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Unfolding Prismatoids as Convex Patches: Counterexamples and Positive Results

Joseph O’Rourke*

Abstract

We address the unsolved problem of unfolding prismatoids in a new context, viewing a “topless prismatoid” as a convex patch—a polyhedral subset of the surface of a convex polyhedron homeomorphic to a disk. We show that several natural strategies for unfolding a prismatoid can fail, but obtain a positive result for “petal unfolding” topless prismatoids. We also show that the natural extension to a convex patch consisting of a face of a polyhedron and all its incident faces, does not always have a nonoverlapping petal unfolding. However, we obtain a positive result by excluding the problematical patches. This then leads to a positive result for restricted prismatoids. Finally, we suggest suggest studying the unfolding of convex patches in general, and offer some possible lines of investigation.

1 Introduction

A prismatoid is the convex hull of two convex polygons $A$ (above) and $B$ (base) in parallel planes. Despite its simple structure, it remains unknown whether or not every prismatoid has a nonoverlapping edge unfolding, a narrow special case of what has become known as Dürer’s Problem: whether every convex polyhedron has a nonoverlapping edge unfolding [DO07, Prob. 21,1, p. 300]. (All polyhedra considered here are convex polyhedra, and we will henceforth drop the modifier “convex,” and consistently use the symbol $P$; we will also use unfolding to mean “nonoverlapping edge unfolding.”) Motivated by the apparent difficulty of placing the top in an unfolding, we explore unfolding topless prismatoids, those with the top $A$ removed. We show that several natural approaches fail, but that a somewhat complex algorithm does succeed in unfolding any topless prismatoid.

This success suggests studying the unfolding of a convex patch more generally: a connected subset of faces of a polyhedron $P$, homeomorphic to a disk. A natural convex patch is an extension of a class studied by Pincu [Pin07]. He proved that the patch that consists of one face $B$ of $P$ and every face that shares an edge with $B$, has a “petal unfolding” (defined below). This edge-neighborhood of a face is itself a natural generalization of “domes,” earlier proven to have a petal unfolding [DO07, p. 323ff]. The generalization we explore is the vertex-neighborhood of a face: $B$ together with every face that shares at least a vertex with $B$. We show that not every vertex-neighborhood patch has a petal unfolding. Note that every topless prismatoid is a vertex-neighborhood of its base. This negative result suggests a restriction that permits unfolding: if $P$ is nonobtusely triangulated, then the vertex-neighborhood of any face does have a petal unfolding. This in turn leads to a proof that triangular prismatoids (top included), composed of nonobtuse triangles, have an unfolding.

Finally, we make a few observations and conjectures about unfolding arbitrary convex patches.

1.1 Band and Petal Unfoldings

There are two natural unfoldings of a prismatoid. A band unfolding cuts one lateral edge and unfolds all lateral faces as a unit, called a band, leaving $A$ and $B$ attached each by one uncut edge to opposite sides of the band (see, e.g., [ADL+07]). Aloupis showed that the lateral edge can be chosen so that band alone unfolds [Alo05], but I showed that, nevertheless, there are prismatoids such that every band unfolding overlaps [O’R07]. The example will be repeated here, as it plays a role in Sec. 4.

The prismatoid with no band unfolding is shown in Fig. 1. The possible band unfoldings are shown in the Appendix, Figs. 16 and 17. Note that this example also establishes that not every topless prismatoid has a band unfolding, simply by interchanging the roles of $A$ and $B$.

The second natural unfolding is a petal unfolding, called a “volcano unfolding” in [DO07, p. 321]. Because Fig. 1 without its base is a edge-neighborhood patch, it can be petal-unfolded by Pincu’s result [Pin07] as noted above: simply cut each lateral edge $a_i b_i$.

Let $P$ be a prismatoid, and assume all lateral faces are triangles, the generic and seemingly most difficult case. Let $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$. Call a lateral face that shares an edge with $B$ a base or $B$-triangle, and a lateral face that shares an edge with $A$ a top or
A triangle. A petal unfolding cuts no edge of \(B\), and unfolds every base triangle by rotating it around its \(B\)-edge into the base plane. The collection of \(A\)-triangles incident to the same \(b_i\) vertex—the \(A\)-fan \(\text{AF}_i\)—must be partitioned into two groups, one of which rotates clockwise (cw) to join with the unfolded base triangle to its left, and the other group rotating counterclockwise (ccw) to join with the unfolded base triangle to its right. Either group could be empty. Finally, the top \(A\) is attached to one \(A\)-triangle. So a petal unfolding has choices for how to arrange the \(A\)-triangles, and which \(A\)-triangle connects to the top. See Fig. 15 in the Appendix. The next section presents the main result: every topless prismatoid has a petal unfolding.

2 Topless Prismatoid Petal Unfolding

An example of a petal unfolding of a topless prismatoid is shown in Fig. 2. Even topless prismatoids present challenges. For example, consider the special case when there is only one \(A\)-triangle between every two \(B\)-triangles. Then the only choice for placement of the \(A\)-triangles is whether to turn each ccw or cw. It is natural to hope that rotating all \(A\)-triangles consistently ccw or cw suffices to avoid overlap, but this can fail, as in Fig. 18, and even for triangular prismatoids, Fig. 19 in the Appendix. A more nuanced approach would turn each \(A\)-triangle so that its (at most one) obtuse angle is not joined to a \(B\)-triangle (resolving Fig. 19), but this can fail also, a claim I will not substantiate.

The proof follows this outline:

1. An “altitudes partition” of the plane exterior to the base unfolding \((B\) plus all \(B\)-triangles) is defined and proved to be a partition.
2. It is shown that both \(\mathcal{P}\) and this partition vary in a natural manner with respect to the separation \(z\) between the \(A\)- and \(B\)-planes.
3. An algorithm is detailed for petal unfolding the \(A\)-triangles for the “flat prismatoid” \(\mathcal{P}(0)\), the limit of \(\mathcal{P}(z)\) as \(z \to 0\), such that these \(A\)-triangles fit inside the regions of the altitude partition.
4. It is proved that nesting within the partition regions remains true for all \(z\).

2.1 Altitude Partition

We use \(a_i\) and \(b_j\) to represent the vertices of \(\mathcal{P}\), and primes to indicate unfolded images on the base plane.

Let \(B_i = \triangle b_i b_{i+1} a'_i\) be the \(i\)-th base triangle. Say that \(BU = B \cup (\bigcup B_i)\) is the base unfolding, the unfolding of all the \(B\)-triangles arrayed around \(B\) in the plane, without any \(A\)-triangles. The altitude partition partitions the plane exterior to the base unfolding.

Let \(r_i\) be the altitude ray from \(a'_i\) along the altitude of \(B_i\). Finally, define \(R_i\) to be the region of the plane incident to \(b_i\), including the edges of the \(B_{i-1}\) and \(B_{i}\) triangles incident to \(b_i\), and bounded by \(r_{i-1}\) and \(r_i\). See Fig. 3.

Lemma 1 No pair of altitude rays cross in the base plane, and so they define a partition of that plane exterior to the base unfolding \(BU\).

\begin{proof}
See Sec. 5.1 in the Appendix. \end{proof}

Our goal is to show that the \(A\)-fan \(\text{AF}_i\) incident to \(b_i\) can be partitioned into two groups, one rotated cw, one ccw, so that both fit inside \(R_i\). (Note that this nesting is violated in Fig. 19 in the Appendix.)

2.2 Behavior of \(\mathcal{P}(z)\)

We will use “(z)” to indicate that a quantity varies with respect to the height \(z\) separating the \(A\)- and \(B\)-planes.
Lemma 2 Let $\mathcal{P}(z)$ be a prismatoid with height $z$. Then the combinatorial structure of $\mathcal{P}(z)$ is independent of $z$, i.e., raising or lowering $A$ above $B$ retains the convex hull structure.

Proof. See Sec. 5.1 in the Appendix.

We will call $\mathcal{P}(0) = \lim_{z \to 0} \mathcal{P}(z)$ a flat prismatoid. Each lateral face of $\mathcal{P}(0)$ is either an up-face or a down-face, and the faces of $\mathcal{P}(z)$ retain this classification in that their outward normals either have a positive or a negative vertical component.

Lemma 3 Let $\mathcal{P}(z)$ be a prismatoid with height $z$, and $BU(z)$ its base unfolding. Then the apex $a_j(z)$ of each $B'_i(z)$ triangle $\triangle b_i b_{i+1} a_j(z)$ in $BU(z)$ lies on the fixed line containing the altitude of $B'_i(z)$.

Proof. See Sec. 5.3 in the Appendix.

Thus the vertices $a_j(z)$ of the base unfolding “ride out” along the altitude rays $r_i$ as $z$ increases (see ahead to Fig. 6 for an illustration). Therefore the combinatorial structure of the altitude partition is fixed, and $R_i$ only changes geometrically by the lengthening of the edges $b_i a_j'$ and $b_{i+1} a_j'$ and the change in the angle gap $\kappa_{b_i}(z)$ at $b_i$.

2.3 Structure of $A$-fans

Henceforth we concentrate on one $A$-fan, which we always take to be incident to $b_2$, and so between $B_1 = \triangle b_1 b_2 a_1$ and $B_2 = \triangle b_2 b_3 a_k$. The $a$-chain is the chain of vertices $a_1, \ldots, a_k$. Note that the plane containing $B_1$ supports $A$ at $a_1$, and the plane containing $B_2$ supports $A$ at $a_k$. Let $\beta = \beta_2$ be the base angle at $b_2$; $\beta = \angle b_1 b_2 b_3$. We state here a few facts true of every $A$-fan.

1. An $a$-chain spans at most “half” of $A$, i.e., a portion between parallel supporting lines (because $\beta > 0$).

2. If an $A$-fan is unfolded as a unit to the base plane, the $a$-chain consists of a convex portion followed by a reflex followed by a convex portion, where any of these portions may be empty. In other words, excluding the first and last vertices, the interior vertices of the chain have convex angles, then reflex, then convex.

3. Correspondingly, an $A$-fan consists of down-faces followed by up-faces followed by down-faces, where again any (or all) of these three portions could be empty.

4. All four possible combinations of up/down are possible for the $B_1$ and $B_2$ triangles.

The second fact above is not so easy to see; its proof is hinted at in Sec. 5.6 in the Appendix. The intuition is that there is a limited amount of variation possible in an $a$-chain. It is the third fact that we will use essentially; it will become clear shortly.

2.4 Flat Prismatoid Case Analysis

How the $A$-fan is proved to sit inside its altitude region $R$ for $\mathcal{P}(0)$ depends primarily on where $b_2$ sits with respect to $A$, and secondarily on the three $B$-vertices $(b_1, b_2, b_3)$. Fig. 4 illustrates one of the easiest cases, when $b_2$ is in $C$, the convex region bounded by the $a$-chain and extensions of its extreme edges. Then all the $A$-faces are down-faces, the $a$-chain is convex, one of the two $B$-faces is a down-face ($B_2$ in the illustration), and we simply leave the $A$-fan attached to that $B$ down-face.

![Figure 3: Partition exterior to the base unfolding by altitude rays $r_i$. In this example both $A$ and $B$ are pentagons; in general there would not be synchronization between the $b_i$ and $a_i$ indices. The $A$-triangles are not shown.](image)

**Figure 3:** Partition exterior to the base unfolding by altitude rays $r_i$. In this example both $A$ and $B$ are pentagons; in general there would not be synchronization between the $b_i$ and $a_i$ indices. The $A$-triangles are not shown.

![Figure 4: Case 1b. Here we have illustrated $b_1 = b_3$ to allow for the maximum $a$-chain extent.](image)

**Figure 4:** Case 1b. Here we have illustrated $b_1 = b_3$ to allow for the maximum $a$-chain extent.
side of the \(\alpha\)-chain. These up-faces must be flipped in the unfolding, reflected across one of the two tangents from \(b_2\) to \(A\). A key point is that not always will both flips be “safe” in the sense that they stay inside the altitude region. An unsafe flip is illustrated in Fig. 22 in the Appendix. Fortunately, one of the two flips is always safe:

**Lemma 4** Let \(b_2\) have tangents touching \(a_x\) and \(a_t\) of \(A\). Then either reflecting the enclosed up-faces across the left tangent, or across the right tangent, is “safe” in the sense that no points of a flipped triangle falls outside the rays \(r_1\) or \(r_k\).

**Proof.** See Sec. 5.4 in the Appendix.

The remaining cases are minor variations on those illustrated, and will not be further detailed. See Fig. 24 in the Appendix.

### 2.5 Nesting in \(\mathcal{P}(z)\) regions

The most difficult part of the proof is showing that the nesting established above for \(\mathcal{P}(0)\) holds for \(\mathcal{P}(z)\). A key technical lemma is this:

**Lemma 5** Let \(\triangle b, a_1(z), a_2(z)\) be an \(A\)-triangle, with angles \(\alpha_1(z)\) and \(\alpha_2(z)\) at \(a_1(z)\) and \(a_2(z)\) respectively. Then \(\alpha_1(z)\) and \(\alpha_2(z)\) are monotonic from their \(z = 0\) values toward \(\pi/2\) as \(z \to \infty\).

**Proof.** See Sec. 5.5 in the Appendix.

I should note that it is not true, as one might hope, that the apex angle at \(b\) of that \(A\)-triangle, \(\angle a_1(z), b, a_2(z)\), shrinks monotonically with increasing \(z\), even though its limit as \(z \to \infty\) is zero. Nor is the angle gap \(\alpha_1(z)\) necessarily monotonic. These nonmonotonic angle variations complicate the proof.

Another important observation is that the sorting of \(ba_i\) edges by length in \(\mathcal{P}(0)\) remains the same for all \(\mathcal{P}(z), z > 0\). More precisely, let \(|ba_i| > |ba_j|\) for two lateral edges connecting vertex \(b \in B\) to vertices \(a_i, a_j \in A\) in \(\mathcal{P}(0)\). Then \(|ba_i(z)| > |ba_j(z)|\) remains true for all \(\mathcal{P}(z), z > 0\) (by reasoning detailed in Lemma 6).

For the nesting proof, I will rely on a high-level description, and one difficult instance. At a high level, each of the convex or reflex sections of the \(\alpha\)-chain are enclosed in a triangle, which continues to enclose that portion of the \(\alpha\)-chain for any \(z > 0\) (by Fact 1, Sec. 5.6). See Fig. 25 in the Appendix for the convex triangle enclosure. The reflex enclosure is determined by the tangents from \(b_2\) to \(A\): \(\triangle a_x b_2 a_t\). So then the task is to prove these (at most three) triangles remain within \(R(z)\). Fig. 6 shows a case where there is both a convex and a reflex section. Were there an additional convex section, it would remain attached to \(B_1(z)\) and would not increase the challenge.

![Figure 5: Case 2a. The \(A\)-triangles between the tangents \(b_2\) to \(a_3\) and \(b_2\) to \(a_6\) are up-faces. (a) shows the up-faces flipped over the left tangent \(b_2a_6\), and (b) when flipped over the right tangent \(b_2a_3\).](image)

![Figure 6: (a) \(z = 0\). \(\triangle a_1a_x a_k\) encloses the convex section, and \(\triangle a_1b_2a_1\) encloses the reflex section. (b) \(z > 0\). Reflex angle \(\alpha_1(z)\) decreases as \(z\) increases.](image)
side with a down-face ($B_2$ in the figure), then $AF'(z) \subseteq R(z)$: the $A$-fan unfolds within the altitude region for all $z$.

**Proof.** See Sec. 5.6 in the Appendix.

I have been unsuccessful in unifying the cases in the analysis, despite their similarity. Nevertheless, the conclusion is this theorem:

**Theorem 7** Every triangulated topless prismatoid has a petal unfolding.

It is natural to hope that further analysis will lead to a safe placement of the top $A$ (which might not fit into any altitude-ray region: see Fig. 18 in the Appendix.

3 Unfolding Vertex-Neighborhoods

Let $N_v(B)$ for a face $B$ of a convex polyhedron $P$ be $B$ plus the set of all faces that share an edge with $B$, and $N_e(B)$ be $B$ plus the set of all faces that share a vertex with $B$. So $N_v(B) \supseteq N_e(B)$. As mentioned previously, Pincu proved that $N_e(B)$ has a petal unfolding. Here we show that $N_v(B)$ does not always have a petal unfolding, even when all faces in the set are triangles.

A portion of the a 9-vertex example $P$ that establishes this negative result is shown in Fig. 7. The $b_1 b_3$ edge of $B$ lies on the horizontal $xy$-plane. The vertices $\{b_2, a_1, a_2, c_1, c_2\}$ all lie on a parallel plane at height $z$, with $b_2$ directly above the origin: $b_2 = (0, 0, z)$.

All of $N_v(B)$ is shown in Fig. 8. The structure in Fig. 7 is surrounded by more faces designed to minimize curvatures at the vertices $b_i$ of $B$. Finally, $P$ is the convex hull of the illustrated vertices, which just adds a quadrilateral “back” face $(p_1, c_1, c_2, p_3)$ (not shown).

The design is such that there is so little rotation possible in the cw and ccw options for the triangles incident to a vertex of $B$, that overlap is forced: see Figs. 9, 10, and 11. The thin $\triangle b_2 a_1 a_2$ overlaps in the vicinity of $a_1$ if rotated ccw, and in the vicinity of $a_2$ is cw (illustrated). Explicit coordinates for the vertices of $P$ are given in Sec. 5.7 of the Appendix.

One can identify two features of the polyhedron just described that led to overlap: low curvature vertices (to restrict freedom) and obtuse face angles (at $a_1$ and $a_2$) (to create “overhang”). Both seem necessary ingredients. Here I pursue excluding obtuse angles.

**Theorem 8** If $P$ is nonobtusely triangulated, then for every face $B$, $N_v(B)$ has a petal unfolding.

A nonobtuse triangle is one whose angles are each $\leq \pi/2$. It is known that any polygon of $n$ vertices has a nonobtuse triangulation by $O(n)$ triangles, which can

![Figure 7: Faces of $P$ in the immediate vicinity of $B$.](image7)

![Figure 8: All faces incident to $N_v(B)$, and one more, the purple quadrilateral $(a_1, c_1, c_2, a_2)$. The red vectors are normal to $B$ and to $\triangle b_1 p_1 c_1$.](image8)

![Figure 9: Complete unfolding of all faces incident to $B$.](image9)

![Figure 10: Zoom of Fig. 9.](image10)

![Figure 11: Zoom of Fig. 10 in vicinity of $a_2$ overlap. The angle gap at $b_3$ is $0.8^\circ$, and the gap at $b_2$ is $2.8^\circ$.](image11)
be found in $O(n \log^2 n)$ time [BMR95]. Open Problem 22.6 [DO07, p. 332] asked whether every nonobtusely triangulated convex polyhedron has an edge unfolding. One can view Theorem 8 as a (very small) advance on this problem.

The nonobtuseness of the triangles permits identifying smaller diamond regions $D_i$ inside the altitude regions $R_i$ used in Sec. 2, such that $D_i$ necessarily contains the A-fan triangles, regardless of how they are grouped. See Fig. 12(a).

Figure 12: (a) $D_i \subset R_i$. (b) Perpendiculars cannot hit $A_i$ or $A_{i-1}$.

A little more analysis leads to a petal unfolding of a (very special) class of prismatoids:

**Corollary 9** Let $P$ be a triangular prismatoid all of whose faces, except possibly the base $B$, are nonobtuse triangles, and the base is a (possibly obtuse) triangle. Then every petal unfolding of $P$ does not overlap.

**Proof.** See Sec. 5.8 in Appendix. \qed

Fig. 13 shows one illustration from the proof, which defines another region $V_i \subset R_i$ which does not overlap the adjacent diamonds $D_{i-1}$ and $D_{i+1}$, and into which it is safe to unfold the top $A$.

Figure 13: The top $A$ of the prismatoid remains inside $V_i$.

4 Unfolding Convex Patches

I believe that unfolding convex patches may be a fruitful line of investigation. For example, notice that the edges cut in a petal unfolding of a topless prismatoid or of vertex-neighborhood of a face, form a disconnected spanning forest rather than a single spanning tree. One might ask: Does every convex patch have an edge unfolding via single spanning cut tree? The answer is no, already provided by the banded hexagon example in Fig. 1. For such a tree can only touch the boundary at one vertex (otherwise it would lead to more than one piece), and then it is easy to run through the few possible spanning trees and show they all overlap.

The term zipper unfolding was introduced in [DDL+10] for a nonoverlapping unfolding of a convex polyhedron achieved via Hamiltonian cut path. They studied zipper edge-paths, following edges of the polyhedron, but raised the interesting question of whether or not every convex polyhedron has a zipper path, not constrained to follow edges, that leads to a nonoverlapping unfolding. This is a special case of Open Problem 22.3 in [DO07, p. 321] and still seems difficult to resolve.

Given the focus of this work, it is natural to specialize this question further, to ask if every convex patch has a zipper unfolding, using arbitrary cuts. I believe the answer is negative: a version of the banded hexagon shown in See Fig. 14 has no zipper unfolding. My argument for this is long and seems difficult to formalize, so I leave the claim as a conjecture that, with effort, the proof could be formalized.

Figure 14: The banded hexagon with a thin band.

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5 Appendix

Figure 15: A triangular prismatoid (top and bottom both triangles), and one petal unfolding. The base \( B \)-triangles are green; the top \( A \)-triangles are yellow.

Figure 16: Apex cuts: each leads to overlap. The highlighted edge is not cut.

Figure 17: Side cuts: each leads to overlap.

5.1 Proof of Lemma 1

**Lemma 1** No pair of altitude rays cross in the base plane, and so they define a partition of that plane exterior to the base unfolding.

**Proof.** Consider three consecutive \( B \) vertices of the prismatoid \( P \), \((b_1, b_2, b_3)\) supporting two base triangles, \( B_1 = \triangle b_1 b_2 a_1 \) and \( B_2 = \triangle b_2 b_3 a_2 \). We will show that \( r_1 \) and \( r_2 \) cannot cross. Let \( \beta_1 = \angle b_1 b_2 a_1 \) and \( \beta_2 = \angle b_2 b_3 a_2 \) be the two angles of the base triangles.

Figure 18: A drum-like prismatoid that results in overlap with consistent ccw rotation of the (yellow) \( A \)-triangles. Here the point \( a_1' \) overlaps the unfolded top \( A' \). This overlap can be removed easily, by rotating the \( A \)-triangle \( \triangle a_1 a_2 b_1 \) cw rather than ccw.

Figure 19: An overhead view of a nearly flat, topless triangular prismatoid. \( A \)-triangles \( \triangle a_2 a_3 b_2 \) and \( \triangle a_3 a_1 b_3 \) are both rotated ccw, about \( b_2 \) and \( b_3 \) respectively. [Figure created in Cinderella.]
incident to $b_2$. (We use $a_2$ for the apex of $B_2$ for simplicity, although there could be intervening $A$ vertices between $a_1$ and $a_2$.) We consider three cases, distinguishing acute and obtuse $\beta_i$ angles.

Figure 20: Only in case (c) could ray $r_1$ cross $r_2$.

If both $\beta_1$ and $\beta_2$ are acute, then the altitudes of $B_1$ and $B_2$ lie on the base edges $b_1b_2$ and $b_2b_3$ respectively, and the lines containing the rays cross behind the rays, as in Fig. 20(a). Similarly, if both $\beta_1$ and $\beta_2$ are obtuse, again the ray lines cross behind the rays, this time exterior to $B$, as in (b) of the figure. Only when one angle is obtuse and the other acute could the rays possibly cross. Without loss of generality, let $\beta_2$ be obtuse and $\beta_1$ acute, as in (c) of the figure. We now concentrate on this case.

Let $H_i$ be the vertical plane containing the altitude of $B'_i$. This plane includes both the unfolded $a'_i$ on the $B$-plane and the vertex $a_i$ on the $A$-plane, because $a'_i$ is the image of $a_i$ rotated about the base edge $b_ib_{i+1}$ to which the altitude of $B_i$ is perpendicular. See Fig. 21. The $B_i$ triangles of $P$ cut the $A$-plane in lines parallel to their base edges $b_ib_{i+1}$, and the top $A$ must fall inside the halfplanes on the $A$-plane bounded by these lines. Examination of the figure shows that this requires $a_1$ to lie on the $A$-plane right of $H_2$ in the figure. But $a'_1$ is necessarily initially left of $H_2$ if $r_1$ is to cross $r_2$, and the rotation of $a'_1$ from the $B$-plane up to the $A$-plane moves it only further left of $H_2$. Thus this last case violates the convexity of $P$, and we have established the lemma for adjacent altitude rays $r_1, r_2$.

(We have shown in the figure $B_1$ and $B_2$ both making an angle less than $\pi/2$ with the base plane, but the argument is not altered if either of those angles exceed $\pi/2$: still the rotation of $a_i$ down to $a'_i$ occurs in the altitude $H_i$ plane.)

Now consider nonadjacent rays, say $r_1$ and $r_2$, based on base triangles $B_1$ and $B_i$. Extend the edges of those triangles in the $B$-plane until they meet at point $\overline{b}$, and form new triangles $\overline{B_1} = \triangle b_1\overline{b}a_1$ and $\overline{B_i} = \triangle b_{i+1}\overline{b}a_i$, sharing $\overline{b}$. (Again we use $a_i$ for the apex of $B_i$ without implying there are exactly $i - 1$ $A$-vertices between $a_1$ and $a_i$.) Notice these triangles are still apexed at $a_1$ and $a_i$ respectively, as the planes containing $B_1$ and $B_i$ support $A$ at these two points. Define $\overline{P}$ as the convex hull of $P \cup \overline{b}$. In $\overline{P}$, the altitudes of the new base triangles $\overline{B_1}$ and $\overline{B_i}$ are exactly the same as the altitudes of the original $B_1$ and $B_i$, because their base edges have been extended while retaining their apaxes on $A$. So the rays $r_1$ and $r_2$ have not changed in the base plane, and we can reapply the argument for adjacent rays.

5.2 Proof of Lemma 2

Lemma 2 Let $P(z)$ be a prismatoid with height $z$. Then the combinatorial structure of $P(z)$ is independent of $z$, i.e., raising or lowering $A$ above $B$ retains the convex hull structure.

Proof. Let $B_1 = \triangle b_1b_2a(z)$ be a $B$-triangle for some $z > 0$. (The argument is the same for an $A$-triangle by inverting $P$.) Let $L(z)$ be the line in the $A$-plane parallel to $b_1b_2$ through $a(z)$, i.e., $L(z)$ is the intersection of the plane containing $B_1$ with the $A$-plane. Then $L(z)$ is a line of support for $A(z)$ in the $A$-plane. As $z$ varies, this line remains parallel to $b_1b_2$, and because $A(z)$ merely translates with $z$ (it does not rotate), $L(z)$ remains a line of support to $A(z)$. Thus the plane containing $B_1(z)$ supports $A(z)$, and of course it supports $B$ because $b_1b_2$ does not move. Therefore, $B_1(z)$ remains a face of $P(z)$ for all $z > 0$.

5.3 Proof of Lemma 3

Lemma 3 Let $P(z)$ be a prismatoid with height $z$, and $BU(z)$ its base unfolding. Then the apex $a'_j(z)$ of each $B'_j(z)$ triangle $\triangle b_ib_{i+1}a'_j(z)$ in $BU(z)$ lies on the fixed line containing the altitude of $B'_i(z)$.

Proof. Recall that $B'_j$ is produced by rotating $B_i$ about its base edge $b_ib_{i+1}$. Thus every point on a line perpendicular to $b_ib_{i+1}$ lying within the plane of $B_i$ unfolds to that line rotated to the base plane. Because $a_j(z)$ lies
on such a line containing $B_i$’s altitude, $a'_j(z)$ is on the line containing the altitude to $B'_i$. \hfill \square

Figure 22: Case 2 gone bad: the chain $(a'_4, a'_5, a'_6)$ leaves $R$ as it crosses $r_1$. The overlap in Fig. 19 can also be understood as caused by an unsafe flip.

5.4 Proof of Lemma 4

**Lemma 4** Let $b_2$ have tangents $a_s$ and $a_t$ to $A$. Then either reflecting the enclosed up-faces across the left tangent, or across the right tangent, is “safe” in the sense that no points of a flipped triangle falls outside the rays $r_1$ or $r_k$.

**Proof.** The rays $r_1$ and $r_k$ are in general below and turned beyond (ccw and cw respectively) the tangency points $a_s$ and $a_t$, but at their “highest” they are as illustrated in Fig. 23. If reflecting $a_s$ to $a'_s$ is not safe as illustrated, then the perpendicular at $a_t$ must hit $b_2a_s$. Because it makes an angle $\beta$ there with $a'_t$, the alternate reflection is safe. \hfill \square

5.5 Proof of Lemma 5

**Lemma 5** Let $\triangle b, a_1(z), a_2(z)$ be an $A$-triangle, with angles $\alpha_1(z)$ and $\alpha_2(z)$ at $a_1(z)$ and $a_2(z)$ respectively. Then $\alpha_1(z)$ and $\alpha_2(z)$ are monotonic from their $z = 0$ values toward $\pi/2$ as $z \to \infty$.

**Proof.** With loss of generality, let $b = (0, 0, 0)$, $a_1(z) = (1, 0, z)$, and $a_2 = (1+x, y, z)$, with $y > 0$. If $x > 0$, then $\alpha_1(z) > \pi/2$ (obtuse), and if $x \leq 0$, then $\alpha_1(z) < \pi/2$ (acute). By symmetry, we need only prove the claim for $\alpha_1(z)$.

The dot-product $(a_1(z) - b) \cdot (a_2(z) - a_1(z))$ determines either $\cos(\alpha_1(z))$ or $\cos(\pi - \alpha_1(z))$, depending on

Figure 23: One of the two reflections must remain above the rays $r_1$ or $r_k$.

Figure 24: Case 2b. Here $B_1$ is an up-face. (a) Flip across the left tangent. (b) Rather than flip the up-A-faces across the right tangent, those faces are flipped while attached to $B_1$—i.e., we treat $B_1$ as joined to those up-A-faces.
whether or not \( \alpha_1(z) \) is acute or not. Direct computation leads to
\[
\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}\sqrt{1 + z^2}}
\]
whose derivative with respect to \( z \) is
\[
-\frac{xz}{\sqrt{x^2 + y^2}(1 + z^2)^{3/2}}.
\]
Because \( z > 0 \), the sign of the derivative is entirely determined by the sign of \( x \). For \( \alpha_1 \) obtuse, \( x > 0 \), the derivative is negative, which corresponds to decreasing \( \alpha_1(z) \), and when \( x < 0 \) and \( \alpha_1 \) is acute, the derivative is positive corresponding to increasing \( \alpha_1(z) \). Thus the claim of the lemma is established.

2. \( \alpha_j(z) \) is monotonic with respect to \( z \), approaching \( \pi \) as \( z \to \infty \) from above (if initially reflex) or below (if initially convex).

The essence of why Fact 1 holds is in Fig. 26. See [O’R12] for proofs. Fact 2 can be established by superimposing neighborhoods of \( a_j \) for two different \( z \)-values \( z_1 < z_2 \), and noting, for reflex \( \alpha_j \), the \( z_2 \)-neighborhood is nested in that for \( z_1 \), and consequently there is a larger curvature \( \kappa_{\alpha_j}(z_2) > \kappa_{\alpha_j}(z_1) \).

**Lemma 6** If the \( a \)-chain consists of a convex and a reflex section, and the safe flip (by Lemma 4) is to a side with a down-face (\( B_2 \) in the figure), then \( A\mathcal{F}'(z) \subset R(z) \): the \( A \)-fan unfolds within the altitude region for all \( z \).

**Proof.** Let \( a_s \) and \( a_t \) be the vertices of the \( a \)-chain so that lines containing \( b_2a_s \) and \( b_2a_t \) are supporting tangents to \( A \) at \( a_s \) and \( a_t \). Thus \( (a_1, \ldots, a_s) \) represents a convex portion of the \( a \)-chain, \( (a_s, \ldots, a_t) \) the reflex portion, and \( (a_t, \ldots, a_k) \) a convex portion. We first assume \( a_s = a_1 \) so we have only a convex and a reflex section, as illustrated in Fig. 6. We also first assume that both \( B_1 \) and \( B_2 \) are down-faces and so do not require flipping. We analyze this case by mixing the convex and reflex approaches in earlier, easier cases not detailed here (but see Fig. 25).

For the reflex chain, we connect \( a_s = a_1 \) to \( a_t \) to form a triangle \( A_{st} = \triangle a_2b_2a_1 \) that encloses the reflex chain. For the convex chain \( (a_1, \ldots, a_s) \) we intersect the line \( L_{23} \) parallel to \( b_2b_3 \) through \( a_k \) (just as in the all-convex case not detailed), and intersect it with the line containing \( b_2a_t \). Let that intersection point be \( a_s \). Then the triangle \( A_s = \triangle b_2a_s a_k \) encloses the convex chain. Under the assumption that \( B_1 \) is a down-face, then \( A_s \) encloses all down-faces, and does not need flipping. \( A_{st} \) does flip, and let us assume the safe flip is across \( b_2a_t \), flipping \( a_s \) to \( a'_s \), with \( A'_s \) the reflected triangle.

Vertex \( a_k(z) \) rides out \( r_2 \). By construction, \( |a_k(z) / a_k(\infty) / r_2 \), as \( a_k \) was defined by \( L_{23} \) parallel to \( r_2 \). Because \( |a_{k}(z)| = |a_{k}(\infty)| \), \( a_{k}(z) \) rides out along a line parallel \( L_{23} \) to \( r_2 \), so \( A'_s \subset R(z) \).

Now the curvature \( \kappa(z) \) at \( b_2 \), i.e., the angle gap in the unfolding, varies in a possibly complex way, but it

![Figure 25: Enclosing a convex chain with a triangle \( \triangle a_1a_2a_k \), where \( a_k \) is the intersection of lines of support at \( a_1 \) and \( a_k \) parallel to \( b_1b_2 \) and \( b_2b_3 \) respectively.](image_url)

**Figure 25:** Enclosing a convex chain with a triangle \( \triangle a_1a_2a_k \), where \( a_k \) is the intersection of lines of support at \( a_1 \) and \( a_k \) parallel to \( b_1b_2 \) and \( b_2b_3 \) respectively.

**5.6 Proof of Lemma 6**

Here we will need two important facts about the unfolded \( a \)-chain:

1. Let \( \alpha_j \) be the angle of the chain at \( a_j \), i.e., the sum of the two incident triangle angles, \( \angle b_2a_ja_{j-1} + \angle b_2a_ja_{j+1} \). If \( \alpha_j \) is convex for \( z = 0 \), it remains convex for all \( z \); and similarly reflex remains reflex, and a sum of \( \pi \) remains independent of \( z \).
remains positive at all times, because clearly \( \mathcal{P}(z) \) is not
flat at \( b_2 \) for any \( z \). Thus \( b_2a'_1(z) \) is rotated ccw from
\( b_2a'_1(z) \). It remains to show that \( b_2a'_1(z) \) cannot cross
\( r_2 \).

By Fact 1 above, the convex angle at \( a_x \) remains con-
 vex at \( a_x(z) \), and therefore \( a_t(z) \) cannot cross \( L_x \) let
alone \( r_2 \). Again by Fact 1, the reflex chain \((a_1, \ldots, a_t)\)
remains a reflex chain with increasing \( z \), and so is con-
tained inside \( A'_t(z) \). This reflex chain straightens, ap-
proaching the segment \( a_t(z)a'_1(z) \).

Because that chain is reflex, the only way that \( A'_t \) can
cross \( r_2 \) is for the segment \( a_t(z)a'_1(z) \) to cross, i.e., for
\( a'_1(z) \) to cross. Notice this requires a highly reflex angle
\( a_t(z) = \angle a'_1(z), a_t(z), a_t(z) \), at least \( 3\pi/2 \) in fact, in
order to cross over the line \( L_x \). Now we have no control
over the initial value of \( a_t \), but we know that the flip
was safe, so initially \( a'_1(z) \) is inside \( r_2 \). If \( a_t \) is convex,
then \( a_t(z) \) remains convex and \( a'_1(z) \) cannot cross \( r_2 \).
So assume \( a_t(z) \) is initially reflex (as illustrated in Fig. 6).
Then by Fact 2, it decreases monotonically toward \( \pi \) as
\( z \) increases. Because it decreases, and needs to be at
least \( 3\pi/2 \) to cross \( r_2 \), it must have started out at least
\( 3\pi/2 \). Now we argue that this is impossible, as the other
flip would have been chosen.

As Fig. 27 shows, if \( a_t > 3\pi/2 \), then the reflection
\( a_t a'_1 \) is already more than \( \pi/2 \) ccw of \( b_2a_t \), which marks
it as an unsafe flip. We would instead have flipped the
reflex portion across \( b_2a_1 = a_s \). And indeed the flip in
Fig. 6 would not have been chosen because it is poten-
tially unsafe (but does not in this case actually place \( a'_1 \)
on the wrong side of \( r_2 \).

Figure 27: In order for \( a_t > 3\pi/2 \), \( a_t a_1 \) must make an
angle more than \( \pi/2 \) with \( b_2a_1 \).

5.8 Proof of Corollary 9

**Corollary 7** Let \( \mathcal{P} \) be a triangular prismatoid all of
whose faces, except possibly the base \( B \), are nonobtuse
triangles, and the base is a (possibly obtuse) triangle.
Then every petal unfolding of \( \mathcal{P} \) does not overlap.

**Proof.** We first let \( B \) be an arbitrary convex polygon.
We define yet another region \( V_i \cap R_i \) incident to \( b_i \),
bound by rays from \( b_i \) through \( a_{i-1} \) and through \( a_i \).
See Fig. 13. Note that these rays shoot at or above the
adjacent diamonds \( D_{i-1} \) and \( D_{i+1} \), and therefore miss
\( A_{i-2} \) and \( A_{i+1} \).

Now we invoke the assumption that \( B \) is a triangle:
In that case, those adjacent diamonds contain all the
remaining \( A \)-triangles, because there are only three \( b_i \)
vertices: \( b_1 \) at which \( V_i \) is incident, and diamonds \( D_2 \)
and \( D_3 \) to either side. (Note there can only be alto-
tgether three \( A \)-triangles, one for each edge of \( A \).) Now
unfold the top \( A \) of \( \mathcal{P} \) attached to some \( A \)-triangle,
without loss of generality a \( A \)-triangle incident to \( b_1 \). Then
because \( A \) is nonobtuse, its altitude, and indeed all of \( A \),
projects into that edge shared with a \( A \)-triangle \( A_1 \). Be-
cause the top of the \( A \)-triangle is inside \( D_1 \), we can see
that \( A \subset V_i \), and we have protected \( A \) from overlapping
any other \( A \)-triangle or any \( A_i \).

It seems quite likely that this corollary still holds with
\( B \) an arbitrary convex polygon, but, were the same
proof idea followed, it would require showing that \( V_i \)
does not intersect nonadjacent diamonds or more dis-
tant \( A_j \) triangles.

| Point  | Coordinates               |
|--------|---------------------------|
| \( b_2 \) | (0, 0, 0.2)              |
| \( a_1, a_2 \) | (±0.603496, 0.0399127, 0.2) |
| \( b_1, b_3 \) | (±2, −0.1, 0)            |
| \( c_1, c_2 \) | (±0.0124876, 0.501659, 0.2) |
| \( p_1, p_3 \) | (±6.03626, −0.4, −0.6)   |

5.7 Vertex-Neighborhood Counterexample Coordinates

The coordinates of the nine vertices comprising \( \mathcal{P} \) in
Fig. 7 are shown in the table below, with \( \{a_2, b_3, c_2, p_3\} \)}