Why does Einasto profile index $n \sim 6$ occur so frequently?

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ABSTRACT

We consider the behavior of spherically symmetric Einasto halos composed of gravitating particles in the Fokker-Planck approximation. This approach allows us to consider the undesirable influence of close encounters in the N-body simulations more adequately than the generally accepted criteria. The Einasto profile with index $n = 6$ is a stationary solution of the Fokker-Planck equation in the halo center. There are some reasons to believe that the solution is an attractor. Then the Fokker-Planck diffusion tends to transform a density profile to the equilibrium one with the Einasto index $n = 6$. We suggest this effect as a possible reason why the Einasto index $n = 6$ occurs so frequently in the interpretation of N-body simulation results. The results obtained cast doubt on generally accepted criteria of N-body simulation convergence.

Key words. Galaxies: structure – Galaxies: formation – astroparticle physics – methods: analytical

1. Introduction

The Einasto density profile was first suggested in 1965 (Einasto 1965) and since then has been extensively used for fitting stellar systems and results of N-body simulations. The profile has the shape of $\rho(r) = \rho_c \exp \left(-2n \left(\frac{r}{r_s}\right)^{1/n}\right)$ where $\rho_c$ is the central density, $r_s$ is the characteristic radius, and $n$ is the profile index. The lower $n$ is, the shallower the central peak is: $n \leq 1$ corresponds to cored profiles, $n > 1$ — to the cuspy ones; $\alpha = 1/n$ is often used instead of $n$.

We will mainly focus on the Einasto profile application in the treatment of N-body simulation results. The profile fits recent simulations even better than the Navarro-Frenk-White (NFW) one; the profile index turns out to be quite universal $n \approx 6$ (Einasto 1965; Gao et al. 2008; Navarro et al. 2010). Currently, $n \approx 6$ is obtained, which corresponds to a cuspy profile, $n > 1$ — to the cuspy ones; $\alpha = 1/n$ is often used instead of $n$.

On the contrary, observations suggest 'cored' profiles for, at least, dark matter (DM) halos of dwarf galaxies (see, for instance, Chemin et al. 2011; Walker & Penarrubia 2011). In particular, the Einasto profile was used to fit observational data for a large data set of various galaxies (Chemin et al. 2011). The Einasto index $n \approx 0.5$ was obtained, which corresponds to a cored profile. The contradiction is well-known as the 'core-cusp problem' and might indicate that the cold noninteracting DM paradigm was inadequate. However, some evidences from theoretical consideration (Baushev 2015) and simulations (Baushev et al. 2017) have been reported that the cusps appearing in the N-body simulations might be just a numerical artifact. We will discuss this point at great length in the two last sections, here restricting ourselves to a short introduction to the problem.

As a rule, the Einasto profile is used to fit systems that may be considered to a first approximation as collisionless. However, the collisions are not always absent completely. Stellar systems and N-body models are constructed by point-mass particles with a newtonian or quasi-newtonian gravitational field. Therefore, they may scatter on the individual potential wells surrounding each particle. On the contrary, real dark matter systems should be completely collisionless, as we shall show below.

The encounters may be classified into close and weak, depending on the ratio between the actual impact parameter $b$ and the 90 degree deflection impact parameter $b_{90}$. Each close one leads to a strong change of momenta of the colliding particles. However, they are quite rare in stellar systems ($b_{90}$ is typically very small). N-body simulations are freed of them by the newtonian potential smoothing. We will not consider close encounters in this paper.

On the contrary, each weak encounter changes the particle momenta only slightly. The combined effect of many weak collisions is a slow diffusion of particles in the phase space. Since the gravitational force is long-acting, the weak encounter effect dominates. Its importance can be roughly estimated with the help of collisional relaxation time (Binney & Tremaine 2008 chapter 1.2.1)

$$\tau_c = \frac{N(r)}{8 \ln \Lambda} \tau_d. \quad (1)$$

Here $N(r)$ is the number of particles inside the radius $r$, $\tau_d = \left(4\pi G\rho(r)/3\right)^{-1/2}$ is the characteristic dynamical time of the system at the radius $r$, $\rho(r)$ is the average density inside $r$, $\Lambda = b_{\max}/b_{\min}$ (where $b_{\max}$ and $b_{\min}$ are the characteristic maximum and minimum values of the impact parameter) is the Coulomb logarithm; $b_{\max}$ is typically comparable with the size of the halo; $b_{\min}$ is the radius where either the assumption of a straight-line trajectory breaks, or the newtonian potential is no longer valid. Since $\ln \Lambda$ depends on $b_{\max}$ and $b_{\min}$ only logarithmically, it is

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1 This definition is equivalent to the standard one, if we introduce $\rho_c = \rho_s \exp (-2n)$ and $\rho = \rho_s \exp \left[-2n \left(\frac{r}{r_s}\right)^{1/n}\right]$.
usually enough to perform a rough estimation of these quantities. For stellar systems \( \Lambda = R_{\text{tot}} |h|_{m} \approx R_{\text{tot}} v^{2} / (Gm) \approx N \). For N-body simulations \( \Lambda = r_{i} / \phi \), where \( \phi \) is the potential softening radius: for instance, \( \Lambda = 3 r_{i} |\phi| \) (Klypin et al. 2013) or \( \Lambda = \min\{N, r_{i} / (4q)\} \) (Farouki & Salpeter 1982).

For stellar systems, the weak encounters are quite physical. On the contrary, the encounters are no more than an undesirable numerical effect in the case of N-body simulations. Indeed, a real DM halo contains some \( \sim 10^{60} \) particles, and its collisional relaxation time \( \tau_{c} \) is gigantic. The number of test bodies in N-body simulations is \( \sim 50 \) orders of magnitude lower. Therefore, their collisions may produce a dynamical friction and other unphysical relaxation processes. One needs to be secured in the negligibility of the collisions to guarantee the reliability of N-body simulations. To put it differently, the ratio \( t_{0} / \tau_{c} \), where \( t_{0} \) is the lifetime of the system, should be low enough. An estimation of the maximum ratio \( t_{0} / \tau_{c} \), wherein N-body simulation results are still not corrupted by the collisions is one of the main aims of N-body convergency tests. The commonly used criteria of N-body simulation convergency are entirely based on merely the density profile stability (for instance, (Power et al. 2003)).

In this paper, we study the influence of the weak encounters in the case of the Einasto profile with the help of the Fokker-Planck (hereafter FP) equation. The results of the study can be interesting for stellar system dynamics. Moreover, with the help of the FP approach, we may consider the behavior of N-body simulation particles: since the strong encounters are totally excluded here by the potential softening, the FP approximation perfectly works in this case. In section 2, we set forth the mathematical formalism and the Fokker-Planck equations for the Einasto profile. In section 3, we discuss the results obtained and suggest a possible reason why the value \( n = (6-7) \) may be distinguished and is so widely occurring in the simulations.

2. Calculations: basic equations

We assume that the system under consideration consists of identical particles of mass \( m \). It can be described by the particle distribution function \( k \).

We define the gravitational potential \( \phi \) as the minus potential energy of a unit mass. The potential at infinity is chosen to be zero. Thus, \( \phi \) is everywhere positive and reaches its maximum in the halo center. We find it convenient to define the particle specific energy \( \varepsilon \) in the gravitational field \( \phi \) as \( \varepsilon = \phi - e^{2} / 2 \). Here, \( e \) is the particle velocity. \( \varepsilon \) is always positive for a bound particle; \( k = 0 \) if \( \varepsilon \leq 0 \). For brevity, the specific energy, the expression to be referred to as “energy.” Since we will never use the real particle energy \( -m \varepsilon \) or potential energy \( -m \phi \), this definition cannot lead to misunderstanding.

The halo mass inside the radius \( r \) is \( M(r) = \int_{0}^{r} 4 \pi r^{2} \rho(r) dr \). It is convenient to introduce a dimensionless radius \( y = r / r_{s} \). We obtain

\[
M(r) = M_{\text{tot}} \left( 1 - \frac{\Gamma(3n, 2ny^{n/3})}{\Gamma(3n)} \right) ; \quad M_{\text{tot}} = \frac{4 \pi \rho_{1} r^{3}_{s} n \Gamma(3n)}{(2n)^{3n}}, \quad N(r) = N_{\text{tot}} \left( 1 - \frac{\Gamma(3n, 2ny^{n/3})}{\Gamma(3n)} \right) ; \quad N_{\text{tot}} = \frac{4 \pi \rho_{1} r^{3}_{s} n \Gamma(3n)}{m(2n)^{3n}},
\]

where \( M_{\text{tot}} \) and \( N_{\text{tot}} \) are the total mass and the number of particles in the halo. \( \Gamma \) is the gamma function

\[
\Gamma(n, y) = \int_{0}^{y} x^{n-1} e^{-x} \, dx; \quad \gamma(n, y) = \int_{0}^{y} x^{n-1} e^{-x} \, dx
\]

\[
\Gamma(n) = \Gamma(n, 0) = \int_{0}^{\infty} x^{n-1} e^{-x} \, dx \quad \Gamma(n, y) + \gamma(n, y) = \Gamma(n)
\]

We denote the value of the halo center potential by \( W = \frac{GM_{\text{tot}} (2n)^{3n} \Gamma(3n)}{r_{s}}, \). Then the potential can be written as

\[
\phi(r) = \frac{W}{\Gamma(2n)} \left( \frac{\gamma(3n, 2n \cdot y^{n/3})}{(2n)^{3n} y} + \Gamma(2n, 2n \cdot y^{n/3}) \right)
\]

In this paper, we perform an accurate consideration of the process with the help of the Fokker-Planck equation (Landau & Lifshitz 1980). We are interested in the central region of the halo \( (r < r_{s}) \), since the density here is the highest, and so the encounters are the most pronounced. We exactly follow the method used by (Evans & Collett 1997) to apply the FP approach in the halo center. They supposed that the system is schemically symmetric and isotropic. The latter supposition is quite natural for the halo center: the velocity distribution becomes isotropic in this region even if it is extremely anisotropic at larger radii (Bauseley 2013). For a spherically symmetric, isotropic system, the particle distribution \( k \) depends only on the energy \( \varepsilon \) and time (Binney & Tremaine 2008) (i.e. the number of particles with energies between \( \varepsilon \) and \( \varepsilon + d \varepsilon \) is \( k(\varepsilon, t) d \varepsilon \)), and the FP equation reads

\[
\frac{\partial k(\varepsilon, t)}{\partial t} = \frac{\partial}{\partial \varepsilon} \left( k(\varepsilon, t) D_{1}(\varepsilon) + \frac{\partial}{\partial \varepsilon} \left[ k(\varepsilon, t) D_{2}(\varepsilon) / 2 \right] \right),
\]

where \( D_{1}(\varepsilon) \) and \( D_{2}(\varepsilon) \) are the diffusion coefficients. The encounters result in a slow diffusion of particles in the phase space. The diffusion flux \( s \) of particles is

\[
s(\varepsilon) = k(\varepsilon, t) D_{1}(\varepsilon) + \frac{\partial}{\partial \varepsilon} \left[ k(\varepsilon, t) D_{2}(\varepsilon) / 2 \right],
\]

and the FP equation can be rewritten in the form of the particle number conservation \( \frac{\partial k(\varepsilon, t)}{\partial t} = -d \varepsilon \) s. In the case of one-dimensional task under consideration, \( s \) is just the number of particles crossing the surface \( \varepsilon = const \) per unit time.

Now we should define the characteristic time \( \tau_{FP}(r) \) in which the Fokker-Planck diffusion changes the Einasto profile significantly. The maximal reasonable estimation of \( \tau_{FP} \) is

\[
\tau_{FP, \text{max}}(r) = \frac{N(r)}{dN(\varepsilon)} = \frac{N(r)}{s(r)}
\]

Indeed, this value corresponds to the time sufficient for the FP diffusion to remove all the particles inside \( r \). However, \( \tau_{FP, \text{max}} \) gives a strongly overestimated value of \( \tau_{FP}(r) \): the number of particles sufficient to change the profile inside this radius is significantly smaller. As an example, let us consider the fraction of mass (or, which is the same, of the test particles) one should add to the \( \rho \propto r^{-1} \) profile in order to transform it into \( \rho \propto r^{-4/3} \) (the importance of this particular instance will be clear from section 3). To be more precise, let us consider a \( \rho \propto r^{-1} \) profile that has density \( \rho_{1} \) at some radius \( l \). Then the profile density is \( \rho = \rho_{1} (r/l)^{-1} \), and the mass inside \( l \) is \( m_{l} = 2 \pi l^{2} \rho_{1} \). What fraction of \( m_{l} \) should be added to the profile in order to transform it into \( \rho \propto r^{-4/3} \) so that the density at \( l \) remains the same? The profile after the modification should be \( \rho = \rho_{1} (r/l)^{-4/3} \), and a trivial integration shows that its mass inside \( l \) is \( 1.2 m_{l} \), i.e., only 20\% of the initial mass should be added.

In a like manner, for each radius \( r \) we can determine the number of particles \( N_{1}(n, r) \) that should be added to the Einasto profile with index \( n \) in order to transform it inside \( r \) into the Einasto profile with index \( n + 1 \), but with the same density at \( r \) and the same value of \( r_{s} \). The final profile has the index \( n + 1 \), and its central density \( \rho_{c} \) is defined by the requirement of the equal density
at \( r; \rho_c \cdot f \exp \left[ -2n(n + 1) \left( r/r_a \right)^{1/(n + 1)} \right] = \rho_c \exp \left[ -2n \left( r/r_a \right)^1 \right] \). Then equation (2) allows us to calculate \( N_{+1} \) as a function of \( n \) and \( r \). It is natural to define \( \tau_{FP}(r) \) as

\[
\tau_{FP}(r) = \frac{N_{+1}(r)}{S(r)}.
\]

(8)

Being so defined, \( \tau_{FP}(r) \) characterizes the time it takes for the FP diffusion \( s(r) \) to bring through the radius \( r \) a number of particles sufficient to change the profile index inside this radius by one.

To show the direction of the FP diffusion, we introduce the sign of \( \tau_{FP}(r) \) and \( \tau_{FP,max}(r) \): it coincides with that of \( s(r) \). The density profile is mainly determined by the particle collisions if \( t \leq \tau_{FP} \). For the important instance of N-body simulations, it means that they can nohow model a real collisionless DM system unless \( t_0 \) is significantly lower than \( \tau_{FP} \).

Equation (1) estimates the relaxation time rather crudely. A much more reliable estimation can be derived from the Fokker-Planck equation (Binney & Tremaine 2008, eqn. 7.106), which yields that

\[
\tau_{r_2} = 0.34 \frac{\sigma^3}{G^2 \mp \ln \Lambda}.
\]

(9)

Here \( \sigma \) is the local one-dimensional velocity dispersion, \( \rho \) is the local density, and \( G \) and \( M \) are the gravitational constant and the particle mass, respectively. We will use both definitions of the relaxation time, \( \tau_{r_1} \) and \( \tau_{r_2} \). The second one is much more precise, but the first one is widely used in literature (for instance, by Power et al. 2003).

The diffusion coefficients read (Evans & Collett 1997)

\[
D_1(\varepsilon) = \frac{16 \pi^2 G^2 m^2 \ln \Lambda}{p(\varepsilon)} \left[ p(\varepsilon) \frac{\pi}{2} - \int_0^\varepsilon f(\varepsilon)P(\varepsilon, \varepsilon)d\varepsilon \right],
\]

(10)

\[
D_2(\varepsilon) = \frac{32 \pi^2 G^2 m^2 \ln \Lambda}{p(\varepsilon)} \left[ q(\varepsilon) \int_0^\varepsilon f(\varepsilon)P(\varepsilon, \varepsilon)d\varepsilon + \frac{w}{\pi} \right].
\]

Here, \( p(\varepsilon) \) and \( q(\varepsilon) \) are the density of states and the total phase-space volume with energy between \( \varepsilon \) and \( W \), respectively

\[
p(\varepsilon) = 16 \pi^2 \int_0^{\varepsilon} \left( 2[\phi(r) - \varepsilon]^{1/2} \sqrt{2} \, dr, \right.
\]

(11)

\[
q(\varepsilon) = \frac{16 \pi^2}{3} \int_0^{\varepsilon} \left( 2[\phi(r) - \varepsilon]^{3/2} \sqrt{2} \, dr, \right.
\]

(12)

The maximum radial digression \( r_{max}(\varepsilon) \) possible for a particle of energy \( \varepsilon \) can be obtained from the energy conservation law

\[
\phi(r_{max}(\varepsilon)) = \varepsilon.
\]

(13)

The phase-space density \( f(\varepsilon) \) of particles reads (Binney & Tremaine 2008, eqn. 4.46b)

\[
f(\varepsilon) = \frac{1}{\sqrt{8\pi^2 M_{tot}}} \int_0^\varepsilon \frac{df}{\sqrt{E - \phi}} \rho d\phi.
\]

(14)

where \( \frac{d^2\rho}{d\phi^2} \) is the second derivative of the halo density \( \rho \) as a function of the gravitational potential \( \phi \). The equations for \( \rho(r) \) and \( \phi(r) \) implicitly define this function. Finally, we need to mention an important relationship between the particle distribution function \( k(\varepsilon, t) \), the phase-space distribution function \( f(\varepsilon, t) \) and the density of states \( p(\varepsilon) \):

\[
k(\varepsilon, t) = f(\varepsilon, t) \cdot p(\varepsilon).
\]

(15)

One can see that \( q, p, s, \) and \( f \) are defined as functions of \( \varepsilon \), while \( N, M, \) and \( \phi \) depend on \( r \). In order to close the system of equations (for instance, to calculate \( \tau_{FP} \) in accordance with (8)), we need to bind \( \varepsilon \) and \( r \) (or \( \varepsilon \) and \( y \)). Theoretically, there is no one-to-one correspondence between the particle energy and the maximal radius of its orbit: it depends on the orbit shape. For a purely radial orbit

\[
\varepsilon = \frac{W}{\Gamma(2n)} \left( \frac{\gamma(3n, 2n \cdot y^{1/n})}{2(2n)y} + \Gamma(2n, 2n \cdot y^{1/n}) \right).
\]

(16)

For a circular orbit

\[
\varepsilon = \frac{W}{\Gamma(2n)} \left( \frac{\gamma(3n, 2n \cdot y^{1/n})}{2(2n)y} + \Gamma(2n, 2n \cdot y^{1/n}) \right).
\]

(17)

The difference between these equations is not very large. Since we consider the case when particle velocity distribution is isotropic, the orbits of the bulk of particles are more or less circular. Therefore, we use relationship (17) between \( \varepsilon \) and \( r \).

3. Calculations: numerical solution

One can see that equations (2)-(13) form a closed system. Here we represent the main course of numerical solution of the equations in the interval \( n \in [1; 12] \).

First of all, it is convenient to turn to dimensionless quantities. We will use \( y = r/r_a \) and \( E = \varepsilon/W \) as coordinates. The dimensionless potential can be introduced as \( \psi = \phi/W \)

\[
\psi(y) = \frac{1}{\Gamma(2n)} \left( \frac{\gamma(3n, 2n \cdot y^{1/n})}{2(2n)y} + \Gamma(2n, 2n \cdot y^{1/n}) \right).
\]

(18)

We can also introduce dimensionless equivalents for \( \rho(E), q(E), f(E), s(E), \) and dynamical time, denoting them by the same letter with a tilde above:

\[
\rho(E) = 16 \sqrt{2\pi^2} r_a^2 \sqrt{W} \tilde{\rho}(E), \quad q(E) = 16 \sqrt{2\pi^2} r_a^2 \sqrt{W} \tilde{q}(E)
\]

\[
f(E) = \frac{\rho_{t}(\Gamma(2n))}{\sqrt{2\pi^2} W \sqrt{W}} \tilde{f}(E), \quad s(E) = \frac{8 \sqrt{W} \ln \Lambda}{\pi^2 n^2 r_a} \tilde{s}(E).
\]

After some trivial but bulky calculations, we obtain from equations (2)-(13):

\[
\tilde{\rho}(E) = \int_0^{\gamma_{max}} y^2 \left( \psi(y) - E \right)^{1/2} dy,
\]

(20)

\[
\tilde{\psi}(E) = \frac{2}{3} \int_0^{\gamma_{max}} y^2 \left( \psi(y) - E \right)^{3/2} dy.
\]

(21)

where \( \gamma_{max} \) is defined by (13)

\[
\psi(\gamma_{max}) = E
\]

(22)

\[
\tilde{f}(E) = \int_0^{E} \frac{df}{\sqrt{E - \psi}} \tilde{\rho}(E) \frac{d\psi}{\psi}
\]

(23)

where

\[
\frac{d^2\tilde{\rho}(y)}{d\psi^2} = (2n)^2 \Gamma(2n) y^{2 + \gamma} \exp(-2n \cdot y^{1/n}) \times \frac{2y^{1/n} - n + 1 + y^{2}(2n)^2 \exp(-2n \cdot y^{1/n})}{n\gamma(3n, 2n \cdot y^{1/n})}
\]

(24)
Fig. 1. The ratio between the relaxation time $\tau_r(r)$ (defined by [1]) and the time in which the Fokker-Planck diffusion can shift the Einasto index $n$ of the profile by one, $\tau_{FP}(r)$ [8], as a function of the radius and $n$. The white line represents the area $[\gamma]$, where the Einasto profiles behave as the power-law $\rho \propto r^{-\gamma}$. The blank region in the low left corner corresponds to the area uncovered by our calculations.

and

$$\bar{s}(E) = \frac{d\bar{f}}{dE} \left( \int_E^1 \bar{q} \cdot \bar{f} dE + \bar{q} \int_0^E \bar{f} dE \right) - \bar{f} \int_E^1 \bar{q} \cdot \bar{f} dE$$

(25)

Equation (25) for $\bar{f}(E)$ can be rewritten as

$$\bar{f}(E) = \int_{\gamma_{\text{min}}(E)}^{\infty} \frac{-d\psi}{\gamma_n(y)} \frac{d\rho}{dy}(y).$$

(27)

where

$$\frac{dy}{d\psi} = \frac{\gamma_n(y)}{2n^2}$$

(28)

To make the integral calculations faster, one may exclude the singularity at $E = \psi$ in (27).

$$\bar{f}(E) = 2 \sqrt{E} \frac{d\rho}{d\psi}(E) + \int_0^1 \frac{d\rho}{d\psi}(\psi(E, \phi)) \frac{d\phi}{d\psi}(E, \phi)$$

(29)

We used the standard Matlab functions for numerical integration [Shampine 2008]. The relative accuracy for the internal integrals (Eq. 20, 21, 22) was set at the level $10^{-7}$ and for the external integral (Eq. 25) — at $10^{-4}$.

4. Results

Figure 1 represents the ratio between the relaxation time $\tau_r(r)$ [1] and the time in which the Fokker-Planck diffusion can shift the Einasto index $n$ of the profile by one, $\tau_{FP}(r)$ [8], as a function of the radius and $n$. The blank region in the low left corner corresponds to the area uncovered by our calculations. We should remind that the sign of $\tau_r/\tau_{FP}$ coincides with the sign of $s(r)$ and demonstrates the direction of the FP diffusion. Figure 2 represents the ratio between the relaxation time $r_{\text{FP}}(r)$ [9] and the maximal estimation [7] of the Fokker-Planck diffusion time $\tau_{FPmax}(r)$, as a function of dimensionless energy $E$ and $n$.

The most remarkable feature in both the Figures is the line $\tau_r/\tau_{FP} = 0$ and the fact that FP diffusion changes its direction on this line. The line defines a stationary solution for the Fokker-Planck equation. Apparently, the Einasto profile cannot have different $n$ at different radii. However, Fig. 1 defines the direction of the profile variation caused by the weak encounters. A similar solution has already been reported in the case of power-law profiles [Evans & Collett 1997; Baushev 2015]. The authors of [Evans & Collett 1997] assumed an isotropic velocity distribution at each point, exactly as we do in this paper. They found that the density profile $\rho \propto r^{-4/3}$, where $\gamma = 4/3$, is a stationary solution. We can easily compare this result with ours. The Einasto profile with index $n$ behaves as the power-law $\rho \propto r^{-n}$ at the radius where $d\log\rho/d\log r = -4/3$. We obtain for the Einasto profile:

$$\log(r/r_s) = n \log \frac{3}{2}.$$  

(30)

This equation defines the white straight line in Fig. 1. One can see that the line $\tau_r/\tau_{FP} = 0$ almost coincides with it. A small discrepancy occurs as a result of the coordinate transformation: equations (2)-(14) define $s$ as a function of $E$. We transform it into the dependence of $r$, supposing that the radius $r$ of the circular orbit of a particle with energy $E$ corresponds to this energy. However, these dependencies are slightly different for the power-law and Einasto profiles, which accounts for the deflection of the curve $\tau_r/\tau_{FP} = 0$ from (30).

Thus, the $\tau_r/\tau_{FP} = 0$ curve that we found is just the solution $\rho \propto r^{-4/3}$ found by [Evans & Collett 1997]. Two conclusions may be drawn from this fact: first, the results of our calculations and [Evans & Collett 1997] confirm each other. Second, there is no other stationary solution of the Fokker-Planck, at least, in the area covered by our simulations. Indeed, the authors of [Evans & Collett 1997] used power-law profiles to investigate FP diffusion. Since the stationary solution turns out to be power-law, they discovered it explicitly. We are less fortunate using the Einasto profiles as test ones. None of them is a stationary solution, and we can see the stationary profile as a slanting line, but we are still able to find it. Thus, even in the case of an unfortunate choice of the model profile to investigate the FP equations, the stationary solution is visible.

A question appears: is the stationary solution $\rho \propto r^{-4/3}$ stable or not? There are some arguments testifying that the solution is an attractor. First, equation 11 in [Evans & Collett 1997] shows that the FP diffusion tends to change any power-law profile with an index $\gamma$ close enough to $4/3$ towards $\rho \propto r^{-4/3}$. If the profile is 'shallower' ($\gamma < 4/3$), the FP diffusion 'pushes' the particles towards the center, making the profile steeper. If the profile is initially steep, the FP diffusion is directed outwards flattening the cusp. Unfortunately, the analysis of [Evans & Collett 1997] supposes from the very beginning that the profiles can be only power-law, and thus this is not a full stability test. Second, as we will see in the next section, a cusp, very similar to the one under consideration routinely occurs in the halo centers in cosmological N-body simulations. Though the particle collisions are already significant in this region [Baushev et al. 2017] (and thus the system may be described by the FP equation), the cusp
is formed in many simulations, being very stable. It would be scarcely possible if $\rho \propto r^{-4/3}$ was an unsolvable solution of the FP equation. Such reasoning, however, cannot replace an exhaustive test of the asymptotical stability of the solution. To summarize: there are strong arguments testifying that $\rho \propto r^{-4/3}$ is an attractor solution of the FP equation in the case of anisotropic velocity distribution of particles. However, we cannot irrefutably prove the attractor nature of the stationary solution, since no full stability test has been performed yet.

5. Why does the Einasto profile with $n \sim 6$ occur so frequently in N-body simulation?

The second important conclusion we can draw from Fig. 1 is that on the main part of the region under consideration, except the vicinity of the curve $\tau_c/\tau_{FP} = 0$, $\tau_{FP}$ is smaller than $\tau_c$, and even more so, Power's time $1.7\tau_c$ (see below). It means that the FP diffusion is, generally speaking, very effective, and the collisions change the profile parameters in a time shorter than $\tau_c$. It questions the generally accepted criteria of N-body simulation convergence since all of them in current use are entirely based on merely the density profile stability. Let us consider, for instance, the most popular criterion $\tau_c \geq 0.6\tau_c$ (Power et al. 2003). The authors used the method for keeping track of how the averaged density inside some radius $r$ from the halo center changes with time. They found that a cuspy density profile (quite similar for various halos) was rapidly formed in the halo center, then the averaged density inside $r$ remained almost constant until, at least, $t_0 \sim 1.7\tau_c$ at this radius, and then a core at the halo center appeared. The authors of the criterion (Power et al. 2003) believed that the profile universality and stability proved the simulation convergence, and the core formation was the first sign of the collision influence. The conclusion that the relaxation has no effect until almost two relaxation times seems rather surprising. However, further tests (conducted with the use of the same profile stability method) showed that the criterion might be even softer, and the collision influence might be negligible until tens of relaxation times (Hayashi et al. 2003 [Klypin et al. 2013]).

The existence of the stationary solution of the FP equation clearly reveals the vulnerability of the convergence criteria based on the profile stability: the collisions themselves may form stationary profiles. The shape of the stationary solution needs not precisely coincide with $\rho \propto r^{-4/3}$: this profile corresponds to the case when the particle velocity distribution is isotropic at each point, the halo is spherically symmetric, and the particle interaction is newtonian or, at least, softened newtonian. All these suppositions are typically not quite true in real N-body simulations: even the particle interaction is calculated with some errors, which probably leads to quite observable effects (see below). However, it seems that the stationary profile in the halo center should lie between $\rho \propto r^{-1}$ (Baushev 2015) and $\rho \propto r^{-4/3}$ (Evans & Collett 1997). Figure 1 shows that any profile that significantly differs from it is significantly shifted by the collisions in a time $t \leq \tau_c$. On the contrary, the profile corresponding to the stationary solution should be standard (the FP coefficients are similar for various N-body codes) and very stable, i.e., survive for tens of relaxation times, as reported in (Hayashi et al. 2003 [Klypin et al. 2013]). However, it may already be created by the test body collisions, and thus be no more than a numerical effect. In this case, the profile universality and stability have nothing to do with the simulation veracity.

However, one may distinguish such a pseudo-convergence from the real one (Baushev 2015). The idea is that the profile stability has different physical reasons in this case. If the collisional contribution is negligible (as it should be) and the density profile is stable, the energies of each particle in it remain constant, and thus the particles have stationary orbits. If the collisions are significant, they lead to the FP diffusion of the particle parameters in the phase space. Thus, the energies of the particles do not conserve; the particle orbits move up and down along the cusp. The profile is stable in the case of the stationary solution because the upward and downward FP streams compensate each other.

An isolated Hernquist halo was simulated in (Baushev et al. 2017). The Hernquist model is close to the Navarro-Frenk-White and behaves the same way ($\rho \propto r^{-1}$) in the central region, but it has the known analytical solution for the stationary velocity distribution. Contrary to standard N-body simulations, the gravitational potential $\Phi(r)$ is exactly constant and spherically symmetric in this case and, therefore, the energy and angular momentum of each particle should be conserved. The simulations with the modern version of the Gadget-3 code (Springel 2005) confirm that the density profile is stable until $t_0 \sim 1.7\tau_c$, in accordance with (Power et al. 2003). However, it turns out that all integrals of motion characterizing individual particles experience strong unphysical variations, revealing an effective interaction between them. Moreover, the authors find Fokker-Planck streams in the cusp region, strong enough to change the shape of the cusp or even to create it. The cusp is stable, because the upward and downward FP streams compensate each other, and the stability has nothing to do with the negligibility of collisional relaxation.

It is probably on account of this fact that the Einasto profile with $n \sim 6$ occurs so frequently in N-body modelling. Indeed, the halo profile occurring in the simulations may be approximated by various phenomenological models at $r \sim r_c$; the Einasto profile with $n \sim 6$ becomes clearly preferable to the NFW one only at $r \sim 10^{-1}r_c$ (see, for instance, Fig. 3 in (Navarro et al. 2010)), or, which is approximately equal at $r \approx 2R_{vir}$ (Fig. 3 in (Dutton & Macciò 2014)). The point $r = 10^{-1}r_c$, $n \approx 6.2$ lies exactly on the stationary solution in Fig. 1. If the influence of the test particle interaction is already significant at these radii, we may surmise that it is just the particle collisions that transform the profile into the Einasto with $n \approx 6$ here. If the

If we quite realistically accept the average ratio $R_{vir}/r_c \equiv c_{vir} \sim 5$. 

profile is a stationary solution of the FP equation, it explains both its phenomenal stability up to many relaxation times and universality: the shape of the profile formed by collisions is defined by the potential of the particle interaction (which is always close to the newtonian one and therefore universal), and not by the properties of initial perturbations or by details of the numerical scheme. On the other hand, probably the profile has nothing to do with the properties of real collisionless dark matter systems. At least, the profile stability says nothing about the simulation convergence and reliability.

However, if $\rho \propto r^{-4/3}$ is a stationary solution, why a core finally appears in the center of the halo? There can be several possible reasons. First, the Fokker-Planck equation is not exact: it can be obtained from the exact Boltzmann equation by expanding the collision integral in a Taylor series and dropping all the terms except the first two ones (Landau & Lifshitz 1980). However, the contribution of the kinetic terms of higher orders also becomes important after several relaxation times, and it can destroy the cusp (see Baushev 2015, eqn. 4, 5 for details). The core formation may be related to short-range, large-angle deflections, which are not taken into account by the Fokker-Planck equation. Finally, the core formation may be caused by some other numerical effects. Whatever the reason is, the core formation cannot be used as the first sign of the collision influence, as was done in Power et al. 2003. The FP diffusion becomes important much earlier.

Apparently, the profile tends to $\rho \propto r^{-4/3}$ only for $r \ll r_\star$; if $r \gg r_\star$ or larger, the influence of collisions is not strong enough since the relaxation time rapidly grows with radius.

Two questions arise: what does the model under consideration predict and is the FP diffusion strong enough to drive the density profiles of halos in numerical simulations towards the stable FP solution? It seems in the framework of the theory that halos resolved with fewer particles (i.e., less massive halos in a cosmological simulation), should have been affected out to larger radii (since $r_\star$ is smaller in this case), and therefore follow the Einasto profile with $n \approx 6$ (hereafter E6 profile) out to larger radii than more massive halos. Furthermore, one should expect that the central regions of N-body halos have a $\rho \propto r^{-4/3}$ cusp, rather than the $n \approx 6$ Einasto profile.

To be specific in answering the questions, let us consider one of the most recent and high-performance simulations (Dutton & Macciò 2014) as an example. Figure 3 in (Dutton & Macciò 2014) shows that the density profile at $(2 \sim 4)R_\mathrm{vir}$ is indeed quite close to E6. The profile is slightly steeper than E6 for small halos and slightly shallower than E6 for the largest halos. Moreover, the profile steepness at 3% $R_\mathrm{vir}$ regularly decreases with the halo mass growth. Though these results agree with our theoretical predictions, the statistics is rather poor and does not allow one to make strong statements, which is quite expectable. Figure 7 in our paper shows that in order to reach the region where the stationary solution gets significantly steeper than E6, one needs to approach closer than 2% $R_\mathrm{vir}$ to the halo center. The masses of the $\rho \propto r^{-1}$ cusp and the $\rho \propto r^{-4/3}$ one differ by 20%. Therefore, we need to have at least $N_m \sim 225$ particles inside 2% $R_\mathrm{vir}$ to distinguish the cusps on the 3σ-level (indeed, $\sigma \approx \sqrt{N_m}$ and $3\sigma/N_m = 3/\sqrt{225} \approx 20\%$). For instance, only 0.46% of the total mass of an NFW halo with $c_\star = 5$ lie inside 2% $R_\mathrm{vir}$. Thus, if a halo contains less than 50000 particles, any profile reconstruction inside 2% $R_\mathrm{vir}$ is certainly impossible.

In fact, this number should be significantly larger in cosmological simulations: the halos formed there are not exactly spherical and experience tidal perturbations; their main structural parameters are significantly scattered (the halos form a random gaussian field, see Dutton & Macciò 2014, Fig. 15)) and in addition to that are measured with an accuracy of $(15 \sim 25)\%$, depending on the algorithm used. The apparently numerical core formation in the halo center is also a serious handicap to the profile reconstruction. As a result of all these factors, the authors of (Dutton & Macciò 2014) report that $\sim 10^4$ test particles are necessary just to obtain reliable $r_\star$ and other Einasto shape parameters. We can conclude that a halo should contain hundreds of thousand particles to distinguish $\rho \propto r^{-1}$ and $\rho \propto r^{-4/3}$ cusps inside 2% $R_\mathrm{vir}$ in real cosmological simulations.

A set of simulations with various particle masses was used in (Dutton & Macciò 2014) in order to increase the halo mass range covered by the simulation and the result reliability (see the paper for details). A single simulation contains $\sim 10^5$ halos holding more than 500 particles and no more than $(600)^3 \approx 2 \times 10^9$ particles in total. Unfortunately, the authors have not specified the mass distribution of the halos. However, we can expect that it obeys the usual law $dn \propto M^{-2}dM$ (Diemand et al. 2005), which implies that each next logarithmic mass interval $[10^{10}M_\odot ; 10^{12}M_\odot]$ contains 10 times less halos than the previous one $[10^9M_\odot ; 10^{11}M_\odot]$. Then among $\sim 10^4$ halos containing $\geq 500$ particles there should be $\sim 10$ containing $\geq 5 \times 10^3$ ones. Thus, even recent and high-performance simulations contain only tens of halos, for which the profile reconstruction below 2% $R_\mathrm{vir}$ is possible. Considering the random gaussian scattering of their properties and other difficulties, it makes it possible to find the above-mentioned tendencies that confirm the predictions of our theory, but the cosmological simulations should contain, at least, $\sim 10$ times more test particles in order to make statements more certain.

Strong influence of the FP diffusion in N-body simulations has been demonstrated in a conclusive way by modelling an isolated halo (Baushev et al. 2017). In contrast with cosmological simulations, the halo in this case is spherically symmetric and does not experience tidal effects. If we properly set the initial velocities, not only the density profile but the particle velocity distribution in each point and even the energy and the vector of angular momentum of each particle should be conserved. Any deviation from this behavior is a numerical effect. Thus, one may collect much more information about the N-body scheme properties than in the case of cosmological simulations. Simulations (Baushev et al. 2017) of an isolated Hernquist halo revealed that the numerical FP streams are strong enough to create the cusps. Recent simulations (van den Bosch et al. 2018) show that some numerical effect stabilizes the central part of subhalos and prevents their tidal destruction in N-body simulations. Indeed, the unphysical cusp formation may be the factor that makes the small halos more robust (Baushev 2016). Finally, our analysis also confirms that the FP diffusion is strong enough to influence the halo profile.

Can we state from our analysis that the FP diffusion in N-body simulations tends to transform the profile in the halo center exactly into $\rho \propto r^{-4/3}$? Certainly not. First, the assumptions applied to deduce the stationary solution in this paper or (Evans & Collett 1997) can fail in the case of N-body simulations, in particular, velocity distribution of the particles need not be isotropic at each point.

A more fundamental problem is that the above-mentioned papers (van den Bosch et al. 2018) and (Baushev et al. 2017) report about significant effects in the halo center, but their origin has been clarified. The numerical effects may have been caused not only by the weak encounters, but also by inaccuracies of the potential calculation, for instance. Equations (10) for the FP coefficients were obtained on the assumption that the particle
interaction is newtonian, or, at least, truncated newtonian. If it is not exactly so, the coefficients may differ from (10).

However, the fact that the universal density profile obtained in simulations falls exactly on the stationary solution of the Fokker-Planck equation in the halo center does not look like a mere coincidence and casts doubt on the convention convergence criteria of N-body simulations. The criteria based on the profile stability are apparently insufficient, and new methods should be developed and used. Some techniques were offered in (Baushev 2015). A better understanding of the possible influence of the inaccuracies of the potential calculation algorithm used in N-body codes is necessary for a reliable interpretation of simulation results.

6. Acknowledgements

We acknowledge the support by the Helmholtz Alliance for Astroparticle Physics HAP funded by the Initiative and Networking Fund of the Helmholtz Association. B.M.V. acknowledges support from the Japan Society for the Promotion of Science (Grant no. 16H00878), NSF grant AST-1306672 and DoE grant de-sc0016369.

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