A GENERAL PRINCIPLE FOR HAUSDORFF MEASURE

MUMTAZ HUSSAIN AND DAVID SIMMONS

Abstract. We introduce a general principle for studying the Hausdorff measure of limsup sets. A consequence of this principle is the well-known Mass Transference Principle of Beresnevich and Velani (2006).

1. Introduction and Statements of main results

A fundamental problem in the theory of metric Diophantine approximation is to determine the ‘size’ of the limsup set

$$\limsup_{i \to \infty} B_i = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i = \{ x \in X : x \in B_i \text{ for infinitely many } i \in \mathbb{N} \}$$

in terms of Lebesgue measure, Hausdorff dimension, or Hausdorff measure. Here and throughout we let $$(B_i)_{i \in \mathbb{N}}$$ be a sequence of open sets in a metric space $$X$$. Let $$f$$ be a dimension function i.e. an increasing continuous function $$f : [0, \infty) \to [0, \infty)$$ with $$f(0) = 0$$. We denote by $$H_f$$ the $$f$$-dimensional Hausdorff measure, which is proportional to the standard Lebesgue measure when $$X = \mathbb{R}^d$$ and $$f(r) = r^d$$. In the case where the dimension function is of the form $$f(r) := r^s$$ for some $$s > 0$$, $$H_f$$ is simply denoted as $$H^s$$.

For the definitions of Hausdorff measure and dimension and their properties we refer to the book [4]. The Hausdorff–Cantelli lemma [2, Lemma 3.10] states that $$H_f(\limsup_{i \to \infty} B_i) = 0$$ if $$\sum_{i=1}^{\infty} f(\text{diam}(B_i)) < \infty$$, where $$\text{diam}(B_i)$$ denotes the diameter of $$B_i$$. Here the emphasis is on using a particular ‘nice’ cover of the limsup set, namely $${B_i : i \in \mathbb{N}}$$, to establish an upper bound for Hausdorff measure. In contrast, proving the $$f$$-dimensional Hausdorff measure to be positive is a challenging task, requiring all possible coverings to be considered and, therefore, represents the main problem in metric Diophantine approximation (in various settings).

Question 1.1. Under what conditions is $$H_f(\limsup_{i \to \infty} B_i)$$ strictly positive?

The following principle commonly known as the Mass Distribution Principle [4, §4.1] has been the go-to method in giving an answer to Question 1.1.

Lemma 1.2 (Mass Distribution Principle). Let $$\mu$$ be a probability measure supported on a subset $$F$$ of $$X$$. Suppose there are positive constants $$c > 0$$ and $$\varepsilon > 0$$ such that

$$\mu(U) \leq c f(\text{diam}(U))$$

for all sets $$U$$ with $$\text{diam}(U) \leq \varepsilon$$. Then $$H_f(F) \geq \mu(U)/c$$.

Specifically, the mass distribution principle replaces the consideration of all coverings by the construction of a particular measure $$\mu$$. Given a sequence of sets $$(B_i)_{i \in \mathbb{N}}$$, if we want to prove that $$H_f(\limsup_{i \to \infty} B_i)$$ is infinite (and in particular strictly positive), one possible strategy is to deploy the mass distribution principle in two steps:

- construct a suitable Cantor type subset $$K \subseteq F = \limsup_{i \to \infty} B_i$$ and a probability measure $$\mu$$ supported on $$K$$.  
- show that for any fixed $$c > 0$$, $$\mu$$ satisfies the condition that for any measurable set $$U$$ of sufficiently small diameter, $$\mu(U) \leq c f(\text{diam}(U))$$.

If this can be done, then by the mass distribution principle, it follows that

$$H_f(F) \geq H_f(K) \geq c^{-1}.$$

Then since $$c$$ is arbitrary, it follows that $$H_f(F) = \infty$$.  

1
The main intricate and substantive part of this entire process is the construction of a suitable Cantor type subset of $F$ which supports a probability measure. We introduce a generalised principle to determine the $f$-dimensional Hausdorff measure of limsup sets which throws out the Cantor type construction from this process.

To state our result, we introduce some notation. Let $X$ be a metric space. For $\delta > 0$, a measure $\mu$ is Ahlfors $\delta$-regular if and only if there exist positive constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that the inequality

$$c_1 r^\delta \leq \mu(B(x,r)) \leq c_2 r^\delta$$

holds for every ball $B := B(x,r)$ in $X$ of radius $r \leq r_0$ centred at $x \in \text{Supp}(\mu)$, where $\text{Supp}(\mu)$ denotes the topological support of $\mu$. The space $X$ is called Ahlfors $\delta$-regular if there is an Ahlfors $\delta$-regular measure whose support is equal to $X$. If $X$ is Ahlfors $\delta$-regular, then so is the $\mathcal{H}^\delta$ measure restricted to $X$ i.e. $\mathcal{H}^\delta \upharpoonright X$. We will frequently be using the notation $B^f := B(x,f(r)^{1/\delta})$. For real quantities $A,B$ that depend on parameters, we write $A \lesssim B$ if $A \leq cB$ for a constant $c > 0$ that is independent of those parameters. We write $A \asymp B$ if $A \lesssim B \lesssim A$.

**Theorem 1.3.** Fix $\delta > 0$, let $(B_i)_{i \in \mathbb{N}}$ be a sequence of open sets in an Ahlfors $\delta$-regular metric space $X$, and let $f$ be a dimension function such that

$$r \mapsto r^{-\delta} f(r)$$

is decreasing, and

$$r^{-\delta} f(r) \to \infty \text{ as } r \to 0.$$ 

Fix $C > 0$, and suppose that the following hypothesis holds:

(*) For every ball $B_0 \subseteq X$ and for every $N \in \mathbb{N}$, there exists a probability measure $\mu = \mu(B_0,N)$ with $\text{Supp}(\mu) \subseteq \bigcup_{i \geq N} B_i \cap B_0$, such that for every ball $B = B(x,\rho) \subseteq X$, we have

$$\mu(B) \lesssim \max \left( \frac{\rho}{\text{diam} B_0} \right)^\delta, \frac{f(\rho)}{C} \right).$$

Then for every ball $B_0$,

$$\mathcal{H}^f \left( B_0 \cap \limsup_{i \to \infty} B_i \right) \geq C.$$ 

In particular, if the hypothesis (*) holds for all $C$, then

$$\mathcal{H}^f \left( B_0 \cap \limsup_{i \to \infty} B_i \right) = \infty.$$ 

The condition (1.2) is a natural condition which implies that $\mathcal{H}^f(B) = \infty$.

1.1. **The Mass Transference Principle.** In a landmark paper [1], Beresnevich and Velani introduced the Mass Transference Principle which has become a major tool in converting Lebesgue measure theoretic statements for limsup sets into Hausdorff measure statements for limsup sets. This is surprising as the Lebesgue measure is the ‘coarser’ notion of ‘size’ than the Hausdorff measure. As one would expect, the Mass Transference Principle has many applications in number theory, such as derivation of Jarník’s theorem from Khintchine’s theorem or the derivation of the Jarník–Besicovitch theorem from Dirichlet’s theorem. Other than that the Mass Transference Principle can be used to determine Hausdorff measure and dimension of limsup sets in the context of dynamical systems such as $\beta$-dynamical systems [3, 7]. This fundamental principle has been extended further to deal with multifractal formalisms in [5].

**Theorem 1.4** (Beresnevich–Velani, 2006). Let $X \subseteq \mathbb{R}^d$ be Ahlfors $\delta$-regular. Let $(B_i)_{i \in \mathbb{N}}$ be a sequence of balls in $X$ with $\text{rad}(B_i) \to 0$ as $i \to \infty$. Let $f$ be a dimension function such that $r \mapsto r^{-\delta} f(r)$ is monotonic. Suppose that for every ball $B \in X$

$$\mathcal{H}^\delta (B \cap \limsup_{i \to \infty} B_i^f) = \mathcal{H}^\delta (B).$$


Then for every ball $B \subseteq X$

$$\mathcal{H}^j(B \cap \limsup_{i \to \infty} B_i) = \mathcal{H}^j(B).$$

The Mass Transference Principle has been significantly used in determining the Hausdorff measure, and as a consequence the Hausdorff dimension, of limsup sets. For instance, using the Mass Transference Principle the Hausdorff measure version of the Duffin-Schaeffer conjecture was formulated in $\mathbb{1}$. By considering $X$ to be the middle third Cantor set, for which $\delta = \log 2/\log 3$, the Mass Transference Principle has been used in $\mathbb{1}$ in establishing a complete metric theory for sets of $\psi$-approximable points and as a consequence proving the existence of very well approximable numbers other than Liouville numbers in the middle third Cantor set, an assertion that was attributed to Mahler.

1.2. Theorem $\mathbb{1.3} \implies$ Theorem $\mathbb{1.4}$ Before establishing this implication we state the following covering lemma, which is a variant of the $K_{G,B}$ lemma of Beresnevich and Velani $\mathbb{1}$. The difference is that we replace the set $B_i^j$ by the set $B_i^{j/C}$, where $C > 0$ is an arbitrary constant, while we need bounds that are independent of $C$. The proof is more involved, and is a variant of the proof of the Vitali covering lemma $\mathbb{8}$ Theorem 2.2].

**Lemma 1.5.** Fix $C > 0$. Let $(B_i = B(x_i, \rho_i))_{i \in \mathbb{N}}$ be a sequence of balls in an Ahlfors $\delta$-regular space $X$ with $\text{rad}(B_i) \to 0$ as $i \to \infty$. Let $f$ be a dimension function such that (1.4) is satisfied for every ball $B$ in $X$. Fix $B_0 \subseteq X$ and $N \in \mathbb{N}$. Then there exists a finite set $I = I(B_0, N) \subseteq \{N, \ldots\}$ such that the collection $\left(\bigcap B_i^{j/C}\right)_{i \in I}$ is disjoint and its union is a subset of $B_0$ satisfying

$$\mathcal{H}^\delta \left( \bigcup_{i \in I} B_i^{j/C} \right) \geq \frac{1}{2} \mathcal{H}^\delta(B_0).$$

**Proof.** Without loss of generality assume that $N = 1$. By rearranging we can without loss of generality assume that $\rho_j \leq 2\rho_i$ whenever $j \geq i$. Let $I^\infty \subseteq \mathbb{N}$ be defined using a greedy algorithm: $j \in I^\infty$ if and only if $B_i^{j/C} \cap B_j^{j/C} = \emptyset$ for all $i < j$ such that $i \in I^\infty$. Fix $M$ and let

$$U = \text{interior}(B_0) \setminus \bigcup_{i \leq M} B_i^{j/C}.$$ 

Fix $x \in U \cap \limsup_{i \to \infty} B_i^j$. Then there exists $i > M$ such that $x \in B_i^j \subseteq U$. If $i \notin I^\infty$, then there exists $j < i$ such that $j \in I^\infty$ and $B_i^{j/C} \cap B_j^{j/C} = \emptyset$. Otherwise, let $j = i$ and the same holds. Since $B_i^{j/C} \subseteq U$, it follows that $j > M$. Moreover, since $\rho_i \leq 2\rho_j$ we have $B_i^j \subseteq 5B_j^j$. Thus,

$$U \cap \limsup_{i \to \infty} B_i^j \subseteq \bigcup_{i \geq M} 5B_i^j.$$ 

Since (1.4) is satisfied, by Ahlfors regularity we have

$$\mathcal{H}^\delta(U) \leq \sum_{i \in I^\infty} \mathcal{H}^\delta(5B_i^j)^{\approx C} \sum_{i \geq M} \mathcal{H}^\delta(B_i^{j/C}) = \mathcal{H}^\delta \left( U \cap \bigcup_{i \in I^\infty} B_i^{j/C} \right).$$

Since $M$ was arbitrary, it follows that

$$\mathcal{H}^\delta \left( B_0 \cap \bigcup_{i \in I^\infty} B_i^{j/C} \right) = \mathcal{H}^\delta(B_0)$$

and thus taking an appropriate finite initial segment of $I^\infty$ completes the proof. $\square$
Let \( I = I(B_0, N) \) be as in Lemma 1.5 and construct the measure \( \mu = \mu(B_0, N) \) as follows:
\[
\mu = \frac{1}{K} \sum_{i \in I} \mathcal{H}^\delta(B_i^{f/C}) \frac{\mathcal{H}^\delta(B_i \cap B)}{\mathcal{H}^\delta(B_i)}
\]
where \( K \) is a constant chosen so that \( \mu(X) = 1 \), i.e. \( K = \sum_{i \in I} \mathcal{H}^\delta(B_i^{f/C}) \). Note that by (1.6),
\( K \asymp \mathcal{H}^\delta(B_0) \).

Also note that by (1.2), by increasing \( N \) if necessary we can assume without loss of generality that \( f(\rho_i)/C \geq (2\rho_i)^\delta \), and thus that \( B_i \subseteq B_i^{f/C} \), for all \( i \geq N \). Since \( B_i^{f/C} \subseteq B_0 \) for all \( i \in I \), we have
\[\text{Supp}(\mu) \subseteq \bigcup_{i \in I} B_i \subseteq \bigcup_{i \geq N} B_i \cap B_0.\]

Fix \( B = B(x, \rho) \); to complete the proof, we need to show that (1.3) holds. Suppose first that \( B \) intersects only one element of the collection \( (B_i)_{i \in I} \), say \( B_i = B(x, \rho_i) \). Then
\[
\mu(B) = \frac{1}{K} \mathcal{H}^\delta(B_i^{f/C}) \frac{\mathcal{H}^\delta(B_i \cap B)}{\mathcal{H}^\delta(B_i)} \leq \frac{1}{\mathcal{H}^\delta(B_0)} \frac{f(\rho_i)}{C} \min \left(1, \left(\frac{\rho}{\rho_i}\right)^\delta\right) \quad \text{(by Ahlfors } \delta\text{-regularity)}
\]
\[
\leq \frac{1}{\mathcal{H}^\delta(B_0)} \frac{f(\rho)}{C} \quad \text{(by (1.1))}
\]

On the other hand, suppose that \( B \) intersects multiple elements of \( (B_i)_{i \in I} \). Since \( (B_i^{f/C})_{i \in I} \) is a disjoint collection, it follows that for all \( i \in I \) for which \( B_i \cap B \neq \emptyset \), we have
\[
2\rho \geq \text{diam}(B) \geq d(B_i, X \setminus B_i^{f/C}) \geq (1/2)(f(\rho_i)/C)^{1/\delta}.
\]

It follows that \( \text{diam}(B_i^{f/C}) \leq 8\rho \) and thus
\[
B_i^{f/C} \subseteq B(x, 9\rho)
\]
so
\[
\mu(B) \leq \frac{1}{K} \sum_{B_i \cap B_i \neq \emptyset} \mathcal{H}^\delta(B_i^{f/C}) \leq \frac{1}{K} \mathcal{H}^\delta(B(x, 9\rho)) \quad \text{(since \( (B_i^{f/C})_{i \in I} \) are disjoint)}
\]
\[
\asymp \left(\frac{\rho}{\text{diam}(B_0)}\right)^\delta \quad \text{(Ahlfors } \delta\text{-regularity)}
\]

This completes the proof.

2. Proof of Theorem 1.3

For each \( n \in \mathbb{N} \), we construct a set \( T^n \subseteq \mathbb{N}^n \), a family of balls \( (B_\omega)_{\omega \in T^n} \), and a family of parameters \( (\rho_\omega)_{\omega \in T^n} \), recursively as follows:

- For \( n = 0 \), we let \( T^0 = \{\emptyset\} \), \( B_{\emptyset} = B(x_{\emptyset}, 1) \), and \( \rho_{\emptyset} = 1 \). Here \( \emptyset \in \mathbb{N}^0 \) is the empty string, and \( x_{\emptyset} \in X \) is chosen arbitrarily.
- Fix \( n \in \mathbb{N} \), and suppose that \( T^n \) and the families \( (B_\omega)_{\omega \in T^n} \) and \( (\rho_\omega)_{\omega \in T^n} \) have been defined. Fix \( \omega \in T^n \), and let \( \mu_\omega = \mu(B_\omega, n) \), where the notation \( \mu(B, N) \) is as in hypothesis (*). Choose
\[
0 < \rho_\omega < d\left(\text{Supp}(\mu_\omega), X \setminus \bigcup_{i \geq n} B_i \cap B_\omega\right)
\]
such that \( \left(\frac{\rho_\omega}{\text{diam}(B_\omega)}\right)^\delta \leq \frac{f(\rho_\omega)}{C} \).
Such a choice is possible by (1.2).

Let \( F_\omega \subseteq K_\omega = \text{Supp}(\mu_\omega) \) be a maximal \( 4\rho_\omega \)-separated set (Here, a set \( F \) is called \( \rho \)-separated if for all \( u_1, u_2 \in F \), we have \( d(u_1, u_2) \geq \rho \). Let \( \tau_\omega : K_\omega \to F_\omega \) be a map such that \( d(x, \tau_\omega(x)) \leq 4\rho_\omega \) for all \( x \in K_\omega \). Let \( k_\omega = \#(F_\omega) \), and let \( (x_\omega)_{1 \leq a \leq k_\omega} \) be an enumeration of \( F_\omega \). Finally, let
\[
T^{n+1} = \{ \omega a : \omega \in T^n, 1 \leq a \leq k_\omega \}.
\]

For each \( \omega a \in T^{n+1} \), let
\[
B_{\omega a} = B(x_\omega, \rho_\omega) \quad \text{and} \quad p_{\omega a} = p_\omega \cdot (\tau_\omega)_*(\mu_\omega)(x_\omega),
\]
where \((\tau_\omega)_*(\mu_\omega)\) denotes the pushforward of a measure \( \mu_\omega \) under the map \( \tau_\omega \).

Note that for all \( \omega \in T = \bigcup_{n \geq 0} T^n \), we have
\[
\sum_{a \leq k_\omega} p_{\omega a} = p_\omega
\]
and for all \( a \leq k_\omega \) we have
\[
B_{\omega a} \subseteq \bigcup_{i \geq |\omega|} B_i \cap B_\omega
\]
where \(|\omega|\) denotes the length of \( \omega \). Moreover, by the definition of \( F_\omega \), the collection \( \{B_{\omega a}\}_{a \leq k_\omega} \) is disjoint.

It follows that there is a unique probability measure \( \mu \) such that
\[
\mu(B_{\omega a}) = p_\omega \quad \text{for all } \omega \in T,
\]
and that \( \text{Supp}(\mu) \subseteq \limsup_{i \to \infty} B_i \). To complete the proof, we need to show that
(2.2)
\[
\mu(B(x, \rho)) \lesssim f(\rho)/C
\]
for any sufficiently small ball \( B(x, \rho) \). This will allow us to apply the mass distribution principle (Lemma 1.2).

Indeed, fix such a ball \( B = B(x, \rho) \), let \( \omega \in T \) be the shortest word such that \( B \cap B_\omega \neq \varnothing \) and \( \rho_\omega \leq \rho \), and let \( \tau \) be the initial segment of \( \omega \) that is one letter shorter than \( \omega \), so that \( \rho_\tau > \rho \). Since \( F_\tau \) is \( 4\rho_\tau \)-separated, for all \( \tau a \neq \omega \) we have \( d(x_\omega, x_\tau a) \geq 4\rho_\tau \) and thus
\[
d(B_{\omega a}, B_{\tau a}) \geq 2\rho_\tau > 2\rho,
\]
from which it follows that \( B \cap B_{\tau a} = \varnothing \). So \( \mu(B) = \mu(B \cap B_\omega) \), and thus
\[
\mu(B) \leq \sum_{a \leq k_\omega} p_{\omega a}
\]
\[
\leq p_\omega \sum_{a \leq k_\omega} (\tau_\omega)_*(\mu_\omega)(x_\omega)
\]
\[
\leq p_\omega \sum_{x_\omega \in B(x, 2\rho)} A_{\omega a} B(x_\omega, 4\rho_\omega)
\]
\[
\leq p_\omega \mu_\omega(B(x, 6\rho)) \lesssim p_\omega \max \left( \left( \frac{\rho}{\text{diam } B_\omega} \right)^{\delta} \cdot \frac{f(\rho)}{C} \right)
\]
by our assumption on \( \mu_\omega \). A similar argument gives
\[
p_\omega = \mu(B_\omega) \lesssim p_\tau \max \left( \left( \frac{\rho_\tau}{\text{diam } B_\tau} \right)^{\delta} \cdot \frac{f(\rho_\tau)}{C} \right).
\]
Combining with (2.1) and using the fact that $p_\tau \leq 1$ gives

$$p_\omega \lesssim \frac{f(\rho_\tau)}{C}$$

and thus since $p_\omega \leq 1$,

$$\mu(B) \lesssim \max\left( \frac{\rho}{\diam B_\omega} \delta f(\rho_\tau), \frac{f(\rho)}{C} \right).$$

Applying (1.1) with $r = \rho_\tau$ and $ar = \rho$ and using the fact that $\diam B_\omega = 2\rho_\tau$ demonstrates (2.2) and completes the proof.

Acknowledgments. The first-named author was supported by La Trobe University’s start-up grant. The second-named author was supported by the EPSRC Programme Grant EP/J018260/1.

References

1. Victor Beresnevich and Sanju Velani, *A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures*, Ann. of Math. (2) 164 (2006), no. 3, 971–992.
2. Vasilii Bernik and Maurice Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, vol. 137, Cambridge University Press, Cambridge, 1999.
3. Michael Coons, Mumtaz Hussain, and Bao-Wei Wang, *A dichotomy law for the diophantine properties in $\beta$-dynamical systems*, Mathematika 62 (2016), no. 3, 884–897. MR 3521359
4. Kenneth Falconer, *Fractal geometry*, third ed., John Wiley & Sons, Ltd., Chichester, 2014, Mathematical foundations and applications. MR 3236784
5. Ai-Hua Fan, Jörg Schmeling, and Serge Troubetzkoy, *A multifractal mass transference principle for Gibbs measures with applications to dynamical Diophantine approximation*, Proc. Lond. Math. Soc. (3) 107 (2013), no. 5, 1173–1219. MR 3126394
6. Jason Levesley, Cem Salp, and Sanju Velani, *On a problem of K. Mahler: Diophantine approximation and Cantor sets*, Math. Ann. 338 (2007), 97–118.
7. Fan Lü and Jun Wu, *Diophantine analysis in beta-dynamical systems and Hausdorff dimensions*, Adv. Math. 290 (2016), 919–937. MR 3451942
8. P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.