Shell Models of Superfluid Turbulence

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Abstract. Superfluid helium consists of two inter-penetrating fluids, a viscous normal fluid and an inviscid superfluid, coupled by a mutual friction. We develop a two-fluid shell model to study superfluid turbulence and investigate the energy spectra and the balance of fluxes between the two fluids in a steady state. At sufficiently low temperatures a 'bottle-neck' develops at high wavenumbers suggesting the need for a further dissipative effect, such as the Kelvin wave cascade.

1. Two-fluid model of liquid helium

Superfluidity is a property of Bose-Einstein condensed systems, such as \(^4\)He, \(^3\)He and ultra-cold atomic gases. Superfluids are notable for their singular vorticity, which is confined to thin filaments.

In this work we are concerned about \(^4\)He. According to Landau’s two-fluid theory, below the critical temperature \(T_c = 2.17K\), liquid helium consists of two inter-penetrating fluid components, a viscous normal fluid and an inviscid superfluid; coupled by a mutual friction. The normal fluid is governed by a Navier-Stokes-like equation and the superfluid by an Euler-like equation. In the presence of a tangle of superfluid vortices the two-fluid equations are given by

\[
\frac{\partial}{\partial t} + \mathbf{v}^N \cdot \nabla \mathbf{v}^N = -\frac{1}{\rho} \nabla p - \frac{\rho^S}{\rho^N} \nabla ST + \frac{\rho^S}{\rho} \mathbf{F}^{NS} + \nu^N \nabla^2 \mathbf{v}^N, \tag{1}
\]

\[
\frac{\partial}{\partial t} + \mathbf{v}^S \cdot \nabla \mathbf{v}^S = -\frac{1}{\rho} \nabla p + \nabla ST - \frac{\rho^N}{\rho} \mathbf{F}^{NS}, \tag{2}
\]

in which superscripts \(N\) and \(S\) denote the normal fluid and superfluid respectively, \(\rho = \rho^N + \rho^S\), \(S\) and \(T\) are total density, entropy and temperature respectively, and \(\nu^N\) is the normal fluid viscosity. Note that the superfluid lacks viscous dissipation. The mutual friction force is \(\mathbf{F}^{NS} = \alpha \kappa L (\mathbf{v}^S - \mathbf{v}^N)\), where \(\alpha\) is the friction coefficient, \(\kappa\) is the quantum of circulation, \(L\) is the vortex line density and (\(\mathbf{v}^S - \mathbf{v}^N\)) is the slip velocity.

There has been limited experimental (1) and extensive numerical (2; 3; 4; 5; 6; 7; 8) evidence, accompanied by some theoretical discussion (9; 10), for a Kolmogorov \(k^{-5/3}\) scaling in the inertial range of the superfluid energy cascade.
2. Two-fluid shell model

Shell models (11) provide a computationally inexpensive method of reproducing the main features of the Navier-Stokes equations: deterministic, quadratically nonlinear, energy conserving and scale invariant. We propose the following shell model for the two-fluid hydrodynamics

\[
\frac{d}{dt} u_N^k = \frac{\nu}{\ell^2} k^2 u_N^k + G_N[u^N] + f\delta_{n,m} + F_{nNS},
\]

\[
\frac{d}{dt} u_S^k = \frac{\nu}{\ell^2} k^2 u_S^k + G_N[u^N] + f\delta_{n,m} - F_{nNS},
\]

where the nonlinear interaction term is

\[
G_N[u^j] = i[a_n u_{n+1}^j u_{n+2}^j + b_n u_{n-1}^j u_{n+1}^j + c_n u_{n-2}^j u_{n-1}^j u_{n-2}^j]^* \quad (j = N, S),
\]

subscript \( n \) (\( 0 \leq n \leq N \)) is the shell index, \( k_n = k_0\lambda^n \) are logarithmically equispaced wavenumbers, \( u_n^j \) are complex velocities, \( k_n^2 \) is the Laplacian in \( k \)-space and \( f \) is the forcing applied at shell \( n = m \). The shell coefficients \( a, b, c \) are chosen so that energy is conserved. The vortex line density, \( L \), is calculated self-consistently from the superfluid enstrophy. We obtain, by dimensional arguments, the average intervortex spacing, \( \ell = 1/\sqrt{L} \), and corresponding wavenumber, \( k_\ell = 1/\ell \). Note that the continuum model is valid only for \( k < k_\ell \).

In line with the classical shell model literature (11) we choose \( N = 18 \), \( a = 1 \), \( b = c = -1/2 \), \( k_0 = 2^{-4} \), \( \lambda = 2 \) and forcing at shell \( m = 4 \).

3. Results

We use the initial condition \( u_n = k_n^2 e^{-k_n^2} \) to build up to a fully-developed spectrum and perform an ensemble average over 500 realisations allowing each realisation to evolve for at least 5 large-eddy turnover times. We repeat this procedure at 3 sample temperatures: 'high' (2.157K), 'medium' (1.96K) and 'low' (1.44K), with density ratios, \( \rho^N/\rho^S \), 10, 1 and 0.1 respectively [cf. (6)] and then reduce temperature to 1.3K (\( \rho^N/\rho^S = 0.05 \)), the lowest temperature at which data is available.

We find that both normal fluid and superfluid velocity obey a Kolmogorov \( k^{-5/3} \) scaling law in the inertial range (see figs. 1 & 2). This result holds true at all temperatures for which data is available. Our results thus confirm the experimental results of Maurer et. al (1) for the total energy spectrum and the numerical results of Roche et. al (6) for the normal fluid and superfluid energy spectra independently.

4. Balance of fluxes

We derive the scale-by-scale energy budgets per unit mass for the normal fluid and superfluids from Eqs. 3 & 4. They are given by

\[
\frac{\partial E^N}{\partial t}(k,t) = T^N(k,t) + D^N(k,t) + M^N(k,t) + \epsilon_{inj}^N\delta_{n,4},
\]

\[
\frac{\partial E^S}{\partial t}(k,t) = T^S(k,t) + M^S(k,t) + \epsilon_{inj}^S\delta_{n,4},
\]

where \( T^N \) and \( T^S \) are the energy transfer rates arising from the triadic interactions between Fourier modes within each fluid, \( D^N \) is the viscous dissipation in the normal fluid, \( M^N \) and \( M^S \)
Figure 1. Ensemble average over 500 realisations of energy spectra for normal fluid at three different temperatures: high (red squares), medium (grey circles) and low (blue triangles). Also shown are $k^{-5/3}$ spectrum (solid black line) and $k_\ell$ for the three temperatures: high, medium and low (left to right black vertical dashed lines).

Figure 2. Ensemble average over 500 realisations of energy spectra for superfluid at three different temperatures: high (red squares), medium (grey circles) and low (blue triangles). Also shown are $k^{-5/3}$ spectrum (solid black line) and $k_\ell$ for the three temperatures: high, medium and low (left to right black vertical dashed lines).
Figure 3. Balance of energy fluxes at high temperature. Flux for the normal fluid (hollow shapes) and superfluid (solid shapes) due to inertial (red squares), mutual friction (grey circles) and viscous terms (blue triangles), and for the algebraic sum (black diamonds).

are the exchange of kinetic energy between the two fluids due to mutual friction and $\epsilon_{inj}^N$ and $\epsilon_{inj}^S$ are the influx of energy due to the forcing terms.

We observe (see fig.3) a balance of energy fluxes in the normal fluid, in which the (triadic) inertial term, $T^N$, is balanced by the viscous dissipation, $D^N$, and the superfluid, in which the (triadic) inertial term, $T^S$, is balanced by the mutual friction, $M^S$, [cf. (6)]. However in the $T = 0$ limit, we observe a 'bottle-neck' of energy at high $k$, indicating that the fluxes do not balance and we cannot attain a steady state. There exists therefore a cut-off point at which the dissipative effect of the Kelvin wave cascade becomes indispensable.

5. Relaxation of turbulence

We also studied decaying turbulence by letting $f = 0$ and starting from saturated spectra as an initial condition. We measured the decay over time of the energy spectra and observed that the spectra maintain their shape during decay (see figs. 4 & 5): $k^{-5/3}$ at lower $k$ and dropping off with dissipation at higher $k$. Furthermore we found that the vortex line density, $L(t)$, and the total turbulent kinetic energy, $E(t)$, where

$$E^j(t) = \sum_n \frac{1}{2} |u_n^j(t)|^2, \ (j = N, S),$$

decay as $E(t) \propto t^{-2}$ and $L(t) \propto t^{-3/2}$, which is in agreement with experiments (12; 13; 14).
**Figure 4.** Decay of energy spectra for normal fluid over 10 realisations. Top to bottom: After 500s, 1000s, 2500s and 5000s. Also shown is the $k^{-5/3}$ spectrum (solid black line).

**Figure 5.** Decay of energy spectra for superfluid over 10 realisations. Top to bottom: After 500s, 1000s, 2500s and 5000s. Also shown is the $k^{-5/3}$ spectrum (solid black line).
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