Generalized ADT charges and asymptotic symmetry algebra

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ABSTRACT

Using the expressions for generalized ADT current and potential in a self consistent manner, we derive the asymptotic symmetry algebra on AdS$_3$ and the near horizon extremal BTZ spacetimes. The structure of symmetry algebra among the conserved charges for asymptotic killing vectors matches exactly with the known results thus establishing the algebraic equivalence between the well known existing formalisms for obtaining the conserved charges and the generalized ADT charges.

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1 Introduction

Conserved charges in general relativity and asymptotic symmetries has recently witnessed a plethora of activities capturing several interesting aspects of AdS/CFT correspondence [1]. The programme initiated with a seminal work of Brown and Henneaux [2] has now become a cornerstone in establishing a concrete relationship between symmetry generators of dual conformal field theory (CFT) and asymptotic symmetries of corresponding bulk anti-de-Sitter (AdS) geometry. The formalism relies on the fact that the algebra among the conserved charges for diffeomorphisms satisfying certain asymptotic boundary conditions is isomorphic to two copies of Virasoro algebra. Indeed, by extracting the central charge of the Virasoro algebra and using Cardy formula, the entropy of Banados-Teitelboim-Zanelli (BTZ) black hole was obtained by Strominger [3]. Since then, there have been much interests in the extension of this work to geometries which are not asymptotically AdS. One such direction has been the study on the asymptotic symmetry algebra in the context of the Kerr/CFT correspondence [4, 5].

Important step in all the above developments is to identify the conserved charges which form a sub-algebra corresponding to the isometry group of the given spacetime. However, in theory of gravity with diffeomorphism symmetry, it is not so straightforward to define conserved charges in an unambiguous manner. There have been several attempts to define conserved charges in gravity. In the well known ADM approach developed in [6], conserved charges are obtained by introducing on-shell vanishing currents by linearizing the Einstein equations on a given asymptotically flat background. After removing certain ambiguity coming from the potential, one can define conserved quantities through the integral of the on-shell potential corresponding to asymptotic killing vectors. Extension of the above method to non-trivial asymptotic boundaries like AdS have been given by Abbott, Deser and Tekin (ADT) [7]. There are also other approaches to derive quasi-local conserved charges without resorting to the linearization of dynamical fields such as Brown and York method [8] and covariant phase space formalism [9, 10, 11, 12, 13]. General theory of conserved charges based on the cohomology principles was developed by Barnich-Brandt and Compere (BBC) [14, 15] (for review see [16, 17]). For exact Killing symmetries, the asymptotic symmetry generators computed within this formalism was shown to be in agreement with those obtained via covariant phase space formalism.

An alternative approach to understand quasi-local conserved charges has been presented recently [18]. A key result of our work was the establishment of one-to-one mapping between off-shell linearized Noether potential and ADT potential. Furthermore, by integrating the linearized Noether potential along a path in phase space, we were able to propose a definition of quasi-local conserved charges which are completely consistent with ADT charges. The method works well for any covariant theory of gravity including the higher curvature theories and also the gravitational Chern-Simons term [19]. Extension of this approach including the asymptotic Killing vectors
has been provided in [20, 21, 22]. The asymptotic Killing vectors or diffeomorphisms play a vital role in determining the structure of asymptotic symmetry algebra. Fixing the asymptotic behaviour of fields in gravity is not straightforward as the choice of appropriate fall-off conditions at spatial infinity is not unique. For instance, the conventional Brown-Henneaux boundary condition given in [2, 3] results into two copies of Virasoro algebra whereas the one used in [4] leads to chiral Virasoro algebra. Other choices for the boundary conditions are also possible leading to interesting physics [23, 24, 25].

Although the application of off-shell ADT method for several interesting geometries is present in the literature [26] (for extensive review see [24]), its role is limited for exact Killing symmetries. Thus, it is not apparent as how to perform asymptotic symmetry analysis using off-shell ADT method. In order to strengthen the robustness and wide applicability of off-shell formalism [18], it is natural to look for the construction of symmetry algebras among the off-shell ADT charges corresponding to the asymptotic diffeomorphisms satisfying various boundary conditions as mentioned in the last paragraph. In this paper, we obtain the asymptotic symmetry algebra entirely within off-shell ADT formalism. To achieve this, we first derive the off-shell expressions for quasi-local ADT charges for the general asymptotic AdS3 spacetime which follows from the Brown-Henneaux boundary conditions [2]. Then, we give detailed analysis of symmetry algebra satisfied by the generalized off-shell ADT charges. As an another illustration, we apply our method to near horizon extremal BTZ black hole. This geometry provides a naive application of Kerr/CFT correspondence capturing all its essential features. The suitable fall-off conditions are the one given in [4]. For both the cases, we find that the asymptotic symmetry algebra among the generalized off-shell ADT charges is in complete agreement with the corresponding results obtained by using the method given in [14, 15].

2 Generalized ADT charges

In this section, we summarize the method for obtaining off-shell ADT currents and charges given in [18] and its generalization [20].

A generic variation of action for any covariant theory of gravity \(S[R, R^2, \cdots]\) is given by

\[
\delta S = \frac{1}{16\pi G} \int d^Dx \frac{\delta}{\delta g} \left( \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{4} (R^\mu)^\sigma_\sigma \delta g^{\mu\nu} \right] \right),
\]

where \(G_{\mu\nu} = 0\) is the equations of motion (EOM) for the metric and \(\Theta\) is the surface term given by

\[
\Theta^\mu(\delta g) = 2\sqrt{-g} [P^{\mu(\nu\rho)}_\sigma \nabla_\sigma \delta g_{\nu\rho} - \delta g_{\nu\rho} \nabla_\sigma P^{\mu(\nu\rho)}_\sigma],
\]

where the \(P\)– tensor is defined as

\[
P_{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}.
\]
Equating the above variation with the corresponding variation caused by any arbitrary diffeomorphism $\xi$, we identify the off-shell conserved Noether current

$$J^\mu_\xi \equiv \partial_\nu J^\mu_{\xi,\nu} = 2\sqrt{-g} G^{\mu\nu} \xi_\nu + \sqrt{-g} \xi^\mu L - \Theta^\mu(\xi). \tag{4}$$

After using Eq. (2) in the last expression the Noether potential $J^\mu_{\xi,\nu}$ can be written as

$$J^\mu_{\xi,\nu} = \sqrt{-g} 2\left[P^{\mu\rho\sigma} \nabla_\rho \xi_\sigma - 2\xi_\sigma \nabla_\rho P^{\mu\rho\sigma}\right]. \tag{5}$$

On the other hand, for exact Killing symmetries $\zeta$, one can introduce linearized off-shell ADT current $\bar{J}^\mu_{ADT}$ and potentials $\bar{J}^\mu_{ADT,\nu}$ as

$$\bar{J}^\mu_{ADT} \equiv \nabla_\nu \bar{J}^\mu_{ADT,\nu} = \delta G^{\mu\nu} \zeta_\nu + \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta} G^{\mu\nu} \zeta_\nu + G^{\mu\nu} \delta g_{\rho\sigma} \zeta_\rho - \frac{1}{2} \zeta_\mu G^{\alpha\beta} \delta g_{\alpha\beta}. \tag{6}$$

Using Bianchi identity and the Killing property of $\zeta$, we can immediately verify the off-shell conservation of $\bar{J}^\mu_{ADT}$. It is important to note that the expression for the off-shell ADT current given above is strictly valid for the exact Killing vector $\zeta$ whereas the off-shell Noether potential Eq. (5) holds true for any diffeomorphism $\xi$.

The relation between the off-shell ADT potential given above and the off-shell Noether potential follows immediately by considering the variation $\delta g_{\mu\nu}$ around some given background admitting the killing vector $\zeta$. Under this variation, the change in the off-shell Noether current Eq. (4) can be written in terms of the off-shell Noether potential as

$$\partial_\nu (\delta J^\mu_{\xi,\nu}) = 2\delta \left(\sqrt{-g} G^{\mu\nu} \zeta_\nu + \zeta^\mu \delta (\sqrt{-g} L) - \delta \Theta^\mu(g, \delta g)\right). \tag{7}$$

Writing Eq. (6) as

$$\partial_\nu (\sqrt{-g} \bar{J}^\mu_{ADT}) = \delta \left(\sqrt{-g} G^{\mu\nu} \zeta_\nu\right) - \frac{1}{2} \sqrt{-g} \zeta^\mu G^{\rho\sigma} \delta g_{\rho\sigma}. \tag{8}$$

and invoking the expression for generic variation of $\sqrt{-g} L$ we get

$$\bar{J}^\mu_{ADT}[g, \delta g, \zeta] = \frac{1}{2\sqrt{-g}} \left[\delta J^\mu_{\xi,\nu} - \zeta^\mu \Theta^\nu + \zeta^\nu \Theta^\mu\right]. \tag{9}$$

Therefore, the infinitesimal change in the conserved ADT charge for a killing vector $\zeta$ is given by

$$\delta \bar{Q}_{ADT}(\zeta) = \frac{1}{8\pi G} \int_{\Sigma} d\Sigma_{\mu\nu} \sqrt{-g} \bar{J}^\mu_{ADT}[g, \delta g, \zeta]. \tag{10}$$

The total quasi-local conserved charge can be determined by integrating the above expression along one parameter path in the phase space as
\[ Q_{ADT}(\zeta) = \frac{1}{8\pi G} \int_{0}^{1} d\alpha \int_{\Sigma} d\Sigma_{\mu\nu} \sqrt{-g} J^{\mu\nu}_{ADT}(g, \delta g, \zeta). \]  

(11)

where \( \Sigma \) is a boundary with co-dimension two.

We may note that the result for off-shell quasi-local ADT charges given by Eq. (11) is limited to exact Killing vectors. However, at the end we are interested in the diffeomorphisms which preserves certain asymptotic boundary conditions. In order to find out the expression for generalized ADT charges corresponding to any asymptotic Killing vector \( \xi \), we consider the right hand side of Eq. (9) augmented with 2–rank antisymmetric tensor \( T_{\mu\nu} \) \( [g, \delta g, \xi] \) and write the generalized ADT potential as

\[ J^{\mu\nu}_{ADT}[g, \delta g, \xi] = \frac{1}{2\sqrt{-g}} \left[ \delta J^{\mu\nu}_{\xi} - \zeta^{\mu} \Theta^{\nu} + \zeta^{\nu} \Theta^{\mu} \right] + \frac{1}{2} T^{\mu\nu}[g, \delta g, \xi]. \]  

(12)

The form for second rank antisymmetric tensor \( T^{\mu\nu} \) appearing in the last equation is restricted by two conditions. First, for exact Killing vector \( J^{\mu\nu}_{ADT} \) must match with \( \bar{J}^{\mu\nu}_{ADT} \). And secondly, the current built from the above expression is conserved at off-shell level. Generic form for \( T^{\mu\nu} \) satisfying these two criterion has been derived in \[20\]

\[ T^{\mu\nu}[g, \delta g, \xi] = -\frac{1}{2} \left( 3 P^{\mu\rho\sigma\nu} g^{\alpha\beta} - 4 g^{\sigma\nu} P^{\mu}[\rho|\alpha)|\beta] \right) \left( \nabla_{(\rho} \xi_{\alpha)} \delta g_{\sigma\beta} - \nabla_{(\sigma} \xi_{\beta)} \delta g_{\alpha\rho} \right) \]  

(13)

Therefore, the total conserved charge for arbitrary diffeomorphism \( \xi \) is given by

\[ Q_{ADT}(\xi) = \frac{1}{8\pi G} \int_{0}^{1} d\alpha \delta Q[g, \delta g, \xi] = \frac{1}{8\pi G} \int_{0}^{1} d\alpha \int_{\Sigma} d\Sigma_{\mu\nu} \sqrt{-g} J^{\mu\nu}_{ADT}(g, \delta g, \xi). \]  

(14)

We shall use this generalized ADT charge valid for any asymptotic Killing diffeomorphism in order to derive the asymptotic symmetry algebra.

### 3 Asymptotic symmetry algebra

We now turn our attention towards computation of asymptotic symmetry algebra satisfied by the generalized ADT charges Eq. (14). We illustrate our construction for the diffeomorphisms which preserves two well known boundary conditions given in \[2\] and \[4\]. For definiteness, we concentrate on 3–d Einstein gravity described by the action

\[ S[g_{\mu\nu}] = \frac{1}{16\pi G} \int d^{3}x \sqrt{-g} \left[ R + \frac{2}{l^{2}} \right]. \]  

(15)

First we note that for the above theory the \( P- \) tensor can be written as

\[ P^{\mu\rho\sigma\nu} = \frac{1}{2} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\rho\nu}). \]  

(16)

Hence, the expression for off-shell Noether potential \( J^{\mu\nu}_{\xi} \) becomes

\[ J^{\mu\nu}_{\xi} = \sqrt{-g} \left[ \nabla^{\mu} \xi^{\nu} - \nabla^{\nu} \xi^{\mu} \right]. \]  

(17)
Taking the $\delta$ variation and denoting $\delta g_{\mu\nu} = h_{\mu\nu}$, we get

$$
\delta J_{\xi}^{\mu\nu} = \sqrt{-g} \left\{ h^{\mu\beta} \nabla_{\beta} \xi^\nu + \frac{1}{2} \nabla_{\mu} \xi_{\nu} + g^{\beta[\mu} \nabla_{\nu[\beta} h_{\rho\alpha]} + \nabla_{\alpha} h_{\beta\rho} - \nabla_{\rho} h_{\beta\alpha} \right\} \xi^\alpha + h \nabla^{[\mu} \xi^{|\nu]} \right\}.
$$

(18)

Also, the surface term in Eq. (2) simplifies to

$$
\Theta^{\mu}[g, \delta g] = \sqrt{-g} \left[ \nabla^{\mu} (g^{\rho\sigma} \delta g_{\rho\sigma}) - \nabla_{\nu} \delta g^{\mu\nu} \right] = \sqrt{-g} \left[ \nabla_{\nu} h^{\mu\nu} - \nabla^{\mu} h \right].
$$

(19)

Therefore, the first term in Eq. (12) can be written as

$$
\delta J_{\xi}^{\mu\nu} - \zeta^{\mu} \Theta^{\nu} + \zeta^{\nu} \Theta^{\mu} = 2 \sqrt{-g} \left[ \xi_{\alpha} \nabla^{[\mu} h^{\nu]\alpha} - h^{\alpha[\mu} \nabla_{\alpha} \xi^\nu + \frac{1}{2} h \nabla^{[\mu} \xi^{|\nu]} + \xi^{[\mu} h^{\nu]} \right].
$$

(20)

Using the above equation in Eq. (12) and substituting Eq. (16) into Eq. (13) we get the expression for generalized ADT potential for Einstein gravity

$$
2 \sqrt{-g} J_{ADT}^{\mu\nu}[g, \delta g, \xi] = 2 \sqrt{-g} \left[ \xi_{\alpha} \nabla^{[\mu} h^{\nu]\alpha} - h^{\alpha[\mu} \nabla_{\alpha} \xi^\nu + \frac{1}{2} h \nabla^{[\mu} \xi^{|\nu]} - \xi^{[\mu} h^{\nu]} \right].
$$

(21)

Note that in this expression for generalized ADT potential the fields $g_{\mu\nu}$ and its perturbations need not have to satisfy the equations of motion. Also, the diffeomorphism $\xi$ is not restricted to exact symmetry.

### 3.1 Brown-Henneaux conditions

Any asymptotically AdS$_3$ solution for Einstein equations near the boundary can be expanded in Feffermann-Graham coordinates as

$$
ds^2 = \ell^2 \frac{dr^2}{r^2} + \frac{r^2}{\ell^2} \left( g^{(0)}_{ab} + \frac{\ell^2}{r^2} g^{(2)}_{ab} + \frac{\ell^4}{r^4} g^{(4)}_{ab} \right) dx^a dx^b,
$$

(22)

with $x^a = x^\pm = \frac{\ell}{2} \pm \phi$ and $\phi = \phi + 2\pi$.

We now look for the diffeomorphisms which change the metric near the asymptotic boundary as

$$
g_{rr} = \frac{1}{r^2} + O(1), \quad g_{tt} = -r^2 + O(1), \quad g_{tr} = O(1),
$$

$$
g_{t\phi} = O(1), \quad g_{r\phi} = O(1), \quad g_{\phi\phi} = r^2 + O(1).
$$

(23)

In Fefferman-Graham coordinates above set of conditions simply become

$$
g^{(0)}_{++} = g^{(0)}_{--} = 0, \quad g^{(0)}_{+-} = g^{(0)}_{-+} = -\frac{1}{2},
$$

(24)
with subleading terms unconstrained. The diffeomorphisms $\xi^\mu$ which preserves Eq. (24) are given by

$$\xi^+(x^+) = f(x^+)\partial_+ - \frac{1}{2}\partial_+ f(x^+)r\partial_r + \frac{\ell^2}{2r^2}\partial_+^2 f(x^+)\partial_-, $$

$$\xi^-(x^-) = g(x^-)\partial_- - \frac{1}{2}\partial_- g(x^-)r\partial_r + \frac{\ell^2}{2r^2}\partial_-^2 g(x^-)\partial_+, $$

where $f(x^+)$ and $g(x^-)$ are arbitrary function of their arguments. Since $\phi$ is $2\pi$ periodic, we can expand the above asymptotic diffeomorphisms in Fourier modes $e^{imx^\pm}$ and write

$$\xi^m_+(x^+) = e^{imx^+}\left[\partial_+ - \frac{imr}{2}\partial_r - \frac{\ell^2m^2}{2r^2}\partial_+\right], $$

$$\xi^m_-(x^-) = e^{imx^-}\left[\partial_+ - \frac{imr}{2}\partial_r - \frac{\ell^2m^2}{2r^2}\partial_+\right]. $$(26)

It is easy to see that the above diffeomorphisms satisfy Diff($S^1$) algebra

$$\{\xi^m_+(x^\pm), \xi^n_+(x^\pm)\} = i(n-m)\xi_{m+n}^\pm $$

(27)

To obtain the general form for $g^{(2)ab}$, we solve the Einstein equation order by order. This fixes the form for components of $g^{(2)ab}$ in terms of two arbitrary functions $T(x^+)$ and $\bar{T}(x^-)$ as

$$g^{(2)+-} = g^{(2)-+} = 0 \quad g^{(2)++} = \ell^2 T(x^+) \quad g^{(2)--} = \ell^2 \bar{T}(x^-), $$

(28)

while the subleading terms are constrained by the leading order ones as

$$g^{(4)ab} = \frac{1}{4}g^{(2)ac}g^{cd}(0)g^{(2)db}. $$

(29)

Therefore, the most general form for solution of Einstein equations respecting the Brown-Henneaux boundary conditions is given by

$$g_{\mu\nu} = \begin{pmatrix}
\ell^2 & 0 & 0 \\
0 & \ell^2 T(x^+) & -\frac{1}{2}\left[r^2 + \frac{\ell^2}{r^2}T(x^+)\bar{T}(x^-)\right] \\
0 & -\frac{1}{2}\left[r^2 + \frac{\ell^2}{r^2}T(x^+)\bar{T}(x^-)\right] & \ell^2 \bar{T}(x^-)
\end{pmatrix} $$

(30)

By setting $T^+ + T^- = 4GM$ and $T^+ - T^- = \frac{4GJ}{\ell}$, we recover the well known BTZ (rotating) black hole metric (in $t, r, \phi$ coordinates), as

$$ds^2 = -N^2dt^2 + \frac{dr^2}{N^2} + r^2(d\phi + N\phi dt)^2 $$

$$N^2 = -8MG + \frac{r^2}{\ell^2} + \left(\frac{4GJ}{r}\right)^2; \quad N\phi = -\frac{4GJ}{r^2} $$

(31)

We would like to stress that for off-shell ADT formalism to be self-consistent one must identify $M$ and $J$ with the conserved charges derived from the off-shell expression Eq. (11) corresponding
to the exact killing vectors $\zeta = \partial_t$ and $\xi = \partial_\phi$, respectively. To this end, we now calculate the mass $M$ for the BTZ solution. For the timelike Killing vector $\zeta = \partial_t$ the Eq. 10 takes the form

$$\delta \tilde{Q}_{\text{ADT}}(\partial_t) = \frac{1}{16\pi G} \int_{\Sigma} d\Sigma_{rt} \left[ \delta \tilde{J}^{rt} + \zeta^i \Theta^r \right].$$

(32)

For Einstein gravity the $P-$ tensor become divergenceless. As a result, the contribution from $\delta J^{t\mu}$ to $\tilde{Q}(\partial_t)$ vanishes. Therefore,

$$\tilde{Q}_{\text{ADT}}(\partial_t) = \frac{1}{16\pi G} \int_0^1 d\alpha \int_{\Sigma} d\Sigma_{rt} \zeta^t \Theta^r [g, \delta g].$$

(33)

Writing the perturbation around the background as

$$\delta g_{\mu\nu} = \partial g_{\mu\nu} \partial_{\alpha} \delta_{\alpha} ; \quad \delta_{\alpha} = G \delta M.$$  

(34)

and evaluating $r$ component of Eq. 2 we get

$$\delta \tilde{Q}_{\text{ADT}}(\partial_t) = \frac{\delta_{\alpha}}{G} = \delta M.$$  

(35)

The total charge is obtained by integrating over $\alpha$ from 0 to $\alpha > 0$ (that is varying $M$ from 0 to any final value), is given by

$$\tilde{Q}_{\text{ADT}}(\partial_t) = M$$

(36)

Similar computation can be performed for the rotational Killing vector to obtain the angular momentum of BTZ black hole as

$$\tilde{Q}_{\text{ADT}}(\partial_\phi) = J.$$  

(37)

Thus, we have obtained the desired expressions for mass and angular momentum for BTZ metric within the off-shell formalism.

We are now in a position to compute the symmetry algebra among the generalized ADT charges. The infinitesimal variation of ADT charges for the diffeomorphisms Eq. 26 is given by

$$\delta Q(\xi_{m}^\pm) \equiv \delta Q_{m}^\pm = \frac{1}{8\pi G} \int_{\Sigma} d\Sigma_{\mu\nu} \sqrt{-g} J_{\text{ADT}}^{\mu\nu}[g, \delta g, \xi_{m}^\pm].$$

(38)

Since the boundary $\Sigma$ is a circle, we only need to compute $(r^+)$ and $(r^-)$ components of $J_{\mu\nu}[g, \delta g]$. By explicitly calculating these relevant components for the diffeomorphism $\xi^+$, we find

$$\delta Q_{m}^+ = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \sqrt{-g} \left( \epsilon_{r+\phi} J^{r+}[g, \delta g, \xi_{m}^+] + \epsilon_{r-\phi} J^{r-}[g, \delta g, \xi_{m}^+] \right)$$

$$= \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left\{ \left[ \frac{1}{2} h^\alpha \nabla_\alpha \nabla_\alpha (\xi_{m}^{+})^+ - (\xi_{m}^{+})^r \nabla_\alpha h_\alpha^+ \right] + \left( \xi_{m}^{+} \right)^r \nabla_\alpha h_\alpha^+ + \left( \xi_{m}^{+} \right)^r \nabla_\alpha h_\alpha^+ \right\} + \left\{ \text{+} \rightarrow \text{+} \right\}.$$  

(39)

\[1\text{From now onwards we drop the subscript ADT from the expressions for ADT charges.}\]
In the above expression, the first term in the curly bracket is coming from \((r+)\) component while the the second curly bracketed term (denoted by \(+ \rightarrow -\)) is due to \((r-)\) component. This term is same as the first one with + component replaced by −. Also, note that in the square bracketed expression in the first curly bracket originates from the first part on the right hand side of Eq. (21).

For the metric Eq. (30) perturbations \(h_{\mu\nu}\) induced by \(\delta T(x^+)\) and \(\delta \bar{T}(x^-)\) can be written as

\[
h_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial T} \delta T(x^+) + \frac{\partial g_{\mu\nu}}{\partial \bar{T}} \delta \bar{T}(x^-). \tag{40}
\]

The nonvanishing components for metric perturbations are given by

\[
h_{++} = \ell^2 \delta T \quad h_{--} = \ell^2 \delta \bar{T} \quad h_{+-} = h_{-+} = -\frac{\ell^4}{2r^2}(\bar{T} \delta T + T \delta \bar{T}). \tag{41}
\]

Substituting these expressions in Eq. (39) we obtain

\[
\delta Q^+_m = \ell \frac{8}{8\pi G} \int_0^{2\pi} d\phi e^{imx^+} \delta T(x^+). \tag{42}
\]

Similarly for \(\xi^-\) we get

\[
\delta Q^-_m = \ell \frac{8}{8\pi G} \int_0^{2\pi} d\phi e^{imx^-} \delta T(x^-). \tag{43}
\]

It is clear that the above expression is integrable. Setting the functions \(T\) and \(\bar{T}\) to zero for the background (zero mass solution) we get the expressions for total charge for the respective asymptotic diffeomorphisms \(\xi^+\) and \(\xi^-\) as

\[
Q^+_m = \frac{\ell}{8\pi G} \int_0^{2\pi} d\phi T(x^+) e^{imx^+} \tag{44}
\]

\[
Q^-_m = \frac{\ell}{8\pi G} \int_0^{2\pi} d\phi \bar{T}(x^-) e^{imx^-}.
\]

For any two diffeomorphisms the Lie bracket among the corresponding conserved charges is defined as

\[
\{Q^\pm_m, Q^{\pm}_n\} = \delta_{\xi^\pm_n} Q^\pm_m \tag{45}
\]

In particular, for \(\xi^+\) we have

\[
\delta_{\xi^+} Q_m = \ell \frac{8}{8\pi G} \int_0^{2\pi} d\phi [\mathcal{L}_{\xi^+} T(x^+)] e^{imx^+}. \tag{46}
\]

The change in \(T(x^+)\) between two nearby solutions \(g\) and \(g + \delta_{\xi^+} g\) is given by

\[
\mathcal{L}_{\xi^+} T(x^+) = e^{imx^+} \left( \partial_+ + 2in + \frac{in^3}{2} \right) T(x^+). \tag{47}
\]
Using Eq. (46) together with Eq. (47) in Eq. (45), we finally get
\[
\{Q^+_m, Q^+_m\} = i(n-m)Q^+_{m+n} + \frac{\ell}{8G}m^3\delta_{m-n}
\]
\[
\{Q^-_m, Q^-_m\} = i(n-m)Q^-_{m+n} + \frac{\ell}{8G}m^3\delta_{m-n}.
\]
(48)

This is nothing but the two copies of the Virasoro algebra with the central charge \(c\) given by
\[
c = \frac{3\ell}{2G}.
\]
(49)

Thus, we have shown that the algebra of generalized ADT charges is consistent with that obtained in [2, 3].

### 3.2 Near horizon extremal BTZ

As an another application of off-shell method, we now construct the symmetry algebra among the symmetry for the near horizon geometry of extremal BTZ black hole spacetime. The asymptotic symmetry analysis of this geometry is well studied in the literature in the context of Kerr/CFT correspondence [4]. The phase space to which the near horizon extremal BTZ geometry belongs is not the same as the one represented by Eq. (30). Consequently, the boundary condition and the corresponding asymptotic diffeomorphisms are quite different as compared to the usual Brown-Henneaux conditions discussed in the previous subsection. Therefore, it is important to study the asymptotic symmetry algebra using generalized ADT charges and to see whether it matches with the known results.

The extremal BTZ black hole is described by the line element in Fefferman-Graham coordinates as
\[
ds^2 = \frac{\ell^2}{r^2}dr^2 - (r \frac{dx^- - 4\ell T(x^+)}{r} \frac{dx^+}{dx^-})(r \frac{dx^- - 4\ell \bar{T}(x^-)}{r} dx^-)
\]
(50)

where \(\ell M = T(x^+) + \bar{T}(x^-)\) and \(J = T(x^+) - \bar{T}(x^-)\). Near horizon limit of the extremal BTZ black hole can be obtained by considering the following coordinate transformations
\[
t = \frac{\tau \sqrt{J}}{\alpha}; \quad r = \ell \sqrt{\alpha} \rho; \quad \tilde{\phi} = \phi - \frac{\tau \sqrt{J}}{\ell \alpha},
\]
(51)

and taking \(\alpha \to 0\) limit. Thus, in the near horizon limit of the extremal BTZ black hole \(x^+ = \tilde{\phi}\) is well defined but \(x^-\) diverges. This can be resolved by setting \(\bar{T}(x^-) = 0\). In the new coordinates, the near horizon extremal BTZ black hole metric is described by
\[
ds^2 = \ell^2 \left[ \frac{d\rho^2}{\rho^2} - \rho^2 d\tau^2 + r_H^2 \left( d\tilde{\phi} - \frac{\rho d\tau}{r_H} \right)^2 \right]; \quad r_H = 4\sqrt{\frac{J}{\ell}}
\]
(52)
This Now we impose the asymptotic boundary conditions

\begin{align*}
g_{\tau\tau} &= O(\rho^2) \quad g_{\tau\rho} = O\left(\frac{1}{\rho^2}\right) \quad g_{\tau\tilde{\phi}} = -\frac{r_H\ell^2\rho}{4} + O(1) \\
g_{\rho\rho} &= \frac{\ell^2}{4\rho^2} + O\left(\frac{1}{\rho^3}\right) \quad g_{\rho\tilde{\phi}} = O\left(\frac{1}{\rho}\right) \quad g_{\tilde{\phi}\tilde{\phi}} = O(1) .
\end{align*}

(53)

These conditions can be regarded as 3- dimensional analogue of the fall-off conditions used in Kerr/CFT correspondence [4, 5]. Recently, it has been argued that the geometry Eq. (52) can be thought of as the decoupling limit of phase space of all Banados geometries with vanishing left (or right) moving modes [29]. Note the the geometry given by Eq. (52) admits \( SL(2, \mathbb{R}) \times U(1) \) isometry group and the boundary conditions Eq. (53) enhances the \( U(1) \) part of the full isometry group. The asymptotic symmetry generator that preserves the above boundary conditions is given by

\[ \xi_m = e^{-im\tilde{\phi}} \left( -\partial_{\tilde{\phi}} - im\rho \partial_{\rho} \right) . \]

(54)

It is easy to show that the above right moving modes \( \xi_m \) satisfy \( \text{Diff}(S^1) \) algebra Eq. (27) as before. Substituting \( \xi_m \) in Eq. (21) and computing the relevant, \((\tau, r)\) components, we find

\[ J^\tau_r = \frac{1}{4} \left[ 2i m \rho h_{\rho\tilde{\phi}} - \left( \frac{h_{\tau\tau}}{\rho^2} + 2h_{\tilde{\phi}\tilde{\phi}} + 2\rho \partial_\rho h_{\rho\tilde{\phi}} \right) \right] + \cdots \]

(55)

with the dots representing the terms which vanish for large \( \rho \). Taking Eq (52) as a background, we explicitly find the expressions for metric perturbations

\begin{align*}
h_{\tau\tau} &= \mathcal{L}_{\xi_m} g_{\tau\tau} = 0 ; \quad h_{\rho\rho} = \mathcal{L}_{\xi_m} g_{\rho\rho} = -\frac{G\ell^2}{2(1 + \rho^2)^2} im e^{-im\tilde{\phi}} \\
h_{\rho\tilde{\phi}} &= \mathcal{L}_{\xi_m} g_{\rho\tilde{\phi}} = -\frac{G\ell^2\rho}{4(1 + \rho^2)^2} m^2 e^{-im\tilde{\phi}} ; \quad h_{\tilde{\phi}\tilde{\phi}} = \mathcal{L}_{\xi_m} g_{\tilde{\phi}\tilde{\phi}} = \frac{G\ell^2}{2} im e^{-im\tilde{\phi}} .
\end{align*}

(56)

Substituting the above expressions in Eq. (55) and integrating the above expression over a circle at \( \rho \to \infty \), we get the central extension as

\[ \{Q(\xi_m), Q(\xi_n)\} - Q(\{\xi_m, \xi_n\}) = \frac{1}{8\pi G} \int_0^{2\pi} d\tilde{\phi} K_{\xi_m}[\mathcal{L}_{\xi_n} g, g] = -i \frac{G\ell m}{8} \left[ m^2 + r_H \right] \]

(57)

This gives the central charge \( c = \frac{3\ell}{2\pi G} \) same as Brown-Henneaux charge discussed earlier. The difference is that in this case the algebra among the charges is chiral, with only one Virasoro symmetry. Our result matches with the previously known results [29, 30] in which the charges were obtained by using the method given in [14, 15].

\footnote{Here we have assumed the infinitesimal the integrability of the surfaces charge \( \delta Q_m \).}
4 Conclusion

In this work, we have given an alternative method based on the off-shell ADT formalism to obtain the asymptotic symmetry algebra. The similarity between the generalized ADT approach initiated in [18] and other existing methods was mentioned in [20]. In particular, the expressions for quasi-local conserved charges for any asymptotic Killing vectors obtained in off-shell ADT formalism bear a close resemblance with the BBC formalism [16]. However, the construction of symmetry algebra within off-shell ADT formalism has not been performed. We have provided explicit equivalence between the two approaches in a consistent manner by computing the asymptotic symmetry algebra among the generalized ADT charges for asymptotic AdS$_3$ and near horizon extremal BTZ geometries. In the first case, we have considered the most general asymptotic AdS$_3$ spacetime dictated by the fall-off conditions given in [2]. Working in the Fefferman-Graham coordinates, we have explicitly computed the expression for generalized ADT surface charges. The algebra among these charges was shown to reproduce two copies of Virasoro algebra. This confirms that the off-shell formalism for quasi-local conserved charges is equivalent to BBC approach [14] [15] even at the level of asymptotic symmetry algebra. It is worthwhile to mention that by construction, the generalized ADT charges is valid for any arbitrary background which may or may not satisfy the field equations. However, while computing the asymptotic symmetry algebra, one has to take background metric as well as its perturbations, both satisfying the equations of motion. The algebraic equivalence between the two methods is mainly because in the Fefferman-Graham gauge, the difference between the relevant components of the generalized ADT potential and Barnich-Brandt’s 2– form potential vanishes. Consequently, the structure of asymptotic symmetry algebra remain unaltered. We would like to point out that the similar simplification occurs in BBC formalism also. In fact, for Einstein gravity one can explicitly show that the difference between the Barnich-Brandt’s surface charge and Noether-Wald surface charge vanishes in the Fefferman-Graham gauge. To further strengthen this equivalence, we have constructed the symmetry algebra for near horizon extremal BTZ spacetime. In this case too, our findings are in exact agreement with existing literature [30] wherein the asymptotic symmetry algebra was obtained by using BBC approach.

Here, we have used Einstein gravity as a prototype example to establish the correspondence between the above mentioned approaches for the computation of quasi-local charges. It is interesting exercise to extend this algebraic correspondence for the gravity theories containing Chern-Simons term. Apart from the boundary conditions used in our work there are other interesting conditions giving rise to deeper understanding of asymptotic symmetry analysis [25] [31]. It would be also interesting to incorporate these boundary conditions within generalized ADT formalism. We would like to address these issues in the near future.
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