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Unphysical energy sheets and resonances in the Friedrichs-Faddeev model

Abstract We consider the Friedrichs-Faddeev model in the case where the kernel of the potential operator is holomorphic in both arguments on a certain domain of \( \mathbb{C} \). For this model we, first, study the structure of the \( T \)- and \( S \)-matrices on unphysical energy sheet(s). To this end, we derive representations that explicitly express them in terms of these same operators considered exclusively on the physical sheet. Furthermore, we allow the Friedrichs-Faddeev Hamiltonian undergo a complex deformation (or even a complex scaling/rotation if the model is associated with an infinite interval). Isolated non-real eigenvalues of the deformed Hamiltonian are called the deformation resonances. For a class of perturbation potentials with analytic kernels, we prove that the deformation resonances do correspond to the scattering matrix resonances, that is, they represent the poles of the scattering matrix analytically continued to the respective unphysical energy sheet.

Keywords Friedrichs-Faddeev model · Unphysical sheets · Resonances · Complex deformation

1 Introduction

In 1938, Kurt Friedrichs \cite{Friedrichs38} considered a model Hamiltonian of the form

\[
H_{\varepsilon} = H_0 + \varepsilon V
\]

with \( H_0 \), the multiplication by the independent variable \( \lambda \),

\[
(H_0 f)(\lambda) = \lambda f(\lambda), \quad \lambda \in (-1, 1) \subset \mathbb{R}, \quad f \in L_2(-1, 1),
\]

\( \varepsilon > 0 \), and \( V \), an integral operator,

\[
(V f)(\lambda) = \int_{-1}^{1} V(\lambda, \mu) f(\mu) d\mu,
\]

where the kernel \( V(\lambda, \mu) \) is a continuous function in \( \lambda, \mu \in [a, b] \) of a Hölder class. Furthermore, he assumed that \( V(-1, \mu) = V(1, \mu) = V(\lambda, -1) = V(\lambda, 1) = 0 \) for any \( \lambda, \mu \in [-1, 1] \).

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The Hermitian (self-adjoint) operator $H_0$ has absolutely continuous spectrum that fills the segment $[-1, 1]$. Friedrichs studied what happens to the continuous spectrum of $H_0$ under the perturbation $\varepsilon V$. He has proven that if $\varepsilon$ is sufficiently small then $H_\varepsilon$ and $H_0$ are similar, which means that the spectrum of $H_\varepsilon$ is also absolutely continuous and fills $[-1, 1]$.

In a 1948 paper [2], Friedrichs has extended this result to the case where the unperturbed Hamiltonian $H_0$ is the multiplication by independent variable in the Hilbert space $H = L_2(\Delta, h)$ of square-integrable vector-valued functions $f : \Delta \to h$ where $\Delta$ is a finite or infinite interval on the real axis, $\Delta = (a, b)$, with $-\infty \leq a < b \leq +\infty$, and $h$ is an auxiliary Hilbert space (finite- or infinite-dimensional). In this case, it is assumed that for every $\lambda, \mu \in \Delta$ the quantity $V(\lambda, \mu)$ is a bounded linear operator on $h$, that $V(\lambda, \mu) = V(\mu, \lambda)^*$, and that $V$ is a Hölder continuous operator-valued function of $\lambda, \mu$. Friedrichs proved that for sufficiently small $\varepsilon$ the perturbed operator $H_\varepsilon = H_0 + \varepsilon V$ is unitarily equivalent to the unperturbed one, $H_0$, and thus the spectrum of $H_\varepsilon$ is absolutely continuous and fills the set $\Lambda$.

In 1958, O.A. Ladyzhenskaya and L.D. Faddeev [3] have completely dropped the smallness requirement on $V$ and considered the model operator

$$H = H_0 + V,$$

$$\begin{align*}
(H_0 f)(\lambda) &= \lambda f(\lambda), \\
(V f)(\lambda) &= \int_\Delta V(\lambda, \mu) f(\mu) d\mu,
\end{align*}$$

with NO small $\varepsilon$ in front of $V$. Instead, they require compactness of the value of $V(\lambda, \mu)$ as an operator in $h$ for any $\lambda, \mu \in \Lambda$.

Proofs of the results in [3] (and their extension) are given in a Faddeev’s 1964 work [4]. In fact, this work presents a complete version of the scattering theory for the model (1.1). The 1964 paper may also be viewed as a relatively simple introduction to the methods and ideas Faddeev used in his celebrated analysis [5] of the three-body problem.

Faddeev’s detail study of the Hamiltonian (1.1) is the first reason why this Hamiltonian is sometimes called the Friedrichs-Faddeev model. One more reason is related to the fact that the 1948 Friedrichs’ paper contains another important $(2 \times 2$ block matrix) operator model that is called “simply” Friedrichs’ model. The second model works, in particular, for the Feshbach resonances (see, e.g., [6], [7] and references therein). For later developments concerning the Friedrichs-Faddeev model proper and its generalizations see [8], [9], [10], [11], [12].

Notice that the typical two-body Schrödinger operator may be viewed as a particular case of the Friedrichs-Faddeev model with $a = 0$ and $b = +\infty$ (see [4]). Simply consider the c.m. two-body Hamiltonian in the momentum $(k)$ space and make the variable change $|k|^2 \to \lambda$; in this case the internal (auxiliary) space is $h = L_2(S^2)$, i.e. the space of square-integrable functions on the unit sphere $S^2 \subset \mathbb{R}^3$.

It turned out that there is a certain gap in the study of analytical properties and structure of the Friedrichs-Faddeev $T$- and $S$-matrices on uphysical sheets of the energy plane. We fill this gap by using the ideas and approach from the author’s works [13], [14]. Namely, we derive representations for the $T$- and $S$-matrices on uphysical sheets that explicitly express them in terms of these same operators considered exclusively on the physical sheet (see Proposition [2] and Corollary [1]). These representations show that the resonances correspond, in fact, to the energies $z$ in the physical sheet where the scattering matrix has eigenvalue zero.

Furthermore, we perform a complex deformation (a generalization of the complex scaling) of the Friedrichs-Faddeev Hamiltonian. Discrete spectrum of the complexly deformed Hamiltonian contains the “complex scaling resonances”. We show that these resonances are simultaneously the scattering matrix resonances. In the case of the Friedrichs-Faddeev model this is done quite easily and illustratively. Recall that, in general, to prove the equivalence of scaling resonances and scattering matrix resonances is rather a hard job (see [13]).

Few words about notation used throughout the article. By $\sigma(T)$ we denote the spectrum of a closed linear operator $T$. Notations $\sigma_p(T)$ and $\sigma_c(T)$ are used for the point spectrum (the eigenvalue set) and continuous spectrum of $T$. By $I_K$ we denote the identity operator on a Hilbert (or Banach) space $K$; the index $K$ may be omitted if no confusion arises.
2 Structure of the $T$- and $S$-matrices on unphysical energy sheets

First, let us recollect the description of the Friedrichs-Faddeev model. We assume that $\mathfrak{h}$ is an auxiliary (“internal”) Hilbert space and $\Delta = (a, b)$, an interval on $\mathbb{R}$, 

$$-\infty < a < b < +\infty.$$ 

Hilbert space of the problem is the space of square-integrable $\mathfrak{h}$-valued functions on $\Delta$, $\mathcal{H} = L_2(\Delta, \mathfrak{h})$, with the scalar product 

$$\langle f, g \rangle = \int_a^b d\lambda \langle f(\lambda), g(\lambda) \rangle_{\mathfrak{h}},$$ 

where $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ denotes the inner product in $\mathfrak{h}$. Surely, the norm on $\mathcal{H}$ is given by 

$$\|f\| = \left( \int_a^b d\lambda \|f(\lambda)\|^2_{\mathfrak{h}} \right)^{1/2},$$ 

where $\| \cdot \|_{\mathfrak{h}}$ stands for the norm on $\mathfrak{h}$.

Unperturbed Hamiltonian $H_0$ and perturbation potential $V$ are given by (1.1) for each $\lambda, \mu \in (a, b)$ the value of $V(\lambda, \mu)$ is a compact operator in $\mathfrak{h}$. We assume that the function $V(\lambda, \mu)$ admits analytic continuation both in $\lambda$ and $\mu$ into some domain $\Omega \subset \mathbb{C}$ containing $\Delta$. More precisely, we assume that 

$$V(\lambda, \mu) \quad \text{is holomorphic in both} \quad \lambda, \mu \in \Omega, \quad (a, b) \subset \Omega. \quad (2.1)$$

In addition, we suppose that $V(\lambda, \mu) = V(\mu, \lambda)^*$ for real $\lambda, \mu \in \Delta$ (for Hermiticity of $V$). This automatically implies $V(\lambda, \mu) = V(\mu^*, \lambda^*)^*$ for any $\lambda, \mu \in \Omega$ such that their conjugates $\lambda^*, \mu^* \in \Omega$ and, hence, the domain $\Omega$ should be mirror symmetric with respect to the real axis. Following Friedrichs [2] and Faddeev [4] we also require 

$$V(a, \mu) = V(b, \mu) = V(\lambda, a) = V(\lambda, b) = 0, \quad \text{in case of finite} a \text{ or} b \quad (2.2)$$
or impose suitable requirements on the rate of decreasing of $V(\lambda, \mu)$ as $|\lambda|, |\mu| \to \infty$, in case of infinite $a$ or $b$. To simplify consideration, in the latter case we assume that $\Omega$ and $V$ are such that 

$$\|V(\lambda, \mu)\| \leq K(1 + |\lambda| + |\mu|)^{-(1+\eta_1)}, \quad \eta_1 > 0; \quad (2.3)$$

$$\|V(\lambda + \alpha, \mu + \beta) - V(\lambda, \mu)\| \leq K(1 + |\lambda| + |\mu|)^{-(1+\eta_1)}(|\alpha|^{\eta_2} + |\beta|^{\eta_2}), \quad \eta_2 > 1/2, \quad (2.4)$$

for some $K > 0$ and for any $\lambda, \mu \in \Omega$ and any $\alpha, \beta$ such that $\lambda + \alpha \in \Omega$, $\mu + \beta \in \Omega$. Since $V(\lambda, \mu)$ is holomorphic in $\Omega$ both in $\lambda$ and $\mu$, the requirement (2.2) with $\eta_2 < 1$ is responsible for the behavior of $V(\lambda, \mu)$ in the neighborhoods of the (finite) end points $a$ and/or $b$. Otherwise, one can replace $\eta_2$ with unity.

As usually, the total Hamiltonian is $H = H_0 + V$. Also we use the standard notation for the resolvents and for the transition operator: For $z$ lying outside the corresponding spectrum $\sigma(H_0)$ or $\sigma(H)$, we introduce 

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H - z)^{-1}, \quad \text{and} \quad T(z) = V - VR(z)V.$$ 

Recall that, at least, for $z \notin \sigma(H_0) \cup \sigma(H)$ 

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z). \quad (2.5)$$

Thus, the spectral problem for the Hamiltonian $H$ is reduced to the study of the transition operator $T(z)$, the kernel of which is less singular than that of $R(z)$.

From Faddeev’s work [4] we know that $T(\lambda, \mu, z)$ is well-behaved function of $\lambda, \mu \in \Delta$ and $z$ on the complex plane $\mathbb{C}$ punctured at $\sigma_p(H)$ and cut along $(a, b)$. More precisely (see [4] Theorem 3.1]), $T(\lambda, \mu, z)$ is of the same class (2.3), (2.4) as $V(\lambda, \mu)$ but with $\eta_1$ and $\eta_2$ replaced by positive $\eta_1' < \eta_1$ and $\eta_2' < \eta_2$ which may be chosen arbitrary close to $\eta_1$ and $\eta_2$, respectively. Furthermore, the kernel $T(\lambda, \mu, z)$ has limits 

$$T(\lambda, \mu, E \pm i0), \quad E \in \Delta \setminus \sigma_p(H),$$

that are (in our case) smooth in $\lambda, \mu \in \Delta \setminus \sigma_p(H)$. The scattering matrix for the pair $(H_0, H)$ is given by 

$$S_+(E) = I_\mathfrak{h} - 2\pi iT(E, E, E + i0), \quad E \in (a, b) \setminus \sigma_p(H).$$
Notice that due to the condition (2.2) and requirement (2.4) the eigenvalue set $\sigma_p(H)$ of $H$ consists of finite number of eigenvalues having finite multiplicities (see [4]; cf. [11]).

Now take a look of the Lippmann-Schwinger equations for the kernel $T(\lambda, \mu, z)$ of the transition operator $T(z)$:

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_a^b d\nu \frac{V(\lambda, v)T(v, \mu, z)}{v - z}, \quad (2.6)$$

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_a^b d\nu \frac{T(\lambda, v, z)V(v, \mu)}{v - z}, \quad (2.7)$$

Clearly, since $V(\lambda, \mu)$ is analytic in $\lambda \in \Omega$, equality (2.6) implies the same analyticity of $T(\lambda, \mu, z)$. Analogously, equality (2.7) yields the holomorphy of $T(\lambda, \mu, z)$ in $\mu \subset \Omega$. Thus we arrive at the following statement.

**Proposition 1** If $z \notin (a,b) \cup \sigma_p(H)$, the kernel $T(\lambda, \mu, z)$ is holomorphic in both $\lambda \in \Omega$ and $\mu \in \Omega$. Furthermore, one can replace $(a,b)$ in (2.6) and (2.7) by arbitrary piecewise smooth Jordan contour $\gamma \subset \Omega$ obtained by continuous deformation from $(a,b)$ provided that the end points are fixed and the point $z$ is avoided during the transformation $(a,b) \rightarrow \gamma$.

In the following $\mathbb{C}^+ = \{ z \in \mathbb{C} | \text{Im} z > 0 \}$ (and $\mathbb{C}^- = \{ z \in \mathbb{C} | \text{Im} z < 0 \}$) stands for the upper (lower) half-plane of $\mathbb{C}$.

![Fig. 1 Holomorphy domain $\Omega$ for the kernel $V(\lambda, \mu)$. The set $\Omega_\gamma$ is bounded by (and contains both) the segment $[a,b]$ and Jordan contour $\gamma$.](image)

To simplify the presentation, in the remainder of this note we usually assume that the real numbers $a$ and $b$ are finite.

Suppose that $\gamma \subset \Omega \cap \mathbb{C}^+$ is a smooth Jordan contour obtained from the interval $(a,b)$ by continuous transformation with fixed end points $a$ and $b$. Then Proposition 1 implies that one can equivalently rewrite (2.6) as

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_\gamma d\nu \frac{V(\lambda, v)T(v, \mu, z)}{v - z}, \quad (2.8)$$

where the set $\Omega_\gamma \subset \mathbb{C}$ is confined by (and containing) the segment $[a,b]$ and the curve $\gamma$ (see Figure 1). One may almost literally repeat for (2.8) the analysis of the Lippmann-Schwinger equation (2.6) performed by Faddeev in [4]. By applying to (2.8) the analytic Fredholm theorem [15] Theorem VI.14] one then concludes that the solution $T(\lambda, \mu, z)$, in the appropriate class of Hölder continuous kernels, exists (and is unique) except for a discrete set of points that consists of the original point spectrum $\sigma_p(H)$ of $H$ and an additional discrete set $\sigma_{res}(\gamma)$ located inside $\Omega_\gamma$. Moreover, the solution $T(\lambda, \mu, z)$ to (2.8) is analytic in $z$ for

$$z \notin \sigma_p(H) \cup \gamma \cup \sigma_{res}(\gamma), \quad (2.9)$$
where the overlining in $\gamma$ means the closure, that is, $\gamma = \gamma \cup \{a\} \cup \{b\}$. Again, because of the holomorphy of $V(\lambda, \mu)$ in $\lambda, \mu \in \Omega$ the solution $T'(\lambda, \mu, z)$ remains analytic in $\lambda, \mu \in \Omega$ for any $z \in \mathbb{C}$ satisfying (2.9). This is proven by the same reasoning as in the proof of the first statement of Proposition 1. The points of $\sigma_{\text{res}}(\gamma)$ give to the solution $T(z)$ poles, residues at which are finite rank operators. Thus, the equation (2.3) gives us an opportunity to pull the argument $z$ of $T(z)$ from the upper half-plane $\mathbb{C}^+$ to the lower one, at least into the interior of the set $\Omega_\gamma$. Surely, during this procedure one should avoid the points of the discrete set $\sigma_{\text{res}}(\gamma)!$

However, if, after such a pulling of $z$, one tries to re-establish in (2.3) the original integration over the interval $(a, b)$, it is necessary to compute the residue at the pole $z$. That is, the Lippmann-Schwinger equation (2.3) changes its form and, hence, for $z \in \Omega \cap \mathbb{C}^-$ the solution $T'(\lambda, \mu, z)$ is taken, in fact, on an unphysical sheet of the Riemann energy surface of $T$. We denote this unphysical sheet by $\Pi_z$; it is attached to the physical sheet via the upper rim of the cut along $(a, b)$. Thus, we are forced to use a different notation, say $T'(\lambda, \mu, z)$ for the continuation of the kernel of $T$ to $\Pi_z$. However for $z$ outside $\Omega_\gamma$ this kernel coincides with the original one, that is, $T'(\lambda, \mu, z) = T(\lambda, \mu, z)$, provided $z \in \mathbb{C} \setminus (\Omega_\gamma \cup \sigma_{\text{res}}(H))$.

In fact we can solve the continued Lippmann-Schwinger equation (2.3) explicitly! To this end assume that $z \in \Omega_\gamma$ but $z \notin \gamma \cup \sigma_{\text{res}}(\gamma)$, and perform a reverse two-step transformation of the contour $\gamma$ (see Figure 1) back to the interval $\Delta = (a, b)$ in the way shown in Figure 2.

$\begin{align*}
&\frac{a}{\circ} \quad z \quad b \quad a \quad \circ \quad b
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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By setting $\lambda = z$ in (2.12) we get $T'(z, \mu, z) = T(z, \mu, z) - 2\pi i T(z, z) T'(z, \mu, z)$, which yields

$$S_-(z) T'(z, \mu, z) = T(z, \mu, z), \quad (2.13)$$

where

$$S_-(z) := I_b + 2\pi i T(z, z), \quad z \in \Omega \cap C^-,$$

is nothing but the scattering matrix for $z$ in the lower complex half-plane. We underline that the argument $z$ of $S_-(z)$ in (2.14), $z \in \Omega \cap C^-$, lies on the physical sheet. From (2.13) it follows that

$$T'(z, \mu, z) = S_-(z)^{-1} T(z, \mu, z), \quad (2.15)$$

provided $z$ is not a point where $S_-(z)$ has an eigenvalue zero. Finally, the relations (2.12) and (2.15) yield

$$T'(\lambda, \mu, z) = T(\lambda, \mu, z) - 2\pi i T(\lambda, z) S_-(z)^{-1} T(z, \mu, z). \quad (2.16)$$

All the entries on the r.h.s. part of (2.16) are taken on the physical sheet.

In a similar way we perform the continuation of $T(\lambda, \mu, z)$ from the lower half-plane $C^-$ to the part $\Omega \cap C^+$ of the unphysical energy sheet $\Pi_\ell$, attached to the physical sheet along the lower rim of the cut $(a, b)$. As a result, we arrive with the following combined statement (for both the sheets $\Pi_\ell$ where the symbol $\ell = \pm 1$ is identified with the respective sign $\pm$ in the previous notation $\Pi_\pm$).

**Proposition 2** The transition operator $T(z)$ admits meromorphic continuation (as a $B(\mathbb{H})$-valued function of the variable $z$) through the cut along $(a, b)$ from both the upper, $C^+$, and lower, $C^-$, half-planes to the respective parts

$$\Omega_{-1} := C^- \cap \Omega \quad \text{and} \quad \Omega_{+1} := C^+ \cap \Omega$$

of the unphysical sheets $\Pi_{-1}$ and $\Pi_{+1}$ adjoining the physical sheet along the upper and lower rims of the mentioned cut. The kernel of the continued operator $T(z)|_{\Pi_\ell \cap \Omega_\ell}$, $\ell = \pm 1$, admits the representation

$$T(\lambda, \mu, z)|_{z \in \Pi_\ell \cap \Omega_\ell} = (T(\lambda, \mu, z) + 2\pi i \ell \ T(\lambda, z) S_\ell(z)^{-1} T(z, \mu, z))|_{z \in \Omega_\ell}, \quad (2.17)$$

where all the terms on the r.h.s. part, including the scattering matrix

$$S_\ell(z) = I_b + 2\pi i \ell T(z, z), \quad (2.19)$$

are taken for $z$ on the physical sheet, and $\sigma_{\text{res}}^{\ell}$ denotes the set of all those points $\zeta \in \Omega \cap C'_{\ell}$ for which $S_\ell(\zeta)$ has eigenvalue zero.

**Remark 1** Whether $\Pi_{-1}$ and $\Pi_{+1}$ represent the same (“second”) unphysical sheet, depends on the analytical properties of $V(\lambda, \mu)$ outside $\Omega$ (if available, cf. [13]).

Continuation formula (2.17) for $T(z)$ implies the following important consequence.

**Corollary 1** Analytic continuation of the scattering matrix $S_{-\ell}(z)$, $\ell = \pm 1$, to the unphysical sheet $\Pi_\ell$ is described by the equality

$$S_{-\ell}(z)|_{z \in \Pi_\ell \cap \Omega_\ell} = S_\ell(z)^{-1}|_{z \in \Omega_\ell} \quad \text{if} \quad z \not\in \sigma_{\text{res}}^{\ell}, \quad (2.20)$$

where the r.h.s. part is taken for $z$ on the physical sheet.
3 Complex scaling and Friedrichs-Faddeev model

Let $\Delta_r$ denote the Laplacian in the variable $r \in \mathbb{R}^3$. In the coordinate space, the standard complex scaling \[16, 17\] means the replacement of the original c.m. two-body Hamiltonian
\begin{equation}
H = -\Delta_r + \hat{V}(r)
\end{equation}
by the non-Hermitian operator
\begin{equation}
H(\theta) = -e^{-2i\theta} \Delta_r + \hat{V}(e^{i\theta} r),
\end{equation}
for a non-negative $\theta \leq \pi/2$, provided that the local potential $\hat{V}(r)$ admits analytic continuation to a domain of complex $C^3$-arguments $r$. Location of the spectrum of $H(\theta)$ is shown schematically in Fig. 3.

Having performed the Fourier transform of (3.2) and then making the change $|kkk| \rightarrow \lambda$ one arrives at the complex version of the Friedrichs-Faddeev model
\begin{equation}
(H(\theta)f)(\lambda) = e^{-2i\theta} \lambda f(\lambda) + e^{-2i\theta} \int_0^\infty V(e^{-2i\theta} \lambda, e^{-2i\theta} \mu) f(\mu) d\mu,
\end{equation}
for every admissible $\lambda, \mu \in \mathbb{C}$ the value of $V(\lambda, \mu)$ is an operator (typically, compact) in $h = L_2(S^2)$.

Surely, in the two-body problem case, one has to assume that $V(\lambda, \mu)$ is analytic in both $\lambda$ and $\mu$ on some domain $\Omega \subset \mathbb{C}$ containing the positive semiaxis $\mathbb{R}^+$. In addition, $\|V(\lambda, \mu)\|$ should decrease sufficiently rapidly as $|\lambda| \rightarrow \infty$ and/or $|\mu| \rightarrow \infty$ (in order to ensure that the integral on the right-hand side of (3.4) defines a reasonable operator in $L_2(\gamma, L_2(S^2))$).

4 Equivalence of the complex rotation resonances and scattering resonances in the Friedrichs-Faddeev model

In the following we consider a family of the Friedrichs-Faddeev Hamiltonians
\begin{equation}
H_{\gamma} = H_0 + V_{\gamma}
\end{equation}
associated with smooth Jordan curves $\gamma \subset \Omega$ originating in $(a, b)$. As before, notation $\Omega$ is used for the holomorphy domain of $V(\lambda, \mu)$ in the variables $\lambda$ and $\mu$; $\Omega$ may or may not include $a$ and/or $b$;

$$(H_{0, \gamma} f)(\lambda) = \lambda f(\lambda) \quad \text{and} \quad (V_{\gamma} f)(\lambda) = \int_{\gamma} V(\lambda, \mu) f(\mu) d\mu, \quad \lambda \in \gamma.$$ 

Here, $f$ is taken from the the Hilbert space $\mathcal{H}_\gamma = L_2(\gamma, \mathfrak{h})$ by which one understands the space of $\mathfrak{h}$-valued functions of the variable $\lambda \in \gamma$ with the scalar product

$$\langle f, g \rangle_\gamma = \int_{\gamma} d\lambda |\langle f(\lambda), g(\lambda) \rangle|_\mathfrak{h}.$$ 

Again assume that both $a$ and $b$ are finite and let $V(\lambda, \mu)$ be as in Section 2. Introduce the transition operator for the pair $(H_{0, \gamma}, H_\gamma)$:

$$T_\gamma(z) = V_\gamma - V_\gamma (H_\gamma - z)^{-1} V_\gamma, \quad z \notin \sigma(H_\gamma). \quad (4.1)$$

For $R_\gamma(z) = (H_\gamma - z)^{-1}$ we have

$$R_\gamma(z) = R_{0, \gamma}(z) - R_{0, \gamma}(z) T_\gamma(z) R_{0, \gamma}(z), \quad (4.2)$$

with $R_{0, \gamma}(z) = (H_{0, \gamma} - z)^{-1}, z \notin \sigma(H_{0, \gamma}).$

Notice that $H_{0, \gamma}$ is a normal operator on $\mathcal{H}_\gamma$ and it has only absolutely continuous spectrum that coincides with the curve $\gamma$. Thus, from (4.2) it follows that the discrete eigenvalues of $H_\gamma$ are nothing but the poles of the operator-valued function $T_\gamma(z)$.

Assume that the above Jordan curve $\gamma$ lies completely in $\Omega_- = \Omega \cap \mathbb{C}^-$ (or completely in $\Omega_+ = \Omega \cap \mathbb{C}^+$). Let again $\Omega_\gamma$ denote the set in the complex plane $\mathbb{C}$ confined by (and containing) the interval $[a, b]$ and the contour $\gamma$.

**Proposition 3** The part of the spectrum of $H_\gamma$ lying outside $\Omega_\gamma$ is purely real and coincides with $\sigma_p(H) \setminus \Delta$. Furthermore, $\sigma_p(H_\gamma) \cap \Delta = \sigma_p(H) \cap \Delta$, that is, the point spectrum eigenvalues of $H_\gamma$ lying on $\Delta$ do not depend on the (smooth) Jordan contour $\gamma$. The spectrum of $H_\gamma$ inside $\Omega_\gamma$ represents the scattering-matrix resonances.

**Proof** Already from (4.1) one may conclude that, for any fixed $z \notin \sigma(H_\gamma)$, the kernel $T_\gamma(\lambda, \mu, z)$ is holomorphic in the variables $\lambda, \mu \in \Omega$ (since $V$ is holomorphic). Indeed, (4.1) means

$$T_\gamma(\lambda, \mu, z) = V(\lambda, \mu) + \int_{\gamma} d\mu' \int_{\gamma} d\lambda' V(\lambda, \mu') R_\gamma(\mu', \lambda', z) V(\lambda', \mu).$$

One may pull $\lambda$ and $\mu$ anywhere in $\Omega$. And this will be true after analytic continuation of $R_\gamma(\mu', \lambda', z)$ in $z$ through $\gamma$.

Now look at the Lippmann-Schwinger equation for $T_\gamma$,

$$T_\gamma(\lambda, \mu, z) = V(\lambda, \mu) - \int_{\gamma} d\nu \frac{V(\lambda, \nu) T_\gamma(\nu, \mu, z)}{V - z}, \quad z \notin \gamma, \quad \lambda, \mu \in \gamma. \quad (4.3)$$

Assume that $z \in \mathbb{C} \setminus \Omega_\gamma$ and consider for such a $z$ also the Lippmann-Schwinger equation for the “original” $T$-matrix — it is associated with the interval $\Delta$:

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_{a}^{b} d\nu \frac{V(\lambda, \nu) T(\nu, \mu, z)}{V - z}, \quad z \notin \Omega_\gamma, \quad \lambda, \mu \in (a, b). \quad (4.4)$$

Since both the kernels $V(\lambda, \cdot, z)$ and $T(\lambda, \cdot, z)$ for fixed $z \notin \Omega_\gamma(\cup \sigma_p(H))$ are holomorphic in $\lambda \in \Omega$, one may transform the interval $[a, b]$ into the contour $\gamma$ and obtain:

$$T(\lambda, \mu, z) = V(\lambda, \mu) - \int_{\gamma} d\nu \frac{V(\lambda, \nu) T(\nu, \mu, z)}{V - z}, \quad z \notin \Omega_\gamma, \quad \lambda, \mu \in (a, b). \quad (4.5)$$

Pull $\lambda, \mu$ on $\gamma$ and then compare (4.3) and (4.5). The uniqueness theorem for the solution to (4.5) implies:

$$T_\gamma(\lambda, \mu, z) = T(\lambda, \mu, z) \quad \text{whenever} \quad \lambda, \mu \in \gamma, z \in \Omega \setminus \Omega_\gamma \text{and} z \notin \sigma_p(H).$$
Fig. 4 The closed set $\Omega_{12}$ bounded by the Jordan contours $\gamma_1$ and $\gamma_2$.

Similarly, we have

$$T_{\gamma}(\lambda, \mu, z) = T_{\gamma}(\lambda, \mu, z) \quad \text{whenever } \lambda, \mu \in \Omega \text{ and } z \notin \Omega_{12} \cup \sigma_p(H_{\gamma}),$$

where $\Omega_{12}$ is the closed set bounded by the curves $\gamma_1$ and $\gamma_2$ (see Figure 4). This also means that

$$\sigma_p(H_{\gamma}) \setminus \Omega_{12} = \sigma_p(H_{\gamma}) \setminus \Omega_{12}.$$

Finally, by the uniqueness principle for analytic continuation, for $z$ inside $\Omega_\gamma$ the kernel $T_{\gamma}(\lambda, \mu, z)$ represents just the analytic continuation of $T(\lambda, \mu, \cdot)$ to the interior of $\Omega_\gamma$ lying in the unphysical sheet. Hence, the poles of $T_{\gamma}(z)$ within $\Omega_\gamma$ represent resonances of the original Friedrichs-Faddeev Hamiltonian (the one that was introduced for the interval $(a, b)$). This also means that the positions of the resonances inside $\Omega_\gamma$ do not depend on $\gamma$. The proof is complete.

Conclusion

For the (analytic) Friedrichs-Faddeev model, we have derived representations that explicitly express the transition operator and scattering matrix on unphysical energy sheets in terms of these same operators considered exclusively on the physical sheet. A resonance on a sheet $\Pi_\ell$, $\ell = \pm 1$, or, more precisely, in the domain $\Pi_\ell \cap \Omega_{12}$ is just a point, for the copy $z$ of which on the physical sheet the corresponding scattering matrix $S_\ell(z)$ has eigenvalue zero, that is,

$$S_\ell(z)\mathcal{A} = 0 \quad \text{for a non-zero vector } \mathcal{A} \in \mathfrak{a}.$$ (4.6)

Furthermore, we have shown that, for the Friedrichs-Faddeev model, the deformation resonances are exactly the scattering matrix resonances.

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