Basis problem for analytic multiple gaps

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Abstract. $k$-gaps are sequences $(C_i)_{i=1}^k$ of pairwise disjoint monotone families of infinite subsets of $\mathbb{N}$ mixed in such a way that we can’t find a partition $\mathbb{N} = M_1 \cup \cdots \cup M_k$ such that $C_i | M_i = \emptyset$ for all $i = 1, \ldots, k$. We say that a $k$-gap $(C_i)_{i=1}^k$ is a reduction of a $k$-gap $(D_i)_{i=1}^k$ and write $(C_i)_{i=1}^k \leq (D_i)_{i=1}^k$ whenever $(C_i)_{i=1}^k$ is isomorphic to a restriction of $(D_i)_{i=1}^k$ to an infinite subset of $\mathbb{N}$. We prove that, relative to this notion of comparison, for every positive integer $k$ there is a finite basis for the class of all analytic $k$-gaps. We also build the fine structure theory of analytic $k$-gaps and give some applications.

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## Contents

**Introduction**  5  
Basic definitions  9  

Chapter 1. The existence of a finite basis  11  
1.1. A partition theorem  11  
1.2. Types in the \(m\)-adic tree  17  
1.3. Finding standard objects  18  
1.4. Projective gaps under determinacy  22  
1.5. Some remarks  26  

Chapter 2. Working in the \(n\)-adic tree  29  
2.1. Normal embeddings  29  
2.2. The max function  31  
2.3. Chain types  39  
2.4. Domination  41  
2.5. Subdomination  45  
2.6. Freely generated minimal gaps  59  
2.7. Strong gaps  60  
2.8. Countable partitions  69  

Chapter 3. Lists of minimal analytic gaps  71  
3.1. Minimal analytic 2-gaps  71  
3.2. Minimal analytic 3-gaps  72  
3.3. Minimal analytic dense gaps  91  
3.4. Higher dimensions  92  

Chapter 4. Applications  97  
4.1. Breaking analytic gaps and the topology of \(\omega^*\)  97  
4.2. Selective coideals  101  
4.3. Sequences in Banach spaces  103  

Bibliography  107  

Index  109
Introduction

In this paper we investigate the following general question for every integer $k \geq 2$.

**Question 1.** Given a sequence $C_1, ..., C_k$ of pairwise disjoint monotone\(^1\) families of infinite subsets of $\mathbb{N}$, is there any combinatorial structure present in the class of its restrictions $C_1|_M, ..., C_k|_M$ to infinite subsets $M$ of $\mathbb{N}$?

To see the relevance of this question, consider a sequence $(x_n)$ of objects (functions, points in a topological space, vectors of a normed space, etc) and let $C_i$ be the collection of all infinite subsets $M$ of $\mathbb{N}$ for which the corresponding subsequence of $(x_n)_{n \in M}$ has some property $P_i$ that is inherited when passing to a subsequence. If $(x_n)$ is a sequence of vectors of some normed space setting, the properties $P_i$ could be for example different incompatible estimates on how the norms are computed in the subsequence (e.g., being $\ell_1$-sequence, $\ell_2$-sequence, $c_0$-sequence, etc.). In the topological setting, we could consider $P_1$ to be the property of being convergent to a point $x$, and $P_2$ not accumulating to $x$, and many other variants. We want to know whether by passing to a subsequence of $(x_n)$ we could get some canonical behavior in each such example. Note that the example in the normed space setting is more concrete, *analytic* as we will call it, and that the general topological example looks rather unmanageable. As we will see, this turns out to be the dividing line between cases where the structure can be found and the cases where the structure is absent.

Before we proceed further we need a concept that helps us in properly stating the somewhat vague Question 1 above. Namely, we say that classes $C_1, ..., C_k$ are *separated* if we can find a partition $\mathbb{N} = M_1 \cup \cdots \cup M_k$ such that $C_i|_{M_i} = \emptyset$ for all $i = 1, ..., k$. If they are not separated, they are said to be *mixed* or that they form a $k$-*gap* (if $k = 2$, we have a 2-gap, which is what is usually called just a gap). What we are after is a combinatorial structure theory that would recognize all the canonical $k$-gaps $C_1, ..., C_k$ modulo the possibility of replacing it by an appropriate restrictions $C_1|_M, ..., C_k|_M$ to infinite subsets $M$ of $\mathbb{N}$. That this project, even in the case $k = 2$, is of a great complexity was first realized by Hausdorff \(^{18}\) more than a century ago after a series of earlier papers of Du Bois-Reymond \(^{11}\) and Hadamard \(^{16}\) showing that the case of countable families $C_i$ presents no difficulties. What Hausdorff showed is that there exist 2-gaps that could code objects far outside the reach of structure theory we would hope to develop. It is interesting that even such a non-structure theorem of Hausdorff would bare some fruits such as, for example, the solution of Kaplanski’s problem (see \(^{20}\)) about the automatic continuity in the context of Banach algebras where the answers essentially depended

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\(^1\)by monotone we mean that $M \subseteq N$ and $N \in C_i$ imply $M \in C_i$

\(^2\)\(C|_M = \{ N \in C : N \subseteq M \}\)
on whether the list of spectra of 2-gaps discovered by Hausdorff [18] is complete or not (see [12], [6], [7]).

When considering problems arising in the mathematical practise, the gaps that one gets are not just arbitrary, but usually have a low descriptive complexity and, in fact, are of low Borel complexity like for example, the ones we have mentioned above in the context of normed spaces. It was the second author [27] who first realized that many of the pathologies that might occur in arbitrary gaps can be ruled out for gaps of certain descriptive complexity, and so, in particular, the pathologies discovered by Hausdorff in [18] and [19] discussed above. It is interesting that even the special structure theory of descriptive 2-gaps developed in [27] has found some substantial applications (see, for example, [28], [2], [10], [9]).

The need to consider more than two families $C_i$ was first encountered in one of our previous papers [3] while trying to give lower estimate on norms of averaging operators. It turns out that a number of new phenomena show up when considering more than two families. The following problem posed in [3] in a different language, has turned out to be a key leading question in our research since then.

Question 2: Suppose that $C_1$, $C_2$, $C_3$ is a 3-gap. Can we find a restriction to an infinite set $C_1|_M$, $C_2|_M$, $C_3|_M$ in which one of the three families is empty and the remaining two form a 2-gap?

This looks like a rather arbitrary question, but in fact it is a question that reveals the key difficulty in any program of building a useful structure theory of descriptive $k$-gaps. In analogy to Hausdorff’s nonstructure theorem discussed above, there is a negative answer found in [3] by considering three classes of sets built out of a certain partition of the real line into non-measurable sets. But what if the classes of subsequences are of reasonable descriptive complexity as these are the classes that one usually finds in the practise? This was left open in [3], and in fact, in this paper we develop a whole new technology to answer Question 2 in the positive.

The new theory of descriptive $k$-gaps that we develop here uses three layers of arguments belonging to different areas of mathematics. The first layer requires a suitable extension of the analytic gap theorems from [27] that gives them a particularly canonical tree-representation. This is done in Section 1.3. The word ‘analytic’ here refers to the descriptive complexity of continuous images of Polish spaces, a complexity that covers essentially all cases in practice. However, we show in Section 1.4 that assuming projective determinacy, all results hold for projective families as well. In practical terms, it means that even coanalytic or any reasonably definable classes $C_i$ of sequences can be studied within this theory.

Once the tree-representation is settled, the second layer is a new Ramsey principle for trees, not covered by the Ramsey theory of strong subtrees of Milliken [25] (used in our previous paper [4]), that we could only develop using some deep reasoning from topological dynamics. We have devoted Section 1.1 to this result, that we feel that it is of independent interest and may have further applications. After
this Ramsey theorem is applied, we are given, for every \( k \), a finite list of basic analytic \( k \)-gaps so that, given any analytic \( k \)-gap, the restriction to a suitable infinite subset of \( \mathbb{N} \) must be isomorphic (in a precise sense, see Definitions 0.0.3 and 0.0.4) to one of the gaps from the finite list. This means that those configurations that appear in that list are possible, but whatever is not found there is a forbidden behavior in the analytic case. So we have just reached the level in the development of our theory where we show that for each positive integer \( k \) canonical examples of pairwise disjoint \( k \)-sequences \( C_1, \ldots, C_k \) of monotone families of infinite subsets of \( \mathbb{N} \) do exist and that, in fact, there are only finitely many of them that could be described in some detail. For example, in Section 3.4 we give an expression for and some lower and upper estimates on the cardinality \( N(k) \) of the irredundant list of basic \( k \)-gaps. For example, we have

\[
2^{J(k-1)-k-1} < N(k) < k^{J(k)-k},
\]

where \( J(k) \) is a function given by certain sum of combinatorial numbers, whose asymptotic behavior is

\[
J(k) \sim \frac{3}{8\sqrt{2\pi k}} \cdot 9^k.
\]

The number \( J(k) \) is in fact of independent interest as it is equal to the cardinality of the set of all oscillation types that are directly used in defining the basic \( k \)-gaps on the index-set \( k^{<\mathbb{N}} \) rather than on \( \mathbb{N} \). They are suggested by our Ramsey-theoretic analysis of analytic \( k \)-gaps that give us embeddings from \( k^{<\mathbb{N}} \) into \( \mathbb{N} \) transferring the basic \( k \)-gaps into restrictions of arbitrary analytic \( k \)-gaps.

It is now clear how to answer Question 2 above: check that all 3-gaps from the finite list satisfy that property, just one by one. The problem is that the information that we are given after this first two layers of arguments is still rough. To get an idea of the difficulties in the structure theory developed up to this point, it would provide a list of \( 3^{58} \) basic 3-gaps. The checking task perhaps could be given in such raw terms to a computer for this particular question, but we are interested in a general understanding of what is possible and what is not in the context of analytic gaps. For this purpose, the information needs to be refined, and this can only be done using finite combinatorial methods in order to study how certain functions act on certain special types of finite subsets of finitely branching trees. This is the third layer of the theory, to which Chapter 2 is devoted. With the three layers developed, we are able to find in Chapter 3 the complete descriptions of the irredundant lists of the minimal analytic 2-gaps and of the minimal analytic 3-gaps and get a considerable grip on the minimal lists for \( k > 3 \).

We must mention that, although we have been guided by multidimensional problems, our theory already gives new and deeper information about classical 2-gaps. So far, only the first layer of the theory for 2-gaps had been considered in [27]. The lists of 5 minimal 2-gaps and of 163 minimal 3-gaps provides a book where to check any three-dimensional question on definable gaps of the sort of Question 2. But the effort to get these somehow exotic lists should not be viewed only as an objective in itself, but such a task has guided us in developing the combinatorial tools from Chapter 2 that unravel the structure of general analytic \( k \)-gaps. These tools however, are not enough for a full understanding of analytic \( k \)-gaps for \( k > 3 \).
In other words, we do not have a precise description of the minimal analytic $k$-gaps for large $k$, so we do not have a general method to solve any given question in higher dimensions. The reason is that the finite combinatorics involved - what we called the third layer - is too intricate. We do not know if it would be possible to go deeper in the understanding of these combinatorial questions and get a general description of the minimal analytic $k$-gaps, or if this might be as hopeless as trying to have a full understanding of, say, how all finite groups look like. We should, however, mention that we do have a substantial partial description of the family of all basic analytic $k$-gaps that is still quite useful. For example, using this theory, we are able to answer questions like Question 2 above for an arbitrary integer $k$ in place of 3.

The reader can get a flavor of the kind of special constrainsthat our results establish on how definable classes $C_i$ can be mixed, by looking at the following sample result: Let $C_1, \ldots, C_k$ be an analytic $k$-gap. Then there exists a an infinite set $M$ where $C_1|_M$ and $C_2|_M$ form a 2-gap, while all but at most 6 of the remaining classes $C_i|_M$ are empty. The number 6 is optimal, and corresponds to the value $6 = J(2) – 2$ of the function $J$ mentioned above. The proof of this fact, the answer to Question 2 and other results of the kind are given in Chapter 4.

We should also mention our previous paper [4] as an important precedent of this work. We consider there similar problems but dealing with countable separation instead of separation, and strong $k$-gaps instead of general $k$-gaps. A similar structure in three layers is present there, but the three of them were much easier. On the first layer, the generalization of the corresponding dichotomy of [27] was easier to figure out, and this was already done in [3]. On the second layer, the Ramsey principle needed was much weaker and followed from Milliken’s theorem [25]. On the third layer, the finite combinatorics involved was less intricate, and we were able to completely analyze them and to give a satisfactory description of the minimal analytic strong $k$-gaps for every positive integer $k$.

As it will be clear, we hope, to anyone reading this paper, the results that follow from the theory that we develop in this paper are easy to state and understand and deal with very basic objects that are of wide interest: sequences and different kind of subsequences. They are however rather difficult facts to prove and very much unexpected. As we mentioned at the beginning of this introduction, the analytic gap dichotomies proved by the second author in [27], which correspond to the first layer of our theory (in dimension 2), have been already found quite useful. It is natural to expect more applications with the deeper and more general theory at hand, and in more areas in mathematics than we presently have.

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Basic definitions

As mentioned above, we introduced the notion of multiple gaps in our previous papers [3, 4]. This time, we shall modify slightly our working definition to gain some generality, we shall work with preideals instead of ideals.

**Definition 0.0.1.** A preideal on a countable set \( N \) is a family \( I \) of subsets of \( N \) such that if \( x \in I \) and \( y \subset x \) is finite, then \( y \in I \).

The preideal ideal \( I \) is analytic if it is analytic as a subset of \( P(N) = 2^N \). We use the symbol \( * \) to denote inclusions modulo finite sets, so \( x \subset^* y \) means that \( x \setminus y \) is finite, and \( x =^* y \) means that \( x \subset^* y \) and \( y \subset^* x \). We say that \( x \) and \( y \) are orthogonal if \( x \cap y =^* \emptyset \). Following the tradition in set-theory, we identify each natural number with its set of predecessors, so that \( n = \{ 0, 1, \ldots, n-1 \} \). The set of natural numbers is written as \( \omega = \{ 0, 1, 2, \ldots \} \). In this way, \( \{ x_i : i \in n \} = \{ x_0, x_1, \ldots, x_{n-1} \} \), while \( \{ x_n \}_{n<\omega} \) denotes an infinite sequence. The letters \( N \) and \( M \) will denote some general countable infinite sets.

**Definition 0.0.2.** Let \( \Gamma = \{ \Gamma_i : i \in n \} \) be a family of \( n \) many preideals on the set \( N \) and let \( \mathcal{X} \) be a family of subsets of \( n \).

1. We say that \( \Gamma \) is separated if there exist subsets \( a_0, \ldots, a_{n-1} \subset N \) such that \( \bigcap_{i \in n} a_i = \emptyset \) and \( x \subset^* a_i \) for all \( x \in \Gamma_i, i \in n \).
2. We say that \( \Gamma \) is an \( \mathcal{X} \)-gap if it is not separated, but \( \bigcap_{i \in A} x_i =^* \emptyset \) whenever \( x_i \in \Gamma_i, A \in \mathcal{X} \).

We say that \( \Gamma \) is analytic if each \( \Gamma_i \) is analytic. In the same way, we can say that \( \Gamma \) is Borel, coanalytic, projective, etc. We will consider only two choices of the family \( \mathcal{X} \), when \( \mathcal{X} = [n]^2 \) is the family of all subsets of \( n \) of cardinality 2, a \( [n]^2 \)-gap will be called an \( n \)-gap, while when \( \mathcal{X} \) consists only of the total set \( n = \{ 0, \ldots, n-1 \} \), then an \( \mathcal{X} \)-gap will be called an \( n \)-gap. The notion of \( n \)-gap is more general than that of a \( n \)-gap, since it does not require the preideals to be pairwise orthogonal. On the other hand, the use of \( n \)-gaps is more natural in some contexts, and for many of the problems that we discuss here, questions about \( n \)-gaps can be reduced to questions about \( n \)-gaps.

In the language of sequences and subsequences that we used in the introduction, if we have an infinite sequence \( \{ x_n \}_{n<\omega} \), and \( \mathcal{C} \) is a hereditary class of subsequences, then \( I = \{ a \subset \omega : \{ x_n \}_{n \in a} \in \mathcal{C} \} \) is a preideal. When we talk about analytic, Borel or projective classes, we mean that the corresponding preideals have that complexity. It is a simple exercise that the notion of separation of preideals stated above is equivalent to the notion of separation of classes that was stated in the introduction. In this way, all the results that we shall produce about gaps can be restated as facts about classes of subsequences and the ways that they can be separated.

Remember that the general question that our theory deals with is the following: Given a gap \( \Gamma \) on \( N \), can we find an infinite set \( M \subset N \) such that the restriction of \( \Gamma \) to \( M \) becomes a gap which is canonical in some sense? The restriction of a preideal \( I \) to \( M \) is the preideal \( I|_M = \{ x \in I : x \subset M \} \), and the restriction of a gap \( \Gamma \) is \( \Gamma|_M = \{ \Gamma_i|_M : i \in n \} \). Notice that \( \Gamma|_M \) may not be in general a gap, as
the preideals may become separated when restricted to $M$.

The orthogonal of $I$ is the family $I^\perp$ consisting of all $x \subset N$ such that $x \cap y = \emptyset$ for all $y \in I$. The orthogonal of the gap $\Gamma$ is $\Gamma^\perp = (\bigcup_{i \in n} \Gamma_i)^\perp$. The gap $\Gamma$ is called dense if $\Gamma^\perp$ is just the family of finite subsets of $N$.

**Definition 0.0.3.** Given $\Gamma$ and $\Delta$ two $n_*$-gaps on countable sets $N$ and $M$, we say that $\Gamma \leq \Delta$ if there exists a one-to-one map $\phi : N \rightarrow M$ such that for every $i \in n$,

1. if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$,
2. if $x \in \Gamma_i^\perp$ then $\phi(x) \in \Delta_i^\perp$.

When $\Gamma$ is a $n$-gap, the second condition can be substituted by saying that if $x \in \Gamma_i^\perp$ then $\phi(x) \in \Delta_i^\perp$. Notice also that if $\Delta$ is a $n$-gap, $\Gamma$ is a $n_*$-gap, and $\Gamma \leq \Delta$, then $\Gamma$ is an $n$-gap. Another observation is that the above definition implies that $\phi(x) \in \Delta_i^{\perp\perp}$ if and only if $x \in \Gamma_i^{\perp\perp}$, and $\phi(x) \in \Delta_i^\perp$ if and only if $x \in \Gamma_i^\perp$. Therefore the gaps $\{\Gamma_i^{\perp\perp} : i < n\}$ and $\{\Delta_i^{\perp\perp} : i < n\}$ are completely identified under the bijection $\phi : N \rightarrow \phi(N)$.

**Definition 0.0.4.** An analytic $n_*$-gap $\Gamma$ is said to be a minimal analytic $n_*$-gap if for every other analytic $n_*$-gap $\Delta$, if $\Delta \leq \Gamma$, then $\Gamma \leq \Delta$.

**Definition 0.0.5.** Two minimal analytic $n_*$-gaps $\Gamma$ and $\Gamma'$ are called equivalent if $\Gamma \leq \Gamma'$ (hence also $\Gamma' \leq \Gamma$).

In this language, one of our main results can be stated as follows:

**Theorem 0.0.6.** Fix a natural number $n$. For every analytic $n_*$-gap $\Gamma$ there exists a minimal analytic $n_*$-gap $\Delta$ such that $\Delta \leq \Gamma$. Moreover, up to equivalence, there exist only finitely many minimal analytic $n_*$-gaps.

The same statements hold for $n$-gaps instead of $n_*$-gaps, the minimal analytic $n$-gaps are a subset of the minimal analytic $n_*$-gaps.
CHAPTER 1

The existence of a finite basis

1.1. A partition theorem

In this section we state and prove a new pigeon hole principle that at the same time incorporates some features of the infinite Hales-Jewett theorem for left-variable words (17, 5) and some features of the Gowers theorem for FIN_k (14); see also [29]. In particular, we shall rely on the Galvin-Glazer method of idempotent ultrafilters on partial semigroups. We refer the reader to the introductory chapters of [29] where this method is explained in details and where both the Gowers theorem and the extension of the Hales-Jewett theorem are proved using this method. The reader will find there also some details and references about the long and intricate way this subject was developed so that we can comment here only about the new ideas in the proof below. First of all, we have to restrict ourselves to idempotent ultrafilters $U_k$ on semigroups of words $W_k$ that besides the usual equations $T(U_k) = U_k$ satisfy the equations $U_k^{-1} U_k = U_k$ rather than the stronger equations $U_k^{-1} U_k = U_k^{-1} U_k = U_k$ for $l \leq k$. The idempotent ultrafilters $U_k$ on $W_k$ are then used to obtain an infinite-dimensional Ramsey statement that involves the notion of an $U_k$-tree. An infinite sequence $\{w_0, w_1, \ldots\}$ generates a partial subsemigroup $Full(w_0, w_1, \ldots)$ in the standard way (see [29], Section 2.5). The crucial lemma here is that for every $U_k$-subtree $\Upsilon$ of $W_k$ there exists a rapidly increasing sequence $w_0, w_1, \ldots$ of elements of $W_k$ such that $(x_0, x_1, \ldots, x_n) \in \Upsilon$ for every $x_0, x_1, \ldots, x_n \in Full(w_0, w_1, \ldots)$ with $\lambda(x_0) < \lambda(x_1) < \cdots < \lambda(x_n)$, where for $x \in Full(w_0, w_1, \ldots)$, by $\lambda(x)$, we denote the maximal index of a term of the sequence $\{w_0, w_1, \ldots\}$ that occurs in the unique concatenation that forms $x$. This allows us to transfer Souslin-measurable colorings of subtrees of $W_k$ of the same shape to colorings of branches of the the tree $W_k^\omega$ and in return get a copy of $W_k$ with all subtrees of the given shape monochromatic. This is quite different from the standard method that involves the Ramsey space of strong subtrees of a given rooted finitely branching tree $U$ of height $\omega$ (see [25, 29], Chapter 6). We expect that this approach will find some other uses.

Given a set $A$, we denote by $A^{<\omega}$ the set of all finite sequences of elements of $A$. Remember that we identify a natural number $m$ with its set of predecessors, $m = \{0, 1, \ldots, m - 1\}$. Thus, $m^{<\omega}$ is the $m$-adic tree. We consider two order relations on $m^{<\omega}$. Consider $t = (t_0, \ldots, t_p)$ and $s = (s_0, \ldots, s_q)$ in $m^{<\omega}$, the tree order is defined by $t < s$ if and only if $p < q$ and $t_i = s_i$ for all $i \leq p$. The linear order relation $\prec$ is given by: $t \prec s$ if and only if either $(p < q)$ or $(p = q$ and $t_{\min \{i : t_i \neq s_i\}} < s_{\min \{i : t_i \neq s_i\}})$. The concatenation of $t$ and $s$ is $t \cdot s = (t_0, \ldots, t_p, s_0, \ldots, s_q)$. We denote by $t \land s$ the infimum of $t$ and $s$ in the order $\prec$,
that is, \( t \wedge s \) is the largest common initial segment of \( t \) and \( s \). If \( t \sim s = r \), then we write \( s = r \setminus t \).

For a fixed natural number \( k \), we denote by \( W_k \) the set of all finite sequences of natural numbers from \( \{0, \ldots, k\} \) that start by \( k \), that is

\[
W_k = \{ (t_0, t_1, \ldots, t_p) : t_0 = k, t_i \in \{0, \ldots, k\}, i = 1, \ldots, p \}
\]

We shall view the set \( W_k \) as a semigroup, endowed with concatenation \( \sim \) as the operation. Define \( T : W_k \to W_{k-1} \) by

\[
T(w)(i) = \max\{0, w(i) - 1\}
\]

That is, \( T(w) \) is a word with the same number of letters as \( w \), and at each place \( T(w) \) has a number one unit less than in \( w \), except for zeros which are preserved.

Let \( T^{(0)} : W_k \to W_k \) be the identity map, \( T^{(1)} = T \) and \( T^{(j)} : W_k \to W_{k-j} \) be the \( j \)-th iterate of \( T \). We will often denote this iteration as \( T_i = T^{(k-i)} : W_k \to W_i \), using the same subindex for \( T_i \) as for the range space \( W_i \).

**Definition 1.1.1.** We will say that a subset \( F \subset m^{<\omega} \) is closed if it satisfies:

1. If \( s, t \in F \), then \( s \wedge t \in F \)
2. If \( s = t \sim r_1 \cdots r_k \) with \( t, s \in F \), \( r_1 \in W_{i_1}, \ldots, r_k \in W_{i_k}, i_1 < i_2 < \cdots < i_k \), then \( t \sim r_1 \in F \) (therefore also \( t \sim r_1 r_2 \in F \), etc.)

Given \( F \subset m^{<\omega} \) we will denote by \( \langle F \rangle \) the intersection of all closed sets which contain \( F \), which is itself a closed set.

**Definition 1.1.2.** Consider sets \( X \subset m^{<\omega} \), \( Y \subset n^{<\omega} \). A function \( f : X \to Y \) is called an equivalence if it is the restriction of a bijection \( g : \langle X \rangle \to \langle Y \rangle \) satisfying the following

1. \( g(t \wedge s) = g(t) \wedge g(s) \) for all \( t, s \in \langle X \rangle \),
2. \( g(t) \prec g(s) \) if and only if \( t \prec s \) for all \( t, s \in \langle X \rangle \),
3. For all \( t, s \in \langle X \rangle \) with \( t \leq s \) and every \( k \), we have that \( s \setminus t \in W_k \) if and only if \( g(s) \setminus g(t) \in W_k \).

Notice that if \( s \) is the immediate successor of \( t \) in \( \langle X \rangle \) (that is, \( t < s \) but there is no \( r \in \langle X \rangle \) with \( t < r < s \)), then \( s \setminus t \in W_k \) for some \( k \), and condition (3) of the above definition can be considered just for pairs of immediate successors. The sets \( X \) and \( Y \) are called equivalent if there is an equivalence between them.

A sequence \( \{w_0, w_1, \ldots\} \subset W_k \) is called rapidly increasing if

\[
|w_i| > \sum_{j<i} |w_j|
\]

for every \( i \). A family \( \{w_s : s \in m^{<\omega}\} \subset W_k \) will be called rapidly increasing if for every \( s \in m^{<\omega} \) we have

\[
|w_s| > \sum_{t<s} |w_t|.
\]

**Definition 1.1.3.** Let \( m \leq n < \omega \). A function \( \psi : m^{<\omega} \to n^{<\omega} \) will be called a nice embedding if there exists a rapidly increasing family \( \{w_s : s \in m^{<\omega}\} \subset W_{m-1} \) such that for every \( t \in m^{<\omega} \) and for every \( i \in m \), we have that \( \psi(t \sim i) = \psi(t) \sim T_i(w_{t \sim i}) \).
Notice that the above implies that \( \psi \) is one-to-one. Along this section we are mostly interested in nice embeddings from the \( m \)-adic tree into itself. The important thing about nice embeddings is that they preserve equivalence.

**Proposition 1.1.4.** If \( \psi : m^{<\omega} \rightarrow n^{<\omega} \) is a nice embedding, then \( X \) is equivalent to \( \psi(X) \) for every set \( X \subset m^{<\omega} \).

The range of a nice embedding \( \psi \) will be call a nice subtree of \( m^{<\omega} \), which is naturally bijected with \( m^{<\omega} \) itself by \( \psi \). For a fixed set \( X_0 \subset m^{<\omega} \), let us say that \( Y \) is an \( X_0 \)-set if \( Y \) is equivalent to \( X_0 \). It is easy to check that the family of all \( X_0 \)-subsets of \( m^{<\omega} \) is closed in the product topology of the Cantor set \( 2^{m^{<\omega}} \), hence this family has a natural Polish topology. This section is devoted to the proof of the following theorem:

**Theorem 1.1.5.** Fix a set \( X_0 \subset m^{<\omega} \). Then for every finite partition of the \( X_0 \)-subsets of \( m^{<\omega} \) into finitely many Suslin-measurable sets, there exists a nice subtree \( T \subset m^{<\omega} \) all of whose \( X_0 \)-subsets lie in the same piece of the partition.

By Suslin measurability, we mean with respect to the \( \sigma \)-algebra generated by analytic sets. This is a partition theorem for trees in a similar spirit as Milliken’s Theorem [25]. Partition theorems are often stated in the language of colorings. Having a finite partition \( X = \bigcup_{i<n} X_i \) of a set \( X \) is equivalent to having a function \( c : X \rightarrow n \), that is called a coloring, and \( c(y) \) is called the color of \( y \). A subset \( Y \subset X \) lies in one piece of the partition if and only if it is monochromatic for the coloring \( c \), meaning that \( Y \subset c^{-1}(i) \) for some \( i \). The simplest case of Theorem [1.1.5] happens when \( X_0 \) is a singleton:

**Corollary 1.1.6.** If we color \( n^{<\omega} \) into finitely many colors, then there is a nice subtree which is monochromatic.

Let \( W_i^* \) be the collection of all nonprincipal ultrafilters on \( l_i \). We extend the concatenation \( \triangleleft \) to an operation on \( W_i^* \) that we also denote by \( \triangleleft \):

\[
A \in U \triangleleft V \iff (\forall x \in U) (\exists y \in V) x \triangleleft y \in A
\]

This\(^1\) makes \((W_i^*, \triangleleft)\) a compact left topological semigroup\(^2\). For every \( U \in W_k^* \) define

\[
T(U) = \{ T(X) : X \in U \} = \{ Y \subset W_{k-1} : T^{-1}(Y) \in U \}
\]

Notice that \( A \in T(U) \) if and only if \( \forall x \in U \) \( T(x) \in A \). The function \( T : W_k^* \rightarrow W_{k-1}^* \) is a continuous onto homomorphism.

We shall construct by induction ultrafilters \( U_k \in W_k^* \) for \( k = 0, 1, \ldots \) which will have the following properties:

1. Each \( U_k \) is a minimal idempotent of \( W_k^* \). Idempotent means that \( U_k \triangleleft U_k = U_k \), and \( U_k \) is minimal among the set of idempotents of \( W_k^* \) in the order given by \( U \leq V \) iff \( U \triangleleft V = V \triangleleft U \). Cf. \([29\) Chapter 2].
2. \( T(U_{k+1}) = U_k \) for every \( k = 0, 1, 2, \ldots \).
3. \( U_k \triangleleft U_l = U_k \) whenever \( l \leq k \).

---

\(^1\)The notation \( U \triangleleft P(x) \) means that \( \{ x : P(x) \} \in U \).

\(^2\)this means that the operation \( U \rightarrow U \triangleleft V \) is continuous for every \( V \in W_i^* \) that we endow with its Stone topology as a set of ultrafilters.
Notice that condition (3) above is just equivalent to $U_{k+1} \preceq U_k = U_{k+1}$ for every $k$. We choose $U_0$ to be any minimal idempotent of $W_0^*$ (see [29 Lemma 2.2]).

Construction of $U_{k+1}$ from $U_k$: Let

$$S = \{ X \in W_{k+1}^* : T(X) = U_k \}.$$ 

Then $S$ is a closed subsemigroup of $W_{k+1}^*$ and

$$S \preceq U_k = \{ X \preceq U_k : X \in S \}$$

is a closed left-ideal of $S$. Using [29 Lemma 2.2] again, we find $U_{k+1} \in S \preceq U_k$ a minimal idempotent of $S \preceq U_k$, which is in turn a minimal idempotent of $S$. Notice that $U_{k+1} \preceq U_k = U_{k+1}$ since $U_{k+1} \in S \preceq U_k$ and $U_k$ is idempotent. Also $T(U_{k+1}) = U_k$. It remains to show that $U_{k+1}$ is a minimal idempotent of $W_{k+1}^*$. Let $V \leq U_{k+1}$ be an idempotent of $W_{k+1}^*$. Since $T$ is a homomorphism and $T(U_{k+1}) = U_k$, we have that $T(V)$ is an idempotent of $W_k^*$ such that $T(V) \leq U_k$. Since $U_k$ was minimal, we conclude that $T(V) = U_k$, hence $V \in S$. Since $U_{k+1}$ was a minimal idempotent of $S$ we conclude that $U_{k+1} = V$.

The construction of the ultrafilters $U_k$ is thus finished. We define a $U_k$-tree to be a nonempty downwards closed subtree $Y$ of $W_k^{<\omega}$ such that

$$\{ x \in W_k : (x_0, \ldots, x_n, x) \in Y \} \in U_k$$

for every $(x_0, \ldots, x_n) \in Y$.

We shall use the following lemma which is a corollary of [29 Theorem 7.42]:

**Lemma 1.1.7.** For every finite Suslin-measurable coloring of the branches of $W_k^{<\omega}$ there exists a $U_k$-tree $Y$ such that the set of branches of $Y$ is monochromatic.

**Definition 1.1.8.** Let $\{w_0, w_1, \ldots\}$ be a rapidly increasing sequence of elements of $W_k$.

$$\text{Full}(w_0, w_1, \ldots) = \{ w_{m_0}^{-1} T_{k_1}(w_{m_1})^{-1} \cdots T_{k_n}(w_{m_n}) : n < \omega, 0 \leq k_1, \ldots, k_n \leq k, m_0 < \ldots < m_n < \omega \}$$

Given $x = w_{m_0}^{-1} T_{k_1}(w_{m_1})^{-1} \cdots T_{k_n}(w_{m_n})$ as above, we denote $\lambda(x) = m_n$ the last subindex of $w_i$ which appears in the expression of $x$. Notice that this is properly defined because the sequence $\{w_0, w_1, \ldots\}$ is rapidly increasing.

**Lemma 1.1.9.** Given $Y$ a $U_k$-tree of $W_k^{<\omega}$ there exists a rapidly increasing sequence $w_0, w_1, \ldots$ of elements of $W_k$ such that

$$(x_0, x_1, \ldots, x_n) \in Y$$

for every $x_0, x_1, \ldots, x_n \in \text{Full}(w_0, w_1, \ldots)$ with $\lambda(x_0) < \lambda(x_1) < \cdots < \lambda(x_n)$.

**Proof.** For every $\bar{x} = (x_0, \ldots, x_n) \in Y$, let $P_\bar{x} = \{ x \in W_k : (x_0, \ldots, x_n, x) \} \in Y$. Along this proof we denote $U_k = U$. We know that $P_\bar{x} \in U$ for every $\bar{x} \in Y$. We can assume without loss of generality that $P_x \supset P_y$ whenever $x < y$ (in the tree order, meaning that $y$ is an end-extension of $x$). We shall construct the sequence $w_0, w_1, \ldots$ by induction.
Construction of \( w_0 \). We know that \( \mathcal{U}x \in P_0 \), and since
\[
\mathcal{U} = \mathcal{U}^* T_{k_1} (\mathcal{U})^* \cdots T_{k_n} (\mathcal{U})
\]
for every \( k_1, \ldots, k_n \leq k \) we have that
\[
\mathcal{U} y_0, \mathcal{U} y_1 \cdots \mathcal{U} y_n \ y_0^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n) \in P_0
\]
In particular, we can choose \( w_0 \in P_0 \) such that
\[
\mathcal{U} y_1 \cdots \mathcal{U} y_n \ w_0^* T_{k_1} (y_1) \cdots T_{k_n} (y_n) \in P_0 \quad (*)
\]
whenever \( 0 \leq k_0 < k_1 < \cdots < k_n \leq k \), since there are only finitely many choices for indices \( k_i \) like this. Notice however, that once \( w_0 \) is chosen in this way, the statement \((*)\) above holds whenever \( 0 \leq k_0, k_1, \cdots, k_n \leq k \) (now infinitely many possibilities). The reason is that if we have an expression
\[
x = w_0^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n)
\]
we can choose \( 1 = i_1 < \ldots < i_m \) such that
\[
k_1 = k_{i_1} < k_{i_2} < \cdots < k_{i_m} = k_m
\]
such that \( k_j \leq k_{i_j} \) whenever \( i_j \leq j < i_{j+1} \). And then, we can rewrite
\[
x = w_0^* T_{k_1} (y_1)^* T^{(k_{i_1} - k_2)} (y_2)^* \cdots T_{k_{i_2}} (y_{k_{i_2}})^* T^{(k_{i_2} - k_{i_3})} (y_{k_{i_3} + 1})^* \cdots
\]

Construction of \( w_m \). Let \( F = Full(w_0, \ldots, w_{m-1}) \) and
\[
G = \{ \bar{x} = (x_0, \ldots, x_\xi) : x_0, \ldots, x_\xi \in F, \lambda(x_0) < \cdots < \lambda(x_\xi) \}.
\]
Notice that \( G \) is finite. Our inductive hypotheses are that for every \( (x_0, \ldots, x_\xi) \in G \) and every \( 0 \leq k_1, \cdots, k_n \leq k \) we have that
\[
x_\xi \in P_{(x_0, \ldots, x_{\xi-1})} \quad (*)
\]
\[
\mathcal{U} y_1 \cdots \mathcal{U} y_n \ x_\xi^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n) \in P_{(x_0, \ldots, x_{\xi-1})} \quad (**)
\]
The hypothesis \((*)\) will prove the statement of the Lemma. The hypothesis \((**)\) is a technical condition necessary for the inductive procedure. On the other hand, like in the case of the construction of \( w_0 \), we have that \( \mathcal{U} y \in P_{\bar{x}} \) for every \( \bar{x} \in G \), and this implies that for every \( \bar{x} \in G \) and every \( 0 \leq k_1, \cdots, k_n \leq k \)
\[
\mathcal{U} y_0 \mathcal{U} y_1 \cdots \mathcal{U} y_n \ y_0^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n) \in P_{\bar{x}}
\]
Therefore in particular, we can find \( w_m \) such that for every \( 0 \leq k_0 \leq k \), every \( 0 \leq k_1 < \cdots < k_n \leq k \) and every \( \bar{x} = (x_0, \ldots, x_\xi) \in G \) we have that
\[
w_m \in P_{\bar{x}}
\]
\[
\mathcal{U} y_1 \cdots \mathcal{U} y_n \ w_m^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n) \in P_{\bar{x}}
\]
\[
x_\xi^* T_{k_0} (w_m)^* T_{k_1} (y_1)^* \cdots T_{k_n} (y_n) \in P_{(x_0, \ldots, x_{\xi-1})}
\]
By the same trick that we used in the case of the construction of \( w_0 \), the above sentences actually hold whenever \( 0 \leq k_0, k_1, \ldots, k_n \leq k \). This completes the proof,
since the statements above imply that the inductive hypotheses (*) and (***) are transferred to the next step.

We proceed now to the proof of Theorem 1.1.5. Without loss of generality we can suppose that $X_0$ is a closed set. Indeed, if $X_0$ is not closed, consider its closure $Y_0 = \langle X_0 \rangle$. If $Y \sim Y_0$, then there is a unique set $X \sim X_0$ such that $Y = \langle X \rangle$, and the correspondence $Y \leftrightarrow X$ is Suslin-measurable. In this way, we reduce the general case to the case of closed $X_0$. We can suppose that $X_0$ is infinite as well. If we prove the theorem for infinite $X_0$, the finite case follows as a corollary, just making $X_0$ infinite by adding zeros above a maximal node. Enumerate $X_0 = \{x_0 \prec x_1 \prec \cdots \}$. Let $k = m - 1$, and we consider the infinite product $W_k^\omega$ that we identify when convenient with the branches of the tree $W_k^\omega$. Let $\tilde{W}_k^\omega \subset W_k^\omega$ be the set of all sequences which are rapidly increasing. We are going to define a function $\Phi$ that associates to each $z \in \tilde{W}_k^\omega$ a $X_0$-set $\Phi(z) \subset m^{<\omega}$. The set $\Phi(z)$ will be the range of a function $\phi_s : X_0 \rightarrow m^{<\omega}$ that establishes an equivalence between $X_0$ and $\Phi(z) = \phi_s(X_0)$. The function $\phi_s$ is defined inductively. As a starting point of the induction, $\phi_s(x_0) = z_0$. Now, suppose that $\phi_s(x_q)$ is defined for $q < p$ and we shall define $\phi_s(x_p)$. Let $x_q$ be the $\leq$-immediate predecessor of $x_p$ in $X_0$, and suppose that $x_p = x_q^\omega r$ with $r \in W_i$, $i \leq k$. Then, define $\phi_s(x_p) = \phi_s(x_q)^\omega T_i(z_p)$.

We consider now a finite partition of $W_k^\omega$, in which one piece is the set $W_k^\omega \setminus \tilde{W}_k^\omega$, while the partition of $\tilde{W}_k^\omega$ is induced by the given partition of the $X_0$-sets of $m^{<\omega}$ through the function $\Phi$. By Lemma 1.1.7 there exists a $\mathcal{U}_k$-tree $\Upsilon \subset W_k^{<\omega}$ all of whose branches lie in the same piece of the partition. This piece cannot be $W_k^\omega \setminus \tilde{W}_k^\omega$ since every $\mathcal{U}_k$ tree has rapidly increasing branches. So what we have is that for each rapidly increasing branch $z$ of $\Upsilon$, the set $\Phi(z)$ has the same color.

Let $\{w_0, w_1, \ldots \}$ be the sequence given by Lemma 1.1.9 applied to the $\Upsilon$ that we found. Let $F = \text{Full}(w_0, w_1, \ldots)$. We reorder $\{w_0, w_1, \ldots \}$ in the form of a rapidly increasing family $\{w_s : s \in m^{<\omega} \}$. We claim that the nice embedding that we are looking for is the one given by $\psi(\emptyset) = w_0$ and $\psi(t^\omega i) = \psi(t^\omega T_i(w_i^\omega i))$. In order to check this, it is enough to prove that for every $X_0$-set $Y$, the set $\psi(Y)$ is of the form $\Phi(z)$ for some $z \in \tilde{W}_k^\omega$ which is a branch of $\Upsilon$. So let $Y = \{y_0 \prec y_1 \prec \cdots \}$ be an $X_0$-set. Let $y_i^\omega$ be the $\leq$-immediate predecessor of $y_i$ in $Y$ and write $\psi(y_i^\omega y_i^\omega i) = T_j(z_i)$ for some $z_i \in W_k$. Then $z_i \in \text{Full}(w_0, w_1, \ldots)$ and $\lambda(z_0) < \lambda(z_1) < \cdots$, so by Lemma 1.1.9 we have that $(z_0, z_1, \ldots)$ is a branch of $\Upsilon$. Moreover $\psi(Y) = \Phi(z)$ and this finishes the proof of Theorem 1.1.5.

We finish this subsection with the following variation of Theorem 1.1.10. We refer to [22] for information on projective sets and the axiom of projective determinacy.

Theorem 1.1.10 (Projective Determinacy). Fix a set $X_0 \subset m^{<\omega}$. Then for every finite partition of the $X_0$-subsets of $m^{<\omega}$ into finitely many projective sets, there exists a nice subtree $T \subset m^{<\omega}$ all of whose $X_0$-subsets lie in the same piece of the partition.

We do not include the proof here since it is out of the scope of this paper. It follows the general lines of Woodin’s proof in [30] that every projective set is
Ramsey in the classical sense (cf. [29]) under the assumption of the projective determinacy.

1.2. Types in the m-adic tree

Among the equivalence classes of subsets of $m^{<\omega}$ to which Theorem 1.1.5 can be applied, we are particularly interested in the minimal equivalence classes of infinite sets, which are described by what we call types.

**Definition 1.2.1.** A type $\tau$ is a triple $\tau = (\tau^0, \tau^1, \triangleleft)$, where $\tau_0$ and $\tau_1$ are finite subsets of $\omega$ with $\tau^0 \neq \emptyset$, $\min(\tau^0) \neq \min(\tau^1)$, together with a linear order relation $\triangleleft$ on the set $(\tau^0 \times \{0\}) \cup (\tau^1 \times \{1\})$ which extends the natural order of $\tau^0$ and of $\tau^1$ and whose maximum is $(\max(\tau^0), 0)$.

Notice that in the above definition we demand that $\tau^0 \neq \emptyset$, but $\tau^1$ may be empty or not. The sentence “extends the natural order of $\tau^0$ and of $\tau^1$” means that $(k, i) \triangleleft (k', i)$ whenever $k < k'$ and $i \in \{0, 1\}$. A type $\tau$ will be represented as a ‘matrix’ where the lower row is $\tau^0$, the upper row is $\tau^1$ and the order $\triangleleft$ is read from left to right (so the rightmost element must be always below). For example

$$\tau = \begin{bmatrix} 6 & 9 \\ 1 & 3 & 6 & 7 \end{bmatrix}$$

would represent the type $((\{1, 3, 6, 7\}, \{6, 9\}, \triangleleft)$ with the order

$$(1, 0) \triangleleft (6, 1) \triangleleft (3, 0) \triangleleft (6, 0) \triangleleft (9, 1) \triangleleft (7, 0)$$

When $\tau^1 = \emptyset$ we will write a ‘matrix’ with just one row.

**Definition 1.2.2.** Consider a type $\tau$ where $\tau^0 = \{k_0 < \cdots < k_n\}$ and $\tau^1 = \{l_0 < \cdots < l_m\}$. We say that a couple $(u, v)$ is a rung of type $\tau$ if the following conditions hold:

1. $u$ can be written as $u_0 \overset{\cdots}{\cdots} u_n$ where $u_i \in W_{k_i}$,
2. $v$ can be written as $v_0 \overset{\cdots}{\cdots} v_m$ where $v_i \in W_{l_i}$,
3. $(k_i, 0) \triangleleft (l_j, 1)$ if and only if $u_0 \overset{\cdots}{\cdots} u_i \triangleleft v_0 \overset{\cdots}{\cdots} v_j$.

In the above definition notice that $v = \emptyset$ if and only if $\tau^1 = \emptyset$.

**Definition 1.2.3.** Consider a type $\tau$. We say that an infinite set $X \subset m^{<\omega}$ is of type $\tau$ if there exists $u \in m^{<\omega}$ and a sequence of rungs $(u_0, v_0), (u_1, v_1), \ldots$ of type $\tau$ such that we can write $X = \{x_0, x_1, \ldots\}$ and

$$x_k = u \overset{\cdots}{\cdots} u_0 \overset{\cdots}{\cdots} u_{k-1} \overset{\cdots}{\cdots} v_k$$

for $k = 0, 1, \ldots$

When $\tau^1 = \emptyset$, subsets of type $\tau$ will be called $\tau$-chains. If $\tau^1 \neq \emptyset$ they will be called $\tau$-combs. A type in $m^{<\omega}$ is a type such that $\tau^0, \tau^1 \in m^{<\omega}$. These are the possible types of subsets of $m^{<\omega}$.

Let us give a couple of examples as illustration. The set

$$\{(00), (00213), (00213213), (00213213213), \ldots\}$$

is a [23]-chain, because it satisfies Definition 1.2.3 for $u = (00)$, and the rungs $(u_i, v_i)$ where $v_i = \emptyset$, and $u_i = (21)\overset{\cdots}{\cdots}(3)$, with $(21) \in W_2$ and $(3) \in W_3$.
On the other hand, the set
\[
\{(005), (002135), (002132135), (002132132135), \ldots\}
\]
is a $[\mathcal{P}_23]$-comb, because if satisfies Definition 1.2.3 for $u = (00)$, and the rungs $(u_i, v_i)$ where $v_i = (5) \in W_5$, and $u_i = (21)^{-1}(3)$, with $(21) \in W_2$ and $(3) \in W_3$.

For a fixed type $\tau$, the sets of type $\tau$ constitute an equivalence class of subsets of $m^{<\omega}$. Every infinite subset of a set of type $\tau$ has again type $\tau$. These facts, together with the following lemma, imply that types can be identified with the minimal equivalence classes of infinite subsets of $m^{<\omega}$.

**Lemma 1.2.4.** If $x$ is an infinite subset of $m^{<\omega}$, then there exists a type $\tau$ and an infinite subset $y \subset x$ of type $\tau$.

**Proof.** Define inductively \( \{t_k : k < \omega\} \subset x \), a chain \( \{s_k : k < \omega\} \subset m^{<\omega} \) and infinite sets \( x = x_0 \supset x_1 \supset \cdots \) in the following way: First, \( x_0 = x \), \( s_0 = \emptyset \) and \( t_0 \in x \) is arbitrary. Given \( t_k, s_k, x_k \), fix a number \( p_k > |t_k| \) and choose \( s_{k+1} \) such that \( |s_{k+1}| = p_k, s_{k+1} > s_k, x_{k+1} = \{t \in x_k : t > s_{k+1}\} \) is infinite\(^3\) and \( t_{k+1} \in x_{k+1} \). The set \( \{t_k : k < \omega\} \subset x \) obtained in this manner may still not be of any type $\tau$ but it is very close. Consider \( r_k = t_k \land t_{k+1} \) which lie in a chain as \( r_k \leq s_{k+1} \). By passing to a subsequence we may suppose that max\( (r_{k+1} \setminus r_k) \) is the same for all \( k \in \omega \), and by passing to a further subsequence we may suppose that the pairs \( (r_{k+1} \setminus r_k, t_k \setminus r_k) \) are all rungs of the same type $\tau$, and then we will get that \( \{t_k : k < \omega\} \) is indeed of type $\tau$. \( \square \)

### 1.3. Finding standard objects

A family of sets $I$ is said to be countably generated in a family $J$ if there exists a countable subset $J_0 \subset J$ such that for every $x \in I$ there exists $y \in J_0$ such that $x \subset y$. The following is restatement of 27 Theorem 3:

**Theorem 1.3.1.** If \( \{\Gamma_0, \Gamma_1\} \) are preideals on $N$ such that $\Gamma_1$ is analytic and is not countably generated in $\Gamma_0^*$, then there exists an injective function $u : 2^{<\omega} \rightarrow N$ such that $u(x) \in \Gamma_1$ whenever $x$ is an $[i]$-chain, $i = 0, 1$.

**Proof.** The actual statement of 27 Theorem 3 says that there is a $\Gamma_0$-tree all of whose branches are in $\Gamma_1$, which means that there is a family $\Sigma$ of finite subsets of $N$ such that

1. $\emptyset \in \Sigma$,
2. $\Sigma_a = \{k \in N : a \cup \{k\} \in \Sigma\}$ is an infinite set in $\Gamma_0$, for every $a \in \Sigma$,
3. if $a_0, a_1, \ldots \in \Sigma$ with $a_0 \subset a_1 \subset \cdots$, then $\bigcup_{k < \omega} a_i \in \Gamma_1$.

We define inductively the function $v : 2^{<\omega} \rightarrow N$ together with a function $a : 2^{<\omega} \rightarrow \Sigma$ such that $v(s) \in \Sigma_{a(s)}$ in the following way: $a(\emptyset) = \emptyset$, $v(\emptyset)$ is some element of $\Sigma_0$, $a(s \uparrow 0) = a(s)$, $a(s \uparrow 1) = a(s) \cup \{v(s)\}$, $v(s \uparrow 0)$ is an element of $\Sigma_{a(s \uparrow 0)} = \Sigma_{a(s)}$ different from all $v(t)$ that have been previously chosen, and finally $v(s \uparrow 1)$ is an element of $\Sigma_{a(s \uparrow 1)}$ different from all $v(t)$ that have been previously chosen. Then, we have

1. If $x = \{s_0 < s_1 < \cdots\}$ is a $[0]$-chain in $2^{<\omega}$, then $a(s_i) = a(s_0)$ for all $i < \omega$, hence $\{v(s_1), v(s_2), \ldots\} \subset \Sigma_{a(s_0)}$ and therefore $v(x) \in \Gamma_0$, since $\Sigma$ was a $\Gamma_0$-tree.

---

\(^3\)The property that $\{t \in x_k : t > s_k\}$ is infinite is assumed inductively on $k$. 

(2) If \( x = \{ s_0 < s_1 < \cdots \} \) is a \([1]\)-chain, then \( a(s_0) \subseteq a(s_1) \subseteq \cdots \) and \( v(x) = \{ v(s_0), v(s_1), \ldots \} \subseteq \bigcup_{i<\omega} a(s_i) \in \Gamma_1 \) since all branches of \( \Sigma \) are in \( \Gamma_1 \).

(3) Finally, \( v \) is injective because at each step we take care that \( v(t) \) is different from all previously chosen values of \( v \).

\[ \square \]

Theorem 1.3.2. If \( \Gamma = \{ \Gamma_i : i \in \mathbb{N} \} \) are analytic preideals on the set \( N \) which are not separated, then there exists a permutation \( \varepsilon : n \rightarrow n \) and a one-to-one map \( u : n^{<\omega} \rightarrow N \) such that \( u(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]\)-chain, \( i \in n \).

Proof. We may assume that \( \Gamma \) is an \( n\)-gap because otherwise the statement of the theorem is trivial. We will prove the theorem by induction on \( n \). At each step, we shall assume that the statement of the theorem holds for smaller \( n \) and we shall find a permutation \( \varepsilon : n \rightarrow n \) and a function \( v : n^{<\omega} \rightarrow \omega \) such that \( v(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]\)-chain, \( i \in n \), but \( v \) will not be one-to-one. Instead, \( v \) will have the property that for every \( s \in n^{<\omega} \), the set \( \{ v(s^0q) : q < \omega \} \) is infinite.

Let us show how to get the one-to-one function \( u \) that we are looking for from a function \( v \) as above. For this we shall consider a one-to-one \( g : n^{<\omega} \rightarrow n^{<\omega} \) and we will make \( u = vg \). The value of \( g(s) \) is defined \( \omega \)-inductively on \( s \): \( g(\emptyset) = \emptyset \) and \( g(s^{-i}) = g(s^{-i}) \cup (0,0,\ldots,0) \) where the number of zeros is chosen so that \( g(g(s^{-i})) \) is different from all \( v(g(i)) \) which have been already defined. Notice that \( g(x) \) is an \([i]\)-chain whenever \( x \) is an \([i]\)-chain and \( u = vg \) is one-to-one and satisfies the statement of the theorem.

Initial case of the induction: \( n = 2 \). In view of Theorem 1.3.1 it is enough to check either \( \Gamma_1 \) is not countably generated in \( \Gamma_0 \) or \( \Gamma_0 \) is not countably generated in \( \Gamma_1 \). So suppose that we had \( x_0 \subset x_1 \subset \cdots \) witnessing that \( \Gamma_1 \) is countably generated in \( \Gamma_\emptyset \), and \( y_0 \subset y_1 \subset \cdots \) witnessing that \( \Gamma_0 \) is countably generated in \( \Gamma_1 \). Then, the elements \( x = \bigcup_{k<\omega} x_k \setminus y_k \) and \( y = \bigcup_{k<\omega} y_k \setminus x_k \) separate \( \Gamma_1 \) and \( \Gamma_0 \). This finishes the proof of the case when \( n = 2 \).

Inductive step: We assume that the theorem holds for \( n - 1 \) and we construct the function \( v \) for \( n \). We say that a family \( I \) of sets is covered by a family \( J \) if for every \( x \in I \) there exists \( y \in J \) such that \( x \subset y \). We say that a set \( a \subset N \) is small if \( \Gamma_a \) is separated. We say that \( I \) covers \( \Gamma \) if \( I \) covers \( \bigcup I \Gamma_i \).

Claim A. \( \Gamma \) cannot be covered by countably many small sets.

Proof of Claim A. Assume that \( \{ a_k : k < \omega \} \) is a sequence of small sets that covers \( \Gamma \). We can suppose that \( a_0 \subset a_1 \subset a_2 \subset \cdots \). For every \( k \), since \( \Gamma_{a_k} \) is separated, there exist sets \( a_k(i), i \in n \) such that \( \bigcap_i a_k(i) = \emptyset \) and \( x \subset^* a_k(i) \) whenever \( x \in \Gamma_{a_k} \). By choosing these sets inductively on \( k \), we can make sure that \( a_k(i) \subset a_{k+1}(i) \) for every \( k, i \). At the end the sets \( a(i) = \bigcup_k a_k(i) \) witness that \( \Gamma \) is separated. This contradiction finishes the proof of Claim A.
By Claim A, we can find \( p \in n \) such that \( \Gamma_p \) is not covered by countably many small sets. Without loss of generality we assume that \( p = n - 1 \). If \( \varepsilon \) is a permutation of \( p \), we say that \( a \subseteq N \) is \( \varepsilon \)-small if there exists no one-to-one function \( u : p^{<\omega} \to a \) such that \( u(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]-\)chain, \( i < p \).

Claim B: There exists a permutation \( \varepsilon : p \to p \) such that \( \Gamma_p \) is not covered by countably many \( \varepsilon \)-small sets.

Proof of the claim: Suppose for contradiction, that \( \Gamma_p \) is countably covered by \( \tau \)-small sets for every permutation \( \tau : p \to p \). Let \( A_\tau \) be a countable family of \( \tau \)-small sets that covers \( \Gamma_p \). Then the family of all intersections of the form \( a = \bigcap_{\alpha \in A_\tau} a_\alpha \) with \( a_\alpha \in A_\tau \) is a countable family that also covers \( \Gamma_p \). Moreover, each such set \( a \) is small by the inductive hypothesis, since we cannot find a permutation \( \tau \) and a one-to-one function \( u : p^{<\omega} \to a \) such that \( u(x) \in \Gamma_{\tau(i)} \) when \( x \) is an \([i]-\)chain, \( i < p \). This contradicts that \( \Gamma_p \) cannot be covered by small sets, and finishes the proof of Claim B.

A tree on the set \( N \times \omega \) is a subset \( \Upsilon \subseteq (N \times \omega)^{<\omega} \) such that if \( t \in \Upsilon \) and \( s < t \) (in the tree-order, meaning that \( s \) is an initial segment of \( t \)) then \( s \in \Upsilon \). A branch of \( \Upsilon \) is an infinite sequence \( (s < t) = (\xi < p^{<\omega} ; r : r \in p^{<\omega}) \). For formal reasons, we consider an imaginary element \( \Upsilon' = (\xi < p^{<\omega} = \emptyset) \). Remember that trees on \( N \times \omega \) characterize analytic families of subsets of \( N \), in the sense that a family \( I \) of subsets of \( N \) is analytic if and only if there exists a tree \( \Upsilon \) on \( N \times \omega \) such that \( I = [\Upsilon]_1 \), where

\[ [\Upsilon]_1 = \{ \{ \xi_k : k < \omega \} : \exists \{ (\xi_k, m_k) : k < \omega \} \in [\Upsilon] \} \]

Since the ideal \( \Gamma_p \) is analytic, we can find a tree \( \Upsilon \) such that \( \Gamma_p = [\Upsilon]_1 \). For \( t \in \Upsilon \) let us denote by \( \Upsilon_t = \{ s \in \Upsilon : s \geq t \text{ or } s \leq t \} \). Let \( \Upsilon' \) be the set of all branches of \( \Upsilon \) that are countably covered by \( \varepsilon \)-small sets. Notice that \( \Upsilon' \) is a downwards closed subtree of \( \Upsilon \). Also, for each \( t \in \Upsilon' \) we have that \( [\Upsilon'_t] \) is not countably covered by \( \varepsilon \)-small sets, since \( [\Upsilon'_t] \) is obtained by removing from \( [\Upsilon_t] \) countably many sets of the form \( [\Upsilon_s] \) which are countably covered by \( \varepsilon \)-small sets.

We shall define the function \( v : n^{<\omega} \to N \) together with a function \( z : n^{<\omega} \to \Upsilon' \). For \( s \in n^{<\omega} \), let \( X_s = \{ s \cup p^{<\omega} r : r \in p^{<\omega} \} \). By induction on \( s \), we shall define \( v|X_s \) and \( z(s) \). For formal reasons, we consider an imaginary element \( \xi \) such that \( \xi^{<p} = \emptyset \). In this way, \( \xi \) is the first step of the induction. We choose \( z(\xi) = \emptyset \). Since \( [\Upsilon'] \) is not covered by countably many \( \varepsilon \)-small sets, in particular \( b = \bigcup [\Upsilon'] \) is not \( \varepsilon \)-small, hence we have a one-to-one function \( v_\xi : X_\xi = p^{<\omega} \to b \) such that \( v(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]-\)chain, \( i < p \). We define \( v|X_\xi = v_\xi \). This finishes the initial step of the inductive definition. We shall suppose along the induction that if \( s \in X_t \) then \( v(s) \in \bigcup [\Upsilon'_z(t)] \setminus z_1(t) \), where \( z_1(t) \) is the set of first coordinates of \( z(t) \): if \( z(t) = \{ (\xi_k, m_k) : k < k_0 \} \) then, \( z_1(t) = \{ \xi_k : k < k_0 \} \).

So suppose that we want to define \( v \) on \( X_s \) and \( z(s) \). Then \( s \in X_t \) for some \( t < s \), \( s = t^{<p} r, r \in p^{<\omega} \). Therefore \( v(s) \in \bigcup [\Upsilon'_z(t)] \setminus z_1(t) \), so there is a branch of \( \Upsilon'_z(t) \) such that \( v(s) \) appears in the first element at some point -higher than the length of \( z(t) \)- in the branch. We pick \( z(s) > z(t) \) to be a node in this
branch which is high enough in order that \( v(s) \) appears in the first coordinate. Let 
\[ b_s = \bigcup \{ \gamma z_i : z_i(s) \} \] which is not \( \varepsilon \)-small, so we get a one-to-one \( v_s : p^\omega \rightarrow b_s \) such that \( v_s(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]-chain\), \( i < p \). For \( \hat{r} = s^*p^\omega r \in X_s \) we define \( v(\hat{r}) = v_s(r) \). This finishes the inductive definition of \( v \).

Let us check that \( v \) has the properties that we were looking for. If \( t \in n^\omega \), then the set \( t = \{ t^*0^k : k < \omega \} \) is contained in some \( X_s \), so since the function \( v|_{X_s} \), obtained from \( v_s \), was one-to-one it is clear that \( v(x) \) is infinite. Suppose that \( x \) is an \([i]-chain\) with \( i < p \). Then \( x \subset X_s \) for some \( s \), and then \( v|_{X_s} \) was given by \( v_s \) which was chosen such that \( v_s(x) \in \Gamma_{\varepsilon(i)} \) whenever \( x \) is an \([i]-chain\), \( i < p \). Finally, suppose that \( x \) is a \([p]-chain\), so that \( x = \{ s_0, s_1, s_2, \ldots \} \) with \( s_k^*p \leq s_{k+1} \) for every \( k < \omega \). Then, by enlarging \( x \) intercalating extra elements if necessary we can suppose that \( s_{k+1} \in X_s \) for every \( k < \omega \). Then, by the way that we chose \( z(s) \) inductively, we have that \( z(s_0) < z(s_1) < \ldots \) and \( v(s_k) \) is the first coordinate of a node of \( z(s_{k-1}) \) above the length of \( z(s_{k-1}) \). It follows that \( \{ v(s_k) : k < \omega \} \subseteq \gamma(z) \) above the length of \( z(s_{k-1}) \). □

**Lemma 1.3.3.** Let \( \Delta_i \) be the set of all \([i]-chain\s\) of \( n^\omega \). Then \( \Delta = \{ \Delta_i : i \in n \} \) is an \( n \)-gap.

**Proof.** The intersection of an \([i]-chain\) and a \([j]-chain\) contains at most one point when \( i \neq j \), so it is clear that the preideals are mutually orthogonal. Let us show that they are not separated. So suppose that we had \( a_i \subset n^\omega \) such that \( x \subset^* a_i \) for every \([i]-chain\) \( x \).

Claim A: For every \( i \in n \) and for every \( s \in n^\omega \), there exists \( t = t(i, s) \in W_i \) such that \( s^*t^*r \in a_i \) for all \( r \in W_i \).

Proof of the claim: If not, we would have \( i \in n \) and \( s \in n^\omega \) such that for every \( t \in W_i \) there exists \( r \in W_i \) with \( s^*t^*r \notin a_i \). But then we can construct by induction a sequence \( \{ r_p : p < \omega \} \subset W_i \) such that \( r_p^* = s^*r_0^*r_1^* \cdots r_p^* \notin a_i \) for every \( p \). But this is a contradiction, because \( \{ r_p^* : p < \omega \} \) is an \([i]-chain\), and we supposed that \( x \subset^* a_i \) for all \([i]-chain\s\).

Using Claim A, define \( s_0 = \emptyset \), and by backwards induction \( s_i = s_{i+1}^*(t(i, s_{i+1})) \) for \( i = n-1, n-2, \ldots, 0 \). In this way \( s_n < s_{n-1} < \cdots < s_0 \) and \( s_i^*r \in a_i \) whenever \( r \in (i+1)^\omega \). At the end, we have that \( s_0^*r \in \bigcap_{j \in n} a_j \) for all \( r \in W_0 \). This shows that \( \bigcap_{j \in n} a_j \neq^* \emptyset \). □

Theorem 1.3.2 is saying that every analytic \( n \)-gap \( \Gamma \) contains -in a sense- a permutation \( \Delta^* \) of the gap \( \Delta \) in Lemma 1.3.3 but it is not saying that \( \Delta^* \leq \Gamma \) because the definition of the order \( \leq \) between gaps is much more demanding, as it requires the one-to-one function to respect the orthogonals as well as each of the preideals. If we want to get \( \Gamma' \leq \Gamma \), we must allow the rest of types to play, not just the simple types \([i]\), and for this we shall need the machinery of Section 1.4

Given a set of types \( S \) in \( m^\omega \) we denote by \( \Gamma_S \) the preideal of all subsets of \( m^\omega \) which are of type \( \tau \) for some type \( \tau \in S \). If \( S \cap S' = \emptyset \), then \( \Gamma_S \) and \( \Gamma_{S'} \) are orthogonal; indeed, if \( x \) and \( y \) have different types, then \( |x \cap y| \leq 2 \).
COROLLARY 1.3.4. If \( \{ S_i : i \in n \} \) are nonempty sets of types in \( n^{<\omega} \) with \( \bigcap_{i \in n} S_i = \emptyset \), and there is some permutation \( \varepsilon : n \rightarrow n \) such that \( [i] \in S_{\varepsilon(i)} \) for every \( i \), then \( \Gamma = \{ \Gamma_{S_i} : i \in n \} \) is an \( n^{*} \)-gap in \( n^{\omega} \). If the sets \( S_i \) are pairwise disjoint, then \( \Gamma \) is an \( n \)-gap.

The existence of the permutation \( \varepsilon \) is not really necessary for Corollary \ref{corollary_1.3.3} to hold, but the proof is more involved and we shall not include it here. A gap of the form \( \{ \Gamma_{S_i} : i \in n \} \) as in Corollary \ref{corollary_1.3.4} above will be called a standard \( n^{*} \)-gap.

When we have an \( n^{*} \)-gap of the form \( \Gamma = \{ \Gamma_{S_i} : i \in n \} \) and a type \( \tau \), we may, in abuse of notation, write \( \tau \in \Gamma_{S_i} \) meaning that \( \tau \in S_i \).

THEOREM 1.3.5. For every analytic \( n^{*} \)-gap \( \Gamma \) there exists a standard \( n^{*} \)-gap \( \Gamma' \) such that \( \Gamma' \leq \Gamma \).

PROOF. First, we obtain \( u : n^{<\omega} \rightarrow \omega \) as in Theorem \ref{theorem_1.3.2}. Now fix a type \( \tau \) and we color the sets of type \( \tau \) into \( 2^n \) many colors by declaring that a set \( x \) of type \( \tau \) has color \( \xi \in n \) if \( u(x) \in \bigcup_{x \in \xi} \Gamma_{\varepsilon(x)} \). This coloring is Suslin-measurable since the ideals \( \Gamma_{\varepsilon(x)} \) are analytic, so by Theorem \ref{theorem_1.1.5} by passing to a nice subtree we can suppose that all sets of type \( \tau \) have the same color. We do this for every type \( \tau \). For \( i \in n \), let \( S_i \) be the set of types for which we got that all sets of type \( \tau \) have a color \( \xi \) with \( i \in \xi \) (notice that \( [i] \in S_i \)). Let us check that under these hypotheses, \( u \) witnesses that \( \{ \Gamma_{S_i} : i \in n \} \leq \Gamma \). It is clear that if \( x \in \Gamma_{S_i} \), then \( u(x) \in \Gamma_i \). Now, take \( x \in \Gamma_{S_i} \) and let us suppose for contradiction that \( u(x) \notin \Gamma_i^{+} \), so that there exists an infinite \( y \subseteq u(x) \) such that \( y \in \Gamma_i \). By Lemma \ref{lemma_1.2.4} we can find an infinite \( z \subseteq u \cdot (y) \) of some type \( \tau \). Since \( z \subseteq x \in \Gamma_{S_i} \), we must have \( \tau \notin S_i \). But this means that all subsets of type \( \tau \) had color \( \xi \neq i \), which implies that \( u(z) \notin \Gamma_i \) which contradicts that \( u(z) \subset y \in \Gamma_i \).

The existence of a finite basis stated in Theorem \ref{theorem_1.0.9} is a corollary of Theorem \ref{theorem_1.3.3} above. There are only finitely many standard \( n^{*} \)-gaps, so if we pick from them those which are minimal among them, that is the finite list of minimal analytic \( n^{*} \)-gaps that lie below any analytic \( n^{*} \)-gap in the order \( \leq \).

### 1.4. Projective gaps under determinacy

Theorem \ref{theorem_1.3.2} below states that Theorem \ref{theorem_1.3.2} holds not only for analytic gaps, but also for gaps of higher complexity, when assuming determinacy axioms. Theorem \ref{theorem_1.4.1} together with Theorem \ref{theorem_1.1.10} imply that the whole theory developed in this paper holds true for projective instead of analytic gaps if one assumes Projective Determinacy. The proof that we provide of Theorem \ref{theorem_1.1.10} consists in a reduction to the analytic case of Theorem \ref{theorem_1.3.2}.

THEOREM 1.4.1 (Projective Determinacy). If \( \Gamma = \{ \Gamma_i : i \in n \} \) are projective preideals on the set \( N \) which are not separated, then there exists a permutation \( \varepsilon : n \rightarrow n \) and a one-to-one map \( u : n^{<\omega} \rightarrow N \) such that \( u(x) \in \Gamma_{\varepsilon(x)} \) whenever \( x \) is an \( [i] \)-chain, \( i \in n \).

PROOF. Consider a game \( \mathcal{G}(\Gamma) \). Player I plays elements \( d_0, d_1, d_2, \ldots \) from \( N \) in such a way that \( d_i \notin \{ d_j : j < i \} \), and Player II responds with \( p_0, p_1, p_2, \ldots \) from \( n \). At the end, we consider \( p_{\infty} = \lim \sup_i p_i \) and \( i_{\infty} = \min \{ i : \forall j \geq i \ p_j \leq p_{\infty} \} \). Player I wins if and only if

\[
\{ d_i : i \geq i_{\infty}, p_i = p_{\infty} \} \in \Gamma_{p_{\infty}}
\]
As far as the families $\Gamma_p$ are projective, this is a projective game, hence determined. It is straightforward to check that Player I having a winning strategy means that there exists a one-to-one map $u : n^{<\omega} \to N$ such that $u(x) \in \Gamma_p$ whenever $x$ is a $[p]$-chain (The strategy immediately gives a function $u$ which may not be one-to-one, but it is easy to make it injective by restricting to a nice subtree).

Claim A: If Player II has a winning strategy in the game $G(\Gamma)$, then there exist Borel preideals $\check{\Gamma} \supset \Gamma$, such that Player II still has a winning strategy in the game $G(\check{\Gamma})$.

Proof of Claim A: Let $S$ be a winning strategy for Player II in the game $G(\Gamma)$.

For $k < n$ and for $\zeta \in N^{<\omega}$, we define the set $V^k(\zeta) \subset N^{<\omega}$ as the family of all $\zeta^\sim \eta = (\zeta_0, \ldots, \zeta_m, \eta_0, \ldots, \eta_l)$ such that if

\[
\begin{array}{lll}
\text{Player I} & \zeta_0 & \ldots & \zeta_m & \eta_0 & \ldots & \eta_l \\
\text{Player II} & p_0 & \ldots & p_m & q_0 & \ldots & q_l
\end{array}
\]

is played according to the strategy $S$, then $q_0, \ldots, q_l \leq k$ and $q_l = k$. We make the convention that $\zeta \in V^k(\zeta)$.

For every $k < n$ and $\zeta \in N^{<\omega}$, we also define

$$x^k_\zeta = \{d \in N : \xi_m \neq d \text{ for all } \xi = (\xi_0, \ldots, \xi_m) \in V^k(\zeta) \setminus \{\zeta\} \}$$

For every $k < n$, we also define $\Upsilon^k$ to be the set of all $\zeta = (\zeta_0, \ldots, \zeta_m) \in N^{<\omega}$ such that if

\[
\begin{array}{lll}
\text{Player I} & \zeta_0 & \ldots & \zeta_m \\
\text{Player II} & p_0 & \ldots & p_m
\end{array}
\]

is played according to the strategy $S$, then there exists $j \in \{0, \ldots, m\}$ such that $p_j = k$ and $p_i < k$ for all $i > j$.

Claim A1: For every $k < n$, for every $\zeta \in \Upsilon^k$ and for every $a \in \Gamma_k$, there exists $\xi \in V^k(\zeta)$ such that $a \subset x^k_\zeta$.

Proof of Claim A1: Fix $k < n$, $\zeta \in \Upsilon^k$ and $a \in \Gamma_k$ for which the statement of Claim A1 fails. Then, it is possible to construct inductively an infinite set \(\{d_1, d_2, d_3, \ldots\} \subset a\) together with elements $\eta_1, \eta_2, \ldots \in N^{<\omega}$ such that

$$\xi^m = \zeta^\sim \eta_1^\sim d_1^\sim \eta_2^\sim d_2^\sim \cdots \eta_m^\sim d_m \in V^k(\zeta)$$

for all $m$. Consider the full infinite round of the game $G(\Gamma)$, in which Player I moves $\zeta < \xi_1 < \xi_2 < \cdots$ and Player II plays according to the strategy $S$. In this case, $p_\infty = k$, and the fact that Player II wins means exactly that \(\{d_1, d_2, \ldots\} \notin \Gamma_k\). This contradicts that \(\{d_1, d_2, \ldots\} \subset a \in \Gamma_k\). This finishes the proof of Claim A1.

For each $k < \omega$, let $\check{\Gamma}_k$ be the family of all sets $a$ that satisfy Claim A1. That is,

$$\check{\Gamma}_k = \{a \subset N : \forall k < n \forall \zeta \in \Upsilon^k \exists \xi \in V^k(\zeta) : a \subset x^k_\zeta\}$$

This is a Borel preideal, and by Claim A1, $\Gamma_k \subset \check{\Gamma}_k$. Now we show that for $\check{\Gamma} = \{\check{\Gamma}_k : k < n\}$, we can find a winning strategy $\check{S}$ for Player II in the game $G(\check{\Gamma})$. 
In order to describe the strategy $S$, let us suppose that Player I plays $d_0, d_1, d_2, \ldots$ and we will describe how Player II must respond. At each move $i < \omega$, we will not only define the number $p_i$ that Player II must play, but also auxiliary numbers $\nu_i^{(k)} < \omega$ for $k < n$.

$$
\begin{array}{c|ccc}
\text{Player I} & d_0 & d_1 & \cdots \\
\hline
\text{Player II} & p_0 & p_1 & \cdots \\
\hline
\nu_0^{(n-1)} & \nu_1^{(n-1)} & \cdots \\
\nu_0^{(n-2)} & \nu_1^{(n-2)} & \cdots \\
\vdots & \vdots & \cdots \\
\nu_0^{(0)} & \nu_1^{(0)} & \cdots \\
\end{array}
$$

For every $k < n$ and every $\zeta \in \Upsilon^k$, let $\{\xi_i^{(k)}(\zeta) : \nu < \omega\}$ be an enumeration of $V^k(\zeta)$. Together with the integers $\nu_i^{(k)}$ we also keep track of elements $\xi_i^{(k)} \in \mathbb{N}^{<\omega}$ defined as follows:

$$
\begin{align*}
\xi_i^{(n-1)} &= \xi_i^{n-1}(\emptyset) \in V^{n-1}(\emptyset) \\
\xi_i^{(k)} &= \xi_i^{k}(\xi_i^{(k+1)}) \in V^k(\xi_i^{(k+1)})
\end{align*}
$$

Notice the following general fact:

Claim A2: Let $\{\xi_i^{(k)} : k < n\} \subset \mathbb{N}^{<\omega}$ be such that $\xi_i^{(n-1)} \in V^{n-1}(\emptyset)$ and $\xi_i^{(k)} \in V^k(\xi_i^{(k+1)})$ for $k < n - 1$. Then $\bigcap_{k<n} x_i^{k(\xi)} = \emptyset$.

Proof of Claim A2: Suppose for contradiction that $d \in \bigcap_{k<n} x_i^{k(\xi)} = \emptyset$. Consider the finite run of the game $G(\Gamma)$ played according to strategy $S$, in which Player I plays the finite sequence $\xi_i^{(0)} \cdot d$. Suppose that Player II responds to the last move $d$ with $p < n$. This would violate that $d_i \in x_i^{p(\xi)}$ and we get a contradiction. This finishes the proof of Claim A2.

The initial input is that $\nu_0^{(k)} = 0$ for all $k < n$. Suppose that we are at stage $i$, that we are given $\nu_i^{(k)}$ (hence also $\xi_i^{(k)}$) for $k < n$, and we describe how Player II must choose $p_i$ and the auxiliary numbers $\nu_i^{(k)}$.

By Claim A2,

$$
\bigcap_{k<n} x_i^{k(\xi)} = \emptyset
$$

so we can choose

$$
p_i = \max \left\{ k < n : d_i \notin x_i^{k(\xi)} \right\}
$$

and then

$$
\begin{align*}
\nu_{i+1}^{(k)} &= \nu_i^{(k)} & \text{if } k > p_i \\
\nu_{i+1}^{(p_i)} &= \nu_i^{(p_i)} + 1 \\
\nu_{i+1} &= 0 & \text{if } k < p_i
\end{align*}
$$

Notice that $V^k(\zeta)$ is nonempty as $\zeta \in V^k(\zeta)$. In case it was finite, an enumeration with repetitions is allowed.
Let us check that this is a winning strategy for Player II in the game $\mathcal{G}(\tilde{\Gamma})$. So consider

| Player I | $d_0$ | $d_1$ | $\cdots$ |
|----------|-------|-------|-----------|
| Player II | $p_0$ | $p_1$ | $\cdots$ |
| $\nu_i^{(n-1)}$ | $\nu_1^{(n-1)}$ | $\cdots$ |
| $\nu_i^{(n-2)}$ | $\nu_1^{(n-2)}$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ |
| $\nu_i^{(0)}$ | $\nu_1^{(0)}$ | $\cdots$ |

a full infinite run of the game, played according to the strategy $\tilde{\sigma}$, and with the auxiliary $\nu_i^{(k)}$ and $\xi_i^{(k)}$ obtained along the run. The first observation is that

$$p_\infty = \max \left\{ k < n : \text{the sequence } \{\nu_i^{(k)} : i < \omega \} \text{ is not eventually constant} \right\}$$

$$i_\infty = \min \left\{ i < \omega : \forall k > p_\infty \text{ the sequence } \{\nu_j^{(k)} : j \geq i \} \text{ is constant} \right\}$$

Let $\zeta$ be the value at which the sequence $\{\xi_j^{(p_\infty+1)} : j < \omega \}$ stabilizes. If $p_\infty = n-1$, we define $\zeta = 0$. Notice that $\zeta \in V^{p_\infty}$ because, in the strategy $S$, if Player I plays $\zeta$, the last move of Player II is $p_\infty$. We have to check that

$$a = \{ d_i : i \geq i_\infty, p_i = p_\infty \} \not\subseteq \tilde{\Gamma}_{p_\infty}$$

Assume for contradiction that $a \in \tilde{\Gamma}_{p_\infty}$. By the definition of the $\tilde{\Gamma}_k$'s, there must exist $\xi \in V^{p_\infty}(\zeta)$ such that $a \subseteq x_{\xi}^{p_\infty}$. This must appear somewhere in the enumeration that we made, so there exists $\nu < \omega$ such that $\xi = \xi_\nu^{p_\infty}(\zeta)$. By the way in which the $p_i$ and the $\nu_i^{(k)}$ are inductively defined, we have that

$$\{\nu_i^{(p_\infty)} : i \geq i_\infty, p_i = p_\infty \} = \{0, 1, 2, 3, \ldots\}$$

But on the other hand, since $a \subseteq x_{\xi}^{p_\infty}$, we have that $d_i \in x_{\xi}^{p_\infty}(\zeta)$ whenever $i \geq i_\infty$, $p_i = p_\infty$, and the definition of $p_i$ and the other numbers then implies $\nu_i^{(p_\infty)} \neq \nu + 1$ for all $i \geq i_\infty$. This is a contradiction, and it finishes the proof of Claim A.

We come back to the proof of the theorem. For every permutation $\sigma : n \rightarrow n$ we consider $\Gamma^{\sigma} = \{ \Gamma_{\sigma(i)} : i < n \}$. If there is a permutation $\sigma$ such that Player I has a winning strategy in the game $\mathcal{G}(\Gamma^{\sigma})$ we are done. Otherwise, Player II has a winning strategy in $\mathcal{G}(\Gamma^{\sigma})$ for every $\sigma$. Consider $\tilde{\Gamma}^{\sigma}$ given by Claim A, and let

$$\Delta_i = \bigcap_{\sigma} \Gamma_{\sigma^{-1}(i)}$$

and we consider again all the permutations $\Delta^{\sigma} = \{ \Delta_{\sigma(i)} : i < n \}$. For every permutation $\sigma$, since $\Delta^{\sigma} \subseteq \tilde{\Gamma}_0^{\sigma}$, and Player II has a winning strategy in $\mathcal{G}(\Delta^{\sigma})$, we conclude that Player II has a winning strategy in $\mathcal{G}(\Delta^{\sigma})$ as well. In particular, Player I does not have a winning strategy, so there is no one-to-one map $u : n^{<\omega} \rightarrow N$ such that $u(x) \in \Delta^{\sigma}$ whenever $x$ is an $[i]$-chain. But the sets $\Delta_i$ are Borel, so we can apply Theorem 1.3.2 and we conclude that $\Delta$ is separated. Since $\Gamma_i \subseteq \Delta_i$, we get that $\Gamma$ is separated as well. $\square$

The proof of Theorem 1.4.1 contains implicitly an asymmetric version of the theorem, in which on one side no permutation is considered, and on the other side separation is substituted by a winning strategy of Player II. In principle, Player II having a winning strategy gives a coanalytic condition, and the proof above is
essentially devoted to transform it into a Borel condition. We found it too technical to state this asymmetric version as a theorem, as it would not look more friendly than referring to the proof of Theorem 1.4.1.

1.5. Some remarks

As we pointed out in the introduction, we have changed a little our language with respect to our previous papers [3] and [4], because there we said that gaps consisted of ideals while here we say that gaps consist of preideals. The reason is that the first applications that we had in mind had to do with Stone duality, so ideals -that correspond to open sets in $\omega^*$- were the natural thing to consider. However, in this work we have found other kind of applications for which the more flexible notion of preideal fits better, and we believe that this is the most natural framework. In this section we shall see that these subtle changes of setting do not alter essentially the theory, and this will be useful in Section 2.7 in order to safely use the results from [4] in our current context. On the way, we shall discuss the alternative use of the orders $\leq'$ and $\leq''$ between gaps, that lead to the same theory of minimal gaps.

Given a preideal $I$, let $I'$ be the ideal generated by $I$, that is the family of all sets $x$ which are contained in a finite union of elements of $I$, and let $I'' = I^\perp \perp$ denote the biorthogonal of $I$, that is the family of all sets $x$ such that every infinite subset of $x$ contains a further infinite subset which belongs to $I$. We have $I \subset I' \subset I''$ and $I^\perp = (I')^\perp = (I'')^\perp$.

Given a preideal $I$, let $I'$ be the ideal generated by $I$, that is the family of all sets $x$ which are contained in a finite union of elements of $I$, and let $I'' = I^\perp \perp$ denote the biorthogonal of $I$, that is the family of all sets $x$ such that every infinite subset of $x$ contains a further infinite subset which belongs to $I$. We have $I \subset I' \subset I''$ and $I^\perp = (I')^\perp = (I'')^\perp$.

It is natural to consider two new orders on $n_\ast$-gaps corresponding to the equivalent columns of the diagram of implications above. We say that $\Gamma \leq' \Delta$ if $\Gamma' \leq \Delta'$, and we say that $\Gamma \leq'' \Delta$ if $\Gamma'' \leq \Delta''$. The order $\leq''$ is a very natural one, notice the following characterization:

**Proposition 1.5.2.** Let $\Gamma$ and $\Delta$ be $n_\ast$-gaps on the countable sets $M$ and $N$. Then $\Gamma \leq'' \Delta$ if and only if there exists a one-to-one function $f : M \rightarrow N$ such that for every $i < n$,

$$x \in \Gamma_i \iff f(x) \in \Delta_i,$$

**Proof.** If $f$ is as above and we want to check that $\Gamma \leq'' \Delta$, it is enough that we check that if $x \in \Gamma_i$, then $f(x) \in \Delta_i$. Namely, if we had $f(x) \notin \Delta_i$, then $f(x)$ contains an infinite set $y \in \Delta_i$. In particular, $y \in \Delta''$, hence $f^{-1}(y) \in \Gamma_i$. But
1.5. SOME REMARKS

Let \( x \in \Gamma_1^+ \) and \( f^{-1}(y) \subset x \), a contradiction. Conversely, suppose that \( \Gamma'' \leq \Delta'' \) and let \( f : M \rightarrow N \) be a function that witnesses it. Let us check that \( x \in \Gamma_2^\perp \) if and only if \( f(x) \in \Delta''_1 \). The implication “\( \Rightarrow \)” is clear. Suppose that \( x \not\in \Gamma_2^\perp \). This means that there exists an infinite subset \( y \subset x \) such that \( y \in \Gamma_2^\perp = (\Gamma_2'')^\perp \). Then \( f(y) \not\in \Delta_1^+ \), so since \( f(y) \subset f(x) \), it follows that \( f(x) \not\in \Delta_1'' \).

\[ \square \]

**Lemma 1.5.3.** Let \( \Gamma \) and \( \Delta \) be analytic \( n_\ast \)-gaps such that \( \Gamma \) is of the form \( \Gamma = \{ \Gamma_S : i < n \} \) for some sets of types \( \{ S_i : i < n \} \) in \( m^{<\omega} \). Then

\[ \Gamma \leq \Delta \iff \Gamma \leq' \Delta \iff \Gamma \leq'' \Delta. \]

**Proof.** Suppose that \( \Gamma \leq'' \Delta \) and let \( f : m^{<\omega} \rightarrow N \) be a function witnessing it. Fix a type \( \tau \in S_i \) for some \( i \). We can color the sets \( x \) of type \( \tau \) into two colors, depending on whether \( f(x) \in \Delta_1 \) or \( f(x) \not\in \Delta_i \). By Theorem 1.1.5, we can find a nice embedding \( u : m^{<\omega} \rightarrow m^{<\omega} \) such that either \( f(u(x)) \in \Delta_1 \) or \( f(u(x)) \not\in \Delta_i \), for all \( x \) of type \( \tau \). The second possibility cannot happen since \( f(y) \in \Delta''_1 \) for every \( y \) of type \( \tau \). After repeating this procedure for every \( i \) and every \( \tau \in S_i \), we obtain that the restriction of \( f \) to a nice subtree witnesses that \( \Gamma \leq \Delta \).

Let us say that an \( n \)-gap \( \Gamma \) is an ideal \( n \)-gap if \( \Gamma = \Gamma' \), that is, if \( \Gamma \) consists of ideals. An ideal \( n \)-gaps is what is called simply an \( n \)-gap in \([3]\) and \([4]\). As a corollary of the results stated in this section, the minimal analytic gaps in any of the orders \( \leq, \leq', \) or \( \leq'' \) are all the same, and the minimal analytic ideal gaps in any of the three orders are given by switching each gap \( \Gamma \) by \( \Gamma' \). The notion of equivalence is moreover the same in all cases. A special property of the order \( \leq'' \) is that minimal analytic gaps are minimal among all gaps in this order relation. That is, if \( \Gamma \) is a minimal analytic \( n_\ast \)-gap and \( \Delta \) is an \( n_\ast \)-gap with \( \Delta \leq'' \Gamma \), then \( \Gamma \leq'' \Delta \), even if \( \Delta \) is not analytic. This is because if \( f : N \rightarrow M \) witnesses that \( \Delta \leq'' \Gamma \), then \( f^{-1} \) witnesses that \( \Gamma_{|f(N)} \leq'' \Delta \), and \( \Gamma_{|f(N)} \) is analytic.
CHAPTER 2

Working in the $n$-adic tree

2.1. Normal embeddings

Theorem [2.3.3] states that, if we are interested in properties of analytic gaps that are preserved under the relation $\leq$ between $n$-gaps, we can restrict our attention to standard gaps. After such a result, the next step is to understand when we have $\Gamma' \leq \Gamma$ for two standard gaps $\Gamma'$ and $\Gamma$ on $n^\omega$ and $m^\omega$ respectively. Remember that this means that there is a one-to-one function $\phi : n^\omega \to m^\omega$, which preserves each of the preideals in the gap as well as the orthogonals. An important observation is that if $u : n^\omega \to n^\omega$ is a nice embedding, and $\phi : n^\omega \to m^\omega$ witnesses that $\Gamma' \leq \Gamma$, then $\phi' = \phi \circ u$ does it as well, since nice embeddings leave standard gaps invariant. In this section we shall show that for every one-to-one function $\phi : n^\omega \to m^\omega$ there exists a nice embedding $u : n^\omega \to n^\omega$ such that the composition $\phi' = \phi \circ u$ is of a very special kind that we call a normal embedding. This will provide the necessary combinatorial tool to analyse when two standard gaps satisfy $\Gamma' \leq \Gamma$.

**Lemma 2.1.1.** Equivalence of sets is determined by 4-tuples. That is, for every set $A \subset m^\omega$, we have that $\{x_\alpha : \alpha \in A\} \sim \{y_\alpha : \alpha \in A\}$ if and only if $\{x_\alpha \wedge x_\beta, x_\gamma, x_\delta\} \sim \{y_\alpha, y_\beta, y_\gamma, y_\delta\}$ for every $\alpha, \beta, \gamma, \delta \in A$.

**Proof.** So suppose that all 4-tuples are equivalent. Then, the function $f : \{x_\alpha \wedge x_\beta : \alpha, \beta \in A\} \to \{y_\alpha \wedge y_\beta : \alpha, \beta \in A\}$ given by $f(x_\alpha \wedge x_\beta) = y_\alpha \wedge y_\beta$ is well defined and preserves both orders $\leq$ and $\prec$. Indeed notice that the sets $\{x_\alpha \wedge x_\beta : \alpha, \beta \in A\}$ and $\{y_\alpha \wedge y_\beta : \alpha, \beta \in A\}$ are already closed under the operation $\wedge$. If $x_\alpha \wedge x_\beta = (x_{\alpha'} \wedge x_{\beta'}) \wedge r_1 \cdots r_m$ with $r_i \in W_k$, $k_1 < k_2 < \cdots < k_m$, then the same expression holds for $y_\alpha \wedge y_\beta$. It follows that we can extend $f$ to a bijection $f : \{x_\alpha : \alpha \in A\} \to \{y_\alpha : \alpha \in A\}$ for which know that all properties of equivalence holds, except for the preservation of $\prec$ out of the set $\{x_\alpha \wedge x_\beta : \alpha, \beta \in A\}$, that we proceed to check now. So suppose that

\begin{align*}
x_\alpha \wedge x_\beta &= (x_{\alpha'} \wedge x_{\beta'}) \wedge r_1 \cdots r_m \text{ with } r_i \in W_k, k_1 < k_2 < \cdots < k_m, \\
x_\gamma \wedge x_\delta &= (x_{\gamma'} \wedge x_{\delta'}) \wedge s_1 \cdots s_p \text{ with } s_i \in W_l, l_1 < l_2 < \cdots < l_p,
\end{align*}

$y_\alpha \wedge y_\beta = (y_{\alpha'} \wedge y_{\beta'}) \wedge \bar{r}_1 \cdots \bar{r}_m$ with $\bar{r}_i \in W_k$, $y_{\gamma'} \wedge y_{\delta'} = (y_{\gamma'} \wedge y_{\delta'}) \wedge \bar{s}_1 \cdots \bar{s}_p$ with $\bar{s}_i \in W_l$.

We must prove that if $(x_{\alpha'} \wedge x_{\beta'}) \wedge \bar{r}_1 \cdots \bar{r}_i \prec (x_{\gamma'} \wedge x_{\delta'}) \wedge \bar{s}_1 \cdots \bar{s}_j$, then $(y_{\alpha'} \wedge y_{\beta'}) \wedge \bar{r}_1 \cdots \bar{r}_i \prec (y_{\gamma'} \wedge y_{\delta'}) \wedge \bar{s}_1 \cdots \bar{s}_j$. Suppose without loss of generality that $x_\alpha \wedge x_\alpha' = x_\beta \wedge x_\beta'$ (if this does not hold, then $x_\beta \wedge x_\alpha' = x_\alpha \wedge x_\beta'$ must hold, and then interchange the role of $\alpha$ and $\beta$) and $x_\gamma \wedge x_\gamma' = x_\delta \wedge x_\delta'$ (similarly as before, this must hold either for $\gamma$ or for $\delta$). If $i < m$ and $j < p$, then it is enough to apply that $\{x_\alpha, x_{\alpha'}, x_{\gamma}, x_{\gamma'}\}$ and $\{y_\alpha, y_{\alpha'}, y_{\gamma}, y_{\gamma'}\}$ are equivalent. If it
happened that $i = m$, then we can apply that \(\{x_\alpha, x_\beta, x_\gamma, x_\delta\}\) and \(\{y_\alpha, y_\beta, y_\gamma, y_\delta\}\) are equivalent, and similarly if $j = p$.

We must take at least 4 elements in Lemma 2.1.1. The families
\[
\{(110), (1111), (20000), (211111)\} \text{ and } \{(100), (1111), (22000), (221111)\}
\]
are not equivalent in \(3^{<\omega}\), but each of their subfamilies are equivalent.

**Definition 2.1.2.** A one-to-one function \(\phi : n^{<\omega} \to m^{<\omega}\) will be called a normal embedding if it has the following properties:

1. If \(x < y\) then \(|\phi(x)| < |\phi(y)|\).
2. Whenever \(\{x_\alpha : \alpha \in A\}\) and \(\{y_\alpha : \alpha \in A\}\) are equivalent families in \(n^{<\omega}\), then \(\{\phi(x_\alpha) : \alpha \in A\}\) and \(\{\phi(y_\alpha) : \alpha \in A\}\) are equivalent families in \(m^{<\omega}\).
3. For every \(x \in n^{<\omega}\), for every \(k < n\) and for every \(w, w' \in W_k\) we have that \(\phi(x) \land \phi(x^{-w}) = \phi(x) \land \phi(x^{-w'})\).

We notice that if \(\phi : n^{<\omega} \to m^{<\omega}\) is a normal embedding and \(\tau\) is a type, then all sets of type \(\tau\) are sent by \(\phi\) to sets of the same type that we denote by \(\phi \tau\). Thus, we have \(\phi : \mathcal{T}_n \to \mathcal{T}_m\), where we denote by \(\mathcal{T}_k\) the set of all types in \(k^{<\omega}\).

**Theorem 2.1.3.** For every one-to-one function \(\phi : n^{<\omega} \to m^{<\omega}\), there is a nice embedding \(u : n^{<\omega} \to n^{<\omega}\) such that \(\phi' = \phi \circ u\) is a normal embedding.

**Proof.** We start by constructing a nice embedding that will guarantee condition (3) of the definition of normal embedding. We define \(v : n^{<\omega} \to n^{<\omega}\) inductively on \(n^{<\omega}\). At a step corresponding to \(y \in n^{<\omega}\), we shall define the value \(v(y)\) and also a nice embedding \(w_y : n^{<\omega} \to n^{<\omega}\) (with its corresponding nice subtree \(T_y\)) so that \(v(z) = w_y(z)\) and \(T_y \subset T_z\) for all \(z \preceq y\). Suppose that \(y = x^{-i}\) for some \(i < n\), and let us define \(v(y)\) and \(w_y\) supposing that \(v(z)\) and \(w_z\) are already defined for all \(z \preceq y\), in particular for \(z = x\). We can color each \(s \in (i + 1)^{<\omega}\) according to the value \(\phi(v(x)) \land \phi(w_x(x^{-i} \cdots s))\). We apply Corollary 1.1.6 and we obtain a nice subtree \(\Upsilon \subset (i + 1)^{<\omega}\) where this coloring is monochromatic. We make \(v(y) = w_x(x^{-r})\) where \(r\) is the root of \(\Upsilon\), and the new nice embedding \(w_y\) is chosen so that \(w_y(x^{-z}) \in \Upsilon\) for all \(z \in W_i\). This finishes the construction of the nice embedding \(v\) which satisfies that \(\phi_0 = \phi \circ v\) has property (3), moreover \(\phi_0 \circ \phi'\) will have property (3) for any further nice embedding \(\phi'\). So we assume without loss of generality that \(\phi\) already has property (3).

Next, we construct a nice embedding \(u_1 : n^{<\omega} \to n^{<\omega}\) such that \(\phi_1 = \phi \circ u_1\) satisfies that if \(x \prec y\) then \(|\phi_1(x)| < |\phi_1(y)|\). This can be achieved just by waiting at each node to define its successors high enough. This property will be preserved after composing with further nice embeddings. For simplicity, we assume that \(\phi_1 = \phi\).

It remains to get a further nice embedding that will ensure property (2) of normal embeddings. Notice that there are only finitely many equivalence classes of 4-sets in \(n^{<\omega}\). Thus, if we fix an equivalence class \(\mathcal{C}\) of 4-subsets of \(m^{<\omega}\), Theorem 1.1.5 provides a nice embedding \(u\) such that all the families of the form
\[
\{\phi u(x_1), \phi u(x_2), \phi u(x_3), \phi u(x_4)\}
\]
are equivalent, for \(\{x_1, x_2, x_3, x_4\} \in \mathcal{C}\). Since there are also only finitely many equivalence classes of 4-subsets of \(m^{<\omega}\), a repeated application of the previous
fact provides a nice embedding $u$ such that $\phi u$ satisfies condition (2) of normal embeddings for all 4-families. This finishes the proof by Lemma 2.1.1. □

2.2. The max function

Given a type $\tau$, $\max(\tau)$ denotes the maximal number which appears in $\tau$. That is,

$$\max(\tau) = \max(\max(\tau^0), \max(\tau^1)) .$$

**Theorem 2.2.1.** For a family $\{\tau_i : i \in n\} \subset \mathbb{T}_m$ the following are equivalent:

1. There exists a normal embedding $\phi : n^{<\omega} \to m^{<\omega}$ such that $\phi[i] = \tau_i$,
2. $\max(\tau_0) \leq \cdots \leq \max(\tau_{n-1})$.

**Proof.** Suppose that item (1) holds, pick $i < j$ and let us check that $\max(\tau_i) \leq \max(\tau_j)$. Let $\alpha = \phi(j) \wedge \phi(ji)$, which implies that $\gamma = \phi(0) \wedge \phi(ji)$, we have similar formulas

\begin{align*}
\text{(I)} & \quad \max(\tau_j) = \max\{\phi(0) \backslash \beta, \phi(j) \backslash \beta\}, \\
\text{(II)} & \quad \max(\tau_j) = \max\{\max\{\phi(0) \wedge \phi(j) \wedge \gamma\}, \max\{\phi(j) \wedge \gamma\}\}.
\end{align*}

We distinguish three cases. The first case is $\beta < \alpha$, which implies that $\gamma = \beta < \alpha$,

so $\max(\phi(j) \backslash \alpha) \leq \max(\phi(j) \backslash \beta)$ and $\max(\phi(ji) \backslash \alpha) \leq \max(\phi(ji) \backslash \gamma)$ so we conclude from the formulas (I), (II) and (III) above that $\max(\tau_i) \leq \max(\tau_j)$ as desired.

The second case is that $\beta = \alpha$, which implies that $\gamma = \beta = \alpha$.

\begin{align*}
\text{(III)} & \quad \max(\tau_j) = \max\{\max\{\phi(0) \wedge \gamma\}, \max\{\phi(j) \wedge \gamma\}\}.
\end{align*}

By formula (I), it is enough to check that $\max(\phi(j) \backslash \alpha) \leq \max(\tau_j)$ and $\max(\phi(ji) \backslash \alpha) \leq \max(\tau_j)$. In this case, $\phi(j) \backslash \alpha = \phi(j) \backslash \beta$ so it is clear that $\max(\phi(j) \backslash \alpha) \leq \max(\tau_j)$ by (II). On the other hand,

$$\phi(ji) \backslash \alpha = (\gamma \backslash \alpha) \wedge (\phi(ji) \backslash \gamma).$$

On one side, $\phi(0) \backslash \beta = (\gamma \backslash \beta) \wedge (\phi(0) \backslash \gamma)$, therefore

$$\max(\gamma \backslash \alpha) = \max(\gamma \backslash \beta) \leq \max(\phi(0) \backslash \beta) \leq \max(\tau_j)$$

by (II), and on the other side $\max(\phi(ji) \backslash \gamma) \leq \max(\tau_j)$ by (III), so we conclude that $\max(\phi(ji) \backslash \alpha) \leq \max(\tau_j)$. By formula (I), this finishes the second case.

The third case is that $\beta > \alpha$, which implies that $\gamma = \alpha < \beta$. 

...
This is solved in a similar way as in the second case, changing the role of \(j\) and \(ji\). By formula (I), it is enough to check that \(\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)\) and \(\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)\). Now, \(\phi(ji) \setminus \alpha = \phi(ji) \setminus \gamma\) so it is clear that \(\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)\) by (III). On the other hand,

\[
\phi(j) \setminus \alpha = (\beta \setminus \alpha) (\phi(j) \setminus \beta)
\]

On one side, \(\phi(\emptyset) \setminus \gamma = (\beta \setminus \gamma) (\phi(\emptyset) \setminus \beta)\) so

\[
\max(\beta \setminus \alpha) = \max(\beta \setminus \gamma) \leq \max(\phi(\emptyset) \setminus \gamma) \leq \max(\tau_j)
\]

by (III), and on the other side \(\max(\phi(j) \setminus \beta) \leq \max(\tau_j)\) by (II). So we conclude that \(\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)\) and this finishes the third case.

Now, suppose that (2) holds.\(^4\) For every \(i\) fix \((u_i, v_i)\) a rung of type \(\tau_i\) and write \(u_i = \tilde{u}_i \sim u_i\) in such a way that \(|\tilde{u}_i| = |v_i|\). When \(\tau_i\) is a chain type, \(v_i = \tilde{u}_i = \emptyset\) and \(\tilde{u}_i = u_i\). When \(\tau_i\) is a comb type we can make the assumption\(^2\) that the last integer of \(\tilde{u}_i\) and the first integer of \(\tilde{u}_i\) are both equal to 0. We shall construct an embedding \(\phi : n^{<\omega} \rightarrow m^{<\omega}\) together with auxiliary functions \(\phi_i, \phi^i : n^{<\omega} \rightarrow m^{<\omega}\) for \(i = 0, \ldots, n - 1\). All of them will be defined by induction on the \(\prec\)-order of \(n^{<\omega}\). We first choose \(\phi(\emptyset), \phi_i(\emptyset), \phi^i(\emptyset)\). Let \(\{j_1, \ldots, j_p\}\) be an enumeration of all indices \(i\) such that \(\tau_i\) is a comb type and such that

\[
\max(\tau^1_{j_1}) \geq \max(\tau^1_{j_2}) \geq \cdots \geq \max(\tau^1_{j_p}),
\]

and moreover, if \(\max(\tau^1_{j_1}) = \max(\tau^1_{j_s})\), then \(j_r < j_s\) if and only if \(r > s\).

We define

\[
\begin{align*}
\phi_{j_1}(\emptyset) & = \emptyset, \\
\phi_{j_k}(\emptyset) & = v^r_{j_1} \cdots v^r_{j_{k-1}} \\
\phi(\emptyset) & = v^r_{j_1} \cdots v^r_{j_p} \\
\phi^i(\emptyset) & = \phi_i(\emptyset) \sim u_i \sim 0^i\text{ if } \tau_i\text{ is a comb type.}\ \\
\phi_i(\emptyset) & = \phi^i(\emptyset) = \phi(\emptyset)\text{ if } \tau_i\text{ is a chain type.}
\end{align*}
\]

The number \(l_i\) of 0’s added to construct \(\phi^i(\emptyset)\) is chosen so that \(\phi^i(\emptyset)\) has length strictly larger than \(\phi(\emptyset)\). Figure 4 represents how \(\phi(\emptyset), \phi_k(\emptyset)\) and \(\phi^k(\emptyset)\) look like in the tree. The pattern reflected in this picture will be repeated for \(\phi(x), \phi_k(x)\) and \(\phi^k(x)\) for any \(x\). It is natural to make the notational convention that \(\phi_{j_{p+1}} = \phi\) and this will avoid repeating some arguments along the proof.

---

1The proof of later Lemma 2.5.5 may be enlightening about the necessity of constructing \(\phi\) in such a complicated way.

2The aim of this assumption is to make sure that the critical nodes of \(u_i\) are far away from the splitting between \(\tilde{u}_i\) and \(\tilde{u}_i\) and to avoid in this way peculiar situations.
We shall see how to define all these functions on $x^\sim k$ once they are defined on all $y \prec x^\sim k$, in particular on $y = x$. We consider

$$q = q(k) = \min\{r : \max(\tau^1_j) < \max(\tau^k) \text{ or } j_r \leq k\}$$

(If there is no $r$ like that we may assign the value $q = p + 1$). The definition of the functions is then made as follows:

$$\phi(x^\sim k) = \phi^k(x)^\sim \bar{u}_k^\sim v_j^\sim v_{j+1}^\sim \cdots v_j^p$$
$$\phi^r_j(x^\sim k) = \phi^r_j(x) \text{ if } r < q$$
$$\phi^r_j(x^\sim k) = \phi^k(x)^\sim \bar{u}_k^\sim v_j^\sim v_{j+1}^\sim \cdots v_{j-r} \text{ if } r \geq q$$
$$\phi^r(x^\sim k) = \phi^r(x)^\sim \bar{u}_i^\sim 0^i \text{ if } \tau_i \text{ is a comb type,}$$
$$\phi^r(x^\sim k) = \phi^r(x)^\sim \bar{u}_i^\sim 0^i \text{ if } \tau_i \text{ is a chain type.}$$

Now, the number $l_i^r$ of 0’s added to construct $\phi^r(x^\sim k)$ is chosen so that $\phi^r(x^\sim k)$ has length larger than $\phi(x^\sim k)$ but also larger than all $\phi(y)$, $\phi_j(y)$, $\phi^j(y)$ that have been already constructed for $y \prec x^\sim k$. A picture of what is going on is given by Figure 2. The point is that both sets \{\phi(x), \phi^k(x), k < \omega\} and \{\phi(x^\sim k), \phi^k(x^\sim k), k < \omega\} must follow the pattern provided by Figure 1, but we make $\phi^r_j(x^\sim k)$ to stay the same as $\phi^r_j(x)$ for $r < q$, while $\phi^r_j(x)$ is moved above $\phi^k(x)^\sim \bar{u}_k$ for $r \geq q$.

Claim 1: For every $x \in n^{\preceq\omega}$ and\footnote{When we say following the same pattern, we mean up to equivalence. Looking at Figure 2, one may wonder if the long path from $\phi^r_{j-1}(x^\sim k)$ till $\phi^r_j(x^\sim k)$ is really equivalent to $v_{j-1}^q$ as Figure 1 suggests. This is the content of Claim 1.} for $r = 1, \ldots, p$,

\[ (*) \phi^r_{j+1}(x) = \phi^r_j(x)^\sim v_j^\sim w \text{ for some } w \text{ such that } \max(w) \leq \max(v_j). \]

Proof of Claim 1: This holds when $x = \emptyset$. We suppose that it holds for $x$ and we prove it for $x^\sim k$. For $r < q = q(k)$ we have that $\phi^r_j(x^\sim k) = \phi^r_j(x)$ while for $r \geq q$ we have that

$$\phi^r_j(x^\sim k) = \phi^k(x)^\sim \bar{u}_k^\sim v_j^\sim v_{j+1}^\sim \cdots v_{j-r}.$$
Thus, we have $\phi_{j_{r+1}}(x) = \phi_{j_r}(x)^{\sim} v_{j_r}$ when either $r < q - 1$ or $r \geq q$. Only the case when $r = q - 1$ deserves special attention. In this case

$$\phi_{j_r}(x) = \phi_{j_{q-1}}(x) = \phi_{j_{q-1}}(x)^{\sim} u_k.$$  

Either $\tau_k$ is a chain type (in which case $\phi_k(x) = \phi(x)$) or $k = j_l$ for some $l$ which must satisfy $l \geq q$ by the definition of $q$. In either case the inductive hypothesis implies that $\phi_{j_l}(x) = \phi_{j_{q-1}}(x)^{\sim} v_{j_{q-1}}^{\sim} w_1$ where $\max(w_1) \leq \max(v_{j_{q-1}})$. If $\tau_k$ is a chain type, then $\phi^k(x) = \phi_k(x)$, so

$$\phi_{j_k}(x) = \phi^k(x)^{\sim} u_k = \phi_{j_{q-1}}(x)^{\sim} v_{j_{q-1}}^{\sim} w_1^{\sim} u_k$$

and this is what we were looking for because $\phi_{j_k}(x) = \phi_{j_{q-1}}(x)^{\sim} v_{j_{q-1}}^{\sim} w_1^{\sim} u_k$.

On the other hand, if $\tau_k$ is a comb type, then $\phi^k(x) = \phi_k(x)^{\sim} \tilde{u}_k^{\sim} \theta^k$, so

$$\phi_{j_k}(x) = \phi^k(x)^{\sim} \tilde{u}_k = \phi_{j_{q-1}}(x)^{\sim} v_{j_{q-1}}^{\sim} w_1^{\sim} \tilde{u}_k^{\sim} \theta^k$$

and this is again what we were looking for, because $\max(\tilde{u}_k), \max(\tilde{u}_k^{\sim}) \leq \max(\tau_k) \leq \max(\tau_{j_{q-1}}^{1}) = \max(v_{j_{q-1}}).

---

5. If $j_l = k$ then in particular $j_l \leq k$ so by the minimality of $q$ in its definition, $q \leq l$.
6. Just apply the formula $(\ast)$ repeatedly for $r = q - 1, q, \ldots$ till arriving at $\phi_k(x)$.
7. The central inequality $\max(\tau_k) \leq \max(\tau_{j_{q-1}}^{1})$ follows from the definition of $q$. 

Figure 2. Passing from $x$ to $x^{\sim k}$
similarly as in the previous case. This finishes the proof of Claim 1.

Claim 2: Suppose that \( \tau_k \) is a chain type. Then for every \( x \in n^{<\omega} \) and every \( w \in W_k \), we have that \( \phi(x \sim w) = \phi(x \sim u_k \sim w' \sim w'' \sim w') \leq \max(\tau_k) \).

Proof of Claim 2: We proceed by induction on the length of \( w \). Together with the statement of the claim, we shall also prove that for every \( i = 0, \ldots, k \), we can write \( \phi_i(x \sim w) = \phi(x \sim u_k \sim w'_i \sim w'' \sim w') \leq \max(\tau_k) \). The first case is that \( w = (k) \). Remember that

\[
\phi(x \sim k) = \phi^k(x) \sim \bar{u}_k \sim v_{j_k} \sim v_{j_{k+1}} \sim \cdots \sim v_{j_p}
\]

and since \( \tau_k \) is a chain type, \( \phi^k(x) = \phi(x) \) and \( \bar{u}_k = u_k \). Moreover, by the definition of \( q = q(k) \) and the way that the sequence \( \{j_r\} \) is chosen we have that

\[
(\star\star) \quad \max(v_{j_p}) \leq \cdots \leq \max(v_{j_q}) \leq \max(\tau_k)
\]

so the expression above is as desired, and the claim is proven for \( w = (k) \). Concerning \( \phi_i(x \sim k) \), if \( \tau_i \) is a chain type, \( \phi_i(x \sim k) = \phi(x \sim k) \) and there is nothing to prove. The other case is that \( i = j_r \) for some \( r \). Then, by the definition of \( q, r \geq q \) since \( j_r = i \leq k \), therefore

\[
\phi_i(x \sim k) = \phi_{j_r}(x \sim k) = \phi^k(x) \sim \bar{u}_k \sim v_{j_k} \sim v_{j_{k+1}} \sim \cdots \sim v_{j_{r-1}}
\]

In the same way as before, by (\( \star\star \)) above, this provides an expression \( \phi_k(x \sim k) = \phi(x \sim u_k \sim w'_i \sim w'' \sim w') \leq \max(\tau_k) \). This finishes the initial step of the inductive proof when \( w = (k) \).

Now we assume that our statement holds for \( w \in W_k \), we fix \( \xi = \{0, \ldots, k\} \) and we shall prove that the statement holds for \( w \sim \xi \). First,

\[
(\star\xi) \quad \phi(x \sim w \sim \xi) = \phi^\xi(x \sim w) \sim \bar{u}_{\xi} \sim v_{j_{\xi}+1} \sim \cdots \sim v_{j_p}
\]

Notice that \( \max(\bar{u}_{\xi}) \leq \max(\tau_{\xi}) \leq \max(\tau_k) \), and in the same way as we had the expression (\( \star\star \)), the defining formula of \( q(\xi) \) implies that

\[
(\star\star}') \quad \max(v_{j_{\xi}+1}) \leq \cdots \leq \max(v_{j_p}) \leq \max(\tau_{\xi}) \leq \max(\tau_k)
\]

so all vectors \( v_{j_p} \) appearing in the expression (\( \star\xi \)) above are bounded by \( \max(\tau_{\xi}) \). Hence, the expression (\( \star\xi \)) above can be rewritten as

\[
\phi(x \sim w \sim \xi) = \phi^\xi(x \sim w) \sim \bar{u}_{\xi} \sim 0^{l_{\xi}}
\]

If \( \tau_{\xi} \) is a chain type, then \( \phi^\xi(x \sim w) = \phi(x \sim w) \) and we are done, by the inductive hypothesis. If \( \tau_{\xi} \) is a comb type, then

\[
\phi^\xi(x \sim w) = \phi_{\xi}(x \sim w) \sim \bar{u}_{\xi} \sim 0^{l_{\xi}}
\]

which also provides the desired form because \( \max(\bar{u}_{\xi} \sim 0^{l_{\xi}}) \leq \max(\tau_{\xi}) \) and we can apply the inductive hypothesis to \( \phi_{\xi}(x \sim w) \).

Finally, we fix \( i \in \{0, \ldots, k\} \) and we prove that also \( \phi_i(x \sim w \sim \xi) \) is of the form \( \phi(x \sim u_k \sim w'_i \sim w'' \sim w') \leq \max(\tau_k) \). If \( \tau_i \) is a chain type, there is nothing to prove because \( \phi_i = \phi \). Otherwise \( \phi_i \) is a comb type, and \( i = j_r \) for some

---

8By the definition of \( q \), either \( \max(v_{j_q}) = \max(\tau_k) \) or \( j_q \leq k \). In the latter case, \( \max(v_{j_q}) \leq \max(\tau_{j_q}) \leq \max(\tau_k) \) by the statement (2) of Theorem 2.2.1 that we are assuming.
r. If \( r < q(\xi) \) then \( \phi_k(x\hat{}^\xi - w\hat{}^\xi) = \phi_k(x\hat{}^\xi) \) and we apply directly the inductive hypothesis. If \( r \geq q(\xi) \), then
\[
\phi_k(x\hat{}^\xi - w\hat{}^\xi) = \phi^k(x\hat{}^\xi - \tilde{u_\xi} - v_{j_{\xi}} - v_{j_{\xi}+1} - \cdots - v_{j_{\xi+1}})
\]
By the expression (**') above, all vectors to the right of \( \phi^k(x\hat{}^\xi) \) are bounded by \( \max(\tau_\xi) \leq \max(\tau_k) \), while
\[
\phi^k(x\hat{}^\xi) = \phi_k(x\hat{}^\xi) - \tilde{u_\xi} - 0^k
\]
is of the form \( \phi(x\hat{}^\xi) - u_k^w' \) with \( \max(w') \leq \max(\tau_k) \), by the inductive hypothesis. This finishes the proof of Claim 2.

Claim 3: Suppose that \( \tau_k \) is a comb type, \( x \in n^{<\omega} \) and \( w \in W_k \). Then
\[
\phi_k(x\hat{}^\xi) = \phi^k(x\hat{}^\xi - \tilde{u_k} - w')
\]
where \( \max(w') \leq \max(\tau_k^0) \).

Proof of Claim 3: Since \( \tau_k \) is a comb type, \( k = j_r \) for some \( r \). We proceed by induction on the length of \( w \). The first case is that \( w = (k) \). Notice that \( r \geq q = q(k) \) because \( j_r = k \leq k \) (by the definition of \( q \)), hence
\[
\phi_k(x\hat{}^k) = \phi^k(x\hat{}^k - \tilde{u_k} - v_j - v_{j+1} - \cdots - v_{j_r-1}).
\]
It is enough to show now that all vectors to the right of \( \tilde{u_k} \) in the expression above are bounded by \( \max(\tau_k^0) \). This is equivalent to show that either \( q = r \) or \( \max(v_j) \leq \max(\tau_k^0) \). Remember that \( \max(v_\xi) = \max(\tau_\xi^1) \) for any \( \xi \). By the definition of \( q \), one of the following two cases must hold:

Case 1: \( \max(\tau_k) \) is a comb type, \( x \in n^{<\omega} \) and \( w \in W_k \). Then, since \( k = j_r \) and \( q \leq r \) we have that
\[
\max(\tau_k) = \max(\tau_k^1)
\]
From the two inequalities above we conclude that \( \max(\tau_k^1) < \max(\tau_k) \), hence \( \max(\tau_k) = \max(\tau_k^0) \). Therefore \( \max(\tau_{j_q}^1) < \max(\tau_k) = \max(\tau_k^0) \) as we wanted to prove.

Case 2: \( \max(\tau_{j_q}^1) \geq \max(\tau_k) \) and \( j_q \leq k \). Now, \( j_q \leq k \) implies that
\[
\max(\tau_{j_q}^1) \geq \max(\tau_k) = \max(\tau_k^1)
\]
hence actually \( \max(\tau_{j_q}^1) = \max(\tau_k) \). If \( \max(\tau_k) = \max(\tau_k^0) \) then we are done, so we suppose that \( \max(\tau_k) = \max(\tau_k^1) > \max(\tau_k^0) \). We combine the two previous equations we get that
\[
\max(\tau_{j_q}^1) = \max(\tau_k) = \max(\tau_k^1) = \max(\tau_{j_r}^1)
\]
but this implies (by the way in which chose the order of the enumeration \( \{j_1, \ldots, j_p\} \) and the fact that \( j_q \leq k = j_r \) assumed in Case 2) that \( r \leq q \), hence \( r = q \) as we wanted to prove. This finishes Case 2, and finishes the proof of initial case \( w = (k) \) as well.

Now we suppose that Claim 3 holds for \( w \), we fix \( \xi \leq k \) and we shall prove that Claim 3 holds for \( w\hat{}^\xi \) as well. If \( r < q(\xi) \) then \( \phi_k(x\hat{}^\xi - w\hat{}^\xi) = \phi_k(x\hat{}^\xi) \) and

\footnote{It should be noticed that since we suppose \( r \geq q \) we cannot have \( q = p + 1 \), so the minimum that defines \( q \) is actually attained at \( q \).}
we apply directly the inductive hypothesis. Hence, we suppose that \( r \geq q(\xi) \) and therefore

\[
(\star) \quad \phi_k(x \wedge w \wedge \xi) = \phi^k(x \wedge w) \sim u_k \sim v_{\beta(i)} \sim v_{\beta(i+1)} \sim \cdots \sim v_{j_{r-1}}.
\]

On the other hand,

\[
\phi^k(x \wedge w) = \phi_k(x \wedge w) \sim u_k \sim v_{j_{r}}
\]

so applying the inductive hypothesis to \( \phi_k(x \wedge w) \), we get that

\[
\phi^k(x \wedge w) = \phi^k(x) \sim u_k \sim w'
\]

with \( \max(w') \leq \max(\tau_k^\phi) \). Looking back at the expression \( (\star) \) above, it is enough to show that all members of that expression to the right of \( \phi \) with \( \max(v) \) (max(\( w \))) are bounded by \( \max(\tau_k^\phi) \). This is equivalent to prove that either \( r = q(\xi) \) or \( \max(\tau_{j(i)}^1) = \max(v_{j(i)}) \leq \max(\tau_k^\phi) \). Let now \( q = q(\xi) \). We distinguish two cases:

Case 1: \( \max(\tau_{j(i)}^1) < \max(\tau_k) \). This case, since \( k = j_r \), we supposed that \( q \leq r \) we have that

\[
\max(\tau_{j(i)}^1) \geq \max(\tau_{j(i)}^1) = \max(\tau_k^1)
\]

From the two inequalities above we conclude that \( \max(\tau_{j(i)}^1) < \max(\tau_k) \), hence \( \max(\tau_k) = \max(\tau_k^1) \). Therefore \( \max(\tau_{j(i)}^1) < \max(\tau_k) = \max(\tau_k^1) \) as we wanted to prove.

Case 2: \( \max(\tau_{j(i)}^1) \geq \max(\tau_k) \). Since \( \xi \leq k \) this implies that \( \max(\tau_{j(i)}^1) \geq \max(\tau_k) \geq \max(\tau_k) \). By the definition of \( q = q(\xi) \), this further implies that \( j_q \leq \xi \). Now, \( j_q \leq k \) implies that

\[
\max(\tau_{j(i)}^1) \leq \max(\tau_{j(i)}) \leq \max(\tau_k)
\]

hence actually \( \max(\tau_{j(i)}^1) = \max(\tau_k) \). If \( \max(\tau_k) = \max(\tau_k^1) \) then we are done, so we suppose that \( \max(\tau_k) = \max(\tau_k^1) > \max(\tau_k^1) \). We combine the previous equations and we get that

\[
\max(\tau_{j(i)}^1) = \max(\tau_k) = \max(\tau_k^1) = \max(\tau_{j(i)})
\]

but this implies (by the way in which chose the order of the enumeration \( \{j_1, \ldots, j_p\} \) and the fact that \( j_q \leq \xi \leq k = j_r \) that we noticed above) that \( r \leq q \), hence \( r = q \) as we wanted to prove. This finishes Case 2, and finishes the proof of Claim 3 as well.

We fix \( k < n \) and we shall prove that if \( Y \subset n^{<\omega} \) is a set of type \( [k] \), then \( \phi(Y) \) is a set of type \( \tau_k \). This will finish the proof of the theorem because, if \( \phi \) was not a normal embedding, we can get a normal embedding by composing with a nice embedding using Theorem 2.1.3.

If \( \tau_k \) is a chain type, then the fact that \( \phi(Y) \) has type \( \tau_k \) follows immediately from Claim 2. So suppose that \( \tau_k \) is a comb type, \( k = j_r \), and \( Y = \{y_1, y_2, y_3, \ldots \} \). If we look at the inductive definition of \( \phi \), and consider the case the case when \( z = x \wedge k \) and \( k = j_r \), notice that then \( r \geq q \) by the definition of \( q \) since \( j_r = k \leq k \), and we can write

\[
\phi(z) = \phi_k(z) \sim v_{j_r} \sim v_{j_{r+1}} \sim \cdots \sim v_p
\]

where \( \max(v) \leq \max(v_{j_r}) = \max(v_k) \) for all \( t = r + 1, \ldots, p \). If we apply this to \( z = y_i \), we can write that

\[
(\star) \quad \phi(y_i) = \phi_k(y_i) \sim v_k \sim w_i
\]
where \( \max(w_i) \leq \max(v_k) \). On the other hand, Claim 3 provides the fact that

\[
(\ast \ast) \quad \phi_k(y_{i+1}) = \phi^k(y_i) - \bar{u}_k^0 w_i' = \phi_k(y_i) - \bar{u}_k^0 \bar{v}_k^0 w_i'
\]

where \( \max(w_i) \leq \max(u_i) \). Remember that in the inductive definition of \( \phi \), the number \( \bar{\varepsilon} \) of 0’s above was chosen so that the length of \( \phi_k(y_i) - \bar{u}_k^0 \bar{v}_k^0 \) is larger than the length of \( \phi(y_i) \). The expressions \((\ast)\) and \((\ast \ast)\) together yield that \( \phi(Y) \) is a set of type \( \tau_k \) with underlying chain \( \{ \phi_k(y_i) : i < \omega \} \), as it is shown in Figure 3.

**Corollary 2.2.2.** If \( \phi : n^{<\omega} \to m^{<\omega} \) is a normal embedding, then \( \max(\tau) \leq \max(\bar{\phi} \tau) \) implies that \( \max(\bar{\phi} \tau') \leq \max(\bar{\phi} \tau') \).

**Corollary 2.2.3.** If \( \{ S_i : i \in n \} \) are pairwise disjoint sets of types in \( m^{<\omega} \), then \( \{ \Gamma S_i : i \in n \} \) is an \( n \)-gap.

**Proof.** The intersection of two sets of different types is finite, so it is clear that the ideals are mutually orthogonal. We have to prove that they cannot be separated. After reordering if necessary, we can find types \( \tau_i \in S_i \) such that \( \max(\tau_0) \leq \max(\tau_1) \leq \cdots \leq \max(\tau_{n-1}) \). By Theorem 2.2.2, there is a normal embedding \( \phi : n^{<\omega} \to m^{<\omega} \) such that \( \bar{\phi}(i) = \tau_i \). Finally, use Lemma 1.3.3.

We can provide now our first example of a minimal analytic \( n \)-gap:

**Corollary 2.2.4.** Let \( M_i \) be the set of all types \( \tau \) in \( n^{<\omega} \) such that \( \max(\tau) = i \). The \( n \)-gap \( M = \{ \Gamma M_i : i < n \} \) in \( n^{<\omega} \) is a minimal \( n \)-gap.

**Proof.** Suppose that \( \Gamma \leq M \) and we must show that \( M \leq \Gamma \). By Theorem 1.3.5, we can suppose that \( \Gamma = \{ \Gamma S_i : i < n \} \) is a standard gap in \( n^{<\omega} \). That is, there is a permutation \( \bar{\varepsilon} : n \to n \) such that \( [i] \in S_{\varepsilon(i)} \). By Theorem 2.1.3 there is a normal embedding \( \phi : n^{<\omega} \to n^{<\omega} \) such that \( \bar{\phi}(i) = \tau_i \) if and only if \( \bar{\phi} \tau \in M_i \). In particular, \( \bar{\phi}(i) \in M_{\varepsilon(i)} \), so \( \max(\bar{\phi}(i)) = \varepsilon(i) \). Since

\[
\max[0] \leq \max[1] \leq \cdots \leq \max[n-1],
\]

Corollary 2.2.2 implies that

\[
\max(\bar{\phi}(i)) \leq \max(\bar{\phi}(i)) \leq \cdots \leq \max(\bar{\phi}(i)),
\]

so \( \varepsilon(0) \leq \varepsilon(1) \leq \cdots \) which implies that \( \varepsilon \) is the identity permutation. Moreover, we claim that \( \Gamma = M \). For pick \( \tau \in M_i \). Then \( \max(\tau) = \max(i) \), so \( \max(\bar{\phi} \tau) = \max(\bar{\phi}(i)) = i \) which implies that \( \bar{\phi} \tau \in M_i \), hence \( \tau \in S_i \). This shows that \( M_i \subset S_i \) for every \( i \). Since the union of the sets \( M_i \) gives all types in \( n^{<\omega} \) this actually implies that \( M_i = S_i \) for every \( i < n \).
For a permutation \( \delta : n \rightarrow n \), let us denote by \( \mathcal{M}^{\delta} = \{ \Gamma_{\mathcal{M}_{\delta}(i)} : i < n \} \) the \( \delta \)-permutation of \( \mathcal{M} \). The minimal gaps \( \mathcal{M}^{\delta} \) are characterized by their extreme asymmetry in the following sense:

**Corollary 2.2.5.** The minimal \( n \)-gap \( \mathcal{M}^{\delta} \) has the following two properties:

1. \( \mathcal{M} \) is dense.
2. The unique permutation \( \varepsilon : n \rightarrow n \) in Theorem 1.3.2 that works for the gap \( \mathcal{M}^{\delta} \) is \( \varepsilon = \delta^{-1} \).

Moreover, if a minimal analytic \( n \)-gap \( \Gamma \) satisfies the two properties above, then \( \Gamma \) is equivalent to \( \mathcal{M}^{\delta} \).

**Proof.** It is clear that \( \mathcal{M} \) is dense. For the second property, if Theorem 1.3.2 holds for the permutation \( \varepsilon \), then by Theorem 2.1.3 there exists a normal embedding \( \phi : n^{<\omega} \rightarrow m^{<\omega} \) such that \( \phi[i] \in \mathcal{M}_{\varepsilon(\delta(i))} \). By Theorem 2.2.1 this implies that \( \varepsilon = \delta^{-1} \). Finally, for the last statement of the theorem, suppose that \( \Gamma = \{ \Gamma_{S_i} : i < n \} \) is standard \( n \)-gap which is a minimal \( n \)-gap with the two properties above. The density means that every type in \( n^{<\omega} \) belongs to some \( S_i \). By Theorem 2.2.1, if \( \Gamma \) satisfies property (2) above, then \( \max(\tau_0) < \cdots < \max(\tau_{n-1}) \) whenever \( \tau_i \in S_{\delta^{-1}(i)} \). This implies that \( S_{\delta^{-1}(i)} = \mathcal{M}_i \) for every \( i < n \), hence \( \Gamma = \mathcal{M}^{\delta} \). \( \square \)

### 2.3. Chain types

Remember that a chain type is nothing else than a finite increasing sequence of natural numbers. We define the composition of two chain types \( \tau = [n_1 < \cdots < n_k] \) and \( \sigma = [m_1 < \cdots < m_l] \) as \( \tau \ast \sigma = [n_1 < \cdots < n_k < m_p < \cdots < m_l] \) where \( p = \min\{ i : m_i > n_k \} \).

**Lemma 2.3.1.** If \( \phi : n^{<\omega} \rightarrow m^{<\omega} \) is a normal embedding and \( \sigma, \tau, \phi \sigma, \phi \tau \) are chain types, then \( \phi(\sigma \ast \tau) \) is also a chain type and \( \phi(\sigma \ast \tau) = \phi \sigma \ast \phi \tau \).

**Proof.** Straightforward. \( \square \)

We investigate now the situation when a normal embedding sends a comb type to a chain type. Lemma 2.3.3 below indicates that this often implies that the normal embedding is trivial in a sense.

**Definition 2.3.2.** A type \( \tau = (\tau^0, \tau^1, \prec) \) will be called a top-comb type if it is a comb type and moreover the penultimate position in the order \( \prec \) is occupied by an element coming from \( \tau^1 \times \{1\} \).

Thus, if \( (u, v) \) is a rung of type \( \tau \) as in Definition 1.2.2, the fact that \( \tau \) is a top-comb type means that \( u_0 \sim \cdots \sim u_{n-1} \prec v_0 \sim \cdots \sim v_n \), see Figure 4. In the matrix representation of types, \( \tau \) is a top-comb type when the second from the right number is in the upper row, so that for instance \( [101] \) is not top-comb, but \( [011] \) is top-comb.

**Lemma 2.3.3.** Let \( \phi : n^{<\omega} \rightarrow m^{<\omega} \) be a normal embedding and let \( k \leq n \). The following are equivalent:

1. There exists a comb type \( \tau \) with \( \max(\tau^1) = k - 1 \) and a chain type \( \sigma \) such that \( \phi(\tau) = \sigma \).

---

\( ^{10} \) Remember the the last position in the order \( \prec \) is always occupied by an element from \( \tau^0 \times \{0\} \), by Definition 1.2.1.
There exists a chain type \( \sigma \) such that \( \bar{\phi} (\tau) = \sigma \) for all types \( \tau \) in \( k^{<\omega} \).

There exists a nice embedding \( u : k^{<\omega} \rightarrow n^{<\omega} \) such that the image of \( \phi u \) is contained in a chain.

There exists a chain type \( \sigma \) and a nice embedding \( u : k^{<\omega} \rightarrow n^{<\omega} \) such that the image of \( \phi u \) is a chain of type \( \sigma \).

**Proof.** The implications \( 4 \Rightarrow 2 \Rightarrow 1 \) are obvious, so it is enough to prove the following two facts:

That \( 3 \Rightarrow 4 \). Let \( u : k^{<\omega} \rightarrow n^{<\omega} \) be the nice embedding and let \( C \subset m^{<\omega} \) be the chain provided by condition (3). For every \( x \in C \) we can consider the set

\[
M(x) = \{ \max(y \setminus x) : y \in C, y > x \},
\]

which is a finite subset of \( m \). By Corollary 1.1.10 we can find a nice embedding \( v_1 : k^{<\omega} \rightarrow n^{<\omega} \) such that \( M \circ \phi \circ u \circ v_1 \) is a constant function equal to \( M_0 \). The next step is to construct a further nice embedding \( v_2 : k^{<\omega} \rightarrow k^{<\omega} \) such that the image of \( \phi \circ u \circ v_1 \circ v_2 \) is a chain \( C_2 \subset C \) such that \( M_0 = \{ \max(y \setminus x) : x < y < x' \} \) whenever \( x, x' \in C_2, x < x' \). This is easy to do, we just have to define \( v_2 \) inductively on \( k^{<\omega} \) and at each step we just need to pick a high enough node. Once we have this, we are done, because \( C_2 \) is a chain of type \( M_0 \), when we view the set \( M_0 \) with its natural order as a chain type.

That \( 1 \Rightarrow 3 \). We distinguish two cases. The first case is that \( \tau \) is a top-comb type. Consider then \( X = \{ x_1, x_2, \ldots \} \) be a set of type \( \tau \) so that \( \phi(X) \) is a chain of type \( \sigma \). Let us consider the branch \( B \) of \( m^{<\omega} \) generated by \( \phi(X) \), \( B = \{ s \in m^{<\omega} : \exists i : s < \phi(x_i) \} \). Consider the nice embedding \( u : k^{<\omega} \rightarrow n^{<\omega} \) given by \( u(s) = x_i \). Because of \( \tau \) being a top-comb type, notice that for every \( s \in k^{<\omega} \) there is a \( p \) such that the set \( X_s = \{ u(s), x_p, x_{p+1}, \ldots \} \) is still\[1\] of type \( \tau \), hence \( \phi(X_s) \) is a chain of type \( \sigma \), and it must be included inside \( B \). Thus, the image of \( \phi u \) is contained in \( B \) as required. Now, we consider the general case, and we will reduce it to the previous case, when \( \tau \) was a top-comb type. We consider again \( X = \{ x_1, x_2, \ldots \} \) a set of type \( \tau \), but now we will choose it in such a way that for all \( s \in k^{<\omega} \) with \( |s| < 10k \) we still have that \( X_s = \{ x_1^{-s}, x_2, x_3, \ldots \} \) is of type \( \tau \). This can be easily achieved simply by intercalating a long sequence of 0’s inside \( x_2 \) at the height of \( x_1 \), so that the relative positions won’t change after adding 10k many numbers above \( x_1 \). Again, \( \phi(X) \) is contained inside a branch \( B \) of \( m^{<\omega} \). Since \( X_s \) is also of type \( \tau \), \( \phi(X_s) \) is a chain as well, and it must be included inside the

\[11\] Notice that if \( \tau \) was not a top-comb type, the type of the set \( X_s \) may not be \( \tau \) because adding elements above \( x_1 \) could change the relative order positions with \( x_p, p > 1 \) required by the order \( \sigma \). But since \( \tau \) is top-comb, this does not happen as far as \( p \) is taken large enough.
same chain $B$. Now for any triple $\{z_1, z_2, z_3\} \subset k^{<\omega}$ we can find $s_1, s_2, s_3 \in k^{<\omega}$, $|s_i| < 10k$, such that $\{x_1^{-1}s_1, x_2^{-1}s_2, x_3^{-1}s_3\}$ is equivalent to $\{z_1, z_2, z_3\}$. Since the first set is mapped by $\phi$ into $B$, it follows that $\bar{\phi}\sigma'$ must be a chain type for every type $\tau'$ in $k^{<\omega}$. In particular we can choose $\tau'$ to satisfy the hypothesis of the first case: it can be taken a top-comb type with $\max((\tau')^1) = k - 1$. In this way we reduce the general case to the first case. □

If $\phi$ satisfies the conditions of Lemma 2.3.3 we shall say that $\phi$ collapses below $k$ (or that $\phi$ collapses up to $k - 1$) into a chain of type $\sigma$. The fact that in condition (1) of Lemma 2.3.3 the maximum of $\tau$ is attained in $\tau^1$ is important, for consider the following example: We can construct a normal embedding $\phi : 3^{<\omega} \rightarrow 2^{<\omega}$ such that for every $x$, $\phi(x^2) \geq \phi(x)^1$, and $\phi(x^i)$ equals $\phi(x)$ followed by a finite sequence of 0’s when $i = 0, 1$. Such an embedding can be constructed inductively so that $x < y$ implies $|\phi(x)| < |\phi(y)|$. Notice that $\bar{\phi}[0, 1, 2] = [01]$ but $\phi$ does not collapse below 3.

2.4. Domination

The notion of top-comb introduced in Definition 2.3.2 and illustrated in Figure 4 is going to be crucial in this section. The key property now will be the following:

**Lemma 2.4.1.** Let $\tau$ be a top-comb type and let $(u, v)$ be a rung of type $\tau$. If $w$ is such that $\max(w) \leq \max(\tau^1)$ and $|v^1w| < |u|$, then $(u, v^1w)$ is also a rung of type $\tau$.

**Proof.** Straightforward. Just look at the left-hand side of Figure 4 □

**Definition 2.4.2.** We say that a type $\tau$ dominates another type $\sigma$, and we will write $\tau \triangleright \sigma$, if $\tau$ is a top-comb type and $\max(\tau^1) \geq \max(\sigma)$.

**Lemma 2.4.3.** Let $\phi : n^{<\omega} \rightarrow m^{<\omega}$ be a normal embedding, and let $\tau \in T_m$ be a type that dominates $\phi\sigma$ for all $\sigma \in T_n$. Then, there exists a normal embedding $\psi : (n + 1)^{<\omega} \rightarrow m^{<\omega}$ such that $\psi\sigma = \bar{\phi}\sigma$ if $\max(\sigma) < n$, and $\psi\sigma = \tau$ if otherwise $\max(\sigma) = n$.

**Proof.** Let $m_0 = \max(\tau^1) + 1$. Without loss of generality we will suppose that $m = m_0$. We can do this because the domination hypothesis implies that all types $\bar{\phi}\sigma$ live in $m_0^{<\omega}$, and therefore we can find $\bar{\phi}_0 : n^{<\omega} \rightarrow m_0^{<\omega}$ such that $\bar{\phi}_0\sigma = \bar{\phi}\sigma$ for all $\sigma$. Let $Y = \{y_0, y_1, \ldots\}$ be an infinite subset of $m^{<\omega}$ of type $\tau$, and let $b : n^{<\omega} \rightarrow \{1, 2, 3, \ldots\}$ be a bijection such that $x < y$ if and only if $b(x) < b(y)$. If $x \in (n + 1)^{<\omega} \setminus n^{<\omega}$, there is a unique way to write $x$ in the form $x = u^1n^1v$ with $u \in (n + 1)^{<\omega}$ and $v \in n^{<\omega}$, by splitting $x$ at the position of the last coordinate equal to $n$. Using this, we can define $\psi : (n + 1)^{<\omega} \rightarrow m^{<\omega}$ as

$$
\psi(v) = y_0^1\phi(v) \quad \psi(u^1n^1v) = y_{b(u)}^1\phi(v)
$$

where $v \in n^{<\omega}$, $u \in (n + 1)^{<\omega}$.

Claim 1: If $X \subset (n + 1)^{<\omega}$ is a set of type $\sigma$ with $\max(\sigma) < n$, then $\psi(X)$ is a set of type $\sigma$.

12One way to do this is to define $\phi_0(t) = (s'_1, \ldots, s'_k)$, where $\phi(t) = (s_0, \ldots, s_k)$, $s'_i = \min(s_i, m_0 - 1)$.
Proof of Claim 1: This is clear, because \( X \) must be either contained in either \( n^{<\omega} \), in which case \( \psi(X) = \phi(X) \), or \( X \) is contained in a set of the form \( \{ u \uparrow n \uparrow v : v \in n^{<\omega} \} \) for some \( v \in n^{<\omega} \), in which case \( \psi(X) = \{ y_{b(u)} \uparrow x : x \in X \} \).

Claim 2: If \( X \subset (n+1)^{<\omega} \) is a set of type \( \sigma \), with \( \max(\sigma) = n \), then \( X \) contains an infinite subset \( X' \) such that \( \psi(X') \) has type \( \tau \).

Proof of Claim 2: Let \( X = \{ x_1, x_2, \ldots \} \), and write \( x_i = u_i \uparrow n \uparrow v_i \) in the form indicated above, with \( v_i \in n^{<\omega} \). Since \( X \) has type \( \sigma \) with \( \max(\sigma) = n \), we must have \( u_i \neq u_j \) for \( i \neq j \). We have that \( \phi(x_i) = y_{b(u_i)} \uparrow \phi(v_i) \). By re-enumerating, let us suppose that \( \phi(x_i) = y_i \uparrow \phi(v_i) \) and remember that \( \{ y_1, y_2, \ldots \} \) has type \( \tau \), so that the set \( \psi(X) \) looks like in Figure 5. Let \( z_i = y_{i+1} \land y_{i+2} \) the root nodes sitting on the chain below \( Y \). By passing to a subsequence, we can suppose that \( |\psi(x_i)| < |z_i| \) for all \( i \), as illustrated in Figure 6. Once we do this, we claim that \( \psi(X) \) has type \( \tau \). We have to check that \( (z_{i+1} \setminus z_i, \psi(x_i) \setminus z_i) \) is a rung of type \( \tau \). We know that \( (z_{i+1} \setminus z_i, y_{i+1} \setminus z_i) \) is a rung of type \( \tau \), since \( Y \) was of type \( \tau \). Remember that \( \psi(x_i) = y_i \uparrow \phi(v_i) \), and we made an assumption at the beginning of the proof that \( m = \max(\tau^1) \geq \max(\phi(v_i)) \). We can apply Lemma 2.4.1 for \( u = z_{i+1} \setminus z_i \), \( v = y_i \setminus z_i \) and \( w = \phi(x_i) \). 

**Theorem 2.4.4.** For \( \{ \tau_i : i \in n \} \subset \Upsilon_m \) pairwise different, the following are equivalent:

1. \( \tau_k \) dominates \( \tau_{k-1} \) for every \( k = 1, \ldots, n-1 \),

\[\]
2.4. DOMINATION

\begin{align*}
0^{p_1} & \leq 0^{p_2} - p_1 \\
0^{q_1} & \leq 0^{q_2} - q_2 \\
0^{p_2} & \leq 0^{q_3} - q_3 \\
0^{q_2} & \leq 0^{q_3} - q_3 \\
0^{p_3} & \leq 0^{q_4} - q_4 \\
\end{align*}

\begin{figure}[h]
\centering
\includegraphics{figure7}
\caption{The nodes $x_{p_n q_n}$ in a sequence with $(\star)$.}
\end{figure}

\begin{align*}
\phi(x_{p_1 q_1}) & \leq \phi(x_{p_2 q_2}) \\
\phi(x_{p_2 q_2}) & \leq \phi(x_{p_3 q_3}) \\
\end{align*}

\begin{figure}[h]
\centering
\includegraphics{figure8}
\caption{The nodes $\phi(x_{p_n q_n})$ as a set of type $\tau_1$ above the branch $B$.}
\end{figure}

(2) there exists a normal embedding $\phi : n^{<\omega} \rightarrow m^{<\omega}$ such that $\bar{\phi}\sigma = \tau_{\max(\sigma)}$ for every $\sigma \in \Sigma_n$.

PROOF. That (1) implies (2) follows from repeated application of Lemma 2.4.3. We prove that (2) implies (1). As a first case, we prove the implication when $n = 2$ and $k = 1$. Thus, we have $\tau_0 \neq \tau_1$ and a normal embedding $\phi : 2^{<\omega} \rightarrow m^{<\omega}$ such that $\bar{\phi}[0] = \tau_0$ and $\bar{\phi}\sigma = \tau_1$ for every type $\sigma \neq [0]$ in $2^{<\omega}$. Notice that $\tau_1$ cannot be a chain type by Lemma 2.3.3. Consider the elements $x_{p_0} = 0^p \cdot 1^q$ in $2^{<\omega}$ (here $0^p$ means a sequence of $p$ many zeros). Notice that whenever $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$ are such that

\begin{align*}
(\star) \quad q_n + 1 & < p_{n+1} - p_n,
\end{align*}

the set $X = \{x_{p_1 q_1}, x_{p_2 q_2}, \cdots\}$ is of type $[1 \ 0]$, see Figure 7. Hence $\phi(X)$ is of (comb) type $\tau_1$, so it looks like in Figure 8. Let $B$ be the underlying branch of this set $\phi(X)$ that we can view in Figure 8 and we can formally define as

\begin{align*}
B = \{t : \exists i \left( \forall j > i \ t < \phi(x_{p_i q_i}) \right)\}.
\end{align*}

Claim A: The branch $B$ does not depend on the choice of the sequences $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$ with property $(\star)$ above. Proof of Claim A: Choose different sequences $p'_1 < p'_2 < \cdots$ and $q'_1 < q'_2 < \cdots$, and consider $X'$ and $B'$ the analogues of the set $X$ and the branch $B$ obtained from this new sequences of integers. Observe that $X$ and $X'$ can be alternated to produce a set of the form

\begin{align*}
Y = \{x_{p_k q_k}, x_{p'_k q'_k}, x_{p_{k+1} q_{k+1}}, x_{p'_{k+1} q'_{k+1}}, \cdots\}
\end{align*}

and the sequence $k_1 < k_2 < \cdots$ can be chosen to grow fast enough so that property $(\star)$ is satisfied, and $Y$ is again a set of type $[1 \ 0]$. Then $\phi(Y)$ is a set of type $\tau_1$ again.
of the form represented in Figure 8 with underlying branch $B_Y$. But $\phi(Y)$ contains both an infinite subsequence contained in $\phi(X)$ and an infinite subsequence contained in $\phi(X')$. This implies that the equality of the underlying branches $B = B_Y = B'$, and finishes the proof of Claim A.

Now, for $p, q < \omega$ let $z_{pq} = \max\{t \in B : t < \phi(x_{pq})\}$. We distinguish two cases:

Case 1: There exists $p < \omega$ and $q_1 < q_2 < \cdots$ such that $z_{pq_0} < z_{pq_1} < z_{pq_2} < \cdots$. In this case, $\{x_{pq_1}, x_{pq_2}, \ldots\}$ has type $[0]$, hence $Z = \{\phi(x_{pq_1}), \phi(x_{pq_2}), \ldots\}$ has type $\tau_0$. But each $\phi(x_{pq_i})$ goes out from the chain $B$ at the node $z_{pq_i}$, so these nodes $\phi(x_{pq_i})$ of the set $Z$ are displayed exactly in the same way as shown in Figure 8 (with now $p = p_1 = p_2 = \cdots$). We argue now that actually $Z$ contains a subsequence of type $\tau_1$, and this derives a contradiction since we said that $Z$ has type $\tau_0$ and we supposed that $\tau_0 \neq \tau_1$. The point is that each node $x_{pq_i}$ is a member of some sequence $\{x_{pq_{i'}}, q'_{i'}\}$ having property $(\ast)$, so each node $\phi(x_{pq_i})$ is a node of some set of type $\tau_1$ with underlying branch $B$. Thus, for high enough $t \in B$, the pair $(t \setminus z_{pq_i}, \phi(x_{pq_i}) \setminus z_{pq_i})$ is a rung of type $\tau_1$. In this way, we can construct a subsequence of $Z$ of type $\tau_1$ as desired.

Case 2: For each $p$ there exists an infinite set $Q_p \subset \omega$ such that $z_{pq} = z_{pq'}$ for all $q, q' \in Q_p$. We denote $z_p = z_{pq}, q \in Q_p$. We can also suppose that $\phi(x_{pq}) > z_p$ for all $q \in Q_p$. The set $Y_p = \{\phi(x_{pq}) : q \in Q_p\}$ is now a set of type $\tau_0$ because it is the image under $\phi$ of a set of type $[0]$. Moreover, all elements of $Y_p$ are above $z_p$. The situation is illustrated in Figure 9. Similarly as in Case 1, we know that each $\phi(x_{pq})$ is an element of a set of type $\tau_1$ with underlying branch $B$, so $(t \setminus z_p, \phi(x_{pq}) \setminus z_p)$ is a rung of type $\tau_1$ for every $p, q$ and high enough $t \in B$. We prove now that $\tau_1$ dominates $\tau_0$. Pick $q_1 < q_2$ in $Q_p$. We have that $\max(\tau_1^1) = \max(\phi(x_{pq_2}) \setminus z_p)$, but since $Y_p$ is of type $\tau_0$,

$$\max(\phi(x_{pq_2}) \setminus z_p) \geq \max(\phi(x_{pq_2}) \setminus \phi(x_{pq_1})) = \max(\tau_0),$$

which proves that $\max(\tau_1^1) \geq \max(\tau_0)$. Finally, we prove that $\tau_1$ is a top-comb type.

We know that $(u, v) = (t \setminus z_p, \phi(x_{pq_1}) \setminus z_p)$ is a rung of type $\tau_1$ for some high enough $t$. Let $h$ be the length of the last critical step of $u$. That is, if $u = u_1 \cdots u_n$ with $u_i \in W_k$, as in Definition [4.2.2] let $h = |z_p \cdot u_1 \cdots u_{n-1}|$. We can pick $q_3 \in Q_p$.

\[14\] $\phi$ is one-to-one so there is at most one $q$ such that $\phi(x_{pq}) = z_p$. 

---

**Figure 9. Sets of type $\tau_0$ over a $\tau_1$-set**
2.5. SUBDOMINATION

Figure 10. rung of a top^2 comb type

such that |φ(x_{pq})| > h. Then (u', v') = (t \setminus z_p, φ(x_{pq}) \setminus z_p) must be again a rung of type τ_1 for high enough t, and we made sure that this rung satisfies the top-comb condition as illustrated in Figure 4.

That finished the proof of the case when n = 2 and k = 1. For the general case, consider a normal embedding ψ: 2^{<ω} → n^{<ω} given by ψ(i_0, ..., i_p) = (k − 1 + i_0, ..., k − 1 + i_p). Then we can apply the case when n = 2 and k = 1 to φ' = φ ◦ ψ, τ'_0 = τ_{k−1} and τ'_1 = τ_k.

□

Corollary 2.4.5. If φ: n^{<ω} → m^{<ω} is a normal embedding, τ ≫ τ' and φτ ≠ φτ', then φτ ≫ φτ'.

Corollary 2.4.6. Let φ: n^{<ω} → m^{<ω} be a normal embedding, τ a top-comb type with max(τ^1) = k, and suppose that φτ is not constant equal to φτ on the set of types of maximum at most k. Then φτ is a top-comb type.

Corollary 2.4.7. Let M be the minimal n-gap of Corollary 2.2.4 and let \{S_i: i < n\} be pairwise disjoint nonempty families of types in m^{<ω}. The following are equivalent:

1. M ≤ \{Γ_{S_i}: i < n\},
2. we can pick τ_i ∈ S_i such that τ_0 ≪ τ_1 ≪ · · · ≪ τ_{n−1}.

2.5. Subdomination

When we remove from domination the condition of being a top-comb, we obtain the notion of subdomination.

Definition 2.5.1. We say that a type τ subdominates another type σ, and we will write τ ≫ σ, if τ = (τ^0, τ^1) is a comb type which is not top-comb, and max(τ^1) ≥ max(σ).

Lemma 2.4.3 says that when a type dominates τ the range of a normal embedding φ, then it is possible to define a new normal embedding ψ whose range equals the range of φ plus the type τ. In this section, we shall see that if τ only subdominates the range of φ, then we can find a normal embedding ψ whose range contains the range of φ, plus the type τ, plus maybe at most five more types, which are formally described in Definition 2.5.2 and illustrated in Figures 11 and 12.

Definition 2.5.2. Given a comb type τ which is not top-comb, we associate to it other comb types:

1. ϣ(τ) is exactly equal to τ except that the last element of τ^1 is moved to the penultimate position in the order ≪ in order to make ϣ(τ) a comb type.
2. WORKING IN THE \( n \)-ADIC TREE

![Diagram of \( \xi(\tau) \) and \( \xi(\tau) \)]

**Figure 11.** Rungs of type \( \tau \) and \( \xi(\tau) \)

![Diagram of \( \tau \) and \( \xi(\tau) \)]

**Figure 12.** Rungs of type \( \tau \), \( p(\tau) \), \( s(\tau) \), \( \xi(\tau) \) and \( \xi(\tau) \)

For example, if \( \tau = \begin{bmatrix} 23 & 167 \end{bmatrix} \), then \( \xi(\tau) = \begin{bmatrix} 2 & 16 \end{bmatrix} \).

(2) \( p(\tau) \) is of the form
\[
p(\tau) = \begin{bmatrix} \mathcal{r}^{0} \mathcal{r}^{1} & 0 \end{bmatrix},
\]
where \( \mathcal{r}^{0} \) and \( \mathcal{r}^{1} \) are the lower and upper rows of \( \tau \) as usual, and
\[
\mathcal{r}^{0} = \{ k \in \tau_{0} : k > \min\{ k' \in \tau_{0} : \forall p \in \tau^{1} (p, 1) \prec (k', 0) \} \}.
\]
For example, if \( \tau = \begin{bmatrix} 4 & 23 \ 9 & 678 \end{bmatrix} \), then \( \mathcal{r}^{0} = \{ 7, 8 \} \), \( \mathcal{r}^{0} \mathcal{r}^{1} = (7, 8) \star (4, 9) = (7, 8, 9) \), and \( p(\tau) = \begin{bmatrix} 7 & 8 \end{bmatrix} \).

(3) \( s(\tau) \) is of the form
\[
s(\tau) = \begin{bmatrix} \mathcal{r}^{0} \mathcal{r}^{1} & 0 \end{bmatrix},
\]
In the same example above, if \( \tau = [14^3_{23}_6] \), then \( s(\tau) = [789_{078}] \).

(4) \( z(\tau) \) has the same lower and upper rows as \( s(\tau) \) but its \( \prec \)-order is different. The reordering is done in the following way: Elements of the upper row coming from \( \tilde{\tau}^0 \), except \( \max(\tilde{\tau}^0) \), form the first block which is placed first. The elements of the lower row, except its maximum, form the second block, which is placed immediately after the first block. The remaining elements of the upper row form the third block, and the maximum of the lower row is -as it must be- the last element. In the same example above, if \( \tau = [14^3_{23}_6] \), then \( s(\tau) = [789_{078}] \) and \( z(\tau) = [7_0^89_8] \).

(5) \( w(\tau) \) is of the form

\[
\begin{align*}
\omega(\tau) &= \left[ \tilde{\tau}^0 \setminus \{ \max(\tilde{\tau}^0) \} \right] + \tau^0 \tau^1 \max(\tilde{\tau}^0).
\end{align*}
\]

In the same example above, if \( \tau = [14^3_{23}_6] \), then \( \omega(\tau) = [7^0_{879}] \).

We notice that \( \epsilon(\tau) \), \( p(\tau) \), \( z(\tau) \) and \( w(\tau) \) are top-combs with the same maximum of the upper row as \( \tau \) or larger, while \( \tau \) and \( s(\tau) \) are not top-combs.

**Definition 2.5.3.** A comb type \( \sigma \) is called a top\(^2\)-comb if the last two positions in the order \( \prec \) before the last one are occupied by numbers of the upper row \( \sigma^1 \). That is, \( \sigma \) is a top\(^2\)-comb type if it can be written like \( \sigma = \left[ \ldots p \ r \cdot \right] \).

**Theorem 2.5.4.** Let \( \phi : n^{< \omega} \rightarrow m^{< \omega} \) be a normal embedding, and let \( \tau \) be a type in \( m^{< \omega} \) that subdominates \( \tilde{\phi} \sigma \) for all \( \sigma \in \Sigma_n \). Then, there exists a normal embedding \( \psi : (n + 1)^{< \omega} \rightarrow m^{< \omega} \) such that \( \psi \sigma = \tilde{\phi} \sigma \) if \( \max(\sigma) < n \), while if \( \max(\sigma) = n \), the following cases occur:

1. \( \psi \sigma = \tau \) if either \( \sigma \) is a chain type, or \( \max(\sigma^1) < n \) and \( \sigma \) is not a top-comb.
2. \( \psi \sigma = \epsilon(\tau) \) if \( \sigma \) is a top-comb with \( \max(\sigma^1) < n \).
3. \( \psi \sigma = p(\tau) \) if \( \max(\sigma^0) < n \).
4. \( \psi \sigma = s(\tau) \) if \( \max(\sigma^0) = \max(\sigma^1) = n \) and \( \sigma \) is not a top-comb.
5. \( \psi \sigma = z(\tau) \) if \( \max(\sigma^0) = \max(\sigma^1) = n \) and \( \sigma \) is a top-comb type which is not top\(^2\)-comb.
6. \( \psi \sigma = w(\tau) \) if \( \max(\sigma^0) = \max(\sigma^1) = n \) and \( \sigma \) is a top\(^2\)-comb type.

**Proof.** Let \( (u, v) \) be a rung of type \( \tau \). The construction follows similar ideas as in the proofs of Theorem 2.2.1 and Theorem 2.4.1 but with some adjustments. As in the proof of Lemma 2.4.3, we can suppose that we have \( m \leq \max(\tau^1) + 1 \). We write \( u = \tilde{u} \uparrow \bar{u} \) in such a way that \( |\bar{u}| > |v| \), \( \max(\tilde{u}) \) is the first element of \( \tau^0 \) which is \( \prec \)-above all elements of \( \tau^1 \), while the first integer of \( \bar{u} \) is the second element of \( \tau^0 \) which is \( \prec \)-above all elements of \( \tau^1 \). The rung of type \( \tau \) in Figure 12 illustrates this, \( \tilde{u} \) corresponding to \( u_0 \uparrow \cdots \uparrow u_k \) and \( u \) being a rung of type \( \tau^0 \). Given \( x \in (n + 1)^{< \omega} \), we write it as \( x = \hat{x} \uparrow \check{x} \) where \( \check{x} \in n^{< \omega} \) and the last number of \( \hat{x} \) equals \( n \) (we make \( \hat{x} = \emptyset \) if \( x \in n^{< \omega} \)). We consider a one-to-one function \( \psi : (n + 1)^{< \omega} \rightarrow m^{< \omega} \) that

\[\text{Footnote 15: The reason for moving the maximal elements of the blocks of these definitions is the way that we defined the order \( \prec \), which was very convenient for the proof of Theorem 1.1.3, but it is unnatural for many purposes, since it compares the heights of next nodes instead of nodes themselves.}\]
we define with the help of two base functions $\beta_0, \beta_1 : (n + 1)^{<\omega} \rightarrow m^{<\omega}$ in the following way:

$$
\begin{align*}
\beta_0(\emptyset) &= \emptyset \\
\beta_1(\emptyset) &= \bar{u} \\
\beta_0(x \concat n) &= \beta_1(x \concat \bar{u}) \\
\beta_0(x) &= \beta_0(\bar{x}) \\
\beta_1(x) &= \beta_0(\bar{x}) \concat \bar{u}^{0 |\phi(\bar{x})|} \\
\psi(x) &= \beta_0(\bar{x}) \concat \bar{v} \concat \phi(\bar{x})
\end{align*}
$$

Notice how these formulas provide a recursive definition of the three functions, so that $\beta_1$ and $\psi$ are one-to-one. Looking just at $\beta_1$, notice that $\beta_1(x \concat i)$ equals $\beta_1(x)$ followed by a sequence of 0’s if $i < n$, while

$$
\beta_1(x \concat n) = \beta_0(x \concat \bar{u}^{0 |\phi(\bar{u})|}) = \beta_1(x \concat \bar{u}^{0 |\phi(\bar{u})|}).
$$

Concerning $\beta_0$, it remains constant when we pass from $x$ to $x \concat i$ for $i < n$, while $\beta_0(x \concat n)$ jumps to $\beta_1(x \concat \bar{u})$. The situation is illustrated in Figure 13.

The restriction of $\psi$ to a nice subtree is a normal embedding, and we shall check that this the normal embedding that we are looking for. It is clear that $\psi \sigma = \phi \sigma$ for any type $\sigma$ in $n^{<\omega}$.

Claim A: For $x \in (n + 1)^{<\omega}$, $s \in n^{<\omega}$ and $w \in W_n$, we have that

$$
\beta_0(x \concat s \concat w) = \beta_0(x) \concat \bar{u}^{0 |\phi(\bar{x})|} \concat \bar{u} \concat \xi
$$

where $\max(\xi) \leq \max(\bar{u})$.

Proof of Claim A: It is enough to prove the claim for $s = \emptyset$, the general case will follow from substituting $x \concat s$ by just $x$. We proceed by induction on the length of $w$. If $|w| = 1$, then $w = (n)$ and we have

$$
\beta_0(x \concat n) = \beta_1(x \concat \bar{u}) = \beta_0(\bar{x}) \concat \bar{u}^{0 |\phi(\bar{x})|} \concat \bar{u} = \beta_0(x \concat \bar{u}^{0 |\phi(\bar{u})|} \concat \bar{u})
$$

as required. Now, we prove the statement for a vector of the form $w \concat i$, supposing that it holds for $w$ and for all vectors of length at most $|w|$. If $i < n$ it is trivial, because $\beta_0(x \concat w \concat i) = \beta_0(x \concat w)$. When $i = n$, using the alternative notation $\bar{x} = \hat{\alpha}(x)$, $\bar{u}(x) = \bar{x}$,

$$
\beta_0(x \concat w \concat n) = \beta_1(x \concat w \concat \bar{u}) = \beta_0(x \concat w \concat \bar{u}^{0 |\phi(\hat{\alpha}(x \concat w))|} \concat \bar{u}).
$$
We can apply the inductive hypothesis and take into account that max(\tilde{u}) ≤ max(\vec{u}), and this finishes the proof of Claim A.

We fix now a type \( \sigma \) with max(\( \sigma \)) = n and we shall see that \( \bar{\psi}_\sigma \) is as stated in the theorem.

**Case I:** \( \sigma \) is a chain type. So let \( X = \{x_0 < x_1 < \cdots \} \) be a chain of such a type \( \sigma \). The basic function \( \beta_0 \) only changes when a number \( n \) is added, while \( \psi(x_i) > \beta_0(x_i) \). Thus, we have \( \beta_0(x_0) < \beta_0(x_1) < \cdots \) is a chain, and \( \psi(x_i) > \beta_0(x_i) \). On the one hand, \( \psi(x_i) = \beta_0(x_i) - \tilde{v} - \phi(\bar{x}_i) \), where max(\( \phi(x_i) \)) ≤ max(\( v \)) since \( \tau \) subdominates all types of the form \( \bar{\phi} \lambda \). On the other hand, by Claim A, \( \beta_0(x_{i+1}) = \beta_0(x_i - s - w) = \beta_0(x_i) - \tilde{u} - 0|\phi(\bar{x}_i - s)| - \tilde{u} - \xi \) where max(\( \xi \)) ≤ max(\( \tilde{u} \)). Thus, \( \psi(X) \) has type \( \tau \) as illustrated in Figure 14, hence \( \bar{\psi}\sigma = \tau \).

**Case II:** \( \sigma \) is a comb type and max(\( \sigma^1 \)) < n (hence max(\( \sigma^0 \)) = n). The situation is similar to \( \sigma \) being a chain type. A set of type \( \sigma \) is now of the form \( Y = \{y_0, y_1, \ldots \} \) and there is a chain \( X = \{x_0, x_1, \ldots \} \) of chain type \( \sigma^0 \), max(\( \sigma^0 \)) = n such that \( y_i > x_i \) and max(\( y_i \setminus x_i \)) < n. Thus, \( \psi(X) \) is, by the previous case, a set of type \( \tau \) which looks like in Figure 14. Since \( y_i > x_i \) and max(\( y_i \setminus x_i \)) < n, we have that \( \bar{x}_i = \bar{y}_i \), hence \( \beta_0(x_i) = \beta_0(y_i) \). Hence, the configuration of the set \( \psi(Y) \) is like in Figure 15. Formally, we have an underlying chain \( \beta_0(x_0) < \beta_0(x_1) < \cdots \), and
\( \psi(y_i) > \beta_0(y_i) = \beta_0(x_i). \) More precisely,
\[
\psi(y_i) = \beta_0(x_i)^{-v^-} \phi(\bar{y}_i) \\
\beta_0(x_{i+1}) = \beta_0(x_i)^{-u^-} \circ \phi(\bar{x}_i^n) = \beta_0(x_i)^{-u^-} \circ \alpha(\bar{x}_i^n)
\]

Thus, \( \psi(Y) \) looks almost like a set of type \( \tau \). We could safely state that \( Y \) is of type \( \tau \) if we knew that

\[
(\star) \ |\psi(y_i)| < |\beta_0(x_i)^{-u^-} \circ \phi(\bar{x}_i^n)|,
\]

that is, if we could draw a horizontal level line in Figure 15 like the one that we draw in Figure 14.

Case II.a: \( \sigma \) is not a top-comb type. We check that the inequality (\( \star \)) above actually happens, so that \( \psi(Y) \) is of type \( \tau \). It is enough to check that \( |\phi(\bar{y}_i)| < |\phi(\bar{x}_i^n)| \) because we know that \( |v| < |\bar{u}| \). The proof is the following: we will have that \( \sigma \) is not top-comb, \( \max(\sigma^0) = n, \max(\sigma^1) < n \). Since \((x_{i+1} \setminus x_i, y_i \setminus x_i)\) is a rung of type \( \sigma \), the first \( n \)-integer of \( x_{i+1} \setminus x_i \) must be above the last integer of \( y_i \setminus x_i \). Writing \( x_{i+1} = x_i^s \circ w \) as above, with \( s \in n^{<\omega}, w \in W_n \), the first \( n \)-integer is indeed the first integer of \( w \). Thus, what we have is that \( |y_i \setminus x_i| < |s| \), so \( |y_i| < |x_i| \). Remember that \( \bar{x}_i = \bar{y}_i \), hence this implies that \( \bar{y}_i < |\bar{x}_i| \), and therefore \( |\phi(\bar{y}_i)| < |\phi(\bar{x}_i^n)| \), which is exactly what we needed in order to ensure that \( \psi(Y) \) is of type \( \tau \).

Case II.b: \( \sigma \) is a top-comb. Then, we can choose our set \( Y \) as above of type \( \tau \) inside the nice subtree where \( \psi \) is a normal embedding, with the extra property that \( |\phi(\bar{y}_i)| > |\phi(\bar{x}_i^n)| \) for every \( i < \omega \). This can be done just by adding as many 0’s as necessary above each \( y_i \), and then passing to a subsequence if necessary. Then, we observethat \( \psi(Y) \) has type \( \forall \tau \). Indeed the set \( \psi(Y) \) would look like in Figure 15 where we could additionally draw a horizontal level line that passes through \( \psi(y_i) \) and cuts \( \xi \).

Case III: \( \max(\sigma^0) < n \). Let \( Y = \{y_0, y_1, \ldots\} \) be a set of type \( \tau \), so that \( y_i > x_i \), where \( \{x_0 < x_1 < \cdots\} \) is a chain of type \( \sigma^0 \), and \( \max(y_i \setminus x_i) = n \). Write \( y_i = x_i^t \circ w \) where \( y_i \in n^{<\omega} \) and \( w \in W_n \). We can suppose that \( |x_0^t| < |x_i^t| < \cdots \). Notice that \( \beta_0(x_i^t) = \beta_0(x_i^{-t_1}^n) \) is the same for all \( x_i \), so let us call \( b = \beta_0(x_i^t) = \beta_0(x_i^{-t_1}^n) \) for all \( i \). We have that

\[
\beta_1(x_i^{-t_1}^n) = b^{-u} \circ \phi(\bar{x}_i^n).
\]

Hence, \( \{\beta_1(x_0^t) < \beta_1(x_1^{-t_1}^n) < \cdots\} \) is a chain of type [0]. On the other hand, notice that \( \bar{y}_i = x_i^{-t_1}^n \), therefore

\[
\psi(y_i) = \beta_0(\bar{y}_i)^{-v} \circ \phi(\bar{y}_i) = \beta_0(x_i^{-t_1}^n)^{-v} \circ \phi(\bar{y}_i) = \beta_1(x_i^{-t_1}^n)^{-u} \circ v \circ \phi(\bar{y}_i)
\]

Therefore, the underlying chain of \( \psi(Y) \) is the chain \( \{\beta_1(x_0^t) < \beta_1(x_1^{-t_1}^n) < \cdots\} \) of type [0], and the basic rungs of \( \psi(Y) \) are of the form

\[
\left( [\phi(\bar{x}_i^n)^{-t_1}], u, v \circ \phi(\bar{y}_i) \right)
\]

16 Once we restrict to a nice subtree where \( \psi \) is a normal embedding, in order to check that \( \psi \sigma = \bar{\tau} \), it is enough to find a particular set \( Y \) of type \( \sigma \) such that \( \psi(Y) \) has type \( \forall \tau \).
17 We can do this precisely because \( \sigma \) is a top-comb type, so adding 0’s above \( y_i \) does not change the type.
It follows that $ψ(Y)$ is a set of type $p(τ)$.

Case IV: $\max(σ^0) = \max(σ^1) = n$. So consider $Y = \{y_0, y_1, \ldots\}$ a set of type $σ$ with underlying chain $\{x_0 < x_1 < \ldots\}$ so that $x_i = y_i \land y_{i+1}$. Let us write $x_{i+1} = \bar{s}_i \bar{w}_i$ and $y_i = x_i \bar{s}_i \bar{w}_i$ in such a way that $s_i, \bar{s}_i \in n^\omega$ and $w_i, \bar{w}_i \in W_n$. We can see in Figure 16 the three possible configurations that we shall distinguish.

Claim B. $β_1(x_{i+1} \sim s_{i+1}) = β_1(x_i \sim s_i) \sim \bar{u} \sim ξ_i$ where $\max(ξ_i) ≤ \max(\bar{u})$.

Proof of Claim B: In the computations below, $ξ$ denotes a finite sequence such that $\max(ξ) ≤ \max(\bar{u}) = \max(u)$, but not necessarily the same finite sequence in every expression.

$$β_1(x_{i+1} \sim s_{i+1}) = β_0(x_{i+1} \sim \bar{u} \sim 0) | x_{i+1} \sim s_{i+1} | = β_0(x_{i+1} \sim ξ)$$
$$= β_0(x_i \sim s_i \sim w_i) \sim ξ = (\text{by Claim A for } x = x_i \sim s_i \sim n)$$
$$= β_0(x_i \sim s_i \sim n) \sim ξ = β_1(x_i \sim s_i) \sim \bar{u} \sim ξ$$

This finishes the proof of Claim B.

Claim C. $ψ(y_i) = β_1(x_i \sim \bar{s}_i) \sim \bar{u} \sim ζ_i \sim v \sim ζ'_i$ where $\max(ζ_i) ≤ \max(\bar{u})$, $\max(ζ'_i) ≤ \max(v)$, and moreover $|ζ_i| ≥ |\{j : [\bar{w}_i]_j = n\}| - 1$.

Proof of Claim C:

$$ψ(y_i) = β_0(y_i) \sim v \sim φ(\bar{y}_i)$$
$$= β_0(x_i \sim \bar{s}_i \sim \bar{w}_i) \sim v \sim ζ'_i = (\text{by Claim A for } x = x_i \sim \bar{s}_i \sim n)$$
$$= β_0(x_i \sim \bar{s}_i \sim n) \sim ζ_i \sim v \sim ζ'_i$$
$$= β_1(x_i \sim \bar{s}_i) \sim \bar{u} \sim ζ_i \sim v \sim ζ'_i$$

We know that $\max(ζ'_i) = \max(φ(\bar{y}_i)) ≤ \max(v) = \max(τ^1)$ by our initial assumption that the range of $φ$ is contained in $(\max(τ^1) + 1)^\omega$, that was possible since $τ$ dominates all types in the range of $φ$. On the other hand we obtained $ζ_i$ as an application of Claim A, and it is easy to check in that claim that the $ζ_i$ that we got satisfies $\max(ζ_i) ≤ \max(\bar{u})$. Moreover, $ζ_i$ was chosen so that

$$(□) \quad β_0(x_i \sim \bar{s}_i \sim \bar{w}_i) = β_0(x_i \sim \bar{s}_i \sim n) \sim ζ_i.$$
Case IV.a: \( \sigma \) is a top\(^2\)-comb type. Then \(|s_i| < |\bar{s}_i|\), so \(|\phi(s_i)| < |\phi(\bar{s}_i)|\) and hence,
\[
\beta_1(x_i^{-} \bar{s}_i) = \beta_1(x_i^{-} s_i) - |\phi(\bar{s}_i)| - |\phi(s_i)|.
\]
Combining this with claims B and C, we get that
\[
\beta_1(x_{i+1}^{-} \bar{s}_{i+1}) = \beta_1(x_i^{-} s_i) - |\phi(\bar{s}_i)| - |\phi(s_i)| - |\phi(\bar{s}_i)| - |\phi(s_i)|.
\]
where \( \max(\xi_i), \max(\zeta_i) \leq \max(\bar{u}) \), \( \max(\zeta'_i) \leq \max(v) \). We can choose our set \( Y \) so that \(|\phi(\bar{s}_i)| - |\phi(s_i)| > |\bar{u}|\). It is clear from this that \( \psi(Y) \) is a set of type \( \mathfrak{m}(\tau) \), as illustrated in Figure 17.

![Figure 17. \( \psi(Y) \) of type \( \mathfrak{m}(\tau) \).](image)

Remember that for any \( x \), \( \beta_0(x) \) remains unchanged when we add a coordinate less than \( n \), while \( \beta_0(x^{-} n) > \beta_0(x) \). Thus, the final estimation on the length of \( \zeta_i \) follows directly from the expression \( \left( \bigotimes \right) \) above. This finishes the proof of Claim C.

Case IV.b: \( \sigma \) is not a top\(^2\)-comb type. Then \(|\bar{s}_i| < |s_i|\), so \(|\phi(\bar{s}_i)| < |\phi(s_i)|\) and hence,
\[
(\star\star) \quad \beta_1(x_i^{-} \bar{s}_i) = \beta_1(x_i^{-} s_i) - |\phi(\bar{s}_i)| - |\phi(s_i)|.
\]
Combining this with claims B and C, we get that
\[
\beta_1(x_{i+1}^{-} \bar{s}_{i+1}) = \beta_1(x_i^{-} s_i) - |\phi(\bar{s}_i)| - |\phi(s_i)| - |\phi(\bar{s}_i)| - |\phi(s_i)| - |\phi(\bar{s}_i)| - |\phi(s_i)|.
\]
where \( \max(\xi_i), \max(\zeta_i) \leq \max(u) \), \( \max(\zeta'_i) \leq \max(v) \).

Case IV.b.1: \( \sigma \) is a top-comb. The set \( Y \) looks like in the central picture of Figure 16. Then we can construct our set \( Y \) of type \( \tau \) inside de nice subtree where \( \psi \) is normal in such a way that, once \( y_{i-1}, x_i \) are given, \( x_{i+1} \) and \( y_{i+1} \) are chosen so that \(|\bar{u}| < |\phi(s_i)| - |\phi(\bar{s}_i)| < |\zeta_i| - |\bar{u}|\). The left-hand side inequality is easy to obtain since \( \bar{u} \) is fixed and we can make \( s_i \) arbitrarily large. Once the \( s_i \) and the \( \bar{s}_i \) are chosen, the right-hand side inequality can be achieved by adding as many \( n \)'s as necessary above \( y_i \), which we can do keeping the type \( \sigma \) since it is a top-comb, and

\[\text{Figure 17. } \psi(Y) \text{ of type } \mathfrak{m}(\tau).\]
As an application of Theorem 2.5.4 we shall provide some new examples of minimal analytic gaps.

**Lemma 2.5.5.** Let \( \phi : 2^{<\omega} \to 2^{<\omega} \) be a normal embedding such that \( \bar{\phi}[0] = [0] \) and \( \bar{\phi}[1] = [1_{01}] \). Then \( \bar{\phi}[01] = \bar{\phi}[1_{01}] = [1_{01}], \bar{\phi}[01] = [0^1_{1}] \) and \( \bar{\phi}\tau = \tau \) for all other types \( \tau \) in \( 2^{<\omega} \).

**Proof.** Since \( \bar{\phi}[1] = [1_{01}], \phi(x) \) and \( \phi(x \sim 1) \) must look like in Figure 18 for any \( x \in 2^{<\omega} \). So let us call

\[
\beta(x) = \phi(x) \land \phi(x \sim 1)
\]

\[
\gamma(x) = \max\{\beta(x) \sim w : w \in W_0, \beta(x) \sim w \leq \phi(x \sim 1)\}
\]

the two relevant nodes in Figure 19 By condition (3) in Definition 2.1.2 we have that \( \beta(x) = \phi(x) \land \phi(x \sim w) \) for every \( w \in W_1 \). In the sequel along the proof, when we say by normality, we mean that we are using condition (2) in Definition 2.1.2 that \( \phi \) transfers equivalent sets into equivalent sets. We shall check how the functions \( \beta, \gamma \) and \( \phi \) change when we pass from a node to its immediate successors, and
once we have that, we will be able to compute the types $\bar{\sigma}$.

Claim A: $\beta(x^{-0}) = \beta(x)$ for every $x$.

Proof of Claim A: Fix $x$. Since $\bar{\sigma}[0] = [0]$, for every $k < \omega$ there exists $k' < \omega$ such that $\phi(x^{-0k}) = \phi(x^{-0k'})$. Since $\phi(x^{-0k})$ and $\beta(x^{-0k})$ must be like in Figure [19] we have

$$\beta(x^{-0k}) < \phi(x^{-0k}) = \phi(x^{-0k'})$$

$$\phi(x^{-0k}) \setminus \beta(x^{-0k}) \in W_1.$$ 

It follows that $\beta(x^{-0k}) < \phi(x)$ for all $k < \omega$. There are infinitely many numbers $k < \omega$ but only finitely many nodes $t < \phi(x)$, therefore $\beta(x^{-0k}) = \beta(x^{-0m})$ for some $k < m$. By normality, we get that $\beta(x) = \beta(x^{-0})$. We write in detail the normality argument this time. Namely, the following two families are equivalent,

$$\{x, x^{-0}, x^{-1}, x^{-01}\},$$

$$\{x^{-0k}, x^{-0m}, x^{-0k1m}, x^{-0m1m}\},$$

therefore their images under $\phi$ are equivalent. Since

$$\phi(x^{-0k}) \land \phi(x^{-0k1m}) = \beta(x^{-0k}) = \beta(x^{-0m}) = \phi(x^{-0m}) \land \phi(x^{-0m1m}),$$

it follows that

$$\beta(x) = \phi(x) \land \phi(x^{-1}) = \phi(x^{-0}) \land \phi(x^{-01}) = \beta(x^{-0}).$$

This finishes the proof of Claim A.

Claim B: $\beta(x^{-1}) > \gamma(x)$ for every $x$. Moreover, $\beta(x^{-1}) \setminus \gamma(x) \in W_1$.

Proof of Claim B: It is enough to prove that $\beta(x^{-1}) \geq \gamma(x)$. If we prove this nonstrict inequality in particular we get $\beta(x^{-1}) > \beta(x)$ for every $x$, and therefore $\beta(x^{-11}) > \beta(x^{-1}) \geq \gamma(x)$. By normality, $\beta(x^{-11}) > \gamma(x)$ actually implies that $\beta(x^{-1}) > \gamma(x)$. So fix $x$ now. We know that $\gamma(x) < \phi(x^{-1})$ and $\beta(x^{-1}) < \phi(x^{-1})$. Hence, $\gamma(x)$ and $\beta(x^{-1})$ are comparable nodes, so if $\beta(x^{-1}) \geq \gamma(x)$ does not hold, we will have

$$\beta(x) \leq \beta(x^{-1}) < \gamma(x) < \phi(x^{-1}).$$

But $\gamma(x)$ was defined so that $\gamma(x) \setminus \beta(x) \in W_0$, so the above implies that $\beta(x^{-1}) \setminus 0 \leq \phi(x^{-1})$. This is a contradiction with the fact that $\phi(x) \setminus \beta(x) \in W_1$ for every $x$, see Figure [19]. It remains to check that $\beta(x^{-1}) \setminus \gamma(x) \in W_1$. For this, just notice that $\gamma(x) < \beta(x^{-1}) < \phi(x^{-1})$ and we know that $\phi(x^{-1}) \setminus \gamma(x) \in W_1$. This finishes the proof of Claim B.

Claim C: For every $x \in 2^{<\omega}$ there exists $k < \omega$ such that $\gamma(x^{-0}) = \gamma(x^{-0k}).$

Proof of Claim C: Fix $x$. We can find a large enough number $p < \omega$ such that $|\phi(x^{-0})| > |\gamma(x)|$. Then, we have $|\gamma(x)| < |\phi(x^{-0p})| < |\gamma(x^{-0p})|$, and moreover $\gamma(x) \setminus \beta(x) \in W_0$ and $\gamma(x^{-0p}) \setminus \beta(x) = \gamma(x^{-0p}) \setminus \beta(x^{-0p}) \in W_0$. It follows that we must have $\gamma(x^{-0p}) = \gamma(x^{-0k})$ for some $k$. By normality, this finishes the proof of Claim C.
can suppose that \( x \) is large enough node. We are now ready to check that \( \beta \) is a nice embedding \( \phi \). We know that \( \phi \) maps elements of type \([101]\) to \( \gamma \) since \( |x_0| < |\gamma(x)| \) whenever \( x < y \). Namely, if \( \phi \) did not have this property, we could construct a nice embedding \( \phi \) such that \( \phi(x) = \gamma \) whenever \( x < y \).

From Claim B and Claim C it follows that \( \gamma(x) > \gamma(y) \) whenever \( x > y \) (remember that \( \gamma(x) > \gamma(y) \)). This property allows to suppose that \( |\phi(x)| < |\gamma(y)| \) and \( |\gamma(x)| < |\phi(y)| \) whenever \( x < y \). Namely, if \( \phi \) did not have this property, we could construct a nice embedding \( \phi \) such that \( \phi(x) = \gamma \) whenever \( x < y \).

On the other hand, by Claim B, \( \gamma(x) > \gamma(y) \) whenever \( x < y \). It follows that \( \gamma(x) > \gamma(y) \) whenever \( x < y \). Namely, if \( \phi \) did not have this property, we could construct a nice embedding \( \phi \) such that \( \phi(x) = \gamma \) whenever \( x < y \).

Case 1: \( \bar{\phi}[01] = [101] \). Let \( X = \{ x_k : k < \omega \} \subset 2^{<\omega} \) be a set of type \([01]\). We can suppose that \( x_0 = \emptyset \) and \( x_{k+1} = x_k^x \). We claim that \( \{ \phi(x_k) : k < \omega \} \) is a set of type \([01]\) with underlying chain \( \{ \beta(x_k) : k < \omega \} \). Claims A and B imply that \( \beta(x) < \beta(y) \) whenever \( x < y \). Moreover, we know that \( \phi(x_k) = \beta(x_k) \wedge w \), \( w \in W_1 \), see Figure 19. On the other hand, we have\(^\text{20}\)

\[
\beta(x_k) = \beta(x_k^x) < \gamma(x_k^x) < \beta(x_k^0) = \beta(x_{k+1})
\]

where, by the general facts illustrated in Figure 19, \( \gamma(x_k^x) \setminus \beta(x_k^0) \in W_0 \), by Claim B, \( \beta(x_k^0) \setminus \gamma(x_k^x) \in W_1 \), and \( |\gamma(x_k^x)| > |\phi(x_k^0)| > |\phi(x_k)| \). All these facts together imply that \( \phi(X) \) is a set of type \([01]\), as illustrated in Figure 20 (left).

Case 2: \( \bar{\phi}[101] = [101] \). Let \( Y = \{ y_k : k < \omega \} \) be a set of type \([01]\) and \( \{ x_k : k < \omega \} \) its underlying chain. We can suppose that \( x_0 = \emptyset \), \( y_k = x_k^x \), \( x_{k+1} = x_k^{y_k} 001 \). We claim that \( \phi(Y) \) is a set of type \([101]\) with underlying chain \( \{ \gamma(x_k) : k < \omega \} \). Claims A and B together imply that \( \gamma(x) < \gamma(y) \) whenever \( x < y \). On the other hand, by Claim B, \( \gamma(x_k) < \beta(x_k^x) < \phi(x_k^x) = \phi(y_k) \), and moreover, \( \phi(x_k^x) \setminus \gamma(x_k) \in W_1 \), we have that \( \phi(y_k) \setminus \gamma(x_k) \in W_1 \). On the other hand, \( \gamma(x_k) < \gamma(x_k^0) < \gamma(x_{k+1}) \)

where \( \gamma(x_k^x) \setminus \gamma(x_k) \in W_0 \) by Claim C, and \( \gamma(x_k^0) \setminus \gamma(x_k^x) \in W_1 \) by Claim B (since \( \gamma(x_k^x) < \beta(x_k^x) \)). Finally, remember that we supposed that \( |\phi(x)| < |\gamma(y)| \) whenever \( x < y \). In particular, \( |\phi(y_k)| = |\phi(x_k^x)| < |\gamma(x_k^x) \rangle \). All these facts together imply that \( \phi(Y) \) has type \([101]\), as illustrated in Figure 20 (right).

Case 3: \( \bar{\phi}[0^1] = [0^1] \). Let \( Y = \{ y_k : k < \omega \} \) be a set of type \([0^1]\) with underlying chain \( \{ x_k : k < \omega \} \). We can suppose that \( x_0 = \emptyset \), \( y_k = x_k^{x_k} \), \( x_{k+1} = x_k^{y_k} 11 \). We claim that \( Y \) is a set of type \([0^1]\) with underlying chain \( \{ \beta(x_k) : k < \omega \} \). On the one hand,

\[
\beta(x_k) = \beta(x_k^x) < \phi(x_k^x) = \phi(y_k)
\]

\(^{20}\)The second inequality follows from Claim C. Remember also that \( x_k^0 = x_{k+1} \).
and we know that $\phi(x_k^{-0}) \setminus \beta(x_k^{-0}) \in W_1$ as in Figure [19] for $x = x_k^{-0}$. On the other hand
\[
\beta(x_k) < \gamma(x_k) < \beta(x_k^{-1}) < \gamma(x_k^{-1}) < \beta(x_k^{-11}) = \beta(x_k+1)
\]
where $\gamma(x_k) \setminus \beta(x_k) \in W_0$ and $\beta(x_k^{-1}) \setminus \gamma(x_k) \in W_1$ by Claim B. Since $|\gamma(x_k)| < |\phi(y_k)|$ as $x_k < y_k$, we have that $\phi(Y)$ is of type $[0^1_1]$.

Case 4: $\tilde{\phi}[0^1_1] = [0^1_1]$. Let $Y = \{y_k : k < \omega\}$ be a set of type $[0^1_1]$ with underlying chain $\{x_k : k < \omega\}$. We can suppose that $x_0 = \emptyset, y_k = x_k^{-11}, x_{k+1} = x_k^{-101}$. We claim that $Y$ is a set of type $[0^1_1]$ with underlying chain $\{\gamma(x_k) : k < \omega\}$. On the one hand, by Claim B, $\varphi(x_k) < \beta(x_k^{-1}) < \phi(x_k^{-11})$ and $\beta(x_k^{-1}) \setminus \gamma(x_k) \in W_1$. On the other hand, using Claims C and B,
\[
\gamma(x_k) < \beta(x_k^{-1}) < \beta(x_k^{-11}) < \beta(x_k^{-101}) < \gamma(x_k^{-101}) = \gamma(x_k+1),
\]
where $\gamma(x_k^{-1}) \setminus \gamma(x_k) \in W_0$ and $\beta(x_k^{-11}) \setminus \gamma(x_k^{-101}) \in W_1$. Since $|\gamma(x_k^{-1})| < |\phi(x_k^{-11})|$ as $x_k^{-1} < x_k^{-11}$, we get that $\phi(Y)$ is of type $[0^1_1]$.

Case 5: $\tilde{\phi}[0^1_0] = [1^0_0]$. Let $Y = \{y_k : k < \omega\}$ be a set of type $[1^0_0]$ with underlying chain $\{x_k : k < \omega\}$. We can suppose that $x_0 = \emptyset, y_k = x_k^{-11}, x_{k+1} = x_k^{-101}$. We claim that $Y$ is a set of type $[1^0_0]$ with underlying chain $\{\gamma(x_k) : k < \omega\}$. On the one hand, the set $\{\gamma(x_k) : k < \omega\}$ is a chain of type $[0]$ by Claim C. On the other hand $\phi(y_k) = \phi(x_k^{-1}) > \gamma(x_k)$ and $\phi(x_k^{-1}) \setminus \gamma(x_k) \in W_1$ as in Figure [19].

Case 6: $\tilde{\phi}[0^1_1] = [1^0_1]$. Let $Y = \{y_k : k < \omega\}$ be a set of type $[0^1_1]$ with underlying chain $\{x_k : k < \omega\}$. We can suppose that $x_0 = \emptyset, y_k = x_k^{-101}, x_{k+1} = x_k^{-1100}$. We claim that $Y$ is a set of type $[1^0_1]$ with underlying chain $\{\gamma(x_k) : k < \omega\}$. On the one hand, by Claim B,
\[
\gamma(x_k) < \beta(x_k^-1) = \beta(x_k^{-100}) < \gamma(x_k^{-100}) = \gamma(x_k+1)
\]
with $\beta(x_k^{-1}) \setminus \gamma(x_k) \in W_1$. On the other hand, by Claim C and by the definition of $\gamma$,
\[
\gamma(x_k) < \gamma(x_k^{-1}) < \phi(x_k^{-1}) = \phi(y_k)
\]
where $\gamma(x_k^{-1}) \setminus \gamma(x_k) \in W_0$ and $\phi(x_k^{-1}) \setminus \gamma(x_k^{-1}) \in W_1$. This means that $\phi(Y)$ is of type $[0^1_1]$. \hfill $\square$

**Lemma 2.5.6.** Let $S_0 = \{[0]\}$ and $S_1 = \{[1], [01], [1^0_1]\}$. Then $\Gamma = \{\Gamma_{S_0}, \Gamma_{S_1}\}$ is a minimal 2-gap in $2^{<\omega}$. Moreover, if $\Delta$ is a standard 2-gap in $2^{<\omega}$ such that $\Delta \leq \Gamma$, then $\Delta = \Gamma$. \hfill $\square$

**Proof.** Suppose that $\Delta \leq \Gamma$. By Theorem [18.53] we can actually suppose that $\Delta = \{\Gamma_{S_0}, \Gamma_{S_1}\}$ is a standard gap, so that $\tilde{S}_i$ is sets of types in $2^{<\omega}$ and there is a permutation $\varepsilon$ of $2$ such that $[\varepsilon(i)] \in \tilde{S}_i$. Let $\varphi : 2^{<\omega} \rightarrow 2^{<\omega}$ be a normal embedding witnessing that $\Delta \leq \Gamma$, that is, such that $\varphi \tau \in S_i$ if and only if $\tau \in S_i$. Since $\tilde{\phi}[\varepsilon(i)] \in \tilde{S}_i$, Theorem 2.2.4 implies that $\varepsilon$ is the identity permutation. Therefore, $\tilde{\phi}[0] \in S_0 = \{[0]\}$ and $\tilde{\phi}[1] \in S_1$. If $\tilde{\phi}[1] = [1]$ or $\tilde{\phi}[1] = [01]$, it is easy to check that $\varphi \tau = \tau$ for all other types in $2^{<\omega}$, and therefore $\tilde{S}_i = S_i$ for $i = 0, 1$. If $\tilde{\phi}[1] = [1^0_1]$, then Lemma 2.5.5 implies that $\tilde{S}_i = S_i$ for $i = 0, 1$ as well. \hfill $\square$
Theorem 2.5.7. Let \( \{ S_i : i < n \} \) be disjoint sets of types in \( m^{<\omega} \), such that \( \Gamma = \{ \Gamma_i : i < n \} \) is a minimal \( n \)-gap in \( m^{<\omega} \). Consider the following sets of types in \( (m + 1)^{<\omega} \):

- \( \mathcal{M}_m \) is the set of all types \( \sigma \) in \( (m + 1)^{<\omega} \) such that \( \max(\sigma) = m \)
- \( \mathcal{N}_m \) is the set of all types \( \sigma \) in \( (m + 1)^{<\omega} \) such that \( \max(\sigma) = m \) and \( \sigma \) is not a top-comb.
- \( \mathcal{O}_m \) is the set of all types \( \sigma \) in \( (m + 1)^{<\omega} \) such that \( \max(\sigma) = m \), \( \sigma \) is not a top-comb, and either \( \sigma \) is a chain type or \( \max(\sigma^1) < m \).

Then, the following are three nonequivalent minimal \( (n + 1) \)-gaps in \( (m + 1)^{<\omega} \):

- \( \Gamma^M = \{ \Gamma_S_0, \ldots, \Gamma_{S_{n-1}}, \Gamma_{M_m} \} \),
- \( \Gamma^N = \{ \Gamma_S_0, \ldots, \Gamma_{S_{n-1}}, \Gamma_{N_m} \} \),
- \( \Gamma^O = \{ \Gamma_S_0, \ldots, \Gamma_{S_{n-1}}, \Gamma_{O_m} \} \).

Proof. Let letter \( X \) denote any of \( M, N, O \), and let \( \Delta \) be a standard \( (n + 1) \)-gap in \( (n + 1)^{<\omega} \) such that \( \Delta \preceq \Gamma^X \), and let \( \phi : (n + 1)^{<\omega} \to (m + 1)^{<\omega} \) be a normal embedding witnessing this. For every \( \tau \in S_i \), \( i < n \) and every \( \tau' \in X_m \) we have that \( \max(\tau) < \max(\tau') = m \). Using Corollary 2.2.2, this implies that \( \max(\phi(\sigma)) = m \) if and only if \( \max(\sigma) = n \). Therefore, the restriction \( \phi_{|_{\Delta^{<\omega}}} \) shows that \( \{ \Delta_i : i < n \} \preceq \prec \Gamma \). By the minimality of \( \Gamma \), there is a normal embedding \( \psi : m^{<\omega} \to n^{<\omega} \) which witnesses that \( \Gamma \preceq \{ \Delta_i : i < n \} |_{n^{<\omega}} \). Now, we distinguish cases:

If \( X = M \), then we pick a top-comb type \( \tau \) in \( (n + 1)^{<\omega} \) such that \( \max(\tau^1) = n \). Since \( \max(\phi(\tau)) = m \), we have that \( \phi(\tau) \in \mathcal{M}_m \), so \( \tau \) belongs to \( \Delta_m \). By Lemma 2.4.3, we can find a normal embedding \( \psi' : (m + 1)^{<\omega} \to (n + 1)^{<\omega} \) which extends \( \psi \) and \( \psi' \sigma = \tau \) for all \( \sigma \) with \( \max(\sigma) = m \). This \( \psi' \) shows that \( \Gamma^M \preceq \Delta \) as required.

For the cases when \( X = N \) or \( X = O \), we are going to consider a function \( \chi : (m + 1)^{<\omega} \to 2^{<\omega} \) defined as follows: Let \( \{ k(s) : s \in (m + 1)^{<\omega} \} \) be natural numbers such that \( k(s) > \sum_{t<s} k(t) \) for every \( s \), and

\[
\begin{align*}
\chi(\emptyset) &= 0 \\
\chi(t \triangleleft m) &= t \triangleleft 0 \uplus 1^{k(t \triangleleft m) - 1} \\
\chi(t \triangleleft i) &= t \triangleleft 0^{k(t \triangleleft i)} & \text{if } i < m
\end{align*}
\]

This \( \chi \) is a normal embedding. The action of \( \bar{\chi} \) is easy to compute and it is related to the classification of types appearing in Theorem 2.5.4.

- If \( \max(\sigma) < m \), then \( \bar{\chi}\sigma = [0] \).
- If \( \max(\sigma) = m \), \( \sigma \) is not a top-comb, and either \( \sigma \) is a chain type, or \( \max(\sigma^1) < m \), then \( \bar{\chi}\sigma = [01] \).
- If \( \sigma \) is a top-comb with \( \max(\sigma^1) < \max(\sigma^0) = m \), then \( \bar{\chi}\sigma = [10] \).
- If \( \max(\sigma^0) < \max(\sigma^1) = m \), then \( \bar{\chi}\sigma = [10] \).
- If \( \max(\sigma^0) = \max(\sigma^1) = m \), and \( \sigma \) is not a top-comb, then \( \bar{\chi}\sigma = [10] \).
- If \( \max(\sigma^0) = \max(\sigma^1) = m \), and \( \sigma \) is a top-comb which is not a top-t2-comb, then \( \bar{\chi}\sigma = [10] \).

21 Namely, if \( \max(\sigma) = n \), then \( \max(\sigma) \geq \max(\sigma') \) for all types \( \sigma' \) in \( (n + 1)^{<\omega} \), hence \( \max(\phi(\sigma)) \geq \max(\phi(\sigma')) \) for all \( \sigma' \), in particular for some \( \phi(\sigma') = \phi(n) \) where \( \max(\phi(\sigma)) = m \). In particular, \( \max(\phi([n])) = m \), so \([n]\) cannot belong to \( \Delta_i, i < n \), it must belong to \( \Delta_n \). For the converse, if \( \max(\sigma) < n \), then \( \max(\phi(\sigma)) \leq \max(\phi([n - 1])) \); since \( \Delta \) is a standard gap, \( [n - 1] \) and \([n]\) cannot belong to the same preideal, so \( \phi([n - 1]) \notin S_i, i < n \), therefore \( \max(\phi([n - 1])) < m \).
If \( \max(\sigma^0) = \max(\sigma^1) = m \), and \( \sigma \) is a top\(^2\)-comb, then \( \tilde{\chi}\sigma = [01] \).

We can rewrite this action by saying that:

- \( \tilde{\chi}\sigma = [01] \) for \( \sigma \in \mathcal{O}_m \),
- \( \tilde{\chi}\sigma = [101] \) for \( \sigma \in \mathcal{N}_n \setminus \mathcal{O}_m \),
- \( \tilde{\chi}\sigma \) is a top-comb type in \( 2^{<\omega} \) for \( \sigma \in \mathcal{M}_m \setminus \mathcal{N}_m \),
- \( \tilde{\chi}\sigma = [0] \) for \( \sigma \notin \mathcal{M}_m \).

Suppose that \( X = N \). We know that \( \tilde{\phi}[n] \in \mathcal{N}_m \). We can consider a normal embedding \( \varphi : 2^{<\omega} \to (n + 1)^{<\omega} \) given by \( \varphi(x^0) = \varphi(x)^0 \), \( \varphi(x^1) = \varphi(x)^n \). Then \( \eta = \chi\varphi : 2^{<\omega} \to 2^{<\omega} \) is a normal embedding such that \( \tilde{\eta}[0] = [0] \) and \( \tilde{\eta}[1] \in \{[01], [101]\} \) because \( \tilde{\eta}[1] = \tilde{\chi}\tilde{\phi}[n] \). Therefore \( \tilde{\eta}[1] = [101] \).

Hence
\[
\tilde{\chi}\tilde{\phi}^n \circ \tilde{\phi} = \tilde{\chi}\tilde{\varphi} = \tilde{\eta} \circ [01] = [101],
\]
which by the way that \( \tilde{\chi} \) acts, implies that \( \tilde{\phi}^n \in \mathcal{N}_m \setminus \mathcal{O}_m \), which in turn implies that \( \tau = [01] \) is a type from \( \Delta_n \), because \( \phi \) witnesses \( \Delta \leq \Gamma^N \). Moreover, \( \tau \) subdominates \( \tilde{\psi}\sigma \) for all types \( \sigma \in n^{<\omega} \), so we have the normal embedding \( \psi' : (m + 1)^{<\omega} \to (n + 1)^{<\omega} \) that extends \( \psi \) provided by Theorem 2.5.4. Let us check that \( \psi' \) witnesses that \( \Gamma^N \leq \Delta \). It is clear that \( \psi \) preserves the ideals \( \Gamma_i \), for \( i \leq n \), since \( \psi \) extends \( \psi \). If \( \sigma \in \mathcal{N}_m \), then \( \tilde{\psi}\sigma = \tau = \tilde{s}(\tau) \) as indicated by Theorem 2.5.4 and \( \tau \) is a type in \( \Delta_n \). Finally, let \( \sigma \) be a type corresponding to the orthogonal of \( \Gamma^N \). If \( \max(\sigma) < m \) then \( \tilde{\psi}\sigma \) also goes to the orthogonal of \( \Gamma^N \) since \( \psi' \) extends \( \psi \). If \( \max(\sigma) = m \), then \( \sigma \in \mathcal{M}_m \setminus \mathcal{N}_m \), so \( \sigma \) is a top-comb, therefore \( \tilde{\psi}\sigma \in \{t(\tau), m(\tau), \tilde{s}(\tau)\} \). By Corollary 2.6.6 \( \tilde{\phi}\psi'\sigma \) is top-comb. Hence \( \tilde{\phi}\psi'\sigma \notin \mathcal{N}_m \), and since \( \phi \) witnesses that \( \Delta \leq \Gamma^N \), we get \( \psi'\sigma \notin \Delta_n \) as required.

Suppose that \( X = O \). We know that \( \tilde{\phi}[n] \in \mathcal{O}_m \). Thus, we have that
\[
\tilde{\chi}\tilde{\phi}[0] = [0], \quad \tilde{\chi}\tilde{\phi}[n - 1] = [0], \quad \tilde{\chi}\tilde{\phi}[n] = [01].
\]

It is easy to check that these three conditions imply that \( \tilde{\chi}\tilde{\phi}^n \circ \tilde{\phi} = [01] \), therefore \( \tilde{\phi}^n \in \mathcal{O}_m \). Since \( \tau = [01] \) subdominates all types \( \psi\sigma \), we can construct an extension \( \psi' : (n + 1)^{<\omega} \to (m + 1)^{<\omega} \) of \( \psi \) as in Theorem 2.5.4. Let us check that this \( \psi' \) witnesses \( \Gamma \leq \Delta \). It is clear that \( \psi' \) sends the ideals \( \Gamma_i \) into the ideals \( \Delta_i \) for \( i < n \) since it extends \( \psi \), and by the same reason, it preserves the types in the orthogonal of \( \Gamma \) with maximum less than \( m \). It remains to see that things work properly for types \( \sigma \) with \( \max(\sigma) = m \): If \( \sigma \in \mathcal{O}_n \), then Theorem 2.5.4 indicates that \( \psi'\sigma = \tau \), and \( \tau \) is a type in \( \Delta_n \) since \( \tilde{\phi}\sigma \in \mathcal{O}_m \). On the other hand, if \( \sigma \) is a type in the orthogonal, then \( \tilde{\psi}\sigma \) is equal either to \( t(\tau) = [0^{n - 1}] \), \( s(\tau) = [0^n] \), \( \tilde{s}(\tau) = [0^n] \), \( m(\tau) = [0^n] \). But since we knew that \( \tilde{\chi}\tilde{\phi}[0] = [0], \tilde{\chi}\tilde{\phi}[n - 1] = [0], \tilde{\chi}\tilde{\phi}[n] = [01] \), it is easy to check that all those types \( [0^{n - 1}], [0^n], [0^n], [0^n] \) are sent to comb types by \( \tilde{\chi}\tilde{\phi} \). Therefore \( \tilde{\chi}\tilde{\phi}\psi'\sigma \) is a comb type. By the way that \( \tilde{\chi} \) acts, we conclude that \( \tilde{\phi}\psi'\sigma \notin \mathcal{O}_m \). Since \( \phi \) witnesses that \( \Gamma^O \leq \Delta \), we conclude that \( \psi'\sigma \notin \Delta_n \), as required.

\[\text{22If } \tilde{\eta}[1] = [101], \text{ this follows from Lemma 2.5.5 while if } \tilde{\eta}[1] = [01] \text{ it is easy to check.}\]
It remains to check that the three gaps are nonequivalent. Suppose that $Γ^X ≤ Γ^M$, and $φ : (m + 1)^{<ω} → (m + 1)^{<ω}$ an embedding witnessing it. Pick $τ ∈ M_m \setminus N_m$. Then, by Corollary 2.6.2, $φτ ∈ M_m$ and this is contradiction if $X \neq M$ because $τ ∉ X_m$. On the other hand, suppose that $Γ^O ≤ Γ^N$. Then, the argument at the beginning of the case $X = N$ above, applied to the particular case when $∆ = Γ^O$ and $n = m$, shows that $τ = [m_{0n}] ∈ O_m$, a contradiction. □

2.6. FREELY GENERATED MINIMAL GAPS

In this section we present another family of minimal analytic $n$-gaps. The following is an easy observation:

**Lemma 2.6.1.** If $m ≤ n$ and $φ : m^{<ω} → n^{<ω}$ is a normal embedding such that $φ[i] = [i]$ for all $i ∈ m$, then $φτ = τ$ for every type $τ$ in $m^{<ω}$.

**Lemma 2.6.2.** Let $φ : m^{<ω} → n^{<ω}$ be a normal embedding and $j < n$. Suppose that there exists a type $τ$ in $m^{<ω}$ such that $φτ = [j]$. Then, there exists $i < m$ such that $φ[i] = [j]$.

**Proof.** If $τ$ is a comb type, it follows from Lemma 2.6.3 that $φ[0] = [j]$. So we suppose that $τ$ is a chain type, and we shall prove that $φ[\min(τ)] = [j]$. Let $X = \{x_0 < x_1 < x_2 < \ldots\}$ be a chain of type $τ$, and intercalate elements in the form $x_0 < y_0 < x_1 < y_1 < \cdots$ in such a way that $y_i \setminus x_i ∈ W_{\min(τ)}$. We know that $φ(X) = \{φ(x_0) < φ(x_1) < \cdots\}$ is a chain of type $[j]$, and $\{φ(y_0) < φ(y_1) < \cdots\}$ is a set of certain type $σ$. For every $i < ω$, let $t_i = \max_{ξ < ω} φ(y_0) ∨ φ(x_ξ)$. Also, $\{x_0, x_1\} ∼ \{z_0, z_1\}$, so by the same reason we should have $φ(x_0) < φ(z)$. But $φ(x_0)$ and $φ(y_0)$ were incomparable, a contradiction.

(3) $φ(x_0) < t_0 < φ(x_1) < t_1 < \cdots$. In this case, we have $φ(x_0) < t_0 < φ(y_0)$ so it’s clear that $φ[\min(τ)]$ is a chain type whose minimum is $j$. On the other hand, since $\max[\min(τ)] ≤ \max(τ)$, we have that $\max(φ[\min(τ)]) ≤ \max(φτ) = j$. Hence $φ[\min(τ)] = [j]$. □

**Theorem 2.6.3.** Let $m ≤ n$ and let $\{S_i : i < n\}$ be nonempty pairwise disjoint sets of types in $m^{<ω}$ such that $S_i = \{[i]\}$ for $i < m$. Then $Γ = \{Γ_{S_i} : i < n\}$ is a minimal $n$-gap. Moreover, all the minimal $n$-gaps of this form are non-equivalent to each other, not even one can be equivalent to a permutation of another.

**Proof.** Let $∆ ≤ Γ$, and suppose that $∆ = Γ_{R_i} = \{Γ_{R_i} : i < n\}$ is a standard gap in $n^{<ω}$ and $φ : n^{<ω} → m^{<ω}$ is a normal embedding witnessing that $∆ ≤ Γ$: $φτ ∈ S_i$ if and only if $τ ∈ R_i$. By Corollary 2.2.2, if we pick $τ_i ∈ R_i$ for $i < m$, then $\max(τ_0) < \max(τ_1) < \cdots < \max(τ_{m-1})$. By Theorem 2.2.1 we can find a normal embedding $ψ : m^{<ω} → n^{<ω}$ such that $ψ[i] = τ_i$ for $i < m$. By Lemma 2.6.1, we get that $ψφτ = τ$ for all types $τ$ in $m^{<ω}$. Hence $ψ$ witnesses that $Γ ≤ ∆$. This
shows that $\Gamma$ is minimal. Before passing to the last statement of the theorem, we prove:

Claim A: For every infinite set $a \subseteq m^{<\omega}$, if $\{\Gamma_S|a : i < m\}$ is a gap, then $\{\Gamma_S|a : i < n\}$ is a gap. Proof: Similarly as we did to prove that $\Gamma$ was minimal, we can get a normal embedding $\phi : m^{<\omega} \to a \subseteq m^{<\omega}$ such that $\phi[i] \in S_i$, so $\phi[i] = [i]$. By Lemma 2.6.1 $\phi \tau = \tau$ for all types in $m^{<\omega}$, so we get the desired conclusion.

Claim B: For every $Z \subseteq n, Z \not\ni m$ there exists an infinite set $a \subseteq m^{<\omega}$ such that $\{\Gamma_S|a : i \in Z\}$ is a gap, but $\{\Gamma_S|a : i < n\}$ is not a gap. Proof: Pick $\tau_i \in S_i$ for $i \in Z$. Using Theorem 2.2.1 we can get a normal embedding $\phi : Z^{<\omega} \to m^{<\omega}$ such that $\phi[i] = \tau_i$ (here $Z$ set is identified with an integer $k$ through a suitable bijection). The range $a$ of $\phi$ satisfies the required properties. Notice that if $i \in m \setminus Z$, then $a$ is orthogonal to $\Gamma_S$, because, by Lemma 2.6.2 there is no type $\tau$ with $\phi \tau = \tau$.

Now, suppose that $\Gamma = \{\Gamma_S : i < n\}$ and $\Gamma' = \{\Gamma_S' : i < n\}$ are like in the theorem, let $\varepsilon : n \to n$ be a permutation, and suppose that $\Gamma^\varepsilon = \{\Gamma_S^\varepsilon(i) : i < n\}$ is equivalent to $\Gamma'$. From claims A and B, it follows that the subgap $\{\Gamma_S : i < m\}$ is characterized by a property which is invariant under equivalence. Therefore $\varepsilon(\{i : i < m\}) = \{i : i < m\}$. Moreover, by Corollary 2.2.2 we must have $\varepsilon(i) = i$ for all $i < m$. Once we have this, indeed we can suppose that $\varepsilon$ is the identity permutation. If we had $\Gamma \leq \Gamma'$ witnessed by some normal embedding $\phi : m^{<\omega} \to m^{<\omega}$, then by Lemma 2.6.1 $\phi \tau = \tau$ for all types $\tau$, hence $\Gamma = \Gamma'$. □

For example, in the case when $m = 2$ and $n = 3$, the minimal gaps provided by Theorem 2.6.3 are of the form $\{\Gamma_0 = \{[0]\}, \Gamma_1 = \{[1]\}, \Gamma_2\}$ where $\Gamma_2$ can be any nonempty family of the remaining types. There are 8 types in $2^{<\omega}$, cf. Section 6.1, hence the total amount of minimal 3-gaps of this form is $2^6 - 1 = 63$. These are the first 61 minimal analytic 3-gaps in the list presented in Section 3.2. We shall see in Section 3.4 that, in general, the number of minimal analytic $n$-gaps that can be constructed in this way is sufficiently large to provide some asymptotic estimation of the total amount of minimal analytic $n$-gaps.

2.7. Strong gaps

Recall that a multiple gap $\Gamma = \{\Gamma_i : i \in n\}$ is called countably separated if there exists a countable family $A$ of sets such that whenever we choose $x_i \in A_i$ for $i \in n$, we can find $a_i \in A$ such that $x_i^+ a_i$ for $i \in n$, and $\bigcap_{i \in n} a_i = \emptyset$. If $\Gamma$ is not countably separated, then it is called a strong gap.

Strong analytic gaps were studied in [41]. In order to use the results there, we need to recall some definitions, and some translation into the language of this work. Given $i < n$, a set $\{x_k : x_k \in m^{<\omega}\}$ is called an $i$-chain if we have $x_k^i i \leq x_k + 1$ for all $k$. Given $i, j < m, i \not= j$, a set $\{y_k : y_k \in m^{<\omega}\}$ is called an $(i, j)$-comb if there is an $i$-chain $\{x_k : x_k \in m^{<\omega}\}$ such that $x_k^i j \leq y_k$ and $|y_k| < |x_k + 1|$ for all $k$. An $(i, j)$-comb is an $i$-chain. Thus, the notion of an $(i, j)$-comb is similar to the notion of set of type $\tau$ but less demanding, taking into account only the first integer after
The analogous of Theorem 1.3.2 for strong gaps is the following Theorem 2.7.1.
The ZFC version for analytic strong gaps was proven in [27] for $n = 2$, and in [3] Theorem 7] for arbitrary $n$. We prove here the projective version under determinacy axioms to complete the picture.

**Theorem 2.7.1 (Projective Determinacy).** For a finite family of projective preideals $\Gamma = \{\Gamma_i : i < n\}$,

1. either they are countably separated
2. or there exists an injective function $u : n^{<\omega} \to \omega$ such that $u(x) \in \Gamma_i$ whenever $x$ is an $i$-chain.

If the $\Gamma_i$ are analytic, then the result holds in ZFC.

**Proof.** Consider an infinite game where, at each step $k$, Player I plays $m_k \in \omega$, and player II plays $i_k \in n$. At the end of the game, Player I wins if and only if $\{m_k : k < \omega\}$ is infinite, and for every $i < n$, we have $\{m_k : i = i_k\} \in \Gamma_i$. If the families $\Gamma_i$ are projective, this is a projective game, so under PD, it is determined.

If Player I has a winning strategy, we check that condition (2) of the theorem holds. For every $s = (s_0, \ldots, s_p) \in n^{<\omega}$ define $u(s) = m_p$ to be the move of Player I under his strategy after Player II has played $s_0, \ldots, s_p$ and Player I followed his strategy. One sees immediately that $u(x) \in \Gamma_i$ whenever $x$ is an $i$-chain, because Player I wins if he follows the strategy. At first, $u$ might not be injective, but since Player I has to produce infinite sets to win, $u(x)$ is infinite whenever $x$ is a branch. Using this, it is easy to restrict to copy of $n^{<\omega}$ inside $n^{<\omega}$ where $u$ becomes injective and has the desired properties.

If Player II has a winning strategy, we show that the families are countably separated. For every $i < n$ and every finite round of the game $\xi = (m_0, i_0, \ldots, m_k, i_k)$ played according to the strategy of Player II, we define $c^i_\xi$ to be the set of all $m$ such that the strategy of Player II does not choose $i$ after $(m_0, i_0, \ldots, m_k, i_k, m)$ is played. Notice that $\bigcap_{i < n} c^i_\xi = \emptyset$. Now, let $a_i \in \Gamma_i$ such that $\bigcap_{i < n} a_i = \emptyset$. We claim that there exists $\xi$ such that $a_i \subset c^i_\xi$. We say that $\xi = (m_0, i_0, \ldots, m_k, i_k)$ is acceptable if $\{m_j : i_j = i, j \leq k\} \subset a_i$ for every $i$. Let $\Xi$ be the family of all acceptable $\xi$’s played according to Player II’s strategy. An acceptable $\xi = (m_0, i_0, \ldots, m_k, i_k) \in \Xi$ is called extensible if there exists $m \notin \{m_0, \ldots, m_k\}$ such that $(m_0, i_0, \ldots, m_k, i_k, m, i) \in \Xi$ for some $i < n$. There must exist some $\xi \in \Xi$ which is not extensible, since otherwise we could produce a complete infinite round played according to the strategy of Player II in which Player I wins. Now, notice that the non-extensibility property implies that $a_i \subset c^i_\xi$. □

As we already mentioned, the ZFC version in the analytic case can be found in [3] Theorem 7]. Also, the proof above provides immediately the Borel case, since Borel determinacy holds in ZFC. Indeed, representing an analytic set as the projection of a closed set in $2^\mathbb{N} \times \omega^\omega$, and making Player I to play pairs $(m_i, s_i) \in N \times \omega^{<\omega}$, one can modify the game in order to use Borel determinacy to provide a ZFC proof of the analytic case. We also remark that the proof of the case $n = 2$ from [27] relies on the open graph theorem of Todorcevic, cf. [26]. This open graph theorem is proved to hold for projective sets in [13] using the projective determinacy, so the
case $n = 2$ of Theorem 2.7.1 follows. For greater dimensions, however, the relation with open graphs is not so clear.

We restate the above result in the language of the theory that we are developing:

**Theorem 2.7.2.** For an analytic $n$-gap $\Gamma$ the following are equivalent

1. $\Gamma$ is a strong gap
2. There exists a one-to-one function $\phi : n^{<\omega} \to N$ such that $\phi(X) \in \Gamma_i$ whenever $X$ is a set of type $\sigma$, where $\sigma$ is a chain type with $\min(\sigma) = i$.
3. There exists a one-to-one function $\varphi : n^{<\omega} \to N$ such that $\varphi(X) \in \Gamma_i$ whenever $X$ is an $i$-chain.

**Proof.** The equivalence of (1) and (3) is exactly the content of Theorem 2.7.1. It is clear that (3) implies (2) because if $X$ has chain type $\sigma$ and $\min(\sigma) = i$, then $X$ is an $i$-chain. On the other hand, if (2) holds, then we can color the $i$-chains $X$ of $n^{<\omega}$ into two colors, depending on whether $\phi(X) \in \Gamma^i$ or not. Since $\Gamma_i$ is analytic, this coloring is Suslin-measurable, and using Milliken's theorem [25], cf. [20] Theorem 6.13 or [4] Theorem 4], we find a strong subtree of $n^{<\omega}$ all of whose $i$-chains have the same color. This must mean that for all $i$-chains $X$ of the subtree we have that $\phi(X) \in \Gamma_i$, the other option being impossible since every strong subtree contains $i$-chains of some type $\tau$ by Lemma 2.7.4. In this way, by restricting to a strong subtree we get a function $\varphi$ like in (3).

Given an $n$-gap $\Gamma = \{\Gamma_i : i \in \{n\}\}$, studying its strength means determining for which subsets $A \subseteq n$, the gap $\{\Gamma_i : i \in A\}$ is strong. We are going to study now the strength of standard gaps.

**Definition 2.7.3.** The strength of a type $\tau$ is the finite set of natural numbers

$$\text{strength} (\tau) = \{\max (\tau^0)\} \cup \{k \in \tau^1 : k > \max(\tau^0)\}$$

Remember that condition (3) in Definition 2.7.2 says that if $\phi$ is a normal embedding, then $\phi(x) \wedge \phi(x^k) = \phi(x) \wedge \phi(x^w)$ for all $k \in n, w \in W_k$. Let us write $\phi_k(x) = \phi(x) \wedge \phi(x^k)$. We shall use the following fact:

**Lemma 2.7.4.** If $\phi : n^{<\omega} \to m^{<\omega}$ is a normal embedding, then for every $x \in n^{<\omega}$ and every $k \in n$, we have

1. $\phi_k(x) < \phi_k(x^k)$;
2. $(\phi_k(x^k) \setminus \phi_k(x), \phi(x) \setminus \phi_k(x))$ is a rung of type $\ddot{\sigma}k$.

**Proof.** Since $X = \{x, x^k, x^k^2, \ldots\}$ is a set of type $[k]$, we have that $\phi(X) = \{\phi(x), \phi(x^k), \phi(x^k^2), \ldots\}$ is a set of type $\ddot{\sigma}k$. Since $\phi_k(x^k^p) = \phi(x^k^p) \wedge \phi(x^k^{p+1})$, the underlying chain of $\phi(X)$ is $\{\phi_k(x), \phi_k(x^k), \phi_k(x^k^2), \ldots\}$. From this fact we get both statements of the Lemma.

**Theorem 2.7.5.** For $\{\tau_i : i \in \{n\}\}$, the following are equivalent:

1. $\text{strength}(\tau_i) = \text{strength}(\tau_j)$ for every $i, j \in \{n\}$,
2. there exists a normal embedding $\phi : n^{<\omega} \to m^{<\omega}$ such that $\ddot{\sigma} \phi \sigma = \tau_{\min(\sigma)}$ for every chain type $\sigma \in \Sigma_n$.

**Proof.** Suppose first that condition (1) of the Theorem holds, so let

$$\eta = \{\eta_1 < \cdots < \eta_d\} = \text{strength}(\tau_i)$$
for all $i$. Let $(u_i, v_i)$ be a rung of type $\tau_i$, for each $i \in n$. Since all the types have the same strength $\eta$, we can find $\vec{v} \neq \emptyset$ and choose these rungs so that we can write $v_i = \vec{v} \iota$ in such a way that $\max(\vec{v_i}) \leq \eta_1$ for all $i$ for which $\tau_i$ is a comb type. Let us also write $u_i = \vec{u} \iota$ in such a way that $|\vec{u}_i| = |\vec{v}_i|$. We can suppose, by just adding zeros in appropriate places, that there is a number $l$ such that

- $|\vec{u}_j| = |\vec{v}_j| = l$ and $|\vec{u}_j| = 2l$ whenever $\tau_j$ is a comb type with $\vec{v}_j \neq \emptyset$,
- $|u_k| = 2l$ whenever $\tau_k$ is a comb type with $\vec{v}_k = \emptyset$,
- $|u_i| = l$ whenever $\tau_i$ is a chain type.

We can also assume, for safety, that the last coordinate of each nonempty $\vec{u}_i$, $\vec{u}_i$, $\vec{u}_i$, $\vec{v}_i$ and $\vec{v}$ is $0$. Let $\{j_1, \ldots, j_p\}$ be an enumeration of all $i \in n$ such that $\tau_i$ is a comb type with $\vec{v}_i \neq \emptyset$, in such a way that

$$\max(\vec{v}_{j_1}) \geq \max(\vec{v}_{j_2}) \geq \cdots \geq \max(\vec{v}_{j_p}).$$

We define by $\prec$-induction an embedding $\phi : n^{< \omega} \to m^{< \omega}$ together with an auxiliary base function $\beta : n^{< \omega} \to m^{< \omega}$ by the following recursive formulas, see Figure 21.

\[
\begin{align*}
\beta(\emptyset) & = \emptyset \\
\phi(x) & = \beta(x) \vec{v}_{j_1} \vec{v}_{j_2} \cdots \vec{v}_{j_p} \vec{v} \\
\beta(x \prec i) & = \phi(x) \vec{u}_i \quad \text{if } \tau_i \text{ is a chain type} \\
\beta(x \prec k) & = \beta(x) \vec{v}_{j_1} \vec{v}_{j_2} \cdots \vec{v}_{j_p} \vec{u}_k \quad \text{if } \tau_k \text{ is a comb type with } \vec{v}_k = \emptyset \\
\beta(x \prec j_r) & = \beta(x) \vec{v}_{j_1} \vec{v}_{j_2} \cdots \vec{v}_{j_{r-1}} \vec{u}_{j_r} \vec{u}^{l(p-r)} \vec{u}_{j_r}. 
\end{align*}
\]

Claim A: For every $x, y \in n^{< \omega}$, if $x \leq y$, then $\beta(x) \leq \beta(y)$ and moreover $\max(\beta(y) \setminus \beta(x))) \leq \eta_1$.

Proof of Claim A: This is rather straightforward if one looks at Figure 21. Formally, it is enough to prove this when $y = x \prec i$ for some $i \in n$. If $\tau_i$ is a comb type this follows immediately\(^ {24}\) from the formulas above for $\beta(x \prec j_r)$ and $\beta(x \prec k)$.

\(^ {23}\)Even if $|\eta| = 1$, $\vec{v}$ can be chosen nonempty by taking $\vec{v}$ to be a harmless sequence of $0$'s. However, we may be forced to take $\vec{v}_i = \emptyset$ in some cases.

\(^ {24}\)Remember that $\eta_i = \max(u_i)$ for all $i$, since strength$(\tau_i) = \eta_i$, and on the other hand we chose $\vec{v}_i$ so that $\max(\vec{v}_i) \leq \eta_1$.
If $\tau_i$ is a chain type, then
\[ \beta(x^{\sim} i) = \beta(x^{\sim} i) \sim \hat{v}_{j_1} \sim \hat{v}_{j_2} \sim \cdots \sim \hat{v}_{j_p} \sim \hat{v}^{\sim} u_i \]
but in this case, the fact that we have a chain type $\tau_i$ with $\text{strength}(\tau_i) = \eta$ implies that $\eta = \{\eta_1\}$ and therefore $\max(v_i) \leq \eta_1$ for all $i$, in particular $\max(\hat{v}) \leq \eta_1$ so we are done again. This finishes the proof of Claim A.

Now, we fix $i$ and a chain type $\sigma$ with $\min(\sigma) = i$ and we prove that the image of a set of type $\sigma$ under $\phi$ is a set of type $\tau_i$. This will finish the proof of the implication $1 \Rightarrow 2$ of the Theorem, because, although $\phi$ is not a normal embedding, we can make it a normal embedding after composing with a nice embedding by Theorem 2.1.3. We first consider the case when $\tau_i$ is a chain type. In this case, it is enough to prove the following Claim B:

Claim B: For every $x, y \in n^{<\omega}$ there exists $z \in m^{<\omega}$ such that $max(z) \leq max(u_i)$ and $\phi(x^{\sim} i \sim y) = \phi(x^{\sim} u_i \sim z)$.

Proof of Claim B: By Claim A above,
\[ \beta(x^{\sim} i) \leq \beta(x^{\sim} i \sim y) \leq \phi(x^{\sim} i \sim y) \leq \beta(x^{\sim} i \sim y) \]
where moreover $\max(\beta(x^{\sim} i \sim y)) = \eta_1$. Hence $z = \phi(x^{\sim} i \sim y) \sim \beta(x^{\sim} i)$ satisfies $\max(z) \leq \eta_1 = \max(u_i)$. We have
\[ \phi(x^{\sim} i \sim y) = \beta(x^{\sim} i) \sim z = \phi(x^{\sim} u_i \sim z), \]
where $\max(z) \leq \eta_1 = \max(u_i)$ as required. This finishes the proof of Claim B.

Next we consider the case when $\tau_i$ is a comb type. So either $i = j_r$ for some $r \in \{1, \ldots, p\}$, or $\hat{v}_1 = 0$, in which case we make the formal conventions that $r = p + 1$, $r - 1 = p$, $j_r = i$ and $l(p - r) = 0$. Let $\{x_1, x_2, x_3, \ldots\}$ be a chain of type $\sigma$. Let $z_q = \phi(x_q) \wedge \beta(x_q^{\sim} i)$. We claim that $\{\phi(x_1), \phi(x_2), \ldots\}$ is a set of type $\tau_i$ with underlying chain $\{z_1, z_2, \ldots\}$. It follows from the definition of $\beta$ and $\phi$, see Figure 24 that
\[ z_q = \beta(x_q) \sim \hat{v}_{j_1} \sim \hat{v}_{j_2} \sim \cdots \sim \hat{v}_{j_{r-1}} \]
\[ \phi(x_q) = z_q \sim \hat{v}_{j_r} \sim \hat{v}_{j_{r+1}} \sim \cdots \sim \hat{v}_{j_p} \sim \hat{v} \]
\[ \beta(x_q^{\sim} i) = z_q \sim \hat{u}_{j_r} \sim 0^{(p-r)} \sim \hat{u}_i \]
The situation is illustrated in Figure 22. This implies that $(\beta(x_q^{\sim} i) \setminus z_q, \phi(x_q) \setminus z_q)$ is a rung of type $\tau_i$. This is because $(\hat{u} \sim \hat{u}_i, \hat{v} \sim \hat{v})$ was a rung of type $\tau_i$, and we can write
\[ \beta(x_q^{\sim} i) \setminus z_q = \hat{u}_i \sim \xi \sim \hat{u}_i \]
\[ \phi(x_q) \setminus z_q = \hat{v}_i \sim \zeta \sim \hat{v} \]
and the intercalated sequences $\xi = 0^{(p-r)}$ and $\zeta = \hat{v}_{j_{r+1}} \sim \cdots \sim \hat{v}_{j_p}$ do not alter the type of the rung because $|\xi| = l(p - r) = |\zeta|$, $\max(\xi) = 0 \leq \max(\hat{u}_i)$ and $\max(\zeta) = \max(\hat{v}_{j_{r+1}}) \leq \max(v_{j_r}) = \max(v_1)$. Once we know that $(\beta(x_q^{\sim} i) \setminus z_q, \phi(x_q) \setminus z_q)$ is
\[ \text{Notice that if we are in the case that } \hat{v}_1 = 0, \text{ then } \xi = \zeta = 0. \]
a rung of type $\tau_i$, the following Claim C will finish the proof that \{$\phi(x_1), \phi(x_2)$\ldots\} is a set of type $\tau_i$ with underlying chain \{$z_1, z_2, \ldots$\}:

Claim C: $z_{q+1} = \beta(x_q^\prec i)^{-} w$ for some $w \in n^{<\omega}$ with $\max(w) \leq \max(u_i)$.

Proof of Claim C: Remember that $\max(u_i) = \eta_i$. Using Claim A and the definition of the nodes $z_q$ we obtain that

$$\beta(x_q^\prec i) \leq \beta(x_{q+1}) \leq z_q \leq \beta(x_{q+1}^\prec i)$$

and $\max(\beta(x_{q+1}^\prec i) \setminus \beta(x_q^\prec i)) \leq \eta_i$. This finishes the proof of Claim C, and also the proof of the implication (1) $\Rightarrow$ (2) in the Theorem.

Now suppose that (2) holds, so we have a normal embedding $\phi : n^{<\omega} \rightarrow m^{<\omega}$ such that $\phi \sigma = \tau_{\min(\sigma)}$ for every chain type $\sigma$. We fix $i, j \in n$ and we have to prove that $\text{strength}(\tau_i) = \text{strength}(\tau_j)$.

Claim D: $\max(\tau_i) = \max(\tau_j)$.

Proof of Claim D: Consider the chain types $[i \ n-1]$ and $[j \ n-1]$ in $n^{<\omega}$. We have that $\phi[i \ n-1] = \tau_i$ and $\phi[j \ n-1] = \tau_j$, so since $\max[i \ n-1] = n-1 = \max[j \ n-1]$, Corollary 2.2.7 implies that $\max(\tau_i) = \max(\tau_j)$, and this finishes the proof of Claim D.

If $\tau$ is a chain type, then $\text{strength}(\tau) = \{\max(\tau)\}$. Hence, if $\tau_i$ and $\tau_j$ are chain types, Claim D already gives that $\text{strength}(\tau_i) = \text{strength}(\tau_j)$. Suppose next that $\tau_i$ is a chain type and $\tau_j$ is a comb type. Then, what we have to prove is that $\max(\tau_i) = \max(\tau_j) = \max(\tau_j^0)$. Let $p$ be a natural number. Since $\phi$ sends chains of type $[i] * [j]$ to chains of type $\tau_i$, we have that $\phi(i^\prec j^p) = \phi(\emptyset)^{\prec} u$ where $u$ is a rung of type $\tau_i$. Using Lemma 2.7.4, we can choose $p$ large enough so that $\phi_j(i^\prec j^p) > \phi(\emptyset)$. Then $\phi(i^\prec j^p) \{m\}$ is a segment inside $u$, and it is also the second entry of a rung of type $\tau_j = \phi(j)$ by Lemma 2.7.4. Therefore

$$\max(\tau_j^1) = \max(\phi(i^\prec j^p) \setminus \phi_j(i^\prec j^p)) \leq \max(u) = \max(\tau_i) = \max(\tau_j)$$

26Lemma 2.7.4 implies that $\phi_j(i^\prec j^p) < \phi_j(i^\prec j^q)$ whenever $p < q$. Since $\phi(\emptyset) \leq \phi(i^\prec j^p)$ and $\phi_j(i^\prec j^p) \leq \phi(i^\prec j^p)$, we shall have that $\phi(\emptyset) < \phi_j(i^\prec j^p)$ as soon as $|\phi(\emptyset)| < |\phi(i^\prec j^p)|$. 

![Figure 22. $z_q$, $\phi(x_q)$ and $\beta(x_q^\prec i)$](image-url)
This implies that $\max(\tau^0_j) = \max(\tau_j)$ as required.

Finally we consider the case when both $\tau_i$ and $\tau_j$ are comb types. Let $\phi_{ij}(x) = \phi(x \sim ij) \wedge \phi(x)$ and $\phi_{ji}(x) = \phi(x \sim ji) \wedge \phi(x)$.

Claim E: For every $x \in n^{<\omega}$ and $w \in \{i,j\}^{<\omega}$,

\[
\phi(x) \wedge \phi(x \sim ij \sim w) = \phi_{ij}(x),
\]

and 

\[
\phi(x) \wedge \phi(x \sim ji \sim w) = \phi_{ji}(x).
\]

Proof of Claim E: We prove the first equality, the other one is the symmetric case. Since there are only finitely many elements below $\phi$, we must find $p < q$ such that $\phi(x) \land \phi(x \sim ij \sim w^p) = \phi(x) \land \phi(x \sim ij \sim w^q)$. Now notice that the families $\{x, x \sim ij, x \sim ij \sim w\}$ and $\{x, x \sim ij \sim w^p, x \sim ij \sim w^q\}$ are equivalent, hence their images under $\phi$ are equivalent as well. This implies the first equality stated in the claim, and finishes the proof of Claim E.

Claim F: For every $x \in n^{<\omega}$ and $w \in \{i,j\}^{<\omega}$,

\[
\phi_{ij}(x) < \phi_{ij}(x \sim ij \sim w), \quad \phi_{ij}(x) < \phi_{ji}(x \sim ij \sim w),
\]

\[
\phi_{ji}(x) < \phi_{ji}(x \sim ji \sim w), \quad \phi_{ji}(x) < \phi_{ij}(x \sim ji \sim w).
\]

Proof of Claim F: We prove the inequalities of the upper line, the lower line is the symmetric case. Let $\xi$ be equal to either $ij$ or $ji$. By Claim E,

\[
\phi(x \sim ij \sim w) \land \phi(x) = \phi_{ij}(x) = \phi(x \sim ij \sim w \sim \xi) \land \phi(x),
\]

hence $\phi_{ij}(x) \leq \phi_{\xi}(x \sim ij \sim w)$. If the inequality was not strict, then by normality\footnote{Use that $\{x, x \sim ij \sim w, x \sim ij \sim w \sim \xi\}$ and $\{x, x \sim ij \sim w^p, x \sim ij \sim w^q \sim \xi\}$ are equivalent families, hence so are their images under $\phi$. Notice that by Claim E, $\phi_{ij}(x) = \phi(x) \land \phi(x \sim ij \sim v)$ and $\phi_{\xi}(x \sim ij \sim v) = \phi(x \sim ij \sim v \sim \xi)$ for $v = w, w'$. Hence the equivalence of the aforementioned triples implies that if $\phi_{ij}(x) = \phi_{\xi}(x \sim ij \sim w)$, then $\phi_{ij}(x) = \phi_{\xi}(x \sim ij \sim w^p)$ as well.} of $\phi$ we would have

\[
(*) \quad \phi_{ij}(x) = \phi_{\xi}(x \sim ij \sim w')
\]

for all $w' \in \{i,j\}^{<\omega}$. In particular, this happens for all $w' = \xi^p, p < \omega$. Since there are infinitely many $p < \omega$ but only finitely many $k \in m$, there must exist $k \in m$ and $p < q$ such that $\phi_{ij}(x) \sim k \leq \phi(x \sim ij \sim \xi^p)$ for $r = p, q$. Then, we have

\[
\phi_{ij}(x) \sim k \leq \phi(x \sim ij \sim \xi^p) \land \phi(x \sim ij \sim \xi^q) = \phi_{\xi}(x \sim ij \sim \xi^p)
\]

where the last equality follows from Claim E. This implies in particular that

\[
\phi_{ij}(x) < \phi_{\xi}(x \sim ij \sim \xi^p),
\]

which is a contradiction with $(*)$ above. This finishes the proof of Claim F.

Since the role of $i$ and $j$ was so far symmetric, we can suppose\footnote{Since $\phi_{ij}(\emptyset) \leq \phi(\emptyset)$ and $\phi_{ji}(\emptyset) \leq \phi(\emptyset)$, we have either $\phi_{ij}(\emptyset) \leq \phi_{ji}(\emptyset)$ or $\phi_{ji}(\emptyset) \leq \phi_{ij}(\emptyset)$.} that $\phi_{ji}(\emptyset) \leq \phi_{ij}(\emptyset)$, which by normality implies that $\phi_{ji}(x) \leq \phi_{ij}(x)$ for all $x$. Consider the types $\sigma_i = [i] \ast [j]$ and $\sigma_j = [j] \ast [i]$. Notice that $\sigma_i$ equals either $[ij]$ or $[ji]$ and $\sigma_j$ equals either $[ji]$ or $[ij]$ depending whether we have $i < j$ or $j < i$. In any case, $\min(\sigma_i) = i$ and $\min(\sigma_j) = j$. Let $\tilde{v} = \phi(\emptyset) \setminus \phi_{ij}(\emptyset)$, $\tilde{u} = \phi_{ij}(\emptyset) \setminus \phi_{ji}(\emptyset)$, $u_i = \phi_{ij}(ij) \setminus \phi_{ji}(ij)$, $u_j = \phi_{ji}(ji) \setminus \phi_{ij}(ij)$. The set $X = \{i, \emptyset, ij, ij, \ldots\}$ has type $\sigma_i$, hence $e(\phi(X)$ has type $\tau_i$. The underlying chain of $\phi(X)$ is $\{\phi_{ij}(\emptyset), \phi_{ij}(ij), \phi_{ij}(ij), \ldots\}$ because
\( \phi_{ij}(x) = \phi(x) \land \phi(x \sim ij) \). By looking at the first three elements of \( \phi(X) \), we see that \((u_i, \bar{v})\) is a rung of type \( \tau_i \), see Figure 23(left). Similarly, \((u_j, \bar{v} \sim \bar{v})\) is a rung of type \( \tau_j \). Namely, the set \( Y = \{\emptyset, ji, ji jj\ldots\} \) has type \( \sigma_j \), hence \( \phi(Y) \) has type \( \tau_j \). The underlying chain of \( \phi(Y) \) is \( \{\phi_{ji}(\emptyset), \phi_{ji}(ji), \phi_{ji}(ji ji), \ldots\} \) because \( \phi_{ji}(x) = \phi(x) \land \phi(x \sim ji) \). By looking at the first three elements of \( \phi(Y) \), we see now that \((u_i, \bar{v} \sim \bar{v})\) is a rung of type \( \tau_j \), see Figure 23(right). So it is enough to prove that \( \max(u_i) = \max(u_j) \) and \( \max(\bar{v}) \leq \max(u_i) \).

We prove first that \( \max(\bar{v}) \leq \max(u_i) \). For this, consider the node \( \phi_{ji}(ij) \). On the one hand, \( \phi_{ji}(ij) \leq \phi_{ij}(ij) \) because we made a choice that \( \phi_{ij}(x) \leq \phi_{ij}(x) \) for all \( x \). On the other hand, \( \phi_{ij}(\emptyset) \leq \phi_{ji}(ij) \) by Claim F. Hence \( \phi_{ij}(\emptyset) \leq \phi_{ji}(ij) \leq \phi_{ij}(ij) \), so \( \phi_{ij}(ij) \backslash \phi_{ji}(ij) \) is a segment inside \( u_i \) and we conclude that

\[
\max(\bar{v}) = \max(\phi_{ij}(ij) \backslash \phi_{ji}(ij)) \leq \max(u_i)
\]

where the first equality follows from the normality of \( \phi \).

We prove now that \( \max(u_i) = \max(u_j) \). Notice that, by Claim F,

\[
\phi_{ij}(\emptyset) < \phi_{ji}(ij) \leq \phi_{ij}(ij) < \phi_{ij}(ij ji) \leq \phi_{ij}(ij ji ji).
\]

Therefore \( \phi_{ij}(ij ji) \backslash \phi_{ij}(\emptyset) \) contains \( \phi_{ji}(ij ji) \backslash \phi_{ij}(ij) \) as a subsegment, and by the normality of \( \phi \),

\[
\max(u_j) = \max(\phi_{ij}(ij ji) \backslash \phi_{ji}(ij)) \leq \max(\phi_{ij}(ij ji) \backslash \phi_{ij}(ij) = \max(u_i).\]

The reverse inequality is proven similarly. Namely, using Claim F we get that

\[
\phi_{ji}(\emptyset) < \phi_{ij}(ji i) \leq \phi_{ij}(ji i j) < \phi_{ij}(ji i j i).
\]

Therefore \( \phi_{ji}(ji i j ji) \backslash \phi_{ij}(\emptyset) \) contains \( \phi_{ij}(ji i j) \backslash \phi_{ij}(ji) \) as a subsegment. Using the normality of \( \phi \), we conclude that

\[
\max(u_i) = \max(\phi_{ij}(ji i ji) \backslash \phi_{ij}(ji)) \leq \max(\phi_{ij}(ji i ji) \backslash \phi_{ij}(\emptyset) = \max(u_j)\]

\[
\square
\]

**Corollary 2.7.6.** If \( \phi : n^{<\omega} \rightarrow m^{<\omega} \) is a normal embedding, then \( \text{strength}(\tau) = \text{strength}(\sigma) \) implies that \( \text{strength}(\phi \tau) = \text{strength}(\phi \sigma) \).

**Corollary 2.7.7.** Let \( \{S_i : i < \omega\} \) be nonempty pairwise disjoint sets of types in \( m^{<\omega} \). The following are equivalent:

1. the \( n \)-gap \( \Gamma = \{\Gamma_{S_i} : i \in n\} \) is strong,
2. we can choose \( \tau_i \in S_i \) for every \( i \in n \) in such a way that \( \text{strength}(\tau_i) = \text{strength}(\tau_j) \) for \( i, j \in n \).

**Proof.** Combine Theorem 2.7.3 and Theorem 2.7.2 \[ \square \]
The analytic strong gaps which are minimal among strong gaps are described in [4] depending on a number of parameters \(A, B, C, D, E, \psi, \mathcal{P}, \gamma\). We shall now describe the minimal analytic gaps which are strong, which correspond exactly to those as above for which the parameters \(C, D, E, \mathcal{P}, \gamma\) are trivial. Fix \(n < \omega\) and consider a partition \(n = A \cup B\) into two sets such that \(A \neq \emptyset\), let \(\langle A \rangle^2 = \{(i, j) \in A^2 : i \neq j\}\) and \(\psi : \langle A \rangle^2 \rightarrow B \cup \{\infty\}\) a function such that for every \(k \in B\) there exists \((i, j) \in \langle A \rangle^2\) such that \(\psi(i, j) = k\). Out of a triple \((A, B, \psi)\) satisfying all these conditions we construct an \(n\)-gap \(\Sigma(A, B, \psi) = \{\Gamma_{S_i} : i \in n\}\) in \(m^\omega\) where \(m = |A|\). Considering \(\xi : m \rightarrow A\) the increasing enumeration of \(A\), the definition is as follows:

\[
S_i = \{\sigma \text{ chain type } : \min(\sigma) = \xi^{-1}(i)\} \text{ if } i \in A
\]

\[
S_i = \{\sigma \text{ comb type } : \psi(\xi^{-1}(\min(\sigma^0)), \xi^{-1}(\min(\sigma^1))) = i\} \text{ if } i \in B
\]

Since we will use the results from [4], we shall use some of the facts discussed in Section 2.7.

**Theorem 2.7.8.** The \(n\)-gaps of the form \(\Sigma(A, B, \psi)\) are representatives of the equivalence classes of the minimal analytic \(n\)-gaps which are strong. More precisely,

1. Each gap \(\Sigma(A, B, \psi)\) is a minimal analytic \(n\)-gap which is moreover strong.
2. Each minimal analytic \(n\)-gap which is strong is equivalent to some \(\Sigma(A, B, \psi)\).  
3. If \(\Sigma(A, B, \psi)\) is equivalent to \(\Sigma(A', B', \psi')\), then \(A = A', B = B'\) and \(\psi = \psi'\).

**Proof.** The fact that the gap \(\Sigma(A, B, \psi) = \{\Gamma_{S_i} : i < n\}\) is strong follows from Corollary 2.7.7 because in each of the sets \(S_i\) we can find a type \(\sigma\) with \(\text{strength}(\sigma) = (m - 1)\). Let us see now that \(\Gamma = \Sigma(A, B, \psi)\) is minimal. Suppose that we have \(\Delta \leq \Gamma\). We can suppose that \(\Delta\) is a standard gap in \(n^\omega\) so that \([\varepsilon(i)] \in \Delta\) for some permutation \(\varepsilon : n \rightarrow n\). Let \(\varphi : n^\omega \rightarrow m^\omega\) be a normal embedding witnessing that \(\Delta \leq \Gamma\). If \(\varepsilon(i) \in A\), then \(\sigma_i = \varphi[\varepsilon(i)] \in S_i\) is a chain type with \(\min(\sigma_i) = \xi^{-1}(i)\).

Consider the function \(\phi : m^\omega \rightarrow n^\omega\) given by

\[
\phi(k_1, \ldots, k_p) = (\varepsilon(\xi(k_1)), \ldots, \varepsilon(\xi(k_p)))
\]

When restricted to a nice subtree, \(\phi\) is a normal embedding that witnesses that \(\Gamma \leq \Delta\).

We prove now (2). Let \(\Delta\) be a minimal \(n\)-gap which is strong, and we will find that \(\Sigma(A, B, \psi) \leq \Delta\) for some \((A, B, \psi)\). We can suppose that \(\Delta\) is a standard \(n\)-gap. Consider \(\Delta'\) its associated ideal \(n\)-gap, as defined in Section 1.7. Then \(\Delta'\) is an ideal strong \(n\)-gap, so we can use the basis for such gaps constructed in [4]. Thus, we have a set of data

\[
\alpha = (A, B, C, D, E, \psi, \mathcal{P}, \gamma)
\]

such that the \(n\)-gap \(\Gamma_{\alpha}\) constructed in [4] Section 6] satisfies \(\Gamma_{\alpha} \leq \Delta'\). Let us briefly recall how this gap looks like. We have that \(n = A \cup B \cup C \cup D \cup E\) is a partition, \(\psi : \langle A \rangle^2 \rightarrow B \cup \{\infty\}\) where \(\langle A \rangle^2\) is the set of all ordered pairs of different elements of \(A\) and \(\psi\) covers \(B\), \(\mathcal{P}\) is a partition of \(C\) into sets of cardinality either 1 or 2, and \(\gamma : D \rightarrow B \cup \{\infty\}\) is a function for which all elements of \(E\) have at least two preimages. One considers then \(M = A \cup \mathcal{P} \cup D\), that we identify with an initial segment of natural numbers by putting an order where \(A < \mathcal{P} < D\), and a function \(f^\alpha : M^2 \rightarrow n \cup \{\infty\}\) that describes the gap \(\Gamma_{\alpha} = \{\Gamma_{S_i} : i < n\}\) so that \(\Gamma_{S_i}\) is the ideal generated by the set of all \((u, v)\)-combs with \((u, v) \in S_i\) = \((f^\alpha)^{-1}(i)\).
The exact definition can be found in [4 Section 6], so let us just point out the
important facts for us now:

- $S_i = \{(i, i)\}$ if $i \in A$
- $S_i = \psi^{-1}(i)$ if $i \in B$
- $S_i$ contains at least a pair $(\xi, \xi')$ with $\xi \neq \xi'$, for $i \in P \cup D$

Consider the gap $\Delta = \{\Gamma_{\bar{S}_i} : i < n\}$, where $\bar{S}_i$ is the set of all comb types $\tau$ such that $(\min(\tau^0), \min(\tau^1)) \in S_i$ and all chain types such that $(\min(\tau), \min(\tau)) \in S_i$.

It is clear that $\Gamma \leq \Gamma_f$, just witnessed by the identity map. We will have that

- If $i \in A$, then $\bar{S}_i$ is the set of all chain types $\sigma$ with $\min(\sigma) = i$,
- If $i \in B$, then $\bar{S}_i$ is the set of all comb types $\sigma$ with $\psi(\min(\sigma^0), \min(\sigma^1)) = i$,
- If $i \in P \cup D$, then $\bar{S}_i$ contains a top-comb type $\tau_i$ that dominates all types in $M^{<\omega}$.

If $C = D = E = \emptyset$, we just have that $\Sigma^{(A, B, \psi)} = \Gamma$. Hence $\Sigma^{(A, B, \psi)} \leq \Gamma'$, so by Lemma 1.5.3 $\Sigma^{(A, B, \psi)} \leq \Gamma$ and we are done.

If some of the sets $C$, $D$ or $E$ is nonempty, then $P \cup D \neq \emptyset$. By Lemma 2.4.3 there exists a normal embedding $\phi : M^{<\omega} \rightarrow M^{<\omega}$ such that $\phi$ preserves all types in $\Lambda^{<\omega}$ while $\phi(\sigma) = \tau_i$ whenever $\max(\sigma) = i \in P \cup D$. This embedding provides a gap $\tilde{\Delta} \leq \Gamma$ which is not strong by Corollary 2.7.7, because $\tilde{\Delta}_i = \{\sigma : \max(\sigma) = i\}$ if $i \in P \cup D$. Since $\tilde{\Delta} \leq \Delta'$, by Lemma 1.6.3 we have that $\tilde{\Delta} \leq \Delta$ and $\Delta$ is not strong, a contradiction.

Finally, we check (3). If we have a normal embedding $\phi : |A|^{<\omega} \rightarrow |A'|^{<\omega}$ that witnesses that $\Sigma^{(A, B, \psi)} \leq \Sigma^{(A, B, \psi')}$, then we can use Lemma 2.3.3 to check that $A = A'$ since the embedding will have to preserve chain types. Once we have that, it is clear that $\phi$ preserves $B$ and $\psi$ as well. □

2.8. Countable partitions

In this section, we include the following lemma:

**Lemma 2.8.1.** Let $\tau$ be a type with $\max(\tau) = \max(\tau^0) = n - 1$. Let $A_\tau$ be the family of all subsets of $n^{<\omega}$ of type $\tau$. Let $c : A_\tau \rightarrow \omega$ be a function such that

1. $c^{-1}(k)$ is Suslin-measurable for each $k < \omega$,
2. If $F \subset n^{<\omega}$ is finite, then $c(a \setminus F) = c(a)$ for every $a \in A_\tau$.

Then, there exists a nice subtree $T$ of $n^{<\omega}$ such that $c$ is constant on the sets of type $\tau$ of $T$.

**Proof.** Suppose we have such a function $c$. Fix $(u, v)$ a rung of type $\tau$. By induction on $p$, we construct $T_0 \supset T_1 \supset \cdots$ nice subtrees of $n^{<\omega}$ with roots $s_0 < s_1 < \cdots$. We start with $T_0 = n^{<\omega}$, hence $s_0 = \emptyset$. Suppose $T_p$ is already defined and we define $T_{p+1}$. Let $\psi_p : n^{<\omega} \rightarrow n^{<\omega}$ the nice embedding such that $T_p = \psi_p(n^{<\omega})$.

We consider a new nice embedding, that restricts our tree above $\psi_p(u)$, that is $\tilde{\psi}_p(u) = \psi_p(u^{<s})$. Now, consider a finite Suslin-measurable coloring $c_p : A_\tau \rightarrow \{0, 1, \ldots, p\}$ given by $c_p(a) = \min(c(\tilde{\psi}_p(a)))$. By Theorem 1.1.5 we can restrict to a nice subtree $T'_p$ where $c_p$ is constant. We define $T_p = \tilde{\psi}_p(T'_p)$. If the constant value is less than $p$, then we are done, because we obtain a nice subtree $T_p$ where $c$ is constant, so we suppose that the constant value of $c_p$ is $p$, which means that
$c(a) \geq p$ for all $a \subseteq T_p$ of type $\tau$. Call $t_p = \psi_p(v)$. The set $a = \{t_0, t_1, \ldots\}$ is a set of type $\tau$. Notice that $\{t_p, t_{p+1}, \ldots\} \subseteq T_p$, hence

$$c(a) = c(\{t_p, t_{p+1}, \ldots\}) \geq p$$

for every $p < \omega$, a contradiction. 

□
CHAPTER 3

Lists of minimal analytic gaps

3.1. Minimal analytic 2-gaps

We claim that the following table is the list of all minimal 2-gaps:

|   | $\Gamma_0$ | $\Gamma_1$ |
|---|---|---|
| 1\,(2) | 0 | all types with $\max(\tau) = 1$ |
| 2\,(2) | 0 | 1 |
| 3\,(2) | 0 | 1, 01 |
| 4\,(1) | 0, 01 | 1 |
| 5\,(2) | 0 | 1, 01, \{1, 01\} |

Each row $i = 1, 2, 3, 4, 5$ represents a minimal standard gap $\Gamma_i = \{\Gamma_i^0, \Gamma_i^1\}$, so that in the $\Gamma_0$-column we put all types belonging to $\Gamma_i^0$ and in the $\Gamma_1$-column all types belonging to $\Gamma_i^1$. The list is given up to permutation, the small number in parenthesis indicating the number of nonequivalent permutations of each of the gaps. The only minimal gap which happens to be equivalent to its permutation is $\Gamma_4$, $\{\Gamma_4^0, \Gamma_4^1\} \sim \{\Gamma_4^1, \Gamma_4^0\}$, so counting permutations there are actually nine minimal analytic 2-gaps. In order to check that this list is correct it is enough to prove the following two facts:

A2. For every analytic 2-gap $\Delta$, there is a gap $\Gamma$ from the list, such that $\Gamma \preceq \Delta$ (this implies that all minimal analytic 2-gaps appear in our list).

B2. If $\Gamma$ and $\Delta$ are different gaps from the list, then $\Gamma \not\preceq \Delta$ (this implies that all the gaps in the list are minimal and never equivalent one to another).

In the sentences above, by the “the list” we mean the list of nine gaps counted with permutations. It is convenient to have in mind the list of types in $2^{<\omega}$ that we summarize in the following table:

| type | strength | top-comb |
|---|---|---|
| 0 | 0 | no |
| 1 | 1 | no |
| 01 | 1 | no |
| 10 | 01 | yes |
| 01 | 1 | no |
| 10 | 01 | yes |
| 11 | 1 | yes |
| 11 | 1 | yes |

Proof of statement A2. By Theorem 1.3.2, we can suppose that $\Delta$ is a standard gap in $2^{<\omega}$ and $[i] \in \Delta_i$ for $i = 0, 1$. Moreover, we can suppose that $\Delta$ is minimal. If $\Delta$ is strong, then by Theorem 2.7.8 it must of the form $\Sigma^{(A,B,\psi)}$, but for $n = 2$ the only possibility is that $A = \{0, 1\}$, $B = \emptyset$, $\psi = \infty$, and then we just get $\Delta = \Sigma^{(A,B,\psi)} = \Gamma^4$. Therefore, we can suppose that $\Delta$ is not strong,
which by Corollary 2.7.7 means that $\Delta_0 \subset \{[0],[1][0]\}$. If $[1][0] \in \Delta_0$, then $[1][0] \in \Delta_0$ dominates $[1] \in \Delta_1$, so applying Theorem 2.4.4 we get that the permuted of $\Gamma^3$ is below $\Delta$: $\{\Gamma^1_3, \Gamma^3_1\} \leq \Delta$. Therefore, we can assume that $\Delta_0 = \{[0]\}$. If $\Delta_1$ contains some top-comb type $\tau$, then $\tau \in \Delta_1$ dominates $[0] \in \Delta_0$, and we can apply again Theorem 2.4.4 showing that $\Gamma^1 \leq \Delta$. Therefore, we can assume that $\Delta_0 = \{[0]\}$ and $\Delta_1 \subset \{[1],[01],[1][01]\}$. If $[01] \in \Delta_1$, then either $\Delta = \Gamma^1$ or $\Delta = \Gamma^3$, while if $\Delta_1 = \{[1]\}$ then $\Delta = \Gamma^2$. Finally, the only remaining case is that $\Delta_0 = \{[0]\}$ and $\Delta_1 = \{[1],[1][01]\}$. Then, consider a normal embedding $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\phi([0]) = [0]$ and $\phi([1]) = [1][01]$, which exists by Theorem 2.2.1. Then, Lemma 2.5.6 implies that $\phi$ witnesses that $\Gamma^5 \leq \Delta$.

Proof of statement B2. Let us denote $\Gamma^{i,0} = \Gamma^i$ and $\Gamma^{i,1} = \{\Gamma^1_3, \Gamma^3_1\}$ its permuted. We notice that $\Gamma^{4,0} \sim \Gamma^{4,1}$ as witnessed by the embedding $\phi(s_0, \ldots, s_k) = (1 - s_0, \ldots, 1 - s_k)$. In the rest of cases,

1. $\Gamma^{i,p} \not\leq \Gamma^{j,q}$ for $i, j = 1, 2, 3, 5$, $p \neq q$. Proof: Notice that $\max(\tau) = j$ when $\tau \in \Gamma^j$ and use Corollary 2.2.2.

2. $\Gamma^4 \not\leq \Gamma^{i,p}$ for $i = 1, 2, 3, 5$, $p = 0, 1$. Proof: $\Gamma^4$ is a strong gap, while the rest are not, by Corollary 2.7.7.

3. $\Gamma^{i,p} \not\leq \Gamma^4$ for $i = 1, 2, 3, 5$, $p = 0, 1$. Proof: We noticed that $\Gamma^4$ is of the form $\Sigma^{(A,B,\psi)}$ as in Theorem 2.7.8 so it is a minimal analytic gap which is moreover strong. This implies that if $\Delta \leq \Gamma^4$ then $\Delta$ must be strong, and $\Gamma^{i,p}$ is not strong if $i \neq 4$.

4. $\Gamma^i \not\leq \Gamma^1$ for $i = 2, 3, 5$. Proof: $\Gamma_1$ is dense and the others are not.

5. $\Gamma^i \not\leq \Gamma^2$ for $i = 1, 3, 5$. Proof: If we had $\Gamma^i \leq \Gamma^2$ witnessed by some normal embedding $\phi$, then $\phi([0]) = [0]$ and $\phi([1]) = [1]$. But then $\phi([01]) = [01] \in (\Gamma^2)^\perp$, so $[01] \in (\Gamma^i)^\perp$ and this is a contradiction if $i = 1, 3, 5$.

6. $\Gamma^i \not\leq \Gamma^3$ for $i = 1, 2, 5$. Proof: If we had $\Gamma^i \leq \Gamma^3$ witnessed by some normal embedding $\phi$, then $\phi([0]) = [0]$ and either $\phi([1]) = [1]$ or $\phi([1]) = [01]$. In either case $\phi([01]) = [01]$ and this gets a contradiction for $i = 2$ since $[01] \in \Gamma^1_3$ but $[01] \not\in \Gamma^i_3$. In either case as well, $\phi$ sends comb types to comb types, and this gives a similar contradiction for $i = 1, 5$.

7. $\Gamma^i \not\leq \Gamma^5$ for $i = 1, 2, 3$. Proof: It follows from Lemma 2.5.6.

3.2. Minimal analytic 3-gaps

We claim that there are 933 minimal analytic 3-gaps (163 if counted up to permutations), and they are the ones listed in the tables that follow. These 3-gaps will be denoted now as $\Delta_i = \{\Delta^i_0, \Delta^i_1, \Delta^i_2\}$ for $i = 1, 2, \ldots, 163$. The small number in parenthesis means again the number of nonequivalent permutations of the gap(s) described in the corresponding row. The list is displayed in four tables.
The first table contains the minimal 3-gaps representable in $2^{<\omega}$:

| $2^{<\omega}$ | $\Delta_0$ | $\Delta_1$ | $\Delta_2$ |
|-------------|-------------|-------------|-------------|
| $\emptyset$ | [0]         | [1]         | any set $A$ of remaining types |

The next two tables display the minimal 3-gaps obtained by supplementing a minimal 2-gap in $2^{<\omega}$ with a set of types in $\mathcal{M}_2$. We denote by $\Gamma^i : i = 1, \ldots, 5$ the five minimal 2-gaps listed in Section 3.1 and by $\mathcal{M}_2$, $\mathcal{N}_2$ and $\mathcal{O}_2$ the three sets of types defined in Section 2.3.

| $\Delta_0 + \Delta_1$ | $\Delta_2$ |
|------------------------|-------------|
| $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4, \Gamma^5$ or $\Gamma^6$ | $\mathcal{M}_2$ |
| $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4, \Gamma^5$ or $\Gamma^6$ | $\mathcal{N}_2$ |
| $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ or $\Gamma^5$ | $\mathcal{O}_2$ |
| $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ or $\Gamma^5$ | [2] |
| $\Gamma^2$ or $\Gamma^4$ | $\mathcal{O}_2$ |
| $\Gamma^2$ | $\mathcal{O}_2$ |

Finally, the rest of minimal 3-gaps:

| $\Delta_0$ | $\Delta_1$ | $\Delta_2$ |
|------------|------------|------------|
| [0]        | [1], [01], [12], [012] | [2] |
| [0]        | [1], [01], [12], [012] | [2], [02] |
| [1]        | [1], [12] | [2], [02], [01] |
| [1]        | [1], [12] | [2], [02], [01] |
| [0], [01], [012], [02] | [1], [12] | [2], [02], [01] |
Let $\Delta$ be a minimal 3-gap, and we shall prove that there is a permutation from a 3-gap in the list above which is below $\Delta$. We suppose that $\Delta$ is a standard gap on $3^{<\omega}$, so that $[\varepsilon(i)] \in \Delta$ for some permutation $\varepsilon : 3 \rightarrow 3$. We can suppose that $\varepsilon(2) = 2$. Then, $\{\Delta_0|_{2^{<\omega}}, \Delta_1|_{2^{<\omega}}\}$ is a 2-gap, so we can find $\{\Delta_0, \Delta_1\} \leq \{\Delta_0|_{2^{<\omega}}, \Delta_1|_{2^{<\omega}}\}$ a minimal 2-gap and a normal embedding $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$ witnessing this. By permuting $\Delta_0$ and $\Delta_1$ if necessary, we can actually suppose that $\{\Delta_0, \Delta_1\} = \Gamma^i$ for some $i = 1, \ldots, 5$ is one of the five minimal 2-gaps, in the right order. We can construct $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ by letting $\psi(x^-2^-s) = \psi(x)^-2^-\phi(s)$ whenever $x \in 3^{<\omega}$, $s \in 2^{<\omega}$. The restriction of $\psi$ to some nice subtree is a normal embedding, and it satisfies that $\psi[2] = [2]$ and $\psi \tau = \tau$ if $\tau$ is a type in $2^{<\omega}$. The embedding $\psi$ allows us to assume, without loss of generality, that our 3-gap $\Delta$ is a standard gap such that $\{\Delta_0|_{2^{<\omega}}, \Delta_1|_{2^{<\omega}}\} = \Gamma^i$ for some $i = 1, \ldots, 5$.

**Case 1.** $\Delta_2$ contains at least one type of maximum less than 2. Then $\{\Delta_0|_{2^{<\omega}}, \Delta_1|_{2^{<\omega}}\} = \Gamma^i$ for some $i = 2, \ldots, 5$ (the case $i = 1$ is excluded because it would leave no room for a type in $2^{<\omega}$ to belong to $\Delta_2$).

**Case 1a.** If $i = 2, 3$, then we are done, because $\Delta|_{2^{<\omega}}$ is one of the gaps $1 - 94$.

**Case 1b.** If $i = 5$, consider a normal embedding $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\phi[0] = [0]$ and $\phi[1] = [01]$, which exists by Theorem 2.2.1. Then, Lemma 2.5.6 says that $\phi[01] = [01], \phi[0] = [01]$, and $\phi[01] = [01]$ for all other types $\sigma$. Thus, if at least some type different from $[01]$ was in $\Delta_2$, this normal embedding $\phi$ shows that some of the gaps 95-102 is below $\Delta|_{2^{<\omega}}$. Otherwise, if $[01]$ is the only type from $2^{<\omega}$ in $\Delta_2$, then since $[01]$ dominates $[0]$, by Theorem 2.4.3, we can find a normal embedding $\chi : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\chi[0] = [0]$ and $\chi[0] = [01]$ whenever $\max(\sigma) = 1$. Since $[01]$ subdominates $[0]$ and $[01]$, we can consider an extension $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ of $\chi$ as in Theorem 2.5.4. This embedding $\psi$ shows that, after permuting $\Delta_1$ and $\Delta_2$, the 3-gap number 110 (that is, $\Gamma_1$ supplemented with $N_2$) is below $\Delta$. That is, $\{\Delta_0|_{110}, \Delta_1|_{110}\} = \{\Gamma_1|_{0}, N_2, \Gamma_1\} \leq \Delta$.

**Case 1c.** If $i = 4$, we can consider an embedding $\psi : 2^{<\omega} \rightarrow 2^{<\omega}$ given by $\psi(x^-0) = \psi(x)^-0$ and $\psi(x^-1) = \psi(x)^-1$. This is, after restricting to a nice subtree, a normal embedding and its action is

$$
\begin{align*}
\bar{\psi}[0] &= [0] \quad \bar{\psi}[1] = [1] \quad \bar{\psi}[01] = [01] \quad \bar{\psi}[01] = [01] \\
\bar{\psi}[0] &= [0] \quad \bar{\psi}[0] = [0] \quad \bar{\psi}[01] = [01] \quad \bar{\psi}[01] = [01]
\end{align*}
$$

**Case 1ca.** If both $[01]$ and $[01]$ are types in $\Delta_2$, then $\psi$ shows that $\Delta|_{2^{<\omega}} \leq \Delta_{2^{<\omega}}$.

**Case 1cb.** If $[01]$ is in $\Delta_2$ but $[01]$ is not, then $\Delta|_{2^{<\omega}} \leq \Delta_{2^{<\omega}}$.

**Case 1cc.** If $[01]$ is in $\Delta_2$ but $[01]$ is not, then we can consider the switch $\phi$ of $\psi$ given by $\phi(x^-0) = \phi(x)^-11$ and $\phi(x^-1) = \phi(x)^-1$. This is, after restricting to a nice subtree, a normal embedding and its action is

$$
\begin{align*}
\bar{\phi}[0] &= [0] \quad \bar{\phi}[1] = [01] \quad \bar{\phi}[01] = [01] \quad \bar{\phi}[01] = [01] \\
\bar{\phi}[0] &= [0] \quad \bar{\phi}[0] = [0] \quad \bar{\phi}[01] = [01] \quad \bar{\phi}[01] = [01]
\end{align*}
$$

so that now $\phi$ shows that a permutation of gap 104 is below $\Delta_{2^{<\omega}}$, namely $\{\Delta|_{104}, \Delta|_{104}, \Delta|_{104}\} \leq \Delta$.

**Case 1cd.** If neither $[01]$ nor $[01]$ belong to $\Delta_2$ but $[01]$ belongs to $\Delta_2$, then since $[01]$ dominates all types in $2^{<\omega}$, by Lemma 2.4.3, we can find $\psi' : 3^{<\omega} \rightarrow 3^{<\omega}$


such that \( \psi' \sigma = \bar{\psi} \sigma \) if \( \max(\sigma) < 2 \) and \( \bar{\psi}' \sigma = [1_0] \) if \( \max(\sigma) = 2 \). This \( \psi' \) shows that \( \Delta^{108} = \{ \Gamma^0_0, \Gamma^1_1, \mathcal{M}_2 \} \leq \Delta \).

**Case 1ce.** If neither \([0^1_1]) \) nor \([^{1_0}]_0 \) nor \([0^1_2] \) are in \( \Delta_2 \) but \( \tau = [0^1_0] \) is in \( \Delta_2 \), then since \( \Delta' \) subdominates all types in \( 2^{<\omega} \), we can find \( \psi' : 3^{<\omega} \to 3^{<\omega} \) such that \( \psi' \sigma = \bar{\psi} \sigma \) if \( \max(\sigma) < 2 \), while for \( \max(\sigma) = 2 \), we have \( \psi' \sigma = \tau, \bar{\xi}(\tau), s(\tau) \), etc, according to the rules given in Theorem 2.5.4. This \( \psi' \) shows that \( \Delta^{113} = \{ \Gamma^0_0, \Gamma^1_1, \mathcal{N}_2 \} \leq \Delta \).

**Case 1cf.** If \( \Delta_2 \) contains just the type \([0^1_1] \), we can consider an embedding \( \chi : 3^{<\omega} \to 2^{<\omega} \) defined with the help of a base function \( \beta : 3^{<\omega} \to 2^{<\omega} \), in the following way: \( \beta(x)\bar{\omega} = \beta(x)\bar{01} \). Since \( \chi(x) = \beta(x)\bar{0} \) is straightforward to check that the action of \( \chi \) satisfies:

- If \( \sigma \) is a chain type with \( \min(\sigma) = i \), then \( \bar{\chi} \sigma \) is a chain type with \( \min(\bar{\chi} \sigma) = i \).
- If \( \sigma \) is a chain type with \( \min(\sigma) = 2 \), then \( \bar{\chi} \sigma = [0^1_1] \).
- If \( \sigma \) is a comb type, then \( \tau = \bar{\chi} \sigma \) is also a comb type with \( \max(\tau) = 1 \).

These facts imply that \( \chi \) witnesses that the 3-gap number 163 is below \( \Delta \).

**Case 2.** We suppose that \( \Delta_2 \subset \mathcal{M}_2 \) and \( \{ \Delta_0|_{2^{<\omega}}, \Delta_1|_{2^{<\omega}} \} = \Gamma^i \) for some \( i = 1, \ldots, 5 \). In every subcase that we consider below, we always assume that none of the previous subcases holds.

**Case 2a.** There is a top-comb \( \tau \) with \( \max(\tau) = 2 \) such that \( \tau \) is in the ideal \( \Delta_k \) for some \( k \in 3 \). Write \( 3 = \{i, j, k\} \). The ideals \( \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \) form a 2-gap, so there exists a normal embedding \( \phi : 2^{<\omega} \to \{i, j\}^{<\omega} \) that witnesses that a permutation of one of the minimal 2-gaps \( \Gamma^1, \ldots, \Gamma^5 \) is below \( \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \). By interchanging \( i \) and \( j \) if necessary, we can actually suppose that \( \phi \) witnesses that \( \Gamma^l \leq \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \) for some \( l = 1, \ldots, 5 \). Since \( \tau \) dominates all types in the range of \( \phi \), Lemma 2.4.3 allows to find \( \psi : 3^{<\omega} \to 3^{<\omega} \) such that \( \psi \sigma = \bar{\phi} \sigma \) if \( \max(\sigma) < 2 \) while \( \psi \sigma = \tau \) if \( \max(\sigma) = 2 \). Then, \( \psi \) shows that there is a 3-gap \( \Delta' \leq \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \) such that \( \mathcal{M}_2 \subset \Delta'_2 \) and \( \{ \Delta'_0|_{2^{<\omega}}, \Delta'_1|_{2^{<\omega}} \} = \Gamma^l \) for some \( l = 1, \ldots, 5 \). If \( \Delta'_2 = \mathcal{M}_2 \) then \( \Delta' \) is one of the gaps 105-109. If not, \( \Delta' \) falls in Case 1.

**Case 2b.** There is a top-comb \( \tau \) with \( \max(\tau) = 1 \) such that \( \tau \) is in the ideal \( \Delta_2 \). Repeat the same argument as in Case 2a for \( k = 2 \).

**Case 2c.** There is a non top-comb, comb type \( \tau \) such that \( \max(\tau) = 2 \) and \( s(\tau) = \tau \) and \( \tau \) is in the ideal \( \Delta_k \) for some \( k \in 3 \). Again, write \( 3 = \{i, j, k\} \). The ideals \( \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \) form a 2-gap, so there exists a normal embedding \( \phi : 2^{<\omega} \to \{i, j\}^{<\omega} \) that witnesses that a permutation of one of the minimal 2-gaps \( \Gamma^1, \ldots, \Gamma^5 \) is below \( \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \). By interchanging \( i \) and \( j \) if necessary, we can actually suppose that \( \phi \) witnesses that \( \Gamma^l \leq \{ \Delta_i|_{\{i, j\}}, \Delta_j|_{\{i, j\}} \} \) for some \( l = 1, \ldots, 5 \). Since \( \tau \) subdominates all types in the range of \( \phi \), Theorem 2.5.4 allows to find \( \psi : 3^{<\omega} \to 3^{<\omega} \) such that

- \( \psi \sigma = \bar{\phi} \sigma \) if \( \max(\sigma) < 2 \),
- \( \psi \sigma = \tau = s(\tau) \) if \( \max(\sigma) = 2 \) and \( \tau \) is not top-comb,
- \( \psi \sigma \) is a top-comb if \( \max(\sigma) = 2 \) and \( \tau \) is a top-comb.

Then, \( \psi \) shows that there is a 3-gap \( \Delta' \leq \{ \Delta_i, \Delta_j, \Delta_k \} \) such that \( \mathcal{M}_2 \cap \Delta'_2 = \mathcal{N}_2 \) and \( \{ \Delta'_0|_{2^{<\omega}}, \Delta'_1|_{2^{<\omega}} \} = \Gamma^l \) for some \( l = 1, \ldots, 5 \). If \( \Delta'_2 = \mathcal{N}_2 \) then \( \Delta' \) is one of the gaps 110-114. If not, \( \Delta' \) falls in Case 1.
Case 2d. There is a non top-comb, comb type $\tau$ such that $\max(\tau^1) = 1$ and
$s(\tau) = \tau$ and $\tau$ is in the ideal $\Delta_2$. Repeat the same argument as in Case 2c for
$k = 2$.

Case 2e. There is a non top-comb, comb type $\tau$ such that $\max(\tau^1) = 2$ and
$s(\tau) \neq \tau$ and $\tau$ is in the ideal $\Delta_k$ for some $k \in 3$. Again, write $3 = \{i, j, k\}$. The
ideals $\{\Delta_i | i, j, k\}$ form a 3-gap, so there exists a normal embedding
$\phi : 2^{<\omega} \rightarrow \{i, j\}^{<\omega}$ that witnesses that a permutation of one of the minimal
2-gaps $\Gamma^1, \Gamma^2, \Gamma^3$ is below $\{\Delta_i | i, j, k\}^{<\omega}$. By interchanging $i$ and $j$ if
necessary, we can suppose that $\Gamma^l \leq \{\Delta_i | i, j, k\}^{<\omega}$ for some $l = 1, \ldots, 5$. If
$\Delta_2 \neq O_2$ then $\Delta^l$ is one of the gaps 115-119. If not, $\Delta^l$ falls in Case 1.

Case 2f. There is a non top-comb, comb type $\tau$ such that $\max(\tau^1) = 1$ and
$s(\tau) \neq \tau$ and $\tau$ is in the ideal $\Delta_2$. Repeat the same argument as in Case 2e for $k = 2$.

Remark 2f⁺. The cases 2a-2f above allow us to assume from now on that all
comb types $\tau$ in $\Delta_0 \cup \Delta_1 \cup \Delta_2$ have $\max(\tau^1) < 2$, and indeed all comb types in $\Delta_2$
have $\max(\tau^1) = 0$.

Case 2g. Both $\Delta_0$ and $\Delta_1$ contain types of maximum 2. Since we are assuming
type $\tau$ in $\Delta_0 \cup \Delta_1 \cup \Delta_2$ with $\max(\tau^1) = 2$, this implies that if $\sigma \in \Delta_0 \cup \Delta_1$
satisfies $\max(\sigma) = 2$, then actually $\max(\sigma^0) = 2$, hence $\text{strength}(\sigma) = \{2\}$. If both $\Delta_0$ and $\Delta_1$ contain types of maximum 2, we
conclude that they contain types of strength $\{2\}$, so by Corollary 2.7.4 $\Delta$ is a
strong 3-gap. We know, by Theorem 2.7.3 that $\Delta$ is equivalent to a 3-gap of the form
$\Sigma^{(A, B, \psi)}$. For $n = 3$ the possibilities for $A, B$ and $\psi$ are the following. The
first possibility is that $A = \{0, 1, 2\}, B = \emptyset$ and $\psi$ is constant equal to $\omega$, in which
case $\Sigma^{(A, B, \psi)} = \Delta^{163}$. The second possibility is that $A = \{i, j\}, B = \{k\}$ and $\psi$
is constant equal to $\omega$. If $i = 0$, $j = 1$ and $k = 2$ then $\Sigma^{(A, B, \psi)} = \Delta^{103}$, and for
other choices of $i, j, k$ we get permutations of $\Delta^{103}$. The third possibility is that
$A = \{i, j\}, B = \{k\}$ and $\psi(i, j) = \omega$ and $\psi(j, i) = k$. If $i = 0$, $j = 1$ and $k = 2$
then $\Sigma^{(A, B, \psi)} = \Delta^{104}$, and for other choices of $i, j, k$ we get permutations of $\Delta^{104}$.

Case 2h. Neither $\Delta_0$ nor $\Delta_1$ contain types of maximum 2, and moreover $\Delta_2$
contains only chain types. Notice that $\{\Delta_0, \Delta_1\} = \Gamma^1$ for some $i = 1, \ldots, 5$.

Case 2ha. $\Delta_2$ only contains the type $[2]$. Then, $\Delta$ is one of the gaps 120-124.

Case 2hb. $\Delta_2$ consists of all chain types of maximum 2: $[2], [02], [12]$ and $[012]$. Say that $\{\Delta_0, \Delta_1\} = \Gamma^1$. If $i = 2, 3, 4$, then $\Delta$ is one of the gaps 125-127.
Suppose that otherwise $i = 1, 5$. Consider a normal embedding $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$
such that $\phi[0] = [0]$ and $\phi[1] = [1_{01}]$, whose action is described in Lemma 2.5.6.
Define $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ by the recursive formula $\psi(x^2 s) = \psi(x)^2 \phi(s)$, for $x \in 3^{<\omega}$, $s \in 2^{<\omega}$. After restricting to a nice subtree, $\psi$ becomes a normal embedding and its action satisfies:

- $\widetilde{\psi} \sigma = \bar{\phi} \sigma$ if max($\sigma$) ≤ 1,
- $\psi[2] = [2]$, $\psi[02] = [02]$,
- If $\sigma$ is a type with max($\sigma$) = 2, $\sigma \not\in \{[2], [02]\}$, then $\tau = \widetilde{\psi} \sigma$ is a comb type with max($\tau$) = 2.

The embedding $\psi$ shows that $\Delta^{128} \leq \Delta$ (if $i = 1$) or that $\Delta^{131} \leq \Delta$ (if $i = 5$).

**Case 2hc.** $\Delta_2$ consists of $[2]$ and $[02]$. If $\{\Delta_0, \Delta_1\} \neq \Gamma^4$, then $\Delta$ is one of the gaps 128-131. If $\{\Delta_0, \Delta_1\} = \Gamma^4$, then we can consider the embedding $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\phi(x^0) = \phi(x)^0$, $\phi(x^1) = \phi(x)^1$, $\phi(x^2) = \phi(x)^2$, which satisfies that $\bar{\phi}[2] = [2]$, $\bar{\phi}[12] = [12]$ and $\bar{\phi}[02] = \bar{\phi}[012] = [012]$, and which shows that $\Delta^{128} \leq \Delta$.

**Case 2hd.** $\Delta_2$ consists of $[2]$ and $[12]$. If $\{\Delta_0, \Delta_1\} \neq \Gamma^2$ or $\Gamma^4$, then $\Delta$ is one of the gaps 132-133. Otherwise, $[01] \in \Delta_1$ and we can consider the embedding $\phi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\phi(x^0) = \phi(x)^0$, $\phi(x^1) = \phi(x)^1$, $\phi(x^2) = \phi(x)^2$, which satisfies that $\bar{\phi}[0] = [0]$, $\bar{\phi}[1] = [01]$, $\bar{\phi}[02] = [02]$ and $\bar{\phi}[12] = \bar{\phi}[012] = [012]$ and shows that one of the gaps 120, 122 or 124 is below $\Delta$, depending whether $i = 1, 3$ or 5.

**Case 2he.** $\Delta_2$ consists of $[2]$, $[02]$ and $[012]$. If $\{\Delta_0, \Delta_1\} = \Gamma^2$ then $\Delta$ is the gap number 134. If $\{\Delta_0, \Delta_1\} = \Gamma^4$ we can consider the embedding $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\psi(x^0) = \psi(x)^1$, $\psi(x^1) = \psi(x)^1$, $\psi(x^2) = \psi(x)^2$, for which $\psi[2] = [2]$, $\psi[12] = [012]$ and $\psi[02] = \psi[012] = [012]$, and one can check that this embedding shows that $\Delta^{133} \subseteq \{\Delta_1, \Delta_0, \Delta_2\}$. If $\{\Delta_0, \Delta_1\}$ is neither $\Gamma^2$ nor $\Gamma^4$, then similarly as in the previous case, since $[01] \in \Delta_1$ we can consider $\phi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\phi(x^0) = \phi(x)^0$, $\phi(x^1) = \phi(x)^1$, $\phi(x^2) = \phi(x)^2$, which satisfies that $\bar{\phi}[2] = [2]$, $\bar{\phi}[02] = [02]$ and $\bar{\phi}[12] = \bar{\phi}[012] = [012]$. This embedding shows that we have $\Delta' \leq \Delta$ and $\Delta'$ falls in Case 2ib above.

**Case 2hf.** $\Delta_2$ consists of $[2]$, $[02]$ and $[12]$. Then, we can consider the embedding $\phi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\phi(x^0) = \phi(x)^0$, $\phi(x^1) = \phi(x)^1$, $\phi(x^2) = \phi(x)^2$, which satisfies that $\bar{\phi}[2] = [2]$, $\bar{\phi}[12] = [12]$ and $\bar{\phi}[02] = \bar{\phi}[012] = [012]$, and it fixes all types in $2^{<\omega}$. This embedding shows that we have $\Delta' \leq \Delta$ which falls into Case 2hd.

**Case 2hg.** $\Delta_2$ consists only of chain types, but it does not fall into any of the previous cases. Then $\Delta_2$ contains the type $[012]$. We can consider the embedding $\phi : 3^{<\omega} \rightarrow 3^{<\omega}$ given by $\phi(x^0) = \phi(x)^0$, $\phi(x^1) = \phi(x)^1$, $\phi(x^2) = \phi(x)^2$, which satisfies that $\bar{\phi}[2] = \bar{\phi}[12] = [12]$ and $\bar{\phi}[02] = \bar{\phi}[012] = [012]$, and it fixes all types in $2^{<\omega}$. This embedding provides $\Delta' \leq \Delta$ such that $\Delta'_i = \Delta_i$ for $i = 0, 1$, and $\Delta'_2$ consists only of chain types, but now $[2]$, $[012]$ and $[02]$ belong to $\Delta_2$. This $\Delta'$ falls into one of the previous cases 2ib or 2he.

**Case 2i.** Neither $\Delta_0$ nor $\Delta_1$ contain types of maximum 2, and moreover $\Delta_2$ contains at least a comb type $\tau$. Since this type must satisfy max($\tau$) = 2 and max($\tau^1$) = 0, there are only three possibilities: $\tau_A = [0]_2$, $\tau_B = [1]_2$ and $\tau_C = [0]_{12}$. We describe now certain embedding $s \psi_A$, $\psi_B$, $\psi_C$, from $3^{<\omega}$ into $3^{<\omega}$ that will

\[1\]We shall define a number of embeddings $3^{<\omega} \rightarrow 3^{<\omega}$ along Case 2i and Case 2j using base functions. We follow similar patterns as in the proofs of Theorem 2.2.1 Lemma 2.3.3 and Theorem 2.5.2. The idea is always that we are given types with max($\tau_0$) ≤ max($\tau_1$) ≤ max($\tau_2$),
be relevant in the discussion of Case 2i. So for \( X = A, B, C \), consider \((u_X, v_X)\) a comb of type \( \tau_X \) and write \( u_X = \vec{u}_X \prec \bar{u}_X \) in such a way that \(|\vec{u}_X| = |v_X|\) and the first coordinate of \( \bar{u}_X \) is 2. Notice that \( v_X \) is just a finite sequence of 0's. Let \( \{n(s) : s \in 3^{<\omega}\} \) be a very rapidly increasing sequence of numbers, say such that \( n(s) > 3|u_X||v_X| \sum_{t < s} n(t) \) for all \( s \). The embedding \( \psi_X : 3^{<\omega} \rightarrow 3^{<\omega} \) is defined recursively with the help of a base function \( \beta : 3^{<\omega} \rightarrow 3^{<\omega} \) as indicated below and illustrated in Figure 1:

\[
\begin{align*}
\beta(\emptyset) &= \emptyset \\
\psi_X(\emptyset) &= v_X \\
\beta(s^\sim 0) &= \beta(s) \\
\beta(s^\sim 1) &= \psi_X(s)^\sim 1^n(s^\sim 1) \\
\beta(s^\sim 2) &= \beta(s)^\sim \bar{u}_X^\sim 0^n(s^\sim 2)^\sim \bar{u}_X \\
\psi_X(s^\sim 0) &= \psi_X(s)^\sim 0^n(s^\sim 0) \\
\psi_X(s^\sim 1) &= \beta(s^\sim 1)^\sim v_X \\
\psi_X(s^\sim 2) &= \beta(s^\sim 2)^\sim v_X
\end{align*}
\]

These are actually normal embeddings, and it is straightforward, with the help of Figure 1, to compute their action, that we summarize in the following table:

| \( \sigma \) in \( 2^{<\omega} \) | \( \tau_A \) | \( \tau_B \) | \( \tau_C \) |
|---|---|---|---|
| 2 | \( \tau_A \) | \( \tau_B \) | \( \tau_C \) |
| 02 | \( \tau_A \) | \( \tau_B \) | \( \tau_C \) |
| 12 | [12] | [12] | [12] |
| 012 | [012] | [012] | [012] |
| \( \tau_A \) | \( \tau_A \) | \( \tau_B \) | \( \tau_B \) |
| \( \tau_B \) | \( \tau_B \) | \( \tau_B \) | \( \tau_B \) |
| \( \tau_C \) | \( \tau_C \) | \( \tau_C \) | \( \tau_C \) |

If the remaining cases, if \( \sigma \) is a comb type with \( \max(\sigma) = 2 \) and \( \max(\sigma^1) \geq 1 \), then \( \tau = \psi_X \sigma \) is again a comb type with \( \max(\tau) = 2 \) and \( \max(\tau^1) \geq 1 \). Remember that in each of the subcases that follow, we assume that the previous subcases do not hold.

and we want to define \( \psi \) so that \( \psi[i] = \tau_i \). We can make the base function \( \beta(s) \) either to increase or to remain constant when passing to \( \beta(s^\sim i) \) depending on the domination or subdomination relations that exist between the types \( \tau_i \).
Case 2ia. If \( \{ \Delta_0, \Delta_1 \} = \Gamma^4 \). We can consider the embedding \( \psi : 3^{<\omega} \to 3^{<\omega} \) given by \( \psi(x) = \psi(x) \). This embedding sends chain types in \( \Delta_0 \) to chain types in \( \Delta_1 \), and vice versa, satisfying \( \psi \circ \sigma = \sigma \circ \psi \) for any \( \sigma \in \Gamma \) and \( \psi \circ \sigma \) is in \( \Phi \). If \( \phi \) is in \( \Delta_0 \), then \( \phi \) is in \( \Delta_1 \) and vice versa. Therefore, \( \Delta \) is one of the gaps 135-146.

Case 2ib. Neither \( \tau_B \) nor \( \tau_C \) is in \( \Delta_2 \). If either \( [12] \) or \( [012] \) is in \( \Delta_2 \), the embedding \( \psi_B \) shows that we can find \( \Delta' \leq \Delta \) which falls into Case 2h. Otherwise, the embedding \( \psi_A \) allows us to suppose that \( \Delta_2 \) consists of \( [2], [02] \) and \( \tau_A \). Thus \( \Delta \) is one of the gaps 153-156.

Case 2ic. If \( \{ \Delta_0, \Delta_1 \} = \Gamma^2 \). By considering either \( \psi_B \) (if \( \tau_B \) is in \( \Delta_2 \)) or \( \psi_C \) (if \( \tau_C \) is in \( \Delta_2 \)) we can suppose that \( [02] \) belongs to \( \Delta_2 \), and also that \( \tau_A \) belongs to \( \Delta_2 \) if and only if \( \tau_B \) belongs to \( \Delta_2 \). Therefore \( \Delta \) is one of the gaps 135-146.

Case 2id. If \( \{ \Delta_0, \Delta_1 \} = \Gamma^3 \), then \( [01] \) is in \( \Delta_1 \), and we can consider an embedding \( \phi : 3^{<\omega} \to 3^{<\omega} \) such that \( \phi(0) = 0 \), \( \phi(1) = 0 \), and \( \phi(2) = 2 \). We know that \( \phi \) preserves \( \Gamma^1 \) and \( \Gamma^3 \), and it is straightforward to check that \( \phi \tau_B = [01] \) and \( \phi \tau_C = [01] \). Hence, \( \phi \) shows that we have \( \Delta' \leq \Delta \) such that \( \{ \Delta_0, \Delta_1 \} = \Gamma^1 \) for \( i = 1, 5 \), and neither \( \tau_B \) nor \( \tau_C \) belong to \( \Delta_2 \). Thus \( \Delta' \) follows in some of the previous cases, indeed either Case 2ib (if \( \tau_A \) is in \( \Delta_2 \)) or Case 2h (if \( \tau_A \) is not in \( \Delta_2 \)).

Case 2j. \( \Delta_1 \) contains types of maximum 2 but \( \Delta_0 \) does not. Remember that in each of the subcases that follow, we assume that all previous subcases do not hold.

Case 2ja. \( \Delta_1 \) contains the type \( [02] \). Then we can restrict the gap \( \Delta \) to \( \{0, 2\}^{<\omega} \). Remember from Remark 2f that any comb type \( \tau \) in \( \Delta_2 \) satisfies \( \max(\tau^1) = 0 \). Therefore, the only type from \( \{0, 2\}^{<\omega} \) that may belong to \( \Delta_2 \) are \( [2], [02] \) and \( [02] \), and we know that \( [2] \) is in \( \Delta_2 \) and \( [02] \) is not in \( \Delta_2 \) since \( [02] \) is in \( \Delta_1 \). Therefore \( \Delta_2 \cap \{0, 2\}^{<\omega} \) consists either of \( \{[2]\} \) or of \( \{[2], [02]\} \). On the other hand, since \( \Delta_0 \) does not contain any type of maximum 2 (we are in Case 2j), we have that \( \Delta_0 \cap \{0, 2\}^{<\omega} \) consists only of \( \{[0]\} \). We can identify \( \{0, 2\}^{<\omega} \) with \( 2^{<\omega} \) by changing the number 2 by 1, and then we get that \( \{\Delta_0, \Delta_2, \Delta_1\} \cap \{0, 2\}^{<\omega} = \Delta^k \) for some \( k \in \{1, 2, \ldots, 94\} \).

Case 2jb. \( \Delta_1 \) contains the type \( [02] \). If \( [02] \) is not in \( \Delta_2 \), then we can apply the same reasoning as in Case 2ja, restricting to \( \{0, 2\}^{<\omega} \). If, on the contrary, \( [02] \) is in \( \Delta_2 \), then we can consider the embedding \( \phi : 2^{<\omega} \to 3^{<\omega} \) given by \( \phi(0) = 0 \) and \( \phi(1) = 0 \). We have that \( \phi(2) = 2 \) and for all other types \( \sigma \) in \( 2^{<\omega} \), the type \( \phi \sigma \) is obtained by changing each 1 by a 2 in \( \sigma \). Remember that comb types \( \tau \) with \( \max(\tau^1) = 2 \) do not belong to \( \Delta_0 \cup \Delta_1 \cup \Delta_2 \) by Remark 2f. Thus, the embedding \( \phi \) witnesses that the 3-gap \( \Delta' \) given by \( \Delta_0 \equiv \{0\}, \Delta_1' \equiv \{[1], [01]\}, \Delta_2' \equiv \{[01]\} \) satisfies \( \Delta' \leq \Delta \). Notice that \( \Delta' \) is one of the gaps 64-94.
Case 2jc. \( \Delta_2 \) contains the type \([0_2] \) and \( \Delta_1 \) does not include any comb type \( \tau \) such that \( \max(\tau) = 2 \). Then \( \Delta_1 \) must include either \([12] \) or \([012] \).

Case 2jca. \( \Delta_1 \) contains the type \([02] \). We consider an embedding \( \psi : 3^\omega \to 3^\omega \) defined in a similar way as in Case 2i. Let \( \{n(s) : s \in 3^\omega\} \) be a very rapidly increasing sequence of numbers, say such that \( n(s) > 3\sum_{t<s} n(t) \) for all \( s \). The function \( \psi \) is defined recursively with the help of a base function \( \beta : 3^\omega \to 3^\omega \) as indicated below.

\[
\begin{align*}
\beta(\emptyset) & = \emptyset \\
\psi(\emptyset) & = (0) \\
\beta(s\uparrow 0) & = \beta(s) \\
\beta(s\uparrow 1) & = \psi(s\uparrow 1)12^n(s\uparrow 1) \\
\beta(s\uparrow 2) & = \beta(s\uparrow 2)2^n(s\uparrow 2) \\
\psi(s\uparrow 0) & = \psi(s\uparrow 0)n(s\uparrow 0) \\
\psi(s\uparrow 1) & = \beta(s\uparrow 1)0 \\
\psi(s\uparrow 2) & = \beta(s\uparrow 2)0
\end{align*}
\]

The picture is the same as for function \( \psi_A \) of Case 2i, illustrated in Figure 1 for \( X = A \). The only difference is the recursive formula for \( \beta(s\uparrow 1) \) that now adds 1 and 2, while we were only adding 1 in the function \( \psi_A \) of Case 2i. The action of \( \psi \) is given by the following table, in which the last column indicates in which \( \Delta_i \) the type \( \psi \bar{\sigma} \) belongs:

| \( \sigma \) | \( \psi \bar{\sigma} \) | \( \Delta_i \) |
|---|---|---|
| \([0] \) | \([0] \) | \( \Delta_0 \) |
| \([1] \) | \([12] \) | \( \Delta_1 \) |
| \([2] \) | \([02] \) | \( \Delta_2 \) |
| \([01] \) | \([012] \) | \( ?^\alpha \) |
| \([01] \) | \([12] \) | \( ?^\beta \) |
| \([02] \) | \([012] \) | \( ?^\alpha \) |
| \([12] \) | \([02] \) | \( ?^\beta \) |
| \([012] \) | \([102] \) | \( ?^\beta \) |
| \([012] \) | \([2] \) | \( \Delta_2 \) |

Notice that \([1_0 2] \notin \Delta_1 \) because we suppose in this case that \( \Delta_1 \) does not contain any comb with maximum 2. Thus, the boxes marked with \( ?^\beta \) are either in \( \Delta_2 \) or in \( \Delta^\bot \). If \([1_0 2] \) or \([012] \) were in \( \Delta_2 \), then \( \psi \) would provide a gap \( \Delta' \leq \Delta \) which falls into Case 1, because we would have \([01] \in \Delta_2 \) or \([01] \in \Delta_2' \), and \( \{\Delta_0', \Delta_1'\} \) is either \( \Gamma^3 \) or \( \Gamma^2 \) depending whether \([012] \) belongs to \( \Delta_1 \) or not. Therefore, the boxes marked with \( ?^\beta \) are indeed types which go to \( \Delta^\bot \) while those with \( ?^\alpha \) go either to \( \Delta^\bot \) or to \( \Delta_1 \). It follows that either \( \Delta_1^{161} \leq \Delta \) or \( \Delta_1^{162} \leq \Delta \).

Case 2jcb. \( \Delta_1 \) contains \([012] \). Consider the embedding \( \phi : 3^\omega \to 3^\omega \) given by \( \phi(x\uparrow 0) = \phi(x)\uparrow 000, \phi(x\uparrow 1) = \phi(x)\uparrow 012, \phi(x\uparrow 2) = \phi(x)\uparrow 222 \). The action of
3.2. MINIMAL ANALYTIC 3-GAPS

Figure 2. Configuration of $\phi$ in Case 2jd.

$\phi$ on chain types is given by

$$
\begin{array}{|c|c|}
\hline
\sigma & \phi\sigma \\
\hline
0 & [0] \in \Delta_0 \\
1 & [012] \in \Delta_1 \\
01 & [012] \in \Delta_1 \\
2 & [2] \in \Delta_2 \\
02 & [02] \in \Delta_1 \\
012 & [012] \in \Delta_1 \\
12 & [012] \in \Delta_1 \\
\hline
\end{array}
$$

Comb types are sent to comb types of maximum 2, which do not belong to neither $\Delta_0$ nor $\Delta_1$. Thus, $\phi$ shows that we have $\Delta' \leq \Delta$, where $\Delta'_0 = \{[0]\}, \Delta'_1 = \{[1],[01],[12],[012]\}$ and $[2] \in \Delta'_2 \subset \mathcal{M}_2$. Therefore, $\Delta'$ falls into Case 2jca.

**Case 2jd.** $\Delta_2$ contains the type $[02]$ and $\Delta_1$ contains a comb type $\tau$ such that $\max(\tau) = 2$. Let $\tau$ be a type in $\Delta_1$ with $\max(\tau) = 2$ (which must satisfy that $\max(\tau^0) = 2$). This time, we are going to define an embedding $\phi : 3^{<\omega} \rightarrow 3^{<\omega}$ following the scheme of the proof of Theorem 2.2.1 in order to make $\bar{\phi}[0] = [0], \bar{\phi}[1] = \tau$ and $\bar{\phi}[2] = [02]$. So let $(u,v)$ be a rung of type $\tau$ with $u = \bar{u} \sim \bar{u}$ in such a way that $|\bar{u}| = |\bar{u}| = |v| = k$. We define functions $\phi, \phi_1, \phi_2, \phi^1, \phi^2$ recursively, that will follow the pattern shown in Figure 2, similar to Figure 1. The initial input is:

$$
\begin{align*}
\phi(\emptyset) & = v^\sim 0^k \\
\phi_1(\emptyset) & = \emptyset \\
\phi^1(\emptyset) & = u^\sim 0^{2k} \\
\phi_2(\emptyset) & = v \\
\phi^2(\emptyset) & = 2^k \sim 0^k
\end{align*}
$$

And the recursive formulas are the following, where $i$ stands for both $i = 1$ and $i = 2$, while $l_0(x), l_1(x)$ and $l_2(x)$ are large enough numbers so that the nodes $\phi(x^\sim 0), \phi^1(x)$ and $\phi^2(x)$ have larger length than any previously defined node along
the recursion:

\[
\begin{align*}
\phi_1(x^0) &= \phi_1(x) \\
\phi_2(x^0) &= \phi_2(x) \\
\phi(x^0) &= \phi(x)^0(x) \\
\phi_1(x^i) &= \phi_i(x)^i \tilde{u} \\
\phi_2(x) &= \phi_1(x)^v \\
\phi(x^i) &= \phi_2(x)^0(x) \\
\phi^3(x) &= \phi_1(x)^{-1}(x) \\
\phi^2(x) &= \phi_2(x)^{-2}(x)
\end{align*}
\]

**Case 2jda.** The type \( \tau \) can be taken to be a top-comb. In this case we could take \( \tilde{u} \) to be just a sequence of 0’s and the action of \( \phi \) can be checked to be described by the following table:

| \( \sigma \) | \( \phi \sigma \) |
|----------------|------------------|
| 0              | 0                | \( \in \Delta_0 \) |
| 1              | \( \tau \)       | \( \in \Delta_1 \) |
| 2              | \( u \)          | \( \in \Delta_2 \) |

\( \max(\sigma) = 1 \)

| \( \sigma \) | \( \phi \sigma \) |
|----------------|------------------|
| 012            | \( \tau \)       | \( \in \Delta_1 \) |
| 12             | \( \tau \)       | \( \in \Delta_1 \) |
| 02             | \( u \)          | \( \in \Delta_2 \) |
| 012            | \( \tau \)       | \( \in \Delta_1 \) |
| 12             | \( \tau \)       | \( \in \Delta_1 \) |
| 02             | \( u \)          | \( \in \Delta_2 \) |

Let \( \Delta' \) be the 3-gap for which \( \phi \) witnesses that \( \Delta' \leq \Delta \). So \( \Delta_0' = \{0\} \), \( \Delta_1' = \{\sigma : \max(\sigma) = 1\} \cup \{[012], [12], [012], [012]\}, \( \Delta_2' = \{[2], [02], [02]\} \). Consider now the embedding \( \chi : 3^{<\omega} \rightarrow 3^{<\omega} \) given by \( \chi(x^0) = \chi(x)^{-00} \), \( \chi(x^1) = \chi(x)^{-22} \) and \( \chi(x^2) = \chi(x)^{-12} \). The embedding \( \chi \) shows that \( \Delta^\leq_{160} \leq (\Delta_0', \Delta_2', \Delta_1') \) because it has the following action:

| \( \sigma \) | \( \phi \sigma \) |
|----------------|------------------|
| 0              | 0                | \( \in \Delta_0' \) |
| 1              | 2                | \( \in \Delta_2' \) |
| 2              | 12               | \( \in \Delta_1' \) |

\( \max(\sigma^i) = 1 \) \( \max(\sigma^i) = 2 \) \( \in (\Delta')^L \)

| \( \sigma \) | \( \phi \sigma \) |
|----------------|------------------|
| 01             | 02               | \( \in \Delta_2' \) |
| 01             | 02               | \( \in \Delta_2' \) |
| 012            | 02               | \( \in \Delta_2' \) |
| 12             | 2                | \( \in \Delta_2' \) |
| 012            | 02               | \( \in \Delta_2' \) |
| 012            | 02               | \( \in \Delta_2' \) |
| 012            | 02               | \( \in \Delta_2' \) |
| 012            | 02               | \( \in \Delta_2' \) |
3.2. MINIMAL ANALYTIC 3-GAPS 83

Case 2jdb. The are no top-combs in $\Delta_1$ with maximum 2, so $\tau$ must be taken to be non top-comb. The action of $\phi$ can be checked to be given now as follows:

| $\sigma$ | $\bar{\phi}\sigma$ | $\in \Delta$ |
|----------|---------------------|--------------|
| [0]      | [0]                 | $\in \Delta_0$ |
| [1]      | $\tau$              | $\in \Delta_1$ |
| [2]      | $\bar{0}_2$         | $\in \Delta_2$ |
| [01]     | $\tau$              | $\in \Delta_1$ |
| [101]    | ?                   | ?            |
| other max($\sigma$) = 1 top-combs, max($\sigma$) = 2 | $\notin \Delta_1$ |
| max($\sigma$) = 2, max($\sigma^1$) $\geq$ 1 max($\sigma^1$) = 2 | $\in \Delta'$ |
| [012]    | $\tau$              | $\in \Delta_1$ |
| [12]     | $\tau$              | $\in \Delta_1$ |
| [02]     | $\bar{0}_2$         | $\in \Delta_2$ |
| $\bar{0}_{12}$ | $\bar{t}(\tau)$     | $\notin \Delta_1$ |
| $\bar{1}_{02}$ | $\bar{t}(\tau)$     | $\notin \Delta_1$ |
| $\bar{2}$ | $\bar{t}(\tau)$     | $\in \Delta_2$ |

The fact that $\bar{t}(\tau) \notin \Delta$ is because $\bar{t}(\tau)$ is a top-comb with maximum 2, and we are assuming in this case that there are no such types in $\Delta_1$. Notice that $\bar{\phi}\sigma \notin \Delta_0$ unless $\sigma = [0]$, because $\max(\bar{\phi}\sigma) \geq \max(\bar{\phi}[1]) = 2$ whenever $\max(\sigma) \geq 1 = \max([1])$. If $\bar{\phi}\sigma \in \Delta_2$ for some type with $\max(\sigma) \leq 1$, then $\phi$ provides $\Delta' \leq \Delta$ which falls into Case 1. Otherwise, $\phi$ provides $\Delta' \leq \Delta$ which falls into Case 2jc above.

Case 2je. $\Delta_2$ does not contain the type $[0_2]$. Let $\tau$ be a type in $\Delta_1$ such that $\max(\tau) = 2$, which must satisfy $\max(\tau^1) \leq 1$ by Remark 2f. By Theorem 2.2.1 we can consider a normal embedding $\psi : 3^{<\omega} \rightarrow 3^{<\omega}$ such that $\psi[0] = [0], \psi[1] = [2]$ and $\psi[2] = \tau$.

Claim A: $\psi[12] = [2]$. Proof: Consider $s = \psi(1) \land \psi(12)$. On the one hand, $\max(\psi(1) \setminus s) = \max(\tau^1) \leq 1$ since $\psi[2] = \tau$. On the other hand, $\psi(\emptyset)^\sim 2 \leq \psi(1)$ since $\psi[1] = [2]$. Both things together imply that $\psi(\emptyset)^\sim 2 \leq s < \psi(12)$ which implies that $\psi[12] = [2]$ as required.

Claim B: If $\max(\sigma^1) \geq 1$ and $\bar{\psi}\sigma = v$, then $\max(v^1) = 2$. Proof: Since $\max(\sigma^1) \geq 1$, we can find a set $\{x_k : k < n\}$ of type $\sigma$ such that $\{x_k^\sim p : k < \omega\}$ is still of type $\sigma$ for either $p = 0$ or $p = 1$. Since $\psi[0] = [0]$ and $\psi[1] = [2]$ we can write $\psi(x_k^0) = \psi(x_k)^\sim v_k$ and $\psi(x_k^1) = \psi(x_k)^\sim w_k$, with $v_k \in W_0$ and $w_k \in W_2$. Thus, we have that the three sets $\{\psi(x_k) : k < \omega\}, \{\psi(x_k)^\sim v_k : k < \omega\}$ and $\{\psi(x_k)^\sim w_k : k < \omega\}$ are of type $v$. The only way that this can happen for three such sets is that $v$ is a comb type and $\max(v^1) = 2$. 

With this information, the following table with the action of $\psi$ is easy to check:

| $\sigma$ | $\psi\sigma$ |
|----------|--------------|
| 0        | 0            | $\in \Delta_0$ |
| 1        | 2            | $\in \Delta_2$ |
| 2        | $\tau$      | $\in \Delta_1$ |
| 01       | 02          | $\notin \Delta_0, \Delta_1$ |
| 12       | 2           | $\in \Delta_2$ |
| 012      | 02          | $\notin \Delta_0, \Delta_1$ |
| $[0]_2$  | $\notin \Delta_0, \Delta_1$ |
| $[0]_2$  | $\notin \Delta_0, \Delta_1$ |
| $[0]_2$  | $\notin \Delta_0, \Delta_1$ |
| $[0]_2$  | $\notin \Delta_0, \Delta_1$ |

Consider the gap $\Delta'$ for which $\psi$ witnesses that $\Delta' \leq \{\Delta_0, \Delta_1, \Delta_2\}$. We shall look at the permuted gap $\Delta'' = \{\Delta''_0, \Delta''_1, \Delta''_2\}$. We summarize the distribution of types in $\Delta''$:

| $\Delta''_0$ | $\Delta''_1$ | $\Delta''_2$ |
|--------------|--------------|--------------|
| 0            | $[0]_2$      | $[0]_2$      |
| $[1], [12]$  | $[02], [0]_2$, $[01] + [012]$ |
| 2            | $[02], [0]_2$ |

We write $[01] + [012]$ because either both types belong to $\Delta''_2$ or no one belongs. If $[02] \in \Delta''_1$, then $\Delta''$ falls into Case 2ja. If $[02] \in \Delta''_2$, then $\Delta''$ falls into Case 2jb. If $[02] \in \Delta''_3$ then $\Delta''$ falls into Case 2jc. After eliminating all these cases, only four possibilities remain for $\Delta''$ depending on whether $[01] + [012]$ belong to $\Delta''$ or not, and depending on whether $[02]$ belongs to $\Delta''_1$ or not. Three of these possibilities correspond to $\Delta''$ being equal to $\Delta^{157}$, $\Delta^{158}$ or $\Delta^{159}$. The remaining case is that $\Delta''_0 = \{[0]\}$, $\Delta''_1 = \{[1], [12]\}$, $\Delta''_2 = \{[2], [02]\}$. In this last case, we have that $\Delta^{157} \leq \{\Delta''_0, \Delta''_2, \Delta''_3\}$, as it is witnessed by the embedding $\phi(x \sim 0) = \phi(x) \sim 0$, $\phi(x \sim 1) = \phi(x) \sim 0$, $\phi(x \sim 2) = \phi(x) \sim 12$.

**Case 2k.** This is the last case. $\Delta_0$ contains types of maximum 2 but $\Delta_1$ does not. Let $\tau$ be a type in $\Delta_0$ with $\max(\tau) = 2$. By Remark 2f, any type of maximum 2 inside $\Delta_0$ must have strength $\{2\}$, like $[2]$. Thus, Corollary 2.7.7 implies that $\{\Delta_0, \Delta_2\}$ is strong. By Theorem 2.2.1 we can find a normal embedding $\bar{\phi} : 3^{<\omega} \rightarrow 3^{<\omega}$ such that $\bar{\phi}[0] = [1]$, $\bar{\phi}[1] = \tau$ and $\bar{\phi}[2] = [2]$. Let $\Delta'$ be the gap for which $\phi$ witnesses that $\Delta' \leq \Delta$, and consider its permutation $\Delta'' = \{\Delta'_i, \Delta''_i, \Delta''_j\}$, which now satisfies $[i] \in \Delta''_i$, for $i = 0, 1, 2$. Notice that, by Corollary 2.2.2, $\max(\bar{\phi}\sigma) = 2$ whenever $\sigma \neq [0]$. In particular, $\bar{\phi}\sigma \notin \Delta_1$ whenever $\sigma \neq [0]$. It follows that $\phi$ shows that such that $\Delta''_0 = \Delta'_1 = \{[0]\}$. In particular, $\{\Delta''_0, \Delta''_2\}$ is not a strong gap. This property is inherited by any $\Delta''' \leq \Delta''$. But then $\Delta''$ must have fallen in some previous case, since we observed that in Case 2k we must have that $\{\Delta_0, \Delta_2\}$ is strong.

Given a permutation $\varepsilon : 3 \rightarrow 3$ and a 3-gap $\Delta$, we denote

$$
\varepsilon \Delta = \{\Delta_{\varepsilon(0)}, \Delta_{\varepsilon(1)}, \Delta_{\varepsilon(2)}\}.
$$

We must show now that $\varepsilon \Delta' \not\leq \Delta'$ for every $i \neq j$ and every permutation $\varepsilon$. Notice that $\varepsilon \Delta \leq \Delta'$ if and only if $\Delta \leq \varepsilon^{-1} \Delta'$, so the side of the inequality in which we write the permutation is irrelevant.
The first observation is that the gaps 103, 104 and 163 are strong, while none of the others is strong, as it is easily checked using the criterion of Theorem 2.5.7. These three are actually the three minimal analytic 3-gaps which are strong that are predicted in Theorem 2.7.8 that correspond to \(|A| = 3\) (gap 163), \(A = \{0, 1\}\) and \(\psi(0, 1) = \psi(1, 0) = 2\) (gap 103), \(|A| = \{0, 1\}\), \(\psi(0, 1) = \infty\), \(\psi(1, 0) = 2\) (gap 104). Thus, for the cases when either \(i\) or \(j\) belongs to \(\{103, 104, 163\}\), it is clear that \(\varepsilon\Delta^i \leq \Delta^j\) implies \(i = j\). We shall view the rest of the list of 3-gaps divided into three main blocks, corresponding to intervals \([1, 102], [105, 156]\) and \([157, 162]\). The proof is structured as a sequence of lemmas.

**Lemma 3.2.1.** If \(\varepsilon\Delta^i \leq \Delta^j\) and \(j \in [105, 156]\), then \(i \in [105, 156]\) as well.

**Proof.** Let us suppose for contradiction that \(\Delta^i \leq \Delta^j\) for some \(i \in [105, 156]\), some \(i \not\in [105, 156]\) and some embedding \(\varepsilon\). Let \(\phi : 3^{<\omega} \rightarrow 3^\omega\) be a normal embedding witnessing it. The common feature of all gaps \(\Delta^i\), \(i \not\in [105, 156]\) is that we can find \(\tau_0 \in \Delta^i\), \(\tau_1 \in \Delta^i\), \(\tau_2 \in \Delta^i\) such that \(\max(\tau_0) < \max(\tau_1) = \max(\tau_2)\). Using Corollary 2.2.2 it follows that \(\max(\phi\tau_0) \leq \max(\phi\tau_1) = \max(\phi\tau_1)\) whenever \(\sigma \in \Delta^i\) and \(\sigma' \in \Delta^j\). Thus, we get a contradiction.

Notice that for \(j \in [105, 156]\), \(\{\Delta^i_j, \Delta^j_i\} = \Gamma^k\) for some \(k = 1, \ldots, 5\) is one of the minimal 2-gaps.

**Lemma 3.2.2.** If \(\varepsilon\Delta^i \leq \Delta^j\) with \(i, j \in [105, 156]\), then \(\{\Delta^i_j, \Delta^j_i\} = \{\Delta^i_j, \Delta^j_i\}\) and \(\varepsilon(2) = 2\).

**Proof.** Consider a normal embedding \(\phi : 3^{<\omega} \rightarrow 3^\omega\) witnessing that \(\varepsilon\Delta^i \leq \Delta^j\). Using again the property that \(\max(\sigma) < \max(\sigma')\) whenever \(\sigma \in \Delta^i_j \cup \Delta^j_i\) and \(\sigma' \in \Delta^j_i\), we conclude that \(\varepsilon(2) = 2\) and \(\phi\) induces by restriction a normal embedding \(2^{<\omega} \rightarrow 2^{<\omega}\) that witnesses that \(\{\Delta^i_j, \Delta^j_i\}\) are the same. Since these are minimal 2-gaps, we conclude that they are the same.

Consider an auxiliary 2-gap \(\Gamma^6\) in \(2^{<\omega}\), where \(\Gamma^6_0 = \{0\}\) and \(\Gamma^6_1 = \{1, [01], [0_1]\}\). This is not a minimal 2-gap, but we only need the following fact:

**Lemma 3.2.3.** \(\Gamma^i \not\subseteq \Gamma^j\) whenever \(i \neq j\), \(i, j \in \{2, 3, 6\}\).

**Proof.** We know that \(\Gamma^2 \not\subseteq \Gamma^3\) and \(\Gamma^3 \not\subseteq \Gamma^2\) since they are nonequivalent minimal 2-gaps. On the other hand, since \([0_1]\) dominates \([0]\), Theorem 2.4.4 implies that \(\Gamma^i \not\subseteq \Gamma^6\) for \(i = 2, 3\). Finally, suppose for contradiction that we have an embedding \(\phi : 2^{<\omega} \rightarrow 2^{<\omega}\) that witnesses \(\Gamma^i \not\subseteq \Gamma^6\) for \(i = 2\) or \(i = 3\), so that \(\phi[0] = [0]\) and \(\phi[1] \in \{[01], [0_1]\}\). In any case, \(\phi[0_1] = [0_1]\) and this is a contradiction since \([0_1] \in \Gamma^6\) but \([0_1] \not\in \Gamma^i\).

**Lemma 3.2.4.** If \(\varepsilon\Delta^i \leq \Delta^j\) with \(j \in [105, 156]\), and \(\{\Delta^i_j, \Delta^j_i\} = \Gamma^2\) and \(\varepsilon(2) = 2\), then \(i = j\).

**Proof.** We know that \(i \in [105, 156]\), \(\{\Delta^i_j, \Delta^j_i\} = \Gamma^2\) and \(\varepsilon(2) = 2\). Since moreover \(\Gamma^2\) is not symmetric, \(\varepsilon\) must be the identity permutation. It is enough to consider the case when \(j > 119\), because we know by Theorem 2.5.7 that the gaps 105–119 are minimal and nonequivalent to each other, so if we had \(\Delta^i \leq \Delta^j\) with \(j \in [105, 119]\) then we also have \(\Delta^j \leq \Delta^i\). Hence, we have to check that \(\Delta^i \not\subseteq \Delta^j\) for \(i \neq j\), \(j = 121, 125, 129, 132, 134–146\) and 154, and for \(i\) the same
gaps plus additionally 106, 111 and 116. So we suppose that we have an embedding \( \phi : 3^{<\omega} \to 3^{<\omega} \) witnessing that \( \Delta^i \leq \Delta^j \). This must satisfy that \( \bar{\phi}[0] = [0] \), \( \bar{\phi}[1] = [1] \), and \( \bar{\phi}[2] \in \{[2], [02], [12], [012], [012], [102], [102] \} \).

Claim A: \( \bar{\phi}[12] = [12] \). Proof of Claim A: If \( \bar{\phi}[2] \in \{[2], [02], [12], [012] \} \), then it is obvious. If \( \tau = \phi[2] \in \{[0], [02], [102] \} \), consider \( s = \phi[1] \land \phi[12] \). Since \( \tau = \phi[2] \) and \( \max(\tau^i) = 0 \), \( \max(\tau^j) = 2 \), we must have that \( \max(\phi[1] \land s) = 0 \) and \( \max(\phi[12] \land s) = 2 \). Since \( \bar{\phi}[1] = [1] \), \( \phi[1] \land \phi[0] \in W_1 \). We conclude that \( \bar{\phi}[0] \cap 1 < s < \phi[12] \), and we observed that \( \max(\phi[12] \land s) = 2 \), hence \( \bar{\phi}[12] = [12] \) as required.

As a consequence of Claim A, we easily check that \( \bar{\phi}[0] = \sigma \) for all \( \sigma \in \{[12], [012], [012], [012], [102] \} \) (notice that \( [102] \in \mathcal{O}_2 \subset 2 \subset 2 \)). Thus, if \( \phi \) witnesses that \( \Delta^i \leq \Delta^j \), all these types must preserved. This covers all the cases that we must check except when \( i, j \in \{121, 129, 154 \} \) because in \( \Delta^{121}, \Delta^{129} \) and \( \Delta^{154} \) all those types are in the orthogonal. In all these cases, we have that for \( k = i, j, \Delta^k = \{[0] \} \) and \( \Delta^k = \{[2], [02] \} \) for \( k = [2], [02] \), and what we have to prove is that \( \Delta^2 = \Delta^1 \). This actually follows from Lemma 3.2.4 applied to \( \Delta^i \leq \Delta^j \) instead of \( \{0, 1\} \leq \omega \).

**Lemma 3.2.5.** If \( \epsilon \Delta^i \leq \Delta^j \) with \( j \in [105, 156] \), and \( \Delta^i, \Delta^j \) is \( \Gamma^3 \), then \( i = j \).

**Proof.** We know that \( i \in [105, 156] \), \( \Delta^i, \Delta^j \) is \( \Gamma^3 \) and \( \epsilon(2) = 2 \). Since moreover \( \Gamma^3 \) is not symmetric, \( \epsilon \) must be the identity permutation. It is enough to consider the case when \( j > 119 \) because we know by Theorem 2.5.7 that the gaps \( 105 - 119 \) are minimal, so if we had \( \Delta^i \leq \Delta^j \) with \( j \in [105, 119] \) then we also have \( \Delta^i \leq \Delta^j \). Hence, we have to check that \( \Delta^i \leq \Delta^j \) for \( i \neq j \).

For every \( i, j \in [122, 129, 154] \), we have \( \phi[0] = 0 \), \( \phi[1] = [1] \), \( \phi[2] \in \{[2], [02], [12], [012], [012], [102], [102] \} \). If \( \phi[1] = [1] \), then \( \phi \) is as in the proof of Lemma 9.2.4, so we would have again that \( \phi \sigma = \sigma \) for all \( \sigma \in \{[12], [012], [012], [012], [102] \} \). This is enough to check that \( i = j \) in all the cases that we are considering now, except for \( i, j \in [122, 130, 155] \). These cases are checked in the same way as we did in the proof of Lemma 3.2.4. We focus now on the case when \( \phi[1] = [01] \).

Claim B: \( \bar{\phi}[12] = [012] \). Proof of Claim B: This is analogous to Claim A in Lemma 3.2.4. If \( \phi[2] \in \{[2], [02], [12], [012] \} \), then it is obvious. If \( \tau = \phi[2] \in \{[0], [02], [102] \} \), consider \( s = \phi[1] \land \phi[12] \). Since \( \tau = \phi[2] \) and \( \max(\tau^i) = 0 \), \( \max(\tau^j) = 2 \), we must have that \( \max(\phi[1] \land s) = 0 \) and \( \max(\phi[12] \land s) = 2 \). Since \( \bar{\phi}[1] = [01] \), \( \phi[1] \land \phi[0] \in W_0 \). We conclude that \( \phi[0] \land w < s < \phi[12] \), with \( w \in W_0 \). We observed that \( \max(\phi[12] \land s) = 2 \), hence \( \bar{\phi}[12] = [012] \) as required.

As a consequence of Claim B, we easily check that \( \bar{\phi}[012] = \bar{\phi}[12] = [012] \), \( \bar{\phi}[0] = \bar{\phi}[02] = \bar{\phi}[102] = [012] \), \( \bar{\phi}[012] = [012] \). These facts are enough to check all the cases, except when \( i, j \in [122, 130, 155] \). In these cases, we have that for \( k = i, j, \Delta^k = \{[0] \} \) and \( \Delta^k = \{[2], [02] \} \) for \( k = [2], [02] \), and what we have to prove is that \( \Delta^2 = \Delta^1 \). This follows from Lemma 3.2.4 applied to \( \Delta^i \leq \Delta^j \) instead of \( \{0, 1\} \).

**Lemma 3.2.6.** If \( \epsilon \Delta^i \leq \Delta^j \) with \( j \in [105, 156] \), and \( \Delta^i, \Delta^j \) is \( \Gamma^1 \) or \( \Gamma^5 \), then \( i = j \).
3.2. MINIMAL ANALYTIC 3-GAPS

Proof. We know that \( \{\Delta_0^i, \Delta_1^i\} = \{\Delta_0^j, \Delta_1^j\} \) and \( \varepsilon(2) = 2 \). Since moreover neither \( \Gamma^3 \) nor \( \Gamma^4 \) is symmetric, \( \varepsilon \) must be the identity permutation. So we suppose that we have a normal embedding \( \phi : 3^{<\omega} \to 3^{<\omega} \) witnessing that \( \Delta_i^i \leq \Delta_j^j \), and \( j > 119 \) as in the previous lemmas. This must satisfy that \( \bar{\phi}[0] = [0] \), max(\( \bar{\phi}[1] \)) = 1, and \( \bar{\phi}[2] \in \{[2], [02], [02]\} \), since the gaps 120, 124, 128, 131, 153, 156 under this case contain only the types 2, [02], [02] in \( \Delta_2 \).

Claim C: The type \( v = \bar{\phi}[1] \) is a comb type with \( \max(v) = 2 \) and \( \max(v^1) \geq 1 \). Proof of Claim C: The fact that \( \max(v) = 2 \) follows from Corollary 2.2.2. The type \( v \) cannot be a chain type by Lemma 2.3.3. Now, we suppose that \( v \) is a comb type with \( \max(v) = 0 \). We can consider a set \( \{x_k : k < \omega\} \) of type \( [1^0_2] \) such that \( \{x_k^0 : k < \omega\} \) is a set of type \( [1^0_2] \). Since \( \bar{\phi}[0] = [0] \), we conclude that \( \bar{v} = \bar{\phi}[1] \) is a comb type with \( \max(\bar{v}) = 0 \) as well. This contradicts Corollary 2.3.5 since \( [1^0_2] \) dominates \( [1] \), but \( \bar{v} = \bar{\phi}[1] \) would not dominate \( \bar{\phi}[1] \) if \( \max(\bar{v}) = 0 \) and \( \max(\bar{\phi}[1]) = 1 \). This finishes the proof of Claim C.

Since \( [1^0_2] \in \mathcal{O}_2 \), Claim C above allows us to suppose that \( i > 119 \). In the rest of cases, we have that for \( k = i, j, \Delta_0^k \in \{[0], [02]\} \) or \( \Delta_0^k \) is either \( \{[2], [02]\} \) or \( \{[2], [02], [012]\} \), and what we have to prove is that \( \Delta_0^i = \Delta_0^j \). This follows from Lemma 3.2.3 applied to \( \{0, 2\}^{<\omega} \) instead of \( \{0, 1\}^{<\omega} \).

Lemma 3.2.7. If \( \varepsilon \Delta^i \leq \Delta^j \) with \( j \in [105, 156] \), and \( \{\Delta_0^i, \Delta_1^i\} = \Gamma^4 \), then \( i = j \).

Proof. We know that \( \{\Delta_0^i, \Delta_1^i\} = \{\Delta_0^j, \Delta_1^j\} \) and \( \varepsilon(2) = 2 \). But now, \( \Gamma^4 \) is symmetric, so we must consider the permutation \( \varepsilon \) that interchanges 0 and 1. The first thing that we are going to do is to compute the gap \( \varepsilon \Delta^i \) in all the relevant cases. For this it is enough to consider the embedding \( \psi : 3^{<\omega} \to 3^{<\omega} \) given by \( \psi(x) = \psi(x) \), \( \psi(x) = \psi(x) \), and \( \psi(x) = \psi(x) \). For \( i = 106, 111, 116, 121, 127 \) this embedding \( \psi \) shows that \( \varepsilon \Delta^i \leq \Delta^i \), while for \( i = 133 \), it shows that the gap \( \Delta^{-133} \) described below satisfies \( \varepsilon \Delta^{133} \leq \Delta^{-133} \leq \varepsilon \Delta^{133} \). Thus, if is enough to show that \( \Delta^i \preceq \Delta^j \) for any two different gaps from the following list:

| \( \Delta_0 + \Delta_1 \) | \( \Delta_2 \) |
|---|---|
| 106 | \( \Gamma^4 \) |
| 111 | \( \mathcal{N}_2 \) |
| 116 | \( \mathcal{O}_2 \) |
| 121 | \( \Gamma^4 \) |
| 127 | \( \{2, [02], [012], [12]\} \) |
| 133 | \( \Gamma^4 \) |
| -133 | \( \{2, [02], [012]\} \) |

As in the previous lemmas, we can suppose that \( j \notin \{106, 111, 116\} \). Thus, we have an embedding \( \phi : 3^{<\omega} \to 3^{<\omega} \) witnessing \( \Delta^i \leq \Delta^j \) with \( \bar{\phi}[0] \in \{[0], [01]\} \), \( \bar{\phi}[1] = [1] \) and \( \bar{\phi}[2] \in \{[2], [02], [012], [12]\} \). In any case, we have that \( \bar{\phi}[0] \) is \( \sigma \) for all \( \sigma \in \{[012], [12], [1^0_2]\} \) and we are done.

Now we are going to deal with the gaps 157-162. Consider now the permutation \( \varepsilon : 3 \to 3 \) such that \( \varepsilon(0) = 0, \varepsilon(1) = 2, \varepsilon(2) = 1 \), and the embedding \( \phi : 3^{<\omega} \to 3^{<\omega} \) such that \( \phi(x) = x, \phi(x) = \phi(x) \), \( \phi(x) = \phi(x) \). For every

\[ \text{For } i = 111, \phi \text{ is not a direct witness that } \varepsilon \Delta^{111} \leq \Delta^{111}, \text{ because } \bar{\phi}[0] = [1], \bar{\phi}[1] \in \mathcal{N}_2, \text{ but } \bar{\phi}[1] \notin \mathcal{N}_2. \text{ But anyway, } \phi \text{ shows that we have } \Delta \leq \varepsilon \Delta^{111} \text{ and } \Delta \text{ is like in Case } 2c \text{ of the first part of this section, because } \tau = [1^0_2] \in \Delta_2. \text{ Arguing like in Case } 2c, \text{ we get that } \Delta^{111} \leq \Delta. \]
\[ \Delta^{-i} \] be the gap such \( \phi \) witnesses \( \Delta^{-i} \leq \epsilon \Delta^i \). It is easy to check that \( \Delta^i = \Delta^{-i} \) for \( i = 158, 159, 160 \) and for the others we have the following table:

| \( i \) | \( \Delta_0 \) | \( \Delta_1 \) | \( \Delta_2 \) |
|-------|--------|--------|--------|
| 157   | 0      | 1, 01, | 12, 012| 2       |
| -157  | 0      | 1, 12  |        | 02      |
| 158   | 0      | 1, 01, | 12, 012| 2, 02   |
| 159   | 0      | 1, 12  |        | 02      |
| 160   | 0      | 1, 01, | 12, 012| 2, 02   |
| 161   | 0      | 1, 12  |        | 02      |
| -161  | 0      | 1, 01, | 12, 012| 2       |
| 162   | 0      | 1, 01, | 12, 012| 2, 02   |
| -162  | 0      | 1, 01, | 12, 012| 2       |

One can check that \( \phi \) witnesses that \( \Delta^{-i} \leq \epsilon \Delta^i \) and also \( \Delta^i \leq \epsilon \Delta^{-i} \).

**Lemma 3.2.8.** If \( i, j \in [157, 162] \) and \( \epsilon \Delta^i \leq \Delta^j \), then \( i = j \).

**Proof.** It is enough to show that if \( \Delta^k \leq \Delta^q \) for some \( k, q \) in the range \( \{157, -157, 158, 159, 160, 161, -161, 162, -162\} \), then \( k = q \). For convenience, let \( \Delta^{-k} = \Delta^k \) if \( k \in \{158, 159, 160\} \). If \( \Delta^k \leq \Delta^q \) then \( \{\Delta_0^k, \Delta_1^k\} \leq \{\Delta_0^q, \Delta_1^q\} \) and by Lemma 3.2.9 this implies that \( \Delta_1^k = \Delta_1^q \). On the other hand \( \Delta^{-k} \leq \epsilon \Delta^k \leq \epsilon \Delta^q \leq \Delta^{-q} \), so applying the same reasoning, we must also have that \( \{\Delta_0^{-k}, \Delta_2^{-k}\} \leq \{\Delta_0^{-q}, \Delta_2^{-q}\} \) and again by Lemma 3.2.9 \( \Delta_2^{-k} = \Delta_2^{-q} \). Looking at the table, these two conditions actually imply that \( k = q \) as required.

**Lemma 3.2.9.** If \( i \in [105, 156] \) and \( j \in [157, 162] \), then \( \epsilon \Delta^i \not\leq \Delta^j \).

**Proof.** As a first case, suppose that \( \{\Delta_0^i, \Delta_1^i\} = \Gamma^4 \) and we have some normal embedding \( \phi \) witnessing that \( \Delta^i \leq \epsilon \Delta^i \). Then \( \phi[0], \phi[1], \phi[2] \in \Delta^i_0 \) and moreover \( \max(\phi[0]) = \max(\phi[1]) \leq \max(\phi[2]) \) and this is contradiction, since \( \Delta^i_0 \) only contains the type \( [0] \) which has a minimal possible maximum. As a second case, \( \{\Delta_0^i, \Delta_1^i\} \neq \Gamma^4 \), so \( \Delta_0^i = \{0\} \) and we must have \( \epsilon(0) = 0 \), and \( \Delta^i \leq \Delta^k \) for some \( k \) in the range \( \{157, -157, 158, 159, 161, -161, 162, -162\} \). This implies that \( \Delta^i \leq \{\Delta_0^{-k}, \Delta_2^{-k}, \Delta_1^{-k}\} \). Let \( \psi \) be a normal embedding witnessing this. Then \( \psi[1] \in \{[2], [02], [02^2]\} \) and \( \psi[2] \in \{[12], [012], [12^2], [012^2]\} \). Since \( \{12 \not\in \Delta^1 \), it is enough to check that \( \psi[12] = \psi[1] \). Consider \( s, t \in 3^{<\omega} \) and \( (u, v), (u', v') \) rungs of type \( \psi[1] \) such that \( \psi(0) = s' \cup v \), \( \psi(1) = s' \cup u' \). Both \( \psi[1] \) and \( \psi[2] \) are types \( \tau \) with \( \max(\tau) = 2 \) and are either chain types or \( \max(\tau) = 0 \). This implies that if we call \( r = \psi(12) \wedge \psi(1) \), then \( r > s \) and \( \max(r \wedge s) = 2 \), and from this we deduce that \( \psi[12] \) is the same type as \( \psi[1] \).

**Lemma 3.2.10.** If \( i \in [1, 102] \) and \( j \in [157, 162] \), then \( \epsilon \Delta^i \not\leq \Delta^j \).

**Proof.** Suppose \( \Delta^i \leq \Delta^j \). We must have \( \epsilon(0) = 0 \) since in all cases the ideal \( \Delta_0 \) contains only \( [0] \), the type with smallest maximum. Reasoning like in Lemma 3.2.9 we must have \( \Delta^i \leq \{\Delta_0^k, \Delta_1^k, \Delta_2^k\} \) for some \( k \) living in the range \( \{157, -157, 158, 159, 161, -161, 162, -162\} \).
Let $\phi : 2^{<\omega} \rightarrow 3^{<\omega}$ be an embedding witnessing this. Then $\phi[0] = [0]$ and $\phi[1] \in \{[2],[02],[01]\}$. But this implies that $\phi_0$ contains only $0's$ and $2's$, for all types $\tau$ in $2^{<\omega}$, hence no type of $\Delta^i_2$ is of the form $\phi_0$, a contradiction. □

Remember that the gaps $103, 104$ and $163$ were already treated at the beginning of the discussion. At this point it only remains to prove that $\Delta^i \leq \varepsilon \Delta^j$ implies $i = j$ when $j \in [1, 102]$.

Lemma 3.2.11. If $i \in [1, 104]$ and $A \subset 2^{<\omega}$ is such that $\{\Delta^i_0|_A, \Delta^i_1|_A\}$ is a $2$-gap, then $\{\Delta^i_0|_A, \Delta^i_1|_A\}$ is a 2-gap.

Proof. By Theorem 1.3.5 we have a permutation $\varepsilon : 2 \rightarrow 2$ and a normal embedding $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\phi(2^{<\omega}) \subset A$, $\phi[0] \in \Delta^i_0(0)$ and $\phi[1] \in \Delta^i_1(1)$. If $i \in \{103, 104\}$, we have two possibilities. If $\varepsilon$ is the identity permutation, then $\phi[0] = [0]$ or $[01]$ and $\phi[1] = [1]$. In any case, $\phi[01] = [01]$ and $\phi[01] = [01]$ so we have at least one type with $\phi_0 \in \Delta^i_2$ and we are done. If $\varepsilon$ is the nontrivial permutation, the $\phi[0] = [1]$ and $\phi[1] = [01]$, hence $\phi[01] = [01]$ and $\phi[01] = [01]$, and again we will have some $\tau$ with $\phi_0 \in \Delta^i_2$. If $i \in \{0, 102\}$ then $\varepsilon$ must be the identity permutation, $\phi[0] = [0]$ and $\phi[1] \in \{[1],[01],[01],[01]\}$. If $\phi[1] = [1]$, then $\phi_0 = \phi_0 \tau$ for all types $\tau$, and picking any $\tau \in \Delta^i_2$ we are done. If $\phi_0 = \phi_0 \tau \in \Delta^i_1$, then similarly $\phi_0 \tau = \phi_0 \tau$ for all types $\tau \in \Delta^i_2$, and picking any $\tau \in \Delta^i_2$ we are done. Finally if $\phi_0 = \phi_0 [01] \in \Delta^i_1$, then we can pick $\tau \in \Delta^i_2$, $\tau \in \{[1],[01],[01],[01]\}$. By Lemma 2.5.5 we have that $\phi_0 = \phi_0 \tau$ and, like in the previous cases, we get a type $\tau$ for which $\phi_0 \tau \in \Delta^i_2$, and finishes the proof. □

Lemma 3.2.12. If $i \in [105, 134] \cup [157, 163]$ and $j \in [1, 104]$, then $\varepsilon \Delta^i \not\leq \Delta^j$.

Proof. It is easy to check in the tables that, for any permutation $\varepsilon$ of 3, $A = \{\varepsilon(0), \varepsilon(1)\}^{<\omega} \in (\Delta^\varepsilon(2))^{<\omega}$ but $\{\Delta^\varepsilon(0)|_A, \Delta^\varepsilon(1)|_A\}$ form a gap. In this way, we get a contradiction with Lemma 3.2.11. □

Lemma 3.2.13. If $i, j \in [1, 104]$ and $\varepsilon \Delta^i \leq \Delta^j$ then $i = j$ and $\varepsilon$ is the identity.

Proof. For every $k \in [101, 104]$ and every $\tau_p \in \Delta^k_p$, $p = 0, 1, 2$, we have that $\max(\tau_0) = 0 < 1 = \max(\tau_1) = \max(\tau_2)$. Therefore, an application of Corollary 2.2.2 yields that $\varepsilon(0) = 0$. As a first case, we suppose that $\varepsilon$ is the identity mapping, so $\Delta^i \leq \Delta^j$. Let $\phi : 2^{<\omega} \rightarrow 2^{<\omega}$ be a normal embedding witnessing this. It also witnesses that $\{\Delta^\varepsilon_0, \Delta^\varepsilon_1\} \leq \{\Delta^\varepsilon_0, \Delta^\varepsilon_1\}$ and since these 2-gaps are minimal, they must be equal. We have $\phi[0] = [0]$ and $\phi[1] \in \{[1],[01],[01],[01]\}$. If $\phi[1] = [1]$, then $\phi_0 = \phi_0 \tau$ for all types $\tau$ in $2^{<\omega}$. Picking a type $\tau$ in the symmetric difference of $\Delta^i_2$ and $\Delta^i_3$ we get a contradiction since $\phi_0 \tau = \phi_0 \tau$. If $\phi[1] = [01] \in \Delta^i_3$, then also $\phi_0 = \phi_0 \tau$ for all types $\tau$ in $2^{<\omega}$ except $[1]$. Similarly as before, picking $\tau$ in the symmetric difference of $\Delta^i_2$ and $\Delta^i_3$ we get a contradiction. If $\phi[1] = [01]$, then $j \in [95, 102]$ so we can pick $\tau$ in the symmetric difference of $\Delta^i_2$ and $\Delta^i_3$, $\tau \in \{[1],[01],[01],[01]\}$. By Lemma 2.5.5 we have that $\phi_0 = \phi_0 \tau$ and, like in the previous cases, we get a contradiction. Now we proceed with the second case, when $\varepsilon(1) = 2$ and $\varepsilon(2) = 1$. If some top-comb belongs to $\Delta^i_3$, then by Theorem 2.4.3 $\Gamma^1 \leq \{\Delta^\varepsilon_0, \Delta^\varepsilon_1\} \leq \{\Delta^\varepsilon_0, \Delta^\varepsilon_1\}$, but $\{\Delta^\varepsilon_0, \Delta^\varepsilon_1\}$ is either $\Gamma^2$, $\Gamma^3$ or $\Gamma^5$, a contradiction. So $\Delta^i_2 \subset \{[01],[01]\}$. If $[01] \in \Delta^i_2$, then $\Delta^i_1 = \{[1]\}$, and we can consider a normal embedding $\psi(\langle x \rangle_0) = \psi(\langle x \rangle_0)$ and $\psi(\langle x \rangle_1) = \psi(\langle x \rangle_0)$, which satisfies $\psi_\tau = \tau$ for all types except $\psi[1] = [01]$. Then the set $A = \psi(2^{<\omega})$ is such that $\{\Delta^\varepsilon_0|_A, \Delta^\varepsilon_2|_A\}$
form a gap, but it is orthogonal to $\Delta^i_1 = \{[1]\}$ because $[1]$ is not of the form $\bar{\psi}\tau$. This contradicts Lemma 3.2.11, since $\varepsilon\Delta^i \leq \Delta^j$. Finally, assume that $\Delta^i_2 = \{[101]\}$.

Now we can consider an embedding $\psi: 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\psi[0] = [0]$ and $\bar{\psi}[1] = [01]$, and for which, by Lemma 2.5.3, neither $[1]$ nor $[01]$ are types of the form $\psi\tau$. This contradicts Lemma 3.2.11 as before. 

This finishes the proof of the fact that if $\varepsilon\Delta^i \leq \Delta^j$, then $i = j$. This proves that the list of gaps $\Delta^1 - \Delta^{163}$ is the list of minimal analytic 3-gaps up to permutation. In order to check that all the information provided in the list is correct, and have a complete list of minimal analytic 3-gaps, it remains to count how many nonequivalent permutations each gap $\Delta^i$ have. That is, it remains to understand for which permutations $\varepsilon: 3 \rightarrow 3$ and which $i \in [1, 163]$ we have $\varepsilon\Delta^i \leq \Delta^i$. We have already analysed most of the cases along the way of the previous arguments, let us simply put all the information together:

(1) If $i \in [1, 102]$, then $\Delta^i$ is not equivalent to any of its permutations, by Lemma 3.2.13. Thus, each $\Delta^i$ has 6 nonequivalent permutations.

(2) If $i = 103, 104$, then $\Delta^i$ has 3 nonequivalent permutations. In fact, the embedding $\phi(x^0) = \phi(x)^011, \phi(x^1) = \phi(x)^101, \phi(x^2) = \phi(x)^222$, shows that $\Delta^i$ is equivalent to $\{\Delta^i_1, \Delta^i_0, \Delta^i_2\}$. On the other hand, any permutation of $\{0, 1, 2\}$ that moves number 2 will produce a nonequivalent permutation of $\Delta^i$. The reason is the following: by Lemma 3.2.11 we have that $\Delta^i|A$ is a 3-gap whenever $\{\Delta^i_0|A, \Delta^i_1|A\}$ is a 2-gap. This is not true if we make a permutation which moves 2, because $\Delta^i_1$ contains a top-com type $\tau$ with $\max(\tau^1) = 1$, so if we pick $p \in \{0, 1\}$ and $\tau \in \Delta^i_0$, by Theorem 2.2.3 we can construct a normal embedding $\psi: 2^{<\omega} \rightarrow 2^{<\omega}$ such that $\psi[0] = \sigma$ and $\psi v = \tau$ for all other types $v$. The set $A = \psi(2^\omega)$ is such that $\{\Delta^i_0|A, \Delta^i_1|A\}$ is a 2-gap, but $A \in (\Delta^i|\neg p)^1$.

(3) If $i \in [105, 156]$ and $\{\Delta^i_0, \Delta^i_1\} \neq \Gamma^4$, then $\Delta^i$ is not equivalent to any of its permutations. Hence each such $\Delta^i$ has 6 nonequivalent permutations. This is a consequence of Corollary 2.2.2 because $\max(\tau) = p$ whenever $\tau$ is a type in $\Delta^i_0$.

(4) If $i \in [104, 156], \{\Delta^i_0, \Delta^i_1\} = \Gamma^4$ and $i \neq 133$, then $\Delta^i$ has three nonequivalent permutations. This is essentially done in Lemma 3.2.7. On the one hand we have that $\max(\tau) = 2$ if $\tau \in \Delta^i_2$ but $\max(\tau) \leq 1$ if $\tau \in \Delta^i_0 \cup \Delta^i_1$, so again by Corollary 2.2.2 any permutation that moves 2 produces a nonequivalent permutation of $\Delta^i$. On the other hand, the embedding $\psi(x^0) = \psi(x)^011, \psi(x^1) = \psi(x)^101, \psi(x^2) = \psi(x)^222$, shows that $\Delta^i$ is equivalent to $\{\Delta^i_1, \Delta^i_0, \Delta^i_2\}$ if $i \neq 133$.

(5) $\Delta^{133}$ has 6 nonequivalent permutations. The same argument as for the previous case shows that a permutation that moves 2 gives nonequivalent permutation of $\Delta^{133}$. It remains to show that $\{\Delta^{133}_1, \Delta^{133}_0, \Delta^{133}_2\} \neq \Delta^{133}$. Indeed, any embedding $\psi$ witnessing this would have to satisfy $\phi[0] = [1], \phi[1] = [01]$, and $\phi[2] = [2]$. But then $\phi[12] = [012]$ which is a contradiction.
since \([12] \in \Delta^1\) but \([012] \notin \Delta^2\). The gap \(\{\Delta^1, \Delta^0, \Delta^2\}\) is equivalent to the gap \(\Delta^{-2}\) in the proof of Lemma \(\ref{lemma:gap_symmetry}\).

(6) \(\Delta^i\) has 6 nonequivalent permutations for \(i = 157, 161, 162\), while \(\Delta^i\) has 3 nonequivalent permutations for \(i = 158, 159, 160\). In any of these cases, \(\max(\tau) = 0\) for any \(\tau \in \Delta^n\), while \(\max(\tau) > 0\) if \(\tau \in \Delta^1 \cup \Delta^2\), so Corollary \(\ref{corollary:max_permutation}\) implies that any permutation of \(\Delta^i\) which moves 0 produces a nonequivalent permutation of \(\Delta^i\). On the other hand, immediately after Lemma \(\ref{lemma:gaps_embedding}\) we constructed the gaps \(\Delta^{-k}\), which are equivalent to \(\{\Delta^0, \Delta^1, \Delta^2\}\) for \(k \in [157, 162]\). Since \(\Delta^{-k} = \Delta^k\) for \(k = 158, 159, 160\), these gaps remain equivalent when permuting 1 and 2. But since we showed in the proof of Lemma \(\ref{lemma:gaps_embedding}\) that \(\Delta^k \not\leq \Delta^{-k}\) for \(k = 157, 161, 162\), we have 6 nonequivalent permutations in those cases.

(7) \(\Delta_{163}^1\) is equivalent to all of its permutations. The embedding given by \(\psi(x^p) = \psi(x)^p\epsilon(p)\) for \(p = 0, 1, 2\), shows that \(\Delta_{163}^1 \leq \epsilon(\Delta_{163}^1)\).

We may notice a relation between symmetry and strength. The more strong subgaps a gap has, the more symmetries it usually has.

### 3.3. Minimal analytic dense gaps

One may be interested in knowing the analytic dense \(n\)-gaps. The following result relates this to knowing the analytic \((n - 1)\)-gaps.

**Theorem 3.3.1.** Every minimal analytic dense \(n\)-gap \(\Delta\) is obtained by one of the following two procedures:

1. **Consider a minimal analytic \((n - 1)\)-gap** \(\Gamma\) which is not dense and extend it to a dense \(n\)-gap by making \(\Gamma^\perp\) the new preideal.

2. **Consider a minimal analytic \((n - 1)\)-gap** \(\Gamma\) in \((n - 1)^\omega\) which is dense, and extend it to a dense \(n\)-gap in \(n^\omega\) by making \(M_{n - 1}\) the new preideal.

**Proof.** We will assume that, when constructing \(\Delta\), the new preideal is \(\Delta^i\) for \(i < n\). First let us check that both procedures lead to minimal analytic dense \(n\)-gaps. For (1), if \(\Delta \leq \Delta\), then in particular \(\{\Delta_0, \ldots, \Delta_{n - 2}\} \leq \Gamma\), so since \(\Gamma\) is minimal, we get that \(\Gamma \leq \{\Delta_0, \ldots, \Delta_{n - 2}\}\), and this implies that \(\Delta \leq \Delta^i\). For (2), if \(\Delta \leq \Delta\) witnessed by some normal embedding \(\phi\), then by the minimality of \(\Gamma\), we get again that \(\Gamma \leq \{\Delta_0, \ldots, \Delta_{n - 2}\}\) witnessed by some normal embedding \(\psi\). On the other hand, by Corollary \(\ref{corollary:max_permutation}\) any type with maximal maximum must be sent by \(\phi\) to \(M_{n - 1} = \Delta^i\). In particular, we can find a type \(\tau \in \Delta^i\) that dominates all types in the range of \(\psi\). Using Lemma \(\ref{lemma:gap_symmetry}\) we find a normal embedding witnessing that \(\Delta \leq \Delta^i\). We prove now that every minimal dense \(n\)-gap is obtained by one of the two procedures. So let \(\Delta\) be a minimal analytic dense \(n\)-gap in \(n^\omega\). By making a permutation, we can suppose that \(\Delta_{n - 1}\) contains a top comb \(\tau\) with \(\max(\tau) = n - 1\). Let \(\Gamma\) be a minimal \((n - 1)\)-gap such that \(\Gamma \leq \{\Delta_0, \ldots, \Delta_{n - 2}\}\), witnessed by some normal embedding \(\psi\). Let \(\Delta\) be the gap obtained from \(\Gamma\) either by procedure 1 (if \(\Gamma\) is not dense) or by procedure 2 (if \(\Gamma\) is dense). In the first case, \(\psi\) already witnesses that \(\Delta \leq \Delta^i\), while in the second case, the type \(\tau\) allows to use Lemma \(\ref{lemma:gap_symmetry}\) to construct an embedding witnessing that \(\Delta \leq \Delta^i\). \(\square\)
3.4. Higher dimensions

In the list of minimal analytic 3-gaps one can see some general patterns that can be generalized to produce minimal analytic gaps of higher dimensions in a recursive way, but we do not figure out at the moment how we could construct the list of all minimal \( n \)-gaps for \( n > 3 \). In this section we shall make some remarks about the value of the number \( N(n) \) of minimal analytic \( n \)-gaps for large \( n \), counted up to permutation. We know the initial values \( N(2) = 5 \), \( N(3) = 163 \). First, we deal with something easier, the number \( J(n) \) of types in \( \mathbb{N}^<\omega \). We shall see in Chapters 4 that the numbers \( J(n) \) play a special role in the combinatorics of analytic multiple gaps, cf. Theorem 4.1.8. They can be computed by the following formula:

\[
J(n) = 2^n - 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{i+j-1}{j} B(i, j, n);
\]

\[
B(i, j, n) = \binom{n}{i} \binom{n}{j} - \sum_{p=0}^{\min(i,j)} \binom{n-p-1}{j-1} \binom{n-p-1}{i-1}.
\]

In the first formula above, \( 2^n - 1 \) is the number of chain types, \( B(i, j, n) \) is the number of pairs \( (\tau^0, \tau^1) \) corresponding to types where \( \tau^0 \) has \( i \) many numbers and \( \tau^1 \) has \( j \) many numbers, while the combinatorial number that multiplies \( B(i, j, n) \) is the number of possible order relations \( \triangleleft \) once \( \tau^0 \) and \( \tau^1 \) are fixed increasing sequences of \( i \) many and \( j \) many numbers. In the second formula, the first summand is the total number of pairs of increasing sequences \( (\tau^0, \tau^1) \) while in each summand of the sum on the right we are subtracting those pairs of sequences that begin with the same integer \( p \), that must be excluded. The first values of \( J(n) \) are the following:

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | \( \cdots \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( J(n) \) | 1 | 8 | 61 | 480 | 3881 | 31976 | 266981 | \( \cdots \) |

It is possible also to find some recursive formulas. But it is perhaps more interesting to analyse its asymptotic behavior, to get an approximated idea of the magnitude of the numbers. This is stated in Proposition 3.4.3, which requires some previous computations along two lemmas. We are not experts in probability, and for the proof of Lemma 3.4.1, we must acknowledge the hint provided in the web math.stackexchange.com (cf. question 353748).

Given \( n < \omega \), and \( p \in \mathbb{Z} \), let \( M_n(p) \) be the set of all \( 2 \times n \) matrices with entries \( \{-1, 0, 1\} \) such that the number of \(-1\)'s in the upper row equals \( p \) plus the number of \(-1\)'s in the lower row.

Given \( u, v \in \{1, -1\} \), let \( M_n(p)_{uv} \) be the set of all matrices from \( M_n(p) \) such that the first (leftmost) nonzero element of the upper row takes value \( u \), and the first nonzero element of the lower row takes value \( v \).

Let \( M_n^u(p)_{uv} \) be the set of all matrices in \( M_n(p)_{uv} \) such that the first nonzero element of the upper row and the first nonzero element of the lower row appear in the same column. Let \( M_n^p(p)_{uv} \) be the set of all matrices in \( M_n(p)_{uv} \) such that the first nonzero element of the upper row and the first nonzero element of the lower row appear in different columns.
LEMMA 3.4.1. For fixed $p$, as $n$ goes to infinity,

$$|M_n(p)| \sim \frac{3 \cdot 9^n}{2\sqrt{2\pi n}}$$

PROOF. We can consider our matrices as random matrices that in each entry take the value 0, 1 or $-1$ with equal probability. Let $X_i$ be the the random variable that provides the difference between number of $-1$’s in the upper row and number of $-1$’s in the lower row, but looking only at column $i$. Thus $X_i$ takes value 0 with probability $5/9$, value 1 with probability $2/9$, and value $-1$ with probability $2/9$. We have that

$$|M_n(p)| = 9^n \cdot P\left(\sum_{i=1}^{n} X_i = p\right)$$

The random variables $X_i$ are independent, equidistributed, have mean $\mu = 0$ and variance $\sigma = 2/3$. Let

$$Y_n = \frac{3}{2\sqrt{n}} \sum_{i=1}^{n} X_i$$

be their standardized sums, which converge to a normal distribution $N(0,1)$. Let $F_n = P(Y_n \leq x)$ be the distribution function of $Y_n$. The way in which the functions $F_n(x)$ converge to the Gaussian $\Phi(x) = \int_{-\infty}^{x} \frac{-t^2}{2\sqrt{\pi}} dt$ is estimated precisely by the Edgeworth expansion,

$$F_n(x) = \Phi(x) + \sum_{j=1}^{\infty} \frac{P_j(x)}{n^{j/2}}.$$

where $P_j(x)$ is a linear combination of the derivatives $\Phi^{(1)}(x), \ldots, \Phi^{(3j)}(x)$ with certain constant coefficients.

$$9^{-n} \cdot |M_n(p)| \sim P\left(Y_n = \frac{3p}{2\sqrt{n}}\right) = F_n\left(\frac{3p}{2\sqrt{n}}\right) - F_n\left(\frac{3(p-1)}{2\sqrt{n}}\right)$$

$$= \Phi\left(\frac{3p}{2\sqrt{n}}\right) - \Phi\left(\frac{3(p-1)}{2\sqrt{n}}\right) + \sum_{j=1}^{\infty} \frac{P_j\left(\frac{3(p-1)}{2\sqrt{n}}\right)}{n^{j/2}}$$

and taking into account that $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} x + o(x)$, we get

$$9^{-n} \cdot |M_n(p)| \sim \frac{3}{2\sqrt{2\pi n}} + o(n^{-1/2}) \sim \frac{3}{2\sqrt{2\pi n}}$$

as desired. \hfill \Box

LEMMA 3.4.2. For fixed $p, u, v$, as $n$ grows to infinity,

$$|M_n^\#(p)[u]| \sim |M_n^\#(p)[v]| \sim \frac{3 \cdot 9^n}{16\sqrt{2\pi n}}$$

PROOF. We are going to proof that $|M_n^\#(p)[u]| \gtrsim \frac{1}{8} M_n(p)$ and $|M_n^\#(p)[v]| \gtrsim \frac{1}{8} M_n(p)$. Since

$$M_n(p) = |M_n^\#(p)[0]| + |M_n^\#(p)[1]| + |M_n^\#(p)[0]| + |M_n^\#(p)[1]| + |M_n^\#(p)[0]| + |M_n^\#(p)[1]| + |M_n^\#(p)[0]| + |M_n^\#(p)[1]|$$

3.4. HIGHER DIMENSIONS 93
we conclude that $|M^m_n(p)^{[u]}| \sim \frac{1}{8} M_n(p)$ and $|M^m_n(p)^{[v]}| \sim \frac{1}{8} M_n(p)$, and the statement of the lemma will follow from Lemma 3.4.4. One corollary of Lemma 3.4.4 that we are going to use in the sequel is that for any fixed $p$ and $q$, $M_{n-k}(q) \sim 9^{-k} M_n(p)$. We start proving that $|M^m_n(p)^{[u]}| \gtrsim \frac{1}{8} M_n(p)$. If we fix $m < n$, we can get a lower bound by counting only those matrices whose first nonzero elements appear in one of the first columns, and we get

$$|M^m_n(p)^{[u]}| \geq \sum_{k=1}^{m} |M_{n-k}(p')|$$

where $p'$ equals either $p, p+1$ or $p-1$, depending on the values of $u$ and $v$. Asymptotically,

$$|M^m_n(p)^{[u]}| \gtrsim \sum_{k=1}^{m} 9^{-k} |M_n(p)|$$

Since this happens for every $m$, and $\sum_{k=1}^{\infty} 9^{-k} = 8^{-1}$, we get that

$$|M^m_n(p)^{[u]}| \gtrsim \frac{1}{8} M_n(p)$$

as desired. We check now that $|M^m_n(p)^{[u]}| \gtrsim \frac{1}{8} M_n(p)$. Given $k, s \geq 1$ with $s+k < n$, the number of matrices in $M^m_n(p)^{[u]}$ whose first nonzero element of the upper row appears at column $s$, and the first nonzero element of the lower row appears at column $s+k$ can be computed as

$$\sum_{\xi \in 3^k} M_{n-s-k}(p\xi)$$

where $\xi$ runs over all possible entries in the upper row between $s+1$ and $s+k$, and $p\xi$ is a number which depends only on $\xi$. Similarly, the number of matrices in $M^m_n(p)^{[u]}$ whose first nonzero element of the lower row appears at column $s$, and the first nonzero element of the upper row appears at column $s+k$ can be computed as

$$\sum_{\xi \in 3^k} M_{n-s-k}(p'\xi)$$

In this way, if we fix $m$, we can estimate for $n > 2m,$

$$|M^m_n(p)^{[u]}| \geq \sum_{k=1}^{m} \sum_{s=1}^{m} \sum_{\xi \in 3^k} M_{n-s-k}(p\xi) + M_{n-s-k}(p'\xi)$$

$$\sim \sum_{k=1}^{m} \sum_{s=1}^{m} \sum_{\xi \in 3^k} \frac{2M_n(p)}{9^{s+k}} \sim 2M_n(p) \left( \sum_{k=1}^{m} \sum_{s=1}^{m} \frac{3^k}{9^s} \right)$$

$$\sim 2M_n(p) \left( \frac{m}{3^e} \right) \left( \frac{m}{9^s} \right)$$

Since this happens for every $m$, and $\sum_{k=1}^{\infty} 3^{-k} = 2^{-1}$ and $\sum_{s=1}^{\infty} 9^{-s} = 8^{-1}$, we get that $|M^m_n(p)^{[u]}| \gtrsim \frac{1}{8} M_n(p)$ as desired.

**Proposition 3.4.3.**

$$J(n) \sim \frac{3}{8\sqrt{2\pi n}} \cdot 9^n$$
3.4. HIGHER DIMENSIONS

Proof. To each type $\tau$ in $n^{<\omega}$ we associate a $2 \times n$ matrix $(a_{ij}^\tau)$ with entries in $\{-1, 0, 1\}$. This time it is convenient to enumerate the rows of the matrix by the indices $i = 0, 1$, and the columns by the indices $j = 0, \ldots, n-1$. The matrix is defined as follows:

- $a_{ij}^\tau = 0$ if $j \notin \tau^i$
- $a_{ij}^\tau = 1$ if $j \in \tau^i$ and the immediate predecessor of $(j, i)$ in $\triangleleft$ is of the form $(k, i)$
- $a_{ij}^\tau = -1$ if $j \in \tau^i$ and the immediate predecessor of $(j, i)$ in $\triangleleft$ is of the form $(k, 1-i)$, or there is no immediate predecessor.

It is a simple exercise to check that

$$\{(a_{ij}^\tau) : \tau \text{ is a comb type in } n^{<\omega}\} = \mathcal{M}^n_{n^0} \cup \mathcal{M}^n_{n^{-1}}$$

The cardinality of the set of chain types in $n^{<\omega}$ is $2^n - 1$, so

$$J(n) = |\mathcal{M}^n_{n^0}| + |\mathcal{M}^n_{n^{-1}}| + 2^n - 1$$

and using Lemma 3.4.2 the proof is over.

Concerning the number $N(n)$ of minimal analytic $n$-gaps counted up to permutation, Theorem 1.3.5 provides an upper bound, the number of standard $n$-gaps,

$$N(n) < n^{J(n)-n}$$

To get an idea how rough this is, for $n = 3$ we have that $N(3) = 163$, while $3^{J(3)-3} = 3^{98}$. In order to get a lower bound, we look at the list of minimal analytic $3$-gaps and we observe that the largest parametrized family is the one formed by the gaps 1-64. This family can be generalized to any dimension in the following way: For any nonempty set of types $A$ in $(n-1)^{<\omega}$ which is disjoint from $\{[0], [1], \ldots, [n-2]\}$, we can consider the $n$-gap $\Delta^A = \{\Gamma_{S_i} : i < n\}$ where $S_i = \{[i]\}$ for $i < n-1$ and $S_{n-1} = A$. By Theorem 2.6.3 these are minimal $n$-gaps and none of them is equivalent to a permutation of another. There are $2^{J(n-1)-n-1} > 2^{J(n-1)/2}$ such gaps $\Delta^A$, so

$$2^{J(n-1)-n-1} < N(n) < n^{J(n)-n} < n^{J(n)}$$

hence,

$$J(n-1) \cdot \log(2) \lesssim \log(N(n)) \lesssim J(n) \cdot \log(n)$$

The number $N^*(n)$ of minimal analytic $n$-gaps counted with permutations satisfies $N^*(n) < n!N(n) < n^n N(n)$, therefore $\log(N^*(n)) < \log(N(n)) + n \log(n)$. Since we know that $9^{n-1}/16 \lesssim J(n-1) \lesssim \log(N(n))$, this means that $\log(N^*(n)) \sim \log(N(n))$. Better approximations should require a better understanding of minimal analytic gaps.
CHAPTER 4

Applications

4.1. Breaking analytic gaps and the topology of \( \omega^* \)

Let \( \Gamma = \{ \Gamma_i : i \in n \} \) be a multiple gap on the countable set \( N \), and \( B \subset n \). We will say that \( \Gamma \) can be \( B \)-broken if there exists an infinite subset \( M \subset N \) such that \( \{ \Gamma_i|_M : i \in B \} \) is a gap, while \( M \in \Gamma_i^\perp \) for \( i \notin B \). In [3] several classes of gaps like jigsaws and clovers are introduced according to the ways that they can be broken. In particular, a clover is an \( n \)-gap which cannot be \( B \)-broken for any proper nonempty subset of \( n \). Examples of clovers are constructed in [3] but it was left as an open problem whether analytic clovers exist. We shall show in this section that the answer to this question is negative. Indeed we shall show that for every analytic \( n \)-gap there is always a large family of subsets \( B \) such that the gap can be \( B \)-broken. Yet, we did not find an optimal result in this direction, and we leave as an open question to characterize the families of subsets \( B \) of \( n \) such that an analytic \( n \)-gap can be found that can be \( B \)-broken by and only by sets \( B \in B \).

The phenomenon of breaking gaps has a nice topological interpretation through Stone duality. Remember from the introduction that to each open subset \( V \) of \( \omega^* = \beta \omega \setminus \omega \) we associate the ideal \( I(V) = \{ a \subset \omega : \pi \setminus a \subset V \} \). We say that \( V \) is analytic if so is \( I(V) \).

**Lemma 4.1.1.** Let \( \{ U_i : i < n \} \) be open subsets of \( \omega^* \).

1. The ideals \( \{ I(U_i) : i < n \} \) are pairwise orthogonal if and only if the open sets \( \{ U_i : i < n \} \) are pairwise disjoint.
2. The ideals \( \{ I(U_i) : i < n \} \) are an \( n \)-gap if and only if the open sets \( \{ U_i : i < n \} \) are pairwise disjoint and \( \bigcap_{i \in n} U_i \neq \emptyset \).
3. For \( B \subset n \) the \( n \)-gap \( \{ I(U_i) : i < n \} \) can be \( B \)-broken if and only if \( \bigcap_{i \in B} U_i \setminus \bigcup_{i \notin B} U_i \neq \emptyset \).

**Proof.** Part (1) is clear. Part (2) was stated without explicit proof in [3]. One implication is obvious: if the ideals are separated, then we can find clopen sets \( V_i \supset U_i \) such that \( \bigcap_{i \in n} V_i = \emptyset \), hence \( \bigcap_{i \in n} \overline{U_i} = \emptyset \). The converse follows from [3] Lemma 9 applied to \( L = \omega^*, \pi = \{ n \} \) and \( B \) be the family of all clopen subsets of \( \omega^* \). For part (3), if the gap can be \( B \)-broken, then we have a set \( M \in \bigcap_{i \notin B} I(U_i)^\perp \) such that \( \{ I(U_i)|_M : i \in B \} \) is a gap. But this means that \( U_i \cap M = \emptyset \) for \( i \notin B \) and \( M \cap \bigcap_{i \in B} U_i \neq \emptyset \). Since \( M \setminus \omega \) is a clopen subset of \( \omega^* \),

\[
\bigcap_{i \in B} \overline{U_i} \setminus \bigcup_{i \notin B} \overline{U_i} \supset M \cap \bigcap_{i \in B} \overline{U_i} \neq \emptyset.
\]

Conversely, if \( \bigcap_{i \in B} \overline{U_i} \setminus \bigcup_{i \notin B} \overline{U_i} \neq \emptyset \) we can pick a point \( x \) in this set, and a clopen neighborhood \( V \) of \( x \) which is disjoint from \( \bigcup_{i \notin B} \overline{U_i} \). This clopen set \( V \) must be
of the form $V = \overline{M} \setminus \omega$ for some $M \subset \omega$, and this set $M$ witnesses that the gap $\{I(U_i) : i < n\}$ can be $B$-broken.

The following is direct consequence of Theorem 1.3.5 and Theorem 2.1.3

**Lemma 4.1.2.** Let $\{S_i : i < \omega\}$ be nonempty pairwise disjoint sets of types in $m^{\leq \omega}$, consider the $n$-gap $\Gamma = \{\Gamma_{S_i} : i < n\}$, and let $B \subset n$. The gap $\Gamma$ can be $B$-broken if and only if there exists some normal embedding $\phi : k^{\leq \omega} \rightarrow m^{\leq \omega}$ such that the range of $\phi$ intersects each $S_i$ with $i \in B$, but it is disjoint from $S_i$ for $i \notin B$.

**Theorem 4.1.3.** For every analytic $n$-gap $\Gamma$, $n \geq 3$, there exist $i \neq j$ such that $\Gamma$ can be $\{i, j\}$-broken.

**Proof.** We can suppose that $\Gamma = \{\Gamma_i = \Gamma_{S_i} : i < n\}$ is a standard gap on $n^{\leq \omega}$ such that $[i] \in S_i$. Moreover, we can restrict to $2^{\leq \omega} \subset m^{\leq \omega}$ where at least $\{\Gamma_0, \Gamma_1\}$ form a gap, so we suppose that $\Gamma$ is a standard gap on $2^{\leq \omega}$. Moreover, by taking a further normal embedding we can suppose that $\{\Gamma_0, \Gamma_1\}$ is a minimal $2$-gap with $[0] \in S_0$, $[1] \in S_1$, so $\{\Gamma_0, \Gamma_1\}$ is one of the $\Gamma^1, \ldots, \Gamma^5$ in the list of minimal $2$-gaps in Section 3.1. If it is $\Gamma^1$ we are done, as we would have found that $\Gamma$ is $\{0, 1\}$-separated. So we suppose that $\{\Gamma_0, \Gamma_1\}$ is one of the gaps $\Gamma^2, \ldots, \Gamma^5$. The types $[0], [1], [0^1], [0^2]$ do not appear in any ideal of the gaps $\Gamma^2, \ldots, \Gamma^5$, hence they belong neither to $\Gamma_0$ nor to $\Gamma_1$. If one of those types $\tau$ belongs to $\Gamma_i$ for $i > 1$, then since such a $\tau$ is a top-comb that dominates $[0]$, by Lemma 2.4.3 we can get a normal embedding $\phi : 2^{\leq \omega} \rightarrow 2^{\leq \omega}$ such that $\phi[0] = [0]$ and $\phi\chi = \tau$ for all types $\chi \neq [0]$. Thus $\Gamma$ would be $\{0, i\}$-broken. Thus, we can suppose that all the types $[1], [0^1], [0^2]$ belong to $\Gamma^\perp$, so the types of $2^{\leq \omega}$ are distributed as follows:

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $\Gamma_0$ | $\Gamma_1$ | $\Gamma^\perp$ | $\Gamma^\perp$ | $\Gamma^\perp$ | $\Gamma^\perp$ |
| $[0]$ | $[1]$ | $[0^1]$ | $[0^2]$ | $[0^1]$ | $[0^2]$ |

Let us consider the normal embedding $\varphi : 2^{\leq \omega} \rightarrow 2^{\leq \omega}$ given by $\varphi([\emptyset]) = [\emptyset]$, $\varphi([s]) = \varphi([s])^{-1}$, $\varphi([0]) = [0]$, $\varphi([s^{-1} 1]) = \varphi([s])^{-1}$. In the following table, the second row indicates how $\varphi$ acts on each of the types of $2^{\leq \omega}$ and the third row indicates how the distribution of ideals will be after composing with $\varphi$.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $[0]$ | $[1]$ | $[0^1]$ | $[0^2]$ | $[0^1]$ | $[0^2]$ |
| $[0^1]$ | $[0^1]$ | $[0^1]$ | $[0^1]$ | $[0^1]$ | $[0^1]$ |

Thus, if $[0^1]$ was originally in some $\Gamma_i$ with $i \neq 1$ (in the above tables, if the symbol $?a$ stands for some $\Gamma_i$, $i \neq 1$), then by composing with $\phi$ we would show that $\Gamma$ was $\{1, i\}$-broken. Thus, we suppose that $[0^1]$ goes either to $\Gamma_1$ or to $\Gamma^\perp$. We can suppose moreover that $[1^0]$ goes to some $\Gamma_i$ for some $i > 1$ (in the tables above, this means that the symbol $?b$ would stand for $\Gamma_i$, $i > 1$) because otherwise all types would go to $\Gamma_0$ or $\Gamma_1$ (both attained) or $\Gamma^\perp$ and we would get that $\Gamma$ can be $\{0, 1\}$-broken. So finally we have the following picture:

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| $\Gamma_0$ | $\Gamma_1$ | $\Gamma_0$ or $\Gamma^\perp$ | $\Gamma^\perp$ | $\Gamma^\perp$ | $\Gamma^\perp$ |

Finally, by Theorem 2.2.1 there exists an embedding $\psi : 2^{\leq \omega} \rightarrow 2^{\leq \omega}$ such that $\psi[0] = [1^0]$, $\psi[1] = [1]$. By Corollary 2.2.2 we have that $\max(\psi \tau) = 1$, 

hence $\bar{\psi} \tau \neq [0]$, for all $\tau$. Thus, the composition with $\psi$ shows that $\Gamma$ can be $\{1,i\}$-broken.

\textbf{Corollary 4.1.4}. Let $\mathcal{U}$ be a finite family of pairwise disjoint analytic open subsets of $\omega^*$, then

(1) either their closures $\{\overline{U} : U \in \mathcal{U}\}$ are also pairwise disjoint,
(2) or there exists a point $x \in \omega^*$ such that $\{|U \in \mathcal{U} : x \in \overline{U}\}| = 2$.

\textbf{Proof}. Suppose that (1) does not hold, and consider $\{U_i : i \in n\}$ a maximal subfamily of $\mathcal{U}$ with $\bigcap_{i \in n} U_i \neq \emptyset$. We can find a clopen set $V \supset \bigcap_{i \in n} U_i$ such that $V \cap \overline{U} = \emptyset$ for all $U \in \mathcal{U}$, $U \neq U_i$, $i < n$. The result follows from Lemma 1.1.1 and Theorem 4.1.3 applied to the gap $\{I(U_i \cap V) : i < n\}$. \hfill $\Box$

\textbf{Corollary 4.1.5}. No analytic clovers exist.

We can improve the result when $n = 3$ to show that there are indeed two pairs $\{i,j\}$ for which a 3-gap can be $\{i,j\}$-broken.

\textbf{Theorem 4.1.6}. For every 3-gap $\Delta$ there exists a permutation $\varepsilon : 3 \to 3$ such that $\Delta$ can be $\{\varepsilon(0),\varepsilon(1)\}$-broken and $\{\varepsilon(0),\varepsilon(2)\}$-broken. In topological terms, if $U_0, U_1, U_2$ are pairwise disjoint analytic open subsets of $\omega^*$ such that $U_0 \cap U_1 \cap U_2 \neq \emptyset$, then at least two of the following three sets are nonempty:

$$(\overline{U_0} \cap \overline{U_1}) \setminus U_2, \quad (\overline{U_0} \cap \overline{U_2}) \setminus U_1, \quad (\overline{U_1} \cap \overline{U_2}) \setminus U_0.$$ 

\textbf{Proof}. We can suppose that $\Delta = \Delta^i$ is one of the 163 minimal 3-gaps of our list. The gaps 105-163 can be $\{0,1\}$, $\{0,2\}$ and $\{1,2\}$-broken simply by restricting to $\{0,1\}^{\omega}$, $\{0,2\}^{\omega}$ and $\{1,2\}^{\omega}$. If $i \in [1,104]$ and $\Delta^i$ contains a top-comb with $\max(\omega^i) = 1$ then Theorem 2.4.4 applies to show that $\Delta^i$ can be $\{1,2\}$ and $\{0,2\}$-broken. This covers in particular the case when $i \in [103,104]$. In the other cases we have $\Delta^i_0 = \{[0]\}$, and for these any normal embedding $\phi : 2^{\omega} \to 2^{\omega}$ such that $\phi[0] \in \Delta^i_0$ and $\phi[1] \in \Delta^i_2$ witnesses that $\Delta^i$ is $\{1,2\}$-broken. It remains to show that $\Delta^i$ is $\{0,2\}$-broken for $i \in [1,102]$ and $\Delta^i \subset \{[01],[1^i01],[0^i1]\}$. If $[0^i1] \in \Delta^i_2$ then $[0^i1]$ dominates $[0]$ so Theorem 2.4.4 implies that $\Delta^i$ is $\{0,2\}$-broken. If $[0^i1] \in \Delta^i_2$ then $\Delta^i_1 = \{[1]\}$ and the embedding $\psi : 2^{\omega} \to 2^{\omega}$ given by $\psi(x) = \bar{x}$, $\psi(x^\omega) = \psi(x)^\omega$ show that $\Delta^i$ is $\{0,2\}$-broken. So suppose finally that $\Delta^i_2 = \{[1], [0^i1]\}$, so that $\Delta^i_1 \subset \{[1], [0^i1]\}$. Then, consider a normal embedding $\psi : 2^{\omega} \to 2^{\omega}$ such that $\phi[0] = [0]$ and $\phi[1] = [0^i1]$. Then, using Lemma 2.5.3 we check that $\psi$ witnesses that $\Delta^i$ is $\{0,2\}$-broken. \hfill $\Box$

We notice that Lemma 3.2.11 implies that the minimal 3-gaps $\Delta^i$ for $i \in [1,104]$ cannot be $\{0,1\}$-broken.

\textbf{Theorem 4.1.7}. For every dense analytic $n$-gap $\Gamma$ there exists $k \in n$ such that $\Gamma$ can be $(B \cup \{k\})$-broken whenever it can be $B$-broken. In particular, $\Gamma$ can be $(i,k)$-broken for all $i \in n$.

In topological terms, whenever $\{U_i : i < n\}$ are pairwise disjoint analytic open subsets of $\omega^*$ such that $\bigcap_{i<n} U_i \neq \emptyset$ and $\bigcup_{i<n} U_i = \omega^*$, then there exists $k < n$ such that for all $B \subset n$

$$\bigcap_{i \in B} U_i \setminus \bigcup_{i \notin B} U_i \neq \emptyset \Rightarrow \bigcap_{i \in B \cup \{k\}} U_i \setminus \bigcup_{i \notin B \cup \{k\}} U_i \neq \emptyset.$$
PROOF. Suppose that $\Gamma$ is a standard $n$-gap in $n^{<\omega}$ and let $k$ be such that $\Gamma_k$ contains a top-comb type $\tau$ with $\max(\tau^1) = n - 1$. The theorem follows from application of Lemma 2.4.3.

For a natural number $n$, remember that $J(n)$ denotes the number of types in $n^{<\omega}$, whose asymptotic behaviour was computed in Proposition 3.4.3

$$J(n) \sim \frac{3}{8\sqrt{2\pi n}} \cdot 9^n.$$

**Theorem 4.1.8.** For every analytic $n$-gap $\Gamma$ and for every set $A \subset n$ with $|A| = k$ there exists $B \supset A$ with $|B| \leq J(k)$ such that $\Gamma$ can be $B$-broken. The number $J(k)$ is optimal in this result.

**Proof.** Let $\phi : k^{<\omega} \to m^{<\omega}$ be a normal embedding witnessing that we have a standard $k$-gap $\Gamma' \leq \{\Gamma_i : i \in A\}$. Since there are only $J(k)$ types in $k^{<\omega}$, at most $J(k)$ ideals from $\Gamma$ can be present after this reduction, so this proves the statement of the theorem. Concerning optimality, let $\{\tau_i : i \in J(k)\}$ be an enumeration of all types in $k^{<\omega}$ where $\tau_i = [i]$ for $i < k$. Consider the $J(k)$-gap $\Gamma = \{\Gamma_{\tau_i} : i \in J(k)\}$ in $k^{<\omega}$. We claim that if $\Gamma$ can be $B$-broken for some $B \supset A = \{i : i < k\}$, then $B$ is the whole $J(k)$. Suppose that we have a normal embedding $\phi : p^{<\omega} \to k^{<\omega}$ witnessing that $\Gamma$ is $B$-broken. Then since $A \subset B$, there must exist types $\sigma_i$ for $i < k$ such that $\phi \sigma_i = [i]$. Since $\max[i] < \max[j]$ when $i < j$, it follows from Corollary 2.2.2 that $\max(\sigma_i) < \max(\sigma_j)$ when $i < j$. By Theorem 2.2.1, we get a normal embedding $\psi : k^{<\omega} \to p^{<\omega}$ such that $\psi[i] = \sigma_i$ for $i < k$. Then, $\phi \psi[i] = [i]$ for $i < k$ and this implies that $\phi \psi \tau = \tau$ for all types $\tau$ in $k^{<\omega}$, hence $B$ must be the whole set $J(k)$. □

If the gap $\Gamma$ in Theorem 4.1.8 is a strong gap, then the bound can be improved to $k^2$ instead of $J(k)$. The argument would be completely analogous but we would use the theory of analytic strong gaps from [4], where the number of possible “strong types” in $k^{<\omega}$ is reduced to $k^2$ (they are called $(i,j)$-combs for $i, j < k$).

**Corollary 4.1.9.** Let $\mathcal{U}$ be a countable family of pairwise disjoint analytic open subsets of $\omega^*$, and let $\{U_i : i \in k\} \subset \mathcal{U}$ be a finite subfamily with $\bigcap_{i<k} U_i \neq \emptyset$. Then, there exists a point $x \in \bigcap_{i<k} U_i$ such that $\{|U \in \mathcal{U} : x \notin U\} \leq J(k)$. Moreover, $J(k)$ is optimal in this result.

**Proof.** First, consider the case when $\mathcal{U}$ is finite, and we write it in the way $\mathcal{U} = \{U_i : i \in n\}$, for some $k \leq n < \omega$. Pick $y \in \bigcap_{i<k} U_i$, and let us suppose as well that we have $k \leq m \leq n$ such that $y \in \bigcap_{i<m} U_i$ but $y \notin \bigcup_{m \leq i < n} U_i$. Let $C$ be a clopen subset of $\omega^*$ such that $y \in C$ and $C \cap \bigcup_{m \leq i < n} U_i = \emptyset$. Since $C$ is a clopen subset of $\omega^*$, it is of the form $C = \bar{e} \setminus \omega$ for some infinite set $c \subset \omega$. By Theorem 4.1.3, we can find $B \supset A = k$ with $|B| \leq J(k)$ such that the gap $\{I(U_i \cap C) : i < m\}$ can be $B$-broken. This implies, by Lemma 4.1.1 that

$$\bigcap_{i \in B} \overline{U_i \cap C} \setminus \bigcap_{i \in m \setminus B} \overline{U_i \cap C} \neq \emptyset.$$

Any point $x$ in the intersection above satisfies that $\{i \in n : x \notin U_i\} \subset B$, so it is as required.

Now, suppose that $\mathcal{U}$ is infinite, and write it as $\mathcal{U} = \{U_i : i < \omega\}$. We define inductively a decreasing sequence of clopen subsets $C_n$ of $\omega^*$ and points $x_n \in
Then, we can choose the finite case proved above, we pick \( x_{n+1} \in C_n \cap \bigcap_{i<k} \overline{U_i} \). Using the finite case proved above, we pick \( x_{n+1} \in C_n \cap \bigcap_{i<k} \overline{U_i} \) (this set is nonempty, since we had \( x_n \) from the previous step) such that \(|\{i < n+1 : x_{n+1} \in \overline{U_i}\}| \leq J(k)\). Then, we can choose \( C_{n+1} \subseteq C_n \) so that \( x_{n+1} \in C_{n+1} \) and \( C_{n+1} \) is disjoint from all \( \overline{U_i} \) such that \( x_{n+1} \notin \overline{U_i} \) and \( i < n+2 \). Let \( x \) be a cluster point of the sequence \( \{x_n : n < \omega\} \). On the one hand, \( x \in \bigcap_{i<k} \overline{U_i} \) since all \( x_n \) belong to that intersection. Suppose for contradiction that \(|\{i < \omega : x \in \overline{U_i}\}| > J(k)\). Find \( n \) such that \(|\{i < n : x \in \overline{U_i}\}| > J(k)\). But the construction of our sequence was done in such a way that \(|\{i < n : y \in \overline{U_i}\}| \leq J(k)\) for all \( y \in C_n \), and \( x \in C_n \) because \( x_m \in C_n \) for all \( m \geq n \). This is a contradiction.

For the optimality, consider the gap on \( \omega \) that witnessed optimality in Theorem 4.1.3. That is, we have \( \Gamma = \{\Gamma_i : i < J(k)\} \) which cannot be \( B \)-broken for any \( B \supset k \), \( B \neq J(k) \). Consider the open sets \( U_i = \bigcup_{a \in \Gamma_i} a \setminus \omega \). These open sets satisfy that \( I(U_i) \) is the ideal generated by \( \Gamma_i \). Hence, the families \( \{I(U_i) : i < J(k)\} \) and \( \{\Gamma_i : i < J(k)\} \) have the same separation properties: they are gaps which cannot be \( B \)-broken whenever \( k \subset B \subset J(k) \), \( B \neq J(k) \). By Lemma 4.1.1, this means that the family \( \{U_i : i < J(k)\} \) witnesses optimality.

### 4.2. Selective coideals

Given an ideal \( I \) of subsets of a set \( N \), let us denote by \( I^+ = \{x \subset N : x \notin I\} \) the corresponding coideal. For analytic ideals, the following two notions are equivalent, cf. [29] Theorem 7.53:

- \( I^+ \) is bisequential; that is: for every ultrafilter \( U \supset I^+ \) there exists a sequence \( \{a_n : n < \omega\} \) of elements of \( U \) such that every \( a \in I \) there exists \( n < \omega \) such that \( a \cap a_n = \emptyset \),

- \( I^+ \) is selective; that is, it has the following two properties, cf. [29], Lemma 7.4:
  1. For every decreasing sequence \( \{a_n : n < \omega\} \subset I^+ \) there exists \( b \in I^+ \) such that \( b \subset a_n \) for all \( n \),
  2. For every \( a \in I^+ \) and every partition \( a = \bigcup_{k<\omega} F_k \) into finite sets there exists \( b \in I^+ \), such that \( b \subset a \) and \(|b \cap F_k| \leq 1 \) for all \( k < \omega \).

For convenience, we shall say that \( I \) is coselective if \( I^+ \) is selective. Mathias [24] proved that selective coideals have the Ramsey property, cf. [29] Corollary 7.23. We shall only need the following two particular consequences of Mathias’ result:

**Theorem 4.2.1.** If \( I^+ \) is selective, then for every \( A \in I^+ \) and every coloring \( c : [A]^2 \to 2 \) of the pairs of \( A \), there exists an infinite set \( B \subset A \), \( B \in I^+ \) such that \( c|_{[B]^2} \) is constant.

**Theorem 4.2.2.** If \( I \) is coselective and analytic, then \( I^+ \) contains an infinite set.

A particular example of coselective analytic ideal is the ideal of subsets of \( 2^{< \omega} \) generated by the branches. Finding this particular example inside a given ideal was the main point in the characterization of non-\( G_4 \) points of Rosenthal compacta given in [28]. We present here a general approach to this idea by looking at how coselective ideals can look like in relation to types.

**Lemma 4.2.3.** Let \( I \) be a coselective ideal on \( \kappa^{< \omega} \) such that for every type \( \tau \), either all sets of type \( \tau \) belong to \( I \) or no set of type \( \tau \) belongs to \( I \). Let \( S \) be the set
of types whose sets belong to \( \mathcal{I} \). If \( S \) contains a top-comb type \( \tau \), then \( S \) contains all types \( \sigma \) with \( \max(\sigma) \leq \max(\tau) \).

**Proof.** Let us suppose for contradiction, that \( \mathcal{I} \) contains all sets of type \( \tau \) but no set of type \( \sigma \), where \( \sigma \) is top-comb and \( \max(\sigma) \leq \max(\tau) \). Fix \( \{t_1, t_2, \ldots\} \) a set of type \( \tau \). For every \( n \), consider

\[
G_n = \{ t \in 2^{<\omega} : \exists r \in 2^{<\omega} : t = t_n \upharpoonright r, \max(r) \leq \max(\tau) \},
\]

\[
F_n = \bigcup_{m>n} G_m.
\]

The set \( F_n \) is not in \( \mathcal{I} \) since it contains sets of type \( \sigma \). Since \( \mathcal{I}^+ \) is a selective coideal, there exists a set \( F \in \mathcal{I}^+ \) such that \( F \subset^* F_n \) for every \( n \). We have that \( F \cap G_n \) is finite, so again, by the selectivity of \( \mathcal{I}^+ \) we can suppose that \( |F \cap G_n| = 1 \) for all \( n \). So \( F = \{s_1, s_2, \ldots\} \notin \mathcal{I} \) with \( s_n \geq t_n \) for all \( n \). If \( \{s_1, s_2, \ldots\} \) was of type \( \tau \) then we would have arrived to a contradiction. Since \( \{t_1, t_2, \ldots\} \) was of type \( \tau \), and \( \tau \) is a top-comb, the only thing that we need to ensure that \( \{s_1, s_2, \ldots\} \) is also of type \( \tau \) is that \( |s_n| < |s_m \wedge s_{m+1}| \) for all \( m \). Color every pair of elements of \( F \) according to whether \( |s_n| < |s_m \wedge s_{m+1}| \) or not. Since \( \mathcal{I}^+ \) is a selective coideal, by Theorem 4.2.4 there exists infinite subset \( F' \subset F \) such that \( F' \notin \text{mathcal{I}}^+ \) and all pairs in \( F' \) have the same color. This \( F' \) is a set of type \( \tau \) and we have arrived to a contradiction.

**Theorem 4.2.4.** Let \( \mathcal{I} \) be an analytic coselective ideal which is not countably generated. Then, there exists a one-to-one function \( u : 2^{<\omega} \to \omega \) such that

1. \( u(x) \in \mathcal{I} \) for every chain \( x \subset 2^{<\omega} \).
2. \( u(x) \in \mathcal{I}^+ \) for every set \( x \subset 2^{<\omega} \) that does not contain any infinite chain.

**Proof.** By Theorem 4.2.2 \( \mathcal{I}^{\perp \perp} = \mathcal{I} \). The hypotheses of Theorem 1.3.1 are satisfied for \( A = \mathcal{I} \) and \( B = \mathcal{I}^+ \), hence we can find an injective function \( \bar{v} : 2^{<\omega} \to \omega \) such that \( \bar{v}(x) \in \mathcal{I} \) whenever \( x \subset 2^{<\omega} \) is a \([1]\)-chain, and \( \bar{v}(x) \in \mathcal{I}^+ \) whenever \( x \subset 2^{<\omega} \) is a \([0]\)-chain. By Theorem 1.1.3 we can suppose that for each type \( \tau \) in \( 2^{<\omega} \), either \( \bar{v}(x) \in \mathcal{I} \) for all \( x \) of type \( \tau \), or \( \bar{v}(x) \in \mathcal{I}^+ \) for all sets \( x \) of type \( \tau \). Since \( \nu \)-images of \([0]\)-chains belong to \( \mathcal{I}^{\perp \perp} \), by Lemma 4.2.3 if \( \tau \) is a top-comb type, then \( \bar{v}(x) \in \mathcal{I}^+ \) for all sets \( x \) of type \( \tau \). Let \( \alpha : 2^{<\omega} \to 2^{<\omega} \) be given by

\[
\alpha(r_0, \ldots, r_n) = (1, r_0, 1, r_1, 1, r_2, \ldots, 1, r_n)
\]

and let \( u : 2^{<\omega} \to \omega \) be given by \( u(r) = \bar{v}(\alpha(r)) \). If \( x \subset 2^{<\omega} \) is a chain, then \( \alpha(x) \) is a \([1]\)-chain, and hence \( u(x) = \bar{v}(\alpha(x)) \in \mathcal{I} \). If \( x \) is a \((0,1)\)-comb, then \( \alpha(x) \) is a set of type \( [0,1] \), and if \( x \) is a \((1,0)\)-comb is a set of type \( [1,0] \). In both cases these are top-comb types, so \( \bar{v}(\alpha(x)) \in \mathcal{I}^+ \). If \( x \) does not contain any infinite chain, then every infinite subset of \( x \) contains either a \((0,1)\)-comb or a \((1,0)\)-comb, hence we conclude that \( u(x) = \bar{v}(\alpha(x)) \in \mathcal{I}^+ \).

**Corollary 4.2.5.** Let \( \mathcal{A} \) be an uncountable analytic almost disjoint family on \( \omega \). Then, there exists an injective function \( u : 2^{<\omega} \to \omega \) such that

1. \( u(x) \) is orthogonal to \( \mathcal{A} \) for every antichain \( x \subset 2^{<\omega} \),
2. for every branch \( x \subset 2^{<\omega} \) there exists an element \( a_x \in \mathcal{A} \) such that \( u(x) \subset a_x \), and \( a_x \neq a_y \) for \( x \neq y \).
4.3. SEQUENCES IN BANACH SPACES

Proof. As it was shown by Mathias [24], the ideal I generated by the almost
disjoint family is coselective, see also [29, Example 7.1.2]. Therefore we can apply
Theorem 1.2.3. Property (1) is satisfied immediately, and on the other hand we get
that for every branch x, u(x) is contained in finitely many elements from A. We can
color the [1]-chains of $2^{<\omega}$ into two colors according to whether $u(x)$ is contained
in one element of A or not. Since A is analytic, we can apply Theorem 1.1.5 and
we can suppose, by passing to subtree, that all [1]-chains have the same color. This
actually means that each [1]-chain is contained in one element of A, since the other
possibility cannot happen. By composing with the function $\alpha : 2^{<\omega} \to 2^{<\omega}$ given
by
$$\alpha(r_0, \ldots, r_n) = (1, r_0, 1, r_1, 1, r_2, \ldots, 1, r_n)$$
which transforms every branch into a [1]-chain, we get a new function u and now
for every branch x there exists $a_x \in A$ such that $u(x) \subseteq a_x$. Notice that by property
(1) that we already ensured, $\{x : a_x = a\}$ is finite for every $a \in A$. Now, color
the pairs $\{x, y\}$ of $2^{<\omega}$ into two colors according to whether $a_x = a_y$ or $a_x \neq a_y$.
This coloring is measurable since A is analytic, so by the perfect set theorem for measurable colorings, cf. [29, Corollary 6.47], there exists a perfect set $P \subseteq 2^{<\omega}$ such
that all pairs $\{x, y\} \subseteq P$ have the same color. This means that $a_x \neq a_y$ whenever
$x \neq y$. Restricting to the subtree induced by P we are done. \[\square\]

Every analytic almost disjoint family gives rise to separable Rosenthal compact
space, whose underlying set is $\omega \cup A \cup \{\infty\}$, the points of $\omega$ being isolated, the basic
neighborhoods of $a \in A$ are $x \cup \{a\} \subset \omega$, $x \subset a$, while the point $\infty$ is added as a
one point compactification of the rest of the space. The point $\infty$ is non-$G_\delta$, and
using this fact, Corollary 1.2.5 can be deduced from known facts concerning non
$G_\delta$-points in Rosenthal compacta, as explained at the beginning of this section, cf.
[28], [29, Section 6.7].

4.3. Sequences in Banach spaces

One natural field of application of this theory would be the study of sequences
of vectors in Banach spaces [21]. Many constructions in Banach space theory have
the form of a basis for which certain subsequences have some property while other
subsequences have a different property. The James tree space [15] might be mentioned
as an example of this, which has a basis $\{e_1\}_{t \in 2^{<\omega}}$ indicated in the dyadic
tree so that branches weakly converge to elements of the double dual, while antichains are equivalent to $\ell_2$-basis. Given a basic sequence $\{e_k\}_{k \in \omega}$, the family of all
$A \subset \omega$ such that $\{e_k\}_{k \in A}$ is equivalent to the $\ell_2$-basis is a Borel family, while the
family of all $A$ such that $\{e_k\}_{k \in A}$ weakly converges to some $x^{**} \in X^{**} \setminus X$ can
be check to be coanalytic. In our language, whenever we will have a basic sequence
where this two classes of subsequences are mixed, we will have a projective 2-gap.
The basis of the James tree space exhibits one of the canonical gaps that arise in our
theory: chain versus antichain, the minimal analytic dense strong 2-gap number
3 in [4, Theorem 22]. Our theory puts this kind of constructions in a context. On
the one hand, it shows that the use of trees is not by chance, as we are proving that
trees provide the canonical place where to mix different classes of subsequences.
On the other hand, our theory also indicates that Banach space theorist have not
been exhaustive in considering all possibilities of creating bases in this way. First of
all, there have been no attempts at mixing more than two classes of subsequences, and even when only two classes have been considered, the patterns used have been either that of a countable sum, corresponding to the minimal analytic dense 2-gap $\Gamma^1$ (like in the basis of $\ell_2(c_0)$ viewed as a mixture of $\ell_2$-sequences and $c_0$-sequences) or the pattern of chains versus antichains as above. But the fact is that we can use any minimal analytic $n$-gap, or in general any $n$-gap given by types, to produce basic sequences where we mix $n$-classes of subsequences according to the gap chosen. The properties of such basic sequences, depending on the minimal gap and on the classes considered ($c_0$-sequences, $\ell_p$-sequences, weakly Cauchy sequences converging to $X^{**}\setminus X$, etc.) are to be explored. Our theory also implies restrictions on the ways that analytic (or even projective) classes of subsequences can be mixed. For example, Theorem 4.1.6 implies that if we have a basic sequence where $c_0$, $\ell_1$ and $\ell_2$ are mixed, it either contains an $\ell_1$-free sequence where $c_0$ and $\ell_2$-sequences are mixed, or it contains an $\ell_2$-free sequence where $c_0$ and $\ell_1$-sequences are mixed. The consequences that this kind of facts might have on the structure of Banach spaces is another field to be explored.

Let us consider a concrete elementary case as an illustration. Remember that, for $p \in [1, \infty)$, a sequence of vectors $x = \{x_k\}_{k<\omega}$ in a Banach space are said to be an $\ell_p$-sequence if there exists a rational number $L > 0$ such that

$$\frac{1}{L} \left( \sum_{i=1}^{m} a_i^p \right)^{1/p} \leq \left\| \sum_{i=1}^{m} a_i x_i \right\| \leq L \left( \sum_{i=1}^{m} a_i^p \right)^{1/p}$$

for all rational numbers $a_1, \ldots, a_m$. It is obvious from this definition that such a class of $\ell_p$-subsequences of a given sequence is Borel. Now, the following proposition says that all minimal analytic $n$-gaps, indeed all gaps that can be expressed in terms of types, can be represented inside a sequence of vectors in a Banach space as the $\ell_p$-subsequences for different $p$’s, and therefore the theory of how we can mix different $\ell_p$-subsequences inside a a given sequence is as complicated as the general theory of analytic $n$-gaps.

**Proposition 4.3.1.** Let $\{S_i : i < n\}$ be pairwise disjoint nonempty sets of types in $m^{<\omega}$, and let $\{p_i : i < n\}$ be numbers with $1 \leq p_i < \infty$. Then there exists a sequence $\{x_k : k < m^{<\omega}\}$ such that $\{x_k : k \in X\}$ is an $\ell_{p_i}$-sequence whenever $X$ is a set of type $\tau \in S_i$.

**Proof.** Let $\{x_k : k \in m^{<\omega}\}$ be the canonical basis of the completion of $c_0(m^{<\omega})$ (the set of all functions $m^{<\omega} \rightarrow \mathbb{R}$ which vanish out of a finite set) endowed with the norm

$$\|f\| = \sup \left\{ \left( \sum_{k<\omega} |f(s_k)|^{p_i} \right)^{1/p_i} : \{s_0, s_1, \ldots\} \text{ is of type } \tau \in S_i \right\}$$

Just take into account that the intersection of two sets of different types has cardinality at most 2. \(\square\)

Remember the language that we used in the introduction: We say that $\ell_{p_1}, \ldots, \ell_{p_n}$-sequences are separated inside the sequence $\{x_k : k < \omega\}$ if we can write $\omega = \bigcup_{i < n} a_i$ in such a way that $\{x_k : k \in a_i\}$ contains no infinite $\ell_{p_i}$-subsequence. If

---

1. Identify $m^{<\omega}$ with $\omega$ by the $\prec$-increasing enumeration
non separated, we say that $\ell_{p_1}, \ldots, \ell_{p_n}$-sequences are mixed, and this corresponds to the families $a \subseteq \omega$ such that $\{x_k : k \in a\}$ is an $\ell_{p_k}$-sequence forming an $n$-gap. There is of course no point in writing here again all the results of the paper in an equivalent form in terms of mixed sequences in Banach spaces. But we can mention a few facts, just to get a flavor. We can say, just in words, that there are 9 canonical forms of mixing the basis of $\ell_1$ and the basis of $\ell_2$ into a single sequence, corresponding to the 9 minimal analytic 2-gaps described in Section 3.1. The mixtures corresponding to the minimal 2-gap $\Gamma^1$ of Section 3.1 and to its permutation would be the familiar canonical bases of $\ell_2(\ell_1)$ and $\ell_1(\ell_2)$. Similarly, there are 933 canonical ways of mixing the bases of $\ell_1$, $\ell_2$ and $\ell_3$. The minimal analytic 3-gap $\Delta^1$ in Section 3.2 where $\Delta^0_1 \equiv \{[0]\}$, $\Delta^1_1 \equiv \{[1]\}$ and $\Delta^1_2$ takes all other types in $3^{<\omega}$, provides an example of a basic sequence where $\ell_1$, $\ell_2$ and $\ell_3$ subsequences are mixed, and such that every subsequence in which $\ell_1$ and $\ell_2$ subsequences are mixed must contain an $\ell_3$-subsequence. But Theorem 4.1.6 implies that it is impossible that such a property holds at the same time for $\ell_1$, $\ell_2$ and $\ell_3$.

More precisely:

**Proposition 4.3.2.** Let $\{x_k\}_{k<\omega}$ be a sequence of vectors in a Banach space in which $\ell_1$, $\ell_2$ and $\ell_3$-subsequences are mixed. Then, at least two of the following three statements hold true:

1. There exists a subsequence $\{x_{k_m}\}_{m<\omega}$ with no further subsequences equivalent to $\ell_1$, but where $\ell_2$ and $\ell_3$ are still mixed.

2. There exists a subsequence $\{x_{k_m}\}_{m<\omega}$ with no further subsequences equivalent to $\ell_2$, but where $\ell_1$ and $\ell_3$ are still mixed.

3. There exists a subsequence $\{x_{k_m}\}_{m<\omega}$ with no further subsequences equivalent to $\ell_3$, but where $\ell_1$ and $\ell_2$ are still mixed.
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Index

(i,j)-comb, 60
A^<\omega, 11
I, 26
I", 26
J(n), 92
N(n), 95
\Sigma(\Lambda, B, \psi), 68
T, 12
T^j, 12
T_i, 12
W_k, 12
W^+_i, 13
\Gamma \leq', \Delta, 26
\Gamma \leq'' \Delta, 26
\Gamma \leq \Delta, 10
\Gamma \mid \bar{M}, 9
\Gamma^M, 57
\Gamma^N, 57
\Gamma^O, 57
\Gamma^+, 10
\Gamma_S, 21
\langle F \rangle, 12
\mathcal{U}x \mathcal{P}(x), 13
\mathcal{M}, 38
\mathcal{M}^A, 38
\mathcal{M}^B, 38
\mathcal{N}_m, 56
\mathcal{O}_m, 57
\mathcal{U}^{-\mathcal{V}}, 13
\mathcal{U}_\Lambda-tree, 14
\mathcal{T}_k, 30
\tau(\tau), 45
p(\tau), 46
s(\tau), 46
w(\tau), 47
j(\tau), 47
\max(\tau), 31
\tau \ast \sigma, 39
i-chain, 60
r \setminus t, 12
t < s, 11
t \prec s, 11
t^{-} s, 11
x \subset^* y, x^{-} y, 9
analytic, 9
analytic gap, 9
analytic open set, 97
broken, 97
chain, 17, 60
closed set, 12
clover, 97, 99
collapse, 41
color, 13
coloring, 13
comb, 17, 60
concatenation, 11, 13
countably separated gap, 60
dense, 10
dominate, 41
equivalent minimal analytic n*-gaps, 10
equivalent sets, 12
gap, \mathcal{X}-gap, 9
gap,n-gap, 9
gap,n*-gap, 9
idempotent, 13
minimal analytic n-gap, 10
minimal analytic n*-gap, 10
minimal analytic dense n-gap, 91
minimal idempotent, 13
nice embedding, 12
nice subtree, 13
normal embedding, 30
orthogonal, 9, 10
preideal, 9
Projective determinacy, 16
rapidly increasing, 12
restriction, 9
separated family, 9
set of type \( \tau \), 17
standargap, 22
strong gap, 60
subdominate, 45
Suslin-measurable, 13

top\(^2\)-comb, 47
top-comb, 39
type, 17
