Chiral potentials, perturbation theory, and the $^1S_0$ channel of $NN$ scattering

Deepshikha Shukla$^1$, Daniel R. Phillips$^2$ and Eric Mortenson$^2$

$^1$Department of Physics, George Washington University, Washington DC 20052.
$^2$Department of Physics and Astronomy, Ohio University, Athens, OH 45701.
(Dated: December 4, 2018)

Abstract

We use nucleon-nucleon phase shifts obtained from experimental data, together with the chiral expansion for the long-distance part of the $NN$ interaction, to obtain information about the short-distance piece of the $NN$ potential that is at work in the $^1S_0$ channel. We find that if the scale $R$ that defines the separation between “long-” and “short-” distance is chosen to be $\lesssim 1.8$ fm then the energy dependence produced by short-distance dynamics is well approximated by a two-term polynomial for $T_{lab} \leq 200$ MeV. We also find that a quantitative description of $NN$ dynamics is possible, at least in this channel, if one treats the long-distance parts of the chiral $NN$ potential in perturbation theory. However, in order to achieve this we have to choose a separation scale $R$ that is larger than 1.0 fm.

PACS numbers: 13.75.Cs, 13.85.-t, 11.30.Rd
I. INTRODUCTION

The chiral symmetry of the strong interaction places significant constraints on hadron-hadron interactions at low energies. Chiral perturbation theory (χPT) implements these constraints in a systematic fashion. (For recent reviews see Refs. [1, 2].) In contrast to the situation in the single-nucleon and mesonic sectors, low-energy nucleon-nucleon interactions do not vanish in the chiral limit, with experimental manifestations of the strength of the NN interaction at low energies being provided by the large np scattering length in the $^1S_0$ channel and the presence of a bound state (the deuteron) in the $^3S_1$ channel.

This complicates the application of perturbation-theory methods to multi-nucleon interactions. Weinberg’s pioneering efforts [3] in the early nineties proposed surmounting this difficulty by making a χPT expansion for the NN potential, $V$, that goes into the Schrödinger equation, i.e. solving:

$$ (E - H_0)|\psi\rangle = V|\psi\rangle, $$

with

$$ V = V^{(0)} + V^{(2)} + V^{(3)} + \ldots. $$

Here the superscripts indicate the power of the (presumably) small quantities $\frac{m_\pi}{\Lambda_{\chi_{SB}}}$, $\frac{p}{\Lambda_{\chi_{SB}}}$ ($p$ is the momentum of the $NN$ collision and $\Lambda_{\chi_{SB}}$ the scale of chiral-symmetry breaking) that is present in that piece of $V$. χPT calculation then reveals that $V^{(0)} = C_0 \delta^{(3)}(r) + V_\pi$ includes one-pion exchange and a short-range interaction, $V^{(2)}$ includes higher-derivative short-range interactions, together with two-pion exchange diagrams constructed from the leading-order χPT Lagrangian, and $V^{(3)}$ involves the so-called “next-to-leading order” two-pion exchange, where the $\pi\pi NN$ vertices are those from $L^{(2)}_{\pi N}$ of χPT. In Weinberg’s original paper there was no distinction made between the power counting for the piece of the potential that is operative at long distances ($r \sim 1/m_\pi$) and that for the delta functions and their derivatives which represent the shorter-range mechanisms ($r < 1/m_\pi$) in this approach. Both are assumed to give a contribution to the $NN$ potential of an order given by naive dimensional analysis in powers of $p$ and $m_\pi$.

The implications of the expansion (2) for $NN$ scattering data were first examined in detail by Ordoñez et al. in their landmark 1996 paper [4]. These authors analyzed $NN$ scattering, not only in the $^1S_0$, but also in a number of other partial waves. Improvements on this analysis were made in the work of Epelbaum and co-authors [5], while peripheral $NN$ partial waves were analyzed in first- and second-Born approximation in Ref. [6]. Two sets of authors have subsequently extended these analyses to fourth order in $V$ [7, 8, 9]. In all these studies the expansion (2) yielded reasonably convergent results for $NN$ phase shifts.

Kaplan, Savage, and Wise (KSW) [10, 11] have demonstrated that the assumption that naive dimensional analysis sets the size of the short-distance pieces of the potential is chirally inconsistent, in the sense that iterations of the leading-order $V$ (2) via the Schrödinger equation (1) generate divergences proportional to $m_\pi^2$ [12]. But, at leading order in Weinberg’s expansion, there is no counterterm to absorb this divergence. KSW proposed an alternative expansion, wherein the delta-function interaction $C_0 \delta^{(3)}(r)$ was promoted to leading-order, but one-pion exchange retained its naive-dimensional-analysis scaling, making it sub-leading in this expansion. This expansion was carefully examined in the $^1S_0$ channel [10, 11, 13], and appears to converge reasonably well there. However, the KSW expansion does not converge in the $^3S_1$ channel for momenta $\gtrsim 100$ MeV [13]. A compromise proposal is to expand
one-pion exchange about its chiral limit value, in which case it is leading in the \(^3S_1\) and sub-leading in \(^1S_0\). The resulting series has been shown to converge, albeit slowly.

However, two-pion-exchange corrections are yet to be considered in light of the analysis of the \(^1S_0\) of Refs. \([12, 14]\). The discussions in these works focused on establishing the correct power counting for the \(NN\) potential \(V\) in the case that the long-range potential is one-pion exchange, i.e. \(V = V^{(0)}\). (For other research that bears on the role of one-pion exchange in this channel see Refs. \([15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]\).) Recent studies, including Refs. \([28, 29, 30]\), have examined the impact of higher-order pieces of \(V\) on the predictions that the \(\chi PT\) approach to nuclear forces makes for phase shifts in the \(^1S_0\) channel.

Like Ref. \([29]\), this paper examines the role of two-pion-exchange corrections to \(V\) in the \(^1S_0\) channel. We seek to answer two questions. First, is there empirical evidence for these two-pion exchange contributions, and, concomitantly, what impact do they have on our understanding of the short-range dynamics in the \(NN\) system? Second, is there any sense in which \(V^{(2)}\) and \(V^{(3)}\) are “small”? I.e. one (or more) of the expansions proposed for \(NN\) dynamics in Refs. \([3, 4, 10, 14]\) is converging. The weakness of one-pion exchange in the \(^1S_0\) channel—thanks to the absence of the tensor pieces of the \(NN\) potential that generate, e.g. deuteron binding—means that these questions are of crucial importance in the development of a quantitative theory of \(NN\) scattering in this channel.

In contrast to Refs. \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 30, 31, 32]\), we do not compare predictions from a treatment of the \(NN\) potential (or amplitude) within a given chiral power counting with data. Instead we examine the extent to which phase-shift data obtained by the Nijmegen group \([33, 34]\) can be used to make inferences regarding the \(NN\) potentials that are at work for \(r < 1/m_\pi\). To do this we invoke \(\chi PT\) with only nucleons and pions as explicit degrees of freedom to obtain \(V\) at distances \(r \sim 1/m_\pi\). (The \(\chi PT\) expansion for \(V\) converges quite well in this domain, see, e.g., Refs. \([6, 35]\).) We then start with the Nijmegen PWA93 phase shift \([33]\) at a fixed energy and use the long-range potential at a given order in \(\chi PT\) to integrate to a finite distance \(R\). The formalism for this treatment is presented in Sec. II. Such an “outside in” approach, that begins with phase shifts, and uses a \(\chi PT\) potential to deduce information about short-distance dynamics in the \(^1S_0\) channel, was already pursued in Refs. \([15, 22, 23, 24]\). However, in these works only the leading-order \(\chi PT\) potential \(V^{(0)}\) was employed for this purpose. And while the NNLO chiral potential \(V^{(0)} + V^{(2)} + V^{(3)}\) was used to do the integration in Ref. \([29]\), only values \(R \ll 1/m_\pi\) were discussed there.

When such an analysis is performed over the range \(0 \leq T_{lab} \leq 200\) MeV it yields information on the energy dependence that the short-range interaction must have if it is to be used in concert with the long-range \(V\) of \(\chi PT\) (at a fixed order) to reproduce data. (This method is very similar to those of the phase-shift analysis itself, although in Ref. \([33]\) a different long-range potential was employed.) Since our effective field theory (EFT) does not contain explicit Delta degrees of freedom we limit ourselves to data at fairly low energies: \(T_{lab} \leq 200\) MeV. The results for the inferred energy dependence due to short-range \(NN\) dynamics are presented in Sec. III. While any energy dependence due to short-range dynamics is in principle possible, short-range potentials with coefficients that are natural with respect to the high scale \(1/R\) will lead to smooth, not rapid, energy dependence on the interval \(0 \leq T_{lab} \leq 200\) MeV. As one might expect, if \(R\) is too large, the energy dependence over this entire energy interval is not smooth, since the \(NN\) collision probes details of the regulator if \(p_{cm} = \sqrt{M T_{lab}}/2\) is greater than or of order \(\frac{\pi}{R}\). However, we find that for \(R \lesssim 1.8\)
fm only smooth energy dependence of the short-range potential is needed, and that this conclusion holds essentially irrespective of the $\chi$PT long-range potential that governs $NN$ dynamics in the region from $r = R$ to $r = \infty$.

The second question is whether dynamics in that region can be understood in perturbation theory. In Section [IV] we show that the long-distance pion-exchange interactions derived from $\chi$PT can indeed be treated in perturbation theory in the region $r > 1.0$ fm. This result is obtained via a perturbative treatment of the chiral potential used in the Schrödinger equation in Sec. [II]. Our perturbation theory preserves the asymptotic wave function, and hence the phase shifts. In this way we can examine the extent to which the energy dependence displayed in Sec. [III] can be understood using perturbation theory. (A similar long-distance perturbation theory was developed in Ref. [29], but there it was used to integrate the Schrödinger equation to $R \approx 0$, whereas here our focus will be on finite $R$.)

In Sec. [V] we present one concrete realization of short-distance physics. We adopt a two-parameter form for the potential in the region $r \leq R$, and obtain values for the relevant parameters that generate the various different energy-dependencies displayed in Sec. [III]. (We stress, however, that the results of Sec. [III] for the boundary condition at $r = R$ do not depend on a particular model of the physics that is operative for $r \leq R$.) Finally, in Sec. [VI] we offer some conclusions and an outlook.

II. SOLVING THE SCHRODINGER EQUATION

We choose to work in co-ordinate space and hence the basic task is to solve the radial Schrödinger equation:

$$\frac{1}{r^2} d \left( \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) \right) + \frac{M}{\hbar^2} (E - V(r)) R(r) = 0.$$  

(3)

Here, $E$ is the c.m. energy of the $NN$ system, and $\frac{M}{2}$ its reduced mass. We adopt $M = 938.918$ MeV. By the substitution $u(r) = R(r)/r$ the equation reduces to

$$\frac{d^2 u_k(r)}{dr^2} + \left( k^2 - U(r) \right) u_k(r) = 0,$$  

(4)

where $k^2 = \frac{ME}{\hbar^2}$, and $U(r) = \frac{MV(r)}{\hbar^2}$. Our objective in this section is to solve this equation in the region $R \leq r$. Because of this, an important feature of our solution is that we do not use the usual boundary condition $u(0) = 0$. Instead, we invert the problem and use the Nijmegen PWA93 [34] extraction of the $^1S_0$ phase shifts as input that provides the boundary condition at $r \to \infty$:

$$u_k(r)|_{r\to\infty} = \frac{\sin(kr + \delta(k))}{\sin \delta(k)},$$  

(5)

$\delta(k)$ being the experimental phase-shift corresponding to c.m.-frame relative momentum $k$. The normalization is done in such a manner that the asymptotic zero-energy solution is given by $1 - r/a$, $a$ being the scattering length. The purpose of solving the Schrödinger equation in this way is to obtain an energy-dependent matching condition at any $R$ (which is naively of the order of or smaller than the range of pion-exchange interaction). This matching condition is defined by the logarithmic derivative:

$$\gamma(k; R) = \left[ \frac{u'_k(r)}{u_k(r)} \right]_{r=R}.$$  

(6)
For example, in the absence of any long-range potential (i.e. the case \( V = 0 \)) we have:

\[ \gamma(k; R) = k \cot(kR + \delta(k)). \]  

(7)

If we incorporate any long-distance potential \( V \) in the analysis for \( r > R \) then the form (7) will no longer apply, and in general, we will have only numerical results for \( \gamma(k; R) \).

Regardless, the energy dependence of \( \gamma(k; R) \) can be fitted with a polynomial in \( k^2 \) as—

\[ \gamma(k; R) = \sum_i A_i(R)(k^2)^i. \]  

(8)

The fact that we represent \( \gamma(k; R) \) in the form (8) is guided by the form of the Lagrangian for an EFT of the \( NN \) interaction. Because of parity and time-reversal symmetries, only even powers of \( k \) can occur in \( NN \) contact interactions. Such local interactions (in the sense of quantum field theory), when smeared over a length scale \( R \), will result in energy dependence that is smooth with respect to the scale \( R \), and so we expect that \( A_i(R) \sim R^{2i-1} \). A lucid description of this philosophy of regulating the short-range part of a two-body potential can be found in [36].

The coefficients in Eq. (8) are then manifestations of the short-range \( NN \) interaction in the \( ^1S_0 \) channel. If one wishes these parameters can in turn be used to construct a model that is a particular realization of that short-distance physics. In other words, once a \( \chi \)PT potential of a given order had been used to obtain information on \( \gamma(k; R) \), we would use that information to extract the coefficients of \( NN \) contact interactions \( C_0, C_2, C_4, \) etc. This is very much in the spirit of the Nijmegen PWA93, where Stoks et al. [33] employed square wells for \( r \leq 1.8 \) fm to produce the necessary energy dependence due to short-distance physics.

But, regardless of what potential generates the coefficients \( A_i \) in Eq. (8), the long-range potential \( (V(r) \text{ for } r > R) \) should produce the faster energy dependence in the phase shifts. Removing this more rapid energy dependence by integrating the wave function from \( r = \infty \) in to \( r = R \) and examining \( \gamma(k; R) \) allows us to look at the behavior of observables with energy that is generated by short-distance dynamics. So, although a polynomial form like (8) is preferred from the EFT point of view, we are, at first, agnostic about the form of \( \gamma(k; R) \), and merely report the results obtained from the integrating-in exercise when \( \chi \)PT potentials of different orders are adopted in the region \( r \geq R \). But, before we do that, we first give the details of the \( \chi \)PT potentials that govern dynamics in that region.

A. The Potential

In our analysis we adopt a \( \chi \)PT expansion for \( V(r) \). Since we will solve the Schrödinger equation in co-ordinate space, we need a co-ordinate-space representation of the corresponding pion-exchange potentials. We have adapted the expressions from Ref. [6]. The potential corresponding to one-pion exchange is the leading-order part of \( V \), \( V^{(0)} \), and can be expressed as

\[ V^{(0)}(r) = (\vec{r}_1 \cdot \vec{r}_2)[\tilde{W}^{(1\pi)}_s(r)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \tilde{W}^{(1\pi)}_t(r)S_{12}(\hat{r})], \]  

(9)

where \( S_{12}(\hat{r}) = 3\vec{\sigma}_1 \cdot \hat{r}\vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \), and

\[ \tilde{W}^{(1\pi)}_s(r) = \frac{g_{\pi NN}^2 m_{\pi}^2}{48\pi M^2} e^{-x} \frac{1}{r}, \]  

(10)
\[ \tilde{W}_T^{(1\pi)}(r) = \frac{g_{\pi N}^2}{48\pi M^2} \frac{e^{-x}}{r^3} (3 + 3x + x^2), \]  
with \( x = m_\pi r \) and \( g_{\pi N} \) the \( \pi NN \) coupling constant. Numerically we choose \( m_\pi = 134.98 \text{ MeV} \) and \( g_{\pi N} = 13.1 \).  

Similarly, the two-pion exchange potentials which yield \( V^{(2)} \) and \( V^{(3)} \) are

\[ V^{(2)}(r) = \text{Re}(\tilde{V}_S(r)(\mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2) + (\mathbf{\tau}_1 \cdot \mathbf{\tau}_2)\tilde{W}_C(r) + \tilde{V}_T(r)S_{12}(\hat{r})), \]

for the “leading-order” pieces of two-pion exchange, which are constructed solely out of vertices in \( \mathcal{L}_{\pi N}^{(1)} \), and

\[ V^{(3)}(r) = \text{Re}(\tilde{V}_C(r) + (\mathbf{\tau}_1 \cdot \mathbf{\tau}_2)\tilde{W}^{(2\pi)}_S(r)(\mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2) + (\mathbf{\tau}_1 \cdot \mathbf{\tau}_2)\tilde{W}^{(2\pi)}_T(r)S_{12}(\hat{r})) \]

for “sub-leading” two-pion exchange, which involves contributions from \( \mathcal{L}_{\pi N}^{(2)} \).

The different coefficient functions \( \tilde{V}_X(r) \) and \( \tilde{W}_X(r) \)—with \( X = C, S, T \) referring to the central, spin-spin and tensor components of the potential—are \[6\]:

\[ \tilde{V}_T(r) = \frac{g_A^4 m_\pi}{128\pi^3 f_\pi^4 r^4} \left\{ -12x K_0(2x) - (15 + 4x^2) K_1(2x) \right\}, \]

\[ \tilde{W}_C(r) = \frac{m_\pi}{128\pi^3 f_\pi^4 r^4} \left\{ \left[ 1 + 2g_A^2(5 + 2x^2) - g_A^4(23 + 12x^2) \right] K_1(2x) + x \left[ 1 + 10g_A^2 - g_A^4(23 + 4x^2) \right] K_0(2x) \right\}, \]

\[ \tilde{V}_S(r) = \frac{g_A^4 m_\pi}{32\pi^3 f_\pi^4 r^4} \left\{ 3x K_0(2x) + (3 + 2x^2) K_1(2x) \right\}, \]

\[ \tilde{W}^{(2\pi)}_S(r) = \frac{g_A^4}{48\pi^2 f_\pi^4} \frac{e^{-2x}}{r^6} \left\{ c_4(1 + x)(3 + 3x + 2x^2) \right\}, \]

\[ \tilde{V}_C(r) = \frac{3g_A^2}{32\pi^2 f_\pi^4} \frac{e^{-2x}}{r^6} \left\{ 2c_1 x^2 (1 + x)^2 + c_3(6 + 12x + 10x^2 + 4x^3 + x^4) \right\}, \]

\[ \tilde{W}^{(2\pi)}_T(r) = \frac{g_A^2}{48\pi^2 f_\pi^4} \frac{e^{-2x}}{r^6} \left\{ -c_4(1 + x)(3 + 3x + x^2) \right\}. \]

Here, and in the calculations we report on subsequently, we have omitted the \( 1/M \) pieces of the potentials derived in Ref. \[6\]. We find that these contributions have a negligible effect on the results for \( R > 1.4 \text{ fm} \). This supports the power counting adopted in Refs. \[5, 9\], where \( M \sim \Lambda^2 \), leading to a sub-dominant role for \( 1/M \) corrections in the \( NN \) potential. As \( R \) is lowered below 1.0 fm, including \( 1/M \) corrections to \( V \) leads to marked differences with our results, which raises questions about the \( M \sim \Lambda^2 \) power counting in that domain. However, for \( R = 1.0 \text{ fm} \) the largest change induced by the addition of \( 1/M \) corrections to \( V \) is only 15%. Thus, even there, they can be regarded as higher order than \( V^{(3)} \).

In what follows we have used \( g_A = 1.29 \) to be consistent with our choice of \( g_{\pi N} \) in the LO potential, and \( f_\pi = 92.4 \text{ MeV} \). For \( c_1, c_3 \) and \( c_4 \) we have used two sets of values corresponding to the low-energy extraction of Rentmeester et. al. \[31\] and Entem et. al. \[8\]. The two sets are tabulated in Table \[\text{I} \].

We now need to project out the potentials that act in the \( ^1S_0 \) channel. This removes the tensor components, leaving us with

\[ V^{(0)}(r) = -3\tilde{W}^{(1\pi)}_S(r), \]
TABLE I: Different values of the low-energy constants \( c_i, i = 1, 3, 4 \) (in GeV\(^{-1}\)).

|                | \( c_1 \) | \( c_3 \) | \( c_4 \) |
|----------------|----------|----------|----------|
| Rentmeester et. al. | -0.76   | -5.08   | 4.70    |
| Entem et. al.     | -0.81   | -3.4    | 3.40    |

which is our leading-order (LO) chiral potential. Similarly,

\[
V^{(2)}(r) = \text{Re}(\tilde{W}_C(r) - 3\tilde{V}_S(r)),
\]

and \( V^{(0)}(r) + V^{(2)}(r) \) is our next-to-leading order (NLO) chiral potential. And finally,

\[
V^{(3)}(r) = \text{Re}(\tilde{V}_C(r) - 3\tilde{W}^{(2\pi)}_S(r)),
\]

yielding \( V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r) \) as our next-to-next-to-leading order (NNLO) chiral potential.

III. THE ENERGY-DEPENDENT MATCHING CONDITION

To examine how the pieces of the long-range \( \chi \)PT potential manifest themselves, we solve the Schrödinger equation (4) using the LO, NLO, and NNLO potentials (20)–(22). The output of this procedure is the energy-dependent logarithmic derivative \( \gamma(k; R) \), at different distances \( R \).

The results for \( \gamma(k; R) \) are displayed as a function of lab. energy \( T_{\text{lab}} = \frac{2p^2k^2}{M} \) for four different values of \( R \) in Fig. 1. As mentioned before, we present results for \( 0 \leq T_{\text{lab}} \leq 200 \text{ MeV} \). The four different panels correspond to choosing the matching point \( R \) to be 1.0 fm, 1.4 fm, 1.8 fm, and 2.2 fm respectively. Each panel itself contains four different curves, corresponding to different choices for the potential \( V \) in the region \( r \geq R \).

Let us first analyze the energy dependence of \( \gamma(k; R) \). Fig. 1 shows that \( \gamma(k; R) \) has a smooth behavior as \( T_{\text{lab}} \) increases from 0 to 100 MeV. For all \( R \)'s examined it could be represented as a polynomial in \( k^2 \) in this range. However, at slightly higher energies, \( T_{\text{lab}} = 140 \text{ MeV} \), there is a singularity in the energy dependence for \( R = 2.2 \text{ fm} \). This is, in fact, an artifact of the singularity of the cotangent derived in Eq. (7). The singularity occurs because \( kR \approx \pi \) at these energies, which means that the lab. energy is high enough that we are probing details of the dynamics at \( r \leq R \). It is therefore not a surprise that \( \gamma(k; R) \) cannot be represented by a polynomial for \( T_{\text{lab}} > 140 \text{ MeV} \). A similar comment applies to the results for \( R = 1.8 \text{ fm} \), although there the singularity is at \( T_{\text{lab}} \) a little above 200 MeV.

A smooth, i.e. flatter, energy dependence is highly desirable because our intention is to use the energy dependence of \( \gamma(k; R) \) to extract information about the short-distance physics per Eq. (8). If we choose \( R = 2.2 \text{ fm} \) or even \( R = 1.8 \text{ fm} \), we become very limited in the energy-range that can be used for extraction of the short-distance physics.

Furthermore, the lower panels in Fig. 1 make it clear that the long-range potentials are not having a significant impact on the energy dependence of \( \gamma(k; R) \) for \( R \geq 1.8 \text{ fm} \). In other words, no matter what order chiral potential we use, or even if we use no chiral potential at all, the result for \( \gamma(k; R) \) at \( R \geq 1.8 \text{ fm} \) looks essentially the same, although there is some effect due to \( V^{(0)} \), i.e. one-pion exchange, in \( \gamma(k; 1.8 \text{ fm}) \) at higher energies.
FIG. 1: $\gamma(k; R) \text{ (MeV)}$ plotted against $T_{lab} \text{ (MeV)}$ for different values of $R$. The dotted (black) curve corresponds to $V(r) = 0$, dashed (red) represents $V(r) = V^{(0)}(r)$, solid (green) represents $V(r) = V^{(0)}(r) + V^{(2)}(r)$ and the dot-dashed (blue) represents $V(r) = V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r)$.

Both of these phenomena can be seen in one plot if we form the dimensionless quantity $\gamma R$, and consider the result as a function of the dimensionless quantity $kR$, as well as dimensionless numbers formed out of scales present in the $NN$ potential:

$$\gamma R = \gamma R(kR, m_{\pi} R, \Lambda_{NN} R, \ldots)$$  \hspace{1cm} (23)

where $\Lambda_{NN} = \frac{16\pi f_{\pi}^4}{9^2 M}$ is a scale that sets the strength of the $NN$ potential at leading order, and the dots indicate the other scales that will appear if $V^{(2)}$ or $V^{(3)}$ is employed in the extraction of $\gamma R$ from data. In Fig. 2 we have plotted $\gamma R$ against the rescaled wave number $kR$. The dotted curves in the figure are generated with $R = 3.0 \text{ fm}$, and we see that they are all close to the black-dotted curve, which encodes (7) at this radius, and so is what is obtained if $V = 0$. In other words, at $R = 3.0 \text{ fm}$ essentially all of the $^1S_0$ phase shift is generated by short-range ($r < 3.0 \text{ fm}$) dynamics. This is hardly a surprise given that OPE is the longest-range part of the $NN$ force, and $m_{\pi} R = 2.1$ at this $R$. But Fig. 2 also shows that as we decrease the scale $R$—the scale that defines the demarcation between “long-” and “short-” distance—there is increased separation between the curve of Eq. (7) and the
FIG. 2: Here we have plotted $\gamma R$ vs $kR$, both dimensionless quantities, for different $R$s. The dotted curves are for $R = 3.0$ fm, the dashed curves for $R = 1.8$ fm and the solid curves for $R = 1.0$ fm. In each case the black curve is the result with $V = 0$, the red curve corresponds to $V = V^{(0)}$, the green curve to $V = V^{(0)} + V^{(2)}$, and the blue curve is what is obtained with $V = V^{(0)} + V^{(2)} + V^{(3)}$.

curves obtained when $V^{(0)}$, $V^{(0)} + V^{(2)}$, and $V^{(0)} + V^{(2)} + V^{(3)}$ are used for the analysis. At a separation scale of $R = 1.8$ fm pionic effects generate a larger portion of the overall phase-shift—although the analysis also shows that this effect comes mostly from one-pion exchange. But at $R = 1.0$ fm we see large effects from pionic dynamics. At this scale we also see significant differences in the results for $\gamma R$, depending on what long-range potential is used for the “renormalization-group evolution” from $R = \infty$ to $R = 1.0$ fm. (The formalism for such a renormalization-group analysis of the phase shift with respect to the scale $R$ was laid out in Refs. [22, 23, 24].)

Indeed, at the separation scale $R = 1.0$ fm, the impact of pion dynamics in the $NN$ potential on $\gamma R$ is largest for the NNLO $V$—which includes so-called “sub-leading” two-pion exchange—and the result obtained for $\gamma(k; R)$ does not appear to be perturbatively close to the (black) no-long-range-potential curve. The departure from this no-long-range-potential result is still visible if one adopts either $V^{(0)}$ or $V^{(0)} + V^{(2)}$ as the long-range potential, but it is nowhere near as large. This casts doubt on whether the hierarchy of effects at this separation scale is still that predicted by $\chi$PT. The issue here, of course, is that both $V^{(2)}$ and $V^{(3)}$ have stronger singularities as $r \to 0$ than does $V^{(0)}$. Thus, as $r$ gets smaller they eventually dominate the evolution of $\gamma(k; R)$ with $R$ and $k$. The $\chi$PT power counting is only a reliable guide to the relative size of contributions provided $R \sim 1/m_\pi$.

For these reasons we believe that the separation of long- and short-distance physics in the $^1S_0$ channel within a $\chi$PT framework will be most effective if $1.0$ fm $< R < 1.8$ fm. If $R > 1.8$ fm there is very little impact from chiral dynamics on $\gamma(k; R)$. Correspondingly, conversion from the variable $kR$ back to $T_{lab}$ produces rapid curvature at disturbingly low values of the laboratory energy. At $R > 1.8$ fm the computation of $\gamma$ is sensitive to details
of the short-distance dynamics (and not just a few coefficients in the expansion (8)) for $T_{\text{lab}}$
only a little above 100 MeV. In contrast, at $R < 1.0$ fm the short-distance dynamics has less
role in generating the energy dependence of the $^1S_0$ phase shift; but the massive attraction
generated by $V^{(3)}$ causes problems of its own, as witnessed by the very large value of $\gamma R$
at $k = 0$ in the presence of “sub-leading two-pion exchange” that is inferred at this separation
scale.

We now take the results of Figs. 1 and 2 and extract the coefficients $A_0$ and $A_1$ in the
expansion (8) of $\gamma(k; R)$ for different choices of the long-range potential $V$. These results
will be used in Section V when we attempt to implement specific short-distance potentials
that reproduce the low-energy behavior of $\gamma(k; R)$. The results for $\gamma(k; R)$ obtained at two
different separation scales $R$ in the range 1.0 fm $< R < 1.8$ fm are presented in Table II.
(For the results tabulated we have used four terms in the expansion (8) to convince
ourselves that the coefficients $A_2$, $A_3$, etc. are indeed much smaller compared to either $A_0$
or $A_1$. For example, $A_2 \sim 10^{-3}$ fm$^{-3}$ and $A_3 \sim 10^{-5}$ fm$^5$ for $R = 1.4$ fm.) The numbers in
Table II reinforce the conclusion that the next-to-leading-order two-pion exchange potential
has the largest effect, with a pronounced impact on $A_0$ and $A_1$ for $R = 1.0$ fm. We note
that $A_2$ and $A_3$ are small, and the values of $A_0$ and $A_1$ shown are natural with respect to
the scale $R$ and the underlying scale of $\chi$PT, $\Lambda_{\chi}\text{SB}$.

TABLE II: The coefficients $A_0$ (in fm$^{-1}$) and $A_1$ (in fm) for $R = 1.0$ fm and $R = 1.4$ fm for different
choices of long-range potential.

| Potential          | R=1.0 fm | R=1.4 fm |
|--------------------|----------|----------|
|                    | $A_0$    | $A_1$    | $A_0$    | $A_1$    |
| No LR pot          | 0.042    | 0.328    | 0.045    | -0.284   |
| $V^{(0)}$          | 0.192    | 0.008    | 0.130    | -0.37    |
| $V^{(0)} + V^{(2)}$| 0.403    | -0.044   | 0.178    | -0.39    |
| $V^{(0)} + V^{(2)} + V^{(3)}$ | 3.101 | -0.648 | 0.496 | -0.518 |

Finally, if sub-leading two-pion-exchange effects are so critical one must be cautious about
the set of low-energy constants chosen for their evaluation. Fig. 3 shows the effect of using
different values of these $c_i$’s in our calculation. The two choices are those tabulated in Table II.
It is obvious that the Nijmegen set of $c_i$’s produces a stronger effect in $\gamma(k; R)$, which is not
surprising given that the combination $c_3 - 2c_4$ sets the size of the leading singularity in $V^{(3)}$
in this channel, and this combination is much larger for the $c_i$’s adopted in Ref. [31]. It
is interesting to note that both choices lead to a similar shape for $\gamma(k; R)$, although the
magnitude is larger in the case of the Rentmeester et al. choice. Presumably the fact that
$\gamma(k; R)$ is smooth in both panels allows either choice of $c_i$’s to yield a reasonable fit for
$NN$ scattering data that is sensitive to the $^1S_0$ phase shift. It is just that the separation
between the long-distance dynamics in $V^{(3)}$ and the short-distance dynamics encoded in $\gamma R$
will differ, depending on the value of the $c_i$’s that is adopted. From this point on all of the
results that we present use the Nijmegen set of $c_i$’s.

IV. ARE PION EXCHANGES PERTURBATIVE?

In the previous section we showed that the logarithmic derivative $\gamma(k; R)$ is a logical way
to look at the short-distance parts of the $NN$ interaction. In this section we turn to the
question of whether the long-range part of \( V \) can be treated perturbatively. To do this we analyze how close a perturbative treatment of the \( \chi \)PT potential is to the full solution of the Schrödinger equation in the region \( r > R \).

Our objective is still to solve the Schrödinger equation:

\[
\frac{d^2u(r)}{dr^2} + (k^2 - U(r))u(r) = 0,
\]

in the region \( r > R \). Here \( U(r) = \frac{MV(r)}{\hbar^2} \) is obtained from the potential in this region of space, which in our case is the LO, NLO or the NNLO potentials generated in \( \chi \)PT. In order to build up the solution, let us first rewrite Eq. (24) as:

\[
-\frac{d^2u_k(r)}{dr^2} + \alpha U(r)u_k(r) = k^2 u_k(r),
\]

where \( \alpha \) is a parameter that is used to describe the perturbative expansion in powers of \( V \) (note that this is not the same as the \( \chi \)PT expansion for \( V \) itself that was described in Sec. [I]), and the subscript \( k \) refers to the wave number of interest. The full solution can now be written as a power series in \( \alpha \) as:

\[
u_k(r) = \sum_{n=0}^{\infty} \alpha^n u_k^{(n)}(r) = u_k^{(0)}(r) + \alpha u_k^{(1)}(r) + \alpha^2 u_k^{(2)}(r) + \ldots,
\]

where, \( u_k^{(0)}(r) \) is the zeroth-order solution (in the absence of \( V(r) \)), \( u_k^{(1)}(r) \) is the first-order-in-\( V \) correction, \( u_k^{(2)}(r) \) is the second-order correction, and so on.

Substituting Eq. (26) into Eq. (25) and equating powers of \( \alpha \) we get:

\[
\frac{d^2u_k^{(0)}(r)}{dr^2} + k^2 u_k^{(0)}(r) = 0,
\]

\[
\frac{d^2u_k^{(n+1)}(r)}{dr^2} + k^2 u_k^{(n+1)}(r) = U(r)u_k^{(n)}(r); \quad n \geq 0.
\]
The solution of Eq. (27) that reproduces the PWA93 $^1S_0$ phase shift $\delta(k)$ is:

$$u_k^{(0)}(r) = \frac{\sin(kr + \delta(k))}{\sin \delta(k)}.$$

Equations (29) and (28) can now be used, together with the Green’s function technique, to calculate the corrections to $u$ at first order and second order in perturbation theory. The Green’s function for Eq. (28) that preserves the form (29) as $r \to \infty$ in the full solution $u_k(r)$ is:

$$G(r, r'; k) = \begin{cases} 
\frac{\sin k(r-r')}{k} & \text{if } r < r', \\
0 & \text{if } r > r'.
\end{cases}$$

Therefore the first-order correction to the wavefunction can be expressed as:

$$u_k^{(1)}(r) = \int_r^\infty G(r, r'; k) U(r') u_k^{(0)}(r') dr'.$$

(31)

The second-order correction to the wavefunction is then calculable as:

$$u_k^{(2)}(r) = \left(\frac{M}{\hbar^2}\right) \int_r^\infty G(r, r'; k) V(r') u_k^{(1)}(r') dr'.$$

(32)

Comparison of results from Eqs. (32) and (31) will allow us to assess the convergence of our “long-distance perturbation theory”. This perturbation theory was set up in Refs. [16, 29], and was used to discuss the convergence of the chiral expansion. In contrast to Ref. [29], which integrated Eqs. (31) and (32) to $R = 0$, we will only integrate them to some finite $R$. Hence, we once again examine the logarithmic derivative at a radius $R$, only this time we define a version that can be computed using long-distance perturbation theory:

$$\gamma^{(n)}(k; R) \equiv \left[ \frac{u_k^{(0)} + u_k^{(1)} + \ldots + u_k^{(n)} - \gamma’(r)}{u_k^{(0)} + u_k^{(1)} + \ldots + u_k^{(n)}(r)} \right]_{r=R}.$$

(33)

Strictly speaking this $\gamma^{(n)}$ includes effects of $V$ which are of an order higher than $n$, but it has the advantage that it is straightforward to compute. It also results in smooth energy dependence when the limit $R \to 0$ is taken [24, 30].

To test the usefulness of Eq. (33) we choose the strongest potential, which, for distances $R < 1.8$ fm, we know to be the NNLO potential $V(r) = V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r)$, and compare the results with those of the previous section. We again adopt the $c_i$’s of Rentmeester et al. [31]. The results are plotted in Fig. 4. The dotted (black) curve is the zeroth-order solution, which corresponds to using the free Schrödinger equation for the integration from $r = \infty$ to $r = R$. This, then, is the same as the “No-long-range potential” case of the previous section. Meanwhile, the dashed (red) curve is the first-order perturbation theory result, the solid (green) curve is the second-order result, and the dot-dashed (blue) curve is from the full solution of the Schrödinger equation obtained in Sec. III.

The right panel of Fig. 4 shows that—as one might expect given the results of that section—at $R = 2.2$ fm this potential has almost no impact on $\gamma(k; R)$, and the result for any finite order of perturbation theory is the same. The left panel of Fig. 4 reiterates that
FIG. 4: $\gamma(k; R)(\text{MeV})$ vs. $T_{\text{lab}}(\text{MeV})$ calculated in perturbation theory and compared to the full solution. The dotted (black) curve is the zeroth order solution, the dashed (red) curve is the first order perturbation theory result, the solid (green) curve is the second order result and the dot-dashed (blue) curve is from the full solution of the Schrödinger equation.

at $R = 1.4$ fm the $\gamma(k; R)$ results with the NNLO potential are very different from the “No-long-range potential case”. But it also shows that this difference is almost entirely due to the first-order perturbation-theory correction. There is a 10% difference between the first-order result, $\gamma^{(1)}$ and the full $\gamma$ at $k \approx 0$, but this 10% difference has almost completely disappeared by $T_{\text{lab}} = 100$ MeV: perturbation theory works better at higher energies. And even at the lower energies where there is some noticeable deviation, the inclusion of the second-order correction brings us to within better than 1% of the full result for $\gamma(k; R)$. This success of perturbation theory implies that the coefficients $A_0$ and $A_1$ defined in Eq. (8) can instead be extracted using the perturbative formulae (29), (31), and (32)—rather than from the full Schrödinger equation solution—without introducing undue error.

TABLE III: The coefficients $A_0$ (in fm$^{-1}$) and $A_1$ (in fm) for $R = 1.4$ fm extracted using Eq. (33). The second and third (fourth and fifth) columns show the $A_0$ and $A_1$ corresponding to $\gamma^{(1)}$ ($\gamma^{(2)}$). The sixth and seventh columns repeat the result of Table III for comparison.

| Potential | $\gamma^{(1)}$ | $\gamma^{(2)}$ | Full soln. |
|-----------|----------------|----------------|------------|
| $V^{(0)}$ | $A_0$ $A_1$    | $A_0$ $A_1$    | $A_0$ $A_1$ |
| $V^{(0)} + V^{(2)}$ | 0.130 -0.369 | 0.130 -0.369 | 0.130 -0.37 |
| $V^{(0)} + V^{(2)} + V^{(3)}$ | 0.569 -0.624 | 0.516 -0.534 | 0.496 -0.518 |

In Table III we do exactly that, using the perturbative formula (33) to determine $A_0$ and $A_1$ for the case $R = 1.4$ fm. The numbers agree well with those in Table II for the “full solution”. A first-order-perturbation-theory extraction is sufficient for $V(r) = V^{(0)}$ and $V(r) = V^{(0)} + V^{(2)}$. For $V(r) = V^{(0)} + V^{(2)} + V^{(3)}$ a second-order-perturbation-theory calculation does a noticeably better job.
Since the potentials get stronger as \( r \to 0 \) the convergence of perturbation theory is rather slow for \( R = 1.0 \text{ fm} \). We find that for \( R = 1.0 \text{ fm} \), \( \gamma^{(1)}(k; R) \) overshoots \( \gamma(k; R) \) by almost a factor of two at \( k = 0 \). The second-order correction brings \( \gamma^{(2)}(k; R) \) to within 10% of the “exact” result at \( k = 0 \), and the agreement is better than 5% at \( T_{\text{lab}} = 100 \text{ MeV} \). So perturbation theory formally converges for \( R = 1.0 \text{ fm} \), but the “chiral” nature of the perturbation theory is very much in question, since—as emphasized in Sec. III—the effects of \( V^{(3)} \) are much larger than those of \( V^{(0)} \) in this region.

We conclude that the impact of the pion-exchange potentials generated in \( \chiPT \) can be calculated in perturbation theory, provided that the region of \( r \) where that perturbation theory is applied is chosen judiciously. It is possible that a more sophisticated perturbation theory works even if \( R \leq 1.0 \text{ fm} \), but what is already clear from these results is that standard “long-distance” perturbation theory is applicable and useful in the domain \( R > 1.0 \text{ fm} \).

V. SHORT-DISTANCE PARAMETERS AND POTENTIAL WELLS

We now change gears and discuss how the information gleaned from the energy-dependent matching condition \( \gamma(k; R) \) can be used to extract useful results regarding the short-distance physics that is operative at \( r < R \).

The coefficients defined in Eq. (8) and listed in Table II for different choices of \( R \) and \( V(r) \) are numbers. But, they can be used to build a short-distance potential that is, in effect, a short-distance regulator for our \( NN \) interaction. Our results suggest that two terms are sufficient to ensure convergence in the expansion in Eq. (8). Thus, any short-range potential we build having two parameters can be correlated to the coefficients \( A_0 \) and \( A_1 \). We have chosen to design our short-range potential to be a well with a bottom that has a slope so that

\[
V_{SR}(r) = -V_0 + \mu r, \quad \text{for } r \leq R. \tag{34}
\]

Here, \( V_0 \) (the strength of the short-distance potential at \( r = 0 \)) and the slope \( \mu \) are the two parameters of our regulator which we denote as short-distance parameters (SDPs). In order to extract these two parameters we solve the Schrödinger equation for \( r \leq R \) such that the following condition is satisfied:

\[
\left[ \frac{du'_m(r)}{u_m(r)} \right]_{r=R} = \gamma(k; R). \tag{35}
\]

The motivation in choosing the short-range potential in this form is that through an appropriate change of variable from \( r \) to \( x \), where

\[
x = \left( \frac{M\mu}{\hbar^2} \right)^{\frac{2}{3}} \left[ r - \frac{E + V_0}{\mu} \right], \tag{36}
\]

the Schrödinger equation for \( r < R \) can be reduced to the Airy equation:

\[
\frac{d^2u(x)}{dx^2} - xu(x) = 0. \tag{37}
\]

The solution to Eq. (37) is a linear combination of Airy functions:

\[
u(x) = a_1 Ai(x) + a_2 Bi(x). \tag{38}
\]
The Airy function $Bi(x)$ diverges at $x \to \infty$, but we are interested in the region $0 < r < R$, and so must keep both solutions. $a_1$ and $a_2$ are arbitrary constants that are evaluated from boundary conditions $u(0) = 0$ and $u'(0) = 1$. Thus changing the variable back to $r$ we obtain

$$u(r) = \pi \left( \frac{\hbar^2}{M \mu} \right)^{\frac{1}{4}} [-Bi(x_0)Ai(\bar{r} + x_0) + Ai(x_0)Bi(\bar{r} + x_0)]$$

(39)

where, $x_0 = -(\frac{M\mu}{\hbar^2})^{\frac{1}{4}} [E - V_{0\mu}]$, $\bar{r} = (\frac{M\mu}{\hbar^2})^{\frac{1}{4}} r$.

From this solution at $r \leq R$ we can easily obtain its logarithmic derivative at $r = R$:

$$\frac{u'(R)}{u(R)} = \left( \frac{\hbar^2}{M \mu} \right)^{\frac{1}{4}} \frac{Bi(x_0)Ai'(y_0) - Ai(x_0)Bi'(y_0)}{Bi(x_0)Ai(y_0) - Ai(x_0)Bi(y_0)},$$

(40)

with $y_0 = (\frac{M\mu}{\hbar^2})^{\frac{1}{4}} [R - \frac{E - V_{0\mu}}{\mu}]$. This logarithmic derivative is a function of our SDPs, $V_0$ and $\mu$. It should be equal to the logarithmic derivative $\gamma(k; R)$ computed in Sec. III. Hence, using the lowest two terms in the Taylor expansion of that $\gamma(k; R)$, $A_0$ and $A_1$, we can extract $V_0$ and $\mu$.

**TABLE IV**: The short-distance parameters $\mu$ (in MeV fm$^{-1}$) and $V_0$ (in MeV) for $R = 1.4$ fm with different choices of long-distance potential. (Note that in our convention $V_0 < 0$ corresponds to a repulsive potential.) The second and third columns give the results that match the Taylor-series coefficients $A_0$ and $A_1$ of $\gamma(k; R)$ defined in Eq. (6). The fourth and fifth (sixth and seventh) columns provide SDPs that match the $A_0$ and $A_1$ for $\gamma^{(1)}(\gamma^{(2)})$ of Eq. (33).

| Potential                  | Full soln. | $\gamma^{(1)}$ | $\gamma^{(2)}$ |
|----------------------------|------------|----------------|----------------|
| No LR pot                  | $\mu$ V0   | 9.85 59.35     | 9.85 59.35     | 9.85 59.35     |
| $V(0)(r)$                  | $\mu$ V0   | -54.5 -10.68   | -54.5 -10.56   | -54.5 -10.56   |
| $V(0)(r) + V(2)(r)$        | $\mu$ V0   | -56.0 -17.39   | -55.5 -18.23   | -56.0 -16.94   |
| $V(0)(r) + V(2)(r) + V(3)(r)$ | $\mu$ V0       | -65.0 -54.4    | -64.0 -55.65   | -65.0 -51.1    |

To demonstrate, we have calculated the short-distance parameters, $V_0$ and $\mu$, in this way for $R = 1.4$ fm. The results are given in Table IV. Results for a different two-parameter short-distance potential and the case of $V = 0$ and $V = V(0)$ can be found in Ref. [15].

We had commented earlier in Sec. IV that $A_0$ and $A_1$ can be extracted from the matching condition calculated in perturbation theory. Consequently, one can then proceed to extract $\mu$ and $V_0$ from these “perturbative” $A_0$ and $A_1$. The results are shown in the third to sixth column of Table IV and are gratifyingly close to those obtained from the full Schrödinger equation solution of Sec. III.

The results of Table IV show that a nice quantitative description of the $NN$ interaction in the $^1S_0$ channel up to $T_{lab} = 200$ MeV can be obtained by using perturbation theory for $r \geq R$ to calculate $\gamma(k; R)$, then using that information to determine the parameters present in a simple short-distance potential such as (34).
VI. CONCLUSION

Our analysis of $NN$ scattering in the $^{1}S_{0}$ channel is similar in its philosophy to that advocated by Lepage \cite{36}, in that we separate the potential into a long-distance part—determined by $\chi$PT—and a short-distance part, which can be parameterized in a variety of ways without affecting the predictions for $NN$ scattering. Indeed, we have gone further than Lepage, and argued that the energy-dependent logarithmic derivative at $R$, $\gamma(k; R)$, is a convenient way to summarize information about the impact of physics that is short-range with respect to the scale $R$. This information on the short-range $NN$ dynamics can be obtained from the $NN$ phase shifts if we know the long-range potential. Combining $\chi$PT long-range potentials (computed to different orders in the chiral expansion) with the experimental phase-shifts from the Nijmegen PWA93 we obtained $\gamma(k; R)$ for a range of scales $R$.

For the extraction of model-independent information on short-distance physics it is desirable that $\gamma(k; R)$ be a smooth function of $k$. This requirement limits the energy range within which one can operate if $R$ is chosen to be greater than 1.8 fm. For values of $R$ in the range from about 1.0 fm to 1.8 fm $\gamma(k; R)$ can be well described by a polynomial in $k^2$ over the entire range $0 \leq T_{lab} \leq 200$ MeV. The coefficients of the terms in this polynomial represent the effect of the short-distance $NN$ interaction. We have found that for $1.0 \text{ fm} < R < 1.8 \text{ fm}$ the first two terms of this polynomial expansion are sufficient to ensure a good representation of the energy dependence of $\gamma$. So, in this range of $R$, details of the short-distance potential are not important: its effects can be summarized in $A_0$ and $A_1$.

The long-range potential $V$ determines the evolution of $\gamma(k; R)$ with the separation scale $R$ (see Refs. \cite{23, 24} for a derivation of the renormalization-group equations associated with this evolution). The scale $R$ in our analysis plays a similar role to the scale $\Lambda$ in the potential $V_{\text{low}}$ \cite{38}. In that approach $\Lambda$ is a cutoff that separates the momentum-space states explicitly included in the Hilbert space from those states whose effects are encoded in $V$. There is then evolution of the $NN$ potential $V$ with $\Lambda$ so that S-matrix elements (which are equivalent to phase shifts) stay the same as the momentum-space cutoff is changed. In our work the short-distance part of the potential also evolves in such a way that the $^{1}S_{0}$ phase shifts always agree with those from the Nijmegen PWA93. This evolution can be traced over a range from $R = \infty$ to $R = 1.0$ fm. The fact that, at the lower end of this range, details of the short-distance potential do not play a key role in describing $NN$ phase shifts below $T_{lab} = 200$ MeV, is presumably a corollary of the existence of a “universal” $V_{\text{low}}$ if the scale $\Lambda$ is lowered to 2 fm$^{-1}$.

In the energy range $0 \leq T_{lab} \leq 200$ MeV we find that the one-pion exchange interaction is the dominant effect in this evolution for $R \geq 1.8$ fm, but its impact on $\gamma(k; R)$ is not sizeable. We obtain a larger shift in $\gamma(k; R)$ compared to its value in the absence of any long-range potential if we consider sub-leading pieces of the $\chi$PT $V$. In particular, if we compute with $V^{(3)}$, and consider $R \leq 1.4$ fm, we see a significant impact of pion-exchange dynamics on the $\gamma(k; R)$ inferred from the $NN$ phase shifts.

But, even in this case, as long as we maintain $R > 1.0$ fm, we find that the evolution of $\gamma(k; R)$ due to long-distance effects can be understood in perturbation theory. Hence we can use perturbation theory in the $\chi$PT potentials in the region $1.0 \text{ fm} < R < \infty$ to connect experimental data to information on the short-distance dynamics in the region $R < 1.0$ fm. The short-distance part then contains non-perturbative dynamics, but these effects are summarized in a few coefficients in the polynomial expansion of the energy-dependent
matching condition. Those coefficients can in turn be matched to an explicit realization of short-distance physics, e.g. the one discussed in Sec. V.

This represents an updated version of the proposal first made by Kaplan, Savage, and Wise in Refs. [10, 11]: that the contact interaction parameterizing short-distance \(NN\) dynamics should be promoted to order \(Q^{-1}\) so that it is a leading-order effect, with pion exchanges then being sub-leading. This proposal was examined in detail for \(NN\) scattering in Refs. [13, 14], and was found to work well in the \(1S_0\) channel as far as the pion potential \(V^{(0)}\) was concerned. We find that \(V^{(2)}\) can also be treated in perturbation theory, and its effects are smaller than \(V^{(0)}\) for \(r > 1.0\) fm. The NNLO piece of \(V\), \(V^{(3)}\), can also be treated in perturbation theory, but only if it is regulated at a scale of order, or well below, the chiral-symmetry breaking scale \(\Lambda_{\chi SB}\). The large size of corrections due to \(V^{(3)}\) raises the question of whether the chiral expansion for \(V\) is behaving as decreed by the \(\chi PT\) power counting.

In particular, some authors have suggested that an expansion in which the Delta-nucleon mass difference, \(\Delta\), is treated as a light scale, might have better convergence properties [4, 39, 40, 41, 42, 43, 44]. Our results support this view—at least if one takes \(R\) to be small enough that chiral dynamics plays a significant role. More recently Robillota has pointed out that the poor convergence of \(\chi PT\) without an explicit Delta for the scalar-isoscalar piece of the \(NN\) potential is related to issues with the description of the nucleon’s scalar form factor [45]. He observes that in that case too, contributions from Delta loops exceed those from nucleon loops once one considers distances \(r\) of 1 fm or less.

We close with some possible implications of our results for future partial-wave analyses (PWAs). Here we have used the Nijmegen-extracted phase shifts [34] and our separation of \(NN\) dynamics into long-range and short-range parts is similar to that adopted in the Stoks et. al. 1993 PWA [33]. However, in Ref. [33] an upgraded version of the Nijm78 potential was used to describe the long-distance dynamics. Minimally, this paper shows that in the \(1S_0\) channel we can use \(\chi PT\) potentials and still get sensible energy dependence in our matching condition which encodes the short-distance \(NN\) dynamics. We have shown that the region \(1.0 \leq R \leq 1.8\) fm yields best results in this regard. This is in accord with the more recent PWA for which details are as yet unpublished [31, 32, 46]. But our results suggest that—at least in the \(1S_0\) channel—future PWAs using these values of \(R\) could employ perturbation theory to compute the effects of the long-distance potentials obtained in \(\chi PT\). It might also be possible to lower the separation scale \(R\) below 1.0 fm and encode the smooth energy dependence due to dynamics at \(r < R\) in energy-dependent square wells, but then the parameters obtained for these wells would be very different depending on whether the LO or NNLO \(NN\) potential were used—or even presumably, if different sets of \(c_i\)'s were used in computing \(V^{(3)}\). Extensions of the methods laid out here to other partial waves are straightforward and would facilitate such a future EFT-based partial-wave analysis.

Acknowledgments

This work was carried out under grant DE-FG02-93ER40756 of the US-DOE (DS, DP, EM) and by NSF grant PHY-0645468 (DS). DP is grateful to Silas Beane and Dick Furnstahl for useful discussions on the topics covered here. We also thank Manuel Pavón Valderrama, Enrique Ruiz Arriola, and Matthias Schindler for stimulating conversations as well as their
A careful reading of, and informative comments on, this manuscript.

[1] V. Bernard, Prog. Part. Nucl. Phys. 60, 82 (2008).
[2] S. Scherer and M. R. Schindler, arXiv:hep-ph/0505265.
[3] S. Weinberg, Phys. Lett. B 251, 288 (1990); Nucl. Phys. B 363, 3 (1991).
[4] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. C 53, 2086 (1996).
[5] E. Epelbaum, W. Gloeckle and U.-G. Meißner, Nucl. Phys. A 671, 295 (2000).
[6] N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A 625, 758–788 (1997).
[7] D. R. Entem and R. Machleidt, Phys. Rev. C 66, 014002 (2002).
[8] D. R. Entem and R. Machleidt, Phys. Rev. C 68, 041001 (2003).
[9] E. Epelbaum, W. Glöckle and U.-G. Meißner, Nucl. Phys. A 747, 362 (2005).
[10] D. B. Kaplan, M. J. Savage and M. B. Wise, Phys. Lett. B 424, 390 (1998).
[11] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B 534, 329 (1998).
[12] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B 478, 629 (1996).
[13] S. Fleming, T. Mehen and I. W. Stewart, Nucl. Phys. A 677, 313 (2000).
[14] S. R. Beane, P. F. Bedaque, M. J. Savage and U. van Kolck, Nucl. Phys. A 700, 377 (2002).
[15] K. A. Scaldeferri, D. R. Phillips, C. W. Kao and T. D. Cohen, Phys. Rev. C 56, 679 (1997).
[16] T. D. Cohen and J. M. Hansen, Phys. Lett. B 440, 233 (1998).
[17] J. V. Steele and R. J. Furnstahl, Nucl. Phys. A 645, 439 (1999).
[18] D. B. Kaplan and J. V. Steele, Phys. Rev. C 60, 064002 (1999).
[19] J. Gegelia, Phys. Lett. B 463, 133 (1999).
[20] T. Frederico, V. S. Timoteo and L. Tomio, Nucl. Phys. A 653, 209 (1999).
[21] J. A. Oller, Nucl. Phys. A 725, 85 (2003).
[22] M. Pavón Valderrama and E. Ruiz Arriola, Phys. Lett. B 580, 149 (2004).
[23] M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 70, 044006 (2004).
[24] M. Pavón Valderrama and E. R. Arriola, Ann. Phys. (in press), arXiv:0705.2952 [nucl-th]
[25] V. S. Timóteo, T. Frederico, A. Delfino and L. Tomio, Phys. Lett. B 621, 109 (2005).
[26] D. Djukanovic, S. Scherer, M. R. Schindler and J. Gegelia, Few Body Syst. 41, 141 (2007).
[27] C. J. Yang, C. Elster and D. R. Phillips, Phys. Rev. C 77, 014002 (2008).
[28] C. H. Hyun, D. P. Min and T. S. Park, Phys. Lett. B 473, 6 (2000).
[29] M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 74, 054001 (2006).
[30] D. R. Entem, E. Ruiz Arriola, M. Pavón Valderrama and R. Machleidt, arXiv:0709.2770 [nucl-th].
[31] M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar and J. J. de Swart, Phys. Rev. Lett. 82, 4992 (1999).
[32] M. C. M. Rentmeester, R. G. E. Timmermans and J. J. de Swart, Phys. Rev. C 67, 044001 (2003).
[33] V. G. J. Stoks, R. A. M. Klomp, M. C. M. Rentmeester and J. J. de Swart, Phys. Rev. C 48, 792–815 (1993).
[34] Partial Wave Analysis from Nijmegen University obtained from world wide web
   http://nn-online.org/NN/
[35] R. Higa, M. R. Robilotta and C. A. da Rocha, Phys. Rev. C 69, 034009 (2004).
[36] G. P. Lepage, arXiv:nucl-th/9706029v1
[37] T. Barford and M. C. Birse, Phys. Rev. C 67, 064006 (2003).
[38] S. K. Bogner, T. T. S. Kuo and A. Schwenk, Phys. Rept. 386, 1 (2003).
[39] E. E. Jenkins and A. V. Manohar, Phys. Lett. B 255, 558 (1991).
[40] T. R. Hemmert, B. R. Holstein and J. Kambor, Phys. Lett. B 395, 89 (1997).
[41] N. Kaiser, S. Gerstendorfer and W. Weise, Nucl. Phys. A 637, 395 (1998).
[42] V. Pascalutsa and D. R. Phillips, Phys. Rev. C 67, 055202 (2003).
[43] V. R. Pandharipande, D. R. Phillips and U. van Kolck, Phys. Rev. C 71, 064002 (2005).
[44] H. Krebs, E. Epelbaum and U.-G. Meißner, Eur. Phys. J. A 32, 127 (2007).
[45] M. Robillota, Talk given at Workshop on Chiral Symmetry in Hadron and Nuclear Physics: Chiral07, Osaka, Japan, 13-16 Nov 2007. [arXiv:0802.2484] [nucl-th].
[46] M. Rentmeester, talk at ECT* workshop “QCD and nuclear forces: never the twain shall meet”, ECT*, Trento, 2005.