Maneuver-based control of the 2-degrees of freedom underactuated manipulator in normal form coordinates

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In this contribution, we provide a constructive way to transform a generic Lagrangian mechanical control system into the well-known Byrnes–Isidori normal form. Then, we restrict ourselves to the underactuated two-joint-manipulator in the horizontal plane. That system fails Brocketts condition and thus achieving point-to-point control is a challenging task. The system is analyzed in normal form coordinates, which allows to design a subordinate sliding mode controller and identify three discrete symmetries. Using these preliminary steps, finally a maneuver-based control scheme is proposed for equilibrium transition. Thereby each maneuver corresponds to a suitable sliding surface defined in normal form coordinates.

Keywords: underactuated manipulator; equilibrium transition; Lagrangian Byrnes–Isidori normalform; discrete symmetry; sliding mode control; maneuver-based control

1. Introduction

Underactuated mechanical systems are an interesting and illustrative field of control theory (Fantoni & Lozano, 2001; Spong, 2000). Many different control approaches have been applied to such systems, see, for example, Graichen, Treuer, and Zeitz (2007), Glück, Eder, and Kugi (2013), Riachy, Orlov, Floquet, Santiesteban, and Richard (2008), and Knoll and Röbenack (2013). Among them, underactuated manipulators (possessing some unactuated joints) are an important subclass, see De Luca, Iannitti, Mattone, and Oriolo (2002) and the references therein. Possible applications include areas where the reduction in weight and/or cost is desired while tasks can still be fulfilled with fewer actuators, like in space robotics (Yesiloglu & Temeltas, 2010; Yoshida, 1997) or when considering simple pick-and-place mechanisms (Oriolo & Nakamura, 1991).

Motivated by pick-and-place tasks, our objective is the equilibrium transition of that system. Due to Oriolo and Nakamura (1991) it is well-known that this manipulator does not fulfill the so-called Brockett condition (Brockett, 1983), which is necessary for the existence of a continuous differentiable feedback that stabilizes an isolated equilibrium point.

Because of the combination of an easy model with challenging control properties this system was subject to many studies before. Some of them, e.g. Arai and Tachi (1991) and Mareczek, Buss, and Schmidt (1999), rely on the presence of a holding brake which, however, might be seen as some kind of actuator. Other approaches assume the presence of considerable static friction, either explicitly (Mahindrakar, Rao, & Banavar, 2006), or implicitly (Scherm & Heimann, 2000) by means of the numerical stability of the algorithm. Like in the present contribution, the model investigated in Nakamura, Suzuki, and Koinuma (1997) and De Luca, Mattone, and Oriolo (2000) is assumed to be frictionless. The two approaches both use periodic input signals leading to high actuator activity, a longer duration and path length of the transition in comparison to the presence of a brake or static friction.

This paper contains two main contributions: Firstly, basing on the partial feedback linearization of a general underactuated system, we introduce its transformation to the “Lagrangian Byrnes–Isidori normal form (LBINF)” and give a simple proof of its global existence in terms of a closed-form formula. This representation allows for
any underactuated system to separate the effect of the input from system-inherent dynamics ("drift"), such that the corresponding vector fields are always orthogonal. In Olfati-Saber (2001, Section 3.7) a related but different approach has been proposed earlier, to transform an underactuated system to Byrnes–Isidori normal form, see also Choukchou-Braham, Cherki, Djemai, and Busawon (2014, Section 4.2). However, in that approach the existence of the transformation depends on an involutivity condition and the calculation depends on the solution of partial differential equations. Therefore, in general it is not possible to express that transformation in closed form, even if it exists. Furthermore, due to the simple structure of our transformation, some of the new state variables can be interpreted as quasi-velocities, see, for example, Cameron and Book (1997). This facilitates the controller design in the following.

The second contribution is, as an extension to Knoll and Röbenack (2010, 2011a), the presentation of a novel approach for the point-to-point control of the underactuated planar 2-DOF manipulator. In contrast to the existing approaches, it does neither rely on a brake or static friction nor on the application of periodic inputs.

This paper is organized as follows: In Section 2, we establish the transformation of a general underactuated mechanical system to the LBINF. Next, in Section 3, the model of the 2-DOF manipulator is stated and transformed into that representation. Section 4 is devoted to the analysis of the manipulator system in the new coordinates. Thereby, a generic subordinate sliding mode control is constructed and three discrete symmetry-mappings are identified, which substantially simplify the control design. Finally, in Section 5, the equilibrium transition is performed via successive execution of suitable maneuvers, each one corresponding to a special sliding surface defined in coordinates of the normal form.

The application of the control strategy proposed in Section 5 depends on a number of details not worthwhile to be included in the written text. Each single detail might be considered as a straightforward consequence of the outlined ideas. However, to facilitate the ability to reproduce and critically examine our results we provide the full source code (using the Python programming language) for the related simulations, see Knoll (2014a). This practice follows the argumentation of Ince, Hatton, and Graham-Cumming (2012).

2. Input–output linearization and Byrnes–Isidori normal form

2.1. Input–output linearization of underactuated systems

In this section, we consider a holonomic mechanical system with $n$ DOF. Its equations of motion, typically obtained from the Lagrangian formalism, read

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{B}(\mathbf{q}) \tau. \quad (1)$$

Thereby, $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ denote the generalized coordinates, velocities and accelerations, respectively, $\mathbf{M}(\mathbf{q})$ is the (positive definite) mass matrix, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ describes the action of centrifugal and Coriolis forces and $\mathbf{K}$ collects the conservative and dissipative forces resulting from potential energy changes and friction. The $m$-dimensional vector $\tau$ contains the generalized forces from the actuators which are assigned to the scalar differential equations of (1) via the $n \times m$-matrix $\mathbf{B}(\mathbf{q})$. Obviously, the quantities $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are time-dependent elements of $\mathbb{R}^n$. However for better readability the time argument is not explicitly written.

We assume that the system is underactuated, that is, $m < n$, and that the actuators are not redundant. Then, $\mathbf{B}(\mathbf{q})$ has full column rank for all $\mathbf{q} \in \mathbb{R}^m$ which implies the existence of $\mathbf{T}(\mathbf{q}) \in \mathbb{R}^{m \times n}$, such that $\mathbf{T}(\mathbf{q}) \mathbf{B}(\mathbf{q}) = (\mathbf{I}_m, \mathbf{0})^T$. By left-multiplying Equation (1) with $\mathbf{T}(\mathbf{q})$ and performing a suitable transformation of the coordinates, the system (1) can be rewritten as

$$
\begin{pmatrix}
\mathbf{M}_{11}(\mathbf{q}) & \mathbf{M}_{12}(\mathbf{q}) \\
\mathbf{M}_{12}^T(\mathbf{q}) & \mathbf{M}_{22}(\mathbf{q})
\end{pmatrix}
\begin{pmatrix}
\ddot{\mathbf{q}}_1 \\
\ddot{\mathbf{q}}_2
\end{pmatrix}
+ 
\begin{pmatrix}
\mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}}) \\
\mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}})
\end{pmatrix}
+ 
\begin{pmatrix}
\mathbf{K}_1(\mathbf{q}, \dot{\mathbf{q}}) \\
\mathbf{K}_2(\mathbf{q}, \dot{\mathbf{q}})
\end{pmatrix}
= 
\begin{pmatrix}
\tau_1 \\
\mathbf{0}
\end{pmatrix}.
\quad (2)
$$

For convenience, we also denote the new generalized coordinates $\mathbf{q} = (\mathbf{q}_1^T, \mathbf{q}_2^T)^T$. In this representation, the whole system splits up into an $m$-dimensionally fully actuated subsystem with joint coordinates $\mathbf{q}_1$ and an $(n-m)$-dimensional subsystem without direct actuation.

For facilitation of the analysis and the controller design often an input–output linearization (also called “partial feedback linearization”) of Equation (2) is performed (see, e.g. De Luca et al., 2002; Sastry, 1999) by applying the nonlinear static feedback

$$
\tau_1 = (\mathbf{M}_{11}(\mathbf{q}) - \mathbf{M}_{12}(\mathbf{q}) \mathbf{M}_{22}^{-1}(\mathbf{q}) \mathbf{M}_{12}^T(\mathbf{q})) \mathbf{a} \\
- \mathbf{M}_{12}(\mathbf{q}) \mathbf{M}_{22}^{-1}(\mathbf{q}) (\mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_2(\mathbf{q}, \dot{\mathbf{q}})) \\
+ \mathbf{C}_1(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_1(\mathbf{q}, \dot{\mathbf{q}}). \quad (3)
$$

The resulting partial linearized system reads

$$
\ddot{\mathbf{q}}_1 = \mathbf{a}, \quad (4a)
\ddot{\mathbf{q}}_2 = -\mathbf{M}_{22}^{-1}(\mathbf{q}) (\mathbf{C}_2(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_2(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{M}_{12}^T(\mathbf{q}) \mathbf{a}). \quad (4b)
$$

According to Equation (4a), the new input $\mathbf{a}$ corresponds to the generalized accelerations of the actuated coordinates. In other words, the feedback (3) can be interpreted as inner control loop for the actuated joints.
In control theory, the input affine state-space representation of a system
\[ \dot{x} = f(x) + g(x)u \] (5)
is very common. In this representation, \( x \) denotes the time-dependent state taking values in \( \mathbb{R}^N \) (or an open subset thereof), \( f \) is the so-called drift vector field and \( g \) is a state-dependent \( N \times m \) matrix assigning the components of the \( m \)-dimensional input \( u \) to the \( N \) differential equations in Equation (5). Clearly, by choosing the state components
\[ x = (q_1^T, q_1^T, q_2^T, q_2^T)^T = : (x_1^T, x_2^T, x_3^T, x_4^T)^T \] (6)
and introducing \( n \) definitional equations of the form \( \dot{x}_i = x_i \) the system of \( n \) second-order ODEs (4) can be rewritten in the form Equation (5) with \( N = 2n \) and \( u = a \):
\[ \begin{align*}
\dot{x}_1 & = x_2, \\
\dot{x}_2 & = a, \\
\dot{x}_3 & = x_4, \\
\dot{x}_4 & = -\tilde{M}_{22}^{-1}(x_1, x_3)(\tilde{C}_2(x) + \tilde{K}_2(x)) + \tilde{M}_{12}^{-1}(x_1, x_3)a.
\end{align*} \] (7d)
The bar over the quantities indicates the reordering of the arguments to take the definition of the state (6) into account.

2.2. Byrnes–Isidori normal form
Equation (7d) shows that except in the special case where \( \tilde{M}_{12}(x_1, x_3) \equiv 0 \) the accelerations of the non-actuated joints are also influenced by \( a \). In terms of the input–output normal form Equations (7c), (7d) represent the internal dynamics. As we will see in Sections 4.1 and 5 system analysis and control design are facilitated by the introduction of new coordinates, say \( z := (z_1, z_2, z_3, z_4) \), along with a state transformation which alters the internal dynamics such that it is not directly influenced by the input.

This corresponds to a transformation to the Byrnes–Isidori normal form (Isidori, 1995), which in general refers to a decomposition of the system into mere integrator chains and an internal dynamics which does not depend on the input. In the scope of underactuated mechanical systems, we can restrict the generality and formulate an adapted definition:

**Definition 1** A dynamical system of the form (5) is said to be in LBINF, if \( f \) and \( g \) are compatible with the representation
\[ \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ 0 \\ \ell_1(z) \\ \ell_1(z) \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \\ 0 \\ 0 \end{pmatrix} a. \] (8)

**Theorem 1** Every mechanical system given in the form (1) can be transformed to the form (8) by a globally invertible change of state coordinates.

**Proof** The equivalence between Equations (1) and (7) is straightforward.\(^2\) Now, the LBINF coordinates are given by
\[ z_i := x_i, \quad i = 1, 2, 3 \] (9a)
and
\[ z_4 := x_4 - \tilde{M}_{22}^{-1}(z_1, z_3)\tilde{M}_{12}^T(z_1, z_3)x_2. \] (9b)
The first three components of the transformation are identical and the last one, given by Equation (9b) can be inverted globally as well as
\[ x_4 = z_4 + \tilde{M}_{22}^{-1}(z_1, z_3)\tilde{M}_{12}^T(z_1, z_3)x_2. \] (10)

To complete the proof, we notice that in the expression for the time derivative of \( z_4 \) the coefficients of \( a \) nullify each other:
\[ \dot{z}_4 = -\tilde{M}_{22}^{-1}(\tilde{C}_2 + \tilde{K}_2) - \tilde{M}_{22}^{-1}\tilde{M}_{12}a \]
\[ + \tilde{M}_{22}^{-1}\tilde{M}_{12}^T a + \left( \frac{d}{dt} \tilde{M}_{22}^{-1}\tilde{M}_{12}^T \right) z_2 \]
\[ = -\tilde{M}_{22}^{-1}(\tilde{C}_2 + \tilde{K}_2) + \left( \frac{d}{dt} \tilde{M}_{22}^{-1}\tilde{M}_{12}^T \right) z_2. \] (11)

**Remark 1** The vector \( z_4 \) is a weighted sum of the actuated and non-actuated velocities and can be interpreted as a quasi-velocity (Cameron & Book, 1997). It obviously has the same physical dimension like \( x_4 \) (\( = q_4 \)). In states with \( x_3 = 0 \), we have \( z_4 = x_4 \), which is interesting for equilibrium transitions (see Section 4.1, Definition 2).

**Remark 2** In contrast to the special Byrnes–Isidori normal form proposed earlier in Olfati-Saber (2001, Section 3.7), here no additional constraints regarding involutivity have to be fulfilled. The transformation to the normal form always exists and is given in closed form by Equation (9). Note that, for a general multi-input system (5) the Byrnes–Isidori normal form exists only if the distribution spanned by the input vector fields is involutive, cf. (Isidori, 1995, Proposition 5.1.2). However, for the mechanical system (7), the input vector fields depend only on \( x_1, x_2 \) but are zero in the first and third block (7a) and (7c), respectively. This structure implies that all Lie brackets between the input vector fields of Equation (7) are zero vector fields, that is, the associated distribution is involutive.

The LBINF representation has two main advantages: The system input \( a \) only occurs in the acceleration-equations of the actuated joints and can therefore be...
eliminated by interpreting the velocities as inputs themselves. This might be reasonable where a subordinate speed control is implemented in order to handle backlash and friction. Additionally, the systems motion through the state space is easier to understand and thus to control which will be illustrated in Section 4.1 and the following ones.

3. The underactuated manipulator

3.1. System description

From now on, we consider the underactuated two-degree-of-freedom horizontal manipulator with a friction-free non-actuated second joint, cf. Figure 1.

The configuration space for this system is considered to be $\mathbb{S}_1 \times \mathbb{S}_1$, that is, for the two angles the value of $2\pi$ is identified with the value 0. For the actual calculations this assumption does not matter, but it simplifies the transition scheme because any desired joint angle can be reached in both rotating directions.

By means of the Lagrangian formalism the equations of motions are obtained in the form of Equation (2). A subsequent partial linearization leads to the model

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\kappa x_2^2 \sin x_3 - (1 + \kappa \cos x_3)a,
\end{align*}
\]

where the only remaining parameter $\kappa$ is dimensionless and describes the distribution of mass on the second arm. Like many other mechanical systems the manipulator model (12) possesses time reversal symmetry (Knoll & Röbenack, 2011a) which will be used later for simplifying the trajectory planning, cf. Section 4.3.

The Jacobian linearization of this system is not controllable and, moreover, the system violates Brockett necessary conditions (Brockett, 1983; Oriolo & Nakamura, 1991). In other words, there exists no continuous differentiable feedback of the state, that is, no continuous differentiable and static control algorithm, which stabilizes an isolated equilibrium point of the system.$^3$ Therefore, despite the simple model (12), the control of this system and especially equilibrium transitions are a challenging problem.

As in the general case, discussed in the previous section, the input $a$ affects both the double integrator subsystem for the active joint and the second subsystem for the passive joint. The transformation to the LBINF and its inverse are given by

\[
\begin{align*}
z_i := x_i & \quad \forall i \in \{1, 2, 3\}, \\
z_4 := x_4 + (1 + \kappa \cos x_3)x_2
\end{align*}
\]

and

\[
x_4 = z_4 - (1 + \kappa \cos x_3)z_2.
\]

This implies

\[
\dot{z}_4 = -\kappa x_2 \sin x_3 (x_2 + x_4) = -\kappa z_2 \sin z_3 (z_4 - \kappa \cos z_3z_2)
\]

and thus, the dynamics in LBINF read

\[
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{pmatrix} = \begin{pmatrix}
z_2 \\
0 \\
z_4 - (1 + \kappa \cos z_3)z_2 \\
-\kappa z_2 \sin z_3 (z_4 - \kappa \cos z_3z_2)
\end{pmatrix} + \begin{pmatrix}0 \\ 0 \\ 0 \\ g(z)\end{pmatrix} f(z).
\]

4. System analysis in normal form

4.1. Projection to the $z_2$-$z_4$-plane

The model representation (15) has the advantage to separate the influence of the input $a$ and the drift. In other words, the two vector fields $f$ and $g$ are always orthogonal to each other. Especially interesting is the projection into the two-dimensional subspace spanned by the second and fourth unit coordinate vector, that is, the $z_2$-$z_4$-plane:

\[
\begin{pmatrix}
\dot{z}_2 \\
\dot{z}_4
\end{pmatrix} = \begin{pmatrix}
0 \\
-\kappa z_2 \sin z_3 (z_4 - \kappa \cos z_3z_2)
\end{pmatrix} + \begin{pmatrix}0 \\ g(z)\end{pmatrix} f(z),
\]

where $z_3$ can be interpreted as a parameter rather than a component of the state. Obviously, $\bar{g}$ and $\bar{f}$ are parallel to the $z_2$ and $z_4$ axes, respectively. From a topological point of view, only considering the direction of the drift vector field $f$, the projected system is similar to a double integrator, see Knoll and Röbenack (2011a) for details. In other words, except for some singular $z_3$-values, in one of the four quadrants the drift always points towards the abscissa (dark gray) and in another one it points away from it (light gray), cf. Figure 2. The remaining two quadrants are crossed by the straight line

\[
z_4 = \kappa \cos z_3z_2
\]

which, together with the $z_4$-axis, indicates where the sign of the drift changes.

This situation makes the system dynamics much more predictable and hence allows the construction of a maneuver-based equilibrium transition.
These maneuvers will be performed by stabilizing the system (15).

**Theorem 2.** The sliding surface given by \( \Phi(z) = 0 \) is asymptotically stabilized by the feedback law

\[ v = \dot{z}_2 - \gamma_1 \text{sign}(\Phi(z)) \mid \Phi(z) \]

with \( \gamma_1 > 0 \) and \( \gamma_2 \in (0, 1) \).

**Proof.** Attractiveness and invariance of the set \( \{z \in \mathbb{R}^4 \mid \Phi(z) = 0\} \) is shown by the Lyapunov argument

\[
\frac{1}{2} \frac{d}{dt} \Phi^2(z) = \Phi(z) \dot{\Phi}(z) = \Phi(z)(\dot{z}_2 - \dot{z}_4) = \Phi(z)(v - \dot{z}_4 f_4(z)) = -\gamma_1 |\Phi(z)|^{1+\gamma_2} \leq 0,
\]

cf. (Slotine & Li, 1991, Chapter 7).

**Remark 3.** If the system is in sliding regime, that is, if \( \Phi(z) = 0 \), then it is governed by the so-called reduced dynamics

\[
\begin{align*}
\dot{z}_3 &= z_4 - (1 + \kappa \cos z_3) \varphi(z_4) =: f_3(z_3, z_4) \quad (22a) \\
\dot{z}_4 &= \kappa \varphi(z_4) \sin z_3 (\kappa \varphi(z_4) \cos z_3 - z_4) =: f_4(z_3, z_4), \quad (22b)
\end{align*}
\]

and therefore behaves like an autonomous system. On the other hand, the particular choice of \( \varphi(\cdot) \) is a remaining degree of freedom to reach the control objective of the respective maneuver.

### 4.3. Symmetry properties

In this section, we identify some properties of the system which allow the direct construction of new solution trajectories from given solution trajectories. As mentioned above, the manipulator possesses time reversal symmetry because it is a conservative mechanical system (see Knoll & Röbenack, 2011b for further explanation). Using the reduced dynamics this is expressed by the observation

\[
\begin{align*}
\bar{f}_3(-z_3, z_4) &= \bar{f}_3(z_3, z_4), \quad (23a) \\
\bar{f}_4(-z_3, z_4) &= -\bar{f}_4(z_3, z_4). \quad (23b)
\end{align*}
\]

**Symmetry Property 1 (time reversal symmetry).** Suppose \( t \mapsto (z_3(t), z_4(t)) \) is a solution of Equation (22) for \( t \in [0, \tau] \). Then, another solution for this interval is given by

\[
\begin{align*}
\dot{z}_3(t) &= -z_3(\tau - t), \quad (24a) \\
\dot{z}_4(t) &= z_4(\tau - t). \quad (24b)
\end{align*}
\]

**Proof.** We differentiate \((\dot{z}_3, \dot{z}_4)\) w.r.t. time, apply the symmetry (23) to change the sign of the respective first arguments and then substitute using Equation (24):

\[
\begin{align*}
\dot{z}_3(t) &= \bar{f}_3(-z_3(\tau - t), z_4(\tau - t)), \quad (25a) \\
\dot{z}_4(t) &= \bar{f}_4(-z_3(\tau - t), z_4(\tau - t)). \quad (25b)
\end{align*}
\]

The result is the system dynamics of Equation (22) but with the newly constructed solution.
This property allows to consider the submaneuvers of leaving and reaching an equilibrium point as essentially the same problem. The only difference is the sign of the time variable. Furthermore, there is a related property associated with Equation (23).

**Symmetry Property 2 (recurrence property).** Let \( t \mapsto (z_3(t), z_4(t)) \) be a solution of Equation (22) for \( t \in [0, \tau] \) and suppose that \( z_3(\tau) = k\pi \) with \( k \in \mathbb{Z} \). Then, another solution for the interval \( [\tau, 2\tau] \) is given by

\[
\hat{z}_3(t) := 2z_3(\tau) - z_3(2\tau - t), \\
\hat{z}_4(t) := z_4(2\tau - t).
\]

The proof of this property follows the same lines as the previous one and is therefore omitted. As a consequence of the recurrence property, in general there are periodic solutions during the sliding regime. The only reasons for non-periodic solutions are: (1) the system reaches an equilibrium, (2) the sliding regime is interrupted, for example, by a change of the switching curve, or (3) the solution ceases to exist due to final escape time. However, the latter case can be excluded if \(|\varphi(\cdot)|\) is bounded.  \(^4\)

Now, we additionally assume point symmetry for the switching curve, that is, \( \varphi(-z_4) = -\varphi(z_4) \), from which immediately follows

\[
\tilde{f}_3(-z_3, -z_4) = -\tilde{f}_3(z_3, z_4), \\
\tilde{f}_4(-z_3, -z_4) = -\tilde{f}_4(z_3, z_4).
\]

**Symmetry Property 3 (quadrant equivalence).** Let \( t \mapsto (z_3(t), z_4(t)) \) be a solution of Equation (22) for \( t \in [0, \tau] \) and suppose \( \varphi(\cdot) \) to be point symmetric. Then, another solution for \( t \in [0, \tau] \) is given by

\[
\tilde{z}_3(t) := -z_3(t), \\
\tilde{z}_4(t) := -z_4(t).
\]

Again, the proof is similar to the argument used in the case of the time reversal symmetry and therefore omitted. The obvious consequence from the quadrant equivalence property is, that submaneuvers (or switching curves) only have to be planned for the first and second quadrant of the \( z_3-z_4 \)-plane and then can be trivially adapted for the third and fourth quadrant, respectively.

### 4.4. Parking regime

Clearly, all equilibrium points of Equation (15) are located at the origin of the \( z_2-z_4 \)-plane. The projected transition-trajectory must thus leave this point and reach this point somehow. Due to the loss of controllability near the equilibria, actually reaching such a state is quite hard. Considering the drift, this difficulty is reflected by the vanishing of the drift on the \( z_4 \)-axis.

**Definition 2** For system (15) all states with \( z_2 = 0 \) are called parking regime.

This denomination results from the fact that whenever the active joint stands still (for finite time), the only non-vanishing time derivative in Equation (15) is \( \dot{z}_3 = \text{const.} \). In the \( z_2-z_4 \)-projection the state then remains unchanged, that is, the projected system is “parking”. However, the drift conditions in the plane of course do change with \( z_3 \). For the motion planning this property allows, simply to wait in the parking regime until suitable drift conditions for the subsequent maneuver are reached.

Another consequence of the investigation of the projected drift vector field is that the tangent of any solution trajectory of Equation (15) containing a parking regime must be parallel to the \( z_2 \)-axis, where \( z_2 = 0 \) holds. Since equilibria also fulfill Definition 2, it is clear that trajectories leaving or reaching the origin must have the tangent line \( z_4 = 0 \), that is, the \( z_2 \)-axis.

### 5. Equilibrium transition via maneuver-based control

#### 5.1. Maneuver overview

We consider a desired transition between the initial state \( z^* \) and the final state \( z^\dagger \), both of which are assumed to be equilibrium points. The transition between these states can be split up into several maneuvers which are connected by time periods where the system is in the parking regime, that is, we have \( z_2(t) = 0 \) and thus \( z_3(t) = z_4(t) = \pm z_4^p \). Therefore, any desired drift condition can be achieved simply by waiting. The maximal waiting time is given by \( 2\pi/z_4^p \) and therefore \( z_4^p \) should be chosen sufficiently large. For the sake of simplicity, we consider \( |z_2^p| \) to be fixed for all occurring parking regimes.

To any of the maneuvers we can associate a curve in the \( z_3-z_4 \)-plane, which can be stabilized by sliding mode control. Obviously, the first maneuver (“A”) has to start in the initial equilibrium \( z^* \) and it ends on the \( z_4 \)-axis in the parking regime. Conversely, the last maneuver (“D”) has to reach the final equilibrium \( z^\dagger \) and it starts on the \( z_4 \)-axis. As will be shown in Section 4.3, maneuvers A and D can be regarded as equivalent due to time reversal symmetry.

For the simplest combinations of initial and final states, these two maneuvers already suffice for equilibrium transition. In general, however, two more maneuvers are necessary. Maneuver B serves to change the sign of \( z_3 \). In other words, it transfers the system from a parking regime with \( z_4 = z_4^p \) to another parking regime with \( z_4 = -z_4^p \), or vice versa. The objective of maneuver C is to adapt \( z_1 \) such that, after the final maneuver D, the active joint is in its desired position. Thereby we exploit the recurrence property, and,
additionally, the fact that the coordinate $z_1$ is cyclic, that is, the dynamics of the system is independent of its value. Figure 3 shows how the switching curves of these four maneuvers might look like.

The further sections require addressing the components of the state at different stages of the transition. To this end, we introduce the following notation: The letters “A”, “B”, “C”, and “D” used as head-index denote the corresponding maneuver, the letter “p” indicates the parking regime and the symbols $*$ and $\dagger$ stand for the beginning and the end of a maneuver, respectively. For example, $z_3^{A}$ is the angle of the second joint in the start equilibrium, $z_4^{A}$ is the $z_4$ value in the parking regime associated with maneuver A and $z_4^{C}$ is the angle of the first joint after the completion of maneuver C. The head index “eq” indicates the equilibrium value of a quantity – either at the start or end of the transition.

5.2. Leaving and reaching equilibrium points

In this section, we consider the transfer of the system from parking regime to an equilibrium (maneuver D), and its time reversal counterpart (maneuver A). Thereby, depending on the $z_3$ value of the concerned equilibrium, two cases can occur. Firstly, we discuss the case $z_3^{eq} \in (\pi/2, \pi) \cup (\pi, 3\pi/2)$, with the maneuvers D1 and A1. After that, we extend the results to $z_3^{eq} \in (0, \pi/2) \cup (3\pi/2, 2\pi)$ for the construction of maneuvers A2 and D2.

From the investigation of the drift in Section 4.1, it becomes obvious that any projected trajectory containing a point with $z_3 = 0$ must have a tangent in this point which is parallel to the $z_2$-axis. Clearly, this also holds true for the switching curve (19). In other words: $\phi(\cdot)$ must have infinite slope at its roots. This property is complied by power functions with exponents smaller then one. From Knoll and Röbenack (2011a), we know that for $z_2, z_4 > 0$ and $z_3 \in (\pi/2, \pi)$.

$$\varphi(z_4):= \begin{cases} (\mu \beta |z_4|)^{1/\beta} & \text{if } \frac{z_4^p - z_4}{z_4^p} > \frac{1}{2}, \\ (\mu \beta |z_4 - z_4|)^{1/\beta} & \text{else} \end{cases}$$ (29)

with $2 < \beta < 3$ is a suitable choice for maneuver D1 in the first quadrant.$^5$

Of course, the state $z_3$ changes during the maneuver which causes two issues: Firstly, the drift conditions change during the execution and secondly, the final value of $z_3$ depends on the $z_3$-start value (of maneuver D1) and this dependency cannot be expressed in closed form. However, for the motion planning a closed-form expression is not necessary. Given the desired final value $z_3^f$, the start value for maneuver D1 can be obtained by backward integration, that is, by stabilizing the system towards the equilibrium. If, however, $z_3$ leaves the interval $(\pi/2, \pi)$ the $z_2-z_4$-projection of the system dynamics are no longer topologically equivalent to the vector field depicted in Figure 2(b). In other words, if the drift changes its sign, the origin cannot be reached. From Equation (15) we extract

$$\dot{z}_3 = z_4 - (1 + \kappa \cos z_3)z_2$$ (30)

and hence, all states with $\dot{z}_3 = 0$ must lay on the straight line

$$z_4 = (1 + \kappa \cos z_3)z_2.$$ (31)

For the sake of simplicity, we assume $\kappa < 1$. This so-called “strong inertial coupling” (Spong, 1998) can be reached for any combination of arm lengths by means of suitable constructional measures (Knoll, Leist, & Röbenack, 2011). Then, the straight line always passes from the third to the first quadrant of the $z_2-z_4$-plane. This means that, the switching curve (29) crosses this line somewhere in the first quadrant, and thus $z_3$ increases at the beginning of the
maneuver and decreases at its end. From this qualitative consideration, we can suspect that the overall change of $z_3$ during maneuver D1 is “relatively small” and the construction (29) hence is a good choice for the sliding surface. This conjecture is confirmed by simulation experiments, see Figure 4.

By applying the quadrant equivalence (Symmetry Property 3) and identifying $z_3 \in (-\pi, -\pi/2)$ with $z_3 \in (\pi, \frac{3}{2}\pi)$, this approach for maneuver D1 can be easily adapted for desired final conditions with $z_3^{\dagger} \in (\pi, \frac{3}{2}\pi)$. From Figure 2(c) follows that the switching line then lies in the third quadrant.

Due to time reversal symmetry it is clear, that (with slight adaption) the results for the deceleration maneuver also hold true for maneuver A1, that is, to leave the start equilibrium and reach a parking regime. In particular, the switching line through the first quadrant ($z_{p4}^b > 0$), which is used to reach equilibrium states with $z_3^{\dagger} \in (\pi/2, \pi)$, also serves to leave equilibria with $z_3^e \in (\pi, \frac{3}{2}\pi)$, while the switching line in the third quadrant (same curve mirrored along both coordinate axes, hence $z_{p4}^b < 0$) can either be used for breaking with $z_3^e \in (\pi/2, \pi)$ or accelerating with $z_3^e \in (\pi, \frac{3}{2}\pi)$.

However, for the remaining two intervals $(0, \pi/2)$ and $(\frac{3}{2}\pi, 2\pi)$ the situation is different. In these cases, the double-integrator-like drift conditions lie in the second and fourth quadrant, see Figure 2(a) and 2(d). From the positive slope of the line (31) implicitly follows that in these quadrants the absolute value of $\dot{z}_3$ is relatively high which causes the drift to change its sign shortly after the maneuver start and therefore inhibits the same kind of maneuver as in the cases above.

The construction of a maneuver-pattern, which incorporates this sign change of the drift, is illustrated best by means of an acceleration-maneuver (“A2”) for $z_3^e \in (\frac{3}{2}\pi, 2\pi)$ (fourth quadrant). The generalizations to the opposite time direction and the other quadrant can then be performed by the Symmetry properties 1 and 3.

The maneuver A2 is divided into two phases, each one associated with one value $^8$ of sign($\dot{z}_4$). In each phase, the system is stabilized onto an suitable sliding surface, see Figures 5 and 6 for an overview and the simulation source code (Knoll, 2014a) for details.

Together the maneuvers A1, A2 and D1, D2 can be used to leave or reach, respectively, almost any equilibrium point. However, the values $z_3 = (k/2)\pi$, $k \in \mathbb{Z}$ must be excluded due to the singular drift conditions in these cases. Additionally, $z_3^e$ values close to these singularities
might cause too long maneuver durations, see Figures 4 and 6.

5.3. Transition of $z_4$ (Maneuver B)

When leaving or reaching an equilibrium, due to the drift conditions (see Figure 2) the equilibrium-angle of the second joint determines the sign of $z_4$ in the associated parking regime. If the desired initial and final states are given such that the $z_4^0$-values, associated to the appropriate maneuvers A and D, do not match, an additional maneuver (“B”) has to perform the transfer $z_4^{A} = \pm |z_4^0| \rightarrow z_4^{D} = \mp |z_4^0|$. As stated above, during parking regime $z_4$ is equivalent to $z_3$, that is, to the angular velocity of the second joint. Thus, the task of maneuver B is the inversion of the rotating direction of that joint.

Although there naturally exists an infinite variety of possible motion patterns for this goal, in the present context it is easily achieved by the combination of maneuvers D1 and A1, described above. With this choice and the assumptions $z_4^A < 0$ and $z_4^D > 0$, we have the following situation: The system is in parking regime before maneuver B begins, which makes it possible to wait until $z_4 = z_4^B \in (\pi, \frac{3}{2}\pi)$. From Figure 2 it is obvious, that the third quadrant can be used for D1-breaking. Furthermore, we can choose a suitable value for $z_4^B$ such that the equilibrium reached by this (intermediate) breaking maneuver has the same $z_4$-value. This is expressed in Figure 4(a) by the intersection (marked with a star) of the simulated curve with the bisecting line of the first quadrant (dash-dotted). After reaching the $z_2$–$z_4$-origin, the manipulator is immediately re-accelerated by maneuver A1 (as second part of velocity B) in the first quadrant. Due to time reversal symmetry this motion is the exact inversion of the first part. Consequently, the $z_3$ value at the beginning and the end are identical as well. Finally, the desired parking regime with $z_4^{D_B}$ is reached with $z_4^{B}\ast \leftarrow z_4^B$.

5.4. Transition of $z_1$ (Maneuver C)

The last missing design step for a transition from almost any desired equilibrium to almost any other is the proper adaption of the actuated joint angle $z_1$, which is the task of maneuver C. As for the previously treated maneuvers we initially assume a certain situation (here: $z_4^B > 0$), and later extend the results by applying the symmetry properties.

For $z_4^D > 0$ Figure 2 implies that the subsequent maneuver D takes place in the first quadrant, thus we have

$$\Delta z_1^D := \int_{t_0}^{t_1} z_2(t) \, dt > 0. \quad (32)$$

That value is known a priori and only depends on $z_3^C$ because $z_1$ is cyclic.

![Figure 8. Value of $\Delta z_1^C$ (displacement of active joint) in dependency of the $z_3$-start value of maneuver C (obtained by numerical simulation). If $z_3^C < \pi$ according to Figure 2 the drift in the second quadrant points upwards, therefore the upper branch of the sliding surface (cf. (35) and Figure 7(a)) is active. For $z_3^C > \pi$, the drift points downwards and thus the lower branch is used. For motion planning this mapping can be inverted piecewise to obtain $z_3^C$ for a given $\Delta z_1^C$.](image-url)
fact, there are two possible solutions for $z_1^*$ corresponding to the two branches of the sliding surface (35). From these, the “nearest” one (considering the $S_1$-Topology) should be chosen, to minimize the waiting time in parking regime. If a bigger displacement is desired, that is, $\Delta z_1^C \in (-2\pi, -\pi)$ is needed, then the simplest approach is to perform two instances of maneuver $C$, each one for achieving a displacement of $\Delta z_1^C/2$. Finally, note that every value from outside the interval $(-2\pi, 0]$ can be mapped to its interior by

$$\tilde{\Delta}z_1 := (\Delta z_1 \mod 2\pi) - 2\pi. \quad (36)$$

Because $z_1$ is the angle of a revolute joint, this mapping does not affect the actual configuration of the manipulator, cf. Section 3.1.

Due to the quadrant equivalence (Symmetry property 1) the extension of maneuver $C$ to $z_2^* < 0$ with a sliding surface in the right half-plane is straightforward.

5.5. Maneuver summary

All four maneuvers (A, B, C, D) enable us to perform almost arbitrary equilibrium transitions. Due to adverse drift conditions equilibria with $z_3$-values close to integral multiples of $\pi/2$ must be excluded. The actual sizes of the excluded $z_3$ intervals depend on the maneuver duration which can be tolerated, cf. for example, Figure 4(b). In Figure 9 a complete equilibrium transition with all maneuvers and parking regimes is depicted. See (Knoll, 2014b) for the respective animations.

Remark 4 Similar to the simple ODE $\dot{x} = \sqrt{x}$ there is an issue with the uniqueness of the solution of the reduced dynamics which is related with the “infinite slope” $dz_2/dz_4$ of the used sliding surfaces for $z_2 = 0$ (see Figure 3 and Equation (29)). The respective intersection point of the sliding surface with the $z_4$-axis (either an equilibrium or an parking regime), cannot be left by sliding mode control. However, if once $|z_2| > 0$ then the solution of the reduced dynamics is unique due to $\psi(\cdot)$ being locally Lipschitz. For the (numerical) realization of the maneuvers this suggests to choose $|z_2(t = 0)| = \varepsilon > 0$. For $\varepsilon$ small enough, its actual value becomes irrelevant to the solution of the dynamics and the input trajectories obtained by this “trick” nevertheless lead the unspoilt system to the desired state.

Remark 5 Of course the proposed motion schemes are by far not unique. In fact, for every single part of the motion (i.e. for every maneuver) there exists an infinite degree of freedom for the shape of the sliding surface, which might be used to optimize the motion w.r.t. overall duration or other suitable criteria. However, the aim of the presented solution is to show the principle applicability of the concept.
Figure 9. Simulation results for a complete equilibrium transition. Left column: Motion in working space. Middle column: Projection of the trajectory (and hence the switching functions) to the $z_2$-$z_4$-subspace. Right column: Time evolution of the joint angles $z_1(t), z_3(t)$. For a video of these results see (Knoll, 2014b).
6. Conclusion and outlook

In this contribution, we defined the LBINF as a special case of the Byrnes–Isidori normal form for generic Lagrangian mechanical systems. Furthermore, we constructively proved its existence for systems given by Equation (1). Then, we analyzed the frictionless underactuated 2-DOF manipulator in that normal form, especially the drift conditions in the $z_2$-$z_4$-subspace. On that basis, the controller (20) was defined which restricts the system dynamics to a submanifold (sliding surface) of the state space, determined by $z_2 = \varphi(z_4)$. Additionally, useful symmetries and the parking regime were identified. Finally, the equilibrium transition was decomposed into several subtasks, each of which is complied by a suitable maneuver, that is, by the suitable choice of the sliding function and appropriate initial conditions (obtained during parking regime).

The feasibility of the proposed controller is demonstrated by detailed simulation studies (Knoll, 2014a, 2014b) and first experimental results (Knoll et al., 2011). Currently, the experimental setup is reworked in order to implement the full equilibrium transition.

Besides the experimental realization there are some more questions for further investigation. One research direction concerns the application of the LBINF to other underactuated control systems and its relation to important control properties like linearizability by static or dynamic feedback, differential flatness and configuration flatness (see Knoll & Röbenack, 2014; Rathinam & Murray, 1998 and the references therein) or accessibility.

Another direction deals with the consequences of the Brockett condition, mentioned in Section 3.1. This result precludes the existence of a continuous differentiable feedback of the state of the original system to stabilize an isolated equilibrium. However, it does not preclude a nonsmooth state feedback or a dynamic extension of the system.

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Notes

1. Although underactuated systems are sometimes referenced as "second-order nonholonomic", for example, in Oriolo and Nakamura (1991), many of them are holonomic in a physical sense.
2. Note the redefinition of $q$ along with the elimination of $B(q)$.
3. In such a case (Oriolo & Nakamura, 1991) suggest either nonsmooth feedback or a different control objective. In particular, the authors propose a smooth controller, which stabilizes a 1-dimensional submanifold of all equilibrium points. The case of a dynamic extension of the state, which might be another possibility to cope with the violation of Brocketts condition, is not discussed.
4. Then $f_3(\cdot, \cdot)$ and $f_4(\cdot, \cdot)$ are globally Lipschitz and hence finite escape time cannot occur.
5. This choice for $\varphi$ is made for the sake of simplicity. In regions which are not near to the $z_4$-axis the curve $\varphi(\cdot)$ could have an arbitrary shape as long as it is continuous, (piecewise) differentiable and takes finite positive values.
6. Note that, due to time reversal symmetry, this also could be interpreted as an execution of maneuver A1 with appropriate adaption of the state.
7. As numerical values $\kappa = 0.9$ and $|z_4|^1 = 1.4s^{-1}$ are used for all simulations.
8. For convenience, we define $\text{sign}(0) := 1$.
9. They are called “upper” and “lower” branch because in the $z_2$-$z_4$-plane the argument of $\varphi_C$ runs vertically.

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