Abstract. This article examines the accuracy for large times of asymptotic expansions from periodic homogenization of wave equations. As usual, $\epsilon$ denotes the small period of the coefficients in the wave equation. We first prove that the standard two scale asymptotic expansion provides an accurate approximation of the exact solution for times $t$ of order $\epsilon^{-2+\delta}$ for any $\delta > 0$. Second, for longer times, we show that a different algorithm, that is called criminal because it mixes different powers of $\epsilon$, yields an approximation of the exact solution with error $O(\epsilon^N)$ for times $\epsilon^{-N}$ with $N$ as large as one likes. The criminal algorithm involves high order homogenized equations that, in the context of the wave equation, were first proposed by Santosa and Symes and analyzed by Lamacz. The high order homogenized equations yield dispersive corrections for moderate wave numbers. We give a systematic analysis for all time scales and all high order corrective terms.

Key words. homogenization, secular growth, dispersive effects, asymptotic crimes, wave equations

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1. Introduction

1.1. Traditional homogenization, secular growth, and long times. This paper studies the long time behavior of the wave equation in an infinite periodic medium,

\begin{equation}
\rho(x/\epsilon) \frac{\partial^2}{\partial t^2} u^\epsilon - \left( \text{div} \ a(x/\epsilon) \ \text{grad} \right) u^\epsilon = f(t, x), \quad u^\epsilon = f = 0 \ \text{for} \ t < 0,
\end{equation}

where $\rho, a$ are periodic functions. The unknown $u^\epsilon$ is real valued as is $\rho \in L^\infty(\mathbb{T}^d)$ while $a \in L^\infty(\mathbb{T}^d)$ has values that are real symmetric matrices ($\mathbb{T}^d$ is the unit torus). The coefficients are postive definite in the sense that there is a constant $m_1 > 0$ so that for all
\( \xi \in \mathbb{R}^d, \)
\[
 a(y)\xi \cdot \xi \geq m_1|\xi|^2, \quad \text{and} \quad \rho(y) \geq m_1, \quad \text{for a.e. } y \in T^d.
\]
The source term \( f \) is smooth with \( \partial^\alpha_{t,x} f \in L^2(\mathbb{R}^{1+d}) \) for all \( \alpha \in \mathbb{N}^{d+1} \) and is supported in \( \{0 \leq t \leq 1\} \) until Section 5.

The motivation comes from the articles, in chronological order, [21], [17], [14], [3], [9] that describe the behavior of solutions on the very long time scale \( t \sim 1/\epsilon^2 \) and beyond (see also the engineering literature, including [7], [15], and the numerical literature [1], [2], [6]). The descriptions for these time scales use modifications of the traditional two scale homogenization ansatz.

The traditional ansatz is \( U(\epsilon, t, x, x/\epsilon) \), where

\[
 U(\epsilon, t, x, y) \sim \sum_{n=0}^{\infty} \epsilon^n u_n(t, x, y), \quad u_n(t, x, y) \text{ periodic in } y.
\]

The right hand side is a formal power series in \( \epsilon \). No convergence is expected (see Appendix A). The sign \( \sim \) represents equality in the sense of formal power series. The coefficient functions \( u_n \) belong to the space of smooth functions of \((t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times T^d \) supported in \( t \geq 0 \) that are periodic in \( y \).

Section 3 proves that the traditional construction (1.2) yields a good approximations on time intervals \( 0 \leq t \leq C\epsilon^{-2+\delta} \) with \( C \) as large and \( 0 < \delta \) as small as one likes. The classical approach [8], [10], [11], [16], [20] proves that the ansatz (1.2) is a good approximation on bounded time intervals. It was first observed by Santosa and Symes in [21], and then proved in [17] (see also [14], [3], [9]), that a different ansatz that we call criminal, yields a good approximation for times of order \( \epsilon^{-2} \). In the elliptic setting, this criminal ansatz was first proposed by Bakhvalov and Panasenko [8].

To analyze the two scale ansatz (1.2), each profile \( u_n \) is written as the sum of its non oscillating contribution \( \pi u_n \) and its oscillating part \( (I - \pi)u_n \), defined as

\[
 u_n(t, x, y) = \pi u_n + (I - \pi)u_n, \quad (\pi u_n)(t, x) := \frac{1}{|T^d|} \int_{T^d} u_n(t, x, y) \, dy.
\]

Introduce the traditional second order partial differential operators

\[
 \mathcal{A}_{yy} := \text{div}_y a(y) \text{grad}_y,
\]
\[
 \mathcal{A}_{xx} := \text{div}_x a(y) \text{grad}_x,
\]
\[
 \mathcal{A}_{xy} := \text{div}_x a(y) \text{grad}_y + \text{div}_y a(y) \text{grad}_x.
\]

Then

\[
 \left[ \rho(x/\epsilon) \partial^2_t - \text{div} a(x/\epsilon) \text{grad} \right] U(\epsilon, t, x, x/\epsilon) \sim W(\epsilon, t, x, x/\epsilon).
\]

\(^1\)This decomposition of the two scale hierarchy emphasizing the projector \( \pi \) follows modern developments in hyperbolic geometric optics, see [19].
where $W$ is the formal Laurent series in $\epsilon$ computed from

$$
(1.4) \quad \left[ \rho(y)\partial^2_t - \frac{1}{\epsilon^2}A_{yy} - \frac{1}{\epsilon}A_{xy} - A_{xx} \right] U(\epsilon, t, x, y) \sim W(\epsilon, t, x, y) := \sum_{n=-2}^{\infty} \epsilon^n w_n(t, x, y).
$$

In all constructions below (the classical one, as well as the new one proposed in this paper) the $u_n$ are chosen so that up to some precision, one has $W = f$. Since the source term $f(t, x)$ does not depend on $y$, it has no oscillating part, $(I - \pi)f = 0$, and thus it is natural to seek the $u_n$ so that $(I - \pi)w_n = 0$. The formal power series $U$, satisfying (1.4), for which $(I - \pi)w_n = 0$ have a very rigid structure that steers our analysis. For $k \geq 1$, we introduce differential operators (see Definition 2.2 for details)

$$
(1.5) \quad \chi_k(y, \partial_t, \partial_x) = \sum_{a \in \mathbb{N}^{d+1}, |a| = k} c_{a,k}(y) \partial^a_{t,x}.
$$

The coefficients $c_{a,k}$ are solutions of periodic cell problems. The coefficients of the pure $x$ derivatives in (1.5) are the classical $k^{th}$ order correctors in elliptic homogenization [8, 10, 20]. Theorem 2.5 proves that the ansatz $U$, which yield profiles $w_n$ satisfying $(I - \pi)w_n = 0$, are so that $u_n$ satisfy,

$$
(1.6) \quad \forall n \geq 0, \quad (I - \pi)u_n = \sum_{k=1}^{n} \chi_k(y, \partial_t, \partial_x) \pi u_{n-k}.
$$

The oscillating part is given in terms of the non-oscillating parts of lower order.

The second structural identity concerning the formal series $U$ satisfying $(I - \pi)w_n = 0$ is a formula for $\pi u_n$ that involves homogenized differential operators $a^*_k(\partial_t, \partial_x)$ with constant coefficients. The operator $a^*_2$ is the standard homogenized wave operator [8, 10, 11, 16, 20]. For $k \geq 3$, the $a^*_k$’s are called high order homogenized operators [8]. By establishing a combinatorial formula for the $a^*_k$’s, Theorem 2.13 proves that the odd order homogenized operators vanish, $a^*_{2n+1} = 0$. This is a classical result for $a^*_1 = 0$ and $a^*_3 = 0$ (see e.g. [4] and references therein). It was already known for all odd orders in the elliptic case [22]. Theorem 2.10 proves that

$$
(1.7) \quad \forall n, \quad (I - \pi)w_n = 0, \quad \iff \quad \forall n, \quad \pi w_n = \sum_{0 \leq 2j \leq n} a^*_{2j+2}(\partial_t, \partial_x) \pi u_{n-2j}.
$$

Equation (1.7) expresses $\pi w_n$ in terms of $\pi u_m$ with $m \leq n$ and having the same parity as $n$. This is the leap frog structure.

The traditional algorithm [8, 10, 20] sets $W = f$. Equivalently, $w_0 = \pi w_0 = f$ and $\pi w_n = 0$ for $n \geq 1$. The first yields the homogenized wave equation $a^*_1(\partial_t, \partial_x)\pi u_0 = f$ whose solution $u_0 = \pi u_0$ has energy independent of $t$ for $t$ beyond the support of $f$. The leading profile $\pi u_0$ does not grow with time. The next equations, $\pi w_n = 0$ for $n \geq 1$, lead to wave equations for each $\pi u_n$ with a source term given in terms of the preceding $\pi u_m$.
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\[ - \pi u \]

that 

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t 

the leap frog structure explains why terms can no longer be understood as correction terms. The slow secular growth from the rate of growth of the profiles 

\[ u \]

\[ u \]

\[ t \]

\[ t^2 \]

\[ \pi u \]

\[ \pi u_2 \]

no faster than \( t^2 \) and \( \pi u_{2k} \) no faster than \( t^k \). The leap frog structure shows that \( \pi u_n \) grows no faster than \( t^{n/2} \). Without the leap frog structure one would have found \( t^n \). The \((2n)^{\text{th}}\) term in the ansatz (1.2) is of size \( \epsilon^{2n} t^n \). For times \( t \sim 1/\epsilon^2 \) the higher order terms can no longer be understood as correction terms. The slow secular growth from the leap frog structure explains why \( t \sim 1/\epsilon^2 \) is a critical time scale for the traditional expansion.

The secular growth estimate implies Theorem 3.1 asserting that for any \( N, \delta > 0 \) choosing a sufficiently large number of terms in the traditional ansatz (1.2) guarantees that the error is \( O(\epsilon^N) \) for times \( t \leq 1/\epsilon^{2-\delta} \).

Appendix A.2 contains an example showing that the classical approximation is not accurate for times \( t \sim 1/\epsilon^{2+\delta} \) for any \( \delta > 0 \).

1.2. Asymptotic crimes and longer times. To find approximate solutions for longer times we abandon the classical ansatz that requires that \( u_0 = f \) and \( w_n = 0 \) for \( n \neq 1 \). In the residual we do not set the coefficients of \( \epsilon^n \) equal to zero. This is an asymptotic crime. In addition to the motivating examples from homogenization theory, asymptotic crimes have a long history in fluid dynamics and geometric optics, see [18]. In order that the computations retain much of the structure from the traditional ansatz we demand that \((1 - \pi)w_n = 0\) for all \( n \). That yields (1.6) and (1.7) and, in particular, the leading term is non oscillating, \( v_0 = \pi v_0 \). To emphasize the fact that the new profiles are not the same as the old ones we call them \( v_n \), and set

\[ V(\epsilon, t, x, y) = \sum_{n=0}^{\infty} \epsilon^n v_n. \]

Then (1.7) implies that the discrepancy \( W - f = \pi(W - f) \) is equal to the sum of the lines,

\[ e^0 [a_2^*(\partial_t, \partial_x)\pi v_0 - f] + \]

\[ e^1 [a_2^*(\partial_t, \partial_x)\pi v_1] + \]

\[ e^2 [a_2^*(\partial_t, \partial_x)\pi v_2 + a_4^*(\partial_t, \partial_x)\pi v_0] + \]

\[ e^3 [a_2^*(\partial_t, \partial_x)\pi v_3 + a_4^*(\partial_t, \partial_x)\pi v_1] + \]

\[ e^4 [a_2^*(\partial_t, \partial_x)\pi v_4 + a_4^*(\partial_t, \partial_x)\pi v_2 + a_6^*(\partial_t, \partial_x)\pi v_0] + \]

\[ e^5 [a_2^*(\partial_t, \partial_x)\pi v_5 + a_4^*(\partial_t, \partial_x)\pi v_3 + a_6^*(\partial_t, \partial_x)\pi v_1] + \cdots. \]
The problems of secular growth came from setting all the rows equal to zero. That yields equations for the corrector terms that have the preceding profiles as sources. The criminal strategy requires only that the sum of the lines vanishes. That can be achieved setting $\pi v_n = 0$ for all $n > 0$ and demanding that

$$\left( a_2^* (\partial_{t,x}) + \epsilon^2 a_4^* (\partial_{t,x}) + \epsilon^4 a_6^* (\partial_{t,x}) + \cdots \right) v_0 = f, \quad v_0 = 0 \quad \text{for} \quad t < 0. \quad (1.9)$$

The coefficients in Equation (1.9) depend on $\epsilon$. To solve this equation with accuracy $O(\epsilon^N)$, $v_0$ must depend on $\epsilon$, $v_0 = v_0(\epsilon, t, x)$. Including oscillatory correction terms, the approximation takes the new form

$$V(\epsilon, t, x, y) \sim \sum_{n=0}^{\infty} \epsilon^n v_n(\epsilon, t, x, y), \quad v_n(\epsilon, t, x, y) \text{ periodic in } y, \quad (1.10)$$

where each profile depends on $\epsilon$. The series is treated as a formal series in $\epsilon$ whose coefficients are functions of several variables including $\epsilon$. The summand $\epsilon^n v_n$ is viewed as a term in $\epsilon^n$ although $v_n$ depends on $\epsilon$.

To construct the profile $v_0$, (1.9) is modified in several ways. The first difficulty is that the terms $\epsilon^{2j-2} a_{2j}^* (\partial_{t}, \partial_{x})$ are typically of order $2^j$ in $\partial_t$. The more terms one keeps the higher order is the equation in $\partial_t$. The truncated operators usually define ill posed initial value problems. The first thing that we do is perform a normal form transformation that converts the operators $a_{2j}^*$ with $j \geq 2$ to operators in $\partial_x$ only. The normal form removes all the time derivatives other than those in $a_2^*$. In Section 4.2 it is proved that there are uniquely determined homogeneous operators $R_{2j}(\partial_{t,x})$ and $\tilde{a}_{2j}(\partial_{x})$ of degree $2j$ so that as formal power series in $\partial_t, \partial_x$,

$$\left[ 1 + \sum_{j=1}^{\infty} R_{2j}(\partial_{t,x}) \right] \left[ \sum_{j=1}^{\infty} a_{2j}^* (\partial_{t,x}) \right] = a_2^*(\partial_{t,x}) + \sum_{j=2}^{\infty} \tilde{a}_{2j}(\partial_{x}). \quad (1.11)$$

The operators $R_{2j}$ and $\tilde{a}_{2j}$ are computable by a rapid recursive algorithm. This step has no analogue in the elliptic context. Multiplying (1.9) by $1 + \sum_{j=1}^{\infty} R_{2j}(\epsilon \partial_{t,x})$ yields the equivalent equation

$$\left[ a_2^*(\partial_{t,x}) + \sum_{j=2}^{\infty} \epsilon^{2j-2} \tilde{a}_{2j}(\partial_{x}) \right] v_0(\epsilon, t, x) = \left[ 1 + \sum_{j=1}^{\infty} \epsilon^{2j} R_{2j}(\epsilon \partial_{t,x}) \right] f. \quad (1.12)$$

One does not need an exact solution. The sums in (1.12) are first truncated to finite sums. The corresponding equation depends on the number of terms retained and the unknown function is denoted $v_0^k$. The truncated equation of order $k$ is, with $R^k := \sum_{j=1}^{k} R_{2j}$,

$$\left[ a_2^*(\partial_{t,x}) + \sum_{j=2}^{k+1} \epsilon^{2j-2} \tilde{a}_{2j}(\partial_{x}) \right] v_0^k(\epsilon, t, x) = \left[ 1 + R^k(\epsilon \partial_{t,x}) \right] f, \quad v_0^k = 0 \quad \text{for} \quad t < 0. \quad (1.13)$$
The initial value problem (1.13) is usually ill posed so does not define a profile \( \tilde{a}_4 \). For example, it is known [13], [17], [4] that, at least when \( \rho(y) \) is constant, the operator \( \tilde{a}_4 \) has the wrong sign so that (1.13) is ill posed for \( k = 2 \). Surprisingly, that is not a fatal flaw.

The idea to overcome this obstacle is the following. The correctors \( \epsilon^{2j-2} \tilde{a}_2(\partial_x) \) added to \( a_2^s(\partial_t, \partial_x) \) are only small compared to \( a_2^s \) when applied to functions whose Fourier transform is supported where \( \epsilon \xi \) is small (\( \xi \) being the Fourier variable). The idea is to filter the source term. Choose \( \psi_1 \in C_*^\infty(\mathbb{R}^d) \) equal to 1 on a neighborhood of the origin. Choose \( 0 < \alpha < 1 \). The operator \( \psi_1(\epsilon^\alpha D) \) is the Fourier multiplier \( g \mapsto \mathcal{F}^{-1} \psi_1(\epsilon^\alpha \xi) \mathcal{F}g \).

Equivalently \( D := (1/i)\partial_x \). The filtered equation is

\[
(1.14) \quad [a_2^s(\partial_t, x) + \sum_{j=2}^{k+1} \epsilon^{2j-2} \tilde{a}_2(\partial_x)]v_0^k = \psi_1(\epsilon^\alpha D)(1 + R^k(\epsilon^\alpha \partial_x))f, \quad v_0^k = 0 \text{ for } t < 0.
\]

The right hand side has Fourier transform supported in \( |\xi| \leq C \epsilon^{-\alpha} \ll 1/\epsilon \).

Equation (1.14) is the one that is solved to determine a profile \( v_0^k \). The filtered equation (1.14) has a unique tempered solution. That solution has spatial Fourier transform supported in \( \epsilon^{-\alpha} \text{supp} \psi_1 \). Energy bounds like those for \( a_2^s \) are proved in Section 4.4. The operator on the left in (1.13) is the sum of the homogenized operator and hopefully small higher order terms. The higher order terms are sometimes thought of as dispersive correctors. This is at least the original interpretation of \( \tilde{a}_4 \) in [21].

The next definition summarizes the recipe for the approximate solution.

**Definition 1.1 (Criminal approximation).** Fix the choice of \( \psi_1 \in C_*^\infty(\mathbb{R}^d), 0 < \alpha < 1, \) and \( 0 \leq k \in \mathbb{N} \). Define profiles \( v_n^k(\epsilon, t, x, y) \) for \( 0 \leq n \leq 2k + 2 \) as follows.

- **Nonoscillatory parts.** For \( 1 \leq n \leq 2k + 2 \), \( \pi v_n^k = 0 \). For \( n = 0 \), \( \pi v_0^k \) is the unique tempered solution of the high order homogenized equation (1.14).

- **Oscillatory parts.** For \( 1 \leq n \leq 2k + 2 \), \( (I - \pi)v_n^k = \chi_n \pi v_0^k \), where \( \chi_n \) is defined by (1.5), and \( (I - \pi)v_0^k = 0 \). Equivalently, \( v_n^k = \pi v_0^k \).

Define

\[
(1.15) \quad V^k(\epsilon, t, x, y) := \sum_{n=0}^{2k+2} \epsilon^n v_n^k(\epsilon, t, x, y) = \left( I + \sum_{n=1}^{2k+2} \epsilon^n \chi_n(y, \partial_t, \partial_x) \right)v_0^k(\epsilon, t, x).
\]

The criminal approximate solution is \( V^k(\epsilon, t, x, x/\epsilon) \).

The main result of the present paper is the following approximation theorem.

**Theorem 1.2 (Criminal error).** Suppose that \( u^\epsilon \) is the exact solution of (1.1) and \( V^k \) is given by (1.15). For each \( k \geq 2 \) there are positive constants \( C, \epsilon_0 \) so that for \( 0 < \epsilon \leq \epsilon_0 \)
and $t \geq 0$, the error in energy satisfies
\[ \| \nabla_{t,x} (u^\epsilon(t,x) - V^k(\epsilon, t, x, x/\epsilon)) \|_{L^2(\mathbb{R}^d)} \leq C \epsilon^{2k+1} \langle t \rangle, \quad \text{with } \langle t \rangle := \sqrt{1 + t^2}. \]

**Remark 1.3.** i. If one wants the error to decrease as $\epsilon^{N_1}$ on time intervals $0 \leq t \leq C/\epsilon^{N_2}$, it suffices to choose $k$ so that $N_1 + N_2 \leq 2k + 1$.

ii. The initial value problem defining $v^k_0$ has constant coefficients. Its spatial Fourier Transform is given by an explicit formula. A spectrally accurate approximate solution is computable by FFT.

Writing $u = \int_0^t u_t \, dt$ yields the following corollary.

**Corollary 1.4.** With the assumptions and notations in Theorem 1.2, the error measured in $L^2(\mathbb{R}^d)$ satisfies
\[ \| u^\epsilon(t) - V^k(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} \leq C \epsilon^{2k+1} \langle t \rangle^2. \]

A more subtle corollary is that the oscillating part of the approximate solution is not necessary for the long time asymptotics if one is content with an error of the order of $\epsilon$.

**Corollary 1.5.** With the assumptions and notations in Theorem 1.2, the error from the leading term $v^k_0$ satisfies
\[ \| u^\epsilon(t) - v^k_0(\epsilon, t, x) \|_{L^2(\mathbb{R}^d)} \leq C \left( \epsilon + \epsilon^{2k+1} \langle t \rangle^2 \right). \]

Corollary 1.5 shows that for $N$ as large as one likes, if one takes $k \geq N$, then uniformly on $0 \leq t \leq C/\epsilon^N$ the $L^2(\mathbb{R}^d)$-error is smaller than $\epsilon$ using only the leading nonoscillatory term $v^k_0(\epsilon, t, x)$ in the approximate solution. Corollary 1.5 was proved in [9] in a more general context (almost periodic or random coefficients) with a proof based on Bloch waves. Theorem 1.2 improves previous results since, not only the approximation error is valid for times as large as one wants, but the error is as high order in $\epsilon$ as one wants. Our results improve those of [17], [14], which were restricted to times of order $1/\epsilon^2$ with an error of order $\epsilon$. The first paper [17] relies on two scale asymptotic expansions, while the second one [14] uses Bloch waves.

The previous works [17], [14], [9] considered (1.1) with $f = 0$ and nonvanishing initial data. In addition in [14] and [9], $\partial_t u(0) = 0$. One of the reasons that we can push the analysis further is that our choice simplifies some things. We next expand a little on this choice. The solutions of (1.1) with $f$ smooth in time with values in $L^2(\mathbb{R}^d)$ and supported in a compact time interval satisfy
\[ \forall j \in \mathbb{N}, \quad \sup_{0 < \epsilon < 1} \sup_{t \in \mathbb{R}} \| \nabla_{t,x} \partial_t^j u^\epsilon(t) \|_{L^2(\mathbb{R}^d)} < \infty. \]
The coefficients vary on the small scale $\epsilon$ but the solutions do not oscillate in time. For smooth solutions with $f = 0$ and Cauchy data $u^\epsilon(0), \partial_t u^\epsilon(0)$ the initial derivatives satisfy
\[ \forall j \geq 0, \ell \in \{0, 1\}, \quad \partial_{t}^{2j+\ell} u^\epsilon(0) = \left( \frac{1}{\rho(x/\epsilon)} (\text{div} a(x/\epsilon) \text{grad}) \right)^j \partial_{t}^{\ell} u^\epsilon(0). \]
These yield formulas for $\nabla_{t,x} \partial_{t}^{j} u^\epsilon \big|_{t=0} := H^\epsilon_j(u^\epsilon(0), \partial_t(u^\epsilon(0)))$. The initial data corresponding to solutions satisfying (1.16) are those so that
\[ (1.17) \quad \forall j, \quad \sup_{\epsilon} \| H^\epsilon_j(u^\epsilon(0), \partial_t(u^\epsilon(0))) \|_{L^2(\mathbb{R}^d)} < \infty. \]
For the $f = 0$ problem with Cauchy data satisfying (1.17) the approximation properties for both classical and criminal strategies are as in our Theorems. For the $f = 0$ problem the accuracy of the approximation is determined by how well the initial data can be approximated by data satisfying (1.17).

The condition (1.17) is awkward to use. For example when $\rho, a$ are just $L^\infty(\mathbb{T}^d)$ and at least one of them is not constant it is true but not immediately obvious that no family of initial data that is independent of $\epsilon$ can satisfy this condition. Without performing a nontrivial computation it is not clear that there are initial data given by two scale expansions that satisfy this condition. The solutions from traditional homogenization with our source term $f(t,x)$ viewed for $t$ beyond $\text{supp} f$ show that there are many such two scale data.

Equation (1.1) shows that solutions that do not oscillate in time are important. Their description via Cauchy data is awkward. We study the natural problem (1.1).

1.3. Outline of the present paper. In Section 2 the classical two scale asymptotic expansion for wave equations is analysed. Theorem 2.13 proves that the odd order homogenized operators vanish. Theorem 2.14 shows that secular growth of the profiles is half as fast as one might expect. Remark 2.6 is a first version of the criminal path.

Section 3 studies the accuracy of the classical expansion. Classical proofs show that for bounded time and any $N$ the error is $O(\epsilon^N)$. We prove that taking more corrector terms one has $O(\epsilon^N)$ accuracy for times of order $\epsilon^{-2+\delta}$, for any $\delta > 0$.

Section 4 presents the details of the derivation of the criminal asymptotic expansion and proves Theorem 1.2.

Section 5 shows that our results for sources $f$ compactly supported in time suffice, by a simple argument, to treat sources that grow at most polynomially in time.

Section 6 discusses second order systems including linear elasticity and Schrödinger’s equation. The place where the argument is not automatic is the proof that the odd order homogenized operators vanish in the systems case (Theorem 6.5).
Appendix A gives an example in dimension $d = 1$ for which the upper bound on the secular growth predicted in Section 2 is attained. For the same example, the classical two scale asymptotic expansion (1.2) does not yield a good approximation for times $t \sim 1/\epsilon^{2+\delta}$.

Appendix B provides a classical a priori estimate for two scale oscillating functions.

Appendix C proves that solutions of the wave equation have finite energy for sources less regular in $x$ but more regular in $t$ than the standard condition $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$.

2. Analysis of the two scale ansatz (1.2)

Revisit the standard method of two scale asymptotic expansions for the wave equation (1.1). We depart from the textbooks [8], [10], [20] in several ways. First, in those books the method is usually applied to an elliptic equation and the wave equation is only said to be treated similarly. Second, we do not content ourselves with computing the first two or three terms and giving a recurrence for the other ones. Exact combinatorial formulas are given for terms of all orders in the ansatz (1.2).

Infinite order asymptotic expansions require that the source term $f$ be infinitely smooth. The periodic coefficients $\rho(y)$ and $a(y)$ are assumed only to be in $L^\infty(T^d)$.

2.1. Ansatz and first hierarchy. Let $A_{yy}, A_{xy}, A_{xx}$ be the second order partial differential operators defined in (1.3). Consider the two scale power series (1.2), and the corresponding formula (1.4) for the right hand side. All terms $u_n(t,x,y)$ and $w_n(t,x,y)$ are periodic in $y$, equivalently defined for $y \in T^d$. The relation (1.4) is equivalent to

$$
\rho(y) \frac{\partial^2}{\partial t^2} - \frac{1}{\epsilon^2} A_{yy} - \frac{1}{\epsilon} A_{xy} - A_{xx} \sum_{n=0}^{\infty} \epsilon^n u_n(t,x,y) = \sum_{n=-2}^{\infty} \epsilon^n w_n(t,x,y)
$$

as formal Laurent series in $\epsilon$. Equation (2.1) at order $\epsilon^n$ reads

$$
\epsilon^{-2} : -A_{yy} u_0 = w_{-2},
$$

$$
\epsilon^{-1} : -(A_{yy} u_1 + A_{xy}) u_0 = w_{-1},
$$

and, for $k \geq 0$, the coefficient of $\epsilon^k$ is

$$
\rho(y) \frac{\partial^2}{\partial t^2} u_k - (A_{yy} u_{k+2} + A_{xy} u_{k+1} + A_{xx} u_k) = w_k.
$$

In both classical and criminal strategies we construct profiles so that the $w_k$ do not depend on the fast variable $y$. The equation (1.1) corresponds to $w_0 = f$ and $w_k = 0, k \neq 0$.

2.2. Projections and the hierarchy. The analysis of (2.2), (2.3) pivots around the second order symmetric elliptic operator $A_{yy} : H^1(T^d) \to H^{-1}(T^d)$. Denote by $\pi$ the $L^2(T^d)$ orthogonal projection on constants,

$$
\pi g := \frac{1}{|T^d|} \int_{T^d} g(y) \, dy.
$$
This operator \( \pi \) coincides with the action of \( g \) as a distribution on the test function 1. It is therefore a well defined operator on all periodic distributions. This operator extends to functions of \( t, x, y \) by acting only on the last variable,

\[
(\pi g)(t, x) := \frac{1}{|T^d|} \int_{T^d} g(t, x, y) \, dy.
\]

**Lemma 2.1 (Cell Problem).** The operators in (1.3) satisfy

\[
\pi A_{yy} = 0, \quad A_{yy} \pi = 0 \quad \text{and} \quad \pi A_{xy} \pi = 0.
\]

The nullspace of \( A_{yy} \) is equal to the space of constant functions, i.e. \( \pi H^1(T^d) \). The image, \( \text{Range} \, A_{yy} \), is the subspace of mean zero functions, i.e. \( (I - \pi)H^{-1}(T^d) \). Therefore \( A_{yy} \) is a bijection \( (I - \pi)H^1(T^d) \to (I - \pi)H^{-1}(T^d) \). \( A_{yy}^{-1} \) denotes its inverse.

**Proof.** A classical application of the Lax-Milgram Lemma. \( \square \)

To solve (2.2) and (2.3), these equations will be projected by \( \pi \) (yielding the non-oscillatory hierarchy) and \((I - \pi)\) (leading to the oscillatory hierarchy) and solved separately.

2.2.1. **The oscillatory hierarchy.** Consider power series \( U \) and \( W \) for which \((I - \pi)w_n = 0\) for all \( n \geq -2 \). Equations (2.2) and (2.3) are multiplied on the left by \((I - \pi)\). Using \((I - \pi)w_{-2} = 0\) the first line of (2.2) becomes

\[
0 = (I - \pi)A_{yy}u_0 = (I - \pi)A_{yy}(I - \pi)u_0.
\]

Lemma 2.1 shows that this is equivalent to \((I - \pi)u_0 = 0\), The oscillatory part of \( u_0 \) vanishes.

Since \( \pi A_{xy} \pi = 0 \), one has \( \pi A_{xy} u_0 = \pi A_{xy} \pi u_0 = 0 \). Thus, the second line of (2.2) shows that \((I - \pi)w_{-1} = 0\) if and only if

\[
(I - \pi)u_1 = -A_{yy}^{-1}(I - \pi)A_{xy} \pi u_0 = -A_{yy}^{-1}A_{xy} \pi u_0 := \chi_1(y, \partial_x)\pi u_0.
\]

Next, derive analogous formulas expressing \((I - \pi)u_k\) in terms of the \( \pi u_j \) with \( j < k \). Since by assumption \((I - \pi)w_k = 0\) for all \( k \geq 0 \), (2.3) leads to

\[
(I - \pi) \left[ \rho(y) \partial_t^2 u_k - A_{yy}u_{k+2} - A_{xy}u_{k+1} - A_{xx}u_k \right] = 0.
\]

By Lemma 2.1 this is equivalent to

\[
(I - \pi)u_{k+2} = -A_{yy}^{-1}(I - \pi) \left[ A_{xy}u_{k+1} + (A_{xx} - \rho(y) \partial_t^2)u_k \right].
\]

Equation (2.5) expresses the oscillatory part of \( u_{k+2} \) in terms of earlier profiles. It can be further simplified by rewriting the earlier profiles as \( u_j = \pi u_j + (1 - \pi)u_j \), the sum of non oscillatory and oscillatory parts. Then express the \((1 - \pi)u_j \) parts in terms of still earlier profiles, and so on. In this way the oscillatory parts can be eliminated yielding a relation determining the oscillatory parts in terms of the nonoscillatory parts that is made explicit.
in Theorem 2.5. In (2.4) an operator $\chi H$ belongs to the range of $A$.

Definition 2.2. Set $\chi_{-1} := 0$, $\chi_0 := I$. For $k \geq 1$ define operators mapping functions of $t, x$ to functions of $t, x, y$ by

$$
(2.6) \quad \chi_k(y, \partial_t, \partial_x) := -A^{-1}_{yy}(I - \pi)\left[A_{xy}\chi_{k-1} + (A_{xx} - \rho(y)\partial_t^2)\chi_{k-2}\right] = (I - \pi)\chi_k.
$$

This recovers the previous definition of $\chi_1 = -A_{yy}^{-1}A_{xy} = -A_{yy}^{-1}(I - \pi)A_{xy}$, where the last equality follows from Lemma 2.1. The operators $\chi_k$ depend on $y$. The $y$-dependence arises only from the coefficients $a(y), \rho(y)$. To show that the above definition makes sense, it suffices to prove that for any smooth function $\varphi \in C^\infty(\mathbb{R}^{1+d})$ and every $(t, x) \in \mathbb{R}^{1+d}$ the argument of $A_{yy}^{-1}$, namely

$$(I - \pi)\left[A_{xy}\chi_{k-1} + (A_{xx} - \rho\partial_t^2)\chi_{k-2}\right] \varphi(t, x),$$

belongs to the range of $A_{yy}$. This is verified in the next Lemma.

Lemma 2.3. For all $k \geq 1$ the following holds. For every function $\varphi \in C^\infty(\mathbb{R}^{1+d})$ and every $(t, x) \in \mathbb{R}^{1+d}$ one has that

$$
\left[A_{xy}\chi_{k-1} + (A_{xx} - \rho\partial_t^2)\chi_{k-2}\right] \varphi(t, x)
$$

belongs to $H^{-1}(\mathbb{T}^d)$. In particular $\chi_k \varphi(t, x) \in (I - \pi)H^1(\mathbb{T}^d)$. Furthermore, for any $k \geq 1$, there exist coefficients $c_{\beta,k} \in (I - \pi)H^1(\mathbb{T}^d)$ such that

$$
(2.7) \quad \chi_k(y, \partial_t, \partial_x) = \sum_{|\beta| = k} c_{\beta,k}(y) \partial_t^\beta \partial_x^\beta.
$$

In particular, $\chi_k$ is a homogeneous operator of degree $k$ in $\partial_t \partial_x$.

Proof. The proof is by induction on $k$. For $k = 1$, $\chi_1$ is a first-order operator in $x$ with $(I - \pi)H^1(\mathbb{T}^d)$ coefficients, since $A_{xy}\varphi$ belongs to $H^{-1}(\mathbb{T}^d)$.

Assume the statement for $k \geq 1$ and prove it for $k + 1$. For a function $\varphi \in C^\infty(\mathbb{R}^{1+d})$ compute

$$
(2.8) \quad \left[A_{xy}\chi_k + (A_{xx} - \rho(y)\partial_t^2)\chi_{k-1}\right] \varphi(t, x) = \text{div}_x(a(y)\text{grad}_y \chi_k \varphi(t, x)) + \text{div}_y(a(y)\text{grad}_x \chi_k \varphi(t, x)) + \text{div}_x(a(y)\text{grad}_x \chi_{k-1} \varphi(t, x)) - \rho(y)\partial_t^2(\chi_{k-1} \varphi(t, x)).
$$

By the induction hypothesis $\chi_k \varphi(t, x)$ and $\chi_{k-1} \varphi(t, x)$ are in $H^1(\mathbb{T}^d)$. Therefore, all terms on the right hand side of (2.8) are in $H^{-1}(\mathbb{T}^d)$. In particular $\chi_{k+1} \varphi(t, x) \in H^1(\mathbb{T}^d)$, which is the claimed result.

Since the operator $A_{xy}$ is homogeneous of degree one and $(A_{xx} - \rho\partial_t^2)$ is homogeneous of degree two, it follows that $\chi_{k+1}$ is homogeneous of degree $k + 1$. □
Remark 2.4. i. If \( \rho \) is independent of \( y \), the fact that \((I - \pi)\chi_0 = 0\) implies that \(\chi_2(y, \partial_t, \partial_x)\) does not depend on \(\partial_t\). ii. In this case an induction on \(k\) shows that for any \(k \geq 1\), \(\chi_k(y, \partial_t, \partial_x)\) contains only time derivatives of order \(\leq k - 2\).

The first structural result concerns formal power series \(U\) for which the oscillatory parts \((I - \pi)w_n\) vanish.

Theorem 2.5. Fix \(k \in \mathbb{Z}\) with \(k \geq -2\). For a formal power series \(U\) and corresponding \(W\) the following are equivalent.

i. For \(-2 \leq j \leq k\) one has

\[
(I - \pi)w_j = 0.
\]

ii. For \(0 \leq \ell \leq k + 2\) one has

\[
(I - \pi)u_\ell = \sum_{n=1}^\ell \chi_n\pi u_{\ell-n}.
\]

Proof. For \(k = -2\) the statement follows directly by recalling that \((I - \pi)w_{-2} = 0\) if and only if \((I - \pi)u_0 = 0\). For \(k = -1\) one has

\[
(I - \pi)w_{-1} = -(I - \pi)(\mathcal{A}_{yy}u_1 + \mathcal{A}_{xy}u_0).
\]

Lemma 2.1 implies that

\[
\pi \mathcal{A}_{yy} = 0, \quad \text{and,} \quad (1 - \pi)\mathcal{A}_{yy}u_k = \mathcal{A}_{yy}(1 - \pi)u_k \quad \text{for all} \quad k \geq 0.
\]

Using (2.12) along with \(u_0 = \pi u_0\) and multiplying (2.11) by \(-\mathcal{A}_{yy}^{-1}\) yields

\[
-\mathcal{A}_{yy}^{-1}(I - \pi)w_{-1} = (I - \pi)u_1 + \mathcal{A}_{yy}^{-1}(I - \pi)\mathcal{A}_{xy}\pi u_0 = (I - \pi)u_1 - \chi_1\pi u_0.
\]

Since \((I - \pi)w_{-1} = 0\) is equivalent to \(-\mathcal{A}_{yy}^{-1}(I - \pi)w_{-1} = 0\), this proves the case \(k = -1\) of the Theorem.

For \(k \geq 0\) reason by induction. Assume the case \(k - 1\) and prove the case \(k\). The induction hypothesis is

\[
(I - \pi)u_\ell = \sum_{n=1}^\ell \chi_n\pi u_{\ell-n} \quad \text{for} \quad 0 \leq \ell \leq k + 1,
\]

if and only if \((I - \pi)w_j = 0\) for \(-2 \leq j \leq k - 1\). For the inductive step need to treat \(j = k\) and \(l = k + 2\). For \(k \geq 0\) one has

\[
(I - \pi)w_k = -(I - \pi)\left(\mathcal{A}_{yy}u_{k+2} + \mathcal{A}_{xy}u_{k+1} + (\mathcal{A}_{xx} - \rho(y)\partial_t^2)u_k\right).
\]

Exploiting (2.12) and multiplying by \(-\mathcal{A}_{yy}^{-1}\) yields

\[
-\mathcal{A}_{yy}^{-1}(I - \pi)w_k = (I - \pi)u_{k+2} + \mathcal{A}_{yy}^{-1}(I - \pi)\left(\mathcal{A}_{xy}u_{k+1} + (\mathcal{A}_{xx} - \rho(y)\partial_t^2)u_k\right).
\]
Expressing each profile in (2.15) as a sum of its oscillatory and non-oscillatory part and recalling the definition $\chi_1 = -A_{yy}^{-1}(I - \pi)A_{xy}$, yields (2.15) as

$$-A_{yy}^{-1}(I - \pi)w_k = (I - \pi)u_{k+2} - \chi_1 \pi u_{k+1} + A_{yy}^{-1}(I - \pi) \left( (A_{xx} - \rho(y)\partial^2_t)\pi u_k \right)$$

(2.16)

$$+ A_{yy}^{-1}(I - \pi) \left( A_{xy}(I - \pi)u_{k+1} + (A_{xx} - \rho(y)\partial^2_t)(I - \pi)u_k \right).$$

Use that $(I - \pi)w_k = 0$ if and only if $A_{yy}^{-1}(I - \pi)w_k = 0$. Thus $(I - \pi)w_k = 0$ if and only if the right hand side of (2.16) vanishes. Using the induction hypothesis, (2.13) holds for $(I - \pi)u_{k+1}$ and $(I - \pi)u_k$. This yields

$$(I - \pi)u_{k+2} = \chi_1 \pi u_{k+1} - A_{yy}^{-1}(I - \pi) \left( (A_{xx} - \rho(y)\partial^2_t)\pi u_k \right)$$

$$- A_{yy}^{-1}(I - \pi) \left( A_{xy} \sum_{n=1}^{k+1} \chi_n \pi u_{k+1-n} + (A_{xx} - \rho(y)\partial^2_t) \sum_{n=1}^{k} \chi_n \pi u_{k-n} \right).$$

By definition $-A_{yy}^{-1}(I - \pi) \left( (A_{xx} - \rho(y)\partial^2_t)\pi u_k + A_{xy} \chi_1 \pi u_k \right) = \chi_2 \pi u_k$. Therefore

$$(I - \pi)u_{k+2}$$

$$= \chi_1 \pi u_{k+1} + \chi_2 \pi u_k - A_{yy}^{-1}(I - \pi) \left( A_{xy} \sum_{n=2}^{k+1} \chi_n \pi u_{k+1-n} + (A_{xx} - \rho(y)\partial^2_t) \sum_{n=1}^{k} \chi_n \pi u_{k-n} \right)$$

$$= \chi_1 \pi u_{k+1} + \chi_2 \pi u_k + \sum_{n=1}^{k} \left( -A_{yy}^{-1}(I - \pi) \right) \left[ A_{xy} \chi_{n+1} + (A_{xx} - \rho(y)\partial^2_t) \chi_n \right] \pi u_{k-n}$$

$$= \chi_1 \pi u_{k+1} + \chi_2 \pi u_k + \sum_{n=1}^{k+2} \chi_{n+2} \pi u_{k-2-n} + \sum_{n=1}^{k} \chi_n \pi u_{k+2-n},$$

where the last line uses the definition (2.6) of $\chi_{n+2}$. The last identity is the desired formula for $(I - \pi)u_{k+2}$. The proof is complete. \(\square\)

Remark 2.6. Theorem 2.3 has a particularly elegant form for profiles so that $(1-\pi)w_\ell = 0$ for all $\ell$. This holds if and only if the formal power series in $\epsilon$ for $u$ is given in terms of the series for $\pi u$ by

$$\sum_{n=0}^{\infty} \epsilon^n u_n = \left( \sum_{\ell=0}^{\infty} \epsilon^\ell \pi u_\ell \right) \left( \sum_{k=0}^{\infty} \epsilon^k \pi u_k \right).$$

The elliptic analogue was observed by Bakhvalov and Panasenko \[8\].
2.2.2. **The nonoscillatory hierarchy.** Next analyse the equations determining the non-oscillatory parts $\pi u_n$ of the profiles. Equations (2.2) and (2.3) are multiplied on the left by $\pi$. Since $\pi A_{yy} = 0$ and $\pi A_{xy} = 0$, one has $\pi w_{-2} = 0$ and $\pi w_{-1} = 0$. For $k \geq 0$ the $A_{yy}$ terms are eliminated and (2.3) with $k \geq 0$ simplifies to

\[(2.17) \quad \pi w_k = \pi \left[ \rho(y) \partial_t^2 u_k - A_{xx} u_k - A_{xy} u_{k+1} \right].\]

Exploiting (2.17) with $k = 0$, writing $u_1 = \pi u_1 + (1 - \pi) u_1$ and using the recurrence from (2.10) yields

\[\pi w_0 = a^*_2(\partial_t, \partial_x) \pi u_0\]

with the homogenized wave operator defined as

\[(2.18) \quad a^*_n(\partial_t, \partial_x) := (\pi \rho) \partial_t^2 - \text{div}_x (\pi a) \text{grad}_x + \pi A_{xy}(I - \pi) A^{-1}_{yy}(I - \pi) A_{xy} \pi.\]

**Remark 2.7.** The homogenized wave operator $a^*_2$ coincides with the formula from classical homogenization theory [8, 10, 16, 20].

**Definition 2.8.** Scalar partial differential operators $a^*_n(\partial_t, \partial_x)$ mapping functions of $t, x$ to functions of $t, x$ are defined for $n \geq 1$ by

\[(2.19) \quad a^*_n(\partial_t, \partial_x) = \pi \left[ (\rho(y) \partial_t^2 - A_{xx}) \chi_{n-2} - A_{xy} \chi_{n-1} \right].\]

**Remark 2.9.** i. The operators $a^*_n$ have constant coefficients. ii. The operator $a^*_n$ is homogeneous of degree $n$. iii. The symbol $a^*_n(\partial_t, \partial_x)$ contains only even powers of $\partial_t$. iv. The definitions of $\chi_0, \chi_{-1}$ imply that $a^*_1 = 0$.

**Theorem 2.10.** Suppose that the formal power series $U$ and corresponding $W$ satisfy the conditions of Theorem 2.5 for some $k \in \mathbb{Z}$ with $k \geq -2$. Then $\pi w_{-2} = \pi w_{-1} = 0$ and for $0 \leq j \leq k + 1$,

\[(2.20) \quad \pi w_j = \sum_{n=0}^{j} a^*_{n+2}(\partial_t, \partial_x) \pi u_{j-n}.\]

**Remark 2.11.** The result is particularly elegant for profiles so that $(1 - \pi) w_n = 0$ for all $n$. In that case the formal power series in $\epsilon$ for the residual is given in terms of the nonoscillating parts by

\[\sum_{j=0}^{\infty} \epsilon^j \pi w_j = \left( \sum_{n=0}^{\infty} \epsilon^n a^*_{n+2} \right) \left( \sum_{m=0}^{\infty} \epsilon^m \pi u_m \right).\]

The elliptic analogue was observed in [8].
Proof. The cases \( k = -2 \) and \( k = -1 \) have already been discussed above. Let \( k \geq 0 \) and fix \( 0 \leq j \leq k + 1 \). Using \( \pi A_{xy} \pi = 0 \) and \( \pi A_{yy} = 0 \) provides
\[
\pi w_j = \pi \left( (\rho(y)\partial_t^2 - A_{xx})\pi u_j + (\rho(y)\partial_t^2 - A_{xx})(I - \pi)u_j - A_{xy}(I - \pi)u_{j+1} \right).
\]
Since we assumed that the conditions of Theorem 2.5 hold for \( \pi \), Regrouping terms and recalling that \( \chi_\pi \)
\[
\pi w_j = \pi \left( (\rho(y)\partial_t^2 - A_{xx})\pi u_j + \pi \left( (\rho(y)\partial_t^2 - A_{xx})\chi_n \pi u_{j-n} - A_{xy} \sum_{n=1}^{j+1} \chi_n \pi u_{j+1-n} \right) \right)
\]
Regrouping terms and recalling that \( \chi_0 = I \) yields
\[
\pi w_j = \pi \sum_{n=0}^{j} \left( (\rho(y)\partial_t^2 - A_{xx})\chi_n - A_{xy}\chi_{n+1} \right) \pi u_{j-n}.
\]
By definition of the effective operators \( a^*_n \), Equation (2.22) is equivalent to
\[
\pi w_j = \sum_{n=0}^{j} a^*_n (\partial_t, \partial_x) \pi u_{j-n}.
\]
This completes the proof. \( \square \)

Remark 2.12. \( a^*_n \) is a homogeneous polynomial of degree \( n \) in \( (\partial_t, \partial_x) \). Formula (2.19) shows that the highest degree of \( \partial_t \) in \( a^*_n \) comes from \( \chi_{n-1} \) or \( \partial_t^2 \chi_{n-2} \). When \( \rho \) is independent of \( y \), Remark 2.4 yields that \( \chi_{n-1} \) and \( \chi_{n-2} \) are of degree \( \leq n - 3 \) and \( \leq n - 4 \), respectively, with respect to time \( t \). Therefore, when \( \rho \) is constant, \( a^*_n \) is of order \( \leq n - 2 \) in \( \partial_t \) for \( n > 2 \).

The next result shows that the equation (2.23) has half as many terms as it seems. The proof depends on a precise combinatorial formula for \( \chi_\pi \). The elliptic analogue of Theorem 2.13 was proved by a quite different variational argument in [22].

Theorem 2.13. For any odd \( n \geq 1 \), the homogenized operator \( a^*_n \) vanishes. That is for \( m \geq 0 \), \( a^*_{2m+1} = 0 \).

Proof. Introduce
\[
C_1 := -A_{yy}^{-1}(I - \pi)A_{xy} \quad \text{and} \quad C_2 := -A_{yy}^{-1}(I - \pi)(A_{xx} - \rho(y)\partial_t^2).
\]
The operator \( A_{yy}^{-1}(I - \pi) \) acts only on the \( y \) variable and is continuous from \( H^{-1}(\mathbb{T}^d) \) to \( H^1(\mathbb{T}^d) \). The operators \( C_j \) are homogeneous polynomials of degree \( j \) in \( (\partial_t, \partial_x) \), whose
coefficients are operators in $y$. In the proof we integrate by parts with respect to $y$ and not with respect to $t,x$. With these $C_j$, (2.6) yields

\begin{equation}
\chi_k = C_1 \chi_{k-1} + C_2 \chi_{k-2}, \quad k \geq 1.
\end{equation}

Replace $\chi_{k-1}$ and $\chi_{k-2}$ using the two earlier instances of the recurrence. Continuing, leads to an expression

\begin{equation}
\chi_k = W_k \chi_0,
\end{equation}

where only the earliest operator $\chi_0$ appears. Equation (2.24) implies that

\begin{equation}
W_k = C_1 W_{k-1} + C_2 W_{k-2}.
\end{equation}

Equation (2.25) implies that $W_k$ is the sum of all words written with the two "letters" $C_1$ and $C_2$ such that the number of letters satisfies $\#C_1 + 2 \#C_2 = k$. Each word is a homogeneous differential operator of degree $k$ in $\partial_x$. Separating the words into two groups, those that end in $C_1$ and those that end in $C_2$ implies that

\begin{equation}
W_k = W_{k-1} C_1 + W_{k-2} C_2.
\end{equation}

Denote with an exponent $T$ the $L^2(\mathbb{T}^d)$ adjoint. Integration by parts in $y$ shows that $A_{xy}^T = -A_{xy}$, while $A_{yy}$ and $A_{xx}$ are selfadjoint. Define operators

\begin{align*}
D_1 &:= -A_{xy} A_{yy}^{-1} (I - \pi) = -C_1^T, \\
D_2 &:= -(A_{xx} - \rho(\partial_t^2) A_{yy}^{-1} (I - \pi) = C_2^T.
\end{align*}

An induction shows that

\begin{equation}
W_k^T = (-1)^k Z_k \quad \text{with} \quad Z_k = D_1 Z_{k-1} + D_2 Z_{k-2}.
\end{equation}

Therefore $Z_k$ is the sum of all words written with the two letters $D_1$ and $D_2$ such that the number of letters satisfy $\#D_1 + 2 \#D_2 = k$.

Introduce $G := A_{yy}^{-1} (I - \pi)$ that satisfies

\begin{align*}
C_1 G &= GD_1, \quad C_2 G = GD_2, \quad W_k G = G Z_k.
\end{align*}

Since $\chi_0(y) = I$, definition (2.19) can be rewritten, by using $A_{xy}^T = -A_{xy}$, $(A_{xx} - \rho \partial_t^2)^T = (A_{xx} - \rho \partial_t^2)$ and $W_k^T = (-1)^k Z_k$, as

\begin{align*}
a_k^*(\partial_t, \partial_x) &= \int_{\mathbb{T}^d} \left( (\rho \partial_t^2 - A_{xx}) W_{k-2} \chi_0(y) - A_{xy} W_{k-1} \chi_0(y) \right) \chi_0(y) \, dy \\
&= \int_{\mathbb{T}^d} \left( W_{k-2} \chi_0(y) (\rho \partial_t^2 - A_{xx}) \chi_0(y) + W_{k-1} \chi_0(y) A_{xy} \chi_0(y) \right) \, dy \\
&= (-1)^k \int_{\mathbb{T}^d} \chi_0(y) \left( Z_{k-2} (\rho \partial_t^2 - A_{xx}) \chi_0(y) - Z_{k-1} A_{xy} \chi_0(y) \right) \, dy.
\end{align*}
The properties of $Z_k$ and $W_k$ imply that
\[-Z_{k-1}A_{xy} = (D_1Z_{k-2} + D_2Z_{k-3})A_{xy} = ((\rho(y)\partial_t^2 - A_{xx})GZ_{k-3} - A_{xy}GZ_{k-2})A_{xy}\]
\[= ((\rho(y)\partial_t^2 - A_{xx}W_{k-3}G - A_{xy}W_{k-2}G)A_{xy}\]
\[= (\rho(y)\partial_t^2 - A_{xx})W_{k-3}C_1 - A_{xy}W_{k-2}C_1.\]

Similarly,
\[Z_{k-2}(\rho(y)\partial_t^2 - A_{xx}) = (D_1Z_{k-3} + D_2Z_{k-4})(\rho(y)\partial_t^2 - A_{xx})\]
\[= ((\rho(y)\partial_t^2 - A_{xx}GZ_{k-4} - A_{xy}GZ_{k-3})(\rho(y)\partial_t^2 - A_{xx})\]
\[= ((\rho(y)\partial_t^2 - A_{xx}W_{k-4}G - A_{xy}W_{k-3}G)(\rho(y)\partial_t^2 - A_{xx})\]
\[= (\rho(y)\partial_t^2 - A_{xx})W_{k-4}C_2 - A_{xy}W_{k-3}C_2.\]

Summing yields
\[Z_{k-2}(\rho(y)\partial_t^2 - A_{xx}) - Z_{k-1}A_{xy} = (\rho(y)\partial_t^2 - A_{xx})(W_{k-3}C_1 + W_{k-4}C_2)\]
\[-A_{xy}(W_{k-2}C_1 + W_{k-3}C_2) + (\rho(y)\partial_t^2 - A_{xx})W_{k-2} - A_{xy}W_{k-1}.\]

Therefore
\[a_k^*(\partial_t, \partial_x) = (-1)^k \int_{\mathbb{T}^d} \chi_0(y)\left((\rho(y)\partial_t^2 - A_{xx})W_{k-2}\chi_0(y) - A_{xy}W_{k-1}\chi_0(y)\right)dy\]
\[= (-1)^k a_k^*(\partial_t, \partial_x).\]

For odd $k$, this implies $a_k^*(\partial_t, \partial_x) = 0$. \hfill \square

Consider the homogenization problem (1.1). The goal is to describe the behavior of the solution $u^\epsilon$ by investigating formal power series $U$ as in (1.2). The classical algorithm is to choose the series $U$ such that $W - f \sim 0$, i.e. to choose the profiles $u_n$ such that for all $t, x, y$
\[w_0(t, x, y) = f(t, x), \quad \forall 0 \neq n \geq -2, \quad w_n = 0.\]
The equation $W - f \sim 0$ is satisfied if and only if $\pi(W - f) \sim 0$ and $(I - \pi)(W - f) \sim 0$. The source term $f(t, x)$ is smooth and non oscillatory, $(I - \pi)f = 0$. For the power series $U$ it follows that $(I - \pi)w_k = 0$ for all $k \geq -2$. According to Theorem 2.5 such a power series satisfies
\[(2.26) \quad (I - \pi)u_{\ell} = \sum_{n=1}^{\ell} \chi_n \pi u_{\ell-n} \quad \text{for all } \ell \geq 0.\]

Next analyse the equations determining the non oscillatory parts $\pi u_n$ of the profiles. Theorem 2.10 shows that if the oscillatory parts satisfy (2.26), then
\[\pi w_j = \sum_{n=0}^{j} a_{n+2}^*(\partial_t, \partial_x)\pi u_{j-n}.\]
Theorem 2.13 implies that only terms of the same parity appear,
\[ \pi w_j = a^*_2(\partial_t, \partial_x)\pi u_j + \sum_{2n+k=j+2} a^*_2n(\partial_t, \partial_x)\pi u_k. \]

The classical algorithm is to set \( \pi w_0 = f \) and \( \pi w_j = 0 \) for \(-2 \leq j \neq 0\), which yields the following hierarchy of equations for the \( \pi u_k \),

\[
\begin{align*}
\epsilon^0 : & \quad a^*_2(\partial_t, \partial_x)\pi u_0 = f \\
\epsilon^1 : & \quad a^*_2(\partial_t, \partial_x)\pi u_1 = 0 \\
\epsilon^2 : & \quad a^*_2(\partial_t, \partial_x)\pi u_2 = -a^*_4(\partial_t, \partial_x)\pi u_0 \\
\epsilon^3 : & \quad a^*_2(\partial_t, \partial_x)\pi u_3 = -a^*_4(\partial_t, \partial_x)\pi u_1 \\
\epsilon^4 : & \quad a^*_2(\partial_t, \partial_x)\pi u_4 = -a^*_4(\partial_t, \partial_x)\pi u_2 - a^*_6(\partial_t, \partial_x)\pi u_0 \\
\epsilon^5 : & \quad a^*_2(\partial_t, \partial_x)\pi u_5 = -a^*_4(\partial_t, \partial_x)\pi u_3 - a^*_6(\partial_t, \partial_x)\pi u_1 \\
\epsilon^6 : & \quad a^*_2(\partial_t, \partial_x)\pi u_6 = -a^*_4(\partial_t, \partial_x)\pi u_4 - a^*_6(\partial_t, \partial_x)\pi u_2 - a^*_8(\partial_t, \partial_x)\pi u_0 \\
\epsilon^7 : & \quad a^*_2(\partial_t, \partial_x)\pi u_7 = -a^*_4(\partial_t, \partial_x)\pi u_5 - a^*_6(\partial_t, \partial_x)\pi u_3 - a^*_8(\partial_t, \partial_x)\pi u_1 \\
\end{align*}
\]

The \( \epsilon^0 \)-order equation yields the classical homogenized wave equation \( a^*_2(\partial_t, \partial_x)\pi u_0 = f \).

2.3. **Leap frog and secular growth.** The equations for the odd subscripts are decoupled from those with even subscripts. The equations repeat in pairs. This is the leap frog structure of the non oscillatory hierarchy. Starting with \( n = 1 \) one concludes by induction in steps of two, that \( \pi u_n = 0 \) for all odd \( n \).

The leap frog structure implies that secular growth is slow. Without the leap frog structure one would have \(|u_n| \lesssim t^n\) instead of the \( t^{n/2} \) in next theorem.

**Theorem 2.14 (Secular growth).** If there is a \( \ell > 0 \) so that \( f = 0 \) for \( t > \ell \); then for each non zero \( \alpha \in \mathbb{N}^{1+d}\setminus\{0\} \) and every \( k = 0, 1, 2, \ldots \) there exists a constant \( C \) depending on \( f, \alpha \) and \( k \) so that for all \( t \geq 0 \),

\[
\| \partial_{t,x}^k u_{2k}(t), \partial_{t,x}^k u_{2k+1}(t) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq C \langle t \rangle^k, \quad \langle t \rangle := \sqrt{1+t^2}.
\]

**Remark 2.15.** Estimate (2.28) provides a bound on the derivatives of the \( u_n \) but not on the \( u_n \) themselves. To estimate \( u_{2k} \) or \( u_{2k+1} \) use \( u = \int_0^t \partial_t u \, dt \) to find

\[
\| u_{2k}(t), u_{2k+1}(t) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} = O(\langle t \rangle^{k+1}).
\]

**Proof of Theorem.** The leading term \( u_0(t,x) \) satisfies

\[ a^*_2(\partial_t, \partial_x)\pi u_0 = f, \quad \pi u_0 = 0 \quad \text{for} \quad t < 0. \]

Since \( f \in C_0^{\infty}(\mathbb{R}; H^s(\mathbb{R}^d)) \) for all \( s \), it follows that for \( 0 \neq \alpha \in \mathbb{N}^{1+d} \), \( \partial_{t,x}^\alpha \pi u_0 \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d \times \mathbb{T}^d)) \). Since \( u_0 = \pi u_0 \) this finishes the analysis of \( u_0 \).

---

*The notation \( A \lesssim B \) means that there is a constant \( C > 0 \), independent of \( A \) and \( B \), so that \( A \leq C B \).*
One has \( \pi u_1 = 0 \). Equation (2.4) implies that the oscillatory part of \( u_1 \) satisfies
\[
\partial_{t,x}^\alpha (I - \pi) u_1 = - \partial_{t,x} A_{yy}^{-1} A_{xy} u_0 \in L^\infty (\mathbb{R}; L^2 (\mathbb{R}^d \times T^d)).
\]
This completes the analysis of \( u_1 \) and therefore the case \( k = 0 \) of the Theorem. The proof is by induction on \( k \). Assuming the result for indices \( \leq k \) it suffices to prove the case \( k + 1 \).

First estimate the \( \pi \) projections. Since \( 2(k + 1) + 1 \) is odd, \( \pi u_{2(k+1)+1} = 0 \). To estimate \( \pi u_{2k+2} \), use the equation
\[
a_2^* (\partial_t, \partial_x) \pi u_{2k+2} = - \sum_{2 \leq n \leq k+2} a_{2n}^* (\partial_t, \partial_x) \pi u_{2j}.
\]
The case \( k \) of (2.28) bounds the right hand side. Since \( a_{2n}^* \) is a sum of derivatives, the inductive hypothesis implies that for all \( \beta \) including \( \beta = 0 \),
\[
\| \partial_{t,x}^\beta \pi u_{2k+2} \|_{L^2 (\mathbb{R}^d \times T^d)} = O (\langle t \rangle^k).
\]
It is important that the right hand side of the equation determining \( a_2^* (\partial_t, \partial_x) \pi u_{2(k+1)} \) involves only derivatives of the earlier profiles and not the profiles themselves. The standard energy estimate for \( a_2^* (\partial_t, \partial_x) \) implies that for \( \alpha \neq 0 \),
\[
\| \partial_{t,x}^\alpha \pi u_{2k+2} \|_{L^2 (\mathbb{R}^d \times T^d)} = O (\langle t \rangle^{k+1}).
\]
It remains to estimate \( (I - \pi) u_n \) for \( n = 2k + 2 \) and \( 2k + 3 \).

Equation (2.5) with index \( 2k + 2 \) in place of \( k + 2 \) expresses \( (I - \pi) u_{2k+2} \) in terms of the profiles with indices \( \leq 2k + 1 \). Those profiles are \( O (\langle t \rangle^k) \) by the inductive hypothesis. This yields
\[
\| \partial_{t,x}^\alpha (I - \pi) u_{2k+2} \|_{L^2 (\mathbb{R}^d \times T^d)} = O (\langle t \rangle^k)
\]
an estimate stronger than the \( O (\langle t \rangle^{k+1}) \) required by the Theorem. Equation (2.5) with index \( 2k + 3 \) in place of \( k + 2 \) expresses \( (I - \pi) u_{2k+3} \) in terms of the profiles with indices \( \leq 2k + 2 \). Those with index \( \leq 2k + 1 \) are \( O (\langle t \rangle^k) \) by the inductive hypothesis. All the derivatives of the profile \( u_{2k+2} \) have just been shown to be \( O (\langle t \rangle^{k+1}) \). It follows that
\[
\| \partial_{t,x}^\alpha (I - \pi) u_{2k+3} \|_{L^2 (\mathbb{R}^d \times T^d)} = O (\langle t \rangle^{k+1}).
\]
\[\square\]

3. High accuracy on \( t \sim \epsilon^{-2+\delta} \) without crimes

This section is devoted to a proof of the correctness of the traditional two scale ansatz for times strictly smaller than \( \epsilon^{-2} \).
Theorem 3.1. For $k \in \mathbb{N}$, define a truncated ansatz, constructed from the first non-oscillating profiles $\pi u_0, \pi u_2, \ldots, \pi u_{2k}$ by

$$U^k(\epsilon, t, x, y) := \sum_{n=0}^{2k} \epsilon^n u_n(t, x, y) + \epsilon^{2k+1}(I - \pi)u_{2k+1} + \epsilon^{2k+2}(I - \pi)u_{2k+2}.$$ 

The approximate solution is $U^k(\epsilon, t, x, x/\epsilon)$. Denote by $u^\epsilon$ the exact solution of (1.1). There is a constant $C$, independent of $0 < \epsilon \leq 1$ and $t \geq 0$, so the energy of the error is bounded by

$$\| \nabla [u^\epsilon(t, x) - U^k(\epsilon, t, x, x/\epsilon)] \|_{L^2(\mathbb{R}^d)} \leq C \epsilon^{2k+1} \langle t \rangle^{k+1}. \quad (3.1)$$

Remark 3.2. i. The energy of $u^\epsilon$ is bounded independently of time so the right hand side of (3.1) estimates the relative energy error.

ii. By choosing $k$ large one gets arbitrarily high order accuracy on time intervals that grow as $1/\epsilon^2 - \delta$ for any $\delta > 0$. Indeed, on the time interval $0 \leq t \leq \epsilon^{-\gamma}$ with $\gamma(k+1) < 2k+1$ the relative error in energy is of order $\epsilon^{2k+1-\gamma(k+1)}$ and tends to zero as $\epsilon \to 0$.

iii. The problem (1.1) is invariant by differentiation in time. The derivative $\partial_t u^\epsilon$ is the solution of the same problem with source term $\partial_t f$. The profiles of the two scale asymptotic solution of that problem are equal to the functions $\partial_t u_n(t, x, y)$. Theorem 3.1 applied to that problem shows that with a constant $C$ depending on $j$ but independent of $t, \epsilon$,

$$\| \partial_t \nabla [u^\epsilon(t, x) - U^k(\epsilon, t, x, x/\epsilon)] \|_{L^2(\mathbb{R}^d)} \leq C \epsilon^{2k+1} \langle t \rangle^{k+1}. \quad (3.2)$$

The proof of Theorem 3.1 has three main ingredients. The first in 3.1 relies on all the work done so far. It is a precise formula for the difference between $f$ and $\rho(x/\epsilon)\partial_t^2 U^k - \text{div } a(x/\epsilon)\text{grad } U^k$. That difference has terms no more regular than $H^{-1}$. They are estimated in 3.2. The error in energy with such singular source terms is bounded using Proposition C.1.

3.1. Formula for the residual. Suppose that $U^k$ is the finite power series from Theorem 3.1. Then

$$\left(\rho(y)\partial_t^2 - A_{xx} - \frac{1}{\epsilon} A_{xy} - \frac{1}{\epsilon^2} A_{yy}\right) U^k(\epsilon, t, x, y) = W^k(\epsilon, t, x, y) = \sum_{n=-2}^{2k+2} \epsilon^n w_n(t, x, y). \quad (3.3)$$

The profiles $w_j$ satisfy

$$w_0 = f, \quad \text{and for } j \neq 0, -2 \leq j \leq 2k, \quad w_j = 0.$$ 

Indeed, for $j \leq 2k-2$ one has

$$w_j = (\rho(y)\partial_t^2 - A_{xx}) u_j - (A_{yy} u_{j+2} + A_{xy} u_{j+1}) = 0.$$
by construction of the profiles $u_j$. For $j = 2k - 1$ use the fact that $\pi u_{2k+1} = 0$ because $2k + 1$ is odd to find that

$$ w_{2k-1} = (\rho(y) \partial_t^2 - A_{xx})u_{2k-1} - (A_{yy}(I - \pi)u_{2k+1} + A_{xy}u_{2k}) $$

$$ = (\rho(y) \partial_t^2 - A_{xx})u_{2k-1} - (A_{yy}u_{2k+1} + A_{xy}u_{2k}) = 0. $$

For $j = 2k$ use $\pi u_{2k+1} = 0$ and $A_{yy} \pi = 0$ to find

$$ w_{2k} = (\rho(y) \partial_t^2 - A_{xx})u_{2k} - (A_{yy}(I - \pi)u_{2k+2} + A_{xy}(I - \pi)u_{2k+1}) $$

$$ = (\rho(y) \partial_t^2 - A_{xx})u_{2k} - (A_{yy}u_{2k+2} + A_{xy}u_{2k+1}) = 0. $$

Therefore

$$ \left[ \rho(y) \partial_t^2 - A_{xx} - \frac{1}{\epsilon} A_{xy} - \frac{1}{\epsilon^2} A_{yy} \right] U^k(\epsilon, t, x, y) - f $$

$$ = \epsilon^{2k+1} w_{2k+1}(t, x, y) + \epsilon^{2k+2} w_{2k+2}(t, x, y) =: r(\epsilon, t, x, y). $$

and thus

$$ \left[ \rho(x/\epsilon) \partial_t^2 - \text{div} \ a(x/\epsilon) \text{grad} \right] U^k(\epsilon, t, x, x/\epsilon) = f = r(\epsilon, t, x, x/\epsilon). $$

Equation (3.3) shows that

$$ w_{2k+1} = (\rho(y) \partial_t^2 - A_{xx})(I - \pi)u_{2k+1} - A_{xy}(I - \pi)u_{2k+2}, $$

$$ w_{2k+2} = (\rho(y) \partial_t^2 - A_{xx})(I - \pi)u_{2k+2}. $$

Therefore

$$ r = \left[ \rho(y) \partial_t^2 - A_{xx} \right] (I - \pi) \left( \epsilon^{2k+1} u_{2k+1} + \epsilon^{2k+2} u_{2k+2} \right) - \epsilon^{2k+1} A_{xy}(I - \pi)u_{2k+2}. $$

The definitions of the operators $A$ yield

$$ r = \epsilon^{2k+1} \left[ (\rho(y) \partial_t^2 - \text{div}_x a(y) \text{grad}_x) (I - \pi)u_{2k+1} + \epsilon \rho(y) \partial^2_t (I - \pi)u_{2k+2} \right] $$

$$ - \epsilon^{2k+1} \left( \text{div}_x a(y) \text{grad}_x + \text{div}_y a(y) \text{grad}_y + \epsilon \text{div}_x a(y) \text{grad}_y \right) (I - \pi)u_{2k+2} $$

$$ := \epsilon^{2k+1} \left( I(\epsilon, t, x, y) + II(\epsilon, t, x, y) \right). $$

In addition, for $\ell = 2k + 1, 2k + 2$, with $\chi_n$ given by (2.7),

$$ (I - \pi)u_\ell = \sum_{n=1, \ell-n \text{ even}} \chi_n \pi u_{\ell-n} = \sum_{n=1, \ell-n \text{ even}} \sum_{|\beta|=n} c_{\beta,n}(y) \partial^\beta_t \pi u_{\ell-n}. $$

3.2. Estimates for the residual. In view of (3.5) and Theorem 2.1, $I(\epsilon, t, x, y)$ in (3.4) is a sum of terms of the form $\epsilon^p c(y)v(t, x)$ with $p \geq 0$, $c \in L^2(\mathbb{T}^d)$ and

$$ \|\partial^\beta_t v\|_{L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^k $$

for $\alpha \in \mathbb{N}^{1+d}$, including $\alpha = 0$. 
Equation (3.5) shows that
\( II - y \) 
Evaluate at
The term \( II \) in \([3.4]\) involves derivatives of \( a(\cdot) \), so is not square integrable. Equation \((3.5)\) shows that \( II(\epsilon, t, x, y) \) is equal to
\[
- (\text{div}_x a(y) \text{grad}_y + \text{div}_y a(y) \text{grad}_x + \epsilon \text{div}_x a(y) \text{grad}_x) \left[ \sum_{n=1}^{2k+2} \sum_{|\beta|=n} c_{\beta,n}(y) \partial_{t,x}^\beta \pi u_{2k+2-n}(t, x) \right].
\]
Evaluate at \( y = x/\epsilon \) to show that, with \( \text{div} \) acting on functions depending on \( x/\epsilon \),
\[
II(\epsilon, t, x, x/\epsilon) = -\epsilon \sum_{n=1}^{2k+2} \sum_{|\beta|=n} \text{div} \left[ a(x/\epsilon)c_{\beta,n}(x/\epsilon) \text{grad} \partial_{t,x}^\beta \pi u_{2k+2-n}(t, x) \right]
\]
\[
+ \sum_{n=1}^{2k+2} \sum_{|\beta|=n} \left( a(x/\epsilon)(\text{grad}_x c_{\beta,n})(x/\epsilon) \cdot \text{grad} \partial_{t,x}^\beta \pi u_{2k+2-n}(t, x) \right)
\]
\[
:= \text{div} \left( II^{(1)}(\epsilon, t, x, x/\epsilon) \right) + II^{(2)}(\epsilon, t, x, x/\epsilon).
\]
Arguing exactly as for \( I \), using that each \( c_{\beta,n} \in H^1(\mathbb{T}^d) \), it follows that \( II^{(1)}(\epsilon, t, x, x/\epsilon) \), \( \partial_t II^{(1)}(\epsilon, t, x, x/\epsilon) \), and, \( II^{(2)}(\epsilon, t, x, x/\epsilon) \) are in \( L^2(\mathbb{R}^d) \) with
\[
\| \partial_t II^{(1)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} + \| II^{(1)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon \langle t \rangle^k
\]
\[
\| II^{(2)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^k.
\]
Combining estimate \((3.8)\) with \((3.9)\) yields the residual
\[
r(\epsilon, t, x, x/\epsilon) = f(\epsilon, t, x) + \text{div} g(\epsilon, t, x),
\]
with \( f(\epsilon, t, x) = \epsilon^{2k+1}(I + II^{(2)})(\epsilon, t, x, x/\epsilon) \) and \( g(\epsilon, t, x) = \epsilon^{2k+1}II^{(1)}(\epsilon, t, x, x/\epsilon) \) and the two estimates,
\[
\| f(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^{2k+1}(t)^k, \quad \| \partial_t g(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} + \| g(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^{2k+2}(t)^k.
\]

### 3.3. End of proof of Theorem 3.3

Denote by \( u^k \) the exact solution and \( U^k \) the approximation from the statement of Theorem 3.3. We have proved that
\[
\left[ \rho(x/\epsilon)\partial_t^2 - \text{div} a(x/\epsilon) \text{grad} \right] (u^k(t, x) - U^k(\epsilon, t, x, x/\epsilon)) := r(\epsilon, t, x, x/\epsilon)
\]
with \( r(\epsilon, t, x, x/\epsilon) \) satisfying \((3.9)\) and \((3.10)\). Apply Proposition [C.1] in the estimate of that Proposition, the \( L^1([0, t]; L^2(\mathbb{R}^d)) \) norms of \( f \) and \( \partial_t g \) are estimated by \( t \) times the \( L^\infty([0, t]; L^2(\mathbb{R}^d)) \) norms, which are controlled by \((3.10)\). This yields the claimed result \((3.11)\). \( \square \)
Remark 3.3. The results concerning the two scale expansions extend with only minor changes in the proofs to the case of coefficients $a(x, x/\epsilon)$ provided that for all $\beta$, $\partial^2_{\beta} a(x, y) \in L^\infty(\mathbb{R}^d \times \mathbb{T}^d)$. In this case the correctors have coefficients $c_{\alpha}(x, y)$ and Lemma 2.3 asserts $\partial^3_{x} c_{\alpha} \in L^\infty(\mathbb{R}^d; H^1(\mathbb{T}^d))$. The proofs of the leap frog structure and slow secular growth are unchanged. The residual estimate for the term $\text{II}$ in (3.7) has a few additional terms treated using this regularity of $c$. For the criminal path, the case $a(x, x/\epsilon)$ is work in progress.

4. The criminal path

The criminal path, briefly presented in Section 1.2, yields approximations valid for times as long as $\epsilon^{-N}$ for arbitrary $N$.

Main idea. The criminal path changes the choice of the nonoscillatory parts $\pi u_n$. The oscillatory parts $(1 - \pi)u_n$ are given in terms of the nonoscillatory parts by (2.10) as in classical homogenization.

We replace the traditional ansatz (1.2) for $U$ by the criminal ansatz (1.10) for $V$. Since the terms $v_n$ in (1.10) depend on $\epsilon$, we commit the asymptotic crime of mixing different orders in $\epsilon$. The terms of order $\epsilon^2$ in the criminal path are introduced in different but related ways in the seminal articles [8], [21], [17].

4.1. Derivation of criminal equations. According to Definition 1.1, the criminal ansatz satisfies $v_0 = \pi v_0$, $\pi v_n = 0$ for $n \geq 1$ and $(I - \pi)v_n = \chi_n v_0$. The leading term $v_0 = \pi v_0$ and profile $V(\epsilon, t, x, y)$ are constructed so that the two formal identities

$$(4.1) \quad \left(a_2^* (\epsilon \partial_{t,x}) + \cdots + a_{2n}^* (\epsilon \partial_{t,x}) + \cdots \right) v_0(\epsilon, t, x) = \epsilon^2 f, \quad v_0 = 0 \text{ for } t < 0,$$

$$(4.2) \quad V(\epsilon, t, x, y) = \left(1 + \sum_{l=1}^{\infty} \epsilon^l \chi_l(y, \partial_{t,x})\right) v_0(\epsilon, t, x),$$

are satisfied up to an acceptable error. Even if (4.1) is truncated to be a finite sum, it is high order in $t$. For each $\epsilon$ it usually defines an ill posed time evolution. In spite of this, the next sections construct functions $v^k$ that satisfy (4.1) with small enough error.

Equation (4.1) can be understood in another way. Theorems 2.5 and 2.10 together with their remarks show that standard homogenization hierarchy is equivalent to the pair of identities in the sense of formal power series,

$$(\sum_{n=1}^{\infty} a_{2n}^* (\epsilon \partial_{t,x}) \left(\sum_{m=0}^{\infty} \epsilon^m \pi u_m \right) = \epsilon^2 f, \quad U = \left(1 + \sum_{l=1}^{\infty} \epsilon^l \chi_l\right) \left(\sum_{n=0}^{\infty} \epsilon^n \pi u_n \right).$$
Proposition 4.1. There are uniquely determined homogeneous operators $R_{2j}(\partial_{t,x})$ and $\tilde{a}_{2j}(\partial_x)$ of degree $2j$, the latter involving only $\partial_x$, so that (4.11) holds as an identity in the sense of formal power series.

The heart of the proof is the following Lemma.

Lemma 4.2. Suppose that $m \geq 2$ and $S_{2m}(\partial_{t,x})$ is homogeneous of degree $2m$ and contains only even powers of $\partial_t$. Then there exists a unique $r_{2m-2}(\partial_{t,x})$, homogeneous of degree $2m - 2$, so that $r_{2m-2}a_2^* + S_{2m}$ is a differential operator in $\partial_x$ only.

Proof. Write

$$r_{2m-2}(\partial_{t,x}) = q_0\partial_t^{2m-2} + q_2(\partial_x)\partial_t^{2m-4} + \cdots + q_{2m-4}(\partial_x)\partial_t^2 + q_{2m-2}(\partial_x).$$

The goal is to determine $q_0, \ldots, q_{2m-2}$ in such a way that $r_{2m-2}a_2^* + S_{2m}$ is a differential operator in $\partial_x$ only. Order the terms in $S_{2m}$ according to the order of the time derivative

$$S_{2m} = s_0(\partial_x)\partial_t^{2m} + s_2(\partial_x)\partial_t^{2m-2} + \cdots + s_{2m-2}(\partial_x)\partial_t^2.$$

Define $\rho := \pi \rho$ and $a_2(\partial_x)$ so that $a_2^*$ from (2.18) satisfies

$$a_2^*(\partial_{t,x}) = \rho \partial_t^2 + a_2(\partial_x).$$

(4.4)

In particular $a_2(\partial_x)$ is second order in $\partial_x$. Then the terms containing time derivatives in $r_{2m-2}a_2^*$ are equal to

$$\rho\left(q_0\partial_t^{2m} + q_2(\partial_x)\partial_t^{2m-2} + \cdots + q_{2m-4}(\partial_x)\partial_t^4 + q_{2m-2}(\partial_x)\partial_t^2\right)$$

$$+ \left((q_0a_2)(\partial_x)\partial_t^{2m-2} + (q_2a_2)(\partial_x)\partial_t^{2m-4} + \cdots + (q_{2m-4}a_2)(\partial_x)\partial_t^4 + (q_{2m-2}a_2)(\partial_x)\partial_t^2\right).$$

(4.5)
Regrouping in order of decreasing powers of $\partial_t$ yields that (4.5) equals
\[
pq 0 \partial_t^{2m} + \left(pq_2 + q_0 a_2\right) (\partial_x) \partial_t^{2m-2} + \cdots \\
+ \left(pq_{2m-4} + q_{2m-6} a_2\right) (\partial_x) \partial_t^2 + \left(pq_{2m-2} + q_{2m-4} a_2\right) (\partial_x) \partial_t^4.
\]
The unique choice eliminating the time derivatives in $r_{2m-2}a_x^2 + S_{2m}$ is given by
\[
q_0 = -\frac{1}{\rho} s_0, \quad \text{and for } 1 \leq j \leq m-1, \quad q_{2j}(\partial_x) = -\frac{1}{\rho} \left(s_{2j}(\partial_x) + (q_{2j-2} a_2)(\partial_x)\right).
\]
This completes the proof of Lemma 4.2.

**Definition 4.3.** Denote by $O_N$ the set of constant coefficient partial differential operators in $\partial_t, x$ that are sums of terms homogeneous of degree at least $N$.

The operators in $O_N$ are those whose symbols vanish to order $N$ at the origin.

**Proof of Proposition 4.1.** In the next expressions $O_N$ represents an element of $O_N$. The identity (1.11) holds if and only if for all $k \geq 2$
\[
(1 + R_2 + R_4 + \cdots + R_{2k-2}) (a_x^2 + a_4^2 + \cdots + a_{2k}^2) = a_x^2 + \tilde{a}_4 + \cdots + \tilde{a}_{2k} + O_{2k+2}.
\]
The goal is to find $R_{2j}$ such that (4.6) holds. For $k = 2$ expanding yields
\[
(1 + R_2) (a_x^2 + a_4^2) = a_x^2 + R_2 a_x^2 + a_4^2 + O_6.
\]
The term of order 4 is $R_2 a_x^2 + a_4^2$. Choose $R_2$ using Lemma 4.2 as the unique homogeneous order 2 operator so that this fourth order term is independent of $\partial_t$. Denote by $\tilde{a}_4$ that differential operator.

The construction is recursive. Suppose that the $R_2, \ldots, R_{2k-2}$ and $\tilde{a}_4, \ldots, \tilde{a}_{2k}$ have been uniquely determined so that (4.6) holds. We show that $R_{2k}$ and $\tilde{a}_{2k+2}$ are uniquely determined so that the case $k + 1$ of (4.6) is satisfied.

In the case $k + 1$ of (4.6) the terms of order $\leq 2k$ on the right are only influenced by $R_2, \ldots, R_{2k-2}$. Separating the lowest order term in $O_{2k+2}$ the right hand side of (4.6) can be written as
\[
a_x^2 + \tilde{a}_4 + \cdots + \tilde{a}_{2k} + p_{2k+2} + O_{2k+4},
\]
where $p_{2k+2}(\partial_{t,x})$ is homogeneous of degree $2k + 2$. To prove the case $k + 1$ one must determine $R_{2k}$ and $\tilde{a}_{2k+2}$ such that
\[
(1 + R_2 + R_4 + \cdots + R_{2k-2} + R_{2k}) (a_x^2 + a_4^2 + \cdots + a_{2k+2}^2) = a_x^2 + \tilde{a}_4 + \cdots + \tilde{a}_{2k+2} + O_{2k+4}.
\]
The term of order $2k + 2$ is $R_{2k} a_x^2 + p_{2k+2}$, where $p_{2k+2}$ is given by (4.8), in terms of the $R_{2j}, \tilde{a}_{2j}$ that are known from the inductive step. Lemma 4.2 shows that there is a unique $R_{2k}$ so that this term of order $2k + 2$ is independent of $\partial_t$. That is the uniquely determined $R_{2k}$ and the operator in $\partial_x$ is $\tilde{a}_{2k+2}$. The recursive construction is complete. \(\square\)
Remark 4.4. The proof yields a recursive algorithm to compute $R_{2j}, \tilde{a}_{2j}$ from the $a^*_2$. The computation of the coefficients of $a^*_2$ requires the solution of $\sim d^{2j}$ cell problems.

Remark 4.5. Proposition 4.1 implies that if \( \epsilon \) instability does not doom the construction of good approximations. Committing an error, the operator in brackets on the left frequently defines an ill posed time evolution. This requires the solution of \( R_{2j} \) cell problems.

Choose cutoff functions \( \psi_j \in C_0^\infty(\mathbb{R}^d) \) for \( j = 1, 2 \) with \( \psi_1 = 1 \) on a neighborhood of the origin and \( \psi_2 = 1 \) on a neighborhood of \( \text{supp} \psi_1 \). Choose \( 0 < \alpha < 1 \). We compute a profile \( \psi_0^k \) that satisfies with \( D := (1/i) \partial_x \),

\[
\left[ a^*_2(\partial_t, \partial_x) + \epsilon^2 \tilde{a}_4(\partial_x) + \cdots + \epsilon^{2k} \tilde{a}_{2k+2}(\partial_x) \right] \psi_0^k = \psi_1(\epsilon^{\alpha} D \left[ 1 + R^k(\epsilon \partial_{t,x}) \right]) f.
\]
The ill posed evolutions remain. However, for the filtered sources on the right in \((4.11)\), there exist nice solutions. Fourier transformation in \(x\) yields ordinary differential equations in time parametrized by \(\xi\) for any tempered solution of \((4.11)\). It shows that a tempered solution must have transform with support in \(\epsilon\psi\). Such a solution satisfies \(\psi_2(\epsilon^a D)v_0^k = v_0^k\). Therefore it also satisfies
\[
(4.12) \quad \left[ a_2^*{\partial}_t - a_4^*{\partial}_x + \cdots + \epsilon^2 a_{2k+2}(\partial_x) \right] \psi_2(\epsilon^a D) v_0^k = \psi_1(\epsilon^a D) \left( 1 + R^k(\epsilon{\partial}_x) \right)f.
\]

4.4. Stability Theorem. The operator applied to \(v_0^k\) in \((4.12)\) is,
\[
a_2^*(\partial_t, \partial_x) + M(\epsilon, k, \partial_x) + M(\epsilon, \xi, \partial_x) := \sum_{j=2}^{k+1} \epsilon^{2j-2} a_{2j}(\partial_x) \psi_2(\epsilon^a D), \quad 0 < \alpha < 1.
\]

The operator \(M\) also depends on \(\alpha\) and \(\psi_2\).

**Theorem 4.7.** There is an \(\epsilon_0 > 0\) so that for each \(\epsilon \leq \epsilon_0, 0 < \alpha < 1, k \in \mathbb{N}\), and \(g_0, g_1 \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) there is a unique solution \(v\) with \(\partial_t^j v \in C(\mathbb{R}; H^{1-j}(\mathbb{R}^d))\) for \(j \geq 0\) to
\[
\left[ a_2^*(\partial_t, \partial_x) + M(\epsilon, k, \partial_x) \right] v = 0, \quad v(0, \cdot) = g_0, \quad \partial_t v(0, \cdot) = g_1.
\]

This solution satisfies with a constant \(C\) independent of \(\epsilon\) and \(v\),
\[
\sup_{t \in \mathbb{R}} \left( \|\nabla_x v(t)\|_{L^2(\mathbb{R}^d)} + \|\partial_t v(t)\|_{L^2(\mathbb{R}^d)} \right) \leq C \left( \|\nabla_x v(0)\|_{L^2(\mathbb{R}^d)} + \|\partial_t v(0)\|_{L^2(\mathbb{R}^d)} \right).
\]

For any \(\alpha_0 < 1\), the bound is uniform for \(0 < \alpha \leq \alpha_0\).

**Remark 4.8.** Duhamel’s principle implies that there is a constant \(C\) so that the unique tempered solution of
\[
\left[ a_2^*(\partial_t, \partial_x) + M(\epsilon, k, \partial_x) \right] v = f, \quad v(0, \cdot) = \partial_t v(0, \cdot) = 0,
\]
satisfies for all \(t, f, \|\nabla_{t,x} v(t)\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^1([0,\epsilon]; L^2(\mathbb{R}^d))}.
\]

**Proof of Theorem 4.7.** \(M\) is bounded from \(H^s \rightarrow H^{s+\sigma}\) for all \(s, \sigma\) with bound independent of \(s\). The bound tends to infinity as \(\epsilon \rightarrow 0\). The boundedness implies the existence statement of the Theorem. That the solutions are bounded independently of \(\epsilon\) is more subtle. This is shown below.

With the notation from \((4.3)\), the equation \((a_2^* + M)v = 0\) has the form \(\sum v_{it} = -\mu(D)v\) with
\[
(4.13) \quad -\mu_v(\xi) := a_2(i\xi) + \psi_2(\epsilon^a \xi) \sum_{2 \leq j \leq k+1} \epsilon^{2j-2} \tilde{a}_{2j}(i\xi).
\]
Each summand on the right is real valued. Theorem 4.7 follows from the following estimate. For each \( \alpha \in [0, 1] \) there is an \( \epsilon_0 > 0 \) and constants \( 0 < c < C \) so that for all \( 0 < \epsilon \leq \epsilon_0, 0 < \alpha \leq \alpha_0 \) and all \( \xi \in \mathbb{R}^d \),
\[
(4.14) \quad c \epsilon |\xi|^2 \leq \mu_{\epsilon}(\xi) \leq C |\xi|^2.
\]
To prove (4.14) it suffices to show that the modulus of the second summand on the right hand side of (4.13) is much smaller than the modulus of the first. In the support of the second summand \( |\xi| \lesssim \epsilon^{-\alpha} \). For such \( \xi \) it holds that
\[
|\epsilon^{2j-2}\tilde{a}_{2j}(i\xi)| \lesssim \epsilon^{2j-2}|\xi|^{2j} = (\epsilon|\xi|)^{2j-2}|\xi|^2 \lesssim \epsilon^{(1-\alpha)(2j-2)}|\xi|^2.
\]
The first factor on the right tends to zero as \( \epsilon \to 0 \), since \( 0 < \alpha < 1 \) and \( j \geq 2 \). This proves the desired inequality.

The spatial Fourier transform of the solution satisfies
\[
\frac{\rho}{2} \frac{\partial^2 \hat{v}}{\partial t^2} + \mu_{\epsilon}(\xi) \hat{v} = 0.
\]
Multiplying by the complex conjugate of \( \partial_t \hat{v} \) and taking the real part proves the conservation laws
\[
\forall \xi \in \mathbb{R}^d, \quad \partial_t \left( \frac{\rho}{2} |\partial_t \hat{v}(t, \xi)|^2 + \mu_{\epsilon}(\xi) |\hat{v}(t, \xi)|^2 \right) = 0.
\]
The estimate (4.14) implies that the conserved quantity
\[
\int_{\mathbb{R}^d} \left( \frac{\rho}{2} |\partial_t \hat{v}(t, \xi)|^2 + \mu_{\epsilon}(\xi) |\hat{v}(t, \xi)|^2 \right) \, d\xi
\]
is uniformly equivalent to \( \|\partial_t v(t)\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla_x v(t)\|_{L^2(\mathbb{R}^d)}^2 \). This completes the proof. \( \square \)

**Corollary 4.9.** Let \( 0 < \alpha_0 < 1, k \in \mathbb{N} \) and \( f \in H^\infty(\mathbb{R} \times \mathbb{R}^d) \) supported in \( [0, 1] \times \mathbb{R}^d \). Then for and all \( 0 < \alpha \leq \alpha_0 \) and \( 0 < \epsilon < \epsilon_0 \) there is a unique solution \( \hat{\zeta} \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}^d)) \) of (4.11). It satisfies \( \text{supp} \hat{\zeta} \subset \epsilon^{-\alpha} \text{supp} \psi_1 \) and
\[
\sup_{t \in [0, \infty]} \left\| \partial_t^\beta \zeta(t) \right\|_{L^2(\mathbb{R}^d)} \leq C(k, f, \beta) < \infty, \quad \beta \neq 0,
\]
\[
(4.16) \quad \left\| \zeta(t) \right\|_{L^2(\mathbb{R}^d)} \leq C(k, f) \langle t \rangle.
\]
**Proof. Uniqueness.** Taking the Fourier transform shows that any solution \( \zeta \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}^d)) \) to (4.11) satisfies \( \text{supp} \hat{\zeta} \subset \epsilon^{-\alpha} \text{supp} \psi_1 \). Therefore \( \zeta \) also satisfies (4.12). The solutions of those equations are uniquely determined thanks to Theorem 4.7.

**Existence.** Define \( \hat{\zeta} \) as the solution to (4.11). The same Fourier transform argument as for uniqueness shows that this function satisfies \( \text{supp} \hat{\zeta} \subset \epsilon^{-\alpha} \text{supp} \psi_1 \). In particular \( \psi_2(e^{\alpha}D)\zeta = \zeta \). This implies that \( \zeta \) satisfies (4.11). Theorem 4.7 implies that it has the additional properties claimed in Corollary 4.9 establishing existence. \( \square \)
4.5. Criminal approximation error. This section performs the computations that are the main ingredients in the proof of Theorem 4.2.

Remark 4.10. The profile \( v^k_0 = \pi v^k_0 \) from Definition 4.1 satisfies \( \psi_2(\epsilon^\alpha D)v^k_0 = v^k_0 \) so is the unique tempered solution supported in \( t \geq 0 \) of

\[
(4.17) \quad \left[ a^*_2(\partial_{t,x}) + \left( \sum_{j=2}^{k+1} \epsilon^{2j-2}a_{2j}(\partial_{t,x}) \right) \psi_2(\epsilon^\alpha D) \right] \pi v^k_0 = \left[ 1 + R^k(\epsilon^{\partial_{t,x}}) \right] \psi_1(\epsilon^\alpha D)f.
\]

(4.17) computes a precise formula for the residual. The entire paper prepares that computation.

4.5.1. Formula for the residual. Recall the criminal approximation \( V^k \) from (1.15).

\[
V^k(\epsilon, t, x, y) = \sum_{n=0}^{2k+2} \epsilon^n v^n_0(\epsilon, t, x, y) = \left( I + \sum_{n=1}^{2k+2} \epsilon^n \chi_n(y, \partial_t, \partial_x) \right) v^k_0(\epsilon, t, x).
\]

Define

\[
Z(\epsilon, t, x, y) := \left[ \rho(y)\partial_t^2 - \frac{1}{\epsilon^2} A_{yy} - \frac{1}{\epsilon} A_{xy} - A_{xx} \right] V^k(\epsilon, t, x, y).
\]

This is regrouped in powers of \( \epsilon \) as if the \( v^n_0 \) did not depend on \( \epsilon \). This yields

\[
(4.19) \quad Z = \sum_{j=-2}^{2k+2} \epsilon^j Z_j = \sum_{j=-2}^{2k} \epsilon^j Z_j + \sum_{j=2k+1}^{2k+2} \epsilon^j Z_j =: \sum_{j=-2}^{2k} \epsilon^j Z_j + \mathcal{E}_1.
\]

The term \( \mathcal{E}_1 \) is the first error term. It is estimated in Lemma 4.11. Theorem 2.5 shows that the definition of the nonoscillatory parts of the \( v^n_0 \) is equivalent to

\[
(4.20) \quad (I - \pi) Z_n = 0, \quad -2 \leq n \leq 2k.
\]

Since \( \pi v^n_0 = 0 \) for \( 1 \leq n \leq 2k + 2 \) and \( v^n_0 = \pi v^n_0 \), Theorem 2.10 yields

\[
\pi \sum_{j=-2}^{2k} \epsilon^j Z_j = \sum_{j=0}^{2k} \epsilon^j a_{j+2}^*(\partial_{t,x})v^k_0 = \epsilon^{-2} \sum_{j=0}^{2k} \epsilon^j a_{j+2}^*(\epsilon^{\partial_{t,x}})v^k_0.
\]

Equation (4.10) of Corollary 4.6 implies

\[
\pi \sum_{j=-2}^{2k} \epsilon^j Z_j = \epsilon^{-2}(1 + \tilde{R}^k(\epsilon^{\partial_{t,x}})) \left[ a^*_2(\epsilon^{\partial_{t,x}}) + \sum_{n=2}^{k+1} a_{2n}(\epsilon^{\partial_{t,x}}) \right] v^k_0 + \epsilon^{-2}\mathcal{O}_{2k+4}(\epsilon^{\partial_{t,x}})v^k_0 + \epsilon^{-2}\mathcal{O}_{2k+4}(\epsilon^{\partial_{t,x}})v^k_0.
\]

\[
\pi \sum_{j=-2}^{2k} \epsilon^j Z_j = \epsilon^{-2}(1 + \tilde{R}^k(\epsilon^{\partial_{t,x}})) \left[ a^*_2(\epsilon^{\partial_{t,x}}) + \sum_{n=2}^{k+1} \epsilon^{2n-2} a_{2n}(\partial_{x}) \right] v^k_0 + \epsilon^{-2}\mathcal{O}_{2k+4}(\epsilon^{\partial_{t,x}})v^k_0.
\]

(That is, the computations are made in the ring of Laurent expansions in \( \epsilon \) whose coefficients are functions of \( \epsilon, t, x, y \). In \( \epsilon^n v^n_0 \), the function \( v^n_0 \) is a coefficient of \( \epsilon^n \). If for instance \( v^n_0 = \epsilon^2 \) the power from \( v^n_0 \) must not be combined with the \( \epsilon^n \), the expression \( \epsilon^n v^n_0 \) is still a term in \( \epsilon^n \).)
Since \( v^k \) satisfies equation (1.11),
\[
\pi \sum_{j=-2}^{2k} e^j Z_j = (1 + \tilde{R}^k(\epsilon \partial_{t,x})) \psi_1(\epsilon^a D)(1 + R^k(\epsilon \partial_{t,x})) f + \epsilon^{-2} O_{2k+4}(\epsilon \partial_{t,x}) v^k_0
\]
\[
= (1 + \tilde{R}^k(\epsilon \partial_{t,x}))(1 + R^k(\epsilon \partial_{t,x})) \psi_1(\epsilon^a D)f + \mathcal{E}_2.
\]

Use \((1 + \tilde{R}^k(\epsilon \partial_{t,x}))(1 + R^k(\epsilon \partial_{t,x})) = 1 + O_{2k+2}\) to continue the computation,
\[
(1 + \tilde{R}^k(\epsilon \partial_{t,x}))(1 + R^k(\epsilon \partial_{t,x})) \psi_1(\epsilon^a D)f = (1 + O_{2k+2}(\epsilon \partial_{t,x})) \psi_1(\epsilon^a D)f
\]
\[
= \psi_1(\epsilon^a D)f + \mathcal{E}_3
\]
\[
= f + (\psi_1(\epsilon^a D) - 1)f + \mathcal{E}_3
\]
\[
= f + \mathcal{E}_4 + \mathcal{E}_3.
\]

Therefore,
\[
Z(\epsilon, t, x, y) - f(t, x) = \mathcal{E}_1(\epsilon, t, x, y) + \sum_{j=2}^{4} \mathcal{E}_j(\epsilon, t, x).
\]

with
\[
\mathcal{E}_1 = \mathcal{E}_2 = \epsilon^{-2} O_{2k+4}(\epsilon \partial_{t,x}) v^k_0,
\]
\[
\mathcal{E}_3 = O_{2k+2}(\epsilon \partial_{t,x}) \psi_1(\epsilon^a D)f,
\]
\[
\mathcal{E}_4 = (\psi_1(\epsilon^a D) - 1)f.
\]

4.6. Residual estimates and proof of Theorem 1.2. The error \( u^\epsilon(t, x) - V^k(\epsilon, t, x, x/\epsilon) \) satisfies
\[
\left[ \rho(x/\epsilon) \partial_t^2 - \text{div} a(x/\epsilon) \text{grad} \right] \left( u^\epsilon(t, x) - V^k(\epsilon, t, x, x/\epsilon) \right)
\]
\[
= \mathcal{E}_1(\epsilon, t, x, x/\epsilon) + \sum_{j=2}^{4} \mathcal{E}_j(\epsilon, t, x).
\]

Lemma 4.11. The error term \( \mathcal{E}_1(\epsilon, t, x, x/\epsilon) \) from (4.22) is of the form
\[
\mathcal{E}_1(\epsilon, t, x, x/\epsilon) = f(\epsilon, t, x) + \text{div} g(\epsilon, t, x)
\]
with \( f, g \) satisfying uniformly in \( t \geq 0 \),
\[
\| f(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} + \| \partial_t g(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} + \| g(\epsilon, t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq \epsilon^{2k+1}.
\]

Proof. As for the residual in the non criminal approximation write
\[
\mathcal{E}_1(\epsilon, t, x, y) = \epsilon^{2k+1} Z_{2k+1}(t, x, y) + \epsilon^{2k+2} Z_{2k+2}(t, x, y)
\]
\[
= \left[ \rho(y) \partial_t^2 - \mathcal{A}_{xx} \right] (I - \pi) \left( \epsilon^{2k+1} v^k_{2k+1} + \epsilon^{2k+2} v^k_{2k+2} \right) - \epsilon^{2k+1} \mathcal{A}_{xy}(I - \pi) v^k_{2k+2}.
\]

Since \( (I - \pi)v_j^k = \chi_j \pi v^k_0 \) for \( 1 \leq j \leq 2k+2 \) and \( \chi_j = \sum_{|\beta|=j} c_{\beta,j}(y) \partial^\beta_{t,x} \), it follows that
\[
\mathcal{E}_1(\epsilon, t, x, y) = \epsilon^{2k+1} (I(\epsilon, t, x, y) + II(\epsilon, t, x, y)),
\]
where
\[ I(\epsilon, t, x, y) := (\rho(y) \partial_t^2 - \text{div}_x a(y) \text{grad}_x) \sum_{|\beta|=2k+1} c_{\beta,2k+1}(y) \partial_{t,x}^\beta \pi v_0^k(\epsilon, x, t) \]
\[ + \epsilon \rho(y) \partial_t^2 \sum_{|\beta|=2k+2} c_{\beta,2k+2}(y) \partial_{t,x}^\beta \pi v_0^k(\epsilon, t, t), \]
\[ II(\epsilon, t, x, y) := -\left( \text{div}_x a(y) \text{grad}_y + \text{div}_y a(y) \text{grad}_x + \epsilon \text{div}_x a(y) \text{grad}_x \right) \]
\[ \times \sum_{|\beta|=2k+2} c_{\beta,2k+2}(y) \partial_{t,x}^\beta \pi v_0^k(\epsilon, t, t). \]

Use that \( c_{\beta,2k+1}, c_{\beta,2k+2} \in H^1(\mathbb{T}^d_y) \) to show that \( I(\epsilon, t, x, y) \) is a sum of terms of the form \( \epsilon^p c(y) v(t, x) \) with \( p \geq 0, c \in L^2(\mathbb{T}^d) \). Corollary 4.9 implies that each \( v \) satisfies, uniformly in \( t \geq 0 \),
\[ \| \partial_{t,x}^\alpha v(t, x) \|_{L^2(\mathbb{R}^d)} \lesssim 1 \]
for \( \alpha \in \mathbb{N}^{1+d} \), including \( \alpha = 0 \). Proposition B.1 in the appendix implies that uniformly for \( t \geq 0 \),
\[ \| I(\epsilon, t, x, y) \|_{L^2(\mathbb{R}^d)} \lesssim 1. \]

The second term, \( II(\epsilon, t, x, y) \), involves derivatives of \( a(\cdot) \). As in (3.7), write
\[ II(\epsilon, t, x, t/\epsilon) = -\epsilon \text{div} \left[ a(x/\epsilon) \sum_{|\beta|=2k+2} c_{\beta,2k+2}(x/\epsilon) \text{grad} \partial_{t,x}^\beta \pi v_0^k(\epsilon, t, x) \right] \]
\[ + a(x/\epsilon) \sum_{|\beta|=2k+2} \left( \text{grad}_y c_{\beta,2k+2}(x/\epsilon) \cdot \text{grad} \partial_{t,x}^\beta \pi v_0^k(\epsilon, t, x) \right) \]
\[ := \text{div} \left( II^{(1)}(\epsilon, t, x, x/\epsilon) \right) + II^{(2)}(\epsilon, t, x, x/\epsilon). \]

As for \( I \) it follows that, uniformly in \( t \geq 0 \),
\[ \| II^{(1)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} + \| \partial_t II^{(1)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon, \]
\[ \| II^{(2)}(\epsilon, t, x, x/\epsilon) \|_{L^2(\mathbb{R}^d)} \lesssim 1. \]

Defining \( f(\epsilon, t, x) := \epsilon^{2k+1} (I + II^{(2)})(\epsilon, t, x, x/\epsilon) \) and \( g(\epsilon, t, x) := \epsilon^{2k+1} II^{(1)}(\epsilon, t, x, x/\epsilon) \) completes the proof of Lemma 4.11. \( \square \)

**End of proof of Theorem 1.2.** Estimate the error terms \( \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \) from (4.22). Since \( v_0^k \) and all of its derivatives are uniformly bounded in \( L^2(\mathbb{R}^d) \), one has
\[ (4.25) \quad \| \mathcal{E}_2(\epsilon, t, x) \|_{L^2(\mathbb{R}^d)} + \| \mathcal{E}_3(\epsilon, t, x) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^{2k+2}. \]
The error from \( \mathcal{E}_4 \) is smaller. For any \( N \) one has
\[ (4.26) \quad \| \mathcal{E}_4(\epsilon, t, x) \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^N. \]
Theorem 1.2 is a consequence of Lemma 4.11, (4.25), (4.26), and Proposition B.1 in the appendix.

Remark 4.12. In the same way as in part iii of Remark 3.2, one finds a constant $C$ depending on $j$ but independent of $\varepsilon \leq 1$, $t \geq 0$ so that

$$\|\partial_t^j \nabla_{t,x}(u^\varepsilon(t) - V^k(\varepsilon, t, x, x/\varepsilon))\|_{L^2(\mathbb{R}^d)} \leq C \varepsilon^{k+1}(t).$$

5. Sources growing polynomially in time

The error estimates for sources with compact support in $0 < t < 32$ (Theorem 3.1 for the classical case and Theorem 1.2 in the criminal case) easily imply similar estimates (5.1) and (5.2) for sources that grow at most polynomially in time. Suppose that

$$\text{Classical homogenization.}$$

This implies

$$\|\nabla_{t,x}[u^\varepsilon(t) - u^\varepsilon_{\text{approx}}(t)]\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{k+1} \langle (t - (j - 1))_+ \rangle^{k+1} j^m.$$

This implies

$$\|\nabla_{t,x}[u^\varepsilon(t) - u^\varepsilon_{\text{approx}}(t)]\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{2k+1} \int_0^t \langle (t - s)_+ \rangle^{k+1} s^m ds \lesssim \varepsilon^{2k+1} \langle t \rangle^{k+2+m}.$$
5.2. Criminal path. Theorem 1.2 shifted in time yields \( e_j^i(t) \lesssim \epsilon^{2k+1} \langle (t - (j - 1))_+ \rangle^m \), so
\[
\| \nabla_{t,x} [u^i(t) - u^i_{\text{approx}}(t)] \|_{L^2(\mathbb{R}^d)} \lesssim \epsilon^{2k+1} \int_0^t \langle (t - s)_+ \rangle^m ds \lesssim \epsilon^{2k+1} (t)^{m+2}.
\]
Choosing \( k \) so large that \( 2k + 1 \geq N + N(m + 2) \) shows that the total error is \( O(\epsilon^N) \) on intervals \( t \leq \epsilon^{-N} \).

6. Systems and Schrödinger equation

6.1. Second order systems. This section considers systems of wave equations including the elastodynamics equations. The unknown \( u(t, x) : \mathbb{R}^{d+1} \to \mathbb{R}^p \) is \( \mathbb{R}^p \) valued. Its gradient is a function with values in \( \mathcal{M}_{p,d} \) the set of \( p \times d \) matrices. The scalar product of matrices is defined by multiplying corresponding components and summing. The systems have the form
\[
\rho(x/\epsilon)\partial_t^2 u - \text{div} \ a(x/\epsilon) \text{grad} u = f(t, x), \quad \rho, a \ \text{periodic}, \quad u = f = 0 \ \text{for} \ t < 0,
\]
with source term \( f : \mathbb{R}^{d+1} \to \mathbb{R}^p \). Assume that for a.e. \( y \in \mathbb{T}^d \) the fourth order tensor \( a(y) \) is a symmetric map from \( \mathcal{M}_{p,d} \) to itself. Assume that \( a \in L^\infty(\mathbb{T}^d) \) is uniformly positive definite in the sense that
\[
\exists m_1 > 0, \ \forall \eta \in \mathcal{M}_{p,d}, \quad a(y)\eta \cdot \eta \geq m_1|\eta|^2 \ \text{a.e.} \ y \in \mathbb{T}^d.
\]
Assume that \( \rho(y) \in L^\infty(\mathbb{T}^d; \mathcal{M}_{p,p}) \) is symmetric and uniformly positive definite.

Keep the notations \( \mathcal{A}_{yy}, \mathcal{A}_{xy}, \mathcal{A}_{xx} \). These operators now map \( \mathbb{R}^p \) valued functions to \( \mathbb{R}^p \) valued functions. They are \( p \times p \) matrices of operators. The operator \( \Pi \) is defined for vector-valued functions as the component wise average over \( \mathbb{T}^d \). The operators \( \chi_k(y, \partial_{t,x}) \) in the next definition have coefficients that belong to \( H^1(\mathbb{T}^d; \mathcal{M}_{p,p}) \). The definition is analogous to the scalar case in Definition 2.2

**Definition 6.1.** Set \( \chi_{-1} := 0 \) and \( \chi_0 := I \), where \( I \) is the \( p \times p \) identity matrix. For \( k \geq 2 \), define operators mapping \( \mathbb{R}^p \)-valued functions of \( t, x \) to \( \mathbb{R}^p \)-valued functions of \( t, x, y \) by
\[
\chi_k(y, \partial_{t,x}) := -\mathcal{A}_{yy}^{-1}(I - \pi) \left[ \mathcal{A}_{xy} \chi_{k-1} + (\mathcal{A}_{xx} - \rho \partial_t^2)\chi_{k-2} \right].
\]
In the above equation, the composition rule uses \( p \times p \) matrix multiplication.

**Remark 6.2.** Denote by \( e_j \) the canonical basis vectors of \( \mathbb{R}^p \). The product of the coefficient of \( \partial_{t,x}^2 \) in \( \chi_k \) and \( e_j \) yields a vector whose components in \( H^1(\mathbb{T}^d) \) are the usual correctors with index \( \alpha \) from periodic homogenization.

**Definition 6.3.** For \( n \geq 2 \) define the \( p \times p \) system of differential operators \( a_n^* \) by,
\[
a_n^*(\partial_t, \partial_x) = \pi \left( \mathcal{A}_{xy} \chi_{n-1}(y, \partial_{t,x}) + (\mathcal{A}_{xx} - \partial_t^2 \rho)\chi_{n-2}(y, \partial_{t,x}) \right).
\]
Then \(a^*_n(\partial_t, \partial_x)\) is homogeneous of degree \(n\) in \((\partial_t, \partial_x)\). It contains only even powers of \(\partial_t\).

**Theorem 6.4.** The cascade of equations for the system wave equation is equivalent to two families of equations. The first

\[
- \sum_{n=2}^{k+2} a^*_n(\partial_t, \partial_x) \pi u_{k+2-n} = f \delta_{0k}
\]

yields wave equations defining the nonoscillatory parts in terms of the source and earlier nonoscillatory profiles. The second

\[
(I - \pi) u_k = \sum_{n=1}^{k} \chi_n(\partial_t, \partial_x, y) \pi u_{k-n}(\tau, \xi),
\]

gives the oscillatory parts in terms of nonoscillatory parts.

**Theorem 6.5.** For any \(n \geq 1\), the coefficients of \(a^*_n(\partial_t, \partial_x)\) are symmetric \(p \times p\) matrices. For any odd \(n \geq 1\), the homogenized operator of order \(n\) vanishes. That is for \(m \geq 1\), \(a^*_{2m+1}(\partial_t, \partial_x) = 0\).

**Proof.** Introduce the operators \(C_1, C_2, D_1, D_2, W_k, Z_k\) as in the proof of Theorem 2.13. Their coefficients are now \(p \times p\) matrices of operators. For given vectors \(\lambda, \mu \in \mathbb{R}^p\), since \(\chi_0 = I\), definition (6.2) implies that

\[
a^*_k(\partial_t, \partial_x) \lambda \cdot \mu
= \int_{\mathbb{T}^d} \left( A_{\xi \theta} W_{k-1}(\tau, \xi) \chi_0(y) \lambda + (A_{\xi \xi} - \tau^2 \rho) W_{k-2}(\tau, \xi) \chi_0(y) \lambda \right) \cdot \chi_0(y) \mu \, dy
\]

\[
= \int_{\mathbb{T}^d} \left( W_{k-1}(\tau, \xi) \chi_0(y) \lambda \cdot (-A_{\xi \theta} \chi_0(y) \mu) + W_{k-2}(\tau, \xi) \chi_0(y) \lambda \cdot ((A_{\xi \xi} - \tau^2 \rho) \chi_0(y) \mu) \right) \, dy
\]

\[
= (-1)^k \int_{\mathbb{T}^d} \chi_0(y) \lambda \cdot \left( Z_{k-1}(\tau, \xi) A_{\xi \theta} \chi_0(y) \mu + Z_{k-2}(\tau, \xi) (A_{\xi \xi} - \tau^2 \rho) \chi_0(y) \mu \right) \, dy.
\]

Integrating by parts as in the proof of Theorem 2.13 using analogous adjoint relations yields

\[
a^*_k(\partial_t, \partial_x) \lambda \cdot \mu = (-1)^k \int_{\mathbb{T}^d} \chi_0(y) \lambda \cdot \left[ A_{\xi \theta} W_{k-1}(\tau, \xi) \chi_0(y) \mu + (A_{\xi \xi} - \tau^2 \rho) W_{k-2}(\tau, \xi) \chi_0(y) \mu \right] \, dy
\]

\[
= (-1)^k \lambda \cdot a^*_k(\partial_t, \partial_x) \mu.
\]

For odd \(k\) this yields \(a^*_k = 0\), and, for even \(k\), the coefficients of \(a^*_k\) are symmetric. \(\square\)

### 6.2. Schrödinger equation

Consider the homogenization of Schrödinger’s equation

\[
i \rho(x/\epsilon) \partial_t u^\epsilon - (\text{div} a(x/\epsilon) \text{grad} u^\epsilon) = f(t, x).
\]

With only the most minor modifications of the proof one finds an analogue of Theorem 2.13. This yields a leap frog structure and slow secular growth for the two scale expansions.
The analogue of Proposition C.1 is that if \( g \) vanishes for \( t > 0 \) and \( g_t \in L^1(\mathbb{R}; L^2(\mathbb{R}^d)) \), then the solution of \( i \rho(x/\epsilon) u_t = \text{div}(a(x/\epsilon) \text{grad} u) + \text{div} g \) vanishing for \( t \leq 0 \) is uniformly bounded in \( L^\infty(\mathbb{R}; H^1(\mathbb{R}^d)) \). The supplementary material to this paper contains a proof of this estimate. The classical approximation is accurate for times \( \sim 1/\epsilon^{-2+\delta} \) for any small \( \delta > 0 \). The criminal approach yields approximations with error \( \leq \epsilon^N \) for times \( 1/\epsilon^N \).

**APPENDIX A. EXAMPLES WITH MAXIMAL SECULAR GROWTH**

A.1. **Saturated secular growth.** Theorem 2.14 gives an upper bound on the growth in time of the profiles. This appendix shows that the upper bound is attained for generic problems in dimension \( d = 1 \). Consider

\[
\partial_t^2 u - \partial_x (a(x/\epsilon) \partial_x u) = f(t, x) \in C_0^\infty([0, 1] \times \mathbb{R}).
\]

with \( a(y) \) 1-periodic and not identically constant. Denote

\[
a_2^*(\partial_t, \partial_x) := \partial_t^2 - c^2 \partial_x^2 = (\partial_x - c\partial_t)(\partial_x + c\partial_t),
\]

the homogenized operator. Then \( u_0 = u_0(t, x) \) is the solution of \( a_2^*(\partial_t, \partial_x)u_0 = f \) that vanishes for \( t < 0 \). Therefore for \( t > 1 \) there are uniquely determined \( g_0, h_0 \in C^\infty(\mathbb{R}) \) with \( g_0(s) = 0 \) for \( s \gg 1 \) and \( h_0(s) = 0 \) for \( s \ll -1 \) so that for \( t > 1 \) one has

\[
u_0 = g_0(x - ct) + h_0(x + ct) .
\]

For most \( f \), both \( g_0 \) and \( h_0 \) are not identically equal to zero. This is true in particular if \( f \geq 0 \) and not identically equal to zero. In that case both \( g_0 \) and \( h_0 \) are non negative and not identically zero. The profile \( \pi u_2 \) satisfies of

\[
a_4^*(\partial_t, \partial_x)\pi u_2 = -a_4^*(\partial_t, \partial_x)u_0 .
\]

The fourth order homogeneous polynomial \( -a_4^*(\tau, \xi) \) contains no odd powers of \( \tau \). For \( t > 1 \), \( u_0 \) satisfies \( a_2^*(\partial_t, \partial_x)u_0 = 0 \). In this domain, replacing systematically \( \partial_t^2 u_0 \) by \( c^2 \partial_x^2 u_0 \) and equivalently \( \tau^2 \) by \( \tau^2 \tau^2 \) yields a new polynomial \( q(\xi) \) so that \( -a_4^*(\partial_t, \partial_x)u_0 = q(\partial_x)u_0 \). As soon as \( a(y) \) is not constant one has \( q(\partial_x) = \gamma \partial_x^4 \) with \( \gamma \neq 0 \) (see [13]). This shows that

\[-a_4^*(\pm c, 1) = \gamma \neq 0 .
\]

The equation for \( \pi u_2 \) for \( t > 1 \) is

\[(A.1) \quad a_2^*(\partial_t, \partial_x)\pi u_2 = (\gamma \partial_x^4 g_0)(x - ct) + (\gamma \partial_x^4 h_0)(x + ct) .
\]

Writing \( a_2^*(\partial_t, \partial_x) = (\partial_t - c\partial_x)(\partial_t + c\partial_x) \) one verifies that the function

\[z_2(t, x) = \frac{1}{4c^2} \left[ (ct + x)(\gamma \partial_x^3 g_0)(x - ct) + (ct - x)(\gamma \partial_x^3 h_0)(x + ct) \right].
\]
satisfies (A.1). Choose a cutoff function $\chi \in C^\infty(\mathbb{R}_t)$ equal to zero for $t < 1/2$ and equal to 1 for $t \geq 1$. Then

$$a_2^\ast(\partial_t, \partial_x)(\pi u_2 - \chi(t)z_2)$$

is compactly supported in $0 \leq t \leq 1$. Therefore

$$\pi u_2 = \chi(t)z_2(t, x) + r_2(t, x), \quad \forall \alpha, \quad \partial_t^\alpha r_2 \in L^\infty(\mathbb{R}^{1+1}).$$

The equation for $\pi u_4$ is

$$a_4^\ast(\partial_t, \partial_x)\pi u_4 = -a_4^\ast(\partial_t, \partial_x)\pi u_2 - a_6^\ast(\partial_t, \partial_x)\pi u_0.$$

Only the $\pi u_2$ term on the right is unbounded. Furthermore, if any of the derivatives in $a_4^\ast(\partial_t, \partial_x)$ fall on the factors $ct \pm x$ in $\pi u_2$ the resulting function is bounded. Therefore

$$a_4^\ast(\partial_t, \partial_x)\left((ct + x)(\gamma \partial^3 g_0)(x - ct) + (ct - x)(\gamma \partial^3 h_0)(x + ct)\right) = (ct + x)(\gamma \partial^3 g_0)(x - ct) + (ct - x)(\gamma \partial^3 h_0)(x + ct) + L^\infty(\mathbb{R}^{1+1}).$$

Reasoning as above yields with $\partial_t^\alpha r_3 \in L^\infty(\mathbb{R}^{1+1})$,

$$\pi u_4 = \frac{1}{(4c^2)^2} \left[\frac{(ct + x)^2}{2}((\gamma \partial^3 g_0)(x - ct) + (ct - x)^2((\gamma \partial^3 h_0)(x + ct)) + \langle t \rangle r_3.\right.$$

An induction yields

$$\pi u_{2n} = \frac{(ct + x)^n}{n!} g_n(x - ct) + \frac{(ct - x)^n}{n!} h_n(x + ct) + \langle t \rangle^{n-1} r_n,$$

$$g_n := \left(\frac{\gamma \partial^3}{4c^2}\right)^n g_0, \quad h_n := \left(\frac{\gamma \partial^3}{4c^2}\right)^n h_0, \quad \partial_t^\alpha r_n \in L^\infty(\mathbb{R}^{1+1}).$$

The $u_n$ saturate the upper bounds of Theorem 2.14.

In addition note that $u_{2n}$ grows with $n$ as the $(3n)^{th}$ derivative of $g_0, h_0$. This implies that for generic real analytic $g_0, h_0$, the series $\sum e^n u_n$ is divergent.

### A.2. Classic homogenization is inaccurate beyond $t = \epsilon^{-2}$.

One could imagine that by including many correctors, the classical algorithm might be accurate for times beyond $\epsilon^{-2}$. The example of the preceding subsection show that that is not the case. For that example the exact solution satisfies $\sup_{0 < c < 1} \sup_{\mathbb{R}^{1+1}} |u| < \infty$. For $c > 0, \delta > 0$ as small as one likes define $t_\epsilon = c/\epsilon^{2+\delta}$. To show the inaccuracy of the classical approximation it suffices to show that for any $0 < N \in \mathbb{Z}$,

$$\lim_{\epsilon \to 0} \left\| \sum_{j=0}^{2N} e^j u_j(t_\epsilon) \right\|_{L^\infty(\mathbb{R}_x)} = \infty.$$

For the example one has for $t$ large

$$\epsilon^{2j} t^j \lesssim \left\| e^{2j} \pi u_{2j}(t) \right\|_{L^\infty(\mathbb{R}_x)} \lesssim \epsilon^{2j} t^j.$$
For \( N \) fixed, formula (1.6) implies that
\[
(A.4) \quad \| e^k (I - \pi) u_k(t) \|_{L^\infty(\mathbb{R}_x)} \lesssim \epsilon^{k-1} t^{k-1}.
\]
Therefore, with \( \lambda_\epsilon := \epsilon^{-\delta} \) one has
\[
\left\| \epsilon^{2N} (I - \pi) u_{2N}(t_\epsilon) + \sum_{k=0}^{2N-1} \epsilon^k u_k(t_\epsilon) \right\|_{L^\infty(\mathbb{R}_x)} \lesssim \lambda_\epsilon^{2N-1} \quad \text{and} \quad \lambda_\epsilon^{2N} \lesssim \| \epsilon^{2N} \pi u_{2N}(t_\epsilon) \|_{L^\infty(\mathbb{R}_x)}.
\]
It follows that with \( C_j > 0 \),
\[
\left\| \sum_{j=0}^{2N} \epsilon^j u_j(t_\epsilon) \right\|_{L^\infty(\mathbb{R}_x)} \geq C_1 \lambda_\epsilon^{2N} - C_2 \lambda_\epsilon^{2N-1}.
\]
The limit \( \epsilon \rightarrow 0 \) yields (A.2).

**Appendix B. Two scale \( L^2 \) estimate**

This appendix contains a proof of a classical estimate for oscillating two scale functions. It is used in the error estimates in Sections 3.2 and 4.5.

**Proposition B.1.** For each integer \( s > d/2 \), there is a constant \( C \) so that for all \( v \in H^s(\mathbb{R}^d) \) and \( c \in L^2(\mathbb{T}^d) \),
\[
(B.1) \quad \int_{\mathbb{R}^d} |v(x) c(x/\epsilon)|^2 \, dx \leq C \|c\|^2_{L^2(\mathbb{T}^d)} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^d} |(\epsilon \partial_x)^\alpha v(x)|^2 \, dx.
\]

**Proof.** Denote by \( Y := [0,1]^d \) the unit box. Then \( \mathbb{R}^d \) is a disjoint union of boxes \( Y_k := k + Y \), \( \mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} Y_k \). Scaling by \( \epsilon \) yields \( \mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} \epsilon Y_k \). Sobolev’s inequality for \( Y_k \) reads
\[
(B.2) \quad \| w \|^2_{L^\infty(\epsilon Y_k)} \leq C(s, d) \sum_{|\alpha| \leq s} \int_{\epsilon Y_k} |\partial_x^\alpha w(y)|^2 \, dy, \quad w \in H^s(\epsilon Y_k).
\]
When \( y \in Y_k, x := \epsilon y \in \epsilon Y_k \). For \( v \in H^s(\epsilon Y_k) \), apply (B.2) to \( w(y) := v(\epsilon y) \in H^s(\epsilon Y_k) \) to find
\[
(B.3) \quad \| v \|^2_{L^\infty(\epsilon Y_k)} \leq C(s, d) \epsilon^{-d} \sum_{|\alpha| \leq s} \int_{\epsilon Y_k} |(\epsilon \partial_x)^\alpha v(x)|^2 \, dx.
\]
Estimate, using (B.3) in the last line,
\[
\int_{\epsilon Y_k} |v(x) c(x/\epsilon)|^2 \, dx \leq \| v \|^2_{L^\infty(\epsilon Y_k)} \int_{\epsilon Y_k} |c(x/\epsilon)|^2 \, dx \leq C(s, d) \epsilon^{-d} \sum_{|\alpha| \leq s} \int_{\epsilon Y_k} |(\epsilon \partial_x)^\alpha v(x)|^2 \, dx \epsilon^d \| c \|^2_{L^2(\mathbb{T}^d)}.
\]
Summing over \( k \) yields (B.1). \( \square \)
Appendix C. Stability estimate for the wave equation

This appendix contains an estimate for wave equations with sources in $L^{\infty}_{\text{loc}}(\mathbb{R}; H^{-1}(\mathbb{R}^d))$. The weak regularity in $x$ is compensated by additional regularity in time. The residuals in the criminal and the non-criminal approximation are of that form. For completeness the proof is included. The systems case is exactly analogous.

Proposition C.1. Suppose that $0 < m_1 < m_2 < \infty$ are real numbers and $a, \rho \in L^{\infty}(\mathbb{R}^d)$ satisfy $m_1 \leq a, \rho \leq m_2$. There is a constant $C > 0$ depending only on $m_1, m_2$ so that for all $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$ and $g \in L^\infty_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}; \mathbb{R}^d))$ with $\partial_t g \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}; \mathbb{R}^d))$ and $f = g = 0$ for $t \leq 0$ the solution of

$$
(\rho \partial_t^2 - \text{div}(a \text{grad})) u = f + \text{div} g, \quad u = 0 \text{ for } t \leq 0,
$$

satisfies for all $t > 0$, 

$$
(C.1) \quad \| \nabla_{t,x} u(t) \|_{L^2(\mathbb{R}^d)} \leq C\left[ \| f \|_{L^1([0,t]; L^2(\mathbb{R}^d))} + \| \partial_t g \|_{L^1([0,t]; L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t]; L^2(\mathbb{R}^d))} \right].
$$

Proof. Approximating $a, \rho, f, g$ by smooth functions it suffices to prove the estimate for solutions and right hand side that belong to $H^s([0, t] \times \mathbb{R}^d)$ for all $s, t > 0$ with a constant that depends only on the $m_j$. Introduce continuous functions

$$
E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \left( \rho(x) |\partial_t u(t, x)|^2 + a(x) |\nabla u(t, x)|^2 \right) dx, \quad \text{and} \quad M(t) := \sup_{0 \leq \tau \leq t} E(\tau).
$$

Testing the equation with $\partial_t u$ yields the standard energy identity

$$
E(t) = \int_{0}^{t} \int_{\mathbb{R}^d} \partial_t u(f + \text{div} g) \, dx \, dt.
$$

Estimate the first of the two summands on the right as

$$
\left| \int_{0}^{t} \int_{\mathbb{R}^d} f \partial_t u \, dx \, dt \right| \leq \| f \|_{L^1([0,t]; L^2(\mathbb{R}^d))} \| \partial_t u \|_{L^\infty([0,t]; L^2(\mathbb{R}^d))} \leq C \| f \|_{L^1([0,t]; L^2(\mathbb{R}^d))} M(t)^{1/2}
$$

with a constant depending only on the $m_j$. For the second summand two integrations by parts yield

$$
\int_{0}^{t} \int_{\mathbb{R}^d} \text{div} g \, \partial_t u \, dx \, dt = - \int_{0}^{t} \int_{\mathbb{R}^d} g \cdot \text{grad} \, \partial_t u \, dx \, dt
$$

$$
= \int_{0}^{t} \int_{\mathbb{R}^d} \partial_t g \cdot \text{grad} u \, dx \, dt - \int_{\mathbb{R}^d} g(t, x) \cdot \text{grad} u(t, x) \, dx.
$$
Therefore
\[
\left| \int_0^t \int_{\mathbb{R}^d} \text{div} g \partial_t u \, dx dt \right| \leq \| \partial_t g \|_{L^1([0,t];L^2(\mathbb{R}^d))} \| \text{grad} u \|_{L^\infty([0,t];L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t];L^2(\mathbb{R}^d))} \| \text{grad} u \|_{L^\infty([0,t];L^2(\mathbb{R}^d))} \leq C \left( \| \partial_t g \|_{L^1([0,t];L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t];L^2(\mathbb{R}^d))} \right) M(t)^{1/2}.
\]
Combining yields
\[
(C.2) \quad E(t) \leq C \left( \| f \|_{L^1([0,t];L^2(\mathbb{R}^d))} + \| \partial_t g \|_{L^1([0,t];L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t];L^2(\mathbb{R}^d))} \right) M(t)^{1/2}.
\]
For each \( t > 0 \), choose \( 0 < t' \leq t \) so that \( E(t') = M(t) \). Estimate \((C.2)\) at time \( t' \) yields
\[
M(t) \leq C \left( \| f \|_{L^1([0,t'];L^2(\mathbb{R}^d))} + \| \partial_t g \|_{L^1([0,t'];L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t'];L^2(\mathbb{R}^d))} \right) M(t')^{1/2} \leq C \left( \| f \|_{L^1([0,t'];L^2(\mathbb{R}^d))} + \| \partial_t g \|_{L^1([0,t'];L^2(\mathbb{R}^d))} + \| g \|_{L^\infty([0,t'];L^2(\mathbb{R}^d))} \right) M(t)^{1/2}.
\]
If \( M(t) \neq 0 \), dividing by \( M(t)^{1/2} \) yields \((C.1)\). If \( M(t) = 0 \), \((C.1)\) holds with \( C = 0 \). \( \square \)

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