Leaves and trajectories for schemes. Application to the comparison of three classical sheaves over the differential spectrum.

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Abstract – In a first part, given a scheme $X$ of characteristic zero and a vector field $\vec{V}$ on $X$, we define what are the leaves of $X$ for $\vec{V}$. We prove that given $x \in X$, there exists a smallest leaf $\eta$ of $X$ passing through $x$, and we call it the trajectory $\text{Traj}_{\vec{V}}(x)$ of $x$. We establish some nice properties of the map $\text{Traj}_{\vec{V}}(-)$, and explain how to deal without any assumption on the characteristic of $X$. In a second part, with the help of these tools, we give a geometrical interpretation and generalize the paper [Car90] of Carrà Ferro. In a third part, we prove the key point of the paper: one can extend, in a unique way, a constant section defined over $U$ to the open set $U^\delta$ generated by $U$ under the action of $\vec{V}$. Finally, in a last part, we use these tools to compare three classical sheaves that have been defined over the differential spectrum. In a appendix, we give some details about the associated sheaf of a presheaf of differential rings, and explain why it commutes with the constant functor.

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1 Introduction

1.1 Differential schemes

Differential schemes have first been introduced by William Keigher in [Kei75]. The aim of introducing such objects is to give to algebro-differential geometry sound foundations, just as scheme theory for algebraic geometry. Nevertheless, and despite the following contributions to this task by A. Buium, G. Carrà Ferro and J. Kovacic, the category of differential schemes is still lacking. Let us explain quickly the several attempts that have been made and why they fail.

One starts with a differential ring \((A, \partial)\) and defines the differential spectrum of \(A\) to be

\[ \text{diff-Spec } A =: \{ p \mid p \text{ is a prime and differential ideal of } A \}. \]

First, this set is endowed with a topology (called the Kolchin topology) whose closed sets are defined as for schemes. Then, one equips it with the restriction sheaf of \(\mathcal{O}_{\text{Spec } A}\) to the subspace \(\text{diff-Spec } A\). The resulting objects are called affine differential schemes. They are differentially ringed spaces, whose stalks are local rings with a maximal ideal that is differential. The category of differential schemes is defined to be the category of such differentially ringed spaces that are locally isomorphic to affine objects.

The first problem with this category (see section 14 of [Kov02a]) is that one does not know if it has fibered products. Kovacic has proved that they exist, but only for AAD differential schemes, i.e. only if these objects satisfy a technical hypothesis that might exclude many cases. The second problem is about global sections. Indeed, unlike for schemes, the natural morphism

\[ A \longrightarrow \hat{A} := \Gamma(\text{diff-Spec } A, \mathcal{O}_{\text{diff-Spec } A}) \]

is neither, in general, injective nor surjective. Under some assumptions (see Theorem 2.6 of [Bui82], Theorem 10.6 of [Kov02a] or Theorem 8 of [Tru09]), one can prove that \(A \longrightarrow \hat{A}\) is an isomorphism of differential rings.

There is another definition to differential schemes. It has been proposed by Carrà Ferro in [Car90] and it is based on another structure sheaf \(\mathcal{O}^{(\text{CF})}_{\text{diff-Spec } A}\) for \(\text{diff-Spec } A\). However, this sheaf has not been used or studied elsewhere. It

---

(1) He then studied them in a series of papers [Kei77, Kei78, Kei81, Kei82a, Kei82b, Kei83].

(2) See the article of Buium and Cassidy in [Kol99] for an excellent survey on algebro-differential geometry. Algebro-differential varieties appear naturally in, for instance, parametrized Galois theory: in this setting, the Galois group is not an algebraic group but a differential-algebraic group. See for instance [CS07], [Cas72], [Lan08], [HS08].

(3) [Bui82]

(4) [Car85, Car90]

(5) [Kov02a, Kov03, Kov06, Kov02b]
seems more complicated than the previous one but it verifies

\[ A \cong \Gamma\left(\text{diff-Spec } A, \mathcal{O}^{(\text{CF})}_{\text{diff-Spec } A}\right). \]

The definition of this sheaf is based on the following lemma:

**Lemma** (Lemma 1.5 of [Car90]). Let \( A \) be a differential scheme. For each open subset \( U \) of \( \text{diff-Spec } A \), there exists an open subset \( U_\Delta \) of \( \text{Spec } A \) such that:

1. \( U_\Delta \cap \text{diff-Spec } A = U \);
2. \( U_\Delta \supset \text{Spec } A \setminus \text{diff-Spec } A \);
3. If \( V \) is an open set of \( \text{Spec } A \) such that \( V \cap \text{diff-Spec } A = U \), then \( V \subset U_\Delta \).

Then, \( \mathcal{O}^{(\text{CF})}_{\text{diff-Spec } A}(U) \) is defined to be \( \mathcal{O}_{\text{Spec } A}(U_\Delta) \). Nevertheless, this construction needs to be clarified.

### 1.2 Content of the paper

The goal of this paper is to bring a new point of view on this problem and these constructions. Our first idea is to not restrict ourselves to differential rings but to conduct our study in the frame of schemes. In this setting, vector fields replace derivations. Schemes with vector field have already been introduced and studied, in particular by Buium in [Bui86] and [Bui92] and Dyckerhoff in [Dyc] but also by Umemura in [Ume96]. Of course, vector fields are also implied in [Gro67], but Grothendieck does not study them extensively\(^6\). Given a scheme \( X \) with a vector field \( \vec{Y} \), we define the **leaves of \( X \) for \( \vec{Y} \)**. It is, intuitively, the elements \( \eta \) of \( X \) that are invariant under the vector field — or, the irreducible closed subsets tangent to \( \vec{Y} \). Then, we prove that given \( x \in X \), there exists a smallest leaf going through \( x \), called the **trajectory of \( x \) under \( \vec{Y} \)**, and we denote it by \( \text{Traj}_{\vec{Y}}(x) \). More precisely, we prove its existence when \( X \) is of characteristic zero and explain how to deal when not. The map \( \text{Traj}_{\vec{Y}} : X \rightarrow X \) satisfies nice and natural properties, and with the help of it, we endow \( X \) with a new topology, that we call the **Carrà Ferro topology**. The open sets of this topology are the Zariski open sets of \( X \) that are invariant under \( \vec{Y} \). In this context, it is very easy to generalize to schemes with vector fields the various constructions done for \( \text{diff-Spec } A \). In particular, we give a new perspective on the sheaf defined by Carrà Ferro in [Car90].

Then, with these tools, we compare the different sheaves that have been defined over \( \text{diff-Spec } A \). Actually, this comparison is valid over the set of leaves.

\(^6\)At the very beginning of [Gro67], Grothendieck writes “Dans ce paragraphe, nous présentons, sous forme globale, quelques notions de calcul différentiel particulièrement utiles en Géométrie algébrique. Nous passons sous silence de nombreux développements, classiques en Géométrie différentielle (connexions, transformations infinitésimales associées à un champ de vecteurs, jets, etc.), bien que ces notions s’écritvent de façon particulièrement naturelle dans le cadre des schémas.”
\(X^{\vec{v}}\) of any scheme \(X\) endowed with a vector field. Three sheaves have been defined. First, the restricted sheaf \(\mathcal{O}\)_{\text{diff-Spec}^A}^{(\text{Keigher})}\), defined in [Kei81]. Second, the sheaf \(\mathcal{O}\)_{\text{diff-Spec}^A}^{(\text{Kovacic})}\), defined à la Hartshorne in [Car85], and used by Kovacic in his several papers. Third, the sheaf \(\mathcal{O}\)_{\text{diff-Spec}^A}^{(\text{CF})}\) defined by Carrà Ferro in [Car90]. We prove that

\[
\mathcal{O}\)_{\text{diff-Spec}^A}^{(\text{Keigher})} \cong \mathcal{O}\)_{\text{diff-Spec}^A}^{(\text{Kovacic})}
\]

for any differential ring \(A\). For the Carrà Ferro sheaf, we prove (see Theorem 5.1):

**Theorem.** Let \(X\) be a reduced \(\mathbb{Q}\)-scheme endowed with a vector field. Then, the Carrà Ferro sheaf and the Keigher sheaf have the same constants:

\[
\forall \, U \text{ open in } X^{\vec{v}}, \quad C(O_{X^{\vec{v}}}(U)) \cong C(O_{X^{\vec{v}}}(U)).
\]

The main ingredient of the proof of this theorem is the following proposition (see Proposition 4.1), which has an interest in itself. It says that given a constant section of \(\mathcal{O} X\) over \(U\), one can extend it to the open set \(U^\delta\) generated by \(U\) under the action of \(\vec{v}\). It is remarkable as one always restricts sections but rarely extends it.

**Proposition.** Let \(X\) be a reduced \(\mathbb{Q}\)-scheme endowed with a vector field \(\vec{v}\). Let \(U\) be an open set of \(X\). Then, for every \(f \in C(O X(U))\), there exists a unique \(\tilde{f}\) in \(C(O X(U^\delta))\) such that \(\tilde{f}|_{U} = f\).

Furthermore, the extension map

\[
\text{ext}_{U \to U^\delta} : C(O X(U)) \to C(O X(U^\delta))
\]

is an isomorphism of rings, whose inverse \(C(O X(U^\delta)) \to C(O X(U))\) is the restriction map.

The proof of this proposition relies on the following fact: if \((a/b)' = 0\) then \(a/b = a'/b'\). Nevertheless, when one wants to state the property we need — something like \(a/b = a^{(n)}/b^{(n)}\) for all \(n\), under the same assumption — one needs to follow quite precisely what happens in the computations. This is done in Proposition 4.2 and Lemma 4.3. In particular, we prove the following statement of commutative algebra.

**Proposition.** Let \(A\) be a differential ring and \(S\) a multiplicative subset of \(A\). If \(s \in S\) and if \(i \in \mathbb{N}\), one has

\[
\left(\frac{a}{s}\right)' = 0 \text{ in } S^{-1}A, \quad s^{(i)} \in S \implies \frac{a}{s} = \frac{a^{(i)}}{s^{(i)}} \text{ in } S^{-1}A.
\]
1.3 Plan of the paper

The paper is organized as follows. Section 1 is this introduction. Section 2 is devoted to the definition of vector fields, leaves, to the construction of \( \text{Traj}_\mathcal{F} \) and to the study of its properties. We also study, as examples, vector fields of \( \mathbb{A}^n \) and \( \mathbb{P}^n \). In particular, we prove that any vector field on \( \mathbb{P}^n \) vanishes at some closed point — another way to say that any vector field on \( \mathbb{P}^n_k \) (\( k \) a field and the vector field constant on \( k \)) has a singular point. We also explain how to define the trajectory in the case of schemes not defined over \( \mathbb{Q} \). In Section 3, we define the Carrà Ferro topology of a scheme with a vector field, as well as the Keigher sheaf and the Carrà Ferro sheaf. Section 4 is devoted to our result on the extension of constant sections. It starts with a result of commutative algebra on constant elements in localized rings. In Section 5, we prove our main results on the comparison of \( \mathcal{O}^{(\text{Keigher})}_{\text{diff-Spec} A} \), \( \mathcal{O}^{(\text{Kovacic})}_{\text{diff-Spec} A} \) and \( \mathcal{O}^{(\text{CF})}_{\text{diff-Spec} A} \). Finally, in an appendix, we explain why the associated sheaf functor, in the case of (pre)sheaves of differential rings, commutes with the functor of constants.

2 Vector fields, leaves and trajectories for schemes

We start this paper with some classical and elementary facts about the possible definitions of vector fields for smooth manifolds. This will motivate our definition for schemes.

2.1 Vector fields

In the case of smooth manifolds, it is well known that one can define global vector fields in various ways. Given \( M \) such a manifold:

a) If the tangent bundle \( TM \) has already been defined, as a smooth manifold, one can say that a global vector field is a section \( s \) of the canonical projection \( \pi : TM \rightarrow M \).

b) It is equivalent to consider a map

\[
\partial : \mathcal{C}^\infty (M, \mathbb{R}) \rightarrow \mathcal{C}^\infty (M, \mathbb{R})
\]

that is \( \mathbb{R} \)-linear and such that

\[
\forall f, g \in \mathcal{C}^\infty (M, \mathbb{R}), \quad \partial (fg) = f\partial(g) + \partial(f)g.
\]

In other words, global vector fields can also be seen as \( \mathbb{R} \)-derivations of the \( \mathbb{R} \)-algebra \( \mathcal{C}^\infty (M, \mathbb{R}) \). The derivation associated to a section \( s \) of \( TM \rightarrow M \) is defined by

\[
\partial_s(f) := \begin{array}{c}
M \\
p \mapsto \mathbb{R}
\end{array} \begin{array}{c}
\mapsto \mathbb{R} \\
\mapsto df_p \cdot s(p)
\end{array}.
\]
c) Actually, given a global vector field, one gets a \( \mathbb{R} \)-derivation \( \partial_U \) of \( C^\infty(U, \mathbb{R}) \) for all open set \( U \) of \( M \). Moreover, these maps are compatible with the restriction maps. Thus, one can attach to a global vector field a \( \mathbb{R} \)-derivation of the structure sheaf \( \mathcal{O}_M \) of \( M \).

This motivates the definition:

**Definition 2.1.** Let \( X \) be a scheme. A vector field \( \vec{V} \) on \( X \) is a derivation of the structure sheaf \( \mathcal{O}_X \) of \( X \).

**Remarks.**

(a) A scheme can always be endowed with the zero-vector field, which corresponds to the derivation \( \partial_U = 0 \) for all open sets \( U \) of \( X \).

(b) If \( (X, \mathcal{O}_X) \) is a scheme, then it is equivalent to consider a vector field \( \vec{V} \) on \( X \) or to endow the sheaf \( \mathcal{O}_X \) with a structure of sheaf of differential rings: \( (X, \mathcal{O}_X, \vec{V}) \) is then what we will call a differentially (locally) ringed space.

(c) In [Gro67], given a \( S \)-scheme \( X \), Grothendieck defines the tangent bundle of \( X/S \). It is a \( S \)-scheme, denoted by \( T_{X/S} \), with a \( S \)-morphism to \( X \):

\[
\begin{array}{ccc}
T_{X/S} & \xrightarrow{\pi} & X \\
\downarrow & & \\
X & & \\
\end{array}
\]

He proves that the \( S \)-section of \( \pi \) correspond to the \( \mathcal{O}_S \)-derivations of \( \mathcal{O}_X \). So, in the case where \( X \) is viewed as a \( \mathbb{Z} \)-scheme, one gets a correspondence between the sections of \( \pi : T_X \to X \) and the group of vector fields of \( X \). The \( \mathcal{O}_X \)-module of \( S \)-sections of \( \pi \) is the dual of \( \Omega^1_{X/S} \). We will denote it by \( \mathcal{T}_{X/S} \) (or by \( \mathcal{T}_X \) when \( S = \text{Spec} \mathbb{Z} \)). ∎

### 2.2 Morphisms and category

If \( \mathcal{X} = (X, \vec{V}) \) and \( \mathcal{Y} = (Y, \vec{W}) \) are two schemes with vector fields, a morphism \( f : \mathcal{X} \to \mathcal{Y} \) will be a morphism \( f : X \to Y \) of schemes such that, for all open set \( U \) of \( Y \), the diagram

\[
\begin{array}{ccc}
\mathcal{O}_X (f^{-1}(U)) & \xleftarrow{f^{\#}_U} & \mathcal{O}_Y (U) \\
\downarrow{\partial_{\vec{V}, f^{-1}(U)}} & & \downarrow{\partial_{\vec{W}_U}} \\
\mathcal{O}_X (f^{-1}(U)) & \xleftarrow{f^{\#}_U} & \mathcal{O}_Y (U) \\
\end{array}
\]

commutes. In other words, \( f \) is a morphism of schemes that is a morphism of differentially ringed spaces. The category of schemes with vector fields will be denoted by \( \text{Sch}^\partial \). Intuitively, as it will be seen in Proposition 2.3, a morphism \( f : \mathcal{X} \to \mathcal{Y} \) pushes the vector field of \( \mathcal{X} \) onto the vector field of \( \mathcal{Y} \).
2.3 Schemes with vector fields and differential rings

If \((A, \partial)\) is a differential ring, then the scheme \(\text{Spec } A\) can be canonically endowed with a vector field \(\vec{V}_A\). As a derivation, this vector field is defined on the basis of open sets \(D(f)\) as the derivation induced on \(A_f\) by \(\partial\). We will denote this scheme with vector field by \(\text{Spec}^\partial A\). Actually, one obtains a functor

\[
\text{Spec}^\partial : (\text{Rng}^\partial)^{\text{op}} \rightarrow \text{Sch}^\partial.
\]

We could have defined the schemes with vector field as differentially ringed spaces locally isomorphic to \(\text{Spec}^\partial A_i\)’s.

**Example.** — Let \(k\) be a field and \(A = k[x]\). The derivation \(\partial^{\text{cst}}\) of \(A\) defined by \(\partial^{\text{cst}}|_k = 0\) and \(\partial^{\text{cst}}(x) = 1\) corresponds to the constant vector field of \(A^1_k\). The derivation \(\partial^{\text{rad}}\) defined by \(\partial^{\text{rad}}|_k = 0\) and \(\partial^{\text{rad}}(x) = x\) corresponds to the radial vector field, as pictured in Figure 1.

![Figure 1: The vector fields of \(A^1_k\) associated to the derivations \(\partial^{\text{cst}}\) and \(\partial^{\text{rad}}\).](image)

As in the non-differential case, one has the following proposition, whose proof is left to the reader:

**Proposition 2.2.** The functors

\[
(\text{Rng}^\partial)^{\text{op}} \xrightarrow{\text{Spec}^\partial} \text{Sch}^\partial
\]

form an adjunction: \(\text{Spec}^\partial(-)\) is a left adjoint to \(\text{Spec}^\partial\).

In particular, the category of affine schemes endowed with vector field is antiequivalent to the category of differential rings. This allows us to describe the vector fields of \(A^n_k\) and \(P^n_k\), as follows.

**Examples.** — (a) Vector fields on \(A^n_k\). Let \(k\) be a ring. Let \(\vec{V}\) be a vector field defined on \(A^n_k\), and constant on \(k\). Then, \(\vec{V}\) corresponds to a \(k\)-derivation of \(k[X_1, \ldots, X_n]\). Such a derivation \(\partial\) is fully determined by the elements \(\partial X_1, \ldots, \partial X_n\). Hence, the abelian group of vector fields on \(A^n_k\) is isomorphic to \((k[X_1, \ldots, X_n], +)^n\).
Let \( k \) be a ring. Then, the vector fields defined on \( \mathbb{P}^n_k \) and constant on \( k \) come from linear vector fields of \( \mathbb{A}^{n+1}_k \). This means, precisely, that for any vector field \( \vec{V} \) defined on \( \mathbb{P}^n_k \) and constant on \( k \), there exists a matrix \( A \in M_{n+1}(k) \) such that the morphism

\[
\pi : (\mathbb{A}^{n+1}_k \setminus \{0\}, \vec{V}_A) \longrightarrow (\mathbb{P}^n_k, \vec{V})
\]

is compatible with the vector fields, where \( \pi : \mathbb{A}^{n+1}_k \setminus \{0\} \longrightarrow \mathbb{P}^n_k \) denotes the canonical projection and where \( \vec{V}_A \) denotes the linear vector field of \( \mathbb{A}^{n+1}_k \) induced by the derivation

\[
\partial_A : k[X_0, \ldots, X_n] \longrightarrow k[X_0, \ldots, X_n]
\]

defined by

\[
\partial_A \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix} = A \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix}.
\]

Indeed, the Euler exact sequence can be written

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1) \longrightarrow \mathcal{T}_{\mathbb{P}^n_k/k} \longrightarrow 0
\]

as in Example 8.20.1 of [Har77]. Hence, one gets an exact sequence in cohomology:

\[
H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)^{n+1}) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{T}_{\mathbb{P}^n_k/k}) \longrightarrow H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}).
\]

But, one knows that \( H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = 0 \) (see for instance [Liu02]). So, the map \( H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)^{n+1}) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{T}_{\mathbb{P}^n_k/k}) \) is surjective. Let us write down explicitly what is this map. To a family \((L_0, \ldots, L_n)\) of linear forms in \( X_0, \ldots, X_n \), it associates the vector field of \( \mathbb{P}^n_k \), defined on each standard open set \( U_i = \text{Spec} \ k[X_0/X_i, \ldots, X_n/X_i] \) by:

\[
\partial(X_k/X_i) = \frac{L_k \cdot X_i - L_i \cdot X_k}{X_i^2}.
\]

So, given a vector field \( \vec{V} \) of \( \mathbb{P}^n_k \), one obtains the required matrix \( A \) by considering the coefficients of the linear forms \( L_0, \ldots, L_n \). \( \triangle \)

### 2.4 Tangent vectors associated to vector fields

Let \( X \) be a scheme. Now, we are going to explain how to associate to a vector field \( \vec{V} \) on \( X \) and to an element \( x \in X \) a (Zariski) tangent vector \( \vec{V}(x) \in T_xX \). First, it is easy to check that a vector field \( \vec{V} \), ie a derivation \( \partial \) of \( \mathcal{O}_X \), induces
a derivation $\partial_x$ of the local ring $\mathcal{O}_{X,x}$. We denote by $\mathfrak{m}_x$, as usual, the maximal ideal of $\mathcal{O}_{X,x}$, and $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$. We then consider the linear map

$$
\begin{array}{c}
\mathfrak{m}_x \\
\downarrow f \\
\kappa(x)
\end{array} \quad (\partial_x f)(x)
$$

This map sends elements of $\mathfrak{m}_x^2$ to zero, since

$$
\partial_x(fg)(x) = (\partial_x f)g + f(\partial_x g)(x) = 0
$$

for $f, g \in \mathfrak{m}_x$. Hence, we got a map

$$
\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \kappa(x),
$$

which is $\kappa(x)$-linear: in other words, we got a element of $T_x X$ the Zariski tangent space of $X$ in $x$. We denote this element by $\vec{V}(x)$. We then have the formula:

**Proposition 2.3.** Let $\mathcal{X} = (X, \vec{V})$ and $\mathcal{Y} = (Y, \vec{W})$ be two schemes with vector fields. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism. Then

$$
\forall x \in X, \quad T_x f \bullet \vec{V}(x) = i_x \circ \vec{W}(f(x)),
$$

where $i_x : \kappa(f(x)) \longrightarrow \kappa(x)$ is the inclusion of residual fields induced by $f$.

**Proof.** — Since the definition of $\vec{V}(x)$ is local, it is sufficient to prove this statement when $X$ and $Y$ are affine. So, let $(A, \partial_A)$ and $(B, \partial_B)$ be two differential rings, and let $\varphi : (A, \partial_A) \longrightarrow (B, \partial_B)$ be a morphism. We denote by $f : \text{Spec}^\partial B \longrightarrow \text{Spec}^\partial A$ the corresponding morphism of schemes with vector fields. Let $x \in \text{Spec} B$, ie let $p_x$ be a prime ideal of $B$. The image of $x$ by $f$ is $p_y := \varphi^{-1}(p_x)$. The morphism $\varphi$ induces, by localisation, an arrow

$$
\vec{\varphi} : A_{p_y} \longrightarrow B_{p_x}.
$$

Better, if we denote by $\mathfrak{m}_x$ and $\mathfrak{m}_y$ the maximal ideals of $A_{p_x}$ and $A_{p_y}$, $\varphi$ induces morphisms

$$
\vec{\varphi} : \mathfrak{m}_y/\mathfrak{m}_y^2 \longrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2 \\
\text{and} \\
i_x : B_{p_y}/\mathfrak{m}_y \longrightarrow A_{p_x}/\mathfrak{m}_x,
$$

this latter being injective. Now, the tangent vector $\vec{V}(x)$ corresponds to the morphism

$$
\begin{array}{c}
\mathfrak{m}_x/\mathfrak{m}_x^2 \\
\downarrow \psi \\
\partial_B(\psi) \text{ mod. } \mathfrak{m}_x
\end{array} \quad \begin{array}{c}
\mathfrak{m}_y/\mathfrak{m}_y^2 \\
\downarrow \partial_x \\
A_{p_x}/\mathfrak{m}_x
\end{array}
$$

and there is a similar description of $\vec{W}(y)$. The image of $\vec{V}(x)$ by the differential $T_x f$ is the map $\partial_y$ making the diagram

$$
\begin{array}{ccc}
\mathfrak{m}_y/\mathfrak{m}_y^2 & \xrightarrow{\vec{\varphi}} & \mathfrak{m}_x/\mathfrak{m}_x^2 \\
\downarrow \partial_y & & \downarrow \partial_x \\
A_{p_x}/\mathfrak{m}_x & \longrightarrow & A_{p_x}/\mathfrak{m}_x
\end{array}
$$
Let \( \psi \in \mathfrak{M}_y \). We have:

\[
\partial_{\psi} (\psi \mod. \mathfrak{M}^2_y) = \partial_x (\hat{\psi}(\psi) \mod. \mathfrak{M}^2_x) = \partial_B (\hat{\psi}(\psi)) \mod. \mathfrak{M}_x = \partial_A (\psi) \mod. \mathfrak{M}_x = i_x (\partial_A(\psi) \mod. \mathfrak{M}_y).
\]

In other words, \( T_x f \circ \hat{V}(x) = i_x \circ \hat{W}(f(x)) \).

## 2.5 Leaves

We are now able to define leaves:

**Definition 2.4.** Let \( X = (X, \vec{V}) \) be a scheme with a vector field. Let \( \eta \in X \). We say that \( \eta \) is a leaf of \( X \) (or a leaf for \( \vec{V} \)) when \( \vec{V}(\eta) = 0 \). The set of leaves of \( X \) will be denoted by \( \mathcal{X}^{\vec{V}} \).

Let us check that the leaves of \( \text{Spec}^\partial A \) correspond to the differential prime ideals of \( A \), when \( A \) is a differential ring. Let \( p \) be a prime ideal of \( A \). Let’s assume that \( p \) is a leaf of \( \text{Spec}^\partial A \). Let \( f \in p \) from \( \vec{V}(p) = 0 \), we deduce that the image of \( f \) under the map

\[
pA_p = \mathfrak{M}_p \to \mathfrak{M}_p/\mathfrak{M}^2_p \xrightarrow{\partial_A} \mathfrak{M}_p/\mathfrak{M}^2_p \to A_p/\mathfrak{M}_p
\]

is zero. Hence, \( \partial_A(f) \in \mathfrak{M}_p = pA_p \) and so, \( f \in p \): the ideal \( p \) is differential. Reciprocally, one can check that if \( p \) is a differential ideal, then it is a leaf of \( \text{Spec}^\partial A \). This fundamental remark shows that

\[
\mathcal{X}^{\vec{V}} \subset X
\]

is the exact non-affine analogue of

\[
\text{diff-Spec} A \subset \text{Spec} A.
\]

**Examples.** — (a) The scheme \( \mathbb{A}^1_k \) endowed with the constant vector field has only one leaf: its generic point \( \eta \). With the radial vector field, it has two leaves: the closed point 0 and \( \eta \).

(b) Let’s consider the ring \( A = k[X_1, \ldots, X_n] \) with a \( k \)-derivation \( \partial \). The derivation \( \partial \) is caracterized par the elements

\[
P_1 := \partial(X_1) \quad P_2 := \partial(X_2) \quad \cdots \quad P_n := \partial(X_n).
\]
One can check that the corresponding vector field $\mathbf{V}$ satisfies, for all $x_1, \ldots, x_n \in k$:

$$\mathbf{V}(x_1, \ldots, x_n) = \begin{pmatrix} P_1(x_1, \ldots, x_n) \\ \vdots \\ P_n(x_1, \ldots, x_n) \end{pmatrix}.$$  

Let’s take $n = 2$ (we denote $\mathbb{A} = k[x, y]$) with the derivation

$$\partial(x) = -2y$$

and

$$\partial(y) = 3x^2.$$  

By a simple computation, one checks that the prime ideal $\eta_c = (x^3 + y^2 - c)$ is differential, for all $c \in k$: consequently, $(\eta_c)_{c \in k}$ is a family of leaves.

(c) As noticed by Buium in [Bui86] (Lemma (2.1) of Chapter 1), if $X$ is a $\mathbb{Q}$-scheme and if $F$ is an irreducible closed set of $X$, then the generic point $\eta_F$ of $F$ is always a leaf, for any vector field $\mathbf{V}$.

(d) Let $k$ be a ring. Any vector field on $\mathbb{P}_k^n$ vanishes on a closed point — a kind of analogue of the hairy ball theorem. Indeed, as explained in (2.3), if $\mathbf{V}$ is a vector field of $\mathbb{P}_k^n$ constant on $k$, then there exists $A \in M_{n+1}(k)$ such that $\mathbf{V}$ comes from the vector field of $\mathbb{A}_k^{n+1}$ defined by

$$\partial \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix} := A \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix}.$$  

Now, let $K$ be a residual field of $k$, i.e., let $\varphi : k \to K$ be a surjective morphism. There exists a finite extension of fields $K \to L$ such that the matrix $A$, when viewed in $L$, has an eigenvector $\mathbf{v}$. This implies that the image of $\mathbf{v}$ under the map $\pi_L : \mathbb{A}_L^{n+1} \to \mathbb{P}_L^n$, denoted by $x_L$, and which is clearly a closed point, is a leaf for $\mathbf{V}_L$ — in other words, $\mathbf{V}_L$ vanishes on $x_L$. By Proposition 2.3, one knows that the image of $x_L$ under the map

$$f : \mathbb{P}_L^n \to \mathbb{P}_K^n \to \mathbb{P}_k^n,$$  

denoted by $x_k$, will also be a leaf for $\mathbf{V}$. Hence, we just need to see why $x_k$ is a closed point. This comes from the following facts:

— First, since $L/K$ is finite, $\text{Spec} L \to \text{Spec} K$ is a proper map and so is $\mathbb{P}_L^n \to \mathbb{P}_K^n$. In particular, it is a closed map.

— Second, since $\text{Spec} K \to \text{Spec} k$ is a closed immersion, the morphism $\mathbb{P}_K^n \to \mathbb{P}_k^n$ is also a closed immersion. In particular, it is a closed map.

— So, $\mathbb{P}_L^n \to \mathbb{P}_k^n$ is a closed map, and sends closed points to closed points: $x_k$ is a closed point.

$\triangle$
2.6 Trajectory of a point

Now, let $\mathcal{X} = (X, \vec{V})$ be in $\text{Sch}^\partial$. We would like to associate to any $x \in X$ “its algebraic trajectory under the vector field $\vec{V}$”. This is possible, thanks to the following theorem, which is an analogue for schemes of the Cauchy-Peano theorem:

**Theorem 2.5.** Let $\mathcal{X} = (X, \vec{V})$ be a scheme with a vector field, defined over $\mathbb{Q}$. Let $x \in X$. Then, the ordered set

$$\left\{ \eta \in X \mid \eta \sim x \right\}$$

has a least element. We denote this element by $\text{Traj}_\mathcal{X}(x)$ and call it the trajectory of $x$ (under $\vec{V}$).

**Remark.** — Here, the order that we consider is: $z \geq y$ if and only if $y \in \{z\}$. In this case, we say that $z$ is a generization of $y$ and that $y$ is a specialization of $z$; we denote $z \sim y$. The properties of this order are classical (see [Gro60], Chapter 0, (2.1.1)). For instance, open sets are stable under generization and, dually, closed sets are stable under specialization. 

**Proof.** — We keep the notations of the theorem. Let $x \in X$ and $U$ an affine neighborhood of $x$. Since all the generizations of $x$ are elements of $U$, one can assume $X$ affine. So, let $A$ be a differential $\mathbb{Q}$-algebra and $\mathfrak{p}$ a prime ideal of $A$. Since the generization order is the opposite of the inclusion order on ideals, one needs to prove that

$$\left\{ q \in \text{Spec } A \mid q \text{ is a differential prime ideal} \right\}$$

has a greatest element. Let’s consider, as in [Kei77], the set

$$\mathfrak{p}^\# := \left\{ f \in A \mid \forall n \geq 0, f^{(n)} \in \mathfrak{p} \right\}.$$ 

Keigher’s Proposition 1.5 says that $\mathfrak{p}^\#$ is a prime ideal (it’s there that $\mathbb{Q} \subset A$ is needed). It is then easy to check that $\mathfrak{p}^\#$ is the required ideal. We will see further a proof of the primality of $\mathfrak{p}^\#$ is a more general context. Let’s remark that, in any case, $I^\#$ is the greatest differential ideal contained in $I$. 

**Examples.** — (a) For all leaf $\eta \in X$, $\text{Traj}_\mathcal{X}(\eta) = \eta$.

(b) Since $X = \mathbb{A}^1_\mathbb{C}$ endowed with the constant vector field has only $\eta$ as a leaf, one has: $\forall x \in X$, $\text{Traj}_\mathcal{X}(x) = \eta$. If we consider the radial vector field, the trajectory of all $x \in X$ but $0$ is $\eta$.

(c) Let’s consider the vector field on $\mathbb{A}^2_\mathbb{C}$ defined by

$$\partial x = 1 - xy^2 \quad \text{and} \quad \partial y = x^2 - y^3.$$
whose smooth real leaves are drawn in Picture 2. Jouanolou proved in [Jou79] that no non-constant smooth leaf of this vector field is algebraic. Thus, the leaves for this vector field are just $\eta$ the generic point and the point $(1, 1)$. △

![Figure 2: Smooth leaves of the vector field defined by $\partial x = 1 - xy^2$ and $\partial y = x^2 - y^3$.](image)

2.7 Properties of the trajectory

First, we prove that the map $\text{Traj}_\mathcal{F}$ is “compatible” with morphisms of $\text{Sch}^\theta$, namely:

**Proposition 2.6.** Let $\mathcal{X} = (X, V)$ and $\mathcal{Y} = (Y, W)$ be two $\mathbb{Q}$-schemes with vector fields, and let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism. Then, for all $x \in X$,

$$f(\text{Traj}_\mathcal{F}(x)) = \text{Traj}_{\mathcal{W}}(f(x)).$$

**Proof.** — By considering affine neighborhoods of $f(x)$ and $x$, it suffices to prove this proposition in the affine case. Hence, let $A$ and $B$ be two $\mathbb{Q}$-differential algebras, let $\varphi : A \to B$ be a morphism of differential rings. Let $p$ be a prime ideal of $B$. We want to prove that

$$\varphi^{-1}(p\#) = (\varphi^{-1}p)_\#.$$
But,
\[
\varphi^{-1}(p_\#) = \{ x \in A \mid \varphi(x) \in p_\# \} = \{ x \in A \mid \forall n \in \mathbb{N}, \varphi(x^{(n)}) \in \mathfrak{p} \} \\
= \{ x \in A \mid \forall n \in \mathbb{N}, x^{(n)} \in \varphi^{-1} \mathfrak{p} \} \\
= (\varphi^{-1} \mathfrak{p})_\#, 
\]
which concludes the proof. ■

The trajectory defines a map
\[
\mathrm{Traj}_{\varphi} : X \rightarrow X^{\vec{\varphi}}. 
\]
Since \(X^{\vec{\varphi}} \subset X\), it is possible to endow the set \(X^{\vec{\varphi}}\) of leaves with the topology induced by the Zariski topology. Then:

**Proposition 2.7.** Let \(X\) be \(\mathbb{Q}\)-scheme endowed with a vector field \(\vec{\varphi}\). Then, \(\mathrm{Traj}_{\varphi} : X \rightarrow X^{\vec{\varphi}}\) is continuous and open.

**Proof.** — First, let’s show that it is continuous. Since this property is local, let’s assume that \(X = \text{Spec}^{\partial} A\), with \(A\) a differential ring. Let \(U = V \cap X^{\vec{\varphi}}\) be an open set of \(X^{\vec{\varphi}}\), where \(V\) is a Zariski open set of \(X\). Let \(I\) be an ideal of \(A\) such that \(V = X \setminus V(I)\). Let’s prove that
\[
(\mathrm{Traj}_{\varphi})^{-1} U = X \setminus V(\langle I \rangle), 
\]
where \(\langle I \rangle\) denotes the differential ideal generated by \(I\). Let \(p\) be a prime ideal of \(A\). Then, one has
\[
\mathrm{Traj}_{\varphi}(p) \in U \iff p_\# \in U \\
\iff p_\# \in V \\
\iff I \subset p_\#. 
\]
But, the latter is equivalent to \(\langle I \rangle \subset p\). Indeed, if \(I \subset p_\#\), since \(p_\#\) is a differential ideal, one has \(\langle I \rangle \subset p_\#\) and since \(p_\# \subset p\), one has indeed \(\langle I \rangle \subset p\). On the other hand, if \(\langle I \rangle \subset p\), since \(p_\#\) is greatest differential ideal contained in \(p\), one has \(\langle I \rangle \subset p_\#\) and so \(I \subset p_\#\). This proves indeed that \((\mathrm{Traj}_{\varphi})^{-1} U = X \setminus V(\langle I \rangle)\).

Let’s prove now that \(\mathrm{Traj}_{\varphi}\) is open. Let \(X\) be a \(\mathbb{Q}\)-scheme endowed with a vector field \(\vec{\varphi}\). Let \(U\) be an open set of \(X\). Since, for all \(\eta \in X^{\vec{\varphi}}\), one has \(\mathrm{Traj}_{\varphi}(\eta) = \eta\), it is easy to check that
\[
\mathrm{Traj}_{\varphi}(U) = U \cap X^{\vec{\varphi}}. 
\]
Hence, the map \(\mathrm{Traj}_{\varphi}\) is indeed open. ■
2.8 The case when the schemes are no more defined over \( \mathbb{Q} \)

A crucial hypothesis in Theorem 2.3 is that the schemes have to be defined over \( \mathbb{Q} \). This comes from the fact that, when differentiating \( f^n \), one gets \( n \cdot f^{n-1} f' \); if \( n \) can be simplified, much more things can be done. In general, Theorem 2.3 is false when the schemes are not defined over \( \mathbb{Q} \). Nevertheless, it is possible to solve this problem by defining Hasse-Schmidt vector fields. Recall that, when \( A \) is a ring, a Hasse-Schmidt derivation of \( A \) is a family \( D = (D_i)_{i \geq 0} \) of map \( A \rightarrow A \) satisfying

(i) for all \( i \geq 0 \), \( D_i : A \rightarrow A \) is an additive map, and \( D_0 = \text{Id}_A \).

(ii) the generalized Leibniz rule: for all \( i \) and all \( f, g \in A \):

\[
D_i(fg) = \sum_{k+\ell=i} D_k(f)D_\ell(g).
\]

(iii) iterativity: for all \( i, j \geq 0 \), \( D_i \circ D_j = (i+j)D_{i+j} \).

If \( A \) is a \( \mathbb{Q} \)-algebra, then, there is a one-to-one correspondance between derivations of \( A \) and Hasse-Schmidt derivations of \( A \) given by

\[
\partial \mapsto D := \left( \frac{\partial^i}{i!} \right)_{i \geq 0}.
\]

Subsequently, if \( X \) is a scheme, we call Hasse-Schmidt vector field of \( X \) any Hasse-Schmidt derivation of the structure sheaf \( \mathcal{O}_X \), ie any family \( (D_U)_U \) of compatible Hasse-Schmidt derivations \( \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U) \) for all open set \( U \). Let’s now define what would be a leaf for a Hasse-Schmidt derivation. The situation is more complicated than for classical vector fields. For any Hasse-Schmidt derivation \( \mathcal{D} \) of \( \mathcal{O}_X \) and any \( x \in X \), it is possible to consider the restriction \( \mathcal{D}_x \) of \( \mathcal{D} \) to the local ring \( \mathcal{O}_{X,x} \): it is a Hasse-Schmidt derivation of \( \mathcal{O}_{X,x} \), and we denote \( \mathcal{D}_x = (\mathcal{D}_{x,i})_{i \geq 0} \). For all \( i \geq 1 \), the map

\[
ev_{x,i} : \frac{\mathfrak{m}_{x,i}^{i+1}}{\mathfrak{m}_{x,i}^i} \rightarrow \mathcal{D}_{x,i}(f)(x)
\]

is well defined, since

\[
\forall f \in \mathfrak{m}_{x,i}^{i+1}, \quad \mathcal{D}_{x,i}(f)(x) = 0.
\]

Indeed, if \( A \) is a ring and if \( D = (D_0, D_1, \ldots) \) is a Hasse-Schmidt derivation of \( A \), the generalized Leibniz rule generalises to

\[
\forall i \geq 0, \forall p \geq 1, \quad D_i(f_1f_2 \cdots f_p) = \sum_{\ell_1, \ldots, \ell_p \geq 0, \ell_1 + \cdots + \ell_p = i} D_{\ell_1}(f_1) \cdots D_{\ell_p}(f_p),
\]

for all \( f_1, \ldots, f_p \in A \). We say that \( x \) is a leaf for this Hasse-Schmidt derivation if the maps \( ev_{x,i} \) are zero for all \( i \geq 1 \). Then we have:
Theorem 2.8. Let $X$ be a scheme endowed with a Hasse-Schmidt vector field $\mathcal{V}$. Let $x \in X$. Then, the ordered set

$$\left\{ \eta \in X \mid \eta \text{ is a leaf for } \mathcal{V} \right\}$$

has a least element.

The proof of this theorem is based on the following proposition:

Proposition 2.9. Let $A$ be a ring and $D = (D_i)_{i \geq 0}$ a Hasse-Schmidt derivation of $A$. Let $p$ be a prime ideal of $A$. Then,

$$p^\# := \{ f \in A \mid \forall i \geq 0, D_i(f) \in p \}$$

is a prime ideal invariant by $D$.

Proof. — The set $p^\#$ is clearly stable under addition. If $f \in p^\#$ and $\lambda \in A$, then the generalized Leibniz rule proves that $\lambda f \in p^\#$. Furthermore, the iterativity of $D$ proves that for all $i \geq 0$, the ideal $p^\#$ is stable under $D_i$. Let’s prove that $p^\#$ is a prime ideal. Let $f, g \in A$ such that $f, g \notin p^\#$. Thus, let $i_0$ and $j_0 \geq 0$ be the least integers such that $D_{i_0}(f) \notin p$ and $D_{j_0}(g) \notin p$.

Let’s prove that $fg \notin p^\#$ by considering

$$D_{i_0+j_0}(fg) = \sum_{k+\ell = i_0+j_0} D_k(f)D_\ell(g).$$

In this sum, the terms split in three parts: the $D_k(f)D_\ell(g)$’s for $k < i_0$, which are in $p$ by definition of $i_0$, the $D_k(f)D_\ell(g)$’s for $\ell < j_0$, which are in $p$ for the same reason, and finally $D_{i_0}(f)D_{j_0}(g)$. This latter isn’t in $p$ for $p$ is a prime ideal. Thus, $D_{i_0+j_0}(fg) \notin p$ and so, $fg \notin p^\#$. This proves that $p^\#$ is a prime ideal. 

3 The Carrà Ferro topology, the Carrà Ferro sheaf and the Keigher sheaf

In this section, we reinterpret the paper [Car90], with the help of vector fields, leaves and trajectories. This new approach allow us to generalize the constructions of Carrà Ferro to the non-affine case and, much more important, to get a geometric understanding of these latter.
3.1 Invariant closed and open sets

We begin this section by defining invariant closed and open sets. For the sake of simplicity, we stick to \(\mathbb{Q}\)-schemes with vector fields but all what follows should work for schemes with Hasse-Schmidt derivations.

**Definition 3.1.** Let \(X\) be a \(\mathbb{Q}\)-scheme with a vector field \(\vec{V}\). A closed set \(F\) of \(X\) will be said invariant under \(\vec{V}\) when

\[
\forall x \in F, \quad \text{Traj}_{\vec{V}}(x) \in F.
\]

An open set \(U\) of \(X\) will be said invariant under \(\vec{V}\) when the closed set \(X \setminus U\) is.

We now prove an analogue of Theorem 2.5 for open sets:

**Proposition 3.2.** Let \(X\) be a \(\mathbb{Q}\)-scheme endowed with a vector field \(\vec{V}\). Let \(U\) be an open set of \(X\). Then, the set

\[
\left\{ V \mid U \subset V \right\}
\]

has a least element. We denote it by \(U^\delta\), and call it the invariant open set associated to \(U\).

**Remark.** Of course, dually, there also exists a greatest invariant closed set included in \(F\), when \(F\) is a closed set of \(X\).

**Proof.** We keep the notations of the statement. Let’s consider the set

\[
V_0 = \{ x \in X \mid \text{Traj}_{\vec{V}}(x) \in U \}.
\]

Since the map \(\text{Traj}_{\vec{V}}\) is continuous, \(V_0\) is an open set of \(X\). Furthermore, \(U \subset V_0\), for we always have \(\text{Traj}_{\vec{V}}(x) \sim x\), and for open sets are stable under generization. Now, let’s check that \(V_0\) is invariant: let \(x \notin V_0\). Then, \(\text{Traj}_{\vec{V}}(x) \notin V_0\), since \(\text{Traj}_{\vec{V}}(\text{Traj}_{\vec{V}}(x)) = \text{Traj}_{\vec{V}}(x) \notin U\). Last, let’s prove that \(V_0\) is the least such set. Let \(V \supset U\) be an invariant open set and \(x \in V_0\). If \(x \notin V\), then, by invariance, one would have \(\text{Traj}_{\vec{V}}(x) \notin V\). But, by definition of \(x \in V_0\), one has \(\text{Traj}_{\vec{V}}(x) \in U\) and thus \(\text{Traj}_{\vec{V}}(x) \in V\). This is absurd. Hence, \(x \in V\) and so, \(V_0 \subset V\).

If \(A\) is a differential ring, if \(X = \text{Spec}^\partial A\) and if \(U\) is the open set of \(X\) defined by an ideal \(I\), then \(U^\delta\) is the open set defined by the differential ideal (\(I\)). Indeed, one has

\[
U^\delta = (\text{Traj}_{\vec{V}})^{-1} U = X \setminus V((I)),
\]

as it has been shown in the proof of Proposition 2.7.
3.2 The Carrà Ferro topology of $\mathcal{X}$

We now prove that the invariant open sets form a topology:

**Proposition 3.3.** Let $\mathcal{X} = (X, \vec{V})$ be a $\mathbb{Q}$-scheme endowed with a vector field. Let $(U_i)_{i \in I}$ be a family of open sets of $X$. Then:

$$\left( \bigcup_{i \in I} U_i \right)^{\delta} = \bigcup_{i \in I} U_i^{\delta} \quad \text{and, when } I \text{ is finite,} \quad \left( \bigcap_{i \in I} U_i \right)^{\delta} = \bigcap_{i \in I} U_i^{\delta}.$$

In particular, the invariant open sets of $X$ form a topology of $X$. We call it the Carrà Ferro topology of $\mathcal{X}$.

**Proof.** — This comes from the fact that for any map $f : E \to F$, $f^{-1}$ commutes with unions and intersections, applied to $f = \text{Traj}_{\vec{V}}$. ■

Consequently, the subset $X^{\vec{V}}$ of $X$ can be endowed with two induced topologies: the one induced by Zariski, and the one induced by Carrà Ferro. They are the same:

**Proposition 3.4.** Let $\mathcal{X} = (X, \vec{V})$ be a $\mathbb{Q}$-scheme endowed with a vector field. Then, the Zariski topology of $X$ and the Carrà Ferro topology of $\mathcal{X}$ induce the same topology on $X^{\vec{V}}$.

**Proof.** — Since, the Carrà Ferro topology is a subtopology of the Zariski topology, it suffices to prove that, if $U$ is Zariski open set of $X$, then, there exists an invariant open set $V$ of $X$ such that:

$$U \cap X^{\vec{V}} = V \cap X^{\vec{V}}.$$

It suffices to take $V := U^{\delta}$. ■

3.3 The Carrà Ferro sheaf and the Keigher sheaf on $\mathcal{X}$

Now, we would like to equip the topological space $X^{\vec{V}}$ with a sheaf. For this, we have three possibilities.

a) First, if we denote by $X_{\text{Zar}}$ the scheme $X$ endowed with the Zariski topology, then the inclusion map

$$i_{\text{Zar}} : X^{\vec{V}} \hookrightarrow X_{\text{Zar}}$$

is a continuous map. Since $X_{\text{Zar}}$ comes with the scheme-structure sheaf $\mathcal{O}_X$, one can consider the pull-back of $\mathcal{O}_X$ by $i_{\text{Zar}}$. In other words, one can consider the restriction of $\mathcal{O}_X$ to the subspace $X^{\vec{V}}$. It is a sheaf denoted by

$$(i_{\text{Zar}})^{-1}\mathcal{O}_X$$
and defined as the sheaf associated to the preasheaf
\[ U \mapsto \colim_{V \text{ open in } X \text{ and } U \subset V} \mathcal{O}_X (V) . \]
Indeed, this latter is not always a sheaf. This sheaf is naturally a sheaf of
differential \( \mathbb{Q} \)-algebras.

b) Second, we can do the same but with the Carrà Ferro topology instead of
the Zariski one. So, if we denote by \( X_{\text{CF}} \) the scheme \( X \) equipped \textit{with the}
Carrà Ferro topology, it is still possible to consider the inclusion map
\[ i_{\text{CF}} : X^\mathbf{\varphi} \longrightarrow X_{\text{CF}} : \]
it is also a continuous map. The sheaf \( \mathcal{O}_X \), defined on \( X_{\text{Zar}} \), induces
naturally a sheaf on \( X_{\text{CF}} \), which we still denote by \( \mathcal{O}_X \). Thus, similarly,
one can consider the sheaf
\[ (i_{\text{CF}})^{-1} \mathcal{O}_X . \]

c) Third, there is another sheaf that one can define on \( X^\mathbf{\varphi} \). Indeed, since
\( \text{Traj}_\mathbf{\varphi} : X_{\text{Zar}} \longrightarrow X^\mathbf{\varphi} \) is a continuous map and since \( X_{\text{Zar}} \) comes with the
sheaf \( \mathcal{O}_X \), one can consider the push-forward of \( \mathcal{O}_X \) by \( \text{Traj}_\mathbf{\varphi} \). It is a sheaf
denoted by
\[ (\text{Traj}_\mathbf{\varphi})_* \mathcal{O}_X \]
and whose definition is simpler than for the pull-back: if \( U \) is a open set
of \( X^\mathbf{\varphi} \), one has, by definition
\[ (\text{Traj}_\mathbf{\varphi})_* \mathcal{O}_X (U) := \mathcal{O}_X ((\text{Traj}_\mathbf{\varphi})^{-1} U) . \]

\textbf{Notation 3.5.} \textit{When \( U \) is a open set of \( X^\mathbf{\varphi} \), we denote}
\[ U_\Delta := (\text{Traj}_\mathbf{\varphi})^{-1} U = \{ x \in X \mid \text{Traj}_\mathbf{\varphi}(x) \in U \} . \]
It is easy to check that \( U_\Delta \) is an invariant open set of \( X \). With this notation,
we have \( (\text{Traj}_\mathbf{\varphi})_* \mathcal{O}_X (U) = \mathcal{O}_X (U_\Delta) \). We have:

\textbf{Proposition 3.6.} \textit{Let} \( \mathcal{X} = (X, \mathbf{\mathbf{\mathbf{\varphi}}}) \) \textit{be} \( \mathbb{Q} \)-\textit{scheme endowed with a vector field.}
\textit{Then,}
\[ (i_{\text{CF}})^{-1} \mathcal{O}_X = (\text{Traj}_\mathbf{\varphi})_* \mathcal{O}_X . \]

\textit{Proof.} — We keep the notations of the proposition. Let \( U \) be an open set
of \( X^\mathbf{\varphi} \). We will prove that \( \mathcal{O}_X (U_\Delta) \) is an inductive limit of the \( \mathcal{O}_X (V) \), for
\( V \) invariant open set of \( X \) such that \( U \subset V \). So, let \( V \) be an invariant open
set containing \( U \). Then, \( U_{\Delta} \subset V \). Hence, the restrictions form a bunch of compatible maps
\[
\psi_V : \mathcal{O}_X (V) \to \mathcal{O}_X (U_{\Delta}).
\]
Let’s prove that these maps make \( \mathcal{O}_X (U_{\Delta}) \) an inductive limit. It’s easy. Let \( A \) be a differential ring, equipped with compatible maps \( \varphi_V : \mathcal{O}_X (V) \to A \) for all invariant open set \( V \) containing \( U \). In particular, there is a map
\[
f := \varphi_{U_{\Delta}} : \mathcal{O}_X (U_{\Delta}) \to A.
\]
What we want to prove is that, for every \( V \), the diagram
\[
\begin{array}{c}
\mathcal{O}_X (V) \\
\downarrow \psi_V \hspace{1cm} \downarrow \varphi_V
\end{array}
\begin{array}{c}
\mathcal{O}_X (U_{\Delta}) \\
\downarrow f
\end{array}
\to A
\]
commutes. This follows from the compatibility of the family \( (\varphi_V)_V \). ■

**Definition 3.7.** Let \( \mathcal{X} = (X, \vec{V}) \) be \( \mathbb{Q} \)-scheme endowed with a vector field. The Keigher sheaf on \( X^{\vec{V}} \) is
\[
\mathcal{O}^{(\text{Keigher})}_{X^{\vec{V}}} := (i_{\text{zar}})^{-1} \mathcal{O}_X.
\]
The Carrà Ferro sheaf on \( X^{\vec{V}} \) is
\[
\mathcal{O}^{(\text{CF})}_{X^{\vec{V}}} := (i_{\text{CF}})^{-1} \mathcal{O}_X = (\text{Traj}_{\vec{V}})_* \mathcal{O}_X.
\]

With these definitions, Corollary 2.4 of [Car90] generalizes to the following:

**Proposition 3.8.** Let \( \mathcal{X} = (X, \vec{V}) \) be \( \mathbb{Q} \)-scheme endowed with a vector field. Then,
\[
\Gamma (X^{\vec{V}}, \mathcal{O}^{(\text{CF})}_{X^{\vec{V}}}) = \Gamma (X, \mathcal{O}_X).
\]
In particular, if \( A \) is \( \mathbb{Q} \)-differential algebra,
\[
\Gamma (\text{diff-Spec} A, \mathcal{O}^{(\text{CF})}_{\text{diff-Spec} A}) = A.
\]

**Proof.** — By definition,
\[
\Gamma (X^{\vec{V}}, \mathcal{O}^{(\text{CF})}_{X^{\vec{V}}}) = \Gamma ((X^{\vec{V}})_\Delta, \mathcal{O}_X).
\]
But, it is clear that \((X^{\vec{V}})_\Delta = X\) and thus, the result follows. ■
3.4 The Kovacic sheaf

When $X$ is affine, a third sheaf has been studied. Although it has been defined for the first time by Carr Ferro in [Car85], we call it the Kovacic sheaf. Indeed, in a series of papers [Kov02a, Kov02b, Kov03, Kov06], Kovacic intensively uses and studies this sheaf. Here is its definition:

**Definition 3.9.** Let $A$ be a differential ring and $U$ an open set of $\text{diff-Spec } A$. The Kovacic sheaf $\mathcal{O}^{(Kov)}_{\text{diff-Spec } A}$ is defined by

$$\mathcal{O}^{(Kov)}_{\text{diff-Spec } A}(U) := \begin{cases} s : U \rightarrow \prod_{p \in U} A_p & (i) \\
\forall p \in U, s(p) \in A_p & \\
\exists (U_i)_{i \in I} \text{ open covering of } U, \exists (a_i)_{i \in I}, (b_i)_{i \in I} \in A^I, & (ii) \\
\forall p \in U, \forall i \in I, p \in U_i \implies (b_i \notin p \text{ and } s(p) = \frac{a_i}{b_i}) & 
\end{cases}$$

We will prove further that $\mathcal{O}^{(Kov)}_{\text{diff-Spec } A}$ and $\mathcal{O}^{(Keigher)}_{\text{diff-Spec } A}$ are isomorphic.

4 Extension of constants

In this section, we prove the following result, which will be our main tool to compare the Keigher sheaf and the Carrà Ferro sheaf. If $A$ is a differential ring, we denote by $C(A)$ the ring of constants of $A$.

**Proposition 4.1.** Let $X = (X, \vec{V})$ be a reduced $\mathbb{Q}$-scheme endowed with a vector field. Let $U$ be an open set of $X$. Then, for every $f \in C(\mathcal{O}_X(U))$, there exists a unique $\tilde{f}$ in $C(\mathcal{O}_X(U^\delta))$ such that $\tilde{f}_{|U} = f$.

Furthermore, this extension map

$$\text{ext}_{U \rightarrow U^\delta} : C(\mathcal{O}_X(U)) \longrightarrow C(\mathcal{O}_X(U^\delta))$$

$\begin{align*}
f \mapsto \tilde{f}
\end{align*}$

is an isomorphism of rings, whose inverse $C(\mathcal{O}_X(U^\delta)) \longrightarrow C(\mathcal{O}_X(U))$ is the restriction map.
4.1 Constants in localized rings

In order to prove Proposition 4.1, we need to study the properties of constant elements in differential rings of the form $S^{-1}A$. If $x = a/s$ is such an element, by differentiating $x$, one gets

$$\frac{a's - s'a}{s^2} = 0 \quad \text{in } S^{-1}A.$$

One would like to derive from this, identities such as

$$a's - s'a = 0 \quad \text{and so} \quad \frac{a}{s} = \frac{a'}{s'} \quad \text{and so} \quad \forall i \in \mathbb{N}, \quad \frac{a}{s} = \frac{a^{(i)}}{s^{(i)}}.$$

Unfortunately, these latters are false, since we don’t have $a's - s'a = 0$ but only $\exists t \in S, \ t \cdot (a's - s'a) = 0$, and since the elements $s^{(i)}$ do not necessarily belong to $S$. Nevertheless, when $s^{(i)} \in S$, we do have $a/s = a^{(i)}/s^{(i)}$ in $S^{-1}A$. This is what tells us the following proposition.

**Proposition 4.2.** Let $A$ be a differential ring. Let $\theta, a, b \in A$ such that

$$\theta \cdot (a'b - ab') = 0.$$

1) Then, for all $N \in \mathbb{N}_{\geq 3}$ and for all $0 \leq i \leq N$, one has

$$\theta \cdot (a'b - ab') = 0$$

$$\theta^2 \cdot (a''b - ab'') = 0$$

$$b^{N-1} \theta^N \cdot \left( b^{(i)} a^{(N-i)} - a^{(i)} b^{(N-i)} \right) = 0.$$

2) In particular, when $S$ is a multiplicative subset of $A$, if $s \in S$ and if $i \in \mathbb{N}$, one has

$$\left( \frac{a}{s} \right)^i = 0 \quad \text{in } S^{-1}A \quad \text{if} \quad s^{(i)} \in S \quad \Rightarrow \quad \frac{a}{s} = \frac{a^{(i)}}{s^{(i)}} \quad \text{in } S^{-1}A.$$

In order to prove this proposition, we need the following lemma :

**Lemma 4.3.** Let $A$ be a differential ring and let $t, A_1, A_2, B_1, B_2 \in A$. Then,

$$t \cdot (A_1B_2 - B_1A_2) = 0$$

$$\downarrow$$

$$t^2 \cdot (B_2^2 \cdot (A_1A_1' - A_1B_1') - B_1^2 \cdot (A_2^2B_2 - B_2^2A_2)) = 0.$$
Proof of Lemma 4.3: Let $A$ be a differential ring and $t, A_1, A_2, B_1, B_2 \in A$. Let’s denote $\Theta = t \cdot (A_1B_2 - B_1A_2)$. A simple computation shows that
\[
t B_1 B_2 \frac{\partial \Theta}{\partial x} - t B_1 B_2 ' \cdot \Theta - t B_1 ' B_2 \Theta - t B_1 B_2 \Theta =
\]
\[
t^2 \cdot (B_2 ^2 \cdot (A_1 ' B_1 - A_1 B_1 ') - B_1 ^2 \cdot (A_2 ' B_2 - A_2 B_2 ')).
\]
Hence, when $\Theta = 0$, one gets the required identity. ■

Now, we can prove Proposition 4.2:

Proof of Proposition 4.2: We keep the notations of the proposition. In particular, we assume that $\theta \cdot (a' b - a b') = 0$. We denote
\[
E_{N,i} := b^{N-1} \theta^N \cdot (b^{(i)} a^{(N-i)} - a^{(i)} b^{(N-i)}).
\]
Let’s begin by showing the assertion 1); we want to prove
\[
\theta \cdot (a' b - a b') = 0
\]
\[
\theta^2 \cdot (a'' b - a b'') = 0
\]
\[
\forall N \geq 3, \forall 0 \leq i \leq N, \quad b^{N-1} \theta^N \cdot (b^{(i)} a^{(N-i)} - a^{(i)} b^{(N-i)}) = 0.
\]
The first identity is our assumption; one gets the second one by differentiating the first one and by multiplying it by $\theta$. For the bunch of next identities, we proceed by induction. For $N = 3$, let’s remark that, when differentiating the second identity and multiplying it by $\theta$, one gets:
\[
\theta^3 \cdot ((a'' b' - a' b'') + (a''' b - a b''')) = 0.
\]
But, by applying Lemma 4.3 with $t = \theta$, $A_1 = a$, $B_2 = b'$, $B_1 = b$ et $A_2 = a'$, one gets
\[
b^2 \theta^2 \cdot (b'' a' - a' b'') = 0;
\]
hence, in particular, one has
\[
b^2 \theta^3 \cdot (b'' a' - a' b'') = 0
\]
and, with (1),
\[
b^2 \theta^3 \cdot (a''' b - a b''') = 0
\]
Now, let’s assume the assertion 1) true for $n \leq N$ and let’s show it for $N + 1$. First, a simple computation shows that
\[
b \theta \cdot \frac{\partial E_{N,i}}{\partial x} = E_{N+1,i+1} + E_{N+1,i}.
\]
Thus, for $0 \leq i \leq N$, one has $E_{N+1,i+1} + E_{N+1,i} = 0$. A consequence of these identities is that, if there exists $i_0$ such that $E_{N+1,i_0} = 0$ then all the $E_{N+1,i}$
are zero. Indeed, in that case, one would have

$$
\begin{pmatrix}
1 & 1 & 0 & \cdots \\
0 & 1 & 1 & \\
\vdots & & & \\
\cdots & & & \\
\cdots & 0 & 1 & 1 \\
\cdots & 0 & 1 & 0 & \\
\cdots & i_0 & 0 & \cdots
\end{pmatrix}

\begin{pmatrix}
E_{N+1,0} \\
E_{N+1,1} \\
\vdots \\
E_{N+1,N+1}
\end{pmatrix}
= 0.
$$

But, developing along the last line, one finds that $\det A = (-1)^{N+i_0}$ and thus that $A$ is invertible. So, it suffices to find $i_0$ such that $E_{N+1,i_0} = 0$. We consider two cases. If $N + 1 = 2k$ is even, then one has

$$
E_{N+1,k} = b^N \theta^{N+1} \cdot \left( b^{(k)} a^{((N+1)-k)} - a^{(k)} b^{((N+1)-k)} \right)
= b^N \theta^{N+1} \cdot \left( b^{(k)} a^{(k)} - a^{(k)} b^{(k)} \right) = 0,
$$

and we can conclude. If $N + 1 = 2k + 1$ is odd, we know, by the induction assumption, that

$$
E_{k,0} = b^{k-1} \theta^k \cdot \left( a^{(k)} b + b^{(k)} a \right) = 0.
$$

Then, if we use Lemma 4.3 with the data

$$
t = \left( b^{k-1} \theta^k \right) \quad A_1 = a^{(k)} \quad B_2 = b \quad B_1 = b^{(k)} \quad A_2 = A,
$$

we get

$$
\left( b^{k-1} \theta^k \right)^2 \cdot \left( b^{(k)} a^{(k+1)} - a^{(k)} b^{(k+1)} \right) + b^{(k)} \cdot (a' b - b' a) = 0.
$$

So, given $\theta \cdot (a' b - b' a) = 0$, we get

$$
b^{2k} \theta^{2k} \cdot \left( b^{(k)} a^{(k+1)} - a^{(k)} b^{(k+1)} \right) = 0.
$$

Multiplying it by $\theta$, we get $E_{N+1,k} = 0$ — and so, all the $E_{N+1,i}$ are zero.

Now, let’s move to the assertion 2). It is an easy consequence of 1). Indeed, let $S$ be a multiplicative subset of $A$ and let $(a, s) \in A \times S$ such that

$$
\left( \frac{a}{s} \right)' = 0 \quad \text{in } S^{-1} A.
$$
It means that there exists $\theta \in S$ such that $\theta \cdot (a's - s'a) = 0$. Let’s assume now that $i \in \mathbb{N}$ verifies $s^{(i)} \in S$. The identity $E_{i,0} = 0$ that we have just shown tells us that

$$\left( s^{i-1} \theta \right) \cdot \left( a^{(i)} s - a s^{(i)} \right) = 0.$$ 

For $s^{(i)} \in S$, this implies

$$\frac{a}{s} = \frac{a^{(i)}}{s^{(i)}} \quad \text{in } S^{-1}A.$$

\section*{4.2 A lemma on stalks and trajectories}

We will also need the following:

**Lemma 4.4.** Let $(X, \mathcal{V})$ be a $\mathbb{Q}$-scheme endowed with a vector field. Let $x \in X$, let $U$ be an open neighborhood of $x$ and let $f \in \mathcal{O}_X(U)$. Then,

(i) $f_{\text{Traj} \mathcal{V}(x)} = 0 \implies \exists n \in \mathbb{N} \mid (f_x)^n = 0.$

(ii) $(f_{\text{Traj} \mathcal{V}(x)} = 0 \text{ and } f' = 0) \implies f_x = 0.$

**Remark.** — This result is false out of the differential context: if $X$ is a scheme, if $x \in X$ and if $\eta \mapsto x$ is a generalization of $x$, then

$$f_{\eta} = 0 \iff \exists n \in \mathbb{N} \mid (f_x)^n = 0.$$ 

To see this, it suffices to consider the closed subscheme of $\mathbb{A}^2_\mathbb{C}$, union of the axes $x = 0$ and $y = 0 : X = \text{Spec } \mathbb{C}[x,y]/(xy)$. In this scheme, the function $y$ is zero in $\mathcal{O}_{X,\eta_x}$ — where $\eta_x$ stands for the generic point of the axis $y = 0$ — although $y$ is not nilpotent in $\mathcal{O}_{X,(0,0)}$. \diamond

**Proof.** — For the property we want to show is local, it suffices to prove it for affine schemes. So, let $A$ be a differential ring. To begin with, let’s prove the small following result :

$$\forall(\theta, f) \in A^2, \quad \theta f = 0 \implies (\forall n \in \mathbb{N}, \quad \theta^{(n)} f^{n+1} = 0).$$

We proceed by induction: if $\theta^{(n)} f^{n+1} = 0$, by differentiating this identity, one gets

$$\theta^{(n+1)} f^{n+1} + (n+1)\theta^{(n)} f' f^n = 0.$$ 

By multiplying the latter by $f$, one gets $\theta^{(n+1)} f^{n+2} = 0$. Now let’s move to assertion (i): let $p$ be a prime ideal of $A$ and let $f \in A$ such that

$$f = 0 \quad \text{in } A_p.$$
This means that there exists $\theta \notin p$ such that $\theta f = 0$. But, $\theta \notin p$ means that there exists $n \in \mathbb{N}$ such that $\theta^{(n)} f \notin p$. Since, we know that $\theta^{(n)} f^{n+1} = 0$, we have $f^{n+1} = 0$ in $A_p$.

Lastly, let’s prove (ii). With the previous notations, we assume, in addition that $f' = 0$. From $\theta f = 0$, one gets, by induction, that $\theta^{(m)} f = 0$ for all $m$. In particular, one has that $\theta^{(n)} f = 0$ and so $f = 0$ in $A_p$. □

### 4.3 Proof of Proposition 4.1

Now, we come to the proof of our result on the extension of constant sections of the structure sheaf. So, let $X$ be reduced $\mathbb{Q}$-scheme, equipped with a vector field $\vec{V}$. Let $U$ be an open set of $X$ and let $f \in C(\mathcal{O}_X(U))$. We start by proving the unicity of an extension of $f$ to $U^\delta$. So, let $\tilde{f}^1, \tilde{f}^2 \in C(\mathcal{O}_X(U^\delta))$ such that $\tilde{f}^1|_U = \tilde{f}^2|_U = f$. Let $x \in U^\delta$. This means that $\text{Traj}_{\vec{V}}(x) \in U$. Let’s denote $y := \text{Traj}_{\vec{V}}(x)$. Thus, one has $\tilde{f}^1_y = \tilde{f}^2_y$.

Consequently, by lemma 4.4 (ii), one has $\tilde{f}^1_x = \tilde{f}^2_x$. Hence, $\tilde{f}^1 = \tilde{f}^2$, what we wanted to show.

Let’s prove now existence of such a extension. Let’s assume that we had shown it in the affine case and let’s show it in the general case. Let $(\Omega_i)_{i \in I}$ be a basis of open affine sets of $X$. We denote $U_i = U \cap \Omega_i$ and $f_i = f|_{U_i}$. According to the affine case, one hence has

$$\tilde{f}_i \in C\left(\mathcal{O}_{\Omega_i}(U_i^\delta_{(\subset \Omega_i)})\right)$$

such that $f_i = \tilde{f}_i|_{U_i}$, where $U_i^\delta_{(\subset \Omega_i)}$ stands for the invariant open set of $\Omega_i$ associated to $U_i$:

$$U_i^\delta_{(\subset \Omega_i)} := \{ x \in \Omega_i \mid \text{Traj}_{\vec{V}}(x) \in U_i \}.$$

Let’s prove that the $\tilde{f}_i$ patch together, so that one can derive from them a function $\tilde{f}$ extending $f$ on $U^\delta$. First, remark that

$$\bigcup_{i \in I} U_i^\delta_{(\subset \Omega_i)} = U^\delta.$$

This follows from

$$U_i^\delta_{(\subset \Omega_i)} = U^\delta \cap \Omega_i.$$

Now, if $i$ and $j$ are such that $\Omega_j \subset \Omega_i$, since $U_j^\delta_{(\subset \Omega_i)} = U_i^\delta_{(\subset \Omega_i) \cap \Omega_j}$ and by unicity of the extension, one has

$$\tilde{f}_j = (\tilde{f}_i)|_{\Omega_j}.$$
Finally, if we denote by \( \tilde{f} \) the patching of the \( \tilde{f}_i \), it is clear that \( \tilde{f}' = 0 \) and that \( \tilde{f}|_U = f \).

Last, but not least, let’s prove the result for affine schemes. For this sake, let’s assume that this following lemma is true. We will prove it after.

**Lemma 4.5.** Let \( A \) be a differential reduced ring and let \( U \) be an open subset of \( \text{diff-Spec} A \). Let \( s \in \mathcal{O}_{\text{diff-Spec} A}(U) \). Then,

(i) there exist a Zariski open set \( W \) of \( \text{Spec} A \), containing \( U \), and \( t \in \mathcal{O}_{\text{Spec} A}(W) \) such that for all \( x \in U \), the stalk \( t_x \) equals \( s(x) \). Moreover, when \( W \subset U_{\Delta} \), this extension \( t \) is unique.

(ii) If, moreover, \( s' = 0 \), this \( W \) can be taken to be equal to \( U_{\Delta} \): there exists a unique \( t \in C(\mathcal{O}_{\text{Spec} A}(U_{\Delta})) \) such that

\[
\forall x \in U, \quad t_x = s(x).
\]

So, let \( A \) be a reduced \( \mathbb{Q} \)-differential algebra, let \( V \) be an open set of \( X := \text{Spec} A \), and let \( f \in \mathcal{O}_X(V) \) a section satisfying \( f' = 0 \). We denote \( U := V \cap \text{diff-Spec} A \). Then, we have \( V^\delta = U_{\Delta} \). If we consider the Hartshorne-like \([\text{Har77}]\) definition of \( f \), then it is clear that \( f \) induces on \( U \) a constant section \( s \) of the Kovacic sheaf. Applying Lemma 4.5 (ii) to \( s \), one gets a constant section \( \tilde{f} \in C(\mathcal{O}_{\text{Spec} A}(U_{\Delta})) \). We just know that \( \tilde{f} \) and \( f \) coincide (in a stalkwise sense) on \( U \). But, since \( X \) is reduced, by Lemma 4.4 this is sufficient to prove that they coincide stalkwisely in \( U_{\Delta} \) and so that they are equal. Now, to conclude the proof of Proposition 4.1, all that remains is to prove Lemma 4.5.

**Proof of Lemma 4.5.** — We keep the notations of the lemma. We start by proving the point (ii). Hence, let \( s \) be a constant section of the Kovacic sheaf. It comes with a covering \((U_i)_{i \in I}\) of \( U \) and two families \((a_i)_{i \in I}\) and \((b_i)_{i \in I}\) fulfilling the required conditions. The unicity is a consequence of Lemma 4.4 (ii), as for Proposition 4.1. For the existence, we use the Hartshorne \([\text{Har77}]\) definition of the structure sheaf of \( \text{Spec} A \). Hence, we look for

\[ \begin{align*}
&\text{a) a family } (t(p))_{p \in U^\delta}, \text{ where } t(p) \in A_p \text{ for each } p, \text{ such that } \\
&\quad \forall p \in U, \quad t(p) = s(p) \quad \text{ and } \quad \forall p \in U^\delta, \quad t(p)' = 0.
\end{align*} \]

\[ \begin{align*}
&\text{b) a covering } (\Omega^\ell)_{\ell} \text{ of } U^\delta \text{ and two families } (\alpha^\ell)_{\ell} \text{ and } (\beta^\ell)_{\ell} \text{ such that } \\
&\quad \forall p \in U^\delta, \quad \forall \ell, \quad \left( p \in \Omega^\ell \quad \Rightarrow \quad \beta^\ell \notin p \right) \text{ and } \quad t(p) = \frac{\alpha^\ell}{\beta^\ell} \text{ in } A_p.
\end{align*} \]

For the item a), we proceed as follows. Let \( p \in U^\delta \): this means that \( p_{\#} \in U \). Hence, one can consider \( s(p_{\#}) \) and denote

\[ s(p_{\#}) = \frac{a_p}{b_p}, \]

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with \( b_p \notin p_{\#} \). This means that there exists an integer \( n \) such that \( b_p^{(n_p)} \notin p \). We will denote the least of these integers by \( n_p \). Then, we define

\[
t(p) := \frac{a_p^{(n_p)}}{b_p^{(n_p)}} \in A_p.
\]

Let’s check that this family fulfill the two required conditions. If \( p \in U \), then \( p_{\#} = p \). So, we want to check that

\[
\frac{a_p}{b_p} = \frac{a_p^{(n_p)}}{b_p^{(n_p)}} \quad \text{in } A_{p_{\#}},
\]

but this follows from the point 2) of Proposition 4.2. Let \( p \in U_\delta \). Since, \( s \) is a constant section of the Kovacic sheaf, we know that \( s(p_{\#})' = 0 \). That means that

\[
\frac{a_p'b_p - a_pb_p'}{b_p^2} = 0 \quad \text{in } A_{p_{\#}},
\]

thus, there exists \( \theta \notin p_{\#} \) such that

\[
\theta \cdot (a_p'b_p - a_pb_p') = 0.
\]

Now, by point 1) of Proposition 4.2 we know:

\[
b_p^{2n_p} \theta^{2n_p+1} \cdot \left( a_p^{(n_p+1)} b_p^{(n_p)} - b_p^{(n_p+1)} a_p^{(n_p)} \right) = 0. \tag{2}
\]

Let’s denote \( c := b_p^{2n_p} \theta^{2n_p+1} \). It is an element that doesn’t belong to \( p_{\#} \) and so, let \( m \in \mathbb{N} \) such that \( c^{(m)} \notin p \). As in the proof of Lemma 4.3, one deduces from (2) that

\[
c^{(m)} \cdot \left( a_p^{(n_p+1)} b_p^{(n_p)} - b_p^{(n_p+1)} a_p^{(n_p)} \right)^{m+1} = 0.
\]

Since \( A \) is reduced, one has

\[
c^{(m)} \cdot \left( a_p^{(n_p+1)} b_p^{(n_p)} - b_p^{(n_p+1)} a_p^{(n_p)} \right) = 0.
\]

This implies

\[
\left( \frac{a_p^{(n_p)}}{b_p^{(n_p)}} \right)' = 0 \quad \text{in } A_p.
\]

In other words, it means that \( t(p)' = 0 \) for all \( p \in U \).

Let’s move now to the item b). For the covering of \( U_\delta \), we choose

\[
V_{i,n} := U_i^\delta \cap D(b_i^{(n)}) \quad \text{for } i \in I \text{ and } n \in \mathbb{N}.
\]
Indeed, let $p \in U^\delta$; this means that $p \notin U$. Hence, let $i \in I$ such that $p_i \in U_i$. Then, since $p_i \in U_i$, we know that $b_i \notin p_i$. Thus, there exists $n \in \mathbb{N}$ such that $b_i^{(n)} \notin p_i$: in other words, $p \in D(b_i^{(n)})$. Hence, we have found a couple $(i, n)$ such that $p \in V_{i,n}$. As families of elements, we choose

$$\alpha_{i,n} := a_i^{(n)} \quad \text{and} \quad \beta_{i,n} := b_i^{(n)}.$$

So, let $p \in U^\delta$ and $(i, n)$ such that $p \in V_{i,n}$. First, we have $\beta_{i,n} / \in p$. So, we have to check that

$$t(p) = \frac{\alpha_{i,n}}{\beta_{i,n}} = \frac{a_i^{(n)}}{b_i^{(n)}} \quad \text{in } A_p$$

By assumption, we have $p \notin U_i$ and so

$$s(p) := \frac{a_p}{b_p} = \frac{a_i}{b_i} \quad \text{in } A_p.$$

Then, since these two elements have a zero derivative, and since, on the other hand, we know that $b_i^{(n)} / \in p$ and $b_i^{(n)} / \in p_i$, Proposition 4.2 tells us that

$$\frac{a_p^{(n_p)}}{b_p^{(n_p)}} = \frac{a_i^{(n)}}{b_i^{(n)}} \quad \text{in } A_p.$$

Then, applying Lemma 4.4 (i) and using the fact that $A$ is reduced, we infer that

$$\frac{a_p^{(n_p)}}{b_p^{(n_p)}} = \frac{a_i^{(n)}}{b_i^{(n)}} \quad \text{in } A.$$

This concludes the proof of (ii).

Now, let’s indicate quickly why (i) is true. We start with a section $s \in \mathcal{O}_{\text{diff-Spec } A}(U)$, a covering $(U_i)_i$ of $U$ and two families $(a_i)_i$ and $(b_i)_i$ of $A$ such that

$$\forall p \in U, \quad p \in U_i \implies \left( b_i \notin p \quad \text{and} \quad s(p) = \frac{a_i}{b_i} \quad \text{in } A_p \right).$$

By definition, one can find Zariski open sets $W_i$ of Spec $A$ such that $U_i = W_i \cap \text{diff-Spec } A$, for all $i$. One can replace the $W_i$’s by the $\tilde{W}_i$ defined by

$$\tilde{W}_i := W_i \cap D(b_i).$$

Then, one considers $W = \bigcup \tilde{W}_i$. This is a Zariski open set of Spec $A$ containing $U$. Let $p \in W$ and assume $p \in \tilde{W}_i$ and $p \in \tilde{W}_j$ for two indexes $i$ and $j$. Then,

$$\frac{a_i}{b_i} = \frac{a_j}{b_j} \quad \text{in } A.$$

Indeed, $p \notin U_i$ and in $U_j$ and so $a_i/b_i = a_j/b_j$ in $A_{p \#}$. But, by Lemma 4.4 and for $A$ is reduced, this implies (3). Then, one can define $t \in \mathcal{O}_{\text{Spec } A}(W)$ by setting $t(x) = a_i/b_i$ when $x \in \tilde{W}_i$. The statement on unicity comes from Lemma 4.4. ■
5 Comparison of the Carr` a Ferro, Keigher and Kovacic sheaves

5.1 Comparison of the Carr` a Ferro and Keigher sheaves

We now come to the main result of this paper:

**Theorem 5.1.** Let $X$ be a reduced $\mathbb{Q}$-scheme endowed with a vector field. Then, the Carr` a Ferro sheaf and the Keigher sheaf have the same constants:

$$\forall U \text{ open in } X, \quad C\left(\mathcal{O}_{\mathcal{X}}^{(\text{Keigher})}(U)\right) \simeq C\left(\mathcal{O}_{\mathcal{X}}^{\text{(CF)}}(U)\right).$$

**Proof.** — We keep the notations of the theorem. The Keigher sheaf is defined as the associated sheaf to a certain presheaf. Hence, thanks to Proposition A.3, the constant of the Keigher sheaf is the same as the associated sheaf to the constant of this certain presheaf. More precisely, one has

$$C\left(\mathcal{O}_{\mathcal{X}}^{(\text{Keigher})}(U)\right) \simeq \left(U \mapsto \operatorname{colim}_{V \text{ open in } X \text{ and } U \subset V} C(\mathcal{O}_{\mathcal{X}}(V))\right)^\dagger.$$

Naturally, one would like to interchange $C(-)$ with colim. This is not possible in general. For instance, the reader might search an example where $C(A_1 \otimes A_2) \neq C(A_1) \otimes C(A_2)$.

But, in this situation, the colimit that we want to compute is of a very special kind: it is a filtered colimit. And, for such colimits, one has

$$C(\operatorname{colim}_i A_i) \simeq \operatorname{colim}_i C(A_i).$$

Indeed, one knows, as explained in chapters 9 and 11 of [Sch72], and after Theorem 11.5.7 of the same book, that in the category $\mathbf{Rng}$ filtered colimits commute with finite limits. But, given a differential ring $A$, its ring of constants can be characterized as a finite limit. More precisely, $C(A)$ can be characterized as the following kernel:

$$C(A) \xrightarrow{1d + \partial(\cdot)} A \xrightarrow{\delta} A[\varepsilon]/\varepsilon^2.$$

Hence, $C(-)$ commutes with filtered colimits. So, one gets that

$$C\left(\mathcal{O}_{\mathcal{X}}^{(\text{Keigher})}(U)\right) \simeq \left(U \mapsto \operatorname{colim}_{V \text{ open in } X \text{ and } U \subset V} C(\mathcal{O}_{\mathcal{X}}(V))\right)^\dagger.$$

Now, let $U$ be an open set of $X$. We will prove that

$$\operatorname{colim}_{V \text{ open in } X \text{ and } U \subset V} C(\mathcal{O}_{\mathcal{X}}(V)) = C(\mathcal{O}_{\mathcal{X}}(U)).$$
To begin with, if $V$ is a Zariski open set of $X$ that contains $U$, one has a map

$$\varphi_V : C(\mathcal{O}_X(V)) \to C(\mathcal{O}_X(U^{\delta})).$$

This comes from Proposition 4.1: $\varphi_V$ is the composition of the extension map $\mathcal{O}_X(V) \to \mathcal{O}_X(V^{\delta})$ with the restriction map to $\mathcal{O}_X(U^{\delta})$. Let’s prove that the $\varphi_V$’s make $\mathcal{O}_X(U^{\delta})$ the sought colimit. Let $A$ be a differential ring together with compatible maps $\psi_V : C(\mathcal{O}_X(V)) \to A$. In particular one has a map $\psi_{U^{\delta}} : C(\mathcal{O}_X(U^{\delta})) \to A$. Let $V$ be a Zariski open set containing $U$. All that we have to prove is that the diagram

$$\begin{array}{ccc}
C(\mathcal{O}_X(V)) & \xrightarrow{\psi_V} & C(\mathcal{O}_X(U^{\delta})) \\
\downarrow{\varphi_V} & & \downarrow{\psi_{U^{\delta}}}
\end{array}$$

commutes. But, in the diagram

$$\begin{array}{ccc}
C(\mathcal{O}_X(V)) & \xrightarrow{\psi_V} & C(\mathcal{O}_X(U^{\delta})) \\
\downarrow{\varphi_V} & & \downarrow{\psi_{U^{\delta}}}
\end{array} \quad \begin{array}{ccc}
C(\mathcal{O}_X(V^{\delta})) & \xrightarrow{\psi_{V^{\delta}}} & A \\
\downarrow{\text{restrict}} & & \downarrow{\psi_{U^{\delta}}}
\end{array}$$

the diagram (1) commutes by definition of $\varphi_V$, the diagram (2) commutes for the $\psi_W$’s form a compatible family, and the big triangle

$$\begin{array}{ccc}
C(\mathcal{O}_X(V)) & \xrightarrow{\psi_V} & A \\
\downarrow{\text{ext}_{V \to V^{\delta}}} & & \downarrow{\psi_{U^{\delta}}}
\end{array} \quad \begin{array}{ccc}
C(\mathcal{O}_X(V^{\delta})) & \xrightarrow{\psi_{V^{\delta}}} & A \\
\downarrow{\text{restrict}} & & \downarrow{\psi_{U^{\delta}}}
\end{array}$$

commutes for the same reason, and for the restriction map and the extension map are inverse one of each other. So, the last triangle

$$\begin{array}{ccc}
C(\mathcal{O}_X(V)) & \xrightarrow{\psi_V} & C(\mathcal{O}_X(U^{\delta})) \\
\downarrow{\varphi_V} & & \downarrow{\psi_{U^{\delta}}}
\end{array}$$
indeed commutes and $C \left( \mathcal{O}_X(U^\delta) \right)$ is the colimit that we wanted to compute.

Now, the end of the proof is easy. Since the constant of a sheaf is still a sheaf, the presheaf

$$U \mapsto C \left( \mathcal{O}_X(U^\delta) \right),$$

which is actually the constant of the Carrà Ferro sheaf, is a sheaf. So, it is its own associated sheaf. ■

Remark. — In general, the sheaves $\mathcal{O}_X(\text{Keigher})$ and $\mathcal{O}_X(\text{CF})$ are not isomorphic. For instance, if $\mathcal{X}$ is $\mathbb{A}_C^1$ with the constant vector field, as we already told, $X^\gamma$ contains only one element, the generic point. The global sections are $\mathbb{C}[t]$ for the Carrà Ferro sheaf and $\mathbb{C}(t)$ for the Keigher sheaf. ◊

5.2 Comparison of the Keigher sheaf and the Kovacic sheaf

We now prove the following proposition, that compares the two classical sheaves over $\text{diff-Spec } A$:

**Proposition 5.2.** Let $A$ be a differential ring. Then,

$$\mathcal{O}_{\text{diff-Spec } A}(\text{Keigher}) \simeq \mathcal{O}_{\text{diff-Spec } A}(\text{Kov}).$$

As an immediate consequence of Theorem 5.1 and of the latter proposition, one gets:

**Corollary 5.3.** Let $A$ be $\mathbb{Q}$-reduced differential algebra. Then,

$$C(\mathcal{O}_{\text{diff-Spec } A}(\text{Keigher})) \simeq C(\mathcal{O}_{\text{diff-Spec } A}(\text{Kov})) \simeq C(\mathcal{O}_{\text{diff-Spec } A}(\text{CF})).$$

**Proof of Proposition 5.2.** — First, let us remark (7) that $\mathcal{O}_{\text{diff-Spec } A}(\text{Keigher})$ and $\mathcal{O}_{\text{diff-Spec } A}(\text{Kov})$ have the same stalks: for all $p \in \text{diff-Spec } A$, the stalks at $p$ are isomorphic to $A_p$. So, to prove that they are isomorphic, it is sufficient to show that there exists a morphism between them, inducing isomorphisms on stalks. Let us indicate how to construct such a morphism $\mathcal{O}_{\text{diff-Spec } A}(\text{Keigher}) \rightarrow \mathcal{O}_{\text{diff-Spec } A}(\text{Kov})$. By the universal property of the associated sheaf, it is sufficient to built a similar morphism

$$\colim_{U \supseteq V \subseteq \text{Spec } A} \mathcal{O}_{\text{Spec } A}(V) \rightarrow \mathcal{O}_{\text{diff-Spec } A}(U),$$

functorial in $U$. But, to define such a morphism, it is sufficient to consider a compatible family of morphisms

$$\mathcal{O}_{\text{Spec } A}(V) \rightarrow \mathcal{O}_{\text{diff-Spec } A}(U)$$

for all Zariski open set $V$ containing $U$. If one consider the Hartshorne-like definition of $\mathcal{O}_{\text{Spec } A}$, it is easy to define these maps, by restriction. ■

(7) It follows, for the Keigher sheaf, from the fact that a sheaf and its restriction to a subset have the same stalks.
A  The associated sheaf in the differential context

The two goals of this appendix are:

(i) to explain why the existence of the associated sheaf \( F^\dagger \) in the context of sheaves and presheaves of sets implies its existence in the context of differential rings;

(ii) to explain why the functor \( F \mapsto F^\dagger \) commutes with the functor of constants.

A.1  Statement of the first result

To begin with, we fix some notations. If \( X \) is a topological space, we denote by

\[
\begin{array}{cccc}
\text{PrSh}(X) & \text{PrSh Ab}(X) & \text{PrSh Rng}(X) & \text{PrSh Rng}^\partial(X) \\
\text{Sh}(X) & \text{Sh Ab}(X) & \text{Sh Rng}(X) & \text{Sh Rng}^\partial(X)
\end{array}
\]

the respective categories of presheaves and sheaves of sets, abelian groups, rings and differential rings. These categories come naturally with forgetful functors. We also denote by

\[
C_{(\text{Sh})} : \text{PrSh Rng}^\partial(X) \to \text{PrSh Rng}(X)
\]

and

\[
C_{(\text{PrSh})} : \text{Sh Rng}^\partial(X) \to \text{Sh Rng}(X)
\]

the functors that associate to a (pre)sheaf \( \mathcal{F} \) of differential rings the (pre)sheaf of rings \( U \mapsto C(\mathcal{F}(U)) \). Finally, we also recall that one denotes

\[
\begin{array}{ccc}
\mathcal{C} & F & \mathcal{D} \\
\downarrow & \uparrow & \\
G & & \\
\end{array}
\]

when \((F, G)\) establishes an adjunction between \( \mathcal{C} \) and \( \mathcal{D} \), ie when \( G \) is left adjoint to \( F \). We want to prove:

**Theorem A.1.** Let \( X \) be a topological space and let \( \mathcal{F} \) be a presheaf of sets over \( X \). Then, the sheaf of sets \( \mathcal{F}^\dagger \), associated to \( \mathcal{F} \), can be endowed with a structure of sheaf of abelian groups (resp. rings, differential rings) when \( \mathcal{F} \) has a structure of presheaf of abelian groups (resp. rings, differential rings).

But, more precisely, what we will prove is the following

**Theorem A.2.** Given a topological space \( X \), there exist four adjunctions

\[
\begin{array}{ccc}
\text{Sh}(X) & \xrightarrow{\omega} & \text{PrSh}(X) \\
\downarrow \ast & & \downarrow \\
\text{Sh Ab}(X) & \xrightarrow{\omega_{\text{Ab}}} & \text{PrSh Ab}(X)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Sh Rng}(X) & \xrightarrow{\omega_{\text{Rng}}} & \text{PrSh Rng}(X) \\
\downarrow \ast_{\text{Rng}} & & \downarrow \\
\text{Sh Rng}^\partial(X) & \xrightarrow{\omega_{\text{Rng}}^\partial} & \text{PrSh Rng}^\partial(X)
\end{array}
\]
making the following diagram commute:

\[
\begin{array}{cccccc}
\mathbf{Sh}_{\operatorname{Rng}}(X) & \xrightarrow{\omega_1} & \mathbf{Sh}_{\operatorname{Rng}}(X) & \xrightarrow{\omega_2} & \mathbf{Sh}_{\operatorname{Ab}}(X) & \xrightarrow{\omega_3} & \mathbf{Sh}(X) \\
\ast_{\operatorname{Rng}} & \downarrow{\omega_{\operatorname{Rng}}} & \ast_{\operatorname{Rng}} & \downarrow{\omega_{\operatorname{Rng}}} & \ast_{\operatorname{Ab}} & \downarrow{\omega_{\operatorname{Ab}}} & \ast \\
\mathbf{PrSh}_{\operatorname{Rng}}(X) & \xrightarrow{\omega_4} & \mathbf{PrSh}_{\operatorname{Rng}}(X) & \xrightarrow{\omega_5} & \mathbf{PrSh}_{\operatorname{Ab}}(X) & \xrightarrow{\omega_6} & \mathbf{PrSh}(X)
\end{array}
\]

Moreover, the functors \(\ast, \ast_{\operatorname{Ab}}, \ast_{\operatorname{Rng}}\) and \(\ast_{\operatorname{Rng}}\) commute with general colimits and finite limits.

With these notations, the left adjoint functor to \(\omega: \mathbf{Sh}(X) \to \mathbf{PrSh}(X)\), denoted here by \(\ast: \mathbf{PrSh}(X) \to \mathbf{Sh}(X)\), is the functor that associates to a presheaf \(\mathcal{F}\) its associated sheaf \(\mathcal{F}^\dagger\).

### A.2 Proof of Theorem A.2

To begin with, we assume that the existence of the associated sheaf, and the fact that it commutes with finite limits, is known for presheaves of sets. It is proved, for instance, in [Har77] or [MM94]. We denote by \(\ast: \mathbf{PrSh}(X) \to \mathbf{Sh}(X)\) the functor that maps \(F\) to its associated sheaf \(F^\dagger\).

Now, let \(\mathcal{C}\) be a category with finite products. We denote by \(1\) a terminal object of \(\mathcal{C}\). Following [MM94] Ch. II, §7, but the interested reader should also consult [Sch72] Ch. 11, we consider the category \(\mathbf{Ab}(\mathcal{C})\) of abelian group objects of \(\mathcal{C}\). It is defined as follows:

- the objects of \(\mathbf{Ab}(\mathcal{C})\) are 4-uplets \((X, a, v, u)\) where \(X \in \text{ob}(\mathcal{C})\) and where \(a: X \times X \to X\), \(v: X \to X\) and \(u: 1 \to X\) are arrows satisfying some conditions. Intuitively, one wants \(a\) to be the addition law, \(v\) to be the subtraction law and \(u\) to be the zero.

- the arrows of \(\mathbf{Ab}(\mathcal{C})\) are arrows \(f: X \to Y\) that commutes with addition.

For instance, \(\mathbf{Ab}(\mathsf{Sets})\) is isomorphic, as a category, to the category of abelian groups. Similarly, for every topological space \(X\), the categories \(\mathbf{Ab}(\mathbf{PrSh}(X))\) and \(\mathbf{Ab}(\mathbf{Sh}(X))\) are isomorphic to \(\mathbf{PrSh}_{\operatorname{Ab}}(X)\) and to \(\mathbf{Sh}_{\operatorname{Ab}}(X)\).

Now, let \(X\) be a topological space and let \(\mathcal{F}\) be presheaf of abelian groups. We want to construct the associated sheaf \(\mathcal{F}^\dagger\) to \(\mathcal{F}\). First, one can see \(\mathcal{F}\) as an object of \(\mathbf{Ab}(\mathbf{PrSh}(X))\): \(\mathcal{F}\) is preasheaf of sets endowed with maps

\[
a: \mathcal{F} \times \mathcal{F} \to \mathcal{F}, \quad v: \mathcal{F} \to \mathcal{F} \quad \text{and} \quad u: \{\ast\} \to \mathcal{F}
\]

where \(\{\ast\}\) denotes the final object of \(\mathbf{PrSh}(X)\). Then, one can apply to \(\mathcal{F}\) and to these maps the functor \(\ast\). Since, \(\ast\) commutes with finite limits, one gets

\[
\ast(a): \mathcal{F}^\dagger \times \mathcal{F}^\dagger \to \mathcal{F}^\dagger, \quad \ast(v): \mathcal{F}^\dagger \to \mathcal{F}^\dagger \quad \text{and} \quad \ast(u): \{\ast\}^\dagger \to \mathcal{F}^\dagger.
\]
Furthermore, these maps still verify the axioms of $\text{Ab}(\mathcal{C})$, since $*$ maps commutative diagrams to commutative diagrams. Therefore, $\mathcal{F}^\dagger$ has naturally a structure of sheaf of abelian groups. One can also verify that $*$ maps additive morphisms of additive morphisms. Thus, one has a functor

$$
*_{\text{Ab}} : \text{PrSh}_{\text{Ab}}(X) \longrightarrow \text{Sh}_{\text{Ab}}(X)
$$

and one can prove that it is the left adjoint that we were looking for. Last, since $*_{\text{Ab}}$ is left adjoint to $\omega_{\text{Ab}}$, one knows that it commutes with all colimits. For finite limits, one proceeds as follows:

1. Let $(\mathcal{F}_i, \varphi_{ij})$ be a finite system of abelian presheaves and let $\varphi_i : \mathcal{F} \longrightarrow \mathcal{F}_i$ be its limit in $\text{PrSh}_{\text{Ab}}(X)$. Then, $\mathcal{F}$, seen as a presheaf of sets is still a limit. This comes, for instance, from the fact that the functor $\omega_{\text{Rng}} : \text{PrSh}_{\text{Ab}}(X) \longrightarrow \text{PrSh}(X)$ has a left adjoint. This adjoint maps a presheaf of sets $\mathcal{G}$ to the presheaf of abelian groups $U \mapsto \mathbb{Z}(\mathcal{G}(U))$.

2. Hence, $\varphi_i : \mathcal{F} \longrightarrow \mathcal{F}_i$ is still a limit, seen in $\text{PrSh}(X)$. For $*$ commutes with finite limits, one gets that $(\varphi_i) : \mathcal{F}^\dagger \longrightarrow \mathcal{F}_i^\dagger$ is a limit in $\text{Sh}(X)$. Furthermore, by definition, $\mathcal{F}_i^\dagger$, $\mathcal{F}^\dagger$ can be seen as sheaves of abelian groups and the morphism $*(\varphi_i)$ are additive.

3. Last, if $\mathcal{G}$ is a sheaf of abelian groups given with morphisms $\psi_i : \mathcal{G} \longrightarrow \mathcal{F}_i^\dagger$, one wants to find an arrow $f : \mathcal{G} \longrightarrow \mathcal{F}^\dagger$ that factorizes the $\psi_i$. For $\mathcal{F}^\dagger$ is a limit in $\text{Sh}(X)$, one can find such an arrow $f$, in $\text{Sh}(X)$. But then, one has to prove that this arrow is additive. This comes from the additiveness of the $\psi_i$ and the unicity of factorizations.

So, this is how one can prove the existence of the left adjoint $*_{\text{Ab}}$ and its properties. The same arguments apply for (pre)sheaves of rings and differential rings: one remarks that it is possible to defines ring objects and differential ring objects in a category $\mathcal{C}$ with finite limits, and that this definition only involves finite products, maps and commutative diagrams.

### A.3 Associated sheaves and constants

Now, we prove

**Proposition A.3.** Let $X$ be a topological space. Then, the diagram

$$
\begin{array}{ccc}
\text{Sh}_{\text{Rng}}(X) & \xrightarrow{\text{C}_{(\text{Sh})}} & \text{Sh}_{\text{Rng}}(X) \\
\downarrow *_{\text{Rng}} & & \downarrow \text{*}_{\text{Rng}} \\
\text{PrSh}_{\text{Rng}}(X) & \xrightarrow{\text{C}_{(\text{PrSh})}} & \text{PrSh}_{\text{Rng}}(X)
\end{array}
$$

commutes, up to isomorphism.
This means that, if $\mathcal{F}$ is a presheaf of differential rings, when one wants to compute the constant of $\mathcal{F}^\dagger$, it suffices to compute the associated sheaf to $C(\mathcal{F})$: in other words, $C(\mathcal{F})^\dagger \simeq C(\mathcal{F}^\dagger)$.

**Proof.** — Let $X$ be a topological space and let $\mathcal{F} \in \text{PrSh}_{\text{Rng}}^\partial(X)$. The constant of $\mathcal{F}$, denoted by $C(\mathcal{F})$ is a finite limit; more precisely, it is a kernel (in $\text{PrSh}(X)$, and in $\text{Sh}(X)$ for sheaves):

$$C(\mathcal{F}) \longrightarrow \mathcal{F} \xrightarrow{\partial} \mathcal{F}.$$  

For $*$ commutes with finite limits, one has

$$C(\mathcal{F})^\dagger \longrightarrow \mathcal{F}^\dagger \xrightarrow{\partial} \mathcal{F}^\dagger;$$

hence, $C(\mathcal{F})^\dagger$ is isomorphic to $C(\mathcal{F}^\dagger)$, as sheaves of sets. But, this enough to infer that they are isomorphic as sheaves of rings. Indeed, first, the map $C(\mathcal{F})^\dagger \longrightarrow \mathcal{F}^\dagger$ is a morphism of sheaves of differential rings; second, if $U$ is any open set, the map $C(\mathcal{F})^\dagger(U) \longrightarrow \mathcal{F}^\dagger(U)$ is injective (this is because it is isomorphic to $C(\mathcal{F}^\dagger)(U) \longrightarrow \mathcal{F}^\dagger$); third, as sets, $C(\mathcal{F}^\dagger)(U)$ and $C(\mathcal{F})^\dagger(U)$ are isomorphic. ■
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