Off-shell Crosscap State and Orientifold Planes with Background Dilatons

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Abstract

We show that a non-trivial dilaton condensation alters the dimensions of orientifold planes. An off-shell crosscap state which naturally interpolates between the usual on-shell crosscap states and their T-duals plays an important role in the analysis. We present an explicit representation of the off-shell crosscap state on an $RP^2$ worldsheet in the gauge in which the worldsheet curvature in the bulk of the fundamental region of the $RP^2$ vanishes. We show that the non-trivial dilaton condensation reproduces the correct descent relation among orientifold plane tensions.

*This work is supported in part by the Grant-in-Aid for Scientific Research (14540264) from the Ministry of Education, Science and Culture, Japan.
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I. Introduction

Orientifold planes (O-planes) as well as D-branes are important objects to reveal non-perturbative effects in unoriented string theory. Recent studies on open-string tachyon condensation have shown that background fields can control the dimensions of D-branes. The relationship between the dimensions of the D-branes and the configuration of the background open-string fields is easily understood from the viewpoint of the worldsheet; the background fields on the boundaries can alter the boundary conditions on the worldsheet. On the other hand, the relationship between the properties of O-planes and background fields has not been understood well as yet.

In the present work, we investigate the connection between the dimensions of O-planes and the configuration of the background dilaton field, in unoriented bosonic string theory. An O-plane is represented as a crosscap on an $RP^2$ worldsheet while a D-brane is represented as a boundary on a disc. Therefore we consider an $RP^2$ worldsheet in the presence of the background dilatons. The reason we consider dilatons is based on the following property. Dilatons couple to the worldsheet curvature and their contribution can be put on any part of the worldsheet in general. However, we show that the contribution of the background dilatons localizes on the crosscap if we choose the gauge in which the worldsheet curvature vanishes in the bulk of the fundamental region of the $RP^2$. This choice of gauge is nice; the bulk part of the $RP^2$ becomes free and all the interactions from the background dilatons appear only on the crosscap. Therefore the effects of the background dilatons can be translated into the modification of the crosscap conditions.

We introduce a new crosscap state which we refer to as an off-shell crosscap state in order to analyze the properties of the $RP^2$ worldsheet. Of course, O-planes couple to only closed strings and we do not have open-string modes on the $RP^2$ worldsheet. Useful tools to analyze the properties of worldsheets in terms of closed-string modes are boundary states and crosscap states \[2, 3, 4, 5\]. Usually, boundary states and crosscap states belong to the closed-string sector which preserves conformal invariance on the worldsheet. Extensions of these boundary states which do not in general maintain conformal invariance have been proposed recently \[6, 7\]. (See also \[8, 9\].) We call them off-shell boundary states in the present article. The off-shell boundary state interpolates between the usual on-shell boundary states and their T-duals. This state is defined on a disc worldsheet with quadratic boundary interactions. The dimension of the corresponding D-brane is controlled by the coupling constants of these interactions. Boundary string field theory (BSFT) \[10, 11\] states that the

\[1\]See, for example, Refs. \[1\] for recent reviews on the related topics.
coupling constants also parameterize the configuration of open-string tachyons \[12, 13, 14, 15, 16, 17\]. Calculating the disc partition function by using these states enables us to obtain the descent relation among the D-brane tensions if we take appropriate on-shell limits of the partition functions.

An attempt to apply the foregoing idea to crosscap states is given in Ref. \[18\], in which the definition of the off-shell crosscap state which interpolates the usual on-shell crosscap states and their T-duals has been proposed. We present an explicit representation of the off-shell crosscap state in the present work. We define the off-shell crosscap state on an \(RP^2\) worldsheet in the presence of quadratic interactions on the crosscap. The dimension of the corresponding O-plane is controlled by the coupling constants of these interactions. The physical meaning of the quadratic interactions on the crosscap is the background dilaton field of quadratic configuration. The behaviour of the off-shell crosscap state shows that the background dilatons control the dimension of the O-plane.

The off-shell crosscap state is a useful tool to obtain correlation functions on the \(RP^2\) worldsheet in the presence of the quadratic dilatons on the crosscap. The exact partition function of the \(RP^2\) worldsheet which is considered to be proportional to the O-plane tension can be calculated exactly by using the two-point function. We show that the condensation of the dilatons of quadratic profile reproduces the correct descent relation among O-plane tensions by taking appropriate on-shell limits of the partition function. Note that we have not attempted to find the dynamical origin of the dilaton condensation.

The paper is organized as follows. In Sec. 2, we construct the \(RP^2\) worldsheet on a complex plane and we show that the contribution of the background dilatons survives on the crosscap alone if we choose the gauge in which the worldsheet curvature vanishes in the bulk of the fundamental region of the \(RP^2\). In Sec. 3 we define the off-shell crosscap state and we obtain its explicit representation. We find that the behaviour of this state signifies that a non-trivial dilaton condensation alters the dimensions of O-planes. We calculate the partition function of the \(RP^2\) by using the off-shell crosscap state. In Sec. 4, we verify that the non-trivial dilaton condensation reproduces the correct descent relation of O-plane tensions, by taking appropriate limits of the partition function of the \(RP^2\) worldsheet. In the final section, we summarize the results of this study. We make comments on some relationship between the \(RP^2\) worldsheet and the supersymmetric disc worldsheet which appears in the analysis of \(D\bar{D}\) systems in Appendix.
II. $RP^2$ worldsheet with background dilatons

Let us consider an $RP^2$ worldsheet with background dilaton field. We show, in this section, that the contribution of the background dilatons localizes on the crosscap if we choose the gauge in which the worldsheet curvature in the bulk of the fundamental region of the $RP^2$ vanishes.

$RP^2$ is a non-orientable Riemann surface of Euler number one with no hole, no boundary and one crosscap. We construct the $RP^2$ worldsheet on a complex $z$-plane by using an involution where we identify $z$ and $-\frac{1}{z}$ on the complex plane. We choose the fundamental region $\Sigma$ to be $\{z = re^{i\sigma}|0 \leq r < 1, 0 \leq \sigma < 2\pi\} \cup \{z = re^{i\sigma}|r = 1, 0 \leq \sigma < \pi\}$. The crosscap $C$, the non-trivial closed loop of the $RP^2$ worldsheet, is represented as half of unit circle $\{z = re^{i\sigma}|r = 1, 0 \leq \sigma < \pi\}$ in this case.

To begin with, we set the metric inside the unit circle ($|z| \leq 1$) on the complex plane as

\[
\begin{align*}
    h_{zz} &= h_{\bar{z}\bar{z}} = 0, \\
    h_{\bar{z}\z} &= h_{z\bar{z}} = \frac{1}{2},
\end{align*}
\]

The metric outside the unit circle ($|z'| \geq 1$) is obtained by the involution $z' = -\frac{1}{z}$, $\bar{z}' = -\frac{1}{\bar{z}}$ as

\[
\begin{align*}
    h_{z'\bar{z}'} &= \frac{\partial \bar{z}}{\partial z'} \frac{\partial z}{\partial \bar{z}'} h_{\bar{z}z} \\
    &= \frac{1}{(z'\bar{z}')^2} h_{\bar{z}z} = \frac{1}{r^4} h_{\bar{z}z},
\end{align*}
\]

where $r^2 = z'\bar{z}'$ for $r \geq 1$. Therefore the metric on the entire complex plane can be written as

\[
\begin{align*}
    h_{zz} &= h_{\bar{z}\bar{z}} = 0, \\
    h_{\bar{z}z} &= h_{z\bar{z}} = \frac{1}{2} e^{\rho},
\end{align*}
\]

where

\[
\rho(r) = (-4 \ln r) \theta(r - 1).
\]

Next, we rewrite the metric in the polar coordinate $(r, \sigma)$ as

\[
g_{ab} = \hat{g}_{ab} e^{\rho},
\]
where $\hat{g}_{rr} = 1, \hat{g}_{\sigma\sigma} = r^2, \hat{g}_{r\sigma} = \hat{g}_{\sigma r} = 0$. The worldsheet curvature $R$ is then given by
\[
\sqrt{\hat{g}} R = -\sqrt{\hat{g}} \hat{\nabla}^2 \rho(r) = -r \left( \partial_r^2 + \frac{1}{r} \partial_r \right) \rho(r) = 4 \{ \delta'(r - 1)r \ln r + (2 + \ln r)\delta(r - 1) \}.
\] (2.6)

Let us calculate the integral $\int_{\Sigma(\epsilon)} dr d\sigma \sqrt{\hat{g}} R \Phi(r, \sigma)$. We define the integral region $\Sigma(\epsilon)$ as \( \{ z = re^{i\sigma} | 0 \leq r \leq 1 + \epsilon, 0 \leq \sigma < \pi \} \cup \{ z = re^{i\sigma} | 0 \leq r \leq 1 - \epsilon, \pi \leq \sigma < 2\pi \} \), where $\epsilon$ is a positive real number. $\Sigma(\epsilon)$ becomes the fundamental region $\Sigma$ of the $\mathbb{R}P^2$ after we take the limit $\epsilon \to 0$. We obtain
\[
\int_{\Sigma(\epsilon)} dr d\sigma \sqrt{\hat{g}} R \Phi(r, \sigma) = \int_0^\pi d\sigma \int_0^{1+\epsilon} dr \sqrt{\hat{g}} R \Phi(r, \sigma) + \int_\pi^{2\pi} d\sigma \int_0^{1-\epsilon} dr \sqrt{\hat{g}} R \Phi(r, \sigma)
\]
\[
= 4 \int_0^\pi d\sigma \int_0^{1+\epsilon} dr \{ \delta'(r - 1)r \ln r + (2 + \ln r)\delta(r - 1) \} \Phi(r, \sigma)
\]
\[
= 4 \int_0^\pi d\sigma \left\{ -\frac{d}{dr} \{ r \ln r \Phi(r, \sigma) \} |_{r=1} + 2\Phi(1, \sigma) \right\}
\]
\[
= 4 \int_0^\pi d\sigma \Phi(1, \sigma),
\] (2.7)

which yields
\[
\frac{1}{4\pi} \int_{\Sigma} dr d\sigma \sqrt{\hat{g}} R \Phi(r, \sigma) = \frac{1}{\pi} \int_0^\pi d\sigma \Phi(1, \sigma).
\] (2.8)

Note that (2.8) gives the correct Euler number of the $\mathbb{R}P^2$ (which is one) if we set $\Phi = 1$. Therefore the contribution of the background dilaton concentrates on the crosscap with the above gauge choice.

### III. Off-shell crosscap state

In this section, we define the off-shell crosscap state and obtain its explicit representation. We apply the off-shell crosscap state to calculate the correlation functions and the partition function of the $\mathbb{R}P^2$ worldsheet with quadratic background dilaton field. We find that the behaviour of the off-shell crosscap state signifies that a non-trivial dilaton condensation alters the dimensions of O-planes.
A. Off-shell crosscap conditions

Let us consider an $RP^2$ worldsheet $\Sigma$ with the following action:

\begin{equation}
I = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \partial X^\mu \partial X_\mu + \frac{1}{\pi} \int_C d\sigma \Phi(\sigma),
\end{equation}

\begin{equation}
\Phi(\sigma) = a + \frac{1}{2\alpha'} \sum_{\mu=1}^{26} u_\mu (X^\mu(\sigma))^2,
\end{equation}

where the interaction $\Phi(\sigma)$ is inserted only on crosscap $C$. Note that the worldsheet action is free in the “bulk” region $\{ z = r e^{i\sigma} | 0 \leq r < 1, 0 \leq \sigma < 2\pi \}$. A closed string propagates freely in the “bulk” toward the crosscap, from the viewpoint of the closed-string channel. In this sense, $X^\mu$ in the “bulk” can be expanded as

\begin{equation}
X^\mu(z, \bar{z}) = X^\mu_0 - \frac{i\alpha'}{2} p^\mu \ln(z\bar{z}) + i \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left( \alpha^\mu_n \frac{z^{-n}}{n} + \tilde{\alpha}^\mu_n \frac{\bar{z}^{-n}}{n} \right).
\end{equation}

The existence of the crosscap, however, makes constraints on the oscillation of the closed string in the neighborhood of the crosscap (in the region $r \to 1$). For example, the constraints when we have no interaction on $C$ are given in Ref. [4] as

\begin{align}
\{ X^\mu(z, \bar{z}) - X^\mu(-1/\bar{z}, -1/z) \} \bigg|_{r \to 1} &= 0, \\
\{ \dot{X}^\mu(z, \bar{z}) + \dot{X}^\mu(-1/\bar{z}, -1/z) \} \bigg|_{r \to 1} &= 0,
\end{align}

where

\begin{align}
\dot{X}^\mu(z, \bar{z}) &\equiv (w \partial_w + \bar{w} \partial_{\bar{w}}) X^\mu(w, \bar{w}) \big|_{w=z, \bar{w}=\bar{z}}, \\
\dot{X}^\mu(-1/\bar{z}, -1/z) &\equiv (w \partial_w + \bar{w} \partial_{\bar{w}}) X^\mu(w, \bar{w}) \big|_{w=-1/\bar{z}, \bar{w}=-1/z}.
\end{align}

Note that (3.4) are equivalent to the following constraints:

\begin{align}
K_0(z, \bar{z}) \big|_{r \to 1} &= 0, \\
\{ (z \partial_z + \bar{z} \partial_{\bar{z}}) K_0(z, \bar{z}) \} \big|_{r \to 1} &= 0,
\end{align}

where

\begin{equation}
K_0(z, \bar{z}) \equiv \dot{X}^\mu(z, \bar{z}) + \dot{X}^\mu(-1/\bar{z}, -1/z).
\end{equation}
These conditions are rewritten in terms of closed-string modes at \( r \to 1 \) as

\[
\alpha_n^\mu + (-1)^n \tilde{\alpha}_{-n}^\mu = 0, \\
p^\mu = 0. 
\] (3.8)

The conditions (3.4), (3.6) or (3.8) are referred to as (on-shell) crosscap conditions.

The aim of this subsection is to extend the on-shell crosscap conditions into the case \( u_\mu \neq 0 \). We should find, in other words, the constraints on the closed-string modes in the neighborhood of \( \mathcal{C} \) in the presence of interaction \( \Phi \). We call these constraints \textit{off-shell crosscap conditions}. We assert that the off-shell crosscap conditions can be written as

\[
\left. K(z, \bar{z}) \right|_{r \to 1} = 0, \\
\left. \left\{ (z \partial_z + \bar{z} \partial_{\bar{z}}) K(z, \bar{z}) \right\} \right|_{r \to 1} = 0, 
\] (3.9)

where

\[
K(z, \bar{z}) \equiv \dot{X}^\mu(z, \bar{z}) + \dot{X}^\mu(-1/\bar{z}, -1/z) + u_\mu \left\{ X^\mu(z, \bar{z}) + X^\mu(-1/\bar{z}, -1/z) \right\} \\
= \left\{ (w \partial_w + \bar{w} \partial_{\bar{w}}) X^\mu(w, \bar{w}) + u_\mu X^\mu(w, \bar{w}) \right\} \bigg|_{w=z, \bar{w}=\bar{z}} \\
+ \left\{ (w \partial_w + \bar{w} \partial_{\bar{w}}) X^\mu(w, \bar{w}) + u_\mu X^\mu(w, \bar{w}) \right\} \bigg|_{w=-1/\bar{z}, \bar{w}=-1/z}. 
\] (3.10)

The right-hand side of (3.10) indicates the meaning of the off-shell crosscap conditions; (3.9) are the conditions so that the \( X^\mu \) in the neighborhood of \( \mathcal{C} \), as well as its image by the involution, connects smoothly with the \( X^\mu \) on \( \mathcal{C} \) which obeys

\[
\left. \left\{ (z \partial_z + \bar{z} \partial_{\bar{z}}) X^\mu(z, \bar{z}) + u_\mu X^\mu(z, \bar{z}) \right\} \right|_{\mathcal{C}} = 0, 
\] (3.11)

given by varying the worldsheet action (3.1).

The off-shell crosscap conditions (3.9) are rewritten in terms of closed-string modes as

\[
-\{ \alpha_n^\mu + (-1)^n \tilde{\alpha}_{-n}^\mu \} + \frac{u_\mu}{n} \{ \alpha_n^\mu - (-1)^n \tilde{\alpha}_{-n}^\mu \} = 0, \\
-i \alpha^\prime p^\mu + u_\mu X^\mu_0 = 0, 
\] (3.12)

where we do not sum over \( \mu \). We can easily check that the conditions (3.12) gives the on-shell crosscap conditions (3.8) in the limit \( u^\mu \to 0 \). We also note that (3.12), in the limit \( u^\mu \to \infty \),

\footnote{Although \( RP^2 \) has no boundary, \((z \partial_z + \bar{z} \partial_{\bar{z}}) X^\mu(z, \bar{z}) \) which comes from the total derivative survives only on the crosscap due to the involution.}
becomes equivalent to the T-dual of (3.8),
\[
\alpha_n - (-1)^n \tilde{\alpha}_n = 0,
\]
\[
X_0^\mu = 0,
\]
which are rewritten as the T-dual of (3.4):
\[
\left\{ X^\mu (z, \bar{z}) + X^\mu (-1/\bar{z}, -1/z) \right\}_{r \to 1} = 0,
\]
\[
\left\{ \dot{X}^\mu (z, \bar{z}) - \dot{X}^\mu (-1/\bar{z}, -1/z) \right\}_{r \to 1} = 0.
\]
(3.14)

Note that the conditions (3.9) in the limit \( u^\mu \to \infty \) is (3.14) itself. Therefore the off-shell crosscap conditions (3.9) or (3.12) naturally interpolate between on-shell crosscap conditions and their T-duals. The coupling constant \( u^\mu \), which is a parameter of the configuration of the background dilaton field, controls the dimension of the corresponding O-plane.

### B. Off-shell crosscap state and partition function

We define off-shell crosscap state \( \langle C(u) | \) using the off-shell crosscap conditions as
\[
\langle C(u) | \left\{ -\{ \alpha_n^\mu + (-1)^n \tilde{\alpha}_n^\mu \} + \frac{u^\mu}{n} \{ \alpha_n^\mu - (-1)^n \tilde{\alpha}_n^\mu \} \right\} = 0,
\]
\[
\langle C(u) | \left\{ -i \alpha' p^\mu + u^\mu X_0^\mu \right\} = 0.
\]
(3.15)

The explicit form of \( \langle C(u) | \) is given as
\[
\langle C(u) | = \langle 0 | C(u),
\]
\[
C(u) \equiv \exp \left( -\frac{1}{2} X_0^\mu A_{\mu\nu} X_0^\nu \right) \exp \left( \sum_{m=1}^{\infty} \tilde{\alpha}_m^\mu C_{\mu\nu}^{(m)} \alpha_m^\nu \right),
\]
(3.16)

where
\[
A_{\mu\nu} \equiv A(u^\mu) \delta_{\mu\nu} = \frac{1}{\alpha'} u^\mu \delta_{\mu\nu}, \quad C_{\mu\nu}^{(m)} \equiv C^{(m)}(u^\mu) \delta_{\mu\nu} = -\frac{(-1)^m}{m} m - u^\mu \delta_{\mu\nu}.
\]
(3.17)

We can easily check that this off-shell crosscap state becomes (the T-dual of) the usual on-shell crosscap state if we take the limit \( u^\mu \to 0 \) (\( u^\mu \to \infty \)). Therefore the off-shell crosscap state naturally interpolates between the crosscap state for a higher-dimensional O-plane and that for a lower-dimensional O-plane.
Next, we show that the off-shell crosscap state is a useful tool to evaluate the quantities on the \( \mathbb{R}P^2 \) worldsheet. For example, we can calculate the Green's function and the partition function on the \( \mathbb{R}P^2 \) worldsheet in the presence of interaction \( \Phi(\sigma) \) on the crosscap. Let us consider one-dimensional target space and omit the superscript \( \mu \) of \( X \) and \( u \) for simplicity. The Green's function for this case is given as

\[
G(z, w) = \frac{\langle C(u) X(z, \bar{z}) X(w, \bar{w}) | 0 \rangle}{\langle C(u) | 0 \rangle} = -\frac{\alpha'}{2} \ln |z - w|^2 - \frac{\alpha'}{2} \ln |1 + z\bar{w}|^2 + \frac{\alpha'}{u} - \alpha'u \sum_{k=1}^{\infty} \frac{1}{k(k + u)} \left[ (-z\bar{w})^k + (-\bar{z}w)^k \right].
\]

(3.18)

In the case \( z = e^{i\sigma} \) and \( w = e^{i\sigma'} \), we obtain

\[
G(e^{i\sigma}, e^{i\sigma'}) = -\frac{\alpha'}{2} \ln |1 - e^{i(\sigma - \sigma')}|^2 - \frac{\alpha'}{2} \ln |1 + e^{i(\sigma - \sigma')}|^2 + \frac{\alpha'}{u} - \alpha'u \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k + u)} [e^{ik(\sigma - \sigma')} + e^{-ik(\sigma - \sigma')}] \] .

(3.19)

We define composite operator \( X^2(\sigma) \) as shown in Ref. [11]:

\[
X^2(\sigma) \equiv \lim_{\epsilon \to 0} (X(\sigma) X(\sigma + \epsilon) - f(\epsilon)) ,
\]

\[
f(\epsilon) = -\frac{\alpha'}{2} \ln |1 - e^{i\epsilon}|^2 + \text{const.}).
\]

(3.20)

We write the constant in (3.20) as \( \alpha' \ln q \) by using a positive constant \( q \). The value for \( q \) is ambiguous at this stage and depends on the renormalization scheme. We will determine the value for \( q \) later. We can then obtain

\[
\langle X^2(\sigma) \rangle = -\frac{\alpha'}{2} \ln 2^2 + \frac{\alpha'}{u} - 2\alpha'u \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k + u)} - \alpha' \ln q
\]

\[
= -\alpha' \ln(2q) - \frac{\alpha'}{u} - 2\alpha' \left[ \Psi \left( \frac{u}{2} \right) - \Psi(u) \right] ,
\]

(3.21)

where

\[
\Psi(u) \equiv \frac{\partial}{\partial u} \frac{\Gamma(u)}{\Gamma(u)}
\]

(3.22)
is a polygamma function for which we have used the relationship
\[ u \sum_{k=1}^{\infty} \left( -\frac{1}{2} \right)^k \frac{1}{k(k+u)} = \frac{1}{u} + \Psi \left( \frac{u}{2} \right) - \Psi(u). \] (3.23)

We can next calculate the partition function of the $RP^2$ worldsheet by using the following relationship:
\[
\frac{d}{du} \ln Z(u) = -\frac{1}{2\pi \alpha'} \int_0^\pi d\sigma \langle X^2(\sigma) \rangle = -\frac{1}{2\alpha'} \langle X^2(\sigma) \rangle \\
= -\frac{1}{2\alpha'} \left\{ -\alpha' \ln(2q) - \frac{\alpha'}{u} - 2\alpha' \left[ \Psi \left( \frac{u}{2} \right) - \Psi(u) \right] \right\} \\
= \frac{1}{2} \left\{ \ln(2q) + \frac{d}{du} \ln u + 2 \left[ \frac{d}{du} \ln \Gamma \left( \frac{u}{2} \right) - \frac{d}{du} \ln \Gamma(u) \right] \right\} \\
= \frac{d}{du} \left( \ln \frac{\sqrt{2q} u^{u/2} \Gamma \left( \frac{u}{2} \right)^2}{\Gamma(u)} + \text{const.} \right). \] (3.24)

We then obtain
\[ Z(u) = \frac{\sqrt{2q} u^{u/2} \Gamma \left( \frac{u}{2} \right)^2}{\Gamma(u)}, \] (3.25)
up to the overall normalization factor. In general, the partition function for 26-dimensional target space with the interaction (3.2) on the crosscap can be written as
\[ Z(a, u) \equiv e^{-a} Z(u) \equiv e^{-a} \prod_{\mu=1}^{26} Z(u_\mu) \equiv e^{-a} \prod_{\mu=1}^{26} \left( \frac{\sqrt{2q} u^{u_\mu} \Gamma \left( \frac{u_\mu}{2} \right)^2}{\Gamma(\mu)} \right), \] (3.26)
up to the overall normalization factor. We note that the partition function (3.26) on the $RP^2$ has an identical representation with the partition function on the supersymmetric disc worldsheet which have been considered in the analysis of open-string tachyon condensation in $DD$ systems [13, 10, 17]. We present some comments on the relationship between the $RP^2$ worldsheet and the supersymmetric disc worldsheet in Appendix.
IV. Derivation of the descent relation among O-plane tensions

In this section, we clarify the physical meaning of the partition function calculated in the previous section. We show that the condensation of the quadratic dilaton field reproduces the correct descent relation among the O-plane tensions.

A. Sigma model approach and $RP^2$ worldsheet

Let us consider our work from the viewpoint of the sigma model approach. The basic idea of the sigma model approach is that the spacetime action for string fields is essentially the renormalized partition function of the worldsheet with corresponding background string fields. In this sense, the spacetime action $S$ for string field $\lambda_i$ may be given as

$$S(\lambda_i) \cong \sum_{\chi} \int_{\Sigma_\chi} [dg_{ab}][dX^\mu]e^{-I_\chi(g_{ab},X^\mu;\lambda_i)} = Z_{sphere}(\lambda_i) + \{Z_{disc}(\lambda_i) + Z_{RP^2}(\lambda_i)\} + \cdots,$$

where $I_\chi$ is the action on the worldsheet $\Sigma_\chi$ of the Euler number $\chi$. Leading term $Z_{sphere}(\lambda_i)$ is the renormalized partition function on the sphere. This term is of order $g_o^{-2}$ where $g_o$ is the open-string coupling constant. Renormalized partition functions $Z_{disc}(\lambda_i)$ on the disc and $Z_{RP^2}(\lambda_i)$ on the $RP^2$ are the loop correction terms of order $g_o^{-1}$. In principle, $Z_{disc}(\lambda_i)$ is proportional to the tension of the corresponding D-brane and $Z_{RP^2}(\lambda_i)$ is proportional to the tension of the corresponding O-plane.

It is known, however, that the right-hand side of (4.1) does not give the correct spacetime action; we need modification of $Z_{sphere}(\lambda_i)$ and $Z_{disc}(\lambda_i)$ in order to obtain the correct off-shell spacetime action. These modifications are closely related to the infinite Möbius volume of the worldsheets; we need to subtract the divergence from the Möbius infinity \[19, 20, 21\]. (See also \[22, 15\].)

The situation is different for $Z_{RP^2}(\lambda_i)$ (and for the partition functions of the worldsheets of $\chi \leq 0$). We should note that the Möbius group of $RP^2$ is $SO(3)$ whose volume is finite, and we have no Möbius infinity from the $RP^2$ worldsheet. Therefore it is natural to assume that partition function $Z_{RP^2}(\lambda_i)$ itself is the exact loop correction term from the $RP^2$ graph. We have calculated the partition function \(3.26\) on the $RP^2$ worldsheet in the presence of quadratic background dilaton field $\Phi = a + (2\alpha')^{-1} \sum_\mu u_\mu X_\mu^2$ on the crosscap. We have fixed

\[3\]we need not modify $Z_{disc}(\lambda_i)$ in superstring theory since the volume of the super-Möbius group of the super-disc is finite \[19\].
the worldsheet metric so that the “bulk” part of the fundamental region of the $RP^2$ becomes flat and the contribution of the dilatons concentrates on the crosscap. Therefore we have to calculate the contribution of the ghost field and the anti-ghost field on the $RP^2$ worldsheet in order to obtain the correct overall normalization of (3.26). We write this overall factor $A$ and then we have the following relationship:

$$Z_{RP^2}(\Phi) = A e^{-a \frac{26}{2}} \prod_{\mu=1}^{26} \left( \frac{\sqrt{2q^i u^\mu \Gamma(u^\mu/2)}}{\Gamma(u^\mu)} \right)$$

$$\equiv Z_{RP^2}(a, u).$$

(4.2)

The sign of $A$ should be minus. Note that factor $e^{-a}$ is equal to $g_{\phi}^{-1}$ when the dilaton is constant.

B. Asymptotic behaviour of $Z(u)$

Let us rewrite $Z(u)$ given in (3.25) as

$$Z(u) \equiv \frac{\sqrt{2q} \sqrt{u \Gamma(u/2)}}{\Gamma(u)} = \frac{4}{\sqrt{u}} \left( \frac{q}{2} \right)^{\nu} F\left(\frac{u}{2}\right),$$

(4.3)

where we have defined function $F(x)$ as

$$F(x) \equiv \frac{4x^2 \Gamma(x)^2}{2\Gamma(2x)}. (4.4)$$

$F(x)$ behaves as follows:

$$F(x) \sim 1 + (2 \ln 2)x + O(x^2) \quad (x \to 0),$$

$$F(x) \sim \sqrt{\pi x} + O(x^{-\frac{1}{2}}) \quad (x \to \infty).$$

(4.5) (4.6)

$Z(u)$ around $u = 0$ is then

$$Z(u) \sim 4 \left( \frac{1}{\sqrt{u}} \right) + O(\sqrt{u}).$$

(4.7)

Thus, $Z(u)$ diverges when $u$ approaches 0. This is an IR divergence which corresponds to the volume of the spacetime.

On the other hand $Z(u)$ around $u = \infty$ is

$$Z(u) \sim \left( \frac{q}{2} \right)^{\nu} \left\{ 4 \sqrt{\frac{\pi}{2}} + O \left( \frac{1}{\sqrt{u}} \right) \right\}.$$
We can obtain a finite and non-zero value of $Z(u)$ in the limit $u \to \infty$ if and only if $q = 2$. Therefore we assign the value 2 to $q$. In other words, we have chosen the renormalization scheme in (3.20) so that we can obtain a finite and non-zero value of $Z(u)$ in the limit $u \to \infty$. $Z(u)$ is then

$$Z(u) = \sqrt[4]{\frac{u \Gamma\left(\frac{u}{2}\right)}{\Gamma(u)}} = \frac{4}{\sqrt{u}} F\left(\frac{u}{2}\right),$$

(4.9)

up to the overall normalization factor and

$$Z_{RP^2}(a,u) = A e^{-a} \prod_{\mu=1}^{26} \left(\frac{4}{u_{\mu}} F\left(\frac{u_{\mu}}{2}\right)\right).$$

(4.10)

C. Ratio of the O-plane tensions

Let us define quantity $S_p$ as follows:

$$Z_{RP^2}(a,u) \to S_p,$$

(4.11)

where the limit is taken as

$$u^1, \ldots, u^{p+1} \to 0,$$

$$u^{p+2}, \ldots, u^{26} \to \infty.$$

(4.12)

We do not touch the parameter $a$ in this subsection. According to the argument in the previous section, $S_p$ is equal to $V_p \times T_p$ where $V_p$ and $T_p$ are the volume and tension of an Op-plane. Here, the dimension of the O-plane is $p + 1$ and is defined as the number of parameters $u^\mu$ which are taken to zero. We can thus write

$$S_{25} = Z_{RP^2}(u_{25} \to 0, u_i \to 0) = \int dx^{25} V_{24} T_{25},$$

$$S_{24} = Z_{RP^2}(u_{25} \to \infty, u_i \to 0) = V_{24} T_{24},$$

(4.13)

where $i \neq 25$. Then

$$\frac{S_{25}}{S_{24}} = \frac{Z(u_{25} \to 0)}{Z(u_{25} \to \infty)} = \frac{\int dx^{25} T_{25}}{T_{24}},$$

(4.14)

where $Z(u)$ is given in (4.9). We can rewrite $Z(u_{25})$ as

$$Z(u_{25}) = \frac{4}{\sqrt{u_{25}}} F\left(\frac{u_{25}}{2}\right)$$
\[ \int dx^{25} e^{-\frac{1}{2\alpha'} u^{25}(x^{25})^2} \frac{1}{\sqrt{2\alpha' \pi}} 4F\left(\frac{u^{25}}{2}\right), \quad (4.15) \]

where \( x^{25} \) is the zero mode of \( X^{25} \). In the second line, we have explicitly rewritten the integral of the zero-mode part of \( Z(u^{25}) \) \[15, 23\]. (This corresponds to \( \sim \frac{1}{u^{25}} \)). We can therefore obtain

\[ \lim_{u^{25} \to 0} Z(u^{25}) = \lim_{u^{25} \to 0} \int dx^{25} e^{-\frac{1}{2\alpha'} u^{25}(x^{25})^2} \frac{1}{\sqrt{2\alpha' \pi}} 4F\left(\frac{u^{25}}{2}\right) = \int dx^{25} \frac{4}{\sqrt{2\alpha' \pi}} \cdot 1. \quad (4.16) \]

We can also obtain

\[ \lim_{u^{25} \to \infty} Z(u^{25}) = \lim_{u^{25} \to \infty} \frac{4}{\sqrt{u^{25}}} F\left(\frac{u^{25}}{2}\right) = \lim_{u^{25} \to \infty} \frac{4}{\sqrt{u^{25}}} \left(\sqrt{\frac{u^{25}}{2}} + O\left(\frac{1}{\sqrt{u^{25}}}\right)\right) \]

\[ = 4 \sqrt{\frac{\pi}{2}}. \quad (4.17) \]

Therefore

\[ \frac{T_{24}}{T_{25}} = \frac{\sqrt{2\alpha' \pi}}{4} \sqrt{\frac{\pi}{2}} = \pi \sqrt{\alpha'}. \quad (4.18) \]

This is precisely the ratio of the tension of an O24-plane and that of an O25-plane. In general, we can show in a similar manner that

\[ \frac{T_p}{T_q} = \left(\pi \sqrt{\alpha'}\right)^{q-p}. \quad (4.19) \]

V. Conclusion

We have considered the relationship between the configuration of the background dilaton field and the dimensions of the O-planes. We showed that the contribution of the dilatons on the \( RP^2 \) worldsheet localizes on the crosscap if we choose the gauge in which the worldsheet curvature in the “bulk” vanishes. This feature enables us to treat dilatons quite easily. We have proposed the off-shell crosscap state which naturally interpolates between the usual crosscap states and their T-duals. The behaviour of the off-shell crosscap state signifies that the non-trivial dilaton condensation alters the dimensions of O-planes. We obtained the correlation functions and partition function on the \( RP^2 \) worldsheet in the presence of the quadratic dilaton field on the crosscap. We showed that the non-trivial dilaton condensation reproduces the correct descent relation among O-plane tensions, by taking the on-shell limits of the partition function of the \( RP^2 \). We found the correspondence between the \( RP^2 \) worldsheet and the supersymmetric disc which is presented in Appendix.
We would like to make some comments. We have studied the effects of the non-trivial dilaton condensation on O-planes. In order to describe the full dynamics of dilatons, we would need open-closed string field theory. Non-perturbative analysis would be necessary too. However the present work can be a step to understand the relationship between the condensation of string fields and the transmutations of O-planes. An extension of the present work into the supersymmetric case is interesting, and studying the relationship between our work and the properties of O-planes described by F-theory is one further direction to pursue. Studying the relationship between the configuration of the dilatons and the O-planes in Type I’ theory is also tempting. It has been shown that non-trivial configurations of dilatons constrain the spacetime positions of D-branes. Effects of the non-trivial dilaton condensation on D-branes are also interesting subjects to study. Recent studies have shown that the condensation of the closed-string tachyons in the twisted sector can alter the topology of the orbifolded spacetime. We expect that the present work can also be a step towards understanding the relationship between the topology of the spacetime and the configuration of the string fields, since the dimensions of O-planes are closely related to the topology of the orientifolded spacetime.

Acknowledgment

We would like to thank H. Kawai, T. Kugo, K. Murakami, T. Nakatsu, A. Tsuchiya and T. Yokono for fruitful discussions. We are grateful to Santo seminar for providing an opportunity for collaboration. S.N. wishes to thank M. Laidlaw, M. Rozali, and G. W. Semenoff for discussions and their hospitality during his stay at University of British Columbia.

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4 An unoriented open-closed string field theory has been proposed in Ref. [24].
Appendix A

Correspondence between the $RP^2$ worldsheet and the supersymmetric disc worldsheet

We comment in this appendix on the relationship between the $RP^2$ worldsheet we have considered and the supersymmetric disc (super-disc) worldsheet which appears in the analysis of $D\bar{D}$ systems. Let us consider a unit disc $\mathcal{M} \{z = re^{i\sigma} | 0 \leq r \leq 1, 0 \leq \sigma < 2\pi\}$ with the following worldsheet action:

$$I_{\text{super-disc}} = \frac{1}{4\pi} \int_{\mathcal{M}} d^2 z \left\{ \frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \bar{\psi}^\mu \partial \bar{\psi}_\mu \right\}$$

$$+ \frac{1}{4\pi} \int_{\partial \mathcal{M}} d\sigma y^\mu \left\{ \frac{2}{\alpha'} X^2_\mu(\sigma) + (\psi^\mu \partial_\mu X^\rho) \right\} \left( \psi^\nu \partial_\nu X_\rho(\sigma) \right).$$

(A.1)

This action has been considered in the context of open-string tachyon condensation in $D\bar{D}$ systems [15, 16, 17]. The partition function of the super-disc is obtained in Ref. [15] as

$$Z(y) \propto \prod_{\mu=1}^{26} \sqrt{\frac{1}{y^\mu}} F(y^\mu),$$

(A.2)

where function $F(x)$ has been defined in (4.4).

Therefore, the partition function (4.10) of the $RP^2$ worldsheet has an identical form to the partition function (A.2) of the super-disc. Comparing (4.10) and (A.2), we find the correspondence

$$\frac{u^\mu}{2} \leftrightarrow y^\mu.$$  

(A.3)

At the level of integrated operators, we find the correspondence

$$O_{RP^2} \leftrightarrow O_{\text{disc}}.$$  

(A.4)

where

$$O_{RP^2} \equiv \int_c d\sigma \frac{2}{\alpha'} X^2_\mu(\sigma)$$

(A.5)

is an operator on the $RP^2$ and

$$O_{\text{disc}}(\sigma) \equiv \int_{\partial \mathcal{M}} d\sigma \left\{ \frac{2}{\alpha'} X^2_\mu(\sigma) + (\psi^\mu \partial_\mu X^\rho) \right\} \left( \psi^\nu \partial_\nu X_\rho(\sigma) \right).$$

(A.6)
is an operator on the super-disc. We should also note that the descent relation among D-brane tensions in the $D\bar{D}$ system can be obtained in the same manner as that we have shown in Sec. 4. The correspondence (A.4) can be explained from the calculation of $\langle X_2^2 \rangle$ for the $RP^2$, in which we have an extra minus sign in the contributions from odd modes. These contributions correspond to those of the fermions on the super-disc, while the even modes behave like the bosonic part on the super-disc.

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5See the third term of the first line on the right-hand side of (5.21).
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