Infinite-dimensional Hamilton-Jacobi theory and
$L$-integrability

Cheng-shi Liu
Department of Mathematics
Daqing Petroleum Institute
Daqing 163318, China
Email: chengshiliu-68@126.com

May 6, 2009

Abstract
The classical Liouvile integrability means that there exist $n$ independent first integrals in involution for $2n$-dimensional phase space. However, in the infinite-dimensional case, an infinite number of independent first integrals in involution don’t indicate that the system is solvable. How many first integrals do we need in order to make the system solvable? To answer the question, we obtain an infinite dimensional Hamilton-Jacobi theory, and prove an infinite dimensional Liouville theorem. Based on the theorem, we give a modified definition of the Liouville integrability in infinite dimension. We call it the $L$-integrability. As examples, we prove that the string vibration equation and the KdV equation are $L$-integrable. In general, we show that an infinite number of integrals is complete if all action variables of a Hamilton system can be reconstructed by the set of first integrals.

Keywords: Hamilton-Jacobi theory, Liouville integrability, the KdV equation, string vibration equation, integrable system.

PACS: 02.30.Ik, 05.45.Yv

1 Introduction

It is well-known that an infinite dimensional Hamilton system such as KdV equation can be considered as a complete integrable system[1,2] if it can be solved by the inverse scattering method[3,4]. In other words, if we can obtain its all action-angle variables, we call this system to be complete integrable and sometimes the Liouville integrable. Finite-dimensional Liouville theorem[5] says that if there exist $n$ independent first integrals in evolution, the Hamilton system is solvable. The so-called liouville integrability is just based on the Liouville theorem. For a given Hamilton system in infinite dimension, a necessary condition in order to make such system integrable is that it posses an infinite number of
constants of motion (or called first integrals). F. Calogero[6] point out that due to the ambiguities in the counting of infinities, this condition is not sufficient. A natural problem is how many constants of motion are sufficient to ensure that such system is solvable. We call this problem the Calogero’s problem. Indeed, we are short of an infinite-dimensional Liouville theorem from which the above question can be solved naturally.

On the other hand, all action variables gives ”all” constants of motion. Therefore, beginning from the action variables, we can understand the essence of the Calogero’s problem on infinite number of constants of motion. But classical Liouville integrability is not equivalent to the solvability of the action-angle variables in infinite dimension. In order to give an equivalent definition, we need to modify the conditions in the classical Liouville integrability such that the new Liouville integrability is equivalent to the solvability of the action-angle variables. At the same time, the Calogero’s problem is solved naturally. This is a way from back to head. But we need a way from head to back!

In the present paper, our aim is to extend the Hamilton-Jacobi theory and Liouville integrability (for example, see Arnold’s book [5]) from the finite-dimensional case to the infinite-dimensional case. We establish an infinite-dimensional Hamilton-Jacobi theory, and prove an infinite-dimensional Liouville theorem. Furthermore, based on the theorem, a modified definition of the Liouville integrability in infinite dimension is given. We call it $L$-integrability. As the application of the theory, we study the string vibration problem in detail. We use the Hamilton-Jacobi theory to solve it. We give an infinite number of first integrals and prove that this is a complete set, that is, the string vibration problem is $L$-integrable. We also discuss the problem about the uncomplete first integrals. Finally, as an important example, we prove that the KdV equation is $L$-integrable. Our results answer the Calogero’s problem.

We must point out that the advantage of our theory is not in technique but in concept since it is difficult to solve Hamilton-Jacobi equation directly by the method of the variables separation.

Other definitions of integrability such as Lax integrability, $C$-integrability, can be found in Ref.[4].

2 Infinite-dimensional Hamilton-Jacobi theory

We consider the case of countably infinite variables. $P = (p_1, \cdots, p_n, \cdots)$ and $Q = (q_1, \cdots, q_n, \cdots)$ are a pair of canonical variables. $H = H(P, Q, t)$ is the Hamilton function. The Hamilton canonical equations are as follows

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},
\]

\[
\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},
\]

for $i = 1, 2, \cdots$. $S$ denote the action function which takes its value on the classical path. We have $p_i = \frac{\partial S}{\partial q_i}$ for $i = 1, 2, \cdots$ and denote them by $P = \frac{\partial S}{\partial q}$. 

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for simplicity. The Hamilton-Jacobi equation is given by

$$\frac{\partial S}{\partial t} = -H(Q, \frac{\partial S}{\partial Q}, t).$$  \hspace{1cm} (3)

If we have a general integral $S = S(Q, \alpha)$ of the H-J equation, where $\alpha = (\alpha_1, \alpha_2, \cdots)$, we can solve the Hamilton canonical equation. A crucial step is that we must can solve out $Q = Q(t, \alpha, \beta)$ from the following system of equations

$$\frac{\partial S}{\partial \alpha_i} = \beta_i,$$  \hspace{1cm} (4)

for $i = 1, 2, \cdots$, where $\beta = (\beta_1, \beta_2, \cdots)$. In the finite dimensional case, this condition can be represented as $\det \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \neq 0$. In the infinite dimensional case, we use the invertible property of the operator $\frac{\partial^2 S}{\partial q_i \partial \alpha_j}$ instead of $\det \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \neq 0$. It is easy to prove the following result.

**Theorem 1.** If the operator $\frac{\partial^2 S}{\partial q_i \partial \alpha_j}$ is invertible,

$$Q = Q(t, \alpha, \beta),$$  \hspace{1cm} (5)

$$P = P(t, \alpha, \beta),$$  \hspace{1cm} (6)

are the solutions of the Hamilton canonical equations (1) and (2).

**Proof.** From $\frac{d}{dt} \frac{\partial S}{\partial \alpha_i} = \frac{d\beta_i}{dt} = 0$, we have

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial \alpha_i \partial q_j} \frac{dq_j}{dt} = 0.$$  \hspace{1cm} (7)

From the Hamilton-Jacobi equation, we have

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial H}{\partial p_j} \frac{\partial q_j}{\partial \alpha_i} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial \alpha_i \partial q_j} \frac{\partial q_j}{dt} = 0,$$  \hspace{1cm} (8)

that is,

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial H}{\partial p_j} \frac{\partial q_j}{\partial \alpha_i} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial \alpha_i \partial q_j} \frac{\partial q_j}{\partial t} + H = \frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial H}{\partial p_j} \frac{\partial q_j}{\partial \alpha_i}.$$  \hspace{1cm} (9)

Comparing (7) with (9), and using the condition that the operator $\frac{\partial^2 S}{\partial q_i \partial \alpha_j}$ is invertible, we obtain $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$. It is just Eq.(1).

From $p_i = \frac{\partial S}{\partial q_i}$, we have

$$\frac{dp_i}{dt} = \frac{\partial^2 S}{\partial q_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial q_j \partial q_i} \frac{dq_j}{dt} = \frac{\partial^2 S}{\partial q_i \partial t} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial q_j \partial q_i} \frac{\partial H}{\partial p_j}.$$  \hspace{1cm} (10)
By H-J equation, we have
\[
\frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial H}{\partial q_i} + \sum_{j=1}^{\infty} \frac{\partial H}{\partial p_j} \frac{\partial p_j}{\partial q_i} = 0,
\]
(11)
that is
\[
\frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial H}{\partial q_i} + \sum_{j=1}^{\infty} \frac{\partial^2 S}{\partial q_i \partial p_j} \frac{\partial H}{\partial p_j} = 0.
\]
(12)
Comparing (10) with (12), we obtain \(\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\). It is just Eq.(2). So we complete the proof.

3 Infinite-dimensional Liouville theorem and L-integrability

We first generalize the Liouville theorem to the infinite dimension.

**Theorem 2.** Suppose that the Hamilton system has an infinite number of first integrals (or motion constants)
\[
f_i(P, Q, t) = \alpha_i, \quad i = 1, 2, \ldots.
\]
(13)
If these first integrals satisfy the following conditions, the Hamilton system is integrable.
1. \([f_i, f_j] = 0\), where \([f_i, f_j] = \sum_{k=1}^{\infty} (\frac{\partial f_i}{\partial q_k} \frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k} \frac{\partial f_j}{\partial q_k})\) is the Poisson bracket.
2. The operator \((\frac{\partial f_i}{\partial p_j})\) is invertible, where \((\frac{\partial f_i}{\partial p_j})\) denotes the infinite-dimensional matrix with the general element \(\frac{\partial f_i}{\partial p_j}\).

**Proof.** According to the condition 2\(^0\), we can solve out
\[
p_i = \phi_i(Q, \alpha, t), \quad i = 1, 2, \ldots,
\]
(14)
from the system of equations (13). If there exists a function \(S = S(Q, \alpha, t)\) such that
\[
dS = \sum_{i=1}^{\infty} p_i dq_i - H^* dt,
\]
(15)
that is,
\[
\frac{\partial S}{\partial q_i} = p_i = \phi_i, \quad i = 1, 2, \ldots,
\]
(16)
\[
\frac{\partial S}{\partial t} = -H^*,
\]
(17)
where
\[
H^*(Q, \alpha, t) = H(Q, \phi, t),
\]
(18)
according to theorem 1, we know that the Hamilton system (1) and (2) is solvable. Indeed, the crucial step of using H-J equation method is to solve out $\alpha$ from the system of equations $p_i = \phi_i(Q, \alpha, t)$, for $i = 1, 2, \cdots$. Since $\alpha$ has been given by Eq.(13), Hamilton system is solvable according to theorem 1. Now what we need is to prove that the differential form $\sum p_i dq_i - H^* dt$ is an exact form. This is equivalent to the following conditions

$$\frac{\partial \phi_i}{\partial q_k} = \frac{\partial \phi_k}{\partial q_i}, \quad i, k = 1, 2, \cdots \quad (19)$$

$$\frac{\partial \phi_i}{\partial t} = -\frac{\partial H^*}{\partial q_i}, \quad i = 1, 2, \cdots \quad (20)$$

We first prove condition (19). Differentiating the expression (13) with respect to $q_i$ yields

$$\sum_{r=1}^{+\infty} \frac{\partial f_r}{\partial q_i} \frac{\partial f_s}{\partial p_i} = 0. \quad (21)$$

We multiply $\frac{\partial f_r}{\partial p_i}$ in two sides of Eq.(21) and take summation for $i$. Then we have

$$\sum_{i=1}^{+\infty} \frac{\partial f_r}{\partial q_i} \frac{\partial f_s}{\partial p_i} \frac{\partial \phi_j}{\partial q_i} = 0. \quad (22)$$

By the same method, we have

$$\sum_{j=1}^{+\infty} \frac{\partial f_r}{\partial q_j} \frac{\partial f_s}{\partial p_j} \frac{\partial \phi_i}{\partial q_j} = 0. \quad (23)$$

By subtraction of Eq.(22) and Eq.(23), and usage of $[f_r, f_s] = 0$, we have

$$\sum_{i,j=1}^{+\infty} \frac{\partial f_r}{\partial q_i} \frac{\partial f_s}{\partial p_i} \frac{\partial \phi_j}{\partial q_i} - \frac{\partial \phi_i}{\partial q_j} = 0. \quad (24)$$

Since the operator $\left( \frac{\partial f_r}{\partial p_i} \right)$ is invertible by condition 20, we have

$$\sum_{j=1}^{+\infty} \frac{\partial f_r}{\partial q_j} \frac{\partial \phi_j}{\partial p_i} = \frac{\partial \phi_i}{\partial q_j}. \quad (25)$$

By the same method, we have

$$\frac{\partial \phi_i}{\partial p_i} = \frac{\partial \phi_i}{\partial p_j}. \quad (26)$$

We next prove the condition (20). From the Hamilton canonical equation, we have

$$- \frac{\partial H}{\partial q_i} = \frac{dq_i}{dt} = \frac{d\phi_i}{dt} + \sum_{j=1}^{+\infty} \frac{\partial \phi_i}{\partial q_j} \frac{dq_j}{dt} = \frac{\partial \phi_i}{dt} + \sum_{j=1}^{+\infty} \frac{\partial \phi_i}{\partial q_j} \frac{dH}{dp_j}. \quad (27)$$
so we have
\[
\frac{\partial \phi_i}{\partial t} = -\frac{\partial H}{\partial q_i} - \sum_{j=1}^{\infty} \frac{\partial \phi_i}{\partial q_j} \frac{\partial H}{\partial p_j} = -\frac{\partial H}{\partial q_i} - \sum_{j=1}^{\infty} \frac{\partial \phi_j}{\partial q_i} \frac{\partial H}{\partial p_j} = -\frac{\partial H^*}{\partial q_i}. \tag{28}
\]

We complete the proof.

Based on the above theorem, we give the following definitions.

**Definition 1.** An infinite number of motion constants (or first integrals) \((13)\) is called a complete set of motion constants if the condition \(2^{0}\) is satisfied.

**Definition 2.** If a Hamilton system has a complete set of motion constants, the system is called to possess the \(L\)-integrability or to be \(L\)-integrable.

**Remark 1.** In \(2n\)-dimensional phase space case, Liouville integrability needs \(n\) independent first integrals in involution, which are not sufficient in infinite dimension since we can take away some first integrals, for example, arbitrary finite number of first integrals, such that two conditions of involution and independence are remained as before. In our new definition 2, we take the completeness instead of involution and independence. Theorem 2 is the theoretical foundation of definition 2. If a soliton equation possesses an infinite number of independent first integrals in involution, what we need is only to verify whether this set of first integrals is complete. We will take some concrete examples as verification.

### 4 The \(L\)-integrability of an infinite-dimensional harmonic oscillator

Consider the string vibration equation
\[
\ddot{u} = \dddot{u}, \tag{29}
\]
\[
u(0, t) = u(2\pi, t) = 0, \tag{30}
\]
\[
u(x, 0) = u_0(x), \quad \nu_t(x, 0) = u_1(x), \tag{31}
\]
which is an infinite-dimensional harmonic oscillator. The corresponding Lagrangian function and Hamilton function are
\[
L = \frac{1}{2} \int_{0}^{2\pi} (\dot{u}^2 - u_x^2) dx, \tag{32}
\]
and
\[
H = \frac{1}{2} \int_{0}^{2\pi} (\dot{u}^2 + u_x^2) dx. \tag{33}
\]
Let \(q = \nu\) and \(p = \nu_t\) be a pair of canonical variables. Then Hamilton function is rewritten as
\[
H(p, q) = \frac{1}{2} \int_{0}^{2\pi} (p^2 + q_x^2) dx. \tag{34}
\]
Therefore the Hamilton canonical system is just Eq.(29). The Hamilton-Jacobi equation is given by

$$\frac{\partial S}{\partial t} = -\frac{1}{2} \int_{0}^{2\pi} \{ (\frac{\delta S}{\delta q})^2 + (q_x)^2 \} dx$$  \hspace{1cm} (35)$$

By the separation of variables, we let

$$S(q, t) = S_0(t) + S_1(q),$$  \hspace{1cm} (36)$$

where $S_1(q)$ is a functional of $q$. Substituting Eq.(36) into Hamilton-Jacobi equation (35) and separating the variables yield

$$S_0'(t) = -\frac{1}{2} \int_{0}^{2\pi} \{ (\frac{\delta S_1}{\delta q})^2 + (q_x)^2 \} dx = -\frac{1}{2} \int_{0}^{2\pi} \beta(x) dx = -E.$$  \hspace{1cm} (37)$$

It follows that

$$S_0(t) = -Et,$$  \hspace{1cm} (38)$$

$$\left( \frac{\delta S_1}{\delta q} \right)^2 + (q_x)^2 = \beta(x),$$  \hspace{1cm} (39)$$

where $\beta(x)$ is an arbitrary function satisfying $\frac{1}{2} \int_{0}^{2\pi} \beta(x) dx = E$. For example, we can take $\beta(x) = 2E$. But we must point out that Eq.(39) is a variational-differential equation and we don't know how to solve it in general case.

Now we adopt another method to deal with these problems. Take Fourier transformation of $u$ with respect to $x$,

$$u(x, t) = \sum_{n=1}^{+\infty} a_n(t) \sin(nx),$$  \hspace{1cm} (40)$$

then

$$u_t(x, t) = \sum_{n=1}^{+\infty} a_n'(t) \sin(nx).$$  \hspace{1cm} (41)$$

Therefore the Hamiltonian function becomes

$$H = \frac{1}{2} \sum_{n=1}^{+\infty} \{ a_n'^2(t) + n^2 a_n^2(t) \}. $$  \hspace{1cm} (42)$$

Taking

$$S(a_1(t), a_2(t), \cdots) = S_0(t) + \sum_{n=1}^{+\infty} S_n(a_n),$$  \hspace{1cm} (43)$$

and substituting it into Hamilton-Jacobi equation yield

$$\left( \frac{dS_n}{da_n} \right)^2 + n^2 a_n^2 = E_n, \hspace{0.5cm} n = 1, 2, \cdots ,$$  \hspace{1cm} (44)$$
where $E'_n$'s are constants and satisfy the following condition

$$\sum_{n=1}^{\infty} E_n = 2E. \quad (45)$$

Solving Eq.(44), we have

$$S_n = \int \sqrt{E_n - n^2a_n^2} da_n. \quad (46)$$

According to the standard steps we can solve out the solutions of $a_n$, for $n = 1, 2, \cdots$. Hence we can use the Hamilton-Jacobi theory to solve the string vibration problem.

We next obtain the $L$-integrability of the string vibration equation. We first give an infinite number of first integrals

$$f_n(u, u_t) = \frac{1}{2} n^2 \left( \int_0^{2\pi} u(x, t) \sin(nx) dx \right)^2 + \frac{1}{2} \left( \int_0^{2\pi} u_t(x, t) \sin(nx) dx \right)^2, \quad (47)$$

for $n = 1, 2, \cdots$. Indeed, we have

$$\frac{df_n}{dt}(u, u_t) = n^2 \int_0^{2\pi} u(x, t) \sin(nx) dx \int_0^{2\pi} u_t(x, t) \sin(nx) dx$$

$$+ \int_0^{2\pi} u_t(x, t) \sin(nx) dx \int_0^{2\pi} u_{tt}(x, t) \sin(nx) dx = 0, \quad (48)$$

where we use $u_{tt} = u_{xx}$ and integration by part in last step. Rewriting the first integrals in terms of variables $a'_n, s_n$, we have

$$f_n = \frac{1}{2} (a_n^2(t) + n^2 a_n^2(t)), \quad n = 1, 2, \cdots. \quad (49)$$

Therefore, every $f_n$ is just the energy of the $n$th mode. The physical picture of these first integrals is very clear.

We now prove these first integrals constitute a complete set. Indeed, in this case, the canonical variables are $q_n = a_n$ and $p_n = a'_n$. From the set of first integrals (49) represented by $f_n = \frac{1}{2} (p_n^2(t) + n^2 q_n^2(t))$ in terms of $q_n$ and $p_n$, we can solve out the $p'_n, s_n$. It follows that this is a complete set. On the other hand, we have

$$\frac{\partial f_n}{\partial q_m} = \delta mn, \quad (50)$$

where $\delta mn$ is the Dirac sign in infinite dimension, that is, the operator (matrix) $(\frac{\partial f_n}{\partial q_m})$ is invertible. Of course, $[f_n, f_m] = 0$ is a simple fact. According to theorem 2, the string vibration problem is $L$-integrable.

If we remove some first integrals in the set (49), the set will be not complete. Indeed, for example, we remove $f_1$, then the remains are also evolutional and independent. But it is easy to see that the remains are not complete since we can’t solve out $p_1$. In other words, the operator (matrix) $(\frac{\partial f_n}{\partial q_m})_{m=1, n=2}^{\infty}$ isn’t invertible.
5 The $L$-integrability of the infinite vibrating string

We consider the Cauchy problem for an infinite vibrating string

\[ u_{tt} = u_{xx}, \quad (51) \]
\[ u(-\infty, t) = u(+\infty, t) = 0, \quad (52) \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \quad (53) \]

We take the Fourier transformation of $u(x, t)$ with respect to the variable $x$,

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(y, t) \sin(xy)dy. \quad (54) \]

It is easy to prove that

\[ f(y, t) = \frac{1}{2} \left\{ a_1^2(y, t) + y^2 a_2^2(y, t) \right\}, \quad (55) \]

or in another form

\[ f(y, t) = \frac{1}{2} \left\{ \left( \int_{-\infty}^{+\infty} u_t(x, t) \sin(xy)dx \right)^2 + y^2 \left( \int_{-\infty}^{+\infty} u(x, t) \sin(xy)dx \right)^2 \right\}, \quad (56) \]

is a first integral for every $y$, that is, $\frac{df(y, t)}{dt} = 0$. This first integral is just the energy of the $y$-th mode. They constitute a set of the first integrals with uncountably infinite elements. In order to construct a countably infinite number of first integrals, we take the Taylor expansion of $f(y, t)$ with respect to the variable y

\[ f(y, t) = \sum_{n=0}^{+\infty} \frac{df(0, t)}{dt^n} y^n. \quad (57) \]

Then every $\frac{df(0, t)}{dt^n}$ is a first integral. Through tedious computation, we obtain

\[ f(y, t) = g_1 y^2 + g_2 y^4 + \cdots + g_k y^{2k} + \cdots, \quad (58) \]

where

\[ g_1 = \left( \int_{-\infty}^{+\infty} xu_t(x, t)dx \right)^2, \quad (59) \]
\[ g_k = \frac{(-1)^{k+1}}{2k-1)!} \int_{-\infty}^{+\infty} xu_t(x, t)dx \int_{-\infty}^{+\infty} x^{2k-1}u_t(x, t)dx \]
\[ + \sum_{m=0}^{k-2} \frac{(-1)^k}{(2m+1)!}(2(k-m)-3)! \left( \int_{-\infty}^{+\infty} x^{2(k-m)-3}udx \int_{-\infty}^{+\infty} x^{2m+1}udx \right) \]
\[ - \frac{1}{(2(k-m)-2)(2(k-m)-1)} \int_{-\infty}^{+\infty} x^{2(k-m)-1}u_tdx \int_{-\infty}^{+\infty} x^{2m+1}u_tdx \quad (60) \]
for $k = 2, 3, \cdots$. Every $g_n$ is a first integral. We will prove that they constitute a complete set of first integrals. For the purpose, we first take the canonical coordinates as

$$q_n = \int_{-\infty}^{+\infty} x^{2n+1} u \, dx, \quad (61)$$

$$p_n = \int_{-\infty}^{+\infty} x^{2n+1} u_t \, dx, \quad (62)$$

for $n = 0, 1, \cdots$. In terms of the canonical coordinates, we rewrite $g_k$ as

$$g_1 = p_2^2, \quad (63)$$

$$g_k = \frac{(-1)^{k+1}}{(2k - 1)!} p_0 p_{k-1} + \sum_{m=0}^{k-2} \frac{(-1)^k}{(2m + 1)! (2(2k - m) - 3)!} \times \{ q_{k-m-2} q_m - \frac{1}{(2(k-m) - 2)(2(k-m) - 1)} p_{k-m-1} p_m \}, \quad (64)$$

for $k = 2, 3, \cdots$. From the system of equations $g_k = \beta_k$ for $k = 0, 1, \cdots$, we can easily solve out the $p_n$ for $n = 0, 1, \cdots$, since these equations all are quadratic. Indeed, we have

$$p_0 = \pm \sqrt{\beta_1}, \quad (65)$$

$$p_k = \frac{(2k + 1)!}{2} \left\{ (-1)^k \beta_{k+1} + \frac{1}{(2k - 1)!} g_0 q_{k-1} - \sum_{m=1}^{k-2} \frac{(-1)^k}{(2m + 1)! (2(k-m) - 3)!} \times \{ q_{k-m-2} q_m - \frac{1}{(2(k-m) - 2)(2(k-m) - 1)} p_{k-m-1} p_m \} \right\}, \quad (66)$$

for $k = 2, 3, \cdots$. Thus we conclude that the infinite vibrating string problem is the $L$-integrable.

We must notice that the expression of $g_k$ is so complicate that we can’t clearly find the physical meanings of these first integrals. On the other hand, the physical meaning of the first integral $f(y, t)$ is very clear. Thus, behind those complicate first integrals, perhaps there is a simple rule such that a clear physical picture can be emerged to us.

**Remark 2.** It is easy to see that

$$g_n(t) = \int_{-\infty}^{+\infty} x^n u_t(x, t) \, dx \quad (67)$$

are first integrals for $n = 0, 1, \cdots$ and they constitute a countably infinite set of first integrals. But this is not a complete set of first integrals.
6 The $L$-integrability of the KdV equation

Consider the following KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \quad (68)$$

The corresponding Schrödinger equation is

$$-\frac{d^2}{dx^2} \phi + u \phi = k^2 \phi. \quad (69)$$

As $x \to \infty$ for $\text{Im} k > 0$ and $u$ satisfies the KdV equation, we let

$$a(k) = \phi(x, k) e^{ikx}, \quad (70)$$

we have (see Ref.[3] for the details on inverse scattering transformation)

$$\frac{d}{dt} a(k) = 0. \quad (71)$$

Let

$$\ln a(k) = \int_{-\infty}^{+\infty} \chi(x, k) dx. \quad (72)$$

Then $\chi$ satisfies the Riccati’s equation

$$\chi_x + \chi^2 - u - 2ik\chi = 0. \quad (73)$$

Furthermore, let

$$\chi = \sum_{m=1}^{+\infty} \frac{\chi_m}{(2ik)^m}, \quad (74)$$

we have $\int \chi_{2m} dx = 0$ and an infinite number of first integrals

$$I_m = \int \chi_{2m-1} dx = 0, \quad (75)$$

for $m = 0, 1, \cdots$. Now we can prove that these first integrals constitute a complete set since we can determine the function $a(k)$ by the values of these first integrals $I_m$ for $m = 0, 1, \cdots$. Indeed, if we know the function $a(k)$, we can obtain the action variables $n(k) = \frac{2\pi}{\ln |a(k)|^2}, k > 0$ and $N_l = k_l^2, k = 1, \cdots, N$, where $ik_l$s are the zeros of $a(k)$. In other words, we can solve out the action variables. We conclude that these first integrals constitute a complete set, that is, the KdV equation is the $L$-integrable. Of course, we can obtain the action-angle variables by solving the Hamilton-Jacobi equation with the Hamiltonian rewritten in terms of the action variables

$$H = -\frac{32}{5} \sum_{l=1}^{N} N_l^{5/2} + 8 \int_0^{+\infty} k^3 n(k) dk. \quad (76)$$
Remark 3. The $L$-integrability of other soliton equations such as nonlinear Schrodinger equation and Sine-Gordon equation, can be established easily.

Remark 4. Magri[7] use a Bi-Hamiltonian structures and the infinitesimal symmetry transformation method to study an infinite number of constants of motion. Wadati[8] also use the infinitesimal symmetry transformation method to obtain many results about conversation laws of KdV equation. The modern development of Bi-Hamiltonian structures method can be seen Ref.[9] and the references therein.

Of course, if an infinite-dimensional Hamilton system can be reformulated in terms of action-angle variables, we usually consider it integrable. Here, we strictly prove that this integrability is just the $L$-integrability. In other words, an infinite number of first integrals represented in terms of all action variables is a complete set. By theorem 2, we can easily prove the following theorem.

Theorem 3. For a given Hamilton system, if its all action variables can be reconstructed by an infinite number of first integrals, the set of first integrals is complete, that is, the system is $L$-integrable.

7 Conclusion

We extend the Hamilton-Jacobi theory and Liouville theorem from the finite-dimensional case to the infinite-dimensional case. We introduce the $L$-integrability which can be considered as a suitable definition of Liouville integrability in infinite dimension. As examples, we study the string vibration problem in detail, and use the Hamilton-Jacobi theory to solve it. We give an infinite number of first integrals and prove that this is a complete set, that is, the string vibration problem is $L$-integrable. We also discuss the problem about the uncomplete first integrals. From our discussion, the physical picture of the $L$-integrability of the string vibration problem is very clear. Finally, we apply our theory to the soliton equation and prove that the KdV equation is $L$-integrable. Of course, the $L$-integrability of other nonlinear evolution equations such as nonlinear Schrodinger equation can also be easily obtained.

Acknowledgments. I would like to thank professor Calogero for his valuable suggestions.

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