BALLOT TILINGS AND INCREASING TREES

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ABSTRACT. We study enumerations of Dyck and ballot tilings, which are tilings of a region determined by two Dyck or ballot paths. We give bijective proofs to two formulae of enumerations of Dyck tilings through Hermite histories. We show that one of the formulae is equal to a certain Kazhdan–Lusztig polynomial. For a ballot tiling, we establish formulae which are analogues of formulae for Dyck tilings. Especially, the generating functions have factorized expressions. The key tool is a planted plane tree and its increasing labelings. We also introduce a generalized perfect matching which is bijective to an Hermite history for a ballot tiling. By combining these objects, we obtain various expressions of a generating function of ballot tilings with a fixed lower path.

1. Introduction

Cover-inclusive Dyck tilings were introduced to study Kazhdan–Lusztig polynomials for the Grassmannian permutations in [8]. Independently, Dyck tilings reappeared in the study of the double-dimer model in [3]. One of the main purposes of this paper is to give bijective proofs of two conjectures by Kenyon and Wilson in [3] (proven by Kim in [4]). The first formula is associated with a Dyck tiling with a fixed lower path and the second one is associated with a Dyck tiling with a fixed upper path. In both formulae, an Hermite history which is bijective to a Dyck tiling plays a central role. We show that the first formula is equal to a certain Kazhdan–Lusztig polynomial by providing a bijection between a term in a Kazhdan–Lusztig polynomial and an Hermite history.

In the study of Kazhdan–Lusztig polynomials, one can study Hermitian symmetric pairs instead of Grassmannian permutations. This replacement corresponds to moving from type A to other types in the classification of Weyl groups. Cover-inclusive ballot tilings appeared in the case of the Hermitian symmetric pair \((B_N, A_{N-1})\) in [7]. A ballot tiling can be regarded as a generalization of a Dyck tiling from type A to type B. Regarding ballot tilings, we have two types of tilings: type BI and type BIII. These two types originate in the classification of Kazhdan–Lusztig bases for the Hermitian symmetric pair \((B_N, A_{N-1})\) [7]. The main purpose of this paper is to give proofs to generalizations of two formulae by Kenyon and Wilson in case of ballot tilings. As in the case of Dyck tilings, two formulae for ballot tilings have factorized form (see Theorems 4.1, 6.5 and 7.4). In [2], two formulae for a Dyck tiling are interpreted in terms of the weak order and the Bruhat order on permutations. We give an analogue of these results for a ballot tiling (see Theorems 4.4, 6.5 and 8.17). Not only factorized expressions, we obtain various expressions of the first formula for a ballot tiling (See Section 8). The key tool is a weakly increasing tree. We also derive several recurrence relations for a generating function of a ballot tiling in terms of trees and solve them explicitly.

The paper is organized as follows. In Section 2, we briefly review cover-inclusive Dyck tilings and introduce two Theorems 2.2 and 2.3. Theorem 2.2 is a formula for Dyck tilings with a fixed lower path and Theorem 2.3 is one for Dyck tilings with a fixed upper path. We provide a simple bijective proof of Theorem 2.3 in Section 2.1. The remaining parts of this section is devoted to the proof of
Theorem 2.2. We start from the notion of planted plane trees in Section 2.2. To compute Kazhdan–Lusztig polynomials, we introduce a labelling of a tree which we call Lascoux–Schützenberger. In Section 2.3, we consider three types of Hermite histories which are bijective to a Dyck tiling. To connect Kazhdan–Lusztig polynomial with an Hermite history bijectively, we introduce an integer array associated to an Hermite history. This integer array plays a central role to give a bijective proof of Theorem 2.2. This bijection is realized by two maps constructed in Section 2.4. We show that the first formula for a Dyck tiling is equal to a Kazhdan–Lusztig polynomial (Theorem 2.17). Since this Kazhdan–Lusztig polynomial is written as a product of quantum integers, we obtain an expression of the first formula in terms of quantum integers (Theorem 2.22), which is a different expression from Theorem 2.2.

In Section 3, we start with two types of incidence matrices $M$ and introduce two types of ballot tilings. The two types are type BI and type BIII. As shown in [3], the entries of the inverse matrix $M^{-1}$ correspond to a cover-inclusive ballot tiling.

In Section 4, we give bijective proofs of the first and second formulae for ballot tilings of type BIII. Theorem 4.1 shows that the first formula for ballot tilings of type BIII factorizes into the product of the first formula for Dyck tilings and an extra factor. In Theorem 4.4, we prove an analogue of the equality between the second and third terms in Theorem 2.3. Here, we consider $(12,12)$-avoiding signed permutations instead of the Weyl group $S_C$ of type C.

Sections 5 to 8 are devoted to analysis of ballot tilings of type BI. In Section 5.1, we start with the notion of planted plane trees which is a generalization of plane trees in Section 2.2. A tree introduced here appeared in [1, 7] as a binary tree to compute Kazhdan–Lusztig polynomials for the Hermitian symmetric pair $(B_N, A_{N-1})$. In Section 5.2, we introduce a generalized perfect matching for ballot tilings to connect them to an Hermite history. In Section 5.3, we give a bijection from a labelling of a plane tree to a ballot tiling. We call this bijection ballot tile strip which is a generalization of Dyck tiling strip (DTS for short) studied in [5]. In Section 5.4, we introduce an Hermite history for a ballot tiling and show a bijection between an Hermite history and a generalized perfect matching. This bijection is realized by two maps constructed in Section 2.4. We show that the first formula for a Dyck tiling is equal to a Kazhdan–Lusztig polynomial (Theorem 5.1). In Section 5.5, we prove an analogue of the equality between the second and third terms in Theorem 2.3. Here, we consider $(12,12)$-avoiding signed permutations instead of the Weyl group $S_C$ of type C.

In Section 6, we consider ballot tilings of type BI with a fixed lower path. One of the main theorems in this paper is Theorem 6.4, which shows that a generating function is factorized. Section 6.1 and 6.2 are devoted to the proof of Theorem 6.4. In Section 6.3, we translate Theorem 6.4 in terms of trees introduced in Section 5.1.

In Section 7, we provide several expressions of the generating function for ballot tilings with a fixed lower path. In Section 7.1 and 7.2, we show recurrence relations for the generating functions. In Section 8.1 and 8.2, we show recurrence relations for the generating functions. In Section 8.3, we prove three expressions for the generating function: Theorem 8.8, Theorem 8.13 and Theorem 8.14. We consider trees without arrows in Section 8.4. The generating function can be expressed in a simple way: Theorem 8.17 and Theorem 8.19. These two theorems can be regarded as an analogue of Theorem 2.2. Further, we have three more different expressions: Theorems 8.20, 8.22 and 8.23. In Section 8.5, we consider a tree $T$ without arrows such that an edge of $T$ does not have a parent edge with a dot (see Section 5.1 for detailed definitions). A generating function for a tree $T$ is expressed as a product of two factors (Corollary 8.26), one of which is the sum of weights for inverse pre-order words introduced in Section 5.
Notation. We introduce the quantum integer \([n] := \sum_{i=0}^{n-1} q^i\), the quantum factorial \([n]! := \prod_{i=1}^{n}[i]\), the \(q\)-analogue of the binomial coefficients
\[
\begin{align*}
\begin{bmatrix} n \\ m \end{bmatrix} &= \frac{[n]!}{[n-m]![m]!}, & \begin{bmatrix} n \\ m \end{bmatrix}_{q^2} &= \frac{[2n]!!}{[2(n-m)]!![2m]!!},
\end{align*}
\]
and the multinomial coefficients
\[
\begin{bmatrix} n \\ k_1, \ldots, k_r \end{bmatrix} := \prod_{i=1}^{r}[k_i]!, & \begin{bmatrix} n \\ k_1, \ldots, k_r \end{bmatrix}_{q^2} := \frac{[2n]!!}{\prod_{i=1}^{r}[2k_i]!!}.
\]
We denote \(q, t\)-integers by \([n]_t := [n-1] + q^{n-2}t\).

2. Dyck Tiling

We recall the definitions of Dyck tiling following [2, 4, 8].

A *Dyck path of length* \(2n\) is a lattice path from the origin \((0,0)\) to \((2n,0)\) with up (or “U”) steps \((1,1)\) and down (or “D”) steps \((1,-1)\), which does not go below the horizontal line \(y = 0\). A sequence of “U” and “D” corresponding to a Dyck path is called Dyck word. The highest (resp. lowest) Dyck path of length \(2n\) is \(U \cdots UD \cdots U\) (resp. \(UDU \cdots UD\)).

A Dyck path \(\lambda\) of length \(2n\) is identified with the Young diagram which is determined by the skew shape \(\lambda/\mu\). We denote by \(\text{inv}(\pi)\) the set of chords of a Dyck path and its length and height (see e.g. Section 1 in [4]). We make a pair of \(U\) and \(D\) which are next to each other. Successively, we make a pair of \(U\) and \(D\) by ignoring paired \(U\) and \(D\). A Dyck path of length \(2n\) consists of \(n\) pairs of \(U\) and \(D\). We call this pair of \(U\) and \(D\) a chord of the Dyck path \(\lambda\). We denote \(\text{Chord}(\lambda)\) by the set of chords of a Dyck path \(\lambda\). We introduce two statistics called area and tiles: the area \(\text{area}(D)\) is the number of boxes in the skew shape \(\lambda/\mu\) and the tiles \(\text{tiles}(D)\) is the number of Dyck tiles of \(D\). The statistic \(\text{art}\) is defined as \(\text{art}(D) = (\text{area}(D) + \text{tiles}(D))/2\).

**Definition 2.1.** Let \(\lambda\) be a Dyck path. We define
\[
F^\text{Dyck}_\lambda := \sum_{D \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(D)}
\]

We introduce a chord of a Dyck path and its length and height (see e.g. Section 1 in [4]). We make a pair of \(U\) and \(D\) which are next to each other. Successively, we make a pair of \(U\) and \(D\) by ignoring paired \(U\) and \(D\). A Dyck path of length \(2n\) consists of \(n\) pairs of \(U\) and \(D\). We call this pair of \(U\) and \(D\) a chord of the Dyck path \(\lambda\). We denote \(\text{Chord}(\lambda)\) by the set of chords of a Dyck path \(\lambda\). We denote two statistics for a chord \(c\) by the length \(l(c)\) and the height \(h(c)\). For a chord \(c \in \text{Chord}(\lambda), l(c)\) is one plus the number of chords in-between \(U\) and \(D\) in \(c\). The height \(h(c)\) is defined to be \(i\) if \(c\) is between the lines \(y = i - 1\) and \(y = i\).

We denote by \(\mathfrak{S}_n\) the symmetric group of degree \(n\). The number of inversions of \(\pi \in \mathfrak{S}_n\) is denoted by \(\text{inv}(\pi)\).
Let \( \pi_0 \) be a permutation associated to a Dyck path \( \lambda \) (see [2] for a detailed definition).

**Theorem 2.2 ([2, 3, 4]).** We have

\[
P^\text{Dyck}_\lambda = \frac{[n]!}{\prod_{c \in \text{Chord}(\lambda)} [l(c)]} = \sum_{\pi \geq_L \pi_0} q^{\text{inv}(\pi) - \text{inv}(\pi_0)}
\]

where \( \geq_L \) is the weak left order on \( S_n \).

Let \( \rho(\mu) \) be a 132-avoiding permutation associated to a Dyck path \( \mu \) (see [2] for a detailed definition).

**Theorem 2.3 ([2, 3, 4]).** We have

\[
\sum_{D \in \mathcal{D}(\ast/\mu)} q^{\text{tiles}(D)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)] = \sum_{\pi \geq \rho(\mu)} q^{\text{inv}(\pi) - \text{inv}(\rho(\mu))}
\]

where \( \geq \) is the Bruhat order on \( S_n \).

In this section, we give a simple bijective proof of Theorem 2.3 and a bijective proof of an another expression of Theorem 2.2.

### 2.1. A bijective proof of Theorem 2.3.

Recall that a chord \( c \) of Dyck path \( \mu \) is a matching pair of an up step \( u \) and a down step. The statistic \( \text{ht}(c) \) for a chord of \( \mu \) is equal to the number of boxes in the southeast direction between \( u \) and the lowest path \( UD UD \ldots UD \) plus 1. We denote by \( n(u) \) this number of boxes, that is, \( \text{ht}(c) = n(u) + 1 \). Thus, the right hand side of Theorem 2.3 is rewritten as \( \prod_{u \text{up step of } \mu} [n(u) + 1] \).

We denote by \( \mu_0 \) the lowest path for \( \mathcal{D}(\ast/\mu) \), which is the zig-zag path ("UD UD \ldots UD"). Fix a path \( \mu \) and an up step \( u \) of \( \mu \). The number of boxes in \( \mu_0/\mu \) in the southeast direction at \( u \) is \( n(u) \). The \( q \)-integer is expanded as \( [\text{ht}(c)] = 1 + q + \ldots + q^{n(u)} \). When we take a term \( q^i \) from \( [\text{ht}(c)] \), we place \( i \) successive boxes in the region between \( u \) and the zig-zag path such that the northwest edge of the northwest box is attached to the up step \( u \). Since a term in \( \prod_q [n(u) + 1] \) is a product of \( q^i \) from \( [n(u) + 1] \), we place boxes as above for each up step (see the left figure in Figure 2.4). We denote by \( D' \) the obtained diagram consisting of boxes.

We will construct a Dyck tiling from \( D' \). We enumerate the up steps of \( \mu \) from right to left by \( u_1, u_2, \ldots, u_n \). We also enumerate the boxes attached to the up step \( u_j \) from northwest to southeast by \( 1, 2, \ldots, r_j \) where \( r_j \) is the number of boxes. Fix a pair \((u_j, i)\) with \( 1 \leq i \leq r_j \) and call the corresponding box \((u_j, i)\) box.

If there is a box to the northeast of \((u_j, i)\) box and it forms a Dyck tile, move to the next pair \((u_j, i + 1)\). Otherwise, there is no box to the northeast of \((u_j, i)\) box. Let \( b \) be a box of \( \mu_0/\mu \) right to the \((u_j, i)\) box such that the translation of \( b \) by \((1,1)\) is either outside of the region \( \mu_0/\mu \) or contained in a Dyck tile. Then, there exists a unique Dyck tile \( D \) such that it starts from \((u_j, i)\) box and ends at the box \( b \) and a box to the north of a box of \( D \) is either outside of the region \( \mu_0/\mu \) or contained in a Dyck tile. We move the boxes \((u_j, k)\) with \( i + 1 \leq k \leq r_j \) to the southeast of the box \( b \). Then, we move to the next pair \((u_j, i + 1)\) for \( 1 \leq i \leq r_j - 1 \) or \((u_{j+1}, 1)\) for \( i = r_j \) (see Figure 2.4).
It is obvious that the obtained Dyck tile has the weight \(q^{\text{tiles}(D)}\). The above operation has an inverse, that is, we can construct a diagram \(D'\) from a Dyck tiling. Thus, we obtain the first equality in Theorem 2.3.

![Figure 2.4. The bijection from a diagram \(D'\) to a Dyck tiling. The statistics \(\text{tiles}(D)\) of the Dyck tiling (right figure) is \(\text{tiles}(D) = 9\).](image)

2.2. **Planted plane tree.** Let \(\mathcal{Z}\) be a set of words consisting of \(U\) and \(D\) such that \(\emptyset \in \mathcal{Z}\), if \(z \in \mathcal{Z}\) then \(UzD \in \mathcal{Z}\) and if \(z_1, z_2 \in \mathcal{Z}\) then the concatenation \(z_1 z_2 \in \mathcal{Z}\). In other words, the set \(\mathcal{Z}\) is a set of Dyck words.

Let \(\lambda_1, \lambda_2\) be two words of length \(2n\) consisting of \(U\) and \(D\). If \(\lambda_1\) is above \(\lambda_2\), we denote it by \(\lambda_1 \geq \lambda_2\). For a word \(\lambda\), we denote by \(||\lambda||\) the length of the word and by \(||\lambda||_\alpha\), \(\alpha = U\) or \(D\), the number of \(\alpha\) in the word \(\lambda\). Let \(\lambda\) be a Dyck word of length \(2n\) and \(\lambda_0 \geq \lambda\) be a word of length \(2n\) consisting of \(U\) and \(D\). Here, \(\lambda_0\) is not necessarily a Dyck word.

Suppose that \(\lambda_0 = \lambda'_0 v w \lambda''_0\) and \(\lambda = \lambda' U D \lambda''\) where \(v, w \in \{U, D\}\) and \(||\lambda'_0|| = ||\lambda'||\). A capacity of the partial word corresponding to \(UD\) in \(\lambda\) is defined by

\[
\text{cap}(UD) := ||\lambda'_0 v||_U - ||\lambda'\lambda''||_U.
\]

The condition \(\lambda \leq \lambda_0\) implies a capacity of \(\lambda\) is non-negative.

We define a tree \(A(\lambda)\) for a Dyck word \(\lambda\). A tree \(A(\lambda)\) satisfies

(\(\bigcirc 1\)) \(A(\emptyset)\) is the empty tree.

(\(\bigcirc 2\)) \(A(D\lambda') = A(\lambda')\).

(\(\bigcirc 3\)) \(A(z\lambda'), z \in D\), is obtained by attaching the tree for \(A(z)\) and \(A(\lambda')\) at their roots.

(\(\bigcirc 4\)) \(A(UzD), z \in \mathcal{Z}\), is obtained by attaching an edge just above the tree \(A(z)\).

We put the capacities of \(\lambda\) with respect to \(\lambda_0\) on leaves of the plane tree \(A(\lambda)\). We denote by \(A(\lambda/\lambda_0)\) a tree \(A(\lambda)\) with capacities.

A labelling of Lascoux–Schützenberger type (labelling of LS type for short) is a set of non-negative integers on the edges of \(A(\lambda)\) satisfying

(LS1) Integers on edges are non-increasing from leaves to the root.

(LS2) An integer on an edge connecting to a leaf is less than or equal to its capacity.

Examples of \(A(\lambda/\lambda_0)\) and its labelling of LS type are shown in Figure 2.6.

For a labelling \(\nu\) of LS type, we denote by \(\sigma(\nu)\) the sum of integers in \(\nu\). Suppose that \(\lambda\) and \(\lambda_0\) be a Dyck word. Define a generating function

\[
P_{\lambda,\lambda_0} := \sum_{\nu} q^{\sigma(\nu)},
\]

where the sum runs over all possible labellings of LS type of \(A(\lambda/\lambda_0)\). Then, Lascoux and Schützenberger proved that

**Theorem 2.5** (Lascoux and Schützenberger [6]). The generating function \(P_{\lambda,\lambda_0}\) is the Kazhdan–Lusztig polynomial for a Grassmannian permutation.
Besides a labelling of LS type, we will consider several labellings of a tree $A(\lambda)$. For this purpose, we introduce an array of integers whose shape is determined by a tree $A(\lambda)$. If two edges $e_1$ and $e_2$ have the same parent edge and $e_2$ is right to $e_1$, we put an integer on $e_2$ right to an integer on $e_1$. If $e_1$ is a parent of $e_2$, then we put an integer on $e_2$ below an integer on $e_1$. We call an array of integers associated to a labelling of LS type an array of integers of LS type for short. An example of an array of integers of LS type is shown in Figure 2.6.

![Figure 2.6](image)

**Figure 2.6.** The left picture is an example of a plane tree $A(\lambda/\lambda_0)$ with capacities $(1, 3, 3, 4, 6)$. The middle picture shows an example of a labelling of LS type associated to the plane tree. The right picture is an array of integers of LS type associated to the middle picture.

### 2.3. Hermite history

We introduce an Hermite history for a Dyck tiling following [4, Section 6] and generalize it. Fix a Dyck tiling $D$ in $D(\lambda/\mu)$. For an up step $u$ of a Dyck word $\lambda$, we denote by $n(u)$ the number of down steps left to the up step $u$ and by $h$ the height of the up step $u$. For example, the height of the third up step from left in a path $UUDUDD$ is two. A *Hermite history of length* $2n$ of type I (resp. type II) is a pair $(\mu, H)$ (resp. $(H, \lambda)$) of a Dyck path $\mu$ (resp. $\lambda$) and a labelling $H$ (resp. $H''$) of $\mu$ (resp. $\lambda$). A labelling $H$ (resp. $H''$) is a set of integers on the up steps of $\mu$ (resp. $\lambda$) such that if the height of an down step $d$ is $h'$ then a label of $d$ is an integer in $[0, h' - 1]$.

A bijection from a Dyck tiling $D$ to $(\mu, H)$ of type I is defined as follows. For a Dyck tile $D$, we define the entry (resp. exit) northwest (resp. southeast) edge of the leftmost (resp. rightmost) cell of $D$. We consider a path starting from an up step $u$ of $\mu$. If $u$ is not a entry of a Dyck tile, then the label of $u$ is zero. If $u$ is an entry of a Dyck tile $D$, then extend a path from the entry to the exit of $D$ until the path arrives at an edge which is not an entry. The labelling of $u$ is a number of tiles which the path starting from $u$ travels. We denote by $|H|$ the sum of labels on up steps of $\mu$. Then, we have $|H| = \text{tiles}(H)$.

A bijection from a Dyck tiling to $(\mu, H'')$ of type III is defined by replacing northwest with northeast, southeast with southwest in the definition of the entry and the exit, and replacing an up step with a down step in the definitions for type I.

A bijection from a Dyck tiling $D$ to $(H', \lambda)$ of type II is defined as follows. We consider the same paths on $D$ as in the case of $(\mu, H)$. A label of an up step $u$ of $\lambda$ is zero if the translation of $u$ by $(0, 2k)$ with $k \geq 0$ is not an exit of a Dyck tile. If the translation of $u$ by $(0, 2k)$ is an exit and its path passes through $r$ Dyck tiles of length $2n_1, 2n_2, \ldots, 2n_r$, the label of $u$ is $\sum_{1 \leq i \leq r} (n_i + 1)$. We denote by $|H'|$ the sum of labels on up steps of $\lambda$. Then, we have $|H'| = \text{art}(D)$. The left figure in Figure 2.7 is an example of Hermite histories of type I and type II.

We associate an array of integers to the pair $(H', \lambda)$. The shape of the array is determined by the tree $A(\lambda)$ as in Section 2.2. We put a label of $H'$ on the array in the up-right order: starting
from the bottom row of the leftmost column to the first row, move to the bottom row of the second-leftmost column and ending at the first row of the rightmost column. We call this array of integers the one of type Hermite history (type Hh for short). The right figure in Figure 2.7 is an example of integer array of type Hh.

Given a Dyck path $\lambda$, recall that the shape of an array of integers $M$ is determined by $\lambda$. We denote by $\lambda_0$ a path which consists of only "U". Then, a capacity of $A(\lambda/\lambda_0)$ is equal to $n(u)$ where $u$ is an up step which corresponds to the bottom row of $M$. We denote by $M(\lambda)$ a set of arrays of integers such that the shape is determined by $\lambda$, integers in a column are weakly increasing from top to bottom and an integer of the bottom row in the column is equal to or less than the corresponding capacity of $A(\lambda/\lambda_0)$. We denote by $M_{\text{LS}}(\lambda)$ the subset of $M(\lambda)$ satisfying the conditions (LS1) and (LS2).

Let $M$ be the following partial array of integers:

$$
\begin{align*}
x_1 \\
x_2 & \quad y_1 \\
& \vdots \quad M' \quad \vdots \quad ' \\
x_n & \quad y_m
\end{align*}
$$

where the $i$-th column is $(x_1, \ldots, x_n)^T$, the $j$-th $(i < j)$ column is $(y_1, \ldots, y_m)^T$ and $M'$ is a partial array such that the non-empty entry of $M'$ is at the same height to or lower than $y_1$. We have no constraint on the height of $x_1$. Let $c_i$ be $i$-th the capacity of $\lambda$ with respect to $\lambda_0$. Let $p$ be the number of $x_k$ which is at the same height to or lower than $y_1$ and $q$ be the number of non-empty entries in $M'$.

Given $x_k$ in the $i$-th column, we define a map $\varphi_i : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ by

$$
\varphi_i(x_k) := c_i - x_k,
$$

where $c_i$ is the capacity for the $i$-th column. We consider the following three conditions on $M$:

(Hh1) There exists an entry $z$ of $M$ such that $z \in M' \cup \{x_1, \ldots, x_p\}$ and $\varphi_i(z) \leq \varphi_j(y_m)$ with $i \leq l \leq j - 1$.

(Hh2) $\varphi_i(z) > \varphi_j(y_m)$ with $i < l < j$ and $c_{i-1} \leq \varphi_j(y_m) < c_i$, when $n = p$ and $z \in M'$.

(Hh3) $x_{p+1} + p + q \geq y_m$, when $n > p$, $\varphi_i(x_{p+1}) \leq \varphi_j(y_m)$ and $\varphi_i(x_k), \varphi_i(z) > \varphi_j(y_m)$ where $1 \leq k \leq p$ and $i < l < j$.

Let $M_{\text{Hh}}(\lambda)$ be the subset of $M(\lambda)$ satisfying either (Hh1), (Hh2) or (Hh3).
Proposition 2.8. There exists a bijection between the pair \((H', \lambda)\) and \(\mathcal{M}_{Hh}(\lambda)\).

Proof. Fix a lower Dyck path \(\lambda\), a Dyck tiling \(\mathcal{D}\). In an integer array \(M\) associated to \(\lambda\), integers in a column are weakly increasing from top to bottom since we have the following three reasons: 1) the upper path of \(\mathcal{D}\) is lower than \(UD\ldots D\), 2) a Dyck tile of length \(2n\) contributes \(n + 1\) in the label \(H'\) and 3) a tiling \(\mathcal{D}\) is cover-inclusive.

We enumerate all steps in \(\lambda\) from left to right by 1, 2, \ldots Suppose that the rightmost box of a Dyck tile of length \(2n\) is on the \(i\)-th up step in \(\lambda\). Then, the leftmost box \(b\) of the Dyck tile is on the \((i - 2n + 2)\)-th up step. To have a Dyck tiling, the existence of a box \(b'\) which is at south west of the box \(b\) is required. It is obvious that the conditions (Hh1), (Hh2) and (Hh3) are equivalent to the conditions for the existence of \(b'\).

\[\square\]

2.4. Maps \(\omega\) and \(\sigma\). Fix \(\lambda_0\) be a path of length \(2n\) consisting of only \(U\). We will give a bijection between an array of integers of type LS associated to a plane tree \(A(\lambda/\lambda_0)\) and an array of integers of type Hh.

Let \(M\) be a partial array of integers in \(\mathcal{M}(\lambda)\). The array \(M\) of integers looks partially an array of integers shown in Eqn.(2.1). We consider the case where \(x_1\) is strictly above \(y_1\) and the entries of \(M'\) are the same height to or lower than \(y_1\). Suppose that \(x_k\) an entry which corresponds to a parent of \(y_1\) in the tree \(A(\lambda)\), that is, \(x_k\) is the lowest entry in \(i\)-th column which is strictly above \(y_1\). An array of integers is said to have a discrepancy of type LS at \(j\)-th column when there exists the pair \((x_k, y_1)\) with \(x_k > y_1\). An array \(M\) of integers is said to have a discrepancy of type Hh at \(j\)-th column when \(M\) violates the conditions from (Hh1) to (Hh3). From the definitions of \(\mathcal{M}(\lambda)\) and \(\mathcal{M}_{LS}(\lambda)\), if \(M\) does not have a discrepancy of type LS, then \(M\) satisfies \(x_k < y_1\). Similarly, a discrepancy of type Hh means the condition such that \(x_p + 1 + p + q < y_m\). Here, \(p\) is the number of \(x_1\) which is at the same height to or lower than \(y_1\) and \(q\) is the number of non-empty entries in \(M'\).

Let \(a\) be the number such that \(x_a - 1 \leq y_1\) and \(x_a > y_1\) and \(b\) be the number such that \(x_b < y_m - p - q\) and \(x_{b+1} \geq y_m - p - q\). We define two maps \(\omega, \sigma: \mathcal{M}(\lambda) \rightarrow \mathcal{M}(\lambda)\) as follows. The image \(\omega(M)\) is obtained by replacing \((x_1, \ldots, x_n)^T\) with \((x_1, \ldots, x_a - 1, y_1, x_a - 1, \ldots, x_{a+p} - 1, x_{a+p+1}, \ldots, x_n)^T\), the entry \(z\) with \(z - 1\) when \(z \in M'\), and \((y_1, \ldots, y_m)^T\) with \((y_2, \ldots, y_m, x_{a+p} + p + q)^T\). In the matrix notation, \(\omega(M)\) is described as

\[
\omega(M) := \begin{pmatrix}
x_1 \\
\vdots \\
x_{a-1} & y_2 \\
\vdots \\
x_{a} - 1 & M' - 1 & y_m \\
\vdots \\
x_{a+p-1} - 1 & x_{a+p} + p + q \\
x_{a+p+1} \\
\vdots \\
x_n
\end{pmatrix}
\]

The image \(\sigma(M)\) is obtained from \(M\) by replacing \((x_1, \ldots, x_n)^T\) with \((x_1, \ldots, x_{b-p-1}, x_{b-p+1} + 1, \ldots, x_b + 1, y_m - p - q, x_{b+1}, \ldots, x_n)^T\), the entry \(z\) with \(z + 1\) when \(z \in M'\), and \((y_1, \ldots, y_m)\) with
In the matrix notation, $\sigma(M)$ is described as

$$
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{b-p-1} \\
    x_{b-p} \\
    x_{b-p+1} + 1 \\
    x_{b+1} \\
    \vdots \\
    y_m - p - q \\
    y_{m-1}
\end{pmatrix}
= 
\begin{pmatrix}
    M' + 1 \\
    \vdots \\
    y_{m-1}
\end{pmatrix},
$$

(2.3)

Given an integer array $N$, we denote by $\text{Dis}^{LS}(N)$ (resp. $\text{Dis}^{Hh}(N)$) be the set of labels of columns in $N$ which have a discrepancy of type LS (resp. Hh). Let $(y_{j_1}, \ldots, y_{j_{n_i}})^T$ be the $j$-th column in $N$. Let $j_1$ be the biggest integer such that $j_1 \in \text{Dis}^{LS}(N)$ and $\varphi_{j_1}(y_1) = \max\{\varphi_i(y_{i,1}) | i \in \text{Dis}^{LS}(N)\}$. Similarly, let $j_2$ be the smallest integer such that $j_2 \in \text{Dis}^{Hh}(N)$ and $\varphi_{j_2}(y_1) = \min\{\varphi_i(y_{i,n_i}) | i \in \text{Dis}^{Hh}(N)\}$. We denote by $M$ the partial array of $N$ involving the $j_1$-th (resp. $j_2$-th) column. The actions of $\omega$ (resp. $\sigma$) on $N$ is defined as $\omega(N) := (N \setminus M) \cup \omega(M)$ (resp. $\sigma(N) := (N \setminus M) \cup \sigma(M)$), i.e., we change a partial matrix $M$ to $\omega(M)$ or $\sigma(M)$ and other entries are not changed.

**Lemma 2.9.** Suppose $M \in \mathcal{M}_{Hh}(\lambda) \cap (\mathcal{M}(\lambda) \setminus \mathcal{M}_{LS}(\lambda))$. Then, $\omega(M) \not\in \mathcal{M}_{Hh}(\lambda)$.

**Proof.** First we show that the non-empty entries of the $i$-th and $j$-th columns of $\omega(M)$ are weakly increasing. Since $x_a > y_1$ implies $x_a - 1 \geq y_1$, the $i$-th column of $\omega(M)$ is weakly increasing. Since the array $M$ is in $\mathcal{M}_{Hh}(\lambda)$, we have a constraint: $y_n \leq x_{p+1} + p + q$. The column $(x_1, \ldots, x_m)^T$ is weakly increasing, which implies $y_n \leq x_{p+1} + p + q \leq x_{a+p} + p + q$. Therefore, the $j$-th column is weakly increasing.

Since we have a discrepancy of type LS at the $j$-th column, we do not have a discrepancy of type LS at the $i$-th column with $i < l < j$. Thus, the integer $z \in M'$ satisfies $z \geq 1$ and $M' - 1$ is well-defined under the action of $\omega$. A column of the partial array $M' - 1$ is weakly increasing since $M'$ is so.

To prove Lemma, it is enough to show that $\omega(M)$ is not in $\mathcal{M}_{Hh}(\lambda)$. The $(p+1)$-th entry of the $i$-th column of $\omega(M)$ is $x_p - 1$. The bottom entry of the $j$-th column of $\omega(M)$ is $x_{a+p} + p + q$. Since $x_p \leq x_{a+p}$, we have $x_p + p + q - 1 < x_{a+p} + p + q$, which is a contradiction against (Hh1), (Hh2) and (Hh3). Therefore, $\omega(M) \not\in \mathcal{M}_{Hh}(\lambda)$.

Let $c$ be an integer such that the $(i, c)$-th entry of $M$ is a parent of the $(j, 1)$-th entry in the corresponding tree.

**Lemma 2.10.** Suppose $M \in \mathcal{M}_{Hh}(\lambda) \cap (\mathcal{M}(\lambda) \setminus \mathcal{M}_{LS}(\lambda))$. Then, we have $\omega^r(M) \not\in \mathcal{M}_{Hh}(\lambda)$ with a positive integer $r \leq \min(x_c, c)$.

**Proof.** A proof is similar to the one of Lemma 2.9. The bottom entry of the $j$-th column of $\omega(M)$ is $x_{a+p+s(r)} + p + q$ with $s(r) \geq r - 1$. The integers $s(r')$ with $0 \leq r' \leq r$ is a weakly increasing. Thus the entries of the $j$-th column is weakly increasing.

The $(p+1)$-th entry of the $i$-th column of $\omega(M)$ is $x_{p-s'} - 1$ with $0 \leq s' \leq r - 1$. We have $x_{p-s'} + p + q - 1 < x_p + p + q \leq x_{a+p+s} + p + q$, which implies the violation of the conditions (Hh1) to (Hh3). Therefore, we have $\omega^r(M) \in \mathcal{M}(\lambda) \setminus \mathcal{M}_{Hh}(\lambda)$.
Lemma 2.11. Suppose that $M \in \mathcal{M}_{Hh}(\lambda) \cap (\mathcal{M}(\lambda) \setminus \mathcal{M}_{LS}(\lambda))$. Then, there exists a positive integer $r \leq \min(x_c, c)$ such that $\omega^r(M) \in \mathcal{M}_{LS}(\lambda)$.

Proof. From Lemma 2.10, $\omega^r(M)$ is in $\mathcal{M}(\lambda)$ with a positive integer $r \leq \min(x_c, c)$. This implies that the first entry of the $j$-th column of $\omega^r(M)$ is weakly increasing. When $r = \min(x_c, c)$, the first entry of the $j$-th column is $y_{r+1}$ or $x_{a+p+s(r)} + p + q$ with some integer $s(r)$. The $c$-th entry of the $i$-th column is equal to or smaller than $y' := \min(y_r, x_{c-r} - r)$. Further, the value $y'$ is equal to or smaller than $y_r$ or $x_{a+p+s(r)} + p + q$. The image $\omega^r(M)$ satisfies the condition (LS1) at the $j$-th column. The $k$-th column with $i < k < j$ also satisfies the condition (LS1). It is obvious that $\omega^r(M)$ satisfies the condition (LS2). Therefore, $\omega^r(M)$ is in $\mathcal{M}_{LS}(\lambda)$.

Lemma 2.12. Suppose $M \in \mathcal{M}_{LS}(\lambda) \cap (\mathcal{M}(\lambda) \setminus \mathcal{M}_{Hh}(\lambda))$. Then, $\sigma^r(M) \in (\mathcal{M}(\lambda) \setminus \mathcal{M}_{LS}(\lambda))$ with a positive integer $r$.

Proof. The integer array $M$ has a discrepancy of type $Hh$ at the $j$-th column. By the definition of $\sigma$, an application of $\sigma$ on a partial integer array $M''$ from the $i$-th column to the $(j - 1)$-th column satisfies one of conditions from (Hh1) to (Hh3). The map $\sigma$ maps $M'$ to $M' + 1$ and some entries in the $i$-th column are increased. Thus, the application of $\sigma^r$ on $M''$ also satisfies the conditions for $\mathcal{M}_{Hh}$. The bottom entry $e$ of the $j$-th column is weakly decreasing if we apply $\sigma$ on $M$. Let $y(n, i)$ be the integer on $e$ after we apply $\sigma$ on $M$ $i$ times. If $y(n, i) \leq p + q$, then $\sigma^r(M)$ is in $\mathcal{M}_{Hh}$. If $y(n, i) > p + q$, we have $y(n, i) \geq y(n, i + 1)$... In this case, there exits an integer $r$ such that $\sigma^r(M)$ satisfies the condition (Hh3). This completes the proof.

For $M \in \mathcal{M}(\lambda)$, we denote by $|M|$ the sum of integers in $M$. From the definitions of $\omega$ and $\sigma$, it is obvious that

Lemma 2.14. We have $|M| = |\omega(M)| = |\sigma(M)|$.

Suppose that $M \in \mathcal{M}_{Hh}(\lambda)$ and $M \notin \mathcal{M}_{LS}(\lambda)$. From Lemma 2.11, there exits a unique smallest positive integer $r_0$ such that $\omega^{r_0}(M) \in \mathcal{M}_{LS}(\lambda)$.

Theorem 2.15. A map $\omega^{r_0}$ is a bijection between $\mathcal{M}_{Hh} \cap (\mathcal{M} \setminus \mathcal{M}_{LS})$ and $\mathcal{M}_{LS} \cap (\mathcal{M} \setminus \mathcal{M}_{Hh})$. The inverse is $\sigma^{r_0}$.

Proof. Suppose that $M \in \mathcal{M}_{Hh} \cap (\mathcal{M}(\lambda) \setminus \mathcal{M}_{LS}(\lambda))$. It is enough to show that $\sigma \circ \omega^r(M) = \omega^{r-1}(M)$ with $1 \leq r \leq r_0$. This follows from the explicit definitions of $\omega$ and $\sigma$. □
Example 2.16. Let $\lambda = UUDDUUDDUDUDDDDD$ be a Dyck path. The shape of an integer array is

\[
\begin{array}{cccc}
* & * & * & * \\
2 & 4 & 5 & \\
\end{array}
\]

where the integers 2, 4 and 5 below a horizontal line are the capacities with respect to $\lambda_0$ and we omit the first column whose capacity and entries are zero. We have $\#\{\nu \mid \nu \in M_{LS}(\lambda) \text{ and } |\nu| = 5\} = 38$. We have 6 labellings out of 38 labellings which have a discrepancy of type $Hh$ and they are

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 \\
0 & 5 & 0 & 4 & 0 & 4 & 0 & 2 \\
\end{array}
\]

The actions of $\sigma$ on these 6 labellings are

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\
\end{array}
\]

and they are in $M_{Hh}(\lambda)$ but have a discrepancy of type $LS$.

2.5. A bijective proof of another expression of Theorem 2.2. Let $\lambda_0$ be a path consisting of only $U$’s.

Theorem 2.17. For a Dyck path $\lambda$, we have

\[
\sum_{D \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(D)} = P_{\lambda, \lambda_0} \tag{2.4}
\]

Proof. From the bijection between a Dyck tiling and $(H', \lambda)$, the left hand side of Eqn.(2.4) is equal to

\[
\sum_{H'} q^{|H'|},
\]

where sum runs over all possible $H'$ of the pair $(H', \lambda)$. From Proposition 2.8, the sum is expressed as

\[
\sum_{M \in M_{Hh}(\lambda)} q^{|M|}.
\]

The set $M_{Hh}(\lambda)$ is equal to $(M_{LS}(\lambda) \cap M_{Hh}(\lambda)) \cup (M_{Hh}(\lambda) \cap (M(\lambda) \setminus M_{LS}(\lambda)))$. We apply Theorem 2.15 and Lemma 2.14 to $M_{Hh}(\lambda) \cap (M(\lambda) \setminus M_{LS}(\lambda))$. Then the sum is equivalent to

\[
\sum_{M \in M_{LS}(\lambda)} q^{|M|},
\]

which implies the sum is equal to the Kazhdan–Lusztig polynomial $P_{\lambda, \lambda_0}$ from Theorem 2.5. $\square$

Let $A(\lambda/\lambda_0)$ be a plane tree associated to a path $\lambda$. We depict the generating function $P_{\lambda, \lambda_0}$ as this tree $A(\lambda/\lambda_0)$ with the capacities. Recall that the tree satisfies the conditions (LS1) and (LS2).
Lemma 2.18. A trivalent tree satisfies the relation:

\[
\begin{align*}
M - m_1 - n_1 &= \left[ \frac{n_1 + m_1}{n_1} \right].
\end{align*}
\]

where \(c_2 := c_1 + n_1\).

Proof. Suppose that a tree \(T\) consisting of \(N\) edges has a single capacity \(c\). The generating function corresponding to the tree \(T\) is \(\left[ N + c \right]_c\). Similarly, if the bottom edge of \(T\) is equal to \(c\), the generating function is \(\left[ N + c \right]_c - \left[ N + c - 1 \right]_c\). From Eqn.\((2.6)\), we have

\[
f'(M, c_1, m_1, n_1) = \sum_{i=0}^{c_1} q^{i(m_1 + n_1 + 1)} \left[ M - m_1 - n_1 + i - 1 \right] \left[ n_1 + c_1 - i \right] \left[ n_1 + m_1 + c_1 - i \right].
\]

From a straightforward calculation, we have

\[
\left[ n_1 + c_1 - i \right] \left[ n_1 + m_1 + c_1 - i \right] = \left[ n_1 + m_1 + c_1 - i \right] \left[ n_1 + m_1 \right].
\]

Substituting Eqn.\((2.7)\) into \(f'(M, c_1, m_1, n_1)\), Eqn.\((2.5)\) is equivalent to the following identity:

\[
\sum_{i=0}^{c_1} q^{i(x+1)} \left[ M - x + i - 1 \right] \left[ M + x - i \right] = \left[ M + c_1 \right],
\]

where \(x = n_1 + m_1\). Let \(f(M, c_1)\) be the left hand side of Eqn.\((2.8)\). By using \(\left[ n+1 \right]_k = q^k \left[ n \right]_k + \left[ n \right]_{k-1}\), we have the recurrence relation:

\[
f(M, c_1) = \sum_{i=0}^{c_1} q^{i(x+1)} \left\{ q^{M-x} \left[ M + i - x - 1 \right] \left[ M - x \right] + \left[ M + i - x - 1 \right] \left[ M - x - 1 \right] \right\} \left[ c_1 + x - i \right] \left[ x \right] = q^{M+1} f(M + 1, c_1 - 1) + f(M, c_1),
\]

with \(f(0, 0) = 1\). This recurrence relation is equivalent to the one for \(q\)-binomial coefficient, which implies that Eqn.\((2.5)\) holds true. \(\square\)

Let \(\lambda\) (resp. \(\lambda'\)) be a Dyck path of length \(2n\) (resp. \(2(n-1)\)).

Lemma 2.19. Suppose a Dyck path \(\lambda\) is written as \(\lambda = UXD\). Then, we have

\[
P_{\lambda, \lambda_0} = P_{\lambda', \lambda_0}
\]
Proof. The tree $A(\lambda/\lambda_0)$ has a unique edge $e$ connecting to the root. Since the leftmost capacity of $A(\lambda/\lambda_0)$ is zero, the integer on the edge $e$ has to be zero. The tree obtained from $A(\lambda/\lambda_0)$ by deleting the edge $e$ is nothing but the tree $A(\lambda'/\lambda_0)$. The sum of labellings in $A(\lambda/\lambda_0)$ is equal to $A(\lambda'/\lambda_0)$. Thus, we have $P_{\lambda,\lambda_0} = P_{\lambda',\lambda_0}$. \hfill \Box

Given two Dyck paths $\lambda_1$ of length $2n_1$ and $\lambda_2$ of length $2m_1$, we denote by $\lambda := \lambda_1 \circ \lambda_2$ the concatenation of two Dyck paths.

**Lemma 2.20.** Suppose $\lambda = \lambda_1 \circ \lambda_2$. Then, we have

$$P_{\lambda,\lambda_0} = \left[\frac{n_1 + m_1}{n_1}\right] P_{\lambda_1,\lambda_0} P_{\lambda_2,\lambda_0}.$$ 

Proof. For a Dyck path $\lambda$ of length $2n$, we denote by $A(\lambda/\lambda_0; c_1)$ the tree $A(\lambda/\lambda_0)$ with capacities shifted by $c_1$. We denote by $P^{c_1}_{\lambda,\lambda_0}$ the generating function corresponding to $A(\lambda/\lambda_0; c_1)$. By successive use of Lemma 2.18, we have

$$P^{c_1}_{\lambda,\lambda_0} = \left[\frac{n + c_1}{c_1}\right] P_{\lambda,\lambda_0}.$$ 

Let $\lambda_1$ (resp. $\lambda_2$) be a Dyck path of length $2n_1$ (resp. $2m_1$). We consider the case where $\lambda = \lambda_1 \circ \lambda_2$. The tree $A(\lambda/\lambda_0)$ is obtained by connecting two trees $A(\lambda_1/\lambda_0)$ and $A(\lambda_2/\lambda_0; n_1)$ at the root. Then, we have

$$P_{\lambda,\lambda_0} = P_{\lambda_1,\lambda_0} P^{n_1}_{\lambda_2,\lambda_0} = \left[\frac{n_1 + m_1}{n_1}\right] P_{\lambda_1,\lambda_0} P_{\lambda_2,\lambda_0},$$

which implies Lemma holds true. \hfill \Box

**Remark 2.21.** If $\lambda = \lambda_1 \circ \lambda_2 \circ \lambda_3$, $\lambda$ can be regarded as $\lambda = (\lambda_1 \circ \lambda_2) \circ \lambda_3$ or $\lambda = \lambda_1 \circ (\lambda_2 \circ \lambda_3)$, which leads to a different expression in Lemma 2.20.

Given a tree $A(\lambda)$, we enumerate the ramification points of the tree $A(\lambda)$ by $1, \ldots, R(\lambda)$ where $R(\lambda)$ is the number of ramification points. Here, a ramification point means that an edge $e$ has two or more edges connecting below $e$. Fix the $i$-th ramification point $A$. At a point $A$, there exists a unique edge $e$ above $A$ and there are $k \geq 2$ edges $e_1, \ldots, e_k$ below $A$. Let $M_i$ be the number of edges of a partial tree connected the point $A$. Let $N_{i,j}$, $1 \leq j \leq k$, be the number of a partial tree connected to the point $A$ such that $e_j$ is the unique edge connected to the root. Thus, we have $M_i = \sum_{1 \leq j \leq k} N_{i,j}$.

**Theorem 2.22.** Let $\lambda$ be a Dyck path and the integers $n_i, m_i$, $1 \leq i \leq R(\lambda)$, defined as above. We have

$$(2.9) \quad \sum_{D \in D(\lambda/\lambda')} q^{\text{art}(D)} = \prod_{i=0}^{R(\lambda)} \left[ \frac{M_i}{N_{i,1}, N_{i,2}, \ldots, N_{i,k}} \right].$$ 

Proof. From Theorem 2.17, the left hand side of Eqn.(2.9) is equal to the Kazhdan–Lusztig polynomial $P_{\lambda,\lambda_0}$. We successively apply Lemma 2.19 and Lemma 2.20 to $P_{\lambda,\lambda_0}$. \hfill \Box
3. Incidence matrix and ballot tilings

3.1. $q,t$-deformed incidence matrix. Let $\lambda$ and $\mu$ be two Dyck paths and $D(\lambda/\mu)$ be the set of cover-inclusive Dyck tilings as in Section 2. In [3], Kenyon and Wilson showed that the inverse matrix $M^{-1}$ of the incidence matrix $M$ is expressed in terms of $|D_{\lambda/\mu}|$. In this subsection, we consider the $q,t$-deformed incidence matrix $M$ whose inverse $M^{-1}$ is expressed in terms of ballot tilings.

Let $\mathcal{P}_n$ be the set of paths of length $n$ which consists of $U$ and $D$. The cardinality of $\mathcal{P}_n$ is $2^n$. We consider operations on a path $\lambda \in \mathcal{P}_n$:

(A) We make a pair between adjacent $U$ and $D$ in this order. Then, connect the pair into a simple arc.

(B) Repeat the procedure (A) until all the $U$'s are to the left of all the $D$'s.

Suppose that $U$ and $D$ are connected by a simple arc and the positions of $U$ and $D$ from left are $i$ and $j$ with $(i < j)$. The size of an arc is defined as $(j - i + 1)/2$.

We define three operations $UD$-flipping, $UU$-flipping and $U$-flipping on $\mathcal{P}_n$ as follows. The $UD$-flipping is an operation to reverse a pair of $U$ and $D$ ($U$ and $D$ are not necessarily adjacent) into $D$ and $U$. Similarly, $UU$-flipping (resp. $U$-flipping) is an operation to reverse a pair of two $U$'s (resp. a single $U$) into two $D$'s (resp. a single $D$).

We consider the following two cases: type BI and type BIII.

Remark 3.1. We call two cases type BI and BIII since an underlying diagram is equivalent to the Kazhdan–Lusztig basis of type BI and BIII studied in [7].

Type BI. In addition to the operations (A) and (B), we have two more operations on a path:

(C) Put a star (★) on the rightmost $U$ if it exists.

(D) For remaining $U$'s, we make a pair of adjacent $U$'s from right to left. Then, we connect this pair into a simple dashed arc.

We define a relation $\leftarrow_I$ on paths in $\mathcal{P}_n$. We say $\lambda_1 \leftarrow_I \lambda_2$ if $\lambda_1$ can be obtained from $\lambda_2$ by $UD$-flippings of the paired $UD$ connected by a simple arc, a $U$-flipping of the $U$ with a star or $UU$-flippings of the $UU$ pairs connected by a simple dashed arc.

Suppose $\lambda_1$ is obtained from $\lambda_2$ by a $U$-flipping on the $U$ with a star and the flipped $U$ is at $(2r + 1)$-th position from right. We define the weight $\text{wt}(\lambda_1 \leftarrow_I \lambda_2) := -q^{r+1}$.

Suppose $\lambda_1$ is obtained from $\lambda_2$ by a $UD$-flipping on $U$ and $D$ connected by an arc of size $m$. We define the weight $\text{wt}(\lambda_1 \leftarrow_I \lambda_2) := -q^m$.

Suppose $\lambda_1$ is obtained from $\lambda_2$ by a $UU$-flipping on two $U$'s connected by a dashed arc of size $m$ and the position of the right $U$ is $2r$ from right. We define the weight $\text{wt}(\lambda_1 \leftarrow_I \lambda_2) := -q^{2r+m}$.

Suppose $\lambda_1 \leftarrow_I \lambda_2$. The weight $\text{wt}(\lambda_1 \leftarrow_I \lambda_2)$ is defined as the product of weights of flippings.

Example 3.2. Let $\lambda_1 = DDUDU$ and $\lambda_2 = UUDDUU$. The path $\lambda_1$ can be obtained from $\lambda_2$ by a $UU$-flipping and a $UD$-flipping. The weight is $\text{wt}(\lambda_1 \leftarrow_I \lambda_2) = q^5$.

Type BIII. We apply operations (A) and (B) to a path. We have unpaired $U$'s and unpaired $D$'s.

We define a relation $\leftarrow_{III}$ on paths in $\mathcal{P}_n$. We say $\lambda_1 \leftarrow_{III} \lambda_2$ if $\lambda_1$ can be obtained from $\lambda_2$ by $UD$-flippings on $U$ and $D$ connected by a simple arc, and by $U$-flippings on unpaired $U$'s.

Suppose that $\lambda_1$ is obtained from $\lambda_2$ by a $U$-flipping on an unpaired $U$ and there are $s$ arcs right to the $U$. We define $\text{wt}(\lambda_1 \leftarrow_{III} \lambda_2) := -q^{s+t}$. 

The weight for $UD$-flipping is the same as the one of type BIII. The weight $\text{wt}(\lambda \leftarrow_{III} \lambda_2)$ is the product of the weights of flippings.

Definition of an incidence matrix $M$. We denote $\leftarrow_I$ or $\leftarrow_{III}$ by $\leftarrow$.

**Definition 3.3.** The incidence matrix $M := (M_{\lambda,\mu})_{\lambda,\mu \in \mathcal{P}_n}$ is defined as

$$M_{\lambda,\mu} := \text{wt}(\lambda \leftarrow \mu) \cdot \delta_{\{\lambda \leftarrow \mu\}}$$

where $\delta_S = 1$ in case of $S$ is true and $\delta_S = 0$ otherwise.

We order the rows and columns according the reversed lexicographic order on paths, i.e., we have $U < D$. Then, the incidence matrix $M$ is lower triangular.

**Example 3.4.** The incidence matrix $M_I, M_{III}$ on $\mathcal{P}_2$ and their inverses are

$$M_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & -q & 1 \end{pmatrix}, \quad M_I^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q & 1 & 0 & 0 \\ q^2 & q & 1 & 0 \\ q^3 & q^2 & q & 1 \end{pmatrix},$$

$$M_{III} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 \\ -t & -q & 1 & 0 \\ t^2 & 0 & -t & 1 \end{pmatrix}, \quad M_{III}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ (1+q)t & q & 1 & 0 \\ qt^2 & qt & t & 1 \end{pmatrix},$$

where the order of bases is $(UU, UD, DU, DD)$.

3.2. **Ballot tiling.** A ballot path of length $(2n,n')$ is a lattice path from the origin $(0,0)$ to $(2n+n',n')$ with $U$ steps and $D$ steps, which does not go below the horizontal line $y = 0$. We denote by $\mathcal{B}_N$ the set of ballot paths of length $N = 2n+n'$. A Dyck path in Section 2 is a ballot path of length $(2n,0)$. The highest path in $\mathcal{B}_N$ is a path consisting of only $U$’s and the lowest path is the zig-zag path $\underbrace{UUD\ldots}_N$.

A ballot path $\lambda$ of length $N$ is identified with the shifted Young diagram which is determined by the path $\lambda$, the line $y = x$ and boxes whose center is on the line $x = N$. We call a box whose center is on the line $x = N$ an anchor box. See Figure 3.5 for an example. If the shifted skew shape $\lambda/\mu$ exists, we call $\lambda$ and $\mu$ the lower path and the upper path.

We denote by $\mathfrak{S}_N^C$ the Weyl group associated to the Dynkin diagram of type $C$. We denote by $s_i$ with $1 \leq i \leq N$ the elementary transposition of $\mathfrak{S}_N^C$. The Weyl group $\mathfrak{S}_N^C$ is isomorphic to the signed permutation group and its order is $2^N \cdot N!$. The elementary transposition $s_i$ is a transposition of $(i, i+1)$ for $1 \leq i \leq N - 1$ and $s_N$ is a transposition $(N, N)$. Here, we use the bar-notation instead of the minus sign. For a reduced word $w := s_{i_1} \ldots s_{i_p}$, we define inversions as follows:

$$\text{inv}_1(w) := \#\{i_j \mid 1 \leq j \leq p, 1 \leq i_j \leq N - 1\},$$

$$\text{inv}_2(w) := \#\{i_j \mid 1 \leq j \leq p, i_j = N\},$$

and $\text{Inv}(w) := \text{inv}_1(w) + \text{inv}_2(w)$.

The shifted Young diagram $\lambda$ gives a word $\omega(\lambda) \in \mathfrak{S}_N^C$ as follows. Fix a path $\lambda \in \mathcal{B}_N$. We enumerate the all $U$’s and $D$’s from right by $1, 2, \ldots, N$. We denote by $\text{Pos}(\lambda)$ the set of the positions of $D$ in $\lambda$. For the $i$-th $D$, we assign a word $\omega_i(\lambda) := s_{N-i+1} \ldots s_N$. Suppose that the set
is written as \( \text{Pos}(\lambda) = \{ i_1 < i_2 < \cdots < i_{N_D} \} \) where \( N_D \) is the number of \( D \) in \( \lambda \). Then, the word \( \omega(\lambda) \) is defined as

\[
\omega(\lambda) := \prod_{i \in \text{Pos}(\lambda)} \omega_i(\lambda) = \omega_{i_1}(\lambda)\omega_{i_2}(\lambda) \cdots \omega_{i_{N_D}}(\lambda),
\]

where the product \( \prod \) is an ordered product. See Figure 3.5 for an example.

Figure 3.5. The shifted Young diagram associated to the path \( \lambda = UDUUD \). The boxes with an asterisk are anchor boxes. The word is \( \omega(\lambda) = s_5s_2s_3s_4s_5 \).

A ballot tile is a ribbon such that the centers of the boxes form a ballot path. A ballot tiling is a tiling of a shifted skew Young diagram \( \lambda/\mu \) by ballot tiles. We consider a cover-inclusive tiling with a condition: the rightmost box of a ballot tile of length \( (2n,n') \) with \( n' \geq 1 \) is on an anchor box of the skew shape \( \lambda/\mu \). We have two types of a ballot tiling: type BI and type BIII.

**Type BI.** We put a constraint on a ballot tiling. The number of ballot tiles of length \( (2n,n') \) is even for \( n' \in 2\mathbb{Z} + 1 \) and zero for \( n' \in 2\mathbb{N}_+ \). The statistics area and tiles for a ballot tile \( B \) of length \( (2n,n') \) with \( n' \geq 1 \) are defined as \( \text{area}(B) := 2n + 1 + n' \) and \( \text{tiles}(B) := 0 \). The statistics on Dyck tile \( D \) (that is a ballot tile of length \( (2n,0) \)) are given by the same as in the case of Dyck tilings, i.e., \( \text{area}(D) = 2n + 1 \) and \( \text{tiles}(D) = 1 \). The statistics art for a ballot tile is defined as \( \text{art}(B) := (\text{area}(B) + \text{tiles}(B))/2 \).

**Type BIII.** Let \( B \) be a ballot tile of length \( (2n,n') \). We denote by \( (S1) \) the following statement:

\( (S1) \) The rightmost box of a ballot tile (including a Dyck tile) is on an anchor box.

We define two statistics area and tiles by \( \text{area}(B) := 2n + 1 \) and \( \text{tiles}(B) := 1 \). Then, statistics art is define by

\[
\text{art}(B) := \begin{cases} 
(\text{area}(B) - \text{tiles}(B))/2, & (S1) \text{ is true}, \\
(\text{area}(B) + \text{tiles}(B))/2, & \text{otherwise}.
\end{cases}
\]

Given a ballot tiling \( T \), we denote by \( \text{tiles}_2(T) \) the number of ballot tiles satisfying the statement \( (S1) \).

We denote by \( B^X(\lambda/\mu) \) with \( X = I \) or III the set of cover-inclusive ballot tilings of type \( X \) in the skew shape \( \lambda/\mu \). We define

\[
B^X(\lambda/*):= \bigcup_{\mu} B^X(\lambda/\mu),
\]

\[
B^X(*/\mu):= \bigcup_{\lambda} B^X(\lambda/\mu).
\]
Definition 3.6. We define generating functions of ballot tilings:

\[ P_{\lambda,\mu}^I := \sum_{B \in \mathcal{B}(\lambda/\mu)} q^{\text{art}(B)}, \]

\[ P_{\lambda,\mu}^{III} := \sum_{B \in \mathcal{B}(\lambda/\mu)} q^{\text{art}(B) + \text{tiles}(B)}, \]

and

\[ P_{\lambda,*}^X := \sum_{\mu} P_{\lambda,\mu}^X, \]

\[ P_{*,\mu}^I := \sum_{B \in \mathcal{B}^I(\ast/\mu)} q^{\text{tiles}(B)}, \]

where \( X = I \) or \( III \).

Example 3.7. We consider the set \( \mathcal{B}^I(\lambda/\ast) \) where \( \lambda = DDUDU \). We have eight ballot tilings in \( \mathcal{B}^I(\lambda/\ast) \) whose statistic art is equal to five:

\[ \begin{array}{c}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \]

By a similar argument to Section 1 in [3], we have the following theorem:

Theorem 3.8. The inverse of the incidence matrix \( M_X \) (\( X = I \) or \( III \)) is given by \( (M_X^{-1})_{\lambda,\mu} = P_{X,\mu}^X \).

4. Ballot tiling of type BI\(I\)

Let \( \lambda \) be a ballot path. The number of \( U \) in \( \lambda \) may not be equal to that of \( D \) in \( \lambda \). A Dyck path can be obtained by a concatenation of \( \lambda \), a sequence of \( U \)’s from left and a sequence of \( D \)’s from right. By abuse of notation, we also denote the obtained Dyck path by \( \lambda \).

We denote by \( N_D(\lambda) \) the number of \( D \) in \( \lambda \).

Theorem 4.1. We have

\[ P_{\lambda,\ast}^{III} = P_{\lambda,*}^{\text{Dyck}} \cdot \prod_{i=1}^{N_D(\lambda)} (1 + q^{i-1}t). \]  

Proof. We give a bijective proof of Eqn.(4.1). A term from the right hand side of Eqn.(4.1) is a product of \( q^{\text{art}(T)} \) for a Dyck tiling \( T \) and a factor \( q^{i_1 + \ldots + i_r - t} \) with \( 1 \leq i_1 < i_2 < \ldots < i_r \leq N_D(\lambda) \).

We divide the shifted shape \( \lambda \) into two domains: the first one is the domain determined by two Dyck paths \( \lambda \) and \( \lambda_H := U_{N-N_D(\lambda)} \ldots U_{N-D_D(\lambda)} D_{N-D_D(\lambda)} \ldots D_{N-D_D(\lambda)} \), and the second one is the domain determined by two ballot paths \( \lambda_H \) and \( \lambda_0 \). The second domain is the staircase of \( N_D(\lambda) \) rows. We put a Dyck tiling
Let $\mu$ be the upper path of the Dyck tiling $T$ and $\text{UP}(\mu)$ be the set of $U$ in $\mu$. We define $\varphi : \text{UP}(\mu) \to \mathbb{N}$ by $\varphi(U)$ as the number of boxes between $U$ and the Dyck path $\lambda_H$ in the $(1,-1)$-direction.

We define an operation on $T$ in the first domain and $i_r + 1$ boxes on the bottom row in the second domain as follows.

Suppose that $\varphi(U) > N_D(\lambda) - i_r - 1$ with $U \in \text{UP}(\mu)$. We translate by $(1,0)$ the boxes (including Dyck tiles) $T'$ between $U \in \text{UP}(\mu)$ and $\lambda$ in the $(1,-1)$-direction. Thus, $T$ is divide into two tilings, i.e., $T = T' \sqcup T''$ where a tiling $T''$ is attached to the path $\lambda$. We denote by $\lambda'$ the upper path of the tiling $T''$.

Let $U_x$ be the up step of $\mu$ such that $\varphi(U_x) \leq N_D(\lambda) - i_r - 1$ and $\varphi(U_{x+1}) > N_D - i_r - 1$. We denote by $D_x$ the down step $D$ in $\mu$ (or equivalently $\lambda'$) such that $D_x$ is $N_D(\lambda) - i_r - \varphi(U_x)$ steps right to $U_x$.

We put ballot tiles along the shape $\lambda'$ from the lowest anchor box $a$ to the down step $D_x$ of $\lambda'$. Let $L$ be the locally lowest point of $\lambda'$ between $a$ and $D_x$. We put a ballot tile from $L$ to the anchor box $a$ along $\lambda'$. We put Dyck tiles from $L$ to $D_x$ along $\lambda'$ such that we have maximal Dyck tiles. If $L$ and $D_x$ are at the same height, we merge the ballot tile from $L$ to $a$ and a Dyck tile from $L$ to $D_x$ into a ballot tile from $D_x$ to $a$. Together with $T'$, this operation gives a ballot tiling above $\lambda'$.

We repeat the above procedure until boxes corresponding to a term $q^1 t$ are transformed into ballot tiles (see the right picture in Figure 4.2). It is easy to see that this operation is invertible and the exponent of a ballot tiling is preserved. Thus, we have a bijection regarding to Eqn.(4.1).

Example 4.3. From the inverse of incidence matrix in Example 3.4, we have

$$P_{U,U,*}^{II} = 1, \quad P_{U,D,*}^{III} = 1 + t, \quad P_{D,U,*}^{III} = (1 + q)(1 + t), \quad P_{D,D,*}^{II} = (1 + t)(1 + qt).$$

Let $\lambda_0$ be a path consisting of only $U$’s. Recall that two paths $\lambda_0$ and $\lambda$ determine the shifted shape $\lambda$. For each $U$ in $\lambda_0$, we define $\text{ht}(U)$ as one plus the number of boxes between $U$ and a path $\lambda$ or an anchor box in the $(1,-1)$ direction. Let $\text{UP}_\circ(\lambda)$ (resp. $\text{UP}_\bullet(\lambda)$) be the set of $U$’s in $\lambda_0$ such that there is an anchor box (resp. no anchor box) in the $(1,-1)$-direction of $U$. 

\[\square\]
We denote by $\mathcal{C}_N(12,12)$ the set of $\{12,12\}$-avoiding signed permutations.

**Theorem 4.4.** Let $\pi_0 = \omega(\mu)$. Then,

\[
\sum_{\pi \leq \pi_0} q^{\text{inv1}(\pi)} t^{\text{inv2}(\pi)} = \prod_{U \in \text{UP}_*(\mu)} \prod_{U \in \text{UP}_*(\mu)} [\text{ht}(U)],
\]

where $\pi \leq \pi_0$ is the Bruhat order on $\mathcal{C}_N$.

**Remark 4.5.** Theorem 4.4 is an analogue of the second and the third terms in Theorem 2.3.

**Proof of Theorem 4.4.** We rotate the shifted shape $\mu$ 45 degrees clockwise. For a term $q^i$, $0 \leq i \leq \text{ht}(U) - 1$, in $[\text{ht}(U)]$ or $[\text{ht}(U)]_t$, we put $i$ boxes from top to bottom in the same column as $U$. Similarly, for a term $q^{\text{ht}(U) - 2t}$ in $[\text{ht}(U)]_t$, we put $\text{ht}(U) - 1$ boxes from top to bottom in the same column as $U$. We denote by $T$ a configuration of boxes in the shifted shape $\mu$. Recall that each box in the shape $\mu$ corresponds to an elementary transposition $s_i$ by $\omega(\mu)$. We assign a word in $\mathcal{C}_N$ to boxes in $\mu$ by reading transpositions in boxes from top to bottom and from right to left, i.e., starting with the rightmost column, reading down transpositions from top to bottom, moving to the column left to the rightmost column, working down from top to bottom and continue the procedure until it ends at the bottom row of the leftmost column. We denote by $\omega_T(\mu)$ the word obtained in this way.

It is obvious that the word $\pi := \omega_T(\mu)$ satisfies $\pi \leq \pi_0$ in the Bruhat order. The existence of two patterns $\underline{12}$ and $\underline{12}$ means that $\pi$ contains the partial sequence $s_{N-1} s_{N} s_{N-1}$. Suppose $\pi$ contains the partial sequence $s_{N-1} s_{N} s_{N-1}$. Then, the existence of a box $b$ corresponding to the right transposition $s_{N-1}$ means that there are boxes left to $b$. Such boxes correspond to a transposition $s_j$ with $j \leq N - 2$. Since we read boxes from top to bottom and from right to left, $\pi$ partially contains a sequence $s_{N-1} w s_N w' s_{N-1}$ where $w$ or $w'$ contains $s_{N-2}$. There is no braid relations such that a reduced word $\pi$ contains $s_{N-1} s_{N} s_{N-1}$. This contradicts the assumption. Similarly, if $\pi$ contains $s_{N-2}$ right to the right $s_{N-1}$, there is $s_{N-3}$ right to the left $s_{N-2}$ in $\omega_T(\mu)$. A reduced word for $\pi$ does not contain a sequence $s_{N-1} s_{N} s_{N-2} s_{N-1} s_{N-2} = s_{N-1} s_{N} s_{N-1} s_{N-2}$. From these observations, $\pi$ is a $\{12,12\}$-avoiding signed permutation. The inversion $\text{inv1}(\pi)$ is equal to the number of boxes except anchor boxes and the inversion $\text{inv2}(\pi)$ is equal to the sum of anchor boxes.

Suppose that a $\{12,12\}$-avoiding signed permutation $\pi$ satisfies $\pi \leq \pi_0$. We show that there exists a unique expression of $\pi$ such that $\pi$ is obtained as $\omega_T(\mu)$ with a certain $T$. Different expressions of a word $\pi \in \mathcal{C}_N$ are obtained by using the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq N - 1$, $s_i s_j = s_j s_i$ for $|i - j| > 1$ and $s_{N-1} s_{N-1} s_{N-1} = s_{N} s_{N-1} s_{N-1}$. Since $\pi$ is a $\{12,12\}$-avoiding signed permutation, $\pi$ contains neither $s_{N-1} s_{N} s_{N-1} s_{N}$, $s_{N} s_{N-1} s_{N} s_{N-1}$ nor $s_{N} s_{N-1} s_{N}$. By a construction of $T$, $\omega_T(\mu)$ do not contain partial sequences $s_i s_j$ with $i < j$ and $s_i s_{i+1} s_i$ with $1 \leq i \leq N - 1$. Thus, given an expression of $\pi$, we have a unique expression of $\pi$ such that $\pi = \omega_T(\mu)$ with some $T$.

We have a correspondence a term in the right hand side of Eqn.\((4.2)\) with $\pi = \omega_T(\mu)$ where $\pi$ is in $\mathcal{C}_N(12,12)$ and satisfies $\pi \leq \pi_0$. This completes the proof. □

## 5. Planted plane tree and Generalized perfect matching

### 5.1. Planted plane tree

We introduce the notion of a tree for a ballot path following [1]. Let $\lambda$ be a ballot path. Recall the definition of $Z$ in Section 2.2. If a path $\lambda = \lambda(U,z)$ with $z \in Z$, we call the underlined $U$ a *terminal* $U$. If a path $\lambda = \lambda(U,z_2 U z_2 \cdots z_3 U z_2 U z_1 U)$ with $r \geq 1$ and
$z_i \in \mathbb{Z}$, we call underlined two $U$’s a $UU$-pair. A $U$ which is not classified above is called an extra $U$.

We define the capacity for a pair $UD$ in the same way as in the case of a Dyck word. If $\lambda = \lambda U$ and $\lambda_0 = \lambda'_0\alpha$ with $\alpha \in \{U, D\}$, the capacity of the underlined $U$ is

$$\text{cap}(U) := ||\lambda_0||_U - ||\lambda'||_U.$$ 

We define a plane tree $A(\lambda)$ for a ballot path $\lambda$. A tree $A(\lambda)$ satisfies $(\diamondsuit 1)$ to $(\diamondsuit 4)$ (see Section 2.2) and the following four conditions $(\diamondsuit 5)$ to $(\diamondsuit 8)$:

$(\diamondsuit 5)$ If the underlined $U$ in $U\lambda'$ is the terminal $U$, the tree $A(U\lambda')$ is obtained by putting an edge just above the tree $A(\lambda')$. We mark the edge with a dot (●).

$(\diamondsuit 6)$ Suppose underlined $U$’s in $UzU\lambda'$ with $z \in \mathbb{Z}$ are a $UU$-pair. Then the tree $A(UzU\lambda')$ is obtained by attaching an edge above the root of $A(z\lambda')$. We mark the edge with a dot (●).

$(\diamondsuit 7)$ If the underlined $U$ in $U\lambda'$ is an extra $U$, we have $A(U\lambda') = A(\lambda')$.

We need to encode an additional information on a tree [1]. Suppose that $\lambda = \lambda'z_2z_1\lambda''$ with $\lambda'' = Uz_2Uz_2' \ldots z_1Uz_0$ and $z_{2r+1} = x_rx_{r-1} \cdots x_1$, $x_i \in \mathbb{Z}$. Here, all $x_i$’s can not be expressed as a concatenation of two non-empty Dyck words. Thus, the tree $A(x_i)$ contains a unique maximal edge (the edge connecting to the root) corresponding to a $UD$ pair. The tree $A(\lambda'')$ also has a unique maximal edge corresponding to a $UU$-pair or a terminal $U$. The maximal edge of $A(x_i)$ (resp. $A(\lambda'')$) is said to immediately precede the maximal edge of $A(x_{i+1})$ (resp. $A(x_1)$) for $1 \leq i \leq s$. Then we put an information on a tree:

$(\diamondsuit 8)$ When an edge $e$ immediately precedes an edge $e'$ in a binary tree $A(\lambda)$, we put a dashed arrow from $e$ to $e'$.

**Remark 5.1.** We can embed a ballot path $\lambda$ into a set of longer ballot paths by adding $U$’s left to $\lambda$. Then, we have a ballot path without $D$’s satisfying $(\diamondsuit 4)$. In other words, all $D$’s are paired with $U$’s. Especially, when a ballot path $\lambda$ consists of only $D$’s, that is, $\lambda = D\ldots D$, we embed $\lambda$ in the set of ballot paths of length $2N$ by adding $N$ $U$’s left to $\lambda$.

**Example 5.2.** When $\lambda = UUDDUUUUUDDDU$, the tree $A(\lambda)$ is depicted as

```
  .
 /|
/ |
/  |
/   |
/    |
```

As in the case of Section 2.2, a labelling of Lascoux–Schützenberger type (type LS for short) is a set of non-negative integers on the edges of $A(\lambda/\lambda_0)$ satisfying (LS1) and (LS2).

**Remark 5.3.** One can compute a Kazhdan–Lusztig polynomial for the Hermitian symmetric pair $(B_N, A_{N-1})$ by a tree $A(\lambda/\mu)$. It requires two more additional information on a tree in addition to (LS1) and (LS2). The extra information comes from the existence of arrows and dotted edges. See [1, 7] for a detailed explanation.

Let $E$ be the number of edges in $A(\lambda)$. A natural labelling of $A(\lambda)$ is a labelling such that integers on edges are increasing from the root to a leaf and an integer in $\{1, \ldots, E\}$ appears exactly once. A
reference natural labelling of $A(\lambda)$ is a natural labelling such that integers on edges are increasing in left-to-right depth-first search. We denote by pre$(L)$ (resp. post$(L)$) the permutation obtained by reading a natural labelling $L$ from left to right using the pre-order (resp. modified post-order). The pre-order (resp. post-order) means that we visit a node before (resp. after) both of its left and right subtrees. The modified post-order means as follows. We visit edges $e$’s with • from top to bottom up to a ramification point soon after all the edges on the subtree which is left to edges $e$’s are visited. Then, we continue visiting remaining edges by following the modified post-order. Let $\pi = \text{post}′(L) := \pi_1 \ldots \pi_E$ be a permutation in the one-line notation. If $\pi_i$ with $1 \leq i \leq E$ is on an edge with a dot, we put an underbar on $\pi_i$. The obtained barred permutation is denote by post$(L)$. Similarly, we obtain pre$(L)$ by adding underbars to pre$′(L)$. We identify post$(L)$ and pre$(L)$ with signed permutations in $S_E$ by regarding an unbarred integer as a negative integer. Note that the pre-order word for the reference natural labelling is the identity.

We define the inversion of a signed permutation $\sigma$ in the signed alphabet $\{\pm 1, \ldots, \pm E\}$ as

$$\text{Inv}(\sigma) := \# \{(i, j) \mid i < j, |\sigma(i)| > |\sigma(j)|\} + 2\# \{(i, j) \mid i < j, 0 < -\sigma(i) < |\sigma(j)|\} + \# \{i \mid \sigma(i) < 0\}.$$

We define the inversion of an inverse pre-order word $\sigma$ as

$$\text{inv}(\sigma) := \# \{(i, j) \mid i < j, \sigma(i) < \sigma(j), \text{ and } i, j \text{ are not underlined}\} + 2\# \{(i, j) \mid i < j, \sigma(i) > \sigma(j), \text{ and } i \text{ or } j \text{ is underlined}\}.$$

Example 5.4. A natural labelling $L$ associated to a path $UUDDUUUUDDUU$ is

$$L = \begin{array}{cccccc}
1 & & & & & 2 \\
& & & & 3 & 4 \\
& & & 5 & & 6 \\
\end{array}$$

Then, we have pre$(L) = 132\underline{4}5\underline{6}$ and post$(L) = 31\underline{2}54\underline{6}$.

5.2. Generalized perfect matching. Given a ballot path $\gamma$, let $\gamma_0$ be the lowest ballot paths such that the skew shape $\gamma_0/\gamma$ exists and $|\gamma|_X = |\gamma_0|_X$ with $X \in \{U, D\}$. The path $\gamma_0$ is a concatenation of a zig-zag path and a path consisting of only $U$’s.

A generalized perfect matching is a diagram consisting of arcs, dashed arcs and a diamond ($\diamond$). We consider two types of generalized perfect matching: type I and type II.

A generalized perfect matching on a path $\gamma$ of type I is defined as follows. An arc connects a $U$ and a $D$ in $\gamma$, a dashed arc connects two $U$’s in $\gamma$ and a diamond is on a $U$ in $\gamma$. Given $\gamma$, we start with making pairs of $U$ and $D$ by connecting them into an arc. Then, we make a pairs of two $U$’s next to each other by connecting them into a dashed arc. We put a diamond $\diamond$ in a $U$ of a remaining $U$ (if it exists). We denote by $\text{PM}_{I}(\gamma)$ the set of generalized perfect matchings of type I for a path $\gamma$. We consider two conditions:

(C1) Two configurations of two dashed arcs

are not admissible.
(C2) There exists no \( U \) right to the diamond (\( \circ \)) such that it forms a dashed arc.

We denote by \( PM'_I(\gamma) \subseteq PM_I(\gamma) \) the set of generalized perfect matchings of \( \gamma \) satisfying (C1) and (C2).

We define a generalized perfect matching of type II as follows. Let \( \lambda \) be a ballot path and \( L \) be a natural labelling of the tree \( A(\lambda) \) and \( \pi = \text{pre}(L) \). Here, we consider only a plane tree without “arrows”. We associate a sequence of (underlined) integers \( p' := (p'_1, \ldots, p'_E) \) to \( L \) which has \( E \) edges. Given an edge \( e \) with an integer \( n_e \), there is a unique sequence of edges from \( e \) to the root. We denote by \( S_e \) the set of such edges. We denote by \( T_e \) the set of edges such that an integer \( n_{e'} \) on the edge \( e' \in T_e \) satisfies \( n_{e'} < n_e \) and \( n_{e'} \) appears to the left to \( n_e \) in \( \pi \). We define \( p' \) as

\[
p'_{n_e} := 2|T_e| - |S_e| + 1,
\]

where \( 1 \leq n_e \leq E \). If an edge \( e \) in \( A(\lambda) \) has a dot, we put an underline on \( p'_{n_e} \).

We define a mini-word \( w \) from \( p' \) as follows. Let \( S_E := [1, 2E] \). For \( 1 \leq i \leq E \), we make a pair between \( (p'_i + 1) \)-th smallest element \( a'_i \) and \( b'_i := \max(S_i) \) in \( S_i \) and define \( S_{i-1} := S_i \setminus \{a'_i, b'_i\} \). When \( p'_i \) is underlined, we put an underline on \( a'_i \). Then, we obtain a sequence of (underlined) integers \( \mathbf{a}' := (a'_1, \ldots, a'_E) \). Let \( e_i \) be an edge with the integer \( i \), and \( m_i, 1 \leq i \leq E \), be

\[
m_i := \#\{a'_j \mid a'_j < a'_i, a'_j \text{ is underlined}, \text{ and } e_j \in S_{e_i}\}.
\]

We define \( \mathbf{a} := (a_1, \ldots, a_E) = (a'_1 + m_1, a'_2 + m_2, \ldots, a'_E + m_E) \) and \( \mathbf{p} := (p_1, \ldots, p_E) = (p'_1 + m_1, p'_2 + m_2, \ldots, p'_E + m_E) \). A sequence of integers \( \mathbf{b} := (b_1, \ldots, b_E) \) is defined as an increasing sequence in \( S_{2E} \setminus \{a_i \mid 1 \leq i \leq E\} \). Then, \( w \) is a \( 2 \times E \) array of integer where the first row is \( \mathbf{a} \) and the second row is \( \mathbf{b} \). By construction, \( b_1, \ldots, b_E \) is an increasing sequence. Further, when \( a_i \) is underlined, we put an underline on \( b_i \). We have a generalized perfect matching from a mini-word \( w \) in the following way. We consider a graph with \( 2E \) nodes which lie in a line from left to right. A pair \((a_i, b_i)\) without underlines corresponds to an arc from the \( a_i \)-th node and the \( b_i \)-th node. If a pair \((a_j, b_j)\) is underlined, then we connect the \( a_j \)-th node and the \( b_j \)-th node by a dashed arc. We call a generalized perfect matching obtained in this way type II. We denote by \( PM_{II}(\lambda) \) the set of generalized perfect matchings of type II corresponding to pre-order words of \( A(\lambda) \).

Example 5.5. We consider the same path and the same natural labelling as in Example 5.4. We have \( \mathbf{p}' = (0, 2, 1, 5, 6, 9) \) and \( \mathbf{a}' = (1, 4, 2, 6, 7, 10) \). Thus we have \( \mathbf{p} = (0, 2, 1, 6, 7, 10) \) and \( w = \frac{1}{3} \frac{4}{5} \frac{2}{6} \frac{7}{9} \frac{8}{10} \frac{11}{12} \). The generalized perfect matching corresponding to \( w \) is depicted as

\[
\begin{array}{c}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} \\
\end{array}
\]

Below, we define two statistics crossing and nesting for a generalized perfect matching in \( PM'_I(\mu) \) and \( PM_I(\mu) \), and nesting for \( PM_{II}(\mu) \). If the length of a generalized perfect matching is \( l \), crossing and nesting are sequences of integers of length \( l \).

We start with the definitions of two statistics crossing and nesting for a generalized perfect matching in \( PM'_I(\mu) \). Suppose that the length of \( \pi \in PM'_I(\mu) \) is \( l \). We assign sequences of integers \( \mathbf{p}_{cr} \) and \( \mathbf{p}_{nes} \) of length \( l \) to \( \pi \).
For crossing, we define a partial integer sequence for a configuration of arcs, dashed arcs and the diamond as follows.

\[
\begin{array}{cccc}
0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

Partial integer sequences for other configurations are zero. Then, a sequence \( p_{\text{cr}} \) is defined by the sum of partial sequences corresponding to diagrams depicted above. We define \( \text{cr}(\pi) := \sum_{i=1}^{l} p_i \) for \( p_{\text{cr}} = p_1 \ldots p_l \).

For nesting, we define a partial integer sequence for a configuration of arcs, dashed arcs and the diamond as follows.

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

Partial integer sequences for other configurations are zero. The sequence \( p_{\text{nes}} \) is defined by adding the partial sequences corresponding to diagrams depicted above. We define \( \text{ne}(\pi) := \sum_{i=1}^{l} p_i \) for \( p_{\text{nes}} = p_1 \ldots p_l \).

**Example 5.6.** Fix a path \( \mu := UUDUUDDUUUD \). Let \( \pi_1 \) be a generalized link pattern in \( \text{PM}_I(\mu) \) depicted as below:

\[
\pi_1 := \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\]

The crossing of \( \pi_1 \) is \( p_{\text{cr}}(\pi_1) := (p_3, p_7, p_8, p_{13}) = (1, 0, 1, 4) \) and other \( p_i \)'s are zero. The nesting of \( \pi_1 \) is \( p_{\text{nes}}(\pi_1) := (p_1, p_2, p_4, p_5, p_6, p_9, p_{10}, p_{11}, p_{12}) = (0, 0, 1, 0, 3, 1, 1, 1, 1) \) and other \( p_i \)'s are zero.

For a perfect matching in \( \text{PM}_I(\mu) \), we define a partial sequence of integers for a configuration of arcs, dashed arcs and the diamond as follows:

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

\( (5.1) \)

For nesting, we define a partial sequence of integers in addition to Eqn.(5.1):

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\]

A partial sequence of integers is zero for other configurations. Similarly for crossing, we define a partial sequence of integers in addition to Eqn.(5.1):

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{array}
\]
A partial sequence of integers is zero for other configurations. Sequences \( p_{\text{nes}} \) and \( p_{\text{cr}} \) are defined as a sum of partial sequences corresponding to diagrams depicted above. The quantities \( \text{nes}(\pi) \) and \( \text{cr}(\pi) \) is defined as the sum of \( p_i \) with \( 1 \leq i \leq l \).

For nesting of a generalized perfect matching in \( \text{PM}_{II}(\lambda) \), we define a partial sequence of integers for a configuration of arcs, dashed arcs (no diamond since we consider a plane tree without arrows) as follows:

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

An integer sequence \( p_{\text{nes}} \) is defined as a sum of partial sequence corresponding to nesting diagrams depicted above. We define \( \text{nes}(\pi) \) as the sum of \( p_i \).

5.3. Ballot tiling strip. We give a bijection called ballot tiling strip (BTS for short) from a natural labelling \( L \) of \( A(\lambda) \) to a ballot tiling. We consider the case where the tree \( A(\lambda) \) has no arrows.

We define two operations \( \text{Dyck spread} \) and \( \text{ballot spread} \) on a ballot path \( \rho \). We first define the Dyck spread following [5, Section 2]. Given a ballot path \( \rho \) and a column \( s \), the Dyck spread of \( \rho \) at \( s \) is the ballot path \( \rho' \) consisting of the following points:

\[
\{(x-1, y) \mid (x, y) \in \rho, x \leq s\} \cup \{(x, y+1) \mid (x, y) \in \rho, x = s\} \cup \{(x+1, y) \mid (x, y) \in \rho, x \geq s\}.
\]

Similarly, the ballot spread of \( \rho \) at \( s \) is the ballot path \( \rho' \) consisting of the following points:

\[
\{(x-1, 0) \mid (x, y) \in \rho, x \leq s\} \cup \{(s, y+1) \mid (s, y) \in \rho\} \cup \{(x+1, y+2) \mid (x, y) \in \rho, x \geq s\}.
\]

We define the spread of a ballot tile at column \( s \) by spreading the ballot path which characterize the ballot tile. Then, the spread of a ballot tiling at column \( s \) by spreading the upper path, the lower path and ballot tiles in-between them.

We define the spread of a ballot tile at column \( s \) by spreading the ballot path which characterize the ballot tile. Then, the spread of a ballot tiling at column \( s \) by spreading the upper path, the lower path and ballot tiles in-between them.

We define a growth process of a ballot tiling \( T \) which we call the strip-grow. Given a ballot tiling \( T \) of length \( N = 2n + n' \) and a column \( s \) with \( 0 \leq s \leq N \), we define the Dyck strip grow \( \text{DSG}(T, s) \) (resp. ballot strip grow \( \text{BSG}(T, s) \)) to be a ballot tiling \( T' \) obtained as follows. Suppose that we obtain a tiling \( T'' \) by (Dyck or ballot) spreading the tiling \( T \) at \( s \). Let \( \mu \) be the upper path of \( T'' \). We put one-box tiles on the up steps of \( \mu \) which is right to the column \( s \). The tiling \( T' \) obtained by the above procedure is denoted by \( \text{DSG}(T, s) \) (resp. \( \text{BSG}(T, s) \)).

Let \( L \) be a natural labelling of a tree \( A(\lambda) \). Recall that one constructs \( \mathbf{p} = (p_1, \ldots, p_E) \) from \( L \). The map BTS from integer sequences \( \mathbf{p} \) to cover-inclusive ballot tilings is given by successive applications of the strip-grow. We define

\[
\text{BTS}(\mathbf{p}) := \begin{cases} 
\emptyset & E = 0, \\
\text{BSG}(\text{BTS}(p_1, \ldots, p_{n-1}), p_n - (n-1)) & n > 0, \ p_E \ \text{is underlined}, \\
\text{DSG}(\text{BTS}(p_1, \ldots, p_{n-1}), p_n - (n-1)) & \text{otherwise}.
\end{cases}
\]

Example 5.7. Let \( L \) be a natural labelling of the following tree:
The growth of ballot tilings are

Given a ballot path \( \lambda \), a sequence of integers \( p \) is bijective to an inverse pre-order word \( \sigma \). Thus we define

\[
\text{BTS}(\lambda, \sigma) := \text{BTS}(p),
\]

where \( p \) is represented by a pair \( (\lambda, \sigma) \).

**Theorem 5.8.** For a ballot path \( \lambda \) and an inverse pre-order word \( \sigma \) associated to a natural labelling \( L \) of the tree \( A(\lambda) \), we have

\[
\text{art}(\text{BTS}(\lambda, \sigma)) = \text{inv}(\sigma).
\]

5.4. **Hermite history for a ballot tiling.** We consider a ballot tiling in the shape \( \lambda/\mu \). We define an Hermite history for a ballot tiling by replacing Dyck with ballot in the definitions in the first three paragraphs of Section 2.3. Though we have three types of Hermite histories, we consider only two types of Hermite histories for a generalized perfect matching in \( \text{PM}_I(\mu) \): type I and type III. For a generalized perfect matching \( \pi \in \text{PM}_I(\lambda) \), we consider only Hermite histories of type I.

The main difference between Hermite history for a Dyck tiling and the one for a ballot tiling is that the statistics tiles \( T \) is zero for a ballot tile of length \((2n, n')\) with \( n' \geq 1 \).

Let \( \pi \) be a generalized perfect matching in \( \text{PM}_I(\mu) \). By definition, a non-zero element in the integer sequence \( \text{pnes}(\pi) \) is on an up step of \( \mu \). Thus a labelling \( H \) can be obtained by deleting zeros, which are on the down steps of \( \mu \), from \( \text{pnes}(\pi) \). The lower path \( \lambda \) can be obtained as follows. Recall that a path \( \mu \) consists of \( U \)'s and \( D \)'s. We put an integer sequence \( \text{pnes} \) on the path \( \mu \). We enumerate the up steps in \( \mu \) from right to left by 1, 2, ... , \( N_U \) where \( N_U \) is the number of up steps in \( \mu \). We perform the following operation starting from the first up step and ending with the \( N_U \)-th up step. Suppose that the \( i \)-th up step has an integer \( n_i \). Let \( \mu_i \) be a partial path in \( \mu \) which is right to the \( i \)-th up step. We perform the operations (A) and (B) (See Section 3.1) on \( \mu_i \). We have paired \( UD \)'s and unpaired \( D \)'s and \( U \)'s. We enumerate the unpaired \( D \)'s and \( U \)'s from left to right by 1, 2, ... and let \( m_j \) be the position of the \( j \)-th \( U \) or \( D \) in \( \mu \). We denote by \( m_0 \) the position of the \( i \)-th up step in \( \mu \). For \( 1 \leq j \leq n_i \), we move the \( j \)-th unpaired \( U \) or \( D \) to the position \( m_{j-1} \) and put an up step at the position \( m_n \). We obtain the lower path \( \lambda \) in this way.

**Example 5.9.** For a ballot path \( \mu = UUUUDDUDUDD \) and \( \text{pnes} = (0, 0, 2, 0, 0) \), the lower path \( \lambda \) is obtained as

\[
\begin{array}{cccccc}
0 & 0 & 2 & 0 & 0 & \rightarrow & 2 \\
U & U & U & D & D & U & D & \rightarrow & U & U & D & U & D & D & U & D
\end{array}
\]

The following theorem is clear from discussions in Section 2.1 and the constructions of \( \text{pnes} \) and the lower path \( \lambda \).

**Theorem 5.10.** The map from \( \text{pnes}(\pi) \) to a labelling \( H \) of type I is a bijection.
Similarly, we can construct a labelling $H''$ of type III from $p_{cr}(\pi)$ by deleting zeros, which are on the up steps of $\mu$, from $p_{cr}(\pi)$. The lower path $\lambda$ can be obtained in a similar way to $p_{res}(\pi)$. We replace $p_{res}$ with $p_{cr}$ an up step with a down step, left with right, right with left in the operation to obtain $\lambda$ from $p_{res}(\pi)$. Then, we have the following theorem.

**Theorem 5.11.** The map from $p_{cr}(\pi)$ to a labelling $H''$ of type III is a bijection.

**Example 5.12.** Let $\pi_1$ be a generalized link pattern as in Example 5.6. We have two types of labellings: $H = (0,0,1,0,3,1,1,1)$ for type I and $H'' = (1,0,1,4)$. The lower paths $\lambda_{res}$ and $\lambda_{cr}$ are given by $\lambda_{res} = UUDDUDUDUUUUU$ and $\lambda_{cr} = UDUUDUDDUUU$. Ballot tilings corresponding to $(\mu,H)$ and $(\mu,H'')$ are depicted as

$$
\begin{pmatrix}
(\mu,H) = & 0 & 1 & 3 & 1 & 1 & 1 \\
0 & 1 & 0 &   &   &   \\
\end{pmatrix}
\quad
\begin{pmatrix}
(\mu,H'') = & 1 & 1 & 4 \\
1 & 0 &   &   \\
\end{pmatrix}
$$

Let $\pi$ be a generalized perfect matching in $PM_I(\mu)$. By definition, a non-zero element in the integer sequences $p_{res}(\pi)$ and $p_{cr}(\pi)$ is on an up step of $\mu$. The labelling $H_{res}$ (resp. $H_{cr}$) can be obtained by deleting zeros, which are on the down steps of $\mu$, from $p_{res}$ (resp. $p_{cr}$). The lower paths $\lambda_{res}$ and $\lambda_{cr}$ can be obtained as follow. Since we have the same algorithm for $p_{res}$ and $p_{cr}$, we abbreviate $p_{res}$ or $p_{cr}$ as $p$. We put the integer sequence $p$ on the path $\mu$. We enumerate the up steps in $\mu$ from right to left by $1,2,\ldots,N_U$ where $N_U$ is the number of up steps in $\mu$. We perform the following operations on $\mu$ starting from the first up step and ending with the $N_U$-th up step. Suppose that the $i$-th up step has an integer $n_i$. Let $\mu_i$ be a partial path in $\mu$ which is right to the $i$-th up step. We perform the operations (A) and (B) on $\mu_i$ as in Section 3.1. We have paired $UD$'s and unpaired $D$'s and $U$'s. For the unpaired $U$'s, we make a pair of two $U$'s from right to left and connect them by a dashed arc. We have at most one $U$ which does not form a pair. We enumerate $U$ in a paired $UU$'s, unpaired $D$'s and $U$ in $\mu_i$ from left to right by $1,2,\ldots$ and let $m_j$ be the position of the $j$-th $U$ or $D$ in $\mu$. Note that we do not enumerate paired $UD$'s. We denote by $m_0$ the position of the $i$-th up step in $\mu$. Let $N(i)$ be an integer such that the sum of the numbers of paired $UU$, unpaired $D$ and unpaired $U$ is $n_i$ in a partial sequence $\nu_i$ in $\mu_i$ which is left to the $(N(i)+1)$-th element. If such $N(i)$ does not exist, then we attach a sequence consisting of only $D$'s to the right of $\mu_i$ and define $N(i)$ as above. For $1 \leq j \leq N(i)$, we move the $j$-th $U$ or $D$ to the position $m_{i-1}$ and put an up step at the position $m_{N(i)}$. Further, if two $U$'s are connected by dashed arc and moved leftward, we flip these two $U$'s into two $D$'s. The obtained path is the lower path $\lambda$.

**Example 5.13.** Let $\mu := UUUDDUDUUUU$ and $p := (0,0,0,0,2,0,0,0,1,1)$. The lower path $\lambda$ is obtained as

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 \\
U & U & U & D & U & U & D & U & U \\
U & U & U & D & U & U & D & U & U \\
U & U & U & D & D & D & D & D & D \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\
U & U & U & D & U & D & U & D & U \\
U & U & U & D & U & D & U & D & U \\
U & U & U & D & D & D & D & D & D \\
\end{array}
\]

\[
\begin{array}{cccccc}
2 \\
U & U & D & U & U & D \\
U & U & D & U & U & D \\
U & U & D & D & D & D \\
\end{array}
\]

\[
\begin{array}{cccccc}
2 \\
U & U & D & U & U & D \\
U & U & D & D & D & D \\
\end{array}
\]
The Hermite history \((\mu, H)\) is given by
\[
(\mu, H) = \begin{pmatrix}
0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Note that the statistics tiles for a ballot tile of length \((2, 1)\) is zero.

Starting with \(\mu\) and \(p := p_{\text{nes}}\) or \(p_{\text{cr}}\), we construct a sequence of lower paths \(\lambda_1, \ldots, \lambda_p\) corresponding to an up step in \(\mu\). One can put a ballot tile (or two ballot tiles of length \((2n, n')\) with \(n' \geq 1\)) in the shape \(\lambda_i/\mu\). Thus, we obtain the pair \((\mu, H)\). Given a pair \((\mu, H)\), \(p\) is obtained by inserting zeros into a label \(H\) at the position of down steps in \(\mu\). Thus, we have the following theorem.

**Theorem 5.14.** The map from \(p_{\text{nes}}\) or \(p_{\text{cr}}\) to a labelling \(H\) of type I is a bijection.

Let \(\pi\) be a generalized perfect matching in \(\text{PM}_{II}(\lambda)\). We construct a pair \((\mu, H)\) of type I from \(\pi\). Recall that the generalized perfect matching \(\pi\) consists of arcs and dashed arcs. If the \(i\)-th node and the \(j\)-th node are connected by an arc, we define \(\mu_i := U\) and \(\mu_j := D\). If the \(i\)-th and the \(j\)-th nodes are connected by a dashed arc, we define \(\mu_i := U\) and \(\mu_j := U\). Then, \(\mu = \mu_1 \ldots \mu_N\) is a ballot path where \(N\) is the number of nodes in \(\pi\). The labelling \(H\) can be obtained by deleting zeros, which are on the down steps of \(\mu\), from \(p_{\text{nes}}\).

**Proposition 5.15.** We have
\[
|H| = \text{nes}(\pi) = \text{tiles}(\text{BTS}(\lambda, \sigma)).
\]

6. **Ballot tiling of type BI with a fixed upper path**

6.1. **Dyck tiling with a fixed upper path.** Let \(\gamma\) be a ballot path of length \((2n, n')\) and \(\gamma_0 := \underbrace{UD\ldots UD}_{2n}U\underbrace{UD\ldots UD}_{n'}\). Since a Dyck path can be obtained from \(\gamma\) by adding \(n'\) \(U\)'s from right, we also denote by \(\gamma\) the obtained Dyck path. Then, the shape \(\gamma'/\gamma\) with \(\gamma_0 \leq \gamma' \leq \gamma\) can be obtained as a region determined by two Dyck paths \(\gamma\) and \(\gamma_0\). We consider the generating function
\[
\tilde{P}_\gamma := \sum_{\gamma_0 \leq \gamma' \leq \gamma} \sum_{D \in \text{D}(\gamma'/\gamma)} q^{\text{tiles}(D)}.
\]

Let \(D(\gamma_0)\) be the set of \(D\) steps in \(\gamma_0\). Given a down step \(d\) in \(\gamma_0\), we denote by \(\text{ht}(d; \gamma)\) one plus the number of boxes in \(\gamma_0/\gamma\) which lie in the \((1, 1)\)-direction from \(d\).

**Theorem 6.1.** We have
\[
\tilde{P}_\gamma = \prod_{d \in D(\gamma_0)} [\text{ht}(d; \mu)]
\]

We omit the proof since one can apply the same argument as the proof of the first equality in Theorem 2.3.

Recall that \(\text{PM}^I(\gamma)\) is the set of generalized perfect matching of type I and satisfying two conditions (C1) and (C2). Then,
Theorem 6.2. We have
\[ \tilde{P}_\gamma = \sum_{\pi \in \text{PM}_I(\gamma)} q^{\text{nes}(\pi)} \]
(6.1)
\[ = \sum_{\pi \in \text{PM}_I(\gamma)} q^{\text{cr}(\pi)} \]

Proof. The generating function \( \tilde{P}_\gamma \) is equal to the sum of \( q^{|H|} \) over all possible Hermite histories \( H \)'s. We have \( |H| = \text{nes}(\pi) \) for a generalized perfect matching \( \pi \) corresponding to the pair \((\gamma, H)\). From Theorem 5.10, we have \( \tilde{P}_\gamma = \sum_{\pi \in \text{PM}_I(\gamma)} q^{\text{nes}(\pi)} \).

The second equality in Eqn. (6.1) follows from the similar argument with Theorem 5.11. \( \square \)

Example 6.3. Let \( \gamma = UUUDDU \). From Theorem 6.1, the generating function \( \tilde{P}_\gamma = [3]^2 \). We have nine generalized perfect matchings in \( \text{PM}_I(\gamma) \):

\[
\begin{array}{c|ccccccc}
\pi & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
\text{nes}(\pi) & 3 & 4 & 2 & 3 & 1 & 11 & 11 \\
\text{cr}(\pi) & 3 & 2 & 4 & 1 & 3 & 3 & 3 \\
\end{array}
\]

From Theorem 6.2, we have \( \tilde{P}_\gamma = \sum_\pi q^{\text{nes}(\pi)} = \sum_\pi q^{\text{cr}(\pi)} \).

6.2. Ballot tiling of type BI with a fixed upper path. Recall that the lowest ballot path of length \( N = 2n + n' \) is the zig-zag path \( \mu_0, \) i.e., \( \mu_0 = \overset{N}{\smile} \). Below, we restrict \( N \) to be odd and consider the generating function \( P_{\ast, \mu}^I \).

Recall that the two paths \( \mu \) and \( \mu_0 \) determine the skew shape \( \mu_0/\mu \). Let \( D(\mu_0) \) be the set of \( D \) steps in \( \mu_0 \). Given a down step \( d \) in \( \mu_0 \), we denote by \( \text{ht}(d; \mu) \) one plus the number of boxes in \( \mu_0/\mu \) which lie in the \((1,1)\)-direction from \( d \).

A permutation \( w \) in \( \mathfrak{S}_{N+1} \) is called a down-up (or alternating) permutation if the one-line notation \( w \) satisfies \( w_{2m-1} > w_{2m} < w_{2m+1} \) for \( m \geq 1 \). Let \( \mathfrak{S}_{N+1}^{\text{DU}} \) be the set of down-up permutations on \( \{1, 2, \ldots, N + 1\} \) for which the entries in the even positions are increasing, i.e., \( w_{2m} < w_{2m+2} \) for \( m \geq 1 \).

We enumerate \( D \) steps in \( \mu_0 \) from left to right by \( 1, 2, \ldots, (N - 1)/2 \). Let \( S_0 := \{1, 2, \ldots, N + 1\} \) and \( \mu \) be a path above \( \mu_0 \). We define a permutation \( \sigma = \sigma_1 \ldots \sigma_{N+1} := \sigma(\mu) \) in \( \mathfrak{S}_{N+1}^{\text{DU}} \) as follows. For \( 1 \leq i \leq (N - 1)/2 \), we define \( \sigma_{2i-1} \) as the \((\text{ht}(d_{2i-1}; \mu) + 1)\)-th smallest element in \( S_{2i-2} \) and \( S_{2i-1} := S_{2i-2} \setminus \{\sigma_{2i-1}\} \). Then, \( \sigma_{2i} \) is the smallest element in \( S_{2i-1} \) and we define \( S_{2i} := S_{2i-1} \setminus \{\sigma_{2i}\} \). We define \( \sigma_N = \max(S_{N-1}) \) and \( \sigma_{N+1} = \min(S_{N-1}) \). By construction, the obtained permutation is in \( \mathfrak{S}_{N+1}^{\text{DU}} \).

Example 6.4. We have \( \text{ht}(d_1; \mu), \text{ht}(d_2; \mu) \) = \((3, 3)\) for \( \mu = UUUDDU \). Thus we have \( \sigma(\mu) = 416253 \).
Theorem 6.5. We have

\[ P_{\pi,\mu}^d = \prod_{d \in D(\mu_0)} [\text{ht}(d; \mu)] \]

(6.2)

\[ = \sum_{\pi \leq \sigma(\mu)} q^{l(\sigma)}\]

where \( \leq \) is the Bruhat order on \( S_{N+1} \) and \( l \) is the inversion on \( S_{N+1} \).

Proof. Fix a ballot path \( \mu \) of odd length and a down step \( d \) of \( \mu \) or a northeast edge of an anchor box in \( \mu_0/\mu \). The number of boxes in \( \mu_0/\mu \) in the southwest direction from \( d \) is \( \text{ht}(d, \mu) \). The \( q \)-integer \( [\text{ht}(d, \mu)] \) is written as a expansion \( 1 + q + \ldots + q^{\text{ht}(d, \mu)-1} \). Thus, when we take a term \( q^i \) from \([\text{ht}(d, \mu)]\), we place \( i \) successive boxes in the region between \( d \) and the zig-zag path such that the northeast edge of the northeast box is attached to the down step \( d \). As in Section 2.1, we place boxes for each down step of \( \mu \) for a product of terms of \( \prod_d [\text{ht}(d; \mu)] \). We denote by \( B' \) the obtained diagram consisting boxes.

We construct a ballot tiling from \( B' \) as follows. We enumerate the down steps of \( \mu \) from left to right by \( d_1, \ldots, d_n \) where \( n \) is the number of down steps in \( \mu \). We also enumerate the boxes attached to the down step \( d_i \) from top to bottom by \( 1, 2, \ldots, r(i) \) where \( r(i) \) is the number of boxes attached to \( d_i \). Given a pair \((d_i, j)\) with \( 1 \leq j \leq r(i) \), we call the corresponding box \((d_i, j)\) box. If there exists a box to the north of \((d_i, j)\) box and it forms a ballot tile, we move to the next pair \((d_i, j+1)\) for \( 1 \leq j \leq r(i) - 1 \) or \((d_{i+1}, 1)\) for \( j = r(i) \). Otherwise, there is no box to the north of \((d_i, j)\) box. Let \( b \) (resp. \( b' \)) be a rightmost box in \( \mu_0/\mu \) left to the \((d_i, j)\) box such that the translation of \( b \) by \((2,0)\) (resp. \((-1,1)\)) is either outside of the region \( \mu_0/\mu \) or contained in a ballot tile. In some cases, \( b \) coincides with \( b' \) but \( b \) and \( b' \) are different boxes in general. By definition, the box \( b \) is right to \( b' \). If \( b \) is the \((d_i, j)\) box, we have a unique ballot tile \( B \) such that it satisfies the cover-inclusive property, it start from \( b \) and ends at the tile \( b' \). We move the boxes \((d_i, k)\) with \( i+1 \leq k \leq r(i) \) to the southwest of the box \( b' \). Then, we move the next pair \((d_i, j+1)\). If \( b \) is not the \((d_i, j)\) box, we have two ballot tiles \( B \) and \( B' \) of length \((2n,n')\) with \( n' \geq 1 \) such that \( B \) is just above \( B' \) and southwest of the leftmost box in \( B \) is \( b \). We also have a unique ballot tile (Dyck tile more precisely) starting from \( b \) and ending at \( b' \) as above. We move the boxes \((d_i, k)\) with \( i+1 \leq k \leq r(i) \) to the southwest of the box \( b' \). Then we move the next pair \((d_i, j+1)\). We obtain a ballot tiling by visiting all boxes \((d_i, j)\) and performing the above-mentioned operation on them. It is obvious that the above operation is invertible, i.e., one can construct a diagram \( B' \) from a ballot tiling. See Fig. 6.6 for an example of this bijection. Thus we obtain a bijective proof of the first equality in Eqn.(6.2).

![Figure 6.6](image-url)
To prove the second equality, we show that there exists a bijection between $B'$ and $\pi \leq \sigma(\mu)$ with $\pi \in S_{N+1}$. It is easy to show that the cardinality of $S_{N+1}$ is given by $|S_{N+1}| = N!!$.

We construct $B''$ from $B'$ by translating the $i$ successive boxes attached to a down step in $\mu$ by $(-1, -1)$ direction such that the southwest edge of the southwest box is attached to the path $\mu_0$. We enumerate down steps of $\mu_0$ from left to right by $1, 2, \ldots, (N - 1)/2$ and denote by $e_j$ the $j$-th down step. Suppose that the down step $e_j$ has $r(j)$ boxes in the $(1, 1)$ direction from $e_j$. We define a product of simple transpositions as $\rho_j := s_{2j+r(j)-1} \cdots s_{2j+1}s_{2j}$. We consider the following ordered product of simple transpositions in $S_{N+1}$:

$$\rho(B') := \rho_1\rho_2 \cdots \rho_{(N-1)/2}.$$ 

The smallest element with respect to the Bruhat order in $S_{N+1}$ is $w^0 := w_1 \cdots w_{N+1}$ with $w_{2j-1} = 2j$ and $w_{2j} = 2j - 1$ in one-line notation. The $\rho_j$ transpose $2j$ with $2j + r(j)$ and $k$ with $k - 1$ for $2j < k \leq 2j + r(j)$. One can easily show that $\rho_{(N-1)/2}w^0$ is in $S_{N+1}$. More in general, we have $\rho(B')w^0 \in S_{N+1}$. The cardinality of possible $B''$'s is given by $\prod_{d \in D(\mu_0)} \text{ht}(d; \mu)$, which implies that $|B'| = N!!$ for $\mu = UU \cdots U$. Thus we have a bijection between $B'$ and a permutation $\pi \in S_{N+1}$. The second equality in Eqn. (6.2) follows from the direct consequence of this bijection.

The generating function $P_{*, \mu}^I$ is related to generalized perfect matchings in $\text{PM}_I(\mu)$ as follows.

**Theorem 6.7.** We have

$$P_{*, \mu}^I = \sum_{\pi \in \text{PM}_I(\mu)} q^{\text{nes}(\pi)} = \sum_{\pi \in \text{PM}_I(\mu)} q^{\text{cr}(\pi)}.$$

**Proof.** From Theorem 6.5, the number of ballot tilings whose fixed upper path is $\mu$ is given by $\prod_{d \in D(\mu_0)} \text{ht}(d; \mu)$. By a direct computation, it is easy to show that the cardinality of the set $\text{PM}_I(\mu)$ is also given by $\prod_{d \in D(\mu_0)} \text{ht}(d; \mu)$. Thus we have a bijection between a ballot tiling $B$ with a fixed upper path $\mu$ and a generalized perfect matching $\pi$ in $\text{PM}_I(\mu)$. Let $(\mu, H)$ be an Hermite history for a ballot tiling with a fixed upper path $\mu$. From tiles$(B) = |H|$, Theorem 5.14 and Proposition 5.15, we have tiles$(B) = |H| = \text{nes}(\pi) = \text{cr}(\pi)$. This completes the proof. \qed

7. Ballot tiling of type BI with a fixed lower path

7.1. Generating function $P(M, N)$. Let $\lambda_{M,N}$ be a path $\lambda_{M,N} := \underbrace{D_N, D_M U \ldots U}_M$. We abbreviate the generating function $P_{\lambda_{M,N}, *}$ of ballot tilings of type BI as $P(M, N)$.

**Lemma 7.1.** The generating function $P(M, N)$ satisfies the following recurrence relation:

$$P(M, N) = \begin{cases} P(M, N - 1) + q^{N}P(M - 1, N), & M: \text{even}, \\ P(M, N - 1) + q^{N}P(M - 1, N) + q^{M+N}P(M + 1, N - 2), & M: \text{odd}. \end{cases}$$

**Proof.** We first consider the case where $M$ is even. Let $Y$ be the shifted shape determined by $\lambda_{M,N}$ as in Section 3. Let $b$ be the leftmost box of $Y$. In other words, the southwest edge of $b$ is attached to the first down step in $\lambda_{M,N}$. Let $B$ be a cover-inclusive ballot tiling of type I in the shape $Y$. The generating function $P(M, N)$ is the sum of two contributions: the first one is that the box $b$ does not form a ballot tile in $B$, and the second one is that the box $b$ forms a ballot tile in $B$. In the first case, we have $P(M, N - 1)$. In the second case, since $\lambda_{M,N}$ does not have a partial sequence
There is no ballot tile of length \((2n, n')\) with \(n \geq 1\) and \(n' \geq 0\) in \(Y\). The box \(b\) forms a ballot tile of length \((0, 0)\) and so do the \(N - 1\) boxes which lie in the southeast direction of \(b\). The second contribution is \(q^NP(M - 1, N)\), which implies Eqn.(7.1) for \(M\) even.

Below, we consider the case where \(M\) is odd. We have three contributions for \(P(M, N)\) and two of them are the same as in case of \(M\) even. The third contribution is that the box \(b\) forms a ballot tile of length \((0, 0)\) and there are two ballot tiles of length \((0, M)\) in \(Y\). Note that the unique lowest box in \(Y\) forms the ballot tile of length \((0, M)\). The contribution of the third configuration is \(q^{M+N}P(M + 1, N - 2)\), which implies Eqn.(7.1).

For positive integers \(m\) and \(N\), we define

\[
a_{2m-1,N} := \frac{[N + 2m]}{[2m]},
\]
\[
a_{2m,N} = \frac{[2N + 2m]}{[N + 2m]}.
\]

The recurrence relations (Lemma 7.1) can be solved by using \(a_{M,N}\).

**Proposition 7.2.** The generating function \(P_{M,N}\) satisfies

\[
P(M, N) = \prod_{1 \leq j \leq N} (1 + q^j) \cdot \prod_{1 \leq j \leq M} a_{j,N}.
\]

**Proof.** We prove Proposition by induction. When \((M, N) = (1, 1)\), we have \(a_{1,1} = [3]/[2]\). Since the generating function is calculated as \(P(1, 1) = [2][3]/[2] = [3]\), Proposition holds true when \((M, N) = (1, 1)\).

Assume that Proposition holds true for \(P(m, n)\) such that \(m + n \leq M + N - 1\). We first consider the case where \(M = 2m\). From the definition of \(a_{M,N}\), we have

\[
\prod_{1 \leq j \leq 2m} a_{j,N-1} = \prod_{1 \leq j \leq m} a_{(2j,N-1)}a_{(2j-1,N-1)}
\]

\[
= \prod_{1 \leq j \leq m} \frac{[2N + 2j - 2]}{[2j]},
\]

and

\[
\prod_{j=1}^{2m-1} a_{j,N} = a_{(2m-1,N)} \prod_{1 \leq j \leq m-1} a_{(2j,N)}a_{(2j-1,N)}
\]

\[
= \frac{[N + 2m]}{[2N + 2m]} \prod_{1 \leq j \leq m} \frac{[2N + 2j]}{[2j]}.
\]
The right hand side of Eqn. (7.1) is written as

\[
P(M, N - 1) + q^N P(M - 1, N) = \prod_{i=1}^{N-1} (1 + q^i) \left\{ \prod_{1 \leq j \leq M} a_{(j,N-1)} + q^N (1 + q^N) \prod_{1 \leq j \leq M-1} a_{(j,N)} \right\}
\]

\[
= \prod_{i=1}^{N-1} (1 + q^i) \prod_{j=1}^{m} \left[ \frac{2N + 2j}{[2j]} \right] + q^N \frac{[2N + [N + 2m]]}{2N + 2m}
\]

\[
= \prod_{i=1}^{N} (1 + q^i) \prod_{j=1}^{m} \left[ \frac{2N + 2j}{[2j]} \right]
\]

\[
= \prod_{i=1}^{N} (1 + q^i) \prod_{1 \leq j \leq M} a_{(j,N)},
\]

where we apply Eqn. (7.3) and Eqn. (7.4) to the first line. This implies Eqn. (7.1) holds for \( P_{2m,N} \).

When \( M = 2m - 1 \), we have

\[
P(2m - 1, N - 1) + q^N P(2m - 2, N) + q^{N+2m-1} P(2m, N - 2)
\]

\[
= \prod_{i=1}^{N-2} (1 + q^i) \left\{ (1 + q^{N-1}) \frac{[N + 2m - 1]}{[2N + 2m - 2]} \prod_{j=1}^{m} \left[ \frac{2N + 2j - 2}{[2j]} \right] + q^N \prod_{j=1}^{m} \left[ \frac{2N + 2j - 4}{[2j]} \right] \right\}
\]

\[
= \prod_{i=1}^{N} (1 + q^i) \prod_{j=1}^{m} \left[ \frac{2N + 2j - 2}{[2j]} \right] + q^N \frac{[N + 2m]}{2N}
\]

\[
= \prod_{i=1}^{N} (1 + q^i) \prod_{1 \leq j \leq M} a_{(j,N)}.
\]

This completes the proof. \( \square \)

**Example 7.3.** The first few values of \( \prod_{1 \leq j \leq M} a_{(j,N)} \) are as follows:

| \( M \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) |
|---|---|---|---|---|---|
| 1 | 3 | 4 | 5 | 6 | 7 |
| 2 | 6 | 8 | 10 | 12 |
| 3 | 2 | 4 | 2 | 4 |
| 4 | 6 | 8 | 10 | 12 |
| 5 | 2 | 4 | 2 | 4 |

**7.2. Factorization of \( P_{\lambda}^I \).** A valley of a ballot path \( \lambda \) is a middle point of a partial path \( DU \) in \( \lambda \). We call the lowest and leftmost valley the minimum point of a path \( \lambda \). A path \( \lambda \) can be written
as a concatenation of two paths \( \lambda_1 \) and \( \lambda_2 \) at the minimum point, \( i.e., \lambda = \lambda_1 \circ \lambda_2 \). Let \( M_2 \) (resp. \( N_2 \)) be the number of \( U \) (resp. \( D \)) in \( \lambda_2 \) and \( N_1 \) be the number of \( D \)'s in \( \lambda_1 \).

**Theorem 7.4.** Let \( \lambda, N_1, M_2 \) and \( N_2 \) be as above. We have

\[
P_{\lambda, s}^I = P_{\lambda_1}^{\text{Dyck}} \cdot P(M_2 + N_2, N_1) \cdot P_{\lambda_2}^I.
\]

Before proceeding with the proof of Theorem 7.4, we introduce some notations and lemmas used later.

We recursively define the set of valleys \( \text{Val}(\lambda) \) from left to right as follows. The leftmost valley is in \( \text{Val}(\lambda) \). Take a valley \( v \). If the rightmost valley in \( \text{Val}(\lambda) \) which is left to \( v \) is lower than \( v \), then \( v \notin \text{Val}(\lambda) \). Otherwise, \( v \in \text{Val}(\lambda) \). The heights of valleys in \( \text{Val}(\lambda) \) are weakly decreasing. The minimum point is the rightmost valley in \( \text{Val}(\lambda) \).

We construct a path \( \overrightarrow{\lambda} \) from \( \lambda \) as follows. The path \( \overrightarrow{\lambda} \) is the same as \( \lambda \) from the starting point to the leftmost valley. From the leftmost valley to the rightmost valley in \( \text{Val}(\lambda) \), the path \( \overrightarrow{\lambda} \) is the highest path which passes through all the valleys in \( \text{Val}(\lambda) \). The path \( \overrightarrow{\lambda} \) is again the same as \( \lambda \) from the rightmost valley in \( \text{Val}(\lambda) \) to the ending point.

Let \( r + 1 \) be the cardinality of the set \( \text{Val}(\lambda) \). We denote by \( \lambda_0 \) the partial path of \( \lambda \) from the starting point to the leftmost valley. The path \( \lambda \) is written as a concatenation of paths \( \lambda_i' \), \( 0 \leq i \leq r \), and \( \lambda_2 \) such that \( \lambda_i' \), \( 1 \leq i \leq r \), is a partial path of \( \lambda \) starting from the \( i \)-th valley in \( \text{Val}(\lambda) \) and ending at the \( (i+1) \)-th valley in \( \text{Val}(\lambda) \), \( i.e., \lambda = \lambda_0' \circ \lambda_1' \circ \ldots \circ \lambda_r' \circ \lambda_2 \). We denote by \( \overrightarrow{\lambda_i} \) the path such that \( \overrightarrow{\lambda} = \overrightarrow{\lambda_0} \circ \lambda_2 \).

Let \( n_0 \) be the number of \( D \) in \( \lambda \) from the starting point of the path to the leftmost valley in \( \text{Val}(\lambda) \). We denote by \( n_i \) (resp. \( m_i \)) the number of \( D \) (resp. \( U \)) in the path \( \lambda_i' \), \( 1 \leq i \leq r \). From the definition of \( \text{Val}(\lambda) \), we have \( n_i \geq m_i \) for \( 1 \leq i \leq r \). Let \( s \) be an integer such that \( n_{s-1} > m_{s-1} \) and \( n_i = m_i \) for \( i \geq s \). Thus, the path \( \overrightarrow{\lambda} \) consists of \( n_0 \) \( D \)'s, followed by \( m_1 \) \( U \)'s, followed by \( n_1 \) \( D \)'s, followed by \( m_2 \) \( U \)'s, \ldots, followed by \( n_r \) \( D \)'s and ending with the path \( \lambda_2 \).

Let \( \mu \) be a partial path of \( \lambda \) from the starting point to \( m_{s-1} \) \( U \)'s. We define

\[
\Box(n_0, m_1, n_1, \ldots, m_{s-1}) := P_{\mu}^{\text{Dyck}},
\]

and \( P_{\mu}^{\text{Dyck}} := P_{\mu}^{\text{Dyck}} \) where the Young diagram \( T \) is determined by the path \( \mu \). We construct sequences of integers \( \mathbf{n}_i, 0 \leq i \leq s-1 \) from \( \mathbf{n} := (n_0, m_1, n_1, \ldots, m_{s-1}) \) as follows. We define \( \mathbf{n}_0 := (n_0 - 1, m_1, n_1, \ldots, m_{s-1}), \) \( \mathbf{n}_1 := (n_0, m_1 - 1, n_1, m_2, \ldots, m_{s-1}), \) and \( \mathbf{n}_i := (n_0 - 1, m_1, n_1, \ldots, n_{i-1} + 1, m_i - 1, \ldots, m_{s-1}) \) for \( 2 \leq i \leq s - 1 \).

**Lemma 7.5.** We have

\[
\Box(\mathbf{n}) = \Box(\mathbf{n}_0) + \sum_{i=0}^{s-2} q^{\text{deg}_i(\mathbf{n})} \Box(\mathbf{n}_{i+1})
\]

where \( \text{deg}_i(\mathbf{n}) := \sum_{0 \leq j \leq i} n_j \).

**Proof.** Recall that a Dyck path \( \mu \) determines the Young diagram \( Y \). We denote by \( b \) the leftmost box in \( Y \). There are two cases: the box \( b \) is empty or forms a Dyck tile. In the first case, we are not able to put a Dyck tile on boxes on the leftmost column (rotate \( Y \) in 45 degrees) of \( Y \). The contribution of this case to \( \Box(\mathbf{n}) \) is given by \( \Box(\mathbf{n}_0) \). In the second case, we have \( (s-1) \) subcases. The first subcase is the case where the boxes \( B_1 \) at the bottom row form a Dyck tile of length zero. This gives a factor \( q^0 \). We consider a Dyck tiling in the remaining diagram \( Y \setminus B_1 \), which gives \( \Box(\mathbf{n}_1) \). Thus, the contribution to \( \Box(\mathbf{n}_0) \) is \( q^0 \cdot \Box(\mathbf{n}_1) \). In the \( i \)-th \( (2 \leq i \leq s-1) \) subcase, we consider the following Dyck tiling. In the bottom row, we have \( n_0 - 1 \) Dyck tiles of length
zero from left to right. We have a Dyck tile of length \( m_j \), \((1 \leq j \leq i - 1)\), which start with the rightmost box in the \((1 + \sum_{1 \leq k \leq j} m_k)\)-th row from the bottom. If two Dyck tiles of length \( m_j \) and \( m_{j+1} \) shares the rightmost box in the \((1 + \sum_{1 \leq k \leq j} m_k)\)-th row from the bottom, we merge them into a larger Dyck tile of length \( m_j + m_{j+1} \). If \( n_j > m_j \) and \( j < i - 1 \), we put \( n_j - m_j - 1 \) Dyck tiles of length zero in the \((1 + \sum_{1 \leq k \leq j} m_k)\)-th row in-between Dyck tiles of length \( m_j \) and \( m_{j+1} \). If \( n_{i-1} > m_{i-1} \), we put \( n_{i-1} - m_{i-1} \) Dyck tiles of length zero right to the Dyck tile of length \( m_{i-1} \). This configuration of Dyck tiles gives the factor \( q^{\deg_i(n)} \). In the remaining region in \( Y \), we consider a Dyck tiling, which gives \( \square(n_i) \). By summing all the contribution, we obtain the right hand side of Lemma 7.5.

\[ \square \]

Let \( T_i \) be the Young diagram determined by \( \overline{\lambda} \) and \( \lambda'_i \). We denote by \( T := (T_1, \ldots, T_r) \) a sequence of Young diagrams. Then, the path \( \lambda \) is characterized a pair \((\overline{\lambda}; T)\). More generally, let \( \text{Val}'(\mu) \) be a subset of the set of valleys and \( r - 1 \) be the cardinality of \( \text{Val}'(\mu) \). We denote by \( \mu'_i, 1 \leq i \leq r \), be the highest path from the \((i - 1)\)-th valley to the \(i\)-th valley in \( \text{Val}'(\lambda) \) where zeroth valley is the starting point of \( \mu \) and \((r + 1)\)-th valley is the ending point of \( \mu \). We denote by \( \overline{\nu} \) the concatenation of paths \( \mu'_1 \circ \cdots \circ \mu'_r \). Let \( \nu_1 \) be a path from the \((i - 1)\)-th valley to the \(i\)-th valley such that the first step of \( \nu_1 \) is \( U \). We denote by \( T'_i \) the Young diagram determined by \( \mu'_i \) and \( \nu_i \). The path \( \mu \) is characterized by the pair \((\overline{\nu}, T')\). Note that there are several choices for \((\overline{\nu}, T')\). The set \( \text{Val}(\lambda) \) is a special choice of \( \text{Val}'(\lambda) \).

We define

\[ Q(\overline{\lambda}; T) := P_{\lambda,*}^f, \]

where \( \lambda \) is characterized by \((\overline{\lambda}; T)\).

We construct paths \( \nu_i, 0 \leq i \leq r + 2 \), from \( \overline{\lambda} \) as follows. The path \( \nu_0 \) is the path obtained from \( \overline{\lambda} \) by replacing \( n_0 \) with \( n_0 - 1 \). The path \( \nu_1 \) is the path obtained from \( \overline{\lambda} \) by replacing \( m_1 \) with \( m_1 - 1 \). The path \( \nu_i, 2 \leq i \leq r \), is the path obtained from \( \overline{\lambda} \) by replacing \( n_0 \) with \( n_0 - 1 \), \( n_{i-1} \) with \( n_{i-1} + 1 \) and \( m_i \) with \( m_i - 1 \). Let \( \nu' \) be a path obtained from \( \overline{\lambda}_1 \) by replacing \( n_0 \) with \( n_0 - 1 \), \( n_r \) with \( n_r + 1 \) and \( \nu'' \) be the path obtained from \( \lambda_r \) by deleting the leftmost \( U \) in \( \lambda_2 \). Then, we define the path \( \nu_{r+1} := \nu' \circ \nu'' \). Let \( \rho \) be a path obtained from \( \overline{\lambda}_1 \) by replacing \( n_0 \) with \( n_0 - 1 \) and \( n_{s-1} \) with \( n_{s-1} - 1 \), and \( \rho' \) be a path \( U \lambda_2 \). Then, we define \( \nu_{r+2} := \rho \circ \rho' \).

**Lemma 7.6.** Let \( \lambda, \nu \) and \( T \) as above and \( M_2 + N_2 \) be even. We have

\[ P_{\lambda,*}^f = Q(\nu_0; T) + \sum_{i=0}^r q^{\deg_i(\lambda)} Q(\nu_{i+1}; T), \]

where \( \deg_i(\lambda) := \sum_{0 \leq j \leq i} n_j \) for \( 0 \leq i \leq r \).

**Proof.** By a similar argument to Lemma 7.5, it is easy to show that Lemma holds true when all \( T_i = \emptyset \) for \( 1 \leq i \leq r \) (compare paths \( v_i \) with \( n_i \)).

Below, we consider the case where at least one \( T_k \neq \emptyset \). The skew shape \( \overline{\lambda}/\nu_1 \) contains only Dyck tiles of length zero and Dyck tiles of length \( 2m_j \) with \( 1 \leq j \leq i - 1 \). Let \( \nu' \) be a path characterized by the pair \((\nu_1, T)\). If \( T_k \neq \emptyset \) with \( 1 \leq k \leq i - 1 \), we deform the Dyck tile of length \( 2m_k \) into the Dyck tile \( D \) of length \( 2m_k \) such that the shape of \( D \) fits \( \nu' \). The weight of Dyck tiles is \( q^{\deg_i(\lambda)} \).

If a ballot tiling contains a Dyck tile of length \( 2n \) with \( n < m_k \) over \( \lambda \), such a configuration is included in the calculation of \( q^{\deg_i(\lambda)} Q(\nu_{i+1}; T) \) with some \( k \leq i \). In the remaining region above \( \nu' \), we can put a ballot tiling without any constraints. Especially, since \( M_2 + N_2 \) is even, the box at the
minimum point are not occupied by a ballot tile of length \((2n, n')\) with \(n' \geq 1\). These imply that we have the generating function \(Q(n, T)\) in the shape \(\nu'\).

The sum of all contributions (the right hand side of Lemma 7.6) is equal to the left hand side of Lemma 7.6.

\[ \Box \]

**Lemma 7.7.** Let \(\lambda, \nu\) and \(T\) as above and \(M_2 + N_2\) be odd. We have

\[
P_{\lambda, s}^{T} = Q(\nu_0; T) + \sum_{i=0}^{r} q^{\deg_i(\lambda)} Q(\nu_{i+1}; T) + q^{\deg_{r+1}(\lambda)} Q(\nu_{r+2}; T),
\]

where \(\deg_i(\lambda) := \sum_{0 \leq j \leq i} n_j\) for \(0 \leq i \leq r\) and \(\deg_{r+1}(\lambda) := \sum_{i=0}^{r} n_i + M_2 + \sum_{i=s}^{r} n_i\).

**Proof.** The proof of Lemma is similar to the one of Lemma 7.7. We have the first two terms in the right hand side of Lemma 7.7 as a contribution to \(P_{\lambda, s}^{T}\).

The difference is that the box at the minimum point can be occupied by a ballot tile \(B_1\) of length \((2n, n')\) with \(n' \geq 1\). From the definition of ballot tilings of type BI, we have another ballot tile \(B_2\) over \(B_1\). Note that the shapes of \(B_1\) and \(B_2\) are the same. Let \(S\) be a valley such that \(S\) is the leftmost valley which is the same height as the minimum height. Then, we consider the configuration such that the leftmost box of \(B_1\) is at the point \(S\). The length of \(B_1\) is \((\sum_{i=s}^{r} n_i, M_2)\). In the remaining shape \((\lambda \setminus (B_1 \cup B_2))/\nu_{r+2}\), we put Dyck tiles like in the case of the term \(q^{\deg_r(\lambda)} Q(\nu_{r+1}; T)\). Thus the contribution is given by \(q^{\deg_{r+1}(\lambda)} Q(\nu_{r+2}; T)\). This completes the proof.

\[ \Box \]

**Lemma 7.8.** Let \(N = n_1 + n_2\). We have

\[
P(n_3, N)\Box(n_1, n_2 - 1) + q^{n_2} P(n_3 - 1, N)\Box(n_1 - 1, n_2) = P(2n_2 + n_3 - 1, n_1) \cdot P(n_3, n_2).
\]

**Proof.** We consider the case where \(n_3\) is even, since the proof for the odd case is essentially the same. Since we have

\[
\Box(r, s) = \begin{bmatrix} r+s \\ r \end{bmatrix},
\]

\[
P(n_3 - 1, N) = \frac{[2N + n_3]}{[N + n_3]} P(n_3, N),
\]

the left hand side of Lemma 7.8 is

\[ (7.5) \quad P(n_3, N)\Box(n_1, n_2 - 1) + q^{n_2} P(n_3 - 1, N)\Box(n_1 - 1, n_2) = \frac{[N]}{[n_1]} \frac{[N + n_2 + n_3 + 1]}{[2N + n_3 + 2]} P(n_3, N). \]

We show that the right hand side of Eqn. (7.5) is equal to the right hand side of Lemma 7.8 by induction on \(n_1\). When \(n_1 = 0\), the both terms are equal to \(P(n_3, n_2)\). Suppose that the statement holds true when \(n_1 = n\). When \(n_1 = n + 1\), the right hand side of Eqn. (7.5) is

\[ (7.6) \quad \frac{[N + 1]}{[n_1 + 1]} \frac{[N + n_2 + n_3 + 1]}{[2N + n_3 + 2]} P(n_3, N + 1) = \frac{[n_2 + 1][N + n_2 + n_3 + 1]}{[n_1 + 1][N + n_2 + n_3 + 2]} \times \frac{[N + 1]}{[n_1]} \frac{[N + n_2 + n_3 + 2]}{[2N + n_3 + 2]} P(n_3, N) \]

\[ = \frac{[n_2 + 1][N + n_2 + n_3 + 1]}{[n_1 + 1][N + n_2 + n_3 + 2]} P(2n_2 + n_3 + 1, n_1) P(n_3, n_2 + 1), \]
where we have used the assumption for \( n_1 = n \) and \( n_2 + 1 \). Since \( n_3 \) is even, we have

\[
(7.7) \quad P(n_3, n_2 + 1) = \frac{[2n_2 + n_3 + 2]}{[n_2 + 1]} P(n_3, n_2),
\]

\[
P(2n_2 + n_3 + 1, n_1) = \frac{[N + n_2 + n_3 + 2][n_1 + 1]}{[N + n_2 + n_3 + 3][2N + n_3 + 2]} P(2n_2 + n_3 + 1, n_1 + 1)
\]

\[
(7.8) \quad = \frac{[N + n_2 + n_3 + 2][n_1 + 1]}{[2n_2 + n_3 + 2][N + n_2 + n_3 + 1]} P(2n_2 + n_3 - 1, n_1 + 1)
\]

Substituting Eqn. (7.7) and (7.8) into the right hand side of Eqn. (7.6), we obtain \( P(2n_2 + n_3 - 1, n_1 + 1) \cdot P(n_3, n_2) \). This completes the proof. \( \square \)

**Proof of Theorem 7.4.** We prove Theorem by induction. We first consider the case where \( M_2 + N_2 \) is even.

Suppose that Theorem holds true up to \(|\lambda| - 1\) and \( M_2 + N_2 \) is even. We calculate the right hand side of Lemma 7.6 with the assumption. From the assumption, we have

\[
Q(\nu_0; \mathbf{T}) = P^1_{\lambda_2, s} \cdot \Box(n_0) \cdot \prod_{j=1}^r P^\text{Dyck}_{T_j} \cdot P(l_s, l'_{s-1} - 1) \cdot \prod_{j=s+1}^{r+1} P(l_j, n_j-1),
\]

\[
Q(\nu_i; \mathbf{T}) = P^1_{\lambda_2, s} \cdot \Box(n_i) \cdot \prod_{j=1}^r P^\text{Dyck}_{T_j} \cdot P(l_s, l'_{s-1}) \cdot \prod_{j=s+1}^{r+1} P(l_j, n_j-1), \quad \text{for } 1 \leq i \leq s - 1,
\]

\[
Q(\nu_i; \mathbf{T}) = P^1_{\lambda_2, s} \cdot \Box(n_0 - 1, m_1, n_1, \ldots, n_{i-1}, n_{i-1} + 1, m_i - 1) \cdot \prod_{j=1}^r P^\text{Dyck}_{T_j}
\]

\[
\times P(l_{i+1}, l'_{i}) \cdot \prod_{j=i+2}^{r+1} P(l_j, n_j-1), \quad \text{for } s \leq i \leq r,
\]

\[
Q(\nu_{r+1}; \mathbf{T}) = P^1_{\lambda_2, s} \cdot \Box(n_0 - 1, m_1, n_1, \ldots, n_{r-1}, m_r) \cdot \prod_{i=1}^r P^\text{Dyck}_{T_i}
\]

where \( l_i := M_2 + N_2 + \sum_{j=1}^r (m_j + n_j) \) and \( l'_i := \sum_{k=0}^i n_k \). We calculate the right hand side of Lemma 7.6. Since \( \deg_l(\lambda) = l'_i \), we have

\[
(7.9) \quad \sum_{i=s-1}^r q^i Q(\nu_{i+1}; \mathbf{T}) = q^{l'-1} P^1_{\lambda_2, s} \cdot \Box(n_0) \cdot \prod_{j=s}^r P(l_{j+1}, n_j) \cdot \prod_{i=1}^r P^\text{Dyck}_{T_i}
\]

where we have used Lemma 7.8 and

\[
\Box(n_0 - 1, m_1, \ldots, n_{i-1} + 1, m_i - 1) = \left[ \sum_{j=0}^{i-1} n_j + m_i - 1 \right] \Box(n_0 - 1, m_1, \ldots, m_{i-1}).
\]

By applying Lemma 7.1 to the sum of Eqn. (7.9) and \( Q(\nu_0; \mathbf{T}) \), we have

\[
(7.10) \quad Q(\nu_0; \mathbf{T}) + \sum_{i=s-1}^r q^i Q(\nu_{i+1}; \mathbf{T}) = P^1_{\lambda_2, s} \cdot \Box(n_0) \cdot \prod_{j=1}^r P^\text{Dyck}_{T_j} \cdot \prod_{j=s}^r P(l_{j+1}, n_j).
\]
By applying Lemma 7.5 to the sum of Eqn.(7.10) and \( \sum_{i=0}^{s-2} q^i Q(\nu_{i+1}; T) \), we obtain
\[
Q(\nu_0; T) + \sum_{i=0}^{s-2} q^{\deg(\lambda)} Q(\nu_{i+1}; T) = \Box(n) \cdot \prod_{j=1}^{r} P_{T_j}^{\text{Dyck}} \cdot P(l_{s-1}) \cdot \prod_{j=s}^{r} P(l_{j+1}, n_j)
\]
\[
= P_{\lambda_2} \cdot P(l_{s-1}) \cdot \prod_{j=s}^{r} P(l_{j+1}, n_j) \cdot P_{\lambda_1}^{\text{Dyck}}.
\]
Thus the statement holds true when the shape is \( \lambda \).

We consider the case where \( M_2 + N_2 \) is odd. From the induction assumption, we have
\[
Q(\nu_{r+2}; T) = P_{\lambda_2} \cdot P(l_{s+1}, n_{s-1} - 2) \cdot \prod_{j=s}^{r} P(l_{j+1}, n_j) \cdot \Box(n_0 - 1, m_1, n_1, \ldots, m_r) \cdot \prod_{j=1}^{r} P_{T_j}^{\text{Dyck}}.
\]
The rest of the proof is essentially the same as in the case of \( M_2 + N_2 \) even except that we use Lemma 7.7 instead of Lemma 7.6. This completes the proof. \( \Box \)

7.3. **Factorization in terms of trees.** Let \( T \) be a tree corresponding to a ballot path \( \lambda \). Theorem 7.4 can be translated into the following operations on a partial tree:

\[
\begin{align*}
\frac{N}{M} \rightarrow \left[ M + N \right] & \cdot \frac{1}{q^M} M + N \\
\frac{N}{M} \rightarrow \left[ M + N \right] & \cdot \prod_{i=1}^{N} (1 + q^i) \cdot \frac{1}{q^M} M + N \\
\frac{N}{M} \rightarrow \left[ M + N \right] & \cdot \prod_{i=1}^{N} (1 + q^i) \cdot \frac{2M + N}{2(M + N)} \cdot \prod_{j=1}^{r} P_{T_j}^{\text{Dyck}}.
\end{align*}
\]

Here (\( \leftrightarrow \)) in the right hand side of the third operation means that if the leftmost top edge in the left hand side of the third operation has an outgoing arrow, we put an outgoing arrow on the top edge of the right hand side.

Then, we define operations on the following trees (not a partial tree):

\[
\begin{align*}
\frac{N}{M} \rightarrow \prod_{i=1}^{N} (1 + q^i), \\
\frac{N}{M} \rightarrow 1.
\end{align*}
\]

Then, we have a map from a tree \( T \) to \( \mathbb{Z}[q, q^{-1}] \) by successive applications of the operations defined above.

8. **Various expressions of the generating function**

In this section, we show various expressions of the generating function \( P_{\lambda}^{\text{Dyck}} \) for a ballot path \( \lambda \). Given a tree \( T := A(\lambda) \), we abbreviate \( P_{\lambda}^{\text{Dyck}} \) by \( T \). It is clear from the context whether \( T \) stands for a tree or a generating function. We denote by \( |T| \) the number of edges in \( T \).

The factorization of the generating function (Theorem 7.4) can be translated into the following operation on a tree. Suppose that the root of a tree \( T := A(\lambda) \) has \( p \) edges and the leftmost edge
$e_1$ has a partial tree $T_1$ ($T_1$ can be an empty tree). We denote by $T_2 := T \setminus (T_1 \cup e_1)$ the partial tree obtained from $T$ by deleting the partial tree $(T_1 \cup e_1)$.

Lemma 8.1. If the edge $e_1$ does not have an incoming arrow, the generating function $T$ satisfies

$$T = P_{T_1}^{\text{Dyck}} \cdot P(2|T_2|, |T_1| + 1) \cdot T_2.$$ 

Similarly, if $e_1$ has an incoming arrow, the generating function $T$ satisfies

$$T = P_{T_1}^{\text{Dyck}} \cdot P(2|T_2| - 1, |T_1| + 1) \cdot T_2.$$ 

8.1. Expansion by tree. Lemma 7.6 and 7.7 can be translated into the following expressions in terms of trees.

Let $T$ be a tree $A(\lambda)$ and $e_1$ be an edge which is the leftmost and lowest in $T$. We denote by $\text{tree}_1(T, e_1)$ the tree obtained from $T$ by deleting $e_1$. We have a unique sequence of edges from the edge $e_1$ to the root. Along the sequence of edges, we enumerate the ramification point from $e_1$ to the root by $1, \ldots, M$ where $M$ is the number assigned to the root. We denote by $r_{i,j}$ the $j$-th edge from left in the $i$-th ramification point and by $r_{M,\bullet}$ an edge with a dot connecting to the root. Let $T'_{i,j}$ be the partial tree in $T$ such that the $i$-th ramification point is the root of $T'_{i,j}$ and edges in $T'_{i,j}$ are left to the edge $r_{i,j+1}$. We denote by $T_{i,j}$ the tree obtained from $T'_{i,j}$ by deleting the edge $e_1$ and adding an edge above the root of $T'_{i,j}$. We define $T \setminus T'_{i,j}$ as a tree obtained from $T$ by deleting the partial tree $T'_{i,j}$. A tree $\text{tree}_1(T, r_{i,j})$ with $(i,j) \neq (M, \bullet)$ is defined as a tree obtained from $T \setminus T'_{i,j}$ and $T_{i,j}$ by putting $T_{i,j}$ below the edge $r_{i,j}$ of $T \setminus T'_{i,j}$ from left. A tree $\text{tree}_1(T, r_{M,\bullet})$ is defined as a tree obtained by concatenating $T \setminus T'_{M,\bullet}$ and $T_{M,\bullet}$ at the root and putting an arrow from the edge $r_{M,\bullet}$ to the unique edge of $T_{M,\bullet}$ connected to the root. We define $\deg_1(T, r_{i,j})$ as the number of edges in $T'_{i,j}$, i.e., $\deg_1(T, r_{i,j}) = |T'_{i,j}|$.

When $T$ does not have arrows on the edges connecting to the root, we have

$$T = \text{tree}_1(T, e_1) + \sum_{i,j} q^{\deg_1(T, r_{i,j})} \text{tree}_1(T, r_{i,j}).$$

Suppose that the $r_{M,j}$-th ($1 \leq j \leq p$) edge in $T$ does not have an arrow and the $r_{M,j}$-th ($p+1 \leq j \leq r$) edge in $T$ has an incoming arrow. The $r_{M,\bullet}$-th edge with a dot has an outgoing arrow. A tree $\text{tree}_1(T, r_{M,p})$ is defined as a tree obtained by concatenating $T_{M,p}$ and $T \setminus T'_{M,p}$ at the root and putting an incoming arrow on the unique edge of $T_{M,p}$ connected to the root. We define

$$S_p := \{(i,j) \mid i \leq M - 1\} \cup \{(M,j) \mid j \leq p\},$$

and $\deg_1(T, r_{M,p}) := |T'_{M,p}|$. We have

$$T = \text{tree}_1(T, e_1) + \sum_{(i,j) \in S_p} q^{\deg_1(T, r_{i,j})} \text{tree}_1(T, r_{i,j}).$$

Suppose that all the edges connecting to the root have arrows in $T$. We define $r_{i,j}$, $\text{tree}_1(T, r_{i,j})$ with $i \leq M$ as above. Let (S2) be the following statement for $T$:

(S2) The depth of the edge $e_1$ is more than or equal to two, and the leftmost edge connected to the root has an incoming arrow.

Here, the depth of an edge $e$ in $T$ is defined as the distance from $e$ to the root of the tree. When $T$ satisfies the statement (S2), we define a tree $\text{tree}_1(T, r_{M+1})$ as follows. Let $T'$ be a tree obtained from $T_{M,1}$ by deleting two successive edges connected to the root. The tree $\text{tree}_1(T, r_{M+1})$ is
obtained by a concatenation of $T'$ and $T \setminus T_{M,1}'$ at the root. We do not put an incoming arrow to the edge of $T'$ connected to the root. We define

$$\deg_1(T, r_{M+1}) := |T_{M,1}'| + 2|T \setminus T_{M,1}'| - 1.$$  

Then, we have

$$T = \text{tree}_1(T, e_1) + \sum_{i,j} q^{\deg_1(T, r_{i,j})} \text{tree}_1(T, r_{i,j}) + \delta_{(S2)} \cdot q^{\deg_1(T, r_{M+1})} \text{tree}_1(T, r_{M+1}),$$

where $\delta_{(S2)} = 1$ if (S2) is true and $\delta_{(S2)} = 0$ otherwise.

8.2. Expansion by $\text{tree}_2$ and $\text{tree}_3$. Let $r_i$, $1 \leq i \leq p+r-1$, be the $i$-th edge without a $\bullet$ from left which is connected to the root of a tree $T$ and $r_{p+r} := r_\bullet$ be the edge with a $\bullet$ which is connected to the root. We consider the tree $T$ where the edges $r_i$ with $1 \leq i \leq p$ do not have incoming arrows and the edges $r_i$ with $p+1 \leq i < p+r$ have incoming arrows. We denote by $T_1$ a partial tree connected to the edge $r_i$ for $1 \leq i \leq p+r$. We define $\text{tree}_2(T, r_i)$ as a tree obtained from $T$ by deleting the edge $r_i$ and connecting two partial trees at the root in-between the $r_{i-1}$-th edge and $r_{i+1}$-th edge. We do not put incoming arrows on the edges left to the edge $r_{i+1}$. Let $E(r_i)$ be the edges left to the $r_i$-th edge. We define

$$\deg_2(T, r_i) := \begin{cases} \# \{ e \mid e \in E(r_i) \} & 1 \leq i \leq p, \\ 2\# \{ e \mid e \in E(r_{p+1}) \} + \# \{ e \mid e \in (E(r_i) \setminus E(r_{p+1})) \} & p+1 \leq i \leq p+r. \end{cases}$$

We introduce two lemmas used later.

**Lemma 8.2.** Let $\lambda_1$ and $\lambda_2$ be Dyck paths. We denote by $\lambda$ the path $U\lambda_1DU\lambda_2D$ and by $|\mu|$ the number of edges for a tree $A(\mu)$. Then, we have

$$P(2(N + |\lambda_2| + 1), |\lambda_1| + 1) \cdot P(2N, |\lambda_2| + 1) \cdot \lambda_1^{\text{Dyck}} \cdot \lambda_2^{\text{Dyck}} = P(2N, |\lambda|) \cdot \lambda^{\text{Dyck}}. \tag{8.1}$$

**Proof.** We have

$$P(2N, |\lambda|) = \prod_{j=1}^{\lfloor |\lambda|/2 \rfloor} \frac{|2N + 2(|\lambda| + 1 - j)|}{|\lambda| + 1 - j} \cdot P(2N, |\lambda_2| + 1),$$

$$P(2(N + |\lambda_2| + 1), |\lambda_1| + 1) = \prod_{j=1}^{\lfloor |\lambda_1|/2 \rfloor} \frac{|2N + 2(|\lambda_1| + 1 - j)|}{|\lambda_1| + 2 - j}.$$  

from Proposition 7.2 and

$$\lambda^{\text{Dyck}} = \left[ \begin{array}{c} |\lambda| \\ |\lambda_1| + 1 \end{array} \right] \lambda_1^{\text{Dyck}} \lambda_2^{\text{Dyck}}$$

from Lemma 2.19 and Lemma 2.20. Substituting these into the left hand side of Eqn.(8.1), we obtain the right hand side of Eqn.(8.1). \qed
Lemma 8.3. We have

\[
\frac{\text{tree}_2(T, r_i)}{T} = \begin{cases} 
\frac{\lvert T_i \rvert + 1}{2(\sum_{1 \leq k \leq p+r} |T_k| + p + r)}, & 1 \leq i \leq p, \\
\frac{2(\sum_{k=p+1}^{p+r} |T_k| + r)}{2(\sum_{k=1}^{p+r} |T_k| + p + r)} \cdot \frac{|T_i| + 1}{|T_i| + 1} \times \prod_{p+1 \leq j \leq i-1} \frac{2(\sum_{k=j+1}^{p+r} |T_k| + p + r - j) + |T_j| + 1}{2(\sum_{k=j+1}^{p+r} |T_k| + p + r - j) + |T_j| + 1}, & \text{for } p+1 \leq i \leq p+r.
\end{cases}
\]

Proof. We prove Lemma for 1 \leq i \leq p since one can apply the similar computation to p + 1 \leq i \leq p + r case.

Let \( r_{i,j} \) 1 \leq i \leq s, be the \( j \)-th edge without a • from left which is connected to the root of a partial tree \( T_i \). We denote by \( T_{i,j} \) a partial tree connected to the edge \( r_{i,j} \).

From the factorization (Lemma 8.1), for 1 \leq i \leq p we have

\[
T = \prod_{j=1}^{i} P_{T_j}^{\text{Dyck}} \cdot \prod_{j=1}^{i} P \left( 2 \left( \sum_{k=j+1}^{p+r} |T_k| + p + r - j \right), |T_j| + 1 \right) \cdot (T \setminus \bigcup_{1 \leq j \leq i} T_j)
\]

\[
\text{tree}_2(T, r_i) = \prod_{j=1}^{i-1} P_{T_j}^{\text{Dyck}} \cdot \prod_{j=1}^{i} P_{T_{i,j}}^{\text{Dyck}} \cdot (T \setminus \bigcup_{1 \leq j \leq i-1} T_j)
\]

\[
\times \prod_{j=1}^{s} P \left( 2 \left( \sum_{j+1 \leq k \leq s} |T_{i,k}| + s - j + \sum_{k=i+1}^{p+r} |T_k| + p + r - i \right), |T_{i,j}| + 1 \right)
\]

\[
\times \prod_{j=1}^{i-1} P \left( 2 \left( \sum_{k=j+1}^{p+r} |T_k| + p + r - j - 1 \right), |T_j| + 1 \right),
\]

where \( P(M, N) \) is defined in Section 7.1. We have

\[
\prod_{j} P_{T_{i,j}}^{\text{Dyck}} \cdot \prod_{j=1}^{s} P \left( 2 \left( \sum_{j+1 \leq k \leq s} |T_{i,k}| + s - j + \sum_{k=i+1}^{p+r} |T_k| + p + r - i \right), |T_{i,j}| + 1 \right)
\]

\[
= P \left( 2 \left( \sum_{k=i+1}^{p+r} |T_k| + p + r - i \right), |T_i| \right)
\]

from the successive use of Lemma 8.2. We also have

\[
P \left( 2 \left( \sum_{k=j+1}^{p+r} |T_k| + p + r - j - 1 \right), |T_j| + 1 \right)
\]

\[
P \left( 2 \left( \sum_{k=j+1}^{p+r} |T_k| + p + r - j \right), |T_j| + 1 \right)
\]

\[
P \left( 2 \left( \sum_{k=i+1}^{p+r} |T_k| + p + r - i \right), |T_i| \right)
\]

\[
P \left( 2 \left( \sum_{k=i+1}^{p+r} |T_k| + p + r - i \right), |T_i| + 1 \right)
\]

Substituting these into the left hand side of Eqn.(8.2), we obtain the right hand side of Eqn.(8.2). \( \square \)
Proposition 8.4. We have
\begin{equation}
T = (1 + q^{\text{deg}_2(T,r_{p+1})}) \sum_{1 \leq i \leq p} q^{\text{deg}_2(T,r_i)} \text{tree}_2(T, r_i) + \sum_{p+1 \leq i \leq p+r} q^{\text{deg}_2(T,r_i)} \text{tree}_2(T, r_i)
\end{equation}

Proof. From Lemma 8.3, we have
\begin{equation}
(1 + q^{\text{deg}_2(T,r_{p+1})}) \sum_{1 \leq i \leq p} q^{\text{deg}_2(T,r_i)} \frac{\text{tree}_2(T, r_i)}{T} = \frac{\left[2 \sum_{k=1}^{p} |T_k| + 2p\right]}{\left[2 \sum_{k=1}^{p+r} |T_k| + p + r\right]},
\end{equation}
\begin{equation}
\sum_{p+1 \leq i \leq p+r} q^{\text{deg}_2(T,r_i)} \frac{\text{tree}_2(T, r_i)}{T} = q^{2 \sum_{k=1}^{p} |T_k| + 2p} \frac{\left[2 \sum_{k=p+1}^{p+r} |T_k| + r\right]}{\left[2 \sum_{k=1}^{p+r} |T_k| + p + r\right]}.
\end{equation}

The sum of these two terms gives Eqn.(8.4). \qed

Let $T_i$ be a partial tree connected to the $r_i$-th $(1 \leq i \leq p + r)$ edge. We define
\begin{equation}
C_i(T) := \begin{cases} q^{\text{deg}_2(T,r_i)}(1 + q^{\text{deg}_3(T,r_i)}), & 1 \leq i \leq p, \\ q^{\text{deg}_3(T,r_i)}, & p + 1 \leq i \leq p + r, \end{cases}
\end{equation}
where
\begin{align*}
\text{deg}_3(T, r_i) &:= 2(|T_{i+1}| + \ldots + |T_{p+r}| + p + r - i) + |T_i| + 1, \\
\text{deg}_4(T, r_i) &:= \sum_{1 \leq j \leq i-1} (|T_j| + 1).
\end{align*}

Then, we have

Proposition 8.5.
\begin{equation}
T = \sum_{1 \leq i \leq p+r} C_i(T) \text{tree}_2(T, r_i).
\end{equation}

Proof. We show that
\begin{equation}
\sum_{1 \leq i \leq p} C_i(T) \frac{\text{tree}_2(T, r_i)}{T} + \sum_{p+1 \leq i \leq p+r} C_i(T) \frac{\text{tree}_2(T, r_i)}{T} = 1.
\end{equation}
We first compute the second sum in the left hand side of Eqn.(8.5). By taking the sum from $i = p + r$ to $i = 1$, we obtain
\begin{equation}
\sum_{p+1 \leq i \leq p+r} C_i(T) \frac{\text{tree}_2(T, r_i)}{T} = q^{2 \sum_{k=1}^{p} |T_k| + 2p} \frac{\left[2 \sum_{i=p+1}^{p+r} |T_i| + r\right]}{\left[2 \sum_{k=1}^{p+r} |T_k| + p + r\right]}.
\end{equation}
We add terms from $i = p$ to $i = 1$ one by one to the right hand side of Eqn.(8.6). We obtain that the left hand side of Eqn.(8.5) is equal to one. \qed

We consider a Dyck path $\lambda$ and the corresponding tree $T^{\text{Dyck}} := A(\lambda)$. We enumerate edges connected to a leaf from left to right by $1, 2, \ldots, p$ and call the $i$-th edge $e_i$. A tree $\text{tree}_2(T^{\text{Dyck}}, e_i)$ is defined as a tree obtained from $T$ by deleting $e_i$. Let $\text{deg}_4(T^{\text{Dyck}}, e_i)$ be the number of edges in $T^{\text{Dyck}}$ strictly left to the edge $e_i$. Here, “strictly left” means that we do not include the edges which are parents of $e_i$. Given an edge $e$ in $T$, we denote by $\text{ht}(e)$ the number of edges below $e$ plus one, that is, the height of the chord corresponding to $e$. 
Lemma 8.6. We have
\[(8.7) \quad \sum_{i=1}^{p} q^{\text{deg}_4(T_{\text{Dyck}}, e_i)} \text{tree}_3(T_{\text{Dyck}}, e_i). \]

Proof. We consider a partial tree $T'$ depicted as below:

The point $R_0$, $R$ is a ramification point and we have $m_{i,j}$ edges with $1 \leq i \leq r$ below $R$ and $m_2$ edges above $R$. We denote by $E$ the set of such edges. Let $T'_i$, $1 \leq i \leq r$, be a tree obtained from $T'$ by deleting the $i$-th leftmost and lowest edge connected to a leaf. We have
\[(8.8) \quad \sum_{i=1}^{r} q^{\text{deg}_4(T_{\text{Dyck}}, e_i)} \text{tree}_3(T_{\text{Dyck}}, e_i) = \left[\frac{m_1 + m_2}{N} \prod_{e \in E} \frac{\text{ht}(e)}{\text{ht}(e) - 1}\right]. \]

The right hand side of Eqn.(8.8) implies that $T'$ can be transformed to a tree with $m_1 + m_2$ edges without a ramification point. By repeating the above procedure for partial trees, we obtain Eqn.(8.7). 

Below, we assume that a tree $T$ does not have arrows. Let $r_i$, $1 \leq i \leq p$ be the $i$-th edge from left which is connected to a leaf of partial tree $T_1, \ldots, T_p$. We define
\[\text{deg}_5(T, r_i) := 2 \left(p + 1 - i + \sum_{i+1 \leq j \leq p+1} |T_j| \right) + \sum_{1 \leq j \leq i} |T_j| + i, \quad 1 \leq i \leq p, \]
\[\text{deg}_5(T, r_{p+1}) := \sum_{1 \leq j \leq p} |T_j| + p. \]

Let $s_i$ be the $i$-th edge from left which is connected to a leaf of partial trees $T_1, \ldots, T_p$. We define
\[\text{deg}_6(T, s_i) := \#\{e | \text{an edge } e \text{ is strictly left to } s_i\}, \]
where “strictly left” means that we do not include the edges which are parents of $s_i$.

Proposition 8.7. Suppose that $T$ has no arrows. We have
\[T = \sum_{1 \leq i \leq p+1} q^{\text{deg}_5(T, r_i)} \text{tree}_2(T, r_i) + \sum_{j} q^{\text{deg}_6(T, s_j)} \text{tree}_3(T, s_j). \]
Proof. From Lemma 8.3, we have an expression\( \text{tree}_2(T, r_i)/T \)

\[
\frac{\text{tree}_2(T, r_{p+1})}{T} = \frac{2! (T_{p+1} + 1)}{2(\sum_{j=1}^{p+1} |T_j| + p + 1)}.
\]

By a straightforward calculation, we have

\[
\sum_{1 \leq i \leq p+1} q^{\deg_5(T, r_i)} \frac{\text{tree}_2(T, r_i)}{T} = q^{\deg_5(T, r_{p+1})} \frac{\sum_{k=1}^{p+1} |T_k| + p + 2(\sum_{j=1}^{p+1} |T_j| + p + 1)}{2(\sum_{j=1}^{p+1} |T_j| + p + 1)}.
\]

We denote by \( e_{i,j} \) the \( j \)-th edge connected to a leaf in a partial tree \( T_i \). We have

\[
\frac{\text{tree}_3(T, e_{i,j})}{T} = \frac{|T_i| + 1}{2(\sum_{k=1}^{p+1} |T_k| + p + 1)} \cdot \frac{P_{\text{Dyck}}(T_i \setminus e_{i,j})}{P_{\text{Dyck}}(T_i)},
\]

for \( 1 \leq i \leq p \), where \( P_{\text{Dyck}}(T) \) is the generating function \( P_{\Delta,s} \) for \( T := A(\lambda) \).

\[
\sum_j q^{\deg_6(T, s_j)} \frac{\text{tree}_3(T, s_j)}{T} = \sum_{1 \leq j \leq p} \sum_{1 \leq l \leq p} q^{\deg_6(T, e_{j,l})} \frac{\text{tree}_3(T, e_{j,l})}{T}
\]

\[
= \sum_{1 \leq j \leq p} \sum_{1 \leq l \leq p} q^{\deg_6(T, e_{j,l})} \frac{|T_j| + 1}{2(\sum_{k=1}^{p+1} |T_k| + p + 1)} \cdot \frac{P_{\text{Dyck}}(T_j \setminus e_{j,l})}{P_{\text{Dyck}}(T_j)}
\]

\[
= \sum_{1 \leq k \leq p} q^{\sum_{1 \leq j \leq k-1} |T_j| + 1} \frac{|T_k| + 1}{2(\sum_{k=1}^{p+1} |T_k| + p + 1)}
\]

where we have used Lemma 8.6 in the third equality. By adding Eqn.(8.9) to Eqn.(8.11), we obtain the desired expression. \( \square \)

8.3. Expressions for a general tree. Let \( L_{\text{ref}} \) be a labelling of a tree \( T \) such that the post'(\( L \)) = \( id \) and \( n_e^{\text{ref}} \) be an integer on an edge \( e \) in \( L_{\text{ref}} \). Given \( L_{\text{ref}} \), we define

\[
S(e) := |T| + 1 - n_e^{\text{ref}}.
\]

We denote by \( \mathcal{E} \) and \( \mathcal{E}_* \) the set of edges in \( T \) and the set of edges with \( * \). Let \( A(T) \subseteq \mathcal{E} \setminus \mathcal{E}_* \) be the set of edges such that an edge \( e \) does not have parents with incoming arrows. We define \( B(T) := \mathcal{E} \setminus (\mathcal{E}_* \cup A(T)) \).

Fix a natural labelling \( L \) in \( T \) and let \( n_e \) be an integer on an edge \( e \). We put a circle on an edge in \( T \) by the following rule:

- (R1) There is no circle on an edge in \( \mathcal{E}_* \).
- (R2) We put a circle on an edge without an incoming arrow.
- (R3) Suppose we have a sequence of edges with arrows \( e(1) \leftarrow \ldots \leftarrow e(p) \leftarrow e \). If \( n_{e(i)} > n_e \) for \( 1 \leq i \leq p \), then we put circles on the edges \( e(1) \) to \( e(p) \).

Let \( \mathcal{E}_o(L) \) be the set of edges with a circle. Note that \( \mathcal{E}_o(L) \) depends on a natural labelling \( L \). From (R2), we have \( A(T) \subseteq \mathcal{E}_o \) and the equality holds when \( T \) does not have arrows. We define

\[
\deg_T(L) := \sum_{e \in \mathcal{E}} \# \{ e' \mid n_{e'} < n_e, \ e' \text{ is strictly right to } e \}.
\]
Theorem 8.8. We have

\[
T = \left( \sum_{L} q^{\deg_{T}(L)} \prod_{e \in B(T) \cap \mathbb{Z}_0(L)} (1 + q^{S(e) - 1}) \right) \cdot \prod_{e \in A(T)} (1 + q^{S(e)}).
\]

Proof. We prove Theorem by induction. When the number of edges in a tree is two, we have six trees. In terms of ballot paths, the six trees are \(UDUD, UUDD, UUD, UUU, UDUU\) and \(UDU\). It is easy to show that Theorem is true for these six trees.

Suppose that \(T\) can be expressed as a concatenation of trees \(T_1, \ldots, T_{p+s}\) where \(T_i, 1 \leq i \leq p+s\), can not be decomposed into a concatenation of trees. There exists a unique edge in \(T_i\) which is connected to the root. We call the \(i\)-th edge from left connected to the root \(r_i\). Further, we assume that the edges \(r_i, 1 \leq i \leq p\), do not have incoming arrows, the edges \(r_i, p+1 \leq i \leq p+s-1\), have incoming arrows and the edge \(r_{p+s}\) has a \(\bullet\). We consider a set \(L(i)\) of natural labellings such that the edge \(r_i\) has the integer one. Let \(\text{tree}_4(L(i))\) be the the following sum:

\[
\text{tree}_4(L(i)) := \left( \sum_{L \in L(i)} q^{\deg_{T}(L)} \prod_{e \in B(T) \cap \mathbb{Z}_0(L)} (1 + q^{S(e) - 1}) \right) \cdot \prod_{e \in A(T)} (1 + q^{S(e)}).
\]

Then, from the induction assumption, we have

\[
\text{tree}_4(L(i)) = \sum_{j=1}^{i-1} |T_j| \text{tree}_2(T, r_i)(1 + q^{|T|}), \quad \text{for } 1 \leq i \leq p,
\]

\[
\text{tree}_4(L(i)) = \sum_{j=1}^{i-1} |T_j| \text{tree}_2(T, r_i) \frac{(1 + q^{|T|})}{(1 + q^{\sum_{j=p+1}^{p+s} |T_j|})}, \quad \text{for } p+1 \leq i \leq p+s.
\]

The term \(\sum_{j=1}^{i-1} |T_j|\) comes from the fact that there are \(\sum_{j=1}^{i-1} |T_j|\) integers bigger than one in \(L(i)\). Note that the number of edges in \(\text{tree}_2(T, r_i)\) is one less than that of \(T\). From Lemma 8.3, we have

\[
\sum_{i=1}^{p+s} \text{tree}_4(L(i)) = T,
\]

which implies that Lemma holds true. \(\square\)

Example 8.9. Let \(T\) be the tree as depicted below:

\[
T := \begin{array}{c}
\text{\begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (1,1);
\draw (1,1) -- (2,2);
\draw (2,2) -- (3,3);
\draw (3,3) -- (4,4);
\draw (4,4) -- (5,5);
\draw (5,5) -- (6,6);
\end{tikzpicture}}
\end{array}
\]

We compute the generating function by Theorem 8.8. We have twelve natural labellings in \(T\) and their weights are give as follows:

\[
\begin{array}{cccccccc}
1 & 3 & 4 & 2 & 1 & 4 & 3 & 1 & 2 & 4 & 3 & 2 & 1 & 4 & 3 & 2 \\
q & q & q & q^2 & q^2 & q^3 & q^3 & q^3 & q^3 & q^4 & q^4 & q^5
\end{array}
\]
The circles give the factor
\[
\begin{array}{c}
\text{\includegraphics[width=2cm]{circle.png}} \\
\text{\includegraphics[width=2cm]{circle.png}}
\end{array}
\rightarrow (1 + q)(1 + q^3)(1 + q^4), \quad
\begin{array}{c}
\text{\includegraphics[width=2cm]{circle.png}} \\
\text{\includegraphics[width=2cm]{circle.png}}
\end{array}
\rightarrow (1 + q^3)(1 + q^4).
\]

The generating function is given by the sum of the contributions, namely we have
\[
T = (1 + q)(1 + q^3)(1 + q^4)(q + q^2 + q^3 + q^4 + q^5)\\n+ (1 + q^3)(1 + q^4)(1 + q + q^2 + q^3 + q^4)
\]
\[
= [3][6][8]/2.
\]

When a tree \( T \) is written as \( T = A(\lambda) \) for a Dyck path \( \lambda \), we abbreviate \( P^\text{Dyck}_\lambda \) as \( p^\text{Dyck}(T) \).

**Lemma 8.10.** Suppose that a tree \( T \) is written as a concatenation of three trees \( T := T_1 \circ T_2 \circ T_3 \) and \( T_1 \) and \( T_2 \) do not have incoming arrows. Then we have the generating function satisfies
\[
P^f(T) = P(2|T_3|, |T_1| + |T_2|) \cdot p^\text{Dyck}(T_1 \circ T_2) \cdot P^f(T_3).
\]

where \( P^f(T) := P^f_L \) for \( T = A(\lambda) \).

**Proof.** Form the factorization (Theorem 7.4), we have
\[
P^f(T) = P(2|T_2 | + 2|T_3 |, |T_1 | + |T_2 |) \cdot p^\text{Dyck}(T_1 \circ T_2) \cdot P^f(T_3).
\]
Substituting Lemma 2.20 and Proposition 7.2 into the above equation, we obtain Eqn.(8.12). \( \square \)

Let \( T^\text{Dyck} \) be a tree for a Dyck path and \( L^D \) be a natural labelling of \( T^\text{Dyck} \). Let \( n_e \) be an integer on the edge \( e \) in the labelling \( L^D \). We define
\[
deg^D(L) := \#\{e' \mid n_{e'} < n_e, e' \text{ is strictly right to } e\}.
\]

**Lemma 8.11.** We have
\[
\sum_L q^{\deg^D(L)} = p^\text{Dyck}(T^\text{Dyck}).
\]

**Proof.** We consider a tree with capacities such that a capacity of a leaf is the number of edges strictly right to the leaf. Let \( L^\text{LS} \) be a labelling of type LS. Let \( n^\text{LS}_e \) be an integer on an edge \( e \) in \( L^\text{LS} \). We have a bijection from \( L \) to \( L^\text{LS} \) such that \( n^\text{LS}_e = \#\{e' \mid n_{e'} < n_e, e' \text{ is strictly right to } e\} \). Let \( \tilde{T} \) be a mirror image of \( T^\text{Dyck} \). Then,
\[
\sum_L q^{\deg^D(L)} = \sum_{L^\text{LS}} q^{\sum_e n^\text{LS}_e} \quad = \quad p^\text{Dyck}(\tilde{T}) \quad = \quad p^\text{Dyck}(T^\text{Dyck}),
\]
where we have used Theorem 2.5 and Theorem 2.17 in the second equality. \( \square \)

We denote by \( T^\text{rev} \) a tree with capacities such that a capacity of a leaf \( l \) is the number of edges strictly right to \( l \). Let \( L^\text{rev} \) be a labelling of \( T^\text{rev} \) satisfying (LS1) and (LS2). We denote by \( L^\text{rev} \) the set of such labellings.

**Lemma 8.12.** Let \( T \) be a tree for a Dyck path \( \lambda \). Then, we have
\[
\sum_{L \in L^\text{LS}(\lambda)} q^{\deg^D(L)} = \sum_{L \in L^\text{rev}(\lambda)} q^{\deg^D(L)}
\]
where \( \deg^D(L) \) is the sum of the labels of a labelling \( L \).
Proof. From Theorem 2.5 and Theorem 2.17, the left hand side of Eqn. (8.13) is equal to $P^\text{Dyck}_\lambda$. A path $\lambda$ is written as $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_{2N}$ with $\lambda_i = U$ or $D$. Then we define $\overline{\lambda} := \overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_{2N}}$ by $\overline{\lambda_i} = U$ for $\lambda_{2N+1-i} = D$ and $\overline{\lambda_i} = D$ for $\lambda_{2N+1-i} = U$. Then, the right hand side of Eqn. (8.13) is equal to $P^\text{Dyck}_{\overline{\lambda}}$. By definition, a path $\overline{\lambda}$ is also a mirror image of $\lambda$. A mirror image of a Dyck tiling $D$ above $\lambda$ is also a Dyck tiling $\overline{D}$ above $\overline{\lambda}$. We also have $\text{art}(D) = \text{art}($ $\overline{D})$. Thus we have $P^\text{Dyck}_\lambda = P^\text{Dyck}_{\overline{\lambda}}$, which implies Eqn. (8.13).

Proposition 8.4 can be translated into the following theorem. We keep the notation in Theorem 8.8. Let $L$ be a natural labelling of $T$, $n_e$ be an integer on the edge in $L$ and $\mathcal{L}(T)$ be the set of natural labellings of $T$. We divide the edges into two blocks: the first block is the set of edges which is strictly left to the edge $r_{p+1}$ and the second block is the set of edges strictly right to the edge $r_p$. We define

$$m_e := 2\# \{ e' \mid n_{e'} < n_e, e' \text{ is strictly right to } e \text{ and in a different block}\}$$

and

$$\deg_g(L) := \sum_e m_e,$$

and $N := \sum_{i=1}^{p} |T_i|$. 

Theorem 8.13. We have

$$T = \left( \sum_{L \in \mathcal{L}(T)} q^{\deg_g(L)} \prod_{e \in B(T) \cap \mathcal{L}_e(L)} (1 + q^{S(e)-1}) \right) \prod_{i=1}^{N} (1 + q^i).$$

Proof. We prove Theorem by induction. We assume that the theorem holds true for a tree with at most $|T| - 1$ edges.

Let $\mathcal{L}(i)$ be the set of natural labellings such that the edge $r_i$ has the integer one. Given a tree $T$, let $T(i)$ be the sum

$$T(i) := \left( \sum_{L \in \mathcal{L}(i)} q^{\deg_g(L)} \prod_{e \in B(T) \cap \mathcal{L}_e(L)} (1 + q^{S(e)-1}) \right) \prod_{i=1}^{N} (1 + q^i).$$

It is obvious that we have $T(i) = (1 + q^{\deg_2(T, r_{p+1})}q^{\deg_2(T, r_1)}\text{tree}_2(T, r_1))$ for $p + 1 \leq i \leq p + s$. From Lemma 8.10 and Lemma 8.11, it is enough to consider the following $T'$. A tree $T'$ is a concatenation of three trees, i.e., $T' = T_1 \circ T_2 \circ T_3$. Further, a tree $T_i$, $1 \leq i \leq 3$, does not have a ramification point, the tree $T_3$ consists of only edges with a $\bullet$, and there is an arrow from the top edge in $T_3$ to the top edge in $T_2$. Let $M_i$, $1 \leq i \leq 3$, be the number of edges in $T_i$. We have

$$\text{tree}_2(T', r_2) = \left[ \frac{M_1 + M_2 - 1}{M_1} \right] \left[ \frac{M_1 + M_2 + M_3 - 1}{M_3} \right] q^2 \prod_{j=1}^{M_1 + M_2 - 1} \frac{[2j]}{[j]^2},$$

and

$$T'(2) = q^{M_1} \left[ \frac{M_2 + M_3 - 1}{M_3} \right] \left[ \frac{M_1 + M_2 + M_3 - 1}{M_1} \right] \prod_{j=1}^{M_1} \frac{[2j]}{[j]} \cdot \prod_{k=M_3+1}^{M_2+M_3} \frac{[2k]}{[k]}.$$
is equal to $T$ by the recurrence relation in Proposition 8.4, which implies Theorem holds true for $T$. □

Let $T$ be a tree with capacities $A(\lambda/\lambda_0)$ where $\lambda_0$ consists of only $U$’s. Let $L^{LS}$ be a labelling of LS type in $T$ and $n_e$ be an integer on an edge $e$. The labels $n_e$ satisfy (LS1) and (LS2). We put an circle on an edge in the following rules:

(R4) There is no circle on an edge with $\bullet$.
(R5) We put a circle on an edge without an incoming arrow.
(R6) Suppose we have a sequence of edges with arrows: $e(0) \leftarrow e(1) \leftarrow \ldots \leftarrow e(p) \leftarrow e$. If $n_{e(i)} \geq n_e$ for $1 \leq i \leq p$ and $n_{e(0)} < n_e$, we put circles on edges $e(1)$ to $e(p)$.

Let $E_0(L^{LS})$ be the set of edges with a circle for a label $L$. We consider the post-order from right on $T$ and let $S'(e)$ be an integer on an edge $e$.

We denote by $E$ the set of edges in $A(\lambda)$. Given a labelling $L^{LS}$ of type LS, we define

$$\deg_9(L^{LS}) := \sum_{e \in E} n_e.$$ 

Theorem 8.14. We have

$$T = \sum_{L^{LS}} q^{\deg_9(L^{LS})} \prod_{e \in E_0(L^{LS})} (1 + q^{S'(e)}).$$

Before proceeding with the proof of Theorem 8.14, we introduce a lemma used later.

Lemma 8.15. We have

$$\sum_{j=0}^{c} q^{j(M+N)} \left[ \begin{array}{c} N + M + c - j - 1 \\ M, N - 1, c - j \end{array} \right] = q^{j(M+N)} \left[ \begin{array}{c} N + M - 1 \\ M \end{array} \right] \left[ \begin{array}{c} N + M + c - I \\ c - I \end{array} \right].$$

Proof. We prove Lemma by induction. When $I = c$, the both sides of Eqn.(8.15) are equal to $q^{c(M+N)} \left[ \begin{array}{c} N + M - 1 \\ M \end{array} \right]$. We assume that Lemma holds true for $I$. Then, we have

$$\sum_{j=0}^{c} q^{j(M+N)} \left[ \begin{array}{c} N + M + c - j - 1 \\ M, N - 1, c - j \end{array} \right] = q^{j(M+N)} \left[ \begin{array}{c} N + M - 1 \\ M \end{array} \right] \left[ \begin{array}{c} N + M + c - I \\ c - I \end{array} \right] + q^{(j-1)(M+N)} \left[ \begin{array}{c} N + M + c - I \\ M, N - 1, c - I + 1 \end{array} \right]$$

$$= q^{(j-1)(M+N)} \left[ \begin{array}{c} N + M - 1 \\ M \end{array} \right] \left[ \begin{array}{c} N + M + c - I + 1 \\ c - I + 1 \end{array} \right],$$

which implies Lemma holds true. □

Proof of Theorem 8.14. To show Theorem is true, it is enough to compute the following two partial trees $T_1$ and $T_2$:

$$T_1 := \begin{array}{c} N \\ C_1 \end{array} \begin{array}{c} M \\ C_2 \end{array}, \quad T_2 := \begin{array}{c} N \\ C_1 \end{array} \begin{array}{c} M \\ C_2 \end{array},$$

where $c_2 := c_1 + N$. 
Let \( g(i, c, N) \) be the generating function for a tree \( T \) such that \( T \) does not have a ramification point, the top edge has the integer \( i \), the capacity is \( c \) and the number of edges in \( T \) is \( N \), i.e.,

\[
g(i, c, N) := q^i \left[ \frac{N + c - 1 - i}{N - 1} \right].
\]

We define \( g_{\leq}(i, c, N) \) (resp. \( g_{>}(i, c, N) \)) in a similar way as \( g(i, c, N) \) except that the integer on the top edge is equal to or bigger than (resp. smaller than) \( i \), i.e.,

\[
g_{\leq}(i, c, N) := \sum_{j=0}^{i} q^j N \left[ \frac{N + c - i - 1}{N - 1} \right],
\]

\[
g_{>}(i, c, N) := \sum_{j=i+1}^{c} q^j N \left[ \frac{N + c - i - 1}{N - 1} \right].
\]

Let \( e_1 \) (resp. \( e_2 \)) be the edge with an incoming (outgoing) arrow in the partial tree \( T_1 \). Since we have an arrow in \( T_1 \), the generating function for \( T_1 \) is the sum of two contributions: the first case is that the integer on \( e_1 \) is smaller or equal to the integer on \( e_2 \), and the second case is the integer on \( e_1 \) is bigger than the integer \( e_2 \). We have

\[
T_1 = \prod_{i=M+1}^{M+N-1} (1 + q^i) \left\{ \sum_{i=0}^{c_1} (1 + q^{M+N}) g(i, c_1, N) g_{\leq}(i, c_2, M) + \sum_{i=0}^{c_1} g(i, c_1, N) g_{>}(i, c_2, M) \right\}.
\]

Since we have \( g_{\leq}(i, c_2, M) + g_{>}(i, c_2, M) = \left[ \frac{M+c_2}{M} \right] \), we have

\[
T_1 = \prod_{i=M+1}^{M+N-1} (1 + q^i) \left\{ (1 + q^{M+N}) \left[ \frac{N + M + c_1}{M, N, c_1} \right] - \sum_{i=0}^{c_1} q^{(i+1)(N+M)+M} \left[ \frac{M + N + c_1 - i - M, N - 1, c_1 - i}{M} \right] \right\}
\]

\[
\quad = \prod_{i=M+1}^{M+N-1} (1 + q^i) \left\{ (1 + q^{M+N}) \left[ \frac{N + M + c_1}{M, N, c_1} \right] - q^{2M+N} \left[ \frac{N + M - 1}{M} \right] \left[ \frac{N + M + c_1}{c_1} \right] \right\}
\]

\[
\quad = \left[ \frac{N + M}{N} \right] q^M \left[ \frac{N + 2M}{2(N + M)} \right] \prod_{i=1}^{N} \left( 1 + q^i \right) \cdot \left[ \frac{1}{c_1} \right] M + N + \frac{c_1}{c_1},
\]

where we have used Lemma 8.15 in the second equality.

By a similar calculation, we have

\[
T_2 = \left[ \frac{N + M + c_1}{N, M, c_1} \right] \prod_{j=M+1}^{N+M} (1 + q^j)
\]

\[
\quad = \left[ \frac{N + M}{N} \right] q^M \prod_{j=1}^{N} \left( 1 + q^j \right) \cdot \left[ \frac{1}{c_1} \right] M + N + \frac{c_1}{c_1}.
\]

From Lemma 2.18 and Section 7.3, it is clear that Eqn.(8.14) gives an expression of the generating function for \( T \). \( \square \)

**Example 8.16.** Let \( T \) be the following tree with capacities:

\[
T := \begin{array}{c}
0 \\
1 \\
2
\end{array}
\]

\[
\begin{array}{cc}
\text{---} & \text{---} \\
| & |
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\]

\[
\begin{array}{c}
\text{---} & \text{---} \\
| & |
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\]

\[
\begin{array}{c}
\text{---} & \text{---} \\
| & |
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\]
We compute the generating function by Theorem 8.14. We have six configurations with weights:

\[
\begin{array}{ccccccc}
\ast & \circ & 0 & 1 & \circ & 0 & 2 \\
1 & q & q^2 & q & q^2 & q^2 & q^3
\end{array}
\]

The circles give the factor
\[
\ast \circ \ast \mapsto (1 + q^2)(1 + q^3), \quad \circ \ast \ast \mapsto (1 + q^3), \quad \ast \circ \ast \mapsto (1 + q^2)
\]

Then, the generating function \( T \) is given by
\[
T = (1 + q)(1 + q^2)(1 + q^3) + (q + q^2)(1 + q^3) + q^2(1 + q^2) + q^3
\]

8.4. **Tree without arrows.** In this subsection, we consider trees without arrow. Given two permutations \( u \) and \( v \) in the set of signed permutations, we define the weak left (Bruhat) order \( \geq_L \) as follows. We have \( v \geq_L u \) if some final subword of some reduced word for \( v \) is a reduced word for \( u \).

**Theorem 8.17.** Let \( L_0 \) be the natural labelling of \( T \) such that \( \text{pre}(L_0) = \text{id} \) and a modified post-order word \( \sigma_0 = \text{post}(L_0) \). We have
\[
P_{\lambda,*}^I = \sum_{\sigma \geq_L \sigma_0} q^{\text{Inv}(\sigma) - \text{Inv}(\sigma_0)}
\]
where \( \geq_L \) is the weak left (Bruhat) order for signed permutations.

**Proof.** We prove Theorem by induction. Theorem holds true when a tree \( T \) has one edge. We assume that Theorem is true at most \( N - 1 \) edges. We have a recurrence formula from Proposition 8.7. We compare \( \sigma := \sigma_1 \ldots \sigma_N \) to the modified post-order word \( \sigma_0 := \sigma_{0,1} \ldots \sigma_{0,N} \) of \( L_0 \). Recall that some \( \sigma_{0,i} \)'s are underlined and correspond to the edges with \( \bullet \). We consider a word \( \sigma \geq_L \sigma_0 \) such that \( \sigma_1 = \sigma_{0,i} \) and \( \sigma_{0,i} \) is not underlined. Since \( \sigma_0 \) is a modified post-order word, such \( \sigma_{0,i} \) corresponds to the edge \( s_j \) in \( L_0 \). Let \( \sigma_0' \) be the post-order word of a labelling \( L_0 \) on the tree \( \text{tree}_3(T, s_j) \). Then, it is obvious that \( i - 1 = \text{deg}_{L_0}(T, s_j) \) and a subword \( \sigma' := \sigma_2 \ldots \sigma_N \) satisfies \( \sigma' \geq_L \sigma_0' \) by induction assumption. Similarly, we consider the case where \( \sigma_{0,i} \) is not underlined and \( \sigma_1 = \sigma_{0,i} \), i.e., \( \sigma_1 \) is underlined. A possible such \( \sigma_{0,i} \) corresponds to the edge \( r_j \). In signed permutations, we need \( \text{deg}_{L_0}(T, r_j) \) simple transpositions to move \( \sigma_{0,i} \) to the \( N \)-th element, put an underline on it, and move forward to the first element in \( \sigma \). Let \( \sigma_0' \) be the post-order word of a labelling \( L_0 \) on the tree \( \text{tree}_2(T, r_j) \). The subword \( \sigma' := \sigma_2 \ldots \sigma_N \) satisfies \( \sigma' \geq_L \sigma_0' \) by induction assumption. The sum of these two contributions implies that Theorem holds true. \( \square \)

**Example 8.18.** Let \( \lambda = UUDDUU \) and \( T = A(\lambda) \). Then, \( L_0 \) is the natural labelling

\[
L_0 :=
\begin{array}{ccc}
1 & \bullet & 3 \\
\ast & 2
\end{array}
\]

The modified post-order word is \( \sigma_0 = \text{post}(L_0) = 213 \). We have twelve signed permutations \( \sigma \)'s satisfying \( \sigma \geq_L \sigma_0 \). They are
\[
\begin{array}{cccccccccccc}
\sigma & 213 & 231 & 321 & 231 & 321 & 213 & 123 & 312 & 132 & 312 & 132 & 123 \\
\text{Inv}(\sigma) & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9
\end{array}
\]

The generating functions is
\[
P_{\lambda,*}^I = (1 + q + 2q^2 + 2q^3 + 2q^4 + 2q^5 + q^6 + q^7) = [4][6]/[2].
\]
The following Theorem is an analogue of Theorem 2.2. Let $N$ be the number of edges in $T$. In $T$, we say an edge $e'$ is a child of an edge $e$ if there exists a unique sequence of edges $e = e_0 \rightarrow e_1 \rightarrow \cdots \rightarrow e_n = e'$ such that $e_{i+1}$ is below $e_i$ in $T$. Given an edge $e$ in $T$, we define

\[ l(e) := \begin{cases} 
\# \{ e' \mid e' \text{ is a child of } e \} + 1, & e \text{ does not have a } \bullet, \\
2\# \{ e' \mid e' \text{ is a child of } e \} + 2, & e \text{ has a } \bullet
\end{cases} \]

**Theorem 8.19.** We have

\[ (8.16) \]

\[ P_{\lambda,*}^I = \frac{[2N]!!}{\prod_{e \in T}[l(e)]}. \]

**Proof.** We use the same notation as Lemma 8.3. The root of $T$ has $p$ edges without $\bullet$ and at most one edge with $\bullet$. We call the edge with $\bullet$ $r_{p+1}$. From Lemma 8.3, we have

\[ \frac{\text{tree}_2(T, r_i)}{T} = \frac{[|T_i| + 1]}{[2N]}, \]

\[ \frac{\text{tree}_2(T, r_{p+1})}{T} = \frac{2(|T_{p+1}| + 1)}{[2N]}. \]

We apply tree$_2$ successively to $T$ and obtain Eqn. (8.16). $\square$

Let $T^{\text{rev}}$ be a tree with following capacities. A capacity on a leaf $l$ is the number of edges strictly right to the edge just above the leaf $l$. Since we can put integers on $T$ satisfying (LS1) and (LS2), we denote by $L^{\text{rev}}$ a labelling of $T$. We denote by $L^{\text{rev}}(\lambda)$ the set of possible labellings $L^{\text{rev}}$s. For a edge $e$ in $T$, we denote by $T_e$ a partial tree connected to the edge $e$ (the root of $T_e$ is connected to $e$ from bottom). Let $c_{\text{min}}$ be the smallest capacity appeared in $T_e$. We define

\[ S^{\text{rev}}(e) := \min_e + |T_e| + 1. \]

where $\min_e$ is the smallest capacity in a partial tree $T_e$.

**Theorem 8.20.** We have

\[ (8.17) \]

\[ T = \left( \sum_{L \in L^{\text{rev}}(\lambda)} q^{\deg_b(L)} \right) \cdot \prod_{e \in E \setminus E_\bullet} \left( 1 + q^{S^{\text{rev}}(e)} \right). \]

**Proof.** Since $T$ does not have arrows, Theorem 8.14 can be reduced to

\[ T = \left( \sum_{L^{\text{rev}}} q^{\deg_b(L)} \right) \cdot \prod_{e \in E \setminus E_\bullet} \left( 1 + q^{S(e)} \right). \]

By comparing the post-order from right in a tree $T$ with a capacity of $T^{\text{rev}}$, we have $S'(e) = S^{\text{rev}}(e)$. Together with Lemma 8.12, we obtain Eqn.(8.17). $\square$

By a straightforward calculation, we have the following lemma.

**Lemma 8.21.** Let $M = \sum_{i=1}^r M_i$. We have

\[ \left[ \begin{array}{c}
M + c \\
M_1, M_2, \ldots, M_r, c
\end{array} \right] \prod_{i=c+1}^{M+c} (1 + q^i) = \left[ \begin{array}{c}
M + c \\
M_1, M_2, \ldots, M_r, c
\end{array} \right] q^2 \prod_{i=1}^r M_i \prod_{j=1}^c (1 + q^j). \]
Let \( \lambda \) be a ballot path and \( \lambda_0 \) be a ballot path consisting of only \( U \)'s. Let \( L \) be a labelling of type \( \text{LS} \) on the tree \( A(\lambda/\lambda_0) \) and \( \mathcal{L}(\lambda) \) be the set of labellings of type \( \text{LS} \). Given a labelling \( L \), we denote by \( n_e \) an integer on an edge \( e \). We define \( N := |E| - |E^*| \).

For an edge \( e \) in a tree \( A(\lambda) \), we define \( S(e) \) as one plus the number of children of \( e \). In other words, \( S(e) \) is one plus the sum of distances from \( e \) to leaves.

**Theorem 8.22.** We have

\[
T = \left( \sum_{L \in \mathcal{L}(\lambda)} q^{2\text{deg}_{10}(L)} \right) \cdot \prod_{e \in E \setminus E^*} (1 + q^{S(e)}).
\]

**Proof.** Starting with the expression Eqn. (8.17) in Theorem 8.20, we apply Lemma 8.21 to ramification points in the tree \( T \). Then we apply Lemma 8.12 to the obtained expression, which implies Theorem holds true. \( \square \)

Given a tree \( T \), \( T \) is a concatenation of trees \( T_1, \ldots, T_M \) at their roots. We call \( T_i, 1 \leq i \leq M \), the \( i \)-th block. Note that only the \( M \)-th block contains edges with a dot. Fix a natural labelling \( L \) of a tree \( T \) and let \( n_e \) be an integer on an edge \( e \). We define \( \text{deg}_{11}(L) \) as

\[
d_{e} := 2\# \{ e' \mid n_{e'} < n_e, e' \text{ is strictly right to } e \text{ and in a different block} \} + \# \{ e' \mid n_{e'} < n_e, e' \text{ is strictly right to } e \text{ and in the same block as } e \},
\]

and

\[
\text{deg}_{10}(L) := \sum_e d_e.
\]

Let \( N_i, 1 \leq i \leq M - 1 \), be the number of edges in the \( i \)-th block.

**Theorem 8.23.** We have

\[
T = \left( \sum_L q^{\text{deg}_{10}(L)} \right) \prod_{i=1}^{M-1} N_i \prod_{j=1}^{M} (1 + q^{j}) \prod_{e \in T_M \cap (E \setminus E^*)} (1 + q^{S_{\text{rev}}(e)}).
\]

**Proof.** We start with the expression in Theorem 8.20. Let \( c_i \) be the smallest capacity in the partial tree \( T_i \) for \( 1 \leq i \leq M - 1 \). From the definition of \( d_e \), if \( n_{e'} \) and \( n_e \) are in the same block and \( e' \) is strictly right to \( e \), the weight for \( n_{e'} < n_e \) is one. We can resolve ramification points by using Lemma 8.11. By this operation, \( T_i \) is a product of \( \text{PDyck}(T_i) \) and a tree \( T'_i \) with \( |T'_i| \) edges and without a ramification point. We apply Lemma 8.21 to each \( T'_i \). The right hand side of Lemma 8.21 implies that we have a weight two if \( n_{e'} < n_e \) and \( e' \) is strictly right to \( e \) and in a different block. This completes the proof. \( \square \)

**8.5 Relation to BTS.** In this subsection, we consider a tree \( T \) without arrows. We put an additional condition on \( T \): an edge \( e \) without a dot does not have a parent edge with a dot.

We denote by \( \mathcal{E} \) the set of edges in \( A(\lambda) \) and by \( \mathcal{E}_* \) the set of edges with a \( \bullet \). We define:

\[
\text{deg}_{11}(L) := \sum_{e \in (\mathcal{E} \setminus \mathcal{E}_*)} n_e + 2 \sum_{e \in \mathcal{E}_*} n_e.
\]

Recall that \( \mathcal{L}(\lambda) \) is the set of labellings of type \( \text{LS} \).
Theorem 8.24. We have

\[ T = \left( \sum_{L \in \mathcal{L}(\lambda)} q^{\deg_{11}(L)} \right) \cdot \prod_{i=1}^{N} (1 + q^i). \]

where \( N \) is the number of edges without a dot in \( T \).

Proof. We assume that \( T \) is written as a concatenation of two trees \( T_1 \) and \( T_2 \) such that \( T_2 \) consists of edges with a dot and without a ramification. Let \( \mathcal{L}_1 \) (resp. \( \mathcal{L}_2 \)) be the set of labellings satisfying (LS1) and (LS2) on \( T_1 \) (resp. \( T_2 \)). We denote by \( \deg_{11}(L_i), i = 1, 2 \) the sum of labels in \( L_i \). By applying Section 7.3 to the generating function \( T \), we have

\[
T = P^{\text{Dyck}}(T_1) \cdot \left[ \frac{|T_1| + |T_2|}{|T_1|} \right] \cdot \prod_{i=1}^{N} (1 + q^i)
\]

\[
= \left( \sum_{L \in \mathcal{L}_1} q^{\deg_0(L_1)} \right) \left( \sum_{L \in \mathcal{L}_2} q^{2\deg_0(L_2)} \right) \cdot \prod_{i=1}^{N} (1 + q^i)
\]

\[
= \left( \sum_{L \in \mathcal{L}(\lambda)} q^{\deg_{11}(L)} \right) \cdot \prod_{i=1}^{N} (1 + q^i)
\]

where we have used Lemma 8.11 in the second equality. This completes the proof. \( \square \)

Let \( L \) be a natural labelling of a tree \( A(\lambda) \) and \( n_e \) be a label of an edge \( e \). \( L^{\text{max}} \) be the unique maximum labelling of LS type and \( n^{\text{max}}_e \) be a label of an edge \( e \). We construct a labelling \( L' \) by replacing \( n_e \) with \( n'_e \) where

\[ n'_e := \# \{ e' \mid n_{e'} > n_e, e' \text{ is strictly left to } e \}. \]

We obtain a labelling of LS type by \( L^{\text{LS}} := L^{\text{max}} - L' \), i.e. \( n^{\text{LS}}_e := n^{\text{max}}_e - n'_e \). We denote by \( \psi \) the map \( \psi : L \to L^{\text{LS}} \). The following lemma is obvious from the definition of \( \psi \). Recall the definition of the inversion for an inverse pre-order word introduced in Section 5.1.

Lemma 8.25. Let \( L \) be a natural labelling of a tree \( T \) and \( \sigma \) be the inverse pre-order word for \( L \). We have

\[ \text{inv}(\sigma) = \deg_{11}(L^{\text{max}}) - \deg_{11}(\psi(L)). \]

Corollary 8.26. Let \( \lambda \) is a ballot path for a tree \( T \) and \( \sigma \) be an inverse pre-order word of the tree \( T \). Then, we have

\[ T = \left( \sum_{\sigma} q^{\text{inv}(\sigma)} \right) \cdot \prod_{i=1}^{N} (1 + q^i)
\]

\[
= \left( \sum_{\sigma} q^{\text{art}(\text{BTS}(\lambda, \sigma))} \right) \cdot \prod_{i=1}^{N} (1 + q^i),
\]

where \( N \) is the number of edges without a dot in \( T \).

Proof. The first equality in Corollary follows from Theorem 8.24 and Lemma 8.25. The second equality follows from Theorem 5.8. \( \square \)
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