Analytical model for laser-assisted recombination of hydrogenic atoms

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Abstract
We introduce a new method that allows one to obtain an analytical cross section for the laser-assisted electron–ion collision in a closed form. As an example, we perform a calculation for the hydrogen laser-assisted recombination. The $S$-matrix element for the process is constructed from an exact electron Coulomb–Volkov wavefunction and an approximate laser-modified hydrogen state. An explicit expression for the field-enhancement coefficient of the process is expressed in terms of the dimensionless parameter $\kappa = \left| \frac{e\varepsilon_0}{q\omega_0} \right|^2$, where $e$ and $q$ are the electron charge and momentum, respectively, and $\varepsilon_0$ and $\omega_0$ are the amplitude and frequency of the laser field, respectively. The simplified version of the cross section of the process is derived and analyzed within a soft photon approximation.

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1. Introduction
Atomic scattering processes in the presence of a harmonic laser field are a significant part of contemporary atomic physics [1]. A simple and accurate theoretical model for laser-assisted atomic scattering processes would allow one to explain their distinctive properties. The standard theoretical approach for laser-assisted electron–ion collisions consists of the construction of the $S$-matrix element for the corresponding process. The electron wavefunction in a combined Coulomb-laser field is given by the well-known Coulomb–Volkov state. The dressed state of the atom is described by the classical time-dependent perturbation series. Within these approximations, the $S$-matrix element was derived and numerically analyzed.

The goal of this paper is the construction of a simple analytical model for an electron–ion collision in a harmonic laser field. Namely, we have developed a new method that allows us to derive an analytical expression for the cross section of a laser-assisted atomic scattering
process in a closed form. The new step consists in using the Bessel generating function as an argument for the Plancherel theorem. This allows one to perform the summation over the number of field harmonics so that the analytical expression for the cross section of the process can be explicitly written. As an example, we perform a calculation for a laser-assisted hydrogen recombination process. An exact expression for the field-enhancement coefficient is given in terms of the dimensionless parameter $\kappa = |e\varepsilon_0/q\omega_0|^2$.

This paper is organized as follows. In section 2, the problem is formulated with the necessary knowledge concerning the standard field-free hydrogen recombination process. In section 3, we give the $S$-matrix element for the hydrogen laser-assisted recombination process based on the electron Coulomb–Volkov wavefunction and the laser-modified hydrogen state. In section 4, we perform a detailed analysis of the Coulomb–Volkov wavefunction. In sections 5 and 6, we obtain the partial and general differential cross sections of the laser-assisted hydrogen recombination. Section 7 is devoted to the analysis of the soft photon approximation, in which a simplified differential cross section of the process is derived and analyzed. Section 8 summarizes the results and advantages of the developed method.

2. Laser-assisted hydrogen recombination

The goal of this paper is to provide an analytical expression for a laser-assisted hydrogen recombination process,

$$p + e + L\hbar\omega_0 \longrightarrow H + \hbar\omega.$$  \hspace{1cm} (1)

The additional term $L\hbar\omega_0$ ($L$ is the number of harmonics) indicates the presence of a laser field,

$$\vec{e} = \vec{e}_0 \sin \omega_0 t,$$  \hspace{1cm} (2)

and points out the conservation of quasienergy. Here $\vec{e}_0$ is the amplitude of the field and $\omega_0$ is the field frequency.

The final result for the analytical cross section of the process (1) can be directly applied to the following laser-assisted recombination processes of

(i) hydrogenic atom (for instance, He$^+$):

$$^2\bar{X}^+ + e + L\hbar\omega_0 \longrightarrow ^2\bar{X} + \hbar\omega,$$  \hspace{1cm} (3)

(ii) antihydrogen atom:

$$\bar{p} + e^+ + L\hbar\omega_0 \longrightarrow \bar{H} + \hbar\omega,$$  \hspace{1cm} (4)

(iii) positronium atom formation:

$$e + e^+ + L\hbar\omega_0 \longrightarrow Ps + \hbar\omega.$$  \hspace{1cm} (5)

Creation of an antihydrogen atom [2] has become a driving force in contemporary atomic physics. Of particular interest in this area is the accuracy of the value of the fine structure constant $\alpha = e^2/\bar{h}c$ which can be improved by comparing atomic spectra of hydrogen and antihydrogen atoms. Exploration of these spectra provides a unique opportunity for studying basic symmetries in physics, including the CPT invariance [3].

The main reaction leading to the formation of the antihydrogen atom in modern experiments [2, 4] is the three-body process which was a subject of recent theoretical calculations [5, 6]. The laser-assisted antihydrogen recombination process (4), being dominated by the three-body process, was studied experimentally [7] and theoretically [8, 9], but the analytical expression for this process has not been found prior to this work.
The differential cross section of the reaction for the standard field-free hydrogen recombination process is well known due to the principle of detailed balance between the differential cross section of photo-recombination, \( \frac{d\sigma_{vf}}{d\Omega_f} \), and that of photo-ionization, \( \frac{d\sigma_{fi}}{d\Omega_i} \) [10]:

\[
\frac{d\sigma_{fi}}{d\Omega_i} = k^2 \frac{d\sigma_{vf}}{d\Omega_f},
\]

where \( k \) and \( q \) are the momenta of the outgoing photon and electron, respectively.

The well-known expression for the differential photo-ionization cross section \( \frac{d\sigma_{fi}}{d\Omega_i} \) is given by [11, 12]

\[
\frac{d\sigma_{fi}}{d\Omega_i} = 2\pi \frac{e^2 \omega}{m \xi^4} \left( \frac{\xi^2}{\xi^2 + 1} \right)^5 \frac{e^{-4\eta \csc \xi}}{1 - e^{-2\pi \xi}} \left( W_0 - \hbar \omega \right)^2,
\]

where \( W_0 \) is the ionization potential of the hydrogen atom from the continuum threshold to the ground state. Here, \( \omega \) is the frequency of the photon emitted at the angle \( \theta \) relative to the incoming electron, and \( m \) and \( e \) are the mass and electrical charge of the electron. The dimensionless parameter \( \xi \) is defined as

\[
\xi = \frac{Z \hbar}{a_0 q} = \frac{\eta}{\bar{q}},
\]

where \( Z \) is the nuclear charge (\( Z = 1 \)), \( a_0 \) is the Bohr radius, and \( \eta = Z \hbar/a_0 \).

3. The S-matrix element

The S-matrix element describing the photo-recombination process (1) in the presence of the laser field (2) is given by [8, 9, 12] \( S = -ie \int dt |\Psi_0^H(\vec{r}, t)\rangle \langle \vec{e} \cdot \nabla | \chi^e(\vec{r}, t)\rangle \),

where \( \chi^e(\vec{r}, t) \) is the Coulomb–Volkov wavefunction of the electron in the field of the proton and external laser field; \( \vec{e}, \vec{k} \) and \( \omega \) are the polarization vector, momentum and frequency of the emitted photon, respectively; and \( \Psi_0^H(\vec{r}, t) \) is the wavefunction of the hydrogen in the laser field \( \vec{E}, \vec{B} \).

Using the standard technique of transformation to the rotating frame, we can obtain the electron wavefunction in the total Coulomb-laser field [13]:

\[
\chi^e(\vec{r}, t) = e^{\frac{i\vec{A} \cdot \vec{r}}{\hbar}} \Gamma(1 - i\xi) F(\xi, 1; i(\vec{q} \cdot \vec{r} - \bar{q} \cdot \bar{r})/\hbar) \times \exp \left[ -\frac{ie^2}{2\hbar mc^2} \int_0^t A^2 d\tau + \frac{i\vec{q} \cdot \vec{r}}{\hbar} - \frac{iE_1t}{\hbar} \right],
\]

where \( \bar{q} \) is the momentum of the incoming electron, \( q = |\vec{q}|, r = |\vec{r}|, F(a, b; x) \) is the confluent hypergeometric function, \( E_i \) is the initial kinetic energy of the incoming electron, and \( c \) is the speed of light.

In the harmonic laser field (2), the vector-potential \( \vec{A} \) is defined as

\[
\vec{A}(t) = \frac{e\vec{E}_0}{\omega_0} \sin \omega_0 t \equiv \vec{A}_0 \sin \omega_0 t.
\]

In what follows, we will use the velocity gauge (or \( A \cdot p \) gauge). Therefore the term in the square brackets in (10), \( (ie^2/2\hbar mc^2) \int_0^t A^2 d\tau \), can be neglected [13]. Evaluation of the remaining integral over \( \tau \) in (10) yields

\[
\chi^e(\vec{r}, t) = \Gamma(1 - i\xi) F(\xi, 1; i(\vec{q} \cdot \vec{r} - \bar{q} \cdot \bar{r})/\hbar) \times \exp \left[ \frac{i\bar{q} \cdot \bar{r}}{\hbar} + \frac{e\vec{q} \cdot \vec{a}_0 \sin \omega_0 t}{m \hbar} \frac{E_1t}{\hbar} - \frac{i\pi \xi}{2} \right],
\]

where \( \vec{a}_0 = \vec{E}_0/\omega_0^2 \).
Using first-order perturbation theory for deriving the hydrogen wavefunction in the presence of the laser field, we obtain [8–10]

\[
\Psi_n^H(\vec{r}, t) = e^{-iW_n t/\hbar} \times \left\{ \psi_n(\vec{r}) - \frac{1}{2} \sum_{m \neq n} \left( \frac{e^{i\omega_{mn} t/\hbar}}{\omega_{mn} + \omega_0} + \frac{e^{-i\omega_{mn} t/\hbar}}{\omega_{mn} - \omega_0} \right) \langle m | e^{\vec{A}_0 \cdot \vec{p}} | n \rangle \psi_m(\vec{r}) \right\},
\]

(13)

where \(\psi_m(\vec{r})\) is the wavefunction for the atomic electron in the field-free state \(|m\rangle\) with energy \(W_m\), and \(\omega_{mn} = (W_m - W_n)/\hbar\). The summation in (13) is extended over the full set of atomic electron states in the absence of the laser field. In the derivation of (13), it was assumed that none of the denominators were close to zero [10].

For the optical frequency of the laser, we have \(\omega_0 \ll \omega_0^0\), and therefore we obtain the following expression for \(\Psi_0^H(\vec{r}, t)\), which holds true even over a broader frequency range [8]:

\[
\Psi_0^H(\vec{r}, t) = e^{-iW_0 t/\hbar} \left( 1 + \frac{i e(\hat{e}_0 \cdot \vec{r})}{\hbar \omega_0} \cos \omega_0 t \right) \psi_0^H(\vec{r}),
\]

(14)

where

\[
\psi_0^H(\vec{r}) = \sqrt{\frac{Z^3}{\pi a_0^3}} e^{-\eta r} \equiv C_0 e^{-\eta r}.
\]

(15)

4. The Coulomb–Volkov wavefunction

In order to perform the time integration in the S-matrix element (9), we decompose the electron wavefunction \(\chi^e(\vec{r}, t)\) given by (12) over the Bessel functions \(J_L(z)\). For this purpose, we introduce the Bessel generating function

\[
\exp(i z \sin u) = \sum_{L=-\infty}^{L=+\infty} J_L(z) \exp(i Lu).
\]

(16)

To simplify the S-matrix element (9), we use the recurrence relation for the Bessel functions:

\[
J_{L+1}(z) + J_{L-1}(z) = \frac{2L}{z} J_L(z).
\]

(17)

Performing the time integration in (9) and with the aid of the Gauss theorem, we obtain the following expression for the S-matrix element in the dipole approximation (\(|\vec{k} \cdot \vec{r}| \ll 1\):

\[
S = -2\pi i \sum_{L=-\infty}^{L=+\infty} f_L \delta(W_0 + \hbar \omega - E_i + L \hbar \omega_0),
\]

(18)

where

\[
f_L = e^{\pi i} \Gamma(1 - i\xi) C_0 \left[ J_L(z) \omega(L) \left( I_1 + \frac{L}{z} I_2 \right) \right].
\]

(19)

Here we have introduced the following notations:

\[
z = \frac{e(\vec{q} \cdot \vec{a}_0)}{m \hbar},
\]

(20)

\[
I_1 = \eta \hbar \int d\vec{r} \psi_0^H(\vec{r}) \frac{(\vec{e} \cdot \vec{r})}{r} \chi^e(\vec{r}),
\]

(21)
\[ I_2 = \frac{ie}{\omega_0} \int d\vec{r} \psi_0^\dagger(\vec{r}) \left( -(\vec{e}_0 \cdot \vec{r}) + \eta \frac{(\vec{e}_0 \cdot \vec{r})(\vec{e} \cdot \vec{r})}{r} \right) \chi(\vec{r}), \]  

(22)

and

\[ \hbar \omega(L) = E_i - W_0 + L\hbar \omega_0 \equiv E_i + L\hbar \omega_0. \]  

(23)

Exploiting the well-known integral involving the confluent hypergeometric function [12, 14]

\[ \int d\vec{r} e^{i(\vec{q} - \vec{p}) \cdot \vec{r} / \hbar - \eta r / \hbar} F(i \xi, 1, i(qr - \vec{q} \cdot \vec{r})) = 4\pi \hbar^2 \left[ \frac{p^2 + (\eta - iq)^2}{(q - p)^2 + q^2} \right]^{1-i\xi}, \]  

(24)

we obtain the following expression for (21):

\[ I_1 = 8\pi \hbar^4 (\vec{e} \cdot \vec{e}_q) \xi (1 - i\xi) q^2 \left( 1 + \xi^2 \right)^2 e^{-2\xi \text{arccot} \xi}, \]  

(25)

and the corresponding expression for (22):

\[ I_2 = -8\pi \hbar^4 \xi e^{-2(1+i\xi) \text{arccot} \xi} \left( (\vec{e}_0 \cdot \vec{e}) - 2(\vec{e}_0 \cdot \vec{e}_q)(\vec{e} \cdot \vec{e}_q) \frac{2 - i\xi}{(1 - i\xi)} \right), \]  

(26)

where \( \vec{e}_q \equiv \vec{q} / q \) is a unit vector.

5. The partial cross section

The partial cross section of reaction (1) with fixed \( L \) is given by

\[ d\sigma_L = 2\pi e^2 |f_L|^2 \delta(W_0 + \omega - E_i + L\hbar \omega_0) \frac{d^3k}{(2\pi)^3}. \]  

(27)

The integration over \( \omega \) yields

\[ \frac{d\sigma_L}{d\Omega} = \frac{e^2 C_0^2}{8\pi^2 q(1 - e^{-2\pi \xi})} (E_i - W_0 + L\hbar \omega_0)^3 \times |J_L(z)|^2 \left( |I_1|^2 + \frac{2\text{Re}(I_1I_2)}{z} L + \frac{L^2}{z^2} |I_2|^2 \right). \]  

(28)

Here the constant \( C_0 \) was defined in (15). The total cross section of reaction (1) in the laser field (2) is obtained with

\[ \frac{d\sigma}{d\Omega} = \sum_{L=-\infty}^{L=+\infty} \frac{d\sigma_L}{d\Omega}. \]  

(29)

6. The summation procedure and cross section

We introduce a new step that allows one to analytically sum up the infinite series (29). The summation over \( L \) in (29) is performed with the aid of the Plancherel theorem [15]:

\[ \frac{1}{2\pi} \int_{-\infty}^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{n=+\infty} |c_n|^2, \]  

(30)

where \( c_n \) are the Fourier coefficients of the function \( f(x) \):

\[ c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) e^{inx} dx. \]  

(31)
The application of the Plancherel theorem to the Bessel generating function (16) leads to the following equalities [16]:

$$\sum_{L=-\infty}^{L=\infty} J^2_L(z) = 1,$$

(32)

$$\sum_{L=-\infty}^{L=\infty} L^2 J^2_L(z) = \frac{e^2}{2},$$

(33)

$$\sum_{L=-\infty}^{L=\infty} L^{2n-1} J^2_L(z) = 0, \quad n \in \mathbb{Z}_+$$

(34)

$$\sum_{L=-\infty}^{L=\infty} L^4 J^2_L(z) = \frac{z^2(4+3z^2)}{8}.$$  

(35)

Performing the summation over the photon polarizations, we obtain the closed analytical expression for the cross section of the process (1):

$$\frac{d\sigma}{d\Omega} = \frac{e^2 C_0^2 E_{i0}^3}{8\pi^2 q(1-e^{-2\pi i})} \left( |I_1|^2 + \left| I_2 \right|^2 \frac{3h\omega_0 z^2}{2E_{i0}^2} \left( \frac{2E_{i0} \cdot \text{Re}(I_1 I_2)}{z} + \hbar \omega_0 |I_1|^2 \right) \right) + \frac{\hbar^2 \omega_0 z^2 (4+3z^2)}{8E_{i0}^2} \left( \frac{3E_{i0} |I_2|^2}{z^2} + 2\hbar \omega_0 \cdot \text{Re}(I_1 I_2) \right),$$

(36)

Here $I_1$ specifies the space integral (25) for the field-free recombination process. $I_2$ is the spatial integral (26) of the laser-assisted recombination process. The parameter $z = e(q \cdot \vec{a}_0)/m \hbar = e(q \cdot \vec{e}_0)/m \hbar \omega_0^2$ was introduced in (20), where $q$, $\omega_0$, and $\omega_0$ denote the momentum of the incoming electron, the amplitude of the laser field, and the field frequency, respectively. The parameter $E_{i0} = E_i - W_0$ is the difference between the kinetic energy of the incoming electron and the hydrogen ground state. The constant $C_0 = \sqrt{Z^3/\pi a_0^3}$ represents the normalization constant (15) of the hydrogen wavefunction (14) in the ground state.

It can be seen that in the zero-field limit, which corresponds to $z \equiv 0$ and $I_2 \equiv 0$, the square brackets in (36), which are responsible for the laser-modified cross section, are identically zero.

To the best of our knowledge, such a closed expression for the laser-assisted hydrogen recombination was not given in the literature up until now.

An explicit summation over the photon polarizations is given by the following equalities:

$$\sum_{\lambda} e_{\mu}^{\lambda} e_{\nu}^{\lambda} = \delta_{\mu\nu},$$

$$\sum_{\lambda} (\vec{a} \cdot \vec{e}^{\lambda})^2 = a^2 (1 - \cos^2 \theta),$$

(37)

where $\theta$ is the angle between the vector $\vec{a}$ and the momentum of an outgoing photon $\vec{k}$.

The corresponding expressions for (25) and (26) with an explicit summation over the photon polarizations are given by the following equalities:

$$\sum_{\lambda} |I_1|^2 = 2^6 \pi^2 \hbar \tilde{q} \frac{\xi^2 e^{-4\tilde{q} \text{arccot} \xi}}{\tilde{q}^2 (1+\xi^2)^3} (1 - \cos^2 \theta),$$

(38)
\[
\sum_{\lambda} |I_2|^2 = 2^6\pi^2\hbar^8 \frac{e^{2\xi^2}}{\omega_0^2} e^{-4(1+\xi)\arctan\xi} \cdot \frac{e^{-2(1+\xi)\arctan\xi}}{q^2(1+\xi^2)^2} \cdot \left[ e_0^2(1 - \cos^2 \phi)(1 + \xi^2) - 4(e_0 \cdot \bar{e}_q)^2(2 + \xi^2) \right] + 4(e_0 \cdot \bar{e}_q)^2(4 + \xi^2)(1 - \cos^2 \theta),
\]

and by the corresponding expression for the interference term

\[
\sum_{\lambda} \text{Re}(I_1I_2) = 2^6\pi^2\hbar^8 \frac{e^{2\xi^2}}{\omega_0^2} e^{-2(1+\xi)\arctan\xi} \cdot \frac{e^{-2(1+\xi)\arctan\xi}}{q^2(1+\xi^2)^2} \cdot (e_0 \cdot \bar{e}_q)(4\cos^2 \theta - 3),
\]

where \(\theta\) is the angle between the momentum of an incoming electron \(\vec{q}\) and the momentum of an outgoing photon \(\vec{k}\). The angle \(\phi\) is formed by the vectors \(\vec{e}_0\) and \(\vec{k}\).

In order to give an estimation for the order of magnitude of the laser field correction to the recombination process, we employ the Keldysh dimensionless tunneling parameter \([17]\)

\[
\gamma = \frac{\omega_0}{e\epsilon_0} \sqrt{2mW_0},
\]

which for characteristic numerical values \(\epsilon_0 \equiv |\vec{e}_0| \sim 5 \times 10^7 \text{ V cm}^{-1}\), and \(h\omega_0 \sim 1 \text{ eV}\), and \(W_0 = 13.6 \text{ eV}\) has the value \(\gamma \sim 4\).

The parameter \(z\) \((20)\) can easily be expressed in terms of the Keldysh tunneling parameter and for the case \(E_i \sim W_0 = 13.6 \text{ eV}\) has the following numerical value:

\[
z = \frac{e(\vec{q} \cdot \vec{a}_0)}{m\hbar} = \frac{2\sqrt{E_iW_0}}{\gamma\hbar\omega_0} \sim 7,
\]

and the parameter \(\xi\) given by \((8)\) can be estimated as

\[\xi = \frac{Zh}{\omega_0q} \sim 1.\]

Finally we obtain the following estimations for the field-influenced terms confined by the square brackets \((36)\):

\[
\frac{|I_2|^2}{|I_1|^2} \sim \kappa \equiv \left| \frac{e\epsilon_0}{q\omega_0} \right|^2 = \frac{1}{\gamma^2} \frac{W_0}{E_i} \sim \frac{1}{16},
\]

\[
\frac{\hbar\omega_0z\text{Re}(I_1I_2)}{E_i} \sim \kappa \sim \frac{1}{16},
\]

\[
\frac{(\hbar\omega_0z)^2|I_2|^2}{E_i^2|I_1|^2} \sim \kappa \sim \frac{1}{16},
\]

\[
\frac{\hbar^2\omega_0^2(4 + 3z^2)|I_2|^2}{E_i^2|I_1|^2} \sim \kappa^2 \sim \frac{1}{256},
\]

\[
\frac{(\hbar\omega_0)^3z(4 + 3z^2)\text{Re}(I_1I_2)}{E_i^3|I_1|^2} \sim \kappa^2 \sim \frac{1}{256}.
\]

The field-influenced terms, \(|I_1|^2\) and \(|I_2|^2\), are positive and accompany the recombination process. According to \((40)\) the interference term, \(\text{Re}(I_1I_2)\), is positive under the following conditions:

\[
(e_0 \cdot \bar{e}_q) > 0, \quad \theta \in \left( -\frac{\pi}{6}; \frac{\pi}{6} \right) \cup \left( \frac{5\pi}{6}; \frac{7\pi}{6} \right);
\]

\[
(e_0 \cdot \bar{e}_q) < 0, \quad \theta \in \left( \frac{\pi}{6}; \frac{5\pi}{6} \right) \cup \left( \frac{5\pi}{6}; \frac{\pi}{6} \right).
\]

In the particular case with \((e_0 \cdot \bar{e}_q) \equiv 0\), the interference terms are absent.
Based on the obtained estimations (44)–(48), we do not see the predicted [9] laser contraction of the hydrogen (antihydrogen) recombination process.

The basic estimations, (44)–(48), show that the field influence terms are directly expressed in terms of the dimensionless parameter

\[ \kappa = \left| \frac{e \varepsilon_0}{q \omega_0} \right|^2. \]

It is proportional to the intensity of the laser field, \( |\varepsilon_0|^2 \), and inversely proportional to the square of the field frequency, \( 1/\omega_0^2 \). This is justified by the chosen electron (12) and hydrogen (14) laser-modified states, where in both cases there is a dependence on the reciprocal field frequency. The spatial integral of the laser-assisted recombination (26) is more sensitive to the momentum of the incoming electron than the field-free spatial integral (25). This results in the inverse proportionality between \( \kappa \) and \( q^2 = \frac{2mE_i}{\gamma^2} \).

The parameter \( \kappa = \left| \frac{e \varepsilon_0}{q \omega_0} \right|^2 = \frac{1}{\gamma^2} \frac{W_0}{E_i} \) governs the laser-assisted recombination process and points out possible ways for increasing the hydrogen laser-assisted recombination rate. An increase of the parameter \( \kappa \) by lowering the parameter \( \gamma^2 \) is the most natural way to improve the recombination rate of a hydrogen atom in a laser field. Although it is experimentally feasible, this path has to be examined carefully due to the arising ionization processes of the produced hydrogen atom. Ionization processes have not been considered in the proposed model, and therefore a quantitative treatment requires a full solution of the time-dependent Schrödinger equation [18].

The binding electron energy \( W_0 \) is a well-defined value and cannot be used in increasing the laser-assisted recombination rate. The ultimate possibility to increase the parameter \( \kappa \) is decreasing the relative kinetic energy between an electron and a proton \( E_i \) for which one can achieve a reasonably small value which would mean improving a laser-assisted recombination rate. This fact has a clear physical explanation: for a quasi-static regime in the inverse to the laser-assisted recombination process, i.e. in the laser-assisted photo-ionization process, the energy of the emitted photo-electrons has a peak at zero value. In our case of the laser-assisted recombination process, this peak corresponds to a maximum value of the photo-recombination rate.

7. The soft photon approximation

The cross section given by (36) is an exact result for the laser-assisted hydrogen recombination process under the proposed approximations for the electron and hydrogen laser-modified wavefunctions. To clarify the meaning of the obtained result, we shall introduce a soft photon approximation. This allows one to reveal the meaning of each term in the expression for the cross section of the process given by (36). With this approximation, the frequency of the emitted photon is independent of the number of the field harmonics \( L \):

\[ \hbar \omega = E_i - W_0 + L \hbar \omega_0 \simeq E_i - W_0. \]  

This is justified as the low-frequency field limit or the soft photon limit [8]. In this case, we are able to produce a simple but non-trivial result. In the soft photon approximation, the corresponding expression for the partial 'soft photon' cross section is given by

\[ \frac{d\sigma_{L}}{d\Omega} = \frac{e^2 C_0^2}{8\pi^2 q(1 - e^{-2\pi i})} (E_i - W_0)^3 \times \left| J_L(z) \right|^2 \left( \left| I_1 \right|^2 + \frac{2\text{Re}(I_1 I_2)}{z} L + \frac{L^2}{z^2} \left| I_2 \right|^2 \right). \]  

Performing the summation (29) over \( L \) by means of the equalities (32)–(34), we obtain the following simple result:

\[ \frac{d\sigma_{\infty}}{d\Omega} = \frac{e^2 C_0^2}{8\pi^2 q(1 - e^{-2\pi i})} E_i^3 \left( \left| I_1 \right|^2 + \left| I_2 \right|^2 \right). \]
Here, the first term $|I_1|^2$ corresponds to the standard laser-free recombination process whereas the second term $|I_2|^2$ is responsible for the field-modified electron (12) and hydrogen (14) states. In the zero-field limit, $I_2 \equiv 0$ and (52) recovers the standard field-free recombination cross section. We have to note that within the limits of the present approximation (50), the interference terms $\text{Re}(I_1 I_2)$ are absent. Thus in this and only this particular case does the square of the $S$-matrix element (18) equal the sum of the squares of its terms. So we have proved the following theorem.

**Theorem.** The $S$-matrix element

$$S = -2\pi i \sum_{L=-\infty}^{L=+\infty} f_L \delta(W_0 + \hbar \omega - E_i + L\hbar \omega_0)$$

with

$$f_L = e^{-\frac{\pi}{4}} \Gamma(1 - i\xi) C_0 \left[ J_L(z) \omega(L) \left( I_1 + \frac{L}{z} I_2 \right) \right]$$

under the approximation

$$\hbar \omega = E_i - W_0 + L\hbar \omega_0 \simeq E_i - W_0 \neq \hbar \omega(L),$$

equals to the sum of the squares of individual terms

$$|S|^2 = |2\pi e^{-\frac{\pi}{4}} \Gamma(1 - i\xi) C_0|^2 \delta^2(W_0 + \hbar \omega - E_i) \omega^2 \left( |I_1|^2 + \frac{|I_2|^2}{2} \right).$$

The theorem reveals the meaning of each term in the expression for the laser-assisted hydrogen photo-recombination given by (36). The first term in the braces (36), $|I_1|^2$, is responsible for the field-free recombination process. The first term in the square brackets (36), $|I_2|^2/2$, reflects the laser-modified electron and hydrogen states and does not take into account the changes in the frequency of the emitted photon, according to the theorem. The four terms in the parentheses (36) are the only terms which contribute to the change of the frequency of the emitted photon in the process.

**8. Summary**

In this paper, we have developed a new method that allows one to obtain an analytical cross section for the laser-assisted electron–ion collision. The standard $S$-matrix formalism is used for describing the collision process. The $S$-matrix element is constructed from the electron Coulomb–Volkov wavefunction in the combined Coulomb-laser field, and the hydrogen laser-modified state. By the aid of the Bessel generating function, the $S$-matrix element is decomposed into an infinite series of the field harmonics.

We have introduced a new step to obtain an analytical expression for the cross section of the process. The main theoretical novelty is the application of the Plancherel theorem to the Bessel generating function. This allows one to obtain an analytical expression for the laser-assisted hydrogen photo-recombination process.

The laser-assisted hydrogen recombination process has been chosen in order to verify the proposed method. The field-enhancement coefficient is evaluated in an analytical way. The final expression for a laser-assisted hydrogen recombination process is presented by the sum of the field-free hydrogen cross section and the laser-assisted addition. The field-dependent terms are expressed through the dimensionless parameter $\kappa \equiv |e\varepsilon_0/q_0\omega_0|^2$.

By introducing a soft photon approximation, based on the assumption of the independence between the frequency of the emitted photon and the field harmonics, the square of the $S$-matrix
element is represented by the sum of the squares of its individual terms. We provide the proof of the corresponding theorem.

The developed method will allow one to reconsider a wide range of problems related to electron–ion collisions in an external field with the goal of obtaining an analytical expression for the cross sections of the corresponding process. The time-dependent problem generated by the infinite series of the Coulomb–Volkov wavefunction is exactly separated from the spatial dependence and thus can be analytically solved by the method.

One of the challenging problems in modern electron–ion collision theory is the calculation of a laser-assisted three-body hydrogen formation process. As far as we know, an analytical solution for this important process has not been found. The developed method allows one to obtain an analytical expression for this process.

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