x^3 y^2$$

$$- (84196 \sqrt{5} + 188268) x^8 y^3 + (544928 \sqrt{5} + 1218496) x^8 y^3 - 726 f^{10} = 0.$$

Furthermore, $C_1'$ is bound by the following two lines parallel to the directrix and approximately $f/25$ apart: $x = (5 + \sqrt{2} - \sqrt{5\sqrt{2} - \sqrt{5}})/2$, and $x = (\sqrt{5\sqrt{2} - \sqrt{5}} - 2\sqrt{2} + 3)/2$.

5. PARABOLA-INSPIRED PENTAGONS

Referring to Figure 12, consider a family of pentagons inscribed in a parabola $P$ of focal distance $f$, and circumscribed about a focus-centered circle $C$ of radius $r$. 
Figure 12. Parabola-inscribed pentagons (green), and their polar polygon (red). The loci of vertex $C_0$, perimeter $C_1$, and area centroids $C_1$ are parabolas (blue) coaxial with $P$ (gold). Over the polar family, $C'_0$ and $C'_2$ are straight lines (dashed orange) perpendicular to the directrix (dashed black). Though the locus of the perimeter centroid $C'_1$ is indistinguishable from a straight line, it is an algebraic curve of degree likely much higher than 10 (since that is the degree for $C'_1$ on $N = 4$).

**Proposition 25.** The pair $P, C$ will admit a Poncelet family of pentagons iff $r/f$ is the only positive root of the following sextic polynomial ($r/f \approx 0.995219$):

$$x^6 + 12x^5 - 28x^4 + 32x^3 + 112x^2 - 64x - 64 = 0.$$

**Proof.** Referring to Figure 13, without loss of generality, let $P$ be the unit parabola $y = x^2$ with focus $F = [0, 1/4]$ and let $C$ be a circle of radius $r$ centered at $F$. Consider the Poncelet pentagon $P_i, i = 1, \ldots, 5$ with $P_4$ at infinity, and $P_1P_2$ horizontal and tangent to $C$ at $[0, 1/4 - r]$. Compute the next Poncelet vertex $P_3 = [x_3, y_3]$ as the intersection of a tangent to $C$ from $P_2$ with $P$. By requiring that $x_3 = r$, we obtain the sextic in the claim. \qed

Referring to Figure 12:

**Conjecture 2.** Over the parabola-inscribed pentagon family, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $P$.

**Conjecture 3.** Over the family of polar polygons to parabola-inscribed pentagons, the locus of vertex and area vertices are lines perpendicular to the directrix while that of the perimeter centroid is an algebraic curve of degree at least four.
6. PARABOLA-INSPIRED HEXAGONS AND SUMMARY

6.1. Hexagons and summary. Referring to Figure 14, we can also consider a family of parabola-inscribed hexagons.

An analogous construction (based on symmetric configurations) was used to obtain \( r/f \) required for convex \( N = 6 \). A summary of all \( r/f \) thus obtained appears in Table 2.

![Figure 13. Construction used to derive \( r/f \) in Proposition 25. Top (resp. bottom) shows the complete picture (resp. a detailed view near the vertex)](image)

| \( N \) | \( r/f \) | \( r/f \) (num.) | Cayley |
|---|---|---|---|
| 3 | \( 2(\sqrt{2} - 1) \) | 0.828427 | 4 |
| 4 | \( 2\sqrt{5} - 2 \) | 0.971737 | 4 |
| 5 | n/a | 0.995219 | 8 |
| 6 | n/a | 0.999183 | 8 |

Table 2. Table of \( r/f \) required for closure of convex \( N \)-gons inscribed in a parabola, and circumscribed about a focus-centered circle. Algebraic expressions (2nd column) are only possible for \( N = 3, 4 \). The last column shows the number of possible solutions for \( r/f \) if one were to include cases where circle and parabola intersect (the Poncelet polygon may be self-intersecting and/or non-convex). For Cayley’s conditions in the general case, see [8].

7. GENERALIZING CENTROIDAL LOCI

Let \( \mathcal{R} \) be a Poncelet family of \( N \)-gons inscribed to a parabola \( \mathcal{P} \), and circumscribed about a focus-centered circle \( \mathcal{C} \). Experimental the evidence for the \( N = 3, 4, 5, 6 \) cases, we propose the following generalizations (reader contributions are encouraged):

**Conjecture 4.** Over \( \mathcal{R} \), for any \( N \geq 3 \), the loci of vertex, perimeter, and area centroids are parabolas coaxial with \( \mathcal{P} \).

**Conjecture 5.** Over \( \mathcal{R} \) and for any \( N \geq 3 \), the loci of vertex and area centroids of the polar polygons with respect to \( \mathcal{P} \) are straight lines parallel to the directrix of \( \mathcal{P} \).

Let \( \mathcal{B}' \) be the conic-inscribed polar image of a generic bicentric family of \( N \)-gons with respect to the bicentric circumcircle (see Appendix B).
Recall that the locus of vertex and area centroids $C_0, C_2$ are conics over any Poncelet family, while that of the perimeter centroid $C_1$ is not, in general, a conic [22]. However, as A. Akopyan reminded us [1]:

**Remark 1.** if a polygon circumscribes a circle (of center $O$), $C_1, C_2, O$ are collinear and $(C_1 - O) = (3/2)(C_2 - O)$.

Therefore, and analogously to [10, Corollary 2]:

**Corollary 2.** Over $B'$, the locus of the perimeter centroid is a conic.

Let $P'$ be the conic to which $B'$ is inscribed.

**Conjecture 6.** Over the polar polygons of $B'$ with respect to $P'$, the locus of the perimeter centroid is never a conic.

### 8. A Conserved Quantity

As in Appendix B, let $B$ denote a bicentric family of $N$-gons inscribed to a circle $C = (O, R)$, and circumscribed about a second, nested circle $C'$. Let $d_i$ denote the perpendicular distance from the bicentric circumcenter $O$ to side $P_i P_{i+1}$.

Referring to Figure 15:

**Lemma 1.** Over $B$, the quantity $\sum d_i$ is conserved.
Figure 15. An $N = 4$ bicentric polygon is shown (blue). Without loss of generality, in the case shown the circumcenter $O$ is interior to the incircle, i.e., the polar family (green) is ellipse-inscribed. Also shown is the pedal polygon (pink) with respect to a point $P$ in the interior of the circumcircle and the unit vectors (brown) along each perpendicular dropped from $P$ onto the sides.

The argument below was kindly provided by A. Akopyan [1].

**Proof.** The above statement is equivalent to stating that over $B$ the sum of unit vectors from a point $P$ in the direction perpendicular to bicentric sides is constant. In turn, the latter is a corollary of the well-known fact that over $B$, the centroid of the touchpoints of sidelines with $C'$ is stationary. □

Let $\theta_i, i = 1, \ldots, N$, denote the angles interior to a polygon $B$.

**Proposition 26.** For all $N$, the porism of polygons polar to $B$ with respect to its circumcenter conserves $\sum_{i=1}^{N} \sin \theta_i / 2 = (1/R) \sum d_i$.

**Proof.** The vertices of the tangential polygon are the poles of each side of $B$ with respect to the circumcircle. Therefore, said vertices are at a distance $D_i = R^2 / d_i$ from the $O$. Since $\sin \theta_i / 2 = R / D_i = d_i / R$, per Lemma 1, the claim follows. □

Note that in general, $\theta_i$ is the directed angle $P_{i-1}P_iP_{i+1}$. In the case when $r < d$, the tangential polygon will be inscribed in two branches of a hyperbola. There are only two cases: Either (i) all vertices lie on a first proximal branch of the hyperbola, or (ii) all but one vertex $P_k$ will lie on said branch, with $P_k$ lying on the distal branch. In case (i), all $\theta_i$ are positive whereas in (ii) all are positive except for $\theta_k$. Furthermore, in this case, the supplement of angles $\theta_{k-1}$ and $\theta_{k+1}$ need to be used
We would like to thank A. Akopyan, B. Gibert, P. Moses, and A. Zaslavsky for their invaluable help with questions and proofs. The second author is fellow of CNPq and coordinator of Project PRONEX/ CNPq/ FAPEG 2017 10 26 7000 508.

Acknowledgements

We would like to thank A. Akopyan, B. Gibert, P. Moses, and A. Zaslavsky for their invaluable help with questions and proofs. The second author is fellow of CNPq and coordinator of Project PRONEX/ CNPq/ FAPEG 2017 10 26 7000 508.

Appendix A. Vertex parametrizations

A.1. Parabola-inscribed triangles. A 3-periodic orbit \( P_i = [x_i, y_i] = [-y_i^2/(4f), y_i] \) is such that

\[
y_2 = \frac{2 \left( 1 - \sqrt{2} \right) (4f y_i + \Delta) f}{8 f^2 \sqrt{2} - 12 f^2 + y_i^2}, \quad y_3 = \frac{2 \left( \sqrt{2} - 1 \right) f \Delta}{8 f^2 \sqrt{2} - 12 f^2 + y_i^2},
\]

where \( \Delta = \sqrt{16(8\sqrt{2} - 11)f^4 + 8f^2y_i^2 + y_i^4} \).

A.2. Hyperbola-inscribed polar triangles. A 3-periodic orbit \( Q_i = [q_1, q_2, i] \) is such that

\[
Q_1 = \left( 1 + \sqrt{2} \right) \left[ \frac{(4f y_i + \Delta) y_i}{2(2\sqrt{2} + 3)y_i^2 - 8f^2}, \frac{(1 + \sqrt{2}) y_i^3 - 4 (1 + \sqrt{2}) f^3 y_i - 2 \Delta f}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} \right],
Q_2 = \left( 1 + \sqrt{2} \right) \left[ \frac{(4f y_i - \Delta) y_i}{2(2\sqrt{2} + 3)y_i^2 - 8f^2}, \frac{(1 + \sqrt{2}) y_i^3 - 4 (1 + \sqrt{2}) f^3 y_i + 2 \Delta f}{2(2\sqrt{2} + 3)y_i^2 - 8f^2} \right],
Q_3 = \left( 1 + \sqrt{2} \right) \left[ \frac{(5 - 3\sqrt{2}) (1 + 2\sqrt{2})y_i^2 - 28f^2) f}{7((3 + 2\sqrt{2})y_i^2 - 4f^2)}, \frac{8f^2y_i}{(3 + 2\sqrt{2})y_i^2 - 4f^2} \right],
\]

where \( \Delta = \sqrt{y_i^4 + 8f^2y_i^2 + 16(8\sqrt{2} - 11)f^4} \).

A.3. Parabola-inscribed quadrilaterals. A 4-periodic orbit \( P_i = [x_i, y_i] = [-1/4f y_i^2, y_i] \) is such that:

\[
y_2 = \frac{\left( 2 \sqrt{5} - 2 \Delta_1 + 4 f y_i (3 - \sqrt{5}) \right) f}{4 f^2 \sqrt{5} - 8 f^2 - y_i^2}, \quad y_3 = \frac{4 (2 - \sqrt{5}) f^2}{y_1},
\]

\[
y_2 = \frac{\left( 2 \sqrt{5} - 2 \Delta_1 + 4 f y_i (\sqrt{5} - 3) \right) f}{4 f^2 \sqrt{5} - 8 f^2 - y_i^2},
\]

where \( \Delta_1 = \sqrt{y_i^4 + 8f^2y_i^2 + 16(9 - 4\sqrt{5})f^4} \).
A.4. **Hyperbola-inscribed polar quadrilaterals.** A 4-periodic orbit $P_i = [p_i, q_i]$ is such that:

\[
p_1 = \frac{\sqrt{5} - 2 \left( \Delta_1 + 6 f y_1 \sqrt{5} \sqrt{5} - 2 + 14 f y_1 \sqrt{5} - 2 \right) y_1}{4 y_1^2 + 2 y_1^4 + 8 f^2 - y_1^2}
\]

\[
q_1 = \frac{2 \sqrt{5} + 2 \Delta_1 f + 4 f^2 y_1 - y_1^3}{2 \left( 4 f^2 \sqrt{5} + 8 f^2 + y_1^2 \right)}
\]

\[
p_2 = \frac{2 \sqrt{5} - 2 (\Delta_1 + 2 \sqrt{5} - 1 y_1) \left( (\sqrt{5} - 2) y_1^2 + 4 f^2 \right) y_1}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

\[
q_2 = \frac{2 \sqrt{5} + 2 \left( y_1 \Delta_1 + 2 \sqrt{5} + 2 f y_1^2 - 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right)}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

\[
p_3 = \frac{-2 \sqrt{5} - 2 (\Delta_1 - 2 \sqrt{5} - 1 f y_1) \left( y_1^2 \sqrt{5} + 4 f^2 - 2 y_1^2 \right) f}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

\[
q_3 = \frac{-2 \sqrt{5} + 2 \left( y_1 \Delta_1 - 2 \sqrt{5} + 2 f y_1^2 + 8 \left( \sqrt{5} - 2 \right)^{3/2} f^3 \right) \left( (\sqrt{5} - 2) y_1^2 + 4 f^2 \right)}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

\[
p_4 = \frac{\sqrt{5} - 2 (\Delta_1 - 2 \sqrt{5} - 1 f y_1) \left( (\sqrt{5} - 2) y_1^2 + 4 f^2 \right) y_1}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

\[
q_4 = \frac{-2 f \Delta_1 \sqrt{5} - 2 + 4 f^2 y_1 - y_1^3}{y_1 \left( 16 f^4 - 16 f^2 y_1^2 - y_1^4 \right)}
\]

---

**Appendix B. Relation to the bicentric family**

Referring to Figure 16, the bicentric family $B$ of $N$-gons is a family of Poncelet $N$-gons inscribed in a circles $C = (O, R_b)$, and circumscribed about another circle $C' = (O', r_b)$. Let $d = |O - O'|$. Relations between $d, R, r_b$ are known for many “low $N$” and are listed in [25, Poncelet’s porism].

---

**Figure 16.** The bicentric family is a family of Poncelet polygons inscribed between two circles. Shown are the $N = 4$ (left) and $N = 5$ (right) convex cases.
Definition 1 (Polar polygon). Given a polygon $P$, its polar polygon $P'$ with respect to a conic $C$ is bounded by the tangents to $C$ at the vertices of $P$.

Proposition 27. The polar family $B'$ of $B$ with respect to $C$ is an ellipse, parabola, or hyperbola-inscribed if $d$ is smaller, equal, or greater than $R'$, respectively ($O$ is interior, on the boundary, or exterior to $C'$, respectively). Furthermore, one of the foci coincides with $O'$.

As shown in Figure 17, when the polar family is hyperbola-inscribed, there are two layouts for its vertices: either (i) all lie on the branch of the hyperbola closest to the incenter of the family, or (ii) all but one lie on said branch, while the remaining one lies on the “other” branch.

Proposition 28. The parabola $P$ which is the polar image of $B$ with $d = r_b$, has focal distance \( f = R_b^2/(2r_b) \).

Proof. Let $O = (0, 0)$. Consider a polygon in $B$ with a vertical side $P_1P_2$ tangent to the incircle at $(2r_b, 0)$. The vertex $V$ of $P$ is the pole of said side which can be obtained as the inversion of point $(2r_b, 0)$ with respect to the circumcircle. This yields the result. □

References

[1] Akopyan, A. (2021). Private communication. Private Communication. 15, 16
[2] Akopyan, A., Schwartz, R., Tabachnikov, S. (2020). Billiards in ellipses revisited. Eur. J. Math. doi:10.1007/s40879-020-00426-9. 2
[3] Akopyan, A. V., Zaslavsky, A. A. (2007). Geometry of Conics. Providence, RI: Amer. Math. Soc. 5, 6
[4] Bisly, M., Tabachnikov, S. (2020). Dan Reznik’s identities and more. Eur. J. Math. doi:10.1007/s40879-020-00426-7. 2
20)

Bos, H. J. M., Kers, C., Raven, D. W. (1987). Poncelet’s closure theorem. *Expo. Math.*, 5: 289–364. 1

[6]

Chavez-Caliz, A. (2020). More about areas and centers of Poncelet polygons. *Arnold Math J*. doi:10.1007/s40598-020-00154-8. 2

[7]

Del Centina, A. (2016). Poncelet’s porism: a long story of renewed discoveries, I. *Arch. Hist. Exact Sci.*, 70(1): 1–122. 1

[8]

Dragović, V., Radnović, M. (2011). *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*. Frontiers in Mathematics. Basel: Springer. 1, 10, 14

[9]

Fierobe, C. (2021). On the circumcenters of triangular orbits in elliptic billiard. *J. of Dyn. and Control Sys.*, 27: 693–705. 2

[10]

Garcia, R., Gheorghe, L., Reznik, D. (2022). Exploring the Steiner–Soddy porism. arXiv:2201.0222. 2, 15

[11]

Garcia, R., Reznik, D. (2021). *Euclidean Properties of Poncelet Triangles: Experimental Discovery and Analysis*. Rio de Janeiro: IMPA. isbn:978-65-89124-43-6. 1

[12]

Garcia, R., Reznik, D. (2021). Family ties: Relating Poncelet 3-periodics by their properties. *J. Croatian Soc. for Geom. & Gr. (KoG)*, 25(25): 3–18. 1

[13]

Garcia, R., Reznik, D., Koiller, J. (2020). Loci of 3-periodics in an elliptic billiard: why so many ellipses? arXiv:2001.08041. 2

[14]

Garcia, R., Reznik, D., Koiller, J. (2021). New properties of triangular orbits in elliptic billiards. *Am. Math. Monthly*, 128(10): 898–910. 1

[15]

Helman, M., Laurain, D., Garcia, R., Reznik, D. (2021). Invariant center power and loci of Poncelet triangles. *J. Dyn. & Contr. Sys*. doi:10.1007/s10883-021-09580-z. 5

[16]

Kimberling, C. (2019). Encyclopedia of triangle centers. *faculty.evansville.edu/ck6/encyclopedia/ETC.html*. 3, 7, 8, 9

[17]

Maplesoft (2019). Maple. A division of Waterloo Maple Inc. 1

[18]

Odehnal, B. (2011). Poristic loci of triangle centers. *J. Geom. Graph.*, 15(1): 45–67. 2

[19]

Pamfilos, P. (2004). On some actions of $D_3$ on a triangle. *Forum Geometricorum*, 4: 157–176. 2

[20]

Reznik, D., Garcia, R., Koiller, J. (2020). The ballet of triangle centers on the elliptic billiard. *Journal for Geometry and Graphics*, 24(1): 079–101. 1

[21]

Romaskevich, O. (2014). On the incenters of triangular orbits on elliptic billiards. *Enseign. Math.*, 60: 247–255. 2

[22]

Schwartz, R., Tabachnikov, S. (2016). Centers of mass of Poncelet polygons, 200 years after. *Math. Intelligencer*, 38(2): 29–34. 1, 2, 15

[23]

Schwartz, R., Tabachnikov, S. (2016). Centers of mass of Poncelet polygons, 200 years after. *Math. Intelligencer*, 38(2): 29–34. 1

[24]

Skutin, A. (2013). On rotation of an isogonal point. *J. Classical Geom.*, 2: 66–67. 2

[25]

Weisstein, E. W. (2002). *CRC concise encyclopedia of mathematics (2nd ed.)*. Boca Raton, FL: Chapman and Hall/CRC. 4, 8, 18

[26]

Wolfram, S. (2019). Mathematica, version 10.0. 1

[27]

Zaslavsky, A. (2021). Private communication. 4

[28]

Zaslavsky, A., Kosov, D., Muzafarov, M. (2003). Trajectories of remarkable points of the poncelet triangle (in russian). *Kvant*, 2: 22–25. 2