Nonlinear Hodge maps

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Abstract

We consider maps between Riemannian manifolds in which the map is a stationary point of the nonlinear Hodge energy. The variational equations of this functional form a quasilinear, nondiagonal, nonuniformly elliptic system which models certain kinds of compressible flow. Conditions are found under which singular sets of prescribed dimension cannot occur. Various degrees of smoothness are proven for the sonic limit, high-dimensional flow, and flow having nonzero vorticity. The gradient flow of solutions is estimated. Implications for other quasilinear field theories are suggested.

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1 Introduction: nonlinear Hodge theory

The original motivation for nonlinear Hodge theory was a conjecture of Bers on the existence of subsonic compressible flow having prescribed periods on a Riemannian manifold. In establishing Bers’ conjecture for a stationary, irrotational, polytropic flow, L. M. and R. J. Sibner were led to more general questions about the properties of differential forms on Riemannian manifolds. In Ref. 2 they introduced a variational principle for the generalized energy functional

\[ E = \frac{1}{2} \int_M \int_0^Q \rho(s) ds \ dM. \]  

(See also Ref. 5, p. 221.) Here \( M \) is a smooth \( n \)-dimensional Riemannian manifold. Denote by \( \omega \) a smooth section of the \( p^{th} \) exterior power of the cotangent bundle on \( M \); then \( Q(\omega) \) is the square of the pointwise norm of \( \omega \) on \( TM \). The \( C^1 \) function \( \rho : \mathbb{R} \to \mathbb{R}^+ \) is assumed to satisfy the conditions

\[ 0 < \frac{d}{dQ} \left( Q \rho^2(Q) \right) < \infty \]  

for \( Q < Q_{\text{crit}} \), and

\[ \lim_{Q \to Q_{\text{crit}}} \frac{d}{dQ} \left( Q \rho^2(Q) \right) = 0. \]  

If \( p = 1 \) and \( \omega \) is the 1-form canonically associated to the velocity field of an adiabatic, isentropic compressible flow on \( M \), then \( \rho \) is given explicitly by the formula

\[ \rho(Q) = \left( 1 - \frac{\gamma_a - 1}{2} Q \right)^{1/(\gamma_a - 1)}, \]

where \( \gamma_a > 1 \) is the adiabatic constant of the medium, and \( Q_{\text{crit}} = 2/\gamma_a + 1 \) is the square of the sonic flow velocity.

The variational equations of \( E \) are the nonlinear Hodge equations

\[ \delta (\rho(Q) \omega) = 0. \]  

If the flow is irrotational then we have an additional equation

\[ d \omega = 0. \]  

Here \( d : \Lambda^p \to \Lambda^{p+1} \) is the (flat) exterior derivative on \( p \)-forms, with adjoint operator \( \delta : \Lambda^p \to \Lambda^{p-1} \).
Applying the converse of the Poincaré Lemma to eq. (5), we find that, locally, there is a $p-1$-form $u$ such that $du = \omega$.

It has been observed that if $\{x_1, ..., x_{n+1}\}$ are coordinates in $\mathbb{R}^{n+1}$, $u$ is a mapping of $M$ into $\mathbb{R}^{n+1}$ such that $x_{n+1} = u(x_1, ..., x_n)$, $p = 1$, and

$$\rho(Q) = (1 + Q)^{-1/2},$$

then eq. (4) can be interpreted as the equation for a family of codimension-1 minimal hypersurfaces having gradients $\nabla u$. The critical value of $Q$ is $Q = \infty$. Of course $E$ does not yield the area functional but rather an indefinite functional

$$\int_M \left( \sqrt{1 + Q} - 1 \right) dM$$

which differs from the area functional by an integration constant.

If $p = 2$, $n = 4$, and $\omega$ denotes an electromagnetic field having electromagnetic potential $u$, then for $\rho = 1$ eqs. (4), (5) reduce to Maxwell’s equations on $M$. If, however, we replace the standard model of electromagnetism with the Born-Infeld model, then we have (taking the energy to be positive-definite)

$$E_{\text{Born-Infeld}} = b^2 \int_M \left( \sqrt{1 + \frac{1}{2b^2} Q} - 1 \right) dM,$$

where $b^2 = mc^2$. The nonzero integration constant observed in (7) arises in (8) from independent physical arguments \[c.f.\] eqs. (1.1) and (1.4) of Ref. 6. Normalizing so that $b^2 = 1/2$, the energy functional (8) becomes identical to the functional (7), and we can choose $\rho$ as in eq. (6) in order to write the variational equations of this common energy functional in the form of eqs. (4). Thus the Born-Infeld model fits naturally into nonlinear Hodge theory as an application for 2-forms. (The equations for a nonparametric codimension-1 minimal surface also have a place in the original gas dynamics context of nonlinear Hodge theory, as the Chaplygin approximation of a compressible flow.\[7\])

Because the bundle $T^*M$ is flat, any connection defined on it will have trivial Lie bracket. For this reason, in comparison to the examples that follow, we call the foregoing cases abelian. In particular, in the example of electromagnetism the vector potential $u$ is identified with a connection 1-form on a bundle over $M$ having abelian structure group $U(1)$. 
Suppose we replace the energy functionals (1) and (7) by the functional
\[ E_h = \frac{1}{2} \int_M \int_{-h}^Q \rho(s) ds dM, \]
where
\[ \rho(s) = (h + s)^{-\alpha}. \]
Here \( h \) and \( \alpha \) are nonnegative parameters. Let \( X \) be a vector bundle over \( M \) having compact structure group \( G \). Define \( Q = \langle F, F \rangle \), where \( \langle , \rangle \) is an inner product on the fibers of the bundle \( \text{ad}X \otimes \Lambda^p(T^*M), p \geq 1 \).

Case 1: Let \( p = 2 \) and let \( A \) be a connection 1-form on the fibers of \( X \); choose \( F \) to be the curvature 2-form \( F_A \) corresponding to \( A \). If \( \alpha = 0 \), \( G = SO(n) \), and \( n = 4 \), then \( E_0 \) is the Yang-Mills functional.

Case 2: If \( p = 2, \alpha = 1/2, X \) is the bundle of orthonormal frames on \( M, G \) is the Lorentz group \( O(1,3) \), and \( n = 4 \), then \( E_0 \) is formally analogous to a (torsion-free) gravitational action functional for the curvature 2-form \( F \).

Case 3: If \( p = 1, \alpha = 1/2, z = A(x,y) \), where \( A \) is the graph of a surface \( \Sigma \) in \( \mathbb{R}^3 \), \( F = \text{grad} A \), \( X = T^*M \), and \( n = 2 \), then \( E_1 \) is the energy functional for a nonparametric minimal surface, as discussed above.

Case 4: If in the last example \( \mathbb{R}^3 \) is replaced by the Minkowski space \( \mathbb{R}^{n,1} \) and if \( p = n - 1 \), then \( E_1 \) is closely related to an energy functional for maximal space-like hypersurfaces.

With the exception of the last two, these examples are characterized by a nonvanishing Lie bracket in \( A \) due to a nonabelian structure group for \( X \). Thus in general these cases are nonabelian. If \( D_A \) represents the exterior covariant derivative associated to a connection 1-form \( A \) on \( \text{ad}(X) \) and if \( D_A^* \) is the formal adjoint of \( D_A \), then the variational equations for \( E_h \) can be written
\[ D_A^* (\rho(Q)F_A) = 0, \tag{9} \]
\[ D_A F_A = 0. \tag{10} \]

The first equation represents a nonabelian version of eq. (4) for curvature 2-forms, and the second equation replaces eq. (5) by the second Bianchi identity.

An intermediate place between the abelian and nonabelian nonlinear Hodge theories is occupied by nonlinear Hodge maps. These are maps \( u \) between Riemannian manifolds such that \( u \) is a critical point of the nonlinear Hodge energy (1). In this case the geometry of the target space is
enriched in comparison to the abelian case but does not have the nontrivial Lie group structure of the nonabelian case; the target space is independent of the base space but is not a curved bundle. In the context of fluid dynamics or the rotation of a nonrigid body, these maps represent flows on a Riemannian manifold $M$ (typically, a domain of $\mathbb{R}^n$) for which the flow potential is constrained to lie on a possibly different Riemannian manifold $N$.

Nonlinear Hodge theory can be viewed as an attempt to extend to the quasilinear field equations of classical physics the unified geometric treatment given linear field equations by the theory of Hodge and Kodaira. The case $\rho = 1$ for the abelian equations (4), (5) reduces to the continuity equation for an incompressible flow ($p = 1$) or the field equations for electromagnetism ($p = 2$). For the nonabelian equations (9), (10) the case $\rho = 1$ reduces to the Yang-Mills equations. In the intermediate case considered in the sequel the case $\rho = 1$ reduces to the equations for harmonic maps (nonlinear sigma-models). In distinction to the approach in Refs. 1-4, our concern is with the geometry of the target space rather than the geometry of the domain.

In the following we denote by $C$ generic positive constants, which may depend on dimension and which may change in value from line to line. We employ the summation convention for repeated indices.

## 2 The variational equations

Consider a map $u : M \to N$ taking a Riemannian manifold $(M, \gamma)$ into a Riemannian manifold $(N, g)$. We are interested in maps which are critical points of the nonlinear Hodge energy (1). Here and throughout we denote by $x = (x^1, \ldots, x^n)$ a coordinate chart on the manifold $M$ having metric tensor $\gamma_{\alpha\beta}(x)$, and we denote by $u = (u^1, \ldots, u^m)$ a coordinate chart on the manifold $N$ having metric tensor $g_{ij}(u)$. (We assume for the moment that the image of $u$ lies in a coordinate chart.) The nonlinear Hodge energy assumes the form

$$E(u) = \frac{1}{2} \int_M \int_0^Q \rho(s)ds \sqrt{\gamma} dx,$$

where

$$Q = \gamma^{\alpha\beta}(x)g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

We have, by the Leibniz rule,

$$\frac{d}{dt} E(u + t\psi)|_{t=0} = \frac{1}{2} \int_M \rho(Q) \frac{d}{dt} Q(u + t\psi)|_{t=0} \sqrt{\gamma} dx,$$

(11)
for arbitrary $\psi \in C_0^\infty(M)$. The construction of the test functions $\psi$ is not entirely straightforward. Use must be made of the Nash Embedding Theorem to embed $N$ in a higher-dimensional euclidean space. One then employs a nearest point projection $\Pi$ of a suitable euclidean neighborhood $O(N)$ onto $N$. If $t$ is small enough and $N$ is a $C^1$ submanifold, the variations $\Pi \circ (u + t\psi)$ will be constrained to lie on $N$, where now $\psi : M \to \mathcal{O}$. A discussion is given in Section 1 of Ref. 11 for the special case of harmonic maps.

Carrying out the indicated operation on the right-hand side of (11), we obtain

$$
\frac{d}{dt}E(u + t\psi)_{t=0} = \int_M \rho(Q)\gamma^{\alpha\beta}(x) g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \psi^j}{\partial x^\beta} \sqrt{\gamma} \, dx
$$

$$
+ \frac{1}{2} \int_M \rho(Q)\gamma^{\alpha\beta}(x) \frac{\partial}{\partial x^k}(g_{ij}(u(x))) \psi^k \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \sqrt{\gamma} \, dx. \quad (12)
$$

But

$$
\int_M \rho(Q)\gamma^{\alpha\beta}(x) g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \psi^j}{\partial x^\beta} \sqrt{\gamma} \, dx =
$$

$$
\int_M \frac{\partial}{\partial x^\beta} \left\{ \rho(Q)\sqrt{\gamma} \gamma^{\alpha\beta}(x) g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \psi^j \right\} \, dx
$$

$$
- \int_M \frac{\partial}{\partial x^\beta} \left\{ \rho(Q)\sqrt{\gamma} \gamma^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \psi^j \right\} g_{ij}(u(x)) \psi^j \, dx
$$

$$
- \int_M \rho(Q)\gamma^{\alpha\beta}(x) \frac{\partial}{\partial u^k} (g_{ij}(u(x))) \psi^j \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} \sqrt{\gamma} \, dx. \quad (13)
$$

Substituting eq. (13) into eq. (12) and taking into account that $\psi$ has compact support in $M$ we obtain, for $\eta^i = g_{ij}\psi^j$, the formula

$$
\frac{d}{dt}E(u + t\psi)_{t=0} = -\int_M \frac{\partial}{\partial x^\beta} \left\{ \rho(Q)\sqrt{\gamma} \gamma^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \right\} \eta^i \, dx
$$

$$
- \frac{1}{2} \int_M \rho(Q)\gamma^{\alpha\beta}(x) \frac{\partial}{\partial u^k} (g_{ij}(u(x))) \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} \right) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} \eta^\ell \sqrt{\gamma} \, dx.
$$

Applying the definition of affine connection, we conclude that stationary maps satisfy the system

$$
\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left\{ \rho(Q)\sqrt{\gamma} \gamma^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \right\} + \rho(Q)\gamma^{\alpha\beta}(x) \Gamma^i_{jk}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = 0, \quad (14)
$$
for $i = 1, \ldots, m$.

We can also approach the variational equations for nonlinear Hodge maps from an intrinsic point of view, defining the nonlinear Hodge tension field $\tau$ by the formula

$$\tau \equiv \text{trace} \nabla_{\text{cov}} (\rho(Q) \omega),$$

where $\nabla_{\text{cov}}$ denotes the covariant derivative in the bundle $T^*M \otimes u^{-1}TN$. If $N = \mathbb{R}^k$, then the equation $\tau = 0$ reduces to the conventional nonlinear Hodge equation (4) for $\omega = du$ [which implies eq. (5)]. In particular, if $\Pi \circ (u + t\psi)$ are the variations described earlier, then we can write the equations for a weak stationary point in the form

$$\frac{1}{2} \int_M \rho(Q) \left[ \langle \nabla u, \nabla \psi \rangle + \nabla_{\psi u} (D\Pi(u) [\nabla u], D\Pi(u) [\nabla u])_N \right] dM = 0, \quad (15)$$

where

$$|\nabla_{\psi u} (D\Pi(u) [\nabla u], D\Pi(u) [\nabla u])_N| \leq C |\psi| |u| |\nabla u|^2; \quad (16)$$

$\nabla = \text{grad } u; \nabla_{\psi u} \langle , \rangle_N$ is the function on $M$ whose value at $x$ is the covariant derivative for $N$ of the metric $\langle , \rangle_N$ in the direction $\psi u(x)$. (See the discussion leading to inequality (1.2) in Ref. 12; that paper considers the regularity of energy minimizing maps for $\rho(Q) = Q^s$.) Here we use the fact that for $u \in \Lambda^0$, $|\omega|$ is the norm of the gradient of $u$ as well as the norm of its differential.

The harmonic map density satisfies

$$e(u)_{\text{harmonic}} = \frac{1}{2} \gamma^{\alpha\beta} \left\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right\rangle |u^{-1}TN| = \frac{1}{2} \langle du, du \rangle |T^*M \otimes u^{-1}TN|.$$ 

That is, in the harmonic map case the energy density is the trace of the pullback, via the map $u$, of the metric tensor $g(u)$ on $N$. In the case of nonlinear Hodge maps the situation is a little more complicated, as the energy density is the integral

$$F(\omega) = \int_0^Q \rho(s) \, ds,$$

which may not be a quadratic form if $\rho \neq 1$. Moreover, the nonlinear Hodge density need not scale like a metric tensor. Thus the geometry of harmonic maps is more transparent than the geometry of nonlinear Hodge maps.
Proposition 1. In order for weak solutions $\omega = du$ of the equations $\tau = 0$ to exist locally on $M$ it is sufficient that $N$ be $\mathbb{R}^k$ there exist a positive constant $K < \infty$ for which

$$K^{-1} \leq \rho(Q) + 2Q\rho'(Q) \leq K. \quad (17)$$

Proof: The argument follows Sec. 1 of Ref. 13. Define $F(\omega)$ as in the preceding paragraph. Then (17) implies that

$$\frac{\partial^2 F}{\partial \omega_\beta \partial \omega_\alpha} > 0$$

and there exist finite positive constants $k_0$ and $k_1$ such that

$$k_0Q \leq \frac{\partial^2 F}{\partial \omega_\beta \partial \omega_\alpha} \omega_\alpha \omega_\beta \leq k_1Q.$$

Moreover, there exist finite positive constants $k_2$ and $k_3$ such that

$$k_2Q \leq F(Q(\omega)) \leq k_3Q.$$

Thus the energy functional $E$ is convex, bounded above and below, and lower semicontinuous with respect to weak $L^2$ convergence. This completes the proof of the proposition.

We will use in several contexts the following pointwise inequality for smooth solutions.

Theorem 2. Let $u : M \to N$ be a $C^2$ stationary point of the nonlinear Hodge energy on $M$, where $M$ is a compact, $n$-dimensional $C^\infty$ Riemannian manifold, $n > 2$, and $N$ is a compact $m$-dimensional $C^\infty$ Riemannian manifold. Then the scalar $Q = |\nabla u|^2$ satisfies an inequality of the form

$$L(Q) + C(Q + 1)Q \geq 0,$$

where the second-order operator $L$ is elliptic whenever condition (17) is satisfied, and the constant $C$ depends on the Ricci curvature of $M$ and the Riemann curvature of $N$.

Proof. Denote by a subscripted $x^\sigma$ differentiation in the direction of the $\sigma^{th}$ coordinate. Differentiation of the metric tensor and Christoffel symbols in the direction of an index is indicated by a comma preceding the subscripted
index. Choose geodesic normal coordinates at the points \( x \in M \) and \( u(x) \in N \). At these points

\[
\gamma^{\alpha\beta}(x) = \delta^{\alpha\beta}(x); \quad g_{ij}(u) = \delta_{ij}(u); \quad \Gamma_{ij}^\eta(x) = \Gamma_{ij}^k(u) = 0.
\]

As in the preceding, greek indices are used for coordinates on \( M \), and latin indices, for coordinates on \( N \).

Write eq. (14) in the form

\[
\gamma^{\alpha\beta} \left\{ \rho(Q) u_{x^\alpha x^\beta}^j + \rho'(Q) Q_{x^\alpha} u_{x^\beta}^j - \rho(Q) \left[ u_{x^\alpha} \Gamma_{x^\alpha x^\beta}^\eta(x) - \Gamma_{x^\alpha x^\beta}^k(u) u_{x^\alpha} u_{x^\eta} \right] \right\} = 0.
\]

Differentiating (18) with respect to \( x^\varepsilon \) and letting \( \alpha = \beta = \sigma \) yields

\[
\rho(Q) u_{x^\sigma x^\sigma x^\varepsilon} = -\rho'(Q) Q_{x^\sigma} u_{x^\varepsilon} - \rho(Q) \left[ u_{x^\alpha} \Gamma_{x^\alpha x^\sigma}^\varepsilon(x) - \Gamma_{x^\alpha x^\sigma}^p(u) u_{x^\varepsilon} u_{x^p} u_{x^\sigma} \right].
\]

Now compute (c.f. Sec. 3.2 of Ref. 14)

\[
\Delta e(u) \equiv \left[ \gamma^{\alpha\beta} g_{ij}(u) \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j \right]_{x^\sigma x^\varepsilon} = \\
-\gamma_{\alpha\beta,\sigma\varepsilon} g_{ij}(u) \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j + \\
\gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j + \left[ \gamma^{\alpha\beta} g_{ij} \rho'(Q) Q_{x^\alpha} u_{x^\beta}^j \right]_{x^\sigma} + \\
\left[ \gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j \right]_{x^\sigma} + \gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j + \\
\gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j + \gamma^{\alpha\beta} g_{ij} \rho'(Q) Q_{x^\alpha} u_{x^\beta}^j + \gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha}^i u_{x^\beta}^j + \gamma^{\alpha\beta} g_{ij} \rho'(Q) Q_{x^\alpha} u_{x^\beta}^j u_{x^\varepsilon} \equiv \sum_{x=1}^{7} T_x.
\]

Here

\[
T_4 = \left[ \gamma^{\alpha\beta} g_{ij} \rho(Q) u_{x^\alpha x^\beta} u_{x^\varepsilon}^j \right]_{x^\sigma} = \left[ \frac{1}{2} \rho(Q) Q_{x^\sigma} \right]_{x^\sigma}.
\]

Subtract this term from the left-hand side of eq. (20). Applying eq. (19) with \( \varepsilon = \alpha, \eta = \beta \), we have

\[
T_1 + T_2 + T_3 + T_4 = \\
\rho(Q) \left( R_{\alpha\beta}^M u_{x^\alpha}^i u_{x^\beta}^j - R_{ijk\ell}^N u_{x^\alpha}^i u_{x^\beta}^k u_{x^\ell} u_{x^\sigma} \right) + \Lambda(Q),
\]

where

\[
\Lambda(Q) \equiv \left\{ \rho'(Q) u_{x^\alpha}^i \left[ Q_{x^\sigma} u_{x^\beta}^j - Q_{x^\alpha} u_{x^\sigma}^j \right] \right\}_{x^\sigma},
\]

9
\( R^M_{\alpha\beta} \) is the Ricci curvature of \( M \), and \( R^N_{\alpha\beta\gamma\delta} \) is the Riemann curvature of \( N \). The last term in (22) results from applying the product rule to the quantity 
\[-u^i_{\alpha} \rho'(Q) Q x^\sigma u^j_{\sigma} \]. We can write 
\[
T_5 = \gamma^\alpha\beta g_{ij} \rho(Q) u^i_{\alpha} u^j_{\beta} = \rho(Q) \langle \partial_\sigma \omega, \partial_\sigma \omega \rangle_{T^* M} \]
and
\[
T_7 = \gamma^\alpha\beta g_{ij} \rho'(Q) Q x^\sigma u^i_{\alpha} u^j_{\sigma} = \sum_\sigma 2\rho'(Q) \langle \partial_\sigma \omega, \omega \rangle_{T^* M}^2.
\]
If \( \rho'(Q) \leq 0 \), then (17) and the Schwarz inequality imply that 
\[
T_5 + T_7 \geq K^{-1} |\nabla \omega|^2.
\]
If \( \rho'(Q) \geq 0 \), then 
\[
T_5 + T_7 = \rho(Q) |\nabla \omega|^2 \geq C |\nabla \omega|^2.
\]
[See the argument leading to (26), below.] In either case we obtain from expressions (20)-(24), defining 
\[
L(Q) = \Delta e(u) - T_4 - \Lambda(Q)
\]
the inequality 
\[
L(Q) \geq \rho(Q) \left( R^M_{\alpha\beta} u^i_{\alpha} u^j_{\beta} - R^N_{\alpha\beta\gamma\delta} u^i_{\alpha} u^j_{\beta} u^k_{\gamma} u^\ell_{\delta} \right) + C |\nabla \omega|^2.
\]
We can write 
\[
L(Q) + C\Phi Q \geq 0,
\]
where \( \Phi = Q + 1 \).

We now show that the operator \( L \) is elliptic whenever condition (17) is satisfied. If \( \rho'(Q) \leq 0 \), then
\[
-\rho'(Q) u^i_{\alpha} \left[ Q x^\sigma u^i_{\sigma} - Q x^\alpha u^i_{\sigma} \right] Q x^\sigma =
\]
\[
|\rho'(Q)| u^i_{\alpha} \left[ Q x^\sigma u^i_{\sigma} - Q x^\alpha u^i_{\sigma} \right] Q x^\sigma =
\]
\[
|\rho'(Q)| \left( Q |\nabla Q|^2 - u^i_{\alpha} u^i_{\sigma} Q x^\alpha Q x^\sigma \right) \geq
\]
\[
|\rho'(Q)| \left( Q \left| \nabla Q \right|^2 - \left| u_{x^\alpha}^i u_{x^\beta}^i Q_{x^\alpha} Q_{x^\beta} \right| \right) \geq \rho'(Q) \left( Q \left| \nabla Q \right|^2 - \frac{1}{2} \left( u_{x^\alpha}^i Q_{x^\alpha}^2 + (u_{x^\beta}^i Q_{x^\beta})^2 \right) \right) = 0,
\]
where in the final step we have applied Young’s inequality. Thus in this case
\[
\left[ \frac{1}{2} \rho(Q) + Q \rho'(Q) \right] \left| \nabla Q \right|^2 \geq \frac{1}{2} K^{-1} \left| \nabla Q \right|^2.
\]
If \( \rho'(Q) \geq 0 \), then \( \rho(Q) \geq \rho(Q_{\min}) = \rho(0) \). At \( Q = 0 \),
\[
\frac{1}{2} \rho(Q) + Q \rho'(Q) = \frac{1}{2} \rho.
\] (26)
Condition (17) then implies that \( \rho \) is bounded below away from zero. Again using Young’s inequality,
\[
\left[ \frac{1}{2} \rho(Q) + Q \rho'(Q) \right] \left| \nabla Q \right|^2 \geq \frac{1}{2} K^{-1} \left| \nabla Q \right|^2.
\]
As \( \rho \) is bounded below away from zero.

Thus for either sign of \( \rho'(Q) \), condition (17) implies that there is a constant \( m_1 > 0 \) such that
\[
\left[ \frac{1}{2} \rho(Q) + Q \rho'(Q) \right] \left| \nabla Q \right|^2 \geq m_1 \left| \nabla Q \right|^2.
\]
and we can write
\[
L(Q) = \frac{\partial}{\partial x^j} \left( a^{ij}(\omega) \frac{\partial Q}{\partial x^i} \right)
\]
for a matrix \( a^{ij} \) satisfying
\[
m_1 \left| \xi \right|^2 \leq a^{ij} \xi_i \xi_j \leq m_2 \left| \xi \right|^2.
\] (27)
(See p. 106 of Ref. 3 or Proposition 1.1 of Ref. 4 for a different proof of ellipticity.) This completes the proof of Theorem 2.

The contribution of the target geometry to the nonlinearity of inequality (25) significantly exceeds that of the geometry of the base manifold, which in the sequel we generally take to be euclidean.
3 Apparent singular sets of prescribed dimension

In general we expect that finite-energy weak solutions to eqs. (15) may be singular, as singularities occur even in the case Ω = 1. It is natural to seek extra conditions under which solutions are actually smooth.

**Theorem 3** Let \( u : \Omega \to N \) be a \( C^2 \) stationary point of the nonlinear Hodge energy on \( \Omega/\Sigma \), where \( \Omega \) is a domain of \( \mathbb{R}^n \), \( n > 2 \); \( N \) is a compact \( m \)-dimensional \( C^\infty \) Riemannian manifold, \( m \leq n \); \( \Sigma \subset B \subset \Omega \) is a compact singular set, completely contained in a sufficiently small \( n \)-disc \( B \), which is itself completely contained in \( \Omega \). Let conditions (16) and (17) hold. If \( n > 4 \), let \( \frac{2n}{(n-2)} < \mu \leq n \), where \( \mu \) is the codimension of \( \Sigma \), and let \( du \in L^\mu(B) \). If \( n = 3,4 \), let \( du \in L^\beta(B) \cap L^q(B) \), where \( \beta = (\mu - \varepsilon) / (\mu - 2 - \varepsilon) \) for \( 2 < \mu \leq n \), \( \varepsilon > 0 \), and \( \frac{1}{2} < q_0 < q \). Then \( du \) is Hölder continuous in \( \Omega \).

**Remarks.** That an \( L^n \) condition is necessary even for \( \rho = 1 \) can be seen by considering the equator map. In Theorem 3 we assume neither that the map \( u \) minimizes energy nor that the energy is small. In distinction to the harmonic map case, we do not obtain higher regularity from the Hölder continuity of \( du \), as the system (15) is not diagonal in its principal part. In the following proof we assume that \( \mu = n \); the extension of the proof to lower values of \( \mu \) can be effected by arguments given in Sec. 3 of Ref. 27.

**Proof.** Initially let \( n \) exceed 4. Integrate inequality (25) against a nonnegative test function \( \zeta \in C_0^\infty(B) \) given by

\[
\zeta = (\eta \psi)^2 \Xi(Q),
\]

where \( B = B_R(x_0) \) is an \( n \)-disc, of radius \( < R \), centered at a point \( x_0 \in \Omega \); assume that \( B \) completely contains \( \Sigma \) and is completely contained in \( \Omega \); \( \eta, \psi \geq 0; \psi(x) = 0 \) in a neighborhood of \( \Sigma \); \( \eta \in C_0^\infty(B') \) where \( B' \subset B \); \( \Xi(Q) = H(Q) H'(Q) \), where \( H(Q) = H_\kappa(Q) \) is the following variant of Serrin's test function\(^{16} \):

\[
H_\kappa(Q) = \begin{cases} 
Q^{[n/(n-2)]^{\kappa n/4}} & \text{for } 0 \leq Q \leq \ell, \\
\frac{\mu - \varepsilon}{\mu - 2 - \varepsilon} \left[ (\ell \cdot Q)^{(\mu - 2 - \varepsilon)/2} \right]^{[n/(n-2)]^{\kappa n/2(\mu - \varepsilon)}} - \frac{2}{\mu - \varepsilon} \ell^{[n/(n-2)]^{\kappa n/4}} & \text{for } Q \geq \ell.
\end{cases}
\]
Iterate the following sequence of elliptic estimates, taking successively \( u \in L^{\alpha(\kappa)}(B) \) for \( \alpha(\kappa) = n[n/(n-2)]^\kappa, \kappa = 0, 1, \ldots \). For all \( \kappa < \infty \),

\[
\int_{B'} a^{ij}(u) \partial_i Q \cdot 2(\eta \psi) \partial_j (\eta \psi) \Xi(Q) \ast 1
\]

\[
+ \int_{B'} a^{ij}(u) (\eta \psi)^2 \Xi'(Q) \partial_i Q \partial_j Q \ast 1 \leq \int_{B'} \Phi Q (\eta \psi)^2 \Xi(Q) \ast 1.
\]

This inequality can be rewritten in the short-hand form

\[ I_1 + I_2 \leq I_3, \quad (28) \]

the integrals of which we estimate individually. Because \( \mu \) exceeds \( 2n/(n-2) \) we have

\[ \Xi'(Q) \geq C(H'(Q))^2. \quad (29) \]

Also

\[ Q \Xi \leq \left( \frac{n}{n-2} \right)^\kappa \frac{n}{4} H^2. \quad (30) \]

Inequality (29) implies, by ellipticity,

\[
I_2 = \int_{B'} a^{ij}(u) (\eta \psi)^2 \Xi'(Q) \partial_i Q \partial_j Q \ast 1 \geq 
\]

\[
C(m_1) \int_{B'} (\eta \psi)^2 (H'(Q))^2 |\nabla Q|^2 \ast 1 = 
\]

\[
C \int_{B'} (\eta \psi)^2 |\nabla H|^2 \ast 1 \equiv i_{21}. \quad (31) \]

Young’s inequality implies

\[
I_1 = \int_{B'} a^{ij}(u) \partial_i Q \cdot 2(\eta \psi) \partial_j (\eta \psi) H(Q) H'(Q) \ast 1 = 
\]

\[
2 \int_{B'} \left( a^{ij}(u) (\eta \psi) (\partial_i H) \right) \partial_j (\eta \psi) H \ast 1 \geq 
\]

\[
-m_2 \left( \varepsilon \int_{B'} (\eta \psi)^2 |\nabla H|^2 \ast 1 + C(\varepsilon) \int_{B'} |\nabla (\eta \psi)|^2 H^2 \ast 1 \right) \equiv -(i_{11} + i_{12}). \quad (32) \]

Using inequality (30) and the Sobolev inequality, we obtain

\[
I_3 = \int_{B'} \Phi Q (\eta \psi)^2 \Xi(Q) \ast 1 \leq \left( \frac{n}{n-2} \right)^\kappa \frac{n}{4} \int_{B'} \Phi (\eta \psi)^2 H^2 \ast 1 \leq 
\]

13
\[
C \|\Phi\|_{n/2} \left( \int_{B'} (\eta\psi H)^{2n/(n-2)} \ast 1 \right)^{(n-2)/n} \leq C' \|\Phi\|_{n/2} \|\eta\psi H\|_{1,2}^2 \\
\leq C \|\Phi\|_{n/2} \left\{ \int_{B'} [\left| \nabla (\eta\psi) \right|^2 + (\eta\psi)^2] H^2 \ast 1 + \int_{B'} (\eta\psi)^2 \left| \nabla H \right|^2 \ast 1 \right\} \\
\equiv i_{31} + i_{32}. \tag{33}
\]

For sufficiently small \( B' \) we have
\[
0 < i_{21} - (i_{32} + i_{11}) \leq C \left( i_{12} + i_{31} \right). \tag{34}
\]

There exists (c.f. Ref. 17, Lemma 2 and p. 73) a sequence of functions \( \xi_\nu \) such that:

a) \( \xi_\nu \in [0, 1] \ \forall \nu; \)

b) \( \xi_\nu \equiv 1 \) in a neighborhood of \( \Sigma \), \( \forall \nu; \)

c) \( \xi_\nu \to 0 \) a.e. as \( \nu \to \infty; \)

d) \( \nabla \xi_\nu \to 0 \) in \( L^{n-\varepsilon} \) as \( \nu \to \infty. \)

Apply the product rule to the squared \( H_{1,2} \) norm in \( i_{31} \) letting \( \psi = \psi_\nu = 1 - \xi_\nu. \) Observing that the cross terms in \( (\nabla \eta) \psi \) and \( (\nabla \psi) \eta \) can be absorbed into the other terms by applying Young’s inequality, we estimate
\[
\lim_{\nu \to \infty} \int_{B'} \eta^2 \left| \nabla \psi_\nu \right|^2 H^2 \ast 1 \leq \lim_{\nu \to \infty} C(\ell) \int_{B'} \left| \nabla \psi_\nu \right|^2 Q^{\frac{\mu-2-\varepsilon}{\mu-\varepsilon}}(\frac{\mu}{n-2})^\frac{n}{2} \ast 1 \\
\leq \lim_{\nu \to \infty} C(\ell) \left\| \nabla \psi_\nu \right\|_{\mu-\varepsilon}^2 \|u\|_{a(\alpha)(\mu-2-\varepsilon)/(\mu-\varepsilon)}^\alpha \equiv 0. \tag{35}
\]

Having shown that the integral on the left in (35) is zero for every value of \( \ell, \) we can now let \( \ell \) tend to infinity. We obtain via Fatou’s Lemma the inequality
\[
\int_{B'} \eta^2 \left| \nabla \left( Q^{\alpha(\mu)/4} \right) \right|^2 \ast 1 \leq \int_{B'} \left| \nabla \eta \right|^2 Q^{\alpha(\mu)/2} \ast 1.
\]

Thus \( Q^{\alpha(\mu)/4} \) is in \( H^{1,2} \) on some smaller disc on which \( \eta = 1. \) But then, because \( u \) is assumed to be \( C^2 \) away from the singularity and \( \Sigma \) is compact, \( Q^{\alpha(\mu)/4} \) must be in \( H^{1,2} \) on the larger disc as well. Apply the Sobolev inequality to conclude that \( u \) is now in the space \( L^{\alpha(\mu+1)}(B). \) Because the sequence \( \{n/(n-2)\}^\kappa \) obviously diverges, we conclude after a finite number of iterations of this argument that \( Q^c \) is in \( H^{1,2}(B) \) for any positive value of \( c. \) A final application of the Sobolev inequality implies that \( \omega \in L^s(B) \) for all \( s < \infty \) and for any small \( B \subset \subset \Omega, \) provided \( n \) exceeds 4.

Now let \( n = 3 \) or 4. Define \( H_n(Q) = \)
\[
14
\]
\[
\begin{cases}
Q^q 	ext{ for } 0 \leq Q \leq \ell, \\
\frac{1}{q} Q^{q'} \left[ q' Q^{q''} - (q_0 - q') \ell^q \right] \text{ for } Q \geq \ell,
\end{cases}
\]

where \( q' = \left[ n / (n - 2) \right]^{\kappa} q \). Arguing as in the higher-dimensional case we obtain, using the Sobolev inequality, \( Q \in L^{2q'n/(n-2)}(B) \). Repeating the argument for \( \kappa = 0, 1, \ldots \), we obtain that \( \omega \in L^s(B) \) for all \( s < \infty \) when \( n \) is 3 or 4.

Now let \( n \) be an arbitrary integer greater than 2. Again let \( \psi_{\nu} = 1 - \xi_{\nu} \), where \( \xi_{\nu} \) satisfies properties \( a)-d) \) above, and let \( \eta \in C_0^\infty(B') \) as before. If \( \zeta = \eta^2 \psi_{\nu}, \) then

\[
\int_{BR} \langle d\zeta, \rho(Q)\omega \rangle * 1 = \int_{BR} \langle \zeta, \rho(Q)b(\omega) \rangle * 1, \tag{36}
\]

with \( b \) given by (15). We have

\[
\int_{BR} \langle d\zeta, \rho(Q)\omega \rangle * 1 = \int_{BR} \langle \eta^2 (d\psi_{\nu}), \rho(Q)\omega \rangle * 1 + \int_{BR} \langle \psi_{\nu} d(\eta^2), \rho(Q)\omega \rangle * 1,
\]

where as \( \nu \) tends to infinity, \( \psi_{\nu} \) tends to 1 a.e. and

\[
\left| \int_{BR} \langle \eta^2 (d\psi_{\nu}), \rho(Q)\omega \rangle * 1 \right| \leq C(K) \| \nabla \psi_{\nu} \|_{\mu-\epsilon} \| \omega \|_{(\mu-\epsilon)/(\mu-\epsilon - 1)} \to 0.
\]

Choosing \( \eta^2(x) \) to equal 1 for \( x \in BR/2 \), we find from (36) that \( \omega \) is locally a weak solution in all of \( \Omega \).

Let the map \( \varphi : BR(x_0) \to \mathbb{R}^k \) satisfy for sufficiently large \( k \) the boundary-value problem

\[
\delta (\rho(Q(d\varphi))d\varphi) = 0 \quad \text{in } BR(x_0), \\
\varphi_{\theta} = u_{\theta} \quad \text{on } \partial B,
\]

where the subscripted \( \theta \) denotes the tangential component of the map in coordinates \( (r, \vartheta_1, \ldots, \vartheta_{n-1}) \). The existence of a \( C^{2,\alpha} \) solution \( \varphi \) to this problem is well known.\(^2\) Moreover, if \( (d\varphi)_{R,x_0} \) denotes the mean value of the 1-form \( d\varphi \) on \( BR(x_0) \), then \( d\varphi \) satisfies a Campanato estimate\(^18\)

\[
\int_{BR(x_0)} |d\varphi - (d\varphi)_{R,x_0}|^2 * 1 \leq CR^{n+2\gamma H}
\]
for some number \( \gamma_H \in (0, 1] \). Then \( u - \varphi \) is an admissible test function, and

\[
\int_{B_R(x_0)} \langle d(u - \varphi) , [\rho(Q(du))du - \rho(Q(d\varphi))d\varphi] \rangle * 1 \\
= \int_{B_R(x_0)} \langle (u - \varphi), \rho(Q)b(u,Du) \rangle * 1
\]

with \( b \) given by (15). Apply to identity (37) Sibner’s mean-value formula (Lemma 1.1 of Ref. 13), which asserts for the unconstrained case that

\[
G^\alpha(\xi, f) - G^\alpha(\eta, h) = A^{\alpha\beta} (f_\beta - h_\beta) + H^\alpha_\beta (\xi^\beta - \eta^\beta)
\]

where

\[
G^\alpha(x, \omega) = \sqrt{\gamma} \frac{\partial F}{\partial \omega^\alpha(x)},
\]

\( A^{\alpha\beta} \) is a positive-definite matrix, and

\[
|H^\alpha_\beta| \leq C (|f(x)| + |h(x)|).
\]

Here \( F \) is the function used in the proof of Proposition 1. Equation (38) extends immediately to our case, as we can estimate the derivative of the metric \( g \) on \( N \) by

\[
\left| \frac{\partial g}{\partial x} \right| \leq \left| \frac{\partial g}{\partial u} \right| \left| \frac{\partial u}{\partial x} \right| \leq C |\omega|
\]

c.f. inequality (1.3c) of Ref. 13; the \( g \) in the above inequalities is not the same object as the \( g \) in Ref. 13, which corresponds to our \( \gamma \).

In formula (38) choose \( \xi = x, \eta = 0, f = \omega, \) and \( h = d\varphi \). We obtain, using (16),

\[
\int_{B_R(x_0)} |d(u - \varphi)|^2 * 1 \leq C \left( \int_{B_R(x_0)} (|\omega| + |d\varphi|) |x| * 1 + \int_{B_R(x_0)} |u - \varphi| \rho(Q) |u| Q * 1 \right).
\]

We can find a number \( s \) sufficiently large so that

\[
\int_{B_R(x_0)} (|\omega| + |d\varphi|) |x| * 1 \leq C (\|\omega\|_s + \|d\varphi\|_s) \left( \int_0^R |x|^{s/(s-1)} |x|^{n-1} d|x| \right)^{(s-1)/s}
\]

16
\[ \leq C(s, n)R^{n+/s} R_{n+1}^{(s-1)/s} \equiv CR^n, \quad (40) \]

where \( \eta > n \) whenever \( s > n \). Also, Young’s inequality yields

\[
R^{-\nu} \int_{B_R(x_0)} |u - \varphi|^2 |u|^2 * 1 \leq R^{-\nu} \int_{B_R(x_0)} Q^2 \rho(Q) * 1 \leq R^{-\nu} \int_{B_R(x_0)} |u - \varphi| |u| * 1 + C(\|Q\|_s, \|\rho\|_\infty) R^{n(s-1)/s+\nu}
\]

for a constant \( \nu \) to be chosen and \( s \) so large that \( \nu s > n \). We have

\[
R^{-\nu} \int_{B_R(x_0)} |u - \varphi| |u| * 1 \leq \int_{B_R(x_0)} \left| \nabla (u - \varphi) \right|^2 * 1 \int_{B_R(x_0)} |u|^n * 1
\]

where \( C_S \) is Sobolev’s constant. The \( L^p \) hypothesis on \( du \) now implies by the Sobolev Theorem (and a trivial application of the Gaffney-Gårding inequality) that for any \( \varepsilon > 0 \) we have

\[
\int_{B_R(x_0)} |u|^n r^{n-1} dr dS \leq |S^n| \left( \int_{B_R(x_0)} |u|^{n+\varepsilon} r^{n-1} dr \right)^{n/(n+\varepsilon)} \left( \int_{B_R(x_0)} r^{n-1} dr \right)^{\varepsilon/(n+\varepsilon)} \leq CR^{\lambda}
\]

for \( \lambda = n\varepsilon/(n+\varepsilon) \). Because of the high \( L^p \) space in which \( u \) sits we have some flexibility: choosing either \( \varepsilon \), \( s \), or \( \nu \) so that \( \nu < \lambda \) allows us to subtract the right-hand side of inequality (41) from the left-hand side of inequality (39). Because the mean value minimizes variance over all location parameters, we find that

\[
\int_{B_R(x_0)} |\omega - (\omega)_{R,x_0}|^2 * 1 \leq \int_{B_R(x_0)} |\omega - (d\varphi)_{R,x_0}|^2 * 1 \leq \int_{B_R(x_0)} |\omega - d\varphi|^2 * 1 + \int_{B_R(x_0)} |d\varphi - (d\varphi)_{R,x_0}|^2 * 1 \leq C \max \left\{ R^{n+2\gamma_{\mu}}, R^n, R^{n(s-1)/s+\nu} \right\}.
\]

Choosing \( x_0 \) so that \( \Sigma \subset B_R(x_0) \), and observing that if \( \mu = n \), this argument works for any smaller positive value of \( R \), completes the proof.
4 The sonic limit

Denote by $\gamma_1$ a closed 1-form having prescribed periods. We add to eqs. (4), (5) the homology condition that $\omega - \gamma_1$ be an exact form. Denote by $M$ a smooth, compact Riemannian manifold and consider a family of maps $u_t : M \to N$ into a smooth, compact Riemannian manifold $N$. We further assume that for each $t : 0 \leq t < t_{\text{crit}}$, $\omega_t = du_t$ is a weak minimizer of the nonlinear Hodge energy on $M$ in the following sense: condition (17) is satisfied, eqs. (14) are weakly satisfied by the vector field canonically associated to $\omega_t$, $\omega_t - t\gamma_1$ is an exact form in $L^2(M)$, and for all other 1-forms $\alpha \in L^2(M)$ such that $\alpha - t\gamma_1$ is exact, the inequality

$$\int_M \int_0^{Q(\omega_t)} \rho(s) \, ds \, dM \leq \int_M \int_0^{Q(\alpha)} \rho(s) \, ds \, dM$$

is satisfied. Borrowing the terminology of fluid dynamics we call weak solutions $\omega_t$, $t \in [0, t_{\text{crit}})$, subsonic. The question is whether such solutions converge, as $t$ tends to $t_{\text{crit}}$, to sonic solutions having velocity $Q_{\text{crit}}$. Ellipticity degenerates in the limit as $Q$ tends to $Q_{\text{crit}}$ [c.f. eq. (3) of Section 1]. In this limit condition (17) fails and is replaced by conditions (2), (3). In the following theorem we replace $M$ by a euclidean domain; but see the remarks at the end of this section.

**Theorem 4** Assume the hypotheses of the preceding paragraph. That is, let $u_t : \Omega \to N$ denote a family of maps between a smooth, compact domain $\Omega$ of $\mathbb{R}^n$ and a coordinate chart on a smooth, compact $m$-dimensional Riemannian manifold $N$, $m \leq n$, where $0 \leq t < t_{\text{crit}}$. Assume that the 1-forms $\omega_t = du_t$ weakly minimize the nonlinear Hodge energy on $\Omega$ over a cohomology class. In particular, let the homology condition of the above paragraph be satisfied for a fixed 1-form $\gamma_1$. Assume that the $C^1$ function $\rho$ satisfies (2), (3) and that

$$Q \leq c \int_0^{Q} \rho(s) \, ds + 1 \quad \forall Q < Q_{\text{crit}}$$

for constant $c$. Then as $t$ tends to $t_{\text{crit}},$

$$\lim_{t \to t_{\text{crit}}} \max_{x \in \text{int} \, \Omega} Q(\omega_t(x)) \to Q_{\text{crit}}.$$  

The conclusion of Theorem 4 implies that $\omega_t$ depends continuously on $t$ in the topology of uniform convergence. This eventually implies Hölder
continuity for weak minimizers at the elliptic degeneracy represented by (3); see Corollary 5.

Proof. The proof is similar to that of Theorem 4.8 of Ref. 13. Denote by \( \{t_\nu\} \) a nonnegative sequence of points in \([0, t_{\text{crit}})\) converging to a limit point. We want to establish a sequence of inequalities satisfied by any subsonic minimizer \( \omega_{t_\nu} \equiv \omega_\nu \). Because \( \omega_\nu \) minimizes energy over a cohomology class we have

\[
\int_\Omega \int_0^{Q(\omega_\nu)} \rho(s) ds * 1 \leq \int_\Omega \int_0^{Q(h_\nu)} \rho(s) ds * 1 \leq C \|h_\nu\|_{L^2(\Omega)}^2,
\]

where \( h_\nu \) is a harmonic form such that \( h_\nu - t_\nu \gamma_1 \) is exact. This gives a uniform bound in \( L^\infty \) on the sequence \( \{\omega_\nu\} \). Now we proceed as in the concluding arguments in the proof of Theorem 3, comparing \( \omega_\nu \) to a \( C^1 \) solution \( \varphi \) of the euclidean nonlinear Hodge equations on \( \Omega \). Conditions (2) and (42) imply that \( u - \varphi \) is an admissible test function. The continuity estimates, starting with formula (37), are also uniform, as the highest bound imposed on \( \omega_\nu \) by these inequalities is in \( L^{n+\varepsilon} \). For example, we can replace inequality (40) by the estimate

\[
\int_{B_R(x_0)} (|\omega| + |d\varphi|) |x| * 1 \leq C \left( \|Q\|_{n/2} + \|d\varphi\|_n \right) \left( \int_0^R |x|^{n/(n-1)} |x|^{n-1} d |x| \right)^{(n-1)/n}
\]

\[
\leq C \left( R^\delta + \left( \int_{B_R(x_0)} |d\varphi|^n |x|^{n-1} d |x| \right)^{1/n} \right)^{1/(n-1)} R^{n-1}
\]

\[
\leq C \left( R^\delta + \|d\varphi\|_{n,s} \left( \int_0^R |x|^{n-1} d |x| \right)^{(s-1)/ns} \right) R^{n} \leq CR^\eta
\]

for some \( \eta > n \). Thus the hypotheses of Theorem 4 imply the concluding Hölder estimate of Theorem 3, from which we obtain equicontinuity for the sequence \( \{\omega_\nu\} \). Now the Arzelá-Ascoli Theorem guarantees uniform convergence of a subsequence to a 1-form satisfying both the equations and the homology condition. This completes the proof of Theorem 4.

Corollary 5 Let the hypotheses of Theorem 4 be satisfied. Then \( \omega \) is Hölder continuous in the interior of \( \Omega \).
Proof. Theorem 4 is the crucial ingredient in the technique of Shiffman regularization, described in the Appendix to Ref. 13. This technique is sufficient to establish the Hölder continuity of \( \omega \) and prove the corollary.

In Ref. 10 a comparison argument similar to (37)-(41) was constructed for solutions of eqs. (9), (10). There we chose an exponential gauge at the origin of coordinates in a euclidean ball \( B_R(0) \) in order to compare solutions of (9), (10) with euclidean solutions of (4), (5). It was shown that the difference of the two solutions is small in a high Campanato space. It was then necessary to show that the gauge transformation to an exponential gauge preserves the Campanato estimate; this allowed us to extend the comparison outside of \( B_R(0) \) and apply a covering argument. The argument of Ref. 10 provides a guide for extending the results of this section to maps of a Riemannian manifold \( M \). The analogy of an exponential gauge is a choice of geodesic normal coordinates in a local coordinate chart. The arguments of Ref. 13 imply that the difference between a comparison map \( \varphi \), taking a euclidean ball into \( \mathbb{R}^k \), and a comparison map \( \varphi' \), taking a Riemannian ball into \( \mathbb{R}^k \), is itself small in a high Campanato space. This is the analogy of our estimates of the gauge transformations in Ref. 10. Now we can extend the local estimate to all of \( M \) by a covering argument. Although in principle this method could be used to extend Theorem 3 to a Riemannian domain as well, in that case no covering argument is needed because \( \Sigma \) is assumed to be small.

5 An application to harmonic maps

We now consider the special case in which \( \Sigma \) is a point, \( \rho \) is constant, and \( n \) exceeds 4. The following result is a special case of a theorem which Liao\(^ {20} \) proved by quite different methods.

**Theorem 6 (Liao).** Let \( u : \Omega \to N \) be a \( C^2 \) stationary point of the nonlinear Hodge energy with \( \rho \equiv 1 \) on \( \Omega - \{p_0\} \), where \( \Omega \) is a domain of \( \mathbb{R}^n \), \( n > 4 \); \( N \) is a compact \( m \)-dimensional \( C^\infty \) Riemannian manifold, \( m \leq n \); \( p_0 \in \Omega \) is a point. If \( Q = |du|^2 \) satisfies the growth condition

\[
Q(x) \leq \frac{\gamma_0}{|x - p_0|^2}
\]

for \( x \in B_R(p_0) \), where \( B_R(p_0) \) is an \( n \)-disc of radius \( R \) centered at \( p_0 \) and \( \gamma_0 \) is a sufficiently small positive constant, then \( du \) is Hölder continuous on \( \Omega \).
Proof. The growth condition guarantees $du \in L^p(B) \forall P < n$. The idea of the proof is to show that $du \in L^n(B)$ and apply Theorem 3. Without loss of generality we take $P_0$ to lie at the origin of coordinates in $\mathbb{R}^n$.

Let $\xi(x) = \zeta(x)\psi(x)$, where $x \in B_R(0) - \{0\}$,
\[
\psi(x) = |x|^{4-n},
\]
and $\zeta$ is chosen so that $\zeta(x) = 1$ if $2\varepsilon < |x| \leq R/2$, and $\zeta(x) = 0$ if $|x| < \varepsilon$ or $|x| > R$. We can find $\zeta$ satisfying the additional conditions that
\[
|\nabla \zeta| \leq \frac{C}{\varepsilon}
\]
and
\[
|\Delta \zeta| \leq \frac{C}{\varepsilon^2}.
\]
Because $L$ is a divergence-form operator and $\nabla \xi$ has compact support in $B_R$, inequality (25) implies that
\[
-\int_{B_R(0)} (\Delta_r \xi) Q * 1 = -\int_{B_R(0)} \xi (\Delta Q) * 1
\]
\[
\leq C \int_{B_R(0)} \xi Q^2 * 1,
\]
where $\Delta_r$ is the Laplacian in radial coordinates. We have
\[
\Delta_r \xi = \Delta \zeta \cdot \psi + 2 \nabla \zeta \cdot \nabla \psi + \zeta \Delta \psi,
\]
where
\[
\Delta \psi = 2(4-n)|x|^{2-n}.
\]
We can write inequality (44) in the form
\[
\int_{B_R(0)} \psi Q \left(\frac{-\Delta \psi}{\psi} - CQ\right) * 1 \leq 2 \int_{B_R(0)} |\nabla \zeta| |\nabla \psi| Q * 1
\]
\[
+ \int_{B_R(0)} |\Delta \zeta| \psi Q * 1.
\]
We are interested in the behavior of this inequality as the constant $\varepsilon$ in the trapezoidal function tends to zero. Write (45) in the form
\[
i_1 \leq 2i_2 + i_3.
\]
Because in $B_R(0) - \{0\}$ we have $Q \leq \gamma_0 |x|^{-2}$, integration in radial coordinates yields

$$i_2 \leq \frac{C}{\varepsilon} \int_{\Gamma} d|x| + C(R),$$

where

$$\Gamma \equiv \{x| \varepsilon \leq |x| \leq 2\varepsilon\}.$$  

Integral $i_2$ is obviously finite as $\varepsilon$ tends to zero. Similarly,

$$i_3 \leq \frac{C}{\varepsilon^2} \int_{\Gamma} |x| d|x| + C(R),$$

which is also finite for every $\varepsilon$. Finally,

$$i_1 \geq \int_{B_R(0)} |x|^{4-n} \zeta Q \left(\frac{-2(4 - n) - C\gamma_0}{|x|^2}\right) * 1.$$

The quantity inside the largest parentheses on the right is positive provided $\gamma_0$ is sufficiently small. In this case

$$\lim_{\varepsilon \to 0} i_1 \geq C \int_{B_{R/2}(0)} Q |x|^{2-n} * 1.$$  

But also,

$$\int_{B_{R/2}(0)} Q^{n/2} * 1 = \int_{B_{R/2}(0)} Q \left(Q^{(n-2)/2}\right) * 1 \leq C \int_{B_{R/2}(0)} Q |x|^{2-n} * 1.$$  

Taken together, these inequalities imply that $\omega$ lies in the space $L^n$ in a neighborhood of the singularity. The hypotheses of Theorem 3 being satisfied, we conclude that $\omega$ is Hölder continuous, which completes the proof of Theorem 6.

### 6 Rotational fields

In this section we study systems of the form

$$\delta (\rho(Q)\omega) = 0, \quad (46)$$

$$d\omega = v \wedge \omega, \quad (47)$$
where $\omega \in \Lambda^p(T^*M)$ for $p \geq 1$; $v \in \Lambda^1(T^*M)$; $M$ is an $n$-dimensional Riemannian manifold; $Q = \langle \omega, \omega \rangle \equiv * (\omega \wedge * \omega)$; $*: \Lambda^p \to \Lambda^{n-p}$ is the Hodge involution; $\rho: \mathbb{R} \to \mathbb{R}^+$ is a $C^1$ function satisfying the condition\(^5\)

$$K^{-1}(Q + k)^q \leq \rho(Q) + 2Q\rho'(Q) \leq K(Q + k)^q$$ \hspace{1cm} (48)

for some positive constant $K$ and nonnegative constants $k, q$.

If $v \equiv 0$ (or if $p = 1$ and $v = \omega$), then condition (47) degenerates to condition (5). If $\omega \in \Lambda^1(T^*M)$ is the 1-form canonically associated to the velocity field of an $n$-dimensional fluid, then condition (5) guarantees that the flow is irrotational: no circulation exists about any curve homologous to zero.

If $\omega \in \Lambda^1(T^*M)$, then condition (47) only guarantees, via the Frobenius Theorem, that $\omega = \ell du$ locally; a potential exists only along the hypersurfaces $\ell = \text{constant}$, and circulation about topologically trivial points is excluded only along these hypersurfaces. (For the extension of this result to exterior products of 1-forms, see, e.g., Ref. 22, Sec. 4-3.) Equations (4), (5) can be used to prescribe a cohomology class for solutions as in Sec. 4, but eqs. (46), (47) will only prescribe a closed ideal.

We have as an immediate consequence of (47) the condition

$$d\omega \wedge \omega = 0.$$ \hspace{1cm} (49)

If $\omega$ denotes tangential velocity of a rigid rotor ($\rho = \rho(x)$ only), eq. (49) corresponds in three euclidean dimensions to the fact that the direction of $\nabla \times \omega$ is perpendicular to the plane of rotation. Condition (49) also arises in thermodynamics.\(^{22,23}\)

As in preceding sections, we replace $M$ by a euclidean domain in proving the technical results. In the general case, the curvature of $M$ enters in a predictable way.\(^{24}\)

**Theorem 7** Let $\omega, v$ smoothly satisfy eqs. (46), (47) on a bounded, open domain $\Omega \subset \mathbb{R}^n$. Assume condition (48). Then the scalar $Q = * (\omega \wedge * \omega)$ satisfies the elliptic inequality

$$L_\omega(Q) + C(Q + k)^q (|\nabla v| + |v|^2) Q \geq 0,$$ \hspace{1cm} (50)

where $L_\omega$ is a divergence-form operator which is uniformly elliptic for $k > 0$.\(\)
Proof. We have (Ref. 5, (1.5)-(1.7))

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle = \partial_i \langle \omega, \partial_i (\rho(Q)\omega) \rangle - \langle \partial_i \omega, \partial_i (\rho(Q)\omega) \rangle = \Delta H(Q) - [\rho(Q) \langle \partial_i \omega, \partial_i \omega \rangle + \rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q],
\]

(51)

where

\[
\Delta H(Q) = \partial_i \left[ \left( \frac{1}{2} \rho(Q) + Q\rho'(Q) \right) \partial_i Q \right],
\]

\(\partial_i = \partial/\partial x^i, x = x^1, ..., x^n \in \Omega\). Observe that \(H\) is defined so that

\[
H'(Q) = \frac{1}{2} \rho(Q) + Q\rho'(Q).
\]

Just as in the derivation of inequality (25), we have

\[
\rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q = \sum_i 2\rho'(Q) \langle \partial_i \omega, \omega \rangle^2.
\]

(52)

If \(\rho'(Q) \geq 0\), then (52) implies that

\[
\rho(Q) \langle \partial_i \omega, \partial_i \omega \rangle + \rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q \geq 
\rho(Q) |\nabla \omega|^2 \geq K^{-1}(Q + k)^q |\nabla \omega|^2.
\]

(53)

In (53) we have used the inequality

\[
\rho(Q) \geq K^{-1}(Q + k)^q,
\]

(54)

which follows from (48) (with a possibly larger constant \(K\)). If \(\rho'(Q) < 0\), then (52) and the Schwarz inequality imply, just as in the derivation of inequality (25), the inequality

\[
\rho(Q) \langle \partial_i \omega, \partial_i \omega \rangle + \rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q \geq 
\rho(Q) |\nabla \omega|^2 + 2\rho'(Q) |\nabla \omega|^2 = \[
\rho(Q) + 2Q\rho'(Q)] |\nabla \omega|^2 \geq K^{-1}(Q + k)^q |\nabla \omega|^2.
\]

(55)

Thus (51) implies, via either (53) or (55) as appropriate, the inequality

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle \leq \Delta H(Q) - K^{-1}(Q + k)^q |\nabla \omega|^2.
\]

(56)

Applying eq. (46) to the left-hand side of (56) yields, for \(\Delta \equiv -(d\delta + \delta d)\),

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle = -\ast [\omega \ast \delta d (\rho(Q)\omega)]
\]

24
\[ (-1)^{n(p+1)+n} \ast [\omega \wedge \ast (\ast d) \ast d(\rho \omega)] \]
\[ = (-1)^{n(n+3)-p} [\omega \wedge d \ast d(\rho \omega)] = \]
\[ (-1)^{n(n+3)} \ast \{ d[\omega \wedge \ast d(\rho \omega)] - [d \omega \wedge \ast d(\rho \omega)] \} \]
\[ = (-1)^{n(n+3)} \ast \{ *d[\omega \wedge \ast d(\rho \omega)] - *\{ \ast \omega \wedge \ast d(\rho \omega) \} \} \equiv \tau_1 - \tau_2. \quad (57) \]

We express the first term in this difference, up to sign, as a divergence in the 1-form \( dQ \), writing
\[ \tau_1 = \ast d[\omega \wedge \ast d(\rho \omega)] = \]
\[ *d[\omega \wedge \ast (\rho'(Q) dQ \wedge \omega)] + *d[\omega \wedge \ast \rho d\omega] \equiv \tau_{11} + \tau_{12}. \quad (58) \]

Equation (47) implies that
\[ \tau_{12} \geq -|\tau_{12}| = -|\ast d[\omega \wedge \ast (\rho v \wedge \omega)]| \]
\[ \geq -C (|\nabla \omega| |v| \rho|\omega| + |\nabla v| \rho Q + |v||\omega||\nabla(\rho \omega)|) \]
\[ \equiv C(-\tau_{12} - \tau_{122} - \tau_{123}). \quad (59) \]

We have, analogously to (54), the inequality \( \rho(Q) \leq K(Q + k)^q \). Using this estimate and Young’s inequality, we write
\[ -\tau_{121} = -\sqrt{\rho} |\nabla \omega| |v| \sqrt{\rho} |\omega| \geq \]
\[ -\varepsilon |\nabla \omega|^2 (Q + k)^q - C(\varepsilon, K)|v|^2 (Q + k)^q Q. \quad (60) \]

Kato’s inequality and (48) yield, using \( |\rho'(Q) \cdot Q| \leq K(Q + k)^q \),
\[ -\tau_{123} = -|v||\omega||\nabla(\rho \omega)| = -|v||\omega||\rho'(Q) \nabla Q \cdot \omega + \rho \nabla \omega| \geq \]
\[ -|v||\omega| (\|2\rho'(Q) |\omega| \nabla|\omega| \cdot \omega| + |\rho(Q) \nabla \omega|) \geq \]
\[ -2|v||\omega||\rho'(Q) \cdot Q| |\nabla|\omega| - |v||\omega||K(Q + k)^q |\nabla \omega| \]
\[ \geq -3|v||\omega| K(Q + k)^q |\nabla \omega| \]
\[ \geq -K(Q + k)^q \left( \varepsilon |\nabla \omega|^2 + C(\varepsilon) |v|^2 Q \right). \quad (61) \]

Substituting (60) and (61) into (59) yields, for a new \( \varepsilon \),
\[ \tau_{12} \geq -|\tau_{12}| \geq \]
\[ -K\varepsilon(Q + k)^q |\nabla \omega|^2 - \left( C(\varepsilon, K)|v|^2 + K|\nabla v| \right) (Q + k)^q Q. \quad (62) \]
Similarly,
$$\tau_2 = * [v \wedge \omega \wedge *d(\rho \omega)] \geq -C|v||\omega||\nabla(\rho \omega)|,$$
which can be estimated by (61). Substituting (62) into (58), (58) into (57),
and (57) into (56), and estimating \(\tau_2\) of (57) by (61) yields, again for a new \(\varepsilon\),
$$*d[\omega \wedge *(\rho'(Q)dQ \wedge \omega)] - K\varepsilon(Q + k)^q|\nabla \omega|^2$$
$$-C(\varepsilon, K)(Q + k)^q \left(|\nabla v| + |v|^2\right) Q \leq \Delta H(Q) - K^{-1}(Q + k)^q|\nabla \omega|^2.$$
We obtain, choosing \(0 < \varepsilon < K^{-2}\),
$$0 \leq (K^{-1} - \varepsilon K)(Q + k)^q|\nabla \omega|^2$$
$$\leq \Delta H(Q) \pm \text{div} (*[\omega \wedge *(\rho'(Q)dQ \wedge \omega)])$$
$$+ C(Q + k)^q \left(|\nabla v| + |v|^2\right) Q \equiv L_\omega(Q) + C(Q + k)^q \left(|\nabla v| + |v|^2\right) Q.$$
The ellipticity of the operator \(L_\omega\) under condition (48) is obvious from
the proof of Theorem 2. This completes the proof of Theorem 7.

**Corollary 8** Let \((\omega, v)\) be a \(C^2\) solution of eqs. (46), (47) on \(\Omega/\Sigma\), where \(\Omega\) is a domain of \(\mathbb{R}^n, n > 2; \Sigma \subset B \subset \Omega\) is a compact singular set, completely contained in a sufficiently small \(n\)-disc \(B\), which is itself completely contained in \(\Omega\). Let condition (48) hold. If \(n > 4, let 2n/(n - 2) < \mu < n, where \mu\) is the codimension of \(\Sigma\), and let \(\omega \in L^\beta(B)\). If \(n = 3, 4, let \omega \in L^{4\beta_0}(B) \cap L^{4\bar{\beta}}(B), where \beta = (\mu - \varepsilon) / (\mu - 2 - \varepsilon) for 2 < \mu \leq n, \varepsilon > 0,\) and \(\frac{1}{2} < \bar{\beta} < \bar{\beta}\). If \((Q + k)^q \left(|\nabla v| + |v|^2\right) \in L^{n/2}(B)\) and \(\nabla v| + |v|^2 \in L^p(B)\) for some \(p\) exceeding \(n/2\), then \(\omega\) is bounded on compact subdomains of \(\Omega\). This bound is uniform for \(k > 0\).

**Proof.** In (25) take \(\Phi = (Q + k)^q \left(|\nabla v| + |v|^2\right)\). Apply the arguments leading to (36) to show that \(Q\) is an \(H^{1,2}\) weak solution. Now choose\(13\)
$$\zeta = (|\omega_k| + \delta)^{2\tau - 2} \eta^2$$
for \(\{\omega_k\}\) an increasing sequence chosen so that \(\lim_{k \to \infty} \omega_k = \omega; \eta \in C_0^\infty(B)\); \(\eta \geq 0; \delta > 0; \tau > 1\). Estimating (36) for this choice of test function implies
in the limit that \(|\omega|^{\tau} \in H^{1,2}(B)\) for some \(\tau > 1\). Also, \((|\omega|^{\tau})^\lambda\) satisfies (25)
for \(\lambda < 2\). Now Theorem 5.3.1 of Ref. 19 implies that \(|\omega|\) is bounded.
7 The heat flow of solutions

Consider the system

\[ -\delta (\rho (Q(x,t)) \omega(x,t)) = \frac{\partial u(x,t)}{\partial t}, \]  
\[ du(x,t) = \omega(x,t), \]  

where \( x \in M, t \in (0, T] \), and exterior differentiation is in the space directions only. Solutions of eqs. (63), (64) describe the heat flow, or gradient flow, of nonlinear Hodge maps. Notice that (64) implies \( d\omega = 0 \).

If \( M \) is compact or if the normal component of \( \omega \) vanishes on \( \partial M \), then the time decay of the energy

\[ E_t(\omega) \equiv \frac{1}{2} \int_M \int_0^{Q(\omega(x,t))} \rho(s) ds dM \]

is given by

\[ \frac{d}{dt} E_t(\omega) = \frac{1}{2} \int_M \rho(Q) \frac{\partial Q}{\partial t} dM = \int_M \rho(Q) \left\langle \frac{\partial \omega}{\partial t}, \omega \right\rangle dM = \int_M \left\langle \frac{\partial \omega}{\partial t}, \rho(Q) \omega \right\rangle dM. \]

Equations (63), (64) imply that

\[ \frac{\partial \omega}{\partial t} = \frac{\partial (du)}{\partial t} = d \left( \frac{\partial u}{\partial t} \right) = -d\delta (\rho(Q)\omega). \]  

(65)

These identities together imply that

\[ \frac{d}{dt} E_t(\omega) = -\int_M \left\langle d\delta (\rho(Q)\omega), \rho(Q)\omega \right\rangle dM \]

\[ = -\int_M \left\langle \delta (\rho(Q)\omega), \delta (\rho(Q)\omega) \right\rangle dM \leq 0. \]  

(66)

We conclude from (66) that a finite energy functional will remain so indefinitely.

The local estimate for \( Q \), taking \( M \) to be a bounded, open domain of \( \mathbb{R}^n \), is similar to its elliptic counterparts in the proof of Theorem 7: If \( \rho(Q(x,t)) \) satisfies inequality (48), then

\[ \partial_t \left[ \left( \frac{1}{2} \rho(Q) + Q\rho'(Q) \right) \partial_t Q \right] - K^{-1}(Q + k)^9 |\nabla \omega|^2 \geq \]
\[
\langle \omega, \Delta (\rho \omega) \rangle = *d [\omega \wedge * (\rho' (Q) dQ \wedge \omega)] + * \left[ \omega \wedge \frac{\partial \omega}{\partial t} \right]
\]

using (63), (64), and (65), and

\[
0 \leq K^{-1} (Q + \kappa)^q |\nabla \omega|^2 \leq L_\omega (Q) - \frac{1}{2} \frac{\partial Q}{\partial t} \equiv \partial_t \left[ \left( \frac{1}{2} \rho (Q) + Q \rho' (Q) \right) \partial_t Q \right] \pm \text{div} * [\omega \wedge * (\rho' (Q) dQ \wedge \omega)] - \frac{1}{2} \frac{\partial Q}{\partial t}. \quad (67)
\]

This inequality is uniformly subparabolic whenever condition (48) is satisfied for \( k > 0 \) or \( q = 0 \).

If \( M \) is a compact Riemannian manifold and \( u : M \times [0, T] \to N \), then we can obtain a global estimate for \( Q \). In place of (14) we have the parabolic system

\[
\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\beta} \left\{ \rho (Q) \sqrt{\gamma} \gamma^{\alpha \beta} \frac{\partial u^i}{\partial x^\alpha} \right\} + \rho (Q) \gamma^{\alpha \beta} \Gamma^i_{jk} (u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} = u^i.
\]

Arguing as in the proof of Theorem 2, we add to the middle and right-hand side of eq. (20) a term of the form

\[
u^i_{x^\alpha} u_{x^\alpha}^i = u^i_{x^\alpha} u^i_{x^\alpha t} = \frac{1}{2} Q_t.
\]

Let the sectional curvature of \( N \) be nonpositive. We obtain as in (67) the inequality

\[
L_\omega (Q) + C_R \rho Q - \frac{1}{2} \frac{\partial Q}{\partial t} \geq 0,
\]

where \( C_R \) depends on the Ricci curvature of \( M \). This inequality is, of course, also uniformly subparabolic whenever condition (17) is satisfied. In fact we have an \textit{a priori} estimate in this case, which strongly depends on the ellipticity of \( L_\omega \).

**Theorem 9** Let \( u(x, t) \) be a mapping of a smooth, compact \( n \)-dimensional Riemannian cylinder \( M \times [0, T] \) into \( N \), where \( N \) is a smooth, compact Riemannian manifold of nonpositive sectional curvature and \( T \) is a finite number. Suppose that \( u \) smoothly satisfies (68) with \( \rho \) small in \( L^{s/2} (M) \), \( s > n \), and with \( \rho' (\bar{s}) \leq 0 \), \( \bar{s} \in [0, Q] \). Let condition (17) hold for each point \((x, t) \in M \times [0, T] \) and let \( Q(x, 0) \leq 1 \). Let the \( H^{1, 2} \) Sobolev inequality hold on
M for constants $S_1$, $S_2$. Then there is a constant $c(s, K, q, M, N, T, S_1, S_2)$ such that for $q > 0$,

$$
\sup_{t \in (0, T]} \left( \sup_{x \in M} Q(x, t) \right) \leq ct^{-n/2(q+1)} \left( E[\omega(x, 0)] \right)^{1/(q+1)},
$$

where $E$ is the nonlinear Hodge energy.

Proof. Multiply inequality (68) by $(Q + \beta)^{r-1}$ for $r > 1$ and $\beta > 0$. Replace the time derivative in (68) by the (identical) time derivative of $Q + \beta$ and integrate over $M$. We obtain

$$
r^{-1} \frac{\partial}{\partial t} \int_M (Q + \beta)^r \, dM \leq \int_M (Q + \beta)^{r-1} \nabla \cdot (a(\omega) \nabla Q) \, dM
$$

$$
+ C \int_M \rho \cdot (Q + \beta)^r \, dM,
$$

(69)

where $\nabla$ is the gradient on $M$ and $a$ is the matrix-valued function of inequality (27). Because $M$ is compact, Stokes’ Theorem implies that

$$
\int_M (Q + \beta)^{r-1} \nabla \cdot (a(\omega) \nabla Q) \, dM = \int_M \nabla \cdot \left( a(\omega) (Q + \beta)^{r-1} \nabla Q \right) \, dM
$$

$$
- \int_M \nabla \left( (Q + \beta)^{r-1} \right) \cdot a(\omega) \nabla Q \, dM = - \int_M (r - 1) (Q + \beta)^{r-2} a(\omega) |\nabla Q|^2 \, dM
$$

$$
\leq -m_1 \int_M (r - 1) (Q + \beta)^{r-2} |\nabla Q|^2 \, dM = -m_1 \int_M \nabla \left( (Q + \beta)^{r-1} \right) \cdot \nabla Q \, dM,
$$

where $m_1$ depends on $k, K$, and $q$. Now

$$
- \left| \nabla \left( (Q + \beta)^{r/2} \right) \right|^2 = - \left| \frac{r}{2} (Q + \beta)^{(r-2)/2} \nabla Q \right|^2 =
$$

$$
- \frac{r^2}{4} (Q + \beta)^{r-2} |\nabla Q|^2 = - \frac{r^2}{4(r-1)} \nabla \left( (Q + \beta)^{r-1} \right) \nabla Q,
$$

so we can write inequality (69) in the form

$$
r^{-1} \frac{\partial}{\partial t} \int_M (Q + \beta)^r \, dM \leq - \frac{4m_1(r - 1)}{r^2} \int_M \left| \nabla \left( (Q + \beta)^{r/2} \right) \right|^2 \, dM
$$

$$
+ C \|\rho\|_{s/2} \|(Q + \beta)^r\|_{s/(s-2)}^2.
$$
Employing the parabolic DeGiorgi-Nash-Moser iteration as in Sec. 4 of Ref. 25, taking $p_0 = q + 1$, we obtain, letting $\beta$ tend to zero, 
\[
\sup_{t \in [0,T]} \left( \sup_{x \in M} Q(x,t) \right) \leq C t^{-n/2(q+1)} \left( \int_M |Q(x,0)|^{q+1} \, dM \right)^{1/(q+1)}.
\]
Because 
\[
\frac{d}{ds} (s\rho(s)) = \rho(s) + s\rho'(s) \leq \rho(s),
\]
we have, for $\rho'(s) \leq 0$, the inequality
\[
Q\rho(Q) = \int_0^Q \frac{d}{ds} (s\rho(s)) \, ds \leq \int_0^Q \rho(s) \, ds.
\]
Thus
\[
2E_{t=0} \geq \int_M \int_0^{Q(x,0)} \rho(s) \, ds \, dM \geq \int_M Q(x,0)\rho(Q,0) \, dM \geq \int_M Q(x,0) [\rho(Q,0) + 2Q(x,0)\rho'(Q)] \, dM \geq K^{-1} \int_M Q(x,0)^{q+1} \, dM.
\]
Taking the $(q + 1)^{st}$ root of this inequality and using (66) completes the proof of Theorem 9.

A local version of Theorem 9 would argue from inequality (67) rather than (68). The initial argument is as in the proof of Theorem 9 except that the integration is against cut-off functions. The Moser iteration is implemented as in Ref. 26.

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