Parametric instability of a rotating shaft containing an elliptical front crack

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Abstract. Compared with the widely used straight front crack model, an elliptical front crack has been found to be more accurate and realistic for modeling the transverse surface crack in rotating machinery. When the shaft rotates, the elliptical crack opens and closes alternatively, due to gravity, and thus a “breathing effect” occurs. This variance in shaft stiffness is time-periodic, and hence a parametrically excited system is expected. Thus, the parametric instability of a rotating shaft containing an elliptical front crack is studied in the paper. The local flexibility due to the crack is derived, and the governing equations of the crack shaft system are established using the assumed modes method. In virtue of discrete state transition matrix (DSTM) method, three typical instability regions of a practical used rotating shaft are determined numerically. The effects of crack parameters (depth, shape factor, position) and damping on the instability regions are respectively considered and discussed.

1. Introduction
Shafts are amongst components subjected to perhaps the most arduous working conditions in high-performance rotating equipment used in process and utility plants. Due to manufacturing flaws or cyclic loading, cracks frequently appear in rotating machinery. The growth of cracks in the rotating components can cause severe accidents if undetected. It is well known that a surface crack introduces local flexibilities, and then alters the vibrational behavior of the shaft. Thus, the dynamic behavior of a rotating shaft containing a transverse surface crack, which is useful for the crack detection in rotating machinery, has been investigated since the 1970s. Early research progress could be found in [1, 2, 3]. Recently, more detailed reviews of the crack modeling and diagnosis were given by Papadopoulos [4] and Bachschmid [5], respectively.

In many of those works, a straight edge is usually considered to idealize the surface crack front [6, 7, 8]. However, it is shown from the experimental observations [9, 10] that the actual surface crack front in shaft under cyclical bending or axial load tends to be elliptical. The elliptical front crack could be more accurate and realistic for modeling the surface crack. As the literature shows, extensive efforts have been devoted to calculating the stress intensity factors (SIF) and then analyzing the fatigue growth of an elliptical surface crack in a shaft or round bar [11, 12, 13, 14, 15], while the dynamic behavior of an elliptical cracked shaft has not gained sufficient attentions. In the pioneering work by Rubio [16], flexibility expressions for cracked shafts having elliptical front cracks under bending or axial tension load were obtained, based on the strain energy release rate (SERR) approach and the polynomial fitting of SIF, taking into account the size and shape of the elliptical cracks. When the shaft rotates, the elliptical crack opens and closes alternatively, due to gravity, and thus a ”breathing effect”
occurs. This variance in shaft stiffness is time-periodic, and hence a parametrically excited system is expected. The parametric excitation from the time-varying shaft stiffness causes instability and severe vibration under certain operating conditions. Determination of operating conditions of parametric instability is crucial to the dynamic analysis of the rotating machinery with elliptical surface cracks.

Therefore, the parametric instability of a rotating shaft containing an elliptical surface crack is studied in the paper. First, local bending flexibilities of an elliptical surface crack in two transverse directions are derived using the method by Rubio [16]. The time-periodic local flexibilities due to shaft rotating are fitted through a similar approach in Ref. [6]. Then, the discrete governing equations of a rotating elliptical cracked shaft are obtained in virtue of the assumed modes method and Lagrange equation. The discrete state transition matrix (DSTM) method is introduced for determining the instability regions. Based upon these, a practical used heavy rotating shaft is taken as a numerical example to study the system parametric instability. Three typical instability regions are considered in the analysis. The effects of various crack parameters (crack depth and shape) upon the position and size of instability region are discussed, respectively. Finally, some useful conclusions are given.

2. Elliptical crack modeling

The cross-section of a circular shaft with an elliptical transverse crack under bending moment $M$ is shown in Fig. 1. The radius of the circular cross-section, major and minor axes of the crack ellipse are respectively denoted as $R$, $b$ and $a$. The points A and B are the intersection points between the crack ellipse and circular. Point C is an arbitrary point on the elliptical arc $s$. The projection lengths of points A and C upon the axis $O\xi$ are expressed as $h$ and $\xi_1$. The characteristic parameters of an elliptical crack are: crack depth $\alpha = a/R$, shape factor $\beta = a/b$ and relative position in the front of the crack $\gamma = \xi_1/h$.

According to the Castigliano theorem, the local flexibility of the crack due to bending moment in the $O\xi$ direction is obtained as

$$c_\xi = \frac{\partial^2}{\partial M^2} \left( \int_{\Delta_c} \Im dA_c \right)$$

where $\Im$ is the energy release rate due to the crack growth (or called strain energy density
function) and $A_c$ is the crack surface area. For the surface crack under bending moment, the SIF with mode I, $K_I$, is the only non-zero one. Considering the formulation of linear fracture mechanics, the relationship between the $\Im$ and $K_I$ in plane strain conditions is

$$\Im = \frac{1 - \nu^2}{E} K^2_I$$  \hspace{1cm} (2)$$

in which $E$ is the Young’s modulus, $\nu$ is Poisson’s ratio. With the geometrical dimensions defined in Fig. 1, the $K_I$ of the elliptical surface crack for the $M$ in the $O\xi$ direction can be expressed in the following form

$$K_I = F_{I,1}(\alpha, \beta, \gamma) \sigma_M \sqrt{\pi a}$$ \hspace{1cm} (3)$$

where $\sigma_M$ is the maximum bending stress in the circular cross-section ($\sigma_M = 4M/(\pi R^3)$), and $F_{I,1}(\alpha, \beta, \gamma)$ is the geometry correction factor. Experimental backtracking technique and finite element analysis have been employed by Shin [15] to evaluate the stress intensities along the front of an elliptical surface crack. A closed-form expression of $F_{I,1}(\alpha, \beta, \gamma)$ by using a multi-parameter fitting technique could be found in [15]. Substituting Eq. (3) into Eq. (2), and then into Eq. (1), one can have

$$c_\xi = \frac{1 - \nu^2}{E} \frac{32}{\pi R^6} \left( \int_{A_c} F^2_{I,1}(\alpha, \beta, \gamma) a dA_c \right)$$ \hspace{1cm} (4)$$

in which $dA_c = ds da$. The elliptical arc differential $ds$ takes the form

$$ds = a \sqrt{\frac{1}{\beta^2} \sin^2 \phi + \cos^2 \phi} d\phi$$ \hspace{1cm} (5)$$

Substituting Eq. (5) into Eq. (4) and considering the symmetrical distribution of $K_I$ about the $o\xi$ axis ($\phi = \pi/2$), one can obtain

$$c_\xi = \frac{1 - \nu^2}{E} \frac{64}{\pi R^6} \int_{\phi_1}^{\pi/2} \int_0^{\alpha_f} F^2_{I,1}(\alpha, \beta, \gamma) a^2 \sqrt{\frac{1}{\beta^2} \sin^2 \phi + \cos^2 \phi} \phi d\phi d\alpha$$ \hspace{1cm} (6)$$

where $\alpha_f$ is the nominal crack depth, $\phi_1$ is the lower limit and $\cos \phi_1 = \bar{h}$, and $\gamma = \frac{1}{h} \cos \phi$ denotes the relative position of the computational point in the crack front. According to the geometry dimensions in Fig. 1, one can have

$$\bar{h} = h/b = \beta \sqrt{\frac{2\sqrt{\beta^2 + (1-\beta^2)\alpha^2} - (1-\beta^2)\alpha^2 - 2\beta^2}{(1-\beta^2)2\alpha^2}} \hspace{1cm} (\beta \neq 1)$$ \hspace{1cm} (7)$$

$$\bar{h} = h/b = \sqrt{1 - \alpha^2/2} \hspace{1cm} (\beta = 1)$$ \hspace{1cm} (8)$$

In order to facilitate the following analysis, the dimensionless flexibility coefficient $c'_\xi$ could be written as

$$c'_\xi = \frac{ER^3}{1 - \nu^2} c_\xi = \frac{64}{\pi} \int_{\phi_1}^{\pi/2} \int_0^{\alpha_f} F^2_{I,1}(\alpha, \beta, \gamma) a^2 \sqrt{\frac{1}{\beta^2} \sin^2 \phi + \cos^2 \phi} \phi d\phi d\alpha$$ \hspace{1cm} (9)$$

When the bending moment $M$ is in the $O\eta$ direction, the SIF along the crack front could be expressed similarly as

$$K_I = F_{I,2}(\alpha, \beta, \gamma) \sigma_M \sqrt{\pi a}$$ \hspace{1cm} (10)$$
where $F_{I,2}(\alpha, \beta, \gamma)$ is the geometry correction factor. For the elliptical crack under bending moment in the $O\eta$ direction, all the numerical results about the $F_{I,2}(\alpha, \beta, \gamma)$ deduced through finite element analysis were displayed in Ref. [11], where $\alpha$ ranges from 0.2 to 1.2, and $\beta$ varies from 0 to 1.2. Here, a closed-form expression of $F_{I,2}(\alpha, \beta, \gamma)$ is obtained by using a multi-parameter fitting technique. Similar with the computation of $c'_\xi$, the flexibility coefficient $c'_\eta$ in the $O\eta$ direction could also be gained as

$$c'_\eta = \frac{ER^3}{1-\nu^2} c_\eta = \frac{32}{\pi} \int_0^{\alpha_f} \left( \int_{\phi_1}^{\pi/2} F_{I,2}^2(\alpha, \beta, \gamma) \alpha^2 \sqrt{\frac{1}{\beta^2} \sin^2 \phi + \cos^2 \phi} d\phi \right) d\alpha$$  \hspace{1cm} (11)$$

Something should be mentioned is that only half of the crack surface is in tension stress condition for the bending moment in the $O\eta$ direction. Thus, $A_c/2$ is adopted in computation.

In virtue of the double numerical integration technique, the flexibility coefficients $c'_\xi, c'_\eta$ due to the elliptical surface crack could be solved with given crack depth $\alpha_f$ and shape factor $\beta$. Fig. 2 gives the variations of $c'_\xi$ with $\alpha_f$ in the range of 0-1 for various shape factors ($\beta = 0, 0.2, 0.5, 1.0$). Obviously, $\beta = 0$ corresponds to a straight crack front. The results of the straight crack front obtained by numerically solving Dimarogonas’ formula [17] are also presented in the figure for comparison with the current results. From Fig. 2, one can see that for $\beta = 0$ (straight front crack), the results of this paper agree well with that of Dimarogonas. As it was expected, and according to the literature results [16], the flexibility, in all the cases under consideration, increases with the crack length and with straight shapes of the crack front.

In the following, the shaft is assumed to rotate at a constant speed $\Omega$. Due to gravity, the crack is opened at its lower position, and closed at its upper one. The extra compliance due to the crack hence depends on its orientation with respect to direction of the gravitational force. The amount goes from zero ($0 \leq \Omega t < \theta_A$, where $\theta_A$ represents $\angle AO_1D$ in Fig. 1) as the crack is closed to $c_\eta$ when it is half-opened, then to its maximum value $c'_\xi$ when the crack is fully opened. This variance is periodic, and hence a periodic time-varying system is expected. For certain crack depth ($\alpha_f = 0.5$), Fig. 3 shows the extra flexibility $c'(t)$ of the shaft due to the crack in one full revolution with various crack shape factors ($\beta = 0, 0.4, 0.8, 1.0$). Since only the values in $\xi - \eta$ directions are available, the curves in between are fitted by spline functions.

**Fig. 2.** The variations of $c'_\xi$ with $\alpha_f$ in the range of 0-1 for various $\beta$. 

$$c'_\eta = \frac{ER^3}{1-\nu^2} c_\eta = \frac{32}{\pi} \int_0^{\alpha_f} \left( \int_{\phi_1}^{\pi/2} F_{I,2}^2(\alpha, \beta, \gamma) \alpha^2 \sqrt{\frac{1}{\beta^2} \sin^2 \phi + \cos^2 \phi} d\phi \right) d\alpha$$  \hspace{1cm} (11)$$
3. Mathematical model of a rotating heavy shaft with elliptical front crack

Figure 4 shows a simply supported heavy shaft with an elliptical surface crack. The shaft is modeled as an Euler-Bernoulli beam with constant angular rotating velocity $\Omega$. The rotor span is $l$, and the position of the elliptical crack on the shaft is $x^*$. Since the energy method is to be adopted for the equations of motion, the kinetic energy $T$ and potential energy $V_s$ of the shaft without crack are expressed as follows

$$T = \frac{1}{2} \int_0^l \rho A \left( \frac{\partial y(x,t)}{\partial t} \right)^2 \, dx, \quad V_s = \frac{1}{2} \int_0^l EI \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)^2 \, dx$$

(12)

where $y(x,t)$ denotes the displacement function of the shaft in the vertical plane, $\rho$ and $A$ are the material density and cross-section area of the shaft, and $EI$ represents shaft bending rigidity. The released potential energy due to the elliptical surface crack could be obtained as

$$V_c = \frac{1}{2} c(t) \left[ EI \frac{\partial^2 y(x^*,t)}{\partial x^2} \right]^2$$

(13)

Thus, the total potential energy of the system is $V = V_s - V_c$.

The assumed modes methods is now chosen for the equations of motion. Since the crack is theoretically of zero thickness, it is realized that the crack imposes significant effects on the local zone: e.g. stress intensity. As to the global behavior, it is believed that the natural
modes of a cracked shaft are merely slightly perturbed from those of an uncracked shaft. It is hence reasonable to chose the natural modes of simply supported beam as a basis upon which to expand the displacement response of the cracked shaft. Then, both the kinetic and potential energies could be discretized with respect to the generalized coordinate. By employing Lagrange’s equation, the discrete governing equations of motion in matrix form are obtained as

$$M \ddot{q}(t) + C \dot{q}(t) + [K_0 - K_c \varphi(t)] q(t) = 0 \tag{14}$$

where $q$ is the vibration displacement vector, and the upper dot denotes differentiation with respect to time. The mass matrix of the shaft system is denoted by $M = \text{diag}[m_i]$. The $K_0 = \text{diag}[k_i]$ and $K_c = [k_{ij}^c]$ are the constant and time-variable parts of the stiffness matrix for the system. If the proportional viscous damping is considered, then the damping matrix could be gained as $C = \varsigma_1 M + \varsigma_2 K_0$, in which $\varsigma_1$ and $\varsigma_2$ are the mass and stiffness damping coefficients, respectively. The elements of the mass and stiffness matrix could be obtained respectively as

$$m_i = \int_0^L \rho A \psi_i(x) \psi_i(x) \, dx \tag{15}$$

$$k_i = \int_0^L E J \psi_i''(x) \psi_i''(x) \, dx \tag{16}$$

$$k_{ij}^c = \frac{(1 - \nu^2)}{4} E I R \frac{\psi_i''(x^*) \psi_j''(x^*)}{\omega_i^2} \tag{17}$$

in which $\psi_i$ and $\psi_j$ are the $i$th and $j$th natural mode shape functions for the simply supported beam. The upper primes in $\psi_i''$ are differentiations with respect to $x$. Due to the existence of time-varying stiffness, the elliptical cracked rotating shaft is a typical parametrically excited system, as shown in Eq. (14). The parametric frequency and amplitude is related to the rotating angular speed and crack depth. Usually, the working speed of rotating machinery is lower than the third order of critical speed. Thus, the first three natural modes of the system is chosen in the following analysis, and the discrete system has three degrees of freedom.

### 4. Parametric instability analysis

Parametric excitations give rise to instabilities when harmonics of the excitation frequencies are close to particular combinations of the natural frequencies. Two types of instability are of most interest [18]: the simple and combination instability regions. The simple instability regions $U_{ni}$ are related to the single natural frequency of the system, and their ranges could be approximately expressed as

$$U_{ni} \sim \frac{2}{n} \bar{\omega}_i \quad (n = 1, 2, 3, \ldots) \tag{18}$$

in which $\bar{\omega}_i$ denotes the $i$th natural frequency of equivalent time-invariant system (the average equivalent frequency). For the combination instability regions of the sum type $U_{m+i+j}$, they are associated with two different average equivalent frequencies and their ranges could be expressed as

$$U_{m+i+j} \sim \frac{1}{n} (\bar{\omega}_i + \bar{\omega}_j) \quad (i \neq j, n = 1, 2, 3, \ldots) \tag{19}$$

The DSTM method is a classical method for the dynamic stability analysis. Presently, many numerical methodologies, such as the direct integration technique and approximation techniques, have been developed for computing the DSTM of parametric system. Here, a numerical method presented by Friedmann [19] is utilized to estimate the DSTM. The characteristic multipliers $\lambda_i$,
which are just the eigenvalues of the DSTM, are utilized to determine whether the parametric system is unstable or not. For an unforced periodic system to be stable, it is sufficient that

$$|\lambda_i| \leq 1 \quad \forall i$$

(20)

If $|\lambda_i| > 1$ for any $i$, then the system is to be unstable and parametric resonance occurs. For given crack parameters and rotating speed, whether the system is unstable or not could be judged by Eq. (20). Thus, the unstable range is determined point-by-point in the parameters’ plane.

5. Computation and discussions

A practical used heavy shaft [6] is taken as a numerical example for parametric instability analysis. The shaft parameters are: $l = 10m$, $R = 0.5m$, $\rho = 7850\text{kg/m}^3$, $E = 2.1 \times 10^{11}\text{Pa}$, $\nu = 0.3$. Without elliptical crack, the first two natural frequencies of the shaft are: $\omega_1 = 127.6\text{rad/s}$ and $\omega_2 = 510.5\text{rad/s}$. A dimensionless parameter $\chi^* = x^*/l$ is defined to describe the relative position of the elliptical crack on the shaft.

From Eqs. (18) and (19), the approximate positions of the simple and combination instability regions in the rotating speed axis could be gained as: $U_1^1 \approx 255.2\text{rad/s}$, $U_2^2 \approx 127.6\text{rad/s}$, $U_1^{1+2} \approx 638.1\text{rad/s}$, and so on. In actual, due to the damping, the high orders of instability regions (usually $n > 2$) are greatly weakened. The above three instability regions would be mainly considered in the following analysis.

When $\beta = 0.2$ and $\chi^* = 0.5$, the variations of the maximum absolute value of the characteristic multipliers Max$(|\lambda_i|)$ with rotating speed $\Omega$ for different crack depths ($\alpha_f = 0.3, 0.5, 1.0$) are plotted in Fig. 5, respectively. From Eq. (20), one can see that the system is unstable once the Max$(|\lambda_i|) > 1$. Thus, the instability regions of the cracked shaft system could be distinguished from Fig. 5. It is shown that increasing $\alpha_f$ would greatly enlarge the instability regions, especially for the $U_1^1$ in Fig. 5(b). This is because the parametric excitation due to the crack is enhanced with the increasing of crack depth. The average natural frequency of the shaft is reduced, and the instability regions are moving towards lower rotating speed regions.

Moreover, the effects of $\beta$, $\chi^*$ and damping ($\zeta_1 = 0.5$ and $\zeta_2 = 2.3e-7$) upon the three instability regions are also analyzed and shown in Figs. 6, 7 and 8, respectively. From Fig. 6, one can see that the instability regions become smaller and shifts to the right with the increasing of $\beta$. For given crack depth, increasing the shape factor reduces the local flexible coefficients. Thus, the parametric excitation amplitude is depressed and the instability regions become small.

When the crack is at the midpoint of the shaft ($\chi^* = 0.5$), the $U_1^2$ and $U_1^1$ have the greater ranges compared with the cases of ($\chi^* = 0.3$ and 0.7), as shown in Fig. 7(a) and (b). However, one can find from Fig. 7(c) that the range of $U_1^{1+2}$ with $\chi^* = 0.3$ or $\chi^* = 0.7$ is greater than that of $\chi^* = 0.5$. This could be explained in the context of nodal location. The $U_1^2$ and $U_1^1$ are related to the first natural mode of the simply supported shaft. When $\chi^* = 0.5$, the elliptical crack position is just the position having the greatest modal deformation. In this case, the crack would have the biggest influence on the $U_1^2$ and $U_1^1$. With the crack location moving towards the supported points ($\chi^* = 0.3$ or $\chi^* = 0.7$), the crack influence is depressed. In the case of $U_1^{1+2}$, the modal deformation of $\chi^* = 0.3$ or $\chi^* = 0.7$ at the second mode shape of the simply supported shaft is bigger than that of $\chi^* = 0.5$. Thus, the $U_1^{1+2}$ with $\chi^* = 0.3$ or $\chi^* = 0.7$ has greater range.

From Fig. 8, one can find that the range of $U_1^2$ is attenuated by the damping obviously, while it seems that the damping has little impact on the ranges of $U_1^1$ and $U_1^{1+2}$. It is indicated that the lower orders of instability region are sensitive to the damping.

6. Conclusions

The investigation on the parametric instability of a rotating shaft with elliptical surface crack is conducted in the paper. The local flexibility due to the crack is derived, and the governing
Fig. 5. The effects of $\alpha_f$ upon the instability region for $\beta = 0.2$ and $\chi^* = 0.5$: (a)$U_2^1$; (b)$U_1^1$; (c)$U_1^1+2$.

Fig. 6. The effects of $\beta$ upon the instability region for $\alpha_f = 0.6$ and $\chi^* = 0.5$: (a)$U_2^1$; (b)$U_1^1$; (c)$U_1^{1+2}$. 
Fig. 7. The effects of $\chi^*$ upon the instability region for $\alpha_f = 0.6$ and $\beta = 0.2$: (a)$U^1_{21}$; (b)$U^1_{11}$; (c)$U^{1+2}_{11}$.

Fig. 8. The effects of damping upon the instability region for $\alpha_f = 0.6$, $\beta = 0.2$ and $\chi^* = 0.5$: (a)$U^1_{21}$; (b)$U^1_{11}$; (c)$U^{1+2}_{11}$. 
equations of the crack shaft system are established using the assumed modes method. In virtue of DSTM method, three typical instability regions of a practical used rotating shaft are determined numerically. The effects of crack parameters (depth, shape factor, position) and damping on the instability regions are respectively considered and discussed. Some useful conclusions are obtained as follows: (1) Increasing the crack depth would enlarge the instability regions. For larger crack depth, the instability regions tends to appear in lower rotating speed regions. (2) The instability regions would be reduced by increasing the shape factor from 0 (straight front) to 1. (3) For the crack at the midpoint of the shaft, the instability regions related to the first natural mode would have greater ranges, while the instability regions related to the second natural mode would have lower ranges. (4) All of the instability regions would be reduced if the damping is considered.

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