Welfare implications of noise traders

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Abstract

We prove the existence of incomplete Radner equilibria in two models with exponential investors and different types of noise traders: an exogenous noise trader and an endogenous noise tracker. In each model, we analyze a coupled system of ODEs and reduce it to a system of two coupled ODEs in order to establish equilibrium existence. As an application, we study the impact of the noise trader types on welfare. We show that the aggregate welfare comparison depends in a non-trivial manner on every equilibrium parameter, and there is no ordering in general. Our models suggest that care should be used when drawing conclusions about welfare effects when noise traders are used.

Keywords: Radner equilibrium, Incompleteness, Welfare, Noise trader

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1 Introduction

We prove global existence of Radner equilibria and study the welfare implications of economies with two different types of noise traders: exogenous and endogenous. In each of the two models, a finite number of utility-maximizing exponential investors and a single noise trader trade in a continuous-time financial market. The exponential investors derive utility from consumption of terminal wealth in a pure-exchange economy.

The use of noise traders is a standard modeling tool in equilibrium theory. Price-inelastic noise traders often model uninformed traders’ demands or serve as a way to express unmodeled demands. These traders are introduced in a wide array of works such as Grossman and Stiglitz [9], Kyle [12], Hellwig [10], Vayanos [14], and the vast literature that has followed. These exogenously-determined, price-inelastic noise traders do not choose demand processes based on a decision problem. In contrast, Sannikov and Skrzypacz [13] and Choi et al. [5] use noise traders who are incentivized to track a noisy target. Such noise trackers maximize their expected wealth from trading but are penalized for deviating from a Brownian motion.

Given the prevalence of noise traders of different varieties in the equilibrium literature, it is natural to consider the welfare implications of such modeling choices. We ask the question:

Which modeling scenario do the utility-maximizing investors prefer – an exogenous noise trader or an endogenous noise tracker?

Our analysis shows that the answer is ambiguous. We quantify the sign and magnitude of the aggregate welfare difference between our two models. Even in our simple setting, the dependence is non-trivial in the individual equilibrium parameters and across multiple parameters. We show that the type of noise trader can have a positive or negative impact on the utility maximizers’ aggregate welfare.

In both of the models presented, we prove the existence of a Radner equilibrium. Incomplete Radner equilibria are difficult to study mathematically because standard
tools and simplifications, such as a representative agent, are often not applicable. Approaches with partial differential equations (PDEs) and backward stochastic differential equations (BSDEs) have been successful in proving abstract existence; see, for example, Choi and Larsen \cite{4}, Xing and Žitković \cite{15}, and Escauriaza et al. \cite{7}. Exogenous noise in equilibrium has been successfully studied in Gârleanu and Pedersen \cite{8} and Bouchard et al. \cite{2} by relaxing the utility maximizers’ problems from exponential investors to linear-quadratic optimizers.

Our approach derives a system of coupled ordinary differential equations (ODEs) that are functions of time. This derivation is possible because we propose a functional form for the equilibrium stock price, which is similar to the Nash equilibrium stock price structure in Chen et al. \cite{3}. In our two models, the ODE systems consist of 9 (exogenous noise) and 15 (endogenous noise) coupled equations. Upon inspection, we reduce each system to two core coupled ODEs, which allows us to prove the existence of global solutions.

The paper is organized as follows. Section 2 describes the shared setting for the model. Section 3 describes the model with exogenous noise trading and presents the corresponding equilibrium existence result. The model with an endogenous noise tracker and its equilibrium existence result are in Section 4. Section 5 compares the utility-maximizing investors’ welfare in the two models and gives some numerics. The proofs are contained in Section 6.

2 Model set-up

For simplicity, we normalize the trading time horizon as 1. Let $\mathbb{P}$ be a probability measure, and $(D_t, Y''_t)_{t \in [0,1]}$ be two independent, one-dimensional Brownian motions under $\mathbb{P}$ with constant values $(D_0, Y''_0)$ at time $t = 0$, zero drifts, and constant volatilities $(\sigma_D, \sigma_{Y''})$. The augmented standard Brownian filtration is denoted by

$$\mathcal{F}_t := \sigma(D_u, Y''_u)_{u \in [0,t]}, \quad t \in [0,1].$$

(2.1)
The market consists of two traded securities: a bank account and stock. The bank account is in zero-net supply with a constant zero interest rate. The stock is in a constant net supply of \( \Sigma \), where \( \Sigma \geq 0 \) is a nonnegative constant. All prices are denominated in units of a single consumption good.

The equilibrium stock price will be determined endogenously in equilibrium as a continuous semimartingale and is denoted \( S = (S_t)_{t \in [0,1]} \). The terminal dividend is modeled by terminal value \( D_1 \) of the Brownian motion \( D \). The terminal equilibrium stock price is exogenously pinned down as:

\[
S_1 = D_1, \quad \mathbb{P}\text{-a.s.} \tag{2.2}
\]

We model the stock positions of a group of \( j \in \{1, ..., I\} \), \( I \in \mathbb{N} \), utility-maximizing investors. Their stock position processes over time are denoted \( \theta_j = (\theta_{j,t})_{t \in [0,1]} \), and investor \( j \) is endowed with an initial stock position of \( \theta_{j,0-} \in \mathbb{R} \) and a zero initial position in the bank account.

There are a range of possibilities when defining admissible trading strategies for exponential investors, such as in Delbaen et. al. [6] and Biagini and Sirbu [1]. Here, we employ the approach of Choi and Larsen [4], who also study an equilibrium with exponential investors. We let \( \mathcal{M} \) denote the collection of \( \mathbb{P} \)-equivalent probability measures under which the stock price process \( S \) is a local martingale. A strategy \( \theta \) is called admissible for \( Q \in \mathcal{M} \) if \( \theta \) is adapted to \( (\mathcal{F}_t)_{t \in [0,1]} \), measurable, \( S \)-integrable, and the wealth process \( X^\theta \) is a \( Q \)-supermartingale, where \( X^\theta \) is define below in (2.3). We denote the collection of admissible strategies with the associated measure \( Q \) by \( \mathcal{A}(Q) \).

The wealth at time \( t \) associated with \( \theta \in \mathcal{A}(Q) \) is denoted by \( X_t^\theta \) where

\[
X_t^\theta := \theta_{0-}S_0 + \int_0^t \theta_u \, dS_u, \quad t \in [0,T]. \tag{2.3}
\]

The utility-maximizing investors seek to maximize their expected utility from ter-
minal wealth. We assume that all investors have identical exponential utility functions

$$-\exp(-ax) \text{ at time } t = 1, \quad x \in \mathbb{R}, \quad (2.4)$$

where \( a > 0 \) is their common risk aversion coefficient. Based on the utility function in (2.4), investor \( j \) seeks to solve

$$\sup_{\theta \in \mathcal{A}(Q)} \mathbb{E} \left[ -\exp \left( -aX^\theta_1 \right) \right]. \quad (2.5)$$

The measure \( Q \) is not investor-specific, unlike in Choi and Larsen \([4]\), because our investors all share a common risk aversion coefficient. It is possible to extend our model to include heterogenous risk aversion coefficients. Because, as we shall see below, our model with identical risk aversion coefficients already produces ambiguous welfare implications, we do not pursue such an extension.

3 Equilibrium with an exogenous noise trader

Our first model introduces a noise trader exogenously, through a given noisy demand in shares of stock. Following Gârleanu and Pedersen \([8]\) and Bouchard et.al. \([2]\), the noise trader’s stock position is exogenously given by

$$Y_t := Y_0 + \int_0^t Y'_u du, \quad t \in [0, 1], \quad (3.1)$$

where we recall that \( Y' \) is a Brownian motion with zero drift and constant volatility \( \sigma_{y'} \).

**Definition 3.1** (Radner equilibrium with exogenous noise trading). Trading strategies \( \hat{\theta}_1, \ldots, \hat{\theta}_I \) and a continuous semimartingale \( S = (S_t)_{t \in [0, 1]} \) form a Radner equilibrium with exogenous noise trading if there exists a measure \( \hat{Q} \) under which \( S \) is a local martingale such that

1. **Strategies are optimal:** For \( j = 1, \ldots, I \), we have \( \hat{\theta}_j \in \mathcal{A}(\hat{Q}) \) solves (2.5) with
measure \( \hat{Q} \), where \( S \) is the corresponding stock price process.

2. Markets clear: We have

\[
\sum_{j=1}^{I} \theta_{j,t} + Y_{t} = \Sigma, \quad t \in [0, 1].
\] (3.2)

The exogenous noise in our model is similar to Gărleanu and Pedersen \[8\] and Bouchard et al. \[2\]; however, our approach to establishing equilibrium existence is different because we use exponential investors instead of their linear-quadratic objectives. The earlier works established existence using pointwise optimization, whereas our existence result relies on analysis of ODEs derived from the investors’ Hamilton-Jacobi-Bellman PDEs.

In order to prove the existence of a Radner equilibrium with exogenous noise trading, we conjecture a form for the exponential investors’ value functions and derive a coupled system of ODEs. Using time \( t \), wealth \( x \), and states \( Y \) and \( Y' \) as state variables for the value function, we conjecture that each investor’s value function will take the form

\[
V(t, x, Y, Y') = -\exp \left( -a \left( x + g_1(t) + g_2(t)Y + g_3(t)Y' + \frac{g_{22}(t)}{2}Y^2 + \frac{g_{23}(t)}{2}YY' + \frac{g_{33}(t)}{2}Y'^2 \right) \right),
\] (3.3)

where \( g_1, g_2, g_3, g_{22}, g_{23}, g_{33} : [0, 1] \rightarrow \mathbb{R} \) are smooth functions of time with terminal conditions

\[
g_1(1) = g_2(1) = g_3(1) = g_{22}(1) = g_{23}(1) = g_{33}(1) = 0.
\] (3.4)

Theorem 3.2 below establishes the existence of a Radner equilibrium with exogenous noise trading and makes the conjectured form of (3.3)-(3.4) rigorous. Theorem 3.2 is the main result of this section.

**Theorem 3.2** (Radner existence with exogenous noise trading). Let \( \Sigma \geq 0, a, \sigma_D^2 > 0 \), and \( \sum_{j=1}^{I} \theta_{j,0-} + Y_0 = \Sigma \). Then, there exists a unique smooth solution to the coupled
system of ODEs for $t \in [0,1]$: \[
\begin{align*}
g'_{33}(t) &= 2a\sigma^2_Y g_{33}(t)^2 - \frac{\beta(t)}{I}, \quad g_{33}(1) = 0, \\
\beta'(t) &= 2a\sigma^2_Y g_{33}(t) \beta(t) - \frac{a\sigma^2_D}{I} (1-t), \quad \beta(1) = 0,
\end{align*}
\]
such that a Radner equilibrium with exogenous noise trading exists. The equilibrium stock price process is given by
\[
S_t := D_t + \mu(t) + \alpha(t)Y_t + \beta(t)Y_t', \quad t \in [0,1],
\]
where for $t \in [0,1]$,
\[
\begin{align*}
\mu(t) &:= -\frac{a\sigma^2_Y}{I} \Sigma (1-t), \\
\alpha(t) &:= \frac{a\sigma^2_D}{I} (1-t).
\end{align*}
\]
Furthermore, there exists $\hat{Q} \in \mathcal{M}$ such that each investor optimally holds $\hat{\theta}_j \in \mathcal{A}(\hat{Q})$ with
\[
\hat{\theta}_{j,t} = \frac{\Sigma - Y_t}{I}, \quad t \in [0,1], \quad j \in \{1,\ldots,I\}. \tag{3.6}
\]
In the class of equilibria that has equilibrium stock prices of the form (3.5) for continuously differentiable functions $\mu, \alpha, \beta$, the equilibrium established in Theorem 3.2 is unique.

4 Equilibrium with an endogenous noise tracker

Our second model replaces the exogenous noise trader by an endogenous noise tracker, similar to Sannikov and Skrzypacz [13]. Rather than hold an exogenously determined number of shares, the endogenous noise tracker is incentivized to track a noisy signal through her decision problem. This trader has linear-quadratic preferences that
encourage her to trade towards the target $Y$ in (3.1). The noise tracker seeks to solve

$$\sup_{\theta \in A_N} \mathbb{E}\left[ X_1 - \int_0^1 \kappa (\theta_t - Y_t)^2 \, dt \right], \quad (4.1)$$

where the constant $\kappa > 0$ measures the motivation to track the noise $Y$, and $A_N$ is the collection of strategies admissible for the noise tracker. The noise tracker’s trading admissibility condition differs from the exponential utility maximizers’ condition since her optimization problem is linear-quadratic rather than exponential. A trading strategy $\theta$ is admissible for the noise tracker if $\theta$ is adapted to $(\mathcal{F}_t)_{t \in [0,1]}$, measurable, and $\mathbb{E}\left[ \int_0^1 \theta_u^2 \, du \right] < \infty$.

We denote the noise trackers’s stock holdings over time by $\theta_{N,t}$. She holds $\theta_{N,0} = Y_0$ shares initially, and, like the utility maximizers, holds a zero position initially in the bank account. With the presence of the endogenous noise tracker, the stock market clearing condition becomes

$$\sum_{j=1}^I \theta_{j,t} + \theta_{N,t} = \Sigma, \quad t \in [0,1]. \quad (4.2)$$

**Definition 4.1** (Radner equilibrium with endogenous noise tracking). Trading strategies $\hat{\theta}_1, \ldots, \hat{\theta}_I$ and $\hat{\theta}_N \in A_N$ and a continuous semimartingale $S = (S_t)_{t \in [0,1]}$ form a **Radner equilibrium with endogenous noise tracking** if there exists a measure $\hat{Q}$ under which $S$ is a local martingale such that

1. **Strategies are optimal**: For $j = 1, \ldots, I$, we have $\hat{\theta}_j \in A(\hat{Q})$ solves (2.5) with measure $\hat{Q}$, and $\hat{\theta}_N \in A_N$ solves (4.1), where $S$ is the corresponding stock price process.

2. **Markets clear**: We have

$$\sum_{j=1}^I \hat{\theta}_{j,t} + \hat{\theta}_{N,t} = \Sigma, \quad t \in [0,1]. \quad (4.3)$$

The value function is again as in (3.3)-(3.4) but for different coefficient functions
compared to Theorem 3.2. Theorem 4.2 establishes the existence of a Radner equilibrium in the second model.

**Theorem 4.2 (Radner existence with endogenous noise tracking).** Let $\Sigma \geq 0$, $a, \sigma_D^2, \kappa > 0$, and $\sum_{j=1}^{I} \theta_{j,0-} + Y_0 = \Sigma$. Then, there exists a unique smooth solution to the coupled system of ODEs for $t \in [0, 1]$:

$$
g_{33}'(t) = \frac{2a\sigma_Y^2 \sigma_D^2 (t + (a\sigma_D^2 + 2\kappa))^2}{(a\sigma_D^2 + a\sigma_Y^2 \beta(t))^2 + 2\kappa} - \frac{2a\kappa \beta(t)}{2\kappa + a\sigma_D^2}, \quad g_{33}(1) = 0,$$

$$
\beta'(t) = \frac{4aI\sigma_Y^2 \sigma_D^2 (t) \beta(t) \kappa}{a\sigma_D^2 + a\sigma_Y^2 \beta(t)^2 + 2\kappa} - \frac{2a\kappa \sigma_D^2}{2\kappa + a\sigma_D^2} (1 - t), \quad \beta(1) = 0,$$

such that a Radner equilibrium with endogenous noise tracking exists. The equilibrium stock price process is given by

$$S_t := D_t + \mu(t) + \alpha(t) Y_t + \beta(t) Y_t', \quad t \in [0, 1],$$

(4.4)

where for $t \in [0, 1]$, we have

$$
\mu(t) := -\frac{2a\kappa \sigma_Y^2 \Sigma}{2\kappa + a\sigma_D^2} (1 - t),
$$

$$
\alpha(t) := \frac{2a\kappa \sigma_Y^2}{2\kappa + a\sigma_D^2} (1 - t).
$$

There exists $\hat{Q} \in \mathcal{M}$ such that the investors’ optimal stock holdings $\hat{\theta}_j \in \mathcal{A}(\hat{Q})$ are given by

$$
\hat{\theta}_{j,t} = \frac{2\kappa (\Sigma - Y_t)}{2\kappa + a\sigma_D^2} - \frac{2a\sigma_Y^2 \beta(t) g_{33}(t) Y_t'}{a\sigma_D^2 + a\sigma_Y^2 \beta(t)^2 + 2\kappa}, \quad t \in [0, 1], \quad j \in \{1, \ldots, I\},
$$

(4.5)

and the noise tracker optimally holds $\hat{\theta}_N \in \mathcal{A}_N$ where

$$
\hat{\theta}_{N,t} = \frac{2\kappa Y_t + a\sigma_Y^2 \Sigma}{2\kappa + a\sigma_D^2} + \frac{2aI\sigma_Y^2 \beta(t) g_{33}(t) Y_t'}{a\sigma_D^2 + a\sigma_Y^2 \beta(t)^2 + 2\kappa}, \quad t \in [0, 1].
$$

(4.6)
Sections 3 and 4 present equilibrium models with exogenous and endogenous noise. The use of both exogenous noise trading and an endogenous noise tracker are prevalent in equilibrium settings. However, welfare comparisons between the two types of noise trading have not been previously studied. Here, we ask the question:

Which type of noise trader do utility maximizers prefer – exogenous noise trading or an endogenous noise tracker?

Based on a welfare analysis of the two models, the answer is that it depends. We measure the utility maximizers’ aggregate welfare by the sum of their certainty equivalents,

$$\sum_{j=1}^{I} CE_j,$$

where the certainty equivalents $CE_j$ are defined implicitly by

$$-e^{-aCE_j} = \sup_{\theta \in \mathcal{A}(\hat{Q})} \mathbb{E} \left[ -\exp \left( -aX_\theta^j \right) \right], \quad j = 1, \ldots, I.$$

The aggregate welfare is defined by

$$\sum_{j=1}^{I} CE_j = S_0 (\Sigma - Y_0) + I \left( g_1(0) + g_2(0)Y_0 + g_3(0)Y_0' + g_{22}(0)Y_0'^2 + g_{23}(0)Y_0Y_0' + g_{33}(0)Y_0'^2 \right),$$

where the functions $g_1$, $g_2$, $g_3$, $g_{22}$, $g_{23}$, and $g_{33}$ come from the functional form of the exponential investors’ value functions in (3.3)-(3.4). For both the exogenous and endogenous noise models, these functions are formally described and defined in the proofs in Section 6. In the $Y_0 = Y_0' = 0$ case, Proposition 5.1 below quantifies the welfare difference between exogenous noise trading and endogenous noise tracking. To distinguish between quantities in the two models, we let the superscripts $ex$ and $en$ represent exogenous and endogenous quantities, respectively.
Proposition 5.1. For \( Y_0 = Y'_0 = 0 \), the welfare difference is given by

\[
I \sum_{j=1}^{l} CE_j^e - I \sum_{j=1}^{l} CE_j^x = \frac{a^3 \Sigma^2 \sigma_D^6}{2I(a \sigma_D^2 + 2 \kappa I)^2} + I \sigma_Y^2 \int_0^1 (g_{33}^e(u) - g_{33}^x(u)) \, du, \tag{5.1}
\]

where \( g_{33}^e \) and \( g_{33}^x \) are the functions \( g_{33} \) with exogenous and endogenous noise from Theorems 3.2 and 4.2, respectively.

The two-ODE systems – \( (g_{33}^e, \beta^e) \) in Theorem 3.2 and \( (g_{33}^x, \beta^x) \) in Theorem 4.2 – do not depend on \( \Sigma \). Therefore, Proposition 5.1 allows us to conclude that the endogenous noise aggregate welfare is greater than the exogenous noise aggregate welfare if and only if

\[
\Sigma^2 > \frac{2I^2 \sigma_Y^2, (a \sigma_D^2 + 2 \kappa I)^2}{a^3 \sigma_D^6} \int_0^1 (g_{33}^e(u) - g_{33}^x(u)) \, du,
\]

where the right-hand side does not depend on \( \Sigma \).

For a sufficiently large stock supply \( \Sigma \), the endogenous noise tracker model is preferable for the exponential investors compared to the exogenous noise trader model when \( Y_0 = Y'_0 = 0 \). For the other equilibrium parameters, we plot the welfare difference (5.1) with \( Y_0 = Y'_0 = 0 \) and observe that the dependence is non-trivial: it is non-monotone in individual and multiple parameters.

Figure 1 plots the welfare difference across a common set of parameters: \( Y_0 = Y'_0 = 1, \Sigma = 1, I = 10, \sigma_D = 1, \sigma_Y = 10, \kappa = 5, \) and \( a \in \{1, 10, 20\} \). By varying these parameters, we observe non-monotone behavior. Amongst the two-dimensional parameter choices \( (\sigma_Y, a), (\sigma_D, a), (\kappa, a), \) and \( (I, a) \), we see similar non-trivial, non-monotone dependencies. The behavior exhibited in these plots shows that the modeling choices involved with noise traders have complex effects on welfare.
6 Proofs

6.1 Proof of Theorem 3.2

**Lemma 6.1.** Let $a, \sigma_D^2 > 0$. Then, the following two-dimensional initial value problem has a unique solution for $t \in [0, \infty)$:

$$
\begin{align*}
    z'_1(t) &= \left(\frac{a \sigma_D^2}{t^2}\right) t - 2a \sigma_Y^2 z_1(t) z_2(t), \\
    z'_2(t) &= \frac{z_1(t)}{t} - 2a \sigma_Y^2 z_2(t)^2,
\end{align*}
\tag{6.1}
$$

*Proof.* The local Lipschitz structure of (6.1) produces a unique solution around the point $t = 0$ by the Picard-Lindelöf theorem (see, e.g., Theorem II.1.1 in Hartman 2002). Furthermore, there exists a maximal interval of existence $0 \in (t, \tilde{t}) \subseteq [-\infty, \infty]$ and we will argue by contradiction to show that $\tilde{t} = \infty$. 
Suppose that $\bar{t} < \infty$. By taking derivatives through (6.1) and inserting the initial values $z_1(0) = z_2(0) = 0$ into these derivatives at $t = 0$, we find

\[
    z_1(0) = z_1'(0) = 0 < z_1''(0) = \frac{a\sigma^2}{T}, \quad (6.2)
\]

\[
    z_2(0) = z_2'(0) = z_2''(0) = 0 < z_2''(0) = \frac{a\sigma^2}{T^2}.
\]

The above equalities and inequalities imply that there exists $\epsilon > 0$ such that $z_1(t) > 0$ and $z_2(t) > 0$ for $t \in (0, \epsilon)$.

To reach a contradiction with $\bar{t} < \infty$, we define $t_0$ as

\[
    t_0 := \inf\{t > 0 : z_1(t) = 0 \text{ or } z_2(t) = 0\}, \quad (6.3)
\]

which is the first time ($z_1$ or $z_2$) reaches zero strictly after time $t = 0$. According to the previous argument, there exists $\epsilon > 0$ such that $t_0 \geq \epsilon$.

If $t_0 < \bar{t}$, then there are two possibilities: (i) In case $z_1(t_0) = 0$, then (6.1) gives $z_1'(t_0) = \left(\frac{a\sigma^2}{T}\right) t_0 > 0$, which contradicts the definition of $t_0$. (ii) In case $z_2(t_0) = 0$ (and $z_1(t_0) > 0$ by (i) above), then (6.1) gives $z_2'(t_0) = \frac{z_1(t_0)}{T} > 0$, which contradicts the definition of $t_0$. Therefore, we conclude that $z_1(t) > 0$ and $z_2(t) > 0$ for $t \in (0, \bar{t})$.

This observation and (6.1) imply

\[
    0 < z_1(t) < \left(\frac{a\sigma^2}{2T}\right) t^2, \quad 0 < z_2(t) < \left(\frac{a\sigma^2}{6T^2}\right) t^3, \quad \text{for} \quad t \in (0, \bar{t}). \quad (6.4)
\]

The above boundedness properties contradicts $\bar{t} < \infty$. \hfill \Box

For the exogenous noise case, the utility-maximizing investors’ value functions are conjectured to have the form (3.3)-(3.4). The investors’ HJB equations and the market clearing condition produce the ODE system for the coefficient functions in (3.3) and
\( \alpha, \beta, \) and \( \mu \) in (3.5):

\[
\begin{align*}
g_1'(t) &= \frac{aI^2 \sigma_Y^2 \gamma_3(t)^2 - a\Sigma^2 (\sigma_D^2 + \sigma_Y^2 \beta(t)^2) - 2I^2 \sigma_Y^2 \gamma_{33}(t)}{2I^2}, \quad g_1(1) = 0, \\
g_2'(t) &= \frac{a(I^2 \sigma_Y^2 \gamma_{23}(t) \gamma_3(t) + \Sigma (\sigma_D^2 + \sigma_Y^2 \beta(t)^2))}{I^2}, \quad g_2(1) = 0, \\
g_3'(t) &= 2a \sigma_Y^2 \gamma_3(t) \gamma_3(t) - g_2(t), \quad g_3(1) = 0, \\
g_22'(t) &= -\frac{a \left( \sigma_Y^2 \beta(t)^2 - I^2 \gamma_{23}(t)^2 \right) + \sigma_D^2}{2I^2}, \quad g_{22}(1) = 0, \\
g_23'(t) &= 2a \sigma_Y^2 \gamma_{23}(t) \gamma_3(t) - 2g_{22}(t), \quad g_{23}(1) = 0, \\
g_33'(t) &= 2a \sigma_Y^2 \gamma_{33}(t)^2 - g_2(t), \quad g_{33}(1) = 0,
\end{align*}
\]

(6.5)

where

\[
\begin{align*}
\alpha'(t) &= -\frac{a \left( \sigma_Y^2 \beta(t)(\beta(t) - Ig_{23}(t)) + \sigma_D^2 \right)}{I}, \quad \alpha(1) = 0, \\
\beta'(t) &= 2a \sigma_Y^2 \gamma_{33}(t) \beta(t) - \alpha(t), \quad \beta(1) = 0, \\
\mu'(t) &= \frac{a \left( \sigma_Y^2 \beta(t)(Ig_3(t) + \Sigma \beta(t)) + \Sigma \sigma_D^2 \right)}{I}, \quad \mu(1) = 0,
\end{align*}
\]

(6.6)

This ODE system will be used to verify optimality of the equilibrium trading strategies in the proof of Theorem 3.2. Lemma 6.2 below establishes the existence and uniqueness of a smooth solution to the system (6.5)-(6.6).

**Lemma 6.2.** Let \( a, \sigma_D^2 > 0 \). Then, there exists a unique smooth solution to the coupled system of ODEs (6.5)-(6.6) for \( t \in [0, 1] \).

**Proof.** Once we show the existence of a solution of (6.5)-(6.6), the local Lipschitz structure of the system ensures the uniqueness of the solution. To show the existence, we construct a solution using Lemma 6.1. We define \( \alpha, \beta, \mu, g_{33}, g_{23}, g_{22}, g_3, g_2, g_1 \) in
terms of \( z_1 \) and \( z_2 \) in Lemma 6.1:

\[
\begin{align*}
\alpha(t) &:= \frac{a\sigma^2 I}{2}(1 - t), \\
\beta(t) &:= z_1(1 - t), \\
\mu(t) &:= -\frac{a\sigma^2 \Sigma}{I}(1 - t), \\
g_{33}(t) &:= z_2(1 - t), \\
g_23(t) &:= \frac{z_1(1 - t)}{I}, \\
g_{22}(t) &:= \frac{a\sigma^2 D}{2I^2}(1 - t), \\
g_3(t) &:= \frac{-\sigma}{I} z_1(1 - t), \\
g_2(t) &:= \frac{-a\sigma^2 \Sigma}{I^2}(1 - t), \\
g_1(t) &:= \frac{a\sigma^2 \Sigma^2}{2I^2}(1 - t) + \sigma^2 \int_t^1 g_{33}(s) ds.
\end{align*}
\]  

(6.7)

By explicit computations using (6.1), we can check that (6.7) satisfies the ODEs in (6.5)-(6.6).

Proof of Theorem 3.2 (Radner existence with exogenous noise trading).

Let \((\alpha, \beta, \mu, g_1, g_2, g_3, g_{22}, g_{23}, g_{33})\) be the unique smooth solution to the coupled system of ODEs (6.5)-(6.6), whose expression is given in (6.7).

First, we prove verification for the utility-maximizing exponential investors via a duality approach. For the function \( V \) given in (3.3), we let the process \( \hat{V} \) be given by \( \hat{V}_t := V(t, X^\theta_t, Y_t, Y_t') \), \( t \in [0, 1] \), where \( \theta \) is defined in (3.6). We drop the subscript \( j \) from \( \hat{\theta} \) since all utility-maximizing investors are identical. For notational simplicity, we denote \( \hat{X}_t := X^\theta_t \). We define the measure \( \hat{Q} \) by

\[
\frac{d\hat{Q}}{dP} := \frac{\hat{V}_1}{\hat{V}_0},
\]

and we reason that \( \hat{Q} \) is a probability measure. By (3.3), the terminal conditions (3.4), and the ODE system (6.5)-(6.6), the dynamics of \( \hat{V} \) are given by

\[
d\hat{V}_t = -a\hat{V}_t \left( \hat{\theta}_t dD_t + \left( \beta(t) \hat{\theta}_t + g_3(t) + 2Y'_t g_{33}(t) + Y_t g_{23}(t) \right) dY'_t \right).
\]

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Since $\hat{\theta}_t$ is affine in $Y_t$, the functions $\beta$, $g_3$, $g_{23}$, and $g_{33}$ are continuous functions of $t$, and $Y_t$ is a progressively measurable functional of $Y'$, we apply Corollary 3.5.16 of [11] to show that $(\hat{V}_t)_{t\in[0,1]}$ is a martingale under $\mathbb{P}$. Thus, $\hat{Q}$ is a probability measure. Since $\hat{V}_1 = -e^{-a\hat{X}_1}$, we also have that $\hat{V}_0 = E\left[-e^{-a\hat{X}_1}\right]$. 

Next, we show that $\hat{X}$ is a $\hat{Q}$-martingale by checking that it is a $\hat{Q}$-local martingale and $E\hat{Q}\left[\int_0^1 \hat{\theta}_t^2 d\langle S\rangle_t\right] < \infty$. Under $\hat{Q}$, $\hat{D}$ and $\hat{Y}'$ are Brownian motions, where 

$$
d\hat{D}_t := dD_t + a\sigma_D^2 \hat{\theta}_t dt, \quad \hat{D}_0 := 0, \\
d\hat{Y}'_t := dY'_t + a\sigma_{Y'}^2 \left( \beta(t)\hat{\theta}_t + g_3(t) + 2Y'_tg_{33}(t) + Y_tg_{23}(t) \right) dt, \quad \hat{Y}'_0 := 0.
$$

By (3.6) and (3.5), the dynamics of $\hat{X}$ are given by 

$$
d\hat{X}_t = \hat{\theta}_tdS_t = \hat{\theta}_t \left( d\hat{D}_t + \beta(t)d\hat{Y}'_t \right).
$$

Since $\beta(t)$ is bounded in $t \in [0,1]$, it suffices to show that $E\hat{Q}\left[\int_0^1 \hat{\theta}_t^2 dt\right] < \infty$ in order to prove that the $\hat{Q}$-local martingale, $\hat{X}$, is a $\hat{Q}$-martingale. Moreover, since $\hat{\theta}_t$ is affine in $Y_t$ and the coefficient functions that appear in the ODE system are all bounded in $t \in [0,1]$, showing that $E\hat{Q}\left[\int_0^1 (Y'_t)^2 dt\right] < \infty$ is sufficient to prove the $\hat{Q}$-martingale property of $\hat{X}$. To this end, we define stopping times 

$$
\tau_k := \inf \left\{ t \geq 0 : |Y'_t| \geq k \right\} \land 1, \quad k \geq 1.
$$

We appeal to the definition of $\hat{Y}'$ in (6.8) to see that there exist constants $C_1, C_2 \geq 0$ that are independent of $t$ and $k$ such that for all $k \geq 1$, 

$$
E\hat{Q}\left[(Y'_{t\land\tau_k})^2\right] \leq C_1 + C_2 E\hat{Q}\left[\int_0^t (Y'_{u\land\tau_k})^2 du\right], \quad t \in [0,1].
$$

Gronwall’s inequality implies that for all $k \geq 1$,

$$
E\hat{Q}\left[(Y'_{t\land\tau_k})^2\right] \leq C_1 e^{C_2 t}, \quad t \in [0,1].
$$
By Fatou’s Lemma, we have

\[
\mathbb{E}^{\hat{Q}} \left[ (Y_t')^2 \right] \leq \liminf_{k \to \infty} \mathbb{E}^{\hat{Q}} \left[ (Y_{t \wedge \tau_k}')^2 \right] \leq C_1 e^{C_2 t}, \quad t \in [0, 1],
\]

and thus,

\[
\mathbb{E}^{\hat{Q}} \left[ \int_0^1 (Y_t')^2 \, dt \right] \leq \int_0^1 C_1 e^{C_2 t} \, dt < \infty.
\]

The exponential investors have utility functions \( U(x) := -\exp(-ax), \ x \in \mathbb{R} \). The Fenchel-Legendre transform of the function \(-U(-x)\) is given by

\[
\bar{U}(y) := \sup_{x \in \mathbb{R}} \{ U(x) - xy \} = -\frac{y}{a} \left( 1 - \log \frac{y}{a} \right), \quad y > 0.
\]

Therefore, for any admissible \( \theta \in \mathcal{A}(\hat{Q}) \) with an associated wealth process \( X^\theta \) beginning with initial wealth \( X_0^\theta = \hat{X}_0 \), we have

\[
\mathbb{E} \left[ U \left( X_1^\theta \right) \right] \\
\leq \mathbb{E} \left[ \bar{U} \left( a e^{-a \hat{X}_1} + a e^{-a \hat{X}_1} X_1^\theta \right) \right] \\
= \mathbb{E} \left[ e^{-a \hat{X}_1} \left( -a \hat{X}_1 - 1 \right) \right] + a \mathbb{E}^{\hat{Q}}[X_1^\theta] \cdot \mathbb{E} \left[ e^{-a \hat{X}_1} \right] \quad \text{by definition of } \bar{U} \text{ and } \hat{Q} \\
= \mathbb{E} \left[ U \left( \hat{X}_1 \right) \right] - a \mathbb{E}^{\hat{Q}}[X_1] \cdot \mathbb{E} \left[ e^{-a \hat{X}_1} \right] + a \mathbb{E}^{\hat{Q}}[X_1^\theta] \cdot \mathbb{E} \left[ e^{-a \hat{X}_1} \right] \\
\leq \mathbb{E} \left[ U \left( \hat{X}_1 \right) \right] \quad \text{since } \hat{X} \text{ is a } \hat{Q}\text{-martingale and } \theta \in \mathcal{A}(\hat{Q}),
\]

which shows that \( \hat{\theta} \) is the optimal strategy for the utility-maximizing investors.

Finally, the form of the optimal strategy in (3.6) ensures the market clearing condition in (3.2). \( \square \)
6.2 Proof of Theorem 4.2

Lemma 6.3. Let $a, \sigma_D^2, \kappa > 0$. Then, the following two-dimensional initial value problem has a unique solution for $t \in [0, \infty)$:

\[ \begin{align*}
  z_1'(t) &= \left( \frac{2a\kappa \sigma_D^2}{2\kappa + a\sigma_D^2} \right) t - \frac{4a\kappa I \sigma_D^2 z_1(t) z_2(t)}{a\sigma_D^2 + a\sigma_D^2 z_1(t)^2 + 2\kappa I}, \quad z_1(0) = 0, \\
  z_2'(t) &= \frac{2a\sigma_D^2}{a\sigma_D^2 + 2\kappa I} - \frac{2a\sigma_D^2 \left( (a\sigma_D^2 + 2\kappa I)^2 + a\sigma_D^2 z_1(t)^2 (a\sigma_D^2 + 4\kappa I) \right) z_2(t)^2}{(a\sigma_D^2 + a\sigma_D^2 z_1(t)^2 + 2\kappa I)^2}, \quad z_2(0) = 0.
\end{align*} \]  
(6.9)

Proof. The proof of Lemma 6.3 follows along nearly identical lines as the proof of Lemma 6.1. Replacing the computations in (6.2) with

\[ \begin{align*}
  z_1(0) &= z_1'(0) = 0 < z_1''(0) = \frac{2a\kappa \sigma_D^2}{2\kappa + a\sigma_D^2}, \\
  z_2(0) &= z_2'(0) = z_2''(0) = 0 < z_2'''(0) = \frac{4a\kappa^2 \sigma_D^2}{(2\kappa + a\sigma_D^2)^2},
\end{align*} \]

allows the proof the proceed as in Lemma 6.1.

\[ \Box \]

The presence of a noise tracker adds considerable complication to the system of equations describing equilibrium. Similar to the exogenous noise model of Section 3, we conjecture the form of the utility maximizers’ and noise tracker’s value functions. The utility maximizing investors’ value functions are conjectured to have the form (3.3)-(3.4). For the noise tracker with state processes time $t$, wealth $x$, and $Y'$, we search for smooth functions $f_1, f_2, f_3, f_{22}, f_{23}, f_{33} : [0, 1] \to \mathbb{R}$ such that

\[ V_N(t, x, Y') = x + f_1(t) + f_2(t) Y + f_3(t) Y' + f_{22}(t) Y^2 + f_{23}(t) Y Y' + f_{33}(t) Y'^2, \quad (6.10) \]

with $f_1(1) = f_2(1) = f_3(1) = f_{22}(1) = f_{23}(1) = f_{33}(1) = 0$. The investors’ HJB equations and the market clearing condition produce the ODE system for the coefficient
functions in (3.3) and (6.10), and \((\alpha, \beta, \mu)\) in (3.5):

\[
g'_1(t) = \frac{\alpha \sigma_{23}^2 g_3(t)^2 \left( \alpha \sigma_{23}^2 \beta(t) (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2 \right)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - \frac{2a \Sigma \kappa (\alpha \sigma_{23}^2 g_3(t) (\beta(t) - \kappa \Sigma) (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) + 2a \Sigma \kappa (\alpha \sigma_{33}^2, \beta(t)^2) - \sigma_{23}^2 g_{33}(t),}
\]

\[
g'_2(t) = \frac{\alpha \sigma_{23}^2 g_{23}(t) \left( g_3(t) (\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2) + 2a \Sigma \beta(t) \kappa (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) \right)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - \frac{2a \kappa (\alpha \sigma_{23}^2, \beta(t)^2) (\alpha \sigma_{23}^2, g_{23}(t) \beta(t) - 2I \kappa)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2},
\]

\[
g'_3(t) = \frac{2a \sigma_{23} g_{23} \left( (\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2) + 2a \Sigma \beta(t) \kappa (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) \right) - g_2(t),}
\]

\[
g''_1(t) = \frac{\alpha \sigma_{23}^2 g_{23}(t) \left( g_3(t) (\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2) + 2a \Sigma \beta(t) \kappa (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) \right)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - \frac{2a \kappa (\alpha \sigma_{23}^2, g_{23}(t) \beta(t) + \kappa) (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) \right)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2},
\]

\[
g''_2(t) = \frac{2a \sigma_{23} g_{23}(t) \left( (\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2) - 2a \kappa (\sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2) \right) - 2g_2(t),}
\]

\[
g''_3(t) = \frac{2a \sigma_{23} g_{23}(t) \left( (\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2) - 2g_2(t),}
\]

\[
g_1(1) = g_2(1) = g_3(1) = g_{22}(1) = g_{23}(1) = g_{33}(1) = 0,
\]

\[
f'_1(t) = -\kappa \left( \frac{\alpha \sigma_{23}^2, \beta(t)^2 (\alpha \sigma_{33}^2 + 4I \kappa) + (\alpha \sigma_{33}^2 + 2I \kappa)^2)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - \sigma_{23}^2 f_{33}(t),
\]

\[
f'_2(t) = -\frac{2a \kappa (\alpha \sigma_{23}^2 g_{23}(t) \beta(t) + 2k) \left( \sigma_{23}^2, \beta(t) (I g_3(t) + \Sigma \beta(t)) + \Sigma \beta(t) \right)}{2 \left( \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2},
\]

\[
f'_3(t) = -\frac{4a^2 I \sigma_{23}^2 g_{33}(t) \beta(t) \kappa (\sigma_{23}^2, \beta(t) (I g_3(t) + \Sigma \beta(t)) + \Sigma \beta(t) \right)}{2 \left( \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - f_2(t),
\]

\[
f''_2(t) = \frac{a \kappa (\sigma_{23}^2, \beta(t) (\beta(t) - I g_{23}(t)) + \sigma_{23}^2) \left( \alpha \sigma_{23}^2, \beta(t) (M g_{23}(t) + \beta(t)) + \alpha \sigma_{33}^2 + 4I \kappa \right)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2},
\]

\[
f''_3(t) = -\frac{4a^2 I^2 \sigma_{23}^2 g_{33}(t) (\beta(t)^2 + 2I \kappa)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - 2 f_{22}(t),
\]

\[
f''_3(t) = -\frac{4a^2 I^2 \sigma_{23}^2 g_{33}(t) (\beta(t)^2 + 2I \kappa)}{2 \left( \alpha \sigma_{33}^2 + \sigma_{23}^2, \beta(t)^2 + 2I \kappa \right)^2} - f_{33}(t),
\]

\[
f_1(1) = f_2(1) = f_3(1) = f_{22}(1) = f_{23}(1) = f_{33}(1) = 0,
\]

(6.11)
where

\[
\begin{align*}
\alpha'(t) &= -\frac{2\alpha_k (\sigma^2_{\beta, g_23}(t) \beta(t) - \sigma^2_D)}{\sigma_D^2 + \sigma^2_{\beta, g_22}(t)^2 + 2\kappa}, \quad \alpha(1) = 0, \\
\beta'(t) &= \frac{4\alpha_k \sigma^2_{\beta, g_33}(t) \beta(t) - \alpha(t)}{\sigma^2_D + \sigma_{\beta, g_23}(t)^2 + 2\kappa}, \quad \beta(1) = 0, \\
\mu'(t) &= \frac{2\alpha_k (\sigma^2_{\beta, g_33}(t) \beta(t) + \Sigma \sigma^2_D)}{\sigma^2_D + \sigma^2_{\beta, g_23}(t)^2 + 2\kappa}, \quad \mu(1) = 0.
\end{align*}
\] (6.12)

This ODE system will be used to verify optimality of the equilibrium trading strategies in the proof of Theorem 4.2. Lemma 6.4 below establishes the existence and uniqueness of a smooth solution to the system (6.11)-(6.12).

**Lemma 6.4.** Let \( a, \sigma^2_D, \kappa > 0 \). Then, there exists a unique smooth solution to the coupled system of ODEs (6.11)-(6.12) for \( t \in [0, 1] \).

**Proof.** Once we show the existence of a solution of (6.11)-(6.12), the local Lipschitz structure of the system ensures the uniqueness of the solution. To show the existence, we construct a solution using Lemma 6.3. In (6.11)-(6.12), we observe that the ODEs for \( \alpha, \beta, \mu, g_{33}, g_{23}, g_{22}, g_3, g_2 \) do not depend on \( g_1, f_1, f_2, f_3, f_{22}, f_{23}, f_{33} \). Hence, we first define \( \alpha, \beta, \mu, g_{33}, g_{23}, g_{22}, g_3, g_2 \) in terms of \( z_1 \) and \( z_2 \) in Lemma 6.3:

\[
\begin{align*}
\alpha(t) &= \frac{2\alpha_k \sigma^2_D}{2\kappa + \sigma^2_D} (1 - t), \\
\beta(t) &= z_1 (1 - t), \\
\mu(t) &= -\frac{2\alpha_k \sigma^2_D \Sigma}{2\kappa + \sigma^2_D} (1 - t), \\
g_{33}(t) &= z_2 (1 - t), \\
g_{23}(t) &= \frac{2\kappa}{2\kappa + \sigma^2_D} z_1 (1 - t), \\
g_{22}(t) &= \frac{2\alpha_k \sigma^2_D}{(2\kappa + \sigma^2_D)^2} (1 - t), \\
g_3(t) &= -\frac{2\kappa \Sigma}{2\kappa + \sigma^2_D} z_1 (1 - t), \\
g_2(t) &= -\frac{4\alpha_k \sigma^2_D \Sigma}{(2\kappa + \sigma^2_D)^2} (1 - t).
\end{align*}
\] (6.13)

By explicit computations using (6.9), we can check that (6.13) satisfies the ODEs for \( \alpha, \beta, \mu, g_{33}, g_{23}, g_{22}, g_3, g_2 \) in (6.11)-(6.12).
Given (6.13), the ODEs for $g_1, f_1, f_2, f_3, f_{22}, f_{23}, f_{33}$ in (6.11)-(6.12) become a linear system of ODEs and we have the following explicit solutions:

\[
\begin{align*}
g_1(t) &:= \frac{2a\kappa^2\sigma_2^2\Sigma^2}{(2I\kappa+a\sigma^2_{\beta, D})^2} (1 - t) + \sigma_Y^2 \int_t^1 g_{33}(s) \, ds, \\
f_2(t) &:= \frac{4aI\kappa^2\sigma_1^2\Sigma}{(2I\kappa+a\sigma^2_{\beta, D})^2} (1 - t), \\
f_3(t) &:= \frac{2aI\kappa^2\sigma_1^2\Sigma}{(2I\kappa+a\sigma^2_{\beta, D})^2} (1 - t)^2 + \int_t^1 \frac{4a^2I\kappa\Sigma^2\sigma_1^2\sigma_2^2 + g_{33}(s)\beta(s)}{(2I\kappa+a\sigma^2_{\beta, D})(2I\kappa+a\sigma^2_{\beta, D} + a\sigma^2_Y, \beta(s))} \, ds, \\
f_{22}(t) &:= \left( \frac{4I^2\kappa^3}{(2I\kappa+a\sigma^2_{\beta, D})^2} - \kappa \right) (1 - t), \\
f_{23}(t) &:= \left( \frac{4I^2\kappa^3}{(2I\kappa+a\sigma^2_{\beta, D})^2} - \kappa \right) (1 - t)^2 + \int_t^1 \frac{8aI^2\kappa^2\sigma_2^2 + g_{33}(s)\beta(s)}{(2I\kappa+a\sigma^2_{\beta, D})(2I\kappa+a\sigma^2_{\beta, D} + a\sigma^2_Y, \beta(s))} \, ds, \\
f_{33}(t) &:= \int_t^1 f_{23}(s) \, ds + \int_t^1 \frac{4a^2I^2\kappa^2\sigma_1^2 + g_{33}(s)^2\beta(s)^2}{(2I\kappa+a\sigma^2_{\beta, D} + a\sigma^2_Y, \beta(s))^2} \, ds, \\
f_{1}(t) &:= \frac{a^2\kappa^2\sigma_1^2\Sigma^2}{(2I\kappa+a\sigma^2_{\beta, D})^2} (1 - t) + \sigma_Y^2 \int_t^1 f_{33}(s) \, ds.
\end{align*}
\]

Note that the integrals above are all finite due to $a, \sigma_Y^2, \kappa > 0. \quad \square$

**Proof of Theorem 4.2 (Radner existence with endogenous noise tracking).**

Let $(\alpha, \beta, \mu, g_1, g_2, g_3, g_{22}, g_{23}, g_{33}, f_1, f_2, f_3, f_{22}, f_{23}, f_{33})$ be the unique smooth solution to the coupled system of ODEs (6.11)-(6.12), whose expression is given in (6.13)-(6.14).

First, we must verify that $\hat{\theta}_j$ in (4.5) is admissible and optimal for the utility-maximizing investors. The proof is largely identical to the verification proof of Theorem 3.2. We only sketch the steps here in order to point out the few proof differences while omitting most of the details.

For the function $V$ given in (3.3), we let the process $\hat{V}$ be given by $\hat{V}_t := V(t, \hat{X}_t, Y_t, Y'_t)$, $t \in [0, 1]$, where we denote $\hat{X}_t := X^\hat{\theta}_j$. We define the measure $\hat{Q}$ by

\[
\frac{d\hat{Q}}{dP} := \frac{\hat{V}_1}{\hat{V}_0}.
\]

The measure $\hat{Q}$ is a probability measure under which $\hat{X}$ is a square-integrable martingale. The argument to show square-integrability relies on $\hat{\theta}_j$ from (4.5) being affine in $(Y_t, Y'_t)$, rather than just affine in $Y_t$, as in the proof of Theorem 3.2. Therefore,
\[ \hat{\theta_j} \in \mathcal{A}(\hat{Q}). \] An identical duality argument to the proof of Theorem 3.2 shows that \( \hat{\theta_j} \) is optimal for the exponential investors.

Second, we show that \( \hat{\theta_N} \) is optimal for the noise tracker. In addition to being adapted and measurable, \( \hat{\theta_N} \) is square-integrable with \( \mathbb{E} \left[ \int_0^1 \hat{\theta}_{N,t}^2 \, dt \right] < \infty \), since \( \beta \) and \( g_{33} \) are continuous functions of time and \( Y \) and \( Y' \) are Gaussian.

Pointwise maximization under the expectation and integral in (4.1) lead to

\[ \hat{\theta}_{N,t} = Y_t + \frac{1}{2\kappa} \left( \mu'(t) + \alpha'(t)Y_t + (\beta'(t) + \alpha(t))Y'_t \right). \]

Plugging in \( \mu, \alpha, \beta, \) and \( g_{33} \) as given in (6.13) agrees with the desired formula for \( \hat{\theta_N} \) in (4.6).

Finally, the form of the optimal strategies in (4.5) and (4.6) ensure the market clearing condition in (4.3).

### 6.3 Proof of Proposition 5.1

We let the superscripts \( \text{ex} \) and \( \text{en} \) represent exogenous and endogenous quantities, respectively. Theorems 3.2 and 4.2 provide us with the existence of equilibria described by the systems of equations given in (6.5)-(6.6) and (6.11)-(6.12), respectively. Using the equilibrium clearing conditions from Definitions 3.1 and 4.1, we see that

\[
\sum_{j=1}^{I} CE_{\text{j}}^{\text{en}} - \sum_{j=1}^{I} CE_{\text{j}}^{\text{ex}} = \Sigma \left( S_0^{\text{en}} - S_0^{\text{ex}} \right) + I \left( g_1^{\text{en}}(0) - g_1^{\text{ex}}(0) \right)
\]

\[
= \Sigma \left( \mu^{\text{en}}(0) - \mu^{\text{ex}}(0) \right) + I \left( g_1^{\text{en}}(0) - g_1^{\text{ex}}(0) \right).
\]

The system of equations (6.5)-(6.6) decouples so that

\[
(g_{1}^{\text{ex}})'(t) = -\frac{a\Sigma^2\sigma_D^2}{2I^2} - \sigma_Y^2 g_{33}^{\text{ex}}(t),
\]

\[
\mu^{\text{ex}}(t) = \frac{a\Sigma^2\sigma_D^2}{I}(t - 1).
\]
and the system of equations (6.11)-(6.12) decouples so that

\[(g_1^{en})'(t) = -\frac{2\alpha \kappa \Sigma^2 \sigma_D^2}{(a\sigma_D^2 + 2\kappa I)^2} - \sigma_Y^2 g_{33}^{en}(t),\]

\[\mu^{en}(t) = \frac{2\alpha \kappa \Sigma \sigma_D^2}{a\sigma_D^2 + 2\kappa I} (t - 1).\]

These calculations show us that in the exogenous case,

\[\Sigma (\mu^{ex})'(0) + I (g_1^{ex})'(0) = \frac{a\Sigma^2 \sigma_D^2}{2I} - g_{33}^{ex}(0)I\sigma_Y^2,\]

and in the endogenous case, we have

\[\Sigma (\mu^{en})'(0) + I (g_1^{en})'(0) = \frac{2\alpha \kappa \Sigma \sigma_D^2 (a\sigma_D^2 + \kappa I)}{(a\sigma_D^2 + 2\kappa I)^2} - g_{33}^{en}(0)I\sigma_Y^2.\]

Therefore, the welfare difference is calculated as

\[\sum_{j=1}^{I} CE_j^{en} - \sum_{j=1}^{I} CE_j^{ex} = \frac{a\Sigma^2 \sigma_D^2}{2I(a\sigma_D^2 + 2\kappa I)^2} + I\sigma_Y^2 \int_0^1 (g_{33}^{en}(u) - g_{33}^{ex}(u)) du,\]

as desired.

\[\diamond\]

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