Commutation matrices and Commutation tensors

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Abstract

The commutation matrix was first introduced in statistics as a transposition matrix by Murnaghan in 1938. In this paper, we first investigate the commutation matrix which is employed to transform a matrix into its transpose. We then extend the concept of the commutation matrix to commutation tensor and use the commutation tensor to achieve the unification of the two formulae of the linear preserver of the matrix rank, a classical result of Marcus in 1971.

keywords: Commutation matrix; commutation tensor; Linear preserver; determinant; transpose.

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1 Introduction

The commutation matrix was introduced by Murnaghan in 1938 in the name of permutation matrices. It is also referred in publications on statistics as the transposition matrix. A commutation matrix is a kind of permutation matrix of order $pq$ expressed as a block matrix where each block is of the same size and has a unique 1 in it. The commutation matrix can be used to describe the relationship of a Kronecker product $A \otimes B$ with $B \otimes A$ where $A, B$ are two arbitrary matrices of any sizes. In this paper, we extend the commutation matrix to a commutation tensor, which is a fourth order tensor, by which we can express the transpose of a matrix as the linear transformation on it. This is further used to deduce some properties of the commutation tensors, and consequently we achieve the unification of the linear preservers of the determinants of matrices.

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For our convenience, we denote \([m]\) for the set \(\{1, 2, \ldots, m\}\) for any positive integer, and \(S_m\) the set of all the permutations on \([m]\). Also we denote by \([a/b]\) the quotient of an integer \(a\) divided by a positive integer \(b\), similarly we denote \([a/b]\) \((\lfloor a/b \rfloor)\) for the upper (resp. lower) quotient of an integer \(a\) divided by a positive integer \(b\). We write \(n(p, q) = pq\) or simply \(n = pq\) when no risk of confusion arises for any positive integers \(p, q\).

Denote \(e_{n,i}\) for the \(i\)th canonical vector of dimension \(n\), i.e., the vector with 1 in the \(i\)th coordinate and 0 elsewhere, and denote \(E_{ij}^{(m,n)} = e_{m,i}e_{n,j}^\top\), and \(E_{ij}^{(n)} = E_{ij}^{(m,n)}\) when \(m = n\). We usually denote them by \(E_{ij}\) when \(m, n\) are known from the context. A permutation matrix \(P = (p_{ij}) \in \mathbb{R}^{n \times n}\) is called a commutation matrix if it satisfies the following conditions:

(a) \(P = (P_{ij})\) is an \(p \times q\) block matrix with each block \(P_{ij} \in \mathbb{R}^{q \times p}\).

(b) For each \(i \in [p], j \in [q]\), \(P_{ij} = (k_{st}^{(i,j)})\) is a \((0, 1)\) matrix with a unique 1 which lies at the position \((j, i)\).

We denote this commutation matrix by \(K_{p,q}\). Thus a commutation matrix is of size \(pq \times pq\).

**Example 1.1.** \(K_{2,3}\) is a \(6 \times 6\) permutation matrix partitioned as a \(2 \times 3\) block matrix, i.e.,

\[
K_{2,3} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \end{pmatrix}
\tag{1.1}
\]

where each block \(H_{ij} = (h_{s,t}^{(i,j)})\) is a \(3 \times 2\) matrix whose unique nonzero entry is \(h_{j,i}^{(i,j)} = 1\). Specifically

\[
K_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\tag{1.2}
\]

Let \(A = K_{p,q} = (A_{st})\) be an \(p \times q\) block matrix with block size \(p \times q\). For each pair \((i, j)\) with \(i \in [p], j \in [q]\), we denote \(A_{ijkl}\) as the entry at the position \((k, l)\) in the block \(A_{ij}\). We denote \(A_{ij}(k, l)\) for the \((i, j)\)-entry of block \(A_{ij}\). Then \(A_{ij}(k, l)\) is the \((s, t)\)-entry of \(A\), denoted \(a_{st}\) where

\[
s = (i - 1)q + k, \quad t = (j - 1)p + l.
\tag{1.3}
\]

Conversely, given an \((s, t)\)th entry of \(K_{p,q}\), we can also find its position according to the block form of \(A\), i.e., \(a_{st} = A_{k_1k_2k_3k_4}\), where

\[
k_1 = \lceil i/q \rceil, \quad k_2 = \lfloor j/p \rfloor, \\
k_3 = i - k_1q \mod \ q, \quad k_4 = j - k_2p \mod \ p.
\]
The results in the following lemma are some fundamental properties for commutation matrices.

Lemma 1.2. Let $K_{p,q}$ be the commutation matrix. Then we have

1. $K_{p,q}^\top = K_{q,p}$ and $K_{p,q}K_{q,p} = I_{pq}$.
2. $K_{p,1} = K_{1,p} = I_p$.
3. $\det(K_{p,p}) = (-1)^{\frac{p(p-1)}{2}}$.

Proof. The first two items are obvious. We need only to prove the last item.

Let $A \in R^{m \times n}, B \in R^{p \times q}$. The Kronecker product of $A, B$, denoted $A \otimes B$, is defined as an $mp \times nq$ matrix in the $m \times n$ block form, i.e., $C = A \otimes B = (C_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ with $C_{ij} = a_{ij}B$. The following propositions on the Kronecker product of matrices will be used in the sequence.

Lemma 1.3. Let $A_i \in R^{m_i \times n_i}, B_i \in R^{n_i \times p_i}$ for $i = 1, 2$. Then

1. $(A_1 \otimes A_2)(B_1 \otimes B_2) = (A_1 B_1) \otimes (A_2 B_2)$.
2. $(A_1 \otimes A_2)' = (A_1)' \otimes (A_2)'$.
3. Let $A, B$ be both invertible. Then $A \otimes B$ is invertible with its inverse $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

The matrix vectorisation, denoted vec($\cdot$), is to turn a matrix into a column vector by vertically stacking all the columns of the matrix in a natural order. More specifically, let $A \in R^{p \times q}$ and $A_{kj}$ be the $j$th column of $A$. Then consequence of the vectorisation of $A$ is an $pq$-dimensional vector vec$(A)$ with

$$\text{vec}(A)^\top := [A'_1, A'_2, \ldots, A'_n] \in R^{pq}$$

Conversely, a vector $x = (x_1, x_2, \ldots, x_n)^\top \in R^n$ with length $n = pq$ ($p, q > 1$) can always be reshaped (matricized) into an $p \times q$ matrix $X$ either by column (i.e., the first $p$ entries of $x$ form the first column, the next $p$ entries form the second column, etc.). Similarly we can also matricize vector $x$ rowisely. Both can be regarded as an 1-1 correspondence between $R^{pq}$ and $R^{p \times q}$. The elements of the $p \times q$ matrix $X = (m_{ij}) \in R^{p \times q}$ obtained from the columnwise matricization is defined by

$$c_{ij} = x_{i+(j-1)p}, \quad \forall 1 \leq i \leq p, 1 \leq j \leq q \quad (1.4)$$

and the elements of the $p \times q$ matrix $X = (m_{ij}) \in R^{p \times q}$ obtained from the rowwise matricization is defined by

$$r_{ij} = x_{j+(i-1)q}, \quad \forall 1 \leq i \leq p, 1 \leq j \leq q \quad (1.5)$$

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We use $\text{vec}_{p,q}^{-1}$ to denote for the columnwise matricization of an $pq$-dimensional vector into an $p \times q$ matrix, and use $\text{vecr}_{p,q}^{-1}$ to denote the rowwise matricization of an $pq$-dimensional vector into an $p \times q$ matrix.

The following property, which can be found in many textbook on the matrix theory, is crucial to the multivariate statistical models.

**Lemma 1.4.** Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$. Then we have

$$\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$$

(1.6)

For $p = m$, we also have

$$\text{Tr}(AB) = \text{vec}(B)'\text{vec}(A)$$

(1.7)

In the next section, we first present some basic properties of the commutation matrices. Section 3 is dedicated to the commutation tensors where we first present the definition of the commutation tensor and study some of its properties. In Section 4 we employ the commutation tensor to study the linear preserving problem (LPP) and extend the LPP to a more general multilinear preserving problem (MLPP). We will also use the relationship between a matrix and its transpose through the commutation matrix (tensor) to unify the form of a linear determinant preserver and linear rank preserver.

2 Commutation matrices

The following result presents a linear relationship between $\text{vec}(A^\top)$ and $\text{vec}(A)$ through the commutation matrix $K_{p,q}$.

**Theorem 2.1.** Let $p, q$ be two positive integers. Then

$$K_{p,q}\text{vec}(X) = \text{vec}(X^\top), \quad \forall X \in \mathbb{R}^{p \times q}$$

(2.1)

Furthermore, $K_{p,q}$ is the unique matrix for (2.1) to be held.

**Proof.** Let $n := pq$ and $X = (x_{ij}) \in \mathbb{R}^{p \times q}$ and denote $y = K_{p,q}\text{vec}(X)$. Then $y \in \mathbb{R}^{n}$. For any $s \in \llbracket n \rrbracket$, $s$ can be written as

$$s = (i - 1)q + j, \quad 1 \leq i \leq p, 0 \leq j < q$$

that is, $i - 1$ and $j$ are respectively the quotient and the remainder of the number $s$ divided by $q$. Thus by definition,

$$y_s = (\sum_{k=1}^{q} H_{ik}X_{k})_j = x_{ij}$$

(2.2)

Here $X_{k}$ denotes the $k$th column of $X$ and $H_{ik}$ is the $(i,k)$th block of $K_{p,q}$ whose unique nonzero entry (equals 1) is at the position $(k,i)$ by definition.
On the other hand, we have $\text{vec}(X^\top)_s = a_{ij}$. Thus (2.1) holds.

Now we suppose there is a matrix $K \in \mathbb{R}^{n \times n}$ such that $K \text{vec}(X) = \text{vec}(X^\top)$ holds for all $X \in \mathbb{R}^{p \times q}$. Then $(K - K_{p,q})\text{vec}(X) = 0$ for all $X \in \mathbb{R}^{p \times q}$. It follows that $K - K_{p,q} = 0$ and consequently $K = K_{p,q}$. □

Theorem 2.1 tells us that the transpose of a matrix $A$ can be regarded as the permutation of $A$ through the commutation matrix, but this linear transformation is established in terms of the matrix vectorisation, which, nevertheless, alters the shape of the matrix. But sometimes we do want to know exactly the relation of $A$ and its transpose while preserving its shape. This will be done in the next section.

We denote $e_{r,s}$ for the $s$th coordinate vector of $\mathbb{R}^r$. The following lemma can be regarded as the rank-1 decomposition of $K_{p,q}$.

**Lemma 2.2.**

$$K_{p,q} = \sum_{i,j} (e_{p,i} \otimes e_{q,j})(e_{q,j} \otimes e_{p,i})^\top$$

(2.3)

Here the summation runs over all $i \in \llbracket p \rrbracket$, $j \in \llbracket q \rrbracket$.

**Proof.** We let $F^{(i,j)} = (G_{st}) \in \mathbb{R}^{pq \times pq}$ be the $p \times q$ block matrix, each block $G_{st} \in \mathbb{R}^{q \times p}$ is a zero block except the $(i, j)$th block $G_{ij} = E_{ji}^\top$. Here $E_{ji} = e_{q,j}e_{p,i}^\top$ is the elementary matrix with the unique 1 at position $(j, i)$. Then it is obvious that

$$K_{p,q} = \sum_{i \in [p], j \in [q]} F^{(i,j)}$$

(2.4)

Note that

$$F^{(i,j)} = e_{p,i}e_{q,j}^\top \otimes e_{q,j}e_{p,i}^\top = (e_{p,i} \otimes e_{q,j})(e_{q,j} \otimes e_{p,i})^\top$$

(2.5)

The last equality of (2.5) follows directly from the first item of Lemma 1.3 and the decomposition (2.3) follows directly from the combination of (2.4) and (2.5). □

The following result, showing an essential role of the commutation matrix in the linear and multilinear algebra, will be employed in the proof of our main result.

**Theorem 2.3.** Let $A \in \mathbb{R}^{pq \times pq}$ where $p, q > 1$ are positive integers. Then

$$A(x \otimes y) = y \otimes x, \quad \forall x \in \mathbb{R}^q, y \in \mathbb{R}^p$$

(2.6)

if and only if $A = K_{p,q}$. 5
Proof. We first prove the sufficiency. Let $A = K_{p,q}$. Then for any $x \in R^q, y \in R^p$, by Lemma 2.2, we have

\[
K_{p,q}(x \otimes y) = \left( \sum_{i,j} (e_{p,i} \otimes e_{q,j})(e_{q,j} \otimes e_{p,i})^\top \right)(x \otimes y) = \left( \sum_{i,j} e_{q,j}^\top x \otimes (e_{p,i}^\top y)(e_{p,i} \otimes e_{q,j}) \right) = \left( \sum_{i,j} x_j y_i e_{p,i} \otimes e_{q,j} \right) = y \otimes x.
\]

The third equality is due to Lemma 1.3.

Conversely, we suppose condition (2.6) holds. We want to show that $A = K_{p,q}$. For each $i \in [p], j \in [q]$, we let $x = e_j \in R^q$ be the $j$th coordinate vector of $R^q$. By the blocking product of $A(x \otimes y)$ and (2.6), we have

\[
A_{ij}y = y_i e_j, \quad \forall i, j \in [p], \forall y \in R^p \quad (2.7)
\]

Denote $E_{ij} \in R^{q \times p}$ for the fundamental matrix, i.e., all of whose entries are zero except the $(i,j)$ entry which is 1. Then we have

\[
E_{ji}y = y_i e_j, \quad \forall i \in [p], j \in [q] \quad (2.8)
\]

where $y = (y_1, y_2, \ldots, y_p)^\top$. Thus we have from (2.7) and (2.8) that

\[
(A_{ij} - E_{ji})y = 0, \quad \forall y \in R^p \quad (2.9)
\]

It follows that $A_{ij} = E_{ji}$ for all $i, j$. Consequently we have $A = K_{p,q}$ by the definition.

An alternative proof to Theorem 2.3 is to employ Theorem 2.1: we denote $X = xy^\top \in R^{p \times q}$. Then

\[
x \otimes y = \text{vec}(X), \quad y \otimes x = \text{vec}(X^\top) = \text{vec}(yx^\top) = y \otimes x
\]

By Theorem 2.1 we have

\[
K_{p,q}(x \otimes y) = K_{p,q} \text{vec}(X) = \text{vec}(X^\top) = y \otimes x.
\]

Since $R^{pq}$ is isometric to $R^p \times R^q$, which is also isometric to $R^q \times R^p$, $K_{p,q}$ can be regarded as a block permutation on $R^{pq}$. Consequently it can be regarded as an automorphism on $R^{n \times n}$ where $n = pq$. The following result, which is an improvement of a known property for the commutation matrices, enhances this point.
Corollary 2.4. Let $A \in R^{p \times p}, B \in R^{q \times q}$ and let $n = pq$ where $p, q$ are positive integers. Then we have

$$A \otimes B = K_{p,q}(B \otimes A)K_{p,q} \quad (2.10)$$

Furthermore, if $p = q$, then $A \otimes B$ is permutation similar to $B \otimes A$.

Proof. Since $K_{p,q}^\top = K_{q,p}$ is an orthogonal matrix by the first item of Lemma 1.2, (2.10) can be equivalently written as

$$K_{q,p}(A \otimes B) = (B \otimes A)K_{q,p} \quad (2.11)$$

For any vector $x \in R^p, y \in R^q$, we have, by Theorem 2.3,

$$K_{q,p}(A \otimes B)(x \otimes y) = K_{q,p}(Ax \otimes By) = By \otimes Ax = (B \otimes A)(y \otimes x) = (B \otimes A)K_{q,p}(x \otimes y)$$

Thus (2.11) holds.

Now if $p = q$, then the commutation matrix $K_p = K_{p,p} = K_p^\top$ by Lemma 1.2. Thus by (2.10) we know that $A \otimes B$ is permutation similar to $B \otimes A$.

For $p = q$, we denote $n = p^2$ and $K_p := K_{p,p}$. Then we have

Theorem 2.5. 1. $K_p$ is a symmetric involution, i.e., $K_p^2 = I_n$.

2. $\text{Tr}(K_p) = p$.

3. $\det(K_p) = (-1)^{\frac{p(p-1)}{2}}$ for any integer $p > 1$.

Proof. (1). The symmetry of $K_p$ follows directly from (1) of Lemma 1.2 and (1-2) of Lemma 1.2 yields the equation $K_p^2 = I_n$.

(2). This is obvious since $\text{Tr}(K_p) = \sum_{i=1}^{p} \text{Tr}(E_{ii}) = p$ where $E_{ii} \in R^{p \times p}$ whose entries are all zeros except the $(i,i)$-entry that is 1.

(3). It is immediate from (1) that $|\det(K_p)| = 1$. By (1), $K_p$ is an orthogonal projection, thus $K_p$ can be decomposed as $K_p = UDU^\top$, where $U \in R^{n \times n}$ is an orthogonal matrix and $D = \text{diag}(I_r, -I_{n-r})$. By (2), we have $p = \text{Tr}(K_p) = r - (n - r) = 2r - n$. It follows that $r = \frac{1}{2}p(p + 1)$. Thus we have

$$\det(K_p) = (-1)^{p^2 - r} = (-1)^{\frac{1}{2}p(p - 1)}$$

Consequently we obtain (3).
3 Commutation Tensors

In this section, we define the commutation tensor and investigate its properties. We use the commutation tensor to obtain an unified form of the linear rank preserver.

Recall that an \(m\)-order tensor \(A = (A_{i_1 i_2 \ldots i_m})\) of size \(N_1 \times N_2 \times \ldots \times N_m\) can be regarded as an \(m\)-way array where the subscripts \((i_1, i_2, \ldots, i_m)\) is taken from the set

\[
S(m, n) := \{\sigma = (i_1, i_2, \ldots, i_m) : i_k \in [N_k], k = 1, 2, \ldots, m.\}
\]

For \(N_1 = N_2 = \ldots = N_m = N\), we call \(A\) an \((m, n)\)-tensor, and denote \(T_{m,n}\) for the set of all \(m\)th order \(n\)-dimensional real tensors. An \((m, n)\)-tensor \(A\) is called symmetric if for any \(\sigma = (i_1, i_2, \ldots, i_m) \in S(m, n)\), we have

\[
A_{i_1 i_2 \ldots i_m} = A_{j_1 j_2 \ldots j_m}
\]

where \((j_1, j_2, \ldots, j_m)\) is a permutation of \(\sigma\). We denote \(ST_{m,n}\) for the set of all \(m\)th order \(n\)-dimensional symmetric tensors.

Given any vector \(x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n\). We generate a rank-1 \((m, n)\)-tensor \(X = (X_{i_1 i_2 \ldots i_m}) := x^m\) where

\[
X_{i_1 i_2 \ldots i_m} = x_1 x_2 \ldots x_m
\]

More generally, a rank-1 \(m\)-order tensor is in form \(\alpha_1 \times \alpha_2 \times \ldots \times \alpha_m\) where \(\alpha_k \in \mathbb{R}^{N_k}\) for \(k \in [m]\). An \((m, n)\)-tensor \(A = (A_{i_1 i_2 \ldots i_m})\) is called positive semidefinite if for each \(x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n\)

\[
f(x) := \sum_{i_1, i_2, \ldots, i_m} A_{i_1 i_2 \ldots i_m} x_1 x_2 \ldots x_m \geq 0
\]

(3.1)

and called positive definite if \(f(x) > 0\) for all \(x \neq 0\). It is easy to see that a nonzero positive (semi-)definite tensor must be of an even order.

Let \(A = (A_{i_1 i_2 \ldots i_m})\) be a tensor of size \(I_1 \times I_2 \times \ldots \times I_m\) and \(B \in R^{I_n \times J_n}\), \(x \in R^{I_n}\). We define the tensor-vector multiplication along mode-\(n\) by

\[
A \times_n x := \sum_{i_n=1}^{I_n} A_{i_1 \ldots i_n \ldots i_m} x_{i_n}
\]

which produces a \((m - 1)\)-order tensor. This definition can also be extended to the multiplication of any two tensors with some consistent dimensions. For example, a tensor-matrix multiplication \(A \times_{3,4} B\) along mode-\(\{3,4\}\) is defined as

\[
(A \times_{3,4} B)_{ij} = \sum_{k,l} A_{ijkl} b_{kl}
\]

(3.2)
where $A \in \mathbb{R}^{m \times n \times p \times q}$, $B \in \mathbb{R}^{p \times q}$. Then $A \times_{3,4} B \in \mathbb{R}^{m \times n}$. As a matrix can be vectorised into a vector, a tensor can be flattened or unfolded into a matrix [12].

Let $A \in \mathcal{T}_{m,n}$ and $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ be a nonzero vector. $x$ is called an eigenvector of $A$ if there exists a scalar $\lambda \in \mathbb{C}$ such that

$$Ax^{m-1} = \lambda x^{m-1}$$

If $x \in \mathbb{R}^n$, then $\lambda \in \mathbb{R}$. We call such a $\lambda$ an H-eigenvalue of $A$ and $x$ the eigenvector of $A$ corresponding to $\lambda$.

Given any positive integer $m, n$. We define the $(m, n)$-commutation tensor $K_{n,m} = (K_{ijkl})$ to be an $(0,1)$ $n \times m \times m \times n$ tensor where $K_{ijkl} = 1$ if and only if $i = l, j = k$ for all $1 \leq i, l \leq n, 1 \leq j, k \leq m$. Note that $K_{n,m}$ can be flattened into the commutation matrix $K_{m,n}$ and that $K(i,j,:) = H_{ij} \in \mathbb{R}^{m \times n}$ for all $1 \leq i \leq n, 1 \leq j \leq m$. $K_{m,n}$ is also called a permutation tensor.

**Example 3.1.** Consider $K_{2,3} = (K_{ijkl})$ with size $2 \times 3 \times 3 \times 2$. Then $K$ has six nonzero entries as

$$K_{1111} = K_{2112} = K_{1221} = K_{2222} = K_{1331} = K_{2332} = 1.$$ 

Given any matrix $X = (x_{ij}) \in \mathbb{R}^{3 \times 2}$. It is easy to see that $K \times_{3,4} X = X^\top$.

An even-order tensor $A = (A_{i_1 \ldots i_m j_1 \ldots j_m})$ is called pair-symmetric if

$$A_{i_\tau(1) \ldots i_\tau(m)} j_\tau(1) \ldots j_\tau(m) = A_{i_1 \ldots i_m j_1 \ldots j_m} \quad (3.3)$$

where $\tau \in S_m$ is an arbitrary permutation. Pair-symmetric tensors have applications in elastic physics [9]. Obvious that $K = K_{n,n}$ is pair-symmetric.

Now we can establish a multi-linear relationship between a matrix and its transpose through the commutation tensor.

**Theorem 3.2.** Let $K = K_{m,n}$ be an $(m,n)$-commutation tensor. Then $K \times_{3,4} X = X^\top$ for any $m \times n$ matrix $X$.

**Proof.** Denote $B := K \times_{\{3,4\}} X = (B_{ij})$. Then $B \in \mathbb{R}^{m \times n}$. By definition [3.2] we have for each $i \in [m], j \in [n]$

$$B_{ij} = \sum_{k,l} K_{ijkl} X_{kl} = K_{ijj} X_{ji} = X_{ji}$$

Thus $B = X^\top$. \qed

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We denote the multiplication $A \times \{3, 4\} B$ simply by $AB$ in the following for our convenience. The definition can also be extended to the case for any two tensors $A, B$ of order $2m$, where $A = (A_{i_1 \ldots i_m, j_1 \ldots j_m})$, $B = (B_{i_1 \ldots i_m, j_1 \ldots j_m})$ are consistent, i.e.,

$$(AB)_{i_1 \ldots i_m, j_1 \ldots j_m} = \sum_{k_1 \ldots k_m} A_{i_1 \ldots i_m, k_1 \ldots k_m} B_{k_1 \ldots k_m, j_1 \ldots j_m} \quad (3.4)$$

or to the case where $A$ is of order $2m$ and $B$ is of order $m$ in a similar way, resulting in an $m$-order tensor.

Let $A, B \in T_{2m,n}$ whose multiplication is defined by (3.4). Then $T_{2m,n}$ is closed under the multiplication. Furthermore

**Lemma 3.3.** $T_{2m,n}$ obeys an associative law under the multiplication defined by (3.4), i.e.,

$$(A \times B) \times C = A \times (B \times C) \quad (3.5)$$

**Proof.** Denote $\mathcal{F} = A \times B = (F_{i_1 \ldots i_m, j_1 \ldots j_m})$, $\mathcal{H} = B \times C = (H_{i_1 \ldots i_m, j_1 \ldots j_m})$, and $\mathcal{G} = \mathcal{F} \times C = (G_{i_1 \ldots i_m, j_1 \ldots j_m})$. For convenience, we denote $\sigma^{(i)} := (i_1 \ldots i_m)$. Thus

$$(\sigma^{(i)}, \sigma^{(j)}) = (i_1 \ldots i_m, j_1 \ldots j_m) \in S(2m,n)$$

and $\mathcal{F} = A \times B = (F_{\sigma^{(i)}, \sigma^{(j)})}$. We have by definition

$$G_{\sigma^{(i)}, \sigma^{(j)}} = \sum_{\sigma^{(k)}} F_{\sigma^{(i)}, \sigma^{(k)}} C_{\sigma^{(k)}, \sigma^{(j)}}$$

$$= \sum_{\sigma^{(k)}} \left( \sum_{\sigma^{(l)}} A_{\sigma^{(i)}, \sigma^{(l)}} B_{\sigma^{(l)}, \sigma^{(k)}} C_{\sigma^{(k)}, \sigma^{(j)}} \right)$$

$$= \sum_{\sigma^{(l)}} A_{\sigma^{(i)}, \sigma^{(l)}} \left( \sum_{\sigma^{(k)}} B_{\sigma^{(l)}, \sigma^{(k)}} C_{\sigma^{(k)}, \sigma^{(j)}} \right)$$

$$= \sum_{\sigma^{(l)}} A_{\sigma^{(i)}, \sigma^{(l)}} H_{\sigma^{(l)}, \sigma^{(j)}}$$

The right hand side of the last equality is exactly the entry $[A \times (B \times C)]_{\sigma^{(i)}, \sigma^{(j)}}$. Thus we complete the proof of (3.5). □

The equality (3.5) in Lemma 3.3 holds for any $2m$-order tensors $A, B, C$ whenever the multiplications in (3.5) make sense. Now we denote $\mathcal{K} = \mathcal{K}_{n,n}$ when no risk of confusion arises for $n$, and $\mathcal{K}^2 = \mathcal{K} \times \mathcal{K}$ is called the square of $\mathcal{K}$. We may also define any power $\mathcal{K}^m$ recursively due to Lemma 3.3 for any positive integer $m$, i.e., $\mathcal{K}^m = \mathcal{K}^{m-1} \mathcal{K}$. Note that the definition of a tensor power can be extended to any even-order tensor.

**Theorem 3.4.** 1. $\mathcal{K}^m = \mathcal{K}$ for any odd number $m$. 

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2. $\mathcal{K}^m = \mathcal{K}^2$ for any positive even number $m$.

Proof. (1). For each $(i, j, k, l) : i, j \in [m], k, l \in [n]$, we have by definition

$$K_{ijkl}^2 = \sum_{i_1, j_1} K_{i_1j_1i} K_{i_1j_1k} = K_{ijij} K_{ikkl} = K_{jikl}$$

(3.6)

It follows that $K_{ijkl}^2 = 1$ if and only if $i = k, j = l$. Thus

$$\mathcal{K}_{ijkl}^3 = \sum_{k', l'} K_{ijkl}^2 K_{k'l'kl} = K_{ijkl}^2$$

(3.7)

for each $(i, j, k, l) : i, j \in [m], k, l \in [n]$, which implies $\mathcal{K}^3 = \mathcal{K}$. Thus $\mathcal{K}^m = \mathcal{K}$ for any odd number $m$ if we apply recursively the fact (3.7).

To prove the second item, we note that from $\mathcal{K}^3 = \mathcal{K}$ it follows that $\mathcal{K}^4 = \mathcal{K}^2$ by the associative law (i.e., Lemma 3.3). Consequently for any even $m = 2k$ ($k \geq 2$) we have by recursion that $\mathcal{K}^m = \mathcal{K}^2$. We note that item (2) can also be proved as in the follows.

$$K_{ijkl}^4 = \sum_{i_1, j_1} K_{ijkl}^2 K_{i_1j_1kl}$$

$$= \sum_{i_1, j_1} K_{ij1j1k1} K_{j1i1kl} = K_{ijkl}$$

where the second equality is due to (3.6). Thus we have $\mathcal{K}^4 = \mathcal{K}$. This in turn follows by $\mathcal{K}^3 = \mathcal{K}^3 \mathcal{K}^4 = \mathcal{K}^3 = \mathcal{K}$. Consequently, we get (2). \qed

Now we let $B = (B_{ijk}) \in R^{n \times n \times m}$ with $B(\cdot, \cdot, k) = B_k \in R^{n \times n}$ corresponding to a permutation $\pi_k \in S_n$. Define $\mathcal{K}^\pi := (K_{i_1 \ldots i_m, j_1 \ldots j_m}) = B_{1} \times B_{2} \times \ldots \times B_{m}$ with $\pi = (\pi_1, \ldots, \pi_m)$ and

$$K_{i_1 \ldots i_m, j_1 \ldots j_m} = B_{i_1j_1} B_{i_2j_2} \ldots B_{i_mj_m}$$

$\mathcal{K}^\pi$ is called a generalised commutation tensor associated with $\pi$, or briefly a $\pi$-GCT. By definition, we have

$$K_{i_1 \ldots i_m, j_1 \ldots j_m} = 1 \iff j_k = i_{r(k)}, \ \forall k, (i_1, \ldots, i_m) \in S(m, n)$$

(3.8)

This is equivalent to

$$\mathcal{K}^\pi = B \times B \times \ldots \times B$$

where $B \in R^{n \times n}$ is a permutation matrix corresponding to $\sigma$. Note that a GCT $K$ becomes a commutation tensor if $m = 2$.

Let $\sigma \in S_n, A \in T_{m,n}$. We will show that tensor $A K^\pi$ (also $\mathcal{K}^\pi A$) can be regarded as a permutation of $A$ whose specific meaning is described in the following. For this purpose, we define tensor $A^\pi = (A_{i_1 \ldots i_m})$ by

$$A_{i_1i_2\ldots i_m} = A_{\sigma(i_1)\sigma(i_2)\ldots\sigma(i_m)}, \ \forall (i_1, \ldots, i_m) \in S(m, n)$$

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It is shown by Comon et al. [4] that an \( m \)-th order \( n \)-dimensional tensor \( A \in T_{m,n} \) can always be decomposed into

\[
A = \sum_{j=1}^{R} \alpha_{1j} \times \alpha_{2j} \times \ldots \times \alpha_{mj} \quad (3.9)
\]

where each \( \alpha_{ij} \in R^n \) is nonzero. The formula (3.9) is called a rank-1 decomposition or a CP decomposition of \( A \), the smallest possible number \( R \) in (3.9) is called the rank of \( A \). Furthermore, \( A \) can be decomposed into form

\[
A = \sum_{j=1}^{R} \alpha_{mj}^{m} \quad (3.10)
\]

if \( A \) is a symmetric tensor, where \( 0 \neq \alpha_{ij} \in R^n \). Given a tensor \( A \in T_{m,n} \) and a matrix \( B \in R^{n\times n} \). We define the right complete product of \( A \) by \( B \), denoted \( A \cdot [B] \), by

\[
A \cdot [B] := A \times_1 B \times_2 B \times_3 \ldots \times_m B.
\]

The complete left product of \( A \) by \( B \), denoted \([B] \cdot A \), is defined analogically. Note that for any \( k \in [m] \), we have

\[
(A \times_k B)_{i_1i_2\ldots i_m} = \sum_{i=1}^{n} A_{i_1\ldots i_{k-1}ii_{k+1}\ldots i_m} B_{ii_k}
\]

and

\[
(B \times_k A)_{i_1i_2\ldots i_m} = \sum_{j=1}^{n} B_{ij_k}A_{i_1\ldots i_{k-1}j_{k+1}\ldots i_m}
\]

Now suppose \( A \) has a CP decomposition (3.9) ( \( 0 \neq \alpha_{ij} \in R^n \)). Then \( A^\sigma \) can also be written as

\[
A^\sigma := \sum_{j=1}^{R} \alpha_{\sigma(1),j} \times \alpha_{\sigma(2),j} \times \ldots \times \alpha_{\sigma(m),j} \quad (3.11)
\]

For \( m = 2 \), \( A^\tau \) is either \( A \) itself (when \( \tau \) is the identity map) or the transpose of \( A \) (when \( \tau = (12) \) is a swap).

**Lemma 3.5.** Let \( A \in T_{m,n} \) be a real tensor with a CP decomposition (3.9) and let \( \tau \in S_m \). Then

\[
A^\tau = A \times K^\tau \quad (3.12)
\]

where \( K^\tau \in T_{2m;n} \) is an \( 2m \)-order GCT defined by (3.8).
Proof. For any \((i_1, i_2, \ldots, i_m) \in S(m, n)\), we let \(\tau(i_1, i_2, \ldots, i_m) = (j_1, j_2, \ldots, j_m)\), i.e., \(j_k = i_{\tau(k)}\) for all \(k \in [m]\). Then by (3.9)
\[
(A^\tau)_{i_1 i_2 \ldots i_m} = \sum_{j=1}^{R} a_{i_1 j}^{(1)} a_{i_2 j}^{(2)} \ldots a_{i_m j}^{(m)}
\]
\[
= \sum_{j=1}^{R} a_{j_1}^{(1)} a_{j_2}^{(2)} \ldots a_{j_m}^{(m)}
\]
\[
= A_{j_1 j_2 \ldots j_m}
\]
On the other side
\[
(A \times K^\tau)_{i_1 i_2 \ldots i_m} = \sum_{k_1, k_2, \ldots, k_m} A_{k_1 \ldots k_m} K_{k_1 \ldots k_m; i_1 \ldots i_m}
\]
\[
= A_{j_1 j_2 \ldots j_m}
\]
Thus \(A^\tau = A \times K^\tau\) for any \(\tau \in S_m\). The proof is completed. \(\square\)

We now denote \(K(\pi, p) = B \times \ldots \times B\), where \(B\) is the permutation matrix associated with \(\pi \in S_n, p \in [m]\). Thus \(K(\pi, m) = K^\pi\). We denote
\[
\mathbb{K}(m, n) := \{K^{\pi, m} : \pi \in S_n\},
\]
The multiplication on \(\mathbb{K}(m, n)\) can be defined by (3.4). Then we have

**Theorem 3.6.** (1) \(\mathbb{K}(m, n)\) is a subgroup of \(T_{2m,n}\) under the tensor multiplication defined by (3.4).

(2) \(K^{(id)} := K^{(id, m)} = I_n \times \ldots \times I_n\) (corresponding to the identity map) is the unique identity element in \(T_{2m,n}\), i.e.,
\[
X \times K^{(id)} = K^{(id)} \times X = X, \quad \forall X \in T_{2m,n} \quad (3.13)
\]

(3) Every element \(K^{\pi} \in \mathbb{K}(m, n)\) is invertible. Furthermore, its inverse is \((K^{\pi})^{-1} = K^{\pi^{-1}}\).

Proof. To prove (1), we let \(K_i = P_i \times \ldots \times P_i\) where \(P_i \in R^{n \times n}\) corresponds resp. to a permutation \(\pi_i \in S_n, i = 1, 2\). Then we have
\[
K_1 K_2 = (P_1 \times P_1 \times \ldots \times P_1)(P_2 \times P_2 \times \ldots \times P_2)
\]
\[
= (P_1 P_2) \times (P_1 P_2) \times \ldots \times (P_1 P_2)
\]
\[
= P \times P \times \ldots \times P
\]

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where \( P = P_1P_2 \in R^{n \times n} \) is also a permutation matrix corresponding to a permutation \( \pi = \pi_1\pi_2 \).

For (2), it suffices to prove the equality \( K^{(id)} \times X = X \) in (3.13) since the other part is similar to it. Denote \( Y = X \times K^{(id)} \). Then

\[
Y_{i_1 \ldots i_m;j_1 \ldots j_m} = \sum_{k_1 \ldots k_m} X_{i_1 \ldots i_m;k_1 \ldots k_m} K_{k_1 \ldots k_m;j_1 \ldots j_m}
\]

\[
= \sum_{k_1 \ldots k_m} X_{i_1 \ldots i_m;k_1 \ldots k_m} \delta_{k_1 j_1} \cdots \delta_{k_m j_m}
\]

\[
= \sum_{k_1 \ldots k_m} X_{i_1 \ldots i_m;j_1 \ldots j_m}
\]

for all possible \((i_1, \ldots , i_m), (j_1, \ldots , j_m) \in S(m, n)\). Thus we have \( X \times K^{id} = X \). The similar argument can be applied to prove \( X = K^{id} \times X \).

For (3), it suffices to verify that

\[
K^{\pi}K^{\pi^{-1}} = K^{id} \tag{3.14}
\]

and

\[
K^{\pi^{-1}}K^{\pi} = K^{id} \tag{3.15}
\]

To prove (3.14), we write \( \sigma := \pi^{-1} \). Then each entry of the left hand side of (3.14) is

\[
(K^{\pi}K^{\sigma})_{i_1 \ldots i_m;j_1 \ldots j_m} = \sum_{k_1 \ldots k_m} K^{\pi}_{i_1 \ldots i_m;k_1 \ldots k_m} K^{\sigma}_{k_1 \ldots k_m;j_1 \ldots j_m}
\]

\[
= \delta_{\sigma([i_1])j_1} \delta_{\sigma([i_2])j_2} \cdots \delta_{\sigma([i_m])j_m}
\]

\[
= \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_m j_m}
\]

the last equality is due to the fact that \( \sigma([k]) = ([\pi^{-1}])(k) = k \) for each \( k \in [n] \). It follows that each entry of the tensor \( K^{\pi}K^{\sigma} \) is either 1 or 0, and that \( (K^{\pi}K^{\sigma})_{i_1 \ldots i_m;j_1 \ldots j_m} = 1 \) if and only if \( i_1 = j_1, i_2 = j_2, \ldots , i_m = j_m \). Thus (3.14) holds. Similar argument applies to (3.15). The proof is completed. \( \square \)

From Theorem 3.6 we can see that \( K^{id} \) is the unique identity element in the group \( T_{2m,n} \). Given any element \( A \in T_{2m,n} \). The invertibility of \( A \) can also be defined as in Item (3) of Theorem 3.6 i.e., \( A \) is invertible if there exists a tensor \( B \in T_{2m,n} \) such that

\[
AB = BA = K^{id} \tag{3.16}
\]

The invertibility of an arbitrary tensor in \( T_{2m,n} \) is too complicated. But if we consider the following set

\[
G_{m,n} := \left\{ A := A \times \ldots \times A : \forall A \in R^{n \times n} \right\}
\]

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Then $G_{m,n} \subset T_{2m,n}$ is isometric to $R^{n \times n}$. Furthermore, we denote

$$A^x := A \times \ldots \times A \in T_{2m,n}$$

and

$$GL_{m,n} := \{ A^x : A \in GL(n) \}$$

where $GL(n)$ is the set of all the nonsingular matrices in $R^{n \times n}$. Then we have

**Theorem 3.7.** Given any tensor $A = A^x \in G_{m,n}$. Then $A$ is invertible if and only if $A$ is invertible. Furthermore, the inverse of $A$ is $(A^{-1})^x$.

**Proof.** This result is immediate from the fact that

$$A^x B^x = (A \times \ldots \times A)(B \times \ldots \times B) = AB \times \ldots \times AB$$

Thus $A^x B^x = K^{id}$ if and only if $AB = I_n$, i.e., $B = A^{-1}$.

Let $w := \{i_1, i_2, \ldots, i_p\}$ be a subset of $[m]$ $(1 \leq p \leq m)$. The incomplete product $K^{\tau,d} \times w A$ can be interpreted as the simultaneous row permutations of $A[n]$ by $\tau$ for each $n \in w$, where $A[n]$ stands for the matrix obtained from the unfolding of $A$ along mode-$n$.

For $m = 2$, there are two commutation tensors $K^{(12)}$ and $K^{id}$, where $\tau = id$ is the identity map on $\{1, 2\}$ and $K := K^{(12)}$ is the commutation tensor $K_{n,n}$. It is obvious that $K^{id} = K^2 := I$.

Given a tensor $A \in T_{2m,n}$. We call a matrix $A = (a_{ij})$ a balance unfolding of $A$, if

$$a_{ij} = A_{i_1i_2\ldots i_m;j_1j_2\ldots j_m}$$

with $i = 1 + \sum_{k=1}^{m} (i_k - 1)n^{k-1}$, $j = 1 + \sum_{k=1}^{m} (j_k - 1)n^{k-1}$, i.e., each row index $(i_1, i_2, \ldots, i_m)$ is turned into a row index, and each column index $(i_1, i_2, \ldots, i_m)$ is turned into a column index of $A$(See e.g. [12]). It is obvious that the balance unfolding of a tensor $A \in T_{2m,n}$ will produce a matrix of size $n^m \times n^m$. A tensor $A \in T_{2m,n}$ is called a balanced permutation tensor (abbrev. BPT) if the consequence of the balance unfolding of $A$ is a permutation matrix. We conclude this section by the following property of a nonnegative tensor in $T_{2m,n}$.

**Theorem 3.8.** Let $m, n > 1$ be two positive integers, and let $A \in T_{2m,n}$ be an entrywise nonnegative tensor. If it has a nonnegative inverse, then the balance unfolding of $A$ is a generalised permutation matrix.
Proof. Let $A$ and $B$ be respectively the balance unfolding of $A$ and $B := A^{-1}$ where $A^{-1}$ stands for the inverse of $A$. Then $A, B \in \mathbb{R}^{m \times m}$ are both entrywise nonnegative by the hypothesis. Since $AB = BA = K_{id}$, it follows that $A, B \in \mathbb{R}^{n \times n}$ are both entrywise nonnegative. By [21], $A$ must be a generalised permutation matrix $A = (A_{ij}) \in \mathbb{R}^{n \times n}$, that is, for each $i, j \in [n]$, there exists a unique nonzero positive entry.

4 From linear preservers to multilinear preservers

In this section, we will use the commutation tensors to deal with the linear preserving problem (LPP). The linear preserving problem has been investigated since the late 20th century by G. Frobenius. There are a lot of work concerning the LPP. We refer the reader to [18] for more detail.

A linear map $T$ on $\mathbb{C}^{n \times n}$ is called a determinant preserver if

$$\det(T(A)) = \det(A) \quad \forall A \in \mathbb{C}^{n \times n} \quad (4.1)$$

A linear transformation $T \in (\mathbb{C}^{n \times n})^*$ (which is called the dual space of $\mathbb{C}^{n \times n}$) is called a rank-1 preserver if $\text{rank}(A) = 1$ always implies $\text{rank}(T(A)) = 1$. Marcus and Moyls showed in 1959 that

**Lemma 4.1.** $T$ is a rank-1 preserver on $\mathbb{C}^{n \times n}$ if and only if there exist invertible matrices $P, Q \in \mathbb{C}^{n \times n}$ such that either

$$T(A) = PAQ \quad \forall A \in \mathbb{C}^{n \times n} \quad (4.2)$$

or

$$T(A) = PA^\top Q \quad \forall A \in \mathbb{C}^{n \times n} \quad (4.3)$$

In 1977, H. Minc showed [18] that a linear map $T$ on $\mathbb{C}^{n \times n}$ is a determinant preserver if and only if there exist invertible matrices $P, Q \in \mathbb{C}^{n \times n}$ with $\det(PQ) = 1$ such that either (4.2) or (4.3) holds. A linear rank preserver is surely a rank-1 preserver, the inverse is also true when the map is invertible. A linear determinant preserver must be a rank-1 preserver (by Lemma 4.1). As the operations on matrices, there are only two kinds, i.e., elementary operations and the transpose. Our aim is to unify them into one formula.

By Theorem 3.2 we know that the transpose of a matrix $A \in \mathbb{R}^{n \times n}$ is associated with $A$ by commutation tensor $K_n$ through $A^\top = KA$.

We denote by $\text{Aut}(V)$ for the set of all the linear automorphisms on a linear space $V$, and let $\pi \in \text{Aut}(\mathbb{R}^n)$. Then $\pi$ is determined by its behaviour on the coordinate vectors $e_1, e_2, \ldots, e_n$, where $e_i$ is the $i$th coordinate vector of $\mathbb{R}^n$. Denote $\hat{e}_i := \pi(e_i)$ for all $i \in [n]$. By the linearity of $\pi$, we have

$$\pi(x) = \sum_{j=1}^{n} x_j \hat{e}_j, \quad x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n \quad (4.4)$$
We are now ready to describe a linear symmetric automorphism \( \phi \in Aut(ST_{m,n}) \). Given any symmetric tensor \( A \in ST_{m,n} \), \( A \) has a symmetric CP decomposition (3.10). Thus

\[
\phi_\pi(A) = \sum_{j=1}^R \hat{\alpha}_j^m
\]

where \( \hat{\alpha}_j = \pi(\alpha_j) \) for each \( j \in [R] \). We call \( \phi \) a positive map if it preserves the nonnegativity of tensors. We have

**Lemma 4.2.** Let \( \phi \in Aut(ST_{m,n}) \) be a linear symmetric rank-1 preserver. Then for any \( 0 \neq x \in \mathbb{R}^n \), there exists a nonzero scalar \( \lambda_1 \in \mathbb{R} \) and \( 0 \neq y \in \mathbb{R}^n \) such that

\[
\phi(x^m) = \lambda y^m
\]

**Proof.** Given any \( 0 \neq x \in \mathbb{R}^n \). Since \( \phi \) is a rank-1 preserver, there exist nonzero vectors, say \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}^n \), such that

\[
\phi(x^m) = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_m
\]

Denote \( A = (A_{i_1 i_2 \ldots i_m}) = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_m \), and \( A = [\alpha_1, \alpha_2, \ldots, \alpha_m] = (a_{ij}) \in R^{n \times m} \). The result is equivalent to \( \text{rank}(A) = 1 \). We first note that \( A \) is a nonzero tensor since \( \text{rank}(A) = 1 \), and thus each \( \alpha_j \neq 0 \). We may assume w.l.o.g. that \( a_{i_k k} \neq 0 \) for \( k = 3, 4, \ldots, n \) (if \( n > 2 \)). Now consider two index

\[
(i_1, i_2, i_3, i_4, \ldots, i_m), \quad (j_1, j_2, i_3, i_4, \ldots, i_m) \in S(m, n)
\]

By the symmetry of \( A \), we have

\[
A_{i_1i_2i_3\ldots i_m} = A_{j_1j_2i_3\ldots i_m}
\]

which, by (4.7), is equivalent to

\[
a_{i_1}a_{i_2}a_{i_3}a_{i_4} \ldots a_{i_m m} = a_{j_1}a_{i_2}a_{i_3}a_{i_4} \ldots a_{i_m m}
\]

It follows that

\[
a_{i_1}a_{i_2} = a_{j_1}a_{j_2}, \quad \forall i, j \in [n]
\]

Thus we have

\[
\frac{a_{i_1}}{a_{i_2}} = \frac{a_{j_1}}{a_{j_2}}, \quad \forall (i, j) : a_{i_1}a_{j_2} \neq 0.
\]

It follows by (4.9) that \( \text{rank}(A) = 1 \). So we may write \( \alpha_k = \mu_k \alpha_1 \) for all \( k \in [m] \) \((0 \neq \mu_j \in \mathbb{R}) \) with \( \mu_1 = 1 \). Denote \( y := \alpha_1 \in \mathbb{R}^n \) which is a nonzero vector, and \( \lambda_j := \mu_1 \mu_2 \ldots \mu_m \). Then (4.6) is proved.

It is easy to see from Lemma 4.2 that a linear symmetric rank-1 preserver \( \phi \) on \( Aut(T_{m,n}) \) can be uniquely determined by a linear positive mapping, say \( \pi \), on \( \mathbb{R}^n \), i.e., \( \pi(x) = \tilde{x} = Mx \) for some invertible matrix \( M = (m_{ij}) \in R^{n \times n} \). Now we express our main result on the multilinear symmetric rank preserver.
Theorem 4.3. Let $\phi$ be a multi-linear map on $ST_{m,n}$. Then $\phi \in Aut(ST_{m,n})$ is a linear symmetric rank preserver if and only if there exists an invertible matrix $B \in R_n^{n \times n}$ such that

$$\phi(X) = [B] \cdot X \quad (4.10)$$

for any symmetric tensor $X \in T_{m,n}$.

Proof. For the necessity, we let $\phi \in Aut(ST_{m,n})$ be a symmetric rank preserver. By Lemma 4.2, we know that $\phi$ is determined by its projection on $R^n$, say $\pi \in Aut(R^n)$. Now we write $B = [\beta_1, \beta_2, \ldots, \beta_n] \in R_n^{n \times n}$ with $\beta_j = \pi(e_j)$ for $j \in [n]$. Here $e_j \in R^n$ is the $j$th coordinate vector of $R^n$. It is obvious that $B$ is invertible since $\pi$ is an automorphism in $R^n$. Now we write $X$ as

$$X = \sum_{i_1, i_2, \ldots, i_m} X_{i_1 i_2 \ldots i_m} e_{i_1} \times e_{i_2} \times \ldots \times e_{i_m}$$

where $e_j \in R^n$ is the $j$th coordinate vector for each $j \in [n]$. By the linearity we get

$$\phi(X) = \sum_{i_1, i_2, \ldots, i_m} X_{i_1 i_2 \ldots i_m} \phi(e_{i_1} \times e_{i_2} \times \ldots \times e_{i_m})$$

$$= \sum_{i_1, i_2, \ldots, i_m} X_{i_1 i_2 \ldots i_m} \pi(e_{i_1}) \times \pi(e_{i_2}) \times \ldots \times \pi(e_{i_m})$$

$$= \sum_{i_1, i_2, \ldots, i_m} X_{i_1 i_2 \ldots i_m} \beta_{i_1} \times \beta_{i_2} \times \ldots \times \beta_{i_m}$$

which is followed by $\phi(X) = [B] \cdot X$.

Now we prove the sufficiency. Suppose that $\phi \in Aut(ST_{m,n})$ is a linear map satisfying (4.10). Denote $\hat{X} := \phi(X) = (\hat{X}_{i_1 i_2 \ldots i_m})$. We let $\pi$ be the projection of $\phi$ on $R^n$ as defined above, and let (3.10) be the CP decomposition of $X$. Then for any $(i_1, i_2, \ldots, i_m) \in S(m,n)$, we have by (4.10)

$$\hat{X}_{i_1 i_2 \ldots i_m} = \sum_{j_1, j_2, \ldots, j_m} X_{j_1 j_2 \ldots j_m} b_{i_1 j_1} b_{i_2 j_2} \ldots b_{i_m j_m}$$

$$= \sum_{j=1}^{R} \left[ \sum_{j_1, j_2, \ldots, j_m} x_{j_1 j_2 \ldots j_m} b_{i_1 j_1} b_{i_2 j_2} \ldots b_{i_m j_m} \right]$$

$$= \sum_{j=1}^{R} (\hat{\alpha}_{i_1 i_2 \ldots i_m})^{j}$$

It follows that

$$\hat{X} = \sum_{j=1}^{R} \hat{\alpha}_{i_1 i_2 \ldots i_m}^{j}$$

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which means that $\phi \in Aut(ST_{m,n})$ is a linear symmetric rank preserver induced by $\pi$. \hfill \square

Let $B^{(k)} = (t_{ij}^{(k)}) \in R^{p_k \times q_k}$ for $k \in [m]$. We define the $2m$-order tensor $B := B(B^{(1)}, \ldots, B^{(m)})$ as

$$B = B^{(1)} \times B^{(2)} \times \ldots \times B^{(m)} = [B_{i_1 \ldots i_m j_1 \ldots j_m}]$$

whose entries are defined by

$$B_{i_1 \ldots i_m j_1 \ldots j_m} = b^{(1)}_{i_1 j_1} b^{(2)}_{i_2 j_2} \ldots b^{(m)}_{i_m j_m} \quad (4.11)$$

$B := B(B^{(1)}, \ldots, B^{(m)})$ is called the tensor product of $(B^{(1)}, \ldots, B^{(m)})$, and is of size $p_1 \times \ldots \times p_m \times q_1 \times \ldots \times q_m$. Then we have

$$[B_1, B_2, \ldots, B_m] : A = B \times A \quad (4.12)$$

For $B_1 = \ldots = B_m = B \in R^{n \times n}$, we write $B := B(B, \ldots, B) \in T_{2m,n}$. By Theorem 4.3 we get

$$\phi(X) = B \times X, \quad \forall X \in ST_{m,n} \quad (4.13)$$

For $m = 2$, (4.13) turns out to be $\phi(X) = BXB^\top$. It is immediate from Theorem 4.3 that each column of $B$ is exactly the image of the projection of $\phi$ on $R^n$. From Theorem 4.3 we obtain

**Corollary 4.4.** Let $\phi \in Aut(C^{n \times n})$. Then $\phi$ is a symmetric rank preserver if and only if there exist invertible matrix $P \in C^{n \times n}$ such that

$$\phi(X) = X \times_1 P^\top \times_2 P, \quad \forall X \in C^{n \times n} \quad (4.14)$$

Formula (4.13) in Corollary 4.4 in the matrix form is $\phi(X) = P^\top XP$ which is exactly the form for a (linear) symmetric rank preserver. Now if $\phi$ is also required to be a positive preserver (i.e., preserving the entrywise nonnegativity of a tensor), the we have

**Corollary 4.5.** Let $\phi \in Aut(ST_{m,n})$ be a nonnegative linear symmetric preserver. Then $\phi$ fixes the identity tensor if and only if $B$ is a permutation matrix in (4.10), if and only if the projection of $\phi$ on $R^n$ preserves the set $\{e_1, e_2, \ldots, e_n\}$.

**Proof.** Let $\phi \in Aut(ST_{m,n})$ be a nonnegative linear symmetric preserver that fixes the identity tensor. Then by Theorem 4.3 there exists an invertible matrix $B = [\beta_1, \ldots, \beta_n] \in R^{n \times n}$ such that (4.10) holds. Thus we have

$$I = I \times_1 B \times_1 B \times_2 \ldots \times_m B \quad (4.15)$$
since \( \phi \) preserves the identity tensor. It follows

\[
\mathcal{I} = \sum_{j=1}^{n} \beta_j^m
\]  

(4.16)

By the linearity of \( \phi \), we have

\[
\mathcal{I} = \phi(\mathcal{I}) = \sum_{j=1}^{n} \phi(\epsilon_j^m) = \sum_{j=1}^{n} \beta_j^m
\]  

(4.17)

Denote \( B = [b_{ij}] \in \mathbb{R}^{n \times n} \). By the nonnegativity of \( \beta_j \) and (4.17), we have

\[
(\beta_j^m)_{i_1i_2...i_m} = b_{i_1j}b_{i_2j}...b_{i_mj} = 0
\]  

(4.18)

for all \( j \in [n] \) and all \( (i_1, i_2, ..., i_m) \in S(m, n) \) where \( i_1, i_2, ..., i_m \) are not identical. It follows that each \( \beta_j \) is a coordinate vector. It turns out that \( B \in \mathbb{R}^{n \times n} \) shall be a permutation matrix since by (4.17) each row of \( B \) has a unique one.

Conversely, we suppose that (4.10) holds with \( B \in \mathbb{R}^{n \times n} \) being a permutation matrix. We may assume that \( B \) corresponds to a permutation \( \tau \in S(n) \), i.e., \( b_{ij} = 1 \) if and only if \( j = \tau(i) \) for each \( i \). Denote \( \hat{\mathcal{I}} := \phi(\mathcal{I}) = (\hat{I}_{i_1i_2...i_m}) \). For each \( (i_1, i_2, ..., i_m) \in S(m, n) \), we have

\[
\hat{I}_{i_1i_2...i_m} = \sum_{j_1,j_2,...,j_m} I_{j_1j_2...j_m} b_{i_1j_1}b_{i_2j_2}...b_{i_mj_m} = I_{\tau(i_1)\tau(i_2)\cdots\tau(i_m)}
\]

which equals 1 if and only if \( \tau(i_1) = \tau(i_2) = \cdots = \tau(i_m) \) by the definition of the identity tensor. It follows that

\[
\hat{I}_{i_1i_2...i_m} = 1 \iff i_1 = i_2 = \cdots = i_m \in [n]
\]

Consequently \( \hat{\mathcal{I}} := \mathcal{I} \). So \( \phi \) fixes the identity tensor.

The second part of the corollary (i.e., the projection of \( \phi \) on \( \mathbb{R}^n \) preserves set \( \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \) can be deduced directly from the proof of Theorem 4.3.

Now we extend the result in Theorem 4.3 and consider a general linear rank preserver in \( \text{Aut}(\mathcal{T}_{m,n}) \).

**Theorem 4.6.** Let \( \phi \in \text{Aut}(\mathcal{T}_{m,n}) \) be a linear map and \( \mathcal{A} \in \mathcal{T}_{m,n} \) be a rank-\( R \) tensor possessing a \( \text{CP} \) decomposition (3.9). Then \( \phi \) is a rank preserver if and only if there exist invertible matrices \( B_j \in \mathbb{R}^n, j \in [m] \) and a permutation \( \tau \in S_m \) such that

\[
\phi(\mathcal{A}) = B \times \mathcal{A} \times \mathcal{K}^\tau
\]  

(4.19)

where \( B := B_1 \times B_2 \times \cdots \times B_m \in \mathcal{T}_{2m,n} \) is an \( 2m \)-order tensor defined by (4.11).
Proof. For the sufficiency, we let \( \phi \) be defined by (eq: phirkpreserv). We want to prove that it is a rank preserver. It suffice to prove that it preserves the rank-1 tensor. But this is an obvious fact when we look at the rank-1 tensor \( A = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_m \) since by (4.19) we have

\[
\phi(A) = \left[ B_1, B_2, \ldots, B_m \right] \cdot A \times K^\tau
\]

where \( i_k = \tau(k) \) and \( \hat{\alpha}_{i_k} = B_k \alpha_{i_k} \in \mathbb{R}^n \) is nonzero due to the nonsingularity of \( B_k \).

Conversely, let \( A \in T_{m,n} \) be a rank-\( R \) tensor with CP decomposition (3.9) and \( \phi \) be a rank preserver. Denote by \( \pi_k \) the projection of \( \phi \) on direction \( k \) \( (k \in [m]) \). Then by Corollary 3.10 of [3] we have

\[
\phi(A) = \sum_{j=1}^R \phi(\alpha_{1j} \times \alpha_{2j} \times \ldots \times \alpha_{mj})
\]

\[
= \sum_{j=1}^R \hat{\alpha}_{i_{1j}} \times \hat{\alpha}_{i_{2j}} \times \ldots \times \hat{\alpha}_{i_{mj}}
\]

\[
= \sum_{j=1}^R \alpha_{i_{1j}} \times \hat{\alpha}_{i_{2j}} \times \ldots \times \hat{\alpha}_{i_{mj}}
\]

with \( i_k = \tau(k) \) where \( \tau \) is a permutation on \([m]\) and \( \hat{\alpha}_j = \pi_j(\alpha_j) \). Now we denote

\[
\hat{A}_j := \alpha_{i_{1j}} \times \hat{\alpha}_{i_{2j}} \times \ldots \times \hat{\alpha}_{i_{mj}}, \quad \forall j \in [m]
\]

By Lemma 3.4 we have \( \hat{A}_j^\tau = \hat{A}_j \times K^\tau \) for each \( j \in [m] \). It follows that

\[
\hat{A} = \phi(A) = \sum_{j=1}^R \hat{A}_j^\tau
\]

\[
= \sum_{j=1}^R \hat{A}_j \times K^\tau
\]

\[
= (\sum_{j=1}^R \hat{\alpha}_{i_{1j}} \times \hat{\alpha}_{i_{2j}} \times \ldots \times \hat{\alpha}_{i_{mj}}) \times K^\tau
\]

\[
= \left[ B_1, \ldots, B_m \right] \cdot A \times K^\tau
\]

\[
= B \times A \times K^\tau
\]

The second last equality is due to the fact that for each \( j \in [m] \) there exists an invertible matrix \( B_j \in \mathbb{R}^{n \times n} \) such that \( \phi_j(x) = B_k x \) for all \( x \in \mathbb{R}^n \) since
\( \phi_j \in Aut(\mathbb{R}^n) \) (the set of linear automorphisms \( Aut(\mathbb{R}^n) \) of \( \mathbb{R}^n \) is isomorphic to the general linear group \( GL_n \) consisting of all invertible matrices in \( \mathbb{R}^{n \times n} \)), and the last equality follows from \[1.12\]. Thus there exist invertible matrices \( B_1, B_2, \ldots, B_m \) such that \( \hat{\alpha}_{kj} = B_k \alpha_k \). The proof is completed. \( \square \)

We call tensor \( \mathcal{B} = B_1 \times \ldots \times B_m \) an associated tensor with \( \phi \) if \( \phi_j(x) = B_jx \) for all \( x \in \mathbb{R}^n \) where \( \phi_j \) is the projection of \( \phi \) on direction \( j \). Note that if we fix a \( \tau \), then \( \phi \) is associated with an \( \mathcal{B} \) uniquely. Conversely, from the argument above, we see that there are \( m! \) linear rank preservers in \( Aut(T_{m,n}) \) associated with an \( 2m \)-order tensor generated by \((B_1, \ldots, B_m)\). Also if we choose \( \tau \) to be the identity map on \( [m] \), then by \[4.19\] the associated linear rank preserver \( \phi \) acts on \( T_{m,n} \) in form

\[
\phi(X) = \mathcal{B} \times X, \quad \forall X \in T_{m,n} \quad (4.20)
\]

In order to describe the linear identity and rank preserver, we let \( B^{(k)}(i_j) \in \mathbb{R}^{n \times n}, \forall k \in [p] \), and let \( \mathcal{B} := (B_{ijk}) \in \mathbb{R}^{m \times n \times p} \) be the 3-order tensor defined by \( B(:, :, k) = B^{(k)} \) for each \( k \in [p] \), and define \( \hat{\mathcal{B}} := (\hat{B}_{i_1 i_2 \ldots i_p}) \) as the Hadamard product of \( (B^{(1)}, B^{(2)}, \ldots, B^{(p)}) \), i.e.,

\[
\hat{B}_{i_1 i_2 \ldots i_p} = \sum_{j=1}^{n} b^{(1)}_{i_1 j} b^{(2)}_{i_2 j} \ldots b^{(1)}_{i_p j}
\]

Here each entry of \( \hat{\mathcal{B}} \) can be regarded as the Hadamard product of the corresponding \( p \) mode-3(along the third direction) slices of tensor \( \mathcal{B} \). When each \( B^{(k)} \in \mathbb{R}^{n \times n} \) is a permutation matrix, say, corresponding to a permutation \( \pi_k \), then we have It follows immediately from that

**Corollary 4.7.** Let \( \phi \) be a linear positive rank preserver on \( T_{m,n} \) associated with the identity map \( \mathcal{I} \). Then it fixes the identity tensor if tensor \( \mathcal{B} \) associated with \( \phi \) is an identity tensor, i.e.,

\[
\mathcal{B} \times \mathcal{I} = \mathcal{I} \quad (4.21)
\]

**Corollary 4.8.** Let \( \phi \in Aut(\mathbb{R}^{n \times n}) \) be a linear map. Then \( \phi \) is a rank preserver if and only if there is invertible matrices \( B_1, B_2 \in \mathbb{R}^{n \times n} \) such that

\[
\phi(X) = X \times_1 B_1 \times_2 B_2 \quad (4.22)
\]

for any real matrix \( X \in \mathbb{R}^{n \times n} \).

Note that formula \[4.22\] is exactly the unified form of two formulae \[1.2\] and \[1.3\].

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