Weyl Equation and (Non)-Commutative $SU(n + 1)$ BPS Monopoles

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Abstract

We apply the ADHMN construction to obtain the $SU(n + 1)$ (for generic values of $n$) spherically symmetric BPS monopoles with minimal symmetry breaking. In particular, the problem simplifies by solving the Weyl equation, leading to a set of coupled equations, whose solutions are expressed in terms of the Whittaker functions. Next, this construction is generalized for non-commutative $SU(n + 1)$ BPS monopoles, where the corresponding solutions are given in terms of the Heun B functions.
1 Introduction

Bogomolny-Prasad-Sommerfield (BPS) monopoles are topological solitons in a Yang-Mills-Higgs gauge theory in three space dimensions. The equation for static monopoles is integrable, and thus a variety of techniques are available for constructing and studying the monopole solutions. Direct construction of solutions of the Bogomolny equation with monopole number greater than one is a difficult task with the exception of spherical symmetry, (see, for example, Ref. [1]-[4] and References therein). To bypass this difficulty a number of intriguing ideas have been put forward rendering this complicated problem more tractable.

A powerful approach introduced by Nahm [5] is the so-called Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction. The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [6], allows the construction of instantons in terms of linear algebras in a vector space, whose dimension is related to the instanton number. Since monopoles correspond to infinite action instantons, then an adaptation of the ADHM construction involving an infinite dimensional vector space might be also possible. Nahm was able to formulate such an adaption in the ADHMN construction. To perform this construction a nonlinear ordinary differential equation (i.e. the Nahm equation), must be solved and its solutions (i.e. the Nahm data), used to define the Weyl equation.
The Nahm equations provide a system of non-linear ordinary differential equations

$$\frac{dT_i(s)}{ds} = \frac{1}{2} \varepsilon_{ijk} [T_j(s), T_k(s)]$$  \hspace{1cm} (1)$$

for three $n \times n$ anti-hermitian matrices $T_i(s)$ (the so-called Nahm data) of functions of the variable $s$, where $n$ is the magnetic charge of the BPS monopole configuration. The tensor $\varepsilon_{ijk}$ is the totally antisymmetric tensor. These equations may be obtained from the self-dual Yang-Mills equations in four dimensions by imposing translation invariance in three dimensions. Thus, equations (1) are completely integrable, and can be solved in terms of abelian functions.

In the ADHMN approach, the construction of $SU(n+1)$ monopole solutions to the Bogomolny equation with topological charge $n$ is translated to the following problem which is known as the inverse Nahm transform [5].

Finding the Nahm data effectively solves the nonlinear part of the monopole construction, however in order to calculate the fields themselves the linear part of the ADHMN construction must be implemented. Given the Nahm data for a $n$-monopole the one-dimensional Weyl equation

$$\left( I_{2n} \frac{d}{ds} - I_n \otimes x_j \sigma_j + iT_j \otimes \sigma_j \right) \mathbf{v}(\mathbf{x}, s) = 0$$  \hspace{1cm} (2)$$

for the complex $2n$-vector $\mathbf{v}(\mathbf{x}, s)$, must be solved. $I_n$ denotes the $n \times n$ identity matrix and $\mathbf{x} = (x_1, x_2, x_3)$ is the position in space at which the monopole fields are to be calculated. Let us choose an orthonormal basis for these solutions, satisfying

$$\int \mathbf{v}^\dagger \mathbf{v} ds = I.$$  \hspace{1cm} (3)$$

Given $\mathbf{v}(\mathbf{x}, s)$, the normalized vector computed from (2) and (3), the Higgs field $\Phi$ and gauge potential $A_i$ are given by

$$\Phi = -i \int s \mathbf{v}^\dagger \mathbf{v} ds,$$  \hspace{1cm} (4)$$

$$A_i = \int \mathbf{v}^\dagger \partial_i \mathbf{v} ds,$$  \hspace{1cm} (5)$$

where the integrations are to be performed over the range spanned by the minimum and maximum eigenvalues of the asymptotic form of $\Phi$. Then the corresponding gauge and Higgs
fields satisfy the Bogomolny equation and the boundary conditions for a charge $n$ monopole, and they are smooth functions of $x$. Although many general results have been obtained few explicit solutions are known. Specifically, many solutions of the Bogomolny equation have been discovered using the inverse Nahm transform, however this does not mean that analytic expressions for the gauge and Higgs fields are known.

One of the main aims of the present investigation is the construction of charge $n$, $SU(n+1)$ monopoles with symmetry breaking from $U(n)$. This means that the asymptotic value of the Higgs field has $n$ equal eigenvalues; which is also referred to as minimal symmetry breaking. Moreover, we construct the non-commutative version of our solutions. As was recently shown, quantum field theories in non-commutative space-time naturally arise as a decoupling limit of the world volume dynamics of D-branes in a constant Neveu-Schwarz-Neveu-Schwarz (NS-NS) two-form background [7]. In particular, we explicitly construct the solutions of the non-commutative BPS equations to the linear order of the non-commutativity scale using a deformed (non-commutative) version of the Nahm equation.

## 2 Weyl Equation and $SU(n + 1)$ BPS Monopoles

In this section, the solutions of the Weyl equation for spherically symmetric $SU(n + 1)$ BPS monopoles (for generic values of $n$) in the minimal symmetry breaking case are obtained explicitly. By spherical symmetry we simply mean here, that one has to deal only with the $r$ dependence, i.e. no azimuthal dependence is considered. To our knowledge the only work within this spirit is presented in [8, 9], but again only for particular values of $n$ (i.e. $n = 2$ and $n = 3$, respectively).

In the minimal symmetry breaking case the Nahm data $T_i$ are defined on a single interval $[-n, 1]$ with the only pole occurring at the left-hand end of the interval. This allows a construction of Nahm data for charge $n$ monopoles in a minimally broken $SU(n + 1)$ theory in terms of rescaled Nahm data for $SU(2)$ monopoles, where the rescaling moves the second pole in Nahm data outside the interval. For convenience we shift $s$ so that the Nahm data are defined in the interval, $[0, n + 1]$. Finally, the boundary conditions require that this representation in the unique irreducible $n$-dimensional representation of $su(2)$ (see e.g. [10]...
for a more detailed discussion). Thus, the Nahm data can be cast as

$$T_i = -\frac{i}{2} f_i \tau_i, \quad i = 1, 2, 3$$

(6)

where \(\tau_i\)'s form the \(n\)-dimensional representation of \(SU(2)\) and satisfy

$$[\tau_i, \tau_j] = 2i \varepsilon_{ijk} \tau_k.$$  

(7)

The \(n\)-dimensional representation of \(SU(2)\) is of the form

$$\tau_1 = \sum_{k=1}^{n-1} C_k \left( e_{kk+1}^{(n)} + e_{k+1k}^{(n)} \right), \quad \tau_2 = i \sum_{k=1}^{n-1} C_k \left( e_{k+1k}^{(n)} - e_{kk+1}^{(n)} \right), \quad \tau_3 = \sum_{k=1}^{n} a_k e_{kk}^{(n)}$$

(8)

where \(e_{ij}^{(n)}\) are \(n \times n\) matrices defined by: \(\left( e_{ij}^{(n)} \right)_{kl} = \delta_{ik} \delta_{jl}\) and

$$a_k = n + 1 - 2k, \quad C_k = \sqrt{k(n-k)}.$$  

(9)

The Nahm data for \(SU(n)\) spherically symmetric monopole of charge \(n - 1\) are given by (6) where \(f_i = f = -\frac{1}{s}\) (see also e.g. \([8, 10]\)). Since the monopole with this Nahm data is spherically symmetric, we may choose \((x_1, x_2, x_3) = (0, 0, r)\), and assume that the vector \(v(x, s)\) is of the form

$$v(x, s) = \sum_{l=1}^{n} h_l(r, s) \hat{e}_l^{(n)} \otimes \left( b_1(r, s) \hat{e}_1^{(2)} + b_2(r, s) \hat{e}_2^{(2)} \right)$$

(10)

where \(\hat{e}_k^{(n)}\) is the \(n\)-dimensional column vector with 1 at the position \(k \in \mathbb{Z}^+\) and 0 elsewhere, i.e. the standard basis of \(\mathbb{R}^n\).

Then the Weyl equation \([2]\) becomes

$$\left[ \frac{d}{ds} + \frac{f}{2} \sum_{k=1}^{n-1} C_k \left( e_{kk+1}^{(n)} + e_{k+1k}^{(n)} \right) \otimes \left( e_{12}^{(2)} + e_{21}^{(2)} \right) - \frac{f}{2} \sum_{k=1}^{n-1} C_k \left( e_{k+1k}^{(n)} - e_{kk+1}^{(n)} \right) \otimes \left( e_{21}^{(2)} - e_{12}^{(2)} \right) \right.$$

$$+ \frac{f}{2} \sum_{k=1}^{n} a_k e_{kk}^{(n)} \otimes \left( e_{11}^{(2)} - e_{22}^{(2)} \right) - r \otimes \left( e_{11}^{(2)} - e_{22}^{(2)} \right) \left] \sum_{l=1}^{n} h_l \hat{e}_l^{(n)} \otimes \left( b_1 \hat{e}_1^{(2)} + b_2 \hat{e}_2^{(2)} \right) \right] = 0.$$  

(11)

To proceed with our computation we exploit the following properties:

$$e_{ij}^{(n)} e_{kl}^{(n)} = \delta_{kj} e_{il}^{(n)}, \quad e_{ij}^{(n)} \hat{e}_k^{(n)} = \delta_{jk} \hat{e}_i^{(n)}.$$  

(12)
With the use of the latter identities and after setting

\[ u_1(r, s) = h_i(r, s) b_1(r, s), \quad w_1(r, s) = h_i(r, s) b_2(r, s), \]  

equation (13) is equivalent to the following first-order system of differential equations

\[
\begin{align*}
\dot{u}_1 - \left( \frac{1}{2s} a_1 + r \right) u_1 &= 0, \\
\dot{u}_{k+1} - \frac{1}{s} C_k w_k - \left( \frac{1}{2s} a_{k+1} + r \right) u_{k+1} &= 0, \\
\dot{w}_k - \frac{1}{s} C_k u_{k+1} + \left( \frac{1}{2s} a_k + r \right) w_k &= 0, \\
\dot{w}_n + \left( \frac{1}{2s} a_n + r \right) w_n &= 0,
\end{align*}
\]

(14)

where \( k = 1, 2, \ldots, n - 1 \). Here, \( \dot{u}_i \) and \( \dot{w}_i \) for \( i = 1, \ldots, n \) are the total derivatives of the functions \( u_i(r, s) \) and \( w_i(r, s) \) with respect to the argument \( s \). Solving these equations is the first step in reconstructing a solution of the Bogomolny equations from Nahm data. Then the problem of recovering the Higgs and gauge fields is linear.

Let us now solve these equations. Assume first that \( r \neq 0 \). The first and last equations may be immediately integrated and their solutions are equal to

\[
\begin{align*}
u_1 &= \kappa_1(r) \sqrt{s a_1} e^{r_s}, \quad w_n = \kappa_2(r) \frac{e^{-r_s}}{\sqrt{s a_n}},
\end{align*}
\]

(15)

where \( \kappa_i(r) \) for \( i = 1, 2 \) are the constants of integration. The coupled equations for \( u_{k+1} \) and \( w_k \) are equivalent (by expressing \( u_{k+1} \) in terms of \( w_k \)) to the single second-order equation

\[
s^2 \ddot{w}_k + 2s \dot{w}_k - \left( r^2 s^2 + (n - 1 - 2k)rs + \frac{n^2 - 1}{4} \right) w_k = 0,
\]

(16)

which may be solved by substituting \( w_k = W_k/s \) and \( z = 2rs \). The latter equation is then reduced to the familiar Whittaker equation:

\[
W_k'' + \left( -\frac{1}{4} + \frac{(2k-n+1)}{z} + \frac{1}{4} - \frac{n^2}{4z^2} \right) W_k = 0,
\]

(17)

where \( W_k'' = \frac{d^2W_k}{dz^2} \).

Therefore, the solutions of (17) are given in a closed form, in terms of the Whittaker functions as

\[
W_k = c_1(r) M \left( -\frac{n}{2} + \frac{1}{2} + k, \frac{n}{2}; 2rs \right) + c_2(r) W \left( -\frac{n}{2} + \frac{1}{2} + k, \frac{n}{2}; 2rs \right)
\]

(18)
where \( c_i(r) \) for \( i = 1, 2 \) are constants (see Appendix, for a brief review on Whittaker functions). In Table 1 and 2 specific examples of the Whittaker \( M(\frac{n}{2} + \frac{1}{2} + k, \frac{n}{2}; 2rs) \) and \( W(\frac{n}{2} + \frac{1}{2} + k, \frac{n}{2}; 2rs) \) functions are presented.

| \( M(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs) \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|
| \( k = 1 \) | \( 12(20) = 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \) | \( 12\sqrt{2} \left[ 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \right] \) | \( 12\sqrt{2} \left[ 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \right] \) |
| \( k = 2 \) | \( 6(60) = 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \) | \( 60\sqrt{2} \left[ 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} + (rs)^{-1/2} \) | \( 60\sqrt{2} \left[ 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} + (rs)^{-1/2} \) |
| \( k = 3 \) | \( 12(20) = 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \) | \( 12\sqrt{2} \left[ 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \right] \) | \( 12\sqrt{2} \left[ 3 \sinh(rs) - rs e^{-rs} + (rs)^{-1/2} \right] \) |

| \( M(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs) \) | \( n = 5 \) | \( n = 6 \) |
|---|---|---|
| \( k = 1 \) | \( 20 \left[ 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} - 12 \sinh(rs) \) | \( 60\sqrt{2} \left[ 15 \sinh(rs) - 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} + (rs)^{-1/2} \) |
| \( k = 2 \) | \( 60\left[ 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} - 12 \sinh(rs) \) | \( 60\sqrt{2} \left[ 15 \sinh(rs) - 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} + (rs)^{-1/2} \) |
| \( k = 4 \) | \( 20 \left[ 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 + r^3 s^3 \right] e^{-rs} \) | \( 60\sqrt{2} \left[ 15 \sinh(rs) - 3 \sinh(rs) + 3 \sinh(rs) + 2r^2 s^2 \right] e^{-rs} + (rs)^{-1/2} \) |

Table 1: Explicit expressions for the Whittaker function \( M(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs) \) for

\( n = 2, \ldots, 6 \)

| \( W(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs) \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|
| \( k = 1 \) | \( (1 + 2r) e^{-rs} \) | \( (1 + rs) e^{-rs} \) | \( (3 + 2r) e^{-rs} \) |
| \( k = 2 \) | \( \left[ 1 + 2rs(1 + rs) \right] e^{-rs} \) | \( \left[ 3 + 2rs(1 + rs) \right] e^{-rs} \) | \( \left[ 3 + 2rs(1 + rs) \right] e^{-rs} \) |
| \( k = 3 \) | \( \left[ 3 + 2rs(1 + rs) \right] e^{-rs} \) | \( \left[ 3 + 2rs(1 + rs) \right] e^{-rs} \) | \( \left[ 3 + 2rs(1 + rs) \right] e^{-rs} \) |
The next step is to derive an orthonormal basis of solutions, and then calculate the Higgs field \( \Phi \). An orthogonal basis of solutions is given by

\[
\mathbf{v}_1 = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_{k+1} = \begin{pmatrix} 0 \\ \vdots \\ w_k \\ \vdots \\ u_{k+1} \\ 0 \end{pmatrix}, \quad \mathbf{v}_{n+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ w_n \end{pmatrix}, \quad k \in \{1, \ldots, n-1\}. \tag{19}
\]

Introduce the inner product

\[
< \mathbf{v}_\kappa, \mathbf{v}_\lambda > = \int_0^{n+1} \mathbf{v}_\kappa^\dagger \mathbf{v}_\lambda \, ds = \mathcal{N}_\kappa \delta_{\kappa\lambda} \tag{20}
\]

then the required solutions of (2) are those which are normalizable with respect to (3). It is clear that the space of normalizable solutions of (2) has dimension \( n + 1 \).

If \( \hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{n+1} \) consist an orthonormal basis for this space then the Higgs field is an \( SU(n+1) \) diagonal matrix whose elements are given by

\[
\Phi_\kappa = -i \int_0^{n+1} (s-n) \hat{\mathbf{v}}_\kappa^\dagger \hat{\mathbf{v}}_\kappa \, ds, \quad \kappa \in \{1, \ldots, n+1\}. \tag{21}
\]
Recall that, the orthogonal vectors $v_1$ and $v_{n+1}$ are given in terms of the functions $u_1$ and $w_n$ in (15). The vectors $v_{k+1}$ for $k \in \{1, \ldots, n-1\}$ are more complicated and are given in terms of the Whittaker functions $M\left(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs\right)$ and $W\left(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs\right)$. In particular, the $w_k$ and $u_{k+1}$ functions are given explicitly by
\[
w_k = \frac{1}{s} W_k, \quad u_{k+1} = \frac{1}{C_k} \left[ \dot{W}_k + \left( \frac{a_k - 2}{2s} + r \right) W_k \right]
\]
where $W_k$ is defined in (18).

Performing the required integrals (20), and using (15), (22), gives
\[
N_1 = \int_0^{n+1} \sqrt{s_{a_1}} e^{rs} ds, \\
N_{k+1} = \frac{1}{C_k^2} \left[ \frac{1}{2} \frac{dW_k^2}{ds} + \left( r + n - 1 - 2k \right) W_k^2 \right]_{s=0}^{n+1}, \quad k \in \{1, \ldots, n-1\}, \\
N_{n+1} = \int_0^{n+1} \frac{e^{-rs}}{\sqrt{s_{a_n}}} ds.
\]
Note that the normalization factors $N_k$, solely consist of boundary terms and thus can be easily evaluated. However, the Whittaker $W\left(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs\right)$ function at $s = 0$ tends to infinity (see Appendix). Hence setting $c_2(r) = 0$ in (18) to avoid the divergencies, the solutions $W_k$ are given in terms of the Whittaker $M\left(\frac{2k-n+1}{2}, \frac{n}{2}; 2rs\right)$ function only.

Then the corresponding diagonal elements of the Higgs field at $(0,0,r)$ are given by
\[
\Phi_1 = -\frac{i}{N_1} \int_0^{n+1} (s-n) \sqrt{s_{a_1}} e^{rs} ds, \\
\Phi_{k+1} = -\frac{i}{N_{k+1}} \frac{C_k^2}{2} \left[ s \frac{dW_k^2}{ds} + \left( rs + \frac{n}{2} - k - 1 \right) W_k^2 \right]_{s=0}^{n+1} - \frac{i}{N_{k+1}} \frac{C_k^2}{2} \int_0^{n+1} \left( \frac{2k-n+1}{2s} - r \right) W_k^2 ds + i n, \quad k \in \{1, \ldots, n-1\}, \\
\Phi_{n+1} = -\frac{i}{N_{n+1}} \int_0^{n+1} (s-n) \frac{e^{-rs}}{\sqrt{s_{a_n}}} ds.
\]
It is a simple task to verify that this solution has indeed the correct asymptotic behavior, and also to recover the results obtained for the special cases $n = 2$ and $n = 3$ in [8, 9], respectively.

To conclude, with the method applied in this section starting from the $n$-dimensional representation of the $SU(2)$ algebra ($n$-monopoles) one ends up with $n + 1$ orthonormal
vectors, which provide the Higgs field corresponding to the minimal symmetry breaking of $SU(n + 1)$. Note that we have focused here on spherically symmetric monopoles located at $(0, 0, r)$. Similarly, the azimuthal dependence can be implemented in a quite straightforward manner via suitable similarity transformations, or by solving the full Weyl equation with the simple Nahm data. Then the gauge potentials can also be recovered. This is a mathematically and physically interesting problem, and will be addressed in full detail in a forthcoming publication.

3 Non-Commutative $SU(n + 1)$ BPS Monopoles

The idea of non-commutative space-time offers a smooth way to introduce nonlocality into field theories without loosing control. Motivated by string theory the investigation of non-Abelian gauge theories defined on non-commutative space-time is of great interest in the last few years. In particular, the dynamics of non-Abelian gauge fields involves non-perturbative field configurations (e.g. instantons, monopoles, etc) in an essential way. Thus, in order to quantize a gauge theory it is mandatory to study its classical solutions and characterize their moduli spaces.

In [11] Bak derived a deformed Nahm equation for the BPS equation in the non-commutative $N = 4$ super-symmetric $U(2)$ Yang-Mills theory. This way, he was able to explicitly construct a monopole solution of the non-commutative BPS equation to the linear order of the non-commutative scale.

In this section, following Bak, the non-commutative $SU(n + 1)$ BPS spherical symmetric monopoles with minimal symmetry breaking, are obtained using a generalization of the ADHMN method. The non-commutative construction goes along the same lines as the commutative one, with the only difference that all the ordinary products are now replaced by the associative (but not commutative) Moyal star product ($\star$-product). The latter is characterized by a constant positive real parameter $\theta$, which appears in the star product of two functions in the following way

$$a(x) \star b(x) = \left( e^{i \theta \partial_\mu \partial'_{\nu}} a(x)b(x') \right) \bigg|_{x=x'}.$$

(25)
For $\theta = 0$ it reduces to the ordinary product of functions. In general, $\theta^{\mu\nu}$ is a constant, real-value antisymmetric ($\theta^{\mu\nu} = -\theta^{\nu\mu}$) matrix of order $d$ (where $d$ is the dimension of space-time). For simplicity, we choose its only non-zero terms to be $\theta_{12} = -\theta_{21} = \theta$.

In the non-commutative case, the derivation goes through once all the product operations are replaced by $\star$-product. Namely, the Weyl equation (2) is modified as

$$\left(-\frac{d}{ds} + I_n \otimes x_j \sigma_j - iT_j \otimes \sigma_j\right) \star \mathbf{v}(x, s) = 0.$$  \hspace{1cm} (26)

$T_i$ satisfy (for a more detailed discussion on these issues see [11]) the deformed Nahm equation:

$$\frac{dT_i}{ds} = \frac{1}{2} \varepsilon_{ijk} [T_j, T_k] - \theta \delta_{i3} I.$$  \hspace{1cm} (27)

Setting $T_i = \tilde{T}_i - \theta s \delta_{i3}$, it is easy to see that $\tilde{T}_i$ satisfy the Nahm equations (1); they have simple poles; and their residues are irreducible representations of $SU(2)$.

The deformed Weyl equation takes then the following form

$$\left(-\frac{d}{ds} + I_n \otimes x_j \sigma_j - iT_j \otimes \sigma_j + \theta \hat{O}\right) \mathbf{v}(\tilde{x}, s) = 0,$$  \hspace{1cm} (28)

where

$$\hat{O} = \frac{i}{2} I_n \otimes (\sigma_1 \partial_2 - \sigma_2 \partial_1) + is I_n \otimes \sigma_3.$$  \hspace{1cm} (29)

Next we construct the actual solutions of the non-commutative BPS equations from the deformed Nahm data $T_i$ (27). As before, we focus on the spherically symmetric case by setting $x_i = (0, 0, r)$, and $\tilde{T}_i$ are the ordinary Nahm data of the form (6) where $f_i = f = -\frac{1}{s}$. In this case, the deformed Weyl equation (28) becomes

$$\left(-\frac{d}{ds} - \frac{f}{2} \tau_i \otimes \sigma_i + \tilde{r} I \otimes \sigma_3\right) \mathbf{v}(\tilde{r}, s) = 0,$$  \hspace{1cm} (30)

where $\tilde{r} = r + i\theta s$.

Similarly to the commutative case, the deformed Weyl equation (30) leads to a first-order system of differential equations given by (14) for $r \rightarrow \tilde{r}$. Then the first and last equations are decoupled and their solutions are given by

$$u_1 = k_1(\tilde{r}) \sqrt{s^{\mu_1}} e^{(rs + \frac{ig}{2}s^2)}, \hspace{1cm} w_n = k_2(\tilde{r}) \frac{e^{-(rs + \frac{ig}{2}s^2)}}{\sqrt{s^{\mu_n}}}.$$  \hspace{1cm} (31)
Finally, the coupled equations for \( u_{k+1} \) and \( w_k \) are equivalent to the \textit{deformed} single second-order differential equation

\[
s^2 \ddot{w}_k + 2s \dot{w}_k - \left( r^2 s^2 - i \theta s^2 + (n - 2k - 1) \dot{r}s + \frac{n^2 - 1}{4} \right) w_k = 0, \tag{32}
\]

for \( k \in \{1, \ldots, n - 1\} \). Note that, the aforementioned equations become the corresponding equations of the previous section when \( \theta = 0 \).

The latter equation is solved in terms of the so-called Heun B functions denoted henceforth as \( H_B \) (see, Appendix, for more information regarding \( H_B \) and the corresponding differential equation). The solution is given by

\[
w_k = c_1(r)\sqrt{s^{n-1}} e^{(rs+i\theta s^2)} H_B \left( n, \frac{2(-1)^{\frac{1}{4}} r}{\sqrt{\theta}}, n - 2k - 2, \frac{(-1)^{\frac{1}{4}} (4k + 2 - 2n)r}{\sqrt{\theta}}; (-1)^{\frac{3}{4}} \sqrt{\theta} s \right) + \frac{c_2(r)}{\sqrt{s^{n+1}}} e^{(rs+i\theta s^2)} H_B \left( -n, \frac{2(-1)^{\frac{1}{4}} r}{\sqrt{\theta}}, n - 2k - 2, \frac{(-1)^{\frac{1}{4}} (4k + 2 - 2n)r}{\sqrt{\theta}}; (-1)^{\frac{3}{4}} \sqrt{\theta} s \right) \tag{33}
\]

where \((-1)^{\frac{i}{4}} = \frac{1+i}{\sqrt{2}}\) and \((-1)^{\frac{3}{4}} = \frac{-1+i}{\sqrt{2}}\). It can be shown (see Appendix) that for the specific form of the solutions given by (33) the corresponding Heun B functions satisfy the conditions

\[
H_B(-n) = \left[ (-1)^{\frac{3}{4}} \sqrt{\theta} s \right]^n H_B(n). \tag{34}
\]

Therefore, the general solution (33) becomes

\[
w_k = c(r, \theta)\sqrt{s^{n-1}} e^{(rs+i\theta s^2)} H_B \left( n, \frac{2(-1)^{\frac{1}{4}} r}{\sqrt{\theta}}, n - 2k - 2, \frac{(-1)^{\frac{1}{4}} (4k + 2 - 2n)r}{\sqrt{\theta}}; (-1)^{\frac{3}{4}} \sqrt{\theta} s \right) \tag{35}
\]

where \( c(r, \theta) \) is an arbitrary function of \( r \) and \( \theta \). Table 3 presents the series expansion of the Heun B function \( H_B \left( n, \frac{2(-1)^{1/4} r}{\sqrt{\theta}}, n - 2 - 2k, \frac{(-1)^{1/4}(4k + 2 - 2n)r}{\sqrt{\theta}}; (-1)^{3/4} \sqrt{\theta} s \right) \) up to third order: \( O(s^3) \).

| \( H_B(n) \propto O(s^3) \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) |
|---|---|---|---|
| \( k = 1 \) | \( 1 - \frac{4}{3} rs + \left( r^2 - \frac{3}{4} I \theta \right) s^2 \) | \( 1 - r s + \frac{2}{5} \left( r^2 - I \theta \right) s^2 \) | \( 1 - \frac{8}{5} rs + \left( \frac{2}{5} r^2 - \frac{1}{2} I \theta \right) s^2 \) |
| \( k = 2 \) | \( 1 - \frac{3}{2} rs + \frac{2}{5} \left( 3r^2 - 2I \theta \right) s^2 \) | \( 1 - \frac{6}{5} rs + \left( \frac{2}{5} r^2 - \frac{2}{5} I \theta \right) s^2 \) | \( 1 - \frac{8}{5} rs + \left( \frac{2}{5} r^2 - \frac{5}{6} I \theta \right) s^2 \) |
| \( k = 3 \) | \( 1 - \frac{5}{2} rs + \frac{1}{10} \left( 5r^2 - 3I \theta \right) s^2 \) | \( 1 - \frac{6}{5} rs + \left( \frac{2}{5} r^2 - \frac{1}{2} I \theta \right) s^2 \) | \( 1 - \frac{8}{5} rs + \left( \frac{2}{5} r^2 - \frac{5}{6} I \theta \right) s^2 \) |
Table 3: The $H_B \left(n, \frac{2(-1)^{1/4}r}{\sqrt{\theta}}, n - 2 - 2k, \frac{(-1)^{1/4}(4k+2-2n)r}{\sqrt{\theta}}; (-1)^{3/4}\sqrt{\theta}s\right)$ function up to order $O(s^3)$ for $n = 2, \ldots, 4$.

Next in order to obtain the deformed Higgs field we follow the procedure of the previous section. First we choose the deformed orthogonal basis similarly to (19). Then the normalization factors are equal to

\[ \mathcal{N}_1 = \int_0^{n+1} \sqrt{s^{a_1}} e^{(rs+\frac{\theta}{2} s^2)} \, ds, \]

\[ \mathcal{N}_{k+1} = \frac{1}{C_k} \left[ \frac{dW_k^2}{ds} + \left( r + i\theta s + \frac{n - 1 - 2k}{2s} \right) W_k^2 \right] \bigg|_{s=0}^{n+1}, \quad k \in \{1, \ldots, n - 1\}, \]

\[ \mathcal{N}_{n+1} = \int_0^{n+1} \frac{e^{-(rs+\frac{\theta}{2} s^2)}}{\sqrt{s^{a_n}}} \, ds, \] (36)

where $W_k = sw_k$ and $w_k$ is given by (35). The corresponding diagonal elements of the Higgs field are then expressed as

\[ \Phi_1 = -\frac{i}{\mathcal{N}_1} \int_0^{n+1} (s - n) \sqrt{s^{a_1}} e^{(rs+\frac{\theta}{2} s^2)} \, ds, \]

\[ \Phi_{k+1} = -\frac{i}{\mathcal{N}_{k+1} C_k} \left[ \frac{dW_k^2}{ds} + \left( rs + i\theta s^2 + \frac{n - 1}{2s} - k - 1 \right) W_k^2 \right] \bigg|_{s=0}^{n+1} \]

\[ -\frac{i}{\mathcal{N}_{k+1} C_k} \int_0^{n+1} \left( \frac{2k - n + 1}{2s} - r - i\theta s \right) W_k^2 \, ds + i n, \quad k \in \{1, \ldots, n - 1\}, \]

\[ \Phi_{n+1} = -\frac{i}{\mathcal{N}_{n+1}} \int_0^{n+1} (s - n) \frac{e^{-(rs+\frac{\theta}{2} s^2)}}{\sqrt{s^{a_n}}} \, ds. \] (37)

As expected, the expressions above reduce to equations (23) and (24) when the deformation parameter vanishes, i.e. in the limit $\theta \to 0$.

4 Discussion

We have been able to derive, using a quite elegant and unifying methodology, explicit expressions for spherically symmetric commutative and non-commutative $SU(n + 1)$ BPS monopoles in the minimal symmetry breaking case. The use of the generic $n$-dimensional representation of $SU(2)$ allowed the explicit construction of solutions of the Weyl equation.
giving rise to sets of simple differential equations. The solutions of the aforementioned differential equations were expressed in terms of confluent (Whittaker) or biconfluent (Heun B) hypergeometric functions for the commutative and the non-commutative case, respectively.

Although we have restricted our investigation in the spherically symmetric case it is important to note that azimuthal dependence may be implemented in the generic situation by means of suitable similarity transformations, however this issue will be discussed in full detail in a forthcoming work. It is also worth noting that the case \( n \to \infty \) merits special investigation, given that the behavior of the discovered solutions in this situation is rather particular, modifying the form of the final expressions for the Higgs field. Physical interpretation of the behavior of the Higgs field is also desirable. Such results may be also of relevance when one considers, within this context, infinite dimensional representations of \( SL(2, R) \) (the non-compact case, e.g. non-trivial spin \( s = 0 \) representation of \( SL(2, R) \)). The solution of the Weyl equation in this case involves a particular polynomial basis, given that the associated representation is expressed in terms of simple differential operators. These are quite intriguing issues, and will be left for future investigations.

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A Whittaker and Heun B functions

The Whittaker functions \( M(k, m; z) \) and \( W(k, m; z) \) solve the Whittaker differential equation (see e.g. [12, 13]):

\[
\frac{d^2W}{dz^2} + \left( \frac{1}{4} + \frac{k}{z} + \frac{1}{4} - \frac{m^2}{z} \right) W = 0
\]  

(38)

where

\[
W = c_1 M(k, m; z) + c_2 W(k, m; z).
\]  

(39)

They can be defined in terms of confluent hypergeometric functions as follows

\[
M(k, m, z) = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} _1F_1\left(\frac{1}{2} + m - k; 1 + 2m; z\right),
\]  

(40)
\[ W(k, m, z) = e^{-\frac{z}{2}} z^{k} \binom{1}{2} F_{0} \left( \frac{1}{2} + m - k, \frac{1}{2} - m - k; -z^{-1} \right), \]  

(41)

where \( \binom{a}{b} F_{1}(a; b; z) = e^{z} \binom{a}{b} F_{1}(b - a; b; -z) \) and \( _{n}F_{m} \) are generalized hypergeometric functions (see e.g. \cite{12, 13, 14}) defined as:

\[ _pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_q; z) = \sum_{l=0}^{\infty} \frac{(a_1)_l (a_2)_l \ldots (a_p)_l}{(b_1)_l (b_2)_l \ldots (b_q)_l} \frac{z^l}{l!}, \quad (a)_l = \frac{\Gamma(a + l)}{\Gamma(a)}. \]  

(42)

In the special case where \( m + \frac{1}{2} - k \) is a positive integer (which is our case), the Whittaker \( W(k, m, z) \) function is defined by the integral function

\[ W(k, m; z) = \frac{e^{-\frac{z}{2}} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_{0}^{\infty} t^{-k-\frac{1}{2}+m} (1 + \frac{t}{z})^{k-\frac{1}{2}+m} e^{-t} dt \]  

(43)

for all values of \( z \) except negative real values. Then, it is obvious that at \( z = 0 \) (corresponding to \( s = 0 \) in our case) it diverges.

Let us now introduce the Heun Biconfluent differential equation

\[ \frac{d^2y}{dz^2} + \left( -2z - b + \frac{1 + a}{z} \right) \frac{dy}{dz} + \left( c - a - 2 - \frac{1}{2} \frac{(a + 1)b + d}{z} \right) y = 0. \]  

(44)

The Heun B function \( H_B(a, b, c, d; z) \) is a Frobenius solution to Heun’s Biconfluent equation \cite{44} computed as a power series expansion around the origin, which is a regular singular point. In this case, the growth of solutions is bounded, in any small sector, by an algebraic function. Because the next singularity is located at infinity, this series converges in the whole complex plane.

Note that the Heun B function can be expressed in terms of the Whittaker \( M \) function for specific values of the parameters, i.e.

\[ H_B(a, 0, c, 0; z) = \frac{1}{z^{\frac{1}{2}+1}} e^{\frac{z^2}{2}} M \left( \frac{c}{4}, \frac{a}{4}; z^2 \right). \]  

(45)

It is thus clear that the results of section 3 reduce to those of section 2, in the limit \( \theta \to 0 \).

A special case occurs when in \( H_B(a, b, c, d, z) \), the third parameter satisfies the condition \( c = 2(k + 1) + a \) where \( k \) is a positive integer (which is our case). In this case the \( k^{th} + 1 \) coefficient in the series expansion is a polynomial of degree \( k \) in \( d \). When \( d \) is a root of this polynomial, the \( k^{th} + 1 \) and subsequent coefficients cancel and the series truncates resulting in a polynomial form of degree \( k \) for Heun B.
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