ON SOME LOCAL COHOMOLOGY MODULES

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1. Introduction

All rings in this paper are commutative and Noetherian. If $R$ is a ring and $I \subset R$ is an ideal, $\text{cd}(R, I)$ denotes the cohomological dimension of $I$ in $R$, i.e. the largest integer $i$ such that the $i$-th local cohomology module $H^i_I(M)$ doesn’t vanish for some $R$-module $M$.

For the purposes of this introduction $R$ is a complete equicharacteristic regular local $d$-dimensional ring with a separably closed residue field and $I \subset R$ is an ideal such that every minimal prime over $I$ has height at most $c$. We quote the following two results.

(i) $\text{cd}(R, I) \leq d - \lfloor \frac{d - 1}{c} \rfloor$ [2] (see Theorem 3.1 below). This bound is sharp for all $d$ and $c$ [12].

(ii) If, in addition, $I$ is prime and $c < d$, then $\text{cd}(R, I) \leq d - \lfloor \frac{d - 2}{c} \rfloor - 1$ [10] 3.8 (see Theorem 3.4 below). This is also sharp for all $d$ and $c$ [10] 5.6.

These two upper bounds, $v = d - \lfloor \frac{d - 2}{c} \rfloor - 1$ and $v' = d - \lfloor \frac{d - 1}{c} \rfloor$, coincide if $c|(d - 1)$, otherwise $v' = v + 1$. This raises the following natural question.

Question. Under what conditions is $\text{cd}(R, I) \leq v$? Equivalently, under what conditions is $H_i^{v+1}(M) = 0$ for every $M$?

We discussed this question in our old paper [10] in the special case that $\lfloor \frac{d - 2}{c} \rfloor = 2$ [10] 5.8-5.12, but were unable to settle even this special case. The purpose of the present paper is to provide a complete answer to this question. Namely, we give a necessary and sufficient condition exclusively in terms of combinatorial properties of the set of subsets $\{i_0, \ldots, i_j\}$ of the indexing set $\{1, \ldots, n\}$ of the minimal prime ideals $I_1, \ldots, I_n$ over $I$, such that the sum $I_{i_0} + \cdots + I_{i_j}$ is $m$-primary. More precisely, we prove the following.

Theorem 1.1. Let $R$ be a complete $d$-dimensional equicharacteristic regular local ring with maximal ideal $m$ and a separably closed residue field $k$. Let $I \subset R$ be an ideal of $R$ and let $M$ be an $R$-module. Let $I_1, \ldots, I_n$ be the minimal primes of $I$. Assume the height of every $I_i$ is at most $c < d$. Let $\Delta$ be the simplicial complex on the vertices $1, \ldots, n$ such that a simplex $\{i_0, \ldots, i_j\}$ is included in $\Delta$ if and only if $I_{i_0} + \cdots + I_{i_j}$ is not $m$-primary. Let $t = \lfloor \frac{d - 2}{c} \rfloor$ and let $v = d - \lfloor \frac{d - 2}{c} \rfloor - 1$. Then $H^{v+1}_I(M)$ is isomorphic to the direct sum of $w$ copies of $H^d_m(M)$, where $w = \dim_k H_{t-1}(\Delta; k)$ and $H_s(\Delta; k)$
is the reduced singular homology of $\Delta$ with coefficients in $k$. In particular, $\text{cd}(R, I) \leq v$ if and only if $\tilde{H}_{t-1}(\Delta; k) = 0$.

If $c = d - 2$, then $t = \lceil \frac{d-2}{c} \rceil = 1$ and the theorem implies the following: if an ideal $I$ is the intersection of prime ideals of heights at most $d - 2$, then $\text{cd}(R, I) \leq d - 2$ if and only if the punctured spectrum of $R/I$ is connected (indeed, the punctured spectrum is connected iff $\Delta$ is connected, i.e. iff $\tilde{H}_0(\Delta; k) = 0$). This is a famous result with a rich history. Various pieces of this result have been proven by Peskine and Szpiro [17, III, 5.5], Hartshorne [7, 7.5], Ogus [10, 2.11] and Huneke and Lyubeznik [10, 2.9]. Theorem 1.1 extends this famous result to higher values of $t = \lceil \frac{d-2}{c} \rceil$.

Theorem 1.1 makes it plain why the sharp bounds $v$ and $v'$ of (i) and (ii) coincide in the case that $c|(d-1)$. Indeed, in this case $d = c(t+1)+1$, where $t = \lceil \frac{d-2}{c} \rceil$, hence the sum of at most $t+1$ primes of height at most $c$ each cannot add up to an $m$-primary ideal, i.e. $\Delta$ contains every single simplex $\{i_0, \ldots, i_p\}$ of dimension $p \leq t$. Hence the reduced singular homology of $\Delta$ in degrees at most $t-1$ coincides with that of the full simplex on $\{1, \ldots, n\}$. But the reduced singular homology of the latter vanishes in all degrees, i.e. $\tilde{H}_{t-1}(\Delta; k) = 0$ for all $I$. Hence $\text{cd}(R, I) \leq v$ for all, not just prime, $I$.

Theorem 1.1 immediately implies an extension of (ii) to the case that $I$ is not necessarily prime, but has a small number of minimal primes. Namely, a sum of fewer than $\frac{d}{c}$ primes of height at most $c$ cannot add up to an $m$-primary ideal in a regular $d$-dimensional ring, hence if $n < \frac{d}{c}$, then the complex $\Delta$ is the full simplex on vertices $\{1, \ldots, n\}$ whose reduced singular homology vanishes in all degrees. Thus we get the following corollary which is a generalization of (ii).

**Corollary 1.2.** Let $R$ be a complete $d$-dimensional equicharacteristic regular local ring with a separably closed residue field. Let $I \subset R$ be an ideal with $n < \frac{d}{c}$ minimal primes each of which has height at most $c$. Then $\text{cd}(R, I) \leq d - 1 - \lceil \frac{d-2}{c} \rceil$.

The bound $n < \frac{d}{c}$ is sharp for all $d$ and $c$ such that $c \not|(d-1)$ (if $c|(d-1)$, then $\text{cd}(R, I) \leq d - 1 - \lceil \frac{d-2}{c} \rceil$ for all $I$; see (i) above). Indeed, if $n$ is the smallest integer $\geq \frac{d}{c}$, then there exist prime ideals $I_1, \ldots, I_n$ of height $c$ each, such that $I_1 + \cdots + I_n$ is $m$-primary in which case $\Delta$ contains every simplex except the top-dimensional one, i.e. $\Delta$ is homeomorphic to the $(n-2)$-sphere, hence $\tilde{H}_{n-2}(\Delta; k) \neq 0$, i.e. $\text{cd}(R, I) > d - 1 - \lceil \frac{d-2}{c} \rceil$ by Theorem 1.1 where $I = I_1 \cap \cdots \cap I_n$.

The principal tool of our proofs is the following Mayer-Vietoris spectral sequence of [11 p. 39] (see Theorem 2.1 below).

$$E_1^{r-p} = \bigoplus_{i_0 < \cdots < i_p} H_{i_0+\cdots+i_p}^q(M) \implies H_{I_1 \cap \cdots \cap I_n}^{q-p}(M).$$

We use it in Section 2 to reduce the proofs of our main results to an auxiliary statement that will be proven in Theorem 3.5, the main theorem of Section 3 (see Corollaries 2.3 and 2.5 and the paragraph following the proof of 2.5).
In the special case that \([4-2] = 2\) that auxiliary statement readily follows from some well-known results which gives a short proof of Theorem \(1.1\) in this special case. We conclude Section 2 by using this special case of Theorem \(1.1\) to gain a complete understanding of an old example that defied our efforts in \([10\ 5.12\) (Example 2.6). The complex \(\Delta\) in this example turns out to be homeomorphic to the real projective plane and consequently \(cd(R,I)\) depends on whether \(char k = 2\).

Section 3 is devoted to a proof of the above-mentioned Theorem \(2.2\)

The final section, Section 4, contains proofs of our main results. We prove our main results for arbitrary local rings containing a field. Theorem \(1.1\) and Corollary \(1.2\) are special cases of Corollaries \(4.4\) and \(4.2\) respectively.

Étale analogues of the results in this paper and their topological applications will be dealt with in an upcoming paper \(14\).

2. The Mayer-Vietoris spectral sequence

The Mayer-Vietoris spectral sequence of \([11\ p. 39\) is stated and proven in \(11\) only for defining ideals of arrangements of subspaces and only for \(M = A\). Yet the same statement\(^1\) holds and the same proof works in general. We reproduce a complete proof to avoid any misunderstanding.

**Theorem 2.1.** Let \(A\) be a commutative ring, let \(I_1, \ldots, I_n \subset A\) be ideals and let \(M\) be an \(A\)-module. There exists a spectral sequence

\[
E_1^{pq} = \oplus_{i_0 < \cdots < i_p} H^q_{I_{i_0} + \cdots + I_{i_p}}(M) \Rightarrow H^{p+q}_{I_1 \cap \cdots \cap I_n}(M).
\]

**Proof.** We recall that for an ideal \(J\) of \(A\) one denotes by \(\Gamma_J(M)\) the submodule of \(M\) consisting of the elements annihilated by some power of \(J\).

If \(J' \subset J\), we let \(\gamma_{J,J'} : \Gamma_J(M) \hookrightarrow \Gamma_{J'}(M)\) be the natural inclusion.

We denote by \(\Gamma^*(M)\) the complex

\[
0 \to \Gamma^{-n+1}(M) \xrightarrow{d} \Gamma^{-n+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Gamma^0(M) \to 0
\]

in the category of \(R\)-modules where \(\Gamma^{-p}(M) = \oplus_{i_0 < \cdots < i_p} \Gamma_{I_{i_0} + \cdots + I_{i_p}}(M)\) and \(d^{-p}(x) = \oplus_{j=0}^{p} (-1)^j \gamma_{J,J'}(x)\) for every element \(x \in \Gamma_J(M) \subset \Gamma^{-p}(M)\), where \(J = I_{i_0} + \cdots + I_{i_p}\) and \(J_j = I_{i_0} + \cdots + I_{j-1} + I_{j+1} + \cdots + I_{i_p}\) and \(\gamma_{J,J'}(x)\) is an element of \(\Gamma_{J'}(M) \subset \Gamma^{-p+1}(M)\).

We view \(\Gamma(-)\) as a functor from the category of \(A\)-modules to the category of complexes of \(A\)-modules; namely, an \(A\)-module map \(f : M \to M'\) induces maps \(f_{i_0, \ldots, i_p} : \Gamma_{I_{i_0} + \cdots + I_{i_p}}(M) \to \Gamma_{I_{i_0} + \cdots + I_{i_p}}(M')\) which induce a map of complexes \(f^* : \Gamma(M) \to \Gamma(M')\) where \(f^{-p} = \oplus_{i_0 < \cdots < i_p} f_{i_0, \ldots, i_p} \).

We claim that if \(M\) is injective, then \(H^t(\Gamma^*(M))\), the \(t\)-th cohomology module of the complex \(\Gamma^*(M)\), is zero if \(t < 0\) and is isomorphic to \(\Gamma_{I_1 \cap \cdots \cap I_n}(M)\) if \(t = 0\). Indeed, an injective module is a direct sum of modules of the form \(E(A/P)\) where \(P\) is some prime ideal of \(A\) and \(E(A/P)\) is

\(^1\)There is a misprint in a key formula of \([11\ p. 39\) which says \(E_1^{-i,j} = \text{Roos}_i(H^{[i]}_{[i]}(E'))\) whereas it should say \(E_1^{-i,j} = \text{Roos}_i(H^{[i]}_{[i]}(R))\).
the injective hull of $A/P$. Since the functors $\Gamma^\bullet(-)$ and $\Gamma_{I_1 \cap \cdots \cap I_n}(-)$ commute with direct sums, it is enough to prove the claim for $M = E(A/P)$ in which case $\Gamma_J(M) = 0$ if $J \not\subseteq P$ and $\Gamma_J(M) = M$ if $J \subseteq P$ while the inclusion maps $\gamma_{J',J}$ for ideals $J \subset J' \subseteq P$ are the identity maps. If none of the ideals $I_1, \ldots, I_n$ are contained in $P$, then the complex $\Gamma^\bullet(M)$ vanishes and so does the module $\Gamma_{I_1 \cap \cdots \cap I_n}(M)$, hence the claim holds. If some of the ideals $I_j$ are contained in $P$, then $\Gamma^\bullet(M) = M \otimes_k K^\bullet$ where $k$ is the fraction field of $A/P$ and $K^\bullet$ is the complex such that upon giving $K^{-p}$ the name $K_p$, the complex $K^\bullet$ turns into a complex $K_\bullet$ which is the standard algebraic complex for the computation of the singular homology with coefficients in $k$ of the full simplex on the set of vertices $\{i|I_i \subset P\}$. Since a full simplex on any finite set of vertices in contractible, the homology of $K_\bullet$ vanishes in positive degrees and its zeroth homology is $k$. Hence the cohomology of $K^\bullet$ vanishes in negative degrees and its zeroth cohomology is $k$. Since tensoring over a field is an exact functor and $M \otimes_k k = M = \Gamma_{I_1 \cap \cdots \cap I_n}(M)$ for $M = E(A/P)$, the claim is proven.

Since $\Gamma(-)$ is a functor, any complex $C^\bullet$ yields a double complex $\Gamma^\bullet(C^\bullet)$ in which the $(-p,q)$-th entry is $\Gamma^{-p}(C^q)$, the vertical map $\Gamma^{-p}(C^q) \to \Gamma^{-p}(C^{q+1})$ is induced by the differential $C^q \to C^{q+1}$ of $C^\bullet$ and the $q$-th horizontal line is $\Gamma^\bullet(C^q)$ (with all the differentials multiplied by $(-1)^q$, to make $\Gamma^\bullet(C^\bullet)$ an honest double complex, rather than a commutative diagram). Since the double complex $\Gamma^\bullet(C^\bullet)$ has just a finite number of non-zero columns, both associated spectral sequences converge to the cohomology of the total complex of $\Gamma^\bullet(C^\bullet)$.

Let $0 \to M \to E^0 \to E^1 \to \ldots$ be an injective resolution of $M$. Setting $C^\bullet = \Gamma_{I_1 \cap \cdots \cap I_n}(E^\bullet)$ we get a double complex $\Gamma^\bullet(C^\bullet)$ as above in which the $q$-th horizontal line is $\Gamma^\bullet(\Gamma_{I_1 \cap \cdots \cap I_n}(E^q))$ with all the differentials multiplied by $(-1)^q$. Since $E^q$ is injective, so is $\Gamma_{I_1 \cap \cdots \cap I_n}(E^q)$, hence according to the above claim, the $q$-th horizontal line is exact except in degree zero where the cohomology is $\Gamma_{I_1 \cap \cdots \cap I_n}(E^q)$. Hence the horizontal cohomology of the double complex is concentrated on the vertical line $p = 0$ where it forms the complex $\Gamma_{I_1 \cap \cdots \cap I_n}(E^\bullet)$ whose cohomology in degree $q$ is $H_{I_1 \cap \cdots \cap I_n}^q(M)$. Thus the abutment in total degree $t$ of each of the two spectral sequences associated with $\Gamma^\bullet(C^\bullet)$ is $H_{I_1 \cap \cdots \cap I_n}^t(M)$.

The $E_1^{-p,q}$ term of one of the two associated spectral sequences is the vertical cohomology of $\Gamma^\bullet(C^\bullet)$ in degree $(-p,q)$. The $(-p)$-th vertical line of $\Gamma^\bullet(C^\bullet)$ is $\Gamma^{-p}(E^0)$ and $\Gamma^{-p}(-) = \oplus_{i_0 < \cdots < i_p} \Gamma_{I_{i_0} \cap \cdots \cap I_{i_p}}(-)$. Hence the vertical cohomology of $\Gamma^\bullet(C^\bullet)$ in degree $(-p,q)$ is $\oplus_{i_0 < \cdots < i_p} H^q(\Gamma_{I_{i_0} \cap \cdots \cap I_{i_p}}(E^\bullet))$. But $H^q(\Gamma_{I_{i_0} \cap \cdots \cap I_{i_p}}(E^\bullet)) \cong H_{I_{i_0} \cap \cdots \cap I_{i_p}}^q(M)$. Hence in the corresponding spectral sequence $E_1^{-p,q} = \oplus_{i_0 < \cdots < i_p} H_{I_{i_0} \cap \cdots \cap I_{i_p}}^q(M)$ and the total degree of $E_1^{-p,q}$ is $t = q - p$. \qed
Remark 2.2. The differentials on the $E_1$ page have bidegree $(1, 0)$, hence $E_2^{-p,q}$ is the $(−p)$-th cohomology of the resulting complex

$$0 \to E_1^{-n+1,d} \to E_1^{-n+2,q} \to \cdots \to E_1^{-1,q} \to E_1^{0,q} \to 0$$

where the differential $E_1^{-p,q} \to E_1^{-p+1,q}$ takes $x \in H^q_{I_{1_0} + \cdots + I_{i_p}}(M) \subset E_1^{-p,q}$ to $\oplus_{j=0}^{p}(-1)^j h^q_{J,j}(x)$. Here $J_j = I_{i_0} + \cdots + I_{i_{j-1}} + I_{i_{j+1}} + \cdots + I_{i_p}$ and $J = I_{i_0} + \cdots + I_{i_p}$ while $h^q_{J,j} : H^q_j(M) \to H^q_j(M)$ is the natural map induced by the inclusion $J_j \subset J$ (hence $h^q_{J,j}(x) \in H^q_j(M) \subset E_1^{-p+1,q}$).

It is not hard to see that if $n = 2$, i.e. there are just two ideals, $I_1$ and $I_2$, the Mayer-Vietoris spectral sequence degenerates at $E_2$. The following two corollaries, 2.3 and 2.5, play a key role in the proofs of our main results. They show that under suitable assumptions on integers $d$ and $t$ the Mayer-Vietoris spectral sequence degenerates at $E_2$ in degree $(-t,d)$, the abutment $H^d_{I,t}(M)$ is isomorphic to $E_2^{-t,d}$ and is expressible in terms of the reduced singular homology of a certain simplicial complex. For integers $d$ and $t$ from the statement of Theorem 1.1 the above-mentioned assumptions will be proven in Theorem 3.5 in the next section.

Corollary 2.3. Let $A$ be a commutative ring, let $I_1, \ldots, I_n \subset A$ be ideals, let $I = I_1 \cap \cdots \cap I_n$, let $M$ be an $A$-module with dimSuppM $\leq d$ and let

$$E_1^{-p,q} = \oplus_{i_0< \cdots < i_p} H^q_{I_{i_0} + \cdots + I_{i_p}}(M) \mapsto H^q_{I,t}(M)$$

be the associated Mayer-Vietoris spectral sequence. Let $t < d$ be a non-negative integer. Assume that if $p < t$, then $H^q_{I_{i_0} + \cdots + I_{i_p}}(M) = 0$ for all $q \geq d - t + p$ and all $\{i_0, \ldots, i_p\}$. Then the spectral sequence degenerates at $E_2$ in degree $(-t,d)$, i.e. $E_2^{-t,d} = E_\infty^{-t,d}$ and $H^d_{I,t}(M) \cong E_2^{-t,d}$.

Proof. The terms $E_1^{-p,q} = \oplus_{j_0< \cdots < j_p} H^q_{I_{j_0} + \cdots + I_{i_p}}(M)$ are zero for $q > d$ since $H^q_{I}(M) = 0$ for any ideal $J$ provided $q > d \geq \dim Supp M$.

The terms $E_1^{-p,q}$ with $q - p = d - t$ are zero if $0 \leq p < t$ for then $H^q_{I_{j_0} + \cdots + I_{i_p}}(M) = 0$ by assumption. They are also zero if $p > t$ for then $q = d - t + p > d$. Thus the only possibly non-zero $E_1^{-p,q}$ with $q - p = d - t$ is $E_1^{-t,d}$. Hence $H^d_{I,t}(M) \cong E_\infty^{-t,d}$.
If $r \geq 2$, the incoming differentials $d^r : E_{r-t-r,d+r-1}^r \to E_{r-t,d}^r$ are zero because $E_{r-t-r,d+r-1}^r$ is a subquotient of $E_{1-t-r,d+r-1}^1$ which vanishes since $d + r - 1 > d$.

If $r \geq 1$, the outgoing differentials $d^r : E_{r-t,d}^r \to E_{r-t+r,d-(r-1)}^r$ are also zero because $E_{1-t+r,d-(r-1)}^1 = 0$ by assumption (in this case $p = t - r < t$ and $q = d - (r - 1)$, hence $q - p = d - t + 1 > d - t$).

Hence $E_{\infty}^{t,d} \cong E_2^{t,d}$. □

Remark 2.4. As is shown above, the differential $d^1 : E_{1-t,d}^1 \to E_{1-t+1,d}^1$ is zero, hence $E_2^{t,d}$ is the cokernel of the map $E_1^{t-1,d} \to E_1^{t,d}$ of the complex $E_1^{\bullet,d}$ of Remark 2.2.

Corollary 2.5. Let $A$ be a complete local ring with maximal ideal $m$. Let $I = I_1 \cap I_2 \cap \cdots \cap I_n$ be the intersection of $n$ ideals $I_1, \ldots, I_n$ of $A$. Let $P$ be a prime ideal of $A$ such that $\dim(A/P) = d$ and let $M$ be an $A$-module supported on $V(P)$. Let

$$E_1^{p,q} = \oplus_{i_0 < \cdots < i_p} H_{i_0 + \cdots + i_p}^q(M) \Longrightarrow H_{i_0 + \cdots + i_p}^q(M)$$

be the associated Mayer-Vietoris spectral sequence. Let $\Delta$ be the simplicial complex on $n$ vertices $\{1, 2, \ldots, n\}$ defined as follows: a simplex $\{j_0, \ldots, j_p\}$ belongs to $\Delta$ iff $I_{j_0} + \cdots + I_{j_p} + P$ is not $m$-primary. For any integer $t \geq 0$ the $E_2^{t,d}$ term is isomorphic to $\tilde{H}_{t-1}(\Delta; H_m^d(M))$, the $(t-1)$-th reduced singular homology group of $\Delta$ with coefficients in $H_m^d(M)$.

Proof. By the Hartshorne-Lichtenbaum local vanishing theorem \[3.1\], $H_m^d(M) \neq 0$ only if $\sqrt{J + P} = m$ in which case $H_m^d(M) \cong H_m^d(M)$. Thus $H_{i_0 + \cdots + i_p}^q(M) \neq 0$ only if $\{j_0, \ldots, j_p\}$ is not a simplex of $\Delta$, in which case $H^{i_0 + \cdots + i_p}(M) \cong H_m^d(M)$. The natural map $h_{i_0 \cdots j_p}^d(M) \to H_{i_0 + \cdots + i_p}^d(M)$ of Remark 2.2 is non-zero only if $\{i_0, \ldots, i_j, \ldots, i_p\}$ is not a simplex of $\Delta$ in which case it is the identity map. Thus the (cohomological) complex $E_1^{\bullet,d}$ of Remark 2.2 upon giving $E_1^{p,d}$ the name $C_p$, turns into a (homological) complex $C_\bullet$ which is the standard complex for the computation of the singular homology of the pair $(S, \Delta)$ with coefficients in the module $H_m^d(M)$, where $S$ is the full $(n - 1)$-simplex on the set $\{1, \ldots, n\}$. Thus $E_2^{t,d} = H^{-t}(E_1^{-\bullet,d}) = H_t(C_\bullet) = H_t^{\text{sing}}(S; \Delta; H_m^d(M))$. There is a long exact sequence $\cdots \to H_{s+1}(S; G) \to H_s(\Delta; G) \to H_s(S; G) \to H_s(S; \Delta; G) \to \cdots$ for every abelian group $G$ which implies (upon setting $G = H_m^d(M)$), that $H_t^{\text{sing}}(S; \Delta; H_m^d(M)) \cong \tilde{H}_{t-1}(\Delta; H_m^d(M))$ since $S$ is contractible. □

The plan of our proof of Theorem 1.1 and the more general results of Section 4 is as follows. In the next section we will prove Theorem 3.5 which among other things implies that if $d$ and $t$ are integers from the statement of Theorem 1.1 and $p < t$, then $H_{i_0 + \cdots + i_p}^q(M) = 0$ for all $q \geq d - t + p$ and all $\{i_0, \ldots, i_p\}$. Corollary 2.3 then implies that $H_{t+1}^{p+1}(M) \cong E_2^{t,d}$, where
$v = d - t - 1$, as in the statement of Theorem 1.1. Now Corollary 2.3 (with $P = 0$) yields $E_{2}^{-t,d} = \tilde{H}_{t-1}^{\text{sing}}(\Delta; H_{m}^{d}(M)) \cong H_{m}^{d}(M) \otimes_{k} \tilde{H}_{t-1}^{\text{sing}}(\Delta; k)$, where $k$ is a coefficient field of $R$ (the last isomorphism holds by the universal coefficients theorem, since we are over a field). This completes the proof of Theorem 1.1 modulo Theorem 3.5.

But in the special case that $t = \lfloor \frac{d-2}{c} \rfloor = 2$ (the smallest value of $t$ for which Theorem 1.1 was not previously known) we can complete the proof of Theorem 1.1 very quickly without appealing to Theorem 3.5. Namely, result (ii) of the Introduction shows that $H_{j}^{q}(M) = 0$ for $q \geq d - 2$, i.e. the hypotheses of Corollary 2.3 hold for $p = 0$. In the next two paragraphs we will prove that $H_{i_{0} + i_{1}}^{q}(M) = 0$ for $q \geq d - 1$, i.e. the hypotheses of Corollary 2.3 hold for $p = 1$ as well. Thus the hypotheses of Corollary 2.3 hold for $p < t = 2$ and the above proof of Theorem 1.1 goes through without any reference to Theorem 3.5.

It remains to show that $H_{i_{0} + i_{1}}^{q}(M) = 0$ for $q \geq d - 1$. Equivalently, it remains to show that every minimal prime of $I_{i_{0}} + I_{i_{1}}$ has dimension at least two and the punctured spectrum of $R/(I_{i_{0}} + I_{i_{1}})$ is connected (this equivalence is due to Ogus [16, 2.11] in characteristic zero and Peskine and Szpiro [17, III, 5.5] in characteristic $p > 0$; a characteristic-free proof has been given by Huneke and Lyubeznik [10, 2.9]).

Since $R$ is regular, the height of every minimal prime over the ideal $I_{i_{0}} + I_{i_{1}}$ is at most the sum of the heights of $I_{i_{0}}$ and $I_{i_{1}}$, i.e. $2c \leq d - 2$. Hence the dimension of every minimal prime over $I_{i_{0}} + I_{i_{1}}$ is at least 2. As for the connectedness, let $k$ be a coefficient field of $R$ and let $X_{1}, \ldots, X_{d}$ generate $m$. Then $R/(I_{i_{0}} + I_{i_{1}}) \cong (R/I_{i_{0}}) \otimes_{k}(R/I_{i_{1}})/\text{Diag}$, where $\otimes_{k}$ is the complete tensor product over $k$ and $\text{Diag}$ is the ideal generated by the $d$ elements $X_{i} \otimes 1 - 1 \otimes X_{i}$. The ring $(R/I_{i_{0}}) \otimes_{k}(R/I_{i_{1}})$ has a unique minimal prime since $k$ is separably closed [10, 4.5]. The sum of the heights of $I_{i_{0}}$ and $I_{i_{1}}$ is at most $2c \leq d - 2$, so the dimension of $(R/I_{i_{0}}) \otimes_{k}(R/I_{i_{1}})$ is at least $d + 2$, hence the Faltings connectedness theorem [3, 4, 9, 3.1, 3.3] implies that the punctured spectrum of $R/(I_{i_{0}} + I_{i_{1}}) \cong (R/I_{i_{0}}) \otimes_{k}(R/I_{i_{1}})/\text{Diag}$ is connected.

We conclude this section by showing that the case $t = \lfloor \frac{d-2}{c} \rfloor = 2$ of Theorem 1.1 that has just been proven quickly leads to a complete understanding of an example that proved intractable in our old paper [10, 5.12] by means of techniques that were known to us then. In that old paper we, assuming that $\lfloor (d - 2)/c \rfloor = 2$, completely analyzed the cases where $I$ has at most five minimal primes by representing $I$ as the intersection of two ideals and then using the standard Mayer-Vietoris long exact sequence [10, 5.11]. But for six minimal primes we found the following example for which the Mayer-Vietoris long exact sequence failed to provide an answer. Below we show that the Mayer-Vietoris spectral sequence works beautifully for this example (via the $\lfloor (d - 2)/c \rfloor = 2$ case of Theorem 1.1) and produces an unexpected result: $\text{cd}(R, I)$ depends on the characteristic of the field $k$. 


Example 2.6. Let $R$ be a complete $d$-dimensional equicharacteristic regular local ring with maximal ideal $m$ and a separably closed residue field $k$. Let $I$ be an ideal of $R$ with six minimal primes $I_1, \ldots, I_6$ such that the height of each $I_i$ equals $c$ where $2c + 2 \leq d$. Set $\Lambda = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{2, 4, 6\}, \{1, 5, 6\}, \{3, 5, 6\}, \{2, 3, 5\}, \{3, 4, 6\}, \{1, 2, 6\}, \{1, 4, 5\}\}$. Suppose $\{i, j, \kappa\} \in \Lambda$ if and only if $I_i + I_j + I_\kappa$ is $m$-primary. Then $H^{d-2}_I(R) = 0$ if and only if $\text{char } k \neq 2$ and $\text{cd}(R, I) \leq d - 3$ also if and only if $\text{char } k \neq 2$.

Proof. It is not hard to check that every four-element subset of $\{1, \ldots, 6\}$ contains some element of $\Lambda$ as a subset. This means that the complex $\Delta$ of Theorem [1] does not contain any 3-simplicies, i.e. it is a union of 2-simplicies. A 2-simplex $\{i, j, \kappa\}$ belongs to $\Delta$ if and only if it is not an element of $\Lambda$. Hence the complete list of the 2-simplicies of $\Delta$ is $\{\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{2, 3, 4\}, \{2, 3, 6\}, \{3, 4, 5\}, \{4, 5, 6\}, \{2, 5, 6\}\}$ and the picture below shows that this collection of simplicies triangulates the real projective plane (which is obtained from the 2-dimensional disc (or hexagon) by identifying pairs of antipodal points on the boundary).

Hence the complex $\Delta$ is homeomorphic to the real projective plane. Thus $\hat{H}_1(\Delta; k)$, the critical singular homology group of $\Delta$, is zero if the characteristic of $k$ is different from 2 and is isomorphic to $k$ otherwise. Now we are done by the $[(d - 2)/c] = 2$ case of Theorem [11].

For a concrete realization of the above example let $a \in k$ be an element such that $a \neq 0, 1, -1$ and $a^2 + a - 1 \neq 0$. Let $R = k[[X_1, X_2, X_3, X_4, X_5, X_6]]$ be the ring of formal power series in six variables $X_1, \ldots, X_6$. Let ideals $I_1, \ldots, I_6$ of height two be defined as follows:

$I_1 = (X_1, X_2), I_2 = (X_3, X_4), I_3 = (X_5, X_6),\nI_4 = (X_1 + X_3 + X_6, \frac{1}{a^2+a-1}X_1 + X_4 + \frac{1}{a}X_6),\nI_5 = (X_1 + X_3 + X_5, X_2 + aX_3 + X_5),\nI_6 = (X_2 + X_4 + X_5, \frac{1}{a}X_2 + aX_4 + X_6).$

It is tedious but straightforward to verify that $I_i + I_j + I_\kappa$ is $m$-primary iff $\{i, j, \kappa\} \in \Lambda$ where $\Lambda$ is the same as in the above example (the assumptions $a \neq 1, -1$, and their consequences $\frac{1}{a} \neq a$, $\frac{1}{a^2+a-1}$ are used in the verification).

Setting $I = I_1 \cap \cdots \cap I_6$ we conclude that $H_1^I(R) = 0$ if $\text{char } k \neq 2$ and $H_1^I(R) \neq 0$ if $\text{char } k = 2$. Also, $\text{cd}(R, I) \leq 3$ if $\text{char } k \neq 2$ and $\text{cd}(R, I) = 4$ if $\text{char } k = 2$. 

\[\begin{array}{cccccc}
& 1 & 2 & 3 & 4 & 5 \\
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
5 & & & & & \\
6 & & & & & \\
\end{array}\]
3. A VANISHING THEOREM FOR $H^q_{I_0+\cdots+I_p}(M)$. 

The main result of this section is Theorem 3.5 which establishes the vanishing of $E^p_{I_0+\cdots+I_p}$ needed in the proofs of our main results. We prove our main results for an arbitrary commutative Noetherian local ring containing a field. This necessitates working not with the height of the minimal primes over $I$, like in the statement of Theorem 1, but with the integer $c(I)$ whose definition we recall below and which is equal to the maximum height of a minimal prime over $I$ if the ring is regular.

Definition. [10, 2.1] If $A$ is a local ring and $I \subset A$ is an ideal, 

$$c(I) = \text{emb.dim} A - \min\{\dim(A/P) | P \text{ is a minimal prime over } I\}.$$ 

In the sequel $\hat{A}$ denotes the completion of $A$ with respect to the maximal ideal. The following result is a rephrasing of Faltings [2, Korollar 2].

Theorem 3.1. Let $A$ be a local ring containing a field, let $c > 0$ and $d \geq 0$ be integers, let $I \subset A$ be an ideal with $c(I\hat{A}) \leq c$ and let $M$ be an $A$-module such that $\dim \text{Supp} M \leq d$. Then $H^q_0(M) = 0$ for $q > d - \lfloor \frac{d-1}{c} \rfloor$.

Proof. [2] Korollar 2] says that if $M$ is finitely generated, then $H^q_0(M) = 0$ for $q > (1 - \frac{1}{c(I\hat{A})}) \cdot \dim M + 1$ provided $c(I\hat{A}) \geq 2$ and $H^q_0(M) = 0$ for $q > c(I\hat{A})$ provided $c(I\hat{A}) < 2$. This immediately implies Theorem 3.1 because $M$ is the direct limit of its finitely generated submodules, the dimension of each finitely generated submodule is at most $d$ and local cohomology commutes with direct limits, and for all integers $d \geq 0$ and $c > 0$ the function $\phi(c, d) = d - \lfloor \frac{d-1}{c} \rfloor = [(1 - 1/c)d + 1]$ is non-decreasing and $\phi(c, d) \geq 1$. □

We need the following generalization.

Theorem 3.2. Let $A$ be a local ring containing a field, let $c$ and $d$ be positive integers, let $J = I_0 + \cdots + I_p$ be the sum of $p + 1$ ideals $I_0, \ldots, I_p$ such that $c(I_j\hat{A}) \leq c$ for each $j$ and let $M$ be an $A$-module such that $\dim \text{Supp} M \leq d$. Then $H^q_J(M) = 0$ for $q > d - \lfloor (d-1)/c \rfloor + p$. In particular, if $\dim A \leq d$, then $\dim(A, I) \leq d - \lfloor (d-1)/c \rfloor + p$.

Proof. We use induction on $p$, the case $p = 0$ being known by Theorem 3.1. Assume $p > 0$ and the result proven for $p - 1$. Since a prime contains the ideal $I_p \cap (I_0 + \cdots + I_{p-1})$ if and only if it contains $J = I_p \cap I_0 + I_p \cap I_1 + \cdots + I_p \cap I_{p-1}$, these two ideals have the same radical, so there is an isomorphism of the corresponding local cohomology functors $H^q_{I_p \cap (I_0 + \cdots + I_{p-1})}(-) \cong H^q_J(-)$. The ideal $J$ (resp. $I_0 + \cdots + I_{p-1}$) is the sum of $p$ ideals $I_p \cap I_0, \ldots, I_p \cap I_{p-1}$ (resp. $I_0, \ldots, I_{p-1}$) such that $c(I_p \cap I_j\hat{A}) \leq c$ (resp. $c(I_j\hat{A}) \leq c$) for every $j \leq p - 1$. Hence by the induction hypothesis $H^q_J(M) \cong H^q_{I_p \cap (I_0 + \cdots + I_{p-1})}(M) \cong 0$ and $H^q_{I_0 + \cdots + I_{p-1}}(M) = 0$ for $q > d - \lfloor (d-1)/c \rfloor + p - 1$. The Mayer-Vietoris exact sequence $H^q_{I_p \cap (I_0 + \cdots + I_{p-1})}(M) \to H^q_J(M) \to H^q_{I_p} \oplus H^q_{I_0 + \cdots + I_{p-1}}(M)$ implies the theorem. □
The following lemma is a consequence of [10, 2.2, 2.3]:

**Lemma 3.3.** Let $A$ be a complete local ring containing a field and let $I$ be an ideal of $A$. Then

(i) $c(I) = c(IB)$ where $B$ is the completion of the strict Henselization of $A$ (i.e. $B \cong K \hat{\otimes}_k \hat{A}$, where $k$ is a coefficient field of $A$, while $\hat{\otimes}_k$ is the complete tensor product over $k$ and $K$ is the separable closure of $k$).

(ii) $c(I A_P) \leq c(I)$ for every prime ideal $P \supset I$, where $A_P$ is the completion of $A_P$.

**Proof.** [10, 2.2] says that if $A$ is a universally catenary local ring containing a field, then $c(I) = c(I \hat{A})$ and $c(I) = c(IB)$ where $B$ is the completion of the strict Henselization of $A$ while [10, 2.3] says that under the same assumptions $c(I_P) \leq c(I)$. But a complete local ring $A$ is universally catenary and so are all its localizations $A_P$. □

The following result is a rephrasing of Huneke and Lyubeznik [10, 3.8].

**Theorem 3.4.** Let $A$ be a local ring containing a field, let $B = \hat{A}^{sh}$ be the completion of the strict Henselization of the completion of $A$, let $c$ and $d$ be positive integers with $c < d$, let $I \subset A$ be an ideal such that $c(I \hat{A}) \leq c$ and $\sqrt{IB}$ is a prime ideal of $B$ and let $M$ be an $A$-module with $\text{dimSupp} M \leq d$. Then $H^q_I(M) = 0$ for $q > d - 1 - \left[ \frac{d - 2}{c} \right]$.

The actual statement of [10, 3.8] is a special case with $M$ finitely generated, $0 < c(I \hat{A}) = c$ and $d = \text{dim} M$. Theorem 3.5 is immediate from this special case just like Theorem 3.1 is immediate from [2, Korollar 2]. The following generalization is the main result of this section.

**Theorem 3.5.** Let $A$ be a local ring containing a field, let $B = \hat{A}^{sh}$ be the completion of the strict Henselization of the completion of $A$, let $c$ be a positive integer and let $J$ be an ideal of $A$ such that $\sqrt{JB} = \sqrt{I_0 + \cdots + I_p}$ where $I_0, \ldots, I_p$ are prime ideals of $B$ with $c(I_j) \leq c$ for each $j$. Let $M$ be an $A$-module with $\text{dimSupp} M \leq d$ where $d > (p + 1)c$. Then $H^q_J(M) = 0$ for $q > d - 1 - \left[ \frac{(d - 2)/c}{p} \right] + p$. In particular, if $(p + 1)c < \text{dim} A$ and $\text{dim} A \leq d$, then $cd(A, I) \leq d - 1 - \left[ \frac{(d - 2)/c}{p} \right] + p$.

**Remarks.** (i) The restriction $d > (p + 1)c$ cannot be omitted. For example, if $A$ is regular of dimension $d = (p + 1)$ and $I_0, \ldots, I_p$ each have height $c$ and add up to the maximal ideal, then $H^d_J(A) \neq 0$ while $d > d - 1 - \left[ \frac{(d - 2)/c}{p} \right] + p$.

(ii) The condition $p < t = \left[ \frac{d - 2}{c} \right]$ implies $d > (p + 1)c$, hence Theorem 3.5 does indeed show that the hypotheses of Corollary 2.3 hold for $t = \left[ \frac{d - 2}{c} \right]$.

Our proof of Theorem 3.5 is considerably longer than the above proof of Theorem 3.2. One cannot deduce Theorem 3.5 from Theorem 3.4 in the same way as Theorem 3.2 was deduced from Theorem 3.1 because the ideals $I_p \cap I_j$ are not formally geometrically irreducible, hence the induction hypotheses does not apply to $J = I_p \cap I_0 + \cdots + I_p \cap I_{p-1}$. 
Proof of Theorem 3.3. The module $H^q_{IH}(B \otimes_A M) \cong B \otimes_A H^q_I(M)$ vanishes if and only if $H^q_I(M)$ vanishes since $B$ is faithfully flat over $A$. Since $\dim\text{Supp} B \otimes_A M \leq d$, replacing the ring $A$ and the module $M$ by $B$ and $B \otimes_A M$ respectively, we may assume that $A$ is complete with separably closed residue field and the $I_\delta$s are prime ideals of $B$.

By one of Cohen’s structure theorems for complete local rings, the ring $A$ contains a coefficient field $k \subset A$ and there exists a surjective $k$-algebra homomorphism $R \to A$ where $R = k[[X_1, \ldots, X_n]]$ is the ring of formal power series in $n$ variables over $k$ and $n$ is the embedding dimension of $A$. The $A$-module $M$ acquires a structure of $R$-module via this homomorphism. If $I \subset A$ is an ideal and $I \subset R$ is the full preimage of $I$, then $H^q_I(M) \cong H^q_{I'}(M)$. Each of the ideals $I_i$ is prime and $c(I_i) = c(I_1)$. Thus replacing $A$ by $R$ and $I_i$ by $I_i$ we may assume that $A = k[[X_1, \ldots, X_n]]$ is a ring of formal power series over a separably closed field $k$.

Let $K$ be an uncountable separably closed field extension of $k$. The inclusion $A = k[[X_1, \ldots, X_n]] \to K[[X_1, \ldots, X_n]] = R'$ induced by the natural inclusion $k \to K$ makes $R'$ an $A$-algebra. Using the Koszul resolution of $A/m \cong k$ on $X_1, \ldots, X_n$ one sees that $\text{Tor}_i^A(k, R') \cong 0$, so $R'$ is flat over $A$ by the local criterion of flatness [15 20.C(3')]. Since the inclusion $A \to R'$ is a local ring homomorphism, $R'$ is faithfully flat over $A$ [15 4.D]. Clearly, $c(I_i R') = c(I_i)$ and $\dim\text{Supp}(R' \otimes_A M) = \dim\text{Supp} M \leq d$. At this point we recall the statement of [10 4.3] which in our notation says the following: if $k$ is separably closed in $K$, then for every prime ideal $P \subset A$ the radical of the ideal $PR'$ of $R'$ is prime. Since in our case $k$ is a separably closed field, the radicals of the ideals $I_i R'$ are prime. Since $R'$ is flat over $A$, $H^q_{I_i R'}(R' \otimes_A M) \cong R' \otimes_A H^q_I(M)$ for every ideal $I$ of $A$ and since $R'$ is faithfully flat, $H^q_{I R'}(R' \otimes_A M) \cong 0$ if and only if $H^q_I(M) \cong 0$. Hence replacing the ring $A$, the ideal $I$, and the module $M$ by $R'$, $\text{rad}(I, R')$ and $R' \otimes_A M$ respectively, we may assume that $A = K[[X_1, \ldots, X_n]]$ is a ring of formal power series over an uncountable separably closed field $K$ and each $I_i$ is prime.

We may assume that $M$ is finitely generated because $H^q_I(M)$ is the direct limit of $H^q_I(M')$ as $M'$ runs over all the finitely generated submodules of $M$ and $\dim M' \leq d$ for all such $M'$.

Let $v = d - 1 - [(d-2)/e]+p$. We may assume that $c \not|(d - 1)$ because otherwise $v = d - [(d-1)/e]+p$ and we are done by Theorem 3.2.

We use induction on $p$, the case $p = 0$ being known by Theorem 3.3. Assume $p > 0$ and let $J' = I_0 + \cdots + I_{p-1}$. The composition of functors $\Gamma_p(\Gamma_{J'}(-)) \cong \Gamma_J(-)$ leads to the spectral sequence

$$E_2^{p,q} = H^p_I(H^q_{J'}(M)) \Rightarrow H^{p+q}_J(M).$$

To prove the theorem it is enough to show that $E_\infty^{p,q} = 0$ provided $p + q > v$, and we are going to show this. For some, but not all, pairs $(p, q)$ with $p + q > v$ we are even going to show that $E_2^{p,q} = 0$. 
To this end, let $p$ and $q$ be non-negative integers with $p+q > v$. There are only two possibilities, either $\dim\text{Supp}H^q_{p}(M) > c$, or $\dim\text{Supp}H^q_{p}(M) \leq c$.

First we consider the case that $\dim\text{Supp}H^q_{p}(M) > c$. We are going to prove that $E_{2}^{p,q} = 0$ in this case. Let $\delta$ be the biggest integer such that $q \leq d - \delta - [(d - \delta - 1)/c] + p - 1$ (it exists since $d - x - [(d - x - 1)/c] + p - 1$ is a non-increasing function of $x$). If $\delta > \delta$, then $q > d - \delta - [(d - \delta - 1)/c] + p - 1$ and if $P$ is a prime of $A$ of dimension $\delta'$, then $c(I\hat{A}_P) \leq c$ by Lemma 3.3 and $\dim\text{Supp}M_P \leq d - \delta'$, so $H^q_{p}(M_P) = 0$ by Theorem 3.2. Hence the module $H^q_{p}(M)$ vanishes at all primes of dimension bigger than $\delta$, which implies that $\delta \geq \dim\text{Supp}H^q_{p}(M) > c$. Now Theorem 3.2 implies that $E_{2}^{p,q} = H^p_{I\hat{A}_P}(H^q_{p}(M)) = 0$ for $p > \delta - 1 - [(\delta - 2)/c]$. Hence it is enough to show that $p > \delta - 1 - [(\delta - 2)/c]$.

This is a formal consequence of the inequalities $p+q > d - 1 - [(d - 2)/c] + p$ and $q \leq d - \delta - [(d - \delta - 1)/c] + p - 1$, which imply

$$p > (d - 1 - [(d - 2)/c] + p) - q$$

$$\geq (d - 1 - [(d - 2)/c] + p) - (d - \delta - [(d - \delta - 1)/c] + p - 1)$$

$$= \delta + [(d - \delta - 1)/c] - [(d - 2)/c],$$

so it is enough to show that

$$\delta + [(d - \delta - 1)/c] - [(d - 2)/c] \geq \delta - 1 - [(\delta - 2)/c],$$

i.e.

$$[(d - \delta - 1)/c] + [(\delta - 2)/c] + 1 - [(d - 2)/c] \geq 0.$$ 

Let $\delta = \kappa c + r$, where $0 \leq r < c$. Then

$$[(d - \delta - 1)/c] + [(\delta - 2)/c] = [(d - r - 1)/c] + [(r - 2)/c].$$

If $r \geq 2$, then $[(r - 2)/c] = 0$, so we need to show that

$$[(d - r - 1)/c] + 1 \geq [(d - 2)/c].$$

But $r \leq c - 1$ implies

$$[(d - r - 1)/c] + 1 \geq [(d - c)/c] + 1 = [d/c] \geq [(d - 2)/c],$$

so we are done in this case. If $r = 0, 1$, then $[(r - 2)/c] = -1$, so

$$[(d - r - 1)/c] + [(r - 2)/c] + 1 = [(d - r - 1)/c] \geq [(d - 2)/c].$$

This completes the proof that $E_{2}^{p,q} = 0$ provided $\dim\text{Supp}H^q_{p}(M) > c$ and $p + q > v$.

It remains to prove that $E_{\infty}^{p,q} = 0$ if $\dim\text{Supp}H^q_{p}(M) \leq c$ and $p + q > v$. If $p > c$, then $p > \dim\text{Supp}H^q_{p}(M)$, hence $E_{2}^{p,q} = H^p_{I\hat{A}_P}(H^q_{p}(M)) = 0$. Thus it remains to consider the case that $p \leq c$.

First we assume that $p + q \geq v + 2$. We are going to show that $E_{2}^{p,q} = 0$ under this assumption. For this it is enough to show, like in the preceding paragraph, that $\dim\text{Supp}H^q_{p}(M) < p$, i.e. $H^q_{p}(M_P) = 0$ for every prime $P$ of dimension $\geq p$. Lemma 3.3 implies that $c(J'\hat{A}_P) \leq c$. We have that $v = d - [(d - 1)/c] + (p - 1)$ since $c \not\mid (d - 1)$. Since $\dim\text{Supp}M_P \leq d - p$,
Theorem 3.2 implies \( H^q_{J^p}(M_P) = 0 \) for \( q > d - p - \lfloor (d - p - 1)/c \rfloor + (p - 1) \) that is, since \( p \leq c \) implies that \( \lfloor (d - p - 1)/c \rfloor \geq \lfloor (d - 1)/c \rfloor \) for\( q > d - \lfloor (d - 1)/c \rfloor + (p - 1) - (p - 1) = v = p + 1 \), i.e. for \( p + q > v + 1 \).

This concludes the proof that \( E^{p,q}_2 = 0 \) provided \( p + q \geq v + 2 \).

It remains to consider the case that \( \dim \text{Supp} H^q_{J^p}(M) \leq c \) while \( p \leq c \) and \( p + q = v + 1 \). This case is the hardest. In this case we are unable to prove that \( E^{p,q}_2 = 0 \), but we are going to prove that \( E^{p,q}_{\infty} = 0 \) by constructing a morphism from our spectral sequence to a different spectral sequence \( \bar{E} \) about which we can prove that for the pairs \( (p, q) \) in question the induced maps \( E^{p,q}_{\infty} \rightarrow \bar{E}^{p,q}_{\infty} \) are isomorphisms and \( \bar{E}^{p,q}_{\infty} = 0 \).

First we claim that if \( q' > v - c \), then \( \dim \text{Supp} H^q_{J^p}(M) \leq c \). Indeed, if \( P \) is a prime of dimension \( > c + 1 \), then \( \dim M_P \leq d - c - 1 \). Lemma 3.3 implies that \( c(J^p \hat{A}_P) \leq c \). By Theorem 3.2 if \( H^q_{J^p}(M_P) \neq 0 \), then \( q' \leq d - c - 1 - \lfloor (d - c - 2)/c \rfloor + (p - 1) = d - 1 - \lfloor (d - 2)/c \rfloor + p = v - c \), i.e. \( q' \leq v - c \). This proves the claim.

We claim there exist elements \( x_1, \ldots, x_c \in I_p \) such that the ideals \( P + I_p \) and \( P + (x_1, \ldots, x_c) \) have the same radical for every minimal prime \( P \) of the support of \( H^q_{J^p}(M) \) as \( q' \) runs through all integers \( q' > v - c \). In fact, by induction on \( c' \) we are going to prove the following statement. There exist \( x_1, \ldots, x_c \in I_p \) such that for each \( c' \leq c \) and each minimal prime \( P \) of some \( H^q_{J^p}(M) \), every prime \( Q \) containing \( P \) and \( x_1, \ldots, x_{c'} \) but not containing \( I_p \), has dimension at most \( c - c' \). This statement for \( c' = c \) implies the claim considering that \( A \) being local has only one prime of dimension zero.

If \( c' = 0 \), we set \( x_0 = 0 \). Since \( \dim \text{Supp} H^q_{J^p}(M) \leq c \), we get \( \dim P \leq c \), so the claim holds if \( c' = 0 \). Assume \( c' > 0 \) and \( x_1, \ldots, x_{c'-1} \) have been found.

To prove the claim it remains to show that there exists an element \( x_{c'} \in I_p \) which does not belong to any prime \( Q \) such that \( Q \) does not contain \( I_p \) and is minimal over some ideal \( (x_1, \ldots, x_{c'-1}) + P \) where \( P \) is minimal in the support of some \( H^q_{J^p}(M) \) with \( q' > v - c \). For then any prime \( Q' \) that does not contain \( I_p \) and contains \( (x_1, \ldots, x_c) + P \) necessarily contains some \( Q \) and \( x_{c'} \notin Q \), hence \( \dim Q' < \dim Q \), but \( \dim Q \leq c - c' + 1 \) by the induction hypothesis and therefore \( \dim Q' \leq c - c' \), as required. The existence of \( x_{c'} \) is shown in the next two paragraphs.

The set of the minimal primes of all the \( H^q_{J^p}(M) \) is countable. This is because \( H^q_{J^p}(M) = \lim_{\rightarrow} \text{Ext}^q_A(A/(J')^n, M) \), so a prime is associated to \( H^q_{J^p}(M) \) only if it is associated to one of the \( \text{Ext}^q_A(A/(J')^n, M) \), but each \( \text{Ext}^q_A(A/(J')^n, M) \) is finitely generated and therefore has a finite number of associated primes. Thus the set of the primes \( Q \) which do not contain \( I_p \) and which are minimal over some ideal \( (x_1, \ldots, x_{c'-1}) + P \), where \( P \) is minimal in the support of some \( H^q_{J^p}(M) \) with \( q' > v - c \), is countable, because the set of the ideals \( P \) is countable.
Let \( y_0, \ldots, y_s \) be a finite system of generators of \( I_p \). For each element \( \kappa \in K \) we set \( y(\kappa) = y_0 + \kappa y_1 + \kappa^2 y_2 + \cdots + \kappa^s y_s \in I_p \). If \( \kappa_0, \kappa_1, \ldots, \kappa_s \in K \) are distinct, the corresponding Vandermonde determinant is non-zero, hence \( y_0, \ldots, y_s \) are linear combinations of \( y(\kappa_0), \ldots, y(\kappa_s) \) with coefficients from \( K \). This implies that each prime \( Q \) of our countable set of primes contains elements \( y(\kappa) \) for at most \( s \) distinct values of \( \kappa \), for otherwise \( y_0, \ldots, y_s \in Q \) and hence \( Q \) contains \( I_p \). Thus the set of all elements \( \kappa \in K \) such that \( y(\kappa) \) belongs to some \( Q \) is countable. Since \( K \) is uncountable, there exists \( \kappa \in K \) such that \( y(\kappa) \in I_p \) does not belong to any \( Q \). We set \( x_c = y(\kappa) \). This completes the proof of the claim. \(^2\)

We claim that the Grothendieck spectral sequence of the composition of functors \( E_2^{p,q} = R^pG(R^qF(A)) \implies R^{p+q}(G \circ F)(A) \) for fixed \( F \) and \( A \) is functorial in \( G \). We were unable to find a reference to this basic fact in the literature, so we provide a proof. Let \( I^\bullet \) be an injective resolution of \( A \) and let \( I^{\bullet, \bullet} \) be an injective double complex resolution of \( I^\bullet \). Let \( G \rightarrow \tilde{G} \) be a natural transformation of functors. The spectral sequences \( E \) and \( \tilde{E} \) associated to the compositions of functors \( G \circ F \) and \( \tilde{G} \circ F \) respectively are the spectral sequences of the filtered total complexes of the double complexes \( G(I^{\bullet, \bullet}) \) and \( \tilde{G}(I^{\bullet, \bullet}) \) respectively. The natural transformation \( G \rightarrow \tilde{G} \) induces a map \( G(I^{p,q}) \rightarrow \tilde{G}(I^{p,q}) \) for every \( p \) and \( q \), hence it induces a morphism of double complexes \( G(I^{\bullet, \bullet}) \rightarrow \tilde{G}(I^{\bullet, \bullet}) \) which in turn induces a morphism of the total complexes that is compatible with the corresponding filtrations. According to \([5, 11.2.3]\), this gives rise to a morphism of associated spectral sequences which proves the claim.

We apply this claim as follows. Setting \( X = (x_1, \ldots, x_c) \) we get a spectral sequence
\[
E_2^{p,\kappa} = H_X^p(H_{J_p}^\kappa(M)) \implies H_X^{p+\kappa}(M).
\]
Since \( X \subset I_p \), we have that \( \Gamma_{I_p}(N) \subset \Gamma_X(N) \) for every module \( N \). This induces a natural transformation of functors \( \Gamma_{I_p}(-) \rightarrow \Gamma_X(-) \) which according to the above claim induces a morphism of spectral sequences \( \phi : E \rightarrow \tilde{E} \).

We claim that \( E_2^{p,\kappa} = H_X^p(H_{J_p}^\kappa(M)) \cong 0 \) for \( p + \kappa \geq v + 2 \). Indeed, this is true if \( p > c \) since \( X \) is generated by \( c \) elements. If \( p \leq c \), it has been shown above that \( \text{dimSupp}H_{J_p}^\kappa(M) < p \) under the additional assumption that \( p + \kappa \geq v + 2 \). This proves the claim.

We claim that the resulting maps \( E_\infty^{p,q} \rightarrow \tilde{E}_\infty^{p,q} \) are isomorphisms if \( p \leq c \) and \( p + q = v + 1 \). This claim is a special case of the following more general claim: If \( p + q = v + 1 - s \), where \( s \geq 0 \) and \( p + s r \leq c \), then the natural map \( E_s^{p,q} \rightarrow \tilde{E}_s^{p,q} \) is an isomorphism. Indeed, if \( p \leq c \) and \( p + q = v + 1 \),

\(^2\)This “countable prime avoidance” argument is necessary because we do not know whether each module \( H_{J_p}^\kappa(M) \) has only a finite number of minimal primes. If we knew this, standard prime avoidance would do and we wouldn’t have even needed to pass from the original possibly countable field \( k \) to the uncountable field \( K \). But it is still an open problem whether the set of the minimal primes of a local cohomology module of a finitely generated module over a Noetherian ring is always finite. One only knows that the set of the associated primes need not be finite \([11, 13]\).
then $s = 0$, hence $p + sr = p \leq c$, so $E_{\phi r}^{p,q} \to \tilde{E}_{\phi r}^{p,q}$ is an isomorphism for all $r$, hence $E_{\phi}^{p,q} \to \tilde{E}_{\phi}^{p,q}$ is an isomorphism.

To prove this more general claim we use induction on $r$. If $r = 2$, we need to show that the natural maps $E_{2}^{p,q} = H_{I_{p}}^{p}(H_{J_{r}}^{q}(M)) \to H_{X}^{p}(H_{J_{r}}^{q}(M)) = \tilde{E}_{2}^{p,q}$ are isomorphisms if $p + q = v + 1 - s$ and $p + 2s \leq c$. The condition $p + 2s \leq c$ implies that $p \leq c - 2s$ and the condition $p + q = v + 1 - s$ implies that $q = v + 1 - s - p \geq v + 1 - s - (c - 2s) = v - c + 1 + s > v - c$. Hence if $P$ is a minimal prime of $H_{J_{r}}^{q}(M)$, then the ideals $P + X$ and $P + I_{p}$ have the same radical. Hence if $M'$ is a finitely generated submodule of $H_{J_{r}}^{q}(M)$, then the ideals $\text{ann}M' + X$ and $\text{ann}M' + I_{p}$ have the same radical, where $\text{ann}M'$ is the annihilator of $M'$. This implies that the natural map $H_{I_{p}}^{p}(M') \to H_{X}^{p}(M')$ is an isomorphism for every finitely generated submodule $M'$ of $H_{J_{r}}^{q}(M)$, hence the natural map $E_{2}^{p,q} = H_{I_{p}}^{p}(H_{J_{r}}^{q}(M)) \to H_{X}^{p}(H_{J_{r}}^{q}(M)) = \tilde{E}_{2}^{p,q}$ is an isomorphism too. This completes the case $r = 2$.

Assume the claim proven for $r - 1$. Let a pair $(p, q)$ be such that $p + q = v + 1 - s$ and $p + sr \leq c$ for some $s \geq 0$. We need to show that the map $\phi_{r} : E_{r}^{p,q} \to \tilde{E}_{r}^{p,q}$ is an isomorphism. We have the following commutative diagram

$$
\begin{array}{ccc}
E_{r-1}^{p-(r-1),q+r-2} & \xrightarrow{d_{r-1}} & E_{r-1}^{p,q} & \xrightarrow{d_{r-1}} & E_{r-1}^{p+r-1,q-r+2} \\
\phi_{r-1} & & \phi_{r-1} & & \phi_{r-1} \\
\tilde{E}_{r-1}^{p-(r-1),q+r-2} & \xrightarrow{d_{r-1}} & \tilde{E}_{r-1}^{p,q} & \xrightarrow{d_{r-1}} & \tilde{E}_{r-1}^{p+r-1,q-r+2}
\end{array}
$$

in which the vertical maps are isomorphisms by the induction hypothesis as we are presently going to show. Indeed, for the map on the left, setting $p' = p - (r - 1)$ and $q' = q + r - 2$ we see that $p' + q' = p + q - 1 = v + 1 - (s + 1)$ and $p' + (s + 1)(r - 1) = p + s(r - 1) \leq p + sr \leq c$, so the induction hypothesis holds for the map on the left. Since $p + s(r - 1) \leq p + sr \leq c$, the induction hypothesis holds for the map in the middle. Hence the vertical maps on the left and in the middle are isomorphisms. For the vertical map on the right we set $p' = p + r - 1$ and $q' = q - r + 2$. If $s = 0$, then $p' + q' = p + q + 1 = v + 2$, so both modules on the right are zero and therefore the map on the right is an isomorphism. If $s \geq 1$, then $p' + q' = p + q + 1 = v + 1 - (s - 1)$ where $s - 1 \geq 0$ and $p' + (s - 1)(r - 1) = p + s(r - 1) \leq p + sr \leq c$, so the induction hypothesis holds for the map on the right. Hence the vertical map on the right also is an isomorphism. Thus all vertical maps are indeed isomorphisms, so the middle map induces an isomorphism $\phi_{r} : E_{r}^{p,q} \to \tilde{E}_{r}^{p,q}$ on the homology modules. This proves the claim.

It remains to show that $\tilde{E}_{\phi}^{p,q} = 0$ if $p + q > v$. Equivalently, it remains to show that $H_{X}^{i}(M) = 0$ for $i > v$. We have a spectral sequence

$$
E_{2}^{p',q'} = H_{J_{r}}^{p'}(H_{X}^{q'}(M)) \implies H_{X}^{p'+q'}(M).
$$

$H_{X}^{i}(M) = 0$ for $q' > c$ since $X$ is generated by $c$ elements. Hence $\tilde{E}_{2}^{p',q'} = 0$ if $q' > c$. 

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Assume \( q' \leq c \). If \( P \) is a prime of dimension \( d - q' \), then \( \dim M_P < q' \), so \( H^d_X(M) = 0 \). Hence \( \dim \text{Supp} H^q_X(M) \leq d - q' \). Since \( d > c(p + 1) \) and \( q' \leq c \), we conclude that \( d - q' > cp \), so by induction on \( p \), \( \tilde{E}_2^{q',q'} = H^q_{\tilde{X}}(H^q_{\tilde{X}}(M)) = 0 \) for \( p' > d - q' - 1 - \lfloor (d - q' - 2)/c \rfloor + p - 1 \), i.e. for \( p' + q' > d - 1 - \lfloor (d - 2)/c \rfloor + p - \lfloor (d - q' - 2)/c \rfloor + 1 - \lfloor (d - 2)/c \rfloor \) and hence for \( p' + q' > d - 1 - \lfloor (d - 2)/c \rfloor + p = v \) as \( q' \leq c \) implies \( \lfloor (d - q' - 2)/c \rfloor + 1 - \lfloor (d - 2)/c \rfloor \geq 0 \).

4. The main results

The following theorem and its corollaries are the main results of this paper.

**Theorem 4.1.** Let \( A \) be a local ring containing a field. Let \( c > 0 \) and \( d > 1 \) be integers, let \( I = I_1 \cap \cdots \cap I_n \) be the intersection of several prime ideals \( I_1, I_2, \ldots \) such that \( c(I_jA) \leq c \) for all \( j \), let \( t = \lfloor d - 2 \rfloor \), let \( v = d - 1 - \lfloor d - 2 \rfloor \), and let \( M \) be an \( A \)-module such that \( \dim \text{Supp} M \leq d \). Assume that \( \sqrt{fB} \) is a prime ideal of \( B \) for all \( j \), where \( B \) is the completion of the strict Henselization of the completion of \( A \). Then \( H^{v+1}_I(M) \) is isomorphic to the cokernel of the map

\[ \Phi_{M,I} : \bigoplus_{j_0 < \cdots < j_t} H^{d}_{I_{j_0} + \cdots + I_{j_t}}(M) \to \bigoplus_{j_0 < \cdots < j_t} H^{d}_{I_{j_0} + \cdots + I_{j_t}}(M) \]

that sends every \( x \in H^{d}_{I_{j_0} + \cdots + I_{j_t}}(M) \) to \( \bigoplus_{s=0}^{t} (-1)^s h_s(x) \) where

\[ h_s : H^{d}_{I_{j_0} + \cdots + I_{j_t}}(M) \to H^{d}_{I_{j_0} + \cdots + I_{j_s} + \cdots + I_{j_{t+1}}}(M) \]

(\( \widehat{I}_{j_s} \) means that \( I_{j_s} \) has been omitted) is the natural map induced by the containment

\[ I_{j_0} + \cdots + I_{j_t} \supset I_{j_0} + \cdots + \widehat{I}_{j_s} + \cdots + I_{j_{t+1}}. \]

We point out that the conditions that \( d > 1 \) and \( d \) is an integer clearly imply \( d \geq 2 \), i.e. \( t \geq 0 \), hence \( H^{d}_{I_{j_0} + \cdots + I_{j_t}}(M) \) makes sense.

**Proof.** \( c(I_jB) = c(I_jA) \leq c \) by Lemma 3.3. If \( p < t = \lfloor d - 2 \rfloor \), then \( d > c(p + 1) \) and Theorem 3.5 with \( p = p \) shows that \( H^d_{I_{j_0} + \cdots + I_{j_p}}(M) = 0 \) for all \( q \geq d - t + p \). Now Corollary 2.3 shows that \( H^{d-t}_I(M) = E^{-t,d}_2 \) and Remark 2.3 completes the proof.

This theorem implies some corollaries for arbitrary commutative Noetherian local rings containing a field. The following corollary deals with the vanishing of \( H^{v+1}_I(M) \) and generalizes Theorem 3.4 to the case that \( \sqrt{fB} \) is not necessarily prime, but is the intersection of a small number (less than \( \frac{d}{c} \), to be precise) of prime ideals. Corollary 1.2 is a specialization of Corollary 1.2 to the case that the ring \( A \) is complete, regular and strictly Henselian.

**Corollary 4.2.** Let \( A \) be a local ring containing a field, let \( c \) be a positive integer and let \( I \) be an ideal of \( A \) with \( c(IA) \leq c \). Let \( B \) be the completion of the strict Henselization of the completion of \( A \) and let \( d > c \) be an integer.
Assume that the ideal $IB$ has $n < \frac{d}{c}$ minimal primes. If $M$ is an $A$-module with $\dim\text{Supp} M \leq d$, then $H^q_I(M) = 0$ for $q > d - 1 - \lfloor \frac{d-2}{c} \rfloor$. In particular, if $\dim A \leq d$, then $cd(A, I) \leq d - 1 - \lfloor \frac{d-2}{c} \rfloor$.

Proof. Let $v = d - 1 - \lfloor \frac{d-2}{c} \rfloor$. Theorem 3.1 implies that $H^q_I(M) = 0$ for all $q > v + 1$. If $c | (d - 1)$, then $v + 1 = d - \lfloor \frac{d-1}{c} \rfloor + 1$, hence Theorem 3.1 implies that $H^{v+1}_I(M) = 0$ in this case. Hence we only need to show that $H^{v+1}_I(M) = 0$ in the case that $c \not| (d - 1)$.

Since $B$ is faithfully flat over $A$ and $B \otimes_A H^v_I(M) \cong H^v_{IB}(B \otimes_A M)$, it is enough to show that $H^v_{IB}(B \otimes_A M) = 0$. The module $H^v_{IB}(B \otimes_A M)$ is isomorphic to the cokernel of the map $\Phi_{B \otimes_A M, IB}$ of Theorem 4.1. The facts that $c \not| (d - 1)$ and $n < \frac{d}{c}$ imply the bound $n \leq \lfloor \frac{d-2}{c} \rfloor = t$. Hence $(t+1)$-tuples $\{j_0, \ldots, j_t\}$ such that $1 \leq j_0 < \cdots < j_t \leq n$ do not exist. Hence the target module of the map $\Phi_{B \otimes_A M, IB}$ vanishes. Therefore the cokernel also vanishes.

The following two corollaries, 4.3 and 4.4, deal with the structure of $H^v_I(M)$ when this module doesn’t necessarily vanish.

In general local cohomology modules of finitely generated modules are not necessarily Artinian, even if they are supported in dimension zero [8, Section 3]. But Corollary 4.3 says, in particular, that $H^v_I(M)$ is Artinian for a finitely generated $M$.

Corollary 4.3. Let $A$ be a local ring containing a field, let $c > 0$ and $d > 1$ be integers, let $v = d - 1 - \lfloor \frac{d-2}{c} \rfloor$, let $I$ be an ideal of $A$ with $c(I \hat{A}) \leq c$ and let $M$ be an $A$-module with $\dim\text{Supp} M \leq d$. Then

(a) $\dim\text{Supp} H^{v+1}_I(M) = 0$. If $M$ is finitely generated, then $H^{v+1}_I(M)$ is Artinian.

(b) Let $B = \hat{A}^{\text{sh}}$ be the completion of the strict Henselization of the completion of $A$, let $k$ be a coefficient field of $\hat{A}$ and let $K$ be the coefficient field of $B$ containing $k$. Then $H^{v+1}_{IB}(B \otimes_A M) = K \otimes_k H^{v+1}_I(M)$.

We note that every coefficient field $k$ of $\hat{A}$ can be uniquely extended to a coefficient field $K$ of $B$ and $K$ is the separable closure of $k$ (in fact, $B \cong K \otimes_k \hat{A}$ where $\otimes_k$ is the complete tensor product over $k$).

Proof of Corollary 4.3. $\dim\text{Supp}(B \otimes_A N) = \dim\text{Supp}(N)$ for every $A$-module $N$ since $B$ is faithfully flat over $A$. Thus $\dim\text{Supp}(B \otimes_A M) \leq d$. This implies that each of the modules $H^d_j(B \otimes_A M)$ appearing in the map $\Phi_{B \otimes_A M, IB}$ of Theorem 4.1 is supported in dimension zero. Indeed, if $P$ is a non-maximal prime ideal of $B$, then $\dim\text{Supp}(B \otimes_A M)_P < d$, hence $H^d_j(B \otimes_A M)_P = H^d_j_{IJ}(B \otimes_A M)_P$ vanishes at $P$.

$c(IB) = c(I \hat{A}) \leq c$ by Lemma 3.3 hence Theorem 4.1 implies that $H^{v+1}_{IB}(B \otimes_A M)$ is the cokernel of the map $\Phi_{B \otimes_A M, IB}$. Thus $H^{v+1}_{IB}(B \otimes_A M)$ is the cokernel of two modules supported in dimension zero and is therefore itself supported in dimension zero. But $B \otimes_A H^{v+1}_I(M) \cong H^{v+1}_{IB}(B \otimes_A M)$
since $B$ is faithfully flat over $A$. Hence $H^{v+1}_i(M)$ also is supported in dimension zero.

Let $N$ be a finitely generated submodule of $H^{v+1}_i(M)$. Then $N$ is annihilated by $m_A^s$ for some $s$, where $m_A$ is the maximal ideal of $A$. The fact that $B/m_A^sB \cong K \otimes_k A/m_A^s$ implies that $B \otimes_A N = K \otimes_k N$. Since $H^{v+1}_i(M)$ is the direct limit of its finitely generated submodules and the tensor product commutes with direct limits, $B \otimes_A H^{v+1}_i(M) \cong K \otimes_k H^{v+1}_i(M)$.

Now assume $M$ is finitely generated. An $A$-module $N$ supported in dimension zero is Artinian if and only if the $B$-module $K \otimes_k N$ is Artinian. Hence it is enough to prove that $H^{v+1}_i(B \otimes_A M) = K \otimes_k H^{v+1}_i(M)$ is Artinian. Since $H^{v+1}_i(B \otimes_A M)$ is the cokernel of a map of two modules each of which is a direct sum of modules of the form $H^{d}_{j}(B \otimes_A M)$, it is enough to prove that each such module is Artinian.

$B \otimes_A M$ is a finitely generated $B$-module. Hence there is a filtration $0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_s = B \otimes_A M$ such that $N_i/N_{i-1} \cong B/P_i$, where the $P_i$s are some prime ideals of $B$. The resulting exact sequences $H^{d}_{j}(N_{i-1}) \rightarrow H^{d}_{j}(N_i) \rightarrow H^{d}_{j}(N_i/N_{i-1})$ imply by induction on $i$ that it is enough to prove that $H^{d}_{j}(N_i/N_{i-1}) = H^{d}_{j}(B/P_i)$ is Artinian for every $i$. By the Hartshorne-Lichtenbaum local vanishing theorem [7, 3.1], $H^{d}_{j}(B/P_i) \neq 0$ only if $\sqrt{J + P_i} = m_B$ (where $m_B$ is the maximal ideal of $B$) in which case $H^{d}_{j}(B/P_i) = H^{d}_{m}(B/P_i)$. But $H^{d}_{m}(N)$ is Artinian for every $j$ and every finitely generated $B$-module $N$.

If $M$ is finitely generated, Theorem 4.3 dispels a great deal of mystery about the module $H^{v+1}_i(B \otimes_A M)$ by presenting it as the cokernel of an explicit map between two Artinian $B$-modules, and then the isomorphism $H^{v+1}_i(B \otimes_A M) = K \otimes_k H^{v+1}_i(M)$ of Corollary 4.3 enables us to understand the $A$-module $H^{v+1}_i(M)$ in terms of the $B$-module $H^{v+1}_i(B \otimes_A M)$. In particular, the length of the annihilator of $m_A^s$ in $H^{v+1}_i(M)$ equals the length of the annihilator of $m_B^s$ in $H^{v+1}_i(B \otimes_A M) = K \otimes_k H^{v+1}_i(M)$ for every $s$.

The following corollary shows that in the case that $M$ is supported on $V(P)$ where $P$ is an ideal of $A$ such that $\sqrt{PB}$ is a prime ideal of $B$, the module $H^{v+1}_i(M)$ can be completely described in terms of the module $H^{d}_{m}(M)$ and the singular homology of a suitable simplicial complex. Theorem 1.1 is a specialization of Corollary 4.3 to the case that the ring $A$ is complete, regular and strictly Henselian.

**Corollary 4.4.** Let $A$ be a local ring containing a field. Let $m$ and $k$ be the maximal ideal and the residue field of $A$. Let $c > 0$ and $d > 1$ be integers, let $I$ be an ideal of $A$ with $c(I\overline{A}) \leq c$ and let $B$ be the completion of the strict Henselization of the completion of $A$. Let $I_1, I_2, \ldots, I_n$ be the minimal primes of $IB$, let $t = [(d-2)/c]$ and let $v = d - 1 - [(d-2)/c]$. Let $P$ be an ideal of $A$ such that $\dim(A/P) = d$ and the ideal $PB$ of $B$ has just one minimal prime. Let $M$ be an $A$-module supported on $V(P)$. Let $\Delta$ be the simplicial complex on $n$ vertices $\{1,2,\ldots,n\}$ defined as follows: a
simplex \(\{j_0, \ldots, j_s\}\) belongs to \(\Delta\) iff \(I_{j_0} + \cdots + I_{j_s} + PB = \text{not } m_B\)-primary, where \(m_B\) is the maximal ideal of \(B\). Let \(w\) be the dimension over \(k\) of \(\hat{H}_{t-1}(\Delta; k)\), the \((t-1)\)-th reduced singular homology group of \(\Delta\) with coefficients in \(k\). Then \(H^{v+1}_t(M)\) is isomorphic to the direct sum of \(w\) copies of \(H^d_m(M)\). In particular, \(H^{v+1}_t(M) = 0\) if and only if either \(H^d_m(M) = 0\) or \(\hat{H}_{t-1}(\Delta; k) = 0\).

**Proof.** If \(N\) is an \(A\)-module supported on \(\{m\}\), then \(\hat{A} \otimes_A N = N\). This implies that \(H^{v+1}_{I\hat{A}}(\hat{A} \otimes_A M) = \hat{A} \otimes_A H^{v+1}_t(M) = H^{v+1}_t(M)\) because \(H^{v+1}_t(M)\) is supported on \(\{m\}\) by Corollary 2.5 and \(\hat{A}\) is flat over \(A\). Hence we can replace \(A\) by \(\hat{A}\) and \(I\) by \(IA\), i.e. we can assume that \(A\) is complete. For the rest of the proof we assume that \(A\) is complete.

Let \(k \subset A\) be a coefficient field of \(A\). Then \(B = K \otimes_k A\) where \(K\) is the separable closure of \(k\). According to [10, 4.2] there exists a finite Galois field extension \(\hat{k} \supset k\) (of course \(K \supset \hat{k}\)) such that upon setting \(\hat{A} = \hat{k} \otimes_k A\), the minimal primes \(\hat{I}_1, \ldots, \hat{I}_n\) of \(\hat{I} = IA\) are in a one-to-one correspondence with the minimal primes of \(IB\), namely, \(\sqrt{I_jB} = I_j\) for all \(j\) (the ring \(\hat{A}\) sits between \(A\) and \(B\)). Clearly, \(\hat{A}\) is a finite free (hence faithfully flat) \(A\)-module and a complete local ring with residue field \(\hat{k}\) which it contains. Since \(B\) is the strict Henselization of \(A\), it follows from Lemma 3.3 that \(c(\hat{I}) = c(IB = IB) = c(I)\), hence \(c(\hat{I}) \leq c\).

If \(N\) is an \(\hat{A}\)-module and \(\mathcal{I}\) is an ideal of \(A\), there is an isomorphism of \(A\)-modules \(A H^t_{I\hat{A}}(N) \cong H^t_t(AN)\), where \(A(-)\) means that the corresponding \(\hat{A}\)-module is viewed as an \(A\)-module via restriction of scalars. Accordingly, in the sequel we often denote \(H^t_{I\hat{A}}(N)\) by \(H^t_t(N)\) and view it as an \(\hat{A}\)-module.

For any \(A\)-module \(N\) we set \(\hat{N} = \hat{A} \otimes_A N = \hat{k} \otimes_k N\). We view \(\hat{N}\) as an \(\hat{A}\)-module in a natural way. Since \(\hat{A}\) is flat over \(A\), there is an isomorphism of \(\hat{A}\)-modules \(H^t_t(\hat{N}) \cong H^t_t(N)\) for any ideal \(\mathcal{I}\) of \(A\).

By Theorem 3.5 if \(p < t\), then \(H^q_{t\hat{I}_{i_0 + \cdots + i_p}}(\hat{M}) = 0\) for all \(q \geq d - t + p\) and all \(\{i_0, \ldots, i_p\}\). Hence by Corollary 2.3 \(H^{d-t+v+1}_{t\hat{I}}(\hat{M}) \cong E_{2-t,d}^{v+1}.\) Clearly, \(\hat{M}\) is supported on \(V(P)\) where \(P = \sqrt{PA}\) is a prime ideal of \(\hat{A}\) such that \(\dim(\hat{A}/P) = d\) and the ideal \(\hat{P}B = PB\) has just one minimal prime. Hence Corollary 2.3 implies \(E_{2-t,d}^{v} \cong \hat{H}_{t-1}(\Delta; H^d_{\hat{M}}(M)) \cong H^d_{\hat{m}}(M) \otimes_k \hat{H}_{t-1}(\Delta; \hat{k})\) where the last isomorphism holds by the universal coefficients theorem since we are over a field. But \(\dim_k \hat{H}_{t-1}(\Delta; \hat{k}) = \dim_k \hat{H}_{t-1}(\Delta; k) = w\) because the field \(\hat{k}\) has the same characteristic as the field \(k\). Hence the module \(H^{v+1}_t(M) \cong \hat{H}^{v+1}_t(M)\) is isomorphic to the direct sum of \(w\) copies of the module \(\hat{H}^d_{\hat{m}}(\hat{M}) \cong \hat{H}^d_{\hat{m}}(M)\), i.e. \(H^{v+1}_t(M) \cong H^d_{\hat{m}}(M)^w\) in the category of \(\hat{A}\)-modules. This completes the proof in the special case that \(A = \hat{A}\), i.e. every minimal prime of \(IB\) is the radical of the extension to \(B\) of a minimal
prime of \( I \). The rest of the proof consists in deducing the general case from this special case. This deduction turns out to be an unexpectedly long story.

Let \( G \) be the Galois group of \( \tilde{k} \) over \( k \). Let \( A[G] = \{ \sum_{\sigma \in G} a_\sigma \sigma | a_\sigma \in A \} \) be the group ring of \( G \) with coefficients in \( A \); every \( a \in A \) commutes with every \( \sigma \in G \). By an \( A[G] \)-module we always mean a left \( A[G] \)-module. If \( N \) is an \( A \)-module, \( \tilde{N} = \tilde{k} \otimes_k N \) acquires a standard structure of \( A[G] \)-module via \( \sigma(c \otimes x) = \sigma(c) \otimes x \) for all \( \sigma \in G, c \in \tilde{k} \) and \( x \in N \). In fact (\( - \)) is a functor from \( A \)-mod to \( A[G] \)-mod, namely, if \( \phi : N \rightarrow N' \) is a morphism of \( A \)-modules, then \( \tilde{\phi} = \text{id} \otimes \phi : \tilde{k} \otimes_k N \rightarrow \tilde{k} \otimes_k N' \) is a morphism of \( A[G] \)-modules.

Let \( J \subset A[G] \) be the ideal generated by the set \( \{ \sigma - \tau | \sigma, \tau \in G \} \). Then \( N \cong \tilde{N}/J\tilde{N} \) for every \( A \)-module \( N \). To complete the proof in the general case it is enough to show the existence of an \( A[G] \)-module isomorphism \( H^{v+1}_I(M) \cong H^d_m(M) \) where the \( A[G] \)-module structures on \( H^{v+1}_I(M) \) and \( H^d_m(M) \) are the standard ones (it has been shown above that the two modules are isomorphic as \( \tilde{A} \)-modules). Indeed, \( H^{v+1}_I(M)/JH^{v+1}_I(M) \cong H^{v+1}_I(M) \) and \( H^d_m(M)/JH^d_m(M) \cong H^d_m(M) \), hence the existence of the above-mentioned \( A[G] \)-module isomorphism would imply that \( H^{v+1}_I(M) \cong H^d_m(M) \) as needed. The rest of the proof is devoted to showing the existence of such an \( A[G] \)-module isomorphism.

If \( N \) is an \( A[G] \)-module, then \( \tilde{H}_I^1(N) \) inherits a structure of \( A[G] \)-module. Namely, the action \( \sigma : N \rightarrow N \), being an \( A \)-module homomorphism, induces the action \( \sigma : \tilde{H}_I^1(N) \rightarrow \tilde{H}_I^1(N) \) for every \( \sigma \in G \). Thus we get a functor \( \tilde{H}_I^1(-) : A[G]-\text{mod} \rightarrow A[G]-\text{mod} \). It is easy to check that for every \( A \)-module \( N \) there is an isomorphism of \( A[G] \)-modules \( \tilde{H}_I^1(\tilde{N}) \cong \tilde{H}_I^1(N) \) where the \( A[G] \)-module structures on \( \tilde{N} \) and \( \tilde{H}_I^1(N) \) are the standard ones and the \( A[G] \)-module structure on \( \tilde{H}_I^1(\tilde{N}) \) is functorially induced from \( \tilde{N} \).

If \( N \) is an \( A \)-module and \( E_A(N) \) is the injective hull of \( N \) in the category of \( A \)-modules, then \( \tilde{E}_A(\tilde{N}) \cong \tilde{E}_A(\tilde{N}) \) where \( \tilde{E}_A(\tilde{N}) \) is the injective hull of \( \tilde{N} \) in the category of \( \tilde{A} \)-modules, i.e. \( \tilde{E}_A(\tilde{N}) \) is an injective \( \tilde{A} \)-module. Since every injective \( A \)-module is a direct sum of modules of the form \( E_A(N) \), we conclude that if \( E \) is an injective \( A \)-module, then \( \tilde{E} \) is an injective \( \tilde{A} \)-module. Hence if the complex \( E^\bullet \) is an injective resolution of \( M \) in the category of \( A \)-modules, then \( \tilde{E}^\bullet \) is both an injective resolution of \( \tilde{M} \) in the category of \( \tilde{A} \)-modules and a complex in the category of \( A[G] \)-modules.

The group \( G \) acts on \( \tilde{A} \) in a standard way (i.e. \( \sigma(c \otimes a) = \sigma(c) \otimes a \) for all \( \sigma \in G, c \in \tilde{k} \) and \( a \in A \)) and the elements of \( G \) permute the ideals \( I_0, \ldots, I_n \). For any \( \sigma \in G \) we write \( \sigma(i) = j \) if \( \sigma(I_i) = I_j \). We recall from the proof of Theorem 2.1 that \( \Gamma^{-\rho}(\tilde{M}) = \oplus_{i_0 < i_1 < \ldots < i_p} \Gamma_{I_{i_0} + \ldots + I_{i_p}}(\tilde{M}) \). This is an \( A[G] \)-module, for if \( x \in \Gamma_{I_{i_0} + \ldots + I_{i_p}}(\tilde{M}) \), then \( \sigma(x) \in \Gamma_{\sigma(i_0) + \ldots + \sigma(i_p)}(\tilde{M}) \)
(indeed, if \( b \in \tilde{A}, x \in \tilde{M} \) and \( \sigma \in G \), then \( \sigma(bx) = \sigma(b)\sigma(x) \), hence an ideal \( \mathcal{T} \) of \( \tilde{A} \) annihilates \( x \) if and only if the ideal \( \sigma(\mathcal{T}) \) annihilates \( \sigma(x) \). It is not hard to see that the differentials in the complex \( \Gamma^*(\tilde{M}) \) are \( A[G] \)-module homomorphisms. Hence the double complex \( \Gamma^*(\Gamma_I(E^*)) \) that induces the Mayer-Vietoris spectral sequence

\[
E_1^{i,p,q} = \oplus_{i_0 < \cdots < i_p} H^q_{I_{i_0} + \cdots + I_{i_p}}(\tilde{M}) \Rightarrow H^{q-p}(\tilde{M}) \cong H^{q-p}_I(M)
\]

(see the proof of Theorem 2.31) is a double complex in the category of \( A[G] \)-modules. Hence this is a spectral sequence in the category of \( A[G] \)-modules.

The map \( \Phi_{\tilde{M},I} \) of Theorem 4.1 is a map on the \( E_1 \) page of the spectral sequence, hence it is an \( A[G] \)-module map. The same argument as in the proof of Theorem 4.1 shows that the abutment of the spectral sequence in degree \( v + 1 \) is isomorphic (in the category of \( A[G] \)-modules) to the cokernel of \( \Phi_{\tilde{M},I} \). But the abutment is \( H^{v+1}_I(M) \) with the standard \( A[G] \)-module structure. Hence it is enough to show that the cokernel of \( \Phi_{\tilde{M},I} \) is isomorphic as an \( A[G] \)-module to \( H^{v+1}_I(M)_w \).

For an integer \( s \) let \( \Lambda_s \) be the set of the subsets \( \{j_0, \ldots, j_s\} \subset \{1, \ldots, n\} \) of cardinality \( s + 1 \) such that \( \sqrt{I_{j_0} + \cdots + I_{j_s} + \tilde{P}} = \tilde{m} \). The Hartshorne-Lichtenbaum local vanishing theorem [7, 3.1] implies that \( H^d_{I_{j_0} + \cdots + I_{j_s}}(\tilde{M}) \) vanishes if \( \{j_0, \ldots, j_s\} \not\subset \Lambda_s \), otherwise \( H^d_{I_{j_0} + \cdots + I_{j_s}}(\tilde{M}) \cong H^d_m(\tilde{M}) \). For \( \lambda = \{j_0, \ldots, j_s\} \in \Lambda_s \) we denote \( H^d_{I_{j_0} + \cdots + I_{j_s}}(\tilde{M}) \) by \( H^d_m(\tilde{M})_\lambda \). The module \( H^d_m(\tilde{M})_\lambda \) is just a copy of \( H^d_m(\tilde{M}) \) indexed by \( \lambda \). The map \( \Phi_{\tilde{M},I} \) of Theorem 4.1 takes the form

\[
\Phi : \oplus_{\lambda \in \Lambda_{s+1}} H^d_m(\tilde{M})_\lambda \to \oplus_{\lambda \in \Lambda_s} H^d_m(\tilde{M})_\lambda.
\]

It follows from the description of the map \( \Phi_{M,I} \) in the statement of Theorem 4.1 that if \( \lambda' \in \Lambda_{s+1} \) and \( \lambda \in \Lambda_s \), then the map \( H^d_m(M)_{\lambda'} \to H^d_m(M)_\lambda \) induced by \( \Phi \) is 0 if \( \lambda \not\subset \lambda' \) and is either id or \(-\)id if \( \lambda \subset \lambda' \) where id denotes the identity map on \( H^d_m(M) \) (we view \( H^d_m(M)_{\lambda'} \) and \( H^d_m(M)_\lambda \) as two different copies of the same module \( H^d_m(M) \)).

Let \( y \in \tilde{k} \) be an element such that the set \( \{\sigma(y) | \sigma \in G \} \) is a \( k \)-basis of \( \tilde{k} \). There is an isomorphism of \( A \)-modules \( \tilde{M} \cong \oplus_{\sigma \in G} (\sigma(y) \otimes_k M) \) and the fact that \( G \) acts on \( \tilde{M} \) in a standard way means that \( \sigma'(\sigma(y) \otimes_k x) = \sigma'(\sigma(y)) \otimes_k x \) for all \( x \in M \) and all \( \sigma, \sigma' \in G \). Hence \( H^d_m(M)_\lambda \cong \oplus_{\sigma \in G} (\sigma(y) \otimes_k H^d_m(M)_\lambda) \) (the only function of the subscript \( \lambda \) in \( \sigma(y) \otimes_k H^d_m(M)_\lambda \) is to indicate that this particular copy of \( \sigma(y) \otimes_k H^d_m(M) \) came from \( H^d_m(M)_\lambda \)). Hence the map \( \Phi \) takes the following form:

\[
\Phi : \oplus_{\sigma \in G, \lambda \in \Lambda_{s+1}} (\sigma(y) \otimes_k H^d_m(M)_\lambda) \to \oplus_{\sigma \in G, \lambda \in \Lambda_s} (\sigma(y) \otimes_k H^d_m(M)_\lambda).
\]
The map \( \sigma'(y) \otimes_k H^d_m(M)_{\lambda'} \rightarrow \sigma(y) \otimes_k H^d_m(M)_{\lambda} \) induced by \( \Phi \) is 0 if \( \sigma' \neq \sigma \) or \( \lambda \not\in \lambda' \) and is either id or \(-id\) if \( \sigma' = \sigma \) and \( \lambda \subset \lambda' \) where id denotes the identity map on \( \sigma(y) \otimes_k H^d_m(M) \).

The above description of the \( G \)-action on \( \Gamma^{-p}(\tilde{M}) \) implies that \( G \) acts on \( \bigoplus_{\lambda' \in \Lambda_0} \bigoplus_{s} \sigma(y) \otimes_k H^d_m(M)_{\lambda}, \) where \( s = t, t + 1, \) as follows. An element \( \sigma' \in G \) sends the \( A \)-module \( \sigma(y) \otimes_k H^d_m(M)_{\lambda} \) isomorphically onto the \( A \)-module \( \sigma' \sigma(y) \otimes_k H^d_m(M)_{\sigma'(\lambda)} \) via the map \( \sigma(y) \otimes x \mapsto \sigma' \sigma(y) \otimes_k x \) for every \( x \in H^d_m(M) \) (if \( \lambda = \{ j_0, \ldots, j_s \}, \) then \( \sigma'(\lambda) = \{ \sigma'(j_0), \ldots, \sigma'(j_s) \} \)).

Let \( O_t \) (resp. \( O_{t+1} \)) be the set of the orbits of the action of \( G \) on the set of the \( A \)-modules \( \{ \sigma(y) \otimes_k H^d_m(M)_{\lambda} | \sigma \in G, \lambda \in \Lambda_t \) (resp. \( \lambda \in \Lambda_{t+1} \)) \}. Since \( \sigma' \sigma(y) = \sigma(y) \) if and only if \( \sigma' \) is the identity element of \( G \), the stabilizer of each \( \sigma(y) \otimes_k H^d_m(M)_{\lambda} \) is trivial. Hence each orbit \( \sigma \) consists of \( |G| \) elements which are \( \{ \sigma(y) \otimes_k H^d_m(M)_{\lambda} | \sigma \in G \} \), where \( \lambda \) is fixed. We denote by \( H^d_m(M)_{\sigma} \) the direct sum of the \( |G| \) elements of \( \sigma \), i.e. \( H^d_m(M)_{\sigma} = \bigoplus_{\sigma \in G} (\sigma(y) \otimes_k H^d_m(M)_{\sigma(\lambda)}) \). Ignoring the subscripts \( \sigma(\lambda) \) and \( \sigma \) (which are there just to keep track of different copies of the same modules) this direct sum is isomorphic to the \( A[G] \)-module \( H^d_m(M) \) with the standard \( A[G] \)-module structure.

The above descriptions of the map \( \Phi \) and the \( G \)-action on its source and target, and the fact that \( \Phi \) commutes with this \( G \)-action, imply that the map \( \Phi : \bigoplus_{\sigma \in O_t+1} H^d_m(M)_{\sigma} \rightarrow \bigoplus_{\sigma \in O_t} H^d_m(M)_{\sigma} \) induced by \( \Phi \) is 0 if \( \sigma' \neq \sigma \), and if \( \sigma' = \sigma \), then this map \( \sigma(y) \otimes_k H^d_m(M)_{\sigma(\lambda)} \rightarrow \sigma(y) \otimes_k H^d_m(M)_{\sigma(\lambda)} \) depends only on \( \lambda' \) and \( \lambda \) but not on \( \sigma \), and is either \( 0 \), or id, or \(-id\) where id denotes the identity map on \( \sigma(y) \otimes_k H^d_m(M) \). Hence every map \( H^d_m(M)_{\sigma'} \rightarrow H^d_m(M)_{\sigma} \) induced by \( \Phi \) is either \( 0 \), or id, or \(-id\) where id denotes the identity map on \( H^d_m(M) \).

Hence the map \( \Phi \) takes the form
\[
\Phi : \bigoplus_{\sigma \in O_t+1} H^d_m(M)_{\sigma} \rightarrow \bigoplus_{\sigma \in O_t} H^d_m(M)_{\sigma}
\]
where each \( H^d_m(M)_{\sigma} \) is isomorphic to the same \( A[G] \)-module \( H^d_m(M) \) and the matrix defining \( \Phi \) is an \( |O_{t+1}| \times |O_t| \)-matrix \( M \) each of whose entries is either \( 0 \), \( 1 \), or \(-1\). Since the entries of \( M \) are all in \( \tilde{k} \) and \( \tilde{k} \) is a field, the cokernel of \( \Phi \) is isomorphic as an \( A[G] \)-module to the direct sum of \( w' \) copies of \( H^d_m(M) \) where \( w' \) is the dimension over \( \tilde{k} \) of the cokernel of the map \( \tilde{k}^{|O_{t+1}|} \rightarrow \tilde{k}^{|O_t|} \) defined by the same matrix \( M \).

It has been shown earlier that the cokernel of \( \Phi \) is isomorphic as an \( A \)-module to a direct sum of \( w \) copies of \( H^d_m(M) \). Hence there is an isomorphism of \( A \)-modules \( H^d_m(M)^w \cong H^d_m(M)^{w'} \). Let \( \ell \) be the dimension over \( k \) of the socle of \( H^d_m(M) \). Since \( H^d_m(M) \) is supported at \( m, \ell = 0 \) if and only if \( H^d_m(M) \). Hence \( \ell = 0 \) implies \( H^d_m(M) = 0 \) because \( \tilde{A} \) is faithfully flat over
A. Hence if \( \ell = 0 \), there is nothing to prove. Now we assume \( \ell \neq 0 \). The dimensions of the socle of the \( A \)-module \( H^d_d(M)^w \cong H^d_m(M)^w \) is, on the one hand, \( \ell w \) and on the other, \( \ell w' \). The equality \( \ell w = \ell w' \) implies \( w = w' \) since \( \ell \neq 0 \). Hence the cokernel of \( \Phi \) is isomorphic as an \( A[V] \)-module to \( H^d_m(M)^w \).

The result of the preceding corollary is based on the fact that if \( M \) is supported on \( V(P) \) where \( P \) is an ideal of \( A \) such that \( \sqrt{PB} \) has just one minimal prime, then the set of the isomorphism classes of the modules \( H^d_{I_0 + \ldots + I_s}(\tilde{M}) \) appearing in the map \( \Phi_{M, i} \) has at most two elements, 0 and \( H^d_m(M) \). But if \( M \) is supported on \( V(P) \) where \( P \subset A \) is such that the ideal \( PB \) has more than one minimal primes, then the set of the isomorphism classes of the modules \( H^d_{I_0 + \ldots + I_s}(\tilde{M}) \) may be quite big. This precludes expressing the module \( H^{d+1}_1(M) \) in terms of a single module \( H^d_m(M) \) and some singular homology. But one can still give a necessary and sufficient condition for \( \text{cd}(A, I) \leq v \) in terms of singular homology alone, as our final corollary shows.

**Corollary 4.5.** Let \( A \) be a \( d \)-dimensional local ring containing a field. Let \( m \) and \( k \) be the maximal ideal and the residue field of \( A \). Let \( c > 0 \) and \( d > 1 \) be integers, let \( t = [(d - 2)/c] \) and let \( v = d - 1 - [(d - 2)/c] \). Let \( I \) be an ideal of \( A \) such that \( c(IA) \leq c \). Let \( B \) be the completion of the strict Henselization of the completion of \( A \). Let \( I_0, I_1, \ldots, I_n \) be the minimal primes of \( IB \) and let \( P_1, P_2, \ldots \) be the minimal \( d \)-dimensional primes of \( B \) (a prime \( P \) of \( B \) is \( d \)-dimensional if \( B/P = d \)). Let \( \Delta_i \) be the simplicial complex on \( n \) vertices \( \{1, 2, \ldots, n\} \) such that a simplex \( \{j_0, \ldots, j_s\} \) belongs to \( \Delta_i \) if and only if \( I_{j_0} + \cdots + I_{j_s} + P_i \) is not \( mB \)-primary. Let \( \tilde{H}_{t-1}(\Delta_i; k) \) be the \((t - 1)\)-th reduced singular homology group of \( \Delta_i \) with coefficients in \( k \). Then \( \text{cd}(A, I) \leq v \) if and only if \( \tilde{H}_{t-1}(\Delta_i; k) = 0 \) for every \( i \).

**Proof.** If \( M \) is a \( B \)-module, then \( H^{t+1}_t(AM) \cong A H^{t+1}_{IB}(M) \) where the subscript \( A \) means that the corresponding \( B \)-module is viewed as an \( A \)-module via restriction of scalars. If \( M \) is an \( A \)-module, \( H^{t+1}_{IB}(B \otimes_A M) \cong B \otimes_A H^{t+1}_t(M) \) vanishes if and only if \( H^{t+1}_t(M) \) vanishes since \( B \) is faithfully flat over \( A \). This implies that \( \text{cd}(A, I) = \text{cd}(B, IB) \).

If \( \tilde{H}_{t-1}(\Delta_i; k) \neq 0 \) for some \( i \), then \( H^{t+1}_t(B/P_i) \neq 0 \) by Corollary 4.4 since \( H^{t}_{MB}(B/P_i) \neq 0 \). Conversely, assume \( \tilde{H}_{t-1}(\Delta_i; k) = 0 \) for every \( i \). We need to show that \( H^{t+1}_{t}(M) = 0 \) for every \( B \)-module \( M \). Since every module is the direct limit of its finitely generated submodules and \( H^{t+1}_{t}(\cdot) \) commutes with direct limits, we can assume that \( M \) is finitely generated in which case there is a finite filtration \( 0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M \) such that \( M_i/M_{i-1} \cong B/Q_i \) where \( Q_i \) is some prime ideal of \( B \). Applying the functor \( H^{t+1}_{t}(\cdot) \) to the exact sequences \( 0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0 \) and
using induction on $i$, it is enough to prove that $H^{v+1}_I(B/Q) = 0$ for every prime ideal $Q$ of $B$.

If $\dim B/Q < d$, then $H^{v+1}_I(B/Q) = 0$ by Theorem 3.1. If $\dim B/Q = d$, then $Q = P_i$ for some $i$ in which case $H^{v+1}_I(B/Q) = 0$ by Corollary 4.4 because $\tilde{H}_{t-1}(\Delta_i; k) = 0$.

□

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