Some results on paracontact metric 
\((k, \mu)\)-manifolds with respect to the Schouten-van Kampen connection

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Abstract

In the present paper we study certain symmetry conditions and some types of solitons on paracontact metric \((k, \mu)\)-manifolds with respect to the Schouten-van Kampen connection. We prove that a Ricci semisymmetric paracontact metric \((k, \mu)\)-manifold with respect to the Schouten-van Kampen connection is an \(\eta\)-Einstein manifold. We investigate paracontact metric \((k, \mu)\)-manifolds satisfying \(\tilde{Q} \cdot \tilde{R}_{\text{cur}} = 0\) with respect to the Schouten-van Kampen connection. Also, we show that there does not exist an almost Ricci soliton in a \((2n + 1)\)-dimensional paracontact metric \((k, \mu)\)-manifold with respect to the Schouten-van Kampen connection such that \(k > -1\) or \(k < -1\). In case of the metric being an almost gradient Ricci soliton with respect to the Schouten-van Kampen connection, then we state that the manifold is either \(N(k)\)-paracontact metric manifold or an Einstein manifold. Finally, we present some results related to almost Yamabe solitons in a paracontact metric \((k, \mu)\)-manifold equipped with the Schouten-van Kampen connection and construct an example which verifies some of our results.

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1. Introduction

Kaneyuki [15] introduced the concept of paracontact metric (for short, pcm) structures in 1985. Recently, pcm manifolds have been studied by many authors, especially after the paper of Zamkovoy [32]. An important class among pcm manifolds is called the \((k, \mu)\)-manifold, which satisfies the nullity condition [6] given by

\[
R_{\text{cur}}(U, W)\xi = \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW),
\]

(1.1)
for all $U, W$ vector fields on $M$, where $\kappa$ and $\mu$ are constants and $h = \frac{1}{2}\mathcal{L}_V \Psi$. This class also includes the para-Sasakian manifolds [15,32], the pcm manifolds satisfying $R_{cur}(U, W)\xi = 0$, for all $U, W$ [33].

Symmetry property is one of the essential tools for investigating the geometry of manifolds. Symmetric Riemannian manifolds, that is Riemannian manifolds admitting $\nabla R_{cur} = 0$, where $R_{cur}$ is the curvature tensor and $\nabla$ is the Levi-Civita (for short, LC) connection, were introduced locally by Shirokov. In 1927, Cartan presented a comprehensive theory of symmetric Riemannian manifolds. If the curvature tensor $R_{cur}$ of a manifold satisfies $R_{cur}(U, W) \cdot R_{cur} = 0$, then it is called a semisymmetric manifold. Here, $R_{cur}(U, W)$ is viewed as a derivation of the tensor algebra at each point of the manifold for the tangent vectors $U, W$. A local classification of semisymmetric manifolds were made by Szabó [27]. In addition, a manifold satisfying $R_{cur}(U, W) \cdot Ric = 0$, where $Ric$ denotes the Ricci tensor of type $(0, 2)$, is called Ricci semisymmetric. Mirzoyan gave a general classification of manifolds of this type in [17]. For certain curvature conditions on pcm $(\kappa, \mu)$-spaces we refer [16].

A pcm $(\kappa, \mu)$-manifold admitting a Ricci tensor satisfying $Ric = \lambda_1 g$ (resp., $Ric = \lambda_1 g + \lambda_2 \eta \otimes \eta$) is called Einstein (resp., $\eta$-Einstein) manifold, where $\lambda_1$ and $\lambda_2$ are constants.

Riemannian manifolds with hyperdistributions and the Schouten van-Kampen (for short, S-vK) connection which is one of the most suitable connection adaptable to the hyperdistributions, were studied by Solov’ev [23–26]. Also see [2,13,21]. Almost pcm manifolds with the S-vK connection and curvature identities of such manifolds were investigated by Olszak [19].

As a generalization of Einstein manifold, an almost Ricci soliton $(M, g, \lambda)$ was defined as a Riemannian manifold endowed with a complete vector field $V$ satisfying

$$\mathcal{L}_V g + 2 Ric + 2 \lambda g = 0,$$

where $\mathcal{L}$ denotes the Lie derivative, $Ric$ is the Ricci tensor on $M$ and $\lambda$ is a differentiable function [12]. If $\lambda$ is negative, zero and positive, then the almost Ricci soliton is called shrinking, steady and expanding, respectively. The concept of the $\eta$-Ricci soliton was introduced in [8].

An almost $\eta$-Ricci soliton is a Riemannian manifold $(M, g, \lambda, \mu)$ admitting a differentiable vector field $V$ such that the Ricci tensor $Ric$ of $M$ satisfies

$$\mathcal{L}_V g + 2 Ric + 2 \lambda g + 2 \beta \eta \otimes \eta = 0,$$

where $\lambda$ and $\mu$ are some differentiable functions. In case of the vector field $V$ is being the gradient of a potential function $-f$, the equation (1.2) reduces to

$$\nabla \nabla f = Ric + \lambda g,$$

and an almost Ricci soliton is said to be an almost gradient Ricci soliton.

It was proved in [12,14] that, for 2-dimensional and 3-dimensional cases, a Ricci soliton on a compact manifold is of constant curvature (see also [9] and [10]). For further read we refer [3,4,20,22].

For solving the Yamabe problem, the Yamabe flows were firstly introduced in [12]. Yamabe solitons are self-similar solutions for Yamabe flows and they seem to be as singularity models. More clearly, the Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively. For further read, we refer [1,5,7,11,18,28–31]. As a generalization of Yamabe solitons, an almost Yamabe soliton is a Riemannian manifold $(M, g)$ endowed with a vector field $V$ satisfying [1]

$$\mathcal{L}_V g - 2(r - \delta) g = 0,$$

where $r$ is the scalar curvature of $M$ and $\delta$ is a differentiable function. An almost Yamabe soliton is called expanding, steady or shrinking, if $\delta < 0$, $\delta = 0$ or $\delta > 0$, respectively. In case of $\delta$ is being a constant, then an almost Yamabe soliton induces to a Yamabe soliton.
Moreover, if the Yamabe soliton is of constant scalar curvature $Sc$, then the Riemannian metric $g$ is said to be a Yamabe metric.

In the present paper, we study certain semisymmetry conditions and some types of solitons in pcm $(\kappa, \mu)$-manifolds. Following the introduction, Section 2 is devoted to some basic concepts that will be need throughout the paper. In Section 3, some properties of pcm $(\kappa, \mu)$-manifolds endowed with the S-vK connection are presented. In section 4, we prove that Ricci semisymmetric pcm $(\kappa, \mu)$-manifold with respect to (for short, wrt) the S-vK connection is an $\eta$-Einstein manifold. In section 5, we study pcm $(\kappa, \mu)$-manifolds satisfying $Q \cdot \bar{R}_{\text{car}} = 0$ wrt the S-vK connection. In section 6, we investigate almost Ricci soliton and almost $\eta$-Ricci soliton types on pcm $(\kappa, \mu)$-manifolds wrt the S-vK connection.

We show that there does not exist an almost Ricci soliton in a pcm $(\kappa, \mu)$-manifold wrt the S-vK connection with $\kappa > -1$ or $\kappa < -1$. Section 7 is devoted to pcm $(\kappa, \mu)$-manifolds $(\kappa \neq -1)$ admitting almost gradient Ricci soliton. In Section 8, we obtain some results related to almost Yamabe solitons in a pcm $(\kappa, \mu)$-manifold and construct an example which verifies some of our results.

2. Preliminaries

Let $M$ be $(2n + 1)$-dimensional differentiable manifold endowed with a tensor field $\Psi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\eta(\xi) = 1, \quad \Psi^2 = I - \eta \otimes \xi,$$

and $\Psi$ induces an almost paracomplex structure on each fibre of ker$(\eta)$. From the last equation, we get $\Psi \xi = 0$, $\eta \circ \Psi = 0$ and note that the rank of endomorphism $\Psi$ is $2n$. If an almost paracontact manifold $M$ admits a pseudo-Riemannian metric $g$ such that

$$g(\Psi U, \Psi W) = -g(U, W) + \eta(U)\eta(W),$$

for all $U, W \in \Gamma(TM)$, then the manifold is said to be an almost pcm manifold. The signature of the pseudo-Riemannian metric $g$ is $(n + 1, n)$ and an orthogonal basis $\{U_i, W_j, \xi\}$, namely a $\Psi$-basis, satisfying $g(U_i, U_j) = \delta_{ij}$, $g(W_i, W_j) = -\delta_{ij}$, $g(U_i, W_j) = 0$, $g(\xi, U_i) = g(\xi, W_j) = 0$, and $W_i = \Psi U_i$, for any $i, j \in \{1, \ldots, n\}$ can always be constructed for an almost pcm manifold.

The fundamental form of the almost pcm manifold is given by $\theta(U, W) = g(U, \Psi W)$. An almost pcm manifold with $d\eta = \theta$ is called a pcm manifold. In a pcm manifold, by help of Lie derivative $L_\xi$ of the fundamental form, a trace-free symmetric operator $h$ can be defined by $h = \frac{1}{2}L_\xi \Psi$. This operator [32] anti-commutes with $\Psi$ and satisfies $h \xi = 0$, $trh = trh \Psi = 0$ and

$$\nabla_U \xi = -\Psi U + \Psi hU,$$

$$\nabla_U \eta W = g(U, \Psi W) - g(hU, \Psi W),$$

where $\nabla$ is the LC connection of the manifold. In addition, $h = 0$ if and only if $\xi$ is Killing vector field, which implies that $(M, \Psi, \xi, \eta, g)$ is said to be a K-paracontact manifold. A normal pcm manifold is said to be a para-Sasakian manifold. Each para-Sasakian manifold is a K-paracontact manifold and but the converse holds only in 3-dimensional case. We also recall that any para-Sasakian manifold satisfies

$$R_{\text{car}}(U, W) \xi = \eta(U)W - \eta(W)U,$$

where $R_{\text{car}}$ is Riemannian curvature operator given by

$$R_{\text{car}}(U, W) Z = \nabla_U \nabla_W Z - \nabla_W \nabla_U Z - \nabla_{[U, W]} Z.$$
3. Pcm \((k, \mu)\)-manifolds with respect to the Schouten-van Kampen connection

A distribution defined by

\[
N(\kappa, \mu) : p \rightarrow N_p(\kappa, \mu) = \left\{ Z \in T_pM \mid R_{\text{cur}}(U, W)Z = \kappa(g(W, Z)U - g(U, Z)W) + \mu(g(W, Z)hU - g(U, Z)hW) \right\},
\]

is called the \((\kappa, \mu)\)-nullity distribution of a pcm manifold \((M, \Psi, \xi, \eta, g)\) for the pair \((\kappa, \mu)\), where \(\kappa\) and \(\mu\) are some real constants. In case of the characteristic vector field \(\xi\) belonging to the \((\kappa, \mu)\)-nullity distribution, from (3.1) we write

\[
R_{\text{cur}}(U, W)\xi = \kappa(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW),
\]

for all \(U, W \in \Gamma(TM)\). We refer [6] for basic results of pcm manifolds with the characteristic vector field satisfying the nullity condition (the condition (3.1)), for some real numbers \(\kappa\) and \(\mu\).

**Lemma 3.1** ([6]). In a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold \((M, \Psi, \xi, \eta, g)\), the followings hold:

\[
h^2 = (\kappa + 1)\Psi^2,
\]

\[
(\nabla_U\Psi)W = -g(U, W)\xi + g(hU, W)\xi + \eta(W)U - \eta(W)hU,
\]

for \(\kappa \neq -1\), \(\mu \neq 0\) (3.2)

\[
(\nabla_Uh)W = -((1 + \kappa)g(U, \Psi W) + g(U, \Psi hW))\xi + \eta(W)hU - \eta(hU)W,
\]

for \(\kappa \neq -1\), \(\mu \neq 0\) (3.3)

\[
(\nabla_U\Psi h)W = g(h^2U - hU, W)\xi + \eta(W)(h^2U - hU) - \kappa\eta(U)hW,
\]

for \(\kappa > -1\), \(\mu \neq 0\) (3.4)

\[
(\nabla_U\Psi h)W = (1 + \kappa)g(U, W)\xi - g(hU, W)\xi + \eta(W)hU - \eta(hU)hW,
\]

for \(\kappa < -1\), \(\mu \neq 0\) (3.5)

\[
QW = (2(1 - n) + n\mu)W + (2(n - 1) + \mu)hW + (2(n - 1) + n(2\kappa - \mu))\eta(W)\xi,
\]

for \(\kappa \neq -1\), \(\mu \neq 0\) (3.6)

\[
Q\xi = 2n\kappa\xi,
\]

\[
(\nabla_U h)W - (\nabla_W h)U = -(1 + \kappa)(2g(U, \Psi W)\xi + \eta(U)\Psi W - \eta(W)\Psi U) + (1 - \mu)(\eta(W)\Psi hU - \eta(U)\Psi U),
\]

for \(\kappa \neq -1\), \(\mu \neq 0\) (3.7)

\[
(\nabla_U\Psi h)W - (\nabla_W\Psi h)U = (1 + \kappa)(\eta(W)U - \eta(U)W) + (1 - \mu)(\eta(W)hU - \eta(U)hW),
\]

for \(\kappa \neq -1\), \(\mu \neq 0\) (3.8)

for any vector fields \(U, W\) on \(M\).

Some important subclasses of pcm \((k, \mu)\)-manifolds are given, regarding (1.1), by para-Sasakian manifolds, and pcm manifolds satisfying \(R_{\text{cur}}(U, W)\xi = 0\). In [33], the authors showed that the pcm manifold \((M^{2n+1}, \Psi, \xi, \eta, g)\) with \(n > 1\) satisfying the last condition is locally isometric to a product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of negative constant curvature \(-4\). From (3.2), note that \(h^2 = 0\) on a pcm \((k, \mu)\)-manifold with \(\kappa = -1\).

On the other hand we have two naturally defined distributions in the tangent bundle \(TM\) of \(M\) as follows:

\[
D^H = \ker \eta, \quad D^V = \text{span}\{\xi\}.
\]

Then we have \(TM = D^H \oplus D^V\), \(D^H \cap D^V = \{0\}\) and \(D^H \perp D^V\). This decomposition allows one to define the S-vK connection \(\tilde{\nabla}\) over an almost paracontact metric structure.
The S-vK connection $\tilde{\nabla}$ on an almost (para) contact metric manifold with respect to LC-connection $\nabla$ is defined by [23]

$$\tilde{\nabla}_UVW = \nabla_UW - \eta(W)\nabla_U\xi + (\nabla_U\eta)(W)\xi.$$ \hfill (3.11)

Thus with the help of the S-vK connection given by (3.11), many properties of some geometric objects connected with the distributions $D^H$ and $D^V$ can be characterized [23–25]. For example $g$, $\xi$ and $\eta$ are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0, \tilde{\nabla}g = 0, \tilde{\nabla}\eta = 0$.

Also the torsion $T\tilde{\nabla}$ of $\tilde{\nabla}$ is defined by

$$T\tilde{\nabla}(U, W) = \eta(U)\nabla_W\xi - \eta(W)\nabla_U\xi + 2d\eta(U, W)\xi.$$ \hfill (3.12)

Now we consider a pcm $(\kappa, \mu)$-manifold wrt the S-vK connection. Firstly, using (2.2) and (2.3) in (3.11), we get

$$\tilde{\nabla}_UVW = \nabla_UW + \eta(W)\Psi_U - \eta(W)\Psi_U - g(U, \Psi W)\eta(U)\xi.$$ \hfill (3.13)

Let $\tilde{R}_{\text{cur}}$ be the curvature tensors of the S-vK connection $\tilde{\nabla}$ given by $\tilde{R}_{\text{cur}}(U, W) = [\tilde{\nabla}_U, \tilde{\nabla}_W] - \tilde{\nabla}_{[U, W]}$. Using (3.13) in the definition of $\tilde{R}_{\text{cur}}(U, W)$, we have

$$\tilde{R}_{\text{cur}}(U, W) = \tilde{\nabla}_UVW + \eta(W)\Psi_U - \eta(W)\Psi_U - g(U, \Psi W)\xi,$$ \hfill (3.14)

Using (3.9), (3.10) and (3.3) in (3.14), we have the relation between $R_{\text{cur}}$ and $\tilde{R}_{\text{cur}}$ on $M$

$$\tilde{R}_{\text{cur}}(U, W) = R_{\text{cur}}(U, W) + g(U, \Psi Z)\Psi W - g(W, \Psi Z)\Psi U + g(hW, \Psi Z)\Psi U - g(hU, \Psi Z)\Psi U - g(hW, \Psi Z)\Psi hU + g(U, \Psi Z)\Psi hU + \eta(U, Z)\eta(W)\xi - \eta(W, Z)\eta(U)\xi + \eta(U, Z)hW - \eta(W, Z)hU.$$ \hfill (3.15)

Now from (3.15), we get

$$g(\tilde{R}_{\text{cur}}(U, W), Z) = g(R_{\text{cur}}(U, W), Z) + g(U, \Psi Z)g(\Psi W, T) - g(W, \Psi Z)g(\Psi U, T) + g(hW, \Psi Z)g(\Psi U, T) - g(hU, \Psi Z)g(\Psi W, T) - g(hW, \Psi Z)g(\Psi hU, T) + \eta(U, Z)\eta(W)\eta(T) - g(W, Z)\eta(U)\eta(T) + g(W, T)\eta(U)\eta(Z) - g(U, T)\eta(W)\eta(Z) + \mu\{g(hU, Z)\eta(W)\eta(T) - g(hW, Z)\eta(U)\eta(T) + g(hW, T)\eta(U)\eta(Z) - g(hU, T)\eta(W)\eta(Z)\}.$$ \hfill (3.16)

If we take $U = T = e_i$, $i = 1, \ldots, 2n + 1$, in (3.16), where $\{e_i\}$ is an orthonormal basis of $\chi(M)$, we get

$$\tilde{R}_{\text{cur}}(W, Z) = Ric(W, Z) - 2n\kappa\eta(W)\eta(Z) - \mu g(hW, Z),$$ \hfill (3.17)
where $\tilde{\text{Ric}}$ and $\text{Ric}$ denote the Ricci tensor of the connections $\tilde{\nabla}$ and $\nabla$, respectively. As a consequence of (3.17), we get for the Ricci operator $\tilde{Q}$

$$\tilde{Q}W = QW - 2n\kappa\eta(W)\xi - \mu hW. \tag{3.18}$$

Also if we take $W = Z = e_i$, $\{i = 1, ..., 2n + 1\}$, in (3.18), we get

$$\tilde{r} = r - 2n\kappa, \tag{3.19}$$

where $\tilde{r}$ and $r$ denote the scalar curvatures of the connections $\tilde{\nabla}$ and $\nabla$, respectively.

4. Ricci semisymmetric pcm $(\kappa, \mu)$-manifolds with respect to the Schouten-van Kampen connection

In this section we study Ricci semisymmetric pcm $(\kappa, \mu)$-manifolds wrt the S-vK connection. Firstly we give the following:

**Definition 4.1.** A semi-Riemannian manifold $(M, g)$, $n > 1$, is said to be Ricci semisymmetric if we have

$$R_{cur}(U, W) \cdot \text{Ric} = 0,$$

holds on $M$ for all $U, W \in \chi(M)$.

Let $M$ be a Ricci semisymmetric pcm $(\kappa, \mu)$-manifold with $(\kappa \neq -1)$ wrt the S-vK connection. Then above equation is equivalent to

$$(\tilde{R}_{cur}(U, W) \cdot \tilde{\text{Ric}})(Z, T) = 0,$$

for any $U, W, Z, T \in \chi(M)$. Thus we have

$$\tilde{\text{Ric}}(\tilde{R}_{cur}(U, W)Z, T) + \tilde{\text{Ric}}(Z, \tilde{R}_{cur}(U, W)T) = 0, \tag{4.1}$$

Using (3.15) in (4.1), we get

$$\kappa \left[ \begin{array}{c} \{g(W, Z) - \eta(W)\eta(Z)\} \tilde{\text{Ric}}(U, T) \\ -\{g(U, Z) - \eta(U)\eta(Z)\} \tilde{\text{Ric}}(W, T) \\ +\{g(W, T) - \eta(W)\eta(T)\} \tilde{\text{Ric}}(U, Z) \\ -\{g(U, T) - \eta(U)\eta(T)\} \tilde{\text{Ric}}(W, Z) \end{array} \right]$$

$$+ \mu \left[ \begin{array}{c} \{g(W, Z) - \eta(W)\eta(Z)\} \tilde{\text{Ric}}(hU, T) \\ -\{g(U, Z) - \eta(U)\eta(Z)\} \tilde{\text{Ric}}(hW, T) \\ +\{g(W, T) - \eta(W)\eta(T)\} \tilde{\text{Ric}}(hU, Z) \\ -\{g(U, T) - \eta(U)\eta(T)\} \tilde{\text{Ric}}(hW, Z) \end{array} \right]$$

$$+ g(U, \Psi Z) \tilde{\text{Ric}}(\Psi W, T) - g(W, \Psi Z) \tilde{\text{Ric}}(\Psi U, T)$$

$$+ g(hW, \Psi Z) \tilde{\text{Ric}}(\Psi U, T) - g(hU, \Psi Z) \tilde{\text{Ric}}(\Psi W, T)$$

$$+ g(W, \Psi Z) \tilde{\text{Ric}}(\Psi hU, T) - g(U, \Psi Z) \tilde{\text{Ric}}(\Psi hW, T)$$

$$+ g(hU, \Psi Z) \tilde{\text{Ric}}(\Psi hW, T) - g(hW, \Psi Z) \tilde{\text{Ric}}(\Psi hU, T)$$

$$+ g(U, \Psi T) \tilde{\text{Ric}}(\Psi W, Z) - g(W, \Psi T) \tilde{\text{Ric}}(\Psi U, Z)$$

$$+ g(hW, \Psi T) \tilde{\text{Ric}}(\Psi U, Z) - g(hU, \Psi T) \tilde{\text{Ric}}(\Psi W, Z)$$

$$+ g(W, \Psi T) \tilde{\text{Ric}}(\Psi hU, Z) - g(U, \Psi T) \tilde{\text{Ric}}(\Psi hW, Z)$$

$$+ g(hU, \Psi T) \tilde{\text{Ric}}(\Psi hW, Z) - g(hW, \Psi T) \tilde{\text{Ric}}(\Psi hU, Z) = 0. \tag{4.2}$$

Putting $U = T = e_i$, $\{i = 1, ..., 2n + 1\}$, in (4.2), we obtain

$$\kappa\tilde{r}\{g(W, Z) - \eta(W)\eta(Z)\} - 2n\kappa\tilde{\text{Ric}}(W, Z) - 2n\mu\tilde{\text{Ric}}(hW, Z) = 0. \tag{4.3}$$

Now putting $W = hW$ in (4.3), we have

$$\kappa\tilde{r}g(hW, Z) - 2n\kappa\tilde{\text{Ric}}(hW, Z) - 2n(\kappa + 1)\mu\tilde{\text{Ric}}(W, Z) = 0. \tag{4.4}$$
Assume that $\kappa \neq -1$ and $\mu \neq 0$. Multiplying with (4.3) by $\kappa$ and (4.4) by $\mu$, then subtract the results, we obtain
\[
2n[\kappa^2 - \mu^2(\kappa + 1)]\bar{R}ic(W, Z) = \kappa^2 R\{g(W, Z) - \eta(W)\eta(Z)\} - \kappa\mu\bar{g}(hW, Z). \tag{4.5}
\]
Using (3.17) in (4.5), we get
\[
2n[\kappa^2 - \mu^2(\kappa + 1)]\{Ric(W, Z) - 2n\kappa\eta(W)\eta(Z) - \mu g(hW, Z)\}
= \kappa^2 R\{g(W, Z) - \eta(W)\eta(Z)\} + (\mu - \kappa\mu\bar{r} - 1)g(hW, Z),
\]
i.e.,
\[
Ric(W, Z) = \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))}g(W, Z)
+ (2n\kappa - \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))})\eta(W)\eta(Z)
+ (\mu - \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))})g(hW, Z). \tag{4.6}
\]
Again using (3.7) in (4.6), we have
\[
Ric(W, Z) = \frac{A_1 - C_1B_2}{1 - A_2C_1}g(W, Z) + \frac{B_1 - C_1C_2}{1 - A_2C_1}\eta(W)\eta(Z), \tag{4.7}
\]
where
\[
A_1 = \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))}, \quad B_1 = \frac{2n\kappa - \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))}}{2n(1 - \mu)}
\]
\[
C_1 = \frac{\mu - \frac{\kappa^2 R}{2n(\kappa^2 - \mu^2(\kappa + 1))}}{2n(1 - \mu)}, \quad A_2 = \frac{2n - 1}{2n(1 - \mu)}
\]
\[
B_2 = \frac{2(1 - n) + n\mu}{2(n - 1) + \mu}, \quad C_2 = \frac{2(1 - n) + n(2\kappa - \mu)}{2(n - 1) + \mu}.
\]
Therefore, from (4.7) we have the following:

**Theorem 4.2.** Let $M$ be a $(2n + 1)$-dimensional pcm $(\kappa, \mu)$-manifold with $\kappa \neq -1$. If $M$ is a Ricci semisymmetric pcm $(\kappa, \mu)$-manifold wrt the S-vK connection then the manifold $M$ is an $\eta$-Einstein manifold wrt the LC connection provided $\mu \neq 2(1 - n)$.

5. Pcm $(\kappa, \mu)$-manifolds satisfying $\bar{\cal{Q}} \cdot \bar{R} = 0$ with respect to the Schouten-van Kampen connection

In this section we study the condition $\bar{\cal{Q}} \cdot \bar{R}_{\text{cur}} = 0$ on pcm $(\kappa, \mu)$-manifolds wrt the S-vK connection. Firstly we give the following:

\[(\bar{\cal{Q}} \cdot \bar{R}_{\text{cur}})(U, W)Z = \bar{\cal{Q}}\bar{R}_{\text{cur}}(U, W)Z - \bar{R}_{\text{cur}}(\bar{\cal{Q}}U, W)Z - \bar{R}_{\text{cur}}(U, \bar{\cal{Q}}W)Z - \bar{R}_{\text{cur}}(U, W)\bar{\cal{Q}}Z = 0.\]

Then we write
\[
g(\bar{\cal{Q}}\bar{R}_{\text{cur}}(U, W)Z, T) - g(\bar{R}_{\text{cur}}(\bar{\cal{Q}}U, W)Z, T)
- g(\bar{R}_{\text{cur}}(U, \bar{\cal{Q}}W)Z, T) - g(\bar{R}_{\text{cur}}(U, W)\bar{\cal{Q}}Z, T) = 0, \tag{5.1}
\]
which infers
\[
g(\bar{R}_{\text{cur}}(U, W)Z, \bar{\cal{Q}}T) + g(\bar{R}_{\text{cur}}(Z, T)W, \bar{\cal{Q}}U)
- g(\bar{R}_{\text{cur}}(Z, T)U, \bar{\cal{Q}}W) + g(\bar{R}_{\text{cur}}(U, W)T, \bar{\cal{Q}}Z) = 0.
\]
So we can write
\[
\bar{R}ic(\bar{R}_{\text{cur}}(U, W)Z, T) + \bar{R}ic(\bar{R}_{\text{cur}}(Z, T)W, U)
- \bar{R}ic(\bar{R}_{\text{cur}}(Z, T)U, W) + \bar{R}ic(\bar{R}_{\text{cur}}(U, W)T, Z) = 0. \tag{5.2}
\]
Now using (3.14) in (5.2), we compute

\[
\kappa \begin{bmatrix}
{g(W, T) - \eta(W)\eta(T)} \bar{\text{Ric}}(U, Z) \\
-{g(U, T) - \eta(U)\eta(T)} \bar{\text{Ric}}(W, Z) \\
+{g(W, T) - \eta(W)\eta(T)} \bar{\text{Ric}}(U, Z) \\
-{g(U, T) - \eta(U)\eta(T)} \bar{\text{Ric}}(W, Z)
\end{bmatrix}
+ \mu \begin{bmatrix}
{g(W, T) - \eta(W)\eta(T)} \bar{\text{Ric}}(hU, Z) \\
-{g(U, T) - \eta(U)\eta(T)} \bar{\text{Ric}}(hW, Z) \\
+{g(W, T) - \eta(W)\eta(T)} \bar{\text{Ric}}(hU, Z) \\
-{g(U, T) - \eta(U)\eta(T)} \bar{\text{Ric}}(hW, Z)
\end{bmatrix}
+ g(U, \Psi Z) \bar{\text{Ric}}(\Psi W, T) - g(W, \Psi Z) \bar{\text{Ric}}(\Psi U, T) + g(hW, \Psi Z) \bar{\text{Ric}}(\Psi U, T)
\]

Putting \( U = T = e_i, \{i = 1, ..., 2n + 1\} \), in (5.3), we have

\[
\kappa(1 - 2n) \bar{\text{Ric}}(W, Z) + \mu(1 - 2n) \bar{\text{Ric}}(hW, Z) + (\kappa + 1) \bar{\text{Ric}}(\Psi W, \Psi Z) + \bar{\text{Ric}}(W, Z) = 0,
\]

which entails

\[
(2n\kappa + 1) \bar{\text{Ric}}(W, Z) + \mu(2n - 1) \bar{\text{Ric}}(hW, Z) - 2(\kappa + 1)(2n - 2 + \mu)g(hW, Z) = 0. \tag{5.4}
\]

Now putting \( W = hW \) in (5.4), we have

\[
(2n\kappa + 1) \bar{\text{Ric}}(hW, Z) + \mu(2n - 1)(\kappa + 1) \bar{\text{Ric}}(W, Z) - 2(\kappa + 1)(2n - 2 + \mu)g(hW, Z) = 0. \tag{5.5}
\]

Multiplying (5.4) by \( 2n\kappa + 1 \) and (5.5) by \( \mu(2n - 1) \), we have

\[
(2n\kappa + 1)^2 \bar{\text{Ric}}(W, Z) + \mu(2n - 1)(2n\kappa + 1) \bar{\text{Ric}}(hW, Z)
\]

\[
-2(\kappa + 1)(2n\kappa + 1)(2n - 2 + \mu)g(hW, Z) = 0,
\]

and

\[
\mu(2n - 1)(2n\kappa + 1) \bar{\text{Ric}}(hW, Z) + \mu(2n - 1)^2(\kappa + 1) \bar{\text{Ric}}(W, Z)
\]

\[
-2(\kappa + 1)(2n - 2 + \mu)\mu(2n - 1)(\kappa + 1)\{g(W, Z) - \eta(W)\eta(Z)\} = 0,
\]

respectively. Subtracting (5.6) from (5.7), we get

\[
\bar{\text{Ric}}(W, Z) = \frac{\lambda_1}{\gamma}g(hW, Z) - \frac{\lambda_2}{\gamma}g(W, Z) + \frac{\lambda_2}{\gamma}\eta(W)\eta(Z), \tag{5.8}
\]
where
\[
\lambda_1 = 2(2n\kappa + 1)(\kappa + 1)(2n - 2 + \mu), \\
\lambda_2 = 2\mu(\kappa + 1)^2(2n - 2 + \mu), \\
\gamma = (2n\kappa + 1)^2 - (2n - 1)^2\mu(\kappa + 1).
\]

Now using (3.7) in (5.8), we obtain
\[
\left(\frac{\lambda_1}{\gamma(B - \mu)} - 1\right)\tilde{Ric}(W, Z) = \left(\frac{\lambda_1 A}{\gamma(B - \mu)} + \frac{\lambda_2}{\gamma}\right)g(W, Z) + \left(\frac{\lambda_1(C - 2n\kappa)}{\gamma(B - \mu)} - \frac{\lambda_2}{\gamma}\right)\eta(W)\eta(Z),
\]
where \(A = (2(1 - n) + n\mu), B = (2(n - 1) + \mu, C = 2(n - 1) + n(2\kappa - \mu).\)

The last equation can be written as
\[
\tilde{Ric}(W, Z) = \rho g(W, Z) + \sigma\eta(W)\eta(Z),
\]
where
\[
\rho = \frac{\lambda_1 A}{\gamma(B - \mu)} + \frac{\lambda_2}{\gamma} - 1, \\
\sigma = -\frac{\lambda_1(C - 2n\kappa)}{\gamma(B - \mu)} - \frac{\lambda_2}{\gamma}.
\]

Thus the manifold \(M\) is an \(\eta\)-Einstein manifold wrt the S-vK connection. Hence we have the following:

**Theorem 5.1.** Let \(M\) be a \((2n + 1)\)-dimensional pcm (\(\kappa, \mu\))-manifold with \(\kappa \neq -1\) satisfying the condition \(\tilde{Q} \cdot \tilde{R}_{\text{cur}} = 0\) wrt the S-vK connection. Then the manifold \(M\) is an \(\eta\)-Einstein manifold wrt the S-vK connection provided \(\frac{\lambda_1}{\gamma(B - \mu)} - 1 \neq 0\).

6. Almost Ricci solitons and almost \(\eta\)-Ricci solitons on pcm (\(\kappa, \mu\))-manifolds with respect to the Schouten-van Kampen connection

In this section we study almost Ricci solitons and almost \(\eta\)-Ricci soliton in pcm (\(\kappa, \mu\))-manifolds wrt the S-vK connection.

In a pcm (\(\kappa, \mu\))-manifold (\(\kappa \neq -1\)) with the S-vK connection, since \(\nabla g = 0\) by using (1.2), we get
\[
(\tilde{\mathcal{L}}_V g)(U, T) = g(\nabla_U V, T) + g(U, \nabla_T V) = (\mathcal{L}_V g)(U, T),
\]
where \(\tilde{\mathcal{L}}\) denotes the Lie derivative on manifold wrt the S-vK connection.

Now we consider an almost Ricci soliton on a pcm (\(\kappa, \mu\))-manifold wrt the S-vK connection. From (1.2), we can write
\[
(\tilde{\mathcal{L}}_V g + 2\tilde{Ric} + 2\tilde{\lambda} g)(U, T) = 0.
\]

Using (6.1) in (6.2), we obtain
\[
\left\{\begin{aligned}
(\mathcal{L}_V g)(U, T) + 2Ric(U, T) + 2\tilde{\lambda} g(U, T) \\
-4nK\eta(U)\eta(T) - 2\mu g(hU, T)
\end{aligned}\right. = 0.
\]

Thus we have the followings:

**Theorem 6.1.** A \((2n + 1)\)-dimensional pcm (\(\kappa, \mu\))-manifold \(M\) bearing an almost Ricci soliton \((V, \lambda, g)\) wrt the S-vK connection admits an almost \(\eta\)-Ricci soliton \((V, \lambda, -2n\kappa, g)\) wrt the LC connection provided the manifold is a \(N(\kappa)\)-pcm manifold.

**Corollary 6.2.** If \(M\) is a \((2n + 1)\)-dimensional pcm (\(\kappa, \mu\))-manifold bearing an almost Ricci soliton \((V, \lambda, g)\) wrt the S-vK connection, then \(M\) admits an almost Ricci soliton \((V, \lambda, g)\) wrt the LC connection provided the manifold is locally isometric to a product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of negative constant curvature equal to \(-4\).
Conversely, assume that a pcm \((\kappa, \mu)\)-manifold admits an almost Ricci soliton \((V, \lambda, g)\) wrt the LC connection. Then, from (1.2) and (3.17), we have
\[
\begin{pmatrix}
(L_V g)(U, T) + 2\tilde{Ric}(U, T) + 2\lambda g(U, T) \\
+ 4n\kappa\eta(U)\eta(T) + 2\mu g(hU, T)
\end{pmatrix} = 0.
\]

Hence we give the followings:

**Theorem 6.3.** Let \(M\) be a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold bearing an almost Ricci soliton \((V, \lambda, g)\) wrt the LC connection. Then \(M\) admits an almost \(\eta\)-Ricci soliton \((V, \lambda, 2n\kappa, g)\) wrt the S-vK connection provided the manifold is a \(N(\kappa)\)-pcm manifold.

**Corollary 6.4.** A \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold bearing an almost Ricci soliton \((V, \lambda, g)\) wrt the LC connection admits an almost \(\eta\)-Ricci soliton \((V, \lambda, 2n\kappa + \beta, g)\) wrt the S-vK connection provided the manifold is locally isometric to a product of a flat \((n + 1)\)-dimensional manifold and an \(n\)-dimensional manifold of negative constant curvature equal to \(-4\).

In case of \(g\) is being an almost \(\eta\)-Ricci soliton wrt the LC connection, we have the following:

**Theorem 6.5.** A \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold bearing an almost \(\eta\)-Ricci soliton \((V, \lambda, \beta, g)\) wrt the LC connection admits an almost \(\eta\)-Ricci soliton \((V, \lambda, 2n\kappa + \beta, g)\) wrt the S-vK connection provided the manifold is a \(N(\kappa)\)-pcm manifold.

**Proof.** Assume that \(M\) is a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold bearing an almost \(\eta\)-Ricci soliton \((V, \lambda, \beta, g)\) wrt the LC connection. From (1.2) and (3.17) we write
\[
(L_V g)(U, T) + 2\tilde{Ric}(U, T) + 2\lambda g(U, T) + 2(2n\kappa + \beta)\eta(U)\eta(T) + 2\mu g(hU, T) = 0.
\]

This completes the proof. \(\Box\)

Now we consider the case of the potential vector field being the structure vector field. Assume that \(M\) is a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold bearing an almost Ricci soliton \((\xi, \lambda, g)\) wrt the S-vK connection. Using (2.2), (3.17) and (6.1) in (6.2), we write
\[
g(U, \Psi hT) + g(QU, T) + \tilde{\lambda}g(U, T) - 2n\kappa\eta(U)\eta(T) - \mu g(hU, T) = 0. \quad (6.4)
\]

From (6.4), we get
\[
\Psi hU + QU + \tilde{\lambda}U - 2n\kappa\eta(U)\xi - \mu hU = 0. \quad (6.5)
\]

By taking covariant derivative of (6.5), we have
\[
(\nabla_X \Psi h) U + \Psi h(\nabla_X U) + (\nabla_X Q) U + Q\nabla_X U + X(\tilde{\lambda})U + \tilde{\lambda}\nabla_X U
- 2n\kappa (g(\nabla_X U, \xi) + g(U, \nabla_X \xi) + \eta(U)\nabla_X \xi)
- \mu (\nabla_X h) U - \mu h\nabla_X U = 0,
\]

which implies that
\[
(\nabla_X \Psi h) U + (\nabla_X Q) U + X(\tilde{\lambda})U
- 2n\kappa (g(U, \nabla_X \xi) + \eta(U)\nabla_X \xi) - \mu (\nabla_X h) U = 0. \quad (6.6)
\]

We have the following cases:

Case 1. Assume that \(\kappa > -1\). By using (3.5), (3.3), (3.4) and (2.2) in (6.6), we have
\[
g(h^2X - hX, U)\xi + \eta(U) (h^2X - hX) - \mu \eta(X)hU
- 2(n - 1)(1 + \kappa)g(X, \Psi U)\xi - 2(n - 1)g(X, \Psi hU)\xi
+ 2(n - 1) \{ \eta(U) (\Psi h^2X - \Psi hX) - \mu \eta(X) \Psi hU \} + X(\tilde{\lambda})U
+ (2(n - 1) - n\mu) \{ -(g(U, \Psi X) - g(U, \Psi hX))\xi - \eta(U) (\Psi X + \Psi hX) \} = 0,
\]

which implies that
(\kappa + 1)g(X, U)\xi - 2(\kappa + 1)\eta(X)\eta(U)\xi - g(hX, U)\xi + (\kappa + 1)\eta(U)X - \eta(U)hX
+ \mu\eta(X)hU - (2\kappa(n - 1) + n\mu)g(X, \Psi U)\xi - n\mu g(X, \Psi hU)\xi
+ (2\kappa(n - 1) + n\mu)\eta(U)\Psi X - n\mu\eta(U)\Psi hX
- 2\mu(n - 1)\eta(X)\Psi hU + X(\lambda)U = 0. \tag{6.7}

By contracting \( X \) in (6.7), we obtain

\[ 2n(\kappa + 1)\eta(U) = -U(\lambda). \tag{6.8} \]

On the other hand, by taking \( U = \xi \) in (6.5), we obtain

\[ \lambda = 0. \tag{6.9} \]

Using (6.9) in (6.8), we conclude that \( \kappa = -1 \), which contradicts with the assumption \( \kappa > -1 \).

Case 2. Assume that \( \kappa < -1 \). By using (3.6), (3.3), (3.4) and (2.2) in (6.6), we get

\[
\begin{align*}
(1 + \kappa)g(X, U)\xi - g(hX, U)\xi + \eta(U)(h^2X - hX) - \mu\eta(X)hU
+ (2\kappa(1 - n) - n\mu)g(X, \Psi U)\xi - n\mu g(X, \Psi hU)\xi
- (2\kappa(1 - n) - n\mu)\eta(U)\Psi X - n\mu\eta(U)\Psi hX
- \mu\eta(X)\Psi hU + X(\lambda)U = 0.
\end{align*}
\tag{6.10}
\]

By contracting \( X \) in (6.10), we have

\[ (2n + 1)(\kappa + 1)\eta(U) = -U(\lambda). \tag{6.11} \]

On the other hand, by taking \( U = \xi \) in (6.5), we get

\[ \lambda = 0. \]

Using the last equation in (6.11), we conclude that \( \kappa = -1 \), which contradicts with the assumption \( \kappa > -1 \).

Hence we give the following:

**Theorem 6.6.** There does not exist an almost Ricci soliton \((\xi, \lambda, \eta, g)\) in a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold \((M, g)\) wrt the S-vK connection with \( \kappa > -1 \) or \( \kappa < -1 \).

Now, we consider \( \kappa = -1 \). In this case we give the following:

**Theorem 6.7.** If a \((2n + 1)\)-dimensional pcm \((\kappa, \mu)\)-manifold \((M, g)\) wrt the S-vK connection admits an almost Ricci soliton \((\xi, \lambda, g)\), then the almost Ricci soliton is steady.

**Proof.** By putting \( U = \xi \) in (6.5), we get \( Q\xi = 2\kappa n\xi - \lambda \). On the other hand, from (3.8) we have \( \bar{Q}\xi = 2\kappa n\xi \). Therefore we obtain \( \lambda = 0 \), which completes the proof. \( \square \)

7. **Almost gradient Ricci solitons on pcm \((\kappa, \mu)\)-manifolds with respect to the Schouten-van Kampen connection**

If the vector field \( V \) is the gradient of a potential function \(-f\), that is \( V = -\text{grad} f \), then \( g \) is called an almost gradient Ricci soliton. In this case equation (1.2) becomes

\[ \nabla \text{grad} f = \text{Ric} + \lambda g, \tag{7.1} \]

where \( \nabla \) is the LC connection.

Now assume that \( M \) is a \((2n + 1)\)-dimensional \((n > 1)\) pcm \((\kappa, \mu)\)-manifold \((\kappa \neq -1)\) wrt the S-vK connection. If we take \( V = -\text{grad} f \) in (6.1), we write

\[ (\tilde{\mathcal{L}}_{\text{grad} f})(U, T) = (\mathcal{L}_{\text{grad} f})(U, T) = g(\nabla_U \text{grad} f, T) + g(U, \nabla_T \text{grad} f). \tag{7.2} \]

We can easily see that

\[ g(\nabla_U \text{grad} f, T) = g(U, \nabla_T \text{grad} f), \]
which implies that
\[ \tilde{\mathcal{L}}_{\text{grad}} g + 2\tilde{R}ic + 2\tilde{\lambda}g = 0, \] (7.3)
that is
\[ g(\nabla U \text{grad}, T) = \tilde{R}ic(U, T) + \tilde{\lambda}g(U, T). \] (7.4)
This reduces to
\[ \nabla U \text{grad} = \tilde{Q}U + \tilde{\lambda}U. \] (7.5)
Now from (7.5), we write
\[ R_{\text{cur}}(U, T)\text{grad} = \nabla U \nabla T \text{grad} - \nabla U \nabla T \text{grad} - \nabla [U, T] \text{grad} \]
\[ = \nabla U \tilde{Q} T + U(\tilde{\lambda})T - \tilde{\lambda} \nabla U T \]
\[ - \nabla T \tilde{Q} U - T(\tilde{\lambda})U - \tilde{\lambda} \nabla T U \]
\[ - \tilde{Q}[U, T] - \tilde{\lambda}[U, T] \]
which implies that
\[ R_{\text{cur}}(U, T)\text{grad} = (\nabla U Q)T - (\nabla T Q)U - 2n\kappa(2g(U, \Psi T) \]
\[ + \eta(T) \nabla U \xi - \eta(U) \nabla T \xi) \]
\[ - \mu((\nabla U h)T + (\nabla T h)U) + U(\tilde{\lambda})T - T(\tilde{\lambda})U. \] (7.6)
Taking covariant derivative of \( Q \) given by (3.7), we have
\[ (\nabla U Q)T = (2(n - 1) + n(2\kappa - \mu)) \left[ g(U, \Psi T)\xi + g(\Psi U, h T)\xi \right] \]
\[ - \eta(T) (\Psi U - \Psi h U) \]
\[ + (2(n - 1) + \mu)(\nabla U h)T. \] (7.7)
Using (7.7) and (2.2) in (7.6), we obtain
\[ R_{\text{cur}}(U, T)\text{grad} = 2(2\kappa - n^2)g(U, \Psi T)\xi \]
\[ + (n^2 + 2\kappa n - 2\kappa)(\eta(T)\Psi U - \eta(U)\Psi T) \]
\[ - (n^2 - 2\mu n + 2\mu)(\eta(T)\Psi h U - \eta(U)\Psi h T) \]
\[ + U(\tilde{\lambda})T - T(\tilde{\lambda})U, \] (7.8)
which implies that
\[ g(R_{\text{cur}}(U, T)\text{grad}, \xi) = 2(2\kappa - n^2)g(U, \Psi T) + U(\tilde{\lambda})\eta(T) - T(\tilde{\lambda})\eta(U). \] (7.9)
If we put \( U = \xi \) in the last equation, we get
\[ g(R_{\text{cur}}(\xi, T)\text{grad}, \xi) = \xi(\tilde{\lambda})\eta(T) - T(\tilde{\lambda}). \] (7.10)
On the other hand, from (1.1) we have
\[ g(R_{\text{cur}}(\xi, T)\text{grad}, \xi) = \kappa g(T, \text{grad} - \xi(f)\xi) + \mu g(h T, \text{grad} f). \] (7.11)
Using (7.10) and (7.11), it follows that
\[ \kappa(\text{grad} f) - \kappa \xi(f)\xi + \mu h(\text{grad} f) - \xi(\tilde{\lambda})\xi + \text{grad} \tilde{\lambda} = 0. \] (7.12)
From (7.8), we get
\[ Q(\text{grad} f) = -2n(\text{grad} f), \]
which infers
\[ 2n\kappa(\text{grad} f) + 2n\mu h(\text{grad} f) = Q(\text{grad} f) + 2n \left( \kappa \xi(f) + \xi(\tilde{\lambda}) \right) \xi, \] (7.13)
via (7.12). Then, by using (3.8) and taking inner product of the last equation with \( \xi \), we obtain
\[ \kappa \xi(f) + \xi(\tilde{\lambda}) = 0. \]
If we put this equation in (7.13), we get
\[ 2n\kappa(\text{grad} f) + 2n\mu h(\text{grad} f) = Q(\text{grad} f). \] (7.14)
Taking $U = \xi$ in (7.5) and using (3.18), we obtain

$$\nabla_{\xi} \text{grad} f = \lambda \xi.$$  

By differentiating (7.14) with respect to $\xi$ and using the last equation we have

$$\mu (\mu (1 - 2n) + 2(n - 1)) h \text{grad} f = 0,$$

which is equal to

$$\mu (\mu (1 - 2n) + 2(n - 1)) \text{grad} f = 0, \quad (7.15)$$

via (3.2). Also taking $\Psi U$ and $\Psi T$ instead of $U$ and $T$, respectively, in (7.9) we write

$$g(R_{\text{cur}}(\Psi U, \Psi T) \text{grad} f, \xi) = (4\kappa - 2n^2)g(\Psi U, T). \quad (7.16)$$

In a pcm $(\kappa, \mu)$-manifold it is well known that $R_{\text{cur}}(\Psi U, \Psi T) \xi = 0$. Then we obtain

$$(4\kappa - 2n^2)g(\Psi U, T) = 0.$$

Because of $d\eta$ is being non-zero, one gets

$$\kappa = \frac{n^2}{2}. \quad (7.17)$$

Hence, considering (7.15) and (7.17) we assume the following three cases:

Case 1. If $\mu = 0$, then we can state that the manifold is a $N(\kappa)$-pcm manifold.

Case 2. If $\Psi \text{grad} f = 0$ and $\mu \neq 0$, then we write

$$\Psi^2 \text{grad} f = \text{grad} f - \eta(\text{grad} f) \xi = 0,$$

that is

$$\text{grad} f = \xi(f) \xi. \quad (7.18)$$

By taking covariant derivative of the above equation along $U$, we have

$$\nabla_{U} \text{grad} f = U(\xi(f)) \xi + \xi(f) (-\Psi U + \Psi h U). \quad (7.19)$$

If we replace $U$ with $\Psi U$ and take inner product with $\Psi T$ in (7.19), we obtain

$$g(\nabla_{\Psi U} \text{grad} f, \Psi T) = -\xi(f) (g(U, \Psi T) + g(h U, \Psi U)), \quad (7.20)$$

which implies

$$g(\nabla_{\Psi T} \text{grad} f, \Psi U) = -\xi(f) (g(T, \Psi U) + g(h T, \Psi U)). \quad (7.21)$$

We know that $d^2 f = 0$ and so, for any vector fields $U$ and $T$, we have $UT(f) - TU(f) - [U,T]f = 0$. It follows that

$$U g(\text{grad} f, T) - T g(\text{grad} f, U) - g(\text{grad} f, [U,T]) = 0,$$

that is

$$\nabla_{U} (\text{grad} f, T) - g(\text{grad} f, \nabla_{U} T) - \nabla_{T} (\text{grad} f, U) - g(\text{grad} f, \nabla_{T} U) = 0.$$ 

Since $g$ is a metric connection then we have

$$g(\nabla_{U} \text{grad} f, T) = g(U, \nabla_{T} \text{grad} f). \quad (7.22)$$

By taking $U = \Psi U$ and $T = \Psi T$ in (7.22), we write

$$g(\nabla_{\Psi U} \text{grad} f, \Psi T) = g(\Psi U, \nabla_{\Psi T} \text{grad} f).$$

Then, from (7.20), (7.21) and the last equation above, we obtain

$$\xi(f) g(U, \Psi T) = 0,$$

which infer $\xi(f) = 0$, since $d\eta \neq 0$. From (7.18) we obtain $\text{grad} f = 0$, that is, $f$ is a constant. Therefore, from (7.5), we get $\hat{\text{Ric}}(U,T) = -\lambda g(U, T)$, which implies that the
manifold is an Einstein manifold with respect to the the S-vK connection. Furthermore, by using (3.17), we have
\[
\text{Ric}(U, T) = -\bar{\lambda}g(U, T) + 2n\kappa\eta(U)\eta(T) + \mu g(hU, T).
\] (7.23)

By using (3.7) in (7.23), we have
\[
\text{Ric}(U, T) = ag(U, T) + b\eta(U)\eta(T),
\]
where \(a = -\frac{\bar{\lambda}(2(n-1)+\mu)+\mu(2(1-n)+n\mu)}{2(n-1)}\) and \(b = \frac{2n\kappa(2(n-1)+\mu)-\mu(2(n-1)+n(2n-n))}{2(n-1)}\), which implies that the manifold is an \(\eta\)-Einstein manifold wrt the LC connection.

Case 3. If \(\mu (1-2n) + 2(n-1) = 0\), then we obtain
\[
\mu = \frac{2(n-1)}{2n-1}.
\] (7.24)

Using (7.14) and (3.7), we get
\[
(2(1-n) + n\mu - 2n\kappa)) (\text{grad} f - \xi(f)\xi) + (2(n-1) + \mu - 2n\mu) h\text{grad} f = 0.
\] (7.25)
Using (7.24) and (7.17) in (7.25), we conclude that \(\text{grad} f = \xi(f)\xi\). So, we get the similar results given in Case 1.

Hence we give the following:

**Theorem 7.1.** Let \((M, g)\) be a \((2n+1)\)-dimensional \((n > 1)\) pcm \((\kappa, \mu)\)-manifold \((\kappa \neq -1)\) bearing an almost gradient Ricci soliton wrt the S-vK connection. Then either the manifold is a \(N(\kappa)\)-pcm manifold, or it is an Einstein manifold wrt the S-vK connection (equivalently, it is an \(\eta\)-Einstein manifold wrt the LC connection).

8. Almost Yamabe solitons on pcm \((\kappa, \mu)\)-manifolds with respect to the Schouten-van Kampen connection

In this section we study almost Yamabe solitons on a pcm \((\kappa, \mu)\)-manifold \((\kappa \neq -1)\) wrt the S-vK connection. Assume that \((M, V, \delta, g)\) is an almost Yamabe soliton on a pcm \((\kappa, \mu)\)-manifold wrt the S-vK connection. Then we write
\[
\frac{1}{2}(\mathcal{L}_V g)(U, T) = (\bar{\delta} - \delta)g(U, T).
\] (8.1)
From (3.19), we write
\[
\frac{1}{2}(\mathcal{L}_V g)(U, T) = (r - 2n\kappa - \delta)g(U, T).
\] (8.2)
Hence, we state the following:

**Theorem 8.1.** An almost Yamabe soliton \((M, V, \delta, g)\) on a \((2n+1)\)-dimensional pcm \((\kappa, \mu)\)-manifold with \(\kappa \neq -1\) is invariant under the S-vK connection if and only if the manifold is a para-Sasakian manifold.

For \(V = \xi\) in (8.2), we get
\[
g(U, \Psi hT) = (r - 2n\kappa - \delta)g(U, T).
\] (8.3)
So we give the followings:

**Theorem 8.2.** Let \(M\) be a \((2n+1)\)-dimensional pcm \((\kappa, \mu)\)-manifold \((\kappa \neq -1)\) bearing a Yamabe soliton \((\xi, \tilde{\delta}, g)\) wrt the S-vK connection. Then, \(M\) is of constant scalar curvature \(2n\kappa + \delta\) wrt the LC connection.

**Corollary 8.3.** An almost Yamabe soliton \((\xi, \tilde{\delta}, g)\) on a \((2n+1)\)-dimensional pcm \((\kappa, \mu)\)-manifold \((\kappa \neq -1)\) wrt the S-vK connection is steady if \(r = 2n\kappa\).

We conclude with an example of pcm \((\kappa, \mu)\)-manifold wrt the S-vK connection such that \(\kappa < -1\).
Example 8.4. Let $g$ be the Lie algebra endowed with a basis $\{E_1, E_2, E_3, E_4, E_5\}$ and non-zero Lie brackets

\[
\begin{align*}
[E_1, E_5] &= \alpha \beta E_1 + \alpha \beta E_2, & [E_2, E_5] &= \alpha \beta E_1 + \alpha \beta E_2, \\
[E_3, E_5] &= -\alpha \beta E_2 + \alpha \beta E_4, & [E_4, E_5] &= \alpha \beta E_3 - \alpha \beta E_4, \\
[E_1, E_2] &= \alpha E_1 + \alpha E_2, & [E_1, E_3] &= \beta E_2 + \alpha E_4 - 2E_5, \quad (8.4) \\
[E_1, E_4] &= \beta E_2 + \alpha E_3, & [E_2, E_3] &= \beta E_1 - \alpha E_4, \\
[E_2, E_4] &= \beta E_1 - \alpha E_3 + 2E_5, & [E_3, E_4] &= -\beta E_3 + \beta E_4,
\end{align*}
\]

where $\alpha, \beta$ are non-zero real numbers such that $\alpha \beta > 0$. Let $G$ be a Lie group whose Lie algebra is $g$. Define on $G$ a left invariant pcm structure $(\Psi, \xi, \eta, g)$ by imposing that, at the identity, $g(E_1, E_1) = g(E_4, E_4) = -g(E_2, E_2) = -g(E_3, E_3) = g(E_5, E_5) = 1$, $g(E_1, E_j) = 0$, for any $i \neq j$, and $\Psi E_1 = E_3, \Psi E_2 = E_4, \Psi E_3 = E_1, \Psi E_4 = E_2, \Psi E_5 = 0, \xi = E_3$ and $\eta = g(\cdot, E_5)$. A very long but straightforward computation shows that

\[
\begin{align*}
\nabla_{E_1}\xi &= \alpha \beta E_1 - \Psi E_1, & \nabla_{E_2}\xi &= \alpha \beta E_2 - \Psi E_2, \\
\nabla_{\Psi E_1}\xi &= -E_1 - \alpha \beta \Psi E_1, & \nabla_{\Psi E_2}\xi &= -E_2 - \alpha \beta \Psi E_2, \\
\nabla_{\xi E_1} &= -\alpha \beta E_2 - \Psi E_1, & \nabla_{\xi E_2} &= -\alpha \beta E_1 - \Psi E_2, \\
\nabla_{\xi \Psi E_1} &= -E_1 - \alpha \beta \Psi E_2, & \nabla_{\xi \Psi E_2} &= -E_2 - \alpha \beta \Psi E_1, \\
\nabla_{E_1 E_1} &= \alpha \beta E_2 - \alpha \beta E_5, & \nabla_{E_1 E_2} &= \alpha \beta E_2 - \alpha \beta E_5, \\
\nabla_{E_2 E_1} &= \alpha E_2, & \nabla_{E_2 E_2} &= -\alpha E_1 + \alpha \beta E_5, \\
\nabla_{\Psi E_1 E_1} &= -\beta E_2 + E_5, & \nabla_{\Psi E_1 E_2} &= -\beta E_1, \\
\nabla_{\Psi E_2 E_1} &= -\beta \Psi E_2 - \alpha \beta E_5, & \nabla_{\Psi E_2 E_2} &= -\beta \Psi E_1, \\
\nabla_{\Psi E_3 E_1} &= -\beta E_2, & \nabla_{\Psi E_3 E_2} &= -\beta E_1 - E_5, \\
\nabla_{\Psi E_3 E_1} &= -\beta \Psi E_2, & \nabla_{\Psi E_3 E_2} &= -\beta \Psi E_1 + \alpha \beta E_5,
\end{align*}
\]

where $\lambda = \alpha \beta$ and $\mu = 2$. Then one can prove that the curvature tensor field of the LC connection of $(G, g)$ satisfies that $(\kappa, \mu)$-nullity condition (1.1), with $\kappa = -1 - (\alpha \beta)^2$ and $\mu = 2$, which implies that $(G, \Psi, \xi, \eta, g)$ is a $5$-dimensional pcm $(\kappa, \mu)$-manifold [6]. Now we shall construct the S-vK connection on $(G, \Psi, \xi, \eta, g)$. Using (8.5), we get

\[
\begin{align*}
\tilde{\nabla}_{E_1} E_1 &= \alpha E_2, & \tilde{\nabla}_{E_1} E_2 &= \alpha E_1, & \tilde{\nabla}_{E_1} E_3 &= \alpha E_4, \\
\tilde{\nabla}_{E_1} E_4 &= \alpha E_3, & \tilde{\nabla}_{E_1} E_5 &= -\alpha E_2, & \tilde{\nabla}_{E_1} E_2 &= -\alpha E_1, \\
\tilde{\nabla}_{E_2} E_3 &= -\alpha E_4, & \tilde{\nabla}_{E_2} E_4 &= -\alpha E_3, & \tilde{\nabla}_{E_2} E_1 &= -\beta E_2, \\
\tilde{\nabla}_{E_2} E_2 &= -\beta E_1, & \tilde{\nabla}_{E_2} E_3 &= -\beta E_4, & \tilde{\nabla}_{E_2} E_4 &= -\beta E_3, \\
\tilde{\nabla}_{E_3} E_1 &= -\beta E_2, & \tilde{\nabla}_{E_3} E_2 &= -\beta E_1, & \tilde{\nabla}_{E_3} E_3 &= -\beta E_4, \\
\tilde{\nabla}_{E_3} E_4 &= -\beta E_3, & \tilde{\nabla}_{E_3} E_5 &= -\alpha \beta E_2 - E_3, \\
\tilde{\nabla}_{E_4} E_1 &= -\beta E_4, & \tilde{\nabla}_{E_4} E_2 &= -\beta E_3, & \tilde{\nabla}_{E_4} E_3 &= -\beta E_4, \\
\tilde{\nabla}_{E_4} E_4 &= -\beta E_3, & \tilde{\nabla}_{E_4} E_5 &= -\alpha \beta E_2 - E_3, \\
\tilde{\nabla}_{E_5} E_2 &= -\alpha \beta E_1 - E_4, & \tilde{\nabla}_{E_5} E_3 &= -E_1 - \alpha \beta E_4, & \tilde{\nabla}_{E_5} E_4 &= -E_2 - \alpha \beta E_3.
\end{align*}
\]
Now using (8.6), we can calculate the non-zero components of its curvature tensor wrt the S-vK connection as follows:

\[
\begin{align*}
\tilde{R}_{\text{cur}}(E_1, E_3)E_1 &= -2E_3, \\
\tilde{R}_{\text{cur}}(E_1, E_3)E_3 &= -2E_1, \\
\tilde{R}_{\text{cur}}(E_1, E_4)E_1 &= 2\alpha\beta E_2, \\
\tilde{R}_{\text{cur}}(E_1, E_4)E_3 &= 2\alpha\beta E_1, \\
\tilde{R}_{\text{cur}}(E_2, E_3)E_1 &= -2\alpha\beta E_2, \\
\tilde{R}_{\text{cur}}(E_2, E_3)E_3 &= -2\alpha\beta E_1, \\
\tilde{R}_{\text{cur}}(E_2, E_4)E_1 &= 2E_3, \\
\tilde{R}_{\text{cur}}(E_2, E_4)E_3 &= 2E_1,
\end{align*}
\]

which imply that the non-zero components of its Ricci tensor wrt the S-vK connection as follows:

\[
\begin{align*}
\tilde{\text{Ric}}(E_1, E_1) = \tilde{\text{Ric}}(E_4, E_4) = 2, \\
\tilde{\text{Ric}}(E_2, E_2) = \tilde{\text{Ric}}(E_3, E_3) = -2.
\end{align*}
\]  

From (8.8), (6.2) and (6.9), one can see that there does not exist an almost Ricci soliton on such a 5-dimensional pcm \((\kappa, \mu)\)-manifold with \(\kappa < -1\).

Furthermore, for \(U = u_1E_1 + u_2E_2 + u_3E_3 + u_4E_4 + u_5E_5\), \(T = t_1E_1 + t_2E_2 + t_3E_3 + t_4E_4 + t_5E_5 \in \chi(G)\), we have

\[
g(U, \Psi hT) = \alpha\beta(u_1t_1 - u_2t_2 + u_3t_3 - u_4t_4).
\]

By using the last equation in (8.3), we say that the 5-dimensional pcm \((\kappa, \mu)\)-manifold \(G\) admits a Yamabe soliton \((\xi, 8 - \alpha\beta, g)\) wrt the S-vK connection. Such a Yamabe soliton is expanding if \(\alpha\beta > 8\), steady if \(\alpha\beta = 8\) and shrinking if \(\alpha\beta < 8\).

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