A simple proof for the multivariate Chebyshev inequality

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Abstract
In this paper a simple proof of the Chebyshev’s inequality for random vectors obtained by Chen (2011) is obtained. This inequality gives a lower bound for the percentage of the population of an arbitrary random vector $X$ with finite mean $\mu = E(X)$ and a positive definite covariance matrix $V = Cov(X)$ whose Mahalanobis distance with respect to $V$ to the mean $\mu$ is less than a fixed value. The proof is based on the calculation of the principal components.

Keywords: Chebyshev (Tchebychev) inequality, Mahalanobis distance, Principal components, Ellipsoid.

1 Introduction
The very well known Chebyshev’s inequality for random variables provides a lower bound for the percentage of the population in a given distance with respect to the population mean when the variance is known. It can be obtained from the Markov’s inequality which can be stated as follows. If $Z$ is a non-negative random variable with finite mean $E(Z)$ and $\varepsilon > 0$, then

$$\varepsilon \Pr(Z \geq \varepsilon) = \varepsilon \int_{[\varepsilon, \infty)} dF_Z(x) \leq \int_{[\varepsilon, \infty)} xdF_Z(x) \leq \int_{[0, \infty)} xdF_Z(x) = E(Z)$$

(where $F_Z(x) = \Pr(Z \leq x)$ is the distribution function of $Z$), that is,

$$\Pr(Z \geq \varepsilon) \leq \frac{E(Z)}{\varepsilon}.$$  (1)

Chebyshev’s inequality is then obtained as follows. If $X$ is a random variable with finite mean $\mu = E(X)$ and variance $\sigma^2 = Var(X) > 0$, then by taking $Z = (X - \mu)^2 / \sigma^2$ in (1), we get

$$\Pr \left( \frac{(X - \mu)^2}{\sigma^2} \geq \varepsilon \right) \leq \frac{1}{\varepsilon}$$  (2)

for all $\varepsilon > 0$. It can also be written as

$$\Pr((X - \mu)^2 < \varepsilon \sigma^2) \geq 1 - \frac{1}{\varepsilon}$$

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or as
\[ \Pr(|X - \mu| < r) \leq 1 - \frac{\sigma^2}{r^2} \]
for all \( r > 0 \).

There are several extensions of these results to the multivariate case (see e.g. Chen (2011); Marshall and Olkin (1960) and the references therein). Recently, Chen (2011) proved the following Chebyshev’s inequality
\[ \Pr((X - \mu)'V^{-1}(X - \mu) \geq \varepsilon) \leq \frac{n}{\varepsilon} \]
for all \( \varepsilon > 0 \) and for all random vectors \( X = (X_1, \ldots, X_n)' \) (w' denotes the transpose of w) with finite mean vector \( \mu = E(X) \) and positive definite covariance matrix \( V = Cov(X) = E((X - \mu)(X - \mu)') \).

Extensions of Chen’s result to Hilbert-space-valued and Banach-space-valued random elements can be seen in Prakasa Rao (2010) and Zhou and Hu (2012), respectively.

In this paper a new (in my knowledge) proof for Chen’s result is given. The proof is based on the calculation of the principal components. The main advantage of the new proof is that it is so simple that it can be can be included in all the basic multivariate analysis text books. Some comments are also included after the proof. In these comments, the case in which \(|V| = 0\) is analyzed. Also some consequences in regression analysis are given.

### 2 Main result

**Theorem 1.** Let \( X = (X_1, \ldots, X_n)' \) be a random vector with finite mean vector \( \mu = E(X) \) and positive definite covariance matrix \( V = Cov(X) \). Then
\[ \Pr((X - \mu)'V^{-1}(X - \mu) \geq \varepsilon) \leq \frac{n}{\varepsilon} \tag{3} \]
for all \( \varepsilon > 0 \)

**Proof.** Let us consider the random variable
\[ Z = (X - \mu)'V^{-1}(X - \mu). \]
As \( V \) is positive definite, then \( Z \geq 0 \). Moreover, as \( V \) is also symmetric, there exists an orthogonal matrix \( T \) such that \( TT' = T'T = I_n \) and \( T'VT = D \), where \( I_n \) is the identity matrix of order \( n \) and \( D = diag(\lambda_1, \ldots, \lambda_n) \) is the diagonal matrix with the ordered eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_n > 0 \). Then \( V = TDT' \) and \( V^{-1} = TD^{-1}T' \). Therefore
\[ Z = (X - \mu)'TD^{-1}T'(X - \mu) = [D^{-1/2}T'(X - \mu)]'[D^{-1/2}T'(X - \mu)] = Y'Y, \]
where
\[ Y = D^{-1/2}T'(X - \mu) \]
and \( D^{-1/2} = diag(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2}) \). The random vector \( Y \) satisfies
\[ E(Y) = E(D^{-1/2}T'(X - \mu)) = D^{-1/2}T'E(X - \mu) = 0 \]
and
\[ Cov(Y) = Cov(D^{-1/2}T'(X - \mu)) = D^{-1/2}T'VD^{-1/2} = D^{-1/2}DD^{-1/2} = I_n. \]
Therefore
\[ E(Z) = E(Y'Y) = E\left(\sum_{i=1}^{n} Y_i^2\right) = \sum_{i=1}^{n} E(Y_i^2) = \sum_{i=1}^{n} \text{Var}(Y_i) = n. \]

Hence, from Markov’s inequality (1), we get
\[ \Pr(Z \geq \varepsilon) = \Pr((X - \mu)'V^{-1}(X - \mu) \geq \varepsilon) \leq \frac{E(Z)}{\varepsilon} = \frac{n}{\varepsilon} \]
for all \( \varepsilon > 0. \)

**Remark 2.** Of course, if \( n = 1 \) in (3), then the univariate Chebyshev inequality (2) is obtained. The vector \( Y = D^{-1/2}T'(X - \mu) \) used in the preceding proof is the vector of the standardized principal components of \( X \). The inequality in (3) can also be written as
\[ \Pr((X - \mu)'V^{-1}(X - \mu) < \varepsilon) \geq 1 - \frac{n}{\varepsilon} \quad (4) \]
for all \( \varepsilon > 0 \). This inequality says that the ellipsoid
\[ E_\varepsilon = \{x \in \mathbb{R}^n : (x - \mu)'V^{-1}(x - \mu) < \varepsilon\} \]
contains at least the \( 100(1 - n/\varepsilon)\% \) of the population for all \( \varepsilon \geq n \) for any random vector \( X \). It is well known that the principal components coincide with the projections to the principal axes of that ellipsoid. For example, for \( \varepsilon = 4n \), we have
\[ \Pr((X - \mu)'V^{-1}(X - \mu) < 4n) \geq 0.75. \]

The inequality can also be written as
\[ \Pr(d_V(X, \mu) < r) \geq 1 - \frac{n}{r^2}, \]
where
\[ d_V(x, y) = \sqrt{(x - y)'V^{-1}(x - y)} \]
is the Mahalanobis distance associated with the positive definite matrix \( V \). Hence (4) gives a lower bound for the percentage of points in spheres “around” the mean in the Mahalanobis distance. A comparison between the volume in these spheres and that in the regions containing the same probability in other multivariate Chebyshev inequalities can be seen in (Chen [2014]).

**Remark 3.** Recall that in the preceding theorem \( X \) is an arbitrary non-singular random vector with finite mean and finite variances-covariances. In particular, if \( X \) has a normal distribution, then the ellipsoid \( E_\varepsilon \) coincides with regions determined by the level curves of the normal probability density function. Moreover, in this case, the exact probability can be obtained by using that \( Y \) is normally distributed with mean \( E(Y) = 0 \) and \( \text{Cov}(Y) = I_n \). Hence \( Y_1, \ldots, Y_n \) are independent and identically distributed with a common standard normal distribution and
\[ Z = Y'Y = Y_1^2 + \cdots + Y_n^2 \]
has a chi-squared distribution with \( n \) degrees of freedom (a well known result, see, e.g., page 39 in Mardia et al. [1979]). For example, for \( n = 2 \), we obtain
\[ \Pr((X - \mu)'V^{-1}(X - \mu) < 8) = \Pr(\chi^2_2 < 8) = 0.9816844 \geq 0.75. \]
Remark 4. If $|V| = 0$ and $X$ is non-degenerate, that is,

$$\lambda_1 \geq \cdots \geq \lambda_{m-1} > \lambda_m = \cdots = \lambda_n = 0$$

for an $m \in \{2, \ldots, n\}$, then we can consider $Y = BT'(X - \mu)$, where $B = \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_{m-1}^{-1/2}, 0, \ldots, 0)$ and by using the preceding theorem we obtain

$$\Pr((X - \mu)'TC(X - \mu) < \varepsilon) \geq 1 - \frac{m - 1}{\varepsilon},$$

where $C = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_{m-1}^{-1}, 0, \ldots, 0)$. This inequality says that the ellipsoid on the region determined by the point $\mu$ and the $m-1$ first principal components contains at least the $100(1-(m-1)/\varepsilon)$% of the population of the random vector $X$.

Remark 5. The inequality in (3) can be applied to conditional random vectors. For example, if we consider $(X, Y)$ where $X = (X_1, \ldots, X_k)$ and $Y = (X_{k+1}, \ldots, X_n)$, $\mu(x) = E(Y|X = x)$ is finite and $V(x) = \text{Cov}(Y|X = x)$ is a positive definite matrix for a fixed $x$, then

$$\Pr([Y - \mu(x)]' [V(x)]^{-1} [Y - \mu(x)] \geq \varepsilon | X = x) \leq \frac{n - k}{\varepsilon}$$

for all $\varepsilon > 0$. This inequality gives a confidence region around the regression map $\mu(x) = E(Y|X = x)$. Similar results can be obtained for other conditional random vectors as, e.g., $(Y|X \geq x)$.

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