\textbf{B}_n\text{-generalized geometry and } G^2_2\text{-structures} \\

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\textbf{Abstract} \\
We introduce the concept of \(G^2_2\)-structure on an orientable 3-manifold \(M\) using the setting of generalized geometry of type \(B_n\), study their local deformation by making use of a Moser-type argument, and give a description of the cone of \(G^2_2\)-structures.

\section{Introduction} \\
Generalized geometry was originally introduced in [Hit03] as, naively, the differential geometry resulting from replacing the tangent bundle \(T\) of a manifold \(M\) with the sum of the tangent and cotangent bundles, \(T \oplus T^*\), which is naturally endowed with an \(\text{SO}(n,n)\)-structure. Classical concepts have then generalized analogues, such as generalized metrics and generalized Calabi-Yau or generalized complex structures. An interesting feature of this geometry is that the bundle of differential forms \(\bigwedge^* T^* M\) becomes a bundle of spinors, in which some of these structures are formulated. For instance, a generalized Calabi-Yau structure is given by a closed section of \(\bigwedge^{ev} T^* M \otimes \mathbb{C}\) or \(\bigwedge^{od} T^* M \otimes \mathbb{C}\) consisting of pure spinors.

The generalized tangent space \(T \oplus T^*\) can be further modified by adding new pieces. The simplest one is the rank 1 trivial bundle over \(M\), which we denote by 1. Since the natural metric of \(T \oplus T^* \oplus 1\) has signature \((n+1,n)\), the group of symmetries becomes \(\text{SO}(n+1,n)\). As this group is of Lie type \(B_n\), we call this geometry generalized geometry of type \(B_n\), from now on \(B_n\)-geometry. Correspondingly, ordinary generalized geometry is called \(D_n\)-geometry. Exceptional geometries based on the split real forms \(E_n\) have also been studied as, for example, in [Hul07].

\(B_n\)-geometry was originally introduced by Baraglia in [Bar12] (Section 2.4). It also arises as a particular case of the more general situation studied
Section 2 of the present work is devoted to stating the basic features of $B_n$-geometry.

In Section 3 we introduce $G_2^3$-structures on an orientable 3-manifold $M$ as suggested by Baraglia. $G_2^3$-structures are defined by analogy with generalized Calabi-Yau structures. They are given by a closed section $\rho$ of $\bigwedge^\bullet T^*$ consisting of non-pure spinors. We consider the existence and equivalence of $G_2^3$-structures on compact orientable 3-manifolds. While $G_2^3$-structures with non-vanishing degree 0 component, $\rho_0 \neq 0$, exist on any 3-manifold, those with $\rho_0 = 0$ only exist on orientable mapping tori. In fact any mapping torus of an orientable surface can be endowed with such a $G_2^3$-structure (Theorem 3.6). In Section 3.2 we show that the Moser argument in symplectic geometry can be modified to obtain the result that a small deformation within the cohomology class does not change the structure up to generalized diffeomorphism (Theorem 3.12). We finish by describing the cone of $G_2^3$-structures inside $H^\bullet(M)$ in Theorem 3.13.

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2 $B_n$-generalized geometry

2.1 The Courant algebroid $T \oplus T^* \oplus 1$

Let $M$ be a differentiable manifold of dimension $n$ with tangent bundle $T$ and cotangent bundle $T^*$. Let 1 denote the trivial bundle of rank 1 over $M$. Define the $B_n$-generalized tangent bundle by $T \oplus T^* \oplus 1$. The sections of this bundle are called generalized vector fields and are naturally endowed with a signature $(n+1, n)$ inner product given by

$$(X + \xi + \lambda, Y + \eta + \mu) = \frac{1}{2}(i_X\eta + i_Y\xi) + \lambda\mu,$$

where $X + \xi + \lambda, Y + \eta + \mu \in C^\infty(T \oplus T^* \oplus 1)$. Together with the canonical orientation on $T \oplus T^* \oplus 1$, this endows $T \oplus T^* \oplus 1$ with the structure of an $SO(n+1, n)$-bundle. We introduce a Courant bracket on $C^\infty(T \oplus T^* \oplus 1)$ via

$$[X + \xi + \lambda, Y + \eta + \mu] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) + \mu d\lambda - \lambda d\mu + (i_Xd\mu - i_Yd\lambda),$$

so that $(T \oplus T^* \oplus 1, (\cdot, \cdot), [\cdot])$ is a Courant algebroid in the sense of [LWX97].
The infinitesimal orthogonal transformations of $T \oplus T^* \oplus 1$ are given by the elements

$$
\begin{pmatrix}
E & \beta & -2\alpha \\
B & -E^t & -2A \\
A & \alpha & 0
\end{pmatrix} \in C^\infty(\mathfrak{so}(T \oplus T^* \oplus 1))
$$

such that $E \in \text{End}(T)$, $\beta \in \bigwedge^2 T$, $B \in \bigwedge^2 T^*$, the $B$-field already present in $D_n$-geometry, $\alpha \in T$ and $A \in T^*$, the $A$-field which will be relevant in $B_n$-geometry. The exponentiation of a $B + A$-field gives the element

$$(B, A) := \exp(B + A) = \begin{pmatrix} 1 & 1 & -2A \\ B - A \otimes A & 1 & -2A \\ A & 1 \end{pmatrix} \in C^\infty(\text{SO}(T \oplus T^* \oplus 1)),
$$

which acts by $(B, A) (X + \xi + \lambda) = X + \xi + i_X B - 2\lambda A - i_X A A + \lambda + i_X A$.

The composition law of these elements in $C^\infty(\text{SO}(T \oplus T^* \oplus 1))$ is

$$(B, A)(B', A') = (B + B' + A \wedge A', A + A').$$

Note that $A$-fields do not commute and their product involves a 2-form.

Their action on the Courant bracket is given by the following result.

**Proposition 2.1.** Let $(B, A) \in C^\infty(\text{SO}(T \oplus T^* \oplus 1))$. For generalized vector fields $u = X + \xi + \lambda$ and $v = Y + \eta + \mu$, we have

$$
[(B, A)u, (B, A)v] = (B, A)[u, v] + i_Y i_X (dB + A \wedge dA) - 2i_Y i_X dA \cdot A + i_Y i_X dA + 2(\lambda i_Y dA - \mu i_X dA).
$$

In particular, the action of $(B, A)$ commutes with the Courant bracket if and only if $A$ and $B$ are closed.

Define the group

$$\Omega^2_{cl} (M) = \{(B, A) \in C^\infty(\text{SO}(T \oplus T^* \oplus 1)) \mid B \in \Omega^2_{cl} (M), A \in \Omega^1_{cl} (M)\}.$$

The group $\Omega^2_{cl} (M)$ is a central subgroup in $\Omega^2_{cl} (M)$, so $\Omega^2_{cl} (M)$ is the central extension $1 \to \Omega^2_{cl} (M) \to \Omega^2_{cl} + 1 (M) \to \Omega^1_{cl} (M) \to 1$.

**Proposition 2.2.** The group of orthogonal transformations of $T \oplus T^* \oplus 1$ preserving the Courant bracket is $\text{Diff} (M) \ltimes \Omega^2_{cl} + 1 (M) =: \text{GDiff} (M)$, called the group of generalized diffeomorphisms of $M$. The product is given by

$$(f \ltimes (B, A)) \circ (g \ltimes (D, C)) = fg \ltimes (g^* B, g^* A)(D, C)$$

$$= fg \ltimes (g^* B + D + g^* A \wedge C, g^* A + C).$$
We describe $\mathfrak{gDiff}(M)$, the Lie algebra of GDif$(M)$, by differentiating the action of a smooth one-parameter family of generalized diffeomorphisms $F_t = f_t \times (B_t, A_t)$ such that $F_t \circ F_s = F_{t+s}$ and $F_0 = \text{id}$. By Proposition 2.2 and $F_t \circ F_s = F_{t+s}$ we have the three equations

$$f_{t+s} = f_t \circ f_s, \quad A_{t+s} = A_s + f_s^* A_t, \quad B_{t+s} = B_s + f_s^* B_t + f_s^* A_t \wedge A_s.$$ 

The first equation says that $\{f_t\}$ is a one-parameter subgroup of diffeomorphisms of $M$. Let $X$ be the corresponding vector field. From the second equation, $A_t = \int_0^t f_s^* a \, ds$, where $a = \frac{dB_t}{dt} \big|_{t=0}$. And from the third equation,

$$\frac{dB_t}{dt} \big|_{t=s} = f_s^* \frac{dB_t}{dt} \big|_{t=0} + f_s^* \frac{dA_t}{dt} \big|_{t=0} \wedge A_s,$$

so $B_t = \int_0^t (f_s^* b + f_s^* a \wedge A_s) \, ds$, where $b = \frac{dB_t}{dt} \big|_{t=0}$ and $A_s$ depends on $a$.

Using the convention $L_X Y = -\frac{d}{dt} \big|_{t=0} f_t^* Y$ for the Lie derivative of a vector field $Y$, we see that the infinitesimal action of the one-parameter subgroup $\{F_t\}$ is

$$-\frac{d}{dt} \bigg|_{t=0} F_{ts}(Y + \eta + \mu) = L_X (Y + \eta + \mu) - i_Y b + 2\mu a - i_Y a,$$

which only depends on the action of $(X, b, a)$. We thus make the identification

$$\mathfrak{gDiff}(M) = \mathcal{C}^\infty(T) \oplus \Omega^1_{cl}(M) \oplus \Omega^1_{cl}(M).$$

Conversely, given an infinitesimal generalized diffeomorphism $(X, b, a)$, we can integrate it to a one-parameter subgroup of generalized diffeomorphisms using the equations above.

**Remark 2.3.** It is also possible to integrate a time-dependent infinitesimal generalized diffeomorphism. From $(X_t, b_t, a_t)$, we get $B_t = \int_0^t (f_s^* b_s + f_s^* a_s \wedge A_s) \, ds$ and $A_t = \int_0^t f_s^* a_s \, ds$, using a method analogous to that used to show Proposition 2.3 in [Gua11].

**Remark 2.4.** We map $\mathcal{C}^\infty(T \oplus T^* \oplus 1)$ to $\mathfrak{gDiff}(M)$ by

$$(X + \xi + \lambda) \mapsto (X, d\xi, d\lambda),$$

so that we regard $X + \xi + \lambda$ as defining an infinitesimal generalized diffeomorphism. Its natural action on sections of $T \oplus T^* \oplus 1$ gives an action of a generalized vector field on generalized vector fields, called the Dorfman product

$$(X + \xi + \lambda)(Y + \eta + \mu) = [X, Y] + L_X \eta + i_X \mu - i_Y \xi + 2\mu d\lambda - i_Y d\lambda.$$

The antisymmetrization of the Dorfman product gives the Courant bracket defined above.
2.2 Differential forms as spinors

By analogy with $D_n$-generalized geometry, the differential forms $\bigwedge^n T^*M$ are a Clifford module over the algebra $C^\infty(Cl(T \oplus T^* \oplus 1))$ with an action defined by

$$(X + \xi + \lambda) \cdot \varphi = i_X \varphi + \xi \wedge \varphi + \lambda \tau \varphi,$$

where $\tau \varphi = \varphi^+ - \varphi^-$ for the even $\varphi^+$ and odd $\varphi^-$ parts of $\varphi$. Thus, $\tau$ defines an involution of $\bigwedge^n T^*M$. The action defined above satisfies the Clifford condition

$$(X + \xi + \lambda)^2 \cdot \varphi = (X + \xi + \lambda)^2 \varphi,$$

as $\tau$ anticommutes with interior and exterior products.

The action of $B$ and $A$ fields, $B, A \in C^\infty(so(T \oplus T^* \oplus 1))$, on $\bigwedge^n T^*M$ via the spinorial representation $\sigma : C^\infty(Spin(T \oplus T^* \oplus 1)) \to \text{Aut}(\bigwedge^n T^*M)$ is given by the Lie algebra action $\sigma(B) \varphi = -B \wedge \varphi$, $\sigma(A) \varphi = -A \wedge \tau \varphi$, and the Lie group action

$$\sigma(\exp(B)) \varphi = -B \wedge \varphi,$$

$$\sigma(\exp(A)) \varphi = -A \wedge \tau \varphi.$$

Since $B$ and $A$ commute, the action of a $B + A$-field is given by

$$\sigma(\exp(B + A)) \varphi = e^{-B} e^{-A \tau} \varphi = e^{-A \tau} e^{-B} \varphi.$$

The Lie derivative of a spinor with respect to a generalized vector field $X + \xi + \lambda$, as also for generalized vector fields in Remark 2.4, is defined by mapping the vector field to the infinitesimal generalized diffeomorphism $(X, d\xi, d\lambda) \in g\text{diff}(M)$ and differentiating the action of the one-parameter subgroup $\{F_t\}$ to which it integrates:

$$L_{X + \xi + \lambda} \varphi = -\frac{d}{dt} {\big|}_{t=0} F_t \varphi = L_X \varphi + d\xi \wedge \varphi + d\lambda \tau \varphi.$$

The Lie derivative of a spinor satisfies a Cartan formula, where the interior product is replaced by the Clifford action, $d((X + \xi + \lambda) \cdot \varphi) + (X + \xi + \lambda) \cdot d\varphi = L_{X + \xi + \lambda} \varphi$.

The differential forms $\bigwedge^n T^*M$ are endowed with an $SO(T \oplus T^* \oplus 1)$-invariant pairing with values in $\bigwedge^n T^*M$ coming from the Chevalley pairing on spinors ([Che54]). Let $\alpha$ be the anti-involution defined by $\alpha(\omega) = (-1)^{(\deg \omega \cdot \frac{n}{2})} \omega$ on forms of pure degree $\omega$ and extended linearly. For $\text{rk} T = \dim M$ odd, the pairing is given by

$$\langle \varphi, \psi \rangle = \left[\alpha(\varphi^-) \wedge \psi^+ - \alpha(\varphi^+) \wedge \psi^-\right]_{\text{top}},$$

while for $\text{rk} T = \dim M$ even, it is given by

$$\langle \varphi, \psi \rangle = \left[\alpha(\varphi^+) \wedge \psi^+ + \alpha(\varphi^-) \wedge \psi^-\right]_{\text{top}}.$$
Remark 2.5. In the case of 3-manifolds,
\[
\langle \varphi, \psi \rangle = [\alpha(\varphi^+) \wedge \psi^- - \alpha(\varphi^-) \wedge \psi^+]_{\text{top}}
\]
\[
= [(\varphi_0 - \varphi_2) \wedge (\psi_1 + \psi_3) - (\varphi_1 - \varphi_3) \wedge (\psi_0 + \psi_2)]_{\text{top}}
\]
\[
= \varphi_0 \psi_3 + \psi_0 \varphi_3 - \varphi_1 \wedge \psi_2 - \psi_1 \wedge \varphi_2,
\]
and, in particular, \( \langle \varphi, \varphi \rangle = 2(\varphi_0^0 \varphi_3^3 - \varphi_1^1 \wedge \varphi_2^2) \), thus defining a quadratic form of signature \((4, 4)\).

3 \( G_2^2 \)-structures on 3-manifolds

In [Hit03], for \( n = 2m \), generalized Calabi-Yau structures are defined by a complex closed form \( \varphi \) that is either even or odd which is a pure spinor and satisfies \( \langle \varphi, \bar{\varphi} \rangle \neq 0 \). This structure defines a reduction to the stabilizer of the spinor field, \( \text{SU}(m, m) \).

In the case of a 3-manifold, we pointwise have a seven-dimensional generalized tangent space with an inner product of signature \((4, 3)\). Its space of spinors is eight-dimensional and equipped with a signature \((4, 4)\) inner product. In this setting, pure spinors correspond to null spinors with respect to the inner product, while non-pure spinors correspond to non-null spinors. Moreover, up to scalar multiplication, there are only two orbits under the action of \( \text{Spin}(4, 3) \): the null ones and the non-null ones. Hence, all non-null spinors have isomorphic stabilizers. While the stabilizer of a non-zero spinor in \( \text{Spin}(7) \) is the compact exceptional Lie group \( G_2 \), for the group \( \text{Spin}(4, 3) \), the stabilizer of a non-null spinor is its non-compact real form \( G_2^2 \). The study of the structure given on a 3-manifold by a section of \( \bigwedge^\bullet T^*M \) consisting of closed non-null spinors motivates the following definition.

**Definition 3.1.** A \( G_2^2 \)-generalized structure on a 3-manifold \( M \) is an everywhere non-null section of the real spinor bundle, \( \rho \in \Omega^\bullet(M) \), such that \( d\rho = 0 \). For the sake of brevity, we call them \( G_2^2 \)-structures.

**Remark 3.2.** Given a section \( \rho \in \Omega^\bullet(M) \) consisting of closed null spinors, its annihilator \( \text{Ann}(\rho) \subset T \oplus T^* \oplus 1 \) defines an integrable real Dirac structure, i.e., a maximal isotropic subbundle of \( T \oplus T^* \oplus 1 \) involutive with respect to the Courant bracket. The involutivity is a consequence of the closedness of \( \rho \), as in Proposition 1 of [Hit03].

3.1 Existence of \( G_2^2 \)-structures

From the non-nullity condition we have that \( \langle \rho, \rho \rangle = 2(\rho_0^0 \rho_3^3 - \rho_1^1 \wedge \rho_2^2) \) defines a volume form on \( M \), so \( G_2^2 \)-structures only exist over orientable manifolds.
In fact, given any volume form \( \omega \), \( c + \omega \) defines a \( G_2 \)-structure for any constant \( c \neq 0 \). Since \( \rho \) is closed, \( \rho_0 \) must be a constant.

From now on, \( M \) will denote a compact orientable 3-manifold. Let \( \text{GDiff}^+(M) \) be the group of orientation-preserving generalized diffeomorphisms.

**Proposition 3.3.** Up to \( \text{GDiff}^+(M) \)-equivalence, a \( G_2 \)-structure \( \rho \) with \( \rho_0 \neq 0 \) on \( M \) is of the form \( c + \omega \) for \( c \neq 0 \) and \( \omega \) a volume form, and

1. is completely determined by the cohomology classes
   \[ ([\rho_0], \langle \rho, \rho \rangle) \in (H^0(M, \mathbb{R}) \setminus \{0\}) \oplus (H^3(M, \mathbb{R}) \setminus \{0\}). \]

**Proof.** Let \( \rho = \rho_0 + \rho_1 + \rho_2 + \rho_3 \) be a \( G_2 \)-structure with \( \rho_0 \neq 0 \). It is equivalent, by the action of the closed \((B + A)\)-field \( \left( -\frac{\rho_2}{\rho_0}, -\frac{\rho_3}{\rho_0} \right) \) to

\[
\rho_0 + \frac{1}{\rho_0} (\rho_0 \rho_3 - \rho_1 \wedge \rho_2) = \rho_0 + \frac{1}{2\rho_0} \langle \rho, \rho \rangle,
\]

which is of the form \( c + \omega \) for \( c \neq 0 \) and \( \omega \) a volume form, as stated in the first part. By Moser’s theorem ([Mos65]), any two volume forms in the same cohomology class are diffeomorphic. \( \square \)

We deal now with the existence of \( G_2 \)-structures with \( \rho_0 = 0 \).

**Proposition 3.4.** If a compact 3-manifold is endowed with a \( G_2 \)-structure such that \( \rho_0 = 0 \), then it is diffeomorphic to the mapping torus of a symplectic surface by a symplectomorphism. Conversely, any such mapping torus can be endowed with a \( G_2 \)-structure with \( \rho_0 = 0 \).

**Proof.** From \( \rho_0 = 0 \) and \( \langle \rho, \rho \rangle \neq 0 \) we get \( \rho_1 \wedge \rho_2 \neq 0 \), so we have nowhere vanishing closed 1-forms and 2-forms \( \rho_1 \) and \( \rho_2 \). We can perform a small deformation on \( \rho_1 \) to give it rational periods (as shown for instance in [Lis70]). A suitable multiple has integral periods and defines a fibration \( \pi : M \to S^1 \).

To define \( \pi \), take a base point \( m \in M \) and let \( \pi(x) = e^{2\pi i \int_{x_0}^x \rho_1 dt} \) where \( c(t) \) is any curve joining \( m \) and \( x \). Let \( X \) be the unique vector field satisfying \( i_X \rho_2 = 0 \) and \( i_X \rho_1 = 1 \) (so it is transversal to the fibration, \( d\pi(X) \neq 0 \)). Integrate the vector field \( X \) to a one-parameter subgroup of diffeomorphisms \( \{f_t\} \) such that \( f_0 = id \). Let \( S \) be the fibre over the point \( m \in M \). By the transversality, we have that \( M \) is diffeomorphic to the mapping torus of \( f_1 \), i.e., the manifold

\[
\frac{S \times [0, 1]}{\{(x, 0) \sim (f_1(x), 1)\}_{x \in S}}.
\]
The diffeomorphism is given by \([(y, t)] \mapsto f_t(y) \in M\). Furthermore, \(\mathcal{L}_X \rho_2 = d(i_X \rho_2) = 0\), so \(f_t^* \rho_2 = \rho_2\) and the fibres have a symplectic structure given by the restriction of \(\rho_2\), which is closed and non-degenerate in every fibre \(f_t(S)\). Thus, \(S\) is a symplectic manifold and \(f_1\) is a symplectomorphism.

For the second part, let \(M_f\) be the mapping torus of an orientable surface \((S, \omega)\) by a symplectomorphism \(f\). We define a 2-form \(\rho_2\) on \(M_f\) as the form which is fibrewise \(\omega\). The form \(\rho_2\) is well defined since \(f^* \omega = \omega\). Let \(\rho_1\) be the pullback of a non-vanishing 1-form over the circle. The form \(\rho_1 + \rho_2\) then defines a \(G_2\)-structure on \(M_f\).

**Lemma 3.5.** The mapping torus of an orientable surface \(S\) by an orientation-preserving diffeomorphism is diffeomorphic to the mapping torus of \(S\) by a symplectomorphism.

**Proof.** Let \(f\) be the orientation-preserving diffeomorphism and let \(\omega\) be a volume form of the surface \(S\). The 2-forms \(f^* \omega\) and \(\omega\) have the same volume and hence define the same cohomology class in \(H^2(S, \mathbb{R})\). We apply Moser’s argument (\[Mos65\]) to the family \(\omega_t = t \omega + (1 - t) f^* \omega\), so we get a family of diffeomorphisms \(\{\varphi_t\}\), with \(\varphi_0 = \text{id}\), such that \(\varphi_t^* \omega_t = \omega\). Then, we have that \((\varphi_1 \circ f)^* = \varphi_1^* f^* \omega = \omega\), i.e., \(\varphi_1 \circ f\) is a symplectomorphism, and \(\{\varphi_t \circ f\}\) defines a diffeotopy between \(f\) and \(\varphi_1 \circ f\) which makes the mapping torus of \(f\) diffeomorphic to the mapping torus of \(\varphi_1 \circ f\).

The following theorem is a consequence of the two previous results.

**Theorem 3.6.** A compact 3-manifold \(M\) admits a \(G_2\)-structure with \(\rho_0 = 0\) if and only if \(M\) is the mapping torus of an orientable surface by an orientation-preserving diffeomorphism.

**Remark 3.7.** From a \(G_2\)-structure with \(\rho_0 = 0\) on a 3-manifold \(M\) we define a symplectic structure on \(M \times S^1\) by \(\rho_2 + \rho_1 \wedge d\theta\), where \(d\theta\) denotes the usual 1-form on \(S^1\) and we really mean the pullbacks of forms on \(M\) and \(S^1\) to \(M \times S^1\). More generally, the condition that a 3-manifold \(M\) fibres over the circle is equivalent to the existence of a symplectic structure on \(M \times S^1\), as addressed in \[FY11\].

**Remark 3.8.** After acting by a generalized diffeomorphism, a \(G_2\)-structure \(\rho\) with \(\rho_0 = 0\) can be written as \(\rho_1 + \rho_2\). This is a co-symplectic structure on the 3-manifold in the sense of \[Lib59\]. In this context, statements similar to the ones in this section have been obtained in \[Li08\].
3.2 Deformation of $G_2^2$-structures

Inspired by the Moser argument for symplectic geometry, we study whether a small perturbation of a $G_2^2$-structure (on a compact 3-manifold $M$) within its cohomology class may change the $G_2^2$-structure up to equivalence by

$$G \text{Diff}_0(M) = \{ f \times (B, A) \in G \text{Diff}(M) \mid f \in \text{Diff}_0(M), B \text{ and } A \text{ are exact} \}.$$ 

Let $\rho^0, \rho^1 \in \Omega^* (M)$ be two $G_2^2$-structures representing the same cohomology class, $\rho^1 - \rho^0 = d\varphi$, and sufficiently close to have that each form $\rho^t = \rho^0 + t(\rho^1 - \rho^0)$ is a $G_2^2$-structure, i.e., $\langle \rho^t, \rho^t \rangle \neq 0$, for $0 \leq t \leq 1$. We would like to have a one-parameter family of generalized diffeomorphisms $\{F_t\}$ such that $F_t^* \rho^t = \rho^0$, making equivalent all the $G_2^2$-structures between $\rho^0$ and $\rho^1$. We will be looking for $\{F_t\}$ coming from a time-dependent generalized vector field $\{X_t + \xi_t + \lambda_t\}$. By differentiating $F_t^* \rho^t = \rho^0$ and using Cartan’s formula, we then have

$$0 = \frac{d}{dt} [F_t^* \rho^t] = F_t^* \left[ \frac{d\varphi}{dt} + L_{X_t + \xi_t + \lambda_t} \rho^t \right] = F_t^* [d\varphi + d((X_t + \xi_t + \lambda_t) \cdot \rho^t)] = 0.$$

So, in order to find such generalized vector fields it will suffice to solve the equation $d((X_t + \xi_t + \lambda_t) \cdot \rho^t) = d(-\varphi)$, or equivalently, to solve the equation $(X_t + \xi_t + \lambda_t) \cdot \rho^t = -\varphi$ where we are allowed to modify $\varphi$ by the addition of a closed form depending on $t$. This latter equation corresponds to $\varphi$ being in the image of the Clifford product of the sections of the rank 7 vector bundle $T \oplus T^* \oplus 1$ by $\rho^t$. The spinor $\rho^t$ defines a map $T \oplus T^* \oplus 1 \to \bigwedge^* T^* M$. Since $\rho^t$ is non-null, this map is injective (the annihilator of a non-null spinor is trivial). From the antisymmetry of the Clifford product with respect to the pairing, $(v_m, \rho^t, \rho^t_m) = 0$, where $v_m$ and $\psi_m$ lie over $m \in M$, and the image is $\{\rho^t\}^\perp = \{ \psi \in \bigwedge^* T^* M \mid \langle \rho^t, \psi \rangle = 0 \}$. Thus, $\rho^t$ defines an isomorphism between the rank 7 vector bundles $T \oplus T^* \oplus 1$ and $\{\rho^t\}^\perp$. Consequently, for the equation $(X_t + \xi_t + \lambda_t) \cdot \rho^t = -\varphi$ to have a solution and then apply the Moser argument, we must have $\varphi \in C^\infty(\{\rho^t\}^\perp)$.

**Proposition 3.9.** Any sufficiently small perturbation $\{\rho^t\}$ within the cohomology class of a $G_2^2$-structure $\rho^0$ such that $\rho^0_0 \neq 0$ is equivalent to $\rho^0$ under the action of the group $G \text{Diff}_0(M)$.

**Proof.** We have that $\rho^t_0 = \rho^0_0 \neq 0$. Since we can add any closed form to $\varphi$, we can arbitarily modify its degree 3 part. The Moser argument applies by setting $\varphi^t_3 = \frac{1}{\rho^0_0} \langle \rho^t, \varphi_o + \varphi_1 + \varphi_2 \rangle$, so that we have $\langle \rho^t, \varphi^t \rangle = 0$.

When $\rho_0 = 0$, the result remains true but involves some technicalities.
Lemma 3.10. Let $\rho$ be a $G_2^3$-structure with $\rho_0 = 0$ and $[\rho_1] \in H^1(M, \mathbb{Q})$. There exists an operator $R : \Omega^*(M) \to \Omega^*_R(M)$ such that $\varphi + R\varphi \in C^\infty(\{\rho\}^\perp)$.

Proof. By considering a multiple of $\rho$ we can consider $[\rho_1] \in H^1(M, \mathbb{Z})$. By Proposition 3.4, $M$ fibres over the circle with fibre $S$. First, define the constant $c = \langle \langle \rho, \varphi \rangle \rangle/[\rho_1 \wedge \rho_2]$. Add the closed form $c\rho_2$ to $\varphi$; then the cohomology class of $\langle \rho, \varphi + c\rho_2 \rangle$ is trivial. Thus, $\langle \rho, \varphi + c\rho_2 \rangle = d\alpha$ for some 2-form $\alpha$. Choose a metric on $M$. Using the Hodge decomposition, the codifferential $d^*$ and the Green operator $G$, we may take $\alpha = d^*G(\rho, \varphi')$. Integrate $\alpha$ over the fibres to get a function $g$ on the circle. Since $\rho_1 \wedge \rho_2 \neq 0$, the fibres are homological and $\rho_2$ is closed, then $\int_S \rho_2 = c' \neq 0$ for any fibre $S$. Let $f = g/c'$.

The 2-form $\alpha_0 = \alpha - f\rho_2$ has zero integral along the fibres. The metric on $M$ induces a metric on any fibre $S$, for which we define the codifferential $d^*_S$, harmonic operator $H_S$ and Green operator $G_S$ such that

$$\alpha_{0|S} = H_S\alpha_{0|S} + d_S(d_S^*G_S\alpha_{0|S}) + d_S^*(d_SG_S\alpha_{0|S}).$$

For degree reasons, $d_SG_S\alpha_{0|S} = 0$, and from $\int_S \alpha_{0|S} = 0$, $H_S\alpha_{0|S} = 0$. We then have, over each fibre $S$, $\alpha_{0|S} = d_S\beta$ where $\beta = d_S^*G_S\alpha_{0|S}$. Since the metric on $M$ determines a smoothly varying family of metrics over the fibres, we have a globally smooth 1-form $\beta$ such that $\alpha_0 - d\beta$ is zero restricted to a fibre.

Let $X$ be the vector field transversal to the fibration such that $i_X\rho_1 = 1$, and let $\gamma = -i_X(\alpha_0 - d\beta)$. We have that $\alpha_0 - d\beta = \gamma \wedge \rho_1$. By differentiating this expression we get

$$d\alpha = d(\alpha_0 + f\rho_2) = df \wedge \rho_2 + \rho_1 \wedge d\gamma.$$

Define $R\varphi = c\rho_2 + df + d\gamma \in \Omega^*_R(M)$. Since $c$, $f$ and $\gamma$ have been uniquely defined, $R$ defines an operator on differential forms. We have by construction that $\langle \rho, \varphi + R\varphi \rangle = 0$, i.e., $\varphi + R\varphi \in C^\infty(\{\rho\}^\perp)$. \hfill \Box

Let $Q\varphi \in C^\infty(T \oplus T^* \oplus 1)$ be the unique generalized vector field such that $Q\varphi \cdot \rho = -(\varphi + R\varphi)$. Thus $Q$ defines an operator $\Omega^*(M) \to C^\infty(T \oplus T^* \oplus 1)$.

Proposition 3.11. Any sufficiently small perturbation $\{\rho'\}$ within the cohomology class of a $G_2^3$-structure $\rho_0$ such that $\rho_0^2 = 0$ is equivalent to $\rho_0$ by $G\text{Diff}_0(M)$.

Proof. When $[\rho_0^2] \in H^1(M, \mathbb{Q})$, we use Lemma 3.10 to produce an operator $R_t$ for each $\rho'$ and we define $\varphi' = \varphi + R_t\varphi$, so that $\langle \rho', \varphi' \rangle = 0$ and the Moser argument applies.

For the general case, we prove an analogous result in a neighbourhood of a $G_2^3$-structure with rational degree 1 part and use a density argument.
We drop the superindex \( t \) for the sake of brevity. Consider \( \rho + \lambda \beta \), with \( \lambda > 0 \) and \( \beta \in \Omega^*_c(M) \) such that \( \beta_0 = 0 \). We want to solve the equation 
\[ u \cdot (\rho + \lambda \beta) = -\varphi \] 
up to addition of closed forms. To do that, consider 
\[ (u_0 + \nu u_1 + \lambda^2 u_2 + \ldots) \cdot (\rho + \lambda \beta) = -(\varphi + R \varphi + \lambda \gamma_1 + \lambda^2 \gamma_2 + \ldots), \] 
for closed forms \( \gamma_i \). We solve it iteratively, starting with \( u_0 \cdot \rho = -\varphi + R \varphi \), which has solution \( u_0 = Q \varphi \). We then have \( u_1 \cdot \rho = -(Q \varphi \cdot \beta + \gamma_1) \). We define the operator \( P : \Omega^*(M) \to \Omega^*(M) \) by \( P \varphi = Q \varphi \cdot \beta \) and consider \( \gamma_1 = RP \varphi \). The equation becomes \( u_1 \cdot \rho = -(P \varphi + RP \varphi) \), whose solution is \( u_1 = QP \varphi \). For \( j \geq 2 \) we have \( u_j \cdot \rho = -u_{j-1} \cdot \beta + \gamma_j = -P^j \varphi + \gamma_j \). By taking \( \gamma_j = RP^j \varphi \), the solution is given by \( u_j = QP^j \varphi \). We thus obtain a formal solution of (\*\*) by 
\[ Q(\varphi + \lambda P \varphi + \lambda^2 P^2 \varphi + \ldots) \cdot (\rho + \lambda \beta) = -\varphi + R(\varphi + \lambda P \varphi + \lambda^2 P^2 \varphi + \ldots). \]
To see the convergence of the series \( \varphi + \sum_{j=1}^{\infty} \lambda^j P^j \varphi \) for \( \lambda \) sufficiently small, we consider Sobolev spaces \( H_s(T \oplus T^* \oplus 1) \) and \( H_s(\Lambda^*(M)) \) with norms \( || \cdot ||_s \). Since the operator \( Q \) is defined in terms of the Green operator and integration over the fibres, it is bounded, and so is the operator \( P \). For \( s \) sufficiently large and any \( \beta \) such that \( ||v \cdot \beta||_s \leq ||v||_s \), there exists some constant \( C_s \) such that \( ||P \varphi||_s \leq C_s ||\varphi||_s \).

Take \( \lambda \) such that \( 0 < \lambda < \frac{1}{2c_\varphi} \). Then, \( \varphi + \sum_{j=1}^{\infty} \lambda^j P^j \varphi \) is a Cauchy sequence and converges to a form \( \Phi \in H_s(\Lambda^*(M)) \). Equation (\*) becomes 
\[ u \cdot (\rho + \lambda \beta) = -(\varphi + R \Phi) \] 
and a solution is given by \( Q \Phi \in H_s(T \oplus T^* \oplus 1) \).

We have that for any \( \rho \) such that \( [\rho_1] \in H^1(M, Q) \), there exists a neighbourhood for which there is a solution in \( H_s(T \oplus T^* \oplus 1) \). Since \( \varphi \in \Omega^*_c(M) \) belongs to \( H_s(\Lambda^*(M)) \) for any \( s \), we have that the solution belongs to \( H_s \) for any \( s \). Thus, the series defines \( \Phi \in C^\infty(\Lambda^*(M)) \), we have that \( Q \Phi \in C^\infty(T \oplus T^* \oplus 1) \) is a solution of \( u \cdot \rho' = -\varphi \) up to closed forms, and the Moser argument applies. Since there exists a solution in an open neighbourhood of any rational form, by density of the rational forms, there exists a solution for any closed form \( \rho \) and the Moser argument applies.

We summarize Propositions 3.9 and 3.11 in the following theorem.

**Theorem 3.12.** Any sufficiently small perturbation \( \{ \rho' \} \) within the cohomology class of a \( G^2 \)-structure \( \rho^0 \) is equivalent to \( \rho^0 \) by \( GDiff_0(M) \).
3.3 The cone of $G_2^3$-structures

Inspired by the cones of Kähler and symplectic structures inside the second cohomology group of a manifold, we raise a similar question for $G_2^3$-structures on compact 3-manifolds. What are the cohomology classes $[\rho] \in H^\bullet(M, \mathbb{R})$ which have a representative in $\Omega^\bullet(M, \mathbb{R})$ defining a $G_2^3$-structure compatible with the orientation of $M$? From the homogeneity of the condition $\langle \rho, \rho \rangle > 0$, it is clear that these elements form an open cone in $H^\bullet(M, \mathbb{R})$.

Consider a mixed degree cohomology class $[\rho] \in H^\bullet(M, \mathbb{R})$ satisfying $[\rho_0][\rho_3] - [\rho_1][\rho_2] > 0 \in H^\bullet(M, \mathbb{R})$. In the case that $[\rho_0] \neq 0$, i.e., $\rho_0 \neq 0$, consider a non-vanishing form $\omega$ representing the degree 3 class $[\rho_0^2 \rho_3 - \rho_1^2 \rho_2]$. Define $\rho' = \rho_0 + \rho_1 + \rho_2 + \frac{1}{\rho_0}(\omega + \rho_1 \wedge \rho_2)$, which satisfies $\langle \rho', \rho' \rangle = 2\omega$ and is thus a $G_2^3$-structure representing $[\rho]$.

On the other hand, for a class $[\rho]$ with $[\rho_0] = 0$, i.e., $\rho_0 = 0$, the condition $\langle \rho, \rho \rangle = -2[\rho_1][\rho_2] > 0$ must be satisfied. Moreover, $[\rho_1]$ and $[\rho_2]$ must be represented by non-vanishing forms. From Theorem 5 in [Thu86], the set of cohomology classes $C_1$ in $H^1(M, \mathbb{R})$ which can be represented by a non-singular closed 1-form constitutes an open set described as follows. Define the norm $X$ for $\omega \in H^2(M, \mathbb{R})$ as the infimum of the negative parts of the Euler characteristics of embedded surfaces defining $\omega$, and extend this definition to $H^1(M, \mathbb{R})$ using Poincaré duality. Namely, the norm of a 1-form $\varphi$ in $M$ is

$$||\varphi||_X = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \varphi\},$$

where $\chi_-(S) = \max\{-\chi(S), 0\}$. The unit ball for this norm is a polytope called the Thurston ball $B_X$. The set of 1-cohomology classes $C_1$ represented by non-vanishing 1-forms consists of the union of the cones on some open faces, so-called fibred faces, of the Thurston ball, minus the origin.

For each element $\alpha = [a] \in C_1$, given by a non-singular $a$, take $h \in H^2(M, \mathbb{R})$ such that $h \cup a > 0$. Lemma 2.2 in [FV12] ensures that we can always find a representative $\Omega$ of the class $h$, such that $\Omega \wedge a > 0$. Hence, if we define

$$C = \{(\alpha, \beta) \in C_1 \oplus H^2(M, \mathbb{R}) \mid \alpha \cup \beta < 0\},$$

we have that the cone of $G_2^3$-structures with $\rho_0 = 0$ in $H^\bullet(M, \mathbb{R})$ is given by $C \oplus H^3(M, \mathbb{R})$. To sum up, we have the following theorem.

**Theorem 3.13.** The cone of $G_2^3$-structures, or $G_2^3$-cone, is given by

$$\{[\rho] \in H^\bullet(M, \mathbb{R}) \mid [\rho_0] \neq 0 \text{ and } [\rho_0][\rho_3] - [\rho_1][\rho_2] > 0\} \cup (C \oplus H^3(M, \mathbb{R})).$$
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