A Certain Conditions on Some Rings Give P.P.Ring

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Abstract. Many new results were obtained in this paper about P.P. ring. Semi-primitive ring $R$ with (dcc) on principal-ideal $P$ is always P.P. ring. Also, St-G.P.P. ring $R$ is given to answer new question; is $B$-rings are P.P. ring. Also regular and Von Neumann regular rings are introduced to find a relationship between P.P. ring and these kind of the rings.

Keywords: P.P. ring, Baer ring, regular ring, weakly regular ring, St-G.P.P. ring.

Some Abbreviation:
weak-regular ring: (W-reg-ring).
prime ideal: Pr-ideal
W-reg-ring: weak regular ring

1. Introduction

All rings are associative with identity. “In [3], Endo introduced general notes about ring $R$ which every ideal in $R$ is principle and projective (P.P. ring)” “In[4], Goodearl studied Von Neumann regular ring in details”. “In [5], Huh, Kim, and Lee achieved the generalization of P.P. ring over non comm. ring”. “In [6], Hong, Kim and Kwak introduced many extension of Baer and P.P. rings”. “In [1], Banh and Le they explained many properties about
P.P.ring”. Every Sem-simple ring is Von Neumann regular ring. Any Noetherian reg-ring is Sem-simple ring. “Any Von Neumann reg-ring is Jacobson-Sem-simple ring” [2]. In this article, we try to find new several relationships between P.P.ring and other kinds of rings.

2. Basic Concepts

In this section we need to state some definitions and lemmas which will be used later to achieve our main goal.

Definition 2.1.[4]. “We say that a ring R is regular (Reg-ring) if for each \( \alpha \in R \), \( \exists \beta \in R \) \( \exists \alpha \beta \alpha = \alpha \).”

Definition 2.2.[7]. “We say that R is a Baer ring if the left Ann(S) = \( e \in R \) s.t \( \emptyset \neq S \subseteq R \) and e is idempotent element.”

Definition 2.3.[8]. “We say that R is P.P.ring if all conditions in Definition 2.2 are true. Equivalently if each principal left ideal of R is projective”.

Definition 2.4.[1]. “Given a group G and a ring R, define the Group Ring RG to be the set of all linear combinations \( \alpha = \sum_{g \in G} a_g g \); where \( a_g \in R \) and where only finitely many of the elements \( a_g \) are non-zero”.

Lemma 2.5.(1), Proposition 1.9). “Suppose R is reg-ring and G is a locally finite group. If the order of finite subgroup of G is a unit in R, then RG is left P.P. ring and so R is P.P. ring”.

Lemma 2.6.[2]. “Any ring R is called semisimple (Sem-simple) if:
1) If M is left R-module has short exact.
2) If M is Sem-simple,
3) If M is f.g. hence is Sem-simple,
4) Cyclic left R-module is Sem-simple,
5) Regular left R-module is Sem-simple”.

Lemma 2.7.[4]. “Let a, b are elements of a ring R. If \( (a – aba) \) is von Neumann regular, then so is a”.

Lemma 2.8. ([1], Proposition 2.15). “If R is a right strictly G.P.P ring with enough idempotents, then R has no nonzero nil right (or left) ideals”.

3. P.P.Ring And Regular property

In this section, we study the direct relationship between two ring, namely, P.P.ring and regular ring (Reg-ring).

Lemma 3.1. Let R be a Ske-field and G be locally finite group. If H is finitely-subgroup (f-subgroup) of R and a unit in R, then R is P.P.ring.

Proof: From definition of sk-field, we can say R is Reg-ring. But R is f-subg with unit property in R leads the group ring (RG) is P.P.ring.

Theorem 3.2. If R is a Sem-simple ring and satisfies all condition in (Lem. 3.1), then R is P.P.ring.

Proof: Since R is a Sem-simple ring; R is a Reg-ring. Hence from (Lem. 3.1), RG is P.P.ring and from (Lem. 2.10), we get the required.

Theorem 3.3. Let R be a ring and G be finitely-locally group. If H finitely-subgroup (f-sub) of G with a unit in R and all cyclic modules over R are projective, then R is P.P.ring.
Proof: We have any cyclic modules over R are projective. Therefore we need to use the previous in order to obtain a Sem-simple character for ring R (see Lem. 2.6), and prove only any short exact:

\[ \{0\} \longrightarrow I \longrightarrow R \longrightarrow \frac{R}{I} \rightarrow \{0\} \]

splits \( \exists I \) is an ideal in R.

Also \( \frac{R}{I} \) is cyclic module over R. So R is Sem-simple ring. So R is Von-reg-ring (R is a Reg-ring). But H is f-subg of G. Then R is P.P. ring.

Theorem 3.4. Let \( R_1, R_2, \ldots, R_n \) be a Sem-simple rings. If R equal the direct product of these rings, then R is P.P. ring.

Proof: Firstly, we say \( R_i = I_{1i} \oplus \cdots \oplus I_{ni} \) (mini-ideals) \( \subseteq R_i \). Let \( R_i \) be a 2-sided ideals of R. Then every \( I_j \) is minimal left R-ideal. Hence

\[ rR = rR \oplus \cdots \oplus rR = \oplus rI_j. \]

Thus \( rR \) is a left Sem-simple ring (Reg-ring). So R has an principal projective ideal.

Corollary 3.5. Let \( R \) be a Sem-simple ring. If \( R \cong M_n(D) \) such that D is skew-field, then R is Reg-ring.

Proof: \( \forall I \in R \ni I \) is a mini-ideal, then \( A_1 \) is a 2-sided ideals of R. Since R is simple, then \( A_1 = R \). Therefore R is a Sem-simple. But this means R is Reg-ring.

Remark 3.6.

1) The center of any Sem-simple ring is also ring with principal projective ideal.

2) Any cyclic-group (Cyc-group) has order two over \( Z_2 \) as \( GR \) as \( Z_2[G] \). Note that Cyc-group has no principal projective ideal, because \( Z_2[C_2] \) is not Sem-simple and so is not Reg-ring.

In the next theorem we shows a good fact which it explain the relationship between Sem-primitive and P.P.rings. But before that we need to introduce the following definition.

Definition 3.7. We say that R is Sem-primitive ring (some time is called J-S-simple if \( J(R) = \{0\} \).

Theorem 3.8. Let \( R \) be a Sem-primitive ring. If \( R \) has (dcc) on Pr-ideal P is always P.P.ring.

Proof: Suppose \( R \) is a Sem-primitive ring and satisfies (dcc) on Pr-ideal. In this case any \( 0 \neq I \) has mini-ideal. We know mini-ideal is \( \overline{R}(i.e. I = \overline{R}) \). Since \( R \) is a Sem-primitive with \( I \neq \{0\} \), then there exists a mini-ideal \( I^* \). So \( 1^* I^* = R \). So \( I \cap I^* = \{0\} \). Finally \( R = I + I^* \). We claim that \( R \) is not Sem-simple ring. In this case, we need to assum \( I_1 \) is mini-ideal belong to R and \( I_1 \) is an ideal s.t \( R = I_1 + I_1^* \). Hence \( I_1 \) is not {0}. and there exists mini-ideal \( I_2 \subseteq I_1^* \). So \( R = I_1 \oplus I_2 \) s.t \( I_1 \) is an ideal in R. Now we take \( I_2 = I_1 \cap I_2^* \). Then \( R = I_1 \oplus I_2 \). We have (dcc) of ideals \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \), because \( I_1^* \) is Pr-ideal and this contradiction. Thus R is Sem-simple ring (Reg-ring) and so is P.P.ring.

Example 3.9. The ring of integer numbers \( Z \) is Sem-primitive and satisfies (dcc) on Pr-ideal P. So \( Z \) is P.P. ring because, \( J(Z) = \cap PZ = \{0\} \); \( Z \) has infinitely many of maximal-ideal PZ.

Example 3.10. Any Ske-field has (dcc) on Pr-ideals is P.P.ring, because Ske-field is Sem-primitive with (dcc) we obtain P.P.ring. (see Th. 3.8.).
Corollary 3.11. Any simple ring $R$ s.t satisfies (dce) on Pr-ideal is P.P.ring.

**Proof:** Suppose that $R$ is simple ring and $0 \neq M$ is an $R$-module. Since $\text{Ann}(M)$ is 2-sided ideal in $R$, then $\text{Ann}(M) = \{0\}$. So $M$ is faithful. Hence $R$ is also faithful simple as a module. Thus $R$ is primitive. This means $R$ is J-Seme-simple (J(R) = \{0\}). So $R$ is a Sem-primitive (P.P.ring).

**Remark 3.12.** Any ring $R$ satisfies all the conditions in (Cor. 3.11) is called prime, because:

Let $0 \neq I \in R \ni IM \neq 0$ and $IM \leq M$. If $M$ is a simple module, then $IM = M$. IF $0 \neq I$ and $J \neq 0$, then $IJM = M$. So $IJ \neq \{0\}$. Thus $R$ is a prime ring.

A ring $R$ is called a generalized right P.P. ring, (G.P.P.ring), if $\forall x \in R$, $\exists n \in \mathbb{N}$ such that $x^nR$ is projective and $R$ is called strictly generalized right PP, (St-G.P.P.ring), if for any nonzero $x \in R$ $\exists n \exists x^n \neq 0$ and the right ideal $x^nR$ is projective.

**Lemma 3.13.** If $R$ is right St-G.P.P.ring and right $C_2$; then $R$ is V-N-Reg-ring.

**Proof.** Let $0 \neq x \in R$. Since $R$ is St-G.P.P.ring, $\exists n \exists x^n \neq 0$ and $r(x^n) = eR$ for some $e^2 = e \in R$. So $x^nR \cong R(r(x^n) \cong (1-e)R)$. Since $R$ fulfills $C_2$, $x^nR \cong R$ and hence, $x^n$ is V-N-Reg-ring. If $n = 1$, then we are done. Otherwise, we show that $x^n$ is also V-N-Reg-ring. This is so, and by induction, $x^n = x^nR$. Indeed, let $e \in R$ be such that $x^n = x^nR$. Put $y = x^{n-1} - x^{n-1}(cx)x^{n-1} = x^{n-1} - x^{n-1}cx^n$. It is easy to check that $y^2 = 0$. In the following two cases, consider:

Case 1: $y = 0$. Then $x^nR = x^{n-1}cx^n$, which implies that $x^{n-1}$ is V-N-Reg-ring.

Case 2: $y \neq 0$. By the equivalent statement of the aforementioned proof and from the aforementioned proofs $y^2 = 0$, We see this is $y$ V-N-Reg-ring. Thus, $x^{n-1}$ is also V-N-Reg-ring(see Lem.2.8).

**Corollary 3.14.** Let $R$ be an abelian ring. If $R$ is a St-G.P.P.ring, then $R$ is P.P.ring.

**Proof.** From (Lem.3.13).

**Corollary 3.15.** Any right St-G.P.P.ring is a Baer ring and so is P.P. ring.

**Proof.** Let $R$ be a right St-G.P.P. ring and $S \subseteq R$, there exists an idempotent $e \in R \ni i(S) = Re \oplus (l(S) \cap R(1-e))$ and $l(S) \cap R(1-e)$ It’s Nile. We have no non-zero nil ideals for $R$ (see Lem2.13). Hence $l(S) = Re$. So $R$ is Baer ring ( P.P. ring).

Finally of this paper we need to study the relationship between P.P. ring and another property namely weak-regular ring (W-reg-ring). See sec. 4.

4. **P.P.Ring and Weak-Regular Property**

In this section, we study another case which it explain same our oue goal but between P.P.ring and W-reg-ring.

**Definition 4.1.** We say that $R$ is weak regular ring (for short W-reg-ring) if any f.g ideals $I$ and $J$ in $R$; $I \subseteq J \subseteq R$ s.t $I = \langle e \rangle$ and $J = \langle e \rangle$ in $R$.

**Lemma 4.2.** Any ring $R$ has exactly two idempotent elements $0$ and $1$ is W-reg-ring.

**Proof.** Suppose that $I$ and $J$ are two f.g ideals. Assume that $I \nsubseteq J$ and $J = \langle e \rangle$. Therefore $J = \{0\}$ and $I = \{0\}$. Thus $R$ is W-reg-ring.
Examples 4.3.
   a) Integral domain is W-reg-ring.
   b) Local ring is W-reg-ring.

Theorem 4.4. Let $\alpha$ be a non unit element in W-reg-ring s.t $1 \neq e$ an idempotent and $\alpha \in \mathbb{R}e$. Hence $R$ is P.P. ring.

Theorem 4.5. Let $R$ be a ring. Then the following are equivalent:
   a) $R$ is Reg-ring.
   b) $R$ is W-reg-ring.

Proof. $a \Rightarrow b$. It is clear.

$b \Rightarrow a$. Suppose that $\alpha$ is non unit element in $R$. Also assume that $e$ is non unit idempotent element in $R$ and $\alpha \in \mathbb{R}e$. $\exists \alpha \in R \alpha^2$. In this case $\alpha = \alpha^2a; a \in R$. So $R$ is reg-ring.

Note that, for example $\mathbb{Z}^n$ is not W-reg-ring s.t $n \geq 2$ because $\mathbb{Z}$ is not Reg-ring. It follows that $\mathbb{Z}^n$ is not W-reg-ring. Finally, $\mathbb{Z}^n$ is not P.P. ring.

Corollary 4.6. If $I$ is Pr-ideal in the ring $R$, then $\frac{R}{I}$ is P.P. ring.

Proof. We have $\frac{R}{I}$ is W-reg-ring. So is reg-ring (P.P. ring).

Now again we return to the condition exactly two idempotent but on another kind of rings namely polynomial ring. See the next theorem.

Theorem 4.7. Let $R[X]$ be a polynomial ring and has exactly two idempotent elements. Then $R$ is P.P. ring.

Proof. Suppose that $R$ has exactly two idempotent elements. If we take $a = a_0 + a_1 + a_2 + \ldots + a_n$ is an idempotent element $I R[X]$, then $(a_0^2) = a_0$. So $a_k = 0$ s.t $k$ belong to the set $\{1, \ldots, n\}$. On the other hand, we have $a_1 = 2a_0a_1$. So $a_0a_1 = a_0 (2a_0a_1)$. Therefore $a_1 = 0$. Also $\sum a_k a_{k-1} = a_k$. Hence $a_k = 2 a_0 a_k$. Finally $a_k = 0$. We can write all idempotent elements in $R[X]$ as $\{a \in R; a^2 = a\}$. So $R[X]$ is W-reg-ring and hence reg-ring. So the polynomial $R[X]$ has principal projective ideal.

Remark 4.8. Note that $R[x_1, \ldots, x_n]$ is a W-reg-ring and so is P.P. ring especially when $R$ has exactly two idempotent elements 0 and 1. Also if $R[X]$ is a ring formal power series in $x$ and all the coefficients belong to $R$, then $R[X]$ is a W-reg-ring and so is P.P. ring.

5. Conclusion

The main results in this paper are any right St-G.P.P. ring is a Baer ring and so is P.P. ring. Also in Corollary 3.11 we obtained new result which say any simple ring $R$ satisfies (dcc) on Principal-ideal is P.P. ring. Finally we introduced a good result about P.P. ring; let $R[X]$ be a polynomial ring and has exactly two idempotent elements. Then $R$ is P.P. ring.

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References

[1] Libo Zan and Jianlong Chen, P. P. Properties of Group Rings, *International Electronic Journal of Algebra*, Vol 3 (2008) **117-124**

[2] E. Jespers, Associative AlgebraAS, 2016-2017, webstek: http://homepages.vub.ac.be/serjesper Hoc: donderdag 09-11 uur, F.4.113].

[3] S. Endo, Note on p.p. rings, *Nagoya Math. J.* 17 (1960) **167–170**.

[4] K. R. Goodearl, Von Neumann Regular Rings, Pitman, London, 1979.

[5] C. Huh, H. K. Kim, and Y. Lee, p.p. rings and generalized p.p. rings, J. Pure Appl. Alg., 167 (2002) **37–52**.

[6] C. Y. Hong, N. K. Kim, and T. K. Kwak, Ore extensions of Baer and p.p. rings, *J. Pure Appl. Alg.*, 151 (2000) **215–226**.

[7] I. Kaplansky, Rings of Operators, Benjamin, New York, (1965).

[8] S. U. Chase, A generalization of the ring of triangular matrices, *Nagoya Math. J.* 18 (1961) **13–25**.