Optimal investment for participating insurance contracts under VaR-Regulation*

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Abstract

This paper studies a VaR-regulated optimal portfolio problem of the equity holder of a participating life insurance contract. In a complete market setting the optimal solution is given explicitly for contracts with mortality risk using a martingale approach for constrained non-concave optimization problems. We show that regulatory VaR constraints for participating insurance contracts lead to more prudent investment than in the case of no regulation. This result is contrary to the situation where the insurer maximizes the utility of the total wealth of the company (without distinguishing between contributions of equity holders and policyholders), in which case a VaR constraint may induce the insurer to take excessive risks leading to higher losses than in the case of no regulation, see [3]. Furthermore, importantly for regulators we observe that for participating insurance contracts both relatively small or relatively large policyholder contributions yield rather risky and volatile strategies. Finally, we also discuss the regulatory effect of a portfolio insurance (PI), and analyze different choices for the parameters of the participating contract numerically.

JEL classification: C61, G11, G18, G31

Key words: Non-concave utility maximization, Value at Risk, optimal portfolio, portfolio insurance, risk management, Solvency II regulation

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1 Introduction

This paper investigates the equity holder’s optimal investment problem of a participating contract under financial regulation. This problem is particularly relevant for insurance companies that operate under Solvency II with a Value-at-Risk (VaR) constraint. Participating contracts are life insurance products which provide the policyholder at least a guaranteed amount in downside market situations and a shared profit in good market scenarios. To participate in the contract, the policyholder pays a premium (participation fee) which is collected together with the (equity holder) insurer’s participation amount in an investment pool. At maturity, the policyholder receives a payoff which is linked to the investment performance. The equity holder’s payoff is determined as the residual amount.

Earlier studies on equity-linked life insurance contracts usually analyze pricing or optimal design problems. Some focus on the policyholder’s perspective assuming specific investment strategies like constant proportion portfolio insurance (CPPI) or generalized constant-mix (see, e.g., [9, 30, 31, 34]). Recently, [14] has incorporated taxes and a so-called fair pricing constraint in the policyholder’s problem.

In this paper, we consider two common contract designs. In the first design we assume that the policyholder is fully protected against an insolvency of the insurance company (i.e., the final payoff to the policyholder always exceeds or equals the guaranteed amount). In the second design, we assume that the equity holder has only limited liability, i.e., the policyholder’s payoff is less than the guaranteed amount in case of a default of the insurance company. Note that in the case of full protection, the equity holder may suffer a negative payoff in case of insolvency. To describe the behavior of the equity holder in the loss domain, we use an $S$-shaped utility function adopted from prospect theory [37, 25, 29].

Risk management and regulations based on a terminal VaR constraint are well-known in banking and insurance regulations. Recall that VaR, defined as an estimate of the maximum portfolio loss given a pre-set significance level, is a quantile measure that controls the tail risk of the terminal portfolio. The problem of utility maximization/optimal asset allocation under VaR-type constraints has been studied extensively in the literature, see e.g. [3, 8, 16, 18, 15].

This paper solves the equity holder’s problem of utility maximization under a regulatory constraint imposed at maturity. We obtain closed-form solutions for various kinds of constraints. We first explicitly solve the problem for the two kinds of contracts mentioned above under a VaR regulation using the martingale approach for non-concave
utility maximization problems with constraints. Second, motivated by the fact that regulators usually affect insurance contract designs by imposing a minimum capital requirement which is used to control adverse events, we also introduce a floor on the minimum guarantee rate. This portfolio insurance (PI) constraint enhances the protection for the policyholder especially when a defaultable put is included in the policyholder’s payoff.

Our theoretical and numerical results show that already in the case of no regulation there is a moral hazard problem since the insurer does not have an incentive to ensure that there is any capital in the loss states where the terminal wealth falls below the minimal guarantee. The reason is that any terminal wealth in those loss states only benefits the policyholder and comes at the expense of a lower terminal wealth in the more prosperous states where the equity holder receives a positive residual. On the other hand, introducing a VaR constraint as in Solvency 2 forces the equity holders to enlarge the proportion of hedged loss states, leading to a genuine improvement for the policyholders. This result is contrary to the situation where the insurer maximizes the utility of the total wealth of the company without distinguishing between equity holders and policyholders, in which case a VaR constraint may induce the insurer to take excessive risk leading to higher losses than in the case of no regulation, see [3]. This more prudent investment behavior described above is more pronounced if a VaR-based regulation is replaced by a PI-based regulation. Furthermore, the introduction of a full protection makes the equity holder’s investment also generally more prudent in bad market scenarios.

Our derivation of the optimal solutions relies on the combination of a martingale approach for non-concave and non-differentiable objective functions and a point-wise optimization technique with constraints. Note that the classical martingale method can not be directly applied for the equity’s problem as the derived utility function is neither concave nor differentiable. The problem of non-concave utility maximization without constraints has been considered by many authors e.g., [13, 36, 20, 12, 35, 17, 28, 4], using a concavification technique. In a more general framework without constraints, [35] proves the existence of an optimal terminal wealth. However, no specific payoff is provided.

When finishing this paper we noticed that [29] has independently investigated a similar problem for power utility function without regulations. Note that for a piece-wise payoff structure, it is very challenging to deal with a VaR constraint because the bindingness should be included in the choice of the corresponding multiplier. Hence, the VaR-constrained problem is more complicated to solve. In this paper, we consider
a more general utility function with (independent) mortality risk. To deal with the constrained non concave problems, we extend the technique developed in [15].

The paper is organized as follows: In Section 2, we introduce the asset model and the parametric family of contract payoffs. We then solve the unregulated problems without mortality in Section 3 in connection with the concavification technique. Section 4 discusses the case with independent mortality. In Section 5, we investigate the constrained problems. The results are numerically illustrated in Section 6. All technical proofs are reported in the Appendix.

2 The financial market and participating contracts

2.1 The financial market

Consider a complete financial market in continuous time without transaction costs that contains one traded risky asset and one risk free asset (the bank account). Let the asset price dynamics for the risky asset $S_t$ and the bank account $B_t$ be given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad S_0 > 0; \quad dB_t = r_t B_t dt, \quad B_0 = 1,$$

(2.1)

where $W_t$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. We assume that $r_t \geq 0$ and that $\mu_t$ and $\sigma_t > 0$ are deterministic and bounded processes. This implies that the stochastic differential equations for $S$ and $B$ have unique solution on $[0, T]$. Denote by $\{\mathcal{F}_t\}_{t \in [0, T]}$ the filtration generated by the Brownian motion. For the moment, ignoring the insurance risk, we are in a complete market setting which implies the existence of a unique state price density

$$d\xi_t = -\xi_t (r_t dt + \theta_t dW_t), \quad \xi_0 = 1,$$

(2.2)

where $\theta_t := \sigma_t^{-1} (\mu_t - r_t)$ is the market price of risk process. Here $\xi_t(\omega)$ is the Arrow-Debreu value per probability (or likelihood) unit of a security which pays out $1$ at time $t$ if the scenario $\omega$ happens, and $0$ else. As this value is high in a recession and low in prosperous times, $\xi_t(\omega)$ has the nice property of directly reflecting the overall state of the economy. Therefore, the functional relationship between the optimal wealth and $\xi_t$ may be used as an interpretation of some of our results. This approached was also used in [33, 3, 8, 18, 15, 2, 23, 24, 21]. See also [17] for an elaborate explanation. We remark that in a consumption based pricing model $\xi_t$ in equilibrium corresponds to a constant times the marginal utility of consumption and is also called pricing kernel or stochastic discount factor.
The insurance company chooses an investment strategy that we describe in terms of the amount \( \pi_t \) (in $) invested in the risky asset at time \( t \). We assume that the remaining fraction of wealth \((1 - \pi_t)\) is invested in the risk-free asset to guarantee that the strategy is self-financing. The wealth process related to the strategy \( \pi_t \) when starting with an initial wealth \( x_0 > 0 \) is then easily seen to satisfy
\[
dX_t = (r_t X_t + \pi_t (\mu_t - r_t)) dt + \pi_t \sigma_t dW_t, \quad X_0 > 0.
\]

**Definition 2.1.** A strategy \((\pi_t)_{t \in [0, T]}\) is said admissible if it is adapted with respect to the natural filtration \( \mathcal{F} \) and \( \mathbb{E}[\int_0^T \pi_t^2 dt] < \infty \). Furthermore, \( X_t \) (the solution of SDE (2.3)) exists and \( X_t \geq 0 \) a.s. for all \( t \in [0, T] \). The set of all admissible strategies is denoted by \( \mathcal{A} \).

In a complete market, it is known from the martingale method that choosing a portfolio is equivalent to choosing a terminal wealth \( X_T \) which can be financed by \( X_0 \). The set of admissible terminal wealth values is defined by
\[
X := \{X \in L_1(\Omega, \mathcal{F}_T, \mathbb{P}) : X \geq 0 \text{ and } \mathbb{E}[\xi_T X] \leq X_0 \}.
\]
Hence, the dynamic problem \( \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T)] \) is equivalent to the static one
\[
\sup_{X_t \in X} \mathbb{E}[U(X_T)],
\]
where \( U \) is some strictly increasing and differentiable concave utility function. This is the classical Merton problem considered by e.g. \([32, 33]\). The solution to this problem is also called the Merton solution. Let us first emphasize that the classical martingale method can not be directly applied when the utility function is neither concave nor differentiable. This issue will be addressed further in the sequel.

### 2.2 Participating contracts and payoffs

We assume that the representative policyholder invests in a single-premium equity-linked life insurance contract with a maturity of \( T \) years with \( T < \infty \). At the initiation of the contract, the policyholder invests a lump sum \( L_0 \); the shareholder provides an initial equity \( E_0 > 0 \). Consequently, the initial portfolio value is given by the sum of both contributions, i.e., \( X_0 := L_0 + E_0 \). We denote by \( \alpha := L_0/X_0 \) the share of the policyholder’s contribution (or equivalently the debt ratio of our insurance company). Below \( \alpha \) is also called the policyholder’s participation rate. At maturity \( T \) and in case of solvency, the policyholder receives some guaranteed amount \( L_T \) (we can, for
example, choose \( L_T = L_0 e^{\gamma T} \), where \( g \in \mathbb{R} \) is the guaranteed rate). If the terminal portfolio value exceeds the guaranteed rate, i.e., if \( X_T > L_T \), the surplus \( X_T - L_T \) is shared between insurance company and policyholder. In this paper, we assume that the policyholder receives a surplus equal to \( \delta \left[ \alpha X_T - L_T \right]^+ \) where the surplus (bonus) rate \( \delta \) is the percentage of surpluses that is credited to the policyholder. If the terminal portfolio value \( X_T \) is less than the guarantee (default event), the policyholder receives the remaining amount. To exclude unrealistic cases, we assume throughout the paper that \( 0 < \alpha < 1 \) and \( 0 \leq \delta < 1 \). To summarize, the policyholder receives the following payoff at time \( T \):

\[
V^p_L(X_T) := \begin{cases} 
X_T & \text{if } X_T \leq L_T, \\
L_T + \delta \left[ \alpha X_T - L_T \right]^+ & \text{else}, 
\end{cases}
= L_T + \delta \left[ \alpha X_T - L_T \right]^+ - [L_T - X_T]^+.
\]

Hence, the policyholder takes a long position in the bonus option and a short position in a defaultable put and benefits from the potential upsides over the final maturity guarantee. This type of defaultable contracts is frequently used in the literature on insurance contracts, see, for example [10, 22, 5]. The equity holder always obtains the residual asset value

\[
V^p_E(X_T) := X_T - V_L(X_T) = \begin{cases} 
0 & \text{if } X_T \leq L_T, \\
X_T - L_T & \text{if } L_T < X_T \leq \tilde{L}_T, \\
X_T - L_T - \delta \left[ \alpha X_T - L_T \right]^+ & \text{else}, 
\end{cases}
\]

\[
= [X_T - L_T]^+ - \delta \left[ \alpha X_T - L_T \right]^+, 
\tag{2.4}
\]

where we introduced \( \tilde{L}_T := L_T / \alpha \) which is the threshold where the participation bonus kicks in. Agreeing on the contract, the insurer takes a long position of the call \([X_T - L_T]^+\) and \(\delta \alpha\) short positions of the bonus call \([X_T - \tilde{L}_T]^+\) with strike price \(\tilde{L}_T\). For our analysis, we introduce

\[
\tilde{\delta} := \alpha \delta; \quad \hat{L}_T := \tilde{L}_T - L_T \quad \text{and} \quad f(x) = (1 - \tilde{\delta}) x - (1 - \delta) L_T. \tag{2.5}
\]

Hence, \( \tilde{\delta} \) is the actual (achieved) bonus rate of the policyholder, \( \hat{L}_T \) is the difference between the bonus threshold and the guarantee, and \( f(x) \) is the payoff that the insurer receives in case that the wealth \( x \) is greater than the bonus threshold \( \hat{L}_T \). Like any profit-seeking company, the life insurance company sets up its investment mix to primarily

\[1[\cdot]^+ \text{ denotes the maximum } \max\{\cdot, 0\}.\]
maximize the benefits of its shareholder. Hereby, we assume that the shareholder values her benefits through a utility function $U$ defined on the positive real line, which is twice differentiable and satisfies the usual Inada and the asymptotic elasticity (AE) (see [27]) conditions

$$\lim_{x \to 0} U'(x) = \infty; \quad \lim_{x \to \infty} U'(x) = 0; \quad \lim_{x \to \infty} \frac{xU''(x)}{U(x)} < 1.$$ (2.6)

We assume furthermore that $U(0) = \lim_{x \to 0} U(x) > -\infty$\footnote{The case where $U(0) = \lim_{x \to 0} U(x) = -\infty$ is easier and can be treated without concavification procedure, see more in Remark 3.}. As usual, we denote by $I$ the inverse of the first derivative of the utility function $U'$. For optimal terminal wealth representations we introduce

$$h(x) := \frac{I(x/(1 - \tilde{\delta})) + (1 - \delta)L_T}{1 - \delta}.$$ (2.7)

Note that $h$ is a decreasing mapping from $(0, (1 - \tilde{\delta})U'(\tilde{L}_T)]$ to $[\tilde{L}_T, \infty)$.

### 2.3 Full protection and S-shaped utility function

In this subsection, we consider the case where the contract at maturity gives the policyholder at least some guaranteed amount $L_T$ (see, among many others, [10, 22]). More precisely, at time $T$ the policyholder receives the following payoff

$$V_{L,p}(X_T) := L_T + \delta \left[\alpha X_T - L_T\right]^+,$$

and the equity holder takes the residual portfolio value

$$V_{E,p}(X_T) := X_T - V_{L}(X_T) = \begin{cases} X_T - L_T & \text{if } X_T \leq \tilde{L}_T, \\ X_T - L_T - \delta \left[\alpha X_T - L_T\right]^+ & \text{if } X_T > \tilde{L}_T. \end{cases}$$

In this full protection case, the insurer takes more risk because she may have a negative payoff in the insolvency case where $X_T - L_T < 0$. Taking potential losses for the insolvency case into account we assume that the insurer evaluates the investment optimality using an S-shaped utility (convex on the loss domain and concave on the gain domain) suggested by prospect theory [37]. In particular, the insurance company’s utility function takes the following form

$$U^S(x) := \begin{cases} -U_0(-x) & \text{if } x < 0, \\ U(x) & \text{if } x \geq 0, \end{cases}$$

where $U_0(x)$ is the risk-free utility function.
where $U_{lo}$ and $U$ are two utility functions. We assume furthermore that $U$ satisfies the Inada and (AE) conditions. For example, we can take $U_{lo}(x) = \eta U(-x)$, for some loss aversion degree $\eta > 1$, see [37]. Note that for such fully protected contracts, the guarantee level $L_T$ is considered as the reference point which is naturally used to distinguish gains and losses.

![Graphs](a) Defaultable contract payoffs (b) Fully protected contract payoffs

Figure 1: Payoffs with $\alpha = 0.8$, $\delta = 0.9$, $L_T = 50e^{0.02\times10}$

### 2.4 Contracts with mortality risk

Mortality is one of the most important risk factors in life insurance as it strongly affects the pricing and premium principles. Note that when non-tradable stochastic mortality is considered, the market becomes incomplete. For detailed discussion on general stochastic mortality models and their applications we refer for instance to [19, 11]. In this section we incorporate mortality risk into the optimal investment problem of the equity holder. For simplicity, we assume that the premature death of the policyholder is modelled by the event $\{d = 1\}$, where $d$ is a binomial random variable which is independent of the financial risk [1, 6] and the reference portfolio value $(X_t)_{t \in [0,T]}$. This assumption allows to employ a separating technique in our optimization problem.

First, consider the case where the policyholder receives full protection. We suppose that the policyholder receives the amount $V_{L}^{mp}(X_T)$ if she is alive at maturity. If she dies before maturity the guarantee $L_T$ will be paid at maturity to the policyholder’s relative (inheritress). So, the policyholder’s payoff is given by

$$V_{L}^{mp,d}(X_T) := V_{L}^{mp}(X_T)1_{d=0} + L_T1_{d=1}.$$
Hence, the equity holder’s payoff will be given by the difference between the terminal portfolio value and the amount paid to the policyholder, i.e.,

\[ V_{np,d}^{np}(X_T) := X_T - V_{np,d}^{np}(X_T) = V_{np}^{np}(X_T)1_{d=0} + (X_T - L_T)1_{d=1}. \]

**Remark 1.** For the sake of simplicity, we have assumed that \( d \) is a Bernoulli variable. This can be generalized to the case where \( d \sim Bin(n, p) \) for some \( p \in (0, 1) \) and \( n \in \mathbb{N} \).

Recall that \( V_{np}^{np} \) and \( V_{np}^{np} \) are the payoffs defined in the absence of mortality risk in Section 2.3.

For the case of a defaultable contract, we still assume that the policyholder receives the amount \( V_p^p(X_T) \) if she is alive at maturity and in case of death the guarantee \( L_T \) will be paid. Therefore, the policyholder’s payoff is given by

\[ V_{L,d}^{p,d}(X_T) := V_{L}^{p}(X_T)1_{d=0} + L_T1_{d=1}, \]

and

\[ V_{E,d}^{p,d}(X_T) := X_T - V_{L,d}^{p,d}(X_T) = V_{E}^{p}(X_T)1_{d=0} + (X_T - L_T)1_{d=1}. \]

### 2.5 Insurer’s optimization objective

The insurer wants to solve the following optimization problem

\[
\sup_{X_T \in \mathcal{X}} \mathbb{E} \left[ \tilde{U}^{S,j,d}(X_T) \right], \quad \text{where} \quad \tilde{U}^{S,j,d} := U^S \circ V_{E}^{j,d}, \quad j \in \{p, np\}. \tag{2.8}
\]

We observe that the payoffs \( V_{L}^{j,d} \) and \( V_{E}^{j,d}, \ j \in \{p, np\} \) admit a path-independent structure which allows to apply the martingale approach. When mortality is ignored we drop \( \tau \) in the superscript and still denote the equity holder’s derived utility function by

\[ \tilde{U}^{S,j} := \tilde{U}^S \circ V_{E}^{j}, \quad j \in \{p, np\}. \]

We remark that for the defaultable contract without mortality the use of an \( S \)-shaped utility is not needed, i.e., \( U^S \equiv U \), and hence \( \tilde{U}^{S,p} \equiv \tilde{U}^p := U \circ V_{E}^{p} \).

We furthermore assume the following integrability condition:

**Assumption (U):** For any \( \lambda \in (0, \infty) \), we have

\[ \mathbb{E}[U((1 - \tilde{\delta})^{-1}I(\lambda \xi_T))] < \infty \quad \text{and} \quad \mathbb{E}[\xi_T I(\lambda \xi_T)] < \infty. \]

\footnote{For a defaultable contract, an alternative assumption is that there is no guaranteed payment in case of premature death. In this situation there is no need to use an \( S \)-shaped utility function as the insurer’s terminal wealth is always non-negative.}
It is straightforward to check that Assumption (U) holds for commonly used utility functions like power, logarithmic, exponential. This condition is needed to guarantee the existence of the Lagrangian multiplier $\lambda$.

Now, the independence of $\tau$ and reference portfolio allows us to write

$$
E \left[ \tilde{U}^{S,j,d}(X_T) \right] = (1 - P(d = 1))E \left[ \tilde{U}^{S,j}(X_T) \right] + P(d = 1)E \left[ U^S(X_T - L_T) \right].
$$

Introducing $\epsilon := P(d = 1)$, the optimization problem (2.8) can be rewritten as

$$
\sup_{X_T \in \mathcal{X}} E \left[ \tilde{U}^{S,j,\epsilon}(X_T) \right],
$$

(2.9)

where

$$
\tilde{U}^{S,j,\epsilon}(x) := (1 - \epsilon)\tilde{U}^{S,j}(x) + \epsilon U^S(x - L_T).
$$

(2.10)

Note that $1 - \epsilon = P(d = 0)$ is the survival probability\(^4\) and $\tilde{U}^{S,j,\epsilon}$ is an $S$-shaped utility function for all $\epsilon \in [0, 1]$. In particular,

$$
\tilde{U}^{S,j,\epsilon}(X_T) :=
\begin{cases}
  -\epsilon j U_{lo}(L_T - X_T) & \text{if } X_T \leq L_T, \\
  U(X_T - L_T) & \text{if } L_T < X_T \leq \tilde{L}_T, \\
  U_\epsilon(X_T) & X_T > \tilde{L}_T,
\end{cases}
$$

(2.10)

where $\epsilon_{np} := 1$ and $\epsilon_p := \epsilon$ and $U_\epsilon(x)$ is defined by

$$
U_\epsilon(x) := (1 - \epsilon)U(f(x)) + \epsilon U(x - L_T).
$$

(2.11)

Hence, $U_\epsilon(x)$ is the utility of the equity holder in case of mortality if $x$ is greater than the bonus threshold. Furthermore, $U_\epsilon(x)$ is a linear combination of $U(f(x))$ and $U(x - L_T)$, weighted by $1 - \epsilon$ and $\epsilon$ respectively. Clearly, $U_\epsilon(x)$ is a strictly increasing and concave function on $[\tilde{L}_T, \infty)$ for all $\epsilon \in [0, 1]$. Its first derivative is given by

$$
U'_\epsilon(x) = (1 - \epsilon)(1 - \delta)U'(f(x)) + \epsilon U'(x - L_T).
$$

(2.12)

Moreover, for any $x \geq \tilde{L}_T$ we have $f(x) \leq x - L_T$, which implies that

$$
U(f(x)) \leq U_\epsilon(x) \leq U(x - L_T), \quad \forall x \geq \tilde{L}_T.
$$

To present the optimal terminal wealth we introduce the inverse marginal utility function $I_\epsilon := [U'_\epsilon]^{-1}$. From (2.11) we observe that for all $y \leq \xi_\epsilon^L = U'_\epsilon(\tilde{L}_T)$ we have

$$
I_\epsilon(y/\tilde{\delta}_\epsilon) + L_T \geq I_\epsilon(y) \geq (1 - \delta)^{-1} \left[ I(y/\tilde{\delta}_\epsilon) - (1 - \delta)L_T \right],
$$

(2.13)

\(^4\)If the policyholder’s age is $x$ then $1 - \epsilon = x_p T$ using the standard actuarial notations.
where $\tilde{\delta}_\epsilon$ is defined by
\[
\tilde{\delta}_\epsilon := (1 - \delta)(1 - \epsilon) + \epsilon. \tag{2.14}
\]
Note that for the case without full protection $\tilde{U}^{S,p,\epsilon}$ coincides with $\tilde{U}^{S,np,\epsilon}$ on the gain domain $[L_T, \infty)$. However, when the portfolio value falls below the liability level $L_T$, the insurer now partially suffers a smaller loss measured by $\epsilon U_{lo}(L_T - X_T)$ due to the presence of the defaultable put. The upper bound of loss is then given by
\[
q_p := \epsilon U_{lo}(L_T) < q_{np} := U_{lo}(L_T). \tag{2.15}
\]

3 Unconstrained problem and concavification

To discuss the solution of the general problem (2.8), we first study the simple case for the payoff $V^p_L$ (given in (2.4)) without mortality, namely
\[
\sup_{X_T \in \mathcal{X}} \mathbb{E}[U(V^p_E(X_T))], \tag{3.1}
\]
where $U$ is a utility function satisfying condition (AE) and the integrability assumption (U). Note that the insurer’s payoff is positive almost surely and hence, the use of an $S$-shaped utility function is not needed. Moreover, as mentioned above, the classical martingale method can not be directly applied for (3.1) because the derived utility function $\tilde{U}^p := U \circ V^p_E$ is neither concave nor differentiable. Motivated by non-linear compensation schemes for a fund manager, the maximizing the utility of terminal wealth for non-concave utility functions has been considered by many authors e.g., [13, 20, 12, 35, 7]. Similar characterisations can be found in [28, 31, 4] where non-linear contract payoffs or changing preferences are considered. Let us briefly summarize the basic ideas of the concavification method which relies on convex analysis.

**Convex conjugate**: Intuitively, the convex conjugate of $\tilde{U}^p$ can be related to a family of upper-half hyperplanes whose intersection equals to the region below the graph of $\tilde{U}^p$. In particular, each member of this family is the smallest affine function of the form $xy + c$ which dominate $\tilde{U}^p$, i.e.,
\[
\tilde{U}^p(x) \leq xy + c, \quad \forall x \in [0, \infty).
\]
Then, for a given slope $y$, the corresponding conjugate of $\tilde{U}^p$ is the smallest constant $c$ being determined by
\[
(\tilde{U}^p)^*(y) := \sup_{x \geq 0} (\tilde{U}^p(x) - xy).
\]
Note that $\tilde{U}^p$ is not differentiable at the utility changing points $L_T$ and $\tilde{L}_T$. Therefore, it is important to first compute the above supremum on each interval $[0, L_T]$, $[L_T, \tilde{L}_T]$ and $(\tilde{L}_T, \infty)$. The convex conjugate $(\tilde{U}^p)^*(y)$ follows from a comparison among these three local maximums. Since $\tilde{U}^p$ is concave on each of these intervals the corresponding supremum defines a convex and decreasing function in $y$. Therefore, $(\tilde{U}^p)^*(y)$ can be seen as a decreasing, convex and differentiable function except on a finite number of points (which are explicitly determined below as the tangency point depending on our parameters). Classical results from convex analysis ensure that the double conjugate $(\tilde{U}^p)^{**}$ is the smallest concave function which dominates $\tilde{U}^p$, and on an interval where the concavification is needed $(\tilde{U}^p)^{**}$ is linear \cite{12, 35}. The optimal terminal wealth is then given by $X_T^* = -((\tilde{U}^p)^{**})'_\Lambda(\lambda\xi_T)$ for some Lagrangian multiplier determined via the budget constraint. For more details, see \cite{13, 20, 12, 35, 7}.

**Remark 2.** We will see later in Section 5 that when a VaR constraint is additionally imposed the optimal solution can not be directly derived from the right derivative of the convex conjugate $(\tilde{U}^p)^*$. As mentioned above, $(\tilde{U}^p)^{**}$ may be linear in some interval which means that the concave hull of the utility function is neither strictly concave nor smooth, and the results in \cite{3} for concave optimization under a VaR constraint cannot be applied.

To obtain explicit solutions, below we use the Lagrangian approach to determine the optimal solution and point out the links to the concavification points of the derived utility function which is determined below.

### 3.1 Concavification

In this section, we determine the concavification of the derived utility function $U_\epsilon(x)$ defined in (2.11). Recall that $U_\epsilon(x)$ is a strictly increasing and concave function on $[\tilde{L}_T, \infty)$ for all $\epsilon \in [0, 1]$ with first derivative $U'_\epsilon(x)$ given by (2.12). Furthermore, let $q \geq 0$ and define

$$\Upsilon^{\epsilon,q}(x) := U_\epsilon(x) - xU'_\epsilon(x) + q. \quad (3.2)$$

The parameter $q$ represents the upper bound of the losses in case of an $S$-shaped utility function and is used to deal with negative wealth. In particular, we set $q = U_{lo}(L_T)$ in the case of full protection and $q = \epsilon U_{lo}(L_T)$ in the case where default put and death probability are considered simultaneously, see Section 4 below.

\footnote{$f'_+\text{ stands for right derivative of } f$}
Concavification of the derived utility function crucially depends on the sign of $\Upsilon^{\epsilon,q}$ at the utility changing point $\tilde{L}_T$. In particular, the concavification area depends on $\Upsilon^{\epsilon,q}(\tilde{L}_T)$ and $\Upsilon^{1,q}(\tilde{L}_T)$, see the Lemma below. From (3.2) we easily observe that
\[
\Upsilon^{\epsilon,q}(\tilde{L}_T) = U(\tilde{L}_T) - \tilde{\delta} U'(\tilde{L}_T) \tilde{L}_T + q \geq U(\tilde{L}_T) - U'(\tilde{L}_T) \tilde{L}_T + q = \Upsilon^{1,q}(\tilde{L}_T),
\]
where $\tilde{\delta}$ is defined by (2.14). The concave hull of the derived utility function is then characterized in the following lemma.

**Lemma 3.1.** Let $\epsilon \in [0,1]$, $q \geq 0$ and $\Upsilon^{\epsilon,q}$ defined as in (3.2). If $\Upsilon^{1,q}(\tilde{L}_T) > 0$ then there exists a positive number $L_T < \tilde{y}^{1,q} < \tilde{L}_T$ satisfying $\Upsilon^{1,q}(\tilde{y}^{1,q}) = 0$, i.e.,
\[
U(\tilde{y}^{1,q} - L_T) - U'(\tilde{y}^{1,q} - L_T)\tilde{y}^{1,q} + q \doteq 0. \tag{3.3}
\]
If $\Upsilon^{\epsilon,q}(\tilde{L}_T) < 0$ then there exists a positive number $\tilde{y}^{\epsilon,q} > \tilde{L}_T$ satisfying $\Upsilon^{\epsilon,q}(\tilde{y}^{\epsilon,q}) = 0$, i.e.,
\[
U_\epsilon(\tilde{y}^{\epsilon,q}) - \tilde{y}^{\epsilon,q} U_\epsilon'(\tilde{y}^{\epsilon,q}) + q \doteq 0. \tag{3.4}
\]

In fact, $\tilde{y}^{1,q}$ is the tangency point of the straight line starting from the point $(0, -q)$ to the curve $U(x - L_T)$ in the first case and $\tilde{y}^{\epsilon,q}$ is the tangency point of the straight line starting from $(0, -q)$ to the curve $U_\epsilon(x)$ in the second case.

**Proof.** We prove the first property. To this end, note that the left hand side of (3.3) is an increasing, continuous function in $\tilde{y}^{1,q}$ due to the concavity of $U$. By Inada’s condition, it takes values in $(-\infty, \Upsilon^{1,q}(\tilde{L}_T)]$ and the conclusion follows from the intermediate value theorem. The second statement can be proved in the same way using the asymptotic elasticity condition of $U$. \(\square\)
Below we solve the optimization problem (3.1) using a Lagrangian approach. The optimal terminal wealth will be expressed as a function of the price density price \( \xi_T \) and a Lagrangian multiplier \( \lambda \) defined via the budget equality.

### 3.2 Unregulated optimal wealth for \( V^p_L \) without mortality

In this section, we ignore mortality risk and assume that the equity holder’s payoff is defined by (2.4). In this context, we take \( q = 0 \) in (3.2) and \( \epsilon = 0 \). Then, the concave hull of the derived utility function \( \widetilde{U} \) intrinsically depends on how to determine the tangent line to the curve \( \tilde{U} \) starting from the origin. In particular, from Lemma 3.1, if \( \Upsilon^{1,0}(\tilde{L}_T) > 0 \) the concave hull is linear in \( [0, \tilde{y}^{1,0}] \) and coincides with the curves \( U(x - L_T) \) and \( U(f(x)) \) in \( [\tilde{y}^{1,0}, \tilde{L}_T] \) and in \( [\tilde{L}_T, \infty) \) respectively. In this case, the utility changing points \( \tilde{y}^{1,0} \) and \( \tilde{L}_T \) play an essential role in the optimal terminal wealth. For the case that \( \Upsilon^{0,0}(\tilde{L}_T) < 0 \), the concave hull is linear in \( [0, \tilde{y}^{0,0}] \) and coincides with the curve \( U(f(x)) \) in \( [\tilde{y}^{0,0}, \infty) \). In this case, only the tangency point \( \tilde{y}^{0,0} \) matters for the optimal terminal wealth.

**Theorem 3.1.** The optimal solution to the unconstrained problem (3.1) is given by

\[
X^p_0 = \begin{cases} 
  h(\xi_T)1_{\xi_T < \xi^0_L} + \tilde{L}_T 1_{\xi^0_L \leq \xi_T < \xi^0_L} + (L_T + I(\lambda \xi_T))1_{\xi^0_L \leq \xi_T < \xi^{1,0}_L} & \text{if } \Upsilon^{1,0}(\tilde{L}_T) > 0, \\
  h(\xi_T)1_{\xi_T < \xi^0_L} + \tilde{L}_T 1_{\xi^0_L \leq \xi_T < \xi^0_U} & \text{if } \Upsilon^{0,0}(\tilde{L}_T) \geq 0 \geq \Upsilon^{1,0}(\tilde{L}_T), \\
  h(\xi_T)1_{\xi_T < \xi^{0,0}_L} & \text{if } \Upsilon^{0,0}(\tilde{L}_T) < 0,
\end{cases}
\]

where

\[
\begin{align*}
\xi^0_L &:= \frac{(1 - \delta)U'(\tilde{L}_T)}{\lambda}, \quad \xi^0_U := \frac{U(\tilde{L}_T)}{L_T \lambda}, \quad \xi^{1,0}_L := \frac{U'(\tilde{y}^{1,0} - L_T)}{\lambda}, \quad \xi^{0,0} := \frac{(1 - \delta)U'(f(\tilde{y}^{0,0}))}{\lambda}.
\end{align*}
\]

The Lagrangian multiplier \( \lambda \) is defined via the budget constraint \( \mathbb{E}[\xi_T X^p_0] = X_0 \).

**Proof.** See Section A.1. \( \square \)

Let us give some comments on Theorem 3.1. First, when \( \Upsilon^{1,0}(\tilde{L}_T) > 0 \) or equivalently, the concavification point of \( \tilde{U} \) lies on the interval \( [L_T, \tilde{L}_T] \), the optimal terminal wealth takes a four-region form determined by the utility changing points \( \tilde{y}^{1,0} \) and \( \tilde{L}_T \). For \( \Upsilon^{0,0}(\tilde{L}_T) < 0 \) or equivalently, for the concavification point being in the interval \( [\tilde{L}_T, \infty) \), the optimal terminal wealth takes a two-region form and is determined by \( \tilde{y}^{0,0} \). When \( \Upsilon^{1,0}(\tilde{L}_T) \leq 0 \leq \Upsilon^{0,0}(\tilde{L}_T) \), concavification is needed on the interval \( [0, \tilde{L}_T] \). The concave hull is then equal to the linear segment connecting zero with \( \tilde{L}_T \) and coincides
with \( U(f(x)) \) on \([\tilde{L}_T, \infty)\). In this case the optimal terminal wealth takes a three-region form. In all cases, the optimal terminal wealth ends up with zero from a certain value of the price density on, i.e., in the worst economic states. This reflects the moral hazard problem that the insurer does not have an incentive to ensure that there is any capital in case the terminal wealth falls below the minimal guarantee. The reason is that any terminal wealth in those loss states only benefits the policyholder and comes at the expense of a lower terminal wealth in the more prosperous states. We will later see that introducing a VaR constraint ameliorates this situation.

**Optimal policy of wealth distribution:** Like the classical concave utility maximization problem, the optimal terminal wealth is given as a function of the state price density \( \xi_T \) at maturity and the Lagrangian multiplier which is determined via the budget equation. In particular, we observe from Theorem 3.1 that for \( \Upsilon^1,0(\tilde{L}_T) > 0 \), the four-region form solution is characterised by wealth levels defined by \([h(\lambda \xi_T), \tilde{L}_T, L_T + I(\lambda \xi_T), 0]\) corresponding to the partition of the terminal market states with boundary points \([\xi^0_L, \xi_L, \hat{\xi}^1, 0] \). In other words, the wealth level at time \( T \) is respectively assigned to \( h(\lambda \xi_T) \) on the sub-interval \((0, \xi^0_L)\), to \( \tilde{L}_T \) on \([\xi^0_L, \xi_L)\), to \( L_T + I(\lambda \xi_T) \) on \([\xi_L, \hat{\xi}^1)\) and finally to zero on \([\hat{\xi}^1, \infty)\). Therefore, it is convenient to represent the terminal wealth in dependence of the state price as

\[
\begin{align*}
\mathbb{W}^{[h, \tilde{L}, I + L, 0]}_{[\xi^0_L, \xi_L, \hat{\xi}^1, 0]} (\lambda, \xi_T) := h(\lambda \xi_T) 1_{\xi_T < \xi^0_L} + \tilde{L}_T 1_{\xi^0_L \leq \xi_T < \xi_L} + (L_T + I(\lambda \xi_T)) 1_{\xi_L \leq \xi_T < \hat{\xi}^1}.
\end{align*}
\]

For simplicity, below we drop the dependence of the terminal wealth on the Lagrangian multiplier and the price density. In the same spirit as above, the optimal wealth for the case \( \Upsilon^1,0(\tilde{L}_T) \leq 0 \leq \Upsilon^0,0(\tilde{L}_T) \) and the case \( \Upsilon^0,0(\tilde{L}_T) < 0 \) can be respectively represented by the three-region and the two-region wealth distributions as

\[
\begin{align*}
\mathbb{W}^{[h, \tilde{L}, I + L]}_{[\xi^0_L, \hat{\xi}^1]} := h(\lambda \xi_T) 1_{\xi_T < \xi^0_L} + \tilde{L}_T 1_{\xi^0_L \leq \xi_T < \xi_L} \quad \text{and} \quad \mathbb{W}^{[h, 0]}_{[\xi^0_L, \hat{\xi}^1]} := h(\lambda \xi_T) 1_{\xi_T < \xi^0_L}.
\end{align*}
\]

These representations of wealth distribution will be used in the rest of the paper for notational convenience.

**Remark 3.** Concavification is not needed for the case where \( U(0) = -\infty \). In this situation, it can be deduced directly from the proof of Theorem 3.1 in Appendix A.1 that the optimal terminal wealth is given by

\[
\begin{align*}
h(\lambda \xi_T) 1_{\xi_T < \xi^0_L} + \tilde{L}_T 1_{\xi^0_L \leq \xi_T < \xi_L} + (L_T + I(\lambda \xi_T)) 1_{\xi_L \leq \xi_T} = \mathbb{W}^{[h, \tilde{L}, I + L]}_{[\xi^0_L, \xi_L]} (\lambda, \xi_T).
\end{align*}
\]
3.3 Full protection and S-shaped utility functions without mortality

We now turn our attention to the case where the policyholder receives a full protection as discussed in Section 2.3 without mortality (i.e., $\epsilon = 0$). Noting that the insurer’s payoff may take negative values, we consider the following unconstrained optimization problem with an S-shaped utility function $U^S$

$$\sup_{X_T \in \mathcal{X}} \mathbb{E} \left[ \tilde{U}^{S, np}(X_T) \right], \quad \text{where} \quad \tilde{U}^{S, np} := U^S \circ V_{E}^{np}. \quad (3.5)$$

We assume that $\tilde{U}^{S, np}$ is convex on $[0, L_T]$ and piecewise concave on each interval $[L_T, \tilde{L}_T]$ and $[\tilde{L}_T, \infty)$. Losses are bounded by $U_{lo}(L_T)$. The concave hull of the derived utility function $\tilde{U}^{S}$ is characterized by Lemma 3.1 which determines the tangent line to the curve $\tilde{U}^{S}$ starting from the $(0, -U_{lo}(L_T))$ with $q = q_l := U_{lo}(L_T)$ in this context.

**Remark 4.** We observe that for any $x \geq \tilde{L}_T$, $\Upsilon^{1, q_l}(x) > \Upsilon^{1, 0}(x)$, which implies that $\hat{y}^{1, q_l} < \hat{y}^{1, 0}$ whenever they both exist. Similarly, $\hat{y}^{0, q_l} < \hat{y}^{0, 0}$ whenever they both exist. This means that the concavification area for the full protection case using an S-shaped utility function will be reduced in comparison to the case of a defaultable contract, assuming that the insurer uses the same concave utility function $U$ for the gain part.

Having discussed the concave hull of the S-shaped utility function we are in a position to present the optimal solution to problem (3.5).

**Theorem 3.2.** The optimal solution to the S-shaped utility unconstrained problem (3.5) is given by

$$X_T^{np,*} = \left[ \frac{[h, \tilde{L}_T + L_T]}{[\xi^0_{U}, \xi^{1, q_l}]} \right] 1_{\Upsilon^{1, q_l}(\tilde{L}_T) > 0} + \left[ \frac{[h, \tilde{L}_T]}{[\xi^0_{U}, \xi^{1, 0}]} \right] 1_{\Upsilon^{1, 0} (\tilde{L}_T) \geq \Upsilon^{1, q_l}(\tilde{L}_T)} + \left[ \frac{[h, 0]}{[\xi^{0, q_l}]} \right] 1_{\Upsilon^{0, q_l}(\tilde{L}_T) < 0},$$

where

$$\xi^0_{U} := \frac{U(\tilde{L}_T) + U_{lo}(L_T)}{L_T \lambda}, \quad \xi^{1, q_l} := \frac{U'(\hat{y}^{1, q_l} - L_T)}{\lambda} \quad \text{and} \quad \xi^{0, q_l} := (1 - \tilde{\delta}) \frac{U'(f(\hat{y}^{0, q_l})))}{\lambda}.$$ 

The Lagrangian multiplier $\lambda$ is defined via the budget constraint $\mathbb{E}[\xi_T X_T^{np,*}] = X_0$.

**Proof.** See Appendix A.1 \[\square\]

We observe from Theorem 3.2 that the optimal terminal wealth takes an all-or-nothing form. Hence, giving a full protection to the policyholder, the insurance company
needs to take the possibility of having negative terminal wealth into account. In line with descriptive decision theory, we assume a convex utility function for the loss area \([0, L_T]\). As numerically shown in Figure 6, the insurer is induced to take a more prudent investment strategy. Mathematically, this follows from \(\hat{y}^{1,0} < \hat{y}^{1,1}\) and \(\hat{y}^{0,0} < \hat{y}^{0,1}\) provided they exist, see Remark 4. Now, using the budget constraint we conclude that the Lagrangian multiplier of the \(S\)-shaped utility problem is greater than the one in the case with a default put. This means that an \(S\)-shaped utility function leads to a shift-to-the-right effect on the optimal terminal wealth.

**Remark 5.** We observe that for \(\alpha = 0\) (i.e., the policyholder has no initial contribution) with a defaultable contract, the problem becomes simpler with a call-option-form payoff \((X_T - L_T)^+\). When \(\alpha = 1\) (i.e., the policyholder fully contributes to the investment pool), the insurer’s payoff also takes a call-option-form payoff \((1 - \delta)(X_T - L_T)^+\). As numerically illustrated in Figure 4, these cases lead to a riskier investment, see more in [13]. For a fully protected contract it can also be observed that the insurer’s payoff is given by \(X_T - L_T\) if \(\alpha = 0\) and \((1 - \delta)(X_T - L_T)\) for \(X_T > L_T\) and \(X_T - L_T\) for \(X_T < L_T\) when \(\alpha = 1\). Note that these situations are not interesting in practice as the contract would in each case not be acceptable to one of the two respective sides.

### 3.4 The optimal strategy

We have determined the optimal terminal wealth as a function of \(\xi_T\) and a multiplier \(\lambda\) that satisfies the budget constraint. For the reader’s convenience, we briefly discuss how to deduce the optimal strategy by applying the martingale representation theorem. Let \(Z = X(\lambda, \xi_T)\) be the optimal terminal wealth and define \(Z_t := E[\xi_T Z | F_t]\). Then, the process \((\xi_t Z_t)_{t \in [0,T]}\) is a martingale under \(\mathbb{P}\). Therefore, by the martingale representation theorem, there exists a square integrable adapted process \((\varsigma_t)\) such that

\[
\xi_t Z_t = X_0 + \int_0^t \varsigma_s dW_s, \quad t \in [0, T],
\]

or equivalently, \(d(\xi_t Z_t) = \varsigma_t dW_t\). On the other hand, applying Itô’s lemma we get

\[
dZ_t = d(\xi_t Z_t \xi_t^{-1}) = \xi_t^{-1} d(\xi_t Z_t) + \xi_t Z_t d\xi_t^{-1} + d(\xi_t Z_t) d\xi_t^{-1} = (\xi_t^{-1} \varsigma_t + Z_t \theta_t) dW_t + ((r_t + \theta_t^2)Z_t + \xi_t^{-1} \varsigma_t \theta_t) dt.
\]

Identifying the last equation with (2.3) we deduce that the optimal strategy \(\pi^*_t\) is given by

\[
\pi^*_t = \sigma_t^{-1} \xi_t^{-1} \varsigma_t + \sigma_t^{-1} Z_t \theta_t, \quad X^*_t = Z_t.
\]

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Note that \( \pi_t^* = \sigma_t^{-1} \xi_{t+1} - \sigma_t^{-1} Z_t \theta_t \) is square integrable and is hence admissible.

The same argument can be applied for the constrained problems below. More explicit representations can be provided for power and logarithmic utility functions.

### 4 Optimal wealth with mortality risk

We now consider Problem (2.8) with mortality, i.e., \( \epsilon > 0 \). Recall that for both payoffs, the derived utility function \( \tilde{U}_{S,j,\epsilon} \), defined in (2.10), takes an \( S \)-shaped form. The concave hull of the derived utility function \( \tilde{U}_{S,j,\epsilon}(x) \), related to the existence of a tangency line starting from \((0, -q_j)\), is now characterized by Lemma 3.1 with \( q = q_j, j \in \{p, np\} \) being defined in (2.15). The optimal terminal wealth of problem (2.9) is characterised by the following theorem.

**Theorem 4.1.** For any \( \epsilon \in [0, 1) \), the optimal terminal wealth of Problem (2.9) is given by

\[
X_{T}^{j,\epsilon,\ast} = \left[ I_{\epsilon}, L T + L, 0 \right]_{[\hat{\xi}^{1,q_{j}}]} 1_{Y^{1,q_{j}}(\tilde{L}_T) > 0} + \left[ I_{\epsilon}, L, 0 \right]_{[\hat{\xi}^{q_{j}}]} 1_{Y^{q_{j}}(\tilde{L}_T) \geq Y^{1,q_{j}}(\tilde{L}_T)} + \left[ I_{\epsilon}, 0 \right]_{[\hat{\xi}^{q_{j}}]} 1_{Y^{q_{j}}(\tilde{L}_T) < 0}, \quad j \in \{p, np\},
\]

where

\[
\begin{align*}
\hat{\xi}^{q_{j}} &:= \frac{U'(\tilde{L}_T)}{L T \lambda}, \\
\hat{\xi}^{q_{j}} &:= \frac{U'(\tilde{L}_T)}{\lambda}, \\
\hat{\xi}^{1,q_{j}} &:= \frac{U(\tilde{L}_T) + q \tilde{L}_T}{L T \lambda} \quad \text{and} \quad \hat{\xi}^{1,q_{j}} := \frac{U'(\tilde{L}_T)}{\lambda}.
\end{align*}
\]

The Lagrangian multiplier \( \lambda \) is defined via the budget constraint.

**Proof.** See Appendix A.1

We remark that the no-mortality case \( \epsilon = 0 \) is already given in Theorem 3.2. Let us now consider the case where \( \epsilon = 1 \), i.e., the policyholder dies almost surely before the maturity of the contract. In this case, assuming that the insurer uses an \( S \)-shaped utility function

\[
U^{S,0}(X_T) := \begin{cases} 
-U_{\omega}(L_T - X_T) & \text{if } X_T \leq L_T, \\
U(X_T - L_T) & \text{if } L_T < X_T,
\end{cases}
\]

we obtain the following optimal solution.

**Corollary 4.1.** Suppose that \( \epsilon = 1 \). The optimal terminal wealth (with and without default put) is given by \( W_{[\hat{\xi}^{1,q_{np}}]}^{[L_T + I, 0]} \), where \( \lambda \) satisfies the budget constraint.
5 Optimal investment under regulations

5.1 The VaR-constrained problem

As seen in the previous section, under the absence of regulation the equity holder will optimally choose a strategy which may lead to insolvency at maturity. In this case, the policyholder suffers severe losses since the terminal portfolio value may be zero for very bad market scenarios. In practice, an appropriate investment must take some regulatory constraints into account. According to Solvency II, the insurance company needs to ensure that a VaR-type regulation (i.e., a default probability constraint) shall be satisfied. In addition, the insurance company is interested in achieving at least a target payment to serve the promised guaranteed amount to the policyholder. Therefore, the equity holder (the insurance company) has to choose an optimal dynamic portfolio under a VaR constraint, which can be stated as

\[
\sup_{X_T \in \mathcal{X}} E \left[ \bar{U}^{S,j\epsilon}(X_T) \right], \quad \text{s.t.} \quad P(X_T < L_T) < \beta, \quad j \in \{p, np\},
\]

for some probability default level \(0 \leq \beta < 1\). It is clear that the introduction of a probabilistic constraint complicates the non-concave problem. In the spirit of the Lagrange method, it is natural to consider the following auxiliary utility function

\[
\bar{U}^{S,j\epsilon}_{\lambda_2}(x) := \bar{U}^{S,j\epsilon}(x) - \lambda_2 1_{x<L_T},
\]

where \(\lambda_2 \geq 0\) is a multiplier. Problem (5.1) boils down to maximize \(E[\bar{U}^{S,j\epsilon}_{\lambda_2}(X_T)]\) over the set \(\mathcal{X}\). However, it is not clear at first sight how to link the optimality of the auxiliary utility function \(\bar{U}^{S,j\epsilon}_{\lambda_2}\) with the optimality of the initial utility function \(\bar{U}\) under the VaR constraint as \(\bar{U}^{S,j\epsilon}_{\lambda_2}\) depends on \(\lambda_2\), and different \(\lambda_2\) will lead to different optimal solutions. However, a good choice for \(\lambda_2\) should reflect the bindingness of VaR constraint. Below we show that this goal can be achieved using a special form of \(\lambda_2\) which follows from a careful comparison of the local maximizers in the unconstrained problem.

**Theorem 5.1.** Let \(\beta \in (0,1)\) and define \(\bar{\xi}\) so that \(P(\xi_T > \bar{\xi}) = \beta\). Suppose that \(X_0 \geq E[\xi_T L_T 1_{\xi_T \leq \bar{\xi}}]\). Then, the VaR-constrained problem (5.1), \(j \in \{p, np\}\), admits the following optimal solution:

- If \(Y_1^{L,j}(\tilde{L}_T) > 0\) then

\[
X_T^{VaR,j\epsilon,*} = \bigcup_{[\xi_{L}, \xi_{\bar{L},\xi}]} 1_{\xi \geq \xi_{1,j}} + \bigcup_{[\xi_{L}, \xi_{\tilde{L},\xi}]} 1_{\xi < \xi_{1,j}}.
\]
• If $\Upsilon^{\epsilon,q_j}(\tilde{L}_T) \geq 0 \geq \Upsilon^{1,q_j}(\tilde{L}_T)$ then

$$X_T^{Var,j,\epsilon,*} = \left[ \frac{|I_\tilde{L},\tilde{L}+L,0|}{|\xi^*_L,\xi^*_L,\xi^*_L|} \right] \mathbf{1}_{\tilde{\xi} \geq \xi^*_L} + \left[ \frac{|I_\tilde{L},\tilde{L}|}{|\xi^*_L,\xi^*_L|} \right] \mathbf{1}_{\tilde{\xi} \leq \xi < \xi^*_L} + \left[ \frac{|I_\tilde{L},\tilde{L}|}{|\xi^*_L,\tilde{\xi}^*_j|} \right] \mathbf{1}_{\xi < \tilde{\xi}^*_j}.$$

• If $\Upsilon^{\epsilon,q_j}(\tilde{L}_T) < 0$ then

$$X_T^{Var,j,\epsilon,*} = \left[ \frac{|I_\tilde{L},\tilde{L}+L,0|}{|\xi^*_L,\xi^*_L,\xi^*_L|} \right] \mathbf{1}_{\tilde{\xi} \geq \xi^*_L} + \left[ \frac{|I_\tilde{L},\tilde{L}|}{|\xi^*_L,\xi^*_L|} \right] \mathbf{1}_{\xi \leq \xi < \xi^*_L} + \left[ \frac{|I_\tilde{L}|}{|\xi^*_L,\tilde{\xi}^*_j|} \right] \mathbf{1}_{\xi^*_j \leq \xi < \tilde{\xi}^*_j} + \left[ \frac{|I_\tilde{L}|}{|\xi^*_L,\tilde{\xi}^*_j|} \right] \mathbf{1}_{\xi < \tilde{\xi}^*_j}.$$

In each case, the Lagrangian multiplier $\lambda$ is defined via the budget constraint.

Proof. See Section A.3.

We observe first that when $\tilde{\xi} < \hat{\xi}^{1,q_j}/\hat{\xi} < \tilde{\xi}^{q_j}/\hat{\xi} < \hat{\xi}^{\epsilon,q_j}$ in the first/second/third case, the VaR constrained is not binding and the corresponding unconstrained solution $X_T^{L,\epsilon,*}$ given by (4.1) is still optimal for (5.1). On the other hand, when the VaR constraint is binding (active) the terminal wealth will be (partially) shifted to the right of the concavification point and we can observe that for $j \in \{n, np\}$, there exists $\xi^*_j > 0$ such that

$$X_T^{Var,j,\epsilon,*}(\xi_T) \geq X_T^{L,\epsilon,*}(\xi_T) \quad \text{for all} \quad \xi_T \geq \xi^*_j,$$

meaning that the VaR-terminal wealth dominates the unconstrained terminal wealth for the most negative loss states (due to the fact that $X_T^{Var,j,\epsilon,*}(\xi_T)$ and $X_T^{L,\epsilon,*}(\xi_T)$ are decreasing in $\xi_T$). Hence, introducing a VaR-constraint, forces the equity holders to enlarge the proportion of hedged loss states, leading to a genuine improvement for the policyholders. This result is contrary to the situation where the insurer maximizes the utility of the total wealth of the company without distinguishing between equity holders and policyholders, in which case a VaR constraint may induce the insurer to take excessive risk leading to higher losses than in the case of no-regulation, see [3]. However, there is still a region of market scenarios in which the optimal terminal wealth equals zero, which means that a VaR regulation does not lead to a full prevention of moral hazard. The intuitive reason is that under a VaR regulation, the equity holder is only required to keep the portfolio value above $L_T$ with a given probability $1 - \beta$. Once the regulation is probabilistically fulfilled the equity holder can push the remaining risk into the tail to seek a higher potential wealth level in good market states. This is consistent with the classical VaR-constrained asset allocation problem with concave utility functions [3]. However, in our case with participating contracts, the use of a VaR constraint does not lead the insurance company to bigger losses than in the case of no
regulations as in the classical VaR problem. On the contrary, our results show that for an insurance company with equity holders, a VaR constraint strictly improves the risk management for the loss states.

5.2 The PI-constrained problem

In this section, we try to better protect the policyholder from the equity holder’s gambling investment strategies by, instead of having a VaR constraint, assuming that the equity holder has to keep the portfolio value almost surely above some given level minimum capital requirement \( l \). Hence the insurance company needs to solve

\[
\sup_{X_T \in \mathcal{X}} \mathbb{E} \left[ \tilde{U}^{S,j,\epsilon}(X_T) \right], \quad \text{s.t.} \quad X_T \geq l \quad \text{a.s.,} \quad j \in \{p, np\}.
\]  

(5.2)

For simplicity we assume that \( l \) is deterministic.

Let us first consider the case \( 0 \leq l \leq \tilde{L}_T \). As before, concavification is characterized by the following generalized version of Lemma 3.1.

**Lemma 5.1** (Tangency point for PI-problem). Let \( \epsilon \in [0, 1] \), \( q \geq 0 \) and \( \Upsilon^{\epsilon,q}_l \) be defined by

\[
\Upsilon^{\epsilon,q}_l(x) := U_\epsilon(x) - U'_\epsilon(x)(x - l) + q.
\]

If \( \Upsilon^{\epsilon,q}_l(\tilde{L}_T) > 0 \) then there exists a positive number \( L_T < \hat{y}^{\epsilon,q}_l < \tilde{L}_T \) satisfying \( \Upsilon^{\epsilon,q}_l(\hat{y}^{\epsilon,q}_l) = 0 \), i.e.,

\[
U(\hat{y}^{\epsilon,q}_l - L_T) - U'(\hat{y}^{\epsilon,q}_l - L_T)(\hat{y}^{\epsilon,q}_l - l) + q = 0.
\]

If \( \Upsilon^{\epsilon,q}_l(\tilde{L}_T) < 0 \) then there exists a positive number \( \hat{y}^{\epsilon,q}_l > \tilde{L}_T \) satisfying \( \Upsilon^{\epsilon,q}_l(\hat{y}^{\epsilon,q}_l) = 0 \), i.e.,

\[
U_\epsilon(\hat{y}^{\epsilon,q}_l) - U'_\epsilon(\hat{y}^{\epsilon,q}_l)(\hat{y}^{\epsilon,q}_l - l) + q = 0.
\]

As before, \( \hat{y}^{\epsilon,q}_l \) is the tangency point of the straight line starting from the point \((l, -q)\) to the curve \( U(x - L_T) \) in the first case, and \( \tilde{y}^{\epsilon,q}_l \) is the tangency point of the straight line starting from \((l, -q)\) to the curve \( U_\epsilon(x) \) in the second case.

We remark that to hedge against the minimum capital level \( l \), the investor must start with at least \( \mathbb{E}[\xi_T l] \). Next, we show in Theorem 5.2 that the optimal terminal wealth under the PI constraint takes a similar form as in the unconstrained case but with bounded losses. Gambling investment behavior is prevented thanks to the additional guarantee \( l \).
Theorem 5.2. Assume that 0 ≤ l < L_T and \( X_0 \geq E[\xi_T l] = le^{-rT} \). Then, the optimal solution to the insurance portfolio problem (5.2) is given by

\[
X_T^{P_I,j,\epsilon,\tau} = \begin{bmatrix}
I_{[\epsilon L, L]} & 1_{Y_i^{\epsilon,q_j}(\tilde{L}) > 0} + I_{\tilde{L}, L]
I_{[\epsilon L, L]} & 1_{Y_i^{\epsilon,q_j}(\tilde{L}) \geq 0} + I_{\tilde{L}, L]}
\end{bmatrix}
\]

where, as in (4.1),

\[
\tilde{\xi}_i^{\epsilon,q} = U'(\tilde{y}_i^{\epsilon,q})/\lambda \quad \text{and} \quad \xi_{U,l}^{q} := \frac{U(\tilde{L}_T)}{(L - l)\lambda}.
\]

The Lagrangian multiplier \( \lambda \) is defined via the budget constraint \( E[\xi_T X_T^{P_I,j,\epsilon,\tau}] = X_0 \).

Proof. See Section A.2.

In this theorem the moral hazard problem shown in Theorem 3.1 is resolved by introducing a minimal bound. However, this comes at the expense of lowering significantly the wealth in the prosperous states and is thus rather costly, see the numerical analyzes below.

Let us now turn to the case where the minimum amount lies between the utility changing points \( L_T \leq l \leq \tilde{L}_T \) (i.e., between the guarantee and the bonus threshold). In this case, the derived utility function \( \tilde{U}_{S,j,\epsilon} \) is strictly increasing and globally concave in the considered domain \([l, \infty)\). However, \( \tilde{U}_{S,j,\epsilon} \) is not smooth at \( \tilde{L}_T \) since \( \lim_{x \rightarrow \tilde{L}_T} U'(x) = ((1 - \epsilon)(1 - \tilde{\delta}) + \epsilon)U''(\tilde{L}_T) \neq \lim_{x \rightarrow \tilde{L}_T} U'(x) = U'(\tilde{L}_T) \), which makes the classical utility maximization result inapplicable. We remark that concavification is not needed because we have global concavity in the optimization domain. The case \( l \geq \tilde{L}_T \) is just the classical portfolio insurance problem with concave utility function \( U_\epsilon \) and can be dealt with similarly. The following result can be directly obtained using the same Lagrangian technique.

Proposition 5.1. Assume that \( X_0 \geq E[\xi_T l] = le^{-rT} \). Then, for \( L_T < l < \tilde{L}_T \), the optimal solution to the insurance portfolio problem (5.2) is given by

\[
\begin{bmatrix}
I_{[\epsilon L, L]} & 1_{\xi_T < \xi_L} + \tilde{L}_T 1_{\xi_L \leq \xi_T < \xi_L} + (L_T + I(\lambda T))(1_{\xi_L \leq \xi_T < \xi_L} + 1_{\xi_T \geq \xi_L})
I_{[\epsilon L, L]} & 1_{\xi_T < \xi_L} + \tilde{L}_T 1_{\xi_L \leq \xi_T < \xi_L} + (L_T + I(\lambda T))(1_{\xi_L \leq \xi_T < \xi_L}
\end{bmatrix}
\]

where \( \xi_l := U'(l - L_T)/\lambda \). When \( l = L_T \) the optimal terminal wealth is

\[
\begin{bmatrix}
I_{[\epsilon L, L]} & 1_{\xi_T < \xi_L} + \tilde{L}_T 1_{\xi_L \leq \xi_T < \xi_L} + (L_T + I(\lambda T))(1_{\xi_L \leq \xi_T < \xi_L}
\end{bmatrix}
\]
Figure 3: Unconstrained optimal terminal wealth with $\delta = 0.6$.

which can be seen the limiting case by sending $\xi_l^T$ in (5.4) to infinity. For $\bar{L}_T \leq l$, the optimal terminal wealth is given by $W_{[l_\xi]} = I_\epsilon(l)\xi_T - \xi^l \epsilon - \lambda \xi_T^l$, where $\xi^l := U'(l)/\lambda$. The Lagrangian multiplier $\lambda$ satisfies the budget constraint.

6 Numerical examples

We assume that the equity holder’s utility function is given by $U(x) := x^{1-\gamma}/(1-\gamma)$ with $\gamma = 0.5$. For the full protection case, we assume that $U_{lo}(x) = \eta U(-x)$. The market coefficients are $\mu = 0.05; r = 0.03; \sigma = 0.3$. The contract has a maturity $T = 10$ and the total contribution (i.e., the initial capital) is fixed with $X_0 = 100$. The guarantee is given by $L_T = 50e^{gT}$, where the guarantee interest rate is $g = 0.02$. Below we look at the investment behavior of the reference portfolio decided by the insurer in order to maximize his expected terminal utility with and without regulations. Note that constant proportion portfolio insurance (CPPI) strategies can be considered as a possible benchmark for comparative analysis, see [29]. We emphasize that the CPPI strategies are not optimal and provide less expected utility for the insurer. For an analytic discussion, it may be useful to compare the insurer’s strategy with the Merton strategy which maximizes the insurer’s utility of the total wealth of the company without
distinguishing between equity and policyholders starting with the same total initial endowment.

Figure 4: Defaultable contract: effect of $\alpha$ and $\delta$ on the unconstrained optimal strategies at $t = 8$.

Figure 3 plots the optimal terminal portfolio with different values of $\alpha$ for the case with a defaultable contract and with a fully protected contract. It can be observed that the optimal terminal wealth can take a two-, three- or four-region form as also shown in Theorem 3.1. The first thing to note from the graph is that in both cases (with a defaultable or a fully protected contract) there is no monotonicity in $\alpha$. Instead, in particular for defaultable contracts, the insurer is most risk seeking for very small and very large values of $\alpha$ which can be seen by the fact that these values in good states (i.e., for small values of $\xi_T$) yield a relatively high terminal wealth while in bad states the terminal wealth is lower, and the unhedged region where the terminal wealth is zero increases. The reason might be that as will be seen below a very high participation rate ($\alpha \approx 1$) means that the insurer contributes very limitedly in the investment pool which he tries to make up by taking on more risks in order to mimic a Merton like terminal wealth in the good states for himself. On the other hand, rather small values of $\alpha$ have a positive effect on the wealth of the insurer by decreasing his payout obligation. This wealth effect induces the insurer to become less risk averse, explaining why also low $\alpha$’s may come with relatively risky investments. Finally, we remark that for fully protected contracts the difference between the optimal terminal wealth for different $\alpha$ is minor in good states. Hence, full protection leads to a harmonization in the gain states of the investment of companies for different contribution levels of the policyholder.
Figure 4 depicts the effect of the participating ratio $\alpha$ and the bonus rate $\delta$ on the relationship between the optimal strategy and the wealth level at time $t = 8$, two years before maturity. As revealed in the figure, the optimal amount invested in the risky asset is always non-negative and exhibits a peak-valley structure. The reason for the peak is that if the wealth becomes low the equity holder is left with nothing or even with negative wealth and will try push the wealth back “into the money” by investing heavily into the risky asset. Once the wealth is above the guarantee level the investment behavior normalizes, i.e., becomes more like the Merton benchmark case. Consistent with what has been observed in Figure 3, the riskiness increases if the policyholder contributes very little or very much to the contract. The effect of the bonus rate $\delta$ in the optimal strategy is presented in the right panel of Figure 4, showing that for a given participation rate the risky investment decreases for low terminal wealth and increases for a larger terminal wealth if the contract provides the policyholder a higher bonus rate $\delta$. Again this is due to two effects: the first effect is that with higher bonus rates the contract becomes less valuable for the equity holder, implicitly lowering his wealth and shifting the curve to the left. The second effect is that in order to mimic the Merton strategy for his personal account the insurer needs to actually increase the riskiness of his position when higher bonus rates go to the policyholder. The graph shows that the first effect outweighs the second effect in bad economic scenarios while the second effect is stronger in good economic scenarios where the strategy overall is less sensitive to shifts in wealth.

The effects of the participation rate $\alpha$, the bonus rate $\delta$ and the loss degree $\eta$ at $t = 8$ for fully protected contracts are illustrated in Figure 5. It can be observed from the left panel that in line what has been discussed in Figure 4 given a fixed bonus rate and a loss degree, changing $\alpha$ does not lead to monotone changes in the optimal strategy. In particular, high and low participation rates may go hand in hand with relatively risky investments in the loss states. Furthermore, the middle panel shows that given a participation rate, an increase in $\delta$ for fully protected contracts does not lead to a significant change in the exposure to risk in bad market scenarios. Hence, in these scenarios the investment of the insurer is rather robust with respect to the bonus rate and investments seem more motivated by the desire to avoid losses. Moreover, the higher $\delta$, the closer the strategy to the Merton strategy in case of good performance.

The right panel also shows that increasing the loss aversion $\eta$ leads the insurer to more prudent investment strategies in case of bad performance. However, the risky investment is almost the same in case of good performance. To explain this, we remark that an increase in $\eta$ only leads to higher losses while the gain part is not influenced.
the case of a fully protected contract, the insurer cares more about losses than gains due to the convexity of the $S$-shaped utility function in the loss domain which economically means that losses hurt more than gains.

Figure 5: Fully protected contract: effect of the participation rate $\alpha$, the bonus rate $\delta$ and the loss degree $\eta$ in the unconstrained strategy at $t = 8$

Figure 6 shows that potential losses of a fully protected contract lead to a more prudent behavior compared to the case of a defaultable contract with the same bonus and participation rates. This numerically confirms the observation in Remark 4.

Let us now compare the unconstrained investment strategy with the constrained one under a VaR or PI constraint. To serve this aim, the minimum guarantee in the PI problem and the default probability are chosen as $l = 0.2L_T$ and $\beta = 0.025$. The comparison of different strategies is presented in Figure 7 from which we can observe that the exposure to risk when the wealth process is small (the market condition gets worse) will be reduced by a VaR constraint. In particular in loss states, Solvency 2 VaR-type constraints for participating insurance contracts lead to more prudent investment than investments which are not regulated (i.e., come from unconstrained optimization problems). This result is contrary to the situation where the insurer maximizes the utility of the total wealth of the company (without distinguishing between equity holder and policyholders), in which case a VaR constraint may induce the insurer to take excessive risk leading to higher losses than in the case of no regulation, see [3].

These
effects are even more significant if a PI regulation is used. We also obtain similar effects on the optimal strategies for the full protection case.

7 Conclusion

We solve a utility maximization problem for the equity holder of a participating life insurance contract under a VaR-type regulatory constraint imposed at maturity. We obtain a closed-form solution extending the martingale approach to constrained non-concave utility maximization problems. Our theoretical and numerical results show that for the case of a defaultable contract, the risk exposure in bad market conditions will be reduced by a VaR constraint. The prudent investment behavior is more significant if a VaR-type regulation is replaced by a PI-type regulation. The effects of the parameters of the contract are also discussed numerically. One interesting extension is to see how a fair pricing constraint influences the investment behavior of the insurance company. Another future research direction would be to consider contracts with more general mortality models.
Figure 7: Comparing different optimal strategies and optimal terminal wealth for defaultable contract with $\delta = 0.6$ and $\alpha = 0.4$.

A Proofs

A.1 Proof of Theorem 4.1 (the unconstrained case)

This section is devoted to the detailed proof for the unconstrained problem (2.8). This problem can be solved by considering the static optimization $\max_{X \geq 0} \Psi(X)$ of the Lagrangian

$$\Psi_j(X) = \tilde{U}^{S,j,\epsilon}(X) - \lambda \xi_T X.$$  \hfill (A.1)

Below we drop the subscript $T$ for simplicity. Notice that $\Psi_j$ is not concave, which makes the problem more challenging to solve. We make use of a modified dual approach which shows that the global maximal value of $\Psi_j$ can be attained at the local maximizers or at the utility changing points $L$ and $\tilde{L}$, or even at the boundary point 0. Note that $\Psi_j$ is not differentiable at $L_T$ and $\tilde{L}_T$. Therefore, it is important to first consider the above supremum on each interval $[0, L]$, $[L, \tilde{L}]$, and $[\tilde{L}, \infty)$. Then the convex conjugate $\Psi_j^*$ can be obtained from a comparison among these three local maximums. Since $\Psi_j$ is concave on each of these intervals the corresponding supremum defines a convex and decreasing function in $\lambda \xi$. Therefore, $\Psi_j^*(\lambda \xi)$ can be seen as a decreasing, convex, differentiable function except at a finite number of points (which are explicitly determined below as the tangency points depending on our parameters). Convex analysis ensures that the double conjugate $\Psi_j^{**}$ is the smallest concave function which dominates $\Psi_j$, and that on an interval where the concavification is needed $\Psi_j^{**}$ is linear. Under smoothness
condition, the optimal terminal wealth of the problem \( \sup_X E[\Psi_j^*(X)] \) is given by the inverse of marginal utility \((\Psi_j^*)^{-1}(\lambda \xi)\), see e.g. [25]. In our setting with piecewise sharing profit structures, \(\Psi_j^*\) is differentiable except for a finite set consisting of the utility changing points \(L, \bar{L}\). For this reason, the solution can be determined by a Lagrangian method. In particular, we will show that the optimal wealth is given by the budget constraint and we will explicitly calculate it. More discussions on concavification methods can be found e.g. in [13, 20, 12, 35, 7].

**Lemma A.1.** For \(\lambda > 0\) and \(\xi > 0\), the unique solution of the problem \(\max_{X \geq 0} \Psi(X)\), for \(j \in \{p, np\}\), is given by

\[
X^{j \ast} (\lambda, \xi) = \left\{ \begin{array}{ll}
[I, \tilde{L}, I + L, 0] & \text{if } \xi > \xi_L,
[\xi_L, \xi_L, \xi^1 \eta_j] & \text{if } \xi \leq \xi_L
\end{array} \right.
\]

\[
\tilde{1}^{\ast, \eta_j}(L_T) > 0 + \left\{ \begin{array}{ll}
[I, \tilde{L}, 0] & \text{if } \xi > \xi_L,
[\xi_L, \xi_L, \xi^1 \eta_j] & \text{if } \xi \leq \xi_L
\end{array} \right.
\]

\[
1^{\ast, \eta_j}(L_T) > 0 \geq \tilde{1}^{\ast, \eta_j}(L_T) + \left\{ \begin{array}{ll}
[I, 0] & \text{if } \xi > \xi_L,
[\xi_L, \xi_L, \xi^1 \eta_j] & \text{if } \xi \leq \xi_L
\end{array} \right.
\]

**Proof.** Note first that we are looking for the global maximizer \(X^\ast\) of a three-part function \(\Psi_j(X) = Q_1(X)1_{X < L} + Q_2(X)1_{L \leq X \leq \bar{L}} + Q_3(X)1_{X > \bar{L}}\), for \(j \in \{p, np\}\) where

\[
Q_1(X) := -\epsilon_j U(L - X) - \lambda \xi X, \quad Q_2(X) := U(X - L) - \lambda \xi X, \quad Q_3(X) := U_e(X) - \lambda \xi X.
\]

Clearly, \(Q_1\) is strictly linearly decreasing whereas \(Q_2, Q_3\) are concave functions having local maximizers \(X_2 := I(\lambda \xi) + L\) and \(X_3 := I_e(\lambda \xi)\) respectively. Below we denote \(Q_{2, \max} := \max_{L \leq X \leq \bar{L}} Q_2(X)\) and \(Q_{3, \max} := \max_{X > \bar{L}} Q_3(X)\) and note that \(Q_{1, \max} = \max_{X < L} Q_1(X) = -\epsilon_j U(L) = -q_j\). The conclusion follows from a suitable comparison between the local optimizer \(X_2, X_3\) with \(X_1 = 0\). From (4.1) we observe first that \(X_2 \in [L, \bar{L}]\) iff \(\xi \geq \xi_L\) and \(X_3 > \bar{L}\) iff \(\xi < \xi^\epsilon_L\). Therefore, we study the Lagrangian on the following subintervals of values of \(\xi\):

(a) For \(\xi < \xi^\epsilon_L\), we have \(X_2 \geq \bar{L}\) and \(X_3 > \bar{L}\) so the Lagrangian \(\Psi_j\) is decreasing from 0 to \(L\) and increasing from \(L\) to \(X_3\) and decreasing again in \((X_3, \infty)\). So global optimality can be attained at 0 or \(X_3\). To conclude, we consider

\[
Q_{3, \max} - Q_{1, \max} = U_e(I_e(\lambda \xi)) - \lambda \xi I_e(\lambda \xi) + q_j := \Delta_a(\xi),
\]

which is a decreasing function in \(\xi\). It follows that \(Q_{3, \max} - Q_{1, \max} \geq \Delta_a(\xi^\epsilon_L) = \tilde{1}^{\ast, \eta_j}(\bar{L}_T)\). If \(\hat{1}^{\ast, \eta_j}(\bar{L}_T) > 0\) then \(Q_{3, \max} - Q_{1, \max} > 0\) and the global maximizer is \(X_3\). When \(\hat{1}^{\ast, \eta_j}(\bar{L}_T) \leq 0\), by Lemma 3.1 there exists \(\hat{\xi}^{\ast, q_j} \geq \xi_L^\epsilon\) such that \(\Delta_a(\hat{\xi}^{\ast, q_j}) = 0\). In this case, the global maximizer can be chosen as \(X_3\) for \(\xi < \hat{\xi}^{\ast, q_j}\), or zero for \(\xi \in (\hat{\xi}^{\ast, q_j}, \xi_L^\epsilon)\).
(b) For $\xi^c_1 \leq \xi < \xi^*_L$, $X_2 \geq \tilde{L}$ but $X_3 \leq \tilde{L}$ so the Lagrangian is decreasing from 0 to $L$ and increasing from $L$ to $\tilde{L}$ and decreasing again in $(\tilde{L}, \infty)$. So global optimality can be attained at 0 or $\tilde{L}$. To conclude, we consider $Q_{2,\max} - Q_{1,\max} = Q_2(\tilde{L}) + q_j = U(\tilde{L} - L) - \lambda \xi \tilde{L} + q_j := \Delta_b(\xi)$, which is a decreasing function in $\xi$. It follows that $\Delta_b(\xi) \in \{Y^1_q, Y^{\infty}_q(\tilde{L})\}$. If $Y^1_q(\tilde{L}_T) > 0$ then $\Delta_b(\xi) > 0$ and the global maximizer is $\tilde{L}$. When $\psi^\infty_q(\tilde{L}_T) \leq 0$, the global maximizer is zero. It remains to consider the case $Y^1_q(\tilde{L}_T) \leq 0 \leq \psi^z_q(\tilde{L}_T)$. By Lemma 3.1, $\Delta_b(\xi^0) = 0$ and $\xi^c_1 \leq \xi^0 \leq \xi^*_L$. This implies that the global maximizer is $\tilde{L}$ for $\xi^c_1 \leq \xi < \xi^*_U$ or is equal to zero for $\xi^0 \leq \xi < \xi^*_L$.

(c) For $\xi \geq \xi^*_L$, $X_2 \leq \tilde{L}$ and $X_3 < \tilde{L}$ so the Lagrangian is decreasing from 0 to $L$ and increasing from $L$ to $X_2$ and decreasing again in $(X_2, \infty)$. So global optimality can be attained at 0 or $X_2$. We need to study $Q_{2,\max} - Q_{1,\max} = Q_2(X_2) - q_j = U(I(\lambda \xi)) - \lambda \xi(I(\lambda \xi) + L) + q_j := \Delta_c(\xi)$, which is a decreasing function in $\xi$. Thus, $\Delta_c(\xi) \leq \Delta_c(\xi^*_L) = Y^1_q(\tilde{L}_T)$. Therefore, zero is the maximizer if $Y^1_q(\tilde{L}_T) \leq 0$. Suppose now that $Y^1_q(\tilde{L}_T) > 0$. By Lemma 3.1, there exists $\hat{\xi}^1_q > \xi^*_L$ such that $\Delta_c(\hat{\xi}^1_q) = 0$ and the global optimality is attained at $X_2$ for $\hat{\xi}^1_q > \xi \geq \xi^*_L$ and at zero for $\xi \geq \hat{\xi}^1_q$, respectively.

Note that $\Delta_a(\hat{\xi}^1_q) = 0$ is equivalent to (3.4) with $\hat{g}^\infty_q = I_1(\lambda \hat{\xi}^1_q)$, whereas $\Delta_c(\hat{\xi}^1_q) = 0$ is equivalent to (3.3) with $\hat{g}^1_q = I(\lambda \hat{\xi}^1_q) + L$. \qed

Remark 6. From the analysis above it follows that $X^{\infty,\star}(\lambda, \xi) = -(\psi^\infty_q)'(\lambda \xi)$.

We now prove Theorem 4.1 and then Theorem 3.1. Let us prove that $X^{\infty,\star}(\lambda, \xi_T)$ defined by (3.1) is the solution to (3.1). For any terminal wealth $X_T \geq 0$ satisfying the budget constraint we have

$$E[\hat{U}^{S,\infty}(X_T)] \leq E[\hat{U}^{S,\infty}(X_T) + \lambda(X_0 - \xi_T X_T)]$$

$$\leq E\left[\sup_{X \geq 0} (\hat{U}^{S,\infty}(X) - \lambda \xi_T X)\right] + X_0 \lambda$$

$$= E[\hat{U}^{S,\infty}(X^{\infty,\star}(\lambda, \xi_T))] + \lambda(X_0 - E[\xi_T X^{\infty,\star}(\lambda, \xi_T)])$$

$$= E[\hat{U}^{S,\infty}(X^{\infty,\star}(\lambda, \xi_T))] + \lambda(X_0 - E[\xi_T X^{\infty,\star}(\lambda, \xi_T)])$$

The last equality follows from $E[\xi_T X^{\infty,\star}(\lambda, \xi_T)] = X_0$. Hence, $X^{\infty,\star}$ is optimal. The existence of the Lagrangian multiplier $\lambda$ follows from the lemma below.

Lemma A.2. For any $\lambda \in (0, \infty)$, we have
1. \( \mathbb{E}[U(I(\lambda \xi_T))] < \infty \implies \mathbb{E}[\tilde{U}^S_{j,\varepsilon}(X_{j,\varepsilon}^*(\lambda, \xi_T))] < \infty; \)

2. \( \mathbb{E}[\xi_T I(\lambda \xi_T)] < \infty \implies \mathbb{E}[\xi_T X_{j,\varepsilon}^*(\lambda, \xi_T)] < \infty; \)

3. Furthermore, if \( \mathbb{E}[\xi_T I(\lambda \xi_T)] < \infty \) for any \( \lambda \in (0, \infty) \), the mapping \( \psi : \lambda \mapsto \mathbb{E}[\xi_T X_{j,\varepsilon}^*(\lambda, \xi_T)] \) is strictly decreasing, continuous and surjective from \( (0, \infty) \) to \( (0, \infty) \).

**Proof.** The first two conclusions follow from the observation that

\[
X_{j,\varepsilon}^*(\lambda, \xi_T) \leq \tilde{L}_T + (1 - \tilde{\delta})^{-1} I(\tilde{\lambda} \xi_T), \quad \text{with} \quad \tilde{\lambda} := [(1 - \tilde{\delta})(1 - \varepsilon) + \varepsilon]^{-1} \lambda,
\]

using (2.13). For the last one, it suffices to notice that \( I \) is strictly decreasing, so for a.s. all \( \omega \in \Omega \), \( \lambda \mapsto X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) \) is strictly decreasing, which implies that \( \psi \) is also decreasing. Now, if \( \mathbb{E}[\xi_T I(\lambda \xi_T)] < \infty \) then \( \psi \) is well defined. Noting that the price density \( \xi_T \) has no atom, (i.e., for all \( a \in \mathbb{R} \), \( \mathbb{P}(\xi_T = a) = 0 \)), we deduce that \( \psi \) is continuous on \( (0, \infty) \). Moreover, for a.s. all \( \omega \) we have \( \lim_{\lambda \to 0} X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) = \infty \) and \( \lim_{\lambda \to \infty} X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) = 0 \) due to the Inada condition and the monotone convergence theorem, which implies that \( \psi \) is surjective. 

### A.2 Proof of Theorem 5.2 (the PI-constraint case)

The proof follows from a modification of Lemma [A.1]. In fact, due to the PI constraint, the problem boils down to consider the global optimality on the interval \([l, \infty)\) of \( \Psi_j(X) \) defined in (A.1).

**Lemma A.3.** For \( \lambda > 0 \) and \( \xi > 0 \), the problem \( \max_{X \geq l} \Psi(X) \), for \( j \in \{p, np\} \), has the solution

\[
X_{j,\varepsilon}^p I(\lambda, \xi_T) \leq \tilde{L}_T + (1 - \tilde{\delta})^{-1} I(\tilde{\lambda} \xi_T), \quad \text{with} \quad \tilde{\lambda} := [(1 - \tilde{\delta})(1 - \varepsilon) + \varepsilon]^{-1} \lambda,
\]

using (2.13). For the last one, it suffices to notice that \( I \) is strictly decreasing, so for a.s. all \( \omega \in \Omega \), \( \lambda \mapsto X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) \) is strictly decreasing, which implies that \( \psi \) is also decreasing. Now, if \( \mathbb{E}[\xi_T I(\lambda \xi_T)] < \infty \) then \( \psi \) is well defined. Noting that the price density \( \xi_T \) has no atom, (i.e., for all \( a \in \mathbb{R} \), \( \mathbb{P}(\xi_T = a) = 0 \)), we deduce that \( \psi \) is continuous on \( (0, \infty) \). Moreover, for a.s. all \( \omega \) we have \( \lim_{\lambda \to 0} X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) = \infty \) and \( \lim_{\lambda \to \infty} X_{j,\varepsilon}^*(\lambda, \xi_T(\omega)) = 0 \) due to the Inada condition and the monotone convergence theorem, which implies that \( \psi \) is surjective.

**Proof.** Global optimality on \([l, \infty)\) of \( \Psi_j \) can be attained at \( X_1 = l \), \( X_2 = L + I(\lambda \xi) \), \( X_3 = h(\lambda \xi) \) or at the utility changing point \( \tilde{L} \). In particular, we need to consider the optimality of \( Q_1(X) \) on the interval \([l, L]\). Thus, \( Q_{1,max} = -q_j - \lambda \xi l \) but \( Q_{2,max} \) and \( Q_{3,max} \) remain unchanged. As a result, the analysis needs to be modified when we compare the local maximizers \( X_2 = L + I(\lambda \xi) \), \( X_3 = I(\lambda \xi) \) and \( \tilde{L} \) to \( X_1 = l \). For example, for the comparison between \( X_3 \) and \( X_1 = l \), we study the difference

\[
Q_{3,max} - Q_{1,max} = \Delta_a(\xi) + \lambda \xi l = U(f(h(\lambda \xi))) - \lambda \xi h(\lambda \xi) + \lambda \xi l + q_j := \Delta_a^*(\xi).
\]

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In the same way, we also have $\Delta^l_s(\xi) := \Delta_a(\xi) + \lambda\xi l$ and $\Delta^l_c(\xi) := \Delta_c(\xi) + \lambda\xi l$. Note that

$$
\Upsilon^j_{1,qj}(\tilde{L}) = \Delta^l_s(\xi^j_{L}) \quad \text{and} \quad \Upsilon^j_{1,qj}(\tilde{L}) = \Delta^l_c(\xi^j_{L}).
$$

(\text{A.3})

Furthermore, $\hat{\xi}^1_j, q_j$ and $\xi^j_U$ (in case they exist) are zero points of $\Delta^l_s$, $\Delta^l_a$ and $\Delta^l_b$ respectively.

The remainder of the proof follows the same lines in Lemma A.1.

\[\square\]

\textbf{A.3 Proof of Theorem 5.1 (the VaR-constrained case)}

To study the constrained problem \[[5.1]\], we again consider the static optimization of the Lagrangian

$$
\max_{X \succeq 0} \Psi_j(X) := U^{S,j_{\epsilon}}(X) - \lambda\xi X - \lambda_2 1_{X < L},
$$

(\text{A.4})

for given $\lambda > 0, \lambda_2 \geq 0$. Again, the subscript $T$ is dropped for simplicity. Note first that $\Psi_j$ is not concave and Problem \[[A.4]\] is more challenging than the unconstrained case due to the presence of the additional indicator function. Below, we show that the optimality can be obtained within a modified dual approach.

The key point is to choose the second multiplier $\lambda_2$ as a function of the first one. This is because we need to check the “bindingness” of the VaR constraint by comparing the unconstrained solution with the VaR-threshold $L_T$. Let $\bar{\xi}$ be defined by $P(\xi_T > \bar{\xi}) = \beta$. From Theorem 3.1 we observe that

$$
P(X^{j_{\epsilon},\epsilon'} < L_T) = \begin{cases} 
P(\xi_T > \hat{\xi}^1_j) & \text{if } \Upsilon^{1,qj}(\tilde{L}_T) > 0, \\
P(\xi_T > \hat{\xi}^c_j) & \text{if } \Upsilon^{c,qj}(\tilde{L}_T) \geq 0 \geq \Upsilon^{1,qj}(\tilde{L}_T), \\
P(\xi_T > \hat{\xi}^b_j) & \text{if } \Upsilon^{c,qj}(\tilde{L}_T) < 0, 
\end{cases}
$$

(\text{A.5})

for $j \in \{p, np\}$. Therefore, to incorporate the bindingness of the VaR constraint, $\lambda_2$ should be chosen such that it is zero when the above probability is smaller than the insolvency level $\beta$. For example, if $\Upsilon^{1,qj}(\tilde{L}_T) > 0$ then $\lambda_2 > 0$ if $\hat{\xi}^{1,qj} < \bar{\xi}$ and $\lambda_2 = 0$ if $\hat{\xi}^{1,qj} \geq \bar{\xi}$. The other cases can be seen similarly. On the other hand, $\lambda_2$ needs to reflect the corresponding comparison of local maxima in the proof of Lemma A.1. Then, a closer inspection suggests defining

$$
\lambda_2 = \begin{cases} 
-\Delta_c(\bar{\xi})1_{\hat{\xi}^{1,qj} < \bar{\xi}}, & \text{if } \Upsilon^{1,qj}(\tilde{L}_T) > 0, \\
-\Delta_b(\bar{\xi})1_{\hat{\xi}^{c,qj} < \xi_L} - \Delta_c(\bar{\xi})1_{\xi_L < \hat{\xi}^{c,qj}}, & \text{if } \Upsilon^{c,qj}(\tilde{L}_T) \geq 0 \geq \Upsilon^{1,qj}(\tilde{L}_T), \\
-\Delta_a(\bar{\xi})1_{\hat{\xi}^{1,qj} < \xi_L} - \Delta_b(\bar{\xi})1_{\xi_L < \hat{\xi}^{c,qj}} - \Delta_c(\bar{\xi})1_{\xi_L < \hat{\xi}^{1,qj}}, & \text{if } \Upsilon^{c,qj}(\tilde{L}_T) < 0.
\end{cases}
$$

(\text{A.6})
Lemma A.4. For \( \lambda > 0 \) and \( \xi > 0 \), the unique solution \( X^{V_{aR,j,\epsilon,\lambda}}(\lambda, \xi) := \arg\max_{X \geq 0} \Psi_j(X) \) is given by:

- If \( \Upsilon^{a,q_j}(\tilde{L}_T) > 0 \) then
  \[
  X^{V_{aR,j,\epsilon,\lambda}}_T = \bigcup_{[\xi^L_j, \xi_L]} \xi \geq \xi^L_j \bigcup_{[\xi^L_j, \xi^U_j]} \xi < \xi^L_j.
  \]

- If \( \Upsilon^{c,q_j}(\tilde{L}_T) \geq 0 \geq \Upsilon^{a,q_j}(\tilde{L}_T) \) then
  \[
  X^{V_{aR,j,\epsilon,\lambda}}_T = \bigcup_{[\xi^L_j, \xi^U_j]} \xi \geq \xi^L_j \bigcup_{[\xi^L_j, \xi^U_j]} \xi < \xi^L_j.
  \]

- If \( \Upsilon^{c,q_j}(\tilde{L}_T) < 0 \) then
  \[
  X^{V_{aR,j,\epsilon,\lambda}}_T = \bigcup_{[\xi^L_j, \xi^U_j]} \xi \geq \xi^L_j \bigcup_{[\xi^L_j, \xi^U_j]} \xi < \xi^L_j.
  \]

Proof. We follow the arguments in the proof of Lemma A.1. The only difference is that \( Q_1(X) := -\epsilon_j U(L - X) - \lambda_2 \), where \( \lambda_2 \) is defined by (A.6). Therefore, \( Q_{1,\max} = -q_j - \lambda_2 \). Note that \( Q_2 \) and \( Q_3 \) remain the same as before. As in (A.3)

\[
\Upsilon^{a,q_j}(\tilde{L}_T) = \Delta_b(\xi^L_j) = \Delta_a(\xi^L_j) \quad \text{and} \quad \Upsilon^{c,q_j}(\tilde{L}_T) = \Delta_b(\xi^L_j) = \Delta_a(\xi^L_j).
\]

We consider the following cases:

**Case 1:** If \( \Upsilon^{a,q_j}(\tilde{L}_T) > 0 \) then \( \lambda_2 = -\Delta_a(\xi^L_j) \). It suffices to suppose that \( \hat{\xi}^1,q_j < \bar{\xi} \), i.e., to consider the case where the VaR constraint is binding.

(a) For \( \xi < \xi^L_j \), we consider \( \Delta := Q_{3,\max} - Q_{1,\max} = \Delta_a(\xi) + \lambda_2 \geq \Delta_a(\xi^L_j) + \lambda_2 = \Upsilon^{c,q_j}(\tilde{L}_T) + \lambda_2 > 0 \). So \( X_3 \) is the global optimizer of the constrained problem for \( \xi < \xi^L_j \).

(b) For \( \xi^L_j \leq \xi < \xi_L \), global optimality can be attained at 0 or \( \tilde{L} \). In this case we have \( \Delta = Q_{2,\max} - Q_{1,\max} = \Delta_a(\xi) + \lambda_2 \). Because \( \Upsilon^{a,q_j}(\tilde{L}_T) > 0 \) we have \( \Delta_a(\xi) > 0 \) as in the proof of Lemma A.1 (b) and the global maximizer is \( \tilde{L} \).

(c) For \( \xi \geq \xi_L \), global optimality can be attained at 0 or \( X_2 \). We need to study \( \Delta = Q_{2,\max} - Q_{1,\max} = \Delta_a(\xi) + \lambda_2 = \Delta_a(\xi) - \Delta_a(\xi) \). Recall that \( \Delta_a \) is decreasing. Therefore, \( \Delta \geq 0 \) if \( \xi_L \leq \xi < \xi \) and the global optimality is attained at \( X_2 \). Similarly, \( \Delta < 0 \) for \( \xi \geq \xi_L \), which means that zero is the optimizer.
Case 2: Suppose $\Upsilon^{a, q_i}(L_T) \geq 0 \geq \Upsilon^{1, q_i}(L_T)$. For $\xi < \xi^c_L$, $\Delta = Q_{3, \text{max}} - Q_{1, \text{max}} = \Delta_a(\xi) + \lambda_2 \geq \Upsilon^{a, q_i}(L_T) + \lambda_2 > 0$ because $\Upsilon^{a, q_i}(L_T) > 0$ and $\lambda_2 > 0$ are strictly positive as in the first case. Hence, $X_3$ is the optimizer.

Assume next that $\xi^e_L \leq \xi < \xi^c_L$. Global optimality can be attained at 0 or $\tilde{L}$.

- Assume that $\xi^{q_i}_U \leq \tilde{\xi} < \xi^c_L$ then $\lambda_2 = -\Delta_b(\tilde{\xi})$. In this case we have $\Delta = Q_{2, \text{max}} - Q_{1, \text{max}} = \Delta_b(\tilde{\xi}) - \Delta_b(\xi)$. As $\Delta_b(\xi)$ decreases in $\xi$ we conclude that $\Delta > 0$ for $\xi^e_L \leq \xi < \tilde{\xi}$, which means $\tilde{L}$ is the global maximizer. For $\tilde{\xi} \leq \xi < \xi^c_L$, $\Delta < 0$ and optimality is attained at zero.

- If $\xi \geq \xi^c_L$ then $\lambda_2 = -\Delta_c(\tilde{\xi})$. From (A.7), we have $\Delta = Q_{2, \text{max}} - Q_{1, \text{max}} = \Delta_b(\tilde{\xi}) - \Delta_c(\tilde{\xi}) \geq \Delta_b(\xi) - \Delta_c(\xi^c_L) = \Delta_b(\xi) - \Delta_b(\xi^c_L) > 0$ since $\xi^c_L \leq \xi < \xi^c_L$, which means that optimality is attained at $\tilde{L}$ for $\xi^e_L \leq \xi < \xi^c_L$.

It remains to consider the case $\xi \geq \xi^c_L$. As before, global optimality can be attained at 0 or $X_2$. We need to study $\Delta = Q_{2, \text{max}} - Q_{1, \text{max}} = \Delta_c(\xi) + \lambda_2$.

- If $\xi_U \leq \tilde{\xi} < \xi^c_L$ then $\lambda_2 = -\Delta_b(\tilde{\xi})$ which implies that $\Delta = \Delta_c(\xi) - \Delta_b(\tilde{\xi})$. Observe that $\Delta_b(\tilde{\xi}) \geq \Delta_b(\xi^c_L) = \Delta_c(\xi^c_L)$ since $\xi^c_L < \xi^c_L$. This leads to $\Delta \leq \Delta_c(\xi) - \Delta_c(\xi^c_L)$. Because $\Delta_c$ is decreasing we conclude that $\Delta \leq 0$ for $\xi \geq \xi^c_L$, which means that zero is the optimizer.

- If $\xi \geq \xi^c_L$ then $\lambda_2 = -\Delta_c(\tilde{\xi})$ and $\Delta = \Delta_c(\xi) - \Delta_c(\tilde{\xi})$. As $\Delta$ is decreasing, it is positive for $\xi^c_L \leq \xi < \tilde{\xi}$, hence, $X_2$ is the optimizer, and negative for $\xi \geq \tilde{\xi}$, which means that optimality is attained at zero.

Case 3: Suppose $\Upsilon^{a, q_i}(L_T) < 0$. Let us first consider the case $\xi < \xi^c_L$. Again, global optimality can be attained at 0 or $X_3$. To conclude, we consider $\Delta = Q_{3, \text{max}} - Q_{1, \text{max}} = \Delta_a(\xi) + \lambda_2$.

- If $\xi^{q_i}_U \leq \tilde{\xi} < \xi^c_L$ then $\lambda_2 = -\Delta_a(\tilde{\xi})$ and hence $\Delta = \Delta_a(\xi) - \Delta_a(\tilde{\xi})$. So the global optimizer is $X_3$ for $\xi < \tilde{\xi}$ and is zero for $\tilde{\xi} \leq \xi < \xi^c_L$.

- If $\xi^c_L \leq \tilde{\xi} < \xi^c_L$ then $\lambda_2 = -\Delta_b(\tilde{\xi})$ and hence $\Delta = \Delta_a(\xi) - \Delta_b(\tilde{\xi}) \geq \Delta_a(\xi) - \Delta_a(\xi^c_L) \geq 0$, which implies that $X_3$ is the global optimizer.

- If $\tilde{\xi} \geq \xi^c_L$ then $\lambda_2 = -\Delta_a(\tilde{\xi})$ and hence $\Delta = \Delta_a(\xi) - \Delta_a(\tilde{\xi}) \geq \Delta_a(\xi^c_L) - \Delta_a(\xi^c_L) = \Delta_b(\xi^c_L) - \Delta_b(\xi^c_L) \geq 0$, which implies that $X_3$ is the global optimizer.

Assume now that $\xi^c_L \leq \xi < \xi^c_L$. The global optimality can be attained at 0 or $\tilde{L}$. In this case we have $\Delta = Q_{2, \text{max}} - Q_{1, \text{max}} = \Delta_b(\xi) + \lambda_2$.  

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• If $\xi_{L}^{a,\gamma} \leq \xi < \xi_{L}^{\delta}$ then $\lambda_{2} = -\Delta_{a}(\xi)$ and hence $\Delta = \Delta_{b}(\xi) - \Delta_{a}(\xi_{L}^{\delta}) = \Delta_{b}(\xi) - \Delta_{a}(\xi_{L}^{a,\gamma}) < 0$. Thus the global optimizer is zero for $\xi < \xi_{L}^{\delta} \leq \xi < \xi_{L}$.

• If $\xi_{L}^{\delta} \leq \xi < \xi_{L}^{\gamma}$ then $\lambda_{2} = -\Delta_{a}(\xi)$ and hence $\Delta = \Delta_{a}(\xi) - \Delta_{a}(\xi_{L}^{\gamma}) = \Delta_{a}(\xi_{L}^{\delta}) - \Delta_{a}(\xi_{L}^{\gamma}) \leq 0$. So the global optimizer is zero.

• If $\xi_{L}^{\gamma} \leq \xi < \xi_{L}$ then $\lambda_{2} = -\Delta_{c}(\xi)$ and hence $\Delta = \Delta_{c}(\xi) - \Delta_{b}(\xi_{L}) = \Delta_{c}(\xi_{L}) - \Delta_{b}(\xi_{L}) \leq 0$, which implies that $0$ is the global optimizer.

• If $\xi \geq \xi_{L}$ then $\lambda_{2} = -\Delta_{c}(\xi)$ and hence $\Delta = \Delta_{c}(\xi) - \Delta_{c}(\xi_{L}) \geq \Delta_{c}(\xi_{L}) - \Delta_{c}(\xi) = \Delta_{b}(\xi_{L}) - \Delta_{b}(\xi_{L}) \geq 0$, which implies that $0$ is the global optimizer.

Thus, the lemma is proved. \hfill \square

Using Lemma A.4 it is straightforward to show that $X^{VaR,j,\epsilon,*}(\lambda, \xi)$ is an optimal solution to constrained problem (5.1), see the arguments in (A.2). The existence of the Lagrangian multiplier $\lambda$ can be seen from Lemma A.2 \hfill \square

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