A New Lower Bound for the Distinct Distance Constant

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Abstract

The reciprocal sum of Zhang sequence is not equal to the Distinct Distance Constant. This note introduces a $B_2$-sequence with larger reciprocal sum, and provides a more precise estimation of the reciprocal sums of Mian-Chowla sequence and Zhang sequence.

1 Introduction

A Sidon sequence, also called $B_2$-sequence, is a sequence of positive integers $a_1 < a_2 < a_3 < \ldots$ such that all the sums $a_i + a_j$ ($i \leq j$) are distinct.

The Distinct Distance Constant (DDC) is the supremum of the set of the reciprocal sums of Sidon sequences. Levine [1] observed that

$$DDC < \sum_{n=0}^{\infty} \frac{1}{1 + \frac{n(n+1)}{2}} = \frac{2\pi}{\sqrt{7}} \tanh \left( \frac{\sqrt{7}}{2} \pi \right) < 2,37366.$$  \hfill (1)

Let $S_A$ be the reciprocal sum of the sequence $A$, $S_A = \sum_{i=1}^{\infty} \frac{1}{a_i}$. It is an open problem to find, if it exists, a Sidon sequence $U$ whose reciprocal sum is equal to the DDC [2]. We only know from Taylor and Yovanof [3] that, if some Sidon sequence achieves the DDC, then it must begin with the values 1, 2, 4.

The $B_2$-sequence with the largest known reciprocal sum had been for long time the one produced by the greedy algorithm, $G = \{1, 2, 4, 8, 13, 21, 31, 45, 66, 81, \ldots\}$. The sequence $G$ is named Mian-Chowla sequence. Lewis found that

$$2.158435 \leq S_G \leq 2.158677,$$  \hfill (2)

where $S_G$ is the reciprocal sum of $G$ [4].
In 1991, Zhang [5] found a Sidon sequence with a reciprocal sum greater than $S_A$. Zhang sequence $Z$ is obtained by running the greedy algorithm for the first 14 terms, setting $z_{15} = 229$, and then going on with the greedy algorithm. Zhang proved that

$$S_Z > 2.1597.$$  \hspace{1cm} (3)

The aim of this note is to show a Sidon sequence $H$ such that $S_H > S_Z$.

## 2 Computation of reciprocal sums

### 2.1 Preliminary considerations

To estimate their reciprocal sum, we will use two basic properties of Sidon sequences: they are growing sequences, and their differences $a_i - a_j$ ($i \geq j$) are all distinct. That is,

$$a_i \geq a_j + (i - j), \ i \geq j$$  \hspace{1cm} (4)

$$a_i > \frac{i(i-1)}{2}.$$  \hspace{1cm} (5)

Therefore, if we know the values of $a_n$ for $1 \leq n \leq k$, we also know that

$$\sum_{n=1}^{k} \frac{1}{a_n} < S_A < \sum_{n=1}^{k} \frac{1}{a_n} + \sum_{n=k}^{\infty} \max \left( \frac{1}{a_k + n - k}, \frac{n(n-1)}{2} \right).$$  \hspace{1cm} (6)

Further assumptions could be made on $a_i$, but for large values of $k$ they would not yield to a significant improvement of the boundaries.

### 2.2 The reciprocal sum of the Mian-Chowla sequence

Let $G$ be the $B_2$-sequence constructed by the greedy algorithm. The values of $g_n$ for $1 \leq n \leq 25000$, computed on my ASUS notebook, are listed in the attached file MianChowla.txt. We get the following boundaries for $S_G$:

$$\sum_{n=1}^{25000} \frac{1}{g_n} < S_G < \sum_{n=1}^{25000} \frac{1}{g_n} + \sum_{n=25001}^{510096} \frac{1}{g_{25000} + n - 25000} + \sum_{n=510097}^{\infty} \frac{2}{n(n-1)},$$  \hspace{1cm} (7)

i.e.,

$$2.15845268 < S_G < 2.15846062.$$  \hspace{1cm} (8)
2.3 The reciprocal sum of the Zhang sequence

The Zhang sequence $Z$ [5] is, at the moment, the known Sidon sequence with the largest reciprocal sum. In the attached file Zhang.txt there are the first 25000 terms of $Z$. Their values allow us to compute the following bounds for $S_Z$:

$$\sum_{n=1}^{25000} \frac{1}{z_n} < S_Z < \sum_{n=1}^{25000} \frac{1}{z_n} + \sum_{n=25001}^{510290} \frac{1}{z_{25000} + n - 25000} + \sum_{n=510291}^{\infty} \frac{2}{n(n-1)}$$

$$S_G < 2.16007769 < S_Z < 2.16008532.$$  

2.4 The sequence $H$ and its reciprocal sum

Definition 1.

$$h_n = \begin{cases} 
1, & \text{if } n = 1; \\
229, & \text{if } n = 15; \\
962, & \text{if } n = 27; \\
\min\{x|\forall i, j, k \leq n, a_i + a_j \neq a_k + x\}, & \text{otherwise}.
\end{cases}$$

The first 26 terms of the sequence $H$ are the same of the Zhang sequence, the 27th term is 962, and from there on the values are provided by the greedy algorithm.

We will now prove that the sequence $H$ satisfies our aim, i.e., $S_H > S_Z$.

We can find the first values of $h_n$ ($1 \leq n \leq 25000$) in the attached file H.txt. It results that

$$\sum_{n=1}^{25000} \frac{1}{h_n} < S_H < \sum_{n=1}^{25000} \frac{1}{h_n} + \sum_{n=25001}^{510140} \frac{1}{h_{25000} + n - 25000} + \sum_{n=510141}^{\infty} \frac{2}{n(n-1)}$$

$$2,16027651 < S_H < 2,16028417.$$  

In conclusion,

$$S_Z < 2,16027651 < S_H \leq DDC,$$

which was what we wanted to prove.

References

[1] Eugene Levine, An extremal result for sum-free sequences, *J. Number Theor.* 12 (1980), 251-257.

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(Concerned with sequence [A005282](https://oeis.org/A005282))