Coincidence sets in quasilinear elliptic problems of monostable type

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Abstract
This paper concerns the formation of a coincidence set for the positive solution of the boundary value problem:

\[-\varepsilon \Delta_p u = u^{q-1} f(a(x) - u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

where \(\varepsilon\) is a positive parameter, \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u),\) \(1 < q \leq p < \infty, f(s) \sim |s|^{\theta-1} s (s \to 0)\) for some \(\theta > 0\) and \(a(x)\) is a positive smooth function satisfying \(\Delta_p a = 0\) in \(\Omega\) with \(\inf_\Omega |\nabla a| > 0\). It is proved in this paper that if \(0 < \theta < 1\) the coincidence set \(O_\varepsilon = \{x \in \Omega : u_\varepsilon(x) = a(x)\}\) has a positive measure and converges to \(\Omega\) with order \(O(\varepsilon^{1/p})\) as \(\varepsilon \to 0\). Moreover, it is also shown that if \(\theta \geq 1\), then \(O_\varepsilon\) is empty for any \(\varepsilon > 0\). The proofs rely on comparison theorems and an energy method for obtaining local comparison functions.

1 Introduction
Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N (N \geq 2)\) with smooth boundary \(\partial \Omega\), and we consider the boundary value problem of quasilinear elliptic equations of monostable type:

\[
\begin{cases}
-\varepsilon \Delta_p u = u^{q-1} f(a(x) - u) & \text{in } \Omega, \\
u \geq 0, u \not= 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (1.1)
where ε is a positive parameter, Δpμ denotes the p-Laplacian \( \text{div}(\nabla_p u) \) with the p-gradient \( \nabla_p u = |\nabla u|^{p-2}\nabla u \), \( 1 < q \leq p < \infty \), \( a : \Omega \to \mathbb{R} \) is a positive and smooth function and \( f \) is a function satisfying the following conditions.

(F1) \( f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \) and \( f(0) = 0 \).

(F2) \( f \) is strictly increasing on \( \mathbb{R} \).

(F3) There exists \( \theta > 0 \) such that \( \lim_{s \to 0} \frac{f(s)}{|s|^{p-1}} = C \) for some \( C > 0 \).

By a solution of (1.1) we mean a function \( u \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega) \) satisfying (1.1) (for details, see Section 2). Applying the theorem of Díaz and Saá [4] and the regularity result of Lieberman [14], we see that if \( \varepsilon < \varepsilon_a \) then (1.1) admits a unique positive solution \( u_\varepsilon \in C^{1,a}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \); if \( \varepsilon \geq \varepsilon_a \) then (1.1) has no solution. Here, \( \varepsilon_a = \infty \) if \( p > q \) and \( \varepsilon_a = 1/\lambda_{f(a)} \) if \( p = q \), where \( \lambda_{f(a)} \) denotes the first eigenvalue of the definite weight eigenvalue problem

\[
\begin{align*}
-\Delta_p u &= \lambda f(a(x))|u|^{p-2}u & \text{in } \Omega, \\
\frac{u}{|\nabla u|} &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

and it can be characterized by

\[
\lambda_{f(a)} = \inf_{u \neq 0 \in W^{1,p}_0(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\int_{\Omega} f(a(x))|u(x)|^p \, dx}.
\]

We define the coincidence set of the positive solution \( u_\varepsilon \) of (1.1) with \( a(x) \) as

\[
O_\varepsilon = \{ x \in \Omega : u_\varepsilon(x) = a(x) \}.
\]

In case \( a(x) \) is constant, problem (1.1) has been already studied by several authors. Let \( a(x) \equiv 1 \) and \( p = q > 2 \). Then, Guedda and Véron [10] for \( N = 1 \) and Kamin and Véron [12] for \( N \geq 2 \) established that there exists a non-empty coincidence set \( O_\varepsilon \) (or a flat core, because the graph of \( u_\varepsilon \) is flat on \( O_\varepsilon \)) for \( \varepsilon \) small enough (when \( \Omega \) is a ball and \( f(s) = s \), Kichenassamy and Smoller [13] had obtained the positive radial solution with a flat core). They and García-Melián and Sabina de Lis [9] proved that if \( 0 < \theta < p - 1 \), then the flat core has a positive measure for small \( \varepsilon \in (0, f(a)/\lambda_{f(a)}) \) and it converges to \( \Omega \) as \( \text{dist}(x, O_\varepsilon) \sim \varepsilon^{1/p} \) (\( \varepsilon \to 0 \)) for any \( x \in \partial \Omega \); while if \( \theta \geq p - 1 \), then the flat core is empty. These earlier results [9, 10, 12, 13] are substantially sharpened by Guo [11].

Moreover, even if \( a(x) \) is constant on a plural subdomain of \( \Omega \), there exists a flat core in each subdomain (see [16]). General references for coincidence set are given in the monographs [3] of Díaz and [15] of Pucci and Serrin.

In this paper we shall investigate the case where \( a(x) \) is variable. It is heuristic that if the coincidence set \( O_\varepsilon \) has an interior point, then \( a(x) \) has to satisfy \( \Delta_p a = 0 \) on its
neighborhood. Inversely, we shall assume \( a(x) \) to be \( p \)-harmonic: \( \Delta \rho a = 0 \) in \( \Omega \), and hence \( a(x) \) satisfies the equation of \((1.1)\). Then, our major finding is that the \( p \)-harmonicity of \( a(x) \) is also a sufficient condition for an appearance of coincidence set.

Before stating the result, we give precise conditions to \( a(x) \):

(A1) \( \inf_{x \in \Omega} a(x) > 0 \),

(A2) \( a \in C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) and \( \Delta \rho a = 0 \) in \( \Omega \), and

(A3) \( \inf_{x \in \Omega} |\nabla a(x)| > 0 \).

We notice that by DiBenedetto \([6]\) and Tolksdorf \([19]\), (A2) follows from, e.g.,

(A2') there exists a domain \( \Omega' \supset \overline{\Omega} \) such that \( a \in W^{1,p}_{\text{loc}}(\Omega') \) and \( \Delta \rho a = 0 \) in \( \Omega' \).

The following theorem suggests that with regard to the coincidence set of positive solution, it is unnecessary to assume \( a(x) \) to be constant as in the past studies.

**Theorem 1.1.** Assume (A1), (A2) and (A3). Let \( 0 < \theta < 1 \). Then, there exist \( L > 0 \) and \( \varepsilon_0 \in (0, \varepsilon_a) \) such that for each \( \varepsilon \in (0, \varepsilon_0) \) the solution \( u_\varepsilon \) of \((1.1)\) satisfies

\[
    u_\varepsilon(x) = a(x) \quad \text{if} \quad \text{dist}(x, \partial \Omega) \geq L\varepsilon^{1/p}.
\]

The corresponding theorem for \( p = 2 \) has been already proved in the author’s paper \([17]\). As mentioned above, the condition \( 0 < \theta < p - 1 \) seems to be valid as a modification to the case \( 1 < p < \infty \), while the condition \( 0 < \theta < 1 \) in the theorem is same as that in case \( p = 2 \). However, this is natural because the principal part of equation of \((1.1)\) is neither degenerate nor singular in \( O_\varepsilon \) when \( a(x) \) satisfies the non-degeneracy condition (A3).

The condition \( 0 < \theta < 1 \) in Theorem 1.1 is optimal in the following sense.

**Theorem 1.2.** Assume \( a(x) \) to be same in Theorem 1.1. Let \( \theta \geq 1 \). Then, for every \( \varepsilon \in (0, \varepsilon_a) \), \( u_\varepsilon < a \) in \( \Omega \), and hence \( O_\varepsilon = \emptyset \).

In our approach, it is significant to study the translation \( -\varepsilon \Delta \rho (v - a) \) of the principal part \( -\varepsilon \Delta \rho v \). Putting \( \Phi_\rho(\nabla v, \nabla a) = \nabla_\rho (v - a) + \nabla_\rho a \) and using (A2), we see that \( \Phi_\rho(0, \nabla a) = 0 \) and that the translation can be represented as the monotone operator \( v \mapsto -\varepsilon \text{div} \Phi_\rho(\nabla v, \nabla a) \). The vector-valued function \( \Phi_\rho(\eta, \nabla a) \) has a different order at \( \eta = 0 \) from what \( \Phi_\rho(\eta, 0) \) has if and only if \( a(x) \) is non-degenerate. This is the reason why the conditions of \( \theta \) in the theorems differ from those in case \( a(x) \) is constant.

Theorems 1.1 and 1.2 are proved in Section 4. In order to show Theorem 1.1 letting the solution \( u_\varepsilon \) be close to \( a(x) \) as \( \varepsilon \to 0 \) (the convergence will be shown in Section 2), we compare \( u_\varepsilon \) with a local comparison function which attains \( a(x) \). Such a comparison function is obtained in Section 3 by means of the energy method developed by Díaz and Véron \([5]\) (see also Díaz \([3]\), and Antontsev, Díaz and Shmarev \([1]\)). In proving Theorem 1.2 we give a Harnack type inequality by Trudinger \([20]\) for an associated differential inequality. Finally, in Section 5, we apply our method to the known case where \( a(x) \) is constant and realize the necessity of modifying the condition of \( \theta \) to \( 0 < \theta < p - 1 \).
The corresponding theorems for $N = 1$ to Theorems [1.1] and [1.2] have been already obtained in the author’s paper [18].

Remark 1.1. If $\Omega = \mathbb{R}^N$, then the corresponding problem to (1.1)

$$-\varepsilon \Delta_p u = u^{q-1} f(a(x) - u) \quad \text{in } \mathbb{R}^N$$

is trivial. Indeed, since $a(x)$ is a positive and $p$-harmonic function in $\mathbb{R}^N$, it is constant by Liouville’s theorem for $p$-Laplacian [15, Corollary 7.2.3] and any nonnegative solution of (1.1) must be the constant (see Du and Guo [7]).

Through the paper, we denote by $C$ positive constants independent of $\varepsilon$ and $\delta$, unless otherwise noted.

2 Convergence to $a(x)$ as $\varepsilon \to 0$

In this section, we show that the solution of (1.1) converges to $a(x)$ uniformly in any compact set of $\Omega$ as $\varepsilon \to 0$.

A function $u = u_\varepsilon \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ is called a solution of (1.1) if $u \geq 0$ a.e. in $\Omega$, $u$ does not vanish in a set of positive measure, and

$$\varepsilon \int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega u^{q-1} f(a(x) - u) \varphi dx$$

for all $\varphi \in W^{1,p}_0(\Omega)$. A function $u = u_\varepsilon \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ is called a supersolution (resp. subsolution) of (1.1) if $u \geq 0$ (resp. $u \leq 0$) a.e. on $\partial \Omega$ and

$$\varepsilon \int_\Omega \nabla u \cdot \nabla \varphi dx \geq \text{ (resp. } \leq \text{)} \int_\Omega u^{q-1} f(a(x) - u) \varphi dx$$

for all $\varphi \in W^{1,p}_0(\Omega)$ satisfying $\varphi \geq 0$ a.e. in $\Omega$. If a function $u$ is not only a supersolution but also a subsolution, then $u$ must be a solution of (1.1).

We denote by $\lambda_1$ the first eigenvalue to the following eigenvalue problem and by $z$ the corresponding eigenfunction to $\lambda_1$ with $||z||_{L^\infty(\Omega)} = \sup_{x \in \Omega} |z(x)| = 1$:

$$\begin{cases}
-\Delta_p z = \lambda |z|^{p-2}z & \text{ in } \Omega, \\
z = 0 & \text{ on } \partial \Omega.
\end{cases}$$

It is well-known that $\lambda_1 > 0$, $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $z > 0$ in $\Omega$. Let $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$, $\Omega_{\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon\}$ and $d = \inf_{x \in \Omega} a(x)/2 > 0$. 


Proposition 2.1. Assume $a(x)$ to satisfy (A1) and (A2). For each $\delta \in (0, 2d)$, there exist $K > 0$ and $\varepsilon_* \in (0, \varepsilon_0)$ such that if $\varepsilon \in (0, \varepsilon_*)$ then the solution $u_\varepsilon$ of (1.1) satisfies

$$a(x) - \delta \leq u_\varepsilon(x) \leq a(x) \quad \text{for all } x \in \Omega_{K\varepsilon^1/p}.$$  

Proof. It is clear from (A2) that $\overline{u} = a$ is a supersolution of (1.1) for every $\varepsilon > 0$.

We shall construct a subsolution of (1.1). From the uniform continuity of $a(x)$ in $\overline{\Omega}$, there exists $r > 0$ such that for every $x_0 \in \Omega$, $a(x) > a(x_0) - \delta/2$ for all $x \in B(x_0, r) \cap \Omega$, and hence for each $x \in B(x_0, r) \cap \Omega$, $a(x) - u > \delta/2$ for all $u \in [0, a(x_0) - \delta]$. Therefore, $f(a(x) - u) \geq \sigma = f(\delta/2)$ for all $x \in B(x_0, r) \cap \Omega$ and $u \in [0, a(x_0) - \delta]$. Let $K > 0$ be a constant satisfying $K^p > \lambda_1 ||a||_{L^\infty(\Omega)}^p/\sigma$ and choose $\varepsilon_* \in (0, \varepsilon_0)$ such that $K\varepsilon_*^{1/p} < r$.

Take any $\varepsilon \in (0, \varepsilon_*)$ and $x_0 \in \Omega_{K\varepsilon_*^{1/p}}$. Changing scaling as $\underline{z}(x) = z((x - x_0)/(K\varepsilon_*^{1/p}))$, we have

$$\begin{cases}
-\varepsilon \Delta_p \underline{z} = \frac{\lambda_1}{K^p} \underline{z}^{p-1} & \text{in } B(x_0, K\varepsilon_*^{1/p}), \\
\underline{z} = 0 & \text{on } \partial B(x_0, K\varepsilon_*^{1/p}).
\end{cases}$$

Then the function

$$\underline{u}(x) = \begin{cases}
(a(x_0) - \delta)\underline{z}(x), & x \in B(x_0, K\varepsilon_*^{1/p}), \\
0, & x \in \Omega \setminus B(x_0, K\varepsilon_*^{1/p})
\end{cases}$$

is a nonnegative subsolution of (1.1). Indeed, $a(x_0) \geq 2d > \delta$, and for every $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$

$$\frac{1}{(a(x_0) - \delta)^{p-1}} \left( \varepsilon \int_{\Omega} \nabla_p \underline{u} \cdot \nabla \varphi \, dx - \int_{\Omega} \underline{u}^{p-1} f(a(x) - \underline{u}) \varphi \, dx \right) \leq -\varepsilon \int_{B(x_0, K\varepsilon_*^{1/p})} (a(x_0) - \delta)^{p-q} \Delta_p \underline{z} \varphi \, dx - \sigma \int_{B(x_0, K\varepsilon_*^{1/p})} \underline{z}^{q-1} \varphi \, dx$$

$$= \int_{B(x_0, K\varepsilon_*^{1/p})} \left( \frac{\lambda_1 (a(x_0) - \delta)^{p-q}}{K^p} \underline{z}^{p-q} - \sigma \right) \underline{z}^{q-1} \varphi \, dx \leq \left( \frac{\lambda_1 ||a||_{L^\infty(\Omega)}^{p-q}}{K^p} - \sigma \right) \int_{B(x_0, K\varepsilon_*^{1/p})} \underline{z}^{q-1} \varphi \, dx \leq 0.$$

Since $\underline{u} < \overline{u}$ in $\Omega$, there exists a solution $u^*$ of (1.1) with $\underline{u} \leq u^* \leq \overline{u}$ in $\Omega$ (e.g., Deuel and Hess [2]). As mentioned in Section 1, the solution of (1.1) is unique. Therefore, $u^* = u_\varepsilon$, and hence $\underline{u} \leq u_\varepsilon \leq \overline{u}$ in $\Omega$. In particular, $a(x_0) - \delta \leq u_\varepsilon(x_0) \leq a(x_0)$ for all $x_0 \in \Omega_{K\varepsilon_*^{1/p}}$ when $0 \leq \varepsilon \leq \varepsilon_*$. \qed
Remark 2.1. Even if (A2) is not assumed, then we can prove that \(|u_e - a| < \delta\). Indeed, we can construct a supersolution of (1.1) close to \(a(x)\) from above. Let \(p \geq 2\) for simplicity, and assume \(\overline{u}\) to be an arbitrary smooth function satisfying \(a + \delta/2 < \overline{u} < a + \delta\). Since

\[-\varepsilon \Delta_p \overline{u} - \overline{u}^{q-1} f(a(x) - \overline{u}) \geq -\varepsilon \Delta_p \overline{u} + C(\overline{u} - a(x))^\theta \geq -\varepsilon \Delta_p \overline{u} + C \left(\frac{\delta}{2}\right)^\theta\]

for all \(x \in \Omega\) and \(\Delta_p \overline{u}\) is continuous in \(\overline{\Omega}\), the last expression can be positive provided \(\varepsilon\) is small enough. For the case \(1 < p < 2\), we refer to [16].

3 Auxiliary problem near \(a(x)\)

In this section, we show that there exists a comparison function with dead core, which satisfies an equation having a subsolution \(a - u_e \geq 0\).

We define the vector-valued function \(\Phi_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N\) as

\[\Phi_p(\eta, \xi) = |\eta - \xi|^{p-2}(\eta - \xi) + |\xi|^{p-2}\xi.\]

In particular, we note that \(\Phi_p(\nabla u, \nabla v) = \nabla_p(u - v) + \nabla_p v\) for gradients.

The following lemma means that for each \(\xi \neq 0\) the function \(\Phi_p(\eta, \xi)\) is of order 1 at \(\eta = 0\).

Lemma 3.1. For all \(\eta, \xi \in \mathbb{R}^N\) with \(|\eta - \xi| + |\xi| > 0\)

\[\Phi_p(\eta, \xi) \cdot \eta \geq \min\{p - 1, 2^{2-p}\}|(\eta - \xi) + |\xi|\}^{p-2}|\eta|, \quad (3.1)\]

\[|\Phi_p(\eta, \xi)| \leq \max\{p - 1, 2^{2-p}\}|(\eta - \xi) + |\xi|\}^{p-2}|\eta|. \quad (3.2)\]

For all \(\eta, \eta', \xi \in \mathbb{R}^N\) with \(|\eta - \xi| + |\eta'| - \xi| > 0\)

\[(\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi)) \cdot (\eta - \eta') \geq \min\{p - 1, 2^{2-p}\}|(\eta - \xi) + |\eta' - \xi|\}^{p-2}|\eta - \eta'|^2, \quad (3.3)\]

\[|\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi)| \leq \max\{p - 1, 2^{2-p}\}|(\eta - \xi) + |\eta' - \xi|\}^{p-2}|\eta - \eta'|. \quad (3.4)\]

Proof. By the mean value theorem, we have

\[(\Phi_p(\eta, \xi), \eta) = (p - 1)|\eta|^2 \int_0^1 |\eta - \xi|^{p-2} dt, \quad (3.5)\]

\[|\Phi_p(\eta, \xi)| = (p - 1)|\eta| \int_0^1 |\eta - \xi|^{p-2} dt. \quad (3.6)\]

Since \(|\eta - \xi| = |t(\eta - \xi) - (1 - t)\xi| \leq |\eta - \xi| + |\xi|\) for all \(t \in [0, 1]\), equation (3.5) yields (3.1) if \(1 < p \leq 2\), while (3.6) yields (3.2) if \(p \geq 2\).
Putting $t_0 = |\xi|/(|\eta - \xi| + |\xi|) \in (0, 1]$, we have

$$|t\eta - \xi| \geq |t| |\eta - \xi| - (1 - t)|\xi| = (|\eta - \xi| + |\xi|)|t - t_0|.$$

If $p > 2$ (resp. $1 < p < 2$), then for every $t_0 \in (0, 1]$ we have that $\int_{t_0}^1 |t - t_0|^{p-2} \, dt \geq (\text{resp. } \leq) 2 \int_0^{1/2} \varepsilon^{p-2} \, dz \equiv 2^{p-2} / (p - 1)$, thus (3.5) (resp. (3.6)) yields (3.1) (resp. (3.2)).

Since $\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi) = \Phi_p(\eta - \eta', \xi - \xi')$, (3.3) and (3.4) follow from (3.1) and (3.2), respectively. \qed

Let $\Lambda$ be a positive constant. Take $x_0 \in \Omega$, $\delta \in (0, 1)$ and $\varepsilon \in (0, 1)$ such that $B = B(x_0, \varepsilon^{1/p}) \subset \Omega$. Consider the boundary value problem

$$\begin{cases}
-\varepsilon \text{div} \Phi_p(\nabla w, \nabla a) + \Lambda |w|^{\sigma-1} w = 0 & \text{in } B, \\
w = \delta & \text{on } \partial B.
\end{cases}$$

(3.7)

For Propositions 3.1 and 3.2 below, we assume only $a \in W^{1,p}(B)$ without (A1), (A2) and (A3).

**Proposition 3.1.** Let $g$ be a non-decreasing function, and suppose that $u, v \in W^{1,p}(B) \cap L^\sigma(B)$, where $\sigma \in [1, \infty]$, satisfy $g(u), g(v) \in L^{\sigma^*}(B)$, where $\sigma^* = \frac{\sigma}{\sigma - 1}$ ($\sigma^* = \infty$ if $\sigma = 1$ and $\sigma^* = 1$ if $\sigma = \infty$), and

$$\begin{cases}
-\text{div} \Phi_p(\nabla u, \nabla a) + g(u) \leq -\text{div} \Phi_p(\nabla v, \nabla a) + g(v) & \text{in } B, \\
u \leq v & \text{on } \partial B.
\end{cases}$$

Then, $u \leq v$ a.e. in $B$.

**Proof.** Using $(u - v)^+ \in W^{1,p}_0(B) \cap L^\sigma(B)$ as a test function, we get

$$\int_D (\Phi_p(\nabla u, \nabla a) - \Phi_p(\nabla v, \nabla a)) \cdot (\nabla u - \nabla v) \, dx \leq -\int_D (g(u) - g(v))(u - v) \, dx \leq 0,$$

where $D = \{x \in B : u(x) > v(x)\}$. On the other hand, the integrand of the left-hand side is non-negative because of (3.5). Thus, we conclude $\nabla u = \nabla v$ a.e. in $D$, and hence $\nabla (u - v)^+ = 0$ a.e. in $B$, which means $(u - v)^+ = 0$ a.e. in $B$. Therefore, $u \leq v$ a.e. in $B$. \qed

**Proposition 3.2.** For any $\varepsilon > 0$, there exists a unique solution $w \in W^{1,p}(B) \cap L^\infty(B)$ of (3.7). Moreover, $0 \leq w \leq \delta$ a.e. in $B$.

**Proof.** We set the $C^1$-energy functional $J$ corresponding to (3.7) as

$$J(u) = \frac{\varepsilon}{p} \int_B |\nabla u - \nabla a|^p \, dx + \varepsilon \int_B \nabla_p a \cdot \nabla u \, dx + \Lambda \int_B |u|^{1+\theta} \, dx,$$
which is defined in

\[ K = \{ u \in W^{1,p}(B) \cap L^{1+\theta}(B) : u - \delta \in W^{1,p}_0(B) \}. \]

Since

\[ |\nabla u| \leq |\nabla a|^{p-1}|\nabla u - \nabla a| + |\nabla a|^p \leq \frac{1}{2p} |\nabla u - \nabla a|^p + C|\nabla a|^p, \]

we have

\[ J(u) \geq \frac{\varepsilon}{2p} \int_B |\nabla u - \nabla a|^p \, dx + \Lambda \int_B |u|^{1+\theta} \, dx - CE \int_B |\nabla a|^p \, dx. \] (3.8)

Then we see that \( J \) is bounded from below and \( J_0 = \inf_{w \in K} J(u) \) exists. It suffices to show that there exists \( w \in K \) such that \( J(w) = J_0 \).

Let \( \{ u_n \} \) be a minimizing sequence such that \( u_n \in K \) and \( J(u_n) \to J_0 \) as \( n \to \infty \). Then, by (3.8) we obtain

\[ \int_B |\nabla u_n - \nabla a|^p \, dx, \quad \int_B |u_n|^{1+\theta} \, dx \leq C, \]

so that \( \{ u_n - \delta \} \) and \( \{ u_n \} \) are bounded in the reflexive Banach spaces \( W^{1,p}_0(B) \) and \( L^{1+\theta}(B) \), respectively. Thus, we can choose a subsequence, which is denoted \( u_n \) again, and \( w \in K \) such that \( u_n \to w \) weakly in \( W^{1,p}(B) \) and weakly in \( L^{1+\theta}(B) \). Thus,

\[ \liminf_{n \to \infty} ||u_n - a||_{W^{1,p}(B)} \geq ||w - a||_{W^{1,p}(B)}, \] (3.9)

\[ \lim_{n \to \infty} \int_B \nabla a \cdot \nabla u_n \, dx = \int_B \nabla a \cdot \nabla w \, dx, \] (3.10)

\[ \liminf_{n \to \infty} ||u_n||_{L^{1+\theta}(B)} \geq ||w||_{L^{1+\theta}(B)}. \] (3.11)

Since \( u_n \to w \) strongly in \( L^p(B) \) by the Poincaré inequality, it follows from (3.9) that

\[ \liminf_{n \to \infty} ||\nabla (u_n - a)||_{L^p(B)} \geq ||\nabla (w - a)||_{L^p(B)}. \] (3.12)

Therefore, by (3.10), (3.11) and (3.12), we conclude that \( J_0 = \liminf_{n \to \infty} J(u_n) \geq J(w) \geq J_0 \), so that \( J(w) = J_0 \). The uniqueness and the boundedness of solutions follow from Proposition 3.1 with \( g(s) = |s|^\theta-1 s \) and \( \sigma = 1 + \theta \).

To show that the solution \( w \) of (3.7) has a dead core for any \( \varepsilon > 0 \), scaling is useful: setting \( y = \varepsilon^{-1/p} (x - x_0), \tilde{w}(y) = \tilde{w}(y; \varepsilon, x_0) = w(x + \varepsilon^1/p y) \) and \( \tilde{a}(y) = \tilde{a}(y; \varepsilon, x_0) = a(x_0 + \varepsilon^{1/p} y) \) in (3.7), we have

\[
\begin{cases}
- \text{div} \Phi_p(\nabla \tilde{w}, \nabla \tilde{a}) + \Lambda \tilde{w}^\theta = 0 & \text{in } B(0, 1), \\
\tilde{w} = \delta & \text{on } \partial B(0, 1).
\end{cases}
\] (3.13)

We shall write \( B_\rho \) to represent \( B(0, \rho) \).
Lemma 3.2. Let $a(x)$ satisfy (A2), and assume $\tilde{w}$ to be the unique solution of (3.13). Then $\tilde{w} \in C^{1,\alpha}(\overline{B_1})$ for some $\alpha \in (0, 1)$ and $\|\nabla(\tilde{w} - \tilde{a})\|_{L^\infty(B_1)} \leq C$, where $C$ is independent of $\varepsilon$, $\delta$ and $x_0$.

Proof. Setting $v(y) = \tilde{w}(y) - \tilde{a}(y)$, we have

$$
\begin{cases}
-\Delta_p v + \Lambda(v + \tilde{a})^\theta = 0 & \text{in } B_1, \\
v = \delta + \tilde{a} & \text{on } \partial B_1.
\end{cases}
$$

Since $\|v + \tilde{a}\|_{L^\infty(B_1)} \leq \delta \leq 1$ by Proposition 3.1 and $\delta + \tilde{a} |_{\partial B_1} \in C^{1,\alpha}(\partial B_1)$ with $\|\delta + \tilde{a}\|_{C^{1,\alpha}(\partial B_1)} \leq 1 + \|\tilde{a}\|_{C^{1,\alpha}(\overline{B_1})}$ (for the norm of $C^{1,\alpha}(\partial B_1)$, see Gilbarg and Trudinger [8, Section 6.2]), it follows from a regularity result of Lieberman [14] that $v \in C^{1,\alpha}(\overline{B_1})$ and $\|v\|_{C^{1,\alpha}(\overline{B_1})} \leq C$ for some $\alpha \in (0, 1)$ and $C > 0$ are independent of $\varepsilon$, $\delta$ and $x_0$. In particular, $\|\nabla v\|_{L^\infty(B_1)} \leq C$. \hfill \Box

Proposition 3.3. Let $a(x)$ satisfy (A2) and (A3), and assume $w$ to be the unique solution of (3.13). If $0 < \theta < 1$, then there exists $M > 0$ independent of $\varepsilon$, $\delta$ and $x_0$ such that $w(x) = 0$ for all $x \in B(x_0, (1 - M\delta^{1+\theta}\gamma)^{1/\tau}e^{1/p})$, where

$$
\gamma = \frac{\frac{1}{1+\theta} - \frac{1}{2}}{N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 1} \in \left(0, \frac{1}{N+2}\right),
$$

$$
\tau = 2N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 2 \in (2, N+2).
$$

In particular, $w(x_0) = 0$ for arbitrary $\varepsilon > 0$ if $\delta^{1+\theta}\gamma < M^{-1}$.

Proof. It is sufficient to prove the existence of dead core for the solution of (3.13). To do this, we follow the energy method developed by Díaz and Véron [5] (see also Díaz [3], and Antontsev, Díaz and Shmarev [1]).

We define the diffusion and absorption energy functions $E_D(\rho)$ and $E_A(\rho)$ in $(0, 1)$ as follows:

$$
E_D(\rho) = \int_{\overline{B_\rho}} \Phi_p(\nabla \tilde{w}(y), \nabla \tilde{a}(y)) \cdot \nabla \tilde{w}(y) \, dy,
$$

$$
E_A(\rho) = \int_{\overline{B_\rho}} |\tilde{w}(y)|^{1+\theta} \, dy.
$$

The total energy function $E_T(\rho)$ is defined as

$$
E_T(\rho) = E_D(\rho) + \Lambda E_A(\rho).
$$
The global total energy $E_T(1)$ is finite. Indeed, (we write $w$, $a$ instead of $\tilde{w}$, $\tilde{a}$, respectively), multiplying the equation of (3.13) by the nonnegative function $\delta - w \in W^1_0(B_1)$ and integrating by parts in $B_1$, we have

$$E_T(1) \leq \Lambda \delta^{1+\theta} |B_1| \leq C \delta^{1+\theta}. \quad (3.14)$$

Multiplying the equation of (3.13) by $w$ and integrating by parts in $B_\rho$, we have also (now we shall write $S_\rho$ to represent $\partial B_\rho$)

$$E_T(\rho) = \int_{S_\rho} \Phi_p(\nabla w(y), \nabla a(y)) \cdot n w(y) \, ds, \quad (3.15)$$

where $n = n(s)$ is the outward normal vector at $y \in S_\rho$. By (3.15), Lemmas 3.1 and 3.2 with (A3)

$$E_T(\rho) = \int_{S_\rho} |\Phi_p(\nabla w, \nabla a)| |w| \, ds$$

$$\leq \left( \int_{S_\rho} |\Phi_p(\nabla w, \nabla a)|^2 \, ds \right)^{1/2} \left( \int_{S_\rho} |w|^2 \, ds \right)^{1/2}$$

$$\leq \left( \int_{S_\rho} (|\nabla w - \nabla a| + |\nabla a|)^{2(p-2)}(\Phi_p(\nabla w, \nabla a) \cdot \nabla w) \, ds \right)^{1/2} \|w\|_{L^2(S_\rho)}$$

$$\leq C \left( \int_{S_\rho} \Phi_p(\nabla w, \nabla a) \cdot \nabla w \, ds \right)^{1/2} \|w\|_{L^2(S_\rho)}. \quad (3.16)$$

On the other hand, by using spherical coordinates $(\omega, r)$ with center $x_0$, we have

$$E_D(\rho) = \int_0^\rho \int_{S_{N-1}} \Phi_p(\nabla w(r\omega), \nabla a(r\omega)) \cdot \nabla w(r\omega) r^{N-1} \, d\omega \, dr.$$}

Hence, $E_D$ is almost everywhere differentiable and

$$\frac{dE_D(\rho)}{d\rho} = \int_{S_{N-1}} \Phi_p(\nabla w(\rho\omega), \nabla a(\rho\omega)) \cdot \nabla w(\rho\omega) \rho^{N-1} \, d\omega$$

$$= \int_{S_\rho} \Phi_p(\nabla w, \nabla a) \cdot \nabla w \, ds. \quad (3.17)$$

Similarly,

$$\frac{dE_A(\rho)}{d\rho} = \int_{S_\rho} |w|^{1+\theta} \, ds. \quad (3.18)$$
Moreover, since $0 < \theta < 1$, we have the following inequality (see Díaz et al. [5, 3, 1]):

$$
\|w\|_{L^2(S, \rho)} \leq C \left( \|\nabla w\|_{L^2(B, \rho)} + \rho^{-\alpha} \|w\|_{L^{1+\theta}(B, \rho)} \right)^{\frac{1}{\beta}} \|w\|_{L^{1+\theta}(B, \rho)}^{1-\frac{1}{\beta}},
$$

where $C = C(N, \theta)$ and

$\alpha = \frac{N(1-\theta) + 2(1+\theta)}{2(1+\theta)} = \frac{N}{1+\theta} - \frac{1}{2} + 1 \in \left(1, \frac{N}{2} + 1\right) \subset (1, \infty),$

$\beta = \frac{N(1-\theta) + 1 + \theta}{N(1-\theta) + 2(1+\theta)} = \frac{N}{1+\theta} - \frac{1}{2} + 1 \in \left(\frac{1}{2}, \frac{N + 1}{N + 2}\right) \subset (0, 1).$

Thus, from (3.1) and Lemma [3.2], we obtain $E_D(\rho) \geq C \|\nabla w\|_{L^2(B, \rho)}^2$, so that

$$
\|w\|_{L^2(S, \rho)}^{1/\beta} \leq C \left( \|\nabla w\|_{L^2(B, \rho)} + \rho^{-\alpha} \|w\|_{L^{1+\theta}(B, \rho)} \right) \|w\|_{L^{1+\theta}(B, \rho)}^{1-1/\beta} \\
= C \left( \|\nabla w\|_{L^2(B, \rho)} \|w\|_{L^{1+\theta}(B, \rho)}^{1/\beta} + \rho^{-\alpha} \|w\|_{L^{1+\theta}(B, \rho)}^{1/\beta} \right) \\
\leq C \rho^{-\alpha} \left( \rho^\theta E_D(\rho)^{\frac{1}{2}} E_A(\rho)^{\frac{1}{2+\theta}} + E_A(\rho)^{\frac{1}{2+\theta}} \right) \\
\leq C \rho^{-\alpha} \left( E_T(\rho)^{\frac{1}{2+\theta}} + E_A(1)^{\frac{1}{2+\theta}} E_A(\rho)^{\frac{1}{2+\theta}} \right) \\
\leq C \rho^{-\alpha} E_T(\rho)^{\frac{1}{2+\theta}}. \quad (3.19)
$$

Here we have used that $E_A(1) \leq C \delta^{1+\theta} < C$ and $0 < \theta < 1$. Combining (3.16)–(3.18) and (3.19), we obtain

$$
E_T(\rho) \leq C \left( \frac{dE_T(\rho)}{d\rho} \right)^{1/2} \rho^{-\alpha} E_T(\rho)^{\frac{1}{2} + \frac{1}{2+\theta}},
$$

that is,

$$
\frac{dE_T(\rho)}{d\rho} \geq C \rho^{\gamma - 1} E_T(\rho)^{1-\gamma},
$$

where

$\gamma = 2(1 - \beta) \left( \frac{1}{1+\theta} - \frac{1}{2} \right) = \frac{\frac{1}{1+\theta} - \frac{1}{2}}{N \left( \frac{1}{1+\theta} - \frac{1}{2} \right) + 1} \in \left(0, \frac{1}{N+2}\right),$

$\tau = 1 + 2\alpha \beta = 2N \left( \frac{1}{1+\theta} - \frac{1}{2} \right) + 2 \in (2, N+2).$
Integrating it on \([\rho, 1]\) and using (3.14), we have

\[
E_T(\rho)^\gamma \leq E_T(1)^\gamma - C(1 - \rho^\gamma) \leq C(\rho^\gamma - (1 - M\delta^{(1+\theta)\gamma}))
\]

for some \(M > 0\), thus \(E_T((1 - M\delta^{(1+\theta)\gamma})^{1/\gamma}) = 0\), i.e., \(\tilde{w}(y) = 0\) for all \(y \in B(0,(1 - M\delta^{(1+\theta)\gamma})^{1/\gamma})\). Scaling back to \(x\), we conclude the assertion. \(\square\)

### 4 Proofs of Theorems

Now we are in a position to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Fix \(\delta \in (0, d)\) such that \(M\delta^{(1+\theta)\gamma} < 1\), where \(M\) and \(\gamma\) are the constants appearing in Proposition 3.3. Thanks to the \(p\)-harmonicity of \(a(x)\), the function \(v = a - u_e\) satisfies that \(-\varepsilon \text{ div } \Phi_p(\nabla v, \nabla a) = -(a(x) - v)^{\theta - 1}f(v)\) in the distribution sense in \(\Omega\). Since

\[
(a(x) - s)^{\theta - 1}f(s) \geq d^{\theta - 1}Cs^\theta =: \Lambda_1s^\theta \quad \text{for all } x \in \Omega \text{ and } s \in (0, \delta]
\]

and by Proposition 2.1, \(\max_{x \in \Omega_{\varepsilon,1/p}} v(x) \leq \delta\) for every \(\varepsilon \in (0, \varepsilon_0)\), we have

\[
-\varepsilon \text{ div } \Phi_p(\nabla v, \nabla a) + \Lambda_1v^\theta \leq 0 \quad \text{in } \Omega_{\varepsilon,1/p}.
\]  

(4.1)

Let \(\varepsilon_0 \in (0, \varepsilon_0)\) be small such that \(\Omega_{(K+1)\varepsilon_0,1/p} \neq \emptyset\). Take any \(\varepsilon \in (0, \varepsilon_0)\) and \(x_0 \in \Omega_{(K+1)\varepsilon_0,1/p}\). Letting \(w\) be the solution of (3.7), we can see

\[
\begin{cases}
-\varepsilon \text{ div } \Phi_p(\nabla w, \nabla a) + \Lambda_1w^\theta = 0 & \text{in } B(x_0, \varepsilon_0,1/p), \\
w = \delta & \text{on } \partial B(x_0, \varepsilon_0,1/p).
\end{cases}
\]  

(4.2)

Since \(B(x_0, \varepsilon_0,1/p) \subset \Omega_{\varepsilon,1/p}\) and \(v \leq \delta = w\) on \(\partial B(x_0, \varepsilon_0,1/p)\), it follows from (4.1) and (4.2) that \(v\) is a subsolution of (4.2). Therefore, Proposition 3.1 gives \(v \leq w\) in \(B(x_0, \varepsilon_0,1/p)\). Proposition 3.3 implies that \(0 \leq v(x_0) \leq w(x_0) = 0\), and hence \(u(x_0) = a(x_0)\) for all \(x_0 \in \Omega_{(K+1)\varepsilon_0,1/p}\). This completes the proof of Theorem 1.1. \(\square\)

**Proof of Theorem 1.2.** Let \(u_e\) be a solution of (1.1). The function \(v = a - u_e \geq 0, \neq 0\), satisfies

\[
-\varepsilon \text{ div } \Phi_p(\nabla v, \nabla a) + \Lambda_2v^\theta \geq 0
\]

for some \(\Lambda_2 > 0\). Since \(u_e \in C^1(\overline{\Omega})\) by the regularity result of Lieberman [14], so is \(v\), and there exists \(k > 0\) such that \(\|\nabla v\|_{L^\infty(\Omega)} \leq k\). We define

\[
M_{p,k} = \sup_{|\eta| \leq k, x \in \Omega} (|\eta - \nabla a(x)| + |\nabla a(x)|)^{p-2},
\]

\[
m_{p,k} = \inf_{|\eta| \leq k, x \in \Omega} (|\eta - \nabla a(x)| + |\nabla a(x)|)^{p-2},
\]

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which are both finite and positive for any $p > 1$ because of (A3). Then, $v$ is also a nonnegative bounded function satisfying

$$-\varepsilon \div \Phi_p(\nabla v, \nabla a) + \Lambda_2 |v|^{\theta-1} v \geq 0,$$

where $\Phi_p(\eta, \nabla a)$ is a vector measurable function as

$$\Phi_p(\eta, \nabla a) = \begin{cases} \Phi_p(\eta, \nabla a) & \text{if } |\eta| \leq k, \\ M_{p,k} \eta & \text{if } |\eta| > k, \end{cases}$$

which satisfies (from (3.2) and (3.1) in Lemma 3.1)

$$|\Phi_p(\eta, \nabla a(x))| \leq M_{p,k} \max\{p-1, 2^{2-p}\} |\eta|,$$

$$\Phi_p(\eta, \nabla a(x)) \cdot \eta \geq m_{p,k} \min\{p-1, 2^{2-p}\} |\eta|^2.$$

Moreover, if $\theta \geq 1$, then there exists $C > 0$ such that $|s|^{\theta-1} |s| \leq C |s|$ if $|s| \leq \|v\|_{L^\infty(\Omega)}$. Thus, the weak Harnack inequality by Trudinger [20, Theorem 1.2] (see also Pucci and Serrin [15, Theorem 7.1.2]) follows: for any $\overline{B}(x_0, 4\rho) \subset \Omega$ and $\gamma \in (0, \frac{N}{N-2})$ ($\gamma \in (0, \infty)$ if $N = 2$), there exists $C = C(N, \gamma, \Lambda_2/\varepsilon, \rho, p, k, M_{p,k}, m_{p,k})$ such that

$$\rho^{-\frac{N}{\gamma}} \|v\|_{L^\gamma(\overline{B}(x_0, 2\rho))} \leq C \inf_{x \in \overline{B}(x_0, 2\rho)} v(x). \quad (4.3)$$

Suppose $v(x_0) = 0$ with $x_0 \in \Omega$. Then the set $O = \{x \in \Omega : v(x) = 0\}$, which is closed relatively to $\Omega$ since $v$ is continuous, is nonempty. Since $v$ is continuous, if $x \in O$ and $\overline{B}(x, 4\delta) \subset \Omega$, then $\inf_{\overline{B}(x, 2\rho)} v = v(x) = 0$. From (4.3) we have that $\|v\|_{L^\gamma(\overline{B}(x, 2\rho))} = 0$ so that $v \equiv 0$ in $\overline{B}(x, 2\rho)$. So $O$ is also open and since $\Omega$ is connected it must be $O = \Omega$, i.e., $v \equiv 0$ in $\Omega$, which is a contradiction. Therefore, $v$ is strictly positive in $\Omega$, i.e., $u_\varepsilon < a$ in $\Omega$. \qed

5 Degenerate case

In this section, we consider the case where $a(x)$ is constant in $\Omega$. As introduced in Section 1, this case has been already treated by several papers [9, 10, 11, 12, 13]. Our approach can be applied to the case.

Since $\nabla a \equiv 0$ in this case, we note $\Phi_p(\nabla w, \nabla a) = \nabla_p w$ and Propositions 3.1, 3.2 and Lemma 3.2 are all satisfied. However, Proposition 3.3 has to be changed as follows.

**Proposition 3.3.** Let $a(x)$ be a constant in $\Omega$, and assume $w$ to be the unique solution of (3.7). If $0 < \theta < p - 1$, then there exists $M > 0$ independent of $\varepsilon$, $\delta$ and $x_0$ such that
\( w(x) = 0 \) for all \( x \in B(x_0, (1 - M\delta^{1+\theta}r^{1/r})^{1/p}) \), where

\[
\gamma = \frac{\frac{1}{1+\theta} - \frac{1}{p}}{N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + 1} \in \left(0, \frac{1}{N + p^*}\right),
\]

\[
\tau = Np^*\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + p^* \in (p^*, N + p^*),
\]

where \( p^* = \frac{p}{p-1} \). In particular, \( w(x_0) = 0 \) for arbitrary \( \varepsilon > 0 \) if \( \delta^{1+\theta}r < M^{-1} \).

**Proof.** It is sufficient to prove the existence of dead core of solution of (3.13). We define the diffusion and absorption energy functions \( E_D(\rho) \) and \( E_A(\rho) \) in \((0, 1)\) as follows:

\[
E_D(\rho) = \int_{B_\rho} |\nabla \tilde{w}(y)|^p \, dy,
\]

\[
E_A(\rho) = \int_{B_\rho} |\tilde{w}(y)|^{1+p} \, dy.
\]

The total energy function \( E_T(\rho) \) is defined as

\[
E_T(\rho) = E_D(\rho) + \Lambda E_A(\rho).
\]

The global total energy \( E_T(1) \) is finite. Indeed, (we write \( w \) instead of \( \tilde{w} \)), multiplying the equation of (3.13) by the nonnegative function \( \delta - w \in W^{1,p}_0(B_1) \) and integrating by parts in \( B_1 \), we have

\[
E_T(1) \leq \Lambda \delta^{1+\theta} |B_1| \leq C \delta^{1+\theta}.
\]

Multiplying the equation of (3.13) by \( w \) and integrating by parts in \( B_\rho \), we have also (now we shall write \( S_\rho \) to represent \( \partial B_\rho \))

\[
E_T(\rho) = \int_{S_\rho} \nabla w(y) \cdot n \, w(y) \, ds,
\]

where \( n = n(s) \) is the outward normal vector at \( y \in S_\rho \). By (5.2)

\[
E_T(\rho) = \int_{S_\rho} |\nabla w| |w| \, ds \leq ||\nabla w||_{L^p(S_\rho)} ||w||_{L^p(S_\rho)}.
\]

On the other hand, by using spherical coordinates \((\omega, r)\) with center \( x_0 \), we have

\[
E_D(\rho) = \int_0^\rho \int_{S^N} |\nabla w(r\omega)|^p \, r^{N-1} \, d\omega \, dr.
\]
Hence, $E_D$ is almost everywhere differentiable and
\[
\frac{dE_D(\rho)}{d\rho} = \int_{S^{N-1}} |\nabla w(r\omega)|^p r^{N-1} d\omega = \int_{S^N} |\nabla w|^p d\sigma. \quad (5.4)
\]

Similarly,
\[
\frac{dE_A(\rho)}{d\rho} = \int_{S^N} |w|^{1+\theta} d\sigma. \quad (5.5)
\]

Moreover, since $0 < \theta < p - 1$, we have the following inequality (see Díaz et al. [5, 3, 1]):
\[
\|w\|_{L^p(S^N)} \leq C \left(\|\nabla w\|_{L^p(B_1)} + \rho^{-\alpha}\|w\|_{L^{1+\eta}(B_1)}\right)^\beta \|w\|_{L^{1+\eta}(B_1)},
\]
where $C = C(N, \theta)$ and
\[
\alpha = \frac{N(p - 1 - \theta) + p(1 + \theta)}{p(1 + \theta)} = N\left(\frac{1}{1 + \theta} - \frac{1}{p}\right) + 1 \in \left(1, \frac{N}{p^*} + 1\right) \subset (1, \infty),
\]
\[
\beta = \frac{N(p - 1 - \theta) + 1 + \theta}{N(p - 1 - \theta) + p(1 + \theta)} = N\left(\frac{1}{1 + \theta} - \frac{1}{p}\right) + 1 \in \left(0, \frac{1}{p} - \frac{N + \frac{1}{p}}{N + p^*}\right) \subset (0, 1).
\]

Thus,
\[
\|w\|_{L^p(S^N)}^{1/\beta} \leq C \left(\|\nabla w\|_{L^p(B_1)} + \rho^{-\alpha}\|w\|_{L^{1+\eta}(B_1)}\right)^{\frac{1}{\beta}} \|w\|_{L^{1+\eta}(B_1)}^{\frac{1-\beta}{\beta}}
\]
\[
= C \left(\|\nabla w\|_{L^p(B_1)}\|w\|_{L^{1+\eta}(B_1)}^{\frac{1-\beta}{\beta}} + \rho^{-\alpha}\|w\|_{L^{1+\eta}(B_1)}^{\frac{1-\beta}{\beta}}\right)
\]
\[
= C\rho^{-\alpha} \left(E_D(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}} + E_A(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}}\right)
\]
\[
\leq C\rho^{-\alpha} \left(E_T(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}} + E_A(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}}\right)
\]
\[
\leq C\rho^{-\alpha} E_T(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}}. \quad (5.6)
\]

Here we have used that $E_A(1) \leq C\delta^{1+\theta} < C$ and $0 < \theta < p - 1$. Combining (5.3)–(5.5) and (5.6), we obtain
\[
E_T(\rho) \leq C \left(\frac{dE_T(\rho)}{d\rho}\right)^{(p-1)/p} \rho^{-\alpha \beta} E_T(\rho)^{\frac{1}{\beta} + \frac{1-\beta}{\beta + \frac{1}{p}}},
\]
that is,
\[
\frac{dE_T(\rho)}{d\rho} \geq C\rho^{\gamma - 1} E_T(\rho)^{1-\gamma},
\]
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where
\[ \gamma = p^*(1 - \beta) \left( \frac{1}{1 + \theta} - \frac{1}{p} \right) = \frac{1}{1 + \theta} - \frac{1}{p} \in \left( 0, \frac{1}{N + p^*} \right), \]
\[ \tau = 1 + p^* \alpha \beta = N p^* \left( \frac{1}{1 + \theta} - \frac{1}{p} \right) + p^* \in (p^*, N + p^*). \]

Integrating it on \([\rho, 1]\) and using (5.1), we have
\[ E_T(\rho)^\gamma \leq E_T(1)^\gamma - C(1 - \rho^\gamma) \leq C(\rho^\gamma - (1 - M \delta^{(1+\theta)\gamma})) \]
for some \(M > 0\), thus \(E_T((1 - M \delta^{(1+\theta)\gamma})^{1/\tau}) = 0\), i.e., \(\tilde{w}(y) = 0\) for all \(y \in B(0, (1 - M \delta^{(1+\theta)\gamma})^{1/\tau}).\) Scaling back to \(x\), we conclude the assertion.

As in Section 4, we obtain the corresponding Theorems 5.1 and 5.2 below to Theorems 1.1 and 1.2, respectively, in the case when \(a(x)\) is constant. For the proof of Theorem 5.2, we have only to use the weak Harnack inequality directly to \(-\varepsilon \Delta u + \Lambda_2 y^\beta \geq 0\) with \(0 < \theta < p - 1\). We note again that these have been already obtained by [12].

**Theorem 5.1.** Assume \(a(x)\) to be a positive constant. Let \(0 < \theta < p - 1\). Then, there exist \(L > 0\) and \(\varepsilon_0 \in (0, \varepsilon_a)\) such that for each \(\varepsilon \in (0, \varepsilon_0)\) the solution \(u_\varepsilon\) of (1.1) satisfies
\[ u_\varepsilon(x) = a(x) \quad \text{if dist}(x, \partial \Omega) \geq L \varepsilon^{1/p}. \]

**Theorem 5.2.** Assume \(a(x)\) to be a positive constant. Let \(\theta \geq p - 1\). Then, for every \(\varepsilon \in (0, \varepsilon_a)\), \(u_\varepsilon < a\) in \(\Omega\), and hence \(O_\varepsilon = \emptyset\).

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