Improved mutual coherence of some random overcomplete dictionaries for sparse representation*

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Abstract

The letter presents a method for the reduction in the mutual coherence of an overcomplete Gaussian or Bernoulli random matrix, which is fairly small due to the lower bound given here on the probability of the event that the aforesaid mutual coherence is less than any given number in (0, 1). The mutual coherence of the matrix that belongs to a set which contains the two types of matrices with high probability can be reduced by a similar method but a subset that has Lebesgue measure zero. The numerical results are provided to illustrate the reduction in the mutual coherence of an overcomplete Gaussian, Bernoulli or uniform random dictionary. The effect on the third type is better than a former result.

Keywords: Sparse representation, Mutual coherence, Gaussian random matrices, Bernoulli random matrices

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1 Introduction

Recently, sparse representation has attracted a lot of scientists from many different fields. If \( \mathbf{b} \in \mathbb{R}^m \) is a given signal which is known to be represented as a linear combination of only a few atoms of a dictionary \( \mathbf{A} \in \mathbb{R}^{m \times n} \) \( (m < n) \), the representation vector \( \mathbf{x} \) can be correctly and effectively computed by some procedures. The desire to find the maximally sparse solutions of an underdetermined linear system \( \mathbf{Ax} = \mathbf{b} \) can be cast as the following optimization problem,

\[
(P_0) \quad \min_{\mathbf{x}} \| \mathbf{x} \|_0 \quad s.t. \quad \mathbf{b} = \mathbf{Ax}
\]  

2 Guarantees for Uniqueness and Stability

It is well known that that for a noiseless signal \( \mathbf{b}_0 = \mathbf{Ax}_0 \), \( \mathbf{x}_0 \) is the unique solution to the problem \((P_0)\) with \( \| \mathbf{x} \|_0 < (1 + 1/\mu(\mathbf{A}))/2 \) \cite{5} and both OMP \cite{8} and BP \cite{4} can find it \cite{5,11}. \( \mu(\mathbf{A}) \) is the largest absolute normalized inner product between different columns from \( \mathbf{A} \). Clearly, the smaller the mutual coherence of \( \mathbf{A} \) \cite{5} is, the better the above result becomes.

An optimized algorithm was proposed in \cite{6} to reduce \( \mu_t \) of the product of a projection and a given dictionary in the context of compressed sensing, a descendant of sparse approximation. \cite{9} presented improved versions of thresholding and OMP for sparse representation with an iterative algorithm which produced a new dictionary at the step of sensing with the cross cumulative coherence. While we work with the mutual coherence of an overcomplete Gaussian or Bernoulli random dictionary to improve the classical results in \cite{5} \cite{8} \cite{4} \cite{11}.

Why we feel an interest in the mutual coherence of such dictionaries? Firstly, an overcomplete Gaussian or Bernoulli random matrix satisfies the RIP condition, another measure of a dictionary, which is widely used and different from the mutual coherence, with a high probability \cite{2}. So they have been commonly used in general numerical experiments to verify new methods. Secondly, the mutual coherence of such a matrix has been studied by some statistics from the limiting laws point of view \cite{10}. But how can a person directly reduces them in the finite dimensional case?

3 Reducing the mutual coherence of an overcomplete Gaussian/Bernoulli random matrix

When \( \mathbf{A} \) is an overcomplete Gaussian or Bernoulli random matrix and has full row rank, the mutual coherence is fairly small \cite{3} but can be reduced by multiplying an inverse matrix \( \mathbf{P} \) generated from \( \mathbf{A} \) itself on the left side. In this section, it is shown that the lower bound of the probability \( \mathbb{P}\{\mu(\mathbf{PA}) \leq \varepsilon\} \) with \( \varepsilon \in (0, 1) \) is larger than the one of \( \mathbb{P}\{\mu(\mathbf{A}) \leq \varepsilon\} \). The two bounds are obtained
by the same way.

A Bernoulli random matrix $A$ means that $a_{ij}$ is independently selected to be $\pm 1/\sqrt{m}$ with the same probability, while $a_{ij}$ of a Gaussian random matrix independently follows a normal distribution $N(0, m^{-1})$. $\alpha_j$ denotes the $j$th column of $A$. From the strong concentration of $\|Ax\|_2^2$ in [2], it is known that the extreme singular values of $A$, $\sigma_1$ and $\sigma_m$, satisfy $0 < \sqrt{1 - \eta} \leq \sigma_m(A) \leq \sigma_1(A) \leq \sqrt{1 + \eta}$ with the probability which is more than $1 - 2\exp(c_0(\eta))$ with $c_0(\eta) = \eta^2/4 - \eta^3/6$, $\eta \in (0, 1)$. So $A$ has full rank with exponentially high probability.

$P = (AA^T)^{-1/2}$ makes $(PA)(PA)^T = I_m$. The intuitive idea to do this is that an orthogonal matrix has the smallest mutual coherence and $PA$ consisting of $m$ orthonormal rows may have a smaller mutual coherence than $\mu(A)$. Consider $PA$ and $A$ in the singular value decomposition domain. The SVD of $A$ is $A = USO^TV^T$ with two orthonormal matrices $U$ and $V$ and a diagonal matrix, $S = \text{diag}\{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m > 0\}$. $V$ can be divided into $V_1 \in \mathbb{R}^{n \times m}$ and $V_2 \in \mathbb{R}^{n \times (n-m)}$. $v_j = [v_j^{(1)}, v_j^{(2)}]$ is the $j$th row of $V$ and $v_j^{(1)}, v_j^{(2)}$ are the $j$th rows of $V_1$ and $V_2$ respectively. The $k$th column of $A = USV^T$ is $\alpha_k = USv_k^{(1)^T}$ and $|\langle \alpha_j, \alpha_k \rangle|$ can be bounded by the Wielandt inequality in [13, 7] as

$$|\langle \alpha_j, \alpha_k \rangle|_{j \neq k} = |\langle USv_k^{(1)^T} (USv_k^{(1)^T}) \rangle| = |v_k^{(1)^T} S^2 v_k^{(1)^T}|$$

$$\leq \left( \frac{\sigma_j^2 - \sigma_m^2}{\sigma_1^2 + \sigma_m^2} + \frac{|\langle v_j^{(1)^T}, v_k^{(1)^T} \rangle|}{\|v_j^{(1)^T}\|_2 \|v_k^{(1)^T}\|_2} \right) \left( 1 + \frac{\sigma_j^2 - \sigma_m^2}{\sigma_1^2 + \sigma_m^2} \frac{|\langle v_j^{(1)^T}, v_k^{(1)^T} \rangle|}{\|v_j^{(1)^T}\|_2 \|v_k^{(1)^T}\|_2} \right)^{-1} \triangleq u_2$$

The $k$th column of $PA$ is $P\alpha_k = UV_k^{(1)^T}$ and $|\langle P\alpha_j, P\alpha_k \rangle|/\|P\alpha_j\|_2 \|P\alpha_k\|_2$ which is equal to $|\langle v_j^{(1)^T}, v_k^{(1)^T} \rangle|/\|v_j^{(1)^T}\|_2 \|v_k^{(1)^T}\|_2 \triangleq u_3$. Clearly, $u_2$ is larger than $u_3$. For the subset of overcomplete Gaussian and Bernoulli random matrices with the aforementioned positive bounded singular values, consider the following events $E_1 = \{(1 - \eta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \eta)\|x\|_2^2, \exists \eta \in (0, 1)\}$, $E_2 = \{\mu(A) \leq \varepsilon, \varepsilon \in (0, 1)\}$ and $E_3 = \{\mu(PA) \leq \varepsilon, \varepsilon \in (0, 1)\}$. It is shown that

$$P\{E_1 \cap E_2\} = P\{E_1\} - P\{E_1 \cap E_2^c\} \quad P\{E_1 \cap E_3\} = P\{E_1\} - P\{E_1 \cap E_3^c\}$$

$$P\{E_1 \cap E_2^c\} \leq n(n - 1)P\{\varepsilon \leq u_2\} \triangleq p_2 \quad P\{E_1 \cap E_3^c\} \leq n(n - 1)P\{\varepsilon \leq u_3\} \triangleq p_3$$

So it indirectly reflects the phenomenon that the mutual coherence of the product of an inverse matrix $P$ and an overcomplete Gaussian or Bernoulli random matrix is smaller than the original mutual coherence from the result that the former has a better lower bound of the probability $P\{\mu(PA) \leq \varepsilon\}$ than the one of $P\{\mu(A) \leq \varepsilon\}, \varepsilon \in (0, 1)$, although they are obtained by the same way.

4 The essential mutual coherence

In this section, we use the above way for reducing the mutual coherence of an overcomplete Gaussian or Bernoulli dictionary on a set $X$ which almost includes
the two types and prove that the newly defined essential mutual coherence on \( \mathbf{X} \) is strictly smaller than the original one but a Lebesgue zero measure subset.

\( \mathbf{X} \triangleq \{ \mathbf{A} \in \mathbb{R}^{m \times n}; m < n, \mu(\mathbf{A}) < 1, 0 \not\in \text{diag}(\mathbf{A}^{T}\mathbf{A}) \}. m < n \) is natural and \( \mathbf{A} \) has no zero columns otherwise there exist some meaningless atoms. It is senseless to multiply an inverse matrix \( \mathbf{P} \) from the left side of \( \mathbf{A} \) for reducing \( \mu(\mathbf{A}) \), if \( \mu(\mathbf{A}) = 1 \). For the Gaussian type, \( \mathbb{P}\{0 \in \text{diag}(\mathbf{A}^{T}\mathbf{A})\} = 0 = \mathbb{P}\{\mu(\mathbf{A}) = 1\} \) if we notice that the independence of each \( a_{ij} \) and the condition on equality of Cauchy Schwarz Inequality. For the Bernoulli type, \( \mathbb{P}\{\mu(\mathbf{A}) = 1\} \leq n(n - 1) \exp(-m/2) \) due to the Hoeffding Inequality in probability. That is to say that the Gaussian or Bernoulli type belongs to \( \mathbf{X} \) with a high probability.

Consider an equivalent problem of \((P_0), (P'_0)\) \( \min_{\mathbf{x}} \|\mathbf{x}\|_0 \ s.t. \mathbf{Pb} = \mathbf{PAx} \) with an invertible matrix \( \mathbf{P} \). A new quantity, essential mutual coherence, is defined as \( \mu_e(\mathbf{A}) \triangleq \inf_{\mathbf{P}} \mu(\mathbf{PA}), \mathbf{P} \in GL_m(\mathbb{R}) = \{ \text{all } m \text{ order real invertible matrices} \} \) and is invariant for the elementary row operations of matrices. In this section we show that \( \mu_e(\mathbf{A}) < 1, \mu(\mathbf{A}) \) holds true almost everywhere on \( \mathbf{X} \) by constructing a matrix \( \mathbf{P} \triangleq \mathbf{E}_m + \varepsilon \mathbf{E}_{m,1} \in GL_m(\mathbb{R}) \) with a proper \( \varepsilon \) such that \( \mu(\mathbf{PA}) < 1, \mu(\mathbf{A}) \). \( \mathbf{E}_m \) is the \( m \) order identity matrix and \( \mathbf{E}_{m,1} \in \mathbb{R}^{m \times m} \) consists of 0 but 1 on the position \((m, 1)\).

How to choose the parameter \( \varepsilon \)? For two different columns of \( \mathbf{A}, \alpha_i \) and \( \alpha_j, i < j \), a polynomial \( f_{ij}(\varepsilon) = A_{ij} + B_{ij}\varepsilon + C_{ij}\varepsilon^2 + D_{ij}\varepsilon^3 + E_{ij}\varepsilon^4 \) can be obtained from

\[
I(\mathbf{P} \alpha_i, \mathbf{P} \alpha_j) \triangleq \frac{|(\mathbf{P} \alpha_i, \mathbf{P} \alpha_j)|}{\|\mathbf{P} \alpha_i\|_2 \|\mathbf{P} \alpha_j\|_2} < 1, \mu(\mathbf{A}) \quad (\text{use } \mu \text{ instead of } \mu(\mathbf{A})).
\]

\[
f_{ij}(0) = A_{ij} = \|\alpha_i\|_2^2 \|\alpha_j\|_2^2 (I(\alpha_i, \alpha_j)^2 - \mu^2) < 0, \text{ for all pairs } (\alpha_i, \alpha_j) \text{ such that } I(\alpha_i, \alpha_j) < 1. \]

There exists a \( \varepsilon_{1(i,j)} \) > 0 such that for all \( \varepsilon \in (-\varepsilon_{1(i,j)}, \varepsilon_{1(i,j)}) \), \( f_{ij}(\varepsilon) < 0 \) owing to continuity of the polynomial \( f_{ij}(\varepsilon) \). A minimum, \( \varepsilon_1 \), can be founded among all the \( \varepsilon_{1(i,j)} > 0 \) for all such pairs.

For all pairs \((\alpha_i, \alpha_j)\) such that \( I(\alpha_i, \alpha_j) = 1\), if \( f_{ij}'(0) = B_{ij} \) is positive or negative, then there exists a \( \varepsilon_{2(i,j)} > 0 \) such that \( f_{ij}(\varepsilon) < 0, \forall \varepsilon \in (-\varepsilon_{2(i,j)}, 0) \) or \( (0, \varepsilon_{2(i,j)}) \). Among all of them there exists a minimum, \( \varepsilon_2 \). So the sign of the final \( \varepsilon \) is attributed to all of the pairs such that \( I(\alpha_i, \alpha_j) = 1 \).

Two exceptions catch our attention. Firstly, it is impossible to choose \( \varepsilon_{1(i,j)} \) or \( \varepsilon_{2(i,j)} \) when both \( A_{ij} \) and \( B_{ij} \) are zero. The set \( \mathbf{X}_1 \triangleq \{ \mathbf{A} \in \mathbf{X}; B_{ij} = 0, A_{ij} = 0 \} \) has measure zero due to \( \mathbf{X}_1 \subset \cup_{i < j} \mathbf{E}_{ij} \) and \( \mathbf{E}_{ij} \) which consists of all matrices that satisfy \( \|\alpha_i\|_2^2 \|\alpha_j\|_2^2 (a_{mi}a_{1j} + a_{1i}a_{mj}) - \langle \alpha_i, \alpha_j \rangle (\|\alpha_i\|_2^2 a_{1j}a_{mj} + \|\alpha_j\|_2^2 a_{1i}a_{mi}) = 0 \) is a Lebesgue measure zero set \( [13] \), if matrix \( \mathbf{A} \in \mathbf{X} \) is seen as a vector in \( \mathbb{R}^{mn} \) in some order. So \( A_{ij} = 0 \) or \( B_{ij} = 0 \) seldom occur at the same time.

Secondly, \( \varepsilon_2 \) can not be obtained if there exist two different pairs such that \( I(\alpha_{i_1}, \alpha_{j_1}) = 1 = I(\alpha_{i_2}, \alpha_{j_2}) \) and \( B_{i_1,j_1}B_{i_2,j_2} < 0 \). Consider a set that contains more elements, \( \mathbf{X}_2 \triangleq \{ \mathbf{A} \in \mathbf{X}; I(\alpha_{i_k}, \alpha_{j_k})_{i_k < j_k}; k > 1 \} \), a subset of \( \cup_{(i_1,j_1) \neq (i_2,j_2)} \mathbf{F}_{(i_1,j_1),(i_2,j_2)} \). Similarly, \( \mathbf{F}_{(i_1,j_1),(i_2,j_2)} = \{ \mathbf{A}, I(\alpha_{i_1}, \alpha_{j_1})^2 \|\alpha_{i_2}\|_2^2 \|\alpha_{j_2}\|_2^2 = \|\alpha_{i_1}\|_2^2 \|\alpha_{j_1}\|_2^2 \} \) is also a measure zero set. So only one pair almost always achieve \( \mu \).

When \( \mathbf{A} \in \mathbf{X} \) is given, an interval \( (-\varepsilon_{1(i,j)}, \varepsilon_{1(i,j)}) \) can be selected for the pairs such that \( I(\alpha_i, \alpha_j) < 1 \) and an interval \( (-\varepsilon_{2(i,j)}, 0) \) or \( (0, \varepsilon_{2(i,j)}) \) can be chosen if \( \alpha_i \),
and $\alpha_j$ satisfy $I(\alpha_i, \alpha_j) = \mu$. So a proper $\varepsilon$ which is in the intersection among all of the above intervals can be selected to obtain $\mu(\mathbf{PA}) < \mu(\mathbf{A})$. All the exceptions that are included in $\mathbf{X}$ fall into $\mathbf{X}_1 \cup \mathbf{X}_2$, which has measure zero. So a proper $\varepsilon$ with plus or minus sign can be chosen to construct $\mathbf{P}$ such that $\mu(\mathbf{PA}) < \mu(\mathbf{A})$ almost everywhere on the set $\mathbf{X}$.

5 Experiments and results

In the first experiment, both $\mu(\mathbf{A})$ and $\mu(\mathbf{PA})$ are calculated for the $m \times 2m$ Bernoulli (Fig. 1) or Gaussian (Fig. 2) random dictionary averaged over 100 times for each value of $m$ in the range $[100, 500]$.

Only Fig. 1 is referred owing to the similarity between the two figures. From about $m = 280$, the bound $0.5(1 + 1/\text{coherence})$ exceeds 3 and keeps increasing till 500 when coherence is taken as $\mu(\mathbf{PA})$ but the one with $\mu(\mathbf{A})$ remains below 3 till 500 although it also becomes larger and larger. That is, $\mu(\mathbf{PA})$ can lead to the recovery of the sparse solution with 3 nonzero elements for both OMP and BP when $m \in [280, 500]$. It is one more than the result obtained by using $\mu(\mathbf{A})$. Both $0.5(1 + 1/\mu(\mathbf{A}))$ and $0.5(1 + 1/\mu(\mathbf{PA}))$ grow alongside the increase in $m$, so only six values of $m$ are listed in Table 1 in order to avoid the similarity between the forms of Fig. 1 and Table 1 which considers all the values in $[1500, 2000]$. As shown in Table 1, the condition $0.5(1 + 1/\mu(\mathbf{A}))$ is below 5 but $0.5(1 + 1/\mu(\mathbf{PA}))$ is above 6 when $m$ is after 1800. The second experiment compares the effect of the above method and the one proposed in [3] (called BEZ here) for decreasing the mutual coherence of the overcomplete uniform random dictionary $\mathbf{D}$ which is obtained by choosing the entries independently from a uniform distribution in $[0, 1]$ and then normalizing the columns to a unit $\ell_1$-norm. The authors in [3] used $\mathbf{P} = (1 - \varepsilon)/100\mathbf{1}^T$. Here, $\mathbf{1} \in \mathbb{R}^{100}$ is of 1’s and $\varepsilon$ satisfies $0 < \varepsilon \ll 1$.

Three values, $0.5(1 + 1/\mu(\mathbf{D}))$, $0.5(1 + 1/\mu(\mathbf{PD}))$ with $\mathbf{P} = (\mathbf{DD}^T)^{-1/2}$ and $0.5(1 + 1/\mu(\mathbf{PD}))$ with $\mathbf{P} = (1 - \varepsilon)/100\mathbf{1}^T$ are obtained for the dictionary $\mathbf{D}$ generated as per instructions in [3] with a fixed size of $100 \times 200$. This is done

![Fig. 1. A comparison of $0.5(1 + 1/\mu(\mathbf{A}))$ and $0.5(1 + 1/\mu(\mathbf{PA}))$ for the Bernoulli random dictionary of size $m \times 2m$, where $m \in [100, 500]$.](image-url)
Fig. 2. A comparison of $0.5(1 + 1/\mu(A))$ and $0.5(1 + 1/\mu(\text{PA}))$ for the Gaussian random dictionary of size $m \times 2m$, where $m \in [100, 500]$.

Table 1
A comparison of $0.5(1+1/\mu(A))$ and $0.5(1+1/\mu(\text{PA}))$ for the Bernoulli random matrix of size $m \times 2m$ when $m \in \text{linspace}(1500, 2000, 100)$ (a MATLAB® notation).

| $m$   | 1500  | 1600  | 1700  | 1800  | 1900  | 2000  |
|-------|-------|-------|-------|-------|-------|-------|
| $0.5(1+1/\mu(A))$ | 4.1531 | 4.2708 | 4.3896 | 4.4893 | 4.5769 | 4.6738 |
| $0.5(1+1/\mu(\text{PA}))$ | 5.6711 | 5.8311 | 5.9798 | 6.0945 | 6.2611 | 6.4046 |

Table 2
A comparison of $0.5(1+1/\mu(A))$ and $0.5(1+1/\mu(\text{PA}))$ for the Gaussian random matrix of size $m \times 2m$ when $m \in \text{linspace}(1500, 2000, 100)$.

| $m$   | 1500  | 1600  | 1700  | 1800  | 1900  | 2000  |
|-------|-------|-------|-------|-------|-------|-------|
| $0.5(1+1/\mu(A))$ | 4.1686 | 4.3185 | 4.4032 | 4.4860 | 4.5952 | 4.6469 |
| $0.5(1+1/\mu(\text{PA}))$ | 5.7126 | 5.8191 | 5.9664 | 6.0822 | 6.2473 | 6.4228 |
100 times and $\epsilon$ is selected to maximize $0.5(1 + 1/\mu(PD))$ with $P$ in [3] over \texttt{linspace}(10^{-4}, 10^{-1}, 1000) per time. In Fig. 3 three lines placed from bottom to up represent the aforementioned three values respectively. $0.5(1 + 1/\mu(D))$ is below 1.5 all the time. In all 100 times, $0.5(1 + 1/\mu(PD))$ obtained by using BEZ in [3] is strictly smaller than 2, but meanwhile, the one generated by applying our method can leap two 86 times.

![Fig. 3. A comparison of $0.5(1+1/\mu(D))$, $0.5(1+1/\mu(PD))$ with $P=(DD^T)^{-1/2}$ and $0.5(1+1/\mu(PD))$ with $P=(1-\epsilon)/10011^T$ for $D$ of size 100 × 200.](image)

6 Conclusion

The mutual coherence of an overcomplete Gaussian or Bernoulli random matrix which is proven to be small can be reduced directly by multiplying an inverse matrix from the left side to improve the traditional umbrella of unique sparse representation and successful reconstruction behavior. Furthermore, the newly proposed essential mutual coherence which is inspired by what has been done is proven to be strictly smaller than the original one on a set except a Lebesgue measure zero set. Numerical results exhibits the decrease in the mutual coherence of an overcomplete Gaussian or Bernoulli random dictionary and also an overcomplete Uniform random matrix which is better than the former result obtained by other authors.

A Tail bound

For Gaussian random matrices, $m||\alpha_j||_2^2 \sim \chi^2(m)$.

- $\mathbb{P}\{Z - m \leq -2\sqrt{mx}\} \leq \exp(-x)$, $\forall x > 0$, for a centralized $\chi^2$-variable $Z$ with $m$ degrees of freedom [12].

- $\mathbb{P}\{|\langle \alpha_j, \alpha_k \rangle|_{j<k} \geq x\} \leq 2\exp(-0.25mx^2/(1 + 0.5x))$, $\forall x > 0$ owing to the independence of $\alpha_j$ and $\alpha_k$, $j < k$ [1].
For all \( a \in (0,1) \), according to the fact that

\[
\mathbb{P}\{\mu(A) \geq \varepsilon\} \leq n(n-1)/2 \left( \mathbb{P}\{|\langle \alpha_j, \alpha_k \rangle|_{j<k} \geq a\varepsilon\} + 2\mathbb{P}\{\|\alpha_j\|_2^2 \leq a\} \right)
\]

we have

\[
\mathbb{P}\{\mu(A) \geq \varepsilon \in (0,1)\} \leq n(n-1)/2 \left[ \exp\left(-\frac{ma^2\varepsilon^2}{4(1+a\varepsilon/2)}\right) + \exp\left(-\frac{m}{4}(1-a)^2\right) \right]
\]

A similar tail bound can be obtained by applying the Hoeffding Inequality in probability for the Bernoulli case.

**B E\(ij\) is a Lebesgue measure zero set**

Without loss of generality, we show that \( m(E_{12}) = 0 \) (\( m \) denotes the Lebesgue measure on \( \mathbb{R}^{nm} \)). \( E_{12} \subset R_{12} \times R^{m\times(n-2)} \triangleq S_{12} = \bigcup_r S_r \). \( R_{12} \) is a set which consists of all \((\alpha_1^T, \alpha_2^T)^T\) that satisfy

\[
\|\alpha_1\|_2^2\|\alpha_2\|_2^2(a_{m1}a_{12} + a_{11}a_{m2}) - \langle \alpha_1, \alpha_2 \rangle(\|\alpha_1\|_2^2a_{12}a_{m2} + \|\alpha_2\|_2^2a_{11}a_{m1}) = 0
\]

\( S_r = S_r^{(1)} \times S_r^{(2)} \) with \( S_r^{(2)} = (-r, r)^{m(n-2)} \). It is known that \( m(S_r) = m(S_r^{(1)}) \cdot m(S_r^{(2)}) = 0 \) due to \( m(S_r^{(1)}) = 0 \) and \( m(S_r^{(2)}) = (2r)^{m(n-2)} \). So \( m(S_{12}) = \lim_{r \to \infty} m(S_r) = 0 \) because \( S_1 \subset S_2 \subset \ldots \subset S_r \subset \ldots \).

\( m(R_{12}) = 0 \) due to a polynomial from \( \mathbb{R}^n \) to \( \mathbb{R} \) is either identically zero or nonzero almost everywhere.

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