ON WEYL MULTIPLIERS OF NON-OVERLAPPING FRANKLIN POLYNOMIAL SYSTEMS

GRIGORI A. KARAGULYAN

Abstract. We prove that $\log n$ is an almost everywhere convergence Weyl multiplier for any orthonormal system of non-overlapping Franklin polynomials. It will also be remarked that $\log n$ is the optimal sequence in this context.

1. Introduction

Recall some definitions well-known in the theory of orthogonal series (see [11]).

Definition 1.1. Let $\Phi = \{\phi_n : n = 1, 2, \ldots\} \subset L^2(0, 1)$ be an orthonormal system. A sequence of positive numbers $\omega(n) \not\to \infty$ is said to be an a.e. convergence Weyl multiplier (C-multiplier) if every series

$$\sum_{n=1}^{\infty} a_n \phi_n(x),$$

with coefficients satisfying the condition $\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty$ is a.e. convergent. If such series converges a.e. after any rearrangement of the terms, then we say $\omega(n)$ is an a.e. unconditional convergence Weyl multiplier (UC-multiplier) for $\Phi$.

The Menshov-Rademacher classical theorem ([12], [16], see also [11]) states that the sequence $\log^2 n$ is a C-multiplier for any orthonormal system. The sharpness of $\log^2 n$ in this theorem was proved by Menshov in the same paper [12]. That is any sequence $\omega(n) = o(\log^2 n)$ fails to be C-multiplier for some orthonormal system. The following inequality is the key ingredient in the proof of the Menshov-Rademacher theorem.

Theorem A (Menshov-Rademacher, [12], [16]). If $\{\phi_k : k = 1, 2, \ldots, n\} \subset L^2(0, 1)$ is an orthogonal system, then

$$\left\| \max_{1 \leq m \leq n} \sum_{k=1}^{m} \phi_k \right\|_2 \leq c \cdot \log n \left\| \sum_{k=1}^{n} \phi_k \right\|_2,$$

where $c > 0$ is an absolute constant.

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Similarly, the counterexample of Menshov is based on the following results. It implies that \( \log n \) on the right of (1.1) is optimal.

**Theorem B** (Menshov, [12]). For any natural number \( n \in \mathbb{N} \) there exists an orthonormal system \( \{ \phi_n : n = 1, 2, \ldots, n \} \subset L^2(0, 1) \), such that

\[
\left\{ x \in (0, 1) : \max_{1 \leq m \leq n} \left| \frac{1}{\sqrt{n}} \sum_{k=1}^{m} \phi_k(x) \right| \geq c \log n \right\} \gtrsim 1,
\]

for an absolute constant \( c > 0 \).

In the sequel the relation \( a \lesssim b (a \gtrsim b) \) stands for the inequality \( a \leq c \cdot b (a \geq c \cdot b) \), where \( c > 0 \) is an absolute constant. Given two sequences of positive numbers \( a_n, b_n > 0 \), we write \( a_n \sim b_n \) if we have \( c_1 \cdot a_n \leq b_n \leq c_2 \cdot a_n, n = 1, 2, \ldots \) for some constants \( c_1, c_2 > 0 \). Throughout the paper, the base of \( \log \) is equal 2.

Let \( \Phi = \{ \phi_k(x), k = 1, 2, \ldots \} \subset L^2(0, 1) \) be an infinite orthonormal system. Given \( g \in L^2(0, 1) \), consider the Fourier coefficients

\[ a_n = \langle g, \phi_n \rangle = \int_0^1 g \phi_n. \]

We say a function \( f \in L^2(0, 1) \) satisfies the relation \( f \prec g \) (with respect to system \( \Phi \)), if \( f = \sum_{n=1}^{\infty} \lambda_n a_n \phi_n \), where \( |\lambda_n| \leq 1 \). Given integer \( n \geq 1 \), We consider the following operator

\[ A_{n,\Phi}(g)(x) = \max_{g_k \prec g, k = 1, \ldots, n} |g_k(x)|, \]  
(1.2)

where the maximum is taken over all the sequences of functions \( g_k, k = 1, 2, \ldots, n \), satisfying \( g_k \prec g \). One can consider in (1.2) only the monotonic sequences of polynomials

\[ g_k = \sum_{j \in G_k} a_j \phi_j \prec g, \quad k = 1, 2, \ldots, n, \]

where \( G_1 \subset G_2 \subset \ldots \subset G_n \subset \mathbb{N} \) and \( a_k \) are the Fourier coefficients of \( g \). Then we will get another operator \( A_{n,\Phi,\text{mon}} \). If we additionally suppose that each \( G_{k+1} \setminus G_k \) consists of a single integer, then we will have the operator \( A_{n,\Phi,\text{sng}} \). For the \( L^2 \)-norms of these operator we clearly have

\[ \|A_{n,\Phi,\text{sng}}\|_{2 \to 2} \leq \|A_{n,\Phi,\text{mon}}\|_{2 \to 2} \leq \|A_{n,\Phi}\|_{2 \to 2}. \]  
(1.3)

Observe that Theorem A implies \( \|A_{n,\Phi,\text{mon}}\|_{2 \to 2} \lesssim \log n \) for every orthonormal system \( \Phi \). On the other hand, applying Theorem B, one can also construct an infinite orthonormal system with the lower bound \( \|A_{n,\Phi,\text{sng}}\|_{2 \to 2} \gtrsim \log n, n = 1, 2, \ldots \). Thus we conclude that for the general orthonormal systems the logarithmic upper bound of \( \|A_{n,\Phi,\text{mon}}\|_{2 \to 2} \) is optimal. As it was remarked in [8] from some results of Nikishin-Ulyanov [13] and Olevskii [14] it follows that \( \|A_{n,\Phi,\text{mon}}\|_{2 \to 2} \gtrsim \sqrt{\log n} \) for any complete orthonormal system \( \Phi \).

The recent papers of author [8–10] highlight the relation of sequences (1.3) in the study of almost everywhere convergence of special orthogonal series. It was proved in [8] that
Theorem C. If $\Phi$ is a martingale difference, then $\|A_{n,\Phi,\text{mon}}\|_2 \lesssim \sqrt{\log n}$.

Theorem D. For any generalized Haar system $H$ we have the relation

$$\|A_{n,H,\text{sng}}\|_2 \sim \|A_{n,H,\text{mon}}\|_2 \sim \sqrt{\log n}. \quad (1.4)$$

The paper \cite{8} also provide corollaries of these results like those considered below. In the case of trigonometric system in \cite{9,10} we prove the following.

Theorem E. If $T$ is the trigonometric system, then we have

$$\|A_{n,T,\text{sng}}\|_2 \sim \|A_{n,T,\text{mon}}\|_2 \sim \sqrt{\log n}. \quad (1.5)$$

Note that the upper bound $\|A_{n,T,\text{mon}}\|_2 \lesssim \log n$ in (1.5) follows from the Menshov-Rademacher theorem. So the novelty here is the estimate $\|A_{n,T,\text{sng}}\|_2 \gtrsim \log n$, which shows that the trigonometric system has no better estimate of the norms of operators $A_n$ that the general orthonormal systems have.

In this paper we prove the analogous of relations (1.4) and corresponding corollaries for the Franklin system $F = \{f_n\}$ of piece-wise linear functions. Moreover, we have involved also a sharp estimation for $A_{n,F}$.

Theorem 1.1. The Franklin system $F$ satisfies the relation

$$\|A_{n,F,\text{sng}}\|_2 \sim \|A_{n,F,\text{mon}}\|_2 \sim \|A_{n,F}\|_2 \sim \sqrt{\log n}. \quad (1.6)$$

Corollary 1.1. The sequence $\log n$ is a C-multiplier for any system of $L^2$-normalized non-overlapping Franklin polynomials

$$p_n(x) = \sum_{j \in G_n} c_j f_j(x), \quad n = 1, 2, \ldots,$$

where $G_n \subset \mathbb{N}$ are finite and pairwise disjoint.

The following particular case of Corollary 1.1 is also new and interesting.

Corollary 1.2. The sequence $\log n$ is a C-multiplier for any rearrangement of the Franklin system.

Corollary 1.3. Let $\{p_n\}$ be a sequence of $L^2$-normalized non-overlapping Franklin polynomials. If $w(n)/\log n$ is increasing and

$$\sum_{n=1}^{\infty} \frac{1}{nw(n)} < \infty, \quad (1.6)$$

then $w(n)$ is an UC-multiplier for $\{p_n\}$.

The only prior result in this context is due to Gevorkyan \cite{5}, who proved that condition (1.6) is a necessary and sufficient for $w(n)$ to be an UC-multiplier for the Franklin system. The optimality of $\log n$ in Corollary 1.2 as well as condition (1.6) in Corollary 1.3 both follows just from the direct combination of this result of Gevorkyan with a result of Ul’yanov-Poleshchuk \cite{15,18}.
The following corollary is an interesting phenomenon of the Franklin series. It is based on the bound $\|A_n, F\|_{2\to 2} \lesssim \sqrt{\log n}$ involved in Theorem 1.1.

**Corollary 1.4.** Let the sequence $a = \{a_k\}$ satisfy $\sum_{k=1}^{\infty} a_k^2 < \infty$. Then for arbitrary sets of indexes $G_k \subset \mathbb{N}$, $k = 1, 2, \ldots, n$ we have

$$\max_{1 \leq m \leq n} \left| \sum_{j \in G_m} a_j f_j \right|_2 \lesssim \sqrt{\log n} \cdot \left( \sum_{k=1}^{\infty} a_k^2 \right)^{1/2}. \quad (1.7)$$

**Remark 1.1.** We do not know if the estimate like (1.5) holds for the Walsh system. Note that the proof of (1.5) is based on a specific argument that is common only for the trigonometric system and it is not applicable in the case of Walsh system. Namely, the proof of (1.5) uses a logarithmic lower bound for the directional Hilbert transform on the plane due to Demeter [4].

**Remark 1.2.** We prove Theorem 1.1 using a good-$\lambda$ inequality due to Chang-Wilson-Wolff [1], which is an extension of classical Azuma-Hoeffding and Bernstein inequalities for martingales. See also [6], where the same method has been first applied in the study of maximal functions of Mikhlin-Hörmander multipliers.

**Remark 1.3.** We will see in the last section that the results of Theorem 1.1 and Corollaries 1.1-1.4 hold also for the Franklin system of periodic type. It will be also proved that for the classical Haar system $H$ as an addition to (1.5) it holds the relation $\|A_n, H\|_{2\to 2} \sim \sqrt{\log n}$ and so an estimate like (1.7).

2. **Notations and the definition of Franklin system**

Recall the definition of the Franklin orthonormal system on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. Given integer $n \geq 2$, write it in the form

$$n = 2^k + j, \quad \text{where} \quad 1 \leq j \leq 2^k, \quad k = 0, 1, \ldots. \quad (2.1)$$

Let $\Pi_n$ be the set of nodes

$$t_{n,i} = \begin{cases} \frac{i}{2^{k+1}}, & \text{if} \quad 1 \leq i \leq 2j, \\ \frac{i-j}{2^k}, & \text{if} \quad 2j < i < n, \end{cases}$$

and suppose also $\Pi_1 = \{t_{1,0} = 0\}$. One can check that $0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n-1} < 1$ and the collection $\Pi_n$ ($n \geq 2$) is obtained from $\Pi_{n-1}$ by adding a single point $t_{n,2j-1}$. Denote by $L_n$ the space of piece-wise linear functions on $\mathbb{T}$ with nodes from $\Pi_n$, which may have a discontinuity point only at $t_{n,0} = 0$. For the dimension of this space we have $\dim L_n = n$. The Franklin functions $f_n(x)$, $n = 0, 1, 2, \ldots$, are defined as follows. We take $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x - 1)$, and if $n \geq 2$, then we let $f_n$ to be an $L^2$-normalized function from $L_n$, which is orthogonal to $L_{n-1}$. Note that $f_n$ is determined uniquely up to the sign.
Our method of proof can not be directly applied to the Franklin functions, since those are not full continuous on the torus $\mathbb{T}$. To handle this we will use the following reconstruction of the Franklin functions $f_n$ defined

$$u_n(x) = \begin{cases} 
    f_n(2x) & \text{if } x \in [0,1/2), \\
    f_n(-2x) & \text{if } x \in (-1/2,0), \\
    f_n(0-) & \text{if } x = -1/2,
\end{cases}$$

for $n = 0, 1, 2, \ldots$. Let $\tilde{L}_n$ be the subspace of $L_n$ consisting of the continuous functions $f$, that is $f(0) = f(0-)$. We have $\dim \tilde{L}_n = n - 1$. Clearly, under the condition (2.1) we have

$$u_n \in \tilde{L}_{2k+2}, \quad \langle f, u_n \rangle = 0, \quad f \in L_{2k+1} \supset \tilde{L}_{2k+1}. \quad (2.2)$$

For $f \in L^2(0,1)$, we consider the Fourier partial sum

$$U_n(f)(x) = \sum_{j=1}^{2^n} \langle f, u_j \rangle u_j(x), \quad n = 0, 1, 2, \ldots,$$

where $\langle f, g \rangle$ denotes the standard scalar product of two functions $f, g \in L^2(\mathbb{T})$. For a given function sequence $G_n, n = 0, 1, 2, \ldots$, we denote $\Delta G_0 = G_0, \Delta G_n = G_n - G_{n-1}, n \geq 1$. Letting $\Lambda_n = \tilde{L}_{2^n}$, from (2.2) we get

$$\Delta U_n(f) = \sum_{j=2^{n-1}+1}^{2^n} \langle f, u_j \rangle u_j, \quad n \geq 1,$$

$$\Delta U_n(f) \in \Lambda_{n+1}, \quad f \in L^2(\mathbb{T}), \quad n = 0, 1, 2, \ldots,$$

$$\Delta U_n(f) \equiv 0 \text{ whenever } f \in \Lambda_n(\mathbb{T}), \quad n = 0, 1, 2, \ldots. \quad (2.3)$$

Now recall the definition of the Haar system. Denote by $\mathcal{S}_n$ the class of right-continuous step functions on $\mathbb{T}$ with discontinuity points in $\Pi_n$. Define $h_1(x) = 1$ and for $n \geq 2$ let $h_n$ be an $L^2$-normalized function from $\mathcal{S}_n$ that is orthogonal to $\mathcal{S}_{n-1}$. We will consider the $\xi$-shifted Haar system $h_{n,\xi}(x) = h_n(x + \xi)$, where $\xi \in \mathbb{T}$. For a function $f \in L^2(\mathbb{T})$ denote

$$H_{n,\xi}(f)(x) = \sum_{j=1}^{2^n} \langle f, h_j,\xi \rangle h_j,\xi(x) \quad (2.4)$$

to be the $2^n$-partial sum of the Fourier series in $\xi$-shifted Haar system. We will also need the $\xi$-shifted Haar square function operator

$$S_{\xi}(f)(x) = \left( \sum_{n=1}^{\infty} |\Delta H_{n,\xi}(f)(x)|^2 \right)^{1/2}. \quad (2.5)$$
3. Exponential estimates and related properties

Recall the well-known exponential estimates of the Franklin functions

\[ |f_n(x)| \lesssim \sqrt{nq^n|x-t_n|}, \quad x \in [0, 1), \quad (3.1) \]

\[ |K_n(x, t)| \lesssim nq^n|x-t|, \quad x, t \in [0, 1). \quad (3.2) \]

where

\[ t_n = t_{n, 2j-1} = \frac{2j-1}{2^{k+1}}, \quad K_n(x, t) = \sum_{k=0}^{2^n} f_k(x)f_k(t). \]

(see [2, 3], or [11] chap. 6). For the next two lemmas we will need the well-known discrete convolution inequality

\[ \left( \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} |a_kb_{n-k}| \right)^2 \right)^{1/2} \leq \left( \sum_{n \in \mathbb{Z}} a_n^2 \right)^{1/2} \cdot \left( \sum_{n \in \mathbb{Z}} |b_n| \right). \quad (3.3) \]

**Lemma 3.1.** For any coefficients \( a_n \) we have

\[ \left\| \sum_{n=2^{k+1}}^{2^{k+1}} a_n u_n \right\|_2 \lesssim \left( \sum_{n=2^{k+1}}^{2^{k+1}} a_n^2 \right)^{1/2}. \quad (3.4) \]

**Proof.** Clearly it is enough to prove (3.4) for the Franklin functions \( f_n \). Chose an arbitrary \( x \in [0, 1) \) and suppose that

\[ x \in \left[ \frac{m-1}{2^k}, \frac{m}{2^k} \right), \quad 1 \leq m \leq 2^k. \]

From (2.1) it easily follows that \( n|x-t_n| \gtrsim |j-m| = |n-2^k-m| \). Thus, using (3.1), we get

\[ \left( \sum_{n=2^{k+1}}^{2^{k+1}} |a_n f_n(x)| \right)^2 \lesssim 2^k \left( \sum_{n=2^{k+1}}^{2^{k+1}} |a_n q_1^{n-2^k-m}| \right)^2, \quad 0 < q_1 < 1. \]

Defining \( a_n = 0 \) if \( n \not\in (2^k, 2^{k+1}] \), and applying (3.3), we obtain

\[ \left\| \sum_{n=2^{k+1}}^{2^{k+1}} a_n f_n \right\|_2^2 \lesssim \sum_{m=1}^{2^k} \left( \sum_{n=2^{k+1}}^{2^{k+1}} |a_n q_1^{n-2^k-m}| \right)^2 \]

\[ \lesssim \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |a_n q_1^{n-m}| \right)^2 \lesssim \sum_{n \in \mathbb{Z}} a_n^2 \]

and so (3.4).
Lemma 3.2. For any interval $J \subset \mathbb{T}$ with the center $c_J$ and an integer $n \geq 0$ there exists a function $\lambda_{J,n}(x) \geq 0$ on $\mathbb{T}$ such that

$$\lambda_{J,n}(x) = 1, \quad x \in J,$$

$$\|\lambda_{J,n}\|_1 \leq |J|,$$  \tag{3.5}

$$\lambda_{J,n}(x) \text{ is increasing on } [c_T - 1/2, c_T), \text{ and decreasing on } [c_T, c_T + 1/2).$$  \tag{3.6}

and for any function $g \in L^\infty(\mathbb{T})$ with $\|g\|_\infty \leq 1$, supp $g \subset J$ we have

$$|\Delta U_n(g)(x)| \lesssim \lambda_{J,n}(x) + \lambda_{J,n}(-x).$$ \tag{3.7}

Proof. Since $\Delta U_n = U_n - U_{n-1}$, it is enough to prove a similar estimate for $U_n(g)$. First we suppose that neither 0 nor 1/2 are in $(a, b)$. Then we will have $J \subset [0, 1/2)$ (or $J \subset [-1/2, 0)$). Set

$$\lambda_{J,n}(x) = \begin{cases} 1 & \text{if } x \in 2J, \\ c|J|nq^{nd(x,c_J)} & \text{if } x \notin 2J, \end{cases}$$

where $d(x, y)$ denotes the distance of two points $x, y$ on the torus $\mathbb{T}$. Clearly this function satisfies conditions (3.5), (3.6) and (3.7) for a small enough absolute constant $c$. To show (3.8), first we let $x \in [0, 1/2)$. Using (3.2) and the definition of $u_n$, one can easily check that

$$|U_n(g)(x)| \leq \int f \cdot K_n(2x, 2t)|dt| \leq \int f q^{2n|x-t|}dt \lesssim \lambda_{J,n}(x), \quad x \in [0, 1/2).$$ \tag{3.8}

Since $U_n(g)$ is an even function, for $x \in [-1/2, 0)$ we will have

$$|U_n(g)(x)| = |U_n(g)(-x)| \lesssim \lambda_{J,n}(-x), \quad x \in [-1/2, 0).$$ \tag{3.9}

Combining (3.9) and (3.10), we get (3.8). If 0 $\in$ $J$, then we consider the intervals $J_1 = J \cap [0, 1/2]$ and $J_2 = J \cap [-1/2, 0]$. Clearly $c_{J_2} \leq c_J \leq c_{J_1}$ and the function

$$\lambda_{J,n}(x) = \begin{cases} \lambda_{J_1,n}(x) & \text{if } x \in [c_J, c_J + 1/2), \\ \lambda_{J_2,n}(x) & \text{if } x \in [c_J - 1/2, c_J), \end{cases}$$

satisfies the conditions of the lemma. The case of $1/2 \in J$ can be considered similarly. \qed

4. The main lemma

We denote by $I_{n,\xi}(x) = [a_{n,\xi}(x), b_{n,\xi}(x))$ the $\xi$-shifted single dyadic interval of the form

$$\left[\xi + \frac{j - 1}{2^n}, \xi + \frac{j}{2^n}\right]$$

containing a given point $x \in \mathbb{T}$. In the case of $\xi = 0$ we will just write $n$ instead of the index $(n, 0)$. For the Haar partial sums (2.4) we can write

$$H_{n,\xi}(f)(x) = \frac{1}{|I_{n,\xi}(x)|} \int_{I_{n,\xi}(x)} f$$
Lemma 4.1. Let the positive function \( \lambda \in L_\infty(\mathbb{T}) \) be increasing on \([r, a)\) and decreasing on \([a, 1 + r)\). Then for any \( f \in L^1(\mathbb{R}) \) it holds the inequality
\[
\left| \int_{\mathbb{R}} f(t) \lambda(t) dt \right| \leq \|\lambda\|_1 \mathcal{M}(f)(a).
\]

Lemma 4.2. If \( g \in \Lambda_m(\mathbb{T}) \), then for any integers \( n \geq m \geq 1 \) it holds the inequality
\[
|\Delta H_{n,\xi}(g)(x)| \lesssim 2^{n-m} \cdot \mathcal{M}(g)(x), \quad x \in [0, 1).
\]

Proof. Given \( x \in [0, 1) \), the function \( g \) is linear on each interval \( I_m(x), I_m^+(x) \) and \( I_m^-(x) \), where \( I_m^+(x) \) and \( I_m^-(x) \) are the left and the right neighbor dyadic intervals of \( I_m(x) \). One can check
\[
\text{OSC}_{I_m,\xi}(g) \leq 2^{n-m} \cdot \text{OSC}_{\Lambda_m}(g) \lesssim 2^{n-m} \mathcal{M}(g)(x).
\]
Without loss of generality we can suppose that \( I_{n+1,\xi}(x) \subset I_{n,\xi}(x) \subset I_m(x) \cup I_m^+(x) \) (or \( \subset I_m(x) \cup I_m^-(x) \)). Thus we obtain
\[
|\Delta H_{n,\xi}(g)(x)| = \left| H_{n+1,\xi}(g)(x) - H_{n,\xi}(g)(x) \right|
= \left| \frac{1}{|I_{n,\xi}(x)|} \int_{I_{n,\xi}(x)} g - \frac{1}{|I_{n+1,\xi}(x)|} \int_{I_{n+1,\xi}(x)} g \right|
\leq \text{OSC}_{I_{n,\xi}}(g)
\lesssim 2^{n-m} \mathcal{M}(g)(x).
\]

Lemma 4.3. If \( f \in L^1(\mathbb{T}) \) and \( I = [p, q) \subset \mathbb{T} \), then for any integer \( m \geq 1/(2(p - q)) \) we have
\[
\left| \int_{\mathbb{T}} f(t) \Delta U_m(1_I)(t) dt \right| \lesssim 2^{-m} \mathcal{M}(f)(p) + \mathcal{M}(f)(-p) + \mathcal{M}(f)(q) + \mathcal{M}(f)(-q)). \tag{4.1}
\]

Proof. Let \( I_m(p) = [a_m(p), b_m(p)) \) and \( I_m(q) = [a_m(q), b_m(q)) \) be the dyadic intervals of length \( 2^{-m} \) containing the points \( p \) and \( q \) respectively. We approximate the function \( 1_I \) by a \( \phi \in \Lambda_m(\mathbb{T}) \) defined
\[
\phi(x) = \begin{cases} 
1 \text{ if } & x \in [b_m(p), a_m(q)], \\
0 \text{ if } & x \notin [a_m(p), b_m(q)], \text{ linear on the intervals } [a_m(p), b_m(p)] \text{ and } [a_m(q), b_m(q)].
\end{cases}
\]

([11], chap. 3). Define the maximal function
\[
\mathcal{M}(f)(x) = \sup_{I : I \supseteq x} \frac{1}{|I|} \int_I |f|, \quad f \in L^1(\mathbb{T}),
\]
where \( \sup \) is taken over all the intervals \( I \subset \mathbb{T} \) containing the point \( x \). We will need the following well-known lemma (see [17], chap. 2).
By (2.3) we have $\Delta U_m(\phi)(x) \equiv 0$, as well as

$$|\phi(x) - \mathbf{1}_I(x)| \leq \mathbf{1}_{I_m(p)}(x) + \mathbf{1}_{I_m(p)}(x).$$

Thus, using Lemma 3.2, we can write

$$|\Delta U_m(\mathbf{1}_I)(t)| = |\Delta U_m(\phi - \mathbf{1}_I)(t)|$$

$$\leq \lambda_m I_m(p)(t) + \lambda_m I_m(p)(-t) + \lambda_m I_m(q)(t) + \lambda_m I_m(q)(-t).$$

Then, combining also Lemma 4.1, we obtain (4.1).

$\square$

Given function $f \in L^2(\mathbb{T})$, we denote

$$\mathcal{M}_n f(x, \xi) = \mathcal{M}(f)(a_n(x, \xi)) + \mathcal{M}(f)(-a_n(x, \xi))$$

$$+ \mathcal{M}(f)(b_n(x, \xi)) + \mathcal{M}(f)(-b_n(x, \xi)).$$

(4.2)

**Lemma 4.4.** If $f \in L^2(0, 1)$ and $m > n \geq 1$, then

$$|H_{n,\xi}(\Delta U_m(f))(x)| \leq 2^{n-m} \mathcal{M}_n f(x, \xi), \quad x \in [0, 1).$$

(4.3)

**Proof.** Using Lemma 4.3, we obtain

$$|H_{n,\xi}(\Delta U_m(f))(x)| = \frac{1}{|I_{n,\xi}(x)|} \left| \int_{I_{n,\xi}(x)} \int_0^1 \Delta K_m(u, t) f(t) dt du \right|$$

$$= 2^n \left| \int_0^1 f(t) \int_{I_{n,\xi}(x)} \Delta K_m(u, t) du dt \right|$$

$$= 2^n \left| \int_0^1 f(t) \Delta U_m(\mathbf{1}_{I_{n,\xi}(x)})(t) dt \right|$$

$$\leq 2^{n-m} \left( \mathcal{M}(f)(a_n(x, \xi)) + \mathcal{M}(f)(-a_n(x, \xi)) + \mathcal{M}(f)(b_n(x, \xi)) + \mathcal{M}(f)(-b_n(x, \xi)) \right),$$

and so (4.3).

$\square$

**Lemma 4.5 (main).** If $f \in L^2(\mathbb{T})$ has a representation $f = \sum_{k=0}^{\infty} b_k u_k$, then there exists a parameter $\xi \in \mathbb{T}$ such that

$$\left\| \sup_{\|\lambda\| \leq 1} S_{\xi} \left( \sum_{k=0}^{\infty} \lambda_k b_k u_k \right) \right\|_2 \lesssim \|f\|_2.$$  \hspace{1cm} (4.4)

where the sup is taken over all the sequences $\lambda = \{\lambda_k\}$ with $|\lambda_k| \leq 1$.

**Proof.** Clearly we can suppose that $b_0 = b_1 = 0$. For a given sequence $\lambda = \{\lambda_k : |\lambda_k| \leq 1\}$, we denote

$$f_\lambda = \sum_{k=0}^{\infty} \lambda_k b_k u_k = \sum_{k=2}^{\infty} \lambda_k b_k u_k.$$
Combining Lemma 4.2 and Lemma 4.4, we can write a pointwise estimation
\[
\sum_{n=1}^{\infty} \left| \Delta H_{n, \xi} \left( \sum_{k=0}^{\infty} \lambda_k b_k u_k \right)(x) \right|^2 \leq \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} |\Delta H_{n, \xi}(\Delta U_m(f))(x)| \right)^2 \\
\lesssim \sum_{n=1}^{\infty} \left( \sum_{m=n+1}^{\infty} 2^{n-m} \mathcal{M}_n(\Delta U_m(f))(x, \xi) \right)^2 \\
+ \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} 2^{m-n} \mathcal{M}(\Delta U_m(f))(x) \right)^2 \\
\lesssim \sum_{n=1}^{\infty} \left( \sum_{m=n+1}^{\infty} 2^{n-m} \mathcal{M}_n \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right)(x, \xi) \right)^2 \\
+ \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} 2^{m-n} \mathcal{M} \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right)(x) \right)^2 \\
= A(f)(x, \xi) + B(f)(x). \tag{4.5}
\]

Applying (3.3), (3.4) and the boundedness of the maximal operator on $L^2$, we get the estimate
\[
\int_{0}^{1} B(f)(x) dx \lesssim \sum_{m=1}^{\infty} \left\| \mathcal{M} \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right) \right\|^2_2 \\
\lesssim \sum_{m=1}^{\infty} \left\| \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right\|^2_2 \lesssim \|f\|^2_2. \tag{4.6}
\]

We can not apply (3.3) to estimate the $A(f)(x, \xi)$, since
\[
d_{n,m}(f)(x, \xi) = \mathcal{M}_n \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right)(x, \xi)
\]
depends both on $n$ and $m$. We proceed as follows

\[
A(f)(x, \xi) = \sum_{n=1}^{\infty} \left( \sum_{m > n} 2^{n-m} d_{n,m}(f)(x, \xi) \right)^2 \\
= \sum_{n=1}^{\infty} \sum_{m, m' > n} 2^{n-m} 2^{n-m'} d_{n,m}(f)(x, \xi) d_{n,m'}(f)(x, \xi) \\
\leq \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m, m' > n} 2^{n-m} 2^{n-m'} ((d_{n,m}(f)(x, \xi))^2 + (d_{n,m'}(f)(x, \xi))^2) \\
\leq \sum_{n=1}^{\infty} \sum_{m > n} 2^{n-m} (d_{n,m}(f)(x, \xi))^2.
\]

Therefore, denoting by $\mathbb{E}$ the integration with respect to $\xi$, from (4.2) and (3.4) we obtain

\[
\mathbb{E} \left( \int_0^1 A(f)(x, \xi) \, dx \right) \lesssim \sum_{n=1}^{\infty} \sum_{m > n} 2^{n-m} \mathbb{E} \left( \int_0^1 (d_{n,m}(f)(x, \xi))^2 \, dx \right) \\
\leq \sum_{n=1}^{\infty} \sum_{m > n} 2^{n-m} \int_0^1 \left( \mathcal{M} \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right) \right)^2 \, dx \\
\leq \sum_{m=2}^{\infty} \left\| \mathcal{M} \left( \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right) \right\|_2^2 \\
\lesssim \sum_{m=2}^{\infty} \left\| \sum_{k=2^{m-1}+1}^{2^m} |b_k u_k| \right\|_2^2 \lesssim \|f\|_2^2.
\]

Based on estimates (4.5), (4.6) and (4.7) we conclude that (4.4) is satisfied for some $\xi$. Lemma is proved.

5. PROOF OF THEOREM 1.1

A key argument in the proof of Theorem 1.1 is the following good-$\lambda$ inequality due to Chang-Wilson-Wolff (see [1], Corollary 3.1):

\[
|\{x \in [0, 1) : \mathcal{M}_\xi^d(f)(x) > \lambda, S_\xi f(x) < \varepsilon \lambda\}| \\
\lesssim \exp \left( -\frac{c}{\varepsilon^2} \right) |\{\mathcal{M}_\xi^d(f)(x) > \lambda/2\}|, \lambda > 0, 0 < \varepsilon < 1,
\]

where $\mathcal{M}_\xi^d$ denotes the $\xi$-shifted dyadic maximal function

\[
\mathcal{M}_\xi^d(f)(x) = \sup_{n \geq 1} \frac{1}{|I_n, \xi(x)|} \int_{I_n, \xi(x)} |f|.
\]

Clearly it is enough to prove Theorem 1.1 for the reconstructed system $U = \{u_k\}$ instead of the Franklin system $F = \{u_k\}$. So let $f = \sum_{j=0}^{\infty} b_j u_j \in L^2(\mathbb{T})$ and the functions
$p_k \in L^2(\mathbb{T})$, $k = 1, 2, \ldots, n$ satisfy $p_k \prec f$ with respect to the system $\mathcal{U}$. It is clear that

$$\mathcal{P}_\xi(x) = \sup_{1 \leq k \leq n} S_\xi(p_k)(x) \leq \sup_{|\lambda_k| \leq 1} S_\xi \left( \sum_{k=0}^{\infty} \lambda_k b_k u_k \right)(x).$$

(5.2)

Thus, according to Lemma 4.5 for a suitable $\xi$ we have

$$\|\mathcal{P}_\xi\|_2 \lesssim \|f\|_2.$$

On the other hand, $|g(x)| \leq M_\xi^d g(x)$ a.e. for any function $g \in L^1$, as well as $S_\xi(p_k)(x) \leq \mathcal{P}_\xi(x)$, $k = 1, 2, \ldots, n$. Thus, applying inequality (5.1) with $\varepsilon_n = (c/ \ln n)^{1/2}$, we obtain

$$\{|\{p_k(x)| > \lambda, \mathcal{P}_\xi(x) \leq \varepsilon_n \lambda\}| \lesssim \exp \left( -\frac{c}{\varepsilon_n^2} \right) |\{M_\xi^d p_k(x) > \lambda/2\}|.$$

(5.3)

For $p^*(x) = \max_{1 \leq m \leq n} |p_m(x)|$ we obviously have

$$\{p^*(x) > \lambda\} \subset \{p^*(x) > \lambda, \mathcal{P}_\xi(x) \leq \varepsilon_n \lambda\}$$

$$\cup \{\mathcal{P}_\xi(x) > \varepsilon_n \lambda\} = A(\lambda) \cup B(\lambda),$$

and thus

$$\|p^*\|_2^2 \leq 2 \int_0^\infty \lambda |A(\lambda)| d\lambda + 2 \int_0^\infty \lambda |B(\lambda)| d\lambda.$$

From (5.3) it follows that

$$\int_0^\infty \lambda |A(\lambda)| d\lambda \leq \sum_{m=1}^{n} \int_0^\infty \lambda \{|p_m| > \lambda, \mathcal{P}_\xi \leq \varepsilon_n \lambda\} |d\lambda$$

$$\leq \exp \left( -\frac{c}{\varepsilon_n^2} \right) \sum_{m=1}^{n} \int_0^\infty \lambda \{|M_\xi^d p_m > \lambda/2\} |d\lambda$$

$$\lesssim \frac{1}{n} \sum_{m=1}^{n} \|M_\xi^d p_m\|_2^2$$

$$\lesssim \frac{1}{n} \sum_{m=1}^{n} \|p_m\|_2^2$$

$$\leq \|f\|_2^2.$$

Combining this and

$$2 \int_0^\infty \lambda |B(\lambda)| d\lambda = \varepsilon_n^2 \|\mathcal{P}_\xi\|_2^2 \lesssim \log n \cdot \|f\|_2^2,$$

we get

$$\|p^*\|_2 = \left\| \max_{1 \leq m \leq n} |p_m(x)| \right\|_2 \lesssim \sqrt{\log n} \cdot \|f\|_2$$

that proves the theorem.
6. Proof of corollaries

**Lemma 6.1** ([7], Theorem 5.3.2). Let \( \{\phi_n(x)\} \) be an orthonormal system and \( w(n) \to \infty \) be a sequence of positive numbers. If an increasing sequence of indexes \( \{n_k\} \) satisfy the bound \( w(n_k) \geq k \), then the condition \( \sum_{k=1}^{\infty} a_k^2 w(k) < \infty \) implies the a.e. convergence of the sums \( \sum_{j=1}^{n_k} a_j \phi_j(x) \) as \( k \to \infty \).

**Proof of Corollary 1.1.** Consider the series
\[
\sum_{k=1}^{\infty} a_k p_k(x)
\]
with coefficients satisfying the condition \( \sum_{k=1}^{\infty} a_k^2 \log k < \infty \) and denote \( S_n = \sum_{k=1}^{n} p_k \). Since \( w(n) = \log n \) satisfies the condition \( w(2^k) \geq k \), from Lemma 4.4 we have a.e. convergence of subsequences \( S_{2^k}(x) \). So we just need to show that
\[
\delta_k(x) = \max_{2^k < n \leq 2^{k+1}} |S_n(x) - S_{2^k}(x)| \to 0 \text{ a.e. as } k \to \infty. \tag{6.1}
\]
We have
\[
\|\delta_k\|_2 \leq K_{2^k}(\mathcal{F}, \text{monotonic}) \left( \sum_{j=2^k+1}^{2^{k+1}} a_j^2 \right)^{1/2} \lesssim \sqrt{k} \left( \sum_{j=2^k+1}^{2^{k+1}} a_j^2 \right)^{1/2}.
\]
So we get
\[
\sum_{k=1}^{\infty} \|\delta_k\|_2^2 \leq \sum_{k=1}^{\infty} k \sum_{j=2^k+1}^{2^{k+1}} a_j^2 \leq \sum_{j=1}^{\infty} a_j^2 \log j < \infty,
\]
which implies (6.1). \( \square \)

To prove the next corollary we will need another lemma.

**Lemma 6.2** ([18], [15]). Let \( u(n) \) be a C-multiplier for any rearrangement of an orthonormal system \( \Phi = \{\phi_n(x)\} \). If an increasing sequence of positive numbers \( \delta(n) \) satisfies the condition
\[
\sum_{k=1}^{\infty} \frac{1}{\delta(k) k \log k} < \infty, \tag{6.2}
\]
then \( \delta(n) u(n) \) turns to be a UC-multiplier for \( \Phi \).

**Proof of Corollary 1.3.** According to Corollary 1.1 \( u(n) = \log n \) is a C-multiplier for the systems of non-overlapping Franklin polynomials and their rearrangements. By the hypothesis of Corollary 1.3 the sequence \( \delta(n) = w(n) / \log n \) is increasing and satisfies (6.2). Thus, the combination of Corollary 1.1 and Lemma 6.2 completes the proof. \( \square \)

**Proof of Corollary 1.4.** The inequality (1.7) immediately follows from the upper bound \( ||A_n, F||_{2 \to 2} \lesssim \sqrt{\log n} \) coming from Theorem 1.1. \( \square \)
7. Final remarks

All the new results formulated in this paper hold also for the Franklin system of periodic type \( \{ \tilde{f}_n(x), n = 1, 2, \ldots \} \). This system is similarly generated from the spaces \( \tilde{L}_n \). Namely, \( \tilde{f}_1(x) = 1 \), and for \( n \geq 2 \) we let \( \tilde{f}_n \) to be an \( L^2 \)-normalized function from \( \tilde{L}_n \), which is orthogonal to \( \tilde{L}_{n-1} \). The functions of the periodic Franklin system can be considered as continuous functions on \( \mathbb{T} \). So our method of proof of Theorem 1.1 can be directly applied to \( \tilde{g}_n(x) \) without any reconstruction in contrast to the classical Franklin system. Likewise (3.1) and (3.2) we have exponential estimates

\[
|\tilde{f}_n(x)| \lesssim \sqrt{nq^{nd(x,t_n)}}, \quad |\tilde{K}_n(x,t)| \lesssim nq^{nd(x,t_n)},
\]

where \( d(x,y) \) denotes the distance of two points \( x, y \in \mathbb{T} \) and

\[
\tilde{K}_n(x,t) = \sum_{k=1}^{2^n} \tilde{f}_k(x)\tilde{f}_k(t).
\]

One can check that all the lemmas proved in the Sections 3 and 4 hold with an absolutely same statements and proofs. One just need to redefine \( \Lambda_n = \tilde{L}_{2n-1} \) instead of \( \Lambda_n = \tilde{L}_{2^n} \) and remove the second term in the sum on the right hand side of (3.8). Thus one can conclude that Theorem 1.1 and Corollaries 1.1-1.4 hold also for the Franklin system of periodic type.

As an addition to Theorems C and D it also holds the following bound.

**Theorem 7.1.** For any \( 1 < p < \infty \) the classical Haar system \( \mathcal{H} \) satisfies the relation

\[
\|A_{n,H}\|_{p \to p} \lesssim c_p \sqrt{\log n}.
\]

**Proof.** The proof is the same as that of Theorem 1.1. For \( f = \sum_{j=1}^{\infty} b_j h_j \in L^p \) we consider a sequence of functions \( p_k \prec f, k = 1, 2, \ldots, n \). Instead of (5.2) one needs to consider the function

\[
\mathcal{P}(x) = \sup_{1 \leq k \leq n} S(p_k)(x) \leq S(f)(x),
\]

where \( S \) is the square function (2.5) corresponding to \( \xi = 0 \). We have \( \|\mathcal{P}\|_p \leq c_p \|f\|_p \), according to the boundedness of the Haar square function operator on \( L^p \) (see [11], chap. 3). Then repeating the argument of the proof of Theorem 1.1 in \( L^p \) setting, we get the bound (7.1). \( \square \)

As a new result this bound provides the analogous of Corollary 1.4 for the classical Haar system. Namely,

**Corollary 7.1.** Let a function \( f \in L^2(\mathbb{T}) \) have Haar representation \( \sum_{k=1}^{\infty} a_k h_k \). Then for arbitrary sets of indexes \( G_k \subset \mathbb{N}, k = 1, 2, \ldots, n \) we have

\[
\left\| \max_{1 \leq m \leq n} \left| \sum_{j \in G_m} a_j h_j \right| \right\|_p \lesssim \sqrt{\log n} \cdot \|f\|_p.
\]
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Faculty of Mathematics and Mechanics, Yerevan State University, Alex Manoogian, 1, 0025, Yerevan, Armenia

E-mail address: g.karagulyan@ysu.am