Abstract. In this note, we extend earlier work by showing that if $X$ and $Y$ are delta-complexes (i.e. simplicial sets without degeneracy operators), a morphism $g: N(X) \to N(Y)$ of Steenrod coalgebras (normalized chain-complexes equipped with extra structure) induces one of 2-skeleta $\hat{g}: X_2 \to Y_2$, inducing a homomorphism $\pi_1(\hat{g}): \pi_1(X) \to \pi_1(Y)$ that is an isomorphism if $g$ is an isomorphism. This implies a corresponding conclusion for a morphism $g: C(X) \to C(Y)$ of Steenrod coalgebras on unnormalized chain-complexes of simplicial sets.

1. Introduction

It is well-known that the Alexander-Whitney coproduct is functorial with respect to simplicial maps. If $X$ is a simplicial set, $C(X)$ is the unnormalized chain-complex and $R S_2$ is the bar-resolution of $\mathbb{Z}_2$ (see [1]), it is also well-known that there is a unique homotopy class of $\mathbb{Z}_2$-equivariant maps (where $\mathbb{Z}_2$ transposes the factors of the target)

$$\xi_X: R S_2 \otimes C(X) \to C(X) \otimes C(X)$$

cohomology, and that this extends the Alexander-Whitney diagonal. We will call such structures, Steenrod coalgebras and the map $\xi_X$ the Steenrod diagonal.

With some care (see appendix A of [3]), one can construct $\xi_X$ in a manner that makes it functorial with respect to simplicial maps although this is seldom done since the homotopy class of this map is what is generally studied. The paper [3] showed that:

Corollary. [3.8] If $X$ and $Y$ are simplicial complexes (simplicial sets without degeneracies whose simplices are uniquely determined by their vertices), any purely algebraic chain map of normalized chain complexes

$$f: N(X) \to N(Y)$$

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that makes the diagram

(1.1) \[
\begin{array}{c}
\xi_X \\
N(X) \otimes N(X) \xrightarrow{\xi_Y f \otimes f} N(Y) \otimes N(Y)
\end{array}
\]

commute induces a map of simplicial complexes

\[\hat{f}: X \to Y\]

If \( f \) is an isomorphism then \( \hat{f} \) is an isomorphism of simplicial complexes — and \( X \) and \( Y \) are homeomorphic.

The note extends that result, slightly, to

**Corollary.** If \( X \) and \( Y \) are delta-complexes, any morphism of their canonical Steenrod coalgebras (see proposition 3.2)

\[ g: N(X) \to N(Y) \]

induces a map

\[ \hat{g}: X_2 \to Y_2 \]

of 2-skeleta. If \( g \) is an isomorphism then \( X_2 \) and \( Y_2 \) are isomorphic as delta-complexes.

and

**Corollary.** If \( X \) and \( Y \) are simplicial sets and \( f: C(X) \to C(Y) \) is a morphism of their canonical Steenrod coalgebras (see proposition 3.2) over their unnormalized chain-complexes, then \( f \) induces a map

\[ \hat{f}: X_2 \to Y_2 \]

of 2-skeleta. If \( f \) is an isomorphism, then \( \hat{f} \) is a homotopy equivalence.

The author conjectures that the last statement can be improved to “if \( f \) is an isomorphism, then \( \hat{f} \) is a homotopy equivalence.”

The author is indebted to Dennis Sullivan for several interesting discussions.

2. **Definitions and Assumptions**

Given a simplicial set, \( X \), \( C(X) \) will always denote its unnormalized chain-complex and \( N(X) \) its normalized one (with degeneracies divided out).

We consider variations on the concept of simplicial set.
Definition 2.1. Let $\Delta_+$ be the ordinal number category whose morphisms are order-preserving monomorphisms between them. The objects of $\Delta_+$ are elements $n = \{0 \to 1 \to \cdots \to n\}$ and a morphism

$$\theta : m \to n$$

is a strict order-preserving map $(i < k \implies \theta(i) < \theta(j))$. Then the category of delta-complexes, $D$, has objects that are contravariant functors $\Delta_+ \to \text{Set}$ to the category of sets. The chain complex of a delta-complex, $X$, will be denoted $N(X)$.

Remark. In other words, delta-complexes are just simplicial sets without degeneracies.

A simplicial set gives rise to a delta-complex by “forgetting” its degeneracies — “promoting” its degenerate simplices to nondegenerate status. Conversely, a delta-complex can be converted into a simplicial set by equipping it with degenerate simplices in a mechanical fashion. These operations define functors:

Definition 2.2. The functor $f : S \to D$ is defined to simply drop degeneracy operators (degenerate simplices become nondegenerate). The functor $d : D \to S$ equips a delta complex, $X$, with degenerate simplices and operators via

$$d(X)_m = \bigsqcup_{m-n} X_n$$

for all $m > n \geq 0$.

Remark. The functors $f$ and $d$ were denoted $F$ and $G$, respectively, in [2]. Equation 2.1 simply states that we add all possible degeneracies of simplices in $X$ subject only to the basic identities that face- and degeneracy-operators must satisfy.

Although $f$ promotes degenerate simplices to nondegenerate ones, these new nondegenerate simplices can be collapsed without changing the homotopy type of the complex: although the degeneracy operators are no longer built in to the delta-complex, they still define contracting homotopies.

The definition immediately implies that
Proposition 2.3. If \( X \) is a simplicial set and \( Y \) is a delta-complex, 
\[ C(X) = N(f(X)), \ N(\partial(Y)) = N(Y), \text{ and } C(X) = N(\partial \circ f(X)). \]

Theorem 1.7 of [2] shows that there exists an adjunction:

\[ \partial: \mathcal{D} \leftrightarrow \mathcal{S}: f \]

The composite (the counit of the adjunction)

\[ f \circ \partial: \mathcal{D} \to \mathcal{D} \]

maps a delta complex into a much larger one — that has an infinite number of (degenerate) simplices added to it. There is a natural inclusion

\[ \iota: X \to f \circ \partial(X) \]

and a natural map (the unit of the adjunction)

\[ g: \partial \circ f(X) \to X \]

The functor \( g \) sends degenerate simplices of \( X \) that had been “promoted to nondegenerate status” by \( f \) to their degenerate originals — and the extra degenerates added by \( \partial \) to suitable degeneracies of the simplices of \( X \).

In [2], Rourke and Sanderson also prove:

Proposition 2.4. If \( X \) is a simplicial set and \( Y \) is a delta-complex then

1. \( |Y| \) and \( |\partial Y| \) are homeomorphic
2. the map \( |g|: |\partial \circ f(X)| \to |X| \) is a homotopy equivalence.
3. \( f: \mathcal{H}\mathcal{S} \to \mathcal{H}\mathcal{D} \) defines an equivalence of categories, where \( \mathcal{H}\mathcal{S} \) and \( \mathcal{H}\mathcal{D} \) are the homotopy categories, respectively, of \( \mathcal{S} \) and \( \mathcal{D} \). The inverse is \( \partial: \mathcal{H}\mathcal{D} \to \mathcal{H}\mathcal{S} \). In particular, if \( X \) is a simplicial set, the natural map

\[ g: \partial \circ f(X) \to X \]

is a homotopy equivalence.

Remark. Here, \( |*| \) denotes the topological realization functors for \( \mathcal{S} \) and \( \mathcal{D} \).

Proof. The first two statements are proposition 2.1 of [2] and statement 3 is theorem 6.9 of the same paper. The final statement follows from Whitehead’s theorem. □
3. **Steenrod Coalgebras**

We begin with:

**Definition 3.1.** A Steenrod coalgebra, \((C, \delta)\) is a chain-complex \(C \in \text{Ch}\) equipped with a \(\mathbb{Z}_2\)-equivariant chain-map

\[
\delta: R\mathbb{S}_2 \otimes C \to C \otimes C
\]

where \(\mathbb{Z}_2\) acts on \(C \otimes C\) by swapping factors and \(R\mathbb{S}_2\) is the bar-resolution of \(\mathbb{Z}\) over \(\mathbb{Z}\mathbb{S}_2\). A morphism \(f: (C, \delta_C) \to (D, \delta_D)\) is a chain-map \(f: C \to D\) that makes the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow{\delta_C} & & \downarrow{\delta_D} \\
C \otimes C & \xrightarrow{f \otimes f} & D \otimes D
\end{array}
\]

commute.

Steenrod coalgebras are very general — the underlying coalgebra need not even be coassociative. The category of Steenrod coalgebras is denoted \(\mathcal{S}\).

Appendix A of [3] shows that:

**Proposition 3.2.** If \(X\) is a simplicial set or delta-complex, then the unnormalized and normalized chain-complexes of \(X\) have a natural Steenrod coalgebra structure, i.e. natural maps

\[
\xi: R\mathbb{S}_2 \otimes N(X) \to N(X) \otimes N(X) \\
\xi: R\mathbb{S}_2 \otimes C(X) \to C(X) \otimes C(X)
\]

**Remark.** If \([\cdot]\) is the 0-dimensional generator of \(R\mathbb{S}_2\), the map \(\xi([\cdot] \otimes \cdot): N(X) \to N(X) \otimes N(X)\) is nothing but the Alexander-Whitney coproduct.

The Steenrod coalgebra structure for \(N(X)\) is a natural quotient of that for \(C(X)\).

Here are some computations of this Steenrod coalgebra structure from appendix A of [3]:

**Fact.** If \(\Delta^2\) is a 2-simplex, then

\[
(3.1) \quad \xi([\cdot] \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2
\]

— the standard (Alexander-Whitney) coproduct — and
\[(3.2) \quad \xi([1, 2]) \otimes \Delta^2) = \Delta^2 \otimes F_0 \Delta^2 - F_1 \Delta^2 \otimes \Delta^2 - \Delta^2 \otimes F_2 \Delta^2 \]

Corollary 4.3 of [3] proves that:

**Corollary 3.3.** Let \( X \) be a simplicial set and suppose
\[
f: N^n = N(\Delta^n) \to N(X)\]
is a Steenrod coalgebra morphism. Then the image of the generator \( \Delta^n \in N(\Delta^n)_n \) is a generator of \( N(X)_n \) defined by an \( n \)-simplex of \( X \).

We can prove a delta-complex (partial) analogue of corollary 4.5 in [3]:

**Corollary 3.4.** Let \( X \) be a delta-complex, let \( n \leq 2 \), and let
\[
f: N(\Delta^n) \to N(X)\]
map \( \Delta^n \) to a simplex \( \sigma \in N(X) \) defined by the simplicial-map \( \iota: \Delta^n \to X \). Then \( f = N(\iota) \).

**Proof.** Let
\[
\xi_i = \xi(e_i \otimes \ast) : N(\Delta^n) \to N(\Delta^n) \otimes N(\Delta^n)\]
denote the Steenrod coalgebra structure, where \( e_i \) is the generator of \( (RS_2)_i \). By hypothesis, the diagram
\[
\begin{array}{ccc}
N(\Delta^n) & \xrightarrow{1 \otimes f} & N(X) \\
\downarrow \xi_i & & \downarrow \xi_i \\
N(\Delta^n) \otimes N(\Delta^n) & \xrightarrow{f \otimes f} & N(X) \otimes N(X)
\end{array}
\]
commutes for all \( i \geq 0 \).

If \( \iota \) is an inclusion (and \( n \) is arbitrary), the conclusion follows from corollary 4.5 in [3]. If \( n = 1 \), and \( \iota \) identifies the endpoints of \( \Delta^1 \), there is a unique morphism from \( N(\Delta^1) \) to \( \text{im} \ N(\iota) \) that sends \( N(\Delta^1)_1 \) to \( \text{im} \ N(\iota)_1 \).

If \( n = 2 \), equation [3.1] implies that
\[
\text{im}(\xi_0(\Delta^2)) = F_2 \Delta^2 \otimes F_0 \Delta^2 \in (N(X)/N(X)_0) \otimes (N(X)/N(X)_0)
\]
Since corollary 3.4 implies that \( f(\Delta^2)_2 = N(\iota)(\Delta^2)_2 \), it follows that the Steenrod-coalgebra morphism, \( f \), must send \( F_i \Delta^2 \) to \( N(\iota)(F_i \Delta^2) \) for \( i = 0, 2 \).
Equation 3.2 implies that
\[ \text{im}(\xi_1(\Delta^2)) = -F_1\Delta^2 \otimes \Delta^2 \in N(X)_1 \otimes (N(X)/N(X)_1) \]
so that \( f(F_1\Delta^2) = N(\iota)(F_1\Delta^2) \) as well. \( \square \)

We define a complement to the \( N(*) \)-functor:

**Definition 3.5.** Define a functor
\[ \text{hom}_{\mathcal{S}}(\bullet, *) : \mathcal{S} \to \mathcal{D} \]
to the category of delta-complexes (see definition 2.1), as follows:
If \( C \in \mathcal{S} \), define the \( n \)-simplices of \( \text{hom}_{\mathcal{S}}(\bullet, C) \) to be the Steenrod coalgebra morphisms
\[ N^n \to C \]
where \( N^n = N(\Delta^n) \) is the normalized chain-complex of the standard \( n \)-simplex, equipped with the Steenrod coalgebra structure defined in.

Face-operations are duals of coface-operations
\[ d_i : [0, \ldots, i-1, i+1, \ldots, n] \to [0, \ldots, n] \]
with \( i = 0, \ldots, n \) and vertex \( i \) in the target is not in the image of \( d_i \).

**Proposition 3.6.** If \( X \) is a delta-complex there exists a natural inclusion
\[ u_X : X \to \text{hom}_{\mathcal{S}}(\bullet, N(X)) \]

**Remark.** This is also true if \( X \) is an arbitrary simplicial set.

**Proof.** To prove the first statement, note that any simplex \( \Delta^k \) in \( X \) comes equipped with a map
\[ \iota : \Delta^k \to X \]
The corresponding order-preserving map of vertices induces an Steenrod-coalgebra morphism
\[ N(\iota) : N(\Delta^k) = N^k \to N(X) \]
so \( u_X \) is defined by
\[ \Delta^k \mapsto N(\iota) \]
It is not hard to see that this operation respects face-operations. \( \square \)

So, \( \text{hom}_{\mathcal{S}}(\bullet, N(X)) \) naturally contains a copy of \( X \). The interesting question is whether it contains more than \( X \):

**Theorem 3.7.** If \( X \in \mathcal{D} \) is a delta-complex then the canonical inclusion
\[ u_X : X \to \text{hom}_{\mathcal{S}}(\bullet, N(X)) \]
defined in proposition 3.6 is the identity map on 2-skeleta.
Proof. This follows immediately from corollary 3.3 which implies that simplices map to simplices and corollary 3.4 which implies that these maps are unique. □

Corollary 3.8. If $X$ and $Y$ are delta-complexes, any morphism of their canonical Steenrod coalgebras (see proposition 3.2)

$$g: N(X) \to N(Y)$$

induces a map

$$\hat{g}: X_2 \to Y_2$$

of 2-skeleta. If $g$ is an isomorphism then $X_2$ and $Y_2$ are isomorphic as delta-complexes.

Proof. Any morphism $g: N(X) \to N(Y)$ induces a morphism of simplicial sets

$$\text{hom}(\star, g): \text{hom}_{\mathcal{S}}(\star, N(X)) \to \text{hom}_{\mathcal{S}}(\star, N(Y))$$

which is an isomorphism (and homeomorphism) of simplicial complexes if $g$ is an isomorphism. The conclusion follows from theorem 3.7 which implies that $X_2 = \text{hom}(\star, N(X))_2$ and $Y_2 = \text{hom}(\star, N(Y))_2$. □

Propositions 2.3 and 2.4 imply that

Corollary 3.9. If $X$ and $Y$ are simplicial sets and $f: C(X) \to C(Y)$ is a morphism of their canonical Steenrod coalgebras (see proposition 3.2) over their unnormalized chain-complexes, then $f$ induces a map

$$\hat{f}: X_2 \to Y_2$$

of 2-skeleta. If $f$ is an isomorphism, then $\hat{f}$ is a homotopy equivalence.

Proof: Simply apply corollary 3.8 to $f(X)$ and $f(Y)$ and then apply $\delta$ and proposition 2.4 to the map

$$\hat{f}: f(X)_2 \to f(Y)_2$$

that results. □

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