Local discrimination of generalized Bell states via commutativity

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We study the distinguishability of generalized Bell states under local operations and classical communication. We introduce the concept of maximally commutative set (MCS), subset of generalized Pauli matrices whose elements are mutually commutative, and there is no other generalized Pauli matrix that commute with all the elements of this set. We find that MCS can be considered as a detector for local distinguishability of set $S$ of generalized Bell states. In fact, we get an efficient criterion. That is, if the difference set of $S$ is disjoint with or completely contain in some MCS, then the set $S$ is locally distinguishable. Furthermore, we give a useful characterization of MCS for arbitrary dimension, which provides great convenience for detecting the local discrimination of generalized Bell states. Our method can be generalized to more general settings which contains lattice qudit basis. Results in [Phys. Rev. Lett. 92, 177905 (2004)], [Phys. Rev. A 92, 042320 (2015)] and a recent work [arXiv: 2109.07390] can be deduced as special cases of our result.

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I. INTRODUCTION

Quantum states discrimination is a fundamental task in quantum information processing. It is well known that a set of quantum states can be perfectly distinguished by global measurement if and only if the states of given set are mutually orthogonal [1]. However, our quantum states are usually distributed in composite systems with long distances, so only local operations and classical communication (LOCC) are allowed. In such setting, a state is chosen from a known orthogonal set of quantum states in a composite systems and the task is to identify the state under LOCC. If the task can be accomplished perfectly, we say that the set is locally distinguishable, otherwise, locally indistinguishable. If an orthogonal set is locally indistinguishable, we also called that the set presents some kind of nonlocality [2] in the sense that more quantum information could be inferred from global measurement than that from local operations. Any two orthogonal multipartite states are showed to be locally distinguishable [3]. Bennett et al. [2] presented the first example of orthogonal product states that are locally indistinguishable which reveals the phenomenon of “quantum nonlocality without entanglement”. Results on the local distinguishability of quantum states have been practically applied in quantum cryptography primitives such as data hiding [4, 5] and secret sharing [6-8].

For general orthogonal sets of quantum states, it is difficult to give a complete characterization of whether they are locally distinguishable or not. Therefore, most studies (See [2, 3, 9–56] for an incomplete list) focus on two extreme cases: sets of product states or sets of maximally entangled states. In this paper, we restrict ourselves to the settings of maximally entangled cases.

Bell states are the most famous maximally entangled states and their local distinguishability has been well understood. In fact, any two Bell states are locally distinguishable but any three or four are not [10]. Nathanson [17] showed that any three maximally entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ can be locally distinguished. Moreover, any $l > d$ maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ are known to be locally indistinguishable [17]. Therefore, it is interesting to consider whether set of maximally entangled states with cardinality $l \leq d$ can be locally distinguishable or not? Interestingly, using the fact that applying local unitary operation does not change the local distinguishability, Fan [16] showed that any $l$ generalized Bell states (GBSs) in $\mathbb{C}^d \otimes \mathbb{C}^d$ are locally distinguishable if $(l - 1)l \leq 2d$ provided that $d$ is a prime number. Fan’s result was extended by Tian et al. to the prime power dimensional quantum system in [57] where they restricted themselves to the mutually commuting qudit lattice states. Since Fan’s result, there has been lots of works [57–64] paid attention to the locally distinguishability of GBSs. However, the complete classification of local distinguishability of GBSs is still difficult to achieve. On the other hand, set of GBSs is an important and special subset of maximally entangled states, which makes the problem of local distinguishability of GBSs important and interesting. Motivated by a recent work [65], we find that the local distinguishability of GBSs can be detected by maximally commutative set (MCS) of GBSs.

The rest of this article is organized as follows. In Sec. II, we introduce the matrices representation of generalized Bell states. Then we give a brief review of some known results on the sufficient conditions of locally distinguishable set of GBSs. In Sec. III, we present the definition of maximally commutative set and show that it is useful for judging the locally distinguishability. After that, we present some examples of MCS and study the properties of general MCS. Finally, we draw a conclusion and presented some questions in Sec. IV.
II. A REVIEW OF LOCAL DISTINGUISHABILITY OF GBSs

Throughout this paper, we will use the following notations. Let $d \geq 2$ be an integer. Denote $\mathbb{Z}_d$ to be the ring defined over $\{0, 1, \cdots, d-1\}$ with the sum operation “+” (here $i + j$ should be equal to the element $(i + j) \mod d$) and multiplication operation (computing the usual multiplication first then taking module $d$). Consider a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$ with both local dimensions equal to $d$. Suppose that $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$ is the computational basis of a single qubit. Under this computational basis, the standard maximally entangled state in this system can be expressed as $|\Psi_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$. Generally, any maximally entangled state can be written in the form $|\Psi_U\rangle = (I \otimes U)|\Psi_{00}\rangle$ for some unitary matrix $U$ of dimensional $d$. We often call $U$ the defining unitary matrix of the maximally entangled state $|\Psi_U\rangle$. To define the generalized Bell states, we define the following two operations

$$X_d = \sum_{i=0}^{d-1} |i + 1 \mod d\rangle \langle i|, \quad Z_d = \sum_{i=0}^{d-1} \omega^i |i\rangle \langle i|,$$

where $\omega = e^{-2\pi i/d}$. Then the following $d^2$ orthogonal MESs are called as generalized Bell states:

$$\{|\Psi_{m,n}\rangle\} = \{|I \otimes X_d^m Z_d^n|\Psi_{00}\rangle\} |m, n \in \mathbb{Z}_d\}.$$ (1)

And matrices in $\{X_d^m Z_d^n\} |m, n \in \mathbb{Z}_d\}$ are called the generalized Pauli matrices (GPMs). For simplicity, we also use $X_d$ and $Z_d$ to represent $X_d$ and $Z_d$ when the dimension is known. Due to the one-one correspondence of MES and its defining unitary matrix, for convenience, we will treat the following three sets equally without distinction

$$\mathcal{S} := \{|\Psi_{m,n}\rangle\}_{i=1}^l = \{X_d^m Z_d^n\}_{i=1}^l = \{(m, n)\}_{i=1}^l.$$ (2)

Our aim in this paper is to provide some sufficient condition such that the set $\mathcal{S}$ is locally distinguishable. Now we gave a brief review of the relative results.

Fan [16] noted that if all $m_i$ ($i = 1, \cdots, l$) are distinct, the set $\mathcal{S}$ can be locally distinguished and set with this property is called $F$-type [64]. For each $\alpha \in \mathbb{Z}_d$, defining $H_\alpha$ be the matrix whose $j, k$ entry is $w^{-j-k-\alpha s_k}/\sqrt{d}$ for $j, k = 1, \cdots, d-1$ and $s_k := \sum_{i=1}^{d-1} i$. Then $H_\alpha$ is a unitary matrix and $H_\alpha \otimes H_\alpha^* \otimes H_\alpha$ transfers $|\Psi_{m,n}\rangle$ to $|\Psi_{am,n+sm-nm}\rangle$. He found that if $d$ is a prime and $(l-1)! \leq 2d$, there exists an $\alpha$ such that $H_\alpha \otimes H_\alpha^* \otimes H_\alpha$ can transfer $\mathcal{S}$ to a set of $F$-type.

There is a useful sufficient condition for local distinguishability of general maximally entangled states. Denote $\mathcal{S}$ as the defining unitary matrices set, if there exists some nontrivial vector $|v\rangle \in \mathbb{C}^d$ such that

$$\langle v| U^\dagger V |v\rangle = 0$$ (3)

for any different $U, V \in \mathcal{S}$, then the set of maximally entangled states corresponding to $\mathcal{S}$ is one-way distinguishable (hence locally distinguishable) [15, 28]. If the set $\mathcal{S}$ is $F$-type, the vector $|v\rangle$ can be chosen as any vector of the computational basis, i.e., $|i\rangle$, $i \in \mathbb{Z}_d$. We define difference set $\Delta \mathcal{S} = \{U_i |i = 1, 2, \cdots, l\}$ as

$$\Delta \mathcal{S} = \{U_i |U_j \ 1 \leq i < j \leq l\}.$$ (4)

Noticing that

$$(X_d^m Z_d^n)|X_d^m Z_d^n\rangle = \omega^{-(m_j-m_i)n_i} X_d^{m_j-m_i} Z_d^{n_j-n_i}.$$ (5)

Up to a phase, we can identify $\Delta \mathcal{S}$ as the set $\{(m_j-m_i, n_j-n_i) |1 \leq i < j \leq l\}$. In order to find some nonzero vector $|v\rangle$ such that Eqs. (2) are satisfied, the following lemma is important (See also in Ref. [65]).

**Lemma 1** For two unitary matrices $U$ and $V$, if they satisfy $UV = zVU$ where $z$ is a complex number and are not commutative, i.e., $z \neq 1$, then each eigenvector $|v\rangle$ of $V$ satisfies $\langle v|V|v\rangle = 0$.

In fact, suppose that $V|v\rangle = \lambda |v\rangle$ where $\lambda \neq 1$. We also have $\langle v|V^\dagger = \overline{\lambda} \langle v|$. Therefore, $\langle v|U|v\rangle = \langle v|U\lambda|v\rangle = \langle v|V^\dagger U V |u\rangle = z \langle v|V |u\rangle$. Hence, $\langle v|U|v\rangle = 0$ as $z \neq 1$. A pair of unitaries that satisfy the first condition are called Weyl commutative.

Fortunately, any pair of generalized Pauli matrices are Weyl commutative. In fact, for two pairs of $(m_i, n_i)$ and $(m_j, n_j)$ in $\mathbb{Z}_d \times \mathbb{Z}_d$, we always have

$$X_d^m Z_d^n |X_d^m Z_d^n\rangle = \omega_m n_i m_n Z_d^m Z_d^n X_d^m Z_d^n.$$ (6)

Moreover, $X_d^m Z_d^n$ and $X_d^m Z_d^n$, are commutative if and only if $m_j n_i - m_i n_j \equiv 0 \mod d$. This condition can be formulated as the determinant equation

$$\left| \begin{array}{c} m_i \\ m_j \\ n_i \\ n_j \end{array} \right| \equiv 0 \mod d.$$ (7)

We also call that $(m_i, n_i)$ and $(m_j, n_j)$ are commutative if this condition is satisfied.

For any GBS set $\mathcal{S}$, if there is a generalized Pauli matrix $V$ which is not commutative to every GPM $U \in \Delta \mathcal{S}$, by Eqs. (2), (3), and Lemma 1, each eigenvector $|v\rangle$ of $V$ satisfies $\langle v|U|v\rangle = 0$ and therefore the set $\mathcal{S}$ is locally distinguishable.

Let $m, n \in \mathbb{Z}_d$, denote the solution set of the following congruence equation by $S(m, n)$,

$$nx - my = 0 \mod d.$$ (8)

Therefore, $S(m, n)$ denote the set of elements in $\mathbb{Z}_d \times \mathbb{Z}_d$ that are commute with $(m, n)$. In order to check whether there is any GPM $V$ that do not commute with all the elements in $\Delta \mathcal{S}$. The authors of Ref. [65] defined the set

$$D(\mathcal{S}) \triangleq (\mathbb{Z}_d \times \mathbb{Z}_d) \setminus \bigcup_{(m, n) \in \Delta \mathcal{S}} S(m, n).$$ (9)

By definition, $D(\mathcal{S})$ denotes the set of all elements in $\mathbb{Z}_d \times \mathbb{Z}_d$ that do not commute with all the elements in $\Delta \mathcal{S}$. Under this definition, they proved the following results.
Theorem 1 (See Ref. [65]) Let $S = \{(m_i, n_i)|4 \leq i \leq l < d\}$ be a GBS set in $\mathbb{C}^d \otimes \mathbb{C}^d$, then the set $S$ is locally distinguishable when any of the following conditions is true.

1. The discriminant set $\mathcal{D}(S)$ is not empty.
2. The set $\Delta S$ is commutative.
3. The dimension $d$ is a composite number, and for each $(m, n) \in \Delta S$, $m$ or $n$ is invertible in $\mathbb{Z}_d$.

The nonemptiness of $\mathcal{D}(S)$ implies the local distinguishability of $S$. Therefore, the set $\mathcal{D}(S)$ can be called a discriminant set of $S$.

III. DETECTOR FOR LOCAL DISTINGUISHABILITY OF GBSs

In the first case of Theorem 1, the nonempty of the discriminant set $\mathcal{D}(S)$ is equivalent to that there exists some $(s, t) \in \mathbb{Z}_d \times \mathbb{Z}_d$ such that

$$ms - nt \neq 0, \quad \forall (m, n) \in \Delta S.$$ 

That is, $X^s Z^t$ is not commute with $X^m Z^n$. Therefore, any nonzero eigenvector $|v\rangle$ of $X^s Z^t$ satisfies

$$\langle v|X^m Z^n|v\rangle = 0$$

from which one can conclude that the set $S$ is locally distinguishable. From this point, we can call $X^s Z^t$ as a detector of local discrimination of GBS. Simply, the ability of the detector $X^s Z^t$ can be defined as the set

$$\mathcal{D}(X^s Z^t) \triangleq (\mathbb{Z}_d \times \mathbb{Z}_d) \setminus S(s, t).$$

This denotes the set of all elements in $\mathbb{Z}_d \times \mathbb{Z}_d$ that do not commute with $(s, t)$. Then the one-way local distinguishability of $S$ can be detected by $X^s Z^t$ if and only if $\Delta S \subseteq \mathcal{D}(X^s Z^t)$.

In fact, we can introduce a stronger detector by the following observation. If a set of detectors $\{X^s Z^t\}_{i=1}^n$ are commutative, they can share a common eigenbasis $\{|v_j\rangle\}_{j=1}^d$. Therefore, if

$$\Delta S \subseteq \bigcup_{i=1}^n \mathcal{D}(X^s Z^t)$$

we can also conclude that the set $S$ is one-way distinguishable. Therefore, the more elements of the detected set, the stronger its distinguishing ability which motivates the following definition.

A subset $\{X^s Z^t\}_{i=1}^n$ of GBSs is call maximally commutative if the elements of the given subset are mutually commute and there is no other GBS which can commute with all the elements of the set. This can be written as the coordinates $\{(s_i, t_i)\}_{i=1}^n \subseteq \mathbb{Z}_d \times \mathbb{Z}_d$ such that $s_i t_j = t_i s_j$ for every $i, j$ but there is no $(s, t) \in \mathbb{Z}_d \times \mathbb{Z}_d$ such that $s_t = t_i s$ for every $i$.

For any maximally commutative set of GBSs $\mathcal{C} := \{X^s Z^t\}_{i=1}^n$, we defined a detector as

$$\mathcal{D}(\mathcal{C}) := \bigcup_{(s, t) \in \mathcal{C}} \mathcal{D}(X^s Z^t).$$

Therefore, one conclude that if $\Delta S \subseteq \mathcal{D}(\mathcal{C})$, the set $S$ is one-way distinguishable. On the other hand, one finds that $\mathcal{D}(\mathcal{C})$ is equal to $\mathcal{P}_d \setminus \mathcal{C}$ where $\mathcal{P}_d := \{X^m Z^n|m, n \in \mathbb{Z}_d\}$. In fact, every element in $\mathcal{P}_d$ but outside $\mathcal{C}$ must be not commute with one of element $X^s Z^t$ in $\mathcal{C}$. That is, it belongs to $\mathcal{D}(X^s Z^t)$. It means that $\mathcal{P}_d \setminus \mathcal{C} \subseteq \mathcal{D}(\mathcal{C})$. Obviously, $\mathcal{D}(\mathcal{C}) \subseteq \mathcal{P}_d \setminus \mathcal{C}$. This, $\mathcal{D}(\mathcal{C}) = \mathcal{P}_d \setminus \mathcal{C}$. Therefore, $\Delta S \subseteq \mathcal{D}(\mathcal{C})$ if and only if $\Delta S \cap \mathcal{C} = \emptyset$. Moreover, if $\Delta S \subseteq \mathcal{C}$, that is, the elements in $\Delta S$ are mutually commutative. By Theorem 1, the set $\mathcal{S}$ is also locally distinguishable.

Theorem 2 Let $S$ be a GBS set in $\mathbb{C}^d \otimes \mathbb{C}^d$ and $\mathcal{C}$ be a set of maximally commutative GBS of dimensional $d$. If $\Delta S \cap \mathcal{C} = \emptyset$ or $\Delta S \subseteq \mathcal{C}$, then the set $S$ is locally distinguishable (see Fig. 1 for an intuitive view of the conditions).

![FIG. 1: Here $\mathcal{C}$ represents a maximally commutative set of GBS and $\Delta S$ is the difference set of $S$. If $\Delta S$ and the detector $\mathcal{C}$ are in one of the above relations, then the set $S$ is locally distinguishable.](image-url)
and suppose that $d = pq$ where $p, q \geq 2$ are two integers. Clearly, $X^p$ is commute with $Z^q$, therefore, they can extend to a maximally commutative set of GBS, said, $C$, if $s$ or $t$ is invertible in $Z_d$, we claim that $X^sZ^t \notin C$. In fact, as $ZX = \omega XZ$, if $s$ is invertible, then $Z^s(X^sZ^t) = \omega^s(X^sZ^t)Z^q \neq (X^sZ^t)Z^q$. If $t$ is invertible, then $(X^sZ^t)X^p = \omega^pX^p(X^sZ^t) \neq X^p(X^sZ^t)$. It also means that $\Delta S \cap C = \emptyset$. Therefore, if $\Delta S$ contains those elements one of whose coordinates is invertible in $Z_d$, then the set $C$ can detect the one-way distinguishability of $S$.

Therefore, it is important to find out all the maximally commutative sets of GBS. Now we present some examples in the low dimensional cases.

Example 1 There are exactly four classes of maximally commute sets of GBS in $C^3 \otimes C^3$.

$C_1 = \{(0,0), (0,1), (0,2)\}$, $C_2 = \{(0,0), (1,0), (2,0)\}$, $C_3 = \{(0,0), (1,1), (2,2)\}$, $C_4 = \{(0,0), (1,2), (2,1)\}$.

Example 2 There are exactly seven classes of maximally commute sets of GBS in $C^4 \otimes C^4$.

$C_1 = \{(0,0), (0,1), (0,2), (0,3)\}$, $C_2 = \{(0,0), (0,2), (2,0), (2,2)\}$, $C_3 = \{(0,0), (0,2), (2,1), (2,3)\}$, $C_4 = \{(0,0), (1,0), (2,0), (3,0)\}$, $C_5 = \{(0,0), (1,1), (2,2), (3,3)\}$, $C_6 = \{(0,0), (1,2), (2,0), (3,2)\}$, $C_7 = \{(0,0), (1,3), (2,2), (3,1)\}$.

Example 3 There are exactly 15 classes of maximally commute sets of GBS in $C^5 \otimes C^5$. Here we do not write out the coordinate (0,0) which belongs to all the 15 sets.

$C_1 = \{(0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7)\}$, $C_2 = \{(0,2), (0,4), (0,6), (4,0), (4,2), (4,4), (4,6)\}$, $C_3 = \{(0,2), (0,4), (0,6), (4,1), (4,3), (4,5), (4,7)\}$, $C_4 = \{(0,4), (2,0), (2,4), (4,0), (4,4), (6,0), (6,4)\}$, $C_5 = \{(0,4), (2,1), (2,5), (4,2), (4,6), (6,3), (6,7)\}$, $C_6 = \{(0,4), (2,2), (2,6), (4,0), (4,4), (6,2), (6,6)\}$, $C_7 = \{(0,4), (2,3), (2,7), (4,2), (4,6), (6,1), (6,5)\}$, $C_8 = \{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0)\}$, $C_9 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$, $C_{10} = \{(1,2), (2,4), (3,6), (4,0), (5,2), (6,4), (7,6)\}$, $C_{11} = \{(1,3), (2,6), (3,1), (4,4), (5,7), (6,2), (7,5)\}$, $C_{12} = \{(1,4), (2,0), (3,4), (4,0), (5,4), (6,0), (7,4)\}$, $C_{13} = \{(1,5), (2,2), (3,7), (4,4), (5,1), (6,6), (7,3)\}$, $C_{14} = \{(1,6), (2,4), (3,2), (4,0), (5,6), (6,4), (7,2)\}$, $C_{15} = \{(1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1)\}$.

Using these MCSs, we can show that Theorem 2 is strictly powerful than Theorem 1 when the dimension $d = 8$. Set $S := \{X^5Z^0, X^6Z^3, X^6Z^5, X^7Z^0\}$ whose difference set $\Delta S$ is

$(5,6), (6,3), (6,5), (7,6), (1,5), (1,7), (2,0), (0,2), (1,3), (1,1)$.

One can check that $D(S) = \emptyset$, $\Delta S$ is non-commutative and neither coordinates of $(2,0)$ are invertible in $Z_p$. Therefore, Theorem 1 fails to detect the distinguishability of this set. However, one finds that $\Delta S \cap C_6 = \emptyset$. That is, the local distinguishability of $S$ can be detected by $C_6$. Moreover, one can check that neither $\Delta S \cap C_i = \emptyset$ nor $\Delta S \subseteq C_i$ when $i \neq 6$. That is, among the 15 classes, $C_6$ is the only detector that can detect the local distinguishability of $S$. More numerical results comparing the power of Theorem 1 and Theorem 2 can be seen in the figure 2 (we randomly generated $N$ sets of $d$ dimensional GBSs with cardinality $n$ and find out the numbers $N_1$ and $N_2$ of sets whose local distinguishability can be detected by Theorem 1 and Theorem 2 respectively. The corresponding successful rates are defined by $N_1/N$ and $N_2/N$).

![Figure 2](image_url)

FIG. 2: For $n = 5, 6, 7, 8, 9, 10, 11$, we randomly generate $N = 100000$ sets of GBSs whose cardinality are all $n$ for $d = 7 \sim 20$ respectively. The solid lines represent the successful rates by Theorem 1 and the dot lines represent the successful rates by Theorem 2. The lines from above to below represent sets with cardinality being from 5 to 11 respectively. Here the starting point of each curve with respect to $n$ is with dimension $d \geq n$.

Proposition 1 Let $p \geq 2$ be a prime. Then there are exactly $p + 1$ classes of maximally commutative sets of GBSs in $C^p \otimes C^p$.

In fact, these sets are characterized by $(0,1)Z_p$, $(1,0)Z_p$, and $(1,i)Z_p, 1 \leq i \leq p - 1$ where $(a,b)Z_p := \{(ai, bi) \mid i \in Z_p\}$. By this proposition and Theorem 2, one would deduce Fan’s result again. That is, if $S$ is a set of $p$ dimensional GBSs with $l$ elements and $l(l - 1)/2 \leq p$, then $S$ is locally distinguishable. In fact, in this setting, the number of elements in $\Delta S$ (which does not contain $(0,0)$) is less or equal than $l(l - 1)/2$. However, $(0,1)Z_p \setminus \{(0,0)\}$, $(1,0)Z_p \setminus \{(0,0)\}$, and $(1,i)Z_p \setminus \{(0,0)\} (1 \leq i \leq p - 1)$ are $p + 1$ classes of mutually disjoint sets. Therefore, there must exists some MCS $C$ such that $C \cap \Delta S = \emptyset$. 
Lemma 2 Each maximally commutative set of GBSs in $\mathbb{C}^d \otimes \mathbb{C}^d$ must be with cardinality less or equal than $d$.

This can be obtained by observing that commutative set of unitary matrices can be simultaneously diagonalized and the elements of GBSs are mutually orthogonal. Moreover, one could easily verify the following lemma.

Lemma 3 Let $C$ be a maximally commutative set of GBSs in $\mathbb{C}^d \otimes \mathbb{C}^d$. If $(i,j)$ belongs to $C$, so does $(ik,jk)$ where $k \in \mathbb{Z}_d$, i.e., $(i,j)\mathbb{Z}_d \subseteq C$. Moreover, if both $(i_1,j_1)$ and $(i_2,j_2)$ belongs to $C$, so does $(i_1 + i_2,j_1 + j_2)$.

From this lemma, one can conclude that each maximally commutative set $C$ can be written as the forms

$$C = \bigcup_{k=1}^{n} (i_k,j_k) \mathbb{Z}_d, \text{ or } C = \bigcup_{k=1}^{n} (i_k,j_k) \mathbb{Z}_d. \quad (7)$$

Here $A + B := \{a + b | a \in A, b \in B\}$ where $A, B$ are subsets of a group.

We find that the number of MCSs of GBSs is related to an interesting function in number theory which is known as sigma function. The sigma function is usually denoted by the Greek letter sigma ($\sigma$). This function actually denotes the sum of all divisors of a positive integer. For examples, $\sigma(6) = 1 + 2 + 3 + 6 = 12$, and $\sigma(16) = 1 + 2 + 4 + 8 + 16 = 31$. Generally, let $d = p_1^{n_1}p_2^{n_2} \cdots p_l^{n_l}$, then

$$\sigma(d) = \prod_{k=1}^{l}(1 + p_k + \cdots + p_k^{n_k}).$$

Theorem 3 (Structure Characterization of MCS)

Let $d \geq 3$ be an integer. For each pair $(i,j)$ in $\mathbb{Z}_d \times \mathbb{Z}_d$ where $i \neq 0$, we define the following set

$$C_{i,j} := \{(x,y) \in \mathbb{Z}_d \times \mathbb{Z}_d | x \equiv y \mod d, x \in i\mathbb{Z}_d\}.$$

Then $C_{i,j}$ is a MCS of GBSs in $\mathbb{C}^d \otimes \mathbb{C}^d$ with exactly $d$ elements. Moreover, if we define $C_{0,0} := \{(0,y) | y \in \mathbb{Z}_d\}$, then every MCS of GBSs in $\mathbb{C}^d \otimes \mathbb{C}^d$ must be one of $C_{i,j}$ with $i \neq 0$ or $C_{0,0}$. There are exactly $\sigma(d)$ classes of MCSs which can be listed as follows

$$\text{MCS}_d := \{C_{i,j} | d = ik, 0 \leq j \leq k - 1\} \cup \{C_{0,0}\}.$$  

Proof. First, we show that the cardinality of each $C_{i,j}$ is equal to $d$. Denote $d_i$ as the greatest common divisor of $i$ and $d$. Then the set $i\mathbb{Z}_d := \{ij | j \in \mathbb{Z}_d\}$ has exactly $d/d_i$ elements. More exactly,

$$i\mathbb{Z}_d = \{ik | k = 0,1,\cdots, \frac{d}{d_i} - 1\}.$$  

For each $x = ik$ ($k = 0,1,\cdots, \frac{d}{d_i} - 1$), there are exactly $d_i$ solutions of $y \in \mathbb{Z}_d$ that satisfies

$$\begin{vmatrix} i & j \\ x & y \end{vmatrix} \equiv 0 \mod d. \quad (8)$$

In fact, the Eq. (8) is equivalent to $i(y - k_j) \equiv 0 \mod d$ whose solutions can be expressed analytically as $y = k_j + \frac{d}{d_i}l$ where $l = 0,1,\cdots, d - 1$. Therefore, the set $C_{i,j}$ can be expressed as

$$\{(ik,k_j + \frac{d}{d_i}l) | k = 0,1,\cdots, \frac{d}{d_i} - 1, l = 0,1,\cdots, d - 1\}.$$

One can check that for two different pairs of $(k_1,l_1)$ and $(k_2,l_2)$, the coordinates $(ik_1,k_j + \frac{d}{d_i}l_1)$ and $(ik_2,k_j + \frac{d}{d_i}l_2)$ are always equal to $0 \mod d$.

Therefore, each $C_{i,j}$ is a commutative set of GBSs with cardinality $d$. By Lemma 2, each $C_{i,j}$ must also be maximally commutative.

Next, we show that for every MCS $C$, it must be one of $C_{i,j}$ with $i \neq 0$ or $C_{0,0}$. For any maximally commutative set $C$ of GBSs in $\mathbb{C}^d \otimes \mathbb{C}^d$, by Eq. (7), there exists $(x_k,y_k) \in C$ ($k = 1,\cdots, n$), such that

$$C = \sum_{k=1}^{n} (x_k,y_k) \mathbb{Z}_d.$$  

If all $x_k$ are equal to zero, one must conclude that $C = C_{0,0}$. If not, let $i$ denote the greatest common divisor of $x_1,x_2,\cdots,x_n$ and $d$, which is not equal to zero in this case. There exist $r_k \in \mathbb{Z}_d$, such that $i = \sum_{k=1}^{n} r_k x_k$ (by Ref. [66], p12, Theorem 1.4, we have $i = R_0 d + \sum_{k=1}^{n} R_k x_k$, $R_i \in \mathbb{Z}$, then taking module $d$). And we define $j = \sum_{k=1}^{n} r_k y_k$. By Lemma 3, we have $(i,j) \in C$. As both $(x_k,y_k)$ and $(i,j)$ are in $C$, by the definition of $i$, for each $k$, the element $x_k \in i\mathbb{Z}_d$. As both $(x_k,y_k)$ and $(i,j)$ are in $C$, we have $iy_k - jx_k \equiv 0 \mod d$. Therefore, by definition of $C_{i,j}$, for each $k$, the element $(x_k,y_k) \in C_{i,j}$. By Lemma 3 again, one have $C \subseteq C_{i,j}$. However, both sets are maximally commutative sets of GBSs. Therefore, $C$ must equal to $C_{i,j}$.

In the following, we show that each $C_{x,y}$ ($x \neq 0$) is in fact lie in one of $\text{MCS}_d$. Set $d_x$ denote the greatest common divisor of $x$ and $d$ (we might assume $x = c_x d_x$ where $c_x \in \mathbb{Z}$). So $d_x = qx + rd$ for some integers $q,r$. There exist a unique $j \in \{0,1,\cdots,k - 1\}$ (where $k_x d_x = d$) such that

$$qy - j \equiv k_x d_x.$$  

That is, $qy - j = k_x l_x$ for some $l_x \in \mathbb{Z}_d$. For this $j$, we have the following equation

$$\begin{vmatrix} d_x & j \\ x & y \end{vmatrix} = \begin{vmatrix} d_x & qy - k_x l_x \\ x & y \end{vmatrix} = (d_x - qx)y + l_x k_x x.$$
which is equal \((ry + l_c c_d) d = 0\) under \(\mod d\). By definition, \(x \in dZd\). Therefore, we have \((x, y) \in C_{d, j}\). As the elements in \(C_{d, j}\) are commute with each other, for any \((x_1, y_1) \in C_{d, j}\), we have

\[
\begin{bmatrix}
  x \\
  y \\
  x_1 \\
  y_1
\end{bmatrix} \equiv 0 \mod d.
\]

Note that \(dZd = xZd\), we have \(x_1 \in xZd\). Therefore, \(C_{d, j} \subseteq C_{x, y}\). By the maximality, we must have \(C_{x, y} = C_{d, j}\). Therefore, one conclude that every \(C_{x, y}\) must be one of the elements in \(MCS_d\).

On the other hand, we need to show that the sets in \(MCS_d\) are mutually different. Clearly, \(C_{0, 0}\) is different from all the other sets. Let \(C_{i_1, j_1}\) and \(C_{i_2, j_2}\) be any two members of \(MCS_d\) where \((i_1, j_1) \neq (i_2, j_2)\) and \(i_1, i_2\) are nonzero. As \(d = i_1 k_1\), if \(i_1 = i_2\), we have \(i_1 j_2 - i_2 j_1 = i_1 (j_2 - j_1)\) which lies between \(-(d - 1)\) and \(d - 1\) but not equal to zero. Hence, \((i_1, j_1)\) and \((i_2, j_2)\) are not commute. Therefore, \(C_{i_1, j_1} \neq C_{i_2, j_2}\). If \(i_1 \neq i_2\), we can assume that \(i_1 < i_2\) without loss of generality. As both \(i_1\) and \(i_2\) are divisors of \(D\), one can check that \(i_1 \notin i_2Zd\). Therefore, by definition, \((i_1, j_1) \notin C_{i_2, j_2}\). Hence, we also have \(C_{i_1, j_1} \neq C_{i_2, j_2}\).

For each divisor \(i(1 \leq i < d)\) of \(d\), it contributes to \(d/i\) classes of MCSs to \(MCS_d\). Therefore,

\[
|MCS_d| = 1 + \sum_{i | d, 1 \leq i < d} \frac{d}{i} = \sum_{i | d} \frac{d}{i} = \sigma(d).
\]

This completes the proof.

From the above theorem, we know that there are \(\sigma(d)\) classes of MCSs of \(d\) dimensional GBSs. Are there any differences in the ability of these MCSs to detect generalized Bell state sets? Is there any redundant MCS in detecting the local discrimination of generalized Bell sets? We present some numerical results for the two questions.

The three solid lines in Fig. 3 imply that the successful rates of all MCSs are almost equal to each other (one should compare this with the detectors defined in Eq. (6), see Fig. 4). The three dot lines imply that each class of MCSs is irredundant in the sense that for each MCS \(C\), there exist some set \(S\) whose local distinguishability can only be detected by \(C\) but not by other MCSs.

![Fig. 3: We consider the cases \(d = 20\) and \(n = 9, 10, 11\) (which correspond to curves with color green, purple and pink respectively). We randomly generate 100000 sets of GBSs whose cardinality are all \(n = 9, 10, 11\) respectively. Each solid curve shows the successful rate for the \(\sigma(20) = 42\) classes of MCSs. The dot lines represent the successful rates for the 42 classes of MCSs such that the local distinguishability of the samples can only detected by one class of MCS itself.](image)

![Fig. 4: We consider the case \(d = 20\) and \(n = 10\). The figure shows the successful rates for each detectors (see Eq. (6)) indicated by the coordinates \((i, j) \in Z_{20} \times Z_{20}\). We randomly generate 100000 sets of GBSs whose cardinality are all \(n = 10\).](image)

### IV. Conclusion and Discussion

In this paper, we studied the problem of local distinguishability of generalized Bell states. Firstly, we gave a review of some important methods for detecting the local distinguishability of GBSs. Motivated by a recent method derived by Yuan et. al. [65], we introduced the concept maximally commutative set of GBSs. Surprisingly, we found that each MCS is useful for detecting the local distinguishability of GBSs. More exactly, given a set \(S\) of GBSs, if there exists some MCS \(C\) such that the difference set of \(S\) is disjoint with or contains in \(C\), then the set \(S\) can be one-way distinguishable. This method is stronger than that in Ref. [65]. This motivates us to find out all the MCSs of given dimension. Indeed, we presented a complete structure characterization of MCS in Theorem 3.

However, MCS only gives a sufficient condition for locally distinguishable, it is not necessary. It is interesting to derive an easy checking condition for local distin-
guishability of GBSs which are both sufficient and necessary. In addition, it is interesting to check whether Fan’s results can be extended to systems without the assumption on the dimension of local systems. A weaker form is that: given any integer \( l \), do there exist some \( D \) (which depends on \( l \)) such that if \( d \geq D \), then any \( l \) GBSs in \( \mathbb{C}^d \otimes \mathbb{C}^d \) are locally distinguishable? As far as we known, this problem is only solved for the case \( l = 3 \). We conjecture that this holds for all other cases.

Note that our method here can be generalized to any maximally entangled basis whose defining unitary matrices \( B \) satisfies: for any \( U, V \in B \) there exists some \( W \in B \) such that \( U \dagger V \propto W \). The lattice qudits basis \([57]\) is such an example. From their proof, any locally distinguishable set of lattices qudits basis that can be detected by Ref. \([57]\) can be always detected by a MCS of lattice qudits basis. Therefore, our method can be also seen as a generalization of theirs. Therefore, it is also interesting to give a complete characterization of the MCS of lattice qudits basis and study its application to local discrimination.

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