THE HERMITIAN CONNECTION AND THE JACOBI FIELDS
OF A COMPLEX FINSLER MANIFOLD

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Abstract. It is proved that all invariant functions of a complex Finsler manifold can be totally recovered from the torsion and curvature of the connection introduced by Kobayashi for holomorphic vector bundles with complex Finsler structures. Equations of the geodesics and Jacobi fields of a generic complex Finsler manifold, expressed by means Kobayashi's connection, are also derived.

1. Introduction.

A complex Finsler manifold \((M, J, F)\) is a complex manifold \((M, J)\) endowed with a complex Finsler metric \(F\), which is a continuous function \(F: TM \rightarrow \mathbb{R}^+\) that is smooth on \(TM \setminus \{\text{zero section}\}\) and verifies the following two properties:

a) \(F(u) > 0\) for any \(u > 0\);

b) \(F(\lambda u) = |\lambda| F(u)\) for any \(u \neq 0\) and \(\lambda \in \mathbb{C}^*\).

In this paper, we will always assume that \(F\) is strictly pseudoconvex, i.e. that any Finsler pseudo-sphere at a point \(x\)

\[ S_x = \{ v \in T_x M : F(v) = 1 \} \]

is strictly pseudoconvex as real hypersurface of \(T_x M \cong \mathbb{C}^n\).

The simplest examples of such Finsler manifolds are the Hermitian manifolds. In fact, if \(g\) is an Hermitian metric on a complex manifold \((M, J)\), the norm function

\[ F_g : TM \rightarrow \mathbb{R}^+ , \quad F(v) = \sqrt{g(v,v)} \quad (1.1) \]

is a strictly pseudoconvex Finsler metric. In what follows, whenever \(F\) is as in (1.1), we will say that \(F\) is associated with the Hermitian metric \(g\).

Other important examples of complex Finsler manifolds are the bounded convex domains in \(\mathbb{C}^n\) with smooth boundary, endowed with their infinitesimal Kobayashi metric (see [Le], [Le1], [Pa], [Fa], [AP], [Sp]). Indeed, the Kobayashi infinitesimal metric of any hyperbolic complex manifold is a "non-smooth" complex Finsler metric.

In ([Sp]), we introduced the concept of adapted linear frames of a complex Finsler manifold \((M, J, F)\). We studied the properties of the bundle \(\pi : U_F(M) \rightarrow M\) of

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all adapted frames and we constructed an absolute parallelism on $U_F(M)$, whose structure functions constitute a complete set of generators for the (local) invariants of $(M, J, F)$.

Such absolute parallelism consists of a finite set of global vector fields \( \{X_1, \ldots, X_{2n}, Y_1, \ldots, Y_p\} \) on $U_F(M)$, which are preserved by any (local) diffeomorphism which is lift of a (local) biholomorphic isometry of $(M, J, F)$. It contains a subset \( \{Y_1, \ldots, Y_p\} \) of vector fields, which span the vertical distribution, and another complementary subset \( \{X_1, \ldots, X_{2n}\} \), whose vector fields span the distribution $H \subset T_{U_F(M)}$ of real subspaces underlying the holomorphic distribution $T^{10}U_F(M) \subset T^C U_F(M)$. By holomorphic distribution $T^{10}U_F(M)$ we use the standard meaning of the set of subspaces $T^{10}_u U_F(M) = T^{10}_u U_F(M) \cap T^{10}_u L^C(M)$, where $T^{10}_u L^C(M)$ is the subspace at $u \in U_F(M)$ generated by the holomorphic vector fields of the bundle of all complex linear frames $L^C(M)$.

The distribution $H$ is complementary to the vertical distribution. This allows to interpret any curve $u_t$ in $U_F(M)$, which is tangent to $H_{\gamma}$ at any $t$, as a 1-parameter family of adapted frames, which represent a ‘parallel transport’ along the curve $\gamma = \pi \circ u : [a, b] \subset \mathbb{R} \rightarrow TM$. Using this ‘parallel transport’, one can define a covariant derivation of functions with values in $TM$ in the directions of vectors tangent to $TM$. Note that, in case $F$ is associated with an Hermitian metric $g$, such covariant derivation reduces to the usual Hermitian covariant derivation of vector fields on $M$ in the direction of vectors tangent to $M$.

At the best of our knowledge, in the literature there exist other three different definitions of Finslerian covariant derivation: namely, the one determined by the absolute parallelism of J.J. Faran in [Fa], the covariant derivation by M. Abate and G. Patrizio given in [AP] and the one defined by S. Kobayashi in [Ko]. Those definitions have the advantage to be defined using the expression of the Finsler metric in complex coordinates and are therefore suitable for explicit computations.

On the other hand, our setting is established in a ‘totally coordinate-free language’ and all objects we deal with (such as torsion and curvature) immediately reduce to the corresponding objects of Hermitian geometry, whenever the Finsler metric is associated with an Hermitian metric.

In this paper, we show that our definition of Finslerian covariant derivation is strictly related with Kobayashi’s definition and we derive the formulae which express the torsion and curvature of Kobayashi’s connection in terms of the torsion and the curvature 2-forms of the absolute parallelism \( \{X_1, \ldots, Y_p\} \) of $U_F(M)$ (see §3 and §4). An important by-product of these formulae is a practical way to evaluate the structure functions of a complex Finsler manifold using coordinates: all computations reduce to use the Kobayashi’s expressions for the Finsler connection and Finsler curvature given in [Ko2] (see also [Ko3]).

We conclude with sections §5 and §6, in which we write the equations of geodesics and of the Jacobi vector fields in terms of the Kobayashi’s Finslerian connection and its torsion and curvature. The equations for geodesics of a complex Finsler space were first derived by H. Rund in [Ru]; alternative presentations are given in [Pa], [Fa], [AP] and [Sp]. The equations for the Jacobi fields were first determined by M. Abate and G. Patrizio in [AP].

The expressions given here have the peculiarity that, with no further arguing or manipulation, they immediately reduce to the corresponding usual formulae of Hermitian geometry in case $F$ is associated with an Hermitian metric.
We believe that a careful investigation of the equations of Jacobi fields of the smoothly bounded convex domains in \( \mathbb{C}^n \) can bring to isolate some crucial properties of the Kobayashi infinitesimal metric, which characterize those domains up to biholomorphisms (for results in this direction, see e.g. [AP], [BD]). A more detailed discussion of this topic will be the content of a forthcoming paper.

2. Preliminaries and Notation.

In the whole paper, we will use greek letters \( \alpha, \beta, \) etc. for indices related to holomorphic vectors, barred greek letters \( \bar{\alpha}, \bar{\beta}, \) etc. for indices related to the conjugated vectors and latin indices \( i, j, k, \) etc. to denote real vectors.

\( J_o \) is the complex structure of \( \mathbb{C}^n \) and \( \langle, \rangle \) is the standard Hermitian product of \( V = \mathbb{C}^n. \)

The elements \( \{ \epsilon_0, \epsilon_1, \ldots, \epsilon_{2n-1} \} \subset \mathbb{C}^n \) constitute the standard real basis of \( V = \mathbb{R}^{2n} = \mathbb{C}^n \) and they are ordered so that \( J_o(\epsilon_{2i}) = \epsilon_{2i+1} \) for any \( i = 0, \ldots, n. \) We set \( \varepsilon_\alpha = \frac{1}{2}(\epsilon_{2\alpha} - \sqrt{-1}\epsilon_{2\alpha+1}), \) \( \alpha = 0, \ldots, n - 1, \) and \( \bar{\varepsilon}_\alpha = \frac{\varepsilon_\alpha}{\sqrt{\varepsilon_\alpha \bar{\varepsilon}_\alpha}}. \) We also use the notation \( \{ \epsilon^i \}, \{ \varepsilon^\alpha \} \) and \( \{ \varepsilon^\alpha \} \) for the dual bases of \( \{ \epsilon_i \}, \{ \epsilon_\alpha \} \) and \( \{ \varepsilon_\alpha \} \), respectively.

We denote by \( M \) a complex manifold with complex structure \( J. \) We also use the notation \( \mathbb{P}T M = T^0 M / \mathbb{C}^*, \) where \( T^0 M = TM \setminus \{ \text{zero section} \}. \)

For any point \( x \in M \) and any \( v \in T_x M, \) the tangent space \( T_v(T_x M) \) is naturally identified with \( T_x M \) and we will use the symbol \( J \) also to denote the complex structure on the tangent spaces \( T_x(T_x M), \) given by the identification with \( T_x M. \)

For any \( v \in T_x M, \) we denote by \( v^{10} \) and \( v^{01} \) the holomorphic and anti-holomorphic parts w.r.t. \( J, \) that is:

\[
v^{10} = \frac{1}{2}(v - \sqrt{-1}Jv), \quad v^{01} = \bar{v}^{10} = \frac{1}{2}(v - \sqrt{-1}Jv).
\]

For any \( x \in M, \) a linear frame is an \( \mathbb{R} \)-linear isomorphism \( u: \mathbb{R}^{2n} \to T_x M. \) A linear frame is called complex linear frame if it is a \( \mathbb{C} \)-linear isomorphism \( u: \mathbb{C}^n = \mathbb{R}^{2n} \to T_x M. \) We always identify a linear frame \( u \) with the corresponding basis \( \{ f_i \} \) in \( T_x M \) defined by

\[
f_i = u(\epsilon_i) \in T_x M.
\]

If a frame \( u \) is complex, we denote by \( u^{10} \) the corresponding holomorphic basis, that is

\[
u^{10} = \{ \epsilon_\alpha = u(\varepsilon_\alpha) = \frac{1}{2}(f_{2\alpha} - \sqrt{-1}f_{2\alpha+1}) \}.
\]

If \( (M, J, F) \) is a complex Finsler manifold, a complex linear frame \( u = \{ f_i \} \) is called adapted if

a) \( f_0 \in S_x \) and \( f_1 = J f_0, \) where \( S_x \) denotes the Finsler pseudo-sphere \( S_x = \{ v \in T_x M : F(v) = 1 \}; \)

b) the vectors \( f_2, \ldots, f_{2n-1} \) span the maximal \( J \)-invariant subspace \( D_{f_0} \) of \( T_{f_0} S_x \subset T_{f_0}(T_x M) \cong T_x M; \)

c) the holomorphic vectors \( \varepsilon_1, \ldots, \varepsilon_{n-1} \) constitute a unitary basis for \( D_{f_0} \) with respect to the Levi form \( L_x \) of \( S_x, \) corresponding to the defining function \( \rho_F = F^2 - 1. \)
The unitary frame bundle of \((M, J, F)\) is the subbundle \(U_F(M) \subset L^C(M)\) given by all the adapted complex linear frames.

It follows from definitions that any fiber of \(U_F(M)\) is invariant under the linear action of \(U_n \times T^1 \subset GL_n(\mathbb{C})\) on \(T_u M\). Moreover, the orbit space of this action can be identified with \(U_F(M)/U_n \times T^1 = \mathbb{P}TM\).

We will use the symbols \(\pi, \hat{\pi}\) and \(\pi'\) to denote the following natural projections

\[
\hat{\pi} : U_F(M) \to U_F(M)/U_{n-1} \times T^1 = \mathbb{P}TM \subset TM , \quad \pi' : \mathbb{P}TM \to M ,
\]

\[
\pi = \pi' \circ \hat{\pi} : U_F(M) \to M .
\]

The non-linear Hermitian connection of \(U_F(M)\) is the unique distribution \(\mathcal{H}\) on \(U_F(M)\), which is complementary to the vertical distribution and which is invariant under the complex structure \(J\) of the complex linear frame bundle \(L^C(M)\). The distribution \(\mathcal{H}\) is equal to the real distribution underlying the holomorphic distribution of the real submanifold \(U_F(M) \subset L^C(M)\) (see proof of Th. 3.9 in [Sp]).

The non-linear Hermitian connection \(\hat{\mathcal{H}}\) is uniquely determined by a connection form \(\omega\), that is by a \(gl_n(\mathbb{C})\)-valued 1-form on \(U_F(M)\) which verifies the following conditions:

a) for any \(u \in U_F(M)\), a vector \(\mathcal{X} \in T_u U_F(M)\) is so that \(\omega_u(\mathcal{X}) = 0\) if and only if \(\mathcal{X} \in \mathcal{H}_u\);

b) if a vector \(\mathcal{X} \in T_u U_F(M)\) is vertical (i.e. \(\pi_u(\mathcal{X}) = 0\)), then \(\omega_u(\mathcal{X}) = E_X\), where \(E_X\) is the unique element in \(gl_n(\mathbb{C})\) which generates an infinitesimal transformation on \(L^C(M)\) assuming the value \(\mathcal{X}\) at the point \(u\).

Let \(\omega_\beta^\alpha, \omega_\beta^\beta, \theta^\alpha\) and \(\theta^\beta\) be the components of the connection form and of the tautological 1-form in the basis \(\{E^\alpha_\beta = \varepsilon_\beta \otimes \varepsilon^\alpha\}\) and \(\{\varepsilon_\alpha\}\) of \(gl_n(\mathbb{C})\) and of \(\mathbb{C}^n\), respectively. In other words let

\[
\theta = \sum_\alpha \varepsilon_\alpha \theta^\alpha + \sum_\beta \varepsilon^\beta \theta^\beta ,
\]

\[
\omega = \sum_{\alpha, \beta} E^\alpha_\beta \omega^\beta_\alpha + E^\beta_\beta \omega^\beta_\beta .
\]

These 1-forms are not linearly independent but, at all points, they generate the whole cotangent bundle \(T^*U_F(M)\). The linear relations between them and the expressions for their exterior differentials are called structure equations of the complex Finsler manifold \((M, J, F)\). We will shortly list all such structure equations.

For this purpose we first have to introduce some special \(\mathbb{C}\)-valued functions on \(U_F(M)\).

For any vector \(X \in T_u M\), denote by \(V^X\) the vector field in \(T(T_u M)\) which assumes the value \(X\) at all points \(U \in T_u M\). For any choice of vectors \(X, Y, Z, W, U \in T_u M\), we define

\[
h_U(X, Y) = V^X \left[V^Y \left(F^2\right)\right]_U ; \quad H_U(X, Y, Z) = V^X \left[V^Y \left[V^Z \left(F^2\right)\right]\right]_U ; \quad (2.1)
\]

\[
H_U(X, Y, Z, W) = V^X \left[V^Y \left[V^Z \left[\hat{V}^W \left(F^2\right)\right]\right]\right]_U . \quad (2.2)
\]
For any adapted frame $u = \{f_i\}$ and corresponding holomorphic frame $u^0 = \{e_\alpha\}$, we set
$$h_{\alpha\beta}(u) = h_{f_\alpha}(e_\alpha, e_\beta), \quad H_{\alpha\beta\gamma}(u) = H_{f_\alpha}(e_\alpha, e_\beta, e_\gamma), \quad H_{\alpha\beta\gamma\delta}(u) = H_{f_\alpha}(e_\alpha, e_\beta, e_\gamma, e_\delta).$$

The symbols $h_{\alpha\beta}(u)$, $H_{\alpha\beta\gamma}(u)$, $H_{\alpha\beta\gamma\delta}(u)$, etc. have analogous meanings.

Finally, in all following formulae, we will assume that the greek indices $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ run between $0, \ldots, n-1$; the indices $\lambda, \mu, \nu, \rho, \sigma$ will instead run between 1 and $n-1$.

The first structure equations are given by the linear equations verified by the 1-forms $\omega^\alpha_\beta$ and $\omega^\alpha_\beta$:
$$\omega^0_0 + \omega^0_0 = 0, \quad \omega^\lambda_0 + h_{\lambda\nu} \omega^\nu_0 = 0, \quad \omega^\lambda_\beta + \omega^\beta_\lambda + H_{\lambda\mu\nu} \omega^\nu_0 + H_{\lambda\mu\nu} \omega^\rho_0 = 0.$$  \hspace{1cm} (2.3)

In order to write down the expressions for the exterior differentials, it is convenient to replace the 1-forms $\omega^\alpha_\beta$, $\omega^\alpha_\beta$ with the following 1-forms $\varpi^\alpha_\beta$ and $\varpi^\alpha_\beta$:
$$\varpi^0_0 = \omega^0_0, \quad \varpi^\lambda_0 = \omega^\lambda_0, \quad \varpi^\lambda_\lambda = -\omega^\lambda_0, \quad \varpi^\mu_\nu = \omega^\mu_\nu + H_{\mu\nu\lambda} \omega^\lambda_0, \quad \varpi^\beta_\beta = \overline{\varpi^\beta_\beta}. \hspace{1cm} (2.4)$$

Then the last structure equations are:
$$d\theta^\alpha + \varpi^\alpha_\beta \wedge \theta^\beta = \Theta^\alpha + \Sigma^\alpha; \hspace{2cm} (2.5)$$
$$d\varpi^\lambda_0 + \varpi^\lambda_\beta \wedge \varpi^\beta_0 = \Omega^\lambda_0 + \Pi^\lambda_0; \hspace{2cm} (2.6)$$
$$d\varpi^\lambda_\beta + \varpi^\lambda_\beta \wedge \varpi^\beta_\lambda = \Omega^\lambda_\beta + \Pi^\lambda_\beta; \hspace{2cm} (2.7)$$
$$d\varpi^\lambda_\mu + \varpi^\lambda_\mu \wedge \varpi^\mu_\lambda = \Omega^\lambda_\mu + \Pi^\lambda_\mu + \Phi^\lambda_\mu; \hspace{2cm} (2.8)$$

where $\Theta^\alpha$, $\Sigma^\alpha$, $\Omega^\lambda_\beta$, $\Pi^\lambda_0$, $\Pi^\lambda_\mu$, $\Phi^\lambda_\mu$ are the following $\mathbb{C}$-valued 2-forms:
$$\Theta^\alpha = \frac{1}{2} T^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma, \quad \Sigma^\alpha = H_{\alpha\mu\lambda} \varpi^\lambda_0 \wedge \theta^\mu, \quad \Omega^\alpha_\beta = R^\alpha_{\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta, \hspace{2cm} (2.9)$$
$$\Pi^\lambda_\beta = -\hat{\epsilon}_\lambda (h_{\lambda\mu}\varpi^\mu_0) \wedge \theta^\gamma, \quad \Pi^\lambda_0 = -\hat{\epsilon}_\lambda (h_{\lambda\mu}) \varpi^\mu_0 \wedge \theta^\gamma, \hspace{2cm} (2.10)$$
$$\Pi^\lambda_\mu = -\hat{\epsilon}_\gamma (H_{\lambda\mu\rho}) \varpi^\rho_0 \wedge \theta^\gamma - \hat{\epsilon}_\gamma (H_{\mu\rho\sigma}) \varpi^\rho_0 \wedge \theta^\gamma, \hspace{2cm} (2.11)$$
$$\Phi^\lambda_\mu = (H_{\lambda\sigma\mu\rho} - h_{\lambda\sigma} h_{\mu\rho} - H_{\nu\lambda\sigma} H_{\nu\mu\rho}) \varpi^\rho_0 \wedge \varpi^\sigma_0. \hspace{2cm} (2.12)$$

for some suitable complex functions $T^\alpha_{\beta\gamma}$ and $R^\alpha_{\beta\gamma\delta}$ on $U_F(M)$.

The 2-forms $\Theta$ and $\Sigma$ are called (pure) torsion form and Finsler torsion form, respectively. The 2-form $\Omega$ is called the (pure) curvature form. A last, we call $\Pi$ and $\Phi$ the oblique Finsler curvature and the vertical Finsler curvature, respectively.

We recall that the Finsler curvature and torsion forms are identically 0 whenever the Finsler metric is associated with an Hermitian metric.

The next concepts will be essential for the discussions of the following sections, where also the motivations for the terminology will appear clear.
Definition 2.1. Let \((M, J, F)\) be a complex Finsler manifold and let \(\varpi_H\) and \(\varpi_K\) be the \(\mathfrak{gl}_n(\mathbb{C})\)-valued 1-forms on \(U_F(M)\) defined as
\[
\omega_H = \sum_{\lambda, \mu=1}^{n} E^\lambda_{\mu} \omega^\lambda_{\mu} + E^\bar{\lambda}_{\bar{\mu}} \varpi^\bar{\lambda}_{\bar{\mu}}, \quad \varpi_K = \sum_{\lambda, \mu=1}^{n} E^\lambda_{\mu} \varpi^\lambda_{\mu} + E^\bar{\lambda}_{\bar{\mu}} \varpi^\bar{\lambda}_{\bar{\mu}} \quad (2.13)
\]
where \(\varpi^\lambda_{\mu}\) and \(\varpi^\bar{\lambda}_{\bar{\mu}}\) are the 1-forms defined in (2.4). Let also \(\mathcal{H}\) the non-linear Hermitian connection on \(U_F(M)\). Then the two distributions on \(U_F(M)\) \(\mathcal{H}'_H\) and \(\mathcal{H}'_K\) defined by
\[
X \in \mathcal{H}'_H \iff \omega_H(X) = 0, \quad X \in \mathcal{H}'_K \iff \varpi_K(X) = 0,
\]
are called semi-Hermitian connection and Kobayashi connection for \((\mathbb{P}TM, J, F)\), respectively.

Note that, at any \(u \in U_F(M)\), the subspaces \(\mathcal{H}_H|_u\) and \(\mathcal{H}_K|_u\) are both containing \(\mathcal{H}_u\) as a proper subspace.

We conclude recalling the definition of connections and Hermitian connections of complex vector bundles (see e.g. [Ko1]).

Let \(p : E \to N\) be a complex vector bundle over a manifold \(N\) and denote by \(\mathcal{A}^p\) denote the space of smooth \(\mathbb{C}\)-valued \(p\)-forms on \(N\). Denote also by \(\mathcal{A}^p(E)\) the space of smooth complex \(p\)-forms with values in \(E\). A connection \(D\) on \(E\) is a \(\mathbb{C}\)-linear homomorphism
\[
D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)
\]
such that
\[
D(f \sigma) = \sigma \otimes df + f \cdot D\sigma
\]
for any \(f \in \mathcal{A}^0\) and \(\sigma \in \mathcal{A}^0(E)\).

In case \(N\) is a complex manifold and \(p : E \to N\) is a holomorphic vector bundle, a connection \(D\) is called holomorphic if
\[
D^{01} = d^{01}
\]
where \(d\) is the usual exterior differential operator and \(D^{01} : \mathcal{A}^0(E) \to \mathcal{A}^{01}(E)\) and \(d^{01} : \mathcal{A}^0(E) \to \mathcal{A}^{01}(E)\) are the components of \(D\) and \(d\), respectively, which transform the sections \(\sigma \in \mathcal{A}^0(E)\) into the \((0, 1)\)-component of \(D(\sigma)\) and \(d(\sigma)\).

In case \(p : E \to N\) is an Hermitian vector bundle (i.e. endowed with a smooth family of Hermitian metrics on the fibers of \(E\)), a connection \(D\) is called Hermitian if it is holomorphic and for any \(\sigma, \rho \in \mathcal{A}^0\)
\[
d(g(\sigma, \rho)) = g(D\sigma, \rho) + g(\sigma, D\rho)
\]
Recall that on any Hermitian vector bundle there exists exactly one Hermitian connection.
3. The Hermitian and the Kobayashi non-linear covariant derivatives of vector fields.

In this section, we introduce the definition of covariant derivation associated with the distributions \( H_H \) and \( H_K \) given in Definition 2.1. As mentioned in the Introduction, this covariant derivations can be defined using the parallel transports along curves in \( \mathbb{P}T^* \), which are determined by the curves in \( U_F(M) \) which are tangent to the horizontal distribution \( H_H \) and \( H_K \), respectively (see also Rmk 3.8 in \[Sp\]; note however that the discussion there concerns only curves \( \gamma \) in \( \mathbb{P}T^* \) for which the vector \( \pi'_\gamma(t) \) is nowhere vanishing).

However, we will adopt here a different approach, which was considered by S. Kobayashi in \[Ko\] and it is equivalent to the previous one, since it is much more suitable for computations and further developments.

Let us denote by \( \tilde{T}(\mathbb{P}T^*) \) the vector bundle \( \tilde{T}(\mathbb{P}T^*) = (\pi')^{-1}(TM) \) defined as the pull-back bundle w.r.t. the projection map \( \pi' \). We thus obtain the following commuting diagram

\[
\begin{array}{ccc}
\tilde{T}(\mathbb{P}T^*) & \xrightarrow{\tilde{\pi}} & TM \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\mathbb{P}T^* & \xrightarrow{\pi'} & M
\end{array}
\]  

(3.1)

It is clear that there exists a unique complex structure on \( \tilde{T}(\mathbb{P}T^*) \) (let us call it \( \tilde{J} \)), which makes \( \tilde{\pi} : \tilde{T}(\mathbb{P}T^*) \to \mathbb{P}T^* \) a holomorphic vector bundle. Moreover, as it was pointed out in \[Ko\], the Finsler metric \( F \) on \( (M, J) \) induces the following natural Hermitian metric on the vector bundle \( \tilde{T}(\mathbb{P}T^*) \):

\[
g_v(X, Y) = 2 \text{Re}(h_U(X^{10}, Y^{01})) = \frac{1}{2} (h_U(X, Y) + h_U(JX, JY))
\]

where \( h_U \) is as in (2.1). It is clear that right hand side of (3.2) is an Hermitian metric on \( T^*_uM \). Moreover from the invariance properties of complex Finsler metrics under \( \mathbb{C}^* \)-multiplications, it follows that for any \( \lambda \in \mathbb{C}^* \)

\[
h_{\lambda v}(X, Y) + h_{\lambda v}(JX, JY) = h_v(X, Y) + h_v(JX, JY)
\]

(3.3)  

(see e.g. \[Sp\] Lemma 2.4 b) and c), or \[Ko\]). This shows that the r.h.s. of (3.2) is indeed an Hermitian metric which depends only on the line \( v = [U] \in \mathbb{P}T^*_uM \).

Let us now denote by \( \mathcal{A}^0(\tilde{T}(\mathbb{P}T^*)) \) the set of all local sections \( X : \mathbb{P}T^* \to \tilde{T}(\mathbb{P}T^*) \). Notice that for any \( X \in \mathcal{A}^0(\tilde{T}(\mathbb{P}T^*)) \) there exists at least one local vector field \( \mathcal{X} \) on \( U_F(M) \), such that

\[
\pi_*(\mathcal{X}_u) = X(v),
\]

for any \( v \in \mathbb{P}T^*_uM \) and any \( u \in \tilde{\pi}^{-1}(v) \subset U_F(M) \). If this is the case, we will say that \( \mathcal{X} \) is \( TM \)-projectable (or, more often, just \( \text{projectable} \)) and we will call \( X \) the projection of \( \mathcal{X} \) in \( \mathcal{A}^0(\tilde{T}(\mathbb{P}T^*)) \).
Similarly, for any local vector field $\tilde{X}$ on $PTM$, there exists some local vector fields $X$ on $U_F(M)$, such that for any frame $u \in U_F(M)$,

$$\tilde{\pi}_*(X_u) = \tilde{X}_{\tilde{\pi}(v)}.$$ 

In this case we will say that $X$ is $PTM$-projectable (or just projectable) and we will call $\tilde{X}$ the projection of $X$ in $PTM$.

Now, the following technical lemma is required.

**Lemma 3.1.** Let $\tilde{X}$ and $Y$ be a (local) vector field of $PTM$ and a (local) section in $A^0(\tilde{T}(PTM))$, respectively, and let $X$ and $\gamma$ two projectable vector fields on $U_F(M)$, such that $X$ is the projection of $\tilde{X}$ on $PTM$ and $Y$ is the projection of $\gamma$ in $A^0(\tilde{T}(PTM))$.

The functions on $U_F(M)$

$$F^{X,Y}(u) = u \left( X_u(\theta(Y)) + \omega_u(X) \cdot \theta_u(Y) \right) \quad (3.4)$$

$$G^{X,Y}(u) = u \left( X_u(\theta(Y)) + \omega_u(X) \cdot \theta_u(Y) \right) \quad (3.5)$$

assume constant values along the fibers $\tilde{\pi}^{-1}(v) \in U_F(M)$ and are independent on the choice of the projectable vector fields $X$ and $Y$.

In particular, $F^{X,Y}$ and $G^{X,Y}$ define elements of $A^0(\tilde{T}(PTM))$, which depend linearly on the value $\tilde{X}_v$, at any $v \in PTM$.

**Proof.** In order to prove that $F^{X,Y}(u)$ is constant along $\pi^{-1}(v)$, it suffices to check that $\tilde{A}_u (F^{X,Y}) \equiv 0$ for any $u \in \pi^{-1}(v)$ and any vertical vector field $\tilde{A}$ on $U_F(M)$.

Indeed, we may consider only vertical vector fields $\tilde{A}$ which are fundamental vector fields, associated with elements $A \in u_n \oplus \mathbb{R}$ (for the Def. of fundamental vector fields, see e.g. [KN] vol.I).

Notice that

$$\tilde{A}(X(\theta(Y)) + \omega(X) \cdot \theta(Y)) =$$

$$= [\tilde{A}, X](\theta(Y)) + X(\mathcal{L}_{\tilde{A}} \theta(Y)) + X\left( \theta([\tilde{A}, Y]) \right) + \mathcal{L}_{\tilde{A}} \omega(X) \cdot \theta(Y) +$$

$$+ \omega([\tilde{A}, X] \cdot \theta(Y)) + \omega(X) \cdot (\mathcal{L}_{\tilde{A}} \theta(Y)) + \omega(X) \cdot \theta([\tilde{A}, Y]) \quad (3.6)$$

On the other hand

$$\tilde{\pi}_*([\tilde{A}, Y]) = [\tilde{\pi}_*(\tilde{A}), \tilde{\pi}_*(Y)] = 0 \quad \tilde{\pi}_*([\tilde{A}, X]) = [\tilde{\pi}_*(\tilde{A}), \tilde{\pi}_*(X)] = 0 \quad .$$

In particular, $[\tilde{A}, Y]$ and $[\tilde{A}, X]$ are both vertical vector fields for the bundle $\tilde{\pi} : U_F(M) \rightarrow PTM$ and we can write that $[\tilde{A}, X]_u = \tilde{B}_u$, where $\tilde{B}$ is a fundamental vector field associated with an element $B \in u_{n-1} \oplus \mathbb{R}$. This implies that

$$\theta([\tilde{A}, Y]) = 0 \quad , \quad [\tilde{A}, X](\theta(Y))_u = -B \cdot \theta(Y)_u = -\omega([\tilde{A}, X] \cdot \theta(Y))_u \quad .$$

Then (3.6) becomes equal to

$$\tilde{A}(X(\theta(Y)) + \omega(X) \cdot \theta(Y)) =$$

From (3.7) and the definition of $F^{X,Y}$, it follows immediately that $\tilde{A}_u\left(F^{X,Y}\right) = 0$ for any $u \in \pi^{-1}(v)$.

Now, consider other two projectable vector fields $X', Y'$, of which $X$ and $Y$ are the corresponding projections. Then, using the structure equations (2.5) - (2.8) we have

$$F^{X',Y'}(u) - F^{X,Y}(u) =$$

$$= (X' - X)(\theta(Y')) + \omega(X' - X) \cdot \theta(Y') + X(\theta(Y' - Y)) + \omega(X) \cdot \theta(Y' - Y) =$$

$$= (X' - X)(\theta(Y')) + \omega(X' - X) \cdot \theta(Y') =$$

$$= d\theta(X' - X, Y') + \omega(X' - X) \cdot \theta(Y') =$$

$$= -\varpi(X' - X) \cdot \theta(Y') + -\varpi(Y') \cdot \theta(X' - X) + \Theta(X' - X, Y') +$$

$$+ \Sigma(X' - X, Y') + \omega(X' - X) \cdot \theta(Y')$$

At this point, we remark that $\varpi(X' - X) = \omega(X' - X)$: in fact $\hat{\pi}_*(X' - X) = 0$ and this implies that the 1-forms $\omega^0_0$ and $\omega^0_1$ vanish on $(X' - X)$, by (5.32) in [Sp]. The same argument implies that $\Theta(X' - X, Y) = \Sigma(X' - X, Y) = 0$ and hence

$$F^{X',Y'}(u) - F^{X,Y}(u) = 0 .$$

This concludes the proof of both claims for the function $F^{X,Y}$. The proof of the corresponding claims for the function $G^{X,Y}$ is based on very similar arguments. □

By means of Lemma 3.1, the following objects are well defined.

**Definition 3.2.** For any $Y \in \mathcal{A}^0(\tilde{T}(\mathbb{PTM}))$, let $\nabla Y$ and $DY$ be the elements in $\mathcal{A}^1(\tilde{T}(\mathbb{PTM}))$ defined by

$$\nabla_{\tilde{X}} Y|_v = u \left( X(\theta(Y)) + \omega_u(X) \cdot \theta_u(Y) \right) ,$$

(3.8)

$$D_{\tilde{X}} Y|_v = u \left( X(\theta(Y)) + \varpi_u(X) \cdot \theta_u(Y) \right) ,$$

(3.9)

for any vector field $\tilde{X}$ on $\mathbb{PTM}$ and any $v \in \mathbb{PTM}$; here $u$ is any frame of $\hat{\pi}^{-1}(v) \subset U_F(M)$ and $X$, $Y$ are two projectable vector fields of $U_F(M)$, whose projections are $\tilde{X}$ and $Y$, respectively.

$\nabla$ and $D$ are connections on the holomorphic vector bundle $\tilde{T}(\mathbb{PTM})$. For any section $Y \in \mathcal{A}^0(\tilde{T}(\mathbb{PTM}))$, we call $\nabla_{\tilde{X}} Y|_v$ and $D_{\tilde{X}} Y|_v$ the non-linear semi-Hermitian covariant derivative and the non-linear Kobayashi covariant derivative, respectively, along $\tilde{X}$ at the point $v$.

**Remark 3.3.** The set of local vector fields on $M$ can be naturally identified with the element in $A^0(\tilde{T}(\mathbb{PTM}))$, which are sections that are constant along the fibers of $\pi' : \mathbb{PTM} \to M$. 
From this and the properties of the distribution $\mathcal{H}$, it follows that whenever $F$ is associated with an Hermitian metric and $Y$ is a vector field on $M$, then both non-linear covariant derivatives $\nabla_X Y|_v$ and $D_X Y|_v$ depend only on $X = \pi'_F(\hat{X}) \in TM$ and on $x = \pi'_F(v) \in M$, and they both coincide with usual linear Hermitian covariant derivative of $Y$ along $X$.

In the following Proposition, we give two useful characterizations of the connections $\nabla$ and $D$. In particular we show that $D$ coincides with the Finslerian connection introduced by S. Kobayashi in [Ko].

**Proposition 3.4.** Let $(M, J, F)$ be a complex Finsler manifold, $\nabla$ and $D$ as in Definition 3.2 and $g$ the Hermitian metric on $\mathcal{T}(\mathbb{F}TM)$ defined in (3.2).

Then for any local vector field $\hat{X}$ on $\mathbb{F}TM$ and any sections $Y, Z \in \mathcal{A}^0(\mathcal{T}(\mathbb{F}TM))$

\[
\nabla_{\hat{X}} JY|_v = J\nabla_{\hat{X}} Y|_v, \quad D_{\hat{X}} JY|_v = JD_{\hat{X}} Y|_v, \quad (3.10)
\]

\[
\hat{X}(g(Y, Z)|_v) + g_v(\nabla_{\hat{X}} Y|_v, Z) + g_v(Y, \nabla_{\hat{X}} Z|_v) = H_u(X, Y^{10}, Z^{01}) + H_v(X, Z^{10}, X^{01}), \quad (3.11)
\]

\[
\hat{X}(g(Y, Z)|_v) + g_v(D_{\hat{X}} Y|_v, Z) + g_v(Y, D_{\hat{X}} Z|_v) = 0. \quad (3.12)
\]

where $U$ is any non-trivial vector in the complex line $v = [U] \in \mathbb{F}TM$, $x = \pi'(v)$ and $H$ is the trilinear function defined in (2.1).

Furthermore, the connection $D$ is holomorphic and it coincides with the Hermitian connection of the Hermitian bundle $\mathbb{H} : \mathcal{T}(\mathbb{F}TM) \to \mathbb{F}TM$.

**Proof.** The Hermitian metric $g_v$ can be conveniently expressed using the components $\theta^\alpha$ of the tautological 1-form of $U_F(M)$. In fact, for given two local sections $X, Y$ in $\mathcal{A}^0(\mathcal{T}(\mathbb{F}TM))$, consider two projectable vector fields $\hat{X}$ and $\hat{Y}$ on $U_F(M)$, which project onto $X$ and $Y$; then for any $v \in \mathbb{F}TM$,

\[
g_v(X, Y) = \sum_{\alpha} (\theta^\alpha(\hat{X}) \theta^{\bar{\alpha}}(\hat{Y}) + \theta^\alpha(\hat{Y}) \theta^{\bar{\alpha}}(\hat{X})) = <\theta_u(\hat{X}), \theta_u(\hat{Y}) > \quad (3.13)
\]

where $u$ is any frame in $\hat{\pi}^{-1}(v)$.

Consider now three vector fields $\hat{X}$, $\hat{Y}$ and $Z$ on $U_F(M)$ which project onto $\hat{X}$, $Y$ and $Z$, respectively. Let also $\hat{Y}^J$ be a vector field on $U_F(M)$ which projects onto the local section $JY \in \mathcal{A}^0(\mathcal{T}(\mathbb{F}TM))$. Then

\[
\nabla_{\hat{X}} JY|_v = u(\hat{X}(\theta(\hat{Y}^J)) + \omega(\hat{X}) \cdot (\hat{Y}^{J'})) = u(J_o \hat{X} \theta(\hat{Y}) + \omega(\hat{X}) \cdot [J_o \theta(\hat{Y})]) =
\]

\[
= J_u(\hat{X}(\theta(\hat{Y}^J)) + \omega(\hat{X}) \cdot (\hat{Y}^J)) = J\nabla_{\hat{X}} Y|_v, \quad (3.14)
\]

In a similar way one can prove that $D_{\hat{X}} JY|_v = JD_{\hat{X}} Y|_v$.

Now, for (3.11), one should observe that

\[
\hat{X}(g(Y, Z)|_v) = \hat{X}(<\theta_u(\hat{Y}), \theta_u(Z)>) =
\]

\[
= <\hat{X}(\theta_u(\hat{Y})), \theta_u(Z) > + <\theta_u(\hat{Y}), \hat{X}\theta_u(Z) > =
\]
On the other hand, (2.4) in place of (2.3), which proves (3.11). (3.12) can be proved in the same way, using the equations (2.3).

By (2.3), we get
\[
\omega_u(X) \cdot \theta_u(Z) + <\omega_u(Y), \theta_u(Z) > = 0 \quad \text{for any } u \in \mathfrak{A}(\mathbb{T}(TM)) \text{ and any vector field } X, Y \text{ on } TM.
\]

By (2.3), we get
\[
<\omega_u(X) \cdot \theta_u(Y), \theta_u(Z) > + <\omega_u(Y), \omega_u(X) \cdot \theta_u(Z) > = 0
\]

which proves (3.11). (3.12) can be proved in the same way, using the equations (2.4) in place of (2.3).

To conclude, we have to show that for any $Y \in \mathfrak{A}(\mathbb{T}(TM))$ and any vector field $X$ in $TM$,
\[
D_{\tilde{X} + \sqrt{-1}J\tilde{X}} Y = (\tilde{X} + \sqrt{-1}J\tilde{X})(Y).
\]

One can verify that this condition is equivalent to show that for any vector field $\tilde{X}$ in $TM$ there exist two projectable vector fields $X$, $\mathcal{X}_j$ on $U_F(M)$ which project onto $\tilde{X}$ and $J\tilde{X}$, respectively, and so that, for any $\lambda, \mu = 1, \ldots, n - 1$,
\[
\omega^\lambda_X(X) = \omega^\lambda_0(X) = 0, \quad \omega^\lambda_{\mathcal{X}_j}(X) = \omega^0_{\mathcal{X}_j}(X) = 0. \tag{3.14}
\]

From (2.4), it is clear that the distribution $\mathcal{B}$ defined by the conditions (3.14) consists of the set of vector spaces
\[
\mathcal{P}_u = \mathcal{H}_u \oplus \tilde{\mathcal{Y}}_u \subset T_u U_F(M),
\]
where $\mathcal{H}$ is the non-linear Hermitian connection of $U_F(M)$ and
\[
\tilde{\mathcal{Y}}_u \overset{\text{def}}{=} \{ X_u : \pi_u(X_u) = 0, \omega^\lambda_u(X_u) = -H_{\lambda\mu} \omega^\lambda_0(X_u) \}.
\]

By the results in [Sp], one can check that the distribution $\mathcal{B}$ is invariant under the action of $U_{n-1} \times T^1$ and, at all points, there exists a complex structure $\tilde{J}_u : \mathcal{P}_u \rightarrow \mathcal{P}_u$ which is $U_{n-1} \times T^1$-invariant and projects onto the complex structure of $T_{\tilde{\pi}(u)}(\mathbb{T}TM)$. In fact, $\mathcal{B}$ is spanned by the vector fields $\text{Re}(\tilde{e}_\alpha)$, $\text{Im}(\tilde{e}_\beta)$, $\text{Re}(\tilde{e}_\lambda)$, $\text{Im}(\tilde{e}_\mu)$, defined in §5.1 - 5.2 in [Sp]. Therefore, by Prop. 5.4 and Prop. 5.5 (1) in [Sp]), it follows that $\mathcal{B}$ and the (almost) complex structure on $\mathcal{B}$ defined by
\[
\tilde{J}_u(\text{Re}(\tilde{e}_\alpha)) = \text{Im}(\tilde{e}_\beta), \quad \tilde{J}_u(\text{Re}(\tilde{e}_\lambda)) = \text{Im}(\tilde{e}_\lambda),
\]
are both invariant under the action of $U_{n-1} \times T^1$.

To conclude the proof, it is enough to take as vector fields $X$ and $\mathcal{X}_j$ there unique vector field $X$ on $\mathcal{B}$, which projects onto $\tilde{X}$, and that the vector field $\mathcal{X}_j = J\mathcal{X}_j$, respectively. $\square$
4. Torsion and curvature of the Kobayashi connection.

We now want to define the torsion and the curvature of the Kobayashi connection of the vector bundle \( \tilde{T}(\mathbb{P}TM) \rightarrow \mathbb{P}TM \) and express them in terms of the Finsler torsions and curvatures of the non-linear Hermitian connection on \( U_F(M) \). This can be done by virtue of the following proposition.

**Proposition 4.1.** Let \( \hat{X}, \hat{Y} \) be local vector fields on \( \mathbb{P}TM \) and \( Z \in A^0(\tilde{T}(\mathbb{P}TM)) \). Let also \( X, Y \) be the sections in \( A^0(\tilde{T}(\mathbb{P}TM)) \) defined by
\[
X(v) = \pi'_v(\hat{X}(v)), \quad Y(v) = \pi'_v(\hat{Y}(v)).
\]
For any \( v \in \mathbb{P}TM \), consider the vectors in \( T_{\pi'(v)}M \) defined by
\[
T_{\hat{X}, \hat{Y}}(v) = D \hat{X} Y |_v - D \hat{Y} X |_v - [X, Y].
\]
(4.1)
\[
R_{\hat{X}, \hat{Y}} Z(v) = D \hat{X} (D \hat{Y} Z |_v) |_v - D \hat{Y} (D \hat{X} Z |_v) |_v - D [\hat{X}, \hat{Y}] Z |_v
\]
(4.2)
Then if \( X, Y \) and \( Z \) are three projectable vector fields on \( U_F(M) \), which project onto \( \hat{X}, \hat{Y} \) and \( Z \), respectively, and if \( u : \mathbb{P}TM \rightarrow U_F(M) \) is any local section of the bundle \( \pi : U_F(M) \rightarrow \mathbb{P}TM \), then \( T_{\hat{X}, \hat{Y}}(v) \) and \( R_{\hat{X}, \hat{Y}} Z(v) \) verify
\[
T_{\hat{X}, \hat{Y}}(v) = u_v(\Theta(X, Y) + \Sigma(X, Y))
\]
(4.3)
\[
R_{\hat{X}, \hat{Y}} Z(v) = u_v(\Omega(X, Y) \cdot \theta(Z) + \Pi(X, Y) \cdot \theta(Z) + \Phi(X, Y) \cdot \theta(Z))
\]
(4.4)
In particular, \( T_{\hat{X}, \hat{Y}}(v) \) and \( R_{\hat{X}, \hat{Y}} Z(v) \) depend only on the values \( \hat{X}_v, \hat{Y}_v \) and \( Z_v \).

**Proof.** By definitions
\[
u^{-1}(D \hat{X} Y |_v - D \hat{Y} X |_v - \pi'_v([\hat{X}, \hat{Y}]_v)) =
\]
\[
\Lambda(\theta_u(\mathcal{Y})) |_{u_v} - \Lambda(\theta_u(\mathcal{X})) |_{u_v} + \omega_{u_v}(\mathcal{X}) \cdot \theta_u(\mathcal{Y}) - \omega_{u_v}(\mathcal{Y}) \cdot \theta_u(\mathcal{X}) - \theta(\mathcal{X}, \mathcal{Y}) u_v =
\]
\[
d \theta_{u_v}(\mathcal{X}, \mathcal{Y}) - \omega_{u_v}(\mathcal{X}) \cdot \theta_u(\mathcal{Y}) - \omega_{u_v}(\mathcal{Y}) \cdot \theta_u(\mathcal{X}) =
\]
\[
= \Theta_{u_v}(\mathcal{X}, \mathcal{Y}) + \Sigma_{u_v}(\mathcal{X}, \mathcal{Y}),
\]
and this proves (4.3). Similarly
\[
u^{-1}(D \hat{X} (D \hat{Y} Z |_v) |_v - D \hat{Y} (D \hat{X} Z |_v) |_v - D [\hat{X}, \hat{Y}] Z |_v) =
\]
\[
= [\mathcal{X}(\omega(\mathcal{Y})) - \omega(\mathcal{X})] \cdot \mathcal{Y}(\theta(Z)) + [\omega(\mathcal{Y}) \cdot \mathcal{X}(\theta(Z)) - \omega(\mathcal{X}) \cdot \mathcal{Y}(\theta(Z))] + [\omega(\mathcal{X}) \cdot \mathcal{Y}(\theta(Z)) - \omega(\mathcal{Y}) \cdot \mathcal{X}(\theta(Z)) + \omega(\mathcal{X}) \cdot \mathcal{Y}(\theta(Z)) + \omega(\mathcal{Y}) \cdot \mathcal{X}(\theta(Z)) =
\]
\[
d \omega(\mathcal{X}, \mathcal{Y}) \cdot \theta(\mathcal{X}, \mathcal{Z}) = \omega(\mathcal{X}) \cdot \omega(\mathcal{Y}) \cdot \theta(\mathcal{Z}) - \omega(\mathcal{Y}) \cdot \omega(\mathcal{X}) \cdot \theta(\mathcal{Z}).
\]
Now, using the structure equations (2.7) and (2.8), it follows that
\[
d \omega(\mathcal{X}, \mathcal{Y}) \cdot \theta(\mathcal{X}, \mathcal{Z}) + \omega(\mathcal{X}) \cdot \omega(\mathcal{Y}) \cdot \theta(\mathcal{Z}) - \omega(\mathcal{Y}) \cdot \omega(\mathcal{X}) \cdot \theta(\mathcal{Z}) =
\]
\[
= [\Omega_{u_v}(\mathcal{X}, \mathcal{Y}) + \Pi_{u_v}(\mathcal{X}, \mathcal{Y}) + \Phi_{u_v}(\mathcal{X}, \mathcal{Y})] |_{u_v} \cdot \theta_u(Z) = R(X, Y, Z, u_v),
\]
and this proves (4.4). \( \square \)

By means of Proposition 4.1, we may define the torsion and the curvature of the Kobayashi connection as follows (see also [KN] or [Ko1]).
The torsion of Kobayashi connection on $\tilde{T}(PTM)$ is the element in $A^2(\tilde{T}(PTM))$, defined by

$$T_{\hat{X},\hat{Y}}(v) = D_{\hat{X}}Y|_v - D_{\hat{Y}}X|_v - [X,Y]$$

for any vector fields $\hat{X}, \hat{Y}$ on $\mathbb{P}T M$; here $X$ and $Y$ are the sections in $A^0(\tilde{T}(PTM))$ defined by $X(v) = \pi'_*(\hat{X}(v))$, $Y(v) = \pi'_*(\hat{Y}(v))$.

The curvature of the Kobayashi connection on $\tilde{T}(PTM)$ is the $\mathbb{C}$-linear operator

$$R : A^0(\tilde{T}(PTM)) \rightarrow A^2(\tilde{T}(PTM))$$

defined by

$$R_{\hat{X},\hat{Y}}(v) = D_{\hat{X}}(D_{\hat{Y}}Z|_v)|_v - D_{\hat{Y}}(D_{\hat{X}}Z|_v)|_v - D_{[\hat{X},\hat{Y}]}Z|_v$$

for any vector fields $\hat{X}, \hat{Y}$ on $\mathbb{P}T M$ and any $Z \in A^0(\tilde{T}(PTM))$.

Note that, by Proposition 4.2, the values $T_{\hat{X},\hat{Y}}(v)$ and $R_{\hat{X},\hat{Y}}(v)$ depends only on the values $\hat{X}|_v$, $\hat{Y}|_v$ and $Z|_v$.

5. Equations of geodesics.

We are going to write down the equations in terms of the Kobayashi connection. It will be a simple corollary of the results of [Sp] and of the previous discussion.

Let $\gamma : [a, b] \rightarrow M$ be a regular curve in $M$ and $v_\gamma$ the corresponding curve in $\mathbb{P}TM$ defined by

$$v_\gamma : [a, b] \rightarrow \mathbb{P}TM ,$$

$$v_\gamma(t) = [\dot{\gamma}_t] \in \mathbb{P}T_{\gamma_t}M$$

It is not difficult to realize that any tangent vector $\dot{v}_\gamma(t)$ depends linearly on the second derivative $\ddot{\gamma}_t$ and non-linearly on the first derivative $\dot{\gamma}_t$.

To simplify the notation, in the following we will assume that the three functions $\dot{\gamma}$, $v_\gamma$ and $\ddot{v}_\gamma$ are always evaluated at the same point $t \in [a, b]$.

We say that a complex Finsler manifold $(M, J, F)$ is geodetically torsion free if for any $v \in \mathbb{P}TM$ and any $0 \neq \hat{X}, \hat{U} \in T_v\mathbb{P}TM$, with $\pi_*(\hat{U}) = U$ so that $U$ is a non-trivial vector in $v = [U],$

$$g_v(T_{\hat{X}\hat{U}}, U) = 0 .$$

(5.1)

Note that from (3.10), (4.3) and the definitions of Finsler torsions $\Theta$ and $\Sigma$, a complex Finsler manifold is geodetically torsion free if and only if it is geodetically torsion free w.r.t. Def. 6.5 of [Sp].

The geodetically torsion free complex Finsler manifolds coincide with the manifold called weakly Kähler by Abate and Patrizio in [AP]. This term is motivated by the fact that, whenever a Finsler metric is associated with an Hermitian metric, it is geodetically torsion free if and only if the corresponding Hermitian metric is torsion free and hence Kähler.
\textbf{Theorem 5.1.} Let \((M, J, F)\) a complex Finsler manifold. Then \(\gamma\) is a geodesic of \(M\) if and only if
\[
D_{\tilde{\gamma}} \tilde{\gamma}_{|v_\gamma} + g_{v_\gamma}(T_{\tilde{\gamma}} v_\gamma, V_\gamma) = 0.
\]
(5.2)
at any \(t \in [a, b]\), where \(V_\gamma\) is any vector in \(T\Pi T v, M\) such that \([\pi'_\gamma(V_\gamma)] = v_\gamma\). In particular, if \((M, J, F)\) is geodetically torsion free, then \(\gamma\) is a geodesic if and only if
\[
D_{\tilde{\gamma}} \tilde{\gamma}_{|v_\gamma} = 0.
\]
(5.3)

\textit{Proof.} From [Sp] Th. 6.2, we have that \(\gamma\) is a geodesic if and only if for any lift \(\tilde{\gamma}: [a, b] \to U_F(M)\)
\[
\tilde{\gamma}_t(\theta^0(\tilde{\gamma}_t)) + \varpi^0_0(\tilde{\gamma}_t) \theta^0(\tilde{\gamma}_t) = 0,
\]
(5.4)
\[
\tilde{\gamma}_t(\theta^0(\tilde{\gamma}_t)) + \varpi^0_0(\tilde{\gamma}_t) \theta^0(\tilde{\gamma}_t) = 0,
\]
(5.5)
\[
\varpi^0_\lambda(\tilde{\gamma}_t) + T_{\lambda_0}^0 \theta^0(\tilde{\gamma}_t) = 0 = \varpi^0_\lambda(\tilde{\gamma}_t) + T_{\lambda_0}^0 \theta^0(\tilde{\gamma}_t).
\]
(5.6)

We recall that a curve \(\tilde{\gamma}: [a, b] \to U_F(M)\) is a lift of \(\gamma\) if and only if \(\pi \circ \tilde{\gamma} = \gamma\) and for any frame \(\tilde{\gamma}_t = \{f_0(t), \ldots, f_{2n-1}(t)\}\), the vector function \(f_0(t)\) belongs to \(f_0(t) \in C^* \tilde{\gamma}_t\).

From the definition, it follows that \(\theta^\lambda(\tilde{\gamma}_t) = \theta^\lambda(\tilde{\gamma}_t) = 0\) for any \(t\) and hence (5.4) - (5.6) are equivalent to
\[
\tilde{\gamma}_t(\theta(\tilde{\gamma}_t)) + \varpi(\tilde{\gamma}_t) \cdot \theta(\tilde{\gamma}_t) + \Theta^0(\tilde{\gamma}_t, e_0) = 0
\]
(5.7)
and this implies (5.7).

Conversely, if (5.1) holds for any \(t\), it follows immediately that (5.7) holds for any lift and hence \(\gamma\) is a geodesic, by [Sp] Th. 6.2. \(\square\)

6. Jacobi fields.

Let \(\gamma: [a, b] \to M\) be a geodesic of \((M, J, F)\) and let \(V: (-\delta, \delta) \times [a, b] \to M\) be a smooth map such that \(V(s, \ast) = \gamma(s): [a, b] \to M\) is a geodesic for any \(s \in [-\delta, \delta]\) and with \(\gamma^{(0)} = \gamma\). We call \(V\) a \textit{1-parameter family of geodesics centered at} \(\gamma\). We recall that a vector field \(I\) on \(\gamma([a, b])\) is called \textit{Jacobi field for} \(\gamma\) if and only if it is of the form
\[
I_{\gamma_t} = \left. \frac{d}{ds}(V(s, t)) \right|_{s=0}
\]
for some 1-parameter family of geodesics centered at \(\gamma\).

The goal of this section is to determine the differential equations which characterize the Jacobi fields, using the Kobayashi connection and its torsion and curvature.

Let \(X\) be a vector field defined on the points of a curve \(\gamma: [a, b] \to M\). We call \textit{standard lift of} \(X\) \textit{along} \(\gamma\) the vector field \(L_{(X, \gamma)}\) along the curve \(v_\gamma: [a, b] \to \mathbb{R}T M\) defined as follows.

Extend \(X\) to a local vector field and let \(\Phi^X_s: M \to M\) the corresponding flow. Then let
\[
V: (-\delta, \delta) \times [a, b] \to M, \quad V(s, t) = \Phi^X_s(\gamma_t)
\]
and let \( v_{\gamma(t)}(t) \) the 1-parameter family of lifted curves

\[
v_{\gamma(t)}(t) = [\dot{\gamma}_t] \in \mathbb{P}TM
\]

where \( \gamma_t = V(s,t) \). We set

\[
L_{(X,\gamma)}|_{v\gamma} \overset{\text{def}}{=} \frac{d}{ds}v_{\gamma(t)}(t)\bigg|_{s=0}.
\]

**Theorem 6.1.** Let \( I \) be a vector field defined on the points of a geodesic \( \gamma \). Then \( I \) is a Jacobi vector field if and only if it verifies the following system of equations at all points of the geodesic:

\[
D_{\dot{\gamma}} \left( D_{\dot{\gamma}} I|_{v\gamma} \right) |_{v\gamma} - R_{\dot{\gamma} I} L_{(I,\gamma)} \dot{\gamma} - D_{\dot{\gamma}} \left( T_{\dot{\gamma} I} L_{(I,\gamma)}(v\gamma) \right) |_{v\gamma} = 0 . \tag{6.1}
\]

**Proof.** Let \( V : (\delta, \delta) \times [a,b] \to M \) be a 1-parameter family of geodesics centered at \( \gamma \) so that

\[
I_{\gamma(t)} = \frac{d}{ds}(V(s,t))\bigg|_{s=0}.
\]

Let also \( \hat{V} : (\delta, \delta) \times [a,b] \to \mathbb{P}TM \) be the associated map such that

\[
\hat{V}(s,t) = [v_{\gamma(t)}(t)].
\]

Then we may consider the vector fields \( \hat{X} = \hat{V}_s(\frac{\partial}{\partial t}) \) and \( \hat{Y} = \hat{V}_t(\frac{\partial}{\partial s}) \) and the associated functions on \((-\delta, \delta) \times [a,b]\) with values in \( TM \) defined by

\[
X(s,t) = \pi'_{\gamma}(\hat{X}(s,t)) , \quad Y = \pi'_{\gamma}(\hat{Y}(s,t)).
\]

Clearly, \([\hat{X}, \hat{Y}] = 0\) as well as \([X,Y] = 0\). Moreover,

\[
\hat{X}_{v\gamma(t)} = \dot{\gamma}(t) , \quad \hat{Y}_{v\gamma(t)} = L_{(I,\gamma)} , \quad X_{\gamma(t)} = \dot{\gamma}_t , \quad Y_{\gamma(t)} = I_{\gamma_1}.
\]

Therefore

\[
D_{\dot{\gamma}} \left( D_{\dot{\gamma}} I|_{v\gamma} \right) |_{v\gamma} = D_{\hat{X}} \left( D_{\hat{X}} Y|_{v\gamma} \right) |_{v\gamma} =
\]

\[
D_{\hat{X}} \left( D_{\hat{Y}} X|_{v\gamma} \right) |_{v\gamma} + D_{\hat{X}} \left( T_{\hat{X}\hat{Y}}(v\gamma) \right) |_{v\gamma} =
\]

\[
D_{\hat{Y}} \left( D_{\hat{X}} X|_{v\gamma} \right) |_{v\gamma} + R_{\hat{X}\hat{Y}} X(v\gamma) + D_{\hat{X}} \left( T_{\hat{X}\hat{Y}}(v\gamma) \right) |_{v\gamma} \tag{6.2}
\]

and this gives the claim since \( D_{\hat{X}} X|_{v\gamma(t)} = D_{\hat{V}_s} \dot{\gamma}|_{v\gamma(t)} \equiv 0 \) for any \( s \).

The converse is proved using suitable modifications of the arguments used for the analogous result in Riemannian or Hermitian geometry (see e.g. the proof of Prop.VII.1.1 in [KN] vol. II). □
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