COHOMOLOGY OF ARBITRARY SPIN CURRENTS
IN $AdS_3$

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Abstract

We study conserved currents of any integer or half integer spin built from massless scalar and spinor fields in $AdS_3$. 2-forms dual to the conserved currents in $AdS_3$ are shown to be exact in the class of infinite expansions in higher derivatives of the matter fields with the coefficients containing inverse powers of the cosmological constant. This property has no analog in the flat space and may be related to the holography of the AdS spaces. “Improvements” to the physical currents are described as the trivial local current cohomology class. A complex of spin $s$ currents $(T^s, D)$ is defined and the cohomology group $H^1(T^s, D) = C^{2s+1}$ is found. This paper is an extended version of hep-th/9906149.

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1 Introduction

The role of anti-de Sitter (AdS) geometry in the high energy physics increased greatly due to the Maldacena conjecture \[1\] on the duality between the theory of gravity in the AdS space and conformal theory on the boundary of the AdS space \[2, 3\]. The holography hypothesis suggests that the two types of theories are equivalent. The same time, AdS geometry plays very important role in the theory of higher spin gauge fields (for a brief review see \[4\]) because interactions of higher spin gauge fields contain negative powers of the cosmological constant \[5\]. The theory of higher spin gauge fields may be considered \[4\] as a candidate for a most symmetric phase of string theory.

The group manifold case of $AdS_3$ is special and interesting in many respects. In particular, the 2d models on the boundary of $AdS_3$ are conformal \[6\]. From the higher spin perspective, a special feature of 3d models is that higher spin gauge fields are not propagating in analogy with the usual Chern-Simons gravitational and Yang-Mills fields. Nevertheless the higher spin gauge symmetries remain nontrivial, like the gravitational (spin 2) and inner (spin 1) symmetries. The higher spin currents can be constructed from the matter fields of spin 0 and spin 1/2. Their couplings to higher spin gauge potentials describe interactions of the matter via higher spin gauge fields.

Schematically, the equations of motion in the gauge field sector have a form

$$R = J(C; W),$$

where $R = dW - W \wedge W$ denotes all spin $s \geq 1$ curvatures built from the higher spin potential $W$, while $C$ denotes the matter fields (precise definitions are given in the sect. \[2\]). The 2-form $J(C; W)$ dual to the 3d conserved current vector field obeys the conservation law

$$DJ(C; W) = 0$$

as a consequence of the equations of motion in the matter field sector. $D$ is the covariant derivative of the (infinite-dimensional) higher spin gauge symmetry algebra \[6, 7\], i.e. $\delta R = D\delta W$, where $\delta W$ is an arbitrary variation of the higher spin gauge potential. To analyze the problem perturbatively, one fixes a vacuum solution $W_0$ that solves

$$R_0 = 0,$$

assuming that $W = W_0 + W_1$ while $C$ starts from the first-order part. When gravity is included, as is the case in the higher spin gauge theories, $W_0$ is different from zero and describes background geometry. In the lowest nontrivial order one gets from \[1\]

$$R_1 \equiv D_0 W_1 = J_2(C^2),$$

where $D_0$ is built from $W_0$ and $J_2(C^2)$ is the part of $J(C; W_0)$ bilinear in $C$. The conservation law \[1, 2\] requires

$$D_0 J_2(C^2) = 0$$

on the free equations of motion of the matter fields.

A nonlinear system of equations of motion describing higher spin gauge interactions for the spin 0 and spin 1/2 matter fields in $AdS_3$ in all orders in interactions has been
formulated both for massless and massive matter fields. An interesting property of the proposed equations discovered in is that there exists a flow generating a mapping of the full nonlinear system to the free one. This mapping is a nonlinear field redefinition having a form of infinite power series in higher derivatives of the matter fields and is therefore generically nonlocal. The coefficients of such expansions contain inverse powers of the cosmological constant (the higher derivative of a matter field the more negative power of the cosmological constant appears) and therefore do not admit a flat limit. We call such expansions in higher derivatives pseudolocal to distinguish them from nonlocal expressions that cannot be represented by power series in higher derivatives.

Comparison of the results of with implies that such a field redefinition exists in a nontrivial model if

$$J_2(C^2) = D_0 U(C^2),$$

where $U$ is some pseudolocal functional of the matter fields. The cohomological interpretation with $D_0$ as de Rahm differential is straightforward because $D_0^2 = R_0 = 0$. Indeed, from it follows that the current $J_2(C^2)$ should be closed on the free equations for matter fields, while implies that it is exact in the class of pseudolocal functionals.

This fact has already been demonstrated for the spin 2 current in where we have found a pseudolocal $U$ for the stress tensor constructed from a massless scalar field. In this paper, we generalize this result to the currents of an arbitrary integer or half integer spin which contain a minimal possible number of spacetime derivatives. The analysis of the currents of an arbitrarily high spin is greatly simplified by a formalism of generating functions developed in this paper. This formalism is based on the so-called unfolded formulation of the relativistic equations which allows us to analyze the problem algebraically, automatically taking into account the on–mass–shell character of the problem.

Exact currents with local $U$ containing at most a finite number of derivatives of the matter fields reproduce “improvements”, i.e., modifications of the currents which are trivially conserved. The new result about AdS space established in this paper is that the true currents can also be treated as “improvements” in the class of pseudolocal expansions. This sounds very suggestive in the context of the holography hypothesis since the corresponding field redefinitions may result in nontrivial boundary terms.

The paper is designed as follows. In sect. 2 we collect some facts about the equations of motion of the Chern-Simons higher spin gauge fields and the “unfolded” formulation of the equations of motion for the massless spin 0 and 1/2 matter fields in $AdS_3$. In sect. 3 we propose a formalism of generating functions to describe differential forms bilinear in derivatives of the matter fields. Then, in sect. 4 we formulate using this method the AdS on-mass-shell complex, and in sect. 5 we study its cohomology, the cohomology of currents. In sect. 6 we discuss what happens in the flat limit.

## 2 Higher Spin and Matter Fields In $AdS_3$

The 3d higher spin gauge fields are described by a spacetime 1-form $W = dx^\mu W_\mu(y, \psi|x)$ depending on the spacetime coordinates $x^\mu$ ($\mu = 0, 1, 2$), auxiliary commuting spinor variables $y_\alpha$ (indices $\alpha, \beta, \gamma = 1, 2$ are lowered and raised by the symplectic form $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$,
\( \epsilon_{12} = \epsilon^{12} = 1, \ A^\alpha = e^{\alpha \beta} A_\beta, \ A_\alpha = A^{\beta} \epsilon_{\beta \alpha}, \) and the central involutive element \( \psi, \ \psi^2 = 1, \)

\[
W_\mu(y, \psi|x) = \sum_{n=0}^{\infty} \frac{1}{2n!} \left[ \omega_{\mu, \alpha(n)}(x) + \lambda \psi \ h_{\mu, \alpha(n)}(x) \right] \ y^{\alpha_1} \ldots y^{\alpha_n} .
\] (2.1)

A constant parameter \( \lambda \) is to be identified with the inverse radius of \( \text{AdS}_3. \)

The higher spin gauge algebra is a Lie superalgebra built via (anti)commutators from the associative algebra spanned by the elements of a form (2.1) with a product law

\[
(f \ast g)(y, \psi) = \frac{1}{(2\pi)^2} \int d^2 u d^2 v \ \exp(iu_\alpha v^\alpha) \ f(y + u, \psi) \ g(y + v, \psi) ,
\] (2.2)

where the integration variables \( u_\alpha \) and \( v_\alpha \) are two-component spinors (in accordance with (2.1) the boson-fermion parity is identified with the parity in the auxiliary variables \( y \)).

This product law yields a particular realization of the Weyl algebra, \([y_\alpha, y_\beta]_\ast = 2i\epsilon_{\alpha \beta}\).

The field strength is \([7, 8]\)

\[
R(y, \psi|x) = dW(y, \psi|x) - W(y, \psi|x) \ast W(y, \psi|x) ,
\] (2.3)

and the equations of motion for the Chern-Simons higher spin gauge fields with a matter source have a form (1.1).

The role of the element \( \psi \) is to make the 3d higher spin superalgebra semisimple \((\text{hs}(2) \oplus \text{hs}(2)) \) in notation of [7]), with simple components singled out by the projectors \( P_\pm = \frac{1}{2} (1 \pm \psi) . \) This is similar to the \( \text{AdS}_3 \) isometry algebra \( o(2, 2) \sim sp(2) \oplus sp(2) . \) The latter is identified with a subalgebra of \( \text{hs}(2) \oplus \text{hs}(2) \) spanned by

\[
L_{\alpha \beta} = \frac{1}{2} \ y_\alpha y_\beta , \quad P_{\alpha \beta} = \frac{1}{2i} \ y_\alpha y_\beta \psi .
\] (2.4)

We therefore identify the \( o(2, 2) \) components of \( W(y, \psi|x) \) (2.1) with the gravitational Lorentz connection 1-form \( \omega^{\alpha \beta}(x) = dx^\mu \omega_{\mu, \alpha \beta}(x) \) and the dreibein 1-form \( h^{\alpha \beta}(x) = dx^\mu h_{\mu, \alpha \beta}(x) . \) Since \( \text{AdS}_3 \) algebra \( o(2, 2) \) is a proper subalgebra of the d3 higher spin algebra it is a consistent ansatz to require the vacuum value of \( W(y, \psi|x) \) to be non-zero only in the spin 2 sector. Then the equation \( R_0 = 0 \) is equivalent to the \( o(2, 2) \) zero-curvature conditions

\[
d\omega_{\alpha \beta} = \omega_{\alpha \gamma} \wedge \omega_{\beta \gamma} + \lambda^2 h_{\alpha \gamma} \wedge h_{\beta \gamma} ,
\] (2.5)

\[
dh_{\alpha \beta} = \omega_{\alpha \gamma} \wedge h_{\beta \gamma} + \omega_{\beta \gamma} \wedge h_{\alpha \gamma} .
\] (2.6)

For the metric interpretation, the dreibein \( h_{\nu, \alpha \beta} \) should be non-degenerate, thus admitting the inverse dreibein \( h^{\nu, \alpha \beta} , \)

\[
h^{\nu, \alpha \beta} h_{\nu, \gamma \delta} = \frac{1}{2} (\delta^{\alpha \gamma} \delta^{\beta \delta} + \delta^{\alpha \delta} \delta^{\beta \gamma})
\] (2.7)

(we use the normalization convention of [11]). Then, (2.6) reduces to the zero-torsion condition which expresses Lorentz connection \( \omega_{\nu, \alpha \beta} \) via dreibein \( h^{\nu, \alpha \beta} \), and (2.3) implies that \( R_{\alpha \beta} = -\lambda^2 h_{\alpha \gamma} \wedge h_{\beta \gamma} , \) where \( R_{\alpha \beta} \) is the Riemann tensor 2-form. Therefore, the
equations (2.5) and (2.6) describe $\text{AdS}_3$ with radius $\lambda^{-1}$, i.e. $\text{AdS}_3$ geometry appears via solution of the vacuum equation (1.3).

The massless Klein-Gordon and Dirac equations in $\text{AdS}_3$ read

$$\Box C = \frac{3}{2} \lambda^2 C \quad \text{and} \quad h_{\mu, \alpha} \partial_\mu C_\beta = 0$$

(2.8)

for the spin 0 boson field $C(x)$ and spin $\frac{1}{2}$ fermion field $C_\alpha(x)$. Here $\Box = \nabla^\mu \nabla_\mu$, where $\nabla_\mu$ is the full covariant derivative with the symmetric Christoffel connection defined via the metric postulate

$$\nabla_\mu h_{\nu, \alpha\beta} = 0.$$  

(2.9)

The world indices $\mu, \nu$ are raised and lowered by the metric tensor $g_{\mu\nu} = h_{\mu, \alpha\beta} h_{\nu, \alpha\beta}$.

The “unfolded” formulation \[11\] of the equations (2.8) in the form of some covariant constancy conditions is most convenient for the analysis of cohomology of currents. To this end one introduces an infinite set of symmetric multispinors $C_{\alpha_1...\alpha_n}$ for all $n \geq 0$. (Following to \[12\] we will assume total symmetrization of indices denoted by the same letter and will use the notation $C_\alpha^{(n)} = C_{\alpha_1...\alpha_n}$ when only a number of indices is important.) As shown in \[11\], the infinite chain of equations

$$D_L C_{\alpha(n)} = \frac{i}{2} \left[ h^{\beta\gamma} C_{\beta\gamma(n)} - \lambda^2 n(n-1) h_{\alpha\alpha} C_{\alpha(n-2)} \right],$$

(2.10)

where $D^L$ is the background Lorentz covariant differential,

$$D^L C_{\alpha(n)} = dC_{\alpha(n)} + n \omega_\alpha \gamma C_{\gamma(n-1)},$$

(2.11)

is equivalent to the equations (2.8) for the lowest rank components $C$ and $C_\alpha$ along with some constraints expressing highest multispinors via highest spacetime derivatives of $C$ and $C_\alpha$ according to

$$C_{\alpha(2n)}(x) = (-2i)^n h^{\nu_1,\alpha_1} h^{\nu_2,\alpha_2} \cdots h^{\nu_n,\alpha_n} \nabla_{\nu_1} \nabla_{\nu_2} \cdots \nabla_{\nu_n} C(x),$$

$$C_{\alpha(2n+1)}(x) = (-2i)^n h^{\nu_1,\alpha_1} h^{\nu_2,\alpha_2} \cdots h^{\nu_n,\alpha_n} \nabla_{\nu_1} \nabla_{\nu_2} \cdots \nabla_{\nu_n} C_\alpha(x),$$

(2.12)

where $\nabla_\mu$ is a full background derivative obeying (2.9) (for multispinors $\nabla_\mu C_{\alpha(n)} = D^L_{\mu} C_{\alpha(n)}$).

Following \[11\], let us introduce the generating function

$$C(y, \psi|x) = \sum_{n=0}^\infty \frac{1}{n!} (\lambda^{-1} \psi)^{\frac{n}{2}} C_{\alpha_1...\alpha_n}(x) y^{\alpha_1} \cdots y^{\alpha_n} = \lambda^{\frac{n}{2}} \pi(C) \tilde{C}(\lambda^{-\frac{1}{2}} y, \psi|x),$$

(2.13)

where $[n + a] = n$, $\forall n \in \mathbb{Z}$ and $0 \leq a < 1$, and the boson-fermion parity $\pi(C) = 0(1)$ for even (odd) functions $C(y)$. The equations (2.10) can be rewritten in the form \[11\],

$$D^L C(y, \psi) = \frac{i\lambda}{2} \psi h^{\alpha\beta} \left[ \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} - y_\alpha y_\beta \right] C(y, \psi),$$

(2.14)

where $D^L = d - \omega^{\alpha\beta} y_\alpha \partial_{y^\beta}$.
Let us note that the definition (2.13) contains inverse powers of $\lambda$ to obtain (2.10) from (2.14) or, equivalently, to have (2.12) with $\lambda$ independent coefficients. Eq. (2.13) is a manifestation of the general property that the higher derivatives in the theory appear together with the negative powers of the cosmological constant.

The fields $C_{\alpha_1...\alpha_n}$ are identified with all on-mass-shell nontrivial derivatives of the matter fields according to (2.12). The condition that the system is on–mass–shell is encoded in the fact that the multispinors $C_{\alpha_1...\alpha_n}$ are totally symmetric. This allows us to work with $C_{\alpha_1...\alpha_n}$ instead of explicit derivatives of the matter fields.

Consider now a function $F[C_{\alpha(n)}(x)]$ of all components of $C_{\alpha_1...\alpha_n}(x)$ at some fixed point $x$. $F$ is not supposed to contain any derivatives with respect to the spacetime coordinates $x$ and therefore looks like a local function of matter fields. One has to be careful however because, when the equations (2.10) hold, (2.12) is true. We will therefore call a function $F[C_{\alpha(n)}]$ pseudolocal if it is an infinite expansion in the field variables $C_{\alpha(n)}(x)$ and local if $F$ is a polynomial with a finite number of nonzero terms.

In terms of the generating functions $C(y,\psi|x)$ this can be reformulated as follows. Let $F(C|x)$ be some functional of the generating function $C(y,\psi|x)$ at some fixed point of spacetime $x$. According to (2.12) its spacetime locality is equivalent on–mass–shell to the locality in the $y$ space. Indeed, from (2.14) it follows that the derivatives in the spinor variables form in a certain sense a square root of the spacetime derivatives. (This is also obvious from (2.12).)

The equation (1.4) for the d3 higher spin system reads (in the rest of the paper we use the symbol $D$ instead of $D_0$)

$$DW_1(y,\psi|x) = J(C^2)(y,\psi|x)$$

with the background AdS covariant differential

$$D = D^L - \lambda\psi h_{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} = d - (\omega^{\alpha\beta} + \lambda\psi h_{\alpha\beta}) y_\alpha \frac{\partial}{\partial y^\beta}. \quad (2.16)$$

That $\omega^{\alpha\beta}(x)$ and $h^{\alpha\beta}(x)$ obey the equations (2.5) and (2.6) guarantees $D^2 = 0$. Thus, our problem is to study the cohomology of $D$ (2.16). Clearly, $D$ commutes with the Euler operator $N = y^\alpha \frac{\partial}{\partial y^\alpha}$. Its eigenvalues are identified with spin $s$,

$$N = 2(s - 1). \quad (2.17)$$

The problem therefore is to be analyzed for different spins independently.

Conserved currents of an arbitrary integer spin in d4 Minkowski spacetime were considered in [13]. For $d = 2$, higher spin conserved currents were constructed in [14]. Also, some currents of spin higher than 2 were recently discussed in [15].

In the case of $AdS_3$, conserved currents of any integer spin $s \geq 1$ built from two massless scalar fields $C$, $C'$ or two massless spinor fields $C_{\alpha}$, $C'_{\alpha}$ have a form

$$J_{\mu,\alpha(2s-2)}^{(s)}(C, C') = \sum_{k=0}^{s-2} \frac{2(-1)^k}{(2k + 1)!(2s - 2k - 3)} h_{\mu\alpha\gamma\alpha(2k+1)} C_{\gamma\alpha(2s-2k-3)} + \sum_{k=0}^{s-1} \frac{(-1)^k}{(2k)!(2s - 2k - 2)} h_{\mu\alpha\gamma\alpha(2k)} C_{\gamma\alpha(2s-2k-2)} C'_{\alpha(2s-2k-2)}$$

$$- C_{\alpha(2k)} C'_{\gamma\alpha(2s-2k-2)} + C_{\alpha(2k)} C'_{\gamma\alpha(2s-2k-2)} - C_{\alpha(2k)} C'_{\gamma\alpha(2s-2k-2)} \right), \quad (2.18)$$
\[ J^{(s)}_{\mu,\alpha(2s-2)}(C_\alpha, C'_\alpha) = \sum_{k=0}^{s-1} \frac{2(-1)^{k+1}}{(2k)!(2s-2k-2)!} h_\mu, \gamma \gamma C_{\gamma \alpha(2k)} C'_{\gamma \alpha(2s-2k-2)} \]
\[ + \sum_{k=0}^{s-2} \frac{(-1)^k}{(2k+1)!(2s-2k-3)!} h_\mu, \gamma \gamma [C_{\gamma \alpha(2k+1)} C'_{\alpha(2s-2k-3)} - C_{\alpha(2k+1)} C'_{\gamma \alpha(2s-2k-3)}] . \quad (2.19) \]

The supercurrent of any half-integer spin \( s \geq \frac{3}{2} \) built from massless scalar \( C \) and spinor \( C'_\alpha \) has a form
\[ J^{(s)}_{\mu,\alpha(2s-2)}(C, C'_\alpha) = \sum_{k=0}^{s-\frac{3}{2}} \left\{ \frac{2(-1)^k}{(2k+1)!(2s-2k-3)!} h_\mu, \gamma \gamma C_{\gamma \alpha(2k)} C'_{\gamma \alpha(2s-2k-3)} \right\} . \quad (2.20) \]

The lowest spin conserved currents read
\[ J^{(1)}_{\mu,\alpha}(C, C'_\alpha) = h_\mu, \gamma \gamma (C_{\gamma \gamma} C'_{\alpha} - C C'_{\gamma \gamma}) , \quad J^{(1)}_{\mu}(C, C'_\alpha) = h_\mu, \gamma \gamma C_{\gamma} C'_{\gamma} \ , \quad (2.21) \]
\[ J^{(3/2)}_{\mu,\alpha}(C, C'_\alpha) = h_\mu, \gamma \gamma (C_{\gamma} C'_{\alpha} - C C'_{\gamma \alpha} + 2C_{\gamma} C'_{\gamma}) \ , \quad (2.22) \]
\[ J^{(2)}_{\mu,\alpha\alpha}(C, C'_\alpha) = \frac{1}{2} h_\mu, \gamma \gamma (C_{\gamma} C'_{\alpha\alpha} - C C'_{\gamma \alpha\alpha} - C_{\gamma \alpha\alpha} C' + C_{\alpha\alpha} C'_{\gamma \gamma} + 4C_{\gamma} C'_{\gamma \alpha}) . \quad (2.23) \]

These currents are all local because any of them contains a finite number of terms (i.e., higher derivatives \( [2.12] \)). The same expressions remain valid in the flat limit with \( \nabla_\mu \to \partial_\mu \) in \( [2.12] \).

## 3 Generating Functions

To analyze the cohomology problem for currents of an arbitrary spin we first elaborate a technique operating with the generating functions \( [2.13] \) rather than with the individual multispinors.

A generic Lorentz covariant spacetime 1-form of spin \( s = n/2 + 1 \) bilinear in two different matter fields \( C \) and \( C' \) and their on–mass–shell nontrivial derivatives is
\[ \Phi_{\alpha(n)}(C, C'|x) = \sum_{k+l=n-2}^{\infty} \sum_{m=0}^{\infty} a(k, l, m) h_{\alpha\alpha} C_{\alpha(k)}^{\beta(m)}(x) C'_{\alpha(l) \beta(m)}(x) \]
\[ + \sum_{k+l=n-1}^{\infty} \sum_{m=0}^{\infty} \left[ b_1(k, l, m) h_{\alpha}^{\gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) C'_{\alpha(l) \beta(m)}(x) \right] \]
\[ + b_2(k, l, m) h_{\alpha}^{\gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) C'_{\alpha(l) \beta(m)}(x), \quad (3.1) \]
\[ + \sum_{k+l=n}^{\infty} \sum_{m=0}^{\infty} \left[ c_1(k, l, m) h^{\gamma \gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) C'_{\alpha(l) \beta(m)}(x) \right] \]
\[ + c_2(k, l, m) h^{\gamma \gamma} C_{\gamma \alpha(k)}^{\beta(m)}(x) C'_{\alpha(l) \beta(m)}(x), \]
where \( a(k, l, m), b_1(k, l, m), \) and \( e_{1,2,3}(k, l, m) \) are arbitrary constants and \( h_{\alpha\alpha} \) is the dreibein 1-form. Introducing

\[
\Phi(y, \psi|x) = \Phi_{\alpha_1...\alpha_n}(\psi|x) y^{\alpha_1} ... y^{\alpha_n},
\]  
(3.2)

one can equivalently rewrite this formula as

\[
\Phi(y, \psi|x) = \frac{1}{(2\pi i)^2} \int dr \int ds \int \tau^{-2} d\tau \int d^2u d^2v \exp \left\{ \frac{i}{\tau} (u_\gamma v^\gamma) \right\}
\times C(u - ry, \psi|x) C'(v + sy, \psi|x) \big[ f_1(r, s, \tau) y^\alpha y^\alpha + f_2(r, s, \tau) y^\alpha u^\alpha + f_3(r, s, \tau) y^\alpha v^\alpha \\
+ f_4(r, s, \tau) u^\alpha u^\alpha + f_5(r, s, \tau) u^\alpha v^\alpha + f_6(r, s, \tau) v^\alpha v^\alpha \big].
\]
(3.3)

Here \( r, s, \) and \( \tau \) are complex variables, \( u_\alpha \) and \( v_\alpha (\alpha = 1, 2) \) are spinor variables. The quantities \( f_i(r, s, \tau), i = 1, ..., 6 \) are polynomials in \( r^{-1} \) and \( s^{-1} \) and formal series in \( \tau^{-1}, \)

\[
f_1(r, s, \tau) = \sum_{0<k, l<p, \ m=1}^{\infty} f_1(k, l, m) r^{-k} s^{-l} \tau^{-m},
\]
(3.4)

\[
f_{2,3}(r, s, \tau) = \sum_{0<k, l<p, \ m=2}^{\infty} f_{2,3}(k, l, m) r^{-k} s^{-l} \tau^{-m},
\]
(3.5)

\[
f_{4,5,6}(r, s, \tau) = \sum_{0<k, l<p, \ m=3}^{\infty} f_{4,5,6}(k, l, m) r^{-k} s^{-l} \tau^{-m}.
\]
(3.6)

The contour integrations are normalized as \( \int \tau^{-n} d\tau = \delta_n^1 \). The Gaussian integrations with respect to \( u_\alpha \) and \( v_\alpha \) should be completed prior the contour integrations.

Inserting (2.13) in the form

\[
C(u - ry, \psi|x) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (\lambda^{-1} \psi)^{\frac{n+m}{2}} C_{\alpha(n)\beta(m)}(x) u^{\alpha_1} ... u^{\alpha_n} (-\tau)^m y^{\beta_1} ... y^{\beta_m},
\]
(3.7)

\[
C'(v + sy, \psi|x) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (\lambda^{-1} \psi)^{\frac{n+m}{2}} C'_{\alpha(n)\beta(m)}(x) v^{\alpha_1} ... v^{\alpha_n} s^m y^{\beta_1} ... y^{\beta_m}
\]
(3.8)

into (3.3) and completing elementary integrations one arrives at (3.11) with

\[
a(k, l, m) = \frac{(-)^{k+m+l} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m}{2}} f_1(k + 1, l + 1, m + 1),
\]
(3.9)

\[
b_1(k, l, m) = \frac{(-)^{k+m+l+1} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m+1}{2}} f_3(k + 1, l + 1, m + 2),
\]
(3.10)

\[
b_2(k, l, m) = \frac{(-)^{k+m+l+1} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m+1}{2}} f_2(k + 1, l + 1, m + 2),
\]
(3.11)

\[
e_1(k, l, m) = \frac{(-)^{k+m+1} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m+1}{2}} f_6(k + 1, l + 1, m + 3),
\]
(3.12)

\[
e_2(k, l, m) = \frac{(-)^{k+m+1} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m+1}{2}} f_5(k + 1, l + 1, m + 3),
\]
(3.13)

\[
e_3(k, l, m) = \frac{(-)^{k+m+1} \lambda^m}{k! l! m!} (\lambda^{-1} \psi)^{\frac{k+m+1}{2}} f_4(k + 1, l + 1, m + 3).
\]
(3.14)
Therefore (3.3) indeed describes a general Lorentz covariant 1-form bilinear in the matter fields. From these expressions we see that the formulas (3.3) produce a spacetime local expression if all the coefficients $f_i$ contain a finite number of terms in (3.4)-(3.6) and pseudolocal if some of the expansions in negative powers of $\tau$ are infinite.

In practice, the following representations of rank $n = 0, 1, 2, 3$ differential forms $\Phi_n(x)$ are shown below to be most convenient,\footnote{\textit{C} \times C} \Phi_0(y, \psi|x) = \psi \left( \frac{1}{2\pi^2} \right) \int \frac{dz}{z} \int \frac{d\bar{z}}{\bar{z}} \int \frac{d\tau}{\tau^2} \int d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q})^2 \right\}

\times C \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2i}(z - \bar{z})y, \psi \right| x \right] C' \left[ \frac{1}{2i}(q - \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x \right] E_0(z, \bar{z}, \tau), (3.15)

\Phi_1(y, \psi|x) = h_{\alpha\alpha} \left( \frac{1}{2\pi^2} \right) \int dz \int d\bar{z} \int \tau^{-2} d\tau \int d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q})^2 \right\}

\times \left\{ \begin{array}{l}
R_1(z, \bar{z}, \tau) y^\alpha y^\alpha + \frac{1}{2\tau z} W_1(z, \bar{z}, \tau) y^\alpha q^\alpha + \frac{1}{2\tau} \bar{W}_1(z, \bar{z}, \tau) y^\alpha q^\alpha \\
\left[ 1 \right]
\end{array} \right. \times C \left[ \frac{1}{2i}(q - \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x \right] E_0(z, \bar{z}, \tau), (3.16)

\Phi_2(y, \psi|x) = -\frac{\lambda}{2} \psi h_{\alpha\beta} \wedge h^{\alpha\beta} \left( \frac{1}{2\pi^2} \right) \int dz \int d\bar{z} \int \tau^{-2} d\tau \int d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q})^2 \right\}

\times C \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2i}(z - \bar{z})y, \psi \right| x \right] C' \left[ \frac{1}{2i}(q - \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x \right] E_0(z, \bar{z}, \tau), (3.17)

\Phi_3(y, \psi|x) = -\frac{\lambda^2}{12} h_{\alpha\beta} \wedge h^{\beta\gamma} \wedge h^{\gamma\alpha} \left( \frac{1}{2\pi^2} \right) \int dz \int d\bar{z} \int \frac{d\tau}{\tau^2} \int d^2q d^2\bar{q} \exp \left\{ -\frac{1}{2\tau}(q, \bar{q})^2 \right\}

\times C \left[ \frac{1}{2}(q + \bar{q}) - \frac{1}{2i}(z - \bar{z})y, \psi \right| x \right] C' \left[ \frac{1}{2i}(q - \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right| x \right] E_3(z, \bar{z}, \tau). (3.18)

Here the factors of $\psi, -\frac{i}{2}\psi$, and $-\frac{\lambda^2}{2}$ are introduced for future convenience. It is not hard to see that the expressions (3.15)-(3.18) reproduce arbitrary Lorentz covariant forms bilinear in the matter fields and their on-mass-shell nontrivial derivatives.

Let $n, \bar{n},$ and $n_\tau$ be the following operators,

\[ n = z \frac{\partial}{\partial z}, \quad \bar{n} = \bar{z} \frac{\partial}{\partial \bar{z}}, \quad n_\tau = \tau \frac{\partial}{\partial \tau} \] (3.19)

(using the same notations for their eigenvalues). The quantities $R_{1,2}(z, \bar{z}, \tau), W_{1,2}(z, \bar{z}, \tau), \bar{W}_{1,2}(z, \bar{z}, \tau), Y_{1,2}(z, \bar{z}, \tau), \bar{Y}_{1,2}(z, \bar{z}, \tau), V_{1,2}(z, \bar{z}, \tau),$ and $E_{0,3}(z, \bar{z}, \tau)$ give a non-zero contribution to (3.15), (3.16), (3.17), and (3.18) when $n, \bar{n},$ and $n_\tau$ satisfy the following
restrictions:

|       | \( n \leq -1 \) | \( \bar{n} \leq -1 \) | \( \tau \leq -1 \) |
|-------|------------------|------------------|------------------|
| \( R_1, R_2 \) | \( n \leq -1 \) | \( \bar{n} \leq -1 \) | \( \tau \leq -1 \) |
| \( W_1, W_2 \) | \( n \leq -1 \) | \( \bar{n} \leq 0 \) | \( \tau \leq -1 \) |
| \( \bar{W}_1, \bar{W}_2 \) | \( n \leq 0 \) | \( \bar{n} \leq -1 \) | \( \tau \leq -1 \) |
| \( Y_1, Y_2 \) | \( n \leq -1 \) | \( \bar{n} \leq 1 \) | \( \tau \leq -1 \) |
| \( \bar{Y}_1, \bar{Y}_2 \) | \( n \leq 1 \) | \( \bar{n} \leq -1 \) | \( \tau \leq -1 \) |
| \( V_1, V_2, E_0, E_3 \) | \( n \leq 0 \) | \( \bar{n} \leq 0 \) | \( \tau \leq -1 \) |

Beyond these regions, the coefficients do not contribute and therefore their values can be fixed arbitrarily. In particular, the quantities \( R_1, W_1, \bar{W}_1, \bar{W}_2, \bar{Y}_1, \bar{Y}_2 \) are defined modulo arbitrary polynomials in \( \tau \)

\[
P(\tau) = \sum_{k=0}^{k_0} P_k \tau^k.
\] (3.21)

The formulae considered in this section make sense for an arbitrary dimension of spinors (discarding the question of the completeness of the basis forms like \( h_{\alpha \beta} \) and \( h_{\alpha \beta} \wedge h^{\beta \alpha} \)). For the two-component spinors there exist additional equivalence relationships due to the fact that antisymmetrization over any three two-component spinor indices gives zero. This is expressed by the identity

\[
a_{\alpha}(b_{\beta}c^{\beta}) + b_{\alpha}(c_{\beta}a^{\beta}) + c_{\alpha}(a_{\beta}b^{\beta}) = 0
\] (3.22)

valid for any three commuting two-component spinors \( a_{\alpha}, b_{\alpha}, \) and \( c_{\alpha} \). As a result, the forms discussed so far are not all independent. The ambiguity in adding any terms which vanish as a consequence of (3.22) can be expressed in a form of some equivalence (gauge) transformations of the coefficients in (3.3), (3.16), and (3.17). We call these equivalence transformations Fierz transformations.

To derive a functional form of a general Fierz transformation it is convenient to rewrite (3.16) as

\[
\Phi_1(y, \psi| x) = h_{\alpha \alpha} \frac{1}{(2\pi)^3} \int dz \int \bar{d}z \int dt \int d^2q d^2\bar{q} \times \exp \left\{ \frac{-t}{2}(q, \bar{q}^\gamma) - \frac{i}{2}z(q, y^\gamma) + \frac{i}{2}\bar{z}(y, q^\gamma) \right\} C \left( \frac{q + \bar{q}}{2}, \psi \right| x) C' \left( \frac{q - \bar{q}}{2}, \psi \right| x) \times \{ f'_1(z, \bar{z}, t) q^\alpha q^\alpha + f'_2(z, \bar{z}, t) y^\alpha q^\alpha + f'_3(z, \bar{z}, t) y^\alpha q^\alpha + f'_4(z, \bar{z}, t) q^\alpha q^\alpha + f'_5(z, \bar{z}, t) q^\alpha q^\alpha \}
\]  

(3.23)

Using the partial integrations w.r.t. \( t, z, \) and \( \bar{z} \) in (3.23) one finds that the transformations,

\[
\delta f'_1 = \partial_t \epsilon, \quad \delta f'_2 = i\partial_z \epsilon, \quad \delta f'_3 = -i\partial_z \epsilon,
\]

\[
\delta f'_3 = \partial_t \eta, \quad \delta f'_5 = i\partial_z \eta, \quad \delta f'_6 = -i\partial_z \eta,
\]

\[
\delta f'_2 = \partial_t \phi, \quad \delta f'_4 = i\partial_z \phi, \quad \delta f'_5 = -i\partial_z \phi,
\]

(3.24)
with arbitrary parameters $\epsilon = \epsilon(z,\bar{z},t)$, $\eta = \eta(z,\bar{z},t)$, and $\phi = \phi(z,\bar{z},t)$ describe all possible Fierz transformations of the 1-form \((3.23)\). From here one derives a form of the Fierz transformations in the representations \((3.15)-(3.18)\).

\[
\begin{align*}
\delta R_{1,2} &= -\partial_\tau \chi_{1,2}, \\
\delta W_{1,2} &= -\partial_\tau \xi_{1,2} + 2i\bar{n}\chi_{1,2}, \\
\delta \bar{W}_{1,2} &= -\partial_\tau \bar{\xi}_{1,2} - 2i\bar{n}\chi_{1,2}, \\
\delta V_{1,2} &= -i(n\xi_{1,2} - \bar{n}\bar{\xi}_{1,2}), \\
\delta Y_{1,2} &= i(n - 1)\xi_{1,2}, \\
\delta \bar{Y}_{1,2} &= -i(n - 1)\bar{\xi}_{1,2},
\end{align*}
\]

with arbitrary parameters $\chi_{1,2}(z,\bar{z},\tau)$, $\xi_{1,2}(z,\bar{z},\tau)$, and $\bar{\xi}_{1,2}(z,\bar{z},\tau)$.

Let us mention that the relation

\[
h_{\alpha\beta} \wedge h_{\gamma\delta} = \frac{1}{4} \left( \varepsilon_{\alpha\gamma} h_{\beta\lambda} \wedge h_{\delta\gamma} + \varepsilon_{\beta\gamma} h_{\alpha\lambda} \wedge h_{\delta\gamma} + \varepsilon_{\beta\delta} h_{\alpha\lambda} \wedge h_{\gamma\lambda} + \varepsilon_{\alpha\delta} h_{\beta\lambda} \wedge h_{\gamma\lambda} \right),
\]

which allows one to use the representation \((3.17)\) for a 2-form, is itself a consequence of \((3.22)\).

## 4 On-Mass-Shell Current Complex

In this section we study the on–mass-shell action of the operator $D$ \((2.10)\) on the differential forms defined in sect. \([3]\). The advantage of the formulation of the dynamical equations in the unfolded form \((2.14)\) is that it expresses the (exterior) spacetime derivative of $C$ via some operators acting in the auxiliary spinor space. As a result, on–mass–shell action of $D$ reduces to some mapping $\mathcal{D}$ acting on the coefficients in the formulae \((3.13)-(3.18)\).

Let us consider the example of a 0-form. Using the Leibnitz rule for $D^L$ and taking into account the equations of motion \((2.14)\) and the zero torsion condition $D^L h_{\alpha\alpha} = 0$ \((2.9)\), one gets

\[
D\Phi_0(y,\psi|x) = \left( D^L - \lambda \psi h_{\alpha\beta} y_{\alpha} \frac{\partial}{\partial y_{\beta}} \right) \Phi_0(y,\psi|x)
\]

\[
= \frac{i\lambda}{2} h^{\alpha\alpha} \frac{1}{(2\pi)^2} \int \frac{dz}{z} \int \frac{d\bar{z}}{\bar{z}} \int \frac{d\tau}{\tau^2} \int d^2q d^2\bar{q} \exp \left\{-\frac{1}{2\tau}(q,\bar{q})\right\} E_0(z,\bar{z},\tau)
\]

\[
\times \left[ 4 \frac{\partial}{\partial q^\alpha} \frac{\partial}{\partial \bar{q}^\alpha} - q_\alpha \bar{q}_\alpha + i y_\alpha (z \bar{q}_\alpha - z \bar{q}_\alpha) - z \bar{z} y_\alpha y_\alpha - 2y_\alpha \left( z \frac{\partial}{\partial q^\alpha} - \bar{z} \frac{\partial}{\partial \bar{q}^\alpha} \right) \right]
\]

\[
\times \left\{ C \left[ \frac{1}{2} (q + \bar{q}) - \frac{1}{2i}(z - \bar{z})y, \psi \right] x \right\} C' \left[ \frac{1}{2i} (q - \bar{q}) + \frac{1}{2}(z + \bar{z})y, \psi \right] x \right\}.
\]

Completing the partial integration w.r.t. $q$ and $\bar{q}$ one arrives at the 1-form $\Phi_1 = D\Phi_0$ with the coefficients $R^D_1(E_0)$, $W^D_1(E_0)$, $\bar{W}^D_1(E_0)$, $Y^D_1(E_0)$, $\bar{Y}^D_1(E_0)$, and $V^D_1(E_0)$ of the form

\[
R^D_1(E_0) = -\frac{i\lambda}{2} E_0(z,\bar{z},\tau),
\]
\( W^P_1(E_0) = i\lambda(1-i\tau)E_0(z, \bar{z}, \tau), \)
\( \bar{W}^P_1(E_0) = i\lambda(1+i\tau)E_0(z, \bar{z}, \tau), \)
\( V^P_1(E_0) = -i\lambda(1+\tau^2)E_0(z, \bar{z}, \tau), \)
\( Y^P_1(E_0) = \bar{Y}^P_1(E_0) = 0. \) (4.2)

Analogously one derives the mapping \( \mathcal{D} : \Phi_i(y) \to \Phi_{i+1}(y) = D\Phi_i(y) \big|_{\text{on-shell}}, i = 1, 2 \) on the coefficients of the differential forms (3.16), (3.17), (3.18),
\[ \mathcal{D} \{ R_1, W_1, \ldots \} = \{ R^P_2, W^P_2, \ldots \}, \quad \mathcal{D} \{ R_2, W_2, \ldots \} = E^P_3, \] (4.3)

with
\[
R^P_2 = -(1-i\tau)nR_1 - (1+i\tau)n\bar{R}_1 + 2R_1 + \frac{i}{4}(1+i\tau)\partial_\tau W_1 - \frac{i}{4}(1-i\tau)\partial_\tau \bar{W}_1 + \frac{1}{4}(W_1 + \bar{W}_1) - \frac{1}{2}(nW_1 + \bar{n}\bar{W}_1), \]
(4.4)
\[
W^P_2 = -\frac{i}{2}\partial_\tau [(1 + \tau^2)W_1] + \frac{3}{2}(1+i\tau)W_1 + 2(1+\tau^2)nR_1 + \frac{1}{2}(1-i\tau)n\bar{W}_1 + (1-2n)Y_1 - \frac{1}{2}(1+i\tau)(\bar{n}-1)W_1 + i(1+i\tau)\partial_\tau Y_1 + \left(\frac{3}{2} - \bar{n}\right)V_1 - \frac{i}{2}(1-i\tau)\partial_\tau V_1, \]
(4.5)
\[
V^P_2 = \frac{1}{2}(1+\tau^2)(nW_1 + \bar{n}\bar{W}_1) + \frac{1}{2}(1+i\tau)(\bar{n}-1)V_1 + \frac{1}{2}(1-i\tau)(n-1)V_1 + V_1 + (1+i\tau)nY_1 + (1-i\tau)n\bar{Y}_1, \]
(4.6)
\[
Y^P_2 = \frac{1}{2}(1+\tau^2)(\bar{n}-1)W_1 - i(1+i\tau)\partial_\tau Y_1 + (1+i\tau)Y_1 + (1-i\tau)nY_1 + \frac{1}{2}(1-i\tau)(\bar{n}-1)V_1, \]
(4.7)
\[
\bar{Y}^P_2 = \frac{1}{2}(1+\tau^2)(n-1)\bar{W}_1 + i(1+\tau^2)\partial_\tau \bar{Y}_1 + (1-i\tau)\bar{Y}_1 + (1+i\tau)n\bar{Y}_1 + \frac{1}{2}(1+i\tau)(n-1)V_1, \]
(4.8)
and
\[
E^P_3 = 4in\bar{n}(1+\tau^2)R_2 + in\bar{n}(W_2 + \bar{W}_2) - 3\tau(nW_2 - \bar{n}\bar{W}_2) - \tau n\bar{n}(W_2 - \bar{W}_2) + \partial_\tau [(1 + \tau^2)(nW_2 - \bar{n}\bar{W}_2)] + 2\partial_\tau (nY_2 - \bar{n}\bar{Y}_2) - (n - \bar{n}) \partial_\tau V_2 + 2i(n\bar{n} - n - \bar{n} + 1)V_2 + i(n + \bar{n} - 2) \tau \partial_\tau V_2 + i(1+\tau^2)\partial_\tau \partial_\tau V_2 - 3i(nW_2 + \bar{n}\bar{W}_2) + (i + \tau) n(n+1)W_2 + (i - \tau) n\bar{n}(n+1)\bar{W}_2 - 4i(nY_2 + \bar{n}\bar{Y}_2) + 2i \tau \partial_\tau (nY_2 + \bar{n}\bar{Y}_2) + 2i[n(n+1)Y_2 + \bar{n}(n+1)\bar{Y}_2]. \] (4.10)

As expected, \( \mathcal{D}^2 = 0 \) and therefore the mapping \( \mathcal{D} \) defines a complex \( (T, \mathcal{D}) \) with
\[
T = \bigoplus_{i=0,1,2,3} T_i, \quad T_{0,3} = \{ E_{0,3} \}, \quad T_{1,2} = \{ R_{1,2}, W_{1,2}, \bar{W}_{1,2}, V_{1,2}, Y_{1,2}, \bar{Y}_{1,2} \}. \] (4.11)
The reformulation of the problem in terms of \((T, \mathcal{D})\) effectively accounts the fact that the fields are on–mass–shell. We identify the cohomology of currents with the cohomology of the operator \(\mathcal{D}\) acting on the space \(T \cup \cdot\). 

The remarkable property of the mapping \(\mathcal{D}\) is that it contains \(z, \bar{z}, \frac{\partial}{\partial z}, \) and \(\frac{\partial}{\partial \bar{z}}\) only via \(n \) and \(\bar{n} \) (3.19), thus implying the separation of variables: the differential \(\mathcal{D}\) leaves invariant eigensubspaces of \(n \) and \(\bar{n}\). As a result one can consider separately functions \(R_{1,2}, W_{1,2}, \ldots \) with equal values \(n \) and \(\bar{n}\). In fact, this is the main reason for using the particular representation (3.13)-(3.18). Needless to say that this property greatly simplifies the study of the cohomology of currents, reducing it to the analysis of functions of a single variable \(\tau\) with two integer parameters \(n \) and \(\bar{n}\). The fact of the existence of such a separation of variables is a consequence of the form of the matter field equations (2.14).

As expected, the system (4.4)-(4.9) is consistent with the Fierz transformations (3.25)-(3.30). Namely, any Fierz transformation of the quantities \(R_1, W_1, \ldots\) leads to the Fierz transformation of the quantities \(R_2^D, W_2^D, \ldots\) with the parameters

\[
\chi_2 \left( \chi_1, \xi_1, \bar{\xi}_1 \right) = -\frac{1}{2} \left[ (n + \bar{n}) - i \tau (n - \bar{n}) - 4 \right] \chi_1
+ \frac{1}{4} \left[ i \partial_\tau (\xi_1 - \bar{\xi}_1) + (1 - \tau \partial_\tau)(\xi_1 + \bar{\xi}_1) - 2(n \xi_1 + \bar{n} \bar{\xi}_1) \right],
\]

(4.12)

\[
\xi_2 \left( \chi_1, \xi_1, \bar{\xi}_1 \right) = (1 + \tau^2) \bar{n} \chi_1 + \frac{1}{2} (1 - i \tau) \bar{n} \bar{\xi}_1
+ \frac{1}{2} \left[ -i (1 + \tau^2) \partial_\tau \xi_1 + (n + 2) \xi_1 - i \tau (n - 2) \xi_1 \right],
\]

(4.13)

\[
\bar{\xi}_2 \left( \chi_1, \xi_1, \bar{\xi}_1 \right) = (1 + \tau^2) n \chi_1 + \frac{1}{2} (1 + i \tau) n \xi_1
+ \frac{1}{2} \left[ i (1 + \tau^2) \partial_\tau \bar{\xi}_1 + (\bar{n} + 2) \bar{\xi}_1 + i \tau (\bar{n} - 2) \bar{\xi}_1 \right],
\]

(4.14)

and any Fierz transformation of \(R_2, W_2, \ldots\) does not affect the parameter \(E_3^D\) (4.10).

Of course the formulae (4.4)-(4.9) are consistent with the ambiguity in adding trivial terms (3.21) to the quantities \(R_1, W_1, \ldots\) in the sense that this transformation leads to the analogous transformation of the quantities \(R_2^D, W_2^D, \ldots\), which does not affect the 2-form \(\Phi_2(y)\).

5 Cohomology of Currents

Following [13] we study the currents containing the minimal possible number of spacetime derivatives for a given spin \(s\). From (2.12) it is clear that this is the case if the number of the contracted indices \(\beta\) in (3.11) is zero. Since the number of contractions is \(-(n_\tau + 1)\) (see sect. 3) we consider 2-forms \(\Phi_{n,\bar{n}}^\alpha\) with \(n_\tau = -1\). Thus we set in (3.17)

\[
R_2 = \alpha_R(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1}, \quad W_2 = \alpha_W(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1}, \quad \bar{W}_2 = \alpha_W(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1},
\]

\[
Y_2 = \alpha_Y(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1}, \quad \bar{Y}_2 = \alpha_Y(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1}, \quad V_2 = \alpha_Y(n, \bar{n}) \, z^n z^{\bar{n}} \tau^{-1}
\]

(5.1)

with some constant parameters \(\alpha_R(n, \bar{n}), \alpha_W(n, \bar{n}), \ldots \sim \lambda[s]\), where \(s = 1 - \frac{1}{2}(n + \bar{n})\). The conservation condition means that \(\Phi_{2,n}^\alpha\) should be \(\mathcal{D}\)-closed. The requirement \(E_3^D = 0\)
modulo terms that do not contribute to (3.18) imposes the following conditions on the parameters in (5.1),

\[ 4n\tilde{n}\alpha_R + (n + \tilde{n} - 2)(n\alpha_W + \tilde{n}\alpha_{\overline{W}}) + 2n(n - 2)\alpha_Y + 2\tilde{n}(\tilde{n} - 2)\alpha_{\overline{Y}} = 0, \]
\[ n\alpha_W - \tilde{n}\alpha_{\overline{W}} + 2(n\alpha_Y - \tilde{n}\alpha_{\overline{Y}}) = 0, \]
\[ \alpha_Y = 0, \]

for \( n \neq 1, \tilde{n} \neq 1 \). For \( n = 1 \) or \( \tilde{n} = 1 \), \( \Phi_n^{\tilde{n}} \) is closed as a consequence of the conditions (3.20) for \( E_3^p \).

Our problem is to investigate whether there exist the coefficients \( R_1, W_1, \ldots \) such that \( R_{2n}^p, W_2^p, \ldots \) have a form (5.1). To this end one has to solve the system (1.4)-(1.9) in terms of the formal series

\[ f(\tau) = \sum_{k=-\infty}^{\mu<\infty} f_k \tau^k. \]

Because of the identities (3.22) there is a freedom in the Fierz transformations (3.25)-(3.30) for \( \Phi_{1,2} \). Also one can use the ambiguity in the exact shifts of \( R_1, W_1, \ldots \) by any \( R_n^p, W_n^p, \ldots \) which do not affect \( \Phi_2 \) because \( D^2 = 0 \). Altogether exact shifts and Fierz transformations of \( \Phi_1 \) produce the following equivalence transformations

\[
\begin{align*}
\delta R_1 & = -\partial\chi_1 + \varepsilon, \\
\delta W_1 & = -\partial\xi_1 + 2i\tilde{n}\chi_1 - 2(1 - i\tau)\varepsilon, \\
\delta \bar{W}_1 & = -\partial\bar{\xi}_1 - 2in\chi_1 - 2(1 + i\tau)\varepsilon, \\
\delta V_1 & = 2(1 + \tau^2)\varepsilon - i(n\xi_1 - \tilde{n}\bar{\xi}_1), \\
\delta Y_1 & = i(\tilde{n} - 1)\xi_1, \\
\delta \bar{Y}_1 & = -i(n - 1)\bar{\xi}_1
\end{align*}
\]

with \( \varepsilon(z, \bar{z}, \tau) = -\frac{ik}{4}E_0(z, \bar{z}, \tau) \) (4.2).

For a given spin \( s \) we consider separately two cases: (i) with \( n = 1, \tilde{n} = 1 - 2s \) or \( \tilde{n} = 1, n = 1 - 2s \) and (ii) with \( n < 1 \) and \( \tilde{n} < 1 \). As shown below, the case (i) corresponds to the nontrivial physical conserved currents, whereas the case (ii) describes all possible “improvements”.

Let us start with the case (i) setting for definiteness \( \tilde{n} = 1 \). The case \( n = 1 \) can be considered analogously. According to (3.20), \( Y_2 \) is the only coefficient giving a non-zero contribution to \( \Phi_2^{Y_{1,2}}(y) \). Obviously, a 2-form with \( \tilde{n} = 1 \) is invariant under the transformations (3.24). The only non-trivial equation is (4.8). With \( Y_2^{Y_{1,2}}(5.1) \) it takes the form

\[ (1 + \tau^2) \frac{\partial}{\partial \tau} Y_1 = -i(1 + i\tau) Y_1 + i(2s - 1)(1 - i\tau) Y_1 + i\alpha_Y(1 - 2s, 1) \frac{z^{1 - 2s}}{\tau} + P(\tau), \]

where \( P(\tau) \) is some polynomial (3.21). As shown in Appendix A, the generic solution of (5.12) is

\[ Y_1(z, \bar{z}, \tau) = -i \alpha_Y(1 - 2s, 1) z^{1 - 2s} \bar{z} (1 - i\tau)(1 + i\tau)^{2s - 1} \ln (1 + \tau^{-2}) + \sigma z^{1 - 2s} \bar{z} (1 - i\tau)(1 + i\tau)^{2s - 1} \ln \left( \frac{1 + i\tau^{-1}}{1 - i\tau^{-1}} \right) + Q(\tau), \]
where \( \sigma \) is an arbitrary constant and \( Q(\tau) \) is some inessential polynomial. The logarithms are treated as power series in \( \tau^{-1} \).

At any \( \sigma \), the solution (5.13) is an infinite series in \( \tau^{-1} \), thus corresponding to some pseudolocal 1-form. Thus, the 2-forms \( \Phi^s_2(y|x) \) constructed with the polynomials \( Y_2 \) at \( n = 1 \) and with \( \tilde{Y}_2 \) at \( n = 1 \) are \( D \)-closed and cannot be represented as \( D\Phi^s_1(y|x) \) with a spacetime local \( \Phi^s_1(y|x) \). We therefore argue that the 2-form \( \Phi^s_2(y) \) describes a physical conserved current of spin \( s \). The currents (2.18)-(2.20) are reproduced via \( Y_2 \) (3.1) with

\[
\alpha_Y (1 - 2s, 1) = 2^{2s-1} (\lambda \psi)^{[s]}.
\] (5.14)

The formula (5.13) solves the problem of reformulation of the physical currents as pseudolocally exact 2-forms.

Let us note that 1- and 2-forms (3.16), (3.17) have the following discrete symmetry permutting \( C \) and \( C' \),

\[
\Phi [C(y), C'(y); y] = (-)^{\pi(C)\pi(C') + 1} \Phi' [C'(iy), C(iy); -y]
\] (5.15)

with \( \Phi'(y) \) defined with the parameters

\[
(R', W', \tilde{W}', V', \tilde{V}', Y')(z, \bar{z}, \tau) = (R, -\bar{W}, -\bar{W}, \bar{V}, \bar{Y})(-\bar{z}, z, \tau).
\] (5.16)

Therefore, the currents generated by \( Y_2(n = 1 - 2s, \bar{n} = 1) \) and \( \tilde{Y}_2(n = 1, \bar{n} = 1 - 2s) \) are equivalent by the interchange \( C \leftrightarrow C' \).

Note that the solution (5.13) is not unique, containing an arbitrary parameter \( \sigma \). Since the transformations (5.6)-(5.11) are trivial for \( Y_1 \) at \( n = 1 \), this one-parametric ambiguity cannot be compensated this way. This means that we have found a pseudolocal 1-form that is \( D \)-closed but not \( D \)-exact, i.e., the cohomology group \( H^1(T, D) \) is nontrivial. The physical meaning of this fact is not completely clear to us. It is however in agreement with the one-parametric ambiguity found in [14] for the spin 2 case.

Let us now consider the case of \( n < 1, \bar{n} < 1 \). Substituting (3.1) into the system (4.4)-(4.9) we show in appendix B that, if the conditions (3.2)-(3.4) guaranteeing that \( \Phi_2 \) is \( D \)-closed are satisfied, then, modulo gauge transformations (3.6)-(3.11), its generic solution is

\[
R_1(z, \bar{z}, \tau) = \frac{z^n \bar{z}^{\bar{n}}}{4\tau n \bar{n}} \left[ n a_W(n, \bar{n}) + \bar{n} a_{\bar{W}}(n, \bar{n}) - \frac{2n}{\bar{n} - 1} \alpha_Y(n, \bar{n}) - \frac{2\bar{n}}{n - 1} \alpha_{\bar{Y}}(n, \bar{n}) \right]
\] (5.17)

\((n < 0, \bar{n} < 0),\)

\[
W_1(z, \bar{z}, \tau) = \frac{\alpha_Y(n, \bar{n})}{\bar{n} - 1} \frac{z^n \bar{z}^{\bar{n}}}{\tau} + \sigma(n, \bar{n}) \frac{z^n \bar{z}^{\bar{n}}}{\tau} \left[ -\frac{i\tau^{-1}}{1 - i\tau^{-1}} + \frac{\bar{n}}{2} \ln \left( \frac{1 + i\tau^{-1}}{1 - i\tau^{-1}} \right) \right],
\] (5.18)

\[
\tilde{W}_1(z, \bar{z}, \tau) = \frac{\alpha_{\bar{Y}}(n, \bar{n})}{n - 1} \frac{z^n \bar{z}^{\bar{n}}}{\tau} + \sigma(n, \bar{n}) \frac{z^n \bar{z}^{\bar{n}}}{\tau} \left[ \frac{i\tau^{-1}}{1 + i\tau^{-1}} - \frac{n}{2} \ln \left( \frac{1 + i\tau^{-1}}{1 - i\tau^{-1}} \right) \right],
\] (5.19)

\[
V_1(z, \bar{z}, \tau) = Y_1(z, \bar{z}, \tau) = \tilde{Y}_1(z, \bar{z}, \tau) = 0,
\] (5.20)

where \( \sigma(n, \bar{n}) \) are free parameters. At \( n = \bar{n} = 0 \) one should set \( \alpha_Y(0, 0) = 0 \) and the solution is a pure gauge.
We observe that at $\sigma(n, \bar{n}) = 0$ the 1-form $\Phi_1^{n, \bar{n}}(y)$ leads to a spacetime local expression since $R_1, W_1, \text{and } W_1$ \((5.17)-(5.19))$ are linear in $\tau^{-1}$. Therefore, $\Phi_2^{n, \bar{n}}(y|x) = D \Phi_1^{n, \bar{n}}(y|x)$ with some local $\Phi_1^{n, \bar{n}}(y|x)$. Thus, it is an “improvement” of the physical current 2-form $J(C^2)$ on the r.h.s. of \((2.13))$, which can be compensated by a local field redefinition of the (higher spin) gauge fields.

Again, the ambiguity related to the parameters $\sigma(n, \bar{n})$ is a manifestation of a non-trivial cohomology. We therefore conclude that $H^1(T^{n, \bar{n}}, D)$ is one-dimensional in each $(n, \bar{n})$ sector. Therefore for a given spin $s = 1 - \frac{1}{2}(n + \bar{n})$, $\dim H^1(T^s, D) = 2s + 1$, what is of course the dimension of the spin $s$ representation of the $d3$ Lorentz algebra $o(2, 1)$.

Thus we have shown that all local $D$-closed forms $\Phi_2^{n, \bar{n}}(y)$ with $n_\tau = -1$ are $D$-exact in the class of pseudolocal expansions. The physical conserved currents are described by $\Phi_2^{1,1-2s}(y)$ or, equivalently, $\Phi_2^{1-2s,1}(y)$. The $D$-closed forms $\Phi_2^{n, \bar{n}}(y)$ with $n, \bar{n} \leq 0$ are locally $D$-exact and therefore describe various “improvements”.

### 6 Dependence on $\lambda$

Generic $p$-forms given by \((3.15)-(3.18))$ depend on $\lambda$ via the expansions \((2.13))$. Such a formulation with $\lambda$-dependent $C(y)$ and $C'(y)$ was convenient for the study of cohomology since it allowed us to use $\lambda$-independent formulae for the mapping $D$ \((4.2), (4.4)-(4.10))$ and the solutions \((5.13), (5.17)-(5.19))$. In the flat limit $\lambda \to 0$ the expansion $C(y)$ becomes meaningless. To investigate what happens in this case one should use generating functions $\tilde{C}(y)$ \((2.13))$. In this variables, the solutions \((5.13), (5.17)-(5.19))$ acquire explicit dependence on $\lambda$. For example, introducing

$$\tilde{\Phi}_{1,2}(\tilde{C}, \tilde{C}'; y) = \lambda^{-2 -\frac{1}{4}\pi(C) -\frac{1}{4}\pi(C')}\Phi_{1,2}(C, C'; \sqrt{\lambda} y), \quad (6.1)$$

we obtain

$$\tilde{\Phi}_{1}(\tilde{C}, \tilde{C}'; y) = h_{\alpha\alpha} \frac{1}{(2\pi)} \int dz \int d\bar{z} \int \tau^{-2} d\tau \int d^2 q d^2 \bar{q} \exp \left\{ -\frac{1}{2\tau} (q, \bar{q}) \right\}$$

$$\times \tilde{C} \left[ \frac{1}{2i} (q + \bar{q}) - \frac{1}{2i} (z - \bar{z}) y, \psi \right| x] \tilde{C}' \left[ \frac{1}{2i} (q - \bar{q}) + \frac{1}{2} (z + \bar{z}) y, \psi \right| x]$$

$$\times \left\{ \tilde{R}_1(z, \bar{z}, \tau) y^\alpha q^\alpha + \frac{1}{2\tau z} \tilde{W}_1(z, \bar{z}, \tau) y^\alpha q^\alpha + \frac{1}{2\tau z} \tilde{Y}_1(z, \bar{z}, \tau) q^\alpha q^\alpha + \frac{1}{2\tau^2 \bar{z} z} \tilde{V}_1(z, \bar{z}, \tau) q^\alpha q^\alpha \right\} \quad (6.2)$$

with

$$\tilde{R}_1(z, \bar{z}, \tau) = R_1(z, \bar{z}, \lambda \tau),$$

$$\tilde{W}_1(z, \bar{z}, \tau) = \lambda^{-1} W_1(z, \bar{z}, \lambda \tau),$$

$$\tilde{W}_1'(z, \bar{z}, \tau) = \lambda^{-1} W_1'(z, \bar{z}, \lambda \tau),$$

$$\tilde{Y}_1(z, \bar{z}, \tau) = \lambda^{-2} Y_1(z, \bar{z}, \lambda \tau),$$

$$\tilde{Y}_1'(z, \bar{z}, \tau) = \lambda^{-2} Y_1'(z, \bar{z}, \lambda \tau),$$

$$\tilde{V}_1(z, \bar{z}, \tau) = \lambda^{-2} V_1(z, \bar{z}, \lambda \tau). \quad (6.3)$$
Therefore, in terms of $\tilde{R}, \tilde{W}, \ldots$ our solutions carry an inverse power of $\lambda$ together with each power of $\tau^{-1}$. Equivalently, every spacetime derivation carries a factor of $\lambda^{-1}$. Hence, a representation of physical current 2-forms $\Phi_2^s(y)$ as some differentials $D\Phi_1^s(y)$ as well as elements of the cohomology group $H^1(T,D)$ become meaningless in the flat limit $\lambda \to 0$. In terms of $\tilde{C}(y)$ the formulae (5.13), (5.18), (5.19) contain the combination $(1 \pm i\lambda^{-1}\tau^{-1})^{-1}$ to be expanded in powers of $\tau^{-1}$. Viewed as analytic functions, these formulae have a radius of convergence equal to $\lambda$, so when the AdS radius $\lambda^{-1}$ tends to infinity, the radius of convergence shrinks to zero.

Conclusion

In this paper, we construct local conserved currents of an arbitrary spin in AdS$_3$ built from scalar and spinor fields. It is shown that they can be treated as “improvements” within the class of infinite power expansions in higher derivatives. In other words, 2-forms $J$ dual to the physical conserved currents are shown to be exact in this class, $J = DU$. The 1-forms $U$ are constructed explicitly what allows us to write down nonlocal field redefinitions compensating matter sources in the equations of motion for the Chern-Simons gauge fields of all spins. The coefficients in the expansion of $U$ in derivatives of the matter fields contain negative powers of the cosmological constant (i.e. positive powers of the AdS radius) and therefore do not admit a flat limit. The existence of $U$ may be related to the holography in the AdS/CFT correspondence since it indicates that local current interactions in the AdS space are in a certain sense trivial and can, up to some surface terms, be compensated by a field redefinition.

Let us note that our analysis with two independent matter fields $C$ and $C'$ in (3.1) covers the case with matter fields belonging to nontrivial representations of the spin 1 Yang-Mills group in the extended systems considered in [9, 10].

To analyze the problem systematically for the currents of all spins we have proposed the formalism of generating functions suitable for the description of differential forms bilinear in the massless scalar and spinor fields. It is based on the “unfolded formulation” of the dynamical equations as certain covariant constancy conditions [11] and allows us to reformulate the problem in terms of a cohomology of some differential $D$ acting in the specific auxiliary spaces encoding the full information on the on–mass–shell matter fields.

Our main result is that the local conserved (i.e. $D$-closed) 2-forms of currents, which belong to a non-trivial cohomology within the class of local expansions, are $D$-exact in the class of pseudolocal expansions (i.e. infinite power series in higher derivatives). Interestingly enough, we have found that the cohomology group $H^1(T,D)$ is nontrivial, implying nonuniqueness of the solution for $U$ already observed in [10] for the case of spin 2. An interesting problems for the future are to find some group theoretical interpretation of the result that the dimension of $H^1(T^s,D)$ in a spin $s$ sector is equal to the dimension of spin $s$ representation of the $d3$ Lorentz algebra $o(2,1)$ and to analyze $H^n(T^s,D)$ for $n \neq 1$. Since in [10] it has been shown that there exists a pseudolocal field redefinition reducing the full nonlinear equations of motion to the free system we expect that $H^2(T,D) = 0$, but it is interesting to analyze the problem independently. In particular, this can shed some light on an appropriate definition of local functionals in the AdS space as some
subspace of the class of formal pseudolocal expansions.

**Note added:** After this paper had been accepted for publication, the interesting paper by D. Anselmi [14] appeared, which contains explicit expressions for higher spin currents in the flat spacetime of any dimension, thus generalizing some of the results of [17] and the present paper to arbitrary dimensions (in the flat background).

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**Appendix A. Solution at \( \bar{n} = 1 \).**

Consider the following differential equation related to (5.12) via \( x = i\tau, \; k = 2s - 1, \; \alpha = i\alpha_Y(n, \bar{n}) \; z^{1-2\tilde{s}}\), and \( Y = Y_1 \),

\[
(1 - x^2) \frac{d}{dx} Y = -(1 + x) Y + k(1 - x) Y + \frac{\alpha}{x} + P(x), \tag{A.1}
\]

where \( k \) and \( \alpha \) are some constants and \( P(x) \) is an arbitrary polynomial. We have to solve (A.1) in terms of formal series (5.5). The essential part of \( Y(x) \) contains negative powers of \( x \), i.e. \( Y(x) \) is defined modulo arbitrary polynomials.

Rewriting (A.1) as

\[
(1 + x)^{k+1}(1 - x)^2 \frac{d}{dx} \left[ (1 + x)^{-k} (1 - x)^{-1} Y \right] = \frac{\alpha}{x} + P(x), \tag{A.2}
\]

we solve it in the form

\[
Y = (1 + x)^k (1 - x) \int_c^x dt \left( 1 + t \right)^{-k-1} \left( 1 - t \right)^{-2} \left( \frac{\alpha}{t} + P(t) \right), \tag{A.3}
\]

with the polynomial \( P'(t) = P(t) - \alpha t^{-1} \left[ (1 + t)^{k+1} (1 - t)^2 - 1 \right] \). Using that

\[
\frac{2}{(1 + t)(1 - t)} = \frac{1}{(1 - t)} + \frac{1}{(1 + t)} \tag{A.4}
\]

one finds that \( P'(t) \) can give a nonpolynomial contribution to \( Y \) only if simple poles in \( (1 \pm t) \) survive in the integral (A.3). Equivalently one can set

\[
P'(t) = (1 + t)^k (1 - t) \left[ \beta (1 - t) + \gamma (1 + t) \right]. \tag{A.5}
\]

Therefore a generic solution of (A.1) is

\[
Y(x) = (1 - x)(1 + x)^k \left[ \alpha \ln x + \beta \ln (1 + x) + \gamma \ln (1 - x) \right] \mod \text{polynomials.} \tag{A.6}
\]
The restriction to the class (5.5) imposes one restriction on the parameters \( \beta \) and \( \gamma \) leading to the final result

\[
Y(x) = -\frac{\alpha}{2}(1-x)(1+x)^k \ln(1-x^{-2}) + \sigma (1-x)(1+x)^k \ln\left(\frac{1-x^{-1}}{1+x^{-1}}\right)
\]

(A.7)

with an arbitrary constant \( \sigma \). Note that the logarithms in (A.7) should be understood as

\[
\ln(1 + x^{-1}) = \infty \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} x^{-m}.
\]

(A.8)

Appendix B. Solution at \( n < 1, \bar{n} < 1 \).

Consider first the case with \( n, \bar{n} < 0 \). To solve the system (4.4)-(4.9) with \( R_2^P, W_2^P, \ldots \) (5.1) it is most convenient to gauge fix the quantities \( R_1, W_1, \ldots \) in the 1-form \( \Phi_1^{n,\bar{n}}(y) \) as follows,

\[
V_1 = Y_1 = \bar{Y}_1 = 0, \quad R_1 = \frac{r}{\tau},
\]

(B.1)

with some \( \tau \)-independent \( r \). Actually, at \( n \neq 1, \bar{n} \neq 1 \) one can always achieve (B.1) using the transformations (5.6)-(5.11). First, one gauges away \( Y_1 \) and \( \bar{Y}_1 \) with an appropriate choice of the parameters \( \xi_1 \) and \( \bar{\xi}_1 \). Taking the parameter \( \varepsilon(\tau) \) in (5.9) in the form

\[
\varepsilon(\tau) = \frac{1}{1 + \tau^2} \left[ E(\tau^{-1}) + \varepsilon_0 \tau \right],
\]

(B.2)

where \( E(\tau^{-1}) \) is some series in the inverse powers of \( \tau \), and \( \varepsilon_0 \) is \( \tau \)-independent, \( V_1 \) can be gauged away by an appropriate choice of \( E(\tau^{-1}) \). The parameter \( \varepsilon_0 \) remains arbitrary since its contribution to \( V_1 \) is proportional to \( \tau \) which is equivalent to zero. From (5.6) it follows that all terms in \( R_1 \) can be gauged away with a choice of \( \chi_2 \) except for the leading term \( \tau^{-1} \). The constant \( r \) in the term \( r\tau^{-1} \) in \( R_1 \) can be gauge fixed by using the ambiguity in \( \varepsilon_0 \), since the corresponding part of \( \varepsilon(\tau) \),

\[
\varepsilon_0 \frac{\tau}{1 + \tau^2} = \varepsilon_0 \sum_{k=0}^{\infty} (-1)^k \tau^{-(2k+1)},
\]

(B.3)

contains the term proportional to \( \tau^{-1} \). At this stage however it is convenient to keep \( r \) as an arbitrary parameter to be fixed later. Note that fixing \( \varepsilon_0 \) completes gauge fixing of the transformations (5.6)-(5.11).

As a result, the system of equations resulting from (4.4)-(4.9) takes the form

\[
\alpha_R(n, \bar{n}) \frac{z^n z^\bar{n}}{\tau} \quad = \quad -(n + \bar{n} - 2) \frac{r}{\tau} + \frac{i}{4}(1 + i\tau)\partial_\tau W_1 - \frac{i}{4}(1 - i\tau)\partial_\tau \bar{W}_1
\]

\[
+ \frac{1}{4}(W_1 + \bar{W}_1) - \frac{1}{2}(nW_1 + \bar{n}\bar{W}_1) - \partial_\tau \chi_2,
\]

(B.4)

\[
\alpha_W(n, \bar{n}) \frac{z^n z^\bar{n}}{\tau} \quad = \quad -\frac{i}{2}\partial_\tau [(1 + \tau^2)W_1] + \frac{3}{2}(1 + i\tau) W_1 + 2\bar{n} \frac{r}{\tau}
\]

\[
+ \frac{1}{2}(1 - i\tau)\bar{n}\bar{W}_1 - \frac{1}{2}(1 + i\tau)(\bar{n} - 1)W_1 - \partial_\tau \xi_2 + 2i\bar{n}\chi_2,
\]

(B.5)
\[ \alpha_W(n, \bar{n}) \frac{z^n \bar{z}^n}{\tau} = \frac{i}{2} \partial_{\tau} [(1 + \tau^2) W_1] + \frac{3}{2} (1 - i\tau) W_1 + 2n \frac{r}{\tau} + \frac{1}{2} (1 + i\tau) n W_1 - \frac{1}{2} (1 - i\tau) (n - 1) W_1 - \partial_{\tau} \bar{\xi}_2 - 2 i n \chi_2, \quad (B.6) \]

\[ \alpha_V(n, \bar{n}) \frac{z^n \bar{z}^n}{\tau} = \frac{1}{2} (1 + \tau^2) (n W_1 + \bar{n} \bar{W}_1) - i(n \xi_2 - \bar{n} \bar{\xi}_2), \quad (B.7) \]

\[ \alpha_Y(n, \bar{n}) \frac{z^n \bar{z}^n}{\tau} = \frac{1}{2} (1 + \tau^2) (\bar{n} - 1) W_1 + i(\bar{n} - 1) \xi_2, \quad (B.8) \]

\[ \alpha_{\bar{Y}}(n, \bar{n}) \frac{z^n \bar{z}^n}{\tau} = \frac{1}{2} (1 + \tau^2) (n - 1) \bar{W}_1 - i(n - 1) \bar{\xi}_2, \quad (B.9) \]

where all the equalities are treated modulo polynomials \([3.21]\) and the variables \(\chi_2, \xi_2,\) and \(\bar{\xi}_2\) account for the ambiguity modulo the Fierz transformations of \(R_2^D, W_2^D, \ldots\).

Introducing the new variables

\[ X = n W_1 - \bar{n} \bar{W}_1, \quad X^+ = n W_1 + \bar{n} \bar{W}_1, \quad (B.10) \]

and expressing the Fierz parameters \(\chi_2, \xi_2,\) and \(\bar{\xi}_2\) in terms of the rest variables, one reduces \([B.7]-[B.3]\) to

\[ (1 + \tau^2) X^+ = \frac{A}{\tau} + P_1(\tau), \quad (B.11) \]

\[ (1 + \tau^2) \frac{\partial}{\partial \tau} X = -2i X^+ - \frac{i(4n \bar{n} r - G)}{\tau} - \frac{B}{\tau^2} + P_2(\tau), \quad (B.12) \]

\[ 4n\bar{n} \frac{\alpha_R}{\tau} + 4n \bar{n}(n + \bar{n} - 2) \frac{r}{\tau} - [\tau(n - \bar{n}) + i(n + \bar{n} - 2)] \partial_{\tau} X + [i(n - \bar{n}) + \tau(n + \bar{n} - 2)] \partial_{\tau} X^+ + 2(n \bar{n} - 1) X^+ + \frac{iK}{\tau^2} + \frac{A - L}{\tau^3} = P_3(\tau), \quad (B.13) \]

where

\[ A = \left( \frac{n}{n - 1} \alpha_Y + \frac{\bar{n}}{\bar{n} - 1} \alpha_{\bar{Y}} + \alpha_Y \right) z^n \bar{z}^n, \]

\[ B = \left( \frac{n}{\bar{n} - 1} \alpha_Y - \frac{\bar{n}}{n - 1} \alpha_{\bar{Y}} \right) z^n \bar{z}^n, \]

\[ G = (n \alpha_W + \bar{n} \alpha_{\bar{W}}) z^n \bar{z}^n, \]

\[ K = (\bar{n} \alpha_W - n \alpha_{\bar{W}}) z^n \bar{z}^n, \]

\[ L = \left( \frac{n}{\bar{n} - 1} \alpha_Y + \frac{\bar{n}}{n - 1} \alpha_{\bar{Y}} - \alpha_Y \right) z^n \bar{z}^n, \quad (B.14) \]

and \(P_1(\tau), P_2(\tau),\) and \(P_3(\tau)\) are arbitrary polynomials.

Using that any polynomial \(P(\tau)\) can be rewritten as

\[ P(\tau) = a + b\tau + (1 + \tau^2)p(\tau) \quad (B.15) \]

with some polynomial \(p(\tau),\) one obtains the general solution of \([B.11]\),

\[ X^+(\tau) = \frac{A}{\tau} + \sigma_1^+ z^n \bar{z}^n \frac{1}{1 + \tau^2} + \sigma_2^+ z^n \bar{z}^n \frac{\tau}{1 + \tau^2} + p(\tau), \quad (B.16) \]
where \( \sigma_1^+ \) and \( \sigma_2^+ \) are arbitrary constants.

Inserting (B.10) into (B.12) and solving it for \( \partial_\tau X \) analogously to (B.11), we get

\[
\frac{\partial}{\partial \tau} X = -\frac{i [4n\bar{n} r - (G - 2A)]}{\tau (1 + \tau^2)} - \frac{B}{\tau^2} - 2i\sigma_1^+ z^n \bar{z}^n \frac{1}{1 + \tau^2} - 2i\sigma_2^+ \bar{z}^n \bar{z}^n \frac{\tau}{(1 + \tau^2)^2} + \sigma_1 z^n \bar{z}^n \frac{1}{1 + \tau^2} + \sigma_2 \bar{z}^n \bar{z}^n \frac{\tau}{1 + \tau^2},
\]

where \( \sigma_1 \) and \( \sigma_2 \) are arbitrary constants. Now it is convenient to set

\[
r = \frac{G - 2A}{4n\bar{n}}.
\]

Also one should set \( \sigma_2 = 0 \) in (B.17) since the term \( \tau (1 + \tau^2)^{-1} \) is not integrable in the form (5.3). As a result, we arrive at the differential equation

\[
\frac{\partial}{\partial \tau} X = -\frac{B}{\tau^2} - 2i\sigma_1^+ z^n \bar{z}^n \frac{1}{(1 + \tau^2)^2} - 2i\sigma_2^+ \bar{z}^n \bar{z}^n \frac{\tau}{(1 + \tau^2)^2} + \sigma_1 z^n \bar{z}^n \frac{1}{1 + \tau^2},
\]

and therefore, modulo polynomials,

\[
X(\tau) = B \frac{\tau}{\tau} - i\sigma_1^+ z^n \bar{z}^n \left[ \frac{\tau^{-1}}{1 + \tau^{-2}} + i \frac{1}{2} \ln \left( \frac{1 + i\tau^{-1}}{1 - i\tau^{-1}} \right) \right] + i\sigma_2^+ \bar{z}^n \bar{z}^n \frac{\tau^{-2}}{1 + \tau^{-2}} + \sigma_1 z^n \bar{z}^n \frac{1}{1 + \tau^{-2}}.
\]

The equation (B.13) is equivalent to the conditions

\[
(2n\bar{n} - n - \bar{n}) \sigma_1^+ - (n - \bar{n}) \sigma_1 = 0,
\]

\[
2(n\bar{n} - n - \bar{n} + 1) \sigma_1^+ + i(n - \bar{n}) \sigma_2^+ - i(n + \bar{n} - 2) \sigma_1 = 0
\]

on the parameters \( \sigma_1, \sigma_1^+ \) and \( \sigma_2^+ \) provided that the conditions (5.2)-(5.4) guaranteeing that \( \Phi_2^{n,\bar{n}}(y) \) is \( D \)-closed are satisfied.

Using (B.4), (B.10), (B.14), (B.16), (B.18), (B.20)-(B.22), one finds the general solution of the system (B.4)-(B.9) in the form (5.17)-(5.20), where \( \sigma = i\sigma_1 \).

Let us turn to the case with \( \bar{n} = 0 \), \( n < 0 \) (the case with \( n = 0 \), \( \bar{n} < 0 \) can be considered analogously). According to (B.20) the only parameters giving a non-zero contribution to \( \Phi_2^{n,\bar{n}}(y) \) are \( W_{1,2}, V_{1,2}, \) and \( Y_{1,2} \), so that the nontrivial equations in the system (B.4)-(B.9) are (B.5), (B.7), and (B.8). Analogously to the case considered above, one gauge fixes

\[
V_1 = Y_1 = 0
\]

and obtains the solution

\[
W_1(z, \bar{z}, \tau) = -\alpha_Y(n, 0) \frac{z^n}{\tau} + \sigma(n, 0) \frac{\tau^{-1}}{1 - i\tau^{-1}}.
\]

The rest case \( n, \bar{n} = 0 \) is trivial. Indeed, in this case the only contributing parameters are \( \alpha_Y(0, 0) \) and \( V_1 \). The parameter \( \alpha_Y(0, 0) \) vanish by the equation (B.7), while \( V_1 \) can be gauged away by the transformation (5.9).
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