The statistics of the entanglement changes generated by the
Hadamard-CNOT quantum circuit

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Abstract

We consider the change of entanglement of formation $\Delta E$ produced by the Hadamard-CNOT circuit on a general (pure or mixed) state $\rho$ describing a system of two qubits. We study numerically the probabilities of obtaining different values of $\Delta E$, assuming that the initial state is randomly distributed in the space of all states according to the product measure recently introduced by Zyczkowski et al. [Phys. Rev. A \textbf{58} (1998) 883].

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Entanglement is one of the most fundamental phenomena of quantum mechanics [1]. It is a physical resource, like energy, associated with the peculiar non-classical correlations that are possible between separated quantum systems. One needs entanglement so as to implement quantum information processes [2,3,4,5,6,7,8,9] such as quantum cryptographic key distribution [10], quantum teleportation [11], superdense coding [12], and quantum computation [13,14,15]. Production of entanglement is the elementary prerequisite for any quantum computation. This basic task is accomplished by unitary transformations $\hat{U}$ (quantum gates) representing quantum evolution acting on the space state of multipartite systems. $\hat{U}$ should describe nontrivial interactions among the degrees of freedom of its subsystems.

One of the fundamental questions about quantum computation is then how to construct an adequate set of quantum gates, and a nice answer can be given: any generic two-qubits gate suffices for universal computation [16]. One would then be legitimately interested in ascertaining just how efficient distinct $\hat{U}$’s are as entanglers. In this respect, much exciting work has recently been performed (see, for instance, [17,18,19,20,21,22,23]).

A state of a composite quantum system is called “entangled” if it cannot be represented as a mixture of factorizable pure states. Otherwise, the state is called separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually [24,25]. A physically motivated measure of entanglement is provided by the entanglement of formation $E[\rho]$ [26]. This measure quantifies the resources needed to create a given entangled state $\rho$. That is, $E[\rho]$ is equal to the asymptotic limit (for large $n$) of the quotient $m/n$, where $m$ is the number of singlet states needed to create $n$ copies of the state $\rho$ when the optimum procedure based on local operations is employed. The entanglement of formation for two-qubits systems is given by Wootters’ expression [27],

$$E[\rho] = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \quad (1)$$

where

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \quad (2)$$
and $C$ stands for the *concurrence* of the two-qubits state $\rho$. The concurrence is given by

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

(3)

$\lambda_i$, $(i = 1, \ldots, 4)$ being the square roots, in decreasing order, of the eigenvalues of the matrix $\rho \tilde{\rho}$, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y).$$

(4)

The above expression has to be evaluated by recourse to the matrix elements of $\rho$ computed with respect to the product basis.

In the present effort we will concern ourselves with one particular quantum circuit: the Hadamard-CNOT circuit, that combines two gates: a single-qubit one (Hadamard’s) with a two-qubits gate (CNOT). Quantum logic gates are unitary evolution operators $\hat{U}$ that act on the states of a certain number of qubits. If the number of such qubits is $m$, the quantum gate is represented by a $2^m \times 2^m$ matrix in the unitary group $U(2^m)$. These gates are reversible: one can reverse the action, thereby recovering an initial quantum state from a final one. We shall work here with $m = 2$. The simplest nontrivial 2-qubits operation is the quantum controlled-NOT, or CNOT (equivalently, the exclusive OR, or XOR). Its classical counterpart is a reversible logic gate operating on two bits: $e_1$, the control bit, and $e_2$, the target bit. If $e_1 = 1$, the value of $e_2$ is negated. Otherwise, it is left untouched. The quantum CNOT gate $C_{12}$ (the first subscript denotes the control bit, the second the target one) plays a paramount role in both experimental and theoretical efforts that revolve around the quantum computer concept. In a given orthonormal basis $\{|0\rangle, |1\rangle\}$, and if we denote addition modulo 2 by the symbol $\oplus$, we have

$$|e_1\rangle |e_2\rangle \rightarrow C_{12} \rightarrow |e_1\rangle |e_1 \oplus e_2\rangle.$$  

(5)

In conjunction with simple single-qubit operations, the CNOT gate constitutes a set of gates out of which *any quantum gate may be built* \[16\]. In other words, single qubit and CNOT gates are universal for quantum computation \[16\].
As stated, the CNOT gate operates on quantum states of two qubits and is represented by a 4x4-matrix. This matrix has a diagonal block form. The upper diagonal block is just the unit 2x2 matrix. The lower diagonal 2x2 block is the representation of the one-qubit NOT gate $U_{\text{NOT}}$, of the form

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
$$

Note that, of course, $C_{12}^2 = 1$. This gate is able to transform factorizable pure states into entangled ones, i.e.,

$$C_{12} : [c_1|0\rangle + c_2|1\rangle]|0\rangle \leftrightarrow c_1|0\rangle|0\rangle + c_2|1\rangle|1\rangle,$$

and this transformation can be reversed by applying the same CNOT operation once more.

The Hadamard transform $T_H$ ($T_H^2 = 1$) is given by

$$T_H = \frac{1}{\sqrt{2}}\sigma_1 + \sigma_3,$$

and acts on the single qubit basis $\{|0\rangle, |1\rangle\}$ in the following fashion

$$T_H|0\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

$$T_H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Consider now the two-qubits uncorrelated basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. If we act with $T_H$ on the members of this basis we obtain

$$\frac{1}{\sqrt{2}} [ |1\rangle - |0\rangle ] \ |0\rangle$$

$$\frac{1}{\sqrt{2}} [ |1\rangle - |0\rangle ] \ |1\rangle$$

$$\frac{1}{\sqrt{2}} [ |0\rangle + |1\rangle ] \ |0\rangle$$

$$\frac{1}{\sqrt{2}} [ |0\rangle + |1\rangle ] \ |1\rangle,$$

so that the posterior action of the CNOT gate yields
\[
\frac{1}{\sqrt{2}} [ |1 > |1 > - |0 > |0 > ]
\]
\[
\frac{1}{\sqrt{2}} [ |1 > |0 > - |0 > |1 > ]
\]
\[
\frac{1}{\sqrt{2}} [ |0 > |0 > + |1 > |1 > ]
\]
\[
\frac{1}{\sqrt{2}} [ |0 > |1 > + |1 > |0 > ]
\]

(11)

i.e., save for an irrelevant overall phase factor in two of the kets, the maximally correlated Bell basis \(|\phi^\pm\rangle, |\psi^\pm\rangle\). We see then that the \(T_H\)-CNOT combination transforms an uncorrelated basis in the maximally correlated one.

Now, the two-qubits systems with which we are going to be concerned here are the simplest quantum mechanical systems exhibiting the entanglement phenomenon and play a fundamental role in quantum information theory. The concomitant space \(\mathcal{H}\) of mixed states is 15-dimensional and its properties are not of a trivial character. While the entanglement of pure states can be regarded as well understood, the entanglement of mixed states still has many properties that deserve further investigation. The reason for this state of affairs lies in the fact the quantum content of the associated correlations is somewhat obscured by the classical correlations in a mixed state. A mixed state which does not violate any Bell inequality can nonetheless exhibit quantum mechanical correlations, as one can distill from it pure maximally entangled states that violate Bell inequalities [6].

There are still then \(\mathcal{H}\)-features, related to the phenomenon of entanglement, that have not yet been characterized in full detail. One such characterization problem will occupy us here. We shall perform a systematic numerical survey of the action of the \(T_H\)-CNOT circuit on our 15-dimensional space in order to ascertain the manner in which \(P(\Delta E)\) is distributed in \(\mathcal{H}\), with \(P\) the probability of generating a change \(\Delta E\) associated to the action of this reversible quantum circuit. This kind of exploratory work is in line with recent efforts towards the systematic exploration of the space of arbitrary (pure or mixed) states of composite quantum systems [29,30,31] in order to determine the typical features exhibited by these states with regards to the phenomenon of quantum entanglement [29,30,31,32,33,34,35]. It is important
to stress the fact that we are exploring a space in which the majority of states are mixed. The exciting investigations reported in [17,18,19,20,21] address mainly pure states. We will try to answer the question: given an initial degree of entanglement of formation $E$, what is the probability $P(\Delta E)$ of encountering a change in entanglement $\Delta E$ upon the action of this circuit?

Our answer will arise from a Monte Carlo exploration of $\mathcal{H}$. To do this we need to define a proper measure on $\mathcal{H}$. The space of all (pure and mixed) states $\rho$ of a quantum system described by an $N$-dimensional Hilbert space can be regarded as a product space $\mathcal{S} = \mathcal{P} \times \Delta$ [29,30]. Here $\mathcal{P}$ stands for the family of all complete sets of orthonormal projectors $\{\hat{P}_i\}_{i=1}^N$, $\sum_i \hat{P}_i = I$ ($I$ being the identity matrix). $\Delta$ is the set of all real $N$-uples $\{\lambda_1, \ldots, \lambda_N\}$, with $0 \leq \lambda_i \leq 1$, and $\sum \lambda_i = 1$. The general state in $\mathcal{S}$ is of the form $\rho = \sum_i \lambda_i \hat{P}_i$. The Haar measure on the group of unitary matrices $U(N)$ induces a unique, uniform measure $\nu$ on the set $\mathcal{P}$ [29,30,36]. On the other hand, since the simplex $\Delta$ is a subset of a $(N-1)$-dimensional hyperplane of $\mathbb{R}^N$, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathbb{R}^{N-1}$ provides a measure for $\Delta$. The aforementioned measures on $\mathcal{P}$ and $\Delta$ lead to a measure $\mu$ on the set $\mathcal{S}$ of quantum states [29,30],

$$\mu = \nu \times \mathcal{L}_{N-1}. \quad (12)$$

We are going to consider the set of states of a two-qubits system. Consequently, our system will have $N = 4$ and, for such an $N$, $\mathcal{S} \equiv \mathcal{H}$. All our present considerations are based on the assumption that the uniform distribution of states of a two-qubit system is the one determined by the measure (12). Thus, in our numerical computations we are going to randomly generate states of a two-qubits system according to the measure (12) and study the entanglement evolution of these states upon the action of our $T_H$-CNOT quantum circuit.

As a first step, we suggest that the reader take a look at Fig. 4a of Ref. [34]. There one finds a plot of the probability $P(E)$ of finding two-qubits states of $\mathcal{H}$ endowed with a given amount of entanglement $E$. In this graph, the solid line corresponds to all states (pure and mixed), while the dashed curve depicts pure state behaviour only. We clearly see...
that our probabilities are of a quite different character when they refer to of pure states than when they correspond to mixed ones. Most mixed states have null entanglement, or a rather small amount of it (see the enlightening discussion in [29]). For pure states it is more likely to encounter them endowed with an intermediate (between null and total) amount of entanglement. It is then important to ascertain how much entanglement the $T_H$-CNOT quantum circuit is able to generate on our 15-dimensional two-qubits space.

We deal with pure states only in Fig. 1. Fig. 1a plots the probability $P(\Delta E)$ of obtaining via the $T_H$-CNOT quantum circuit a final state with entanglement change $\Delta E = E_F - E_0$. In 1b we are concerned with the average value $\langle E_F \rangle$ pertaining to final states that result from the gate-operation on initial ones of a given (fixed) entanglement $E_0$ (solid line). The horizontal line is plotted for the sake of reference. It corresponds to the average entanglement of two-qubits pure states, equal to $1/(3 \ln 2)$. The diagonal line $\langle E_F \rangle = E_0$ is also shown (dashed line). $E_F$ is a decreasing function of $E_0$ although the quantum circuit considered increases the mean final entanglement approximately up to 0.5 for states with $E_0$ lying in the interval $[0, 0.5]$.

The same analysis, but involving now all states (pure and mixed), is summarized in Fig. 2. The graph 2a is the counterpart of 1a, while 2b is that of 2a. The dashed line of 2b, given for the sake of visual reference, if just the line $\langle E_F \rangle = E_0$. The two Figs. allow one to appreciate the fact that it is quite unlikely that we may generate, via the $T_H$-CNOT quantum circuit, a significant amount of entanglement if the initial state is separable. In Fig 2 we see that the mean final entanglement $\langle E_F \rangle$ rises rapidly near the origin, from zero, with $E_0$. The rate of entanglement-growth decreases steadily with $E_0$ and the interval in which $\langle E_F \rangle$ is greater than $E_0$ is significantly smaller that the one corresponding to pure states (Fig. 1b).

The $P(\Delta E)$ vs. $\Delta E$ plots exhibit a nitid peak at $\Delta E = 0$. The peak is enormously exaggerated if mixed states enter the picture (2a). Thus, if the initial state has null entanglement, our survey indicates that the most probable circumstance is that the circuit will leave its entanglement unchanged.
We performed a systematic survey, in the space of all two-qubits states, concerning the entanglement changes associated with the action of the $T_H$-CNOT circuit. We found that the probability distribution of entanglement changes obtained when the circuit acts on pure states is quite different from the distribution obtained when the circuit acts on general mixed states. The probability of entangling mixed states turns out to be rather small. On average, the $T_H$-CNOT transformation is more efficient, as entangler, when acting upon states with small initial entanglement, specially in the case of pure states.

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FIGURE CAPTIONS

Fig. 1-a) $P(\Delta E)$ vs. $\Delta E$ for pure states. The change of entanglement $\Delta E$ arises as a result of the action of the $T_H$-CNOT quantum circuit. b) Probability of obtaining, via the $T_H$-CNOT transformation, a final state with entanglement $E_F$, when the initial state is endowed with a given entanglement $E_0$ (solid line). The horizontal line depicts the mean entanglement of all pure states. The diagonal (dashed line) is drawn for visual reference.

Fig. 2 The same as in Fig. 1 for all states (pure and mixed).
fig. 1

Panel (a): A plot of $P(\Delta E)$ versus $\Delta E$.

Panel (b): A plot of $\langle E_F \rangle$ versus $E_0$.
fig. 2

(A) $P(\Delta E)$ vs. $\Delta E$

(B) $\langle E_F \rangle$ vs. $E_0$