New subclass of harmonic univalent functions defined by integral operator

G. M. Birajdar
School of Mathematics & Statistics,
Dr. Vishwanath Karad MIT World Peace University,
Pune (M.S) India 411038
Email: gmbirajdar28@gmail.com

N. D. Sangle
Department of Mathematics,
D. Y. Patil College of Engineering & Technology,
Kasaba Bawda, Kolhapur
(M.S.) India 416006
Email: navneet_sangle@rediffmail.com

Abstract
In this paper, we introduce the subclass $S^{−m}(\alpha, \beta)$ using integral operator and give sufficient coefficient conditions for normalized harmonic univalent function in the subclass $S^{−m}(\alpha, \beta)$. These conditions are also shown to be necessary when the coefficients are negative.

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1 Introduction
The class $S_H$ investigated by Clunie and Sheil-Small [1]. They studied geometric subclasses and established some coefficient bounds. Several researchers have worked on the class $S_H$ and its subclasses. By introducing new subclasses, Silverman [13], Silverman and Silvia [14], Jahangiri [7], Dixit and Porwal [4] etc. presented a systematic study of harmonic univalent functions.
Motivating the research work done by Jahangiri [2, 3], Purohit et al. [6], Darus and Sangle [17], Ravindar et.al [16], Bhoosnurmath and Swamy [5], Yalcin [10], Al-Shaqsi

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et.al [11], Sangle and Birajdar [18], Murugusundaramoorthy [15], we introduce new subclasses of harmonic mappings using integral operator. Also, we determine extreme points, coefficient estimates of $SHP^{-m}(\alpha, \beta)$ and $THP^{-m}(\alpha, \beta)$. Suppose $A$ be family of analytic functions defined in unit disk $U$ and $A^0$ be the class of all normalized analytic functions. Let the functions $h \in A$ be of the form

$$h(z) = z + \sum_{v \geq 2} a_v z^v$$  \hspace{1cm} (1.1)$$

The integral operator $I^m$ in [3] for the above functions $h$ defined as

$$I^0 h(z) = h(z).$$

$$I^1 h(z) = I(z) = \int_0^z h(t) t^{-1} dt;$$

$$I^m h(z) = I \left( I^{m-1} h(z) \right), m \in N = \{1, 2, 3, \ldots \}$$

$$I^m h(z) = z + \sum_{k=2}^\infty [v]^{-m} a_v z^v$$  \hspace{1cm} (1.2)$$

The harmonic functions can be expressed as $f = h + \overline{g}$ where $h \in A^0$ is given by (1.1) and $g \in A$ has power series expansion:

$$g(z) = \sum_{v \geq 1} \infty b_v z^v, |b_1| < 1$$

Clunie and Sheil-Small [11] defined function of form $f = h + \overline{g}$ that are locally univalent, sense-preserving and harmonic in $U$. A sufficient condition for the functions $f$ to be univalent in $U$ is $|h'(z)| \geq |g'(z)|$ in $U$.

A function $f(z) = h + \overline{g}$ is harmonic starlike [9] for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta} (\arg (f(re^{i\theta}))) = \Re \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > 0.$$ 

The integral operator in [3] defined for the harmonic functions $f$ by

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)}$$  \hspace{1cm} (1.3)$$

where $I^m$ is defined by (1.2).

Now for $0 \leq \alpha < 1, m \in N_0$ and $z \in U$, suppose that $SHP^{-m}(\alpha, \beta)$ denote the family of harmonic univalent function $f$ of the form $f(z) = h(z) + \overline{g(z)}$ where

$$h(z) = z + \sum_{v \geq 2} a_v z^v, \quad g(z) = \sum_{v \geq 1} b_v z^v, |b_1| < 1$$  \hspace{1cm} (1.4)$$

$$\Re \left\{ (1 - \alpha) \frac{I^m h(z) + (-1)^m \overline{I^m g(z)}}{z} + \alpha \left[ I^m h(z) + (-1)^m \overline{I^m g(z)} \right] \right\} \geq \beta$$  \hspace{1cm} (1.5)
We further denote by $T H P^{-m}(\alpha, \beta)$ subclass of $S H P^{-m}(\alpha, \beta)$ consisting harmonic functions $f = h + \overline{g}$ in $T H P^{-m}(\alpha, \beta)$ so that $h$ and $g$ are of the form

$$h(z) = z - \sum_{v \geq 2} |a_v| z^v \text{ and } g(z) = \sum_{v \geq 1} |b_v| z^v.$$  \hspace{1cm} (1.6)

In this paper, we will give the sufficient condition for functions $f = h + \overline{g}$ to be in the subclass $S H P^{-m}(\alpha, \beta)$. It is shown that the coefficient condition is also necessary for the functions in the class $T H P^{-m}(\alpha, \beta)$. Coefficient and distortion bounds, extreme points, convolution conditions and convex combination of this class are obtained.

2 Main Results

We begin by proving some sharp coefficient inequalities contained in the following theorem.

**Theorem 2.1.** Let the function $f = h + \overline{g}$ be such that $h$ and $g$ are given by (1.6), furthermore, let

$$\sum_{v \geq 1} |v|^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) \leq 1 - \beta.$$  \hspace{1cm} (2.1)

Then $f(z)$ is harmonic univalent, sense preserving in $U$ and $f(z) \in S H P^m_\alpha(\alpha, \beta)$.

**Proof:** For $|z_1| \leq |z_1| < 1$, we have by equation (2.1)

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$\geq |z_1 - z_2| \left( 1 - \sum_{v \geq 2} v |a_v| |z_2|^{v-1} - \sum_{v \geq 1} v |b_v| |z_2|^{v-1} \right)$$

$$\geq |z_1 - z_2| \left( 1 - \sum_{v \geq 2} v (|a_v| + |b_v|) |z_2|^{v-1} + |b_1| \right)$$

$$\geq |z_1 - z_2| \left( 1 - \sum_{v \geq 2} v (|a_v| + |b_v|) + |b_1| \right)$$

$$\geq |z_1 - z_2| \left( 1 - \sum_{v \geq 2} v |v|^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) + |b_1| \right)$$

$$\geq |z_1 - z_2| (1 - (1 - \beta) - |b_1| + |b_1|)$$

$$\geq |z_1 - z_2| |\beta| \geq 0.$$
Hence, $f(z)$ is univalent in $U$. $f(z)$ is sense preserving in $U$. This is because

$$|h'(z)| \geq 1 - \sum_{v \geq 2}^\infty |a_v| |z|^{v-1}$$

$$> 1 - \sum_{v \geq 2}^\infty v |a_v|$$

$$> 1 - \sum_{v \geq 2}^\infty |v|^{-m} (1 - \alpha - \alpha v) |a_v|$$

$$\geq \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha - \alpha v) |b_u|$$

$$\geq \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha - \alpha v) |b_v| |z|^{v-1}$$

$$> \sum_{v \geq 1}^\infty v |b_v| |z|^{v-1}$$

$$|h'(z)| \geq |g'(z)|.$$  

Now, we show that $f(z) \in \text{SH}^P(\alpha, \beta)$. Using the fact that $\Re \{w\} \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$, its sufficient to show that

$$\left|1 - \beta + \Re \left\{ (1 - \alpha) \frac{I^m h(z) + (-1)^m \overline{I^m g(z)}}{z} + \alpha \left[ I^m h(z) + (-1)^m \overline{I^m g(z)} \right]' \right\} \right|$$

$$- \left|1 + \beta - \Re \left\{ (1 - \alpha) \frac{I^m h(z) + (-1)^m \overline{I^m g(z)}}{z} - \alpha \left[ I^m h(z) + (-1)^m \overline{I^m g(z)} \right]' \right\} \right| \geq 0.$$  

(2.2)

$$= 2 - \beta + \sum_{v \geq 2}^\infty |v|^{-m} (1 - \alpha + \alpha v) a_v z^{v-1} + (-1)^m \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha + \alpha v) b_u z^{v-1}$$

$$- \beta - \sum_{v \geq 2}^\infty |v|^{-m} (1 - \alpha + \alpha v) a_v z^{v-1} - (-1)^m \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha + \alpha v) b_u z^{v-1}$$

$$\geq 2 \left| (1 - \beta) - \sum_{v \geq 2}^\infty |v|^{-m} (1 - \alpha + \alpha v) a_v |z|^{v-1} + \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha + \alpha v) b_u |z|^{v-1} \right|$$

$$> 2 \left| (1 - \beta) - \sum_{v \geq 2}^\infty |v|^{-m} (1 - \alpha + \alpha v) a_v + \sum_{v \geq 1}^\infty |v|^{-m} (1 - \alpha + \alpha v) b_u \right|.$$  

This last expression is non-negative by (2.1).

The harmonic mappings

$$f(z) = z + \sum_{v \geq 2}^\infty \frac{1 - \beta}{|v|^{-m} (1 - \alpha + \alpha v)} x_v z^v + \sum_{v \geq 1}^\infty \frac{1 - \beta}{|v|^{-m} (1 - \alpha + \alpha v)} y_v z^v$$  

(2.3)
where \( \sum_{v \geq 2}^\infty |x_v| + \sum_{v \geq 1}^\infty |y_v| = 1 \) shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in \( \text{SHP}^{-m}(\alpha, \beta) \) because

\[
\sum_{v \geq 1}^\infty [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) = 1 + (1 - \beta) \sum_{v \geq 2}^\infty |x_v| + \sum_{v \geq 1}^\infty |y_v| = 2 - \beta.
\]

**Theorem 2.2.** Let the function \( f = h + g \) be such that \( h \) and \( g \) are given by (1.6) then \( f(z) \in \text{TH}P^{-m}(\alpha, \beta) \) if and only if

\[
\sum_{v \geq 1}^\infty |v|^m (1 - \alpha + \alpha v) (|a_v| + |b_v|) \leq 2 - \beta.
\]

**Proof:** The ‘if part’ follows from Theorem 2.1 upon noting that the functions \( h(z) \) and \( g(z) \) in \( f(z) \in \text{SHP}^{-m}(\alpha, \beta) \) are of the form (1.6), then \( f(z) \in \text{TH}P^{-m}(\alpha, \beta) \).

For the ‘only if’ part, we show that if \( f(z) \in \text{TH}P^{-m}(\alpha, \beta) \) then the condition (2.4) holds. Note that a necessary and sufficient condition for \( f = h + g \) given by (1.6) be in \( \text{TH}P^{-m}(\alpha, \beta) \) is that

\[
\text{Re} \left\{ (1 - \alpha) \frac{I_p^m h(z) + (-1)^m I_p^m g(z)}{z} + \alpha \left[ I_p^m h(z) + (-1)^m I_p^m g(z) \right]^\prime \right\} > \beta
\]

or, equivalently

\[
\text{Re} \left\{ 1 - \sum_{v \geq 2}^\infty [v]^{-m} (1 - \alpha + \alpha v) |a_v| z^{v-1} - \sum_{v \geq 1}^\infty [v]^{-m} (1 - \alpha + \alpha v) |b_v| z^{v-1} \right\} > \beta.
\]

If we choose \( z \) to be real and \( z \to 1^- \), we get

\[
1 - \sum_{v \geq 2}^\infty [v]^{-m} (1 - \alpha + \alpha v) |a_v| - \sum_{v \geq 1}^\infty [v]^{-m} (1 - \alpha + \alpha v) |b_v| \geq \beta
\]

this is precisely the assertion of (2.4).

**Theorem 2.3.** If \( f(z) \in \text{TH}P^{-m}(\alpha, \beta) \), \( |z| = r < 1 \) then

\[
|f(z)| \leq (1 + |b_1|) r + \frac{1}{[2]^{-m} (1 + \alpha)} (1 - |b_1| - \beta) r^2
\]

and

\[
|f(z)| \geq (1 - |b_1|) r - \frac{1}{[2]^{-m} (1 + \alpha)} (1 - |b_1| - \beta) r^2
\]

(2.5)
Proof: Taking the absolute values of $f(z)$, we obtain

$$|f(z)| \leq (1 + |b_1|) r + \sum_{v \geq 2} \frac{1}{[v]^{-m}} (1 - \alpha + \alpha v) \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) r^v$$

$$\leq (1 + |b_1|) r + \sum_{v \geq 2} (|a_v| + |b_v|) r^v$$

$$\leq (1 + |b_1|) r + \frac{1}{[2]^{-m}} (1 + \alpha) \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) r^v$$

$$\leq (1 + |b_1|) r + \frac{1}{[2]^{-m}} (1 + \alpha) (1 - |b_1| - \beta) r^2$$

and

$$|f(z)| \geq (1 - |b_1|) r - \sum_{v \geq 2} (|a_v| + |b_v|) r^v$$

$$\geq (1 - |b_1|) r - \sum_{v \geq 2} (|a_v| + |b_v|) r^v$$

$$\geq (1 - |b_1|) r - \frac{1}{[2]^{-m}} (1 + \alpha) \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) r^v$$

$$\geq (1 - |b_1|) r - \frac{1}{[2]^{-m}} (1 + \alpha) (1 - |b_1| - \beta) r^2.$$

For the functions

$$f(z) = z + |b_1| \bar{z} - \frac{1}{[2]^{-m}} (1 + \alpha) (1 - |b_1| - \beta) \bar{z}^2$$

and

$$f(z) = z - |b_1| z - \frac{1}{[2]^{-m}} (1 + \alpha) (1 - |b_1| - \beta) z^2.$$

For $|b_1| \leq 1 - \beta$ shows that the bounds given in Theorem 2.3 are sharp.

Next, we determine the extreme points of the closed convex hulls of $THP^{-m}(\alpha, \beta)$ denoted by $clcoTHP^{-m}(\alpha, \beta)$.

Theorem 2.4. A function $f(z) \in clco THP^{-m}(\alpha, \beta)$ if and only if

$$f(z) = \sum_{v \geq 1} (\mu_v h_v + \eta_v g_v)$$

(2.6)

where

$$h_v(z) = z - \frac{1 - \beta}{[v]^{-m}} (1 - \alpha + \alpha v) z^v, \ (v = 2, 3, ...)$$
Theorem 2.5. Each member of $T H P^{-m}(\alpha, \beta)$ is convex and closed, so $\text{clco} THP^{-m}(\alpha, \beta) = THP^{-m}(\alpha, \beta)$. Then the statement of Theorem 2.4 is really for $f(z) \in THP^{-m}(\alpha, \beta)$.

**Theorem 2.5.** Each member of $T H P^{-m}(\alpha, \beta)$ maps $U$ on to a starlike domain.

**Proof:** We only need to show that if $f(z) \in THP^{-m}(\alpha, \beta)$ then

$$\text{Re} \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > 0.$$ 

Using the fact that $\text{Re} \{w\} > 0$ if and only if $|1 + w| > |1 - w|$, it suffices to show that

$$ \left| h(z) + g(z) + zh'(z) - zg'(z) \right| - \left| h(z) + g(z) + zh'(z) + zg'(z) \right|$$

In particular, the extreme points of $T H P^{-m}(\alpha, \beta)$ are $\{h_v\}$ and $\{g_v\}$.

**Proof:** For the functions $f(z)$ of the form (2.6), we have

$$f(z) = \sum_{v \geq 1} (\mu_v h_v + \eta_v g_v)$$

then

$$\sum_{v \geq 1} \frac{1 - \beta}{[v]^{-m} (1 - \alpha + \alpha v)} \mu_v z^v + \sum_{v \geq 1} \frac{1 - \beta}{[v]^{-m} (1 - \alpha + \alpha v)} \eta_v z^v$$

and so $f(z) \in \text{clco} THP^{-m}(\alpha, \beta)$.

Conversely, suppose that $f(z) \in \text{clco} THP^{-m}(\alpha, \beta)$, consider

$$\mu_v = \frac{[v]^{-m} (1 - \alpha + \alpha v)}{1 - \beta} |a_v|, \quad (v = 2, 3, \ldots)$$

and

$$\eta_v = \frac{[v]^{-m} (1 - \alpha + \alpha v)}{1 - \beta} |h_v|, \quad (v = 1, 2, 3, \ldots)$$

Then note that by Theorem 2.2 $0 \leq \mu_v \leq 1$ $(v = 2, 3, \ldots)$, and $0 \leq \eta_v \leq 1$ $(v = 1, 2, 3, \ldots)$.

We define $\mu_1 = 1 - \sum_{v \geq 2} \mu_v + \sum_{v \geq 1} \eta_v$ and note that, by Theorem 2.2 $\mu_1 \geq 0$.

Consequently, we obtain

$$f(z) = \sum_{v \geq 1} (\mu_v h_v + \eta_v g_v)$$

as required.
Theorem 2.6. Following Ruscheweyh [8], we call the set 
monic starlike function.

\[ \sum_{v \geq 2} (v + 1) |a_v| z^v + \sum_{v \geq 1} (v - 1) |b_v| z^{-v} \geq 2 |z| - \sum_{v \geq 2} (v + 1) |a_v| z^v - \sum_{v \geq 1} (v - 1) |b_v| z^{-v} \]

\[ \geq 2 |z| - \sum_{v \geq 2} (v + 1) |a_v| z^v + \sum_{v \geq 1} (v - 1) |b_v| z^{-v} \]

\[ \geq 2 |z| \left( 1 - \left( \sum_{v \geq 2} v |a_v| |z|^{v-1} + \sum_{v \geq 1} v |b_v| |z|^{v-1} \right) \right) \]

\[ \geq 2 |z| \left( 1 - \left( \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) |a_v| + \sum_{v \geq 1} [v]^{-m} (1 - \alpha + \alpha v) |b_v| \right) \right) \]

\[ \geq 2 |z| |1 - (1 - \beta)| = 2 |z| \beta \]

\[ \geq 0. \]

Theorem 2.7. If \( f(z) \in THP^{-m}(\alpha, \beta) \) then \( f(z) \) is convex in the disc

\[ |z| < \min_v \left[ \frac{1 - \beta - |b_1|}{|v|} \right]^{1/v}, \quad v = 2, 3, ..., 1 - \beta > |b_1| \]

Proof: Let \( f(z) \in THP^{-m}(\alpha, \beta) \) and let \( r \) be fixed such that \( 0 < r < 1, \text{then if } r^{-1} f(rz) \in THP^{-m}(\alpha, \beta) \) and we have

\[ \sum_{v \geq 2} v^2 (|a_v| + |b_v|) r^{v-1} = \sum_{v \geq 2} v (|a_v| + |b_v|) (vr^{v-1}) \]

\[ \leq \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|) (vr^{v-1}) \]

\[ \leq 1 - \beta - |b_1|. \]

Provided \( vr^{v-1} \leq 1 - \beta - |b_1| \), which is true

\[ r < \min_v \left[ \frac{1 - \beta - |b_1|}{|v|} \right]^{1/v}, \quad v = 2, 3, ..., 1 - \beta > |b_1|. \]

Following Ruscheweyh [8], we call the set

\[ N_\delta f(z) = \left\{ G : G(z) = z - \sum_{v \geq 2} |C_v| z^v - \sum_{v \geq 1} |D_v| z^{-v} \quad \text{and} \quad \sum_{v \geq 1} u (|a_v - C_v| + |b_v - D_v|) \leq \delta \right\} \]

(2.7)

as the \( \delta \)-neighborhood of \( f(z) \). From (2.7) we obtain

\[ \sum_{v \geq 1} v (|a_v - C_v| + |b_v - D_v|) = |b_1 - D_1| + \sum_{u \geq 2} v (|a_v - C_v| + |b_v - D_v|) \leq \delta. \]  

(2.8)

Theorem 2.7. Let \( f(z) \in THP^{-m}(\alpha, \beta) \) and \( \delta \leq \beta \). If \( G \in N_\delta(f) \), then \( G \) is a harmonic starlike function.

Proof: Let \( G(z) = z - \sum_{v \geq 2} |C_v| z^v - \sum_{v \geq 1} |D_v| z^{-v} \in N_\delta f(z) \), we have

\[ \sum_{v \geq 2} v (|C_v| + |D_v| + |D_1|) \leq \sum_{v \geq 2} v (|a_v - C_v| + |b_v - D_v|) + \sum_{v \geq 2} v (|a_v| + |b_v|) + |D_1 - b_1| + |b_1| \]

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\[
\leq \sum_{v \geq 2} [v]^{-m} \left(1 - \alpha + \alpha v \right) (|a_v| - |b_v| + |b_v - D_v|) + |D_1 - b_1| + |b_1| + \sum_{v \geq 2} [v]^{-m} (1 - \alpha + \alpha v) (|a_v| + |b_v|)
\]

\[
\leq \delta + |b_1| + (1 - \beta - |b_1|)
\]

Hence, \( G(z) \) is a harmonic starlike function.

For the next theorem, we require to define the convolution of two harmonic functions. For harmonic functions of the form

\[
f(z) = z - \sum_{v \geq 2} |a_v| z^v - \sum_{v \geq 1} |b_v| z^v
\]

and

\[
G(z) = z - \sum_{v \geq 2} |C_v| z^v - \sum_{v \geq 1} |D_v| z^v
\]

we define the convolution of two harmonic functions \( f(z) \) and \( G(z) \) as

\[
(f * G)(z) = f(z) * G(z) = z - \sum_{v \geq 2} |a_v| |C_v| z^v - \sum_{v \geq 1} |b_v| |D_v| z^v
\]

Using the above definition, we show that the class \( THP^{-m}(\alpha, \beta) \) is closed under convolution.

**Theorem 2.8.** For \( 0 \leq \alpha_1 \leq \alpha_2, 0 \leq \beta_1 \leq \beta_2 \) let \( f(z) \in THP^{-m}(\alpha_2, \beta_2) \) and \( G(z) \in THP^{-m}(\alpha_1, \beta_1) \). Then

\[
(f * G)(z) \in THP^{-m}(\alpha_2, \beta_2) \subset THP^{-m}(\alpha_1, \beta_1)
\]

**Proof:** Let

\[
f(z) = z - \sum_{v \geq 2} |a_v| z^v - \sum_{v \geq 1} |b_v| z^v \in THP^{-m}(\alpha_2, \beta_2)
\]

\[
G(z) = z - \sum_{v \geq 2} |C_v| z^v - \sum_{v \geq 1} |D_v| z^v \in THP^{-m}(\alpha_1, \beta_1).
\]

Then the convolution \((f * G)\) is given by (2.9). We wish to show that the coefficient of \((f * G)\) satisfies the required condition given in Theorem 2.2.

For \( G(z) \in THP^{-m}(\alpha_1, \beta_1) \), we note that \( |C_v| < 1 \) and \( |D_v| < 1 \). Now, for the convolution function \( f \ast G \), we obtain

\[
\sum_{v \geq 2} [v]^{-m} \frac{(1 - \alpha_1 + v \alpha_1)}{1 - \beta_1} |a_v| |C_v| + \sum_{v \geq 1} [v]^{-m} \frac{(1 - \alpha_1 + v \alpha_1)}{1 - \beta_1} |b_v| |D_v|
\]

\[
\leq \sum_{v \geq 2} [v]^{-m} \frac{(1 - \alpha_1 + v \alpha_1)}{1 - \beta_1} |a_v| + \sum_{v \geq 1} [v]^{-m} \frac{(1 - \alpha_1 + v \alpha_1)}{1 - \beta_1} |b_v|
\]

\[
\leq \sum_{v \geq 2} [v]^{-m} \frac{(1 - \alpha_2 + v \alpha_2)}{1 - \beta_2} |a_v| + \sum_{v \geq 1} [v]^{-m} \frac{(1 - \alpha_2 + v \alpha_2)}{1 - \beta_2} |b_v| \leq 1.
\]

Since \( 0 \leq \alpha_1 \leq \alpha_2, 0 \leq \beta_1 \leq \beta_2 \) let \( f(z) \in THP^{-m}(\alpha_2, \beta_2) \), thus \( (f \ast G)(z) \in THP^{-m}(\alpha_2, \beta_2) \subset THP^{-m}(\alpha_1, \beta_1) \).
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