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Recommended Citation
Budarina, Natalia; Bernik, Vasilii; and O'Donnell, Hugh, "New estimates for the number of integer polynomials with given discriminants" (2020). Articles. 89.
https://arrow.tudublin.ie/ittsciart/89

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New estimates for the number of integer polynomials with given discriminants

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Received September 25, 2018; revised August 4, 2019

Abstract. In this paper, we propose a new method of upper bounds for the number of integer polynomials of the fourth degree with a given discriminant. By direct calculation similar results were established by H. Davenport and D. Kaliada for polynomials of second and third degrees.

MSC: 11J83, 11J68

Keywords: Diophantine approximation, discriminant of polynomials

1 Introduction

Denote by $\mathcal{P}_n$ the class of integer polynomials $P$ of degree $n$. In what follows, we use the Vinogradov symbols \(\ll\) (and \(\gg\)) where \(a \ll b\) means that there exists a constant \(C\) such that \(a \leq Cb\). If \(a \ll b \ll a\), then we write \(a \asymp b\). We denote the cardinality of a set \(B\) by \(#B\). Positive constants that depend only on \(n\) will be denoted by \(c(n)\); where necessary, these constants will be numbered \(c_j(n)\), \(j = 1, 2, \ldots\).

The discriminant of a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}_n$ is defined by

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ are the roots of $P$. Let $H(P) = \max_{0 \leq j \leq n} |a_j|$ denote the standard (naive) height of $P = \sum_{i=0}^{n} a_i x^i$. Given a parameter \(Q \in \mathbb{N}_{>1}\), let

$$\mathcal{P}_n(Q) = \{ P(x) \in \mathcal{P}_n : H(P) \leq Q \}$$

denote the set of integer polynomials $P$ of degree $n$ and height $H(P) \leq Q$. If $P$ has no repeated roots, then $D(P) \neq 0$. It is well known [16] that $D(P)$ can be represented as a determinant of order $2n-1$, which consists...
of the coefficients of \( P \). Hence, whenever \( D(P) \neq 0 \), we have that \( |D(P)| \geq 1 \) and \( |D(P)| \) is bounded from above in terms of the height and degree of the polynomial \( P \). We easily verify that for every \( n \geq 2 \), there exists a constant \( c_1 > 0 \) that depends on \( n \) only such that for any \( P \in \mathcal{P}_n(Q) \), we have that

\[
1 \leq |D(P)| < c_1 Q^{2n-2}.
\] (1.1)

The properties and estimates for \( D(P) \) imply the estimates for \( |x - \alpha_1| \), where \( x \in \mathbb{R} \), and \( \alpha_1 \) is the root of \( P \) closest to \( x \) (see [9, 10, 15]). These estimates were crucial to prove Mahler’s conjecture in the case \( n = 2, 3 \). In a more systematic way, the relation between \( |x - \alpha_1| \) and \( D(P) \) was investigated by Sprindzuk [15] and others [2, 3, 4, 5, 6, 11, 12, 13, 14]. In recent years, the problem of counting polynomials with a small discriminant \( D(P) \) has become a new branch of the theory of Diophantine approximations.

Given \( v \in \mathbb{R}_{\geq 0} \), define the subset of \( \mathcal{P}_n(Q) \) as follows:

\[
\mathcal{P}_n(Q, v) = \{ P(x) \in \mathcal{P}_n(Q) : 1 \leq |D(P)| < Q^{2n-2-2v} \}.
\]

Establishing the correct lower and upper bounds for \( \#\mathcal{P}_n(Q, v) \) is the goal of this branch of Diophantine approximations. We now briefly recall the results obtained to date. In the case of quadratic polynomials, it was shown in [13] that

\[
\#\mathcal{P}_2(Q, v) \asymp Q^{3-2v}, \quad 0 < v < \frac{3}{4}.
\]

In the case of cubic polynomials, it was established in [14] that

\[
\#\mathcal{P}_3(Q, v) \asymp Q^{4-5v/3}, \quad 0 \leq v < \frac{3}{5}.
\]

Establishing the correct lower bounds for arbitrary \( n \) has been the subject of numerous papers including [2, 3, 6, 13, 14]. The most general and best estimate was found in [3], where it was shown that

\[
\#\mathcal{P}_n(Q, v) > c_2 Q^{n+1-(n+2)v/n}, \quad 0 \leq v \leq n - 1.
\] (1.2)

The lower bound (1.2) for the full range of \( v, 0 \leq v \leq n - 1 \), was obtained for the polynomials that have all \( \alpha_2, \ldots, \alpha_n \) roots close to \( \alpha_1 \) and \( x \). The method for constructing a large number of polynomials \( P \in \mathcal{P}_n(Q, v) \) is based on the results from [1]. Moreover, the following two propositions are key elements of the method for obtaining the lower bound (1.2).

**Proposition 1.** (See [3].) Let \( n \geq 2 \), and let \( v_0, v_1, \ldots, v_n \) be a collection of real numbers such that

\[
v_0 + v_1 + \cdots + v_n = 0 \quad \text{and} \quad v_0 \geq v_1 \geq \cdots \geq v_n \geq -1.
\]

Then there are positive constants \( c_3 \) and \( c_4 \) depending on \( n \) only with the following property. For any interval \( J \subset [1/2, 1/2] \), there is a sufficiently large \( Q_0 \) such that for all \( Q > Q_0 \), there is a measurable set \( G_J \subset J \) satisfying \( |G_J| \geq |J|/2 \) such that for every \( x \in G_J \), there are \( n + 1 \) linearly independent primitive irreducible polynomials \( P \in \mathbb{Z}[x] \) of degree exactly \( n \) such that

\[
c_3 Q^{-v_0} \leq |P(x)| \leq c_4 Q^{-v_0}, \quad c_3 Q^{-v_j} \leq |P^{(j)}(x)| \leq c_4 Q^{-v_j}, \quad 1 \leq j \leq n.
\] (1.3)
Proposition 2. (See [3].) Let \( n \) and \( v_j \) be as in Proposition 1. Let

\[
d_j = v_{j-1} - v_j, \quad 1 \leq j \leq n.
\]

Suppose that \( d_1 \geq d_2 \geq \cdots \geq d_n \geq 0 \) and that for some \( x \in \mathbb{C} \) and \( Q > 1 \), inequalities (1.3) are satisfied by some polynomial \( P \) over \( \mathbb{C} \) of degree \( \deg P = n \). Then there are roots \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) of \( P \) such that

\[
|x - \alpha_j| \leq c_{5,j}Q^{-d_j}, \quad 1 \leq j \leq n,
\]

where

\[
c_{5,1} = nc_4c_3^{-1},
\]

\[
c_{5,j+1} = \max \left( \frac{2c_4n!}{c_3(j+1)!(n-j-1)!}, \frac{2c_{5,j}n!}{j!(n-j)!} \right), \quad 1 \leq j \leq n-1.
\]

It is much harder to get upper bounds for \( \#\mathcal{P}_n(Q, v) \) with arbitrary \( n \). Note that the range of \( v \) depends on the number of roots of the polynomial close to \( \alpha_1 \). For example, if only one root \( \alpha_2 \) is close to \( \alpha_1 \), then the range for \( v \) is \( 0 \leq v \leq n/2 \).

For results in the \( p \)-adic case, see [7]. The upper and lower bounds for the number of polynomials having small discriminants in terms of the Euclidean and \( p \)-adic metrics simultaneously are obtained in [5,11].

Let \( \alpha_1, \ldots, \alpha_n \) be the roots of the polynomial \( P \in \mathcal{P}_n \). An upper bound for the number of integer cubic polynomials with a given discriminant is obtained in [4], where it is established that

\[
\#\mathcal{P}_3'(Q, v) \ll Q^{4-5v/3+\epsilon}, \quad 0 \leq v \leq 2, \quad \forall \epsilon > 0,
\]

where \( \mathcal{P}_3'(Q, v) \) is a subclass of \( \mathcal{P}_3(Q, v) \) with a special distribution of roots. The first step of the proof is the ordering the roots \( \alpha_1, \alpha_2, \alpha_3 \) with respect to one of them \( \alpha_j \), which will denote by \( \alpha_1 \), in such way that

\[
|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3|, \quad |\alpha_1 - \alpha_3| \asymp |\alpha_2 - \alpha_3|.
\]

In the case of the polynomials of fourth degree, we will have another principal case for the ordering of the roots:

\[
|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3| \leq |\alpha_1 - \alpha_4|, \quad |\alpha_1 - \alpha_2| \ll |\alpha_3 - \alpha_4| \ll |\alpha_2 - \alpha_3| \ll |\alpha_1 - \alpha_3|.
\]

Other cases are similar to (1.4).

Let \( \alpha_{1,j}, \ldots, \alpha_{n,j} \) be the roots of the polynomial \( P_j \in \mathcal{P}_n \) ordered according to (1.4) or (1.5) depending on the degree of \( P_j \). For \( n = 3 \), the polynomials \( P_j \) are expanded into Taylor series in a neighbourhood of \( \alpha_{1,j} \), and the absolute values of \( P_j \) are estimated from above. Then we form the new polynomials \( R_{j+1} = P_{j+1} - P_j \) of degree \( \deg R_{j+1} < n \) from the polynomials \( P_j \) with the same oldest coefficients.

For the polynomials of fourth degree, in case (1.4), from the estimates \( |P_j| \) in a neighbourhood of \( \alpha_{1,j} \) we cannot get strong estimates for \( |P_j| \) in a neighbourhood of \( \alpha_{3,j} \). Therefore the expansion into Taylor series must be carried out in a neighbourhood of \( \alpha_{1,j} \) and in a neighbourhood of \( \alpha_{3,j} \).

The partition of the roots \( \alpha_j \) into the clusters is possible for \( n = 5,6 \), but for the arbitrary \( n \), we did not find a convenient method to classify the roots. Therefore, from now on, \( n = 4 \) and the roots \( \alpha_j \) satisfy (1.5). Let \( \mathcal{P}_4'(Q, v) \) denote the set of polynomials \( P \in \mathcal{P}_4(Q, v) \) with distinct roots satisfying (1.5). In this paper, we obtain an upper bound for the number of polynomials \( P \in \mathcal{P}_4'(Q, v) \).

Theorem 1. For any \( \epsilon > 0 \) and any sufficiently large \( Q \), we have the estimate

\[
\#\mathcal{P}_4'(Q, v) < Q^{5-3v/2+\epsilon}, \quad 0 \leq v \leq 1.
\]
2 Auxiliary statements

Let $P \in \mathcal{P}_4(Q, v)$ have complex distinct roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let

$$|\alpha_1 - \alpha_i| = Q^{-\rho_i}, \quad 2 \leq i \leq 4, \quad \rho_2 \geq \rho_3 \geq \rho_4,$$

and

$$|\alpha_3 - \alpha_4| = Q^{-\rho_6}. \quad (2.1)$$

Similar to other problems of the metric theory regarding polynomials, we assume that $|a_4(P)| \gg H(P)$. If the polynomial $P$ does not satisfy the last condition, then the transformation $S(x) = P(x + m)$ for some $0 \leq m \leq 4$ can be performed followed by an inversion to obtain $U(x) = x^4S(1/x)$. Therefore this new polynomial $U(x) = \sum_{j=0}^4 b_jx^j$ satisfies $|b_4| \gg H(S) \gg H(P)$. For more details, see [15]. If $P$ satisfies $|a_4(P)| \gg H(P)$, then $|\alpha_i| \leq c_6$, $1 \leq i \leq 4$, and $|\alpha_i - \alpha_j| \leq 2c_6$ for $1 \leq i < j \leq 4$. Therefore $\rho_i \geq \epsilon_1$, $2 \leq i \leq 5$, for any $\epsilon_1 > 0$ and any sufficiently large $Q$.

For a given number $\epsilon_1 > 0$, let $T = \lceil \epsilon_1^{-1} \rceil + 1$, where $\lceil a \rceil$ is the integer part of $a \in \mathbb{R}$. For a polynomial $P \in \mathcal{P}_4(Q, v)$, the real numbers $\rho_i$, $i = 2, 3, 4, 5$, were defined in (2.1). Also define the integers $l_i$ by

$$\frac{l_i - 1}{T} < \rho_i \leq \frac{l_i}{T}, \quad i = 2, 3, 4, 5.$$

It is not difficult to show that the number of vectors $\bar{\ell} = (l_2, l_3, l_4, l_5)$ is finite, depends only on $\epsilon_1$, and does not depend on $Q$ and $H(P)$.

In order for the polynomial $P(x)$ to belong to the class $\mathcal{P}_4(Q, v)$, it is necessary and sufficient that the inequality

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \geq v \quad (2.3)$$

holds. Note that inequality (2.3) follows from (1.1), (1.5), (2.1), (2.2), and the triangle inequalities for the roots of the polynomial $P$. For (2.3), the inequality

$$\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + \frac{l_5}{T} \geq v + 6\epsilon_1$$

is sufficient. By (1.1), (2.1), and (2.2) we have

$$\rho_2 + 2\rho_3 + 2\rho_4 + \rho_5 \leq 3. \quad (2.4)$$

For the roots of $P \in \mathcal{P}_4$, we define the sets

$$S(\alpha_j) = \{x \in \mathbb{R}: |x - \alpha_j| = \min_{1 \leq i \leq 4} |x - \alpha_i|\}, \quad 1 \leq j \leq 4.$$

**Lemma 1.** Let $\alpha_1$ be a complex root of an integer polynomial $P \in \mathcal{P}_4$, and let $x \in S(\alpha_1)$. Then

$$|x - \alpha_1| \leq \min_{2 \leq j \leq 4} \left(2^{4-j}|P(x)||P'(\alpha_1)|^{-1}\prod_{k=2}^{j} |\alpha_1 - \alpha_k|^{1/j}\right)$$

for $P'(\alpha_1) \neq 0$.

Lemma 1 is proved in [10].
Lemma 2. Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(x), T(x) \in \mathcal{P}_k(Q), k \leq 4$, have the same vector $\bar{l}$ and have no common roots. Let $I$ denote interval of length $|I| = Q^{-\gamma}$ with $\gamma \in \mathbb{R}_+$. If there exists a real number $\tau > 0$ such that for all $x \in I$,
\[
\max_{x \in I} (|P(x)|, |T(x)|) < Q^{-\tau},
\]
then
\[
\tau + 1 + 2 \sum_{j=1}^{k} \max(\tau + 1 - j\gamma, 0) < 2k + \delta.
\]

Lemma 2 can be proved similarly to Lemma 3 in [5]. In this case, we need to add the summands related to the root $\alpha_4$.

To prove Theorem 1, we need to consider a generalization of Lemma 2 for the simultaneous approximations of the polynomials on two intervals (see Lemma 3). We consider a new classification of the roots $\alpha_i, 1 \leq i \leq 4$, of $P \in \mathcal{P}_4(Q)$ with respect to $\alpha_1$ (as before) and $\alpha_3$ simultaneously. We obtain
\[
|\alpha_1 - \alpha_2| \leq |\alpha_1 - \alpha_3| \leq |\alpha_1 - \alpha_4|,
|\alpha_3 - \alpha_4| \leq |\alpha_3 - \alpha_2| \leq |\alpha_3 - \alpha_1|.
\]

(2.5)

Let $|\alpha_3 - \alpha_2| = Q^{-\rho_6}$ and define the integer $l_6$ by $(l_6 - 1)/T < \rho_6 \leq l_6/T$. It is not difficult to see that by (1.5)
\[
\rho_4 \leq \rho_3 \leq \rho_2, \quad \rho_3 \leq \rho_6 \leq \rho_5,
\]
where $\rho_1, 2 \leq i \leq 5$, are defined in (2.1)–(2.2). We also define the vector $\bar{l}' = (\bar{l}, l_6)$. Define the class $\mathcal{P}_4(Q, v)$ consisting of the polynomials $P \in \mathcal{P}_4(Q, v)$ corresponding to a vector $\bar{l}'$.

Lemma 3. Fix $\delta > 0$ and $Q > Q_0(\delta)$. Suppose that the polynomials $P(t), T(t) \in \mathcal{P}_k(Q), k \leq 4$, have the same vector $\bar{l}'$ and have no common roots in the rectangle $I_1 \times I_2$, where $|I_1| = Q^{-\gamma_1}$ and $|I_2| = Q^{-\gamma_2}$ with $\gamma_j \in \mathbb{R}_+, j = 1, 2$. Furthermore, let $P(t)$ and $T(t)$ satisfy the system of inequalities
\[
\max_{x \in I_1} (|P(x)|, |T(x)|) < Q^{-\gamma_1}, \quad \max_{y \in I_2} (|P(y)|, |T(y)|) < Q^{-\gamma_2}.
\]

(2.7)

Then for any $\delta > 0$ and $Q > Q_0(\delta)$, we have the inequality:
\[
\tau_1 + \tau_2 + 2 + l_2 + 2l_3 + 3l_4 + l_5 < 2k + \delta.
\]

(2.8)

The proof of Lemma 3 follows from the new classification (2.5) of the roots of polynomials, using inequalities (2.6) and (2.7), and can be proved similarly to Lemma 2 in [8].

3 Proof of Theorem 1

Assume that estimate (1.6) does not hold, so that
\[
\# \mathcal{P}_4(Q, v) \geq Q^{5-3\nu/2+\epsilon}.
\]

(3.1)

Consider two intervals $I_1, I_2 \subset \mathbb{R}$ with $|I_1| = Q^{-l_2}/T$ and $|I_2| = Q^{-l_6}/T$. We will say that the polynomial $P$ belongs to $M = I_1 \times I_2$ if $(\alpha_1, \alpha_3) \in M$, where $\alpha_1$ and $\alpha_3$ are the roots of $P$ in the ordering (1.5). From (3.1) it follows that there exist rectangles $I_1 \times I_2$ that contain at least
\[
\Delta = Q^{5-3\nu/2-l_2}/T-l_6/T+\epsilon
\]

Lith. Math. J., 60(1):1–8, 2020.
polynomials $P \in \mathcal{P}'_4(Q, v)$ satisfying (2.7). Fix one of these rectangles, say $M$. Since $\#l' \ll 1$, there exists a vector $l'$ satisfying (2.3) such that

$$
\#\mathcal{P}'_{4,l'}(Q, v, M) \gg Q^{5-3v/2+\epsilon-l_2/T-l_3/T+\epsilon},
$$

where $\mathcal{P}'_{4,l'}(Q, v, M)$ denotes the subset of $\mathcal{P}'_{4,l'}(Q, v)$ consisting of polynomials $P$ belonging to $M$. Fix the vector $l'$ and set

$$
h = 5 - \frac{3v}{2} - \frac{l_2}{T} - l_5/T + \epsilon/2.
$$

By (2.4) we have

$$
\frac{l_2}{T} + \frac{2l_3}{T} + \frac{2l_4}{T} + l_5/T \leq 3.
$$

From (3.2) we obtain that $h > 0$ for $v \leq 4/3$.

Expand the polynomial $P \in \mathcal{P}'_{4,l'}(Q, v, M)$ into its Taylor series in a neighbourhood of $\alpha_1$ to obtain

$$
P(x) = P(\alpha_1) + P'(\alpha_1)(x - \alpha_1) + \frac{1}{2}P''(\alpha_1)(x - \alpha_1)^2 + \frac{1}{6}P'''(\alpha_1)(x - \alpha_1)^3 + \frac{1}{24}P^{(4)}(\alpha_1)(x - \alpha_1)^4.
$$

Estimating each term gives

$$
|P'(\alpha_1)(x - \alpha_1)| \leq |a_4| \cdot |\alpha_1 - \alpha_2| \cdot |\alpha_1 - \alpha_3| \cdot |\alpha_1 - \alpha_4| \cdot |x - \alpha_1| \\
\leq Q^{1-\rho_2-\rho_3-\rho_4-l_2/T} < Q^{1-2l_2/T-l_3/T-\epsilon/3},
$$

$$
|P''(\alpha_1)(x - \alpha_1)| \leq 6|a_4| \max(|\alpha_1 - \alpha_2|, |\alpha_1 - \alpha_3|, |\alpha_1 - \alpha_4|, |\alpha_1 - \alpha_3|) \cdot |x - \alpha_1|^2 \\
< 6Q^{1-2l_2/T-l_3/T-\epsilon/4},
$$

$$
|P'''(\alpha_1)(x - \alpha_1)| \leq 18|a_4| \max(|\alpha_1 - \alpha_2|, |\alpha_1 - \alpha_3|, |\alpha_1 - \alpha_4|) \cdot |x - \alpha_1|^3 \\
< 18Q^{1-3l_2/T-l_4/T+\epsilon/3},
$$

$$
|P^{(4)}(\alpha_1)(x - \alpha_1)| \leq 24|a_4| |x - \alpha_1|^4 \leq 24Q^{1-4l_2/T}
$$

for $x \in I_1$. Thus

$$
|P(x)| \ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon}, \quad x \in I_1.
$$

Also develop the polynomial $P$ as Taylor series on the interval $I_2$ at the point $\alpha_3$ and obtain the upper bounds for all terms in the series. Thus we obtain

$$
|P(y)| \ll Q^{1-2l_2/T-l_3/T-l_4/T+3\epsilon}, \quad y \in I_2.
$$

Further, for $Q^h$ polynomials $P$, we use the Dirichlet box principle. We will assume that the fractional part of $h$ does not exceed $\epsilon_1$. If the last condition is not satisfied, then we rewrite $h$ as $h = [h] + \{h\}$. As a result, using the number $Q^{[h]}$, we reduce the degree of polynomials, and using the number $Q^{\{h\}}$, we reduce the height.
of polynomials $R_{j+1}(t) = P_{j+1}(t) - P_1(t)$, $j = 1, 2, \ldots$, as in [5]. Therefore the new polynomials $R_j$ satisfy

$$|R_j(x)| \ll Q^{1-2l_j/T - l_3/T - l_4/T + 3\epsilon_1}, \quad x \in I_1,$$

$$|R_j(y)| \ll Q^{1-2l_j/T - l_3/T - l_6/T + 3\epsilon_1}, \quad y \in I_2,$$

$$H(R_j) \ll Q^{1-\epsilon_1}, \quad \deg R_j \leq 4 - \left(20 - \frac{3v}{2} - \frac{l_2}{T} - \frac{l_5}{T} + \frac{\epsilon}{2} - \frac{\epsilon_1}{2}\right).$$

(3.3)

(3.4)

If there exist two polynomials $R_1$ and $R_2$ with no common roots, then Lemma 3 can be applied. The values of $\tau_1$ and $\tau_2$ are found from estimates (3.3) and (3.4). Thus

$$\tau_1 = \frac{-1 + 2l_2/T + l_3/T + l_4/T - 3\epsilon_1}{1 - \epsilon_1} \quad \text{and} \quad \tau_2 = \frac{-1 + 2l_5/T + l_3/T + l_6/T - 3\epsilon_1}{1 - \epsilon_1}.$$

The left-hand side of (2.8) is equal to

$$\frac{3l_2/T + 4l_3/T + 4l_4/T + 3l_5/T + l_6/T - 6\epsilon_1}{1 - \epsilon_1}.$$

This leads to a contradiction in (2.8) for $v \leq 1$ and $\delta \leq \epsilon - 2\epsilon_1$.

If, among polynomials $R_j(t)$, there exist no two polynomials without common roots, then the polynomials $R_j(t)$ are reducible. It is not difficult to see that $\deg R_j \leq 2$ for $v \leq 1$. Thus the polynomials $R_j(t)$ are decomposed into the product of two linear polynomials. Again, as, for example, in [4], we will use Lemma 2 to get a contradiction.

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