FACT SHEET RESEARCH ON BAYESIAN DECISION THEORY

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Abstract. In this fact sheet we give some preliminary research results on the Bayesian Decision Theory. This theory has been under construction for the past two years. But what started as an intuitive enough idea, now seems to have the makings of something far more fundamental.

1. Introduction

Thanks to the endeavors of Knuth and Skilling, it has been shown that the product and sum rules of both the Bayesian probability and the Bayesian information theories are derivable by consistency constraints on the lattices of, respectively, statements and questions, [14, 41, 58]; the implication being that in our plausibility and relevancy judgments we humans have a preference for consistency, or, equivalently, rationality. Moreover, Knuth is now researching if the very laws of Nature themselves may be derived by way of consistency constraints on lattices of events, [43].

What we now perceive to be the laws of physics are, in the final analysis, nothing more than conjectures. These conjectures have obtained the status of laws because of, on one hand, their close correspondence with empirical fact, and, on the other hand, their power to predict physical phenomena other than the ones that guided us to these conjectures in the first place. Knuth, and his MaxEnt-colleague, are now in the process of deriving the theorems of Nature from, what then would be, the primary first principle of Nature itself, namely consistency.

In light of both these exciting new developments and the fact that these authors, after two years of continuous research, have reached the point that they have come to trust their Bayesian decision theory, to almost the same extent as they have grown to trust the Bayesian probability and information theories; these authors have come to entertain the notion that maybe their Bayesian decision theory, which initially started as an intuitive enough Bayesian alternative for the paradigm of behavioral economics, might actually be Bayesian in the strictest sense of the word; that is, an inescapable consequence of the desideratum of consistency.

1MaxEnt-Bayesians are, as a rule, physicists that trace their statistical lineage from Jaynes, back to Jeffreys, back to Laplace.
2The former being their field of expertise, and the latter being the subject matter of the first author’s current thesis work.
In this fact sheet we will present the case for the Bayesian decision theory, that led us to this daring conjecture, together with the lattice theoretical proof of the Bernoulli law, also known as the Weber-Fechner law of sense perception, which, in our mind, would seem to be the only possible degree of freedom in the Bayesian decision theoretical algorithm.

2. THE BAYESIAN DECISION THEORY

The Bayesian decision theory is very simple in structure. Its algorithmic steps are the following:

1. Use the product and sum rules of Bayesian probability theory to construct outcome probability distributions.

2. If our outcomes are monetary in nature, then by way of the Bernoulli law we may map utilities to the monetary outcomes of our outcome probability distributions.

3. Maximize the sum of the lower and upper bounds of the resulting utility probability distributions.

This, then, is the whole of the Bayesian decision theory.

3. CONSTRUCTING OUTCOME PROBABILITY DISTRIBUTIONS

In the Bayesian decision theory each problem of choice is understood to consist of a set of decisions from which we must choose. Each possible decision, when taken, has its own set of possible outcomes, and each outcome, for a given decision, has its own plausibility of occurring relative to the other outcomes under that same decision. So, each decision in our problem of choice admits its own outcome probability distribution.

We will demonstrate in this section how to construct outcome probability distributions using the rules of Bayesian probability theory. In our hypothetical problem of choice, the possible decisions $D_i$, for $i = 1, 2$, under consideration are whether or not to wear seat belts:

$$ D_1 = \text{Seat belts}, \quad D_2 = \text{No Seat belts}. $$

3 For we share Jaynes’ weary and wariness, when he states, [29]: “[W]e have seen enough ambitious but short-lived efforts with the generic title: ‘A New Foundation For . . .’ to become a bit weary of them. And we have seen enough putative ‘foundations’ develop a fluid character unlike real foundations and themselves to the unyielding practical realities, to become a bit wary of them.”

4 Or, as Bernoulli called them, moral values, [6].

5 See also Appendix A.
The relevant events $E_j$, for $j = 1, 2, 3$, when driving a car, as perceived by the decision maker, are

$$E_1 = \text{No Accident}, \quad E_2 = \text{Small Accident}, \quad E_3 = \text{Severe Accident}.$$ 

The perceived outcomes $O_k$, for $k = 1, 2, 3$, are

$$O_1 = \text{No Harm}, \quad O_2 = \text{Some Bruises}, \quad O_3 = \text{Broken Bones}.$$ 

For this particular case, the decisions taken do not modulate the probabilities of an event. So, we have that the probability for an event conditional on the decision taken is the same for both decisions, say:

$$P(E_1 | D_i) = 0.950, \quad P(E_2 | D_i) = 0.049, \quad P(E_3 | D_i) = 0.001, \quad (3.1)$$

for $i = 1, 2$. However, the conditional probability distributions of the outcomes given an event are modulated by the decision taken.

We first consider the case where the decision maker is considering to wear seat belts, that is, $D_1$. Say, we have the following conditional probabilities:

$$P(O_1 | E_1, D_1) = 1.00, \quad P(O_2 | E_1, D_1) = 0.00, \quad P(O_3 | E_1, D_1) = 0.00,$$

$$P(O_1 | E_2, D_1) = 0.75, \quad P(O_2 | E_2, D_1) = 0.25, \quad P(O_3 | E_2, D_1) = 0.00,$$

$$P(O_1 | E_3, D_1) = 0.20, \quad P(O_2 | E_3, D_1) = 0.70, \quad P(O_3 | E_3, D_1) = 0.10. \quad (3.2)$$

Then by way of the product rule, [30],

$$P(A) P(B | A) = P(AB) = P(B) P(A | B), \quad (3.3)$$

we may combine the probability of an event, (3.1), with the corresponding conditional probability distributions of some outcome given that event, (3.2), and obtain the probabilities of an event $E_j$ and an outcome $O_k$ given decision $D_i$:

$$P(E_j, O_k | D_i) = P(E_j | D_i) P(O_k | E_j, D_i). \quad (3.4)$$

We may present all these probabilities (3.4) in a table and so get the corresponding bivariate probability distribution, Table 1.
\[ P(E_j, O_k | D_1) \]

| \( E_1 = \text{No Accident} \) | \( O_1 = \text{No Harm} \) | \( O_2 = \text{Some Bruises} \) | \( O_3 = \text{Broken Bones} \) |
|--------------------------|-----------------|-----------------|-----------------|
| 0.9500                  | 0.0000          | 0.0000          | 0.0000          |
| \( E_2 = \text{Small Accident} \) | 0.0370          | 0.0120          | 0.0000          |
| \( E_3 = \text{Severe Accident} \) | 0.0002          | 0.0007          | 0.0001          |

Table 1. Bivariate event-outcome probability distribution for \( D_1 \)

Let \( A_i = \{A_1, \ldots, A_n\} \) be a set of \( n \) mutually exclusive and exhaustive propositions, that is, one and only one of the \( A_i \) is necessarily true. Let \( B_j = \{B_1, \ldots, B_m\} \) be another set of \( m \) mutually exclusive and exhaustive propositions. Then, by way of the generalized sum rule, \( [30] \), we have

\[
\sum_{i=1}^{n} P(A_i, B_j) = P(A_1, B_j) + \cdots + P(A_n, B_j) = P(B_j), \tag{3.5}
\]

where \( \sum_j P(B_j) = 1 \). Using this generalized sum rule, we may ‘marginalize’ the event-outcome probabilities \( P(E_j, O_k | D_1) \) over the events \( E_j \), that is,

\[
P(O_k | D_i) = \sum_{j=1}^{m} P(E_j, O_k | D_i), \tag{3.6}
\]

and so get the marginalized outcome probability distribution, Table 2.

| \( O_1 = \text{No Harm} \) | \( O_2 = \text{Some Bruises} \) | \( O_3 = \text{Broken Bones} \) |
|-----------------|-----------------|-----------------|
| 0.9872          | 0.0127          | 0.0001          |

Table 2. Marginalized outcome probability distribution for \( D_1 \)
We now consider the case where the decision maker is considering not to wear seat belts, that is, \( D_2 \). Say we have the following conditional probabilities:

\[
\begin{align*}
P(O_1 | E_1, D_2) &= 1.00, & P(O_2 | E_1, D_2) &= 0.00, & P(O_3 | E_1, D_2) &= 0.00, \\
P(O_1 | E_2, D_2) &= 0.25, & P(O_2 | E_2, D_2) &= 0.75, & P(O_3 | E_2, D_2) &= 0.00, \\
P(O_1 | E_3, D_2) &= 0.10, & P(O_2 | E_3, D_2) &= 0.30, & P(O_3 | E_3, D_2) &= 0.60.
\end{align*}
\]

Then, using (3.3), we may combine the probability of an event, (3.1), with the corresponding conditional probability distributions of some outcome given that event, (3.7), and so get the corresponding bivariate probability distribution, Table 3.

| \( E_j \) | \( O_1 = \text{No Harm} \) | \( O_2 = \text{Some Bruises} \) | \( O_3 = \text{Broken Bones} \) |
|---|---|---|---|
| \( E_1 = \text{No Accident} \) | 0.9500 | 0.00 | 0.00 |
| \( E_2 = \text{Small Accident} \) | 0.0120 | 0.0370 | 0.00 |
| \( E_3 = \text{Severe Accident} \) | 0.0001 | 0.0003 | 0.0006 |

Table 3. Bivariate event-outcome probability distribution for \( D_2 \)

Marginalizing the event-outcome probabilities \( P(E_j, O_k | D_2) \) over the events \( E_j \), (3.5), we get the marginalized outcome probability distribution, Table 4.

| \( O_k \) | \( O_1 = \text{No Harm} \) | \( O_2 = \text{Some Bruises} \) | \( O_3 = \text{Broken Bones} \) |
|---|---|---|---|
| \( P(O_k | D_1) \) | 0.9621 | 0.0373 | 0.0006 |

Table 4. Marginalized outcome probability distribution for \( D_2 \)

In its most abstract form, we have that each problem of choice consists of a set of potential decisions

\[
D_i = \{ D_1, \ldots, D_n \}.
\]
Each decision $D_i$ we make may give rise to a set of possible events

$$E_{j_i} = \{E_{1i}, \ldots, E_{mi}\}.$$  

These events $E_{j_i}$ are associated with the decisions $D_i$ by way of the conditional probabilities $P(E_{j_i} \mid D_i)$. Furthermore, each event $E_{j_i}$ allows for a set of potential outcomes

$$O_{k_{ji}} = \{O_{1ji}, \ldots, O_{l_{ji}}\}.$$  

These outcomes $O_{k_{ji}}$ are associated with the events $E_{j_i}$ by way of the conditional probabilities $P(O_{k_{ji}} \mid E_{j_i})$.

By way of the product rule, we compute the bivariate probability distribution of an event and an outcome conditional on the decision taken:

$$P(E_{j_i}, O_{k_{ji}} \mid D_i) = P(E_{j_i} \mid D_i) P(O_{k_{ji}} \mid E_{j_i}).$$  

The outcome probability distribution is then obtained by marginalizing, over all the possible events

$$P(O_{k_{ji}} \mid D_i) = \sum_{j_i=1}^{m_i} P(E_{j_i}, O_{k_{ji}} \mid D_i).$$  

The outcome probability distributions, for $i = 1, \ldots, n$, are the information carriers which represent our state of knowledge in regards to the consequences of our decisions.

At first sight the added event space in the abstract form given here may seem somewhat superfluous. But it was felt that further down the line, in decision theoretical problems more complex than the ones given in this fact sheet, this added space may help one in the construction of complex outcome probability distributions.

### 4. Translating Monetary Outcomes to Utilities

The Bernoulli law in the field of psycho-physics is called Weber-Fechner law. Seeing that we will use in this section the psycho-physical point of view of money increments as a stimulus, we will refer in what follows to the Bernoulli law as the Weber-Fechner law. But both names point to the same law.

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6As this event space hardly is used in this fact sheet.

7If we want to collect all the possible outcomes in one probability distribution, under a given decision, then we may first gather the different events, which precede these outcomes, in probability distribution. We foresee that may greatly help in the construction of complex outcome probability distributions; cascading events, event trees, etc...

8Psycho-physics is the experimental field of psychology that studies sense perception.

9In the Appendices B, C, and D we first will discuss the Bernoulli law and its ubiquitousness, then we will give its corollary of the negative Bernoulli law, after which we will proceed to formally derive this law, by way of a consistency quantification on the lattice of ordering; thus, demonstrating why it is that the Bernoulli law is so ubiquitous in the field psycho-physics.
The translation of monetary stimuli to utilities is analogous to the case where we are asked to translate loudness to a numerical value. According to Weber-Fechner law, postulated in the 19th century\footnote{Which is just Bernoulli’s law, postulated in the 18th century; see Appendix~\ref{app:bernoulli}.} by the experimental psychologist Fechner, intuitive human sensations tend to be logarithmic functions of the difference in stimulus.\footnote{\label{footnote:weber}Which is just Bernoulli’s law, postulated in the 18th century; see Appendix~\ref{app:bernoulli}.} So, we do not perceive stimuli in isolation, rather we perceive the relative change in stimuli, case in point being the decibel scale of sound.

Let $S_1$ and $S_2$ be two stimuli which are to be compared. Then the Weber-Fechner law tells us that the Relative Change (RC) is the difference of the logarithms of the stimuli:

$$RC = c \log_d S_2 - c \log_d S_1 = c \log_d \frac{S_2}{S_1},$$

(4.1)

where $c$ is some scaling factor and $d$ some base of the logarithm. From (4.1), we have that if stimuli $S_1$ and $S_2$ are indistinguishable, that is, of the same strength, then their RC is 0. If $S_2$ increases relative to $S_1$, then RC $> 0$. If $S_2$ decreases relative to $S_1$, then RC $< 0$.

The Weber-Fechner law allows for one degree of freedom. This can be seen as follows. Since

$$\log_d x = \frac{\log x}{\log d},$$

we can rewrite (4.1) as

$$RC = q \log \frac{S_2}{S_1},$$

(4.2)

where

$$q = \frac{c}{\log d}.$$  

(4.3)

Let $\Delta S$ be an increment, either positive or negative, in a monetary stimulus $S$. Then we may define the utility of a monetary increment $\Delta S$ to be the perceived relative change in the initial wealth $S$ due to that increment $\Delta S$, (4.2):

$$u(\Delta S | S) = q \log \frac{S + \Delta S}{S}, \quad -S < \Delta S < \infty.$$  

(4.4)

If $\Delta S = -S$, then (4.4) tells us that a loss of all one’s initial wealth $S$ would have a utility of minus infinity. This is clearly not realistic. So, in order to model such a loss, we must introduce the threshold of income which is still significant $\gamma$, \footnote{Which is just Bernoulli’s law, postulated in the 18th century; see Appendix~\ref{app:bernoulli}.}, where $\gamma > 0$. The threshold of income has the following interpretation.

Even for the homeless person there is some minimum amount of money that is still significant. This may be one dollar for a bag of potato chips, or three dollars for a packet of cigarettes. If the loss of money breaks through the limit of the minimum significant amount $\gamma$, the homeless person is left with an amount of money which, for all intents and purposes, is worthless. Using the concept of the threshold of
income, we may modify (4.4) as

\[ u(\Delta S|S) = q \log \frac{S + \Delta S}{S}, \quad -S + \gamma < \Delta S < \infty. \] (4.5)

If we want to give a graphical representation of (4.5), then the scaling constant \( q \), also known as the Weber constant, must be set to some numerical value.

Say, we have a monthly expendable income of a thousand dollars, for groceries and the like, then introspection\(^{11}\) would suggest that a loss or gain of an amount less than ten dollars would not move us that much.

So, \( \Delta S = 10 \) constitutes a just noticeable difference, or, equivalently, 1 utile, for an initial wealth of \( S = 1000 \), (4.5):

\[ 1 \text{ utile} = q \log \frac{1000 + 10}{1000}. \] (4.6)

If we then solve for the unknown Weber constant \( q \), we find

\[ q = \frac{1}{\log 1010 - \log 1000} \approx 100. \] (4.7)

Note that utiles represent the utility of the monetary outcomes, much like decibels represent the perceived intensity of sound\(^{12}\).

Suppose we have a student who has three hundred dollars per month to spend on groceries and the like and who stands to lose or to gain up to two hundred dollars. Then, by way of (4.5) and (4.7), we obtain the following mapping of monetary outcomes to utilities, Figure 1.

\[ Figure 1. \text{ Utility plot for initial wealth 300 dollars} \]

\(^{11}\)Introspection being the starting point of all psychological experimentation.

\(^{12}\)Note that for the decibel scale the Weber constant has been determined to be \( q = 10/\log 10 = 4.34 \).
For the case of the rich man who has one million dollars to spend on groceries and the like and who stands to lose or to gain up to a hundred thousand dollars, we obtain the alternative mapping, Figure 2.

Figure 2. Utility plot for initial wealth 1.000.000 dollars

Loss aversion is the phenomenon that losses may loom larger than gains. Comparing Figures 1 and 2 we see that the Weber-Fechner law of experimental psychology, which is just Bernoulli’s law, captures both the loss aversion of the poor student, that is, asymmetry in gains and losses, as well as the linearity of the utility of relatively small gains and losses for the rich man.

5. The Criterion Of Choice

In this section we will discuss the maximization of the sum of the lower and upper bounds, the third step in the Bayesian decision algorithm, as a criterion of choice. This criterion is highly non-intuitive, as it admits no interpretation. This is because it is a ‘corollary’, or, to be more precise, an algebraic reshuffling, of a criterion which is intuitive, but less succinct in its expression.

Let $D_1$ and $D_2$ be two decisions we have to choose from. Let $O_i$, for $i = 1, \ldots, n$, and $O_j$, for $j = 1, \ldots, m$, be the monetary outcomes associated with, respectively, decisions $D_1$ and $D_2$.

In the Bayesian decision analysis, we first construct the two outcome distributions that correspond with these decisions:

$$P(O_i | D_1), \quad P(O_j | D_2),$$

(5.1)

where, if $n = m$, the outcomes $O_i$ and $O_j$ may or not may be equal for $i = j$.

We then proceed, by way of the Bernoulli law, or, equivalently, the Weber-Fechner law, to map utilities to the monetary outcomes $O_i$ and $O_j$ in (5.1). This leaves us
with the utility probability distributions:
\[ P(u_i \mid D_1), \quad P(u_j \mid D_2). \] (5.2)

Our most primitive intuition regarding the utility probability distributions (5.2) is that the decision which corresponds with the utility probability distribution which lies more to the right will also be the decision that promises to be the most advantageous. So, when making a decision we compare the positions of the utility probability distribution on the utility axis. This utility axis goes from minus infinity to plus infinity. Hence, the more-to-the-right criterion of choice.

Now, the confidence bounds of (5.2), say:
\[ [LB(u_i \mid D_1), UB(u_i \mid D_1)], \quad [LB(u_j \mid D_2), UB(u_j \mid D_2)], \] (5.3)
may provide us with a numerical handle on the concept of more-to-the-right.

For example, if we have that both
\[ LB(u_i \mid D_1) > LB(u_j \mid D_2), \quad UB(u_i \mid D_1) > UB(u_j \mid D_2). \] (5.4)
Then we will have an unambiguous preference for decision \( D_1 \) over decision \( D_2 \); seeing that under both the still probable worst and best case we will be better if we opt for \( D_1 \).

Likewise, if we have that either
\[ LB(u_i \mid D_1) = LB(u_j \mid D_2), \quad UB(u_i \mid D_1) > UB(u_j \mid D_2), \] (5.5)
or
\[ LB(u_i \mid D_1) > LB(u_j \mid D_2), \quad UB(u_i \mid D_1) = UB(u_j \mid D_2). \] (5.6)
Then, again, we will have an unambiguous preference for decision \( D_1 \) over decision \( D_2 \). In the constellation (5.5), we stand, all other things being equal, to be better of under the still probable best case scenario; while in the constellation (5.6), we stand, all other things being equal, to be less worse of under the still probable worst case scenario.

However, things become more ambiguous when, say, under decision \( D_1 \), we have to make a trade-off between either a gain in the upper bound and a loss in the lower bound
\[ LB(u_i \mid D_1) < LB(u_j \mid D_2), \quad UB(u_i \mid D_1) > UB(u_j \mid D_2), \] (5.7)
or a gain in the lower bound and a loss in the upper bound
\[ LB(u_i \mid D_1) > LB(u_j \mid D_2), \quad UB(u_i \mid D_1) < UB(u_j \mid D_2). \] (5.8)

We postulate here that a rational criterion of choice in the respective trade-off situations (5.7) and (5.8), would be to pick that decision whose gain in either the lower or upper bound exceeds the loss in the corresponding upper or lower bound.
So, if, say, under $D_1$ we stand to gain more in the still probable best case scenario than we stand to lose under the still probable worst case scenario, that is, (5.7):

$$LB(u_i|D_2) - LB(u_i|D_1) < UB(u_i|D_1) - UB(u_j|D_2),$$

then we will choose $D_1$ over $D_2$. Likewise, if under $D_1$ we stand to gain more in the still probable worst case scenario than we stand to lose under the still probable best case scenario, that is, (5.8):

$$LB(u_i|D_1) - LB(u_j|D_2) > UB(u_j|D_2) - UB(u_i|D_1),$$

then again we will choose $D_1$ over $D_2$.

Note that the gains and losses in this discussion pertain to gains and losses on the utility dimension, not on the monetary outcome dimension. On the utility dimension the phenomenon of loss aversion, that is, the phenomenon that monetary losses may weigh heavier than equal monetary gains, has already been accounted for. Stated differently, the utility scale is a linear loss-aversion corrected scale for the moral value of monies.

Now, if we look at the scenarios (5.7) and (5.8), and the corresponding postulated rational, because intuitive, criteria of choice (5.9) and (5.10), then we see that we will choose $D_1$ over $D_2$ whenever we have that

$$LB(u_i|D_1) + UB(u_i|D_1) > LB(u_j|D_2) + UB(u_j|D_2).$$

Moreover, this single criterion of choice is also consistent with the choosing of $D_1$ over $D_2$ in the scenarios (5.4), (5.5), and (5.6).

This, then, is the rationale behind the non-intuitive, because it admits no interpretation, criterion of choice that we should maximize the sum of the lower and upper bounds of the utility probability distributions, in order to come to the optimal decision.\footnote{Instead of the criterion (5.11), one may also use a lower bound maximization, that is, choose $D_1$ whenever

$$LB(u_i|D_1) > LB(u_j|D_2),$$

A possible lower bound maximizer might be the regulator who is not that interested in a bank’s potential profit, but only has an eye for the potential catastrophic losses that, were they to materialize, could destabilize the entire financial system. Or, alternatively, one may use an upper bound maximization, that is, choose $D_1$ whenever

$$UB(u_i|D_1) > UB(u_j|D_2),$$

where the upper bound maximizer is someone who, as a matter of principle, is willing to go down in a blaze of glory, as he dares it all on a single throw of the dice.}

Note that if the decision inequality (5.11) goes to an equality:

$$LB(u_i|D_1) + UB(u_i|D_1) = LB(u_j|D_2) + UB(u_j|D_2).$$

Then we have that we will be undecided when it comes to the decisions $D_1$ and $D_2$.\footnote{Instead of the criterion (5.11), one may also use a lower bound maximization, that is, choose $D_1$ whenever

$$LB(u_i|D_1) > LB(u_j|D_2),$$

A possible lower bound maximizer might be the regulator who is not that interested in a bank’s potential profit, but only has an eye for the potential catastrophic losses that, were they to materialize, could destabilize the entire financial system. Or, alternatively, one may use an upper bound maximization, that is, choose $D_1$ whenever

$$UB(u_i|D_1) > UB(u_j|D_2),$$

where the upper bound maximizer is someone who, as a matter of principle, is willing to go down in a blaze of glory, as he dares it all on a single throw of the dice.}
Also note that for $k$-sigma bounds (5.3) translates to
\[ E(u_i \mid D_1) \pm k \: \text{std}(u_i \mid D_1), \quad E(u_j \mid D_2) \pm k \: \text{std}(u_j \mid D_2), \] (5.13)
which, if substituted in (5.11), gives the inequality
\[ E(u_i \mid D_1) > E(u_j \mid D_2), \] (5.14)
which brings us right back to Bernoulli’s expected utility theory, as proposed in 1738, [6]. And it follows that Bernoulli’s expected utility theory is a special case of the Bayesian decision theory, where the lower and upper bound of the utility probability distributions are constructed as $k$-sigma intervals.

However, if we use skewness corrected intervals, which will be given in the next section, then, for non-zero skewness, the $k$-sigma skewness corrected upper and lower bounds will be asymmetric around the expectation value. As a consequence, the criterion of choice of the expected utility theory, (5.14), will no longer be valid. As is borne out by the infamous Ellsberg and Allais paradoxes.

6. The Skewness Confidence Interval

We give here the skewness interval, which is a generalization of the standard sigma interval. It will be found that the structure present in these skewness intervals will greatly enrich the Bayesian decision theory, in that additional, pertinent, information regarding the utility probability distributions can now be taken into account.

Any probability distribution is determined by its cumulants, [23]. If we only focus on the means and standard deviations, that is, the first two cumulants, of the utility probability distributions, then the Bayesian decision theory, (5.11) and (5.13), collapses back to Bernoulli’s expected utility theory, (5.14). This is problematic, because of the aforementioned Ellsberg and Allais paradoxes of Bernoulli’s expected utility theory. So we now will also take the effect of the skewness, that is, the third cumulant, of the utility probability distributions into account.

The skewness is the scaled third order central moment of a distribution and a measure of its asymmetry, [23]:
\[ \text{skew}(u \mid D_i) = \frac{\int [u - E(u \mid D_i)]^3 p(u \mid D_i) \: du}{[\text{std}(u \mid D_i)]^3}. \] (6.1)
If we let, for notational convenience,
\[ \mu = E(u \mid D_i), \quad \sigma = \text{std}(u \mid D_i), \quad \gamma = \text{skew}(u \mid D_i). \] (6.2)

14These paradoxes will be discussed later on, when we have the decision theoretical framework in place, needed to address them.
Then the traditional 1-sigma confidence interval may be written down as:

\[(\mu - \sigma, \mu + \sigma)\]  \hspace{1cm} (6.3)

If we let the following three simple considerations be our guide:

1. The corrected confidence interval should for \( \gamma = 0 \), this being the skewness of the normal distribution, revert back to (6.3); as it is only by such a property that the new skewness corrected confidence interval may encompass the standard confidence interval (6.3) as a special limit case.

2. The corrected confidence interval should take into account the skewness \( \gamma \) in such a way that for \( \gamma > 0 \) it would compress the lower bound while elongating the upper bound; as this is the qualitative way in which, relative to (6.3), positive skewness ought to be corrected.

3. The corrected confidence interval should have a coverage for skewed probability distributions that approaches 0.68; as this is the coverage of the sigma confidence interval (6.3) for the non-skewed normal distribution.

Then we may find\(^1\) that the corresponding skewness corrected 1-sigma confidence intervals are given as, for a skewness of \( \gamma > 0 \):

\[
\left[ \mu - \frac{\sigma}{\sqrt[3]{1 + \gamma + \frac{1}{1+\gamma}}} \mu + \left( 1 + \frac{\sqrt[3]{\gamma}}{1 + \gamma + \frac{1}{1+\gamma}} \right) \sigma \right], \hspace{1cm} (6.4)
\]

and for a skewness of \( \gamma < 0 \):

\[
\left[ \mu - \left( 1 - \frac{\sqrt[3]{\gamma}}{1 - \gamma + \frac{1}{1-\gamma}} \right) \sigma, \mu + \frac{\sigma}{\sqrt[3]{1 - \gamma + \frac{1}{1-\gamma}}} \right], \hspace{1cm} (6.5)
\]

where it is understood that the third square root of a negative returns a negative.

Now, where for (6.3) the sum of the lower and upper bound collapses to 2\(\mu\), giving us (5.14), we have that for (6.4), that is, positive skewness, the sum of the

\(^1\)See Appendix E.
lower and upper bound evaluates to

\[ 2\mu + \sigma \left[ \left( 1 + \frac{\sqrt{\gamma}}{1 + \gamma + \frac{1}{1+\gamma}} \right) - \frac{1}{1 + \frac{\sqrt{\gamma}}{1+\gamma}} \right], \quad (6.6) \]

whereas for (6.5), that is, negative skewness, the sum of the lower and upper bound evaluates to

\[ 2\mu + \sigma \left[ \frac{1}{1 - \frac{\sqrt{\gamma}}{1 - \gamma + \frac{1}{1-\gamma}}} - \left( 1 - \frac{\sqrt{\gamma}}{1 - \gamma + \frac{1}{1-\gamma}} \right) \right], \quad (6.7) \]

For both (6.6) and (6.7), we have that the effect of the second and third cumulants do not cancel out for non-zero skewnesses. This added structure, relative to expected utility theory, (5.14), will enable the Bayesian decision theory to accommodate the Ellsberg and Allais paradoxes, which are no longer paradoxical. Because their results are predicted, from first principles, by the Bayesian decision theory.

7. THE ELLSBERG PARADOX

In this section we will demonstrate how to construct a non-trivial outcome distribution, by way of the product and sum rules, and how to map outcomes to their corresponding utilities by way of the product and sum rules.

Ellsberg found that the willingness to bet on an uncertain event depends not only on the degree of uncertainty but also on its source. He observed that people prefer to bet on an urn containing equal numbers of red and green balls, rather than on an urn that contains red and green balls in unknown proportions. Ellsberg called this observed phenomenon source dependence.

Tversky and Kahneman [63], state that source dependence constitutes one of the minimal challenges that must be met by any adequate descriptive theory of choice. As our theory of choice is Bayesian, we will proceed to give a Bayesian treatment of this phenomenon.

16 Note that we do not claim that decision makers will derive the Bayesian equations which will follow ad verbatim. Rather, we state that Bayesian inference is common sense amplified, having a much higher probability resolution than our human brains can ever hope to achieve. So, being common sense amplified, the results of the Bayesian analysis should be commensurate with our intuitions; rather than the analysis itself. See also Appendix F.
7.1. **Constructing outcome probability distributions.** Say, we have a large urn consisting of 1000 balls of which 500 are red and 500 green. We tell our subject that of the $N = 1000$ balls $R = 500$ are red and $N - R = 500$ green, and that he is to draw a ball $n = 100$ times. After each draw he will get a dollar if the ball is red and nothing if the ball is green, after which the ball is to be put back in the urn. The subject is also told that for the privilege to partake in this bet an entrance fee of 50 dollars is to be paid.

The probability of drawing $r$ red balls in the first bet $D_1$ may be modeled by way of a binomial distribution:

$$p(r|n, R, N, D_1) = \frac{n!}{r!(n-r)!} \left( \frac{R}{N} \right)^r \left( 1 - \frac{R}{N} \right)^{n-r}. \quad (7.1)$$

Now as the net return, say, $u$ is in dollars, having as its value number of red balls minus entrance fee, we have

$$o = r - 50, \quad (7.2)$$

where, as there are $n = 100$ draws, $-50 \leq o \leq 50$. We then make a simple change of variable, using (7.2),

$$r = o + 50 \quad (7.3)$$

and substitute (7.3) into (7.1), so as to get the probability function of the net return

$$p(o|n, R, N, D_1) = \frac{n!}{(o+50)!(n-50)!} \left( \frac{R}{N} \right)^{o+50} \left( 1 - \frac{R}{N} \right)^{n-o-50}. \quad (7.4)$$

The probability distribution of the net return for bet $D_1$, that is,

$$p(o|n = 100, R = 500, N = 1000, D_1),$$

then can be plotted as, Figure 3.
This probability distribution has a mean, standard deviation, and skewness of, respectively,

\[ E(o|D_1) = 0, \quad \text{std}(o|D_1) = 5, \quad \text{skew}(o|D_1) = 0. \]

(7.5)

For the second bet, we tell our subject that the urn holds \( N = 1000 \) balls which are either red or green. Again, for every red ball drawn there will be a dollar payout. There will be \( n = 100 \) draws and after each draw the ball is to be placed in the urn again. The entrance fee of the bet is 50 dollars. However, we are also told that the number of red balls is neither zero nor thousand \( 17 \), that is, \( 0 < R < N \), thus, precluding the certainty outcomes of \( o = -50 \) and \( o = 50 \) for, respectively, \( R = 0 \) and \( R = 1000 \).

As the subject does not know the actual number of red balls \( R \), the Bayesian thing to do is to weigh the probability of drawing \( r \) red balls over all plausible values of \( R \), the total number red balls in the urn. Based on the background information, we assign an uniform prior probability distribution to \( R \), the unknown number of red balls in the urn:

\[ p(R) = \frac{1}{N-1}, \]

(7.6)

where \( R = 1, \ldots, N-1 \). So, by way of the product and the generalized sum rules, \( (3.3) \) and \( (3.5) \), the probability of drawing \( r \) red balls in the second bet \( D_2 \) translates to

\[ p(r|n,N,D_2) = \sum_{R=1}^{N-1} p(r|R)n,N,D_2) = \sum_{R=0}^{N} p(R) p(r|n,R,N,D_2). \]

(7.7)

Again making a change of variable from the number of red balls drawn \( r \) to the net return \( o \), we substitute \( (7.1) \) and \( (7.6) \) into \( (7.7) \) and make a change of variable by way of \( (7.3) \). This results in the probability distribution of the net return \( o \):

\[ p(o|n,N,D_2) = \sum_{R=1}^{N-1} \frac{1}{N-1} \frac{n!}{(o+50)! (n-o-50)!} \left( \frac{R}{N} \right)^{o+50} \left( 1 - \frac{R}{N} \right)^{n-o-50}. \]

(7.8)

The probability distribution of the net return for bet \( D_2 \), that is,

\[ p(o|n = 100, N = 1000, D_2), \]

then can be plotted as, Figure 4.

\[ \text{We could let the range of the red balls be } 0 \leq R \leq N. \text{ But then the resulting outcome probability distribution would become unwieldy, because of this added structure.} \]
This probability distribution has a mean, standard deviation, and skewness of, respectively,

\[ E(o|D_2) = 0, \quad \text{std}(o|D_2) = 29, \quad \text{skew}(o|D_2) = 0. \]  

(7.9)

In Figure 5 we give both probability distributions, Figures 6.1 and 6.2, together.

7.2. Constructing utility probability distributions. Let \( D_1 \) stand for the decision to choose the first Ellsberg bet and \( D_2 \) for the decision to choose the second Ellsberg bet. Then, by substituting, where appropriate, the values \( n = 100, R = 500, N = 1000 \) into (7.4) and (7.8), we may obtain the corresponding outcome probability
distributions:

\[
p(o \mid n = 100, R = 500, N = 1000, D_1) = \frac{100!}{(o + 50)! (50 - o)!} \left(\frac{1}{2}\right)^{100},
\]

\[
p(o \mid n = 100, N = 1000, D_2) = \frac{100!}{(o + 50)! (50 - o)!} \sum_{R=1}^{999} \frac{1}{999} \left(\frac{R}{1000}\right)^{o+50} \left(1 - \frac{R}{1000}\right)^{50-o}.
\]

In order to map the outcomes \(o\) in (7.10) to values on the utility dimension \(u\), we introduce the conditional probability distribution \(p(u \mid o)\). Combining (7.10) with \(p(u \mid o)\), by way of the product rule (3.3), and marginalizing over the possible outcomes \(o\), by way of the generalized sum rule (3.5), we may get the utility probability distribution of interest, that is,

\[
p(u \mid D_i) = \sum_{o} p(u, o \mid D_i) = \sum_{o} p(u \mid o) p(o \mid D_i).
\]

Now, if we assign as our utility function the Bernoulli law, then the conditional utility probability distribution to be employed in the Ellsberg example is:

\[
p(u \mid o, m, q) = \begin{cases} 
1, & u = q \log \frac{m+o}{m} \\
0, & u \neq q \log \frac{m+o}{m}
\end{cases}
\]

for \(o = -50, -49, \ldots, 50\), and where \(m\) is our initial wealth and \(q\) is the scaling constant of the Bernoulli law. This probability distribution takes us from the \(o\)-dimension, which is the dimension of the monetary outcomes, to the dimension of the utility \(u\), which is the dimension of the moral value of these monetary outcomes.

From (7.12), we see that every outcome \(o\) admits only one utility value \(u\); that is, (7.12) is of the Dirac delta form:

\[
p(u \mid o, q, m) = \delta \left( u - q \log \frac{m+o}{m} \right),
\]

where \(\delta\) is the Dirac delta function for which

\[
\delta(u - c) \, du = \begin{cases} 
1, & u = c \\
0, & u \neq c
\end{cases}
\]

Because of (7.14), we have that

\[
\int \delta(u - c) \, f(u) \, du = f(c).
\]
Applying both (7.11) and the Dirac delta (7.13) to the outcome distributions (7.10), we obtain the utility probability distributions:

\[
p(u| m, q, D_1) = \sum_{o=-50}^{50} \delta\left(u - q \log \frac{m + o}{m}\right) p(o| D_1),
\]

\[
p(u| m, q, D_2) = \sum_{o=-50}^{50} \delta\left(u - q \log \frac{m + o}{m}\right) p(o| D_2),
\]

where in the probability distributions we only conditionalize on that which is not yet specified.

7.3. **Applying the criterion of choice.** The \(k\)th-order moments of the utility probability distributions (7.16) may be evaluated by way of the integrals:

\[
E(a^k| D_i) = \int a^k p(u| m, q, D_i) \, du.
\]

If we assume a modest initial wealth, that is, monthly expendable income, of two-hundred dollars, or, equivalently, \(m = 200\). Then the mean, standard deviation, and skewness of the utility probability distributions (7.16) are given as

\[
E(u| q, D_1) = -0.0003 q,
\]

\[
\text{std}(u| q, D_1) = 0.025 q,
\]

\[
\text{skew}(u| q, D_1) = -0.0744,
\]

and

\[
E(u| q, D_2) = -0.0108 q,
\]

\[
\text{std}(u| q, D_2) = 0.1479 q,
\]

\[
\text{skew}(u| q, D_2) = -0.1784.
\]

Seeing that the utility probability distributions have negative skewness, we compute the sum of the lower and upper bounds of the utility probability distributions (7.16) by way of (6.7), which for (7.18) gives

\[
LB(u| q, D_1) + UB(u| q, D_1) = -0.0102 q,
\]

and for (7.19) gives

\[
LB(u| q, D_2) + UB(u| q, D_2) = -0.0948 q.
\]

If we compare (7.20) and (7.21), then we find that decision \(D_1\) is more advantageous than decision \(D_2\):

\[
LB(u| q, D_1) + UB(u| q, D_1) > LB(u| q, D_2) + UB(u| q, D_2),
\]
since

\[-0.0102 \, q > -0.0948 \, q, \quad (7.23)\]

for any positive scaling constant \( q \).

As an aside the unknown scaling constant \( q \) will always fall away in the decision theoretical inequalities. This is because the mean and standard deviations of the utility probability distributions are linear in \( q \), whereas the \( q \) in the skewness cancels out. Stated differently, it may be checked that, for the stochastics \( X \) and \( Y \), and a positive constant \( q \), \([35]\):

\[
E(qX) = q \, E(X), \quad \text{std}(qX) = q \, \text{std}(X), \quad \text{skew}(qX) = \text{skew}(X). \quad (7.24)
\]

So, unless we want to interpret our decisions in terms of net utility gain, as we do below, then we may, without any loss of generality, set the unknown scaling constant \( q \) to one.

Now, if we are willing to commit ourselves to the scaling constant value of \( q = 100 \) as the Weber constant for monetary stimuli, \([47]\). Then we may, by way of \([510]\), interpret \([7.22]\) as follows. Under decision \( D_1 \), betting on an urn with an equal number of red and green balls, there is a gain of 16.9 utiles, in terms of loss mitigation, and a loss of 8.45 utiles, in terms of gain reduction, relative to decision \( D_2 \), betting on an urn with an unknown proportion of red balls. This makes \( D_1 \), with a net utility gain of 8.45 utiles, more attractive a choice than \( D_2 \).

The setting of the scaling constant give the utility probability distributions under \( D_1 \) and \( D_2 \), Figure 6.

![Figure 6. Probability utility distributions for bets 1 and 2](image)

We summarize, in a Bayesian decision theoretical analysis we first construct the outcome probability distributions on a monetary unit scale under the decisions \( D_1 \) and \( D_2 \), Figure 3. We then construct, by way of the Bernoulli law, the corresponding utility probability distributions on an (un)scaled utile scale, Figure 6. We then compare, for a given decision, the gain/loss in the lower bound relative to the
corresponding loss/gain in the upper bound; or, equivalently, in the algebraic sense of the word, we compare the sums of the upper and lower bounds under the decisions $D_1$ and $D_2$.

The results of this Bayesian decision theoretical analysis is in correspondence with the Ellsberg finding that people prefer to bet on an urn containing equal numbers of red and green balls, rather than on an urn that contains red and green balls in unknown proportions. [11]

In closing, we may envisage decision problems in which we are uncertain regarding the actual utility of a given outcome $o$. Such an occasion may arise when we do not know the initial wealth $m$ of the subject under investigation. In those cases we will want to assign probability distributions $p(u|o)$ less dogmatic than the Dirac delta function to our utilities.

8. The Psychological Certainty Effect, Part I

Risk seeking refers to a specific pattern in betting behavior. Uncertain larger gains are preferred over sure smaller gains and uncertain larger losses are preferred over sure smaller losses. The psychologists Kahneman and Tversky state that risk seeking constitutes one of the minimal challenges that must be met by any adequate descriptive theory of choice, [63].

The observation that large gains are preferred over sure much smaller gains is commensurate with the fact that we may prefer high-risk, high-yield investment opportunities over low-risk, low-yield ones. Likewise, the observation that uncertain larger losses are preferred over sure smaller, though still substantial, losses is in accordance with those instances in the past where traders incurred hundreds of millions in losses, in their attempts to make good on their previous losses.[18]

If the signs of the outcomes in the risk seeking betting scenarios are reversed, then the preferences between the bets will also reverse. This is called the reflection effect, [34]. So, risk seeking in the positive domain is accompanied by risk aversion in the negative domain. Conversely, risk seeking in the negative domain is accompanied by risk aversion in the positive domain.

8.1. Risk Seeking I. We first give an example of risk seeking in the case of a small probability of winning a large prize, that is, risk seeking in the positive domain. This case of risk seeking represents our tendency to profit maximization and demonstrates that we will be willing to invest in a long shot if the pay-out is high enough.

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18 As, for example, happened to Nicholas William Leeson, a trader for the Barings Bank in the nineties. Though we believe that Leeson would have acted less recklessly had he been investing his own money, instead of the deposit holders. That is, we expect that his Weber constant for his own money, say, $q$, was markedly larger than his Weber constant for the deposit holders money, say, $q_0$, where $0 < q_0 < < q$. 


The outcome probability distributions for the respective bets in our risk seeking example are

\[ p(O|D_1) = \begin{cases} 
0.001, & O = 5000 \\
0.999, & O = 0 
\end{cases} \] (8.1)

and

\[ p(O|D_2) = \begin{cases} 
1.0, & O = 5 
\end{cases} \] (8.2)

It is found that 72% of \( N = 72 \) subjects prefer decision \( D_1 \) over \( D_2 \), \[34\]. Even though both bets have the same expectation value of

\[
E(O|D_1) = 0.001 \times 5000 = 5 = 1.0 \times 5 = E(O|D_2).
\]

We now interpret this finding in terms of the Bayesian decision theoretic framework.

Kahneman and Tversky state that the median net monthly income for a family is about 3000 Israeli pounds, \[34\], being that the subjects were all students we will assume an initial amount of money of \( m = 1000 \) Israeli pounds. This gives us the following utility probability distributions, (4.5), (8.1), and (8.2):

\[ p(u|D_1) = \begin{cases} 
0.001, & u = q \log \frac{6000}{1000} \\
0.999, & u = 0 
\end{cases} \] (8.3)

and

\[ p(u|D_2) = \begin{cases} 
1.0, & u = q \log \frac{1005}{1000} 
\end{cases} \] (8.4)

Using the identities, \[14\]:

\[
E(X) = \sum_i X_i P_i, \quad \text{std}(X) = \sqrt{\sum_i \left[ X_i - E(X) \right]^2 P_i}, \quad (8.5)
\]

and, \[23\]:

\[
\text{skew}(X) = \frac{\sqrt{\sum_i \left[ X_i - E(X) \right]^3 P_i}}{\left[ \text{std}(X) \right]^3}, \quad (8.6)
\]

and, depending on the sign of (8.6), either the skewness confidence interval (6.4) or (6.5), we may construct the skewness corrected intervals:

\[ [\text{LB}(u|p_1, D_1), \text{UB}(u|p_1, D_1)] \],

(8.7)

and

\[ [\text{LB}(u|p_2, D_2), \text{UB}(u|p_2, D_2)] \].

(8.8)

Letting \( D_2 \) be the decision to choose for the certain gain or loss, that is, \( p_2 = 1 \), and relabeling \( p_1 = p \), we may construct the general decision theoretical equality, (8.7) and (8.8):

\[
[\text{LB}(u|1, D_2) - \text{LB}(u|p, D_1)] = \text{UB}(u|p, D_1) - \text{UB}(u|1, D_2)]. \quad (8.9)
\]
If we solve this equality for \( p \), then we find the probability \( p \) of the uncertainty bet \( D_1 \), for which the bets \( D_1 \) and \( D_2 \) are undecided; that is, that probability \( p \) for which \( D_1 \) and \( D_2 \) are in fair, in that a larger lower loss bound under \( D_1 \) is compensated with a commensurate larger upper gain bound.

If we solve for the probability \( p \), we find the fair probability under bet \( D_1 \):

\[
p = 0.0000288
\]

So, if the probability of the uncertain events exceeds the lower bound

\[
p > 0.0000288,
\]

then we will accept the uncertain bet \( D_1 \). As the gain in the utility upper bound under \( D_1 \) will dominate the loss in the utility lower bound under \( D_2 \).

It is found that 72\% of \( N = 72 \) subjects prefer decision \( D_1 \) over \( D_2 \), \([34]\), even though both bets have the same outcome expectation values. The phenomenon of utility upper bound dominance for gains constitutes risk seeking in the positive domain.

We may plot the fair probability \( p \) for a certainty bet as a function of the initial wealth \( m \), where we let \( 200 < m < 10.000 \), Figure 7.

![Figure 7. Fair probability as function of initial wealth.](image)

As the initial wealth \( m \rightarrow \infty \), the utility of an increment in wealth in the range of \( 5 < \Delta m < 5000 \) will both become linear and tend to zero, and the fair probability will converge to

\[
p \rightarrow 0.0000039.
\]

(8.10)

Note that if we commit ourselves to a value of the constant \( q \), this constant being the appropriate utility scaling factor for monetary outcomes\([19]\), we may construct

\[\text{One may obtain a value for } q \text{ by either personal introspection, or by psychological experimentation, where subjects are asked to report their introspection.}\]
attraction maps, with increments, both positive and negative, of, say, ten utilities, relative to the probability fairness baseline in Figure 7.

8.2. Risk Aversion I. The above analysis may also be performed for the case when there is a small probability of loosing a large sum of money. We then will see a reversal in the preference for bet $D_1$ over bet $D_2$ to a preference for bet $D_2$ over bet $D_1$. Risk aversion in the negative domain represents our tendency to hedge against large and catastrophic losses.

The outcome probability distributions for the respective bets are:

\[ p(O|D_1) = \begin{cases} 0.001, & O = -5000 \\ 0.999, & O = 0 \end{cases} \]  
(8.11)

and

\[ p(O|D_2) = \begin{cases} 1.0, & O = -5 \end{cases} \]  
(8.12)

It is found that 83% of $N = 72$ subjects preferred the bet $D_2$ over $D_1$, \[34\].

We now will imagine that the students of the Kahneman and Tversky experiments, who were asked to perform imaginary bets, have an imaginary initial amount of money of $m = 6000$ Israeli pounds\[21\]. Assuming the utility function (4.5), we get the following utility probability distributions:

\[ p(u|D_1) = \begin{cases} 0.001, & u = q \log \frac{1000}{6000} \\ 0.999, & u = 0 \end{cases} \]  
(8.13)

and

\[ p(u|D_2) = \begin{cases} 1.0, & u = q \log \frac{5995}{6000} \end{cases} \]  
(8.14)

If we solve (8.9) for $p$, then we find the probability $p$ of the uncertainty bet $D_1$, for which the bets $D_1$ and $D_2$ are undecided:

\[ p = 0.0000008. \]

So, if the probability of the uncertain events exceeds the lower bound

\[ p > 0.0000008, \]

then we will accept the certain bet $D_2$. As the gain in the utility lower bound under $D_2$ will dominate the loss in the utility upper bound under $D_1$.

It is found that 83% of $N = 72$ subjects prefer decision $D_1$ over $D_2$, \[34\], even though both bets have the same outcome expectation values. The phenomenon of utility lower bound dominance for losses constitutes risk aversion in the negative domain.

\[20\]Compare with (8.1) and (8.2).

\[21\]Kahneman and Tversky do not take the initial wealth $m$ into account in their discussion of their experimental results; see Appendix \[G\].
We may plot the fair probability \( p \) for a certainty bet as a function of the initial wealth \( m \), where we let \( 5200 < m < 10,000 \), Figure 8.

As the initial wealth \( m \to \infty \), the utility of an increment in wealth in the range of \(-5000 < \Delta m < -5\) will both become linear and tend to zero, and the fair probability will converge to the convergence of the symmetrical case, where the outcomes are positive, (8.10),

\[
p \to 0.0000039.
\]

8.3. Risk Seeking II. We now give an example of risk seeking when people must choose between a sure loss and a substantial probability of a larger loss, that is, risk seeking in the negative domain. This case of risk seeking represents our tendency to try to evade large and catastrophic losses.

The outcome probability distributions for the respective bets in our risk seeking example are

\[
p(O \mid D_1) = \begin{cases} 0.5, & O = -1000 \\ 0.5, & O = 0 \end{cases} \quad (8.15)
\]

and

\[
p(O \mid D_2) = \begin{cases} 1.0, & O = -500 \end{cases} \quad (8.16)
\]

It is found that 69% of \( N = 68 \) subjects preferred the bet \( D_2 \) over \( D_1 \), [34].

We now imagine an initial amount of money of \( m = 1500 \) Israeli pounds. Assuming the utility function (4.5), we get the utility probability distributions:

\[
p(u \mid D_1) = \begin{cases} 0.5, & u = q \log \frac{500}{1500} \\ 0.5, & u = 0 \end{cases} \quad (8.17)
\]

and

\[
p(u \mid D_2) = \begin{cases} 1.0, & u = q \log \frac{1000}{1500} \end{cases} \quad (8.18)
\]
If we solve (8.9) for \( p \), then we find the probability \( p \) of the uncertainty bet \( D_1 \), for which the bets \( D_1 \) and \( D_2 \) are undecided:

\[ p = 0.191. \]

We may plot the fair probability \( p \) for a certainty bet as a function of the initial wealth \( m \), where we let \( 1500 < m < 10.000 \), Figure 9.

![Figure 9. Fair probability as function of initial wealth.](image)

As the initial wealth \( m \to \infty \), and Bernoulli law converges to

\[ q \log \left( \frac{m - O}{m} \right) \to q \frac{O}{m}, \]

the utility of an increment in wealth in the range of \(-1000 < \Delta m < -500\) will both become linear and tend to zero, and the fair probability will converge, because of the skewness correction, to an interval of fair values,

\[ p \to (0.342, 0.658). \]

The range (8.20) represents the probability interval for which the outcome interval of the uncertainty bet, for all intents and purposes, is

\[ (0, 1000), \]

with equality holding at the probabilities

\[ p = 0.342, \quad p = 0.500, \quad p = 0.658. \]

First, as the probability \( p \) approaches \( p = 0.5 \) from \( p = 0.342 \), where (8.21) holds, the skewness interval, in the absence of a kurtosis correction, under shoots the outcome upper bound, with a factor 0.04 of the outcome upper bound.\(^{22}\) Then, as \( p \) crosses the \( p = 0.5 \) point, the skewness correction transitions from (6.5) to (6.4).

\(^{22}\)Note that there will also be a commensurate overshoot of the outcome lower bound. But as we know this lower bound to be zero, we already have correct for this overshoot in the confidence bound construction phase.
and \( p = 0.5 \), where the skewness is zero. \((8.21)\) holds again. Finally, as \( p \) approaches \( p = 0.658 \), where \((8.21)\) holds, the skewness interval, in the absence of a kurtosis correction, slightly under shoots the outcome lower bound, with a factor 0.04 of the outcome upper bound.

But if we forgo of the skewness interval, and use the sigma interval, and solve the corresponding \((8.9)\) for \( p \). Then, as the initial wealth \( m \to \infty \), the fair probability will converge to just the one value,

\[
p \to 0.5.
\]

So, if it is found that 69\% of \( N = 68 \) subjects prefer decision \( D_1 \) over \( D_2 \), \([34]\), even though both bets have the same outcome expectation values, then this is because people tend to want to mitigate their losses.

Note that the decision theoretical phenomenon of loss aversion is generally understood to point to the concave down curvature of the Bernoulli law, \([4.5]\). But we have here loss aversion on a meta-level, where, all things being equal, in terms of utility upper and lower bounds, people tend to prefer a possible mitigation of a sure loss.

If we have a certain loss of \(-250\) and an uncertain loss of \(-1000\), then we will be willing to take the uncertain bet, for an initial wealth of \( m = 1500 \), if the probability of the uncertain event is smaller than

\[
p = 0.050.
\]

For an initial wealth of \( m \to \infty \), this probability converges to

\[
p = 0.098. \tag{8.22}
\]

If we take such an uncertainty bet, then we adhere to an utility upper bound dominance for losses, which constitutes risk seeking in the negative domain.

8.4. Risk Aversion II. The previous analysis may also be performed for the opposite case of a sure gain and a substantial probability of a larger gain. We then will see a reversal in the preference for bet \( D_1 \) over bet \( D_2 \) to a preference for bet \( D_2 \) over bet \( D_1 \). Risk aversion in the positive domain represents our tendency to secure our profits.

The outcome probability distributions for this problem of choice are

\[
p(O|D_1) = \begin{cases} 
0.5, & O = 1000 \\
0.5, & O = 0 
\end{cases} \tag{8.23}
\]

\( ^{23} \)Compare with \((8.15)\) and \((8.16)\).
and

\[ p(O|D_2) = \begin{cases} 1.0, & O = 500 \\ \end{cases} \]  

(8.24)

Seeing that this is just another incarnation of Allais’ paradox\(^{24}\), we know that people will tend to prefer bet \(D_2\) over \(D_1\), \([4]\); and indeed, 80\% of \(N = 95\) subjects preferred bet \(D_2\) over \(D_1\), \([24]\).

Assuming an initial wealth of \(m = 1000\) and by way of (4.5), we find corresponding utility probability distributions:

\[ p(u|D_1) = \begin{cases} 0.5, & u = q \log \frac{2000}{1000} \\ 0.5, & u = 0 \\ \end{cases} \]  

(8.25)

and

\[ p(u|D_2) = \begin{cases} 1.0, & u = q \log \frac{1500}{1000} \\ \end{cases} \]  

(8.26)

If we solve (8.9) for \(p\), then we find the probability \(p\) of the uncertainty bet \(D_1\), for which the bets \(D_1\) and \(D_2\) are undecided:

\[ p = 0.764 \]

We may plot the fair probability \(p\) for a certainty bet as a function of the initial wealth \(m\), where we let \(200 < m < 10.000\), Figure 10.

![Figure 10. Fair probability as function of initial wealth.](image)

As the initial wealth \(m \to \infty\), the utility of an increment in wealth in the range of \(-1000 < \Delta m < -500\) will both become linear and tend to zero, and the fair probability will converge, because of the skewness correction, to an interval of fair values,

\[ p \to (0.342, 0.658), \]  

(8.27)

which is the same interval as (8.20).\(^{24}\)

\(^{24}\)See Appendix [1].
So, if it is found that 84% of $N = 70$ subjects prefer decision $D_2$ over $D_1$, even though both bets have the same outcome expectation values, then this is because people tend to want to secure their gains.

If we have a certain gain of 250 and an uncertain gain of 1000, then we will be prefer the certain bet, for an initial wealth of $m = 1000$, if the probability of the uncertain event is smaller than

$$p = 0.151.$$  

For an initial wealth of $m \rightarrow \infty$, this probability converges to, (8.22),

$$p = 0.098.$$  

If we take such a certainty bet, then we adhere to an utility lower bound dominance for gains, which constitutes risk aversion in the positive domain.

9. The Psychological Certainty Effect, Part II

In the previous section we defined, for certainty bets, fairness as the decision theoretical equality, (8.9):

$$[LB(u|1, D_2) - LB(u|p, D_1) = UB(u|p, D_1) - UB(u|1, D_2)],$$  

where $D_1$ and $D_2$ correspond, respectively, with the uncertainty and certainty bets.

Let $O_c$ and $O_u$, respectively, be the certainty and the uncertainty outcomes, where $O_c < O_u$. If, for a certainty bet having positive outcomes, we solve (9.1) for the fair probability $p$, assuming a linear utility for money, we find that the fair probability $p$ maps to the outcome intervals

$$(0, 2O_c), \quad O_c \leq \frac{O_u}{2},$$  

which is intuitively fair for the takers of decision $D_1$, relative to the certainty offer of $O_c$, and

$$\left[2 \left(O_c - \frac{O_u}{2}\right), O_u\right), \quad O_c > \frac{O_u}{2},$$  

which is intuitively fair for the providers of decision $D_1$, relative to the certainty offer of $O_c$.

If for an uncertainty pay out of either 0 or $O_u = 5000$, we plot the solution of (9.1) for the fairness probability $p$, assuming a linear utility for monetary outcomes, as a function of the certainty outcome $O_c$, we obtain Figure [11].
If we again, but now neglecting the skewness correction, we solve (9.1) for the fairness probability $p$, assuming a linear utility for monetary outcomes, as a function of the certainty outcome $O_c$, and add this curve to Figure 11 we obtain Figure 12.

We may construct, again assuming a linear utility for monetary outcomes, the same graph for the fair probability $p$ of certainty bets involving negative outcomes, where $O_c > O_u$, Figure 13.
If we rescale the $x$-axes of Figures 12 and 13 as the ratio $O_c/O_u$, where $|O_c| \leq |O_u|$, and reverse the axes, we obtain the alternative Figure 14:

Now, those readers who are familiar with the cumulative prospect theory may recognize in Figure 14 Kahneman and Tversky’s Figures 1, 2, and 3 of their [63]. But Kahneman and Tversky obtained their figures, not from first principles, as we have, but through experimentation, in which subjects where asked to decide on certainty bets of the type we discussed in the previous section. So, it would seem that Kahneman and Tversky, inadvertently, for they are outspoken anti-Bayesian, have provided the Bayesian decision theory with a very strong supporting contact.

Kahneman and Tversky see in the empirical observation of the typical S-curve of Figure 14 another justification for their probability weighing function:

$$w^+(p) = \frac{p\gamma}{p\gamma + (1 - p)^\delta}, \quad (9.4)$$

25See Appendix J.
26See Appendix J for a discussion of the initial justification.
27Note that Kahneman and Tversky’s $\gamma$ and $\delta$ are not our $\gamma$ and $\delta$. 

Figure 13. Fair probability, sigma and skewness intervals, negative outcomes

Figure 14. Rescaled and rotated figure for sigma and skewness intervals
and

\[ w^-(p) = \frac{p^\delta}{p^\delta + (1 - p)^\delta}, \]

(9.5)

which overweighs small probabilities and underweighs large probabilities. Moreover, Kahneman and Tversky offer up the implied underweighing of small probabilities, in order to explain the general popularity of lotteries and insurances.

We, on the other hand, see in the empirical observation of the typical \( S \)-curve of Figure 14 a confirmation of the intuitive relevancy of the skewness intervals, (6.4) and (6.5).

As we progressed in our research on the Bayesian decision theory, it became obvious to us that the sigma interval, (6.3), though still superior to the ‘interval’ of expected utility theory, \((\mu, \mu)\), left out pertinent symmetry information.

All our initial case studies involved extreme outcomes having small probabilities of occurring, which leaves our probability distributions highly skewed. The presence of skewness leads for sigma intervals to confidence interval coverages which are sub-par, as it will lead to both an under and over shooting of the actual confidence bounds. This is why we felt compelled to search for the skewness interval, (6.4) and (6.5); as this interval promised us more realistic, that is, better informed, criterions of action.

If we drop the assumption of a linear utility of monetary outcomes in the neighborhood of \(-5000 < \Delta m < 5000\), and for initial wealths of \( m = 1000 \) and \( m = 6000 \) for certainty bets involving, respectively, positive and negative outcomes. Then we may assign, by way of the Bernoulli law, (4.5), utilities to the monetary outcomes. By doing so, we obtain the following fairness ratio outcomes for a given probability \( p \) of the uncertain proposition, Figures 15 and 16.

![Figure 15. Rescaled and rotated figure for positive outcomes](image-url)
Comparing Figures 15 and 14 we see that by taking into account the initial wealth $m$, through the Bernoulli law, (4.5), the fair outcome ratios, as a function of the probability $p$ for the positive uncertainty outcome $C_u$, are adjusted downward, relative to Figure 14. Furthermore, the fairness symmetry point $p = 0.5$ has been adjusted downward in Figure 15.

Comparing Figures 16 and 14 we see that by taking into account the initial wealth $m$, through the Bernoulli law, (4.5), the fair outcome ratios, as a function of the probability $p$ for the negative uncertainty outcome $C_u$, are adjusted upward, relative to Figure 14. Furthermore, the fairness symmetry point $p = 0.5$ has been adjusted upward in Figure 16. These adjustments make nothing but sense.

If we have a small initial wealth, and we stand to gain more than we initially would have gained. Then, for given outcome ratios, we will be more inclined to accept the possibility of gaining nothing, relative to the case where we have a large initial wealth, as the pay-out, in terms of subjective consequences, is relatively larger.

But if we have a small initial wealth, and we stand to lose more than we initially would have lost. Then, for given outcome ratios, we will be less inclined to accept the possibility of losing even more, relative to the case where we have a large initial wealth, as the penalty, in terms of subjective consequences, is relatively larger.

As our initial wealth tends to infinity, and our utility for money becomes linear, we will perceive both problems to be symmetric, as monetary losses are weighed the same as monetary gains, Figure 14.

Furthermore, the differences in the Figures 15 and 16 are commensurate with the fact that Kahneman and Tversky found that their weighing functions for probabilities, (9.4) and (9.5), differed for certainty bets involving positive and negative outcomes.
10. Discussion

In this fact sheet we have presented the case for the Bayesian decision theory. It is our belief that this decision theory, just like the Bayesian probability and information theories, is Bayesian in the strictest sense in the word; that is, an inescapable consequence of the desideratum of consistency. This belief led us to consider that the Bernoulli law, the only remaining degree of freedom in our decision theory, might be more fundamental then we initially had thought.

Because our initial justification for the Bernoulli law had come from the observation that this law, in the guise of the Weber-Fechner and the Steven’s power law, had been demonstrated by psycho-physics to be an appropriate model for the way we humans perceive the increments in sensory stimuli, in terms of sensation strength. So, if monetary outcomes are considered to be a sensory stimuli, in the most abstract sense of the word, then it would follow the Bernoulli law would be the most appropriate model for the way we humans perceive the increments in monetary wealth, in terms of their utilities.

The tipping point, where the Bayesian decision theory transitioned from an intuitive idea to something more fundamental, came for us when, having found the skewness intervals, we were re-analyzing the Kahnmenan and Tversky data on the psychological preferences in certainty bets.

It was found that Bayesian decision algorithm confirmed most of the reported preferences. Nonetheless, some preferences were forcefully rejected.

For example, for the certainty bet where we have to choose between a certain gain of 3000, and a probability $p = 0.8$ of gaining 4000 and a probability of $1 - p = 0.2$ of gaining nothing, it is found that 80% of the $N = 95$ subjects preferred the certain outcome. For an initial wealth of $m = 1000$, it is found that, by way of the Bernoulli law and skewness interval, that the fair probability for this certainty bet is $p = 0.963$. So, only for probabilities larger than this fair probability, will those with a modest income feel inclined to consider the uncertainty choice, which is in correspondence with the observed preference for the certainty choice.

However, for the certainty bet where we have to choose between a certain loss of 3000, and a probability $p = 0.8$ of losing 4000 and a probability of $1 - p = 0.2$ of losing nothing, it is found that 92% of the $N = 95$ subjects preferred the uncertain outcome. For an initial wealth of $m = 5000$, it is found that, by way of the Bernoulli law and skewness interval, that the fair probability for this certainty bet is $p = 0.747$. So, only for probabilities smaller than this fair probability, will those with a modest income feel inclined to consider the uncertainty choice, which is in strong contradiction with the observed preference for the uncertainty choice.
Now, for an initial wealth that tends to infinity, both fair probabilities will tend to \( p = 0.902 \), which is in correspondence with both observed preferences. Nonetheless, we felt that human intuition had erred in the latter experiment, as the both the subjects and we ourselves\(^{28}\) do not have initial wealths that tend to infinity. In the study of Kahneman and Tversky’s work, we had learned to doubt somewhat the infallibility of the experimental method of hypothetical betting choices\(^{29}\). As a consequence, we were put in the position that we put more faith in the Bayesian decision theory, than the reported preferences in the hypothetical betting choices. That is, we trusted the Bayesian decision algorithm to teach our intuition, in those instances where the intuitive ‘resolution’ is lacking to make clear and crisp choices\(^{30}\).

Especially so, since we, on the one hand, had eliminated the confounding effect of the skewness of the utility probability distributions, by way of the skewness intervals, and, on the other hand, had taken painstaking care to search out those ‘unyielding practical realities’, as mentioned in the introduction, that would put our foundations to the test. And it had been found that all these practical realities fell nicely in line with the proposed foundations of the Bayesian decision theory.

So, by analogy, Jaynes’ reasoning computer of Bayesian probability theory, \(^{30}\), had become a decision making computer. And this decision making computer, not much unlike a veteran stock broker, knew when to take his losses and not to throw good money after bad.

This then put the burden on us to provide a proof of the fundamentalness of the Bayesian decision theory. Because the history of Bayesian probability theory has taught us that the usefulness of a theory, in terms of its practical and beautifully intuitive results, in the absence of a compelling axiomatic basis, provides no safeguard against attacks by those who choose to close their eyes to this usefulness\(^{31}\).

It may be read in Jaynes’ \(^{30}\), that to the best of his knowledge, there are as of yet no formal principles at all for assigning numerical values to loss functions; not even when the criterion is purely economic, because the utility of money remains ill-defined. In the absence of these formal principles, Jaynes final verdict was that decision theory can not be fundamental.

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\(^{28}\) For we were, in choice, among the 92% who opted for the uncertainty bet.

\(^{29}\) See Appendix K.

\(^{30}\) Just like we have learned, having been Bayesians for the past ten years, to trust the Bayesian probability algorithm to teach our intuition, in those instances where the intuitive resolution is lacking to make clear and crisp plausibility assessments.

\(^{31}\) Note that this historical fact explains why Bayesians have their axiomatic house in such good order. This process started with the work of Cox, \(^{8}\), was expanded upon by Jaynes, \(^{30}\), which was then further refined by the work of Knuth and Skilling, \(^{41}\). Moreover, the more general axiomatic framework of the latter has enabled them, amongst other things, \(^{43}\), to bring some order to the field of quantum theory, by showing why this theory is forced to use a complex arithmetic. \(^{22}\).
We believe that Jaynes would have approved, would he have been told that his direct descendents, Knuth and Skilling, would be the ones that would provide the Bayesian community with the formal principles with which to assign numerical values to loss functions\[32\].

For that is what Knuth and Skilling have done, by providing the lattice theoretical framework, on which quantifications are derived, by way of associativity symmetries on the underlying lattice algebra, which inherits its meaning from the join and meet of its constituting lattice elements.

The Bernoulli law, initially derived by Bernoulli, by way of common sense first principles, has now been derived by way of a quantification on the lattice of ordering\[33\] thus, removing the one remaining degree of freedom of the Bayesian decision theory, and, in the process, demonstrating why it is that Bernoulli’s law has proved to be so ubiquitous in the field of psycho-physics. Simply, because consistency demands it.

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\[32\] Though we think that the research along the lines of [22] would have pleased him even more. Because the deepest driving motivation behind all of Jaynes’ work on statistical theory was not just the desire for more powerful practical methods of inference. It was rather the conviction that progress in basic understanding of physical law, prevented for fifty years by the positivist Copenhagen philosophy, could be resumed only by a drastic modification of the view of the world then taught to physics. Jaynes was of the belief that the mathematics of quantum theory described in part physical law, in part human inference, all scrambled together in such a way that nobody had seen how to separate them. Jaynes had become convinced that this unscrambling would require that the probability theory itself should be reformulated along the lines of the Bayesian probability theory. [28, 29].

\[33\] See Appendix D.
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FACT SHEET, PART I

The whole of Bayesian probability theory flows forth from two simple rules. The product rule,

\[ P(A) P(B | A) = P(AB) = P(B) P(A | B), \]  

and the sum rule

\[ P(\overline{A}) = 1 - P(A). \]  

By way of the product and the sum rule, we may derive the generalized sum rule,

\[ P(A + B) = P(A) + P(B) - P(AB). \]  

If we have that the propositions are exhaustive and mutually exclusive, that is, \( B = \overline{A} \), we have that, by way of (A.2) and (A.3),

\[ P(A + \overline{A}) = P(A) + P(\overline{A}) = 1, \]  

we may obtain a probability distribution. This probability distribution then may be further generalized to the bivariate probability distribution:

\[ \sum_i \sum_j P(A_i B_j) = 1, \]  

which allows us to ‘marginalize’ over parameter, say, \( B_j \), which is of no direct interest:

\[ P(A_i) = \sum_j P(A_i B_j), \]
where
\[ \sum_i P(A_i) = 1. \] (A.7)

We may let \( i \) and \( j \), that is, the number of propositions \( A_i \) and \( B_j \), tend to infinity. By doing so, we go from discrete to continuous probability distributions. Furthermore, we may add propositions \( C_k, D_l, E_m, \) etc..., and so get higher variate distributions.

Now, to a non-Bayesian it may seem to be somewhat surprising, that the whole of Bayesian probability theory flows forth from the product and rules. But the whole of Boolean logic on an operational level, is also captured by the AND- and NOT-operations. These operations correspond, respectively, with \( (A.1) \) and \( (A.2) \); as these operators combine, with the negation of a NAND-operation, in the OR-operation, which corresponds with \( (A.3) \).

Moreover, it may be shown that Boolean logic is just a special limit case of the more general Bayesian probability theory. The operators of Boolean logic combine in a like manner as the operators in Bayesian probability theory. But in Boolean logic propositions can have only the truth values true or false. Whereas in Bayesian probability theory propositions can have plausibility values in the interval \([0, 1]\), where 0 and 1, respectively, correspond with false and true.

So Boolean logic is the language of deduction, whereas Bayesian probability theory is the language of both induction and deduction; the former being a limit case of the latter, in which we have absolute knowledge about the propositions in play.

Now, on the conceptual level Bayesian probability theory is very simple. However, on a technical level, when doing an actual data-analysis, it may be quite challenging, which, as an aside, makes it fun to be a Bayesian. And we refer the reader to [56], for a first cursory glimpse on what it entails to be a Bayesian statistician.

**APPENDIX B. THE UBQUITOUS BERNOULLI LAW**

We now will give the derivations of the Bernoulli, the Weber-Fechner, and Steven’s power laws. It will be seen all that these three laws are equivalent.

**B.1. The Bernoulli Law.** The utility of a given outcome is the perceived worth of that outcome. If we take the utilities that monetary outcomes hold for us to be an incentive for our decisions, then we may perceive money to be a stimulus.

For the rich man ten dollars is an insignificant amount of money. So, the prospect of gaining or losing 10 dollars will fail to move the rich man, that is, an increment of ten dollars for him has an utility which tends to zero.

\[^{34}\text{We use the term ‘Boolean algebra’ in its meaning as referring to two-valued logic in which symbols like ‘A’ stand for propositions, [30].}\]
For the poor man ten dollars is two days worth of groceries and, thus, a significant amount of money. So, the prospect of gaining or losing ten dollars will most likely move the poor man to action. It follows that an increment of ten dollars for him has an utility significantly greater than zero.

Consider persons $A$ and $a$, with $A$ having a fortune of 100,000 full-ducats, and with $a$ a fortune of 100,000 semi-ducats, a semi-ducat being the half of a full-ducat. Let $f_A$ and $f_a$ be the moral value functions, defined on, respectively, the monetary full-ducat axis $x$ and the semi-ducat axis $\tilde{x}$. Let $x_A$ and $\tilde{x}_a$ stand for the initial wealths of $A$ and $a$, respectively; where $x_A$ and $\tilde{x}_a$ are points on the monetary axes $x$ and $\tilde{x}$, respectively.

Bernoulli derived his law by way of three simple symmetry considerations for the moral functions $f_A$ and $f_a$, [6, 54]:

1. For an arbitrary increment $c$ in wealth, the moral movement of this increment will be less for the rich man, than for the poor man; that is, if we make for $f_a$ the appropriate change of variable, from $\tilde{x}$ to $x$, then we have that

$$
\frac{d}{dx} f_A(x) \bigg|_c < \frac{d}{dx} f_a(x) \bigg|_c.
$$

From which it follows that effect of $c$ on a given $f$ decreases as the initial wealth increases.

2. It is proposed that the movement in a general moral value function $f$, for a given positive increment $dx$, is proportional to the value of this increment; that is,

$$
\frac{d}{du} f(u) \bigg|_{c=dx} \propto dx,
$$

as this is the simplest function for which $f$ increases as a function of an increment in $x$.

3. Furthermore, it is proposed that this movement in $f$ is inversely proportional to the value of the initial wealth $x$; that is,

$$
\frac{d}{du} f(u) \bigg|_{c=dx} \propto \frac{1}{x},
$$

where ‘$\propto$’ is the proportionality sign.

Bernoulli arrived at his third consideration, using the following reasoning. The change in moral value of $c$ full-ducats for $A$ will be half the change in moral value of $c$ full-ducats for $a$. Only if either $a$ sees his fortune increased to 100,000 semi-ducats, or, equivalently, 100,000 full-ducats, or if $A$ sees his fortune reduced to 50,000 full-ducats, or, equivalently, 100,000 semi-ducats, only then will $a$ have the same change in moral value as $A$ for $c$ full-ducats.
We then have that, if we make for \( f_a \) the appropriate change of variable from \( \tilde{x} \) to \( x \),
\[
\left. \frac{d}{dx} f_A (x) \right|_c = \frac{x_a}{x_A},
\]
(B.1)
where \( x_a \) is the initial fortune of \( a \), translated from the semi-ducat \( \tilde{x} \)-axis to the full-ducat \( x \)-axis.

It follows from (B.1) that we have, in general, that the change in moral value is inversely proportional to the initial we hold, that is,
\[
\left. \frac{d}{dx} f (x) \right|_c \propto \frac{1}{x},
\]
(B.2)
which is Bernoulli’s third consideration.

If we combine the second and the third consideration, we obtain the differential equation
\[
f' (x) = q \frac{dx}{x},
\]
(B.3)
which, if solved for the boundary condition that for a given person with an initial wealth of \( x_0 \) an increment of zero holds no utility, either negative or positive, gives
\[
f (x) = q \log \frac{x}{x_0},
\]
(B.4)
which may be rewritten as
\[
f (\Delta x | x_0) = q \log \frac{x_0 + \Delta x}{x_0}.
\]
(B.5)

B.2. The Weber-Fechner Law. Let \( S \) signify stimuli intensity and let \( Q \) signify sensation strength. Weber’s law states that the increment \( \Delta S \) needed to elicit a judgment that \( S + \Delta S \) is just noticeably different from \( S \) is proportional to \( S \):
\[
\Delta S = wS,
\]
(B.6)
where \( w \) is a positive constant dependent upon the specific type of sensory stimulus offered and \( \Delta S \) is understood to be the stimulus increment corresponding with a just noticeable difference.

Fechner generalized the experimental Weber law by stating that all differences in sensational strength, and not only the ones that are just noticeable, are proportional to the relative change \( \Delta S / S \), that is,
\[
\Delta Q = q \frac{\Delta S}{S}.
\]
(B.7)
where \( k \) is a positive constant dependent upon the specific type of sensory stimulus offered and \( \Delta S \) is now understood to be the stimulus increment corresponding with the increment in sensation strength \( \Delta Q \).
Dividing both sides of (B.7) by $\Delta S$ gives

$$\frac{\Delta Q}{\Delta S} = q \frac{1}{S}.$$  \hspace{1cm} (B.8)

Fechner then makes the assumption that, just as a physically small quantity $\Delta S$ can be reduced without limit to the differential $dS$, so a small quantity of sensation can be reduced without limit to the differential $dQ$. By way of this assumption, we may let (B.8) tend to the differential equation

$$\frac{dQ}{dS} = q \frac{1}{S}.$$  \hspace{1cm} (B.9)

The general solution of this differential equation is

$$Q = q \log S + c,$$  \hspace{1cm} (B.10)

where $c$ is some constant of integration.

Introducing an initial value condition for (B.9) that says that at stimulus value $S_0$ there is no sensation strength, that is, $Q(S_0) = 0$, leaves us with the Weber-Fechner law

$$Q(S|S_0) = q \log \frac{S}{S_0},$$  \hspace{1cm} (B.11)

or, equivalently,

$$Q(\Delta S|S_0) = q \log \frac{S_0 + \Delta S}{S_0}.$$  \hspace{1cm} (B.12)

The Weber-Fechner law, (B.12), is identical to the utility function which had been proposed a century earlier by Bernoulli, (B.5).

Fechner himself was aware of this equivalence. Nonetheless, he believed his derivation to be the more general. Fechner argued that Bernoulli’s derivation only applied to the special case of utility, whereas his law, though identical, applied to all sensations, as it invokes Weber’s law.

However, as pointed out in [54], Fechner failed to provide any compelling reason why the principles employed in Bernoulli’s derivation of the subjective value of objective monies should not be extendible to sensations in general. Nonetheless, we do believe that Fechner acted in good faith, in denying Bernoulli scientific primacy.

First of all, Fechner called the Weber-Fechner law, when he first published it, the Weber law. Second of all, Fechner had a deep spiritual need for some kind of harmony between the physical and mental universes, and the Weber-Fechner law provided him with this harmony, for this law spoke of the basic oneness of the physical and mental universes, [13].

The Weber-Fechner law demonstrated that both universes adhered to seemingly mechanistic laws. It then followed that the freedom of the latter universe, in terms of free will and volition, implied, by way of analogy, a commensurate freedom of the
former; thus, opening the way for the possibility of a besouled physical universe. Which had become Fechner’s only hope for spiritual salvation. \[15\].

We can imagine that Fechner might have felt that a law that assigned subjective values to objective monies was too arbitrary and sordid a foundation for the lofty purpose he wished it to serve. In contrast, the initial Weber law allowed Fechner to forgo of the money argument and derive a law, which though in form identical to Bernoulli’s, differed in that it applied to all human sensations.

B.3. Steven’s Power Law. Steven’s power law is based on the observation, that it is the ratio $\Delta Q/Q$, rather than the difference $\Delta Q$, that is proportional to $\Delta S/S$, \[60\]. This observation leads to the equality

$$\frac{\Delta Q}{Q} = q \frac{\Delta S}{S}. \tag{B.13}$$

Letting the differences in $Q$ and $S$ go to differentials, we may rewrite (B.13) as

$$\frac{dQ}{Q} = q \frac{dS}{S}. \tag{B.14}$$

This equation has its general solution

$$\log Q = q \log S + c'. \tag{B.15}$$

Taking the exponent of both sides of (B.15), we get the power law for stimulus perception

$$Q = c S^q, \tag{B.16}$$

where $c = \exp(c')$.

Stevens found the power law to hold for several sensations; binaural and monaural loudness, brightness, lightness, smell, taste, temperature, vibration duration, repetition rate, finger span, pressure on palm, heaviness, force of hand grip, autophonic response, and electric shock, \[60\].

The power law is applied by letting subjects compare the sensation ratio of $Q_1$ to $Q_0$ for corresponding stimuli strengths $S_1$ and $S_0$:

$$\frac{Q_1}{Q_0} = \left(\frac{S_1}{S_0}\right)^q. \tag{B.17}$$

Let $S_1 = S_0 + \Delta S$, where $\Delta S$ is some increment, then we may rewrite (B.17) as

$$\frac{Q_1}{Q_0} = \left(\frac{S_0 + \Delta S}{S_0}\right)^q. \tag{B.18}$$

For an increment of $\Delta S = 0$, the ratio of perception stimuli will be $Q_1/Q_0 = 1$. Taking the log of the ratio (B.18) we may map the ratio of perceived stimuli to a corresponding utility scale where a zero increment $\Delta S$ corresponds with a zero
utility:
\[
Q' (\Delta S | S_0) = \log \frac{Q_1}{Q_0} = q \log \frac{S_0 + \Delta S}{S_0}
\]

(B.19)

But this is just the Weber-Fechner law, (B.13).

B.4. Summary. The Weber-Fechner law gives us just noticeable differences on a log scale, (B.13). The power law gives us ratios of sensation strengths, (B.18). Taking the log of the ratio of sensation strengths, we may obtain the just noticeable differences again, (B.19). But the Weber-Fechner for just noticeable differences is just the Bernoulli law for utilities, (B.5).

We refer the reader to [54], for a discussion of Thurnstone’s derivation of the satisfaction law. This law, which takes as its input the increment in the number of items of commodity, is also of the form of Bernoulli’s law.

APPENDIX C. THE NEGATIVE BERNOULLI LAW

In this appendix we present the negative Bernoulli law for debts, which is a corollary of the Bernoulli law for income. The negative Bernoulli law predicts that for the very poor, having a small initial wealth and large initial debts, a large loss of direct income will be more devastating, than an increase of, say, twice that loss in their long-term debt. This law also explains why, for these poor, having a small initial wealth and large initial debts, the temptation to take out loans, if offered the opportunity, will be quite great, [25].

Until now we have treated only the case were the maximal loss did not exceed the initial wealth \( m \). However, in real life we may lose more than we actually have, by way of debt. So, we now proceed to assign utilities to increments in debt.

According to the Weber-Fechner law we cannot lose more money than we initially had. Otherwise we may have that the ratio in the logarithm in the Weber-Fechner Law, (4.4),
\[
u(\Delta S | S) = q \log \frac{S + \Delta S}{S},
\]

may become negative, leading to a breakdown of the logarithm.

However, whenever we incur a debt we lose more money than we have. Furthermore, we can have a debt and an income, both at same time. So, we propose that there are two different monetary stimuli dimensions in play; the first dimension being an actual income dimension and the second dimension being a debt dimension.

We propose to model the debt utilities by way of the negative Weber-Fechner law:
\[
u(\Delta D | D) = -b \log \frac{D + \Delta D}{D},
\]

(C.2)

where we let \( D \) be the initial debt, \( \Delta D \) the increment in debt, and \( b \) the the Weber constant of a monetary debt.
The rationale behind (C.2) is as follows. If we view a debt increment as a stimulus, then it follows that we may use the psycho-physical Weber-Fechner law in the determination of the moral value of a given debt increment.

For positive increments $\Delta D$, there is an increase in current debt, whereas for negative increments $\Delta D$, there is a decrease in current debt. In order to assign both a negative utility to an increase in current debt and positive utility to a decrease in debt, we need to multiply the Weber-Fechner law times minus one, (C.2).

If we have no initial debt, that is, $D = 0$, then (C.2) tells us that any positive increment in debt $\Delta D$ would have a utility of minus infinity. This is clearly not realistic. So, in order to model an increment in debt for those who are without debt, we must introduce a minimum significant amount of debt which is equal to minimum significant amount of income, $\gamma$.

The threshold amount of debt, $\gamma$, may also be used in the case of $\Delta D = -D$, in order to prevent an infinite utility being assigned to a full repaying of one’s debts. Using the concept of the minimum significant amount of debt stimulus, we may modify (C.2) as

$$u(\Delta D \mid D) = -b \log \frac{D + \Delta D}{D}, \quad -D + \gamma < \Delta D < \infty,$$

(C.3)

If we want to give a graphical representation of (C.3), then the Weber constant $b$, must be set to some numerical value.

Say, we have a total debt of forty thousand dollars, in the form of a student loan, which we eventually will have to pay back, but not right now. Then introspection would suggest that a increment or decrement of an amount less than a thousand dollars would not move us that much.

So, $\Delta D = 1000$ constitutes one utile, or, equivalently, a just noticeable difference in debt for an initial debt of $D = 40,000$, that is, (C.4):

$$1 \text{ utile} = -b \log \frac{40,000 - 1000}{40,000}.$$

If we then solve for the unknown Weber constant $b$ of debt stimuli,

$$b = -\frac{1}{\log 390000 - \log 40000} \approx 40,$$

(C.5)

we find this Weber constant to be smaller by a factor of 2.5 than the Weber constant $q$ of income stimuli, (4.7).

It is well possible that this difference in Weber constants can be attributed to the difference in abstractness of the concepts. The losing of actual monies is quite concrete, whereas the accruing of a debt, repayable somewhere in a distant future, is somewhat more abstract.
But there is always a chance that these authors were off in their introspection and that both Weber constants should be approximately equal. We leave this issue, together with the psychological reality of the phenomenon of debt relief, given below, for future psychological experimentation, as we proceed with our discussion of the debt utilities.

Suppose that a student has a student loan which has accumulated to forty thousand dollars. Then, by way of (C.2) and (C.5), we obtain the following mapping of increments in debt to utilities, Figure 17.

![Figure 17. Utility plot for initial debt 40,000 dollars](image)

As stated previously, loss aversion is the phenomenon that losses may loom larger than gains. In Figure 17 we see the phenomenon that debt reduction may loom larger than debt increase. We will call this corollary of the psycho-physical Weber-Fechner law: ‘debt relief’, the relief of loosing one’s debts.

Now, does the phenomenon of debt relief correspond with a real psychological phenomenon? We belief that it actually does.

Say, we have a debt of a thousand dollars. Then we can imagine ourselves feeling greatly relieved, were we to be released of our debt. Now, were our debt, instead, to be doubled to two thousand dollars, then we can also imagine ourselves feeling unhappy about this. But this feeling of unhappiness about the doubling of our debt would be of a lesser intensity than the corresponding relief of having our debt acquitted.

Note that actual value of the Weber constant $b$ of the debt stimuli has no direct bearing on any of the results given in this fact sheet; save the handful of examples which are given in this section, in order to demonstrate the qualitative behavior of the negative Weber-Fechner law, or, equivalently, the negative Bernoulli law.
We will now look at the practical implications of the negative Bernoulli law, \( (C.2) \), and its Weber constant \( b \), \( (C.5) \).

A student loan initially represents a gain in debt stimulus. This debt makes itself felt, in terms of actual loss of income, only after graduation, the moment the monthly payments have to be paid and take a considerable chunk out of one’s actual income.

Say, that the student of Figure 17, having become a PhD, and having a net income of fifteen hundred dollars, is called upon to make good on his loan, by way of monthly payments of five hundred dollars. Then these payments represent both a loss in income, having a negative utility of, \( (C.1) \) and \( (4.7) \):

\[
u^{\text{income-loss}} = 100 \log \frac{1500 - 500}{1500} = -41.5, \tag{C.6}
\]

as well as a decrements in debt, having a positive utility of, \( (C.2) \) and \( (C.5) \):

\[
u^{\text{debt-decrease}} = -40 \log \frac{40000 - 500}{40000} = 0.5, \tag{C.7}
\]

It follows that our PhD can find little to no comfort in the fact that he is paying of his debt, as he acutely feels the sting of loss of income. This is, together with the difference in Weber constants \( (4.7) \) and \( (C.5) \), reflective of the fact that his utility function for income is highly non-linear in the neighborhood of the increment, whereas his utility function for debt is highly linear in that region.

Now, say that we have another PhD, who during his student days lived a more frugal life style and, consequently, only has a debt of two thousand dollars. For this PhD student, when called upon to make good on the loan, the loss of income will be felt just as keenly, with a negative utility of \( u = -40.5 \), \( (C.6) \). However, he will find more satisfaction in the fact that he is paying of his debts, \( (C.2) \) and \( (C.5) \):

\[
u^{\text{debt-decrease}} = -40 \log \frac{2000 - 500}{2000} = 11.5, \tag{C.8}
\]

seeing that he has a more curved utility function for debt than our previous PhD student.

Nonetheless, the first PhD student may feel, after a couple of years of monthly repayments, when his loan has been reduced to twenty thousand dollars, for the first time, as if he has an actual stake in the repayment of his debt, \( (C.2) \) and \( (C.5) \):

\[
u^{\text{debt-decrease}} = -40 \log \frac{20000 - 500}{20000} = 1.0, \tag{C.9}
\]

as his debt repayment utility crosses the threshold of the just noticeable difference.

The negative Bernoulli law also gives an explanation why for the very poor, having a minimum monthly wage of seven hundred euros, and already having a

\[36\] A difference which accounts only for a factor of 2.5 in the observed differences of the utilities \( (C.6) \) and \( (C.7) \).
large debt of, say, twenty thousand euros, a loss of income of, say, five hundred euros, is perceived to be so much more devastating than an increase in debt of, say, a thousand euros.

For this poor person, the loss of actual income has a negative utility of $-125$ utiles and the gain of an increase of has a negative utility of only $-2$ utiles.$^{37}$

Likewise, the temptation for the very poor, if offered the opportunity, to take out a loan of a thousand euros will be quite great.

As for this poor person, the immediate gain of a direct increase of a thousand euros in income will have a positive utility of $+89$ utiles, whereas the negative utility of an increase in debt of a thousand dollars will have a negative utility of only $-2$ utiles.$^{38}$

APPENDIX D. A CONSISTENCY PROOF OF THE BERNOULLI LAW

In the Bayesian decision theory, we start by constructing our outcome probability distributions, by way of the product, sum, and generalized rules of the Bayesian probability theory. We then proceed to assign utilities to the outcomes of these probability distributions, by way of the Bernoulli law, in order to construct our utility probability distributions. Finally, we compare the location of these utility probability distributions by way of some function of the cumulants of the utility probability distributions.

The product, sum, and generalized rules of the Bayesian probability theory are the only consistent operators on probabilities.$^{8,30,41}$ So, consistency wise, we have no choice but to use these rules to construct our outcome probability distributions.

The cumulant function initially proposed by Bernoulli was the identity function for the first cumulant, that is, the expectation value of the utility probability distribution. But this proposal, though sufficient enough in many cases, may, nonetheless, lead to Ellsberg and Allais ‘paradoxes’, which is an indication that the information of the higher order cumulants should also be taken into account.

If we take as our function of the cumulants of the utility probability distribution the skewness intervals, then we find that all these paradoxes fall away and, moreover, leave us with a decision theoretical algorithm, which is both surprisingly rich in structure and eminently intuitive.

In order to map monetary outcomes of the outcome probability distributions to their corresponding utilities, and so construct the utility probability distributions, we make use the Bernoulli law.

$^{37}$Even if $b = 100$, (C.5), this negative utility will only be $-5$ utiles.

$^{38}$Idem.
This law is the one remaining degree in the Bayesian decision theory. In this section we will give the derivation of the Bernoull law, by way of consistency constraints on the lattice of ordering.

D.1. Lattice Theory and Quantification. Two elements of a set are ordered by comparing them according to a binary ordering relation, that is, by way of ‘≤’, which may be read as ‘is included by’. Elements may be comparable, in which case they form a chain, or they may be incomparable, in which case they form a antichain. A set consisting of both inclusion and incomparability are called partially ordered sets, or posets for short. [38]

Given a set of elements in a poset, their upper bound is the set of elements that contain them. Given a pair of elements \( x \) and \( y \), the least element of the upper bound is called the join, denoted \( x \lor y \). The lower bound of a pair of elements is defined dually by considering all the elements that the pair of elements share. The greatest elements of the lower bound is called the meet, denoted \( x \land y \).

A lattice is a partially ordered set where each pair of elements has a unique meet and unique join. There often exist elements that are not formed from the join of any pair of elements. These elements are called join-irreducible elements. Meet-irreducible elements are defined similarly. We can choose to view and join and meet as algebraic operations that take any two lattices elements to a unique third lattice element. From this perspective, the lattice is an algebra.

An algebra can be extended to a calculus by defining functions that take lattice elements to real numbers. This enables one to quantify the relationships between the lattice elements.

A valuation \( v \) is a function that takes a single lattice element \( x \) to a real number \( v(x) \) in a way that respects the partial order, so that, depending on the type of algebra, either \( v(x) \leq v(y) \) or \( v(y) \leq v(x) \), if in the poset we have that \( x \leq y \). This means that the lattice structure imposes constraints on the valuation assignments, which can be expressed as a set of constraint equations, [40].

The Ordering Space. The set of all possible orderings is called the ordering space.

The lattice of ordering is generated by taking the power set, which is the set of all possible subsets of the set of all order elements, say, \( x, y, z, \) etc..., where \( x < y < z < \) etc..., and ordering them according to Polya’s min-max rule, [39], where the meet \( \land \) is defined as

\[
x \land y = \min (x, y) = x,
\]

and the join is defined as

\[
x \lor y = \max (x, y) = y.
\]
The ordering relation of the min-max rule naturally encodes ordering, such that an ordering element higher up on the lattice is always greater or equal than all the connecting elements below it. Likewise, an ordering element further down on the lattice is always smaller or equal than all the connecting elements below it.

For example, \( y \) is greater than the lower lattice element \( x \lor y, x \lor y \lor z \), to which it is directly connected, by way of \( x \lor y \) and \( y \lor z \). Likewise, \( y \) is greater than the higher lattice element \( x \land y, x \land y \lor z \), to which it is directly connected, by way of \( x \land y \) and \( y \land z \). In this sense the lattice of ordering is an algebra.

In what follows we derive a measure, called the Bernoulli law, that quantifies the degree of ordering.

D.2. The general sum rule. We begin by considering a special case of elements \( x \) and \( y \) with join \( x \lor y \). In Figure 18 we give the graphical representation of this simple lattice.

\[
\begin{align*}
\frac{a \lor b}{a} & \quad \frac{b}{b}
\end{align*}
\]

**Figure 18.** Lattice of \( x \lor y \)

The value we assign to the join \( x \lor y \), written \( v(x \lor y) \), must be a function of the values we assign to both \( x \) and \( y \), \( v(x) \) and \( v(y) \). Since, if there did not exist any functional relationship, then the valuation could not possibly reflect the underlying lattice structure; that is, valuation must maintain ordering, in the sense that \( x \leq x \lor y \) implies either \( v(x) \leq v(x \lor y) \) or \( v(x) \geq v(x \lor y) \).

So, we write this functional relationship in Figure 18 in terms of an unknown binary operator \( \oplus \):

\[
v(x \lor y) = v(x) \oplus v(y). \quad (D.3)
\]

We now consider another case where we have three elements \( x, y, \) and \( z \), Figure 19.
Because of the associativity of the join, we have that the least upper bound of these three elements, \( x \lor y \lor z \), can be obtained in these two different ways:

\[
x \lor (y \lor z) \quad \text{and} \quad (x \lor y) \lor z.
\] (D.4)

By applying (D.3) to (D.4), the value we assign to this join can also be obtained in two different ways:

\[
v(x) \oplus [v(y) \oplus v(z)] \quad \text{and} \quad [v(x) \oplus v(y)] \oplus v(z).
\] (D.5)

Consistency then demands that the equivalent assignments (D.5) have the same value:

\[
v(x) \oplus [v(y) \oplus v(z)] = [v(x) \oplus v(y)] \oplus v(z).
\] (D.6)

This the functional equation for the operator \( \oplus \), for which the general solution is given by, [1]:

\[
f[v(x \lor y)] = f[v(x)] + f[v(y)],
\] (D.7)

where \( f \) is an arbitrary invertible function, so that many valuations are possible. We define the valuation \( u \) as

\[
u(x) \equiv f[v(x)],
\]

and rewrite (D.7) as

\[
u(x \lor y) = u(x) + u(y).
\] (D.8)

Now that we have a constraint on the valuation for our simple example, we seek the general solution for the entire lattice. To derive the general case, we consider the lattice in Figure 20.
If we apply (D.8) to both the elements \( y \) and \( x \lor y \), we get
\[
u(y) = \nu(x \land y) + \nu(z),
\]
and, since \( y \) is just the join of the part it shares with \( x \) joined with \( z \), where, for the lattice of ordering, \( z \) is understood to be the meet, (D.1), with \( y \) and some other ordering element to the right,

\[
u(x \lor y) = \nu(x) + \nu(z).
\]

Substituting for \( \nu(z) \) in (D.9) and in (D.10), we get the general sum rule:
\[
u(x \lor y) = \nu(x) + \nu(y) - \nu(x \land y)
\]
(D.11)

In general, for bi-valuations we have
\[
w(x \lor y|t) = w(x|t) + w(y|t) - w(x \land y|t),
\]
(D.12)
for any context \( t \), [41].

Note that the sum rule is not focused solely on joins since it is symmetric with respect to interchange of joins and meets.

At this point we have derived additivity of the measure, which is considered to be an axiom of measure theory. This is significant in that associativity constrains us to have additive measures - there is no other option, [41].

If we apply (D.1) and (D.2), which are the operators of this particular lattice, to (D.12), we are left with the platitude
\[
w(y|t) = w(x|t) + w(y|t) - w(x|t) = w(y|t),
\]
(D.13)
which, nonetheless, is very consistent.
So, we find that on the lattice of ordering the general sum rule provides no other constraint than that the quantification \( w \) should assign the same value to the same argument, which we intended to do anyway.

Now, for both the lattice of statements and questions, which quantify, respectively, to the Bayesian probability and information theories, we have that the general sum rule \((D.12)\) is a highly non-trivial operator, as it gives rise to the general sum rule of the measures of probability and relevancy, respectively.

**Chain Rule.** We now focus on bi-valuations and explore changes in context \[40\].

We begin with a special case and consider four ordered elements \( x \leq y \leq z \leq t \).

The relationship \( x \leq z \) can be divided into the two relations \( x \leq y \) and \( y \leq z \). In the event that \( z \) is considered to be the context, this sub-division implies that the context can be considered in parts. The bi-valuation we assign to \( x \) with respect to the context \( z \), that is, \( w(x|z) \), must be related to both the bi-valuation we assign to \( x \) with respect to the context \( y \), that is, \( w(x|y) \), and the bi-valuation we assign to \( y \) with respect to the context \( z \), that is, \( w(y|z) \).

So, there exists a binary operator \( \otimes \) that relates the bi-valuations assigned to the two steps to the bi-valuation assigned to the one step:

\[
w(x|z) = w(x|y) \otimes w(y|z). \tag{D.14}
\]

By extending \((D.14)\) to three steps, and considering the bi-valuation \( w(x|t) \), relating \( x \) and \( t \), via intermediate contexts \( y \) and \( z \), we get Figure 21.

![Figure 21. Context lattice of t](image)

This figure leads to the associativity relationship:

\[
[w(x|y) \otimes w(y|z)] \otimes w(z|t) = w(x|y) \otimes [w(y|z) \otimes w(z|t)]. \tag{D.15}
\]
By way the associativity theorem, \[41\], we have that any operator, be it \(\oplus\), \(\otimes\), or \(\odot\), has a scale on which associativity relations \[D.6\] and \[D.15\] are additive, which would seem to solve our \[D.15\] trivially.

However, once we have fixed the behavior of \(w\) to be additive with respect to either the arguments before the solidus or the arguments behind the solidus, we can not regrade to that scale anymore. We then will have to infer additivity on some other grade, say, \(\Theta (w)\), \[41\].

For example, in the quantification of the lattice of statements we are forced to infer additivity on the alternative grade \(\Theta\); seeing that we have lost the degree of freedom of addition on the grade \(w\) when we find a non-trivial generalized sum rule \[D.15\], \[41\].

So, for the lattice of statements, we have that the chain rule for context change forces us to use addition on the alternative grade \(\Theta\), which leaves us with the equality:

\[
\Theta [w (x \| z)] = \Theta [w (x \| y)] + \Theta [w (x \| y)], \quad (D.16)
\]

If we solve \[D.16\], it is found that \(\Theta\) is the logarithmic function times some constant \(q\), which we may set to one, if we so like, \[41\].

Since we have that the inverse of the logarithmic function is the exponential function, we may label this inverse as, say, \(\Psi\). We then return to our original grade \(w\), on which we have derived the general sum rule, by way of inversion:

\[
w (x \| z) = \Psi \{\Theta [w (x \| y)] + \Theta [w (x \| y)]\}
\]

\[= e^{\log [w(x \| y)] w (x \| y)} \quad (D.17)\]

\[= w (x \| y) w (x \| y)\]

which is the product rule of Bayesian probability theory.

Now, seeing that for the Bernoulli law addition on the \(w\) grade is still allowed, \[D.13\], we have that by way of the associativity theorem, \[D.15\] results in a constraint equation for non-negative bi-valuations involving changes in context \[41\]:

\[
w (x \| z) = w (x \| y) + w (y \| z), \quad (D.18)
\]

where the grade \(w\) in \[D.18\] is the same as the grade \(\Theta (w)\) in \[D.16\].

It then follows, by way of \[41\], that \(w (x)\) is of the form \(q \log (x)\), which leaves with the change of context rule of decreasing orderings

\[
q_d \log (x \| z) = q_d \log (x \| y) + q_d \log (y \| z). \quad (D.19)
\]

where \(q_d\) is the chain rule constant for decreasing orderings.
Alternatively, if $x$ is considered to be the context, rather than $z$, then the subdivision of $x \leq z$ in relations $x \leq y$ and $y \leq z$ also implies that the context can be considered in parts.

The bi-valuation we assign to $z$ with respect to the context $x$, that is, $w(z|x)$, must be related to both the bi-valuation we assign to $z$ with respect to the context $y$, that is, $w(z|y)$, and the bi-valuation we assign to $y$ with respect to the context $x$, that is, $w(y|x)$.

So, there again exists a binary operator $\otimes$ that relates the bi-valuations assigned to the two steps to the bi-valuation assigned to the one step

$$w(z|x) = w(z|y) \otimes w(y|x).$$

(D.20)

By extending (D.20) to three steps, and considering the bi-valuation $w(t|x)$, relating $t$ and $x$, via intermediate contexts $z$ and $y$, we get Figure 22:

![Figure 22. Context lattice of t](image)

This figure leads to the associativity relationship:

$$w(t|z) \otimes [w(z|y) \otimes w(y|x)] = [w(t|z) \otimes w(z|y)] \otimes w(y|x).$$

(D.21)

This relationship then leads us, by way of (D.16), (D.17), and (D.18), to the change of context rule of increasing orderings

$$q_i \log(z|x) = q_i \log(z|y) + q_i \log(y|x).$$

(D.22)

where $q_i$ is the chain rule constant for increasing orderings.

D.3. Deriving the Bernoulli Law. We now will apply chain rule (D.19) to the lattice of ordering in Figure 23 where the elements $x$, $y$, $z$ are understood to be orderings.
We focus on the small diamond in Figure 23 defined by \( x, x \lor y, y, \) and \( x \land y \). If we consider the context to be \( x \lor y \), then the chain rule (D.19) for this diamond may be written down as:

\[
q_d \log (x \land y | x \lor y) = q_d \log (x \lor y | y) + q_d \log (y | x \lor y). 
\] (D.23)

which reduces to, by way of (D.1) and (D.2),

\[
q_d \log (x | y) = q_d \log (x | y) + q_d \log (y | y). 
\] (D.24)

which implies that

\[
q_d \log (y | y) = 0. 
\] (D.25)

It follows from (D.25) and the properties of the logarithm that

\[
q_d \log (x | y) = q_d \log \frac{x}{y}. 
\] (D.26)

We again focus on the small diamond in Figure 23 defined by \( x, x \lor y, y, \) and \( x \land y \). If we now consider the context to be \( x \land y \), then the chain rule (D.22) for this diamond may be written down as:

\[
q_i \log (x \lor y | x \land y) = q_i \log (x \lor y | y) + q_i \log (y | x \land y). 
\] (D.27)

which, by way of (D.1) and (D.2), reduces to

\[
q_i \log (y | x) = q_i \log (y | y) + q_i \log (y | x). 
\] (D.28)

which implies that

\[
q_i \log (y | y) = 0. 
\] (D.29)

It follows from (D.29) and the properties of the logarithm that

\[
q_i \log (y | x) = q_i \log \frac{y}{x}. 
\] (D.30)
Now, we may go, by way of (D.30), from an ordering $x$ to $y$, and then, by way of (D.26), go from $y$ back to $x$ again. The taking of this path should be consistent, in that the net gain in ordering is zero, which leaves with the equality:

$$q_i \log \frac{y}{x} + q_d \log \frac{x}{y} = 0, \quad (D.31)$$

or, equivalently,

$$q_i \log \frac{y}{x} = -q_d \log \frac{x}{y} = q_d \log \frac{y}{x}. \quad (D.32)$$

It follows from (D.32) that for the lattice of order, consistency demands that the constants $q_i$ and $q_d$ must be equal, which leaves us with the Bernoulli law, which for $x < y$, assigns the valuation

$$q \log \frac{y}{x}. \quad (D.33)$$

for an increase in ordering, and

$$q \log \frac{x}{y}. \quad (D.34)$$

for a decrease in ordering.

If the ordering elements $x_1, x_2, \text{etc.}$, are numbers on the positive real, where $x_1 < x_2 < \text{etc.\ldots}$, then $x_i$ will tend to $\infty$, as $i \to \infty$. However, if the ordering elements $x_1, x_2, \text{etc.}$, are numbers on the negative real, where $x_1 < x_2 < \text{etc.\ldots}$, then $x_i$ will tend to 0, as $i \to \infty$.

This then explains why loss aversion, a phenomenon belonging to the positive Bernoulli law, in which losses are weighted heavier than commensurate gains, in the negative Bernoulli law, changes to the phenomenon of debt relief in which gains are weighted heavier than commensurate losses.

**APPENDIX E. THE DERIVATION OF THE SKEWNESS CORRECTED CI**

To the best of our knowledge, no generalization of the time proven interval (6.3), in the form of the intervals (6.4) and (6.5), is to be found in the statistical literature. So, we can sympathize if, at a first glance, these intervals might seem somewhat arbitrary.

In order to take away from this possible sense of arbitrariness, we shall now share here the reasoning process that led us to our discovery of the skewness corrected confidence intervals.

The search of (6.4) started with two simple considerations. Firstly, we were looking for a skewness corrected confidence interval which for $\gamma = 0$, this being the

\[\text{Note that Cornish-Fisher expansions are cumulant corrected confidence bounds for sampling statistics, which are obtained by way of series expansions in the sample size } n. \text{ Here we do not have sampling statistics, samples, or, for that matter, sample sizes; as we ourselves quickly came to realize, in our initial search for the skewness corrected confidence interval. Furthermore, the adjusting terms in the Cornish-Fisher expansions, which are summated, are of the form: power of a cumulant times a polynomial function; which is not the form which we have here, [23].} \]
skewness of the normal distribution, would revert back to (6.3); as it is only by such a property that the new skewness corrected confidence interval may encompass the standard confidence interval (6.3) as a special limit case. Secondly, we desired from our corrected confidence interval that it should take into account the skewness $\gamma$ in such a way that for $\gamma > 0$ it should compress the lower bound while elongating the upper bound; as this is the qualitative way in which, relative to (6.3), positive skewness ought to be corrected.

These considerations led us, for positive skewness, to the initial proposal:

$$\left(\mu - \frac{\sigma}{1 + \gamma}, \mu + (1 + \gamma) \sigma\right).$$  \hfill (E.1)

But it was found that with this proposal the corrected confidence interval of the Bernoulli distributions, for $p \geq 0.5$ and outcomes $C_1$ and $C_2$, where $C_1 < C_2$, was approximately constant:

$$\left(\mu - \frac{\sigma}{1 + \gamma}, \mu + (1 + \gamma) \sigma\right) \approx (C_1, C_2),$$  \hfill (E.2)

with equality holding in the limits $p \to 0.5$ and $p \to 1$.

It was also found that for the binomial distributions, having outcomes $i = 0, 1, \ldots, n$, the interval (E.1) converged to the interval $(0, 1)$, as $p$, the probability of a success, tended to zero.

This meant that our proposal would not do, for it followed that (E.1) as a confidence interval would lead to a loss of the probabilistic element in our decision theoretical analyses.

Nonetheless, on the up-side, for the exponential distribution,

$$p(x|\lambda) = \lambda \exp (-\lambda x), \quad 0 \leq x < \infty$$  \hfill (E.3)

which has a mean, standard deviation, and skewness of, respectively,

$$\mu = \frac{1}{\lambda}, \quad \sigma = \frac{1}{\lambda}, \quad \gamma = 2,$$  \hfill (E.4)

there were some encouraging results to report.

The traditional confidence interval of the exponential distribution, as found by way of (E.4) and the unadjusted (6.3), is $(0, \frac{2}{\lambda})$ and has a coverage of:

$$\int_{0}^{\frac{2}{\lambda}} \lambda \exp (-\lambda x) \, dx = 0.86.$$  \hfill (E.5)

It was found that (E.1), by way of (E.4), translated to the interval $(\frac{2}{3\lambda}, \frac{4}{\lambda})$. This interval had a coverage of:

$$\int_{\frac{2}{3\lambda}}^{\frac{4}{\lambda}} \lambda \exp (-\lambda x) \, dx = 0.50.$$  \hfill (E.6)
Now, if we compare the coverages (E.6) and (E.5) with the coverage of the standard sigma interval \((\mu - \sigma, \mu + \sigma)\) for the normal distribution:

\[
\int_{\mu - \sigma}^{\mu + \sigma} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2} (x - \mu)^2\right] dx = 0.68,
\]

then it would seem that the adjusted (E.6) was no worse than the traditional (E.5). Moreover, the lower bound of the adjusted interval was no longer the trivial zero, while the upper bound had been elongated by, what would seem to be a reasonable factor.

These modest successes for the confidence interval of the continuous exponential distribution, in terms of qualitative behavior and actual coverage, managed to give us a sense of being on the right track somehow.

We then contemplated that the standard deviation \(\sigma\) is the square root of the second order central moment; the square root being the operation by which we translate the second-order information about the spread in our probability distribution to the first-order dimension, which is the dimension in which our propositions of interest reside.

So maybe we had to take the third central moment \(m^{(3)}\), (6.1) and (6.2):

\[
m^{(3)} = \int (x - \mu)^3 p(x|\theta) dx = \gamma \sigma^3,
\]

take its third square root, and then replace the \(\gamma\)'s in (E.1) with that root.

This led us to our second proposal

\[
(\mu - (1 + \sqrt[3]{\gamma}) \sigma, \mu + \frac{\sigma}{1 + \sqrt[3]{\gamma}} \sigma),
\]

But it was found that with this second proposal the corrected confidence interval of the Bernoulli distributions, for \(p \geq 0.5\) and outcomes \(C_1\) and \(C_2\), where \(C_1 < C_2\), resulted in an unwanted factor \(C_2 - C_1\) in the \(\sigma\)'s following \(\sqrt[3]{\gamma}\).

Without this factor (E.9) seemed to work quite well for the Bernoulli distributions, with corrected confidence intervals that were probabilistic; that is, intervals whose bounds converged to the expectation values as we approached certainty. So, the question then became: How to loose this factor \(C_2 - C_1\) in a non-arbitrary manner?

If we could express the factor \(C_2 - C_1\) as a function of the cumulants of the Bernoulli distribution, then we could, on the one hand, divide this disruptive factor out and, on the other hand, obtain the, apparently, necessary cumulant correction for our skewness confidence interval.

We then remembered that our initial proposal (E.1), when applied to Bernoulli distributions, resulted in the non-probabilistic interval \((C_1, C_2)\), which has a range of \(C_2 - C_1\). This range being equal the factor that we wished to see eliminated.
Rewriting the interval \([E.1]\) as a range, we arrived at the ‘support’:

\[
\mu + (1 + \gamma) \sigma - \left(\mu - \frac{\sigma}{1 + \gamma}\right) = (1 + \gamma) \sigma + \frac{\sigma}{1 + \gamma}.
\]  
(E.10)

Substituting \((E.10)\) into \((E.9)\), in such a way that the factor \(C_2 - C_1\) was lost, we then obtained our final proposal \((6.4)\):

\[
\left[\mu - \frac{\sigma}{1 + \frac{3\sqrt{\gamma}}{1 + \gamma}}, \mu + \left(1 + \frac{3\sqrt{\gamma}}{1 + \gamma + \frac{1}{1 + \gamma}}\right) \sigma\right].
\]

Having found \((6.4)\), it was then easy enough to find, by way of symmetry arguments, the skewness corrected interval \((6.5)\) for \(\gamma < 0\).

**E.1. Supporting contacts for the skewness confidence interval.** The interval \((6.4)\), together with \((E.4)\), translates for the exponential distribution, \((E.3)\),

\[
p(x|\lambda) = \lambda \exp(-\lambda x), \quad 0 \leq x < \infty
\]

which has a mean, standard deviation, and skewness of, respectively, \((E.4)\),

\[
\mu = \frac{1}{\lambda}, \quad \sigma = \frac{1}{\lambda}, \quad \gamma = 2,
\]

to the skewness corrected interval

\[
\left(\frac{3\sqrt{2}}{(10 + 3\sqrt{2}) \lambda}, \frac{20 + 3\sqrt{2}}{10\lambda}\right).
\]  
(E.11)

This interval has a coverage of:

\[
\int_{\frac{20 + 3\sqrt{2}}{10 + 3\sqrt{2} \lambda}}^{\frac{20 + 3\sqrt{2}}{10\lambda}} \lambda \exp(-\lambda x) \, dx = 0.67.
\]  
(E.12)

which is very close to the benchmark coverage value of 0.68 of the 1-sigma confidence interval of the normal distribution, \((E.7)\),

\[
\int_{\mu - \sigma}^{\mu + \sigma} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2} (x - \mu)^2\right] \, dx = 0.68.
\]

We conjecture that the missing 0.01 probability density coverage in \((E.12)\), relative to \((E.7)\), is a function of the kurtosis and the other higher order cumulants of the exponential distribution\(^{40}\), seeing that any probability distribution is wholly determined by its moments.

\(^{40}\) We also conjecture that the kurtosis correcting term will be exponentially larger than the skewness correcting term; seeing the progression from the standard deviation ‘correcting’ term, to the skewness correcting term:

\[
\sigma \quad \text{to} \quad 1 + \frac{\sqrt{\gamma}}{1 + |\gamma| + \frac{1}{1 + |\gamma|}}.
\]
In contrast, the traditional confidence interval of the exponential distribution, as found by way of (E.4) and the unadjusted (6.3), is
\[ \left( 0, \frac{2}{\lambda} \right) \]  
\text{(E.13)}
and has a coverage of, (E.5),
\[ \int_0^{\frac{2}{\lambda}} \lambda \exp(-\lambda x) \, dx = 0.86. \]
So, it would seem that the adjusted (E.11) is much closer to the mark than the traditional (E.13).

The beta distribution is defined as:
\[ p(\theta | r, n) = \frac{(n - 1)!}{(r - 1)! (n - r - 1)!} \theta^{r-1} (1 - \theta)^{n-r-1}, \quad 0 \leq \theta \leq 1 \]  
\text{(E.14)}
For \( r = 5 \) and \( n = 10 \), where we have a symmetrical beta distribution with \( \gamma = 0 \), and an expectation value of
\[ E(\theta) = \mu = \frac{r}{n} = 0.5, \]  
\text{(E.15)}
we have that (6.4) collapses, by construction, to (6.3), giving a shared confidence interval of
\[ (0.35, 0.65), \]
which corresponds with a coverage of
\[ \int_{0.35}^{0.65} \frac{(n - 1)!}{(r - 1)! (n - r - 1)!} \theta^{r-1} (1 - \theta)^{n-r-1} d\theta = 0.66. \]  
\text{(E.16)}
A coverage which, for \( r = n/2 \), will converge to the benchmark coverage (E.7), as \( n \) goes to infinity.

For the more severe case of \( r = 1 \) and \( n = 10 \), where we have a skewed beta distribution, \( \gamma = 1.47 \), and an expectation value
\[ E(\theta) = \mu = \frac{r}{n} = 0.1, \]
we find that (6.4) will give the corrected confidence interval
\[ (0.04, 0.23), \]
which corresponds with a coverage of
\[ \int_{0.04}^{0.23} \frac{(n - 1)!}{(r - 1)! (n - r - 1)!} \theta^{r-1} (1 - \theta)^{n-r-1} d\theta = 0.63, \]  
\text{(E.17)}
which is still very close to the benchmark (E.7).

For comparison, for \( r = 1 \) and \( n = 10 \) the uncorrected (6.3) will give a confidence interval,
\[ (0.01, 0.19), \]
whose lower bound is four times closer to the trivial zero, and as a consequence, as the bulk of the probability density of a positively skewed distribution lies to the left, will give an inflated coverage of

\[
\int_{0.01}^{0.19} \frac{(n-1)!}{(r-1)!(n-r-1)!} \theta^{r-1} (1-\theta)^{n-r-1} d\theta = 0.77. \quad (E.18)
\]

Furthermore, for \( r = 1 \) and \( n \to \infty \), the skewness of the beta distribution converges to

\[
\gamma \to 2.
\]

As a consequence, the coverage of unadjusted interval \((6.3)\) diverges from the benchmark coverage \((E.7)\), with a ‘limit’ of 0.86, for \( n = 10^6 \). In contrast, the coverage of \((6.4)\) converges to the coverage \((E.12)\), with a ‘limit’ of 0.67, for \( n = 10^6 \); where we note that the exponential distribution \((E.3)\) has a skewness of \( \gamma = 2 \), \((E.4)\), and a convergence, for the sigma interval of \((E.5)\).

So, we find that the skewness corrected confidence intervals, for both exponential and beta distributions, give us excellent coverages which are extremely close to the benchmark coverage of the normal distribution, or tend to do so, in some well-defined limit. And the same holds for the gamma, chi-square distributions. Also, the 1.96 skewness corrected confidence intervals tend to a coverage of 0.95.

**Appendix F. Bayesian Inference**

Bayesian statistics is not only said to be common sense quantified, but also common sense amplified\(^{41}\) having a much higher ‘probability resolution’ than our human brains can ever hope to achieve.\(^{30}\)

This statement is in accordance with the Kahneman and Tversky finding that, if presented with some chance of a success, \( p \), subjects fail to draw the appropriate binomial probability distribution of the number of successes, \( r \), in \( n \) draws. Even though subjects manage to find the expected number of successes, they fail to accurately determine the probability spread of the \( r \) successes. Kahneman and Tversky see this as evidence that humans are fundamentally non-Bayesian in the way they do their inference.\(^{32}\)

We instead propose that human common sense is not hard-wired for problems involving sampling distributions. Otherwise there would be no need for such a thing as data-analysis, as we only would have to take a quick look at our sufficient statistics after which we then would draw the probability distributions of interest.

However, humans do seem to be hard-wired for more ‘Darwinian’ problems of inference.

\(^{41}\)If Bayesian inference were not common sense amplified, then it could not ever hope to enjoy the successes it currently enjoys in the various fields of science; astronomy, astrophysics, chemistry, image recognition, etc...
For example, if we are told that our burglary alarm has gone off, after which we are also told that a small earthquake has occurred in the vicinity of our house around the time that the alarm went off. Then common sense would suggest that the additional information concerning the occurrence of a small earthquake will somehow modify our probability assessment of there actually being a burglar in the house.

We may use Bayesian probability theory to examine how the knowledge of a small earthquake having occurred translates to our state of knowledge regarding the plausibility of a burglary. The narrative we will formally analyze is taken from [53]:

Fred lives in Los Angeles and commutes 60 miles to work. Whilst at work, he receives a phone-call from his neighbor saying that Fred’s burglar alarm is ringing. While driving home to investigate, Fred hears on the radio that there was a small earthquake that day near his home.

Let

\[ B = \text{Burglary} \]
\[ \overline{B} = \text{No burglary} \]
\[ A = \text{Alarm} \]
\[ \overline{A} = \text{No alarm} \]
\[ E = \text{Small earthquake} \]
\[ \overline{E} = \text{No earthquake} \]

Furthermore, we will distinguish between two states of knowledge:

\[ I_1 = \text{State of knowledge where hypothesis of earthquake is also entertained} \]
\[ I_2 = \text{State of knowledge where hypothesis of earthquake is not entertained} \]

We assume that the neighbor would never phone if the alarm is not ringing and that the radio report is trustworthy too; thus, we know for a fact that the alarm is ringing and that a small earthquake has occurred near the home. Furthermore, we assume that the occurrence of an earthquake and a burglary are independent. We also assume that a burglary alarm is almost certainly triggered by either a burglary or a small earthquake or both, that is,

\[ P(A|BEI_1) = P(A|\overline{B}EI_1) = P(BEI_1) \rightarrow 1, \quad (F.1) \]
whereas alarms in the absence of both a burglary and a small earthquake are extremely rare, that is,

$$P(A|\overline{B}\overline{E}I_1) \to 0. \quad (F.2)$$

But if, in our state of knowledge, we do not entertain the possibility of an earthquake, then (F.1) and (F.2) collapse, respectively, to

$$P(A|BI_2) \to 1, \quad (F.3)$$

and

$$P(A|\overline{B}I_2) \to 0. \quad (F.4)$$

Let

$$P(E) = e, \quad P(B) = b. \quad (F.5)$$

Then we have, by way of the sum rule (A.2),

$$P(E) = 1 - e, \quad P(E) = 1 - b. \quad (F.6)$$

In we are in a state of knowledge where we allow for an earthquake, we have, by way of the product rule (A.1), as well as (F.1), (F.2), (F.5), and (F.6), that

$$P(ABE|I_1) = P(A|BEI_1)P(B)P(E) \to b(1 - e), \quad (F.7)$$

$$P(A\overline{B}E|I_1) = P(A|\overline{B}EI_1)P(E) \to (1 - b)e, \quad (F.7)$$

$$P(ABE|I_1) = P(A|BEI_1)P(B)P(E) \to be,$$

$$P(AB\overline{E}|I_1) = P(A|\overline{B}\overline{E}I_1)P(\overline{B})P(\overline{E}) \to 0.$$

By way of ‘marginalization’, that is, an application of the generalized sum rule, (A.3), we obtain the probabilities

$$P(AB|I_1) = P(ABE|I_1) + P(AB\overline{E}|I_1) \to (1 - b)e, \quad (F.8)$$

$$P(AB|I_1) = P(ABE|I_1) + P(AB\overline{E}|I_1) \to b,$$

and

$$P(A|I_1) = P(AB|I_1) + P(AB|I_1) \to b + e - be, \quad (F.9)$$
and
\[ P(AE | I_1) = P(ABE | I_1) + P(A\overline{B}E | I_1) \rightarrow b \, (1 - e), \] (F.10)
\[ P(AE | I_1) = P(ABE | I_1) + P(A\overline{B}E | I_1) \rightarrow e. \]

But if we are in a state of knowledge where we do not allow for an earthquake, we have, by way of the product rule (A.1), as well as (F.3), (F.4), (F.5), and (F.6), that
\[ P(AB | I_2) = P(A | BI_2) P(B) \rightarrow b, \] (F.11)
\[ P(A\overline{B} | I_2) = P(A | \overline{BI}_2) P(\overline{B}) \rightarrow 0, \]

By way of ‘marginalization’, that is, an application of the generalized sum rule, (A.3), we obtain the probability
\[ P(A | I_2) = P(AB | I_2) + P(A\overline{B} | I_2) \rightarrow b. \] (F.12)

The moment Fred hears that his burglary alarm is going off, then there are two possibilities.

One possibility is that Fred may be new to Los Angeles and, consequently, overlook the possibility of a small earthquake triggering his burglary alarm, that is, his state of knowledge is \(I_2\), which will make his prior probability of his alarm going off, go to (F.12).

Fred then assesses, by way of the product rule (A.1), (F.11), and (F.12), the likelihood of a burglary to be
\[ P(B | AI_2) = \frac{P(AB | I_2)}{P(A | I_2)} \rightarrow \frac{b}{b} = 1, \] (F.13)
which leaves him greatly distressed, as he drives to his home to investigate.

Another possibility is that Fred is a veteran Los Angeleno and, as a consequence, instantly will take into account the hypothesis of a small tremor occurring near his house, that is, his state of knowledge is \(I_1\).

Fred then assesses, by way of the product rule (A.1), (F.8), and (F.9), the likelihood of a burglary to be
\[ P(B | AI_1) = \frac{P(AB | I_1)}{P(A | I_1)} \rightarrow \frac{b}{b + e - be} \approx \frac{b}{b + e}, \] (F.14)
seeing that \(b + e >> be\).

If earthquakes are somewhat more common than burglaries, then Fred, based on his (F.13), may still hope for the best, as he drives home to investigate.
Either way, the moment that Fred hears on the radio that a small earthquake has occurred near his house, around the time when the burglary alarm went off, then, by way of the product rule (A.1) and (F.7) and (F.10), Fred updates the likelihood of a burglary to be

$$P(B|AEI_1) = \frac{P(ABE|I_1)}{P(AE|I_1)} \rightarrow be = b.$$

(F.15)

In the presence of an alternative explanation for the triggering of the burglary alarm, that is, a small earthquake occurring, the burglary alarm has lost its predictive power over the prior probability of a burglary, that is, (F.5) and (F.15),

$$P(B|AEI_1) \rightarrow P(B).$$

(F.16)

Consequently, Fred’s fear for a burglary, as he rides home, after having heard that a small earthquake did occur, will only be dependent upon his assessment of the general likelihood of a burglary occurring. If we assume that Fred lives in a nice neighborhood, rather than some crime-ridden ghetto, then we can imagine that Fred will be, if not greatly, then at least somewhat, relieved.

One of the arguments made against Bayesian probability theory as a normative model for human rationality is that people are generally numerical illiterate. Hence, the Bayesian model is deemed to be too numerical a model for human inference, [59].

However, note that the Bayesian analysis given here was purely qualitative, in that no actual numerical values were given to our probabilities, apart from (F.1), (F.3), (F.2), and (F.4), which are limit cases of certainty and, hence, in a sense, may also be considered to be qualitative.

Moreover, the result of this qualitative analysis seems to be intuitive enough. Indeed, the qualitative correspondence of the product and sum rules with common sense has been noted and demonstrated time and again by many distinguished scientists, including Laplace [44], Keynes [36], Jeffreys [31], Polya [51, 52], Cox [9], Tribus [61], de Finetti [17], Rosenkrantz [55], and Jaynes [30].

As an aside, Jaynes warns us that in any problem we must conditionalize on our state of knowledge $I_i$. These authors were very well aware of this warning. Nonetheless, in our initial discussion of this toy problem we did heed Jaynes’ warning. Because of the perceived gain in notational compactness. But to ignore Jaynes’ warnings is to invite confusion, as we ourselves found out.

This object lesson notwithstanding, we will not conditionalize on our state of knowledge in our further discussion of the Bayesian decision theory. Our excuse for this lack in Bayesian rigor is that our decisions will be assumed to be based on an ‘universal’ prior state of knowledge, which will allow us, for the sake of notational compactness, to drop the extra symbol $I_i$. 
Kahneman dedicates in his Nobel lecture a whole section on ‘Bernoulli’s error’ and on how prospect theory may remedy this error. In Kahneman’s Nobel lecture we may read the following on Bernoulli’s error:

The idea that decision makers evaluate outcomes by the utility of final asset positions has been retained in economic analyses for almost 300 years. This is rather remarkable, because the idea is easily shown to be wrong; I call it Bernoulli’s error. Bernoulli’s model is flawed because it is reference-independent: it assumes that the value that is assigned to a given state of wealth does not vary with the decision maker’s initial state of wealth.

So, Bernoulli’s model is claimed to be in error in that it would evaluate outcomes by the utility of final asset positions alone, without taking into account the initial wealth of the decision maker.

But it may be checked, Paragraph 10 of [6], that Bernoulli gives an utility function of the form:

\[ u(S|S_0) = q \log \frac{S}{S_0} \]  \hspace{1cm} (G.1)

where, adopting the Kahneman’s terminology, \(S\) and \(S_0\) are, respectively, the final and initial asset states. Let the asset increment \(\Delta S\) be defined as

\[ \Delta S = S - S_0. \]  \hspace{1cm} (G.2)

Then, by substituting (G.2) into (G.1), we obtain the equivalent utility function, (4.5):

\[ u(\Delta S|S_0) = q \log \frac{S_0 + \Delta S}{S_0}. \]  \hspace{1cm} (G.3)

It follows that in Bernoulli’s expected utility theory asset increments \(\Delta S\), be they positive or negative, are evaluated relative to the initial wealth \(S_0\) of the decision maker.

This then begs the question: What was it, that led Kahneman to the misguided belief that in Bernoulli’s model the gains and losses are not the carriers of utility?

Kahneman and Tversky state the first two tenets of expected utility theory to be, respectively, the tenets of expectation and asset integration. The tenet of expectation is

\[ U(x_1, p_1; \ldots; x_n, p_n) = p_1 u(x_1) + \cdots + p_n u(x_n). \]  \hspace{1cm} (G.4)

\(^{42}\)The third, and last, tenet is that of loss aversion, which states that \(u\) must be some concave down function. The Bernoulli law is concave down.
The tenet of asset integration states that if $w$ is our current asset position, that is, our initial wealth, then we will accept an uncertain prospect having outcomes $x_i$ if

$$U(w + x_1, p_1; \ldots; w + x_n, p_n) > u(w).$$  \hspace{1cm} (G.5)$$

By substituting (G.4) into (G.5), we obtain the implied asset integration tenet:

$$p_1 u(w + x_1) + \cdots + p_n u(w + x_n) > u(w).$$  \hspace{1cm} (G.6)$$

But then we have that the tenets of expectation and asset integration, as stated by Kahneman and Tversky, are incompatible with Bernoulli’s expected utility theory.

For the expectation tenet (G.4), Bernoulli’s expected utility theory [6] implies the function $u$:

$$u(x) = q \log \frac{w + x}{w},$$  \hspace{1cm} (G.7)$$

whereas, under the asset integration tenet (G.6), the implied function $u$ would be

$$u(x) = q \log x.$$  \hspace{1cm} (G.8)$$

By way of (G.7), (G.8), and the fact that

$$q \log \frac{w + x}{w} \neq q \log x,$$

it is then demonstrated that the two tenets, as proposed by Kahneman and Tversky in their [34], are incompatible with Bernoulli’s expected utility theory.

By dropping the expectation tenet (G.4), while retaining the asset integration in its form of (G.6), we may, very easily, do away with the Kahneman and Tversky inconsistency.

By substituting the implied (G.8) into the asset integration tenet (G.6), we find

$$p_1 [q \log(w + x_1)] + \cdots + p_n [q \log(w + x_n)] > q \log(w),$$  \hspace{1cm} (G.9)$$

Then, by way of the properties of the log function, we may rewrite (G.9) into the equivalent

$$p_1 \left[ q \log \frac{w + x_1}{w} \right] + \cdots + p_n \left[ q \log \frac{w + x_n}{w} \right] > 0,$$  \hspace{1cm} (G.10)$$

which for any psychologist should be recognizable as the weighted sum of Weber-Fechner utilities [44].

Especially, if those psychologists, like Kahneman and Tversky, explicitly state that the facts of perceptual adaptation were in their minds when they began their joint research on decision making under risk, [35].

---

43Note that Laplace [44] discusses Bernoulli’s suggestion by way of the equivalent inequality: 

$$p_1 \log(w + x_1) + \cdots + p_n \log(w + x_n) > \log(w).$$

44The Weber-Fechner law is used, amongst other things, to determine the decibel scale of human sound perception, where the Weber constant has been experimentally determined as $q = \frac{10}{\log 10}$. 

So, if it is claimed by Kahneman and Tversky \cite{34} that Bernoulli’s model is in error, as it would evaluate outcomes by the utility of final asset states alone, rather than gains or losses. Then we may infer that Kahneman and Tversky did not fully realize \cite{45} that, according to Bernoulli \cite{6}, their abstract (G.6) necessarily implies the concrete (G.10).

At the end of the quote on Bernoulli’s error \cite{46} we may find the following cryptic footnote by Kahneman:

> What varies with wealth in Bernoulli’s theory is the response to a given change of wealth. This variation is represented by the curvature of the utility function for wealth. Such a function cannot be drawn if the utility of wealth is reference-dependent, because utility then depends not only on current wealth but also on the reference level of wealth.

We now will offer up an interpretation of what is stated in this footnote; as it may shed some further light on the Kahneman and Tversky position.

Let us assume that Kahneman and Tversky had at least some sense of what is written in, and here we quote Kahneman, \cite{35}, ‘the brilliant essay that introduced the first version of expected utility theory (Bernoulli, 1738)’; that is, we assume that they were aware of the fact that Bernoulli proposes to use the log function, in some shape or form, in order to assign utilities to outcomes.

Then it may well have been that Kahneman and Tversky were under the wrongful impression that Bernoulli’s utility function is given as

\[ u = Q \log (w + x), \quad \text{(G.11)} \]

The first two sentences in the footnote then may be interpreted as expressing the idea that, with differing levels of wealth \( w \), the supposed utility function (G.11) will be more or less linear in a given change of wealth \( x \).

If in the third sentence we let ‘current wealth’ stand for change in wealth and ‘reference level of wealth’ for initial state of wealth, then we may interpret it as stating that (G.11) misses the necessary structure to take into account the initial state of wealth in its utility assignments.

\[ p_1 [\log(w + x_1)] + \cdots + p_n [\log(w + x_n)] > \log(w). \]

But it was professor Han Vrijling, to whom we owe a debt of gratitude, who first pointed us to Bernoulli’s \cite{6}, and the equivalence of the Weber-Fechner law and the Bernoulli law. It was only then, that we realized that Laplace’s formulation is equivalent to (G.9).

In all fairness, we ourselves initially thought that we had improved on the insurance example given by Jaynes in his \cite{30}, by using the Weber-Fechner law, which we still remembered from our psychology days, rather than Laplace’s

\[ p_1 [\log(w + x_1)] + \cdots + p_n [\log(w + x_n)] > \log(w). \]

But it was professor Han Vrijling, to whom we owe a debt of gratitude, who first pointed us to Bernoulli’s \cite{6}, and the equivalence of the Weber-Fechner law and the Bernoulli law. It was only then, that we realized that Laplace’s formulation is equivalent to (G.9).\cite{46}

Given at the beginning of the chapter.
Bernoulli’s supposed error then would be that he proposed as his utility function (G.11). But Bernoulli proposed (G.7) instead of (G.11).

The erroneous utility function (G.11) is problematic in that it cannot assign a value of zero to a change of wealth of $x = 0$. In Kahneman and Tversky’s prospect theory we have that changes in wealth $x$ are assigned values by way of the value function $v$, where

$$v(x) = 0, \quad \text{for} \ x = 0. \quad \quad \text{(G.12)}$$

So, if we read in [35]:

Preferences appeared to be determined by attitudes to gains and losses, defined relative to a reference point, but Bernoulli’s theory and its successors did not incorporate a reference point. We therefore proposed an alternative theory of risk, in which the carriers of utility are gains and losses - changes of wealth rather than states of wealth. Prospect theory (Kahneman & Tversky, 1979) embraces the idea that preferences are reference-dependent, and includes the extra parameter that is required by this assumption.

Then we may interpret is as saying that the erroneous $u$ cannot assign zero utilities to zero outcomes, whereas Kahneman and Tversky’s value function $v$ can.

It would then follow that that which is embraced by prospect theory is the constraint (G.12) on the value function $v$. But this constraint also holds, trivially, for Bernoulli’s utility function (G.7).

Now, after having established some tentative understanding into the reasoning process that might have led Kahneman and Tversky to their misunderstanding of Bernoulli’s position, and after having provided a possible interpretation of Kahneman’s footnote, we may start to wonder: We know how the initial wealth $w$ factors into Bernoulli’s expected utility theory, but how does this initial wealth factor into prospect theory?

We quote Kahneman and Tversky [34]:

The emphasis on changes as the carriers of value should not be taken to imply that the value of a particular change is independent of initial position. Strictly speaking, value should be treated as a function in two arguments: the asset position that serves as a reference point, and the magnitude of the change (positive or negative) from that reference point.

Note, Kahneman and Tversky’s two-part value function is given as, [63]:

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda (-x)^\beta, & x < 0 \end{cases}$$

Constraints are not the same as parameters.
And we could not agree more, though we ourselves would have dropped the ‘strictly speaking’ modifier, as it weakens the imperative. Kahneman and Tversky continue:

However, the preference order of prospects is not greatly altered by small or even moderate variations in asset position. ... Consequently, the representation of value as a function in one argument generally provides a satisfactory approximation.

So, it is Kahneman himself rather than Bernoulli, who does not take explicitly into account the decision maker’s initial state of wealth.\(^{49}\)

**Appendix H. Variance Preferences**

In the Bayesian decision theory bounds of confidence intervals are compared with each other. So, in the Bayesian framework the expectation values, standard deviations, and skewnesses of the utility probability distributions are taken into account in the making of decisions.

It turns out that the suggestion to take into the cumulants of the utility probability distributions, beyond just the expectation value, was also made by both Allais \(^{3, 4, 2}\) and Georgescu-Roegen\(^{50}\).

Moreover, Allais constructed his famous paradox for the sole purpose of demonstrating the psychological reality of ‘variance preferences’\(^{5}\). An Allais paradox may go as follows. Assuming linear utilities for the value of money, we have bets \(D_1\) and \(D_2\), which have the utility probability distributions:

\[
P(u|D_1) = \begin{cases} 
1, & u = 1.000.000 \\
0.000.000 & \end{cases} \quad \text{(H.1)}
\]

and

\[
P(u|D_2) = \begin{cases} 
0.5, & u = 0 \\
0.5, & u = 4.000.000 \\
\end{cases} \quad \text{(H.2)}
\]

Based on the utility probability distributions (H.1) and (H.2), people tend to prefer bet \(D_1\) over \(D_2\), even though the utility expectation value under bet \(D_1\) is much smaller than under bet \(D_2\),

\[
E(u|D_1) = 1.000.000 < 2.000.000 = E(u|D_2), \quad \text{(H.3)}
\]

which is in contradiction with the basic premise of expected utility theory that people will choose that bet which maximizes the expected utility.

Allais contributed this finding to the fact that the variance under bet \(D_1\) is zero, while under bet \(D_2\) it is much greater than zero; what holds for the variances, also

\(^{49}\)As may be checked in the previous footnote.

\(^{50}\)We were unable to find Georgescu-Roegen’s article, referenced in [10], on-line.
holds for standard deviations:

\[ \text{std}(u|D_1) = 0 \ll 2.000.000 = \text{std}(u|D_2) . \]  

(H.4)

These standard deviations, together with their corresponding means, [H.3], convey that \( D_1 \) assuredly will lead to a great gain in utility; whereas under \( D_2 \) there is a very real chance of not winning anything at all.

People not only try to maximize the expectation value of utility, they also take into account the variances of the respective utility probability distributions. Hence, the name variance preferences, that is, preferences between decisions based upon the variance, or, equivalently, the standard deviations of the utility probability distributions\(^{51}\).

Allais’ paradox stands prominent among the paradoxes which are used to dismiss excepted utility theory, [48]. This is somewhat ironic, as it is Allais himself who showed us the way out by pointing out that, together with the expected value, the variances and higher order moments of the utility probability distributions should also be taken into account, we quote\(^{52}\):

In the Theory of Games, von Neumann and Morgenstern presented both a method for determining cardinal utility and a rational rule of behavior. Both are based on the consideration of an index which may be called the neo-Bernoullian utility index\(^{53}\). The theory devised by von Neumann and Morgenstern demonstrates the existence of this index from a system of postulates, and they identified it with cardinal utility in Jevons’ sense. According to them, in order to be rational, any operator must maximize the mathematical expectation of this index.

\(^{51}\)Allais states that decisions ought to be taken on the basis of all the information present in the utility probability distribution, \( \psi(\gamma) \), by way of some function \( h \), Eq.6, [4]. However, Allais does not proceed to give suggestions as to the form and shape of this function \( h \), at least, as far as we are aware. The lower and upper bounds of the utility probability distributions, as used in this fact sheet, are possible examples of such functions \( h \).

\(^{52}\)Italics are by Allais himself.

\(^{53}\)Note that the method for determining cardinal utility, mentioned by Allais, refers to the utility measurement scheme which is proposed by von Neumann and Morgenstern in their [49]. This measurement scheme is very much different from the one that was originally proposed by Bernoulli [6]. So much so, that we would opt to designate the resulting utility index to be ‘non-Bernoullian’, rather than neo-Bernoullian. Bernoulli derives his utility function, or, equivalently, the Weber-Fechner law, by way of three simple considerations. This utility function is then used to compute the expectation of utilities (as opposed to the expectation of monetary outcomes). Von Neumann and Morgenstern, however, postulate, as we believe, a more opaque axiomatic system, from which they then derive that the utility indices, which are to be compared, necessarily must take the form of expectation values. A result which is in contradiction with empirical observations, as Ellsberg and Allais have shown with their paradoxes. See [10] for a simple application of the von Neumann and Morgenstern utility assignment scheme.
This stance struck me as being unacceptable because it amounts to neglecting the probability distribution of psychological values around their mean, which precisely represents the fundamental psychological element of the theory of risk.

I illustrated my argumentation through counter-examples; one of them became famous as the ‘Allais Paradox’. In fact, the ‘Allais Paradox’ is paradoxical in appearance only, and it merely corresponds to a very profound psychological reality, the preference for security in the neighborhood of certainty.

The main reason that the concept of variance preferences never caught on is probably because Allais failed to provide an explicit function by which monetary outcomes could be transformed to utilities. Thus, preventing Allais to proceed with the constructing of utility probability distributions and the computation of their variances.

We can only guess as to why Allais disqualified Bernoulli’s law as a possible candidate utility function. It may be that Allais disqualified Bernoulli’s function because of the latter’s oversight to realize the importance of the variances of the utility probability distributions as a criterion of action.

Or it may be that he thought the problem to be intractable, as also perceived to need to assign subjective probability values to the ‘objective’ frequentistic probabilities of orthodox statistics.

This then would not only constitute another oversight on the part of Bernoulli, as Bernoulli himself had not perceived this need, but it would also compound the problem of assigning moral values to objective monies, seeing that one also would have to assign moral values to objective frequencies.

APPENDIX I. THE CASE AGAINST BAYES; A REPRESENTATIVE EXAMPLE.

The psychological paradigm of heuristics and biases originated as a reaction to the shortcomings of the mathematical expected utility theory.

In the 1950’s it was found that expected utility theory, the then dominant decision theory, failed to adequately model human decision making in certain instances, leading to such paradoxes as the Ellsberg and Allais paradox. Consequently, Edwards and his research team of PhD-students and post-docs took it upon themselves to remedy the situation.

Kahneman and Tversky, both post-docs under Edwards, proposed to construct a systematic theory about the psychology of uncertainty and judgment. In this

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54This is because Bernoulli regarded probabilities, just like Laplace who came after him, to be a state of knowledge, rather than a limiting frequency of an imaginary infinity of replications of some experiment. And a state of knowledge is always ‘subjective’, [30].
psychological theory a handful of heuristics would replace the mathematical laws of chance as a more realistic model for subjective judgment of uncertainty. But what started as a parsimonious theory of human inference, consisting of only three heuristics and their associated biases, \cite{35}, has proliferated into 20 heuristics and an impressive 170+ biases\footnote{Source: Wikipedia, search ‘heuristics’ and ‘list of cognitive biases’}.

Heuristics are said to be mental short cuts or ‘rules of thumb’ humans use to do inference. It is theorized that, as we do not always have the time or resources to compare all the information at hand, we use heuristics to do inference quickly and efficiently.

However, or so we are warned, even though these mental short cuts will be helpful most of the times, if used carelessly heuristics may lead to heuristic-induced biases, that is, systematic errors in reasoning, \cite{35}.

For example, when we use the representativeness heuristic then we estimate the likelihood of an event by comparing it to an existing prototype that already exists in our minds, \cite{33}.

Our prototype is what we think is the most relevant or typical example of a particular event or object. The bias associated with the representativeness heuristic is that when making judgments based on representativeness we are likely to overestimate the likelihood that the representative event will occur; just because an event or object is representative does not mean that it is more likely to occur.

In order to demonstrate this bias Kahneman and Tversky performed the following experiment, \cite{33}. They divided the participants in their study up into three groups.

The base-rate group was asked to guess the percentage of all first-year graduate students in the following nine fields of specialization: business administration, computer science, engineering, humanities and education, law, library science, medicine, physics, and social sciences.

The base-rate group estimated the highest percentage of graduate students, with 20%, to be in humanities and education, and the second lowest percentage, with 7%, to be in computer sciences.

The similarity group was presented with the following profile:

\begin{quote}
Tom W. is of high intelligence, although lacking in true creativity. He has a need for order and clarity, and for neat and tidy systems in which every detail finds its appropriate place. His writing is rather dull and mechanical, occasionally enlivened by somewhat corny puns and by flashes of imagination of the sci-fi type. He has a strong drive for competence. He seems to feel little sympathy for other
\end{quote}
people and does not enjoy interacting with others. Self-centered, he nonetheless has a deep moral sense.

After which they were asked to rate how similar Tom was perceived to be to the typical graduate student in each of the nine graduate specializations.

The similarity group assigned computer science the highest ranking position, with a mean similarity of 2.1, whereas humanities and education was assigned the second lowest ranking position, with a mean similarity ranking of 7.2.

The prediction group was given the same personality sketch of Tom as the similarity group, with the following additional information:

The preceding personality sketch of Tom W. was written during Tom’s senior year in high school by a psychologist, on the basis of projective tests. Tom W. is currently a graduate student.

Then they were asked to rank the nine fields of graduate specialization in order of the likelihood that Tom was now a graduate student in each of these fields.

It was found that 95% of the prediction group judged that Tom was more likely to study computer science than humanities and education.

Since the likelihood rankings of the prediction group closely follow the similarity rankings of the similarity group, whereas they do not follow the base-rate estimates of the base-rate group, Kahneman and Tversky conclude that the representativeness heuristic must have been used by the participants in the prediction group. [33].

However, the use of representativeness does not necessarily imply the representativeness heuristic. We quote Kahneman and Tversky on the representativeness heuristic, [32]:

Our thesis is that, in many situations, an event \( A_1 \) is judged more probable than an event \( A_2 \) whenever \( A_1 \) appears more representative than \( A_2 \). In other words, the ordering of events by their subjective probabilities coincides with their ordering by representativeness.

We now will give a formal analysis of the representativeness heuristic, as proposed by Kahneman and Tversky.

Let

\[
A_1 = \text{Computer Science Student} \\
A_2 = \text{Humanities and Education Student} \\
B = \text{Profile}
\]

If Tom’s psychological profile, \( B \), is more representative of computer science students, \( A_1 \), than of humanities students, \( A_2 \), then

\[
P(B|A_1) > P(B|A_2), \tag{1.1}
\]
or, equivalently,
\[
P( B \mid A_1) > 1.
\] (I.2)

So, the representativeness heuristic is build upon the thesis:
\[
P( B \mid A_1) / P( B \mid A_2) > 1 \iff P( A_1 \mid B) / P( A_2 \mid B) > 1,
\] (I.3)

where ‘\( \iff \)’ is the symbol for logical implication.

However, thesis (I.3) is unfounded in that it does not follow directly from the rules of plausible reasoning. Moreover, Kahneman and Tversky seem to intuit as much; seeing that they warn us for the bias of base rate neglect, when using their representativeness thesis, [33].

By taking the conclusion part in the thesis (I.3) as the starting point of a formal Bayesian analysis, we may find, by way of the product rule
\[
P(A) P( B \mid A) = P(AB) = P(B) P( A \mid B),
\]

that
\[
P( A_1 \mid B) / P( A_2 \mid B) = P( B) P( A_1 \mid B) / P( B) P( A_2 \mid B)
\]

\[
= P( A_1 B) / P( A_2 B)
\]

\[
= P( A_1) / P( A_2) \cdot P( B \mid A_1) / P( B \mid A_2).
\] (I.4)

It follows that the correct thesis, which actually does take into account the base rate, would be
\[
P( B \mid A_1) / P( B \mid A_2) > P( A_2) / P( A_1) \iff P( A_1 \mid B) / P( A_2 \mid B) > 1.
\] (I.5)

Kahneman and Tversky make in [33] the implicit assumption that the reported use of representativeness, that is, an evaluation and use of the odds (I.2), as reported by the participants of the prediction group, necessarily implies their thesis (I.3). This leads them to conclude that the participants must have used their representativeness heuristic.

However, it can be seen that this assumption is incorrect, as the Bayesian thesis (I.5) also makes use of the odds in (I.2) and, thus, representativeness. Moreover, based on the reported use of representativeness and the experimental data, one could make the case that the participants in the experiment intuitively made use of the correct (I.5), instead of the erroneous (I.3).
Kahneman and Tversky report that the prior odds for humanities and education against computer science were estimated by the participants to be, \[ P(A_2) \approx 3. \tag{I.6} \]

Now, (I.4), or, equivalently, thesis (I.5), tells us that (I.6) together with \[ \frac{P(B|A_1)}{P(B|A_2)} > 3 \tag{I.7} \]

implies the conclusion \[ \frac{P(A_1|B)}{P(A_2|B)} > 1, \tag{I.8} \]
or, equivalently, \[ P(A_1|B) > P(A_2|B). \tag{I.9} \]

So, if 95% of the participants in the third group judged that Tom was more likely to study computer science than humanities, then we may infer that 95% of the participants deemed the odds-inequality (I.7) to hold, which in our opinion is not that far-fetched\(^56\).

Kahneman and Tversky, however, by way of an informational accuracy argument, are of the opinion that the plausibility judgments of the participants in the third group ‘drastically violate the rules of the normative [that is, Bayesian] rules of prediction’, seeing that the following considerations were ignored by the participants in the prediction group, \[33\]:

First given the notorious invalidity of projective personality tests, it is very likely that Tom W. was never in fact as compulsive and as aloof as his description suggests. Second, even if the description was valid when Tom W. was in high school, it may not longer be valid now that he is in graduate school. Finally, even if the description is still valid, there are probably more people who fit that description among students of humanities and education than among students of computer science, simply because there are so many more students in the former than in the latter field.

As to the last consideration, it follows from the Bayesian ‘heuristic’ \(15\) that the inference of Tom being a computer science student implies the corollary inference that among all the graduate students who fit Tom’s description there will be more students of computer science than students of humanities and education. Even if

\(^{56}\) Indeed, the ranking of computer sciences was, with a mean similarity of 2.1, more than three times higher than the ranking of humanities, which had a mean similarity ranking of only 7.2, \[33\]. Even though rankings do not translate easily to probabilities, as a \textit{sine qua non}, this ordering of similarity rankings still constitutes corroborating evidence for inequality (I.7) to have held for the participants in the Kahneman and Tversky experiment.
there are many more students in the field of humanities and education than in the field of computer science.

The odds \([1.6]\) represent the ratio of humanities and education students to computer science students. Whereas the odds \([1.7]\) represent the ratio of computer students having a profile like Tom’s to humanities and education students having a like profile. If the latter odds dominate the former, then we must conclude, by way of \([1.4]\), that there are probably more people who fit Tom’s description among students of computer science than among students of humanities and education; that is, \(P(A_1B) > P(A_2B)\). Which invalidates Kahneman’s and Tversky’s last consideration.

Moreover, we have also come to doubt Kahneman and Tversky’s competency somewhat, when it comes to matters of the normative rules of prediction.

This, then, leaves us with the following arguments for the thesis that people tend to neglect the base rate, when taking the mental shortcut \([1.3]\), thus, violating the rules of the normative [that is, Bayesian] rules of prediction:

- (1) given the notorious [that is, clinical] invalidity of projective personality tests, it is very likely that Tom W. was never in fact as compulsive and as aloof as his description suggests.
- (2) even if the description was valid when Tom W. was in high school, it may not longer be valid now that he is in graduate school.

Now, it would seem that these arguments pertain to some other thesis, namely, that the participants of the prediction group should have disregarded the description of Tom altogether, as no data was actually given. But this alternative thesis, apart from it not being the issue\(^\text{57}\) is not all that compelling.

Because, if, in answer to the first argument, we filter out those qualifications which might point to compulsiveness and aloofness, then we are left with the following profile for Tom:

- high intelligence,
- dull and mechanical writing,
- lacking in true creativity,
- corny puns,
- flashes of imagination of the sci-fi type,
- strong drive for competence,
- deep moral sense,

which tells us quite a lot.

\(^{57}\)Though Kahneman and Tversky have made it so, by way of their informational accuracy argument.
It tells us that Tom is very bright, does not like to read, as he apparently is not that lyrical in his writing, has a sense of humor, has a passion for sci-fi, is disciplined, and has a sense of justice.

As to the second argument, which is also the hardest to answer. It is taught at the psychology courses, that personality traits tend to be stable over long periods of time. So, if Tom did not like to write in high school than chances are that he would pick a field of specialization in which he would not have to read and write a lot. Which would make humanities and education less probable a field of choice, and computer sciences more probable.

But, lest we forget, the original thesis under discussion was that normative rules of prediction tend to be neglected, as people tend to neglect the base rate; not the alternative thesis that the prediction group should have disregarded the description of Tom, because of the clinical invalidity of projective personality tests and the possibility that Tom might have changed his personality over the course of the few years between high school and college.

If Kahneman and Tversky wish to prove their initial thesis, then at this point, having presented their experimental data, they should proceed to demonstrate that their subjects could not possibly have used the normative, that is, Bayesian, rules of prediction, as those rules would have implied results other than those that were observed. Which they do not.

Having come to the end of our discussion of the representative heuristic, we find that the reported plausibility judgments by the prediction group are not inconsistent with a possible use of the Bayesian ‘heuristic; seeing that the odds may be assumed to lie in the realm of the possible. This leaves the Kahneman and Tversky experiment inconclusive.

**Appendix J. Non-Linear Preferences**

Tversky and Kahneman, state that non-linear preferences constitute one of the minimal challenges that must be met by any adequate descriptive theory of choice. We shall explain.

If we have a bet which has the following outcome probability distributions

\[
p(O|D_1) = \begin{cases} 
1, & O = 1.000.000 \\
0.000.000, & O = 5.000.000 \\
0.000.000, & O = 0 
\end{cases} \tag{J.1}
\]

and

\[
p(O|D_2) = \begin{cases} 
0.99, & O = 1.000.000 \\
0.01, & O = 0 
\end{cases} \tag{J.2}
\]

then people will typically prefer the bet \(D_1\) over \(D_2\).

---

58 Moreover, what is left out also gives us some tentative information on Tom’s psychologist.
59 A liking, admittedly, is not a personality trait, but still.
In contrast, if we have a bet which has the following outcome probability distributions

\[ p(O|D_1) = \begin{cases} 
0.90, & O = 1.000.000 \\
0.10, & O = 0 
\end{cases} \quad (J.3) \]

and

\[ p(O|D_2) = \begin{cases} 
0.89, & O = 5.000.000 \\
0.11, & O = 0 
\end{cases} \quad (J.4) \]

then people will typically prefer the bet \( D_2 \) over \( D_1 \).

Allais gave this example, in a slightly altered form, to demonstrate that Savage’s fifth axiom of independence does not hold. \[4\]. According to Savage’s independence axiom, which we do not endorse, we may add for both \( J.1 \) and \( J.2 \) a probability mass of 0.10 to the proposition \( u = 0 \), while subtracting that same probability mass for the respective propositions \( u = 5.000.000 \) and \( u = 1.000.000 \), leading to \( J.3 \) and \( J.4 \), and still maintain the same problem of choice. But this assumption, as one would hope, is shown to be incorrect by the observed reversal in preferences from bet \( D_1 \) over \( D_2 \) to bet \( D_2 \) over \( D_1 \).

Now, according to Kahneman and Tversky the example by Allais not only refutes Savage’s axiom of independence, but it also shows that the difference between probabilities of 0.99 and 1.00 has more impact on preferences than the difference between 0.10 and 0.11.

Kahneman and Tversky find this observation to be so full of meaning that they deem it to be a psychological phenomenon in and of itself, and proceed to label it as the ‘certainty effect’, \[34\], which later turns into ‘non-linear preferences’, \[63\]. But for Bayesians the phenomenon of non-linear preferences is not that special and not that new\[60\].

---

60The particular U-shape of the non-informative Jeffreys’ prior for the parameter \( \theta \) of the beta distribution,

\[ p(\theta) \propto \theta^{-1} (1 - \theta)^{-1}, \]

is a consequence of the non-linearity of probabilities. If we make a change of variable from \( \theta \) to the log-odds \( \omega = \log[\theta/(1 - \theta)] \), then it may be found that the implied non-informative prior of the log-odds \( \omega \) is uniform,

\[ p(\omega) \propto \text{constant}. \]

So, log-odds, having the whole infinity of the \( x \)-axis at their disposal, are linear; whereas probabilities, being forced to lie within the heavily constricted interval \([0, 1]\), are not.
While working on the German enigma code, during World War II, Good and Turing introduced the ‘deciban’ measure which is measured in decibans:

$$\text{deciban}(P) = 10 \log_{10} \frac{P}{1-P}, \quad (J.5)$$

and which gives the plausibility of a proposition being true, relative to it not being true. [21].

For undecidedness, that is, for a fifty-fifty change of some hypothesis $A$ being true, we have

$$P = 1 - P = 0.5. \quad (J.6)$$

Substituting (J.6) in (J.5), we find that undecidedness, (J.6), corresponds with

$$\text{deciban}(0.5) = 10 \log_{10}(1) = 0. \quad (J.7)$$

Just as 1 db sound represents the just noticeable difference relative to silence, so a ±1 deciban change in probability represents the just noticeable difference relative to undecidedness, [20].

Using (J.5), we find that the decibans associated with the probabilities 0.99 and 1.00 are, respectively,

$$\text{deciban}(0.99) = 10 \log_{10} \frac{0.99}{0.01} = 19.96 \quad (J.8)$$

and

$$\text{deciban}(1.00) = 10 \log_{10} \frac{1.00}{0.00} \to \infty. \quad (J.9)$$

Now, (J.9) tells us that a probability 1.00 is a limit case of absolute certainty. Whereas a probability of 0.99 is not, representing just under 20 decibans of evidence for proposition $A$ being true.

So, the difference in evidence for proposition $A$ being true for the probabilities 0.99 and 1.00 is much more than 1 deciban:

$$\text{deciban}(1.00) - \text{deciban}(0.99) = \infty \gg 1. \quad (J.10)$$

It may be checked that, (J.5), the probabilities of 0.10 and 0.11 correspond with a less than 1 deciban difference in evidence:

$$\text{deciban}(0.11) - \text{deciban}(0.10) = 0.46 < 1. \quad (J.11)$$

---

61Good wrote about 2000 articles on Bayesian statistics, found throughout the statistical and philosophical literature starting in 1940. Workers in the field generally granted that every idea in modern statistics can be found expressed by him in one or more of these articles; but their sheer number made it impossible to find or cite them, and most are only one or two pages long, dashed off in an hour and never developed further. So, for many years, whatever one did in Bayesian statistics, one just conceded priority to Jack Good by default, without attempting the literature search, which would have required days. Finally, in 1983, a bibliography was provided of most of the first 1517 of these articles with a long index, so it is now possible to give proper acknowledgments of his works up to 1983, [30].
So, if we find that subjects prefer the second bet in the second collection of bets, equations (J.3) and (J.4). Then this also may be interpreted as follows. Subjects are indifferent to the difference in probabilities 0.10 and 0.11, as this difference represents a change of less than 1 deciban in the plausibility of hypothesis $A$ being true, (J.11). So, all things being equal, subjects choose the bet with the highest potential payout of 5,000,000 dollars.

In closing, the deciban, (J.5), represents the intuitive scale on which the plausibility of proposition $A$ being true, relative to it not being true, is judged; that is, the deciban is the scale of the numerically coded plausibilities. Whereas the probability,

$$P = \frac{10^{\text{deciban}(P)}}{1 + 10^{\text{deciban}(P)}} \quad (J.12)$$

represents the non-intuitive ‘technical’ scale, which follows from the quantification of our common sense. [11, 58].

So, if the difference between probabilities of 0.99 and 1.00 has more impact on preferences than the difference between 0.10 and 0.11, as is found in psychological experiments, then this is reflective of the fact that the qualitative properties of the intuitive deciban-scale, up to a certain point, are retained in the transformation to the more technical probability-scale.

We say up to a certain point, because probability theory, which makes use of the technical probabilities, is common sense amplified, having a much higher probability resolution than our human brains can ever hope to achieve. More concretely, we expect that for human probability perception the range of meaningful decibans is bounded somewhere around, say, $\pm 40$ deciban. [30].

APPENDIX K. THE FRAMING EFFECT

The assumption that preferences are not affected by variations of irrelevant features of options or outcomes is called invariance, [2].

According to Kahneman and Tversky invariance is an essential aspect of rationality, which is violated in demonstrations of framing effects, [35]. Now, in order to discuss these framing effects, we will first have to discuss the topic of loss and gain adaptation. [32]

Imagine a person who is involved in a business venture, who has lost 2000, and now is facing a choice between a sure gain of 1000 and an even chance to win 2000 or nothing, [34]. If he has not adapted to his loss, he is likely to add this loss to all the outcomes and, consequently, by way of the Weber-Fechner law, code the

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[30] What we will call loss and gain adaptation, is called shifts of reference by Kahneman and Tversky in their [34].
problem as a choice between the following utility distributions

\[ p(u|D_1) = \begin{cases} 
1.0, & u = q \log \frac{S_0 - 1000}{S_0} 
\end{cases} \quad (K.1) \]

and

\[ p(u|D_2) = \begin{cases} 
0.5, & u = q \log \frac{S_0 - 2000}{S_0} \\
0.5, & u = q \log \frac{S_0 - 0}{S_0} 
\end{cases} \quad (K.2) \]

where \( S_0 \) is the pre-loss asset position and \( q \) is the Weber constant for money.

Looking at the increments in assets, it is predicted that our business man will tend to prefer \( D_2 \) over \( D_1 \), as this is the preferred choice under risk seeking in the negative domain. It follows that a failure to adapt to losses may induce risk seeking in the negative domain, [34]. Stated differently, a person who has not made peace with his losses is likely to accept gambles that would be unacceptable to him otherwise. We may find support for this hypothesis by the observation that the tendency to bet on long shots will increase in the course of a betting day, [47].

However, if our business man has adapted to his loss, then he will update his pre-loss asset position \( S_0 \) to an adjusted asset position \( S_{0(\text{adj.})} \) in which the loss is discounted:

\[ S_{0(\text{adj.})} = S_0 - 2000 \]

and code the problem as a choice between the utility distributions

\[ p(u|D_1) = \begin{cases} 
1.0, & u = q \log \frac{S_{0(\text{adj.})} + 1000}{S_{0(\text{adj.})}} 
\end{cases} \quad (K.3) \]

and

\[ p(u|D_2) = \begin{cases} 
0.5, & u = q \log \frac{S_{0(\text{adj.})} + 0}{S_{0(\text{adj.})}} \\
0.5, & u = q \log \frac{S_{0(\text{adj.})} + 2000}{S_{0(\text{adj.})}} 
\end{cases} \quad (K.4) \]

where \( S_{0(\text{adj.})} \) is the post-loss asset position and \( q \) is the Weber constant for money.

Again looking at the increments in assets, we see that the signs of these increments have reversed. By this reversal in the sign of the asset increments, we go from a risk seeking in the negative domain scenario to a risk aversion in the positive domain scenario, [34]. So, it is now predicted that our business man, having already adapted to his loss, will tend to reverse his preferences, and choose \( D_1 \) over \( D_2 \).

Having introduced the concepts of loss and gain adaptation and the adjusted initial wealth \( S_{0(\text{adj.})} \), we now may turn to the discussion of the framing effect.

Consider the following problems, which were presented to two different groups of subjects, [34].
**Group 1:** In addition to whatever you own, you have been given 1000. You are now asked to choose between

\[
p(O|D_1) = \begin{cases} 
0.5, & O = 0 \\
0.5, & O = 1000
\end{cases}
\]

and

\[
p(O|D_2) = \begin{cases} 
1.0, & O = 500
\end{cases}
\]

**Group 2:** In addition to whatever you own, you have been given 2000. You are now asked to choose between

\[
p(O|D_1) = \begin{cases} 
0.5, & O = -1000 \\
0.5, & O = 0
\end{cases}
\]

and

\[
p(O|D_2) = \begin{cases} 
1.0, & O = -500
\end{cases}
\]

It is found that 84% of \( N = 70 \) subjects in Group 1 prefer bet \( D_2 \) over bet \( D_1 \); whereas 69% of \( N = 68 \) subjects in Group 2 prefer bet \( D_1 \) over bet \( D_2 \), indicating risk seeking in the negative domain, \[34\].

This result may indicate that the subjects in both groups adapted to the respective gifts of 1000 and 2000, by adjusting their initial \( S_0 \) into a new \( S_0^{(adj)} \), before choosing between the two bets. Since we would have expected to see the same preferences for both groups had the gifts been discounted in the outcomes\[63\]. But instead, we observe a reversal in preferences. So, we conclude that the gifts must have been discounted in initial wealth, rather than in the outcomes.

Furthermore, based on the unadjusted outcomes alone, as given in the corresponding outcome probability distributions, we would expect both the observed risk aversion in the positive domain in Group 1, that is, the observed preference of \( D_2 \) over \( D_1 \), and the observed risk seeking in the negative domain in Group 2, that is, the observed preference of \( D_1 \) over \( D_2 \). This then also points to an adjustment of the initial wealth, rather an adjustment of the outcomes.

However, an alternative, more parsimonious, explanation for this framing effect, or, equivalently, the observed reversal in preferences in both groups, would be that the respective gifts of 1000 and 2000 were neglected by the subjects, \[34\].

\[63\] It may be checked that a discounting of the respective gifts in the corresponding outcomes would have resulted in identical outcome probability distributions for both Groups 1 and 2:

\[
p(O|D_1) = \begin{cases} 
0.5, & O = 1000 \\
0.5, & O = 2000
\end{cases}
\]

and

\[
p(O|D_2) = \begin{cases} 
1.0, & O = 1500
\end{cases}
\]
And we would tend to agree with Kahneman and Tversky on this one. For, when reviewing these hypothetical choices, we ourselves overlooked these gifts too.

But where Kahneman and Tversky see the framing effect as an indication that such monetary gifts, as a rule, will not factor into our real-life decisions, we quote, [34]:

The apparent neglect of a bonus [our gift] that was common to both options [our decisions $D_1$ and $D_2$] in Problems 11 and 12 [our Groups 1 and 2] implies that the carriers of value or utility are changes of wealth, rather than final asset positions that include current wealth. This conclusion is the cornerstone of an alternative theory of risky choice [their prospect theory].

We, instead, propose that this neglect of the gifts point to the limitations of the experimental method of hypothetical choices, as employed by Kahneman and Tversky.

Speaking strictly for ourselves, the receiving of a real-life gift of either 1000 or 2000 euros would be quite the momentous occasion. Consequently, we can hardly imagine neglecting such a substantial sum of money, or, for that matter, not factoring its occurrence in our every monetary decision[64].

So, it would seem, at least based on the above [34] quotation, that Kahneman and Tversky build their prospect theory around a phenomenon which, as our introspection would suggest, is nothing but an experimental artifact of the method of hypothetical choices.

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[64] The question if such a gift would be discounted into our initial wealth $S_0$, or into the specific outcomes $\Delta S$ of some set of decisions $D_i$, would be dependent on the particular context in which the gift was received.