About calculation of the Hankel transform using 
preliminary wavelet transform

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Abstract
The purpose of this paper is to present an algorithm for evaluating Hankel 
transform of the null and the first kind. The result is the exact analytical 
representation as the series of the Bessel and Struve functions multiplied to 
the wavelet coefficients of the input function. Numerical evaluation of the 
test function with known analytical Hankel transform illustrates proposed 
algorithm.

The Hankel transform is a very useful instrument in a wide range of physical 
problems which have an axial symmetry [1]. The influence of the Laplasian on a 
function in a cylindrical coordinates is equal to the product of the parameter of the 
transformation squared and the transform of the function:

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) f(r) \leftrightarrow -p^2 F_0(p) \\
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) f(r) \leftrightarrow -p^2 F_1(p)
\]

The Hankel transform of the null \((n = 0)\) and the first \((n = 1)\) kind are repre-
sented as

\[
F_n(p) = \int_{0}^{\infty} f(r) J_n(pr) rdr,
\]
\[
f_n(p) = \int_{0}^{\infty} F(p) J_n(pr) p dp.
\]

Besides that integrals like (2) are connected with the problems of geophysics 
and cosmology, for example [2, 3].

But practical calculation of direct and inverse Hankel transform is connected 
with two problems. The first problem is based on the fact that not every transform 
in the real physical situation has analytical expression for result of inverse Hankel 
transform. The second one is determination of functions as a set of their values for 
numerical calculations. Large bibliography on those issues can be found in [4]. The 
classical trapezoidal rule, Cotes rule and other rules connected with replacement 
of integrand by sequence of polynoms have high accuracy if integrand is a smooth 
function. But \(f(r) J_n(pr)r\) (or \(F_p(p) J_n(pr)p\)) is a quick oscillating function if \(r\) 
(or \(p\)) is large. There are two general methods of the effective calculation in this 
area. The first is the Fast Hankel transform [4]. The specification of that method 
is transforming the function to the logarithmical space and fast Fourier transform 
in that space. This method needs a smoothing of the function in log-space. The 
second method is based on the separation of the integrand into product of slowly
varying component and a rapidly oscillating Bessel function \[7\]. But it needs the
smoothness of the slow component for its approximation by low-order polynoms.
The goal of this article is to apply wavelet transform with Haar bases to \[2\].
The both direct and inverse transforms \[2\] are symmetric. Let us consider only
one of them, for example, direct transform. Let’s denote \(f(r)\) as \(g(r)\). Then Hankel transform is
\[F_{0,1}(p) = \int_0^\infty g(r) J_{0,1}(pr) dr\] (3)

The expansion \(g(r) \in L^2(R)\) into wave series with the Haar bases is \[8\] :
\[g(r) = \sum_{n=0}^\infty c_n \varphi_n(r) + \sum_{n=0}^\infty \sum_{k=0}^\infty d_{jk} \psi_{jk}(r),\] (4)

\[\varphi_n(r) = \varphi^H(r - k), \psi_{jk} = 2^{j/2} \varphi^H(2^j r - k),\]

\[\varphi^H(t) = \begin{cases} 1, & t \in (0, 1) \\ 0, & t \notin (0, 1) \end{cases}, \psi^H(t) = \begin{cases} 1, & t \in (0, 1/2) \\ -1, & t \in (1/2, 0) \\ 0, & t \notin (0, 1) \end{cases}.\]

After substitution (4) into (3) one has
\[F_{0,1}(p) = \sum_{n=0}^\infty c_n \int_0^\infty \varphi_n(r) J_{0,1}(pr) dr + \sum_{n=0}^\infty \sum_{k=0}^\infty d_{jk} \int_0^\infty \psi_{jk}(x) J_{0,1}(pr) dr.\]

Making use integrals from (2) we have as a result
\[F_0(p) = \sum_{k \in Z} c_k [(k + 1) J_0(p(k + 1)) - k J_0(pk)] + \]
\[\frac{\pi}{2} [(k + 1) D(p(k + 1)) - kD(pk)] + \]
\[\sum_{j=0}^\infty \sum_{k \in Z} d_{jk} [2(k + \frac{1}{2}) J_0(p(k + \frac{1}{2})) - (k + 1) J_0(p(k + 1)) - k J_0(pk) - \]
\[\frac{\pi}{2} [2(k + \frac{1}{2}) D(p(k + \frac{1}{2})) - (k + 1) D(p(k + 1)) - kD(pk)]\]
\[F_1(p) = \frac{1}{p} \left\{ \sum_{k \in Z} c_k [J_0(pk) - J_0(p(k + 1))] + \right. \]
\[\left. \sum_{j=0}^\infty \sum_{k \in Z} d_{jk} [2 J_0(p(k + \frac{1}{2}) 2^{-j}) - J_0(p(k + 1) 2^{-j}) - J_0(pk 2^{-j})] \right\},\] (6)

where \(D(\xi) = H_0(\xi) J_1(\xi) - H_1(\xi) J_0(\xi)\) and \(H_{0,1}\) is Struve function of the null
and the first kind.
The most sufficient result is that equations (5) and (6) are exact. They can be
used in any analytical expressions. Especially it’s useful for Hankel transform of
the first kind because (5) contains only a combination of Bessel functions, and one can
use such their properties as orthogonality etc. The coefficients \(c_{ok}\) means average
value of \(g(r)\) at the range \([k, k + 1]\) :
\[c_{ok} = \int_k^{k+1} g(r) dr\]
Figure 1: Original function (left) and Hankel transform (right)

The detail coefficients are

\[ d_{jk} = 2^{j/2} \left\{ \int_{2^{-j}k}^{2^{-j}(k+1)} g(r) dr - \int_{2^{-j}k}^{2^{-j}(k+1/2)} g(r) dr \right\}. \]

The formulas (5) and (6) allow us to get full analytical solution if integrals above have close form solution. In the opposite case the solution must be numerical but this method provides an effective algorithm for that. It's obvious that \( d_{jk} \) decrease very quickly if \( g(r) \) is a smooth function. One can practically use \( d_{jk} > \varepsilon \), where \( \varepsilon \) is small. But if \( g(r) \) has steps, sharp vertices or discontinues then the detail coefficients concentrate around these points and one can appropriate they are equal to the zero in other areas.

Let us consider for example a function with known analytical Hankel transform

\[ \int_0^\infty e^{-a^2 r^2} r J_1(pr) r dr = \frac{p}{4a^3} e^{-p^2/4a^2}. \] (7)

The approximation and detail coefficients may be calculated analytically in closed form

\[ c_{0k} = \frac{\sqrt{\pi \text{erf}(r) - 2ar e^{-a^2 r^2}}}{4a^4} \left| \frac{d}{k} \right|^{(k+1)} \]
\[ d_{jk} = 2^{j/2} \frac{\sqrt{\pi \text{erf}(r) - 2ar e^{-a^2 r^2}}}{4a^4} 2^{-j/2} \left| \frac{d}{k} \right|^{2^{-j}(k+1/2)} - \]
\[ 2^{j/2} \frac{\sqrt{\pi \text{erf}(r) - 2ar e^{-a^2 r^2}}}{4a^4} 2^{-j/2} \left| \frac{d}{k} \right|^{2^{-j}(k+1/2)}. \] (8)

Thus (6) with the coefficients (8) is the exact representation of the Hankel transform. Let us consider the approximate solution. Suppose the function is known only in the segment \([0, h] \). Then there is the series instead of (4):

\[ g(r) = c_0 \varphi_0(r) + \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} d_{jk} \psi_{jk}(r). \] (9)

If \( J \to \infty \) then (9) is exact for this truncated function. But practically we only use several first levels. For example we can see the original function (the replacement \( r \) to \( x = r/h \) is used) and the transform at the Fig.1. One can see that exact transform (solid line) and the transform at level \( J = 3 \) (dotted line) coincide at
this figure. The absolute errors between the exact transform and the approximate transform at the levels $J = 2$ (solid line), $J = 3$ (dashed line), and $J = 4$ (dotted line) are represented at the Fig.2 (left). It’s oblivious that the error is small in comparison with the values of the $F_1(p)$. The absolute error at the level $J = 3$ in a wide range of $p$ is plotted in the right side of the Fig.2. One can see that this error has quasi-periodic oscillations because the function is truncated. But they decrease with the growth of $p$ (and $J$) when oscillations in classical Fast Hankel Transform increase.

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