Neutral Color Superconductivity and Pseudo-Goldstone Modes

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Four of the five expected Goldstone modes, which will be eaten up by gauge fields, in neutral two-flavor color superconductor are actually pseudo-Goldstone modes, and their degenerated mass is exactly the magnitude of the color chemical potential, which is introduced to guarantee the color neutrality at moderate baryon density.

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Since the attractive interaction in the antitriplet quark-quark channel in quantum chromodynamics (QCD), the cold and dense quark matter is believed to favor the formation of diquark condensate and in the superconducting phase [1]. In the idealized case at asymptotically high baryon density, the color superconductivity with two massless flavors and the color-flavor-locking (CFL) phase with three degenerated massless quarks have been widely discussed from first principle QCD calculations [2]. For physical applications, one is more interested in the moderate baryon density region which may be realized in compact stars and, in very optimistic cases, even in heavy-ion collisions. To have a stable and macroscopic color superconductor, one should take into account the electric and color charge neutrality condition [3,4] which lead to a new phase, the gapless color superconductivity [5] or the breached pairing phase [6]. The most probable temperature for this new phase is finite but not zero [7]. In two flavor case, the color neutrality can be satisfied by introducing a color chemical potential $\mu_8$ in the four-fermion interaction theory at moderate baryon density [4], or by a dynamic generation of a condensation of gluon field $A^g_0$ in the frame of perturbative QCD at extremely high baryon density [8] where the back ground gluon field ($A^g_0$) plays the role of the color chemical potential $\mu_8$.

It is generally accepted that in the two-flavor color superconductor, there will be five massless Goldstone bosons, corresponding to the spontaneously broken color symmetry from $SU_C(3)$ to $SU_C(2)$. At moderate baryon density, if the charge neutrality condition is not taken into account, there are only three Goldstone modes [9]. Since the degenerated mass of two diquarks of the rest five collective modes is proportional to the net color charge $Q_8$ of the system, it is expected that [9] one can recover the five massless Goldstone bosons by requiring color neutrality. While the Goldstone bosons will finally be eaten up by gauge fields through Higgs mechanism [10], it is necessary to check whether the five expected Goldstone modes are actually pseudo-Goldstone modes, their degenerated mass is exactly the magnitude of the color chemical potential $\mu_8$ which is used to guarantee the color neutrality of the system.

The pseudo-Goldstone bosons were generally discussed in theories with spontaneously broken local symmetries thirty years ago [11]. The higher-order correction leads to spinless bosons which behavior like Goldstone bosons but have a small mass.

For a neutral two-flavor color superconductor, the quark chemical potential matrix

$$\mu = \text{diag}(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)$$

in color and flavor space can be expressed in terms of baryon chemical potential $\mu_b$, electrical chemical potential $\mu_e$, and color chemical potential $\mu_8$,

$$\begin{align*}
\mu_1 &= \mu_2 = \mu_b/3 - 2\mu_e/3 + \mu_8/3, \\
\mu_3 &= \mu_b/3 - 2\mu_e/3 - 2\mu_8/3, \\
\mu_4 &= \mu_5 = \mu_b/3 + \mu_e/3 + \mu_8/3, \\
\mu_6 &= \mu_b/3 + \mu_e/3 - 2\mu_8/3, \\
\end{align*}$$

where $\mu_b$ controls the baryon number density, and $\mu_e$ and $\mu_8$ have to be introduced to ensure the electric and color charge neutrality. If the $SU_C(3)$ color symmetry is not broken, the color chemical potential $\mu_8$ has to vanish, otherwise it would break $SU_C(3)$ explicitly. However, when the color symmetry is broken spontaneously by a color-charged diquark condensate, $\mu_8$ does not need to be zero [12].

It is well-known that a nonzero quark-antiquark condensate $\langle \bar{\psi}\psi \rangle$ spontaneously breaks the chiral symmetry $SU_L(2) \otimes SU_R(2)$ of the system, and the corresponding Goldstone bosons are the three pion mesons. However, when the current quark mass $m \neq 0$, the pions get a small mass proportional to $m$, due to the explicit chiral symmetry breaking. Similarly, when $\mu_8 = 0$, a nonzero diquark condensate $\langle \bar{\psi}^c_i \gamma^5 \beta^j \epsilon^{ij} \epsilon^{a\beta\gamma} \psi_3 \rangle$ spontaneously breaks down the color symmetry $SU_C(3)$ to $SU_C(2)$, and causes five Goldstone bosons corresponding to the broken generators $T_4, T_5, T_6, T_7$ and $T_8$, where $\epsilon^{ij}$ and $\epsilon^{a\beta\gamma}$ are totally antisymmetric tensors in flavor and color space, respectively, and it is assumed that only the first two colors participate in the condensate, while the third one does not. However, in presence of a nonzero $\mu_8$, the $SU_C(3)$ symmetry is explicitly broken down to $SU_C(2) \otimes U_C(1)$ with broken generators $T_4, T_5, T_6$ and $T_7$. Therefore, the four Goldstone bosons corresponding to the broken generators $T_4, T_5, T_6$ and $T_7$ will become
massive, and only the Goldstone boson corresponding to the broken generator $T_8$ remains massless.

Here we have not considered the color chemical potential $\mu_3$, since the neutrality for the color charge $Q_3$ is automatically satisfied in the current case with diquark condensation in the third color direction.

To investigate quantitatively the diquarks as pseudo-Goldstone bosons, we have to choose a suitable model in describing color superconductivity at moderate baryon densities. It is generally accepted that the Nambu–Jona-lasinio model (NJL) [13] applied to quarks [14] offers a simple but effective scheme to study chiral symmetry restoration [15], color symmetry spontaneously breaking [4,5,7,9,16,17] and isospin symmetry spontaneously breaking [18–20]. In the mean field approximation to quarks and random phase approximation (RPA) to mesons, one can obtain the hadronic mass spectra and the static properties of mesons remarkably well [14], especially the Goldstone modes corresponding to the chiral symmetry spontaneously breaking [15] and to the isospin symmetry spontaneously breaking [20].

The flavor $SU(2)$ NJL model is defined through the Lagrangian density

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu + \mu \gamma_0) \psi + G_S \left[ (\bar{\psi}\psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right] + G_D (\bar{\psi}^a i \gamma_5 \tau^a \psi) \left( \bar{\psi}^a i \gamma_5 \tau^a \psi \right),$$

(3)

where $G_S$ and $G_D$ are coupling constants in color singlet channel and color anti-triplet channel, respectively, $\psi^c = C \bar{\psi}^T$ and $\psi^c = \psi^T C$ are charge-conjugate spinors, $C = i \gamma^2 \gamma^0$ is the charge conjugation matrix.

Since we focus in this Letter on the color symmetry spontaneously breaking and the corresponding Goldstone modes, to simply the notation, we consider in the following only the color symmetry spontaneously breaking phase with nonzero diquark condensate

$$\Delta = -2G_D (\bar{\psi}^a i \gamma_5 \tau^a \psi),$$

(4)

and assume that the chiral symmetry is restored in this phase.

In the mean field approximation, the quarks behavior like quasi-particles, and the diquark condensate is controlled by the gap equation [4]

$$1 - 2G_D I_\Delta = 0,$$

(5)

with the function

$$I_\Delta = 4 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{\epsilon = \pm} \frac{1 - f(E_{\pm}^\epsilon) - f(E_{-}^\epsilon)}{E_\Delta},$$

(6)

where the quasi-particle energies are defined as $E_{\pm}^\epsilon = E_3^\pm + \delta \mu \pm \mu_3$ and $E_{\pm}^\epsilon = \sqrt{((|\mathbf{p}| \pm \bar{\mu})^2 + \Delta^2}$ with the two effective chemical potentials $\bar{\mu}$ and $\delta \mu$ given by $\bar{\mu} = \mu_3/3 - \mu_3/6 + \mu_8/3$, and $\delta \mu = \mu_3/2$, and $f(x) = 1/\left(e^{x/T} + 1\right)$ is the Fermi-Dirac distribution function.

To consider the color and electric charge neutralities we need to calculate the color and electric charge densities which can also be expressed as summations of quasiparticle contributions [4],

$$Q_8 = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{\epsilon = \pm} \epsilon \left[ \frac{E_0^\epsilon}{E_\Delta} (1 - f(E_{+}^\epsilon) - f(E_{-}^\epsilon)) + (f(E_{u3}) + f(E_{d3})) \right] = 0,$$

(7)

$$Q_\epsilon = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{\epsilon = \pm} \epsilon \left[ \frac{E_0^\epsilon}{E_\Delta} (1 - f(E_{+}^\epsilon) - f(E_{-}^\epsilon)) + 3 \left( f(E_{+}^\epsilon) - f(E_{-}^\epsilon) \right) - \epsilon \left( 2 f(E_{u3}) - f(E_{d3}) \right) \right] - \frac{\mu_3^3}{2\pi^2} = 0,$$

with quark energies $E_0^\pm = |\mathbf{p}| \pm \bar{\mu}$, $E_{u3}^\pm = |\mathbf{p}| + \mu_3$ and $E_{d3}^\pm = |\mathbf{p}| \pm \mu_3$, where the last term of $Q_\epsilon$ is the contribution from the electron gas. Note that the quarks with color 3 are not involved in the diquark condensation, they behave like free quarks in the color superconductivity phase. The diquark condensate $\Delta$ and color and electric chemical potentials $\mu_8$ and $\mu_3$, as functions of temperature $T$ and baryon chemical potential $\mu_3$ are determined self-consistently by the above three coupled equations.

We now investigate diquark and meson properties at finite temperature and chemical potentials. In the NJL model, the diquark and meson modes are regarded as quantum fluctuations above the mean field. The meson modes can be calculated in the frame of RPA [14,15]. When the mean field quark propagator is diagonal in color, flavor, and Nambu-Gorkov [21] space, for instance the case with only chiral condensation, the summation of bubbles in RPA selects its specific channel by choosing at each stage the same proper polarization function, a meson mode which is determined by the pole of the corresponding meson propagator is related to its own polarization function $\Pi_{MM}(k)$ [14,15] only,

$$1 - 2G_S \Pi_{MM}(k) = 0.$$

(8)

However, for the quark propagator with off-diagonal elements, like the cases of $\eta$ and $\eta'$ meson spectrum [14], pion superfluidity [20], and color superconductivity considered here, we must consider carefully all possible channels in the bubble summation in RPA.

In the two-flavor NJL model there are four meson modes, the scalar meson $\sigma$ and the three pseudoscalar mesons $\pi_+, \pi_0, \pi_-$. In the current case considering color symmetry spontaneously breaking in the third direction in color space, there are six kinds of diquarks, $D_1, D_2, D_3$, $D_1'$, $D_2'$, $D_3'$ constructed, respectively, by colors 2 and 3, 1 and 3, 1 and 2, 2 and 3, 1 and 3, and 1 and 2.

Since we restrict ourselves in the color symmetry spontaneously breaking phase, the mixture among the different channels in the bubble summation in RPA is greatly
reduced. The dispersion relations for the meson modes \( \sigma \) and \( \pi_0 \) are determined by their own polarization functions,

\[
1 - 2G_S\Pi_{\sigma}(k) = 0 , \\
1 - 2G_S\Pi_{\pi_0}(k) = 0 ,
\]

while the ones for the other mesons \( \pi_+ \) and \( \pi_- \) and all the diquarks satisfy the coupled equations,

\[
[1 - 2G_S\Pi_{\pi_+}(k)] [1 - 2G_S\Pi_{\pi_-}(k)] = 0 , \\
[1 - 2G_D\Pi_{D_1}(k)] [1 - 2G_D\Pi_{D_2}(k)] = 0 , \\
[1 - 2G_D\Pi_{D_3}(k)] [1 - 2G_D\Pi_{D_4}(k)] = 0 , \\
\det \left( 1 - 2G_D\Pi_{D_1}(k) - 2G_D\Pi_{D_2}(k) \right) = 0 .
\]

For the meson and diquark masses computed through the above dispersion relations at \( k_0^2 = M^2 \) and \( k = 0 \), one needs to know the explicit expressions \( \Pi(k_0, 0) \) only. Performing a relatively complicated but straightforward calculation including Matsubara frequency summation, they can be written as

\[
\Pi_{\sigma}(k_0) = \Pi_{\pi_0}(k_0) = J_1(k_0^2) , \\
\Pi_{\pi_+}(k_0) = \Pi_{\pi_-}(k_0) = J_2(\mu_c - k_0) , \\
\Pi_{D_1}(k_0) = \Pi_{D_2}(k_0) = \Pi_{D_3}(k_0) = \Pi_{D_4}(k_0) = 0 \text{, and } \\
\Pi_{D_3}(k_0) = \Pi_{D_4}(k_0) = \Pi_{D_5}(k_0) = \Pi_{D_6}(k_0) = 0 ,
\]

where the functions \( J_1, J_2 \) and \( J_3 \) related to the diquarks are defined as

\[
K_1 = \int \frac{d^3p}{(2\pi)^3} \sum_{c=\pm} \left[ \left( \frac{1}{F_1} - \frac{1}{F_2} \right) \left( f(E^+_{\pi_0}) + f(E^-_{\pi_0}) - 1 \right) \right] F_1^c(k_0, |p|) , \\
K_2 = \int \frac{d^3p}{(2\pi)^3} \sum_{c=\pm} \frac{1}{F_3^c} \left( f(E^+_{\pi_0}) + f(E^-_{\pi_0}) - 1 \right) , \\
K_3 = - \int \frac{d^3p}{(2\pi)^3} \sum_{c=\pm} \frac{E_0^c}{F_3^c} \left( f(E^+_{\pi_0}) + f(E^-_{\pi_0}) - 1 \right) ,
\]

with

\[
F_1^\pm(k_0, |p|) = k_0 + \mu_8 \mp E_0^\pm \mp E_\Delta^\pm , \\
F_2^\pm(k_0, |p|) = k_0 + \mu_8 \mp E_0^\pm \mp E_\Delta^\pm , \\
F_3^\pm(k_0) = E_\Delta \left( k_0^2 - 4 (E_\Delta)^2 \right) .
\]

Since we will not discuss the meson masses in detail in this Letter, the explicit expressions of the functions \( J_1 \) and \( J_2 \) related to the mesons are not listed here. However, the relations among the meson dispersion relations shown in (11) can help us to understand the meson mass splitting in chiral symmetry restoration phase. From the first equation of (11) it is clear that the masses of \( \sigma \) and \( \pi_0 \) mesons become degenerate in the color superconductivity phase,

\[
M_\sigma = M_{\pi_0} .
\]

Remember that in presence of a nonzero electric chemical potential \( \mu_e \), the chiral symmetry \( SU_L(2) \otimes SU_R(2) \) is explicitly broken down to \( U_L(1) \otimes U_R(1) \) with generator \( \tau_3 \), and the masses of the three pion mesons are not the same in the chiral restoration phase. This can be easily seen from the difference among the three pion polarization functions shown in (11). Only in the case of \( \mu_c = 0 \), the relation \( J_2(k_0) = J_1(k_0) \) results in degenerated meson mass \( M_\sigma = M_{\pi_0} = M_{\pi_+} = M_{\pi_-} \).

Considering the gap equation (5) in the color superconductivity phase and the relations among the polarization functions \( \Pi_{D_1}, \Pi_{D_2}, \Pi_{D_3}, \Pi_{D_4}, \Pi_{D_5} \) and \( \Pi_{D_6} \) shown in (11), the two mass equations for \( D_1, D_2, D_3 \) and \( D_4 \) are both simplified as

\[
(k_0^2 - \mu_8^2)K_1(k_0)K_1(-k_0) = 0 .
\]

Obviously, one solution is \( k_0^2 = \mu_8^2 \), and the other is determined by

\[
H(k_0^2) = K_1(k_0)K_1(-k_0) = 0 .
\]

From the comparison with the color charge density \( Q_8 \), one can easily prove

\[
K_1(k_0 = -\mu_8) = 2 \frac{Q_8}{\Delta} .
\]

Taking into account the color charge neutrality condition \( Q_8 = 0 \), one has

\[
H(k_0^2 = \mu_8^2) = 0 .
\]

Therefore, the masses of the four diquarks, \( D_1, D_2, D_3 \) and \( D_4 \), are degenerate and exactly equal to the magnitude of the color chemical potential,

\[
M_{D_1} = M_{D_2} = M_{D_3} = M_{D_4} = |\mu_8| ,
\]

in the color superconductivity phase. The lower panel of Fig.1 shows \( \mu_8 \) as a function of baryon chemical potential \( \mu_b \) at zero temperature, calculated through solving the gap equation (5) and the color and electrical charge neutrality condition (7). There are three parameters in the NJL model (3) in chiral limit. The momentum cutoff \( \Lambda \) and the coupling constant \( G_S \) can be fixed by fitting the pion decay constant and chiral condensate in the vacuum [14,15], and the coupling constant \( G_D \) in the diquark channel is taken as \( G_D = 3G_S/4 \) [4] in our calculation.
In this case the color superconductivity phase starts at $\mu_b/3 = 330\text{MeV}$. Since the above analytic discussion for the meson and diquark masses does not depend on whether one considers electrical charge neutrality or not, we show also in the upper panel of Fig.1 the color chemical potential $\mu_b$ without considering electrical charge neutrality, namely we take $\mu_e = 0$ in the calculation. We see that in any case the magnitude of $\mu_b$ is small. Especially, when we take the both charge neutralities, $\mu_b$ is only a few MeV in a wide region. That means, the four degenerated pseudo-Goldstone bosons are almost massless in neutral color superconductor.

The relations among the polarization functions $\Pi_{D_3,D_3}, \Pi_{D_3,D_3}, \Pi_{D_3,D_3}$ and $\Pi_{D_3,D_3}$ and their explicit expressions shown in (11) reduce the mass equation for the modes $D_3$ and $\tilde{D}_3$ which are linear combinations of $D_3$ and $\tilde{D}_3$ to

$$k_0^2 \left[ (k_0^2 - 4\Delta^2)K_2^3(k_0^2) - 4K_3^3(k_0^2) \right] = 0 .$$

One of its solution is of course

$$M_{D_3} \left( M_{D_3} \right) = 0 ,$$

and the other massive mode $\tilde{D}_3 \left( \tilde{D}_3 \right)$ is calculated numerically through

$$(k_0^2 - 4\Delta^2)K_2^3(k_0^2) - 4K_3^3(k_0^2) = 0 .$$

FIG. 1. The color chemical potential $\mu_b$ as a function of baryon chemical potential $\mu_b$ at zero temperature in the color superconductivity phase. The upper and lower panels are, respectively, the results without ($\mu_e = 0$) and with electrical charge neutrality.

We have investigated the quantum fluctuations in the neutral two-flavor color superconductivity phase in mean field approximation to quarks together with the random phase approximation to mesons and diquarks in the frame of NJL model. We have shown analytically that, there is only one massless Goldstone boson, and the other four expected Goldstone bosons are actually pseudo-Goldstone bosons, and their degenerated mass is exactly the magnitude of the color chemical potential $\mu_b$ which is introduced in the Lagrangian density to guarantee the color neutrality of the system and breaks explicitly the color symmetry from SU($C$(3) to SU($C$(2) $\otimes$ U$_C$(1)). By self-consistently determining the diquark condensate, color and electric chemical potentials, the pseudo-Goldstone mass is only a few MeV, the same order like the current quark mass which breaks explicitly the chiral symmetry of QCD. It is necessary to note that, while the one Goldstone and four pseudo-Goldstone modes will finally be eaten up by gauge fields, the nonzero color chemical potential $\mu_b$ is also coupled to the meson and the heavy diquark modes which will then be reflected in the measurable properties of the system.

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