Shape Changes of Deformable Spherical Membranes with \( n \)-atic Order

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We present the existence of the Kosterlitz-Thouless (KT) transition for \( n \)-atic tangent-plane order on a deformable spherical surface and investigate the development of quasi-long range \( n \)-atic order and the continuous shape changes of a deformable spherical surface below the KT transition in the low temperature (mean field) limit. The \( n \)-atic order parameter \( \psi = \exp[i n \Theta] \) describes, respectively, vector, nematic, and hexatic order for \( n = 1, 2, \) and 6. We derive a Coulomb gas Hamiltonian to describe it. We then convert it into the sine-Gordon Hamiltonian and find a linear coupling between a scalar field and the Gaussian curvature on the fluctuating spherical surface. After integrating over the shape fluctuations, we obtain the massive sine-Gordon Hamiltonian, where the interaction between vortices is screened. We find, for \( n^2 K_n/\kappa \gg 1/4 \) where \( K_n \) and \( \kappa \) are \( n \)-atic and bending rigidity, respectively, the KT transition is suppressed altogether. On the other hand, if \( n^2 K_n/\kappa \ll 1/4 \), there is an effective KT transition. In the ordered phase, tangent-plane \( n \)-atic order expels the Gaussian curvature. In addition, the total vorticity of orientational order on a surface of genus zero is two. Thus, the ordered phase of an \( n \)-atic on such a surface will have \( 2n \) vortices of strength \( 1/n \). \( 2n \) zeros in its order parameter, and a nonspherical equilibrium shape. Our calculations are based on a phenomenological model with a gauge-like coupling between \( \psi \) and curvature, and our analysis follows closely the Abrikosov treatment of a type II superconductor just below \( H_{c2} \).

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I. INTRODUCTION

When surfactant molecules, or amphiphiles which consist of molecules that combine both polar and non-polar parts, are dissolved in a single solvent such as water, these molecules tend to form bilayer membranes where the hydrocarbon parts of each monolayer are aggregated in the middle of the bilayer to reduce the contact between the water and the non-polar parts of the molecule due to the hydrophobic interactions of the hydrocarbon chains. These membranes are not covalently bonded, but rather stabilized by the weaker hydrophobic interaction. These membranes can have characteristic sizes which may be much larger than those of a single molecule. The sizes and shapes of these membranes can change as a function of temperature, salinity, and/or surfactant concentration. Depending on the physical conditions of the system, these membranes can form random extended surfaces, regular periodic structures, or closed vesicles separating an interior from an exterior [1]. Furthermore, some bilayer membranes are prototypes of biological systems, although it should be noted that true biological membranes have several components and greater complexity.

In a wide class of membranes, molecules move freely within the membrane forming a 2-dimensional fluid offering little resistance to changes in membrane shape. Such membranes are physical examples of random surfaces, which can undergo violent shape changes. Molecules in membranes can also exhibit varying degrees of orientational and positional order including tilt (smectic-C), hexatic, and crystalline orderings similar to those found in free standing liquid crystal films [2][3]. These ordered membranes provide fascinating laboratories for the study of the coupling between order and geometry, analogous in many ways to the coupling between matter and geometry in general relativity. The underlying cause of this coupling is as follows. A field describing orientational order cannot be parallel everywhere if it is forced to lie on a surface, such as that of a sphere, that curves in two directions and thus has nonzero Gaussian curvature. It could lower its energy by flattening the surface. An additional complication arises when order develops on closed surfaces. A closed surface can be classified according to its genus \( g \); the number of handles. Orientational order on a closed surface necessarily has topological defects (vortices) with total strength (vorticity) equal to the Euler characteristic \( 2(1 - g) \) of the surface [4]. Tangent-plane order on a sphere (torus) will have vorticity 2 (0), since a sphere (torus) has genus 0 (1), respectively. The continuous development of vector order on a deformable surface of genus zero will be accompanied by a continuous change from spherical to ellipsoidal shape [4]. Since vortices are energetically costly, it may be favorable for a closed physical membrane to transform into an open cylindrical structure when tangent-plane order develops in response to changes in temperature or other control variables [4]. Indeed, there are a number of experimental examples of shape changes that may be explained by the development of tangent-plane order.
Orientational orders for smectic-C and hexatic liquid crystals are described by an $n$-atic order parameter $\psi = \langle \exp[\imath n \theta] \rangle$ with $n = 1$ and $n = 6$, respectively. In this paper, we present the existence of the KT transition of $n$-atic order on a closed surface of genus zero and investigate the development of $n$-atic order below the KT transition temperature and the concomitant change in shape from spherical to non-spherical. An $n$-atic order parameter can have vortices of strength $1/n$, and, since it is generally favorable to form vortices of minimum strength, we expect $2n$ maximally separated vortices of strength $1/n$ to be present in the ordered phase (low temperature phase). Thus for $n = 1$, we expect two antipodal vortices, and respectively for $n = 2, n = 3$ and $n = 6$, we expect vortices at the vertices of a tetrahedron, an octahedron, and an icosahedron. Indeed calculations on a rigid sphere confirm this conjecture \[8\]. We also expect the vesicle shape to change from spherical to ellipsoidal, tetrahedral, octahedral, or icosahedral in the four cases above.

Our calculations are based on a phenomenological Hamiltonian for a complex order parameter field whose coupling to shape occurs via a covariant derivative and via changes in the metric tensor \[9\]. The model is almost identical to the Landau-Ginzburg theory of superconductivity except that vorticity is fixed by surface topology rather than energetically determined by an external magnetic field. The ordering transition we find is very similar to the transition from a normal metal to the Abrikosov vortex lattice in a superconductor, and indeed our analysis follows very closely that of Abrikosov \[10\]. We find the order parameter, which is necessarily spatially inhomogeneous because of the topological constraint on the total vorticity, from among a highly degenerate set of functions that diagonalize the harmonic Hamiltonian on a rigid sphere. This degenerate set has exactly $2n$ zeros at arbitrary positions on the sphere and is very similar in form to fractional quantum Hall wave functions \[11\].

This paper is organized as follows. In Sec. II, we review fluid membranes in some detail and describe their free energy as a function of curvature. We introduce a tangent-plane orientational order parameter and its coupling to shape changes in Sec. III. The model Hamiltonian for a deformable spherical membrane with $n$-atic order is presented in the normal gauge representation and the existence of the KT transition is described in Sec. IV. Minimization scheme and the calculated shapes for temperatures below the KT transition temperature to the ordered $n$-atic phase for $n = 1, 2, 3, 4$, and $6$ are given in Sec. V, and we give concluding remarks in Sec. VI.

II. FLUID MEMBRANES

In this paper, we use the term membrane to denote a thin film of one material that separates two similar materials (bilayer membrane). We focus on fluid membranes and their possible degrees of orientational order. In fluid membranes, there is no in-plane shear modulus and the only in-plane deformations are compressions/expansions. An $n$-atic membrane, although fluid, will sustain a local orientational order defined modulo rotations by $2\pi/n$, which is described by a local order parameter $\psi = \exp[\imath n \theta]$.

Although fluid membranes can be composed of many different types of chemical and molecular species, their behavior (shapes, fluctuations, thermodynamics) can be understood from a unified point of view that considers their free energy of deformation. If the membrane were constrained to lie in a plane, the only relevant energy would be the compression of the molecules, that is, change of the average area per molecule. This is analogous to sound waves in a 3-dimensional fluid; there is no low-frequency response of the system to shear. However, since the membrane can also deform in the normal direction, there is an additional set of modes describing the conformations of the film. These out-of-plane deformations are known as bending or curvature modes and the free energy associated with such modes is known as the curvature free energy. While a general deformation of the membrane involves changes in both volume and curvature, we shall see that the lowest energy deformations usually involve only curvature. In most systems, changing the average volume of the membrane is a higher energy process and hence is less important when effects involving the thermal behavior of the membrane are considered. For a membrane with finite thickness, we denote as pure curvature deformations those perturbations of the membrane that do not change the overall membrane volume, but where there may be local stretching and compression of different parts of the film. The location of the interface within a membrane can be chosen to lowest order in the curvature energy so that the surface which defines the interface undergoes no stretching or compression (neutral surface). For long-wavelength curvature deformations whose wave lengths are much larger than the membrane thickness, the exact position of the neutral surface within the membrane is not crucial.

The membrane is characterized by the area per molecule, $\Sigma$, its thickness, $\lambda$, and its curvature. For simplicity, we assume that the equation of state of the membrane determines the thickness as a function of the area per molecule. A simple example is the case of an incompressible molecule where the product of $\lambda \Sigma$ is constrained to equal the molecular volume so that $\lambda \sim 1/\Sigma$. We thus take the curved membrane to be characterized only by the area per molecule and its curvature. Consider a locally flat, isolated interface. Saturation occurs when the free energy achieves a minimum with respect to $\Sigma$: $\partial f_0/\partial \Sigma = 0$ where $f_0$ is the free energy per molecule for a flat layer and $\Sigma$ is the area.
per molecule. The free energy per molecule is minimized when \( \Sigma = \Sigma_0 \), the optimal value of the area per molecule which arises from a balance of terms such as the entropy and the tension terms or attractions. The entropy favors a larger area per molecule because of the greater number of center-of-mass positions and chain conformations, while the tension terms and attractions favor a small value of \( \Sigma \) to reduce the contact of the hydrocarbon chains with the water. There can be deviations in the area per molecule from this optimal value and the energy cost of such a compression or expansion is

\[
\Delta f_0 = \frac{1}{2} f_0'' (\Sigma - \Sigma_0)^2
\]

(2.1)

where the primes signify a derivative with respect to \( \Sigma \). However, these deformations are typically of higher energy than the curvature deformations. A membrane can change its shape or size with a much lower free energy cost than that required to compress or expand it. It is important to remember, therefore, that for insoluble amphiphiles it is the energy cost of saddle-like deformation. The spontaneous curvature describes the tendency of the bilayer membrane to bend. It is viewed to arise from the fact that the two layers of the bilayer do not have areas per molecule which are exactly equal, nor are their curvatures equal and opposite. This can result in a vesicle that is more stable than a flat, lamellar bilayer since it may be that the favorably curved outer layer has a smaller value of the area per molecule and hence more molecules in it than the inner layer. The bending moduli, \( \kappa \) and \( \bar{\kappa} \), arise from the elastic constants determined by the head-head and tail-tail interactions. It is expected that these moduli are sensitively dependent on the surfactant chain length but only weakly dependent on the head-head interaction strength. The parameters \( H_0, \kappa, \) and \( \bar{\kappa} \) can be derived from a simple microscopic model that incorporates both the change in the area per molecule and the curvature. We note that a stable membrane will always have \( \kappa > 0 \). However, the sign of \( \bar{\kappa} \) can be either positive
or negative. Membranes that prefer isotropic shapes where the Gaussian curvature $K > 0$ such as spheres or planes will have $\kappa > 0$, while membranes that prefer saddle shapes where the Gaussian curvature $K < 0$ will have $\kappa < 0$. One can show that the requirement of a positive-definite quadratic term implies that membranes are only stable if $\kappa - 2\kappa > 0$; otherwise higher order curvature terms are needed to stabilize the system. We consider the spherical membrane whose topology is fixed. Thus the Gaussian curvature energy gives constant contribution by Gauss-Bonnet theorem $\int \sqrt{g}K = 2\pi \chi$.

III. ORIENTATIONAL ORDER

In a fluid membrane, molecules can flow freely to adapt themselves to any particular shape of the surface. If correlations among the molecular positions of the molecules forming the membrane exist, the molecules may exhibit in-plane crystalline order and form a kind of two-dimensional solid. On the other hand, the molecules may exhibit a weaker order in which the orientations are correlated at long distance scale. Orientational order means that to each point on the membrane is associated a preferred direction within the tangent plane of the membrane. For example at high temperature, the stable phase of the membrane is generally the liquid phase with no translational or orientational order if we are considering the hexatic order, or the smectic-A phase in which molecular axes are normal to the surface for the vector order. At lower temperature, the membrane can condense into an hexatic phase in which there is quasi-long-range 6-fold bond-angle order or into a smectic-C phase in which molecules tilt relative to the surface normal. To describe Sm-C (vector) and/or hexatic order, we introduce at each point $X(u)$ on the membrane a unit vector $m(u)$ in the tangent plane of the membrane. For Sm-C order, $m(u)$ is a true vector, invariant under rotations of $2\pi p$ ($p$ is an integer) about the unit surface normal $N(u)$ erected at $X(u)$. For hexatic order, rotations of $m(u)$ by $2\pi/n$ lead to physically equivalent states. More generally, we consider “$n$-atic” order in which rotations of $m(u)$ by $2\pi/p$ produce physically equivalent states. A two-dimensional nematic with an in-plane symmetric traceless tensor order parameter is an example of a 2-atic. Although we know of no physical realizations of other $n$-atics yet, we find it instructive to consider how the development of such order affects morphological changes in spherical vesicles.

To describe tangent-plane $n$-atic order, we introduce orthonormal unit vectors $e_1$ and $e_2$ at each point on the membrane. $e_1(u) \cdot m(u) = \cos \Theta(u)$ defines a local angle $\Theta(u)$. $n$-atic order is then described by the order parameter $\psi(u) = \exp[in\Theta(u)]$, which can be related to the $n$-th rank symmetric traceless tensor constructed from the unit vector $m$. The $n$-th rank symmetric traceless tensors are the $n$-th rank spherical tensors. They are listed in Table I for $n = 1, 2, 3, 4,$ and $6$. In general, there are $2^n$ components in the $n$-th rank spherical tensor $Q_{i_1i_2\cdots i_n}^{(n)}$ since $i_1, i_2, \ldots, i_n$ can be either 1 or 2. By permutational symmetry, there are only $(n + 1)$ possible independent components. However, there are $(n - 1)$ additional traceless conditions. Hence, there are only 2 independent components in $Q_{i_1i_2\cdots i_n}^{(n)}$. The linear combination of these $n$-th rank spherical tensors

$$\sum_{i_1,i_2,\cdots,i_n}^{1,2} i^k Q_{i_1i_2\cdots i_n}^{(n)},$$

(3.7)

where $k$ is the number of 2’s in $(i_1, i_2, \ldots, i_n)$, becomes $(m_1 + im_2)^n$ where $m_1 = e_1 \cdot m = \cos \Theta$ and $m_2 = e_2 \cdot m = \sin \Theta$. Thus $n$-atic order parameter is described by

$$\psi = \sum_{i_1,i_2,\cdots,i_n}^{1,2} i^k Q_{i_1i_2\cdots i_n}^{(n)} = e^{in\Theta}. $$

(3.8)

Note that since $\Theta(u)$ depends on the choice of orthonormal vectors $e_1$ and $e_2$, the order parameter $\psi(u)$ does as well. This means that any spatial derivatives in a phenomenological Hamiltonian for $\psi$ must be covariant derivatives.

We now describe the Hamiltonian of the membrane with orientational order in a reparametrization-invariant way. For a flat membrane the vector Hamiltonian corresponds to the usual XY model

$$\mathcal{H} = \frac{1}{2} K_n \int d^2 x \partial_\alpha m \cdot \partial_\alpha m. $$(3.9)

Using the fact that any spatial derivatives must be covariant derivatives and

$$D_\alpha m = (D_\alpha m^\beta) \kappa_\beta $$

(3.10)
is the tangential component of \( \partial_\mu \mathbf{m} \), one can show that the only possible free energy term for \( \mathbf{m} \) which respects this \( Z_1 \) symmetry is

\[
\mathcal{H} = \frac{1}{2} K_1 \int d^2 u \sqrt{g} D_\mu m^\alpha D^\mu m_\alpha. \tag{3.11}
\]

The dimensionless coupling constant \( K_1 \) is the vector-order stiffness and measures the strength of the coupling between the orientations of neighboring vectors. We then generalize this XY-like Hamiltonian for the vector order to the \( n \)-atic Hamiltonian using the \( n \)-th rank spherical tensors introduced in Table 1. In general, the \( n \)-atic Hamiltonian can be written as

\[
\mathcal{H} = \frac{1}{2} K_n \int d^2 u \sqrt{g} D_\mu Q^{(n)\alpha_1 \cdots \alpha_n} D^\mu Q^{(n)\alpha_1 \cdots \alpha_n}, \tag{3.12}
\]

where \( K_n \) is the \( n \)-atic rigidity. Using the spherical tensors in Table 1, we find

\[
Q^{(n)\alpha_1 \cdots \alpha_n} Q^{(n)\alpha_1 \cdots \alpha_n} = \frac{1}{2(n-1)}, \tag{3.13}
\]

and

\[
D_\mu Q^{(n)\alpha_1 \cdots \alpha_n} D^\mu Q^{(n)\alpha_1 \cdots \alpha_n} = \frac{n^2}{2(n-1)} D_\mu m^\alpha D^\mu m_\alpha, \tag{3.14}
\]

for \( n > 1 \). Hence, the Hamiltonian for the \( n \)-atic order becomes

\[
\mathcal{H}_n = \frac{1}{2} n^2 K_n \int d^2 u \sqrt{g} D_\mu m^\alpha D^\mu m_\alpha. \tag{3.15}
\]

In this form of \( \mathcal{H}_n \), we neglected all terms that are irrelevant at large distance by power counting. Other terms such as \( m_\alpha K_3 K^{3\gamma} m_\gamma \) which couple \( \mathbf{m} \) to the principal directions of curvature of the membrane are not invariant under the global rotation by \( 2\pi/n \) of \( \mathbf{m} \), and are irrelevant at large distance. Thus, this free energy has a full \( O(2) \) rotational symmetry. This is similar to the fact a 2-dimensional crystal with hexagonal or triangular structure has isotropic elastic properties at large distance scale. For \( n \)-atics with \( n \geq 3 \), there is only one elastic constant \( K_n \). For \( n = 1 \) or \( n = 2 \), there are in general two elastic constants. For simplicity, we will consider the single elastic constant approximation for all \( n \)-atics.

To describe this Hamiltonian in terms of a local angular order parameter \( \Theta \), at each point we introduce two orthonormal vectors \( \mathbf{e}_a(u)(a = 1, 2) \) tangent to the membrane. This is equivalent to introduce a 2-bein \( e^i_a(u) \) compatible with the induced metric \( g_{\alpha\beta}(u) \). In components,

\[
\mathbf{e}_a = e_a^\alpha t_\alpha \tag{3.16}
\]

and the orthonormality \( \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab} \) implies

\[
e_a^\alpha e_b^\beta g_{\alpha\beta} = \delta_{ab} ; \quad e_a^\alpha e_b^\beta g_{\alpha\beta} = g^{\alpha\beta}. \tag{3.17}
\]

The angular order parameter is frustrated by the rotation of tangent vectors that occurs under parallel transport on a curved surface. The amount of frustration is given by the gauge field \( A_\alpha \), i.e. the covariant derivative of \( e_\alpha \) in direction \( \alpha \) defines the gauge field \( A_\alpha \). Under parallel transport in direction \( du^\alpha \), each \( e_\alpha \) is rotated by an angle \( A_\alpha du^\alpha \). Thus the gauge field \( A_\alpha \) is defined by

\[
D_\alpha e_\alpha = -A_\alpha e_\alpha \theta_\beta, \tag{3.18}
\]

where \( \varepsilon_{ab} \) is the antisymmetric tensor with \( \varepsilon_{12} = -\varepsilon_{21} = 1 \) and \( A_\alpha \varepsilon_{ab} \) is called the spin-connection and describes how the basis vector \( \mathbf{e}_a \) rotates under parallel transport according to the Gaussian curvature \( K \) of the surface. In fact, \( A_\alpha \) is related to \( K \). The curl of the gauge field \( A_\alpha \) is the Gaussian curvature;

\[
\gamma^{\alpha\beta} D_\alpha A_\beta = K, \tag{3.19}
\]

where \( \gamma^{\alpha\beta} \) is the antisymmetric tensor defined via
We investigate the effect of thermal shape fluctuations of a genus zero surface on the KT transition in the low temperature limit $\beta \kappa \gg 1$. In this limit, we can parametrize the surface by its radius vector as a function of standard polar coordinates $\mathbf{u} = (\theta, \phi) \equiv \Omega$.

\[
\gamma_{\alpha\beta} = \mathbf{N} \cdot (\mathbf{t}_\alpha \times \mathbf{t}_\beta) = \sqrt{g} \varepsilon_{\alpha\beta},
\]
\[
\gamma^{\alpha\beta} = g^{\alpha\alpha'} g^{\beta\beta'} \gamma_{\alpha'\beta'}. \tag{3.20}
\]

The covariant derivative of $\mathbf{m} = \cos \Theta \mathbf{e}_1 + \sin \Theta \mathbf{e}_2$ writes

\[
D_\alpha \mathbf{m} = (D_\alpha m_\alpha) \mathbf{e}_\alpha + m_\alpha (D_\alpha \mathbf{e}_\alpha) \\
= (D_\alpha m_\alpha) \mathbf{e}_\alpha - m_\alpha A_\alpha \varepsilon_{\alpha\beta} \mathbf{e}_\beta \\
= (D_\alpha \Theta)(- \sin \Theta \mathbf{e}_1 + \cos \Theta \mathbf{e}_2) \\
- A_\alpha(\cos \Theta \mathbf{e}_2 - \sin \Theta \mathbf{e}_1) \\
= (D_\alpha \Theta - A_\alpha) \mathbf{m}_\perp. \tag{3.21}
\]

where $\mathbf{m}_\perp = - \sin \Theta \mathbf{e}_1 + \cos \Theta \mathbf{e}_2$ satisfying $\mathbf{m} \cdot \mathbf{m}_\perp = 0$. Then the $n$-atic Hamiltonian writes

\[
\mathcal{H}_n = \frac{1}{2} n^2 K_n \int d^2 u \sqrt{g} g^{\alpha\beta} (\partial_\alpha \Theta - A_\alpha)(\partial_\beta \Theta - A_\beta). \tag{3.22}
\]

This form of the free energy is invariant under local transformations $\Theta(\mathbf{u}) \rightarrow \Theta(\mathbf{u}) + \Lambda(\mathbf{u})$, $A_\alpha(\mathbf{u}) \rightarrow A_\alpha(\mathbf{u}) + \partial_\alpha \Lambda(\mathbf{u})$. This gauge invariance corresponds to a local rotation of the reference frame $\mathbf{e}_\alpha$. In terms of the complex order parameter $\psi$, the Hamiltonian for $n$-atic order becomes

\[
\mathcal{H}_n = \frac{1}{2} K_n \int d^2 u \sqrt{g} g^{\alpha\beta} D_\alpha \psi^* D_\beta \psi \tag{3.23}
\]

where $D_\alpha \psi = (\partial_\alpha - i n A_\alpha) \psi$. In this description, all $n$-atics have the same long-wavelength elastic energy. Their properties can differ however, because their topological excitations are characterized by different winding numbers. Since we assume that in the disordered state the membrane forms a spherical shape and there is no topological deformation of the membrane shape, order in $\psi$ is necessarily accompanied by the topological excitations called vortices. As we can see in Ref. [3], the total winding vorticity of a vector field on a closed surface with $h$ handles must be $2(1 - h)$. Thus, a vector field on the surface of a sphere ($h = 0$) has total vorticity 2. The minimum winding number disclination for an $n$-atic is $1/n$. The energy of an individual disclination on both flat and curved surfaces is proportional to the square of its winding number. It is therefore always favorable to form disclinations with the lowest possible winding number. In addition, disclinations with the same sign repel each other. These considerations imply that the ground state of a sphere with surface $n$-atic order will have $2n$ maximally separated disclinations of winding number $1/n$.

### IV. EXISTENCE OF THE KT TRANSITION

In this section, we present that in a certain region in the parameter space $(\kappa, K_n)$ there is an effective KT transition. Ovrut and Thomas have discussed the structure of the KT transition of a vortex-monopole Coulomb gas on a rigid sphere and have shown it is the same as in the planar case. They have shown the KT transition temperature on a rigid sphere is the same as that on the Euclidean plane, i.e. $T_{\text{sphere}}^{KT} = T_{\text{plane}}^{KT} = \pi K_A/2$, where $K_A$ is the square of the vortex charge in their discription [3]. Recently, we have extended the study of the structure of the KT transition to the hexatic Hamiltonian on the fluctuating spherical surfaces and have shown the hexatic Hamiltonian undergoes the effective KT transition at the same temperature as in the planar case [4]. Following the procedure in Ref. [4], we generalize a study of the KT transition to the $n$-atic Hamiltonian and present the region in the parameter space $(\kappa, K_n)$ where the KT transition exists.

From the previous sections, the $n$-atic Hamiltonian on the fluctuating surfaces writes $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_n$:

\[
\mathcal{H}_c = \frac{1}{2} \int d^2 u \sqrt{g} (H - H_0)^2 \tag{4.24}
\]
\[
\mathcal{H}_n = \frac{1}{2} n^2 K_n \int d^2 u \sqrt{g} g^{\alpha\beta} (\partial_\alpha \Theta - A_\alpha)(\partial_\beta \Theta - A_\beta). \tag{4.25}
\]

We investigate the effect of thermal shape fluctuations of a genus zero surface on the KT transition in the low temperature limit $\beta \kappa \gg 1$. In this limit, we can parametrize the surface by its radius vector as a function of standard polar coordinates $\mathbf{u} = (\theta, \phi) \equiv \Omega$;
where \( \mathbf{e}_r \) is the radial unit vector and \( \rho(\Omega) \) measures deviation from sphericity with radius \( R \). This parametrization is a “normal gauge”. To make the equations simple, we map this parametrization onto the unit sphere parametrization with \( R = 1 \). Since we are interested in the limit \( \beta k \gg 1 \) and in this limit \( \mathcal{H}_c \) dominates, we first minimize \( \mathcal{H}_c \) over the shape fluctuation field \( \rho \) which gives \( \rho(\Omega) = 0 \) and then we minimize \( \mathcal{H}_\alpha \) over \( \Theta \) with \( \rho(\Omega) = 0 \) and find

\[
\frac{\delta \mathcal{H}_\alpha^0}{\delta \Theta(\Omega)} \bigg|_{\Theta = \Theta^0} = \frac{1}{\sqrt{g^0}} \partial_q \delta^{\alpha \beta}(\partial_q \Theta^0 - A^0_\alpha) = 0,
\]

(4.27)

where the superscript \(^0\) stands for the rigid sphere with \( \rho(\Omega) = 0 \). In Ref. [8], Lubensky and Prost show that in the ground state 2\( n \) disclinations of strength \( 2\pi/n \) are arranged at the vertices of polyhedron inscribed in the sphere. A disclination at \( \mathbf{u} = \mathbf{u}_i \) with strength \( q_i \) gives rise to a singular contribution \( \Theta^{\text{sing}}_\alpha \) to \( \Theta^0 \) satisfying

\[
\oint \mathbf{r} \cdot \partial_\alpha \Theta^{\text{sing}} = q_i,
\]

(4.28)

where \( \Gamma \) is a contour enclosing \( \mathbf{u}_i \). Thus, in general \( \partial_\alpha \Theta^0 = \partial_\alpha \Theta^0 + v_\alpha^0 \) where \( \Theta^0 \) is nonsingular, \( v_\alpha^0 = \partial_\alpha \Theta^{\text{sing}}_\alpha \), and

\[
\gamma^{\alpha \beta} D_\alpha v_\beta^0 = s^0(\Omega),
\]

(4.29)

where

\[
s^0(\Omega) = \frac{2\pi}{n} \sum_{i=1}^{2n} \delta(\Omega - \Omega_i),
\]

(4.30)

where is the disclination density in the ground state and \( \Omega_i \)'s are the coordinates of the vertices of icosahedron. Since \( (\partial_\alpha \Theta^0 - A^0_\alpha) \) satisfies \( D^\alpha (\partial_\alpha \Theta^0 - A^0_\alpha) = 0 \), it is divergence-less and purely transverse. Accordingly \( (\partial_\alpha \Theta^0 - A^0_\alpha) \) can be written in terms of the curl of scalar fields and by applying the operator \( \gamma^{\beta \alpha} D_\beta \) to \( (\partial_\alpha \Theta^0 - A^0_\alpha) \), we find these scalar fields to be related to the Gaussian curvature \( K_0 \) of the rigid sphere and the ground state disclination density on the rigid sphere,

\[
\gamma^{\beta \alpha} D_\beta (\partial_\alpha \Theta^0 - A^0_\alpha) = \gamma^{\beta \alpha} D_\beta v_\beta^0 - \gamma^{\beta \alpha} D_\beta A^0_\alpha = s^0 - K_0,
\]

(4.31)

where \( K_0 \) is a Gaussian curvature of the rigid sphere and \( s^0 \) is the disclination density in the ground state.

Now taking into account the bond angle fluctuations around \( \Theta^0 \) and the shape fluctuations around the sphere,

\[
\Theta = \Theta^0 + \tilde{\Theta}, \quad A_\alpha = A^0_\alpha + \delta A_\alpha,
\]

(4.32)

the full Hamiltonian writes \( \mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H} \)

\[
\mathcal{H}_0 = \frac{1}{2} n^2 K_0 \int d\Omega (\delta^{\alpha} \Theta^0 - A^0_\alpha) (\partial_\alpha \Theta^0 - A^0_\alpha),
\]

\[
\delta \mathcal{H} = \frac{1}{2} \int d\Omega ((\nabla^2 + 2)\rho)^2
\]

\[
+ \frac{1}{2} n^2 K_0 \int d\Omega (\delta^{\alpha} \tilde{\Theta} - \delta A^0_\alpha) (\partial_\alpha \tilde{\Theta} - \delta A_\alpha) + O(\rho^3).
\]

(4.33)

The angle fluctuation field \( \tilde{\Theta}(\Omega) \) can also have disclinations of strength \( q = 2\pi(k/n) \) where \( k \) is an integer, due to the thermal fluctuation [3]. Thus, \( \partial_\alpha \tilde{\Theta} \) can be decomposed into singular and nonsingular parts; \( \partial_\alpha \tilde{\Theta} = \partial_\alpha \Theta^{\text{sing}} + v_\alpha \) where \( \Theta^{\text{sing}} \) is nonsingular, \( v_\alpha = \partial_\alpha \Theta^{\text{sing}} \) and

\[
\gamma^{\alpha \beta} D_\alpha v_\beta = s(\Omega), \quad s(\Omega) = \sum_i q_i \delta(\Omega - \Omega_i),
\]

(4.34)

where \( s(\Omega) \) is the thermally-excited disclination density with disclinations of strength \( q_i \) at \( \Omega_i \). The vector \( v_\alpha \) can always be chosen so that it is purely transverse, i.e. \( D_\alpha v_\alpha = 0 \). In the hexatic Hamiltonian, \( \partial_\alpha \Theta \) always occurs in the
combination \((\partial_\nu \hat{\Theta} - \delta A_\alpha)\). The spin-connection \(\delta A_\alpha\) can and will in general have both a longitudinal and a transverse component. However, one can always redefine \(\Theta^\parallel\) to include the longitudinal part of \(\delta A_\alpha\). This amounts to choosing locally rotated orthonormal vectors \(e_1(u)\) and \(e_2(u)\) so that \(D_\beta \delta A^\alpha = 0\). Thus we may take both \(v_\alpha\) and \(\delta A_\alpha\) to be transverse and the hexatic Hamiltonian

\[
\frac{1}{2} n^2 K_n \int d\Omega (\partial^\alpha \Theta^\parallel + v^\alpha - \delta A^\alpha \partial^\alpha \Theta^\parallel + v_\alpha - \delta A_\alpha)
\]

\[
= \mathcal{H}^\parallel + \mathcal{H}^\perp, \tag{4.35}
\]

can be decomposed into a regular longitudinal part,

\[
\mathcal{H}^\parallel = \frac{1}{2} n^2 K_n \int d\Omega \partial^\alpha \Theta^\parallel \partial^\alpha \Theta^\parallel,
\]

and a transverse part,

\[
\mathcal{H}^\perp = \frac{1}{2} n^2 K_n \int d\Omega (v^\alpha - \delta A^\alpha)(v_\alpha - \delta A_\alpha),
\]

where the cross term \(\int d\Omega (v_\alpha - \delta A_\alpha) \partial^\alpha \Theta^\parallel\) is dropped since \(D^\alpha(v_\alpha - A_\alpha) = 0\).

It costs an energy \(\epsilon_\alpha(k)\) to create the core of a disclination of strength \(k\). (We assume for the moment that the core energies of the positive and negative disclinations are the same. See, however, Refs. [16–18].) Thus, partition sums should be weighted by a factor \(y_k = e^{-\beta \epsilon_\alpha(k)}\) for each disclination of strength \(k\). Since \(\epsilon_\alpha(k) \sim k^2\), we may at low temperature restrict our attention to configurations in which only configurations of strength \(\pm 1\) appear. Let \(N_{\pm}\) be the number of disclinations of strength \(\pm 1\) and let \(u_{\alpha \pm}\) be the coordinate of the core of the disclination with strength \(\pm 1\) labeled by \(i\). The hexatic membrane partition function can then be written as

\[
Z(\kappa, K_A, y) = \text{Tr}_v y^N \int \mathcal{D}R \int \mathcal{D}\Theta^\parallel e^{-\beta \mathcal{H}_A - \beta (\mathcal{H}^\parallel + \mathcal{H}^\perp)}, \tag{4.38}
\]

where \(y = y_1\), and \(N = N_+ + N_-\). \(\mathcal{H}^\perp\) depends on all of the disclination coordinates \(\Omega_{\nu \pm}\) where \(\nu = 1, 2, \ldots, N_{\pm}\), and \(\text{Tr}_v\) is the sum over all possible disclination distribution with the topological constraint \[B\] :

\[
\text{Tr}_v = \sum_{N_{\pm}, N_{\mp}} \delta_{N_{\pm}, N_{\mp}} N_{\pm}! N_{\mp}! \prod_{\nu^+} \frac{d\Omega_{\nu^+}}{a^2} \prod_{\nu^-} \frac{d\Omega_{\nu^-}}{a^2}, \tag{4.39}
\]

where \(a^2\) is a molecular solid angle. The Kronecker factor \(\delta_{N_{\pm}, N_{\mp}}\) in \(\text{Tr}_v\) imposes the topological constraint that the total disclination strength on a sphere is 2 since with \(N_{\pm} = N_-\) we have \(2n\) ground state disclinations with the strength \(1/n\) giving the total disclination strength \(2n \times (1/n) = 2\).

The hexatic model of Eq. (4.38) can easily be converted to a Coulomb gas model using

\[
\gamma^{\alpha\beta} D_\alpha (v_\beta - \delta A_\beta) = s - \delta K \equiv C, \tag{4.40}
\]

which follows from Eq. (3.19) and Eq. (4.34) where \(\delta K\) is the deviation of the Gaussian curvature from the rigid sphere. The quantity \(C = s - \delta K\) is a “charge” density with contributions arising both from disclinations and from Gaussian curvature. Equation (4.40) implies

\[
v_\alpha - \delta A_\alpha = -\gamma^{\alpha\beta} D_\beta \frac{1}{\Delta} C, \tag{4.41}
\]

where we used \(\gamma_{\alpha\lambda} D^{\lambda\alpha\beta} D_\alpha = -\Delta\) and \(\Delta = D^\alpha D_\alpha = (1/\sqrt{g}) \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta\) is the Laplacian on a surface with metric tensor \(g_{\alpha\beta}\) acting on a scalar. Recall [Eq. (3.20)] that \(\gamma_\alpha^\beta\) rotates a vector by \(\pi/2\) so that \((v_\alpha - \delta A_\alpha)\) is perpendicular to \(D_\beta (-\Delta)^{-1} C\) and is thus manifestly transverse. Using Eq. (4.41) in Eq. (4.37), we obtain

\[
Z = \text{Tr}_v y^N \int \mathcal{D}R \int \mathcal{D}\Theta^\parallel e^{-\beta \mathcal{H}_A - \beta \mathcal{H}^\parallel - \beta \mathcal{H}^\perp}, \tag{4.42}
\]

where
is the Coulomb Hamiltonian associated with the charge $C$. Since the longitudinal variable $\Theta^\parallel$ appears only quadratically in $H^\parallel$, the trace over $\Theta^\parallel$ can be done directly giving the Liouville action \[19\] arising from the conf ormal anomaly;

$$
\int D\Theta^\parallel e^{-\beta H^\parallel} = e^{-\beta H_L},
$$

where

$$
\beta H_L = \frac{1}{8\pi a^2} \int d\Omega - \frac{1}{24\pi} \int d\Omega d\Omega' K(\Omega) \left(-\frac{1}{\Delta} \delta(\Omega - \Omega')\right) K(\Omega').
$$

The Coulomb gas partition function can thus be written

$$
Z = \text{Tr}_c y^N \int \mathcal{D}R e^{-\beta H_* - \beta H_L - \beta H_C}.
$$

The Coulomb gas model can be converted following standard procedures into a sine-Gordon model. The first step is to carry out a Hubbard-Stratonovich transformation on $\beta H_C$:

$$
e^{-\beta H_C} = e^{\beta H_L} \int D\Phi e^{-\frac{1}{2}(\beta n^2 K_n)^{-1} \int d\Omega \beta^* \Phi \partial_n \Phi} e^{i \int d\Omega \Phi},
$$

where the Liouville factor $e^{\beta H_L}$ is needed to ensure that $e^{-\beta H_C}$ be one when $C = 0$. Inserting this in Eq. \[4.46\], we obtain

$$
Z = \text{Tr}_c y^N \int \mathcal{D}R \mathcal{D}\Phi e^{-\beta H_* - \beta H_L \Phi} e^{i \int d\Omega(\delta K)\Phi},
$$

where

$$
\beta H_\Phi = \frac{1}{2}(\beta n^2 K_n)^{-1} \int d\Omega \partial_n \Phi \partial_n \Phi.
$$

The only dependence on disclinations is now in the term linear in $n$. Thus to carry out $\text{Tr}_c$, we need only to evaluate

$$
\text{Tr}_c y^N e^{i \int d\Omega \Phi}
$$

$$
\sum_{N_+, N_-} \frac{1}{N_+! N_-!} \delta_{N_+, N_-} y^{N_+ + N_-} \left(\int \frac{d\Omega}{a^2} e^{2\pi i \Phi(\Omega)/n}\right)^{N_+} \left(\int \frac{d\Omega}{a^2} e^{-2\pi i \Phi(\Omega)/n}\right)^{N_-}
$$

$$
\sum_{N_+, N_-} \frac{1}{N_+! N_-!} \int \frac{d\omega}{2\pi} \left(y \int \frac{d\Omega}{a^2} e^{i(2\pi [\Phi(\Omega)/n] - \omega)}\right)^{N_+} \left(y \int \frac{d\Omega}{a^2} e^{-i(2\pi [\Phi(\Omega)/n] - \omega)}\right)^{N_-}
$$

$$
\int \frac{d\omega}{2\pi} e^{(2y/a^2) \int d\Omega \cos[2\pi (\Phi/n) - \omega]}.
$$

Thus

$$
Z = \int \frac{d\omega}{2\pi} \int D\Phi \int \mathcal{D}R e^{-\beta H_*} e^{-\beta H_L \Phi} e^{(2y/a^2) \int d\Omega \cos[2\pi (\Phi/n) - \omega]} e^{-i \int d\Omega \delta K}.
$$

We can now change variables, letting $\Phi = (n/2\pi)(\Phi' + \omega)$. The term linear in the Gaussian curvature then becomes

$$
-i \int d\Omega \delta K \frac{n}{2\pi} (\omega + \Phi') = -i \frac{p}{2\pi} \int d\Omega \delta K,
$$

\[4.52\]
where we used $\int d\Omega \delta K = 0$. The integral over $\omega$ in Eq. (4.51) is now trivial, and dropping the prime we obtain

$$Z = \int \mathcal{D} \Phi \int \mathcal{D} R e^{-\beta \mathcal{H}^{\star} e^{-\mathcal{L}}},$$

(4.53)

where

$$\mathcal{L} = \frac{1}{2} (\beta n^2 K_n)^{-1} \left( \frac{n}{2\pi} \right)^2 \int d\Omega \partial^a \Phi \partial_a \Phi - 2y \alpha \int d\Omega \cos \Phi - \frac{i}{2\pi} \int d\Omega \Phi \delta K$$

(4.54)

is the sine-Gordon action on a fluctuating surface of genus zero. The first two terms of this action are the gradient and cosine energies present on a rigid sphere. The final term provides the principal coupling between $\Phi$ and fluctuations in the metric. It is analogous to the dilaton coupling [20] of string theory though here the coupling constant is imaginary rather than real. Note that the Liouville action is not explicitly present in Eq. (4.53).

In the regime $\beta \kappa \gg 1$, we can truncate the higher order terms in $\rho$. In the normal gauge, the partition function becomes

$$Z = \int \mathcal{D} \rho \mathcal{D} \Phi \exp \left[ -\frac{1}{2} \beta \kappa \int d\Omega (\nabla^2 + 2) \rho \right]$$

$$- \frac{1}{2} \beta \Gamma \int d\Omega (\nabla \Phi)^2 + \frac{2y}{a^2} \int d\Omega \cos \Phi$$

$$+ \frac{n}{2\pi} \int d\Omega \Phi (\nabla^2 + 2) \rho \right]$$

(4.55)

where $\beta \Gamma \equiv 1/(4\pi^2 \beta K_n)$ and we used $\delta K = (\nabla^2 + 2) \rho$. To lowest order in $\rho$, the shape fluctuation field $\rho$ is linearly coupled to the scalar field $\Phi$ which is the conjugate field to the disclinations. In Ref. [16], we have shown the similar coupling in the fluctuating flat membrane is quadratic in the shape fluctuation field.

Integrating over the shape fluctuation field $\rho$ gives the effective Hamiltonian for the conjugate field to the disclinations

$$Z = \int \mathcal{D} \Phi \exp \left[ -\frac{1}{2} \beta \Gamma \int d\Omega (\nabla^2 + \mu^2 \Phi^2) \right]$$

$$+ \frac{2y}{a^2} \int d\Omega \cos \Phi$$

(4.56)

with $\mu^2 = n^2 K_n/\kappa$. This is the massive sine-Gordon theory. The shape fluctuations induce the mass term for $\Phi$ field and screen the Coulombic interaction between the disclinations giving the Yukawa interaction between them. This partition function is equivalent to that of the Yukawa gas Hamiltonian on the rigid sphere with radius $R$:

$$\mathcal{H}_{\text{Yukawa}} = \frac{1}{2} \beta K_A \int d\Omega s(\Omega) \frac{1}{(-\nabla^2 + \mu^2)} s(\Omega)$$

$$= \frac{1}{2} \beta K_A \sum_{i,j} q_i q_j G(\Omega_i - \Omega_j),$$

(4.57)

where

$$G(\Omega_i - \Omega_j) = \sum_l \frac{2l + 1}{l(l + 1) + \mu^2} P_l(\cos \omega_{ij})$$

$$- \frac{\pi}{\cos ((\nu + \frac{1}{2})\pi)} P_{\nu}(\cos \omega_{ij}),$$

(4.58)

where $P_{\nu}(\cdot)$ is the Legendre polynomial with degree $\nu = -1/2 \pm (\sqrt{1 - 4\mu^2})/2$ and $\omega_{ij}$ is the angle between two disclinations at $\Omega_i$ and $\Omega_j$. For $0 \leq \mu \leq 1/2$, degree of Legendre polynomial, $\nu$, is real and the length scale introduced by $\lambda_{\nu} \equiv (\mu/R)^{-1}$ is larger than the system size, $2R$ after recovering the original length scale by mapping the unit sphere back to the sphere with the radius $R$. On the other hand, if $\mu > 1/2$, $\nu$ is a complex number, $\nu = -1/2 \pm i\tau$
where \( \tau = (\sqrt{4\mu^2 - 1})/2 \) and \( \lambda_d < 2R \). The length scale introduced by \( \lambda_d = R/\mu \) may be interpreted as the Debye screening length arising from shape fluctuations.

The interaction energy between two disclinations \( i \) and \( j \) at positions \( \Omega_i \) and \( \Omega_j \) with strength \( q_i \) and \( q_j \) is given by \( G(d_{ij}) \) where \( d_{ij} = 2R\sin(\omega_{ij}/2) \) is the chordal distance between two disclinations on the sphere with radius \( R \). The interaction \( G(d_{ij}) \) has the following limiting forms:

\[
G(d_{ij}) \simeq \begin{cases} -\frac{1}{2} \ln(d_{ij}/2), & \mu^{-1} \gg 2, \ d_{ij} \ll 2R \\ -\frac{1}{2} \ln(d_{ij}/2), & \mu^{-1} \ll 2, \ d_{ij} \ll \lambda_d \\ e^{-d_{ij}/\lambda_d}, & \mu^{-1} \ll 2, \ d_{ij} \gg \lambda_d \end{cases} \tag{4.59}
\]

Following the analogy of the 2D Coulomb gas, when the screening length is much larger than the system size, \( \lambda_d \gg 2R \), the induced mass term arising from shape fluctuations is irrelevant for the KT transition and the KT transition temperature is given by \( T_C = \pi K_n/2 \) for \( \mu \ll 1/2 \). However, for \( \mu \gg 1/2 \), the screening length is shorter than the system size, \( \lambda_d \ll 2R \), and the mass term is relevant for the KT transition and changes the universality class of the system. There is no KT transition at non-zero temperature. The disclinations are always unbound at non-zero temperature and the KT transition temperature vanishes. Thus, when \( n^2K_n/\kappa \ll 1/4 \), the effect of shape fluctuations is irrelevant and the KT transition occurs at the finite temperature.

V. SHAPE CHANGES BELOW \( T_C \)

For \( n^2K_n/\kappa \ll 1/4 \), there exists a KT transition at the finite temperature \( T_C = \pi K_n/2 \). To describe the phase transition and the corresponding shape changes, we introduce the magnitude of the complex \( n \)-atic order parameter in \( \psi = \psi_0 e^{in\phi} \) and consider the simplest long-wavelength Landau-Ginzburg Hamiltonian for the \( n \)-atic order parameter \( \psi \):

\[
\mathcal{H}_{LG} = \int d^2 u \sqrt{g} \left[ r|\psi|^2 + \frac{1}{2} u|\psi|^4 \right] + \frac{1}{2} K_n \int d^2 u \sqrt{g} g^{\alpha\beta}(\partial_\alpha - inA_\alpha)\psi(\partial_\beta + inA_\beta)\psi^*. \tag{5.60}
\]

This Landau-Ginzburg \( n \)-atic Hamiltonian is similar to the Landau-Ginzburg Hamiltonian for a superconductor in an external magnetic field,

\[
\mathcal{H}_{LG} = \int d^3 x \left[ r|\psi|^2 + C[(\nabla - ie^*A)\psi]^2 + \frac{1}{2} u|\psi|^4 + \frac{1}{8\pi}(\nabla \times A - \mathbf{H})^2 \right], \tag{5.61}
\]

where \( e^* = 2e/\hbar c \). Both have a complex order parameter \( \psi \) with covariant derivatives providing a coupling between \( \psi \) and a “vector potential” \( A \) or \( A_n \). In a magnetic field, the superconductor can undergo a second order mean-field transition from a normal metal to the Abrikosov vortex lattice phase with a finite density of vortices determined energetically by temperature and the magnetic field \( \mathbf{H} \). The magnetic field is conjugate to the vortex number \( N_v \) since \( \int d^3 x (\nabla \times \mathbf{H}) = LN_v \phi_0 \), where \( L \) is the length of the sample along \( \mathbf{H} \) and \( \phi_0 = h c/2e \) is the flux quantum. On a closed surface with \( n \)-atic order, there is a second-order mean-field transition to a state with vortex number determined by topology rather than conjugate external field. Thus, the mean-field \( n \)-atic transition on a closed surface is analogous to the transition to an Abrikosov phase with a fixed number of vortices rather than fixed field conjugate to vortex number. However, a Meissner phase with zero vortices does not exist because we do not have an analogue for magnetic intensity \( \mathbf{H} \), rather topology fixes vortex number.

The complete Hamiltonian for \( n \)-atic order on a deformable surface is \( \mathcal{H} = \mathcal{H}_A + \mathcal{H}_{LG} \). The field \( \rho(\Omega) \) in \( \mathbf{R}(\Omega) = R_0(1 + \rho(\Omega))\mathbf{e} \) can be expanded in spherical harmonics. Any isotropic change in \( \mathbf{R} \) can be described by \( R_0 \). In addition, uniform translations, which change neither the shape nor the energy of the vesicle, correspond to distortions in \( \rho \) with \( l = 1 \) and can be absorbed by fixing the center of mass of the vesicle. These considerations imply that \( \rho \) will have no \( l = 0 \) or \( l = 1 \) components in spherical harmonic expansion:

\[
\rho(\Omega) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \rho_{lm} Y_l^m(\Omega). \tag{5.62}
\]

The shape and size of the vesicle are determined entirely by the parameters \( R_0 \) and \( \rho_{lm} \). The reduced tensors \( g_{\alpha\beta} \) and \( K_{\alpha\beta} \) do not depend on \( R_0 \). Therefore, \( R_0 \) can be expressed as a function of \( A \) and \( \rho_{lm} \) via the relation
\[ A = \int d^2 u \sqrt{g} = R_0^2 \int d^2 u \sqrt{g} \]
\[ = R_0^2 \int d\Omega \left[ (1 + \rho)^2 + \frac{1}{2} \left( (\partial_\theta \rho)^2 + \frac{(\partial_\phi \rho)^2}{\sin^2 \theta} \right) + O(\rho^4) \right] \]
\[ = 4\pi R_0^2 \left[ 1 + \frac{1}{2} \frac{1}{4\pi} \sum_{l=2}^{\infty} \sum_{m} (l(l+1) + 2) |\rho_m|^2 + O(\rho^4) \right]. \quad (5.63) \]

In the disordered phase above \( T_C \), \( \rho = 0 \) and \( R = R_0 \). We will use the Hamiltonian \( \mathcal{H} \) expressed in terms of reduced parameters and the constant area \( A \) in our calculations of shape changes below the second-order disordered-to-n-atic transition.

Using the spherical polar coordinates with origin at the center of mass of the vesicle, the metric tensor \( g_{\alpha\beta} \) can be written as

\[ g_{\alpha\beta} = \partial_\alpha R \cdot \partial_\beta R = R_0^2 g_{\alpha\beta} \]
\[ \bar{g}_{\alpha\beta} = (1 + \rho)^2 \tilde{g}_{\alpha\beta} + (\partial_\rho \rho)(\partial_\rho \rho) \]
\[ \bar{g}^0_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} \]

and the curvature tensor \( K_{\alpha\beta} \) is

\[ K_{\alpha\beta} = N \cdot D_\alpha D_\beta R = R_0 \bar{K}_{\alpha\beta} \]
\[ \bar{K}_{\alpha\beta} = \frac{1}{\sqrt{1 + (\nabla \rho)^2}} \left( (1 + \rho) - \frac{\rho_0^2}{\sin \theta} + 2(\partial_\rho \rho)^2 \right) - \sin \theta \partial_\rho \frac{\partial_\phi \rho}{\sin \theta} + 2 \partial_\phi \rho \partial_\rho \rho \]
\[ = \frac{1}{\sqrt{1 + (\nabla \rho)^2}} \left( (1 + \rho) - \frac{\rho_0^2}{\sin \theta} + 2(\partial_\rho \rho)^2 \right) \]
\[ = \frac{1}{\sqrt{1 + (\nabla \rho)^2}} \left( (1 + \rho) - \frac{\rho_0^2}{\sin \theta} + 2(\partial_\rho \rho)^2 \right). \quad (5.68) \]

where \((\nabla \rho)^2 = (\partial_\rho \rho)^2 + \frac{1}{\sin \theta}(\partial_\phi \rho)^2\). The Hamiltonian is a functional of \( \psi(\Omega) \) and \( \rho(\Omega) \) at the fixed area \( A \). To find the equilibrium form of \( \psi(\Omega) \) and \( \rho(\Omega) \), we need to minimize \( \mathcal{H} \) over \( \psi(\Omega) \) and \( \rho(\Omega) \). There is a considerable simplification if we restrict our attention to the neighborhood of the transition temperature \( T_C \). Near \( T_C \), to order \( \psi^4 \), the variation of the curvature energy is of order \( \kappa \rho^2 \). When this is comparable to the Landau-Ginzburg free energy for \( \psi \), we only need to keep couplings of order \( \rho \psi^2 \). To this order, the Landau-Ginzburg Hamiltonian for \( \psi \) becomes

\[ \mathcal{H}_{LG} = \int d\Omega \left( r|\psi|^2 + \frac{1}{2} \bar{u}|\psi|^4 \right) + \frac{1}{2} K_n \int d\Omega g^{\alpha\beta}(\partial_\alpha - inA_\alpha^0)\psi(\partial_\beta + inA_\beta^0)\psi^* \]
\[ + \int d\Omega \rho \left( r|\psi|^2 + \frac{1}{2} \bar{u}|\psi|^4 \right) + \int d\Omega J^\alpha \delta A_\alpha, \]
\[ \quad (5.69) \]

where \( A_\beta^0 = 0 \) and \( A_\phi^0 = -\cos \theta \). In this formula,

\[ J^\alpha = \frac{\delta \mathcal{H}}{\delta A_\alpha} \]
\[ = \frac{nK_n}{2i} g^{\alpha\beta}(\psi \partial_\beta \psi^* - \psi^* \partial_\beta \psi + 2inA_\beta^0|\psi|^2) \]
\[ \quad (5.70) \]

has been evaluated at \( \rho_{lm} = 0 \), \( R_0 = R \) and \( \delta A_\alpha = \gamma_\alpha^\beta \partial_\alpha \rho \). In terms of \( \rho \), the curvature energy can be written as

\[ \mathcal{H}_c = \frac{1}{2} \kappa \int d\Omega \left( (\nabla^2 + 2)^2 \rho \right), \]
\[ \quad (5.71) \]

where

\[ \nabla^2 \rho = -\left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta \rho + \frac{1}{\sin^2 \theta} \partial_\phi^2 \rho \right). \]
\[ \quad (5.72) \]

Hence, to order \( \psi^4 \), we can write \( \mathcal{H} \) as \( \mathcal{H} = \mathcal{H}_0 + \delta \mathcal{H} \) with

\[ \mathcal{H}_0 = \frac{A}{4\pi} \int d\Omega \left( r|\psi|^2 + \frac{1}{2} \bar{u}|\psi|^4 \right) \]
\[ + \frac{1}{2} K_n \int d\Omega g^{\alpha\beta}(\partial_\alpha - inA_\alpha^0)\psi(\partial_\beta + inA_\beta^0)\psi^*, \]
\[ \quad (5.73) \]
which does not depend on $\rho(\Omega)$ and

$$
\delta \mathcal{H} = \frac{A}{4 \pi} \int d\Omega \rho \left( r|\psi|^2 + \frac{1}{2} u|\psi|^4 \right) - \int d\Omega \frac{\rho}{\sin \theta} \partial_\alpha (\gamma_\alpha^\beta J^\beta) + \frac{1}{2} \kappa \int d\Omega \left( (\nabla^2 + 2) \rho \right)^2,
$$

(5.74)

which contains $\rho(\Omega)$. This can be rewritten as

$$
\mathcal{H} = \mathcal{H}^0 + \int d\Omega \rho(\Omega) \tilde{\Phi}(\Omega) + \frac{1}{2} \kappa \int d\Omega d\Omega' \rho(\Omega) M(\Omega, \Omega') \rho(\Omega')
$$

(5.75)

with

$$
\tilde{\Phi}(\Omega) = \frac{A}{2 \pi} \left( r|\psi|^2 + \frac{1}{2} u|\psi|^4 \right) - \frac{1}{\sin \theta} \partial_\alpha (\gamma_\alpha^\beta J^\beta)
$$

(5.76)

$$
M(\Omega, \Omega') = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} Y_l^m(\Omega)(l-1)^2(l+2)^2 Y_l^m(\Omega').
$$

(5.77)

Minimizing over $\rho(\Omega)$, we obtain

$$
\rho(\Omega) = -\frac{1}{\kappa} \int d\Omega' M^{-1}(\Omega, \Omega') \tilde{\Phi}(\Omega'),
$$

(5.78)

where $M^{-1}(\Omega, \Omega')$ satisfies

$$
\int d\Omega_3 M(\Omega_1, \Omega_3) M^{-1}(\Omega_3, \Omega_2) = \delta_{(l \geq 2)}(\Omega_1 - \Omega_2)
$$

(5.79)

and $\delta_{(l \geq 2)}(\Omega_1 - \Omega_2)$ is the Dirac delta function in spherical harmonics space with $l \geq 2$. By substituting this for $\rho(\Omega)$ into $\mathcal{H}$, we have

$$
\mathcal{H}_{\text{eff}} = \frac{A}{4 \pi} \int d\Omega \left( r|\psi|^2 + \frac{1}{2} u|\psi|^4 \right) + \frac{1}{2} K_n \int d\Omega \tilde{\Phi}_{lm}^\alpha (\partial_\alpha - inA_0^\alpha)(\psi \partial_\beta + inA_0^\beta)\psi^* - \frac{1}{2\kappa} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{(l-1)^2(l+2)^2} |\tilde{\Phi}_{lm}|^2,
$$

(5.80)

where

$$
\tilde{\Phi}_{lm} = \int d\Omega Y_l^m(\Omega) \tilde{\Phi}(\Omega)
$$

(5.81)

$$
\tilde{\Phi}(\Omega) = \frac{Ar_c}{2\pi} |\psi|^2 - \frac{1}{\sin \theta} \partial_\alpha (\gamma_\alpha^\beta J^\beta).
$$

(5.82)

Before proceeding with our analysis of the $n$-atic transition on a deformable sphere, it is useful to recall Abrikosov’s calculation of the transition to the vortex state. The first step is to calculate the eigenfunctions of the harmonic part of $\mathcal{H}_{\text{GL}}$ when $\nabla \times \mathbf{A} = \mathbf{H}$. These can be divided into highly degenerate sets separated by an energy gap, $\hbar \omega_c = 2C e^* H$. In the Landau gauge with $\mathbf{A} = (0, Hx, 0)$, the eigenfunctions in the lowest energy manifold are $\psi_k = e^{ik_k y - e^* H |x - x_k|^2}$, where $x_k = k/e^* H$. The order parameter $\psi(x)$ of the ordered state is expressed as a linear combination $\psi(x) = \sum C_k \psi_k$, where the complex parameters $C_k$ are determined by minimization of $\mathcal{H}_{\text{GL}}$.

With this analogy in mind, we will follow Abrikosov’s treatment of the superconducting transition near $H_{C_2}$ to study the development of $n$-atic order on a sphere. We first diagonalize $\mathcal{H}$ in the harmonic level when $\rho = 0$ and
$A_\theta = A_0^\theta$, $A_\phi = A_0^\phi$, that is, we determine the functions $\psi$ that satisfy $K A \partial_\alpha D^\alpha \psi = \varepsilon \psi$, for $\rho = 0$, $A_\alpha = A_0^\alpha$. Nonlinearities arise from the $| \psi |^4$ term in Landau-Ginzburg Hamiltonian for $\psi$ and from the fact that $A_\alpha$ and $g_{\alpha\beta}$ depend nonlinearly on the deviation from the ideal spherical shape. To minimize $H_{\text{eff}}$, first we seek the lowest energy configurations of the operators corresponding to the harmonic terms only. Then, we linearly combine these eigenfunctions to get the function which has the lowest energy for $H_{\text{eff}}$. That is, the manifold of the lowest energy eigenstates of the harmonic terms is highly degenerate. Nonlinear terms pick out the lowest energy state which is the linear combination of the degenerate eigenstates of the harmonic terms and resolve the degeneracy. We divide $H_{\text{eff}}$ into the harmonic part, $H_{\text{har}}$, and the nonlinear part, $H_{\text{nl}}$:

$$H_{\text{har}} = \frac{A}{4\pi} \int d\Omega |\psi|^2 + \frac{1}{2} K_n \int d\Omega \bar{g}^{\alpha\beta} (\partial_\alpha - inA_\alpha^0) \psi (\partial_\beta + inA_\beta^0) \psi^*$$

$$H_{\text{nl}} = \frac{A}{4\pi} \int d\Omega \frac{1}{2} u |\psi|^4 - \frac{1}{2\kappa} \sum_{l=-2}^{\infty} \sum_{m=-l}^{l} \frac{|\bar{\Phi}_{nl}|^2}{(l-1)^2(l+2)^2}$$

The differential operator for $H_{\text{har}}$ is

$$\frac{1}{\sqrt{\bar{g}^0}} \frac{\delta H_{\text{har}}}{\delta \psi^*} = \left( \frac{A}{4\pi} + \frac{K_n}{2} \Delta \right) \psi,$$

where

$$\Delta = -\frac{1}{\sin \theta} \left[ \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin \theta} \partial_\phi^2 - n^2 \cos^2 \theta \sin \theta + 2in \cos \theta \sin \theta \partial_\phi \right].$$

One possible form of the eigenfunctions of $\Delta$ has the structure of $\sin^n \theta$.

$$\Delta \sin^n \theta = n \sin^n \theta.$$  (5.88)

The complete spectrum and the eigenfunctions are derived in the Appendix. Introducing the projection representation, we can reparametrize the sphere using a $(z, \bar{z})$ coordinate system defined by

$$z = \tan \frac{\theta}{2} e^{i\phi}; \quad \bar{z} = \tan \frac{\theta}{2} e^{-i\phi}.$$  (5.89)

The inverse transformation gives

$$\theta = 2 \tan^{-1} \sqrt{z \bar{z}}; \quad \phi = \frac{1}{2i} \ln \frac{z}{\bar{z}}.$$  (5.90)

The corresponding metric tensor, spin connection, and the differential operator, $\Delta$, in this representation can be written as follows

$$\bar{g}^{\alpha\beta} = \frac{2}{(1 + |z|^2)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Omega_z^0 = -\frac{1}{2i} \frac{1 - |z|^2}{1 + |z|^2}; \quad \Omega_{\bar{z}}^0 = \frac{1}{2i} \frac{1 - |z|^2}{1 + |z|^2}$$

$$\Delta = -(1 + |z|^2)^2 [(\partial_z \partial_{\bar{z}} - A_0^z A_0^{\bar{z}}) - 2in(A_0^z \partial_{\bar{z}} + A_0^{\bar{z}} \partial_z)].$$

The eigenfunction $\sin^n \theta$ can be written as

$$f(|z|^2) = \left( \frac{2|z|}{1 + |z|^2} \right)^n.$$  (5.94)
Thus, the full eigenfunctions of $\Delta/n$ of strength 1 are

$$\triangle f(|z|^2) = nf(|z|^2)$$

$$\Rightarrow \triangle f(|z|^2)Q(z) = nf(|z|^2)Q(z).$$

(5.95)

Thus, the full eigenfunctions of $\Delta$ are the product of $f(|z|^2)$ and the arbitrary function $Q(z)$. In particular, for $n = 1$, we have

$$f(|z|^2) = \frac{2|z|}{1 + |z|^2}.$$  

(5.96)

By taking $Q(z) = \frac{(z-z_1)(z-z_2)}{z}$,

$$\psi = (\text{Const.}) \frac{2|z|}{1 + |z|^2} \frac{(z-z_1)(z-z_2)}{z},$$

(5.97)

which means that $\psi$ has the vortices at $z = z_1$ and $z = z_2$ and that $\psi$ goes to $z_1 z_2 e^{-i\phi}$ as $z$ goes 0, $\psi$ goes to $e^{i\phi}$ as $z$ goes to $\infty$. This means there is an enormous degeneracy in the lowest energy manifold of the harmonic terms. In the lowest energy manifold of the harmonic terms, $\psi$ will have exactly $2n$ zeros specifying the vortex positions, each of strength $1/n$.

In view of this vortex strength and distribution, we choose $Q(z)$ as

$$Q(z) = \frac{\alpha}{z^n} \prod_{i=1}^{2n} (z - z_i),$$

(5.98)

where $\alpha, z_i$ are arbitrary and $z_i \neq z_j$ if $i \neq j$. By minimizing $\mathcal{H}_{\text{eff}}$ with the degenerate eigenfunctions of $\Delta$, we have

$$\psi = \alpha \left( \frac{2|z|}{1 + |z|^2} \right) \frac{1}{z^n} \prod_{i=1}^{2n} (z - z_i),$$

(5.99)

the non-linear terms pick out the set $\{ z_i | 1 \leq i \leq 2n \}$ that gives the lowest energy for $\mathcal{H}_{\text{eff}}$. This set $\{ z_i \}$ also gives $|\psi|$, vortex positions, and the shape change $\mathbf{R}$ of the membrane. In terms of spherical coordinates,

$$\psi = (\text{Const.}) \prod_{i=1}^{2n} \left( \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{i(\phi - \phi_i)/2} - \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{-i(\phi - \phi_i)/2} \right) \equiv \psi_0 P(\Omega)$$

(5.100)

and

$$|\psi|^2 = \frac{\psi_0^2}{N} \prod_{i=1}^{2n} (1 - \cos \gamma_i)$$

(5.101)

where $\cos \gamma_i = \cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos (\phi - \phi_i)$, $\Omega_i = (\theta_i, \phi_i)$ specifies the position of the $i$-th zero of $\psi$, and $N$ is the normalization factor which makes

$$\int d\Omega |\psi|^2 = \psi_0^2.$$  

(5.102)

Since $\psi$ of Eq. (5.100) is the most general function in the lowest energy manifold, the order parameter and vesicle shape just below the transition temperature are obtained by minimizing $\mathcal{H}_{\text{eff}}$ over $\psi_0$ and the positions of zeros. Inserting this expression for $\psi$, the free energy becomes

$$\mathcal{H}_{\text{eff}} = \frac{A}{4\pi} \int d\Omega \left( (r - r_c)|\psi|^2 + \frac{1}{2} i u |\psi|^4 \right)$$

$$- \frac{1}{2\kappa} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{|\Phi_{lm}|^2}{(l-1)^2(l+2)^2}.$$  

(5.103)
where $r_c = -4\pi nK_A/A$ and
\[
\tilde{\Phi}(\Omega) = -nK_n\psi_0^2 \prod_{i=1}^{2n} (1 - \cos \gamma_i) \left[ 1 + \sum_{i=1}^{2n} \frac{\cos \gamma_i}{(1 - \cos \gamma_i)} + \sum_{i>j} \frac{\cos \gamma_{ij} - \cos \gamma_i \cos \gamma_j}{(1 - \cos \gamma_i)(1 - \cos \gamma_j)} \right] \\
\equiv \psi_0^2 \Phi(\Omega).
\]
(5.104)

Minimization over $\psi_0$ leads to $\psi_0^2 = -\frac{1}{2} r[\{\Omega_i\}] / u[\{\Omega_i\}]$ and the effective free energy density
\[
f[\{\Omega_i\}] = -\frac{1}{2} r[\{\Omega_i\}] / u[\{\Omega_i\}],
\]
(5.105)
depending only on the positions $\{\Omega_i\}$ of the zeros of $\psi(\omega)$ where
\[
r[\{\Omega_i\}] = (r - r_c) \int d\Omega |P(\Omega)|^2
\]
(5.106)
and
\[
u[\{\Omega_i\}] = u \int d\Omega |P(\Omega)|^4 - \frac{4\pi}{\kappa_A} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{|\Phi_{lm}|^2}{(l-1)^2(l+2)^2}.
\]
(5.107)

The next step is to minimize $f[\{\Omega_i\}]$ over $\Omega_i$. Shape changes are described by $\rho(\Omega) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \rho_{lm} Y_m^l(\Omega)$ where
\[
\rho_{lm} = -\frac{1}{\kappa(l-1)^2(l+2)^2} \int d\Omega Y_m^l(\Omega) \tilde{\Phi}(\Omega)
\]
(5.108)
\[
\equiv -\frac{\psi_0^2}{\kappa(l-1)^2(l+2)^2} \tilde{\Phi}(\Omega)
\]
with $\{\Omega_i\}$ that minimize $f[\{\Omega_i\}]$. In the disordered phase above the transition temperature, $\psi_0 = 0$. Hence, $\rho = 0$ and $R_0 = R$. The vesicle shape is spherical. In the ordered phase below the transition temperature, we have been able to evaluate $f[\{\Omega_i\}]$ analytically for $n = 1$ and $n = 2$. For these two cases, we find as expected that the minimum energy configurations are, respectively, those with zeros of $\psi(\Omega)$ at opposite poles and at the vertices of a tetrahedron. The shape function $\rho(\Omega) = (nK_n/\kappa)\psi_0^2 \tilde{\Phi}(\Omega)$ associated with $n$-atic order is proportional to $\psi_0^2 \sim (r - r_c)$ to the order of our calculations. In general, the Legendre decomposition of $\tilde{\Phi}(\Omega)$ will contain Legendre polynomials of order $2n$. For $n = 1$ and $n = 2$, we find
\[
\tilde{\Phi}^{(1)}(\Omega) = \frac{1}{6} \sqrt{\frac{\pi}{5}} Y_2^0(\Omega),
\]
(5.109)
\[
\tilde{\Phi}^{(2)}(\Omega) = \frac{8}{135} \sqrt{\frac{\pi}{7}} \left[ Y_3^0(\Omega) - \sqrt{\frac{5}{7}} (Y_3^3(\Omega) - Y_3^{-3}(\Omega)) \right] + \frac{4}{3645} \sqrt{\pi} \left[ Y_4^0(\Omega) + \sqrt{\frac{10}{7}} (Y_4^4(\Omega) - Y_4^{-4}(\Omega)) \right].
\]
(5.110)

Figure 8 shows the shapes described by these functions. The transformations from the initial spherical shape to distorted shapes occur continuously. Our calculations for the shape are valid to order $(r - r_c)$. As temperature is lowered, higher-order terms in $(r - r_c)$ and higher-order spherical harmonics are needed to describe the equilibrium shape. In Ref. 6, a variational function for $\psi$ and spherical harmonics up to order 8 were used to calculate the shape for $n = 1$ for temperatures well below the transition. We use symmetry considerations to treat $n = 3, 4$, and 6. For $n = 3$ and $n = 6$, we expect the zeros of $\psi(\Omega)$ to lie, respectively, at the vertices of an octahedron and an icosahedron. We find the shape functions for $n = 3$ and $n = 6$
\[
\tilde{\Phi}^{(3)}(\Omega) = \frac{1}{297} \sqrt{\pi} \left[ Y_4^0(\Omega) + \sqrt{\frac{5}{14}} (Y_4^4(\Omega) + Y_4^{-4}(\Omega)) \right] + \frac{1}{60060} \sqrt{13\pi} \left[ Y_6^0(\Omega) - \sqrt{\frac{7}{2}} (Y_6^4(\Omega) + Y_6^{-4}(\Omega)) \right],
\]
(5.111)
\[
\bar{\rho}^{(6)}(\Omega) = \frac{15488}{275559375} \sqrt{13\pi} \left[ Y_6^0(\Omega) + \frac{7}{11} (Y_6^5(\Omega) - Y_6^{-5}(\Omega)) \right] \\
+ \frac{256}{692803125} \sqrt{21\pi} \left[ Y_{10}^0(\Omega) + \frac{33}{13} (Y_{10}^5(\Omega) - Y_{10}^{-5}(\Omega)) + \frac{187}{247} (Y_{10}^{10}(\Omega) + Y_{10}^{-10}(\Omega)) \right] \\
- \frac{512}{6834953125} \sqrt{\pi} \left[ Y_{12}^0(\Omega) - \frac{1}{3} \frac{286}{119} (Y_{12}^5(\Omega) - Y_{12}^{-5}(\Omega)) + \frac{247}{357} (Y_{12}^{10}(\Omega) + Y_{12}^{-10}(\Omega)) \right]
\] (5.112)

Figure 3 shows the shapes described by these functions. For \( n = 4 \), following the calculations of Palffy-Muhoray [21], we expect the zeros of \( \psi(\Omega) \) to lie at the vertices of a distorted cube obtained by rotating its top face about its fourfold axis by \( \pi/4 \) and compressing opposite faces. For this case, we minimized the energy over a single parameter determining the separation between opposite rotated faces and obtain

\[
\bar{\rho}^{(4)}(\Omega) = -0.01846311 Y_6^0(\Omega) - 0.305216 Y_6^5(\Omega) 0.033585 (Y_6^4(\Omega) + Y_6^{-4}(\Omega)) \\
+ 0.0201099 Y_6^0(\Omega) + 0.00721177 (Y_6^4(\Omega) + Y_6^{-4}(\Omega)) \\
- 0.00159671 \left[ Y_8^0 - 0.578489 (Y_8^4(\Omega) + Y_8^{-4}(\Omega)) \right].
\] (5.113)

Figure 3 shows the shape for \( n = 4 \).

VI. CONCLUSIONS

We have presented the general \( n \)-atic Hamiltonian in terms of the \( n \)-th rank spherical tensors and have shown there exists an effective KT transition for \( n^2 K_n/\kappa \ll 1/4 \). Also we have presented here an analysis of the mean-field transition to \( n \)-atic order on a fixed-area surface of genus zero and the corresponding continuous shape changes below the KT transition temperature in this parameter region. We are really dealing with two kinds of order: \( n \)-atic and the positional order of vortices. In mean-field theory, these two kinds of order develop simultaneously. Our investigation.

APPENDIX A: COMPLETE SPECTRUM OF \( \Delta \)

We present the complete spectrum of the differential operator, \( \Delta \), for the harmonic part, \( \mathcal{H}_{\text{har}} \). The operator \( \Delta \) in the spherical polar representation,

\[
\Delta = -\frac{1}{\sin \theta} \left( \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin \theta} (\partial_\phi + i \cos \theta)^2 \right)
\] (1.1)

is similar to the differential operator \( -(\nabla - i A)^2 \) for the superconductivity where...
\[ \nabla = \theta \partial_{\theta} + \phi \frac{1}{\sin \theta} \partial_{\phi}, \]  

and

\[ \mathbf{A} = \theta \mathbf{A}_{\theta} + \phi \mathbf{A}_{\phi}. \]  

The Schrödinger equation on a sphere in the field of a Dirac magnetic monopole of strength \( g \) at the origin plus an infinitely long and thin solenoid carrying flux \( F \) along the \( z \) axis is described by

\[ \frac{1}{2} (\nabla - i \mathbf{A})^2 \psi = E \psi \]  

with

\[ A_{\theta} = 0; \quad A_{\phi} = \frac{g}{\sin \theta} (1 - \cos \theta) + \frac{F}{2\pi \sin \theta}. \]  

The monopole has been chosen to have a Dirac string singularity along the negative \( z \) axis. Setting \( g = n \), \( F = -2\pi n \), we find

\[ A_{\theta} = 0; \quad A_{\phi} = -\frac{n \cos \theta}{\sin \theta} \]  

and \(- (\nabla - i \mathbf{A})^2\) is exactly identical to \( \Delta \). Hence, the differential operator, \( \Delta \), is equivalent to the Hamiltonian describing the field of a Dirac magnetic monopole of strength \( n \) at the origin plus an infinitely long and thin solenoid carrying flux \( 2\pi n \) along the positive \( z \) axis. Following the method in Ref. [24], we define the formal differential operator

\[ \mathbf{J} \equiv -ir \times (\nabla - i \mathbf{A}) - nr \]  

with components

\[ J_\pm = J_x \pm i J_y = \frac{e^{\pm i\phi}}{2} \left[ \pm \partial_{\theta} + i \cot \theta (\partial_{\phi} + in) - \frac{n \sin \theta}{1 + \cos \theta} \right] \]  

\[ J_z = -i(\partial_{\phi} + in) - n \]  

which obeys the formal commutation relations

\[ [\mathbf{J}, \mathcal{H}] = 0; \quad [J_i, J_j] = i\epsilon_{ijk} J_k, \]  

for \( \sin \theta \neq 0 \). We find

\[ J^2 = n^2 + \Delta \]  

\[ \Rightarrow \Delta = J^2 - n^2. \]  

Let us denote

\[ J^2 \Psi(\Omega) = l(l + 1) \Psi(\Omega), \]  

where we set

\[ \Psi(\Omega) = \exp[im\phi]P(\cos \theta) \quad ; \quad m = 0, \pm 1, \pm 2, \ldots. \]  

Then

\[ J_z \Psi(\Omega) = m \Psi(\Omega). \]  

The eigenvalues of \( J^2 \) are found to be given by \( l(l + 1) \) with

\[ l \equiv k + \frac{1}{2} (|m + n| + |m - n|) \quad ; \quad k = 0, 1, 2, 3, \ldots \quad ; \quad m = 0, \pm 1, \pm 2, \ldots. \]  

The corresponding orthonormal eigenfunctions are
$$\Psi(k,m)(\Omega) = C_{km} \left( \frac{1 - \cos \theta}{2} \right)^{\frac{|m|+n}{2}} \left( \frac{1 + \cos \theta}{2} \right)^{\frac{|m| - n}{2}} \cdot P^{|m+n|,|m-n|}_k(\cos \theta)e^{im\phi}, \quad (1.16)$$

where $P^{(a,b)}_k$ denote Jacobi polynomials, and

$$C_{km} = \left( \frac{k! \Gamma(|m+n| + |m-n| + k + 1) \Gamma(|m+n| + |m-n| + 2k + 1)}{4\pi \Gamma(|m+n| + |m-n| + k + 1) \Gamma(|m-n| + k + 1)} \right)^{\frac{1}{2}}. \quad (1.17)$$

The complete spectrum, $\delta$, of the operator $\Delta$ is

$$\delta = \left( k + \frac{1}{2} (|m+n| + |m-n|) \right) (k + 1 + \frac{1}{2} (|m+n| + |m-n|)) - n^2$$
$$k = 0, 1, 2, 3, \ldots; \quad m = 0, \pm 1, \pm 2, \ldots. \quad (1.18)$$

For $|m| \leq n$,

$$\delta = (n + k)^2 - n^2 + (n + k). \quad (1.19)$$

For $|m| \geq n$,

$$\delta = (|m| + k)^2 - n^2 + (|m| + k). \quad (1.20)$$

Thus,

$$\delta = (\text{max}(n, |m|) + k)(\text{max}(n, |m|) + k + 1) - n^2. \quad (1.21)$$

The ground state energy is generated when $k = 0, n \geq |m|$.

$$\delta = n(n + 1) - n^2 = n. \quad (1.22)$$

The corresponding ground state orthonormal eigenfunction has the form

$$\Psi^{(0,m)}(\Omega) = C_{0m} \left( \sin \frac{\theta}{2} \right)^{n+m} \left( \cos \frac{\theta}{2} \right)^{n-m} e^{im\phi} \quad (1.23)$$

which is the same as that in Eq. (5.100).
TABLE I. The $n$th rank symmetric traceless tensors, $Q^{(n)}$, for $n = 1, 2, 3, 4$, and 6 in 2-dimensional space.

| $Q^{(1)}$ | $m_\alpha$ = $Q^{(1)}_\alpha$ |
| $Q^{(2)}_{\alpha_1\alpha_2}$ | $m_\alpha m_{\alpha_2} - \frac{1}{2} \delta_{\alpha_1 \alpha_2}$ |
| $Q^{(3)}_{\alpha_1\alpha_2\alpha_3}$ | $m_\alpha m_{\alpha_2} m_{\alpha_3} - \frac{1}{4} (\delta_{\alpha_1 \alpha_2} m_{\alpha_3} + \delta_{\alpha_1 \alpha_3} m_{\alpha_2} + \delta_{\alpha_2 \alpha_3} m_{\alpha_1})$ |
| $Q^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ | $m_\alpha m_{\alpha_2} m_{\alpha_3} m_{\alpha_4} - \frac{1}{6} (\delta_{\alpha_1 \alpha_2} m_{\alpha_3} m_{\alpha_4} + 5$ more terms by permutations$)$ + $\frac{1}{2}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} + 2$ more terms by permutations$)$ |
| $Q^{(6)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}$ = ($\delta_{\alpha_1 \alpha_2} m_{\alpha_3} m_{\alpha_4} m_{\alpha_5} m_{\alpha_6}$ + 14 more terms by permutations$)$ + $\frac{1}{4}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} m_{\alpha_5} m_{\alpha_6} + 44$ more terms by permutations$)$ − $\frac{1}{10}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} \delta_{\alpha_5 \alpha_6} + 14$ more terms by permutations$)$ |

\[ m_\alpha = Q^{(1)}_\alpha \]
\[ m_\alpha m_{\alpha_2} = Q^{(2)}_{\alpha_1\alpha_2} + \frac{1}{2} \delta_{\alpha_1 \alpha_2} \]
\[ m_\alpha m_{\alpha_2} m_{\alpha_3} = Q^{(3)}_{\alpha_1\alpha_2\alpha_3} + \frac{1}{4}(\delta_{\alpha_1 \alpha_2} Q^{(1)}_{\alpha_3} + \delta_{\alpha_1 \alpha_3} Q^{(1)}_{\alpha_2} + \delta_{\alpha_2 \alpha_3} Q^{(1)}_{\alpha_1}) \]
\[ m_\alpha m_{\alpha_2} m_{\alpha_3} m_{\alpha_4} = Q^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4} + \frac{1}{6}(\delta_{\alpha_1 \alpha_2} Q^{(2)}_{\alpha_3\alpha_4} + 5$ more terms by permutations$)$ + $\frac{1}{2}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} + 2$ more terms by permutations$)$ |
\[ m_\alpha m_{\alpha_2} m_{\alpha_3} m_{\alpha_4} m_{\alpha_5} m_{\alpha_6} = Q^{(6)}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} + \frac{1}{10}(\delta_{\alpha_1 \alpha_2} Q^{(4)}_{\alpha_3\alpha_4\alpha_5\alpha_6} + 14$ more terms by permutations$)$ + $\frac{1}{4}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} Q^{(2)}_{\alpha_5\alpha_6} + 44$ more terms by permutations$)$ − $\frac{1}{10}(\delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} \delta_{\alpha_5\alpha_6} + 14$ more terms by permutations$)$ |
FIG. 1. Mean-field shapes of deformable surfaces of genus zero with vector order \((n = 1)\) on the left and nematic order \((n = 2)\) on the right. Above the mean-field transition temperature, the equilibrium shape is spherical for all \(n\). Below the transition temperature, the equilibrium shape depends on \(n\) and has an ellipsoidal form with 2 ground state vortices with strength 1 located at opposite poles for \(n = 1\) and a tetrahedral form with 4 ground state vortices with strength \(1/2\) located at the vertices of a tetrahedron for \(n = 2\).

FIG. 2. Mean-field shapes of deformable surfaces of genus zero with tetric order \((n = 3)\) on the left and hexatic order \((n = 6)\) on the right. Below the transition temperature, the equilibrium shape has an octahedral form with 6 ground state vortices with strength \(1/3\) located at the vertices of a octahedron for \(n = 3\) and an icosahedral form with 12 ground state vortices with strength \(1/6\) located at the vertices of a icosahedron for \(n = 6\).
FIG. 3. Mean-field shapes of deformable surfaces of genus zero with quartic order ($n = 4$). Below the transition temperature, the equilibrium shape has a distorted cubic form with 8 ground state vortices with strength $1/4$ located at the vertices of a distorted cube obtained by rotating its top face about its 4-fold axis by $\pi/4$ and compressing opposite faces.