Numerical over-approximation of invariance entropy via finite abstractions

M. S. Tomar\textsuperscript{a}, C. Kawan\textsuperscript{b}, M. Zamani\textsuperscript{a,b}

\textsuperscript{a}Computer Science Department, University of Colorado Boulder, USA
\textsuperscript{b}Institute of Informatics, LMU Munich, Germany

Abstract

For a closed-loop system with a digital channel between the sensor and controller, invariance entropy quantifies the smallest average rate of information above which a compact subset \( Q \) of the state set can be made invariant. There exist different versions of invariance entropy for deterministic and uncertain control systems, which are equivalent in the deterministic case. In this paper, we present the first numerical approaches to obtain rigorous upper bounds of these quantities. Our approaches are based on set-valued numerical analysis and graph-theoretic constructions. We combine existing algorithms from the literature to carry out our computations for several linear and nonlinear examples. A comparison with the theoretical values of the entropy shows that our bounds are of the same order of magnitude as the actual values.

Keywords: Invariance entropy; finite abstractions; numerical methods.

1. Introduction

In classical control theory, sensors and controllers are usually connected through point-to-point wiring. In networked control systems (NCS), sensors and controllers are often spatially distributed and involve wireless digital communication networks for data transfer. Compared to classical control systems, NCS provide many advantages such as reduced wiring, low installation and maintenance costs, greater system flexibility and ease of modification. NCS find applications in many areas such as car automation, intelligent buildings, and transportation networks. However, the use of communication networks in feedback control loops makes the analysis and design of NCS much more complex. In NCS, the use of digital channels for data transfer from sensors to controllers limits the amount of data that can be transferred per unit of time. This introduces quantization errors that can affect the control performance adversely.

The problem of stabilizing or observing a system over a communication channel with a limited bit rate has attracted a lot of attentions in the past two decades. In this context, a classical result, often called the data-rate theorem, states that the minimal bit rate or channel capacity above which a linear system can be stabilized or observed is given by the log-sum of the unstable eigenvalues. This result has been proved under various assumptions on the system model, channel model, communication protocol, and stabilization/estimation objectives. Comprehensive reviews of results on data-rate-limited control can be found, e.g., in the surveys \cite{2,10,28} and the books \cite{39,26}.

For nonlinear systems, the smallest bit rate of a channel between the coder and the controller, to achieve some control task such as stabilization or invariance, can be characterized in terms of certain notions of entropy which are defined in terms of the open-loop system and are independent of the choice of the coder-controller. In spirit, they are similar to classical entropy notions used in the theory of dynamical systems to quantify the rate at which a system generates information, see \cite{15}.

In this paper, we first consider deterministic systems and focus on the notion of invariance entropy (IE) introduced in \cite{8} as a measure for the smallest average data rate above which a compact controlled invariant subset \( Q \) of the state set can be made invariant. We present the first attempt to compute upper bounds of IE numerically. Our approach combines different algorithms. First, we compute a symbolic abstraction of the given control system over the set \( Q \) and the corresponding invariant controller using the tool SCOTS \cite{30}. This results in a fine box partition of \( Q \) with a set of admissible control inputs assigned to each box for maintaining invariance of \( Q \). In the second step, we use the tool dtControl \cite{3} that converts the controller from a look-up table into a decision tree. Each leaf node of the tree represents a group of boxes to which the same single control input is assigned. The set of groups constitute a coarse partition of \( Q \). Finally, in the third step, an algorithm that was proposed in \cite{12} for estimation of topological entropy is adopted. Its output serves as an upper bound for the IE.

In addition, we also develop a method to approximate the IE of uncertain control systems, as introduced in \cite{31}[34], that generalizes the IE of deterministic systems. If the IE of a set \( Q \) (for an uncertain system) is finite \cite{31} Sec. 4.2], an upper bound can be computed from a graph constructed using a finite abstraction of the system \cite{31} Sec. 6]. However, the number of vertices in the graph is of the order of \( 2^n \), where \( n \) is the number of states in the finite abstraction. In this paper, we present an upper bound
for the IE of uncertain systems that can be computed from a weighted directed graph constructed from an invariant partition (a pair of a finite partition of \( Q \) and a map that assigns a control input to every partition element). Our main result characterizes the entropy of the invariant partition in terms of the weights of the graph and establishes that it is the same as the maximum cycle mean of the graph. We should highlight that the number of vertices in this graph is not larger than \( n \). Our proposed procedures may still suffer from the curse of dimensionality due to constructing finite abstractions of control systems. Moreover, at this point, we are not able to quantify the gap between the upper bounds and the actual values of the IE.

**Brief literature review.** The notion of invariance entropy for deterministic systems is equivalent to topological feedback entropy that has been introduced earlier in [27]; see [9] for a proof. Various notions of invariance entropy have been proposed to tackle different control problems or other classes of systems, see for instance [5] (exponential stabilization), [18] (invariance in networks of systems), [31] (invariance for uncertain systems), [6] [37] (measure-theoretic versions of invariance entropy) and [21] (stochastic stabilization). An over-approximation of invariance entropy through a compositional approach, for networks of uncertain control systems, was also discussed in [15]. Also the problem of state estimation over digital channels has been studied extensively by several groups of researchers. As it turns out, the classical notions of entropy used in dynamical systems, namely measure-theoretic and topological entropy (or variations of them), can be used to describe the smallest data rate or channel capacity above which the state of an autonomous dynamical system can be estimated with an arbitrarily small error, see [32, 22, 33, 38, 20]. Motivated by the observation that estimation schemes based on topological entropy suffer from a lack of robustness and are hard to implement, the authors of [24] [25] introduce a suitable notion of restoration entropy which characterizes the minimal data rate for so-called regular and fine observability. Finally, algorithms for state estimation over digital channels have been proposed in several works, particularly in [22, 24, 12, 19].

**Related work:** In [13], the authors consider linear uncertain control systems and provide an algorithm to compute an invariant cover, the cardinality of which serves as an upper bound for the invariance entropy. In contrast, our proposed procedure here is applicable to nonlinear systems as well.

**Notation:** We write \( \mathbb{N} = \{1, 2, 3, \ldots \} \) for the natural numbers, \( \mathbb{Z} \) for the set of integers and \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \). By \( \mathbb{R} \), we denote the set of real numbers and define \( \mathbb{R}_+ := \{ r \in \mathbb{R} : r \geq 0 \} \) and \( \mathbb{R}_{>0} := \mathbb{R}_+ \setminus \{0\} \). By \( \{a; b\} = \mathbb{Z} \cap \{a, b\} \) and \( \{a; b\} = \mathbb{Z} \setminus \{a, b\} \), we denote closed and right-open discrete intervals. We write \( |A| \) for the cardinality of a set \( A \) and \( p(R) \) for the spectral radius of a square matrix \( R \). The notation \( Y^X \) is used for the set of all functions \( f : X \rightarrow Y \). For \( \tau \in \mathbb{Z}_+ \), we use \( X^\tau \) to denote \( X^{[0, \tau)} \). By \( f : X \cong Y \), we denote a set-valued map from \( X \) to \( Y \). A cover \( \mathcal{A} \) of set \( Q \) is a family of subsets of \( Q \) such that \( \bigcup_{A \in \mathcal{A}} A = Q \). A cover \( \mathcal{A} \) is called a partition if for all \( A_1, A_2 \in \mathcal{A}, A_1 \neq 0 \) and \( A_1 \cap A_2 = 0 \). We write \( f|_M \) for the restriction of a map \( f \) to a subset \( M \subseteq X \).

### 2. Background on invariance entropy

In this section, we provide the necessary theoretical background for our proposed numerical methods.

**A deterministic discrete-time control system** is given by

\[
\Sigma : \quad x_{t+1} = f(x_t, u_t),
\]

where \( f : X \times U \rightarrow X \), \( X \subseteq \mathbb{R}^n \), \( U \subseteq \mathbb{R}^m \), is a (not necessarily continuous) map. The transition map \( \varphi : \mathbb{Z}_+ \times X \times \mathbb{Z}_+^\tau \rightarrow X \) of \( \Sigma \) is defined as

\[
\varphi(t, x, \omega) := \begin{cases} 
  x & \text{if } t = 0, \\
  f(\varphi(t-1, x, \omega), \omega_{t-1}) & \text{if } t > 0.
\end{cases}
\]

Now, consider a compact set \( Q \subseteq X \) which is controlled invariant, i.e., for each \( x \in Q \) there is \( u \in U \) with \( f(x, u) \in Q \). For any \( \tau \in \mathbb{N} \), a set \( S \subseteq U^\tau \) is called \( (\tau, Q) \)-spanning if for each \( x \in Q \) there is \( \omega \in S \) with \( \varphi(t, x, \omega) \in Q \) for \( 0 \leq t \leq \tau \). We write \( r_{\text{inv}}(\tau, Q) \) for the minimal cardinality among all \( (\tau, Q) \)-spanning sets and define the invariance entropy (IE) of \( Q \) as

\[
h_{\text{inv}}(Q) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q),
\]

if \( r_{\text{inv}}(\tau, Q) \) is finite for all \( \tau \); otherwise, \( h_{\text{inv}}(Q) = \infty \). The existence of the limit follows from the subadditivity of the sequence \( (\log_2 r_{\text{inv}}(\tau, Q))_{\tau \in \mathbb{N}} \), using Fekete’s subadditivity lemma (see [9] Lem. 2.1) for a proof.

The method we propose to estimate \( h_{\text{inv}}(Q) \) is based on an alternative characterization of this quantity that we will now describe. A triple \((\mathcal{A}, \tau, G)\) is called an invariant partition of \( Q \) if \( \mathcal{A} \) is a finite partition of \( Q \), \( \tau \in \mathbb{N} \), and \( G : \mathcal{A} \rightarrow U^\tau \) is a map satisfying \( \varphi(t, x, \omega) \subseteq Q \) for every \( A \in \mathcal{A} \) and \( 0 \leq t \leq \tau \) (note that \( \varphi(t, x, \omega) \) only depends on \( \omega_{[0,\tau)} \)). For a given \( C = (\mathcal{A}, \tau, G) \), we define

\[
T_C : Q \rightarrow Q, \quad T_C(x) := \varphi(x, \omega, G(A_i)),
\]

where \( A_i \in \mathcal{A} \) is such that \( x \in A_i \). Since \( \mathcal{A} \) is a partition of \( Q \), \( T_C \) is well-defined.

Now, let \( C = (\mathcal{A}, \tau, G) \) be an invariant partition. For each \( N \in \mathbb{N} \), we introduce the set

\[
W_N(T_C) := \{ a \in \mathbb{A}^N : \exists x \in Q \text{ s.t. } T_C^i(x) \in A_{\alpha_i}, 0 \leq i < N \},
\]

which is constituted by all such \( N \)-length sequences in \( \mathcal{A} \) that there exists a trajectory of \( T_C \) that follows the sequence. Next we define

\[
h^*(T_C) := \lim_{N \to \infty} \frac{1}{N} \log_2 |W_N(T_C)|.
\]

Again, subadditivity guarantees the existence of the limit. Then, by [16] Thm. 2.3], the IE satisfies

\[
h_{\text{inv}}(Q) = \inf_{C=(\mathcal{A},\tau,G)} \frac{1}{\tau} h^*(T_C),
\]

\footnote{For a set \( A \subseteq X \), by \( \varphi(t, A, G(A)) \) we refer to \( \bigcup_{C \in \mathcal{A}} \varphi(t, x, G(A)) \).}
where the infimum is taken over all invariant partitions of \( Q \). In particular, \( h_{inv}(Q) < \infty \) if and only if an invariant partition of \( Q \) exists [16] Prop. 2.20, Lem. 2.3.

An uncertain discrete-time control system is given by

\[
\Sigma : \quad x_{t+1} \in F(x_t, u_t),
\]

where \( \mathcal{X} \subseteq \mathbb{R}^n, U \subseteq \mathbb{R}^m \), and \( F : \mathcal{X} \times U \Rightarrow \mathcal{X} \) is a set-valued map satisfying \( F(x, u) \neq \emptyset \) for all \((x, u) \in \mathcal{X} \times U\).

Consider a compact set \( Q \subseteq \mathcal{X} \) which is controlled invariant, i.e., for each \( x \in Q \) there is \( u \in U \) with \( F(x, u) \subseteq Q \). We define the invariance entropy of \( Q \) in a quite different manner as in the deterministic case. However, in the special case when \( F \) is single-valued, i.e., when \( \Sigma \) is deterministic, it coincides with the previous notion.

A pair \((\mathcal{A}, G)\) is called an invariant cover of \( Q \) w.r.t. \( \Sigma \) if \( \mathcal{A} \) is a finite cover of \( Q \) and \( F(A, G(A)) \subseteq Q \) for all \( A \in \mathcal{A} \). In the case when \( \mathcal{A} \) is a partition, we call \((\mathcal{A}, G)\) an invariant partition, analogous to the deterministic case. \(^4\) For \( \tau \in \mathbb{N} \), let \( \mathcal{J} \subseteq \mathcal{A}^{(\mathbb{N}, \tau)} \) be a set of sequences in \( \mathcal{A} \) of length \( \tau \). For \( \alpha \in \mathcal{J} \) and \( t \in \{0; \tau-2\} \), define

\[
P_{\mathcal{J}}(\alpha|_{[0:t]}) := \{ A \in \mathcal{A} : \alpha|_{[0:t]}A = \hat{\alpha}t_{[0:t+1]} \}
\]

as the set of immediate successor cover elements \( A \) of \( \alpha|_{[0:t]} \) in \( \mathcal{J} \), and for \( t = \tau - 1 \), define

\[
P_{\mathcal{J}}(\alpha|_{[0:t]}) := \{ A \in \mathcal{A} : \alpha = \hat{\alpha}(0) \}
\]

as the set of the first components of the sequences in \( \mathcal{J} \). Although this set does not depend on \( \alpha \), for consistency reasons, we still use the same notation as in \([4]\). A set \( \mathcal{J} \subseteq \mathcal{A}^{(\mathbb{N}, \tau)} \) is called \((\tau, Q)\)-spanning in \((\mathcal{A}, G)\) if \( P_{\mathcal{J}}(\alpha) \) covers \( Q \) and for all \( \alpha \in \mathcal{J} \) and all \( t \in \{0; \tau - 2\} \),

\[
F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A \in P_{\mathcal{J}}(\alpha|_{[0:t]})} A'.
\]

In this case, we associate to \( \mathcal{J} \) its expansion number

\[
N(\mathcal{J}) := \max_{\alpha \in \mathcal{J}} \sum_{t=0}^{\tau-1} |P_{\mathcal{J}}(\alpha|_{[0:t]})|,
\]

and with \( r_{inv}(\tau, Q, \mathcal{A}, G) \) for the smallest expansion number among all \((\tau, Q)\)-spanning sets in \((\mathcal{A}, G)\), i.e.,

\[
r_{inv}(\tau, Q, \mathcal{A}, G) := \min\{N(\mathcal{J}) \mid \mathcal{J} \text{ is } (\tau, Q)\text{-spanning in } (\mathcal{A}, G)\}.
\]

The entropy of an invariant cover \((\mathcal{A}, G)\) is then defined as

\[
h(\mathcal{A}, G) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{inv}(\tau, Q, \mathcal{A}, G).
\]

The existence of the limit follows again by subadditivity. The invariance entropy of \( Q \) is now defined as

\[
h_{inv}(Q) := \inf_{(\mathcal{A}, G)} h(\mathcal{A}, G),
\]

where the infimum is taken over all invariant covers of \( Q \). Although this definition does not seem to have much similarity with the definition(s) for deterministic systems, \( h_{inv}(Q) \) reduces to \( h_{inv}(V) \) in the case when \( F \) is single-valued, see [34] Thm. 4.

3. Upper bounds: deterministic case

In this section, we explain how to obtain a computable upper bound for \( h_{inv}(Q) \), based on \([2]\). Suppose that we have an invariant partition \( C = (\mathcal{A}, \tau, G) \) with \( \mathcal{A} = \{A_1,\ldots,A_q\} \) at our disposal. Then any upper bound on \( h^*(T_C) \) will yield an upper bound on \( h_{inv}(Q) \).

Let us first select a refinement \( B = \{B_1,\ldots,B_r\} \) of \( \mathcal{A} \), i.e., a partition of \( Q \) such that each \( B \in B \) is contained in some \( A \in \mathcal{A} \). Now we define

\[
W_N(\mathcal{B}, \mathcal{A}) := \{\alpha \in \mathcal{A}^N : \exists \beta \in \mathcal{B}^N \text{ with } T_C(\beta_j) \cap \beta_{j+1} \neq \emptyset \forall j \in \{0; N - 2\} \text{ s.t. } \beta_j \leq \alpha_i \forall i \in \{0; N - 1\}\}.
\]

From [12] Sec. 2.2, we have

\[
h(\mathcal{B}, \mathcal{A}) := \lim_{N \to \infty} \frac{\log_2 |W_N(\mathcal{B}, \mathcal{A})|}{N} \geq h^*(T_C).
\]

Moreover, assuming compactness of the partition sets and continuity of the map, it can be shown that \( h(\mathcal{B}, \mathcal{A}) \) converges to \( h^*(T_C) \) as the maximal diameter of the elements of \( B \) tends to zero, see [12] Thm. 4.16.

The paper [12] describes an algorithm for the exact computation of \( h(\mathcal{B}, \mathcal{A}) \), based on symbolic dynamics. First, we associate a transition matrix to \( B \) via

\[
\Gamma_{ij} := \begin{cases} 1 & \text{if } T_C(B_i) \cap B_j \neq \emptyset, \quad i, j = 1, \ldots, r \quad (7) \\ 0 & \text{otherwise} \end{cases}
\]

Then one constructs a directed labeled graph \( \mathcal{G} \) from the transition matrix \( \Gamma \). The set of nodes is \( B \) and \( \Gamma_{ij} = 1 \) indicates that there is a directed edge from \( B_i \) to \( B_j \). To this edge, we assign the edge label

\[
L(B_i) := j, \text{ where } j \text{ is such that } B_i \subset A_j.
\]

Elements of \( W_N(\mathcal{B}, \mathcal{A}) \) are thus generated by concatenating labels along walks of length \( N \) on \( \mathcal{G} \). To compute \( h(\mathcal{B}, \mathcal{A}) \), a right-resolving graph \( \mathcal{G} \) needs to be determined (see [23] §3.3), such that the subset of \( \mathbb{N}^\mathcal{G} \) generated by concatenation of edge labels along walks in the graph is same for both \( \mathcal{G} \) and \( \mathcal{G} \). Each node in the right-resolving graph \( \mathcal{G} \) is some subset of

\(^3\) Of course, such an assumption can, in general, not be satisfied. We expect that the result still holds true if only a negligibly small amount of the exponential orbit complexity of the closed-loop dynamics is concentrated on the boundaries of the sets \( A_i \). If \( T_C \) was continuous on \( Q \), this could be formalized by requiring that these boundaries have measure zero w.r.t. any \( T_C \)-invariant Borel probability measure.

\(^4\) In the language of symbolic dynamics, the matrix \( \Gamma \) defines a \textit{subshift of finite type} over the alphabet \([1, \ldots, r]\).

\(^5\) A labeled graph is right-resolving if, for each vertex, all the outgoing edges have different labels.

\(^6\) The subset of \( \mathbb{N}^\mathcal{G} \) generated by concatenating edge labels along all walks on \( \mathcal{G} \) forms a sofic shift whose topological entropy equals \( h(\mathcal{B}, \mathcal{A}) \).
Consider the linear control system

\[ x_{t+1} = Ax_t + Bu_t, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \]

with \( x_t \in \mathbb{R}^2 \) and \( u_t \in [-1, 1] \). For the compact controlled invariant set \( Q = [-1, 1] \times [-2, 2] \), see \[7\] Ex. 21, we intend to compute an upper bound of \( h_{\text{max}}(Q) \).

Given a discrete-time system \( \Sigma \) as in (1) and a compact controlled invariant set \( Q \subseteq X \), we proceed according to the following steps.

1. Compute a symbolic invariant controller for the set \( Q \).

We use SCOTS to compute an invariant controller for \( \Sigma \) with \( Q \) as the state set and \( \eta_1, \eta_2 \) as the grid parameters for the state and input sets, respectively. A smaller value for \( \eta_1 \) results in a finer grid on the state set, which typically results in a better upper bound. We denote the set of boxes in the domain of the computed controller by \( B = \{ B_1, \ldots, B_r \} \) and put

\[ \hat{Q} := \bigcup_{i=1}^r B_i \subseteq Q. \]

The set \( \hat{Q} \) is our approximation of \( Q \).

2. The controller obtained in the previous step is, in general, non-deterministic, i.e., different control inputs are assigned to the same state. In this step, we determine the obtained controller. We denote the closed-loop system (\( \Sigma \) with the determined controller \( C \)) by \( \Sigma_C \). To determine the controller, we used the state-of-the-art toolbox dtControl \[8\], which utilizes the decision tree learning algorithm. This also provides the required coarse partition \( \mathcal{A} \) of which \( B \) is a refinement.

**Example 1 (Continued).** We used SCOTS with the state set \( Q_X = Q \) and the state and input set grid parameters \( \eta_1 = [2/3, 4/3] \) and \( \eta_2 = 1 \). This results in a state set grid with 9 boxes, \( B = \{ B_1, \ldots, B_9 \} \) and \( Q = \hat{Q} \) (see Fig. 7).

3. For the dynamical system \( \Sigma_C \), we obtain the transition matrix \( \Gamma \) (defined in (7)) for the boxes in \( \hat{Q} \).

4. We obtain the edge labels map \( L(B_i) \) as in (8).

**Example 1 (Continued).** For any \( B_i \in B \),

\[ L(B_i) = \begin{cases} 1 & \text{if } i = 1 + 3t, 0 \leq t \leq 2, \\ 2 & \text{if } i = 2 + 3t, 0 \leq t \leq 2, \\ 3 & \text{if } i = 3 + 3t, 0 \leq t \leq 2. \end{cases} \]

5. We construct a directed labeled graph \( G \) with \( B \) as the set of nodes. If \( \Gamma_{ij} = 1 \), there is a directed edge from the node \( B_i \) to \( B_j \) with label \( L(B_i) \).

6. We determine the strongly connected components of \( G \).
Example 1 (Continued). \(G\) is strongly connected. Figure 2 shows the constructed graph \(\bar{G}\).

(7) For every strongly connected component \(G_k\) of \(G\), we find a right-resolving graph \(\bar{G}_k\). The directed graph \(\bar{G}_k\) is deterministic in the sense that for every node no two outgoing edges have the same label.

Example 1 (Continued). Right-resolving graph of \(G\) with nodes \(r_1 = \{B_2 : i \in \{7, 8\}\}, \ r_2 = \{B_2 : 4 \leq i \leq 6\}, \ \ r_3 = \{B_2 : i \in \{2, 3\}\}, \ r_4 = \{B_2 : 4 \leq i \leq 9\}, \ r_5 = \{B_2 : i \in \{2, 3, 5, 6\}\}, \ r_6 = \{B_2 : 1 \leq i \leq 6\},\) and \(r_7 = \{B_2 : i \in \{4, 5, 7, 8\}\}.\) The constructed right-resolving graph \(\bar{G}\) is shown in Figure 2.

(8) Using \(\bar{G}_k\), we construct an adjacency matrix \(R^k\) by \(R^k_{ij} := l\), where \(l\) is the number of edges from node \(i\) to node \(j\) in \(\bar{G}_k\).

Example 1 (Continued). From \(\bar{G}_k\), we obtain

\[
R = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix},
\]

with \(\rho(R) = 2.41421\) and \(\log_2(2.4142) = 1.2716\).

4. Upper bounds: uncertain case

In this section, we explain how to obtain a computable upper bound for the IE of an uncertain system.

Suppose again that we know an invariant partition \((\mathcal{A}, G)\) and recall that the time step \(\tau\) is always set to 1 for uncertain systems. We define a set-valued map \(T : \mathcal{A} \rhd \mathcal{A}\) by \(T(x) := F(x, G(A_x))\), where \(x \in A\), \(A \in \mathcal{A}\). We also define a weighted directed graph \(G\) with \(\mathcal{A}\) as its set of nodes. The graph \(G\) contains an edge from \(A\) to \(A'\), denoted by \(e_{AA'}\), if \(A \cap A' \neq \emptyset\). We define maps \(D : \mathcal{A} \rhd \mathcal{A}\) and \(w : \mathcal{A} \to \mathbb{R}_+\) by

\[
D(A) := \{A' \in \mathcal{A} : T(A) \cap A' \neq \emptyset\},
\]

\[
w(A) := \log_2 |D(A)|.
\]

The weight of the edge \(e_{AA'}\) is defined to be \(w(A)\). We observe that \(T(A) \subseteq \bigcup_{A \in D(A)} \hat{A}\).

Given \(\tau \in \mathbb{N} \cup \{\infty\}\), we let \(W_\tau(G)\) denote the set of all (node) paths \((A_i)_{i=0}^{\tau}\) in \(G\) of length \(\tau\).

Consider a cycle \(c = (e_{A_k A_{k+1}})_{k=1}^{\tau},\) where \(A_k = A_1\) in \(G\). The mean cycle weight of \(c\) is defined as \(w_m(c) := \frac{1}{k} \sum_{i=1}^{k} w(A_i)\).

The maximum cycle mean weight is then defined as \(w^*_m(G) := \max_c w_m(c)\), the maximum taken over all cycles in \(G\) (the maximum exists because, due to the finiteness of the graph, it suffices to take the maximum over finitely many cycles).

Our algorithm is based on the following theorem, which yields a characterization of the entropy of an invariant partition in terms of the associated graph \(G\).

**Theorem 1.** For an uncertain control system \(\Sigma\) as in (5), a compact controlled invariant set \(Q \subseteq X\) and an invariant partition \((\mathcal{A}, G)\), we have

\[
\tilde{h}(\mathcal{A}, G) = \lim_{\tau \to \infty} \frac{1}{\tau} \max_{A \in D(A)} w_m(c) = w^*_m(G).
\]

**Remark 1.** In [11], the authors show that the logarithm of the joint spectral radius of a finite set of rank one matrices equals the maximum cycle mean in a directed complete graph. The result of the paper can be used to establish that, for the case of non-complete graph, the entropy of an invariant partition is upper bounded by the maximum cycle mean.
The rest of this section is devoted to the proof of the theorem. We start with two lemmas.

**Lemma 1.** $W_t(G)$ is a $(\tau, Q)$-spanning set in $(\mathcal{A}, G)$.

**Proof.** Since $(\mathcal{A}, G)$ is an invariant cover, we have $D(A) \neq \emptyset$ for every $A \in \mathcal{A}$. Thus, for every node in $G$ there is at least one outgoing edge. Hence, for all $A \in \mathcal{A}$ and $\tau \in \mathbb{N}$, there is at least one path of length $\tau$ starting from $A$. It follows that

$$[\alpha(0) : \alpha \in W_t(G)] = \mathcal{A}.$$  

Consider any $\alpha \in W_t(G)$ and $t \in [0; \tau - 1]$. By the definition of $G$, we have an edge from $\alpha(t)$ to every $A \in D(\alpha(t))$. Thus, for every $t \in [0; \tau - 2]$ we have

$$P_{W_t(G)}(\alpha|_{[0,t]}) = D(\alpha(t)).$$  

Using (10), we conclude that $W_t(G)$ satisfies (5), and hence is a $(\tau, Q)$-spanning set in $(\mathcal{A}, G)$. \hfill \Box

**Lemma 2.** For any $(\tau, Q)$-spanning set $\mathcal{J}$ in $(\mathcal{A}, G)$, $W_t(G) \subseteq \mathcal{J}$.

**Proof.** Let $\mathcal{J}$ be a $(\tau, Q)$-spanning set in $(\mathcal{A}, G)$. Then, since $\mathcal{A}$ is a partition, $[\alpha(0) : \alpha \in \mathcal{J}] = \mathcal{A}$. If $\alpha \in \mathcal{J}$ and $t \in [0; \tau - 1]$, then from (5) it follows that $P_{\mathcal{J}}(\alpha|_{[0,t]})$ covers $F(\alpha(t), G(\alpha(t))) = T(\alpha(t))$. Since $\mathcal{A}$ is a partition, $D(\alpha(t))$ must be contained in every subset of $\mathcal{A}$ which covers $T(\alpha(t))$. Thus, $\mathcal{J}$ is $(\tau, Q)$-spanning, from (5) we know that $T(\alpha(t))$ is covered by $P_{\mathcal{J}}(\alpha(t))$, implying $\beta(0) \in \mathcal{A} = [\alpha(0) : \alpha \in \mathcal{J}]$, implying $\beta(0) = \alpha(0)$ for some $\alpha \in \mathcal{J}$. From (12), we have $P_{W_t(G)}(\beta(0)) = D(\beta(0))$. Similarly to the reasoning above, since $\mathcal{A}$ is a partition, $D(\beta(0))$ is contained in every subset of $\mathcal{A}$ which covers $T(\beta(0))$. As $\mathcal{J}$ is $(\tau, Q)$-spanning, from (5) we know that $T(\alpha(t))$ is covered by $P_{\mathcal{J}}(\alpha(t))$, implying $P_{\mathcal{J}}(\alpha(t)) \supseteq D(\beta(0))$. From the definition of $\mathcal{G}$, we obtain $\beta(1) \in D(\beta(0))$, which leads to $\beta(1) \in P_{\mathcal{J}}(\alpha(t))$. Thus, there exists an $\alpha \in \mathcal{J}$ with $[\alpha|_{[0,t]}] = [\beta|_{[0,t]}]$. Inductively, we obtain the existence of $\alpha \in \mathcal{J}$ with $\alpha = \beta$, which concludes the proof. \hfill \Box

We can now prove Theorem 1.

**Proof.** (of Theorem 1) From (6) and Lemma 2, we conclude that for every $(\tau, Q)$-spanning set $\mathcal{J}$ in $(\mathcal{A}, G)$, the inequality $N(W_t(G)) \leq N(\mathcal{J})$, holding that

$$h_{\text{sn}}(\tau, Q, \mathcal{A}, G) = N(W_t(G)) \quad \text{for all } \tau \in \mathbb{N}.  \quad (13)$$

By taking logarithms on both sides of (6) and using (12) and (9), we obtain

$$\log_2 N(W_t(G)) = \max_{\alpha \in W_t(G)} \sum_{i=0}^{\tau-2} w(\alpha(t)) + \log_2 |\mathcal{A}|. \quad (14)$$

Putting (13) and (14) together, it follows that

$$h(\mathcal{A}, G) = \lim_{\tau \to \infty} \frac{1}{\tau} \max_{\alpha \in W_t(G)} \sum_{i=0}^{\tau-2} w(\alpha(t)).$$

Observing that the elements of $W_t(G)$ are restrictions of elements of $W_\infty(G)$ to $[0; \tau - 1]$, the first equality in (11) follows.

For the proof of the second equality in (11), let $\mathcal{A} = \{A_1, \ldots, A_q\}$ and consider an arbitrary $\alpha \in W_\infty(G)$. From (36, Lem. 3), we know that for each $\tau$ we can write

$$\sum_{i=0}^{\tau-2} w(\alpha(t)) = \sum_{i=0}^{\tau-2} w(\beta(t)) + \sum_{i=0}^{\tau-2} l_i w_m(\sigma_i),$$

for some $\beta \in W_\infty(G)$, $a < n - 1$ and proper cycles $\sigma_i$ of length $l_i$ so that $\tau - 1 = a + 1 + \sum_{i=1}^{\tau-2} l_i$. It thus follows that

$$\sum_{i=0}^{\tau-2} w(\alpha(t)) \leq n \max_{\alpha \in \mathcal{A}} w(\alpha(t)) + w_m \sum_{i=1}^{\tau-2} l_i \leq n \log_2 n + \tau w_m,$$

leading to

$$\lim_{\tau \to \infty} \frac{1}{\tau} \max_{\alpha \in W_\infty(G)} \sum_{i=0}^{\tau-2} w(\alpha(t)) \leq w_m.$$

To show the converse inequality, consider an $\alpha \in W_\infty(G)$ that traces a proper cycle with mean weight equal to the maximum cycle mean $w_m^*$. Let $\tau$ be the length of the cycle and write $\tau - 1 = r l + a$ for any $\tau > 1$, where $r \geq 0$ and $0 \leq a < l$ are integers. This implies

$$\frac{1}{r} \sum_{i=0}^{r-2} w(\alpha(t)) \geq \frac{1}{r} w_m^*,$$

and hence

$$\frac{1}{\tau} \sum_{i=0}^{\tau-2} w(\alpha(t)) \geq \frac{1}{\tau} w_m^* (\tau - 1 - n).$$

It now easily follows that $h(\mathcal{A}, G) \geq w_m^*$, which concludes the proof. \hfill \Box

5. Relationship between the upper bounds

In this section, we prove that in the deterministic case, where the obtained upper bound of the IE for deterministic systems and the one for uncertain ones both apply, these bounds are related by an inequality.

Consider a deterministic system $\Sigma$ as in (11), a compact controlled invariant set $Q \subseteq X$, and an invariant partition $(\mathcal{A}, G)$ with $\mathcal{A} = \{A_1, \ldots, A_q\}$. Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a refinement of $\mathcal{A}$ and construct the weighted directed graph $G$ as described in Section 3. The sets $W_t(\mathcal{B}, \mathcal{A})$ and $h(\mathcal{B}, \mathcal{A})$ are defined as in (7). For simplicity, we assume that $\mathcal{G}$ is strongly connected, in which case we know that

$$h(\mathcal{B}, \mathcal{A}) = \lim_{N \to \infty} \frac{|W_N(\mathcal{B}, \mathcal{A})|}{N} = \log_2 \rho(R),$$

where $R$ is the adjacency matrix associated with a right-resolving graph.
Proposition 1. Given the invariant partition \((\mathcal{A}, G)\), for any refinement \(\mathcal{B}\) of \(\mathcal{A}\), it holds that

\[
h(\mathcal{A}, G) \geq h(\mathcal{B}, \mathcal{A}) = \log_2 \rho(R).
\]

Proof. We use \(\mathcal{W}_\tau(\mathcal{A})\) to refer to the set \(\mathcal{W}_\tau(\mathcal{B}, \mathcal{A})\), which is defined in Section 5 for the case when \(\mathcal{B} = \mathcal{A}\) and \(\tau = N\), i.e., \(\mathcal{W}_\tau(\mathcal{A}) := \{ \alpha \in \mathcal{A}^N : T_C(\alpha) \cap \alpha_{i+1} \neq \emptyset \ \forall i \in \{0, N - 2\}\)\). Constructing the graph \(\mathcal{G}\) associated with \(\mathcal{A}\) as in Section 4 leads to

\[
W_t(\mathcal{G}) = W_t(\mathcal{A}) \quad \text{for all } t \in \mathbb{N}.
\]

From [34] Lem. 2 and [13], we obtain

\[
|W_t(\mathcal{A})| = |W_t(\mathcal{G})| \leq N(W_t(\mathcal{G})) = r_{inv}(\tau, Q, \mathcal{A}, G).
\]

Then (15) yields

\[
h(\mathcal{A}) := \lim_{\tau \to \infty} \frac{\log_2 |W_t(\mathcal{A})|}{\tau} \leq h(\mathcal{A}, G).
\]

It is clear that \(h(\mathcal{B}, \mathcal{A}) \leq h(\mathcal{A})\). Hence,

\[
\log_2 \rho(R) = h(\mathcal{B}, \mathcal{A}) \leq h(\mathcal{A}) \leq h(\mathcal{A}, G).
\]

This concludes the proof. \(\square\)

6. Examples

In this section, we illustrate the effectiveness of our proposed results on some case studies.

6.1. A linear discrete-time system

Consider the following linear control system obtained from a similarity transformation applied to the system in Example 1:

\[
x_{k+1} = A x_k + 0.9463 1.051 u_k, \quad A = \begin{bmatrix} 2 & 0.0784 \\ 0.0784 & 0.5041 \end{bmatrix},
\]

with \(x_k \in \mathbb{R}^2\) and \(u_k \in U = [-1, 1]\). Consider the set \(Q\) given by the inequality

\[
\begin{bmatrix} 0.0261 & -0.4993 \\ 0.9986 & 0.0523 \\ -0.0261 & 0.4993 \\ -0.9986 & -0.0523 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^2,
\]

which is compact and controlled invariant.

To compute an upper bound on the IE of \(Q\), we put \(Q_x := [-1.2, 1.2] \times [-2.1, 2.1], \eta_x := [0.04, 0.08]^T\) and \(\eta_x := 0.2\).

For the parameters \(\eta_x = [0.04, 0.08]^T\), \(\eta_x = 0.2\), Table 1 lists the values of \(h(\mathcal{B}, \mathcal{A})\) for different selections of the coarse partition \(\mathcal{A}\). For the same values of \(\eta_x\) and \(\eta_x\), the obtained value for the bound in Theorem 1 is \(w_{inv}^c(\mathcal{G}) = 2.5849\) (with computation time 0.048 sec). For dtControl parameters Classifier = ‘logreg’ and Determinizer = ‘maxfreq’, Table 2 shows the variation of the upper bound \(h(\mathcal{B}, \mathcal{A})/\tau\) with increasing control sequence length \(\tau\).

6.2. A scalar continuous-time nonlinear control system

Consider the following scalar continuous-time control system discussed in [16] Ex. 7.2:

\[
\mathbf{\Sigma} : \dot{x} = (-2b \sin x \cos x - \sin^2 x + \cos^2 x) + u \cos^2 x,
\]

where \(u \in [-\rho, \rho]\), \(b > 0\) and \(0 < \rho < b^2 + 1 =: a\). The equation describes the projectivized linearization of a controlled damped mathematical pendulum at the unstable position, where the control acts as a reset force. The following set is controlled invariant:

\[
Q = \left[ \arctan(-b - \sqrt{a - \rho}), \arctan(-b - \sqrt{a - \rho}) \right].
\]

In fact, \(Q\) is the closure of a maximal set of complete approximate controllability. With \(T_S \in \mathbb{R}_{>0}\) as the sampling time, we first obtain a discrete-time system as in [1]. Theory suggests that the following formula holds, see [16] Ex. 7.2: \(h_{inv}(Q) = \frac{2}{\ln 2} \sqrt{a - \rho}\).

Discretizing the given system with sampling time \(T_s\), results in a discrete-time system \(\Sigma^T\), that satisfies

\[
h_{inv}(Q; \Sigma^T) \geq T_s \cdot h_{inv}(Q) = \frac{2T_s}{\ln 2} \sqrt{a - \rho}.
\]

The inequality is due to the fact that continuous-time open-loop control functions are lost due to the sampling (since only the piecewise constant control functions, constant on each interval of the form \([kT_S, (k + 1)T_S), k \in \mathbb{Z}_+\), are preserved under sampling). Since \(Q\) can be made invariant by constant control inputs only, \(Q\) is also a controlled invariant set of \(\Sigma^T\). Table 3 and 4 list the values of \(h(\mathcal{B}, \mathcal{A})/\tau\) for different choices of the sampling time with the parameters \((\rho = 1, b = 1, \eta_x = 10^{-6})\).

\[
\text{Table 1: Entropy estimates for Example 6.1 with different choices of the determination options in dtControl. Here, } h_{inv}(Q) = 1.003.
\]

| Classifier | Determinizer | \(| \mathcal{A} |\) | \(h(\mathcal{B}, \mathcal{A})\) | time |
|------------|--------------|-------------|----------------|------|
| cart       | maxfreq     | 10          | 1.2133         | 10 sec|
| logreg     | maxfreq     | 9           | 1.1802         | 10 sec|
| linsvm     | maxfreq     | 10          | 1.2133         | 10 sec|
| cart       | minnorm     | 135         | 1.7848         | 9 sec |
| logreg     | minnorm     | 111         | 1.8015         | 11 sec|
| linsvm     | minnorm     | 143         | 1.8300         | 10 sec|

\[
\text{Table 2: Entropy estimates for Example 6.1 with control sequences of length } \tau. \text{ Here, } h_{inv}(Q) = 1.003.
\]

| \(\tau\) | \(h(\mathcal{B}, \mathcal{A})/\tau\) | time |
|----------|-------------------------------|------|
| 1        | 1.1802                        | 9.6 sec |
| 2        | 1.0688                        | 16.7 sec |
| 3        | 1.0588                        | 1 min 11 sec |

\[\hat{\text{The factor ln(2) appears due to the choice of the base-2 logarithm instead of the natural logarithm, which is typically used for continuous-time systems.}}\]
Table 3: Entropy estimates for Example 6.2 with \( \rho = 1, b = 1 \) and different choices of the sampling time \( T_\tau \). Here, \( h_m(\mathcal{Q}) = 2.8854 \).

| \( T_\tau \) | \(|\mathcal{A}|\) | \( h(\mathcal{B}, \mathcal{A})/T_\tau \) | time |
|---|---|---|---|
| 0.8 | 11 | 4.0207 | 21.23 hr |
| 0.5 | 6 | 4.0847 | 2.98 hr |
| 0.1 | 2 | 4.744 | 3.33 min |
| 0.01 | 2 | 5.1994 | 55 sec |
| 0.001 | 2 | 24.7 | 60 sec |

Table 4: Entropy estimates for Example 6.2 with \( \rho = 50, b = 10 \) and different choices of the sampling time \( T_\tau \). Here, \( h_m(\mathcal{Q}) = 20.6058 \).

| \( T_\tau \) | \(|\mathcal{A}|\) | \( h(\mathcal{B}, \mathcal{A})/T_\tau \) | time |
|---|---|---|---|
| 0.11 | 15 | 28.5012 | 1.9 hr |
| 0.1 | 11 | 29.1723 | 1.35 hr |
| 0.01 | 2 | 34.4707 | 13 sec |
| 0.001 | 2 | 55.5067 | 12 sec |
| 0.0001 | 2 | 1.5635e+03 | 31 sec |

\( \eta_t = 0.2\rho \) and \( (\rho = 50, b = 10, \eta_t = 10^{-6}, \eta_\tau = 0.2\rho) \), respectively. In both tables, the dtControl parameters are Classifier = ‘cart’ and Determinizer = ‘maxfreq’. Table 5 shows the values of \( h(\mathcal{B}, \mathcal{A})/T_\tau \) for different selections of the coarse partition \( \mathcal{A} \) with the parameters \( T_\tau = 0.01, \eta_t = 10^{-6}, \eta_\tau = 0.2\rho, \rho = 1, b = 1 \). For the same selection of parameters as in Table 3 with \( T_\tau = 0.01, \) Table 6 presents the variation of the upper bound \( h(\mathcal{B}, \mathcal{A})/(\tau T_\tau) \) with increasing length \( \tau \) of the control sequences.

### 6.3. A 2d uniformly hyperbolic set

Consider the map

\[
f(x, y) := (5 - 0.3y - x^2, x), \quad f : \mathbb{R}^2 \to \mathbb{R}^2,
\]

from the Hénon family, one of the most-studied classes of dynamical systems that exhibit chaotic behavior. We extend \( f \) to a control system with additive control:

\[
\Sigma : \begin{bmatrix} x_{r+1} \\ y_{r+1} \end{bmatrix} = \begin{bmatrix} 5 - 0.3y_t - x_t^2 + u_t \\ x_t + v_t \end{bmatrix},
\]

with \( \max[|u_t|, |v_t|] \leq \varepsilon \). It is known that \( f \) has a non-attracting uniformly hyperbolic set \( \Lambda \), which is a topological horseshoe. This set is contained in the square centered at the origin with side length \( 20 \) [29 Thm. 4.2]

\[
r := 1.3 + \sqrt{1.3^2 + 20} = 5.9573.
\]

Table 5: Entropy estimates for Example 6.2 with different choices of dtControl parameters. Here, \( h_m(\mathcal{Q}) = 2.8854 \).

| Classifier | Determinizer | \(|\mathcal{A}|\) | \( h(\mathcal{B}, \mathcal{A})/T_\tau \) | time |
|-----------|-------------|---|---|---|
| cart      | maxfreq    | 2 | 5.1994 | 55 sec |
| logreg    | maxfreq    | 2 | 5.1994 | 65 sec |
| linsvm    | maxfreq    | 2 | 5.1994 | 61 sec |
| cart      | minnorm    | 11 | 6.4475 | 57 sec |
| logreg    | minnorm    | 11 | 6.4475 | 74 sec |

Table 6: Upper bound \( h(\mathcal{B}, \mathcal{A})/(\tau T_\tau) \) for Example 6.2 with control sequences of length \( \tau \). Classifier ‘cart’, and Determinizer = ‘maxfreq’ in dtControl. Here, \( h_m(\mathcal{Q}) = 2.8854 \).

| \( \tau \) | \( h(\mathcal{B}, \mathcal{A})/(\tau T_\tau) \) | time |
|---|---|---|
| 1 | 5.1994 | 57 sec |
| 2 | 5.0036 | 7.5 sec |
| 3 | 4.9547 | 1.91 hr |
| 4 | 4.9266 | 27.27 hr |

Table 7: Entropy estimates for Example 6.3 with different selections of dtControl options. Here, \( h_m(\mathcal{Q}) = 0.696 \).

| Classifier | Determinizer | \(|\mathcal{A}|\) | \( h(\mathcal{B}, \mathcal{A}) \) | time |
|-----------|-------------|---|---|---|
| cart      | maxfreq    | 573 | 2.3884 | 0.95 min |
| linsvm    | maxfreq    | 567 | 2.3956 | 1.82 min |
| logreg    | maxfreq    | 454 | 2.3994 | 1.4 min |
| cart      | minnorm    | 1921 | 2.9342 | 1 min |
| logreg    | minnorm    | 1533 | 2.9215 | 2 min |
| linsvm    | minnorm    | 1923 | 2.9376 | 2.15 min |

If the size \( \varepsilon \) of the control range is chosen small enough, the set \( \Lambda \) is “blown up” to a compact controlled invariant set \( Q^c \) with nonempty interior which is not much larger than \( \Lambda \), see [17 Sec. 6]. Moreover, the theory suggests that as \( \varepsilon \downarrow 0, h_m(\mathcal{Q}^c) \) converges to the negative topological pressure of \( f|_{\Lambda} \) w.r.t. the negative unstable log-determinant on \( \Lambda \); see [4] for definitions. A numerical estimate for this quantity, obtained in [11] Table 2 via Ulam’s method, is 0.696.

We select \( \bar{Q} = [-r/2, r/2]^2 \). For \( \varepsilon = 0.08 \), using SCOTS with parameter values \( \eta_t = [0.009, 0.009]^\top \) and \( \eta_\tau = [0.01, 0.01]^\top \), through iteration, we obtain an all-time controlled invariant set \( Q \subset \bar{Q} \). In the iteration, we begin with the set \( \bar{Q} \) and, as the first step, we compute an invariant controller for the system \( \Sigma \).

Let \( Q_1 \) be the domain of the obtained controller. Consider the time-reversed system

\[
\Sigma^r : \begin{bmatrix} x_{r+1} \\ y_{r+1} \end{bmatrix} = \begin{bmatrix} y_t - v_t \\ \frac{1}{\varepsilon}(5 - x_t^2 + u_t - x_t) \end{bmatrix}.
\]

In the second step, we compute an invariant controller for \( \Sigma^r \) in the set \( Q_1 \), and denote the controller domain by \( Q_2 \). In the third step, we compute an invariant controller for \( \Sigma \), but in the set \( Q_2 \), and denote the controller domain by \( Q_3 \). The steps are repeated until \( Q = Q_{r+1} =: Q \). In this way, we hope to approximate \( Q^c \).

Figure 4 shows the set \( Q \). For the parameter values \( \varepsilon = 0.08 \), \( \eta_t = [0.009, 0.009]^\top \) and \( \eta_\tau = [0.01, 0.01]^\top \), Table 6 lists the values of \( h(\mathcal{B}, \mathcal{A}) \) for different choices of the coarse partition \( \mathcal{A} \). For the same values of \( \varepsilon, \eta_t, \) and \( \eta_\tau \), the obtained value for the bound in Theorem 1 is \( w^\ast(\mathcal{Q}) = 3.5646 \) (with computation time 2.51 sec).

6.4. An uncertain linear system

We consider an uncertain linear control system

\[
\Sigma : \begin{bmatrix} x_{r+1} \\ y_{r+1} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -0.4 & 0.5 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t + W
\]
with \( x_i \in \mathbb{R}^2 \), \( u_i \in U := [-1, 1] \), and the disturbance set \( W := [-0.1, 0.1]^2 \). For a set \( Q \subseteq [-1, 1] \times [-2, 2] \), we compute an upper and a lower bound of \( \bar{h}_{inv}(Q) \). We used SCOTS to obtain an invariant controller for the state set \([-1, 1] \times [-2, 2] \) with \([0.2, 0.2]^\top\) and 0.05 as the state and input set grid parameters, respectively. The set \( Q \) is taken to be the domain of the obtained controller that consists of 109 state grid cells each of size 0.2 \times 0.2. Figure 5 shows the set \( Q \).

**Computation of the lower bound:** We utilize [34, Thm. 7] to compute a lower bound. From [34, Rem. 2], we know that the lower bound in [34, Thm. 7] is invariant under coordinate transformations. After a similarity transformation which diagonalizes the dynamical matrix, we have

\[
\begin{bmatrix}
1.6531 & 0 \\
0 & 0.8469
\end{bmatrix}
\begin{bmatrix} z_i \\ V^{-1} \end{bmatrix} + V^{-1} W,
\]

where \( V = \begin{bmatrix} 0.9448 & -0.6552 \\ -0.3277 & 0.7555 \end{bmatrix} \). For \( i = 1, 2 \), let \( \pi_i \) denote the canonical projection to the \( i \)th coordinate. Then, \( \pi_1 V^{-1} Q = [-2.1207, 2.1207] \), \( \pi_2 V^{-1} Q = [-3.4, 3.4] \), \( \pi_1 V^{-1} W = [-0.2827, 0.2827] \) and \( \pi_2 V^{-1} W = [-0.2550, 0.2550] \). By [34, Thm. 7], one obtains

\[
0.9316 \leq \bar{h}_{inv}(Q).
\]

**Computation of the upper bound:** We construct an invariant partition \((\mathcal{A}, G)\) of \( Q \) by selecting the set of grid cells in the domain of the controller obtained from SCOTS as the cover \( \mathcal{A} \).

Let \( C : \mathcal{A} \rightarrow U \) denote the controller from SCOTS. For \( A \in \mathcal{A} \), \( C(A) \) is the list of control inputs in the controller assigned to cell \( A \) such that each of the control inputs in the list ensures invariance of the states in \( A \) w.r.t. the set \( Q \). For each \( A \in \mathcal{A} \), we define \( G(A) := u \in C(A) \), where \( u \) is chosen such that \( F(A, u) \) has nonempty intersection with a minimum number of elements of \( \mathcal{A} \). If there are multiple such control values, then one of them is selected randomly. Using \((\mathcal{A}, G)\) and the transition function \( F \) of the system, we construct a weighted directed graph \( G \) as described in Section 4. We used the LEMON library to compute the maximum cycle mean weight for the graph \( G \) and obtained \( w_{\text{m}}^*(G) = 3.3219 \) with computation time 0.027 sec. Thus, \( \bar{h}_{inv}(Q) \leq 3.3219 \).

**Discussion on the selection of partition:** A better upper bound is expected when the number of outgoing edges, for every node in the graph, is smaller. As a heuristic, gradually smaller values of the state grid parameter \( \eta_i \) can be tried. But very small \( \eta_i \) that make width of the grid cell smaller than that of the disturbance set should be avoided, because in that case, the number of outgoing edges for any cell will begin to rise. This can also be observed from Table 8.

### 7. Software tools and pseudo-code

In this section, we provide brief descriptions of the used software tools and summarize our algorithms in terms of pseudo-code (cf. Figures 6 and 7).

**Description of the computation of the maximum cycle mean (MCM) using the LEMON library:** The maximum cycle mean of a directed weighted graph can be computed by Karp’s algorithm which runs in \( O(nm) \) time, where \( n \) and \( m \) are the number of nodes and edges in the graph, respectively. For MCM, we utilize the LEMON library which is a C++ library that provides efficient implementations of algorithms related to graphs. LEMON provides the implementation of Karp’s algorithm in the class LEMON::MCM. The class constructor requires two arguments: Digraph and CostMap. Digraph specifies the type of

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Table 8: Entropy estimates for Example 6.4 with \( \eta_i = 0.05 \).

| \( \eta_i \) | \( w_{\text{m}}^* \) | time (sec) |
|------------|---------------|------------|
| 0.03       | 6.4594        | 1.112      |
| 0.06       | 5             | 0.129      |
| 0.09       | 4.2811        | 0.051      |
| 0.1        | 4.3923        | 0.033      |
| 0.2        | 3.3219        | 0.027      |

https://lemon.cs.elte.hu/trac/lemon
A corresponds to some leaf node in the decision tree obtained from \( \text{dtControl} \). \( G_{\text{sc}} \) is the set of strongly connected components of the graph \( G \), and it can be computed in \( O(n + E) \) time, where \( n \) and \( E \) are the number of nodes and edges, respectively, in the graph. A right-resolving graph can be computed by the power-set construction in \( O(2^n |\mathcal{A}|) \). The adjacency matrix and the right-resolving graph are computed simultaneously. Thus, the procedure in Fig. 6 runs in \( O(n2^n) \).

![Figure 7](https://github.com/mahendrasinghtomar/Invariance_Entropy_upper_bounds)

**Figure 7** Procedure for computing an upper bound of IE in the uncertain case

### Input: \( \eta_s, \eta_t, Q_x, Q, U, F \)

### Output: \( w_{\eta_s}^\text{max}(G) \)

1. \( C \leftarrow \text{SCOTS}(\eta_s, \eta_t, Q_x, Q, U, F, 1) \)
2. \( \hat{C} \leftarrow \text{determinize}(C) \)
3. \( G \leftarrow \text{graph}(F, \hat{C}) \)
4. \( w_{\eta_s}^\text{max}(G) \leftarrow \text{maxCycleMean}(G) \)

In Fig. 7, the controller \( C : \mathcal{B} \equiv U \) can be determined through the selection of such control inputs that result in minimum number of successor state-cells, in \( O(\bar{n}m) \) time, where \( \bar{n} = |\mathcal{B}| \). The maximum cycle mean by Karp’s algorithm can be computed in \( O(\bar{m}m) \), where \( \bar{m} \) is the number of edges in the graph, which in the worst case will be \( \bar{n}^2 \). Thus, the procedure in Fig. 7 runs in \( O(n^m m + n^3) \).

**Remark 2.** To reduce computational complexity, for the uncertain case, one can leverage the proposed compositionality results in [34] for the computation of an overapproximation of the invariance entropy for a large scale interconnected system in a divide and conquer manner by computing overapproximations for subsystems using the method proposed here. Thus complexity breaks down to the level of subsystems.

The code is publicly accessible at https://github.com/mahendrasinghtomar/Invariance_Entropy_upper_bounds

### Quality of upper bounds of IE:

For uncertain nonlinear systems, because of the absence of any theory providing a lower bound (better than zero) in the literature, we do not know how far our computed upper bounds are from the actual values. For uncertain linear systems with additive disturbance, one can comment on this gap based on the availability of a lower bound [34, Thm. 7]. In the deterministic nonlinear case, the gap is not yet quantified as well.

### 8. Conclusion and future work

Our first contribution is the combination of three different algorithms designed for different purposes to numerically compute an upper bound of the invariance entropy of deterministic control systems. The second contribution is a procedure to numerically compute an upper bound for the invariance entropy of uncertain control systems. We also describe the relationship between the two upper bounds and thus the need for the second bound. Finally, we illustrate the effectiveness of the proposed procedures on four examples. Open questions for future work

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[https://scikit-learn.org/ stable/modules/tree.html#complexity](https://scikit-learn.org/ stable/modules/tree.html#complexity)
include the selection of entropy-minimizing partitions and the computation of lower bounds of IE for uncertain nonlinear systems.

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