Nonlinear Effects in the Cosmic Microwave Background *

Roy Maartens†

Relativity and Cosmology Group, Division of Mathematics and Statistics,
Portsmouth University, Portsmouth PO1 2EG, England.

Major advances in the observation and theory of cosmic microwave background anisotropies have opened up a new era in cosmology. This has encouraged the hope that the fundamental parameters of cosmology will be determined to high accuracy in the near future. However, this optimism should not obscure the ongoing need for theoretical developments that go beyond the highly successful but simplified standard model. Such developments include improvements in observational modelling (e.g. foregrounds, non-Gaussian features), extensions and alternatives to the simplest inflationary paradigm (e.g. non-adiabatic effects, defects), and investigation of nonlinear effects. In addition to well known nonlinear effects such as the Rees-Sciama and Ostriker-Vishniac effects, further nonlinear effects have recently been identified. These include a Rees-Sciama-type tensor effect, time-delay effects of scalar and tensor lensing, nonlinear Thomson scattering effects and a nonlinear shear effect. Some of the nonlinear effects and their potential implications are discussed.

1. INTRODUCTION

In 1948, Alpher and Herman predicted a cosmic microwave background (CMB) radiation, which remained a theoretical possibility until the serendipitous discovery by Penzias and Wilson in 1965. A potential crisis for cosmology, arising from the apparent perfect isotropy of the CMB, was turned into a dramatic success when in 1992 the COBE satellite detected small anisotropies at the level predicted by the standard inflationary cold dark matter model. The confluence of theoretical and observational successes has opened up a new era of precision-tested cosmology, exemplified by the upcoming launch of the MAP and Planck satellites (see e.g. [1]). Together with parallel advances in galactic, supernovae, lensing and other observations, this has promoted the hope that the fundamental parameters of the standard cosmological model ($\Omega_m$, $\Omega_\Lambda$, $h$, $n$, ...) will be determined to high accuracy, and that in some sense, cosmology will be “solved” up to details of fine-tuning.

However, physics rarely turns out be as simple as this, and optimism needs to be tempered by a realisation that future observations could entail not only the resolution of old problems, but also the generation of new and unexpected ones (see also [4]). The new “golden age” of cosmology is perhaps better seen as the end of the beginning, rather than the beginning of the end.

*Invited contribution to Relativistic Cosmology, a symposium celebrating GFR Ellis’ 60th, Cape Town, February 1999

†roy.maartens@port.ac.uk
It is clearly necessary to refine and develop the predictions of the standard model in readiness for the emerging observational data, and much effort has been put into this (see e.g. [3–7]). But it is also necessary to develop beyond the simplified standard model, in recognition of the rich complexity of the universe, and in anticipation of new surprises from future observations. This encompasses a range of theoretical advances:

- The modelling of observations needs significant development in relation to foreground extraction, non-Gaussian features, data analysis, etc. (see e.g. [8,9]).
- Extensions and alternatives to the simple inflationary paradigm need to be explored, including investigation of non-adiabatic effects (during inflation and reheating), of string cosmology (see e.g. [10]), and of defect theories (see e.g. [11,12]). Such exploration can lead to unexpected results; for example, it has recently been shown [13] that resonant amplification of metric perturbations in preheating after inflation could re-process the primordial power spectrum, and leave an imprint on CMB anisotropies.
- Nonlinear effects in CMB anisotropies require further study.

This article focuses on the latter aspect, in particular on recently identified nonlinear effects. Well-known nonlinear effects include the Ostriker-Vishniac [14] and kinetic Sunyaev-Zeldovich [15] effects, which are local effects that arise from the modulation of the Doppler effect with inhomogeneities in the optical depth. These effects could restore small-scale power via hot cluster gas or re-ionisation (see e.g. [16]). The Rees-Sciama effect [17] arises as a second-order contribution to the integrated Sachs-Wolfe term. Nonlinear integrated effects can in principle be significant because of the long photon paths after last scattering. In practice the Rees-Sciama effect is typically very small. More recently, it has been shown [18] that a larger effect follows from gravitational lensing of CMB photons by density perturbations.

Considerable effort has been put into calculating the imprint of these particular nonlinear effects on CMB anisotropies, and this work is of great importance for isolating the secondary anisotropies and revealing the primordial spectrum [19]. What has received less attention is a systematic approach to nonlinear effects in general. This is important because it could reveal new sources of secondary anisotropies that may be significant. Furthermore, a systematic analysis of second-order effects forms the basis for estimating the theoretical errors associated with the linear analysis on which CMB science rests.

Mollerach and Matarrese [20] recently developed a systematic second-order Sachs-Wolfe analysis, building on previous work by Pyne and Carroll [21]. They found the second-order metric perturbations to a matter-dominated flat universe in the Poisson gauge by transforming the known synchronous gauge solutions, using the methods of Bruni et al. [22]. Then these solutions were used to find the second-order Sachs-Wolfe effect. The main relevant effects which they identified are:

- in addition to the Rees-Sciama effect, another nonlinear integrated Sachs-Wolfe effect representing corrections to the linear gravitational wave contribution;
- in addition to gravitational lensing by linear density perturbations, gravitational lensing by linear gravitational wave modes;
• time delay effects of scalar and tensor lensing;
• a coupling of the velocity at last scattering with the perturbed photon wave vector.

An alternative approach to nonlinear effects is developed in [23], and will be briefly described here. Instead of starting from a background model and perturbing, this approach starts from the fully nonlinear model of the inhomogeneous universe. The approach is thus more general, although this generality comes at the cost of greater difficulty in quantifying the effects on CMB anisotropies. More importantly, the alternative approach covers nonlinearity not only in the metric, but also in Thomson scattering, and in relative motions of the particle species. (Second-order scattering effects have previously been investigated via a different approach by Hu et al. [24].) It thus offers a foundation for a comprehensive second- and higher-order analysis, taking into account all sources of nonlinearity. (Nonlinear polarisation has not been treated, although key elements of such a treatment can be found in Challinor’s linear analysis [25].)

The key local effects identified in the new approach are:
• at each multipole order $\ell$, a coupling of baryonic bulk velocity to the radiation brightness multipoles of order $\ell \pm 1$;
• a coupling of the acceleration, vorticity and shear at order $\ell$ to the brightness multipoles of order $\ell$, $\ell \pm 1$, $\ell \pm 2$;
• for $\ell \gg 1$, the shear coupling is dominant and could be important.

These local effects need to be integrated to evaluate their impact on CMB anisotropies, but a qualitative understanding has been achieved as a first step.

2. NONLINEAR DYNAMICS

The covariant Lagrangian, or 1+3 covariant, approach is based on a physical choice of a 4-velocity vector field $u^a$. All variables are in principle physically measurable by comoving observers. This approach is inherently nonlinear. It starts from the inhomogeneous and anisotropic universe, without a priori restrictions on the degree of inhomogeneity and anisotropy, and then applies the Friedmann limit when required. The basic theoretical ingredients are: the covariant Lagrangian hydrodynamics of Ehlers and Ellis [26,27], and the perturbation theory of Hawking [28] and Ellis and Bruni [29] which is derived from it; the covariant Lagrangian approach to kinetic theory of Ellis et al. [30]; and the covariant analysis of temperature anisotropies introduced by Maartens et al. [31] and developed by Challinor and Lasenby [32,23] and Gebbie et al. [33].

The projected symmetric tracefree (PSTF) parts of vectors and rank-2 tensors are

$$V_{(a)} = h_a{}^b V_b, \quad S_{(ab)} = \left\{ h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h^{cd} h_{ab} \right\} S_{cd},$$

where $h_{ab} = g_{ab} + u_a u_b$ is the projector, with $g_{ab}$ the spacetime metric. The skew part of a projected rank-2 tensor is spatially dual to the projected vector $S_a = \frac{1}{2} \varepsilon_{abc} S^{[bc]}$, where $\varepsilon_{abc} = \eta_{abcd} u^d$ is the projection of the spacetime alternating tensor. Any projected rank-2 tensor has the irreducible decomposition
\[ S_{ab} = \frac{1}{3} S h_{ab} + \varepsilon_{abc} S^c + S_{(ab)} , \]
where \( S = S_{cd} h^{cd} \) is the spatial trace. A covariant vector product and its generalization to PSTF rank-2 tensors are
\[ [V, W]_a = \varepsilon_{abc} V^b W^c , \quad [S, Q]_a = \varepsilon_{abc} S^b Q^{cd} . \]
The covariant derivative \( \nabla_a \) produces time and spatial derivatives
\[ J^{a\cdots b} = u^c \nabla_c J^{a\cdots b} , \quad D_a J^{a\cdots b} = h^d_c h^a_e \cdots h^f_d \nabla_d J^{e\cdots f} . \]
The projected derivative \( D_a \) further splits irreducibly into a spatial divergence and curl
\[ \text{div} V = D^a V_a , \quad (\text{div} S)_a = D^b S_{ab} , \]
\[ \text{curl} V_a = \varepsilon_{abc} D^b V^c , \quad \text{curl} S_{ab} = \varepsilon_{cd(a} D^c S_{b)} d . \]
Relative motion of comoving observers is encoded in the kinematic quantities: the expansion \( \Theta = D^a u_a \), the 4-acceleration \( A_a \equiv u_a = A_{(a)} \), the vorticity \( \omega_a = -\frac{1}{3} \text{curl} u_a \), and the shear \( \sigma_{ab} = D_{(a} u_{b)} \). The dynamic quantities describe the sources of the gravitational field: the (total) energy density \( \rho = T_{ab} u^a u^b \), isotropic pressure \( p = \frac{1}{3} h_{ab} T^{ab} \), energy flux \( q_a = -T_{(a)b} u^b \), and anisotropic stress \( \pi_{ab} = T_{(ab)} \), where \( T_{ab} \) is the total energy-momentum tensor. The locally free gravitational field, i.e. the part of the spacetime curvature not directly determined locally by dynamic sources, is given by the Weyl tensor \( C_{abcd} \). This splits irreducibly into the gravito-electric and gravito-magnetic fields
\[ E_{ab} = C_{abcd} u^c u^d = E_{(ab)} , \quad H_{ab} = \frac{1}{3} \varepsilon_{acd} C^{cd} b e u^e = H_{(ab)} , \]
which provide a covariant Lagrangian description of tidal forces and gravitational radiation.

The Ricci identity for \( u^a \) and the Bianchi identities \( \nabla^d C_{abcd} = \nabla_a (\nabla_b C_{cd} + \frac{1}{6} R g_{bc}) \) produce the fundamental evolution and constraint equations governing the above covariant quantities [26,27]. Einstein’s equations are incorporated via the algebraic replacement of the Ricci tensor \( R_{ab} \) by \( T_{ab} = \frac{1}{2} T^c_g g_{ab} \). These equations, in fully nonlinear form and for a general source of the gravitational field, are:

**Evolution:**

\[
\begin{align*}
\dot{\rho} + (\rho + p) \dot{\Theta} + \text{div} q &= -2 A^a q_a - \sigma_{ab} \pi_{ab} , \\
\dot{\Theta} + \frac{\dot{\omega}}{3} \Theta^2 + \frac{1}{2} (\rho + 3p) - \text{div} A &= -\sigma_{ab} \sigma^{ab} + 2 \omega_a \omega^a + A_a A^a , \\
\dot{q}_{(a)} + \frac{4}{3} \Theta q_a + (\rho + p) A_a + D_a p + (\text{div} \pi)_a &= -\sigma_{ab} q^b + [\omega, q]_a - A^b \pi_{ab} , \\
\dot{\omega}_{(a)} + \frac{2}{3} \Theta \omega_a + \frac{1}{2} \text{curl} A_a &= \sigma_{ab} \omega^b , \\
\dot{\pi}_{(ab)} + \frac{2}{3} \Theta \sigma_{ab} + E_{ab} - \frac{1}{2} \pi_{ab} - D_{(a} A_{b)} &= -\sigma_{c(a} \sigma_{b)}^c + \omega_{(a} \omega_{b)} + A_{(a} A_{b)} , \\
E_{(ab)} + \Theta E_{ab} - \text{curl} H_{ab} + \frac{1}{2} (\rho + p) \sigma_{ab} + \frac{1}{2} \pi_{(ab)} + \frac{1}{6} \Theta \pi_{ab} + \frac{1}{2} D_{(a} q_{b)} &= -A_{(a} q_{b)} + 2 A^e \varepsilon_{cd(a} H^d_{b)} + 3 \sigma_{c(a} \sigma_{b)}^c + \omega_{(a} \omega_{b)} + \sigma_{c(a} \pi_{b)}^c + \frac{1}{2} \omega^c \varepsilon_{cd(a} E^d_{b)} - \frac{1}{2} \sigma_{c(a} \pi_{b)}^c - \frac{1}{2} \omega^c \varepsilon_{cd(a} \pi_{b)}^d , \\
H_{(ab)} + \Theta H_{ab} + \text{curl} E_{ab} - \frac{1}{2} \text{curl} \pi_{ab} &= 3 \sigma_{c(a} H_{b)}^c - \omega^c \varepsilon_{cd(a} H^d_{b)} - 2 A^c \varepsilon_{cd(a} E^d_{b)} - \frac{3}{2} \omega_{(a} q_{b)} + \frac{1}{2} \sigma_{c(a} \pi_{b)}^c .
\end{align*}
\]
The dynamic quantities as measured in the overall frame. The inverse velocity relation is

\[
\text{div } \omega = A^a \omega_a ,
\]

(8)

\[
(\text{div } \sigma)_a - \text{curl } \omega_a - \frac{2}{3} D_a \Theta + q_a = -2[\omega, A]_a ,
\]

(9)

\[
\text{curl } \sigma_{ab} + D_{(a} \omega_{b)} - H_{ab} = -2 A_{(a} \omega_{b)} ,
\]

(10)

\[
(\text{div } E)_a + \frac{1}{2} (\text{div } \pi)_a - \frac{1}{3} D_a \rho + \frac{1}{3} \Theta q_a = [\sigma, H]_a
\]

\[
- 3 H_{ab} \omega^b + \frac{1}{2} \sigma_{ab} q^b - \frac{3}{2} [\omega, q]_a ,
\]

(11)

\[
(\text{div } H)_a + \frac{1}{2} \text{curl } q_a - (\rho + p) \omega_a = -[\sigma, E]_a
\]

\[
- \frac{1}{2} [\sigma, \pi]_a + 3 E_{ab} \omega^b - \frac{1}{2} \pi_{ab} \omega^b .
\]

(12)

If the universe is almost Friedmann, then quantities that vanish in the Friedmann limit are \(O(\epsilon)\), where \(\epsilon\) is a dimensionless smallness parameter, and the quantities are suitably normalised (e.g. \(\sqrt{\epsilon} \sigma^a b / \Theta < \epsilon\), etc.). Linearisation reduces all the right hand sides of the evolution and constraint equations to zero.

For a given choice of fundamental frame \(u^a\), each of the species \(I\) which source the gravitational field has relative velocity \(v^a\) and 4-velocity

\[
u^a = \gamma^a \left( u^a + v^a \right), \quad v^a u_a = 0.
\]

(13)

If the 4-velocities are close, i.e. if the frames are in non-relativistic relative motion, then \(O(v^2)\) terms may be dropped from the equations, except if we include nonlinear kinematic, dynamic and gravito-electric/magnetic effects, in which case, for consistency, we must retain \(O(\epsilon v^2)\) terms such as \(\rho v^2\), which are of the same order of magnitude in general as \(O(\epsilon^2)\) terms. This is the physically relevant nonlinear regime, i.e. the case where only nonrelativistic bulk velocities are considered, but no restrictions are imposed on non-velocity terms, and we neglect only terms \(O(\epsilon v^2, v^3)\). Second-order effects involve neglecting also terms \(O(\epsilon^3)\), while linearisation means neglecting terms \(O(\epsilon^2, \epsilon v, v^2)\).

The dynamic quantities in the evolution and constraint equations (11)-(12) are the total quantities, with contributions from all dynamically significant particle species. Thus

\[
T^{ab} = \sum_I T^{ab}_I = \rho u^a u^b + ph^{ab} + 2 q^{(a}_u u^{b)} + \pi^{ab} ,
\]

(14)

\[
T^{ab}_I = \rho_i u^a u^b + p_i h^{ab} + 2 q^{(a}_i u^{b)} + \pi^{ab}_I .
\]

(15)

The dynamic quantities \(\rho_i, \cdots\) in equation (13) are as measured in the \(I\)-frame. For cold dark matter (CDM), baryons, photons and neutrinos

\[
\rho_C = 0 = \pi_C^{ab} , \quad q_C^a = 0 = \pi_C^{ab} , \quad p_R = \frac{1}{3} \rho_R , \quad p_N = \frac{1}{3} \rho_N ,
\]

where we have chosen the unique 4-velocity in the CDM and baryonic cases which follows from modelling these fluids as perfect. The cosmological constant is characterized by

\[
\rho_v = -\rho_v = -\Lambda , \quad q_v^a = 0 = \pi_v^{ab} , \quad v_v^a = 0 .
\]

The conservation equations for the species are best given in the overall \(u^a\)-frame, as are the evolution and constraint equations above. This requires the expressions for the partial dynamic quantities as measured in the overall frame. The inverse velocity relation is
The total energy flux is given by

$$u^a = \gamma_i \left( u_i^a + v_i^* a \right), \quad v_i^* a = -\gamma_i \left( v_i^a + v_i^2 u_i^a \right),$$

where $v_i^* a u_i a = 0$, and $v_i^* a v_i a = v_i^a v_i a$. Then the dynamic quantities of species $I$ as measured in the overall $u^a$-frame are

$$\rho_i^* = \rho_i + \left\{ \gamma_i^2 u_i^2 (\rho_i + p_i) + 2\gamma_i q_i^a v_i a + \pi_i^{ab} v_i a v_i b \right\},$$  \hspace{1cm} (16)

$$p_i^* = p_i + \frac{1}{3} \left\{ \gamma_i^2 u_i^2 (\rho_i + p_i) + 2\gamma_i q_i^a v_i a + \pi_i^{ab} v_i a v_i b \right\},$$  \hspace{1cm} (17)

$$q_i^{* a} = q_i^a + (\rho_i + p_i) v_i a$$

$$+ \left\{ (\gamma_i - 1) q_i^a - \gamma_i q_i^b v_i b u_i a + \gamma_i^2 v_i^2 (p_i + p_i) v_i a$$

$$+ \pi_i^{ab} v_i b - \pi_i^{bc} v_i b v_i c u_i a \right\},$$  \hspace{1cm} (18)

$$\pi_i^{* ab} = \pi_i^{ab} + \left\{ -2 u_i (a \pi_i^{b c}) v_i c + \pi_i^{bc} v_i b v_i c u_i a u_i b \right\}$$

$$+ \left\{ -\frac{1}{3} \pi_i^{cd} v_i c v_i d h_i^{ab} + \gamma_i^2 (p_i + p_i) v_i^a v_i^b + 2\gamma_i v_i^a (v_i^a) \right\}. \hspace{1cm} (19)$$

These are the nonlinear generalisations of well-known linearised results, which correspond to removing all terms in braces, dramatically simplifying the expressions. To linear order, there is no difference in the dynamic quantities when measured in the $I$-frame or the fundamental frame, apart from a simple velocity correction to the energy flux. But in the general nonlinear case, this is no longer true.

The total dynamic quantities are simply given by

$$\rho = \sum \rho_i^* , \quad p = \sum p_i^* , \quad q^a = \sum q_i^{* a} , \quad \pi^{ab} = \sum \pi_i^{* ab}.$$  \hspace{1cm}

A convenient choice for each partial 4-velocity $u_i^a$ is the energy frame, i.e. $q_i^a = 0$ for each $I$. In the fundamental frame, the partial energy fluxes do not vanish, i.e. $q_i^{* a} \neq 0$, and the total energy flux is given by

$$q^a = \sum \left[ (\rho_i + p_i) v_i^a + \pi_i^{ab} v_i b + O(\epsilon u_i^a, v_i^3) \right].$$

Then the dynamic quantities of CDM as measured in the fundamental frame are

$$\rho_C^* = \gamma_C^2 \rho_C , \quad p_C^* = \frac{1}{3} \gamma_C^2 v_C^2 \rho_C , \hspace{1cm} (20)$$

$$q_C^{* a} = \gamma_C^2 \rho_C v_C^a , \quad \pi_C^{* ab} = \gamma_C^2 \rho_C \langle v_C^{(a} v_C^{b)} \rangle , \hspace{1cm} (21)$$

while for baryonic matter:

$$\rho_B^* = \gamma_B^2 \left( 1 + w_B v_B^2 \right) \rho_B , \quad p_B^* = \left[ w_B + \frac{1}{3} \gamma_B^2 v_B^2 (1 + w_B) \right] \rho_B , \hspace{1cm} (22)$$

$$q_B^{* a} = \gamma_B^2 (1 + w_B) \rho_B v_B^a , \quad \pi_B^{* ab} = \gamma_B^2 (1 + w_B) \rho_B \langle v_B^{(a} v_B^{b)} \rangle , \hspace{1cm} (23)$$

where $w_B \equiv p_B / \rho_B$. For radiation and neutrinos, the dynamic quantities relative to the $u^a$-frame are found directly via kinetic theory below.

The total energy-momentum tensor is conserved, i.e. $\nabla_b T_i^{ab} = 0$, which is equivalent to the evolution equations (1) and (3). The partial energy-momentum tensors obey

$$\nabla_b T_i^{ab} = J_i^a = U_i^a u^a + M_i^{* a}, \hspace{1cm} (24)$$
where \( U^a \) is the rate of energy density transfer to species \( I \) as measured in the \( u^a \)-frame, and \( M^{\alpha a}_I = M^{(a)}_I \) is the rate of momentum density transfer. Thus

\[
J^a_C = 0 = J^a_N, \quad J^a_R = -J^a_B = U_T u^a + M^a_T,
\]

where the Thomson rates are, to \( O(\epsilon v^2_B, v^3_B) \),

\[
U_T = n_e \sigma_T \left( \frac{4}{3} \rho^*_R v^2_B - q^*_R v_{ba} \right),
\]

\[
M^a_T = n_e \sigma_T \left( \frac{4}{3} \rho^*_R v^a_B - q^*_R + \pi^{a,ab}_R v_{ba} \right),
\]
as shown below. Here \( n_e \) is the free electron number density, and \( \sigma_T \) is the Thomson cross-section. Note that beyond linear order there is energy transfer, i.e. \( U_T \neq 0 \).

Using equations (20)–(23) in (24), we find that, to \( O(\epsilon v^2, v^3) \), for CDM

\[
\dot{\rho}_C + \Theta \rho_C + \rho_C \text{div} v_C = - \left( \rho_C v^2_C \right) - \frac{4}{3} \epsilon v^2 \rho v^2_B \\
- v^a_C D_a \rho_C - 2 \rho_C A_a v^a_C,
\]

(25)

\[
\dot{v}^a_C + \frac{1}{3} \Theta v^a_C + A^a = A^a_B \nu^b C u^a - \sigma^a_b v^b_C + [\omega, v_C]^a - v^b_C D_b v^a_C,
\]

(26)

and for baryonic matter

\[
\dot{\rho}_B + \Theta (1 + w_B) \rho_B + (1 + w_B) \rho_B \text{div} v_B = - \left[ (1 + w_B) \rho_B v^2_B \right] \\
- \frac{4}{3} \epsilon v^2_B \Theta (1 + w_B) \rho_B - v^a_B D_a \left[ (1 + w_B) \rho_B \right] \\
- 2 (1 + w_B) \rho_B A_a v^a_B - n_e \sigma_T \left( \frac{4}{3} \rho^*_R v^2_B - q^*_R v_{ba} \right),
\]

(27)

\[
(1 + w_B) v^a_B + \left( \frac{1}{3} - c^2_B \right) \Theta v^a_B + (1 + w_B) A^a_B + \rho_B^{-1} D^a p_B \\
+ \rho_B^{-1} n_e \sigma_T \left( \rho^*_R v^a_B - q^*_R \right) = (1 + w_B) A^a_B \nu^b C u^a_B \\
- (1 + w_B) \sigma^a_b v^b_B + (1 + w_B) [\omega, v_B]^a - (1 + w_B) v^b_B D_b v^a_B \\
+ c^2_B (1 + w_B) (\text{div} v_B) v^a_B - \rho_B^{-1} n_e \sigma_T \pi_{R}^{a,ab} v_{bb},
\]

(28)

where \( c^2_B \equiv \dot{\rho}_B / \rho_B \) (this equals the adiabatic sound speed only to linear order). These are the nonlinear energy conservation and relative velocity equations for matter. Linearization reduces the right hand sides to zero, dramatically simplifying the equations. The conservation equations for the massless species (radiation and neutrinos) are found below.

### 3. NONLINEAR THOMSON SCATTERING

In covariant Lagrangian kinetic theory \[30\] the photon 4-momentum is split as

\[
p^a = E (u^a + e^a), \quad e^a e_a = 1, \quad e^a u_a = 0,
\]

where \( E = -u_a p^a \) is the energy and \( e^a = p^{(a)} / E \) is the direction, as measured by a co-moving (fundamental) observer. The photon distribution function is expanded in covariant harmonics.
\[ f(x,p) = f(x,E,e) = F + F_a e^a + F_{ab} e^a e^b + \cdots = \sum_{\ell \geq 0} F_{A\ell}(x,E)e^{(A\ell)}, \]

where \( e^{A\ell} \equiv e^{a_1} e^{a_2} \cdots e^{a_\ell} \) and

\[ F_{a\cdots b} = F_{(a\cdots b)} \Leftrightarrow F_{a\cdots b} = F_{(a\cdots b)}^* , \quad F_{a\cdots b} u^b = 0 = F_{a\cdots b} h^{bc} . \]

The first 3 multipoles arise from the radiation energy-momentum tensor,

\[ T^{ab}_R(x) = \int p^a p^b f(x,p) d^3 p = \rho^* u^a u^b + \frac{1}{3} p^* h^{ab} + 2 q^*_{(a} u^{b)} + \pi^{*ab} . \]

The radiation brightness multipoles are

\[ \Pi_{a_1 \cdots a_\ell} = \int E^3 F_{a_1 \cdots a_\ell} dE , \]

so that (dropping asterisks) \( \Pi = \rho_R / 4\pi , \quad \Pi^a = 3q^a_R / 4\pi \) and \( \Pi^{ab} = 15\pi^{ab}_R / 8\pi \). These multipoles define the temperature fluctuations [31].

The Boltzmann equation is

\[ \frac{df}{dv} \equiv \frac{p^a}{\partial x^a} - \Gamma_{bc}^a p^b p^c \frac{\partial f}{\partial p^a} = C[f] = b + b_a e^a + b_{ab} e^a e^b + \cdots , \]

where the collision term \( C[f] \) determines the rate of change of \( f \) due to emission, absorption and scattering processes. For simplicity, the effects of polarization are neglected (see [25] for a linear treatment), and

\[ C[f] = \sigma_x n_x E_B \left[ \tilde{f}(x,p) - f(x,p) \right] , \]

where \( E_B = -p_a u^a_B \) is the photon energy relative to the baryonic frame \( u^a_B \), and [32]

\[ \tilde{f}(x,p) = \frac{3}{16\pi} \int f(x,p') \left[ 1 + \left( e^{a'}_{b' \alpha} e^{\alpha}_{a'} \right)^2 \right] d\Omega_{b'} . \quad (29) \]

Here \( e^{\alpha}_{b' \alpha} \) is the initial and \( e^{a}_{b} \) is the final direction, so that

\[ p'^a = E_B \left( u^a_B + e^a_B \right) , \quad p^a = E_B \left( u^a_B + e^a_B \right) , \]

where we have used \( E'_B = E_B \), which follows since the scattering is elastic. The exact forms of the photon energy and direction in the baryonic frame are

\[ E_B = E_{\gamma_B} \left( 1 - v^a_B e^a_B \right) , \]

\[ e^a_B = \frac{1}{\gamma_B (1 - v^a_B e^a_B)} \left[ e^a + \gamma_B^2 \left( v^b_B e^c_B - v^2_B \right) u^a + \gamma_B^2 \left( v^b_B e^c_B - 1 \right) v^a_B \right] . \]

Since the baryonic frame will move non-relativistically relative to the fundamental frame in all cases of physical interest, it is sufficient to linearize only in \( v^a_B \), and not in the other quantities. Thus we drop terms in \( O(\epsilon v^2_B, v^2_B) \) but do not neglect terms that are \( O(\epsilon^0 v^2_B, \epsilon v^2_B) \) or \( O(\epsilon^2) \) relative to the background. It follows from equation (29) that
\[ 4\pi \int \tilde{f} E_B^3 \, dE_B = (\rho_R)_B + \frac{3}{4}(\pi^{ab}_R)_B e_B^a e_B^b, \]

where the dynamic radiation quantities are evaluated in the baryonic frame. Transforming back to the fundamental frame, we find, to \( O(\epsilon v_B^2, v_B^3) \),

\[
(\rho_R)_B = \rho_R \left[ 1 + \frac{4}{3} v_B^2 \right] - 2 q_R a v_B, \\
(\pi^{ab}_R)_B = \pi_R^{ab} + 2 v_B c \pi^{(a}_R u^{b)} - 2 q_R^{(a}_B v^{b)} + \frac{4}{3} \rho_R v^{(a}_B v^{b)}. 
\]

Using the above equations and various identities \([33]\), and defining the energy-integrated scattering multipoles

\[ K_{A\ell} = \int E^2 b_{A\ell} \, dE, \]

we find that to \( O(\epsilon v_B^2, v_B^3) \)

\[
K = n_E \sigma_T \left[ \frac{4}{3} \Pi_B^{\ell} v_B^2 - \frac{1}{3} \Pi_B^{a} v_B^{a} \right], \\
K^a = -n_E \sigma_T \left[ \Pi^a - 4 \Pi_B^{a} - \frac{4}{3} \Pi_B^{b} v_B^{ab} \right], \\
K^{ab} = -n_E \sigma_T \left[ \frac{4}{9} \Pi_B^{ab} - \frac{1}{2} \Pi_B^{(a} v_B^{b)} - \frac{3}{7} \Pi_B^{abc} v_B^{bc} - 3 \Pi_B^{(a} v_B^{b)} \right], \\
K^{abc} = -n_E \sigma_T \left[ \Pi_B^{abc} - \frac{2}{3} \Pi_B^{(ab} v_B^{c)} - \frac{4}{9} \Pi_B^{abc} v_B^{d} \right],
\]

and, for \( \ell > 3 \):

\[ K^{A_\ell} = -n_E \sigma_T \left[ \Pi^{A_\ell} - \Pi^{(A_{\ell-1} v_B^{a})} - \left( \frac{\ell + 1}{2\ell + 3} \right) \Pi^{A_{\ell} a} v_B^{a} \right]. \]

Equations (30)–(34) are a nonlinear generalisation of the linearised Thomson scattering results \([33]\). They show clearly the coupling of baryonic bulk velocity to the radiation multipoles, arising from local nonlinear effects in Thomson scattering.

The multipoles of \( E^{-1} \, df/\, dv \) are derived in \([30]\) and \([33]\), using different methods. The result is

\[
\begin{align*}
\left. \mathcal{F}_{(A_\ell)} = & -\frac{3}{3} \Theta E F_{(A_\ell)} + D_{(a_\ell} F_{A_{\ell-1})} + \frac{2}{2\ell + 3} \Pi^{a_\ell} F_{A_\ell} \\
- & \frac{(\ell + 1)}{(2\ell + 3)} E^{-(\ell + 1)} \left[ E^{\ell + 2} F_{A_\ell} \right] A^a - E^\ell \left[ E^{1 - \ell} F_{(A_{\ell-1})} \right] A_{a_\ell} \\
- & \ell (a_\ell \epsilon^{bc} (a_\ell F_{(A_{\ell-1})}^c) - \frac{(\ell + 1)(\ell + 2)}{(2\ell + 3)(2\ell + 5)} E^{-(\ell + 2)} \left[ E^{\ell + 3} F_{ab} \right] A_{a_\ell}^{bc} \\
- & \frac{2\ell}{(2\ell + 3)} E^{-(\ell + 1)} \left[ E^{3/2} F_{b(A_{\ell-1})} \right] \sigma_{a_\ell}^b - E^{2 - \ell} \left[ E^{2 - \ell} F_{(A_{\ell-2})} \right] \sigma_{a_{\ell-1} a_\ell} 
\end{align*}
\]

where a prime denotes \( \partial/\partial E \). This result is exact and holds for any photon or (massless) neutrino distribution in any spacetime. Integrating, it leads to
The quadrupole evolution equation is
\[ K_{A\ell} = \Pi_{(A\ell)} + \frac{4}{3} \Theta \Pi_{A\ell} + A_{(a\ell} \Pi_{A\ell-1)} + \frac{(\ell + 1)}{(2\ell + 3)} D^b \Pi_{bA\ell} \]
\[ - \frac{(\ell + 1)(\ell - 2)}{(2\ell + 3)} A^b \Pi_{bA\ell} + (\ell + 3) A_{(a\ell} \Pi_{A\ell-1)} - \ell \omega^b \varepsilon_{bc(a\ell} \Pi_{A\ell-1)c} \]
\[ - \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{(2\ell + 3)(2\ell + 5)} \sigma^{bc} \Pi_{bcA\ell} + \frac{5\ell}{(2\ell + 3)} \sigma^{b(a\ell} \Pi_{A\ell-1)b} \]
\[ - (\ell + 2) \sigma^{(a\ell \ell-1) A\ell-2)} \cdot \] (35)

For decoupled neutrinos, \( K_{A\ell} = 0 \), while for (unpolarised) photons undergoing Thomson scattering, the left hand side of equation (35) is given by equations (30)–(34), which are exact in the kinematic and dynamic quantities, but first order in the baryonic bulk velocity. These equations show the coupling of acceleration, vorticity and shear to the radiation multipolesthat arises at nonlinear level.

The monopole and dipole of equation (33) imply photon conservation of energy and momentum density (to \( O(\nu_B^2, v_B^3) \)):
\[ \dot{\rho}_\nu + \frac{4}{3} \Theta \rho_R + D_a q^a_R + 2A_\nu q^a_R + \sigma_{ab} \pi^{ab}_R = n_\nu \sigma_T \left( \frac{4}{3} \rho_R v_B^2 - q^a_R v_B^a \right), \] (36)
\[ \dot{q}^{(a)}_R + \frac{4}{3} \Theta q^a_R + \frac{4}{3} \rho_R A^a + \frac{1}{5} D^a \rho_R + D_b \pi^{ab}_R + \sigma^a_b \pi^b_R - [\omega, q^a_R] + A_\nu \pi^{ab}_R = n_\nu \sigma_T \left( \frac{4}{3} \rho_R v_B^a - q^a_R + \pi^{ab}_R v_B^b \right). \] (37)

The quadrupole evolution equation is
\[ \dot{\pi}^{(ab)}_R + \frac{4}{3} \Theta \pi^{ab}_R + \frac{8}{15} \rho_R \sigma^{ab} + \frac{2}{5} D^{(ab} q^{b)}_R + \frac{8\pi}{35} D_c \Pi^{abc} \]
\[ + 2A^{(ab} q^{b)}_R - 2\omega^c \varepsilon_{cd} (a \pi^{b)}_R) + \frac{2}{7} \sigma^c (a \pi^{b)}_R) - \frac{32\pi}{315} \sigma_{cd} \Pi^{abcd} \]
\[ = n_\nu \sigma_T \left[ \frac{9}{10} \pi^{ab}_R - \frac{1}{5} \pi^{(a} v_B^{b)} - \frac{8\pi}{35} \Pi^{abc} v_{bc} - \frac{2}{5} \rho_R v^{(a} v_B^{b)} \right], \] (38)

and the higher multipoles (\( \ell > 3 \)) evolve according to
\[ \Pi^{(A\ell)} + \frac{4}{3} \Theta \Pi^{A\ell} + D^{(a\ell} \Pi^{A\ell-1)} + \frac{(\ell + 1)}{(2\ell + 3)} D_b \Pi^{bA\ell} \]
\[ - \frac{(\ell + 1)(\ell - 2)}{(2\ell + 3)} A_b \Pi^{bA\ell} + (\ell + 3) A^{(a\ell} \Pi^{A\ell-1)} - \ell \omega^b \varepsilon_{bc(a\ell} \Pi^{A\ell-1)c} \]
\[ - \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{(2\ell + 3)(2\ell + 5)} \sigma_{bc} \Pi^{bcA\ell} + \frac{5\ell}{(2\ell + 3)} \sigma^b (a\ell \Pi^{A\ell-1)b} \]
\[ - (\ell + 2) \sigma^{(a\ell \ell-1) A\ell-2)} \cdot \] (39)

For \( \ell = 3 \), the second term in square brackets on the right of equation (39) must be multiplied by \( \frac{2}{7} \). The temperature fluctuation multipoles are determined by the radiation brightness multipoles \( \Pi_{A\ell} \).

These equations show in a transparent and covariant form precisely which physical effects are directly responsible for the evolution of CMB anisotropies in an inhomogeneous universe.
They apply at second and higher-order, and are readily specialised to the linear case. They show how the matter generates anisotropies: directly through interaction with the radiation, as encoded in the Thomson scattering terms on the right of equations (36)–(39); and indirectly through the generation of inhomogeneities in the gravitational field via the field equations (1)–(12) and the evolution equation (28) for the baryonic velocity $v_B^a$. This in turn feeds back into the multipole equations via the kinematic quantities, the baryonic velocity $v_B^a$, and the spatial gradient $D_a \rho_b$ in the dipole equation (37). The coupling of the multipole equations themselves provides an up and down cascade of effects, shown in general by equation (39). Power is transmitted to the $\ell$-multipole by lower multipoles through the dominant (linear) distortion term $D^{(a\ell} \Pi_{\ell^{-1}}^{a\ell)}$, as well as through nonlinear terms coupled to the 4-acceleration ($A^{(a\ell} \Pi_{\ell^{-1}}^{a\ell)}$), baryonic velocity ($v_B^{(a\ell} \Pi_{\ell^{-1}}^{a\ell)}$), and shear ($\sigma^{(a\ell} \Pi_{\ell^{-2}}^{a\ell)}$). Simultaneously, power cascades down from higher multipoles through the linear divergence term $(\text{div} \, \Pi)^A\ell$, and the nonlinear terms coupled to $A^a$, $v_B^a$ and $\sigma^{ab}$. The vorticity coupling does not transmit across multipole levels.

4. Conclusion

A range of nonlinear effects is identified via a systematic covariant analysis. These include the following.

- **Nonlinear relative velocity** corrections, as exemplified in the dynamic quantities in equations (16)–(19) and in the bulk velocity equations (26) and (28).

- **Nonlinear Thomson scattering** corrections affect the baryonic and radiation conservation equations, entailing a coupling of the baryonic bulk velocity $v_B^a$ to the radiation energy density, momentum density and anisotropic stress. The evolution of the radiation quadrupole $\pi_{ab}^R$ also acquires nonlinear Thomson corrections, which couple $v_B^a$ to the radiation dipole $q_{\ell}^a$ and octopole $\Pi_{abc}$. Linearization, by removing these terms, has the effect of removing the nonlinear contribution of the radiation multipoles $\Pi_{A\ell \pm 1}$ to the collision multipole $K^{A\ell}$.

- **Nonlinear kinematic** corrections introduce additional acceleration and shear terms. Vorticity corrections are purely nonlinear, i.e. a linear approach could give the false impression that vorticity has no direct effect at all on the evolution of CMB anisotropies. However, for very high $\ell$, i.e. on very small angular scales, the nonlinear vorticity term could in principle be non-negligible. The general evolution equation (39) for the radiation brightness multipoles $\Pi_{A\ell}$ shows that **five successive multipoles**, i.e. for $\ell-2, \cdots, \ell+2$, are linked together in the nonlinear case. The 4-acceleration $A_{\ell}$ couples to the $\ell \pm 1$ multipoles, the vorticity $\omega_{\ell}$ couples to the $\ell$ multipole, and the shear $\sigma_{\ell}$ couples to the $\ell \pm 2$ and $\ell$ multipoles. All of these couplings are nonlinear, except for $\ell = 1$ in the case of $A_{\ell}$, and $\ell = 2$ in the case of $\sigma_{\ell}$. These latter couplings that survive linearisation are shown in the dipole equation (37) (i.e. $\rho_{\ell} A^a$) and the quadrupole equation (38) (i.e. $\rho_{\ell} \sigma^{ab}$). The latter term drives Silk damping during the decoupling process. The disappearance of most of the kinematic terms upon linearisation is further reflected in the fact that the linearised equations link only **three** successive moments, i.e. $\ell, \ell \pm 1$. 

11
A crucial feature of the nonlinear kinematic terms is that some of them scale like $\ell$ for large $\ell$, as already noted in the case of vorticity. There are no purely linear terms with this property, which has an important consequence, i.e. that for very high $\ell$ multipoles (corresponding to very small angular scales in CMB observations), certain nonlinear terms may reach the same order of magnitude as the linear contributions. (Note that the same effect applies to the neutrino background.) The relevant nonlinear terms in Eq. (39) are (for $\ell \gg 1$):

$$- \ell \left[ \frac{1}{4} \sigma_{bc} \Pi^{bc} A_\ell + \sigma^{(a_l a_{l-1})} \Pi^{2 A_{l-2}} \right] + A^{(a_l} \Pi_{A_{l-1})} + \frac{1}{2} A_b \Pi^{b A_\ell} + \omega^b \varepsilon_{bc} \langle a_l \Pi^{A_{l-1}} c \rangle .$$

Any observable imprint of this effect will be made after last scattering. In the free-streaming era, it is reasonable to neglect the vorticity relative to the shear. We can remove the acceleration term by choosing $u^a$ as the dynamically dominant cold dark matter frame (i.e. choosing $v^a_C = 0$). It follows that the nonlinear correction to the rate of change of the linearised fluctuation multipoles is

$$\delta(\dot{\Pi}^{A_\ell}) \sim \ell \left[ \frac{1}{4} \sigma_{bc} \Pi^{bc} A_\ell + \sigma^{(a_l a_{l-1})} \Pi^{2 A_{l-2}} \right] \text{ for } \ell \gg 1 .$$

The linear solutions for $\Pi_{A_\ell}$ and $\sigma_{ab}$ can be used in equation (40) to estimate the correction to second order. Its effect on observed anisotropies will be estimated by integrating $\delta(\Pi^{A_\ell})$ from last scattering to now.

Further quantitative analysis of the new nonlinear effects in [20, 23] is needed to identify more clearly which effects could be observationally significant. This would also clarify the relationship between various nonlinear effects that have been derived under different assumptions and using different approaches [14, 15, 17, 20, 23, 24].

Acknowledgements

It is a pleasure to acknowledge the inspiration provided by the energy, enthusiasm and ideas of George Ellis, whose influence is widely felt in relativistic cosmology.
REFERENCES

[1] Bahcall, N., Ostriker, J.P., Perlmutter, S., and Steinhardt, P.J. (1999). Science, 284, 1481.
[2] Scott, D., astro-ph/9810330.
[3] Hu, W. and Sugiyama, N. (1995). Astrophys. J. 444, 489; ibid. (1995). Phys. Rev. D51, 2599.
[4] Ma, C.P., and Bertschinger, E. (1995). Astrophys. J. 455, 7.
[5] Zaldarriaga, M., Seljak, U., and Bertschinger, E. (1998). Astrophys. J. 494, 491.
[6] Hu, W., Seljak, U., White, M., and Zaldarriaga, M. (1998). Phys. Rev. D 57, 3290.
[7] Durrer, R. and Kahniashvili, T. (1999). Helv. Phys. Acta 71, 445.
[8] Tenorio, L., Jaffe, A.H., Hanany, S., and Lineweaver, C.H., astro-ph/9903206.
[9] Verde, L., Wang, L., Heavens, A.F., and Kamionkowski, M., astro-ph/9906301.
[10] Melchiorri, A., Vernizzi, F., Durrer, R., and Veneziano, G., astro-ph/9905237.
[11] Uzan, J-P., Deruelle, N., and Riazuelo, A., astro-ph/9810313.
[12] Contaldi, C., Hindmarsh, M., and Magueijo, J. (1999). Phys. Rev. Lett. 82, 2034.
[13] Bassett, B.A., Kaiser, D.I., and Maartens, R. (1999). Phys. Lett. B455, 84; Bassett, B.A., Tamburini, F., Kaiser, D.I., and Maartens, R., hep-ph/9901319.
[14] Ostriker, J.P., and Vishniac, E.T. (1986). Astrophys. J. 306, L51.
[15] Sunyaev, R.A., and Zeldovich, Ya.B. (1970). Astrophys. Space Sci. 7, 3.
[16] Haiman, Z., and Knox, L., astro-ph/9902311.
[17] Rees, M., and Sciama, D.W. (1968). Nature 519, 611.
[18] Seljak, U. (1996). Astrophys. J. 463, 1.
[19] Refregier, A., Spergel, D.N., and Herbig, T., astro-ph/9806349.
[20] Mollerach, S., and Matarrese, S. (1997). Phys. Rev. D56, 4494.
[21] Pyne, T., and Carroll, S.M. (1996). Phys. Rev. D53, 2920.
[22] Bruni, M., Matarrese, S., Mollerach, S., and Sonego, S. (1997). Class. Quantum Grav. 14, 2585.
[23] Maartens, R., Gebbie, T., and Ellis, G.F.R. (1999). Phys. Rev. D59, 083506.
[24] Hu, W., Scott, D., and Silk, J. (1994). Phys. Rev. D49, 648.
[25] Challinor, A.D., this volume (astro-ph/9903283).
[26] Ehlers, J. (1993). Gen. Rel. Grav. 25, 1225 (translation of 1961 article).
[27] Ellis, G.F.R. (1971). In General Relativity and Cosmology, R.K. Sachs, ed. (Academic, New York).
[28] Hawking, S.W. (1966). Astrophys. J. 145, 544.
[29] Ellis, G.F.R., and Bruni, M. (1989). Phys. Rev. D40, 1804.
[30] Ellis, G.F.R., Matravers, D.R., and Treciokas, R. (1983). Ann. Phys. 150, 455; Ellis, G.F.R., Treciokas, R., and Matravers, D.R. (1983). Ann. Phys. 150, 487.
[31] Maartens, R., Ellis, G.F.R., and Stoeger, W.R. (1995). Phys. Rev. D51, 1525.
[32] Challinor, A.D., and Lasenby, A.N. (1998). Phys. Rev. D58, 023001; ibid. (1999). Astrophys. J. 513, 1.
[33] Gebbie, T., and Ellis, G.F.R., astro-ph/9804310; Gebbie, T., Dunsby, P.K.S., and Ellis, G.F.R., astro-ph/9904408.
[34] Maartens, R. (1997). Phys. Rev. D55, 463.