Unified theory of double-time quantum transport and quantum diffusion

P. Kleinert
Paul-Drude-Institut für Festkörperlektronik, Hausvogteiplatz 5-7, 10117 Berlin, Germany
E-mail: kl@pdi-berlin.de

Abstract. Based on the nonequilibrium Green function technique, a unified theory of quantum transport and quantum diffusion is developed from quantum-kinetic equations of the full carrier distribution function \( f(<k, \kappa|T, t>) \). The rigorous two-time approach is equally applicable to quantum transport as well as quantum diffusion that proceeds either via extended states or by hopping between localized carriers. The approach focuses on the entanglement between the microscopic and macroscopic time regimes and on the related non-Markovian evolution in both time channels. For a one-dimensional superlattice model subject to quantizing dc and ac electric fields, both phononless transport and diffusion mechanisms are identified that have a strict double-time origin. The experimental verification of this intricate quantum effect seems to be feasible by studying quantum-box superlattices at low temperatures.

1. Introduction
Diffusion implying spreading of particles, of probabilities, or of information is a paradigm, which is widely used in physical, biological, geological, and social sciences. The widespread application of the diffusion equation is caused by fundamental physical and mathematical findings dating back to the early studies by Fourier and Laplace about two hundred years ago [1]. At that time, the physical approach focused on the heat propagation through a solid. Fourier derived a parabolic partial differential equation in time and three spatial coordinates, which became later known as the diffusion equation. In a completely other field, namely the theory of probability, the notion of diffusion played also an important role. Laplace proved the central limit theorem, which states that the sum of independent and identically distributed random variables asymptotically approaches the normal distribution, which is likewise a solution of the diffusion equation. In 1905, both facets of diffusion, namely its physical and probabilistic aspects, were put together by Einstein in his famous molecular kinetic theory. By combining the microscopic, stochastic diffusion of particles with the phenomenological description of diffusion on the macroscale, he was able to determine Avogadro’s number from the phenomenological diffusion parameter. This ingenious achievement provided further strong evidence for the molecular structure of matter.

Another related topic appeared about four decades later in solid state physics, namely hot electron transport that is theoretically accounted for by the kinetic Boltzmann equation [2]. An extension of this semiclassical theory was soon anticipated, when Martin and Schwinger communicated in their pioneering work the theory of nonequilibrium quantum statistics [3]. Especially on the basis of the Keldysh nonequilibrium Green function approach, the theory of high-field quantum transport has been developed to a high level of sophistication and has
been reviewed not only in many papers but also in textbooks. Compared with these activities, there are only few contributions to the related theory of quantum diffusion in solids. One reason for this imbalance could be the famous Einstein relation, which is a consequence of the steady-state fluctuation theorem (for a review, see Ref. [4]) that relates the diffusion coefficient to the mobility. Consequently, near thermal equilibrium, an independent measurement of the diffusion coefficient does not add any information that is not already available from the mobility. However, under nonequilibrium conditions, quantum diffusion unfolds its own independence that is mirrored in a respective general quantum-mechanical description. Indeed, there are research fields, in which quantum diffusion is a central notion. We mention the study of atomic migration in crystalline solids, when tunneling becomes essential. Examples encompass tunneling in quantum crystals such as the diffusion of He$^3$ in solid He$^4$, hydrogen subbarrier motion in insulators and metals, the tunneling motion of atoms in insulating glasses, and the kinetics of heavy electrons.

To the best of the authors knowledge, there seems to be no systematic study of quantum diffusion of carriers under nonequilibrium conditions in semiconductor structures. This disparity is strange in view of many applications that found the notion of quantum diffusion, for instance, in the theory of weak localization, in the field of spintronics, or in device simulations based on drift-diffusion equations. It is therefore the aim of this paper to reduce the gap between the compelling achievements in the field of nonlinear quantum transport on the one hand and quantum diffusion of carriers in semiconductors on the other hand. The unified approach covers nonequilibrium quantum transport and quantum diffusion on the same footing and can profit thereby from well established methods developed in the nonequilibrium quantum transport theory.

The treatment is based on the above mentioned Einstein methodology to mutually interchange the descriptions via physical and stochastic diffusion. Accordingly, the analysis starts from the Fokker-Planck equation for the conditional probability $P$ to find a particle at the space-time point $\mathbf{r}$, $t$, provided it occupied the lattice site $\mathbf{r}_0$ at an earlier time $t = 0$. From the Laplace transformed equation for the probability of propagation along the $z$ axis

$$sP(\mathbf{r} - \mathbf{r}_0 \mid s) = \delta(\mathbf{r} - \mathbf{r}_0) + v_z(s) \frac{\partial}{\partial z} P(\mathbf{r} - \mathbf{r}_0 \mid s) + D_{zz}(s) \frac{\partial^2}{\partial z^2} P(\mathbf{r} - \mathbf{r}_0 \mid s),$$

(1)

the transport coefficients namely the drift velocity $v_z(s)$ and the diffusion coefficient $D_{zz}(s)$ can be expressed by the moments of the probability propagator $P$, which is therefore the main quantity of the approach. The pure stochastic Kolmogorov forward Eq. (1) is transferred to a physical picture for quantum transport by a reinterpretation of the conditional probability $P$ in terms of a vacuum expectation value [5]

$$P(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4|s) = \frac{1}{Z} \int_0^\infty dt e^{-st} \text{Tr}_{\text{ph}} \left\{ \langle 0|\psi(\mathbf{r}_2) e^{iHt/\hbar} \psi^\dagger(\mathbf{r}_4) \psi(\mathbf{r}_3) e^{-iHt/\hbar} \psi^\dagger(\mathbf{r}_1)|0 \rangle \right\},$$

(2)

in which Fermionic creation $\psi^\dagger(\mathbf{r})$ and annihilation $\psi(\mathbf{r})$ operators appear for carriers at the lattice site $\mathbf{r}$. An average over the heat-bath Hamiltonian $H_{\text{ph}}$ is carried out. $Z$ denotes the partition function and $H$ the total Hamiltonian including the contribution of the electric field. After this physical reinterpretation of the stochastic approach, a more general two-particle correlation function $P$ appears that satisfies a Bethe-Salpeter equation, from which all needed information about quantum transport as well as quantum diffusion are obtained within a unified approach. The most familiar result, which can be derived by this straightforward exact procedure, is the quantum-kinetic Boltzmann equation with a field-dependent scattering term that gives rise to intra-collisional field effects [6, 7]. It is worth mentioning that this unified point...
of view elucidates in turn general features of the well established transport theory. The main conclusion of this type refers to the principal equivalence of representations that are designed to account for transport via localized and extended states. This equivalence of both different descriptions holds also for the quantum diffusion of carriers in semiconductors. The ability to cover quantum transport and quantum diffusion both via extended and localized states by using principally equivalent pictures is a general character of the unified theory of transport coefficients [8].

Extensions of the unified approach sketched above are possible in many directions. A quantum extension, which we will treat in the next Section, and which goes completely beyond the semiclassical approximation, takes into account the double-time character of the diffusion propagator. It is shown that interesting quantum effects are due to this generic two-time dependence of nonequilibrium one-particle Green functions.

2. Double-time approach

The consideration of transport coefficients that account for double-time effects could likewise start from a Bethe-Salpeter equation for the general four-point probability propagator $P$, which depends on four time variables. However, a simplified derivation is possible and profits from a procedure that is in a sense opposite to the one-time approach because it starts from double-time Green functions with the related distribution functions and provides expressions for transport coefficients by generalizing the one-time results. From the analysis of the unified one-time approach, it becomes evident that in general the current density is expressed by the time derivative of the dipole operator. In order to calculate this observable within a quantum-statistical framework, exactly two independent two-point nonequilibrium Green functions are needed. From a physical point of view, most attractive are the statistical and spectral components $G^{\Sigma}(k, \kappa |T, t)$ of the Green matrix, which count loosely speaking the available states as well as their occupancy. For simplicity, let us treat a non-degenerate electron gas in a semiconductor as their occupancy. For simplicity, let us treat a non-degenerate electron gas in a semiconductor.

$$\begin{align*}
\left[ i\hbar \frac{\partial}{\partial t} - \varepsilon(k) + ieE(t)\nabla_k \right] G^{\Sigma}(k t | k' t') &= \pm \hbar \int dk_1 \left\{ \int_{t'}^{t} dt_1 \Sigma^{\Sigma}(k t | k_1 t_1) G^{\Sigma}(k_1 t_1 | k' t') \\
+ \int_{-\infty}^{t'} dt_1 \Sigma^{\Sigma}(k t | k_1 t_1) G^{\Sigma}(k_1 t_1 | k' t') - \int_{-\infty}^{t} dt_1 \Sigma^{\Sigma}(k t | k_1 t_1) G^{\Sigma}(k_1 t_1 | k' t') \right\},
\end{align*}$$

(3)

in which the self-energies $\Sigma^{\Sigma}$ encompass elastic and inelastic scattering on impurities and phonons. To illustrate the general approach, we treat the self-consistent Born approximation with a spatially nonlocal, but temporally local coupling. The kinetic energy of carriers is denoted by $\varepsilon(k)$, and the applied electric field $E(t)$ is expressed by the vector potential $A(t)$ via the equation $dA(t)/dt = eE(t)/\hbar$. To make the approach more transparent, center of mass and relative coordinates are introduced according to the shift of variables $k \rightarrow k + \kappa/2$, $k' \rightarrow k - \kappa/2$ and $t \rightarrow T - t/2$, $t' \rightarrow T + t/2$. In the new representation, the $\kappa$ and $t$ dependence do not appear, when the model system is homogeneous with respect to space and time, respectively. The derivation proceeds by separating out a phase factor and by introducing the nonequilibrium carrier distribution function $f^{<}(k, \kappa|T, t)$ from the equation

$$G^{<}(k, \kappa|T, t) = G^{<}(k, \kappa + A(T, t)|T, t) = if^{<}(k \pm A(T, t)/2, \kappa|T, t)$$

(4)
be replaced by the carrier distribution function dipole operator, the current density is expressed by the Keldysh Green Function \(G\) valid for \(t\) in which an effective drift velocity appears that depends on both time scales as well as both \(W\) current density is obtained \(G\) the Dyson Eq. (3) for \(f\) only the particular distribution function of “excess carriers” is monitored. Indeed, the diffusion of an initial \(\delta\) the system. Such a dependence usually occurs in the treatment of diffusion, when the spreading \(\kappa\) appearance of the second wavevector \(k\) within the unified approach, a similar dependence on both quasi-momenta \(k\), \(\kappa\) also for the field-mediated drift velocity or current density. This more general definition of the carrier density requires for its theoretical description the quantity \(\nabla\)\(\kappa\) \(\nabla\kappa f_<^<(k,\kappa|T,t)\) at \(\kappa=0\), which refers to a homogeneous system. Within the unified approach, a similar dependence on both quasi-momenta \(k\) and \(\kappa\) appears also for the field-mediated drift velocity or current density. This more general definition of the current density is necessary to simultaneously account for transport via localized and extended electronic states.

Exploiting the quantum-kinetic equation for the quantity \(f_<^<(k,\kappa|T,t)\), which is derived from the Dyson Eq. (3) for \(G_<\) by taking into account Eq. (4), another equivalent expression for the current density is obtained

\[
j(T) = ien \sum_k \nabla \kappa \frac{\partial}{\partial T} f_<^<(k,\kappa|T,t) \bigg|_{\kappa=0},
\]

(5)

where \(n\) denotes the carrier density. What is surprising and unexpected in Eq. (5) is the appearance of a second wavevector \(\kappa = k - k'\), which refers to a possible inhomogeneity of the system. Such a dependence usually occurs in the treatment of diffusion, when the spreading of ”excess carriers” is monitored. Indeed, the diffusion of an initial \(\delta\)-like inhomogeneity in the carrier density requires for its theoretical description the quantity \(\nabla \kappa f_<^<(k,\kappa|T,t)\) \(n=0\) and not only the particular distribution function \(f_<^<(k,\kappa = 0|T,t)\) that refers to a homogeneous system. Within the unified approach, a similar dependence on both quasi-momenta \(k\) and \(\kappa\) appears also for the field-mediated drift velocity or current density. This more general definition of the current density is necessary to simultaneously account for transport via localized and extended electronic states.

in which an effective drift velocity appears that depends on both time scales as well as both wave vectors \(k\) and \(\kappa\)

\[
v_{\text{eff}}(k) = v(k) + i \int_{-\infty}^{\infty} dt_1 \frac{f_<^<(k|T,-t_1)}{f_<^<(k|T,0)} \sum_{k'} W_1(k,k'|T,t_1).
\]

(7)

The vectorial scattering contribution \(W_1(k,k'|T,t_1)\) is given by \(\nabla \kappa W(k,k',\kappa|T,t)|_{\kappa=0}\), with \(W\) denoting the electric field dependent scattering probability [10]. The second contribution on the right-hand side of Eq. (7) vanishes, whenever the interaction Hamiltonian commutes with the dipole operator. Fortunately, this happens, for instance, for the Fröhlich electron-phonon coupling and the Coulomb interaction. However, this condition is not always satisfied. A famous counterexample is the transport of small polarons, where the \(W_1\) contribution to the current even dominates. Again, it is noted that in general not only the distribution functions \(f_<^<(k,\kappa = 0|T,t)\) enter the approach, but in addition also the \(\kappa\) gradient of the full function \(f_<^<(k,\kappa|T,t)\) at \(\kappa = 0\). The \(\kappa\) dependence reappears in Eq. (7) via the vector field \(W_1\). Without the second term on the right-hand side of Eq. (7), there is no current, when the states do not exhibit any dispersion \(v(k) = \nabla \kappa \epsilon(k)/\hbar = 0\). Consequently, the theory cannot cope with hopping transport in this case. It is just the \(W_1\) contribution in Eq. (7), which is needed for the exact derivation of an equivalent representation that is designed for the treatment of hopping [8].
A second feature of Eqs. (6) and (7) consists in its double-time nature. Even the distribution function \( f^<(k|T,0) \), which enters Eq. (6), depends effectively on both time scales \( T \) and \( t \). This fact is due to the non-Markovian time behavior of the quantum-kinetic equations [10], which are obtained from the Dyson Eq. (3). The mixing of the "macroscopic" \( (T) \) and "microscopic" \( (t) \) time channels is a generic property of nonequilibrium dynamics. In the conventional understanding, there is only a one-dimensional time flow from the past to the future. However, the more fundamental quantum-statistical point of view reveals an entangled time hierarchy that constitutes in the background, mediated by a second time variable, the evolution of real macroscopic objects. The entanglement of these histories is avoided by adopting the generalized Kadanoff-Baym ansatz [11, 12, 13], by which the dependence on the time parameter \( t \) is effectively disregarded [9].

To give a specific result for the quantum-kinetic equation, let us treat a non-degenerate electron gas in the self-consistent Born approximation. Assuming that \( t \) is effectively disregarded [9]. To give a specific result for the quantum-kinetic equation, let us treat a non-degenerate electron gas in the self-consistent Born approximation. Assuming that the slow time scale is inert against fast microscopic fluctuations \((T+\Delta t \rightarrow T)\), the double-time "Boltzmann" equation has the form

\[
\left[ \frac{\partial}{\partial t} + \frac{i}{\hbar} \left( \varepsilon(k + \kappa) - \varepsilon(k - \kappa) \right) - \frac{\partial A(T,t)}{\partial t} \nabla_k + \frac{\partial A(T,t)}{\partial T} \nabla_\kappa \right] f^<(k, \kappa|T,t) = \sum_{k_1} \int_{-\infty}^{\infty} dt_1 f^<(k_1, \kappa|T,t-t_1) W(k_1, k, \kappa|T,t,t_1),
\]

where the scattering probability \( W \) includes scattering-in and scattering-out contributions. This equation, which reduces to the well-known results in the one-time approximation [6], reminds on the gauge-invariant quantum-kinetic equation, in which memory effects due to collisions are taken into account [14, 15].

The diffusion coefficient is obtained by a similar double-time approach, which is also based on the method of moments applied to the Fokker-Planck equation, by which both transport coefficients are defined. By collecting both the regular and the irregular part of the diffusion tensor \( \hat{D} \), we obtain

\[
\hat{D}(T) = \sum_k \left\{ v(k) \otimes \varphi(k|T,0) + i \int_{-\infty}^{\infty} dt_1 \varphi(k|T,-t_1) \otimes \sum_{k'} W_1(k, k'|0, t_1) \right\}
- \frac{1}{2} \sum_k \int_{-\infty}^{\infty} dt_1 f^<(k|T,-t_1) \sum_{k'} \tilde{W}_2(k, k'|0, t_1),
\]

which has a similar structure as Eq. (6). There is an additional contribution to \( \hat{D} \), which is due to the second derivative with respect to \( \kappa \): \( \tilde{W}_2 = \nabla_{\kappa_0} W_{\kappa=0} \). The first term on the right-hand side of Eq. (9) mainly differs from the result for the current density in Eqs. (6) and (7) by the appearance of the vector field \( \varphi(k|T,t) \), which takes over the role played by the distribution function \( f^< \) in Eq. (6). This quantity satisfies the sum rule

\[
\sum_k \varphi(k|T,t) = 0,
\]

and is related to the gradient of the distribution function \( g(k|T,t) = i \nabla_\kappa f^<(k, \kappa|T,t)|_{\kappa=0} \) via the equation

\[
\varphi(k|s,t) = g(k|s,t) - \frac{1}{s} f^<(k|s,t) \sum_{k'} v_{\text{eff}}(k'|s,t) f^<(k'|s,t).
\]
The distinct character of quantum diffusion under nonequilibrium conditions is reflected by this quantity that satisfies its own kinetic equation. For the sake of comparison, let us treat the same model that leads to the kinetic Eq. (8). The corresponding double-time ”quantum Boltzmann” equation for the vector field $\varphi$ has the form

$$
\left[ \frac{\partial}{\partial T} - \frac{\partial A(T, t)}{\partial t} \nabla_k \right] \varphi(k|T, t) = \sum_{k_1} \int_{-\infty}^{\infty} dt_1 \varphi(k_1|T, t-t_1) W(k_1, k|T, t, t_1) $$

$$+ v(k) f^<(k|T, t) - \sum_{k_1} v(k_1) f^<(k_1|T, t) \right) (12)

+ i \sum_{k_1} \int_{-\infty}^{\infty} dt_1 f^<(k_1|T, t-t_1) \left[ W_1(k_1, k|T, t, t_1) - \sum_{k_2} W_1(k_1, k_2|T, t, t_1) \right].
$$

Again, it is concluded that there is no closed equation for the one-time function $\varphi(k|T, t = 0)$, which underlines the strict double-time character of the approach. In addition, Eqs. (9) to (12) are sufficiently general so that they are likewise applicable to quantum diffusion via extended and localized electronic states. Furthermore, all remarks concerning the entangled time evolution as well as the $\kappa$ dependence that apply to the current density are also valid for Eqs. (9) to (12), which describe double-time quantum diffusion of carriers in semiconductors.

As mentioned, in the hopping regime and in one-time approximation, both the drift velocity and the diffusion coefficient vanish in the absence of any inelastic scattering. However, taking into account the double-time character of the quantum statistics, a phononless transport mechanism can be identified in the regime of Wannier-Stark localization. This is shown in the next Section.

3. Application: phononless transport

The double-time dependence of kinetic coefficients has to be taken into account, for instance, in ultra-short-time laser spectroscopy of semiconducting specimens [16, 17], in the consideration of aging phenomena as well as for the treatment of the nonexponential relaxation in glassy systems that break time translation symmetry [18, 19, 20].

In this Section, a robust high-field, steady-state transport mechanism in a semiconductor superlattice is studied that strictly originates from the double-time dependence of the correlation functions. The model is a one-dimensional, weakly-coupled semiconductor superlattice with the dispersion relation

$$
\varepsilon(k) = \frac{\Delta}{2} (1 - \cos(kd)), \quad (13)
$$

where $\Delta$ denotes the miniband width and $d$ the lattice constant of the superlattice. Along the superlattice axis, sufficiently strong dc and ac electric fields are applied

$$
E(t) = E_{dc} + E_{ac} \cos(\omega_{ac} t), \quad (14)
$$

so that Wannier-Stark localization occurs due to Bloch oscillations ($\Omega_{dc,ac} \tau \gg 1$, with $\tau$ being an effective scattering time and $\Omega_{dc,ac} = eE_{dc,ac} d/\hbar$ are the Bloch frequencies). Both the periodicity in the time domain and the Stark-ladder structure of the energy spectrum are accounted for by the discrete Fourier representation

$$
f^\Xi(k|T, t) = \sum_{l,m=-\infty}^{\infty} f_1^\Xi(m, t)e^{i(kd + lm\omega_{ac} T)}, \quad (15)
$$
which is used to solve the quantum-kinetic equations for \( f^< \) by taking into account elastic scattering on white-noise potential fluctuations. According to Eq. (5), the current density is calculated from the equation

\[
j = en \Delta d \frac{1}{2 \hbar^2} \left[ f^<_{i=0} (m = 0, t = 0) - f^<_{i=1} (m = 0, t = 0) \right]. \tag{16}
\]

Under the condition of strong quantizing electric fields, Wannier-Stark localization takes place, and the quantum-kinetic equation for the Fourier coefficients \( f^<_i (T, 0) \) reduces to the form

\[
\left\{ \frac{\partial}{\partial T} + i l \Omega_{dc} + il \Omega_{ac} \cos(\omega_{ac} T) \right\} f^<_i (T, 0) = U \sum_{k, q} e^{-ilkd} \int_0^\infty dt_1 \left[ \Phi_{0t_1} (k, q) - \Phi_{0t_1}^* (k, q) \right] \\
\times \left\{ f^>_0 (T - \frac{t_1}{2}, -t_1) f^>_0 (T - \frac{t_1}{2}, t_1) - f^<_0 (T - \frac{t_1}{2}, t_1) f^<_0 (T - \frac{t_1}{2}, -t_1) \right\},
\]

where \( U \) denotes the strength of the short-range impurity potential. The phase factor \( \Phi_{t_1t_1} (k, q) \), which enters this equation, is given by

\[
\Phi_{t_1t_1} (k, q) = \exp \left\{ \frac{t_1}{\hbar} \right\} \tau f \left[ \varepsilon \left( k + q + A(\tau + T - \frac{t_1}{2} - t_1) - A(T - \frac{t_1}{2}) \right) \right] - \varepsilon \left( k + A(\tau + T - \frac{t_1}{2} - t_1) - A(T - \frac{t_1}{2}) \right). \tag{18}
\]

From Eq. (17), it is found that no current can flow through the superlattice, when the double-time dependence is discarded \( [f^>_0 (T, t) \rightarrow f^>_0 (T)] \). This observation confirms the fact that under Wannier-Stark localization there is no current without any inelastic scattering. However, within the two-time approach, there exists a phononless current contribution even in the Wannier-Stark regime. An analytical solution is found by a Fourier transformation with respect to the additional time dependence and by adopting the ansatz

\[
f^<_{i=0} (m = 0, \omega) = f^<_{i=0} (m = 0, \omega) f(\omega), \tag{19}
\]

in which the frequency dependent function \( f(\omega) \) accounts for the time variable \( t \), which appears only in the double-time approach, when time translation invariance is broken. Using the formal solution of the kinetic Eq. (17), the current density of the one-dimensional superlattice is given by

\[
j = e n v_d, \quad v_d = \frac{\pi U \Delta^2 d J_{\nu} (\nu') J_{-\nu} (\nu')}{8 \hbar^2 \omega_{ac}^2 \sin (\pi \nu)} \sum_k \frac{J_k^2 (\nu')}{k + \nu} \times \int \frac{d\omega}{2\pi} f^>_0 (0, \omega) f^>_0 (0, \omega + \omega_{ac} (k + \nu)) [f(\omega + \omega_{ac} (k + \nu)) - f(\omega)],
\]

with the frequency parameters \( \nu = \Omega_{dc}/\omega_{ac} \) and \( \nu' = \Omega_{ac}/\omega_{ac} \). \( J_k \) denotes the Bessel function. From Eq. (20), it is concluded that the phononless current disappears in the strict one-time approach \( f(\omega) = \text{const.} \), which is an approximation completely in the spirit of the Kadanoff-Baym ansatz. The specific phononless current contribution only appears in the Wannier-Stark
regime, when the combined double-time history is taken into account in the equation of motion. Consequently, this steady-state current is a signature of the double-time character of quantum statistics. The result for the current density in Eq. (20) that is valid under quantizing electric fields has a familiar structure consisting of the combined density of states multiplied by the difference of occupation numbers for the final and initial states. It should be noted, however, that the \( \omega \) dependence in this equation has its strict origin on fast quantum effects on the microscopic time scale, which are completely disregarded in the one-time approximation. An experimental study of this intricate phononless quantum transport should be possible by studying a strongly biased semiconductor superlattice, which is subject to a strong perpendicular magnetic field that quantizes the in-plane carrier motion. To effectively suppress inelastic scattering due to phonons, the lattice temperature of this quantum-box superlattice should be extremely low.

The treatment of the diffusion coefficient proceeds in a similar manner. Starting from Eq. (9), we obtain for the one-dimensional model

\[
D = \sum_k \frac{1}{\hbar} \frac{\mathrm{d} \varepsilon(k)}{\mathrm{d}k} \frac{2\pi}{\omega_{ac}} \int_0^{\omega_{ac}} d\omega \varphi(k|T, t = 0),
\]

which is complemented by the quantum-kinetic equation for the Fourier coefficients \( \varphi_l(T, t) \) given by

\[
\left\{ \frac{\partial}{\partial T} + il\Omega_{dc} + il\Omega_{ac} \cos(\omega_{ac}T) \cos\left(\frac{\omega_{ac}t}{2}\right) \right\} \varphi_l(T, t) = i\frac{\Delta d}{4\hbar} \left\{ f^{<}_{l+1}(T, t) - f^<_{l-1}(T, t) - (f^>_{l}(T, t) - f^>_{l-1}(T, t)) \right\}
\]

\[
+ \frac{d}{2\pi} \int_0^{2\pi/d} \frac{d\epsilon}{\epsilon} \sum_{k_1} \int_{-\infty}^{\infty} df \varphi(k_1|T, t - t_1) W(k_1, k|T, t, t_1).
\]

The formal solution of this equation is used to derive an analytic expression for the diffusion coefficient in Eq. (21). For a weakly coupled superlattice, the result has the Tien-Gordon [21] form

\[
D(\Omega_{dc}, \Omega_{ac}) = \sum_{k=\infty}^{\infty} J_k \left( \frac{\Omega_{ac}}{\Omega_{dc}} \right) \tilde{D}(\Omega_{dc} + k\omega_{ac}, \Omega_{ac})
\]

\[
\tilde{D}(\Omega_{dc}, \Omega_{ac}) = \frac{d}{2} v_d(\Omega_{dc}, \Omega_{ac}) \coth \left( \frac{\hbar \Omega_{dc}}{2\Delta} \right),
\]

in which, however, the drift velocity \( v_d \) depends both on the Bloch frequency of the dc and ac electric field. This suggestive result applies to the Wannier-Stark hopping regime under the condition that carriers thermalize more quickly in a given well than they need to escape by tunneling. The double-time quantum character of this high-field phononless diffusion mechanism has the same origin as the phononless current treated above. It appears only in an approach that goes beyond the Kadanoff-Baym ansatz. The simple model of a one-dimensional superlattice has been adopted here to derive analytical results that describe phononless quantum transport and quantum diffusion on an equal footing. The double-time effect itself seems to be sufficiently robust that it should survive in more realistic calculations.

Further progress in the development of a unified approach of carrier drift and diffusion in semiconductors is expected from the treatment of collective diffusion in the presence of many-body interaction as well as from a study of quantum transport and quantum diffusion in finite semiconductor devices. Another promising direction of research includes anomalous diffusion in the unified quantum picture of carrier drift and diffusion.
4. Conclusion

The Fokker-Planck equation, which has many applications both in physics and mathematics, serves as a suitable basis for the development of a unified theory of carrier drift and diffusion. By interpreting the main quantity of this equation, namely the probability propagator, in terms of an averaged vacuum expectation value, a quantum theory of transport coefficients is straightforwardly derived. The general results that refer to carrier drift and diffusion in semiconductors are equally applicable to the treatment of localized and extended states. By an exact transformation, the hopping transport picture is converted into a description that is more designed for band transport problems and vice versa. Based on the theory of nonequilibrium Green functions, the unified approach is extended to account for the double-time dependence of the one-particle propagators. Within the self-consistent Born approximation with respect to scattering, double-time kinetic equations are obtained that have the structure of a non-Markovian Boltzmann equation, where the scattering rates depend on the electric field. As an application of the unified quantum approach, a phononless current contribution is identified that has its origin in the double-time dependence of Green functions.

References

[1] Narasimhan T N 2009 Physics Today July 48
[2] Davydov B I 1937 Zh. exper. teor. Fiz. 7 1069
[3] Martin P and Schwinger J 1959 Phys. Rev. 115 1342
[4] Evans D J and Searles D J 2002 Adv. Phys. 51 1529
[5] Kudinov E K and Firsov Y A 1966 Sov. Phys. JETP 22 603
[6] Haug H and Jauho A P 1996 Quantum Kinetics in Transport and Optics of Semiconductors (Springer Series in Solid-State Sciences vol 123) (Berlin, Heidelberg, New York: Springer Verlag)
[7] Bryksin V V and Kleinert P 2003 J. Phys.: Condens. Matter 15 1415
[8] Kleinert P and Bryksin V V 2003 Phys. Letters A 317 315
[9] Bryksin V V and Kleinert P 1999 J. Phys. A: Math. Gen. 32 2731
[10] Kleinert P 2009 Phys. Rev. E 79 051107
[11] Lipavský P, Čpička V and Velický B 1986 Phys. Rev. B 34 6933
[12] Čpička V and Lipavský P 1995 Phys. Rev. B 52 14615
[13] Velický B, Kalová A and Čpička V 2008 Phys. Rev. B 77 041201
[14] Kremp D, Bornath T, Bonitz M and Schlanges M 1999 Phys. Rev. E 60 4725
[15] Bonitz M, Bornath T, Kremp D, Schlanges M and Kraeft W D 1999 Contrib. Plasma Phys. 39 329
[16] Bonitz E M 2000 Progress in Nonequilibrium Green’s Functions (Singapore: World Scientific)
[17] Eckstein M and Kollar M 2008 Phys. Rev. B 78 205119
[18] Rubí E M and Pérez-Vicente C 1997 Complex Behavior of Glassy Systems (Berlin: Springer Verlag)
[19] Jaroszyński J and Popović D 2006 Phys. Rev. Lett. 96 037403
[20] Jaroszyński J and Popović D 2007 Phys. Rev. Lett. 99 046405
[21] Tien P R and Gordon J P 1963 Phys. Rev. 129 647