The approximation neural-network method for solving nonlinear multi-criteria inverse problems of geophysics

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Abstract. A multi-criteria inverse problem is reduced to a system of operator equations on compact sets. It is shown that some a posteriori error estimates of solutions of the multi-criteria problem decrease with an increasing number of the criteria used. The approximation neural-network method for solving the multi-criteria inverse problem is presented. An example of a numerical solution of the two-criterion problem of geoelectrics is given, and the a posteriori error estimates are calculated.

1. Introduction
Many inverse problems of geophysics are reduced to solving a nonlinear operator equation of the first kind of the form \[ Ag = f, \quad g \in G_1 \subset G, \quad f \in F, \] (1) where \( G, F \) are the normalized spaces of solutions (the sought characteristics \( g \) of the medium) and data (the observed characteristics \( f \) of the geophysical field), respectively, \( A \) is a continuous operator acting from \( G \) into \( F \); \( G_1[\Omega] \) is a compact set of the functions that are defined in the studied domain \( \Omega \) and that determine the properties of the medium within this domain, with natural a priori constraints taken into account. If \( A \) is a one-to-one operator, then the problem of solving equality (1) refers to a class of the conditionally well-posed inverse problems and is a real mathematical model for interpreting the geophysical data. For a conditionally well-posed problem on a compact set, estimates of the error of an approximate solution can be obtained [2-6].

2. Multi-Criteria Inverse Problems
We consider a problem in which it is required to determine two characteristics \( g_1, g_2 \) of the medium from two different types of characteristics \( f_1, f_2 \) of geophysical fields of the same or different physical nature, i.e., a problem of two-criterion (joint) inversion. This problem is defined by a system of two operator equations [7]:

\[ \begin{cases} A g_1 = f_1, & g_1 \in G_1 \subset G, \quad f_1, f_2 \in F, \\ A g_2 = f_2, & g_2 \in G_2 \subset G, \quad f_1, f_2 \in F, \end{cases} \] (2)
where \( A_1, A_2 \) are the continuous one-to-one operators. It is assumed that solutions \( g_1, g_2 \) belong to the same functional space \( G \) and the same applies to the elements of the data space \( f_1, f_2 \in F \). This condition is achieved through the appropriate normalization of the characteristics of the medium and field.

Since \( A_1, A_2 \) are one-to-one operators, each equation can be solved independently of the other. However, the physical characteristics of a medium are typically interdependent, and at a given value of one characteristic, another can only have quite determinate values [8]. Due to this property, the inverse problem for each characteristic can be solved on the narrower sets of the admissible solutions which, as a rule, reduces practical ambiguity (an error) of the approximate solutions. In many actual situations the interdependence of characteristics \( g_1, g_2 \) of the medium can be described based on the a priori information with the use of some given continuous functionals \( u_i(g_1, g_2) \) in the form of the criterion constraints [5,7]:

\[
1 \leq g_1, g_2 \leq 0, \quad i = 1, ..., I, \quad g_1 \in G_1, \quad g_2 \in G_2. \tag{3}
\]

We assume that if \( g_2 \in G_2' \subseteq G_2 \), where \( G_2' \) is a closed subset, then the criterion constraints (3) define a certain closed subset \( G_1' = \{ g_1 : g_1 \in G_1, \; u_i(g_1, g_2) \leq 0, \; i = 1, ..., I, \; g_2 \in G_2' \subseteq G_2 \} \).

Let us consider a system of equations (2) under condition (3). Since the inverse problem for each criterion has a unique solution, for solving the first equation of system (2), we can use the second equation combined with (3) as the additional constraints for the set of the admissible solutions for the first equation and reduce system (2) to one operator equation. Let \( f_1, f_2 \) be the fixed right-hand sides of the first and second equations of (2) specified with errors \( \delta_1, \delta_2 \) and \( g_{1\delta}, g_{2\delta} \) - the exact solutions of system (2) under condition (3).

The problem of solving the first equation of system (2) with the constraints (3) and provided that the solution of the second equation \( g_2 \in G_{2\delta} \), where \( G_{2\delta} = \{ g_2 : g_2 \in G_2, \; \| A_2 g_2 - f_2 \|_G \leq \delta_2 \} \), is reduced to the following operator equation:

\[
A_1 g_1 = f_1, \quad g_1 \in G_{1\delta}, \quad f_1 \in F, \tag{4}
\]

where

\[
G_{1\delta} = \{ g_1 : g_1 \in G_1, \; u_i(g_1, g_2) \leq 0, \; i = 1, ..., I, \; g_2 \in G_{2\delta} \subseteq G_2 \}. \tag{5}
\]

Note that by definition (3) the subset \( G_{1\delta} \subseteq G_1 \) is closed and not empty (\( g_{1\delta} \in G_{1\delta} \)). So the problem (5) is well-posed.

2.1. Error estimation for Multi-Criteria Problems

We suppose that there are approximate solutions \( g_{1\delta}, g_{2\delta} \) of problem (2) under condition (3):

\[
\| A_1 g_{1\delta} - f_1 \|_G \leq \delta_1, \quad \| A_2 g_{2\delta} - f_2 \|_G \leq \delta_2 \quad (\text{in the general, solutions } g_{1\delta}, g_{2\delta} \text{ of problem (2) under condition (3) may not exist when the right-hand sides } f_1, f_2 \text{ are known with error } \delta > 0)).
\]

This means that \( g_{1\delta} \) satisfies equation (4). The classical error estimate \( \beta_1(\delta_1, g_{1\delta}) \) of solution \( g_{1\delta} \) of the first equation in the system (2) considered independently of the second equation, is determined from the solution of the extremum problem [4]:

\[
\beta_1(\delta_1, g_{1\delta}) = \max \{ \| g' - g_{1\delta} \|_G : g' \in G_{1\delta}, \| A_1 g_1 - f_1 \|_G \leq \delta_1 \}, \tag{6}
\]

where \( G_{1\delta} = \{ g_1 : g_1 \in G_1, \; \| A_1 g_1 - f_1 \|_G \leq \delta_1 \} \).

The error estimate of the same solution \( g_{1\delta} \) under the condition that \( g_{1\delta} \) satisfies equation (4) is determined as follows:

\[
\beta_{12}(\delta_1, \delta_2, g_{1\delta}) = \max \{ \| g' - g_{1\delta} \|_G : g' \in G_{1\delta} \cap G_{1\delta}, \| A_1 g_1 - f_1 \|_G \leq \delta_1 \}. \tag{7}
\]
The set \( G_{1,\delta} \cap G_{12,\delta} \) is not empty because \( g_{1,\delta} \in G_{1,\delta}, G_{12,\delta} \). Problems (6) and (7) differ in the admissible sets on which a maximum of the functional is sought. Since \( G_{1,\delta} \cap G_{12,\delta} \subseteq G_{1,\delta} \), then
\[
\beta_{12}(\delta_1,\delta_2, g_{1,\delta}) \leq \beta_1(\delta_1, g_{1,\delta}). \tag{8}
\]

Error \( \|g_{1,\delta} - g_{1}\|_G \) of solution \( g_{1,\delta} \) is estimated by the inequality
\[
\|g_{1,\delta} - g_{1}\|_G \leq \beta_{12}(\delta_1, \delta_2, g_{1,\delta}) \leq \beta_1(\delta_1, g_{1,\delta}). \tag{9}
\]

In the general case, the solution \( \hat{g}_{1,\delta} \in G_{1,\delta} \) of a single-criterion problem obtained from the first equation independently of the second equation does not coincide with the solution \( g_{1,\delta} \in G_{12,\delta} \) of the two-criterion problem, and for the estimates \( \beta_1(\delta_1, \hat{g}_{1,\delta}) \), \( \beta_{12}(\delta_1, \delta_2, g_{1,\delta}) \), type (8) inequality may not be fulfilled because estimates of this type depend on a fixed approximate solution. Let us consider other types of estimates for the single-criterion and two-criterion problems:
\[
\beta'_1(\delta_1) = \max \left\{ \|g' - \hat{g}_{1,\delta}\|_G : g' \cdot g_{1,\delta} \in G_{1,\delta} \right\},
\beta'_{12}(\delta_1, \delta_2) = \max \left\{ \|g' - g_{1}\|_G : g' \cdot g_{1,\delta} \in G_{1,\delta} \cap G_{12,\delta} \right\}. \tag{10}
\]

These estimates define the diameters of sets \( G_{1,\delta}, G_{1,\delta} \cap G_{12,\delta} \) of the solutions comparable inaccuracy with the input data. For these estimates, the inequality similar to (8) holds:
\[
\beta'_{12}(\delta_1, \delta_2) \leq \beta'_1(\delta_1). \tag{12}
\]

For the solution \( \hat{g}_{1,\delta} \in G_{1,\delta} \) of the single-criterion problem, the following estimate is valid:
\[
\|\hat{g}_{1,\delta} - g_{1}\|_G \leq \beta'_1(\delta_1),
\]
whereas the corresponding estimate for the solution \( g_{1,\delta} \in G_{12,\delta} \) of a two-criterion problem is
\[
\|g_{1,\delta} - g_{1}\|_G \leq \beta'_{12}(\delta_1, \delta_2) \leq \beta'_1(\delta_1). \tag{11}
\]

In the special case when a two-criterion inverse problem is to determine one characteristic \( g \) of the medium from two criteria \( f_1, f_2 \) of the same or different physical nature, constraints (3) have the form \( g_1 = g_2 = g \) and the problem is reduced to a simpler system of equations:
\[
\begin{align*}
A_1 g & = f_1, \quad g \in G_1 \subset G, \quad f_1, f_2 \in F.
A_2 g & = f_2,
\end{align*}
\tag{13}
\]

A problem of type (13) was analyzed in [7, 9-12]. Error estimates of the solutions of problem (13) were considered in [7, 11].

The results presented above can be generalized to the case of local error estimates over the subdomains of the studied domain \( \Omega \) [11, 14]. The methods for calculating different types of error estimates based on the Monte Carlo algorithms are discussed in [13, 14].

### 3. The Approximation Neural-Network Method for Solving a Multi-Criteria Problem

We consider a two-criterion problem of type (13) in the class of finite-parametric models of the media:
\[
\begin{align*}
A_1 s_1 & = e_1, \quad s_1 \in S_N \subset R^N, \quad e_1, e_2 \in F, \quad M \geq N,
A_2 s_2 & = e_2.
\end{align*}
\tag{14}
\]

where \( A_1, A_2 \) are the continuous one-to-one operators, which act from \( R^N \) to \( F \); \( s_1 = (s_1^1, \ldots, s_1^N) \), \( s_2 = (s_2^1, \ldots, s_2^N) \) are the parameter vectors of the medium corresponding to the inversion criteria; \( S_N \) is a bounded closed set of a finite-dimensional space (a closed cube): \( S_N = \{ s_{mn} \leq s_n \leq s_{mm} + D_n \}, \quad n = 1, \ldots, N \); \( e_1 = e_1 (\hat{e}_1^1, \ldots, \hat{e}_1^M), e_2 = e_2 (\hat{e}_2^1, \ldots, \hat{e}_2^M) \) are the elements of data space \( F \) which are constructed using the approximation–interpolation procedures applied to the finite sets (vectors) of the data \( \hat{e}_1 = (\hat{e}_1^1, \ldots, \hat{e}_1^M), \hat{e}_2 = (\hat{e}_2^1, \ldots, \hat{e}_2^M) \) defined on some grid.
The approximation method for solving a single-criterion problem is considered in [14, 15]. The underlying idea of this method (e.g., for the first equation of system (14)) is as follows. On the base of the preliminarily constructed finite set \(B(s,e)\) of the known solutions of the direct and inverse problems, the continuous vector function \(\Psi_1(\hat{a}_1,x_1,\ldots,x_M) = (\psi_1^1,\ldots,\psi_1^N)\) of \(M\) variables \(x_1,\ldots,x_M\) defined in the analytic form (inversion approximator) is build. This function makes it possible to obtain the approximate solutions of the first equation of system (14) in explicit form for any data vector \(e_i \in R^M\). The matrix of coefficients \(\hat{a}_i\) is determined by the solution of the extremum problem (training the approximator)

\[
\sum_{q=1}^{l_b} \left( \left\| \Psi_1(\hat{a}_i,e_i) - s_{iq} \right\|^2 \right) \rightarrow \min, \tag{15}
\]

where \((s_{iq},e_{iq}) \in B(s,e), e_{iq} = A_i s_{iq}, q = 1,\ldots,l_b\). The error of the solutions of the equation that are obtained using the approximator is evaluated based on the independent testing set of the known solutions. The solution error can be reduced by improving the interpolation properties of the constructions used for building the approximator, by increasing the number of the reference solutions and additional iterations based on the approximation-iteration method [14, 15].

If the coordinate functions \(\psi_i^q\) of the approximator are the artificial neural networks type of MLP perceptron [16], then the method described above is called the approximation neural-network method.

In the case of the two-criterion problem (14), a second training set \(B_{2b}(s_{2q},e_{2q})\), \(e_{2q} = A_2 s_{2q}\), \(q = 1,\ldots,l_{b2}\) is additionally constructed on the base of the second equation of system (14). The matrix of coefficients \(\hat{a}_{i2}\) of the inversion approximator \(\Psi_{12}(\hat{a}_{i2},e_i,e_2)\) for the system (14) is determined by the solution of the extremum problem

\[
\sum_{q=1}^{l_{b2}} \left( \left\| \Psi_{12}(\hat{a}_{i2},e_{iq}) - s_{iq} \right\|^2 \right) + a_w \sum_{q=1}^{l_{b2}} \left( \left\| \Psi_{12}(\hat{a}_{i2},e_{2q}) - s_{2q} \right\|^2 \right) \rightarrow \min, u_i(s_i,s_2) \leq 0, i = 1,\ldots,I_c, \tag{16}
\]

where \(a_w \geq 1\) is a given weighting coefficient; \(u_i(s_i,s_2) \leq 0, i = 1,\ldots,I_c\) are the criterion constraints. The numerical methods for solving a problem (16) do not fundamentally differ from those for solving a problem (15).

4. Numerical Example

We present an example of solving a two-criterion 2D problem of geoelectrics (MT sounding) in which it is required to determine the resistivity of the medium from two different characteristics (\(Z_{TE}\) and \(Z_{TM}\) impedances) of the magnetotelluric field corresponding to the TE and TM fields [9]. The problem is solved in a class \(G_0[\Omega]\) of piecewise-constant functions specified in the studied domain \(y,z \in \Omega\) (figure 1a) and reduced to the system (14) at \(s_1 = s_2\). The sought parameters \(s_n\) of the problem are logarithms of the medium resistivities \(\rho\) in the cells of a given parameterization grid \(\theta_n\) of size \(N\) covering the domain \(\Omega: s_n = \lg \rho_n, n = 1,\ldots,N\). The model of the medium is shown in figure 1a. The number of unknown parameters of the inverse problem \(N = 155\), the range of variations of the parameters \(D = 4\). Any additional a priori information (for instance, that we search for local bodies in a homogeneous medium and resistivity of this medium) was not used. The input data \(Z_{TE}(y,\omega), Z_{TM}(y,\omega)\) are specified on a spatial-frequency grid \(\{(y,\omega)\}\) that is located on the Earth’s
Figure 1. The example of solving a two-criterion 2D inverse problem of geoelectrics. The model is a group of local objects with resistivity of \(10\, (\Omega \cdot m)\) in a homogeneous half-space with resistivity of \(200\, (\Omega \cdot m)\); (a) true model; (b) the results of the two-criterion (TE + TM) inversion; (c), (d) the results of the single-criterion TE, TM inversions.

Table 1. The error estimates of the inversion.

| No. of grid layer | TE+TM | TE | TM |
|-------------------|-------|----|----|
| \(\varepsilon_i\), % | \(\beta_i\), % | \(\overline{\varepsilon_i}\), % | \(\overline{\beta_i}\), % | \(\varepsilon_i\), % | \(\beta_i\), % |
| 1                 | 0.06  | 0.03| 8.77| 11.19| 0.05 | 0.34 |
| 2                 | 2.64  | 1.86| 7.97| 10.78| 10.67| 10.22|
| 3                 | 1.78  | 1.71| 2.58| 5.95 | 6.28 | 7.77 |
| 4                 | 7.88  | 7.78| 11.54| 15.11| 8.31 | 8.19 |
| 5                 | 8.68  | 7.93| 9.17| 10.85| 4.41 | 4.00 |
| Average           | 4.21  | 3.86| 8.01| 10.78| 5.94 | 6.10 |
| Residual \((\overline{\delta})\) | 1.0   | 4.8 | 3.9 |
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