COHOMOLOGY OF GENERALIZED DOLD SPACES

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Abstract. Let \((X,J)\) be an almost complex manifold with a (smooth) involution \(\sigma : X \to X\) such that \(\text{Fix}(\sigma) \neq \emptyset\). Assume that \(\sigma\) is a complex conjugation, i.e., the differential of \(\sigma\) anti-commutes with \(J\). The space \(P(m,X) := \mathbb{S}^m \times X / \sim\) where \((v,x) \sim (-v,\sigma(x))\) was referred to as a generalized Dold manifold. The above definition admits an obvious generalization to a much wider class of spaces where \(X,S\) are arbitrary topological spaces. The resulting space \(P(S,X)\) will be called a generalized Dold space.

When \(S\) and \(X\) are CW complexes satisfying certain natural requirements, we obtain a CW-structure on \(P(S,X)\). Under certain further hypotheses, we determine the mod 2 cohomology groups of \(P(S,X)\).

We determine the \(\mathbb{Z}_2\)-cohomology algebra when \(X\) is (i) a torus manifold whose torus orbit space is a homology polytope, (ii) a complex flag manifold. One of the main tools is the Stiefel-Whitney class formula for vector bundles over \(P(S,X)\) associated to \(\sigma\)-conjugate complex bundles over \(X\) when the \(S\) is a paracompact Hausdorff topological space, extending the validity of the formula, obtained earlier by Nath and Sankaran, in the case of generalized Dold manifolds.

1. Introduction

The classical Dold manifold \(P(m,n)\) is defined as the orbit space of the \(\mathbb{Z}/2\mathbb{Z}\)-action on \(\mathbb{S}^m \times \mathbb{C}P^n\) generated by the involution \((v,[z]) \mapsto (-v,\bar{z})\), \(v \in \mathbb{S}^m, [z] \in \mathbb{C}P^n\). Here \([\bar{z}]\) denotes \([\bar{z}_0 : \cdots : \bar{z}_n]\) when \([z] = [z_0 : \cdots : z_n] \in \mathbb{C}P^n\). See [D].

Let \(\sigma : X \to X\) be a complex conjugation on an almost complex manifold \((X,J)\), that is, \(\sigma\) is an involution with non-empty fixed point set such that, for any \(x \in X\), the differential \(T_x\sigma : T_xX \to T_{\sigma(x)}X\) satisfies the equation \(J_{\sigma(x)} \circ T_x\sigma = -T_x\sigma \circ J_x\). See [CF, §24]. The generalized Dold manifold \(P(m,X)\) was introduced in [NS] as the quotient of \(\mathbb{S}^m \times X\) under the identification \((v,x) \sim (-v,\sigma(x))\). The main focus in [NS] was the study of manifold-properties of \(P(m,X)\) such as the description of its tangent bundle, a formula for its total Stiefel-Whitney class, the (stable) parallelizability and related properties, and its cobordism class.

Here, our aim is to compute the mod 2-cohomology algebra of the generalized Dold manifolds. While studying the homotopical/homological properites, it is natural to do away with stringent requirements such as \(X\) to be an almost complex manifold. Also we replace \((\mathbb{S}^m, \text{antipode})\), by a pair \((S,\alpha)\) where \(S\) is, say, a paracompact Hausdorff topological space and \(\alpha : S \to S\) a fixed point free involution. Likewise, \(X\) is any

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Hausdorff topological space with an involution $\sigma : X \to X$ having a non-empty fixed point set. Then $P(S, \alpha, X, \sigma)$ (or more briefly $P(S, X))$ is the space $S \times_{Z_2} X = S \times X/\sim$ where $(v, x) \sim (\alpha(v), \sigma(x))$ which is the quotient of $S \times X$ by the free $Z_2$-action generated by $\alpha \times \sigma$. When $S, X$ are CW complexes satisfying certain additional hypotheses, we show that, as $Z_2$-vector spaces, $H^*(P(S, X); Z_2) \cong H^*(Y; Z_2) \otimes H^*(X; Z_2)$ where $Y = S/Z_2$.

We also obtain the same result when $H^*(X; Z_2)$ is generated by the Stiefel-Whitney classes of finitely many $\sigma$-conjugate (complex) vector bundles. See Proposition 3.2. The notion of a $\sigma$-conjugate vector bundle will be defined in §2.

We obtain a formula for Stiefel-Whitney classes of the real vector bundle over $P(S, X)$ associated to a $\sigma$-conjugate vector bundle over $X$ when $H^1(X; Z_2) = 0$. This formula is applied to obtain the ring structure of $H^*(P(S, X); Z_2)$ when $X$ is a torus manifold whose torus quotient is a homology polytope, or is a complex flag manifold. See Theorem 4.4 and Theorem 4.8. Note that the class of such torus manifolds contains the class of all quasi-toric manifolds as well as that of all smooth complete toric manifolds. Finally, as an application, we compute the equivariant cohomology ring $H^*_\mathbb{Z}_2(X; \mathbb{Z}_2)$ (see Theorem 4.10).

Recently, Sarkar and Zvengrowski [SZ] have constructed smooth manifolds which are a simultaneous generalization of projective product spaces due to Davis [D] and of Dold manifolds and call them generalized projective product spaces. These are $Z_2$-quotients $P(M, N)$ of products $M \times N$ by the diagonal action where $M, N$ are manifolds admitting $Z_2$-actions where the action on $M$ is assumed to be free. When the fixed point set for $Z_2$-action on $N$ is non-empty, they are also generalized Dold spaces in our sense. Sarkar and Zvengrowski also obtain results on $Z_2$-cohomology in many special cases, including when $M$ is a product of spheres and $N$ is a quasi-toric manifold, besides results on manifold-properties of generalized product spaces.

2. Generalized Dold spaces

Let $\alpha : S \to S$ be a fixed point free involution and let $\sigma : X \to X$ be involution with non-empty fixed point set. We assume that $S, X$ are connected, locally path connected, paracompact, Hausdorff topological spaces. We denote by $Y$ the quotient space $S/\sim$ where $v \sim \alpha(v), v \in S$. The space $P(S, \alpha, X, \sigma)$ (or more briefly $P(S, X)$ when there is no risk of confusion) is defined to be the quotient space $S \times X/\sim$ where $(v, x) \sim (\alpha(v), \sigma(x))$ and is called the generalized Dold space. It is a (smooth) manifold if $S, X$ (and $\alpha, \sigma$) are smooth. The quotient maps $q : S \to Y$ and $p : S \times X \to P(S, X)$ are covering projections with deck transformation groups isomorphic to $Z_2$ generated by $\alpha$ and $\alpha \times \sigma$ respectively. We denote by $[v] \in Y$ the element $q(v) = \{v, \alpha(v)\}$ and similarly $[v, x] = p(v, x) = \{(v, x), (\alpha(v), \sigma(x))\} \in P(S, X)$. The first projection $S \times X \to Y$ induces a map $\pi : P(S, X) \to Y$ which is the projection of a locally trivial bundle with fibre space $X$. Denote by $X^\sigma \subset X$ the fixed points of $\sigma$. For each $x \in X^\sigma := \text{Fix}(\sigma)$, we have a cross-section $s_x = s : Y \to P(S, X)$ where $s([v]) = [v, x]$. In fact, we have a well-defined map $i : Y \times X^\sigma \to P(S, X)$ defined as $i([v], x) = [v, x]$. We observe that $i$ is an embedding.
The above construction is functorial in \((S, \alpha)\) and in \((X, \sigma)\) for maps \((S', \alpha') \to (S, \alpha)\) and \((X', \sigma') \to (X, \sigma)\) that are \(\mathbb{Z}_2\)-equivariant maps.

2.1. \(\sigma\)-conjugate vector bundles. Let \(\sigma : X \to X\) be an involution on a path connected paracompact Hausdorff topological space. We assume that \(X^\sigma\) is a non-empty proper subset of \(X\). Let \(\omega\) be a complex vector bundle over \(X\). An involutive bundle map \(\hat{\sigma} : E(\omega) \to E(\omega)\) on the total space of \(\omega\) that covers \(\sigma\) and is conjugate complex linear on the fibres of the projection \(\pi_\omega : E(\omega) \to X\). If such a bundle involution exists, we call \((\omega, \hat{\sigma})\) (or more briefly \(\omega\)) a \(\sigma\)-conjugate bundle.

Let \((\omega, \hat{\sigma})\) be a \(\sigma\)-conjugate vector bundle over \(X\) with bundle projection \(E(\omega) \to X\). The zero-cross section \(X \to E(\omega)\) is \(\sigma\)-equivariant. In particular, \(\text{Fix}(\hat{\sigma})\) is non-empty. We obtain a real vector bundle \(P(S, \omega)\), more briefly denoted \(\hat{\omega}\) when there is no danger of confusion, over \(P(S, X)\) with projection \(P(S, E(\omega), \hat{\sigma}) \to P(S, X)\) defined as \([v, e] \mapsto [v, \pi_\omega(e)]\).

Note that \(\hat{\sigma}\) is also a \(\sigma\)-conjugation on the conjugate complex vector bundle \(\bar{\omega}\) and we have an isomorphism of real vector bundles \(\bar{\omega} \cong \hat{\omega}\).

The above construction of \(\hat{\omega}\) over \(P(S, X)\), as prolongation of a \(\sigma\)-conjugate complex vector bundle over \(X\), can be extended to real vector bundles as follows. Let \(\eta\) be any real vector bundle over \(X\) with a bundle involution \(\hat{\eta}\) that covers \(\sigma\). We denote by \(\hat{\eta}\) the vector bundle over \(P(S, X)\) with total space \(P(S, E(\eta)) = P(S, E(\eta), \hat{\sigma})\) and bundle projection \(\pi_{\hat{\eta}} : P(S, E(\eta)) \to P(S, X)\) defined as \([v, e] \mapsto [v, \pi_\eta(e)]\) \(\forall v \in S, e \in E(\eta)\).

2.2. Functoriality. Suppose that \(\alpha_1 : S_1 \to S_1\) is a fixed point free involution and that \(f : S_1 \to S\) is \(\mathbb{Z}_2\)-equivariant, i.e., \(f(\alpha_1(v_1)) = \alpha f(v_1), v_1 \in S\). By the functoriality, we obtain a map \(F : P(S_1, X) \to P(S, X)\). Explicitly \(F\) is induced by the \(\mathbb{Z}_2\)-equivariant map \(f \times \text{id} : S_1 \times X \to S \times X\). Also, we have a morphism of real vector bundles \(P(S_1, \omega) \to P(S, \omega)\) where the map between the total spaces \(\bar{F} : P(S_1, E(\omega), \hat{\sigma}) \to P(S, E(\omega), \hat{\sigma})\) is again got by functoriality. Explicitly, \(\bar{F}([v_1, e]) = [f(v_1), e] \forall v_1 \in S_1, e \in E(\omega)\). We have the following commuting diagram:

\[
\begin{array}{ccc}
P(S_1, E(\omega)) & \xrightarrow{\bar{F}} & P(S, E(\omega)) \\
\downarrow & & \downarrow \\
P(S_1, X) & \xrightarrow{F} & P(S, X)
\end{array}
\]

It follows that

\[
F^*(P(S, \omega)) = P(S_1, \omega). \tag{1}
\]

We have the following isomorphism of real vector bundle for any \(\sigma\)-conjugate complex vector bundle \((\omega, \hat{\sigma})\) over \(X\):

\[
P(S, \omega) \cong \xi_\alpha \otimes P(S, \omega). \tag{2}
\]

An explicit bundle isomorphism can be obtained as follows: Write \(\hat{\omega} := P(S, \omega)\). The total space \(E(\xi_\alpha \otimes \hat{\omega})\) of \(\xi_\alpha \otimes \hat{\omega}\) is a subspace of the quotient of \(S \times X \times E(\epsilon_\mathbb{R} \otimes \omega)\) under the identification \((v, x, t \otimes w) \sim (\alpha(v), \sigma(x), -t \otimes \hat{\sigma}(w))\). We write the equivalence class
of \((v, x, t \otimes w)\) as \([v, x, t \otimes w]\). The space \(E(\xi_\alpha \otimes \omega)\) consists of triples \([v, x, t \otimes w]\) where 
\[
\pi_{\xi_\alpha \otimes \omega}(t \otimes w) = x \in X.
\]
A bundle map \(f : E(\xi_\alpha \otimes \hat{\omega}) \to E(P(S, \omega))\) that covers the identity is obtained as \([v, x, t \otimes w] \mapsto [v, x, \sqrt{-1}tw]\). The routine verification is left to the reader. (Cf. [U, Proposition 1.4(iii)], [NS, Lemma 2.7].)

2.3. The Stiefel-Whitney classes of \(\hat{\omega}\). Assuming that \(H^1(X; \mathbb{Z}_2) = 0\), a formula for Stiefel-Whitney classes of \(\hat{\omega}\) was obtained in [NS] in the case of generalized Dold manifolds \(P(S^m, X, \sigma)\). The proof used functorial properties of the Dold construction and the splitting principle for complex vector bundles over \(X\). These properties hold for any space \(X\) as the manifold properties of \(X\) play no role in the proof of the Stiefel-Whitney class formula. Specifically, the following formula holds for \(\sigma\)-conjugate vector bundles over \(P(S^n, X)\) for any CW complex \(X\) with \(H^1(X; \mathbb{Z}_2) = 0\).

There exist suitable cohomology classes \(\tilde{c}_j(\omega) \in H^{2j}(P(S^m, X); \mathbb{Z}_2)\) which restricts to the mod 2 Chern class \(c_j(\omega) = w_{2j}(\omega) \in H^{2j}(X)\) along the fibres of \(P(S^m, X) \to \mathbb{R}P^m\).

One has the following formula for the \(i\)th Stiefel-Whitney class of \(P(S^m, \omega) = \hat{\omega}\) in terms of the \(\tilde{c}_j(\omega)\): (cf. [NS, Prop. 2.11]):

\[
w_i(\hat{\omega}) = \sum_{0 \leq j \leq r} \binom{r-j}{i-2j} x^{i-2j} \tilde{c}_j(\omega)
\]

(3)

where \(x := w_1(\xi_\alpha) \in H^1(P(S^m, X); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2\) is non-zero and \(r = \text{rank}_\omega\). (It is understood that the binomial coefficient \(\binom{a}{b}\) is 0 if \(b > a\) or if \(a < 0\) or \(b < 0\).) See [NS].

We have \(\tilde{c}_1(\omega) = w_2(\hat{\omega}) + \binom{r}{2} x^2\) and \(w_1(\hat{\omega}) = rx\). It can be seen easily, using (2) and induction, that \(\tilde{c}_j(\omega)\) is expressible as a polynomial \(\tilde{c}_j(\omega) = Q_j(x^2, w_2(\hat{\omega}), w_4(\hat{\omega}), \ldots, w_{2j}(\hat{\omega}))\) for \(1 \leq j \leq r\). Therefore, for \(0 \leq j < r\), \(w_{2j+1}(\hat{\omega})\) can be expressed as a polynomial in \(x\) and even Stiefel-Whitney classes \(w_{2i}(\hat{\omega}), 0 \leq i \leq j\):

\[
w_{2j+1}(\hat{\omega}) = xP_j(x^2, w_2(\hat{\omega}), w_4(\hat{\omega}), \ldots, w_{2j}(\hat{\omega}))
\]

(4)

for a suitable polynomial \(P_j = P_j(x^2, w_2, w_4, \ldots, w_{2j})\) of total degree \(2j\) where \(\text{deg}(w_i) = i\).

As noted already, \(P_0 = r \in \mathbb{Z}_2\). Equations (3) and (4) are still valid when \(X\) is any connected CW complex as long as \(H^1(X; \mathbb{Z}_2) = 0\).

We need to extend Equation (4) to the more general context of the bundle \(\hat{\omega} := P(S, \omega)\) over \(P(S, X, \sigma)\) under the assumption that \(Y\) (and hence \(S\)) and \(X\) are CW complexes (although one can relax these restrictions even further).

We proceed as follows: First, extend the validity of (3) to the case when \(S = S^\infty\) with antipodal involution \(-id\) so that \(Y = \mathbb{R}P^\infty\). Then, any double cover \(S \to Y\) can be classified by a map \(\bar{f} : Y \to \mathbb{R}P^\infty\) and so we have an \(\mathbb{Z}_2\)-equivariant map \(f : (S, \alpha) \to (S^\infty, -id)\). Hence we obtain formula (3) with \(x = w_1(\xi_\alpha)\) for the bundle \(P(S, \omega)\), using the naturality of Stiefel-Whitney classes and the isomorphism (1). So, it remains only to consider the case \((S, \alpha) = (S^\infty, -id)\).

Note that \(P(S^\infty, X) = \bigcup_{m \geq 1} P(S^m, X)\). Moreover, the inclusion \(P(S^m, X) \hookrightarrow P(S^\infty, X)\) is an \((m - 1)\)-equivalence (as can be seen using the homotopy exact sequence). Since
the spaces involved are CW complexes $i$ induces an isomorphism in cohomology up to
dimension $m - 1$. Given any $\sigma$-conjugate complex vector bundle $\omega$ over $X$ of rank $r$ we
merely choose $m > 2r$. It follows that the formula (3) holds for $P(S^\infty, \omega)$.

Using (4), or directly, we see that $P(S, \omega)$ is orientable if and only if $\text{rank}_{\mathbb{C}}(\omega)$ is even.

3. Cell structure and cohomology

Let $S, X$ to be locally finite CW complexes. Under appropriate hypotheses on the cell-
structures on $S$ and $X$, we obtain a nice CW structure on $P(S, X)$ which has the property
that the mod 2 cellular boundary map vanishes. We shall see that there are many classes
of smooth manifolds $S, X$ where there are such CW structures.

Denote by $(C_*(A), \partial^A)$ the cellular chain complex of a CW complex $A$ over $\mathbb{Z}_2$. Every
cell $e$ has a unique orientation mod 2 and so defines a basis element, again denoted by $e$,
of $C_q(A; \mathbb{Z}_2) = H_q(A(q), A^{(q-1)}; \mathbb{Z}_2)$. When $A$ is clear from the context we write $\partial$ instead
of $\partial^A$. In fact, we shall denote by the same symbol $\partial$ for the differential in any chain
complexes when there is no danger of confusion which one is meant.

Let $X$ be a connected CW complex. Assume that $\sigma : X \to X$ is an involution which
stabilizes each cell of $X$. In particular the zero cells are contained in $\text{Fix}(\sigma)$. Since $\sigma$
stabilizes each cell of $X$, we have $\sigma_*(e) = \pm e = e$ for each cell $e$ of $X$ and so $\sigma_* : C_*(X; \mathbb{Z}_2) \to C_*(X; \mathbb{Z}_2)$ is the identity map.

Let $S$ be a $\mathbb{Z}_2$-equivariant CW complex where $\alpha : S \to S$ is the generator of the
$\mathbb{Z}_2$-action on $S$. Since $\alpha$ is fixed point free involution, no open cell is mapped to itself
under $\alpha$. The CW-structure on $(S, \alpha)$ yields a CW structure on $Y$ consisting of one cell
$q(e) = q(\alpha(e))$ for each pair of cells $e, \alpha(e)$ of $S$. On the other hand, any CW structure
on $Y$ lifts to an equivariant CW-structure on $S$.

Also we obtain the product CW structure on $S \times X$ which is $\mathbb{Z}_2$-equivariant where the
$\mathbb{Z}_2$-action is generated by $\tau := \alpha \times \sigma$. It consists of cells which are products $e \times d$ where
e, $d$ are cells of $S$ and $X$ respectively. The induced CW structure on $P(S, X)$ consists of
one cell $(e, d)$ for each pair of cells $e \times d, \tau(e \times d) = \alpha(e) \times \sigma(d) = \alpha(e) \times d$.

Suppose that the cell structure on $X$ is perfect mod 2, that is, the differential in the
cellular chain complex of $X$ with $\mathbb{Z}_2$-coefficients vanishes in all dimensions. Since $\partial^X = 0$
in all dimensions, we have $H_*(X; \mathbb{Z}_2) \cong C_*(X; \mathbb{Z}_2)$. Our assumptions are trivially valid
when $X$ has no odd dimensional cells. In this case the cellular boundary map $\partial$ vanishes
even with $\mathbb{Z}$-coefficients.

Fix $r \geq 0$. We have $C_r(S \times X; \mathbb{Z}_2) \cong \oplus_{p+q=r} C_p(S; \mathbb{Z}_2) \otimes C_q(X; \mathbb{Z}_2)$. Since $\partial^X = 0$, we
have $\partial(e \otimes d) = \partial(e) \otimes d$. Moreover $\tau_*(e \otimes d) = \alpha_*(e) \otimes \sigma_*(d) = \alpha(e) \otimes d$. Recall that $p$
denotes the covering projection $S \times X \to P(S, X)$. Since $p \circ \tau = p$ we have the following
commuting diagram of chain complexes:

\[
\begin{array}{ccc}
C_r(S \times X; \mathbb{Z}_2) & \xrightarrow{p_*} & C_r(P(S, X); \mathbb{Z}_2) \\
\partial & & \partial
\end{array}
\]

We denote by \((e, d)\) the cell \(p(e \times d) = p(\alpha(e) \times d)\) in \(P(S, X)\). Let \(\dim e = p, \dim d = q\). Suppose that \(\partial^S(e) = \sum_{j \in J(e)} e_j \in C_{p-1}(S; \mathbb{Z}_2)\) where \(J(e)\) is a suitable subset of the indexing set for cells of \(S\). Then, \(\partial(e \times d) = \sum e_j \otimes d\) and so

\[
\partial((e, d)) = \sum_{j \in J(e)} p_*(e_j \otimes d) = \sum_{j \in J(e)} (e_j, d) \in C_{r-1}(P(S, X); \mathbb{Z}_2).
\]

Since \(q_*(\alpha(e_j)) = q_*(e_j) \in C_{p-1}(Y)\), we have \((e_j, d) = (\alpha(e_j), d)\) for any cell \(d\) of \(X\). If \(\alpha(J(e)) = J(e)\), then both \((e_j, d)\) and \((\alpha(e_j), d)\) occur in the sum (5) and cancel each other out and we conclude that \(\partial((e, d)) = 0\). (Note that \(e_j, \alpha(e_j)\) are distinct cells of \(S\).)

We are ready to prove the following proposition. For the notion of cohomology extension of fibre we refer the reader to [Sp, Theorem 9, §7, Chapter 5].

**Proposition 3.1.** Suppose that \((S, \alpha)\) is any \(\mathbb{Z}_2\)-equivariant CW complex such that the induced cell structure on \(Y\) is perfect mod 2. Suppose that \(X\) is a CW complex such that (i) each skeleton \(X^{(k)}\), \(k \geq 1\) is finite, (ii) the cell structure is perfect mod 2, and, (iii) each cell is mapped to itself by \(\sigma : X \rightarrow X\). Then the CW structure on \(P(S, X)\) induced by the product CW structure on \(S \times X\) is perfect mod 2. In particular, one has an isomorphism \(H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)\) of \(H^*(Y; \mathbb{Z}_2)\)-modules.

**Proof.** Let \(\partial^S(e) = \sum_{j \in J(e)} e_j\). Since the induced cell structure on \(Y\) is perfect, we have \(0 = \partial^S(q_*(e)) = q_*(\partial^S(e)) = q_* \sum_{j \in J(e)} e_j\). Since \(q_*(\partial^S(e)) = 0\) we see that \(J(e)\) is stable by \(\alpha\). Hence, from Equation (5), for any cell \(d\) of \(X\), we have \(\partial((e, d)) = 0\). Thus the induced cell-structure on \(P(S, X)\) is perfect mod 2.

Assume that \(X\) is a finite CW complex. We have \(H_k(P(S, X); \mathbb{Z}_2) \cong C_k(P(S, X); \mathbb{Z}_2) \cong \bigoplus_{i+j=k} C_i(Y; \mathbb{Z}_2) \otimes C_j(X; \mathbb{Z}_2) \cong \bigoplus_{i+j=k} H_i(Y; \mathbb{Z}_2) \otimes H_j(X; \mathbb{Z}_2)\). Consequently \(H_* (P(S, X); \mathbb{Z}_2) \cong H_*(Y; \mathbb{Z}_2) \otimes H_*(X; \mathbb{Z}_2)\). Also \(H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)\) is a free module over \(H^*(Y; \mathbb{Z}_2)\) via \(\pi^* : H^*(Y; \mathbb{Z}_2) \rightarrow H^*(P(S, X); \mathbb{Z}_2)\).

In the general case, the inclusion \(X^{(k)} \hookrightarrow X\) induces an inclusion \(P(S, X^{(k)}) \hookrightarrow P(S, X)\) which covers the identity map of \(Y\). The proposition follows from the observation that, for any \(n \geq 1\), the inclusion-induced homomorphism \(H^n(P(S, X); \mathbb{Z}_2) \rightarrow H^n(P(S, X^{(k)}); \mathbb{Z}_2)\) is an isomorphism for all \(k > n\).

We shall obtain another situation where the \(X\)-bundle \(\pi : P(S, X) \rightarrow Y\) admits a \(\mathbb{Z}_2\)-cohomology extension of fibre. As observed already, since \(X^e \neq \emptyset\), we have a cross-section \(Y \rightarrow P(S, X)\) of the \(X\)-bundle and so it follows that \(\pi^* : H^*(Y; \mathbb{Z}_2) \rightarrow H^*(P(S, X); \mathbb{Z}_2)\) is a monomorphism. We have the following result. For the notion of cohomology extension of fibre we refer the reader to [Sp, Theorem 9, §7, Chapter 5].

**Proposition 3.2.** Let \((\omega_j, \hat{\sigma}_j), 1 \leq j \leq r\) be \(\sigma\)-conjugate complex vector bundles over \((X, \sigma)\) such that the cohomology algebra \(H^*(X; \mathbb{Z}_2)\) is generated by the mod 2 Chern classes
$c_q(\omega_j) \in H^{2q}(X;\mathbb{Z}_2), 1 \leq q \leq \text{rank}(\omega_j), 1 \leq j \leq r$. Assume that $X$ satisfies any of the following: (i) $\dim_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2) < \infty$, (ii) $X$ is a CW complex with finite $k$-skeleton for each $k \geq 1$ where each cell is stable by $\sigma$. Then we have an isomorphism of $\mathbb{Z}_2$-vector spaces:

$$H^r(P(S,X);\mathbb{Z}_2) \cong \bigoplus_{p+q=r} H^p(Y;\mathbb{Z}_2) \otimes H^q(X;\mathbb{Z}_2). \quad (6)$$

In particular, $H^*(P(S,X);\mathbb{Z}_2)$ is a free $H^*(Y;\mathbb{Z}_2)$-module with basis $B$ where $B$ is a $\mathbb{Z}_2$-basis for $H^*(X;\mathbb{Z}_2)$.

**Proof.** Let $e_0 \in S$. Set $\hat{w}_j := P(S,\omega_j), 1 \leq j \leq r$. Then $\iota^*(\hat{w}_j) \cong \omega_j$ where $\iota$ is the inclusion $\iota : X \to P(S,X)$ is defined as $x \mapsto [e_0, x]$. So $w_{2i}(\hat{w}_j)$ restricts to $w_{2i}(\omega_j) = c_i(\omega_j) \in H^{2i}(X;\mathbb{Z}_2)$. Since the $c_i(\omega_j), 1 \leq i \leq \text{rank}(\omega_j), 1 \leq j \leq r$, generate the cohomology algebra $H^*(X;\mathbb{Z}_2)$, it follows that the homomorphism $H^*(P(S,X);\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$ is surjective. Thus, the $X$-bundle $(P(S,X),Y,\pi)$ admits $\mathbb{Z}_2$-cohomology extension of the fibre. If $\dim_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2)$ is finite, such as when $X$ is a finite CW complex, by Leray-Hirsch theorem we conclude that $H^*(P(S,X);\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2)$ as $H^*(Y;\mathbb{Z}_2)$-modules. This proves (i).

Suppose that $X$ is a CW complex such that for any $k \geq 1$ the $k$-skeleton $X^{(k)}$ of $X$ is a finite CW complex. Moreover, $\sigma$ restricts to $X^{(k)}$. It follows from part (i) of the proposition that $H^*(P(S,X^{(k)});\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(X^{(k)};\mathbb{Z}_2)$, which establishes the isomorphism (6). The last assertion follows readily from (6). \hfill $\square$

### 3.1. Sphere bundles.

Let $X = S^n, n \geq 1$, and let $\sigma : S^n \to S^n$ be the reflection map that sends $e_{n+1} \to -e_{n+1}$ and pointwise fixes $S^{n-1} \subset \mathbb{R}^n = \{e_{n+1}\}^\perp$. The cell structure on $S^n$ with one 0-cell $d_0 = \{e_1\}$ and one (closed) $n$-cell $d_n = S^n$ is stable by $\sigma$. With $(S,\alpha)$ as in Proposition 3.1, we have $H^*(P(S,S^n);\mathbb{Z}_2) \cong H^*(Y;\mathbb{Z}_2) \otimes H^*(S^n;\mathbb{Z}_2)$ which is a free $A := H^*(Y;\mathbb{Z}_2)$-module of rank 2. We shall determine $H^*(P(S,S^n);\mathbb{Z}_2)$ as an $A$-algebra. The two cases $n \geq 2$ and $n = 1$ need to be treated separately.

**Proposition 3.3.** Let $n \geq 2$. Suppose that $S$ satisfies the hypothesis of Proposition 3.1. Let $A = H^*(Y;\mathbb{Z}_2)$. Then $H^*(P(S,S^n);\mathbb{Z}_2) \cong A[u]/\langle u^2 \rangle$ as an $A$-algebra where $\deg u = n$.

**Proof.** We first prove the claim when $S = S^m$, then extend it to the case $S = S^\infty$. The general case will be shown to follow from the case $S = S^\infty$.

Let $S = S^m$, $\alpha = -id$ so that $Y = \mathbb{R}P^m$. The Proposition is obvious when $m < n$ and so we assume that $m \geq n$. We shall denote $P(S^m, S^n)$ by $P_{m,n}$. One has an inclusion $P_{k,n} \subset P_{m,n}$ for $0 \leq k \leq m$, where $k = 0$ corresponds to the fibre inclusion $S^n \hookrightarrow P_{m,n}$.

Let $y$ be the non-zero element in $H^1(P_{m,n};\mathbb{Z}_2) \cong H^1(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Then $A = H^*(\mathbb{R}P^m;\mathbb{Z}_2) = \mathbb{Z}_2[y]/\langle y^{m+1} \rangle$. Observe that $y$ is Poincaré dual to the submanifold $P_{n-1,n} \hookrightarrow P_{m,n}$. Also, one may obtain $P_{k,n}$ as the intersection of $(m-k)$-copies of $P_{m-1,n}$ in general position and so $y^{m-k}$ is the Poincaré dual of $P_{k,n}$. In particular the Poincaré dual of the fibre $S^n \hookrightarrow P_{m,n}$ equals $y^m$. 
Any \( x \in \mathbb{S}^{n-1} = \text{Fix}(\sigma) \) defines a cross-section \( s : \mathbb{R}P^{m} \to P_{m,n} \) of the sphere bundle projection \( P_{m,n} \to \mathbb{R}P^{m} \). Any two such sections are isotopic since \( \mathbb{S}^{n-1} \) is connected. (Here we use the hypothesis that \( n \geq 2 \).) We set \( u = u_{n} \) to be the Poincaré dual of the submanifold \( s : \mathbb{R}P^{m} \to P_{m,n} \). Since the intersection \( s(\mathbb{R}P^{m}) \cap \mathbb{S}^{n} \) is transverse and is exactly one point, we see that \( y^{m}u \in H^{m+n}(P_{m,n}; \mathbb{Z}_{2}) \cong \mathbb{Z}_{2} \) is non-zero. Also, taking two distinct fixed points \( x, x' \in \mathbb{S}^{n-1} \), we obtain cross-sections \( s, s' \) where \( s(\mathbb{R}P^{m}) \cap s'(\mathbb{R}P^{m}) = \emptyset \). This shows that \( u_{n}^{2} = 0 \), completing the proof of the proposition in this case.

Next, let \( S = \mathbb{S}^{\infty} \). Choose \( m > 2n \) so that the inclusion \( j : P(S^{m}, S^{n}) \hookrightarrow P(\mathbb{S}^{\infty}, S^{n}) \) is an \((m-1)\)-equivalence. Let \( U_{n} \in H^{n}(P(S^{m}, S^{n}); \mathbb{Z}_{2}) \cong H^{n}(P(S^{m}, S^{n}); \mathbb{Z}_{2}) \) be the element that corresponds to \( u_{n} \). Since \( u_{n}^{2} = 0 \) and since \( j^{*} \) is an isomorphism in dimension \( 2n \), it follows that \( U_{n}^{2} = 0 \). Thus taking \( u = U_{n} \), the claim holds for \( S = \mathbb{S}^{\infty} \).

In the general case of \((S, \alpha)\), we have a classifying map \( \bar{f} : Y \to \mathbb{R}P^{\infty} \) which classifies the double covering \( S \to Y \). Let \( f : S \to \mathbb{S}^{\infty} \) be a lift of \( \bar{f} \). Then we obtain a morphism of \( S^{n}\)-bundles \( F : P(S, S^{n}) \to P(\mathbb{S}^{\infty}, S^{n}) \) that covers \( \bar{f} \) by functoriality. Let \( u_{n} = F^{*}(U_{n}) \). Since \( H^{*}(P(\mathbb{S}^{\infty}, S^{n}); \mathbb{Z}_{2}) \to H^{*}(P(S, S^{n}); \mathbb{Z}_{2}) \) is a ring homomorphism, we have \( u_{n}^{2} = 0 \). Setting \( u := u_{n} \), the Claim follows.

We turn to the case \( n = 1 \). Note that when \( m = 1 \), \( P_{m,1} \) is the Klein bottle. It can be seen that, as in the special case of the Klein bottle, \( P_{m,1} \) is a connected sum \( \mathbb{R}P^{m+1} \# \mathbb{R}P^{m+1} \). To see this, we let \( J^{+}, J^{-} \subset S^{1} \subset \mathbb{R}^{2} \) to be the closed arcs with end points \( e_{2}, -e_{2} \) where \( e_{1} \in J^{+}, -e_{1} \in J^{-} \). Note that \( \sigma \) stabilizes \( J^{+} \) and \( J^{-} \). Then \( P(S^{m}, J^{+}), P(S^{m}, J^{-}) \subset P_{m,n} \) are twisted \( I \)-bundles over \( \mathbb{R}P^{m} \) with common boundary \( S^{m} \). The projections of the \( I \)-bundle restricted to \( S^{m} \) is the double cover \( S^{n} \to \mathbb{R}P^{m} \). Since \( \mathbb{R}P^{m+1} \) is got by attaching an \( m \)-cell via the double cover \( S^{m} \to \mathbb{R}P^{m} \), it follows that \( P_{m,1} \) is the connected sum \( \mathbb{R}P^{m+1} \# \mathbb{R}P^{m+1} \). Hence \( H^{*}(P_{m,1}) \cong \mathbb{Z}_{2}[a, b]/\langle a^{m+2}, b^{m+2}, ab \rangle \).

The projection of the circle bundle \( \pi : P_{m,1} \to \mathbb{R}P^{m} \) induces \( \pi^{*} : H^{*}(\mathbb{R}P^{m}; \mathbb{Z}_{2}) \to H^{*}(P_{m,1}; \mathbb{Z}_{2}) \), defined by \( y \mapsto a + b \), where \( y \) is the generator of \( A = H^{*}(\mathbb{R}P^{m}; \mathbb{Z}_{2}) \). (Note that \( y = a + b \in H^{1}(P_{m,n}; \mathbb{Z}_{2}) \) is the only element that satisfies the conditions \( y^{m+1} = 0, y^{m} \neq 0 \).) As an \( A \)-algebra, \( H^{*}(P_{m,1}; \mathbb{Z}_{2}) \) is isomorphic to \( A[a]/\langle a^{2} - ay \rangle \). When \( m = \infty \), \( A \) is a polynomial algebra in \( y \) and we have \( H^{*}(P_{\infty,1}; \mathbb{Z}_{2}) \cong A[a]/\langle a^{2} - ay \rangle \).

When \((S, \alpha)\) is as in Proposition 3.1, \( P(S, S^{1}) = Y^{+} \cup Y^{-} \) where \( Y^{\pm} = P(S, J^{\pm}) \) are total spaces of twisted \( I \)-bundles \( \gamma^{\pm} \) over \( Y \) where \( Y^{+} \cap Y^{-} = P(S, S^{0}) \cong S \). We have a continuous surjection \( \eta : P(S, S^{1}) \to P(S, J^{+})/S = T(\gamma^{+}) \) where \( T(\gamma^{+}) \) stands for the Thom space of \( \gamma^{+} \). We let \( a \in H^{1}(P(S, S^{1}); \mathbb{Z}_{2}) \) be the image of the Thom class \( a \in H^{1}(T(\gamma^{+}); \mathbb{Z}_{2}) \cong \mathbb{Z}_{2} \) under the homomorphism induced by \( \eta \). Arguing as in the proof of the above proposition, we obtain the following.

**Proposition 3.4.** Let \((S, \alpha)\) be as in Proposition 3.1. Then \( H^{*}(P(S, S^{1}); \mathbb{Z}_{2}) = A[a]/\langle a^{2} - ay \rangle \) where \( \deg a = 1 = \deg y \).

\(\square\)
4. \( \mathbb{Z}_2 \)-cohomology algebra of certain generalized Dold spaces

In this section we determine the cohomology ring of \( P(S,X) \) (i) \( X \) is a torus manifold under certain mild assumptions, which are satisfied by quasi-toric manifolds, (ii) \( X \) is a complex flag manifolds and \( S \) is paracompact. Basic facts concerning torus manifolds that are required for our purposes will be recalled in §4.1.

We begin with families of compact connected smooth manifolds with involutions \( (X, \sigma) \) satisfying the hypotheses of Proposition 3.2. The basic examples are: (a) complex Grassmann manifolds, or more generally, complex flag manifolds \( U(n)/(U(n_1) \times \cdots \times U(n_r)) \) where \( \sum n_j = n \), (b) smooth complete toric varieties, [F] and, as we shall show below, their topological analogues quasi-toric manifolds [DJ] and torus manifolds (satisfying certain additional hypotheses) [MP]. As for examples of \( (S, \alpha) \) satisfying the hypothesis of Proposition 3.1, we may take \( S \) to be oriented (real) Grassmann manifolds \( G_{n,k} = SO(n)/SO(k) \times SO(n-k) \) where \( \alpha: G_{n,k} \rightarrow G_{n,k} \) is the involution which reverses the orientation on each oriented \( k \)-plane in \( G_{n,k} \). Note that when \( k = 1 \), \( \alpha \) is the antipodal map of the sphere \( G_{n,1} = S^{n-1} \). The quotient space \( \mathbb{R}G_{n,k} \) is the real Grassmann manifold which admits a perfect cell-decomposition over \( \mathbb{Z}_2 \). See [MS, Chapter 6].

We begin with the following lemma which will be needed in the sequel.

Let \( X \) be an oriented compact smooth manifold on which the torus \( T \cong (\mathbb{S}^1)^k \) acts smoothly and effectively. Suppose that \( H \subset T \) is a circle subgroup such that \( F := X^H = \{ x \in X \mid t.x = x \ \forall t \in H \} \) is an oriented connected submanifold of codimension 2. Then \( T \) stabilizes \( F \). Let \( \nu \) be the normal bundle of \( F \hookrightarrow X \). We put a Riemannian metric on \( X \) that is invariant under \( T \). Then \( \nu \) is the orthogonal complement of the tangent bundle \( \tau F \) in \( \tau X|_F \). Moreover, \( \nu \) is a \( T \)-equivariant bundle over \( F \). Note that \( H \) acts on \( \nu \) as bundle automorphisms since \( F \) is point-wise fixed by \( H \). Since \( F \) and \( X \) are oriented, so is \( \nu \) and we may (and do) regard \( \nu \) as a complex line bundle.

**Lemma 4.1.** With notations as above, the complex line bundle \( \nu \) is the restriction to \( F \) of a \( T \)-equivariant complex line bundle \( \omega \) over \( X \). Moreover, one has a \( T \)-equivariant cross-section \( s : X \rightarrow E(\omega) \) which vanishes precisely on \( F \). We have

\[
\begin{align*}
    c_1(\omega) &= [F] \in H^2(X; \mathbb{Z}) \\
    \text{where } [F] &\text{ denotes the the cohomology class dual to } F \hookrightarrow X.
\end{align*}
\]

**Proof.** Without the equivariance part, the existence of \( \omega \) and \( s \) were proved in [Sa] and the equality \( c_1(\omega) = [F] \) was deduced. So we need only construct \( \omega \) and \( s \) as \( T \)-equivariant objects. As in the discussion before the statement of the lemma, we put a Riemannian metric on \( X \) that is invariant under \( T \).

Denote by \( \omega_0 \) the pull-back of \( \nu \) to \( D(\nu) \) via \( \pi \) where \( \pi : D(\nu) \rightarrow F \) is the projection of the unit disk bundle associated to \( \nu \). Then \( \omega_0 := \pi^*(\nu) \) is a \( T \)-equivariant complex line bundle which admits an equivariant cross-section \( s_0 : D(\nu) \rightarrow E(\omega_0) \) defined as follows: Recall that \( E(\omega_0) \) is the fibre product \( \{ (v, w) \in D(\nu) \times E(\nu) \mid \pi(v) = \pi(w) \} \subset D(\nu) \times E(\nu) \) and that \( T \) acts on \( E(\omega_0) \) diagonally: \( t.(v, w) = (t.v, t.w) \). We have \( s_0(v) := (v, v) \ \forall v \in D(\nu) \). Note that \( s_0 \) vanishes precisely when \( v = 0 \), i.e., on the zero-cross section \( F \rightarrow D(\nu) \).
Moreover, \( s_0(t.v) = (t.v, t.v) = t.(v, v) = t.s_0(v) \) for all \( v \in D(\nu), t \in T \), showing that \( s_0 \) is \( T \)-equivariant. Let \( S(\nu) = \partial D(\nu) \). We have an isomorphism of complex line bundles \( \phi : E(\omega_0|S(\nu)) \rightarrow S(\nu) \times \mathbb{C} \) defined as \( \phi(v, zv) = (v, z||v||) \) \( \forall v \in S(\nu), z \in \mathbb{C} \). The section \( \phi \circ s_0 \) corresponds to the constant function 1 (i.e., \( \phi \circ s_0(v) = (v, 1) \) \( \forall v \in S(\nu) \)). Since the Riemannian metric on \( X \) is \( T \)-invariant, we have \( \phi(t.(v, zv)) = \phi(t.v, zt.v)) = (t.v, z||t.v||) = t.(v, z) = t.\phi(v, zv) \forall (v, w) \in E(\omega_0), t \in T \). This shows that, with the trivial \( T \)-action on \( \mathbb{C} \) understood, \( \phi \) is \( T \)-equivariant.

Let \( N \subset X \) be an equivariant tubular neighbourhood of \( F \subset X \) which is \( T \)-equivariantly diffeomorphic to \( D(\nu) \). (See [B, §2, Chapter VI].) Identifying \( N \) with \( D(\nu) \) via such a diffeomorphism, we obtain a \( T \)-equivariant complex line bundle, again denoted \( \omega_0 \), on \( N \) which restricts to \( \nu \) on \( F \), and a \( T \)-equivariant cross-section \( s_0 : N \rightarrow E(\omega_0) \) which vanishes precisely on \( F \). Moreover, we have an isomorphism \( \phi : E(\omega_0|\partial N) \rightarrow \partial N \times \mathbb{C} \). We glue \( E(\omega_0) \) and \( (X \setminus \text{int}(N)) \times \mathbb{C} \), the total space of the trivial line bundle \( \epsilon_\mathbb{C} \), along \( S(\nu) \times \mathbb{C} \) using \( \phi \) to obtain a \( T \)-equivariant complex line bundle \( \omega \) on \( X \). The section \( s_0 \) extends to a \( T \)-equivariant section \( s : X \rightarrow E(\omega) \) such that \( s|_{X \setminus \text{int}(N)} \) corresponds to the constant function \( x \mapsto 1 \in \mathbb{C} \). In particular, \( s \) vanishes precisely on \( F \).  

4.1. Torus manifolds. The notion of torus manifolds is due to Hattori and Masuda [HM], [M]. We shall use the definition as given in Masuda and Panov [MP]. In fact, we consider a restricted class of torus manifolds (in the sense of Masuda-Panov), namely, those torus manifolds where the torus action is locally standard and the orbit space is a homology polytope. This restricted class itself is a generalization of the notion of quasitoric manifolds due to Davis and Januskiewicz [DJ], where the orbit space is a simple convex polytope. We begin by recalling the basic definitions.

A torus manifold is an even dimensional smooth compact orientable manifold \( X \) on which an \( n \)-dimensional torus \( T \cong U(1)^n \) acts smoothly and effectively with non-empty fixed point set where \( \dim X = 2n \). One says that the \( T \)-action on \( X \) is locally standard if \( X \) is covered by open sets \( \{U\} \) that are equivariantly diffeomorphic to an open subset of \( \mathbb{C}^n \) with the standard \( T \cong U(1)^n \)-action. This means that, for each \( U \), there exists an automorphism \( \psi : T \rightarrow T \) and an embedding \( f : U \rightarrow \mathbb{C}^n \) such that \( f(t.u) = \psi(t)f(u) \forall u \in U, t \in T \). The orbit space \( Q := X/T \) is then an \( n \)-dimensional manifold with corners (i.e., is modelled on \( \mathbb{R}_{>0}^n \)). It turns out that set of \( T \)-fixed points of \( X \) is finite and their images are the vertices of \( Q \). It is easy to see (using the irreducible characters of the isotropy representation) that each \( T \)-fixed point is a connected component of the intersection of exactly \( n \) distinct \( T \)-stable submanifolds each having codimension 2 in \( X \). There are only finitely many \( T \)-stable codimension 2 submanifolds in \( X \), say \( X_i, 1 \leq i \leq m \), and each of these are fixed by certain circle subgroup \( S_i \) of \( T \). These are the characteristic submanifolds of \( X \). Their images, \( Q_i := X_i/T \) in \( Q \) are the facets of \( Q \), which have codimension 1 in \( Q \). The characteristic submanifolds of \( X \) are all orientable and are again torus manifolds under the \( T/S_i \)-action with orbit space \( Q_i \).

A preface of \( Q \) is a non-empty intersection of a facets of \( Q \). A face of \( Q \) is a connected component of a preface. We regard \( Q \) itself as (the improper) face; all the other faces
are proper. $Q$ is said to be face-acyclic if all its faces (including $Q$ itself) are acyclic, i.e., have the integral homology of a point. If $Q$ is face acyclic, then every face contains a vertex of $Q$. If $Q$ is face-acyclic and if every preface is a face, then $Q$ is called a homology polytope. For example, when $X$ is a quasi-toric manifold, then $Q = X/T$ is a simple convex polytope, which is evidently a homology polytope. (A (compact) convex polytope of dimension $n$ is simple if exactly $n$ facets meet at each vertex.)

A characteristic submanifold $X_i \hookrightarrow X$ determines a circle subgroup $S_i \subset T$; namely, the subgroup of $T$ that fixes each point of $X_i$. Choosing an isomorphism $f_i : S^1 \cong S_i$ amounts to choosing a primitive vector $v_i \in \text{Hom}(S^1, T) \cong \mathbb{Z}^n$ whose image equals $S_i$. Note that $v_i$ is determined up to sign corresponding to two isomorphisms $S^1 \rightarrow S_i$ namely $f_i, f_i \circ \iota$ where $\iota(z) = z^{-1} \forall z \in S^1$. The sign is determined once an orientation on $S_i$ is fixed.

We shall denote the group of 1-parameter subgroups $\text{Hom}(S^1, T)$ by $N$ and the group of characters $\text{Hom}(T, S^1)$ by $N^\vee \cong \mathbb{Z}^n$. One has a natural pairing $\langle \cdot, \cdot \rangle : N^\vee \times N \rightarrow \mathbb{Z}$ defined by $u \circ v(z) = z^{\langle u, v \rangle}$.

An omni-orientation of $X$ is a choice of an orientation on $X$ and on each characteristic submanifold $X_i, 1 \leq i \leq m$. The orientations on $X, X_i$ determine a unique orientation on the normal space to $T_xX_i \subset T_xX$ for any $x \in X_i$ which in turn leads to an orientation on $S_i$. This determines a unique primitive vector $v_i \in N$ whose image is $S_i$.

Fix an omni-orientation on $X$ and assume that $Q$ is a homology polytope. Denote by $Q$ the set of all facets of $Q$. We obtain map $\Lambda : Q \rightarrow N$ defined as $\Lambda(Q_i) = v_i, 1 \leq i \leq m$. Local standardness of the $T$ action implies that $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_n})$ is a $\mathbb{Z}$-basis of $N$ whenever $Q_{i_1}, \ldots, Q_{i_n} \in Q$ meet at a vertex of $Q$. The map $\Lambda$ is called the characteristic function of $X$. The pair $(Q, \Lambda)$ determines $X$ up to equivariant homeomorphism assuming the vanishing of $H^2(Q; \mathbb{Z})$. In fact, let $X(Q, \Lambda)$ denote the space $T \times Q/\sim$ where $(t, q) \sim (t', q')$ if and only if $q = q'$ and $t^{-1}t'$ belongs to the subgroup of $T$ generated by the one parameter subgroups $\Lambda(Q_{i_1}), \ldots, \Lambda(Q_{i_k})$ where $q$ is in the interior of the face $Q_{i_1} \cap \cdots \cap Q_{i_k}$. Then $X(Q, \Lambda)$ is a a smooth manifold on which $T$ acts on the left with orbit space $Q$. Further $Q$ is naturally embedded in $X(Q, \Lambda)$ via the map $q \mapsto [1, q]$. We regard $Q$ as a subspace of $X$. The quotient map $X(Q, \Lambda) \rightarrow Q$ is therefore a retraction. It was shown in [MP, Lemma 4.5], assuming the vanishing of $H^2(Q; \mathbb{Z})$, that $X(Q, \Lambda)$ is equivariantly homeomorphic to $X$. We identify $X$ with $X(Q, \Lambda)$.

Let $\sigma : X(Q, \Lambda) \rightarrow X(Q, \Lambda)$ be the involution $[t, q] \mapsto [t^{-1}, q]$. (It is readily verified that this definition is meaningful.) Then $X^\sigma = \{[t, q] \mid t^2 \in S_q, \forall q \in Q\}$, which is the analogue of a small cover in the context of quasi-toric manifolds; see [DJ]. Suppose that $X_i$ is a characteristic submanifold of $X$, fixed by a subgroup $S_i \cong S^1$ of $T$. Let $q_i$ be in the interior of $Q_i$ (i.e., $q_i \in Q_i$ but not in any proper face of $Q_i$). Then $S_i$ is the isotropy subgroup at $q_i$, and $X_i$ is the closure of the $T$-orbit of $Q_i$. Since $\sigma(Q_i) = Q_i$, it follows that $\sigma(X_i) = X_i.$
Lemma 4.2. (i) The bundle map \( \hat{\pi} \) is orientation reversing on \( \nu_i \).

(ii) The involution \( \tilde{\sigma}_{i,0} : E(\nu_i,0) \to E(\nu_i,0) \) is a complex conjugation that covers \( \sigma|_{N_i} \).

Proof. (i) It suffices to show that \( \sigma \) is orientation reversing on the fibre of \( N_i \) over a point \( q_i \in Q_i \). Since any neighbourhood of \( Q_i \) meets the interior of \( Q \), we see that \( N_i \cap \text{int}(Q) \neq \emptyset \). Let \( q \in N_i \cap \text{int}(Q) \). Since \( \sigma(q) = q \), and since \( \sigma \) is fibre preserving on \( N_i \to X_i \), it follows that \( q \) is in the fibre \( D_{q_i} \) over a \( q_i \in Q_i \). Let \( t_0 \in S_i \) be the unique order two element. Then \( [t_0, q] \in D_{q_i} \) and \( \sigma([t_0, q]) = [t_0, q] \neq [1, q] = q \) and, moreover, no other point in the \( H_i \)-orbit of \( q \) is fixed by \( \sigma \). This shows that \( \sigma \) is the reflection of the disk \( D_{q_i} \) about the ‘diameter’ of the disk \( D_{q_i} \) through \( q \). Hence \( \sigma \) is orientation reversing.

(ii) Fixing orientation on \( \nu_i \), we obtain a reduction of the structure group of \( \nu_i \) to \( SO(2) \cong U(1) \), making \( \nu_i \) a complex line bundle. Since \( T\sigma|_{E(\nu_i)} \) preserves the Euclidean metric on \( \nu_i \), and it is an involution, we conclude that it is a complex conjugation covering \( \sigma|_{X_i} \). The same argument applied to the pull-back of \( \nu_i \) via the projection \( \pi_i : N_i \to X_i \) of the disk bundle shows that \( (p, v) \mapsto (\sigma(p), T_{\pi_i(p)}\sigma(v)) \) is a complex conjugation of \( \pi_i^*(\nu_i) = \omega_{i,0} \) covering \( \sigma|_{N_i} : N_i \to N_i \).

Taking \( F = X_i \subset X \), a characteristic submanifold in Lemma 4.1, we obtain a \( T \times \langle \sigma \rangle \)-equivariant complex line bundle \( \omega_i \) over \( X \) that extends the normal bundle \( \nu_i \) over \( X_i \), and a \( T \)-equivariant cross-section \( s_i : X \to E(\omega_i) \) which vanishes precisely on \( X_i \). In fact \( \omega_i|_{N_i} = \omega_{i,0} \) and \( \omega_i|_{X \setminus \text{int}(N_i)} = \varepsilon_{\mathbb{C}} \), the trivial complex line bundle.

We claim that the complex conjugation \( \tilde{\sigma}_{i,0} \) on \( \omega_{i,0} \) extends to a complex conjugation on \( \omega_i \) that covers \( \sigma : X \to X \). Explicitly, we let \( \tilde{\sigma} \) to be the standard complex conjugation on the trivial bundle over \( (X \setminus \text{int}(N_i)) \). It remains to show that, under the identification of \( \omega_{i,0}|_{\partial N_i} \) with the trivial bundle via the cross-section \( s_0 : N_i \to E(\omega_i) \), the restriction \( \tilde{\sigma}_{i,0}|_{\partial N_i \times \mathbb{C}} \) is the same as the standard complex conjugation on \( E(\omega_{i,0}|_{\partial N_i}) \). This is clear since \( s_0|_{\partial N_i} \) corresponds to the constant function \( x \mapsto 1 \) as noted in the proof of Lemma 4.1. Thus we have proved the following.

Lemma 4.3. With the above notations, one has a \( \sigma \)-conjugation \( \hat{\sigma} : E(\omega_i) \to E(\omega_i) \) for each \( 1 \leq i \leq m \). Moreover, the Chern class \( c_1(\omega_i) \in H^2(X; \mathbb{Z}) \) equals the cohomology class Poincaré dual to \( X_i \hookrightarrow X \), i.e., \( c_1(\omega_i) = [X_i] \in H^2(X; \mathbb{Z}) \). \( \square \)
In view of Lemma 4.3 and the fact that the cohomology algebra $H^*(X; \mathbb{Z}_2)$ is generated by the classes $[X_i], 1 \leq i \leq m$, the hypotheses of Proposition 3.2 are satisfied. This leads to a description of the chomology $H^*(P(S, X); \mathbb{Z}_2)$ as a module over $H^*(Y; \mathbb{Z}_2)$ for any pair $(S, \alpha)$ where $S$ is a paracompact space and $Y = S/\mathbb{Z}_2$. Our aim is to describe $H^*(P(S, X); \mathbb{Z}_2)$ as an $H^*(Y; \mathbb{Z}_2)$-algebra in terms of generators and relations.

Recall from [MP] that the integral cohomology ring $H^*(X; \mathbb{Z})$ is the quotient of the polynomial algebra $\mathbb{Z}[x_1, \ldots, x_m]$ modulo the ideal $I$ generated by the following two types of elements:

(i) $x_{j_1} \cdots x_{j_r} = 0$ whenever $Q_{j_1} \cap \cdots \cap Q_{j_r} = \emptyset$,

(ii) $\sum_{1 \leq j \leq m} \langle u, v_j \rangle x_j = 0 \ \forall u \in \text{Hom}(T, S^1)$,

where $v_j = \Lambda(Q_j) \in \mathbb{N}$. The element $x_j$ corresponds to $[X_j] \in H^2(X; \mathbb{Z})$. In particular, $X$ satisfies the hypothesis of Proposition 3.2 and we have $H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y, \mathbb{Z}_2) \otimes H^*(X; \mathbb{Z}_2)$.

For $u \in \mathbb{N}^N$, consider the complex line bundle $\omega_u := \otimes_{1 \leq j \leq m} \omega_j^{a_j}$ where $a_j = \langle u, v_j \rangle \in \mathbb{Z}$. Then $\omega_u$ is isomorphic to the trivial complex line bundle since $c_1(\omega_u) = \sum a_j c_1(\omega_j) = \sum a_j[X_j] = 0$ in view of the relation (ii) above. Using the fact that $\bar{\eta} \cong \text{Hom}_C(\eta, \epsilon_C)$ and $\eta \otimes \epsilon_1$ are $\sigma$-conjugate complex vector bundles when $\eta, \epsilon_1$ are $\sigma$-conjugate vector bundles yields that $\omega_u$ is a $\sigma$-conjugate line bundle. (See [NS, Example 2.2(iv)].) In fact, in the case of $\bar{\eta}$, the $\sigma$-conjugation $\tilde{\sigma}$ of $\eta$ is also a $\sigma$-conjugation of $\bar{\eta}$.

We have a $T \times \mathbb{Z}_2$-equivariant cross-section $s_j : X \to E(\omega_j)$ whose zero locus equals $X_j$. We write $s_j$ to denote the corresponding cross-section of $\omega_j^{a_j}$ for $a \geq 1$; when $a < 0$, we set $s^a_j$ as $s_j^{[a]}$, regarded as a section of $\omega_j^{[a]}$. (Note that $E(\omega) = E(\bar{\omega})$.) When $a = 0$, $\omega_0$ is the trivial bundle and $s_j^0$ corresponds to the constant map $X \to \mathbb{C}$ to $x \mapsto 1$.

Let $\tilde{x}_j := w_2(\tilde{\omega}_j) \in P(S, X)$ where $\tilde{\omega}_j := P(S, \omega_j)$. Then $\sum b_j \tilde{x}_j$ restricts to $\sum b_j x_j = w_2(\otimes \omega_j^{b_j})$ along the fibres of the $X$-bundle $P(S, X) \to Y$. Let $p \in \text{Fix}(\sigma)$. Denote by $s_p$ the cross section $Y \to P(S, X)$ defined as $[v] \mapsto [v, p]$. Since $H^1(X; \mathbb{Z}) = 0$, we have $H^2(P(S, X); \mathbb{Z}_2) \cong H^2(Y; \mathbb{Z}_2) \oplus H^2(X; \mathbb{Z}_2)$ by Proposition 3.2.

Claim: $s_p^*(\tilde{x}_j) = 0$ in $H^2(Y; \mathbb{Z}_2)$.

To see this, note that $s_p : Y \to P(S, X)$ factors as follows: $Y \xrightarrow{\tilde{\omega}} P(S, p) \hookrightarrow P(S, X)$. Now $\tilde{\omega}_j|_{P(S, p)} = \xi_0 \oplus \epsilon_\mathbb{R}$ since $\omega_j|_{P(S, p)} \cong \epsilon_\mathbb{C} = \{x_0\} \times \mathbb{C}$ (with standard complex conjugation). So $s_p^*(\tilde{x}_j) = w_2(\xi_0 \oplus \epsilon_\mathbb{R}) = 0$, as claimed.

It follows from the above Claim that $w_2(\otimes \omega_j^{b_j}) = \sum b_j \tilde{x}_j$. Taking $b_j = \langle u, v_j \rangle$ and using the isomorphism $P(S, \otimes \omega_j^{b_j}) \cong P(S, \omega_u) \cong P(S, \epsilon_C)$ where the trivial complex line bundle has the standard conjugation, we obtain that, for any $u \in \mathbb{N}^N$,

$$\sum_{1 \leq j \leq m} \langle u, v_j \rangle \tilde{x}_j = 0. \quad (7)$$

Next, suppose that $\cap_{1 \leq q \leq r} Q_{j_q} = \emptyset$. The Whitney sum $\omega := \oplus_{1 \leq q \leq r} \omega_{j_q}$ admits a cross-section $s : X \to E(\omega)$ given by $s(x) = (s_{j_1}(x), \ldots, s_{j_r}(x))$. Clearly $s$ vanishes along
$\cap_{1 \leq q \leq r} X_{j_q} = \emptyset$, that is, $s$ is nowhere vanishing and so we obtain a splitting $\omega \cong \eta \oplus \epsilon_C$. The $\sigma$-conjugations on each summand $\omega_{j_q}$ of $\omega$ put together yields a $\sigma$-conjugation on $\omega$. Since $s$ is $T \rtimes \mathbb{Z}_2$-equivariant, the $\sigma$-cojugation on $\omega$ restricts to $\sigma$-conjugations on $\eta$ and $\epsilon_C$, and, on the latter it is the standard conjugation. It follows that $\tilde{\omega} = \oplus_{1 \leq q \leq r} \tilde{\omega}_{j_q} = \hat{\eta} \oplus \epsilon_R \oplus \xi_\alpha$.

Hence the top Stiefel-Whitney class of $\tilde{\omega}$ is zero. That is,

$$\prod_{1 \leq q \leq r} \tilde{x}_{j_q} = 0 \text{ whenever } Q_{j_1} \cap \cdots \cap Q_{j_r} = \emptyset. \quad (8)$$

Let $A = H^*(Y; \mathbb{Z}_2)$ and let $A[\tilde{x}_1, \ldots, \tilde{x}_m]$ denote the polynomial algebra in the indeterminates $\tilde{x}_1, \ldots, \tilde{x}_m$. As a consequence of (7) and (8) we obtain the following.

**Theorem 4.4.** Let $X = X(Q, \Lambda)$ be a $T$-torus manifold where $X/T = Q$ is a homology polytope with $m$ facets. Let $\sigma : X \to X$ be the involution $[t, q] \mapsto [t^{-1}, q]$. Then, with the above notations, $H^*(P(S, X); \mathbb{Z}_2)$ is isomorphic, as an $A = H^*(Y; \mathbb{Z}_2)$-algebra, to the quotient $R(Q, \Lambda) := A[\tilde{x}_1, \ldots, \tilde{x}_m]/I$ where the ideal $I = I(Q, \Lambda)$ is generated by the following two types of elements:

(i) $\sum_{1 \leq j \leq m} (u, v_j) \tilde{x}_{j}$, $u \in \mathbb{N}^\vee$, and,

(ii) $\prod_{1 \leq q \leq r} \tilde{x}_{j_q}$ whenever $Q_{j_1} \cap \cdots \cap Q_{j_r} = \emptyset$.

The isomorphism is given by $\tilde{x}_j \mapsto w_2(\tilde{\omega}_j) \in H^2(P(S, X); \mathbb{Z}_2)$.

**Proof.** From the description of the cohomology ring of $X$ and the definition of $R(Q, \Lambda)$ it is clear that one has an isomorphism $R(Q, \Lambda) \cong H^*(X; A) \cong A \otimes_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2)$ of graded $A$-modules (by the universal coefficient theorem). In particular $R(Q, \Lambda)$ is a free $A$-module of rank equal to $\dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2) < \infty$. In fact, any $\mathbb{Z}_2$-basis consisting of monomials in $w_2(\omega_j)$ lifts an $A$-basis for $R(Q, \Lambda)$ got by replacing $w_2(\omega_j)$ by $\tilde{x}_j$.

We have a well-defined $A$-algebra homomorphism $\theta : R(Q, \Lambda) \to H^*(P(S, X); \mathbb{Z}_2)$ defined by $\tilde{x}_j \mapsto w_2(\tilde{\omega}_j), 1 \leq j \leq m$, in view of Equations (7) and (8). By Proposition 3.2, we see that $\theta$ is a surjective homomorphism of $A$-modules. By the observation made above, as an $A$-module homomorphism, $\theta$ maps an $A$-basis to an $A$-basis and hence is an isomorphism. \hfill \Box

### 4.2. Grassmann manifolds and related spaces.

In this section our aim is to describe the $\mathbb{Z}_2$-cohomology ring of generalized Dold manifold $P(S, G_{n,k})$, which is fibre by the complex Grassmann manifold $X := G_{n,k} = G_k(\mathbb{C}^n)$. The fixed point set of the (usual) complex conjugation $\sigma$ on $G_{n,k}$ is the real Grassmann manifold $\mathbb{R}G_{n,k}$. Let $\xi = \xi_\alpha$ denote the line bundle associated to the double cover $S \to Y$. We shall also denote by $\xi$ the line bundle over $P(S, G_{n,k})$ obtained as the pull back $\pi^*(\xi)$ via the projection $\pi : P(S, X) \to Y$ of the $G_{n,k}$-bundle. Denote by $\gamma_{n,k}, \beta_{n,k}$ the tautological $k$-plane bundle and its orthogonal complement bundle, which is of rank $(n - k)$. Note that the complex conjugation on $\mathbb{C}^n$ yields $\sigma$-conjugations on $\gamma_{n,k}$ and on $\beta_{n,k}$. Moreover, we have an isomorphism

$$\gamma_{n,k} \oplus \beta_{n,k} \cong \epsilon_C^n.$$

This yields an isomorphism of real vector bundles: (cf. [NS, Example 2.4(ii)])

$$\hat{\gamma}_{n,k} \oplus \hat{\beta}_{n,k} \cong n\xi_\alpha \oplus n\epsilon_R \quad (9)$$
where \( \hat{\omega} := P(S, \omega) \). Consequently, the following relation among the Stiefel-Whitney classes holds, where \( y = w_1(\xi_\alpha) \):
\[
w(\dot{\gamma}_{n,k}) \cdot w(\dot{\beta}_{n,k}) = (1 + y)^n.
\]
We rewrite the above relation in terms of Stiefel-Whitney polynomials:
\[
w(\dot{\beta}_{n,k}, t) = (1 + yt)^n \cdot w(\dot{\gamma}_{n,k}, t)^{-1} = \sum_{j \geq 0} a_j t^j = a(t)
\]
where \( a_j := a_j(y, w_1(\dot{\gamma}_{n,k}), \ldots, w_j(\dot{\gamma}_{n,k})) \) is the homogeneous polynomial of (total) degree \( j \) in \( (1 + y)^n, w(\dot{\gamma}_{n,k})^{-1} \).

Since \( H^1(P(S, G_{n,k}); \mathbb{Z}_2) \cong H^1(Y; \mathbb{Z}_2) = \mathbb{Z}_2 y \), it is clear that \( w_1(\dot{\gamma}_{n,k}), w_1(\dot{\beta}_{n,k}) \in \mathbb{Z}_2 y \). Moreover, \( w_1(\dot{\gamma}_{n,k}) = 0 \) (resp. \( w_1(\dot{\beta}_{n,k}) = 0 \)) if and only if \( k \) is odd (resp. \( n - k \) is even) as a consequence of Equation (4) (or by a direct argument). Hence we see that, \( w_1(\dot{\beta}_{n,k}) = ny + w_1(\dot{\gamma}_{n,k}) \in \mathbb{Z}_2 y \). This also follows from the above equation. Using Equation (4) and induction, we see that the Stiefel-Whitney class \( w_j(\dot{\beta}_{n,k}) = a_j \) is expressible as a polynomial in \( y, w_2(\dot{\gamma}_{n,k}), \ldots, w_j(\dot{\gamma}_{n,k}) \) for all \( j \) where \( i = [j/2] \). We note that, by degree considerations, \( a_j \) is divisible by \( y \) when \( j \) is odd.

We may view (10) as defining \( w_j(\dot{\beta}_{n,k}), 1 \leq j \leq 2(n - k) \), as the polynomial \( a_{2j} \). Since \( w_j(\dot{\beta}_{n,k}) = 0 \) for \( j > 2n - 2k \), Equation (10) leads to the relations
\[
a_j = a_j(y, w_2(\dot{\gamma}_{n,k}), \ldots, w_{2k}(\dot{\gamma}_{n,k})) = 0, j > 2n - 2k.
\]

Let \( I \subset \mathbb{Z}_2[y, w_2(\dot{\gamma}_{n,k}); 1 \leq i \leq k] \) be the ideal generated by \( a_j, j > 2n - 2k \). Suppose that the height of \( y \) equals \( N \in \mathbb{N} \). Then \( (1 + yt)^{-1} = \sum_{0 \leq j \leq N} y^j t^j \) and we have \( a(t)(\sum_{0 \leq j \leq N} y^j t^j)w(\dot{\gamma}_{n,k}, t) = 1 \). It follows that \( a_{N+2n+j} \) is in the ideal generated by \( a_{2n-2k+i}, 1 \leq i \leq N + 2k \), for all \( j \geq 1 \) and so \( I \) is generated by \( a_{2n-2k+i}, 1 \leq i \leq N + 2k \). Moreover, it is easily seen that \( a_{2n+i} \) is in the ideal generated by \( y, a_j, 2n - 2k + 1 \leq j \leq 2n \) for all \( i \geq 1 \).

Consider the graded polynomial algebra \( R := \mathbb{Z}_2[y, \hat{w}_2; 1 \leq j \leq k] \) in the indeterminates \( y, \hat{w}_2, 1 \leq j \leq k \), where \( \deg y = 1 \), \( \deg \hat{w}_2 = 2j \). We regard \( a_{2j} = a_{2j}(y, \hat{w}_2, \ldots, \hat{w}_2) \) as elements of \( R \).

**Lemma 4.5.** The elements \( y, a_{2j} \in R, n - k < j \leq n \), form a regular sequence in \( R \) and \( R/\langle y, a_{2j}, n - k < j \leq n \rangle \cong H^*(G_{n,k}; \mathbb{Z}_2) \).

**Proof.** To see this, it suffices to show that \( \tilde{a}_{2j}, n - k < j \leq n \) is a regular sequence in \( \tilde{R} := R/\langle y \rangle \) where \( \tilde{a}_{2j} := a_{2j} \mod \langle y \rangle \tilde{R} \). Note that \( \tilde{a}_{2j} = h_{2j}(\tilde{w}) \) where \( h_{2j}(\tilde{w}) \) denotes the complete symmetric polynomial of degree \( 2j \) in \( \tilde{w}_2, \ldots, w_{2k} \), that is, \( h_{2j} \) equals the coefficient of \( t^{2j} \) in \( (1 + \hat{w}_2 t^2 + \cdots + \hat{w}_{2k} t^{2k})^{-1} \). One can show, as in [BT, §23], that \( \tilde{a}_{2j}, n - k < j \leq n \), is a regular sequence in \( \tilde{R} \) and that \( \tilde{R}/\langle a_{2j}, n - k < j \leq n \rangle \cong H^*(G_{n,k}; \mathbb{Z}_2) \).

It follows from the above lemma that \( y^{m+1}, a_{2j}, n - k < j \leq n \), is a regular sequence in \( R \) and that the quotient \( R/\langle y^{m+1}, a_{2j}, n - k < j \leq n \rangle \) is isomorphic to \( \mathbb{Z}_2[y]/\langle y^{m+1} \rangle \otimes
$H^*(G_{n,k}; \mathbb{Z}_2) = H^*(\mathbb{R}P^m; \mathbb{Z}_2) \otimes H^*(G_{n,k}; \mathbb{Z}_2)$ as graded $\mathbb{Z}_2$-vector spaces. We are ready to prove the following proposition. Recall that $\xi_\alpha = \xi$ denotes the pull-back of the Hopf line bundle over $\mathbb{R}P^m$ via the projection of the $X$-bundle $P(S^m, X) \to \mathbb{R}P^m$.

**Proposition 4.6.** With the above notations, the $\mathbb{Z}_2$-cohomology algebra $H^*(P(S^m, G_{n,k}); \mathbb{Z}_2)$ is isomorphic to $R/I$ where $I$ is the ideal generated by $y^{m+1}, a_{2j}(\hat{w}), n - k < j \leq n$, where $\hat{w}_{2j}$ corresponds to $w_{2j}(\hat{\gamma}_{n,k})$ and $y$ to $w_1(\xi)$.

*Proof.* Consider the homomorphism $\eta : R \to H^*(P(S^m, G_{n,k}); \mathbb{Z}_2)$ of rings defined as $\eta(\hat{w}_{2j}) = w_{2j}(\hat{\gamma}_{n,k})$ and $\eta(y) = w_1(\xi)$. By Equation (10) and Proposition 3.2, $\eta$ is surjective. It follows from Equation (11) that $\eta$ factors as $R \to R/I \xrightarrow{\bar{\eta}} H^*(P(S^m, G_{n,k}); \mathbb{Z}_2)$. By the discussion preceding the statement of the proposition, we see that $\bar{\eta}$ is an isomorphism since $R/I$ and $H^*(P(S^m, G_{n,k}); \mathbb{Z}_2) = H^*(\mathbb{R}P^m; \mathbb{Z}_2) \otimes H^*(G_{n,k}; \mathbb{Z}_2)$ have the same dimension. □

Next we prove the following theorem.

**Theorem 4.7.** We keep the above notations. Suppose that $S$ is paracompact. The cohomology algebra $H^*(P(S, G_{n,k}); \mathbb{Z}_2)$ is isomorphic, as an $H^*(Y; \mathbb{Z}_2)$-algebra, to $H^*(Y; \mathbb{Z}_2)[\hat{w}_2, \ldots, \hat{w}_{2k}]$, where $I$ is generated by $a_{2j}, n - k < j \leq n$, under an isomorphism that maps $\hat{w}_{2j}$ to $w_{2j}(\hat{\gamma}_{n,k})$.

*Proof.* Set $\mathcal{R} := H^*(Y; \mathbb{Z}_2)[\hat{w}_2, \ldots, \hat{w}_{2k}]$. In view of Equation (11), it is clear that we have a surjective $H^*(Y; \mathbb{Z}_2)$-algebra homomorphism $\mathcal{R}/I \to H^*(P(S, X); \mathbb{Z}_2)$ where $\hat{w}_{2j}$ maps to $w_{2j}(\hat{\gamma}_{n,k})$. Therefore it suffices to show that $\mathcal{R}/I$ is isomorphic to $H^*(P(S, X); \mathbb{Z}_2)$ as an $H^*(Y; \mathbb{Z}_2)$-module.

Let $A$ denote the $\mathbb{Z}_2$-subalgebra of $H^*(Y; \mathbb{Z}_2)$ generated by $y = w_1(\xi)$ and let $R = A[\hat{w}_{2j}; 1 \leq j \leq k]$. Then $\mathcal{R} = H^*(Y; \mathbb{Z}_2) \otimes_A R$. Let $I \subset R$ denote the ideal generated by $a_{2j}, n - k < j \leq n$. Then $\mathcal{R}/I \cong H^*(Y; \mathbb{Z}_2) \otimes_A (R/I)$.

Suppose that $(S, \alpha) = (S^m, -id)$. By Proposition 4.6, we have $R/I \cong H^*(P(S^m; G_{n,k}); \mathbb{Z}_2) \cong A \otimes H^*(G_{n,k}; \mathbb{Z}_2)$. So $\mathcal{R}/I \cong H^*(Y; \mathbb{Z}_2) \otimes_A A \otimes H^*(G_{n,k}; \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(G_{n,k}; \mathbb{Z}_2)$.

Next we suppose that $(S, \alpha) = (S^\infty, -id)$. The inclusion $S^m \hookrightarrow S^\infty$ defines an inclusion $j_m : P(S^m, X) \hookrightarrow P(S^\infty, X)$ which induces a $H^*(Y; \mathbb{Z}_2)$-algebra homomorphism in cohomology. Moreover, $j_m^*$ is an isomorphism up to dimension $m - 1$. From this observation it follows that the theorem holds for $(S^\infty, -id)$. In the general case, the result follows from functoriality and the fact that the $X$-bundle $P(S, X)$ arises as a pull-back of the $X$-bundle $P(S^\infty, X)$ via a classifying map $S \to \mathbb{R}P^\infty$. This completes the proof. □

Proposition 4.6 and Theorem 4.7 are valid when the Grassmann manifold is replaced by complex flag manifolds. More precisely, let $\nu := n_1, \ldots, n_r$ be an increasing sequence of positive numbers and let $n = \sum_{1 \leq j \leq r} n_j$. Denote by $F_\nu$ the complex flag manifold whose elements are complex vector subspaces $\mathcal{U} := (U_1, \ldots, U_r)$ of $\mathbb{C}^n$ where $U_i \perp U_j$ for $i \neq j$ and $\dim U_j = n_j, 1 \leq j \leq r$. Then $F_\nu \cong U(n)/(U(n_1) \times \cdots \times U(n_r))$ has a natural structure of a complex manifold given by the usual inclusion of $U(n) \subset GL_n(\mathbb{C})$.
so that $F_\nu \cong GL_n(\mathbb{C})/P_\nu$ where $P_\nu$ is the subgroup which is block upper-triangular, where the diagonal sizes are $n_1, \ldots, n_r$. (Under this identification, $U$ corresponds to the sequence $U_1 \subset U_1 + U_2 \subset \cdots \subset U_1 + \cdots + U_r = \mathbb{C}^n$.) It is well-known that $F_\nu$ has a CW-structure given by Schubert cells which are all even dimensional. The Schubert cells are obtained as $B$-orbits of $T$-fixed points where $T \subset GL_n(\mathbb{C})$ is the diagonal subgroup and $B \subset P_\nu$ is the group of upper triangular matrices. Their closures are the Schubert varieties in $F_\nu$. The complex conjugation in $\mathbb{C}^n$ induces a complex conjugation $\sigma$ on $F_\nu$, and, moreover, $\sigma$ stabilizes each Schubert variety. The fixed points of $\sigma$ is the real flag manifold $\mathbb{R}F_\nu \cong O(n)/(O(n_1) \times \cdots \times O(n_r))$ consisting of $\underline{U}$ where $U_j \cap \mathbb{R}^n$ is $n_j$-dimensional for all $j$.

We denote by $\gamma_{\nu,j}$ (or more briefly $\gamma_j$), the complex vector bundle over $F_\nu$ of rank $n_j$ whose fibre over $\underline{U}$ is the vector space $U_j$. This is the pull-back of the bundle $\gamma_{n,n_j}$ on $G_{n,n_j}$ via the projection $F_\nu \rightarrow G_{n,n_j}$ that sends $\underline{U}$ to $U_j$. The complex conjugation $\sigma$ leads to a $\sigma$-conjugation on $\hat{\sigma}_j$ of $\gamma_j$. Also one has a natural isomorphism of vector bundles

$$\bigoplus_{1 \leq j \leq r} \gamma_j \cong n_\xi \mathbb{C}$$

(12)

which respects $\sigma$-conjugation, as in the case of Grassmann manifolds. The integral cohomology ring of $F_\nu$ is generated by $c_{i,j} := c_i(\gamma_j), 1 \leq i \leq n_j, 1 \leq j \leq r$, where the only relations among the $c_{i,j}$ are generated by the following (inhomogeneous) relation: $\prod_{1 \leq j \leq r} c(\gamma_j) = 1$. It follows that $w_{2i,j} = w_{2i}(\gamma_j), 1 \leq i \leq n_j, 1 \leq j \leq r$, generate $H^*(F_\nu; \mathbb{Z}_2)$ and the relations among these generators are all consequences of $\prod_{1 \leq j \leq n_r} w(\gamma_j, t) = 1$.

Suppose that $(S, \alpha)$ is a paracompact space where $\alpha$ is a fixed point free involution. Then we have the real vector bundles $\tilde{\gamma}_j = P(S, \gamma_j)$ over the generalized Dold space $P(S, F_\nu)$. One has the following isomorphism of real vector bundles, resulting from the isomorphism (12):

$$\bigoplus_{1 \leq j \leq r} \tilde{\gamma}_j \cong n_\xi \alpha \oplus n_\xi \mathbb{R}$$

where $\xi_\alpha$ is the real line bundle associated to the double cover $S \times F_\nu \rightarrow P(S, F_\nu)$. Therefore we obtain

$$w(\tilde{\gamma}_r, t) = (1 + yt)^n, \prod_{1 \leq j < r} w(\tilde{\gamma}_j, t)^{-1}.$$  

(13)

Then $H^*(P(S, F_\nu); \mathbb{Z}_2)$ is isomorphic to $H^*(Y; \mathbb{Z}_2) \otimes H^*(F_\nu; \mathbb{Z}_2)$ as an $H^*(Y; \mathbb{Z}_2)$-module, by Proposition 3.2. Arguing as in the proof of Theorem 4.7, we observe that $H^*(P(S, F_\nu); \mathbb{Z}_2)$ is generated by as $H^*(Y; \mathbb{Z}_2)$-algebra by $\tilde{\omega}_{2i,j} := \tilde{w}_{2i}(\tilde{\gamma}_j), 1 \leq i \leq n_j, 1 \leq j < r$. Using Equation (13), we obtain a regular sequence $a_{2s}, n_r < s \leq n$, in the polynomial algebra $R_\nu = \mathbb{Z}_2[\tilde{w}_{2i,j} \mid 1 \leq i \leq n_j, 1 \leq j \leq r]$ that correspond to the coefficient of $t^{2s}$ in $(1 + ty)^n(\prod \tilde{w}_{2i+1})$ (rewriting $\tilde{w}_{2i+1}$ in terms of $y, \tilde{w}_{2i}, l \leq i$, using Equation (4)).

We set $R_\nu$ to the polynomial algebra over $H^*(Y; \mathbb{Z}_2)$ generated by ‘indeterminates’ $\tilde{w}_{2i,j}, 1 \leq i \leq n_j, 1 \leq j < r$, and let $I_\nu$ be the ideal of $R_\nu$ generated by the elements
\[ a_{2i} = a_{2i}(y^2, \hat{w}_{2i,j}), \; n_r < i \leq n. \] Then \( a_{2i}, \; n_r < i \leq n, \) form a regular sequence in \( \mathcal{R}_\nu. \) The proof of the following theorem is analogous to that of Theorem 4.7.

**Theorem 4.8.** Suppose that \((S, \alpha)\) is a paracompact space with a fixed point free involution \(\alpha\) and let \(\nu = n_1, n_2, \ldots, n_r\) be a sequence of positive numbers. With notations as above, we have an isomorphism \( \mathcal{R}_\nu / \mathcal{I}_\nu \rightarrow H^*(P(S, F_\nu); \mathbb{Z}_2) \) of \( H^*(Y; \mathbb{Z}_2) \)-algebras defined by \( \hat{w}_{2i,j} \mapsto w_{2i}(\hat{\gamma}_j). \)

We remark that, setting \( k = n - n_r, \) the projection \( \pi : F_\nu \rightarrow G_{n,k} \) defined as \( U \mapsto U_1 + \cdots + U_{r-1} \) is \( \mathbb{Z}_2 \)-equivariant and pulls back \( \gamma_{n,k} \) (resp. \( \beta_{n,k} \)) to \( \bigoplus_{1 \leq j < \gamma_j} \) (resp. \( \gamma_j \)). Hence \( P(S, \pi)^*(\hat{\gamma}_{n,k}) = \bigoplus_{1 \leq j < \gamma_j} \hat{\gamma}_j. \) Moreover, using Proposition 3.2 and the fact that \( \pi^* : H^*(G_{n,k}; \mathbb{Z}_2) \rightarrow H^*(F_\nu; \mathbb{Z}_2) \) is a monomorphism, we see that \( P(S, \pi) : P(S, F_\nu) \rightarrow P(S, G_{n,k}) \) induces a monomorphism in \( \mathbb{Z}_2 \)-cohomology.

We conclude this section with the following remark.

**Remark 4.9.** Let \( \sigma : S \rightarrow S \) be a fixed point free involution. Let \( X \hookrightarrow F_\nu \) be a Schubert variety in a complex flag manifold \( F_\nu. \) As had already been commented, \( X \) is stable by complex conjugation \( \sigma \) on \( F_\nu \) and \( X^\sigma = X \cap \mathbb{R}F_\nu \) is non-empty. Moreover, \( X \) admits a cell-decomposition having cells only in even dimensions, where the cells are the Schubert cells of \( F_\nu \) contained in \( X. \) Hence the inclusion \( X \hookrightarrow F_\nu \) induces a surjection \( H^*(F_\nu; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z}). \) It follows that the (mod 2) cohomology of \( X \) is generated by Chern classes (mod 2) of complex vector bundles on \( X. \) So Proposition 3.2 is applicable to the \( X \)-bundle \( P(S, X) \rightarrow Y \) and we have \( H^*(P(S, X); \mathbb{Z}_2) \cong H^*(Y; \mathbb{Z}_2) \otimes H^*(X) \) as \( H^*(Y; \mathbb{Z}_2) \)-modules. As for the \( H^*(Y; \mathbb{Z}_2) \)-algebra structure, it is determined by the surjection \( H^*(P(S, F_\nu); \mathbb{Z}_2) \rightarrow H^*(P(S, X); \mathbb{Z}_2). \) We omit the details.

### 4.3. Equivariant cohomology of \((X, \sigma)\)

As an application of Theorems 4.4, 4.8 we obtain the \( \mathbb{Z}_2 \)-equivariant cohomology \( H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \) of \((X, \sigma)\) when \( X \) is either a torus manifold whose torus quotient is a homology polytope, or, is a complex flag manifold \( F_\nu. \) At least in the case of complex flag manifolds, this result is perhaps known to experts but we could not find an explicit reference.

When \( S = S^\infty \) with antipodal action, the space \( P(S^\infty, X) \) is identical to the Borel construction \( S^\infty \times_{\mathbb{Z}_2} X \) (since \( S^\infty \) is contractible). Therefore the equivariant cohomology algebra \( H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \) equals \( H^*(P(S^\infty, X); \mathbb{Z}_2). \) When \( H^*(X; \mathbb{Z}_2) \) is generated by mod 2 reduction of Chern classes of finitely many \( \sigma \)-conjugate vector bundles \((\omega_j, \sigma_j),\) Proposition 3.2 is applicable and we obtain that \( H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \) is isomorphic to \( \mathbb{Z}_2[y] \otimes H^*(X; \mathbb{Z}_2) \) as a \( H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[y] \)-module. The inclusion \( S^m \hookrightarrow S^\infty \) induces an inclusion \( P(S^m, X) \hookrightarrow P(S^\infty, X) \) which is an \( (m - 1) \)-equivalence. It follows that \( H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \cong H^i(P(S^\infty, X); \mathbb{Z}_2) \rightarrow H^i(P(S^m, X); \mathbb{Z}_2) \) induced by inclusion is an isomorphism for all \( i < m. \) Therefore \( H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \) is isomorphic to the inverse limit of graded \( \mathbb{Z}_2 \) algebras \( \{H^*(P(S^m, X); \mathbb{Z}_2)\}_{m \geq 2}. \) As an illustration, we obtain the following result as an immediate consequence of Theorems 4.4 and 4.8.
Theorem 4.10. We keep the above notations. 
(i) Let $X = X(Q, \Lambda)$ be a $T$-torus manifold where $Q = X/T$ is a homology polytope. Then $H^*_\mathbb{Z}_2(X; \mathbb{Z}_2)$ is isomorphic to the $A$-algebra $R(Q, \Lambda)$ where $A = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[y]$. 
(ii) Let $\nu = n_1 < \cdots < n_r, n = \sum n_j$. Then $H^*_\mathbb{Z}_2(F_\nu; \mathbb{Z}_2)$ is isomorphic to $R_\nu/I_\nu$ where $R_\nu = A[\hat{w}_{2i,j}; 1 \leq i \leq n_j, 1 \leq j < r]$ where $A = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[y]$ and $I_\nu \subset R_\nu$ is the ideal generated by the $a_{2i} \in R_\nu, n_r < i \leq n$. □

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