MINIMAL DFAS FOR TESTING DIVISIBILITY

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1. Statement of the Problem

The following exercise is typical in introductory texts on deterministic finite automata (DFAs): “produce an automaton that recognizes the set of binary strings that, when interpreted as binary numbers, are divisible by \( k \).” For example, exercise 1.30 in [2] asks the student to prove that the language \( \{ x \mid x \text{ is a binary number that is a multiple of } k \} \) is regular for each \( k \geq 1 \); explicitly presenting an automaton is the easiest solution.

The traditional (and correct) answer constructs a \( k \)-state automaton that keeps track not only of divisibility by \( k \), but also the current residue modulo \( k \). For example, if the input read was 1101, the machine would remember “13 mod \( k \)”. The transitions between states are simple: if the automaton’s current state is “\( r \) mod \( k \)”, and the input symbol read is “0”, it moves to state \( (2r) \mod k \); if the input symbol read is “1”, it moves to state \( (2r + 1) \mod k \).

(This example also generalizes to bases other than binary. Furthermore, even if the input string is encoded in base \( b \), the canonical DFA will still have \( k \) states. It will, however, contain \( b \) transitions from each state.)

The traditional answer, unfortunately, in general fails to produce a minimal DFA. This paper addresses the considerably more difficult question of “how many states does a minimal DFA that recognizes the set of base-\( b \) numbers divisible by \( k \) have?” We denote this number by \( f_b(k) \) and derive a closed-form expression; in the proof, we also describe the states of the minimal DFA in more detail.

The function \( f_b(k) \) may be computed by algorithmic means. The author used two implementations of the Hopcroft minimization algorithm: an original Perl program and the highly-optimized AT&T FSM Package\textsuperscript{TM}. According to experts in the field, no prior work addresses the general case of this problem except through such computational alleys.

2. Interesting Patterns

The function \( f_b(k) \) exhibits very curious behavior. One interesting pattern considers \( f_b(k) \) with \( b \) fixed and \( k = x \cdot y^z \) for increasing values of \( z \).

**Example.** Table of \( f_b(k) \) for \( b = 6 \) and \( k = 2^z \). (That is, \( x = 1, y = 2, \) and \( z \) ranges from 0 to 10.)

| \( z \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|--------|---|---|---|---|---|---|---|---|---|---|----|
|        | \( 2^z \) | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
|        | \( f_b(2^z) \) | 1 | 2 | 3 | 5 | 8 | 12 | 20 | 29 | 45 | 72 | 104 |
|        | \( f_b(2^{z+1}) - f_b(2^z) \) | 1 | 1 | 2 | 3 | 4 | 8 | 9 | 16 | 27 | 32 | 64 |

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The successive differences of $f_6(2^z)$ are the powers of 2 and 3, sorted in increasing order!

**Example.** Table of $f_b(k)$ for $b = 2^2 \cdot 5 = 20$ and $k = 30 \cdot 5^z$. (That is, $x = 30$, $y = 5$, and $z$ ranges from 0 to 6.)

| $z$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|----|
| $30 \cdot 5^z$ | 30 | 150 | 750 | 3750 | 18750 | 93750 | 468750 |
| $f_{20}(30 \cdot 5^z)$ | 4  | 6  | 14 | 26 | 58 | 118 | 246 |
| $f_{20}(30 \cdot 5^{z+1}) - f_{20}(30 \cdot 5^z)$ | 2  | 8  | 12 | 32 | 60 | 128 | 300 |

Here, the successive differences of $f_{20}(30 \cdot 5^z)$ come in increasing order from two sequences: 

\[
\{2 \cdot 4^m\} = \{2, 8, 32, 128, \ldots\}
\]

and

\[
\{12 \cdot 5^m\} = \{12, 60, 300, \ldots\}.
\]

We observe that the function $f_b(k)$ manages to pick terms, in increasing order, from two unrelated sequences! At first, it is hard to imagine a formula that would produce such a function. Investigating this bizarre behavior was the starting point for this study.

## 3. Main Result

**Theorem 1.** Let $l(x, y) = \frac{x}{\text{gcd}(x, y)}$. Then

\[
f_b(k) = l(k, b^\infty) + \sum_{\alpha=0}^{\infty} \min \left\{ l(b^\alpha, k), l(k, b^\alpha) - l(k, b^{\alpha+1}) \right\}
\]

\[
= \min_{A \geq 0} \left\{ l(k, b^A) + \sum_{\alpha=0}^{A-1} l(b^\alpha, k) \right\}
\]

\[
= l(k, b^{A_0}) + \sum_{\alpha=0}^{A_0-1} l(b^\alpha, k),
\]

where $A_0$ is the smallest nonnegative integer $\alpha$ satisfying $l(k, b^\alpha) - l(k, b^{\alpha+1}) < l(b^\alpha, k)$.

**Remarks.** The function $l(x, y)$ is not symmetric; indeed, $l(x, y) = l(y, x)$ if and only if $x = y$.

We use the notation $l(k, b^\infty)$ to denote $l(k, b^\alpha)$ for sufficiently large $\alpha$; similarly, the infinite sum can be truncated when $l(k, b^\alpha) - l(k, b^{\alpha+1}) = 0$. This equality certainly holds for $\alpha \geq \log_2 k$.

Lemma 6 shows that the three expressions in the theorem are equivalent.

To understand the expressions of $f_b(k)$ in the theorem, we may draw a table listing $\alpha$, $l(b^\alpha, k)$, $l(k, b^\alpha)$, and $l(k, b^\alpha) - l(k, b^{\alpha+1})$. The first and third expressions may be understood fairly simply as written. However, the second expression is more difficult; it states that $f_b(k)$ is the minimal sum one can obtain by summing zero of more elements of the form $l(b^\alpha, k)$ (as $\alpha$ ranges from 0 to $A - 1$) and then the following value of $l(k, b^\alpha)$ (that is, $\alpha = A$).

**Example.** $b = 6, k = 16 = 2^4$: We can calculate $f_b(k)$ with any of the expressions above (for the third, use $A_0 = 2$). The minimal terms of the first expression appear underlined below; simultaneously, the minimal “path” $8 = 1 + 3 + 4$ (in terms of the second formula above) is indicated in boldface. Note that other paths such as $15 = 1 + 3 + 9 + 2, 9 = 1 + 8, \text{and } 16 = 16$ (the trivial path $A = 0$) yield non-minimal sums.
Corollary 2. The following are upper bounds for $f_b(k)$:

$$f_b(k) \leq k = l(k, b^0)$$

$$f_b(k) \leq 1 + \frac{k}{\gcd(k, b)} = l(b^0, k) + l(k, b^1)$$

$$f_b(k) \leq 1 + \frac{\gcd(b, k)}{\gcd(k, b^2)} + \frac{k}{\gcd(k, b^2)} = l(b^0, k) + l(b^1, k) + l(k, b^2)$$

Proof. These follow immediately from the second expression in Theorem 1. □

Corollary 3. The canonical DFA described in Section 1 is minimal if and only if $\gcd(k, b) = 1$ or $k = 2$.

Proof. The canonical DFA has $k$ states and hence we must determine when $f_b(k) = k$.

If $\gcd(k, b) = 1$ or $k = 2$, the first expression of Theorem 1 immediately gives $f_b(k) = k$. Otherwise, we have $\frac{k}{\gcd(k, b)} < k - 1$, and by the previous corollary,

$$f_b(k) \leq 1 + \frac{k}{\gcd(k, b)} < k.$$ □

Corollary 4. The successive differences of $f_b(2^z)$ are powers of 2 and 3, sorted in increasing order.

Proof. Manipulation of the result of the theorem yields

$$f_b(2^z) = l(2^z, 6^\infty) + \sum_{\alpha=0}^{\infty} \min \left\{ l(6^\alpha, 2^z), l(2^z, 6^\alpha) - l(2^z, 6^{\alpha+1}) \right\}$$

$$= 1 + \sum_{\alpha=0}^{\infty} \min \left\{ 3^\alpha \cdot \left\lfloor 2^{z-\alpha} \right\rfloor, 2^{z-\alpha-1} \right\}$$

$$= 1 + \sum_{\alpha=0}^{z-1} \min \left\{ 3^\alpha, 2^{z-\alpha-1} \right\}.$$ 

It is not difficult to see that as one increments $z \mapsto z + 1$, a new term of the form $\min \left\{ 3^\alpha, 2^{z-\alpha-1} \right\}$ is added, and the desired property holds. □

Remark. A similar approach may be applied to the general case of $f_b(x \cdot y^z)$ for increasing values of $z$. In particular, we can easily prove the pattern we noticed in Section 2 for $f_b(30 \cdot 5^z)$.

Corollary 5. If $b = p^n$ (not necessarily prime, but see the remark) and $k = p^m \cdot x$ with $\gcd(x, p) = 1$, then $f_b(k) = x + \left\lfloor \frac{m}{n} \right\rfloor$. 
Proof. We use the first expression of the theorem:

\[ f_b(k) = l(k, b^\infty) + \sum_{\alpha=0}^{\infty} \min \left\{ l(b^\alpha, k), l(k, b^\alpha) - l(k, b^{\alpha+1}) \right\} \]

\[ = x + \sum_{\alpha=0}^{\infty} \min \left\{ \left\lfloor \frac{p^\alpha}{m} \right\rfloor, \left\lfloor \frac{m}{n} \right\rfloor \cdot x - \left\lfloor \frac{p^\alpha}{n} \right\rfloor \cdot x \right\} \]

As long as \( n\alpha < m \), \( \left\lfloor \frac{p^\alpha}{m} \right\rfloor = 1 \) and \( \left\lfloor \frac{p^\alpha}{n} \right\rfloor \cdot x > \left\lfloor \frac{p^\alpha}{n} \right\rfloor \cdot x \). There are precisely \( \left\lfloor \frac{m}{n} \right\rfloor \) such \( \alpha \) (since \( 0 \leq \alpha < \frac{m}{n} \)), so we have

\[ f_b(k) = x + \sum_{\alpha=0}^{\left\lfloor \frac{m}{n} \right\rfloor - 1} \{1\} + \sum_{\alpha=\left\lfloor \frac{m}{n} \right\rfloor}^{\infty} \{0\} \]

\[ = x + \left\lfloor \frac{m}{n} \right\rfloor, \]

as desired. \( \square \)

Remark. If \( p \) is prime, and thus \( b \) is a prime power, this corollary completely characterizes \( f_b(k) \), as all \( k \) can be represented in the form \( p^m \cdot x \) with \( \gcd(x, p) = 1 \).

5. PROOF OF THE MAIN RESULT

Lemma 6. The three expressions of Theorem 1 are equivalent.

Proof. By looking at the powers of a fixed prime, we see that \( l(b^\alpha, k) \) and \( \gcd(k, b^\alpha) \) are increasing (not necessarily strictly) functions of \( \alpha \). It is also easy to show that \( \gcd(k, b^{\alpha+1})/\gcd(k, b^\alpha) \) is decreasing, which immediately implies that \( l(k, b^\alpha) - l(k, b^{\alpha+1}) \) is decreasing. Therefore, in the sum

\[ \sum_{\alpha=0}^{\infty} \min \left\{ l(b^\alpha, k), l(k, b^\alpha) - l(k, b^{\alpha+1}) \right\}, \]

one takes \( A_0 \) elements from the first sequence \( \{l(b^\alpha, k)\} \) and then infinitely many from the second sequence \( \{l(k, b^\alpha) - l(k, b^{\alpha+1})\} \). Telescoping the latter, one gets the other two expressions of the theorem. (The cut-off \( A_0 \) is the smallest nonnegative integer \( \alpha \) satisfying \( l(k, b^\alpha) - l(k, b^{\alpha+1}) < l(b^\alpha, k) \).) \( \square \)

Proof of Theorem 1. Constructing a DFA directly, as in Section 1, is often difficult because one must describe the transitions between states in addition to the states themselves. We will use the Myhill-Nerode Theorem and the accompanying theory of extension invariant equivalence relations to work with the states of the automaton only.

Definition. Given a language (set of strings) \( L \) over an alphabet \( \Sigma \), we define the extension invariant equivalence relation \( \sim_L \) associated with \( L \) as follows: strings \( x \) and \( y \) in \( \Sigma^* \) are equivalent \( (x \sim_L y) \) if for any suffix \( z \in \Sigma^* \), \( xz \in L \) if and only if \( yz \in L \). (As is customary, \( \Sigma^* \) denotes the set of all finite strings over \( \Sigma \). Later, we use \( \Sigma^+ = \Sigma^* \setminus \{\epsilon\} \) to denote the set of nonempty strings over \( \Sigma \).)

The Myhill-Nerode Theorem [1 Thms 3.9–10] establishes that the minimal-state automaton accepting \( L \) has, up to isomorphism, one state corresponding to each equivalence class of \( \sim_L \). Therefore, the minimal-state automaton has exactly the number of states as the index of \( \sim_L \). (In particular, a language \( L \) is regular if and
only if \( \sim_L \) has finite index.) In addition, any DFA recognizing \( L \) can be altered by identifying ("gluing") some states together to obtain the minimal-state automaton.

In this proof, we let \( \Sigma \) be the set of base-\( b \) digits and \( L \) the set of base-\( b \) numbers divisible by \( k \). In addition, since we work with only one language at a time, we may write \( x \sim y \) rather than \( x \sim_L y \).

To begin, we will restate the problem equivalently in a way that will allow us to utilize modular arithmetic. Because the canonical DFA accepting \( L \) has a state for each residue modulo \( k \), the Myhill-Nerode Theorem implies that the minimal-state DFA will contain states that correspond to groups of residues modulo \( k \). Therefore, in the pursuing analysis, rather than considering strings of digits, we discuss residues; in a way, we are projecting \( \Sigma^* \) onto \( \mathbb{Z}_k \) (in the natural manner). For example, \( L \) now becomes very simple: instead of containing all numbers divisible by \( k \), it contains the single residue \( 0 \) (mod \( k \)). To complete the reduction, we need only bother ourselves with one further

**Definition.** Let \( r \in \mathbb{Z}_k \) be a residue modulo \( k \) and \( d \in \Sigma \) a base-\( b \) digit. We define the concatenation \( rd \) to be the residue \( b \cdot r + d \) (mod \( k \)). Similarly, if \( d = d_{n-1} \cdots d_0 \in \Sigma^* \) is a nonempty string of digits, let the concatenation \( rd \) be what is obtained by successively concatenating individual digits:

\[
rd \equiv b \cdot (b \cdot (\ldots (b \cdot r + d_{n-1}) \cdots) + d_1) + d_0 \equiv b^n \cdot r + \overline{d_{n-1} \cdots d_1 d_0} \pmod{k},
\]

where \( \overline{d} \) denotes \( d \) interpreted as an integer. Of course, if \( d = \epsilon \), the empty string, \( rd = re \equiv r \).

Finally, extend \( \sim \) onto \( \mathbb{Z}_k \): residues \( x, y \in \mathbb{Z}_k \) are equivalent if for any string \( z \in \Sigma^* \), \( xz \equiv 0 \) (mod \( k \)) if and only if \( yz \equiv 0 \) (mod \( k \)).

Now, suppose \( \mathcal{A} \) is a nonnegative integer. We will describe

\[
(*) \quad l(k, b^\mathcal{A}) + \sum_{\alpha=0}^{\mathcal{A}-1} l(b^\alpha, k)
\]

pre-equivalence classes, each a group of residues, which will be a refinement of the equivalence classes of \( \sim_L \).

The pre-equivalence classes we define naturally present themselves in packages, a term we borrow from computer programming to indicate collections of classes. Altogether, there are \( \mathcal{A}+1 \) distinct packages, which we number \( 0, \ldots, \mathcal{A} \); in addition, we will sometimes refer to package \( \mathcal{A} \) as the distinctive package etcetera. These packages come in the sizes anticipated from \( (*) \): if \( 0 \leq \alpha < \mathcal{A} \), package \( \alpha \) contains \( l(b^\alpha, k) \) pre-equivalence classes, while package \( \mathcal{A} \) contains \( l(k, b^\mathcal{A}) \) pre-equivalence classes.

We now define the packages. Suppose \( 0 \leq \alpha < \mathcal{A} \). Package \( \alpha \) will consist of those residues \( r \) such that there exists a string \( d \) of length \( \alpha \) such that \( rd \equiv 0 \) and no smaller \( \alpha \) works; furthermore, these residues will be grouped according to their corresponding \( d \)'s. Mathematically, for each \( 0 \leq c < b^\alpha \) such that gcd\( (b^\alpha, k) | c \), package \( \alpha \) contains the pre-equivalence class \( \{ x \mid b^\alpha \cdot x + c \equiv 0 \} \), except those \( x \) that appeared in package \( \alpha - 1 \) or earlier. (Note that the equation \( b^\alpha \cdot x + c \equiv 0 \) has a solution \( x \) iff gcd\( (b^\alpha, k) | c \).) Because there are precisely \( b^\alpha / \gcd(b^\alpha, k) = l(b^\alpha, k) \) such \( c \) in the desired range, these packages have the stated sizes. Before we proceed, note that the union of the pre-equivalence classes in packages \( 0 \) through \( \alpha \) consists of all residues \( x \) satisfying \( b^\alpha \cdot x + c \equiv 0 \) with \( 0 \leq c < b^\alpha \), and no others.
Package etcetera consists of the leftovers; mathematically, it is similar, but there is no restriction on $c$: for each $0 \leq c < k$ (only to avoid duplication modulo $k$), package $A$ contains the pre-equivalence class $\{x \mid b^{\alpha} \cdot x + c \equiv 0\}$, except those $x$ that have appeared previously. Once again, we have the necessary number of classes, since $k/\gcd(k, b^{\alpha}) = l(k, b^{\alpha})$.

Example. $b = 6$, $k = 16 = 2^4$: the pre-equivalence classes for $A = 2$. This value of $A$ was chosen so that these groups correspond to the states in the minimal DFA. Strikeouts indicate that the given value of $x$ satisfies $b^{\alpha} \cdot x + c \equiv 0$ but already appeared in a previous package.

| Package 0 | Package 1 | Package 2 (etcetera) |
|-----------|-----------|----------------------|
| $c$ | $\{x\}$ | $c$ | $\{x\}$ | $c$ | $\{x\}$ |
| 0 | $\{0\}$ | 0 | $\{\square, 8\}$ | 0 | $\{\square, 4, 12\}$ |
| 2 | $\{5, 13\}$ | 4 | $\{3, 7, 11, 15\}$ |
| 4 | $\{2, 10\}$ | 8 | $\{\square, 6, 14\}$ |
| 12 | 1 | 9 | |

Recall once more from the statement of the theorem that $A_0$ is the smallest nonnegative integer $\alpha$ satisfying $l(k, b^{\alpha}) - l(k, b^{\alpha+1}) < l(b^{\alpha}, k)$.

We make three separate claims:

(1) for any $A$, our pre-equivalence classes coincide with the equivalence classes of $\sim_L$ with two possible exceptions: some pre-equivalence classes may be empty and some pre-equivalence classes in package etcetera may actually be equivalent (both of these would produce an overcount);

(2) for $A \leq A_0$, all the pre-equivalence classes are nonempty; and

(3) for $A \geq A_0$, the classes of package etcetera are actually inequivalent.

It follows that for $A = A_0$, our pre-equivalence classes are precisely the Myhill-Nerode equivalence classes of $\sim_L$.

We begin by affirming (1): if two residues $r$ and $s$ are in the same class of package $\alpha$, there exists no string $d$ of length less than $\alpha$ such that $rd \equiv 0$ or $sd \equiv 0$. In addition, $r \cdot b^{\alpha} \equiv s \cdot b^{\alpha}$, so for any string $d$ of length at least $\alpha$, we have $rd \equiv sd$. Therefore, $r$ and $s$ are equivalent, and the pre-equivalence classes are a refinement of those of $\sim_L$.

Moreover, if $r$ and $s$ are in different classes and at least one of $r$ and $s$ is not in package etcetera, then $r \not\sim s$. Indeed, if $r$ and $s$ are in different packages, the result is obviously true. If $r$ and $s$ are in different classes of the same package $\alpha$ with $\alpha < A$, we can also conclude that $r \not\sim s$ because $r$ and $s$ satisfy $b^{\alpha} \cdot x + c \equiv 0$ for different values of $c$; therefore, there exists a string $d$ (namely, the $d$ such that $d = c$) of length $\alpha$ such that $rd \equiv 0$ but $sd \equiv 0$.

Before continuing, we note the significance of $A_0$. If $\alpha \leq A_0$, then

$$l(k, b^{\alpha-1}) - l(k, b^{\alpha}) \geq l(b^{\alpha-1}, k) \iff k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^{\alpha})} \leq k - b^{\alpha-1},$$

and if $\alpha > A_0$, then

$$l(k, b^{\alpha-1}) - l(k, b^{\alpha}) < l(b^{\alpha-1}, k) \iff k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^{\alpha})} > k - b^{\alpha-1}.$$
Equipped, we proceed in order to (2). Suppose \( A \leq A_0 \); then, we claim that for any fixed \( 0 < \alpha \leq A \) and \( c \) such that \( \gcd(k, b^\alpha) \mid c \), there exists an \( x \) satisfying
\[
 b^\alpha \cdot x + c \equiv 0 \tag{†}
\]
which does not satisfy \( b^{\alpha-1} \cdot x + c' \equiv 0 \) with \( 0 \leq c' < b^{\alpha-1} \). Indeed, consider all \( x \) satisfying (†) and note that these \( x \) are spaced apart equally with \( \frac{k}{\gcd(k, b^\alpha)} \) separation between consecutive solutions. Multiplying these \( x \) by \( b^{\alpha-1} \) yields (possibly duplicate) residues \( b^{\alpha-1} \cdot x \) spaced \( k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^\alpha)} \) apart. But, because \( \alpha \leq A \leq A_0 \),
\[
 k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^\alpha)} \leq k - b^{\alpha-1},
\]
whence there exists an \( x \) satisfying (†) such that \( (b^{\alpha-1} \cdot x) \mod k \) is in between 1 and \( k - b^{\alpha-1} \), and such an \( x \) cannot satisfy \( b^{\alpha-1} \cdot x + c' \equiv 0 \) with \( 0 \leq c' < b^{\alpha-1} \). Therefore, all of the classes of packages 0 through \( A \) are nonempty.

We finish with (3). Suppose \( A \geq A_0 \); it suffices to show that if \( r \sim s \) and \( \alpha \) is the minimal \( \alpha \) such that \( b^\alpha \cdot r \equiv b^\alpha \cdot s \), then \( \alpha \leq A \). Assume the contrary: \( \alpha > A \). Then, \( r \) and \( s \) are both solutions of (†) for a fixed \( c \). To derive a contradiction, we again focus on the spacing of solutions of (†). So, consider all \( x \) satisfying (†); they are spaced \( \frac{k}{\gcd(k, b^\alpha)} \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^\alpha)} \) apart. As before, the residues \( b^{\alpha-1} \cdot x \) for \( x \) satisfying (†) are spaced \( k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^\alpha)} \) apart. However, because \( \alpha > A \geq A_0 \),
\[
 k \cdot \frac{\gcd(k, b^{\alpha-1})}{\gcd(k, b^\alpha)} > k - b^{\alpha-1}
\]
and thus there is not enough room for two distinct \( (b^{\alpha-1} \cdot x) \mod k \) in between 1 and \( k - b^{\alpha-1} \). Therefore, either \( b^{\alpha-1} \cdot r \equiv b^{\alpha-1} \cdot s \) or one of \( r \) and \( s \) satisfies \( b^{\alpha-1} \cdot x + c' \equiv 0 \) with \( 0 \leq c' < b^{\alpha-1} \). The former contradicts the minimality of \( \alpha \), and the second is impossible as well: without loss of generality, \( r \) satisfies such an equation. But then, there exists a string \( d \) of length \( \alpha - 1 \) such that \( rd \equiv 0 \). Because \( r \sim s \), it follows that \( rd \equiv sd \equiv 0 \) for a string of length \( \alpha - 1 \), once again contradicting the minimality of \( \alpha \). We have reached a contradiction in all cases, therefore our assumption was false and \( \alpha \leq A \). Therefore, any two residues are “distinguished” at or before \( \alpha = A \), and it follows that any \( r \) and \( s \) in the package etcetera are equivalent if and only if they are in the same pre-equivalence class.

At last, we are done. \( \square \)

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For large-scale computations, when speed was crucial, AT&T Research’s FSM Package was used to compute \( f_b(k) \), to complement the author’s own programs.

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