Correlation functions of the one-dimensional attractive Bose gas

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The zero-temperature correlation functions of the one-dimensional attractive Bose gas with delta-function interaction are calculated analytically for any value of the interaction parameter and number of particles, directly from the integrability of the model. We point out a number of interesting features, including zero recoil energy for large number of particles, analogous to a Mössbauer effect.

Recent experiments on trapped one-dimensional (1D) atomic gases [1, 2] provide a unique opportunity to study the effects of quantum correlations and fluctuations for one of the paradigms of strongly-correlated systems: the Lieb-Liniger interacting Bose gas [3]. One of the striking features of these experiments is that the effective 1D coupling $c$ can be tuned to essentially any (positive or negative) value. On the theoretical side the model is exactly solvable, and therefore physics beyond mean-field can be reliably investigated. Although the repulsive regime is well-understood, the attractive case is less commonly discussed in the literature. This case is unusual because the atoms can bind together: in fact, the ground state of $N$ particles of mass $m$ in infinite space is a clump whose binding energy is $E_0 = -mc^2N(N^2 - 1)/6\hbar^2$ [4]. This clump behaves like a particle of mass given by $Nm$.

Our purpose here is to provide a detailed analysis of the dynamics of this system, by calculating observable correlation functions analytically. We give expressions for the one-body function and dynamical structure factor (DSF), which are experimentally accessible using ballistic expansion and Bragg spectroscopy, and highlight their important features. The attractive regime has found renewed experimental interest with the observation of bright solitons analogous to a M"ossbauer effect.

The Hamiltonian of the Lieb-Liniger model is given by

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - 2\bar{c} \sum_{\langle i,j \rangle} \delta(x_i - x_j)$$

(1)

with $\bar{c} = -c > 0$ the interaction parameter, $m$ the mass of the particles (atoms), and the sum runs over all pairings. In terms of experimental parameters the 1D coupling constant is $\bar{c} = \hbar^2/ma_{1D}$, where $a_{1D}$ is the effective 1D scattering length which can be tuned via Feshbach resonance or transverse confinement [5]. For definiteness, we consider a system of length $L$ with periodic boundary conditions. From now on we fix $\hbar = 2m = 1$.

Hamiltonian (1) is diagonalized by the Bethe Ansatz [3]: the eigenfunctions are superpositions of plane waves,

$$\Psi = \sum_{\lambda} A_{\lambda} \prod_{i=1}^{N} e^{i\lambda_{\lambda_j}}$$

over all permutations $P$ of the momenta (rapidities) $\lambda$. The $A_{\lambda}$ coefficients are functions of the two-particle scattering phase shifts obtained from (1), and periodicity of the wavefunction requires the set of rapidities $\{\lambda\}$ to be solution to the Bethe equations

$$e^{i\lambda_{\lambda_a}} = \prod_{a \neq b} \frac{\lambda_a - \lambda_b - i\bar{c}}{\lambda_a - \lambda_b + i\bar{c}}, \quad a = 1, \ldots, N.$$  

(2)

For the repulsive case, all solutions are real [6], and the wavefunctions are scattering states of $N$ atoms. For the attractive case $\bar{c} > 0$, however, the physics is completely different. Complex solutions to the Bethe equations are allowed: atoms can therefore bind and form new types of particles which are stable under scattering. A general eigenstate is therefore made by partitioning the $N$ atoms into a set of $N_j$ bound-states. The rapidities associated to a bound-state of $j$ atoms form a regular pattern in the complex plane which is called a string [7]:

$$\lambda_{\alpha}^j = \lambda_{\alpha} + \frac{i\bar{c}}{2}(j + 1 - 2\alpha) + i\delta_{\alpha}^j.$$  

(3)

Here, $\alpha = 1, \ldots, j$ labels the rapidities within the string, while $\alpha = 1, \ldots, N_j$ labels the set of $N_j$ strings of length $j$. Note that $N = \sum_{j} N_j$, with the total number of strings given by $N_s = \sum_{j} N_j$. $\delta_{\alpha}^j$ are deviations which fall off exponentially with system size $L$. Perfect strings (i.e. with $\delta = 0$) are then exact eigenstates in the limit $L \to \infty$ for arbitrary $N$. Thus we consider the limit $L \to \infty$ at fixed $N$ [18].

Such an eigenstate will have momentum and energy

$$K = \sum_{\langle j,\alpha \rangle} j\lambda_{\alpha}^j, \quad E = \sum_{\langle j,\alpha \rangle} \left[ j\lambda_{\alpha}^j - \frac{\bar{c}^2}{12}(j^2 - 1) \right],$$

(4)

where $\sum_{\langle j,\alpha \rangle}$ represents the sum over strings in the eigenstate. The set of $N_s$ rapidities $\lambda_{\alpha}^j$ obey a set of reduced Bethe equations obtained by substituting (3) into (2).

The ground-state corresponds to a single $N$-string centered on zero. Excited states can be classified in terms of their string content. One-particle (or one string $N_s = 1$) states are obtained by giving a finite momentum $K$ to the ground state. Their dispersion relation is $\omega = K^2/N$, meaning that in the large $N$ limit they form a flat dispersion band degenerate with the ground state. Two-particle states ($N_s = 2$) are composed of two strings made of $M$ and $N - M$ particles respectively. Multiparticle states with $N_s > 2$ are similarly constructed.
Form factors and correlation functions. We consider the zero-temperature density-density correlation function \( S^0(\rho(x,t) \rho(0,0)) \) with \( \rho(x,t) = \sum_{\mu=1}^{N} \delta(x-x_{\mu}) \), whose Fourier transform is known as the dynamical structure factor (DSF), and the one-body dynamical correlation function of the canonical Bose field \( \Psi(x,t) \), \( S^0(\Psi(x,t) \Psi(0,0)) \). By Fourier transform \( \mathcal{O}(k,\omega) = \int_{0}^{\pi} \frac{dx}{\omega} e^{i(\omega t-kx)} \mathcal{O}(x,t) \), these can be written as a sum over intermediate states \( \mu \),

\[
S^0(k,\omega) = 2\pi L \sum_{\mu} \frac{|\Sigma_{\mu}|^2}{||\Psi(0,0)||^2} \delta(\omega - E_{\mu} + E_0),
\]

where \( GS \) denotes the ground state, and the form factor (FF) \( \Sigma_{\mu}^0 = \langle \mu | \mathcal{O}(0,0) | GS \rangle = \frac{1}{2} \langle \mu | \mathcal{O}_{K_0} | GS \rangle \) depends on the operator \( \mathcal{O} \) and on the state \( \mu \) (\( \mathcal{O}_{K} \) is the Fourier transform of the operator \( \mathcal{O} \) at momentum \( K \)). \( ||\mu|| \) denotes the norm of state \( \mu \). For the Lieb-Liniger model, FFs and norms were calculated for general Bethe states [9–12]. They are given by the determinant of the norm of state \( \mu \).

In the repulsive case, the Bethe equations cannot be solved analytically. It is however possible to solve the Bethe equations numerically at finite \( N \) and \( L \), determine the corresponding FFs, and perform the sum over intermediate states by selectively scanning the Fock space, thereby obtaining very precise results for the correlation functions [12, 13]. In the present case, however, the velocities can be determined to exponential precision for all states such that \( N_{s} \ll N \). The remaining \( N_{s} \) equations for the string centers yield rapidities which in the limit \( L \to \infty \) are quantized as for free particles (full details will be published elsewhere [8]). This allows us to go analytically much further than in the repulsive case.

One technical difficulty is that the elements of the norm and FF matrices are singular when calculated on exact strings with \( \delta = 0 \). A more careful analysis however shows that all divergences cancel [8], and that it is possible to obtain closed forms for all norms and form factors. Single-particle and density FFs between the ground state and any other string state have the rather simple forms

\[
|\Sigma_{\mu}^\Psi| = (\frac{2\pi}{\epsilon})^N N!(N-1)! \prod_{j,\alpha} H_j(\mu_{\alpha}^j/\epsilon),
\]

\[
|\Sigma_{\mu}^\rho| = \frac{K^2}{\epsilon} N!(N-1)! \prod_{j,\alpha} H_j(\mu_{\alpha}^j/\epsilon),
\]

where we have defined the fundamental block \( H_M(x) = |\prod_{j}(\frac{\alpha_j^+ \alpha_{j+1}^+ + \beta_j^+ \beta_{j+1}^+)}{\prod_{j}(\frac{\alpha_j^+ \alpha_{j+1}^- + \beta_j^- \beta_{j+1}^-)}|\). These expressions are exact in the limit of large \( L \) for any finite \( N \) [14]. The general expression for the norms is

\[
||\mu||^2 = (\tilde{\epsilon} \tilde{c})^N \prod_j 2^{N_j} \prod_{(j,\alpha)>(k,\beta)} F^{jk}(\mu^j_{\alpha} - \mu^k_{\beta})
\]

where \( F^{jk}(\mu) = (\frac{\mu^2 + (j+k)^2}{\epsilon_{\mu}^2 + (j+k)^2}) \).

Eqs. (6), (7) and (8) give an exact representation of each term in the expansion (5) of the desired correlation functions for \( L \to \infty \) for any value of \( N \) and \( \tilde{\epsilon} \). The sums over intermediate states can now be analytically performed for various families of excited states, starting from the simplest ones.

Dynamical structure factor. Let us start here with the simplest excited states, which are composed of one \( N \)-string. For arbitrary \( N \), these states give a single coherent peak,

\[
S^0(\rho,\omega) = 2\pi L \delta(\omega - \frac{k^2}{\tilde{\epsilon}^2}) \prod_{a=1}^{N-1} [1 + k^2/\alpha^2 \tilde{c}^2 N^2]^{-2}.
\]

The leading multiparticle contributions come from two-particle \( N - M : M \) states, for which we find

\[
S^0_M(\rho,\omega) = \frac{\Theta(\omega - \omega_M^\rho) k^4 t^4(N)}{[\omega - \omega_M^\rho]^2 \tilde{c}^2 C^N_N} \prod_{\sigma=\pm} H_{M-M}^2(H_{M}^2/\epsilon^2)
\]

\[
[\omega - \omega_M^\rho(k)^2] \prod_{\sigma=\pm} H_{M-M}^2(H_{M}^2/\epsilon^2)
\]

\[
\left( \frac{\omega - \omega_M^\rho(k)^2}{\omega - \omega_M^\rho(k)^2} \right)^2 \prod_{\sigma=\pm} H_{M-M}^2(H_{M}^2/\epsilon^2)
\]

\[
\left( \frac{\omega - \omega_M^\rho(k)^2}{\omega - \omega_M^\rho(k)^2} \right)^2 \prod_{\sigma=\pm} H_{M-M}^2(H_{M}^2/\epsilon^2)
\]

\[
[\omega - \omega_M^\rho(k)^2] \prod_{\sigma=\pm} H_{M-M}^2(H_{M}^2/\epsilon^2)
\]
that the coherent mode contribution becomes
\[ S_c^0(k, \omega) = \frac{2\pi N^2}{L} \frac{(\pi k/g)^2}{\sinh^2(\pi k/g)} \delta(\omega - k^2/N). \] (13)
and for multiparticle states (the leading \( M = 1 \) part),
\[ S_1^0(k, \omega) = \frac{N}{L} \frac{k^4}{2g} \frac{\Theta(\omega - g^2/4)}{[\omega - g^2/4]^2} \sum_{\sigma=\pm} \int_0^\infty F_2^2(x) \, dx \] (14)
with \( F_2(x) = \frac{x}{\cosh(\pi x)} \) and \( x_{\pm} = (\omega - g^2/4)^{1/2} \pm k/g \).

To certify our results, a number of checks can be made. The first is to compute the second density moment \( \langle \rho^2 \rangle \), which is given by the Hellmann-Feynman theorem as
\[ \frac{1}{L} \sum \frac{\partial E}{\partial \rho_0} = \frac{\pi^3}{6L} + \ldots \] Integrating shows that the one-particle part \( S_0^0(k, \omega) \) completely saturates this to leading order in \( N \). To go further, we can study the \( f \)-sumrule, stating that at fixed \( k \) the integral \[ \int_0^\infty \frac{d\omega}{2\pi} \rho \sigma(k, \omega) \] equals \( k^2N/L \). Thus, in the large \( N \) limit, the one \( N \)-string states contribute to the \( f \)-sumrule as
\[ f_0(k) = -\frac{(\pi k/g)^2}{\sinh^2(\pi k/g)} \frac{N}{L} k^2 < \frac{N}{L} k^2, \] (15)
and therefore to achieve complete saturation, we here need higher states (this is natural, since we are computing a first frequency moment). The leading \( M = 1 \) two-particle states contribute to the \( f \)-sumrule as
\[ f_1(k) = \frac{NK^2}{L} \left( 1 - \frac{(\pi k/g)^2}{\sinh^2(\pi k/g)} \right) \] (16)
and therefore completely saturate the remaining part of the \( f \)-sumrule when \( N \) string contributions have been taken into account. Higher frequency moments further suppress the \( S_0^0 \) part; in general, the full DSF is therefore well approximated by \( S_0^0(k, \omega) = S_0^0(k, \omega) + S_0^0(k, \omega) \).

The static structure factor is the integral of the dynamical one and can be written as
\[ S^0(k) = \int \frac{d\omega}{2\pi} S^0(k, \omega) = S_0^0(k) + S_1^0(k), \quad \text{with } S_0^0(k) \text{ trivial and} \]
\[ S_1^0(k) = \frac{2NK^4}{Lg^4} \left[ \frac{g^2}{k^2} - \frac{\pi^2}{\sinh^2(\pi k/g)} - \Re \psi_2 \left( \frac{i k}{g} \right) \right] \] (17)
where \( \psi_2(z) \) is the polygamma function (second derivative of the logarithm of the Gamma function). For large \( k \) the static structure factor is exponentially suppressed and is dominated by the two-particle term. The latter shows a peak at \( k/g \sim 1.2 \) reminiscent of what is obtained for the super-Tonks-Girardeau gas-like regime [16].

For the one-body function, we similarly find
\[ S_0^1(k, \omega) = \frac{2\pi^3}{g \cosh^2(\pi k/g)} \frac{N}{L} \delta(\omega - k^2/N) \] (18)
with two-particle states with \( M = 1 \) giving the leading contribution in the continuum,
\[ S_1^1(k, \omega) = \frac{g^2}{2L} \frac{\Theta(\omega - g^2/4)}{[\omega - g^2/4]^2} \omega^2 \sum_{\sigma=\pm} F_2^2(x_{\sigma}) \] (19)
with \( F_2(x) = \pi x/\sinh \pi x \) and \( x_{\pm} \) as defined above. To check the relative weight of these states, we use the sumrule that the momentum and frequency integral of this must equal the density \( N/L \), and obtain
\[ \int_0^\infty \frac{dk}{2\pi} \int_0^\infty \frac{d\omega}{2\pi} S^1(k, \omega) = \frac{N}{L}, \]
i.e. the one-particle states completely saturate this sumrule. However, as we have seen with the DSF, calculating higher frequency moments would necessitate taking the multiparticle terms into account, starting with the \( M = 1 \) part. This correlation function is therefore well-approximated by
\[ S^0(k, \omega) = S_0^0(k, \omega) + S_0^0(k, \omega) \]

The sum rules we have studied are saturated using only either the one-particle intermediate states, or the simplest two-particle excitations formed by breaking the original string in two pieces, one of which is a single rapidity. For large \( N \), the contributions from higher excitations are suppressed by progressively higher powers of \( 1/N \), and can be neglected for most practical purposes. The importance of the simplest two-particle excitations is highlighted by calculating higher frequency moments.

**Discussion of the results.** Both correlation functions considered here show similar features. They are dominated by single-particle coherent contributions, lying on lines \( \omega = k^2/N \) which become dispersionless in the large \( N \) limit. There however exist multiparticle continua starting from finite mass gap thresholds \( (\omega \sim g^2/4) \) for the leading two-particle ones) and extending to arbitrarily high energies. At the lower threshold, the correlators are square-root singular. At high energies, the correlators decay exponentially, which is reminiscent of massive integrable quantum field theories [17].

The fact that the flat band single-particle excitations account for most of the correlation functions can interestingly be interpreted as a Mössbauer-like effect, where the recoil energy (say upon scattering with a photon in Bragg spectroscopy) vanishes because the system acts as a single particle of mass \( N \gg 1 \) (the ground state string can in fact be viewed as a crystal in rapidity space). It should therefore be feasible to observe an analogue of the Mössbauer effect on this system.

The two-particle \( (N-1:1) \) part of the DSF is plotted in Figure 1 for large \( N \), with \( g = 1 \). At the lower threshold, the DSF diverges as \( (\omega - g^2/4)^{1/2} \). For \( \omega > g^2/4 \) the DSF is a monotonous decreasing function of \( \omega \) as long as \( k/g < x_c = 1.0565 \ldots \) whereas for \( k/g > x_c \) it shows a characteristic broad peak, whose position grows like \( k^2 \) for large \( k \) and its amplitude decreases like \( k^{-1} \). Away from this peak the correlator decays exponentially. Four fixed momentum cuts are given in Figure 2, showing these features in more detail. For the one-body correlator, the maximum lies at the lower threshold, with monotonous exponential decay at higher energies.

**Concluding remarks.** In this letter, we showed how integrability could be used to compute the zero-temperature dynamical structure factor (Eqs (13), (14)) and the dynamical one-body correlation function (Eqs
functions are given (for fixed integrable quantum field theories, namely the correlation mechanical system has many similarities with massive illustrating that this experimentally-realizable quantum-a subset of our results. The results obtained clearly responding static correlation functions are obtained as (18),(19)) for the 1D attractive Bose gas. The cor-
ing of the square-root singularity at higher momentum, and FIG. 2: Fixed momentum cuts of Fig. 1, showing the weaken-
ing of the square-root singularity at higher momentum, and the displacement of the maximum towards higher energies.

(18),(19)) for the 1D attractive Bose gas. The corre-
spending static correlation functions are obtained as a subset of our results. The results obtained clearly illustrate that this experimentally-realizable quantum-mechanical system has many similarities with massive integrable quantum field theories, namely the correlation functions are given (for fixed k, as a function of ω) by a coherent peak followed by a multiparticle continuum. Additionally, we have quantified an interesting zero recoil energy effect for large particle numbers, which is analogous to the Møssbauer effect in solids.

The methods and results obtained in this letter can be adapted to describe many other features of the attractive Lieb-Liniger gas. For example, our expressions could lead to a calculation of finite temperature correlation functions, allowing the description of the quantum to classical crossover in this strongly interacting model.

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[18] This limit is not trivial (as it is in the repulsive regime): here, the N particles remain strongly correlated and bound to one another even when L → ∞.
[19] In general, the contribution from a family of P particle excited states is a P − 2-fold integral. All terms have 1/L dependence, which can be traced to our choice of normalizing the intermediate states to 1 rather than L.