Non-uniqueness of Leray-Hopf solutions for a dyadic model

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Abstract

The dyadic model \( \dot{u}_n + \lambda^{2n} u_n - \lambda^{\beta n} u_{n-1}^2 + \lambda^{\beta(n+1)} u_n u_{n+1} = f_n, u_n(0) = 0 \), is considered. It is shown that in the case of non-trivial right hand side the system can have two different Leray-Hopf solutions.

Introduction

Consider the following system of ordinary differential equations

\[
\begin{aligned}
\dot{u}_n(t) + \lambda^{2n} u_n(t) &- \lambda^{\beta n} u_{n-1}^2(t) + \lambda^{\beta(n+1)} u_n(t) u_{n+1}(t) = f_n(t), \quad t \in [0; T], \\
u_n(0) &= a_n, \quad n = 1, 2, \ldots.
\end{aligned}
\] (0.1)

Here \( u_0 \equiv 0; \lambda > 1, \beta > 0 \) are parameters, \( u_n, f_n \) are real valued functions. We assume that initial data \( \{a_n\} \in l_2 \); and right-hand sides \( f_n \in L_2(0, T) \), the behaviour of \( f_n \) while \( n \to \infty \) will be described later.

System (0.1) is similar to the system of the Navier-Stokes equations

\[
\begin{aligned}
\partial_t u - \Delta u + P ((u, \nabla) u) &= f \quad \text{in} \quad [0, T] \times \mathbb{T}^d, \\
\text{div} u &= 0, \quad u|_{t=0} = a(x).
\end{aligned}
\] (0.2)

Here \( \mathbb{T}^d \) is a \( d \)-dimensional torus, \( P \) is an orthogonal projector in \( L_2(\mathbb{T}^d) \) on the subspace of solenoidal functions. Both systems can be written in an abstract way

\[
\begin{aligned}
\dot{u} + Au + B(u, u) &= f, \quad t \in [0, T], \\
u(0) &= a.
\end{aligned}
\] (0.3)

Function \( u(t) \) here takes values in a Hilbert space \( \mathcal{H} \), where \( \mathcal{H}_{(0.1)} = l_2 \) in the case (0.1), and

\[
\mathcal{H}_{(0.2)} = \{u \in L_2(\mathbb{T}^d, \mathbb{R}^d) : \text{div} u = 0\}
\]

for the system (0.2). \( A \) is a self-adjoint non-negative unbounded operator in \( \mathcal{H} \),

\[ A_{(0.3)} \{ u_n \} = \{ \lambda^{2n} u_n \}. \]

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and $A_{\sigma_2} = -\Delta$. Finally, $B$ is a bilinear unbounded map $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$,

$$B(u_n, v_n) = -\lambda^{\beta n} u_{n-1}(t)v_{n-1}(t) + \lambda^{\beta(n+1)} u_n(t)v_{n+1}(t),$$

$$B(u, v) = P((u, \nabla)v).$$

The map $B$ has two important properties.

1) Ortogonality:

$$(B(u, v), v)_{\mathcal{H}} = 0$$

for a dense set of "good" $u$ and $v$. For the system (0.1) we have

$$\sum_{n=1}^{\infty} \left( -\lambda^{\beta n} u_{n-1}(t)v_{n-1}(t) + \lambda^{\beta(n+1)} u_n(t)v_{n+1}(t) \right) = 0,$$

if all the series converge. For (0.2) using the condition div $u = 0$ one can get

$$\int_{T^d} \sum_{j,k=1}^{3} u_j \partial_j v_k v_k \, dx = -\frac{1}{2} \int_{T^d} \sum_{j=1}^{3} \partial_j u_j |v|^2 \, dx = 0,$$

if all the integrals converge.

2) Estimate

$$\|B(u, u)\|_{\mathcal{H}} \leq C \|A^{\sigma_1} u\|_{\mathcal{H}} \|A^{\sigma_2} u\|_{\mathcal{H}}.$$

Exponents $\sigma_1$ and $\sigma_2$ can take any value, but their sum is fixed. For the system (0.1)

$$\sigma_1 + \sigma_2 = \frac{\beta}{2}.$$

And for (0.2) we use the Cauchy inequality

$$\int_{T^d} |(u, \nabla)u|^2 \, dx \leq \left( \int_{T^d} |u|^4 \, dx \right)^{1/2} \left( \int_{T^d} |\nabla u|^4 \, dx \right)^{1/2}$$

and embedding theorems

$$W^{d/4}_2 \subset L_4, \quad W^{d/4+1}_2 \subset W^1.$$

Thus, one could take

$$\sigma_1 = \frac{d}{8}, \quad \sigma_2 = \frac{d+4}{8}, \quad \text{and get} \quad \sigma_1 + \sigma_2 = \frac{d+2}{4}.$$

We consider the system (0.1) to be a model for Navier-Stokes equations. The space dimension $d = 2$ in Navier-Stokes system corresponds to the value of the parameter $\beta = 2$ in dyadic model (0.1), and the dimension $d = 3$ corresponds to the value $\beta = 5/2$. The explicit value of the parameter $\lambda > 1$ has no importance for us.

System (0.1) originates from the work [3] as a model of turbulence in hydrodynamics.

**Definition 0.1.** Suppose $\{a_n\} \in l_2, f_n \in L_2(0,T)$ for all $n$, and $\sum_{n=1}^{\infty} \lambda^{-2n} \int_0^T f_n(t)^2 dt < \infty$.

A sequence of functions $\{u_n(t)\}_{n=1}^{\infty}$ is called a Leray-Hopf solution for system (0.1) if

- $u_n \in W^1_2(0,T)$ and (0.1) hold for all $n$. 

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Theorem 0.2. Suppose a Leray-Hopf solution to the system (0.1)

The condition

Remark 0.3. condition \( f \)

Theorem 0.4. Leray-Hopf solution was proved under the following assumptions.

Theorem 0.5. Let us note some results concerning strong solutions, although it is not directly related to our paper. The word ”strong” here means fast decreasing \( u_n(t) \) with \( n \to \infty \). Different authors give various definitions of strong solutions. In all cases the existence of a strong solution guarantees the uniqueness of the Leray-Hopf solution.

Theorem 0.6. Let \( \lambda > 1, \beta \leq 2, a_n = 0 \) for all \( n \). There exists \( T > 0 \) and functions \( \{f_n(t)\} \) such that \( \sum_{n=1}^{\infty} \lambda^{-2n} \int_0^T f_n(t)^2 dt < \infty \), but system (0.1) has two different Leray-Hopf solutions.

Remark 0.7. As will be seen from the proof, the energy conservation holds for constructed solutions. So, for all \( t \in [0,T] \) we have the equality in (0.4), see (3.8) below.

Remark 0.8. The problem with non-zero but rapidly decreasing with \( n \) right-hand sides is still open.

System (0.1) and similar ones were considered also in works [1, 2, 6, 7, 9, 10].

Let us note some results concerning strong solutions, although it is not directly related to our paper. The word ”strong” here means fast decreasing \( u_n(t) \) with \( n \to \infty \). Different authors give various definitions of strong solutions. In all cases the existence of a strong solution guarantees the uniqueness of the Leray-Hopf solution.

Theorem 0.9 ([2]). If \( \beta \leq 2, \sum_{n=1}^{\infty} \lambda^{2n} a_n^2 < \infty, f_n(t) \geq 0, \sum_{n=1}^{\infty} \int_0^T f_n(t)^2 dt < \infty. Then there exists solution of (0.1), such that estimate

\[
\sup_{t \in [0,T]} \sum_{n=1}^{\infty} \lambda^{2n} u_n(t)^2 < \infty
\]
2) Let \( \beta > 3, \varepsilon > 0, f_n \equiv 0 \). Then there exists such a number \( M(\varepsilon) \), that if \( a_n \geq 0 \) and \( \sum_{n=1}^{\infty} \lambda^{2\beta n} a_n^2 > M \), then for any solution \( \{u_n\} \) of the system (0.1)

\[
\int_{0}^{T} \left( \sum_{n=1}^{\infty} \lambda^{2(\varepsilon+1/3)\beta n} u_n(t)^2 \right)^{3/2} dt = +\infty
\]

holds for some finite \( T \). For example, one could take all \( a_n = 0 \) for \( n \geq 2 \) if \( a_1 \) is big enough.

The second part of this theorem in addition to the theorem 0.4 means that for \( \beta > 3 \) the Leray-Hopf solutions can be not strong.

Theorem 0.10 ([1]). Let \( \lambda = 2, 2 < \beta \leq \frac{5}{2} \). Suppose \( \{a_n\} \in l_2, a_n \geq 0, f_n \equiv 0 \) for all \( n \). Then there exists a solution to (0.1), such that

\[ u_n(t) = O(\lambda^{-\gamma n}), \quad n \to \infty, \quad \forall \gamma, \quad \forall t > 0. \]

The question of existence of the strong solution with arbitrary (not necessarily non-negative) ”good” initial data and right-hand sides remains open. For example, one could take all \( a_n = 0 \) for \( n \geq 2 \) if \( a_1 \) is big enough.

All three works [1, 2, 4] are using the following property of positivity conservation in the absence of the right-hand sides (or with non-negative right-hand sides):

if

\[ \dot{u}_n + \lambda^{2n} u_n + \lambda^{\beta(n+1)} u_n u_{n+1} = \lambda^{\beta n} u_{n-1}^2, \]

and \( u_n(t_0) \geq 0 \), then \( u_n(t) \geq 0 \) for all \( t > t_0 \).

This property follows from the explicit formula

\[
u_n(t) = u_n(t_0) \exp \left( - \int_{t_0}^{t} (\lambda^{2n} + \lambda^{\beta(n+1)} u_{n+1}(s)) ds \right) + \int_{t_0}^{t} \exp \left( - \int_{s}^{t} (\lambda^{2n} + \lambda^{\beta(n+1)} u_{n+1}(\sigma)) d\sigma \right) \lambda^{\beta n} u_{n-1}(s)^2 d\sigma.
\]

However, the conservation of positivity is a random property in a sense that firstly, Navier-Stokes equations do not have any analogous property, and secondly, that it is destroyed when a right-hand side is considered in the system (0.1). This gave the authors the idea to build an example of non-uniqueness of Leray-Hopf solutions by choosing an appropriate right-hand side.

**Idea of the proof**

The proof of the theorem 0.6 is based on an idea, originating from K. Golovkin. Now we get back to the abstract setting (0.3). Suppose that system (0.3) has two different solutions. We denote them as \( u^\pm \) and rewrite them in a form

\[ u^\pm(t) = v(t) \pm g(t), \]

where \( v \) and \( g \) are half-sum and half-difference of \( u^\pm \) respectively. Then the system (0.3) is equivalent to:

\[
\begin{align*}
\dot{v} + A v + B(v,v) + B(g,g) &= f, \\
v(0) &= a, \\
\dot{g} + A g + B(v,g) + B(g,v) &= 0, \\
g(0) &= 0.
\end{align*}
\] (0.5)
Note that the system on \( g \) becomes linear. Now we need to calibrate the coefficient \( v \) in a way that the system on \( g \) has a non-trivial solution. After that using the recently found \( v \) and \( g \) we calculate \( f \), the right-hand side of the first equation in (1.5), and make sure that it satisfies the requirements.

In our case the system (1.5) takes form

\[
\begin{align*}
\dot{v}_n + \lambda^{2n} v_n - \lambda^{\beta n} v_{n-1}^2 - \lambda^{\beta n} g_{n-1}^2 + \lambda^{\beta(n+1)} v_n v_{n+1} + \lambda^{\beta(n+1)} g_n g_{n+1} &= f_n, \\
v_n(0) &= a_n, \\
\dot{g}_n + \lambda^{2n} g_n - 2 \lambda^{\beta n} v_{n-1} g_{n-1} + \lambda^{\beta(n+1)} v_n g_{n+1} + \lambda^{\beta(n+1)} v_{n+1} g_n &= 0,
\end{align*}
\]

(0.6)

One can see that with \( v_n \equiv 0 \) the third equation of the system (0.6) becomes trivial. So we will split up \([0; T]\) into a set of intervals and put \( v_n = 0 \) on most of them. After that we need only to solve the problem on \( g_n \) on a few intervals.

Using scaling we are able to transform all the equations on \( g_n \) to the unified form — system of three equations on \([0; 1]\) (see (3.1) below). To make \( g_n \) continuous one needs to add "gluing conditions" to the system. Existence of the solution of the system with such conditions is the subject of theorem 3.1.

**Plan of the paper.** In §1 we prove that Leray-Hopf solution always exists. In §2 we prove that Leray-Hopf solution is unique when \( \beta \leq 2 \). In §3 we formulate theorem 3.1 and derive the main result (theorem (0.6)) from it. In §§4 and 5 we prove the theorem 3.1.

1 \ Existence of a Leray-Hopf solution

For the sake of completeness we provide the proof of the existence of Leray-Hopf solutions.

We introduce Galerkin solutions for the problem (1.1). For any \( N \in \mathbb{N} \) consider the problem on the segment \([0; T]\)

\[
\begin{align*}
\dot{v}_n^{(N)} + \lambda^{2n} v_n^{(N)} - \lambda^{\beta n} \left( v_{n-1}^{(N)} \right)^2 + \lambda^{\beta(n+1)} v_n^{(N)} v_{n+1}^{(N)} &= f_n, \\
v_n^{(N)}(0) &= a_n, \\
v_n^{(N)}(0) &= v_0^{(N)} \equiv v_{N+1}^{(N)} = 0.
\end{align*}
\]

(1.1)

It is equivalent to the system of integral equations

\[
v_n^{(N)}(t) = a_n + \int_0^t \left( f_n(\tau) - \lambda^{2n} v_n^{(N)}(\tau) + \lambda^{\beta n} v_{n-1}^{(N)}(\tau)^2 - \lambda^{\beta(n+1)} v_n^{(N)}(\tau) v_{n+1}^{(N)}(\tau) \right) d\tau, \quad n = 1, \ldots, N,
\]

(1.2)

or one equation in \( \mathbb{R}^N \)

\[
v^{(N)}(t) = a_{(N)} + \int_0^t F_N(v^{(N)}(\tau), \tau) d\tau,
\]

where \( a_{(N)} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \),

\[
|F_N(y, \tau)| \leq C_N \left( |y| + |y|^2 + |\tau|_{(N)} \right)
\]

(1.3)
and

\[ |F_N(y, \tau) - F_N(z, \tau)| \leq C_N(1 + |y| + |z||y - z|. \]  

Denote

\[ R_N = 2|a(N)| + 2 \int_0^T |f(t(N))| \, dt, \]

\[ \delta_N = \frac{1}{2C_N(R_N + 1)} = \frac{1}{C_N \left( 4|a(N)| + 4 \int_0^T |f(t(N))| \, dt + 2 \right)}. \]

In the space of continuous functions \( C([0, \delta_N], \mathbb{R}^N) \) consider the closed ball

\[ M_N := \{ v \in C([0, \delta_N], \mathbb{R}^N) : \| v \|_C \leq R_N \} \]

and the map

\[ (K_N v)(t) = a(N) + \int_0^t F_N(v(\tau), \tau) \, d\tau. \]

It maps \( M_N \) to itself due to (1.3) and (1.5), and it is a contraction due to (1.4) and (1.5). Thus, systems (1.2) and (1.1) have a solution on \([0, \delta_N]\), where \( \delta_N \) is defined by (1.5). It is clear that \( v^{(N)}_n \in W^1_2(0, \delta_N) \). Multiplying (1.1) with \( 2v^{(N)}_n \), summing for all \( n \) and integrating we get

\[ \sum_{n=1}^N v^{(N)}_n(t)^2 + 2 \sum_{n=1}^N \lambda^{2n} \int_0^t v^{(N)}_n(\tau)^2 \, d\tau = \sum_{n=1}^N a_n^2 + 2 \sum_{n=1}^N \lambda^{2n} \int_0^t f_n(\tau)v^{(N)}_n(\tau) \, d\tau. \]  

Using Cauchy inequality for the last addend in the right-hand side, we arrive at the estimate

\[ \sum_{n=1}^N v^{(N)}_n(t)^2 + \sum_{n=1}^N \lambda^{2n} \int_0^t v^{(N)}_n(\tau)^2 \, d\tau \leq \sum_{n=1}^N a_n^2 + \sum_{n=1}^N \lambda^{-2n} \int_0^t f_n(\tau)^2 \, d\tau. \]

So the following lemma is now proven.

**Lemma 1.1.** System (1.1) has a solution on the segment \([0, t_1]\),

\[ t_1 = \frac{1}{C_N \left( 4|a(N)(0)| + 4 \int_0^T |f(t(N))| \, dt + 2 \right)}. \]

and

\[ |v^{(N)}(t_1)| \leq |v^{(N)}(0)| + \left( \sum_{n=1}^N \lambda^{-2n} \int_0^T f_n(t)^2 \, dt \right)^{1/2}. \]

After that we construct the solution on time intervals \([t_1, t_2], [t_2, t_3]\) and so on. And we have

\[ |v^{(N)}(t_k)| \leq |a(N)| + k \left( \sum_{n=1}^N \lambda^{-2n} \int_0^T f_n(t)^2 \, dt \right)^{1/2}, \]

\[ t_{k+1} - t_k \geq \frac{1}{C_N \left( 4|a(N)| + 4k \left( \sum_{n=1}^N \lambda^{-2n} \int_0^T f_n(t)^2 \, dt \right)^{1/2} + 4 \int_0^T |f(t(N))| \, dt + 2 \right)}. \]
Since the series $\sum_{k=1}^{\infty} (t_{k+1} - t_k)$ diverges, we get that system (1.1) has a solution on the entire interval $[0, T]$. It also satisfies (1.7) for all $t \in [0, T]$, and hence

$$\sum_{n=1}^{N} v^{(N)}_n(t)^2 + \sum_{n=1}^{N} \lambda_n^2 \int_0^t v^{(N)}_n(\tau)^2 d\tau \leq \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} \lambda_n^{-2n} \int_0^t f_n(\tau)^2 d\tau, \quad \forall t \in [0, T]. \quad (1.8)$$

This inequality and the equation (1.2) imply that the sequence $\{v^{(N)}_n\}_{N=n}^{\infty}$ is bounded in $W^1_2(0, T)$ for any $n$. Therefore there exists a sequence $\{v^{(N)}_n\}$, converging in $C[0, T]$ while $N_k \to \infty$. Using a diagonal process we get the sequence of numbers $M_k$, such that

$$v^{(M_k)}_n \xrightarrow{k \to \infty} u_n \quad \text{in} \quad C[0, T] \quad \forall n \in \mathbb{N}.$$ 

We now show that the constructed sequence $\{u_n\}$ is a Leray-Hopf solution. Indeed, substituting $N = M_k$ in (1.2) and going to the limit $k \to \infty$, one can get that the sequence $\{u_n(t)\}_{n=1}^{\infty}$ satisfies the system (0.1). Besides, $u_n \in W^1_2(0, T)$ for all $n$. Next, (1.8) yields

$$\sum_{n=1}^{N} \left( v^{(M_k)}_n(t)^2 + \lambda_n^2 \int_0^t v^{(M_k)}_n(\tau)^2 d\tau \right) \leq \sum_{n=1}^{\infty} \left( a_n^2 + \lambda_n^{-2n} \int_0^T f_n(\tau)^2 d\tau \right) \quad \text{with} \quad M_k \geq N.$$ 

Taking a limit $k \to \infty$, we get

$$\sum_{n=1}^{\infty} \left( u_n(t)^2 + \lambda_n^2 \int_0^t u_n(\tau)^2 d\tau \right) \leq \sum_{n=1}^{\infty} \left( a_n^2 + \lambda_n^{-2n} \int_0^T f_n(\tau)^2 d\tau \right).$$

Due to the arbitrariness of $N$, this estimate guarantees that $\sum_{n=1}^{\infty} u_n(t)^2$ is bounded and that the series $\sum_{n=1}^{\infty} \lambda_n^2 \int_0^T u_n(\tau)^2 d\tau$ converges.

Now the only thing left to prove is the energy estimate. We introduce the notation

$$v^{(N)} = \{v^{(N)}_n\}_{n=1}^{\infty}, \quad \text{where} \quad v^{(N)}_n = 0 \quad \text{with} \quad n > N.$$ 

It follows from (1.8) that the sequence $\{v^{(N)}_n\}_{N=1}^{\infty}$ is bounded in the Hilbert space of sequences of functions with a norm $\left( \sum_{n=1}^{\infty} \lambda_n^2 \int_0^T v_n(t)^2 dt \right)^{1/2}$. Without loss of generality, one can suppose that the sequence $\{v^{(M_k)}_n\}_{k=1}^{\infty}$ weakly converges in the mentioned space. Also, all $v^{(M_k)}_n$ weakly converge $L_2(0, T)$, and therefore the limit coincides with the sequence $u = \{u_n(t)\}_{n=1}^{\infty}$. Moreover, this weak convergence implies that

$$\sum_{n=1}^{\infty} \int_0^t f_n(\tau) v^{(M_k)}_n(\tau) d\tau \xrightarrow{k \to \infty} \sum_{n=1}^{\infty} \int_0^t f_n(\tau) u_n(\tau) d\tau \quad \forall t \in [0, T]. \quad (1.9)$$

Finally, with $M_k \geq N$ we have

$$\sum_{n=1}^{N} \left( v^{(M_k)}_n(t)^2 + 2\lambda_n^2 \int_0^t v^{(M_k)}_n(\tau)^2 d\tau \right) \leq \sum_{n=1}^{\infty} \left( v^{(M_k)}_n(t)^2 + 2\lambda_n^{2n} \int_0^t v^{(M_k)}_n(\tau)^2 d\tau \right)$$

$$= \sum_{n=1}^{M_k} a_n^2 + 2 \sum_{n=1}^{\infty} \int_0^t f_n(\tau) v^{(M_k)}_n(\tau) d\tau,$$
where we used (1.6) in the second equality. Taking the limit \( k \to \infty \) again, we get
\[
\sum_{n=1}^{N} \left( u_n(t)^2 + 2\lambda^{2n} \int_0^t u_n(\tau)^2 \, d\tau \right) \leq \sum_{n=1}^{\infty} \left( a_n^2 + 2 \int_0^t f_n(\tau) u_n(\tau) \, d\tau \right),
\]
here we used the convergence (1.9). Because of the arbitrariness of \( N \), the estimate (1.4) follows. Theorem 0.2 is proven.

2 Uniqueness of the solution in the case \( \beta \leq 2 \)

Lemma 2.1. Let \( u \) be a Leray-Hopf solution to the problem (0.1), \( \beta \leq 2 \). Then for any \( \varepsilon > 0 \) there exists \( N \) such that
\[
\sum_{n=N}^{\infty} u_n(t)^2 \leq \varepsilon \quad \forall \, t \in [0, T].
\]

Proof. As \( \beta \leq 2 \), series
\[
\sum_{n=1}^{\infty} \lambda^{\beta n} \int_0^T |u_{n-1}(t)|^2 u_n(t) | \, dt \leq \sup_{n \in \mathbb{N}, t \in [0, T]} |u_n(t)| \cdot \sum_{n=1}^{\infty} \lambda^{2n} \int_0^T u_{n-1}(t)^2 \, dt < \infty
\]
converges by the definition of a Leray-Hopf solution. So, multiplying (0.1) by \( 2u_n \), integrating with respect to \( t \) and summing up with respect to \( n \) from \( N \) to \( \infty \), we get
\[
\sum_{n=N}^{\infty} \left( u_n(t)^2 + 2\lambda^{2n} \int_0^t u_n(\tau)^2 \, d\tau \right) - 2\lambda^{\beta N} \int_0^t u_{N-1}(\tau)^2 u_N(\tau) \, d\tau
\]
\[
= \sum_{n=N}^{\infty} \left( a_n^2 + 2 \int_0^t f_n(\tau) u_n(\tau) \, d\tau \right). \tag{2.1}
\]

It is clear that
\[
\sum_{n=N}^{\infty} a_n^2 \xrightarrow{N \to \infty} 0,
\]
\[
\sum_{n=N}^{\infty} \left| \int_0^t f_n(\tau) u_n(\tau) \, d\tau \right| \leq \left( \sum_{n=N}^{\infty} \lambda^{-2n} \int_0^T f_n(\tau)^2 \, d\tau \right)^{1/2} \left( \sum_{n=N}^{\infty} \lambda^{2n} \int_0^T u_n(\tau)^2 \, d\tau \right)^{1/2} \xrightarrow{N \to \infty} 0,
\]
and
\[
\lambda^{\beta N} \left| \int_0^t u_{N-1}(\tau)^2 u_N(\tau) \, d\tau \right| \leq C \lambda^{2N} \int_0^T u_{N-1}(\tau)^2 \, d\tau \leq C \lambda^2 \sum_{n=N-1}^{\infty} \lambda^{2n} \int_0^T u_n(\tau)^2 \, d\tau \xrightarrow{N \to \infty} 0.
\]

This convergence and (2.1) give the result. \( \blacksquare \)

Remark 2.2. In particular, Lemma 2.1 implies that if \( u \) is a Leray-Hopf solution to (0.1) and \( \beta \leq 2 \), then \( u \in C([0, T]; l_2) \).
Proof of Theorem 3.1. Suppose \( u^\pm \) are two Leray-Hopf solutions of the system (0.1). We write them in the form \( u^\pm = v \pm g \). Then

\[
\sup_{t \in (0, T)} \sum_{n=1}^{\infty} (v_n(t)^2 + g_n(t)^2) < \infty, \quad \sum_{n=1}^{\infty} \lambda^{2n} \int_0^T (v_n(t)^2 + g_n(t)^2) \, dt < \infty,
\]

(2.2)

Due to the last Lemma,

\[
\forall \varepsilon \exists N : \sup_{t \in (0, T)} \sum_{n=N}^{\infty} v_n(t)^2 < \varepsilon.
\]

(2.3)

Functions \( v_n, g_n \) satisfy the system (0.6), therefore

\[
\dot{g}_n g_n + 2 \lambda v_n g_n - 2 \lambda \beta \nu v_{n-1} g_{n-1} + \lambda \beta v_n g_n g_{n+1} + \lambda \beta v_{n+1} g_n^2 = 0.
\]

Integrating over \([0, t]\), summing up with respect to \( n \) from 1 to \( \infty \) (all the series converge due to (2.2)) and using that \( g_n(0) = 0 \), we get

\[
\sum_{n=1}^{\infty} \left( \frac{g_n(t)^2}{2} + \lambda^{2n} \int_0^t g_n(\tau)^2 \, d\tau \right) = \sum_{n=1}^{\infty} \left( \lambda^{2n} \int_0^t v_{n-1} g_{n-1} \, d\tau - \lambda \beta \int_0^t v_n g_n g_{n+1} \, d\tau \right)
\]

\[
\leq C_1 \sum_{n=1}^{\infty} \lambda^{2n} \left( \|v_{n-1}\|_C \int_0^t (g_{n-1}^2 + g_n^2) \, d\tau + \|v_n\|_C \int_0^t g_n^2 \, d\tau \right)
\]

\[
= C_1 \sum_{n=1}^{\infty} \lambda^{2n} \left( \|v_{n-1}\|_C + 2 \|v_n\|_C + \|v_{n+1}\|_C \right) \int_0^t g_n^2 \, d\tau.
\]

By virtue of (2.3) one can find such \( N \) that

\[
\|v_{n-1}\|_C + 2 \|v_n\|_C + \|v_{n+1}\|_C \leq \frac{1}{C_1} \quad \forall \, n > N.
\]

Then

\[
\sum_{n=1}^{\infty} g_n(t)^2 \leq C_2 \sum_{n=1}^{N} \int_0^t g_n(\tau)^2 \, d\tau.
\]

Function \( \Phi(t) = \sum_{n=1}^{\infty} g_n(t)^2 \) is bounded due to (2.2). Applying the Gronwall inequality to the function \( \Phi \) we arrive at \( g_n(t) \equiv 0 \) for all \( n \). Thus, \( u^+ = u^- \).}

3 Reduction to the system of three ODE

We want to construct a non-trivial solution to the system (0.6) with initial data \( a_n = 0 \). The following theorem plays a key role in this construction.

**Theorem 3.1.** Let \( \lambda > 1, \beta > 2, R > 0 \). Consider on \([0; 1]\) the system of ODE:

\[
\begin{aligned}
\dot{h}_1(\tau) &= (\lambda^{-2} + q(\tau)) h_1(\tau) - p(\tau) h_2(\tau), \\
\dot{h}_2(\tau) &= 2 p(\tau) h_1(\tau) - h_2(\tau) + \lambda^2 q(\tau) h_3(\tau), \\
\dot{h}_3(\tau) &= -2 \lambda^2 q(\tau) h_2(\tau) - \lambda^2 h_3(\tau), \\
h_1(0) &= 0, \quad h_2(0) = y, \quad h_3(0) = z.
\end{aligned}
\]

(3.1)
There exist functions \( p, q \in C_0^\infty(0,1) \) and numbers \( y, z \in \mathbb{R}, y^2 + z^2 \neq 0 \), such that the only solution \( h \in C^\infty([0,1]; \mathbb{R}^3) \) of the system (3.1) with given \( p, q, y, z \) has the properties

\[
h_1(1) = \rho y, \quad h_2(1) = \rho z, \quad \text{where} \quad |\rho| > R. \tag{3.2}
\]

This theorem is proven in [3]. Now we take \( T = \frac{1}{\lambda^2-1} \) and divide the interval \((0,T)\) into an infinite set of subintervals. Let \( t_n = \frac{1}{(\lambda^2-1)^2n} \), then

\[
t_{n-1} - t_n = \lambda^{-2n}, \quad (0, T) = \bigcup_{n=1}^{\infty} [t_n, t_{n-1}).
\]

Suppose the functions \( p, q, h_1, h_2, h_3 \) are given by the theorem [3.1] with sufficiently large \( \rho \); the value of \( \rho \) will be chosen later. Functions \( v_n \) and \( g_n \) will "start" not at 0, but at the moment of time \( t_{n+1} \). Namely, we take

\[
v_n(t) = \begin{cases} 
0, & t < t_{n+1}, \\
\lambda^{(2-\beta)(n+1)}p(\lambda^{2n+2}(t-t_{n+1})), & t_{n+1} < t < t_n, \\
-\lambda^{(2-\beta)n}q(\lambda^{2n}(t-t_n)), & t_n < t < t_{n-1}, \\
0, & t > t_{n-1};
\end{cases} \tag{3.3}
\]

\[
g_n(t) = \begin{cases} 
0, & t < t_{n+1}, \\
\rho^{-n}h_1(\lambda^{2n+2}(t-t_{n+1})), & t_{n+1} < t < t_n, \\
\rho^{-n}h_2(\lambda^{2n}(t-t_n)), & t_n < t < t_{n-1}, \\
\rho^{-n+1}h_3(\lambda^{2n-2}(t-t_{n-1})), & t_{n-1} < t < t_{n-2}, \\
\rho^{-n+1}h_3(1)e^{-\lambda^{2n}(t-t_{n-2})}, & t > t_{n-2}.
\end{cases} \tag{3.4}
\]

It is clear that \( v_n \in C^\infty(0, T) \) and that the functions \( g_n \) are piecewise smooth, continuous at \( t_{n+1}, t_n, t_{n-1}, t_{n-2} \) due to (3.1), (3.2), and therefore, \( g_n \in W_1^2(0, T) \). It is also clear that

\[ v_n(0) = g_n(0) = 0 \quad \text{for all} \ n, \]

and that

\[ v_n(t) = O(\lambda^{(2-\beta)n}), \quad \dot{v}_n(t) = O(\lambda^{(4-\beta)n}), \quad g_n(t) = O(\rho^{-n}), \quad n \to \infty. \tag{3.5} \]

**Lemma 3.2.** Suppose functions \( v_n \) and \( g_n \) are defined by the formulas (3.3) and (3.4) respectively. Then

\[ \dot{g}_n + \lambda^{2n}g_n - 2\lambda^{3n}v_{n-1}g_{n-1} + \lambda^{(n+1)}v_nv_{n+1} + \lambda^{(n+1)}v_{n+1}g_n = 0, \]

i.e. the third equation from (0.6) is satisfied.

**Proof.** Let us denote the left-hand side as \( G_n(t) \). One can see that \( G_n(t) = 0 \) while \( t < t_{n+1} \).

While \( t_{n+1} < t < t_n \) we introduce a new variable \( \tau = \lambda^{2n+2}(t-t_{n+1}) \in [0; 1] \). By definition \( v_{n-1} \equiv 0 \) on this time interval, so

\[
G_n(t) = \lambda^{2n+2}\rho^{-n-1}\dot{h}_1(\tau) + \lambda^{2n}\rho^{-n-1}h_1(\tau) \\
+ \lambda^{(n+1)}\lambda^{(2-\beta)(n+1)}p(\tau)\rho^{-n-1}h_2(\tau) - \lambda^{(n+1)}\lambda^{(2-\beta)(n+1)}q(\tau)\rho^{-n-1}h_1(\tau) \\
= \lambda^{2n+2}\rho^{-n-1}\left(\dot{h}_1(\tau) + \lambda^{-2}h_1(\tau) + p(\tau)h_2(\tau) - q(\tau)h_1(\tau)\right) = 0
\]

due to the first equation in (3.1).
While $t_n < t < t_{n-1}$ we introduce a new variable $\tau = \lambda^{2n}(t - t_n) \in [0; 1]$. On this time interval $v_{n+1} \equiv 0$, so
\[
G_n(t) = \lambda^{2n}\rho^{-n} \left( h_2(\tau) + h_2(\tau) - 2p(\tau)h_1(\tau) - \lambda^2 q(\tau)h_3(\tau) \right) = 0
\]
due to the second equation in (3.1).

While $t_{n-1} < t < t_{n-2}$ we introduce $\tau = \lambda^{2n-2}(t - t_{n-1}) \in [0; 1]$; here $v_n \equiv v_{n+1} \equiv 0$, and therefore
\[
G_n(t) = \lambda^{2n-2}\rho^{-n+1} \left( h_3(\tau) + \lambda^2 h_3(\tau) + 2\lambda^2 q(\tau)h_2(\tau) \right) = 0
\]
due to the third equation in (3.1).

And finally, while $t > t_{n-2}$
\[
G_n(t) = \dot{g}_n(t) + \lambda^{2n}g_n(t) = \rho^{-n-1}h_3(1) \left( -\lambda^2 e^{-\lambda^2(t-t_{n-2})} + \lambda^2 e^{-\lambda^2(t-t_{n-2})} \right) = 0.
\]

Lemma 3.3. We define functions $f_n$ from the first equation in (0.6) with $v_n$ and $g_n$, given by formulas (3.3) and (3.4). If $|\rho| \geq \lambda^2$, then
\[
f_n(t) = \begin{cases} 0, & t < t_{n+1}, \\ O(\lambda^{(4-\beta)n}), & t_{n+1} < t < t_{n-2}, \\ O(\lambda^{-\beta n}), & t > t_{n-2}. \end{cases}
\]

Proof. While $t < t_{n+1}$ we have $f_n(t) = 0$ by construction. The estimate $f_n(t) = O(\lambda^{(4-\beta)n})$ follows from (3.3). The last estimate follows from (3.3) and the fact that
\[
f_n(t) = -\lambda^\beta g_{n-1}(t)^2 + \lambda^{\beta(n+1)}g_n(t)g_{n+1}(t) \quad \text{with} \quad t > t_{n-2}.
\]

Corollary 3.4. Under the conditions of Lemma 3.3
\[
\sum_{n=1}^{\infty} \lambda^{-2n} \int_0^T f_n(t)^2 dt < \infty.
\]

Proof. Using the previous lemma, we get
\[
\int_0^T f_n(t)^2 dt = \int_{t_{n+1}}^{t_{n-2}} O(\lambda^{(8-2\beta)n}) dt + \int_{t_{n-2}}^T O(\lambda^{-2\beta n}) dt = O(\lambda^{(6-2\beta)n}).
\]

Therefore the series $\sum_{n=1}^{\infty} \lambda^{-2n} \int_0^T f_n(t)^2 dt$ converges because $\beta > 2$.

Remark 3.5. Moreover the series $\sum_{n=1}^{\infty} \lambda^{-\gamma n} \int_0^T f_n(t)^2 dt$ converges for all $\gamma > 6 - 2\beta$.

Proof of the theorem 0.1. By the theorem 3.1 with $R = \lambda^3$ we find functions $p$, $q$, $h_1$, $h_2$, $h_3$, satisfying (3.1) and (3.2). Using them, we construct functions $v_n$ and $g_n$ by formulae (3.3) and (3.4). Let $u_n^\pm(t) = v_n(t) \pm g_n(t)$, functions $f_n(t)$ we define from the first equation of (0.6). Then by Lemma 3.2 the system (0.6) is satisfied, and thus, (1.1) is satisfied for functions $u_n^\pm$.
with $a_n = 0$. Furthermore, $\sum_{n=1}^{\infty} \lambda^{-2n} \int_0^T f_n(t)^2 dt < \infty$ due to the corollary 3.4. All functions $u_n^\pm \in W^1_2(0, T)$ and

$$u_n^\pm(t) = \begin{cases} 0, & t < t_n+1; \\ O(\lambda^{(2-\beta)n}), & t_{n+1} < t < t_{n-1}, \\ O(\lambda^{-\beta n}), & t > t_{n-1}, \\ \end{cases}$$

due to (3.5). Therefore,

$$\sup_{t \in [0,T]} \sum_{n=1}^{\infty} u_n^\pm(t)^2 < \infty,$$

and

$$\int_0^T u_n^\pm(t)^2 dt = \int_{t_{n+1}}^{t_{n-1}} O(\lambda^{(4-2\beta)n}) dt + \int_{t_{n-1}}^{T} O(\lambda^{-2\beta n}) dt = O(\lambda^{(2-2\beta)n}),$$

wherefrom

$$\sum_{n=1}^{\infty} \lambda^{2n} \int_0^T u_n^\pm(t)^2 dt < \infty.$$

We are left with the energy estimate.

Multiplying (0.1) by $u_n$, substituting $u_n = u_n^\pm$ and integrating over the time, we get

$$\frac{1}{2} u_n^\pm(t)^2 + \lambda^{2n} \int_0^t u_n^\pm(\tau)^2 d\tau - \lambda \int_0^t u_{n-1}^\pm(\tau)^2 u_n^\pm(\tau) d\tau$$

$$+ \lambda^{\beta(n+1)} \int_0^t u_n^\pm(\tau)^2 u_{n+1}^\pm(\tau) d\tau = \int_0^t f_n(\tau) u_n^\pm(\tau) d\tau. \quad (3.7)$$

By virtue of (3.6)

$$\lambda^{\beta n} \int_0^T |u_{n-1}^\pm(\tau)^2 u_n^\pm(\tau)| d\tau = \lambda^{\beta n} \int_{t_{n+1}}^{t_{n-1}} O(\lambda^{(6-3\beta)n}) d\tau + \int_{t_{n-2}}^{T} O(\lambda^{-2\beta n}) d\tau = O(\lambda^{(4-3\beta)n}),$$

and hence we can sum up the equation (3.7) with respect to $n$ from 1 to $\infty$, due to the absolute convergence of all the series. We get

$$\sum_{n=1}^{\infty} \left( u_n^\pm(t)^2 + 2\lambda^{2n} \int_0^t u_n^\pm(\tau)^2 d\tau \right) = 2 \sum_{n=1}^{\infty} \int_0^t f_n(\tau) u_n^\pm(\tau) d\tau. \quad (3.8)$$

So, $\{u_n^\pm(t)\}_{n=1}^\infty$ and $\{u_n^\pm(t)\}_{n=1}^\infty$ are Leray-Hopf solutions of the problem (0.1). They are distinct because $g_n \not\equiv 0$. ■

Now, Theorem 3.1 is the only thing left to prove.

## 4 The case of constant coefficients

In this section we consider the system (3.1) with constant coefficients $p$ and $q$, moreover with $p = q/2$, and prove an analog of Theorem 3.1 for this case. In the next section we show that the statement of the theorem 3.1 is continuous with respect to changes of $p$ and $q$ in $L_1$-norm, and thus prove it for some functions $p, q \in C^\infty_0(0, 1)$. 

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If $p$ and $q$ are constant, then the system (3.1) can be transformed into

$$
\begin{aligned}
\dot{h}(\tau) &= Mh(\tau), \\
h(0) &= v,
\end{aligned}
$$

(4.1)

where

$$
\begin{bmatrix}
h_1(\tau) \\
h_2(\tau) \\
h_3(\tau)
\end{bmatrix}, \quad v = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,
$$

$$
M = \begin{pmatrix} -\lambda^2 + q & -p & 0 \\ 2p & -1 & \lambda^\beta q \\ 0 & -2\lambda^\beta q & -\lambda^2 \end{pmatrix}.
$$

(4.2)

Then $h(1) = e^Mv$. To satisfy (3.2), we need to find values of $p$ and $q$ such that there are sufficiently large numbers in the spectrum of matrices $M$ and $e^M$.

The following fact is well known, the proof can be found for example in [5, Chapter II].

**Theorem 4.1.** Suppose $T$ is a real $n \times n$ matrix with a simple spectrum,

$$
\sigma(T) = \{\lambda_1(T), \ldots, \lambda_n(T)\}, \quad \lambda_j(T) \neq \lambda_k(T) \text{ for } j \neq k.
$$

Then for any $\varepsilon > 0$ there exists $\delta > 0$, such that if $\|T - S\| < \delta$, then $|\lambda_j(S) - \lambda_j(T)| < \varepsilon$ with appropriate numeration of the spectrum $S$. Moreover, if all the eigenvectors $v_j(T)$, $j = 1, \ldots, n$, have $v_j(T)_1 = 1$, then all the corresponding eigenvectors $v_j(S)$ of matrix $S$ can be chosen in such a way that $v_j(S)_1 = 1$ and $|v_j(T) - v_j(S)| < \varepsilon$.

We remind that in our case $\lambda > 1$, $\beta > 2$. Components of a three-dimensional vector $v_j$ will be denoted by $x_j$, $y_j$, $z_j$.

**Lemma 4.2.** Consider the matrix

$$
A_0 = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & 0 & \lambda^\beta \\ 0 & -2\lambda^\beta & 0 \end{pmatrix}.
$$

The spectrum of $A_0$ is simple and

$$
\sigma(A_0) = \{x^0, w^0, \bar{w}^0\},
$$

where $x^0$ is real, moreover $3/4 < x^0 < 1$; $w^0$ is not real, $0 < \text{Re} w^0 < \frac{1}{8}$, $\text{Im} w^0 > 0$. The corresponding eigenvectors can be chosen in a form

$$
v_1^0, \quad v_2^0 + iv_3^0, \quad v_2^0 - iv_3^0,
$$

with $v_1^0, v_2^0, v_3^0 \in \mathbb{R}^3$,

$$
x_1^0 = x_2^0 = 1, \quad x_3^0 = 0, \quad y_1^0 > 0, \quad y_3^0 < 0, \quad z_3^0 > 0.
$$
Proof. The characteristic polynomial of the matrix $A_0$ has the form
$$
\chi(\alpha) = \alpha^3 - \alpha^2 + \left(\frac{1}{2} + 2\lambda^2\right) \alpha - 2\lambda^2.
$$
Its derivative is positive everywhere on $\mathbb{R}$:
$$
\chi'(\alpha) = 3\alpha^2 - 2\alpha + \frac{1}{2} + 2\lambda^2 \geq \alpha^2 + 2\lambda^2 > 0 \quad \forall \alpha \in \mathbb{R}.
$$
Hence, matrix $A_0$ has only one real eigenvalue and a complex conjugate pair of eigenvalues. We denote them $\kappa_0, w_0, \bar{w}_0$, with $\text{Im} \ w_0 > 0$. By virtue of simple estimates
$$
\chi(1) = \frac{1}{2} > 0, \quad \chi\left(\frac{3}{4}\right) = \frac{15}{64} - \frac{1}{2}\lambda^2 < 0,
$$
one has $\kappa_0 \in (3/4, 1)$. By Vieta’s formulas, $\kappa_0 + 2 \text{Re} \ w_0 = 1$, therefore $\text{Re} \ w_0 \in (0, 1/8)$.

One can easily see that the first components of eigenvectors cannot be zero, so they can be chosen unitary, i.e. $x_1^0 = x_2^0 = 1, x_3^0 = 0$. Next, it follows from $A_0v_1^0 = \kappa_0 v_1^0$ that
$$
y_1^0 = 2(1 - \kappa_0) > 0.
$$
Finally, the equality $A_0(v_2^0 + iv_3^0) = w_0^0(v_2^0 + iv_3^0)$ implies
$$
y_3^0 = -2 \text{Im} \ w_0 < 0 \quad \text{and} \quad z_3^0 = \text{Im} \left(\frac{4\lambda^2(w_0 - 1)}{w_0^0}\right) = \frac{4\lambda^2 \text{Im} \ w_0}{|w_0^0|^2} > 0.
$$

Corollary 4.3. There exist numbers $q_0, \mu > 1$ and $\nu \in (0, 1)$, such that for $q > q_0$, the matrix
$$
A = \begin{pmatrix}
1 - \frac{\lambda^2}{q} & -\frac{1}{2} & 0 \\
1 & -\frac{1}{q} & \lambda^2 \\
0 & -2\lambda^2 & -\frac{\lambda^2}{q}
\end{pmatrix}
$$
has a simple spectrum
$$
\sigma(A) = \{\kappa, w, \bar{w}\}, \quad \frac{3}{4} < \kappa < 1, \quad 0 < \text{Re} \ w < \frac{1}{8}, \quad \text{Im} \ w > 0,
$$
and the corresponding eigenvectors $v_1, v_2 + iv_3, v_2 - iv_3$ can be chosen in such a way that
$$
x_1 = x_2 = 1, \quad x_3 = 0, \quad (4.3)
$$
$$
|v_1| + |v_2| + |v_3| \leq \mu, \quad z_3 - y_1y_3 \geq \nu. \quad (4.4)
$$

Proof. Note that $\|A - A_0\| = O(\frac{1}{q})$. Therefore for a sufficiently large $q_0$ all the inequalities follow from Lemma 4.2 and Theorem 4.1. For $\mu$ and $\nu$ one can take
$$
\mu = 2\left(|v_1^0| + |v_2^0| + |v_3^0|\right), \quad \nu = \min\left(\frac{z_3^0}{2} - \frac{y_1^0y_3^0}{4}, \frac{1}{2}\right).
$$
Lemma 4.4. Let $A$, $\mu$, $\nu$ be the matrix and the numbers from Corollary 4.3. Let $R > 0$. There exists a number $q_1$ such that for $q > q_1$ the matrix

$$B = e^{qA} = \exp \left( \begin{pmatrix} q - \lambda^2 & -q/2 & 0 \\ q & -1 & \lambda^3 q \\ 0 & -2\lambda^3 q & -\lambda^2 \end{pmatrix} \right)$$

has the same eigenvectors $v_1$, $v_2 + iv_3$, $v_2 - iv_3$, as the matrix $A$, and the eigenvalues $e^{\alpha} = k$, $e^{\beta w} = a + ib$, $e^{\beta \bar{w}} = a - ib$.

Moreover,

$$|a| < \omega k, \quad |b| < \omega k,$$

where

$$\omega < \frac{\nu^2}{100\mu^4} < \frac{\nu}{100\mu^2} < \frac{1}{100}.$$  \tag{4.8}$$

Proof. As $\lambda > 3/4$, one has $k > e^{3q/4}$. Next, Re $w < 1/8$ yields

$$\max(|a|, |b|) \leq |e^{q\alpha}| = e^{q\text{Re} w} \leq e^{q/8} \leq e^{-5q/8} k.$$  

Therefore $\omega$ can be chosen to be $\omega = e^{-5q/8}$. If one chooses $q_1$ so big that

$$e^{3q_1/4} > \frac{5\mu^3 R}{2\nu} \quad \text{and} \quad e^{-5q_1/8} < \frac{\nu^2}{100\mu^4},$$

then the conditions (4.6), (4.7) and (4.8) will be fulfilled.

We fix $2p = q > q_1$, where $q_1$ is a number from Lemma 4.4. Suppose $h$ is a solution to the system \((4.1), (4.2)\) with such $p$ and $q$. Then the condition (3.2) is equivalent to the system

$$\begin{pmatrix} B \ 0 \\ y \\ z \end{pmatrix}_1 = \rho y, \quad \begin{pmatrix} B \ 0 \\ y \\ z \end{pmatrix}_2 = \rho z,$$  \tag{4.9}$$

where $B = e^{M}$ is a matrix from Lemma 4.4. We denote

$$\tilde{B} = \begin{pmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{pmatrix}.$$  \tag{4.10}$$

One can see that the existence of a non-trivial solution $\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$ of a system (4.9) is equivalent to $\rho \in \sigma(\tilde{B})$.

Lemma 4.5. Let $R > 0$. Suppose $q_1$ and $B$ are defined by $R$ as in Lemma 4.4. Then for $q > q_1$ the matrix $B$ has two different eigenvalues $\rho_1$, $\rho_2$, and $\max(|\rho_1|, |\rho_2|) > R$.  

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Proof. We will search for a solution of (4.9) in a form
\[ v = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = c_1 v_1 + c_2 v_2 + c_3 v_3, \]
where \( v_1, v_2, v_3 \) satisfy (4.5). \( c_1, c_2, c_3 \in \mathbb{R} \). Then we get the following system
\[
\begin{cases}
  c_1 x_1 + c_2 x_2 + c_3 x_3 = 0, \\
  c_1 (\rho y_1 - k x_1) + c_2 (\rho y_2 - a x_2 + b x_3) + c_3 (\rho y_3 - b x_2 - a x_3) = 0, \\
  c_1 (\rho z_1 - k y_1) + c_2 (\rho z_2 - a y_2 + b y_3) + c_3 (\rho z_2 - b y_2 - a y_3) = 0,
\end{cases}
\]
of three linear equations on \( c_i \). Its determinant is equal to
\[
\det = x_1 ((\rho y_2 - a x_2 + b x_3)(\rho z_2 - b y_2 - a y_3) - (\rho z_2 - a y_2 + b y_3)(\rho y_3 - b x_2 - a x_3)) \\
- x_2 ((\rho y_1 - k x_1)(\rho z_2 - b y_2 - a y_3) - (\rho z_2 - k y_1)(\rho y_3 - b x_2 - a x_3)) \\
+ x_3 ((\rho y_1 - k x_1)(\rho z_2 - a y_2 + b y_3) - (\rho z_1 - k y_1)(\rho y_2 - a x_2 + b x_3))
\]
\[ = U \rho^2 + V \rho + W, \]
where
\[
U = \det(v_1, v_2, v_3) \neq 0,
\]
\[
V = (k - a)(z_3 - y_1 y_3) + b(y_1 y_2 - y_2^2 - y_3^2 + z_2 - z_1),
\]
\[
W = (a^2 + b^2)y_3 - k(a y_3 + b y_2 - b y_1);
\]
in the last two equalities we used the relations (4.3). Using (4.4), (4.7) and (4.8) we get
\[
|U| \leq |v_1||v_2||v_3| \leq \mu^3;
\]
\[
V \geq \frac{9k\nu}{10} - 5|b|\mu^2 \geq \frac{4k\nu}{5},
\]
\[
|W| \leq \mu(a^2 + b^2 + k|a| + 2k|b|) \leq \mu(2\omega^2 + 3\omega)k^2 \leq \frac{\nu^2k^2}{25\mu^3}.
\]
Therefore
\[
V^2 - 4UW \geq \frac{16k^2\nu^2}{25} - \frac{4k^2\nu^2}{25} > 0,
\]
which means that the discriminant is positive and the equation
\[
U \rho^2 + V \rho + W = 0
\]
has two distinct roots. By Vieta’s formulas, one can see that
\[
|\rho_1 + \rho_2| = \frac{V}{|U|} \geq \frac{4k\nu}{5\mu^3} > 2R
\]
due to (4.6). Thus, there is a root \( \rho \) such that \( |\rho| > R \). \( \blacksquare \)
5 Proof of Theorem 3.1

Let $A \in L_1 ((a, b); \text{Mat}(n \times n, \mathbb{R}))$. Consider on the interval $(a, b)$ the following Cauchy problem

$$\begin{cases}
\dot{h}(t) = A(t)h(t), \\
h(a) = h_0;
\end{cases}$$

here $h(t) \in \mathbb{R}^n$. It is equivalent to the integral equation

$$h(t) = h_0 + \int_a^t A(\tau)h(\tau) \, d\tau.$$  \hfill (5.1)

It is well known (see for example [8, Chapter III, §31]), that there exists a unique solution $h \in C([a, b]; \mathbb{R}^n)$ (and therefore $h \in W^{1,1}_{\mathbb{R}^n}$). Iterating (5.1), we get

$$h(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(a; t)^m} \mathcal{T}(A(\tau_1), \ldots, A(\tau_m)) \, d\tau_1 \ldots d\tau_m h_0,$$  \hfill (5.2)

where the symbol $\mathcal{T}$ denotes a chronological ordering

$$\mathcal{T}(A(\tau_1), \ldots, A(\tau_m)) = A(\sigma_1) \ldots A(\sigma_m),$$

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$ is a non-increasing permutation of the arguments $\tau_1, \ldots, \tau_m$. From (5.2) follows a well known (can be found in [8]) estimate

$$\|h\|_{C} \leq e^{\|A\|_{L_1}} |h_0|.$$  

We denote by $B$ a linear operator mapping initial data $h_0$ to the final value $h(b)$. Matrix $B \in \text{Mat}(n \times n, \mathbb{R})$ is a $T$-exponent of the matrix function $A$:

$$B = \text{T-exp}(A) \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(a; b)^m} \mathcal{T}(A(\tau_1), \ldots, A(\tau_m)) \, d\tau_1 \ldots d\tau_m.$$  

The map

$$\text{T-exp} : L_1 ((a, b); \text{Mat}(n \times n)) \rightarrow \text{Mat}(n \times n)$$

is continuous.

**Theorem 5.1.** Let $A_1, A_2 \in L_1 ((a, b); \text{Mat}(n \times n))$, $B_1 = \text{T-exp}(A_1)$, $B_2 = \text{T-exp}(A_2)$. Then

$$\|B_1 - B_2\| \leq \exp \left( \max(\|A_1\|_{L_1}, \|A_2\|_{L_1}) \right) \int_a^b \|A_1(\tau) - A_2(\tau)\| \, d\tau.$$  

However, we could not find the reference in the literature. For the convenience of the reader the proof of the theorem is given at the end of this section.

Let us come back to the proof of the main result.
Proof of Theorem 3.1. Lemmas 4.4 and 4.5 provide us with numbers \( q^* \) and \( \rho^* \), \(|\rho^*| > R\), and a non-zero vector \( \begin{pmatrix} y^* \\ z^* \end{pmatrix} \in \mathbb{R}^2 \), such that

\[
\tilde{B}_s \begin{pmatrix} y^* \\ z^* \end{pmatrix} = \rho^* \begin{pmatrix} y^* \\ z^* \end{pmatrix},
\]

where matrix \( \tilde{B}_s \) is defined by the formula (4.10) using the matrix

\[
B_s = e^{M_s}, \quad M_s = \begin{pmatrix} q^* - \lambda^{-2} & -q^*/2 & 0 \\ q^* & -1 & \lambda^b q^* \\ 0 & -2\lambda^b q^* & -\lambda^2 \end{pmatrix}.
\]

Note that the eigenvalues of \( \tilde{B}_s \) are distinct.

Now we find functions \( p, q \in C_0^\infty(0, 1) \), that are close to constants \( q^*/2 \) and \( q^* \) in the sense of \( L_1(0, 1) \)-norm. We construct the corresponding matrix function \( M(t) \) using (4.2) and matrix \( B = T\exp(M) \). By Theorem 5.1 the norm of the difference \( \| B - B_s \| \) (and therefore the norm \( \| \tilde{B} - \tilde{B}_s \| \)) can be made arbitrarily small by taking the functions \( p \) and \( q \) close to the numbers \( q^*/2 \) and \( q^* \). Now Theorem 4.1 guarantees that the spectrum of the matrix \( \tilde{B} \) consists of two real numbers, and one them is such \( \rho \) that \(|\rho| > R\),

\[
\tilde{B} \begin{pmatrix} y \\ z \end{pmatrix} = \rho \begin{pmatrix} y \\ z \end{pmatrix}, \quad y^2 + z^2 > 0.
\]

Therefore, the solution to the system (3.1) with such \( p, q, y, z \) satisfies the condition (3.2).

5.1 Continuity of \( T \)-exponent

Proof of Theorem 5.1. We have

\[
A_1(\tau_1) \ldots A_1(\tau_m) - A_2(\tau_1) \ldots A_2(\tau_m)
= \sum_{k=1}^m A_1(\tau_1) \ldots A_1(\tau_{k-1}) (A_1(\tau_k) - A_2(\tau_k)) A_2(\tau_{k+1}) \ldots A_2(\tau_m),
\]

wherefrom

\[
\left\| \int_{[a;b]^m} \left( T(A_1(\tau_1), \ldots, A_1(\tau_m)) - T(A_2(\tau_1), \ldots, A_2(\tau_m)) \right) d\tau_1 \ldots d\tau_m \right\|
\leq \sum_{k=1}^m \int_{[a;b]^m} \| A_1(\tau_1) \| \ldots \| A_1(\tau_{k-1}) \| \| A_1(\tau_k) - A_2(\tau_k) \| \| A_2(\tau_{k+1}) \| \ldots \| A_2(\tau_m) \| d\tau_1 \ldots d\tau_m
= \sum_{k=1}^m \| A_1 \|_{L_1}^{k-1} \| A_1 - A_2 \|_{L_1} \| A_2 \|_{L_1}^{m-k} \leq mL^{m-1} \| A_1 - A_2 \|_{L_1},
\]
with $L = \max(\|A_1\|_{L^1},\|A_2\|_{L^1})$. Therefore

$$\|B_1 - B_2\| \leq \sum_{m=1}^{\infty} \frac{1}{m!} \left\| \int_{\triangle^m} (T(A_1(\tau_1), \ldots, A_1(\tau_m)) - T(A_2(\tau_1), \ldots, A_2(\tau_m))) d\tau_1 \ldots d\tau_m \right\| \leq \sum_{m=1}^{\infty} \frac{L^{m-1}}{(m-1)!} \|A_1 - A_2\|_{L^1} = e^L \|A_1 - A_2\|_{L^1}.$$  

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