Ground Rings and Their Modules in 2D Gravity with $c \leq 1$ Matter

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All solvable two-dimensional quantum gravity models have non-trivial BRST cohomology with vanishing ghost number. These states form a ring and all the other states in the theory fall into modules of this ring. The relations in the ring and in the modules have a physical interpretation. The existence of these rings and modules leads to nontrivial constraints on the correlation functions and goes a long way toward solving these theories in the continuum approach.

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Recently, using matrix model techniques, a number of non-critical string models have been solved exactly [1] [2] [3] [4] [5]. Some of these models were shown to be equivalent to certain topological field theories [6] and they exhibit unexpected relation to integrable systems. Despite some progress in the continuum Liouville description of these theories [7] [8] [9] [10] [11] [12], their surprising integrability is yet to be understood. In the flat space version of these theories the null vectors in degenerate Virasoro representations lead to Ward identities and to the solution of the theories [13]. This fact has led many people to conjecture that the simplicity and solvability of these quantum gravity models should be associated with these null vectors. In this paper we take a step towards a complete solution of the models using the null vectors.

The first consequence of the existence of degenerate Virasoro representations in the matter sector of these theories is the appearance of infinitely many new states in the BRST cohomology [14] [15]. The standard physical fields have the form $T = \bar{c}cOe^{\alpha\phi}$ where $O$ is a matter primary field. In the $(p,p')$ (with $p > p'$) minimal models there are $(p-1)(p'-1)/2$ such fields

$$T_{n,n'} = \bar{c}cO_{n,n'}e^{[1+\frac{p}{p'}-\frac{pn'-p'n}{p'}]2\phi}$$  \hspace{1cm} (1)

$(\gamma = \sqrt{\frac{2p'}{p}})$ labeled by $n = 1,...,p-1$ and $n' = 1,...,p'-1$ with $pn' - p'n > 0$. In the (non-compact) $c = 1$ theory there is a continuous set of operators

$$T_q = \bar{c}c e^{iqX/\sqrt{2}}e^{(2-|q|)\frac{2}{2}\phi}$$  \hspace{1cm} (2)

$(\gamma = \sqrt{2})$ referred to as tachyons, labeled by the momentum $q$ and infinitely many ‘special states’ for integer $q$ labeled by an integer $s \geq 1$

$$D_{q,s} = \bar{c}c e^{iqX/\sqrt{2}}P_{q,s}(\partial X,\ldots)\bar{P}_{q,s}(\bar{\partial} X,\ldots)e^{[2-(|q|+2s)]\frac{2}{2}\phi}$$  \hspace{1cm} (3)

where $P_{q,s}$ is a polynomial in derivatives of $X$ of dimension $|q|s + s^2$. We will refer to a tachyon with integer momentum $T_q = D_{q,s=0}$ as a special tachyon. In the expressions for the operators (1)-(3) we used the bound on the Liouville exponent of $\phi$.

Rings

In the interesting papers [14] [15] Lian and Zuckerman have shown that the null vectors in the matter and Liouville representations lead to more states with other ghost numbers. In the $c = 1$ system, these have the same $X$ and Liouville momenta as (3) but have vanishing ghost numbers (in our convention the physical states have ghost number one).
In fact, there are three sets of such states with ghost numbers \((1, 0)\), \((0, 1)\) and \((0, 0)\). The states with \((1, 0)\) and \((0, 1)\) lead to conserved currents [16] and the \((0, 0)\) states lead to a ring [16]. We will denote these operators \(R_{q,s}\).

In the minimal models each highest weight state has two primitive null vectors, so the ground ring has twice as many elements as the number of matter primaries. The vanishing ghost number operators \(R_{n,n'}\) are labeled by \(n = 1, \ldots, p-1\) and \(n' = 1, \ldots, p'-1\) (without the standard identification of \((n, n')\) with \((p - n, p' - n')\)). They are given by polynomials in Virasoro generators of Liouville and matter sectors, as well as the modes of the ghost number current, acting on \(\exp[-((n - 1) + \frac{p}{p'}(n' - 1))\frac{\gamma}{2}\phi]O_{n,n'}\). Note that \(O_{n,n'} \equiv O_{p-n,p'-n'}\) but \(R_{n,n'} \neq R_{p-n,p'-n'}\) due to the different null vectors used in their construction. The first of them \(R_{1,1}\) is the identity operator. Explicit construction of some of the other operators was given in [17]

\[
R_{2,1} = |bc - \frac{1}{\gamma}(L_{-1} - M_{-1})|^2 e^{-\gamma\phi/2} O_{2,1} \\
R_{1,2} = |bc - \frac{\gamma}{2}(L_{-1} - M_{-1})|^2 e^{-(p/p')\gamma\phi/2} O_{1,2}
\]

(4)

For example, in pure gravity \((p = 3, p' = 2)\) \(R_{2,1} = |bc - \frac{1}{\gamma}\partial\phi|^2 e^{-\frac{\gamma}{2}\phi}\). There are also operators with arbitrarily larger negative ghost number. Note that unlike the \(c = 1\) system there are no \((1, 0)\) or \((0, 1)\) operators and hence there are no conserved currents.

As pointed out by Witten [16] the vanishing ghost number operators are special because they lead to a ring structure. The ring multiplication is obtained by considering the operator product expansion of two vanishing ghost number operators \(R_m\) and \(R_{m'}\) in the BRST cohomology. Since the product is BRST invariant, and has vanishing ghost number, it can be written as

\[
R_m(z)R_{m'}(w) = \sum_{m''} f_{m,m'}^{m''} R_{m''}(z) + [Q, O]
\]

(5)

for some operator \(O\) depending on \(m, m', z\) and \(w\). Here we have used the fact that \(R_m(z)\) has dimension zero. Ignoring the BRST commutator in (5) we find a ring with structure constants \(f_{m,m'}^{m''}\). Below we will examine whether these BRST commutators can be dropped in correlation functions.

Treating the Liouville field as free Witten [16] has shown that in the non-compact \(c = 1\) system the ring is generated by

\[
a_+ = R_{1,1} = |bc - \frac{1}{\gamma}(\partial\phi - i\partial X)|^2 e^{-\frac{\gamma}{2}(\phi - iX)}
\]

(6)
and its conjugate
\[ a_- = R_{-1,1} = |bc - \frac{1}{\gamma}(\partial\phi + i\partial X)|^2 e^{-\frac{\gamma}{2}(\phi + iX)} \]  

(7)
i.e. \( R_{n,s} = a_+^{(|n|+n)/2+s-1}a_-^{(|n|-n)/2+s-1} \) and it has no relations. The generators \( a_\pm \) have a beautiful interpretation \([16]\) as the phase space coordinates of the free fermions of the matrix model description of this model, and the whole ring is then identified as functions on phase space. Note that the scaling of \( a_\pm \) is \( e^{-\gamma\phi/2} \); i.e. they scale like inverse length. This is precisely the expected scaling of \( \lambda \) and its time derivative \( \dot{\lambda} \). There is at least one matrix model operator with these quantum numbers \([18]\) \[ \frac{1}{g_{str}} \int dX \psi^\dagger \lambda^3 \psi e^{\pm iX/\sqrt{2}} \] but there may be others. The power of \( \lambda \) in the operator does not lead to the wrong scaling behavior because of the factor of the string coupling in the vertex which scales like the square of the length.

The operators \( J_{q,s} \) and \( \tilde{J}_{q,s} \) related to \( R_{q,s} \) with ghost numbers \((1,0)\) and \((0,1)\), are almost conserved. Their divergences \( \partial J_{q,s} \) and \( \partial \tilde{J}_{q,s} \) are BRST commutators. If these commutators can be ignored, these operators are holomorphic and anti-holomorphic currents \([16]\) and lead to a symmetry \( W' \). \( W' \) is the subalgebra of the algebra \( W \) of area preserving diffeomorphisms of the \( a_\pm \) plane that preserves the lines \( a_+ = 0 \) and \( a_- = 0 \). \( W \) transformations closely related to those of this symmetry were first noted in the matrix model in \([19]\). They were modified and identified as symmetries of the matrix model in \([18]\) where their relation to the special states was also explained (see also \([20]\)). In the continuum approach this symmetry was related to the special states also in \([21]\). The elements of \( W' \) do not have to preserve the lines \( a_\pm = 0 \) pointwise but only as a set. These lines were interpreted in \([16]\) as the Fermi surface of the matrix model and \( W' \) is then the subalgebra of the matrix model symmetry \( W \) which is preserved by the ground state \([16]\).

We now return to the minimal models. The matter content of the ground ring operator \( R_{n,n'} (\mathcal{O}_{n,n'}) \), and the CFT fusion rules constrain the multiplication table of the ring. Assuming that the Liouville field is free and examining the Liouville momenta of \( R_{n,n'} \), it appears that the ring is generated by \( R_{1,2} \) and \( R_{2,1} \)

\[ R_{n,n'} = R_{2,1}^{n-1} R_{1,2}^{n'-1} \]  

(8)
(We did not check this expression explicitly in the most general case.) Unlike the \( c = 1 \) system, \( n \) and \( n' \) are bounded, and therefore there must be some relations in the ring. Examining the Liouville momentum we conclude \( R_{1,2}^{n'-1} = 0 \) and \( R_{2,1}^{n-1} = 0 \). It is amusing to note that the relation \( R_{1,2}^{n'-1} = 0 \) is the relation in the underlying chiral ring in the LG
description of the topological field theory at the point \( p = 1 \). It would be interesting to understand the role of the other relation in that context.

The generator \( R_{2,1} \) scales like the eigenvalue of the matrix model (inverse length) and the other generator \( R_{1,2} \) scales like the conjugate momentum (length to the power \( -p/p' \)). Motivated by the interpretation of the ring at \( c = 1 \) and this scaling behavior, we would like to interpret these generators as the eigenvalue and its conjugate momentum in the matrix model. These are precisely the operators \( Q \) and \( P \) in Douglas’ derivation of the string equation. We therefore propose the identification \( R_{2,1} = Q \) and \( R_{1,2} = P \) and the finite ring as functions on this “phase space.” Note that this “phase space” is not standard because \( Q \) and \( P \) can be raised only to finite powers. It should be pointed out that the matrix model operators corresponding to \( R_{2,1} \) and \( R_{1,2} \) are generally believed to be given by the fractional powers \( Q_{+}^{\frac{2p+p'}{p'}} \) and \( Q_{+}^{\frac{2p+2p'}{p'}} \) of \( Q \) and not by \( Q \) and \( Q_{+}^{p/p'} \) respectively. The apparent discrepancy with the scaling properties is again resolved by recalling the extra factor of the string coupling, which scales as the \( \left( \frac{p+p'}{p'} \right)^{th} \) power of \( Q \). It is curious that both the \( c = 1 \) and the minimal models have an operator which scales like inverse length. Such an operator, \( e^{-\frac{1}{2} \gamma \phi} \), plays a fundamental role in the Backlund transformation in Liouville theory \([22, 23]\) and provides the relation to its \( SL(2, R) \) symmetry \([23, 24]\).

The discussion above generalizes to the fermionic string. Again, the BRST cohomology can be analyzed, and when there are degenerate representations there is nontrivial BRST cohomology at ghost number zero (and all negative ghost numbers at \( c < 1 \)) \([25]\), except that in this case the generating elements are in the Ramond sector. For \( c < 1 \) the matter highest weights \( O_{n,n'} \) are Ramond for \( n-n' \) odd and Neveu-Schwarz for \( n-n' \) even. Hence \( R_{1,2} \) and \( R_{2,1} \) are Ramond operators. At \( c = 1 \) the special states are easily constructed using super-\( SU(2) \) current algebra as in \([16]\) (where the odd half-integer spin states are in the Ramond sector). The ring multiplication table is identical to the bosonic case, the standard \( Z_2 \) symmetry of Ramond-Neveu-Schwarz is identical to the \( Z_2 \) of even vs. odd polynomials in \( \lambda \). Since the generators are Ramond fields, any representation (see below) contains both Neveu-Schwarz and Ramond states.

Note that the construction of the ground ring uses very little of the structure of the theory, simply that it consists of two sectors: Liouville and matter, and the matter sector has degenerate representations. From a null state, general BRST arguments of the type given by Witten \([16]\) predict the existence of BRST cohomology at ghost number zero. In fact one can interpret the program of \([23]\) for \( c > 1 \) as a study of this sector of the string Hilbert space. Indeed, the Liouville momenta studied there are precisely at the special
values given by the Kač formula. Clearly, these operators form a closed operator algebra. It is crucial that the Liouville exponent for these fields is always real even for $c > 1$. It is not clear to us why one is allowed to ignore all the other states in these theories and it remains to be seen what the physical interpretation of these states is for $c > 1$.

**Modules**

Now, consider the other operators in the BRST cohomology. The operators of fixed ghost number form a module (a representation) of the ring. To see that, consider the operator product expansion of $\mathcal{R}_m$ and an operator in the cohomology

$$\mathcal{R}_m(z)V_i(w) = \sum_{i'} T_{m,i}^{i'}V_{i'}(z) + [Q, \mathcal{O}]$$

where the sum over $i'$ is over the fields in the cohomology with the same ghost number as $V_i$. As in (5), we first ignore the BRST commutator on the right hand side and conclude that the coefficients $T_{m,i}^{i'}$ represent the ring multiplication. It is sometimes the case that this representation is not faithful; i.e. the matrices $T_{m,i}^{i'}$ satisfy more relations than the underlying ring and represent a quotient of it.

We now examine the various modules which are present in these theories. We start with the $c = 1$ system and consider a tachyon state $\mathcal{T}_q$ with generic (not integer) momentum $q$. An easy free field calculation shows that for every fractional part of $q$ and every sign of $q$ there is a separate module. For $q > 0$

$$a_+ \mathcal{T}_q = q^2 \mathcal{T}_{q+1} + [Q, \mathcal{O}_{q+1}^+]$$
$$a_- \mathcal{T}_q = 0 + [Q, \mathcal{O}_{q-1}^-]$$

and for $q < 0$

$$a_+ \mathcal{T}_q = 0 + [Q, \mathcal{O}_{q+1}^+]$$
$$a_- \mathcal{T}_q = q^2 \mathcal{T}_{q-1} + [Q, \mathcal{O}_{q-1}^-]$$

None of these modules is faithful. For $q > 0$ the ring generator $a_-$ is represented by zero and for $q < 0$ $a_+$ is zero. This fact has a simple interpretation in the matrix model. As explained by Polchinski [26], the tachyons can be thought of as ripples on the Fermi surface. Therefore, they satisfy the equation of the Fermi surface which for vanishing cosmological constant are $a_+ = 0$ for $q < 0$ and $a_- = 0$ for $q > 0$.

Similarly, the tachyons $\mathcal{T}_q$ are not in a faithful representation of the symmetry algebra $\mathcal{W}'$. Since the anti-holomorphic part of $J_{q,s}$ is the anti-holomorphic parts of $\mathcal{R}_{q,s} = \mathcal{R}_{q,s}^\ast$.
\[ a_+^{(|q|+q)/2+s-1} a_-^{(|q|-q)/2+s-1}, \] only \( J_{q,s=1} \) act non-trivially and even of these, the negative \( q \) \( J \)'s annihilate the positive momentum tachyons and vice versa. In terms of the underlying phase space the interpretation of this fact is interesting. The \( J \)'s generate the algebra \( \mathcal{W} \) of reparametrizations of the filled Fermi sea. It has a subalgebra \( \mathcal{W}'' \) of transformations which leave the Fermi surface invariant pointwise. Since the tachyons \( T_q \) “live” on the Fermi surface, \( \mathcal{W}'' \) acts trivially on them and the tachyons represent only the quotient \( \mathcal{W}/\mathcal{W}'' \) which is essentially a Virasoro algebra.

For integer values of \( q \) the relations are different than (10)(11). For \( q \) positive

\[
\begin{align*}
a_+ T_q &= q^2 T_{q+1} + [Q, O_{q+1}^+] \\
a_- T_q &= D_{q+1,s=1} + [Q, O_{q-1}^-] \\
a_+ D_{q,s} &= A_{q,s}^+ D_{q+1,s} + [Q, O_{q+1,s}^+] \\
a_- D_{q,s} &= A_{q,s}^- D_{q-1,s+1} + [Q, O_{q-1,s}^-]
\end{align*}
\] (12)

where \( A_{q,s}^\pm \) are calculable coefficients. Similar relations hold for \( q < 0 \). Note that the zero momentum tachyon \( T_{q=0} \) is annihilated both by \( a_+ \) and by \( a_- \). However, the cosmological constant operator \( \phi T_{q=0} \) is in the same module with the special tachyons and the special states. We conclude that the special states, the special tachyons and the cosmological constant are all in one module. Unlike the tachyon module, here the relation \( a_+ a_- = 0 \) is not satisfied. This relation was interpreted on the tachyon module as a consequence of the fact that the tachyons “live” on the Fermi surface. Similarly we would like to argue that since it is not satisfied for the special tachyons and the special states, these are not ripples on the Fermi surface. We conclude that some of the deformations of the potential cannot be represented as a change in the state of the system. This observation is consistent with the Minkowski space interpretation of this theory. Rotating \( X \) to Minkowskian signature, all the states in the theory are deformations of the Fermi surface [26]. Indeed, for Minkowskian \( X \) momentum and for macroscopic Liouville states there are no special states in the BRST cohomology.

For \( c < 1 \), the ghost number zero states are at values of \( h \) such that there are null vectors in their Verma modules as well, leading to ghost number \(-1\) BRST cohomology via the same argument that produced the ground ring. This structure repeats at each stage, leading to \((p-1)(p'-1)\) dimensional cohomology at all negative ghost numbers related to the tower of inclusions of null modules inside one another in the matter sector [14] (note that these physical states are not dressed null states). The ground ring acts within the
ghost number $n$ Hilbert space; thus it is a representation (module) of the ground ring modulo BRST commutators. In the fermionic string there is again a tower of inclusions of null modules, and hence BRST cohomology at every ghost number.

Each BRST module at negative ghost number is a faithful representation of the ground ring $\mathcal{R}$. There are as many states in the BRST module as elements of $\mathcal{R}$, which acts in a nondegenerate way. This is not so for the physical state module at ghost number one, which is half the size due to the identification $\mathcal{O}_{n,n'} \equiv \mathcal{O}_{p-n, p'-n'}$. The physical state module $\mathcal{P}$ cannot be a faithful representation and there must be extra relations defining the action of $\mathcal{R}$ on $\mathcal{P}$.

These take the form

$$\mathcal{R}_{1,2}^a \mathcal{R}_{2,1}^b = 0 \ , \quad a + b = [(p' - 1)p/p'] .$$

There are two interesting submodules of this module. The ring action on the fields $\mathcal{T}_{1,n'}$ with $n' = 1, \ldots, p' - 1$ are annihilated up to BRST commutators by $\mathcal{R}_{2,1}$ and satisfy $\mathcal{R}_{1,2} \mathcal{T}_{1,n'} = \mathcal{T}_{1,n'+1} + [Q, \mathcal{O}]$. Similarly, $\mathcal{T}_{n,p'-1}$ are annihilated up to BRST commutators by $\mathcal{R}_{1,2}$ and satisfy $\mathcal{R}_{2,1} \mathcal{T}_{n,p'-1} = \mathcal{T}_{n-1, p'-1} + [Q, \mathcal{O}]$ for $1 \leq n < p - \frac{p}{p'}$. These are analogous to the tachyon modules in the $c = 1$ system (10)(11).

**Correlation Functions**

If it is legitimate to drop the BRST commutators in (9) in correlation functions, we derive a set of identities for the amplitudes:

$$< \mathcal{R}_m V_i \cdots V_{i_n} > = \sum_{i'} T_{m,i_1}^{i'} < V_{i'} V_{i_2} \cdots V_{i_n} > = \sum_{i'} T_{m,i_2}^{i'} < V_{i_1} V_{i'} \cdots V_{i_n} > = \ldots$$

Note that these are not Ward identities. The latter would involve a sum of $n$ terms in each of which one of the operators in the correlation function is modified. Here we have an equality between pairs of correlation functions. As we will see, (14) is not always satisfied. Correspondingly, the BRST commutators in (9) and (10) do not necessarily decouple. The standard proof of their decoupling proceeds by moving the BRST charge from the BRST commutator to all the other operators in the correlation function. This has the effect of generating total derivatives on moduli space. The original BRST commutator fails to decouple when these total derivatives do not integrate to zero. This phenomenon can be equivalently described in terms of contact terms at the boundaries of moduli space. It
leads to violations of (14); the modules are deformed, with the structure constants $T_{jk}^i$ acquiring dependence on the couplings one turns on in the action.

Due to the structure of the ring for non-compact $c = 1$ described above, it is enough to consider in this case

$$A^{(\pm)}(q_1, ..., q_n) \equiv \langle a_\pm(z) T_{q_1} ... T_{q_n} \rangle$$  \hspace{1cm} (15)$$

where $\sum q_i \pm 1 = 0$. It is implied in (15) that $n - 3$ of the positions of $T_{q_i}$ are integrated over, and the appropriate $T$ are stripped of the $c\bar{c}$ factors in (2). The strategy for extracting information from (15) is to note that on general grounds $A^{(\pm)}(q_i)$ is independent of $z$; therefore we can compare its value as $a_\pm$ approaches two different (unintegrated) $T_{q_i}$. This will give a set of relations between different amplitudes (14).

It is convenient to consider first amplitudes, in which the $\{q_i\}$ satisfy a “resonance” condition $\sum (2 - |q_i|) = 5$. Such amplitudes are proportional to the volume of space-time and possess integral representations which have been studied before [27] [10] [11]. We will now show that many of their properties are simple consequences of the action of the ring on the tachyon modules.

Consider the general such correlation function $\langle T_{q_1} ... T_{q_n} T_{p_1} ... T_{p_m} \rangle$ where $q_i > 0$, $p_i < 0$. It is known [10] [11] that for $n, m \geq 2$ these amplitudes vanish. To derive this fact from the ring we evaluate

$$A_{n,m}(q_i, p_j) = \langle a_\pm(z) T_{q_1}(0) T_{p_1}(1) T_{q_2}(\infty) \prod_{i=3}^{n} d^2 z_i T_{q_i}(z_i) \prod_{j=2}^{m} d^2 w_i T_{p_j}(w_j) \rangle$$  \hspace{1cm} (16)$$

in two different limits. As $z \to 0$ we can replace $a_+ T_{q_1}$ by $T_{q_1+1}$ using (10). It is important that the BRST commutator in (11) does not contribute to (16). Commuting $Q$ to $T_{q_i} (T_{p_j})$ we find a total derivative in $z_i (w_j)$. One can show that it integrates to zero; indeed, it is readily verified that near all boundaries of moduli space the integrand goes to zero in an appropriate region in momentum space (and is analytically continued to vanish everywhere else). Hence, as $z \to 0$ we find

$$A_{n,m}(q_i, p_j) = q_1^2 \langle T_{q_1+1} \prod_{i=2}^{n} T_{q_i} \prod_{j=1}^{m} T_{p_j} \rangle$$  \hspace{1cm} (17)$$

On the other hand, as $z \to 1$, we use (11) (again, one can explicitly verify that the boundary terms due to the BRST commutators vanish) and conclude that $A_{n,m}(q_i, p_j) = 0$ (for $n, m \geq 2$). Comparing to (17) we find the desired result.
The cases $n = 1$ (any $m$), and $m = 1$ (any $n$) have to be discussed separately:

1) $m = 1$: In this case, momentum conservation and the “resonance” condition enforce $p = p_1 = -(n-2)$. As we saw before, for integer $p_1$ (11) should be replaced by (12) i.e. acting with $a_+$ produces one of the special states (3). Although one can proceed this way, a much more useful relation is obtained by exchanging $T_{q_1} \leftrightarrow T_{q_2}$, such that (16) takes the form

$$A_{n,1}(q_i) = \langle a_+(z) T_{q_1}(0) T_{q_2}(1) T_{p_1}(\infty) \prod_{i=3}^{n} d^2 z_i T_{q_i}(z_i) \rangle$$

In this case it is easy to see that we can use the naive form of (14) to find

$$F(q_1, \ldots q_n) = \langle T_{q_1} \ldots T_{q_n} T_{p} \rangle$$

Redefining $T_q = \frac{\Gamma(1-|q|)}{\Gamma(|q|)} \tilde{T}_q$, we conclude that

$$F(q_1+1, q_2, \ldots) = F(q_1, q_2+1, \ldots) = \ldots$$

An explicit evaluation [11] yields $F(q_i) = \text{const}$, but one cannot determine the periodic function $F$ from the action of the ring. The algebraic reason for this ambiguity is that the theory has a number of different modules. The relation of the ring cannot determine the “reduced matrix elements” of different modules.

2) $n = 1$: In this case we have to be careful with the BRST commutators in (10), (11). As $z \to 0$, one can use (10) naively; hence, $A_{1,m} = \langle T_{q_1+1} \prod_{i=1}^{m} T_{p_i} \rangle$. On the other hand, as $z \to 1$, we find a BRST commutator which does not decouple. The point is that since $q_1$ is fixed kinematically ($q_1 = m - 2$), an on shell tachyon arises in the channel where all $w_i$ simultaneously approach zero. This can be shown to lead to a finite boundary contribution of the appropriate total derivative. Hence here $a_+ T_p \neq 0$ ($p < 0$). For a quantitative analysis it is more convenient to replace $a_+$ by $a_-$ in (16), and imitate the procedure of the first case.

To emphasize the ambiguity of the ring relations (14) in $c = 1$ by a periodic function (21), it is useful to consider the open $c = 1$ string theory on the disk [28]. The qualitative considerations used above are valid there as well. Equation (14) takes the form (for $q > 0$)

$$a_+ T_q = q T_{q+1} + [Q, V^+]$$
$$a_- T_q = [Q, V^-]$$

(22)
and similarly for $q < 0$. The analysis of which correlation functions vanish is quite different in this case; its conclusions are in agreement with 28. The case $\langle a_+ \prod_{i=1}^n T_{q_i} T_p \rangle$ with $q_i > 0$, $p < 0$ leads, as above, to

$$\langle \prod_{i=1}^n T_{q_i} T_p \rangle = \prod_{i=1}^n \frac{1}{\Gamma(q_i)} G(q_1, \ldots, q_n)$$

(23)

where $G$ is a periodic function of the $q_i$. $G$ is actually a complicated function of momenta 28: $G(q_i) = \prod_{l=1}^{n-1} \frac{1}{\sin \pi (q_1 + q_2 + \ldots + q_l)}$. It does not seem to be obtainable from the action of the ring.

So far we have only discussed “resonant” amplitudes in which $\mu$ is in a sense zero. In generic amplitudes (“finite $\mu$”) the situation is more involved. The BRST commutator terms in (9) - (11) cannot be ignored. One can still study the deformations of the Fermi surface in the presence of tachyon perturbations. As conjectured in 16, the equation $a_+ a_- = 0$ should be modified for non-zero $\mu$ to $a_+ a_- = \mu$. (Note that this relation is not an operator relation in the theory.) Unlike 16, from our point of view the relation $a_+ a_- = 0$ is obtained as a relation in the tachyon module (10)(11). Following 16, we conjecture that it is also modified to $a_+ a_- = \mu$. Indeed, one can show (using the methods of 11) that the three tachyon amplitude with generic momenta $q_i$ satisfies

$$\langle (a_+ a_- - \mu) \tilde{T}_{q_1} \tilde{T}_{q_2} \tilde{T}_{q_3} \rangle = 0$$

(24)

In the presence of more tachyons the operator $(a_+ a_- - \mu)$ does not vanish. For example

$$\langle (a_+ a_- - \mu) \tilde{T}_{q_1} \tilde{T}_{q_2} \tilde{T}_{q_3} \tilde{T}_{q_4} \rangle = \mu^{\frac{1}{2}} \sum |q_i|^{-1}$$

(25)

and more complicated expressions for higher $n$ point functions. This fact has an obvious interpretation in the spirit of 26 and 16. In the presence of more tachyons the Fermi surface is deformed and no longer satisfies $a_+ a_- = \mu$. As explained after equation (14), from the world-sheet point of view, this deformation can be understood as a contact term leading to non-zero correlation functions for the BRST commutators in (9) - (11).

To summarize, the main point in this note is the importance of the relations in the ring and in its non-faithful modules. These relations constrain the correlation functions. However, in order to fully utilize the ring and its relations, we have to get better control of the contact terms at the boundaries of moduli space. In the infinite radius $c = 1$ system the tachyons are small deformations of the Fermi surface in the matrix model, and
therefore, the relations in the tachyon module have a natural interpretation as determining the location of the Fermi surface. The contact terms should therefore be associated with the deformation of the Fermi surface due to the presence of other tachyons. We expect them to appear as multi tachyon states in the right hand side of $a_{\pm}T_q$. An important open problem is to explicitly determine these contact terms.

In the minimal models the ring and its relations should be more powerful than in the $c = 1$ system. The ambiguity in the correlation functions in the $c = 1$ theory stems from the existence of infinitely many modules and the relations in the ring cannot determine the “reduced matrix elements.” In the minimal models the physical, ghost number one fields are all in one module and therefore a similar ambiguity is not present. Unfortunately, for these theories we do not have an interpretation of the relations in the module analogous to the Fermi surface at $c = 1$. An interesting relation $R_{1,2}R_{2,1} = 0$ in the submodules of $\mathcal{T}_{1,n'}$ and $\mathcal{T}_{n,p'\pm 1}$ is similar to the equation $a_+a_- = 0$ in the tachyon module. An amusing possibility is that this relation will be deformed to $R_{1,2}R_{2,1} = g_{\text{str}}$ which is reminiscent of the tree level string equation $Q_0P_0 = g_{\text{str}}$ in terms of the constant (zeroth order in derivatives) terms in the KdV operators $P$ and $Q$. Since we know that the string equation is analytic in the matrix model coupling constants, $t_k$, and that these are analytic in the conformal field theory couplings [12], we expect that the string equation can be computed perturbatively in these couplings using free field techniques. The non-analyticity of the solution will arise only from the solution of this equation. We hope that a better understanding of this issue will lead to the entire KdV structure and the Virasoro and $W$ constraints of these theories and will make the connection of Liouville theory to the matrix model and to topological field theory complete.

Note added: After the completion of this work we learned that I. Klebanov and A. Polyakov had obtained some of our $c = 1$ results using another approach and that P. Bouwknegt, J. McCarthy and K. Pilch had independently found the ground ring in the $\hat{c} \leq 1$ fermionic system.

It is a pleasure to thank T. Banks, M. Douglas, B. Lian, G. Moore, A. Polyakov, S. Shenker, C. Vafa, H. Verlinde, E. Witten, A.B. Zamolodchikov and G. Zuckerman for useful discussions. This work was supported in part by DOE grants DE-FG05-90ER40559, DE-AC02-76ER-03072 and DE-AC02-80ER-10587 and an NSF Presidential Young Investigator Award.
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