ON THE COMPLEXITY OF COMPUTING KRONECKER COEFFICIENTS

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Abstract. We study the complexity of computing Kronecker coefficients \( g(\lambda, \mu, \nu) \). We give explicit bounds in terms of the number of parts \( \ell \) in the partitions, their largest part size \( N \) and the smallest second part \( M \) of the three partitions. When \( M = O(1) \), i.e. one of the partitions is hook-like, the bounds are linear in \( \log N \), but depend exponentially on \( \ell \). Moreover, similar bounds hold even when \( M = e^{O(\ell)} \). By a separate argument, we show that the positivity of Kronecker coefficients can be decided in \( O(\log N) \) time for a bounded number \( \ell \) of parts and without restriction on \( M \). Related problems of computing Kronecker coefficients when one partition is a hook, and computing characters of \( S_n \) are also considered.

1. Introduction and main results

The study of Kronecker coefficients \( g(\lambda, \mu, \nu) \) of the symmetric group \( S_n \) has rare qualities of being classical, highly technical, and largely mysterious. The area was initiated by Murnaghan 75 years ago [Mu1], and continued to be active for decades, with scores of interesting connections to other areas. Despite a large body of work on the Kronecker coefficients, both classical and very recent, it is universally agreed that “frustratingly little is known about them” [Bir]. The problem of finding a combinatorial interpretation for \( g(\lambda, \mu, \nu) \), can be restated as whether computing the coefficients is in \( \#P \). It remains a major open problem, one of the oldest unsolved problems in Algebraic Combinatorics.

More recently, the interest in computing Kronecker coefficients has intensified in connection with Geometric Complexity Theory, pioneered recently as an approach to the \( P \) vs. \( NP \) problem (see [M2, MS, R2]). With Valiant’s theory of determinant computations as its starting point, their approach relies, among other things, on the (conjectural) ability to decide in polynomial time the positivity of Kronecker coefficients and their plethystic generalizations. Envisioned as a far reaching mathematical program requiring over 100 years to complete [F2], this approach led to a flurry of activity in an attempt to understand and establish some critical combinatorial and computational properties of Kronecker coefficients (see [KOR1, BI1, CDW, Ike, M1]). This paper is a new advance in this direction.

While we present several algorithmic and complexity results, they are centered around a single unifying problem. We are trying to understand what exactly makes the Kronecker coefficients hard to compute. Since the problem is \( \#P \)-hard in general (see [BI1]), a polynomial time algorithm for deciding positivity is unlikely to exist. On the other hand, the problem can be viewed as a generalization of LR coefficients \( c_{\beta\gamma}^\alpha \), another \( \#P \)-complete problem. In view of a nice geometric interpretation of the latter, it can be computed using Barvinok’s algorithm in polynomial time for any fixed \( \ell \) (see [S, L]). We show that a similar analysis applies to Kronecker coefficients. In other words, it is not the large part sizes that make an obstacle, but the number \( \ell \) of parts in the partitions.

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Our main result is the following theorem, which introduces a new parameter $M$, the smallest number among the second parts of the three partitions. Our complexity bound is general, but is especially sharp for triples of partitions when one is *hook-like*. We state the theorem here in a somewhat abbreviated form, as we postpone the definitions and details.

**Main Theorem 3.5.** Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

To illustrate the result, consider several special cases of the theorem. First, when $\ell$ is fixed, the recent breakthrough in [CDW] gives a polynomial time algorithm for computing Kronecker coefficients. The main theorem in this case is Main Lemma 5.4 which gives explicit bounds on the dependence of $\ell$. Curiously, even for a simpler problem of deciding positivity $g(\lambda, \mu, \nu) > 0$ this gives the best known general bound. Our Theorem 6.1 uses the *semigroup property* of Kronecker coefficients to give a surprisingly powerful linear bound, but without giving explicitly the dependence on $\ell$.

Second, when we have $\lambda_2, \mu_2, \nu_2 \leq M$ and $N$ is large compared to $(M\ell)$, the Kronecker coefficients stabilize to reduced Kronecker coefficients, which generalize LR coefficients and are believed to be easier to compute (see §8.9). Our Main Theorem lends further support to this conjecture.

Finally, when $M = O(1)$ the theorem gives a new type complexity bound of computing $g(\lambda, \mu, \nu) > 0$ when $\nu$ is *hook-like*. This may seem surprising, as already the case of hooks (i.e. when $M = 1$), received considerable attention in the literature (see e.g. [Las, Rem, Ros]). There, even in the simplest cases, the resulting formulas for Kronecker coefficients seem rather difficult, and the recent combinatorial interpretation by Blasiak unsuitable for efficient computation [Bla]. Curiously, we use Blasiak’s combinatorial interpretation of Kronecker coefficients to show that computing $g(\lambda, \mu, \nu)$ is in $\#P$ when $\nu$ is a hook (Theorem 4.3).

**Corollary 1.1** In the notation of the Main Theorem, suppose

$$\log M, \ell = O\left(\frac{(\log \log N)^{1/3}}{(\log \log \log N)^{2/3}}\right).$$

Then there is a polynomial time algorithm to compute $g(\lambda, \mu, \nu)$.

The proofs are based on two main tools. The first is the *Reduction Lemma* (Lemma 5.1), which implies that when $\nu_2$ is small, we either immediately have $g(\lambda, \mu, \nu) = 0$, or else there are partitions $\varphi(\lambda), \varphi(\mu), \varphi(\nu)$ of size $O(\ell^3 \nu_2)$, such that $g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu))$. In other words, we reduce the problem from binary input to unary input, and apply Lemma 5.4 to the latter case.

The second tool is the *Main Lemma* 5.4 mentioned above, which gives an effective bound on the complexity of computing Kronecker coefficients without any restrictions on $M$. It coincides with the Main Theorem 3.5 when $M = N$, and states that the Kronecker coefficients can be computed in time $\text{Poly}((\ell \log N)^{\ell^3 \log \ell})$. This is achieved by separating the algebraic and complexity parts; the latter is reduced to counting integer points in certain 3-way statistical tables via Barvinok’s algorithm (see [2.5] and [8.4]).
The rest of this paper is structured as follows. We begin with basic definitions in Section 2 and proceed to state our new complexity results in Section 3. In Section 4, we discuss Blasiak’s combinatorial interpretation and its implications. This section is largely separate from the rest of the paper and uses some background in Algebraic Combinatorics.

The main results of this paper, notably the Main Lemma and the Reduction Lemma are proved in Section 5. We follow with two short sections 6 and 7 discussing the case of bounded \( \ell \) and the complexity of computing the characters of \( S_n \), respectively. Namely, we prove that the problem of deciding whether \( \chi^\lambda[\nu] = 0 \) is NP-hard, extending earlier easy results by Hepler [Hep] (see §8.10 for the connection with Kronecker coefficients). We conclude with final remarks and open problems in Section 8.

2. Definitions and background

We briefly remind the reader of basic definitions, standard notations and several claims which will be used throughout the paper. For more background on the representation theory of the symmetric group and related combinatorics, see e.g. [Mac, Sag, Sta].

2.1. Partitions and characters. Let \( \lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \) be a partition of \( n \), and let \( P_n \) denote the set of partitions of \( n \). Denote by \( \lambda' \) the conjugate partition of \( \lambda \). Denote by \( \ell(\lambda) = \lambda'_1 \) the number of parts in \( \lambda \). We use Young diagram \([\lambda]\) to represent a partition \( \lambda \). Partitions \((n - k, 1^k)\) are called hooks; partitions \((n - k, k)\) with two parts will also play a major role. We also define the union and intersection of two partitions as the union or intersection of their Young diagrams. In other words, \( \pi = \lambda \cup \mu \) mean that \( \pi_i = \max(\lambda_i, \mu_i) \), and \( \rho = \lambda \cap \mu \) means that \( \rho_i = \min(\lambda_i, \mu_i) \). Denote by \( \lambda+\mu \) the partition \((\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots)\).

We denote by \( \chi^\lambda, \lambda \vdash n \), the irreducible character of the symmetric group \( S_n \), and \( \chi^\lambda[\mu] \) be its value \( \chi^\lambda(u) \) on any permutation \( u \) of cycle type \( \mu \). For a general character \( \eta \), the multiplicity of \( \chi^\lambda \) in \( \eta \) is given by the scalar product:

\[
c(\chi^\lambda, \eta) = \frac{1}{n!} \sum_{u \in S_n} \chi^\lambda(u) \eta(u).
\]

We use “sign” to denote character corresponding to partition \((1^n)\).

The characters can be computed by the Murnaghan–Nakayama rule (see e.g. [Sag, Sta]). Briefly, it says that

\[
\chi^\lambda[\mu] = \sum_{B \in \text{MN}_\mu^\lambda} (-1)^{ht(B) - \ell(\mu)},
\]

where \( \text{MN}_\mu^\lambda \) is the set of all border-strip tableaux of shape \( \lambda \) and type \( \mu \) and \( ht(B) \) is the sum of the number of rows (height) in each border-strip of \( B \). A border-strip is a skew connected Young diagram which does not contain a \( 2 \times 2 \) square of boxes. A border-strip tableaux of shape \( \lambda \) and type \( \mu \) is a filling of the Young diagram of \( \lambda \) with \( \mu_1 \) integers 1, \( \mu_2 \) integers 2, etc., such that the entries along each row and down column are weakly increasing, and such that all squares with the same number form a border-strip.

2.2. Kronecker coefficients. We use \( \chi \otimes \eta \) to denote the tensor product of characters. The Kronecker coefficients \( g(\lambda, \mu, \nu) \), where \( \lambda, \mu, \nu \vdash n \) are given by

\[
\chi^\lambda \otimes \chi^\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) \chi^\nu.
\]
It is well known that
\[ g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\omega \in S_n} \chi^\lambda(\omega) \chi^\mu(\omega) \chi^\nu(\omega). \]
This implies that Kronecker coefficients have full $S_3$ group of symmetry:
\[ g(\lambda, \mu, \nu) = g(\mu, \lambda, \nu) = g(\lambda, \nu, \mu) = \ldots \]
In addition, recall that $\chi^\lambda \otimes \text{sign} = \chi^{\lambda'}$. This implies
\[ g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu) = g(\lambda', \mu, \nu') = g(\lambda, \mu', \nu'). \]

2.3. Symmetric functions. We denote by $h_n$ and $h_\lambda = h_{\lambda_1}h_{\lambda_2} \ldots$ the homogeneous symmetric functions, and by $s_\lambda$ the Schur functions (see e.g. [Mac, Sta]). The Littlewood–Richardson (LR) coefficients are denoted by $LR(\lambda, \mu, \nu) = c_{\mu, \nu}^\lambda$, where $|\lambda| = |\mu| + |\nu| = n$. They are given by
\[ s_\mu \cdot s_\nu = \sum_{\lambda \vdash n} c_{\mu, \nu}^\lambda s_\lambda. \]
The integers $c_{\mu, \nu}^\lambda$ have a combinatorial interpretation in terms of certain semi-standard Young tableaux (see [Sag, Sta]) and BZ triangles (see e.g. [PV1]).

Define the Kronecker product of symmetric functions as follows:
\[ s_\mu \ast s_\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) s_\lambda. \]

The following Littlewood’s identity (see [LI]) is crucial for our study:
\[ s_\lambda \ast (s_\tau \ast s_\theta) = \sum_{\alpha \vdash |\tau|, \beta \vdash |\theta|} c_{\alpha, \beta}^\lambda (s_\alpha \ast s_\tau)(s_\beta \ast s_\theta). \]

We also need the generalized Cauchy’s identity (see [Mac Ex I.7.10] and [Sta Ex 7.78])
\[ \sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,\ell} \frac{1}{1-x_i y_j z_\ell}. \]

Given a power series $F = a_0 + a_1 t + a_2 t^2 + \ldots$, denote by $[t^n] F$ the coefficient $a_n$. Similarly, when $F$ is a symmetric function and $A$ is a Schur function, denote by $[A] F$ the coefficient of $A$ in the expansion of $F$ in the linear basis of Schur functions. By a slight abuse of notation, we use $[A] F$ for other bases of symmetric functions as well.

2.4. Semigroup property. The triples $(\lambda, \mu, \nu)$ for which $g(\lambda, \mu, \nu) > 0$ form a semigroup in the following sense.

Theorem 2.1 (Semigroup property). Suppose $\lambda, \mu, \nu, \alpha, \beta, \gamma$ are partitions of $n$, such that $g(\lambda, \mu, \nu) > 0$ and $g(\alpha, \beta, \gamma) > 0$. Then $g(\lambda + \alpha, \mu + \beta, \nu + \gamma) > 0$.

This leads to the following definition of the Kronecker semigroup. Let $K_\ell$ denote the set of triples of partitions $(\lambda, \mu, \nu)$ written as vectors
\[ (\lambda_1, \ldots, \lambda_\ell, \mu_1, \ldots, \mu_\ell, \nu_1, \ldots, \nu_\ell), \]
such that $g(\lambda, \mu, \nu) > 0$. The theorem implies that $K_\ell$ is a semigroup under addition.
Corollary 2.2. The Kronecker semigroup $K_\ell$ is finitely generated.

Both results are proved in [CHM] (see also [S7] and [Ike §4.4]). We use them crucially in Section 6.

2.5. Barvinok’s algorithm. Let $P \subset \mathbb{R}^d$ be a convex polytope given by a system of linear equations and inequalities over integers. Denote by $L$ the size of the input.

Theorem 2.3 (Barvinok). For every fixed $d$, there is a polynomial algorithm computing the number of integer points in $P$. Furthermore, for general $d$, the algorithm works in $L^{O(d \log d)}$ time.

The original algorithm by Barvinok required $L^{O(d^2)}$ time, which was subsequently reduced to that in the theorem. We refer to [Bar, BP] for the proof, detailed surveys and further references (see also [DHTY, DK]).

3. Complexity problems

3.1. Decision problems. We are interested in deciding whether Kronecker coefficients $g(\lambda, \mu, \nu)$ are strictly positive.

Positivity of Kronecker coefficients (KP):
Input: Integers $N, \ell$, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Decide: whether $g(\lambda, \mu, \nu) > 0$.

Recall that the two ways to present the input: in binary and in unary. The difference is in the input size, denoted size$(\lambda, \mu, \nu)$: in the binary case we have size$(\lambda, \mu, \nu) = \Theta(\ell \log N)$, and in the unary case size$(\lambda, \mu, \nu) = \Theta(\ell N)$. Throughout the paper we assume the input is in binary, unless specified otherwise. The problem then becomes a well known conjecture:

Conjecture 3.1 (Mulmuley). KP $\in P$.

Note that it is not even known whether KP $\in \text{NP}$, except in a few special cases, such as when one of the input partitions is a hook. This case is considered in Section 4. Here we consider various subproblems like the case when $\ell$ is fixed, denoted KP$(\ell)$, and the case when one partition is a hook, denoted KP(hook).

3.2. Counting problems. There are analogous problems about computing the exact values of the coefficients mentioned above.

Kronecker coefficients (Kron):
Input: Integers $N, \ell$, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Compute: the Kronecker coefficient $g(\lambda, \mu, \nu)$.

Analogously to the KP case, we consider also the subproblems Kron$(\ell)$ when $\ell$ is fixed and Kron(hook) when one partition is a hook.

The main complexity result in the area is the following recent theorem.

Theorem 3.2 (Bürgisser–Ikenmeyer). Kron $\in \text{GapP}$.

\footnote{Occasionally, this complexity is reported as $L^{O(d)}$, but it seems a more careful accounting gives the bound as in the theorem (A. Barvinok, personal communication).}
Here GapP is a class of functions obtained as a difference of two functions in #P. We give a different proof of the theorem in Section 5 (cf. [CDW]). We conclude with two more conjectures by Mulmuley [M1].

**Conjecture 3.3** (Mulmuley). \( \text{Kron} \in \#P \).

**Conjecture 3.4** (Mulmuley). The Kronecker coefficient \( g(\lambda, \mu, \nu) \) is equal to the number of integer points in convex polytope \( P(\lambda, \mu, \nu) \) with a polynomial description.

This conjecture is the counting version of Conjecture 3.1. It extends the classical result by Gelfand and Zelevinsky, expressing LR coefficients as the number of integer points in convex polytopes (see e.g. [Zel]).

The main result of the paper is the following theorem (the proof is in Section 5).

**Theorem 3.5** (Main Theorem). Consider the problem \( \text{Kron} \), where the input is integers \( N, \ell \) and partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell), \mu = (\mu_1, \mu_2, \ldots, \mu_\ell), \nu = (\nu_1, \ldots, \nu_\ell) \), such that \( \lambda_1, \mu_1, \nu_1 \leq N \) and \( |\lambda| = |\mu| = |\nu| \). Suppose that \( \nu_2 \leq M \). Then \( g(\lambda, \mu, \nu) \) can be computed in time

\[
O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.
\]

**Corollary 3.6** ([CDW]). We have \( \text{KP}(\ell) \in \text{P} \) and \( \text{Kron}(\ell) \in \text{FP} \), for every fixed \( \ell \).

Here FP is a class of functions computable in polynomial time, a counterpart of \( \text{P} \) for decision problems. Note that in Section 6, we use the semigroup property to prove the first part of the corollary.

4. THE CASE OF A HOOK

Here we consider separately the complexity of \( \text{KP}(\text{hook}) \) and \( \text{Kron}(\text{hook}) \) when one of the partitions involved is a hook.

Let \( \nu = (n-t, 1^t) \) be a hook shape, and \( \lambda, \mu \vdash n \), such that \( \ell(\lambda), \ell(\mu) \leq k \) and \( \lambda_1, \mu_1 \leq N \). Theorem 3.5 implies the following result in this case.

**Corollary 4.1.** Let \( \nu = (n-t, 1^t) \) be a hook, \( \lambda_1, \mu_1 \leq N \) and \( \ell(\lambda), \ell(\mu) \leq k \). Then \( g(\lambda, \mu, \nu) \) can be computed in time

\[
O(k^2 \log N) + k^{O(k^5 \log k)}.
\]

**Proof.** Recall that when \( \ell(\lambda)\ell(\mu) < \ell(\nu) \) we have \( g(\lambda, \mu, \nu) = 0 \), see [Dvir]. Thus, when \( t > k^2 \) this implies \( g(\lambda, \mu, \nu) = 0 \). On the other hand, when the height of the hook \( t \leq k^2 \), apply Theorem 3.5 with \( \ell = k^2 \) and \( M = 1 \), to obtain the result. \( \square \)

**Remark 4.2.** Note that Lemma 5.3 and subsequently Lemma 5.4 below can be easily modified for partitions of different lengths. This more careful analysis in the proofs of lemmas can reduce the exponent in Corollary 4.1 from \( k^6 \) to \( k^4 \). We omit this improvement for the sake of clarity.

We use the recent combinatorial interpretation by Blasiak in [Bla] as outlined below, to prove the following result.

**Theorem 4.3.** We have \( \text{KP}(\text{hook}) \in \text{NP} \) and \( \text{Kron}(\text{hook}) \in \#\text{P} \).
We consider barred and unbarred entries $\overline{1}, \overline{2}, \ldots, 1, 2, \ldots$, which we use to fill a Young tableau. Such tableau is called semi-standard if the entries are weakly increasing in both rows and columns, with no two equal barred numbers in the same row, and no two equal unbarred numbers in the same column. The content of a tableau $A$ is a sequence $(m_1, m_2, \ldots)$, where $m_r$ is the total number of $r$- and $\overline{r}$-entries in $A$. Two orders are considered:

natural order: $\overline{1} < 1 < \overline{2} < 2 < \ldots$

small barred order: $1 < \overline{2} < \ldots < 1 < 2 < \ldots$

An example of two tableaux with different orders but the same shape and content is given in Figure 1.

![Figure 1. Semi-standard Young tableaux of shape (6, 5^2, 4, 2) with small barred and natural order, of the same content (7, 6, 5, 3, 2).](image)

There is a natural tableau switching bijection $A \leftrightarrow \pi(A)$ between semi-standard tableaux with natural and small bar ordering, preserving the shape and content. The idea is to make a number of jeu-de-taquin slides exchanging barred and unbarred entries in order to convert the tableau from one order to the other. Specifically, a jeu-de-taquin slide is the following local operation: given an “out-of-order” entry $c$, i.e. such that the element to its left $a$ and/or the element above it $b$ is larger, then we exchange $c$ with the larger among $a, b$:

$$
\begin{align*}
&\text{with } c > a \text{ and/or } c > b \\
&\begin{cases}
&b \\
&a \\
&c
\end{cases}
\rightarrow
\begin{cases}
&c \\
&a \\
&b
\end{cases}
\end{align*}
$$

Tableau-switching is then the process in which we start with a tableau in the small-bar order and sort it into the natural order, by applying the jeu-de-taquin described above starting with the smallest and left-most unbarred entry which is out-of-order with respect to the natural ordering and moving it “up” until both entries above and left of it are smaller or equal. Then we continue with the next smallest left-most out-of-order entry and so on until the tableau is a natural SSYT. It is a well known that this entire process is well-defined (see [Bla, BSS]).

Note that one can view a small bar order as a pair of two semi-standard Young tableaux $A_1$ and $A_2$, one of shape $\lambda/\nu$ and one of shape $\nu'$, respectively. Here $A_1$ is the subtableau consisting of the unbarred entries of $A$, and $A_2$ is the conjugate of the subtableau of barred entries of $A$. Denote by $w(A)$ the word obtained by reading $A_2$ right to left, top row to bottom row, concatenated with the word obtained by reading $A_1$ top to bottom, right to left. For example, for $A$ as in the figure, we have

$$w(A) = 11.21.3311.4222.33.2.134.1245.$$
By \( m_i(w) \) denote the number of \( i \)-s in \( w \). Word \( w \) is called a ballot sequence if in its every prefix \( w' \) we have \( m_1(w') \geq m_2(w') \geq \ldots \). For example, \( w(A) \) as above is not a ballot sequence as can be seen for the prefix \( w' = 112133 \), with \( m_2(w') = 1 < m_3(w') = 2 \).

**Theorem 4.4** (Blasiak). Let \( \lambda, \mu \vdash n \) and \( \nu = (n-k,1^k) \). Then \( g(\lambda, \mu, \nu) \) is equal to the number of small bar semi-standard tableaux \( A \) of shape \( \lambda \), content \( \mu \), with \( k \) barred entries, such that \( w(A) \) is a ballot sequence, and such that the lower left corner of \( \pi(A) \) is unbarred.

Although we never defined tableaux switching, as originally defined in [BSS] only gives a pseudo-polynomial time algorithm. Below we show that one can speed up the tableaux switching to be able to check the validity of Blasiak’s tableaux in polynomial time (cf. [PV2]).

**Proof of Theorem 4.3** First, we check whether \( \ell(\nu) = k+1 \) is greater than \( \ell(\lambda)\ell(\mu) \). It follows from [Dvir] that in the case we have \( g(\lambda, \mu, \nu) = 0 \).

Now, if this is not the case, we must have \( k \leq \ell(\lambda)\ell(\mu) \), so \( k \) is polynomial in the size of the input. Proceed as follows.

The polynomial witness for KP(hook) is a small-bar semi-standard tableaux \( A \) of shape \( \lambda \), content \( \mu \) and \( k \) barred entries. We encode the tableaux as a pair of one \( \ell(\mu) \times \ell(\lambda) \) matrix and one \( 2 \times k \) matrix. In the first matrix, the entry at position \((i,j)\) records the number of \( i \)'s in row \( j \). In the second matrix, each column corresponds to a single barred letter and contains the coordinates of its position in the tableaux. This encoding requires \( \ell(\mu)\ell(\lambda) \log N + 2k \log N \) bits. As noted above, this is polynomial in the input size.

We show that tableau-switching can be done in polynomial time. Indeed, switching a single \( i \) and all unbarred letters \( j \) can be done in one step, as follows. If in the process, \( i \) becomes adjacent to a letter \( j \) for the first time then the letter \( i \) should move either to the row below or to the end of the horizontal strip of letters \( j \). Since there are \( k = \ell(\nu) - 1 \) barred letters, the tableaux-switching is done in polynomially many operations.

By Theorem 4.4 Kron(hook) counts the number of tableaux as in the theorem. From the above argument, we can verify that they satisfy the condition in the Theorem in polynomial time. Moreover, the number of such tableaux is at most exponential in the input size. This implies the result. \( \square \)

5. Complexity of Kron and the proofs

We turn towards the computational complexity of Kron as defined in Section 3. Here we prove Theorem 4.4 and related results. We first establish our main tool for this, the Reduction Lemma.

5.1. The Reduction Lemma. In order to prove these statements we will need the following Lemma. Informally, it states that under the conditions of Theorem 3.5 we can either conclude that \( g(\lambda, \mu, \nu) = 0 \) or reduce the computation of \( g(\lambda, \mu, \nu) \) to the computation of a Kronecker coefficient for much smaller partitions. To describe these partitions and state the lemma we construct the following reduction map \( \varphi \) on partitions, which depends on a fixed triple of partitions \( (\lambda, \mu, \nu) \) for which \( |\lambda_i - \mu_i| \leq n - \nu_1 \).

Let \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell \), set \( t = n - \nu_1 \) and suppose that and \( |\lambda_i - \mu_i| \leq t \) for all \( i \leq \ell \) (otherwise, the map \( \varphi \) is undefined). Denote \( \omega = \lambda \cup \mu \) and \( \rho = \lambda \cap \mu \). Let \( I = \{ i : \rho_i \geq \omega_{i+1} + t, 1 \leq i \leq \ell \} \cup \{ \ell + 1 \} \), where \( \omega_{\ell+1} = 0 \). For all indices \( j \), set \( i_j = \min \{ i \in I, i \geq j \} \) and let \( ind_I(i) = \# \{ i' \in I : \ell \geq i' \geq i \} \) be the position of \( i \) in \( I \) when sorted in decreasing order (without counting the entry \( \ell + 1 \)) and set \( ind_I(\ell + 1) = 1 \).
For any partition \( \theta \) with at most \( \ell \) parts and with \( \rho \subset \theta + (t \ell) \), define the partition \( \varphi(\theta) \) via its parts by

\[
\varphi(\theta)_j = \theta_j - \rho_i + t(\ell - i_j + \text{ind}_I(i_j)), 1 \leq j \leq \ell.
\]

Let \( r = |\varphi(\lambda)| = |\varphi(\mu)| \), where the latter equality follows by construction. Define also \( \varphi(\nu) = (r - s, \nu_2, \nu_3, \ldots) \vdash r \).

**Lemma 5.1** (Reduction Lemma). Given integers \( n, \ell \) and partitions \( \lambda, \mu, \nu \vdash n \) such that \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell \), let \( t = n - \nu_1 \). We have the following two cases:

(i) If \( |\lambda_i - \mu_i| > n - \nu_1 \) for some \( i \), then \( g(\lambda, \mu, \nu) = 0 \).
(ii) If \( |\lambda_i - \mu_i| \leq n - \nu_1 \) for all \( i = 1, \ldots, \ell \), then \( g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu)) \).

Moreover we have that \( |\varphi(\lambda)| = |\varphi(\mu)| = |\varphi(\nu)| \leq 2(n - \nu_1)\ell^2 \).

**Example 5.2.** Let \( \lambda = (19, 15, 12, 5, 1), \mu = (16, 16, 14, 3, 3) \) and \( t = 3 \). These partitions give \( \omega = (19, 16, 14, 5, 3) \) and \( \rho = (16, 15, 12, 3, 1) \). The relevant partitions are displayed in the top picture of Figure 2: the dark blue area is the skew shape \( \omega/\rho \), whereas all blue areas represent the connected components of \( \omega/(\rho - 3) \). We have that \( I = \{3, 6\} \), \( \text{ind}_I(3) = 1 \) and \( \text{ind}_I(6) = 1 \), and we see that the map \( \varphi \) shifts the top 3 parts of any partition 3 boxes to the left and leaves the bottom parts intact, giving \( \varphi(\omega) = (16, 13, 11, 5, 3) \) and \( \varphi(\rho) = (13, 12, 9, 3, 1) \). Now, for \( \theta = (14, 14, 10, 2) \), we have \( \varphi(\theta) = (11, 11, 7, 2) \), as in the figure.

The rest of this subsection is the proof of this lemma.

**Preliminaries.** We start with a few observations and a setup for the proof.

Clearly, by the definition of \( \ast \), we have \( s_\alpha \ast s_{n-r} = s_\alpha \) for all \( \alpha \vdash n - r \). Then Littlewood’s identity \([1]\), gives

\[
[s_\tau](s_\sigma \ast (s_{n-r} s_\zeta)) = \sum_{\alpha \vdash n-r, \beta \vdash r} c^\sigma_{\alpha \beta}[s_\tau] s_\alpha(s_\beta \ast s_\zeta)
\]

for any three partitions \( \tau, \sigma, (n-r, \zeta) \vdash n \). Expressing the remaining Kronecker products as Schur functions via

\[
s_\beta \ast s_\zeta = \sum_{\gamma \vdash r} g(\beta, \zeta, \gamma) s_\gamma,
\]

the rest of the proof is as follows.
we conclude that the coefficient of \( s_\pi \) can be obtained as

\[
[\sigma_t](s_\sigma \ast (s_{n-r}s_\zeta)) = \sum_{\alpha \vdash n-r, \beta \vdash r, \gamma \vdash r} c_\alpha^\sigma c_\beta^\tau g(\beta, \zeta, \gamma)
\]

Let \( \pi = (\nu_2, \ldots, \nu_\ell) \), so that \( \nu = (n-t, \pi) \).

**Proof of part (i).** Suppose that \( |\lambda_i - \mu_i| > t \) for some \( i \) and assume without loss of generality that \( \lambda_i > \mu_i \). Then we must have

\[
|\lambda \cap \mu| \leq \sum_{i \neq i} \lambda_j + \mu_i = n - \lambda_i + \mu_i < n - t.
\]

In particular, there are no partitions \( \alpha \vdash n-t, \beta \vdash t \), such that \( c_\alpha^\lambda \neq 0 \) and \( c_\beta^\mu \neq 0 \).

Otherwise by the Littlewood-Richardson rule, we must have \( \alpha \subset \lambda, \mu \), so \( \alpha \subset \lambda \cap \mu \) and then \( n - t = |\alpha| \leq |\lambda \cap \mu| < n - t \), contradicting the inequality above.

Substituting \( \tau \leftarrow \mu, \sigma \leftarrow \lambda, r \leftarrow s \) and \( \zeta \leftarrow \pi \) in equation (3), we then get

\[
[\mu]\(s_\lambda \ast (s_{n-t}s_\pi)) = \sum_{\alpha \vdash n-s, \beta \vdash r-s, \gamma \vdash s} c_\alpha^\lambda c_\beta^\mu g(\beta, \pi, \gamma) = 0.
\]

The Pieri rule [Sta, §7.15] gives

\[
s_{n-s}s_\pi = s_{(n-s, \pi)} + \sum_{\eta \in P_\nu} s_\eta,
\]

where \( P_\nu \) is the set of partitions \( \eta \) obtained by adding a horizontal strip of length \( n-s \) to \( \pi \) (i.e. adding \( n-s \) boxes to Young diagram of \( \pi \), so that no column obtains more than one box) and such that \( \eta \neq \nu \). Taking Kronecker product with \( s_\lambda \), extracting the coefficient of \( s_\mu \) and comparing with equation (4) gives

\[
0 = [\mu](s_\lambda \ast (s_{n-s}s_\pi)) = g(\lambda, \mu, \nu) + \sum_{\eta \in P_\nu} g(\lambda, \mu, \eta).
\]

Since \( g(\lambda, \mu, \eta), g(\lambda, \mu, \nu) \geq 0 \), they must all be equal to 0, so in particular \( g(\lambda, \mu, \nu) = 0 \).

**Proof of part (ii).** Suppose now that \( |\lambda_i - \mu_i| \leq t \) for all \( i \). As in the definition of \( \varphi \) above, let \( \omega = \lambda \cup \mu \) and \( \rho = \lambda \cap \mu \). Then we have \( \rho \subseteq \omega \) and \( \omega_i - \rho_i \leq t \). The idea of \( \varphi \) is to shift to the left the connected components in the Young diagram of \( \omega/\rho \) and then perform the same shifts of parts in \( \theta \) for any partition \( \theta \), so that the resulting partitions are much smaller, but the skew shape \( \theta/(\rho - t^\ell) \) is preserved under \( \varphi \). The set \( I \) indicates the rows where the skew shape \( \omega/(\rho - t^\ell) \) is disconnected and is used to divide the skew shapes in the corresponding connected components.

Observe that \( \varphi(\theta) = \theta - \rho + \varphi(\rho) \) and so given any partition \( \alpha \) of at most \( \ell \) parts, we can construct the inverse \( \varphi^{-1}(\alpha) = \alpha + \rho - \varphi(\rho) \). Since \( \rho - \varphi(\rho) \) is also a partition, the map \( \varphi \) is then a bijection from the set of partitions containing \( \rho - (t^\ell) \) to all partitions.

Let \( r \) be as in the preliminaries and consider equation (3). The idea is to first reduce the Littlewood-Richardson coefficients appearing there. Let \( r \leq t \). We have \( \alpha \vdash n-r \) and \( \beta, \gamma \vdash r \). Observe that it suffices to consider only nonzero terms in the rhs of (3). Then, the condition \( c_{\alpha \beta}^\lambda > 0 \) implies that \( \alpha \subseteq \lambda \cap \mu = \rho \). Since \( |\rho| - |\alpha| \leq |\rho| - n + r \leq r \leq t \), we also have \( \rho \subseteq \alpha + (t^\ell) \) and we can construct \( \varphi(\alpha) \).

Note that \( c_{\alpha \beta}^\lambda \) is equal to the number of Littlewood-Richardson tableaux of shape \( \lambda/\alpha \) of type \( \beta \). Note that the shapes \( \lambda/\alpha \) and \( \varphi(\lambda)/\varphi(\alpha) \) have identical connected components.
This follows from the fact that in the horizontal shifts performed by \( \varphi \) at the rows in \( I \) the component ending in row \( i_j \) is not shifted beyond the end of the component right below, since

\[
\varphi(\alpha)_{i_j} - \varphi(\lambda)_{i_j+1} = \alpha_{i_j} - \rho_{i_j} - (\lambda_{i_j+1} - v\rho_{i_j+1}) + t(i_j+1 - i_j) + t(\text{ind}_I(i_j) - \text{ind}_I(i_j+1)) \geq 0.
\]

Then the skew tableaux \( \lambda/\alpha \) and \( \varphi(\lambda)/\varphi(\alpha) \) are identical, so we have the same number of LR tableaux of fixed type and thus

\[
c_\alpha^\lambda = c_{\varphi(\alpha)}^\varphi(\lambda) \quad \text{and} \quad c_\gamma^\mu = c_{\varphi(\alpha)}^\varphi(\lambda).
\]

Similarly, using the inverse of \( \varphi \), when \( |\theta| \geq |\varphi(\lambda)| - t \), then

\[
c_\gamma^\mu = c_{\varphi^{-1}(\theta)}^\varphi(\lambda) \quad \text{and} \quad c_\beta^\nu = c_{\varphi^{-1}(\theta)}^\varphi(\lambda).
\]

Applying \( \varphi \) and the above observations, we can now rewrite (3) with \( \tau \leftarrow \mu, \sigma \leftarrow \lambda, \zeta \leftarrow \eta \) and then apply it again for \( \tau \leftarrow \varphi(\mu), \sigma \leftarrow \varphi(\lambda) \). We have

\[
[s_\mu] \left( (s_\lambda * (s_{n-r}s_\eta)) \right) = \sum_{\alpha^\lambda_\beta^\gamma \gamma^r \beta^r \gamma \gamma^r} c_\alpha^\lambda c_\beta^\mu g(\beta, \eta, \gamma)
\]

\[
= \sum_{\alpha^\lambda_\beta^\gamma \gamma^r \beta^r \gamma \gamma^r} c_{\varphi(\alpha)}^\varphi(\lambda) c_{\varphi(\alpha)}^\varphi(\lambda) g(\beta, \eta, \gamma)
\]

\[
= \sum_{\theta^\varphi(\lambda) \beta^r \gamma \gamma^r} c_{\theta^\varphi(\alpha)}^\varphi(\lambda) c_{\theta^\varphi(\alpha)}^\varphi(\lambda) g(\beta, \eta, \gamma)
\]

\[
= [s_{\varphi(\mu)}] \left( (s_{\varphi(\lambda)} * (s_{\varphi(n-r)s_\eta})) \right).
\]

We now show that this leads to part (ii) in the lemma. The idea is that \( s_\nu \), for \( \nu = (n - t, \pi) \), can be expressed as a linear combination of terms of the form \( s_{n-r}s_\eta \) with \( r \leq s \leq t \). We then apply (3) to obtain the desired equality for the Kronecker coefficients.

By the Jacobi-Trudi identity, for any partition \( \xi \), we have

\[
s_\xi = \det \left[ h_{\xi_j-i+j} \right]_{i,j=1}^{\ell} = \sum_{j=1}^{\ell} (-1)^{j-1} h_{\xi_{i-1}+j} \det \left[ h_{\xi_i-i+\ell} \right]_{i=2,k=1,\ell \neq j}
\]

\[
= \sum_{j=0}^{\ell-1} (-1)^j h_{\xi_1+j} \left( \sum_{\eta \neq s-j} c_\eta s_\eta \right) = \sum_{j=0}^{\ell-1} \sum_{\eta \neq s-j} (-1)^j c_\eta s_{\xi_1+j} s_\eta.
\]

Here we expand the \((\ell - 1) \times (\ell - 1)\) determinants of homogenous symmetric functions as sums of Schur functions, so \( c_\eta \) are the coefficients with which they appear in the sum.
Finally, applying (3) with $\nu = (n - s, \pi)$ and (7) with $\xi \leftarrow \nu$, we have

$$g(\lambda, \mu, \nu) = [s_\mu] (s_\lambda \ast s_\nu) = [s_\mu] \left\{ s_\lambda \ast \left( \sum_{j=0}^{\ell-1} \sum_{\eta \not\in \eta-n-j} (-1)^j \ c_\eta \ s_{n-s+j} \ s_\eta \right) \right\}$$

$$= \sum_{j=0}^{\ell-1} \sum_{\eta \not\in \eta-n-j} (-1)^j \ c_\eta \ [s_\mu] \ (s_\lambda \ast (s_{n-s+j} \ s_\eta))$$

$$= \sum_{j=0}^{\ell-1} \sum_{\eta \not\in \eta-n-j} (-1)^j \ c_\eta \ [s_{\varphi(\mu)}] \ (s_{\varphi(\lambda)} \ast (s_{\varphi(n-s+j)} \ s_\eta))$$

$$= [s_{\varphi(\mu)}] \ (s_{\varphi(\lambda)} \ast \sum_{j=0}^{\ell-1} \sum_{\eta \not\in \eta-n-j} (-1)^j \ c_\eta \ s_{\varphi(n-s+j)} \ s_\eta)$$

$$= [s_{\varphi(\mu)}] \ (s_{\varphi(\lambda)} \ast s_{(\varphi(n-s), \pi)}) = g(\varphi(\mu), \varphi(\lambda), \varphi(\nu)).$$

Here the penultimate equality comes from application of (7) with $\xi_1 = \varphi(n - s)$. The last equality follows from the fact that $\varphi(\nu) = (\varphi(n - s), \pi)$, since $|\pi| \leq t$. This completes the proof of the lemma.

## 5.2. Proofs of complexity results

The following result (Main Lemma) gives a bound on the computational complexity of Kronecker coefficients in the general case. Together with the Reduction Lemma it gives Theorem 3.5. Incidentally, its proof can be used to derive Theorem 3.2, as we explain.

Before we proceed, we need the following technical result. For integer vectors $a, b, c \in \mathbb{Z}_{\geq 0}^\ell$, denote by $C(a, b, c)$ the number of three dimensional contingency arrays (3-way statistical tables) with 2-way marginals $a, b, c$.

**Lemma 5.3.** In the notation above, let $R = \max\{a_i, b_i, c_i \mid i = 1, \ldots, \ell\}$. Then the number $C(a, b, c)$ can be computed in time $(\ell \log R)^{O(\ell^3 \log \ell)}$.

**Proof.** Note that these contingency arrays are just the integer points in a polytope in $\mathbb{R}^{\ell^3} = \{(\ldots, x_{ijk}, \ldots) \mid i, j, k = 1, \ldots, \ell\}$, defined by the $\ell^3$ inequalities

$$x_{ijk} \geq 0, \ \text{for} \ 1 \leq i, j, k \leq \ell$$

and the $3\ell$ equations

$$\sum_{j,k} x_{ijk} = a_i, \ \sum_{i,k} x_{ijk} = b_j, \ \sum_{i,j} x_{ijk} = c_k.$$ 

Recall Barvinok’s algorithm (Theorem 2.3), which computes the number of integer points in a polytope of dimension $d$ and input size $L$ in time $L^{O(d \log d)}$. Here we have $L = O(\ell^3 \log R)$ and $d = O(\ell^3)$, which gives the result. \hfill \Box

**Lemma 5.4** (Main Lemma). Let $\alpha, \beta, \gamma \vdash n$ be partitions of the same size, such that $\alpha_1, \beta_1, \gamma_1 \leq m$ and $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq \ell$. Then $g(\alpha, \beta, \gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$. 


Proof. Consider three sets of variables, $x = (x_1, \ldots, x_\ell)$, $y = (y_1, \ldots, y_\ell)$ and $z = (z_1, \ldots, z_\ell)$. We use the determinantal formula for Schur functions:

$$s_\theta(x_1, \ldots, x_\ell) = \frac{\det \left[ x_i^{\theta_i + \ell - j} \right]_{i,j=1}^{\ell}}{\Delta(x)},$$

where $\Delta$ is the Vandermonde determinant.

Thus, for a symmetric function $F(x_1, \ldots, x_\ell)$, we have

$$[s_\theta] F = [ x_1^{\theta_1 + \ell - 1} \cdots x_\ell^{\theta_\ell} ] \Delta(x) F(x),$$

i.e. the coefficient of $s_\theta$ in the expansion of $F$ as a linear combination of Schur functions is equal to the coefficient of the respective monomial in the expansion of the polynomial $\Delta(x)F(x)$ as a sum of monomials. Applying this to identity (2) and using the staircase partition $\delta = (\ell-1, \ldots, 1, 0)$, gives

$$g(\alpha, \beta, \gamma) = [s_\alpha(x)s_\beta(y)s_\gamma(z)] \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = [x^{\alpha + \delta} y^{\beta + \delta} z^{\gamma + \delta}] \Delta(x) \Delta(y) \Delta(z) \frac{1}{1 - x_i y_j z_k}$$

$$= [x^{\alpha + \delta} y^{\beta + \delta} z^{\gamma + \delta}] \prod_{1 \leq i < j \leq \ell} (x_i - x_j)(y_i - y_j)(z_i - z_j) \prod_{i,j,k=1}^{\ell} (1 + x_i y_j z_k + \cdots + (x_i y_j z_k)^{m+\ell}).$$

Here we truncated the infinite sums $(1 - x_i y_j z_k)^{-1}$ to the maximal powers which could be involved in computing the necessary coefficients.

Using the notation from Lemma 5.3, we have that

$$\prod_{i,j,k=1}^{\ell} (1 + x_i y_j z_k + \cdots + (x_i y_j z_k)^{m+\ell}) = \sum_{a,b,c \in \{0,1,\ldots,m+\ell\}^\ell} C(a,b,c).$$

Equation (8) then gives

$$g(\alpha, \beta, \gamma) = \sum_{\sigma^1, \sigma^2, \sigma^3 \in S_\ell} \text{sgn}(\sigma^1 \sigma^2 \sigma^3) C(\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3),$$

where the sum goes over triples of permutations on $[1, \ldots, \ell]$ and

$$\alpha + 1 - \sigma^1 = (\alpha_1 + 1 - \sigma^1_1, \ldots, \alpha_\ell + 1 - \sigma^1_\ell),$$

and similarly for the other terms. Applying Lemma 5.3 for each of the summands in the above sum, we get that the Kronecker coefficient is computed in time

$$\left(\ell^2 \log(m + \ell)\right)^{O(\ell^3 \log \ell)} = \left(\ell \log m\right)^{O(\ell^3 \log \ell)},$$

as desired. \hfill \square

Proof of Theorem 3.2 Notice that (9) can be presented as the following difference: the number of contingency 3d arrays with marginals of the form $\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3$, where $\text{sgn}(\sigma^1 \sigma^2 \sigma^3) = 1$, minus the number of contingency 3d arrays with marginals of the form $\alpha + 1 - \sigma^1, \beta + 1 - \sigma^2, \gamma + 1 - \sigma^3$, where $\text{sgn}(\sigma^1 \sigma^2 \sigma^3) = -1$. Since each of these two
numbers counts polynomially verifiable objects, and there are \((\ell !)^3/2\) many of them, we conclude the numbers are in \(\#P\). This implies the result.

Proof of Theorem 3.5. Let \(n = |\nu|\) and \(t = n - \nu_1\). Then we have \(t \leq \ell M\). In at most \(\ell\) steps we can check whether \(|\lambda_i - \mu_i| > t\) for some \(i\), in which case part (i) of Lemma 5.1 immediately implies that \(g(\lambda, \mu, \nu) = 0\). Otherwise, part (ii) of the Reduction Lemma 5.1 implies that \(g(\lambda, \mu, \nu) = g(\varphi(\lambda), \varphi(\mu), \varphi(\nu))\). Note that, by the definition of \(\varphi\), we can compute \(\varphi(\theta)\) in at most \(3\ell\) steps.

In the Main Lemma 5.4, let \(\alpha = \varphi(\lambda), \beta = \varphi(\mu), \gamma = \varphi(\nu)\) and \(m \leq 2t\ell^2 \leq 2M\ell^3\). Then the lemma implies that the Kronecker coefficients can be found in time

\[
O(\ell \log N) + (\ell^2 \log M)^O(\ell^3 \log \ell),
\]

where the term \(O(\ell \log N)\) comes from the initial comparison of the parts of the partitions and the application of \(\varphi\).

\[\square\]

6. Partitions of fixed lengths

Here we consider the Kronecker Positivity problem \(KP\) when \(\ell\) is fixed. While the Main Lemma 5.4 already gives that \(KP \in P\) in this case, we present a different proof and a new complexity bound for this result.

Theorem 6.1. The problem \(KP(\ell) \in P\) for every fixed \(\ell\). Moreover, for \(\lambda_1, \mu_1, \nu_1 \leq N\) and \(\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell\), there is an algorithm which decides whether \(g(\lambda, \mu, \nu) > 0\) in time \(O(\ell \log N)\).

Proof of Theorem 6.1. Let \(v_1, \ldots, v_k \in \mathbb{Z}_{3\ell}^+\) be the basis of generators of the Kronecker semigroup \(K_{3\ell}\) (see Section 2.4). Let \(A\) be the \((3\ell) \times k\) matrix whose columns are the vectors \(v_1, \ldots, v_k\), i.e. \(A = [v_1 \ldots v_k]\). Deciding whether \(g(\lambda, \mu, \nu) > 0\) is equivalent to deciding whether the vector \(X = (\lambda, \mu, \nu)^T\) is a nonnegative integral combination of the vectors \(v_1, \ldots, v_k\). In other words, we need to decide whether \(AY = X\) has a solution \(Y \in \mathbb{Z}_{3\ell}^+\). Since \(\ell\) and \(k\) are fixed constants, the matrix \(A\) is also fixed. The first part of the theorem follows from Lenstra’s theorem stating that this integer linear program (ILP) can be solved in polynomial time in the size of the input (i.e. \(X\)) for every fixed dimension (see e.g. [Sch]). The second part follows from more recent results on the complexity of feasibility of ILP (see [Eis]).

\[\square\]

Remark 6.2. Observe that the dependence on \((\log N)\) in the theorem is linear, rather than polynomial of degree \((\ell^3)\) in the Main Lemma 5.4 making this algorithm more efficient. Note also that the proof of Corollary 2.2 of the finite generation of \(K_{3\ell}\) given in [CHM] is inexplicit. Thus we have no control over the size \(k = k(\ell)\) as in the proof, not even whether this is a computable function of \(\ell\). Getting any bounds in this direction would be interesting, as it would make effective the constant implied by the \(O(\cdot)\) notation.
7. Complexity of deciding whether a character is zero

We now consider an analogue of the KP problem, which we call CharP. Note that complexity of characters has been studied before, see [Hep] for a treatment of the problem when the input is in unary and a conclusion that the decision problem is then PP-complete.

**Problem**: \( \chi^\lambda[\nu] = 0 \) (CharP):

- **Input**: Integers \( N, \ell \), partitions \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), \( \nu = (\nu_1, \ldots, \nu_\ell) \), where \( 0 \leq \lambda_i, \nu_i \leq N \), and \( |\lambda| = |\nu| \).
- **Decide**: whether \( \chi^\lambda[\nu] = 0 \).

The main result of this section is the following theorem.

**Theorem 7.1.** The problem CharP is NP-hard.

The proof follows from the following observation, implying that CharP is at least as hard as the Knapsack problem.

**Proposition 7.2.** Suppose that \( \ell(\lambda) = 2 \) and \( \nu_i \equiv 0 \pmod{2} \). Then CharP is NP-hard.

**Proof.** We reduce to the classical NP-complete Knapsack problem:

- **Knapsack**: Given the input \( (k, a_1, \ldots, a_\ell) \), determine whether there are \( \epsilon_i \in \{0, 1\} \) for \( i = 1, \ldots, \ell \), such that
  \[
  k = \sum_{i=1}^{\ell} \epsilon_i a_i.
  \]

Consider the CharP problem in the special case when \( \lambda = (n - 2k, 2k) \) is a two-row partition and set \( \nu = (2a_1, 2a_2, \ldots) \) (assume the sequence is weakly decreasing, otherwise it can be sorted). Frobenius’ formula (Jacobi-Trudi identity) gives

\[
\chi^\lambda = \chi^{(n-2k)} \circ \chi^{(2k)} - \chi^{(n-2k+1)} \circ \chi^{(2k-1)} = \chi^{(n, 2k)/(2k)} - \chi^{(n, 2k-1)/(2k-1)}.
\]

We evaluate the characters for skew shapes by the usual Murnaghan–Nakayama rule. In this case the height of each border strip has to be 1 to fit into skew tableaux consisting of disconnected rows. For any multiset of positive integers \( R = \{r_1, \ldots, r_q\} \) and any integer \( m \), denote by \( P_R(m) \) the number of ways to write \( m \) as a sum of entries from \( R \). In other words

\[
P_R(m) = \# \{(i_1, i_2, \ldots) : 1 \leq i_1 < i_2 < \cdots \leq q, \text{ s.t. } r_{i_1} + r_{i_2} + \cdots = m\}.
\]

For any \( a \) and \( b \) the Murnaghan–Nakayama rule gives

\[
\chi^{(a+b, a)}/(a)[\nu] = P_\mathcal{X}(a),
\]

where \( \mathcal{X} = \{\nu_1, \ldots, \nu_\ell\} \). Hence

\[
\chi^\lambda[\nu] = P_\mathcal{X}(2k) - P_\mathcal{X}(2k - 1).
\]

Since all elements in \( \mathcal{X} \) are even, we have that \( P_\mathcal{X}(2k - 1) = 0 \), so \( \chi^\lambda[\nu] = 0 \) if and only if \( P_\mathcal{X}(2k) = 0 \). Determining whether \( P_\mathcal{X}(2k) = 0 \) is the same as the Knapsack problem above.

\[\square\]
8. Final remarks

8.1. As mentioned in the introduction, the Main Lemma implies that one can compute Kronecker coefficients in time polynomial in the size of the parts, but exponential in the number of parts. This phenomenon is similar to the well known distinction between weak and strong NP-completeness (see e.g. [GJ, Pap, Vaz]), corresponding to the input given in binary and in unary. It applies to other counting problems as well. For example, counting the number of perfect matchings in graphs with large multiple edges and fixed number $n$ of vertices is easily polynomial, while for large $n$ the problem is classically #P-complete even when edge multiplicities are 0 or 1. More generally, the number of BIN PACKING solutions is strongly #P-complete, via a standard reduction to #TRIPARTITE MATCHINGS problem (see e.g. [Pap]). Of course, for other counting problems this phenomenon fails. For example, the counting of KNAPSACK solutions is polynomial when the input is in unary.

From this point of view, we believe that the bounds in the Main Lemma cannot be substantially improved.

Conjecture 8.1. The Kronecker coefficients $g(\lambda, \mu, \nu)$ and the LR coefficients $c^\lambda_{\mu \nu}$ are strongly #P-hard.

Of course, the second part of the conjecture implies the first part, via Murnaghan’s reduction (see below). The final reduction in [Nar] proving #P-completeness of LR coefficients, is from contingency tables (see [DG] for an introduction). It is easy to see that the decision problem for existence of contingency tables with given marginals is polynomial when the input is in unary, and NP-complete when the input is in binary. However, despite the large body of literature, it seems open whether the counting problem is strongly #P-complete. Note, however, that De Loera and Onn proved that for the three-way statistical tables (see §5.2), the counting problem is strongly #P-complete, even when one dimension $m \geq 2$ is bounded, see [DO].

8.2. The Littlewood–Richardson coefficients are much better understood than the Kronecker coefficients. In fact, the LR coefficients are actually a special case of the Kronecker coefficients:

$$c^\lambda_{\mu \nu} = g((n - |\lambda|, \lambda), (n - |\mu|, \mu), (n - |\nu|, \nu)),$$

for any partitions $\lambda, \mu, \nu$, such that $|\lambda| = |\mu| + |\nu|$, and any sufficiently large $n$. This equality is due to Murnaghan and Littlewood [L2, Mu2, Mu3]. Suppose that $\nu_1 \leq M$. Then the Reduction Lemma applied to the partitions in (10), gives the following alternative for the LR coefficients. When $|\lambda_i - \mu_i| > |\nu|$ for some $i$, we have $c^\lambda_{\mu \nu} = 0$; otherwise, there exist partitions $\psi(\lambda), \psi(\mu)$ of sizes at most $\ell M$, such that $c^\lambda_{\mu \nu} = c^{\psi(\lambda)}_{\psi(\mu) \nu}$. Note that these results can also be obtained directly from the Littlewood–Richardson rule, as in the proof of the Reduction Lemma. Let us mention also that for $i = 1$, part (i) of the Reduction Lemma is proved in [Kl2] (see also [JK, §2.9]).

The analogue of Theorem 3.5 for LR coefficients is the following result.

Corollary 8.2. When $\nu_1 \leq M, \lambda_1, \mu_1 \leq N$ and $\ell(\mu), \ell(\lambda), \ell(\nu) \leq \ell$, the Littlewood–Richardson coefficient $c^\lambda_{\mu \nu}$ can be computed in time $O(\ell \log N) + (\ell \log M)^O(\ell^3 \log \ell)$.

This result seems already nontrivial and hard to establish directly. It would be nice to improve the complexity in the corollary (cf. §8.4 below). In a different direction, it would
be interesting to extend the Main Theorem 3.3 to plethystic constants $a_{\lambda\mu}^\pi$ (see e.g. [JK, M1, Sta]).

8.3. An analogous to KP, yet much simpler, is the positivity problem for the Littlewood–Richardson coefficients:

**LRP Problem:** Given the input $(\lambda, \mu, \nu)$ as in KP, decide whether $c_{\lambda\mu}^{\nu} > 0$.

Knutson and Tao’s proof of the *Saturation Conjecture* [KT] implies that this decision problem is in $P$, since it reduces to a feasibility problem of a linear program with $O(\ell^2)$ inequalities and constraints of size $O(N)$, see [MNS, BI2]. Such problems can be solved in polynomial time in the size $s$ of the input, $s = \Theta(\ell \log N)$.

Unfortunately, the saturation theorem does not hold for Kronecker coefficients (see e.g. [Kir, §2.5]). Mulmuley’s original approach to KP was via Conjecture 3.3 and a weak version of the Saturation Conjecture. The original version of the latter was disproved in [BOR1], and modified by Mulmuley in the appendix to [BOR1] (see also [M1]).

While the decision problem for the positivity of LR–coefficients is in $NP$ even without the Knutson-Tao theorem, conjectures 3.1 and 3.3 remain out of reach. As of now, there are no combinatorial interpretation for Kronecker coefficients $g(\lambda, \mu, \nu)$ except for a few special cases (see the references in [PP3, PPV]).

8.4. For the LR coefficients, one can apply Barvinok’s algorithm for counting integer points in polytopes of BZ triangles, see [DM]. In notation of Corollary 8.2 these polytopes have dimension $d = o(\ell^2)$ and input size $L = O(\ell \log N)$. By Theorem 2.3 the resulting algorithm has cost

$$L^{O(d \log d)} = (\ell \log N)^{O(\ell^2 \log \ell)}.$$ 

This is roughly comparable with the result of Corollary 8.2 larger in some cases and smaller in other.

Note also that in light of Barvinok’s algorithm, one can view the Main Lemma 5.4 as an evidence in support of Mulmuley’s Conjecture 3.4.

8.5. A positive combinatorial interpretation for the Kronecker coefficients, analogous to the LR–rule, would likely show that the decision problem is in $NP$ and the counting problem in $\#P$. Such interpretation would also imply a combinatorial interpretation for the difference between the number of partitions of $k$ and the number of partitions of $k - 1$, which fit into an $\ell \times m$ rectangle (see [PP1]). Formally, this difference is equal to $g(m^\ell, m^\ell, (n - k, k))$; in full generality its combinatorial interpretation is already highly nontrivial and will appear in [PP3] (see also [BO]).

8.6. The known results so far do not prove Mulmuley’s Conjecture 3.1 even when the input is in unary. As the current results suggest, the computational complexity comes from two sources – the length $\ell$ of the partitions, and the size $N$ of their parts. While it is often possible to reduce the problem to one where the size of the parts is $O(\log N)$, the exponential dependence on $\ell$ cannot be reduced with current methods. As suggested by the proof of Lemma 5.4 and the equivalent formulas through inverse Kostka coefficients (see [V1]), the Kronecker coefficients are given by alternating sums over all permutations in $S_\ell$, whose number is $O(\ell^\ell)$. 
8.7. The semigroup property (Theorem 2.1) was conjectured by Klyachko in 2004, and recently proved in [CHM] (see also [Man]). It is the analogue of the semigroup property of LR coefficients proved by Brion and Knop in 1989 (see [Zel] for the history and the related results).

8.8. Throughout the paper, we are rather relaxed in our treatment of the algorithm timing. Our time complexity is in the cost of arithmetic operations with integers as in the input.

8.9. The reduced Kronecker coefficients, see e.g. [BDO, BOR2], are defined as

\[(11) \tilde{g}(\lambda, \mu, \nu) = g((n - |\lambda|, \lambda), (n - |\mu|, \mu), (n - |\nu|, \nu)) \text{ for } n \text{ large enough.}\]

Note that equation (11) generalizes equation (10) when there is no constraint \(|\lambda| = |\mu| + |\nu|\).

The fact that \(\tilde{g}(\lambda, \mu, \nu)\) are well defined has been established by Murnaghan [Mu1] (see also [Bri, V]), but determining effective bounds on \(n\) for which the sequence stabilizes is still an active area. The Reduction Lemma 5.1 immediately implies Murnaghan’s result that they stabilize. Its proof, and more specifically the map \(\varphi\), also gives the following upper bound.

**Corollary 8.3.** Equation (11) holds for all triples of partitions \((\lambda, \mu, \nu)\), such that \(|\nu| \leq |\mu| \leq |\lambda|\) and \(n \leq \max\{\lambda_1, \mu_1\} + |\nu| + |\lambda|\).

This result matches the bound in [V] in the cases when \(\lambda_1 \geq \mu_1\), but is slightly weaker otherwise. This result is also comparable to the result of [BOR2, Theorem 1.4], where, for example, in the case \(\lambda = \mu = \nu\) they coincide, but in general is also slightly weaker.

**Sketch of proof.** Apply the map \(\varphi\) from Lemma 5.1 with \(t = |\nu|\) to the partitions \((n - |\lambda|, \lambda)\) and \((n - |\mu|, \mu)\). When \(n \geq \max\{\lambda_1, \mu_1\} + |\nu| + |\lambda|\), we have \(1 \in I\). Then, for any partition \(\theta\) as in the proof, the first part \(\varphi(\theta)\) does not depend on \(n\) anymore. Thus \(\varphi(n - |\lambda|, \lambda), \varphi(n - |\mu|, \mu), \varphi(n - |\nu|, \nu)\) are also independent of \(n\). The rest follows from the proof of Lemma 5.1. \(\square\)

It would be interesting to see if the Reduction Lemma can be further extended to imply better bounds, or whether there is a plethystic generalization.

8.10. Note that the elementary construction in the proof of Theorem 7.1 also implies that computing characters is \(#P\)-hard, a result obtained earlier in [Hep]. In [PPV], the authors prove that positivity of certain Kronecker coefficients is a consequence of nonzero character values, which are easier to establish via the Murnaghan–Nakayama rule in certain large cases. Unfortunately, the fact that CHAP is \(NP\)-hard (Theorem 7.1), implies that this approach is unlikely to have complexity implications. Similarly, using characters to efficiently compute the Kronecker coefficients via the formula in §2.2 is most likely going to be futile (cf. [BCS, §13.5]).

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