A Closed Queueing Maintenance Network with Two Batch Policies

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Abstract

This paper discusses a maintenance network with failed items that can be removed, repaired, redistributed, and reused under two batch policies: one for removing the failed items from each base to a maintenance shop and the other for redistributing the repaired items from the maintenance shop to bases. This maintenance network can be considered a virtual closed queueing network, and the Markov system of each node is described as an elegant block-structured Markov process whose stationary probabilities can be computed by the RG-factorizations. The structure of this maintenance network is novel and interesting. To compute the closed queueing network,
we set up a new nonlinear matrix equation to determine the relative arrival rates, in which the nonlinearity comes from two different groups of processes: the failure and removal processes and the repair and redistribution processes. This paper also extends a simple queueing system of a node to a more general block-structured Markov process which can be computed by the RG-factorizations. Based on this, the paper establishes a more general product-form solution for the closed queueing network and provides performance analysis of the maintenance network. Our method will open a new avenue for quantitative evaluation of more general maintenance networks.

**Keywords:** Stochastic processes; Maintenance system; Closed queueing network; RG-factorization; Block-structured Markov process

1 Introduction

With the development of the sharing economy, various sharing systems are prevalent; for example, there are bike-sharing systems, car-sharing systems, shared power banks, umbrella-sharing systems, shared sleep warehouses, etc. These sharing systems provide customers with convenience and value of service. However, with the operation of these systems, maintenance problems become critical for the sustainable development. Since stations or sites are scattered throughout a city, the removal, repair, and redistribution processes are labor-consuming. Finding the efficient removing and reusing process is vital to these sharing systems.

To perform a quantitative analysis, this paper develops a closed queueing maintenance network from these sharing systems. It is found that the structure of the closed queueing network is novel and interesting. Due to the complicated and special structure, the modeling and analysis of the maintenance network is quite difficult. To the best of authors’ knowledge, there are no existing methodologies or results along such a research line. We consider two reasonable and practical batch policies, which make the removing and redistributing more efficient, although it introduces the complexity to the model.

For such a maintenance network, there are some interesting practical problems that need to be addressed. First, how to estimate the effect of item failure on the quality of service and on the system performance. Second, how to develop effective removal policies of failed items such that the system performance can further be improved. To answer these questions, this paper derives the product-form solution of the closed queueing network by
solving the non-linear matrix equation system for the relative arrival rates combined with
the RG-factorizations of block-structured Markov processes. Based on the product-form
solution, the performance measures for the maintenance network can be computed.

Comparing with the existing studies in the literature of maintenance networks, our
analysis has the following four key features.

A novel maintenance network: For the maintenance network of sharing systems, items
may fail at any base, failed items are batch-removed from bases to the maintenance shop,
and repaired items are batch-redistributed for reuse from the maintenance shop to bases.
So far, there have been no studies of using analytical models on such a complicated
maintenance network.

A new class of virtual closed queueing networks: Compared with the previous studies
on the closed queueing network Li et al. (2016, 2017a,b). The features of the items’ fail-
ure, removal, repair, distribution, and reuse processes, together with two batch policies,
can substantially change the physical structure of the virtual closed queueing network.
This paper is the first to analyze closed queueing networks with such an interesting and
complicated structure.

The block-structured Markov process: This paper also contributes to the literature of
closed queueing networks by extending and generalizing a simple queueing system (e.g.,
the M/M/1 queue, the M/M/C queue, the M/G/1 queue, and others) of a node to a
more general block-structured Markov process. We show that such a block structure is
established by either items’ failure and batch removal processes in a base or items’ repair
and batch redistribution processes in the maintenance shop.

A nonlinear routing matrix equation: In the theory of closed queueing networks, it
is well known that the relative arrival rates can uniquely be determined by a system
of linear equations $\mathbf{e} P = \mathbf{e}$, where $\mathbf{e}$ is the relative arrival rate vector and $P$ is the
routing matrix, e.g., see Bolch et al. (2006) and Serfozo (2012) for details. However, from
the maintenance network of sharing systems, we find a new fundamental result: The
relative arrival rate vector should be determined by a nonlinear routing matrix equation
$\mathbf{e} P (\mathbf{e}) = \mathbf{e}$, where some entries of the routing matrix $P (\mathbf{e})$ depend on the relative arrival
rate vector. Therefore, the nonlinearity of the routing matrix equation opens a new
research avenue in the study of closed queueing networks.

This paper extends and generalizes the product-form solution of the closed queueing
networks to a more general case that the Markov systems of the nodes are described as
the block-structured Markov processes. The methodology and results given in this paper shed light on the study of more general maintenance networks and open a new research direction.

The remainder of this paper is organized as follows. Section 2 provides a literature review. Section 3 presents a model description for a maintenance network with failed items. Section 4 formulates the maintenance network as a closed queueing network, and provides a detailed analysis for the relative arrival rates, the service processes, the block-structured Markov processes, and the nonlinear routing matrix equation. Section 5 derives the product-form solution of the closed queueing network which is used for performance analysis of the maintenance network. Section 6 concludes the paper with a summary.

2 Literature Review

Our current work is related to three streams of literature: maintenance networks and reliability analysis, closed queueing networks, and the RG-factorizations. We review some related literature in these areas.

Maintenance networks and reliability analysis: Heidergott and Farenhorst-Yuan (2010) studied the gradient estimation for multicomponent maintenance systems with age-replacement policy. You (2019) provided a maintenance policy for maintenance scheduling with the help of both time-based maintenance and condition-based maintenance techniques. Rawat and Lad (2020); Rawat et al. (2020) presented a joint optimization approach of reliability design and level of repair analysis for fleet systems with multi-machine and multi-indenture. Lin et al. (2020) studied the optimal maintenance plan by examining the relationship between facility reliability and lifespan. There has been much research on manufacturing engineering with maintenance and reliability, such as manufacturing systems with repairable machines by Buzacott and Yao (1986) and Gregory (2015), production systems with repairable bases by Li and Meerkov (2008), and parts inventory systems with repairable parts by Park and Lee (2011, 2014). In contrast to the literature, this paper studies a maintenance network with a special and novel structure.

Closed queueing networks: George and Xia (2011) modeled the vehicle rental systems as a closed queueing network to determine the optimal number of parking places in each rental location. Waserhole and Jost (2010); Waserhole et al. (2013) used closed queueing networks, combined with the fluid approximation, to establish Markov decision
models to determine the optimal policy of the bike-sharing system. As a closely related application of closed queueing networks, the sharing systems have drawn much research attention. Important examples include Adelman (2007), George (2012), Fanti et al. (2014), Samet et al. (2018), and so forth. Also, closed queueing networks are used in the research of maintenance systems. Park and Lee (2011, 2014) established a multi-class closed queueing maintenance network model with a parts inventory system. Gross et al. (1983) and Madu (1988) used a closed queueing network to decide the number of repairable items and the capacity of repair depot. This paper models a maintenance network as a closed queueing network and extends some nodes from simple queueing system to general block-structured Markov processes.

**RG-factorizations:** For block-structured Markov processes, Li (2010) provided a unified effective computational framework of the RG-factorizations, which includes stationary performance analysis, transient solution, the first passage times, reward processes, quasi-stationary distribution, and so forth. For further details on RG-factorization and their usefulness in stochastic modeling, readers can refer to Wang et al. (2007), Li et al. (2009), Li and Zhad (2005), Wang et al. (2010), Yu and Alfa (2015), Samanta and Nandi (2021), and Das and Samanta (2021).

This paper studies a maintenance system with a novel structure by applying the RG-factorizations of block-structured Markov processes to the closed queueing networks and contributes to the literature by providing a more general product-form solution.

3 Model Description

In this section, we describe a maintenance network with $N$ bases and one maintenance shop. The $N$ bases are assumed to be different and the capacity of each base is sufficiently large. Items are kept and may fail in the bases. The total number of items in the maintenance network is fixed at $K$. See Figure 1 for a pictorial illustration.

We denote base $i$ as Node $i$, $i = 1, \ldots, N$ and express the maintenance shop as Node 0. There are transfer path nodes between any two nodes, which can be divided into the following two kinds.

(i) *Path nodes between bases.* Let Node $i \rightarrow j$ denote the path node from base $i$ to base $j$. Note that Node $i \rightarrow j$ and Node $j \rightarrow i$ may be different due to practical factors.

Denote all the nodes beginning from base $i$ for $1 \leq i \leq N$ as $R_B(i) = \{\text{Node } i \rightarrow j :$
Figure 1: The physical structure of the maintenance network

\[ j \neq i, 1 \leq j \leq N \}. \] Write all the bases in the downlink of base \( i \) as \( \Theta_i = \{ \text{Node } j : \text{Node } i \rightarrow j \in R_B (i) \} \).

(ii) Path nodes between bases and the maintenance shop. There are two classes of path nodes: (ii-1) the path nodes for removing failed items from a base to the maintenance shop, denoted as \( R_E (0) = \{ \text{Node } i \rightarrow 0 : 1 \leq i \leq N \} \); (ii-2) and the path nodes for redistributing repaired items from the maintenance shop to bases, written as \( R_B (0) = \{ \text{Node } 0 \rightarrow i : 1 \leq i \leq N \} \).

An outside user arrives at base \( i \) in order to rent an item. If there is no usable item (either the base \( i \) is empty or all the items in base \( i \) are failed), then the user immediately leaves the system. If there is at least one usable item in base \( i \), then the user rents an item and gets the service on Node \( i \rightarrow j \) with probability \( p_{i,j} \) for \( j \neq i, 1 \leq j \leq N \) and \( \sum_{j \in \Theta_i} p_{i,j} = 1 \) for each \( i = 1, 2, \ldots, N \). The arrivals of outside users at base \( i \) follow a Poisson process with arrival rate \( \lambda_i > 0 \) for \( 1 \leq i \leq N \). We assume that the service times on Node \( i \rightarrow j \) are i.i.d. and exponential with rate \( \mu_{i,j} > 0 \). We assume that all the items are identical, and the lifetime of an item is exponential with failure rate \( \alpha > 0 \). For the
maintenance problem, we propose the following two batch policies.

A batch removal policy: Once the number of failed items at any base reaches a positive integer $M$, the $M$ failed items are removed in a batch and transported to the maintenance shop. We assume the transportation time on Node $i \rightarrow 0$ is exponentially distributed with rate $\mu_{i,0} > 0$.

A batch redistribution policy: Once the number of repaired items in the maintenance shop reaches a given positive integer $Z$, the $Z$ repaired items are taken in a batch away from the maintenance shop to some bases, in which $Z_i$ repaired items are sent to the base $i$ with $\sum_{i=1}^{N} Z_i = Z$. Let $\beta_i = Z_i / Z$. The transportation time from the maintenance shop to base $i$ is exponentially distributed with rate $\mu_{0,i} > 0$.

If a seriously damaged item is scrapped after a repair, then a new item is added to the system immediately. Thus, the total number of items in the system is constant. We assume that there are $r$ repairmen in the maintenance shop, and the repair time of each failed item is exponentially distributed with rate $w > 0$. For convenience of expression, we assume that $Z = \psi M$, where $\psi$ is a given positive integer. In addition, let $\phi = \lfloor K/M \rfloor$.

We assume that all the random variables mentioned above are independent of each other. The notation is summarized in Table 1.

4 A Virtual Closed Queueing Network

In this section, we describe the maintenance network as a closed queueing network, explain its physical structure, and introduce mathematical notations.

Since the total number of items in the maintenance network is fixed, we can formulate the system as a closed queueing network as follows.

(1) Virtual customers: Since items are either kept in bases, transferred on path nodes, or repaired in the maintenance shop, they can be viewed as virtual customers. The items have two states: usable and failed. We use $G$ ($G$: Good) and $B$ ($B$: Bad) to denote the usable and failed items, respectively.

(2) Virtual nodes: Note that bases, path nodes, and the maintenance shop have different physical attributes such as functions and geographical structures. Thus they are considered as different classes of nodes.

(3) An irreducible path graph: The set of all the virtual nodes in the maintenance network is given by $\Theta = \{\text{Node } i : 0 \leq i \leq N\} \cup \bigcup_{i=1}^{N} R_B (i) \cup R_B (0) \cup R_E (0)$. We assume
Table 1: Summary of Notation

| Symbol | Description |
|--------|-------------|
| $N$    | Number of bases; |
| $K$    | Total number of items; |
| $\Theta_i$ | Set of bases in the downlink of base $i$, for $1 \leq i \leq N$; |
| $\lambda_i$ | User arrival rate at base $i$, for $1 \leq i \leq N$; |
| $\mu_{i,j}$ | Service rate on path node $i \rightarrow j$, for $1 \leq i \leq N, j \in \Theta_i$; |
| $\mu_{i,0}$ | Service rate on path node $i \rightarrow 0$, for $1 \leq i \leq N$; |
| $\mu_{0,i}$ | Service rate on path node $0 \rightarrow i$, for $1 \leq i \leq N$; |
| $p_{i,j}$ | Transition probability from base $i$ to path node $i \rightarrow j$, for $1 \leq i \leq N, j \in \Theta_i$; |
| $\alpha$ | Failure rate of an item; |
| $w$ | Repair rate of an failed item; |
| $r$ | Number of repairmen in the maintenance shop; |
| $M$ | Batch size of removal failed items from any base; |
| $Z$ | Batch size of redistributing repaired items from the maintenance shop; |
| $Z_i$ | Number of repaired items redistributed to base $i$, for $1 \leq i \leq N$; |
| $\beta_i$ | $Z_i/Z$; |
| $\phi$ | $\lfloor K/M \rfloor$ |
| $\psi$ | $Z/M$, a positive integer. |

that all the path nodes of the system are connected as an irreducible path graph whose nodes are in the set $\Theta$. In this case, we call the maintenance network path irreducible.

Let $Q_G^{(i)}(t)$ and $Q_B^{(i)}(t)$ denote the numbers of usable items and failed items kept in base $i$ at time $t \geq 0$ for $1 \leq i \leq N$, respectively; and $Q_G^{(0)}(t)$ and $Q_B^{(0)}(t)$ the numbers of usable items and failed items in the maintenance shop at time $t \geq 0$, respectively. Let $R_{i,j}(t)$ be the number of items transferred on Node $i \rightarrow j$ at time $t$ for $1 \leq i \leq N$ and $j \in \Theta_i$. $R_{i,0}(t)$ and $R_{0,i}(t)$ are numbers of items on Node $i \rightarrow 0$ and Node $0 \rightarrow i$ at time $t$ for $1 \leq i \leq N$, respectively.

Denote the state vector by $X(t) = (L_0(t), L_1(t), L_2(t), \ldots, L_{N-1}(t), L_N(t))$, where $L_0(t) = (Q_G^{(0)}(t))$, $Q_B^{(0)}(t); R_{i,0}(t), R_{0,i}(t) : 1 \leq i \leq N$, and for $1 \leq i \leq N$, $L_i(t) = (Q_G^{(i)}(t), Q_B^{(i)}(t), R_{i,j}(t) : j \in \Theta_i)$. Obviously, $\{X(t) : t \geq 0\}$ is a continuous-time irreducible Markov process. The
state space of the Markov process \( \{X(t) : t \geq 0\} \) is given by

\[
\Omega = \left\{ \mathbf{n} : \sum_{i=0}^{N} (n_{G}^{(i)} + n_{B}^{(i)}) + \sum_{i=1}^{N} \sum_{j \in \Theta_i} m_{i,j} + \sum_{i=1}^{N} (m_{i,0} + m_{0,i}) = K, \right. \\
0 \leq n_{B}^{(i)} \leq M, 0 \leq n_{G}^{(i)}, m_{i,j} \leq K, 0 \leq n_{G}^{(0)}, 0 \leq n_{B}^{(0)} \leq \phi M, \\
\left. n_{G}^{(0)} + n_{B}^{(0)} = kM, m_{i,0} = kM, m_{0,i} = lZ, \text{for } 1 \leq i \leq N, j \in \Theta_i, \right.
\]

where \( \mathbf{n} = (n_{0}, n_{1}, n_{2}, \ldots, n_{N-1}, n_{N}) \), and \( \mathbf{n}_0 = (n_{G}^{(0)}, n_{B}^{(0)}; m_{i,0}, m_{0,i} : 1 \leq i \leq N) \), for \( 1 \leq i \leq N \), \( \mathbf{n}_i = (n_{G}^{(i)}, n_{B}^{(i)}; m_{i,j} : j \in \Theta_i) \).

Note that \( n_{G}^{(i)} \) and \( n_{B}^{(i)} \) are the numbers of usable items and failed items kept in base \( i \) for \( 1 \leq i \leq N \), respectively; \( n_{G}^{(0)} \) and \( n_{B}^{(0)} \) are the numbers of usable items and failed items in the maintenance shop, respectively; \( m_{i,j} \) is the number of items ridden on Node \( i \to j \) for \( 1 \leq i \leq N, j \in \Theta_i \); and \( m_{i,0}, m_{0,i} \) denote the numbers of failed items and repaired items transported on Node \( i \to 0 \) and Node \( 0 \to i \) for \( 1 \leq i \leq N \), respectively. Figure 2 illustrates the physical structure of the closed queueing network.

To study the closed queueing network, we analyze the relative arrival rates in Subsection 4.1, the Markov processes of any base in Subsection 4.2, the Markov processes of the maintenance shop in Subsection 4.3, and the routing matrix in Subsection 4.4.

### 4.1 The relative arrival rates

Note that the relative arrival rates play a key role in deriving the product-form solution of queueing networks, e.g., see Bolch et al. (2006) and Serfozo (2012) for details.

We denote by \( e_i \) the relative arrival rate of Node \( i \) for \( 0 \leq i \leq N \), \( e_{i,j} \) the relative arrival rate of Node \( i \to j \) for \( 1 \leq i \leq N \) and \( j \in \Theta_i \), \( e_{i,0} \) and \( e_{0,i} \) the relative arrival rates of Node \( i \to 0 \) and Node \( 0 \to i \), respectively. Let

\[
E = \{ \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N-1}, \mathbf{e}_N \},
\]

where \( \mathbf{e}_0 = (e_0; e_{i,0}, e_{0,i} : 1 \leq i \leq N) \), and for \( 1 \leq i \leq N \), \( \mathbf{e}_i = (e_i; e_{i,j} : j \in \Theta_i) \).

Note that the relative arrival rates only depend on the physical structure of the closed queueing network, while they are independent of the states of the Markov process \( \{X(t) : t \geq 0\} \).

To determine the relative arrival rates, we first analyze the Markov processes of various bases and of the maintenance shop in Subsections 4.2 and 4.3, respectively. Then we
establish the routing matrix in Subsection 4.4, so that we can set up a nonlinear routing matrix equation to determine the relative arrival rates.

4.2 Markov process of any base

When the relative arrival rates are introduced to various nodes in the closed queueing network, all the nodes are isolated from each other so that the Markov processes of the nodes are independent of each other. Therefore, the Markov system of each node is a block-structured Markov process, which is based on the numbers of usable items and failed items in either any base or the maintenance shop.

To set up the block-structured Markov process of a base, it is necessary to analyze the state transitions of the Markov process for the base. Note that $Q^{(i)}_G(t)$ and $Q^{(i)}_B(t)$ are the numbers of usable items and failed items at base $i$ at time $t$, respectively. Under the batch removal policy of failed items with parameter $M$, it is easy to see that

$$\left\{ (Q^{(i)}_G(t), Q^{(i)}_B(t)) : t \geq 0 \right\}$$

is a two-dimensional Markov process with finite levels and phases, where $Q^{(i)}_G(t)$ is the phase variable while $Q^{(i)}_B(t)$ is the level variable. Figure 3 depicts the state transition relations of the Markov process. Thus the infinitesimal generator
Q of the Markov process is given by

\[
Q = \begin{pmatrix}
Q_{0,0} & Q_{0,1} & & & \\
& Q_{1,1} & Q_{1,2} & & \\
& & \ddots & \ddots & \\
& & & Q_{M-1,M-1} & Q_{M-1,M} \\
Q_{M,0} & & & & Q_{M,M}
\end{pmatrix}.
\]

(1)

All the elements of the infinitesimal generator \( Q \) are given in Appendix A.

Figure 3: State transition relations of the Markov process in base \( i \)

The structure of the matrix \( Q \) is bidiagonal blocks with a special block \( Q_{M,0} \) in the lower-left corner. Obviously, the matrix-geometric method (see Neuts (1994) or Latouche and Ramaswami (1999)) does not apply to such a Markov process. Thus, to compute the stationary probabilities of the Markov process, we need to use the UL-type RG-factorization (see Li (2010) for details). To this end, we write \( Q^{[\leq M]}_{M,M} = Q_{M,M} \), and for \( 1 \leq k \leq M - 1 \),

\[
Q^{[\leq k]} = \begin{pmatrix}
Q_{0,0} & Q_{0,1} & & & \\
& Q_{1,1} & Q_{1,2} & & \\
& & \ddots & \ddots & \\
& & & Q_{k-1,k-1} & Q_{k-1,k} \\
Q_{k,0} & & & & Q_{k,k}
\end{pmatrix}.
\]
Lemma 1 We have $Q_{M,M}^{[\leq M]} = Q_{M,M}$, for $1 \leq k \leq M - 1$,

$$Q_{k,0}^{[\leq k]} = \prod_{l=k}^{M-1} Q_{l,l+1} (-Q_{l+1,l+1})^{-1} Q_{M,0}$$

and

$$Q^{[\leq 0]} = Q_{0,0} + Q_{0,1} (-Q_{1,1})^{-1} \left[ \prod_{l=1}^{M-1} Q_{l,l+1} (-Q_{l+1,l+1})^{-1} \right] Q_{M,0}. \tag{3}$$

The proof is given in Appendix B.

Let

$$\Psi_n = Q_{n,n}^{[\leq n]}, \ 0 \leq n \leq M, \tag{4}$$

$$R_{i,j} = Q_{i,j}^{[\leq j]} (-\Psi_j)^{-1}, \ 0 \leq i < j \leq M \tag{5}$$

and

$$G_{i,j} = (-\Psi_i)^{-1} Q_{i,j}^{[\leq i]}, \ 0 \leq j < i \leq M. \tag{6}$$

The following theorem gives the UL-type RG-factorization of the Markov process $Q$.

Theorem 1 The UL-type RG-factorization of the continuous-time Markov process $Q$ is given by

$$Q = (I - R_U) \Psi_D (I - G_L),$$

where

$$R_U = \begin{pmatrix}
0 & R_{0,1} & & \\
& 0 & R_{1,2} & \\
& & 0 & R_{2,3} \\
& & & \ddots & \ddots \\
& & & & 0 & R_{M-2,M-1} \\
& & & & & 0 & R_{M-1,M} \\
& & & & & & 0
\end{pmatrix},$$

$$\Psi_D = \text{diag}(\Psi_0, \Psi_1, \Psi_2, \ldots, \Psi_{M-1}, \Psi_M),$$
and

\[ G_L = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
G_{1,0} & 0 & \cdots & 0 \\
G_{2,0} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
G_{M-2,0} & 0 & \cdots & 0 \\
G_{M-1,0} & 0 & \cdots & 0 \\
G_{M,0} & 0 & \cdots & 0
\end{pmatrix}. \]

The proof is provided in Appendix C.

Since the Markov process is irreducible, with a finite state space, and satisfies \( Q \mathbb{1} = 0 \) where \( \mathbb{1} \) is a column vector of ones, it must be positive recurrent. Let \( \pi^{(i)}_B = (\pi^{(i)}_{B,0}, \pi^{(i)}_{B,1}, \pi^{(i)}_{B,2}, \ldots, \pi^{(i)}_{B,M}) \) be the stationary probability vector of the Markov process, where

\[
\begin{align*}
\pi^{(i)}_{B,0} &= (\pi^{(i)}_{G,0:B,0}, \pi^{(i)}_{G,1:B,0}, \pi^{(i)}_{G,2:B,0}, \ldots, \pi^{(i)}_{G,K-1:B,0}, \pi^{(i)}_{G,K:B,0}), \\
\pi^{(i)}_{B,1} &= (\pi^{(i)}_{G,0:B,1}, \pi^{(i)}_{G,1:B,1}, \pi^{(i)}_{G,2:B,1}, \ldots, \pi^{(i)}_{G,K-2:B,1}, \pi^{(i)}_{G,K-1:B,1}), \\
& \quad \vdots \\
\pi^{(i)}_{B,M} &= (\pi^{(i)}_{G,0:B,M}, \pi^{(i)}_{G,1:B,M}, \pi^{(i)}_{G,2:B,M}, \ldots, \pi^{(i)}_{G,K-M-1:B,M}, \pi^{(i)}_{G,K-M:B,M}).
\end{align*}
\]

Then using the UL-type RG-factorization, we obtain

\[
\begin{align*}
\pi^{(i)}_{B,0} &= \kappa x_0, \\
\pi^{(i)}_{B,k} &= \pi^{(i)}_{B,k-1} R_{k-1,k}, \quad 1 \leq k \leq M,
\end{align*}
\]

where \( x_0 \) is the stationary probability vector of the censored Markov chain \( \Psi_0 \) to level 0, the scalar \( \kappa \) is uniquely determined by \( \sum_{k=0}^{M} \pi^{(i)}_{B,k} \mathbb{1} = 1 \). Note that the censored Markov process \( Q_{\leq 0} \) has the stationary probability vector \( x_0 \), and thus we have

\[
\begin{align*}
x_0 Q_{\leq 0} &= 0, \\
x_0 \mathbb{1} &= 1.
\end{align*}
\]

### 4.3 Markov process of the maintenance shop

The repair behavior of the maintenance shop can be analyzed by a two-dimensional continuous-time Markov process \( \{ (Q_B^{(0)}(t), Q_G^{(0)}(t)) : t \geq 0 \} \), where \( Q_B^{(0)}(t) \) is the phase variable and \( Q_G^{(0)}(t) \) is the level variable. When the number of repaired items amounts to \( Z \), the \( Z \) repaired items are taken away in batch from the maintenance shop to bases. We assume that the proportion of repaired items redistributed to base \( i \) is \( \beta_i \), where \( \beta_i = Z_i / Z \) for \( 1 \leq i \leq N \) and the transportation time of the \( Z_i \) repaired items is exponential with
rate $\mu_{0,i}$. We write $\mu_0 = \sum_{i=1}^{N} \beta_i \mu_{0,i}$. Note that there are $r$ repairmen at the maintenance shop, and the repair time of each failed item is exponential with repair rate $w$. The repair rate of the maintenance shop is given by

$$\varpi \left(n_B^{(0)}\right) = \min\left\{n_B^{(0)}, r\right\} w,$$

where $n_B^{(0)}$ is the number of failed items in the maintenance shop.

Figure 4 depicts the state transition relations of the Markov process $\{(Q_B^{(0)}(t), Q_G^{(0)}(t)) : t \geq 0\}$ in the maintenance shop. Obviously, this Markov process is irreducible and positive recurrent. The infinitesimal generator $T$ of the Markov process $\{(Q_B^{(0)}(t), Q_G^{(0)}(t)) : t \geq 0\}$ is given by

$$T = \begin{pmatrix}
T_{0,0} & T_{0,1} \\
T_{1,1} & T_{1,2} \\
& \ddots \\
T_{M,M} & T_{M,M+1} \\
T_{M+1,M+1} & T_{M+1,M+2} \\
& \ddots \\
& & & \ddots \\
T_{Z-1,Z-1} & T_{Z-1,Z} \\
T_{Z,Z}
\end{pmatrix},$$

whose block elements are given in Appendix D.

For the two-dimensional Markov process, we write $T_{Z,Z}^{[\leq Z]} = T_{Z,Z}$. For $1 \leq k \leq Z - 1$, we can iteratively get

$$T_{\leq k}^{[\leq]} = \begin{pmatrix}
T_{0,0} & T_{0,1} \\
T_{1,1} & T_{1,2} \\
& \ddots \\
& & \ddots \\
& & & T_{k-1,k-1} & T_{k-1,k} \\
& & & & T_{k,k}
\end{pmatrix},$$

where $\Xi_k = \prod_{l=k}^{Z-1} \left[T_{l,l+1} (-T_{l+1,l+1})^{-1}\right] T_{Z,0}$. Note that $T_{\leq 0}^{[\leq]} = T_{0}^{[\leq]}$, and we obtain

$$T_{\leq 0}^{[\leq]} = T_{0,0} + T_{0,1} (-T_{1,1})^{-1} \left[\prod_{l=1}^{Z-1} T_{l,l+1} (-T_{l+1,l+1})^{-1}\right] T_{Z,0}. \quad (10)$$

Let

$$U_n = T_{n,n}^{[\leq n]}, \quad 0 \leq n \leq Z,$$

$$R_{i,j} = T_{i,j}^{[\leq]} (-U_j)^{-1}, \quad 0 \leq i < j \leq Z.$$
Figure 4: State transition relations of the Markov process in the maintenance shop

and

\[ G_{i,j} = (-U_i)^{-1} T_{i,j}^{[\leq i]}, \ 0 \leq j < i \leq Z. \]

Then the UL-type RG-factorization of the Markov process is given by

\[ T = (I - R_U) U_D (I - G_L), \]

where

\[
R_U = \begin{pmatrix}
0 & R_{0,1} & 0 & R_{1,2} & 0 & R_{2,3} & \cdots & \cdots & 0 & R_{Z-2,Z-1} & 0 \\
0 & 0 & R_{1,2} & 0 & R_{2,3} & \cdots & \cdots & 0 & 0 & R_{Z-1,Z} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & R_{Z-2,Z-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & R_{Z-1,Z} \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[ U_D = \text{diag} (U_0, U_1, \ldots, U_{Z-1}, U_Z), \]
and
\[
G_L = \begin{pmatrix}
0 & & & \\
G_{1,0} & 0 & & \\
G_{2,0} & 0 & & \\
& \ddots & \ddots & \\
G_{Z-1,0} & 0 & & \\
G_{Z,0} & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.
\]

Since the Markov process is irreducible and positive recurrent, there exists the stationary probability vector. Let \( \pi_G^{(0)} = \left( \pi_{G,0}^{(0)}, \pi_{G,1}^{(0)}, \pi_{G,2}^{(0)}, \ldots, \pi_{G,Z-1}^{(0)}, \pi_{G,Z}^{(0)} \right) \) be the stationary probability vector of the Markov process, where
\[
\pi_{G,k}^{(0)} = \begin{cases}
\pi_{B,0;G,k}^{(0)}, & \text{for } k = lM, \ 0 \leq l \leq \psi,
\pi_{B,l+1M-k;G,k}^{(0)}, & \text{for } lM + 1 \leq k < (l+1)M, \ 0 \leq l \leq \psi - 1.
\end{cases}
\]

Then by using the UL-type RG-factorizations, we obtain
\[
\begin{cases}
\pi_{G,0}^{(0)} = \sigma y_0, \\
\pi_{G,k}^{(0)} = \pi_{G,k-1}^{(0)} R_{k-1,k}, \ 1 \leq k \leq Z,
\end{cases}
\]
where \( y_0 \) is the stationary probability vector of the censored Markov process \( U_0 \) to level 0, and the scalar \( \sigma \) is uniquely determined by \( \sum_{k=0}^{Z} \pi_{G,k}^{(0)} 1 = 1 \). The censored Markov process \( T[\leq 0] \) has the stationary probability vector \( y_0 \) such that
\[
\begin{cases}
y_0 T[\leq 0] = 0, \\
y_0 1 = 1.
\end{cases}
\]

### 4.4 The routing matrix

The routing probabilities of the closed queueing network are complicated since they depend on the states of each node by considering the items’ failure, removal, repair, redistribution, and reuse processes under two batch policies. Figure 5 depicts the routing probabilities for the three classes of nodes.

For the three classes of nodes, we write the routing matrix as \( P = (f_{i,j}) \), where \( f_{i,j} \) is given by
Figure 5: Routing probabilities of nodes

\[
f_{i,j} = \begin{cases} 
q_{0,i}, & \text{Node 0 to Node 0} \rightarrow i, 1 \leq i \leq N, \\
q_{i,0}, & \text{Node } i \text{ to Node } i \rightarrow 0, 1 \leq i \leq N, \\
q_{i,i}, & \text{Node } i \text{ to Node } i, 0 \leq i \leq N, \\
q_{i,j}, & \text{Node } i \text{ to Node } i \rightarrow j, j \in \Theta_i, 1 \leq i \leq N, \\
1, & \text{Node } i \rightarrow j \text{ to Node } j, j \in \Theta_i, 1 \leq i \leq N, \\
1, & \text{Node } i \rightarrow 0 \text{ to Node } 0, 1 \leq i \leq N, \\
1, & \text{Node } 0 \rightarrow i \text{ to Node } i, 1 \leq i \leq N, \\
0, & \text{otherwise}, 
\end{cases}
\]

where

\[
q_{0,0} = 1 - \left( \sum_{l=0}^{\phi-\psi} \pi_{B,1M;G,Z}^{(0)} \right),
\]

for \(1 \leq i \leq N\)

\[
q_{0,i} = \left( \sum_{l=0}^{\phi-\psi} \pi_{B,1M;G,Z}^{(0)} \right) \beta_i, \quad q_{i,0} = \sum_{k=0}^{K-M} \pi_{G,k;B,M}^{(i)}, \quad q_{i,i} = \sum_{k=0}^{M-1} \pi_{G,0;B,k}^{(i)},
\]

and for \(1 \leq i \leq N, j \in \Theta_i,\)

\[
q_{i,j} = (1 - q_{i,0} - q_{i,i}) p_{i,j}.
\]

Note that the routing probabilities are expressed by the stationary probabilities of the two Markov processes \(\{(Q_G^{(i)}(t), Q_B^{(i)}(t)) : t \geq 0\}\) and \(\{(Q_B^{(0)}(t), Q_G^{(0)}(t)) : t \geq 0\}\), both of which are determined by the relative arrival rate vector \(e\). Thus the routing matrix depends on \(e\) and can be written as \(P(e)\).
Then the relative arrival rate vector \( \mathbf{e} \) satisfies a nonlinear routing matrix equation

\[
\mathbf{e} = \mathbf{e} \mathbf{P} (\mathbf{e}).
\]  
(13)

To compute the relative arrival rates, we develop the Algorithm 1 to give an iterative approximate solution to Equation (13).

### Algorithm 1 An Iterative Algorithm for Computing the Relative Arrival Rates

**Step 0: Initialization**

Give an initial vector \( \mathbf{e}_0 \).

**Step 1: The first iterative computation**

(a) Using \( \mathbf{e}_0 \), compute stationary probability vector \( \pi_B^{(i)} \) for \( 1 \leq i \leq N \) and \( \pi_G^{(0)} \) by UL-type RG-factorization.

(b) Obtain the routing matrix \( \mathbf{P} (\mathbf{e}_0) \) based on \( \pi_B^{(i)} \) and \( \pi_G^{(0)} \).

(c) Obtain \( \mathbf{e}_1 \) by the equation \( \mathbf{e}_1 = \mathbf{e}_0 \mathbf{P} (\mathbf{e}_0) \).

**Step 2: The second iterative computation**

(a) Using \( \mathbf{e}_1 \), compute stationary probability vector \( \pi_B^{(i)} \) for \( 1 \leq i \leq N \) and \( \pi_G^{(0)} \) by UL-type RG-factorization.

(b) Obtain routing matrix \( \mathbf{P} (\mathbf{e}_1) \) based on \( \pi_B^{(i)} \) and \( \pi_G^{(0)} \).

(c) Obtain \( \mathbf{e}_2 \) by the equation \( \mathbf{e}_2 = \mathbf{e}_1 \mathbf{P} (\mathbf{e}_1) \).

**Step 3: The \((k + 1)\)st iterative computation for \( k \geq 2 \)**

(a) Using \( \mathbf{e}_k \), compute stationary probability vector \( \pi_B^{(i)} \) for \( 1 \leq i \leq N \) and \( \pi_G^{(0)} \) by UL-type RG-factorization.

(b) Obtain routing matrix \( \mathbf{P} (\mathbf{e}_k) \) based on \( \pi_B^{(i)} \) and \( \pi_G^{(0)} \).

(c) Obtain \( \mathbf{e}_{k+1} \) by the equation \( \mathbf{e}_{k+1} = \mathbf{e}_k \mathbf{P} (\mathbf{e}_k) \).

**Step 4: Convergence check**

If there exists a relative arrival rate vector \( \mathbf{n} \mathbf{e} \) such that

\[
\sqrt{\left( n_{e_0} - n_0\right)^2 + \left( n_{e_1} - n_1\right)^2 + \cdots + \left( n_{e_N} - n_N\right)^2 } < \varepsilon,
\]

(called a stop condition), for a given precision \( \varepsilon = 1\varepsilon - 10 \), then the computation stops. In this case, \( n_{e_{k+1}} \approx n_\mathbf{e} \). Otherwise, return to **Step 3** until this stop condition is satisfied.

**Step 5: Output**

Obtain the relative arrival rate vector \( n_\mathbf{e} \).
5 A Product-Form Solution and Performance Analysis

In this section, we obtain the product-form solution for the stationary joint probabilities of queue lengths in the closed queueing network. Based on this, we can define and compute some useful performance measures of the maintenance network.

5.1 The product-form solution

The numbers of usable items and failed items in any base and in the maintenance shop are described as block-structured Markov processes, and thus the stationary probabilities of Markov processes of the two classes of nodes can be computed by the UL-type RG-factorization.

For the closed queueing network, Theorem 2 provides the product-form solution of the stationary joint probabilities $\pi(n)$ of queue lengths for $n \in \Omega$.

**Theorem 2** For the closed queueing network of the maintenance network, the stationary joint probability $\pi(n)$ is given by

$$
\pi(n) = \frac{1}{C} \prod_{i=0}^{N} H\left(n_{G}^{(i)}, n_{B}^{(i)}\right) \prod_{j=1}^{N} H\left(m_{i,j}\right) \prod_{i=1}^{N} H\left(m_{i,0}\right) H\left(m_{0,i}\right),
$$

where

$$
H\left(n_{G}^{(i)}, n_{B}^{(i)}\right) = \begin{cases}
\pi^{(i)}_{G,n_{G}^{(i)},n_{B}^{(i)}}, & 1 \leq i \leq N, 0 \leq n_{G}^{(i)} \leq K, 0 \leq n_{B}^{(i)} \leq M, \\
\pi^{(0)}_{B,n_{B}^{(0)},n_{G}^{(0)}}, & i = 0, 0 \leq n_{B}^{(0)} \leq Z, 0 \leq n_{G}^{(0)} \leq \phi M,
\end{cases}
$$

$$
H\left(m_{i,j}\right) = \left(\frac{e_{i,j}}{\mu_{i,j}}\right)^{m_{i,j}} \frac{1}{m_{i,j}!}, 0 \leq m_{i,j} \leq K,
$$

$$
H\left(m_{i,0}\right) = \left(\frac{e_{i,0}}{\mu_{i,0}}\right)^{m_{i,0}} \frac{1}{m_{i,0}!}, m_{i,0} = kM \text{ for } 0 \leq k \leq \phi,
$$

$$
H\left(m_{0,i}\right) = \left(\frac{e_{0,i}}{\mu_{0,i}}\right)^{m_{0,i}} \frac{1}{(m_{0,i})!}, m_{0,i} = lZ_i \text{ for } 0 \leq l \leq \phi / \psi,
$$

and $C$ is a normalization constant, given by

$$
C = \sum_{n \in \Omega} \prod_{i=0}^{N} H\left(n_{G}^{(i)}, n_{B}^{(i)}\right) \prod_{j=1}^{N} H\left(m_{i,j}\right) \prod_{i=1}^{N} H\left(m_{i,0}\right) H\left(m_{0,i}\right).
$$

The proof is given in Appendix E.

The following theorem establishes relations between the marginal stationary probabilities and the joint stationary probabilities.
Theorem 3 For the closed queueing network, the marginal probabilities of the system can be determined by the joint stationary probabilities as follows:

1. For Node \( i, 1 \leq i \leq N, 0 \leq l \leq M, 0 \leq k \leq K - l \),

\[
\pi \left( n : n_G^{(i)} = k, n_B^{(i)} = l \right) = \sum_{n \in \Omega \atop n_G^{(i)} = k, n_B^{(i)} = l} \pi(n) = \pi^{(i)}_{G,k;B,l} \frac{\tilde{C}(n_G^{(i)} = k, n_B^{(i)} = l)}{C}.
\]

where

\[
\tilde{C}(n_G^{(i)} = k, n_B^{(i)} = l) = \sum_{n \in \Omega \atop n_G^{(i)} = k, n_B^{(i)} = l} \prod_{j=0}^{N} H \left( n_G^{(j)}, n_B^{(j)} \right) \prod_{j=h}^{N} H \left( m_j, l \right) H \left( m_0 \right).
\]

2. For Node \( i, i = 0, 0 \leq k \leq Z, k + l = hM, 0 \leq h \leq \phi \),

\[
\pi \left( n : n_G^{(0)} = k, n_B^{(0)} = l \right) = \sum_{n \in \Omega \atop n_G^{(0)} = k, n_B^{(0)} = l} \pi(n) = \pi^{(0)}_{B,0;G,k} \frac{\tilde{C}(n_G^{(0)} = k, n_B^{(0)} = l)}{C},
\]

where

\[
\tilde{C}(n_G^{(0)} = k, n_B^{(0)} = l) = \sum_{n \in \Omega \atop n_G^{(0)} = k, n_B^{(0)} = l} \prod_{j=1}^{N} H \left( n_G^{(j)}, n_B^{(j)} \right) \prod_{j=h}^{N} H \left( m_j, l \right) H \left( m_0 \right).
\]

3. For Node \( i \to j, 1 \leq i \leq N, j \in \Theta_i, 0 \leq h \leq K \),

\[
\pi \left( n : m_{i,j} = h \right) = \sum_{n \in \Omega \atop m_{i,j} = h} \pi(n) = \frac{1}{h!} \left( \frac{e_{i,j}}{\mu_{i,j}} \right)^h \tilde{C}(m_{i,j} = h) \frac{\tilde{C}(n_G^{(0)} = k, n_B^{(0)} = l)}{C},
\]

where

\[
\tilde{C}(m_{i,j} = h) = \sum_{n \in \Omega \atop m_{i,j} = h} \prod_{k=0}^{N} H \left( n_G^{(k)}, n_B^{(k)} \right) \prod_{k=1}^{N} H \left( m_{k,i} \right) \prod_{k=1}^{N} H \left( m_{k,j} \right) H \left( m_{0,k} \right).
\]

4. For Node \( i \to 0, 1 \leq i \leq N, 0 \leq h \leq \phi \),

\[
\pi \left( n : m_{i,0} = hM \right) = \sum_{n \in \Omega \atop m_{i,0} = hM} \pi(n) = \frac{1}{(hM)!} \left( \frac{e_{i,0}}{\mu_{i,0}} \right)^{hM} \tilde{C}(m_{i,0} = hM) \frac{\tilde{C}(n_G^{(0)} = k, n_B^{(0)} = l)}{C},
\]

where

\[
\tilde{C}(m_{i,0} = hM) = \sum_{n \in \Omega \atop m_{i,0} = hM} \prod_{k=0}^{N} H \left( n_G^{(k)}, n_B^{(k)} \right) \prod_{k=1}^{N} H \left( m_{k,i} \right) \prod_{k=1}^{N} H \left( m_{k,j} \right) H \left( m_{0,k} \right).
\]

5. For Node \( 0 \to i, 1 \leq i \leq N, 0 \leq l \leq \phi/\psi \),

\[
\pi \left( n : m_{0,i} = lZ_i \right) = \sum_{n \in \Omega \atop m_{0,i} = lZ_i} \pi(n) = \frac{1}{(lZ_i)!} \left( \frac{e_{0,i}}{\mu_{0,i}} \right)^{lZ_i} \tilde{C}(m_{0,i} = lZ_i) \frac{\tilde{C}(n_G^{(0)} = k, n_B^{(0)} = l)}{C},
\]

20
where
\[ \tilde{C} (m_{0,i} = lZ_i) = \sum_{n \in \Omega} \prod_{k=0}^{N} H \left( n_{G}^{(k)}, n_{B}^{(k)} \right) \prod_{k=1}^{N} H (m_{k,i}) \prod_{k=1}^{N} H (m_{0,k}) \prod_{k=1}^{N} H (m_{0,k}) . \]

The proof is immediate by Section 7 in Bolch et al. (2006), and hence it is omitted here.

5.2 Performance Analysis

In this subsection, we provide performance measures for the maintenance network by using the stationary joint probability vector \( \pi (n) \) for \( n \in \Omega. \)

(a) The stationary proportion of failed items in the maintenance network

In the maintenance network, we are interested in the stationary proportion of failed items that are distributed in the \( N \) bases, the maintenance shop, and the path nodes from bases to the maintenance shop. The average number of failed items \( E [3] \) can be computed by the marginal probabilities as follows:

\[ E [3] = \sum_{i=1}^{N} \sum_{l=0}^{M} l \pi (n : n_{i}^{(i)} = l) + \phi \sum_{i=1}^{N} \sum_{l=0}^{M} l \pi (n : n_{B}^{(0)} = l) + \phi M \sum_{i=1}^{N} \sum_{l=0}^{M} l \pi (n : m_{i,0} = lM) , \]

where the three terms are the average numbers of failed items in the \( N \) bases, the maintenance shop, and Node \( i \rightarrow 0 \) for \( 1 \leq i \leq N \), respectively. Based on Theorem 3, we obtain

\[ \pi (n : n_{i}^{(i)} = l) = \frac{1}{\tilde{C}} \sum_{k=0}^{K-l} \left[ \pi_{G,k,B,l}^{(i)} \tilde{C} \left( n_{G}^{(i)} = k, n_{B}^{(i)} = l \right) \right] , \]

\[ \pi (n : n_{B}^{(0)} = l) = \frac{1}{\tilde{C}} \sum_{h=0}^{\phi} \sum_{k=0}^{Z} \sum_{l=0}^{hM} \left[ \pi_{B,l,G,k}^{(0)} \tilde{C} \left( n_{G}^{(0)} = k, n_{B}^{(0)} = l \right) \right] , \]

and

\[ \pi (n : m_{i,0} = lM) = \frac{1}{lM!} \left( \frac{e_{i,0}}{\mu_{i,0}} \right)^{lM} \tilde{C} (m_{i,0} = lM) . \]

Thus the proportion of failed items in the maintenance network with respect to the total number \( K \) of items in the maintenance network is given by

\[ \eta = \frac{E [3]}{K} . \]

(b) The stationary proportion of usable items in either various bases or the path nodes

The path nodes with usable items have two different cases. We denote the average number \( E [\overline{\omega}] \) of usable items in either the bases or the path nodes by the marginal
probabilities as follows:

\[ E[\pi] = \sum_{i=1}^{N} \sum_{k=0}^{K} k \pi (n : n_G^{(i)} = k) + \sum_{i=1}^{N} \sum_{j \in \Theta_i} \sum_{k=0}^{K} k \pi (n : m_{i,j} = k) + \sum_{i=1}^{N} \sum_{k=0}^{K} k \beta_i Z \pi (n : m_{0,i} = k \beta_i Z), \]

where the three terms are the average number of usable items in the \( N \) bases, the Node \( i \to j \), and Node \( 0 \to i \) for \( 1 \leq i \leq N \) and \( j \in \Theta_i \), respectively. Based on Theorem 3, we obtain

\[ \pi (n : n_G^{(i)} = k) = \frac{1}{C} \sum_{l=0}^{\min(M,K-k)} \left[ \pi_G^{(i)}(k) \tilde{C} \left( n_G^{(i)} = k, n_B^{(i)} = l \right) \right], \]

\[ \pi (n : m_{i,j} = k) = \frac{1}{k!} \left( \frac{e_{i,j}}{\mu_{i,j}} \right)^k \tilde{C} \left( m_{i,j} = k \right), \]

and

\[ \pi (n : m_{0,i} = k \beta_i Z) = \frac{1}{(k \beta_i Z)!} \left( \frac{e_{0,i}}{\mu_{0,i}} \right)^{k \beta_i Z} \tilde{C} \left( m_{0,i} = k \beta_i Z \right). \]

Thus the stationary proportion of usable items in either bases or the path nodes with respect to the total number \( K \) is given by

\[ \xi = \frac{E[\pi]}{K}. \]

Note that the stationary proportion \( \xi \) measures the availability of items that can directly serve the arriving users.

**c) The busy probability of the maintenance shop**

Let \( F_0 \) be the stationary probability that there is no failed item in the maintenance shop. Then, we have

\[ F_0 = \pi (n : n_B^{(0)} = 0) = \frac{1}{C} \sum_{k=0}^{\psi} \left[ \pi_B^{(0)}(k) \tilde{C} \left( n_G^{(0)} = kM, n_B^{(0)} = 0 \right) \right], \]

where

\[ \tilde{C} \left( n_G^{(0)} = kM, n_B^{(0)} = 0 \right) = \sum_{n \in \Omega : n_G^{(0)} = kM, n_B^{(0)} = 0} \prod_{j=1}^{N} H \left( n_G^{(i)}, n_B^{(i)} \right) \prod_{j=1}^{N} H \left( m_{j,h} \right) \prod_{j=1}^{N} H \left( m_{j,0} \right). \]

Let

\[ F_A = 1 - F_0. \]
Then the probability $F_A$ is the busy probability of the maintenance shop, which can be used to measure the maintenance process and the sizes of the two batch policies.

(d) Two useful stationary proportions

Let $\gamma_1$ be the proportion of the repaired items with respect to all the items in the maintenance shop. Then

$$\gamma_1 = \frac{\sum_{k=0}^{Z} k \pi \left( n : n_G^{(0)} = k \right)}{\sum_{h=0}^{\phi} \sum_{l=0}^{k+hM} (k+l) \pi \left( n : n_G^{(0)} = k, n_B^{(0)} = l \right)},$$

where

$$\pi \left( n : n_G^{(0)} = k, n_B^{(0)} = l \right) = \frac{1}{C} \pi \left( n_G^{(0)} = k, n_B^{(0)} = l \right),$$

and

$$\pi \left( n : n_G^{(0)} = k \right) = \frac{1}{C} \sum_{h=0}^{\phi} \sum_{l=0}^{k+hM-k} \left[ \pi \left( n_G^{(0)} = k, n_B^{(0)} = l \right) \right].$$

Let $\gamma_2$ denote the proportion of the failed items in the maintenance shop with respect to all the failed items in the maintenance network

$$\gamma_2 = \frac{\sum_{l=0}^{M} l \pi \left( n : n_B^{(0)} = l \right)}{E[3]},$$

where

$$\pi \left( n : n_B^{(0)} = l \right) = \frac{1}{C} \sum_{h=0}^{\phi} \sum_{k=0}^{k+hM-l} \left[ \pi \left( n_G^{(0)} = k, n_B^{(0)} = l \right) \right].$$

The two proportions $\gamma_1$ and $\gamma_2$ can be used to measure the repair ability of the maintenance shop and the removal ability of failed items in the maintenance network, respectively.

The computational method developed in this paper and the two batch policies can be applied to improve the quality of service, system design, and operations management of the maintenance network.

6 Concluding Remarks

This paper studies a maintenance network and develops a new computational method by applying the RG-factorizations of block-structured Markov processes in the closed queueing network. First, we describe the maintenance network as a closed queueing network that characterizes the items’ failure, removal, redistribution, and reuse processes under
two batch policies. Then we show that the relative arrival rates satisfy a nonlinear routing matrix equation. The Markov systems in any base and the maintenance shop are treated by the block-structured Markov processes whose stationary probability vectors can be computed by the RG-factorizations. Finally, we provide a more general product-form solution for the closed queueing network by extending and generalizing from simple queueing systems of the nodes to general block-structured Markov processes. Based on this, we provide performance measures of the maintenance network with failed items.

We hope the methodology and results given in this paper open a new research direction on the closed queueing network and shed light on the analysis of more general maintenance networks. Along the research line, there are still a number of interesting directions for future research. First, it is significant to develop periodic policies both for removing the failed items and redistributing the repaired items. The second issue is to develop new algorithms to solve the nonlinear matrix equation system satisfied by the relative arrival rates. Third, the optimization problem of the batch policies for the maintenance network are worth further exploration.

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Appendices

Appendix A. The elements of matrix Q

We provide expression for the block element $Q_{m,n}$ of matrix $Q$ in Equation (1), where $m$ and $n$ are two level variables with $0 \leq m, n \leq M$. We consider base $i$. For $0 \leq n \leq M - 1$

$$Q_{n,n} = \begin{pmatrix}
-e_i & e_i \\
\lambda_i & -(\lambda_i + e_i + \alpha) & e_i \\
\ddots & \ddots & \ddots \\
\lambda_i & -(\lambda_i + e_i + (K - n - 1) \alpha) & e_i \\
\lambda_i & -\lambda_i & -(\lambda_i + (K - n) \alpha)
\end{pmatrix},$$

whose size is $(K - n + 1) \times (K - n + 1)$, and

$$Q_{n,n+1} = \begin{pmatrix}
0 & 2\alpha \\
\alpha & \ddots \\
& \ddots & \ddots \\
& & (K - n - 1) \alpha & (K - n) \alpha
\end{pmatrix},$$

for $n = M$

$$Q_{M,M} = \begin{pmatrix}
-(e_i + \mu_{i,0}) & e_i \\
\lambda_i & -(\lambda_i + e_i + \mu_{i,0}) & e_i \\
\ddots & \ddots & \ddots \\
\lambda_i & -(\lambda_i + e_i + \mu_{i,0}) & e_i \\
\lambda_i & -\lambda_i & -(\lambda_i + \mu_{i,0})
\end{pmatrix},$$

whose size is $(K - M + 1) \times (K - M + 1)$, and

$$Q_{M,0} = \begin{pmatrix}
\mu_{i,0} & \mu_{i,0} & \cdots & \mu_{i,0} \\
\mu_{i,0} & \cdots & \mu_{i,0} & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\mu_{i,0} & 0 & \cdots & 0
\end{pmatrix},$$

whose size is $(K - M + 1) \times (K + 1)$. 25
Appendix B. Proof of Lemma 1

Proof: It is obvious that $Q_{M,M}^{[\leq M]} = Q_{M,M}$. We use the inductive method to prove that for the censoring matrix $Q^{[\leq k]}$, Equation (2) is true for $0 \leq k \leq M - 1$. Our proof contains three steps as follows:

Step one. For $k = M - 1$, referring to Section 2 of Li (2010), we get

$$Q^{[\leq M-1]} = \begin{pmatrix} Q_{0,0} & Q_{0,1} \\ Q_{1,1} & Q_{1,2} \\ & \ddots & \ddots \\ & & Q_{M-2,M-2} & Q_{M-2,M-1} \\ & & & Q_{M-1,M-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q_{M-1,M} \end{pmatrix} + \left( -Q_{M,M} \right)^{-1} \begin{pmatrix} Q_{M,0} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

in which $Q_{M,0}^{[\leq M-1]} = Q_{M-1,M} \left( -Q_{M,M} \right)^{-1} Q_{M,0}$. Thus when $k = M - 1$, Equation (2) is true.

Step two. We assume that when $k = m$ for $2 \leq m \leq M - 2$, Equation (2) is true, i.e.,

$$Q^{[\leq k]} = Q^{[\leq m]} = \begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} \\ Q_{1,1} & Q_{1,2} & \ddots & \ddots \\ & \ddots & \ddots \\ & & & Q_{m-1,m-1} & Q_{m-1,m} \\ & & & & Q_{m,m} \end{pmatrix},$$
in which \( Q_{m,0}^{[\leq m]} = \prod_{l=m}^{M-1} [Q_{l,l+1} (-Q_{l+1,l+1})^{-1}] Q_{M,0} \). Now, we consider the case with \( k = m - 1 \) for \( 2 \leq m \leq M - 2 \),

\[
Q_{m,0}^{[\leq m-1]} = \begin{pmatrix}
Q_{0,0} & Q_{0,1} \\
Q_{1,1} & Q_{1,2} \\
& \ddots & \ddots \\
& & Q_{m-2,m-2} & Q_{m-2,m-1} \\
& & & Q_{m-1,m-1}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
(Q_{m,m}^{[\leq m-1]})^{-1} \begin{pmatrix}
Q_{m,0}^{[\leq m]} & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
Q_{0,0} & Q_{0,1} & \cdots & \cdots \\
Q_{1,1} & Q_{1,2} & & \\
& \cdots & \ddots & \ddots \\
& & & Q_{m-2,m-2} & Q_{m-2,m-1} \\
Q_{m-1,m} & & & & Q_{m-1,m-1}
\end{pmatrix}
\]

in which \( Q_{m,0}^{[\leq m-1]} = Q_{m-1,m}^{[\leq m]} = (Q_{m,m}^{[\leq m]})^{-1} Q_{m,0}^{[\leq m]} \). Since \( Q_{m,0}^{[\leq m]} = \prod_{l=m}^{M-1} [Q_{l,l+1} (-Q_{l+1,l+1})^{-1}] Q_{M,0} \), it is easy to check that \( Q_{m,0}^{[\leq m-1]} = \prod_{l=m-1}^{M-1} [Q_{l,l+1} (-Q_{l+1,l+1})^{-1}] Q_{M,0} \). Thus when \( k = m - 1 \), Equation (2) is also true. Therefore, Equation (2) is true for for \( 1 \leq k \leq M - 1 \).

Step three. It is easy to see from Step two that

\[
Q_{[\leq 1]} = \begin{pmatrix}
Q_{0,0} & Q_{0,1} \\
Q_{1,0}^{[\leq 1]} & Q_{1,1}
\end{pmatrix}
\]

in which \( Q_{1,0}^{[\leq 1]} = \prod_{l=1}^{M-1} [Q_{l,l+1} (-Q_{l+1,l+1})^{-1}] Q_{M,0} \). Then

\[
Q_{[\leq 0]} = Q_{0,0} + Q_{0,1} (-Q_{1,1})^{-1} Q_{1,0}^{[\leq 1]} = Q_{0,0} + Q_{0,1} (-Q_{1,1})^{-1} \prod_{l=1}^{M-1} [Q_{l,l+1} (-Q_{l+1,l+1})^{-1}] Q_{M,0}.
\]

Based on the above three steps, Equation (3) is true for \( 0 \leq k \leq M \). This completes the proof.
Appendix C. Proof of Theorem 1

Proof: The proof can easily be completed by means of Chapter 2 of Li (2010). Here, it is necessary to fix the two special matrices $R_U$ and $G_L$ as follows:

The $R$-measures: From Equations (4) and (5), we obtain that $R_{i,j} = Q_{i,j}^{[\leq j]} \left(-Q_{j,j}^{[\leq j]}\right)^{-1}$ for $0 \leq i < j \leq M$. It is seen from Equation (2) that $Q_{i,i+1}^{[\leq i+1]} > 0$, while the element $Q_{i,j}^{[\leq i+1]}$ is zero for $i + 2 \leq j \leq M$. Thus for $0 \leq i < j \leq M$

$$R_{i,j} = \begin{cases} R_{i,i+1}, & j = i + 1, \\ 0, & i + 2 \leq j \leq M. \end{cases}$$

Thus we have

$$R_U = \begin{pmatrix} 0 & R_{0,1} \\ 0 & R_{1,2} \\ & \ddots & \ddots \\ & 0 & R_{M-2,M-1} \\ & & 0 & R_{M-1,M} \\ & & & 0 \end{pmatrix}.$$

The $G$-measures: From Equations (4) and (6), we obtain that $G_{i,j} = \left(-Q_{i,i}^{[\leq i]}\right)^{-1} Q_{i,j}^{[\leq i]}$, $0 \leq j < i \leq M$. From Equation (2), we can see $Q_{i,0}^{[\leq i]} > 0$, while the elements $Q_{i,j}^{[\leq i]}$ is zero for $1 \leq j \leq i - 1$. Thus for $0 \leq j < i \leq M$

$$G_{i,j} = \begin{cases} G_{i,0}, & j = 0, \\ 0, & 1 \leq j \leq i - 1. \end{cases}$$

Then we get

$$G_L = \begin{pmatrix} 0 \\ G_{1,0} \\ G_{2,0} \\ \vdots \\ G_{M-2,0} \\ G_{M-1,0} \\ G_{M,0} \end{pmatrix}.$$

Based on the above analysis, this completes the proof. ■
Appendix D. The block elements of Matrix T

Note that \( \omega(n_B^{(0)}) = \min \{ n_B^{(0)}, r \} \) \( w \). For \( 0 \leq k \leq \psi - 1 \) and Equation (9), we have

\[
T_{kM,kM} = \begin{pmatrix}
-e_0 & e_0 \\
-\omega(M) + e_0 & e_0 \\
\ddots & \ddots \\
-\omega((\phi - k - 1)M) + e_0 & e_0 \\
-\omega((\phi - k)M)
\end{pmatrix},
\]

\[
T_{kM,kM+1} = \begin{pmatrix}
0 \\
\omega(M) \\
\omega(2M) \\
\ddots \\
\ddots \\
\omega((\phi - k - 1)M) \\
\omega((\phi - k)M)
\end{pmatrix};
\]

For \( 0 \leq k \leq \psi - 1 \), \( 1 \leq j \leq M - 1 \)

\[
T_{kM+j,kM+j} = \begin{pmatrix}
-\omega(M - j) + e_0 & e_0 \\
\ddots & \ddots \\
-\omega((\phi - k - 1)M - j) + e_0 & e_0 \\
-\omega((\phi - k)M - j)
\end{pmatrix},
\]

\[
T_{kM+j,kM+j+1} = \begin{pmatrix}
\omega(M - j) \\
\ddots \\
\omega((\phi - k - 1)M - j) \\
\omega((\phi - k)M - j)
\end{pmatrix};
\]

\[
T_{Z,Z} = \begin{pmatrix}
-\mu_0 \\
-\mu_0 \\
\ddots \\
-\mu_0 \\
-\mu_0
\end{pmatrix},
\]
\[
T_{Z,0} = \begin{pmatrix}
\mu_0 & \mu_0 & \cdots \\
\mu_0 & \mu_0 & \\
& \ddots & \ddots \\
& & \mu_0 & 0 & \cdots & 0
\end{pmatrix}.
\]

**Appendix E. The proof of the Theorem 2**

We first classify the nodes in the closed queueing network as three classes: Node \(i\) for \(0 \leq i \leq N\); Node \(i \rightarrow j\) for \(1 \leq i \leq N\) and \(j \in \Theta_i\); and Node \(i \rightarrow 0\) and Node \(0 \rightarrow i\) for \(1 \leq i \leq N\). Based on this, we prove the product-form solution from the following three different parts.

**Part one**: Node \(i\) for \(0 \leq i \leq N\). By using the stationary probabilities of block-structured Markov processes of base \(i\) with \(1 \leq i \leq N\) given in Section 4.2, we get

\[
H \left( n_G^{(i)}, n_B^{(i)} \right) = \pi_G^{(i)}(n_G^{(i)}; B, n_B^{(i)}),
\]

Similarly, from the stationary probabilities of block-structured Markov process of the maintenance shop given in Section 4.3, we obtain

\[
H \left( n_G^{(0)}, n_B^{(0)} \right) = \pi_B^{(0)}(n_B^{(0)}; G, n_G^{(0)}).
\]

Thus we have

\[
H \left( n_G^{(i)}, n_B^{(i)} \right) = \begin{cases} 
\pi_G^{(i)}(n_G^{(i)}; B, n_B^{(i)}), & 1 \leq i \leq N, 0 \leq n_G^{(i)} \leq K, 0 \leq n_B^{(i)} \leq M, \\
\pi_B^{(0)}(n_B^{(0)}; G, n_G^{(0)}), & i = 0, 0 \leq n_B^{(0)} \leq Z, 0 \leq n_G^{(0)} \leq \phi M.
\end{cases}
\]

**Part two**: For Node \(i \rightarrow j\) with \(1 \leq i \leq N\) and \(j \in \Theta_i\).

Since the number of servers in Node \(i \rightarrow j\) may be infinite, applying Gordon and Newell (1967) and Subsection 7.35 in Bolch et al. (2006), the function \(H(m_{i,j})\) is given by

\[
H(m_{i,j}) = \left( \frac{e_{i,j}}{\mu_{i,j}} \right)^{m_{i,j}} \frac{1}{m_{i,j}!}, \quad 0 \leq m_{i,j} \leq K.
\]

**Part three**: For Node \(i \rightarrow 0\) (resp. Node \(0 \rightarrow i\)) with \(1 \leq i \leq N\).

The failed items (resp. the repaired items) are transported on path nodes by the batch removal (resp. batch redistribution) policy. Refer to Henderson and Taylor (1990), we get

\[
H(m_{i,0}) = \left( \frac{e_{i,0}}{\mu_{i,0}} \right)^{m_{i,0}} \frac{1}{m_{i,0}!}, \quad m_{i,0} = kM \text{ for } 0 \leq k \leq \phi,
\]

\[
H(m_{0,i}) = \left( \frac{e_{0,i}}{\mu_{0,i}} \right)^{m_{0,i}} \frac{1}{m_{0,i}!}, \quad m_{0,i} = \phi M \text{ for } 0 \leq \phi.
\]
and
\[
H(m_{0,i}) = \left( \frac{e_{0,i}}{\mu_{0,i}} \right)^{m_{0,i}} \frac{1}{(m_{0,i})!}, \quad m_{0,i} = lZ_i \text{ for } 0 \leq l \leq \frac{\phi}{\psi}.
\]

Finally, for the closed queueing network, we can express the product-form solution in Theorem 2, and the normalization constant is given by
\[
C = \sum_{n \in \Omega} \prod_{i=0}^{N} H\left(n_G^{(i)}, n_B^{(i)}\right) \prod_{j=1}^{i} \prod_{i=1}^{N} H(m_{i,j}) \prod_{i=1}^{N} H(m_{i,0}) H(m_{0,i}).
\]

This completes the proof. ■

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