$Q$-learning with Logarithmic Regret

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Abstract

This paper presents the first non-asymptotic result showing that a model-free algorithm can achieve a logarithmic cumulative regret for episodic tabular reinforcement learning if there exists a strictly positive sub-optimality gap in the optimal $Q$-function. We prove that the optimistic $Q$-learning studied in [Jin et al. 2018] enjoys an $O\left(\frac{SA\text{poly}(H)}{\text{gap}_{\text{min}}} \log(ST)\right)$ cumulative regret bound, where $S$ is the number of states, $A$ is the number of actions, $H$ is the planning horizon, $T$ is the total number of steps, and $\text{gap}_{\text{min}}$ is the minimum sub-optimality gap. This bound matches the information theoretical lower bound in terms of $S, A, T$ up to a $\log(ST)$ factor. We further extend our analysis to the discounted setting and obtain a similar logarithmic cumulative regret bound.

1 Introduction

$Q$-learning [Watkins and Dayan, 1992] is one of the most popular classes of methods for solving reinforcement learning (RL) problems. $Q$-learning tries to estimate the optimal state-action value function ($Q$-function). With a $Q$-function, at every state, one can greedily choose the action with the largest $Q$ value to interact with the RL environment while achieving near optimal expected cumulative rewards in the long run. Compared to another popular classes of methods, e.g., model-based RL, $Q$-learning algorithms (or more generally, model-free algorithms) often enjoy better memory and time efficiency. These are the main reasons why $Q$-learning is applied in solving a wide range of RL problems [Mnih et al., 2015].

While model-free methods are widely applied in practice, most theoretical works study model-based RL. In one of the most fundamental RL frameworks, tabular RL, which is the focus of this paper, the majority of works study model-based algorithms [Kearns and Singh, 1999, Kakade, 2003, Singh and Yee, 1994, Azar et al., 2013, 2017, Dann and Brunskill, 2015, Dann et al., 2017, Agarwal et al., 2018, Simeonov and Jamison, 2019] with a few exceptions [Strehl et al., 2006, Jin et al., 2018, Dong et al., 2019, Zhang et al., 2020]. From a regret minimization point of view, the state-of-the-art analysis demonstrates that one can achieve a $\sqrt{T}$-type regret bound where $T$
the number of episodes. Although these bounds are sharp in the worst-case scenario, they do not reveal the favorable structures of the environment, which can significantly reduce the regret.

One such structure is the existence of a strictly positive sub-optimality gap, i.e., for every state, there is a strictly positive value gap between the optimal action(s) and the rest (cf. Definition 2.1). In practice, arguably, nearly all environments with a finite action set satisfy some sub-optimality gap conditions. For instance, for board games, e.g., tic-tac-toe, Chess, or even Go, most states have zero rewards except for the winning states. Hence, every optimal action has a Q-value 1 and the rest actions have value 0 or some number strictly less than 1. In Atari-games, e.g., Freeway, the optimal action has a value that is usually very distinctive from the rest of actions. In many other environments with finite number of actions, e.g., those control environments in OpenAI gym [Brockman et al., 2016], the gap condition usually holds. Similar gap conditions can be observed in other environments (see e.g. Kakade [2003]).

Theoretically, the sub-optimality gap is extensively investigated in the bandit problems, which can be viewed as RL problems with the planning horizon being 1. With this structure, one can drastically reduce the \( \sqrt{T} \)-type regret to \( \log T \)-type regret [Bubeck and Cesa-Bianchi, 2012, Lattimore and Szepesvári, 2018, Slivkins, 2019]. For RL, most existing works that can leverage this structure require additional assumptions about the environment, such as finite hitting time and ergodicity [Jaksch et al., 2010, Tewari, 2007, Ok et al., 2018] or access to a generator [Zanette et al., 2019]. Recently, Simchowitz and Jamieson [2019] presented a systematic study of episodic tabular RL with the gap structure. They presented a novel algorithm which achieves the near-optimal \( \sqrt{T} \)-type regret in the worst scenario and \( \log T \)-type regret if there exists a strictly positive sub-optimality gap. Furthermore, they also provided instance-dependent lower bounds for a class of reasonable algorithms. See Section 1.2 for more detailed discussions.

However, to our knowledge, all existing works that obtain \( \log T \)-type regret bounds are about model-based algorithms. It remains open whether model-free algorithms such as Q-learning can achieve \( \log T \)-type regret bounds. Indeed, this is a challenging task: as discussed in [Simchowitz and Jamieson, 2019], their analysis framework cannot be applied to model-free algorithms directly. Later in this section, we also provide some technical explanations on why their approach is difficult to adopt.

Our Contributions We answer the aforementioned open problem by proving that the optimistic Q-learning algorithm studied in [Jin et al., 2018] enjoys a \( O\left(\frac{SAH}{\text{gap}_{\min}} \log (SAH)\right) \) cumulative regret, where \( S \) is the number states, \( A \) is the number of actions, \( H \) is the planning horizon and \( \text{gap}_{\min} \) is the minimum sub-optimality gap. To our knowledge, this is the first result showing model-free algorithms can achieve \( \log T \)-type regret bounds. Furthermore, our bound matches the lower bound by [Simchowitz and Jamieson, 2019] in terms of \( S, A \) and \( T \) up to a \( \log (SA) \) factor. Second, we extend our analysis to the infinite-horizon discounted setting with the regret defined in [Liu and Su, 2020], for which we show the optimistic Q-learning achieves \( O\left(\frac{SA}{\text{gap}_{\min}(1-\gamma)} \log \left(\frac{SAH}{\text{gap}_{\min}(1-\gamma)}\right)\right) \) regret where \( 0 < \gamma < 1 \) is the discount factor.

1.1 Main Challenges and Technique Overview

Here we explain the main challenge of using existing analyses and give an overview of our main techniques. The existing proof in [Jin et al., 2018] bounds the regret in terms of a weighted sum of
the estimation error of $Q$-function. Note the estimation error scales $1/\sqrt{T}$ which in turn gives a $\sqrt{T}$-type regret, but cannot give a log $T$-type regret bound.

For model-based algorithms, Simchowitz and Jamieson [2019] introduced a novel notion, optimistic surplus (cf. Equation (26)), which can be bounded by the estimation error of the transition probability. The logarithmic regret bound can be proved via a clipping trick on top of the optimistic surplus.

Unfortunately, as acknowledged by Simchowitz and Jamieson [2019], their analysis is highly tailored to model-based algorithms. First, model-free algorithms do not estimate the probability transition, so we cannot bound the optimistic surplus via this approach. Secondly, although we can also obtain a formula for the optimistic surplus in each episode using the update rules of the $Q$-learning algorithm, the formula depends on the estimation error of $Q$-function in previous episodes. This dependency makes it difficult to bound the optimistic surplus. See Section D for more technical details.

In this paper, we adopt an entirely different counting approach. We first write the total regret as expected sum over sub-optimality gaps appearing in the whole learning process, then use the estimation error of $Q$-function and the definition of sub-optimality gap to upper bound the number of times the algorithm takes suboptimal actions. To obtain a sharp dependency on $\text{gap}_{\min}$, we divide the interval $[\text{gap}_{\min}, H]$ (the range of all gaps) into multiple subintervals, and we bound the sum of learning error in each subinterval by its maximum value times the number of steps falling into this subinterval.

Organization This paper is organized as follows. In Section 1.2 we discuss related works. In Section 2 we introduce necessary definitions and backgrounds. In Section 3 we present our main results. In Section 4 we give the proof of our theorem on the episodic setting. We conclude in Section 5 and leave some technical proofs to the appendix.

1.2 Related Work

Gap-independent Finite-horizon and Infinite-horizon Discounted RL

There is a long list of results about regret or sample complexity of tabular RL, dating back to Singh and Yee [1994]. One line of works requires access to a simulator where the agent can query samples freely from any state-action pair of the environment and therefore the agent does not need to design a strategy to explore the environment. [Kearns and Singh, 1999, Kakade, 2003, Singh and Yee, 1994, Azar et al., 2013, Sidford et al., 2018b, Agarwal et al., 2019, Zanette et al., 2019, Li et al., 2020].

Another line of works drop the simulator assumption and thus the agent needs to use advanced techniques, such as upper confidence bound (UCB) to explore the state space. [Azar et al., 2017, Dann and Brunskill, 2015, Dann et al., 2017, 2019, Jin et al., 2018, Strehl et al., 2006, Zhang et al., 2020, Simchowitz and Jamieson, 2019, Zanette and Brunskill, 2019, Dong et al., 2019]. In terms of the regret, the state-of-art result shows one can achieve $\tilde{O}\left(\sqrt{SAH^2T} + \text{poly}\left(S, A, H\right)\right)$ regret for which the first term nearly match the $\Omega\left(\sqrt{SAH^2T}\right)$ up to logarithmic factors [Dann and Brunskill, 2013, Osband and Roy, 2016]. Among these results, only a few are for model-free algorithms [Strehl et al., 2009].

There is another line of works on gap-independent infinite-horizon average-reward setting. This setting is beyond the scope of this paper.

In this paper, we study the same setting as in Jin et al. [2018] where the reward at each level is in [0, 1], and the transition probabilities at each level can be different. In another setting, the total reward is bounded by 1 and...
and only very recently, Jin et al. [2018], Zhang et al. [2020] showed Q-learning can achieve \( \sqrt{T} \)-type regret bounds.

**Sub-optimality Gap** The results about gap-dependent regret bounds for MDP algorithms can be categorized into asymptotic bounds and non-asymptotic bounds. Asymptotic bounds are only valid when the total number of steps \( T \) is large enough. These bounds often suffer from the worst-case dependency on some problem-specific quantities, such as diameter and worst-case hitting time. Under the infinite-horizon average-reward setting, Auer and Ortner [2007] provided a logarithmic regret algorithm for irreducible MDPs. Besides dependency on hitting times, their regret also depends inversely on \( \text{gap}_1^2 \), the squared distance between optimal and second-optimal policy. Note that the \( \text{gap}_1 \) requirement is much stronger than our requirement as our sub-optimality gap only depends on actions of very state. Along this direction and improving over previous algorithm of Burnetas and Katehakis [1997], Tewari and Bartlett [2008] proposed an algorithm called Optimistic Linear Programming (OLP). OLP is proved to have \( C(P) \log T \) regret asymptotically in \( T \), where \( C(P) \) depends on some diameter-related quantity as well as the sum over reciprocals of gaps for \((x, a)\) inside a critical set.

For non-asymptotic bounds, Jaksch et al. [2010] introduced UCLR2 algorithm, which enjoys \( \tilde{O}(D^2S^2A \log T) \) regret where \( D \) is the diameter. More recently, Ok et al. [2018] derived problem-specific lower bounds for both structured and unstructured MDPs. This lower bound scales as \( SA \log T \) for unstructured MDP and \( c \log T \) for structured MDP, where this \( c \) depends on both the minimal action sub-optimality gap and the span of bias function, which can be bounded by diameter \( D \). For non-asymptotic bounds, Simchowitz and Jamieson [2019] proved that model-based optimistic algorithm strongEULER has gap-dependent regret bound that holds uniformly over \( T \). Moreover, their bounds depend only on \( H \) and not on any term such as hitting time or diameter. In Section 3 we compare our result with the one in Simchowitz and Jamieson [2019] in more detail.

### 2 Preliminaries

In this section we introduce necessary notions and definitions.

**Episodic MDP** An episodic Markov decision process (MDP) is a tuple \( \mathcal{M} := (S, A, H, P, r) \), where \( S \) is the finite state space with \( |S| = S \), \( A \) is the finite action space with \( |A| = A \), \( H \in \mathbb{Z}_+ \) is the planning horizon, \( P_h : S \times A \to \Delta(S) \) is the transition operator at step \( h \) that takes a state-action pair and returns a distribution over states, and \( r_h : S \times A \to [0, 1] \) is the deterministic reward function at step \( h \). Each episode starts at an initial state \( x_1 \in S \) picked by an adversary. In this paper, we focus on deterministic policies. A deterministic policy \( \pi \) is a sequence of mappings \( \pi_h : S \to A \) for \( h = 1, \ldots, H \). Given a policy \( \pi \), for a state \( x \in S \), the value function of state \( x \in S \) at the \( h \)-step is defined as

\[
V^\pi_h(x) := \mathbb{E} \left[ \sum_{h' = h}^H r_{h'}(x_{h'}, \pi(x_{h'})) \, | \, x_h = x \right],
\]

the transition probabilities at each level are the same. The latter setting is more challenging to analyze and the worst-case sample complexity is still open [Jiang and Agarwal, 2018, Wang et al. 2020].
Our paper investigates what structures of the MDP enable us to improve the Sub-optimality Gap \( Q \pi \) and the associated Regret. Jin et al. [2019] proved that Algorithm 1 enjoys the regret bound based on the observed data. In this paper we focus on bounding the expected regret \( \mathbb{E}[\text{Regret}(K)] \) where the expectation is over the randomness from the environment.

**Q-learning Algorithm** In this paper we focus on model-free Q-learning algorithms. Formally, by model-free algorithms, we mean the space complexity of the algorithm scales at most linearly in \( S \) in contrast to the model-based algorithms whose space complexity often scales quadratically with \( S \) [Strehl et al., 2006, Sutton and Barto, 1998, Jin et al., 2018]. For episodic MDP, we will analyze the Q-learning with UCB-Hoeffding algorithm studied in Jin et al. [2018] (cf. Algorithm 1). At a high level, this algorithm maintains an upper bound of \( Q^* \) for every \((s, a)\) pair and choose the action greedily at every episode. The algorithm uses a carefully designed step size sequence \( \alpha_k \) to update the upper bound based on the observed data. Jin et al. [2019] proved that Algorithm 1 enjoys the regret bound \( \sqrt{H^4SAT \log(SAT)} \) which is the first \( \sqrt{T} \)-type bound for model-free algorithms.

**Sub-optimality Gap** Our paper investigates what structures of the MDP enable us to improve the \( \sqrt{T} \)-type bound. In this paper we focus on the positive sub-optimality gap condition [Simchowitz and Jamieson, 2019, and Du et al., 2019].

**Definition 2.1** (Sub-optimality Gap). Given \( h \in [H] \), \((x, a) \in S \times A\), the suboptimality gap of \((x, a)\) at level \( h \) is defined as \( \text{gap}_h(x, a) := V^*_h(x) - Q^*_h(x, a) \).
Assumption 2.1 (Strictly Positive Minimum Sub-optimality Gap). Denote by $\text{gap}_{\text{min}}$ the minimum non-zero gap: $\text{gap}_{\text{min}} := \min_{h,x,a} \{ \text{gap}_h(x,a) : \text{gap}_h(x,a) \neq 0 \}$. We assume $\text{gap}_{\text{min}} > 0$.

In Section 1.1 we have discussed why many MDPs admit this structure. Our main result is a logarithmic regret bound of Algorithm 1 under Assumption 2.1.

Infinite-horizon Discounted MDP In this paper we also study infinite-horizon discounted MDP, which is a tuple $M := (S,A,\gamma,P,r)$, where every step shares the same transition operator $P$ and reward function $r$. Here $\gamma$ denotes the discount factor, and there is no restart during the entire process. Let $C = \{ x \times A \times [0,1]\}^* \times S$ be the set of all possible trajectories of any length. A non-stationary deterministic policy $\pi: C \rightarrow A$ is a mapping from paths to actions. The $V$ function and $Q$ function are defined as below ($c_i := (x_i, a_i, r_i, \cdots, x_i)$).

$$
V^\pi(x) := \mathbb{E} \left[ \sum_{i=1}^{\infty} \gamma^{i-1} r(x_i, \pi(c_i)) \right| x_1 = x],
$$

$$
Q^\pi(x,a) := r(x,a) + \mathbb{E} \left[ \sum_{i=2}^{\infty} \gamma^{i-1} r(x_i, \pi(c_i)) \right| x_1 = x, a_1 = a].
$$

Consider a game that starts at state $x_1$. A learning algorithm $\text{Alg}$ specifies an initial non-stationary policy $\pi_1$. At each time step $t$, the player takes action $\pi_t(x_t)$, observes $r_t$ and $x_{t+1}$, and updates $\pi_t$ to $\pi_{t+1}$. The total regret of $\text{Alg}$ for the first $T$ steps is thus defined as $\text{Regret}(T) = \sum_{t=1}^{T} (V^* - V^\pi(x_t))$. This definition was studied in Liu and Su [2020], which follows the sample complexity definition in Kakade [2003]. For this setting, we study Algorithm 2. This is a simple adaptation of Algorithm 1 that takes $\gamma$ into account, so we defer it to the appendix. We prove Algorithm 2 enjoys logarithmic regret bound under Assumption 2.1.

3 Main Theoretical Results

Now we present our main results.

Main Result for Episodic MDP The following theorem characterizes the performance of Algorithm 1 for episodic MDP under Assumption 2.1. To our knowledge, this is the first theoretical result showing a model-free algorithm can achieve logarithmic regret of tabular RL.

**Theorem 3.1** (Logarithmic Regret Bound of $Q$-learning for Episodic MDP). Under Assumption 2.1, the expected regret of Algorithm 1 for episodic tabular MDP is upper bounded by

$$
E[\text{Regret}(K)] \leq O \left( \frac{H^6SA}{\text{gap}_{\text{min}}} \log (SAT) \right).
$$

Proposition 2.2 in Simchowitz and Jamieson [2019] suggested that any reasonable algorithms, i.e. algorithms with sub-linear regret in the worst case, suffer $\Omega \left( \sum_{(x,a),\text{gap}_1(x,a) > 0} \frac{H^2}{\text{gap}_1(x,a)} \log T \right)$ expected regret. Therefore, for the environment where there are $SA$ state-action pairs whose gap is $\text{gap}_{\text{min}}$, the lower bound becomes $\Omega \left( \frac{SAH^2}{\text{gap}_{\text{min}}} \log T \right)$. Thus, for this class of environments, our upper bound is tight in terms of the dependencies on $S,A,\text{gap}_{\text{min}}$ and $T$ up to a $\log (SA)$...
factor. However, our dependency on \( H \) is not tight. We leave it as a future work for improving the dependency on \( H \).

An interesting advantage of our theorem is adaptivity. Note the algorithm we analyze is exactly the same algorithm studied in [Jin et al. 2019], which has been shown to achieve the worst-case \( \sqrt{T} \)-type regret bound. Theorem 3.1 suggests that one does not need to modify the algorithm to exploit the strictly positive minimum sub-optimality gap structure, Algorithm 1 automatically adapts to this benign structure.

We compare Theorem 3.1 with the regret bound for model-based algorithm in Simchowitz and Jamieson [2019] (in big-\( O \)-form):

\[
\left( \sum_{(x,a): \exists h \in [H], \text{gap}_h(x,a) > 0} \frac{H^3}{\min_h \text{gap}_h(x,a)} + \frac{SH^3}{\text{gap}_\min} + H^4 S A \max (S, H) \log \left( \frac{SAH}{\text{gap}_\min} \right) \right) \log (SAHT)
\]

First recall our bound is for a model-free algorithm which is more space-efficient than the model-based algorithm in Simchowitz and Jamieson [2019]. In terms of the regret bound, Theorem 3.1's dependency on \( H \) is worse than that in their bound. We remark that simple model-free algorithms may have a worse dependency on \( H \) compared to model-based algorithms (e.g., see Jin et al. [2018]), and more advanced algorithmic ideas are needed to improve the dependency on \( H \) [Zhang et al., 2020].

In the sequel, we focus on the dependency on \( S, A \), and gap. The regret bound in Simchowitz and Jamieson [2019] can be viewed as a more fine-grained characterization that its first term depends on \( \min_h \text{gap}_h(x,a) \) where our bound only depends on \( \text{gap}_\min \). Unfortunately, their second term has an \( \frac{S}{\text{gap}_\min} \) dependency and they showed this is unavoidable for optimistic algorithms, which include both their algorithm and Algorithm 1 (see Theorem 2.3 in Simchowitz and Jamieson [2019]).

Now let us consider an environment where there are \( \sim SA \) state-action pairs whose gap is \( \text{gap}_\min \). Then the bound in Simchowitz and Jamieson [2019] becomes

\[
\left( \frac{H^3SA}{\text{gap}_\min} + H^4 SA \max (S, H) \log \left( \frac{SAH}{\text{gap}_\min} \right) \right) \log (SAHT)
\]

In this regime, both Theorem 3.1 and their bound have an \( \frac{SA}{\text{gap}_\min} \) term. Their bound also has an additional \( H^4 SA \max (H, S) \log \left( \frac{SAH}{\text{gap}_\min} \right) \) burn-in term which our bound does not have. When \( S \) is large compared to \( H \) and \( \text{gap}_\min \), this term scales \( S^2 \) and can dominate other terms. In this regime, our bound is better. The technical reason behind this phenomenon is that Algorithm 1 is model-free and does not require \( O(S^2) \) samples to estimate the \( Q \)-function, which has a complexity proportional to \( O(S) \).

Main Result for Infinite-horizon Discounted MDP We also obtain a logarithmic regret bound for infinite-horizon discounted MDP.

**Theorem 3.2** (Logarithmic Regret Bound of \( Q \)-learning for Infinite-horizon Discounted MDP). Under Assumption 2.7, the expected regret of Algorithm 2 for infinite-horizon discounted MDP is upper bounded by

\[
\mathbb{E}[\text{Regret}(T)] \leq O \left( \frac{SA}{\text{gap}_\min (1 - \gamma)^6} \log \frac{SAT}{\text{gap}_\min (1 - \gamma)} \right).
\]
Theorem 3.2 suggests that model-free algorithms can achieve logarithmic regret even in the infinite-horizon discounted MDP setting. The main difference from Theorem 3.1 is that $H$ is replaced by $\frac{1}{1-\gamma}$. By analogy, we believe the dependencies on $S, A, T$ and $\text{gap}_{\text{min}}$ are nearly tight and the dependency $\frac{1}{1-\gamma}$ can be improved. The proof of Theorem 3.2 is deferred to Appendix.

4 Proof of Theorem 3.1

In this section, we prove Theorem 3.1.

Notations For every variable $X$ maintained by Alg, let $X^k$ denote the value of $X$ right before the $k$-th episode. Let $\mathbb{I}[]$ denote the indicator function. Let $\tau_h(x, a, i) := \max \{k : N_h^k(x, a) = i - 1\}$ be the episode $k$ at which $(x_h^k, a_h^k) = (x, a)$ for the $i$-th time. We will abbreviate $N_h^k(x_h^k, a_h^k)$ for $n_h^k$ when no confusion can arise.

Proof of Theorem 3.1 Our proof starts with the observation that the regret of each episode can be rewritten as the expected sum of sub-optimality gaps for each action:

$$
(V_1^* - V_1^\pi_k)(x_1^k) = V_1^*(x_1^k) - Q_1^*(x_1^k, a_1^k) + (Q_1^* - Q_1^\pi_k)(x_1^k, a_1^k) = \text{gap}_1(x_1^k, a_1^k) + \mathbb{E}_{s' \sim P_1}\left[(V_2^* - V_2^\pi_k)(s')\right] = \cdots = \mathbb{E}\left[\sum_{h=1}^{H} \text{gap}_h(x_h^k, a_h^k) \bigg| a_h = \pi_k(x_h^k)\right].
$$

(1)

Before proceeding to bound \text{gap}_h(x_h^k, a_h^k) by learning error $(Q_h^k - Q_h^*) (x_h^k, a_h^k)$, we refer to Jin et al. [2018] for the following lemma that establishes bounds on the estimation error on $Q$-function via a concentration argument.

Lemma 4.1 (Bounded Learning Error). Let $\beta_i = 4c\sqrt{\frac{H\beta_i}{t}}$. Then the event $\mathcal{E}_{\text{conc}}$, which is defined as

$$
\mathcal{E}_{\text{conc}} := \left\{ (x, a, h, k) : 0 \leq (Q_h^k - Q_h^*)(x, a) \leq \alpha_{n_h}^H H + \sum_{i=1}^{n_h} \alpha_{n_h}^i \left(V_{h+1}^* (x_h^i, a) - V_1^* (x_h^i, a)\right) \leq \beta_h^k\right\},
$$

occurs w.p. at least $1 - \frac{1}{t}$.

Lemma 4.1 suggests optimism holds on $\mathcal{E}_{\text{conc}}$. Combining with the greedy choice of actions yields

$$
V_h^* (x_h^k) = Q_h^* (x_h^k, a^*) \leq Q_h^k (x_h^k, a_h^k) \leq Q_h^k (x_h^k, a_h^k).
$$

(2)

To bound $\text{gap}_h(x_h^k, a_h^k)$, the following notion introduced in Simchowitz and Jamieson [2019] is convenient. If we define $\text{clip}_{[x, \delta]} := x \cdot \mathbb{I}[x \geq \delta]$, then $\text{gap}_h(x_h^k, a_h^k)$ can be bounded by clipped estimation error:

$$
\text{gap}_h(x_h^k, a_h^k) = \text{clip}_{[V_h^* (x_h^k) - Q^* (x_h^k, a_h^k), \text{gap}_{\text{min}}]} \leq \text{clip}_{[(Q_h^k - Q^*) (x_h^k, a_h^k), \text{gap}_{\text{min}}]}.
$$

(3)
As we have discussed in Section 1.1, our main technique to get $1/\text{gap}_{\min}$ instead of $1/\text{gap}_{\min}^2$ regret bound is to classify gaps of state-action pairs to different intervals and count them separately. Note the gap can range from $\text{gap}_{\min}$ to $H$. Thus, we divide the interval $[\text{gap}_{\min}, H]$ into $N$ disjoint intervals: $[\text{gap}_{\min}, 2\text{gap}_{\min}), \cdots, [2^{N-1}\text{gap}_{\min}, 2^N\text{gap}_{\min}]$, where $N = \lceil \log_2 (H/\text{gap}_{\min}) \rceil$.

Lemma 4.2 below is our main technical lemma which upper bounds the number of steps Algorithm 1 chooses a sub-optimal action whose suboptimality is in a certain interval.

Lemma 4.2 (Bounded Number of Steps in Each Interval). Under $E_{\text{conc}}$, we have for every $n \in [N]$, 

$$C^{(n)} := \left\{ (k, h) : (Q^k_h - Q^*) (x^k_h, a^k_h) \in [2^{n-1}\text{gap}_{\min}, 2^n\text{gap}_{\min}) \right\} \leq O\left( \frac{H^6SA \iota}{4^n\text{gap}_{\min}^2} \right)$$

where $\iota = \log (SAT^2)$.

Before we give the proof for Lemma 4.2, we first show how to use Lemma 4.2 to prove Theorem 3.1.

Proof of Theorem 3.1 Since the trajectories inside $E_{\text{conc}}$ have bounded empirical regret, and complementary event $E_{\text{conc}}$ happens with sufficiently low probability,

$$\mathbb{E}[\text{Regret}(K)] = \mathbb{E} \left[ \sum_{k=1}^{K} \sum_{h=1}^{H} \text{gap}_h (x^k_h, a^k_h) \right]
\leq \sum_{\text{traj} \in E_{\text{conc}}} P(\text{traj}) \sum_{h=1}^{H} \sum_{k=1}^{K} \text{clip} \left[ (Q^k_h - Q^*) (x^k_h, a^k_h | \text{traj}) \right] \text{gap}_{\min} + \sum_{\text{traj} \in E_{\text{conc}}} P(\text{traj}) \cdot TH
\leq P(E_{\text{conc}}) \sum_{n=1}^{N} 2^n \text{gap}_{\min} C^{(n)} + P(E_{\text{conc}}) \cdot TH
\leq \sum_{n=1}^{N} O\left( \frac{H^6SA \iota}{2^n\text{gap}_{\min}^2} \right) + \frac{1}{T} \cdot TH \leq O\left( \frac{H^6SA \iota}{\text{gap}_{\min}^2} \log(SAT) \right). \quad (4)$$

Above, ① is because Ineq (2) and ③ show that for trajectories inside $E_{\text{conc}}$, gaps can be bounded by clipped learning error; whereas for trajectories outside of $E_{\text{conc}}$, sub-optimality gaps never exceed $H$. ② follows from adding an outer summation for state-action pairs over the $N$ disjoint subintervals, then bounding the estimation error in each subinterval by its maximum value times the number of steps it contains. ③ comes from a sum of numbers in a geometric progression generated by Lemma 4.2 and the fact that $P(E_{\text{conc}}) \leq 1/T$ from Lemma 4.1. In the final step, we notice that $\iota = \log(SAT^2) = O(\log(SAT))$.

Proof of Lemma 4.2 The proof of Lemma 4.2 relies on a general lemma characterizing a weighted sum of the estimation error of $Q$-function in terms of the properties of this sequence of weights. Then we choose a particular sequence of weights to prove Lemma 4.2. We remark that this general idea has appeared in [Jin et al. 2018], [Dong et al. 2019], [Zhang et al. 2020].

Formally, we use the following definition.
Definition 4.1 ((C,w)-Sequence (Definition 3 in Dong et al. [2019])). A sequence \( \{w_k\}_{k \geq 1} \) is called a \((C,w)\)-sequence if \( 0 \leq w_k \leq w \) for all \( k \) and \( \sum_k w_k \leq C \).

Using the properties of \( \alpha_t \), we can prove the following lemma. The proof of Lemma 4.3 is rather technical, so we defer it to the appendix.

Lemma 4.3 (Weighted Sum of Learning Errors). On event \( \mathcal{E}_{\text{conc}} \), the following holds for every \( h \in [H] \) and \((C,w)\)-sequence \( \{w_{k,h}\}_{k \in [K]} \):

\[
\sum_{k=1}^{K} w_{k,h} (Q_h^k - Q_h^*) \left( x_h^k, a_h^k \right) \leq ewSH^2 + 10c\sqrt{ewSACH^5t}.
\]

With Lemma 4.3, we can easily prove Lemma 4.2 by choosing a particular \((C,w)\)-sequence.

Proof of Lemma 4.2. For every \( n \in [N] \) and \( h \in [H] \), let

\[
w_{k,h}^{(n)} := \indic \left( (Q_h^k - Q_h^*) \left( x_h^k, a_h^k \right) \in [2^{n-1}\text{gap}_{\min}, 2^n\text{gap}_{\min}] \right),
\]

\[
C_{h}^{(n)} := \sum_{k=1}^{K} w_{k,h}^{(n)} = \left| \left\{ k : (Q_h^k - Q_h^*) \left( x_h^k, a_h^k \right) \in [2^{n-1}\text{gap}_{\min}, 2^n\text{gap}_{\min}] \right\} \right|.
\]

By definition, \( \{w_{k,h}^{(n)}\}_{k \in [K]} \) is a \((C_{h}^{(n)},1)\)-sequence. Combining lemma 4.3 and the definition of \( w_{k,h}^{(n)} \) we have:

\[
(2^{n-1}\text{gap}_{\min}) \cdot C_{h}^{(n)} \leq \sum_{k=1}^{K} w_{k,h}^{(n)} (Q_h^k - Q_h^*) \left( x_h^k, a_h^k \right) \leq eSAH^2 + 10c\sqrt{eSAC_{h}^{(n)}H^5t}.
\]

Solving inequality (7) for \( C_{h}^{(n)} \), we obtain \( C_{h}^{(n)} \leq O \left( \frac{H^5SA\text{gap}_{\min}}{4^n} \right) \). Finally, taking summation \( C^{(n)} = \sum_{h=1}^{H} C_{h}^{(n)} \) proves the lemma.

5 Conclusion and Future Directions

This paper gives the first logarithmic regret bound for \( Q \)-learning in tabular RL. Below we list some future directions.

H dependence The dependency on \( H \) in our regret bound for episodic RL is \( H^6 \), which we believe is suboptimal. As discussed in Simchowitz and Jamieson [2019], improving the \( H \) dependence is often a challenging task. Recently, Zhang et al. [2020] showed a model-free algorithm can achieve near-optimal regret in the worst case using the idea of reference value function. It would be interesting to apply this idea to improve the \( H \) dependence in our logarithmic regret bound.
Function Approximation  Lastly, we note that recently researchers found the sub-optimality gap assumption is crucial for dealing with large state-space RL problems where function approximation is needed. Du et al. [2019c] presented an algorithm that enjoys polynomial sample complexity if there is a sub-optimality gap and the environment satisfies a low-variance assumption. Du et al. [2019a, 2020] further showed this assumption is necessary in certain settings. There is another line of works putting certain low-rank assumptions on MDPs [Krishnamurthy et al., 2016, Jiang et al., 2017, Dann et al., 2018, Du et al., 2019b, Sun et al., 2018, Misra et al., 2019]. It would be interesting to extend our analysis to these settings and obtain logarithmic regret bounds.

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A Proofs for Episodic MDP

Before presenting the proof of Lemma 4.3, we refer the readers to Jin et al. [2018] for Lemma A.1 below, which summarizes the properties of $\alpha_t^i$ that will be useful in our proof.

**Lemma A.1** (Properties of $\alpha_t^i$). Let $\alpha_t = \frac{H+1}{H-t}$, $\alpha^0_t = \prod_{j=1}^t (1-\alpha_j)$ and $\alpha_t^i = \alpha_t \prod_{j=i+1}^t (1-\alpha_j)$.

(i) $\sum_{i=1}^t \alpha_t^i = 1$ and $\alpha^0_t = 0$ for every $t \geq 1$, $\sum_{i=1}^t \alpha_t^i = 0$ and $\alpha_t^0 = 1$ for $t = 0$.

(ii) $\sum_{i=1}^\infty \alpha_t^i = 1 + \frac{1}{H}$ for every $i \geq 1$.

**Proof of Lemma A.1.** We will recursively bound the weighted sum of step $h$ by its next step $(h+1)$, and unroll for $(H-h+1)$ steps for the desired bound. As suggested by Lemma 4.1, upper bounds of learning error holds under $\mathcal{E}_{\text{conc}}$. Thus we have

$$
\sum_{k=1}^K w_{k,h} \left( Q_h^k - Q_{h+1}^k \right) \left( x_h^k, a_h^k \right)
\leq \sum_{k=1}^K w_{k,h} \left( \alpha^0_{n_h^k} H + \sum_{i=1}^{n_h^k} \alpha_{t_h^k}^i \left( V_{h+1}(s,a,i) - V^* \right) \left( x_{h+1}^i, a_{h+1}^i \right) + \beta_{n_h^k} \right)
= \sum_{k=1}^K w_{k,h} H + \sum_{k=1}^K w_{k,h} \beta_{n_h^k} + \sum_{k=1}^K \left( V_{h+1}^k - V_{h+1}^* \right) \left( x_{h+1}^k, a_{h+1}^k \right) \sum_{i=1}^{n_h^k+1} \alpha_{t_h^k}^i \left( V_{h+1}(s,a,i), h \right)
\leq SAHw + \sum_{k=1}^K w_{k,h} \beta_{n_h^k} + \sum_{k=1}^K w_{k,h+1} \left( Q_{h+1}^k - Q_{h+1}^* \right) \left( x_{h+1}^k, a_{h+1}^k \right),
$$

The second term of Ineq (8) can be bounded from above in the following way:

$$
\sum_{k=1}^K w_{k,h} \beta_{n_h^k} = \sum_{s,a} \sum_{k=1}^K w_{k,h} \beta_{n_h^k} \theta(s,a) = 4c\sqrt{H^3t} \sum_{s,a} \sum_{i=1}^{N_h^k(s,a)} \frac{w_{r(s,a,i),h}}{\sqrt{t}} \left( \beta_t = 4c \sqrt{\frac{H^3t}{l}} \right)
\leq 4c\sqrt{H^3t} \sum_{s,a} \sum_{i=1}^{\left[ \frac{c^h_s}{\sqrt{t}} \right]} \frac{w}{\sqrt{t}} \left( \text{Rearrangement inequality, } C^h_{s,a} := \sum_{i=1}^{\left[ \frac{c^h_s}{\sqrt{t}} \right]} w_{r(s,a,i),h} \right)
\leq 10c\sqrt{H^3t} \sum_{s,a} \sqrt{C^h_{s,a}} \leq 10c\sqrt{SACwBH^3t}. \quad (9)
$$
Algorithm 2: Infinite Q-learning with UCB-Hoeffding

1: **Initialized:** $Q(x,a) \leftarrow \frac{1}{1-\gamma} \text{ and } N(x,a) \leftarrow 0 \text{ for all } (x,a) \in \mathcal{S} \times \mathcal{A}$.

2: **Define** $\epsilon(k) \leftarrow \log (SAT(k + 1)(k + 2))$, $H \leftarrow \frac{\ln(2)}{\ln(1/\gamma)} \text{ and } \alpha_k = \frac{H+1}{H+k}$.

3: **for** step $t \in [T]$ **do**

4: Take action $a_t \leftarrow \arg\max_a Q(x_t,a)$, observe $x_{t+1}$.

5: $k = N(x_t,a_t) \leftarrow N(x_t,a_t) + 1$, \hspace{1cm}  \triangleright \ c_2 \text{ is a constant that can be set to } 4\sqrt{2}.

6: $b_k \leftarrow \frac{C}{1-\gamma} \sqrt{H^t(k)/k}$,

7: $\hat{V}(x_{t+1}) \leftarrow \max_{a' \in \mathcal{A}} \tilde{Q}(x_{t+1}, a')$,

8: $Q(x_t,a_t) \leftarrow (1-\alpha_k) Q(x_t,a_t) + \alpha_k \left[r(x_t,a_t) + b_k + \gamma \hat{V}(x_{t+1})\right]$,

9: $\tilde{Q}(x_t,a_t) \leftarrow \min\left\{\tilde{Q}(x_t,a_t), Q(x_t,a_t)\right\}$.

For the third term in Ineq (8), we notice that $V^k_{h+1}(x^k_{h+1}) = Q^k_{h+1}(x^k_{h+1}, a^k_{h+1})$ due to greedy choice of actions and $V^*_{h+1}(x^k_{h+1}) \geq Q^*_{h+1}(x^k_{h+1})$ by definition. For the weights we let

$$w_{k,h+1} = \sum_{i=n^k_h+1}^{N^k_h(x^k_{h+1}, a^k_{h+1})} w_{\tau_h(x^k_{h+1}, a^k_{h+1}, i), h} n^k_h.$$ 

It then follows directly from Lemma 4.1 that \{w_{k,h+1}\}_{k \in [K]} is a $(C, (1+1/H)w)$-sequence, as derived below.

$$w_{k,h+1} \leq w \sum_{i=n^k_h}^{N^k_h(x^k_{h+1}, a^k_{h+1})} \frac{n^k_i}{w_{\tau_h(x^k_{h+1}, a^k_{h+1}, i), h}} \leq \left(1 + \frac{1}{H}\right) w, \quad \sum_{k=1}^{K} w_{k,h+1} = \sum_{k=1}^{K} \sum_{i=1}^{n^k_h} \alpha_i n^k_h = \sum_{i=1}^{K} w_{i,h} \leq C. \quad (10)$$

Plugging Ineq (9) and (10) back to Ineq (8) gives

$$\sum_{k=1}^{K} w_{k,h} \left(Q^k_h - Q^*_h\right) x^k_h, a^k_h \right) \leq SAH w + 10c \sqrt{SACwH^31} + \sum_{k=1}^{K} w_{k,h+1} \left(Q^k_{h+1} - Q^*_h\right) x^k_{h+1}, a^k_{h+1} \right), \quad (11)$$

where the third term is a weighted sum of learning errors in level $h+1$, with weights \{w_{k,h+1}\}_{k \in [K]} being $(C, (1+1/H)w)$-sequence. Recursing this argument for $h+1, h+2, \cdots, H$ yields

$$\sum_{k=1}^{K} w_{k,h} \left(Q^k_h - Q^*_h\right) x^k_h, a^k_h \right) \leq \sum_{h' \geq 0}^{H-h} \left(SAH \left(1+1/H\right)^{h'} w + 10c \sqrt{SAC \left(1+1/H\right)^{h'} wH^3}\right) \leq H \left(SAH w + 10c \sqrt{SACwH^3}\right). \quad (12)$$

which is the desired conclusion.

\[\square\]

### B Algorithm for Discounted MDP

The pseudocode is listed in Algorithm 2.
C Proofs for Discounted MDP

Proof of Theorem 3.2 We shall decompose the regret of each step as the expected sum of discounted gaps using the exact same argument as Eq (1), where the expect runs over all the possible infinite-length trajectories taken by Algorithm 2:

\[(V^* - V^{\pi_t})(s_t) = \mathbb{E} \left[ \sum_{h=0}^{\infty} \gamma^h \text{gap}(x_{t+h}, a_{t+h}) \right] \mid a_{t+h} = \pi_t(s_{t+h}) \].

(13)

Based on this expression, the expected total regret over first \(T\) steps can be rewritten as

\[ \mathbb{E}[\text{Regret}(T)] = \mathbb{E}\left[ \sum_{t=1}^{T} (V^* - V^{\pi_t})(x_t) \right] = \mathbb{E}\left[ \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{h=0}^{\infty} \gamma^h \text{gap}(x_{t+h}, a_{t+h}) \right] \right] \]

\[= \mathbb{E}\left[ \sum_{t=1}^{\infty} \sum_{h' = t}^{\infty} \gamma^{h' - t} \text{gap}(x_{h'}, a_{h'}) \right] \tag{14} \]

Our next lemma is borrowed from Dong et al. [2019], which shows that Algorithm 2 satisfies optimism and bounded learning error with high probability. By abuse of notation, we still use \(E_{\text{conc}}\) to denote the successful concentration event in this setting. Recall that Algorithm 2 specifies \(\iota(t) = \log (SAT(t + 1)(t + 2))\) and \(\beta_t = c\frac{\theta}{\gamma} \sqrt{H_i(t)}\).

Lemma C.1 (Bounded Learning Error). Under Algorithm 2, event \(E_{\text{conc}}\) occurs w.p. at least \(1 - \frac{1}{2T}:

\[E_{\text{conc}} := \left\{ \forall (x,a,t) \in S \times A \times \mathbb{N}_+: 0 \leq (\hat{Q}^t - Q^*) (x,a) \leq (Q^t - Q^*) (x,a) \right\} \]

\[\leq \frac{\alpha_0}{1 - \gamma} \sum_{i=1}^{n} \gamma \alpha_i (V^* - V^t) (x_t, a_t) + \beta_t \right\}. \]

Then we proceed to present an analog of Lemma 4.3 that bounds the weighted sum of learning error in the discounted setting.

Lemma C.2 (Weighted Sum of Learning Errors). Under \(E_{\text{conc}}\), for every \((C,w)\)-sequence \(\{w_t\}_{t\geq 1}\), the following holds.

\[\sum_{t\geq 1} w_t (\hat{Q}^t - Q^*) (x_t, a_t) \leq \frac{HC}{1 - \gamma} + O \left( \frac{\sqrt{SAH_i(C)} + wSA}{(1 - \gamma)^2} \right) \tag{15} \]

Proof. Recall that Lemma C.1 bounds the learning error \((\hat{Q}^t - Q^*) (x_t, a_t)\) on \(E_{\text{conc}}\). Thus we have:

\[\sum_{t\geq 1} w_t \frac{\alpha_d^t}{1 - \gamma} \leq \sum_{t\geq 1} [n' = 0] \cdot \frac{w}{1 - \gamma} = \frac{SAw}{1 - \gamma}; \tag{16} \]

\[\sum_{t\geq 1} w_t \beta_{n'} = \sum_{s,a} \sum_{i=1}^{N^i(s,a)} w_{\gamma(s,a,i)} \beta_i = c\frac{\sqrt{H_i}}{1 - \gamma} \sum_{s,a} \sum_{i}^{N^i(s,a)} w_{\gamma(s,a,i)} \sqrt{\frac{i(i)}{i}} \]

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Eq (10). The second term of Ineq (18) comes from the following observation:

Moreover,

Moreover,

\[
\sum_{t \geq 2} \sum_{i=1}^{n_t} \gamma \alpha_{it}^i (\hat{\nu}(x,a,i) - V^*) (x_{t*(x,a,i)})^2 \left( \sum_{i \geq n_t+1} w_{\gamma(x,a,i)} \alpha_{it}^i \right)
= \gamma \sum_{t \geq 2} w_t \left( \hat{\nu}^t - V^* \right)(x_t) + \gamma \sum_{t \geq 2} w_{t+1} \left( \hat{\nu}^t - \hat{\nu}^{t+1} \right)(x_t)
\leq \gamma \sum_{t \geq 2} w_t \left( \hat{\nu}^t - Q^* \right)(x_t) + \frac{\gamma(1 + 1/H)wS}{1 - \gamma}.
\] (18)

It can be easily verified that \( \{ w_t' \}_{t \geq 2} \) is a \( (C, (1 + 1/H)w) \)-sequence, using a similar argument to Eq (10). The second term of Ineq (18) comes from the following observation:

\[
\gamma \sum_{t \geq 2} w_{t+1} \left( \hat{\nu}^t - \hat{\nu}^{t+1} \right)(x_t) \leq \gamma (1 + 1/H)w \sum_s \sum_{t \geq 1} (\hat{\nu}^t - \hat{\nu}^{t+1})(s) \leq \frac{\gamma(1 + 1/H)wS}{1 - \gamma}.
\]

Plugging Ineq (16), (17) and (18) back into \( \sum_{t \geq 1} w_t (\hat{Q}^t - Q^*)(x_t, a_t) \), we obtain

\[
\sum_{t \geq 1} w_t \left( \hat{Q}^t - Q^* \right)(x_t, a_t)
\leq \sum_{t \geq 1} w_t \left( \frac{\alpha_0^t}{1 - \gamma} + \beta_{t+1} + \gamma \sum_{i=1}^{n_t} \alpha_{it}^i (\hat{\nu}^{(s,a,i)} - V^*) (x_{t*(s,a,i)}) \right) \leq \frac{SAw}{1 - \gamma} + \frac{2c_3}{1 - \gamma} \sqrt{SAH C w_t(C)} + \frac{\gamma(1 + 1/H)wS}{1 - \gamma} + \gamma \sum_{t \geq 2} w_t \left( \hat{Q}^t - Q^* \right)(x_t, a_t)
\] (19)

Note that the last term in Ineq (19) is another weighted sum of learning errors starting from step 2, where the weights form a \( (C, (1 + 1/H)w) \)-sequence. We can therefore repeat this unrolling argument for \( H \) times. This choice of \( H \) guarantees not only the bounded blow-up factor of weights, but also sufficiently small contribution of learning error after step \( H \).

\[
\sum_{t \geq 1} w_t \left( \hat{Q}^t - Q^* \right)(x_t, a_t)
\leq \sum_{h=0}^{H} \gamma^h C \left( \frac{\sqrt{wSAH C_t(C) + wSA}}{1 - \gamma} \right) + \gamma^H \sum_{t \geq H+1} w_t \left( \hat{Q}^t - Q^* \right)(x_t, a_t)
\]

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\[ \text{Proof.} \quad \text{For every} \quad C \quad \text{then} \quad \sum_{t \geq H+1} w_t^{(H)}. \]

Using the fact that the weights after \( H \) unrolling \( \{w_t^{(H)}\} \) is a \((C, (1+1/H)^H w \leq \epsilon w)\)-sequence completes the proof. \( \square \)

Note that we have clarified in Ineq (3) that on \( \mathcal{E}_{\text{conc}} \) where optimism holds, sub-optimality gaps can be bounded by clipped learning error of \( Q \)-function. Again we divide its range \([\text{gap}_{\min}, \frac{1}{1-\gamma}]\) into disjoint subintervals and bound the sum inside each subinterval independently.

**Lemma C.3.** Let \( N = \lceil \log_2 \left( \frac{1}{\text{gap}_{\min}(1-\gamma)} \right) \rceil \). On \( \mathcal{E}_{\text{conc}} \), for every \( n \in [N] \),

\[ C^{(n)} := \left\{ t \in \mathbb{N}_+ : \left( Q^t - Q^* \right)(x_t, a_t) \in [2^{n-1}\text{gap}_{\min}, 2^n\text{gap}_{\min}] \right\} \]

\[ \leq \mathcal{O} \left( \frac{SA}{4^n\text{gap}_{\min}^2 (1-\gamma)^\delta \ln \left( \frac{SAT}{(1-\gamma)\text{gap}_{\min}} \right)} \right). \]

Again, based on Lemma C.1 we prove Lemma C.3 by choosing a particular sequence of weights.

**Proof.** For every \( n \in [N] \), let

\[ w_t^{(n)} := I[(Q^t - Q^*)(x_t, a_t) \in [2^{n-1}\text{gap}_{\min}, 2^n\text{gap}_{\min}]], \]

then \( C^{(n)} = \sum_{t=1}^\infty w_t^{(n)} \) and \( \{w_t^{(n)}\} \) is a \((C^{(n)}, 1)\)-sequence. According to Lemma C.2

\[ (2^{n-1}\text{gap}_{\min}) \cdot C^{(n)} \leq \sum_{t \geq 1} w_t^{(n)} (Q^t - Q^*)(x_t, a_t) \]

\[ \leq \frac{\gamma H C^{(n)}}{1-\gamma} + \mathcal{O} \left( \frac{\sqrt{SHC^{(n)}t(C^{(n)})} + SA}{(1-\gamma)^2} \right) \]

\[ = \frac{\text{gap}_{\min}}{2} C^{(n)} + \mathcal{O} \left( \frac{\sqrt{SHC^{(n)}t(C^{(n)})} + SA}{(1-\gamma)^2} \right). \]

Now we proceed to solve the above inequality for \( C^{(n)} \). For simplicity, let \( \delta = 2^{n-2}\text{gap}_{\min} \) and \( C^{(n)} = SAC' \). Then we have the following:

\[ \delta \cdot SAC' \leq \left( 2^{n-1} - \frac{1}{2} \right) \text{gap}_{\min} C^{(n)} \leq \mathcal{O} \left( \frac{SA}{\sqrt{HC^{(n)}t(SAC') + 1}} \right), \]

\[ \frac{\delta C'}{(1-\gamma)^\delta} \leq \mathcal{O} \left( \frac{\sqrt{C'}}{(1-\gamma)^5 \ln (SATC')} \left[ \frac{1}{\text{gap}_{\min} (1-\gamma) \ln (SATC')} \right] \right), \]

\[ C' \leq \mathcal{O} \left( \frac{1}{\delta^2 (1-\gamma)^5 \ln (SATC')} \right), \]

\[ \text{(22)} \]

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where \(1\) comes from the definition \(H = \frac{\ln(2/\text{gap}_{\text{min}}(1-\gamma))}{\ln(1/\gamma)}\). Solving Ineq (22) yields

\[
C' \leq \frac{1}{\delta^2 (1-\gamma)^6} \ln \left( \frac{S_A}{\text{gap}_{\text{min}}(1-\gamma)} \right).
\]

Finally, substituting \(C(n) = S_A'\) and \(\delta = 2^{-2}\text{gap}_{\text{min}}\) yields the desired formula.

**Proof of Theorem 3.2** We continue the calculation based on the regret decomposition in Eq (14). For every infinite trajectory \(\text{traj} \in \mathcal{E}_{\text{conc}}\),

\[
\sum_{t=1}^{T} \sum_{h'=t}^{\infty} \gamma^{h'-t} \text{gap}(x_{h'}, a_{h'}) \equiv \sum_{h=1}^{\infty} \text{gap}(x_h, a_h) \sum_{t=1}^{\min\{T, h\}} \gamma^t \leq \frac{1}{1-\gamma} \sum_{h=1}^{\infty} \text{gap}(x_h, a_h)
\]

\[
\leq \frac{1}{1-\gamma} \sum_{t \geq 1} \text{clip} \left[ (Q^t - Q^*) (x_t, a_t) \right] \text{gap}_{\text{min}}
\]

\[
\leq \frac{1}{1-\gamma} \sum_{n=1}^{N} 2^n \text{gap}_{\text{min}} C(n)
\]

\[
\leq \mathcal{O} \left( \frac{S_A}{\text{gap}_{\text{min}}(1-\gamma)^6} \ln \left( \frac{S_A}{p\epsilon(1-\gamma)\text{gap}_{\text{min}}} \right) \right). \tag{23}
\]

For the above inequalities, \(2\) comes from an interchange of summations, \(3\) is by optimism of estimated \(Q\)-values, \(4\) is because we can add an outer summation over subintervals \(n \in [N]\) and bound each of them by their maximum value times the number of steps inside. Finally, \(5\) follows directly from Lemma C.3.

On the other hand, for trajectories outside of \(\mathcal{E}_{\text{conc}}\), since sub-optimality gaps are upper bounded by \(1/1-\gamma\), we have:

\[
\sum_{t=1}^{T} \sum_{h'=t}^{\infty} \gamma^{h'-t} \text{gap}(x_{h'}, a_{h'}) \leq \sum_{t=1}^{T} \sum_{h'=t}^{\infty} \gamma^{h'-t} \frac{1}{1-\gamma} \leq \frac{T}{(1-\gamma)^2}. \tag{24}
\]

Therefore, combining Ineq (23) and (24) gives us

\[
\mathbb{E}[\text{Regret}(T)] = \mathbb{E} \left[ \sum_{t=1}^{T} \sum_{h'=t}^{\infty} \gamma^{h'-t} \text{gap}(x_{h'}, a_{h'}) \right]
\]

\[
\leq \mathbb{P}(\mathcal{E}_{\text{conc}}) \cdot \mathcal{O} \left( \frac{S_A}{\text{gap}_{\text{min}}(1-\gamma)^6} \ln \left( \frac{S_A}{(1-\gamma)\text{gap}_{\text{min}}} \right) \right) + \mathbb{P}(\mathcal{E}_{\text{conc}}) \cdot \frac{T}{(1-\gamma)^2}
\]

\[
\leq \mathcal{O} \left( \frac{S_A}{\text{gap}_{\text{min}}(1-\gamma)^6} \ln \left( \frac{S_A}{(1-\gamma)\text{gap}_{\text{min}}} \right) \right), \tag{25}
\]

where the last step is comes from \(\mathbb{P}(\mathcal{E}_{\text{conc}}) \leq 1/2T\). Ineq (25) is precisely the assertion of Theorem 3.2. \(\square\)
D Difficulty in Applying Optimistic Surplus

The closest related work is by Simchowitz and Jamieson [2019] who proved the logarithmic regret bound for a model-based algorithm. Simchowitz and Jamieson [2019] introduced a novel property characterizing optimistic algorithms, which is called *optimistic surplus* defined as

\[
E_{k,h}(x,a) := Q^k_h(x,a) - \left[ r_h(x,a) + P_h(x,a)^T V^k_{h+1} \right].
\]

(26)

Under model-based algorithm with bonus term \( b^k_h \), surplus can be decomposed as follows, where \( \hat{P} \) is the estimated transition probability:

\[
E_{k,h}(x,a) = \left( \hat{P}_h^T(x,a) - P_h^T(x,a) \right) V^*_h + \left( \hat{P}_h^T(x,a) - P_h^T(x,a) \right) \left( V^k_{h+1} - V^*_h \right) + b^k_h.
\]

The analysis of model-based algorithms is to first bound the regret \((V^* - V^{\pi_k})\) by a sum over surpluses that are clipped to zero whenever being smaller than some gap-related quantities, then combine the concentration argument and properties of specially-designed bonus terms \( b^k_h \) to provide high probability bound for surpluses. However, for model-free algorithms, estimates of transition probabilities are no longer maintained, so \( \hat{P}_h \) is a one-hot vector reflecting only the current step’s empirical sample drawn from the real next-state distribution. In this scenario, concentration argument of \( \left( \hat{P} - P \right) \) cannot give us \( \log T \) regret.

Following the update rule of \( Q \)-learning with learning rate \( \alpha_i \) and upper confidence bound \( b_i \), the surplus becomes

\[
E_{k,h}(x,a) = \alpha_i^0 H + \left( \sum_{i=1}^{t} \alpha_i V_{h+1}^{k_i}(x_{h+1}^{k_i}) - P_h(x,a)^T V^k_{h+1} \right) + \sum_{i=1}^{t} \alpha_i b_i,
\]

in which \( t = n_h^k(x,a) \) is the number of times \((x,a)\) has been visited, and \( \alpha_i = \alpha \prod_{j=i+1}^{t} (1 - \alpha_j) \) is the equivalent weight associated with the \( i \)-th visit of pair \((x,a)\). This indicates that the surplus of an episode is closely correlated with estimates of value functions during previous episodes. The correlation makes the analysis more difficult. Therefore, we use a very different approach to analyze \( Q \)-learning in this paper.