Fully resonant scalars on asymptotically AdS wormholes

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Abstract

In this work we show the existence of asymptotically AdS wormhole geometries where the scalar probe has an equispaced, fully resonant spectrum, as that of a scalar on AdS spacetime, and explore its dynamics when non-linearities are included. The spacetime is a solution of Einstein-Gauss-Bonnet theory with a single maximally symmetric vacuum. Introducing a non-minimal coupling between the scalar probe and the Ricci scalar remarkably leads to a fully resonant spectrum for a scalar field fulfilling reflective boundary conditions at both infinities. Applying perturbative methods, which are particularly useful for unveiling the dynamics at time scales of order $\varepsilon^{-2}$ (where $\varepsilon$ characterizes the amplitude of the initial perturbation), we observe both direct and inverse energy cascades between modes. This motivates us to explore the energy returns in the case in which the dynamics is dominated by a single mode. We find numerical and perturbative evidence that near exact returns do exist in this regime. We also provide some comments on the fully backreracting case and provide a proof of the universality of the weakly non-linear dynamics around AdS, in the context of Lovelock theories with generic couplings, up to times of order $\varepsilon^{-2}$.

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I. INTRODUCTION

General Relativity in dimensions higher than four can be extended, still fulfilling the requirements of second-order field equations and diffeomorphism invariance. In general, in dimensions \( D \geq 5 \), precise combination of higher curvature terms can be added to the Einstein-Hilbert action leading also to second order field equations \(^{[1]}\). These combinations are dimensional continuations of the Euler densities of the lower, even dimensions. These combinations may appear as low energy effective actions in string theory, as it is the case for the \( R^2 \) term in the Heterotic and Bosonic string theories \(^{[2]}\). The simplest deformation from GR is obtained in five dimensions, where the Einstein-Gauss-Bonnet theory has the following action principle

\[
I [g_{\mu\nu}] = \frac{1}{16\pi G_5} \int \left[ R - 2\Lambda + \alpha \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\sigma}R^{\mu\nu\gamma\sigma} \right) \right] \sqrt{-g} d^5 x, \tag{1}
\]

Note that the coupling \( \alpha \) has mass dimension \(-2\), and while here it represents a free coupling.

For generic values of the couplings, the theory admits a local Lorentz invariance which can be made manifest in the first order formulation where the vielbein and the spin connection transform as a vector and connection of \( SO(4,1) \), respectively. When \( \alpha \Lambda = -3/4 \), the local symmetry group is enlarged to \( SO(4,2) \) \(^{[3]}\), the theory admits a unique maximally symmetric AdS solution and it has the maximum number of propagating degrees of freedom \(^{[4]}\). At this particular point, the space of solutions is also enlarged and contains, in addition to black holes \(^{[3]}\) and analytic rotating solutions \(^{[6]},^{[7]}\), asymptotically locally AdS wormholes \(^{[8]}\). The line element of the wormhole metric reads

\[
ds^2 = \ell^2 \left[ - \cosh^2 \rho dt^2 + d\rho^2 + \cosh^2 \rho \left( d\varphi^2 + d\Sigma_2^2 \right) \right], \tag{2}
\]

where \(-\infty < t < +\infty\), \(-\infty < \rho < +\infty\), \(0 < \varphi \leq 2\pi\) and \(d\Sigma_2\) stands for the line element of a compact, smooth quotient of the pseudo-sphere with radius \( 3^{-1/2} \). Here \( \ell^2 = 4\alpha = -3/\Lambda \). The two asymptotically locally AdS\(_5\) regions \( \rho \to \pm\infty \), are connected by a traversable throat located at \( \rho = 0 \) and the spacetime is symmetric under the reflection \( \rho \to -\rho \). This spacetime being devoid of singularities and horizons, represents a soliton in the non-linear Einstein-Gauss-Bonnet theory \(^1\).

\(^1\) It is interesting to notice that a similar situation occurs for Einsteinian gravities \(^{[9]}\), where it was shown
The propagation of a scalar probe on the background geometry was originally explored in [11], where the focus was on the computation, in a closed form, of the normal frequencies fulfilling different possible boundary conditions. In [12], this problem was partially revisited and it was shown that for a particular value of a non-minimal coupling with the scalar curvature, the propagation of the scalar is controlled by an effective Schrödinger problem in a Rosen-Morse potential, for which the energies are proportional to the square of the frequencies of the scalar probe. Since the eigenvalues of the Schrödinger operator in Rosen-Morse potentials are quadratic in the mode number [13], the spectrum of the purely radial scalar probe turn out to be equispaced or fully resonant. There is evidence that fully resonant, equispaced spectra may play an important role in the turbulent energy transfer leading to the non-perturbative AdS instability [14]-[19]. This leads to a rich phenomenology that also appears in non-linear models of different physical nature as in self-gravitating scalars on a spherical cavity in 3+1 [20], on systems describing Bose-Einstein condensates [21], and vortex precession [22], as well as in the conformal dynamics on the Einstein Universe [23]. A program to classify all the spacetimes in which a minimally coupled scalar probe may lead to an exactly solvable effective Schroedinger problem, suggestively dubbed “Klein-Gordonization”, was initiated in [24].

In this paper we go beyond the linear level and perturbativelly explore different aspects of the non-linear dynamics of a fully resonant, self-interacting scalar probe on the wormhole spacetime [2]. In Section II, we revisit with detail the linear propagation of a scalar probe with a precise non-minimal coupling with the Ricci scalar, and define the “wormhole oscillons”. After introducing a self-interaction, in Section III we construct the system of equations that control the dynamics of the infinite oscillators in the Two-Time-Framework (TTF) [25] or the time averaged system [26]-[27]. This approach has been particularly useful in the context of the non-perturbative instability of AdS, since it captures the dynamics at times of order $\varepsilon^{-2}$, where $\varepsilon$ characterizes the energy content of the initial perturbation. By truncating the system of oscillators we study the energy transfer between modes, and show that there are direct and inverse energy cascades. In particular, when the dynamics is dom-

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that a wormhole exists for a particular value of the coupling constants. Note also that recently the first wormhole solution in GR in vacuum has been constructed in [10] which connects two asymptotically AdS geometries through a traversable throat.
inated by a single mode, we find evidence of near exact energy returns, which is confirmed in Section IV analytically. Section V is devoted to conclusions and further comments on the universality of the weakly non-linear dynamics on AdS for Lovelock theories with generic couplings.

II. THE LINEAR SCALAR PROBE

Let us consider the equation for a scalar probe on the wormhole geometry (2), with a fixed non-minimal coupling

\[
\left(\Box - m^2 - \frac{3}{8} \mathcal{R}\right) \phi_1(x^\mu) = 0. 
\]

We will see below that even though the Ricci scalar is a non-trivial function of the radial coordinate

\[
R = -\frac{20}{\ell^2} + \frac{6}{\ell^2 \cosh^2(\rho)},
\]

the equation for the scalar probe can be solved analytically. Hereafter, for simplicity, we fix \(\ell = 1\).

Introducing a mode separation and considering only a radial spatial dependence

\[
\phi_1(t,\rho) = e^{-i\omega t} R(\rho), 
\]

and the radial coordinate \(\rho^* = 2 \arctan(e^{\rho})\) which maps \(\rho \in ]\infty, \infty[\) to \(\rho^* \in ]0, \pi[\), we obtain

\[
- \frac{d^2 S(\rho^*)}{d\rho^{*2}} + U(\rho^*)S(\rho^*) = \omega^2 S(\rho^*)
\]

where \(S(\rho) = \cosh(\rho)^{3/2} R(\rho)\) and the effective potential reads

\[
U(\rho^*) := \frac{1}{4} \frac{4m^2 - 15}{\sin^2(\rho^*)}.
\]

In terms of the coordinate \(\rho^*\), the metric is manifestly conformal to the product of \(\mathbb{R}_t \times \mathbb{R} \times S^1 \times \Sigma_2\), and reads

\[
ds^2 = \frac{1}{\sin^2(\rho^*)} [-dt^2 + d\rho^{*2} + d\varphi^2 + d\Sigma_2^2],
\]

and the wormhole boundaries are located at the divergences of the conformal factor.

\(^2\) Note that this is not the conformal coupling since in general \(\xi_{conf} = \frac{D-2}{4(D-1)}\).
The equation (6) is that of a quantum particle moving in a Rosen-Morse potential. For the following analysis it is convenient to introduce the coordinate \( z \) such that
\[
\tanh(\rho) = 1 - 2z ,
\]
which maps \( \rho \in \mathbb{R}, \mathbb{R} \) to \( z \in [1, 0] \). The wave equation (3) leads to
\[
\ddot{\phi}_1 + L \phi_1 = 0
\]
where the operator \( L \) is defined as
\[
L := -z^2(1-z)^2 \frac{d}{dz} \left( \frac{1}{z(1-z)} \frac{d}{dz} \right) + \frac{m^2 - 15/2}{4z(1-z)} + \frac{9}{4}.
\]
This operator admits the following asymptotic behaviors
\[
R(z) \xrightarrow{z \to 0} D_1 z^{\Delta_+} + D_2 z^{\Delta_-},
\]
\[
R(z) \xrightarrow{z \to 1} \tilde{D}_1 (1-z)^{\Delta_+} + \tilde{D}_2 (1-z)^{\Delta_-},
\]
where \( \Delta_\pm = 1 \pm \frac{1}{2} \sqrt{m^2 - 7/2} \). Note that \( \Delta_+ > 0 \), while \( \Delta_- \leq 0 \) for \( m^2 \geq 15/2 \). Assuming \( m^2 \geq 15/2 \) (which in global AdS would correspond to \( m^2 \geq 0 \)), we impose reflective boundary conditions at both boundaries \( z = 1 \) and \( z = 0 \), setting \( D_2 = \tilde{D}_2 = 0 \), and consequently the operator \( L \) is essentially self-adjoint on \( L^2([1,0], -\frac{1}{z(1-z)^2}) \). This differential eigenvalue problem therefore leads to the following normal frequencies and normal modes
\[
\omega_j^2 = \left( j + \frac{1}{2} + \sqrt{m^2 - \frac{7}{2}} \right)^2
\]
\[
e_j(z) = C_j z^{1+\frac{1}{2} \sqrt{m^2 - \frac{7}{2}}} (1-z)^{1-\frac{1}{2} \sqrt{m^2 - \frac{7}{2}}}
\]
\[
\times {}_2F_1 \left( -j - \sqrt{m^2 - \frac{7}{2}}, 1 + j + \sqrt{m^2 - \frac{7}{2}}; 1 + \sqrt{m^2 - \frac{7}{2}}; z \right).
\]
where \( C_j \) are normalization constants that depend on the mass of the scalar probe. Hereafter we refer to (14) as “wormhole oscillons”. The general solution to the linear problem is given by an arbitrary superposition of the modes (14), leading to
\[
\phi_1(t, z) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j) e_j(z).
\]
For simplicity, and mimicking the massless case in AdS, we set \( m^2 = 15/2 \). In this case the normalization constants in (14) fulfill
\[
2C_j = [(j + 1)(j + 2)(j + 3)(j + 4)(5 + 2j)]^{1/2}
\]
leading to \((e_i(z), e_j(z)) = \delta_{ij}\). The frequencies which were already equispaced in (13), further reduce to
\[
\omega_j = \frac{5}{2} + j
\]
with \(j = 0, 1, 2, \ldots \).

Note that we have been able to find a fully resonant, equispaced spectrum for a scalar probe propagating on a spacetime with non-trivial topology. Below, we introduce a self-interaction on the scalar to characterize the energy transfer between modes. As mentioned above, it has been shown that such problem captures some features of the backreacting, massless scalar in AdS (see e.g. [28], [29]).

**III. SELF-INTERACTING SCALAR PROBE**

Now we will introduce a non-linearity in the scalar probe we discussed in the previous section. In particular we will focus on
\[
\Box \phi - \left( m^2 + \frac{3}{8} R \right) \phi - \frac{\lambda}{3!} \phi^3 = 0 ,
\]
setting \(m^2 = 15/2\), since this value leads to \(\Delta_- = 0\) in (12) and mimics the massless case in AdS. Here \(\lambda\) is a constant with mass dimension \(-1\). Following the TTF we introduce the slow time \(\tau = \epsilon^2 t\) and the perturbative ansatz
\[
\phi(t, \tau, z) = \sum_{j=0}^{\infty} \epsilon^{2j+1} \phi_{2j+1}(t, \tau, z) .
\]
Note that a direct perturbative approach leads to resonant terms, some of which could be perturbatively absorbed by a Poincare-Lindstedt shift. The TTF helps dealing with this feature, and even more its validity is ensured at least up to times of order \(\epsilon^{-2}\). Naturally, at first order in \(\epsilon\) one re-obtains the linear problem
\[
\partial_t^2 \phi_1 + L \phi_1 = 0,
\]
where the operator \(L\) is given in (11), leading to
\[
\phi_1(t, \tau, x) = \sum_{l=0}^{\infty} \left( A_l(\tau) e^{-i\omega_l t} + \bar{A}_l(\tau) e^{i\omega_l t} \right) e_l(x),
\]
where \( A_i(\tau) \) are arbitrary functions of the slow time \( \tau \). Here \( \tilde{A}_i(\tau) \) stands for the complex conjugate of \( A_i(\tau) \). At the next perturbative order in \( \epsilon \) one obtains

\[
\partial_t^2 \phi_3 + L\phi_3 + 2\partial_t \partial_\tau \phi_1 = S(t, \tau, x), \tag{22}
\]

with the source given by the lower order term \( S = -\frac{1}{24} \frac{\phi_1^4}{1-z} \). Here one proposes a solution for \( \phi_3 \) of the form:

\[
\phi_3(t, \tau, x) = \sum_{n} \left( B_n(t, \tau) + \tilde{B}_n(t, \tau) \right) e_n(x). \tag{23}
\]

Projecting equation (22) on the basis of wormhole oscillons one obtains

\[
\partial_t^2 \left( B_j + \tilde{B}_j \right) + \omega_j^2 \left( B_j + \tilde{B}_j \right) - 2i\omega_j (\partial_\tau A_j e^{-i\omega_j t} - \partial_\tau \tilde{A}_j e^{i\omega_j t}) = (e_j, S), \tag{24}
\]

with

\[
(e_j, S) = \sum_{n,l,m} \frac{S_{jnlm}}{3} \left( A_l A_m A_n e^{-i(\omega_l + \omega_m + \omega_n)t} + 3A_l A_m \tilde{A}_n e^{-i(\omega_l + \omega_m - \omega_n)t} \right.
\]
\[
+ 3A_l \tilde{A}_m A_n e^{-i(\omega_l - \omega_m - \omega_n)t} + \tilde{A}_l A_m \tilde{A}_n e^{-i(\omega_l - \omega_m - \omega_n)t} \right), \tag{25}
\]

where the interaction integrals \( S_{jnlm} \in \mathbb{R} \) are defined as

\[
S_{jnlm} = \frac{1}{8} \int_1^0 \frac{e_j(z)e_n(z)e_l(z)e_m(z)}{z^3(1-z)^3} dz. \tag{26}
\]

The TTF equations are obtained by imposing that the functions \( A(\tau) \) that appear in the l.h.s. of equation (24), exactly cancel the resonant terms coming from the r.h.s. of the same equation. This leads to

\[
-2i\omega_j \partial_\tau A_j = \sum_{n,l,m} \frac{S_{jnlm}}{3} \left[ A_l A_m A_n \delta_{\omega_j, \omega_l + \omega_m + \omega_n} + 3A_l A_m \tilde{A}_n \delta_{\omega_j, \omega_l + \omega_m - \omega_n} \right.
\]
\[
+ 3A_l \tilde{A}_m A_n \delta_{\omega_j, \omega_l - \omega_m - \omega_n} + \tilde{A}_l A_m \tilde{A}_n \delta_{\omega_j, \omega_l - \omega_m - \omega_n} \right]. \tag{27}
\]

We observe that the integrals \( S_{jnlm} \), are non-vanishing only in the channel \( \omega_j + \omega_n = \omega_l + \omega_m \) (or equivalently \( j + n = l + m \)), leading to the TTF equations

\[
-2i\omega_j \partial_\tau A_j = \sum_{j+n=m+l} S_{jnlm} A_l A_m \tilde{A}_n. \tag{28}
\]

Observing the symmetries in the indices of the overlap integrals \( S_{jnlm} \), and the fact they vanish unless \( j + n = m + l \), one can deduce that the total energy \( E = \sum_j \omega_j^2 \tilde{A}_j(\tau) A_j(\tau) \).
and the “particle number” \( N = \sum_j \omega_j \bar{A}_j(\tau) A_j(\tau) \) are conserved \([28],[27],[30]\). The conservation of these quantities is particularly useful for monitoring the stability of the numerical integration of the truncated version of the system \([28]\).

In what follows we will solve the system of oscillators by truncating the sum up to order \( j = j_{\text{max}} \), for different initial data. We monitor the convergence by increasing \( j_{\text{max}} \) and study the energy transfer between modes induced by the non-linearities.

\[
\ln(E_j) \quad \tau = 100 \quad \tau = 1000 \quad \tau = 5000 \quad \tau = 10000 \quad \tau = 20000 \quad \tau = 100000 \quad \tau = 200000
\]

**FIG. 1**: Evolution of the energy per mode as a function of the mode number \( j \), for \((E_0(0), E_1(0)) = (3/4, 1/4)\) (left-panel) and \((E_0(0), E_1(0)) = (1/2, 1/2)\) (right-panel). For late times the energy per mode is exponentially suppressed for large \( j \), i.e. \( E_j \sim e^{-j} \).

In Figure 1 we plot the evolution of the spectrum, showing energy transfer induced by the non-linearities, for different initial conditions. We have evolved the truncated TTF system with \( j_{\text{max}} = 50 \). The spectra stabilize after some time, showing an exponential suppression of the energy as a function of the mode number. As it occurs for non-backreacting probes in AdS, these spectra suggest the absence of a turbulent phenomenology.

Figure 2 shows the actual time evolution of the energy per mode, for different initial conditions with \( j_{\text{max}} = 50 \). Even though the energy is initially distributed only in the fundamental and first excited modes, the non-linearities transfer energy to the higher harmonics. The plots suggest energy returns after a finite time. In the next section we provide perturbative evidence of near exact energy returns for situations as that depicted in the upper left panel, in which the dynamics is clearly dominated by the fundamental mode.

It is also illustrative to consider initial data with three and four modes turned on. Figure 3 shows the time evolution of the energy content as well as the evolution of the stabilized spectra. Note that also in this case, the energy in modes with large \( j \) are exponentially suppressed.
FIG. 2: The plots present the evolution of the energy per mode, for different initial excitations in the fundamental and first excited mode. The upper left panel corresponds to $(E_0(0), E_1(0)) = (3/4, 1/4)$, for the upper right panel we have used $(E_0(0), E_1(0)) = (3/5, 2/5)$, the lower left panel corresponds to the two-mode equal energy initial date $(E_0(0), E_1(0)) = (1/2, 1/2)$ and finally, for the lower right panel $(E_0(0), E_1(0)) = (1/4, 3/4)$.

IV. NEAR EXACT ENERGY RETURNS

Closely following reference [31], a perturbative argument can be given to have an analytic understanding of the (near) exact energy returns suggested by Figure 1. We will focus on the situation such that the dynamics is dominated by the fundamental mode, as well as in the case where the first excited mode dominates.

A. Fundamental mode dominating the dynamics

In particular in the case in which the dynamics is dominated by the fundamental mode, we can introduce the scaled oscillators $q_j$ such that

$$A_j(\tau) = \frac{q_j(\tau)}{\sqrt{\omega_j}}\delta^j,$$  \hspace{1cm} (29)
FIG. 3: Time evolution of the energy and spectra for three (upper panel) and four (lowe panel) modes with equal energy as initial conditions.

with \( \delta \) a small, perturbative parameter. The time-averaged system now reads

\[
i \partial_\tau q_j = \sum_{m=0}^{\infty} \sum_{k=0}^{j+m} \delta^{2m} C_{j,m,k,j+m-k} q_k q_{n+m-k}, \tag{30}
\]

where

\[
C_{j,n,l,m} = -\frac{1}{2} \frac{S_{jnlm}}{\omega_j \omega_n \omega_l \omega_m}, \tag{31}
\]

and the overlap integrals have been defined in (26). At leading order in \( \delta \) we obtain the non-linear system

\[
i \partial_\tau q_j = \bar{q}_0 \sum_{k=0}^{j} C_{j,0,k,j-k} q_k q_{n-k}. \tag{32}
\]

For \( j = 0 \), the system leads to a decoupled, non-linear equation for \( q_0 \), which is solved in a closed form, giving a constant modulus and a time dependent phase for \( q_0 \in \mathbb{C} \). The global symmetries of the system can be used to set the absolute value of \( q_0 \) to 1. Then, for \( j \geq 1 \), one obtains a set of linear equations that can be solved in a recursive manner, where the \( q_{k<j} \)'s appear as sources. The homogeneous equations depend only on the coefficients of the form \( C_{j0j0} \) (Figure 4 depicts these integrals up to \( j \sim 500 \)). A further use of the symmetries of the system allows to set

\[
q_0(\tau) = e^{-iC_{0000} \tau} \quad \text{and} \quad q_1(\tau) = e^{-2iC_{1010} \tau}. \tag{33}
\]
FIG. 4: The scaled overlap integral $C_{j0j0}$ that determine the dynamics when the fundamental mode dominates.

With this in mind one can compute the time periods of the energies in the higher modes by computing the periods $T_j$ of $E_j \sim q_j \bar{q}_j$. Using our overlap integrals we obtain $T_2 = 24024\pi$ and $T_3 = 17T_2$, $T_4 = 19T_3$, $T_5 = T_4$, $T_6 = 23T_5$. Note that the ratios of the frequencies are relatively simple fractions (a simple, pictorial method to see the exact and near exact returns is outlined in Figure 5). The commensurability of the periods of the energy, ensure exact energy returns at finite time within this perturbative approach. Nevertheless it must be noted that the periods $T_j$ are an increasing function of the mode number $j$, and therefore as more modes are included in the analysis one should have to wait longer for observing the recurrence. Note even though that higher modes are suppressed as $\delta^j$.

**B. First excited mode dominating the dynamics**

The lower-right panel of Figure 2 depicts a case in which the first excited mode is dominating the dynamics of the energy content in the system. We can analytically explore such case in a perturbative manner by introducing the ansatz

$$A_{0} = \frac{q_{0}(\tau)}{\sqrt{\omega_{0}}} \delta \quad \text{and} \quad A_{j \geq 1}(\tau) = \frac{q_{j}(\tau)}{\sqrt{\omega_{j}}} \delta^{j-1}. \quad (34)$$

Retaining the leading contributions as $\delta$ goes to zero, one obtains

$$i\dot{q}_{0} = C_{0211}q_{2}q_{1}^{2} + 2C0101|q_{1}|^{2}q_{0}, \quad (35)$$

and

$$i\dot{q}_{n} = \bar{q}_{1} \sum_{k=1}^{n} C_{n,1,k,n+1-k}q_{k}q_{n+1-k} + \bar{q}_{0} \sum_{k=1}^{n-1} C_{n,0,k,n-k}q_{k}q_{n-k}. \quad (36)$$
FIG. 5: The figures depict the exact and near exact energy returns for the modes with \( j = 3, \, 4 \) and 6. Since in the initial condition these modes do not have energy, the exact returns are obtained once the dots intersect the horizontal line.

where the couplings \( C \)'s have been defined in (31). As before using the global symmetries, the time dependence of the leading oscillator appear in its phase, while the time dependence of the first two subleading oscillators leads to equal periods of their energy content given by \( T_0 = T_2 = 14586\sqrt{14/635}\pi \). For the sub-subleading modes, after some manipulation (see Section 4.2 of [31]), one obtains that \( q_3(\tau) \) is a superposition of non-commensurable oscillations. Therefore in this case the returns may be less exact than in the previous scenarios (see Figure 5).

V. FURTHER COMMENTS

In this work we have shown the existence of a spacetime with non-trivial topology on which the linear dynamics of a scalar probe turn out to be fully resonant, leading to a rich phenomenology when non-linearities are included. The five-dimensional wormhole geometry explored in this work, can be generalized to arbitrary odd dimensions \( D = 2n + 1 \)

\[
ds^2 = \ell^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \cosh^2 \rho d\Sigma_{D-2}^2 \right], \tag{37}
\]
FIG. 6: This plot exhibits the near exact returns of the energy in the third mode. Even though some point seem to lie on the horizontal axis, the actually do not touch it since for example for $n = 31$, $|q_3(nT_2)|$ is of the order $10^{-7}$.

as a solutions of Lovelock theory in the Chern-Simons case [3], provided the manifold $d\Sigma_{D-2}^2$ fulfils a suitable scalar constraint [8,32,33,34]. It was shown in [11] that a linear, non-minimally coupled scalar probe, fulfilling reflective boundary conditions has the following spectrum

$$\omega_n^2 = \left(n + \frac{1}{2} + \sqrt{\left(\frac{D-1}{2}\right)^2 + m_{\text{eff}}^2 l^2}\right)^2 - \left(\frac{D-2}{2}\right)^2 + Q + \xi \left[(D - 1)(D - 2) + \tilde{R}\right]. \tag{38}$$

where $m_{\text{eff}}^2 := m^2 l^2 - D(D - 1)\xi$, $\tilde{R}$ is the Ricci scalar of the Euclidean manifold $\Sigma_{D-2}$ (which we assume constant) and $Q$ stands for an eigenvalue of the Laplace operator on such Euclidean manifold, normalized as $\nabla^2_\Sigma Y = -Q Y$ ($Q$ being positive if $\Sigma$ is compact and without boundary). Generically, this spectrum will be only asymptotically resonant, nevertheless, in the particular case in which

$$\left(\frac{D-2}{2}\right)^2 - Q - \xi \left[(D - 1)(D - 2) + \tilde{R}\right] = 0 , \tag{39}$$

the spectrum will be exactly equispaced. For a given rotational dependence of the scalar probe, i.e. for a fixed value of $Q$, resonance can be achieved for a particular value of the non-minimal coupling parameter $\xi$, providing a new, infinite family of gravitational backgrounds with fully resonant, equispaced spectrum for scalars probes.

- On the universality of the weakly non-linear dynamics in Lovelock theories
Some comments on the backreacting case are in order. For the Einstein-Gauss-Bonnet theory, the scalar field collapse in AdS has been explored [35]. As shown below, remarkably, one can give a general analysis of the perturbative TTF, time averaged approach in a generic Lovelock theory, in AdS. The field equations of Lovelock theories coupled to a massless scalar (the analysis can be trivially extended to the massive case) read

$$\mathcal{E}_{\mu\nu} := \sum_{k=0}^{[D/2]} \alpha_k E^{(k)}_{\mu\nu} - T_{\mu\nu} = 0 ,$$  \hspace{1cm} (40)

where $T_{\mu\nu}$ stands for the stress-energy tensor of the minimally coupled scalar field $\phi$ and the Lovelock tensor of order $k$ is defined as

$$E^{(k)}_{\mu\nu} := - \frac{1}{2k+1} g_{(\mu|\sigma} g^{\rho_1 \cdots \rho_{2k}}_{\nu)} R^{\gamma_1 \gamma_2}_{\rho_1 \rho_2} \cdots R^{\gamma_{2k-1} \gamma_{2k}}_{\rho_{2k-1} A_{2k}} .$$  \hspace{1cm} (41)

Here the couplings $\alpha_k$ are dimensionfull.

Consider a metric of the form

$$d s^2_n = g_{\mu\nu}d x^\mu d x^\nu = g_{ab}^{(2)} (y^c) d y^a d y^b + F^2 (y) d \Omega^2_{S^{n-2}} ,$$  \hspace{1cm} (42)

where $d \Omega_{S^{n-2}}$ stands for the line element of the $(n - 2)$-sphere, $g_{ab}^{(2)}$ is a metric on a two-dimensional, Lorentzian manifold $M_2$, and $F (y)$ is a scalar on $M_2$.

The components of the $p$–th Lovelock tensor along the two-dimensional manifold $M_2$ were explicitly computed in [36] and read

$$E^{(k)}_{ab} = - \frac{k (n - 2)!}{(n - 2k - 1)!} D_a D_b F - g_{cd}^{(2)} D^c F D^d F g_{ab}^{(2)} \left( \frac{1 - g_{ef}^{(2)} D^e F D^f F}{F^2} \right)^{k-1}$$

$$- \frac{(n - 2)!}{2 (n - 2k - 2)!} g_{ab}^{(2)} \left( \frac{1 - g_{ef}^{(2)} D^e F D^f F}{F^2} \right)^k ,$$

where $D_a$ is the Levi-Civita covariant derivative on $M_2$. As usual, due to diffeomorphism invariance and spherical symmetry the Lovelock equations along the angles in $S^{n-2}$, are a consequence of the equation along $M_2$ and the equation for the scalar field. The expression for the Lovelock equations on the metric of our interest

$$d s^2 = \frac{L^2}{\cos^2 (x)} \left[ -e^{-2f(t,x)} A(t,x) dt^2 + \frac{dx^2}{A(t,x)} + \sin^2 (x) d \Omega^2_{S^{n-2}} \right] ,$$  \hspace{1cm} (43)

can be directly obtained by setting $F(y^a) = \tan x$ and $g_{ab}^{(2)}$ the metric along the $(t, x)$ directions in (43). Following [37], here we will consider the scaled slow time

$$\tau = s_1 \varepsilon^2 t ,$$  \hspace{1cm} (44)
where \( s_1 \) and \( s_2 \) (below) are finite constants to be fixed at convenience. We will consider the expansions

\[
A(t, \tau, x) = 1 + s_2 \varepsilon^2 A_2(t, \tau, x) + \mathcal{O}(\varepsilon^4),
\]

(45)

\[
f(t, \tau, x) = s_2 \varepsilon^2 f_2(t, \tau, x) + \mathcal{O}(\varepsilon^4),
\]

(46)

\[
\phi(t, \tau, x) = \varepsilon \phi_1(t, \tau, x) + s_2 \varepsilon^3 \phi_3(t, \tau, x) + \mathcal{O}(\varepsilon^5).
\]

(47)

At the lowest order, the Lovelock equations determine the AdS radius \( L \) in terms of the couplings \( \alpha_p \), through the equation

\[
P[L^{-2}] := \sum_{p=0}^{[D/2]} \frac{\alpha_p (-1)^p}{(n-2p-1)!} \left( \frac{1}{L^2} \right)^p = 0,
\]

(48)

which defines the polynomial \( P[\xi] \). The equation for the scalar field, at the lowest order, determine \( \phi_1(t, x) \) as an arbitrary superposition of \( n \)-dimensional AdS oscilons. At order \( \varepsilon^2 \) one obtains the following expression for the \( E_t \), \( E_x \) and \( E_{tx} \) equations in (40):

\[
-\frac{(n-2)!}{2L^2} \frac{dP[L^{-2}]}{d\xi} \left( \frac{A^2_2}{\tan x} - \frac{2\cos^2 x - n + 1}{\sin^2 x} A_2 \right) = -\frac{\cos^2 x}{2L^2} \left( \dot{\phi}_1^2 + \phi_1^2 \right),
\]

(49)

\[
-\frac{(n-2)!}{2L^2} \frac{dP[L^{-2}]}{d\xi} \frac{\dot{A}_2}{\tan x} = -\frac{\cos^2 x}{L^2} \dot{\phi}_1^2 \dot{\phi}_1,
\]

(50)

\[
-\frac{(n-2)!}{2L^2} \frac{dP[L^{-2}]}{d\xi} \left( \frac{A^2_2}{\tan x} - \frac{2\cos^2 x - n + 1}{\sin^2 x} A_2 - \frac{2f_2'}{\tan x} \right) = \frac{\cos^2 x}{2L^2} \left( \ddot{\phi}_1^2 + \phi_1^2 \right).
\]

(51)

where (') and (·) denote derivative with respect to \( x \) and \( t \), respectively. No derivatives with respect to the slow time appear at this order. From these equations one can solve \( A_2(t, x) \) and \( f_2(t, x) \) as in GR. Then, substituting this at the next non-trivial order in the Klein-Gordon equation, one obtains

\[
\dddot{\phi}_3 + \mathcal{L}[\phi_3] + \frac{2s_1}{s_2} \partial_t \partial_x \phi_1 = \phi_1^2 A_2 + \phi_1 A_2' - \phi_1 \dot{f}_2 - \phi_1 f_2' + 2 \left( f_2 - A_2 \right) \mathcal{L}[\phi_1]
\]

(52)

where the action of the operator \( \mathcal{L}[\phi_i] \) is defined as

\[
\mathcal{L}[\phi_i] := -\frac{\partial^2 \phi_i}{\partial x^2} - \frac{(n-2)}{\cos x \sin x} \frac{\partial \phi_i}{\partial x}.
\]

(53)

Therefore, as in reference [37] for the Einstein-Gauss-Bonnet case in five dimensions, but now in the whole family of Lovelock theories, setting

\[
s_1 = s_2 = \left( \frac{dP[L^{-2}]}{d\xi} \right)^{-1}
\]

(54)
we see that the equations for the TTF approach, for a generic Lovelock theory \((49)-(52)\) take exactly the same functional form than the equations in GR, provided we are expanding about AdS with a curvature corresponding to a simple zero of the polynomial \((48)\), for which \(\frac{dP}{d\xi} L^{-2}\) is non-vanishing. It is interesting to see that this polynomial, completely controls the perturbative dynamics in Lovelock theories. On the other hand, for the wormhole studied in this work, the asymptotic AdS curvature radius exactly cancels the derivative of the polynomial (this occurs for any Chern-Simons theory within the Lovelock family) and therefore, the perturbative approach, in the backreacting situation does not apply. This is why we have focused on the probe limit of the scalar. Notwithstanding, the equations presented above for Lovelock theories are a signal of the universality of the weakly-nonlinear dynamics up to times of order \(\varepsilon^{-2}\), captured by the TTF, for generic values of the couplings, when the higher curvature curvature terms belong to the Lovelock family. This was shown in \([37]\) (see also \([38]\)) for the Einstein-Gauss-Bonnet case, and here we have shown that such results extend to the whole family of Lovelock theories for generic values of the couplings. These results extend also for the family of Quasitopological Gravities \([39]\), \([40]\), \([41]\), \([42]\), since the dynamics on the spherically symmetric dynamical scenarios with matter, can be obtained from the same formulae \((49)-(51)\), by including extra terms in the polynomial. All the mentioned theories admit Birkhoff’s theorems, therefore on a spherically symmetric scenario, the dynamics is completely driven by the scalar.

Finally, it is well-known that for Lovelock theories containing a \(k\)-th order term in dimension \(n = 2k + 1\), the maximally symmetric AdS vacuum is gapped with respect to the smallest black hole (see e.g. \([43]\), \([44]\)), as it occurs for the BTZ black hole \([45]\). It is interesting to note that the TTF dynamics, at times \(\varepsilon^{-2}\), being universal for Lovelock theories with generic couplings, does not capture this gap\(^3\).

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\(^3\) In \([46]\) it was noted that in a direct perturbative at order \(\varepsilon^4\) the presence of a Gauss-Bonnet term cannot be scaled out.
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