On Irregularities of Fourier Transforms of Regular Holonomic $\mathcal{D}$-Modules *

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Abstract

We study Fourier transforms of regular holonomic $\mathcal{D}$-modules. By using the theory of Fourier-Sato transforms of enhanced ind-sheaves developed by Kashiwara-Schapira and D’Agnolo-Kashiwara, a formula for their enhanced solution complexes will be obtained. Moreover we show that some parts of their characteristic cycles and irregularities are expressed by the geometries of the original $\mathcal{D}$-modules.

1 Introduction

The theory of Fourier transforms of $\mathcal{D}$-modules is a beautiful subject in algebraic analysis. They interchange algebraic $\mathcal{D}$-modules on complex vector spaces $\mathbb{C}^N$ with those on their duals. Especially, the case $N = 1$ has been studied precisely by many mathematicians such as Bloch-Esnault [BE04], Mochizuki [Moc10], Sabbah [Sab08] etc. On the other hand, after a groundbreaking development in the theory of irregular meromorphic connections by Kedlaya [Ked10, Ked11] and Mochizuki [Moc09, Moc10], in [DK16] D’Agnolo and Kashiwara established the Riemann-Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules. For this purpose, they introduced enhanced ind-sheaves extending the classical notion of ind-sheaves introduced by Kashiwara-Schapira [KS01]. Subsequently, in [KS16a] Kashiwara and Schapira adapted this new notion to the Fourier-Sato transforms of Tamarkin [Tam08] and developed a new theory of Fourier-Sato transforms for enhanced ind-sheaves which correspond to those for algebraic holonomic $\mathcal{D}$-modules. Recently, in [DK17] by making use of these results effectively, D’Agnolo and Kashiwara succeeded in studying Fourier transforms of holonomic $\mathcal{D}$-modules on $\mathbb{C}$ very precisely. Note that in this particular case $N = 1$ Fourier transforms of regular holonomic $\mathcal{D}$-modules were studied successfully in the previous paper D’Agnolo-Hien-Morando-Sabbah [DHMS17] by a different method. In conclusion, thanks to the new theory of enhanced ind-sheaves of [DK16], we now understand the Fourier transforms of $\mathcal{D}$-modules on $\mathbb{C}$ more clearly than before. However, we know only little in the higher-dimensional case $N \geq 2$. The following beautiful theorem is due to Brylinski [Bry86]. Set $X = \mathbb{C}^N$ and recall that a constructible sheaf $\mathcal{F} \in \mathbb{D}^b_{\text{c}-c}(\mathbb{C}X)$ on it is called monodromic if its cohomology sheaves are locally constant on each $\mathbb{C}^*$-orbit in $\mathbb{C}^N$. Let $Y \simeq \mathbb{C}^N$ be the dual of $X = \mathbb{C}^N$.  

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Theorem 1.1 ([Brylinski [Bry86]]). Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Assume that its solution complex $\text{Sol}_X(\mathcal{M})$ is monodromic. Then its Fourier transform $\mathcal{M}^\wedge$ is regular and $\text{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.

To the best of our knowledge, except for this theorem not much is known for Fourier transforms of $\mathcal{D}$-modules on $\mathbb{C}^N$ for $N \geq 2$. Recall also that some topological results related to them were obtained in Kashiwara-Schapira [KS97]. Thus the higher-dimensional case $N \geq 2$ still remains very mysterious.

In this paper, we clarify this situation in the light of the more sophisticated theories of Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. Especially, we study the Fourier transforms of regular holonomic $\mathcal{D}$-modules on $X = \mathbb{C}^N$ for $N \geq 2$. For this purpose, we make use of the theory of Fourier-Sato transforms of enhanced ind-sheaves developed by Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. In particular, we obtain the following result. For an algebraic regular holonomic $\mathcal{D}$-module $\mathcal{M}$ on $X = \mathbb{C}^N$ denote by $\text{char}(\mathcal{M}) \subset T^*X \simeq X \times Y$ its characteristic variety. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections. Then we define a (Zariski) open subset $\Omega \subset Y = \mathbb{C}^N_w$ by:

$$w \in \Omega \iff \begin{cases} 
\text{there exists an open neighborhood } U \text{ of } w \text{ in } Y \\
\text{such that the restriction } q^{-1}(U) \cap \text{char}(\mathcal{M}) \to U \\
\text{of } q : X \times Y \to Y \text{ is an unramified covering.}
\end{cases}$$

Since $\text{char}(\mathcal{M})$ is $\mathbb{C}^*$-conic, $\Omega \subset Y = \mathbb{C}^N_w$ is also $\mathbb{C}^*$-conic. Denote by $k \geq 0$ the degree of the covering $q^{-1}(\Omega) \cap \text{char}(\mathcal{M}) \to \Omega$. For a point $w \in \Omega \subset Y = \mathbb{C}^N$, let $\{\mu_1(w), \ldots, \mu_k(w)\} = q^{-1}(w) \cap \text{char}(\mathcal{M}) \subset T^*X$ be its fiber by $q^{-1}(\Omega) \cap \text{char}(\mathcal{M}) \to \Omega$. For $1 \leq i \leq k$ set

$$\alpha_i(w) := p(\mu_i(w)) \in X = \mathbb{C}^N.$$

denote by $m_i > 0$ the multiplicity of $\mathcal{M}$ at $\mu_i(w) \in \text{char}(\mathcal{M})$. Let $i_Y : Y = \mathbb{C}^N \hookrightarrow \overline{Y} = \mathbb{P}^N$ be the projective compactification of $Y$. We extend the Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(D_Y)$ to a holonomic $\mathcal{D}$-module $\mathcal{M}^\wedge := i_{Y*}(\mathcal{M}^\wedge) \simeq D_{i_Y*}(\mathcal{M}^\wedge)$ on $\overline{Y}$. Let $\overline{\text{Y}}^\text{an}$ be the underlying complex manifold of $\overline{Y}$ and define the analytification $\overline{\mathcal{M}^\wedge}^\text{an} \in \text{Mod}_{\text{hol}}(D_{\overline{\text{Y}}^\text{an}})$ of $\mathcal{M}^\wedge$ by $\overline{\mathcal{M}^\wedge}^\text{an} = \mathcal{O}_{\overline{\text{Y}}^\text{an}} \otimes_{\mathcal{O}_{\overline{\text{Y}}}} \mathcal{M}^\wedge$. Then we have the following formula for the enhanced solution complex

$$\text{Sol}_Y^E(\overline{\mathcal{M}^\wedge}) := \text{Sol}_{\overline{\text{Y}}^\text{an}}(\overline{\mathcal{M}^\wedge}^\text{an}) \in \mathcal{E}^b(\mathcal{I}\mathcal{C}_{\overline{\text{Y}}^\text{an}})$$

of $\overline{\mathcal{M}^\wedge}^\text{an}$.

Theorem 1.2. Let $U \subset \Omega \subset Y = \mathbb{C}^N$ be a connected and simply connected open subset of $\Omega$. Then we have an isomorphism

$$\pi^{-1} \mathcal{C}_U \otimes \left(\text{Sol}_Y^E(\overline{\mathcal{M}^\wedge})\right) \simeq \bigoplus_{i=1}^k \pi^{-1} \mathcal{C}_U \otimes \left(\text{lim}_{a \to +\infty} \mathcal{C}_{\{i \geq \text{Re}(\alpha_i(w)) + a\}}^{\oplus m_i}\right)$$

in $\mathcal{E}^b(\mathcal{I}\mathcal{C}_{\overline{\text{Y}}^\text{an}})$, where $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C}$ is the canonical paring. In particular, the restriction $\mathcal{M}^\wedge|_{\Omega}$ of the Fourier transform $\mathcal{M}^\wedge$ to $\Omega$ is an algebraic integrable connection of rank $\sum_{i=1}^k m_i$. 

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Namely the regularity of $\mathcal{M}$ implies the $C^*$-conicness of the smooth locus $\Omega$ of the Fourier transform $\mathcal{M}^\wedge$ (in fact, we can also show that $\text{Sol}_X(\mathcal{M}^\wedge)$ is monodromic as we see in [IT18]). Note that a different expression for the rank of $\mathcal{M}^\wedge$ at generic points of $Y = \mathbb{C}^N$ was given also by Brylinski in [Bry86 Corollaire 8.6]. Our proof of Theorem 1.2 is based on the arguments in that of Esterov-Takeuchi [ET15, Theorem 5.5] (for their applications see [AET15]) and relies on some careful observation of the geometric situation at infinity of the perverse sheaf $\mathcal{F} := \text{Sol}_X(\mathcal{M})[N] \in \mathbf{D}^b_{\mathbb{C}^*} \mathbb{C}_X^{an}$) (see the proof of Theorem 4.4 and Lemma 3.21). Recall that in Adolphson introduced confluent [IT18]). Note that a different expression for the rank of $\mathcal{M}$ normal slice showed also that they have (non-empty) non-confluent (i.e. regular) ones of Gelfand-Kapranov-Zelevinsky [GKZ89], [GKZ90]. He also that they also have (non-empty) $C^*$-conic smooth loci in $\mathbb{C}^N$. Recently, in Saito [Sai11], Schultz-Walther [SW09], [SW12] and Esterov-Takeuchi [ET15], the authors found that Adolphson’s confluent $A$-hypergeometric systems are Fourier transforms of some special regular holonomic $\mathcal{D}$-modules. This motivated us to formulate Theorem 1.2 for Fourier transforms of general regular holonomic $\mathcal{D}$-modules. In the proof of Theorem 1.2 we first rewrite the Fourier-Sato transforms of enhanced ind-sheaves as in D’Agnolo-Kashiwara [DK17] to obtain the geometric situation similar to the one in the proof of Esterov-Takeuchi [ET15, Theorem 5.5]. Then we apply the Morse theoretical argument in it to the solution complex of $\mathcal{M}$. By Theorem 1.2 at generic points $v \in Y \setminus \Omega$ where $D := Y \setminus \Omega$ is a smooth hypersurface we obtain also the irregularity and the exponential factors of $\mathcal{M}^\wedge$ along it. More precisely, let $D_{\text{reg}} \subset D$ be the smooth part of $D$ and $v \in D_{\text{reg}}$ such a generic point. Take a subvariety $M \subset Y$ of $Y = \mathbb{C}^N$ which intersects $D_{\text{reg}}$ at $v$ transversally. We call it a normal slice of $D$ at $v$. By definition $M$ is smooth and of dimension 1 on a neighborhood of $v$. Let $i_M : M \hookrightarrow Y = \mathbb{C}^N$ be the inclusion map and set $\mathcal{K} = i_M^* \mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_M)$. Then we can describe the irregularity $\text{irr}(\mathcal{K}(*\{v\}))$ of the meromorphic connection $\mathcal{K}(*\{v\})$ on $M$ along $\{v\} \subset M$ as follows. Shrinking the normal slice $M$ if necessary we may assume that $M = \{u \in \mathbb{C} \mid |u| < \varepsilon \}$ for some $\varepsilon > 0$, $\{v\} = \{u = 0\}$ and $M \setminus \{v\} \subset \Omega$. Let $i_0 : M \setminus \{v\} \hookrightarrow \Omega$ be the inclusion map and define (possibly multi-valued) holomorphic functions $\varphi_i : M \setminus \{v\} \to \mathbb{C}$ $(1 \leq i \leq k)$ by

$$\varphi_i(u) = \langle \alpha_i(i_0(u)), i_0(u) \rangle.$$ 

Then it is easy to see that $\varphi_i(u)$ are Laurent Puiseux series of $u$ (see Kirwan [Kir92, Section 7.2] etc.). For each Laurent Puiseux series $\varphi_i(u) = \sum_{a \in \mathbb{Q}} c_i,a u^a \quad (c_i,a \in \mathbb{C})$

set $r_i = \min \{a \in \mathbb{Q} \mid c_i,a \neq 0\}$ and define its pole order $\text{ord}_{\{v\}}(\varphi_i) \geq 0$ by

$$\text{ord}_{\{v\}}(\varphi_i) = \begin{cases} -r_i & \text{if } r_i < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In [Sab93, p35] Sabbah introduced the classical Hukuhara-Levelt-Turrittin theorem as “This result is analogous to Puiseux theorem for plane algebraic (or algebroid) curves”.

It is indeed the case for Fourier transforms of regular holonomic $\mathcal{D}$-modules as we see in the following theorem.
Theorem 1.3. The exponential factors appearing in the Hukuhara-Levelt-Turrittin decomposition of the meromorphic connection $\mathcal{K}(\{v\})$ at $v \in M$ are the pole parts of $-\varphi_i$ ($1 \leq i \leq k$). Moreover for any $1 \leq i \leq k$ the multiplicity of the pole part of $-\varphi_i$ is equal to $m_i$. In particular we have

$$\text{irr}(\mathcal{K}(\{v\})) = \sum_{i=1}^{k} m_i \cdot \text{ord}_{\{v\}}(\varphi_i).$$

This result would be useful for the study of the irregularities of confluent $A$-hypergeometric functions. Recall that the irregularity $\text{irr}(\mathcal{K}(\{v\}))$ of $\mathcal{K}(\{v\})$ is a non-negative integer and equal to $-\chi_v(Sol_M(\mathcal{K}(\{v\}))$, where

$$\chi_v(Sol_M(\mathcal{K}(\{v\})) := \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j Sol_M(\mathcal{K}(\{v\}))_v$$

is the local Euler-Poincaré index of $Sol_M(\mathcal{K}(\{v\}))$ at the point $v \in M$ (see Sabbah [Sab93] etc.). Moreover by Theorem 1.2 for linear subspaces $L \cong \mathbb{C}$ of the dual $Y = \mathbb{C}^N$ such that $L \cap \Omega = L \setminus \{0\}$ we obtain a formula for the exponential factors at infinity of the restrictions $\mathcal{M}^\wedge\mid_L$ of $\mathcal{M}^\wedge$ to them. See Theorem 4.6 for the details. This result extends (some part of) our previous one [ET15, Theorem 5.5] for confluent $A$-hypergeometric systems to Fourier transforms of general regular holonomic $D$-modules $\mathcal{M}$. In the course of the proof of these results, we use the following technical result which may be of independent interest.

Proposition 1.4. Let $X$ be a complex manifold, $D \subset X$ a normal crossing divisor in it and $\mathcal{M}_i$ ($i = 1, 2$) (analytic) holonomic $D_X$-modules. Denote by $\varpi_X : \tilde{X} \to X$ the real blow-up of $X$ along $D$. Let $V \subset X \setminus D$ be an open sector in $X$ along $D$ and assume that we have an isomorphism

$$\pi^{-1}C_V \otimes Sol^E_X(\mathcal{M}_1) \simeq \pi^{-1}C_V \otimes Sol^E_X(\mathcal{M}_2).$$

Let $W \subset \tilde{X}$ be an open subset of $\tilde{X}$ such that $W \cap \varpi_X^{-1}(D) \neq \emptyset, \overline{W} \subset \text{Int}(\varpi_X^{-1}V)$. Then we have an isomorphism

$$\mathcal{M}_1\mid_W \simeq \mathcal{M}_2\mid_W$$

of $D_{\tilde{X}}$-modules on $W$ (for the definition of $D_{\tilde{X}}$, etc., see Section 3).

This result follows from a more essential one in Theorem 3.12. Namely we can reconstruct the $D_{\tilde{X}}$-module structure of $\mathcal{M}^\wedge$ on $W \subset \tilde{X}$ by the enhanced ind-sheaf $\pi^{-1}C_V \otimes Sol^E_X(\mathcal{M})$. We regard it as a directional (or sectorial) refinement of the irregular Riemann-Hilbert correspondence of [DK16]. In fact Proposition 1.4 is a consequence of the extended Riemann-Hilbert correspondence of Kashiwara-Schapira [KS16a, Theorem 4.5]. For the converse of Proposition 1.4 see Theorem 3.8 If a meromorphic connection $\mathcal{M}$ along $D$ admits a good lattice in the sense of Mochizuki [Moc11], we can also know its exponential factors from $\pi^{-1}C_V \otimes Sol^E_X(\mathcal{M})$. See Theorem 3.20 for the details.

Let $\mathcal{M}$ be an algebraic regular holonomic $D$-module on $X = \mathbb{C}^N$. Then by our formula for the enhanced solution complex $Sol^E(\mathcal{M}^\wedge)$ we can calculate also some part of the characteristic cycle of the Fourier transform $\mathcal{M}^\wedge$. To explain a special case of this result, first we define a “conification” of the perverse sheaf $\mathcal{F} = Sol_X(\mathcal{M})[N] \in D^{b}_{c-}\mathbb{C}(\mathbb{C}^{m\times})$ as
follows. Let $j = i_X : X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N$ be the projective compactification of $X = \mathbb{C}^N$ and $h$ the (local) defining equation of the hyperplane at infinity $H_\infty := \overline{X} \setminus X \simeq \mathbb{P}^{N-1}$ in $\overline{X}$ such that $H_\infty = h^{-1}(0)$. Moreover let $\gamma : X \setminus \{0\} = \mathbb{C}^N \setminus \{0\} \rightarrow H_\infty = \mathbb{P}^{N-1}$ be the canonical projection. Then

$$\mathcal{G} := \gamma^{-1} \psi_h(j_! \mathcal{F}) \in D^b_{\mathcal{C}, c}(\mathbb{C}_{X \setminus \{0\}})$$

is a perverse sheaf on $X \setminus \{0\}$. More precisely, the nearby cycle sheaf $\psi_h(j_! \mathcal{F})$ is defined globally on $H_\infty$ by the corresponding $\mathcal{D}$-module on it. We call it the conification of $\mathcal{F}$. In particular, $\mathcal{G}$ is monodromic. We extend it to a monodromic perverse sheaf on the whole $X = \mathbb{C}^N$ and denote it also by $\mathcal{G}$. Let $\mathcal{N} \in \text{Mod}_{\mathcal{D}_X}$ be the regular holonomic $\mathcal{D}_X$-module such that $\text{Sol}_X(\mathcal{N})[N] \simeq \mathcal{G}$. Now we recall the well-known relationship between the characteristic cycle $\text{CC}((\mathcal{N}))$ of $\mathcal{N}$ and that of its Fourier transform $\mathcal{N}^\wedge$. Take $\mathbb{C}^*$-conic subvarieties $V_i \subset X$ of $X$ and positive integer $n_i > 0$ ($1 \leq i \leq r$) such that

$$\text{CC}(\mathcal{N}) = \sum_{i=1}^r n_i \cdot [T^*_V X].$$

Then by the natural identification $T^*X \simeq X \times Y \simeq T^*Y$ for any $1 \leq i \leq r$ there exists a $\mathbb{C}^*$-conic subvariety $W_i \subset Y$ of $Y = \mathbb{C}^N$ such that $T^*_V X = T^*_V Y$ (see Gelfand-Kapranov-Zelevinsky [GKZ94, §1.3]). In this situation it is well-known that

$$\text{CC}(\mathcal{N}^\wedge) = \sum_{i=1}^r n_i \cdot [T^*_W Y].$$

For the Fourier transform $\mathcal{M}^\wedge$ of the original $\mathcal{M}$ we obtain the following result.

**Theorem 1.5.** Assume that $d_{W_i} = N - 1$ and $\mathcal{F} = \text{Sol}_X(\mathcal{M})[N] \in D^b_{\mathcal{C}, c}(\mathbb{C}^{\infty})$ is moderate at infinity over a neighborhood of a generic point $v \in (W_i)_{\text{reg}}$ in $Y \setminus \{0\}$ (see Definition 4.15). Then the multiplicity $\text{mult}_{T^*W_i Y} \mathcal{M}^\wedge \geq 0$ of the Fourier transform $\mathcal{M}^\wedge$ along $T^*_W Y$ is given by

$$\text{mult}_{T^*_W Y} \mathcal{M}^\wedge = \text{mult}_{T^*_V Y} \mathcal{N} + \text{irr}(\mathcal{K}(\{v\}))$$

( $\geq \text{mult}_{T^*_V Y} \mathcal{N} = n_i > 0$).

In particular, the conormal bundle $T^*_W Y$ is contained the characteristic variety $\text{char}(\mathcal{M}^\wedge)$ of $\mathcal{M}^\wedge$ and we have $W_i \subset D = Y \setminus \Omega$.

For a more general formula, see Theorem 4.12.

**2 Preliminary Notions and Results**

In this section, we briefly recall some basic notions and results which will be used in this paper. We assume here that the reader is familiar with the theory of sheaves and functors in the framework of derived categories. For them we follow the terminologies in [KS00] etc. For a topological space $M$ denote by $D^b(\mathbb{C}_M)$ the derived category consisting of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on it.
2.1 Ind-sheaves

We recall some basic notions and results on ind-sheaves. References are made to Kashiwara-Schapira [KS01] and [KS06]. Let $M$ be a good topological space (which is locally compact, Hausdorff, countable at infinity and has finite soft dimension). We denote by $\text{Mod}(\mathbb{C}_M)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on it and by $\text{IC}_M$ that of ind-sheaves. Then there exists a natural exact embedding $\iota_M : \text{Mod}(\mathbb{C}_M) \to \text{IC}_M$ of categories. We sometimes omit it. It has an exact left adjoint $\alpha_M$, that has in turn an exact fully faithful left adjoint functor $\beta_M$:

$$\text{Mod}(\mathbb{C}_M) \xrightarrow{\iota_M} \text{IC}_M \xleftarrow{\beta_M} \alpha_M.$$ 

The category $\text{IC}_M$ does not have enough injectives. Nevertheless, we can construct the derived category $\mathbb{D}^b(\text{IC}_M)$ for ind-sheaves and the Grothendieck six operations among them. We denote by $\otimes$ and $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}$ the operations of tensor products and internal homs respectively. If $f : M \to N$ be a continuous map, we denote by $f^{-1}, Rf_*, f^!$ and $Rf!!$ the operations of inverse images, direct images, proper inverse images and proper direct images respectively. We set also $R\mathcal{H}\mathcal{o}\mathcal{m} := \alpha_M \circ \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}$. We thus obtain the functors

- $\iota_M : \mathbb{D}^b(\mathbb{C}_M) \to \mathbb{D}^b(\text{IC}_M)$,
- $\alpha_M : \mathbb{D}^b(\text{IC}_M) \to \mathbb{D}^b(\mathbb{C}_M)$,
- $\beta_M : \mathbb{D}^b(\mathbb{C}_M) \to \mathbb{D}^b(\text{IC}_M)$,
- $\otimes : \mathbb{D}^b(\text{IC}_M) \times \mathbb{D}^b(\text{IC}_M) \to \mathbb{D}^b(\text{IC}_M)$,
- $R\mathcal{H}\mathcal{o}\mathcal{m} : \mathbb{D}^b(\text{IC}_M)^{\text{op}} \times \mathbb{D}^b(\text{IC}_M) \to \mathbb{D}^b(\text{IC}_M)$,
- $Rf_* : \mathbb{D}^b(\text{IC}_M) \to \mathbb{D}^b(\text{IC}_N)$,
- $f^{-1} : \mathbb{D}^b(\text{IC}_N) \to \mathbb{D}^b(\text{IC}_M)$,
- $Rf!! : \mathbb{D}^b(\text{IC}_M) \to \mathbb{D}^b(\text{IC}_N)$,
- $f^! : \mathbb{D}^b(\text{IC}_N) \to \mathbb{D}^b(\text{IC}_M)$.

Note that $(f^{-1}, Rf_*)$ and $(Rf!!, f^!)$ are pairs of adjoint functors. We may summarize the commutativity of the various functors we have introduced in the table below. Here, “$\circ$” means that the functors commute, and “$\times$” they do not.

|       | $\otimes$ | $f^{-1}$ | $Rf_*$ | $f^!$ | $Rf!!$ | $\lim\rightarrow$ | $\lim\leftarrow$ |
|-------|----------|----------|--------|-------|--------|-------------------|------------------|
| $\iota$ | $\circ$   | $\circ$  | $\circ$ | $\circ$ | $\times$ | $\times$          | $\circ$          |
| $\alpha$| $\circ$   | $\circ$  | $\times$ | $\circ$ | $\circ$ | $\times$          | $\circ$          |
| $\beta$ | $\circ$   | $\circ$  | $\times$ | $\times$ | $\circ$ | $\times$          |                  |
| $\lim\rightarrow$| $\circ$ | $\circ$ | $\times$ | $\circ$ | $\circ$ |                  |                  |
| $\lim\leftarrow$| $\times$ | $\times$ | $\circ$ | $\times$ | $\times$ |                  |                  |
2.2 Ind-sheaves on Bordered Spaces

For the results in this subsection, we refer to D’Agnolo-Kashiwara [DK16]. A bordered space is a pair \( M_\infty = (M, \bar{M}) \) of a good topological space \( \bar{M} \) and an open subset \( M \subset \bar{M} \).

A morphism \( f : (M, \bar{M}) \to (N, \bar{N}) \) of bordered spaces is a continuous map \( f : M \to N \) such that the first projection \( \bar{M} \times \bar{N} \to \bar{M} \) is proper on the closure \( \Gamma_f \) of the graph \( \Gamma_f \) of \( f \) in \( \bar{M} \times \bar{N} \). If also the second projection \( \Gamma_f \to \bar{N} \) is proper, we say that \( f \) is semi-proper. The category of good topological spaces embeds into that of bordered spaces by the identification \( M = (M, M) \). We define the triangulated category of ind-sheaves on \( M_\infty = (M, \bar{M}) \) by

\[
D^b(\mathcal{I}C_{M_\infty}) := D^b(\mathcal{I}C_{\bar{M}}) / D^b(\mathcal{I}C_{\bar{M}\setminus M}).
\]

The quotient functor

\[
q : D^b(\mathcal{I}C_{\bar{M}}) \to D^b(\mathcal{I}C_{M_\infty})
\]

has a left adjoint \( l \) and a right adjoint \( r \), both fully faithful, defined by

\[
l(qF) := \mathbb{C}_M \otimes F, \quad r(qF) := R\mathcal{I}hom(\mathbb{C}_M, F).
\]

For a morphism \( f : M_\infty \to N_\infty \) of bordered spaces, the Grothendieck’s operations

\[
\otimes : D^b(\mathcal{I}C_{M_\infty}) \times D^b(\mathcal{I}C_{M_\infty}) \to D^b(\mathcal{I}C_{M_\infty}),
\]

\[
R\mathcal{I}hom : D^b(\mathcal{I}C_{M_\infty})^{op} \times D^b(\mathcal{I}C_{M_\infty}) \to D^b(\mathcal{I}C_{M_\infty}),
\]

\[
Rf_* : D^b(\mathcal{I}C_{M_\infty}) \to D^b(\mathcal{I}C_{N_\infty}),
\]

\[
f^{-1} : D^b(\mathcal{I}C_{N_\infty}) \to D^b(\mathcal{I}C_{M_\infty}),
\]

\[
Rf_{!!} : D^b(\mathcal{I}C_{M_\infty}) \to D^b(\mathcal{I}C_{N_\infty}),
\]

\[
f^1 : D^b(\mathcal{I}C_{N_\infty}) \to D^b(\mathcal{I}C_{M_\infty})
\]

are defined by

\[
q(F) \otimes q(G) := q(F \otimes G),
\]

\[
R\mathcal{I}hom(q(F), q(G)) := q(R\mathcal{I}hom(F, G)),
\]

\[
Rf_*(q(F)) := q(Rpr_{2*}R\mathcal{I}hom(\mathbb{C}_{\Gamma_f}, pr_1^*F)),
\]

\[
f^{-1}(q(G)) := q(Rpr_{1!!}(\mathbb{C}_{\Gamma_f} \otimes pr_2^{-1}G)),
\]

\[
Rf_{!!}(q(F)) := q(Rpr_{2!!}(\mathbb{C}_{\Gamma_f} \otimes pr_1^{-1}F)),
\]

\[
f^1(q(G)) := q(Rpr_{1*}R\mathcal{I}hom(\mathbb{C}_{\Gamma_f}, pr_2^*G))
\]

respectively, where \( pr_1 : \bar{M} \times \bar{N} \to \bar{M} \) and \( pr_2 : \bar{M} \times \bar{N} \to \bar{N} \) are the projections. Moreover, there exists a natural embedding

\[
D^b(\mathbb{C}_M) \to D^b(\mathcal{I}C_{M_\infty}).
\]
2.3 Enhanced Sheaves

For the results in this subsection, see Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. Let $M$ be a good topological space. We consider the maps

$$M \times \mathbb{R}^2 \xrightarrow{p_1:p_2:\mu} M \times \mathbb{R} \xrightarrow{\pi} M$$

where $p_1, p_2$ are the first and the second projections and we set $\pi(x, t) := x$ and $\mu(x, t_1, t_2) := (x, t_1 + t_2)$. Then the convolution functors for sheaves on $M \times \mathbb{R}$ are defined by

$$F_1 \ast F_2 := \mathbb{R} \mu!(p_1^{-1} F_1 \otimes p_2^{-1} F_2),$$

$$\mathbb{R} \mathcal{H}om^+(F_1, F_2) := \mathbb{R} p_1! \mathbb{R} \mathcal{H}om(p_2^{-1} F_1, \mu_! F_2).$$

We define the triangulated category of enhanced sheaves on $M$ by

$$\mathcal{E}^b(C_M) := \mathcal{D}^b(C_{M \times \mathbb{R}})/\pi^{-1} \mathcal{D}^b(C_M).$$

Then the quotient functor

$$Q : \mathcal{D}^b(C_{M \times \mathbb{R}}) \to \mathcal{E}^b(C_M)$$

has fully faithful left and right adjoints $L^E, R^E$ defined by

$$L^E(Q F) := (\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}) \hat{\otimes} F, \quad R^E(Q G) := \mathbb{R} \mathcal{H}om^+(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, G),$$

where $\{t \geq 0\}$ stands for $\{(x, t) \in M \times \mathbb{R} \mid t \geq 0\}$ and $\{t \leq 0\}$ is defined similarly. The convolution functors are defined also for enhanced sheaves. We denote them by the same symbols $\ast, \mathbb{R} \mathcal{H}om^+$. For a continuous map $f : M \to N$, we can define naturally the operations $E f^{-1}, Ef_*, Ef^!, Ef!$ for enhanced sheaves. We have also a natural embedding $\varepsilon : \mathcal{D}^b(C_M) \to \mathcal{E}^b(C_M)$ defined by

$$\varepsilon(F) := Q(\mathbb{C}_{\{t \geq 0\}} \otimes \pi^{-1} F).$$

For a continuous function $\varphi : U \to \mathbb{R}$ defined on an open subset $U \subset M$ of $M$ we define the exponential enhanced sheaf by

$$E^\varphi_{\mid U} := Q(\mathbb{C}_{\{t + \varphi \geq 0\}}),$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x, t) \in M \times \mathbb{R} \mid x \in U, t + \varphi(x) \geq 0\}$.

2.4 Enhanced Ind-sheaves

We recall some basic notions and results on enhanced ind-sheaves. References are made to D’Agnolo-Kashiwara [DK16] and Kashiwara-Schapira [KS16b]. Let $M$ be a good topological space. Set $\mathbb{R}_\infty := (\mathbb{R}, \mathbb{R})$ for $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$, and let $t \in \overline{\mathbb{R}}$ be the affine coordinate. We consider the maps

$$M \times \mathbb{R}_\infty^2 \xrightarrow{p_1:p_2:\mu} M \times \mathbb{R}_\infty \xrightarrow{\pi} M$$
where $p_1, p_2$ and $\pi$ are morphisms of bordered spaces induced by the projections. And $\mu$ is a morphism of bordered spaces induced by the map $M \times \mathbb{R}^2 \ni (x, t_1, t_2) \mapsto (x, t_1 + t_2) \in M \times \mathbb{R}$. Then the convolution functors for ind-sheaves on $M \times \mathbb{R}_\infty$ are defined by

$$F_1 \otimes F_2 := R\mu_!(p_1^{-1}F_1 \otimes p_2^{-1}F_2),$$
$$R\mathcal{I}\text{hom}^+(F_1, F_2) := Rp_1\ast R\mathcal{I}\text{hom}(p_2^{-1}F_1, \mu^!F_2).$$

Now we define the triangulated category of enhanced ind-sheaves on $M$ by

$$E^b(\mathcal{I}C_M) := D^b(\mathcal{I}C_{M \times \mathbb{R}_\infty})/\pi^{-1}D^b(\mathcal{I}C_M).$$

Note that we have a natural embedding of categories

$$E^b(\mathcal{I}C_M) \hookrightarrow E^b(\mathcal{I}C_M).$$

The quotient functor

$$Q: D^b(\mathcal{I}C_{M \times \mathbb{R}_\infty}) \to E^b(\mathcal{I}C_M)$$

has fully faithful left and right adjoints $L^E, R^E$ defined by

$$L^E(QK) := (C_{\{t \geq 0\}} \oplus C_{\{t \leq 0\}}) \otimes K, \quad R^E(QK) := R\mathcal{I}\text{hom}^+(C_{\{t \geq 0\}} \oplus C_{\{t \leq 0\}}, K),$$

where $\{t \geq 0\}$ stands for $\{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, t \geq 0\}$ and $\{t \leq 0\}$ is defined similarly.

The convolution functors are defined also for enhanced ind-sheaves. We denote them by the same symbols $\otimes, R\mathcal{I}\text{hom}^+$. For a continuous map $f : M \to N$, we can define also the operations $Ef^{-1}, Ef_*, Ef^!, Ef_!$ for enhanced ind-sheaves. For example, by the natural morphism $\bar{f} : M \times \mathbb{R}_\infty \to N \times \mathbb{R}_\infty$ of bordered spaces associated to $f$ we set $Ef_*(QK) = Q(R\bar{f}_*(K))$. The other operations are defined similarly. We thus obtain the six operations $\otimes, R\mathcal{I}\text{hom}^+, Ef^{-1}, Ef_*, Ef^!, Ef_!$ for enhanced ind-sheaves. Moreover we denote by $D^E_M$ the Verdier duality functor for enhanced ind-sheaves. We have outer hom functors

$$R\mathcal{I}\text{hom}^E(K_1, K_2) := R\pi_\ast R\mathcal{I}\text{hom}(L^EK_1, L^EK_2) \simeq R\pi_\ast R\mathcal{I}\text{hom}(L^EK_1, R^EK_2),$$
$$R\mathcal{H}\text{om}^E(K_1, K_2) := \alpha_M R\mathcal{I}\text{hom}^E(K_1, K_2),$$
$$R\mathcal{H}\text{om}^E(K_1, K_2) := R\Gamma(M; R\mathcal{H}\text{om}^E(K_1, K_2)),$$

with values in $D^b(\mathcal{I}C_M), D^b(\mathcal{I}C_M)$ and $D^b(\mathcal{I}C_M)$, respectively. Moreover for $F \in D^b(\mathcal{I}C_M)$ and $K \in E^b(\mathcal{I}C_M)$ the objects

$$\pi^{-1}F \otimes K := Q(\pi^{-1}F \otimes L^EK),$$
$$R\mathcal{I}\text{hom}(\pi^{-1}F, K) := Q(R\mathcal{I}\text{hom}(\pi^{-1}F, R^EK)).$$

in $E^b(\mathcal{I}C_M)$ are well-defined. Set $C^E_M := Q\left(\lim_{a \to +\infty} \lim_{a \to +\infty} C_{\{t \geq a\}} \right) \in E^b(\mathcal{I}C_M)$. Then we have natural embeddings $\varepsilon, e : D^b(\mathcal{I}C_M) \to E^b(\mathcal{I}C_M)$ defined by

$$\varepsilon(F) := Q(C_{\{t \geq 0\}} \otimes \pi^{-1}F)$$
$$e(F) := C^E_M \otimes \pi^{-1}F \simeq C^E_M \otimes \varepsilon(F).$$
For a continuous function $\varphi : U \to \mathbb{R}$ defined on an open subset $U \subset M$ of $M$ we define the exponential enhanced ind-sheaf by

$$E^\varphi_{U|M} := \mathbb{C}^E_M \otimes E^\varphi_{U|M} = \mathbb{C}^E_M \otimes \mathbb{Q}C_{\{t + \varphi \geq 0\}} = \mathbb{Q}\left( \lim_{a \to +\infty} \right) \mathbb{C}_{\{t + \varphi \geq a\}}$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, x \in U, t + \varphi(x) \geq 0\}$.

2.5 $\mathcal{D}$-Modules

In this subsection we recall some basic notions and results on $\mathcal{D}$-modules. References are made to [Bjö93, HTT08, KS01, §7, DK16, §8, 9, KS103, §3, 4, 7] and [Kas16, §4, 5, 6, 7, 8]. For a complex manifold $X$ we denote by $d_X$ its complex dimension. Denote by $\mathcal{O}_X, \Omega_X$ and $\mathcal{D}_X$ the sheaves of holomorphic functions, holomorphic differential forms of top degree and holomorphic differential operators, respectively. Let $\mathcal{D}^b(\mathcal{D}_X)$ be the bounded derived category of left $\mathcal{D}_X$-modules and $\mathcal{D}^b(\mathcal{D}_X^{op})$ be that of right $\mathcal{D}_X$-modules. Moreover we denote by $\mathcal{D}^b_{coh}(\mathcal{D}_X), \mathcal{D}^b_{good}(\mathcal{D}_X), \mathcal{D}^b_{hol}(\mathcal{D}_X)$ and $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ the full triangulated subcategories of $\mathcal{D}^b(\mathcal{D}_X)$ consisting of objects with coherent, good, holonomic and regular holonomic cohomologies, respectively. For a morphism $f : X \to Y$ of complex manifolds, denote by $\otimes, \mathcal{R}Hom_{\mathcal{D}_X}, \mathcal{D}f_\ast, \mathcal{D}f^\ast$ the standard operations for $\mathcal{D}$-modules. We define also the duality functor $\mathcal{D}_X : \mathcal{D}^b_{coh}(\mathcal{D}_X)^{op} \xrightarrow{\sim} \mathcal{D}^b_{coh}(\mathcal{D}_X)$ by

$$\mathcal{D}(\mathcal{M}) := \mathcal{R}Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X].$$

Note that there exists an equivalence of categories $(\cdot)^\ast : \text{Mod}(\mathcal{D}_X) \xrightarrow{\sim} \text{Mod}(\mathcal{D}_X^{op})$ given by

$$(\mathcal{M})^\ast := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

The classical de Rham and solution functors are defined by

$$DR_X : \mathcal{D}^b_{coh}(\mathcal{D}_X) \to \mathcal{D}^b(\mathbb{C}_X), \quad \mathcal{M} \mapsto \Omega_X^L \otimes_{\mathcal{D}_X} \mathcal{M},$$

$$Sol_X : \mathcal{D}^b_{coh}(\mathcal{D}_X)^{op} \to \mathcal{D}^b(\mathbb{C}_X), \quad \mathcal{M} \mapsto \mathcal{R}Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Then for $\mathcal{M} \in \mathcal{D}^b_{coh}(\mathcal{D}_X)$ we have an isomorphism $Sol_X(\mathcal{M})[d_X] \simeq DR_X(\mathcal{D}(\mathcal{M}))$. For a closed hypersurface $D \subset X$ in $X$ we denote by $\mathcal{O}_X(\ast D)$ the sheaf of meromorphic functions on $X$ with poles in $D$. Then for $\mathcal{M} \in \mathcal{D}^b(\mathcal{D}_X)$ we set

$$\mathcal{M}(\ast D) := \mathcal{M} \otimes \mathcal{O}_X(\ast D).$$

For $f \in \mathcal{O}_X(\ast D)$ and $U := X \setminus D$, set

$$\mathcal{D}_X e^f := \mathcal{D}_X/\{P \in \mathcal{D}_X \mid Pe^f|_U = 0\},$$

$$\mathcal{E}_U e^{f} := \mathcal{D}_X e^f(\ast D).$$

Note that $\mathcal{E}_U e^{f}$ is holonomic and there exists an isomorphism

$$\mathcal{D}_X(\mathcal{E}_U e^{f})(\ast D) \simeq \mathcal{E}_U e^{f}.$$
Namely $\mathcal{E}^t_{\Omega X}$ is a meromorphic connection associated to $d + df$.

One defines the ind-sheaf $\mathcal{O}_X$ of tempered holomorphic functions as the Dolbeault complex with coefficients in the ind-sheaf of tempered distributions. More precisely, denoting by $\bar{X}$ the complex conjugate manifold to $X$ and by $X_\mathbb{R}$ the underlying real analytic manifold of $X$, we set

$$\mathcal{O}_X := \mathcal{R}hom_{\mathcal{D}X}(\mathcal{O}_{\bar{X}}, \mathcal{D}b^t_{X_\mathbb{R}}),$$

where $\mathcal{D}b^t_{X_\mathbb{R}}$ is the ind-sheaf of tempered distributions on $X_\mathbb{R}$ (for the definition see [DK16, Definition 7.2.5]). Moreover, we set

$$\Omega^t_X := \beta_X \Omega_X \otimes \beta_X \mathcal{O}_X \mathcal{O}_X.'$$

Then the tempered de Rham and solution functors are defined by

$$DR^t_X : \mathcal{D}b_{\mathcal{D}X} \to \mathcal{D}b^t(\mathbb{IC}_X), \quad \mathcal{M} \mapsto \Omega^t_X \otimes \mathcal{D}_X \mathcal{M},$$

$$Sol^t_X : \mathcal{D}b_{\mathcal{D}X}^{op} \to \mathcal{D}b^t(\mathbb{IC}_X), \quad \mathcal{M} \mapsto \mathcal{R}hom_{\mathcal{D}X}(\mathcal{M}, \mathcal{O}^t_X).$$

Note that we have isomorphisms

$$Sol_X(\mathcal{M}) \cong \alpha_X Sol^t_X(\mathcal{M}),$$

$$DR_X(\mathcal{M}) \cong \alpha_X DR^t_X(\mathcal{M}),$$

$$Sol^t_X(\mathcal{M})[d_X] \cong DR^t_X(\mathcal{D}_X^t \mathcal{M}).$$

Let $i : X \times \mathbb{R}_\infty \to X \times \mathbb{P}$ be the natural morphism of bordered spaces and $\tau \in \mathbb{C} \subset \mathbb{P}$ the affine coordinate such that $\tau|_\mathbb{R}$ is that of $\mathbb{R}$. We then define objects $\mathcal{O}^E_X \in \mathcal{E}^b(\mathbb{ID}_X)$ and $\Omega^E_X \in \mathcal{E}^b(\mathbb{ID}^{op}_X)$ by

$$\mathcal{O}^E_X := \mathcal{R}hom_{\mathcal{D}X}(\mathcal{O}_{\bar{X}}, \mathcal{D}b_{\mathbb{R}_\infty})$$

$$\simeq i'((\mathcal{E}^t_{\mathbb{C} \mathbb{P}}) \otimes \mathcal{D}_X \mathcal{O}^t_{X \times \mathbb{P}})[1] \simeq i'\mathcal{R}hom_{\mathcal{D}X}(\mathcal{E}_{\mathbb{C} \mathbb{P}}, \mathcal{O}^t_{X \times \mathbb{P}})[2],$$

$$\Omega^E_X := \Omega_X \otimes \mathcal{O}_X \mathcal{O}^E_X \simeq i'(\Omega^t_X \otimes \mathcal{D}_X \mathcal{E}^t_{\mathbb{C} \mathbb{P}})[1],$$

where $\mathcal{D}b_{\mathbb{R}_\infty}$ stand for the enhanced ind-sheaf of tempered distributions on $X_\mathbb{R}$ (for the definition see [DK16, Definition 8.1.1]). We call $\mathcal{O}^E_X$ the enhanced ind-sheaf of tempered holomorphic functions. Note that there exists an isomorphism

$$i_0^* \mathcal{R}^E \mathcal{O}^E_X \cong \mathcal{O}_X,'$$

where $i_0 : X \to X \times \mathbb{R}_\infty$ is the inclusion map of bordered spaces induced by $x \mapsto (x, 0)$.

The enhanced de Rham and solution functors are defined by

$$DR^E_X : \mathcal{D}b_{\mathcal{D}X} \to \mathcal{E}^b(\mathbb{IC}_X), \quad \mathcal{M} \mapsto \Omega^E_X \otimes \mathcal{D}_X \mathcal{M},$$

$$Sol^E_X : \mathcal{D}b_{\mathcal{D}X}^{op} \to \mathcal{E}^b(\mathbb{IC}_X), \quad \mathcal{M} \mapsto \mathcal{R}hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^E_X).$$

Then for $\mathcal{M} \in \mathcal{D}b_{\mathcal{D}X}$ we have isomorphism $Sol^E_X(\mathcal{M})[d_X] \simeq DR^E_X(\mathcal{D}_X \mathcal{M})$ and $Sol^E_X(\mathcal{M}) \simeq i_0^* \mathcal{R}^E \mathcal{S}ol^E_X(\mathcal{M})$. We recall the following results of [DK16].
Theorem 2.1. (i) For $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$ there is an isomorphism in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_X)$

$$\mathcal{D}^E_X(D\mathcal{R}^E_X(\mathcal{M})) \simeq \text{Sol}^E_X(\mathcal{M})[d_X].$$

(ii) Let $f : X \to Y$ be a morphism of complex manifolds. Then for $\mathcal{N} \in \mathcal{D}^b_{\text{good}}(\mathcal{D}_Y)$ there is an isomorphism in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_X)$

$$\text{Sol}^E_X(Df^*\mathcal{N}) \simeq \mathcal{E}f^{-1}\text{Sol}^E_Y(\mathcal{N}).$$

(iii) Let $f : X \to Y$ be a morphism of complex manifolds and $\mathcal{M} \in \mathcal{D}^b_{\text{good}}(\mathcal{D}_X) \cap \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$. If $\text{supp}(\mathcal{M})$ is proper over $Y$ then there is an isomorphism in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_Y)$

$$\text{Sol}^E_Y(Df_*\mathcal{M})[d_Y] \simeq \mathcal{E}f_*\text{Sol}^E_X(\mathcal{M})[d_X].$$

(iv) For $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, there exists an isomorphism in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_X)$

$$\text{Sol}^E_X(\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2) \simeq \text{Sol}^E_X(\mathcal{M}_1) \overset{+}{\otimes} \text{Sol}^E_X(\mathcal{M}_2).$$

(v) If $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$ and $D \subset X$ is a closed hypersurface, then there are isomorphisms in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_X)$

$$\text{Sol}^E_X(\mathcal{M}(\ast D)) \simeq \pi^{-1}\mathcal{C}_X \otimes \text{Sol}^E_X(\mathcal{M}),$$

$$\mathcal{D}^E_X(\mathcal{M}(\ast D)) \simeq R\text{Hom}(\pi^{-1}\mathcal{C}_X \otimes \mathcal{D}^E_X(\mathcal{M})).$$

(vi) Let $D$ be a closed hypersurface in $X$ and $f \in \mathcal{O}_X(\ast D)$ a meromorphic function along $D$. Then there exists an isomorphism in $\mathcal{E}^b(\mathcal{I}_\mathcal{C}_X)$

$$\text{Sol}^E_X(f^\phi \lambda_{D/X}) \simeq \mathcal{E}^{\text{Re}\phi}_{\lambda_{D/X}}.$$
3 Some Auxiliary Results on Meromorphic Connections etc.

In this section we prepare some auxiliary results on meromorphic connections etc. First we recall some notions and results on $D^b_X$ in [DK16, §7]. Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Denote by $\varpi_X : \tilde{X} \to X$ the real blow-up of $X$ along $D$ (sometimes we denote it simply by $\varpi$). Then we set

$$O^\lambda_X := R\mathcal{H}om_{\varpi^{-1} \mathcal{D}_X}(\varpi^{-1} \mathcal{O}_X, \mathcal{D}^b_{\varpi^{-1} \mathcal{O}_X}),$$

$$\mathcal{A}_X := \alpha \mathcal{O}^\lambda_X,$$

$$D^A_X := \mathcal{A}_X \otimes_{\varpi^{-1} \mathcal{O}_X} \varpi^{-1} \mathcal{D}_X,$$

where $\mathcal{D}^b_{\varpi^{-1} \mathcal{O}_X}$ stands for the ind-sheaf of tempered distributions on $\tilde{X}$ (for the definition see [DK16, Notation 7.2.4]). Recall that a section of $\mathcal{A}_X$ is a holomorphic function having moderate growth at $\varpi^{-1}_X(D)$. Note that $\mathcal{A}_X$ and $D^A_X$ are sheaves of rings on $\tilde{X}$. We define also enhanced ind-sheaves $O^E_X \in \mathbf{E}^b(D^A_X)$ and $\Omega^E_X \in \mathbf{E}^b(D^A_X)^{\text{op}}$ by

$$O^E_X := R\mathcal{I}hom_{\varpi^{-1} \mathcal{D}_X}(\varpi^{-1} \mathcal{O}_X, \mathcal{D}^b_{\varpi^{-1} \mathcal{O}_X})$$

$$\simeq k^!((\mathcal{E}^\Sigma_{\mathbf{C}^P})^L \otimes_{\mathcal{D}_P} \mathcal{O}^k_{\widetilde{X} \times \mathbf{P}})[1] \simeq k^!\mathcal{I}hom_{\mathcal{D}_P}(\mathcal{E}_{\mathbf{C}^P}^\Sigma, \mathcal{O}^k_{\widetilde{X} \times \mathbf{P}})[2],$$

$$\Omega^E_X := \varpi^{-1} \Omega_X \otimes_{\varpi^{-1} \mathcal{O}_X} \mathcal{O}^E_X \simeq k^!(\Omega^k_{\widetilde{X} \times \mathbf{P}} \otimes_{\mathcal{D}_P} \mathcal{E}^\Sigma_{\mathbf{C}^P})[1],$$

where $k : \tilde{X} \times \mathbb{R}_\infty \to \tilde{X} \times \mathbf{P}$ is the natural morphism of bordered spaces and $\mathcal{D}^b_{\varpi^{-1} \mathcal{O}_X}$ stands for the enhanced ind-sheaf of tempered distributions on $\widetilde{X}_\mathbb{R}$ (for the definition see [KSI16] (7.6.11)).

For $\mathcal{M} \in \mathbf{D}^b(D_X)$ we define an object $\mathcal{M}^A \in \mathbf{D}^b(D^A_X)$ by

$$\mathcal{M}^A := D^A_X \otimes_{\varpi^{-1} \mathcal{D}_X} \varpi^{-1} \mathcal{M} \simeq \mathcal{A}_X \otimes_{\varpi^{-1} \mathcal{O}_X} \varpi^{-1} \mathcal{M}.$$

Note that if $\mathcal{M}$ is a holonomic $D_X$-module such that $\mathcal{M} \sim \mathcal{M}(\ast D)$ and $\text{sing}\text{.supp}(\mathcal{M}) \subset D$, then one has $\mathcal{M}^A \simeq D^A_X \otimes_{\varpi^{-1} \mathcal{D}_X} \varpi^{-1} \mathcal{M}$ (see [DK16, Lemma 7.3.2]). Moreover we have an isomorphism $\mathcal{M}^A \sim \mathcal{M}(\ast D)^A$ for any holonomic $D_X$-module $\mathcal{M}$ (see [DK16, Lemma 7.2.2]). For $\mathcal{M} \in \mathbf{D}^b(D_X)$ we define the enhanced de Rham and solution functors on $\tilde{X}$ by

$$\text{Sol}^E_X(\mathcal{M}) := R\mathcal{I}hom_{D^A_X}(\mathcal{M}, \mathcal{O}^E_X),$$

$$\text{DR}^E_X(\mathcal{M}) := \Omega^E_X \otimes_{D^A_X} \mathcal{M},$$

respectively. Recall that we have isomorphisms

$$\text{DR}^E_X(\mathcal{M}^A) \simeq E\varpi^! \text{DR}^E_X(\mathcal{M}(\ast D)) \simeq E\varpi^! R\mathcal{I}hom(\pi^{-1} \mathcal{C}_{X\setminus D}, \text{DR}_X(\mathcal{M})), $$

$$\text{Sol}^E_X(\mathcal{M}^A) \simeq E\varpi^! R\mathcal{I}hom(\pi^{-1} \mathcal{C}_{X\setminus D}, \text{Sol}^E_X(\mathcal{M})).$$
and 
\[
E\pi_+DR_X^E(M^A) \simeq DR_X^E(M(*D)) \simeq R\mathcal{I}hom(\pi^{-1}C_{X,D}, DR_X^E(M)),
\]
\[
E\pi_+Sol_X^E(M^A) \simeq R\mathcal{I}hom(\pi^{-1}C_{X,D}, Sol_X^E(M))
\]
for \(M \in D^b_{\text{hol}}(D_X)\) (see [DK16] Corollary 9.2.3, p191 and Theorem 9.2.2]).

**Definition 3.1.** Let \(X\) be a complex manifold and \(D \subset X\) a normal crossing divisor in it. Then we say that a holonomic \(D_X\)-module \(M\) has a normal form along \(D\) if
(i) \(M \xrightarrow{\sim} M(*D)\)
(ii) sing.\(\text{supp}(M) \subset D\)
(iii) for any \(\theta \in \pi^{-1}(D) \subset \bar{X}\), there exist an open neighborhood \(U \subset X\) of \(\bar{\omega}(\theta)\), finitely many \(\varphi_i \in \Gamma(U; \mathcal{O}_X(*D))\) and an open neighborhood \(V\) of \(\theta\) with \(V \subset \pi^{-1}(U)\) such that 
\[
\mathcal{M}|_V \simeq \left( \bigoplus_i (\mathcal{E}_{U \setminus D[U]}^* \mathcal{A}^i) \right) |_V.
\]

**Lemma 3.2.** Let \(X\) be a complex manifold and \(D \subset X\) a normal crossing divisor in it and \(M\) a holonomic \(D_X\)-module. Then for the dual \((\mathcal{M}^A)^* := R\mathcal{H}om_{D_X^A}(\mathcal{M}^A, \mathcal{D}_X^A) \otimes_{\pi^{-1}\mathcal{O}_X}\pi^{-1}\Omega_X^{0,-1}[d_X]\) of the \(\mathcal{D}_X^A\)-modules \(\mathcal{M}^A\) and the holonomic \(D_X\)-module \(\mathbb{D}_X(M)(*D)\) we have an isomorphism 
\[
(\mathcal{M}^A)^* \simeq \left( \mathbb{D}_X(M)(*D) \right)^A.
\]

In particular, there exists an isomorphism
\[
DR_X^E\left(\left( \mathbb{D}_X(M)(*D) \right)^A \right) \simeq Sol_X^E(M^A)[d_X].
\]

If moreover \(M\) has a normal form along \(D\), then the holonomic \(D_X\)-module \(\mathbb{D}_X(M)(*D)\) has also a normal form along \(D\).

**Proof.** Let 
\[
0 \to \mathcal{D}_X^{N_k} \to \cdots \to \mathcal{D}_X^{N_i} \to \mathcal{D}_X^{N_0} \to M \to 0
\]
be a (local) free resolution of \(M\). Set 
\[
\mathcal{L}^* := [0 \to \mathcal{D}_X^{N_k} \to \cdots \to \mathcal{D}_X^{N_0} \to 0]
\]
so that we have a quasi-isomorphism \(\mathcal{L}^* \xrightarrow{\sim} M\). Hence we obtain an isomorphism 
\[
\mathbb{D}_X(M) \simeq \mathcal{K}^* := \mathcal{H}om_{D_X}(\mathcal{L}^*, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{0,-1}[d_X].
\]
By applying the exact functor \((-)(*D) = (-) \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)\) to it, we obtain also a quasi-isomorphism 
\[
\mathcal{N} := \mathbb{D}_X(M)(*D) \simeq \mathcal{K}^*(*)D).
\]
Obviously we have an isomorphism 
\[
(\mathcal{L}^*)^A \simeq \mathcal{M}^A.
\]
We thus obtain the desired isomorphism 
\[
(\mathcal{M}^A)^* = R\mathcal{H}om_{D_X^A}(\mathcal{L}^*, \mathcal{D}_X^A) \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega_X^{0,-1}[d_X] \simeq (\mathcal{K}^*)^A \simeq N^A.
\]
The remaining assertion can be shown easily by using this isomorphism and [DK16, Lemma 6.1.2].

\[\square\]
A ramification of $X$ along $D$ on a neighborhood $U$ of $x \in D$ is a finite map $p : X' \to U$ of complex manifolds of the form $z' \mapsto z = (z_1, z_2, \ldots, z_n) = p(z') = (z_1^{m_1}, \ldots, z_r^{m_r}, z_{r+1}', \ldots, z_n')$ for some $(m_1, \ldots, m_r) \in (\mathbb{Z}_{>0})^r$, where $(z_1', \ldots, z_n')$ is a local coordinate system of $X'$ and $(z_1, \ldots, z_n)$ is that of $U$ such that $D \cap U = \{z_1 \cdots z_r = 0\}$.

**Definition 3.3.** Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Then we say that a holonomic $D_X$-module $\mathcal{M}$ has a quasi-normal form along $D$ if it satisfies the conditions (i) and (ii) above, and if for any $x \in D$ there exists a ramification $p : X' \to U$ on a neighborhood $U$ of it such that $Dp^*(\mathcal{M}|_U)$ has a normal form along $p^{-1}(D \cap U)$.

Note that $Dp^*(\mathcal{M}|_U)$ as well as $Dp_Dp^*(\mathcal{M}|_U)$ is concentrated in degree zero and $\mathcal{M}|_U$ is a direct summand of $Dp_Dp^*(\mathcal{M}|_U)$. The following fundamental result is due to Kedlaya and Mochizuki.

**Theorem 3.4 ([Ked10] [Ked11] [Moc09] [Moc11]).** For a holonomic $D_X$-module $\mathcal{M}$ and $x \in X$, there exist an open neighborhood $U$ of $x$, a closed hypersurface $Y \subset U$, a complex manifold $X'$ and a projective morphism $f : X' \to U$ such that

(i) $\text{sing supp}(\mathcal{M}) \cap U \subset Y$,

(ii) $D := f^{-1}(Y)$ is a normal crossing divisor in $X'$,

(iii) $f$ induces an isomorphism $X \setminus D \xrightarrow{\sim} U \setminus Y$,

(iv) $(Df^*\mathcal{M})(*D)$ has a quasi-normal form along $D$.

This is a generalization of the classical Hukuhara-Levelt-Turrittin theorem to higher dimensions.

**Proposition 3.5.** Let $X$ be a complex manifold and $D \subset X$ a normal crossing divisor in it. Assume that a holonomic $D$-module $\mathcal{M}$ has a quasi-normal form along $D$ for a ramification map $f : Y \to X$ and set $D' := f^{-1}(D) \simeq D$. Denote by $\varpi_X : \tilde{X} \to X$ (resp. $\varpi_Y : \tilde{Y} \to Y$) the real blow-up of $X$ (resp. $Y$) along $D$ (resp. $D'$). For a point $y_0 \in \varpi^{-1}_Y(D')$, let $W \subset \tilde{Y}$ be its sufficiently small open neighborhood for which there exits an open subset $U$ of $\varpi_Y(W)$ and $\varphi_i \in \Gamma(U; \mathcal{O}_Y(*D'))$ ($1 \leq i \leq m$) such that we have an isomorphism

$$
(Df^*\mathcal{M})^A|_W \simeq \left( \bigoplus_{i=1}^m (\mathcal{E}^\varphi_i|_{U 
abla D'})^A \right)|_W
$$

of $\mathcal{D}^A$-modules on $W$. Let $V' \subset Y \setminus D'$ be an open sector in $Y$ along $D'$ such that $\varpi_Y^{-1}(V') \subset W$ and set $V = f(V') \subset X \setminus D$. Finally, for $1 \leq i \leq m$ let $\tilde{\varphi}_i \in \Gamma(V; \mathcal{O}_X)$ be a holomorphic function on the sector $V$ along $D$ such that $\tilde{\varphi}_i \circ (f|_V) = \varphi_i|_V$. Then we have an isomorphism

$$
\pi^{-1}\mathcal{C}_V \otimes \text{Sot}^E_X(\mathcal{M}) \simeq \bigoplus_{i=1}^m (\pi^{-1}\mathcal{C}_V \otimes E_{V|X}^{\text{Re} \tilde{\varphi}_i}).
$$

**Proof.** Let $g : \tilde{Y} \to \tilde{X}$ be the lift of $f : Y \to X$ i.e. the unique continuous map for which
we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{g} & \tilde{X} \\
\pi_Y & & \pi_X \\
Y & \xrightarrow{f} & X.
\end{array}
\]

For the sufficiently small \( W \subset \tilde{Y} \) it induces a homeomorphism \( g|_W : W \to g(W) \). Then by [DK16, Theorem 9.1.2(ii) and Corollary 9.2.3] we have an isomorphism

\[
\text{E}(g|_W)_*(DR^E_Y((Df^\ast \mathcal{M})^A)|_{\pi^{-1}(W)}[d_Y]) \simeq DR^E_X(\mathcal{M}^A)[\pi^{-1}(g(W))[d_X]] \quad (3.1)
\]

(in this case we have \( d_X = d_Y \)). For the Verdier duality functor \( D^E_X : \text{B}^b(\text{IC}_X)^{\text{op}} \to \text{E}^b(\text{IC}_X) \) we also obtain a chain of isomorphisms

\[
D^E_X(\pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_X(\mathcal{M})) \simeq \text{R} \text{Hom}^+(\pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_X(\mathcal{M}), \omega^E_X) \\
\simeq \text{R} \text{Hom}(\pi^{-1}\mathcal{C}_V, \text{R} \text{Hom}^+(\text{Sol}^E_X(\mathcal{M}), \omega^E_X)) \\
\simeq \text{R} \text{Hom}(\pi^{-1}\mathcal{C}_V, DR^E_X(\mathcal{M})[d_X]) \\
\simeq \text{E} \text{Hom}_X, \text{R} \text{Hom}(\pi^{-1}\mathcal{C}_\varphi^{-1}(V), DR^E_X(\mathcal{M}^A)[d_X]),
\]

where in the third (resp. forth) isomorphism we used [DK16, Corollary 9.4.9] (resp. [DK16, Corollary 9.2.3]). Combining it with (3.1), it follows from our assumption that there exist isomorphisms

\[
D^E_X(\pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_X(\mathcal{M})) \simeq \text{E} \text{Hom}_X, \text{E} g_* \text{R} \text{Hom}
\left( \pi^{-1}\mathcal{C}_\varphi^{-1}(V), DR^E_Y((Df^\ast \mathcal{M})^A)[d_Y]\right) \\
\simeq \text{E} f_* \text{E} \varphi_V, \text{R} \text{Hom}
\left( \pi^{-1}\mathcal{C}_\varphi^{-1}(V), \bigoplus_{i=1}^m DR^E_Y((\mathcal{E}^{\varphi_i}_{U \setminus D^i[U]})^A)[d_Y]\right) \\
\simeq \bigoplus_{i=1}^m \text{E} f_* D^E_Y(\pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_Y(\mathcal{E}^{\varphi_i}_{U \setminus D^i[U]}) \\
\simeq \text{D}^E_X(\bigoplus_{i=1}^m \text{E} f_* (\pi^{-1}\mathcal{C}_V \otimes \text{E}^{\text{Re} \varphi_i}_{U \setminus D^i[U]}) \\
\simeq \text{D}^E_X(\bigoplus_{i=1}^m \text{E} f_* (\pi^{-1}\mathcal{C}_V \otimes \text{E}^{\text{Re} \varphi_i}_{V \setminus X}) \\
\simeq \text{D}^E_X(\bigoplus_{i=1}^m (\pi^{-1}\mathcal{C}_V \otimes \text{E}^{\text{Re} \varphi_i}_{V \setminus X})
\]

where in the last step we used the projection formula. By applying the functor \( D^E_X \) to the both sides, we obtain the desired isomorphism. \( \square \)

**Remark 3.6.** In Proposition 3.5 we assumed that \( \mathcal{M} \xrightarrow{\sim} \mathcal{M}(\ast D) \). However this condition is not really necessary. Indeed by Theorem 2.1 (v), for the holonomic \( D_X \)-module \( \mathcal{N} = \mathcal{M}(\ast D) \) there exists an isomorphism

\[
\pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_X(\mathcal{M}) \simeq \pi^{-1}\mathcal{C}_V \otimes \text{Sol}^E_X(\mathcal{N}).
\]
Moreover, obviously we have also an isomorphism \((\mathbf{D} f^* \mathcal{M})^A \simeq (\mathbf{D} f^* \mathcal{N})^A\).

By this Remark \[3.6\] and the classical Hukuhara-Levelt-Turrittin theorem we obtain the following corollary.

**Corollary 3.7.** Let \( X \) be a Riemann surface and \( D \subset X \) a point in it. Let \( \mathcal{M} \) be a holomorphic \( \mathcal{D}_X \)-module. Then for any direction \( \theta \in S_D X = \check{T}_D X / \mathbb{R}_{>0} \simeq S^1 \) there exists its sectorial neighborhood \( V_\theta \subset X \setminus D \) and some Puiseux series \( \psi_i \in \Gamma(V_\theta; \mathcal{O}_X) \) \((1 \leq i \leq m)\) for which we have an isomorphism

\[
\pi^{-1} C_{V_\theta} \otimes \text{Sol}^E_\chi(\mathcal{M}) \simeq \bigoplus_{i=1}^m (\pi^{-1} C_{V_\theta} \otimes \text{Res}_{V_\theta} \mathcal{M}).
\]

Note that this result is stated without proof in D’Agnolo-Kashiwara [DK17]. Proposition 3.5 can be deduced also from the following theorem.

**Theorem 3.8.** Let \( X \) be a complex manifold and \( D \) a normal crossing divisor in it. For \( \mathcal{M} \in \mathbf{D}_0(\mathcal{D}_X) \) and an open subset \( W \) of \( \check{X} \) such that \( W \cap \check{\omega}^{-1}(D) \neq \emptyset \), we set \( \mathcal{K} := E|_W \pi^{-1} \text{Sol}^E_\chi(\mathcal{M}) = \text{Sol}^E_\chi(\mathcal{M}^A|_W) \), where \( i_W : W \hookrightarrow \check{X} \) is the inclusion map.

Then for any sector \( V \subset X \setminus D \) along \( D \) such that \( \check{V} := \check{\omega}^{-1}(V) \subset W \), there exists an isomorphism

\[
\pi^{-1} C_V \otimes \text{Sol}^E_\chi(\mathcal{M}) \simeq E\omega_*(\pi^{-1} C_{\check{\omega}^{-1}(V)} \otimes Ei_{\check{V}}^* E_j^{-1} \mathcal{K})
\]

in \( \mathbf{E}^b(\mathcal{I} C_X) \), where \( j : \check{V} \hookrightarrow W \) and \( i_{\check{V}} : \check{V} \hookrightarrow \check{X} \) are the inclusion maps.

**Proof.** First, we shall prove the isomorphism

\[
\pi^{-1} C_{X \setminus D} \otimes K \sim \pi^{-1} C_{X \setminus D} \otimes R\text{Ihom}(\pi^{-1} C_{X \setminus D}, K)
\]

for an object \( K \) of \( \mathbf{E}^b(\mathcal{I} C_X) \). By [DK16] Proposition 3.2.9 (iii)] there exist isomorphisms

\[
R\text{Ihom}(\pi^{-1} C_X, K) \simeq K, \quad \pi^{-1} C_{X \setminus D} \otimes R\text{Ihom}(\pi^{-1} C_D, K) \simeq 0.
\]

Hence by applying the contravariant functor

\[
\pi^{-1} C_{X \setminus D} \otimes R\text{Ihom}(\pi^{-1} \cdot, K)
\]

to the distinguished triangle

\[
\begin{array}{c}
C_{X \setminus D} \\
\to \quad C_X \\
\to \quad C_D \\
\end{array}
\]

we obtain the desired isomorphism

\[
\pi^{-1} C_{X \setminus D} \otimes K \sim \pi^{-1} C_{X \setminus D} \otimes R\text{Ihom}(\pi^{-1} C_{X \setminus D}, K).
\]

On the other hand, for \( L \in \mathbf{E}^b(\mathcal{I} C_X) \) we have

\[
R\text{Ihom}(\pi^{-1} C_{X \setminus D}, E\omega_* E\omega^1 L) \simeq R\text{Ihom}(\pi^{-1} (R\omega_* \omega^{-1} C_{X \setminus D}), L) \simeq R\text{Ihom}(\pi^{-1} C_{X \setminus D}, L).
\]
We thus obtain a sequence of isomorphisms

\[ \pi^{-1}C_V \otimes Sol^E_X(\mathcal{M}) \simeq \pi^{-1}C_V \otimes (\pi^{-1}C_{X \setminus D} \otimes Sol^E_X(\mathcal{M})) \]
\[ \simeq \pi^{-1}C_V \otimes (\pi^{-1}C_{X \setminus D} \otimes R\mathcal{I}hom(\pi^{-1}C_{X \setminus D}, Sol^E_X(\mathcal{M}))) \]
\[ \simeq \pi^{-1}C_V \otimes R\mathcal{I}hom(\pi^{-1}C_{X \setminus D}, E\mathcal{O}_X \otimes Sol^E_X(\mathcal{M})) \]
\[ \simeq \pi^{-1}C_V \otimes E\mathcal{O}_X \otimes R\mathcal{I}hom(\pi^{-1}C_{X \setminus D}, Sol^E_X(\mathcal{M})). \]

Therefore by using the isomorphism

\[ Sol^E_X(\mathcal{M}^A) \simeq E\mathcal{O}_X \otimes R\mathcal{I}hom(\pi^{-1}C_{X \setminus D}, Sol^E_X(\mathcal{M})) \]

(see [DK16, p191]) we obtain isomorphisms

\[ \pi^{-1}C_V \otimes Sol^E_X(\mathcal{M}) \simeq \pi^{-1}C_V \otimes E\mathcal{O}_X \otimes Sol^E_X(\mathcal{M}^A) \]
\[ \simeq E\mathcal{O}_X(\pi^{-1}C_{X \setminus D}) \otimes Sol^E_X(\mathcal{M}^A)) \]
\[ \simeq E\mathcal{O}_X(\pi^{-1}C_{X \setminus D}) \otimes (\pi^{-1}C_{X \setminus D} \otimes Sol^E_X(\mathcal{M}^A)) \]
\[ \simeq E\mathcal{O}_X(\pi^{-1}C_{X \setminus D}) \otimes E\mathcal{O}_X(\pi^{-1}C_{X \setminus D} \otimes Sol^E_X(\mathcal{M}^A)) \]
\[ \simeq E\mathcal{O}_X(\pi^{-1}C_{X \setminus D}) \otimes E\mathcal{O}_X(\pi^{-1}C_{X \setminus D} \otimes Sol^E_X(\mathcal{M}^A)). \]

We can prove the converse of this theorem see Theorem \ref{thm:3.12}

**Theorem 3.9.** Let \( \mathcal{X} \) be a complex manifold, \( D \subset \mathcal{X} \) a normal crossing divisor in it and \( \hat{\mathcal{X}} \) the real blow-up of \( \mathcal{X} \) along \( D \). Let \( \delta : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \times \hat{\mathcal{X}} \) be the diagonal map. Then for \( \mathcal{M} \in D^b_{hol}(\mathcal{D}_X) \) we have isomorphisms

\[ \mathcal{M}^A \xrightarrow{\sim} R\mathcal{H}om^E_Sol^E_{\hat{\mathcal{X}}}(\mathcal{M}^A, O^E_{\hat{\mathcal{X}}}) \]
\[ \mathcal{M}^A \xrightarrow{\sim} R\mathcal{H}om^E(C^E_{\hat{\mathcal{X}}}, E\mathcal{O}^E(\mathcal{D}^E_{\hat{\mathcal{X}}}(\mathcal{M}^A) \boxtimes O^E_{\hat{\mathcal{X}}})) [d_X] \]

in \( E^b(\mathcal{D}^A_{\hat{\mathcal{X}}}) \).

**Proof.** First, we shall prove the isomorphism

\[ \mathcal{M}^A \xrightarrow{\sim} R\mathcal{H}om^E_Sol^E_{\hat{\mathcal{X}}}(\mathcal{M}^A, O^E_{\hat{\mathcal{X}}}) \]

Recall that we have already the canonical morphism

\[ \mathcal{M}^A \to R\mathcal{H}om^E_Sol^E_{\hat{\mathcal{X}}}(\mathcal{M}^A, O^E_{\hat{\mathcal{X}}}) \]

in [DK17, proof of Lemma 9.6.6]. We shall prove that this morphism is an isomorphism. Let \( \tilde{\omega} : \hat{\mathcal{X}} \times \mathbb{R}_\infty \to \mathcal{X} \times \mathbb{R}_\infty \) be the natural morphism of bordered spaces. By the isomorphisms

\[ E\tilde{\omega}^{-1}Sol^E_{\hat{\mathcal{X}}}(\mathcal{M}) \simeq \tilde{\omega}^{-1}\pi^{-1}C_{X \setminus D} \otimes Sol^E_{\hat{\mathcal{X}}}(\mathcal{M}^A) \]
\[ O^E_{\hat{\mathcal{X}}} \simeq E\tilde{\omega} R\mathcal{I}hom(\pi^{-1}C_{X \setminus D}, O^E_X) \]
(see [DK16, (9.6.6)]), we obtain a sequence of isomorphisms
\[
\mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right) \simeq \mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{R} \text{I} \text{hom}(\tilde{\omega}^{-1} \pi^{-1} \mathcal{C}_{X \setminus D}, \mathcal{O}^E_X)\right)
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\tilde{\omega}^{-1} \pi^{-1} \mathcal{C}_{X \setminus D} \otimes \mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right)
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\mathcal{E} \omega^{-1} \mathcal{S}o\ell^E_X(\mathcal{M}), \mathcal{O}^E_X\right)
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\mathcal{E} \omega^{-1} \mathcal{S}o\ell^E_X(\mathcal{M}), \mathcal{E} \omega^1 \mathcal{R} \text{I} \text{hom}(\pi^{-1} \mathcal{C}_{X \setminus D}, \mathcal{O}^E_X)\right)
\]
\[
\simeq \alpha_X \mathcal{R} \omega^1 \mathcal{R} \text{I} \text{hom}\left(\mathcal{S}o\ell^E_X(\mathcal{M}), \mathcal{R} \text{I} \text{hom}(\pi^{-1} \mathcal{C}_{X \setminus D}, \mathcal{O}^E_X)\right)
\]
\[
\simeq \alpha_X \mathcal{R} \omega^1 \mathcal{R} \text{I} \text{hom}\left(\mathcal{C}_{X \setminus D}, \mathcal{R} \text{I} \text{hom}(\mathcal{S}o\ell^E_X(\mathcal{M}), \mathcal{O}^E_X)\right).
\]

Moreover, by Theorem 2.2 we have an isomorphism
\[
\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}^1_X \cong \mathcal{R} \text{I} \text{hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}), \mathcal{O}^E_X\right).
\]

We thus obtain isomorphisms
\[
\mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right) \simeq \alpha_X \mathcal{R} \omega^1 \mathcal{R} \text{I} \text{hom}(\mathcal{C}_{X \setminus D}, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}^1_X)
\]
\[
\simeq \alpha_X \mathcal{R} \omega^1 \left(\mathcal{R} \text{I} \text{hom}(\mathcal{C}_{X \setminus D}, \mathcal{O}^1_X) \otimes_{\mathcal{O}_X} \mathcal{M}\right)
\]
\[
\simeq \alpha_X \left(\mathcal{O}^1_X \otimes_{\mathcal{O}_X} \mathcal{C}_{X \setminus D} \otimes_{\mathcal{O}_X} \mathcal{M}\right)
\]
\[
\simeq \alpha_X \left(\mathcal{O}^1_X \otimes_{\mathcal{O}_X} \mathcal{C}_{X \setminus D} \otimes_{\mathcal{O}_X} \mathcal{M}\right)
\]
\[
\simeq \mathcal{M}^A
\]

where in the third isomorphism we used [DK16, Theorem 7.2.7].

Next, we shall prove the isomorphism
\[
\mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right) \simeq \mathcal{R} \text{Hom}^E\left(\mathcal{C}^E_X, \mathcal{E} \delta^1(\mathcal{D} \mathcal{R}^E_X(\mathcal{M}^A), \mathcal{O}^E_X)\right)[d_X].
\]

By [DK16] Proposition 4.9.23 we have an isomorphism
\[
\mathcal{R} \text{Hom}^E\left(\mathcal{C}^E_X, \mathcal{E} \delta^1(\mathcal{D} \mathcal{R}^E_X(\mathcal{M}^A), \mathcal{O}^E_X)\right)[d_X] \simeq \mathcal{R} \text{Hom}^E\left(\mathcal{D} \mathcal{R}^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right)[d_X].
\]

Since there exists an isomorphism
\[
\mathcal{D} \mathcal{R}^E_X(\mathcal{M}^A) \simeq \mathcal{E} \omega^{-1} \mathcal{S}o\ell^E_X(\mathcal{M}) \simeq \tilde{\omega}^{-1} \pi^{-1} \mathcal{C}_{X \setminus D} \otimes \mathcal{S}o\ell^E_X(\mathcal{M}^A)[d_X]
\]
we obtain a sequence of isomorphisms
\[
\mathcal{R} \text{Hom}^E\left(\mathcal{D} \mathcal{R}^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right)[d_X]
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\tilde{\omega}^{-1} \pi^{-1} \mathcal{C}_{X \setminus D} \otimes \mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right)
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{R} \text{I} \text{hom}(\tilde{\omega}^{-1} \pi^{-1} \mathcal{C}_{X \setminus D}, \mathcal{O}^E_X)\right)
\]
\[
\simeq \mathcal{R} \text{Hom}^E\left(\mathcal{S}o\ell^E_X(\mathcal{M}^A), \mathcal{O}^E_X\right).
\]
The following proposition is useful to know the exponential types of holonomic $D$-modules from their enhanced solution complexes.

**Proposition 3.10.** Let $X$ be a complex manifold, $D \subset X$ a normal crossing divisor in it and $\mathcal{M}_i$ $(i = 1, 2)$ holonomic $D_X$-modules. Let $V \subset X \setminus D$ be an open sector in $X$ along $D$ and assume that we have an isomorphism

$$\pi^{-1}C_V \otimes \text{Sol}_X^E(\mathcal{M}_1) \simeq \pi^{-1}C_V \otimes \text{Sol}_X^E(\mathcal{M}_2).$$

Let $W \subset \tilde{X}$ be an open subset of the real bow-up $\tilde{X}$ such that $W \cap \bar{\omega}^{-1}(D) \neq \emptyset$ and $\bar{W} \subset \text{Int}(\bar{\omega}^{-1}(V))$. Then we have an isomorphism

$$\mathcal{M}_1^A|_W \simeq \mathcal{M}_2^A|_W$$

of $\mathcal{D}_X^A$-modules on $W$.

**Proof.** Let $W \subset \tilde{X}$ be an open subset of $\tilde{X}$ such that $W \cap \bar{\omega}^{-1}(D) \neq \emptyset$ and $\bar{W} \subset \text{Int}(\bar{\omega}^{-1}(V))$. Let $\delta : \tilde{X} \hookrightarrow \tilde{X} \times \tilde{X}$ be the diagonal map. Then by Theorem 3.9 there exist isomorphisms

$$\mathcal{M}_i^A \simeq R\text{Hom}^E(C_X^E, E\delta^!(DR_X^E(\mathcal{M}_i^A) \otimes O_X^E))[d_X] \quad (i = 1, 2).$$

Hence it suffices to show that for any $G \in E^b(\text{IC}_W)$ we have an isomorphism

$$\text{Hom}_{E^b(\text{IC}_W)}(G, DR_X^E(\mathcal{M}_i^A)|_{\delta^{-1}(W)}) \simeq \text{Hom}_{E^b(\text{IC}_W)}(G, DR_X^E(\mathcal{M}_j^A)|_{\delta^{-1}(W)}).$$

Let $j : W \hookrightarrow \tilde{X}$ be the inclusion map. Then for $i = 1, 2$ there exist isomorphisms

$$\text{Hom}_{E^b(\text{IC}_W)}(G, DR_X^E(\mathcal{M}_i^A)|_{\delta^{-1}(W)}) \simeq \text{Hom}_{E^b(\text{IC}_W)}(G, E\delta^! DR_X^E(\mathcal{M}_i^A))$$

$$\simeq \text{Hom}_{E^b(\text{IC}_X)}(E\delta^! G, DR_X^E(\mathcal{M}_i^A))$$

$$\simeq \text{Hom}_{E^b(\text{IC}_X)}(E\varepsilon^! E\delta^! G, DR_X^E(\mathcal{M}_i((D))))$$

$$\simeq \text{Hom}_{E^b(\text{IC}_X)}(E\varepsilon^! G, DR_X^E(\mathcal{M}_i((D)))).$$

Set $G' := E\varepsilon^! G \in E^b(\text{IC}_X)$. Then for $i = 1, 2$ we obtain isomorphisms

$$\text{Hom}_{E^b(\text{IC}_W)}(G, DR_X^E(\mathcal{M}_i^A)|_{\delta^{-1}(W)}) \simeq \text{Hom}_{E^b(\text{IC}_X)}(G', R\text{Ihom}(\pi^{-1}C_X \setminus D, DR_X^E(\mathcal{M}_i)))$$

$$\simeq \text{Hom}_{E^b(\text{IC}_X)}(\pi^{-1}C_X \setminus D \otimes G', DR_X^E(\mathcal{M}_i))$$

$$\simeq \text{Hom}_{E^b(\text{IC}_X)}(\pi^{-1}C_V \otimes G', DR_X^E(\mathcal{M}_i))$$

$$\simeq H^0R\Gamma(X, R\text{Hom}^E(\pi^{-1}C_V \otimes G', DR_X^E(\mathcal{M}_i))),$$

where in the third isomorphism we used $G' \simeq \pi^{-1}C_{\bar{\omega}(W)} \otimes G'$ and $(X \setminus D) \cap \bar{\omega}(W) = V \cap \bar{\omega}(W)$. Since we have

$$R\text{Hom}^E(\pi^{-1}C_V \otimes G', DR_X^E(\mathcal{M}_i)) \simeq \alpha_X R\pi_* R\text{Ihom}(\pi^{-1}C_V \otimes G', DR_X^E(\mathcal{M}_i))$$

$$\simeq \alpha_X R\pi_* R\text{Ihom}(G', R\text{Ihom}(\pi^{-1}C_V, DR_X^E(\mathcal{M}_i))),$$

20
it remains for us to prove the isomorphism
\[ \mathcal{R}I\text{hom}(\pi^{-1}\mathcal{C}_V, D^b_{\mathcal{X}}(\mathcal{M}_1)) \simeq \mathcal{R}I\text{hom}(\pi^{-1}\mathcal{C}_V, D^b_{\mathcal{X}}(\mathcal{M}_2)). \]

But this follows immediately by applying the Verdier duality functor \( D^b_{\mathcal{X}} \) to the isomorphism
\[ \pi^{-1}\mathcal{C}_V \otimes \text{Sol}_{\mathcal{X}}^E(\mathcal{M}_1) \simeq \pi^{-1}\mathcal{C}_V \otimes \text{Sol}_{\mathcal{X}}^E(\mathcal{M}_2) \]
(see the proof of Proposition 3.5). This completes the proof. \( \square \)

By Proposition 3.10 (and the proof of Corollary 3.7) we obtain the following result.

**Corollary 3.11.** Let \( X \) be a Riemann surface and \( D \subset X \) a point in it. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be holonomic \( D_X \)-modules. Assume that for a point \( \theta \in S_D X \simeq \omega^{-1}(D) \simeq S^1 \) there exists its sectorial neighborhood \( V_\theta \subset X \setminus D \) such that we have an isomorphism
\[ \pi^{-1}\mathcal{C}_{V_\theta} \otimes \text{Sol}_{\mathcal{X}}^E(\mathcal{M}_1) \simeq \pi^{-1}\mathcal{C}_{V_\theta} \otimes \text{Sol}_{\mathcal{X}}^E(\mathcal{M}_2). \]
Then there exists an open neighborhood \( W \) of \( \theta \) in the real blow-up \( \tilde{X} \) on which we have an isomorphism
\[ \mathcal{M}_1^A|_W \simeq \mathcal{M}_2^A|_W \]
of \( D^\mathcal{A}_X \)-modules.

We can prove Proposition 3.10 also using the following theorem.

**Theorem 3.12.** Let \( X \) be a complex manifold and \( D \) a normal crossing divisor in it. For \( \mathcal{M} \in D^b_{\text{hol}}(D_X) \) and a sector \( V \subset X \setminus D \) along \( D \) we set \( K := \pi^{-1}\mathcal{C}_V \otimes \text{Sol}_{\mathcal{X}}^E(\mathcal{M}) \). Then for any open subset \( W \) of \( \tilde{X} \) such that \( W \cap \omega^{-1}(D) \neq \emptyset, \bar{W} \subset \text{Int}(\omega^{-1}(V)) \), there exists an isomorphism
\[ \mathcal{M}^A|_W \simeq \mathcal{R}\text{Hom}_E\left( (E\omega^! \mathcal{R}I\text{hom}(\pi^{-1}\mathcal{C}_{X \setminus D}, K))|_W, O_{\mathcal{X}}^E|_W \right) \]
in \( D^b(D^\mathcal{A}_X) \).

**Proof.** By \( \mathcal{M}^A|_W \simeq \mathcal{R}\text{Hom}_E(\text{Sol}_{\mathcal{X}}^E(\mathcal{M}^A)|_W, O_{\mathcal{X}}^E|_W) \) (see Theorem 3.9), it is enough to show
\[ \text{Sol}_{\mathcal{X}}^E(\mathcal{M}^A)|_W \simeq \left( E\omega^! \mathcal{R}I\text{hom}(\pi^{-1}\mathcal{C}_{X \setminus D}, K) \right)|_W. \]
We consider the following diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow{i_W} & & \downarrow{j} \\
W & \xrightarrow{\omega|_W} & \omega(W).
\end{array}
\]
Let \( \tilde{i}_W : W \times \mathbb{R}_\infty \rightarrow \tilde{X} \times \mathbb{R}_\infty \) and \( \tilde{\varpi} : \tilde{X} \times \mathbb{R}_\infty \rightarrow X \times \mathbb{R}_\infty \) be the natural morphisms of bordered spaces. Then we obtain a sequence of isomorphisms:

\[
\text{Sol}_X^E(\mathcal{M}^A)|_W \simeq Ei_W^{-1}E\varpi^1R\text{Thom}(\pi^{-1}\mathcal{C}_{X,D}, \text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq Ei_W^{-1}R\text{Thom}(\tilde{\varpi}^{-1}\pi^{-1}\mathcal{C}_{X,D}, E\varpi^1\text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq Ei_W^{-1}R\text{Thom}(\tilde{\varpi}^{-1}\pi^{-1}\mathcal{C}_{X,D}, E\varpi^{-1}\text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq R\text{Thom}(\tilde{i}_W^{-1}\tilde{\varpi}^{-1}\pi^{-1}\mathcal{C}_{X,D}, Ei_W^{-1}E\varpi^{-1}\text{Sol}_X^E(\mathcal{M}))
\]

where the third isomorphism follows from the fact that \( \varpi \) is an isomorphism over \( X \setminus D \).

Since we may assume \( \mathcal{M} \rightarrow \mathcal{M}(\ast D) \), there exists a sequence of isomorphisms:

\[
Ei_W^{-1}E\varpi^{-1}\text{Sol}_X^E(\mathcal{M}) \simeq Ei_W^{-1}E\varpi^{-1}(\pi^{-1}\mathcal{C}_{X,D} \otimes \text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq E(\varpi|_W)^{-1}Ej^{-1}(\pi^{-1}\mathcal{C}_{X,D} \otimes \text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq E(\varpi|_W)^{-1}(\pi^{-1}\mathcal{C}_{j^{-1}(X \setminus D)} \otimes Ej^{-1}\text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq E(\varpi|_W)^{-1}(\pi^{-1}\mathcal{C}_{j^{-1}(V)} \otimes Ej^{-1}\text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq E(\varpi|_W)^{-1}Ej^{-1}(\pi^{-1}\mathcal{C}_V \otimes \text{Sol}_X^E(\mathcal{M}))
\]

\[
\simeq Ei_W^{-1}E\varpi^{-1}K.
\]

Therefore we obtain isomorphisms

\[
\text{Sol}_X^E(\mathcal{M}^A)|_W \simeq R\text{Thom}(\tilde{i}_W^{-1}\tilde{\varpi}^{-1}\pi^{-1}\mathcal{C}_{X,D}, Ei_W^{-1}E\varpi^{-1}K)
\]

\[
\simeq \left( E\varpi^{-1}R\text{Thom}(\pi^{-1}\mathcal{C}_{X,D}, K) \right)|_W.
\]

\(\square\)

From now, until the end of this section, let \( X \) be a smooth algebraic variety and \( Z \subset X \) a subvariety in it. Set \( U := X \setminus Z \) and let \( X^\text{an}, Z^\text{an}, U^\text{an} \) be the underlying complex analytic spaces of \( X, Z, U \) respectively. If there is no risk of confusion, we sometimes denote them simply by \( X, Z, U \) for short. Let

\[
Z \overset{i_Z}{\rightarrow} X \overset{i_U}{\leftarrow} U
\]

be the inclusion maps. We denote the corresponding morphisms of complex analytic spaces by the same symbols \( i_Z \) and \( i_U \). For an algebraic coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X) \) on \( X \), by using its analytification \( \mathcal{M}^\text{an} = \mathcal{O}^\text{an} \otimes_{\mathcal{O}_X} \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_{X^\text{an}}) \) we set \( \text{Sol}_X(\mathcal{M}) := \text{Sol}_{X^\text{an}}(\mathcal{M}^\text{an}) \in \mathbf{D}^b(\mathbb{C}_{X^\text{an}}) \). We define \( \text{Sol}_X^E(\mathcal{M}) \in \mathbf{D}^b(\mathbb{IC}_{X^\text{an}}), \text{Sol}_X^E(\mathcal{M}) \in \mathbf{E}^b(\mathbb{IC}_{X^\text{an}}) \), etc. Similarly. Recall that there exists an isomorphism

\[
\text{Sol}_X(\mathcal{M})|_Z := i_Z^{-1}\text{Sol}_X(\mathcal{M}) \simeq i_Z^{-1}a_Xi_U^1R^E\text{Sol}_X^E(\mathcal{M}).
\]

**Lemma 3.13.** For \( F \in \mathbf{E}^b(\mathbb{IC}_{X^\text{an}}) \) we have an isomorphism

\[
i_Z^{-1}a_Xi_U^1R^E(ei_Z, ei_Z^1F) \simeq a_ZR\pi, R\text{Thom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, ei_Z^1F).
\]

**Proof.** Let \( \sigma : X^\text{an} \times \mathbb{R}^2 \rightarrow X^\text{an} \times \mathbb{R}_\infty \) be a morphism of bordered spaces induced by the map \( X^\text{an} \times \mathbb{R}^2 \ni (x, t_1, t_2) \mapsto (x, t_2 - t_1) \in X^\text{an} \times \mathbb{R} \). We define also a morphism


\[ j_0 : X^\alpha \times \mathbb{R}_\infty \to X^\alpha \times \mathbb{R}_\infty^2 \] of bordered spaces by the map \( X^\alpha \times \mathbb{R} \ni (x,t) \mapsto (x,0,t) \in X^\alpha \times \mathbb{R}^2 \). Then we have isomorphisms

\[ i_2^{-1}\alpha_Xi_0^!R\mathbb{F}(Ei_Z,F) \]
\[ \simeq i_2^{-1}\alpha_Xi_0^!R \mathcal{H}om(\sigma^{-1}(C_{t\geq 0} + C_{t\leq 0}),p_1^!Ei_Z,F) \]
\[ \simeq i_2^{-1}\alpha_XR\pi_*i_0^!R \mathcal{H}om(\sigma^{-1}(C_{t\geq 0} + C_{t\leq 0}),p_2^!Ei_Z,F) \]
\[ \simeq i_2^{-1}\alpha_XR\pi_*R \mathcal{H}om(C_{t\geq 0} + C_{t\leq 0},Ei_Z,F) \]
\[ \simeq i_2^{-1}\alpha_XR\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_Z,F) \]
\[ \simeq i_2^{-1}\alpha_XR\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_Z,F) \]
\[ \simeq \alpha_2R\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_Z,F). \]

Applying this lemma to \( F = \text{Sol}_X^E(M) \in \mathbb{E}^b(\mathcal{I}C_{X^\alpha}) \) and the distinguished triangle

\[ \pi^{-1}C_{X^\alpha \setminus \bar{Z}^\alpha} \otimes F \to F \to \pi^{-1}C_{\bar{Z}^\alpha} \otimes F \simeq Ei_Z,F \xrightarrow{+1} \]

associated to it, we obtain a distinguished triangle

\[ i_2^{-1}\alpha_Xi_0^!R\mathbb{F}(\pi^{-1}C_{X^\alpha \setminus \bar{Z}^\alpha} \otimes \text{Sol}_X^E(M)) \to \text{Sol}_X(M)|_D \to \]
\[ \alpha_2R\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_Z^{-1}\text{Sol}_X^E(M)) \xrightarrow{+1} . \]

From now assume moreover that \( Z \) is a smooth hypersurface \( D \) in \( X \) and that \( M \in \text{Mod}_{\text{hol}}(\mathcal{D}_X) \). Then by Theorem [2.1(v)] we have isomorphisms

\[ i_0^{-1}\alpha_Xi_0^!R\mathbb{F}(\pi^{-1}C_{X^\alpha \setminus \bar{Z}^\alpha} \otimes \text{Sol}_X^E(M)) \simeq i_0^{-1}\alpha_Xi_0^!R\mathbb{F}(\text{Sol}_X^E(M(*D))) \]
\[ \simeq \text{Sol}_X(M(*D))|_D. \]

We thus obtain a distinguished triangle

\[ \text{Sol}_X(M(*D))|_D \to \text{Sol}_X(M)|_D \to \]
\[ \alpha_2R\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_D^{-1}\text{Sol}_X^E(M)) \xrightarrow{+1} . \]

Since we have a distinguished triangle

\[ \text{RG}_D(M) \to \mathcal{M} \to i_U\alpha_U^{-1} \mathcal{M} \simeq \mathcal{M}(*D) \xrightarrow{+1} \]

for the algebraic \( \mathcal{D}_X \)-module \( \text{RG}_D(M) = Di_D,Di_D^*\mathcal{M}[-1] \) (see [HTT08 Proposition 1.7.1 (iii)]), in this case there exist also isomorphisms

\[ \text{Sol}_X(\text{RG}_D(M))|_D \simeq \text{Sol}_X(Di_D,Di_D^*\mathcal{M}[-1])|_D \]
\[ \simeq i_0^{-1}\alpha_Xi_0^!R\mathbb{F}(\text{Sol}_X^E(Di_D,Di_D^*\mathcal{M})[1]) \]
\[ \simeq i_0^{-1}\alpha_Xi_0^!R\mathbb{F}(Ei_D,Ei_D^{-1}\text{Sol}_X^E(M)) \]
\[ \simeq \alpha_2R\pi_*R \mathcal{H}om(C_{t\geq 0} \oplus C_{t\leq 0},Ei_D^{-1}\text{Sol}_X^E(M)). \]

The following Proposition is useful to calculate

\[ \text{Sol}_X(M(*D))|_D \simeq i_0^{-1}\alpha_X\text{Sol}_X(M(*D)). \]
Proposition 3.14. Assume that $X$ is a Riemann surface and $D$ is a point in it. For a meromorphic function $\varphi \in \mathcal{O}_X(\ast D)$ having a pole of order $k > 0$ in $D \in X$, we consider $E^\varphi_{X \setminus D} \in \text{Mod}_\text{hol}(\mathcal{D}X^{an})$. Let $u \in \mathbb{C}$ be a local coordinate of $X$ at $D$ such that $D = \{u = 0\}$ and for $a \gg 0$ set

$$Q_a := \{u \in X \mid u \neq 0, \text{Re}(\varphi(u)) \geq a\} \subset X.$$

Then we have isomorphisms

$$H^j \text{Sol}^E_X(E^\varphi_{X \setminus D} \mid X) \simeq\begin{cases} \quad \lim_{a \to +\infty} \mathbb{C}_{X \setminus Q_a} & (j = 0) \\ \mathbb{C}^\oplus_D & (j = 1) \\ 0 & \text{(otherwise)}. \end{cases}$$

Moreover we have

$$\dim H^j \text{Sol}_X(E^\varphi_{X \setminus D} \mid X)_D = \begin{cases} k & (j = 1) \\ 0 & \text{(otherwise)}. \end{cases}$$

Proof. We set $N := E^\varphi_{X \setminus D} \in \text{Mod}_\text{hol}(\mathcal{D}X^{an})$. Recall that by (the proof of) Lemma 3.13 we have isomorphisms

$$\text{Sol}^E_X(N) \simeq i_0^! R^E \text{Sol}_X^E(N) \simeq R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, \text{Sol}^E_X(N)).$$

Moreover by Theorem 2.1 (vi) there exists also an isomorphism

$$\text{Sol}^E_X(N) \simeq \left. \lim_{a \to +\infty} \right. \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}.$$

For any sufficiently large $a \gg 0$ we can easily show that

$$R\text{Hom}(\mathbb{C}_{\{t \geq 0\}}, \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}) \simeq R\Gamma_{\{t \geq 0\}} \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}} \simeq \mathbb{C}_{\{t > 0, t \geq -\text{Re}\varphi + a\}}.$$

Similarly for $a \gg 0$ we have

$$R\text{Hom}(\mathbb{C}_{\{t \leq 0\}}, \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}) \simeq \mathbb{C}_{\{t < 0, t \geq -\text{Re}\varphi + a\}}.$$

Let $\pi : X \times \mathbb{R} \to X$ be the projection and $j : X \times \mathbb{R} \hookrightarrow X \times \mathbb{R}$ the inclusion map. Then for $a \gg 0$ it is easy to see that

$$R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \leq 0\}}, \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}) \simeq R\pi_* R\text{Ihom}_{\pi}^X(\mathbb{C}_{X \times \mathbb{R}}, \mathbb{C}_{\{t \leq 0, t \geq -\text{Re}\varphi + a\}}) \simeq R\pi_* Rj_* \mathbb{C}_{\{t \leq 0, t \geq -\text{Re}\varphi + a\}} \simeq 0.$$

We thus obtain an isomorphism

$$R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \leq 0\}}, \text{Sol}^E_X(N)) \simeq 0.$$
For $a \gg 0$ let us calculate
\[ R\pi_!R\mathcal{I}hom(\mathbb{C}_{\{t \geq 0\}}, \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}) \simeq R\pi_!Rj_*\mathbb{C}_{\{t > 0, \ t \geq -\text{Re}\varphi + a\}}. \]
The stalk of this complex at the point $D \in X$ is isomorphic to
\[ R\Gamma(D \times \mathbb{R}; Rj_*\mathbb{C}_{\{t > 0, \ t \geq -\text{Re}\varphi + a\}}). \]
We also see that
\[ (Rj_*\mathbb{C}_{\{t > 0, \ t \geq -\text{Re}\varphi + a\}})_{(D, +\infty)} \simeq \mathbb{C}^k[-1]. \]
Indeed, for $b \gg 0$ let us set
\[ Rb := \{ u \in X \mid u \neq 0, \ \text{Re}(\varphi(u)) \geq -b \} \subset X. \]
Then for a sufficiently small open neighborhood $U$ of the point $(D, +\infty)$ in $X \times \mathbb{R}$ the set
\[ U \cap \{ (u, t) \in X \times \mathbb{R} \mid t > 0, \ u \neq 0, \ t \geq -\text{Re}\varphi(u) + a \} \]
is homotopic to $Rb$. This implies that the stalk at $D \in X$ is isomorphic to $\mathbb{C}^k[-1]$. Moreover its stalk at a point $P \in X \setminus D$ is isomorphic to $\mathbb{C}$ (resp. $0$) if $P \in X \setminus Q_a$ (resp. $P \in Q_a$). For $j \in \mathbb{Z}$ we thus obtain an isomorphism
\[ H^jR\pi_!R\mathcal{I}hom(\mathbb{C}_{\{t \geq 0\}}, \mathbb{C}_{\{t \geq -\text{Re}\varphi + a\}}) \simeq \begin{cases} \mathbb{C}_{X \setminus Q_a} & (j = 0) \\ \mathbb{C}^k_D & (j = 1) \\ 0 & \text{(otherwise)} \end{cases}. \]
The remaining assertion follows from the isomorphism $\alpha_X \text{Sol}^0_X(\mathcal{E}_{X \setminus D|X}^\varphi) \simeq \text{Sol}_X(\mathcal{E}_{X \setminus D|X}^\varphi)$. \qed

Note that a special case of this proposition was proved in Kashiwara-Schapira [KS03 Proposition 7.3 and Remark 7.4]. In the situation of Proposition 3.14, for a meromorphic connection $\mathcal{N}$ on $X$ along $D \subset X$ i.e. a holonomic $\mathcal{D}_{X, \text{an}}$-module $\mathcal{N}$ such that $\mathcal{N}(*D) \simeq \mathcal{N}$ we define its irregularity $\text{irr}(\mathcal{N}) \in \mathbb{Z}$ by
\[ \text{irr}(\mathcal{N}) = \dim H^1\text{Sol}_X(\mathcal{N})_D \mid \dim H^0\text{Sol}_X(\mathcal{N})_D. \]
We know that it is a non-negative integer (see Sabbah [Sab93 Chapter II, the proof of Theorem 1.3.10] etc.). Moreover the meromorphic connection $\mathcal{N}$ is regular if and only if $\text{irr}(\mathcal{N}) = 0$. The last assertion in Proposition 3.14 implies that $\text{irr}(\mathcal{E}_{X \setminus D|X}^\varphi) = k$. We can generalize this result as follows.

**Proposition 3.15.** In the situation of Proposition 3.14, let $\varpi_X : \tilde{X} \to X$ be the real blow-up of $X$ along $D$. For a meromorphic connection $\mathcal{N}$ on $X$ along $D \subset X$ assume that there exist meromorphic functions $\varphi_i \in \mathcal{O}_X(*D)$ ($1 \leq i \leq m$) such that for any point $\theta \in S_D X \simeq \varpi_X^{-1}(D) \simeq S^1$ there exists its sectorial neighborhood $V_\theta \subset X \setminus D$ for which we have an isomorphism
\[ \pi^{-1}C_{V_\theta} \otimes \text{Sol}_X^E(\mathcal{N}) \simeq \pi^{-1}C_{V_\theta} \otimes \bigoplus_{i=1}^m \text{Sol}_X^E(\mathcal{E}_{X \setminus D|X}^{\varphi_i}). \]

25
For $1 \leq i \leq m$ such that $\varphi_i$ has a pole along $D$ (resp. is holomorphic at $D$) we denote by $\text{ord}_D(\varphi_i) > 0$ its pole order (resp. we set $\text{ord}_D(\varphi_i) = 0$). Then we have

$$\text{irr}(\mathcal{N}) = \sum_{i=1}^{m} \text{ord}_D(\varphi_i).$$

In particular, if $\text{ord}_D(\varphi_i) = 0$ for any $1 \leq i \leq m$, then $\mathcal{N}$ is regular.

Proof. The proof is similar to that of Proposition 3.14. Shrinking $X$ if necessary we may assume that $X = \{u \in \mathbb{C} \mid |u| < \varepsilon\}$ for some $\varepsilon > 0$, $D = \{u = 0\}$ and $X \setminus D$ is covered by some sectors $V_{a_1}, V_{a_2}, \ldots, V_{a_l} \subset X \setminus D$ for which we have isomorphisms

$$\pi^{-1}\mathbb{C}V_{a_j} \otimes \text{Sol}^E_X(\mathcal{N}) \simeq \pi^{-1}\mathbb{C}V_{a_j} \otimes \left( \bigoplus_{i=1}^{m} \text{Sol}^E_X(\mathcal{E}^{e\varphi_i}_{X \setminus D\setminus X}) \right) \quad (1 \leq j \leq l).$$

Then by the Mayer-Vietoris exact sequences associated to the open covering $X \setminus D = \bigcup_{j=1}^{l} V_{a_j}$ of $X \setminus D$ we can modify the proof of Proposition 3.14 to our case. \hfill \Box

We have also the following result. We call a finite sum

$$\varphi(u) = \sum_{a \in Q, u \leq 0} c_a u^a \quad (c_a \in \mathbb{C})$$

a Puiseux polynomial of $u^{-1}$. Here we regard it as a function on a sector $V \subset \mathbb{C} \setminus \{0\}$ by fixing the branches of the monomials $u^a$.

**Proposition 3.16.** In the situation of Proposition 3.14, let $\varpi_X : \tilde{X} \rightarrow X$ be the real blow-up of $X$ along $D = \{u = 0\}$. Let $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m$ be Puiseux polynomials of $u^{-1}$ without constant terms. Assume that for a point $\theta \in S_D X \simeq \varpi_X^{-1}(D) \simeq S^1$ there exists its open neighborhood $U$ in $\tilde{X}$ on which we have an isomorphism

$$\Phi : \bigoplus_{j=1}^{m} \mathcal{A}_{\tilde{X}}^{e\varphi_j} \sim \bigoplus_{i=1}^{m} \mathcal{A}_{\tilde{X}}^{e\psi_i}$$

of $\mathcal{D}^{A}_{\tilde{X}}$-modules, where $\mathcal{A}_{\tilde{X}}^{e\varphi_j} \simeq \mathcal{A}_{\tilde{X}}^{e\psi_i} (1 \leq j \leq m)$ and $\mathcal{A}_{\tilde{X}}^{e\varphi_j} \simeq \mathcal{A}_{\tilde{X}}^{e\psi_i} (1 \leq i \leq m)$ are the natural $\mathcal{D}^{A}_{\tilde{X}}$-modules associated to the functions $e^{\varphi_j}$ $(1 \leq j \leq m)$ and $e^{\psi_i}$ $(1 \leq i \leq m)$, respectively. Then after reordering $\varphi_j$'s and $\psi_i$'s for any $1 \leq j \leq m$ we have $\varphi_j = \psi_j$.

Proof. For a sector $V \subset X \setminus D$ we define binary relations $\succ$ and $\equiv$ on the set of Puiseux polynomials of $u^{-1}$ by

$$\varphi \succ \psi \iff \text{Re}\varphi > \text{Re}\psi \text{ on } V, \quad \varphi \equiv \psi \iff \text{Re}\varphi = \text{Re}\psi \text{ on } V.$$

We set also

$$\varphi \succ \psi \iff \varphi \succ \psi \text{ or } \varphi \equiv \psi.$$

Note that the condition $\text{Re}\varphi = \text{Re}\psi$ on $V$ for two Puiseux polynomials of $u^{-1}$ without constant terms implies $\varphi = \psi$. Hence the relation $\succ$ defines a partial order on the set
of such Puiseux polynomials. We can choose a point \( \theta' \in S_D X \cap U \) and its small and narrow sectorial neighborhood \( V \subset X \setminus D \) so that after reordering \( \varphi_j \) (\( 1 \leq j \leq m \)) and \( \psi_i \) (\( 1 \leq i \leq m \)) we have

\[
\varphi_1 \succ_{V} \varphi_2 \succ \cdots \succ_{V} \varphi_m \succ_{V} \psi_1 \text{ and } \psi_1 \succ_{V} \psi_2 \succ \cdots \succ_{V} \psi_m.
\]

We can also choose \( V \) so that for any \( 1 \leq j \leq m \) and \( 1 \leq i \leq m \) one of the conditions \( \varphi_j \succ_{V} \psi_i \), \( \varphi_j =_{V} \psi_i \) and \( \varphi_j \prec_{V} \psi_i \) is satisfied. Then for any \( 1 \leq j \leq m \), \( 1 \leq i \leq m \) and connected open subset \( W \subset \tilde{X} \) such that \( W \cap \varpi^{-1}(D) \neq \emptyset \), \( W \subset \text{Int}(\varpi^{-1}(V)) \) we have an isomorphism

\[
\Gamma(W; \text{Hom}_{D}^{\omega_{\tilde{X}}}(\mathbb{A}_{\tilde{X}} e^{\varphi_j}, \mathbb{A}_{\tilde{X}} e^{\psi_i})) \simeq \begin{cases} 
\mathbb{C} & (\varphi_j \preceq_{V} \psi_i) \\
0 & (\varphi_j \succ_{V} \psi_i) .
\end{cases}
\]

Moreover if \( \varphi_j \preceq_{V} \psi_i \) the 1-dimensional vector space

\[
\Gamma(W; \text{Hom}_{D}^{\omega_{\tilde{X}}}(\mathbb{A}_{\tilde{X}} e^{\varphi_j}, \mathbb{A}_{\tilde{X}} e^{\psi_i})) \simeq \mathbb{C}
\]

is generated by the morphism

\[
1 \cdot e^{\psi_i} \mapsto e^{\varphi_j - \psi_i} \cdot e^{\psi_i}.
\]

This implies that the restriction of the isomorphism

\[
\Phi : \bigoplus_{j=1}^{m} \mathbb{A}_{\tilde{X}} e^{\varphi_j} (\simeq \mathbb{A}_{\tilde{X}}^{\oplus m}) \twoheadrightarrow \bigoplus_{i=1}^{m} \mathbb{A}_{\tilde{X}} e^{\psi_i} (\simeq \mathbb{A}_{\tilde{X}}^{\oplus m})
\]

to \( W \) is represented by the invertible matrix

\[
F(u) := (f_{ij}(u))_{1 \leq i, j \leq m} \in M_m(\mathbb{A}_{\tilde{X}}(W)),
\]

where we have

\[
f_{ij}(u) = C_{ij} e^{\varphi_j - \psi_i} \in \mathbb{A}_{\tilde{X}}(W)
\]

for some constants \( C_{ij} \in \mathbb{C} \) such that \( C_{ij} = 0 \) if \( \varphi_j \npreceq_{V} \psi_i \). First let us consider the case where we have \( \varphi_{m-1} \succ_{V} \varphi_{m} \succeq_{V} \psi_{m} \). Then the matrix \( F(u) \) is a block upper triangular matrix of the form

\[
F(u) = \begin{pmatrix}
\ast \\
\vdots \\
\ast \\
0 \ldots 0 & \ast
\end{pmatrix} \in M_m(\mathbb{A}_{\tilde{X}}(W)).
\]

For it to be invertible, we have \( C_{mm} \neq 0 \) and hence \( \varphi_{m} = \psi_{m} \) on \( V \). Since \( \det F(u) \in \mathbb{A}_{\tilde{X}}(W) \) is invertible in \( \mathbb{A}_{\tilde{X}}(W) \), the same is true also for \( \det \tilde{F}(u) \in \mathbb{A}_{\tilde{X}}(W) \). Then the morphism

\[
\tilde{\Phi} : \bigoplus_{j=1}^{m-1} \mathbb{A}_{\tilde{X}} e^{\varphi_j} \twoheadrightarrow \bigoplus_{i=1}^{m-1} \mathbb{A}_{\tilde{X}} e^{\psi_i}
\]
of $D^A_X$-modules on $W$ induced by the matrix $\tilde{F}(u)$ is an isomorphism. Hence the induction on the rank $m$ proceeds.

Next let us consider the general case where we have

$$\varphi_{m-k} \succ V \varphi_{m-k+1} \succ \cdots \succ V \varphi_m (\succ V \psi_m)$$

for some $k \geq 1$. Then the matrix $F(u)$ has the form

$$F(u) = \begin{pmatrix} * & * \\ 0 \cdots 0 & * \cdots * \end{pmatrix} \in M_m(A_\tilde{X}(W)).$$

For it to be invertible, we have

$$\left( \varphi_{m-k+1} \succ V \varphi_{m-k+2} \succ \cdots \succ V \varphi_m \right) \varphi_m = \psi_m$$

and

$$(C_{mm-k+1}, C_{mm-k+2}, \ldots, C_{mm}) \neq (0, 0, \ldots, 0).$$

Assume that the condition

$$\psi_{m-k+1} \succ V \psi_{m-k+2} \succ \cdots \succ V \psi_m$$

is not satisfied. Then $\psi_{m-k+1} \succ V \psi_m$ and the isomorphism

$$\Psi := \Phi^{-1} : \bigoplus_{i=1}^{m} A_{\tilde{X}} e^{\psi_i} \sim \rightarrow \bigoplus_{j=1}^{m} A_{\tilde{X}} e^{\psi_j}$$

is represented by a block upper triangular matrix

$$G(u) := (g_{ij}(u))_{1 \leq i, j \leq m} \in M_m(A_{\tilde{X}}(W))$$

of the form

$$G(u) = \begin{pmatrix} \tilde{G}(u) & * \\ O & O * \end{pmatrix} \begin{pmatrix} m-k \\ k \end{pmatrix} \in M_m(A_{\tilde{X}}(W)).$$

This is a contradiction. Hence we have

$$\psi_{m-k+1} \succ V \psi_{m-k+2} \succ \cdots \succ V \psi_m$$

and the matrix $F(u)$ has the form

$$F(u) = \begin{pmatrix} \tilde{F}(u) & * \\ O & B \end{pmatrix} \in M_m(A_{\tilde{X}}(W)).$$

for an invertible constant matrix $B \in M_k(\mathbb{C})$. Similarly to the case $k = 1$ the function $\det \tilde{F}(u) \in A_{\tilde{X}}(W)$ is invertible in $A_{\tilde{X}}(W)$ and the induction on the rank $m$ proceeds. \hfill \Box
By this proposition we obtain the following useful result. In the situation of Proposition 3.14 let \( N \) be a meromorphic connection of rank \( m \) along the point \( D = \{ u = 0 \} \subset X \). Denote by \( \hat{O}_{D|X} \) the formal completion of \( O_X \) along \( D \subset X \). Then by the Hukuhara-Levelt-Turrittin theorem after a ramification the formalization
\[
\hat{N} := \hat{O}_{D|X} \otimes_{O_X,D} N_D \in \text{Mod}(\hat{O}_{D|X})
\]
of \( N \) along \( D \) admits a decomposition by some Puiseux polynomials \( \psi_1(u), \ldots, \psi_m(u) \) of \( u^{-1} \) without constant terms. We call them the exponential factors of \( N \).

**Corollary 3.17.** In the situation as above, assume that for Puiseux polynomials \( \varphi_1(u), \ldots, \varphi_m(u) \) of \( u^{-1} \) without constant terms and a point \( \theta \in S_D X \simeq \varpi_X^{-1}(D) \simeq S^1 \) there exists its open neighborhood \( U \) in \( \tilde{X} \) on which we have an isomorphism
\[
N^A \simeq \bigoplus_{j=1}^m A_{\tilde{X}} e^{\varphi_j}
\]
of \( \mathcal{D}^A_{\tilde{X}} \)-modules. Then after reordering \( \varphi_j \)'s and \( \psi_i \)'s for any \( 1 \leq j \leq m \) we have \( \varphi_j = \psi_j \). Namely \( \varphi_1(u), \ldots, \varphi_m(u) \) are the exponential factors of \( N \) counting with multiplicities.

**Proof.** By the Hukuhara-Levelt-Turrittin theorem after shrinking \( U \) we obtain an isomorphism
\[
N^A \simeq \bigoplus_{i=1}^m A_{\tilde{X}} e^{\psi_i}
\]
of \( \mathcal{D}^A_{\tilde{X}} \)-modules on \( U \). Then the assertion follows from Proposition 3.16.

By this corollary and Corollary 3.11 we obtain the following result.

**Theorem 3.18.** In the situation as above, assume that for convergent Laurent Puiseux series \( \varphi_1, \ldots, \varphi_m \) of \( u \) and a point \( \theta \in S_D X \simeq \varpi_X^{-1}(D) \simeq S^1 \) there exists its sectorial neighborhood \( V_0 \subset X \setminus D \) for which we have an isomorphism
\[
\pi^{-1}C_{V_0} \otimes \text{Sol}^E_{\tilde{X}}(N) \simeq \bigoplus_{j=1}^m E^{\varphi_j}_{V_0|X}.
\]
Then after reordering \( \varphi_j \)'s and \( \psi_i \)'s for any \( 1 \leq j \leq m \) the pole part of \( \varphi_j \) coincides with \( \psi_j \). In particular, we have
\[
\text{irr}(N) = \sum_{i=1}^m \text{ord}_D(\varphi_i).
\]

Note that Proposition 3.15 is a very special case of this theorem. Theorem 3.18 could be deduced also from [DK17, Lemma 6.3.4]. Our arguments can be applied to exponential factors of meromorphic connections in higher dimensions as follows. Let \( X \) be a complex manifold and \( D \subset X \) a normal crossing divisor in it. Let us take local coordinates \((u_1, \ldots, u_l, v_1, \ldots, v_{d_X-l})\) of \( X \) such that \( D = \{ u_1 u_2 \cdots u_l = 0 \} \). We define a partial order \( \leq \) on the set \( \mathbb{Z}^l \) by
\[
a \leq a' \iff a_i \leq a'_i \quad (1 \leq i \leq l).
\]
Then for a meromorphic function \( \varphi \in \mathcal{O}_X(*D) \) on \( X \) along \( D \) by using its Laurent expansion

\[
\varphi = \sum_{a \in \mathbb{Z}^l} c_a(\varphi)(v) \cdot u^a \in \mathcal{O}_X(*D)
\]

with respect to \( u_1, \ldots, u_l \) we define its order \( \text{ord}(\varphi) \in \mathbb{Z}^l \) to be the minimum

\[
\min \left( \{ a \in \mathbb{Z}^l \mid c_a(\varphi) \neq 0 \} \cup \{0\} \right)
\]

if it exists. In [Moc11] Chapter 5 Mochizuki defined the notion of good sets of irregular values on \((X,D)\) to be finite subsets \( S \subset \mathcal{O}_X(*D)/\mathcal{O}_X \) satisfying some properties. We do not recall here its precise definition. Just recall that for any values on \( (X,D) \) it exists. In [Moc11] Chapter 5 Mochizuki defined the notion of good sets of irregular values on \((X,D)\). Let us call a finite subset \( S \subset \mathcal{O}_X(*D)/\mathcal{O}_X \) satisfying only this property a quasi-good set of irregular values on \((X,D)\).

**Proposition 3.19.** In the situation as above, let \( \varpi_X : \tilde{X} \to X \) be the real blow-up of \( X \) along the normal crossing divisor \( D \). Assume that \( \varphi_1, \ldots, \varphi_m \) (resp. \( \psi_1, \ldots, \psi_m \)) \( \in \mathcal{O}_X(*D)/\mathcal{O}_X \) form a quasi-good set of irregular values on \((X,D)\). Assume also that for any \( v \in Y = \{ u_1 = u_2 = \cdots = u_l = 0 \} \subset D \). Let us call a finite subset \( S \subset \mathcal{O}_X(*D)/\mathcal{O}_X \) satisfying only this property a quasi-good set of irregular values on \((X,D)\).

Proof. The proof being very similar to that of Proposition 3.16 we shall freely use the notations etc. in it. As in the case \( l = 1 \) we can choose a sector \( V \subset X \setminus D \) along \( D \) such that \( \varpi_X^{-1}(V) \subset U \) so that after reordering \( \varphi_j \) (1 \( \leq j \leq m \)) and \( \psi_i \) (1 \( \leq i \leq m \)) we have

\[
\varphi_1 \geq_V \varphi_2 \geq_V \cdots \geq_V \varphi_m \quad \text{and} \quad \psi_1 \geq_V \psi_2 \geq_V \cdots \geq_V \psi_m.
\]

Suppose that \( \varphi_m \neq \psi_m \) in \( \mathcal{O}_X(*D)/\mathcal{O}_X \). Then there exists a weight vector \( b = (b_1, \ldots, b_l) \in \mathbb{Z}_{>0}^l \) such that

\[
L := \min \{ \langle a, b \rangle \mid a \in \mathbb{Z}^l, c_a(\varphi_m - \psi_m) \neq 0 \} < 0
\]

and the set \( \{ a \in \mathbb{Z}^l \mid c_a(\varphi_m - \psi_m) \neq 0, \langle a, b \rangle = L \} \) consists of a single point. Define a subset \( K \subset V \) of the sector \( V \) by

\[
K = \{ (s^{b_1}e^{i\theta_1}, \ldots, s^{b_l}e^{i\theta_l}, v_1, \ldots, v_{d_{x_1}-1}) \in V \mid 0 < s < 1 \} \subset V.
\]

Then the restriction of the function \( \text{Re}(\varphi_m - \psi_m) \) to \( K \) tends to \( +\infty \) or \( -\infty \) as \( s \to +0 \). Replacing \( \Phi \) with \( \Phi^{-1} \) if necessary, we may assume that the restriction of the function \( e^{\varphi_m - \psi_m} \) to \( K \) increases rapidly as \( s \to +0 \). Hence we have \( C_{m1} = C_{m2} = \cdots = C_{mm} = 0 \). This contradicts to our assumption that \( \Phi \) is an isomorphism. Then we can continue the arguments in the proof of Proposition 3.16. \( \Box \)
In the situation of Proposition 3.19 let \( \mathcal{N} \) be a meromorphic connection of rank \( m \) on \( X \) along the the normal crossing divisor \( D \). If it admits an unramified good lattice (at a point \( v \in Y = \{ u_1 = u_2 = \cdots = u_l = 0 \} \subset D \) in the sense of Mochizuki [Moc11], we call the elements of \( \mathcal{O}_X(\ast D)/\mathcal{O}_X \) appearing in the formal decomposition the exponential factors of \( \mathcal{N} \). Recall that they form a good set of irregular values on \( (X, D) \). Then by Propositions 3.10 and 3.19 we obtain the following result.

**Theorem 3.20.** In the situation as above, assume that the meromorphic connection \( \mathcal{N} \) admits an unramified good lattice. Assume also that \( \varphi_1, \ldots, \varphi_m \in \mathcal{O}_X(\ast D)/\mathcal{O}_X \) form a quasi-good set of irregular values on \( (X, D) \) and for a point \( \theta \in \varpi_X^{-1}(Y) \subset \varpi_X^{-1}(D) \) there exists its sectorial neighborhood \( V_{\theta} \subset X \setminus D \) for which we have an isomorphism

\[
\pi^{-1}C_{V_{\theta}} \otimes \text{Sol}_X^{E}(\mathcal{N}) \simeq \bigoplus_{j=1}^{m} E_{V_{\theta}|X}^{\text{Ref}_j}.
\]

Then \( \varphi_1, \ldots, \varphi_m \in \mathcal{O}_X(\ast D)/\mathcal{O}_X \) are the exponential factors of \( \mathcal{N} \) counting with multiplicities.

Obviously by ramification maps we can generalize this theorem to meromorphic connection admitting good lattices in the sense of Mochizuki [Moc11].

The following lemma will be used in the proof of Theorem 4.4.

**Lemma 3.21.** Let \( M \) be the complex vector space \( \mathbb{C}^n \) with the standard coordinates \( x = (x_1, x_2, \ldots, x_n) \) and \( N, H \subset M \) its linear subspaces defined by

\[
N := \{ x_1 = x_2 = 0 \} \subset H := \{ x_1 = 0 \} \subset M.
\]

For \( F \in D^b(\mathbb{C}_{M\setminus H}) \) assume that there exists sufficiently small \( 0 < \varepsilon \ll 1 \) such that its micro-support \( SS(F) \subset T^*(M\setminus H) \) does not intersect

\[
U_{\varepsilon} = \left\{ (x, \xi) \in T^*(M\setminus H) \mid \xi_1 \in \mathbb{C}, \sqrt{\sum_{i=3}^{n} |\xi_i|^2} < \varepsilon |\xi_2| \right\}.
\]

Let \( \rho : M^N \rightarrow M \) be the (complex) blow-up of \( M \) along \( N \), \( E \subset M^N \) its exceptional divisor and \( H' \subset M^N \) the proper transform of \( H \) in it. Let \( f : M^N \rightarrow \mathbb{P}^1 \) be the holomorphic map induced by the meromorphic function \( \frac{x_2}{x_1} \) on \( M \) and \( \iota : M\setminus H \hookrightarrow M^N \) the inclusion map. Then for any point \( Q \in E \setminus H' \) we have the vanishing

\[
\text{R}^\ast_{(\text{Ref})^{-1}(i \in \mathbb{R}|t \geq \text{Ref}(Q))}(t, F)_Q \simeq 0.
\]

**Proof.** There exists an affine open subset \( W \simeq \mathbb{C}^n \) of \( M^N \) with the coordinates \( y = (y_1, y_2, \ldots, y_n) \) such that the restriction of the morphism \( \rho : M^N \rightarrow M \) to it is given by

\[
(y_1, y_2, \ldots, y_n) \mapsto (x_1, x_2, \ldots, x_n) = (y_1, y_1 y_2, y_3, \ldots, y_n)
\]

and \( E \setminus H' = E \cap W = \{ y \in W \mid y_1 = 0 \} \). Moreover the restriction \( f|_W \) of \( f \) to it is given by \( (f|_W)(y) = y_2 \). By the isomorphism \( W \setminus E \simeq M \setminus H \) we regard \( F \) as an object of \( D^b(\mathbb{C}_{W\setminus E}) \). Then it is easy to see that its micro-support does not intersect

\[
U'_{\varepsilon} = \left\{ (y, \eta) \in T^*(W\setminus E) \mid \eta_1 \in \mathbb{C}, \sqrt{\sum_{i=3}^{n} |\eta_i|^2} < \varepsilon |\eta_2| / |y_1| \right\}.
\]

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In particular, the micro-support of the restriction of $\mathcal{F} \in D^b(\mathbb{C}^n_{W\setminus E})$ to $(W\setminus E) \cap \{ |y_1| < 1 \}$ does not intersect

$$V_{\varepsilon} = \left\{ (y, \eta) \in T^* ((W\setminus E) \cap \{ |y_1| < 1 \}) \mid \eta_1 \in \mathbb{C}, \sqrt{\sum_{i=3}^{n} |\eta_i|^2} < \varepsilon |\eta_2| \right\}.$$ 

Let $\varpi_{\text{tot}} : \widetilde{W}_{E}^{\text{tot}} = \{ \zeta \in \mathbb{C} \mid |\zeta| = 1 \} \times \mathbb{R}_+ \times \mathbb{C}^{n-1} \rightarrow W = \mathbb{C}^n$ be the total real blow-up of $W = \mathbb{C}^n$ along $E \cap W = \{ y_1 = 0 \}$ of D’Agnolo-Kashiwara (see [DK16, § 7.1]) defined by

$$(\zeta, t, y_2, \ldots, y_n) \mapsto (t \zeta, y_2, \ldots, y_n)$$

and identify $W \setminus E$ with the open subset $\{ t > 0 \}$ of $\widetilde{W}_{E}^{\text{tot}}$. Let $j : W \setminus E \hookrightarrow \widetilde{W}_{E}^{\text{tot}}$ be the inclusion map and for a point $Q = (0, b_2, \ldots, b_n) \in E \setminus H' = E \cap W = \{ y_1 = 0 \}$ set

$$Z = (\text{Re} f)^{-1}(\{ t \in \mathbb{R} \mid t \geq \text{Re} f(Q) \}) = \{ y \in W \mid \text{Re} y_2 \geq \text{Re} b_2 \}.$$ 

Then we have isomorphisms

$$\text{R} \Gamma Z \langle t, \mathcal{F} \rangle_Q \simeq (\text{R} \Gamma Z \langle \varpi_{\text{tot}} \rangle_{\langle j; \mathcal{F} \rangle})_Q \simeq (\text{R} (\varpi_{\text{tot}}), \text{R} \Gamma (\varpi_{\text{tot}})^{-1}(Z) \langle j; \mathcal{F} \rangle)_Q \simeq \text{R} \Gamma (\langle \text{Re}_{\text{tot}} \rangle^{-1}(Q); \text{R} \Gamma (\varpi_{\text{tot}})^{-1}(Z) \langle j; \mathcal{F} \rangle).$$

Hence it suffices to show the vanishing

$$(\text{R} \Gamma (\varpi_{\text{tot}})^{-1}(Z) \langle j; \mathcal{F} \rangle)_P \simeq 0$$

at each point $P \in (\varpi_{\text{tot}})^{-1}(Q) \simeq \{ \zeta \in \mathbb{C} \mid |\zeta| = 1 \} \simeq S^1$. By using the above estimate of the micro-support of the restriction of $\mathcal{F} \in D^b(\mathbb{C}^n_{W\setminus E})$ to $(W\setminus E) \cap \{ |y_1| < 1 \}$ we can show it by [KS90, Theorem 6.3.1]. This completes the proof.\hfill \Box

4 Main Theorems

First we recall Fourier transforms of algebraic $\mathcal{D}$-modules. Let $X = \mathbb{C}^N_z$ be a complex vector space and $Y = \mathbb{C}^N_w$ its dual. We regard them as algebraic varieties and use the notations $\mathcal{D}_X$ and $\mathcal{D}_Y$ for the rings of “algebraic” differential operators on them. Denote by $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ (resp. $\text{Mod}_{\text{hol}}(\mathcal{D}_X)$, $\text{Mod}_{\text{rh}}(\mathcal{D}_X)$) the category of coherent (resp. holonomic, regular holonomic) $\mathcal{D}_X$-modules. Let $W_N := \mathbb{C}[z, \partial_z] \simeq \Gamma(X; \mathcal{D}_X)$ and $W_N^* := \mathbb{C}[w, \partial_w] \simeq \Gamma(Y; \mathcal{D}_Y)$ be the Weyl algebras over $X$ and $Y$, respectively. Then by the ring isomorphism

$$W_N \xrightarrow{\sim} W_N^* \quad (z_i \mapsto -\partial_{w_i}, \partial_z \mapsto w_i)$$

we can endow a left $W_N$-module $M$ with a structure of a left $W_N^*$-module. We call it the Fourier transform of $M$ and denote it by $M^\wedge$. For a ring $R$ we denote by $\text{Mod}_f(R)$ the category of finitely generated $R$-modules. Recall that for the affine algebraic varieties $X$ and $Y$ we have the equivalences of categories

$$\text{Mod}_{\text{coh}}(\mathcal{D}_X) \simeq \text{Mod}_f(\Gamma(X; \mathcal{D}_X)) = \text{Mod}_f(W_N),$$

$$\text{Mod}_{\text{coh}}(\mathcal{D}_Y) \simeq \text{Mod}_f(\Gamma(Y; \mathcal{D}_Y)) = \text{Mod}_f(W_N^*).$$
(see e.g. [HTT08, Propositions 1.4.4 and 1.4.13]). For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X) \) we thus can define its Fourier transform \( \mathcal{M}^\wedge \in \text{Mod}_{\text{coh}}(\mathcal{D}_Y) \). It follows that we obtain an equivalence of categories

\[
(\cdot)^\wedge : \text{Mod}_{\text{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \text{Mod}_{\text{hol}}(\mathcal{D}_Y)
\]

between the categories of holonomic \( \mathcal{D} \)-modules. Let

\[
X \xleftarrow{p} X \times Y \xrightarrow{q} Y
\]

be the projections. Then by Katz-Laumon [KL85], for a holonomic \( \mathcal{D}_X \)-module \( \mathcal{M} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X) \) we have an isomorphism

\[
\mathcal{M}^\wedge \simeq D_q^*(D^p\mathcal{M} \otimes \mathcal{O}_{X \times Y} e^{-(z,w)}) ,
\]

where \( D^p, D_q, \otimes \) are the operations for algebraic \( \mathcal{D} \)-modules and \( \mathcal{O}_{X \times Y} e^{-(z,w)} \) is the integral connection of rank one on \( X \times Y \) associated to the canonical paring \( \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{C} \). In particular the right hand side is concentrated in degree zero. Let \( \overline{X} \simeq \mathbb{P}^N \) (resp. \( \overline{Y} \simeq \mathbb{P}^N \)) be the projective compactification of \( X \) (resp. \( Y \)). By the inclusion map \( i_X : X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N \) we extend a holonomic \( \mathcal{D}_X \)-module \( \mathcal{M} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X) \) on \( X \) to the one \( \overline{\mathcal{M}} := i_X^*\mathcal{M} \simeq D^p i_X^*\mathcal{M} \) on \( \overline{X} \). Denote by \( \overline{X}^\text{an} \) the underlying complex manifold of \( \overline{X} \) and define the analytification \( \mathcal{M}^\text{an} \in \text{Mod}_{\text{hol}}(\mathcal{D}_{\overline{X}^\text{an}}) \) of \( \mathcal{M} \) by \( \mathcal{M}^\text{an} = \mathcal{O}_{\overline{X}^\text{an}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{M} \). Then we set

\[
\text{Sol}^E_{\overline{X}}(\overline{\mathcal{M}}) := \text{Sol}^E_{\overline{X}^\text{an}}(\overline{\mathcal{M}}^\text{an}) \in \mathbf{E}^b(\mathbb{IC}_{\overline{X}^\text{an}}).
\]

Note that by Theorem \([2.1](v)\) there exists an isomorphism

\[
\text{Sol}^E_{\overline{X}}(\overline{\mathcal{M}}) \simeq \pi^{-1}\mathcal{C}_{\overline{X}^\text{an}} \otimes \text{Sol}^E_{\overline{X}}(\overline{\mathcal{M}}).
\]

Similarly for the Fourier transform \( \mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_Y) \) we define \( \text{Sol}^E_{\overline{Y}}(\mathcal{M}^\wedge) \in \mathbf{E}^b(\mathbb{IC}_{\overline{Y}^\text{an}}) \).

**Remark 4.1.** Let \( j_X : X^\text{an}_{\infty} = (X^\text{an}, \overline{X}^\text{an}) \to \overline{X}^\text{an} \) be the canonical morphism of bordered spaces and

\[
\text{Sol}^E_{X_{\infty}} : \text{D}^b_{\text{hol}}(\mathcal{D}_X) \to \text{D}^b_{\text{hol}}(\mathcal{D}_{X_{\infty}}) \to \mathbf{E}^b(\mathbb{IC}_{X_{\infty}})
\]

the functor defined in [KS16a, Definition 4.14]. Then for \( \mathcal{M} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X) \) there exists an isomorphism

\[
\text{Sol}^E_{X_{\infty}}(\mathcal{M}) \simeq \mathcal{E}_j X^{-1} \text{Sol}^E_{\overline{X}}(\overline{\mathcal{M}}).
\]

Let

\[
\overline{X}^\text{an} \xleftarrow{p} \overline{X}^\text{an} \times \overline{Y}^\text{an} \xrightarrow{q} \overline{Y}^\text{an}
\]

be the projections. Then the following theorem is essentially due to Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. For \( F \in \mathbf{E}^b(\mathbb{IC}_{\overline{Y}^\text{an}}) \) we set

\[
^LF := \mathcal{E}_{q_!}(\mathcal{E}_{p^{-1}!}^L F \otimes \mathcal{E}_{X \times Y \to \overline{X \times Y}}^{-\text{Re}(z,w)}[N]) \in \mathbf{E}^b(\mathbb{IC}_{\overline{Y}^\text{an}})
\]

(here we denote \( X^\text{an} \times Y^\text{an} \) etc. by \( X \times Y \) etc. for short) and call it the Fourier-Sato (Fourier-Laplace) transform of \( F \).
Theorem 4.2. For $\mathcal{M} \in \text{Mod}_{hol}(\mathcal{D}_X)$ there exists an isomorphism

$$\text{Sol}^E_{\mathcal{F}}(\overline{\mathcal{M}}^\wedge) \simeq L^\star \text{Sol}^E_X(\overline{\mathcal{M}}).$$

Proof. Let

$$\overline{X} \xleftarrow{\overline{p}} \overline{X} \times \overline{Y} \xrightarrow{\overline{q}} \overline{Y}$$

be the projections and $i_Y : Y = \mathbb{C}^N \hookrightarrow \overline{Y} = \mathbb{P}^N$, $i_{X \times Y} : X \times Y \hookrightarrow \overline{X} \times \overline{Y}$ the inclusion maps of algebraic varieties. Then by [HTT08, Theorem 1.7.3 and Corollary 1.7.5] there exists an isomorphism

$$\overline{\mathcal{M}}^\wedge \simeq i_Y \ast \text{Sol}_{X \times Y} e^{-(z,w)}.$$ 

Moreover we have $(i_{X \times Y} \ast \mathcal{O}_{X \times Y} e^{-(z,w)})^{an} \simeq \mathcal{E}_{X \times Y}^{-(z,w)}$. Then the assertion follows from [HTT08, Proposition 4.7.2] and Theorem 2.1.

Form now on, we focus our attention on Fourier transforms of regular holonomic $\mathcal{D}_X$-modules. For such a $\mathcal{D}_X$-module $\mathcal{M}$, by [HTT08, Theorem 7.1.1] we have an isomorphism

$$\text{Sol}^E(X) \simeq i_{X!} \text{Sol}_X(\mathcal{M}),$$

where the right hand side $i_{X!} \text{Sol}_X(\mathcal{M}) \in D^b(\mathbb{C}^n)$ is the extension by zero of the classical solution complex of $\mathcal{M}$ to $\mathbb{C}^n$. Moreover by [DK16, Proposition 9.1.3 and Corollary 9.4.9] there exists an isomorphism

$$\text{Sol}^E_X(\overline{\mathcal{M}}) \simeq \mathcal{E}_{X^{an}}^\star \otimes \varepsilon(i_{X!} \text{Sol}_X(\mathcal{M})).$$

For an enhanced sheaf $F \in \mathcal{E}^b(\mathcal{C}^{an})$ on $\mathbb{C}^{an}$ we define its Fourier-Sato (Fourier-Laplace) transform $L^F \in \mathcal{E}^b(\mathbb{C}^{an})$ by

$$L^F := \mathcal{E}^\ast_{\overline{\mathcal{M}}} (E^\ast \mathcal{F} \otimes E^\ast (z,w)) \in \mathcal{E}_X^{an}[\mathcal{F}].$$

Since we have

$$L(\mathcal{E}^\ast_{X^{an}} \otimes (\cdot)) \simeq \mathcal{E}^\ast_{X^{an}} \otimes L(\cdot)$$

it suffices to study the Fourier-Sato transform of the enhanced sheaf $\varepsilon(i_{X!} \text{Sol}_X(\mathcal{M})) \in \mathcal{E}^b(\mathbb{C}^{an})$ on $\mathbb{C}^{an}$. Fix a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ and denote by $\text{char}(\mathcal{M}) \subset T^\ast X \simeq X \times Y$ its characteristic variety.

Definition 4.3. We define a (Zariski) open subset $\Omega \subset Y = \mathbb{C}^N_w$ by:

$$w \in \Omega \iff \begin{cases} \text{there exists an open neighborhood } U \text{ of } w \text{ in } Y \\ \text{such that the restriction } q^{-1}(U) \cap \text{char}(\mathcal{M}) \to U \\ \text{of } q : X \times Y \to Y \text{ is an unramified covering.} \end{cases}$$

Since $\text{char}(\mathcal{M})$ is $\mathbb{C}^*$-conic, $\Omega \subset Y = \mathbb{C}^N_w$ is also $\mathbb{C}^*$-conic. Note that $\Omega$ is dense in $Y$. Denote by $k \geq 0$ the degree of the covering $q^{-1}(\Omega) \cap \text{char}(\mathcal{M}) \to \Omega$. For a point $w \in \Omega \subset Y = \mathbb{C}^N_w$, let $\{\mu_1(w), \ldots, \mu_k(w)\} = q^{-1}(w) \cap \text{char}(\mathcal{M}) \subset T^\ast X$ be its fiber by $q^{-1}(\Omega) \cap \text{char}(\mathcal{M}) \to \Omega$. Then in a neighborhood of each point $\mu_i(w) \in q^{-1}(w) \cap \text{char}(\mathcal{M})$
the characteristic variety \( \text{char}(\mathcal{M}) \) is smooth and hence there exists a (locally closed) complex submanifold \( S_i \subset X \) such that \( \mu_i(w) \in \text{char}(\mathcal{M}) = T_{S_i}^*X \). For \( 1 \leq i \leq k \) set
\[
\alpha_i(w) := p(\mu_i(w)) \in S_i \subset X = \mathbb{C}^N.
\]

By our definition of \( \Omega \subset Y = \mathbb{C}^N \), it is easy to see that the restriction of the linear function
\[
\ell(w) : X = \mathbb{C}^N \to \mathbb{C} \quad (z \mapsto \langle z, w \rangle)
\]
to \( S_i \) has a non-degenerate (complex Morse) critical point at \( \alpha_i(w) \in S_i \) (see e.g. Kashiwara-Schapira [KS85, Lemma 7.2.2]). For \( 1 \leq i \leq k \) denote by \( m_i > 0 \) the multiplicity of \( \mathcal{M} \) at \( \mu_i(w) \in \text{char}(\mathcal{M}) \).

**Theorem 4.4.** Let \( U \subset \Omega \subset Y = \mathbb{C}^N \) be a connected and simply connected open subset of \( \Omega \). Then we have an isomorphism
\[
\pi^{-1}C_U \otimes \left( \text{Sol}_X^E(\mathcal{M}^\alpha) \right) \simeq \bigoplus_{i=1}^k \pi^{-1}C_U \otimes \left( \text{"lim "}_{a \to +\infty} \mathbb{C}^{\oplus m_i}_{\{t \geq \text{Re}(\alpha_i(w)), w \}} \right)
\]
in \( \mathcal{E}^b(\mathcal{I}C_\mathcal{T}) \).

**Proof.** It suffices to prove that there exists an isomorphism
\[
L^1(\mathcal{E}(i_X!\text{Sol}_X(\mathcal{M}))) \simeq \bigoplus_{i=1}^k \mathbb{C}^{\oplus m_i}_{\{t \geq \text{Re}(\alpha_i(w)), w \}}
\]
of enhanced sheaves on \( U \subset \Omega \subset Y \). Let
\[
X \times \mathbb{R}_s \leftarrow^{p_1} (X \times \mathbb{R}_s) \times (Y \times \mathbb{R}_t) \to^{p_2} Y \times \mathbb{R}_t
\]
be the projections. Then by D’Agnolo-Kashiwara [DK17, Lemma 7.2.1] on \( Y \subset \mathcal{Y} \) we have an isomorphism
\[
L^1(\mathcal{E}(i_X!\text{Sol}_X(\mathcal{M}))) \simeq \mathcal{Q}(p_2^*(p_1^!(\mathcal{C}_{s \geq 0}) \otimes \pi^{-1}\text{Sol}_X(\mathcal{M})) \otimes \mathcal{C}_{\{t - s - \text{Re}(z, w) \geq 0\}}[N])\),
\]
where \( \mathcal{Q} : \mathcal{D}^b(\mathcal{C}_\mathcal{T}^{\text{an}} \times \mathbb{R}) \to \mathcal{E}^b(\mathcal{C}_\mathcal{T}^{\text{an}}) \) is the quotient functor. For a point \( (w, t) \in Y^{\text{an}} \times \mathbb{R} \) we have also isomorphisms
\[
\begin{align*}
(\text{R}p_2^*(p_1^!(\mathcal{C}_{s \geq 0}) \otimes \pi^{-1}\text{Sol}_X(\mathcal{M})) \otimes \mathcal{C}_{\{t - s - \text{Re}(z, w) \geq 0\}}[N])((w, t)) & \simeq \text{R}\Gamma_c(\{z, s \in X^{\text{an}} \times \mathbb{R} \mid t - s - \text{Re}(z, w) \geq 0, s \geq 0\}; \pi^{-1}\text{Sol}_X(\mathcal{M})[N]) \\
& \simeq \text{R}\Gamma_c(X^{\text{an}}; \text{R}\pi_1(\mathcal{C}_{\{t - s - \text{Re}(z, w) \geq 0, s \geq 0\}} \otimes \pi^{-1}\text{Sol}_X(\mathcal{M})[N])) \\
& \simeq \text{R}\Gamma_c(X^{\text{an}}; (\text{R}\pi_1(\mathcal{C}_{\{t - s - \text{Re}(z, w) \geq 0, s \geq 0\}}) \otimes \text{Sol}_X(\mathcal{M})[N]) \\
& \simeq \text{R}\Gamma_c(\{z \in X^{\text{an}} \mid \text{Re}(z, w) \leq t\}; \text{Sol}_X(\mathcal{M})[N]),
\end{align*}
\]
where we used
\[
\text{R}\pi_1(\mathcal{C}_{\{t - s - \text{Re}(z, w) \geq 0, s \geq 0\}}) \simeq \mathbb{C}_{\{\text{Re}(z, w) \leq t\}}
\]
in the last isomorphism. Fix \( w \in U \subset \Omega \subset Y = \mathbb{C}^N \). Then by an argument similar to the one in the proof of Esterov-Takeuchi [ET15, Theorem 5.5] we can prove the vanishing
\[
\text{R}\Gamma_c(\{z \in X^{\text{an}} \mid \text{Re}(z, w) \leq t\}; \text{Sol}_X(\mathcal{M})[N]) \simeq 0
\]
for $t \ll 0$ as follows. Let $H_\infty := \overline{X}\setminus X \simeq \mathbb{P}^{N-1}$ be the hyperplane at infinity of $X = \mathbb{P}^N$ and

$$\ell(w) : X = \mathbb{C}^N \to \mathbb{C} \quad (z \mapsto \langle z, w \rangle)$$

the linear function defined by $w$. Then the meromorphic extension of $\ell(w)$ to $\overline{X} = \mathbb{P}^N$ has points of indeterminacy in the complex submanifold $H(w) := \ell(w)^{-1}(0) \cap H_\infty \simeq \mathbb{P}^{N-2}$ of $\overline{X} = \mathbb{P}^N$. Let $\rho : \overline{X}^{H(w)} \to \overline{X}$ be the complex blow-up of $\overline{X}$ along $H(w)$ and $\iota : X \hookrightarrow \overline{X}^{H(w)}$ the inclusion map. Then the meromorphic extension $f$ of $\ell(w)$ to $\overline{X}^{H(w)}$ has no point of indeterminacy and we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\iota} & \overline{X}^{H(w)} \\
\ell(w) \downarrow & & \downarrow f \\
\mathbb{C} & \xrightarrow{i} & \mathbb{P}^1
\end{array}
$$

of holomorphic maps, where $i : \mathbb{C} \hookrightarrow \mathbb{P}^1$ is the inclusion map. See the proof of Lemma 3.21 or [MTF3, Theorem 3.6]. Note that $f$ is proper and set

$$L(Sol_X(\mathcal{M})) := Rp_2! (p_1^{-1}(\mathbb{C}_{\{s \geq 0\}} \otimes \pi^{-1}Sol_X(\mathcal{M})) \otimes \mathbb{C}_{\{t-s-\Re(z,w) \geq 0\}}[N]) \in D^b(C_{Y \times \mathbb{R}}).$$

Then for $t \in \mathbb{R}$ the stalk

$$(L(Sol_X(\mathcal{M})))_{(w,t)} \simeq R\Gamma_c(\{z \in X^{an} \mid \Re(z,w) \leq t\}; Sol_X(\mathcal{M})[N])$$

is calculated as follows:

$$R\Gamma_c(\{z \in X^{an} \mid \Re(z,w) \leq t\}; Sol_X(\mathcal{M})[N]) \\
\simeq R\Gamma_c(\overline{X}^{H(w)}; \mathbb{C}_{\{z \in X^{an} \mid \Re(z,w) \leq t\}} \otimes \iota_! Sol_X(\mathcal{M})[N]) \\
\simeq R\Gamma_c(\overline{X}^{H(w)}; f^{-1}\mathbb{C}_{\{\tau \in \mathbb{C} \mid \Re\tau \leq t\}} \otimes \iota_! Sol_X(\mathcal{M})[N]) \\
\simeq R\Gamma_c(\mathbb{P}; \mathbb{C}_{\{\tau \in \mathbb{C} \mid \Re\tau \leq t\}} \otimes Rf_! \iota_! Sol_X(\mathcal{M})[N]) \\
\simeq R\Gamma_c(\{\tau \in \mathbb{C} \mid \Re\tau \leq t\}; Rf_! \iota_! Sol_X(\mathcal{M})[N]).$$

Since the direct image $Rf_! \iota_! Sol_X(\mathcal{M})[N]$ of the perverse sheaf $\iota_! Sol_X(\mathcal{M})[N] \in D^b_{\mathbb{C}_{\mathcal{M}}} (\mathbb{C}_{\overline{X}^{H(w)}})$ is constructible, for $t \ll 0$ the restrictions of its cohomology sheaves to the closed half space $\{\tau \in \mathbb{C} \mid \Re\tau \leq t\} \subset \mathbb{C} \subset \mathbb{P}^1$ of $\mathbb{C}$ are locally constant. Thus for $t \ll 0$ we obtain the vanishing

$$R\Gamma_c(\{z \in X^{an} \mid \Re(z,w) \leq t\}; Sol_X(\mathcal{M})[N]) \simeq 0.$$

Let $H'_\infty = f^{-1}(\{\infty\}) \subset \overline{X}^{H(w)}$ be the proper transform of $H_\infty$ by the blow-up $\rho$ and $g : \overline{X}^{H(w)} \setminus f^{-1}(\{\infty\}) \to \mathbb{C}$ the restriction of $f$ to $\overline{X}^{H(w)} \setminus f^{-1}(\{\infty\})$. Then by our construction of $\overline{X}^{H(w)}$ for any $t \in \mathbb{R}$ the real hypersurface $(\text{Reg})^{-1}(t)$ intersects the exceptional divisor $E = \rho^{-1}(H(w)) \subset \overline{X}^{H(w)}$ of $\rho$ transversally (see the proof of Lemma 3.21). Hence it is

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compact and smooth in $\bar{X}^{H(w)}$. This implies that the morphism $\text{Reg} : \bar{X}^{H(w)} \setminus f^{-1}(\{\infty\}) \to \mathbb{R}$ is proper. Let $U : X \hookrightarrow \bar{X}^{H(w)} \setminus f^{-1}(\{\infty\})$ be the inclusion map. Then for any $t \in \mathbb{R}$ we have isomorphisms

$$\text{RHom}_c(\{z \in X^\text{an} \mid \text{Re}(z, w) \leq t\}; \text{Sol}_X(\mathcal{M})[N])$$

$$\simeq \text{RHom}_c(\{x \in \bar{X}^{H(w)} \setminus f^{-1}(\{\infty\}) \mid \text{Re}(x) \leq t\}; t_!\text{Sol}_X(\mathcal{M})[N])$$

$$\simeq \text{RHom}_c(\{s \in \mathbb{R} \mid s \leq t\}; R(\text{Reg})_*t_!\text{Sol}_X(\mathcal{M})[N]).$$

For $1 \leq i \leq k$ set $c_i := \text{Re}(\alpha_i(w), w) = \text{Re}(\ell(w))(\alpha_i(w))$. For the fixed $w \in \Omega$, after reordering $\alpha_1(w), \ldots, \alpha_k(w) \in X$ we may assume that

$$c_1 \leq c_2 \leq \cdots \leq c_k.$$

If for some $1 \leq i < j \leq k$ such that $c_i < c_j$ the open interval $(c_i, c_j) \subset \mathbb{R}$ does not intersect the set $\{c_1, c_2, c_3, \ldots, c_k\}$ (of the stratified critical values of $\text{Re}(\ell(w)) : X^\text{an} \to \mathbb{R}$), then by Lemma 3.21 for any $t_1, t_2 \in \mathbb{R}$ such that $c_i < t_1 < t_2 < c_j$ we obtain an isomorphism

$$\text{RHom}_c(\{z \in X^\text{an} \mid \text{Re}(z, w) \leq t_2\}; \text{Sol}_X(\mathcal{M})[N])$$

$$\simeq \text{RHom}_c(\{z \in X^\text{an} \mid \text{Re}(z, w) \leq t_1\}; \text{Sol}_X(\mathcal{M})[N]).$$

Indeed, we can prove the equivalent one

$$\text{RHom}_c(\{s \in \mathbb{R} \mid s \leq t_2\}; R(\text{Reg})_*t_!\text{Sol}_X(\mathcal{M})[N])$$

$$\simeq \text{RHom}_c(\{s \in \mathbb{R} \mid s \leq t_1\}; R(\text{Reg})_*t_!\text{Sol}_X(\mathcal{M})[N])$$

as follows. Let

$$\Lambda_{\ell(w)} = \{(z, \text{grad}(\ell(w))(z)) \in T^*X \simeq X \times Y \mid z \in X\} \subset T^*X$$

be the (non-homogeneous) Lagrangian submanifold of $T^*X$ associated to the function $\text{Re}(\ell(w)) : X \to \mathbb{C}$. Since $\Lambda_{\ell(w)} \simeq X \times \{w\} \subset X \times Y$, the condition $w \in \Omega$ implies that $\Lambda_{\ell(w)}$ intersects $\partial\mathcal{M}$ only over finitely many points in $X$. Hence, for any $\tau \in \mathbb{C}$ the conormal bundle of the level set $\ell(w)^{-1}(\tau) \simeq \mathbb{C}^{N-1} \subset X = \mathbb{C}^N$ of $\ell(w)$ in $X$ intersects the micro-support of the perverse sheaf $\text{Sol}_X(\mathcal{M})[N]$ only in the zero-section $T^*_XX \simeq X$ of $T^*X$ on a neighborhood of $H(w) \subset \bar{X}$. Note also that $\ell(w)^{-1}(\tau) \simeq \mathbb{P}^{N-2}$ and different from the special one $H_\infty \simeq \mathbb{P}^{N-1}$. The same is true also for any small perturbation of $w$ in $\Omega$. This implies that the assumption of Lemma 3.21 is satisfied for the submanifolds $N = H(w) \subset H = H_\infty$ in the above context. Then we obtain the desired isomorphism by applying Kashiwara’s non-characteristic deformation lemma (see [KS90, Proposition 2.7.2]) to the complex $R(\text{Reg})_*t_!\text{Sol}_X(\mathcal{M})[N]$ on the real line $\mathbb{R}$. Since $\text{Sol}_X(\mathcal{M})[N]$ is a perverse sheaf on $X$, by Kashiwara-Schapira [KS95, Theorem 9.5.2], for any $1 \leq i \leq k$ there exists an isomorphism

$$\text{Sol}_X(\mathcal{M})[N] \simeq \mathbb{C}_S^{\oplus m_i}[d_i]$$

in the localized category $\text{D}^b(\mathbb{C}_S; \mu_i(w))$ of $\text{D}^b(\mathbb{C}_S)$ at $\mu_i(w) \in T^*X$ (see [KS90, Definition 6.1.1]). Moreover the restriction of the function $\text{Re}(\ell(w)) : X^\text{an} \to \mathbb{R}$ to the submanifold
$S_i \subset X$ has a non-degenerate (real Morse) critical point of Morse index $d_{S_i}$ at $\alpha_i(w) \in S_i$. It follows that for the closed subset

$$Z_i := \{ z \in X^{\text{an}} \mid \text{Re}(z, w) \geq c_i \} \subset X^{\text{an}}$$

of $X^{\text{an}}$ and $j \in \mathbb{Z}$ we have

$$H^j \Gamma Z_i(S\ell_X(M)[N])_{\alpha_i(w)} \simeq H^j \Gamma Z_i(\mathbb{C}^{\oplus m_i}[d_{S_i}])_{\alpha_i(w)}$$

$$\simeq \begin{cases} 
\mathbb{C}^{m_i} & (j = 0) \\
0 & \text{(otherwise).}
\end{cases}$$

Hence for any $t = c_i = \text{Re}(\alpha_i(w), w) \in \mathbb{R}$ ($1 \leq i \leq k$) there exists $0 < \varepsilon \ll 1$ such that

$$\Gamma_c(\{z \in X^{\text{an}} \mid \text{Re}(z, w) \leq t + \varepsilon \}; S\ell_X(M)[N])$$

$$\simeq \Gamma_c(\{z \in X^{\text{an}} \mid \text{Re}(z, w) \leq t \}; S\ell_X(M)[N])$$

$$\simeq \Gamma_c(\{z \in X^{\text{an}} \mid \text{Re}(z, w) \leq t - \varepsilon \}; S\ell_X(M)[N]) \oplus \mathbb{C}^{m_i}.$$ 

This implies that the restriction of $L(S\ell_X(M))$ to the fiber $\pi^{-1}(w) \simeq \mathbb{R}$ of $w \in U \subset \Omega$ is isomorphic to the sheaf

$$\bigoplus_{i=1}^{k} \mathbb{C}^{\oplus m_i}_{\{t \geq \text{Re}(\alpha_i(w)), w\}}.$$ 

Since the subsets $(U \times \mathbb{R}) \cap \{t \geq \text{Re}(\alpha_i(w)), w\} \simeq U \times \mathbb{R}_{\geq 0}$ of $U \times \mathbb{R}$ are connected and simply connected, we can extend this isomorphism to $U \times \mathbb{R} \subset Y^{\text{an}} \times \mathbb{R}$. This completes the proof. \[\square\]

**Corollary 4.5.** The restriction of the Fourier transform $\tilde{M} \in \text{Mod}_{\text{hol}}(D_Y)$ of $M$ to $\Omega \subset Y$ is an integrable connection. Moreover its rank is equal to $\sum_{i=1}^{k} m_i$.

**Proof.** By Theorem 4.4 locally on $\Omega$ there exists an isomorphism

$$\text{Sol}^g(\tilde{M}^\wedge) \simeq \bigoplus_{i=1}^{k} \text{“lim}_{a \to +\infty} \mathbb{C}^{\oplus m_i}_{\{t \geq \text{Re}(\alpha_i(w)), w+a\}}.$$ 

Let $i_0 : \tilde{Y}^{\text{an}} \hookrightarrow \tilde{Y}^{\text{an}} \times \mathbb{R}$ be the map given by $y \mapsto (y, 0)$. Then locally on $\Omega$ we have isomorphisms

$$\text{Sol}_Y(M^\wedge) \simeq \alpha_{\tilde{M}^\wedge} \cap_{i_0} \text{Sol}^g(\tilde{M}^\wedge)$$

$$\simeq \alpha_{\tilde{M}^\wedge} \cap_{i_0} \text{Sol}^g(\tilde{M}^\wedge)$$

$$\simeq \alpha_{\tilde{M}^\wedge} \cap_{i_0} \left( \bigoplus_{i=1}^{k} \text{“lim}_{a \to +\infty} \mathbb{C}^{\oplus m_i}_{\{t < \text{Re}(\alpha_i(w)), w+a\}} \right) \oplus [1].$$

On the other hand, for $a \gg 0$ the sheaf $\mathbb{C}_{\{t < \text{Re}(\alpha_i(w)), w+a\}}$ being isomorphic to the constant sheaf $\mathbb{C}_{Y^{\text{an}} \times \mathbb{R}}$ locally on a neighborhood of $i_0(\Omega) \subset \Omega \times \mathbb{R}$, there exist also an isomorphism

$$i_0^{\wedge} \left( \bigoplus_{i=1}^{k} \text{“lim}_{a \to +\infty} \mathbb{C}^{\oplus m_i}_{\{t < \text{Re}(\alpha_i(w)), w+a\}} \right) \simeq \mathbb{C}^{\oplus m_i}_{\Omega}.$$
Since we fixed the point \( w \) locally on \( \Omega \). Then by \( \text{SS}(\text{Sol}_Y(M^\wedge)) = \text{char}(M^\wedge) \) the characteristic variety \( \text{char}(M^\wedge) \) of \( M^\wedge \) is contained in the zero-section of \( T^*Y^{an} \) on \( \Omega \). Now all the assertions are clear. \( \square \)

Next fix a point \( w \in \Omega \) such that \( w \neq 0 \) and set
\[
\mathbb{L} := \mathbb{C}w = \{ \lambda w \mid \lambda \in \mathbb{C} \} \subset Y = \mathbb{C}^N.
\]
Then \( \mathbb{L} \) is a complex line isomorphic to \( \mathbb{C}_\lambda \). By the \( \mathbb{C}^* \)-conicness of \( \Omega \) its open subset \( \mathbb{L}\{0\} \simeq \mathbb{C}_\lambda^* \) is contained in \( \Omega \). Note that \( \alpha_i(\lambda w) = \alpha_i(w) \) (\( 1 \leq i \leq k \)) for any \( \lambda \in \mathbb{C}^* \). Since we fixed the point \( w \in \Omega \) we set \( \alpha_i = \alpha_i(w) \) (\( 1 \leq i \leq k \)) for short. Let \( \overline{\mathbb{P}} := \mathbb{L} \sqcup \{ \infty \} \subset \overline{Y} = \mathbb{P}^N \) be the projective compactification of \( \mathbb{L} \). We extend \( M^\wedge|_{\mathbb{L}\{0\}} \) to a meromorphic connection on \( \mathbb{P} \) and denote it by \( \mathcal{L} \in \text{Mod}_{\text{hol}}(\mathcal{D}_\mathbb{P}) \). Note that \( \mathcal{L} \) is isomorphic to the restriction of \( M^\wedge \) to \( \mathbb{P} \subset \overline{Y} \) on a neighborhood of the point \( \infty \in \mathbb{P} \). The following result is a generalization (of some part) of Esterov-Takeuchi \cite[Theorem 5.6]{ET15}. Let \( \varpi_\mathbb{P} : \overline{\mathbb{P}} \to \mathbb{P} \) be the real blow-up of \( \mathbb{P} \) along the divisor \( \{ \infty \} \subset \mathbb{P} \).

**Theorem 4.6.** For any point \( \theta \in \varpi_\mathbb{P}^{-1}(\{ \infty \}) \simeq S^1 \) there exists its open neighborhood \( W \) in \( \overline{\mathbb{P}} \) such that we have an isomorphism
\[
\mathcal{L}^A|_W \simeq \left( \bigoplus_{i=1}^k \left( (\mathcal{E}_{\mathbb{L}|\overline{\mathbb{P}}}^{-\langle \alpha_i,w \rangle \lambda})^{\wedge m_i} \right) \right)|_W
\]
of \( \mathcal{D}_\mathbb{P}^A \)-modules on \( W \). In particular, the functions \( -\langle \alpha_i,w \rangle \lambda \) of \( \lambda \) are the exponential factors of the meromorphic connection \( \mathcal{L} \) at the point \( \infty \in \mathbb{P} \) (the multiplicity of \( -\langle \alpha_i,w \rangle \lambda \) is equal to \( m_i \)). Moreover the meromorphic connection \( \mathcal{L} \) is regular at the origin \( 0 \in \mathbb{L} \subset \mathbb{P} \).

**Proof.** By \cite[Corollary 9.4.12]{DK16} we have isomorphisms
\[
\text{Sol}_\mathbb{P}^E(\mathcal{E}_{\mathbb{L}|\overline{\mathbb{P}}}^{-\langle \alpha_i,w \rangle \lambda}) \simeq \mathbb{E}_{\mathbb{L}|\overline{\mathbb{P}}}^{-\text{Re}(\langle \alpha_i,w \rangle \lambda)} \simeq \lim_{a \to +\infty} \mathbb{C}_{\{ t \geq \text{Re}(\langle \alpha_i,w \rangle \lambda) + a \}}^{\wedge m_i}.
\]
On the other hand, by Theorem 4.4 for any \( \theta \in \varpi_\mathbb{P}^{-1}(\{ \infty \}) \simeq S^1 \) there exists its sectorial neighborhood \( V_\theta \subset \mathbb{P}\{ \infty \} \) such that we have an isomorphism
\[
\pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}^E(\mathcal{L}) \simeq \bigoplus_{i=1}^k \pi^{-1}C_{V_\theta} \otimes \left( \lim_{a \to +\infty} \mathbb{C}_{\{ t \geq \text{Re}(\langle \alpha_i,w \rangle \lambda) + a \}}^{\wedge m_i} \right).
\]
We thus obtain an isomorphism
\[
\pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}^E(\mathcal{L}) \simeq \pi^{-1}C_{V_\theta} \otimes \text{Sol}_\mathbb{P}^E\left( \bigoplus_{i=1}^k (\mathcal{E}_{\mathbb{L}|\overline{\mathbb{P}}}^{-\langle \alpha_i,w \rangle \lambda})^{\wedge m_i} \right).
\]
Then the first assertion follows from Corollary 3.11. The second one follows from Theorem 3.18.

We can show the remaining one by using the last part of Proposition 3.15. \( \square \)
Remark 4.7. In Esterov-Takeuchi [ET15] for the Fourier transform $(\cdot)^\wedge : \text{Mod}_{\text{hol}}(\mathcal{D}_X) \to \text{Mod}_{\text{hol}}(\mathcal{D}_Y)$ the authors used the kernel $\mathcal{O}_{X \times Y}e^{(z,w)}$ instead of the one $\mathcal{O}_{X \times Y}e^{-(z,w)}$ used in this paper. This is the reason why we obtain $\mathcal{E}_{\mathbb{L}^\bullet P}^{(\alpha_i,w)\lambda}$ instead of $\mathcal{E}_{\mathbb{L}^\bullet P}^{-\langle \alpha_i,w \rangle \lambda}$ in Theorem 4.6.

As in Esterov-Takeuchi [ET15, Remark 5.7], by Theorems 4.4 and 4.6 we easily obtain the Stokes lines of the meromorphic connection $\mathcal{L} \in \text{Mod}_{\text{hol}}(\mathcal{D}_P)$ at $\infty \in \mathbb{P}$ as follows. We define a subset $J$ of $\{1,2,\ldots,k\}^2$ by

$$J = \{(i,j) \in \{1,2,\ldots,k\}^2 \mid \langle \alpha_i, w \rangle \neq \langle \alpha_j, w \rangle \}.$$ 

Then the union of the Stokes lines is equal to the set

$$\bigcup_{(i,j) \in J} \{ \lambda \in \mathbb{C} \simeq \mathbb{L} \mid \text{Re} \langle \alpha_i - \alpha_j, \lambda w \rangle = 0 \}$$

in $\mathbb{L} \simeq \mathbb{C}$.

Now we shall study the enhanced ind-sheaf $\text{Sol}^E(\mathcal{M}) \simeq \text{L} \text{Sol}^E(\mathcal{M})$ on the remaining set $D := Y \setminus \Omega \subset Y = \mathbb{C}^N$. Fix a point $w \in D = Y \setminus \Omega$ such that $w \neq 0$ and define $\ell(w) : X = \mathbb{C}^N \to \mathbb{C}$, $H(w) := \ell(w)^{-1}(0) \cap H_{\infty} \simeq \mathbb{P}^{N-2}$, $f : X^H(w) \to \mathbb{P}^1$ etc., as before. Then there exists finite points $P_1, P_2, \ldots, P_t \in \mathbb{C}$ in $\mathbb{C}$ such that the restriction

$$\overline{X}^H(w) \setminus f^{-1}(\{\infty, P_1, P_2, \ldots, P_t\}) \to \mathbb{P}^1 \setminus \{\infty, P_1, P_2, \ldots, P_t\}$$

of $f$ is a stratified fiber bundle with respect to a stratification associated to the perverse sheaf $\text{Sol}_X(\mathcal{M})[N] \in D^b_{\mathbb{C}^{-\epsilon}}(\mathbb{C}_X^{u(w)})$ (see e.g. Goersky-Macpherson [GM88, P43]). Recall that we set

$$L(\text{Sol}_X(\mathcal{M})) := \text{R}p_{2!}(p_{1!}(\mathbb{C}_{s \geq 0} \otimes \pi^{-1}\text{Sol}_X(\mathcal{M})) \otimes \mathbb{C}_{t-s-\text{Re}(z,w) \geq 0}[N]) \in D^b(\mathbb{C}_{X \times \mathbb{R}}).$$

For $t \in \mathbb{R}$ its stalk $\left(L(\text{Sol}_X(\mathcal{M}))\right)_{(w,t)}$ is calculated as in the case $w \in \Omega$. Moreover for $t \ll 0$ we have also the vanishing

$$\text{R}\Gamma_{(\cdot)}(\{z \in X^an \mid \text{Re}(z,w) \leq t\}; \text{Sol}_X(\mathcal{M})[N]) \simeq 0.$$ 

For $1 \leq i \leq \ell$ define a function $h_i : \mathbb{C}_\tau \to \mathbb{C}$ by $h_i(\tau) = \tau - P_i$ so that we have $h_i^{-1}(0) = \{P_i\} \subset \mathbb{C}$. Let us set

$$K_i := \{\tau \in \mathbb{C} \mid \text{Re}(h_i(\tau)) \geq 0\} \subset \mathbb{C}. $$

Then for $t \in \mathbb{R}$ and $0 < \varepsilon \ll 1$ there exists a distinguished triangle

$$\bigoplus_{i : \text{Re} P_i = t} \text{R}\Gamma_{K_i}(\text{Rf}_{*i!i^*}\text{Sol}_X(\mathcal{M})[N])_{P_i} \to \text{R}\Gamma_{\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t + \varepsilon\}; \text{Sol}_X(\mathcal{M})[N]} \to \text{R}\Gamma_{\{\tau \in \mathbb{C} \mid \text{Re} \tau \leq t - \varepsilon\}; \text{Sol}_X(\mathcal{M})[N]} \to .$$

Moreover for each $1 \leq i \leq \ell$ such that $\text{Re} P_i = t$ we have an isomorphism

$$\text{R}\Gamma_{K_i}(\text{Rf}_{*i!i^*}\text{Sol}_X(\mathcal{M})[N])_{P_i} \simeq \phi_{h_i}(\text{Rf}_{*i!i^*}\text{Sol}_X(\mathcal{M})[N])[-1],$$

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where $\phi_{h_i} : D^b(C) \to D^b(C_{h_i^{-1}(0)}) = D^b(C_{(P_i)})$ is the vanishing cycle functor associated to $h_i$ (see Kashiwara-Schapira [KS90] and Dimca [Dim04] etc.). Since $f$ is proper, we have also an isomorphism

$$\phi_{h_i}(Rf_!t_iSol_X(M)[N])[-1] \simeq R\Gamma(f^{-1}(P_i); \phi_{h_{iof}}(t_iSol_X(M)[N])[-1]).$$

Note that $\phi_{h_{iof}}(t_iSol_X(M)[N])[-1]$ is a perverse sheaf on $(h_i \circ f)^{-1}(0) = f^{-1}(P_i) \subset X^{H(w)}$. Moreover for the inclusion map $i_{(w)} : \{w\} \hookrightarrow Y = \mathbb{C}^N$ we have the following result.

**Lemma 4.8.** For $a \gg 0$ we have isomorphisms

$$Sol_{(w)}(D^i_{(w)}M^\wedge) \simeq (L(Sol_X(M)))_{(w,a)}$$

$$\simeq R\Gamma_{\epsilon}(\{z \in X^{an} | Re(z, w) \leq a\}; Sol_X(M)[N]).$$

**Proof.** First note that we have isomorphisms

$$Sol_{(w)}(D^i_{(w)}M^\wedge) \simeq \alpha_{(w)}t_0R^ESol^E_{(w)}(D^i_{(w)}M^\wedge)$$

$$\simeq \alpha_{(w)}t_0R^ESol^E_{(w)}M^\wedge.$$
of the interval \( \{t_1 \leq t\} \subset \mathbb{R} \) the support of \((L(Sol_X(\mathcal{M})))_{\{w\} \times \mathbb{R}} \) is divided into those of \((F_i)_{\{t_i \leq t < t_{i+1}\}} \) and \(G_{\{t_k \leq t\}} \). By (the proof of) Lemma \[3.13\] for \(1 \leq i \leq k - 1\) we have

\[
i_0^! \mathbb{R}^E \left( \operatorname{lim}_{a \to +\infty}^{\wedge} (C_{\{t \geq a\}} \otimes (F_i)_{\{t_i \leq t < t_{i+1}\}}) \right)
\]

\[
\simeq \mathbb{R} \pi_* \mathcal{R} \text{Thom} \left( C_{\{t \geq 0\}} \otimes C_{\{t \leq 0\}}, \operatorname{lim}_{a \to +\infty}^{\wedge} (F_i)_{\{t_i + a \leq t < t_{i+1} + a\}} \right)
\]

\[
\simeq \mathbb{R} \pi_* \mathcal{R} \text{Thom} \left( C_{\mathbb{R}}, \operatorname{lim}_{a \to +\infty}^{\wedge} (F_i)_{\{t_i + a \leq t < t_{i+1} + a\}} \right) \simeq 0.
\]

We thus obtain an isomorphism

\[
i_0^! \mathbb{R}^E \left( \operatorname{lim}_{a \to +\infty}^{\wedge} (C_{\{t \geq a\}} \otimes L(Sol_X(\mathcal{M})))_{\{w\} \times \mathbb{R}} \right)
\]

\[
\simeq i_0^! \mathbb{R}^E \left( \operatorname{lim}_{a \to +\infty}^{\wedge} C_{\{t \geq a\}} \otimes G_{\{t_k \leq t\}} \right).
\]

Moreover there exists an isomorphism

\[
i_0^! \mathbb{R}^E \left( \operatorname{lim}_{a \to +\infty}^{\wedge} C_{\{t \geq a\}} \otimes G_{\{t_k \leq t\}} \right)
\]

\[
\simeq \mathbb{R} \pi_* \mathcal{R} \text{Thom} \left( C_{\{t \geq 0\}} \otimes C_{\{t \leq 0\}}, \operatorname{lim}_{a \to +\infty}^{\wedge} G_{\{t_k + a \leq t\}} \right)
\]

\[
\simeq \mathbb{R} \pi_* \mathcal{R} \text{Thom} \left( C_{\mathbb{R}}, \operatorname{lim}_{a \to +\infty}^{\wedge} G_{\{t_k + a \leq t\}} \right) \simeq G.
\]

Then the assertion immediately follows. \( \square \)

By the proof of this lemma, we see that \( Sol_Y(\mathcal{M}^\wedge) \) is monodromic. Indeed, for \( \lambda \in \mathbb{R}_+ \) we have \( \Re \ell(\lambda w) = \lambda \cdot \Re \ell(w) \). This implies that \( Sol_Y(\mathcal{M}^\wedge) \) is \( \mathbb{R}_+ \)-conic (see Ito-Takeuchi \[IT18\] for a precise proof). For a general theory of conic ind-sheaves see \[Pre11\]. We can rewrite Lemma \[4.8\] more geometrically as follows.

**Proposition 4.9.** For any \( \tau \in \mathbb{C} \setminus \{P_1, P_2, \ldots, P_l\} \) we have

\[
\chi_w(Sol_{\{w\}}(\mathcal{D}i_{\{w\}}^*(\mathcal{M}^\wedge))) = \chi(\Gamma_c(\mathcal{X}; Sol_X(\mathcal{M})[N])) - \chi(\Gamma_c(\ell(w)^{-1}(\tau); Sol_X(\mathcal{M})[N])).
\]

**Proof.** For \( a \gg 0 \) the restriction of \( \ell(w) : X = \mathbb{C}^N \to \mathbb{C} \) to the open subset \( \{\tau \in \mathbb{C} \mid \Re \tau > a\} \subset \mathbb{C} \) is a stratified fiber bundle with respect to a stratification associated to the perverses sheaf \( Sol_X(\mathcal{M})[N] \in \mathcal{D}_{\text{con}}(\mathbb{C}_X) \). Then we obtain the assertion by applying the Künneth formula to Lemma \[4.8\]. \( \square \)

Note that a special case of Proposition \[4.9\] was obtained by Brylinski \[Bry86\] Corollaire 8.6.

From now, let us calculate (some part of) the characteristic cycle of the Fourier transform \( \mathcal{M}^\wedge \). Let \( \cup_{i=1}^m D_i \) be the irreducible decomposition of \( D = Y \setminus \Omega \subset Y = \mathbb{C}^N \). For \( 1 \leq i \leq m \) such that \( d_{D_i} = N - 1 \) we shall calculate the multiplicity

\[
\text{mult}_{T_{\mathbb{C}^N} \cap \mathcal{M}^\wedge} \geq 0
\]

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of the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ along $T^*_{D_i}Y$. For this purpose, first we calculate the local Euler-Poincaré index

$$\chi_v(Sol_y(\mathcal{M}^\wedge)) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j Sol_y(\mathcal{M}^\wedge)_v$$

of $Sol_y(\mathcal{M}^\wedge)$ at generic smooth points $v \in D_{\text{reg}}$. Fix $1 \leq i \leq m$ and let $v \in D_i \cap D_{\text{reg}}$ be such a generic point. Let $M \subset Y$ be a subvariety of $Y$ which intersects $D_i$ at $v$ transversally. We call it a normal slice of $M$ characteristic for $v$. Let $i_M : M \hookrightarrow Y = \mathbb{C}^N$ be the inclusion map. Then it is non-characteristic for $\mathcal{M}^\wedge$ and hence we obtain an isomorphism

$$Sol_y(\mathcal{M}^\wedge)|_M \simeq Sol_M(Di^*_M\mathcal{M}^\wedge).$$

Let us consider the special case where $d_{D_i} = N - 1$. In this case we have $d_M = 1$. For the holonomic $D$-module $\mathcal{K} := Di^*_M\mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(D_M)$ on the normal slice $M$ of $D_i$ at $v$, consider the distinguished triangle

$$\text{R}\Gamma_v(\mathcal{K}) \rightarrow \mathcal{K} \rightarrow \mathcal{K}(\{v\}) \rightarrow.$$  

Then by the results in Section 3 we obtain

$$\chi_v(Sol_y(\mathcal{M}^\wedge)) = \chi_v(Sol_M(\mathcal{K}))$$

$$= \chi_v(\text{Sol}_M(\text{R}\Gamma_v(\mathcal{K}))) + \chi_v(\text{Sol}_M(\mathcal{K}(\{v\})))$$

$$= \chi_v(Sol_v(\text{Di}^*_v\mathcal{M}^\wedge)) + \chi_v(Sol^{-}(\mathcal{K}(\{v\}))).$$

Combining this with Proposition 4.9 we finally obtain the following theorem.

**Theorem 4.10.** Assume that $d_{D_i} = N - 1$. Then for $|\tau| \gg 0$ we have

$$\chi_v(Sol_y(\mathcal{M}^\wedge)) = \chi(\text{R}\Gamma_c(X; \text{Sol}_X(\mathcal{M})[N])) - \chi(\text{R}\Gamma_c(\ell(\tau)^{-1}(\tau); \text{Sol}_X(\mathcal{M})[N]))$$

$$+ \chi_v(\text{Sol}_M(\mathcal{K}(\{v\}))).$$

For the meromorphic connection $\mathcal{K}(\{v\})$ on the Riemann surface $M$ we can calculate $\chi_v(Sol_M(\mathcal{K}(\{v\})))$ by Proposition 3.15 as follows. Recall that $-\chi_v(Sol_M(\mathcal{K}(\{v\})))$ is equal to the irregularity $\text{irr}(\mathcal{K}(\{v\})) \geq 0$ of $\mathcal{K}(\{v\})$. Let $\varpi_M : \tilde{M} \rightarrow M$ be the real blow-up of $M$ along $\{v\} \subset M$. Shrinking the normal slice $M$ if necessary we may assume that $M = \{u \in \mathbb{C} \mid |u| < \varepsilon\}$ for some $\varepsilon > 0$, $\{v\} = \{u = 0\}$ and $M \setminus \{v\} \subset \Omega$. Then we define Laurent Puiseux series $\varphi_i(u) (1 \leq i \leq k)$ and their pole orders $\text{ord}_{v_i}(\varphi_i) \geq 0$ as in Section 1. Moreover by Theorems 4.4 and 2.1 (ii) for any point $\theta \in S_{v_i}M \simeq \mathbb{C}^{*}\mathbb{Q}_{\tau}^{-1}(\{v\}) \simeq S^1$ there exists its sectorial neighborhood $V_{\theta} \subset M \setminus \{v\}$ for which we have an isomorphism

$$\pi^{-1}C_{V_{\theta}} \otimes \text{Sol}_M^E(\mathcal{K}(\{v\})) \simeq \bigoplus_{i=1}^k \left( E_{\hat{V}_{\theta}M}^{\text{Re}_i} \right)^{\oplus m_i}.$$

Then by Theorem 3.18 we obtain the following result.
Theorem 4.11. The exponential factors appearing in the Hukuhara-Levitt-Turrittin decomposition of the meromorphic connection $\mathcal{K}(\ast \{v\})$ at $v \in M$ are the pole parts of $-\varphi_i$ ($1 \leq i \leq k$). Moreover for any $1 \leq i \leq k$ the multiplicity of the pole part of $-\varphi_i$ is equal to $m_i$. In particular we have

$$\text{irr}(\mathcal{K}(\ast \{v\})) = \sum_{i=1}^{k} m_i \cdot \text{ord}_{\{v\}}(\varphi_i).$$

Theorem 4.12. Assume that $d_{D_i} = N - 1$ and let $v_0 \in \Omega$ be a point of $\Omega$. Then for $|\tau| \gg 0$ and a generic point $v \in D_i \cap D_{reg}$ the multiplicity $\text{mult}_{T^*_D Y} \mathcal{M}^\wedge \geq 0$ of the Fourier transform $\mathcal{M}^\wedge$ along $T^*_D Y$ is given by

$$\text{mult}_{T^*_D Y} \mathcal{M}^\wedge = \chi(\mathcal{R} \Gamma_c(\ell(v)^{-1}(\tau); \text{Sol}_X(\mathcal{M})[N]))$$

- $\chi(\mathcal{R} \Gamma_c(\ell(v_0)^{-1}(\tau); \text{Sol}_X(\mathcal{M})[N])) + \text{irr}(\mathcal{K}(\ast \{v\})).$

Proof. By Kashiwara's local index theorem for holonomic $\mathcal{D}$-modules in [Kas83 Corollary 6.3.4], we have

$$\text{mult}_{T^*_D Y} \mathcal{M}^\wedge = \chi_{v_0}(\text{Sol}_M(\mathcal{M}^\wedge)) - \chi_{v}(\text{Sol}_M(\mathcal{M}^\wedge)).$$

Note that for $|\tau| \gg 0$ and $v_0 \in \Omega$ the equality

$$\chi_{v_0}(\text{Sol}_M(\mathcal{M}^\wedge)) = \chi(\mathcal{R} \Gamma_c(X; \text{Sol}_X(\mathcal{M})[N])) - \chi(\mathcal{R} \Gamma_c(\ell(v_0)^{-1}(\tau); \text{Sol}_X(\mathcal{M})[N]))$$

holds (see Brylinski [Bry86 Corollaire 8.6]). Then the assertion follows from Theorem 4.10. Recall that we have $\text{irr}(\mathcal{K}(\ast \{v\})) = -\chi_{v}(\text{Sol}_M(\mathcal{K}(\ast \{v\}))).$ \hfill $\square$

Example 4.13. Let us consider the special case where the Fourier transform $\mathcal{M}^\wedge$ is a confluent $A$-hypergeometric system on $Y = \mathbb{C}^2$. For the subset $A = \{2, 3\}$ of the 1-dimensional lattice $\mathbb{Z} \subset \mathbb{R}$ consider the embedding

$$i_T : T = \mathbb{C}^* \hookrightarrow X = \mathbb{C}^2_{x,y}, \quad s \mapsto (s^2, s^3)$$

of the 1-dimensional torus $T = \mathbb{C}^*$ associated to it. For a complex number $c$ such that $c \notin \mathbb{Z}$ set $\mathcal{L} = O_T s^{c-1} \in \text{Mod}_{\text{rh}}(\mathcal{D}_T)$ and $\mathcal{M} = D_{i_T} \mathcal{L}$. Then $c$ is non-resonant in the sense of Adolphson [Ado] and we have $\mathcal{M} \in \text{Mod}_{\text{rh}}(\mathcal{D}_X)$. Set $Z = i_T(T) = \{x^3 = y^2\} \subset X$. Then we can easily see that

$$\text{char}\mathcal{M} = T^*_0 X \cup T^*_Z X$$

and

$$\text{mult}_{T^*_0 X} \mathcal{M} = 2, \quad \text{mult}_{T^*_Z X} \mathcal{M} = 1.$$
By Theorem 4.12 we thus obtain

\[ Y \]

Similarly for \( D \). Then by Theorem 4.4 it is easy to show that the restriction of \( \text{Sol}_Y(M^\wedge) \approx L \text{Sol}_Y(M^\wedge) \)
to \( M \cap \Omega \simeq \mathbb{C}_u^* \) is isomorphic to

\[
\left( \lim_{a \to +\infty} \mathbb{C}^{\mathbb{Z}}_{(t \geq a)} \right) \oplus \left( \lim_{a \to +\infty} \mathbb{C}_{(t \geq \Re(\varphi(u)) + a)} \right),
\]

where we set \( \varphi(\eta) := \frac{4}{27\pi^2} \). By Theorem 4.11 this implies that the irregularity of the
meromorphic connection obtained by restricting \( M^\wedge \) to the normal slice \( M \) is equal to 2.

Set \( v = (1, 0) \in M \cap D_1 \) and \( v_0 = (1, \varepsilon) \in M \cap \Omega = M \setminus D_1 \) \( (\varepsilon \neq 0) \). Then for \( |\tau| \gg 0 \) we have

\[
\chi(\Gamma_c(\ell(v)^{-1}(\tau); \mathcal{F})) - \chi(\Gamma_c(\ell(v_0)^{-1}(\tau); \mathcal{F})) = (-2) - (-3) = 1.
\]

By Theorem 4.12 we thus obtain

\[
\text{mult}_{T_{D_1}Y} M^\wedge = 1 + 2 = 3.
\]

Similarly for \( D_2 = \{(w_1, w_2) \in Y \mid w_1 = 0\} \) we can show \( \text{mult}_{T_{D_2}Y} M^\wedge = 0 \). Hence \( M^\wedge \)
is an integrable connection on \( Y \setminus D_1 \supset \Omega \). In fact, for \( A = \{2, 3\} \) this follows from
Adolphson’s result in [Ado].

**Example 4.14.** For the smooth hypersurface \( Z = \{z \in X = \mathbb{C}^N \mid z_1^2 + \cdots + z_{N-1}^2 + z_N = 1\} \subset X \) consider the perverse sheaf \( \mathcal{F} = \mathbb{C}_Z[N - 1] \in \mathbf{D}^b_{c, \epsilon}(\mathbb{C}_X^{an}) \) on \( X = \mathbb{C}^N \). Let \( M \in \text{Mod}_{hol}(\mathcal{D}_X) \) be the regular holonomic \( D_X \)-module such that \( \text{Sol}_X(M)[N] = \mathcal{F} \).

Then \( M \) is not monodromic and we have

\[
\text{char} M = T^*_Z X
\]

\[
= \{(z, \zeta(2z_1, \ldots, 2z_{N-1}, 1)) \in T^*_X X \simeq X \times Y \mid \zeta \in \mathbb{C}, z_1^2 + \cdots + z_{N-1}^2 + z_N = 1\}.
\]

It follows that for the projection \( q : T^*_X X \simeq X \times Y \to Y \) and a point \( w \in Y \setminus \{0\} = \mathbb{C}^N \setminus \{0\} \) we have

\[ w \in q(T^*_Z X) \iff w_N \neq 0. \]

Moreover if \( w \in q(T^*_Z X) \setminus \{0\} \) then the set \( q^{-1}(w) \cap T^*_Z X \) is explicitly calculated as

\[
q^{-1}(w) \cap T^*_Z X = \left\{ (z, w) \in T^*_X X \simeq X \times Y \mid z_i = \frac{w_i}{2w_N} (i = 1, \ldots, N - 1), z_N = 1 - \frac{1}{4w_N^2} (w_1^2 + \cdots + w_{N-1}^2) \right\}.
\]

This shows that the open subset \( \Omega \subset Y = \mathbb{C}^N \) is given by

\[ \Omega = \{w \in Y = \mathbb{C}^N \mid w_N \neq 0\} \]

and the morphism \( q^{-1}(\Omega) \cap \text{char} M \to \Omega \) induced by \( q \) is a covering of degree 1. By
Corollary 4.5 the restriction of the Fourier transform \( M^\wedge \in \text{Mod}_{hol}(\mathcal{D}_Y) \) of \( M \) to \( \Omega \subset Y \) is
an integrable connection of rank 1. We can also easily see that \( M^\wedge \) has some irregularities
at infinity by using Theorem 4.6. Set \( D = Y \setminus \Omega = \{w \in Y = \mathbb{C}^N \mid w_N = 0\} \) and take its
normal slice

\[ M = \{(a_1, \ldots, a_{N-1}, u) \in Y = \mathbb{C}^N \mid u \in \mathbb{C} \} \subset Y \]
at the generic point \((a_1, \ldots, a_{N-1}, 0) \in D \setminus \{0\}\). Then for a point \(v = (a_1, \ldots, a_{N-1}, u) \in M \setminus D = M \cap \Omega \ (u \neq 0)\) we have
\[
q^{-1}(v) \cap T^*_Z X = \left\{ (z, v) \in T^* X \simeq X \times Y \mid z_i = \frac{a_i}{2u} \ (i = 1, \ldots, N-1), \ z_N = 1 - \frac{1}{4u^2} (a_1^2 + \cdots + a_{N-1}^2) \right\}
\]
Since the Puiseux series
\[
\varphi(u) = \langle z, v \rangle = u + \frac{1}{4u^2}(a_1^2 + \cdots + a_{N-1}^2)
\]
of \(u\) start from the negative degree \(-1\), by Theorem 4.11 the irregularity of meromorphic connection obtained by restricting \(M^\wedge\) to the normal slice \(M \subset Y\) is equal to 1. Set \(v = (a_1, \ldots, a_{N-1}, 0) \in M \cap D\) and \(v_0 = (a_1, \ldots, a_{N-1}, \varepsilon) \in M \cap \Omega = M \setminus D \ (\varepsilon \neq 0)\). Note that there exists an isomorphism \(Z \simeq \mathbb{C}^{N-1}\) induced by the projection \(X = \mathbb{C}^N \to \mathbb{C}^{N-1}\), \(z \mapsto (z_1, \ldots, z_{N-1})\). Then for \(|\tau| > 0\) we have \(\ell(v)^{-1}(\tau) \cap Z \simeq \mathbb{C}^{N-2}\), \(\ell(v_0)^{-1}(\tau) \cap Z \simeq \{ (\lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{C}^{N-1} \mid \lambda_1 + \cdots + \lambda_{N-1} = 1 \}\) and hence
\[
\chi(R\Gamma_c(\ell(v)^{-1}(\tau); \mathcal{F})) - \chi(R\Gamma_c(\ell(v_0)^{-1}(\tau); \mathcal{F})) = (-1)^{N-1} - \{ (-1)^{N-1} + (-1)^{N-1}(-1)^{N-2} \} = 1.
\]
By Theorem 4.12 we thus obtain
\[
\text{mult}_{T^*_Z Y} M^\wedge = 1 + 1 = 2.
\]
We shall rewrite Theorem 4.12 more explicitly. For this purpose, we introduce a “conification” of the perverse sheaf \(\mathcal{F} = Sol_X(M)[N] \in \mathbf{D}^b_{\mathbb{C}=c}(\mathbb{C}_{X^{\text{an}}})\) as follows. Let \(j = i_X : X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N\) be the projective compactification of \(X = \mathbb{C}^N\) and \(h\) the (local) defining equation of the hyperplane at infinity \(H_\infty = \overline{X} \setminus X\) in \(\overline{X}\) such that \(H_\infty = h^{-1}(0)\). Moreover let \(\gamma : X \setminus \{0\} = \mathbb{C}^N \setminus \{0\} \to H_\infty = \mathbb{P}^{N-1}\) be the canonical projection. Then
\[
\mathcal{G} := \gamma^{-1} \psi_h(j_* \mathcal{F}) \in \mathbf{D}^b_{\mathbb{C}=c}(\mathbb{C}_{X \setminus \{0\}})
\]
is a perverse sheaf on \(X \setminus \{0\}\). We call it the conification of \(\mathcal{F}\). In particular \(\mathcal{G}\) is monodromic in the sense of [Ver83] and [Bry86]. We extend it to a perverse sheaf on the whole \(X\) and denote it also by \(\mathcal{G}\). Let \(\mathcal{N} \in \text{Mod}_{\mathbb{H}}(\mathcal{D}_X)\) be the regular holonomic \(\mathcal{D}_X\)-module such that \(\text{Sol}_X(\mathcal{N})[N] \simeq \mathcal{G}\). Now we shall recall the well-known relationship between the characteristic cycle \(CC(\mathcal{N})\) of \(\mathcal{N}\) and that of its Fourier transform \(\mathcal{N}^\wedge\). For a subvariety \(V \subset X\) of \(X\) set
\[
T^*_V X := T^*_{V_{\text{reg}}} X \subset T^* X,
\]
where \(V_{\text{reg}} \subset V\) stands for the regular part of \(V\). Then there exist some \(\mathbb{C}^*-\text{conic}\) subvarieties \(V_i \subset X\) of \(X\) and positive integer \(n_i > 0\) \((1 \leq i \leq r)\) such that
\[
CC(\mathcal{N}) = \sum_{i=1}^r n_i \cdot [T^*_V X].
\]
By the natural identification \(T^*_X \simeq T^*_Y\) for any \(1 \leq i \leq r\) there exists a \(\mathbb{C}^*-\text{conic}\) subvariety \(W_i \subset Y\) of \(Y = \mathbb{C}^N\) such that \(T^*_W X = T^*_W Y\). Note that their projectivizations
\( \mathbb{P}(V_i) \subset \mathbb{P}(X) \simeq \mathbb{P}^{N-1} \) and \( \mathbb{P}(W_i) \subset \mathbb{P}(Y) \simeq \mathbb{P}^{N-1} \) are dual varieties in the classical theory of projective duality (see Gelfand-Kapranov-Zelevinsky [GKZ91], §1.3). Then we have

\[
\text{CC}(N^w) = \sum_{i=1}^r n_i \cdot [T_{W_i} Y].
\]

This equality can be easily seen also by using the arguments for the enhanced micro-supports \( SS^b(\cdot) \) in D’Agnolo-Kashiwara [DK17]. We will show also that \( W_i \subset D = Y \setminus \Omega \) for \( 1 \leq i \leq r \) satisfying some condition and \( d_{W_i} = N - 1 \). By \( G_0 := \psi_h(j_i \mathcal{F}) \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{H^w}) \) we can rewrite our results as follows.

**Definition 4.15.** Let \( w \neq 0 \) be a point of \( Y = \mathbb{C}^N \). Then we say that \( \mathcal{F} \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is moderate at infinity over \( w \) if there exists a (complex analytic) Whitney stratification \( \overline{X} = \bigcup_{a \in A} S_a \) of \( \overline{X} = \mathbb{P}^N \) adapted to \( j_i \mathcal{F} \) and subdividing the one \( \overline{X} = X \cup H_{\infty} \) such that for any stratum \( S_a \subset X = \mathbb{C}^N \) the set \( T^*_S \overline{X} \cap T^*_H \overline{X} \) is contained the zero section of \( T^* \overline{X} \) over a neighborhood of \( H(w) \) in \( \overline{X} \). Moreover for a subset \( B \subset Y \setminus \{0\} \) we say that \( \mathcal{F} \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is moderate at infinity over \( B \) if it is so over any point \( w \in B \) in it.

Obviously, if \( \mathcal{F} \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is monodromic then it is moderate at infinity over any point \( w \neq 0 \) of \( Y \). Moreover the set of the points \( w \neq 0 \) over which \( \mathcal{F} \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is moderate at infinity is \( C^*_\text{conic} \) in \( Y = \mathbb{C}^N \).

**Example 4.16.** Let \( f(z) \in \mathbb{C}[z_1, z_2, \ldots, z_N] \) be a polynomial on \( X = \mathbb{C}^N \) such that the hypersurface of \( \mathbb{P}^{N-1} \) defined by its top degree part is smooth. Then the constructible sheaf \( \mathcal{F}_{f-1}(0) \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is moderate at infinity over any point \( w \neq 0 \) of \( Y = \mathbb{C}^N \).

**Lemma 4.17.** Assume that \( \mathcal{F} \in \mathcal{D}^b_{\mathbb{C},c}(\mathbb{C}_{X^{\infty}}) \) is moderate at infinity over a point \( w \neq 0 \) of \( Y = \mathbb{C}^N \). Then for \( |\tau| \gg 0 \) we have

\[
\chi(R\Gamma_c(\ell(w)^{-1}(\tau) ; \mathcal{F})) = \chi(R\Gamma_c(H_\infty \setminus H(w) ; \psi_h(j_i \mathcal{F}))).
\]

**Proof.** For \( w \in Y \setminus \{0\} \) and \( \tau \neq 0 \) let \( \gamma(w, \tau) : \ell(w)^{-1}(\tau) \sim H_\infty \setminus H(w) \simeq \mathbb{C}^{N-1} \) be the isomorphism induced by the projection \( \gamma : X \setminus \{0\} = \mathbb{C}^N \setminus \{0\} \to H_\infty = \mathbb{P}^{N-1} \). For \( \varepsilon > 0 \) let \( U_\varepsilon \supset H(w) \) be the open neighborhood of \( H(w) \simeq \mathbb{P}^{N-2} \) in \( H_\infty \simeq \mathbb{P}^{N-1} \) consisting of points whose distances from \( H(w) \) (with respect to the Fubini-Study metric of \( H_\infty \)) are less than \( \varepsilon \). Then by our assumption there exist \( 0 < \varepsilon < 1 \) and \( C \gg 0 \) such that for any \( \tau \in \mathbb{C} \) with \( |\tau| \geq C \) we have

\[
\chi(R\Gamma_c(U_\varepsilon \setminus H(w) ; \psi_h(j_i \mathcal{F}))) = \chi(R\Gamma_c(\gamma(w, \tau)^{-1}(U_\varepsilon \setminus H(w)) ; \mathcal{F})).
\]

Moreover the compact subset \( K_\varepsilon := H_\infty \setminus U_\varepsilon \subset H_\infty \setminus H(w) \simeq \mathbb{C}^{N-1} \) is a closed ball with real analytic boundary \( \partial K_\varepsilon \). Take an affine chart \( W = \mathbb{C}^N_x \) of \( X = \mathbb{P}^N \) such that \( W \cap H_\infty = H_\infty \setminus H(w) = \{x_N = 0\} \subset W = \mathbb{C}^N_x \), \( \ell(w)^{-1}(\tau) = \{x_N = \frac{1}{\tau}\} \) for any \( \tau \neq 0 \) and the restriction \( \gamma|_W : W \to W \cap H_\infty \) of \( \gamma \) to it is given by \( x \mapsto (x_1, \ldots, x_{N-1}) \). For the Whitney stratification of \( X = \mathbb{P}^N \) adapted to \( j_i \mathcal{F} \), by the microlocal Bertini-Sard theorem (see [KS90], Proposition 8.3.12] if \( \varepsilon > 0 \) is generic enough the real analytic hypersurface \( (\gamma|_W)^{-1}(\partial K_\varepsilon) \) of \( W = \mathbb{C}^N_x \) intersects all its strata transversally on a neighborhood of \( H_\infty \). Then by the analytic curve selection lemma it is easy to show that for \( |\tau| \) large enough we have an isomorphism

\[
R\Gamma(K_\varepsilon ; \psi_h(j_i \mathcal{F})) \simeq R\Gamma(\gamma(w, \tau)^{-1}(K_\varepsilon) ; \mathcal{F}).
\]

Then the assertion immediately follows.
Now we can rewrite Proposition 4.18 as follows.

**Proposition 4.18.** Assume that perverse sheaf $\mathcal{F} = \text{Sol}_X(\mathcal{M})[N] \in D^b_{\mathbb{C}^\times}(\mathbb{C}_X^{an})$ is moderate at infinity over a point $w \neq 0$ of $Y = \mathbb{C}^N$. Then for $|\tau| \gg 0$ we have

$$\chi_w((\text{Sol}_{\{w\}}(\mathcal{D}_{\{w\}}^*\mathcal{M}^\wedge))) = \chi(R\Gamma_c(X; \mathcal{F})) - \chi(R\Gamma_c(H_\infty \setminus H(w); \psi_h(j_{\tau}\mathcal{F}))).$$

Finally we obtain the following result.

**Theorem 4.19.** Assume that $d_{W_i} = N - 1$ and perverse sheaf $\mathcal{F} = \text{Sol}_X(\mathcal{M})[N] \in D^b_{\mathbb{C}^\times}(\mathbb{C}_X^{an})$ is moderate at infinity over a neighborhood of a generic point $v \in (W_i)_{\text{reg}}$ in $Y \setminus \{0\}$. Take a normal slice $M$ of $W_i$ at $v$ and consider the meromorphic connection $\mathcal{K}(\ast\{v\})$ on it. Then the multiplicity $\text{mult}_{T_{W_i}y} \mathcal{M}^\wedge \geq 0$ of the Fourier transform $\mathcal{M}^\wedge$ along $T_{W_i}Y$ is given by

$$\text{mult}_{T_{W_i}Y} \mathcal{M}^\wedge = \text{mult}_{T_{V_i}X} \mathcal{N} + \text{irr}(\mathcal{K}(\ast\{v\}))$$

$$(\geq \text{mult}_{T_{V_i}X} \mathcal{N} = n_i > 0).$$

In particular, the conormal bundle $T_{W_i}Y$ is contained the characteristic variety $\text{char}(\mathcal{M}^\wedge)$ of $\mathcal{M}^\wedge$ and we have $W_i \subset D = Y \setminus \Omega$.

**Proof.** By our assumption there exists a point $v_0 \in \Omega$ over which $\mathcal{F} \in D^b_{\mathbb{C}^\times}(\mathbb{C}_X^{an})$ is moderate at infinity. Then by Theorem 4.12 and Lemma 4.17 we obtain

$$\text{mult}_{T_{W_i}Y} \mathcal{M}^\wedge = \chi\left(R\Gamma_c(H_\infty \setminus H(v); \psi_h(j_{\tau}\mathcal{F}))\right) - \chi\left(R\Gamma_c(H_\infty \setminus H(v_0); \psi_h(j_{\tau}\mathcal{F}))\right)$$

$$+ \text{irr}(\mathcal{K}(\ast\{v\})).$$

Note that for the conification $\mathcal{G}$ of $\mathcal{F}$ we have $\mathcal{G}_0 = \psi_h(j_{\tau}\mathcal{G}) = \psi_h(j_{\tau}\mathcal{G})$. Then by replacing $\mathcal{M}^\wedge$ with the regular holonomic $D_Y$-module $\mathcal{N}^\wedge$ we obtain also

$$\text{mult}_{T_{W_i}Y} \mathcal{N}^\wedge = \chi\left(R\Gamma_c(H_\infty \setminus H(v); \psi_h(j_{\tau}\mathcal{F}))\right) - \chi\left(R\Gamma_c(H_\infty \setminus H(v_0); \psi_h(j_{\tau}\mathcal{F}))\right).$$

From this the assertion immediately follows. \qed

**Remark 4.20.** We define a $\mathbb{Z}$-valued function $\phi : \mathbb{P}(Y) \to \mathbb{Z}$ on $\mathbb{P}(Y) \simeq \mathbb{P}^{N-1}$ by

$$\phi([w]) = \chi\left(R\Gamma(H(w); \mathcal{G}_0)\right)$$

$$(|w| \in \mathbb{P}(Y)).$$

This is the topological Radon transform of the constructible function $\chi(\mathcal{G}_0)$ on $\mathbb{P}(X) = H_\infty \simeq \mathbb{P}^{N-1}$ studied by many mathematicians. Since the characteristic cycle of $\chi(\psi_h(j_{\tau}\mathcal{F}))$ is a sum of the conormal bundles of $\mathbb{P}(V_i) \subset \mathbb{P}(X)$ and the zero-sections $\mathbb{P}(X)$, by Ernström [Ern94 Corollary 3.3] (see also Matsui-Takeuchi [MT07] for a new proof to it and a generalization to the real case) the function $\phi$ is constant on $(W_i)_{\text{reg}} \setminus \bigcup_{j \neq i} W_j \subset (W_i)_{\text{reg}}$. This fact would be very useful to apply our Theorem 4.19.

**Example 4.21.** For the smooth hypersurface $Z = \{z \in X = \mathbb{C}^N | z_1^2 + \cdots + z_N^2 = 1\} \subset X$ consider the perverse sheaf $\mathcal{F} = \mathbb{C}_Z[N - 1] \in D^b_{\mathbb{C}^\times}(\mathbb{C}_X^{an})$ on $X = \mathbb{C}^N$. Let
\[ \mathcal{M} \in \text{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \] be the regular holonomic \( \mathcal{D}_X \)-module such that \( \text{Sol}_X(\mathcal{M})[N] = \mathcal{F} \). Then \( \mathcal{M} \) is not monodromic and we have

\[
\text{char} \mathcal{M} = T^*_Y X = \{(z, (\zeta_1, \ldots, \zeta_N)) \in T^*X \cong X \times Y \mid \zeta \in \mathbb{C}, z_1^2 + \cdots + z_N^2 = 1\}.
\]

It follows that for the projection \( q : T^*X \cong X \times Y \to Y \) and a point \( w \in Y \setminus \{0\} = \mathbb{C}^N \setminus \{0\} \) we have

\[
w \in q(T^*_Y X) \iff \ z_1^2 + \cdots + z_N^2 \neq 0.
\]

Moreover if \( w \in q(T^*_Y X) \setminus \{0\} \) then the set \( q^{-1}(w) \cap T^*_Y X \) is explicitly calculated as

\[
q^{-1}(w) \cap T^*_Y X = \left\{ \left( w \frac{1}{\zeta}, w \right) \in T^*X \cong X \times Y \mid \zeta^2 = w_1^2 + \cdots + w_N^2 \right\}.
\]

This shows that the open subset \( \Omega \subset Y = \mathbb{C}^N \) is given by

\[
\Omega = \{ w \in Y = \mathbb{C}^N \mid w_1^2 + \cdots + w_N^2 \neq 0 \}
\]

and the morphism \( q^{-1}(\Omega) \cap \text{char} \mathcal{M} \to \Omega \) induced by \( q \) is a covering of degree 2. By Corollary 4.5 the restriction of the Fourier transform \( \mathcal{M}^\wedge \in \text{Mod}_{\mathrm{hol}}(\mathcal{D}_Y) \) of \( \mathcal{M} \) to \( \Omega \subset Y \) is an integrable connection of rank 2. We can also easily see that \( \mathcal{M}^\wedge \) has some irregularities at infinity by using Theorem 4.6. Set \( D = Y \setminus \Omega = \{ w \in Y = \mathbb{C}^N \mid w_1^2 + \cdots + w_N^2 = 0 \} \) and take its normal slice

\[
M = \{(1 + u, \sqrt{-1}, 0, \ldots, 0) \in Y = \mathbb{C}^N \mid u \in \mathbb{C} \} \subset Y
\]

at the point \( (1, \sqrt{-1}, 0, \ldots, 0) \in D \setminus \{0\} \). Then for a point \( w = (1 + u, \sqrt{-1}, 0, \ldots, 0) \in M \setminus D = M \cap \Omega \) \( (u \neq 0) \) we have

\[
q^{-1}(w) \cap T^*_Y X = \left\{ \left( w \frac{1}{\zeta}, w \right) \in T^*X \cong X \times Y \mid \zeta = \pm \sqrt{u(2 + u)} \right\}.
\]

Since the Puiseux series

\[
\varphi_\pm(u) = \left( \frac{\pm w}{\sqrt{u(2 + u)}} \right) = \pm \sqrt{u(2 + u)}
\]

of \( u \) start from the positive degree \( \frac{1}{2} \), by Theorem 4.11 the meromorphic connection obtained by restricting \( \mathcal{M}^\wedge \) to the normal slice \( M \subset Y \) is regular along the point \( (1, \sqrt{-1}, 0, \ldots, 0) \in M \). On the other hand, by Example 4.16 the perverse sheaf \( \mathcal{F} = \mathbb{C}[N - 1] \in \mathbf{D}^b_{\text{c-c}}(\mathbb{C}X^{an}) \) is moderate at infinity over any point \( w \neq 0 \) of \( Y \setminus \{0\} \). Set \( V = \{ z \in X = \mathbb{C}^N \mid z_1^2 + \cdots + z_N^2 = 0 \} \subset X \) and \( W = D = \{ w \in Y = \mathbb{C}^N \mid w_1^2 + \cdots + w_N^2 = 0 \} \subset Y \). Then by the natural identification \( T^*_X \cong T^*Y \) we have \( T^*_X = T^*_Y \). Take a conification \( \mathcal{G} \in \mathbf{D}^b_{\text{c-c}}(\mathbb{C}X^{an}) \) of \( \mathcal{F} \) such that \( \mathcal{G}|_{X \setminus \{0\}} \cong \mathbb{C}|_{Y \setminus \{0\}}[N - 1] \), and let \( \mathcal{N} \in \text{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \) be the (monodromic) regular holonomic \( \mathcal{D}_X \)-module such that \( \text{Sol}_X(\mathcal{N})[N] = \mathcal{G} \). Then by Theorem 4.19 we obtain

\[
\text{multi}_{T^*_X Y} \mathcal{M}^\wedge = \text{multi}_{T^*_X Y} \mathcal{N} + 0 = 1.
\]
Example 4.22. Let us consider the case where $\mathcal{M}^\wedge$ is a confluent $A$-hypergeometric system on $Y = \mathbb{C}^3$. Define a subset $A$ of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ by

$$A = \left\{ a(1) = \left( \begin{array}{c} 2 \\ -1 \end{array} \right), a(2) = \left( \begin{array}{c} 1 \\ 1 \end{array} \right), a(3) = \left( \begin{array}{c} -2 \\ 0 \end{array} \right) \right\} \subset \mathbb{Z}^2.$$

Then by the condition $\sum_{i=1}^3 Za(i) = \mathbb{Z}^2$ the morphism

$$i_T : (\mathbb{C}^*)^2 \hookrightarrow X = \mathbb{C}^3, \quad s = (s_1, s_2) \mapsto (s_{a(1)}, s_{a(2)}, s_{a(3)})$$

associated to it of the 2-dimensional torus $T = (\mathbb{C}^*)^2$ is a closed embedding. For $c = (c_1, c_2) \in \mathbb{C}^2$ set $\mathcal{L} = \mathcal{O}_T s_1^{c_1-1} s_2^{c_2-1} \in \text{Mod}_h(D_T)$ and $\mathcal{M} = D_{i_T} \mathcal{L} \in \text{Mod}_h(D_X)$. In this case, $Z = i_T(T) \subset X = \mathbb{C}^3$ is a closed hypersurface and explicitly given by

$$Z = \{ z \in X = \mathbb{C}^N | z_1^2 z_2^2 z_3^3 = 1 \}.$$

Hence we have

$$\text{char}\mathcal{M} = T^*_Z X = \{ (z, 2\zeta z_1 z_2 z_3^2, 2\zeta z_1 z_2 z_3^3, 3\zeta z_1^3 z_2^2 z_3^2) \in T^* X \simeq X \times Y | \zeta \in \mathbb{C}, \zeta_1^2 z_2^2 z_3^3 = 1 \}$$

and $\text{mult}_{T^*_Z X} \mathcal{M} = 1$. It follows that for the projection $q : T^* X \simeq X \times Y \to Y$ and a point $w \in Y \setminus \{ 0 \} = \mathbb{C}^N \setminus \{ 0 \}$ we have

$$w \in q(T^*_Z X) \iff w_1 w_2 w_3 \neq 0.$$

Moreover if $w \in q(T^*_Z X) \setminus \{ 0 \}$ then the set $q^{-1}(w) \cap T^*_Z X$ is explicitly calculated as

$$q^{-1}(w) \cap T^*_Z X = \left\{ \left( \frac{2\zeta}{w_1}, \frac{2\zeta}{w_2}, \frac{3\zeta}{w_3}, w \right) \in T^* X \simeq X \times Y | \zeta^7 = \frac{w_1^2 w_2^3 w_3^3}{4 \cdot 4 \cdot 27} \right\}.$$

This shows that the open subset $\Omega \subset Y = \mathbb{C}^N$ is given by

$$\Omega = \{ w \in Y = \mathbb{C}^N | w_1 w_2 w_3 \neq 0 \}$$

and the morphism $q^{-1}(\Omega) \cap \text{char}\mathcal{M} \to \Omega$ induced by $q$ is a covering of degree 7. By Corollary 4.5 the restriction of the Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_h(D_Y)$ of $\mathcal{M}$ to $\Omega \subset Y$ is an integrable connection of rank 7. This coincides with Adolphson’s result in [Ado]. Indeed, the normalized volume of the convex hull $\Delta \subset \mathbb{R}^2$ of $\{ 0 \} \cup A \subset \mathbb{R}^2$ is equal to 7. We see also that $\mathcal{M}^\wedge$ has some irregularities at infinity by Theorem 4.6. Set $D = Y \setminus \Omega = \{ w \in Y = \mathbb{C}^N | w_1 w_2 w_3 = 0 \}$ and take its normal slice

$$M = \{ (1, 1, u) \in Y = \mathbb{C}^N | u \in \mathbb{C} \} \subset Y$$

at the point $(1, 1, 0) \in D \setminus \{ 0 \}$. Set

$$\left\{ \zeta \in \mathbb{C} | \zeta^7 = \frac{1}{4 \cdot 4 \cdot 27} \right\} = \{ \zeta_1, \zeta_2, \ldots, \zeta_7 \}.$$

Then for a point $w = (1, 1, u) \in M \setminus D = M \cap \Omega$ $(u \neq 0)$ we have

$$q^{-1}(w) \cap T^*_Z X = \left\{ \left( \frac{2\zeta_i}{w_1} u^\frac{i}{2}, \frac{2\zeta_i}{w_2} u^\frac{i}{2}, \frac{3\zeta_i}{w_3} u^\frac{i}{2}, w \right) \in T^* X \simeq X \times Y | 1 \leq i \leq 7 \right\}.$$
Since the Puiseux series
\[ \varphi_i(u) = 7\zeta_i u^3 \quad (1 \leq i \leq 7) \]
of \( u \) start from the positive degree \( \frac{3}{7} \), by Theorem 4.11 the meromorphic connection obtained by restricting \( M^\wedge \) to the normal slice \( M \subset Y \) is regular along the point \( (1, 1, 0) \in M \). On the other hand, we can easily see that the perverse sheaf \( \mathcal{F} = \text{Sol}_X(M)[N] \in D^b_{\text{C},c}(\mathbb{C}_X) \) is moderate at infinity over any point \( w \neq 0 \) of \( Y \setminus \{0\} \). Set \( V_1 = \{ z_2 = z_3 = 0 \}, V_2 = \{ z_1 = z_3 = 0 \}, V_3 = \{ z_1 = z_2 = 0 \} \subset X \) and \( W_1 = \{ w_1 = 0 \}, W_2 = \{ w_2 = 0 \}, W_3 = \{ w_3 = 0 \} \subset Y \). Then by the natural identification \( T^*X \cong T^*Y \) for any \( 1 \leq i \leq 3 \) we have \( T^*_i X = T^*_i Y \). Take a conification \( G \in D^b_{\text{C},c}(\mathbb{C}_X) \) of \( \mathcal{F} \) such that \( G|_{X \setminus \{0\}} \cong \mathcal{F}|_{X \setminus \{0\}} \) and let \( \mathcal{N} \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \) be the (monodromic) regular holonomic \( \mathcal{D}_X \)-module such that \( \text{Sol}_X(\mathcal{N})[N] = G \). Then for any singular point \( z \neq 0 \) of the normal crossing divisor \( \{ z_1 z_2 z_3 = 0 \} \subset X \) we have \( \chi_z(G) = 0 \). This follows from the well-known fact that the Euler characteristic of the Milnor fiber of the function \( z_1^2 z_2 z_3^3 \) at such a point is equal to zero. By Kashiwara’s local index theorem for holonomic \( \mathcal{D} \)-modules in [Kas83, Corollary 6.3.4], we thus obtain \( \text{multi}_{T^*_i Y} M^\wedge = \text{multi}_{T^*_i X} N + 0 = \text{multi}_{T^*_i X} N \).

Moreover by Theorem 4.19 for any \( 1 \leq i \leq 3 \) we have

\[ \text{multi}_{T^*_i Y} M^\wedge = \text{multi}_{T^*_i X} N + 0 = \text{multi}_{T^*_i X} N. \]

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