Regularity properties in a state-constrained expected utility maximization problem

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Abstract We consider a stochastic optimal control problem in a market model with temporary and permanent price impact, which is related to an expected utility maximization problem under finite fuel constraint. We establish the initial condition fulfilled by the corresponding value function and show its first regularity property. Moreover, we can prove the existence and uniqueness of an optimal strategy under rather mild model assumptions. This will then allow us to derive further regularity properties of the corresponding value function, in particular its continuity and partial differentiability. As a consequence of the continuity of the value function, we will prove a dynamic programming principle without appealing to the classical measurable selection arguments. This permits us to establish a tight relation between our value function and a nonlinear parabolic degenerated Hamilton–Jacobi–Bellman (HJB) equation with singularity. To conclude, we show a comparison principle, which allows us to characterize our value function as the unique viscosity solution of the HJB equation.

Keywords Expected utility maximization problem · Value function · Price impact · Optimal strategy · Dynamic programming principle · Bellman’s principle · Hamilton–Jacobi–Bellman equation · Viscosity solution · Comparison principle

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1 Introduction

The purpose of this paper is to investigate the connection between the value function associated to an expected utility maximization problem with finite fuel constraint, i.e., where initial and terminal conditions are imposed on processes we consider, which originates from a portfolio liquidation problem, and solutions of a Hamilton–Jacobi–Bellman (HJB) equation with singularity. Our particular focus will be on utility functions with bounded Arrow–Pratt coefficient of absolute risk aversion. We show the existence and uniqueness of the corresponding optimal strategy, which is no longer deterministic in this general setting. This result then helps us to derive regularity properties of the associated value function. With this at hand, we will see that the value function and the corresponding optimal strategy are tied up with the solution of a certain stochastic differential equation (SDE), which might be useful for numerical purposes (see Remark 4.6). Finally, we will show that the value function in the more general case, where no smoothness assumptions are imposed, can be regarded as the (unique) viscosity solution of the corresponding HJB-equation.

A dynamic execution strategy that minimizes expected cost was first derived in Bertsimas and Lo (1998). However, as illustrated, for instance, by the 2008 Société Générale trading loss, we have to add to execution costs the volatility risk incurred when trading. This extension and the corresponding mean–variance maximization problem was treated in Almgren and Chriss (2001), in a discrete-time framework, where the execution costs are assumed to be linear and are split into a temporary and a permanent price impact component. Nevertheless, linear execution costs do not seem to be a realistic assumption in practice, as argued in Almgren (2003), and it may be reasonable to consider a nonlinear temporary impact function. As opposed to the temporary impact, the permanent impact has to be linear in order to avoid quasi-arbitrage opportunities, as shown in Huberman and Stanzl (2004). The mean–variance approach can also be regarded as an expected-utility maximization problem for an investor with constant absolute risk aversion, which was in part solved by Schied et al. (2010), where the existence and uniqueness of an optimal trading strategy, which is moreover deterministic, is proved. The latter one can be computed by solving a nonlinear Hamilton equation. Furthermore, the corresponding value function is the unique classical solution of a nonlinear degenerated Hamilton–Jacobi–Bellman equation with singular initial condition.

In this paper, we generalize this framework by considering utility functions that lie between two exponential utility functions (which are also called CARA utility functions). This case was already studied for infinite-time horizons in a one-dimensional framework with linear temporary impact without drift; see Schied and Schöneborn (2009), as well as Schöneborn (2008), where the optimal trading strategy is characterized as the unique bounded solution of a classical fully nonlinear parabolic equation. It was shown that the optimal liquidation strategy is Markovian and a feedback form was given. Moreover, the optimal strategy is deterministic if and only if the utility function is an exponential function. The derivation of the above results is due to the fact that, when considering infinite time horizon, the (transformed) optimal strategy solves a classical parabolic PDE, because the time parameter does not appear in the equation. In this article, we address the question of deriving the optimal liquidation
strategy for the finite-time horizon. Here, we face the difficulty that commonly used change of measure techniques, involving the Doléans-Dade exponential, simply go out the window. Due to this failure, we have to think differently and to extend our consideration to solutions that are no longer classical ones.

Our first main result deals with the existence and uniqueness of the optimal strategy. The proof of this result is mainly an analytical one and only requires the boundedness of the Arrow–Pratt coefficient of risk aversion of the utility function. As a direct consequence of this theorem, we can show that the associated value function is continuous and also continuously differentiable in its revenues parameter (and even twice continuously differentiable if the utility function is supposed to have a convex and decreasing derivative; this condition is fulfilled if, e.g., the utility function is a convex combination of exponential utility functions or, more generally, if \((-u)\) is a complete monotone function). With this at hand, we will state a so-called dynamic programming principle (also known as Bellmann Principle). In its proof we face measurability issues, and we have to restrict ourselves to considering the Wiener space to make matters clearer. This will be carried out without referring to measurable selection arguments, typically used in proofs of the dynamic programming principle where no a priori regularity of the value function is known to hold; see, e.g., Meyer (1966) or Wagner (1979), Rieder (1978). Note that in most of the literature where the Bellman principle is related to stochastic control problems, its (rigorous) proof is simply omitted, or the reader is referred to the above literature. When the value function is supposed to be continuous, an easier version of its proof can be found in Krylov (2009) or Bertsekas and Shreve (1978): this is however not directly applicable in our context, since we have to deal, among others, with our finite fuel constraint. Further, we will show a relation between the optimal strategy, the value function, and the solution of an SDE, which might be useful for numerical computations. Ideas of the proofs are classical, however there are some issues that make it impossible for us to follow straightforwardly the classical ideas. The boundary conditions imposed in our strategies, the singularity in our initial condition, a quotient term in our SDE, as well as the exponential growth of our expected utilities require further techniques to complete the proofs. Our second main result deals with the value function in the more general case, where no smoothness assumptions are required. We will see that the value function is not only a viscosity solution of the HJB equation, but also the unique one, by using a comparison principle. This comparison principle will be proved without the use of the Crandall–Ishii lemma, only by applying Taylor expansion on some test functions. It is worth mentioning that the continuity of the value function established previously will enable us to overcome some difficulties we will face.

After setting up our framework in Sect. 2.1 and making clearer our definition of utility functions with exponential growth, we prove the concavity property and the initial condition fulfilled by the value function (Sect. 2.2). Our main results on the existence and uniqueness of an optimal strategy is given in Theorem 2.4. The derivation of both results is split into several technical steps (see Sect. 2.3). With this at hand, we start in Sect. 3 by deriving the differentiability property of the value function in the revenues parameter (Theorem 3.3). The relatively involved proof of the continuity property (stated in Theorem 3.11) will also follow from Theorem 2.4. Using the continuity property of the value function, we conclude by establishing the underlying
Bellman principle (Theorem 3.12). In Sect. 4, we derive the HJB equation satisfied by
the value function when the value function is smooth enough. In Sect. 4.1 we state a
verification theorem (Theorem 4.5), allowing us to infer that the value function is the
unique classical solution of the HJB equation in this case (under some conditions).
Dropping the smoothness assumption, we turn to viscosity solutions in Sects. 4.2
and 4.3. Theorem 4.10 establishes that the value function is a viscosity solution of the
HJB equation, and Theorem 4.17 establishes a strong comparison principle without
appealing to the well-known Crandall–Ishii’s lemma. To simplify matters, most of the
proofs of this work are to be found in the appendix section.

2 Introducing the value function and proving its first main properties

2.1 Modeling framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual
conditions. Taking $X_0 \in \mathbb{R}^d$, we consider a stochastic process $X_t = (X^1_t, \ldots, X^d_t)$
starting in $X_0$ at time $t = 0$ that has to fulfill the boundary condition $X_T = 0$. For
example, we can think of a basket of shares in $d$ risky assets an investor can choose to
liquidate a large market order, where we describe by $X^i_t$ the number of shares of the
$i$-th asset held at time $t$. Following the notation in Schied and Schöneborn (2008), we
denote by

$$R^X_T = R_0 + \int_0^T X^\top_t \sigma \, dB_t + \int_0^T b \cdot X_t \, dt - \int_0^T f(\dot{X}_t) \, dt$$

(2.1)

the revenues over the time interval $[0, T]$ associated to the process $X$. Here $R_0 \in \mathbb{R}$,
$B$ is a standard $m$-dimensional Brownian motion starting in 0 with drift $b \in \mathbb{R}^d$ and
volatility matrix $\sigma = (\sigma^{ij}) \in \mathbb{R}^{d \times m}$, and the nonnegative, strictly convex function $f$
has superlinear growth and satisfies the two conditions

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty \quad \text{and} \quad f(0) = 0.$$ 

Further, we assume that the drift vector $b$ is orthogonal to the kernel of the covariance
matrix $\Sigma = \sigma \sigma^\top$, which guarantees that there are no arbitrage opportunities
for a ‘small investor’ whose trades do not move asset prices. The revenues processes
can be interpreted economically: $R_0$ can be viewed as the face value of the portfolio
(which can include a permanent price impact component), the stochastic integral models
the accumulated volatility risk, whereas the second integral represents the linear
drift applied to our state process. The last term stands for the cumulative cost of the
temporary price impact. Further, by

$$\mathcal{X}_{\text{det}}(T, X_0) = \left\{ X : [0, T] \to \mathbb{R}^d \text{ absolutely continuous, } X_0 \in \mathbb{R}^d \text{ and } X_T = 0 \right\}$$
we denote the set of the deterministic processes whose speed liquidation processes $\dot{X}_t$ are defined $\lambda$-a.e., where $\lambda$ is the Lebesgue-measure on $[0, T]$. Analogously, by

$$\mathcal{X}(T, X_0) := \{(X_t)_{t \in [0, T]} \text{ adapted, } t \mapsto X_t \in \mathcal{X}_{det}(T, X_0), \text{ a.s., and } \sup_{0 \leq t \leq T} |X_t| \in L^\infty(\mathbb{P})\}$$

we denote the set of the $\mathbb{P} \otimes \lambda$-a.e. bounded stochastic processes whose speed liquidation processes $\dot{X}_t$ can be defined $\mathbb{P} \otimes \lambda$-a.e., due to absolute continuity.

**Remark 2.1** From a hedging point of view, the absolute continuity of $X$ seems to be very restrictive, since this does not englobe the Black–Scholes Delta hedging, for example. However, from a mathematical point of view, this serves as a reasonable starting point for developing a theory of optimal control problems for functions with bounded variation.

It will be convenient to parametrize elements in $\mathcal{X}(T, X_0)$ as in Schied and Schöneborn (2008). Toward this end, for $\xi$ progressively measurable and $\xi_t$ with values in $\mathbb{R}^d$, for $t \leq T$, let us denote by

$$\dot{X}_0(T, X_0) = \{\xi \mid X_t = X_0 - \int_0^t \xi_s \, ds \text{ a.s. for } X \in \mathcal{X}(T, X_0)\}$$

the set of control processes or speed processes of a given process $X$. From now on we will write $\mathcal{R}^\xi$ for the revenues process associated to a given $\xi \in \dot{X}_0(T, X_0)$, to insist on the dependence on $\xi$. The pair $(X^\xi, \mathcal{R}^\xi)$ is then the solution of the following controlled stochastic differential equation:

$$\begin{cases} 
\dot{R}_t^\xi = X_t^\top \sigma dB_t + b \cdot X_t \, dt - f(-\xi_t) \, dt, \\
\dot{X}_t = -\xi_t \, dt, \\
R^\xi_{t=0} = R_0 \text{ and } X_{t=0} = X_0.
\end{cases} \quad (2.2)$$

We denote by $\dot{\mathcal{X}}(T, X_0)$ the subset of all control processes $\xi \in \dot{X}_0(T, X_0)$ that satisfy the additional requirement

$$\mathbb{E} \left[ \int_0^T (X_t^\xi)^\top \Sigma X_t^\xi + |b \cdot X_t^\xi - f(\xi_t)| + |\xi_t| \, dt \right] < \infty. \quad (2.3)$$

For convenience, we enlarge the preceding set $\dot{\mathcal{X}}(T, X_0)$ by introducing the notation $\dot{\mathcal{X}}^1(T, X_0)$ for the set of the liquidation strategies whose paths satisfy (2.3), but are not necessarily uniformly bounded:

$$\dot{\mathcal{X}}^1(T, X_0) := \{\xi \mid \left(X_t^\xi := X_0 - \int_0^t \xi_s \, ds\right)_{t \in [0, T]} \text{ adapted, } t \mapsto X_t^\xi(\omega) \in \mathcal{X}_{det}(T, X_0) \mathbb{P}\text{-a.s.}\} \cap \{\xi \mid \mathbb{E} \left[ \int_0^T (X_t^\xi)^\top \sigma X_t^\xi + |b \cdot X_t^\xi - f(\xi_t)| + |\xi_t| \, dt \right] < \infty\},$$
which is clearly a subset of $\mathcal{X}(T, X_0)$. The maximization problem can thus be written in the form

$$\sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}\left[ u\left( R_T^\xi \right) \right]. \quad (2.4)$$

In this paper, we will consider a special class of utility functions. These functions will have a bounded Arrow–Pratt coefficient of absolute risk aversion, i.e., we will suppose that there exist two positive constants $A_i, i = 1, 2$, such that

$$0 < A_1 \leq -\frac{u''(x)}{u'(x)} \leq A_2, \quad \forall x \in \mathbb{R}. \quad (2.5)$$

This inequality implies that we can assume w.l.o.g. that $0 < A_1 < 1 < A_2$, which gives us the following estimates

$$\exp(-A_1 x) \leq u'(x) \leq \exp(-A_2 x) + 1 \quad \text{for} \quad x \in \mathbb{R}. \quad (2.6)$$

and

$$u_1(x) := \frac{1}{A_1} - \exp(-A_1 x) \geq u(x) \geq -\exp(-A_2 x) =: u_2(x). \quad (2.7)$$

From Schied et al. (2010) we know that for exponential utility functions (that is, utility functions of the form $a - b \exp(-c x)$, where $a \in \mathbb{R}$ and $b, c > 0$) there exists a unique deterministic and continuous strategy solving the maximization problem (2.4). Moreover, the corresponding value function, i.e., the value function generated by the exponential expected-utility maximization problem, is the unique continuously differentiable solution of a Hamilton–Jacobi–Bellman equation. We will use this strong result to establish the existence of an optimal control under the condition (2.7). Here, we will study the regularity properties of the following value function:

$$V(T, X_0, R_0) = \sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}\left[ u\left( R_T^\xi \right) \right], \quad (2.8)$$

where the utility function $u$ satisfies (2.7). Note that the corresponding estimates yield the following bounds for our value function

$$\sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}\left[ u_1\left( R_T^\xi \right) \right] \geq \sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}\left[ u\left( R_T^\xi \right) \right] \geq \sup_{\xi \in \mathcal{X}(T, X_0)} \mathbb{E}\left[ u_2\left( R_T^\xi \right) \right], \quad (2.9)$$

whence

$$V_1(T, X_0, R_0) = \mathbb{E}\left[ u_1\left( R_T^{\xi^*_1} \right) \right] \geq V(T, X_0, R_0) \geq \mathbb{E}\left[ u_2\left( R_T^{\xi^*_2} \right) \right] = V_2(T, X_0, R_0), \quad (2.10)$$

where $V_i, i = 1, 2$, denote the corresponding exponential value functions and $\xi^*_i, i = 1, 2$, are the corresponding optimal strategies.
2.2 Concavity property and initial condition satisfied by the value function

The aim of this subsection is to prove that the map

$$(X, R) \mapsto V(T, X, R)$$

is concave, for fixed $T \in [0, \infty[$, and to derive the initial condition satisfied by $V$, where $V$ is the value function of the optimization problem as defined in (2.8). These are fundamental properties of the value function of the considered maximization problem.

We start by proving the following proposition which establishes the first regularity property of the value function: the concavity of the value function in the revenues parameter, with $T, X_0 \in ]0, \infty[ \times \mathbb{R}^d$ being fixed. This will enable us later to prove the differentiability of the value function in the revenues parameter, other parameters being fixed, with the help of the existence of an optimal strategy.

**Proposition 2.2** For fixed $T \in ]0, \infty[$,

$$(X, R) \mapsto V(T, X, R)$$

is a concave function.

**Proof** Toward this end, let $X, \overline{X} \in \mathbb{R}^d$, $R, \overline{R} \in \mathbb{R}$ and $\lambda \in ]0, 1[$. Further, consider the strategies $\xi \in \dot{X}^1(T, X)$ and $\overline{\xi} \in \dot{X}^1(T, \overline{X})$. Note that $\lambda \xi + (1 - \lambda) \overline{\xi} \in \dot{X}(T, \lambda X + (1 - \lambda) \overline{X})$. Let us denote

$$R_T^{\lambda \xi + (1 - \lambda) \overline{\xi}} := \int_0^T \left( X_t^{\lambda \xi + (1 - \lambda) \overline{\xi}} \right)^\top \sigma dB_t$$

$$+ \int_0^T b \cdot X_t^{\lambda \xi + (1 - \lambda) \overline{\xi}} dt - \int_0^T f(-\lambda \xi + (1 - \lambda) \overline{\xi}_t) dt.$$ 

We then have for fixed $\xi, \overline{\xi}$:

$$V(T, \lambda X + (1 - \lambda) \overline{X}, \lambda R + (1 - \lambda) \overline{R})) \geq \mathbb{E}\left[ u(\lambda R + (1 - \lambda) \overline{R} + R_T^{\lambda \xi + (1 - \lambda) \overline{\xi}}) \right]$$

$$\geq \mathbb{E}\left[ u(\lambda R + (1 - \lambda) \overline{R}) + \lambda R_T^{\overline{\xi}} + (1 - \lambda) R_T^{\xi} \right]$$

$$\geq \lambda \mathbb{E}\left[ u(R + R_T^{\overline{\xi}}) \right] + (1 - \lambda) \mathbb{E}\left[ u(R + R_T^{\xi}) \right],$$

where the first inequality is due to the definition of the value function $V$ at $(\lambda X + (1 - \lambda) \overline{X}, \lambda R + (1 - \lambda) \overline{R})$, and the second one follows from the fact that $\xi \mapsto R_T^{\xi}$ is concave (due to the fact that $f$ is convex) and $u$ is increasing. Finally, the third one is due the concavity of $u$. Taking now the supremum over $\xi$ ($\overline{\xi}$ being fixed), we obtain

$$V(T, \lambda X + (1 - \lambda) \overline{X}, \lambda R + (1 - \lambda) \overline{R})) \geq \lambda V(T, X, R) + (1 - \lambda) \mathbb{E}\left[ u(R + R_T^{\overline{\xi}}) \right].$$
Taking the supremum over $\xi$ in the preceding equation, we obtain

$$V(T, \lambda X + (1 - \lambda)\overline{X}, \lambda R + (1 - \lambda)\overline{R}) \geq \lambda V(T, X, R) + (1 - \lambda)V(T, X, \overline{R}),$$

which yields the assertion. \hfill \Box

Further, we establish the initial condition fulfilled by the value function. The intuition behind the singularity in the initial condition is that a strategy that does not lead to a complete liquidation of the portfolio within a given time period is highly penalized (even if holding a nonzero portfolio position at the end of liquidation period, for example by holding a positive position, where the drift is positive in all its components and where stock prices are moving upwards, would lead to a higher expected utility than selling out the position within the remaining period).

**Proposition 2.3** Let $V$ be the value function of the maximization problem (2.8). Then $V$ fulfills the following initial condition

$$V(0, X, R) = \lim_{T \downarrow 0} V(T, X, R) = \begin{cases} u(R), & \text{if } X = 0, \\ -\infty, & \text{otherwise}. \end{cases} \quad (2.11)$$

**Proof** We first note that if $X \neq 0$, then

$$\lim_{T \to 0} V(T, X, R) = -\infty,$$

because $V$ is supposed to lie between two CARA value functions which tend to $-\infty$ as $T$ goes to zero, if $X \neq 0$ (see Schied et al. 2010). Suppose now that $X = 0$. We want to show that

$$\lim_{T \to 0} V(T, 0, R) = u(R).$$

Observe first that

$$V(T, 0, R) \geq \mathbb{E}\left[u\left(\mathcal{R}_{T}^{\xi}\right)\right] = u(R),$$

by choosing the strategy $\xi_t = 0$ for all $t \in [0, T]$. $T > 0$. Since $V$ is increasing in $T$, for fixed $X, R$, the limit $\lim_{T \to 0} V(T, X, R)$ exists, which implies that

$$\lim_{T \to 0} V(T, 0, R) \geq u(R).$$

We now prove the reverse inequality

$$\lim_{T \to 0} V(T, 0, R) \leq u(R). \quad (2.12)$$
Let $\xi$ be a round trip starting from 0 (i.e.: $\xi \in \mathcal{X}^1(T, 0)$). Applying Jensen’s inequality to the concave utility function $u$, we get

$$\mathbb{E}[u(\mathcal{R}_T^\xi)] \leq u\left( R + \mathbb{E}\left[ \int_0^T b \cdot X^\xi_t \ dt - \int_0^T f(-\xi_t) \ dt \right] \right).$$

We have to show now

$$\limsup_{T \downarrow 0} \mathbb{E}\left[ \int_0^T b \cdot X^\xi_t \ dt - \int_0^T f(-\xi_t) \ dt \right] \leq 0. \quad \text{(2.13)}$$

To this end we use the integration by parts formula to infer

$$\mathbb{E}\left[ \int_0^T b \cdot X^\xi_t \ dt - \int_0^T f(-\xi_t) \ dt \right] = \mathbb{E}\left[ \int_0^T tb \cdot \xi_t - f(-\xi_t) \ dt \right] \leq \int_0^T f^*(-bt) \ dt,$$

where $f^*$ designates the Fenchel–Legendre transformation of the convex function $f$. Note that $f^*$ is a finite convex function, due to the assumptions on $f$ (see Theorem 12.2 in Rockafellar 1997), and in particular continuous, so that

$$\int_0^T f^*(-bt) \ dt \xrightarrow{T \downarrow 0} 0,$$

which proves (2.13). Finally, using that $u$ is continuous and nondecreasing, we get

$$\lim_{T \to 0} V(T, 0, R) \leq \liminf_{T \to 0} \sup_{\xi \in \mathcal{X}^1(T, 0)} u\left( R + \mathbb{E}\left[ \int_0^T b \cdot X^\xi_t \ dt - \int_0^T f(-\xi_t) \ dt \right] \right) \leq u(R).$$

\[\square\]

### 2.3 Existence and uniqueness of an optimal strategy

In this section we aim at investigating the existence and uniqueness of an optimal strategy for the maximization problem

$$\sup_{\xi \in \mathcal{X}^1(T, X_0)} \mathbb{E}[u(\mathcal{R}_T^\xi)],$$

where $u$ is strictly concave, increasing and satisfies (2.7). The quantity $\mathcal{R}_T^\xi$ denotes the revenues associated with the liquidation strategy $\xi$ over the time interval $[0, T]$. The next theorem establishes the main result of the current section.
Theorem 2.4 Let \((T, X_0, R_0) \in ]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}\), then there exists a unique optimal strategy \(\xi^* \in X(T, X_0)\) for the maximization problem (2.8), which satisfies

\[
V(T, X_0, R_0) = \sup_{\xi \in X(T, X_0)} \mathbb{E}[u(R^\xi_T)] = \mathbb{E}[u(R^\xi^*_T)].
\] (2.14)

The main idea of the proof is to show that a sequence of strategies \((\xi^n)\) such that the corresponding expected utilities converge from below to the supremum, i.e.,

\[
\mathbb{E}[u(R^\xi^n_T)] \uparrow \sup_{\xi \in X(T, X_0)} \mathbb{E}[u(R^\xi_T)],
\]

lies in a weakly sequentially compact subset of \(X(T, X_0)\), due to the fact that the function \(u\) satisfies the inequalities (2.7). Then we can choose a subsequence that converges weakly to the strategy \(\xi^*\). The uniqueness of the optimal strategy will follow from the strict concavity of the map \(\xi \mapsto -\mathbb{E}[u(R^\xi_T)]\).

Remark 2.5 Note that due to inequality (2.10), we can w.l.o.g suppose that the above sequence verifies

\[
\mathbb{E}\left[\exp(-A_1 R^\xi^n_T)\right] \leq 1 + 1/A_1 - V_2(T, X_0, R_0), \quad \text{for all } n \in \mathbb{N}, \quad (2.15)
\]

where \(V_2\) denotes the following CARA value function:

\[
V_2(T, X_0, R_0) = \sup_{\xi \in X(T, X_0)} \mathbb{E}\left[-\exp\left(-A_2 R^\xi_T\right)\right].
\]

We will split the proof into several steps. First, we will prove a weak compactness property of certain subsets of \(X(T, X_0)\). Let us start by recalling some fundamental functional analysis results. The first one is a classical characterization of convex closed sets (see, e.g., Föllmer and Schied 2011, Theorem A.60).

Theorem 2.6 Suppose that \(E\) is a locally convex space and that \(C\) is a convex subset of \(E\). Then \(C\) is weakly closed if and only if \(C\) is closed with respect to the original topology of \(E\).

Corollary 2.7 Let \(\varphi : E \to ]-\infty; \infty]\) be a lower semi-continuous convex function with respect to the original topology of \(E\). Then \(\varphi\) is lower semi-continuous with respect to the weak topology \(\sigma(E', E)\), where \(E'\) denotes the dual space of \(E\). In particular, if \((x_n)\) converges weakly to \(x\), then

\[
\varphi(x) \leq \lim \inf \varphi(x_n). \quad (2.16)
\]

Proof See, e.g., Brezis (2011). \(\square\)
Corollary 2.8 Let \((S, S, \mu)\) be a measurable space, \(F : \mathbb{R}^d \to \mathbb{R}\) a convex function bounded from below, and \((x_n) \subset L^1((S, S, \mu); \mathbb{R}^d)\). Suppose that \((x_n)\) converges to \(x\), weakly. Then

\[
\int F(x) d\mu \leq \lim \inf \int F(x_n) d\mu.
\]

Further, if we suppose that \(F : \mathbb{R}^d \to \mathbb{R}\) is concave and bounded from above, we have an analogous conclusion, i.e.,

\[
\int F(x) d\mu \geq \lim \sup \int F(x_n) d\mu.
\]

Proof Use the preceding corollary combined with Fatou’s Lemma.  

With this at hand, we can show the following lemma, which will be also useful for us to prove the continuity of the value function.

Lemma 2.9 Let \((X^n_0, T^n) \subset \mathbb{R}^d \times \mathbb{R}\) be a sequence that converges to \((X_0, T)\) and set \(\bar{T} := \sup_n T^n\). Moreover, consider a sequence \((\zeta^n)\) in \(\dot{X}^1(T^n, X^n_0)\) and take a constant \(c > 0\) such that

\[
\mathbb{E} \left[ \int_0^T f(-\zeta^n_t) \, dt \right] \leq c. \tag{2.17}
\]

Suppose that \((\zeta^n)\) converges to \(\zeta\) with respect to the weak topology in

\[
L^1 := L^1 \left( \Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), (\mathbb{P} \otimes \lambda) \right).
\]

Then \(\zeta \in \dot{X}^1(T, X_0)\) and

\[
\mathbb{E} \left[ \int_0^T f(-\zeta_t) \, dt \right] \leq c. \tag{2.18}
\]

We can now prove a weak compactness property of a certain family of subsets of \(\dot{X}^1(T, X_0)\).

Proposition 2.10 For \(c > 0\), let

\[
K_c := \left\{ \xi \in \dot{X}^1(T, X_0) \middle| \mathbb{E} \left[ \int_0^T f(-\xi_t) \, dt \right] \leq c \right\}.
\]

Then \(K_c\) is a weakly sequentially compact subset of

\[
L^1 := L^1 \left( \Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]), (\mathbb{P} \otimes \lambda) ; \mathbb{R}^d \right).
\]

Proof We first prove that \(K_c\) is a closed convex set with respect to the strong topology of \(L^1\).
The convexity of $K_c$ is a direct consequence of the convexity of the map

$$
\xi \mapsto \mathbb{E} \left[ \int_0^T f(-\xi_t) \, dt \right].
$$

To show that $K_c$ is closed, let $\xi^n$ be a sequence in $K_c$ that converges strongly to $\xi$. Then, in particular, $\xi^n$ converges to $\xi$ weakly and we are in the setting of Lemma 2.9, which proves that $\xi \in K_c$. Thus, $K_c$ is convex and closed in $L^1$. Hence, it is also closed with respect to the weak topology, as argued in Theorem 2.6. To prove that $K_c$ is weakly sequentially compact, it remains to show that $K_c$ is uniformly integrable, by the Dunford–Pettis theorem (Dunford and Schwartz 1988, Corollary IV.8.11).

To this end, take $\varepsilon > 0$ and $\xi \in K_c$. There exists a constant $\alpha > 0$ such that $|\xi_t| f(-\xi_t) \leq \varepsilon c$ for $|\xi_t| > \alpha$, due to the superlinear growth property of $f$. Because $f(x) = 0$ if and only if $x = 0$, the quantity $1/f(-\xi_t)$ is well-defined on $\{|\xi_t| > \alpha\}$ and we obtain

$$
\mathbb{E} \left[ \int_0^T \mathbb{1}_{\{|\xi_t| > \alpha\}} |\xi_t| \, dt \right] \leq \mathbb{E} \left[ \int_0^T \mathbb{1}_{\{|\xi_t| > \alpha\}} f(-\xi_t) \, dt \right] \leq \frac{\varepsilon}{\alpha},
$$

which proves the uniform integrability of $K_c$. \qed

In the next lemma, we give a lower and an upper bound for the non-stochastic integral terms that appear in the revenue process.

**Lemma 2.11** Suppose that $b \neq 0$, and let $\xi \in \mathcal{X}^1(T, X_0)$ and $t^1, t^2 \in [0, T]$. Then there exists a constant $C > 0$, depending on $f$, $b$ and $T$, such that

$$
-\frac{5}{4} \int_{t^1}^{t^2} f(-\xi_t) \, dt - |b|CT^2/2 - b \cdot (t_1X^\xi_{t^1} - t_2X^\xi_{t^2})
$$

$$
\leq \int_{t^1}^{t^2} \left( b \cdot X^\xi_t - f(-\xi_t) \right) \, dt \leq -\frac{3}{4} \int_{t^1}^{t^2} f(-\xi_t) \, dt
$$

$$
+ |b|CT^2/2 - b \cdot (t_1X^\xi_{t^1} - t_2X^\xi_{t^2}).
$$

The subsequent lemma shows that a sequence of strategies in $\mathcal{X}^1(T, X_0)$ such that the corresponding expected utilities converge to the supremum in (2.14) can be chosen in a way that it belongs to some $K_m$, for $m$ large enough. This will be crucial for proving the existence of an optimal strategy. Here, we will use the fundamental property (2.15) satisfied by the sequence $(\xi^n)$.

**Lemma 2.12** Let $(\xi^n)$ be a sequence of strategies such that

$$
\xi^n \in \mathcal{X}^1(T, X_0) \quad \text{and} \quad \mathbb{E} \left[ u \left( R^\xi^n_T \right) \right] \nearrow \sup_{\xi \in \mathcal{X}^1(T, X_0)} \mathbb{E} \left[ u \left( R^\xi_T \right) \right]. \quad (2.19)
$$
Then there exists a constant $m > 0$ such that
\[
\xi^n \in \overline{K}_m = \left\{ \xi \in \hat{X}^1(T, X_0) \mid \mathbb{E}\left[ \int_0^T f(-\xi_t) \, dt \right] \leq m \right\},
\]
for every $n \in \mathbb{N}$.

**Remark 2.13** Due to the preceding lemma, we can w.l.o.g assume that the supremum in (2.14) can be taken over strategies that belong to the set $\overline{K}_m$, for suitable $m$. More precisely, (2.14) becomes
\[
V(T, X_0, R_0) = \sup_{\xi \in \hat{X}^1(T, X_0)} \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right] = \sup_{\xi \in \overline{K}_m} \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right],
\]
where $m$ has to be chosen such that
\[
m \geq \frac{4}{3} \left( \frac{-V_2(T, X_0, R_0)}{A_1} + R_0 + N \right).
\]

In the following, we will prove a fundamental property of the map $\xi \mapsto \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right]$, which we will also use to prove the continuity of the value function for the underlying maximization problem.

**Proposition 2.14** The map $\xi \mapsto \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right]$ is upper semi-continuous on $\hat{X}^1(T, X_0)$ with respect to the weak topology in $L^1$.

**Proof** Direct consequence of Corollary 2.7, since the map $\xi \mapsto \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right]$ is concave, and Corollary 2.8. \hfill \Box

Now we are ready for the proof of the existence and uniqueness of the optimal strategy.

**Proof of Theorem 2.4** Let $(\xi^n)_{n \in \mathbb{N}}$ be such that
\[
\xi^n \in \hat{X}^1(T, X_0, R_0) \quad \text{and} \quad \mathbb{E}\left[ u\left( R_{T}^{\xi^n} \right) \right] \geq \sup_{\xi \in \hat{X}^1(T, X_0)} \mathbb{E}\left[ u\left( R_{T}^{\xi} \right) \right].
\]

Lemma 2.12 implies that there exists a subsequence $(\xi^{n_k})$ of $(\xi^n)$ and some $\xi^* \in \hat{X}^1(T, X_0)$ such that $\xi^{n_k} \rightharpoonup \xi^*$, weakly in $L^1$. Due to Proposition 2.14, we get
\[
V(T, X_0, R_0) = \limsup_k \mathbb{E}\left[ u\left( R_{T}^{\xi^{n_k}} \right) \right] \leq \mathbb{E}\left[ u\left( R_{T}^{\xi^*} \right) \right],
\]
which proves that $\xi^*$ is an optimal strategy for the maximization problem (2.8). The uniqueness of the optimal strategy is a direct consequence of the convexity of $\hat{X}^1(T, X_0)$ and (strict) concavity of $\xi \mapsto \mathbb{E}[u(R_{T}^{\xi})]$. \hfill \Box
It is established in Schied et al. (2010) that the optimal strategies for CARA value functions are such that the corresponding revenues have finite exponential moments, i.e., \( \mathbb{E}\left[ \exp\left(-\lambda R^* T\right) \right] < \infty \), for all \( \lambda > 0 \), where \( \xi^{*,i} \) are the optimal strategies for the value functions with respective CARA coefficients \( A_1 \) and \( A_2 \). This is due to the fact that the optimal strategies are deterministic, and hence \( \int_0^T (X^\xi_{i,t})^\top \sigma dB_t \) have finite exponential moments. However, for the optimal strategy in (2.14), we only have \( \mathbb{E}\left[ \exp\left(-\lambda R^* T\right) \right] < \infty \) if \( \lambda \leq A_1 \). For \( \lambda > A_1 \), it is not clear whether or not the analogue holds. Thus, in order to avoid integrability issues, we will have to make the following assumptions.

**Assumption 2.15** We suppose that the moment generating function of the revenues of the optimal strategy, denoted by \( M_{R^* T} \), is defined for \( 2A_2 \), where we set \( M_{R^* T}(A) := \mathbb{E}\left[ \exp(-AR^* T) \right] \). Thus, we will restrict ourselves to the following set of strategies:

\[
\dot{\mathcal{X}}_{2A_2}^1(T, X_0) := \left\{ \xi \in \dot{\mathcal{X}}^1(T, X_0) \mid \mathbb{E}\left[ \exp(-2A_2 R^* T) \right] \leq M_{R^* T}(2A_2) + 1 \right\}.
\]

**Proposition 2.16** The set \( \dot{\mathcal{X}}_{2A_2}^1(T, X_0) \) is a closed convex set with respect to the strong topology in \( L^1 \) (and hence with respect to the weak topology).

**Proof** Due to the convexity of the map \( \xi \mapsto \mathbb{E}[\exp(-A(R^* T))] \), the preceding set is convex. To show that it is closed in \( L^1 \), we take a sequence \( (\xi^n) \) in \( \dot{\mathcal{X}}_{2A_2}^1(T, X_0, R_0) \) that converges to \( \xi \) in \( L^1 \). Since \( \xi^n \) in particular converges weakly to \( \xi \), we can use Corollary 2.8 to obtain

\[
\mathbb{E}\left[ \exp(-2A_2 R^* T) \right] \leq \liminf \mathbb{E}\left[ \exp(-2A_2 R^* T^n) \right] \leq M_{R^* T}(2A_2) + 1,
\]

which completes the proof.

**Remark 2.17** As argued before, if \( M_{R^* T}(2A_2) < \infty \), then we also have

\[
M_{R^* T}(A) < \infty \quad \text{for all} \quad 0 < A < 2A_2.
\]

Note that if we suppose that \( u \) is a convex combination of CARA utility functions, then \( M_{R^* T} \) is defined on \( [A_1, A_2] \). However, we need \( M_{R^* T}(2A_2) \) to be well-defined, since we will have to apply the Cauchy-Schwarz inequality to prove the continuity of the value function.

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3 Regularity properties of the value function and the dynamic programming principle

3.1 Partial differentiability of the value function

In this section, we will establish that the value function $V$ is continuously differentiable with respect to the parameter $R \in \mathbb{R}$, for fixed $(T, X) \in ]0, \infty[ \times \mathbb{R}^d$. Surprisingly, we just need the existence and uniqueness of the optimal strategy to prove it. Compared to the proof of the continuity of the value function in its parameters, this one is essentially easier, due to fact that, for fixed $T, X_0$, the value function is concave as showed in Proposition 2.2.

Further, we need the following result.

**Proposition 3.1** Let $\xi \in \mathcal{X}^{1}_{2A_2}(T, X_0)$. Then, the map $R_0 \mapsto -\mathbb{E}[u(R_T^\xi + R_0)]$ is twice differentiable on $\mathbb{R}$ with first and second derivative given by $\mathbb{E}[u'(R_T^\xi)]$ and $\mathbb{E}[u''(R_T^\xi)]$, respectively.

Before beginning with the proof, we need to prove the following lemma.

In our case, the optimal strategy depends on the parameter $R$ without, a priori, any known control of this dependence. Since the concavity property of the value function will be the key to establishing the desired regularity properties, we consider now a family of concave $C^1$-functions $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ and define

$$f(x) = \sup_{\alpha} f_\alpha(x).$$

Note that the supremum is not necessarily concave. However, if $f$ is concave in a neighborhood of a point $t$, then the following proposition gives us a sufficient condition under which $f$ is differentiable at this point.

**Lemma 3.2** Consider a family $(f_\alpha)_{\alpha \in A}$ of concave $C^1(\mathbb{R})$-functions that are uniformly bounded from above. Define

$$f(x) = \sup_{\alpha \in A} f_\alpha(x).$$

Suppose further that there exist $t \in \mathbb{R}$ and $\eta > 0$ such that $f$ is concave on $]t-\eta, t+\eta[$ and $\alpha^*_t \in A$ such that $f(t) = f_{\alpha^*_t}(t)$. Then, $f$ is differentiable at $t$ with derivative

$$f'(t) = f'_{\alpha^*_t}(t).$$

If we suppose moreover that $\alpha^*_t$ is uniquely determined, then $f'$ is continuous at $t$.

We can now state and show the main result of this subsection.

**Theorem 3.3** The value function is continuously partially differentiable in $R$, and we have the formula

$$V_r(T, X, R) = \mathbb{E}[u'(R_T^x)],$$
where $\xi^*$ is the optimal strategy associated to $V(T, X, R)$.

**Proof** The proof is a direct consequence of Lemma 3.2, when applied to the family of concave functions ($R \mapsto \mathbb{E}[u(R_T^\xi + R)]_{\xi \in \mathcal{A}_1(T, X_0)}$. Indeed, this is a family of concave $C^1$-functions (due to Proposition 3.1). The existence and uniqueness of an optimal strategy (Theorem 2.4) and the concavity of the map $R \mapsto V(T, X, R)$, for fixed $T, X$ (Lemma 2.2), yield that the remaining conditions of the preceding lemma are satisfied.

**Corollary 3.4** Suppose that $u'$ is convex and decreasing. Then, the value function is twice differentiable with second partial derivative

$$V_{rr}(T, X, R) = \mathbb{E}[u''(R_T^\xi)],$$

where $\xi^*$ is the optimal strategy associated to $V(T, X, R)$.

**Proof** The proof is similar to the one of Theorem 3.3 and is obtained by applying Lemma 3.2 to $u'$ and Proposition 3.1.

**Remark 3.5** We are in the setting of the preceding corollary if, e.g., $u$ is a convex combination of exponential utility functions or, more generally, if $(-u)$ is a complete monotone function, i.e., if $\forall n \in \mathbb{N}^* : (-1)^n (-u)^{(n)} \geq 0$. According to the Hausdorff–Bernstein–Widder’s theorem (cf. Widder 1941 or Donoghue 1974, Chapter 21), this is equivalent to the existence of a Borel measure $\mu$ on $[0, \infty[$ such that

$$-u(x) = \int_0^\infty e^{-xt} d\mu(t).$$

Note that a utility function which is a convex combination of exponential utility functions is *not* an exponential utility function in general.

### 3.2 Continuity of the value function

The proof of the continuity of our value function will be split in two propositions. We will first prove its upper semi-continuity and then its lower semi-continuity. To prove the upper semi-continuity we will use the same techniques as are used to prove the existence of the optimal strategy for the maximization problem (2.8). The main idea to prove the lower semi-continuity is to use a convex combination of the optimal strategy for (2.8) and the optimal strategy of the corresponding exponential value function at a certain well-chosen point. Here, we have to distinguish between two cases; the case where the value function is approximated from above, and the case where the value function is approximated from below in time. In the sequel, for $\xi \in \mathcal{A}_1(T, X_0)$ we will automatically set $\xi_i = 0$ for $i \geq T$.

**Proposition 3.6** The value function is upper semi-continuous on $]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$. 

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In the following, we will prove the lower semi-continuity of the value function $V$. Contrarily to the proof of the upper semi-continuity of $V$, we will have to consider two cases; when the sequence of time converges from above and from below to a fixed time $T$. For the latter case, we will first need to derive a certain lower semi-continuity property of the value function within time, for fixed $X_0, R_0$. The difficult part of the proof of the lower semi-continuity is due to the fact that accelerating the strategy when we approximate the time from below cannot be useful to prove the result, since we are then facing measurability issues. Therefore we will have to use other techniques.

We first need to prove the following lemma, which gives a sufficient condition to ensure that the expected utilities $\mathbb{E}[u(R_{\eta T}^n)]$ converge to $\mathbb{E}[u(R_{\eta T}^n)]$, when $R_{\eta T}^n$ converges to $R_{\eta T}^n$, in probability.

**Lemma 3.7** Let $\eta^n \in \dot{X}^1(T, X_0)$ be a sequence of strategies such that $R_{\eta T}^n$ converges to $R_{\eta T}^n$, in probability, where $\eta \in \dot{X}^1(T, X_0)$.

Suppose moreover that $(\exp(-2A_2R_{\eta T}^n))_n$ is uniformly bounded in $L^2$. Then we have

$$\mathbb{E}\left[u\left(R_{\eta T}^n\right)\right] \longrightarrow n \rightarrow \infty \mathbb{E}\left[u\left(R_{\eta T}^n\right)\right].$$ (3.1)

**Proof** We need to prove that $(u(R_{\eta T}^n))_n$ is uniformly bounded in $L^2$. But this is a direct consequence of the fact that $(\mathbb{E}[u^+(R_{\eta T}^n)])_n$ is bounded and that, for all $n \in \mathbb{N}$, $\mathbb{E}[(u^-(R_{\eta T}^n))^2] \leq \mathbb{E}[\exp(-2A_2R_{\eta T}^n)]$, due to inequality (2.7). Since $\mathbb{E}[\exp(-2A_2R_{\eta T}^n)] < \infty$, applying Vitali’s convergence theorem we conclude that

$$\mathbb{E}\left[u\left(R_{\eta T}^n\right)\right] \longrightarrow n \rightarrow \infty \mathbb{E}\left[u\left(R_{\eta T}^n\right)\right].$$

\[\square\]

The next lemma is a direct consequence of the integration by parts formula for the stochastic integral.

**Lemma 3.8** Let $\xi^n \in \dot{X}^1(T, X_0)$ converge to some $\xi \in \dot{X}^1(T, X_0)$ in the $L^1[0, T]$-weak convergence sense, $\mathbb{P}$-a.s. Then

$$\int_0^T (X_t^{\xi^n})^\top \sigma \, dB_t \longrightarrow \int_0^T (X_t^{\xi})^\top \sigma \, dB_t \quad \mathbb{P}\text{-a.s.}$$

Now we are ready to state and prove the following proposition.

**Proposition 3.9** Let $(T, X_0, R_0) \in [0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$ and $T^n$ be a sequence of positive real numbers that converges from below to $T$, i.e., $T^n \uparrow T$. Then we have

$$\lim \inf_n V(T^n, X_0, R_0) \geq V(T, X_0, R_0).$$ (3.2)
Proof In the following, we will need Assumption 2.15. Let \((T, X_0, R_0) \in ]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}\) and \(\xi \in \mathcal{X}_{2A_2}^1(T, X_0)\). Define

\[
\varphi^\xi : ]0, \infty[ \longrightarrow \mathbb{R}
\]

\[
T \longmapsto \mathbb{E}[u(R_T^\xi)].
\]

Note that the map \(\varphi^\xi\) is constant on \([T, \infty[\). We show that \(\varphi^\xi\) is continuous at \(T\). To this end, it is sufficient to take a sequence \((T_n)\) such that \(T_n \uparrow T\) and to prove that

\[
\varphi^\xi(T^n) \longrightarrow \varphi^\xi(T) \quad (3.3)
\]

or, equivalently,

\[
\mathbb{E}[u(R_{T_n}^\xi)] \longrightarrow \mathbb{E}[u(R_T^\xi)].
\]

We easily have the convergence

\[
R_{T_n}^\xi = \int_0^{T_n} (X_t^\xi)^\top \sigma dB_t + \int_0^{T_n} b \cdot X_t^\xi dt - \int_0^{T_n} f(-\xi_t) dt \longrightarrow_{n \to \infty} R_T^\xi \quad \mathbb{P}\text{-a.s.} \quad (3.4)
\]

Because \(u\) is continuous, we then obtain

\[
\lim_n u(R_{T_n}^\xi) = u(R_T^\xi) \quad \mathbb{P}\text{-a.s.} \quad (3.5)
\]

Now, we have to prove the boundedness of the sequence \((\mathbb{E}[\exp(-2A R_{T_n}^\xi)])_n\). For this matter, we write

\[
\mathbb{E}\left[ \exp\left( -2A R_{T_n}^\xi \right) \right]
\leq K \mathbb{E}\left[ \exp\left( -2A \left( \mathbb{E}\left[ \int_0^T (X_t^\xi)^\top \sigma dB_t + \int_0^T b \cdot X_t^\xi dt - \int_0^T f(-\xi_t) dt \mid \mathcal{F}_{T_n} \right] \right) \right]
\leq K \mathbb{E}\left[ \exp\left( -2A \left( \int_0^T (X_t^\xi)^\top \sigma dB_t + \int_0^T b \cdot X_t^\xi dt - \int_0^T f(-\xi_t) dt \right) \right) \right]
\leq K \mathbb{E}\left[ \exp\left( -2A \left( \int_0^T (X_t^\xi)^\top \sigma dB_t + \int_0^T b \cdot X_t^\xi dt - \int_0^T f(-\xi_t) dt \right) \right) \right] < \infty,
\]

where \(K = \exp(T |b| \|X^\xi\|_{L^2})\) is obtained using Hölder’s inequality, and where the finiteness of the last term follows with \(\xi \in \mathcal{X}_{2A_2}^1(T, X_0)\). Thus, the sequence \((u(R_{T_n}^\xi))\) is uniformly bounded in \(L^2\), whence using Vitali’s convergence theorem we infer

\[
\mathbb{E}[u(R_{T_n}^\xi)] \longrightarrow_{n \to \infty} \mathbb{E}[u(R_T^\xi)],
\]

which proves \((3.3)\). Hence, \(\varphi^\xi\) is continuous at \(T\), and \(\sup_{\xi \in \mathcal{X}_{2A_2}^1(T, X_0)} \varphi^\xi\) is lower semi-continuous at \(T\), because it is the supremum of a family of (lower semi-) continuous functions. Since

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this proves in particular that for every sequence of time $T^n$ that converges from below to $T$, we have

$$\lim \inf_n \sup_{\xi \in \dot{X}^{1}_{2, A}(T, X_0)} \varphi^\xi(T^n) \geq \sup_{\xi \in \dot{X}^{1}_{2, A}(T, X_0)} \varphi^\xi(T) = V(T, X_0, R_0),$$

which proves (3.2).

We can now derive the lower semi-continuity of the value function $V$.

**Proposition 3.10** The value function is lower semi-continuous on $]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$.

As a consequence of Propositions 3.6 and 3.10, we obtain the following fundamental result.

**Theorem 3.11** The value function $V$ is continuous on $]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$.

### 3.3 The Bellman principle and the construction of $\varepsilon$-maximizers

In this section we prove the Bellman principle of optimality underlying our maximization problem (2.8). To this end, we use $\varepsilon$-maximizers constructed on a bounded region. Their existence is proved by using an approximating sequence of strategies. Thus, we avoid here the use of a measurable selection theorem, which appears typically in optimal control theory. The dynamic programming principle is a key result to prove both a verification theorem and a theorem stating that the value function is a solution, in the viscosity sense, of a Hamilton–Jacobi–Bellman equation. From now on, for a fixed time $T \in ]0, \infty[$, we will consider the time-reversed value function: $t \mapsto V(T - t, X_0, R_0)$, and we will assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical Wiener Space.

**Theorem 3.12** (Bellman Principle) Let $(T, X_0, R_0) \in ]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$. Then we have

$$V(T, X_0, R_0) = \sup_{\xi \in \dot{X}^{1}(T, X_0)} \mathbb{E}[V(T - \tau, X^\xi_\tau, R^\xi_\tau)]$$

for every stopping time $\tau$ taking values in $]0, T[$.

**Remark 3.13** Note that Bouchard and Touzi (2011) developed a weak formulation of the dynamic principle, which can be used to derive the viscosity property of the corresponding value function, in some optimal control problems. However, this requires the following concatenation property (Assumption A) of the strategies: for $\xi, \eta \in \dot{X}^{1}(T, X_0)$ and a stopping time $\tau \in [0, T[$, we must have that $\xi 1_{[0, \tau[} + \eta 1_{[\tau, T[} \in \dot{X}^{1}(T, X_0)$, which is however not the case in general, and therefore is not usable in our work. In Bouchard and Nutz (2012), another weak formulation of the dynamic principle with generalized state constraints is formulated. Here
again, a concatenation property (Assumption B) in the following form is required: for \( \xi, \eta \in \dot{X}^1(T, X_0) \) and a time \( s \in [0, T] \), it must hold that \( X^\xi_t = X^\xi_s - \int_s^t \eta_u \, du \), for \( t \leq s \), which is again not the case in general, and thus cannot be directly applied here.

The proof of Theorem 3.12 is split in two parts. For ease of reference, let us first make the following assumption on \( f \).

**Assumption 3.14** From now on, we suppose that \( f \) has at most a polynomial growth of degree \( p \), i.e., there exists \( C > 0 \) such that

\[
f(x) \leq C(1 + |x|^p) \quad \text{for all } x \in \mathbb{R}^d.
\]

Further, in order to avoid measurability issues, we need to suppose that for \( T \in [0, \infty) \), \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T)}, P) \) is the canonical Wiener space. Taking this perspective, let us start with proving some measurability results. Here also, we will restrict our attention to strategies that lie in \( \dot{X}^2(T, X_0, R_0) \), as mentioned in Assumption 2.15.

**Lemma 3.15** For \( \omega \in \Omega \), define the map \( \phi_\omega : \Omega \to \Omega \) by

\[
\phi_\omega(\tilde{\omega}) = \begin{cases} 
\omega(s), & \text{for } s \in [0, \tau(\omega)], \\
\omega(\tau(\omega)) + \tilde{\omega}(s) - \tilde{\omega}(\tau(\omega)), & \text{for } s \in [\tau(\omega), T],
\end{cases}
\]

where \( \tau \) is as in (3.7). Moreover, for \( \xi \in \dot{X}^1(T, X_0) \) we define

\[
\xi^{\omega}_t(\tilde{\omega}) := \xi_t \circ \phi_\omega(\tilde{\omega}).
\]

Then, for \( \mathbb{P}\)-a.e. \( \omega \),

\[
\mathbb{E}\left[u\left(\mathcal{R}^\xi_T\right)|\mathcal{F}_\tau\right](\omega) = \mathbb{E}\left[u\left(\mathcal{R}^\xi_T + \mathcal{R}^{\xi^{\omega}}_{T, \tau}\right)|\mathcal{F}_\tau\right](\omega) = \mathbb{E}\left[u\left(\mathcal{R}^\xi_T(\omega) + \mathcal{R}^{\xi^{\omega}}_{\tau(\omega), T}\right),
\]

where \( \mathcal{R}^{\xi}_{T, \tau} \) denotes the revenues generated by the strategy \( \xi^{\omega} \) during the time period \([t, T]\), i.e:

\[
\mathcal{R}^{\xi}_{T, \tau} = \int_t^T (X^\xi_s)^T \sigma \, dB_s + \int_t^T b \cdot X^\xi_s \, ds - \int_t^T f(-\xi_s) \, ds.
\]

The following lemma yields an upper bound for an exponential value function at some stopping time with values in \([0, T]\). It uses the notations of Lemma 3.15. For \( d = 1 \), an analogous result can be found in Schied and Schöneborn (2008).

**Lemma 3.16** Let \( \overline{V}(T, X_0, R_0) = \inf_{\xi \in \dot{X}^1(T, X_0)} \mathbb{E}\left[\exp(-AR^\xi_T)\right] \) and \( \tau \) be a stopping time with values in \([0, T]\). We then have

\[
\overline{V}(T - \tau, X^\xi_\tau, \mathcal{R}^\xi_\tau) \leq \mathbb{E}\left[\exp(-AR^\xi_T)|\mathcal{F}_\tau\right] \quad \mathbb{P}\text{-a.s.}
\]

for every \( \xi \in \dot{X}^1(T, X_0) \).
We wish now to prove the following fundamental proposition:

**Proposition 3.17** Let \( \xi \in \mathcal{X}_{2A_2}^1 (T, X_0) \) and \( \tau \) be a stopping time with values in \([0, T]\). Then we have

\[
V(T, X_0, R_0) \geq \mathbb{E} \left[ V(T - \tau, X_\tau^\xi, R_\tau^\xi) \right].
\]  

(3.10)

This proposition will follow from the subsequent lemma and the theorem on the existence of \( \varepsilon \)-maximizers on a bounded region. The latter one will be proved without the use of a measurable selection argument, by simply using the continuity of the value function and the existence of an optimal strategy for the maximization problem (2.8).

The next lemma allows us to restrict our problem to a region where the parameters \( T, X_0 \) and \( R_0 \) are bounded. Indeed, outside this region (with the bound of the parameters having to be taken large enough), the following result proves that the right-hand side term of (3.10) can be chosen smaller than \( \varepsilon \).

**Lemma 3.18** Let \( \xi \in \mathcal{X}_{2A_2}^1 (T, X_0) \). Under the assumptions and notations of Proposition 3.17, there exists \( N = N_\varepsilon \in \mathbb{N} \) such that

\[
\mathbb{E} \left[ \left| V(T - \tau, X_\tau^\xi, R_\tau^\xi) \right| \mathbb{1}_{\left\{ \left| X_\tau^\xi \right| \vee \left| R_\tau^\xi \right| > N \} \right] \leq \varepsilon. \]  

(3.11)

We can now state the following fundamental theorem of this subsection.

**Theorem 3.19** (Existence of the \( \varepsilon \)-maximizers on a bounded region) With the notations of Proposition 3.17, Lemmas 3.15 and 3.18, there exists a progressively measurable process \( \tilde{\xi}_\tau, \varepsilon \in \mathcal{X}_{2A_2}^1 (T - \tau, X_\tau^\xi) \) such that for \( \mathbb{P} \)-a.e. \( \omega \in \left\{ X_\tau^\xi \wedge R_\tau^\xi \leq N \right\} \),

\[
V(T - \tau(\omega), X_\tau^\xi(\omega), R_\tau^\xi(\omega)) \leq \mathbb{E} \left[ u \left( R_\tau^\xi(\omega) + R_{\tilde{\xi}_\tau(\omega)}^\omega, T \right) \right] + \varepsilon. \]  

(3.12)

We can now turn to proving Proposition 3.17

**Proof of Proposition 3.17** Lemma 3.18 and Theorem 3.19 imply for \( \xi \in \mathcal{X}_{2A_2}^1 (T, X_0) \):

\[
\mathbb{E} \left[ V(T - \tau, X_\tau^\xi, R_\tau^\xi) \right] \\
= \mathbb{E} \left[ V(T - \tau, X_\tau^\xi, R_\tau^\xi) \mathbb{1}_{\left\{ \left| X_\tau^\xi \right| \vee \left| R_\tau^\xi \right| > N \} \right] \right] + \mathbb{E} \left[ V(T - \tau, X_\tau^\xi, R_\tau^\xi) \mathbb{1}_{\left\{ \left| X_\tau^\xi \right| \vee \left| R_\tau^\xi \right| \leq N \} \right] \right] \\
\leq \varepsilon + \int_\Omega \mathbb{E} \left[ u \left( R_\tau^\xi + R_{\tilde{\xi}_\tau(\omega)}^\omega, T \right) \right] \mathbb{P}(d\omega) + \varepsilon \\
= 2\varepsilon + \int_\Omega \mathbb{E} \left[ u \left( R_{\tilde{\xi}_\tau(\omega)}^\omega, T \right) \right] \mathbb{P}(d\omega) \\
= 2\varepsilon + V(T, X_0, R_0),
\]

due to Lemma 3.15, whereby the process \( \xi_{\tau, \varepsilon} \) is defined as

\[
\xi_{\tau, \varepsilon}(\omega) = \begin{cases} 
\xi_t(\omega) & \text{for } t \in [0, \tau(\omega)) \\
\tilde{\xi}_t(\omega) & \text{for } t \in [\tau(\omega), T],
\end{cases}
\]

\( \varepsilon \) Springer
and the definition of $V(T, X_0, R_0)$. □

In Proposition 3.17 we have proved the inequality $'' \geq''$ of Eq. (3.7). Now it remains to prove the reverse inequality. To this end, we need the following proposition, which uses the notion of the essential supremum of a set $\Phi$ of random variables, denoted by $\text{ess sup}_\Phi$.

**Proposition 3.20** With the notations of Lemma 3.15, we have

$$V\left(-\tau(\omega), X_\xi^{\tau(\omega)}, R_\xi^{\tau(\omega)}\right) = \text{ess sup}_{\xi^{\omega} \in \hat{X}_{2A_2}^1(T-\tau(\omega), X_{\tau}^{\xi(\omega)})} \mathbb{E}\left[u(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega}}) \mid \mathcal{F}_\tau\right](\omega)$$

for $\mathbb{P}$-a.e. $\omega$ on $\{ |X_{\tau}^{\xi}| \wedge |R_{\tau}^{\xi}| \leq N \}$.

**Proof** We recall the $\mathbb{P}$-a.s. equality fulfilled by $V(T-\tau, X_{\tau}^{\xi}, R_{\tau}^{\xi})$,

$$V\left(T-\tau(\omega), X_{\tau}^{\xi(\omega)}, R_{\tau}^{\xi(\omega)}\right) \geq \sup_{\xi^{\omega} \in \hat{X}_{2A_2}^1(T-\tau(\omega), X_{\tau}^{\xi(\omega)})} \mathbb{E}\left[u(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega}}) \mid \mathcal{F}_\tau\right](\omega) \quad \mathbb{P}\text{-a.s.},$$

where $\xi^{\omega}$ is defined as in Lemma 3.15. Hence, this permits us to write

$$V(T-\tau(\omega), X_{\tau}^{\xi(\omega)}, R_{\tau}^{\xi(\omega)}) \geq \mathbb{E}\left[u\left(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega}}\right) \mid \mathcal{F}_\tau\right](\omega) \quad \mathbb{P}\text{-a.s.}$$

for all $\xi^{\omega} \in \hat{X}_{2A_2}^1(T-\tau(\omega), X_{\tau}^{\xi(\omega)})$. Using the definition of the essential supremum (see, e.g., Föllmer and Schied 2011, Definition A.34), it follows then

$$V(T-\tau(\omega), X_{\tau}(\omega), R_{\tau}(\omega)) \geq \text{ess sup}_{\xi^{\omega} \in \hat{X}_{2A_2}^1(T-\tau(\omega), X_{\tau}^{\xi(\omega)})} \mathbb{E}\left[u\left(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega}}\right) \mid \mathcal{F}_\tau\right](\omega),$$

which proves the inequality $'' \geq''$ of (3.13). For the converse inequality, let $\tilde{\xi}^{\omega,\tau,e}$ be as in Theorem 3.19. We have on $\{ |X_{\tau}^{\xi}| \wedge |R_{\tau}^{\xi}| \leq N \}$ :

$$\mathbb{E}\left[u\left(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega,\tau,e}}\right) \mid \mathcal{F}_\tau\right](\omega) \geq V(T-\tau(\omega), X_{\tau}^{\xi(\omega)}, R_{\tau}^{\xi(\omega)}) - \varepsilon \quad \mathbb{P}\text{-a.s.}$$

And therefore

$$\text{ess sup}_{\xi^{\omega} \in \hat{X}_{2A_2}^1(T-\tau(\omega), X_{\tau}^{\xi(\omega)})} \mathbb{E}\left[u\left(R_{\tau}^{\xi(\omega)} + \mathbb{R}_{\tau,T}^{\xi^{\omega}}\right) \mid \mathcal{F}_\tau\right](\omega) \geq V(T-\tau(\omega), X_{\tau}^{\xi(\omega)}, R_{\tau}^{\xi(\omega)}) - \varepsilon \quad \mathbb{P}\text{-a.s.}$$

Letting $\varepsilon$ go to 0 gives us the required inequality. □

We can now prove Theorem 3.12.
Proof of Theorem 3.12 Thanks to Proposition 3.17, it remains to show only the inequality “≤” in (3.7). Let \( \xi \in \tilde{\mathcal{X}}_{A_2}^T(T, X_0) \) and set \( \tilde{\xi}_s = \xi_{s+t} \in \tilde{\mathcal{X}}_{A_2}^T(T - \tau, X_\tau) \) for \( s \geq \tau \) and \( t \geq 0 \). The definition of the essential supremum, in conjunction with Proposition 3.20 and Lemma 3.18, yields

\[
\mathbb{E}\left[u(R_{T}^{\xi})\right] = \mathbb{E}\left[u(R_{T}^{\xi} + R_{\tau,T}^{\xi})\right] = \mathbb{E}\left[\mathbb{E}\left[u(R_{T}^{\xi} + R_{\tau,T}^{\xi})|\mathcal{F}_\tau\right]\right] \\
= \mathbb{E}\left[\mathbb{E}\left[u(R_{T}^{\xi} + R_{\tau,T}^{\xi})|\mathcal{F}_\tau\right]\left(\mathbb{1}\{X_\tau^{\xi} \mid |R_{\tau}^{\xi}| > N\} + \mathbb{1}\{X_\tau^{\xi} \mid |R_{\tau}^{\xi}| \leq N\}\right)\right] \\
\leq \varepsilon + \mathbb{E}\left[V(T - \tau, X_\tau^{\xi}, R_{\tau}^{\xi})\mathbb{1}\{X_\tau^{\xi} \mid |R_{\tau}^{\xi}| \leq N\}\right].
\]

Taking the supremum over \( \xi \) and then sending \( \varepsilon \) to zero (which implies sending \( N \) to infinity), shows the assertion. \( \square \)

4 The Hamilton–Jacobi–Bellman equation and its tight connection with the value function

In the sequel, we bring to light a strong relationship between \( V \) and a Hamilton–Jacobi–Bellman (HJB) equation, which is obtained via a classical heuristic derivation. We first suppose that \( V \in \mathcal{C}^{1,1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R} \). To simplify matters, let us introduce the following linear second-order operator \( \mathcal{L}^\eta \), where for \( \eta \in \mathbb{R}^d \),

\[
\mathcal{L}^\eta v(T, X, R) := \left(\frac{X^\top \Sigma X}{2} v_{rr} + b \cdot X v_r - \left(\eta^\top \nabla_X v + f(-\eta)v_r\right)\right) (T, X, R).
\]

Note that this operator is continuous in \( \eta \), due to the continuity of \( f \). Classical heuristic derivations as well as Proposition 2.3 suggest that \( V \) should satisfy

\[
-V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V = 0, \tag{4.2}
\]

\[
V(0, X, R) = \lim_{T \downarrow 0} V(T, X, R) = \begin{cases} u(R), & \text{if } X = 0 \\ -\infty, & \text{otherwise.} \end{cases} \tag{4.3}
\]

Remark 4.1 (i) Since \( f \) is positive and \( \lim_{|x| \to \infty} f(x) = \infty \), Eq. (4.2) makes sense only when \( V_t(t, x, r) > 0 \) for every \( (t, x, r) \in ]0, T] \times \mathbb{R}^d \times \mathbb{R} \). This is however in concordance with Theorem 3.3, which implies that the value function has a strictly positive partial derivative in its third argument.

(ii) Let us denote by

\[
f^*(z) := \sup_{x}(x \cdot z - f(x))
\]
the Fenchel–Legendre transformation of \( f \), which is a finite convex function, due to the assumptions on \( f \) (see Theorem 12.2 in Rockafellar 1997). With this at hand, Eq. (4.2) can be written equivalently as

\[
- V_t + b \cdot X_t V_r + \frac{X^\top \Sigma X}{2} V_{rr} + V_r f^* \left( \frac{\nabla_x V}{V_r} \right) = 0.
\]

(4.4)

We now suppose that \( V \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \). The next theorem shows that it is a classical solution of (4.2).

**Theorem 4.2** Let \( V \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \) be the value function of the maximization problem (2.8). Then \( V \) is a classical solution of (4.2) with initial condition (4.3).

The proof follows from the two propositions.

**Proposition 4.3** Let \( V \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \) be the value function of the maximization problem (2.8). Then \( V \) is a supersolution of (4.2), i.e., \( V \) fulfills the inequality

\[
\left( - V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \right) (t, x, r) \leq 0 \quad \text{for all} \quad (t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}.
\]

(4.5)

**Proof** The proof uses classical argumentations (see, e.g., Crandall et al. 1992) by making some adaptations, due to our constraint condition on strategies \( \xi \) as well as our blow up initial condition for \( V \). To this end, set \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \eta \in \mathbb{R}^d, \varepsilon > 0 \) be such that \( t + \varepsilon < T \), define \( \xi \in \tilde{X}^{1}_{A_2}([t, T], x) \) as

\[
\xi_s := \begin{cases} 
\eta, & \text{if } s \in [t, t + \varepsilon[, \\
-x \frac{\varepsilon \eta}{T-(t+\varepsilon)}, & \text{if } s \in [t + \varepsilon, T],
\end{cases}
\]

and consider the corresponding processes \((X^\xi, \mathcal{R}^\xi)\) that verify \( X^\xi_t = x, \mathcal{R}^\xi_t = r \). Now working step by step through argumentation as done in, e.g., Touzi (2004) Proposition 1.1 proves the statement.

**Proposition 4.4** Let \( V \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \) be the value function of the maximization problem (2.8). Then \( V \) is a subsolution of (4.2), i.e., \( V \) fulfills the inequality

\[
\left( - V_t + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \right) (t, x, r) \geq 0 \quad \text{for all} \quad (t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}.
\]

(4.6)

**Proof** Here again, the proof is a classical one and we can follow the ideas of Touzi (2004) or Touzi (2013), Proposition 1.2 or respectively Proposition 3.5) by making some adaptations due to our constraint condition on strategies \( \xi \) as well as our blow up initial condition for \( V \).
4.1 Verification theorem

In the next step we give sufficient conditions under which a smooth function \( w \) satisfying (4.2) with initial condition (4.3) coincides with our value function \( V \). This so-called verification argument relies essentially on Itô’s lemma (see, for example, Touzi 2013 or Pham 2009 for further details). Due to the existence and uniqueness of the optimal control for the value function \( V \), we will only need the existence of a strong solution to an associated SDE in order to ensure that \( w = V \). As suitable growth condition, we will assume as before that \( w \) lies between two CARA value functions.

**Theorem 4.5** Let \( T > 0 \) and \( w \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \cap C([0, T] \times \mathbb{R}^d \times \mathbb{R}) \) be such that the following inequalities hold

\[
V_2(t, x, r) \leq w(t, x, r) \leq V_1(t, x, r)
\]

where \( V_i, \ i = 1, 2, \) is as in (2.10). We then have the following two statements, depending on what additional assumptions we require.

(i) Suppose that

\[
0 \geq -w_t(T - t, x, r) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi w(T - t, x, r)
\]

for all \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) and

\[
\lim_{t \downarrow 0} w(t, x, r) = \begin{cases} 
  w(0, 0, r) \geq u(r), & \text{if } X = 0 \\
  -\infty, & \text{otherwise}
\end{cases}
\]

on \([0, T] \times \mathbb{R}^d \times \mathbb{R}\). Then \( w \geq V \) on \([0, T] \times \mathbb{R}^d \times \mathbb{R}\).

(ii) Suppose that

\[
0 = -w_t(T - t, x, r) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi w(T - t, x, r)
\]

for all \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) and

\[
\lim_{T \downarrow 0} w(t, x, r) = \begin{cases} 
  u(r), & \text{if } X = 0 \\
  -\infty, & \text{otherwise}
\end{cases}
\]

Moreover, assume

\[
w_t(T - t, x, r) > 0 \quad \text{for all } t, x, r \text{ on } [0, T[ \times \mathbb{R}^d \times \mathbb{R}].
\]

(a) Then, the continuous function \( \hat{\xi} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}^d \) defined by

\[
\hat{\xi}(t, x, r) := \nabla f^* \left( \frac{\nabla_x w(t, x, r)}{w_t(t, x, r)} \right)
\]
satisfies
\[ - w_t(T-t,x,r) + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi w(T-t,x,r) = - w_t(T-t,x,r) + \mathcal{L}^\hat{\xi} w(T-t,x,r) = 0 \] (4.14)

for every \((t,x,r)\) on \([0,T] \times \mathbb{R}^d \times \mathbb{R}\).

(b) If we furthermore assume that there exists a strong solution \((\hat{X}, \hat{R})\) to the SDE
\[
\begin{cases}
    d\hat{R}_t = (X_t)^	op \sigma dB_t + b \cdot X_t dt - f (-\hat{\xi}(t,X_t,\hat{R}_t)) dt, \\
    dX_t = -\hat{\xi}(t,X_t,\hat{R}_t) dt, \\
    \hat{R}_{|t=0} = R_0 \text{ and } X_{|t=0} = X_0,
\end{cases}
\] (4.15)
such that \(\hat{\xi}(\cdot, \hat{X}, \hat{R}) \in \dot{X}^*_{2A_2}(T,X_0)\), then we have \(w = V\) on \([0,T] \times \mathbb{R}^d \times \mathbb{R}\).

The solution of the preceding SDE is unique and given by \((X^*_{t}, R^*_{t})\), where \(\hat{\xi}^*\) denotes the optimal liquidation strategy for the value function \(V(T,X_0,R_0)\).

Moreover, the optimal control is given in feedback form by
\[ \hat{\xi}^*_t = \hat{\xi}(T-t, \hat{X}_t, \hat{R}_t), \quad (\mathbb{P} \otimes \lambda)-\text{a.s.} \]

Remark 4.6 (i) In the special case where the utility function \(u\) is a convex combination of exponential utility functions, i.e., \(u(x) = \lambda u_1(x) + (1-\lambda)u_2(x)\) with \(\lambda \in [0,1]\) and \(u_i, \ i = 1,2\), exponential utility functions, it can be easily proved that \(w := \lambda V_1 + (1-\lambda)V_2\), where \(V_i\) denotes the corresponding value function w.r.t. \(u_i\) satisfies (4.8) as well as the boundary condition
\[ \lim_{t \downarrow 0} w(t,x,r) = \begin{cases} w(0,0,r) = \lambda u_1(r) + (1-\lambda)u_2(r), \quad \text{if } X = 0 \\
-\infty, \quad \text{otherwise} \end{cases} \]
on \([0,T] \times \mathbb{R}^d \times \mathbb{R}\). However, inequality (4.8) satisfies by \(w\) previously defined is strict in general.

(ii) Proving the existence (and uniqueness) of a strong solution of (4.15) can be very challenging, since
\[ \nabla f^* \left( \frac{\nabla_x w(t,x,r)}{w_r(t,x,r)} \right) \]
is at most supposed to be continuous and does not satisfy any global Lipschitz-continuity, due to the quotient term and the fact that \(\nabla f^*\) can be superlinear.

(iii) With formula (4.13) we have a way to numerically compute the optimal liquidation strategy. However, this would require to first compute the gradient of the value function, which is not an easy task, in general. Moreover, as mentioned above, the coefficients in the SDE do not satisfy any (global) Lipschitz condition,
and thus (up to our knowledge) no known converging method can be applied to solve the SDE (4.15).

\[ \square \]

### 4.2 Viscosity solutions of the HJB-equation

So far, we have established connections between our maximization problem (2.8) and classical solutions of the HJB equation (4.2). Unfortunately, this method works out only if our value function \( V \) is smooth enough, which, however, may not be satisfied even in the deterministic case (see, e.g., Yong and Zhou 1999, Chapter 4, Example 2.3). To overcome this difficulty, we will use in the following the notion of viscosity solutions. Since our value function is continuous, we will restrict our framework to the class of continuous viscosity solutions. Note that a more general definition (in the class of locally bounded functions) can be found, for instance, in Fleming and Soner (2006). With this definition, however, a strong comparison principle would imply that \( V \) is again continuous.

#### 4.2.1 The value function as viscosity solution of the HJB equation

Let us start with introducing an abstract definition of viscosity solutions (see, e.g., Touzi 2013 or Fleming and Soner 2006). Consider a nonlinear second-order degenerate partial differential equation

\[
F(T - t, x, r, v(T - t, x, r), v_t(t, x, r), v_x(t, x, r), v_{xx}(t, x, r)) = 0,
\]

where \( F \) is a continuous function on \([0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \) taking values in \( \mathbb{R} \), with a fixed \( T > 0 \) and \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \). We have to impose the following crucial assumption on \( F \).

**Assumption 4.7** (Ellipticity) For all \((t, x, r, q, p, s, m) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \) and \( a, b \in \mathbb{R} \), we assume

\[
F(T - t, x, r, q, p, s, m, a) \leq F(T - t, x, r, q, p, s, m, b) \text{ if } a \geq b.
\]

**Definition 4.8** Let \( v : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function.

1. We say that \( v \) is a viscosity subsolution of (4.16) if for every \( \varphi \in C^{1,1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R} \) and every \((r^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\), when \( v - \varphi \) attains a local maximum at \((T - t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\), we have

\[
F(, v, \varphi_t, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \leq 0.
\]

2. We say that \( v \) is a viscosity supersolution of (4.16) if for every \( \varphi \in C^{1,1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R} \) and every \((r^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\), when \( v - \varphi \) attains a local minimum at \((T - t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\), we have

\[
F(, v, \varphi_t, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \geq 0.
\]
We say that \( v \) is a \textit{viscosity solution} of the Eq. (4.16) if \( v \) is a viscosity subsolution and supersolution.

\textbf{Remark 4.9} It may be interesting to note that the above definition is unchanged if the maximizer (or minimizer) \((T - t^*, x^*, r^*)\) is global and/or strict (see Barles 2013 for more details). Moreover, we can suppose w.l.o.g. that \( v(T - t^*, x^*, r^*) = \varphi(T - t^*, x^*, r^*) \). The function \( \varphi \) is called a test function for \( v \).

The following result justifies the introduction of this notion.

\textbf{Theorem 4.10} The value function \( V \) is a viscosity solution of the Hamilton–Jacobi–Bellman equation (4.2) with initial condition (4.3).

\textit{Proof} As for the classical case, the proof is a classical one (see, e.g., Touzi 2013) and is obtained by making some adaptations due to our constraint condition on strategies \( \xi \) as well as our blow up initial condition for \( V \). \( \square \)

\section{4.3 Comparison principles and uniqueness results}

In order to prove that our value function is the \textit{unique} viscosity solution of (4.2) with initial condition (4.3), it will be convenient to add a linear term in (4.2). We begin first by briefly analysing the classical case. To this end, we use the transformed equation

\[
\left( -V_t + \beta V + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \right) (T - t, x, r) = 0, \tag{4.20}
\]

where \( \beta < 0 \) and \((T - t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \).

\textbf{Definition 4.11} A function \( U \) (resp., \( V \)) \( \in C^{1,1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R} \) is called a subsolution (resp., supersolution) of (4.20) if \( U \) (resp., \( V \)) fulfills the following inequality:

\[
0 \leq \left( -U_t + \beta U + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi U \right) (T - t, x, r), \quad \text{resp.,} \quad 0 \geq \left( -V_t + \beta V + \sup_{\xi \in \mathbb{R}^d} \mathcal{L}^\xi V \right) (T - t, x, r)
\]

for all \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \).

The next lemma shows that one may consider w.l.o.g. the HJB equation in this useful form.

\textbf{Lemma 4.12} Assume that \( U \) (resp., \( V \)) \( \in C^{1,1,2}(0, T] \times \mathbb{R}^d \times \mathbb{R} \) is a subsolution (resp., supersolution) of (4.2). Then, \( \overline{U}(T - t, x, r) := \exp(\beta(T - t))U(T - t, x, r) \) (resp., \( \overline{V}(T - t, x, r) := \exp(\beta(T - t))V(T - t, x, r) \)) is a subsolution (resp., supersolution) of (4.20).

\textit{Proof} Through straightforward calculations. \( \square \)
Remark 4.13 In the classical case, the common argument, which consists in penalizing the supersolution and then working toward a contradiction (see, e.g., Pham 2009 for the polynomial case) does not seem to work out here. If we followed the idea of the previously mentioned work, we would be looking for a function $\varphi$ such that for every $\varepsilon > 0$, $U$ subsolution, and $V$ supersolution it should hold

$$\lim_{|x|, |r| \to \infty} \sup_{[0,T]} (U - V_\varepsilon)(T-t, x, r) \leq 0 \quad \text{for all} \quad \varepsilon > 0,$$

where $V_\varepsilon = \varepsilon \varphi + V$ is a supersolution (as linear positiv combination of two supersolutions). However, $(V_\varepsilon)_r$ has to be strictly positive in order for $V_\varepsilon$ to be a supersolution, and this seems to be difficult (even impossible) to obtain when (4.21) is satisfied (recall also the growth condition imposed on $U$ and $V$ and the singularity in the initial condition).

4.3.1 Strong comparison principle for viscosity solutions

Since our value function is continuous, we can restrict the associated comparison principle to continuous functions (i.e., we do not deal here with definitions of lower or upper semi-continuous functions). Note that there are several comparison principles for unbounded viscosity solutions; for instance, the comparison principle for nonlinear upper semi-continuous functions) Note that there are several comparison principles for unbounded viscosity solutions; for instance, the comparison principle for nonlinear degenerate parabolic equations of Koike and Ley (2011). Nevertheless, this methodology cannot be applied here, since the requirements (13), (14) and (15) in Koike and Ley (2011) are not satisfied in our case.

In order to prove the strong comparison principle in our framework, we first need to introduce an equivalent definition of viscosity solution, with the help of superjets and subsjets (see, e.g., Pham 2009).

Definition 4.14 Let $U$ be a continuous function on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$. The second-order superjet of $U$ at a point $(t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ is the set $J^{2,+}U(T-t^*, x^*, r^*)$ of elements $(\tilde{q}, \tilde{p}, \tilde{s}, \tilde{m}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ satisfying

$$U(T-t, x, r) \leq U(T-t^*, x^*, r^*) + \tilde{q}(t-t^*) + \tilde{p} \cdot (x-x^*) + \tilde{s}(r-r^*) + \frac{1}{2} \tilde{m}(r-r^*)^2 + o(|t-t^*| + |x-x^*| + |r-r^*|^2).$$

(4.22)

Analogously, we can define the second-order subjet of a continuous function $V$, defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$, at a point $(t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$: this is the set of elements $(\tilde{q}, \tilde{p}, \tilde{s}, \tilde{m}) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ satisfying

$$V(T-t, x, r) \geq V(T-t^*, x^*, r^*) + \tilde{q}(t-t^*) + \tilde{p} \cdot (x-x^*) + \tilde{s}(r-r^*) + \frac{1}{2} \tilde{m}(r-r^*)^2 + o(|t-t^*| + |x-x^*| + |r-r^*|^2).$$

(4.23)

We denote this set by $J^{2,-}V(T-t^*, x^*, r^*)$.
Remark 4.15 Let \((t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) be a local minimizer of \(V - \varphi)(T - t, x, r)\), where \(\varphi \in C_{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})\). Then, a second-order Taylor expansion of \(\varphi\) yields

\[
V(t - T, x, r) \geq V(t - t^*, x^*, r^*) - \varphi(t - t^*, x^*, r^*) + \varphi(t - t, x, r) \\
= V(t - t^*, x^*, r^*) - \varphi_1(T - t^*, x^*, r^*)(t - t^*) + \nabla_x \varphi(t - t^*, x^*, r^*)(x - x^*) \\
+ \varphi_r(T - t^*, x^*, r^*)(r - r^*) + \frac{1}{2}\varphi_{rr}(T - t^*, x^*, r^*)(r - r^*)^2 \\
+ o(|t - t^*| + |x - x^*| + |r - r^*|^2),
\]

which implies that

\[
(\varphi_1, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \in J^2_- V(T - t^*, x^*, r^*). \tag{4.25}
\]

Similarly, for \(U\) we consider \((t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) to be a local maximizer of \((U - \varphi)(T - t, x, r)\). Then,

\[
U(t - T, x, r) \leq U(t - t^*, x^*, r^*) - \varphi(t - t, x, r) - \varphi(t - t^*, x^*, r^*) \\
= U(t - t^*, x^*, r^*) - \varphi_1(T - t^*, x^*, r^*)(t - t^*) + \nabla_x \varphi(t - t^*, x^*, r^*)(x - x^*) \\
+ \varphi_r(T - t^*, x^*, r^*)(r - r^*) + \frac{1}{2}\varphi_{rr}(T - t^*, x^*, r^*)(r - r^*)^2 \\
+ o(|t - t^*| + |x - x^*| + |r - r^*|^2),
\]

implying

\[
(\varphi_1, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) \in J^2_+ U(T - t^*, x^*, r^*). \tag{4.27}
\]

Actually, the converse property also holds: for any \((q, p, s, m) \in J^2_+ U(T - t^*, x^*, r^*)\), there exists \(\varphi \in C_{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})\) such that

\[
(\varphi_1, \nabla_x \varphi, \varphi_r, \varphi_{rr})(T - t^*, x^*, r^*) = (q, p, s, m).
\]

See Lemma 4.1 in Fleming and Soner (2006) for a construction of such a \(\varphi\). \(\Box\)

The next lemma provides an alternative characterization of a viscosity solution of the equation (4.20).

Lemma 4.16 Let \(v\) be a continuous function on \([0, T] \times \mathbb{R}^d \times \mathbb{R}\).

(i) Then, \(v\) is a viscosity subsolution of (4.20) on \([0, T] \times \mathbb{R}^d \times \mathbb{R}\) if and only if for all \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) and all \((q, p, s, m) \in J^2_+ v(T - t, x, r)\) we have

\[
0 \leq q + \beta v(T - t, x, r) + \frac{x^\top \sum x}{2} m + b \cdot x s + \sup_{\xi \in \mathbb{R}^d} \left(\xi^\top p - sf(\xi)\right). \tag{4.28}
\]
(ii) Respectively, $v$ is a viscosity supersolution of (4.20) on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$ if and only if for all $(t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ and all $(q, p, s, m) \in \mathcal{J}_2^2$, $v(T - t, x, r)$ we have

$$0 \geq q + \beta v(T - t, x, r) + \frac{x^\top \Sigma x}{2} m + b \cdot x s + \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top p - sf(\xi) \right). \tag{4.29}$$

With this at hand, the following strong comparison principle can be established. The first part of its proof, which is given in the next section, will be similar to what can be found in Pham (2009), requiring a few adaptations because of growth and boundary conditions. Moreover, since we use the local definition of viscosity solution, and since the considered functions are continuous, we do not need to penalize the supersolution. In particular, we do not need to use the Crandall–Ishii lemma in the last part of our proof: indeed, in our HJB equation, the second derivative term is only one-dimensional and we thus only have to apply the Taylor formula to find adequate elements of the sub- and superjet of $U$ and $V$, respectively, to work toward a contradiction.

**Theorem 4.17** Let $U$ (resp., $V$) be a continuous viscosity subsolution (resp., continuous viscosity supersolution) of (4.20), defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$, satisfying the growth conditions

$$V_2(t, x, r) \leq v(t, x, r) \leq V_1(t, x, r) \quad \text{for all} \quad (t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \tag{4.30}$$

(where $v$ can be chosen to be $U$ or $V$). Moreover, suppose that $U$ and $V$ satisfy the boundary condition

$$\limsup_{t \to 0} (U(t, x, r) - V(t, x, r)) \leq 0, \quad \text{for fixed} \quad x, r \in \mathbb{R}^d \times \mathbb{R}. \tag{4.31}$$

Then $U \leq V$ on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$.

The following uniqueness result directly follows from the above theorem (Fig. 1).

**Corollary 4.18** The value function defined in (2.8) is the unique viscosity solution of (4.2) with initial condition (4.3).

**Remark 4.19** In the one-dimensional framework, adding a term of the form $\varepsilon V_{xx}$ in Eq. (4.20), with $\varepsilon > 0$, does not change the conclusion of the preceding theorem: indeed, we can apply step by step the same arguments as in the proof of Theorem 4.17 to obtain the analogous conclusion for the strong comparison result. This allows us to approximate our degenerate parabolic equation through non-degenerate parabolic ones, which also fulfill a strong comparison result. The corresponding setting in our optimal control problem consists in adding an $\varepsilon$-noise to the controlled process $X$, by setting:

$$dX_t = -\xi_t + \varepsilon dW_t,$$
where \((W_t)\) is a Brownian motion independent of \((B_t)\). With this at hand, we can derive the corresponding non-degenerate HJB equation

\[
-V_t + \frac{X^2 \sigma^2}{2} V_{rr} + \varepsilon V_{xx} + b \cdot X V_r + \sup_{\xi \in \mathbb{R}^d} (\xi \cdot \nabla_x V - f(\xi)V_r).
\]

In the d-dimensional framework, things can become more complicated, and we have to use among others Crandall–Ishii’s lemma to find the corresponding sub- and superjet associated with the second-order terms in order to prove a comparison result for the non-degenerate parabolic equation.

\[\Box\]

Appendix A: Proofs of Lemmas and Theorems

A.1 Proofs of the main results from Sect. 2

Proof of Lemma 2.9 First note that we have the canonical inclusion \(\hat{\mathcal{X}}^1(T^n, X^0_n) \subseteq \hat{\mathcal{X}}^1(T, X^0_n)\), by setting \(\zeta^n = 0\) on \([T^n, T]\). Now, we wish to prove that \(\int_0^T \zeta \, dt = X_0\).

Suppose by way of contradiction that \(\int_0^T \zeta \, dt \neq X_0\). Then, there exists a component \(\zeta^i\) such that \(\int_0^T \zeta^i \, dt \neq X^i_0\). Thus, we can assume without loss of generality that \(d = 1\) and work toward a contradiction. Under this assumption, there exists a measurable set \(\mathcal{A}\) with \(\mathbb{P}(\mathcal{A}) > 0\), such that \(\int_0^T \zeta_t \, dt > X_0\) on \(\mathcal{A}\), or \(\int_0^T \zeta_t \, dt < X_0\) on \(\mathcal{A}\). Without loss of generality, we can assume that

\[\int_0^T \zeta_t \, dt > X_0 \quad \text{on } \mathcal{A}. \tag{A.1}\]
Because $\zeta^n \in \hat{X}^1(T^n, X^n_0)$ converges to $\zeta$, weakly in $L^1$, we have

\[
0 = \mathbb{E} \left[ \left( X^n_0 - \int_0^{T^n} \zeta^n_t \, dt \right) \mathbb{1}_A \right] = \mathbb{E} \left[ \left( X^n_0 - \int_0^{T} \zeta^n_t \, dt \right) \mathbb{1}_A \right] \rightarrow \mathbb{E} \left[ \left( X_0 - \int_0^{T} \zeta_t \, dt \right) \mathbb{1}_A \right] = 0.
\]

If $\overline{T} = T$ the result is proved, because the expectation on the right-hand side has to be negative, due to the assumption (A.1); this is a contradiction.

Suppose now that $\overline{T} > T$. It is sufficient to show that $\zeta = 0$ on $[T, \overline{T}]$. To this end, set

\[
\eta_t(\omega) := \mathbb{1}_{\{\zeta_t(\omega) > 0\}} \mathbb{1}_{[T, \overline{T}]}(t).
\]

Analogously, we get

\[
0 = \mathbb{E} \left[ \int_{T^n}^{\overline{T}} \zeta^n_t \eta_t \, dt \right] \rightarrow \mathbb{E} \left[ \int_{T}^{\overline{T}} \zeta_t \eta_t \, dt \right] = 0,
\]

due to the weak convergence of $\zeta^n$ to $\zeta$, the fact that $\eta \in L^\infty((\Omega \times [0, \overline{T}], F \otimes B([0, \overline{T}]), (\mathbb{P} \otimes \lambda); \mathbb{R}^d))$, and $\zeta_n = 0$ on $[T^n, \overline{T}]$. Thus, $\{\zeta_t(\omega) > 0; t \in [T, \overline{T}]\}$ is a null set. Taking $\eta_t(\omega) := \mathbb{1}_{\{\zeta_t(\omega) > 0\}} \mathbb{1}_{[T, \overline{T}]}(t)$, we can prove in the same manner that $\{\zeta_t(\omega) < 0; t \in [T, \overline{T}]\}$ is a null set. Hence, $\zeta = 0$ on $[T, \overline{T}]$ and therefore $\int_0^{\overline{T}} \zeta \, dt = X_0$.

Using Corollary 2.8 we infer

\[
\mathbb{E} \left[ \int_0^{T} f(-\zeta_t) \, dt \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \int_0^{T} f(-\zeta^n_t) \, dt \right] \leq c,
\]

which concludes the proof. \qed

Proof of Lemma 2.12 Set $\overline{M} := \overline{M}(T, X_0, R_0) = 1 + 1/A_1 - V_2(T, X_0, R_0)$. We first note that, due to (2.15), we have

\[
\mathbb{E} \left[ e^{-A_1 \left( R_0 + \int_0^{T} (X^n_t)^\top \sigma \, dB_t + \int_0^{T} b \cdot X^n_t \, dt - \int_0^{T} f(-\xi^n_t) \, dt \right)} \right] \leq 1/A_1 - V_2(T, X_0, R_0) = \overline{M}.
\]

We want to show that

\[
\xi^n \in \tilde{K}_\alpha := \left\{ \xi \in \hat{X}^1(T, X_0) | \mathbb{E} \left[ \int_0^{T} -b \cdot X_t^\xi + f(-\xi_t) \, dt \right] \leq \alpha \right\}, \quad \text{(A.2)}
\]
for $\alpha \geq \frac{M-1}{A_1} + R_0$. To prove (A.2), we use the fact that $e^x \geq 1 + x$, for all $x \in \mathbb{R}$, as well as the martingale property of $Y_T := \int_0^T (X_t^\xi)^\top \sigma \, dB_t$ [which is satisfied, due to (2.3)], whence we infer

$$\overline{M} \geq \mathbb{E} \left[ -A_1 \left( R_0 + \int_0^T b \cdot X_t^\xi \, dt - \int_0^T f(-\xi_t^n) \, dt \right) \right] + 1.$$

Then

$$\mathbb{E} \left[ \int_0^T -b \cdot X_t^\xi + f(-\xi_t^n) \, dt \right] \leq \frac{M-1}{A_1} + R_0,$$

and therefore (A.2) is true.

Using now Lemma 2.11 we obtain (when setting $N := \|b\|CT^2$):

$$\alpha \geq \frac{M-1}{A_1} + R_0 \geq \mathbb{E} \left[ \int_0^T -b \cdot X_t^\xi + f(-\xi_t^n) \, dt \right] \geq \frac{3}{4} \mathbb{E} \left[ \int_0^T f(-\xi_t^n) \, dt \right] - N.$$

Finally, for $m \geq \frac{4}{3}(\alpha + N)$ we get

$$\mathbb{E} \left[ \int_0^T f(-\xi_t^n) \, dt \right] \leq m,$$

which shows that $\xi^n \in \mathcal{K}_m$. □

**Proof of Lemma 2.11** Set $\gamma := \frac{1}{4\|b\|T}$. Because $\lim_{|x| \to \infty} \frac{|x|}{f(x)} = 0$, there exists a constant $C_\gamma = C > 0$ such that $\frac{|y|}{f(y)} \leq \gamma$ for $|y| > C$. Consider now the set $A_t := \{ \|\xi_t\| \leq C \}$. Then we have using integration by parts:

$$\int_{t_1}^{t_2} \left( -b \cdot X_t^\xi + f(-\xi_t) \right) \, dt$$

$$\geq b \cdot (t_1 X_{t_1}^\xi - t_2 X_{t_2}^\xi) - \int_{t_1}^{t_2} 1_{A_t} |b \cdot \xi_t| t \, dt + \int_{t_1}^{t_2} 1_{A_t} f(-\xi_t) \, dt$$

$$\geq b \cdot (t_1 X_{t_1}^\xi - t_2 X_{t_2}^\xi) + \int_{t_1}^{t_2} 1_{A_t} f(-\xi_t) \left( 1 + \frac{b \cdot \xi_t}{f(-\xi_t)} \right) \, dt$$

$$\geq b \cdot (t_1 X_{t_1}^\xi - t_2 X_{t_2}^\xi) + \frac{1}{4} \int_{t_1}^{t_2} 1_{A_t} f(-\xi_t) \, dt + \frac{3}{4} \int_{t_1}^{t_2} f(-\xi_t) \, dt - |b|CT^2/2,$$

using the above estimates. This proves the lower inequality. To prove the upper inequality, it is sufficient to follow step by step the preceding arguments and to give an upper bound of the corresponding terms, instead of a lower bound. □
A.2 Proofs of the main results from Sect. 3

To prove proposition 3.1, we need first to prove the following lemma

Lemma A.1 Let \( g \) be a real-valued locally integrable function on \([0, \infty[\) such that

\[
\int_0^x g(t) \, dt \geq 0, \quad \text{for all } x > 0. \quad (A.3)
\]

Then \( \lim sup_{x \to \infty} g(x) \geq 0 \).

**Proof** Suppose in contrary that there exists \( \varepsilon > 0 \) such that \( \lim sup_{x \to \infty} g(x) < -2\varepsilon \).

Then there exists \( x_0 > 0 \) such that \( g(x) \leq -\varepsilon \) for all \( x \geq x_0 \), whence we get

\[
\int_0^x g(t) \, dt \leq \int_0^{x_0} g(t) \, dt - \varepsilon(x - x_0) < 0 \quad \text{for } x \text{ large enough},
\]

which is in contradiction with (A.3). \( \square \)

**Proof of Proposition 3.1** By translating \( u \) horizontally if necessary, we can assume without loss of generality that \( R_0 = 0 \). Hence it is sufficient to prove

\[
\mathbb{E}[u'(R_T^\xi - 1)] < \infty. \quad (A.4)
\]

Due to inequalities (2.7), we get

\[
\exp(A_2x) + u(-x) = \int_0^x \left( \frac{1}{A_2} \exp(A_2x) - u'(-x) \right) dx + u(0) - \frac{1}{A_2} \geq 0, \quad x \geq 0.
\]

Hence, by translating \( u \) vertically if necessary, the conditions of Lemma A.1 apply with \( g(x) = \frac{1}{A_2} \exp(A_2x) - u'(-x) \) on \([0, \infty[\). Therefore, we can find a constant \( C > 0 \) such that

\[
u'(-x) \leq C(\exp(A_2x) + 1) \quad \text{for all } x \geq 0.
\]

Thus,

\[
\mathbb{E}[u'(R_T^\xi - 1)] \leq C(\mathbb{E}[\exp(-A_2R_T^\xi)] + 1) + \mathbb{E}[u'(R_T^\xi - 1) 1_{R_T^\xi \leq 0}] < \infty,
\]

since \( u' \) is bounded on \([0, \infty[\) and \( \mathbb{E}[\exp(-A_2R_T^\xi)] < \infty \), due to the assumption on \( \xi \). This shows the assertion for the first derivative. For the second one, we take \( 0 < \eta < 1 \) and \( r \in ]-\eta, \eta[ \). We wish to prove that

\[
\sup_{r \in ]-\eta, \eta[} \mathbb{E}[u''(R_T^\xi + r)] < \infty. \quad (A.5)
\]
To this end, we use inequality (2.5) to obtain

\[ \mathbb{E}[|u''(R_T^\xi + r)|] \leq \mathbb{E}[A_2u'(R_T^\xi - 1)] < \infty, \]

which completes the proof. \(\square\)

**Proof of Lemma 3.2** By translating the function \(f\) if necessary, we can suppose without loss of generality that \(t = 0\). Because \(f\) is concave in a neighborhood of \(t = 0\), we only have to prove that \(f'_+(0) \geq f'_-(0)\). To this end, let \(\varepsilon > 0\) and \(\alpha^*_0 \in A\) be such that \(f(0) = f_{\alpha^*_0}(0)\). Because \(f_{\alpha^*_0}\) is concave and differentiable at 0, for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for all \(0 < h < \delta\), we have

\[ \frac{f_{\alpha^*_0}(h) - f_{\alpha^*_0}(0)}{h} \geq \frac{f_{\alpha^*_0}(-h) - f_{\alpha^*_0}(0)}{-h} - \varepsilon. \]

Thus we get

\[ \frac{f(h) - f(0)}{h} \geq \frac{f_{\alpha^*_0}(-h) - f_{\alpha^*_0}(0)}{-h} - \varepsilon \geq \frac{f(-h) - f(0)}{-h} - \varepsilon, \]

by the definition of \(f\). Sending \(h\) to zero we infer \(f'_+(0) \geq f'_-(0) \geq f'_-(0) - \varepsilon\) for every \(\varepsilon > 0\), and hence \(f\) is differentiable.

Assume now that \(\alpha^*_t\) is uniquely determined, and suppose to the contrary that \(f'\) is not continuous at \(t\). Since \(f\) is concave on \(|t - \eta, t + \eta|\) and hence \(f'\) is nonincreasing on \(|t - \eta, t + \eta|\), the left- and right-hand limits at \(t\) exist, and we infer

\[ f'(t^-) = f'_{\alpha^*_t}((t^-)) > f'(t^+) = f'_{\alpha^*_t}(t^+), \]

where \(\alpha^*_t, \alpha^*_t \in A\). Using the continuity of \(f'_{\alpha^*_t}\) at \(t\), we must have, on the one hand, \(\alpha^*_t \neq \alpha^*_t\). However, we must equally have, on the other hand,

\[ f(t) = f_{\alpha^*_t}(t) = f(t^+) = f_{\alpha^*_t}(t^+) = f_{\alpha^*_t}((t^-)), \]

as a direct consequence of the definition of \(\alpha^*_t\) and the continuity of \(f\). Therefore, the uniqueness of \(\alpha^*_t\) implies \(\alpha^*_t = \alpha^*_t = \alpha^*_t\), which is clearly a contradiction. \(\square\)

**Proof of Proposition 3.6** Take \((T, X_0, R_0) \in ]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}\) and let \(\{T^n, X^n_0, R^n_0\}_n\) be a sequence that converges to \((T, X_0, R_0)\). We have to show that

\[ \limsup_n V(T^n, X^n_0, R^n_0) \leq V(T, X_0, R_0). \]  

(A.6)

Since \((T^n, X^n_0, R^n_0)\) and \(V(T^n, X^n_0, R^n_0)\) are bounded, it follows that \(\limsup_n V(T^n, X^n_0, R^n_0) < \infty\), in conjunction with (2.10). Taking a subsequence if necessary, we can suppose that \((V(T^n, X^n_0, R^n_0))\) converges to \(\limsup_n V(T^n, X^n_0, R^n_0)\). Let \(\xi^n\) be the optimal strategy associated to \(V(T^n, X^n_0, R^n_0)\), which exists for every \(n \in \mathbb{N}\), Springer
due to Theorem 2.4. In the sequel we prove, as in Lemma 2.12, that the sequence $\xi^n$ lies in a weakly sequentially compact set. Note that this proposition can be proved without using Assumption 2.15.

**First step** We set $\tilde{T} := \sup_n T^n$. We will show that, for every $n \in \mathbb{N}$, we have $\xi^n \in \overline{K}_m$, provided that $m$ is large enough, where

$$
\overline{K}_m = \left\{ \xi \in \overline{C}(\dot{X}^1(T^n, X^n_0))_n \mid \mathbb{E}\left[ \int_0^{\tilde{T}} f(-\xi_t) \, dt \right] \leq m \right\},
$$

and where $\overline{C}(\dot{X}^1(T^n, X^n_0))_n$ denotes the closed convex hull of the sequence of sets $(\dot{X}^1(T^n, X^n_0))_n$. To this end, we use Remark 2.13, noting that we can choose $\xi^n \in \overline{K}_m$, where $m_n$ has to be chosen such that

$$
m_n \geq \frac{4}{3} \left( \frac{-V_2(\tilde{T}, X^n_0, R^n_0)}{A_1} + R^n_0 + N \right),
$$

and $N$ depends only on $f$, $b$ and $\tilde{T}$. Take now $m \in \mathbb{R}$ such that $m \geq \sup_n m_n$. Note that such $m$ exists, because $(X^n_0, R^n_0)$ is bounded and $V_2$ is continuous. Then it follows that

$$
\mathbb{E}\left[ \int_0^{\tilde{T}} f(-\xi^n_t) \, dt \right] \leq m \quad \text{for all} \quad n \in \mathbb{N}.
$$

Taking now the convex hull of the sequence of sets $(\dot{X}^1(T^n, X^n_0))_n$, we conclude that $\xi^n \in \overline{K}_m$ for all $n \in \mathbb{N}$.

**Second step** We will prove that $\overline{K}_m$ is weakly sequentially compact. To this end, we will first prove that it is a closed convex set in $L^1$.

The set $\overline{K}_m$ is convex, because the map $\xi \mapsto \mathbb{E}\int_0^{\tilde{T}} f(-\xi_t) \, dt$ is convex (due to the convexity of $f$) and defined on the convex set $\overline{C}(\dot{X}^1(T^n, X^n_0))_n$. We will show that it is closed with respect to the $L^1$-norm. Denote by $\overline{C}(X^n_0)_n$ the closed convex hull of the sequence $(X^n_0)_n$, which is bounded in $\mathbb{R}^d$. We show that for $\xi \in \overline{K}_m$ there exists $\tilde{X}$ in $\overline{C}(X^n_0)_n$ such that $\xi \in \dot{X}^1(\tilde{T}, \tilde{X})$. To this end, we write $\xi$ as a convex combination of $\xi^{n_i} \in \dot{X}^1(T^{n_i}, X^{n_i}_0)$,

$$
\xi = \lambda_1 \xi^{n_1} + \cdots + \lambda_s \xi^{n_s},
$$

where $\sum_{i=1}^s \lambda_i = 1$, $\lambda_i \geq 0$. By expressing then the constraint on $\xi^{n_i}$, we get

$$
\lambda_i \int_0^{T^{n_i}} \xi_t^{n_i} \, dt = \lambda_i X_0^{n_i},
$$
which implies

$$\int_0^T \xi_i \, dt = \sum_{i=1}^s \lambda_i \int_0^{T_i} \xi_{ni} \, dt = \sum_{i=1}^s \lambda_i X_{ni}^0 = \tilde{X}.$$  

Take now a sequence $(\tilde{\xi}^q)_q$ of $\overline{K}_m$ that converges in the $L^1$-norm to a liquidation strategy $\tilde{\xi}$. We prove that $\tilde{\xi} \in \mathcal{X}^1(\tilde{T}, \tilde{X})$ for $\tilde{X} \in \overline{C}(X_0^0)_n$. As previously remarked, there exists a sequence $(\tilde{X}^q)_q \subset \overline{C}(X_0^0)_n$ such that $\tilde{\xi}^q \in \mathcal{X}^1(\tilde{T}, \tilde{X}^q)$. Hence, we have

$$\int_0^T \tilde{\xi}^q \, dt = \tilde{X}^q, \quad \mathbb{P}\text{-a.s.}$$

Replacing $(\tilde{X}^q)_q$ by a subsequence if necessary, we can suppose that it converges to some $\tilde{X}$, because this sequence is bounded. Moreover, $\tilde{X}$ lies in $\overline{C}(X_0^0)_n$. Since $(\tilde{\xi}^q)_q$ converges weakly to $\tilde{\xi}$, we are now in the setting of Lemma 2.9, which ensures that $\tilde{\xi} \in \mathcal{X}^1(\tilde{T}, \tilde{X})$, as well as $\mathbb{E}[\int_0^T f(-\tilde{\xi}_i)] \leq M$. Hence, this proves that $\overline{K}_m$ is a closed subset of $L^1$. Since $\overline{K}_m$ is convex, it is also closed with respect to the weak topology of $L^1$. Thus, it is sufficient to prove that $\overline{K}_m$ is uniformly integrable. To this end, take $\varepsilon > 0$ and $\xi \in \overline{K}_m$. There exists $\alpha > 0$ such that $f(\xi_i) \leq \frac{\varepsilon}{\alpha}$, for $|\xi_i| > \alpha$, due to the superlinear growth property of $f$. Because $f(x) = 0$ if and only if $x = 0$, the term $1/f(-\xi_i)$ is well-defined on $\{|\xi_i| > \alpha\}$, hence

$$\mathbb{E} \left[ \int_0^T \mathbb{1}_{|\xi_i| > \alpha} |\xi_i| \, d\xi_i \right] \leq \mathbb{E} \left[ \int_0^T \mathbb{1}_{|\xi_i| > \alpha} f(-\xi_i) \, dt \right] \frac{\varepsilon}{c} \leq \varepsilon,$$

which proves the uniform integrability of $\overline{K}_m$.

**Last step** We have proved that $(\xi^n)_n$ is a sequence in the weakly sequentially compact set $\overline{K}_m$. Thus, there exist a subsequence $\xi^n_{nk}$ of $\xi^n$ and some $\tilde{\xi} \in \overline{K}_m$ such that $\xi^n_{nk}$ converges to $\tilde{\xi}$, weakly in $L^1$. We are here again in the settings of Lemma 2.9, which allows us to deduce that $\tilde{\xi} \in \mathcal{X}^1(T, X_0)$. Finally, because $\tilde{\xi} \mapsto \mathbb{E}[u(R^\tilde{\xi}_T)]$ is upper semi-continuous with respect to the weak topology of $L^1$, due to Proposition 2.14, we get

$$\limsup_n V(T^n, X^n_0, R^n_0) = \limsup_k \mathbb{E} \left[ u \left( R^{\xi_{nk}}_T \right) \right] \leq \mathbb{E} \left[ u \left( R^T_{\tilde{\xi}} \right) \right] \leq V(T, X_0, R_0),$$

where the last inequality is due to the definition of $V$ at $(T, X_0, R_0)$ and the fact that $\tilde{\xi} \in \mathcal{X}^1(T, X_0)$. This concludes the proof of the upper semi-continuity of $V$. \hfill $\square$

**Proof of Proposition 3.10** Let $(T, X_0, R_0) \in [0, \infty[ \times \mathbb{R}^d \times \mathbb{R}$ and $(T^n, X^n_0, R^n_0)_n$ be a sequence that converges to $(T, X_0, R_0)$. We have to show that

$$\liminf_n V(T^n, X^n_0, R^n_0) \geq V(T, X_0, R_0). \quad (A.7)$$
We split the proof of (A.7) in two parts; first we will assume that $T_n \downarrow T$, second we will assume that $T_n \uparrow T$ (for this latter case, we will use Proposition 3.9).

First case Suppose that $T_n \downarrow T$. We set

$$\lambda_n := \begin{cases} |X_0^n - X_0|, & \text{if } |X_0^n - X_0| \neq 0, \\ \frac{1}{n}, & \text{otherwise}, \end{cases} \quad (A.8)$$

which belongs to $[0, 1[$, for $n$ large enough. Let now $\widehat{X}_0^n \in \mathbb{R}^d$ be such that $X_0^n = (1 - \lambda_n)X_0 + \lambda_n \widehat{X}_0^n$ and consider the sequence of strategies

$$\xi^n_t := (1 - \lambda_n)\xi^*_t + \lambda_n \widehat{\xi}_n^t,$$

where $\xi^*$ is the optimal strategy associated to $V(T, X_0, R_0)$, and $\widehat{\xi}^n$ is the optimal strategy associated to $V_2(T_n, \widehat{X}_0^n, R_n^0)$. Note that, due to the choice of $\lambda_n$, the vector $\widehat{X}_0^n$ is bounded: indeed, we have

$$\widehat{X}_0^n = \frac{X_0^n - X_0}{\lambda_n} + \lambda_n + X_0,$$

which is bounded, due to the boundedness of $X_0^n$ and the definition of $\lambda_n$. Hence, $V_2(T_n, \widehat{X}_0^n, R_n^0)$ is bounded in $n$, which implies that $\int_0^{T_n} f(-\widehat{\xi}_t^n) \, dt$ is again bounded in $n$. Since $f$ has superlinear growth and is positive, the integral $\int_0^{T_n} |-\widehat{\xi}_t^n| \, dt$ is also bounded in $n$.

Observe that

$$\int_0^{T_n} \xi_t^n \, dt = (1 - \lambda_n) \int_0^{T_n} \xi_t^* \, dt + \lambda_n \int_0^{T_n} \widehat{\xi}_t^n \, dt = (1 - \lambda_n)X_0 + \lambda_n \widehat{X}_0^n = X_0^n,$$

where the last equality follows with $T_n \geq T$ and the fact that $\xi_t^* = 0$ for $t \geq T$. Moreover, $\xi^n$ verifies (2.3), due to the convexity of $f$ and the boundedness of $\widehat{\xi}^n$, whence $\xi^n \in \mathcal{X}_{2,A_2}^{1}(T^n, X_0^n)$.

We now show that

$$\mathcal{R}_{T_n}^{\widehat{\xi}^n} = \int_0^{T_n} (X_t^{\widehat{\xi}^n})^\top \sigma \, dB_t + \int_0^{T_n} b \cdot X_t^{\widehat{\xi}^n} \, dt - \int_0^{T_n} f(-\xi_t^n) \, dt \xrightarrow{n \to \infty} \mathcal{R}_T^{\xi^*}, \quad \mathbb{P}\text{-a.s.,} \quad (A.9)$$

by individually considering each term, starting from the left.

Because $\int_0^{T_n} |\widehat{\xi}_t^n| \, dt$ is uniformly bounded, $\xi^n$ converges to $\xi^*$ in $L^1[0, T], \mathbb{P}$-a.s. Indeed, we write

$$\mathbb{E}\left[ \int_0^{T_n} |\xi_t^n - \xi_t^*| \, dt \right] = \lambda_n \left( \mathbb{E}\left[ \int_0^{T_n} |\widehat{\xi}_t^n| \, dt \right] + \mathbb{E}\left[ \int_0^{T_n} |\xi_t^n| \, dt \right] \right) \xrightarrow{n \to \infty} 0.$$
Therefore, Lemma 3.8 yields

\[
\int_0^T (X^k_n)_t \sigma \, dB_t \xrightarrow{n \to \infty} \int_0^T (X^\xi^*_n)_t \sigma \, dB_t.
\]

Due to

\[
X^k_n(t) = (1 - \lambda_n)X^\xi^*_n(t) + \lambda_n X^\hat{\xi}^n(t) \quad \mathbb{P}\text{-a.s. for all } t \in [0, T],
\]

we can express the second integral in (A.9) as follows:

\[
\int_0^T b \cdot X^k_n \, dt = (1 - \lambda_n) \int_0^T b \cdot X^\xi^*_n \, dt + \lambda_n \int_0^T b \cdot X^\hat{\xi}^n \, dt,
\]

which converges \(\mathbb{P}\)-a.s. to \(\int_0^T b \cdot X^k_n \, dt\), because \(\int_0^T b \cdot X^\hat{\xi}^n \, dt\) is uniformly bounded and \(\lambda_n\) is a null sequence.

We now prove that

\[
\int_0^T f \left( -\left(1 - \lambda_n\right)\xi^*_t - \lambda_n \hat{\xi}^n_t \right) \, dt \xrightarrow{n \to \infty} \int_0^T f \left( -\xi^*_t \right) \, dt, \quad \mathbb{P}\text{-a.s. (A.10)}
\]

Due to the continuity of \(f\), we have

\[
f \left( -\left(1 - \lambda_n\right)\xi^*_t - \lambda_n \hat{\xi}^n_t \right) \xrightarrow{\text{P-a.s.}} f \left( -\xi^*_t \right).
\]

Because \(f\) is convex, we further get

\[
0 \leq f \left( -\left(1 - \lambda_n\right)\xi^*_t - \lambda_n \hat{\xi}^n_t \right) \leq (1 - \lambda_n) f \left( -\xi^*_t \right) + \lambda_n f \left( -\hat{\xi}^n_t \right).
\]

Since \(\int_0^T f \left( -\hat{\xi}^n_t \right) \, dt\) is uniformly bounded in \(n\), the dominated convergence theorem of Lebesgue implies (A.10). Therefore, (A.9) is established, whence again

\[
\lim_n u \left( \mathcal{R}^n_T \right) = u \left( \mathcal{R}^*_T \right) \quad \mathbb{P}\text{-a.s., (A.11)}
\]

using the continuity of \(u\).

Further, with \(L := \sup_n V_2(T, \hat{X}^n_0, \mathcal{R}^n_T)\), we obtain

\[
\exp(-2A_2 \mathcal{R}^n_T) \leq \left( (1 - \lambda_n) \exp(-2A_2 \mathcal{R}^\xi^*_T) + \lambda_n \exp(-2A_2 \mathcal{R}^\hat{\xi}^n_T) \right) \leq \left( (1 - \lambda_n)M_{\mathcal{R}^\xi^*_T}(2A2 + \lambda_n L) < \infty, \right.
\]

because \(\xi \mapsto \exp(-2A\mathcal{R}^\xi_T)\) is convex and \(T^n \geq T\), in conjunction with Assumption 2.15. Therefore, applying Lemma 3.7 gives

\[
\mathbb{E}[u \left( \mathcal{R}^n_T \right)] \xrightarrow{n \to \infty} \mathbb{E}[u \left( \mathcal{R}^\xi^*_T \right)].
\]
Finally, we can write
\[
\liminf_n V(T^n, X_0^n, R_0^n) \geq \liminf_n \mathbb{E} \left[ u \left( R_T^{\xi^n} \right) \right] = \mathbb{E} \left[ u \left( R_T^{\hat{\xi}^n} \right) \right] = V(T, X_0, R_0),
\]
which proves (A.7) when \( T^n \downarrow T \).

**Second case** Suppose now that \( T^n \uparrow T \). We let \( \lambda_n \) and \( \hat{X}_0^n \in \mathbb{R}^d \) as in (A.8) and consider the following sequence of strategies
\[
\xi^n_t := (1 - \lambda_n)\xi^{*,n}_t + \lambda_n \hat{\xi}^n_t,
\]
where \( \xi^{*,n} \) is the optimal strategy associated to \( V(T^n, X_0^n, R_0^n) \) and \( \hat{\xi}^n \) is the optimal strategy associated to \( V_2(T^n, \hat{X}_0^n, R_0^n) \).

As above, we can show that \( \xi^n \in \mathcal{A}_2(T^n, X_0^n) \), wherefore
\[
\liminf_n V(T^n, X_0^n, R_0^n) \geq \liminf_n \mathbb{E} \left[ u \left( (1 - \lambda_n)\xi^{*,n} + \lambda_n \hat{\xi}^n \right) T^n \right] \\
\geq \liminf_n (1 - \lambda_n) V(T^n, X_0^n) + \liminf_n \lambda_n V_2(T^n, \hat{X}_0^n, R_0^n) \\
\geq V(T, X_0, R_0).
\]

Here, we have used the concavity of \( \xi \mapsto \mathbb{E} \left[ u \left( R_T^\xi \right) \right] \) for the second inequality, inequality (2.10) for the third one, and Proposition 3.9, in conjunction with the fact that \( V_2(T^n, X_0^n, R_0^n) \) is bounded and \( \lambda_n \) is a null sequence, for the last one. This proves (A.7) when \( T^n \uparrow T \).
We can now prove Lemma 3.15

**Proof of Lemma 3.15** First, note that
\[
\mathcal{R}_T^\xi \circ \phi_\omega (\tilde{\omega}) = \mathcal{R}_{\tau}^\xi (\omega) + \mathcal{R}_{\tau(\omega)}^{\bar{\nu}_T} (\tilde{\omega}) \]
for \( \mathbb{P} \)-a.e. \( \tilde{\omega} \in \Omega \). Due to the fact that \( u \) is bounded from above, we can apply the preceding Lemma to \( H := -u (\mathcal{R}_T^\xi) \) (by translating \( u \) vertically if necessary), and we finally get (when dropping the minus sign in front of \( u \))
\[
\mathbb{E}[u (\mathcal{R}_T^\xi)|\mathcal{F}_\tau] (\omega) = \mathbb{E}[u (\mathcal{R}_T^\xi + \mathcal{R}_{\tau(\omega), T}^{\bar{\nu}_T})],
\]
which proves the lemma. \( \square \)

**Proof of Lemma 3.16** Let \( \tau \leq T \) be a stopping time, \( \xi \in \dot{\mathcal{X}}^1 (T, X_0) \), and denote by
\[
\mathcal{R}_{s,T}^\xi = \int_s^T (X_t^\xi) \sigma d B_t + \int_s^T b \cdot X_t^\xi dt - \int_s^T f(-\xi_t) dt \tag{A.13}
\]
the revenues generated by \( \xi \) over the time interval \([s, T]\). In Schied et al. (2010), there is another convenient formulation of \( \bar{V} \): for every \( \omega \in \Omega \),
\[
\bar{V} (T - \tau(\omega), X_\tau^\xi (\omega), \mathcal{R}_\tau^\xi (\omega)) = \exp \left(-A \mathcal{R}_\tau^\xi (\omega) + A \inf_{\tilde{\xi} \in \dot{\mathcal{X}}_{det}(T - \tau(\omega), X_\tau^\xi (\omega))} \int_\tau^T \mathcal{L}(X_t^{\tilde{\xi}}, \tilde{\xi}_t) dt \right).
\]
Let us next set
\[
Y^\xi = e^{-A \int_\tau^T (X_t^\xi)^\top \sigma d B_t - \frac{1}{2} \int_\tau^T A^2 (X_t^\xi)^\top \Sigma X_t^\xi dt}.
\]
We then have for every \( \xi \in \dot{\mathcal{X}}^1 (T, X_0) \) and almost every \( \omega \in \Omega \):
\[
\mathbb{E}\left[ \exp(-A \mathcal{R}_{\tau,T}^\xi)|\mathcal{F}_\tau\right] (\omega) = \mathbb{E}\left[ Y^\xi \exp \left(A \int_\tau^T \mathcal{L}(X_t^\xi, \xi_t) dt\right) |\mathcal{F}_\tau\right] (\omega) \\
\geq \mathbb{E}\left[ Y^\xi \exp \left(A \inf_{\tilde{\xi} \in \dot{\mathcal{X}}_{det}(T - \tau(\omega), X_\tau^\xi (\omega))} \int_\tau^T \mathcal{L}(X_t^{\tilde{\xi}}, \tilde{\xi}_t) dt\right) |\mathcal{F}_\tau\right] (\omega) \\
= \mathbb{E}\left[ Y^\xi e^{A \mathcal{R}_\tau^\xi (\omega)} \bar{V} (T - \tau(\omega), X_\tau^\xi (\omega), \mathcal{R}_\tau^\xi (\omega)|\mathcal{F}_\tau\right] (\omega) \\
= \exp \left(A \mathcal{R}_\tau^\xi (\omega)\right) \bar{V} (T - \tau(\omega), X_\tau^\xi (\omega), \mathcal{R}_\tau^\xi (\omega)) \mathbb{E}\left[ Y^\xi |\mathcal{F}_\tau\right] (\omega).
\]
Here, we have used (A.13) for the first equality and the monotonicity property of the conditional expectation for the inequality.

It remains to show that
\[ \mathbb{E}\left[Y^\xi | \mathcal{F}_\tau\right] = 1 \quad \mathbb{P}\text{-a.s.} \] (A.14)
Indeed, this will prove the result, because we also have that
\[ \mathbb{E}\left[ \exp\left(-AR^\xi_T\right) | \mathcal{F}_\tau\right](\omega) = \mathbb{E}\left[ \exp\left(-A(R^\xi_{\tau,T} + R^\xi_T(\omega))\right) | \mathcal{F}_\tau\right](\omega) \\
= \exp\left(-AR^\xi_T(\omega)\right)\mathbb{E}\left[ \exp\left(-AR^\xi_{\tau,T}(\omega)\right) | \mathcal{F}_\tau\right](\omega), \]
by using (3.8). To prove (A.14), let us define the following process
\[ Z^\xi_t = e^{-A\int_0^t (X^\xi_u)\top \sigma \, dB_u - \frac{1}{2} \int_0^t A^2(X^\xi_u)\top \Sigma X^\xi_u \, du}, \]
which is a true martingale, due to Girsanov’s theorem (X^\xi fulfills (2.3), due to the assumption on \( \xi \)). Therefore, we have
\[ \mathbb{E}\left[Z^\xi_T | \mathcal{F}_\tau\right] = \mathbb{E}\left[Y^\xi Z^\xi_T | \mathcal{F}_\tau\right] = Z^\xi_T , \]
which proves (A.14) and hence also our lemma.

\[ \square \]

**Proof of Lemma 3.18** We first prove that
\[ \mathbb{E}\left[|V_2(T - \tau, X^\xi_T, R^\xi_T)|\right] < \infty, \] (A.15)
where we have \(|V_2(T, X_0, R_0)| = \inf_{\xi \in \hat{X}(T,X_0)} \mathbb{E}\left[ \exp(-A_2R^\xi_T) \right]\). This is a direct consequence of Lemma 3.16. Indeed, we can write
\[ \mathbb{E}\left[|V_2(T - \tau, X^\xi_T, R^\xi_T)|\right] \leq \mathbb{E}\left[ \exp(-A_2R^\xi_T) | \mathcal{F}_\tau\right] \]
\[ = \mathbb{E}\left[ \exp(-A_2R^\xi_T) \right] < \infty. \]
Here, the first inequality is due to (3.9), and the last one follows from the fact that \( \xi \in \hat{X}_{2A_2}(T, X_0) \). Thus (A.15) follows, and hence, there exists \( N \in \mathbb{N} \) such that
\[ \mathbb{E}\left[(|V_2(T - \tau, X^\xi_T, R^\xi_T)| + 1/A_1)^\frac{1}{2} \mathbb{I}\{|X^\xi| > N\}] \leq \varepsilon. \]
Using
\[ |V(T, X_0, R_0)| \leq |V_2(T, X_0, R_0)| + 1/A_1, \quad (T, X_0, R_0) \in ]0, \infty[ \times \mathbb{R}^d \times \mathbb{R}, \]
which is due to (2.10), we infer (3.11).
**Proof of Theorem 3.19** (existence of the $\varepsilon$-maximizers on a bounded region)

The proof of this result is split in several steps. Let us first consider a simple process $\xi$ which is allowed to take only countably many values and a discrete stopping time $\tau$. The existence of the $\varepsilon$-maximizers is easier to prove in this case, because we are not facing any measurability problems.

In the second step, we consider an arbitrary process $\xi \in \mathcal{X}^1_{2A_2}(T, X_0)$ and a stopping time $\tau$ taking values in $[0, T]$. The process $\xi$ can then be approximated by simple processes as in the first step, with respect to the topology of the $L^p$-norm, where $p$ has to be chosen such that $f(x) \leq C(1 + |x|^p)$ (see Assumption 3.14).

In the third step, we show by compactness arguments that the corresponding sequence of $\varepsilon$-maximizers (as obtained in the first step) converges weakly to a process $\xi_{\tau, \varepsilon}$.

In the last step, we show that $\xi_{\tau, \varepsilon}$ is the $\varepsilon$-maximizer we were looking for.

As observed in Remark 2.13, we will use the fact that a process $\xi \in \mathcal{X}^1_{2A_2}(T, X_0)$ lies, in particular, in the set $K_m(T, X_0)$ for a constant $m > 0$, with $K_m(T, X_0) = \{ \xi \in \mathcal{X}^1_{2A_2}(T, X_0) | \mathbb{E}\left[ \int_0^T f(-\xi_t) \, dt \right] \leq m \}$.

**First step** Let $\varepsilon > 0$. For $L \in \mathbb{N}$ and $i \in \{0, \ldots, 2L\}$, define

$$t_i = \frac{i T}{2L},$$

and $\xi \in \mathcal{X}^1_{2A_2}(T, X_0)$ as follows:

$$\xi_i(\omega) = \sum_{i=1}^{2L} \xi_i(\omega) \mathbb{I}_{[t_i, t_{i+1}]}(t), \quad (A.16)$$

where $\xi_i$ takes values in the set $\{z_{i,p} | p \in \mathbb{N}, z_{i,p} \in \mathbb{R}^d\}$. Moreover, let $\tau$ be a stopping time taking values in the set $\{t_0, t_1, \ldots, t_{2L}\}$, and set $\Omega_{i,p_i} := \{\xi_t = z_{i,p_i}\}$, $\Gamma_j := \{\tau = t_j\}$. Note that $\Gamma_j$ and $\Omega_{i,p_i}$ can be empty. For every $t \in [0, T]$, we have

$$X^\xi_t = X_0 - \sum_{i=1}^{k-1} \xi_i(t_{i+1} - t_i) - \xi_k(t - t_k), \quad (A.17)$$

where $k$ is such that $t \in [t_k, t_{k+1}]$. We can therefore write for every $\omega \in \bigcap_{i=1}^{q} \Omega_{i,p_i} \cap \Gamma_q$,

$$X^\xi_t(\omega) = X_0 - \sum_{i=1}^{q-1} z_{i,p_i}(t_{i+1} - t_i). \quad (A.18)$$

Because $V$ and $u$ are continuous (see Theorem 3.11), $V$ is uniformly continuous on $C_N := [t_1, T] \times \overline{B}(0, N) \times [-N, N]$ (where $\overline{B}(0, N)$ denotes the $d$-dimensional
Regularity properties in a state-constrained expected… 229

euclidian closed ball with radius $N$), and $u$ is uniformly continuous on $[-N, N]$. Therefore, we can find $\delta_N$ such that for every $t^i, x^i, r^i, i = 1, 2$, we have

$$|(t^1-t^2, x^1-x^2, r^1-r^2)| < \delta_N \Rightarrow |V(t^1, x^1, r^1) - V(t^2, x^2, r^2)| \vee |u(r^1) - u(r^2)| < \varepsilon.$$  

Further, take $L \in \mathbb{N}$ such that

$$\frac{N}{2L} < \delta_N,$$

and introduce

$$G := \{(1, p_1), \ldots, (q, p_q) | q \in \{0, \ldots, 2^L\}, p_1, \ldots, p_q \in \mathbb{N}\}.$$  

Setting

$$r_j := -N + \frac{jN}{2L}, \quad x_g := X_0 - \sum_{i=1}^{q-1} z_{i, p_i} (t_{i+1} - t_i),$$

$$j \in \{1, \ldots, 2^{L+1}\}, \quad g \in G,$$

we can now define the following grid:

$$\Gamma_N = \{(t_i, x_g, r_j) | i \in \{0, \ldots, 2^L\}, j \in \{0, \ldots, 2^{L+1}\}, g \in G\} \cap C_N.$$  

When

$$\left(\tau(\omega), X^\xi(\omega), R^\xi(\omega)\right) \in \{t_i\} \times \{x_g\} \times [r_l, r_{l+1}[ \cap C_N,$$

we set

$$\gamma_N(\omega) := (T - t_i, x_g, r_j).$$  

Note that $\gamma_N$ is $\mathcal{F}_\tau$-measurable. Let us denote by $\xi^*, \gamma_N(\omega)$ the optimal strategy associated to $V(\gamma_N(\omega))$ [which exists, due to Theorem (3.11)]. Then, the process $\xi^*, \gamma_N(\omega)$ is well-defined for every $\omega \in \{X^\xi, R^\xi \leq N\}$. Moreover, it belongs to the set $\hat{X}_{2A_2}^1(T-t_i, x_g) = \hat{X}_{2A_2}^1(T - \tau(\omega), X^\xi(\omega))$. (Note that if $\tau(\omega) = T$ and $x_g = 0$, then $\gamma_N(\omega) = (0, 0, r_l)$, for some $r_l$, which implies that $V(\gamma_N(\omega)) = u(r_l)$, and therefore $\xi^*, \gamma_N(\omega) = 0$ is well-defined in this case, too.) Furthermore, we have by construction

$$V(T - t_i, x_g, r_l) = \mathbb{E}\left[u\left(r_l + \mathcal{R}^{\xi, \gamma_N(\omega)}_{\tau(\omega), T}\right)\right],$$

(A.19)
hence we obtain on \( \{ |X_t^\xi| \wedge |R_t^\xi| \leq N \} \):

\[
\begin{align*}
|V(T - \tau(\omega), X_t^\xi(\omega), R_t^\xi(\omega)) &- \mathbb{E}\left[u\left(R_t^\xi(\omega) + R_t^{\xi, \gamma_N(\omega)}(\omega)\right)\right]| \\
&\leq |V(T - \tau(\omega), X_t^\xi(\omega), R_t^\xi(\omega)) - V(\gamma_N(\omega))| \\
&+ |V(\gamma_N(\omega)) - \mathbb{E}\left[u\left(R_t^\xi(\omega) + R_t^{\xi, \gamma_N(\omega)}(\omega)\right)\right]| \\
&= |V(T - t_i, x_g, R_t^\xi(\omega)) - V(T - t_i, x_g, r_t)| \\
&+ |\mathbb{E}\left[u\left(r_t + R_t^{\xi, \gamma_N(\omega)}(\omega)\right)\right] - u\left(R_t^\xi(\omega) + R_t^{\xi, \gamma_N(\omega)}(\omega)\right)| \\
&\leq \varepsilon + \varepsilon \\
&= 2\varepsilon,
\end{align*}
\]

due to the uniform continuity of \( V \) and of \( u \). Thus, we have found a process \( \xi^{\ast, \gamma_N(\cdot)} = ~\tilde{\xi}^{\ast, \xi, \varepsilon} \in \dot{\mathcal{X}}^1_{2A_Z}(T - \tau(\cdot), X_t^\xi(\cdot)) \) such that (3.12) holds for every \( \omega \in \{ |X_t^\xi| \wedge |R_t^\xi| \leq N \} \). Moreover,

\[
\tilde{\xi}^{\ast, \xi, \varepsilon} \in \overline{K}_m^{\varepsilon}(T - \tau(\cdot), X_t^\xi(\cdot)),
\]

where \( m^{\varepsilon} \) has to be chosen as in (2.21).

Second step Let \( \xi \) and \( \tau \) be arbitrary. We can find a sequence of processes \( \xi^k \) as in the first step such that \( \xi^k \) converges to \( \xi \) in \( L^p \), i.e.,

\[
\mathbb{E}\left[ \int_0^T |\xi^k_t - \xi_t|^p dt \right] \longrightarrow 0,
\]

where \( p \) is chosen according to Assumption 3.14. Moreover, this sequence of processes may be chosen to lie in \( \dot{\mathcal{X}}^1_{2A_Z}(T, X_0) \), as argued in Assumption 2.15. We will prove that

\[
R_t^{\xi^k} \longrightarrow R_t^\xi \quad \text{in probability. (A.20)}
\]

Due to Lemma 3.8, we have that

\[
\int_t^T (X_s^{\xi^k})^\top \sigma dB_s \longrightarrow \int_t^T (X_s^\xi)^\top \sigma dB_s \quad \mathbb{P}\text{-a.s.}
\]

We have moreover, as a direct consequence of the \( L^p \) convergence of \( \xi^k \) to \( \xi \),

\[
\int_t^T b \cdot X_s^{\xi^k} ds \longrightarrow \int_t^T b \cdot X_s^\xi ds \quad \mathbb{P}\text{-a.s.}
\]

and

\[
\int_t^T f(-\xi_s^{\xi^k}) ds \longrightarrow \int_t^T f(-\xi_s^\xi) ds \quad \text{in } L^1
\]

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(due to the growth condition imposed on \( f \) in Assumption 3.14), and hence in probability. This establishes (A.20).

**Third step** We can find a sequence of stopping times \( (\tau_k) \) (with values in \([0, T]\)) as in the first step such that \( \tau_k \downarrow \tau \) \( \mathbb{P}\)-a.s. As can be seen in the first step above, for each \( k \in \mathbb{N} \), we can find \( \tilde{\xi}_{\tau_k, \varepsilon} \in \overline{K}_{m^\varepsilon}(T - \tau_k(\cdot), X_{\tau_k}^{\varepsilon}(\cdot)) \) such that

\[
V(T - \tau_k(\omega), X_{\tau_k}^{\varepsilon}(\omega), R_{\tau_k}^{\varepsilon}(\omega)) \leq \mathbb{E} \left[ u \left( R_{\tau_k}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau_k, \varepsilon}}^{\omega, \tau_k, \varepsilon}(\omega, T) \right) \right] + \varepsilon \quad (A.21)
\]

for \( \mathbb{P}\)-a.e \( \omega \in \{|X_{\tau_k}^{\varepsilon}| \land |R_{\tau_k}^{\varepsilon}| \leq N\} \). Moreover, we have that \( \tilde{\xi}_{\tau_k, \varepsilon} \in \overline{K}_{m^\varepsilon} \), with

\[
\overline{K}_{m^\varepsilon} = \left\{ \xi \in \mathcal{C}(\hat{X}_{2A}^1(T - \tau_k(\cdot), X_{\tau_k}^{\varepsilon}(\cdot)), k) \mid \left| \int_{\tau(\cdot)}^{T} f(-\xi_t) \, dt \right| \leq m^\varepsilon \right\},
\]

where \( \mathcal{C}(\hat{X}_{2A}^1(T - \tau_k(\cdot), X_{\tau_k}^{\varepsilon}(\cdot)), k) \) denotes the closed convex hull of the sequence of sets \( (\hat{X}_{2A}^1(T - \tau_k(\cdot), X_{\tau_k}^{\varepsilon}(\cdot))) k \). Recall that we set here

\[
\zeta_t = 0 \quad \text{for} \quad t \in [\tau(\cdot), \tau_k(\cdot)] \quad \text{when} \quad \zeta \in \hat{X}_{2A}^1(T - \tau_k(\cdot), X_{\tau_k}^{\varepsilon}(\cdot)),
\]

since \( \tau(\cdot) \leq \tau_k(\cdot), \) \( \mathbb{P}\)-a.s.

Because \( \overline{K}_{m^\varepsilon} \) is weakly sequentially compact, as proved in Proposition 3.6, there exists \( \tilde{\xi}_{\tau, \varepsilon} \in \overline{K}_{m^\varepsilon} \) such that by passing to a subsequence if necessary, \( \tilde{\xi}_{\tau_k, \varepsilon} \) converges to \( \tilde{\xi}_{\tau, \varepsilon} \) weakly in \( L^1 \). Using now Lemma 2.9, we have that \( \tilde{\xi}_{\tau, \varepsilon} \in \overline{K}_{m^\varepsilon} \) \( \mathbb{P}\)-a.s. on \( \{|X_{\tau_k}^{\varepsilon}| \land |R_{\tau_k}^{\varepsilon}| \leq N\} \).

**Last step** Notice first that we have

\[
\limsup_k \mathbb{E} \left[ u \left( R_{\tau_k}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau_k, \varepsilon}}^{\omega, \tau_k, \varepsilon}(\omega, T) \right) \right] \leq \mathbb{E} \left[ u \left( R_{\tau}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau, \varepsilon}}^{\omega, \tau, \varepsilon}(\omega, T) \right) \right] \quad (A.22)
\]

for \( \mathbb{P}\)-a.e \( \omega \in \{|X_{\tau_k}^{\varepsilon}| \land |R_{\tau_k}^{\varepsilon}| \leq N\} \). Indeed, similarly to how it was established for \( \tilde{\xi} \mapsto \mathbb{E} \left[ u \left( R_{\tau}^{\varepsilon}(\omega) \right) \right] \), we can prove that \( (r, \eta) \mapsto \mathbb{E} \left[ u(r + R_{\tau, \varepsilon}^{\eta}(\omega, T)) \right] \) is concave and thus we can apply Corollary 2.8, which proves (A.22). (Note that we cannot simply apply Fatou’s lemma to prove (A.22), since it is not known whether or not.)

\[
\limsup_k u \left( R_{\tau_k}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau_k, \varepsilon}}^{\omega, \tau_k, \varepsilon}(\omega, T) \right) \leq u \left( R_{\tau}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau, \varepsilon}}^{\omega, \tau, \varepsilon}(\omega, T) \right),
\]

because we only have a weak convergence of \( \tilde{\xi}_{\tau, \varepsilon} \) to \( \tilde{\xi}_{\tau, \varepsilon} \) going back to (A.21) and passing to the limit superior on both sides of the inequality, we finally get for \( \mathbb{P}\)-a.e. \( \omega \in \{|X_{\tau}^{\varepsilon}| \land |R_{\tau}^{\varepsilon}| \leq N\} \),

\[
\limsup_k u \left( R_{\tau_k}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau_k, \varepsilon}}^{\omega, \tau_k, \varepsilon}(\omega, T) \right) \leq u \left( R_{\tau}^{\varepsilon}(\omega) + R_{\tilde{\xi}_{\tau, \varepsilon}}^{\omega, \tau, \varepsilon}(\omega, T) \right),
\]
Equation (4.8) then implies

\[ V(T - \tau(\omega), X^{\xi}_t(\omega), \mathcal{R}^{\xi}_t(\omega)) = \limsup_k V(T - \tau_k(\omega), X^{\xi_k}_t(\omega), \mathcal{R}^{\xi_k}_t(\omega)) \]

\[ \leq \limsup_k \mathbb{E}\left[ u\left( \mathcal{R}^{\xi_k}_t(\omega) + \mathcal{R}^{\xi_{\omega_t,\tau_k,\epsilon}}_{\tau_k(\omega),T} \right) \right] + \varepsilon \]

\[ \leq \mathbb{E}\left[ u\left( \mathcal{R}^{\xi}_t(\omega) + \mathcal{R}^{\xi_{\omega_t,\tau,\epsilon}}_{\tau(\omega),T} \right) \right] + \varepsilon, \]

where the first equality is due to the continuity of \( V \) in its arguments. This shows (3.12).

**Proof of Theorem 4.5** To prove (i), let \( \xi \in \hat{\mathcal{X}}_{2A_2}^1(T, X_0), t \in ]0, T[, \) and \( \tau_k \) be defined as follows

\[ \tau_k := \inf \left\{ s > 0, |w_r(T - s, X^\xi_s, \mathcal{R}^{\xi}_s)| > k \right\} \wedge t. \]

Note that \( \tau_k \longrightarrow t, \) a.s., when \( k \longrightarrow \infty. \) Itō’s formula then yields

\[
w(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}) - w(T, X_0, R_0) = \int_0^{\tau_k} \left( -w_r(T - s, X^\xi_s, \mathcal{R}^{\xi}_s) + \mathcal{L}^\xi w(T - s, X^\xi_s, \mathcal{R}^{\xi}_s) \right) ds \]

\[ + \int_0^{\tau_k} (X^\xi_s)^\top \sigma w_r(T - s, X^\xi_s, \mathcal{R}^{\xi}_s) dB_s, \]

where the last term is a true martingale (due to definition of \( \tau_k \) and integrability property of \( X^\xi \)). Hence, by taking expectations on both sides we obtain

\[
\mathbb{E}\left[ w(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}) \right] - w(T, X_0, R_0) = \mathbb{E}\left[ \int_0^{\tau_k} \left( -w_r(T - s, X^\xi_s, \mathcal{R}^{\xi}_s) + \mathcal{L}^\xi w(T - s, X^\xi_s, \mathcal{R}^{\xi}_s) \right) ds \right].
\]

Equation (4.8) then implies

\[
\mathbb{E}\left[ w(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}) \right] \leq w(T, X_0, R_0).
\]

(A.23)

In order to send \( k \) to infinity on the left-hand side, we need to establish the uniform integrability of the sequence \((w(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))\). Since \( w \) is bounded from above, it is sufficient to prove the boundedness of the sequence \((w^-(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))\) in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \). To this end, we write

\[
(w^-(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))^2 \leq (V_2(T - \tau_k, X^{\xi}_{\tau_k}, \mathcal{R}^{\xi}_{\tau_k}))^2 \]

\[ \leq \mathbb{E}\left[ \exp(-A_2\mathcal{R}^{\xi}_{\tau})|\mathcal{F}_{\tau_k} \right]^2 \]

\[ \leq \mathbb{E}\left[ \exp(-2A_2\mathcal{R}^{\xi}_{\tau})|\mathcal{F}_{\tau_k} \right],
\]

\( \odot \) Springer
where the first inequality follows with (4.7), the second one with Lemma 3.16, and the last one with Jensen’s inequality. Since moreover \( \xi \in \mathcal{X}^{1}_{2A_{2}}(T, X_{0}) \), we thus have

\[
\mathbb{E}\left[ \mathbb{E}\left[ \exp(-2A\mathcal{R}_{T}^{k}) | \mathcal{F}_{T} \right] \right] = \mathbb{E}\left[ \exp(-2A\mathcal{R}_{T}^{k}) \right] \leq M_{\mathcal{R}_{T}^{k}}(2A2) + 1,
\]

and hence \( ((w - (T - \tau_{k}, X^{\xi}_{T_{k}}, \mathcal{R}_{\tau_{k}}^{k})) \) is bounded in \( L^{2}(\Omega, \mathcal{F}, \mathbb{P}) \). The sequence \( (w(T - \tau_{k}, X^{\xi}_{T_{k}}, \mathcal{R}_{\tau_{k}}^{k})) \) is therefore uniformly integrable, and by using Vitali’s convergence theorem we obtain

\[
\lim_{k \to \infty} \mathbb{E}\left[ w(T - \tau_{k}, X^{\xi}_{T_{k}}, \mathcal{R}_{\tau_{k}}^{k}) \right] = \mathbb{E}\left[ w(T - t, X^{\xi}_{t}, \mathcal{R}_{t}^{k}) \right] \leq w(T, X_{0}, R_{0}). \quad (A.24)
\]

Now we want to send \( t \) from below to \( T \). To this end, we consider the following sequence of stopping times

\[
\sigma_{k} := \inf \left\{ t \geq 0 \left| (T - t) f \left( \frac{X^{\xi}_{T - t}}{T - t} \right) \geq k \right\} \wedge T.
\]

Note that \( \sigma_{k} \to T \), a.s., when \( k \) goes to infinity. We want to show that

\[
\mathbb{E}\left[ w(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \to 0. \quad (A.25)
\]

From (4.7) we have that \( \mathbb{E}\left[ w(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \) lies between \( \mathbb{E}\left[ V_{1}(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \) and \( \mathbb{E}\left[ V_{2}(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \). It is hence sufficient to show that

\[
\mathbb{E}\left[ V_{1}(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \to 0. \quad (A.26)
\]

Now, Lemma 3.16 implies

\[
\mathbb{E}\left[ V_{i}(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \leq \mathbb{E}\left[ \mathbb{E}\left[ \exp(-A_{i}\mathcal{R}_{T}^{k}) | \mathcal{F}_{\sigma_{k}} \right] \mathbb{1}_{\{\sigma_{k} < T\}} \right] = \mathbb{E}\left[ \exp(-A_{i}\mathcal{R}_{T}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right].
\]

By using the Lebesgue dominated convergence theorem, we then get

\[
\mathbb{E}\left[ \exp(-A_{i}\mathcal{R}_{T}^{k}) \mathbb{1}_{\{\sigma_{k} < T\}} \right] \to 0,
\]

which proves (A.26). On the other hand, we have

\[
\mathbb{E}\left[ w(T - \sigma_{k}, X^{\xi}_{\sigma_{k}}, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} = T\}} \right] = \mathbb{E}\left[ w(0, 0, \mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} = T\}} \right] \geq \mathbb{E}\left[ u(\mathcal{R}_{\sigma_{k}}^{k}) \mathbb{1}_{\{\sigma_{k} = T\}} \right] = \mathbb{E}\left[ u(\mathcal{R}_{T}^{k}) \right],
\]
where we used (4.9) in the inequality. Hence, (A.24) implies
\[
\mathbb{E}
\left[
 u(R_T^k) \mathbb{1}_{[\sigma_k = T]}
\right] + \mathbb{E}
\left[
 w \left( T - \sigma_k, X_{\sigma_k}^k, R_{\sigma_k}^k \right) \mathbb{1}_{[\sigma_k < T]}
\right] \leq w(T, X_0, R_0),
\]
and sending \( k \) to infinity yields
\[
\mathbb{E}[u(R_T^k)] \leq w(T, X_0, R_0).
\]
In the last step, taking the supremum over \( \xi \in \dot{X}^1_{2A_2} (T, X_0) \) we infer
\[
V(T, X_0, R_0) \leq w(T, X_0, R_0),
\]
which proves (i).

We now turn to proving (ii). Thanks to Remark 4.1, in conjunction with assumption (4.12), we can rewrite (4.10) as follows
\[
0 = \left( -w_t + x^\top \Sigma x + b \cdot x w_r + \frac{1}{w_r} f^* \left( \frac{\nabla_x w}{w_r} \right) \right) (T - t, x, r).
\]
Then, Theorem 26.5 in Rockafellar (1997) (note that \( f \) has superlinear growth, is strictly convex, and continuously differentiable on \( \mathbb{R}^d \)) implies that \( (\nabla f)^{-1} = \nabla f^* \) is well-defined and continuous. Hence, setting
\[
\hat{\xi}(t, x, r) := \nabla f^* \left( \frac{\nabla_x w(t, x, r)}{w_r(t, x, r)} \right)
\]
we obtain that \( \hat{\xi} \) is also continuous in \( t, x, \) and \( r \) and fulfills (4.14), which proves part (a) in (ii).

To prove part (b), suppose that there exists a strong solution \( (X, R) \) to the SDE
\[
\begin{aligned}
\frac{dR_t}{\sigma dB_t} + b \cdot x dX_t dt - f \left( -\hat{\xi}(t, X_t, R_t) \right) dt,
\end{aligned}
\]
\[
\begin{aligned}
\frac{dX_t}{t} = -\hat{\xi}(t, X_t, R_t) dt,
\end{aligned}
\]
\[
\begin{aligned}
R_{|t=0} = R_0 \text{ and } X_{|t=0} = X_0.
\end{aligned}
\]

Setting \( \tau_k \) as before, we infer with Itô's formula
\[
\begin{aligned}
w(T - \tau_k, X_{\tau_k}, R_{\tau_k}) - w(T, X_0, R_0)
&= \int_0^{\tau_k} \left( -w_t(T - s, X_s, R_s) + L^\xi w(T - s, X_s, R_s) \right) ds
+ \int_0^{\tau_k} (X_s)^\top \sigma w_r(T - s, X_s, R_s) dB_s,
\end{aligned}
\]
where the last term is a true martingale (see the above argumentation). Thus, taking expectations yields
\[
\mathbb{E}[w(T - \tau_k, X_{\tau_k}, R_{\tau_k})] - w(T, X_0, R_0) = \mathbb{E} \left[ \int_0^{\tau_k} \left( - w_t(T - s, X_s, R_s) + L\xi w(T - s, X_s, R_s) \right) ds \right],
\]
and by using (4.14), this gives us
\[
\mathbb{E}[w(T - \tau_k, X_{\tau_k}, R_{\tau_k})] = w(T, X_0, R_0).
\]

The same arguments as above permit us to send \( k \) to infinity, whence we obtain
\[
\mathbb{E}[w(T - t, X_t, R_t)] = w(T, X_0, R_0).
\]

Analogously, the same arguments as above also allow us to set \( t = T \). Equation (4.11) implies that we necessarily have \( X_T = 0 \) in order to be able to establish
\[
V(T, X_0, R_0) \geq \mathbb{E} \left[ u(R_T) \right] = \mathbb{E} \left[ w(0, 0, R_T) \right] = w(T, X_0, R_0),
\]
where the first equality follows from (4.11). Hence, we have shown that \( w \leq V \). Using the reverse inequality established in (i), we finally get \( w = V \). Therefore it follows that \( (X, R) = (X^{\xi^*}, R^{\xi^*}) \), due to the uniqueness of the optimal strategy (Theorem 2.4). Moreover,
\[
\xi_t^* = \hat{\xi}(T - t, X_t^{\xi^*}, R_t^{\xi^*}), \quad (\mathbb{P} \otimes \lambda)\text{-a.s.},
\]
which concludes the proof. \( \square \)

### A.3 Proofs of the main results from Sect. 4.1

**Proof of Lemma 4.16** We prove only (i). Suppose that \( v \) fulfills the inequality (4.28) for all \((t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) and all \((q, p, s, m) \in \mathcal{J}^{2,+}v(T - t, x, r)\). Take \( \varphi \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})\) and consider \((t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) such that \((V - \varphi)(T - t, x, r)\) has a local maximum at \((t^*, x^*, r^*)\). Due to (4.24) in Remark 4.15 above, \( \varphi \) fulfills (4.18), which implies that \( v \) is a viscosity subsolution.

Suppose now that \( v \) is a viscosity subsolution and let \((q, p, s, m) \in \mathcal{J}^{2,+}v(T - t^*, x^*, r^*)\). As mentioned in Remark 4.15 above, there exists \( \varphi \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})\) such that
\[
(-\varphi_t, \nabla_x \varphi_x, \varphi_r, \varphi_{rr}) (T - t^*, x^*, r^*) = (q, p, s, m).
\]
By using (4.23) together with (4.24), we obtain that \((T - t^*, x^*, r^*)\) is a local maximizer of \( v - \varphi \). Thus \( \varphi \) fulfills (4.18), which proves that \((q, p, s, m)\) fulfills (4.28). \( \square \)
Proof of Theorem 4.17 Assume that (4.31) is true and suppose by way of contradiction that there exists \((t^*, x^*, r^*) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\) such that \((U - V)(T - t^*, x^*, r^*) > 0\). Since \(U - V\) is continuous on \([0, T] \times \mathbb{R}^d \times \mathbb{R}\), we can suppose w.l.o.g. that the supremum of \(U - V\) on a compact subset is attained at some \((T - t^*, x^*, r^*)\), i.e.,

\[
\bar{m} = \sup_{K \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}} (U - V)(T - t, x, r) = (U - V)(T - t^*, x^*, r^*) > 0, \quad (A.30)
\]

where \(K\) is compact with non-empty interior. In the following, we will use the doubling of variables technique, developed first by Kružkov (1970). For any \(\varepsilon > 0\), consider the functions

\[
\Phi_{\varepsilon}(t, t', x, x', r, r') := U(t, x, r) - V(t', x', r') - \varphi_{\varepsilon}(t, t', x, x', r, r'), \quad (A.31)
\]

\[
\varphi_{\varepsilon}(t, t', x, x', r, r') := \frac{1}{\varepsilon} \left( |t - t'|^2 + |x - x'|^2 + |r - r'|^2 \right). \quad (A.32)
\]

Let \([0, \eta] \times \bar{B}(0, r) \times [r^* - \alpha, r^* + \alpha] \subset K\) be a compact neighborhood of \((t^*, x^*, r^*)\), where \(0 < \eta < T, 0 < \alpha < r^*,\) and \(r > 0\). The continuous function \(\Phi_{\varepsilon}\) attains its maximum on the compact neighborhood \([0, \eta]^2 \times \bar{B}(0, r)^2 \times [r^* - \alpha, r^* + \alpha]^2\), denoted by \(m_{\varepsilon}\), at some \((T - t_{\varepsilon}, T - t'_{\varepsilon}, x_{\varepsilon}, x'_{\varepsilon}, r_{\varepsilon}, r'_{\varepsilon})\). We will show that

\[
m_{\varepsilon_n} \to \bar{m} \quad \text{and} \quad \varphi(T - t_{\varepsilon_n}, T - t'_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}, r_{\varepsilon_n}, r'_{\varepsilon_n}) \to 0, \quad (A.33)
\]

for a sequence \((\varepsilon_n)\) with \(\varepsilon_n \to 0\). First, note that

\[
\bar{m} = \Phi_{\varepsilon}(T - t^*, T - t^*, x^*, x^*, r^*, r^*)
\]

\[
= (U - V)(T - t^*, x^*, r^*) - \varphi_{\varepsilon}(T - t^*, T - t^*, x^*, x^*, r^*, r^*)
\]

\[
\leq U(T - t_{\varepsilon}, x_{\varepsilon}, r_{\varepsilon}) - V(T - t'_{\varepsilon}, x'_{\varepsilon}, r'_{\varepsilon}) - \varphi_{\varepsilon}(T - t_{\varepsilon}, T - t'_{\varepsilon}, x_{\varepsilon}, x'_{\varepsilon}, r_{\varepsilon}, r'_{\varepsilon}) \quad (A.34)
\]

\[
= m_{\varepsilon} \leq U(T - t_{\varepsilon}, x_{\varepsilon}, r_{\varepsilon}) - V(T - t'_{\varepsilon}, x'_{\varepsilon}, r'_{\varepsilon}). \quad (A.35)
\]

Since \(((T - t_{\varepsilon}, T - t'_{\varepsilon}, x_{\varepsilon}, x'_{\varepsilon}, r_{\varepsilon}, r'_{\varepsilon}))_{\varepsilon > 0}\) belongs to the compact set \([0, \eta]^2 \times \bar{B}(0, r)^2 \times [r^* - \alpha, r^* + \alpha]^2\), we can find a sequence \((T - t_{\varepsilon_n}, T - t'_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}, r_{\varepsilon_n}, r'_{\varepsilon_n})\), where \(\varepsilon_n \downarrow 0\), which converges to some \((T - \tilde{t}, T - \tilde{t}', \tilde{x}, \tilde{x}', \tilde{r}, \tilde{r}')\), as \(n \to \infty\). The boundedness of the sequence \((U(T - t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n}) - V(T - t'_{\varepsilon_n}, x'_{\varepsilon_n}, r'_{\varepsilon_n}))\) implies that \((\varphi_{\varepsilon_n}(T - t_{\varepsilon_n}, T - t'_{\varepsilon_n}, x_{\varepsilon_n}, x'_{\varepsilon_n}, r_{\varepsilon_n}, r'_{\varepsilon_n}))\) is also bounded (from above), due to inequality (A.34). Therefore, by using (A.32), we must have

\[
T - \tilde{t} = T - \tilde{t}', \quad \tilde{x} = \tilde{x}', \quad \tilde{r} = \tilde{r}',
\]

as well as

\[
\bar{m} = U(T - \tilde{t}, \tilde{x}, \tilde{r}) - V(T - \tilde{t}, \tilde{x}, \tilde{r}),
\]
applying inequality (A.35) and the definition of $\tilde{m}$. We can thus suppose w.l.o.g. that $\tilde{t} = t^*, \tilde{x} = x^*, \tilde{r} = r^*$. Letting $\varepsilon_n$ go to 0 in (A.35), we get

$$\tilde{m} \leq \lim_{n \to \infty} m_{\varepsilon_n} \leq (U - V)(T - t^*, x^*, r^*) = \tilde{m},$$

and thus (A.33) is proved.

Furthermore, we have that $\varphi_\varepsilon \in C^{1,1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})$ and

$$(T - t_\varepsilon, x_\varepsilon, r_\varepsilon)$$

is a local maximum of

$$(t, x, r) \mapsto U(T - t, x, r) - \varphi_\varepsilon(T - t, T - t'_\varepsilon, x, x'_\varepsilon, r, r'_\varepsilon), \quad (A.36)$$

resp.,

$$(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon)$$

is a local minimum of

$$(t', x', r') \mapsto V(T - t', x', r') + \varphi_\varepsilon(T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon). \quad (A.37)$$

Our purpose now is to use formulas (4.26) and (4.24) to find adequate elements of $\mathcal{J}^{2,+} U(T - t_\varepsilon, x_\varepsilon, r_\varepsilon)$ and $\mathcal{J}^{2,-} V(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon)$. To this end, we compute the following derivatives:

\[
(\varphi_\varepsilon)_t (T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon) = \frac{2}{\varepsilon} (t_\varepsilon - t'_\varepsilon),
\]

\[
(\varphi_\varepsilon)_r (T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon) = \frac{2}{\varepsilon} (r_\varepsilon - r'_\varepsilon),
\]

\[
\nabla_x (\varphi_\varepsilon)(T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon) = \frac{2}{\varepsilon} (x_\varepsilon - x'_\varepsilon),
\]

\[
(\varphi_\varepsilon)_{rr} (T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon) = \frac{2}{\varepsilon}.
\]

Because $\frac{r^* - r_\varepsilon}{\varepsilon}, \frac{x^* - x_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon$ goes to 0, due to (A.33), we can choose a neighborhood $[0, \eta] \times \overline{B}(0, r) \times [r^* - \alpha_\varepsilon, r^* + \alpha_\varepsilon]$ of $(t^*, x^*, r^*)$ such that $\frac{r_\varepsilon}{\varepsilon} \to 0$, as $\varepsilon$ goes to 0. Using this and (A.36), and inserting the derivatives of $(t, x, r) \mapsto \varphi_\varepsilon(T - t, T - t'_\varepsilon, x, x'_\varepsilon, r, r'_\varepsilon)$ at $(T - t_\varepsilon, x_\varepsilon, r_\varepsilon)$ in (4.26), we obtain

\[
U(T - t, x, r) - U(T - t_\varepsilon, x_\varepsilon, r_\varepsilon)
\leq -\varphi_\varepsilon(T - t_\varepsilon, T - t'_\varepsilon, x_\varepsilon, x'_\varepsilon, r_\varepsilon, r'_\varepsilon) + \varphi_\varepsilon(T - t, T - t'_\varepsilon, x, x'_\varepsilon, r, r'_\varepsilon)
\]

\[
\leq -\frac{2}{\varepsilon} (t_\varepsilon - t'_\varepsilon)(t - t_\varepsilon) + \frac{2}{\varepsilon} (x_\varepsilon - x'_\varepsilon)(x - x_\varepsilon) + \frac{2}{\varepsilon} (r_\varepsilon - r'_\varepsilon)(r - r_\varepsilon)
\]

\[
- \frac{1}{3\varepsilon} (r - r_\varepsilon)^2 + o(|t - t_\varepsilon| + |x - x_\varepsilon|)
\]

\[
\leq -\frac{2}{\varepsilon} (t_\varepsilon - t'_\varepsilon)(t - t_\varepsilon) + \frac{2}{\varepsilon} (x_\varepsilon - x'_\varepsilon)(x - x_\varepsilon) + \frac{2}{\varepsilon} (r_\varepsilon - r'_\varepsilon)(r - r_\varepsilon)
\]

\[
+ \frac{2\alpha_\varepsilon}{3\varepsilon} (r - r_\varepsilon)^2 + o(|t - t_\varepsilon| + |x - x_\varepsilon| + |r - r_\varepsilon|).\]
Using Remark 4.15 we have thus proved that

$$\left( -\frac{2}{\varepsilon}(t_e - t'_e), -\frac{2}{\varepsilon}(x_e - x'_e), \frac{2}{\varepsilon}(r_e - r'_e), \frac{2\alpha_e}{3\varepsilon} \right) \in J^{2, -}_V(T - t'_e, x'_e, r'_e). \quad (A.38)$$

In the next step we look for an adequate element of $J^{2, -}_V(T - t'_e, x'_e, r'_e)$. To this end, as before, we compute the derivatives of $(t', x', r') \mapsto \varphi_\varepsilon(T - t_e, T - t_e, x_e, x'_e, r_e, r_e)$ at $(T - t'_e, x'_e, r'_e)$. Inserting them at $(T - t'_e, x'_e, r'_e)$ into (4.24), we have in conjunction with (A.37):

$$V(T - t, x, r) - V(T - t'_e, x'_e, r'_e) \geq \varphi_\varepsilon(T - t_e, T - t'_e, x_e, x'_e, r_e, r_e) - \varphi_\varepsilon(T - t_e, T - t_e, x_e, x'_e, r_e, r_e)$$

$$= \frac{2}{\varepsilon}(t'_e - t_e)(t - t'_e) - \frac{2}{\varepsilon}(x'_e - x_e)(x - x'_e) - \frac{2}{\varepsilon}(r'_e - r_e)(r - r'_e)$$

$$+ \frac{1}{3\varepsilon}(r' - r'_e)^2 + o(|t'_e - t_e| + |x'_e - x_e| + |r'_e - r_e|).$$

This shows that

$$\left( -\frac{2}{\varepsilon}(t_e - t'_e), -\frac{2}{\varepsilon}(x_e - x'_e), \frac{2}{\varepsilon}(r_e - r'_e), -\frac{2\alpha_e}{3\varepsilon} \right) \in J^{2, -}_V(T - t'_e, x'_e, r'_e), \quad (A.39)$$

thanks to Remark 4.15. Applying Lemma 4.16 to the viscosity subsolution $U$ we thus obtain

$$0 \leq -\frac{2}{\varepsilon}(t_e - t'_e) + \beta U(T - t_e, x_e, r_e) + \frac{2}{\varepsilon}(r_e - r'_e) b \cdot x_e + \frac{\alpha_e x_e^\top \Sigma x_e}{3\varepsilon}$$

$$+ \frac{1}{\varepsilon} \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top (x_e - x'_e) - (r_e - r'_e) f(\xi) \right), \quad (A.40)$$

in conjunction with (A.38). Analogously, using the viscosity supersolution property of Lemma 4.16 for $V$, as well as (A.39), we get

$$0 \geq -\frac{2}{\varepsilon}(t_e - t'_e) + \beta V(T - t'_e, x'_e, r'_e) + \frac{2}{\varepsilon}(r_e - r'_e) b \cdot x'_e - \frac{\alpha_e x'_e^\top \Sigma x'_e}{3\varepsilon}$$

$$+ \frac{1}{\varepsilon} \sup_{\xi \in \mathbb{R}^d} \left( \xi^\top (x_e - x'_e) - (r_e - r'_e) f(\xi) \right). \quad (A.41)$$
By subtracting (A.40) from (A.41), we have
\[
0 \leq \beta(U(T - t_\varepsilon, x_\varepsilon, r_\varepsilon) - V(T - t'_\varepsilon, x'_\varepsilon, r'_\varepsilon)) + \frac{2}{\varepsilon}(r_\varepsilon - r'_\varepsilon) b \cdot (x_\varepsilon - x'_\varepsilon) \\
+ \frac{\alpha_\varepsilon}{3\varepsilon}(x_\varepsilon \top \Sigma x_\varepsilon + x'_\varepsilon \top \Sigma x'_\varepsilon).
\]

Sending now \(\varepsilon\) to 0 and using the fact that \(\frac{\alpha_\varepsilon}{\varepsilon}, r_\varepsilon - r'_\varepsilon, |x_\varepsilon - x'_\varepsilon| \to 0,\) when \(\varepsilon \to 0,\) we get
\[
0 \leq \beta(U - V)(T - t^*, x^*, r^*). \tag{A.42}
\]

Because \(\beta < 0,\) (A.42) is in contradiction with (A.30). Thus, we have shown that \(U \leq V\) on \([0, T] \times \mathbb{R}^d \times \mathbb{R} \). \(\square\)

**Proof of Corollary 4.18** Let \(U\) be another solution of (4.2) with initial condition (4.3) satisfying the growth condition
\[
V_2(t, x, r) \leq U(t, x, r) \leq V_1(t, x, r), \quad \text{for all } (t, x, r) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}.
\]

Then we have
\[
\lim_{t \to 0} (U(t, x, r) - V(t, x, r)) = 0, \quad \text{for fixed } x, r \in \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\} \times \mathbb{R},
\]

which can be extended to \(\mathbb{R}^d \times \mathbb{R}\). Hence, by using Theorem 4.17 we deduce that \(U \leq V\). Since both \(U\) and \(V\) are viscosity sub- and supersolution, respectively, we conclude by reversing the preceding inequality. \(\square\)

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