Two neutrino positron double beta decay of $^{106}$Cd for $0^+ \rightarrow 0^+$

transition

A. Shukla and P. K. Raina
Department of Physics and Meteorology,
IIT Kharagpur-721302, India.

R. Chandra and P. K. Rath
Department of Physics,
University of Lucknow,
Lucknow-226007, India.

J. G. Hirsch
Instituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México,
A.P. 70-543 México 04510 D.F.
Abstract

The two neutrino positron double beta decay of $^{106}$Cd for $0^+ \rightarrow 0^+$ transition has been studied in the Hartree-Fock-Bogoliubov model in conjunction with the summation method. In the first step, the reliability of the intrinsic wave functions of $^{106}$Cd and $^{106}$Pd nuclei has been tested by comparing the theoretically calculated results for yrast spectra, reduced $B(E2:0^+ \rightarrow 2^+)$ transition probabilities, quadrupole moments $Q(2^+)$ and gyromagnetic factors $g(2^+)$ with the available experimental data. In the second step, the nuclear transition matrix element $M_{2\nu}$ and the half-life $T_{1/2}^{2\nu}$ for $0^+ \rightarrow 0^+$ transition have been calculated with these wave functions. Moreover, we have studied the effect of deformation on nuclear transition matrix element $M_{2\nu}$.

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*Electronic address: ashukla@phy.iitkgp.ernet.in
I. INTRODUCTION

The nuclear double beta ($\beta\beta$) decay, one of the rarest processes of the nature, is characterized by two modes. They are the two neutrino double beta ($2\nu \beta\beta$) decay and the neutrinoless double beta ($0\nu \beta\beta$) decay. These modes can be classified into double electron ($\beta^-\beta^-$) emission, double positron ($\beta^+\beta^+$) emission, electron-positron conversion ($\beta^+EC$) and double electron capture ($ECEC$). The later three processes are energetically competing and we shall refer to them as positron double beta decay ($e^+DBD$) modes. If the $0\nu \beta\beta$ decay is observed, the $e^+DBD$ processes would play a crucial role in discriminating the finer issues like dominance of Majorana neutrino mass or the right handed current. The theoretical implications and experimental aspects of $e^+DBD$ modes have been widely reviewed over the past years [1-8].

The half-lives of many $\beta^-\beta^-$ emitters are shorter, compared with the other modes, due to a larger available phase space. For this reason they were the natural choice for the experimental observation to start with. However, the experimental sensitivity of $\beta^-\beta^-$ decay mode gets limited because of the presence of electron background. On the other hand, from the experimental point of view, the $e^+DBD$ modes are relatively easier to be separated from the background contaminations. Moreover, the $e^+DBD$ modes are also attractive due to the possibility to detect the coincidence signals from four $\gamma$-rays, two $\gamma$-rays and one $\gamma$-ray for $\beta^+\beta^+$, $\beta^+EC$, $ECEC$ modes respectively. The $Q$-value for the $2\nu ECEC$ mode can be large enough (up to 2.8 MeV) but the detection of the $0^+ \rightarrow 0^+$ transition is difficult since only X-rays are emitted.

There have been very few experimental attempts for determining the half-lives of $2\nu e^+DBD$ modes even for the best candidate $^{106}$Cd [9-16] but one of the latest observations is very close to the predictions for $\beta^+EC$ mode [14]. With improved sensitivity in detection systems of the planned bigger Osaka-OTO experiment [17], it is expected that $2\nu e^+DBD$ modes will be in observable range in near future [18]. Hence, a timely reliable prediction of the half-life of $^{106}$Cd decay will be helpful in designing of an experimental set up and analysis of data.

Rosen and Primakoff were the first to study the $2\nu e^+DBD$ modes theoretically [2]. Later on, Kim and Kubodera estimated the half-lives of all the three modes with modified nuclear transition matrix elements (NTMEs) and non-relativistic phase space factors [19]. Abad et
al. performed similar calculations using relativistic Coulomb wave functions [20]. In the meantime, the QRPA emerged as the most successful model in explaining the quenching of NTMEs by incorporating the particle-particle part of the effective nucleon-nucleon interaction in the proton-neutron channel and the observed $T^{2\nu}_{1/2}$ of several $2\nu \beta \beta$ decay emitters were reproduced successfully [6]. Staudt et al. used QRPA model for evaluating $2\nu \beta^+ \beta^+$ decay transition matrix elements [21]. Subsequently, the $2\nu e^+\text{DBD}$ modes were studied in QRPA and its extensions [13, 22-26], SU(4)$_{\sigma \tau}$ [27] and SSDH [28].

A vast amount of data concerning the level energies as well as electromagnetic properties have been compiled through experimental studies involving in-beam $\gamma$-ray spectroscopy over the past years. Hence, there is no need to study the $\beta\beta$ decay as an isolated nuclear process. The availability of data permits a rigorous and detailed critique of the ingredients of the microscopic framework that seeks to provide a description of nuclear $\beta\beta$ decay. However, most of the calculations of $e^+\text{DBD}$ transition matrix elements performed so far but for the work of Barabash et al. [13] and Suhonen et al. [25] do not fully satisfy this criterion.

The nuclear structure in the mass region $A=100$ offers a nice example of shape transition i.e. sudden onset of deformation at neutron number $N=60$. The nuclei are soft vibrators for $N < 60$ and quasi rotors for $N > 60$. The nuclei with neutron number $N=60$ are transitional nuclei. Hence, it is expected that deformation degrees of freedom will play some crucial role in the structure of $^{106}\text{Pd}$ and $^{106}\text{Cd}$ nuclei. Further, the pairing of like nucleons plays an important role in all $\beta\beta$ decay emitters, which are even-$Z$ and even-$N$ nuclei. Hence, it is desirable to have a framework for the study of $\beta\beta$ decay in which the pairing and deformation degree of freedom are treated on equal footing in its formalism. The Projected Hartree-Fock-Bogoliubov (PHFB) model is a very reasonable choice which fulfills these requirements. The successful study of shape transition vis-à-vis electromagnetic properties of various nuclei in PHFB model [29-32] using pairing plus quadrupole-quadrupole (PPQQ) [33] interaction motivates us to apply the HFB wave functions to study the $2\nu e^+\text{DBD}$ modes of $^{106}\text{Cd}$.

Further, it has been shown that there exists an inverse correlation between the Gamow-Teller strength and the quadrupole moment [34, 35]. It is well known that the pairing degree of freedom accounts for the preference of nuclei to have a spherical form, whereas the quadrupole-quadrupole ($QQ$) interaction increases the collectivity in the nuclear intrinsic wave functions and makes the nucleus deformed. Hence, the PHFB model using the PPQQ interaction is a convenient choice to examine the explicit role of deformation on NTME $M_{2\nu}$. 


Our aim is to study the $2\nu e^+DBD$ transition of $^{106}$Cd $\rightarrow$ $^{106}$Pd for $0^+ \rightarrow 0^+$ transition together with other observed nuclear properties using the PHFB model. In PHFB model, the NTME $M_{2\nu}$ is usually calculated using the closure approximation. In the present calculation, we have avoided the closure approximation by making use of the summation method \[36\].

In Sec. II, we briefly outline the theoretical framework. In Sec. III, the reliability of the wave functions is first established by calculating the yrast spectra, reduced $B(E2 : 0^+ \rightarrow 2^+)$ transition probabilities, static quadrupole moments $Q(2^+)$ and $g$-factors $g(2^+)$ of both parent $^{106}$Cd and daughter $^{106}$Pd nuclei and by comparing them with the available experimental data. The half-lives of $2\nu e^+DBD$ modes for $0^+ \rightarrow 0^+$ transition have been given as prediction. The role of deformation on NTME $M_{2\nu}$ has also been studied. We present the conclusions in Sec. IV.

II. THEORETICAL FRAMEWORK

The theoretical formalism to calculate the half-lives of $2\nu e^+DBD$ modes has been given by Doi et al. [5] and Suhonen et al. [6]. Hence, we briefly outline steps of the above derivations for clarity in notation following Doi et al. [5]. Details of the mathematical expressions used to calculate electromagnetic properties are given by Dixit et al. [37].

The half-life of the $2\nu e^+DBD$ mode for the $0^+ \rightarrow 0^+$ transition is given by

$$\left[ T_{1/2}^{2\nu}(0^+ \rightarrow 0^+) \right]^{-1} = G_{2\nu} |M_{2\nu}|^2 \quad (2.1)$$

where the integrated kinematical factor $G_{2\nu}$ can be calculated with good accuracy [5] and the NTME $M_{2\nu}$ is given by

$$M_{2\nu} = \sum_N \frac{\langle 0_F^+ |\sigma_- | 1_N^+ \rangle \langle 1_N^+ |\sigma_- | 0_I^+ \rangle}{E_N - (E_I + E_F)/2} \quad (2.2)$$

$$= \sum_N \frac{\langle 0_F^+ |\sigma_- | 1_N^+ \rangle \langle 1_N^+ |\sigma_- | 0_I^+ \rangle}{E_0 + E_N - E_I} \quad (2.3)$$

where

$$E_0 = \frac{1}{2} (E_I - E_F)$$

$$= \frac{1}{2} Q_{\beta\beta} + m_e = \frac{1}{2} W_0 \quad (2.4)$$
Here, $W_0$ is the total energy released and is given by

$$W_0 = E_I - E_F \quad (2.5)$$

$$W_0(\beta^+ \beta^+) = Q_{\beta^+ \beta^+} + 2m_e \quad (2.6)$$

$$W_0(\beta^+ EC) = Q_{\beta^+ EC} + e_b \quad (2.7)$$

$$W_0(EC EC) = Q_{EC EC} - 2m_e + e_{b1} + e_{b2} \quad (2.8)$$

The summation over intermediate states can be completed using the summation method and the $M_{2\nu}$ can be written as

$$M_{2\nu} = \frac{1}{E_0} \left\langle 0^+_F \left| \sum_m (-1)^m \Gamma_{-m} F_m \right| 0^+_I \right\rangle \quad (2.9)$$

where the Gamow-Teller (GT) operator $\Gamma_m$ is given by

$$\Gamma_m = \sum_s \sigma ms \tau_s^- \quad (2.10)$$

and

$$F_m = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{E_0^\lambda} D\lambda \Gamma_m \quad (2.11)$$

with

$$D\lambda \Gamma_m = [H, [H, ........, [H, \Gamma_m] ........]^{(\lambda \text{ times})} \quad (2.12)$$

In the present work, we use a Hamiltonian with PPQQ type of effective two-body interaction. Explicitly, the Hamiltonian is written as

$$H = H_{sp} + V(P) + \chi_{qq} V(QQ) \quad (2.13)$$

where $H_{sp}$ denotes the single particle Hamiltonian. The pairing part of the effective two-body interaction $V(P)$ is written as

$$V(P) = - \left( \frac{G}{4} \right) \sum_{\alpha\beta} (-1)^{j_\alpha + j_\beta - m_\alpha - m_\beta} a^\dagger_\alpha a^\dagger_\beta a_\beta a_\alpha \quad (2.14)$$

where $\alpha$ denotes the quantum numbers $(nljm)$. The state $\bar{\alpha}$ is same as $\alpha$ but with the sign of $m$ reversed. The $QQ$ part of the effective interaction $V(QQ)$ is given by

$$V(QQ) = - \left( \frac{\chi}{2} \right) \sum_{\alpha,\beta,\gamma,\delta} \sum_\mu (-1)^\mu \langle \alpha | q^2_\mu | \gamma \rangle \langle \beta | q^2_{-\mu} | \delta \rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma \quad (2.15)$$
where
\[ q_\mu^2 = \left( \frac{16\pi}{5} \right)^{1/2} r^2 Y_\mu^2(\theta, \phi) \]  

(2.16)

The \( \chi_{qq} \) is an arbitrary parameter and the final results are obtained by setting the \( \chi_{qq} = 1 \). The purpose of introducing \( \chi_{qq} \) is to study the role of deformation by varying the strength of \( QQ \) interaction.

When the GT operator commutes with the effective two-body interaction, the Eq. (2.12) can be further simplified to

\[ M_{2\nu} = \sum_{\pi, \nu} \frac{\langle 0^+_F || \sigma \cdot \sigma \tau^- \tau^- || 0^+_I \rangle}{E_0 + \varepsilon(n_\nu, l_\nu, j_\nu) - \varepsilon(n_\pi, l_\pi, j_\pi)} \]  

(2.17)

In the case of pseudo SU(3) model [38-40], the GT operator commutes with the two-body interaction and the energy denominator is a well-defined quantity without any free parameter. It has been evaluated exactly for \( 2\nu \beta^- \beta^- \) [38, 39] and \( 2\nu ECEC \) modes [40] in the context of pseudo SU(3) scheme. However, in the present case, the model Hamiltonian is not isospin symmetric. Hence, the energy denominator is not as simple as in Eq. (2.17). But the violation of isospin symmetry for the QQ part of our model Hamiltonian is negligible, as will be evident from the parameters of the two-body interaction given later and the violation of isospin symmetry for the pairing part of the two-body interaction is presumably small. Under these assumptions, the expression to calculate the NTME \( M_{2\nu} \) of \( e^+ \text{DBD} \) modes for \( 0^+ \rightarrow 0^+ \) transition in PHFB model is obtained as follows.

The essential idea behind the HFB theory is to transform particle coordinates to quasiparticle coordinates through general Bogoliubov transformation such that the quasiparticles are relatively weakly interacting. Essentially, the Hamiltonian \( H \) is expressed as

\[ H = E_0 + H_{qp} + H_{qp-int} \]  

(2.18)

where \( E_0 \) is the energy of the quasiparticle vacuum, \( H_{qp} \) is the elementary quasiparticle excitations and \( H_{qp-int} \) is a weak interaction between the quasiparticles. In HFB theory, the interaction between the quasiparticles is usually neglected and the hamiltonian \( H \) is approximated by an independent quasiparticle hamiltonian. In time dependent HFB (TDHFB) or the quasiparticle random phase approximation (QRPA), some effects of quasiparticle interaction can be included. The axially symmetric intrinsic HFB state with \( K=0 \) can be written
as

\[ |\Phi_0\rangle = \prod_{im} (u_{im} + v_{im} b_{im}^\dagger b_{im}^\dagger) |0\rangle \quad (2.19) \]

where the creation operators \( b_{im}^\dagger \) and \( b_{im}^\dagger \) are given by

\[
b_{im}^\dagger = \sum_\alpha C_{i\alpha,m} a_{\alpha}^\dagger \quad \text{and} \quad b_{im}^\dagger = \sum_\alpha (-1)^{l_j-m} C_{i\alpha,m} a_{\alpha,-m} \quad (2.20)
\]

Using the standard projection technique, a state with good angular momentum \( J \) is obtained from the HFB intrinsic state through the following relation.

\[
|\Psi^J_{MK}\rangle = P^J_{MK}|\Phi^K\rangle = \left[ \frac{(2J+1)}{8\pi^2} \right] \int D^J_{MK}(|\Phi^K\rangle \, d\Omega
\]

where \( R(\Omega) \) and \( D^J_{MK}(\Omega) \) are the rotation operator and the rotation matrix respectively.

Finally, one obtains the following expression for NTMEs of e\(^+\)DBD modes

\[
M_{2\nu} = \sum_{\pi,\nu} \frac{\langle \Psi^{J=0}_{00} | \sigma \cdot \sigma_{\tau^-} \tau^- | \Psi^{J=0}_{00} \rangle}{E_0 + \varepsilon(n_{\nu}, l_{\nu}, j_{\nu}) - \varepsilon(n_{\pi}, l_{\pi}, j_{\pi})}
\]

\[
= \left[ n_{Z-2,N+2}^{J=0} n_{Z,N}^{J=0} \right]^{-1/2} \times \sum_{\alpha\beta\gamma\delta} \frac{\langle \sigma_{1} \sigma_{2} \tau^- \tau^- \rangle \gamma \delta}{E_0 + \varepsilon_{\alpha}(n_{\nu}, l_{\nu}, j_{\nu}) - \varepsilon_{\gamma}(n_{\pi}, l_{\pi}, j_{\pi})} \sum_{\varepsilon\eta} \left[ (1 + F^{(\nu)}_{Z,N}(\theta) F^{(\pi)}_{Z-2,N+2}) \right]^{-1} (F^{(\nu)*}_{Z-2,N+2}) \varepsilon_{\beta} \gamma_{\eta} \sin \theta \, d\theta
\]

where

\[
n^{J} = \int_0^\pi \{ \det[1 + F^{(\pi)}(\theta) F^{(\pi)*}] \}^{1/2} \times \{ \det[1 + F^{(\nu)}(\theta) F^{(\nu)*}] \}^{1/2} \, d\theta \sin(\theta) \, d\theta \quad (2.23)
\]

and

\[
n_{(Z,N),(Z-2,N+2)}(\theta) = \{ \det[1 + F^{(\pi)}_{Z,N}(\theta) F^{(\pi)*}_{Z-2,N+2}] \}^{1/2} \times \{ \det[1 + F^{(\nu)}_{Z,N}(\theta) F^{(\nu)*}_{Z-2,N+2}] \}^{1/2} \quad (2.24)
\]
with
\[
[F_{Z,N}(\theta)]_{\alpha\beta} = \sum_{m'_\alpha, m'_\beta} d^{m'_\alpha}_{m_\alpha, m'_\alpha}(\theta) d^{m'_\beta}_{m_\beta, m'_\beta}(\theta) f_{\alpha,m'_\alpha,\beta,m'_\beta} \tag{2.25}
\]
and
\[
[f_{Z,N}]_{\alpha\beta} = \sum_i C_{ij,ma} C_{ij,mb} \left( v_{ima}/u_{ima} \right) \delta_{ma,-m_b} \tag{2.26}
\]
Here \(\pi(\nu)\) stands for the proton (neutron) of nuclei involved in \(2\nu e^+DBD\). The results of PHFB calculations which are summarized by the amplitudes \((u_{ima}, v_{ima})\) and the expansion coefficients \(C_{ij,m}\) are used to setup the matrices for \([F_{Z,N}(\theta)]_{\alpha\beta}\) and \([f_{Z,N}]_{\alpha\beta}\) given by Eqs. (2.25) and (2.26) respectively. Finally, the required NTME \(M_{2\nu}\) is calculated in a straightforward manner using Eq. (2.22) with 20 point gaussian quadrature points in the range \((0, \pi)\).

III. RESULTS AND DISCUSSIONS

The model space, single particle energies (SPE’s) and two-body interactions are same as our earlier calculation on \(2\nu \beta\beta\) decay of \(^{100}\)Mo for \(0^+ \to 0^+\) transition \([37]\). We include a brief discussion of them in the following for convenience. We have treated the doubly even nucleus \(^{76}\)Sr (\(N=Z=38\)) as an inert core with the valence space spanned by the orbits \(1p_{1/2}, 2s_{1/2}, 1d_{3/2}, 1d_{5/2}, 0g_{7/2}, 0g_{9/2}\) and \(0h_{11/2}\) for protons and neutrons. The orbit \(1p_{1/2}\) has been included in the valence space to examine the role of the \(Z=40\) proton core vis-a-vis the onset of deformation in the highly neutron rich isotopes.

The set of single particle energies (SPE’s) used here are (in MeV) \(\varepsilon(1p_{1/2})=-0.8, \varepsilon(0g_{9/2})=0.0, \varepsilon(1d_{5/2})=5.4, \varepsilon(2s_{1/2})=6.4, \varepsilon(1d_{3/2})=7.9, \varepsilon(0g_{7/2})=8.4\) and \(\varepsilon(0h_{11/2})=8.6\) for proton and neutrons. This set of SPE’s but for the \(\varepsilon(0h_{11/2})\), which is slightly lowered, has been employed in a number of successful shell model \([41, 42]\) as well as variational model \([29-32]\) calculations for nuclear properties in the mass region \(A=100\). The strengths of the pairing interaction is fixed through the relation \(G_p=30/A \) MeV and \(G_n=20/A \) MeV, which are same as used by Heestand et al. \([43]\) to explain the experimental \(g(2^+\nu)\) data of some even-even Ge, Se, Mo, Ru, Pd, Cd and Te isotopes in Greiner’s collective model \([44]\). The strengths of the like particle components of the \(QQ\) interaction are taken as: \(\chi_{pp} = \chi_{nn} = \)
0.0105 MeV $b^{-4}$, where $b$ is oscillator parameter. The strength of proton-neutron ($pn$) component of the $QQ$ interaction $\chi_{pn}$ is varied so as to reproduce the experimentally observed excitation energy of the $2^+$ state $E_{2^+}$ of $^{106}$Cd and $^{106}$Pd as closely as possible. The $\chi_{pn}$ has been fixed to be 0.0151 and 0.0145 MeV $b^{-4}$ for $^{106}$Cd and $^{106}$Pd respectively. Thus for a given model space, SPE’s, $G_p$, $G_n$ and $\chi_{pp}$, we have fixed $\chi_{pn}$ through the experimentally available energy spectra. These values for the strength of the $QQ$ interaction are comparable to those suggested by Arima on the basis of an empirical analysis of the effective two-body interactions [45].

We have varied the $\chi_{pn}$ to obtain the yrast spectra of $^{106}$Cd and $^{106}$Pd in optimum agreement with experimental results [46]. We have taken the theoretical spectra to be the optimum if the excitation energy of the $2^+$ state $E_{2^+}$ is reproduced as closely as possible in comparison to the experimental results. Theoretically calculated intrinsic quadrupole moments $\langle Q^2_0 \rangle$ and yrast energies for the $E_{2^+}$ to $E_{6^+}$ levels of $^{106}$Cd and $^{106}$Pd for $\chi_{pn} = 0.0142$ to 0.0154 are presented in Table I. In the case of $^{106}$Cd, the $\langle Q^2_0 \rangle$ increases by 5.6267 units and the $E_{2^+}$ decreases by 0.1880 MeV as the $\chi_{pn}$ is varied from 0.0142 to 0.0154 MeV $b^{-4}$. For the same variation in $\chi_{pn}$, the $\langle Q^2_0 \rangle$ increases by 3.7314 units and the $E_{2^+}$ decreases by 0.1109 MeV in the case of $^{106}$Pd. This observed inverse correlation between $\langle Q^2_0 \rangle$ and $E_{2^+}$ is understandable as there is an enhancement in the collectivity of the intrinsic state with the increase of $|\chi_{pn}|$, the $E_{2^+}$ decreases. This is known as Grodzins’s rule [47]. The theoretically calculated $E_{2^+}$ for $^{106}$Cd is 0.6220 MeV corresponding to $\chi_{pn} = 0.0151$ MeV $b^{-4}$ in comparison to the experimentally observed value 0.6327 MeV. In case of $^{106}$Pd, the theoretically calculated $E_{2^+}$ for $\chi_{pn} = 0.0145$ MeV $b^{-4}$ is 0.5036 MeV in comparison to the observed value of 0.5119 MeV. All these input parameters are kept fixed for calculation of spectroscopic properties as well as the NTMEs discussed below.

The calculated as well as the experimentally observed values of the reduced $B(E2:0^+ \rightarrow 2^+)$ transition probabilities, static quadrupole moments $Q(2^+)$, and the gyromagnetic factors $g(2^+)$ have been presented in Table II. We have calculated $B(E2)$ values for effective charges $e_{eff} = 0.40$, 0.50, and 0.60, which are displayed in columns 2 to 4, respectively. The experimentally observed values are displayed in column 5. It is noticed that the calculated and the observed $B(E2)$ [48] values are in excellent agreement for $e_{eff} = 0.5$. The theoretically calculated $Q(2^+)$ are tabulated in columns 6 to 8 for the same effective charges as given above. The experimental $Q(2^+)$ results [49] are given in column 9. It can be seen that for
the same effective charge 0.5, the calculated values are close to the experimental limit in case of $^{106}$Pd while the agreement between the calculated and experimental values is off for $^{106}$Cd. The $g(2^+)$ values are calculated with $g_1^\pi = 1.0$, $g_1^\nu = 0.0$, and $g_s^\pi = g_s^\nu = 0.60$. The calculated $g(2^+)$ is 0.370 nm and 0.466 nm for $^{106}$Cd and $^{106}$Pd respectively. The theoretically calculated and experimentally observed $g(2^+)$ values are in good agreement for $^{106}$Cd and slightly off by 0.047 nm for $^{106}$Pd from the upper limit given by Raghavan [49]. The overall agreement between the calculated and observed electromagnetic properties of $^{106}$Cd and $^{106}$Pd suggests that the PHFB wave functions generated by fixing $\chi_{pn}$ to reproduce the yrast spectra are quite reliable.

The $2\nu e^+\mathrm{DBD}$ of $^{106}$Cd for the $0^+ \rightarrow 0^+$ transition has been investigated by very few experimental groups, whereas some theoretical investigations have been made using the QRPA and its extensions [13,21-26], SU(4)$_{\sigma\tau}$ [27] and SSDH [28]. In Table III, we have compiled all the available experimental [9-16] and theoretical results [13,21-28] along with our calculated $M_{2\nu}$ and corresponding half-life $T_{1/2}^{2\nu}$. We have used phase space factors $G_{2\nu} = 4.991 \times 10^{-26}$ yr$^{-1}$, $1.970 \times 10^{-21}$ yr$^{-1}$ and $1.573 \times 10^{-20}$ yr$^{-1}$ for $2\nu \beta^+\beta^+$, $2\nu \beta^+\mathrm{EC}$ and $2\nu \mathrm{ECEC}$ modes respectively as given by Doi et al. [5]. The phase space integral has been evaluated for $g_A = 1.261$ by Doi et al. [5]. However, in heavy nuclei it is more justified to use the nuclear matter value of $g_A$ around 1.0. Hence, the theoretical $T_{1/2}^{2\nu}$ are calculated for $g_A = 1.0$ and 1.261. We have presented only the theoretical $T_{1/2}^{2\nu}$ for those models for which no direct or indirect information about $M_{2\nu}$ or $G_{2\nu}$ is available to us.

In column 3 of Table III, we have presented the experimentally observed limits on half-lives $T_{1/2}^{2\nu}$. In comparison to the theoretically predicted $T_{1/2}^{2\nu}$, the present experimental limits for $0^+ \rightarrow 0^+$ transition of $^{106}$Cd are smaller by a factor of $10^5-7$ in case of $2\nu \beta^+\beta^+$ mode but are quite close for $2\nu \beta^+\mathrm{EC}$ and $2\nu \mathrm{ECEC}$ modes. The half-life $T_{1/2}^{2\nu}$ calculated in PHFB model using the summation method differs from all the existing calculations. The presently calculated NTME $M_{2\nu}$ is smaller than the recently given results in QRPA(WS) model of Suhonen and Civitarese [25] by a factor of 2 approximately for all the three modes. The theoretical $M_{2\nu}$ values of PHFB model and SU(4)$_{\sigma\tau}$ [27] again differ by a factor of 2 approximately for the $2\nu \beta^+\mathrm{EC}$ and $2\nu \mathrm{ECEC}$ modes. On the other hand, the $M_{2\nu}$ calculated in our PHFB model is smaller than the values of Hirsch et al. [23] by a factor of 3 approximately in case of $2\nu \beta^+\beta^+$ and $2\nu \mathrm{ECEC}$ modes while for $2\nu \beta^+\mathrm{EC}$ mode the results differ by a factor of 4 approximately. All the rest of the calculations predict NTMEs,
which are larger than our predicted $M_{2\nu}$ approximately by a factor of 7 to 10.

We have studied the role of deformation on $\langle Q_0^2 \rangle$ and $M_{2\nu}$ vis-a-vis the variation of the strength of $pn$ part of the $QQ$ interaction $\chi_{qq}$. The results are tabulated in Table IV. The $\langle Q_0^2 \rangle$ of $^{106}$Cd and $^{106}$Pd remain almost constant as the $\chi_{qq}$ is varied from 0.0 to 0.80. The $M_{2\nu}$ also remains almost constant as the $\chi_{qq}$ is changed from 0.0 to 0.80. As the $\chi_{qq}$ is further changed from 0.80 to 1.20, the $\langle Q_0^2 \rangle$ increases while the $M_{2\nu}$ decreases to 0.0417 having a fluctuation at 1.05. To quantify the effect of deformation on $M_{2\nu}$, we define a quantity $D_{2\nu}$ as the ratio of $M_{2\nu}$ at zero deformation ($\chi_{qq} = 0$) and full deformation ($\chi_{qq} = 1$). The $D_{2\nu}$ is given by

$$D_{2\nu} = \frac{M_{2\nu}(\chi_{qq} = 0)}{M_{2\nu}(\chi_{qq} = 1)}.$$  \hspace{1cm} (3.1)

The value of $D_{2\nu}$ is 2.09, which suggests that the $M_{2\nu}$ is quenched by a factor of approximately 2 due to deformation effects.

It is evident from the above discussions that it is difficult to establish the validity of different nuclear models presently employed to study $2\nu e^+DBD$ due to limiting values in experimental results as well as uncertainty in $g_A$. Further work is necessary both in the experimental as well as the theoretical front to judge the relative applicability, success and failure of various models used so far for the study of $2\nu e^+DBD$ processes before they can have better predictive power for the $0\nu e^+DBD$ modes.

IV. CONCLUSIONS

We have tested the quality of HFB wave functions by comparing the theoretically calculated results for a number of spectroscopic properties namely yrast spectra, reduced $B(E2:0^+ \rightarrow 2^+)$ transition probabilities, quadrupole moments $Q(2^+)$ and $g$-factors $g(2^+)$ of $^{106}$Cd and $^{106}$Pd with the available experimental data. The same HFB wave functions are employed to calculate the NTME $M_{2\nu}$ and the half-life $T_{1/2}^{2\nu}$ of $^{106}$Cd for $2\nu \beta^+\beta^+$, $2\nu \beta^+EC$ and $2\nu ECEC$ modes. The values of $T_{1/2}^{2\nu}$ calculated in the PHFB model with the summation method are larger than the previous calculations. The presently calculated NTME $M_{2\nu}$ is smaller than the recently given results in QRPA(WS) model of Suhonen and Civitarese [25] by a factor of 2 approximately for all the three modes. The proton-neutron part of the PPQQ interaction that reflects the deformations of intrinsic ground state, plays an important role in the quenching of $M_{2\nu}$ by a factor of 2 approximately in this particular case. The
calculated $2\nu e^+DBD$ decay half-lives are very close to the experimentally observable limits for $2\nu \beta^+EC$ and $2\nu ECEC$ modes. It is hoped that the calculated $T_{1/2}^{2\nu}$, which is of the order of $10^{21-23}$ yrs can be reached experimentally for $2\nu \beta^+EC$ mode in near future [14].

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TABLE I: Variation in intrinsic quadrupole moment $\langle Q^2_0 \rangle$ and excitation energies (in MeV) of $J^\pi =2^+, 4^+$, and $6^+$ yrast states of $^{106}\text{Cd}$ and $^{106}\text{Pd}$ nuclei with change in $\chi_{pn}$, keeping fixed $G_p=30/A$ MeV, $G_n=20/A$ MeV, $\chi_{pp} = \chi_{nn} =0.0105$ MeV $b^{-4}$ and $\varepsilon(0h_{11/2}) = 8.6$ MeV.

| Nucleus | $\chi_{pn}$ | Theo. $\langle Q^2_0 \rangle$ | Theo. $E_{2^+}$ | Theo. $E_{4^+}$ | Theo. $E_{6^+}$ | Expt. $[46]$ $\langle Q^2_0 \rangle$ | Expt. $E_{2^+}$ | Expt. $E_{4^+}$ | Expt. $E_{6^+}$ |
|---------|-------------|-------------------------------|-----------------|-----------------|-----------------|---------------------------------|----------------|----------------|----------------|
| $^{106}\text{Cd}$ | 0.0142 | 43.3772 | 0.7749 | 1.9024 | 3.2993 | 44.7355 | 0.7339 | 1.8728 | 3.0302 |
| | 0.0145 | 46.0289 | 0.6797 | 1.8022 | 3.2389 | 47.3807 | 0.5869 | 1.6690 | 3.1411 |
| | 0.0148 | 49.0039 | 0.6220 | 1.7129 | 3.1089 | 49.0039 | 0.6327 | 1.4939 |
| | 0.0151 | | | | | | | |
| | 0.0154 | | | | | | | |
| $^{106}\text{Pd}$ | | 51.4360 | 0.5524 | 1.5706 | 2.8526 | 52.4295 | 0.4819 | 1.4269 | 2.6655 |
| | | 53.4325 | 0.4500 | 1.3554 | 2.5652 | 54.2709 | 0.4415 | 1.3435 | 2.5620 |
| | | 55.1674 | 0.5119 | 1.2292 | | | | | |

TABLE II: Comparison of the calculated and experimentally observed reduced transition probabilities $B(E2:0^+ \rightarrow 2^+)$, static quadrupole moments $Q(2^+)$ and $g$-factors $g(2^+)$. Here $B(E2)$ and $Q(2^+)$ are calculated in units of $e^2 b^2$ and $e b$, respectively for effective charge $e_p = 1+e_{eff}$ and $e_n = e_{eff}$. The $g(2^+)$ has been calculated in units of nuclear magneton for $g^{\pi}_{l}=1.0$, $g^{\nu}_{l}=0.0$ and $g^{\pi}_{s}=g^{\nu}_{s}=0.60$. Corresponding references for experimentally observed values are given in parentheses.

| Nucleus | $B(E2: 0^+ \rightarrow 2^+)$ Theo. | $Q(2^+)$ Theo. | $g(2^+)$ Theoretical | Expt. $[48]$ | $Q(2^+)$ Expt. $[49]$ | $g(2^+)$ Expt. $[49]$ | $e_{eff}$ | $e_{eff}$ |
|---------|----------------------------------|-----------------|----------------------|----------------|------------------------|----------------------|---------|---------|
| $^{106}\text{Cd}$ | 0.334 | 0.531 | 0.410±0.020 | 0.40 | 0.59 | 0.28±0.08 | 0.370 | 0.40±0.10 |
| | | | -0.52 |-0.66 |-0.28±0.08 | 0.370 | 0.40±0.10 |
| | | | 0.386±0.05 | |
| $^{106}\text{Pd}$ | 0.407 | 0.647 | 0.610±0.090 | 0.58 | 0.65 | 0.56±0.08 | 0.466 | 0.398±0.021 |
| | | | -0.52 |-0.73 |-0.56±0.08 | 0.466 | 0.398±0.021 |
| | | | 0.656±0.035 | | 0.51±0.08 | 0.30±0.06 | |
TABLE III: Experimental limits on half-lives $T_{1/2}^{2\nu}(0^+ \rightarrow 0^+)$, theoretically calculated $M_{2\nu}$ and corresponding $T_{1/2}^{2\nu}(0^+ \rightarrow 0^+)$ for $2\nu \beta^+\beta^+$, $2\nu \beta^+EC$ and $2\nu ECEC$ modes of $^{106}$Cd. The numbers corresponding to (a) and (b) are calculated for $g_A=1.261$ and 1.0 respectively.

| Decay | Experiment | Mode | Ref. | $T_{1/2}^{2\nu}$ (yr) | Ref. | Models | $|M_{2\nu}|$ | $T_{1/2}^{2\nu}$ (yr) |
|-------|------------|------|------|----------------------|------|--------|----------|-------------------|
| $\beta^+\beta^+$ | [15] | $> 5.0 \times 10^{18}$ | Present | PHFB | 0.081 | a) | $307.58 \times 10^{25}$ |
| | [14] | $> 2.4 \times 10^{20**}$ | | | | | | $777.71 \times 10^{25}$ |
| | [13] | $> 1.0 \times 10^{19*}$ | [26] | SQRPA(l.b.) | 0.61 | a) | 5.38$\times 10^{25}$ |
| | [12] | $> 9.2 \times 10^{17}$ | | | | | | 13.60$\times 10^{25}$ |
| | [10] | $> 5.0 \times 10^{17}$ | | SQRPA(s.b.) | 0.57 | a) | 6.16$\times 10^{25}$ |
| | [9] | $> 2.6 \times 10^{17*}$ | | | | | | 15.58$\times 10^{25}$ |
| | [25] | | | QRPA(WS) | 0.166 | a) | 72.71$\times 10^{25}$ | b) | 183.84$\times 10^{25}$ |
| | | | | QRPA(AWS) | 0.722 | a) | 3.84$\times 10^{25}$ | b) | 9.72$\times 10^{25}$ |
| | [13] | | | QRPA(WS) | 0.840 | a) | 2.84$\times 10^{25}$ | b) | 7.18$\times 10^{25}$ |
| | | | | QRPA(AWS) | 0.780 | a) | 3.29$\times 10^{25}$ | b) | 8.33$\times 10^{25}$ |
| | [23] | | | QRPA | 0.218 | a) | 42.2$\times 10^{25}$ | b) | 106.6$\times 10^{25}$ |
| | [21] | | | QRPA | | | | 4.94$\times 10^{25}$ |
| $\beta^+ EC$ | Present PHFB | $a)$ $77.925 \times 10^{21}$ | $b)$ $197.03 \times 10^{21}$ |
|-----------------|-------------|----------------------|----------------------|
| $[15]$ | $1.2 \times 10^{18}$ | 0.081 | |
| $[14]$ | $> 4.1 \times 10^{20}$ | | |
| $[13]$ | $> 0.66 \times 10^{19}$ | SQRPA(l.b.) 0.61 | a) $1.36 \times 10^{21}$ |
| $[12]$ | $> 2.6 \times 10^{17}$ | | b) $3.44 \times 10^{21}$ |
| $[9]$ | $> 5.7 \times 10^{17}$ | SQRPA(s.b.) 0.57 | a) $1.56 \times 10^{21}$ |
| | | | b) $3.94 \times 10^{21}$ |
| $[25]$ | QRPA(WS) 0.168 | a) $17.99 \times 10^{21}$ | b) $45.48 \times 10^{21}$ |
| | QRPA(AWS) 0.718 | a) $0.98 \times 10^{21}$ | b) $2.49 \times 10^{21}$ |
| $[27]$ | SU(4)$_{\sigma \tau}$ 0.1947 | a) $13.39 \times 10^{21}$ | b) $33.86 \times 10^{21}$ |
| $[24]$ | RQRPA(WS) 0.550 | a) $1.68 \times 10^{21}$ | b) $4.24 \times 10^{21}$ |
| $[24]$ | RQRPA(AWS) 0.560 | a) $1.62 \times 10^{21}$ | b) $4.09 \times 10^{21}$ |
| $[13]$ | QRPA(WS) 0.840 | a) $0.72 \times 10^{21}$ | b) $1.82 \times 10^{21}$ |
| | QRPA(AWS) 0.780 | a) $0.83 \times 10^{21}$ | b) $2.11 \times 10^{21}$ |
| $[23]$ | QRPA 0.352 | a) $4.1 \times 10^{21}$ | b) $10.4 \times 10^{21}$ |
| $[22]$ | QRPA(WS) 0.493-0.660 | a) $(2.09-1.16) \times 10^{21}$ | b) $(5.28-2.95) \times 10^{21}$ |
| Method          | Value  | Uncertainty | Half-life Limit |
|-----------------|--------|-------------|-----------------|
| $E_{\text{CEC}}$ | $5.8 \times 10^{17}$ | Present PHFB 0.081 | a) $97.593 \times 10^{20}$ |
| $E_{\text{CEC}}$ | $1.0 \times 10^{18}$ | b) $246.76 \times 10^{20}$ |
| $E_{\text{CEC}}$ | $5.8 \times 10^{17}$ | SQRPA(l.b.) 0.61 | a) $2.6 \times 10^{20}$ |
|                 |        | SQRPA(s.b.) 0.57 | b) $6.57 \times 10^{20}$ |
|                 |        | QRPA(WS) 0.168 | a) $22.52 \times 10^{20}$ |
|                 |        | QRPA(AWS) 0.718 | b) $56.95 \times 10^{20}$ |
|                 |        | SSDH(Theo) 0.280 | a) $8.11 \times 10^{20}$ |
|                 |        | SSDH(Exp) 0.170 | b) $20.50 \times 10^{20}$ |
|                 |        | SU(4) $\sigma T$ 0.1947 | a) $16.77 \times 10^{20}$ |
|                 |        | RQRPA(WS) 0.550 | b) $42.40 \times 10^{20}$ |
|                 |        | RQRPA(AWS) 0.560 | a) $2.03 \times 10^{20}$ |
|                 |        | QRPA(WS) 0.840 | b) $5.31 \times 10^{20}$ |
|                 |        | QRPA(AWS) 0.780 | a) $2.28 \times 10^{20}$ |
|                 |        | QRPA 0.270 | b) $1.05 \times 10^{20}$ |
|                 |        | QRPA(WS) 0.493-0.660 | a) $(2.62-1.46) \times 10^{20}$ |
|                 |        | b) $(6.61-3.69) \times 10^{20}$ |

* and ** denote half-life limit for $0\nu+2\nu$ and $0\nu+2\nu+0\nu\beta$ modes respectively.
| $\chi qq$ | $^{106}$Cd |          |          | $^{106}$Pd |          |          | $|M_{2\nu}|$ |
|---------|-----------|---------|---------|-----------|---------|---------|-----------|
|         | $\langle Q_0^2 \rangle_{\pi}$ | $\langle Q_0^2 \rangle_{\nu}$ | $\langle Q_0^2 \rangle$ | $\langle Q_0^2 \rangle_{\pi}$ | $\langle Q_0^2 \rangle_{\nu}$ | $\langle Q_0^2 \rangle$ |         |
| 0.00    | 0.0       | 0.0     | 0.0     | 0.0       | 0.0     | 0.0     | 0.1689    |
| 0.05    | -0.0025   | 0.0039  | 0.0015  | -0.0008   | 0.0057  | 0.0048  | 0.1709    |
| 0.20    | -0.0087   | 0.0169  | 0.0082  | 0.1067    | 0.2100  | 0.3168  | 0.1636    |
| 0.40    | -0.0100   | 0.0442  | 0.0342  | 0.0099    | 0.0701  | 0.0800  | 0.1624    |
| 0.60    | 0.0218    | 0.1261  | 0.1479  | 0.0483    | 0.1617  | 0.2100  | 0.1655    |
| 0.70    | 0.0683    | 0.2193  | 0.2876  | 0.0892    | 0.2455  | 0.3347  | 0.1682    |
| 0.80    | 0.1416    | 0.3594  | 0.5010  | 0.4521    | 0.8958  | 1.3479  | 0.1713    |
| 0.85    | 0.2227    | 0.5053  | 0.7280  | 11.2116   | 18.4734 | 29.6850 | 0.1432    |
| 0.90    | 11.63     | 20.0514 | 31.6814 | 15.0534   | 25.0116 | 40.0650 | 0.1218    |
| 0.95    | 14.9910   | 26.1956 | 41.1866 | 17.5444   | 29.7372 | 47.2816 | 0.0935    |
| 1.00    | 17.4655   | 29.9152 | 47.3807 | 19.2454   | 33.1840 | 52.4295 | 0.0807    |
| 1.05    | 22.3626   | 34.0604 | 56.4230 | 20.4735   | 35.9085 | 56.3820 | 0.0831    |
| 1.15    | 31.6509   | 38.4407 | 70.0915 | 22.9707   | 39.9774 | 62.9481 | 0.0638    |
| 1.20    | 33.9053   | 39.6519 | 73.5572 | 24.8922   | 41.7666 | 66.6589 | 0.0417    |