The extrinsic holonomy Lie algebra of a parallel submanifold

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Abstract

We investigate parallel submanifolds of a Riemannian symmetric space \( N \). The special case of a symmetric submanifold has been investigated by many authors before and is well understood. We observe that there is an intrinsic property of the second fundamental form which distinguishes full symmetric submanifolds from arbitrary full parallel submanifolds of \( N \), usually called “1-fullness of \( M \)”. Furthermore, for every parallel submanifold \( M \subset N \) we consider the pullback bundle \( TN|_M \) with its induced connection, which admits a distinguished parallel subbundle \( OM \), usually called the “second osculating bundle of \( M \)”. If \( M \) is a complete parallel submanifold of \( N \), then we can describe the corresponding holonomy Lie algebra of \( OM \) by means of the second fundamental form of \( M \) and the curvature tensor of \( N \) at the origin. If moreover \( N \) is simply connected and \( M \) is even a full symmetric submanifold of \( N \), then we will calculate the holonomy Lie algebra of \( TN|_M \) in an explicit form.

1 Introduction

In this article, \( N \) denotes a Riemannian symmetric space. For an isometric immersion \( f : M \to N \), let \( TM, \perp f, h : TM \times TM \to \perp f \) and \( S : TM \times \perp f \to TM \) denote the tangent bundle of \( M \), the normal bundle of \( f \), the second fundamental form and the shape operator, respectively. Let \( \nabla^M \) and \( \nabla^N \) denote the Levi Civita connection of \( M \) resp. of \( N \) and \( \nabla^\perp \) the usual connection on \( \perp f \) (obtained by projection). The equations of Gauß and Weingarten state for \( X,Y \in \Gamma(TM), \xi \in \Gamma(\perp f) \)

\[
\nabla^N_X Tf Y = Tf(\nabla^M_X Y) + h(X,Y) \quad \text{and} \quad \nabla^N_X \xi = -Tf(S_\xi(X)) + \nabla^\perp_X \xi.
\]

On the vector bundle \( L^2(TM, \perp f) \) there is a connection induced by \( \nabla^M \) and \( \nabla^\perp \) in a natural way, often called “Van der Waerden-Bortolotti connection”.

Definition 1. \( f \) is called parallel if its second fundamental form \( h \) is a parallel section of the vector bundle \( L^2(TM, \perp M) \).

In a similar fashion, we define parallel submanifolds of \( N \) (via the isometric immersion given by the inclusion map \( \iota^M : M \to N \)).

Example 1 (Circles). A unit speed curve \( c : J \to N \) is parallel if and only if it satisfies the equation

\[
\nabla^N_\partial \nabla^N_\partial \dot{c} = -\kappa^2 \dot{c}
\]

for some constant \( \kappa \in \mathbb{R} \). For \( \kappa = 0 \) these curves are geodesics; otherwise, due to Nomizu and Yano in [NY], \( c \) is called an (extrinsic) circle. One can show that for every pair \( (u,v) \in T_pN \times T_pN \) with \( ||u|| = 1 \) there exists a unique solution \( c \) of (2) defined on the whole real line with \( \dot{c}(0) = u, \nabla^N_\partial \dot{c}(0) = v \). It is obtained as the envelopment of some straight line or some circle in \( T_pN \), see also [JR].
So far, a classification of parallel isometric immersions has been achieved only if the ambient space is a rank-1 symmetric space. (see [BCO], Ch. 9.3). Nevertheless, even if $N$ is of higher rank, then the special case of a symmetric submanifold is completely understood by the work of H. Naitoh and others (for an overview on the classification of symmetric submanifolds of symmetric spaces see [BCO], Ch. 9.4).

**Definition 2.** $M$ is called a symmetric submanifold of $N$ if $M$ is a symmetric space (whose geodesic symmetries are denoted by $\sigma_p^M$ ($p \in M$)) and for every point $p \in M$ there exists an involutive isometry $\sigma_p^\perp$ of $N$ such that

- $\sigma_p^\perp(M) = M$,
- $\sigma_p^\perp|_M = \sigma_p^M$,
- and the differential $T_p\sigma_p^\perp$ is the linear reflection in the normal space $\perp_pM$.

Then we also say that $M$ is extrinsically symmetric in $N$. The family $\sigma_p^\perp (p \in M)$ is unique (if it exists) and is called the extrinsic symmetries of $M$.

In fact, symmetric submanifolds of $N$ are parallel, but the converse is not true. So far there seems to be not much known about arbitrary (i.e. not necessarily extrinsically symmetric) parallel submanifolds of an irreducible symmetric space $N$ of higher rank, except for a result of K. Tsukada [Ts1] on parallel Kähler submanifolds of Hermitian symmetric spaces (which, in case $N$ is of higher rank, can be interpreted as a negative result) and the analogue in [ADM] for parallel submanifolds of Kählerian type in a quaternionic-Kähler symmetric space of non-vanishing scalar curvature.

The aim of this article is three-fold:

- First, we will relate the extrinsic symmetry of a full parallel submanifold of $N$ to an intrinsic property of the second fundamental form called “1-fullness of $M$” (see Definition 3 and Theorem 1).
- Second, for every complete parallel submanifold $M \subset N$ we will introduce the extrinsic holonomy Lie algebra of $M$ resp. of its second osculating bundle (see Definition 5), and we will be able to express the latter Lie algebra only in terms of the second fundamental form of $M$ and the curvature tensor of $N$ at the origin (see Theorem 3).
- Third, for the full symmetric submanifolds of the simply connected symmetric spaces we will calculate their extrinsic holonomy Lie algebras in an explicit form (up to certain exceptions, see Theorem 6).

The precise definitions and the statement of the theorems can be found in the next Section.

In a forthcoming paper [J1], the extrinsic homogeneity of (arbitrary) parallel submanifolds in an ambient symmetric space of possibly higher rank will be studied, for which Theorem 3 of this article will serve as a useful tool; moreover, it seems possible that the explicit calculations in the extrinsically symmetric case could also be helpful for the further study of arbitrary parallel submanifolds in symmetric spaces.

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1.1 Overview

This section gives a detailed, self contained overview on the results presented in this article, the necessary notation included. In Section 2 we recall some well known properties of parallel submanifolds, and we consider certain relevant examples. Given an isometric immersion $f : M \to N$, in order to keep our notation as simple as possible, here and in the following we implicitly identify the tangent space $T_pM$ with the “first osculating space” $Tf(T_pM)$ by means of the injective linear map $T_pf$ for each $p \in M$. Then we introduce for each $p \in M$ the first normal space

$$\perp_p^1f := \{h(x,y) \mid x, y \in T_pM\} \subset \mathbb{R},$$

(3)
and the second osculating space 
\[ O_p f := T_pM \oplus \perp_{\perp_p} f, \] 
seen as a linear subspace of \( T_{f(p)}N \). If \( M \subset N \) is actually a (smoothly embedded) submanifold, then the first normal space \( \perp_{\perp_p} M \) and the second osculating space \( O_p M \) are defined as before via the isometric immersion \( \iota : M \hookrightarrow N \).

**Definition 3.** (a) In accordance with [BCO], Ch. 2.5, an isometric immersion \( f : M \to N \) is called **full** if \( f(M) \) is not contained in any proper, totally geodesic submanifold \( \tilde{N} \subset N \).

(b) In accordance with [LS], an isometric immersion \( f : M \to N \) is called 1-**full** if always the first normal space \( \perp_{\perp_p} f \) coincides with the normal space \( \perp_{\perp_p} f \).

Note that there always exists a smallest complete, totally geodesically submanifold \( \tilde{N} \subset N \) which contains \( f(M) \), and then \( \tilde{N} \) is a symmetric space and \( f : M \to \tilde{N} \) is a full isometric immersion. However, 1-fullness is a somehow more intrinsic property of \( f \).

As a consequence of the Gauss Equation, we see that 1-fullness implies fullness, but the converse is not true even for parallel isometric immersions:

**Definition 4.** Let \( R^N \) denote the curvature tensor of \( TN \). A linear subspace \( V \subset T_pN \) is curvature invariant if \( R^N(V, V) V \subset V \).

**Example 2.** There exists a full circle \( c : \mathbb{R} \to \mathbb{C}P^2 \) (see Example [1]) which is not 1-full and whose normal spaces are not curvature invariant.

**Proof.** For \( p := (1 : 0 : 0) \in \mathbb{C}P^2 \) let \( u, v \in T_p\mathbb{C}P^2 \) be two vectors with \( ||u|| = 1 \) and the property that \( \{ u, v \}_{\mathbb{R}} \) is neither a totally real nor a complex linear space. Then there exists a circle \( c : \mathbb{R} \to N \) with the initial conditions
\[ \dot{c}(0) = u \quad \text{and} \quad \nabla^N c(0) = v. \]

Suppose that \( \tilde{N} \subset \mathbb{C}P^2 \) is a totally geodesic submanifold such that \( c(\mathbb{R}) \subset \tilde{N} \). Thus \( T_p\tilde{N} \) is a curvature invariant linear subspace with \( \{ u, v \} \subset T_p\tilde{N} \). Since all curvature invariant linear subspaces of \( \mathbb{C}P^2 \) are either totally real or complex, it follows by construction that \( T_p\tilde{N} = T_p\mathbb{C}P^2 \); thus \( c \) is full. The last statement follows, because the normal spaces of \( c \) are three-dimensional (and hence neither totally real nor complex subspaces).

Section 3 deals with the proof of the following theorem, which can not be found in the literature so far.

**Theorem 1.** (a) The first normal spaces \( \perp_{\perp_p} f \) of a parallel isometric immersion are always curvature invariant.

(b) Let a simply connected symmetric space \( N \) and a submanifold \( M \subset N \) be given. \( M \) is a full symmetric submanifold of \( N \) if and only if \( M \) is a 1-full, complete parallel submanifold of \( N \).

(c) Let \( N \) be a simply connected symmetric space, which has no Euclidian factor (in the sense of the “de Rham decomposition theorem”, see [BCO], p. 290). If \( M \) is a full symmetric submanifold of \( N \), then at each point \( p \in M \) the second fundamental form \( h_p \) is a non-degenerate symmetric bilinear form.

*When I talked about my results at Augsburg, I learned that the result described in Part (a) of Theorem 1 could also be found in an unpublished paper by E. Heintze.*

†According to Theorem 7 of [JR], for every (not necessarily complete) parallel submanifold \( M_{loc} \subset N \) there exists a simply connected Riemannian symmetric space \( M \), a parallel isometric immersion \( f : M \to N \) and an open subset \( U \subset M \), such that \( f|U : U \to M_{loc} \) is covering. Hence, loosely said, all parallel submanifolds can be “extended” to simply connected, complete, immersed parallel submanifolds and therefore the completeness assumption in the above theorem is not too striking.
If \( V \) is a curvature invariant subspace of \( T_pN \), then \( \exp^N(V) \subset N \) (where \( \exp^N \) denotes the exponential spray defined on \( TM \)) is a totally geodesic submanifold by a result due to E. Cartan. The following result on the “reduction of the codimension” (in the sense of \( [Es] \)) is well known (cf. Lemma 2.1 of \( [Ts1] \)); it is in fact a consequence of Theorem 3.4 in \( [L] \) combined with Part (d) of Proposition 5 further below:

**Theorem 2** (Dombrowski). If \( f : M \to N \) is parallel and if at one point \( p \in M \) the second osculating space \( O_pf \) is contained in some curvature invariant subspace \( V \subset T_pN \); then \( f(M) \subset \bar{N} \), where \( \bar{N} \) denotes the totally geodesic submanifold \( \exp_p(V) \subset N \) (which again is a symmetric space).

Combining Theorem \( [L] \) with Theorem \( [2] \) we hence obtain:

**Corollary 1.** If at one point \( p \in M \) the linear space \( O_pM \) of a parallel submanifold is a curvature invariant subspace of \( T_pN \) and if then \( \bar{N} := \exp^N(O_p,M) \) is simply connected, then \( M \) is an extrinsically symmetric submanifold of \( \bar{N} \).

Corollary \( [L] \) should be compared with Lemma 2.2 of \( [Ts1] \). But note that the second osculating spaces of a parallel isometric immersion are not *always* curvature-invariant (see Example \( [2] \)); hence Corollary \( [L] \) is not always applicable.

Parallel submanifolds are sometimes also called “weakly locally symmetric submanifolds” (cf. \( [NT] \)). Theorem \( [2] \) in Section \( [3] \) will give a geometric reason for that notion, as follows:

For every parallel isometric immersion \( f : M \to N \) we introduce the pullback bundle \( f^*TN := \bigcup_{p \in M} \{ p \} \times T_{f(p)}N \) (which is a vector bundle over \( M \)); moreover, \( \nabla^N \) defines a connection on \( f^*TN \). According to Proposition \( [5] \) the *second osculating bundle* \( O_f := \bigcup_{p \in M} \{ p \} \times O_pf \)

is a \( \nabla^N \)-parallel subbundle of \( f^*TN \), hence \( O_f \) is equipped with the connection \( \nabla^{O_f} \) induced by restriction of \( \nabla^N \). If \( f : M \to N \) is a parallel isometric immersion defined on a simply connected symmetric space \( M \) (cf. Proposition \( [4] \)), then we can proof the existence of certain distinguished vector bundle involutions on \( O_f \); in this way, we finally come to the conclusion that \( M \) is “extrinsically symmetric in \( O_f \)” (in a weak sense). However, due to its technical nature the precise statement of Theorem \( [2] \) is skipped at this point of the paper. As a first consequence of Theorem \( [9] \) we will see that \( \perp^1f \) is a *homogeneous vector bundle* over \( M \) (Proposition \( [10] \)).

We now introduce the extrinsic holonomy Lie algebras of a parallel isometric immersion \( f : M \to N \) and of its second osculating bundle with respect to some base point \( o \in M \). For each differentiable curve \( c : [0,1] \to N \) let \( \frac{1}{0} c \) denote the parallel displacement in \( TN \) along \( c \) and consider the Holonomy groups of \( TN \) with respect to \( \nabla^N \) and of \( O_f \) with respect to \( \nabla^{O_f} \) (the connection which was introduced above), respectively:

\[
\text{Hol}(N) := \left\{ (\frac{1}{0} c)^N | c : [0,1] \to N \text{ is a loop with } c(0) = f(o) \right\}, \quad (5)
\]

\[
\text{Hol}(f^*TN) := \left\{ (\frac{1}{0} f \circ c)^N | c : [0,1] \to M \text{ is a loop with } c(0) = o \right\}, \quad (6)
\]

\[
\text{Hol}(O_f) := \left\{ (\frac{1}{0} f \circ c)^{O_f} | c : [0,1] \to M \text{ is a loop with } c(0) = o \right\}. \quad (7)
\]

Then \( \text{Hol}(N) \) and \( \text{Hol}(f^*TN) \) are known to be Lie subgroups of \( \text{SO}(T_{f(o)}N) \), and \( \text{Hol}(O_f) \) is a Lie subgroup of \( \text{SO}(O_o, f) \); the corresponding Lie algebras are denoted by \( \mathfrak{hol}(N) \) resp. by \( \mathfrak{hol}(f^*TN) \). Moreover,
\( \text{Hol}(f^*TN) \subset \text{Hol}(N) \) is a Lie subgroup and hence \( \mathfrak{hol}(f^*TN) \) is a Lie subalgebra of \( \mathfrak{hol}(N) \). If \( M \subset N \) is a parallel submanifold, then we define the pullback bundle \( TN|M \) and the second osculating bundle of \( M \) via the isometric immersion \( f = \iota^M \). Then the Lie groups \( \text{Hol}(TN|M) \) and \( \text{Hol}(OM) \) and their Lie algebras \( \mathfrak{hol}(TN|M) \) and \( \mathfrak{hol}(OM) \) are defined in a similar fashion.

**Definition 5.** (a) We will call \( \mathfrak{hol}(f^*TN) \) resp. \( \mathfrak{hol}(TN|M) \) the **extrinsic holonomy Lie algebra** of the immersion \( f \) resp. of the submanifold \( M \).

(b) \( \mathfrak{hol}(Of) \) resp. \( \mathfrak{hol}(OM) \) will be called the **extrinsic holonomy Lie algebra** of \( Of \) resp. of \( OM \).

**Example 3.** Let \( M \) be a totally geodesic submanifold of \( N \). Since both the vector subbundle \( TM \subset TN|M \) and the curvature tensor \( R^N \) are parallel with respect to \( \nabla^N \), the Theorem of Ambrose/Singer implies that \( \mathfrak{hol}(TN|M) = \{ R^N(x,y) | x,y \in T_pM \} \mathbb{R} \).

**Remark 1.** By means of the Theorem of Ambrose/Singer, a (parallel) isometric immersion \( f : M \to N \) is curvature isotropic (i.e. \( R^N(x,y) = 0 \) for all \( x,y \in T_pM \) and \( p \in M \), cf. [FP] ) if and only if \( \mathfrak{hol}(f^*TN) = \{ 0 \} \). Therefore, briefly said, \( \mathfrak{hol}(f^*TN) \) measures "how far \( f \) is away from being curvature isotropic".

The next theorem describes the general structure of \( \mathfrak{hol}(Of) \) only in terms of the curvature tensor \( R^N \) at \( f(o) \) and the second fundamental form of \( f \) at \( o \). For this we will need the following notation:

For an arbitrary Euclidean vector space \( V \) and some subspace \( W \subset V \) let \( \sigma^\perp \in O(V) \) denote the linear reflection in \( W^\perp \) and \( \text{Ad}(\sigma^\perp) : \mathfrak{so}(V) \to \mathfrak{so}(V), A \mapsto \sigma^\perp \circ A \circ \sigma^\perp \) the induced involution on \( \mathfrak{so}(V) \). Let \( \mathfrak{so}(V)^\perp \) resp. \( \mathfrak{so}(V)^\perp \) be the \(+1\)- resp. \(-1\)-eigenspaces of \( \text{Ad}(\sigma^\perp) \), i.e.

\[
\mathfrak{so}(V)^\perp := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathfrak{so}(W), B \in \mathfrak{so}(W^\perp) \right\},
\]

\[
\mathfrak{so}(V)^\perp := \left\{ \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \middle| C \in L(W,W^\perp) \right\}.
\]

Then the rules for \( \mathbb{Z}/2\mathbb{Z} \) graded Lie algebras hold, i.e.

\[
[\mathfrak{so}(V)^\perp, \mathfrak{so}(V)^\perp] \subset \mathfrak{so}(V)^\perp, \quad \text{and} \quad [\mathfrak{so}(V)^\perp, \mathfrak{so}(V)^\perp] \subset \mathfrak{so}(V)^\perp.
\]

For an isometric immersion \( f : M \to N \) and some \( p \in M \) we will apply this construction with \( W = T_pM \) and \( V = Opf \) resp. \( V = Tf(p)N \); then we obtain the induced splitting

\[
\mathfrak{so}(Tf(p)N) = \mathfrak{so}(Tf(p)N)^\perp \oplus \mathfrak{so}(Tf(p)N)^\perp \quad \text{and} \quad \mathfrak{so}(Opf) = \mathfrak{so}(Opf)^\perp \oplus \mathfrak{so}(Opf)^\perp.
\]

**Definition 6.** For each \( p \in M \) let \( h : T_pM \to \mathfrak{so}(Tf(p)N) \) be the linear map defined by

\[
\forall x, y \in T_pM, \xi \in \perp_pM : h(x)(y + \xi) := h(x,y) - S_\xi x;
\]

note that \( h \) and \( \mathfrak{h} \) are equivalent objects.

In the following, \( \mathfrak{so}(Opf) \) is seen as a Lie subalgebra of \( \mathfrak{so}(Tf(p)N) \) in a natural way:

\[
\mathfrak{so}(Opf) \ni \{ A \in \mathfrak{so}(Tf(p)N) | A(Opf) \subset Opf, A(Opf)^\perp = 0 \} ;
\]

then we have

\[
\mathfrak{so}(Opf)^\perp = \mathfrak{so}(Tf(p)N)^\perp \cap \mathfrak{so}(Opf),
\]

\[
\forall x \in T_pM : h(x) \in \mathfrak{so}(Opf)^\perp.
\]

For a submanifold \( M \subset N \) the linear spaces \( \mathfrak{so}(OpM)^\perp \subset \mathfrak{so}(T_pN)^\perp \) and the linear map \( h : T_pM \to \mathfrak{so}(OpM)^\perp \) are defined by means of \( \iota^M \).
Throughout this paper, we will make use of the following convention: Given two points \( p, q \in M \) and a linear map \( \ell : T_f(p)N \to T_f(q)N \) with \( \ell(O_f) \subset O_qf \) we put
\[
\ell^O := \ell|O_f : O_o \to O_qf.
\] (15)

**Theorem 3.** Let \( f : M \to N \) be a parallel isometric immersion defined on a symmetric space \( M \). The extrinsic holonomy Lie algebra of \( O_f \) is characterized by the following properties:

(a) There is the splitting
\[
\mathfrak{hol}(O_f) = \mathfrak{hol}(O_f)_+ \oplus \mathfrak{hol}(O_f)_-,
\] (16)
with \( \mathfrak{hol}(O_f)_\pm := \mathfrak{hol}(O_f) \cap \mathfrak{so}(O_o f)_\pm \).

(b) We have \( R^N(x,y)O_o f \subset O_o f \) and \( R^N(\xi,\eta)O_o f \subset O_o f \) for all \( x, y \in T_o M, \xi, \eta \in \bot_o f \), and the splitting (16) is given by
\[
\mathfrak{hol}(O_f)_+ = \{(R^N(x,y))O | x, y \in T_o M\}_R + \{(R^N(\xi,\eta))O | \xi, \eta \in \bot_o f\}_R,
\] (17)
\[
\mathfrak{hol}(O_f)_- = \{[h(x), A] | x \in T_o M, A \in \mathfrak{hol}(O_f)_+\}_R.
\] (18)

Furthermore, for all \( x \in T_o M \) we have
\[
[h(x), \mathfrak{hol}(O_f)] \subset \mathfrak{hol}(O_f).
\] (19)

If moreover \( f \) is a full immersion, then \( \mathfrak{hol}(O_f) \cong \mathfrak{hol}(f^*TN) \), more precisely:

(c) We have \( A(O_o f) \subset O_o f \) for all \( A \in \mathfrak{hol}(f^*TN) \), and the linear map \( \mathfrak{hol}(f^*TN) \to \mathfrak{hol}(O_f), A \mapsto A^O \) is a Lie algebra isomorphism.

The proof of Theorem 3 is given in Section 5.

In Section 6, for every full symmetric submanifold \( M \) of some simply connected symmetric spaces \( N \) the extrinsic holonomy Lie algebra is calculated in an explicit way. Because of the following result, thereby it is always enough to consider the case when \( N \) is an irreducible Riemannian space:

**Theorem 4.** [Na] Let \( N \) be a simply connected symmetric space, \( N \cong R^d \times N_1 \times \cdots \times N_k \) its “de Rham decomposition” (see [KN], Ch. IV, Theorem 6.2) and \( M \subset N \) a symmetric submanifold. Then there exist symmetric submanifolds \( M_0 \subset R^d \) and \( M_i \subset N_i \) for \( i \geq 1 \) such that \( M \cong M_0 \times M_1 \times \cdots \times M_k \) (as a submanifold).

In the irreducible case, one knows the following result, which is a consequence of Proposition 9.3.3 combined with Theorem 9.3.4 from [BCO]:

**Theorem 5 (Naitoh).**

If \( N \) is a simply connected, irreducible symmetric space and \( M \subset N \) is a full symmetric submanifold with \( o \in M \), then only the following possibilities can occur:

- \( N \) is a real space form.
- \( N^{2n} \) is a complex space form \((n \geq 2)\) and \( M \) is a complex submanifold.
- \( N^{2n} \) is a complex space form \((n \geq 2)\) and \( M^n \) is a Lagrangian submanifold.
- \( N^{4n} \) is a quaternionic space form \((n \geq 2)\) and \( M^{2n} \) is a totally complex submanifold.
- The rank of \( N \) is larger than 1, \( N \) admits a symmetric R-space (see Definition 7) and \( M \) belongs to the family of symmetric submanifolds associated therewith (in the sense of Definition 8).  

\^As was shown in [Ko], [NT] and [Ts], actually there do not exist any full parallel submanifolds in a complex or quaternionic hyperbolic space.
We will prove:

**Theorem 6.** Let \( N \) be a simply connected, irreducible symmetric space and \( M^m \) be a full symmetric submanifold of \( N \) through \( o \) with \( m \geq 2 \). The extrinsic holonomy Lie algebra of \( M \) is given as follows:

(a) Suppose that \( N \) is a Hermitian symmetric space. Here we have \( \mathfrak{hol}(N) = [\mathfrak{hol}(N), \mathfrak{hol}(N)] \oplus \mathbb{R} j \), where \( j \) denotes the complex structure of \( N \) at \( o \) and \( [\mathfrak{hol}(N), \mathfrak{hol}(N)] \) is the commutator ideal of \( \mathfrak{hol}(N) \). If \( M \) is a Lagrangian submanifold of \( N \), then we have \( \mathfrak{hol}(TN|M) = [\mathfrak{hol}(N), \mathfrak{hol}(N)] \).

(b) Suppose that \( N \) is the quaternionic projective space \( \mathbb{HP}^n \) with \( n \geq 2 \). Here we have \( \mathfrak{hol}(N) = \mathfrak{sp}(T_o N) \oplus \mathcal{Q} \), where \( \mathcal{Q} \subset \mathfrak{so}(T_o N) \) denotes the quaternionic structure at \( o \). For each complex submanifold \( M^2n \subset N \) (in the sense of \( [Ts^2] \)) it is possible to choose a canonical basis \( \{i, j, k\} \) of \( \mathcal{Q} \) (i.e. \( \{i, j, k\} \) is a basis of \( \mathcal{Q} \) such that the usual quaternionic relations \( i^2 = j^2 = k^2 = -1 \), \( i o j = -j o i = k \) hold, see \( [Ts^2] \)). Definition 2.2 such that \( i(T_o M) = T_o M \) and \( j(T_o M) = T_o M \). In this situation, we have \( \mathfrak{hol}(TN|M) = \mathfrak{sp}(T_o N) \oplus \mathbb{R}j \).

(c) Suppose that \( M^2 \) is a submanifold of a 4-dimensional space form (this “exceptional” case actually occurs, see Remark \( \S \) below). Then we have \( \mathfrak{hol}(N) \cong \mathfrak{so}(4) \) and either \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N) \) or \( \dim_{\mathbb{R}}(\mathfrak{hol}(TN|M)) = 4 \) holds.

(d) In all other cases we have \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N) \).

Moreover, if in Case (a) the rank of \( N \) is larger than 1, then there exists some \( x \in T_o M \) with \( h(x) = j \). The proof of Theorem \( \S \) can be found in Section \( \S \). Let us “apply” Theorem \( \S \) to the relevant cases (actually, Theorem \( \S \) gets proved the other way around):

**Example 4.** In the following we assume that \( n, m \geq 2 \).

(a) Full symmetric submanifolds \( M^m \) of the Euclidian sphere \( S^n \) resp. of the hyperbolic space \( \mathbb{H}^n \) where classified in \( [Fl] \) resp. in \( BR \) and \( Ta \). In accordance with Theorem \( \S \), here we have \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N) = \mathfrak{so}(T_o N) \) unless \( (n, m) = (4, 2) \).

(b) Full, complex symmetric submanifolds \( M \subset \mathbb{CP}^n \) where classified in \( [NaTa] \); cf. also Table 9.1 of \( [BCO] \). A prominent example is given by the image of the “Segre embedding” \( f : \mathbb{CP}^m \times \mathbb{CP}^n \to \mathbb{CP}^N \) with \( N + 1 = (m + 1)(n + 1) \), given by

\[
([z_0 : \cdots : z_m], [w_0 : \cdots : w_n]) \mapsto [z_0 w_0 : z_0 w_1 : \cdots : z_m w_n] \text{ (all possible combinations)}.\]

In accordance with Theorem \( \S \), here we always have \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N) = \mathfrak{u}(T_o N) \).

(c) Full, Lagrangian symmetric submanifolds \( M \subset \mathbb{CP}^n \) where classified in \( [N1] \); cf. also Table 9.2 of \( [BCO] \). In accordance with Theorem \( \S \), here we always have \( \mathfrak{hol}(TN|M) = \mathfrak{su}(T_o N) \), which is strictly contained in \( \mathfrak{hol}(N) \).

(d) Full, totally complex symmetric submanifolds \( M^2n \subset \mathbb{HP}^n \) where classified in \( [Ts^2] \); cf. also Table 9.4 of \( [BCO] \).

(e) Let \( N \) be an irreducible Hermitian symmetric space which admits a symmetric \( R \)-space and let \( \{M_c\} \) (where \( c \) ranges over \( \mathbb{R} \)) denote the family of symmetric submanifolds associated therewith, see \( \S \). As a consequence of \( \S \) \( M_c \) is a full submanifold of \( N \) unless \( c = 0 \). Moreover, \( M_c \) is a Lagrangian submanifold of \( N \) for each \( c \in \mathbb{R} \), by virtue of \( \text{Lemma 13} \) therefore, in accordance with \( \text{Theorem 6} \) the extrinsic holonomy Lie algebra \( \mathfrak{hol}(TN|M_c) \) is given by \( [\mathfrak{hol}(N), \mathfrak{hol}(N)] \) for each \( c \neq 0 \).

(f) Let \( N \) be an irreducible symmetric space which is not of Hermitian type and admits a symmetric \( R \)-space. As above, let \( \{M_c\} \) denote the family of symmetric submanifolds associated therewith; then \( \mathfrak{hol}(TN|M_c) = \mathfrak{hol}(N) \) for each \( c \neq 0 \).

\(^5\)In this case, the result is not very satisfying. However, the methods developed in this paper are not suitable to obtain a better result. Possibly, here a “case by case” argument would shed some more light on the subject.

\(^*\)Hence we see that in this case the condition \( h(T_o M) \subset \mathfrak{hol}(TN|M) \) is violated.
Remark 2. In $S^4$ there exists a symmetric submanifold $M^2$ which is isometric to $\mathbb{R}P^2$; it is congruent to a standard embedded symmetric R-space which is a (symmetric) orbit of the isotropy representation of the five dimensional symmetric space $SU(3) / SO(3)$. By the results of [BR] and [T], one knows that there also exist certain full symmetric submanifolds $M^2$ in $S^4$. More precisely, $M$ is an extrinsic product (in the sense of [BR], Definition 4) which is isometric to $\mathbb{R} \times S^1(r)$, $S^1(r) \times S^1(s)$ or $S^2(r)$ (where $r,s > 0$ are arbitrary).

As a conclusion of Theorem 6, we notice that for every full symmetric submanifold $M$ of some simply connected, irreducible symmetric space $N$ the subspace $\mathfrak{hol}(TN|M) \subset \mathfrak{hol}(N)$ is surprisingly large; always its codimension is 0, 1 or 2. Moreover, Parts (c) and (d) of Lemma 11 in Section 6 show that in each case $\mathfrak{hol}(TN|M)$ is “as large as possible” (maybe the last assertion is not true for the exceptional case described in Part (c) of Theorem 6).

2 Symmetric submanifolds

We aim to review the relation between parallel and symmetric submanifolds of a symmetric space $N$. Let $I(N)$ denote the Lie group of isometries on $N$ (see [He], Ch. IV, § 2 and § 3), $I^0(N)$ its connected component and $i(N)$ the corresponding Lie algebra. For each $X \in i(N)$ we have the one-parameter subgroup $\psi^X_t := \exp(t \cdot X)$ of isometries on $N$; the corresponding “fundamental vector field” $X^*$ on $N$ (in the sense of [KN]) defined by

$$X^*(p) := \frac{d}{dt} \bigg|_{t=0} \psi^X_t(p)$$

is a Killing vector field on $N$ such that $\psi^X_t$ ($t \in \mathbb{R}$) is the flow of $X^*$ \textsuperscript{1} The isotropy subgroup of $I^0(N)$ at some fixed origin $o \in N$ is by definition

$$K := \{ g \in I^0(N) \mid g(o) = o \}.$$  

The isotropy representation is given by

$$K \to \text{SO}(T_o N), g \mapsto T_o g.$$  

Let $\mathfrak{k}$ denote the Lie algebra of $K$ and $\pi_2 : \mathfrak{k} \to \mathfrak{so}(T_o N)$ the linearized isotropy representation, i.e.

$$\forall X \in \mathfrak{k}, u \in T_o N : \pi_2(X) = \frac{d}{dt} \bigg|_{t=0} T_o \psi^X_t(u).$$  

Theorem 7 (Strübing-Naitoh-Eschenburg). For a submanifold $M$ of a simply connected symmetric space $N$ the following assertions are equivalent:

(a) $M$ is a symmetric submanifold.

(b) $M$ is a complete parallel submanifold, such that all normal spaces are curvature invariant.

(c) $M$ is a complete, parallel submanifold, at one point $p \in M$ the normal space $\mathfrak{n}_p M$ is curvature invariant, and

$$\mathfrak{h}(T_p M) \subset \pi_2(\mathfrak{k}).$$  

For a proof of (a) $\iff$ (b) see [N3], Corollary 1.4, for the other directions see [E2], Theorem 4. Notice that if $N$ has constant curvature, then every subspace of $T_p N$ is curvature invariant; therefore, as a consequence of Theorem 7 in a space form every complete parallel submanifold is extrinsically symmetric; and the converse is also true:

\textsuperscript{1}By the map $X \mapsto X^*$ the vector space $i(N)$ is identified with the Lie algebra of Killing vector fields on $N$; but one should be aware that $[X^*, Y^*] = -[X, Y]^*$, where the bracket on the l.h.s. is the Lie bracket for vector fields and on the r.h.s. is the bracket of $i(N)$. 

---

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Proposition 1. If every complete parallel submanifold of $N$ is extrinsically symmetric, then $N$ is of constant curvature.

Proof. A particular example for parallel submanifolds are geodesic lines, see Example 1. Suppose that all complete geodesic lines of $N$ would be extrinsically symmetric. Then according to Theorem 7 for each $p \in M$ any linear hyperplane of $T_pN$ is curvature invariant (since it can be realized as the normal space of some geodesic through $p$). By a result due to E. Cartan, $N$ is a space of constant curvature, see [182].

Example 5. (a) Let $\mathbb{R}P^n$ be canonically embedded in $\mathbb{C}P^n$ and let $M$ be a (proper) symmetric submanifold of $\mathbb{R}P^n$. Then $M$ is parallel in $\mathbb{C}P^n$, but not extrinsically symmetric in $\mathbb{C}P^n$.

(b) The circle mentioned in Example 2 is a covering onto a full parallel submanifold $M \subset \mathbb{C}P^2$. Moreover, $M$ is the orbit of a subgroup of $\mathbb{I}(\mathbb{C}P^2)$, but $M$ is not a symmetric submanifold of $\mathbb{C}P^2$.

Proof. For (a): Of course, $M$ is also parallel in $\mathbb{C}P^n$. On the other hand, the normal spaces $\perp_p M (p \in M)$ are neither complex nor totally real subspaces. It is well known that therefore they are not curvature invariant. Thus $M$ is not extrinsically symmetric in $\mathbb{C}P^n$ as a consequence of Theorem 7.

For (b): By a result of [MT], $c$ is the orbit of a one-parameter subgroup of $\mathbb{I}(\mathbb{C}P^2)$ and hence $c$ is a covering onto a full, extrinsically homogenous parallel submanifold of $\mathbb{C}P^2$, which can be not be extrinsically symmetric in $\mathbb{C}P^2$ according to Example 2 in combination with Theorem 7.

2.1 Irreducible symmetric R-spaces

For this section cf. [BCO], Ch. 3.7 and A.4, [BENT] and [EH]. Let $N$ be a simply connected, irreducible symmetric space; hence $N$ is of compact type or of non-compact type. Let $o \in N$ be some origin, $K \subset \mathbb{I}(N)$ the isotropy subgroup, $i(N) = \mathfrak{t} \oplus \mathfrak{p}$ the Cartan decomposition and $B$ the Killing form of $i(N)$. We consider the adjoint representation $\text{Ad} : \mathbb{I}(N) \to \text{Gl}(i(N))$ and its linearization $\text{ad} : i(N) \to \text{gl}(i(N))$.

Let $\epsilon \in \{-1, 1\}$ be chosen such that the restriction of $\epsilon B$ to $\mathfrak{p} \times \mathfrak{p}$ is a positive definite inner product; hence $\epsilon = 1$ if and only if $N$ is of non-compact type. Then $\text{Ad}$ induces a faithful orthogonal representation of $K$ on $\mathfrak{p}$, by restriction; the corresponding infinitesimal action is given by $\text{ad}_{\mathfrak{p}} : \mathfrak{t} \to \mathfrak{so}(\mathfrak{p})$. In this section, we will consider certain $\text{Ad}(K)$-orbits of $\mathfrak{p}$, so called standard embedded irreducible symmetric R-spaces.

As was shown in [171], these objects are the fundamental examples of parallel submanifolds in a Euclidean space; moreover, they also give rise to families of symmetric submanifolds in $N$ as will be explained in the next section.

Remark 3. Let $N^\ast$ denote the dual symmetric space (cf. [BCO], A.4), which again is a simply connected irreducible symmetric space such that $\epsilon^\ast = -\epsilon$. Then $K$ is also the isotropy group of $N^\ast$ and $i(N^\ast) = \mathfrak{t}^\ast \oplus \mathfrak{p}^\ast \cong \mathfrak{t} \oplus \mathfrak{i} \mathfrak{p}$ (seen as a Lie subalgebra of the complexified Lie algebra $i(N) \otimes \mathbb{C}$) is the Cartan decomposition for $N^\ast$.

Lemma 1. For each $X \in \mathfrak{p}$ with $\text{ad}(X)^3 = \epsilon \text{ad}(X)$ we have:

(a) $\text{ad}(iX)^3 = -\epsilon \text{ad}(iX)$ on $\mathfrak{t} \oplus \mathfrak{i} \mathfrak{p}$ (in the sense of Remark 3).

(b) $\text{ad}(X)$ is diagonalizable over $\mathbb{C}$ with $\text{Spec}(\text{ad}(X)) \subset \{-1, 0, 1\}$ (in case $\epsilon = 1$) resp. $\text{Spec}(\text{ad}(X)) \subset \{-i, 0, i\}$ (otherwise). Hence, there exist decompositions

$$\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_c := \text{Kern}(\text{ad}(X)|\mathfrak{t}) \oplus \{ Y \in \mathfrak{t} \mid \text{ad}(X)^2Y = \epsilon Y \} ,$$

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_c := \text{Kern}(\text{ad}(X)|\mathfrak{p}) \oplus \{ Y \in \mathfrak{p} \mid \text{ad}(X)^2Y = \epsilon Y \} .$$

Then (25) is an orthogonal splitting and we have

$$\text{ad}_p(\mathfrak{t}_0) \subset \mathfrak{so}(\mathfrak{p})_+ \quad \text{and} \quad \text{ad}_p(\mathfrak{t}_c) \subset \mathfrak{so}(\mathfrak{p})_- .$$

(c) We have $\text{ad}(X) \mathfrak{t}_c \subset \mathfrak{p}_c$ , $\text{ad}(X) \mathfrak{p}_c \subset \mathfrak{t}_c$ and $J := -\epsilon \text{ad}(X)|\mathfrak{t}_c \oplus \mathfrak{p}_c$, seen as an endomorphism of $\mathfrak{t}_c \oplus \mathfrak{p}_c$, satisfies $J^2 = \epsilon \text{id}$; moreover,

$$J|\mathfrak{p}_c : \mathfrak{p}_c \to \mathfrak{t}_c \quad \text{is a linear isomorphism} .$$
Proof. (a) is obvious. Therefore, by switching between \( N \) and the dual symmetric space \( N^* \), we may assume that \( \epsilon = 1 \) (i.e. \( N \) is of non-compact type); then the other results are in accordance with [BCO]. Example 7.7.

Proposition 2. Let \( N \) be a simply connected symmetric space of non-compact type, suppose that there exists \( X \in \mathfrak{p} \) with \( X \neq 0 \) and \( \text{ad}(X)^3 = \epsilon \text{ad}(X) \) and consider the orbit \( M := \text{Ad}(K)X \). Then we have \( \text{ad}(X) = [\mathfrak{k},X] \) and \( \text{ad}(X)^3 = \epsilon \text{ad}(X) \).

Proof. We have

\[
\forall Y \in \mathfrak{p}_e : h(Y) = \text{ad}(JY)\mathfrak{p} : \mathfrak{p} \to \mathfrak{p} ,
\]

where \( J \) is the linear map from Lemma 1. Moreover, \( M \) is a 1-full symmetric submanifold of \( \mathfrak{p} \) and \( h_o \) is non-degenerate.

\[
\text{ad}(\mathfrak{p}) X = \mathfrak{k}, X \]

Thus \( \text{ad}(\mathfrak{p}) = \mathfrak{k} \) and hence \( \perp \mathfrak{p} = \mathfrak{p}_0 \), since the splitting \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) is an orthogonal sum.

For (28), For each \( Y \in \mathfrak{p}_e \), we have \( JY \in \mathfrak{g} \), according to (27); hence the linear map \( A_Y := \text{ad}(\mathfrak{p})(JY) \) (seen as a linear vector field on \( \mathfrak{p} \)) is tangent to \( M \). We have (by the Gauß equation) for all \( Z \in \mathfrak{p}_e \)

\[
h(Y) Z = \epsilon h(JY^2 Z) = \epsilon h([X,[X,Y]], Z) = -\epsilon h(A_Y X, Z) = -\epsilon (A_Y Z)_{+} = [JY, Z]_{+} \]

therefore (28) follows in view of (10), (26). Now the non-degeneracy of \( h \) follows, since \( J|\mathfrak{p}_e \) in injective and \( \text{ad}(\mathfrak{p}) \) is a faithful representation. Furthermore, it is well known that \( M \) is a symmetric submanifold of \( \mathfrak{p} \), cf [BCO], Proposition 3.7.7 or EH, Theorem 2. To see that \( M \) is 1-full in \( \mathfrak{p} \), let \( Z \in \perp \mathfrak{p} = \mathfrak{p}_0 \) be given. I claim that \( S_Z = 0 \) already implies that \( Z = 0 \):

Thereby, without loss of generality we may assume that \( \epsilon = 1 \). According to (27), we have \( [X, Z] = 0 \).

Therefore we may choose a maximal Abelian subspace \( a \subset \mathfrak{p} \) with \( \{X, Z\} \subset a \). Then the adjoint action of \( a \) on \( \mathfrak{n} \) is simultaneously diagonalizable. Let \( \Sigma \) denote the corresponding set of weights, usually called the “restricted roots”, choose some ordering of \( a \) such that \( X \) lies in the closure of the Weyl Chamber where the roots are positive (cf. [HC], Ch. VII, Remark after Lemma 2.20) and let \( \Sigma^+ \) denote the set of positive roots. Put \( \Sigma^+_n := \{ \lambda \in \Sigma^+ | \lambda(X) = n \} \) for \( n = 0, 1 \); then, since \( \text{Spec}(\text{ad}(X)) \subset \{-1,0,1\} \), \( \Sigma^+ \) is the disjoint union of \( \Sigma^+_0 \) and \( \Sigma^+_1 \). In compliance with a result of [BCO] (see p. 64 there), the set of eigenvalues for \( S_Z \) is given by \( \{ \lambda(Z) | \lambda \in \Sigma^+_1 \} \). Therefore, if \( S_Z = 0 \), then \( \lambda(Z) = 0 \) for all \( \lambda \in \Sigma^+_1 \). I claim that this already implies that \( \lambda(Z) = 0 \) for all \( \lambda \in \Delta^+ \):

For this, note that \( \Sigma^+ \) is the intersection of \( \Sigma^+ \) with the hyperplane \( \{ \lambda \in \mathfrak{a}^* | \lambda(X) = 0 \} \). Therefore, since \( \Sigma \) is a root system, we see that \( \Sigma^+_1 \) spans a vector subspace \( V \subset \mathfrak{a}^* \) which is invariant under all reflections in the various elements of \( \Sigma^+ \). Since the (abstract) Weyl group of \( \Sigma \) is generated by the reflections in the various elements of \( \Sigma^+ \), we hence conclude from the irreducibility of \( N \) that \( V = \mathfrak{a}^* \) holds, which immediately gives our claim.

Hence \( \lambda(Z) = 0 \) for all \( \lambda \in \Delta^+ \). To see that this already implies the vanishing of \( Z \), we proceed as follows: Let \( \mathfrak{h} \) be a maximal Abelian subspace of \( \mathfrak{g}^* \) with \( a \subset \mathfrak{h} \) and let \( \mathfrak{h}_C \) denote its complexification; then \( \mathfrak{h}_C \) is a Cartan subalgebra of \( \mathfrak{g}_C \), according to [HC], Ch. VI, Lemma 3.2. Let \( \Delta \) denote the corresponding set of roots; hence \( \lambda : \mathfrak{h}_C \to \mathbb{C} \) is a linear function with \( \lambda(\mathfrak{h}) \subset \mathbb{R} \) for each \( \lambda \in \Delta \). As explained in [HC], Ch. VI, § 3, for each \( \lambda \in \Delta \) we either have \( \lambda(a) = 0 \), or \( \lambda|a \in \Sigma \) holds; therefore, since \( \Delta \) spans the dual space \( \mathfrak{h}_C^* \) (cf. [HC], Ch. III, § 4, Proof of Theorem 4.2, Equation (2)), we obtain that \( \lambda(Z) = 0 \) for all \( \lambda \in \Sigma \) implies \( Z = 0 \).

Thus \( \perp X M \to \text{End}(\text{ad}(X)), Z \mapsto S_Z \) is an injective map, and hence a straightforward calculation shows that the second fundamental form of \( M \) spans \( \perp X M \). This finishes the proof.

Definition 7. Let \( N \) be a simply connected, irreducible symmetric space of compact type or of non-compact type, \( o \in N \) some origin, \( K \subset \mathfrak{g}(N) \) the isotropy group and \( i(N) = \mathfrak{k} \oplus \mathfrak{p} \) the Cartan decomposition. If there exists \( X \in \mathfrak{p} \) with \( X \neq 0 \) and \( \text{ad}(X)^3 = \epsilon \text{ad}(X) \), then \( M := \text{Ad}(K)X \) is called a (standard embedded, irreducible) symmetric R-space and we say that “\( N \) admits a symmetric R-space”.
Because of Remark 3 and Lemma 11 on the level of symmetric R-spaces, it is always enough to consider the case when \( N \) is of compact type, i.e. \( \epsilon = -1 \). Then the following theorem gives the classification of irreducible symmetric R-spaces, in accordance with \[2] and Tables A.6 and A.7:

**Theorem 8** ([KONa]). The irreducible symmetric spaces \( N \) which are of compact type and admit a symmetric R-space \( M \) are given as follows:

(a) If additionally \( N \) is of Hermitian type

| \( N \)                                      | \( M \)                                      | Remarks |
|---------------------------------------------|---------------------------------------------|---------|
| \( SU(2n) / (SU(n) \times U(n)) \)          | \( U(n) \)                                  | \( n \geq 2 \) |
| \( SO(n + 2) / SO(2n) \times SO(n) \)       | \( (S^1 \times S^{n-1}) / \mathbb{Z}_2 \)   | \( n \geq 3 \) |
| \( Sp(n) / U(n) \)                          | \( U(n) / SO(n) \)                          | \( n \geq 3 \) |
| \( SO(4n) / U(2n) \)                        | \( U(2n) / Sp(n) \)                        | \( n \geq 3 \) |
| \( E_7 / T \cdot E_6 \)                     | \( (T \cdot E_6) / F_4 \)                   | --      |

(b) Otherwise

| \( N \)                                      | \( M \)                                      | Remarks |
|---------------------------------------------|---------------------------------------------|---------|
| \( Spin(n) \)                               | \( SO(n) / (SO(2n) \times SO(n - 2)) \)     | \( n \geq 5 \) |
| \( Spin(2n) \)                              | \( SO(2n) / U(n) \)                         | \( n \geq 3 \) |
| \( SU(n) \)                                 | \( SU(n) / (SU(p) \times U(n - p)) \)       | \( n \geq 2, 1 \leq p \leq \frac{n}{2} \) |
| \( Sp(n) \)                                 | \( Sp(n) / U(n) \)                          | \( n \geq 2 \) |
| \( E_6 \)                                   | \( E_6 / T \cdot Spin(10) \)               | --      |
| \( E_7 \)                                   | \( E_7 / T \cdot E_6 \)                     | --      |
| \( SU(n) / SO(n) \)                         | \( G_p(\mathbb{R}^n) \)                    | \( n \geq 3, 1 \leq p \leq \frac{n}{2} \) |
| \( SU(2n) / Sp(n) \)                        | \( G_p(\mathbb{H}^n) \)                    | \( n \geq 2, 1 \leq p \leq \frac{n}{2} \) |
| \( SO(2n) / SO(n) \times SO(n) \)           | \( SO(n) \)                                | \( n \geq 5 \) |
| \( Sp(2n) / Sp(n) \times Sp(n) \)           | \( Sp(n) \)                                | \( n \geq 2 \) |
| \( E_6 / Sp(4) \)                           | \( G_2(\mathbb{H}^4) / \mathbb{Z}_2 \)     | --      |
| \( E_6 / F_4 \)                             | \( \mathbb{O} P^2 \)                        | --      |
| \( E_7 / SU(8) \)                           | \( (SU(8) / Sp(4)) / \mathbb{Z}_2 \)       | --      |
| \( SO(n) / SO(p) \times SO(n - p) \)        | \( (S^{p-1} \times S^{n-p-1}) / \mathbb{Z}_2 \) | \( n \geq 3, 3 \leq p \leq \frac{n}{2} \) |

2.2 Symmetric submanifolds associated with irreducible symmetric R-spaces

Continuing with the notation from the last section, we now introduce certain symmetric submanifolds of \( N \) which were already mentioned in Theorem 5. In the following, note that the linear map \( \pi_1 : i(N) \rightarrow T_o N, X \mapsto X^* (o) \) is surjective and that we have

\[
\mathfrak{t} = \{ Y \in i(N) | \pi_1(Y) = 0 \};
\]

hence \( \pi_1 | p \) induces a linear isomorphism \( p \cong T_o N \).

**Proposition 3.** Let \( N \) be a symmetric space of compact type or of non-compact type and suppose that there exists \( X \in p \) with \( X \neq 0 \) and \( \text{ad}(X)^3 = \epsilon \text{ad}(X) \). Then there exists a family of symmetric submanifolds \( M_c \subset N \), where \( c \) ranges over \( \mathbb{R} \), uniquely determined by the following properties:

For each \( c \in \mathbb{R} \) we have \( o \in M_c, T_o M_c = \pi_1(p_c), \perp_{o} M_c = \pi_1(p_0) \), and the second fundamental form \( h \) of \( M_c \) is characterized by

\[
\forall Y \in p_c : h(\pi_1(Y)) = c \pi_2(JY),
\]

where \( J \) is the linear map from Lemma 11. Therefore, \( M_0 \) is totally geodesic in \( N \), whereas for \( c \neq 0 \) the submanifold \( M_c \) is \( 1 \)-full in \( N \) with non-degenerate second fundamental form at \( o \).

**Proof.** In case \( N \) is of non-compact type, the existence of the family \( \{ M_c \} \) is established in Theorem 2.3 of \[5\] (likewise, cf. \[2] and Prop. 9.3.8). The compact case can be handled by similar methods, cf. \[2\], Ch. 9.3. Furthermore, it is well known that a symmetric submanifold is uniquely determined by its “2-jet” \( (T_p M, h_p) \) at one point \( p \in M \), cf. \[3\]; thus \( M_c \) is uniquely determined. For the last
conclusion of the above theorem, notice that, as a result of Proposition 2 both \( M_e \) and the symmetric R-space \( \text{Ad}(K) \) \( X \) have the same tangent space at \( o \) and at \( X \), respectively (by means of identification \( p \cong T_o N \) via \( \pi_1 \)). Using also that \( \pi_1 \) is an equivariant map of \( \ell \)-modules, i.e.

\[
\forall X \in \ell, Y \in p : \pi_1(\text{ad}(X) Y) = \pi_2(X) \pi_1(Y),
\]

we now see from comparing (28) with (30) that (again by means of the identification \( M \)) the second fundamental form of \( M_e \) at \( o \) equals \( c \) times the second fundamental form of \( \text{Ad}(K) \) \( X \) at \( X \); hence the result follows, again as a consequence of Proposition 2.

**Definition 8.** In the situation of Definition 7, suppose that there exists \( X \in p \) with \( X \neq 0 \) and \( \text{ad}(X)^3 = c \ \text{ad}(X) \). Then \( N \) admits a symmetric R-space, and the family of submanifolds \( M_e \subset N \) from Proposition 3 will be called “the family of symmetric submanifolds associated therewith”.

## 3 Some intrinsic properties of \( O_f \)

Throughout this section, \( f : M \to N \) is a parallel isometric immersion. Then for each \( p \in M \) the linear map \( T_p f : T_p M \to O_f) \) \( x \mapsto T_p f x \) induces an injective vector bundle homomorphism \( TM \to f^*TN \) (whose image \( Tf(TM) \) is usually called the “first osculating bundle of \( f^* \)"; hence we have the corresponding orthogonal splitting

\[
f^*TN = Tf(TM) \oplus \perp f \cong TM \oplus \perp f.
\]

In the following, in order to keep to our convention that “\( T_p M \) is seen as a linear subspace of \( f^*(p) N \)” for each \( p \in M \), we suppress the vector bundle isomorphism \( Tf(TM) \to Tf(TM) \); for convenience, the reader may assume that \( M \subset N \) is a submanifold and \( f = \ell M \).

**Definition 9** (Split-parallelity). (a) The split connection is by definition the linear connection \( \nabla^{sp} := \nabla^M \oplus \nabla^\perp \) on \( f^*TN = TM \oplus \perp f \). A section of \( f^*TN \) will be called split-parallel if it is parallel with respect to \( \nabla^{sp} \).

(b) For a curve \( c : J \to M \) let \( t \mapsto c^{\perp} : T_{c(t_1)} N \to T_{c(t_2)} N \) denote the corresponding split parallel displacement along \( c \) (where \( (t_1, t_2) \) varies over \( J \times J \)), which is the family of linear isometries characterized by the following properties:

- \( [t_2 \mapsto t_1] \) \( X(t_1) = X(t_2) \) for any \( \nabla^M \)-parallel section \( X : J \to TM \) along \( c \),
- \( [t_2 \mapsto t_1] \) \( \nabla^\perp \xi(t_1) = \xi(t_2) \) for any \( \nabla^\perp \)-parallel section \( \xi : J \to \perp M \) along \( c \).

Now the equations of Gauß and Weingarten can formally be combined to

\[
\forall X \in \Gamma(TM), V \in \Gamma(f^*(TN)) : \nabla^N_X V = \nabla^{sp} X V + h(X) V.
\]

\( R^{sp} \) and \( R^\perp \) will denote the curvature tensors of \( f^*TN \) and \( \perp f \) with respect to \( \nabla^{sp} \) and \( \nabla^\perp \), respectively. Since \( f \) is parallel, also the curvature equations of Gauß, Codazzi and Ricci can formally be combined to

\[
\forall x, y \in T_p M : R^N(x, y) = R^{sp}(x, y) + [h(x), h(y)].
\]

The following lemma is proved in a straightforward manner:

**Lemma 2.** Let \( V \) be a Euclidian vector space and \( W \subset V \) a linear subspace. Recall the splitting \( \mathfrak{so}(V) = \mathfrak{so}(V)^+ \oplus \mathfrak{so}(V)^- \) defined by (8) and (9).

(a) We have \( A \in \mathfrak{so}(V)^+ \) if and only if \( A(W) \subset W \).
Proposition 4. Let a parallel isometric immersion \( f : M \rightarrow N \) be given.

(a) \( T_p M \) is a curvature invariant subspace of \( T_{f(p)} N \), and we have
\[
\forall x, y \in T_p M : \ R^N(x, y) \in \mathfrak{so}(T_x N)_+. \tag{35}
\]

(b) \( M \) is locally symmetric, i.e. \( R^M \) is parallel.

(c) If \( M \) is a complete, simply connected, parallel submanifold, then \( M \) is a symmetric space.

(d) \( h \) satisfies a second order tensorial property known as “semiparallelity”:
\[
\forall x_1, x_2, y_1, y_2 \in T_p M : \ R^N(x_1, x_2) h(y_1, y_2) = h(R^M(x_1, x_2) y_1, y_2) + h(y_1, R^M(x_1, x_2) y_2). \tag{36}
\]

(e) Equation (36) is equivalent to
\[
\forall x, y, z \in T_p M : \ h(R^M(x, y) z) = [R^N(x, y) - [h(x), h(y)], h(z)]. \tag{37}
\]

Proof. (a) follows from the Codazzi Equation and Lemma 2. For the proof of (b) one needs assertion (a) and the curvature equation of Gauß. If \( M \) is simply connected and complete, then it is even globally symmetric (cf. [He], Ch. IV, § 6, Theorem 5.6). The proof of Equation (36) is straightforward, see for example [Ts1] or [F2]. For (37), note that both sides of this equation are elements of \( \mathfrak{so}(T_p N)_- \). Thus by virtue of Lemma 2 it is enough to verify that (37) holds on \( T_p M \). For this let \( \tilde{y} \in T_p M \) be given; then (34) implies:
\[
[R^N(x, y) - [h(x), h(y)], h(z)] \tilde{y} = R^N(x, y) h(z) \tilde{y} - h(z) R^M(x, y) \tilde{y};
\]
now use (36). \( \Box \)

Proposition 5. Let a parallel isometric immersion \( f : M \rightarrow N \) be given.

(a) \( \perp^1 f \subset \perp f \) is a \( \perp^1 \)-parallel vector subbundle. In particular,
\[
\forall x, y \in T_p M : \ R^N(x, y)(\perp^1 f) \subset \perp^1 f. \tag{38}
\]

(b) \( \mathcal{O} f \) is a split-parallel vector subbundle of \( f^* T N \).

(c) If \( c : J \rightarrow M \) is a curve, \( X(t), Y(t) \) are parallel sections of \( T M \) and \( \xi(t) \) is a parallel section of \( \perp^1 f \) along \( c \), then \( R^N(X(t), Y(t)) \xi(t) \) is a parallel section of \( \perp^1 f \) along \( c \).

(d) \( \mathcal{O} f \) is a \( \nabla^N \)-parallel vector subbundle of \( f^* T N \). Hence \( \nabla^N \) induces a connection on \( \mathcal{O} f \), as already described in Section 1.1. Therefore
\[
\forall t_1, t_2 \in J : \left( \left[ \frac{t_2}{t_1} \right] c \right)^N(\mathcal{O}_{c(t_1)} f) = \mathcal{O}_{c(t_2)} f, \tag{39}
\]
\[
\forall x, y \in T_p M : \ R^N(x, y)(\mathcal{O}_p f) \subset \mathcal{O}_p f; \tag{40}
\]

and the corresponding parallel displacement resp. curvature tensor are given by
\[
\left( \left[ \frac{t_2}{t_1} \right] c \right)^{\mathcal{O} f} = \left( \left[ \frac{t_2}{t_1} \right] c \right)^N^{\mathcal{O} f} \quad (\text{see } [E]), \tag{41}
\]
\[
R^{\mathcal{O} f}(x, y) = (R^N(x, y))^N^{\mathcal{O} f}. \tag{42}
\]

Moreover, we have
\[
R^{\mathcal{O} f}(x, y) \in \mathfrak{so}(\mathcal{O}_p f)_+. \tag{43}
\]
Moreover, using the canonical identification by means of (14) and (49) we have

**Proposition 5.**

The transport of the velocity vector field $\dot{\gamma}$.

**Lemma 3.** In the situation of Definition 9, we have for all $x, y \in T_{c(t_1)}M$:

\[ h((\|c\|)^\mu x, (\|c\|)^\mu y) = ((\|c\|)^\mu h(x, y), \] \hspace{1cm} (44)

\[ h((\|c\|)^\mu x) = ((\|c\|)^\mu h(x) = (\|c\|)^\mu h(x) \circ (\|c\|)^\mu, \] \hspace{1cm} (45)

\[ \forall \tau \in J : (\|c\|)^\mu(\mathcal{O}_{c(t_1)} f) = \mathcal{O}_{c(t_2)} f. \] \hspace{1cm} (46)

**Proof.** (44) follows by definition of the parallelity of the second fundamental form. In particular, (45) holds on $T_{c(t_1)}M$; moreover both sides of Equation (45) are elements of $\mathfrak{so}(T_{c(t)}N)$ (because of Equation (14)), and since $(\|c\|)^\mu$ respects the two splittings $T_{f(c(t_1))}N = T_{c(t_1)}M \oplus \perp_{c(t_1)}M$ and $T_{f(c(t_2))}N = T_{c(t_2)}M \oplus \perp_{c(t_2)}f$, and thus (45) follows from Lemma 2. Equation (46) follows immediately from Part (b) of Proposition 5.

Because of (45), the following proposition is a consequence of Lemma 3 in [JR]:

**Proposition 6.** For each curve $c : J \to M$ with $0 \in J$ and $c(0) = p$ let $X$ denote the backward parallel transport of the velocity vector field $\dot{c}$, i.e.

\[ X(t) := (\|c\|)^\mu(\dot{c}(t)) \in T_pM. \] \hspace{1cm} (47)

Then the function

\[ \mu_c : J \to SO(T_{c(0)}N), \quad t \mapsto (\|c\|)^\mu(\dot{c}(t)) \] \hspace{1cm} (48)

solves the linear differential equation

\[ \mu_c(t) = \mu_c(t) \circ h(X(t)) \quad \text{with} \quad \mu_c(0) = id. \] \hspace{1cm} (49)

Equations (46) and (49) imply for a curve $c : [0, 1] \to M$ as above:

\[ \forall t \in J : \mu_c(t)(\mathcal{O}_p f) = \mathcal{O}_p f. \] \hspace{1cm} (50)

Moreover, using the canonical identification

\[ SO(\mathcal{O}_p f) \cong \{ g \in SO(T_pM) \mid g(\mathcal{O}_p f) = \mathcal{O}_p f \quad \text{and} \quad g(\mathcal{O}_p f) = \text{Id} \}, \]

by means of (14) and (49) we have

\[ \forall t \in J : \mu_c(t) \in SO(\mathcal{O}_p f). \] \hspace{1cm} (51)

**Example 6.** (a) If $c$ denotes the geodesic $\gamma_x : J \to M$ with $\gamma_x(0) = x$, then

\[ \mu_c(t) = \exp(t \dot{h}(x)). \] \hspace{1cm} (52)
(b) We can construct \( \mu_c \) as in Proposition 3 also if we only assume that \( c \) is continuous and piecewise differentiable; then \( \mu_c \) will be continuous and piecewise differentiable, too. Now let \( x, y \in T_p M \) be given, consider the corresponding smooth geodesic line \( \gamma_x : \mathbb{R} \to M \) and put

\[
\tilde{\gamma} := (||\gamma_x||^N)_y \in T_{\gamma_x(1)} M ;
\]

hence we also have the corresponding smooth geodesic line \( \gamma_{\tilde{\gamma}} : \mathbb{R} \to M \). Let \( \gamma_{(x,y)} : [0,2] \to M \) be the broken geodesic line characterized by

\[
\forall t \in [0,2] : \gamma_{(x,y)}(t) = \begin{cases} 
\gamma_x(t) & \text{for } t \leq 1 \\
\gamma_{\tilde{\gamma}}(t-1) & \text{for } t \geq 1
\end{cases}
\]  

(53)

Then for the curve given by \( c(t) := \gamma_{(x,y)}(t) \) we have

\[
\mu_c(2) = \exp(h(x)) \circ \exp(h(y)) .
\]

Proof. For (a): Here the function \( X(t) \) of Equation (47) is constant equal to \( x \); therefore the solution of (46) is the one-parameter subgroup given by (52).

For (b): By virtue of Equation (45) and using Part (a) several times, we obtain

\[
\mu_c(2) = \frac{0}{0} 2^N \circ (||c||^N) = \frac{0}{0} 1 \circ (||\gamma_{\tilde{\gamma}}||^N) \circ (||\gamma_{\tilde{\gamma}}||^N) \circ (||\gamma_{\tilde{\gamma}}||^N) \circ (||\gamma_x||^N) \circ (||\gamma_x||^N) = \exp(h(x)) \circ \exp((||\gamma_x||^N) \circ h(y) \circ (||\gamma_x||^N)) .
\]

The result follows from the previous together with (50).

Lemma 4. In the situation of Proposition 3 for any choice of vectors \( y_1, y_2 \in T_p M \) and \( v \in \mathcal{O}_p f \) we have

\[
R^N(\mu_c(t) y_1, \mu_c(t) y_2) v \in \mathcal{O}_p f ,
\]

and the following two equalities hold:

\[
R^N(\mu_c(t) y_1, \mu_c(t) y_2) v = (||c||^N \circ R^N((||c||^N) y_1, (||c||^N) y_2) \circ (||c||^N) v ,
\]

(55)

\[
R^N(\mu_c(t) y_1, \mu_c(t) y_2) v = \mu_c(t) \circ R^N( y_1, y_2) \circ \mu_c(t)^{-1} v .
\]

(56)

Proof. Using the \( \nabla^N \)-parallelity of \( R^N \) and Part (e) of Proposition 3 we have

\[
R^N(\mu_c(t) y_1, \mu_c(t) y_2) \mu_c(t) v = (||c||^N \circ R^N((||c||^N) y_1, (||c||^N) y_2) \circ (||c||^N) v
\]

\[
= (||c||^N) \circ R^N(y_1, y_2) v = \mu_c(t)(R^N(y_1, y_2)) v .
\]

The result follows with (50).

For every Euclidian vector space \( V \) and every \( A \in \mathfrak{so}(V) \), let \( A^{(2)} : \bigwedge^2 V \to \bigwedge^2 V \) denote the induced Endomorphism, i.e.

\[
A^{(2)} u \wedge v := A u \wedge v + u \wedge A v = \frac{d}{dt} \bigg|_{t=0} \exp(t A) u \wedge \exp(t A) v .
\]

(57)
Furthermore, one knows that for $A, B \in \mathfrak{s}o(V)$

$$\frac{d}{dt}|_{t=0} \exp(tA) \circ B \circ \exp(tA)^{-1} = [A, B].$$

(58)

**Lemma 5.** For arbitrary $p \in M$, $x_1, x_2 \in T_p M$, $y_1, y_2 \in T_p M$ and $v \in \mathcal{O}_p f$ we have:

$$[h(x_1), R^N(y_1, y_2)] v = R^N(h(x_1)(y_1 \wedge y_2)) v,$$

(59)

$$[h(x_1), [h(x_2), R^N(y_1, y_2)]] v = R^N(h(x_1)h(x_2)(y_1 \wedge y_2)) v.$$

(60)

**Remark 4.** Suppose that $i(N)$ is semisimple and let $\mathfrak{k}$ denote the Lie algebra of the isotropy group at $p$. In accordance with [He], Ch. V, Theorem 4.1., we have

$$\pi_2(\mathfrak{k}) = \{ A \in \mathfrak{so}(T_p N) | \forall u, v \in T_p N : [A, R^N(u, v)] = R^N(Au, v) + R^N(u, Av) \}.$$

(61)

Hence, if $M$ is a symmetric submanifold of $N$, then, according to (23), Equation (59) (and therefore also (60)) holds for all $y_1, y_2 \in T_p N$.

**Proof of Lemma 5.** To derive Equation (59), let $\gamma : J \to M$ denote the geodesic with $\gamma(0) = x_1$. Then $\mu_\gamma(t) = \exp(t h(x_1))$ holds by means of Equation (52); considering also Equations (57) and (58) (with $A = h(x_1)$) (59) therefore follows by taking the derivative $\frac{d}{dt}|_{t=0}$ on both sides of Equation (56).

To derive Equation (60), let $(s, r) \in \mathbb{R} \times \mathbb{R}$ be fixed elements and $c$ be the broken geodesic $\gamma(s, x_1, r x_2) : [0, 2] \to M$, as described in Equation (69). Then $\mu_c$ satisfies $\mu_c(2) = \exp(\gamma(s, x_1)) \circ \exp(r h(x_2)) =: f(s, r)$, according to [H]; hence Equation (60) gives

$$(R^N(f(s, r) y_1, f(s, r) y_2)) v = (f(s, r) \circ R^N(y_1, y_2)) v.$$

Now (60) follows by taking the derivatives $\frac{\partial}{\partial r} \frac{\partial}{\partial s}|_{r=s=0}$ on both sides of this equation.

$$\Box$$

3.1 Curvature invariance of the first normal spaces

**Proposition 7.** For arbitrary $p \in M$, $x, y \in T_p M$ we have $h(x) h(y)(T_p M) \subset T_p M$ and the following equation holds on $\mathcal{O}_p f$:

$$R^N(h(x, x), h(y, y)) = [h(x), [h(y), R^N(x, y)]] - R^N(h(x) h(y) x, y) - R^N(x, h(x) h(y) y).$$

(62)

Moreover, for all $\xi, \eta \in \perp^1_p f$ the curvature endomorphism

$$R^N(\xi, \eta) : T_p N \to T_p N, v \mapsto R^N(\xi, \eta) v$$

has the following property:

$$R^N(\xi, \eta)(\mathcal{O}_p f) \subset \mathcal{O}_p f$$

and

$$R^N(\xi, \eta) \in \mathfrak{so}(\mathcal{O}_p f)_+.$$

(63)

(64)

**Proof.** Let us first verify Equation (62) on $\mathcal{O}_p f$: According to (60), we have

$$\forall v \in \mathcal{O}_p f : [h(x), [h(y), R^N(x, y)]] v = R^N(h(x) h(y) h(x) x \wedge y) v,$$

and furthermore (using (57) twice and the symmetry $h(y, x) = h(x, y)$):

$$h(x) h(y) x \wedge y = h(x) h(y) x \wedge y + h(x) x \wedge h(y) y + h(y) y \wedge h(x) x + h(x) h(y) y.$$
Therefore, Equation (62) holds on $O_p f$. For the proof of (63), it is enough to assume that there exist $x, y \in T_p M$ with $\xi = h(x, x)$, $\eta = h(y, y)$, because $h$ is a symmetric bilinear map. Furthermore, note that $h(x) h(y) z = -S_{h(z, z)} x$ for all $z \in T_p M$, hence $h(x) h(y) (T_p M) \subset T_p M$ and therefore the linear space $O_p f$ is invariant under each of the three terms on the right hand side of (62), in accordance with (12), (13), (10); which implies Equation (63). To conclude also (64), just note that (after projection to $\mathfrak{so}(O_p f)$) each of the three terms on the right hand side of (62) is an element of $\mathfrak{so}(O_p f)$, according to (14), (12) and (13) and the rules for $\mathbb{Z}_2$-graded Lie algebras.

As a consequence of (8), (63) and (64) we have:

**Corollary 2.** $\perp f$ is a curvature invariant subspace of $T f(p) N$.

Motivated by Lemma 2 in [E1], we are now able to generalize Part (e) of Proposition 5:

**Proposition 8.** If $f : M \rightarrow N$ is parallel, then the tensor of type $(0, 4)$ on $O f$ defined by

$$R^f(v_1, v_2, v_3, v_4) := \langle R_N(v_1, v_2)v_3, v_4 \rangle \quad \text{for} \quad v_1, \ldots, v_4 \in O_p f$$

(65)

is split-parallel; which means: For a curve $c : J \rightarrow M$ and split-parallel sections of $O f$ along $c$, $V_i(t) (i = 1, \ldots, 4)$, the function $f(t) := R^f(V_1(t), V_2(t), V_3(t), V_4(t))$ is constant.

**Proof.** Note that $V_i(t) = X_i(t) + \xi_i(t)$, where $X_i$ resp. $\xi_i$ are split-parallel sections of $TM$ resp. of $\perp f$. Thus it is enough to consider the following two cases.

First case: Exactly one of the sections $V_i$ is a section of $\perp f$ (resp. of $TM$) and the other ones are sections of $TM$ (resp. of $\perp f$). Then $f(t) = 0$, since $T_c(t) M$ (resp. $\perp f$) are curvature invariant subspaces of $T_c(t) N$; see Proposition (a) and the previous Lemma. Note that for this argument it was not used that the sections are split-parallel.

Second case: An even number of $V_1, V_2, V_3, V_4$ are sections of $TM$, and the other ones are sections of $\perp f$. Then, by the equations of Gauß and Weingarten, for each $i \in \{ 1, \ldots, 4 \}$ an odd number of the sections $V_1, \ldots, V_4$ are sections of $TM$, and the other ones are sections of $\perp f$. It follows from the parallelity of $R^N$ and the considerations made for the first case that $f(t) = 0$.

**Lemma 6.** Equations (55), (56), (59) and (60) also hold if one replaces “$y_1, y_2 \in T_p M$” with “$y_1, y_2 \in O_p f$” in each of these Equations.

**Proof.** The result follows by repeating the proofs of (55), (56), (59) and (60), but now using Proposition 8 instead of Proposition 5 (e).

Lemma 3 (applied to (60)) will be needed for the proof of Theorem 8 in Section 5.

**Corollary 3.** For all $x \in T_p M$ and $v_1, \ldots, v_4 \in O_p f$ we have

$$\sum_{i=1}^4 R^f(v_1, \ldots, h(x) v_i, \ldots, v_4) = 0$$

(66)

**Proof.** From Lemma 3 (applied to Equation (54)) we obtain

$$R^f(\mu_c(t) v_1, \mu_c(t) v_2, \mu_c(t) v_3, \mu_c(t) v_4) = \text{const}$$

(67)

for each curve $c : \mathbb{R} \rightarrow M$. If $c = \gamma_x$ is the geodesic considered in Example 6 then (63) follows with (52) by taking the derivative $\frac{d}{dt} \big|_{t=0}$ of (67).
3.2 Proof of Theorem 1

At the end of this section, we will give the proof of Theorem 1.

Lemma 7. Let a full parallel submanifold $M \subset N$ with $o \in M$ be given and $\mathfrak{hol}(M^T)$ be the holonomy Lie algebra of the totally geodesic submanifold $M^T := \exp N(T_oM)$ with respect to the base point $o$. Then $\text{Kern}(h^o_M)$ is a $\mathfrak{hol}(M^T)$-invariant subspace; hence, if $M^T$ is an irreducible symmetric space, then $h^o_M$ is a non-degenerate symmetric bilinear form.

Proof. According to the Theorem of Ambrose/Singer, since $M^T$ is a totally geodesic submanifold of $N$, $\mathfrak{hol}(M^T)$ is the linear subspace of $\mathfrak{so}(T_oM)$ given by

$$\{ R^N(x,y) | T_oM \to T_oM | x, y \in T_oM \}_R .$$

I claim that $\text{Kern}(h_o)$ is a subspace of $T_oM$ which is invariant under the action of $\mathfrak{hol}(M^T)$; hence, if $\mathfrak{hol}(M^T)$ acts irreducible on $T_oM$, then $h$ is injective, because $h = 0$ is not possible for a full submanifold.

For this, let $z \in T_oM$ be given and assume that $h(z) = 0$. I claim that then also $h(A(z)) = 0$ for each $A \in \mathfrak{hol}(M^T)$.

For this: By the previous, we may assume that there exist $x, y \in T_oM$ with $A = R^N(x,y)|T_oM : T_oM \to T_oM$. Then r.h.s. of (37) vanishes, therefore $h(R^M(x,y)z) = 0$ and thus we have (by the Gauß equation for the curvature)

$$h(R^N(x,y)z) = h(h(x)(h(y,z)) - h(y)(h(x,z))) = 0 .$$

The result follows. $\square$

Lemma 8. For the following types of parallel submanifolds their second osculating spaces are curvature invariant:

(a) $M$ is a submanifold of a real space form.

(b) $M^m$ is a totally real submanifold of the complex projective space $\mathbb{C}P^n$ and $m > 1$, see [N2], Lemma 2.1.

(c) $M$ is a complex submanifold of $\mathbb{C}P^n$ (since here the second osculating spaces are complex subspaces).

(d) $M$ is a totally complex submanifold of the quaternionic projective space $\mathbb{H}P^n$ which is locally of “Kählerian type” (in the sense of [TS], Definition 2.12), see [ADM], Prop. 5.6.

Furthermore, according to Proposition 2.11 of [TS], a totally complex submanifold $M^{2m} \subset \mathbb{H}P^n$ with $m \geq 2$ is already locally of Kählerian type.

Proof of Theorem 1. Part (a) of Theorem 1 stands in accordance with Corollary 2.

For Part (b) and (c): A 1-full, complete parallel submanifold of $N$ is even a symmetric submanifold as a consequence of Part (a) combined with Theorem 1.

Conversely, let $M$ be a full symmetric submanifold of $N$; hence the subgroup of $I(N)$ generated by the extrinsic symmetries of $M$ acts transitively $M$ and therefore $M$ is a complete Riemannian manifold. Moreover, its second fundamental form is parallel according to Theorem 7. It remains to show that $M$ is 1-full and, in case $N$ has no Euclidian factor, that $h$ is non-degenerate.

For this, because of Theorem 1, it is enough to assume that $N$ is an Euclidian space or an irreducible symmetric space. If $N$ is a Euclidian space or if $N$ is irreducible and the rank of $N$ is 1, then by virtue of Theorem 5 combined with Lemma 8 the second osculating spaces of $M$ are curvature invariant; thus $M$ is even 1-full by means of “reduction of the codimension” (Theorem 2). Furthermore, if the rank of $N$ is 1, then for each $p \in M$ the totally geodesic submanifold $M^T(p)$ defined in Lemma 7 is either a real space form (in Cases (a) and (b) of Lemma 8) or a complex space form (in Cases (c) and (d) of Lemma 8), and therefore the non-degeneracy of $h_o$ is given by Lemma 7. If $N$ is irreducible and the rank of $N$ is larger
than 1, then by the strength of Theorem 5, \( N \) admits a symmetric R-space and \( M \) belongs to the family of symmetric submanifolds associated therewith, i.e. \( M = M_c \) for some \( c \in \mathbb{R} \). Then \( c \neq 0 \) (since \( M \) is not totally geodesic in \( N \)) and hence \( M \) is a 1-full submanifold of \( N \) and \( h_0 \) is non-degenerate according to Proposition 6.

4 Symmetry of \( \mathcal{O}f \)

Throughout this section, \( f : M \to N \) is a parallel isometric immersion defined on a simply connected Riemannian symmetric space \( M \) (cf. Part (c) of Proposition 4). The corresponding geodesic symmetries of \( M \) will be denoted by \( \sigma^M_p \) \((p \in M)\). Remember that \( \mathcal{O}f \subset f^*TN \) is a \( \nabla^N \)-parallel vector subbundle, according to Part (d) of Proposition 5, and hence \( \mathcal{O}f \) is equipped with the connection \( \nabla^{\mathcal{O}f} \) induced by restriction of \( \nabla^N \). Furthermore, continuing with the notation from Section 3, there is the splitting \( \mathcal{O}f = TM \oplus \perp f \); but note that in general \( \nabla^{\mathcal{O}f} \) is not the split-connection (introduced in Definition 9) restricted to \( \mathcal{O}f \). In Section 4.1 we will prove the following result:

**Theorem 9.** For each \( p \in M \) there exists a unique involutive map \( \Sigma_p : \mathcal{O}f \to \mathcal{O}f \) characterized by the following properties:

(a) \( \Sigma_p \) is a fibrewise isometric vector bundle homomorphism along \( \sigma^M_p \), i.e. the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{O}f & \xrightarrow{\Sigma_p} & \mathcal{O}f \\
\downarrow & & \downarrow \\
M & \xrightarrow{\sigma^M_p} & M
\end{array}
\]

and for each \( q \in M \) the map \( \Sigma_p|_{\mathcal{O}q} f : \mathcal{O}q f \to \mathcal{O} \sigma^M_q (q) f \) is a linear isometry.

(b) \( \Sigma_p \) is a \( \nabla^{\mathcal{O}f} \)-parallel vector bundle isomorphism of \( \mathcal{O}f \).

(c) \( \Sigma_p|_{\mathcal{O}q} f \) is the linear reflection in \( \perp f \).

Moreover:

(d) \( \forall q \in M, x \in T_q M : \Sigma_p x = T \sigma^M_p x \).

\( \forall q \in M, \forall x, y \in T_q M : \Sigma_p h(x, y) = h(T \sigma^M_p x, T \sigma^M_p y) \).

(e) For every smooth geodesic line \( \gamma \) of \( M \) with \( \gamma(0) = p \) we have \( \sigma^M_p (\gamma(-1)) = \gamma(1) \),

\[
\Sigma_p |_{T\gamma(-1) M} = -\frac{1}{\| \gamma \|^M},
\]

\[
\Sigma_p |_{\perp\gamma(-1) f} = \left( \frac{1}{\| \gamma \|^1} \right) |_{\perp\gamma(-1) f}.
\]

(f) If \( M \) is a symmetric submanifold of \( N \) (with extrinsic symmetries \( \sigma^\perp_p \) \((p \in M)\)), then we have \( T \sigma^\perp_p (\mathcal{O}M) \subset \mathcal{O}M \) and

\[
\Sigma_p|_{\mathcal{O}M} = T \sigma^\perp_p |_{\mathcal{O}M} : \mathcal{O}M \to \mathcal{O}M.
\]

Comparing the last theorem with Definition 2, we hence see that the family \( \Sigma_p \) \((p \in M)\) can be seen as sort of “weak extrinsic symmetries” of \( M \), and hence \( M \) is (at least) “extrinsically symmetric in \( \mathcal{O}f \).

**Definition 10.** For each geodesic \( \gamma \) of \( M \) with \( \gamma(0) = p \) and each \( t \in \mathbb{R} \) we define

\[
\theta_\gamma (t) := \sigma^M_{\gamma(t/2)} \circ \sigma^M_p \quad \text{and} \quad \Theta_\gamma (t) := \Sigma_{\gamma(t/2)} \circ \Sigma_p.
\]
Then \( \theta_\gamma(t) \) \((t \in \mathbb{R})\) is a family of isometries defined on \( M \) and \( \Theta_\gamma(t) \) \((t \in \mathbb{R})\) is a family of isometric, parallel vector bundle isomorphism of \( O_f \) (by virtue of Theorem 9), and the following diagram is commutative:

\[
\begin{array}{ccc}
O_f & \xrightarrow{\Theta_\gamma(t)} & O_f \\
\downarrow & & \downarrow \\
M & \xrightarrow{\theta_\gamma(t)} & M
\end{array}
\]  

(73)

**Corollary 4.** In the situation of Definition 12, let \( \gamma^\parallel_0 \) denote the corresponding split-parallel displacement along \( \gamma \) as introduced in Definition 4. For each \( t \in \mathbb{R} \) we have \( \theta_\gamma(t)(p) = \gamma(t) \) and

\[
\Theta_\gamma(t)|O_p f = \left( \gamma^\parallel_0 \right)|O_p f : O_p f \to O_{\gamma(t)} f .
\]

(74)

**Proof.** Using Definition 12 and Part (c) of Theorem 9, we have \( \theta_\gamma(t)(p) = \sigma_{\gamma(t/2)}(p) = \gamma(t) \) and for all \( x+\xi \in T_p M \oplus \perp_{1/2}^f \)

\[
\Theta_\gamma(t)(x+\xi) = \Sigma_{\gamma(t/2)}(-x+\xi) \left( \gamma^\parallel_0 \right)(x+\xi),
\]

which yields the stated result.

\[ \Box \]

## 4.1 Certain involutions on the first normal bundle

At the end of this section we will give the proof of Theorem 9. But first we have to verify the existence of certain involutions on \( \perp_{1/2}^f \), for which purpose we will now state some general facts about the existence of parallel sections of some vector bundle \( E \) over a simply connected Riemannian manifold \( M \) equipped with a connection. Let \( o \in M \) be a fixed “origin” and \( s_0 \in E_o \) considered as “initial condition”.

**Lemma 9.** Suppose that the curvature tensor \( R^E \) is parallel (considered as a section of \( L(\Lambda^2(TM), \text{End}(E)) \), where the latter space is equipped with the induced connection). Then there exists a parallel section \( s \) of \( E \) with \( s(o) = s_0 \) if and only if

\[
\forall x, y \in T_o M : R^E(x, y) s_0 = 0 .
\]

(75)

**Proof.** Let \( \text{Hol}(E) \) denote the holonomy group of \( E \) with respect to the base point \( o \), defined by

\[
\text{Hol}(E) := \{ (\frac{1}{0} c)^\parallel : [0, 1] \to M \text{ is a curve with } c(0) = c(1) = o \} ,
\]

where \( (\frac{1}{0} c)^\parallel \) means the parallel displacement along \( c \) in \( E \). It is known that \( \text{Hol}(E) \) is a Lie subgroup of \( \text{GL}(E_o) \), its Lie algebra, \( \mathfrak{h}o(\mathbb{E}) \subset \text{End}(E_o) \), is called the holonomy Lie algebra of \( E \). The “Theorem of Ambrose/Singer” shortly states that \( \mathfrak{h}o(\mathbb{E}) \) is generated by the curvature of \( E \); more exactly it is generated (as a vector space over \( \mathbb{R} \)) by the elements

\[
(\frac{0}{1} c)^\parallel \circ R^E(x, y) \circ (\frac{0}{1} c)^\parallel ,
\]

where \( c \) runs over all curves \([0, 1] \to M \) with \( c(0) = o \) and \( x, y \in T_o M \). If \( R^E \) is parallel, then we have

\[
(\frac{0}{1} c)^\parallel \circ R^E(x, y) \circ (\frac{0}{1} c)^\parallel = R^E((\frac{0}{1} c)^M x, (\frac{0}{1} c)^M y) ;
\]

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and therefore
\[ \mathfrak{hol}(\mathbb{E}) = \{ R^\mathbb{E}(x, y) | x, y \in T_0 M \} \mathbb{R} . \]  
(76)

Let \( s \in \Gamma(\mathbb{E}) \) be a section with \( s(o) = s_0 \). Then \( s \) is parallel if and only if for every curve \( c: [0, 1] \to M \) with \( c(0) = o \) we have
\[ s(c(1)) = \frac{1}{0} s_0 . \]  
(77)

Thus, if there exists a parallel section with \( s(o) = s_0 \), then in particular \( s_0 \) is a fix point of \( \text{Hol}(\mathbb{E}) \). And if \( s_0 \) is a fix point of \( \text{Hol}(\mathbb{E}) \), then one defines a section \( s \) via \((77)\), which is then parallel with \( s(o) = s_0 \).

Since \( \text{Hol}(\mathbb{E}) \) is connected (because \( M \) is simply connected), \( s_0 \) is a fixed point with respect to the action of \( \text{Hol}(\mathbb{E}) \) if and only if
\[ \forall A \in \mathfrak{hol}(\mathbb{E}) : A s_0 = 0 . \]  
(78)

The lemma follows from \((76)\) and \((78)\).

We will now apply Lemma 9 to deduce the following result:

**Proposition 9.** Let a parallel isometric immersion \( f: M \to N \) be given. For each \( p \in M \) there exists a unique involutive map \( I_p: \perp^1 f \to \perp^1 f \), characterized by the following properties:

(a) \( I_p \) is a fibrewise isometric vector bundle homomorphism along \( \sigma^M_p \), i.e. the following diagram is commutative,
\[ \begin{array}{ccc}
\perp^1 f & \xrightarrow{I_p} & \perp^1 f \\
\downarrow & & \downarrow \\
M & \xrightarrow{\sigma^M_p} & M
\end{array} \]
and for each \( q \in M \) the map \( I_p|\perp^1_q f : \perp^1_q f \to \perp^1_{\sigma^M_p(q)} f \) is a linear isometry.

(b) \( I_p \) is parallel.

(c) \( I_p|\perp^1 f \) is the identity on \( \perp^1 f \).

Moreover:

(d) For every smooth geodesic line \( \gamma: [-1, 1] \to M \) with \( \gamma(0) = p \) we have
\[ I_p|\perp^1_{\gamma(-1)} f = \tau.h.s. \text{ of } (71) . \]  
(79)

(e) We also have for each \( q \in M \):
\[ \forall x, y \in T_q M : I_p(h(x, y)) = h(T\sigma^M_p x, T\sigma^M_p y) . \]  
(80)

**Proof.** The uniqueness of \( I_p \) follows immediately from its parallelity together with (c). For its existence let, we consider the origin \( o = p \) and put \( \sigma := \sigma^M_o \). To prove the existence of \( I_o \), we will apply Lemma 9 with \( \mathbb{E} := L(\perp^1 f, \sigma^* \perp^1 f) \) (where \( \sigma^* \perp^1 f \) is the pullback bundle, whose fiber at \( q \in M \) is given by \( \perp^1_{\sigma(q)} f \)). Thus \( \mathbb{E} \) is a vector bundle over \( M \) with fibers \( \mathbb{E}_q = L(\perp^1 f, \perp^1_{\sigma(q)} f) \), whose sections correspond in a natural way to the vector bundle homomorphisms of \( \perp^1 f \) along \( \sigma \). To get a connection on \( \mathbb{E} \), note that \( \nabla^\perp \) defines a connection on \( \perp^1 f \) (since \( \perp^1 f \subset \perp f \) is a parallel vector subbundle). The pullback of this connection via \( \sigma^* \) gives a connection on \( \sigma^* \perp^1 f \); thus we obtain the induced connection on \( \mathbb{E} \), such that parallel sections of \( \mathbb{E} \) correspond to parallel vector bundle homomorphisms. Its parallel displacement of an element \( \ell \in \mathbb{E}_q \) along a curve \( c: [0, 1] \to N \) with \( c(0) = q \) is given by
\[ \frac{1}{0} \left( [\frac{1}{c} c^\perp(\ell) \xi = [\frac{1}{0} [\sigma \circ c]^\perp \circ \ell \circ [\frac{0}{c} c^\perp] \xi \right) \text{ for all } \xi \in \perp^1_{c(1)} f . \]  
(81)

According to Part (c) of Proposition 5 the curvature tensor of \( \perp^1 f \) (which is the restriction of \( R^\perp \) to \( \perp^1 f \)) is a parallel tensor; thus the curvature tensor of \( \sigma^* \perp^1 f \) is given by
\[ R^\perp(T\sigma x, T\sigma y) \xi \text{ for all } x, y \in T_q M \text{ and } \xi \in \perp^1_{\sigma(q)} f . \]
is also parallel (since $\sigma$ is an isometry of $M$). Therefore, the induced curvature tensor of $E$ given for all $x, y \in T_q M$, $\ell \in E_q$ by
\[
R^E(x, y) \ell = R^E(T_q \sigma x, T_q \sigma y) \circ \ell - \ell \circ R^E(x, y)
\]
is parallel, too. As $E_o = L(\perp_{o} f, \perp_{o} f)$, we obtain for $s_0 := Id := Id_{\perp_{o} f}$
\[
\forall x, y \in T_0 M : R^E(x, y) s_0 = [R^E(x, y), Id] = 0 ,
\]
hence Equation (75) holds. Thus there exists a unique parallel section $s$ of $L(\perp_{o} f, \sigma^* \perp_{o} f)$ with $s(o) = Id$. Let $I_o$ denote the corresponding vector bundle homomorphism. To verify (79), notice that $\sigma \circ \gamma$ is the inverse curve $\gamma^{-1} : t \mapsto \gamma(-t)$. Because of
\[
I_o(\gamma(1)) = (1 \sigma^{-1}_f(\gamma^-1))(Id) (\perp) \gamma^{-1}_f(\gamma^-1) \circ Id \circ (1 \sigma^{-1}_f(\gamma^-1))^-f = (1 \sigma^{-1}_f(\gamma^-1))^-f,
\]
Equation (78) follows; we also see from the last Equation that $I_o(p)$ is an isometry for each $p \in M$, since $p$ can be joint with $o$ through some geodesic. We have $I_o^2 = Id$ on $\perp_{o} f$, thus $I_o^2 = Id$ follows from the parallelity of $I_o$.

To prove (80), let $c : [0, 1] \to M$ be a curve with $c(0) = p$ and $c(1) = q$, put $\sigma := \sigma^M_p$. Let $X, Y$ be parallel sections of $TM$ along $c$ with $X(1) = x$ and $Y(1) = y$. Consider the two sections $S_1$ and $S_2$ of $\perp_{o} f$ along the curve $\sigma \circ c$ defined by $S_1(t) := I_o(h(X(t), Y(t)))$ and $S_2(t) := h(T \circ X(t), T \sigma Y(t))$. Using (c) and the parallelity of $h$, we see that $S_1$ is a parallel section. $S_2$ is parallel, too, because $\sigma$ is an isometry of $M$. Furthermore $S_1(0) = S_2(0)$ holds, since we have (with $\tilde{x} := X(0)$, $\tilde{y} := Y(0)$):
\[
S_1(0) = I_o h(\tilde{x}, \tilde{y}) \overset{(c)}{=} h(\tilde{x}, \tilde{y}) = h(-\tilde{x}, -\tilde{y}) = h(T_p \sigma \tilde{x}, T_p \sigma \tilde{y}) = S_2(0) ;
\]
therefore $S_1 = S_2$, in particular (80) holds.

**Remark 5.** Even if $f$ is not parallel, then nevertheless it may happen that the involution $I_o$ described above exists. But one can easily show that (80) in addition implies the parallelity of $f$.

**Proof of Theorem 4.** The uniqueness of the described map on $Of$ follows immediately from Properties (b) and (c) described in Theorem 3. To prove its existence, we consider for each $p \in M$ the unique vector bundle isomorphism of $Of$ given by
\[
\forall q \in M, \forall x \in T_q M, \xi \in \perp_{o} f : \Sigma_p(x + \xi) = T_p \sigma^M_p(x) + I_p(\xi) ,
\]
where $I_p$ was defined in Proposition 9. Then $\Sigma_p$ is an involution of $Of$ and a fibrewise isometric, split-parallel (cf. Definition 9) vector bundle homomorphism along $\sigma^M_p$ according to (82) and Parts (a), (b) of Proposition 9. Furthermore, (82) combined with Part (c) of Proposition 9 and the equality $T_p \sigma^M_p = -Id_{T_q M}$ implies Part (c) of Theorem 9 whereas (83) and (80) follow from (82) combined with the well known facts that we have $\sigma^M_p(\gamma(1)) = \gamma(1)$ and
\[
T_{\gamma(1)} \sigma^M_p = -(\perp_{o} f) .
\]
(60) is an immediate consequence of (82) combined with (80). (71) follows immediately from (82) in combination with Part (d) of Proposition 9. It remains to establish Assertion (b) of Theorem 4.

For this, Equation (83) (the Gauß-Weingarten equation) and the split-parallelity of $\Sigma_p$ implies that $\Sigma_p$ is a parallel if and only if for all $q \in M$
\[
\forall x \in T_q M, v \in Of : \Sigma_p(h(x) v) = h(T \sigma^M_p x)(\Sigma_p v) .
\]
The result follows, since Equations (83) and (69) are equivalent as a consequence of Lemma 2.

Now suppose that $M$ is even a symmetric submanifold of $N$ with extrinsic symmetries $\sigma^M_p$ ($p \in M$), according to Definition 2. Then we have for all $x, y \in T_q M : T \sigma^M_p(h(x, y)) = h(T \sigma^M_p(x), T \sigma^M_p(y))$, hence $T \sigma^M_p(OM) \subset OM$ and thus $\Sigma_p := T \sigma^M_p | OM : OM \to OM$ satisfies Properties (a)-(c) stated in Theorem 9. Hence $\Sigma_p = \Sigma_p$, by uniqueness.
4.2 Homogeneity of $\perp^1 f$

In this section, $M$ is a simply connected Riemannian symmetric space which is isometric to a Riemannian product $M_1 \times \cdots \times M_r$ of irreducible symmetric spaces $M_i$ and $f : M \to N$ is a parallel isometric immersion. We aim to prove that $\perp^1 f$ is a homogeneous vector bundle over $M$. Let $i(M) = T^M \oplus p^M$ denote the Cartan decomposition, and let $\text{Sym}(M)$ denote the subgroup of $I(M)$ generated by its geodesic symmetries $\sigma_p^M$, where $p$ ranges over $M$. One can show that $\text{Sym}(M)$ is actually a Lie subgroup of $I(M)$ with $I(M)^0 \subset \text{Sym}(M)$ (in case $M$ is irreducible, this fact is explained in Sec. 3.3. of [J1]).

**Definition 11.** We will call a vector bundle $\mathbb{E}$ over $M$ a *homogeneous vector bundle* if there exists an action $\alpha : I(M)^0 \times \mathbb{E} \to \mathbb{E}$ by vector bundle isomorphisms such that the bundle projection of $\mathbb{E}$ is equivariant.

In the above situation, we consider $O f$ as a vector bundle over $M$ equipped with the connection $\nabla^{O f}$ described at the beginning of Section 4.

**Proposition 10.** (a) There exists a natural action $\alpha : \text{Sym}(M) \times O f \to O f$ where $\text{Sym}(M)$ acts through isometric, parallel vector bundle isomorphisms, characterized as follows: For each point $p$ of $M$ we have

$$\forall v \in \mathcal{O}_p f : \alpha(\sigma_p^M, v) = \Sigma_p(v) .$$

(b) $\alpha$ splits into two actions on $T M$ resp. on $\perp^1 f$ (denoted by $\perp^\top$ resp. by $\perp^\bot$), i.e.

$$\forall g \in \text{Sym}(M), x + \xi \in T_p M \oplus \perp^1 f : \alpha(g, x + \xi) = \alpha^\top(g, x) + \alpha^\bot(g, \xi) \in T M \oplus \perp^1 f ;$$

and we have for all $g \in \text{Sym}(M), x \in T_p M$

$$\alpha^\bot(g, x) = T_p g x .$$

Furthermore, the second fundamental form $h$ is $\alpha$-invariant in the following sense:

$$\forall g \in \text{Sym}(M), x, y \in T_p M : \alpha^\bot(g, h(x, y)) = h(\alpha(g, x), \alpha(g, y)) .$$

(c) $\perp^1 f$ is a homogeneous vector bundle over $M$ via the action of $\alpha^\bot$ restricted to $I(M)^0$.

(d) One can also show that the normal connection on $\perp^1 f$ is the canonical connection induced by the Cartan decomposition as described in Section 2.1 of [J1] (without proof).

**Proof.** Put $G := \text{Sym}(M)$, and let $\tilde{G}$ denote the subgroup of vector bundle isomorphisms on $O f$ generated by all $\Sigma_p$ with $p \in M$ (see Theorem 10); thus we have the natural action $\tilde{\alpha} : \tilde{G} \times O f \to O f$ and a surjective group homomorphism $\pi : \tilde{G} \to G$ such that $\pi(\Sigma_p) = \sigma_p^M$ for each $p \in M$, hence by Equation (88)

$$\forall g \in \tilde{G} : \tilde{\alpha}_g | T M = T \pi(g) .$$

Moreover, by means of (89) we have for arbitrary $g \in \tilde{G}$

$$\forall x, y \in T M : \tilde{\alpha}(g, h(x, y)) = h(T \pi(g) x, T \pi(g) y) .$$

which implies that $\pi$ is also injective. Therefore $\pi$ is an isomorphism; thus we may pointwise define $\alpha$ via $\alpha_{\pi(g)} = \tilde{\alpha}_g$ for $g \in \tilde{G}$. (88) and (89) imply (84)-(87). It remains to show that $\alpha$ is differentiable. By means of (88) and (89), this will be clear if $\alpha^\bot$ is differentiable. For this, let an arbitrary (differentiable) section $\xi$ of $\perp^1 f$ be given. For each point $p \in M$ there exists an open neighbourhood $U$ of $p$ in $M$, vector fields $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ on $U$ and $C^\infty$-functions $\lambda_1, \ldots, \lambda_k$ on $U$ such that $\xi|U = \sum_{i=1}^k \lambda_i h(X_i, Y_i)$. Then we have

$$\alpha^\bot(g, \xi)|U = \sum_{i=1}^k \lambda_i h(T g X_i, T g Y_i),$$

which is a differentiable function on $G \times U$. It follows that $\alpha^\bot$ is differentiable. For (c): Since $G$ is generated by the reflections $\sigma_p^M$, the equivariance of $\alpha^\bot$ follows from (73) combined with the construction of $\alpha$. Thus $\perp^1 f$ is a homogeneous vector bundle over $M$. \qed
It is planned to investigate parallel isometric immersions \( f : M \to N \) defined on a symmetric space \( M \) as above in a forthcoming paper \([J2]\).

## 5 Proof of Theorem 3

In this section, we will give the proof of Theorem 3. Let \( f : M \to N \) be a parallel isometric immersion defined on a simply connected symmetric space \( M \). Without loss of generality, we can assume that \( M \) is simply connected, for the following reason: Let \( \tau : M \to M \) denote the universal covering, and consider the isometric immersion \( \tilde{f} := f \circ \tau \) and the corresponding holonomy group \( \text{Hol}(\tilde{f}^*)TN \) with respect to some point \( \tilde{o} \in \tau^{-1}(o) \). Then it is well known that the connected components of \( \text{Hol}(f^*TN) \) and \( \text{Hol}(\tilde{f}^*TN) \) are equal, and thus the holonomy Lie algebras \( \mathfrak{hol}(f^*TN) \) and \( \mathfrak{hol}(\tilde{f}^*TN) \) are equal, too. Moreover, for the sake of an easier notation, we assume that \( M \) is a submanifold of \( N \) and \( f = \iota^M \).

For Part (c). Let us first lead the discussion on the level of the corresponding Holonomy groups. According to \([6], (7)\) combined with \([11]\), by \( \text{Hol}(f^*TN) \to \text{Hol}(\mathcal{O}_f) \), \( g \to g^{\mathcal{O}_f} \) is defined a surjective Lie group homomorphism. This map is even an isomorphism, which is seen as follows: Suppose \( g^{\mathcal{O}_f} = \text{Id} \) on \( \mathcal{O}_f \) for some \( g \in \text{Hol}(f^*TN) \). Using the \( \nabla^N \) parallelity of \( R^N \), we have

\[
\forall u, v, w \in T_o N : g(R^N(u, v) w) = R^N(gu, gv)(gw) ;
\]

hence the linear space \( V := \{ v \in T_o N \mid gv = v \} \) is curvature invariant. Therefore, because \( \mathcal{O}_f \subset V \) by assumption, \( f \) maps into the totally geodesic submanifold defined by \( V \), according to Theorem 2; thus \( g = \text{Id} \) by the fullness of \( f \). Switching to the level of the Lie algebras, the result hence follows. \( \square \)

For Part (a). Let \( \sigma^\perp : \mathcal{O}_f \to \mathcal{O}_f \) denote the linear reflection in \( \perp_o f \). We have to show that

\[
\text{Ad}(\sigma^\perp)(\mathfrak{hol}(\mathcal{O}_f)) = \mathfrak{hol}(\mathcal{O}_f). \tag{90}
\]

Let \( \Sigma_o \) denote the symmetry of \( \mathcal{O}_f \) at the point \( o \) described in Theorem 3 and let \( c : [0, 1] \to M \) be a loop with \( c(0) = o \). Remember that \( \Sigma_o \) is a \( \nabla^{\mathcal{O}_f} \)-parallel vector bundle isomorphism of \( \mathcal{O}_f \) along \( \sigma^\perp_o \), in accordance with Theorem 2; hence

\[
\sigma^\perp \circ (\|c\|^{\mathcal{O}_f}) = (\|c\|^{\mathcal{O}_f} \circ c)^{\mathcal{O}_f} \circ \sigma^\perp. \tag{91}
\]

From the last line we conclude that \( \text{Hol}(\mathcal{O}_f) \) is invariant by group conjugation with \( \sigma^\perp \); thus \( 90 \) holds.

For Equation 19. Let \( \text{ad} : \mathfrak{so}(\mathcal{O}_f) \to \text{End}(\mathfrak{so}(\mathcal{O}_f)) \), \( A \mapsto [A, \cdot] \) denote the adjoint representation of \( \mathfrak{so}(\mathcal{O}_f) \); thus \( 19 \) is equivalent to

\[
\forall x \in T_o M : \text{ad}(h(x)) (\mathfrak{hol}(\mathcal{O}_f)) \subset \mathfrak{hol}(\mathcal{O}_f). \tag{92}
\]

Let \( x \in T_o M \) and \( \gamma \) be the geodesic of \( M \) with \( \gamma(0) = o \), \( \gamma(0) = x \), and let \( \Theta_\gamma(t) \) \( (t \in \mathbb{R}) \) denote the family of vector bundle isomorphisms on \( \mathcal{O}_f \) along \( \theta(t) \) from Definition 10. Because \( \Theta_\gamma(t) \) is a \( \nabla^{\mathcal{O}_f} \)-parallel vector bundle isomorphism of \( \mathcal{O}_f \) along \( \theta(t) \) for each \( t \in \mathbb{R} \) (Corollary 4), we obtain for each loop \( c : [0, 1] \to M \) with \( c(0) = o \)

\[
\Theta_\gamma(t) \circ (\|c\|^{\mathcal{O}_f}) \circ \Theta_\gamma(t)^{-1} = (\theta(t) \circ c)^{\mathcal{O}_f} ; \tag{93}
\]

and therefore (by virtue of \([59], 48 \) and \([51]\))

\[
\mu_\gamma(t) \circ (\|c\|^{\mathcal{O}_f}) \circ \mu_\gamma(t)^{-1} = (\|c_t\|^{\mathcal{O}_f}) , \tag{94}
\]

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where \( c_t \) means the loop at \( o \) defined by going first along \( \gamma \) from \( o \) to \( \gamma(t) \), then along the loop \( \theta \gamma(t) \circ c \) centered at \( \gamma(t) \) and then back from \( \gamma(t) \) to \( o \) along the inverse curve \( \gamma^{-1} \). In accordance with Equations (6) and (7), this shows that l.h.s. of (94) is an element of \( \text{Hol}(O_f) \). Therefore, we have for each \( t \in \mathbb{R} \)

\[
\mu_t(t) \circ \text{Hol}(O_f) \circ \mu_t(t)^{-1} \subset \text{Hol}(O_f) , \quad \text{hence} \quad \text{Ad}(\mu_t(t))(\text{hol}(O_f)) \subset \text{hol}(O_f) \quad (95)
\]

(where \( \text{Ad} \) means the adjoint representation of \( \text{SO}(O_o,f) \)). Since \( \mu_t(t) = \exp(t \, \mathbf{h}(x)) \), Equation (92) follows by taking the derivative in (95) with respect to \( t \) at \( t = 0 \).

**For Part (b).** In the following, the simple relations between the parallel displacement resp. the curvature tensor of \( O_f \) and \( f^*TN \) described in (39)-(42) will be used without further reference. Taking into account (14) and (63), we can define the following linear subspaces of \( \text{so}(O_o,f) \):

\[
j_+ := \{ \mathbf{h}(x_1), \mathbf{h}(x_{i-1}), \ldots, \mathbf{h}(x_1), R^{OF}(y_1, y_2), \ldots, \}|x_1, \ldots, x_i \in T_o M, y_1, y_2 \in T_o M \}_{\mathbb{R}},
\]

\[
 j := \sum_{i=0}^3 j_i ,
\]

\[
 j_+ := \text{r.h.s. of (14)},
\]

\[
 j_- := \text{r.h.s. of (18)} .
\]

Because of Equations (63) and (64), we have \( j_+ \subset \text{so}(O_o,f)_+ \); and hence \( j_- \subset \text{so}(O_o,f)_- \), according to (14) and the rules for \( \mathbb{Z}/2\mathbb{Z} \) graded algebras. Let us now see that \( \text{hol}(O_f) = \sum_{i=0}^3 j_i = j_+ \oplus j_- \) holds; the proof will be divided into three steps.

**First step.** Let us see that we have \( j_+ \oplus j_- \subset j \subset \text{hol}(O_f) \): As a consequence of the Theorem of Ambrose/Singer, we have \( j_0 \subset \text{hol}(O_f) \) and hence \( j_i \subset \text{hol}(O_f) \) for each \( i = 0, \ldots, 3 \), according to (19); thus \( j \subset \text{hol}(O_f) \). Moreover, Proposition 7 implies that

\[
 \forall \xi, \eta \in \bigoplus^f \mathbb{R}^3 : (R^N(\xi, \eta))^O_{\bigoplus^f} \in j_0 + j_2 ;
\]

thus also \( j_+ \subset j_0 + j_2 \) and \( j_- \subset j_1 + j_3 \).

**Second step.** I claim that \( j_+ \oplus j_- \subset \text{so}(O_o,f) \) is a vector space invariant by \( \text{ad}(\mathbf{h}(x)) \) for each \( x \in T_o M \):

It suffices to show that

\[
 [\mathbf{h}(x), j_-] \subset j_+ ,
\]

which means for all \( z_1, z_2 \in T_o M , \xi, \eta \in \bigoplus^f \mathbb{R}^3 \):

\[
 [\mathbf{h}(x), [\mathbf{h}(y), (R^N(\xi, \eta))_{\bigoplus^f}]] \in j_+ \quad \text{and} \quad [\mathbf{h}(x), [\mathbf{h}(y), (R^N(\xi, \eta))_{\bigoplus^f}]] \in j_+ .
\]

(96) holds because of Proposition 7. For (97) choose \( v \in O_o f \); then by means of Lemma 9 (applied to (60)) we get

\[
 [\mathbf{h}(x), [\mathbf{h}(y), R^N(\xi, \eta)]] v = R^N (\mathbf{h}(x) \mathbf{h}(y) \xi, \eta) v + R^N (\xi, \mathbf{h}(x) h(y) \eta) v
\]

\[
 + R^N (\mathbf{h}(x) \xi, \mathbf{h}(y) \eta) v + R^N (\mathbf{h}(y) \xi, \mathbf{h}(x) \eta) v ,
\]

(98)

with \( U := \bigoplus^f \mathbb{R}^3 , W := T_o M \); which proves (97).

**Third step.** \( \text{hol}(O_f) \subset j_+ \oplus j_- \) is finally proved as follows: By virtue of the Theorem of Ambrose/Singer, the vector space \( \text{hol}(O_f) \) is generated by elements of the form

\[
 (\mathbf{h})_{\bigoplus^f} . R^{OF}(y_1, y_2) \circ (\mathbf{h})_{\bigoplus^f}
\]

(99)
for various curves \(c : [0, 1] \to M\) with \(c(0) = o\) and \(y_1, y_2 \in T_c(1)M\). Therefore, given such a curve \(c\) and \(y_1, y_2 \in T_c(1)M\), we introduce

\[
\forall t \in [0, 1] : \tilde{R}(t) := \left(\left\langle c, c' \right\rangle \circ R^{\text{O}f}(y_1, y_2) \circ \left(\left\langle t, c \right\rangle \circ f\right) \in \mathfrak{so}(\mathcal{O}_o f) \right).\]

Of course, by the previous, it suffices to show that

\[
\forall t \in [0, 1] : \tilde{R}(t) \in j_+ \oplus j_-. \tag{100}
\]

For this: Let \(\mu_c\) be the function defined in Equation (103). From Lemma 3 we get

\[
\tilde{R}(t) = \text{Ad}(\mu_c(t)) R^{\text{O}f}(\tilde{y}_1, \tilde{y}_2) , \tag{101}
\]

with \(\tilde{y}_i := \left(\left\langle c, c' \right\rangle \circ y_i\right)\) for \(i = 1, 2\). Introduce the linear space

\[
n_{\text{ad}}(j_+ \oplus j_-) := \{ A \in \mathfrak{so}(\mathcal{O}_o f) \mid \text{Ad}(A)(j_+ \oplus j_-) \subset j_+ \oplus j_- \}, \tag{102}
\]

which is actually the Lie algebra of the Lie subgroup of \(\mathcal{O}_o f\) given by

\[
N_{\text{Ad}}(j_+ \oplus j_-) := \{ g \in \mathcal{O}_o f \mid \text{Ad}(g)(j_+ \oplus j_-) = j_+ \oplus j_- \}. \tag{103}
\]

By means of the second step, we have \(h(X(t)) \in n_{\text{ad}}(j_+ \oplus j_-)\) for each \(t \in \mathbb{R}\) (where \(X : [0, 1] \to T_c M\) denotes the function defined by (17)), and therefore the left invariant vector field \(\tilde{X}\) defined on the Lie group \(\mathcal{O}_o f\) by \(\forall g \in \mathcal{O}_o f : \tilde{X}_t(g) := g \circ h(X(t))\) is tangential to the submanifold \(N_{\text{Ad}}(j_+ \oplus j_-)\). By means of (100), the curve \(\mu_c\) solves the ODE

\[
\dot{\mu}_c(t) = \tilde{X}_t(\mu_c(t)) \quad \text{with} \quad \mu_c(0) = \text{Id}. \tag{104}
\]

Thus we find that in fact \(\mu_c\) is a curve in \(N_{\text{Ad}}(j_+ \oplus j_-)\). Since \(\tilde{y}_i \in T_c M\) for \(i = 1, 2\) (in accordance with Definition 3), we moreover have \(R^{\text{O}f}(\tilde{y}_1, \tilde{y}_2) \in j_+\). From the previous, we finally conclude that \(\forall t \in [0, 1] : t \mapsto \text{Ad}(\mu_c(t)) R^{\text{O}f}(\tilde{y}_1, \tilde{y}_2)\) actually describes a curve into \(j_+ \oplus j_-\), which together with (101) proves (100).

\section{The extrinsic holonomy Lie algebra of a full symmetric submanifold...}

In this section, \(M^m\) is a full symmetric submanifold of a simply connected, irreducible symmetric space \(\mathcal{N}\), and \(o \in M\) is some origin. We will now calculate as explicitly as possible the extrinsic holonomy Lie algebra of \(M\).

\begin{proposition}
In the above situation, the Lie algebras \(\mathfrak{k}\) and \(\mathfrak{hol}(N)\) are isomorphic via \(\pi_2\).
\end{proposition}

\begin{proof}
It is well known that \(\pi_2\) is a faithful representation of \(\mathfrak{k}\) on \(T_N \mathcal{N}\); thus it is sufficient to verify that \(\pi_2(\mathfrak{k}) = \mathfrak{hol}(N)\) holds. By the Theorem of Ambrose/Singer we have \(\mathfrak{hol}(N) = \{ R_{N(x, y)} | x, y \in T_c \mathcal{N} \}_{x, y} \mathbb{R}\). Since \(\mathcal{N}\) is irreducible, the Lie algebra \(\mathfrak{i}(N)\) is semisimple (cf. [13, Ch. V, Prop. 4.2]) and hence we can apply [13, Ch. V, Part (iii) of Theorem 4.1].

In accordance with Definition 2 let \(\sigma_o^+ \in \mathfrak{i}(N)\) be the corresponding extrinsic symmetry of \(M\) at \(o\). Since \(\sigma_o^+\) is an isometry of \(N\) with \(\sigma_o^+(o) = o\) and \(T_o \sigma_o^+ = \sigma_o^+\), we have

\[
\sigma_o^+ \circ \text{Hol}(N) \circ \sigma_o^+ = \text{Hol}(N) \quad \text{and} \quad \text{Ad}(\sigma_o^+) \mathfrak{hol}(N) = \mathfrak{hol}(N) . \tag{104}
\]

(Where \(\sigma_o^+ : T_o N \to T_o N\) denotes the linear reflection in \(\perp_o M\); therefore the splitting \(T_o N = T_o M \oplus \perp_o M\) induces the splitting

\[
\mathfrak{hol}(N) = \mathfrak{hol}(N)_+ \oplus \mathfrak{hol}(N)_- \quad \text{with} \quad \mathfrak{hol}(N)_\pm := \mathfrak{hol}(N) \cap \mathfrak{so}(T_o N)_\pm . \tag{105}
\]

\end{proof}
Lemma 10. We have

$$\mathfrak{hol}(N)_+ = \{ R^N(x,y) | x,y \in T_o M \}_R + \{ R^N(\xi,\eta) | \xi,\eta \in \perp_o M \}_R \quad \text{and} \quad (107)$$

$$\mathfrak{hol}(N)_- = \{ R^N(x,\xi) | x \in T_o M, \xi \in \perp_o M \}_R . \quad (108)$$

Proof. On the one hand, $\sigma^+ R^N(u,v) \sigma^+ = R^N(\sigma^+ u, \sigma^+ v)$ for all $u,v \in T_o N$, thus $R^N(u,v) \in \mathfrak{so}(T_o N)_+$ (resp. $R^N(u,v) \in \mathfrak{so}(T_o N)_-$) if $u$ and $v$ are both contained in $T_o M$ or both in $\perp_o M$ (resp. if $u \in T_o M$ and $v \in \perp_o M$). On the other hand, $\mathfrak{hol}(N) = \{ R^N(x,y) | x,y \in T_o N \}_R$ by the Theorem of Ambrose/Singer (since $R^N$ is a parallel tensor). \hfill \Box

Proposition 12. We have $\mathcal{O}M = TN|M$. Consequently, we can introduce the splitting $\mathfrak{hol}(TN|M) = \mathfrak{hol}(TN|M)_+ \oplus \mathfrak{hol}(TN|M)_-$, in accordance with (10). The Lie algebra $\mathfrak{hol}(TN|M)_+$ coincides with $\mathfrak{hol}(N)_+$. Moreover, $\mathfrak{hol}(TN|M)_-$ is an ad($\mathfrak{hol}(N)_+$)-invariant linear subspace of $\mathfrak{hol}(N)_-$ i.e. we have

$$[\mathfrak{hol}(N)_+, \mathfrak{hol}(TN|M)_-] \subset \mathfrak{hol}(TN|M)_- . \quad (109)$$

Proof. $M$ is a 1-full, parallel submanifold of $N$ by virtue of Theorem (1) thus $\mathcal{O}M = TN|M$ holds. Comparing (17) and (107), we hence see that we have $\mathfrak{hol}(TN|M)_+ = \mathfrak{hol}(N)_+$. Since $\mathfrak{hol}(TN|M)$ is a Lie algebra, the last assertion now follows from the rules of $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras. \hfill \Box

Let $\text{Hol}(N)_+ := \{ g \in \text{Hol}(N) | \sigma^+ g o \sigma^+ = g \}$; then, according to (107), $(\text{Hol}(N), \text{Hol}(N)_+)$ is a symmetric pair (in the sense of $\text{He}$, Ch. 4, § 3), and, in accordance with (106), the Lie algebra of $\text{Hol}(N)_+$ is given by $\mathfrak{hol}(N)_+$. Moreover, with respect to the natural action of $\text{SO}(T_o N)$ on the Grassmannian $G_{m}(T_o N)$, the isotropy subgroup of $\text{Hol}(N)$ at $T_o M$ is given by $\text{Hol}(N)_+$; hence the quotient space $L := \text{Hol}(N)/\text{Hol}(N)_+$ is equipped with a natural inclusion $L \hookrightarrow G_m(T_o N)$, $g \circ \text{Hol}(N)_+ \hookrightarrow T_o g(T_o M)$. This maps $L$ onto a totally geodesic submanifold of the symmetric space $G_m(T_o N)$; then the metric on the tangent space $T_{[c]} L \cong \mathfrak{hol}(N)_-$ is given by $(A,B) = -1/2 \cdot \text{trace}(A \circ B)$. In this way, $L$ becomes a Riemannian symmetric space, and, moreover, (105) defines an orthogonal symmetric Lie algebra (in the sense of $\text{He}$, Ch. IV, § 1) such that the symmetric pair $(\text{Hol}(N), \text{Hol}(N)_+)$ is associated therewith (in the sense of $\text{He}$, Ch. IV, definition preceding Prop.3.6).

Lemma 11. (a) Let $\hat{L}$ denote the universal covering space of $L$. Then there exist symmetric spaces $L_1$ and $L_2$ such that $\hat{L} \cong L_1 \times L_2$ with $T_o \hat{L} \cong \mathfrak{hol}(TN|M)_-$ and $T_o \hat{L} \cong \mathfrak{hol}(TN|M)_+$ . In particular, if neither $\mathfrak{hol}(TN|M)_- = \mathfrak{hol}(N)_-$ nor $\mathfrak{hol}(TN|M) = \{ 0 \}$, then $L$ is reducible.

(b) For each subspace $V \subset \mathfrak{so}(T_o N)$ we introduce its centralizer in $\mathfrak{so}(T_o N)$, via

$$\mathfrak{c}(V) := \{ A \in \mathfrak{so}(T_o N) | \forall B \in V : A \circ B = B \circ A \} .$$

If $\dim (\mathfrak{c}(\mathfrak{hol}(N)_+)) \cap \mathfrak{hol}(N)_- < m$, then $\mathfrak{hol}(TN|M)_- \neq \{ 0 \}$.

(c) Suppose that $N$ is a Hermitian symmetric space and that $M \subset N$ is a Lagrangian submanifold; let $j$ denote the complex structure of $T_o M$. Then $j$ is orthogonal to $\mathfrak{hol}(TN|M)$.

(d) Suppose that $N^{4n}$ is a quaternionic K" ahler symmetric space with $n \geq 2$ and that $M^{2n}$ is a totally complex submanifold. Let a canonical basis $\{ i,j,k \}$ of the quaternionic structure of $T_o N$ be given such that $i(T_o M) = T_o M$ and $j(T_o M) = \perp_o M$ holds. Then both $j$ and $k$ are orthogonal to $\mathfrak{hol}(TN|M)$.

Proof. For (a): As before, we consider the orthogonal symmetric Lie algebra $g := \mathfrak{hol}(N)$. Suppose first that $\hat{L}$ is of compact type. Then, in accordance with Proposition (12)

$$[\mathfrak{hol}(N)_+, \mathfrak{hol}(TN|M)_-] \subset \mathfrak{hol}(TN|M)_- \quad \text{and} \quad [\mathfrak{hol}(N)_+, (\mathfrak{hol}(TN|M)_-)] \subset (\mathfrak{hol}(TN|M)_-) \quad \text{.}$$

*In accordance with $\text{BCQ}$, p. 290, we use the following convention: A Riemannian manifold $M$ is called “reducible” if its universal covering splits as a (non-trivial) product of two Riemannian spaces; otherwise $M$ is called “irreducible.”
therefore, Proposition 11 in combination with [He], Ch. V, Part (i) of Theorem 4.1, (applied to $\hat{L}$) shows that both $\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_-$ and $(\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_-)^\perp$ are $\hat{\mathfrak{h}}(\hat{L})$-invariant subspaces of $\hat{\mathfrak{h}}(\hat{N})_- \cong T_{[e]}\hat{L}$. Thus the result follows from the decomposition theorem of de Rham.

In the general case, note that the sectional curvature of $L$ is non-negative (because $L$ is totally geodesically embedded in $G_m(T_0 N)$), hence, according to [He], Theorem 3.1, in combination with Proposition 4.2, $\hat{L}$ is a product of a Euclidian space and a symmetric space of compact type (in fact, it may happen that $\hat{L}$ splits off a Euclidian factor; for example, if $N$ is a complex space form and $M$ is a Lagrangian submanifold of $N$ – cf. the proof of Theorem 6 in the next section); however, using a decomposition theorem for orthogonal symmetric Lie algebras (see [He], Theorem 1.1), and switching to the level of symmetric pairs, we easily reduce the problem to the case when already $L$ is of compact type (cf. the proof of [He], Proposition 4.1).

For (b): Assume that $\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_- = \{0\}$ holds. Then we have $[h(x), \hat{\mathfrak{h}}(N)_+] = \{0\}$ according to 18 and Proposition 12 hence $h(x) \in c(\hat{\mathfrak{h}}(N)_+)$ for each $x \in T_n M$. Furthermore, by virtue of 23 combined with Proposition 11 for each $x \in T_n M$ we have

$$h(x) \in \pi_2(t) \cap \mathfrak{s}o(T_n N)_- = \hat{\mathfrak{h}}(N)_-.$$  

Thus $h(T_n M)$ is actually a subspace of $c(\hat{\mathfrak{h}}(N)_+) \cap \hat{\mathfrak{h}}(N)_-$ and therefore $\dim c(\hat{\mathfrak{h}}(N)_+) \cap \hat{\mathfrak{h}}(N)_- \geq m$, as a consequence of Part (c) of Theorem 1.

For (c): If $M \subset N$ is a Lagrangian submanifold, then $J_p$ maps the tangent space of $T_p N$ onto the normal space $\perp_p M$ and vice versa; thus we have $J_p \in \mathfrak{s}o(T_p N)_-$ in accordance with 19, whereas the curvature invariance of $T_p M$ implies that $R^N(x, y) \in \mathfrak{s}o(T_p N)_+$ for all points $p \in M$ and $x, y \in T_p M$; hence $\text{trace}(J_p \circ R^N(x, y)) = 0$. Therefore, using the parallelity of $J$ in combination with the Theorem of Ambrose/Singer, we see that $\text{trace}(J \circ A) = 0$ holds for every $A \in \hat{\mathfrak{h}}(T\hat{N}|\hat{M})$, i.e. $j \in \hat{\mathfrak{h}}(T\hat{N}|\hat{M})^\perp$.

For (d): Let $Q$ denote the $\nabla^N$-parallel subbundle of $\mathfrak{s}o(T_n N)$ which defines the Quaternionic Kähler structure of $N$. Since $T_n M$ is a totally complex subspace of $T_n N$, according to Definition 2.7 of 13 there exists a canonical basis $\{i, j, k\}$ of $Q$, with the additional property that $i(T_0 M) = T_0 M$ and $j(T_0 M) = \perp_0 M$. Such a canonical basis is not unique; however, if $\{\bar{i}, \bar{j}, \bar{k}\}$ is a second canonical basis of $Q$, with this property, then we have $\bar{i} = \pm i$ and there exists some $\varphi \in [0, 2\pi]$ such that $\bar{j} = \cos(\varphi) j - \sin(\varphi) k$ and $\bar{k} = \sin(\varphi) j + \cos(\varphi) k$. Furthermore, by definition of a totally complex submanifold (see 13, Definition 2.8), the pullback bundle $Q|M$ is locally spanned by three sections $I, J, K$ which satisfy the usual quaternionic relations such that additionally $I_p(T_p M) = T_p M$ and $J_p(T_p M) = \perp_p M$ for all $p$. By the previous, without loss of generality we may assume that $I_0 = i$, $J_0 = j$ and $K_0 = k$ holds. Since $n \geq 2$, $I$ is even a $\nabla^N$-parallel section of $Q|M$, according to Lemma 2.10 of 13; hence the vector subbundle $Q$ which is locally spanned by $\{I, J, K\}$ is even a globally well defined, $\nabla^N$-parallel subbundle of $Q|M$. Furthermore, $Q_p \subset \mathfrak{s}o(T_p N)_-$ for each $p \in M$ (like in the Lagrangian case); now a proof which uses the same ideas as for Part (c) shows that $j$ and $k$ both belong to the orthogonal complement of $\hat{\mathfrak{h}}(T\hat{N}|\hat{M})$ in $\mathfrak{s}o(T_0 N)$.  

6.1 . . . in an ambient symmetric space of rank 1

Proof of Theorem 4 in case the rank of $N$ is 1. According to Theorem 3 it suffices to consider the following possibilities.

N is a real space form of non-vanishing sectional curvature: Here we have $\hat{\mathfrak{h}}(N) = \mathfrak{s}o(T_0 N)$ and we claim that $\hat{\mathfrak{h}}(T\hat{N}|\hat{M}) = \mathfrak{s}o(T_0 N)$ holds, unless $(n, m) = (4, 2)$ and $\dim(\hat{\mathfrak{h}}(T\hat{N}|\hat{M})) = 4$.

For this, remember that $\mathfrak{s}o(T_0 N)_+ \cong \mathfrak{s}o(T_0 M) \oplus \mathfrak{s}o(\perp_0 M)$ (see 8), and hence straightforward considerations show that $c(\mathfrak{s}o(T_0 N)_+) \cap \mathfrak{s}o(T_0 N)_- = \{0\}$ holds (since $m \geq 2$ by assumption). Thus the possibility $\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_- = \{0\}$ is excluded, as a consequence of Part (b) of Lemma 11. Furthermore, the symmetric space $L$ (described in the last section) corresponds to the real Grassmannian $G_m(T_0 N)$, which is an irreducible symmetric space unless $(n, m) \neq (4, 2)$. Therefore, if $(n, m) \neq (4, 2)$, then we have $\hat{\mathfrak{h}}(T\hat{N}|\hat{M}) = \mathfrak{h}(N)$ pursuant to Part (a) of Lemma 11. For $(n, m) = (4, 2)$, $G_m(T_0 N)$ is a 4-dimensional reducible symmetric space; its universal covering splits into two 2-dimensional factors. From Part (a) of Lemma 11 combined with the previous, we conclude that in this case $\dim(\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_-) \in \{2, 4\}$ and $\hat{\mathfrak{h}}(T\hat{N}|\hat{M})_+ = \mathfrak{s}o(T_0 N)_+$ holds. We thus obtain $\dim(\hat{\mathfrak{h}}(T\hat{N}|\hat{M})) \in \{4, 6\}$; moreover, in case $\dim(\hat{\mathfrak{h}}(T\hat{N}|\hat{M})) = 6$ we necessarily have $\hat{\mathfrak{h}}(T\hat{N}|\hat{M}) = \mathfrak{s}o(T_0 N)$.
\textbf{N is a complex space form and M is a complex submanifold:} Here we have $\mathfrak{hol}(N) = u(T_oN) = \mathbb{R} j \oplus \mathfrak{su}(T_oN)$, where $j$ denotes the complex structure of $T_oN$. We will show that always $\mathfrak{hol}(TN|M) = u(T_oN)$ holds, as follows:

We have $\mathfrak{hol}(N)_+ = u(T_oN) \cap \mathfrak{so}(T_oN)_+ \cong \mathfrak{su}(T_oM) \oplus \mathfrak{su}(\mathbb{R}_o M) \oplus j$ and thus we easily verify that $c(\mathfrak{hol}(N)_+) \cap \mathfrak{so}(T_oN)_- = \{0\}$ holds. Moreover, $L$ is isomorphic to the Grassmannian manifold of complex $m$-planes in $T_oN$, which is an irreducible symmetric space; therefore, by combining Parts (a) and (b) of Lemma 11 we obtain $\mathfrak{hol}(TN|M) = \mathfrak{hol}(N)$.

\textbf{N is a complex space form and M is a Lagrangian submanifold of N:} Here we have $\mathfrak{hol}(N) = u(T_oN) = \mathfrak{su}(T_oN) \oplus \mathbb{R} j$ and we aim to prove the equality $\mathfrak{hol}(TN|M) = \mathfrak{su}(T_oN)$.

For this, notice that $\mathfrak{hol}(N)_+$ resp. $\mathfrak{hol}(N)_-$ is given by $u(T_oN) \cap \mathfrak{so}(T_oN)_+$ resp. by $\mathbb{R} j \oplus u(T_oN) \cap \mathfrak{so}(T_oN)_-$, and hence the linear maps given by $\mathfrak{hol}(N)_+ \to \mathfrak{so}(T_oM)$, $A \mapsto A|T_oM$ and $\mathfrak{hol}(N)_- \to \mathfrak{so}(\mathbb{R}_o M)$, $A \mapsto A|\mathbb{R}_o M$ both are isomorphisms. Therefrom, we easily verify that $c(\mathfrak{hol}(N)_+) \cap \mathfrak{so}(T_oN)_- = \mathbb{R} j$ holds, hence we have dim $(c(\mathfrak{hol}(N)_+) \cap \mathfrak{hol}(N)_-) = 1$ and thus $\mathfrak{hol}(TN|M)_- = \{0\}$ is not possible by virtue of Part (b) of Lemma 11. Moreover, $L$ is isomorphic to the Grassmannian manifold of Lagrangian planes in $T_oN$, whose universal covering space is a product of an irreducible symmetric space and a 1-dimensional factor: The corresponding decomposition of “de Rham type” is given by $\mathfrak{hol}(N)_- = V_1 \oplus V_2$ with $V_1 := \mathbb{R} j$ and $V_2 := \mathfrak{su}(T_oN) \cap \mathfrak{so}(T_oN)_-$. Thus Part (a) of Lemma 11 implies that $\mathfrak{hol}(TN|M)_- = \{0\}$ is equal to one of the spaces $\mathbb{R} j$, $\mathfrak{su}(T_oN)_-$ or $\mathbb{R} j \oplus \mathfrak{su}(T_oN)_-$. But $j$ is orthogonal to $\mathfrak{hol}(TN|M)$, as a consequence of Part (c) of Lemma 11 therefore the only remaining possibility is $\mathfrak{hol}(TN|M) = \mathfrak{su}(T_oN)$.

\textbf{N}^{2n} is a quaternionic space form (with $n \geq 2$) and $M^{2n}$ is a totally complex submanifold of $N$: Here the holonomy Lie algebra $\mathfrak{hol}(N)$ is given by $\mathfrak{sp}(T_oN) \oplus \mathbb{Q}$, where $\mathbb{Q}$ denotes the quaternionic structure at $o$. Choose a canonical basis $\{i, j, k\}$ with $i(T_oM) = T_oM$, $j(T_oM) = \mathbb{R}_{o} M$ and $k(T_oM) = \mathbb{R}_{o} M$. Let us see that that $\mathfrak{hol}(TN|M) = \mathfrak{sp}(n) \oplus \mathbb{R} j$ holds:

We notice that $\mathfrak{hol}(N)_+ = \mathfrak{sp}(T_oN)_+ \oplus \mathbb{R} i$ and $\mathfrak{hol}(N)_- = \mathfrak{sp}(T_oN)_- \oplus \mathbb{R} j \oplus \mathbb{R} k$ with $\mathfrak{sp}(T_oN)_{\pm} := \mathfrak{sp}(T_oN) \cap \mathfrak{so}(T_oN)_{\pm}$ and that the linear maps $\mathfrak{sp}(T_oN)_+ \to u(T_oM)$, $A \mapsto A|T_oM$ and $\mathfrak{sp}(T_oN)_- \to u(\mathbb{R}_{o} M)$, $A \mapsto A|\mathbb{R}_{o} M$ both are isomorphisms. Hence $c(\mathfrak{sp}(T_oN)_+ \cap \mathfrak{so}(T_oN)_- = \{i, j\}_\mathbb{R}$, thus $c(\mathfrak{hol}(N)_+) \cap \mathfrak{so}(T_oN)_- = \{0\}$, since $i$ does not commute with $j$ or with $k$. Therefore, $\mathfrak{hol}(TN|M)_- = \{0\}$ again is not possible. Moreover, $L$ is isomorphic to the Grassmannian manifold of totally complex $2n$-planes in $T_oN$, whose universal covering is a product of two irreducible factors: The corresponding decomposition of “de Rham type” is given by $\mathfrak{hol}(N)_- = V_1 \oplus V_2$ with $V_1 := \mathfrak{sp}(T_oN)_-$ and $V_2 := \{i, j\}_\mathbb{R}$. Now the result follows by means of Part (d) of Lemma 11 combined with similar arguments as in the Lagrangian case.

\section*{6.2 \ldots in an ambient symmetric space of higher rank}

In this section, we will prove Theorem\cite{6} in case the ambient space $N$ is of higher rank. As usual, let $o \in N$ be an origin, $K$ denote the isotropy subgroup of $1^\circ (N)$ at $o$, $i(N) = \mathfrak{t} \oplus \mathfrak{p}$ the corresponding Cartan decomposition and $\pi_2 : \mathfrak{t} \to \mathfrak{so}(T_o N)$ the linearized isotropy representation (see Section\cite{2}).

Suppose that $M \subset N$ is a full symmetric submanifold with $o \in M$. Then Theorem\cite{6} ensures that there exists some $X \in \mathfrak{p}$ with $\text{ad}(X)^3 = \text{ad}(X)$ (in the non-compact case) resp. $\text{ad}(X)^3 = -\text{ad}(X)$ (in the compact case) such that $M$ belongs to the family of symmetric submanifolds associated with the symmetric R-space $\text{Ad}(K)X$ according to Definition\cite{8}: hence $M = M_c$ for some $c \neq 0$. Then proposition\cite{3} states that $T_oM = \pi^1(p_c), \mathbb{R}_o M = \pi_1(p_0)$ and $\forall Y \in \mathfrak{p}_c : h(\pi_1(Y)) = c \pi_2(J(Y))$ (where $J$ is the linear map defined in Lemma\cite{11}).

Lemma 12. By means of the identification $\mathfrak{t} \cong \mathfrak{hol}(N)$ from Proposition\cite{17}, the splittings $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_c$ (see \cite{24}) and $\mathfrak{hol}(N) = \mathfrak{hol}(N)_+ \oplus \mathfrak{hol}(N)_-$ (see \cite{105}, \cite{106}) are in correspondence with each other.

\textbf{Proof.} The result follows from the previous in combination with Equations\cite{26}, \cite{106} and \cite{31}. \hfill $\square$
Proposition 13. We always have \( h(T_oM) = \mathfrak{hol}(TN|M)_- \). Furthermore, the decomposition \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(TN|M)_+ \oplus \mathfrak{hol}(TN|M)_- \) mentioned in Proposition 12 is given by \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N)_+ \oplus \mathfrak{hol}(N)_- \). Therefore, \( \mathfrak{hol}(TN|M) \) contains the ideal \( \mathfrak{hol}(N, \mathfrak{hol}(N)) \).

Proof. Because of \( (27) \) combined with Proposition 3 we have \( h(T_oM) = \pi_2(\mathfrak{t}_o) = \mathfrak{hol}(N)_- \), where the second equality follows from Lemma 12. Therefore, as a consequence of \( (18) \) and Proposition 12 we have \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N)_+ \oplus \mathfrak{hol}(N)_- \mathfrak{hol}(N)_+ \). The last assertion is now seen from

\[
[\mathfrak{hol}(N), \mathfrak{hol}(N)] = [\mathfrak{hol}(N)_+, \mathfrak{hol}(N)_+] + [\mathfrak{hol}(N)_-, \mathfrak{hol}(N)_-] + [\mathfrak{hol}(N)_-, \mathfrak{hol}(N)_+] \\
\subset \mathfrak{hol}(N)_+ \oplus [\mathfrak{hol}(N)_-, \mathfrak{hol}(N)_+] .
\]

Suppose that \( N \) is a Hermitian symmetric space. Then we have the splitting \( \mathfrak{t} = \mathfrak{c} \oplus [\mathfrak{k}, \mathfrak{k}] \) with a one-dimensional factor \( \mathfrak{c} \), the center of \( \mathfrak{t} \). Moreover, there exists \( Z \in \mathfrak{c} \) with \( \pi_2(Z) = j \), where the latter denotes the complex structure of \( T_oN \) (cf. [BCO], A. 4).

Lemma 13. We have \( \mathfrak{c} \subset \mathfrak{t}_o \). Furthermore, \( M \) is a Lagrangian submanifold of \( N \).

Proof. Let \( \sigma_o^\perp \) denote the extrinsic symmetry of \( M \) at \( o \) and consider the (second) involution \( \tau := \text{Ad}(\sigma_o^\perp) \) on \( i(N) \). Then we have \( \tau(\mathfrak{t}) = \mathfrak{t} \) and (in the notation of Section 6)

\[
\forall X \in \mathfrak{t} : \pi_2(\tau X) = \sigma^\perp \circ \pi_2(X) \circ \sigma^\perp .
\]

Thus Lemma 12 implies that the splitting \( \mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{c} \) is also the decomposition of \( \mathfrak{t} \) into the +1 and -1 eigenspaces of \( \tau \); therefore, and since \( \tau \) is a Lie algebra involution, it maps \( \mathfrak{c} \) onto itself, hence either \( \mathfrak{c} \subset \mathfrak{t}_0 \) or \( \mathfrak{c} \subset \mathfrak{c} \). By contradiction, if we had \( \mathfrak{c} \subset \mathfrak{t}_0 \), then \( [X, Z] = 0 \) according to \( (24) \), thus

\[
j(\pi_1(X)) = \pi_2(Z) \pi_1(X) = \pi_1(\text{ad}(Z)X) = 0
\]

(because \( Z \) belongs to the center of \( \mathfrak{t} \)); therefore, and since j is the complex structure of \( T_oN \), we have \( \pi_1(X) = 0 \) and hence \( X = 0 \), which is not possible. Thus we obtain \( \mathfrak{c} \subset \mathfrak{t} \), and therefore \( j \in \pi_2(\mathfrak{c}) \subset \mathfrak{so}(T_oN)_- \), which, by virtue of \( (9) \), implies that \( j \) maps \( T_oM \) to \( \perp_oM \) and vice versa. Since \( M \) is an extrinsically homogeneous submanifold of \( N \), we hence see that \( M \) is already a Lagrangian submanifold.

Proof of Theorem 4 in case \( N \) is of higher rank. If \( N \) is of Hermitian type, then \( M \) is a Lagrangian submanifold of \( N \) according to Lemma 13. Furthermore, Proposition 13 implies that \( [\mathfrak{hol}(N), \mathfrak{hol}(N)] \subset \mathfrak{hol}(TN|M) \subset \mathfrak{hol}(N) = [\mathfrak{hol}(N), \mathfrak{hol}(N)] \oplus \mathbb{R} j \). Because the complex structure \( j \) is orthogonal to \( \mathfrak{hol}(TN|M) \) as a result of Lemma 11 in fact we have \( \mathfrak{hol}(TN|M) = [\mathfrak{hol}(N), \mathfrak{hol}(N)] \). Moreover, \( j = \pi_2(j) \in \pi_2(\mathfrak{t}_o) = h(T_oM) \) by means of Lemma 13 combined with Proposition 13.

If \( N \) is not of Hermitian type, then \( \mathfrak{t} \) is semisimple (because \( N \) is not of Hermitian type and hence the center of \( i(N) \) is trivial) and thus we have \( \mathfrak{t} = [\mathfrak{t}, \mathfrak{t}] \). Therefore, \( \mathfrak{hol}(TN|M) = \mathfrak{hol}(N) \) follows by virtue of Proposition 13.

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