Reduction of cluster iteration maps to symplectic maps

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Abstract

We study iteration maps of recurrence relations arising from mutation periodic quivers of arbitrary period. Combining tools from cluster algebra theory and (pre)symplectic geometry, we show that these cluster iteration maps can be reduced to symplectic maps on a lower dimensional submanifold, provided the matrix representing the quiver is singular. The reduced iteration map is explicitly computed for several new periodic quivers.

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1 Introduction

Recurrence relations arise in a natural way from periodic quivers via Fomin-Zelevinsky cluster mutations. We call this type of relations cluster recurrence relations. The iteration map of such recurrence relations is birational and defined as the composition of mutations and permutations. In [FoMa] the notion of mutation-periodicity of a quiver is used to show that 1-periodic quivers give rise to recurrence relations on the real line while to higher periodic quivers correspond recurrence relations on higher dimensional spaces.

Since the introduction of cluster algebras by Sergey Fomin and Andrei Zelevinsky in [FoZe02], the theory of cluster algebras has grown in many research directions. Relevant to our work are the relations between cluster algebras and Poisson/symplectic geometry whose main achievements are surveyed in [GeShVa10]. The presymplectic structures considered in the context of cluster algebra theory are known as log-canonical presymplectic structures. We will refer to these forms just as log presymplectic forms.

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Our starting motivation was to understand the relevance of presymplectic and Poisson structures compatible with a cluster algebra to the recurrence relations arising from such algebras. Some steps in this direction were taken in [FoHo11] and [FoHo12], where the integrability of some Somos-type sequences (associated to 1-periodic quivers) was obtained by reduction of the iteration map to a symplectic map. These references, based on the classification of 1-periodic quivers obtained in [FoMa], prove that iteration maps arising from a 1-periodic quiver can be reduced to symplectic maps.

Our main result, Theorem 2, shows that any iteration map arising from a quiver of arbitrary period \( m \) can be reduced to a symplectic map on a \( 2k \)-dimensional space with respect to a log symplectic form, where \( 2k \) is the rank of the matrix representing the quiver. The proof of this theorem does not rely on the classification of periodic quivers, which is unknown for periods higher than 1. The main ingredients of the proof are: (i) the invariance of the standard log presymplectic form under the iteration map, which is proved in Theorem 1; (ii) a classical theorem of G. Darboux (or of E. Cartan for the linear version) for the reduction of an arbitrary presymplectic form to a symplectic form.

To illustrate the main result, we explicitly compute the reduced symplectic iteration map of some cluster recurrence relations. We would like to point out the relevance of the constructive proof of E. Cartan’s theorem to the computations carried out. In fact, this construction provides explicit Darboux coordinates necessary to the computation of the reduced iteration map.

We note that iteration maps are not intrinsic objects of study in cluster algebra theory since they only appear when one considers cluster algebras associated to periodic quivers. Several results presented here provide new insights into the role of some cluster algebras structures, in particular of the so-called secondary cluster manifold. In fact, Theorem 2 can be interpreted as reduction of an extra structure - the iteration map - to the secondary cluster manifold with its Weil-Petersson form (see [GeShVa03], [GeShVa05] and [GeShVa10]).

The organization of the paper is as follows. In Section 2 we introduce the basic notions of the theory of cluster algebras necessary to subsequent sections, in particular, the definition of periodic quivers and the construction of cluster recurrence relations from periodic quivers. The next section is devoted to the proof of Theorem 1 which shows the invariance, under the iteration map, of the log presymplectic form whose coefficient matrix is (up to a constant) the matrix representing the periodic quiver. In Section 4 we prove the main result, Theorem 2, on the reduction of cluster iteration maps to symplectic maps. The paper ends with examples illustrating the previous results.

2 Mutation-periodic quivers and cluster iteration maps

Here we introduce the notions of the theory of cluster algebras necessary to the following sections. We will work in the context of coefficient free cluster algebras
\( \mathcal{A}(B) \) where \( B \) is a (finite) skew-symmetric integer matrix.

In this work quiver means an oriented graph with \( N \) nodes and no loops nor 2-cycles. It will be represented by an \( N \)-sided polygon whose vertices are the nodes of the quiver and will be labelled by 1, 2, \ldots, \( N \) in clockwise direction. To each oriented edge of the polygon one associates a weight which is the positive integer representing the number of arrows between the corresponding nodes of the quiver. A quiver can also be identified with a skew-symmetric matrix \( B = [b_{ij}] \), being \( b_{ij} \) the number of arrows from node \( i \) to node \( j \) minus the number of arrows from \( j \) to \( i \). We denote by \( B_Q \) the \( N \times N \) skew-symmetric matrix representing a quiver \( Q \) with \( N \) nodes.

To each node \( i \) of a quiver \( Q \) one attaches a variable \( u_i \) called cluster variable. The pair \((B_Q, u)\) is called the initial seed and \( u = (u_1, \ldots, u_N) \) the initial cluster.

The basic operation of the theory of cluster algebras is called a mutation. A mutation \( \mu_k \) in the direction of \( k \) (or at node \( k \)) acts on a given seed \((B, u)\) with \( B = [b_{ij}] \) and \( u = (u_1, \ldots, u_N) \), as follows:

- \( \mu_k(B) = [b'_{ij}] \) with
  
  \[
  b'_{ij} = \begin{cases} 
  -b_{ij}, & (k-i)(j-k) = 0 \\
  b_{ij} + \frac{1}{2} ([b_{ik}|b_{kj} + b_{ik}|b_{kj}]), & \text{otherwise.} 
  \end{cases}
  \]  
  (1)

- \( \mu_k(u_1, \ldots, u_N) = (u'_1, \ldots, u'_N) \), with
  
  \[
  u'_i = \begin{cases} 
  \frac{u_i}{\prod_{j:b_{kj}>0} u_j^{b_{kj}}} + \prod_{j:b_{kj}<0} u_j^{-b_{kj}}, & i \neq k \\
  u_k^{b_{ki}}, & i = k. 
  \end{cases}
  \]  
  (2)

When one of the products in (2) is taken over an empty set, its value is assumed to be 1.

Formulae (1) and (2) are called (cluster) exchange relations. It is easy to see that the exchange relations have the properties: (i) if \( B \) is skew-symmetric, then \( B' = \mu_k(B) \) is again skew-symmetric; (ii) \( \mu_k \) is an involution, that is \( \mu_k \circ \mu_k = \text{Id} \).

Given a matrix \( B \) and an initial cluster \( u \) we can apply a mutation \( \mu_k \) to produce another cluster, and then apply another mutation \( \mu_p \) to this cluster to produce another cluster and so on. The cluster algebra (of geometric type) \( \mathcal{A}(B) \) is the subalgebra of the field of rational functions in the cluster variables, generated by the union of all clusters.

The notion of mutation periodicity of a quiver introduced in [FoMa] enables to associate a recurrence relation to a periodic quiver. More precisely, to an \( m \)-periodic quiver one associates a recurrence relation on an \( m \)-dimensional space. Let us recall the definition of periodic quiver.

Consider the permutation \( \sigma : (1, 2, \ldots, N) \mapsto (2, 3, \ldots, N, 1) \) and \( \sigma Q \) the quiver in which the number of arrows from node \( \sigma(i) \) to node \( \sigma(j) \) is the number
of arrows in $Q$ from node $i$ to node $j$. Equivalently, the action of $\sigma$ on the polygon representing $Q$ leaves the weighted edges fixed and moves the vertices in the counterclockwise direction.

If $B_Q$ is the skew-symmetric matrix representing the quiver $Q$, the action $Q \mapsto \sigma Q$ corresponds to the conjugation $B_Q \mapsto \sigma^{-1} B_Q \sigma$, that is,

$$B_{\sigma Q} = \sigma^{-1} B_Q \sigma,$$

where, slightly abusing notation, $\sigma$ also denotes the matrix representing the permutation, that is

$$\sigma = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (3)$$

**Definition 1.** Let $Q$ be a quiver with $N$ nodes, $B_Q$ the skew-symmetric matrix representing $Q$ and

$$\sigma : (1, 2, \ldots, N) \mapsto (2, 3, \ldots, N, 1).$$

$Q$ is said to have period $m$ if $m$ is the smallest positive integer such that

$$\mu_m \circ \cdots \circ \mu_1(B_Q) = \sigma^{-m} B_Q \sigma^m. \quad (4)$$

The notion of mutation periodicity means that if a quiver is $m$-periodic, then after applying $\mu_m \circ \cdots \circ \mu_1$ we return to a quiver which is equivalent (up to a certain permutation) to the original quiver, and so mutating this quiver at node $m + 1$ produces an exchange relation identical in form to the exchange relation at node 1 but with a different labeling. The next mutation at node $m + 2$ will produce an exchange relation whose form is identical to the exchange relation at node 2 with a different labeling, and so on. This process produces a list of exchange relations, which is interpreted as a recurrence relation.

We note that the definition of a periodic quiver could also be stated in terms of the action of a mutation on a quiver which is defined by specifying a set of rules for mutating the arrows of the quiver (see for instance [FoMa] and [Ke]).

We now explain with a running example how to construct the recurrence relation (and the respective iteration map) corresponding to a periodic quiver.

**Example 1.** Consider an initial cluster $u = (u_1, u_2, u_3, u_4, u_5, u_6)$ and the quiver with 6 nodes represented by the matrix

$$B = \begin{bmatrix} 0 & -r & s & -p & s & -t \\ r & 0 & -t - rs & s & -p - rs & s \\ -s & t + rs & 0 & -r - s(t - p) & s & -p \\ p & -s & r + s(t - p) & 0 & -t - rs & s \\ -s & p + rs & -s & t + rs & 0 & -r \\ t & -s & p & -s & r & 0 \end{bmatrix}. \quad (5)$$
where \( r, s, t, p \) are positive integers. Using (1) and (3) it is easy to check that one has \( \sigma^{-1}B\sigma = \mu_1(B) \) if \( r = t \) and \( \sigma^{-2}B\sigma^2 = \mu_2 \circ \mu_1(B) \) if \( r \neq t \). That is, the quiver is 1-periodic when \( r = t \) and 2-periodic otherwise.

1. Case \( r = t \)

According to (2), the mutation at node 1 produces the cluster \( \mu_1(u) = (u_7, u_2, u_3, u_4, u_5, u_6) \) with

\[
    u_7u_1 = u_2^{r}u_6^{r} + u_3^{s}u_5^{s},
\]

and \( u_7 = u_1' \). The exponents of the right hand side monomials were read directly from the first row of \( B \). As the quiver is 1-periodic, mutating now at node 2 produces an exchange relation which is a shift by 1 of the relation \( (6) \). Indeed, \( \mu_2 \circ \mu_1(u) = (u_7, u_8, u_3, u_4, u_5, u_6) \) with

\[
    u_8u_2 = u_3^{r}u_5^{r}u_7 + u_4^{s}u_6^{s},
\]

where the exponents of the right hand side monomials can now be read from the second row of \( \mu_1(B) \) which is \((-r, 0, -r, s, -p, s)\). Mutating successively at consecutive nodes we obtain the following sixth order cluster recurrence relation on the real line:

\[
    u_{n+6}u_n = u_{n+1}^{r}u_{n+3}^{r}u_{n+5}^{p} + u_{n+2}^{s}u_{n+4}^{s}, \quad n = 1, 2, \ldots
\]  

(7)

2. Case \( r \neq t \)

The quiver represented by \( B \) is now 2-periodic. Mutating the initial cluster at node 1 produces the cluster \( \mu_1(u) = (u_7, u_2, u_3, u_4, u_5, u_6) \) with

\[
    u_7u_1 = u_2^{r}u_6^{r} + u_3^{s}u_5^{s},
\]

(8)

The second row of \( \mu_1(B) \) is now \((-r, 0, -t, s, -p, s)\), and so the cluster \( \mu_2 \circ \mu_1(u) = (u_7, u_8, u_3, u_4, u_5, u_6) \) satisfies the relation

\[
    u_8u_2 = u_3^{r}u_5^{r}u_7 + u_4^{s}u_6^{s},
\]

(9)

with \( u_7 \) given by (3). Unlike the previous case, the exchange relation (7) is not a shift of the relation (3). However, as the quiver is 2-periodic, mutating at node 3 gives an exchange relation which is a shift by 2 of (3) and the next mutation at node 4 gives an exchange relation which is a shift by 2 of (3). That is, the 2-periodic quiver gives rise to the following third order cluster recurrence relation on the plane:

\[
\begin{align*}
    x_{n+3}x_n &= y_{n+1}^{r}y_{n+2}^{p} + x_{n+1}^{s}x_{n+2}^{s} \\
    y_{n+3}y_n &= x_{n+1}^{d}x_{n+2}^{p} + y_{n+1}^{a}y_{n+2}^{s},
\end{align*}
\]

where \( x_n = u_{2n-1} \) and \( y_n = u_{2n} \).
Remark 1. The form of the matrices representing 1-periodic quivers was obtained in [FoMa], however the classification of quivers of higher period is still unknown. To the best of our knowledge the 2-periodic quiver with 6 nodes represented by the matrix $B$ in (5) is new.

From the construction of the recurrence relation associated to an $m$-periodic quiver, it is easy to see that the corresponding cluster iteration map is given by

$$\varphi = \sigma^m \circ \mu_m \circ \cdots \circ \mu_2 \circ \mu_1.$$  \hfill (11)

In particular, the iteration maps for the recurrence relations (7) and (10) are, respectively,

$$\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = \left( u_2, u_3, u_4, u_5, u_6, \frac{u_2^5 u_4^5 u_6^5 + u_4^5 u_6^5}{u_1} \right), \quad \hfill (12)$$

$$\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = \left( u_3, u_4, u_5, u_6, \frac{u_3^5 u_5^5 u_7^5 + u_5^5 u_7^5}{u_1}, \frac{u_2^5 u_4^5 u_6^5 + u_4^5 u_6^5}{u_8} \right). \hfill (13)$$

3 Log presymplectic forms and periodic quivers

Presymplectic structures (and quadratic Poisson structures) associated to a cluster algebra $A(B)$ were introduced in [GeShVa03] and [GeShVa05]. The presymplectic forms considered in the context of cluster algebras are of the type $\omega = \sum_{i<j} w_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}$. We will call these forms log presymplectic forms and the functions $u_i$ log coordinates with respect to $\omega$. Such log presymplectic form is said to be compatible with a cluster algebra $A(B)$ if all the clusters in $A(B)$ give log coordinates with respect to $\omega$, with eventually different coefficients $\omega'_{ij}$.

It was shown in [GeShVa05] that any cluster algebra carries a compatible log presymplectic structure, and whenever the matrix $W = [w_{ij}]$ has not full rank there exits a rational (symplectic) manifold of dimension $2k = \text{rank} W$ called the secondary cluster manifold. The symplectic form on this manifold is called the Weil-Petersson form associated to the cluster algebra $A(B)$. The reason for this name is its relation with the Weil-Petersson form on a Teichmüller space (see [GeShVa05] or [GeShVa10] for details).

Our main aim is to understand the relevance of these log presymplectic structures to the recurrence relations arising from $m$-periodic quivers. As we will show in the next section, if the matrix representing the quiver is singular the recurrence’s iteration map can be reduced to a symplectic map with respect to a log symplectic form. The key property behind this symplectic reduction is precisely the invariance of the standard log presymplectic structure (14) under the iteration map.
If \((B, u)\) is the initial seed with \(B = [b_{ij}]\), we call the 2-form
\[
\omega = \sum_{1 \leq i < j \leq N} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j},
\] (14)
the \textit{standard log presymplectic form} associated to the cluster algebra \(A(B)\). The skew-symmetric matrix \(B\) will be called the \textit{coefficient matrix} of \(\omega\).

The 2-form \(\omega\) is known in the literature as log-canonical presymplectic form, since in the coordinates \(v_i = \log u_i\) it has the following \textit{canonical} form:
\[
\omega = \sum_{i < j} b_{ij} dv_i \wedge dv_j.
\]

Although we do not use explicitly the notion of compatibility of a presymplectic form with a cluster algebra, we remark that the standard log presymplectic form (14) is in fact compatible with the cluster algebra \(A(B)\) (see for instance Theorem 6.2 in [GeShVa10] which characterizes such compatible forms).

**Theorem 1.** Let \((B, u)\) be an initial seed, and \(\omega\) the standard log presymplectic form (14) associated to \(A(B)\). Then, the following are equivalent:

1. The matrix \(B\) represents an \(m\)-periodic quiver, that is
\[
\mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1(B) = \sigma^{-m} B \sigma^m.
\]

2. \(\varphi^* \omega = \omega\), where \(\varphi = \sigma^m \circ \mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1\).

The proof of this theorem relies on the following two lemmas.

**Lemma 1.** Let \((B, u)\) be an initial seed, \(\omega\) the standard log presymplectic form (14) and \(\sigma\) the permutation (3). Then, the pullback of \(\omega\) by \(\sigma\) is given by
\[
\sigma^* \omega = \sum_{i < j} (\sigma^{-1} B \sigma)_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}.
\] (15)

**Proof.** As \(\sigma(u_1, u_2, \ldots, u_N) = (u_2, \ldots, u_N, u_1)\), the pullback of \(\omega\) by \(\sigma\) is given by
\[
\sigma^* \omega = \sum_{1 \leq i \leq N-1} b_{i+1,N} \frac{du_{i+1}}{u_{i+1}} \wedge \frac{du_1}{u_1} + \sum_{1 \leq i < j \leq N-1} b_{ij} \frac{du_{i+1}}{u_{i+1}} \wedge \frac{du_{j+1}}{u_{j+1}} - \sum_{2 \leq k \leq N} b_{k-1,N} \frac{du_k}{u_k} \wedge \frac{du_1}{u_1} + \sum_{2 \leq k < l \leq N} b_{k-1,l-1} \frac{du_k}{u_k} \wedge \frac{du_l}{u_l}.
\] (16)

Therefore the coefficient matrix of \(\sigma^* \omega\) is
\[
\hat{B} = \begin{bmatrix}
0 & -b_{1,N} & -b_{2,N} & \cdots & -b_{N-1,N} \\
-b_{1,N} & 0 & b_{1,2} & \cdots & b_{1,N-1} \\
0 & b_{2,N} & 0 & \cdots & b_{2,N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{N-2,N} & -b_{N-1,N} & -b_{N-2,N} & \cdots & 0 \\
b_{N-1,N} & -b_{N-1,N} & -b_{2,N-2} & \cdots & 0
\end{bmatrix}.
\] (17)
A straightforward computation shows that \( \sigma^T B = \hat{B} \sigma^T \). As \( \sigma \) is orthogonal, this is equivalent to \( \sigma^{-1} B \sigma = \hat{B} \), which concludes the proof. \( \square \)

**Lemma 2.** Let \((B, u)\) be an initial seed, \(\omega\) the standard log presymplectic form \((14)\) and \(\mu_k\) the mutation in the direction \(k\) given by \((1)\) and \((2)\). Then, the pullback of \(\omega\) by \(\mu_k\) is given by

\[
\mu_k^* \omega = \sum_{i<j} (\mu_k(B))_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}. \tag{18}
\]

**Proof.** Recall from \((2)\) that

\[
\mu_k(u_1, u_2, \ldots, u_N) = \left( u_1, u_2, \ldots, u_{k-1}, \frac{A^+ + A^-}{u_k}, u_{k+1}, \ldots, u_N \right),
\]

with

\[
A^+ = \prod_{l:b_{kl}>0} u_l^{b_{kl}}, \quad A^- = \prod_{l:b_{kl}<0} u_l^{-b_{kl}}.
\]

The pullback of \(\omega\) by \(\mu_k\) is then given by

\[
\mu_k^* \omega = \sum_{k \neq i < j \neq k} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j} + \sum_{j > k} b_{kj} \frac{du'_j}{u'_j} \wedge \frac{du_j}{u_j} + \sum_{i < k} b_{ik} \frac{du_i}{u_i} \wedge \frac{du_i}{u_i}. \tag{19}
\]

As

\[
du'_j = -\frac{du_k}{u_k} + \frac{A^+}{A^+ + A^-} \sum_{l:b_{kl}>0} b_{kl} \frac{du_l}{u_l} - \frac{A^-}{A^+ + A^-} \sum_{l:b_{kl}<0} b_{kl} \frac{du_l}{u_l},
\]

substituting into \((19)\) and re-arranging all the terms, we obtain

\[
\mu_k^* \omega = \sum_{i<j} \hat{b}_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}. \tag{20}
\]

with \(\hat{B} = [\hat{b}_{ij}]\) given by:

a) for \(i < k\) (resp. for \(j > k\)): \(\hat{b}_{ik} = -b_{ik}\) (resp. \(\hat{b}_{kj} = -b_{kj}\));

b) for \(i < j < k\): 

\[
\hat{b}_{ij} = b_{ij} + \begin{cases} 
   \frac{A^+(b_{ik}b_{kj}-b_{ik}b_{kj})}{A^+ + A^-} = 0, & \text{if } b_{ki} > 0, b_{kj} > 0 \\
   -\frac{A^-(b_{ik}b_{kj}-b_{ik}b_{kj})}{A^+ + A^-} = 0, & \text{if } b_{ki} < 0, b_{kj} < 0 \\
   \frac{A^+b_{ik}b_{kj} + A^-b_{ik}b_{kj}}{A^+ + A^-} = b_{ik}b_{kj}, & \text{if } b_{ki} < 0, b_{kj} > 0 \\
   -\frac{A^-b_{ik}b_{kj} + A^+b_{ik}b_{kj}}{A^+ + A^-} = -b_{ik}b_{kj}, & \text{if } b_{ki} > 0, b_{kj} < 0 
\end{cases}
\]
where the equalities inside the above bracket follow from the skew-symmetry of $B = [b_{ij}]$.

Using similar arguments, the remaining entries $\hat{b}_{ij}$ are as follows.

c) for $i < k < j$:

$$\hat{b}_{ij} = b_{ij} + \begin{cases} 
\frac{A^+(b_{ik}b_{kj}b_{ji})}{A^++A^-} = 0, & \text{if } b_{ki} > 0, b_{kj} > 0 \\
-\frac{A^-(b_{ik}b_{kj}b_{ji})}{A^++A^-} = 0, & \text{if } b_{ki} < 0, b_{kj} < 0 \\
\frac{A^+b_{ik}b_{kj}b_{ji} - A^-b_{ik}b_{kj}b_{ji}}{A^++A^-} = b_{ik}b_{kj}, & \text{if } b_{ki} < 0, b_{kj} > 0 \\
-\frac{A^-b_{ik}b_{kj}b_{ji} - A^+b_{ik}b_{kj}b_{ji}}{A^++A^-} = -b_{ik}b_{kj}, & \text{if } b_{ki} > 0, b_{kj} < 0 
\end{cases}$$

\[21\]

Summing up, if $i = k$ or $j = k$ then $\hat{b}_{ij} = -b_{ij}$, and in any other case

$$\hat{b}_{ij} = b_{ij} + \begin{cases} 
0, & \text{if } b_{ik}b_{kj} < 0 \\
b_{ik}b_{kj}, & \text{if } b_{ik} > 0, b_{kj} > 0 \\
-b_{ik}b_{kj}, & \text{if } b_{ik} < 0, b_{kj} < 0 
\end{cases}$$

It is now easy to check that each $\hat{b}_{ij}$ coincides with $b'_{ij}$ given in [1], which shows that $\hat{B} = [\hat{b}_{ij}]$ is precisely $\mu_k(B)$. \qed

**Remark 2.** Although we do not require compatibility of $\omega$ with the cluster algebra $\mathcal{A}(B)$, the proof of the above lemma could follow from the proof of Theorem 6.2 in [GeShVa10] which characterizes log presymplectic structures compatible with a cluster algebra. However, to the best of our knowledge the statement in Lemma 2, in particular the expression (18), is new and not completely obvious from the literature in cluster algebras.
Proof of Theorem 1. Using properties of the pullback operation and the identities (15) and (18) from Lemmas 1 and 2, it follows that
\[
\varphi^* \omega = (\sigma^m \circ \mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1)^* \omega = \mu_1^* \circ \cdots \circ \mu_m^* (\sigma^m)^* \omega
\]
\[
= \sum_{i<j} (\mu_1(\cdots \mu_{m-1}(\mu_m(\sigma^{-m} B\sigma^m)))) \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}.
\]
Therefore \(\varphi^* \omega = \omega\) if and only if
\[
\mu_1 \circ \cdots \circ \mu_{m-1} \circ \mu_m(\sigma^{-m} B\sigma^m) = B.
\]
As any mutation is an involution we have the following equivalence
\[
\varphi^* \omega = \omega \iff \sigma^{-m} B\sigma^m = \mu_m \circ \cdots \circ \mu_1(B),
\]
which concludes the proof.

4 Symplectic reduction of cluster iteration maps

In this section we prove that whenever the matrix \(B\) representing an \(m\)-periodic quiver is singular, the corresponding iteration map can be reduced to a symplectic map with respect to a \emph{log symplectic form}.

Our main result, Theorem 2, is a generalization for quivers of arbitrary period \(m\) of Theorem 2.6 in \cite{FoHo12}. The strategy we follow to prove this theorem is completely different from the one in \cite{FoHo12}, which was based on the classification of 1-periodic quivers. In fact our proof relies on a classical theorem of G. Darboux for presymplectic forms and on the invariance of the standard presymplectic form under the iteration map proved in Theorem 1.

Theorem 2. Let \(Q\) be an \(m\)-periodic quiver with \(N\) nodes, \((B_Q, u)\) the associated initial seed and \(\varphi\) the iteration map given in (11). If \(\text{rank}(B_Q) = 2k < N\) then there exist

i) a submersion \(\pi : \mathbb{R}^N_+ \rightarrow \mathbb{R}^{2k}_+\),

ii) a map \(\hat{\varphi} : \mathbb{R}^{2k}_+ \rightarrow \mathbb{R}^{2k}_+\)

such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{R}^N_+ & \xrightarrow{\varphi} & \mathbb{R}^N_+ \\
\pi \downarrow & & \pi \downarrow \\
\mathbb{R}^{2k}_+ & \xrightarrow{\hat{\varphi}} & \mathbb{R}^{2k}_+ \\
\end{array}
\]

Furthermore \(\hat{\varphi}\) is symplectic with respect to the canonical log symplectic form
\[
\omega_0 = \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} + \cdots + \frac{dy_{2k-1}}{y_{2k-1}} \wedge \frac{dy_{2k}}{y_{2k}},
\]
that is \(\hat{\varphi}^* \omega_0 = \omega_0\).
The proof of this theorem relies on the next proposition which in turn relies
on Darboux’s theorem for closed 2-forms (or presymplectic forms) of constant
rank.

**Proposition 1.** Let \( \omega \) be a closed 2-form of constant rank \( 2k \) on a manifold \( M \)
and \( x_0 \in M \). Then, there exists a set \( \{ g_1, g_2, \ldots, g_{2k} \} \) of \( 2k \) locally independent
functions on \( M \) such that

\[
\omega = dg_1 \wedge dg_2 + \cdots + dg_{2k-1} \wedge dg_{2k}.
\]  

(23)

Moreover, if \( \varphi \) is a local diffeomorphism preserving \( \omega \), that is \( \varphi^* \omega = \omega \), then
each of the functions

\[
\psi_1 = g_1 \circ \varphi, \quad \psi_2 = g_2 \circ \varphi, \quad \ldots, \quad \psi_{2k} = g_{2k} \circ \varphi
\]
depends only on \( \{ g_1, g_2, \ldots, g_{2k} \} \).

**Proof.** The first statement is just Darboux’s theorem for closed 2-forms of con-
tant rank (see for instance [AbMa] and [SS]).

To prove that the functions \( \psi_i = g_i \circ \varphi \) just depend on the set \( \{ g_1, g_2, \ldots, g_{2k} \} \),
let \( \omega^{(k)} \) be the \( k \)th exterior power of \( \omega \). As \( \varphi^* \omega = \omega \) then

\[
\varphi^* \omega^{(k)} = \omega^{(k)}.
\]  

(24)

Using the expression (23) for \( \omega \), it turns out that

\[
\omega^{(k)} = (-1)^{\lfloor k/2 \rfloor} k! \, dg_1 \wedge dg_2 \wedge \cdots \wedge dg_{2k-1} \wedge dg_{2k},
\]
and so (24) is equivalent to

\[
d\psi_1 \wedge d\psi_2 \wedge \cdots \wedge d\psi_{2k-1} \wedge d\psi_{2k} = dg_1 \wedge dg_2 \wedge \cdots \wedge dg_{2k-1} \wedge dg_{2k}.
\]  

(25)

Now complete the set \( \{ g_1, g_2, \ldots, g_{2k} \} \) to a full set of coordinates \( \{ g_1, g_2, \ldots, g_N \} \)
on \( M \), and consider the Jacobian matrix of \( \psi = (\psi_1, \psi_2, \ldots, \psi_{2k}) \),

\[
J = \begin{bmatrix}
\frac{\partial \psi_1}{\partial g_1} & \cdots & \frac{\partial \psi_1}{\partial g_{2k}} & \frac{\partial \psi_1}{\partial g_{2k+1}} & \cdots & \frac{\partial \psi_1}{\partial g_N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial \psi_{2k}}{\partial g_1} & \cdots & \frac{\partial \psi_{2k}}{\partial g_{2k}} & \frac{\partial \psi_{2k}}{\partial g_{2k+1}} & \cdots & \frac{\partial \psi_{2k}}{\partial g_N}
\end{bmatrix}.
\]

Condition (25) implies that the determinant of the leftmost \( (2k) \times (2k) \) subma-
trix of \( J \) is equal to 1 and all the other \( (2k) \times (2k) \) determinants of \( J \) are equal
to 0.

Linear algebra arguments assure that the rightmost \( (2k) \times (N-2k) \) submatrix
of \( J \) is the zero matrix, which concludes the proof. \( \square \)
Proof of Theorem 2. Let \( \omega \) be the standard log presymplectic form on \( \mathbb{R}^N_+ \)

\[
\omega = \sum_{1 \leq i < j \leq N} b_{ij} \frac{du_i}{u_i} \wedge \frac{du_j}{u_j}.
\]

Since \( \omega \) has rank \( 2k \), Darboux’s theorem (see Proposition 1) implies the existence of functions \( g_1, \ldots, g_{2k} \) such that \( \omega \) is given by \( (23) \). Considering

\[
\pi : \mathbb{R}^N_+ \rightarrow \mathbb{R}^{2k}_+ \quad \text{u} \mapsto (\exp(g_1(u)), \ldots, \exp(g_{2k}(u))),
\]

and \( \omega_0 \) given by \( (22) \), we have

\[
\pi^* \omega_0 = \omega.
\] (26)

By Theorem 1 the iteration map \( \varphi \) preserves \( \omega \), that is \( \varphi^* \omega = \omega \). Then, again by Proposition 1, \( \pi \circ \varphi(u) \) depends only on \( \pi(u) \), since each \( g_i \circ \varphi \) depends only on \( \{g_1, \ldots, g_{2k}\} \). This is equivalent to say that \( \varphi \) exists and makes the diagram commutative.

It remains to prove that \( \hat{\varphi} \) is symplectic. For this purpose, we note that the commutativity of the diagram is equivalent to

\[
(\pi \circ \varphi)^* \omega_0 = (\hat{\varphi} \circ \pi)^* \omega_0 \iff \varphi^* (\pi^* \omega_0) = \pi^* (\hat{\varphi}^* \omega_0).
\]

Using (26) and the fact that \( \varphi \) preserves \( \omega \), we have

\[
\varphi^* (\pi^* \omega_0) = \pi^* (\hat{\varphi}^* \omega_0) \iff \pi^* (\omega_0 - \hat{\varphi}^* \omega_0) = 0.
\]

As \( \pi \) is a submersion, this implies \( \hat{\varphi}^* \omega_0 = \omega_0 \).

Remark 3. In the proof of Theorem 2 we can use any (nonzero) multiple of \( \omega \). In fact, if \( B \) comes from an \( m \)-periodic quiver then \( \lambda \omega \) is also preserved by \( \varphi \) (direct consequence of Theorem 1). Proposition 1 can still be used with \( \lambda \omega \), and reduction is achieved in an entirely analogous way. This fact will be used in Example 4 to improve the form of the reduced iteration map.

We note that without the assumption of periodicity of the quiver \( Q \) (that is, without the iteration map \( \varphi \)), Theorem 2 may be seen as an alternative proof of the existence of the secondary cluster manifold with its Weil-Petersson form (see for instance [GeShVa10]). However, when taking into account the iteration map, Theorem 2 has stronger consequences since it allows to reduce recurrence relations to the secondary manifold.

5 Computation of the reduced iteration map

In this section we compute the reduced iteration map \( \hat{\varphi} \) as well as the reduced variables \( \pi(u) \) for three periodic quivers. To the best of our knowledge this reduction has not been performed before for quivers of period higher than 1.
Our approach to the computation of the reduced symplectic iteration map \( \hat{\varphi} \) is based on the construction of the functions \( g_1, \ldots, g_{2k} \) appearing in Proposition \[\text{Proposition} \] We note that these functions are given by Darboux’s theorem whose proof is not constructive. However due to the particular form of the log presymplectic forms it is possible to change coordinates in such a way that one can apply a theorem of Cartan (Theorem 2.3 in \[\text{LibMa} \]) whose proof is constructive, and then obtain explicit reduced variables \( f_i = \exp(g_i(u)) \).

Let us consider the standard log presymplectic form \( \omega \) written in coordinates \( v_i = \log u_i \),

\[
\omega = \sum_{1 \leq i < j \leq N} b_{ij} dv_i \wedge dv_j,
\]

and assume that its rank is \( 2k < N \).

Cartan’s Theorem says that there exist \( 2k \) functions \( g_i \) (depending linearly on the \( v_i \) variables) such that

\[
\omega = \sum_{1 \leq i < j \leq 2k} dg_i \wedge dg_j.
\]

We recall the main steps of the proof of Cartan’s Theorem, which explicitly produces the functions \( g_i \).

Reordering if necessary the \( v_i \)-coordinates, we can assume that \( b_{12} \neq 0 \). Let

\[
g_1 = \frac{1}{b_{12}} \sum_{k=1}^{N} b_{1k} v_k \quad \text{and} \quad g_2 = \sum_{k=1}^{N} b_{2k} v_k,
\]

so that

\[
dg_1 \wedge dg_2 = b_{12} dv_1 \wedge dv_2 + \sum_{i=3}^{N} b_{1i} dv_i \wedge dv_1 + \sum_{j=3}^{N} b_{2j} dv_2 \wedge dv_j + \alpha
\]

where \( \alpha \) depends only on \( \{v_3, \ldots, v_N\} \). Then the 2-form

\[
\tilde{\omega} = \omega - dg_1 \wedge dg_2
\]

is a closed 2-form on the \((N - 2)\)-dimensional vector space, with coordinates \( \{v_3, \ldots, v_N\} \), and \( \text{rank}(\tilde{\omega}) = 2k - 2 \).

If \( \text{rank}(\omega) = 2 \) then \( \tilde{\omega} = 0 \) and \( \omega = dg_1 \wedge dg_2 \). Otherwise, the previous procedure is repeated, replacing \( \omega \) by \( \tilde{\omega} \). After \( k \) steps all the functions \( g_1, \ldots, g_{2k} \) will have been obtained.

As each function \( g_i \) is a linear function of the variables \( v_i = \log u_i \), it can be written in the form

\[
g_i(u_1, \ldots, u_N) = \log(f_i(u_1, \ldots, u_N)), \quad i = 1, 2, \ldots N.
\]
The submersion $\pi$ in Theorem 2 is then given by

$$\pi(u_1, \ldots, u_N) = (f_1(u), \ldots, f_{2k}(u)),$$

and the functions $f_1, \ldots, f_{2k}$ are reduced variables. Again by Theorem 2, $\hat{\phi} \circ \pi = \pi \circ \varphi$ and so the reduced iteration map is

$$\hat{\phi}(f_1, \ldots, f_{2k}) = (f_1 \circ \varphi, \ldots, f_{2k} \circ \varphi).$$

In the next three examples we consider quivers represented by the matrix $B$ in (5) and we compute the respective reduced iteration map. In the first two examples the quivers are 2-periodic and the computation of the reduced iteration map follows step by step the described proof of Cartan’s theorem. The last example is a 1-periodic quiver and aims to illustrate the possibility of modifying Cartan’s construction to choose a more convenient set of reduced variables.

Recall that $B$ in (5) represents a 1-periodic quiver if $r = t$ and a 2-periodic quiver otherwise. Furthermore, if $p = r + t$ then $B$ fails to have maximal rank and so the iteration map $\varphi$ can be reduced to a symplectic map (by Theorem 2).

**Example 2.** A 2-periodic quiver $Q$ with 6 nodes and rank $B_Q = 2$.

When $r = 2, t = 5, p = 7, s = 13$ the matrix $B$ in (5) has rank 2. From (13) the iteration map is

$$\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_3, u_4, u_5, u_6, u_7, u_8)$$

with

$$u_7 = \frac{u_3^{13}u_5^{13} + u_2^2u_4^7u_6^5}{u_1}, \quad u_8 = \frac{u_4^{13}u_6^{13} + u_3^5u_5^7u_7^2}{u_2}. \quad (27)$$

The first and second rows of $B$ are respectively

$$(0, -2, 13, -7, 13, -5) \quad \text{and} \quad (2, 0, -31, 13, -33, 13),$$

and so (27) gives $\omega = dg_1 \wedge dg_2$, with

$$g_1 = v_2 - \frac{13}{2}v_3 + \frac{7}{2}v_4 - \frac{13}{2}v_5 + \frac{5}{2}v_6 = \log \left( \frac{u_2u_4^{7/2}u_6^{5/2}}{u_3^{13/2}u_5^{13/2}} \right)$$

and

$$g_2 = 2v_1 - 31v_3 + 13v_4 - 33v_5 + 13v_6 = \log \left( \frac{u_1^{2}u_4^{13}u_6^{13}}{u_3^{3}u_5^{3}} \right).$$

The reduced log symplectic form is $\hat{\omega} = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2}$, with

$$f_1 = \frac{u_2u_4^{7/2}u_6^{5/2}}{u_3^{13/2}u_5^{13/2}}, \quad f_2 = \frac{u_1^{2}u_4^{13}u_6^{13}}{u_3^{3}u_5^{3}}.$$

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To obtain the reduced iteration map $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2)$ we just have to compute the compositions $f_i \circ \varphi$ as functions of $f_1$, $f_2$, which produces

$$
\hat{\varphi}_1 = f_1 \circ \varphi = \frac{f_2^{3/4} (f_2 + (1 + f_2^2))^{5/2}}{f_1^{5/2}(1 + f_1^{2})^{3/2}},
$$

$$
\hat{\varphi}_2 = f_2 \circ \varphi = \frac{f_2^{1/2} (f_2 + (1 + f_2^2)^{13}}{f_1^{1/3}(1 + f_1^{2})^{33}}.
$$

**Example 3.** A 2-periodic quiver $Q$ with 6 nodes and rank $B_Q = 4$.

Taking $r = 1$, $s = 1$, $t = 2$, $p = 3$ in (5), we have rank $B = 4$. In this case, one needs to go one step further in the proof of Cartan’s Theorem in order to compute the reduced functions $f_i$.

The iteration map is $\varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_3, u_4, u_5, u_6, u_7, u_8)$ with

$$
u_7 = \frac{u_3 u_5 + u_2 u_4^2 u_6}{u_1}, \quad u_8 = \frac{u_4 u_6 + u_2 u_4^2 u_7}{u_2}.
$$

Using Cartan’s construction we get $\omega = dg_1 \wedge dg_2 + dg_3 \wedge dg_4$ with

$$
dg_1 \wedge dg_2 = d(v_2 - v_3 + 3v_4 - v_5 + 2v_6) \wedge d(v_1 - 3v_3 + v_4 - 4v_5 + v_6),
$$

and

$$
dg_3 \wedge dg_4 = d(v_4 + v_6) \wedge d(8v_3 + 8v_5).
$$

From the above expressions, we obtain

$$
f_1 = \frac{u_2 u_4^3 u_6}{u_3 u_5}, \quad f_2 = \frac{u_1 u_4 u_6}{u_2 u_5^3}, \quad f_3 = u_4 u_6, \quad f_4 = u_3^8 u_5.
$$

The reduced iteration map is $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4)$ with

$$
\hat{\varphi}_1 = f_1 \circ \varphi = \frac{f_2^3 (1 + f_1 + f_2)^2}{f_1^2 f_2 (1 + f_1)},
$$

$$
\hat{\varphi}_2 = f_2 \circ \varphi = \frac{f_3 f_4 (1 + f_1 + f_2)}{f_1 (1 + f_1)^4},
$$

$$
\hat{\varphi}_3 = f_3 \circ \varphi = \frac{f_3^2 (1 + f_1 + f_2)}{f_1 f_2 f_4^{1/8}},
$$

$$
\hat{\varphi}_4 = f_4 \circ \varphi = \frac{f_3^5 (1 + f_1)^8}{f_2^5 f_4^2}.
$$

**Example 4.** A 1-periodic quiver with 6 nodes and rank $B_Q = 2$.

Let us consider the matrix $B$ in (5) with $r = t = 2$, $p = 4$ and $s = 6$, which represents a 1-periodic quiver. The rank of $B$ is 2 and its first and second rows are, respectively, $(0, -2, 6, -4, 6, -2)$ and $(2, 0, -14, 6, -16, 6)$. 

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The iteration map is \( \varphi(u_1, u_2, u_3, u_4, u_5, u_6) = (u_2, u_3, u_4, u_5, u_6, u_7) \) with
\[
   u_7 = \frac{u_2^2 u_4^2 + u_5^6 u_6}{u_1}.
\]

The reduction of this iteration map was done in [CrSD] and also in [FoHo11] (using a different approach). We include it here in order to illustrate the possibility of using Cartan’s proof to obtain a more convenient set of reduced variables. The strategy of the previous examples is modified as follows.

We start by considering the presymplectic form \( \omega' = \omega'(\varphi) \) (see Remark 3). That is,
\[
   \omega' = d(v_2 - 3v_3 + 2v_4 - 3v_5 + v_6) \wedge d(-v_1 + 7v_3 - 3v_4 + 8v_5 - 3v_6).
\]

Next, noting that \( \omega' = dg_1 \wedge dg_2 = dg_1 \wedge d(g_2' + 3g_1') \) and taking \( g_1 = -g_2' - 3g_1' \) and \( g_2 = g_1' \) we obtain
\[
   \omega' = dg_1 \wedge dg_2 = d(v_1 - 3v_2 + 2v_3 - 3v_4 + v_5) \wedge d(v_2 - 3v_3 + 2v_4 - 3v_5 + v_6).
\]

The reduced variables are then
\[
   f_1 = \frac{u_1 u_2^2 u_5}{u_2^3 u_4}, \quad f_2 = \frac{u_2 u_4^2 u_6}{u_3^2 u_5^2},
\]
and the reduced map \( \tilde{\varphi} \), which is symplectic with respect to the form \( \tilde{\omega} = \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \), is given by
\[
   \tilde{\varphi}(f_1, f_2) = \left( f_2, \frac{1 + f_2}{f_1 f_2} \right).
\]

Therefore the sixth order recurrence relation (7)
\[
   u_{n+6}u_n = u_{n+1}^2 u_{n+3} u_{n+5} + u_{n+2}^6 u_{n+4}, \quad n = 1, 2, \ldots
\]
reduces to the following relation
\[
   f_{n+2} = \frac{1 + f_{n+1}^2}{f_n f_{n+1}}^2, \quad n = 1, 2, \ldots
\]
which is a recurrence relation of order 2.

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