Measurement-Prepared Quantum Criticality: from Ising model to gauge theory, and beyond

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Measurements allow efficient preparation of interesting quantum many-body states exhibiting long-range, topological, or fractonic orders. Here, we prove that the so-called conformal quantum critical points (CQCP) can be obtained by performing general single-site measurements in appropriate basis on the cluster states in $d \geq 2$. The equal-time correlators of the said states are described by correlation functions of certain $d$-dimensional classical model at critical temperature, featuring spatial conformal invariance. This establishes an exact correspondence between the measurement-prepared critical states and conformal field theories of a range of critical spin models, including familiar Ising model and gauge theories among others. Furthermore, by mapping the correlations of the measured quantum state into the statistical mechanics problem, we establish the stability of long-range or topological orders with respect to measurements deviating from the ideal setting, without any post-selection. Therefore, our findings suggest a novel mechanism in which a quantum critical wavefunction emerges, providing new practical ways to study quantum phases and conformal quantum critical points.

I. INTRODUCTION

Recently, the perplexing and exciting effects of measurements on the evolution of quantum many-body states are attracting growing interests from both condensed matter and quantum information communities. There are two branches of studies. In the first branch, one focuses on how quantum entanglements propagate and builds up under random measurements and unitary dynamics. Initiated by the discovery of the transition in entanglement structure under random measurements and circuit evolution [1–3], there has been extensive works along this directions [4–16]. In the second branch, one focuses on more structured measurements in static situation, preparing quantum states by performing measurements on a subsystem of a so-called resource state, which is related to the idea of measurement-based quantum computation (MBQC). Remarkably, various families of quantum states with long range entanglement, such as Greenberger-Horne-Zeilinger (GHZ) [17] state, those with certain topological and fracton orders [18–22], and more generally an entire family of stabilizer CSS code states [23–25] can be prepared through measurements on the so-called cluster states. The viability to prepare these quantum states is demonstrated to be intimately related to the symmetry-protected topological (SPT) properties of the cluster states [26] that may involve generalized symmetries [27].

Practically, many of the measurement-based state preparation require a precise control of the measured operators and a specific measurement outcome. For instance, to generate the GHZ and toric code states from the cluster states in 1$d$ and 2$d$, single-spin measurements of the Pauli $X$ operators on the sites are needed in the paradigmatic example [19]. However, to adopt the measurement scheme to prepare these quantum states in experiments, it is important to understand whether measurements deviating from the $X$-axis can still produce a quantum state with the same topological or entanglement properties as noises in controlling the measurement angle are unavoidable in experiments. Motivated by that, in this work, we explore the effects of general single-site measurements on the cluster states, namely measurements of single spin along arbitrary directions, which will shed lights on the stability of many measurement-based state preparation schemes.

With the general single-qubit measurements, a natural and perhaps more exciting question to ask is whether we can tune the resulting state through a phase transition, i.e. access certain quantum critical states, by tuning the measurement directions. The primary result of this pa-
per is that a family of quantum critical states at so-called conformal quantum critical points (CQCP) [28, 29] can be realized by measuring a subset of spins on in cluster states in the direction rotated away from X-axis by a special angle \( \theta_c \). We prove that the series of quantum states labeled by the measurement angle \( \theta \) in \( d \) space dimension has wavefunction amplitude given by the Boltzmann weight of a corresponding classical spin model in \( d \)-dimension at inverse temperature \( \beta(\theta) = \tanh^{-1}(\cos \theta) \). In \( d \geq 2 \), certain quantum states can undergo a phase transition at a critical measurement angle \( \theta_c \). At the criticality, the spatial correlation functions of the state exhibit conformal invariance [28]. Strikingly, we argue that such a critical quantum state can be obtained even without post-selections, a special unstable fixed point with random universality class, facilitating the experimental preparation without much overhead. Our discovery provides a complementary viewpoint in essential ways to the critical steady states in monitored quantum dynamics either with both unitary evolution and measurements or with measurements only [1–16]. The preparation of critical state in our study can be thought of as a shallow depth unitary circuit followed by a uniform single-gate measurements on a subsystem, as illustrated in Fig. 1.

The rest of the paper is organized as follows.

In Sec. II, we provide a detailed analysis of the 1D cluster state under measurements in a rotated basis. By calculating correlation functions of the post-measurement state, we provide a rigorous understanding of the stability of long-range entanglement there. Furthermore, we establish that the post-measurement wavefunction amplitude is proportional to the Boltzmann weight of a classical 1D Ising model. More generally, the post-measurement state can be expressed as a product state in X-basis evolved by a certain quantum Hamiltonian in imaginary time. With this observation, we build a parent Hamiltonian for the measured state, parametrized by the measurement angle \( \theta \).

In Sec. III, we generalize our idea and construction to the 2D cluster state defined on vertices and edges of a square lattice. By measuring edges in a rotated basis, we obtain the classical 2D Ising model as the classical Hamiltonian describing the post-measurement wavefunction amplitudes, while by measuring vertices, we obtain the 2D Ising gauge theory as the classical correspondence. Interestingly, we find that the long-range entanglement structure of the GHZ state in the case of measuring edges is robust against a finite angle deviation from the X-basis. The transition of the post-measurement state from a symmetry breaking phase to a disordered phase is found at a finite measurement angle \( \theta_c \) with critical behaviors. On the other hand, the long-range entanglement structure of the toric code is unstable as soon as we deviate from the X-basis. We show that the stability of these long-range entanglement structures can be understood from the phase transitions of the corresponding classical models.

In Sec. IV, we further proceed our constructions to three dimensions, where we can construct two different cluster states, namely symmetry protected topological (SPT) phases associated with \( \mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(2)} \) and \( \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)} \) symmetries. We show that the post-measurement state of the first case corresponds to the ordinary 3D Ising model and the 3D 2-form Ising model\(^1\) when measured on the vertices and edges respectively, while the second case gives the 3D Ising gauge theory. By mapping to the classical partition functions, we show that the 3D toric code topological order with 1-form symmetry breaking is stable under the measurement away from the X-axis, while the 3D toric code with 2-form symmetry breaking is unstable.

\(^1\) Ising model and Ising gauge theory can be thought of as a model with 0-form and 1-form \( \mathbb{Z}_2 \) symmetries. A model where six spins at faces interact has a 2-form \( \mathbb{Z}_2 \) symmetry, which we call 2-form Ising model.
In Sec. V, we discuss the issue of post-selection using the 2D and 3D Ising models as examples. We show that measurement outcomes are under a correlated probability distribution, which can be decoded by mapping into a specific gauge. By explicitly constructing a gauge fixing procedure, we establish a correspondence between the cluster states measured at variable angles without post-selection and a random bond Ising model or random plaquette Ising gauge theory on the Nishimori line [30–32], a special manifold in the parameter space of random spin models. Therefore, we establish the stability of long-range entanglement and correlation for generic post-measurement cluster states away from the X-basis without post selections. Based on the gauge fixing argument, we also outline an experimental blueprint to detect the long-range order and phase transitions without post-selection.

In Sec. VI, we provide more examples of measuring cluster states with subsystem symmetries and fracton physics. In Sec. VII, we conclude with a discussion on conformal quantum critical states and outlook.

II. WARM-UP: MEASUREMENTS ON 1D CLUSTER STATES

As a warm-up, we start our discussion from measurements on 1D cluster states. Consider a chain of qubits which are in a 1D cluster state \( |\psi\rangle \) stabilized by the following Hamiltonian: \( H^{1D}_{\text{cluster}} = -\sum_n Z_{n-1}X_nZ_{n+1} \). The cluster state has interesting properties upon measurements, which can be understood from its nature as a decorated domain wall SPT [33]. The Hamiltonian has \( G = Z_2 \times Z_2 \) symmetry where \( g_1 = \prod_n X_{2n+1} \) and \( g_2 = \prod_n X_{2n} \) are the two generators. Note that \( Z_{2n-1}Z_{2n+1} \) measures the \( g_1 \) domain wall, while \( X_{2n} \) measures the \( g_2 \) charge. Therefore, the ground state can be understood as a superposition of all possible \( g_1 \) domain wall configurations where a \( g_2 \) charge is attached to a \( g_1 \) domain wall. This special structure of the wavefunction indicates that measuring the \( Z_2 \) charges on the even sites will specify the domain wall structure of the \( g_1 \) symmetry. If the measurement outcomes are \( X_{2n} = 1 \), the resulting wavefunction will have no \( g_1 \) domain walls. Since the measurements on the even sites commute with the \( Z_2 \) \((g_1)\) symmetry defined on the odd sites, this measurement generates a GHZ state on the odd sites, i.e. \(|\uparrow\uparrow...\rangle + |\downarrow\downarrow...angle\), a symmetric superposition of spontaneously symmetry-breaking (SSB) states of the \( g_1 \) symmetry. On the other hand, if we measure qubits on the even sites in the Z-basis and get all +1 outcomes, this proliferates the domain walls of the \( g_1 \) symmetry resulting in a disordered wavefunction on the odd sites, namely \(|+ + ...\rangle\).

A. General single-site measurements

Seeing that measuring the subset of qubits along the X-direction and Z-direction give us states with completely different characteristics, one may wonder what would happen if a general measurement is conducted along the axis rotated away from X-axis by an angle \( \theta \). Will the resulting state have the same universal properties as the GHZ state for small \( \theta \)? Is there a transition from the GHZ state to the paramagnetic state at a certain angle? To answer these questions, let us consider the following projective measurement operator of the \( n^{th} \) qubit \( P_n \) along the spin axis \( \hat{n} \equiv (\cos \theta, \sin \theta \sin \phi, \sin \theta \cos \phi) \),

\[
P_n = \frac{1}{2} [I + s_n (X_n n_x + Y_n n_y + Z_n n_z)], \quad \mathcal{P}^2_n = P_n
\]

where \( s_n = \pm 1 \) is the measurement outcome. For now, we assume \( \phi = 0 \), which would not change the physics of primary interest. Let \( \mathcal{P} = \prod_n P_{2n} \) denote the projection operator for measurements on all the even sites. To characterize the resulting state \( \mathcal{P} |\psi\rangle \equiv |\mathcal{P}\psi\rangle \), we will calculate correlation functions, in particular, \( \langle Z_1 Z_{2n+1} \rangle_{\mathcal{P}\psi} \). To that end, the following lemma is useful throughout the work:

Lemma Consider a \( d \)-dimensional stabilizer SPT state protected by symmetry groups \( G_1^{(n)} \times G_2^{(d-n-1)} \), which have a mixed anomaly. Here the superscripts denote \( n \) and \( (d-n-1) \) form symmetries, while \( G_1 \) and \( G_2 \) act on qubits on two different sublattices. Then, the expectation value of an operator defined on a given sublattice is non-vanishing only if the operator is a symmetry action on that sublattice.

The lemma can be understood quite intuitively. As the ground state is stabilized by local stabilizer terms, if a certain operator does not commute with all the stabilizers, its expectation value should vanish. However, if it commutes with all the stabilizers, it simply means that the operator is nothing but a symmetry of the given stabilizer Hamiltonian. For example, for the 1D cluster state \( |\psi\rangle \), for any operator \( O \) defined on the even sublattice, \( \langle \psi| O |\psi\rangle \) vanish unless \( O \) is the identity or \( \prod_n X_{2n} \) (See Appendix. A).

With this lemma, we can show that the correlation function in the measured state with outcomes \( \{s_{2n}\} \) has the following form

\[
\langle Z_1 Z_{2n+1} \rangle_{\mathcal{P}\psi} = \frac{\langle \mathcal{P} \psi | Z_1 Z_{2n+1} | \mathcal{P} \psi \rangle}{\langle \mathcal{P} \psi | \mathcal{P} \psi \rangle} = \frac{\prod_{m=1}^n s_{2m} (\cos \theta)^m + \prod_{m=n+1}^N s_{2m} (\cos \theta)^{N-n}}{[1 + \prod_{m=1}^N s_{2m} (\cos \theta)^N]} \xrightarrow{N \to \infty} \left( \prod_{m=1}^n s_{2m} \right) e^{-n/\xi}, \quad \text{with } \xi = |\ln \cos \theta|^{-1}.
\]

Here \( N \) (even) is the number of unmeasured sites and we have assumed the periodic boundary condition. In
this derivation, we employed both the lemma and the equality: \(Z_1 Z_{2n+1} |\psi\rangle = \prod_{n=1}^{2n+1} X_{2n} |\psi\rangle\). The correlation length only depends on \(\theta\) characterizing the deviation of the measurement angle from the \(\hat{x}\)-axis. This means that the long-range correlation of the GHZ state disappears at any finite \(\theta\) in the thermodynamic limit. Practically, one would get an approximate GHZ state for system size smaller than the length scale \(\xi(\theta)\).

### B. Connection to Classical Partition Function

A keen reader may have noticed that Eq. (2) closely resembles the low temperature series expansion for the 1D classical Ising model. Based on this observation, one can show that the norm of the post-measurement wavefunction is proportional to the partition function \(\sum_n \xi_{\mu=1}^{\mu=1} Z_2 = 1\) or \(1\), which is the imaginary time evolution of the product state \(\otimes_{n=1}^{N} |X_{2n+1} = s_{2n} s_{2n+2}\rangle\) by the Hamiltonian \(H\). This form of the wavefunction immediately implies that in the limit \(\beta \to \infty\) as \(\theta \to 0\), the system should relax into the ground state of \(H\).

Let us describe the derivation. The pre-measurement cluster state wavefunction is written as the equal weight superposition of all domain-wall configurations with charges attached accordingly:

\[
|\psi\rangle = \frac{1}{\sqrt{2^{N-1}}} \sum_{|d_{2n}\rangle} \langle d_{2n} |_{ddw}
\]

where \(|d_{2n}\rangle = \pm\rangle\), and \(d_{2m} = -1\) denotes a domain wall between sites \(2m - 1\) and \(2m + 1\). Here the subscript \(ddw\) stands for the state that is a decorated domain-wall basis, where domains (charges) are defined on odd (even) sites. The summation is over \(2^{N-1}\) configurations since with periodic boundary conditions the domain walls are under the constraint \(\prod_{n=1}^{N} d_{2n} = 1\). Here,

\[
\langle d\rangle|_{ddw} = \langle \{d\}_{\text{odd}} | \{d\}_{\text{even}} \rangle
\]

In this definition, the state labeled by \(|d_{2n}\rangle_{ddw}\) is the cat state of two different spin configurations giving the same domain-wall configuration. For example, the state with no domain wall, namely \(|d_{2n} = 1\rangle_{ddw}\), would be the GHZ state on odd sites, and accordingly, all \(|+\rangle\) states on even sites:

\[
\frac{1}{\sqrt{2}} \left[ |\uparrow\downarrow\cdots\rangle + |\downarrow\uparrow\cdots\rangle \right]_{\text{odd}} \otimes [+]^{N}_{\text{even}}.
\]

With an explicit representation in hand for \(|\psi\rangle\), we now want to obtain the amplitude of the post-measurement wavefunction, \(P|\psi\rangle\), which can be written as

\[
P|\psi\rangle = \sum_{|d_{2n}\rangle} C(|d_{2n}\rangle) \langle \{d\}_{\text{even}} | M \rangle
\]

where now \(|M\rangle = \otimes_{n=1}^{N} |d_{2n}\rangle\) stands for the measured component on even sites. To obtain \(C(|\{d\}\rangle)\), we first decompose the kets \(|\pm\rangle = [1, \pm1]^T / \sqrt{2}\) into the measurement basis:

\[
|\pm\rangle = a_{\pm} |M_+\rangle + b_{\pm} |M_-\rangle,
\]

with

\[
|M_\pm\rangle = \frac{1}{\sqrt{2(1 \pm \sin \theta)}} \left( \sin(\theta) \pm 1 \cos(\theta) \right)
\]

satisfying \(O_\theta |M_\pm\rangle = \pm |M_\pm\rangle\) with measurement operator \(O_\theta = X \cos \theta + Z \sin \theta\) (assuming \(\phi = 0\)). The coefficients follow from \(a_{\pm} = \langle M_+ \pm \rangle\) and \(b_{\pm} = \langle M_- \pm \rangle\).

### C. The post-measurement wavefunction

In this section, we directly derive the wavefunction after measurements from the decorated domain-wall construction \cite{33}. The result shows an intriguing structure for the wavefunction on the unmeasured odd sites:

\[
|P\psi\rangle_{\text{odd}} \propto e^{-\beta \hat{H}} \left[ \otimes_{n=1}^{N} |X_{2n+1} = s_{2n} s_{2n+2}\rangle \right]
\]

\[
\hat{H} = - \sum_{n} s_{2n} Z_{2n-1} Z_{2n+1},
\]

which is the imaginary time evolution of the product state \(\otimes_{n=1}^{N} |X_{2n+1} = s_{2n} s_{2n+2}\rangle\) by the Hamiltonian \(H\).
Then, note that for a measurement outcome \( \{s_{2n}\} \), the projection is defined as
\[
\mathcal{P} \mapsto 
\otimes_{n=1}^{N} |M_{s_{2n}} \rangle \langle M_{s_{2n}} |.
\] (11)
The coefficient can then be obtained by
\[
C(\{d_{2n}\}) = \frac{\langle M | \otimes_{n=1}^{N} |d_{2n} \rangle}{\sqrt{2^{N-1}}} = \prod_{n=1}^{N} \langle M_{s_{2n}} |d_{2n} \rangle / \sqrt{2^{N-1}},
\] (12)
where (see Appendix. B)
\[
\langle M_{s_{2n}} |d_{2n} \rangle = \varphi_{2n} \sqrt{\frac{(1 + s_{2n}d_{2n}\cos \theta)}{2}} = \varphi_{2n} e^{\frac{\beta}{2}s_{2n}d_{2n} / \sqrt{2 \cosh \beta}}.
\] (13)
with \( \varphi_{2n} \equiv (-1)^{(1-s_{2n})(1-d_{2n})/4} = \varphi_{2n}^{(1-s_{2n})/2} \). Using \( d_{2n} = \sigma_{2n-1}\sigma_{2n+1} \), we have (at \( \phi = 0 \))
\[
|\mathcal{P}\psi\rangle_{\text{odd}} \propto \sum_{\{\sigma\}} e^{\frac{\beta}{2} \sum_{n} s_{2n}\sigma_{2n-1}\sigma_{2n+1}} \prod_{n} \varphi_{2n} \{\{\sigma_{2n}\}, \}
\] (14)
Note that the last factor of the above wavefunction can be simplified as
\[
\sum_{\{\sigma\}} \left( \prod_{n} (\sigma_{2n-1}\sigma_{2n+1})(1-s_{2n})/2 \right) \{\{\sigma_{2n}\}, \} \nonumber \\
= \prod_{n} \left( Z_{2n-1} Z_{2n+1} \right) \left( 1-s_{2n} \right) / 2 \left( \otimes_{n=1}^{N} |+\rangle_{2n-1} \right) \nonumber \\
= \prod_{n} \left( Z_{2n-1} Z_{2n+1} \right) \left( 1-s_{2n} \right) / 2 \left( \otimes_{n=1}^{N} |+\rangle_{2n-1} \right) \nonumber \\
= \otimes_{n=1}^{N} |X_{2n-1} = -s_{2n-2}s_{2n} \rangle
\] (15)
Putting everything together, we finally obtain the wavefunction in Eq. (4). This expression gives consistent results for the norm \( \langle \mathcal{P} \psi | \mathcal{P} \psi \rangle \). For \( \phi \neq 0 \), we can obtain non-trivial complex phase factors as detailed in the Appendix. B. Although these phase factors can affect the expectation values when we measure correlations of \( Y \) or \( X \) operators, they do not change any physics in the \( Z \) correlations.

There are two solvable limits: \( \theta = 0 \) and \( \theta = \pi/2 \). At \( \theta = 0 \), the result simply implies that unless \( s_{2n} = d_{2n} \), the wavefunction component is zero. Therefore, the only surviving component would be \( \{|\sigma_{s}\rangle\} \), whose explicit form is defined in Eq. (6). At \( \theta = \pi/2 \), the wavefunction amplitudes become uniform with phase factors \( \varphi_{2n} \) depending on the measurement comes. For \( s_{2n} = 1 \), it gives \( |+\rangle_{\otimes N} \), and for \( s_{2n} = -1 \), it gives \( |-\rangle_{\otimes N} \). This aligns with the expectation from the stabilizer correlation \( Z_{2n}X_{2n+1}Z_{2n+2} |\psi \rangle \propto |\psi \rangle \).

We remark that the derivation here is completely general for any wavefunction constructed by a decorated domain wall method. This is because decorated domain wall method completely specifies the relation between measured and unmeasured sites in a simple manner. In a higher dimensional case, we can show that the measurement-projected amplitude in Eq. (13) can be calculated in terms of domain wall variable, which can be converted into the operator action on unmeasured sites. The expression for the product state to be time-evolved in Eq. (15) can also be derived in a similar manner, where the state in \( X \)-basis would be given by the product of measurement outcomes neighboring an unmeasured site.

D. Parent Hamiltonian for the measured states

Interestingly, we find that the family of states \( |\mathcal{P}_{\theta,\phi=0}\psi \rangle \) on the odd sites with post-selection \( s_{2n} = 1 \) is the ground state of the following Hamiltonian \( H = \sum n H_{n} \):
\[
H_{n} = -\left[ X_{2n-1} - \cos^{2} \theta Z_{2n-3} X_{2n-1} Z_{2n+1} \right. \nonumber \right. \\
+ \cos \theta (Z_{2n-3} Z_{2n-1} + Z_{2n-1} Z_{2n+1}) \left. \right].
\] (16)

We derive the Hamiltonian following the Witten conjugation method [34–36]. It is a simple procedure that later allows us to generate the parent Hamiltonians for our post-measurement states in various settings. The crucial premise is that \( |\mathcal{P}\psi \rangle \) is given by the imaginary time evolution of a certain product state Eq. (4) for the inverse temperature \( \beta = \beta(\theta) \). This is to say, at \( \beta = 0 \), the state is the ground state of \( H_{0} = -\sum n s_{2n} s_{2n+1} X_{2n+1} \), and at \( \beta > 0 \), the state can be thought of as the evolution of the ground state under the non-unitary operator \( M_{\beta} = e^{\frac{\beta}{2} \sum n s_{2n} Z_{2n-1} Z_{2n+1}} \).

To proceed further, let us perform a Kramers-Wannier duality, where we can write \( X_{2n-1} \rightarrow X_{2n-1}' X_{2n}' \) and \( Z_{2n-1} Z_{2n+1} \rightarrow Z_{2n}' \). The state becomes
\[
|\mathcal{P}\psi \rangle \propto M_{\beta}' |\Psi_{0} \rangle', \nonumber \right.
\[M_{\beta}' = \prod_{n} e^{\frac{\beta}{2} s_{2n} Z_{2n}'}, \nonumber \left.
|\Psi_{0} \rangle' = |\{X_{2n}' X_{2n+2}' = s_{2n} s_{2n+2}, \text{ for all } n\} \rangle'.
\] (17)
The Hamiltonian for \( |\Psi_{0} \rangle \) is
\[
H_{0}' = \sum \frac{1}{2} \left( 1 - s_{2n} s_{2n+2} X_{2n}' X_{2n+2}' \right) \nonumber \\
= \sum \Gamma(n) |G_{n} \rangle \langle G_{n}| \nonumber \\
\Gamma(n) = \frac{1}{2} \left( s_{2n} X_{2n}' - s_{2n+2} X_{2n+2}' \right). \nonumber \left.
\] (18)
where we intentionally write \( H_{0}' \) in positive semi-definite structure. Then it follows that one choice of the Hamiltonian for \( |\mathcal{P}\psi \rangle \)' is
\[
H' = \sum \Gamma_{\beta}(n) |\Pi_{\beta} \rangle \langle \Pi_{\beta}| \nonumber \\
|\Pi_{\beta} \rangle = M_{\beta}' |\Pi_{\beta} \rangle \nonumber . \nonumber \left.
\] (19)
As \( e^{-\beta Z} = \cosh \beta (1 - Z \tanh \beta) = \cosh \beta (1 - Z \cos \theta) \), up to an overall constant prefactor of \( \cosh \beta(\theta) \),

\[
\Gamma'_\beta(n) \propto (s_{2n}X_{2n}^2 - s_{2n+2}X_{2n+2}^2) - i \cos \theta (Y_{2n} - Y_{2n+2}^2) \quad (20)
\]

Thus, \( H'_n \) reads

\[
H'_n = -s_{2n}s_{2n+2}X_{2n}^2X_{2n+2} + \cos^2 \theta Y_{2n}Y_{2n+2} - \cos \theta (s_{2n}Z_{2n} + s_{2n+2}Z_{2n+2}) + \text{const}. \quad (21)
\]

As \( H'_n \) is a positive semi-definite Hamiltonian that annihilates \( |\mathcal{P}\psi\rangle \), \( \sum_n H'_n \) is a valid parent Hamiltonian whose ground state is \( |\mathcal{P}\psi\rangle \). Once we reverse the Kramers-Wannier duality, we obtain

\[
H_n = -s_{2n}s_{2n+2}X_{2n}X_{2n+1} + \cos^2 \theta Z_{2n-1}X_{2n+1}Z_{2n+3} - \cos \theta (s_{2n}Z_{2n-1}Z_{2n+1} + s_{2n+2}Z_{2n+1}Z_{2n+3}) \quad (22)
\]

For the measurement outcome \( s_{2n} = 1 \) for all \( n \), we obtain the Hamiltonian Eq. (16). Furthermore, note that applying X-gate (basis flip) to a set of sites is equivalent to flipping \( \{s\} \) that are emanating from the set of sites. In fact, if \( \prod s_{2n} = 1 \), we can always find a basis where the model is entirely ferromagnetic. Even when \( \prod s_{2n} = -1 \), we can find a basis where only a single bond is antiferromagnetic and the other bonds are ferromagnetic.

The spectrum of this Hamiltonian is known to be gapless at \( \theta = 0 \) [37], which is a multicritical point neighboring the paramagnet, ferromagnet (\( \mathbb{Z}_2^{(0)} \) SSB), and \( \mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^T \) SPT (i.e., the cluster state) [38] as illustrated in Fig. 3. The gaplessness at \( \theta = 0 \) is easily seen from Eq. (21), where the terms are nothing but the XY model with perpendicular magnetic field [36] (which can be written in terms of free fermions under a Jordan-Wigner transformation). Although the Hamiltonian is gapless, the ground state entanglement entropy does not diverge; this can be directly inferred from the fact that the initial 1D cluster state is described by a matrix product state (MPS) with bond dimension \( \chi = 2 \), and the measurement projection cannot change the MPS structure. This seemingly inconsistent behavior can be resolved by realizing that the criticality is not captured by a 2D conformal field theory but by a critical theory of free fermions with a dynamic critical exponent \( z = 2 \) [39]. In fact, the ground state trajectory of Eq. (16) is a paradigmatic example of a 1D phase transition that can be expressed by a simple MPS with bond dimension \( \chi = 2 \) [40].

III. 2D CLUSTER STATES

Now that we have a thorough understanding of the post-measurement states in one dimension, we move onto two spatial dimensions, where higher form symmetries [27] become important. Consider the 2D cluster state Hamiltonian where qubits reside at vertices and edges of 2d square lattice as illustrated in Fig. 4(a):

\[
H = -\sum_v \left( X_v \prod_{e \ni v} Z_e \right) - \sum_e \left( X_e \prod_{v \ni e} Z_v \right). \quad (23)
\]

Bolded symbols \( Z \) and \( X \) act on edges, and unbolded symbols \( Z \) and \( X \) act on vertices. Here, all operators commute with others and the groundstate satisfy each term to be 1. This implies that \( B_p \equiv \prod_{e \ni p} X_e = 1 \) for any plaquette \( p \). There are two symmetries in this Hamiltonian:

\[
\begin{align*}
\mathbb{Z}_2^{(0)} & \text{ 0-form: } g = \prod_v X_v \\
\mathbb{Z}_2^{(1)} & \text{ 1-form: } h_\gamma = \prod_{e \in \gamma} X_e
\end{align*} \quad (24)
\]

where \( \gamma \) is any closed loop along the bonds. Again, note that the ground state of Eq. (23) has the decorated domain wall (defect) structure: the creation of a pair of 1-form charged object by \( \prod_{e \ni p} Z_e \) is accompanied with the creation of 0-form domain walls by \( X_e \); also, the creation of 1-form domain walls by \( X_e \) is accompanied with the creation of a pair of 0-form charges by \( Z_eZ_{e'} \).

For the 2D cluster states, one can choose to measure the spins either on the vertices or on the edges. The two measurement schemes exhibit qualitatively different physics as we will show below.

A. Measurements on vertices: Ising gauge theory

We apply the projective measurement in Eq. (1) on every vertex spin. At \( \theta = 0 \) with all measurement outcomes being +1, we would expect the resulting state
have the following constraint, \( A_v = \prod_{e \in v} Z_e = 1 \) and \( B_p = \prod_{e \in p} X_e = 1 \), giving rise to a topological order of the 2D toric code model [41]. In this state, the operator \( C_{\gamma}^X = \prod_{j \in \gamma \perp} X_j \) defined on the contractible loop \( \gamma \perp \) is \( \sim = 1 \), which is a signature of the spontaneously broken 1-form symmetry as \( C_{\gamma}^Z \) counts the 1-form symmetry defects enclosed by the loop. In a complementary point of view, we can embed the system into the torus geometry, and consider a logical qubit operator \( C_{\gamma}^Z \) for \( \gamma \perp \) being a non-contractible loop along the cycle. Here, we can show that \( \langle C_{\gamma}^Z \rangle = 0 \). As \( C_{\gamma}^X = \prod_{e \in \gamma} X_e = 1 \) for any non-contractible loop \( \gamma \) due to the stabilizer structure and \( \{ C_X, C_Z \} = 0 \), the post-measurement state must have \( \langle C_{\gamma}^Z \rangle = 0 \). This implies that the resulting quantum state is the symmetric superposition of four different configurations under \( C^Z \) operators. However, we can show that the correlation function of two non-contractible loops separated by the distance \( l \) is constant, i.e., the state develops a long-range order with a spontaneously broken 1-form symmetry.

We want to detect whether the topological order is robust when the measurements are moved away from \( \theta = 0 \). To do so, we calculate the expectation value of \( C_{\gamma}^Z \) defined on the boundary of a surface \( S \). Using the same formalism, we can show that (See Appendix. A 2 b)

\[
\langle C_{\gamma}^Z \rangle_{\psi} = \frac{\langle P \psi | C_{\gamma}^Z | \psi \rangle}{\langle P \psi | P \psi \rangle} \sim (\cos \theta)^{|S|} \tag{25}
\]

where we employed both the lemma and the equality \( \prod_{e \in S} Z_e = \prod_{e \in S} X_e \). The area law of the loop expectation value, instead of the perimeter law, indicates that 1-form symmetry is intact in the thermodynamic limit. Therefore, the \( |P \psi \rangle \) cannot be topologically ordered. A complementary fact in support of this observation is that the expectation value of \( \prod_{e \in l} X_l \), an operator that creates a pair of anyons on the two ends of a open string \( l \), is non-vanishing, namely

\[
\langle \prod_{e \in l} X_l \rangle_{\psi} \approx \langle \prod_{v \in \partial l} Z_v \rangle_{\psi} \approx (\sin \theta)^2. \tag{26}
\]

Non-vanishing expectation value for any open string operator is the signature of the anyon condensation which gives a trivial symmetric phase for \( \theta > 0 \). However, we remark that even though the state is not the SSB of 1-form symmetry, Eq. (25) gives a quantitative answer for how far it exhibits the correlation structure that can be approximated as being topologically ordered.

We observe that the above loop expectation value can be mapped to the area-law correlation function in the 2D classical Ising gauge theory at finite temperature, where the 1-form symmetry exactly maps to the local gauge symmetry. More precisely, the corresponding Ising gauge theory is defined on the edges of the dual lattice, and the local gauge transformation is defined as flipping the spins on the edges emanating from the set of dual vertices. Such a gauge transformation is equivalent to the 1-form symmetry action \( \prod X \) along the loop defined on the boundary of the set of dual sites. Similar to the 1D example in Eq. (4), the post-measurement wavefunction can be expressed in the following form:

\[
|P \psi \rangle \propto e^{-\frac{\theta}{2} \hat{H}} \prod_{n=1}^{N} |X_e = \prod_{v \in e} s_v \rangle

\hat{H} = - \sum_v s_v \prod_{e \in v} Z_e = - \sum_p s_p \prod_{t \in p} Z_t, \tag{27}
\]

where the tilde subscript is for the dual lattice label. Expanded in \( Z_e \) basis in the dual lattice, the above expression gives rise to the wavefunction amplitude given by the Boltzmann weight of the 2D Ising gauge theory (see Appendix. A 2 b). 2D Ising gauge theory is exactly solvable and known to enter a trivial phase with area law loop expectation value at any finite temperature, which aligns with Eq. (25).

We remark that in Eq. (27), \( |P \psi \rangle \) remains the same if one replaces the imaginary time evolution \( e^{-\beta \hat{H}/2} \) by \( e^{-\beta \hat{H}_{2D}} \) since \( B_p = \prod_{e \in p} X_e \) commutes with all other terms in \( \hat{H} \) and acts trivially on the state. Therefore, at \( \beta \to \infty (\theta \to 0) \), we expect the projected wavefunction to be the toric code ground state. Based on this observation, we can find a series of parent Hamiltonian which stabilizes the wavefunction in Eq. (27) parametrized by \( \beta (\theta) \). The parent Hamiltonian on the dual lattice reads [36]

\[
H = H_0 + \cos^2 \theta H_{\text{SP}} + 4 \cos \theta H_{\text{toric}},
\]

\[
H_0 = - \sum_e \left[ \prod_{p \ni e} s_p \right] X_e
\]

\[
H_{\text{SP}} = \sum_e X_e \prod_{e' \in \partial(e)} Z_{e'},
\]

\[
H_{\text{toric}} = - \sum_p s_p \prod_{e \in p} Z_e. \tag{28}
\]
Here \( \hat{n}(\hat{c}) \) is the set of neighboring edges that are boundaries of two dual plaquettes sandwiching \( \hat{c} \). The derivation is a direct generalization of the procedure described in Sec. II D.

This parent Hamiltonian is gapless at \( \theta = 0 \). Similar to the 1D case, the gapless point is multicritical, neighboring a \( \mathbb{Z}_2 \) topological order, a \( \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(2)} \) SPT, and a trivial paramagnetic phase as illustrated in Fig. 3. Interestingly, the ground state at \( \theta = 0 \) gapless point is exactly the toric code ground state as our construction demonstrates. However, for any \( \theta > 0 \), the Hamiltonian enters the trivial confined phase. At the multicritical point, the model has \( U(1) \) pivot symmetry [36] and the low energy effective theory can be shown to be dual to a dilute interacting Bose gas at zero density in 2d [42–44]. This critical theory is known to have exact dynamical exponent \( z = 2 \) as the self energy correction from the boson self-interaction vanishes at every order of perturbation theory due to the absence of particles in the vacuum state [44].

### B. Measurements on edges: Ising model

Next, we apply the projective measurement in Eq. (1) on every edge spin. At \( \theta = 0 \) with all measurement outcomes \( s_e = 1 \), we would expect the resulting state to have \( Z_vZ'_v = 1 \) for any edge \( e = (v,v') \). Therefore, the post-measurement state would be the GHZ state. At \( \theta > 0 \), we can show that the two-point correlation \( \langle Z_iZ_j \rangle_{\psi^\theta} \) is given by the two-point correlation function of classical 2D Ising model at temperature \( \beta(\theta) \) (See Appendix A 2 c). Furthermore, the wavefunction on the unmeasured sites is expressed as:

\[
|\psi\rangle \propto e^{-\frac{\beta}{2} \hat{H}} \left| \bigotimes_{n=1}^N X_v \right| \prod_{e \in n(v)} s_e \rightangle
\]

\[
\hat{H} = - \sum_{e = (v,v')} s_e Z_v Z_{v'}.
\]  

(29)

If \( s_e = 1 \) for all measurements, the physical properties of this wavefunction can be understood from the 2D classical ferromagnetic Ising model. Unlike in the 1D case, the 2D Ising model has a finite temperature ordering transition which implies that long-range entanglement is robust for \( \theta \neq 0 \) and that we can prepare a quantum critical state at a specific measurement angle \( \theta_c = \cos^{-1} \tanh(\beta_c) \). Since the 2D Ising model has an exact self-duality, we obtain that the transition happens when \( \beta_c = \sqrt{2} - 1 \), i.e. \( \theta_c \approx 65^\circ \). Therefore, by measuring the 2D cluster state, we can prepare the wavefunction which goes through the phase transition across this measurement angle. In particular, the ordered state will appear as the GHZ state of two SSB configurations, which is long-range entangled and robust up to a finite \( \theta \) as illustrated in Fig. 2(a).

At \( \theta = \theta_c \), we have a quantum phase transition, where the critical state has an area law entanglement since the pre-measurement cluster state is parameterized by 2D tensor-network state, called projected entangled pair states (PEPS), with bond dimension \( \chi = 2 \) [39]. As its correlation functions in \( Z \)-basis are determined by the critical 2D Ising model, the wavefunction has a spatial conformal structure with power-law decaying correlation functions. This turns out to be a specific example of a conformal quantum critical point (CQCP)[28, 29]. For a CQCP with a known statistical weight, one can always construct a parent Hamiltonian for the critical state, which is the generalization of the RK Hamiltonian [45]. Such a parent (quantum) Hamiltonian will have a dynamic critical exponent, \( z \), to be equal to the dynamic exponent for relaxational critical dynamics for the corresponding classical statistical model. Thereby, we can compute the dynamic critical exponent for a given CQCP Hamiltonian. For CQCPs with a \( U(1) \) symmetry, the critical theory is analytically known to be a quantum Lifschitz theory with a dynamic critical exponent \( z = 2 \). However, the present model does not have any \( U(1) \) symmetry. Indeed, a numerical analysis reveals that its dynamic critical exponent is \( z \approx 2.2 \) [46, 47].

Similar to previous sections, we can obtain the parent Hamiltonian for the post-measurement state using the general method outlined in Sec. II. For measurement outcomes \( s_e = 1 \), \( H \) is given by

\[
H = -\alpha_X \sum_i X_i - \alpha_{ZZ} \sum_{\langle ij \rangle} Z_i Z_j - \alpha_{Z^2 Z} \sum_i Z_i^2 Z_j
- \alpha_{\text{body}} \sum_i X_i B_i - \alpha_{\text{5 body}} \sum_i X_i \prod_{j \in n(i)} Z_j
\]

(30)

where \( B_i = \sum_{j,k \in n(i), j \neq k} Z_j Z_k \) and \( n(i) \) is the set of sites neighboring the site \( i \). The coefficients are graphically shown in Fig. 5. At the critical point \( \theta_c, \alpha_X \approx 1.1 \), while the other coefficients \( \alpha_2, \alpha_3, \alpha_5 \) are relatively small. For a comparison, the critical point of the 2d transverse field Ising model is given by \( (\alpha_2, \alpha_3)_{\text{TFIS}} \approx 1.5[48] \).

In the case without post selection, the state after measurements is still the ground state of a local Hamiltonian, whose structure is similar to Eq. (30) but with the signs of

![FIG. 5. The coefficients of the parent Hamiltonian in Eq. (30) as a function of measurement angle \( \theta \in [0, \frac{\pi}{2}] \). The red dashed line marks the position of \( \theta_c \). Coefficients are normalized in such a way that the strength of the transverse field term \( \alpha_X \) is 1 throughout the range.](image)
the coefficients depending on the measurement outcomes (See Appendix. C1). With randomly signed interaction coefficients, one may wonder whether there still exists a phase transition in this Hamiltonian as a function of $\theta$. If exists, what is the nature of the transition? We will discuss these questions in Sec. V.

IV. 3D CLUSTER STATES

A. 3D SPT with $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ symmetry

The generalizations of the previous discussions into three spatial dimensions are more diverse as there can be different types of cluster states. There are straight-forward generalizations of the low dimensional stories. Moreover, in 3D, we can consider a new type of cluster states, illustrated in Fig. 4(b), which has a generalization of decorated domain wall (defect) construction for two 1-form symmetries. In this model, qubits are defined on the edges and faces of a cubic lattice. Note that qubits on the faces can be thought to be defined on the edges of the dual lattice. This particular 3D cluster stabilizer Hamiltonian is written as the following:

$$H_{3D \text{ SPT}} = -\sum_{f \in \partial V} X_f \prod_{f \in \partial V} Z_f - \sum_f X_f \prod_{e \in f} Z_e$$  \hspace{1cm} (31)

where $f$ runs for all faces of the cubic lattice. Bolded symbols act on faces, and unbolded symbols act on edges. Note that by multiplying stabilizers, we obtain that $\prod_{f \in c} X_f = 1$ for any cube $c$ and $\prod_{e \in v} X_e = 1$ for any vertex $v$. Here, generators of two 1-form symmetries are defined on two-dimensional surfaces as the following:

$$\mathbb{Z}_2^{(1)} 1\text{-form: } h_{\partial V} \equiv \prod_{f \in \partial V} X_f$$

$$\mathbb{Z}_2^{(1)} 1\text{-form: } g_{\partial V} \equiv \prod_{e \in \partial V} X_e$$  \hspace{1cm} (32)

where $V$ is a certain three-dimensional volume enclosed by cubic faces, and $\hat{V}$ is a infinitesimally inflated version of $V$ which intersects with edges emanating from $V$. Therefore, $\partial V$ is a set of faces, while $\partial \hat{V}$ is a set of edges. Without loss of generality, if we measure all faces in $X$-basis, then we obtain that the resulting state has $\prod_{f \in \partial V} Z_f = 1$ and $\prod_{e \in v} X_e = 1$ for all $f$ and $v$, which gives the 3D toric code ground state.

Now, let us measure the qubits on the faces at angle $\theta$ (due to the dual nature of the system, we get the same physics by measuring edges). In this case, the post-measurement state in $Z$-basis has its amplitudes given by the Boltzman weight of the 3d Ising gauge theory (c.f. Eq. (3)). One can show that the 1-form symmetry of the unmeasured model maps into the local gauge symmetry in the 3d Ising gauge theory. With measurement output

$$\langle \mathcal{P} \psi \rangle \propto e^{-\frac{\beta}{2} \hat{H}} \left[ \otimes_{f=1}^{N} |X_e = \prod_{f \in e} s_f \rangle \right]$$

$$\hat{H} = -\sum_f s_f \prod_{e \in f} Z_e.$$  \hspace{1cm} (33)

Assume we shift $\hat{H}$ by the sum of local terms $A_e \equiv \prod_{c \ni e} X_e$ to convert it into $H_{3D \text{ toric}}$. Still, if we define $\langle \mathcal{P} \psi \rangle$ as above with this new Hamiltonian $H_{3D \text{ toric}}$, the state would be the same because $A_e$ commutes with $\hat{H}$ and $A_e$ acts trivially on the initial state.\footnote{2 This is because $\prod_{e \ni v} (\prod_{f \in e} s_f) = 1$}

Therefore, the above equation is nothing but an imaginary time evolution by 3D toric code Hamiltonian $H_{3D \text{ toric}}$. On the other hand, note that $\hat{H}$ itself corresponds to the 3D Ising gauge theory with a random interaction sign $s_f$ at $\beta = \tanh^{-1}(\cos \theta)$. Since the 3D Ising gauge theory has a finite temperature transition at $\beta_{3D \text{ gauge}} = 0.76$, the correspondence implies that the confinement transition would happen at $\theta \approx 50^\circ$ with post-selection $s_f = 1$.

For $\theta < \theta_c$, the expectation of the Wilson loop operator $W_\Gamma = \prod_{e \in \Gamma} X_e$ over a loop $\Gamma$ decays exponentially with an perimeter law, which can be predicted based on the correspondence to the 3d Ising gauge theory (See Appendix. A 2 f). As a result, the preparation of three-dimensional deconfined phase is robust. For $\theta > \theta_c$, $\langle W_\Gamma \rangle$ decays exponentially with an area law, which implies that the phase belongs to the trivial confined phase. At $\theta = \theta_c$, the state becomes critical. Its dynamical critical exponent can be obtained from the dynamics of 3d classical Ising model [47, 49], which can be calculated by various methods. The parent Hamiltonian for the case without post-selection is in Appendix. C2.

In comparison, the transition between the same two gapped phases is more commonly described by a 3d toric code model in a single transverse field, where the transverse field term generates flux loop excitations. In that model, the direct transition between the two phases are mapped to Wegner’s 4-dimensional lattice gauge theory [50], and it is known to be of first order [51, 52].

B. 3D SPT with $\mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(2)}$ symmetry

Note that one can also consider a different geometry for a 3D cluster state, where qubits reside at vertices and edges of a cubic lattice. This cluster state has a decorated domain wall construction of a $\mathbb{Z}_2$ 0-form and a $\mathbb{Z}_2$ 2-form symmetries. By measuring vertices in $X$-basis with $s_v = 1$, one can get the 3D toric code state. However, upon measuring vertices at angle $\theta$, the resulting topological order becomes unstable, which get mapped
into a 3D 2-form Ising model (See Appendix. A.2.d). The post-measurement wavefunction is given by
\[ |\mathcal{P}\psi\rangle \propto e^{i\sum s_i \sum_{e \in V} Z_e} |\bar{\psi}_e\rangle |X_e = \prod_{e \in V} s_e\rangle. \] (34)

At finite \( \theta \), the expectation value of Wilson surface operator \( M_{\partial V} \equiv \prod_{e \in \partial V} Z_e \) that measures 2-form symmetry defects decays with the volume enclosed by the surface,
\[ \frac{\langle \mathcal{P}\psi \middle| \prod_{e \in \partial V} Z_e |\mathcal{P}\psi\rangle}{\langle \mathcal{P}\psi |\mathcal{P}\psi\rangle} \sim (\cos \theta)^{|\partial V|}. \] (35)

For the state to be a topologically ordered, the above quantity should decay at most exponentially with the surface area. Therefore, there is no SSB of the 2-form symmetry nor long-range entanglement. In a complimentary point of view, we can show that \( \langle \prod_{e \in \mathcal{L}} X_e \rangle_{\mathcal{P}\psi} \sim (\sin \theta)^2 (\mathcal{L} \text{ is an open string}) \), which indicates the anyon condensation occurs for any \( \theta > 0 \).

In fact, the \( \theta = 0 \) is again a multicriticality point with the 3D toric code state as the ground state. As one may guess from its similarity to the 2D toric code preparation case, this multi-critical point is captured by the Hamiltonian with \( U(1) \) pivot symmetry generated by \( H_{\text{3D toric}}^{\text{3D}} \):
\[ H = H_0 + \cos^2 \theta H_{\text{SPT}} + 6 \cos \theta H_{\text{3D toric}}, \] (36)
whose detailed structure is illustrated in Appendix. C.3. Similar to the 1D and 2D examples, the state has parent Hamiltonian which is the interpolation of disordered, \( \mathbb{Z}_2^{(2)} \times \mathbb{Z}_2 \) SPT, and topologically ordered states as illustrated in Fig. 3.

On the other hand, if we measure edges of the model, we obtain the wavefunction with the amplitudes from 3D Ising model (See Appendix. A.2.c). In this case, the long-range entanglement, i.e., cat-ness of the state, is robust as long as \( \theta < \theta_c \approx 78^\circ \) if we post-select on \( s_e = 1 \).

V. CORRELATED RANDOMNESS

A. No post-selection at \( \theta = 0 \)

So far, we have assumed that the measurement outcome \( s_i = 1 \). However, in general, the two measurement outcomes, \( \pm 1 \), are equally probable since for an operator \( O^\theta \equiv X \cos \theta + Z \sin \theta, \langle O \rangle = 0 \) in any cluster state. Without post-selection, one might guess that the resulting classical partition function we obtain would be a random bond Ising model with the probability of having a ferromagnetic bond being \( p_e = 1/2 \), since the coefficients \( s_e \) in Eq. (29) are equally probable for \( \pm 1 \). However, this is not the case due to the correlation among measurement outcomes at different bonds.

To demonstrate the simplest yet interesting case, consider measuring the qubits on the edges of a 2D cluster state at \( \theta = 0 \). The following operator for each plaquette \( p \), which is a product of stabilizers of the cluster state,
\[ B_p = \prod_{e \in p} (X_e \prod_{v \in e} Z_v) = \prod_{e \in p} X_e, \] (37)
equals to 1 when acting on the cluster state. Since \( B_p \) commutes with the operators to be measured, after the measurement, it remains to be a stabilizer with the same classical value,
\[ B_p = \prod_{e \in p} s_e = 1. \] (38)

Intriguingly, this is to say, the outcomes of measurements on the cluster state are not completely independent.

One implication of this discussion is that at \( \theta = 0 \), there is no true randomness in a gauge-invariant sense. Here, we define a gauge transformation as redefining the \( Z \)-basis and measurement outcomes in the following way
\[ \tilde{s}_i = t_i \sigma_i, \quad \tilde{s}_e = \{ij\} = t_i t_j, \quad t_i = \pm 1. \] (39)

Note that this transformation leaves the partition function of the corresponding classical model invariant\(^3\).

\(^3\) In a more general setting, the gauge transformation is defined as \( \tilde{s}_i = \prod_{e \in n(i)} t_e s_e \) where \( n(i) \) is the unmeasured sites neighboring to the measured site \( i \).

\[ \text{FIG. 6. Four different probability distribution of } p_i(\{s_e\} \text{)} \text{ for bonds to be } +1 \text{ or } -1. \text{ If we define a gauge transformation as changing signs of all bonds emanating from a selected set of sites, then the two distributions (a) and (b) can be exchanged under the gauge transformation on the sublattice. In the gauge choice (c), we have all horizontal bonds and single vertical bonds to have all positive signs. The probability distribution of remaining bonds are completely determined by the correlations } E[\prod_{e \in \gamma_{\text{toric}}} s_e] = (\cos \theta)\langle \gamma \rangle, \text{ which gives the distribution 4 in (d). Note that distributions 1,2,3 can all be gauge fixed to the configuration in (c), which gives the distribution 4. Therefore, if there is a physics that only depends on the gauge-invariant structure of the probability distribution of bonds, these four distribution must give the same physics. Note that for a gauge fixing configuration of bonds, it works as long as gauge-fixed bonds forms a graph without cycle, i.e., tree structure. In principle, there are many such configurations and any of those works.} \]
More concretely, the post-measurement wavefunction can be mapped into the same statistical model independent of measurement outcomes after a proper gauge transformation which redefines the basis in the unmeasured sites. To see this, let us rewrite the Eq. (4) as

$$|\mathcal{P}\psi\rangle \propto e^{-\frac{\beta}{2}H([s])} |\mathcal{P}\psi([s])\rangle_{\beta=0}$$

$$\hat{H}([s]) = -\sum s_{2n}Z_{2n-1}Z_{2n+1} \quad (40)$$

where $|\mathcal{P}\psi([s])\rangle_{\beta=0}$ is the superposition of all possible spin configurations with irrelevant signs $\eta_i$ that depends on measurement outcomes:

$$|\mathcal{P}\psi([s])\rangle_{\beta=0} = \sum_{\{\sigma_i=\pm 1\}} \eta_i(\{s\})|\sigma_i\rangle \quad (41)$$

Therefore, we observe that the wavefunction norm

$$\langle \mathcal{P}\psi | \mathcal{P}\psi \rangle = \sum_{\{\sigma_i=\pm 1\}} \langle \{\sigma_i\} | e^{-\beta \hat{H}([s])} | \{\sigma_i\} \rangle \quad (42)$$

is independent of the gauge transformation in Eq. (39).

Along with this gauge transformation, the constraints in Eq. (38) imply that distinct values of $\{s_e\}$ are pure gauge degrees of freedom; by choosing an appropriate set of $\{t_i\}$, we can always make $\bar{s}_{ij} = s_{ij}t_iti_j = 1$ for all edge $e = (ij)$. In other words, as long as random outcomes are under the constraint in Eq. (38), one can always find a gauge where the transformed Hamiltonian $\hat{H}([\bar{s}])$ is a fully ferromagnetic Ising model.

The same argument goes through for all the other models at $\theta = 0$. For example, in a 3D cubic lattice, by performing a gauge transformation on selected sites, one can always make Ising interactions to be ferromagnetic due to these constraints. The same holds for the gauge theory, where in 2D we have a single constraint $\prod_i X_v = 1$. Since a pair of measurement outcomes $s_{ij} = -1$ can always be made into $s_v = +1$ without changing other outcomes by the change of $Z$-basis, which is flipping spins along the open string along the dual lattice connecting two lattice sites, it implies that all plaquette interaction terms can be made ferromagnetic. The similar argument holds in 3D. This implies that in order to achieve true randomness for $\theta = 0$, one has to start with stabilizers that are randomly distributed between $\pm 1$. For example, this can be achieved by starting with the product state of randomly distributed $|\pm\rangle$ states, which can be obtained by measuring $|\up\rangle \otimes N$ in $X$-basis.

**B. True randomness at $\theta \neq 0$**

Next, consider a measurement along the general operator $O^\theta = X \cos \theta + Z \sin \theta$ on edges of the 2D SPT example we considered. Note that

$$\langle \prod_{e \in \gamma_{\text{hoop}}} O_e^\theta \rangle_{\psi} = (\cos \theta)^4 \Rightarrow \mathbb{E}[\prod_{e \in \gamma_{\text{hoop}}} s_e] = (\cos \theta)^4$$

$$\langle \prod_{e \in \gamma_{\text{open}}} O_e^\theta \rangle_{\psi} = 0 \Rightarrow \mathbb{E}[\prod_{e \in \gamma_{\text{open}}} s_e] = 0 \quad (43)$$

The second equality implies that the probability $p_+$ for the bond outcome to be ferromagnetic (+1) is exactly 1/2. However, the first equality implies that the joint probability between different bonds have a nontrivial structure. More generically, Eq. (43) implies that all non-gauge-invariant quantities must have zero expectation values due to the correlation structure, and only gauge-invariant quantities take non-zero expectation values. Therefore, our probability distribution $P(\{s\})$ of measurement outcomes $\{s\}$ must be gauge invariant in a sense that any two gauge equivalent configurations should have the same probability. Indeed, we confirm this with an explicit calculation of the probability distribution for measurement outcomes (D6) in Appendix. D.

How do we gain more physical insight on this correlated randomness? As discussed in the previous section, the physics of the post-measurement state is independent of the gauge transformation in Eq. (39). This implies that two different sets of signs of the bonds $\{s\}$ and $\{s'\}$ give the same physics if they are related by the gauge transformation, i.e., $\{s\} \sim \{s'\}$. For example, the physics of the random bond Ising model (RBIM) where the probability for a ferromagnetic bond $p_+ = (1 + \cos \theta)/2$ (distribution 1 in Fig. 6(a)) is equivalent to the physics of the RBIM where $p_+ = (1 - \cos \theta)/2$ (distribution 2 in Fig. 6(b)). This is because under the gauge transformation with $t_i = -1$ on one sublattice, we can exchange ferromagnetic and antiferromagnetic bonds. This example gives an insight that in the probability distribution of bonds, the only important part is gauge-invariant structure, i.e., the distribution of the gauge-invariant frustrated.

Indeed, consider a partition function for the corresponding classical spin Hamiltonian defined for the measurement outcome $\{s\}$:

$$Z(\{s\}) = \langle \mathcal{P}\psi | \mathcal{P}\psi \rangle = \sum_{\{\sigma\}} e^{\beta \sum_{(i,j)} s_{ij}\sigma_i\sigma_j} \quad (44)$$

Then, $Z(\{s\}) = Z(\{\bar{s}\})$ if $\{s\} \sim \{\bar{s}\}$. Therefore, in some sense, in the probability distribution 1, many of the configurations are gauge equivalent.

In order to extract the gauge-invariant information of a given probability distribution $P(\{s\})$, we fix the gauge for all measurement outcomes as in Fig. 6(e), where all horizontal rows and a single vertical column is made to be ferromagnetic. For any measurement outcomes $\{s\}$, there is a unique gauge transformation $G_{s}(e)$ that maps $\{s\}$ into the configuration in Fig. 6(e) where all bonds along the backbone of the lattice, illustrated by red lines, is ferromagnetic. We denote the configuration for unfixed bonds as $\{\bar{s}\}_{gf}$. Conditioned on performing a gauge
transformation \( G_s \) for every measurement outcomes, we can obtain the probability distribution \( P^\text{st}((\tilde{s})_t) \) for the unfixed bonds in the gauge fixed configuration in Fig. 6(e) from an original probability distribution \( P(\{s\}) \) as the following:

\[
P^\text{st}((\tilde{s})_t) = \sum_{G(\{s\}) = (\tilde{s})_t} P(\{s\}). \tag{45}
\]

Therefore, for any probability distribution \( P \), we can map it to a new probability distribution \( P^\text{st} \) defined on the fixed gauge. If any two distributions \( P \) and \( P' \) map to the same \( P^\text{st} \), two distributions should be gauge equivalent and give the same partition function and other gauge-invariant physics.

Now, let us understand the distribution of measurement outcomes \( \{s\} \) in our protocol. The distribution is characterized by expectation values of all possible curves, either closed or open as in Eq. (43), referred as distribution 3 in Fig. 6(c). At first, its randomness seems qualitatively different from the distribution 1 or 2. However, the closed loop expectation values in the distribution 1 and 2 are also given as

\[
\mathbb{E} \left[ \prod_{e \in \gamma} s_e \right] = \sum_{n=0}^{l} \binom{l}{n} (-1)^{-n} p^n_+ (1 - p_+)^{-n} = (p_+ - (1 - p_+))^l = \left( 2 \cos \theta \right)^l \tag{46}
\]

where \( l \) is the length of the loop \( \gamma \). Since three distributions 1, 2, and 3 agree on the expectation values of all gauge-invariant object (products along the closed loop), it already hints that they are equivalent in terms of the gauge-invariant structure.

In Appendix E, we rigorously prove that the new distribution \( P^\text{st} \) for unfixed bonds under the gauge-fixing in Eq. (45) is uniquely specified by the expectation values of all loops as in Eq. (46), which we denote by the distribution 4 in Fig. 6(d). This, in turns, implies that the gauge-invariant structure of the distribution (1-4) should be the same, and these four distribution must give the same physics. In this language, we remark that the well-known Mattis’ model \([53]\), where \( H = - \sum_{ij} J_{ij} \sigma_i \sigma_j \) with the structure \( J_{ij} = \epsilon_i \epsilon_j \) for random \( \epsilon_i = \pm 1 \), is trivially mapped to the RBIM with \( p_+ = 1 \), i.e. pure ferromagnetic Ising model, since its random bonds have no frustration: \( \prod_{e} s_e = \prod_{e} (\epsilon_i \epsilon_j) = 1 \).

Therefore, when it comes to the gauge-invariant physics, we can freely choose any probability distribution from 1 to 4 to describe our post-measurement wavefunction. In particular, using distribution 1 (or 2), we find our post-measurement wavefunction should be described by the RBIM with the probability of ferromagnetic bond

\[
p_+ = \frac{1}{2} (1 + \cos \theta). \tag{47}
\]

at temperature \( \beta = \tanh^{-1}(\cos \theta) \). Note that RBIM on square lattice is symmetric under \( p_+ \leftrightarrow 1 - p_+ \).

For a 2D RBIM, the 2D Ising model limit is obtained at \( p_c = 1 \) where the transition happens at \( \beta^{-1} = 2.27 \). At zero temperature, the phase transition can occur from the ferromagnetic (FM) phase to spin glass (SG) phase at \( p_c = 0.104 \) \([55]\). Note that the SG phase is unstable at \( T > 0 \), immediately transitioning into a paramagnetic phase. The general phase boundary is illustrated in Fig. 7(a) whose values are taken from \([54–56]\). Here, the red line is the trajectory our measurement protocol traverse without post-selection, i.e., \( (p, \beta) = \left((1 + \cos \theta)/2, \tanh^{-1}(\cos \theta)\right) \). The trajectory happens to coincide with the Nishimori line \([30]\). Therefore, without additional frustration from the initial stabilizer configurations, we will get a critical behavior for the unstable fixed point denoted by red circles.

![FIG. 7. We plot phase diagrams for the random bond Ising model in (a) two and (b) three dimensions, and for the (c) random plaquette gauge model in three dimension. \( \theta_c \) denoted in the diagram represents the transition angle \( \theta_c \) where the long-range entanglement disappears when we do not post-select outcomes. These phase boundaries are taken from \([54–56]\). Here, the red line is the trajectory our measurement protocol traverse without post-selection, i.e., \( (p, \beta) = \left((1 + \cos \theta)/2, \tanh^{-1}(\cos \theta)\right) \). The trajectory happens to coincide with the Nishimori line \([30]\). Therefore, without additional frustration from the initial stabilizer configurations, we will get a critical behavior for the unstable fixed point denoted by red circles.](attachment:image.png)
the concentration of the frustrated plaquettes, if we can somehow manipulate the number of frustrated plaquettes in the measured state, we can deviate from the red trajectory at a given inverse temperature. There are several ways to achieve, but to name a few, one can post-select on configurations with slightly smaller or larger concentration of frustrated plaquettes. Or, one may even start from specific stabilizer signs (as briefly discussed in the previous subsection) to distort the probability distribution under Eq. (43). Therefore, we have another tuning knob to access different universality classes. If the measured state ends up flowing into the random Ising model universality, then its dynamic critical exponent would be \( z \approx 3.11 \) [61]. For the 2D random plaquette gauge model (RPGM) which can arise if we measure vertices of the 2D cluster state, we remark that there is no transition anyway at finite angle even without any frustration, and the presence of quenched disorder only makes the stability worse.

For 3D, the situation is similar in that our protocol would go through the Nishimori line crossing the unstable fixed point for both 3D RBIM (from measuring edges of \( Z_2^{(0)} \times Z_2^{(2)} \) SPT) and 3D RPGM (from measuring faces or edges of \( Z_2^{(1)} \times Z_2^{(1)} \) SPT). Their phase diagrams are illustrated in Fig. 7(b,c) [54, 57, 62]. Again, depending on how we perturb away from the Nishimori line, we can access either 3d Ising universality or 3d random bond or random plaquette universality. At the 3D random bond Ising universality, its dynamic critical exponent is given by \( z \approx 2.11 \) [63], which is much larger than the 3d Ising universality value. For the 3D RPGM, we remark that the phase transition behavior is closely related to the robust storage of quantum information in the surface codes [64].

The numerical study shows that the RPGM is within the perimeter law phase (deconfined) for \( p < 0.03 \), and transitions into the area law phase (confined) for \( p > 0.03 \) [55, 56], which implies that for \( \theta_c^{3dRPGM} = 19.9^\circ \). As long as \( \theta < \theta_c \), the preparation of the 3D topological order is robust. However, we note that its dynamic critical exponent has not been studied in the literature. The clear understanding of the critical theory would require a detailed future numerical work.

In summary, the stability of the random interaction models at \( \theta > 0 \) implies that the stable long-range correlation or topological structure is robust even without post selections. Furthermore, these results guarantee that we can prepare some quantum critical states with or without post-selections.

Now, there comes an important question: how do we experimentally verify this physics? First of all, notice that a vanilla correlation function or susceptibility in \( Z \)-basis will give a gauge-dependent physics, which will vanish under averaging over different measurement outcomes\(^4\). In order to overcome this issue, we define a gauge-invariant object as the following: for a predefined path \( \gamma \) connecting two sites \( v \) and \( v' \), one can define a gauge-invariant correlation function:

\[
\langle Z_v Z_{v'} \rangle^\text{G-inv}_{\psi} = \langle Z_v \left[ \prod_{e \in \gamma} s_e \right] Z_{v'} \rangle_{\psi}, \quad (48)
\]

Note that once we perform a gauge transformation \((Z_v \rightarrow \tilde{Z}_v)\) into the configuration in Fig. 6(e), the vanila correlation function in terms of new variable \( \tilde{Z} \) would be equivalent to the gauge-invariant correlation function defined for the set of paths \( \{\gamma_{v,v'}\} \) that only goes through the red ferromagnetic backbone, which exists for any pair \((v, v')\). In other words,

\[
\langle \tilde{Z}_v \tilde{Z}_{v'} \rangle_{\psi} = \langle \tilde{Z}_v \left[ \prod_{e \in \gamma_{v,v'}} \tilde{s}_e \right] \tilde{Z}_{v'} \rangle_{\psi}, \quad (49)
\]

where we employed that \( \prod_{e \in \gamma_{v,v'}} \tilde{s}_e = 1 \) along the ferromagnetic red path in Fig. 6(e). The model written in terms of \( \tilde{Z} \) is simply another RBIM at a specific disorder realization whose gauge-invariant disorder (frustration) content is same as the one generated by the distribution 1 with \( p_+ = (1 + \cos \theta)/2 \) (the average number of gauge-invariant frustration should only depend on \( \theta \), as illustrated in Fig. 8b). Therefore, the singularity of its partition function occurs at \( \theta_c^{RBIM} \), and it must coincide with the singular behavior of its susceptibility defined as

\[
\frac{1}{N} \sum_{v,v'} \langle \tilde{Z}_v \tilde{Z}_{v'} \rangle_{\psi} \propto \begin{cases} N & \theta < \theta_c \quad \text{(FM)} \\ \text{const} & \theta > \theta_c \quad \text{(PM)} \end{cases} \quad (50)
\]

which diverges or stays constant depending on which phase it belongs to. Therefore, without any gauge transformation, if we simply calculate the gauge-invariant version of the correlation function defined along the path

---

\(^4\) Especially in our case with distribution 3, which is a gauge-invariant probability distribution, any gauge-dependent quantity should be averaged to zero. A simple way to see this is that \( p_+ = p_- = 1/2 \) here when other bonds are marginalized.
within the red backbone in Fig. 6(e), $\frac{1}{2} \sum_{ij} (Z_i Z_j)^{\text{G-inv}}_{P \Psi}$ should diverge with $N$ in the ferromagnetic phase, and give an experimental way to access phase transitions in the quantum states governed by distribution 3. Finally, we remark that the choice of gauge fixing in Fig. 6(e) is not unique. If we perform a gauge fixing into ferromagnetic signs for bonds along any graph without cycle (tree), this will serve as a prescription equally good as all the others.

Unfortunately, there is another complication; every time we go through the circuit and perform measurements in a rotated basis Fig. 1, we obtain sets of different measurement outcomes $\{s\}$ which can belong to different gauge-equivalent classes unless $\theta = 0$. Without post-selection into the same gauge-equivalent class, we cannot obtain the wavefunction in a specific gauge-equivalent disorder realization of a random bond (or plaquette) Hamiltonian. In other words, each experimental snapshot of the post-measurement wavefunction corresponds to a different gauge-equivalent class of disorder realizations. Therefore, if we average over these snapshots, it would be different from obtaining expectation values for a specific quenched disorder configuration. However, in principle, we remark that as the number of iteration $M \to \infty$, we expect the averaging over different snapshots would approach to the disorder averaging over expectation values for a specific disorder realization, which should behave in a desired way (c.f. Eq. (50)).

VI. GENERALIZATION

In general, the framework we developed can be applied for any stabilizer-based SPT states, such as subsystem SPT (SSPT). For example, consider a 2D cluster state with qubits defined on the vertices of a square lattice with $H = -\sum_v X_v \prod_{v' \in v} Z_{v'}$. The square lattice is bipartite, and we can decompose the lattice into two sublattices $A$ and $B$. The system has two subsystem symmetries $G_A$ and $G_B$ defined by the application of the product of $X$ operators along any diagonal direction for a corresponding sublattice. Upon measuring the sublattice $A$ in $X$-basis, we obtain the ground state of Xu-Moore model [65] on the other sublattice, described by $H_{\text{Xu-Moore}} = -\sum_{v \in A} s_v \prod_{v' \in \Lambda(v)} Z_{v'}$. With the subsystem symmetry $G_B$, the ground state manifold has an extensive degeneracy, and the post-measurement state should be the superposition of exponentially-many ($\sim 2^{2L-1}$) SSB configurations of the subsystem symmetry. As expected, this exponentially many superposition of SSB configuration, which is the generalization of the GHZ state, is not robust if $\theta \neq 0$. Indeed, the corresponding classical partition function and correlation functions are those of 2d plaquette (gonohedric) Ising model [66], which is different from 2d Ising gauge theory as spins reside at vertices. This classical model exhibits an exotic correlation structure, which can be decomposed into decoupled 1d Ising models. As it maps to the stacked 1d Ising model, the system is disordered for any finite $\theta$ (or any $\beta^{-1} > 0$) and the long-range entanglement is not robust.

For 3D SSPT, one can consider the cluster state whose qubits are defined on vertices and faces of the cubic lattice, described by $H = -\sum_f X_f \prod_{v \in f} Z_v - \sum_v X_v \prod_{f \ni v} Z_f$. The model has two symmetries. One is subsystem symmetries $G_A$ acting as $\prod_{\text{plane}} X$ for qubits on vertices, and there are $3L$ planes for $L \times L \times L$ sites. The other is the one-form symmetry $G_B$ acting as $\prod_{f \in S} X_f$ for qubits on faces. By construction, the model has $\prod_{f \in F(\text{dual})} X_f = 1$, where $F(\text{dual})$ is the set of four faces penetrated by the cross in the dual lattice. Upon measuring faces in $X$-basis, one specifies $\prod_{v \in f} Z_v = s_f$ for any face, which is the ground state of the 3D-version of Xu-Moore model with a degeneracy $\sim 2^{3L-2}$ [67]. Post-selecting outcomes without any frustration, the corresponding 3D Ising plaquette model exhibits a first-order phase transition at finite temperature $\beta_c \approx 0.55$ [67]. It implies that the long-range entanglement of the superposition of exponentially many SSB configurations for the post-measurement state is robust up to $\theta_c \approx 60^\circ$ with post-selection $s_f = 1$. Even without any post-selection, the long-range entanglement is expected to be robust for $\beta^{-1} > 0$ with some frustrated interactions as long as the subsystem symmetry is intact, similar to Fig. 7. Therefore, similar to the RBIM and RPGM, we expect the transition to occur at finite $\theta$. The exact phase diagram of 3d random plaquette Ising model (RPIM) at finite temperature is left for future study.

Upon measuring vertices in $X$-basis, we obtain $\prod_{f \ni v} Z_f = s_v$, which gives rise to a X-cube fracton state [68] in the dual lattice. Again, the resulting state is a superposition of $\sim 2^{3L-3}$ degenerate ground states. If we thought of qubits on the faces (vertices) as edges (cubes, denoted by $\mathbb{E}$) of the dual lattice, we get $\prod_{v \in \mathbb{E}} Z_v = s_v$. Moving away from the $X$-basis with an angle $\theta$, the wavefunction is given by $\langle P \psi \rangle \propto e^{-\beta H_{\text{X-cube}}/2} \otimes \ket{X_f = \prod_{v \in f} s_v}$ where $H$ is defined as the following on the dual lattice:

$$H_{\text{X-cube}} = -\sum_{v \in \mathbb{E}} s_v \left( \prod_{\mathbb{E} \ni v} Z_v \right) - \sum_{+} \prod_{\mathbb{E} \ni v} X_v$$

The properties of the post-measurement state at angle $\theta$ can be argued to be trivial based on the observation that the corresponding 12-body spin model has no phase transition at $T > 0$. Following our previous strategy to calculate the norm of the post-measurement state, we can show that $\langle P \psi | P \psi \rangle \sim \frac{1}{\beta^4} \left( 1 + 2^4 L \langle \cos \theta \rangle^{O(L^2)} \right) \sim c_0 + e^{-c_2 L - c_1 \cos \theta} L^2$ where the summation is taken for all possible intersecting planes $(2^3 L)$, i.e., elements of the subsystem symmetry $G_A$. Since the second term gets exponentially suppressed for any $\theta \neq 0$ ($c_2 > 0$) in a large system size, it implies the absence of the phase transition. Therefore, the fracton order is unstable in our scheme for any $\theta \neq 0$. 

\[14\]
It is also worth pointing out the connection between two different Hamiltonians resulting from measuring different sublattices of a given cluster SPT. Above, measuring one sublattice gives the 3D Xu-Moore Hamiltonian, while measuring the other sublattice gives the X-cube model. As illustrated in [68], these two models are related by a generalized duality via gauging. This is precisely what is happening through the cluster entangler followed by measurements [26]. We note that the wavefunction of the form in Eq. (3) with the Boltzmann weight from a corresponding spin-model in [68] at a certain $\beta$ would be realized through our protocol by measuring the state obtained by a fracton state coupled with ancilla qubits through cluster entangler.

VII. SUMMARY AND OUTLOOK

In this work, we revealed the fate of quantum states obtained by measuring cluster states in the rotated basis $O_\theta = X \cos \theta + Z \sin \theta$, which is equivalent to applying a certain shallow depth unitary circuit to the product of $|+\rangle$ and perform measurements in the same basis. We showed that any post-measurement state is expressed by a certain product state in $X$-basis under the imaginary time evolution $e^{-\beta H}$ by the Hamiltonian depending on the measurement outcomes, where $\beta = \text{tanh}^{-1}(\cos \theta)$. As a result, any post-measurement state has its amplitudes given by the Boltzmann weights of various corresponding classical spin models, ranging from Ising model and gauge theories [69] to plaquette model and even beyond, at the temperature determined by the measurement angle.

At specific angles, the post-measurement wavefunction exhibits quantum criticality, where the wavefunction exhibits spatial conformal symmetry due to the amplitude structure. Constrained by the finite amount of entanglement the shallow circuit can infuse, the resulting state has a constrained entanglement structure, giving rise to a special family of quantum critical states. A family of found quantum criticalities is called conformal quantum critical point (CQCP), and we found that the dynamical exponent $z \geq 2$ for all the examples discussed in this work, which is consistent with the analytical bound for the dynamical exponent $z = 2$ argued in several literature [28, 47]. In particular, in any dimensions, we found a family of post-measurement states whose parent Hamiltonian is generated by a so-called pivoting structure [36] with the phase diagram in Fig. 3. From the Kramers-Wannier duality, the CQCPs for this family of states maps into the Bose-Einstein condensation transition of hardcore bosons with $z = 2$ [42–44]. We remark that this class of CQCPs with $z = 2$ has non-local $U(1)$ symmetry [36], analogous to quantum Lifschitz transitions and famous Rohksar-Kivelson model [28, 70, 71] with $z = 2$. We also found nontrivial examples with $z > 2$ where there is no extra $U(1)$ symmetry that protects the dynamical critical exponent [47]: $z \approx 2.16$ (2D) and $z \approx 2.02$ (3D) for Ising CQCPs with post-selections.

Without post-selections, the measured wavefunction can simulate random interaction models, e.g. 2D and 3D random bond Ising models, along the Nishimori line, whose critical behaviors are determined by unstable fixed points [32]. By post-selecting configurations slightly away from the Nishimori line, one can access either Ising universality classes or random universality classes with $z \approx 3.11$ (2D RBIM) and $z \approx 2.11$ (3D RBIM) among others.

Interestingly, we found that the post-measurement wavefunctions with $z = 2$ CQCPs are all unstable in their long-range entanglement structure under $\theta \neq 0$. This is intimately tied to the observation that the original cluster state is described by a tensor network with a finite bond dimension with an area-law entanglement capacity, and measurements cannot change the underlying tensor-network structure. Indeed, for all the examples presented, the groundstate at $\theta = 0$ already saturates the entanglement capacity. As the post-measurement state parameterized by $\theta$ already saturates its entanglement entropy at $\theta = 0$ which can only decrease with $\theta > 0$, it is natural for its parent Hamiltonian naturally to host a (multi)critical point at $\theta = 0$.

Our results have several implications. For example, it answers the robustness of the measurement-based quantum state preparation in a rigorous way by mapping the problem into the concrete statistical mechanics problem. Furthermore, it provides an experimental guide to prepare an exotic family of conformal quantum critical states. Excitingly, we believe our framework is generalizable for various quantum phases. Although we have considered $\mathbb{Z}_2$ higher-form symmetries, in general we can consider $\mathbb{Z}_N$ symmetries, which we conjecture to give rise to general Potts model and $\mathbb{Z}_N$ gauge theories among others. Furthermore, our method would allow experimental preparation of a so-called skeleton states [40, 72, 73] to plaquette model and even beyond, at the temperature determined by the measurement angle.

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[1] Y. Li, X. Chen, and M. P. A. Fisher, Quantum zeno effect and the many-body entanglement transition, Phys. Rev. B 98, 205136 (2018).
[2] B. Skinner, J. Ruhman, and A. Nahum, Measurement-induced phase transitions in the dynamics of entanglement, Phys. Rev. X 9, 031009 (2019).
[3] A. Chan, R. M. Nandkishore, M. Pretko, and G. Smith, Unitary-projective entanglement dynamics, Phys. Rev. B 99, 224307 (2019).
[4] Y. Li, X. Chen, and M. P. A. Fisher, Measurement-driven entanglement transition in hybrid quantum circuits, Phys. Rev. B 100, 134306 (2019).
[5] R. Vasseur, A. C. Potter, Y.-Z. You, and A. W. W. Ludwig, Entanglement transitions from holographic random tensor networks, Phys. Rev. B 100, 134203 (2019).
[6] X. Cao, A. Tilloy, and A. D. Luca, Entanglement in a fermion chain under continuous monitoring, SciPost Phys. 7, 24 (2019).
[7] M. J. Gullans and D. A. Huse, Dynamical purification of phase transition induced by quantum measurements, Phys. Rev. X 10, 041020 (2020).
[8] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Quantum error correction in scrambling dynamics and measurement-induced phase transition, Phys. Rev. Lett. 125, 030505 (2020).
[9] Q. Tang and W. Zhu, Measurement-induced phase transition: A case study in the nonintegrable model by density-matrix renormalization group calculations, Phys. Rev. Research 2, 013022 (2020).
[10] C.-M. Jian, Y.-Z. You, R. Vasseur, and A. W. W. Ludwig, Measurement-induced criticality in random quantum circuits, Phys. Rev. B 101, 104302 (2020).
[11] J. Lopez-Piqueres, B. Ware, and R. Vasseur, Mean-field entanglement transitions in random tree tensor networks, Phys. Rev. B 102, 064202 (2020).
[12] Y. Bao, S. Choi, and E. Altman, Theory of the phase transition in random unitary circuits with measurements, Phys. Rev. B 101, 104301 (2020).
[13] D. Rossini and E. Vicari, Measurement-induced dynamics of many-body systems at quantum criticality, Phys. Rev. B 102, 035119 (2020).
[14] R. Fan, S. Vijay, A. Vishwanath, and Y.-Z. You, Self-organized error correction in random unitary circuits with measurements, Phys. Rev. B 103, 174309 (2021).
[15] Y. Li, X. Chen, A. W. W. Ludwig, and M. P. A. Fisher, Conformal invariance and quantum nonlocality in critical hybrid circuits, Phys. Rev. B 104, 104305 (2021).
[16] D. Ben-Zion, J. McGreavy, and T. Grover, Disentangling quantum matter with measurements, Phys. Rev. B 101, 115131 (2020).
[17] H. J. Briegel and R. Raussendorf, Persistent entanglement in arrays of interacting particles, Phys. Rev. Lett. 86, 910 (2001).
[18] R. Raussendorf, S. Bravyi, and J. Harrington, Long-range quantum entanglement in noisy cluster states, Phys. Rev. A 71, 062313 (2005).
[19] M. Aguado, G. K. Brennen, F. Verstraete, and J. I. Cirac, Creation, manipulation, and detection of abelian and non-abelian anyons in optical lattices, Phys. Rev. Lett. 101, 260501 (2008).
[20] A. Bolt, G. Duclos-Cianci, D. Poulin, and T. M. Stace, Foliated quantum error-correcting codes, Phys. Rev. Lett. 117, 070501 (2016).
[21] D. J. Williamson and T. Devakul, Type-II fractons from coupled spin chains and layers, Phys. Rev. B 103, 155140 (2021).
[22] R. Verresen, N. Tantivasadakarn, and A. Vishwanath, Efficiently preparing Schrödinger’s cat, fractons and non-Abelian topological order in quantum devices, arXiv e-prints , arXiv:2112.03061 (2021).
[23] A. R. Calderbank and P. W. Shor, Good quantum error-correcting codes exist, Phys. Rev. A 54, 1098 (1996).
[24] A. Steane, Multiple-particle interference and quantum error correction, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 452, 2551 (1996).
[25] A. Bolt, G. Duclos-Cianci, D. Poulin, and T. M. Stace, Foliated quantum error-correcting codes, Phys. Rev. Lett. 117, 070501 (2016).
[26] N. Tantivasadakarn, R. Thorngren, A. Vishwanath, and R. Verresen, Long-range entanglement from measuring symmetry-protected topological phases, arXiv e-prints , arXiv:2112.01519 (2021).
[27] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized global symmetries, Journal of High Energy Physics 2015, 172 (2015).
[28] E. Ardonne, P. Fendley, and E. Fradkin, Topological order and conformal quantum critical points, Annals of Physics 310, 493 (2004).
[29] E. Fradkin and J. E. Moore, Entanglement entropy of 2d conformal quantum critical points: Hearing the shape of a quantum drum, Phys. Rev. Lett. 97, 050404 (2006).
[30] H. Nishimori, Internal Energy, Specific Heat and Correlation Function of the Bond-Random Ising Model, Progress of Theoretical Physics 66, 1169 (1981).
[31] H. Nishimori, Geometry-induced phase transition in the ±J Ising model, Journal of the Physical Society of Japan 55, 3305 (1986).
[32] A. Honecker, M. Picco, and P. Pujol, Universality class of the nishimori point in the 2d ±J random-bond Ising model, Phys. Rev. Lett. 87, 047201 (2001).
[33] X. Chen, Y.-M. Lu, and A. Vishwanath, Symmetry-protected topological phases from decorated domain walls, Nature Communications 5, 3507 (2014).
[34] E. Witten, Constraints on supersymmetry breaking, Nuclear Physics B 202, 253 (1982).
[35] J. Wouters, H. Katsura, and D. Schuricht, Interrelations among frustration-free models via Witten's conjugation, SciPost Phys. Core 4, 027 (2021).

[36] N. Tantivasadakarn, R. Thorngren, A. Vishwanath, and R. Verresen, Pivot Hamiltonians as generators of symmetry and entanglement, arXiv e-prints , arXiv:2110.07599 (2021).

[37] C. Fernández-González, N. Schuch, M. M. Wolf, J. I. Cirac, and D. Pérez-García, Frustration free gapless hamiltonians for matrix product states, Communications in Mathematical Physics 333, 299 (2015).

[38] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry protected topological orders and the group cohomology of their symmetry group, Phys. Rev. B 87, 155114 (2013).

[39] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Criticality, the area law, and the computational power of projected entangled pair states, Phys. Rev. Lett. 96, 220601 (2006).

[40] M. M. Wolf, G. Ortiz, F. Verstraete, and J. I. Cirac, Quantum phase transitions in matrix product systems, Phys. Rev. Lett. 97, 110403 (2006).

[41] A. Kitaev, Anyons in an exactly solved model and beyond, Annals of Physics 321, 2 (2006).

[42] D. Uznov, On the zero temperature critical behaviour of the nonideal bose gas, Physics Letters A 87, 11 (1981).

[43] D. S. Fisher and P. C. Hohenberg, Dilute bose gas in two dimensions, Phys. Rev. B 37, 4996 (1988).

[44] M. P. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Boson localization and the superfluid-insulator transition, Physical Review B 40, 546 (1989).

[45] C. L. Henley, From classical to quantum dynamics at rokhasar–kivelson points, Journal of Physics: Condensed Matter 16, S891 (2004).

[46] M. P. Nightingale and H. W. J. Blöte, Monte carlo computation of correlation times of independent relaxation modes at criticality, Phys. Rev. B 62, 1089 (2000).

[47] S. V. Isakov, P. Fendley, A. W. W. Ludwig, S. Trebst, and M. Troyer, Dynamics at and near conformal quantum critical points, Phys. Rev. B 83, 125114 (2011).

[48] H. W. J. Blöte and Y. Deng, Cluster monte carlo simulation of the transverse ising model, Phys. Rev. E 66, 066110 (2002).

[49] C. Castelnovo, C. Chamon, C. Mudry, and P. Pujol, From quantum mechanics to classical statistical physics: Generalized rokhasar–kivelson hamiltonians and the “stochastic matrix form” decomposition, Annals of Physics 318, 316 (2005).

[50] F. J. Wegner, Duality in generalized ising models and phase transitions without local order parameters, Journal of Mathematical Physics 12, 2259 (1971).

[51] R. Balian, J. Drouffe, and C. Itzykson, Gauge fields on a lattice. iii. strong-coupling expansions and transition points, Physical Review D 11, 2104 (1975).

[52] M. Creutz, L. Jacobs, and C. Rebbi, Experiments with a gauge-invariant ising system, Physical Review Letters 42, 1390 (1979).

[53] D. Mattis, Solvable spin systems with random interactions, Physics Letters A 56, 421 (1976).

[54] J. D. Reger and A. Zippelius, Three-dimensional random-bond ising model: Phase diagram and critical properties, Phys. Rev. Lett. 57, 3225 (1986).

[55] C. Wang, J. Harrington, and J. Preskill, Confinement-higgs transition in a disordered gauge theory and the accuracy threshold for quantum memory, Annals of Physics 303, 31 (2003).

[56] T. Ohno, G. Arakawa, I. Ichinose, and T. Matsui, Phase structure of the random-plaquette z2 gauge model: accuracy threshold for a toric quantum memory, Nuclear Physics B 697, 462 (2004).

[57] H. Rieger and A. P. Young, Zero-temperature quantum phase transition of a two-dimensional ising spin glass, Phys. Rev. Lett. 72, 4141 (1994).

[58] S. Cho and M. P. A. Fisher, Criticality in the two-dimensional random-bond ising model, Phys. Rev. B 55, 1025 (1997).

[59] I. A. Gruzberg, N. Read, and A. W. W. Ludwig, Random-bond ising model in two dimensions: The nishimori line and supersymmetry, Phys. Rev. B 63, 104422 (2001).

[60] P. Le Doussal and A. B. Harris, Location of the ising spin-glass multicritical point on nishimori's line, Phys. Rev. Lett. 61, 625 (1988).

[61] H. J. Luo, L. Schülke, and B. Zheng, Short-time critical dynamics of the two-dimensional random-bond ising model, Phys. Rev. E 64, 036123 (2001).

[62] K. Hukushima, Random fixed point of three-dimensional random-bond ising models, Journal of the Physical Society of Japan 69, 631 (2000).

[63] W. Xiong, F. Zhong, W. Yuan, and S. Fan, Critical behavior of a three-dimensional random-bond ising model using finite-time scaling with extensive monte carlo renormalization-group method, Phys. Rev. E 81, 051132 (2010).

[64] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, Journal of Mathematical Physics 43, 4452 (2002).

[65] C. Xu and J. E. Moore, Strong-weak coupling self-duality in the two-dimensional quantum phase transition of p + ip superconducting arrays, Phys. Rev. Lett. 93, 047003 (2004).

[66] M. Mueller, D. A. Johnston, and W. Janke, Exact solutions to plaquette ising models with free and periodic boundaries, Nuclear Physics B 914, 388 (2017).

[67] M. Mueller, W. Janke, and D. A. Johnston, Nonstandard finite-size scaling at first-order phase transitions, Phys. Rev. Lett. 112, 200601 (2014).

[68] S. Vijay, J. Haah, and L. Fu, Fracton topological order, generalized lattice gauge theory, and duality, Phys. Rev. B 94, 235157 (2016).

[69] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Rev. Mod. Phys. 51, 659 (1979).

[70] D. S. Rokhsar and S. A. Kivelson, Superconductivity and the quantum hard-core dimer gas, Phys. Rev. Lett. 61, 2376 (1988).

[71] R. Moessner, S. L. Sondhi, and E. Fradkin, Short-ranged resonating valence bond physics, quantum dimer models, and ising gauge theories, Phys. Rev. B 65, 024504 (2001).

[72] D. Perez-Garcia, F. Verstraete, J. I. Cirac, and M. M. Wolf, PEPS as unique ground states of local Hamiltonians, arXiv e-prints , arXiv:0707.2260 (2007).

[73] N. G. Jones, J. Bibo, B. Jobst, F. Pollmann, A. Smith, and R. Verresen, Skeleton of matrix-product-state-solvable models connecting topological phases of matter, Phys. Rev. Research 3, 033265 (2021).

[74] G.-Y. Zhu, N. Tantivasadakarn, A. Vishwanath, S. Trebst, R. Verresen, To Appear.
Appendix A: Correlation functions

In this appendix, we demonstrate the following lemma in the main text and its consequence on calculating the correlation function and norm of the post-measurement wavefunction.

Lemma Consider a stabilizer $d$-dim SPT state with $G_1^{(n)} \times G_2^{(d-n-1)}$ mixed anomaly, where the superscript represents they are $n$ and $(d-n-1)$ form symmetries. Here, $G_1$ and $G_2$ act on different sublattices. Then, the expectation value of an operator defined on a certain sublattice does not vanish only if the operator is a symmetry action on the corresponding sublattice.

1. Understanding SPT and Lemma

First, let us demonstrate the simplest example in 1D cluster state defined by the following Hamiltonian

$$
H = -\sum_n Z_{n-1}X_n Z_{n+1},
$$

(A1)

whose ground state is an 1D SPT protected by $G = G_1 \times G_2 = Z_2 \times Z_2$, defined by product of $X$s in even and odd sites respectively. More precisely, we have two generators for $G$: $g_1 = \prod_{n=1}^N X_{2n-1}$ and $g_2 = \prod_{n=1}^N X_{2n}$. Here, $X$ measures $Z_2$ charge, while $Z$ creates $Z_2$ charge. (Think about $U(1)$ symmetry, whose transformation is $e^{i\theta Q}$. In this sense, $X$ indeed measures charge) Note that $ZZ$ measures whether the $G_1$ domain wall exists, and whenever there is a $G_1$ domain wall, the term enforces $X$ in between to take nontrivial value, i.e., nontrivial $G_2 = Z_2$ charge. Note that $0d$ $G_2$ SPT is distinguished by the charge. Therefore, we call such an SPT phase to have a mixed anomaly between $G_1 = Z_2^{(0)}$ and $G_2 = Z_2^{(0)}$ (superscript implies they are 0-form symmetries). Note that it can be interpreted in the other way – instead of prescribing an energy penalty for certain configurations, each term can be interpreted as creating and annihilation certain configurations. To elaborate further, $Z$ term creates non-trivial SPT by flipping $X$-basis, while $X$ term creates two domain walls next to it (and annihilate). Therefore, $ZXZ$ term can be interpreted as creating two domain walls next to $X$, and then each $Z$ creates a non-trivial SPT at each domain wall. The ground state would be the superposition of all configurations that is invariant under the action of such operations, which fits into the decorated domain wall construction picture [33]. The later perspective will be used for the generalization.

For this 1D cluster state, we want to demonstrate the Lemma. Let $L_1$ ($L_2$) be the odd (even) sublattice where $G_1$ ($G_2$) is acting on. Without loss of generality, consider an operator $O$ defined on $L_1$. Then, we can show that its expectation value under $|\psi\rangle$ disappears if it involves $Z_{2n+1}$ or $Y_{2n+1}$:

$$
\langle \psi | O | \psi \rangle = \langle \psi | (Z_{2n}X_{2n+1}Z_{2n+2})O_1(Z_{2n}X_{2n+1}Z_{2n+2}) | \psi \rangle = - \langle \psi | O_1 | \psi \rangle = 0
$$

(A2)

Here, we can pull $Z_{2n}X_{2n+1}Z_{2n+2}$ from $|\psi\rangle$ since the ground state is stabilized by it. Furthermore, as two $Z$s on $L_2$ in the stabilizer simply commutes with any operator defined on $L_1$, while $X_{2n+1}$ anti-commutes with $Z_{2n+1}$ or $Y_{2n+1}$ inside $O$. Therefore, for the expectation value of $O$ to not vanish, it has to be made of either $I$ or $X$ operators. Now, we claim that it is one of two cases: either $O = I$ or $O = \prod_{n=1}^N X_{2n+1}$. Assume that $O \neq I, \prod X$. Then, there must be a neighboring two odd sites $2m-1$ and $2m+1$ where $O|_{2m-1} = I$ while $O|_{2m+1} = X$. Then,

$$
\langle \psi | O_1 | \psi \rangle = \langle \psi | (Z_{2m-1}X_{2m}Z_{2m+1})O_1(Z_{2m-1}X_{2m}Z_{2m+1}) | \psi \rangle = - \langle \psi | O_1 | \psi \rangle = 0
$$

(A3)

This concludes the proof, i.e., $\langle O \rangle = 0$ for $O$ defined on $L_1$ if $O$ is not the element of $G_1 = \{I, \prod_{n=1}^N X_{2n+1}\}$. In fact, this is not a coincidence. For a given commuting stabilizer Hamiltonian, the ground state has a vanishing expectation value against any operator that anti-commutes with any stabilizer. However, if a given operator commutes with all stabilizers, if simply means that the operator is nothing but a symmetry action of a given Hamiltonian.

Now we want to generalize the claim for higher dimensional cluster state SPTs with $G_1^{(n)} \times G_2^{(d-n-1)}$ mixed anomaly, where $G_1$ is an $n$-form symmetry acting on the sublattice $L_1$ and $G_2$ is a $(d-n-1)$-form symmetry acting on the sublattice $L_2$. Such a cluster state SPT Hamiltonian consists of two local terms:

$$
H = -\sum_i h_{1,i} - \sum_j h_{2,j}, \quad h_{1,i} = O_{1,i}^{\text{charge}} O_{2,i}^{\text{d.w.}} \quad h_{2,j} = O_{1,j}^{\text{d.w.}} O_{2,j}^{\text{charge}}
$$

(A4)

where $O_{1}^{\text{charge}}$ ($O_{2}^{\text{d.w.}}$) creates the charge (domain wall) of the symmetry $G_1^{(n)}$ in a symmetric fashion.

For example, if $G_1$ is a $Z_2$ 1-form symmetry, the generators of $G_1$ is given by $h_{\gamma} = \prod_{e \in \gamma} X_e$ for any closed loop $\gamma$ defined on $L_1$. This is the symmetry that randomly creates or annihilates $Z = -1$ closed loops. For this symmetry,
In general, if a Pauli operator \( O \) defined on the sublattice \( L_1 \) does not to vanish for this SPT, it has to satisfy
\[
\forall i, ~ h_{1,i} O h_{1,i} = O \quad \text{and} \quad \forall j, ~ h_{2,j} O h_{2,j} = O \quad (A5)
\]
These conditions imply that \( O \) is invariant under the conjugation by \( O_{1,i}^{d.w.} \) and \( O_{1,i}^{s.w.} \). In fact, this already implies that \( O \) is in fact a symmetry of the system. Since \( O \) is restricted to \( L_1 \), it means that \( O \in G_1 \).

\[\text{2. Application of Lemma}\]

Below, we evaluate the correlation function for various post-measurement cluster states, which can be thought of as \( \mathcal{P} | \psi \rangle \) where the measurement-projection operator \( \mathcal{P} \) on the sublattice \( L_i \) is defined as
\[
\mathcal{P} = \prod_{n \in L_i} P_n, \quad P_i = \frac{1}{2} [I + s_i (X_i n_x + Y_i n_y + Z_i n_z)], \quad P_i^2 = P_i \quad (A6)
\]
where \( s_i \) is the measurement outcome.

\[\text{a. 1D Cluster State}\]

In this case, the even-sited operator’s expectation value with respect to \( | \psi \rangle \) is non-vanishing only if the operator is identity or product of \( X \)-s in all even sites. Then, we can calculate the correlation
\[
\langle \mathcal{P} | \psi \rangle Z_i Z_{2n+1} | \mathcal{P} | \psi \rangle = \langle \psi | Z_i Z_{2n+1} | \mathcal{P} | \psi \rangle = \langle \psi | X_2 X_4 \ldots X_{2n} | \mathcal{P} | \psi \rangle
\]
\[
= \frac{1}{2^N} \left( \prod_{n=1}^{N} s_{2m} \right) \left( \prod_{m=n+1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N-n}
\]
\[
= \left( \prod_{m=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N} \left( 1 + \prod_{m=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N-n}
\]
\[
\langle \mathcal{P} | \psi \rangle \langle \mathcal{P} | \psi \rangle = \frac{1}{2^N} \left( \prod_{n=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N} \left( 1 + \prod_{m=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N-n} \left( \prod_{n=1}^{N} X_{2n} \right) \quad (A7)
\]
since \( [\mathcal{P}, Z_{2n+1}] = 0 \) and \( Z_i Z_{2n+1} | \psi \rangle = X_2 \ldots X_{2n} | \psi \rangle \). Here we used that
\[
X_2 X_4 \ldots X_{2n} \mathcal{P} = \text{(Vanishing terms)}
\]
\[
+ \frac{1}{2^N} \left( \prod_{m=1}^{N} s_{2m} \right) \cos \theta \cdot I + \left[ \prod_{m=n+1}^{N} s_{2m} \right] \cos \theta \cdot \left[ \prod_{n=1}^{N} X_{2n} \right] \quad (A8)
\]
Note that the norm of \( | \mathcal{P} | \psi \rangle \) can be similarly calculated as
\[
\langle \mathcal{P} | \psi \rangle \langle \mathcal{P} | \psi \rangle = \frac{1}{2^N} \left( \prod_{m=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N} \left( 1 + \prod_{m=1}^{N} s_{2m} \right) \left( \cos \theta \right)^{N-n} = \frac{1}{2 \cosh \beta} \sum_{\sigma} e^{s_{2n} \beta \sum_{\sigma} \sigma_{2n-1} \sigma_{2n+1},} \quad \tanh \beta = \cos \theta \quad (A9)
\]
for a system with \( 2N \) sites. Here I used that \( e^{a \sigma_{i,j}} = \cosh a \sigma_{i,j} + \tanh a \sigma_{i,j} \) for \( \sigma_{i,j} = \pm 1 \). We can also calculate other correlation functions involving \( X \) or \( Y \) by rewriting them in terms of operators defined on even sites. For example, note that the expectation value of \( X_{2n+1} \) can be calculated as
\[
\langle \mathcal{P} | \psi \rangle X_{2n+1} | \mathcal{P} | \psi \rangle = \langle \psi | X_{2n+1} | \mathcal{P} | \psi \rangle = \langle \psi | Z_{2n} Z_{2n+2} | \mathcal{P} | \psi \rangle
\]
\[
= \frac{1}{2^N} [\eta_1^2 + \eta_2^2 \eta_1^{N-2}] \quad (A10)
\]
where \( \eta_1 \equiv \sin \theta \sin \phi \) and \( \eta_2 \equiv \sin \theta \cos \phi \). Expectation values of more complicated operators can be calculated in a similar way.
b. 2D Cluster State, measurement on vertices

A 2D cluster state becomes a 2D toric code state when measured on vertices in X-basis. The Lemma implies that the expectation value of operators defined on vertices disappear unless it is an element of the 0-form symmetry $Z_2^{(0)}$, i.e., a product of $X_i$ over the all vertices. Then, the post-measurement wavefunction norm is given by

$$
\langle P\psi|P\psi \rangle = \langle \psi|P|\psi \rangle = \frac{1}{2N} \left( 1 + \prod_v s_v (\cos \theta)^N \right) \tag{A11}
$$

To detect the spontaneous symmetry breaking of the 1-form symmetry, one can measure the product of $Z$s along the boundary of a certain region $S$ as the following:

$$
\langle P\psi| \prod_{e \in \partial S} Z_e |P\psi \rangle = \langle \psi| \prod_{v \in S} X_v |P\psi \rangle = \frac{1}{2N} \left( \prod_{v \in S} s_v (\cos \theta)^n + \prod_{v \notin S} s_v (\cos \theta)^{N-n} \right) \tag{A12}
$$

In the limit $N \to \infty$, the correlation function measuring the 1-form symmetry breaking becomes:

$$
\left\langle \prod_{e \in \partial S} Z_e \right\rangle_{P\psi} \propto (\cos \theta)^{|S|} \tag{A13}
$$

This 1-form symmetry breaking looks robust only up to a finite region. Such 1-form symmetry becomes restored in a long-enough length scale, and the resulting state must become 1-form symmetric.

c. 2D Cluster State, measurement on edges

The Lemma implies that the expectation value of operators defined on edges disappear unless it is an element of the 1-form symmetry $Z_2^{(1)}$, i.e., a product of $X_i$ on the closed loop. Let $C$ be the set of closed loops one can draw on a given lattice. Then, the post-measurement wavefunction norm is given by

$$
\langle \psi|P|\psi \rangle = \frac{1}{2N} \sum_{l \in C} \left\langle \prod_{e \in l} s_e \right\rangle (\cos \theta)^{|l|} \tag{A14}
$$

as when we expand $P$, terms vanish unless it forms a product of $X$ along a closed loop. For given two vertices $v$ and $v'$, let $p$ be a path between them. Then, the correlation between two vertices is given as

$$
\frac{\langle P\psi|Z_v Z_{v'} |P\psi \rangle}{\langle \psi|P|\psi \rangle} = \frac{\langle \psi| \prod_{l \in p} X_l |P\psi \rangle}{\langle \psi|P|\psi \rangle} = \frac{1}{\langle \psi|P|\psi \rangle} \left[ \frac{1}{2N} \left( \sum_{\bar{p} \text{ s.t. } \bar{p} + p \in C} \left\langle \prod_{e \in \bar{p}} s_e \right\rangle (\cos \theta)^{|\bar{p}|} \right) \right] \tag{A15}
$$

which is nothing but an expression for the 2D Ising model with the sign of the Ising interaction at the edge $e$ is given by $s_e$. From this structure, we can immediately infer that the amplitudes of the wavefunction in the $Z$ basis should be proportional to the Boltzmann weights.

d. 3D Cluster State with $Z_2^{(0)} \times Z_2^{(2)}$: measurements on vertices

Here we prepare a 3D cluster state defined on the cubic lattice where qubits reside at both vertices ($V$) and edges ($E$). This cluster state has the mixed anomaly of 0-form and 2-form symmetries, and by measuring vertices in X-basis,
one can get the 3D toric code state. The stabilizer Hamiltonian is given by

\[ H = - \sum_v X_v \prod_{e \in n(v)} Z_e - \sum_e \left( X_e \prod_{v \in \partial e} Z_v \right) \]  

(A16)

where \( n(v) \) is the set of edges neighboring the vertex \( v \). Here, all operators commute and we have two symmetries:

0-form: \( g = \prod X_v \)

2-form: \( h_\gamma = \prod_{e \in \gamma} X_e, \quad \gamma \) is the loop along the bonds \( \) (A17)

When measured on vertices, the above 3D cluster state becomes a 3D toric code state, spontaneously breaking the 2-form symmetry. The Lemma implies that the expectation value of operators defined on edges disappear unless it is an element of the 0-form symmetry \( Z_2^{(0)} \), i.e., a product of \( X_i \) over the all vertices. Then, the post-measurement wavefunction norm is given by

\[ \langle P \psi | P \psi \rangle = \frac{1}{2^N} \left( 1 + \left[ \prod_v s_v \right] (\cos \theta)^N \right) \]  

(A18)

To detect the spontaneous symmetry breaking of the 2-form symmetry, one can measure the product of \( Z_s \) along the boundary of a certain region \( V \) as the following:

\[ \langle P \psi | \prod_{e \in \partial V} Z_e | P \psi \rangle = \langle \psi | \prod_{e \in \partial V} Z_e | \psi \rangle = \langle \psi | \prod_{v \in V} X_v | \psi \rangle \]

\[ = \frac{1}{2^N} \left( \left[ \prod_v s_v \right] (\cos \theta)^n + \left[ \prod_{v \not\in V} s_v \right] (\cos \theta)^{N-n} \right) \]  

(A19)

In the limit \( N \to \infty \), the correlation function measuring the 2-form symmetry breaking becomes:

\[ \left\langle \prod_{e \in \partial V} Z_e \right\rangle_{P \psi} \propto (\cos \theta)^{|V|} \]  

(A20)

This 2-form symmetry breaking looks robust only up to a finite region for \( \theta \neq 0 \), implying that the resulting 3D topological order is unstable under the deviation from the \( X \)-basis measurements. The corresponding classical model is a Ising membrane theory (or Ising 2-form symmetric theory), which is defined by

\[ H(\{ \sigma \}) = - \sum_v \prod_{f \in n(v)} \sigma_f \]  

(A21)

where \( f \) is the face of the dual cubic lattice, and \( n(v) \) is the set of dual faces neighboring to the vertex \( v \) in the original cubic lattice. Note that each term is that the product of Ising spins defined on six faces. In fact, this is a natural extension of Ising model (Ising 0-form symmetric theory) and Ising gauge theory (Ising 1-form symmetric theory). Note that this classical partition function is also exactly solvable.

e. \quad \textbf{3D Cluster State with } \mathbb{Z}_2^{(0)} \times \mathbb{Z}_2^{(2)} \textbf{: measurements on edges}

Here, we studied the same model as above but measuring edges. The Lemma implies that the expectation value of operators defined on edges disappear unless it is an element of the 2-form symmetry \( Z_2^{(2)} \), i.e., a product of \( X_i \) on the closed loop. Let \( C \) be the set of closed loops one can draw on the cubic lattice. Then, the post-measurement wavefunction norm is given by

\[ \langle \psi | P | \psi \rangle = \frac{1}{2^N} \sum_{l \in C} \left[ \prod_{e \in l} s_e \right] (\cos \theta)^{|l|} \]  

(A22)
as when we expand $\mathcal{P}$, terms vanish unless it forms a product of $X$ along a closed loop. For given two vertices $v$ and $v'$, let $p$ be a path between them. Then, the correlation between two vertices is given as

$$
\langle \mathcal{P} | \psi \rangle \langle \psi | \mathcal{P} \rangle \equiv \frac{1}{\langle \psi | \mathcal{P} | \psi \rangle} \left[ \frac{1}{2^N} \sum_{\bar{p} \text{ s.t. } p + \bar{p} \in C} \left( \prod_{e \in \bar{p}} s_e \right) \cos \theta \right] \] ^{\bar{p}}
$$

(A23)

which is nothing but an expression for the 3D Ising model with the sign of the Ising interaction at the edge $e$ is given by $s_e$. From this structure, we can immediately infer that the amplitudes of the wavefunction in the $Z$ basis should be proportional to the Boltzmann weights.

\section*{f. 3D Cluster State with $Z_2^{(1)} \times Z_2^{(1)}$: measurements on edges}

Another 3D cluster Hamiltonian is written as the following:

$$
H_{3D \ SPT} = - \sum_e X_e \prod_{f \ni e} Z_f - \sum_f X_f \prod_{e \ni f} Z_e
$$

(A24)

where $f$ runs for all faces of the cubic lattice. Bolded symbols act on faces, and unbolded symbols act on edges. Note that by multiplying stabilizers, we obtain that $\prod_{f \ni e} X_f = 1$ for any cube $c$ and $\prod_{e \ni f} X_e = 1$ for any vertex $v$. Here, generators of two 1-form symmetries are defined on two-dimensional surfaces as the following:

$$
Z_2^{(1)} \ 1\text{-form: } h_{\partial V} \equiv \prod_{f \in \partial V} X_f
$$

$$
Z_2^{(1)} \ 1\text{-form: } g_{\partial \bar{V}} \equiv \prod_{v \in \partial \bar{V}} X_e
$$

(A25)

where $V$ is a certain three-dimensional volume enclosed by cubic faces, and $\bar{V}$ is a infinitesimally inflated version of $V$ which intersects with edges emanating from $V$. Therefore, $\partial V$ is a set of faces, while $\partial \bar{V}$ is a set of edges. Without loss of generality, if we measure all faces in $X$-basis, then we obtain that the resulting state has $\prod_{e \in f} Z_e = 1$ and $\prod_{f \ni c} X_c = 1$ for all $f$ and $v$, which gives the 3D toric code ground state.

When the measurement direction deviates from the $\hat{x}$-direction, the plaquette terms $B_p = \prod_{l \in p} Z_l$ are no longer stabilizers. Nevertheless, the star terms $A_s = \prod_{l \ni s} X_l$ is still a stabilizer, as we illustrated in Fig. 9.

Assume we measured faces with angles. The norm of the wavefunction is given by

$$
\langle \mathcal{P} | \psi \rangle = \langle \psi | \prod_{e \in \partial S} Z_e | \mathcal{P} | \psi \rangle = \langle \psi | \prod_{f \in S} X_f | \mathcal{P} | \psi \rangle = \frac{1}{2^N} \sum_{V \subseteq \partial \bar{V}} \left[ \prod_{f \in \partial V} s_f \right] \cos \theta ^{\partial V}
$$

(A26)

where the summation of $V$ is over any three-dimensional volume. In fact, one can notice that this is the partition function of 3D Ising gauge theory with plaquette signs given by $\{s_f\}$. To detect the spontaneous symmetry breaking
of the 1-form symmetry (topological orders) one can measure the product of $Z$s along the boundary of a certain surface $S$ as the following:

$$\langle \mathcal{P} \psi \prod_{e \in \partial S} Z_e | \mathcal{P} \psi \rangle = \langle \psi \prod_{e \in \partial S} Z_e | \mathcal{P} \psi \rangle = \langle \psi \prod_{f \in S} X_f | \mathcal{P} \psi \rangle$$

$$= \frac{1}{2^N} \sum_{V \text{ s.t. } \partial V = S} \left[ \prod_{f \in \partial V} \left( \cos \theta \right) \right]^{\partial V}$$

which agrees with the expressions for the loop correlation functions in 3D Ising gauge theory. Depend on $\theta$, the loop correlation decay exponentially either by $|\partial S|$ or $|S|$.

**Appendix B: Post-measurement wavefunctions**

One can directly write down the wavefunction after measurements in Eq. (3) using the decorated domain-wall construction. The cluster state wavefunction is written as the equal superposition of all domain wall configurations with charges attached accordingly:

$$|\psi\rangle = \frac{1}{\sqrt{2^N-1}} \sum_{\{d_{2n}\}} |\{d_{2n}\}\rangle_{\text{ddw}}$$

where the subscript ddw stands for that the state is a decorated domain wall basis, where domains (charges) are defined on odd (even) sites. For example, the basis without a domain wall, $|\{d_{2n} = 1\}\rangle$ would be the GHZ state on odd sites (and accordingly, all $|+\rangle$ states on even sites):

$$\frac{1}{\sqrt{2}} \left[ |\uparrow\uparrow\cdots\rangle + |\downarrow\downarrow\cdots\rangle \right]_{\text{odd}} \otimes \left[ |+\rangle \otimes |\rangle \right]_{\text{even}}$$

where the odd sites define spin configurations and even sites are charged based on whether the domain wall exists in neighboring odd sites. The summation is over $2^N-1$ configurations since domain walls are under the constraint $\prod d_{2n} = 1$. Also, the domain-wall basis is the cat state of two different spin configurations giving the same domain-wall configuration. Then, the wavefunction norm can be calculated as

$$\langle \mathcal{P} \psi | \mathcal{P} \psi \rangle = \frac{1}{2^N} \langle \psi | \prod_n (1 + s_{2n} \cos \theta X_{2n}) | \psi \rangle$$

$$= \frac{1}{2^{2N-1}} \sum_{\{d_{2n}\}} \prod_n (1 + s_{2n} \cos \theta d_{2n})$$

$$= \frac{1}{2^N} (1 + \prod_{m=1}^N s_{2m} \cos \theta)^N = Z$$

where we used the fact that in the expansion of $\mathcal{P}$, any terms involving $Y$ or $Z$ disappears under the Lemma. The measured wavefunction can be written as

$$|\mathcal{P} \psi\rangle = \sum_{\{d_{2n}\}} C(\{d_{2n}\}) |\sigma(\{d_{2n}\})\rangle \otimes |M\rangle$$

where now $|M\rangle = \otimes_{n=1}^N |M_{s_{2n}}\rangle$ stands for the measured component on even sites. The structure of the $Z$ already implies that

$$|C(\{d_{2n}\})| = \frac{1}{\sqrt{2^{2N-1}}} \left( \prod_{n=1}^N (1 + s_{2n} \cos \theta d_{2n}) \right)^{1/2}$$

$$= \frac{1}{\sqrt{2^{2N-1}}} [\cosh(\beta/2)]^{-N} e^{\frac{\beta}{2} \sum_n s_{2n} d_{2n}}$$

where we used $e^{a\sigma_i \sigma_j} = \cosh(a) (1 + \tanh(a) \sigma_i \sigma_j)$. Although the calculated magnitude agrees with the square root of the Boltzmann weight, in order to calculate the phase factor, one has to proceed with details. First, decompose $|\pm\rangle$ state into the measurement basis:

$$|\pm\rangle = a_+ |M_+\rangle + b_\pm |M_-\rangle$$

(B6)
Then, we can obtain that \( a_\pm = \langle M_+ | \pm \rangle \) and \( b_\pm = \langle M_- | \pm \rangle \), where \( | \pm \rangle = [1, \pm 1]^T/\sqrt{2} \). Here, for a different parametrization \( \hat{n} = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta) \), we get

\[
| M_\pm \rangle = \frac{1}{\sqrt{2(1 \pm \sin \theta)}} \left( \sin(\theta) \pm 1 \right) e^{i\phi} \cos(\theta) \tag{B7}
\]

Then, note that for a measurement outcome \( \{s_{2n} \} \), the projection is defined as

\[
P \mapsto \bigotimes_{n=1}^N | M_{s_{2n}} \rangle \langle M_{s_{2n}} | \tag{B8}
\]

Then, we see that

\[
C(\{d_{2n}\}) = \frac{\langle M | \bigotimes_{n=1}^N | d_{2n} \rangle}{\sqrt{2^{N-1}}} = \prod_{n=1}^N \langle M_{s_{2n}} | d_{2n} \rangle \tag{B9}
\]

Note that

\[
\langle M_{s_{2n}} | d_{2n} \rangle = \frac{s_{2n}}{\sqrt{1 + s_{2n}^2}} \left[ s_{2n} \sin \theta + 1 + s_{2n}d_{2n}e^{i\phi} \cos \theta \right] = \frac{s_{2n}}{1 + s_{2n} \sin \theta / 2} \left[ (\cos \theta / 2 + s_{2n} \sin \theta / 2)^2 + s_{2n}d_{2n}e^{i\phi}(\cos^2 \theta / 2 - \sin^2 \theta / 2) \right] = \frac{s_{2n}}{2} \left[ (1 + s_{2n}d_{2n}e^{i\phi}) \cos(\theta / 2) + (s_{2n} - d_{2n}e^{i\phi}) \sin(\theta / 2) \right] \tag{B10}
\]

For \( \phi = 0 \), one can show that

\[
\langle M_{s_{2n}} | d_{2n} \rangle = \begin{cases} s_{2n} \cos(\theta / 2) = s_{2n} \sqrt{(1 + \cos \theta) / 2}, & \text{if } s_{2n}d_{2n} > 0, \phi = 0 \\ \sin(\theta / 2) = \sqrt{(1 - \cos \theta) / 2}, & \text{if } s_{2n}d_{2n} < 0, \phi = 0 \end{cases}
\]

\[
\prod_n \langle M_{s_{2n}} | d_{2n} \rangle = \left( \prod_n \varphi_{2n} \right) \sqrt{\prod_n (1 + s_{2n}d_{2n} \cos \theta) / 2^N} \tag{B11}
\]

which agrees with the expression obtained from the norm \( \langle P | \psi \rangle | \langle P | \psi \rangle \). Now, for \( \phi \neq 0 \), we obtains that

\[
\langle M_{s_{2n}} | d_{2n} \rangle = e^{i\eta_{2n}} \sqrt{1 + s_{2n}d_{2n} \cos \theta \cos \phi / 2}, \quad \eta_{2n} = \text{Arg}(\sin \theta + s_{2n} + s_{2n}d_{2n} \cos \theta e^{i\phi}) \tag{B12}
\]

which agrees with the norm calculation (here the prefactor for \( X \) is \( \cos \theta \cos \phi \)).

Although these phase factors can affect the expectation values when we measure correlations of \( Y \) or \( Z \) operators, they should not change any essential physics. The result implies that if we measure in \( xz \)-plane without \( y \)-component, the wavefunction is real. Then, it simply implies that the wavefunction weight would be given by a Gibbs weight. However, even if there is a phase factor, it would simply correspond to some basis rotation; as long as we do not measure correlations in \( Y \) or \( Z \), such basis rotation along the \( X \)-axis should not matter. After such rotation, we should obtain a real wavefunction weight again.

### Appendix C: Parent Hamiltonians

We use the strategy described in Sec. II D [34–36] to generate the parent Hamiltonians for our post-measurement states in higher dimensions. In the cases described below, minor modifications are needed compared to the derivation for 1D.

1. **2d measurement on edges**

After the measurement, the wavefunction is

\[
| \Psi \rangle = M_\beta | \Psi_0 \rangle, \quad | \Psi_0 \rangle = \left| \{ X_i = \prod_{l \geq i} s_l \} \right>, \quad M_\beta = \prod_{\langle ij \rangle} e^{\frac{\theta}{2} s_{ij} Z_i Z_j}, \tag{C1}
\]
One choice of its parent Hamiltonians is $H = \sum_i H_i$, and

$$H_i = -\cos \theta (1 + \cos^2 \theta) \sum_{\langle i,j \rangle} s_{ij} Z_i Z_j + \frac{2}{3} \cos^2 \theta B_i - X_i \prod_{j \in i} s_l \left( 1 + \cos^4 \theta \prod_{j \in n(i)} Z_j - \frac{\cos^2 \theta}{3} B_i \right),$$

$$B_i = \sum_{j,k \in n(i), j \neq k} s_{ij} s_{ik} Z_i Z_k. \tag{C2}$$

where $n(i)$ is the set of sites neighboring the site $i$. We derive the Hamiltonian (C2) by first performing a Kramers-Wannier duality of the wavefunction, which becomes

$$|\Psi'\rangle = M_\beta' |\Psi'_0\rangle, \quad |\Psi'_0\rangle = \left( \prod_{i \in i} s_{ij} X_i' = 1 \right), \quad M_\beta' = \prod_l e^{\frac{\beta}{3} s_l Z_l'}. \tag{C3}$$

Adapting the strategy in Subsection II.D, the parent Hamiltonian for this state is found to be

$$H' = \frac{1}{3} \sum_l \left( \sum_{j=2}^4 (M_\beta' \Gamma'_{1j}(l) M_\beta')' (M_\beta' \Gamma'_{1j}(l) M_\beta') \right), \tag{C4}$$

where

$$\Gamma'_{12}(l) = \frac{1}{2} \left( s_2 X_1 X_2 - s_4 X_3 X_4 \right), \quad \Gamma'_{13}(l) = \frac{1}{2} \left( 1 - \prod_{i=1}^4 s_i X_i' \right), \tag{C5}$$

and similarly for $\Gamma'_{13,14}$. The result is $H' = \sum_i H'_i$, where

$$H'_i = -\cos \theta (1 + \cos^2 \theta) \sum_{\langle i,j \rangle} s_{ij} Z_i' + \frac{2}{3} \cos^2 \theta B_i' - A_i' \left( 1 + \cos^4 \theta \prod_{j \in i} s_l Z_j' + \frac{\cos^2 \theta}{3} B_i' \right),$$

$$A_i' = \prod_{l \in i} s_l X_l', \quad B_i' = \sum_{l \in i, m \in i, l \neq m} s_{lm} Z_l' Z_m'. \tag{C6}$$

After reversing the KW duality, we obtain the aforementioned parent Hamiltonian.

### 2. 3d measurement on plaquettes of the $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ SPT

One choice of the parent Hamiltonian $H = \sum_l H_l$ is

$$2H_l = -\cos \theta (1 + \cos^2 \theta) \sum_{p \in l} s_p B_p + \frac{2}{3} \cos^2 \theta B_l - X_l \prod_{p \in l} s_p \left[ 1 + \cos^4 \theta \prod_{q \in l, p \neq q} s_q Z_q - \frac{\cos^2 \theta}{3} B_l \right]$$

$$B_p = \prod_{l \in p} Z_p, \quad B_l = \sum_{p \in l, q \in l, p \neq q} s_p s_q B_p B_q, \tag{C7}$$

together with the gauge constraint $\prod_{l \in s} X_l = 1$ for each vertex $s$.

We derive it in this way. The state post-measurement is

$$|\Psi\rangle = M_\beta |\Psi_0\rangle, \quad |\Psi_0\rangle = \{|X_l = \prod_{p \in l} s_p\rangle\}, \quad M_\beta = \prod_p e^{\beta s_p Z_p' Z_p Z_p Z_p'}, \tag{C8}$$

where $Z_p', i = 1, 2, 3, 4$ act on the links on the boundary of the plaquette $p$. Under a Kramers-Wannier duality from links to plaquettes, the state is mapped to

$$|\Psi'\rangle = M_\beta' |\Psi'_0\rangle, \quad |\Psi'_0\rangle = \{|X_1 X_2 X_3 X_4 = s_1 s_2 s_3 s_4\rangle\}, \quad M_\beta' = \prod_p e^{\beta s_p Z_p'}, \tag{C9}$$
where $X_i^j, i = 1, 2, 3, 4$ represent the Pauli-$X$ operators that act on the four plaquettes neighboring the link $l$, and $s_i, i = 1, 2, 3, 4$ are the measurement outcomes on those plaquettes. $|\Psi'^0\rangle$ is the ground state of the Hamiltonian

$$H'_0 = \sum_l \frac{1 - s_1^l s_2^l s_3^l s_4^l X_1^l X_2^l X_3^l X_4^l}{2},$$

which can be rewritten as

$$H'_0 = \frac{1}{3} \sum_l \left[ \Gamma'_{12}(l) + \Gamma'_{13}(l) + \Gamma'_{14}(l) \right].$$

where

$$\Gamma'_{12}(l) = \frac{1}{2} (s_1^l s_2^l X_1^l X_2^l - s_3^l s_4^l X_3^l X_4^l),$$

and similarly for $\Gamma'_{13}$ and $\Gamma'_{14}$. In particular, $\Gamma'_{ij}(l)|\Psi'^0\rangle = 0, ij = 12, 13, 14$. Then it follows that the state $|\Psi'\rangle$ is the ground state of the following Hamiltonian,

$$H'_0 = \frac{1}{3} \sum_l \sum_{j=2,3,4} \left( M_{\beta}^l \Gamma'_{1j}(l) M_{\beta}^{-1} \right) \left( M_{\beta}^l \Gamma'_{1j}(l) M_{\beta}^{-1} \right).$$

More explicitly,

$$\Gamma'_{12}(l) = \frac{1}{2} \left( s_1 s_2 e^{\beta(s_1 Z_1^1 + s_2 Z_2^1)} X_1^l X_2^l - s_3 s_4 e^{\beta(s_3 Z_3^1 + s_4 Z_4^1)} X_3^l X_4^l \right),$$

where the Pauli operators act on the four plaquettes around the link $l$.

$$2\Gamma'_{12}(l)\Gamma'_{12}(l) = - \cos \theta (1 + \cos^2 \theta) \sum_{p=1}^4 s_p Z_p + 2 \cos^2 \theta (s_1 s_2 Z_1^1 Z_2^1 + s_3 s_4 Z_3^1 Z_4^1)$$

$$- \left( \prod_{p=1}^4 s_p X_p \right) \left[ 1 + \cos^4 \theta \prod_{p=1}^4 s_p Z_p + \cos^2 \theta (s_1 s_2 Z_1^1 Z_2^1 + s_3 s_4 Z_3^1 Z_4^1) \right]$$

$$- \cos^2 \theta (s_1 s_2 Z_1^1 Z_3^1 + s_1 s_4 Z_1^1 Z_4^1 + s_3 s_4 Z_3^1 Z_4^1 + s_2 s_3 Z_3^1 Z_4^1).$$

Similar results hold for $\Gamma'_{1j}(l)$ and $\Gamma'_{1j}(l)$ for $j = 3, 4$. Here, we have scaled Eq. (C15) by an overall constant $\cosh^2 \beta$, such that at $\theta = \frac{\pi}{2}$, the parent Hamiltonian is $-\frac{1}{2} \sum l X_l \prod_{p \in l} s_p$.

Then it follows that

$$2H'_0 = - \cos \theta (1 + \cos^2 \theta) \sum_{p=1}^4 s_p Z_p + \frac{2}{3} \cos^2 \theta B'_0 - A'_0 \left[ 1 + \cos^4 \theta \prod_{p=1}^4 s_p Z_p - \frac{\cos^2 \theta}{3} B'_0 \right].$$

$$A'_0 = \prod_{p \in l} s_p X_p, \quad B'_0 = \sum_{p \in l, q \in l, p \neq q} s_p s_q Z_p Z_q.$$

These local terms, after reversing KW duality, becomes Eq. (C7).

3. 3d measurement on sites of the $S_2^{(1)} \times S_2^{(2)}$ SPT

A computation similar as in 2d case leads us to the following parent Hamiltonian, up to an unimportant constant and total prefactor, $H = \sum_l H_l$.

$$H_l = -X_l \prod_{v \in l} s_v + \cos^2 \theta X_l \prod_{v \in l} Z_v - \cos \theta \sum_{v \in l} s_v \prod_{l \ni v} Z_l,$$

together with gauge contraints, $\prod_{l \ni p} X_l = 1$. 


Let us rewrite it in a more illuminating way, for the case that the outcomes are $s_v = 1$,

$$
H = H_0 + 6 \cos \theta H_{t.c.} + \cos^2 \theta H_{SPT},
$$

$$
H_0 = -\sum_l X_l, \quad H_{t.c.} = -\sum_v \prod_{i \geq v} Z_l, \quad H_{SPT} = \sum_l X_l \prod_{e \in l} \prod_{m \geq e} Z_m,
$$

(C18)

together with gauge contraints, $\prod_{l \in p} X_l = 1$. Under a Kramers-Wannier duality from links to vertices, the model becomes

$$
H = -\sum_{\langle ij \rangle} (X_i X_j + \cos^2 \theta Y_i Y_j) - 6 \cos \theta \sum_i Z_i.
$$

(C19)

At $\theta = 0$, the model has a $U(1)$ symmetry.

Appendix D: Probability distribution of measurement outcomes is gauge invariant

Consider measuring $O^\theta = X \cos \theta + Z \sin \theta$ on the edges of the cluster state on a square lattice, for the measurement outcomes $s_l \in \{0, 1\}$, the probability distribution after the measurements is

$$
P(\{s_l\}) = \sum_{\{s_l\}} \prod_l \left[ a + (-1)^{\sum_{i \in e_l} g_i} b \right]^{2-2s_l} \left[ a - (-1)^{\sum_{i \in e_l} g_i} b \right]^{2s_l}.
$$

(D1)

where $a = \sqrt{1+\sin^2 \theta}/2$, $b = \sqrt{1-\sin^2 \theta}/2$.

Based on this formula, we can show that the distribution is invariant under gauge transformation,

$$
P(\{s_{ij}\}) = P(\{s_{ij}' = t_i + s_{ij} + t_j \mod 2\}),
$$

(D2)

for any gauge configuration $\{t_i \in \{0, 1\}\}$, where $i,j$ labels the sites, and $\langle ij \rangle$ is a link. So any two configurations that are gauge equivalent has the same probability to be an outcome. Meanwhile, the probability distribution satisfies that $\prod_{l \in C} s_l = |\cos \theta|^{|C|}$ for any loop $C$.

The derivation of Eq. (D1) is the following. Before measurement, the cluster state is

$$
|\Psi\rangle \propto \sum_{g_i=0,1} \sum_{h_i=0,1} \sum (-1)^{g_i} \prod_{i \in e_l} h_i |\tilde{g}_i, \tilde{h}_i\rangle.
$$

(D3)

Let $|\theta_+\rangle$ and $|\theta_-\rangle$ represent the eigenstates of $O^\theta = X \cos \theta + Z \sin \theta$ with the eigenvalue $+1$ and $-1$, respectively. Then the eigenstates of Pauli-Z satisfying $Z|h\rangle = (-1)^h |h\rangle$, can be written as

$$
|0\rangle = a|\theta_+\rangle - b|\theta_-\rangle, \quad |1\rangle = b|\theta_+\rangle + a|\theta_-\rangle,
$$

(D4)

where $a, b$ are given above.

Then we relate the eigenstates with possible measurement outcomes $s_l$, by $|\theta_+\rangle = |s_l = 0\rangle$ and $|\theta_-\rangle = |s_l = 1\rangle$, and expanded the wavefunction (Eq. (D3)) in the measurement basis,

$$
|\Psi\rangle = \sum_{\{s_l\}} \sum_{\{g_l\}} \prod_l \left[ a + (-1)^{\sum_{i \in e_l} g_i} b \right]^{1-s_l} \left[ a - (-1)^{\sum_{i \in e_l} g_i} b \right]^{s_l} |\tilde{g}_l\rangle |\{s_l\}\rangle.
$$

(D5)

By projecting this wavefunction to the state $|\{s_l\}\rangle$, one can find the probability distribution in Eq. (D1).

Combining the properties of correlation functions in Eq. (43), and that the probability distribution is gauge invariant, we have enough information to fully determine the probability distribution. In fact, there is only a unique probability distribution that can satisfy the above two conditions, which is a result that can be proved similarly as the one shown in Appendix E. The simplest way to describe it is the following. Define the uncorrelated distribution such that on each link, $p(s_l = \pm 1) = (1 \pm \cos \theta)/2$. Then the probability distribution is to start with the uncorrelated probability distribution, and then to take the average over the gauge transformations,

$$
P(\{s_l\}) = \frac{1}{2^N} \sum_{\{t_l = \pm 1\}} \prod_{\langle ij \rangle} p(t_i s_{ij} t_j),
$$

(D6)

with $N$ the number of sites. This shows that the wavefunction $|\mathcal{P}\psi\rangle$ without post-selection, is related to the random bond Ising model, through averaging the uncorrelated probability distribution over gauge transformations.
Appendix E: Gauge Invariant Probability Distribution

In this section, we want to discuss the gauge-invariant structure of the probability distribution of random bonds in the square $L \times L$ lattice. First, note that the partition function

$$Z = \sum_{\{\sigma\}} e^{\beta \sum_{e=(v,v')} s_e \sigma_v \sigma_{v'}} \quad \text{(E1)}$$

is invariant under the gauge transformation

$$G(\{t_v\}) : s_e \mapsto s_e t_{v} t_{v'} \quad \text{for } e = (v, v') \quad \text{(E2)}$$

for any set of $\{t_v = \pm 1\}$. Therefore, when we discuss the random bond Ising model with the probability distribution $P(\{s_e\})$ for the random bonds, the only important part in the discussion of phase transition is the gauge invariant structure of $P(\{s_e\})$. In general, note that $P(\{s_e\})$ itself is not gauge-invariant.

In order to understand the gauge-invariant structure of a given probability distribution, we define the complete gauge fixing condition as in Fig. 6(e) in the case of 2D. We remark that there are many different choices for the complete gauge fixing. We have $N = L^2$ sites to freely assign $t_v$, which means that we can always make $L^2 - 1$ edges to be always ferromagnetic ($s_e = 1$). Indeed, the procedure in Fig. 6(e) satisfy this bound. Once we move in this gauge, we note that the expectation value of the gauge-invariant object

$$E \left[ \prod_{e \in \gamma_{\text{loop}}} s_e \right] = (\cos \theta)^{|\gamma|} \quad \text{(E3)}$$

for all closed loops $\gamma$ completely determines the probability distribution for unfixed edges. We prove this in an inductive way. Note that in the first column, all bonds except for the last one is gauge-fixed to be one. Also in each row, all bonds except for the one that connects the last column and the first column in the periodic boundary condition is undetermined. There are total $L^2 + 1$ bonds whose signs are now governed by $\tilde{P}$. We label random edges in this gauge by the site it is coming out from, and whether it is vertical (up) or horizontal (right). For example, $s^v_{n,m}$ is the edge coming out from the site $(n, m)$ upward, and $s^h_{n,m}$ is the edge coming out from the site $(n, m)$ to the right.

To show that $\tilde{P}$ is completely fixed by the correlation in Eq. (E3), we proceed as the following:

- Get all $\tilde{P}_e(s_e)$, which is the probability distribution of each bond where the others are marginalized. Due to the gauge choice, this is not homogeneous anymore. Using Eq. (E3), we can show that for undetermined vertical bonds, we get

$$\tilde{P}(s^v_{n,m}) = \begin{cases} \frac{1}{2}(1 + \cos 2m \theta), & \text{if } 1 \leq n < L \\ \frac{1}{2}(1 + \cos (L+2(m-1) \theta)), & \text{if } n = L \end{cases} \quad \text{(E4)}$$

This is possible because for any single undetermined edge, we can find a loop that goes through the ferromagnetic fixed edges except for that edge. Now for $L$ horizontal bonds $s^h_{n,L}$, we can calculate that

$$\tilde{P}(s^h_{n,L}) = \frac{1}{2}(1 + \cos L \theta) \quad \text{(E5)}$$
• We get all one-edge probability distribution. Based on this information, we can get all two-edges probability distributions between any two undetermined edges. For example, for any \( s_1 \equiv (s_{n,m}^a) \) and \( s_2 \equiv (s_{n',m'}^a) \), we can find a loop that goes through the ferromagnetic fixed edges except for those two edges with length \( l \). Then, we can establish the set of equations:

\[
\begin{align*}
\left( \cos \theta \right)^l &= \sum_{s_1, s_2} (s_1 s_2) \tilde{P}(s_1, s_2) \\
\tilde{P}(s_1) &= \sum_{s_2} \tilde{P}(s_1, s_2) \\
\tilde{P}(s_2) &= \sum_{s_1} \tilde{P}(s_1, s_2) \\
1 &= \sum_{s_1, s_2} \tilde{P}(s_1, s_2)
\end{align*}
\]

(E6)

Since \( \tilde{P}(s_1, s_2) \) consists of \( 4 = 2^2 \) variables and we have 4 equations, we can completely determine the two-point probability distribution. Actually we can show that any two vertical bonds separated more than one row \( |m - m'| > 1 \) are independent, while within the same row or neighboring rows, two vertical bonds are highly correlated. Similarly, we can do this for horizontal bonds as well. For any two horizontal bonds \( (s_{n,L}^a, s_{n',L}^a) \), they are uncorrelated. However, \( (s_{n,L}^a, s_{n',m'}^a) \) are correlated if \( n' = n \) or \( n' = n - 1 \).

• The above proof can be simply extended for any three-edges probability distribution, and so on. More generally, for a given 1, 2, ..., \( k - 1 \)-edges probability distributions and the correlation structure Eq. (E3), we can obtain \( k \)-edges probability distribution. For a given \( \tilde{P}(\{s_i\}_{i=1}^k) \), it satisfies the set of equations as the following:

\[
\begin{align*}
\left( \cos \theta \right)^l &= \sum_{\{s\}_k} \left[ \prod_{i=1}^k s_i \right] \tilde{P}(\{s\}_k) \\
\tilde{P}(s_i) &= \sum_{\{s\}_k \setminus \{s_i\}} \tilde{P}(\{s\}_k) \\
\tilde{P}(s_i, s_j) &= \sum_{\{s\}_k \setminus \{s_i, s_j\}} \tilde{P}(\{s\}_k) \\
&\vdots \\
\tilde{P}(\{s\}_k \setminus \{s_i\}) &= \sum_{s_i} \tilde{P}(\{s\}_k) \\
1 &= \sum_{\{s\}_k} \tilde{P}(\{s\}_k)
\end{align*}
\]

(E7)

where \( l \) is the length of the loop that goes through \( k \) edges (note that the loop does not have to be connected – it can be disconnected loop as well). Since we already know that such a loop exist for each \( s_i \), the resulting loop would be nothing but a product of all such loops, where edges would cancel at where they intersect. Note that the number of equations are simply given by the binomial expansion:

\[
\# \text{ of equations} = 1 + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1} + 1 = 2^k
\]

(E8)

Therefore, since we have \( 2^k \) independent equations, it should completely determines \( \tilde{P}(\{s_i\}_{i=1}^k) \) with \( 2^k \) variables.

• By induction, since we know how to get all \( \tilde{P} \) for 1-edge, we can obtain \( \tilde{P} \) for all undetermined \( L^2 + 1 \) edges. This implies that the loop correlation function is enough to completely specify the probability distribution at this gauge.

Appendix F: Removing complex phase factor by a shallow quantum circuit

Let us show that the difference between the measurement at \( \phi \neq 0 \), comparing to the case that \( \phi = 0 \). If \( \mathcal{P}|\psi\rangle \) is the pure state measured at \( (\theta, \phi = 0) \), then \( \mathcal{P}_\phi|\psi\rangle \) measured at angle \( (\theta, \phi) \) only differs from \( \mathcal{P}|\psi\rangle \) by the \( U(1) \) phase in their wavefunctions.
Without loss of generality, let us consider we perform measurements on all links on the cluster state on a $d$-dimensional square lattice.

\[
\mathcal{P}_\phi = R_{x,\phi}^{-1} \mathcal{P} R_{x,\phi}, \quad R_{x,\phi} = \prod_{\text{links } l} \left( \cos \frac{\phi}{2} 1_l + i \sin \frac{\phi}{2} X_l \right) \quad \text{(F1)}
\]

\[
\mathcal{P} |\psi\rangle = \sum_{\{\sigma_i = \pm 1\}} \omega_{\{\sigma_i\}} |\{\sigma_i\}\rangle \otimes |s_l\rangle \quad \text{(F2)}
\]

It follows that

\[
\mathcal{P_\phi} |\psi\rangle = R_{x,\phi}^{-1} \mathcal{P} R_{x,\phi} |\psi\rangle = R_{x,\phi}^{-1} \mathcal{P} \prod_{\text{links } l} \left( \cos \frac{\phi}{2} 1_l + i \sin \frac{\phi}{2} \prod_{i \in l} Z_i \right) |\psi\rangle \quad \text{(F3)}
\]

Since the projectors are operators on links,

\[
\mathcal{P}_{\phi} |\psi\rangle = R_{x,\phi}^{-1} \prod_{\text{links } l} \left( \cos \frac{\phi}{2} 1_l + i \sin \frac{\phi}{2} \prod_{i \in l} Z_i \right) \mathcal{P} |\psi\rangle
\]

\[
= R_{x,\phi}^{-1} \prod_{\text{links } l} \left( \cos \frac{\phi}{2} 1_l + i \sin \frac{\phi}{2} \prod_{i \in l} Z_i \right) \sum_{\{\sigma_i = \pm 1\}} \omega_{\{\sigma_i\}} |\{\sigma_i\}\rangle \otimes |s_l\rangle
\]

\[
= \sum_{\{\sigma_i = \pm 1\}} \omega_{\{\sigma_i\}} e^{i \frac{\phi}{2} \sum_{\text{links } l} \prod_{i \in l} \sigma_i} |\{\sigma_i\}\rangle \otimes R_{x,\phi}^{-1} |s_l\rangle. \quad \text{(F4)}
\]

Therefore,

\[
|\mathcal{P_{\phi}}\psi\rangle = U_Z |\mathcal{P}\psi\rangle, \quad U_Z = \prod_l e^{i \frac{\phi}{2} \prod_{i \in l} Z_i}. \quad \text{(F5)}
\]

The two states are related by a single-depth local unitary.

Similar results also hold when we measure the qubits on sites. For example, if we start with the $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ SPT state and measure the sites, the post-measurement state are related by

\[
|\mathcal{P_{\phi}}\psi\rangle = U_Z |\mathcal{P}\psi\rangle, \quad U_Z = \prod_i e^{i \frac{\phi}{2} \prod_{j \in i} Z_i}. \quad \text{(F6)}
\]