SINGULARITY FORMATION FOR BURGERS EQUATION WITH TRANSVERSE VISCOSITY

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Abstract. We consider Burgers equation with transverse viscosity
\[
\partial_t u + u \partial_x u - \partial_{yy} u = 0, \quad (x, y) \in \mathbb{R}^2, \quad u : [0, T) \times \mathbb{R}^2 \to \mathbb{R}.
\]
We construct and describe precisely a family of solutions which become singular in finite time by having their gradient becoming unbounded. To leading order, the solution is given by a backward self-similar solution of Burgers equation along the \(x\) variable, whose scaling parameters evolve according to parabolic equations along the \(y\) variable, one of them being the quadratic semi-linear heat equation. We develop a new framework adapted to this mixed hyperbolic/parabolic blow-up problem, revisit the construction of flat blow-up profiles for the semi-linear heat equation, and the self-similarity in the shocks of Burgers equation.

1. Introduction

1.1. Setting of the problem and motivations

We consider Burgers equation with transverse viscosity
\[
\left\{ \begin{array}{l}
\partial_t u + u \partial_x u - \partial_{yy} u = 0, \quad (x, y) \in \mathbb{R}^2, \\
\quad u_{t=0} = u_0,
\end{array} \right.
\]
eq:burgers

for \(u : [0, T) \times \mathbb{R}^2 \to \mathbb{R}\). The present study is motivated by the following. This model reduces to the classical inviscid Burgers equation for solutions that are independent of the transverse variable \(u(t, x, y) = U(t, x)\), which is a classical example of a nonlinear hyperbolic equation for which initially smooth solutions can become singular in finite time, see for example [10, 25]. The effects of viscosity in the streamwise direction, namely the equation \(\partial_t u + u \partial_x u - \epsilon \partial_{xx} u = 0\), have been extensively studied, see [17, 18] and references therein. This work aims at understanding precisely the consequence of an additional effect, here the transverse viscosity, on a blow-up dynamics that it does not prevent. Moreover, this new effect changes the nature of the equation which is of a mixed hyperbolic/parabolic type. Handling these two issues, our result then extends known ones for blow-ups in a new direction, as well as raising new interesting problems, see the comments after the main Theorem 3.

More importantly, the study of (1.1) is motivated by fluid dynamics, from the fact that it is a simplified version of Prandtl’s boundary layer equations. Solutions to Prandtl’s equations might blow up in finite time [7, 11, 20] but a precise description of the singularity formation is still lacking. The present work is a step towards that goal and this issue will be investigated in a forthcoming work. Finally, let us mention that there has been recent progress on other models...
for singular solutions in fluid dynamics, see [4, 5, 26] and references therein.

The existence of smooth enough solutions to (1.1) follows from classical arguments, and there holds the following blow-up criterion. The solution $u$ blows up at time $T > 0$ if and only if

$$\limsup_{t \to T} \|\partial_x u\|_{L^\infty(\mathbb{R}^2)} = +\infty.$$  \hspace{1cm} (1.2)  

Before stating the main theorem, let us give the structure of the singularities of Burgers equation, and of the ones of the parabolic system encoding the effects of the transverse viscosity.

### 1.2. Self-similarity in shocks for Burgers equation

Burgers equation

$$\begin{cases}
\partial_t U + U \partial_x U = 0, & x \in \mathbb{R}, \\
U_{t=0} = U_0,
\end{cases}$$  \hspace{1cm} (1.3)  

admits solutions becoming singular in finite time in a self-similar way:

$$U(t,x) = \mu^{-1}(T-t)^{\frac{1}{2}} \Psi (\mu x - \frac{x_0 - c(T-t)}{(T-t)^{1+\frac{1}{2}}}),$$

where $(\Psi_i)_{i \in \mathbb{N}^*}$ is a family of analytic profiles (see [12] for example), and where $\mu > 0$ is a free parameter. They are at the heart of the shock formation, a fact that is rarely emphasised, which lead us to give a precise and concise study in Section 2. Self-similar and discretely self-similar blow-up profiles for Burgers equation are classified in Proposition 4. Different scaling laws are thus possible, depending on the initial condition via its behaviour near the characteristic where the shock will form, which has to do with the fact that the scaling group of (1.3) is two-dimensional, see Section (2). This possibility of several scaling exponents is referred to as self-similarity of the second kind [1]. For each $i \in \mathbb{N}^*$, $\Psi_i$, defined in Proposition 5, is an odd decreasing profile, which is nonnegative and concave on $(-\infty,0]$ and such that $\partial_X \Psi_i$ is minimal at the origin with asymptotic $\Psi_i(X) = -X + X^{2i+1}$ as $X \to 0$. One has in particular the formula

$$\Psi_1(X) := \left( -\frac{X}{2} + \left( \frac{1}{27} + \frac{X^2}{4} \right)^{\frac{1}{3}} \right) + \left( -\frac{X}{2} - \left( \frac{1}{27} + \frac{X^2}{4} \right)^{\frac{1}{3}} \right),$$  \hspace{1cm} (1.4)  

for the fundamental one [8]. As in the above formula, all these profiles are unbounded at infinity but they emerge nonetheless from well localised initial data. A precise description of these profiles is given in Proposition 5. Any regular enough non-degenerate solution $v$ to (1.3) that forms a shock at $(T,x_0)$ is equivalent to leading order near the singularity to a self-similar profile $\Psi_i$ up to the symmetries of the equation

$$U(t,x) \sim (T-t)^{\frac{1}{2}} \mu^{-1} \Psi_i \left( \mu \frac{x - (x_0 - c(T-t))}{(T-t)^{1+\frac{1}{2}}} \right) + c \text{ as } (t,x) \to (T,x_0),$$  \hspace{1cm} (1.5)  

see Proposition 9. The blow-up dynamics involving the concentration of $\Psi_1$ is a stable one for smooth enough solutions. The scenario corresponding to the concentration of $\Psi_i$ for $i \geq 2$ is an unstable one. For a suitable topological functional space, the set of initial conditions leading to the concentration of $\Psi_i$ for $i \geq 2$ is located at the boundary of the set of initial condition leading to the concentration of $\Psi_1$, and admits $2(i-1)$ instability directions yielding to one or several shocks formed by $\Psi_j$ for $j < i$. The linearised dynamics is described in Proposition 8.
1.3. Blow-up for the reduced parabolic system

For a solution \( u \) to \((1.1)\) that is odd in \( x \), the behaviour on the transverse axis \( \{x = 0\} \) is encoded by a closed system, which is the motivation for this symmetry assumption. It admits solutions blowing up simultaneously with a precise behaviour. Indeed, assume \( \partial_x^j u_0(0, y) = 0 \) for all \( y \in \mathbb{R} \) for \( 2 \leq j \leq 2i \) for some integer \( i \in \mathbb{N} \). This remains true for later times and the trace of the derivatives

\[
\xi(t, y) := -\partial_x u(t, 0, y) \quad \text{and} \quad \zeta(t, y) = \partial_x^{2i+1} u(t, 0, y)
\]

solve the parabolic system

\[
\begin{align*}
&(NLH) \quad \xi_t - \xi^2 - \partial_{yy} \xi = 0, \\
&(LFH) \quad \zeta_t - (2i + 2)\xi\zeta - \partial_{yy} \zeta = 0.
\end{align*}
\]

Solutions to the nonlinear heat equation \((NLH)\) might blow up in finite time, a dynamics that can be detailed precisely, see [24] for an overview. There exists a stable fundamental blow-up [2, 3, 16, 22], and a countable number of unstable “flatter” blow-ups [3, 14], all driven to leading order by the ODE \( f' = f^2 \). For the present work, we had to show additional weighted estimates than those showed in these articles. In particular, we revisited the proof in [3, 14, 22] and obtained a true improvement for the “flat” unstable blow-ups, see the comment below. For the solutions \( \xi \) to \((NLH)\) below the solution to the linearly forced heat equation \((LFH)\) may also blow-up in finite time with precise asymptotic that we detail later on.

**Theorem 1.** Let \( J \in \mathbb{N} \). There exists an open set for a suitable topology of even solutions to \((NLH)\) blowing up in finite time \( T > 0 \) with

\[
\xi(t, y) = \frac{1}{T - t} + \frac{1}{1 + \frac{y^2}{8(T - t)\log(T - t)}} + \tilde{\xi},
\]

where the remainder \( \tilde{\xi} \) satisfies for \( 0 \leq j \leq J \) for some constant \( C > 0 \):

\[
|\partial_y^j \tilde{\xi}| \leq C \frac{1}{(T - t)\log(T - t)} \left( 1 + \frac{y^2}{8(T - t)\log(T - t)} \right)^{\frac{1}{4}} \left( \sqrt{(T - t)\log(T - t)} + |y| \right)^J.
\]

For any \( k \in \mathbb{N}, k \geq 2, a > 0 \), there exists \( T^* > 0 \), such that for any \( 0 < T < T^* \) there exists an even solution to \((1.7)\) blowing up at time \( T \) with

\[
\xi(t, y) = \frac{1}{T - t + ay^2} + \tilde{\xi},
\]

where the remainders \( \tilde{\xi} \) satisfies for \( j = 0, ..., J \) for some constant \( C > 0 \):

\[
|\partial_y^j \tilde{\xi}| \leq C \frac{1}{(T - t)^{\frac{1}{2k}}} \left( (T - t)^{\frac{1}{2k}} + |y| \right)^J.
\]

**Comments on the result.**

The even assumption is not necessary, it is here to fit the even assumption on \((1.1)\). The construction that we give here for the second case of the unstable blow-ups is not a copy of the seminal previous ones [3, 14, 22], but a bit simpler and more precise. In particular, we extensively use the fact that these profiles are perturbations of the smooth unstable self-similar profiles of the quadratic equation \( f_t = f^2 \), and that away from the origin in self-similar variables the problem is a perturbation of the renormalised quadratic equation \( f_s + f - (Z/2k)f_Z - f^2 = 0 \). We avoid the use of maximum principle as in [14] or of Feynman-Kac formula as in [3, 22], and
obtain a sharp estimate. Namely, the convergence of the solution to the blow-up profile holds in a spatial region that is of size one in original y variables which is the estimate $\text{(1.9)}$. For example, this estimate directly implies the existence of a profile at blow-up time $u(t, y) \rightarrow U^*(y)$ as $t \rightarrow T$ for $y \neq 0$, and that it satisfies $U^*(y) \sim (ay^{2k})^{-1}$ as $y \rightarrow 0$ (this fact would not be obtained directly in previous works).

**Proof of Theorem 1.** The second part, concerning the unstable blow-ups, is proved in Section 4. The proof of the first part for the stable blow-up is very similar, and though our method is a bit simpler than [3, 22] it does not yield truly improved estimates, hence we just sketch the proof in Section 5. \hfill \Box

### 1.4. Statement of the result

The main result of this paper shows how, in a case with symmetries, the viscosity affects the shock formation of Burgers equation, resulting in a concentration of a self-similar shock $\Psi_i$ along the vertical axis ($x = 0$), with scaling parameters that are related to the solution of the parabolic system $(NLH) - (LFH)$. As a consequence, any blow-up solutions to the two one-dimensional equations can be combined to obtain a two-dimensional blow-up. Note that the solutions below can be chosen initially with compact support, and that we are only able to construct them around an initially concentrated blow-up profile. The first theorem involves the stable blow-up of $(NLH)$.

**Theorem 2.** For any $i \in \mathbb{N}^*$ and $b > 0$, there exists solution to $\text{(1.1)}$ blowing up at time $T$ with

$$u(t, x, y) = b^{-1} \lambda^{-\frac{1}{2k}}(t, y)\Psi_i \left( b^{1+\frac{1}{2k}}(t, y)x \right) + \tilde{u}(t, x, y)$$

where $\Psi_i$ is defined by $\text{(2.5)}$ and the transverse scale satisfies

$$\lambda(t, y) = \frac{1}{T - t} \frac{1}{1 + \frac{1}{8(T-t)|\log(T-t)|}} ,$$

and one has the convergence in self-similar variables $(X, Z)$

$$(T-t)^{-\frac{1}{2k}} u \left( (T-t)^{1+\frac{1}{2k}} X, \sqrt{(T-t)|\log(T-t)|} Z \right) \rightarrow b^{-1}(1 + Z^2/8)^{\frac{1}{2k}} \Psi_i \left( b \frac{X}{(1 + Z^2/8)^{1+\frac{1}{4}}} \right)$$

in $C^1$ on compacts sets and for some constants $C > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \leq C(T-t)^{-1}|\log(T-t)|^{-\frac{1}{2}}.$$  \hfill \text{(1.10)}

The second theorem involves the unstable "flat" blow-ups of $(NLH)$.

**Theorem 3.** For any $k, i \in \mathbb{N}^*$, $k \geq 2$, $a, b > 0$, there exists $T^* > 0$, such that for any $0 < T < T^*$ there exists a solution $u$ to $\text{(1.1)}$ odd in $x$ and even in $y$ blowing up at time $T$ with

$$u(t, x, y) = b^{-1} \lambda^{-\frac{1}{2k}}(t, y)\Psi_i \left( b^{1+\frac{1}{2k}}(t, y)x \right) + \tilde{u}(t, x, y)$$

where $\lambda(t, y) = (T - t + ay^{2k})^{-1}$ and one has the convergence in self-similar variables $(X, Z)$

$$(T-t)^{-\frac{1}{2k}} u \left( (T-t)^{1+\frac{1}{2k}} X, (T-t)^{\frac{1}{2k}} Z \right) \rightarrow b^{-1}(1 + aZ^{2k})^{\frac{1}{2k}} \Psi_i \left( b \frac{X}{(1 + aZ^{2k})^{\frac{1}{2}}} \right)$$

in $C^1$ on compact sets and for some constants $C, \eta > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \leq C(T-t)^{-1+\eta}.$$  \hfill \text{(1.13)}
Proof. Theorem 3 is proved in Section 3. It is a consequence of Proposition 13 and Lemma 14. The proof of Theorem 2 is very similar, and is just sketched in Section 5.

1.5. Comments on the result and open problems

1. Stability and symmetry assumptions. For \( i = 1 \) the solutions of Theorem 2 are stable in the symmetry class of functions that are odd in \( x \) and even in \( y \), for a suitable weighted \( C^4 \) topology. These solutions are however unstable by the odd/even symmetry breaking, as can be seen at the linear level in Proposition 12. As there exist also large global solutions of (1.1) of the form \( u = V(t, x - \epsilon y) \) where \( V \) solves the viscous Burgers equation \( V_t + VV_x - \epsilon^2 V_{xx} = 0 \), the generic behaviour is an interesting question.

2. Anisotropy. Very few results concerning a precise description of anisotropic singularity formation exist, despite its fundamental relevance in fluid dynamics. We see that here a wide range of different scaling laws in the \( x \) and \( y \) variables are possible. The formation of shocks for two-dimensional extensions of Burgers equation is studied in [23]. Let us also mention that in [9, 21] anisotropic blow-ups were constructed for the energy supercritical semi-linear heat equation.

3. Connexions between self-similar blow-ups. (1.1) appears to be a good candidate to study connexions between self-similar profiles. As the concentration of \( \Psi_i \) for \( i \geq 2 \) for Burgers is unstable with instabilities yielding to the concentration of \( \Psi_j \) for \( j < i \), and as the same should hold for the unstable blow-ups of (NLH) (see [15] for the genericity result), one interesting result would be to prove rigorously that solutions to (1.1) concentrating the \( i \)-th profile of Burgers and the \( k \)-th of (NLH) (see [15] for the genericity result), one interesting result would be to prove rigorously that solutions to (1.1) concentrating the \( i \)-th profile of Burgers and the \( k \)-th of (NLH) are unstable with instabilities yielding to the concentration of the \( j \)-th profile of Burgers and the \( \ell \)-th of (NLH) for \((j, \ell) < (i, k)\).

4. Continuation after blow-up. The inviscid Burgers equation possesses global weak solutions that can be obtained using a viscous approximation and that are unique under a suitable entropy condition. The investigation of the analogous problem for (1.1) is natural. In particular, if the solution can be continued and has jumps, what is the set of points with discontinuities and its dynamics?

1.6. Ideas of the proof and Organisation of the paper

The result relies on the extension of a lower-dimensional blow-up along a new spatial direction, as in [9, 21]. Self-similar blow-up in Burgers equation is completely studied via direct computations, without technical difficulties. It is an easy setting to understand properties of blow-ups, for example regularity and stability issues and discretely self-similar singularities. The extension along the transverse direction is studied through modulation equations (1.7), which for the first time are non-trivial PDEs. To obtain weighted estimates for (NLH) we adapt [22] and use a new exterior Lyapunov functional in Lemma 28, see the comments below Theorem 1. The blow-up of the solution to (NFL) can then be studied in the same analytical framework. The core of the paper is the 2-d analysis. The ideas are somewhat similar to those used in other contexts of blow-up through a prescribed profile, but are specific to the problem at hand and we hope that they will have other applications in transport and mixed hyperbolic/parabolic problems. We derive a blow-up profile with well-understood properties and linearisation, and build an approximate blow-up profile using modulation to neutralise growing modes. We then construct a solution in its vicinity via a bootstrap argument. We use solely weighted energy
estimates, which are robust and reminiscent of a duality method for the asymptotic linear operator, and derivatives are taken along adapted vector fields to commute well with the equation.

The paper contains two independent sections devoted to Burgers equation and the modulation system, and another one proving the main theorem which can also be read separately as it uses their results as a black box. Section 2 concerns the self-similarity in the blow-ups of Burgers equation. Section 3 is devoted to the proof of Theorem 3, assuming some results for the derivatives on the vertical axis, Theorem 1 and Proposition 10. The blow-up profile and the linearised dynamics are studied in Lemma 11 and Proposition 12, and the heart of the proof is a bootstrap argument in Proposition 13. Section 4 deals with the two Propositions 1 and 10 admitted in Section 3, and concerns in particular the flat blow-ups for the semi-linear heat equation. Finally in Section 5 we sketch how the proof of Theorem 3 can be adapted to prove Theorem 2.

1.7. Notations
We use the Japanese bracket notation
\[ \langle Y \rangle = (1 + Y^2)^{1/2}. \]

For functions having in argument a rescaling of the variable \( X \), we use the general notation \( \tilde{X} \) for their variable, as in \( (\tilde{X} + \tilde{X}^2)(cX) = cX + (cX)^2 \) for example. Depending on the context, \( \tilde{X} \) will also refer to the main renormalised variable
\[ \tilde{X} = \frac{X}{(1 + Z^{2k})^{\frac{1}{2}}} \]
and there should not be confusions. We write \( a \lesssim b \) if there exists a constant \( C \) independent of the other constants of the problem such that \( a \leq Cb \). We write \( a \approx b \) if \( a \lesssim b \) and \( b \lesssim a \). Generally, \( C \) will denote a constant that is independent of the parameters used in the proof, whose value can change from one line to another. When its value depends on some parameter \( p \), we will specify it by the notation \( C(p) \). To perform localisations, the function \( \chi \) is a smooth nonnegative cut-off function, \( \chi = 1 \) on \([-1, 1]\) and \( \chi = 0 \) outside \([-2, 2]\).

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2. Self similarity in shocks for Burgers equation
This section is devoted to the formation of shocks for Burgers equation
\[ U_t + UU_x = 0 \]
This simple equation appears as a toy model for blow-up issues involving self-similar behaviours. However, we did not find works in which this was emphasised apart from [12] (though implicit in some other works) where the existence of smooth self-similar singularities and their linearised dynamics are briefly studied, the usual point of view being geometrical [6]. Everything is explicit, which is convenient as the picture described in Subsection 1.2 shares many similarities with other equations. In particular one sees the link between the regularity of the solution and its blow-up behaviours (this issue appearing in other hyperbolic equations as in [19]).
2.1. Invariances

If \( U(t, x) \) is a solution, then the following function is again a solution by time and space translation, galilean transformation, space and time scaling invariances:

\[
\frac{\mu}{\lambda} U \left( \frac{t - t_0}{\lambda}, \frac{x - x_0 - ct}{\mu} \right) + c.
\]

In particular for \( \lambda \in \mathbb{R}_+^* \) and \( \alpha \in \mathbb{R} \), \( \lambda^{\alpha - 1} U \left( t/\lambda, x/\lambda^\alpha \right) \) is also a solution. The associated infinitesimal generators of the above transformations are\(^1\)

\[
\Lambda_\mu := \text{Id} - x \partial_x, \quad \tilde{\Lambda}_\lambda^{(\alpha)} := -(1 - \alpha) \text{Id} - t \partial_t - \alpha x \partial_x, \quad \check{\Lambda}_c := -t \partial_x + 1, \quad \Lambda_{x_0} := -\partial_x, \quad \Lambda_{t_0} := -\partial_t \tag{2.2}
\]

and there holds the commutators relations

\[
[\Lambda_\mu, \tilde{\Lambda}_\lambda^{(\alpha)}] = 0, \quad [\tilde{\Lambda}_c, \tilde{\Lambda}_\lambda^{(\alpha)}] = -(\alpha - 1) \tilde{\Lambda}_c + 1, \quad [\Lambda_{x_0}, \tilde{\Lambda}_\lambda^{(\alpha)}] = -\alpha \Lambda_{x_0}, \quad [\Lambda_{t_0}, \tilde{\Lambda}_\lambda^{(\alpha)}] = -\Lambda_{t_0}. \tag{2.3}
\]

The tilde comes from the fact that we will use their spatial counterparts:

\[
\Lambda_\alpha := (1 - \alpha) \text{Id} + \alpha X \partial_X, \quad \Lambda_c := \partial_X + 1, \quad \Lambda := -1 + X \partial_X \tag{2.4}
\]

2.2. Self-similar and discretely self-similar solutions

Important solutions are those who constantly reproduce themselves to smaller and smaller scales. To measure their regularity, let us define the following Hölder spaces. For \( i \in \mathbb{N} \) and \( \delta \in (0, 1) \), \( C^{i + \delta} \) is the set of functions \( f \in C^i(\Omega) \) such that

\[
\lim_{x \to x_0} \frac{\partial_x^i f(x) - \partial_x^i f(x_0)}{|x - x_0|^{\delta}} \quad \text{and} \quad \lim_{x \to x_0} \frac{\partial_x^i f(x) - \partial_x^i f(x_0)}{|x - x_0|^{\delta}}
\]

are well-defined for all \( x_0 \in \mathbb{R} \). We then use the notation \( C^{r+} = \cup_{r'>r} C^{r'} \) and \( C^{r-} = \cup_{r'<r} C^{r'} \). Assume \( U \) is a \( C^1 \) solution to Burgers equation becoming singular at a singular point \((t_0, x_0)\). Then one can always use gauge invariance to map it to a solution defined on some domain \((T, 0) \times \mathbb{R} \) with \( T < 0 \), that becomes singular at \((0, 0)\) and such that \( u(t, 0) = 0 \) for all \( t \in (T, 0) \). In particular, \( u_x(t, \cdot) \) is minimal at the origin with \( u_x(t, 0) = -1/t \). The subgroup of the invariances \( \mathbb{R}^3 \times (\mathbb{R}^+)^2 \) preserving these properties is

\[
g = (\lambda, \mu) \in \mathcal{G} := (0, +\infty)^2, \quad g.U : (t, x) \mapsto \frac{\mu}{\lambda} U \left( \frac{t}{\lambda}, \frac{x}{\mu} \right)
\]

Let \( \Omega := (-\infty, 0) \times \mathbb{R} \). The stabiliser of \( u \in C^1(\Omega) \) is the subgroup \( \mathcal{G}_s(U) := \{ g \in \mathcal{G}, g.U = U \} \). Solutions with invariances can be classified according to their regularity.

**Proposition 4** (Classification of self-similar solutions). Let \( U \in C^1(\Omega) \) be a solution to (2.1) with \( U(-1, 0) = 0, \inf_{\mathbb{R}} U_x(-1, \cdot) = U_x(-1, 0) = -1 \) and such that \( \mathcal{G}_s \) is nontrivial. Then three scenarios only are possible and exist, the profiles \( \Psi \in C^1(\mathbb{R}) \) below being defined in Propositions 5 and 6 and in (2.13).

- Analytic self-similarity: \( U \) is analytic and there exists \( i \in \mathbb{N} \) and \( \mu > 0 \) such that

\[
U(t, x) = \mu^{-1}(-t)^{\frac{i}{2}} \Psi_i \left( \mu \frac{x}{(-t)^{1 + \frac{i}{2}}} \right),
\]

or\( U(t, x) = \Psi_\infty(x/(−t)) = x/t. \)

\(^1\)Here \( \text{Id} \) stands for the identity and \( 1 \) for the function with constant value 1.
- Non-smooth self-similarity: There exists \( i, \mu, \mu' > 0 \) with \( i \notin \mathbb{N} \) and \( \mu = \mu' \) (resp. \( i > 0 \) and \( \mu \neq \mu' \)) such that

\[
U(t, x) = (-t)^{\frac{i}{2}} \Psi_{(i, \mu, \mu')}(\frac{x}{(-t)^{1+\frac{i}{2i}}}),
\]

where \( \Psi_{(i, \mu, \mu')} \) is defined by \eqref{eq:psi}, and \( \Psi_{(i, \mu, \mu')} \in C^{1+2i}(\mathbb{R}), \Psi_{(i, \mu, \mu')} \notin C^{1+2i}(\mathbb{R}) \) (resp. \( \Psi_{(i, \mu, \mu')} \in C^{1+2i}(\mathbb{R}), \Psi_{(i, \mu, \mu')} \notin C^{1+2i}(\mathbb{R}) \)).

- Non-smooth discrete self-similarity: There exists \( i > 0 \) and \( \lambda > 1 \) such that \( U \notin C^{1+2i}(\Omega) \) (there exist such solutions with any regularity between \( C^1 \) and \( C^{1+2i-} \)), that for all \( k \in \mathbb{Z} \):

\[
U(t, x) = \lambda^\frac{i}{t} U\left(\frac{t}{\lambda^i}, \frac{x}{\lambda^{k(1+\frac{i}{2i})}}\right),
\]

and that there exists \( (t, x) \in \Omega \) such that \( U(t, x) \neq (-t)^{1/(2i)}U(-1, x/(-t)^{1+1/(2i)}) \).

Before proving the above Proposition 4, let us present the self-similar and discretely self-similar solutions.

**Proposition 5** (Self-similar solutions \cite{12}). There exists a set \( \{\Psi_i, i \in \mathbb{N}^*\} \cup \{\Psi_\infty\} \) of analytic functions on \( \mathbb{R} \) with the following properties. One has \( \Psi_\infty(X) = -X \). For \( i \in \mathbb{N}^* \), the function \( \Psi_i \) is odd, decreasing, and concave on \( (-\infty, 0] \), satisfy the implicit equation

\[
X = -\Psi_i(X) - (\Psi_i(X))^{1+2i}
\]  

(2.5) \hspace{1cm} \text{id:implicit}

and have the following asymptotic expansions:

\[
\Psi_i^{(i)}(X) = -X + X^{2i+1} + \sum_{k=2}^{+\infty} c_{i,k} X^{2ki+1} \text{ as } X \to 0,
\]

(2.6) \hspace{1cm} \text{id:dvptUic}

\[
\Psi_i(X) = -\text{sgn}(X)|X|^{\frac{1}{1+2i}} + \text{sgn}(X)\frac{|X|^{1+\frac{2}{2i+1}}}{2i+1} + O(|X|^{-2i+\frac{3}{2i+1}}) \text{ as } |X| \to +\infty.
\]

(2.7) \hspace{1cm} \text{id:asUinfty}

Moreover, it solves the equation

\[
-\frac{1}{2i} \Psi_i + \frac{2i+1}{2i} X \partial_X \Psi_i + \Psi_\partial_X \Psi_i = 0
\]

(2.8) \hspace{1cm} \text{eq:smoothselfsim}

and any other globally defined \( C^1 \) solution is of the form \( \Psi = \mu^{-1}\Psi_i(\mu X) \) for some \( \mu > 0 \) or is \( -X \) or \( 0 \). The functions \( U^{(\infty)}(t, x) = x/t \) and \( U^{(\mu)}(t, x) = \mu^{-1}(-t)^{1/(2i)}\Psi_i(\mu x/(-t)^{1+1/(2i)}) \) are solutions to \eqref{eq:smoothselfsim}.  

**Proof.** Consider the function \( \phi(\Psi) = -\Psi - \Psi^{2i+1} \) which is an analytic diffeomorphism on \( \mathbb{R} \). Its inverse \( \Psi_i := \phi^{-1} \) satisfies \eqref{eq:smoothselfsim}, \eqref{eq:psi}, \eqref{eq:asUinfty} and the other properties of the proposition from direct computations. Since

\[
\frac{2i}{(\phi^{-1})'(X)} \left( \frac{1}{2i} \phi^{-1}(X) + \frac{2i+1}{2i} X (\phi^{-1})'(X) + \phi^{-1}(X)(\phi^{-1})'(X) \right)
\]

\[
= \frac{1}{(\phi^{-1})'(X)} (-\phi^{-1}(X) + (2i+1)X (\phi^{-1})'(X) + 2i\phi^{-1}(X)(\phi^{-1})'(X))
\]

\[
= -\frac{(2i+1)(\phi^{-1}(X))^{2i+1} + (2i+1)X + 2i\phi^{-1}(X)}{(\phi^{-1})'(X)}
\]

\[
= -\phi^{-1}(X)(-1 - (2i+1)(\phi^{-1}(X))^{2i+1}) + (2i+1)X + 2i\phi^{-1}(X)
\]

\[
= -(2i+1)(-\phi^{-1}(X) - (\phi^{-1}(X))^{2i+1}) + (2i+1)X = 0,
\]
it solves the equation (2.8). Since it solves this equation, \( U(t, x) = (-t)^{(2/\alpha)} \Psi_\alpha(x / (-t)^{1+1/2\alpha}) \) solves (2.1) since
\[
U_t + UU_x = -\alpha(t - 1)(-t)^{-\alpha - 2}\Psi_\alpha \left( \frac{x}{(-t)^\alpha} \right) + \alpha(t - 1)^{\alpha - 2} \frac{x}{(-t)^\alpha} \partial_x \Psi_\alpha \left( \frac{x}{(-t)^\alpha} \right) \\
+ (-t)^{-\alpha - 1}\Psi_\alpha \left( \frac{x}{(-t)^\alpha} \right) (-t)^{-1} \partial_x \Psi_\alpha \left( \frac{x}{(-t)^\alpha} \right) = (-t)^{\alpha - 2} (-\alpha(t - 1)\Psi_\alpha + \alpha(t - 1)\Psi_\alpha \partial_x \Psi_\alpha) \left( \frac{x}{(-t)^\alpha} \right) = 0.
\]
The same reasoning applies for \( \mu^{-1}\Psi_\alpha(\mu X) \) since (2.8) is invariant under the transformation \( \Psi \mapsto \mu^{-1}\Psi(\mu X) \). If \( \Psi \) is another solution to (2.8) with \( -1 < \Psi(1) < 0 \) then using this invariance \( \Psi = \mu^{-1}\Psi(\mu X) \) for some \( \mu > 0 \). If \( \Psi(1) < -1 \) or \( \Psi(1) > 0 \) it is easy to check that the solution is not globally defined.

\[
\square
\]

There also exist solutions reproducing themselves to a smaller scale, but in a somewhat periodic manner, unlike self-similar solutions. They have a fractal behaviour near the origin and are never smooth.

\[\text{Proposition 6} \text{ (Non-smooth discretely self-similar blow-up).} \text{ Let } \alpha > 1, \lambda > 1, X_0, X_1 \in (-\infty, 0) \text{ with } \lambda^{1-\alpha}X_0 < X_1 < \lambda^{-\alpha}X_0 \text{ and consider a function } V \in C^1([X_0, X_1], \mathbb{R}) \text{ satisfying} \]
\[X_1 = \lambda^{-\alpha}X_0 + (\lambda^{-\alpha} - \lambda^{1-\alpha})V(X_0), \]
\[V(X) \in (0, -X) \text{ and } V_X(X) \in (-1, 0) \text{ on } [X_0, X_1], \]
and
\[\lim_{X \to X_1} V(X) = \lambda^{1-\alpha}V(X_0), \lim_{X \to X_1} V_X(X) = \frac{\lambda V_X(X_0)}{1 - (\lambda - 1)V_X(X_0)}. \tag{2.9}\]

There exists a unique odd function \( W \in C^1(\mathbb{R}) \) such that for all \( X \in \mathbb{R}, \)
\[W(X) = \lambda^{1-\alpha}U \left( \lambda^\alpha X + (\lambda^\alpha - \lambda^{1-\alpha})W(X) \right) \tag{2.10}\]
and \( W = V \) on \([X_0, X_1]\). One has \( W(X) \in (0, -X) \) and \( W_X(X) \in (-1, 0) \) for all \( X \in (-\infty, 0) \), and its derivative is minimal at the origin with value \( W_X(0) = -1 \). Let \( i = 1/(2(\alpha - 1)) \). Then
\[0 < \liminf_{X \to 0} \frac{-W(X) - X}{|X|^{1+2i}} \leq \limsup_{X \to 0} \frac{-W(X) - X}{|X|^{1+2i}} < +\infty \]
with equality if and only if \( W(X) = \mu^{-1}\Psi_\alpha(\mu X) \) for some \( \mu > 0 \) where \( \Psi_\alpha \) is given by (2.5). Therefore, unless \( W = \mu^{-1}\Psi_\alpha(\mu X) \) one has \( W \notin C^{2\alpha+1} \). There exist such solutions of regularity \( C^{2\alpha+1-\epsilon} \) for any \( \epsilon > 0 \). Moreover, the solution \( U \) defined on \((0, 0) \times \mathbb{R} \) as the solution to (2.1) with \( U(-1, x) = W(x) \) satisfies
\[U(t, X) = \frac{1}{\lambda^{k(1-\alpha)}} U \left( \frac{t}{\lambda^k}, \frac{X}{\lambda^\alpha} \right) \tag{2.11}\]
for all \((t, X, k) \in (-\infty, 0) \times \mathbb{R} \times \mathbb{Z} \).

\[\text{Remark 7.} \text{ If } (1-\alpha)W + \alpha X W_X + W W_X \neq 0, \text{ then } U \text{ is not of the form } U = (-t)^{\alpha-1}W(x / (-t)^\alpha), \text{ implying that the set of all } k \in \mathbb{R} \text{ such that } (2.11) \text{ hold is isomorphic to } \mathbb{Z} \text{ and that the solution is not continuously self-similar.} \]

\[\text{2The set of such functions is non empty and it contains profiles which do not satisfy } (1-\alpha)v + \alpha X v + X v \neq 0.\]
Proof. We proceed in two steps. First we extend \( V \) in a periodic manner, and then we show the regularity properties.

**Step 1 Construction.** Consider the mapping \( \phi : [X_0, X_1] \to \mathbb{R} \) defined by
\[
\phi(X) = \lambda^{-\alpha}X + (\lambda^{-\alpha} - \lambda^{1-\alpha})V(X).
\]
One has \( \phi(X_0) = X_1 \) and since \( \lambda, \alpha > 1 \) and \( V_X \in (-1, 0) \) one computes
\[
\phi_X(X) = \lambda^{-\alpha} + (\lambda^{-\alpha} - \lambda^{1-\alpha})V_X(X) > \lambda^{-\alpha} > 0
\]
and hence \( \phi \) is a \( C^1 \) diffeomorphism onto its image. Define
\[
X_2 = \lim_{X \to X_1} \phi(X) = \lambda^{-\alpha}X_1 + (\lambda^{-\alpha} - \lambda^{1-\alpha})\lambda^{1-\alpha}V(X_0).
\]
and for \( X \in [X_1, X_2] \) extend \( V \) by
\[
W(X) = \lambda^{1-\alpha}V(\phi^{-1}(X)).
\]

**Claim:** One has \( X_1 < X_2 < 0 \), that \( U \) is \( C^1 \) on \([X_0, X_2]\) and that restricted to \([X_1, X_2]\) it satisfies
the condition of the proposition. Moreover for all \( X \in [X_1, X_2] \), \( \lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1})W(X) \in [X_0, X_1] \) and
\[
W(X) = \lambda^{1-\alpha}W(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1})W(X)).
\]
The proof of this claim involves only basic computations that we safely omit here.

From Claim 1 we see that we can repeat the construction a countable number of times. If \( (X_k)_{k \in \mathbb{N}} \) denotes the set of points coming from the construction then by induction
\[
X_k = \lambda^{-k\alpha}X_0 + (\lambda^{-k\alpha} - \lambda^{k(1-\alpha)})W(X_0)
\]
hence \( X_k \to 0 \). The construction then provides a \( C^1 \) extension \( W \) of \( \phi \) on \((X_0, 0)\) such that
for all \( X \) in this set, \( 0 < W(X) < -X \), \(-1 < W_X < 0 \), and for all \( X_1 \leq X < 0 \), \( \lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1})W(X) \in [X_0, 0] \) and
\[
W(X) = \lambda^{1-\alpha}W(\lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1})W(X)).
\]

**Step 2: Properties.** From the definition of the extensions one has
\[
\sup_{X \in [X_{k+1}, X_{k+2}]} |W(X)| = \lambda^{1-\alpha} \sup_{X \in [X_k, X_{k+1}]} |W(X)|
\]
and therefore \( \lim_{X \to 0} W(X) = 0 \). From (2.10) one sees that
\[
\partial_X W(\lambda^{-\alpha}X + (\lambda^{-\alpha} - \lambda^{1-\alpha})W(X)) = f(\partial_X W(X)), \quad f(a) = \frac{\lambda a}{1 - (1 - \lambda^\alpha).}
\]
f has two fixed points \(-1\) and 0, is increasing on \((-1, 0)\) with \(-1 < f(a) < 0\). Therefore,
\[
-1 < \inf_{[X_{k+1}, X_{k+2}]} \partial_X W = f(\inf_{[X_{k+1}, X_{k+2}]} \partial_X W) \leq \sup_{[X_{k+1}, X_{k+2}]} \partial_X W = f(\sup_{[X_{k+1}, X_{k+2}]} \partial_X W) \to -1
\]
implying that \( \partial_X W(X) \to -1 \) as \( X \to 0 \), and in particular \( \partial_X W \) is minimal at the origin with
\( \partial_X W(0) = -1 \). We now prove the absence of regularity at the origin. Take any \( z_0 \in [X_0, X_1] \)
and define the sequence \( z_k \) by induction following
\[
z_k = \lambda^{-\alpha}z_{k-1} + (\lambda^{-\alpha} - \lambda^{1-\alpha})W(z_{k-1}).
\]
It follows that \( W(z_k) = \lambda^{1-\alpha}W(z_{k-1}) \). By induction,
\[
z_k = \lambda^{-k\alpha}z_0 + (\lambda^{-k\alpha} - \lambda^{k(1-\alpha)})W(z_0) = -\lambda^{k(1-\alpha)}W(z_0)(1 + O(\lambda^{-k}))
\]
as \( k \to +\infty \) since \( \lambda, \alpha > 1 \), with the constant in the \( O \) uniform in \( z_0 \in [X_0, X_1] \). By induction,
\[
-z_{k+1} - W(z_{k+1}) = \lambda^{-k\alpha}(-z_0 - W(z_0)).
\]
Therefore,
\[
\frac{-z_k - W(z_k)}{|z_k|^{\frac{\alpha}{\alpha - 1}}} \to \frac{-z_0 - W(z_0)}{|W(z_0)|^{\frac{\alpha}{\alpha - 1}}} > 0
\]
as \( k \to +\infty \). One then deduces that since the convergence is uniform for \( z_0 \) taken in \([X_0, X_1]\),
\[
0 < \liminf_{X \to 0} \frac{-X - W(X)}{|X|^{\frac{\alpha}{\alpha - 1}}} \leq \limsup_{X \to 0} \frac{-X - W(X)}{|X|^{\frac{\alpha}{\alpha - 1}}} < +\infty.
\]
Therefore the solution is not \( C^{\frac{\alpha}{\alpha - 1}} \) if the equality does not hold. Assume now the equality. This means that there exist a constant \( c > 0 \) such that for any \( X \in [X_0, X_1] \) one has
\[
\frac{-X_0 - W(X_0)}{|W(X_0)|^{\frac{\alpha}{\alpha - 1}}} = c, \quad \text{i.e.} \quad X = -W - c|W|^{\frac{\alpha}{\alpha - 1}}.
\]
\( W \) is then the self-similar profile\(^3\) of Proposition 5, and is not discretely self-similar.

One can apply the same extension technique to define \( W \) on the other side \((-\infty, X_0)\). The uniqueness of the extension follows from an induction, using the fact that if \( W \) is given on some \([X_k, X_{k+1}]\) then it has to be given on \([X_{k+1}, X_{k+2}]\) by the construction we provided. We leave to the reader to prove that if \( V \in C^\gamma \) for some \( 1 < \gamma < \alpha/(\alpha - 1) \) then so is \( W \).

\[\]
Self-similar and discretely self-similar solutions having been presented in Propositions 5 and 6, we can now give the proof of the classification Proposition 4.

**Proof of Proposition 4.** We only sketch the proof, since either the computations involved are rather easy or they are very similar to what can be found in the proofs of Proposition 5 and 6. The stabiliser of \( U \) is closed in \( G \) from the regularity of \( U \). One identifies \( G_s \) to a closed subgroup of \( \mathbb{R}^2 \) via \((z_1, z_2) = (\log(\lambda), \log(\mu))\), and recall that closed subgroups of \( \mathbb{R}^2 \) are isomorphic to one of the following groups: \( \mathbb{Z}, \mathbb{R}, \mathbb{Z} \times \mathbb{Z}, \mathbb{R} \times \mathbb{Z} \) or \( \mathbb{R}^2 \).

**Case 1, \( G_s \cong \mathbb{Z} \):** in that case \( G_s = \{ (\lambda^k, \mu^k), \ k \in \mathbb{Z} \} \) for \( \lambda, \mu \neq (1, 1) \) meaning that
\[
U(t, x) = \frac{\mu^k}{\lambda^k} U \left( \frac{t}{\lambda^k}, x, \mu^k \right), \quad \forall k \in \mathbb{Z}.
\]
One can check that if \( \lambda = 1 \) then \( u = c(t)x \), and if \( \mu = 1 \) then \( U = 0 \), which are contradictions. Hence \( \lambda, \mu \neq 1 \) and we define \( \alpha \in \mathbb{R} \), by \( \mu = \lambda^\alpha \) giving \( G_s = \{ (\lambda^k, \lambda^{k\alpha}), \ k \in \mathbb{Z} \} \). For all \( k \in \mathbb{Z} \),
\[
U(t, x) = \frac{1}{\lambda^{(1-\alpha)k}} U \left( \frac{t}{\lambda^k}, x, \lambda^{k\alpha} \right)
\]
and since \( G_s \not\cong \mathbb{R} \) there exists \((t, x)\) such that \( U(t, x) \neq (-t)^{\alpha-1} U(-1, x/(-t)^\alpha) \). One can always take \( \lambda > 1 \). We take \( t = -1, k = 1, \) to obtain
\[
U \left( \frac{-1}{\lambda}, x \right) = \lambda^{1-\alpha} U(-1, \lambda^\alpha x).
\]
\(^3\)For \( \alpha \neq 1 + 1/(2i) \) for \( i \in \mathbb{N} \) the profiles \( \Psi_{t/(2(\alpha-1))} \) defined in Proposition 5 still exist and have all the corresponding properties, they just are no longer smooth.
From the relation on characteristics
\[ U(-1, x) = U \left( \frac{1}{\lambda}, x + \left( 1 - \frac{1}{\lambda} \right) U(-1, x) \right). \]

Introducing the profile \( W(X) := U(-1, X) \) one deduces that it satisfies
\[ W(X) = \lambda^{1-\alpha} W \left( \lambda^\alpha X + (\lambda^\alpha - \lambda^{\alpha-1}) W(X) \right) \tag{2.12} \]
and that \( W \) is \( C^1 \) with \( W(0) = 0 \) and \( W_X \) minimal at zero with \( W_y(0) = -1 \). We claim that if \( \alpha > 1 \) then \( W \) is a function as described in Proposition 6 in which the above functional equation was studied. We claim that the case \( \alpha = 1 \) is impossible and that if \( \alpha < 1 \) the function is not \( C^1 \) by looking at its behaviour at the origin. Therefore Case 1 corresponds to Proposition 6.

**Case 2**, \( \mathcal{G}_s \simeq \mathbb{R} \): in that case \( \mathcal{G}_s = \{ (\lambda^a, \mu^a), \ a \in \mathbb{R} \} \) for \( (\lambda, \mu) \neq (1, 1) \) meaning that
\[ U(t, x) = \frac{\mu^a}{\lambda^a} U \left( \frac{t}{\lambda^a}, \frac{x}{\mu^a} \right), \ \forall a \in \mathbb{R}. \]

This group of transformation contains the cases \( a \in \mathbb{Z} \), and we have seen in case 1 that one cannot have \( \lambda = 1 \) or \( \mu = 1 \). Hence \( \lambda \neq 1 \) and \( \mu \neq 1 \). Define \( \alpha \) by \( \mu = \lambda^\alpha \) giving (up to an abuse of notation) \( \mathcal{G}_s = \{ (\lambda, \lambda^\alpha), \ \lambda > 0 \} \) and that for all \( \lambda > 0 \),
\[ U(t, x) = \frac{1}{\lambda(1-\alpha)} U \left( \frac{t}{\lambda}, \frac{x}{\lambda^\alpha} \right). \]

In particular, \( u \) is invariant by the transformation
\[ U(t, x) = \frac{1}{\lambda^{k(1-\alpha)}} U \left( \frac{t}{\lambda^k}, \frac{x}{\lambda^{k\alpha}} \right), \]
for any fixed \( \lambda > 1 \) and \( k \in \mathbb{Z} \). We have seen in the study of Case 1 that one cannot have \( \alpha < 1 \) for such an invariance, hence \( \alpha > 1 \). We now write
\[ U(t, x) = \frac{1}{(-t)^{1-\alpha}} U \left( -1, \frac{x}{(-t)^\alpha} \right). \]

Hence the profile \( W(X) = U(-1, X) \) satisfies the equation
\[ (1 - \alpha) W + \alpha X W_X + WW_X = 0. \]

Solutions to this equation with \( W(0) = 0 \), \( W_X \) minimal at 0 with \( W_X(0) = -1 \) have been classified when \( 1/(2(\alpha - 1)) \in \mathbb{N}^* \) in Proposition 5. It is straightforward to check that if \( 1/(2(\alpha - 1)) \notin \mathbb{N}^* \) the profiles \( \Psi_{1/(2(\alpha - 1))} \) defined in Proposition 5 exist, have all the corresponding properties, and are \( C^{\alpha/(\alpha-1)} \). Any self-similar shocks can then be written in the form
\[ \Psi_{(i, \mu, \mu')} (X) = \begin{cases} \mu^{-1} \Psi_i (\mu X) & \text{if } X \leq 0, \\ \mu^{-1} \Psi_{i'} (\mu' X) & \text{if } X \geq 0, \end{cases} \tag{2.13} \]
for \( i \in \mathbb{R}, \ i > 0 \) where \( \Psi_i \) is given by (2.5). When \( \alpha = 1 \), the only solution to \( XW_X + WW_X = 0 \) with \( W(0) = 0 \) and \( W_X(0) = -1 \) is \( W(X) = -X \) which is a contradiction.

**Case 3** If \( \mathcal{G}_s \simeq \mathbb{Z}^2 \) or \( \mathcal{G}_s \simeq \mathbb{Z} \times \mathbb{R} \), one can check that there exists a subgroup of \( \mathcal{G}_s \) of the form \( \{ (\lambda^k, \lambda^{ka}), \ k \in \mathbb{Z} \} \) with \( \lambda > 0 \) and \( \alpha < 0 \). From the study of Case 1, such an invariance is impossible. If \( \mathcal{G}_s \simeq \mathbb{R}^2 \), one can check that \( u(t, x) = x/t \).

\[ \square \]
2.3. Stability and convergence at blow-up to self-similar solutions

The suitable framework for the stability of \( \Psi_i \) is that of self-similar variables where the linearised operator is

\[
H_X := \Lambda_{\alpha_i} + \Psi_i \partial_X + \partial_X \Psi_i = (1 - \alpha_i) + \partial_X \Psi_i + (\alpha_i X + \Psi_i) \partial_X. \tag{2.14}
\]

Proposition 8 (Spectral properties of \( H_X \)). The point spectrum of \( H_X \) on smooth functions is

\[
\gamma(H_X) = \left\{ \frac{j - 2i - 1}{2i}, \quad j \in \mathbb{N} \right\}. \tag{2.15}
\]

The eigenfunctions related to symmetries are

\[
H_X \Lambda_{x_0} \Psi_i = -\alpha_i \Lambda_{x_0} \Psi_i, \quad H_X(\Lambda_{\alpha_i} \Psi_i) = -\Lambda_{\alpha_i} \Psi_i, \quad H_X(\Lambda_c \Psi_i) = -(\alpha_i - 1) (\Lambda_c \Psi_i), \quad H_X \Lambda_{\mu} \Psi_i = 0. \tag{2.16}
\]

More generally, the eigenfunctions are given by the formula:

\[
H_X (\phi_{X,j}) = \frac{j - 2i - 1}{2i} \phi_{X,j}, \quad \phi_{X,j} := \frac{(-1)^k \Psi_i}{1 + (2i + 1) \Psi_i^{2i}}. \tag{2.17}
\]

They have the following asymptotic behaviour:

\[
\phi_{X,j}(X) = X^j - (j + 2i + 1) X^{j+2i} + O(X^{j+4i}) \quad \text{as} \quad X \to 0, \tag{2.18}
\]

\[
\phi_{X,j}(X) = \frac{1}{2i + 1} X^{\frac{j-2i}{2i}} + O(X^{\frac{j-2i}{2i}+\frac{2i}{2i}+\frac{2}{2}}) \quad \text{as} \quad X \to +\infty. \tag{2.19}
\]

Proof. Step 1 Proof of (2.16). Let \( U(t,x) := (-t)^{\alpha_i-1} \Psi_i(x/(-t)^{\alpha_i}) \) which solves (2.1) and by invariance, \( (\tau_c^{(3)} U)_t = -\tau_c^{(3)} \partial_x (\tau_c^{(3)} U) \) for any \( c \in \mathbb{R} \). Differentiating with respect to \( c \) one obtains \( (\tilde{\Lambda}_c U)_t = -\Lambda_c \partial_x U - U \partial_x (\Lambda_c U) \), which evaluated at \( t = -1 \) yields:

\[
\partial_t (\tilde{\Lambda}_c U) (-1, \cdot) = -\Psi_i \partial_x (\Lambda_c \Psi_i) - \partial_X \Psi_i \Lambda_c \Psi_i. \tag{2.20}
\]

Self-similarity implies from (2.2) that \( \tilde{\Lambda}_c^{(\alpha_i)} u = 0 \) hence \( \tilde{\Lambda}_c^{(\alpha_i)} \tilde{\Lambda}_c u + [\tilde{\Lambda}_c, \tilde{\Lambda}_c^{(\alpha_i)}] u = 0 \). This identity reads from the commutator relation (2.3):

\[
\tilde{\Lambda}_c^{(\alpha_i)} \tilde{\Lambda}_c u = (\alpha_i - 1) \tilde{\Lambda}_c u.
\]

At time \( t = -1 \) the above identity yields from (2.2) and (2.4):

\[
\partial_t (\tilde{\Lambda}_c u)(-1, \cdot) - (1 - \alpha_i) \Lambda_c \Psi_i - \alpha_i X \partial_X \Lambda_c \Psi_i = (\alpha_i - 1) \Lambda_c \Psi_i.
\]

From (2.20) the left hand side in this identity is \(-H_X \Lambda_c \Psi_i\), ending the proof of (2.16). The proof for the eigenfunctions related to the other symmetries (2.16) is exactly the same.

Step 2 Proof of (2.15) and (2.17). Assume \( f \) solves \( H_X f = \nu f \). Then using the implicit equation (2.5) one obtains:

\[
\frac{\partial f}{\partial \Psi_i} = f \left[ \frac{\alpha_i + \nu + (\alpha_i - 1 + \nu)(2i + 1) \Psi_i^{2i}}{(\alpha_i - 1) \Psi_i + \alpha_i \Psi_i^{2i+1}} \right] = f \left[ \frac{2i + 1 + 2\nu + (1 + 2i\nu)(2i + 1) \Psi_i^{2i}}{(\alpha_i - 1) \Psi_i + (2i + 1) \Psi_i^{2i+1}} \right]
\]

whose solution is of the form

\[
f \in \text{Span} \left( \frac{\Psi_i^{2i+1 + 2\nu}}{1 + (2i + 1) \Psi_i^{2i}} \right)
\]

From (2.6) the above formula defines a smooth function if and only if \( \nu = (j - 2i - 1)/(2i) \) for some \( j \in \mathbb{N} \).
The smooth self-similar profiles are the asymptotic attractors of all smooth and non-degenerate shocks in the following sense.

**Proposition 9.** Let \( U_0 \in C^\infty(\mathbb{R}) \) be such that \( \partial_x U_0 \) is minimal at \( x_0 \) with
\[
U_0(x_0) = c, \quad \partial_x U_0(x_0) < 0, \quad \partial_{xj}^2 U_0(x_0) = 0 \quad \text{for} \quad j = 2, \ldots, 2i, \quad \text{and} \quad \partial_{x}^{2i+1} U_0(x_0) > 0
\]
for some \( i \in \mathbb{N}^* \). Then \( u \) blows up at time \( T = -1/U_x(x_0) \) at the point \( x_\infty = x_0 + cT \) with:
\[
U(t,x) = \mu^{-1}(T-t)^{\frac{1}{2i}} \Psi_i \left( \frac{x - x_0 - ct}{(T-t)^{1+\frac{1}{2i}}} \right) + c + w(t,x)
\]
where \( \Psi_i \) is defined by Proposition 5, where \( \mu = \left( \frac{\partial_{x}^{2i+1} u(x_0)}{(2i+1)!(-\partial_x U(x_0))^{2i+1}} \right)^{\frac{1}{2}} \) and where
\[
\frac{w}{(T-t)^{\frac{1}{2i}} \Psi_i} \left( \frac{x - x_0 - ct}{(T-t)^{1+\frac{1}{2i}}} \right) \to 0 \quad \text{as} \quad (x,t) \to (x_\infty, T).
\]

**Proof.** Without loss of generality, up to the symmetries of the equation we consider the case \( x_0 = 0, \ U(0) = 0, \ U_x(0) = -1 \) and \( \partial_{x}^{2i+1} U_0(0) = (2i+1)! \), i.e. \( T = 1 = b, \ c = 0 \). For \( 0 \leq t < 1 \) and \( x \in \mathbb{R} \) we have the formula using characteristics for \( |y| \leq 1 \):
\[
U(x,t) = U_0(\phi_t^{-1}(x)), \quad \phi_t(y) = y + tU_0(y) = (1-t)y + y^{2i+1} + O(y^{2i+2}) + O(|y|^{2i+1}|1-t|),
\]
\( \phi_t \) defining a diffeomorphism on \( \mathbb{R} \) for all \( 0 \leq t < 1 \). Given \( (t,x) \) close to \((1,0)\) we look for an inverse \( \phi_t^{-1}(x) \) of the form \(-1-t)^{1/(2i)}\Psi_i(x(1+h)/(1-t)^{1+1/(2i)})\). Since \( |\Psi_i(x)| \lesssim |x|^{1/(2i+1)} \) for \( x \in \mathbb{R} \), we compute using (2.5):
\[
\phi_t \left( -(1-t)^{\frac{1}{2i}} \Psi_i \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) \right)
= - (1-t)^{1+\frac{1}{2i}} \left( \Psi_i + \Psi_i^{2i+1} \right) \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) + O \left( (1-t)^{1+\frac{2}{2i}} \Psi_i^{2i+2} \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) \right)
+ O \left( (1-t)^{2+\frac{1}{2i}} |\Psi_i^{2i+1}| \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) \right)
= x(1+h) + O \left( (1-t)^{1+\frac{2}{2i}} \left( \frac{|x||1+h|}{(1-t)^{1+1/(2i)}} \right)^{1+\frac{1}{2i}} \right) + O \left( (1-t)^{2+\frac{1}{2i}} \frac{|x||1+h|}{(1-t)^{1+1/(2i)}} \right)
= x(1+h) + O(|x|^{1+\frac{1}{2i+1}}|1+h|) + O((1-t)|x||1+h|).
\]
From the intermediate values theorem, there exists \( h = O(|x|^{1/(2i+1)} + (1-t)) \) such that there holds the inverse formula
\[
\phi_t \left( -(1-t)^{\frac{1}{2i}} \Psi_i \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) \right) = x
\]
and there holds
\[
\Psi_i \left( \frac{x(1+h)}{(1-t)^{1+1/(2i)}} \right) = \Psi_i \left( \frac{x}{(1-t)^{1+1/(2i)}} \right) + \int_{\mu=1}^{\mu=1+1} \frac{d\mu}{\mu} \left( \hat{X} \partial_x \Psi_i \right) \left( \frac{x\mu}{(1-t)^{1+1/(2i)}} \right)
= \Psi_i \left( \frac{x}{(1-t)^{1+1/(2i)}} \right) + O \left( |h| \Psi_i \left( \frac{x}{(1-t)^{1+1/(2i)}} \right) \right).
\]
Injecting \( U_0(y) = -y + O(|y|^{2i+1}) \) in (2.23), using (2.24), the bound on \( h \), the above bound and \( (1-t)^{1/(2i)}|\Psi_i|(x/(1-t)^{1+1/(2i)}) \lesssim |x|^{1/(2i+1)} \), one obtains (2.22).
3. Proof of the main Theorem 3

To ease notations we consider the case \( i = 1 \) corresponding to the \( \Psi_1 \) profile for Burgers, the proof being the same for \( i \geq 2 \). Recall the notation for the derivatives on the transverse axis (1.6) and the corresponding system (1.7) that they solve under the odd in \( x \) and even in \( y \) symmetry assumption. Solutions to \((NLH)\) in (1.7) might blow up according to a dynamic described in Theorem 1. The following proposition then describes how the singularity formation for \( \xi \) makes some solutions to the other equation \((LFH)\) in (1.7) blow up in finite time with a precise behaviour. Its proof and that of Theorem 1 are relegated to Section 4 and we prove here Theorem 3 admitting them.

**Proposition 10.** Let \( i = 1 \). For any \( k \in \mathbb{N} \) with \( k \geq 2 \), \( a, b > 0 \) and \( J \in \mathbb{N} \), there exists \( T^* > 0 \) such that for any \( 0 < T < T^* \), there exists \( \xi \) a solution to (1.7) satisfying (1.8) and (1.9), and \( \zeta_0 \) such that the corresponding solution to \((LFH)\) blows up at time \( T \) with

\[
\zeta = \frac{b}{(T-t+ay^{2k})^4} + \tilde{\zeta},
\]

where the remainders \( \tilde{\zeta} \) satisfy for \( j = 0, \ldots, J \) for some constant \( C > 0 \):

\[
|\partial_y^j \tilde{\zeta}| \leq C \frac{1}{((T-t)^{\frac{1}{4k}} + |y|)^{8k+j}} \left( (T-t)^{\frac{1}{4k}} + |y| \right)^{\frac{1}{2}}.
\]

**Proof of Theorem 1 and of Proposition 10.** Section 4 is devoted to their proof. Proposition 24 and the estimates (4.28) for \( \xi \), and Proposition 31 and the estimates (4.48) for \( \zeta \), indeed imply that Theorem 1 and Proposition 10 hold for one particular value of \( a > 0 \) and of \( b > 0 \). One then obtains the general result for any value of \( a \) and \( b \) by using the symmetries of the equation. Namely, \((NLH)\) and \((LFH)\) are invariant by time translation, (1.7) is invariant by the scaling transformation \( \lambda \mapsto \lambda^2 \xi (\lambda^2 t, \lambda y) \) for any \( \lambda > 0 \) and \((LFH)\) is invariant by homothesy since it is linear.

The infinitesimal behaviour near the origin along the transverse axis being understood, we need to ”extend” it along the \( x \) variable. The typical scale along the \( y \) variable is from Theorem 1 and Proposition 10 \( |y| \sim (T-t)^{1/(2k)} \). The typical scale for diffusive effects in a blow-up at the origin at time \( T \) is \( \sqrt{T-t} \). Formally, since \( k \geq 2 \) the diffusive effects are negligible. A reasonable guess is that the blow-up of a solution to (1.1) is given by a shock of Burgers equation \( \lambda^{3/2} \mu \Psi_1 (\lambda^{-3/2} \mu^{-1} x) \) whose two parameters are dictated by (1.7). Let us first give additional properties of \( \Psi_1 \) than those contained in Subsection 1.2. From Proposition 5 it solves

\[
-\frac{1}{2} \Psi_1 + \frac{3}{2} X \partial_X \Psi_1 + \Psi_1 \partial_X \Psi_1 = 0 \tag{3.1}
\]

and has the asymptotic behaviour

\[
\Psi_1 (X) \xrightarrow{X \to 0} -X + X^3 + O(X^5), \quad \Psi_1 (X) \xrightarrow{|X| \to +\infty} -\text{sgn}(X) |X|^\frac{1}{4} + O(|X|^{-\frac{1}{4}}). \tag{3.2}
\]

We define the self-similar variables:

\[
X := \sqrt{\frac{b}{6}} \frac{x}{(T-t)^{\frac{1}{2}}}, \quad Y := a^\frac{1}{4k} \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := e^{-\frac{s}{2k}} Y = \frac{a^\frac{1}{2k} y}{(T-t)^{\frac{1}{2k}}} \tag{3.3}
\]

and

\[
u(t, x, y) = \frac{b}{6} (T-t)^{\frac{1}{2}} v (s, Y)
\]
where the renormalisation factors $\sqrt{b/6}$ and $a^{1/2k}$ will simplify notations. To ease the analysis, since the value of $a$ and $b$ will never play a role in this section, we take

$$a = 1 = b$$  \hspace{1cm} (3.4) \hspace{1cm} \text{eq:def a b}

without loss of generality for the argument. Then $v$ solves from the choice (3.4):

$$v_s - \frac{1}{2} v + \frac{3}{2} X \partial_X v + \frac{1}{2} Y \partial_Y v + v \partial_X v - \partial_Y v = 0.$$  \hspace{1cm} (3.5) \hspace{1cm} \text{main:eqvautosim}

We define accordingly

$$f(s,Y) := -\partial_X v(s,0,Y) = (T-t) \xi(t,y), \quad g(s,Y) := \partial^3_X v(s,0,Y) = (T-t)^{\frac{4}{3}} b \zeta(t,y),$$  \hspace{1cm} (3.6) \hspace{1cm} \text{main:eq:def fg}

which from Theorem 1 satisfy:

$$f(s,Y) = F_k(Z) + \tilde{f}, \quad F_k(Z) := \frac{1}{1 + Z^{2k}}, \quad |\partial^j_Z \tilde{f}| \lesssim e^{-\frac{1}{4k} s}(1 + |Z|)^{\frac{1}{2} - 2k - j}, \quad j = 0,...,J,$$  \hspace{1cm} (3.7) \hspace{1cm} \text{eq:def tildedef f}

$$g(s,Y) = \frac{6}{(1 + Z^{2k})^4} + \tilde{g}, \quad |\partial^j_Z \tilde{g}| \lesssim e^{-\frac{1}{4k} s}(1 + |Z|)^{\frac{1}{2} - 8k - j}, \quad j = 0,...,J,$$  \hspace{1cm} (3.8) \hspace{1cm} \text{eq:def tildedef g}

and solve the system from (1.7) and (3.4):

$$\begin{cases} f_s + f + \frac{k}{2} \partial_Y f - f^2 - \partial_Y f = 0, \\ g_s + 4g + \frac{Y}{2} \partial_Y g - 4f g - \partial_Y g = 0. \end{cases}$$  \hspace{1cm} (3.9) \hspace{1cm} \text{eq:f}

In (3.3), the variable $Y$ is adapted to the viscosity effects whereas the variable that is adapted to the blow-up profile is $Z$. The renormalised function $w(s,X,Z) = v(s,X,Y)$ solves in fact

$$w_s - \frac{1}{2} w + \frac{3}{2} X \partial_X w + \frac{1}{2k} Z \partial_Z w + w \partial_X w - e^{-\frac{1}{k} s} \partial_Z w = 0.$$  \hspace{1cm} (3.10) \hspace{1cm} \text{eq:w}

Since $w$ is a global solution to (3.10) whose derivatives up to third order on the axis $\{X = 0\}$ converge to some fixed profiles from (3.7) and (3.8) one can believe that $w$ converges as $s \to +\infty$ to a profile $w_\infty$ which then has to solve the asymptotic stationary self-similar\(^4\) equation

$$-\frac{1}{2} w_\infty + \frac{3}{2} X \partial_X w_\infty + \frac{1}{2k} Z \partial_Z w_\infty + w_\infty \partial_X w_\infty = 0.$$  \hspace{1cm} (3.11) \hspace{1cm} \text{eq:Q}

**Lemma 11.** For any $a, b > 0$, equation (3.11) admits the following solution that is odd in $X$ and even in $Z$:

$$\Theta[a,b](X,Z) := b^{-1} F_k^{-\frac{1}{2}}(aZ) \Psi_1 \left( b F_k^{\frac{3}{2}}(aZ) X \right)$$

**Proof.** This is a direct computation. First notice that the equation is invariant by the scaling $z \mapsto aZ, x \mapsto bX$ and $w_\infty \mapsto b^{-1} w_\infty$, so that we take $a = b = 1$ without loss of generality. From

\(^4\)Self-similarity is here with respect to the equation (1.1) without viscosity.
\( (3.7) \) and \( (3.1) \):

\[
\begin{align*}
- \frac{1}{2} \Theta[1,1] + \frac{3}{2} \tilde{X} \partial_X \Theta[1,1] + \frac{1}{2k} Z \partial_Z \Theta[1,1] + \Theta[1,1] \partial_X \Theta[1,1] \\
= F_k^{-\frac{1}{2}}(Z) \left( - \frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1 \right) (F_k^{\frac{3}{2}}(Z)X) \\
+ \frac{1}{2k} Z \partial_Z F_k(Z) F_k^{-\frac{3}{2}}(Z) \left( - \frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1 \right) (F_k^{\frac{3}{2}}(Z)X) + F_k^{\frac{3}{2}}(Z)(\Psi_1 \partial_X \Psi_1)(F_k^{\frac{3}{2}}(Z)X) \\
- \frac{1}{2} \partial_X (F_k^{\frac{3}{2}}(Z)X) + \frac{1}{2} Z \partial_Z (F_k^{\frac{3}{2}}(Z)X) \\
= F_k^{-\frac{1}{2}} \left( - \frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1 \right) (F_k^{\frac{3}{2}}(Z)X) \\
+ (F_k^{\frac{1}{2}} - F_k^{-\frac{1}{2}}) \left( - \frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1 \right) (F_k^{\frac{3}{2}}(Z)X) - \frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1 \left( F_k^{\frac{3}{2}}(Z)X \right) = 0.
\end{align*}
\]

The choice \( (3.4) \) implies that the good candidate for \( (3.11) \) is

\[
\Theta(X, Z) := \Theta[1,1](X, Z) = F_k^{-\frac{1}{2}}(Z) \Psi_1 \left( F_k^{\frac{3}{2}}(Z)X \right).
\]  \( (3.12) \)

eq def Theta

The main order linearised operator corresponding to \( (3.10) \) near \( \Theta \) is consequently

\[
L_Z := - \frac{1}{2} + \frac{3}{2} \tilde{X} \partial_X + \frac{1}{2k} Z \partial_Z + \Theta \partial_X + \partial_X \Theta \\
= - \frac{1}{2} + \frac{3}{2} \tilde{X} \partial_X + \frac{1}{2k} Z \partial_Z + F_k^{-\frac{1}{2}} \Psi_1 \left( F_k^{\frac{3}{2}}(Z)X \right) \partial_X + F_k(Z) \partial_X \Psi_1 \left( F_k^{\frac{3}{2}}(Z)X \right).
\]

We claim that its spectral structure can be understood through the spectral analysis of two linearised operators, \( H_X \) for Burgers equation studied in Proposition 8 and \( H_Z \) for the semi-linear heat equation studied in Proposition 21.

\textbf{Proposition 12.} Let \( k \in \mathbb{N}, k \geq 2 \). For any \( (j, \ell) \in \mathbb{N}^2, (j - 3)/2 + (\ell - 2k)/2k + 1 \) is an eigenvalue of the operator \( L_Z : C^1(\mathbb{R}^2) \to C^0(\mathbb{R}^2) \) associated to the eigenfunction

\[
\varphi_{j,\ell}(X, Z) = \phi_{Z,\ell}(Z) F_k^{-1 - \frac{4}{Z}}(Z) \phi_{X,j} \left( F_k^{\frac{3}{2}}(Z)X \right) = Z^j F_k^{1 - \frac{4}{Z}}(Z) \times \frac{(-1)^j \Psi_1 F_k^{\frac{3}{2}}(Z)X}{1 + 3 \Psi_1^2 F_k^{\frac{3}{2}}(Z)X}, \quad \text{(3.13) id:phi0j} \]

where \( \phi_{X,j} \) and \( \phi_{Z,\ell} \) are defined by \( (2.17) \) and \( (4.3) \).
Proof. This is a direct computation. From (4.2), (2.17) and (4.3) one has:

\[
\mathcal{L}_Z \phi_{j,k} = \phi_{Z,\ell} F_k^{-1-\frac{j}{2}} \left( -\frac{1}{2} \phi_{X,j} + \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \phi_{X,j} \right) (F_k^{\frac{3}{2}} X) + \frac{1}{2k} Z \partial_Z \phi_{Z,\ell} F_k^{-1-\frac{j}{2}} \phi_{X,j} (F_k^{\frac{3}{2}} X) + \frac{1}{2k} Z \partial_Z \phi_{Z,\ell} F_k^{2-\frac{j}{2}} (-\frac{j}{2} \phi_{X,j} + \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \phi_{X,j}) (F_k^{\frac{3}{2}} X) + \phi_{Z,\ell} F_k^{-1-\frac{j}{2}} \left( -\frac{1}{2} \phi_{X,j} + \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \phi_{X,j} \right) (F_k^{\frac{3}{2}} X) + \phi_{Z,\ell} \left( F_k^{\frac{j}{2}} - F_k^{-1-\frac{j}{2}} \right) \left( (-\frac{j}{2} \phi_{X,j} + \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \phi_{X,j}) (F_k^{\frac{3}{2}} X) \right) + \phi_{Z,\ell} F_k^{-\frac{j}{2}} \left( \frac{j}{2} - \frac{1}{2} \phi_{X,j} - \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \phi_{X,j} \right) (F_k^{\frac{3}{2}} X) = \left( \frac{j}{2} - \frac{3}{2} \right) F_k^{-1-\frac{j}{2}} \phi_{X,j} (F_k^{\frac{3}{2}} X),
\]

\]
since we are no more in the blow-up zone, and the appropriate profile is 0 rather than \( \Psi_1 \). We set for \( 0 < d \ll 1 \) a cut-off function (note that \( |Y| \leq 2de^{k/2} \) is equivalent to \( |y| \leq 2d \))

\[
\chi_d(s,Y) := \chi \left( \frac{Y}{de^\frac{k}{2}} \right)
\]

and then decompose our solution to (3.5) according to:

\[ v(s,X,Y) = Q + \varepsilon = \chi_d(s,Y)\tilde{\Theta} + (1 - \chi_d(s,Y))\Theta_e + \varepsilon, \quad Q = \chi_d(s,Y)\tilde{\Theta} + (1 - \chi_d(s,Y))\Theta_e \quad (3.18) \]

where \( \tilde{\Theta} \) is the approximate blow-up profile in the interior zone

\[
\tilde{\Theta}(s,X,Y) := \mu^{-1}(s,Y)f^{-\frac{3}{2}}(s,Y)\Psi_1 \left( f^\frac{3}{2}(s,Y)\mu(s,Y)X \right) = \sqrt{6}g^{-\frac{1}{2}}f^\frac{3}{2}\Psi_1 \left( \frac{g^\frac{1}{2}f^{-\frac{1}{2}}}{\sqrt{6}}X \right) \quad (3.19)
\]

where \( f \) and \( g \) are defined in (3.6) and

\[
\mu(s,Y) := \left( \frac{g(s,Y)}{6f^3(s,Y)} \right)^\frac{1}{2},
\]

(3.20)

(notice that for \( d \) small enough and for \( Y \) in the support of \( \chi_d(s,\cdot) \), the functions \( f \) and \( g \) do not vanish from (3.7) and (3.8), and hence \( \mu \) and \( \mu^{-1} \) are well-defined), and where \( \Theta_e \) is the profile for the external zone

\[
\Theta_e(s,X,Y) := \left( -Xf(s,Y) + X^3g(s,Y) \right)e^{-\tilde{X}^4}. \quad (3.21)
\]

To estimate the remainder \( \varepsilon \), we will use weighted Sobolev norms, and to control its derivatives we will use vector fields that commute well with \( \partial_s + \mathcal{L}_Z \):

\[
A := \left( \frac{3}{2}X + \bar{F}_k^{-\frac{3}{2}}(Z)\Psi_1(\bar{F}_k^\frac{3}{2}(Z)X) \right)\partial_X, \quad \partial_Z \quad \text{and} \quad Z\partial_Z \quad (3.22)
\]

and that are equivalent to usual vector fields, see Lemma 41. We claim that \( \varepsilon \) decays thanks to the following bootstrap argument, which is the heart of our proof of Theorem 3.

**Proposition 13.** Let \( \xi \) and \( \zeta \) be given by the second part of Theorem 1 and Proposition 10. For any \( 0 < \kappa \ll 1 \) small enough, there exist large enough constants \( q \in \mathbb{N} \), \( K_{j_1,j_2} \gg 1 \) for integers \( j_1, j_2 \) with \( 0 \leq j_1 + j_2 \leq 2 \), and \( K_{j_1,j_2} \gg 1 \) for integers \( j_1, j_2 \) with \( 0 \leq j_1 + j_2 \leq 2 \) and \( j_2 \geq 1 \), and \( s_0 \gg 1 \) such that if initially the solution is given by (3.18) with \( \varepsilon(s_0) = \varepsilon_0 \) satisfying

\[
\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \frac{(\partial^j_z A^{j_2} \varepsilon_0)^{2q} + ((Y\partial_Y)j_1 A^{j_2} \varepsilon_0)^{2q} dX dY}{\varphi^q_{4,0}(X,Z)} |X| |Y| \leq e^{-2q(\frac{1}{2} - \kappa)s_0} \quad (3.23)
\]

then the solution to (3.5) is global and satisfies for \( 0 \leq j_1 + j_2 \leq 2 \):

\[
\left( \int_{\mathbb{R}^2} \frac{(\partial^j_z A^{j_2} \varepsilon)^{2q} dX dY}{\varphi^q_{4,0}(X,Z)} |X| |Y| \right)^{\frac{1}{2q}} \leq K_{j_1,j_2}e^{-(\frac{1}{2} - \kappa)s}, \quad (3.24)
\]

and for \( 0 \leq j_1 + j_2 \leq 2 \) and \( j_2 \geq 1 \):

\[
\left( \int_{\mathbb{R}^2} \frac{(Y\partial^j_z A^{j_2} \varepsilon)^{2q} dX dY}{\varphi^q_{4,0}(X,Z)} |X| |Y| \right)^{\frac{1}{2q}} \leq K_{j_1,j_2}e^{-(\frac{1}{2} - \kappa)s}. \quad (3.25)
\]
The proof of the above Proposition 13 follows a classical bootstrap reasoning. Namely, throughout the remaining part of this section we assume that $v$ is a solution to (3.5) defined on $[s_0, s_1]$ and such that the decomposition (3.18) satisfies (3.23), (3.24) and (3.25). All the results below will show that (3.24) and (3.25) are in fact strict at time $s_1$, what will allow us to conclude the proof of Proposition 13 by a continuity argument at the end of this section.

First, notice that the bounds of Proposition 13 imply pointwise control by weighted Sobolev embedding.

**Lemma 14.** There holds on $[s_0, s_1]$ with constants in the inequalities depending on the bootstrap constants $K_{j_1, j_2}$ and $K_{j_1, j_2}$:

\[
|\epsilon| \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} (1 + |Z|)^{2k} |X|^4 (1 + |\tilde{X}|)^{\frac{2}{5} - 4} \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} |X|^4 (1 + |Z|)^{3k} + |X|^\frac{2}{5} - 4 \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} |X|,
\]

(3.26)

\[
|\partial_X \epsilon| \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} (1 + |Z|)^{-k} |\tilde{X}|^{3} (1 + |\tilde{X}|)^{\frac{2}{5} - 4} \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} |X|^3 (1 + |Z|)^{3k} + |X|^\frac{2}{5} - 4 \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s}
\]

(3.27)

\[
|\partial_Z \epsilon| \lesssim e^{-\left(\frac{4}{5} - \kappa\right) s} (1 + |Z|)^{-2k - 1} |\tilde{X}|^{4} (1 + |\tilde{X}|)^{\frac{2}{5} - 4}
\]

(3.28)

**Proof.** Recall that $Y \partial_Y = Z \partial_Z$. **Step 1** Proof of (3.26). From the identity

\[
|\langle Y \rangle \partial_Y \epsilon| \lesssim |\partial_Y \epsilon| + |Y \partial_Y \epsilon| = e^{-k \frac{s}{2k}} |\partial_Z \epsilon| + |Z \partial_Z \epsilon|
\]

(3.29)

and the equivalence between vector fields (B.6) we infer that

\[
|\epsilon| + |X \partial_X \epsilon| + |\langle Y \rangle \partial_Y \epsilon| \lesssim |\epsilon| + |A \epsilon| + |\partial_Z \epsilon| + |Z \partial_Z \epsilon|
\]

and therefore the bootstrap bounds (3.24) and (3.25) imply in particular that:

\[
\int_{\mathbb{R}^2} \frac{\epsilon^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY + \int_{\mathbb{R}^2} \frac{(X \partial_X \epsilon)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY + \int_{\mathbb{R}^2} \frac{(|\langle Y \rangle \partial_Y \epsilon|)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY \lesssim e^{-2q \left(\frac{4}{5} - \kappa\right) s}
\]

and the weighted Sobolev embedding (B.2) implies that $|\epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} |\varphi_{4,0}|$ which gives (3.26) using (3.16).

**Step 2** Proof of (3.27). The very same reasoning as in Step 1 show that the bootstrap bounds (3.24) and (3.25) imply $|A \epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} |\varphi_{4,0}|$. From (B.4) and (B.6) we infer that $|A \epsilon| \approx |X \partial_X \epsilon|$, implying then that $|\partial_X \epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} |X|^{-1} |\varphi_{4,0}|$, which yields (3.27) using (3.16).

**Step 3** Proof of (3.28). Using (3.29) and (B.3) we obtain that

\[
|\langle Y \rangle \partial_Y \partial_Z \epsilon| \lesssim |\partial_Z^2 \epsilon| + |Z \partial_Z \epsilon| \lesssim |\partial_Z \epsilon| + |Z \partial_Z \epsilon| \lesssim |\partial_Z \epsilon| + |Z \partial_Z \epsilon| + |(Z \partial_Z)^2 \epsilon|
\]

and from (B.6) that $|X \partial_X \partial_Z \epsilon| \lesssim |A \epsilon| + |\partial_Z A \epsilon|$. Therefore, we infer from (3.24) and (3.25) that

\[
\int_{\mathbb{R}^2} \frac{(\partial_Z \epsilon)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY + \int_{\mathbb{R}^2} \frac{(X \partial_X \partial_Z \epsilon)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY + \int_{\mathbb{R}^2} \frac{(|\langle Y \rangle \partial_Y \partial_Z \epsilon|)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY \lesssim e^{-2q \left(\frac{4}{5} - \kappa\right) s}
\]

implying using (B.2) that $|\partial_Z \epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} |\varphi_{4,0}|$. The very same reasoning applies for the function $Z \partial_Z \epsilon$, yielding $|Z \partial_Z \epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} |\varphi_{4,0}|$. Therefore, $|\partial_Z \epsilon| \lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} (1 + |Z|)^{-1} |\varphi_{4,0}|$ which yields (3.28) using (3.16).
We start by investigating the infinitesimal behavior of $\varepsilon$ near the line $\{X = 0\}$. This corresponds to establishing the so-called modulation equation for the parameters describing the blow-up profile $\Psi_1$ and the external profile $\Theta_e$ on each fixed line $\{Y = Cte\}$. We claim that $\varepsilon$ vanishes on the axis up to the third order.

**Lemma 15.** For all $s \geq 0$ and $Y \in \mathbb{R}$ one has that

$$\partial_X^j \varepsilon(s,0,Y) = 0, \quad j = 0, 1, 2, 3, 4. \quad (3.30)$$

**Proof.** This is a direct computation. First since the profile is odd in $X$ and even in $Y$ one has that $\varepsilon$, $\partial_X^2 \varepsilon$ and $\partial_X^3 \varepsilon$ vanish on the vertical axis $\{X = 0\}$. Then, one has by definition (3.6) of $f$ that $\partial_X v(s,0,Y) = - f(s,Y)$ and from (3.18), (3.19) and (3.21) that

$$\partial_X v(s,0,Y) = - \chi_d f - (1 - \chi_d) f + \partial_X \varepsilon(s,0,Y) = - f + \partial_X \varepsilon(s,0,Y).$$

Therefore $\partial_X \varepsilon(s,0,Y) = 0$ for all $s \geq s_0$ and $Y \in \mathbb{R}$. Similarly, by (3.6), $\partial_X^2 v(s,0,Y) = g(s,Y)$ and from (3.18), (3.19) and (3.21) one has

$$\partial_X^2 v(s,0,Z) = \chi_d b^2 f^4 + (1 - \chi_d) g + \partial_X^2 \varepsilon(s,0,Z) = g + \partial_X^2 \varepsilon(s,0,Z).$$

Therefore $\partial_X^3 \varepsilon(s,0,Z) = 0$ for all $s \geq s_0$ and $Y \in \mathbb{R}$ which ends the proof of the lemma.

The time evolution of $\varepsilon$ is given by:

$$\varepsilon_s + \mathcal{L} \varepsilon + \tilde{L} \varepsilon + R + \varepsilon \partial_X \varepsilon = 0 \quad (3.31)$$

where

$$\mathcal{L} := - \frac{1}{2} + \partial_X \Theta + \left( \frac{3}{2} X + \Theta \right) \partial_X + \frac{1}{2} Y \partial_Y - \partial_{YY} = \mathcal{L}_Z + \frac{k - 1}{2k} Z \partial_Z - \partial_{YY},$$

$$\tilde{L} \varepsilon = (Q - \Theta) \partial_X \varepsilon + (\partial_X Q - \partial_X \Theta) \varepsilon; \quad (3.32)$$

and

$$R = Q_s - \frac{1}{2} Q + \frac{3}{2} X \partial_X Q + \frac{1}{2} Y \partial_Y Q + Q \partial_X Q - \partial_{YY} Q. \quad (3.33)$$

We first investigate the linear dynamics and find an energy estimate that mimics (3.17) in the presence of dissipation.

**Lemma 16.** Assume that $u$ and $\Xi$ are smooth and satisfy:

$$u_s + \mathcal{L} u = \Xi \quad (3.34)$$

on $[s_0, s_1]$, that for some $i_0 \in \mathbb{N}$, for $i = 0, \ldots, i_0$, one has the cancellation on the axis $\{X = 0\}$

$$\partial_X^i u(s,0,Y) = 0. \quad (3.35)$$

and that for $j \in \mathbb{R}$, $0 \leq j < i_0$, and $q \in \mathbb{N}^*$:

$$\int_{\mathbb{R}^2} \frac{u(s_0)^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} < +\infty \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{|\Xi(s,X,Y)|^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} \in L^1([s_0,s_1])$$

Then there exists $C > 0$ independent of $q$ such that for $s_0$ large enough the following energy identity holds:

$$\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{u^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} \right) \leq - \left( \frac{j - 3}{2} - \frac{C}{q} \right) \int_{\mathbb{R}^2} \frac{u^{2q}}{\varphi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} - \frac{2q - 1}{q^2} \int \frac{|\partial_Y (u^q)|^2}{\varphi_{j,0}^{2q}} \frac{dX dY}{|X| |Y|} + \int \frac{u^{2q-1} \Xi}{\varphi_{j,0}^{2q}} \frac{dX dY}{|X| |Y|} \quad (3.36)$$
Proof. This is a direct computation. One computes from the evolution equation (3.34), performing integration by parts,

\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} u^{2q} \, dX dY \right) = \int_{\mathbb{R}^2} \frac{u^{2q} \partial_s u}{\phi_{j,0}^2(X, Z)} \, dX dY - \int_{\mathbb{R}^2} u^{2q} \partial_s \phi_{j,0}(X, Z) \, dX dY
\]

where we used Proposition (12). The integration by parts are legitimate near the axis \( \{X = 0\} \) because of the cancellation (3.35) and since \( \Psi(X) \sim -X \) and \( \phi_{j,0} \sim X^j \) as \( X \to 0 \), and at infinity thanks to the finiteness of the quantity being differentiated with time. The last terms are lower order ones. Indeed, one has:

\[
\left| \partial_Y \left( \frac{Y}{\langle Y \rangle} \right) \right| = \frac{1}{\langle Y \rangle^3}
\]

and

\[
\partial_X \left( \frac{F_k^{-\frac{2}{3}}(Z) \Psi_1(F_k^{\frac{3}{2}}(Z))}{X} \right) = \frac{F_k(Z) \partial_X \Psi_1(F_k^{\frac{3}{2}}(Z))}{X} + \frac{F_k^{-\frac{2}{3}}(Z) \Psi_1(F_k^{\frac{3}{2}}(Z))}{X |X|}.
\]

For the first term in the above identity, one has that \( |F_k(Z)| = (1 + Z^{2k})^{-1} \leq 1 \) and that \( |\partial_X \Psi_1| = |1/(1 + 3\Psi_1^2)| \leq 1 \). For the second, one has that \( |\Psi_1(X)| \leq |X| \). Therefore,

\[
\left| \partial_X \left( \frac{F_k^{-\frac{2}{3}}(Z) \Psi_1(F_k^{\frac{3}{2}}(Z))}{X} \right) \right| \leq \frac{2}{|X|}.
\]
Next, since $|\partial_Z\varphi_{4,0}(X, Z)| \lesssim (1 + |Z|)^{-j}|\varphi_{4,0}(X, Z)|$ from (3.13) and $\partial_Y = e^{-(k-1)s/(2k)}\partial_Z$:

$$\left| \partial_{YY} \left( \frac{1}{\varphi_{4,0}(Y)} \right) \right| \lesssim \frac{(1 + q^2 e^{-k-1/k}s)}{\varphi_{4,0}(Y)}.$$ 

Therefore,

$$\left| \frac{1}{2q} \int \frac{u^{2q}}{\varphi_{3,0}^2} \left( \frac{1}{2} \partial_Y \left( \frac{Y}{Y} \right) - \partial_X \left( \frac{\Psi_1}{|X|} \right) \right) \right| dX dY$$

$$\leq C \left( 1 + q^2 e^{-k-1/k}s \right) \int \frac{u^{2q}}{\varphi_{3,0}^2} dX dY,$$

which, injected in the previous energy identity yields the desired result.

The linear evolution in (3.31) being understood in the previous Lemma (16), we now give estimates for the error $R$ defined by (3.33).

**Lemma 17.** Let $q \in \mathbb{N}$, $q \geq 1$, and $3 + 1/k \leq j \leq 5 - 2/k$. Then one has the estimate

$$\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \frac{\partial_{j_1}^2 A_{j_2}^2 R^{2q} + \left( (Y \partial_Y) \right)^{j_1} A_{j_2}^2 R^{2q}}{\varphi_{4,0}(X, Z)} dX dY \lesssim \rho e^{-q^2}. \tag{3.37}$$

**Proof.** Recall that $Y \partial_Y = Z \partial_Z$. From (3.33), (3.19) and (3.21) we compute:

$$R = R_1 + R_2 + R_3 + R_4 + R_5$$

where

$$R_1 := \chi_d \left( \tilde{\Theta} - \frac{1}{2} \tilde{\Theta} + \frac{3}{2} X \partial_X \tilde{\Theta} + \frac{1}{2} Y \partial_Y \tilde{\Theta} + \tilde{\Theta} \partial_X \tilde{\Theta} - \partial_Y \tilde{\Theta} \right),$$

$$R_2 := (1 - \chi_d) \left( \partial_{\epsilon} \Theta_{\epsilon} - \frac{1}{2} \Theta_{\epsilon} + \frac{3}{2} X \partial_X \Theta_{\epsilon} + \frac{1}{2} Y \partial_Y \Theta_{\epsilon} + \Theta_{\epsilon} \partial_X \Theta_{\epsilon} - \partial_Y \Theta_{\epsilon} \right),$$

$$R_3 := (\tilde{\Theta} - \Theta_{\epsilon}) (\partial_{\epsilon} \chi_d + \frac{1}{2} Y \partial_Y \tilde{\Theta}_{\epsilon} - \partial_Y \chi_d), \quad R_4 := -2 \partial_Y \chi_d \partial_Y (\tilde{\Theta} - \Theta_{\epsilon}),$$

$$R_5 := \chi_d (1 - \chi_d) (\tilde{\Theta} - \Theta_{\epsilon}) \partial_X (\tilde{\Theta} - \Theta_{\epsilon}).$$

We now prove the corresponding bounds for all terms $R_i$.

**Step 1** Estimate for $R_1$. All the computations are performed in the domain of $\chi_d$, $|Y| \lesssim 2de^{s/2}$, where $f, g > 0$. We compute from (3.19), (3.20), (3.9), (3.9) and Lemma 11 that:

$$\tilde{\Theta} - \frac{1}{2} \tilde{\Theta} + \frac{3}{2} X \partial_X \tilde{\Theta} + \frac{1}{2} Y \partial_Y \tilde{\Theta} + \tilde{\Theta} \partial_X \tilde{\Theta} - \partial_Y \tilde{\Theta}$$

$$= -\sqrt{6} (\partial_Y g)^{2} f^{-\frac{5}{2}} f^{\frac{3}{2}} \left[ \left( \frac{3}{2} + \frac{1}{2} \tilde{X} \partial_{\tilde{X}} \right)(-\frac{1}{2} \tilde{\Psi}_1 + \frac{1}{2} \tilde{X} \partial_{\tilde{X}} \tilde{\Psi}_1) \right] \left( \frac{g^{\frac{1}{2}} f^{-\frac{1}{2}}}{\sqrt{6}} X \right)$$

$$-\sqrt{6} \partial_Y g \partial_Y f^{-\frac{5}{2}} f^{\frac{1}{2}} \left[ \frac{3}{2} \tilde{\Psi}_1 + \frac{3}{2} \tilde{X} \partial_{\tilde{X}} \tilde{\Psi}_1 - \frac{1}{2} \tilde{X}^2 \partial_{\tilde{X}} \tilde{\Psi}_1 \right] \left( \frac{g^{\frac{1}{2}} f^{-\frac{1}{2}}}{\sqrt{6}} X \right)$$

$$-\sqrt{6} g^{-\frac{1}{2}} (\partial_Y f)^{2} f^{-\frac{1}{2}} \left[ \frac{1}{2} - \frac{1}{2} \tilde{X} \partial_{\tilde{X}} \right] \left( \frac{g^{\frac{1}{2}} f^{-\frac{1}{2}}}{\sqrt{6}} X \right)$$
We only treat the first term, the proof being the same for the others. First, from (3.7) and (3.8):
\[(1 + |Z|)^{-3k} \lesssim |g_\frac{f}{f} (s, Y) f^{-\frac{j}{2}} (s, Y)| \lesssim (1 + |Z|)^{-3k}, \quad |(Y \partial_Y)^j \chi_d| \lesssim 1\]
on the support of \(\chi_d\). For any \(j \in \mathbb{N}\), from (3.2) and (3.2) one has that
\[
\left| \left( \hat{X} \partial_{\hat{X}} \right)^j \left( -\frac{3}{2} + \frac{1}{2} \hat{X} \partial_{\hat{X}} \right) (\hat{X}) \right| \lesssim |\hat{X}|^5 (1 + |\hat{X}|)^{\frac{1}{2} - 5}
\]
and from (3.7) and (3.8) and for \(j_1 + j_1' \leq J\):
\[
\frac{|(Y \partial_Y)^j \partial_{\hat{Z}}^j (g_\frac{f}{f} f^{-\frac{1}{2}})|}{g_\frac{f}{f} f^{-\frac{1}{2}}} \lesssim (1 + |Z|)^{-j'}, \quad |(Y \partial_Y)^j \partial_{\hat{Z}}^j ((\partial_Y g)^2 g^{-\frac{f}{2}} f^{\frac{1}{2}})| \lesssim e^{-\frac{k-1}{s} (1 + |Z|)^{j - j'}}
\]
since \(\partial_Y = e^{-(k-1)s/(2k)} \partial_Z\). We therefore infer that:
\[
\left| (Y \partial_Y)^j (\partial_{\hat{Z}}^j (X \partial_X)^j) \left( \left( -\frac{3}{2} + \frac{1}{2} \hat{X} \partial_{\hat{X}} \right) (\hat{X}) \right) \right| \lesssim |\hat{X}|^5 (1 + |\hat{X}|)^{\frac{1}{2} - 5} \lesssim \frac{|X|^5 ((1 + |Z|)^{3k} + |X|)^{\frac{1}{2} - 5}}{(1 + |Z|)^k},
\]
and in turn, using (3.16):
\[
\left| (Y \partial_Y)^j (\partial_{\hat{Z}}^j (X \partial_X)^j) \left( (\partial_Y g)^2 g^{-\frac{f}{2}} f^{\frac{1}{2}} \left( -\frac{3}{2} + \frac{1}{2} \hat{X} \partial_{\hat{X}} \right) (\hat{X}) \right) \right| \lesssim \frac{e^{-\frac{k-1}{s} |X|^{5-j} (1 + |Z|)^{-2} ((1 + |Z|)^{3k} + |X|)^{j - \frac{1}{2} - \frac{5}{2}}}}{|\varphi_{j,0}(X, Z)|}
\]
Therefore, one has the following estimate, performing two changes of variables, the first one being \(X = (1 + Z^{2k})^{3/2} \hat{X}\) and the second one \(Z = e^{-(k-1)s/(2k)} Y\), with \(dX/|X| = d\hat{X}/|\hat{X}|, dY/|Y| = dZ/|Z|, \) since \(|Y|/|Y| \leq 1\) and \(|(Y \partial_Y)^j \partial_{\hat{Z}}^j \chi_d| \lesssim 1:\)
\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \left( (Y \partial_Y)^j (\partial_{\hat{Z}}^j (X \partial_X)^j) \chi_d (\partial_Y g)^2 g^{-\frac{f}{2}} f^{\frac{1}{2}} \left( -\frac{3}{2} + \frac{1}{2} \hat{X} \partial_{\hat{X}} \right) (\hat{X}) \right) \right)^{2q} dX dY
\]
\[
\lesssim e^{-2q \frac{k-1}{s}} \int_{\mathbb{R}^2} \left( |X|^{5-j} (1 + |Z|)^{-2} ((1 + |Z|)^{3k} + |X|)^{j - \frac{1}{2} - \frac{5}{2}} \right)^{2q} dX dY
\]
\[
\lesssim e^{-2q \frac{k-1}{s}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( (1 + |Z|)^{3k(1 - \frac{1}{2} - \frac{5}{2})} |\hat{X}|^{5-j} \right) \right)^{2q} d\hat{X} \left( \frac{dY}{|Y|} \right)
\]
\[
\lesssim e^{-2q \frac{k-1}{s}} \int_{|Y| \leq e^{-\frac{k-1}{s} / 2k}} \frac{dY}{|Y|} + \int_{|Y| \geq e^{-\frac{k-1}{s} / 2k}} \left( (1 + |Z|)^{3k(1 - \frac{1}{2} - \frac{5}{2})} \right)^{2q} dY \left( \frac{dY}{|Y|} \right)
\]
\[
\lesssim e^{-2q \frac{k-1}{s}} \left( s + \int_{|Z| \geq 1} \left( (1 + |Z|)^{3k(1 - \frac{1}{2} - \frac{5}{2})} \right)^{2q} \frac{dZ}{|Z|} \left( \frac{dY}{|Y|} \right) \right) \lesssim s e^{-2q \frac{k-1}{s}}
\]
provided that \(3 < j < 5\). We claim that the very same estimates for the other terms in the expression of \(R_1\) holds, and that they can be proved performing the same computation, thanks
to the same fundamental cancellations
\[
\left[ -\frac{3}{2} \Phi_1 + \frac{3}{2} \hat{X} \partial_\hat{X} \Phi_1 - \frac{1}{2} \hat{X}^2 \partial_\hat{X}^2 \Phi_1 \right] (\hat{X}) + \left[ \frac{1}{2} - \frac{1}{2} \hat{X} \partial_\hat{X} \right] \left( \frac{3}{2} - \frac{1}{2} \hat{X} \partial_\hat{X} \Phi_1 \right) (\hat{X}) \lesssim |\hat{X}|^5 (1 + |\hat{X}|)^{\frac{1}{2} - 5},
\]
implying that for \( j = 4 \), since \( k \geq 2 \):
\[
\sum_{0 \leq j_1 + j_2 \leq 4} \int_{\mathbb{R}^2} \left( (Y \partial_Y)^{j_1} (X \partial_X)^{j_2} R_4 \right)^{2g} dX dY \lesssim e^{-2q(k-1)\gamma} \lesssim e^{-qs},
\]
which can be rewritten using the equivalence (B.3), (B.4) and (B.5):
\[
\sum_{0 \leq j_1 + j_2 \leq 4} \int_{\mathbb{R}^2} \left( (Y \partial_Y)^{j_1} A^{j_2} \right)^{2g} dX dY \lesssim e^{-qs}.
\]

**Step 2:** Estimate for \( R_2 \): Note that \( |Z| \geq de^{\gamma/(2k)} \gg 1 \) on the support of \( 1 - \chi_d \). We first compute using (3.21), (3.9) and (3.9):

\[
\partial_\theta_\epsilon - \frac{1}{2} \Theta_\epsilon + \frac{3}{2} X \partial_X \Theta_\epsilon + \frac{1}{2} Y \partial_Y \Theta_\epsilon + \Theta_\epsilon \partial_X \Theta_\epsilon - \partial_Y \Theta_\epsilon
\]
\[
= \frac{1}{12} X^5 g^2 e^{-2\hat{X}^4} + (-X f + X^3 g/6) (\partial_\theta + \frac{3}{2} X \partial_X + \frac{1}{2} Y \partial_Y - \partial_Y) (e^{-\hat{X}^4})
\]
\[
+ (X f^2 - \frac{4}{12} X^3 g f) e^{-\hat{X}^4} \left( e^{-\hat{X}^4} - 1 \right)
\]
\[
+ (-X f + \frac{X^3 g}{6} g^2) e^{-\hat{X}^4} \partial_X \left( e^{-\hat{X}^4} \right) - 2 \partial_Y (-X f + \frac{X^3 g}{6}) \partial_Y \left( e^{-\hat{X}^4} \right)
\]
\[
(3.39)
\]

In what follows \( 0 < \gamma \ll 1 \) denotes a small constant whose value can change from one line to another. For the first term, (3.8) and (3.8) imply that on the support of \( 1 - \chi_d \) for \( j_1, j_1', j_2 \in \mathbb{N} \) with \( j_1 + j_1' \leq J \):

\[
\left| (Y \partial_Y)^{j_1} \partial_\theta_\epsilon (X \partial_X)^{j_2} \left( X^5 g^2 e^{-2\hat{X}^4} \right) \right| \lesssim e^{-\frac{q}{2k}(1 + |Z|)^{1-k+j_1} \hat{X}^5 e^{-\gamma \hat{X}^4}}.
\]

Therefore, using (3.16):

\[
\left| (Y \partial_Y)^{j_1} \partial_\theta_\epsilon (X \partial_X)^{j_2} \left( X^5 g^2 e^{-2\hat{X}^4} \right) \right| \lesssim e^{-\frac{q}{2k}(1 + |Z|)^{1-k+j_1} \hat{X}^5 e^{-\gamma \hat{X}^4}}.
\]

Since \( dX/|X| = d\hat{X}/|\hat{X}| \) and \( |Z| \gtrsim e^{\frac{1}{2k}q} \) on the support of \( 1 - \chi_d \) and \( |(Y \partial_Y)^{j_1} \partial_\theta_\epsilon \chi_d| \lesssim 1 \) one then infers that:

\[
\int_{\mathbb{R}^2} \left( (Y \partial_Y)^{j_1} \partial_\theta_\epsilon (X \partial_X)^{j_2} \left( (1 - \chi_d) X^5 g^2 e^{-2\hat{X}^4} \right) \right)^{2q} dX dY \lesssim e^{-\frac{q}{4k}} \int_{|Z| \geq e^{\frac{1}{2k}q} \gamma} \left( 1 + |Z| \right)^{1-k+j_1-3} \hat{X}^{5-j} e^{-\gamma \hat{X}^4} \lesssim e^{-\frac{q}{4k}} \int_{|Z| \geq e^{\frac{1}{2k}q} \gamma} |Z|^{1-k+j_1-3} 2^q dZ \lesssim e^{-\frac{q}{4k}} \int_{|Z| \geq e^{\frac{1}{2k}q} \gamma} |Z|^{1-k+j_1-3} \lesssim e^{-\frac{q}{4k}} \int_{|Z| \geq e^{\frac{1}{2k}q} \gamma} |Z|^{1-k+j_1-3} \leq 5 \lesssim |\hat{X}|^4 e^{-\gamma \hat{X}^4}.
\]
Therefore, from (3.7), (3.7), (3.8), (3.8) and (3.16) we obtain that:

\[
\left| \frac{(Y \partial_Y)_{j_1} \partial^j_Z (X \partial X)_{j_2} \left( (-X f + X^3 \frac{q}{6})(\partial_s + \frac{3}{2} X \partial_X + \frac{1}{2} Y \partial_Y - \partial_Y Y) \left(e^{-\hat{X}^4}\right)\right)}{\varphi_{4,0}(X, Z)} \right| \\
\lesssim e^{-\frac{4}{5}s}(1 + |Z|)^{\frac{1}{2} - k(j-3)}|\hat{X}|^{-5-j}e^{-\gamma \hat{X}^4}
\]
Since \(dX/|X| = d\hat{X}/|\hat{X}|, |Z| \geq e^{\frac{4}{5}s}\) and \(|(Y \partial_Y)_{j_1} \chi_d| \lesssim 1\) on the support of \(1 - \chi_d\) one then infers that:

\[
\left| \frac{(Y \partial_Y)_{j_1} \partial^j_Z (X \partial X)_{j_2} \left( (1 - \chi_d)(-X f + X^3 \frac{q}{6})(\partial_s + \frac{3}{2} X \partial_X + \frac{1}{2} Y \partial_Y - \partial_Y Y) \left(e^{-\hat{X}^4}\right)\right)}{\varphi_{j,0}(X, Z)} \right|^{2q} \\
\lesssim e^{-\frac{4}{5}s} \int_{|Z| \geq e^{\frac{4}{5}s}} \left( (1 + |Z|)^{\frac{1}{2} - k(j-3)}|\hat{X}|^{-5-j}e^{-\gamma \hat{X}^4}\right)^{2q} \frac{d\hat{X}dZ}{|X||Y|} \\
\lesssim e^{-\frac{4}{5}s} \int_{|Z| \geq e^{\frac{4}{5}s}} |Z|^{\frac{1}{2} - k(j-3)2q} \frac{dZ}{|Z|} \lesssim e^{-(j-3)qs}
\]
provided that \(3 + 1/(2k) < j < 5\). We claim that all the other remaining terms in (3.38) can be treated verbatim the same way, yielding for \(j = 4\):

\[
\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \left| \frac{(Y \partial_Y)_{j_1} (X \partial X)_{j_2} R_2^{2q} + (\partial^j_Z (X \partial X)_{j_2} R_2^{2q})}{\varphi_{4,0}^{2q}(X, Z)} \frac{dXdY}{|X||Y|} \right| \lesssim e^{-2q\frac{k-1}{k} s} \lesssim e^{-qs},
\]
which can be rewritten using the equivalence (B.3), (B.4) and (B.5):

\[
\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \left| \frac{(Y \partial_Y)_{j_1} A^{j_2} R_2^{2q} + (\partial^j_Z A^{j_2} R_2^{2q})}{\varphi_{4,0}^{2q}(X, Z)} \frac{dXdY}{|X||Y|} \right| \lesssim e^{-qs}.
\]

**Step 3 Estimate for \(R_3\):** one first notices that on the support of this term \(de^{s/2} \leq |Y| \leq 2de^{s/2}\), which will then be assumed throughout this step, and that for \(j_1, j_1' \in \mathbb{N}\):

\[
|(Y \partial_Y)_{j_1} \partial^j_Z (\partial_s \chi_d + \frac{1}{2} Y \partial_Y \chi_d - \partial_Y \chi_d)| \lesssim 1. \tag{3.40}
\]
Also, \(\partial^j_X \tilde{\Theta} = \partial^j_X \Theta\) for \(j = 0, ..., 4\) on the axis \(\{X = 0\}\). From this, the formulas (3.19) and (3.21), and the estimate (3.7), (3.7), (3.8) and (3.8) one obtains that if \(j_1 + j_1' \leq J\):

\[
|(Y \partial_Y)_{j_1} \partial^j_Z (X \partial X)_{j_2} (\tilde{\Theta} - \Theta_e)| \lesssim (1 + |Z|)^{k-j'}|\hat{X}|^{5-j}(1 + |\hat{X}|)^{\frac{1}{2} - 5}
\]
giving using (3.16) the estimate:

\[
\frac{|(Y \partial_Y)_{j_1} \partial^j_Z (X \partial X)_{j_2} (\tilde{\Theta} - \Theta_e)|}{|\varphi_{j,0}(X, Z)|} \lesssim (1 + |Z|)^{-k(j-3)}|\hat{X}|^{-5-j}(1 + |\hat{X}|)^{-4+j-\frac{1}{2}}
\]
The above estimate and (3.40) therefore imply, since \(|Z| \sim e^{\frac{4}{5}s}\) and since \(dX/|X| = d\hat{X}/|\hat{X}|\):

\[
\left| \frac{(Y \partial_Y)_{j_1} \partial^j_Z (X \partial X)_{j_2} R_3^{2q}}{\varphi_{j,0}^{2q}(X, Z)} \frac{dXdY}{|X||Y|} \right| \lesssim \int_{e^{\frac{4}{5}s} \leq |Z| \leq 2e^{\frac{4}{5}s}} \left( (1 + |Z|)^{-k(j-3)}|\hat{X}|^{-5-j}(1 + |\hat{X}|)^{-4+j-\frac{1}{2}} \right)^{2q} \frac{d\hat{X}dZ}{|X||Z|} \lesssim e^{-(j-3)qs}
\]
provided that $3 < j < 5$. Taking $j = 4$ and using (B.3), (B.4) and (B.5) this gives:

$$\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \frac{\left(\nabla Y\right)^{j_1} A^{j_2} R_3}{\varphi_{4,0}^q(X, Z)} dX dY \lesssim e^{-q s}.$$ 

**Step 4 Estimate for $R_4$ and $R_5$:** These estimates can be proved along the very same lines as we just estimated $R_1$, $R_2$ and $R_3$. We safely leave the proof to the reader in order to keep the present article short.

We now estimate the lower order linear term in (3.31).

**Lemma 18.** There holds on $[s_0, s_1]$, for $j_1 \leq J$ and $j_2 \in \mathbb{N}^*$:

$$|\nabla_2 \nabla^2 (Q - \Theta)| \lesssim e^{-\frac{1}{4} s} |X| (1 + |Z|)^{-j_1} \quad \text{and} \quad |\nabla_2 \nabla_1 (Q - \Theta)| \lesssim e^{-\frac{1}{4} s} (1 + |X|)(1 + |Z|)^{-j_1}.$$  

(3.41)

**Proof. Step 1 Inner estimate.** In the zone $|Y| \leq de^{s/2}$, from (3.18) and (3.12):

$$Q - \Theta = \Theta - \Theta = b^{-1} f^{-\frac{1}{2}} \Psi (b^{-1} f^{-\frac{1}{2}} X) - F_k^{-\frac{1}{2}} (Z) \Psi (F_k^{-\frac{1}{2}} (Z) X).$$

Therefore, using (3.7), (3.8) and (3.20) (implying that in this zone $\mu$ and $f/F_k$ are close to 1):

$$|Q - \Theta| = \left| f^{-\frac{1}{2}} \Psi (\mu f^{-\frac{1}{2}} X) - f^{-\frac{1}{2}} \Psi (f^{-\frac{1}{2}} X) + f^{-\frac{1}{2}} \Psi (f^{-\frac{1}{2}} X) - F_k^{-\frac{1}{2}} (Z) \Psi (F_k^{-\frac{1}{2}} (Z) X) \right|$$

$$= \left| f^{-\frac{1}{2}} \int_{\bar{\mu} = 1}^\mu \mu^{-\frac{1}{2}} \Psi (1 + \bar{X} \partial_X \Psi (\bar{\mu} f^{-\frac{1}{2}} X)) d\mu + F_k^{-\frac{1}{2}} \int_{\lambda = 1}^{F_k^{-\frac{1}{2}}} \lambda^{-\frac{1}{2}} \Psi (1 + \frac{3}{2} 
abla \partial_X \Psi (F_k^{-\frac{1}{2}} X)) d\lambda \right|$$

$$\lesssim \left| f^{-\frac{1}{2}} \Psi (\mu f^{-\frac{1}{2}} X) - f^{-\frac{1}{2}} \Psi (f^{-\frac{1}{2}} X) \right| \lesssim e^{-\frac{1}{4} s} (1 + |Z|)^{-k+\frac{1}{2}} |X|^{-\frac{1}{4} + \frac{1}{2}} \lesssim e^{-\frac{1}{4} s} |X|.$$

One computes similarly that

$$|\partial_X (Q - \Theta)| = \left| f \partial_X \Psi (\mu f^{-\frac{1}{2}} X) - f \partial_X \Psi (f^{-\frac{1}{2}} X) + f \partial_X \Psi (f^{-\frac{1}{2}} X) - F_k (Z) \partial_X \Psi (F_k^{-\frac{1}{2}} (Z) X) \right|$$

$$= \left| f \int_{\bar{\mu} = 1}^\mu \bar{\mu}^{-\frac{1}{2}} \partial_X \partial_X \Psi (\bar{\mu} f^{-\frac{1}{2}} X) d\bar{\mu} + F_k \int_{\lambda = 1}^{F_k^{-\frac{1}{2}}} \lambda^{-\frac{1}{2}} \partial_X \partial_X \Psi (F_k^{-\frac{1}{2}} X) d\lambda \right|$$

$$\lesssim \left| f \partial_X \Psi (\mu f^{-\frac{1}{2}} X) - f \partial_X \Psi (f^{-\frac{1}{2}} X) \right| \lesssim e^{-\frac{1}{4} s} (1 + |Z|)^{-2k+\frac{1}{2}} |X|^{-\frac{1}{4} + \frac{1}{2}} \lesssim e^{-\frac{1}{4} s}.$$ 

The proof for higher order derivatives is a direct generalisation of the above computations, that we safely omit here, giving (3.41) and (3.41) in this zone.
**Step 2 Outer estimate.** Let $0 < \gamma \ll 1$ be a small constant whose value can change from one line to another. We now turn to the zone $de^{s/2} \leq |Y| \leq 2de^{s/2}$ or equivalently $de^{s/(2k)} \leq |Z| \leq 2de^{s/(2k)}$. We perform brute force estimates on the identity (3.18) using (3.7) and (3.8):

\[
|Q - \Theta| = |\chi_d \Theta + (1 - \chi_d) \Theta_e - \Theta| \leq |\Theta| + |\Theta_e| + |\Theta|
\]

\[
\lesssim (1 + |Z|)^k |\tilde{X}|^k + (1 + |Z|)^k |\tilde{X}|(1 + |\tilde{X}|)^{s-1} + (1 + |Z|)^k |\tilde{X}|e^{-\gamma \tilde{X}^4}
\]

\[
\lesssim e^{-\frac{s}{4}} (1 + |Z|)^{k+\frac{1}{2}} |\tilde{X}|(1 + |\tilde{X}|)^{s-1} \lesssim e^{-\frac{s}{4}} (1 + |Z|)^{-2k+\frac{1}{2}} |X|(1 + |\tilde{X}|)^{s-1}
\]

and similarly

\[
|\partial_X (Q - \Theta)| \lesssim |\partial_X \Theta| + |\partial_X \Theta_e| + |\partial_X \Theta|
\]

\[
\lesssim (1 + |Z|)^{-2k} (1 + |\tilde{X}|)^{-\frac{s}{2}} + (1 + |Z|)^{-2k} (1 + |\tilde{X}|)^{-\frac{s}{2}} + (1 + |Z|)^{-2k} e^{-\gamma \tilde{X}^4}
\]

\[
\lesssim e^{-\frac{s}{4}} (1 + |Z|)^{-2k+\frac{1}{2}} (1 + |\tilde{X}|)^{-\frac{s}{4}} \lesssim e^{-\frac{s}{4}}
\]

Again, the generalisation of this argument for higher order derivatives is direct, yielding (3.41) and (3.41) in this zone.

**Step 3 Outer estimate.** We now turn to the zone $|Y| \geq 2de^{\frac{s}{2}}$ where $Q - \Theta = \Theta_e - \Theta$. We perform the very same computations as in Step 2, estimating $\Theta$ and $\Theta_e$ separately, giving (3.41) and (3.41) in this zone and ending the proof of the Lemma.

\[\square\]

We can now perform energy estimates in the bootstrap regime of Proposition 13 and improve the bootstrap bounds.

**Lemma 19.** There exists $K_{0,0}$ large enough independent of the other constants in the bootstrap argument such that at time $s_1$ there holds:

\[
\left( \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}(s_1)}{\varphi^{2q}_{4,0} (X, Y)} dX dY \right)^{\frac{1}{2q}} \leq \frac{K_{0,0}}{2} e^{-(\frac{1}{2} - \kappa)s_1}.
\]

**Proof.** We compute from (3.31) and (3.36):

\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi^{2q}_{4,0} (X, Z)} dX dY \right) \leq - \left( \frac{1}{2} - \frac{C}{q} \right) \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi^{2q}_{4,0} (X, Z)} dX dY - \frac{2q - 1}{q^2} \int \frac{dY}{\varphi^{2q}_{4,0} (X, Y)} dX dY
\]

\[
+ \frac{1}{2q} \int \varepsilon^{2q} \partial_X \left( \frac{Q - \Theta}{\varphi^{2q}_{4,0} |X|} \right) dX dY - \int \frac{\varepsilon^{2q-1}}{\varphi^{2q}_{4,0}} (\varepsilon \varepsilon_X + R + \partial_X (Q - \Theta) \varepsilon) \frac{dX dY}{|X| |Y|}.
\]

We now estimate the last terms. First,

\[
\partial_X \left( \frac{Q - \Theta}{\varphi^{2q}_{4,0} |X|} \right) = \partial_X (Q - \Theta) \frac{\varphi^{2q}_{4,0} |X|}{\varphi^{2q}_{4,0}} - \frac{Q - \Theta}{\varphi^{2q}_{4,0} |X|} \partial_X \left( \frac{1}{\varphi^{2q}_{4,0}} \right).
\]
One has from (3.13) that $|\partial_X \varphi_{4,0}(Z, X)| \lesssim |\varphi_{4,0}(X, Z)|/|X|$. From this fact, from (3.41) and (3.41) one infers that:

$$\left| \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^{2q}|X|} \right) \right| \lesssim q e^{-\frac{1}{4}s} \frac{1}{\varphi_{4,0}^{2q}(X, Z)|X|} \tag{3.43}$$

From the above estimate, (3.27) and (3.41) we infer that:

$$\left| \frac{1}{2q} \int \varepsilon^{2q} \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^{2q}|X|} \right) dX dY \langle Y \rangle - \int \varepsilon^{2q-1} \varphi_{4,0}^{2q} (\varepsilon \varepsilon_X + \partial_X (Q - \Theta) \varepsilon) dX dY \langle Y \rangle \right| \leq \frac{C}{q} \left( 1 + q e^{-\frac{1}{4}s} + q e^{-\left(\frac{1}{2} - \kappa\right)s} \right) \int \varepsilon^{2q} dX dY \langle Y \rangle \leq \frac{C}{q} \int \varepsilon^{2q} dX dY \langle Y \rangle$$

if $s_0$ has been chosen large enough. Applying Hölder and using (3.37):

$$\left| \int \frac{\varepsilon^{2q-1}}{\varphi_{4,0}^{2q}} R dX dY \langle Y \rangle \leq \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} |X| \langle Y \rangle \right| \left| \int \frac{R^{2q}}{\varphi_{4,0}^{2q}} dX dY \langle Y \rangle \right| \leq \frac{1}{q} \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} dX dY \langle Y \rangle + C q e^{-qs}.$$

We then obtained

$$\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X, Z)} |X| \langle Y \rangle \right) \leq - \left( \frac{1}{2} - \frac{C}{q} \right) \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X, Z)} |X| \langle Y \rangle \right| - \frac{2q - 1}{q^2} \int \frac{\partial_Y (\varepsilon^q)^2}{\varphi_{4,0}^{2q}} dX dY \langle Y \rangle + C s e^{-qs}.$$

We take $q$ large enough so that $|C/q| \leq \kappa$, and reintegrate until the time $s_1$ the above estimate, yielding from (3.23):

$$\int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X, Z)} |X| \langle Y \rangle \leq e^{-2q(\frac{1}{2} - \kappa)(s_1 - s_0)} \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X, Z)} dX dY \langle Y \rangle + C e^{-2q(\frac{1}{2} - \kappa)s_1} \int_{s_0}^{s} \tilde{e}^{2q(\frac{1}{2} - \kappa)s - q \tilde{s}} d\tilde{s} \leq C e^{-2q(\frac{1}{2} - \kappa)s_1} \leq \frac{K_{0,0}^{2q}}{2q} e^{-2q(\frac{1}{2} - \kappa)s_1}$$

if $K_{0,0}$ has been chosen large enough independently of the other bootstrap constants. This ends the proof of the Lemma.

We now perform similar weighted energy estimates for the derivatives of $\varepsilon$.

**Lemma 20.** There exists a choice of constants $K_{j_1,j_2} \gg 1$ for $0 \leq j_1 + j_2 \leq 2$ and $(j_1, j_2) \neq (0,0)$, and $\tilde{K}_{j_1,j_2} \gg 1$ for $0 \leq j_1 + j_2 \leq 2$ and $j_2 \geq 1$, such that at time $s_1$, for $0 \leq j_1 + j_2 \leq 2$ and $(j_1, j_2) \neq (0,0)$:

$$\left( \int_{\mathbb{R}^2} \frac{(\partial_Z \varepsilon^{j_1} |j_2| \varepsilon^{j_2}(s_1))^{2q}}{\varphi_{4,0}^{2q}} dX dY \langle X \rangle \right)^{\frac{1}{2q}} \leq \frac{K_{j_1,j_2} e^{-\left(\frac{1}{2} - \kappa\right)s_1}}{2}, \tag{3.44}$$
and for $0 \leq j_1 + j_2 \leq 2$ and $j_2 \geq 1$:

\[
\left( \int_{\mathbb{R}^2} \frac{\left( (Y \partial_Y^j) A^{j_2} \varepsilon(s_1) \right)^{2q}}{\varphi^q_{4,0} |X(Y)|} \, dX dY \right) \frac{1}{2q} \leq \frac{K_{j_1j_2}}{2} e^{-(\frac{1}{2}-\kappa)s_1}.
\] (3.45)

**Proof.**

**Step 1** Proof for $A \varepsilon$.
Recall (3.22) and let $w := A \varepsilon$. Then $A$ commutes with the transport part of the flow:

\[
\left[ A, \partial_s + \left( \frac{3}{2} X + \Theta \right) \partial_X + \frac{1}{2} Z \partial_Z \right] = 0.
\]

Indeed, we compute the commutator using (4.2):

\[
\left[ \left( \frac{3}{2} X + F_k^{-\frac{3}{2}}(Z) \Theta \right) \partial_X, \partial_s + \left( \frac{3}{2} X + \Theta \right) \partial_X + \frac{1}{2} Z \partial_Z \right]
= \left( - \partial_s F_k^{-\frac{3}{2}}(Z) \Psi_1(F_k^{\frac{3}{2}} X) + \left( \frac{3}{2} X + \partial_X F_k^{-\frac{3}{2}}(Z) \Psi_1(F_k^{\frac{3}{2}} X) \right) \partial_X \left( \frac{3}{2} X + F_k^{-\frac{3}{2}}(Z) \Psi_1(F_k^{\frac{3}{2}} X) \right) \right) \partial_X
= \frac{3}{2} \left( - \frac{1}{2k} Z \partial_Z F_k F_k^{-\frac{3}{2}} + F_k^{\frac{1}{2}} - F_k^{\frac{3}{2}} \right) \left( - \Psi_1 + \tilde{X} \partial_X \Psi_1 \right) (F_k^{\frac{3}{2}} X) \partial_X = 0.
\]

We compute from (3.22), (3.31) and the above cancellation the evolution of $w$:

\[
w_s + \mathcal{L} w - \partial_{YY} w + \tilde{\mathcal{L}} w + (A \partial_X \Theta) \varepsilon + AR + [A, \tilde{\mathcal{L}}] \varepsilon + w \varepsilon_X + \varepsilon \partial_X w + \varepsilon [A, \partial_X] \varepsilon - [A, \partial_{YY}] \varepsilon = 0.
\]

Using the linear energy identity (3.36) we infer that:

\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi^q_{4,0}(X,Y)} \, dX dY \right)
\leq - \left( \frac{1}{2} - \frac{C}{q} \right) \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi^q_{4,0}(X,Y)} \, dX dY - \frac{2q-1}{q^2} \int \frac{\varphi^q_{4,0}(X,Y)}{\varphi^q_{4,0}(X)} \, dX dY
+ \frac{1}{2q} \int w^{2q-1} \left( \frac{Q - \Theta}{\varphi^q_{4,0}(X)} \right) \, dX dY
- \frac{1}{2q} \int \varphi^q_{4,0} (Q - \Theta) w \, dX dY \frac{1}{|X(Y)|}.
\]

From (3.43), and (3.41), one has:

\[
\left| \frac{1}{2q} \int \varphi^q_{4,0} (Q - \Theta) w \, dX dY \frac{1}{|X(Y)|} \right| \leq e^{-\frac{\kappa s}{2}} \int \frac{w^{2q}}{\varphi^q_{4,0}(X,Y)} \, dX dY.
\]

Moreover, since

\[
A \partial_X \Theta = \frac{3}{2} F_k(Z) (\tilde{X} \partial_X^2 \Psi_1)(F_k^{\frac{3}{2}}(Z) X) + F_k(Z)(\partial_X \Psi_1)^2 \Psi_1(F_k^{\frac{3}{2}}(Z) X)
\]

and since both $F_k$ and $X \partial_X^2 \Psi_1$ and $\Psi_1 \partial_X^2 \Psi_1$ are bounded, one has that

\[
|A \partial_X \Theta| \lesssim 1.
\]
and therefore using the bootstrap bound (3.24) on $\varepsilon$ one deduces from Hölder:

$$\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} (A \partial_X \Theta) \varepsilon \, dX dY \right| \leq C \left| \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \, dX dY \right| \frac{2q}{2q-1} \left| \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} \, dX dY \right|^{\frac{1}{2q}} \leq CK_{0,1}^{-1} K_{0,0} e^{-2q(\frac{1}{2}-\kappa)s}.$$ 

We compute from (3.22) and (3.32) that

$$[A, \tilde{\mathcal{L}}] \varepsilon = \left( A(Q - \Theta) - (Q - \Theta) \right) \frac{3}{2} + \partial_X \Psi_1 \left( F_k^3(X) \right) \partial_X \varepsilon + (A \partial_X (Q - \Theta)) \varepsilon = \frac{A(Q - \Theta) - (Q - \Theta) \frac{3}{2} + \partial_X \Psi_1 \left( F_k^3(X) \right)}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z) F_k^3(Z)X} w + (A \partial_X (Q - \Theta)) \varepsilon.$$ 

From (3.22), (B.10), (3.14) and (3.41) we then obtain

$$[A, \tilde{\mathcal{L}}] \varepsilon \leq Ce^{-\frac{1}{2q}(\varepsilon + |\varepsilon|)}$$

for $C$ independent of the bootstrap bounds, which implies using Hölder and (3.24):

$$\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} [A, \tilde{\mathcal{L}}] \varepsilon \frac{dX dY}{|X| \langle Y \rangle} \right| \leq Ce^{-\frac{1}{2q}} \left( \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right) \left( \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right)^{\frac{1}{2q}} \leq Ce^{-\frac{1}{2q}} \left( \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right) + CK_{0,1}^{-1} K_{0,0} e^{-2q(\frac{1}{2}-\kappa)s}.$$ 

One then deduces from (B.10), (3.22), (3.37) and Hölder:

$$\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} AR \frac{dX dY}{|X| \langle Y \rangle} \right| \leq \left| \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right| \left| \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right| \left( \int \frac{(AR)^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right)^{\frac{1}{2q}} \leq K_{0,0}^{2q-1} s^{\frac{1}{2q}} e^{-2q(\frac{1}{2}-\kappa) s}.$$ 

Using (3.27) we infer that

$$\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} \varepsilon \frac{dX dY}{|X| \langle Y \rangle} \right| \approx e^{-\frac{1}{2q}(\frac{1}{2} - \kappa) s} \left( \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \right).$$

Integrating by parts, one has the identity:

$$\int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} \varepsilon \partial_X w \frac{dX dY}{|X| \langle Y \rangle} = - \frac{1}{2q} \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \partial_X \varepsilon - \frac{1}{2q} \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \varepsilon \partial_X \left( \frac{1}{\varphi_{4,0}^{2q}} \frac{X}{|X| \langle Y \rangle} \right) dX dY.$$ 

From (3.16) one has that $|\partial_X \varphi_{4,0}(X, Z)/\varphi_{4,0}(X, Z)| \leq |X|^{-1}$. Therefore, using (3.26) we obtain that:

$$\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} \varepsilon \partial_X w \frac{dX dY}{|X| \langle Y \rangle} \right| \leq C \left( \| \partial_X \varepsilon \|_{L^\infty} + \frac{\varepsilon}{|X|} \right) \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle} \leq Ce^{-\frac{1}{2q}(\frac{1}{2} - \kappa) s} \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X| \langle Y \rangle}.$$
Next, from the identity
\[ [A, \partial_X] = -\left( \frac{3}{2} + \partial_X \Psi_1(F_k^{3/2}(Z)X) \right) \partial_X = -\frac{\frac{3}{2} + \partial_X \Psi_1(F_k^{3/2}(Z)X)}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X)} A, \]
since \( F_k \) and \( \partial_X \Psi_1 \) are uniformly bounded, from (B.10) and (3.26) we obtain that:
\[
\left| \int w^{2q-1} \frac{\varepsilon[A, \partial_X] \varepsilon}{\varphi_{4,0}^q} dX dY \right| = \left| \int w^{2q} \frac{\varepsilon}{\varphi_{4,0}^q} F_k(Z) \partial_X \Psi_1(F_k^{3/2}(Z)X) \partial_X \varepsilon dX dY \right| \lesssim \frac{\varepsilon}{X} \int w^{2q} \frac{dX dY}{\varphi_{4,0}^q[X|Y]} \lesssim e^{-\frac{(1-\kappa)s}{2}} \int w^{2q} \frac{dX dY}{\varphi_{4,0}^q[X|Y]}.
\]
Finally, one computes that:
\[
[A, \partial_Y] \varepsilon = -2 \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \partial_Y \varepsilon - \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \partial_X \varepsilon
\]
\[
= -2 \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \partial_Y \left( \frac{w}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X)} \right) - \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \partial_X \varepsilon
\]
\[
= F_1 w + F_2 \partial_Y w
\]
where
\[
F_1 := \left( 2 \frac{\left( \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right)^2}{\left( \frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right)^2} - \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right)
\]
and
\[
F_2 := -2 \frac{\partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right)}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X)}.
\]
One has that \( \partial_Z F_k/F_k \) is bounded, and that \( |\Psi_1(\tilde{X})| + |\tilde{X} \partial_X \Psi_1(\tilde{X})| \lesssim |\tilde{X}| \). Therefore,
\[
\left| \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right| = e^{-\frac{3}{2X}} \left| \partial_Z F_k F_k^{-\frac{3}{2}} \left( -\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1(F_k^{3/2}(Z)X) \right) \right| \lesssim e^{-\frac{3}{2X}} |X|.
\]
The same computation can be performed for the second term in \( F_1 \), giving from (B.10):
\[
\left| \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right|^2 + \left| \partial_Y \left( F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right| \lesssim e^{-\frac{3}{2X}} |X|
\]
and hence:
\[
\left| \int w^{2q-1} F_1 w \frac{dX dY}{\varphi_{4,0}^q[X|Y]} \right| \lesssim e^{-\frac{3}{2X}} \int w^{2q} \frac{dX dY}{\varphi_{4,0}^q[X|Y]}.
\]
From the above estimate one obtains similarly that:
\[
|F_2| \lesssim e^{-\frac{3}{2X}}, \quad |\partial_Y F_2| \lesssim e^{-\frac{3}{2X}}.
\]
so that integrating by parts and using (3.16)

\[ \left| \int \frac{w^{2q-1}}{\varphi_{2q,4,0}} F_2 \partial_Y w \frac{dXdY}{X|Y} \right| = \frac{1}{2q} \left| \int w^{2q} \partial_Y \left( \frac{F_2}{\varphi_{2q,4,0}(X,Z)} \right) \frac{dXdY}{|X|} \right| \leq e^{-\frac{k-1}{2q} s} \int \frac{w^{2q}}{\varphi_{2q,4,0}|X|} dXdY. \]

One has then proven that

\[ \left| \int \frac{w^{2q-1}}{\varphi_{2q,4,0}} [A, \partial_Y] \varepsilon dXdY \right| \leq e^{-\frac{k-1}{2q} s} \int \frac{w^{2q}}{\varphi_{2q,4,0}|X|} dXdY. \]

From the collection of the above estimates, one infers that for \( q \) and \( s_0 \) large enough:

\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{2q,4,0}(X,Z)} \frac{dXdY}{|X|} \right) \\
\leq - \left( \frac{1}{2} - C \left( \frac{1}{q} + e^{-\frac{1}{2q} s} + e^{-\frac{1}{2} - \kappa s} + e^{-\frac{k-1}{2q} s} \right) \right) \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{2q,4,0}(X,Z)} \frac{dXdY}{|X|} \\
+ CK_{0,1}^{2q-1} K_{0,0} e^{-2q(\frac{1}{2} - \kappa) s} + CK_{0,1}^{2q-1} K_{0,0} e^{-2q(\frac{1}{2} - \kappa) s} + e^{-\frac{k-1}{2q} s} \int_{s_0}^{s} e^{\kappa q s} d\tilde{s} \\
\leq - \left( \frac{1}{2} - \kappa \right) \int_{\mathbb{R}^2} \frac{e^{2q}}{\varphi_{2q,4,0}(X,Z)} \frac{dXdY}{|X|} + CK_{0,1}^{2q-1} K_{0,0} e^{-2q(\frac{1}{2} - \kappa) s} + e^{-\frac{k-1}{2q} s} \int_{s_0}^{s} e^{\kappa q s} d\tilde{s} \\
\leq C(1 + K_{0,1}^{2q-1} K_{0,0}) e^{-2q(\frac{1}{2} - \kappa) s} \leq \frac{K_{0,1}^{2q}}{2q} e^{-2q(\frac{1}{2} - \kappa) s}.
\]

If \( K_{1,0} \) has been chosen large enough depending on \( K_{0,0} \).

**Step 2** Proof for \( \partial_Z \). Define \( w = \partial_Z \varepsilon = e^{(k-1)s/(2k)} \partial_Y \varepsilon \), which from (3.31) solves:

\[
0 = w_s + \frac{1}{2k} w + L \varepsilon - \partial_Y w + \partial_Y \partial_Z \Theta X \varepsilon + \partial_Z \Theta \varepsilon + \partial_Z (Q - \Theta) \partial_X \varepsilon + \partial_X (Q - \Theta) \varepsilon \\
+ \partial_Z R + w \partial_X \varepsilon + \varepsilon \partial_X w,
\]

and hence obeys the energy identity from (3.36):

\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{2q,4,0}(X,Z)} \frac{dXdY}{|X|} \right) \\
\leq - \left( \frac{1}{2} + \frac{1}{2k} - C \right) \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{2q,4,0}(X,Z)} \frac{dXdY}{|X|} - \frac{2q - 1}{q^2} \int_{\mathbb{R}^2} \frac{|\partial_Y (w^q)|^2}{\varphi_{2q,4,0}^2|X|} dXdY \\
+ \frac{1}{2q} \int \frac{w^{2q-1}}{\varphi_{2q,4,0}} \left( \partial_Z \Theta \partial_X \varepsilon + \partial_Z \Theta \varepsilon + \partial_Z (Q - \Theta) \partial_X \varepsilon + \partial_X (Q - \Theta) \varepsilon \\
+ \partial_Z R + w \partial_X \varepsilon + \varepsilon \partial_X w + \partial_X (Q - \Theta) w \right) \frac{dXdY}{|X|}. \]
Using (3.43), and (3.41) we infer that:
\[
\left| \frac{1}{2q} \int w^{2q} \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^{2q}|X|} \right) \, dX dY - \int w^{2q} \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^{2q}} \right) dX dY \right| \lesssim e^{-\frac{1}{4q}s} \int w^{2q} \, dX dY.
\]
Next one computes that
\[
\partial_Z \partial_X \varepsilon = \frac{\partial_Z F_k(Z) F_k^{-\frac{1}{2}}(Z)(-\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1)(F_k^{\frac{3}{2}}(Z)X)}{F_k(Z)} \frac{3}{2} X + F_k^{-\frac{1}{2}}(Z)(\Psi_1(F_k^{\frac{3}{2}}(Z)X)) A \varepsilon.
\]
In the above formula, $\partial_Z F_k(Z)/F_k(Z)$ is uniformly bounded, and as $F_k$ is bounded and $|(-1/2 \Psi_1 + 3/2 \tilde{X} \partial_X \Psi_1)(\tilde{X})| \lesssim |\tilde{X}|$ we obtain from (B.10) that:
\[
\left| \frac{\partial_Z F_k(Z) F_k^{-\frac{1}{2}}(Z)(-\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_X \Psi_1)(F_k^{\frac{3}{2}}(Z)X)}{F_k(Z)} \frac{3}{2} X + F_k^{-\frac{1}{2}}(Z)(\Psi_1(F_k^{\frac{3}{2}}(Z)X)) \right| \lesssim 1.
\]
From Hölder, (3.24) and (3.25) we then infer that:
\[
\left\| \int w^{2q} \partial_X \partial_Z \varepsilon \, dX dY \right\|_{\mathcal{L}} \lesssim C \int \left( \int w^{2q} \, dX dY \right) \left( \int \left( \frac{A \varepsilon}{\varphi_{4,0}^{2q}} \right) \, dX \right) \lesssim CK_{0,1} e^{-2q(\frac{1}{2} - \kappa)s}.
\]
Similarly, since $F_k, \partial_Z F_k/F_k, \partial_X \Psi_1$ and $\tilde{X} \partial_X^2 \Psi_1$ are uniformly bounded,
\[
|\partial_Z \Psi_1| = \left| \frac{\partial_Z F_k(Z)}{F_k(Z)} F_k(\partial_X \Psi_1 + 3/2 \tilde{X} \partial_X^2 \Psi_1)(F_k^{\frac{3}{2}}(Z)X) \right| \lesssim 1,
\]
and from Hölder, (3.24) and (3.25) we then infer that:
\[
\left\| \int w^{2q} \partial_X \partial_Z \varepsilon \, dX dY \right\|_{\mathcal{L}} \lesssim C \int \left( \int w^{2q} \, dX dY \right) \left( \int \varepsilon \, dX dY \right) \lesssim CK_{0,1} e^{-2q(\frac{1}{2} - \kappa)s}.
\]
Then, from (3.41) and (B.10) we infer that:
\[
|\partial_Z (Q - \Theta) \partial_X \varepsilon| = \left| \frac{\partial_Z (Q - \Theta)}{\frac{3}{2} X + F_k^{-\frac{1}{2}}(Z) \Psi_1(F_k^{\frac{3}{2}}(Z)X)} \right| \lesssim e^{-\frac{1}{4q}s} A \varepsilon
\]
and therefore from Hölder, (3.24) and (3.25):
\[
\left\| \int w^{2q} \partial_X (Q - \Theta) \partial_X \varepsilon \, dX dY \right\|_{\mathcal{L}} \lesssim e^{-\frac{1}{4q}s} \left( \int w^{2q} \, dX dY \right) \left( \int \varepsilon \, dX dY \right) \lesssim C(K_{0,1}) e^{-2q(\frac{1}{2} - \kappa)s} e^{-\frac{1}{4q}s}.
\]
Using (3.41) one has that
\[
|\partial_X (Q - \Theta)| \lesssim e^{-\frac{1}{4q}s}.
\]
Therefore, one infers by Hölder, (3.24) and (3.25):
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^q} \partial_{ZX} (Q - \Theta) \varepsilon dX dY \right| \leq e^{-\frac{1}{4s}} \left| \int \frac{w^{2q}}{\varphi_{4,0}^q} \frac{dX dY}{|X|} \right| \frac{1}{2q} \leq C(K_{0,0}, K_{1,0}) e^{-2q(\frac{1}{2} - \kappa)s} e^{-\frac{1}{4s}}.
\]

Next, from Hölder, (3.37) and (3.24):
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^q} \partial Z R \frac{dX dY}{|X|} \right| \leq C \left| \int \frac{w^{2q}}{\varphi_{4,0}^q} \frac{dX dY}{|X|} \right| \frac{1}{2q} \leq C K_{1,0}^{-1} s \frac{1}{2q} e^{-2q(\frac{1}{2} - \kappa + \frac{s}{2})s}.
\]

From (3.27) one has that:
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^q} w \partial X \varepsilon \frac{dX dY}{|X|} \right| \leq \| \partial X \varepsilon \|_{L^\infty} \int \frac{w^{2q}}{\varphi_{4,0}^q} \frac{dX dY}{|X|} \leq e^{-\frac{1}{2} - \kappa s} \int \frac{w^{2q}}{\varphi_{4,0}^q} \frac{dX dY}{|X|}.
\]

We finally perform an integration by parts to obtain:
\[
\int \frac{w^{2q-1}}{\varphi_{4,0}^q} \varepsilon \partial X w \frac{dX dY}{|X|} = \int \frac{w^{2q}}{\varphi_{4,0}^q} \partial X \left( \frac{\varepsilon}{\varphi_{4,0}^q(X, Z)} \right) \frac{dX dY}{|X|}.
\]

From (3.26), (3.27), (3.16) one has:
\[
\left| \partial X \left( \frac{\varepsilon}{\varphi_{4,0}^q(X, Z)} \right) \right| \leq \frac{e^{-\frac{1}{4}}}{\varepsilon_{\varphi_{4,0}^q(X, Z)} |X|}
\]

so that:
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^q(X, Z)} \varepsilon \partial X w \frac{dX dY}{|X|} \right| \leq e^{-\frac{1}{4} - \kappa s} \int \frac{w^{2q}}{\varphi_{4,0}^q} \frac{dX dY}{|X|}.
\]

From the collection of the above estimates, the energy identity becomes:
\[
\frac{d}{ds} \left( \frac{1}{2q} \int \frac{w^{2q}}{\varphi_{4,0}^q(X, Z)} \frac{dX dY}{|X|} \right) \leq - \left( \frac{1}{2} + \frac{1}{2k} - C \left( \frac{1}{q} + e^{-\frac{1}{2} - \kappa s} + e^{-\frac{1}{4s}} \right) \right) \int \frac{w^{2q}}{\varphi_{4,0}^q(X, Z)} \frac{dX dY}{|X|}
\]

\[
+C(K_{0,1} + K_{1,0})K_{1,0}^{-1} e^{-2q(\frac{1}{2} - \kappa)s} + C K_{1,0}^{-1} s^{\frac{1}{2q}} e^{-2q(\frac{1}{2} - \kappa + \frac{s}{2})s} + C(K_{0,0}, K_{1,0}, K_{1,0}) e^{-2q(\frac{1}{2} - \kappa)s} e^{-\frac{1}{4s}}
\]

\[
\leq - \frac{1}{2} \int \frac{w^{2q}}{\varphi_{4,0}^q(X, Z)} \frac{dX dY}{|X|} + C(K_{0,1} + K_{0,0})K_{1,0}^{-1} e^{-2q(\frac{1}{2} - \kappa)s}
\]

if \( q \) has been chosen large enough and then \( s_0 \) large enough. From the initial size (3.23) the above differential inequality yields:
\[
\int \frac{w^{2q}(s)}{\varphi_{4,0}^q(X, Z)} \frac{dX dY}{|X|} \leq e^{-q(s-s_0)} \int \frac{w^{2q}}{\varphi_{4,0}^q(X, Z)} \frac{dX dY}{|X|} + C K_{1,0}^{-1} e^{-qs} \int_{s_0}^{s} (K_{0,0} + K_{0,1}) e^{2q\alpha s} d\tilde{s}
\]

\[
\leq C(1 + K_{1,0}^{-1}(K_{0,0} + K_{0,1})) e^{-2q(\frac{1}{2} - \kappa)s} \leq \frac{K_{1,0}^{2q}}{2q} e^{-2q(\frac{1}{2} - \kappa)s}
\]

if \( K_{1,0} \) has been chosen large enough depending on \( K_{0,0} \) and \( K_{0,1} \).
Step 3 Proof for $Y \partial Y$. Let $w = Z \partial_Z \varepsilon = Y \partial_Y \varepsilon$. From (3.31) one obtain the evolution of $w$:

$$
0 = w_s + \mathcal{L} \varepsilon - \partial_Y w + \mathcal{L} w + Z \partial_Z \Theta \partial X \varepsilon + Z \partial_Z \Theta X \varepsilon + 2 \partial_Y \varepsilon + Z \partial_Z (Q - \Theta) \partial X \varepsilon + Z \partial_Z X (Q - \Theta) + Z \partial_Z R + w \partial_X \varepsilon + \varepsilon \partial_X w.
$$

Using (3.36) yields the energy identity:

$$
\frac{d}{ds} \left( \frac{1}{2 q} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{4,0}^2(X, Z)} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \right) \\
\leq - \left( \frac{1}{2} - \frac{C}{q} \right) \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{4,0}^2(X, Z)} \frac{dX dY}{\langle X \rangle \langle Y \rangle} - 2 q - 1 \frac{2q}{q^2} \int \frac{\partial_Y (w^q)}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \\
+ \frac{1}{2q} \int w^{2q-1} \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^2} \right) \frac{dX dY}{\langle Y \rangle} \\
- \int w^{2q-1} \left( Z \partial_Z \Theta \partial X \varepsilon + Z \partial_Z X \Theta \varepsilon + 2 \partial_Y \varepsilon + Z \partial_Z (Q - \Theta) \partial X \varepsilon + Z \partial_Z X (Q - \Theta) \right) \frac{dX dY}{\langle X \rangle \langle Y \rangle}.
$$

Using (3.43), and (3.41) we infer that:

$$
\left| \frac{1}{2q} \int w^{2q} \partial_X \left( \frac{Q - \Theta}{\varphi_{4,0}^2} \right) \frac{dX dY}{\langle Y \rangle} - \int w^{2q-1} \partial_X (Q - \Theta) w \frac{dX dY}{\langle X \rangle \langle Y \rangle} \right| \lesssim e^{-\frac{1}{4 s}} \int w^{2q} \frac{dX dY}{\langle X \rangle \langle Y \rangle}.
$$

Next,

$$
Z \partial_Z \Theta \partial X \varepsilon = \frac{Z \partial_Z F_k(Z) F_k^{-\frac{1}{2}}(Z)(-\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_Z \Psi_1)(F_k^{\frac{1}{2}}(Z)X)}{F_k(Z)} A \varepsilon.
$$

In the above formula, $Z \partial_Z F_k(Z)/F_k(Z)$ is uniformly bounded, and as $F_k$ is bounded and $|(-1/2 \Psi_1 + 3/2 \tilde{X} \partial_Z \Psi_1)(\tilde{X})| \lesssim |\tilde{X}|$ we obtain from (B.10) that:

$$
\left| \frac{Z \partial_Z F_k(Z) F_k^{-\frac{1}{2}}(Z)(-\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_Z \Psi_1)(F_k^{\frac{1}{2}}(Z)X)}{F_k(Z)} \right| \lesssim \frac{3}{2} X + F_k^{-\frac{1}{2}}(Z)(\Psi_1(F_k^{\frac{1}{2}}(Z)X)) \lesssim 1.
$$

From Hölder, (3.24) and (3.25) we then infer that:

$$
\int \frac{w^{2q-1}}{\varphi_{4,0}^2} Z \partial_Z \Theta \partial X \varepsilon dX dY \lesssim \left| \int \frac{w^{2q}}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \right| \lesssim C \int \frac{w^{2q}}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \left| \int \frac{dX dY}{\varphi_{4,0}^2} \right| \frac{A \varepsilon}{\varphi_{4,0}^2} \lesssim \frac{K^{2q-1}}{K_{0,1}^0} e^{-2q(\frac{1}{4} - \kappa)s}.
$$

Similarly, since $F_k$, $\partial_Z F_k/F_k$, $\tilde{X} \partial_Z \Psi_1$ and $\tilde{X} \partial_Z^2 \Psi_1$ are uniformly bounded,

$$
|Z \partial_Z X \Theta| = \left| \frac{Z \partial_Z F_k(Z) F_k^{-\frac{1}{2}}(Z)(-\frac{1}{2} \Psi_1 + \frac{3}{2} \tilde{X} \partial_Z \Psi_1)(F_k^{\frac{1}{2}}(Z)X)}{F_k(Z)} \right| \lesssim 1,
$$

and from Hölder, (3.24) and (3.25) we then infer that:

$$
\int \frac{w^{2q-1}}{\varphi_{4,0}^2} Z \partial_Z X \Theta \varepsilon dX dY \lesssim C \int \frac{w^{2q}}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \lesssim C \int \frac{w^{2q}}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \left| \int \frac{w^{2q}}{\varphi_{4,0}^2} \frac{dX dY}{\langle X \rangle \langle Y \rangle} \right| \frac{1}{\varphi_{4,0}^2} \lesssim \frac{K^{2q-1}}{K_{0,0}^0} e^{-2q(\frac{1}{4} - \kappa)s}.
$$
We then integrate by parts:

\[
\int \frac{w^{2q-1}}{\varphi^{2q}_{4,0}(X,Z)} \partial_Y \varepsilon \frac{dX}{|X|}\left\langle Y \right\rangle = -\frac{2q-1}{q} \int \partial_Y (w^q) w^{q-1} \frac{dX}{|X|}\left\langle Y \right\rangle - \int w^{2q-1} \partial_Y \varepsilon \partial_Y \left( \frac{1}{\varphi^{2q}_{4,0}(X,Z)} \right) \frac{dX}{|X|}.
\]

For the first term we use the generalised Hölder inequality, (3.24) and (3.25):

\[
\left\| \int \frac{\partial_Y (w^q) w^{q-1}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq e^{-\frac{k-1}{2k}s} \left\| \int \partial_Y (w^q) \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \left\| \int w^{2q} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \left\| \int \frac{\partial_Y (w^q) w^{q-1}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq e^{-\frac{k-1}{2k}s} \kappa \int \frac{|\partial_Y (w^q)|^2}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle + C(\tilde{K}_{1,0}, K_{1,0})e^{-2q(\frac{1}{2} - \kappa)s - \frac{k-1}{2k}s}.
\]

For the second term, from (3.16) we infer that

\[
\left\| \partial_Y \left( \frac{1}{\varphi^{2q}_{4,0}(X,Z)} \right) \right\| \lesssim \frac{1}{\varphi^{2q}_{4,0}(X,Z)}.
\]

and therefore, from Hölder, (3.24) and (3.25):

\[
\left\| \int w^{2q-1} \partial_Y \varepsilon \partial_Y \left( \frac{1}{\varphi^{2q}_{4,0}(X,Z)} \right) \frac{dX}{|X|} \right\| \leq e^{-\frac{k-1}{2k}s} \left\| \int \frac{w^{2q}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \left\| \int \frac{\partial_Y (w^q) w^{q-1}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq C(\tilde{K}_{1,0}, K_{1,0})e^{-2q(\frac{1}{2} - \kappa)s - \frac{k-1}{2k}s}.
\]

From (B.10) and (3.41):

\[
|Z \partial_Z (Q - \Theta) \partial_X \varepsilon| = \left\| \frac{Z \partial_Z (Q - \Theta)}{\frac{3}{2} X + F_k(x) (Z) \Psi_1 (x) (Z) X} \right\| |A\varepsilon| \lesssim e^{-\frac{k-1}{2k}} |Z| |1 + (1 + |X|)^{\frac{3}{2}}| |1 + |X||^{1 - \frac{1}{2}} |A\varepsilon| \lesssim e^{-\frac{k-1}{2k}} |A\varepsilon|
\]

and therefore from Hölder, (3.24) and (3.25):

\[
\left\| \int \frac{w^{2q-1}}{\varphi^{2q}_{4,0}} Z \partial_Z (Q - \Theta) \partial_X \varepsilon \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq e^{-\frac{k-1}{2k}s} \left\| \int \frac{w^{2q}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \left\| \int \frac{(A\varepsilon)^{2q}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq C(K_{0,1}, \tilde{K}_{1,0})e^{-\frac{k-1}{2k}s} e^{-2q(\frac{1}{2} - \kappa)s}.
\]

Similarly, from (3.41):

\[
|Z \partial_X Z (Q - \Theta)| \lesssim e^{-\frac{k-1}{2k}s} |Z| |1 + |Z||^{-\frac{3}{2}} |1 + |X||^{1 - \frac{1}{2}} \lesssim e^{-\frac{k-1}{2k}}
\]

and from Hölder, (3.24) and (3.25):

\[
\left\| \int \frac{w^{2q-1}}{\varphi^{2q}_{4,0}} Z \partial_Z (Q - \Theta) \partial_X \varepsilon \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq e^{-\frac{k-1}{2k}s} \left\| \int \frac{w^{2q}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \left\| \int \frac{(A\varepsilon)^{2q}}{\varphi^{2q}_{4,0}(X,Z)} \frac{dX}{|X|}\left\langle Y \right\rangle \right\| \leq C(K_{0,0}, \tilde{K}_{1,0})e^{-\frac{k-1}{2k}s} e^{-2q(\frac{1}{2} - \kappa)s}.
\]
Next, from Hölder, (3.37) and (3.25):
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} Z \partial_z R \frac{dX dY}{|X \langle Y|} \right| \leq \left| \int \frac{w^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} \right| \left| \int \frac{(Z \partial_z R)^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} \right|^{\frac{1}{2q}} \\
\leq C K_{1,0}^{2q-1} s^\frac{1}{2q} e^{-2q(\frac{1}{2} - \kappa + s)} s.
\]
Performing an integration by parts, and then using (3.27) and (3.46) we finally obtain:
\[
\left| \int \frac{w^{2q-1}}{\varphi_{4,0}^{2q}} (w \partial_X \varepsilon + \varepsilon \partial_X w) \frac{dX dY}{|X \langle Y|} \right| = \left| \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \partial_X \varepsilon \frac{dX dY}{|X \langle Y|} - \frac{1}{2q} \int w^{2q} \partial_X \left( \frac{\varepsilon}{\varphi_{4,0}^{2q}|X|} \right) \frac{dX dY}{|X \langle Y|} \right| \\
\leq \left( \| \partial_X \varepsilon \|_{L^\infty} + \| X \varphi_{4,0}^{2q}(X,Z) \partial_X \left( \frac{\varepsilon}{\varphi_{4,0}^{2q}|X|} \right) \|_{L^\infty} \right) \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X \langle Y|} \\
\lesssim e^{-\left(\frac{1}{2} - \kappa\right) s} \int \frac{w^{2q}}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X \langle Y|}.
\]
From the collection of the above estimates, as \((k - 1)/(2k) \geq 1/(4k)\) one deduces that:
\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} \right) \\
\leq - \left( \frac{1}{2} - C \left( \frac{1}{q} + e^{-\left(\frac{1}{2} - \kappa\right) s} + e^{-\frac{1}{2q} s} \right) \right) \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} - \left( \frac{2q - 1}{q^2} - \kappa \right) \int \frac{|\partial_Y (w^q)|^2}{\varphi_{4,0}^{2q}} \frac{dX dY}{|X \langle Y|} \\
+ C(K_{0,0}, K_{1,0}, K_{0,1}, K_{1,0}) e^{-2q(\frac{1}{2} - \kappa) s} e^{-\frac{1}{2q} s} + C K_{1,0}^{2q-1} (K_{0,0} + K_{0,1}) e^{-2q(\frac{1}{2} - \kappa) s} \\
+ C K_{1,0}^{2q-1} s^\frac{1}{2q} e^{-2q(\frac{1}{2} - \kappa + s)} s \\
\leq - \left( \frac{1}{2} - \frac{\kappa}{2} \right) \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} + C K_{1,0}^{2q-1} (K_{0,0} + K_{0,1}) e^{-2q(\frac{1}{2} - \kappa) s} 
\]
if \(\kappa\) has been chosen small enough, then \(q\) large enough and then \(s_0\) large enough. From the initial size (3.23) the above differential inequality yields:
\[
\int_{\mathbb{R}^2} \frac{w^{2q}(s_1)}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} \leq e^{-2q(\frac{1}{2} - \kappa)(s-s_0)} \int_{\mathbb{R}^2} \frac{w^{2q}}{\varphi_{4,0}^{2q}(X,Z)} \frac{dX dY}{|X \langle Y|} \\
+ C K_{1,0}^{2q-1} (K_{0,0} + K_{0,1}) e^{-2q(\frac{1}{2} - \kappa) s} \int_{s_0}^{s} e^{\eta s} d\tilde{s} \\
\leq C(1 + K_{1,0}^{2q-1}(K_{0,0} + K_{0,1})) e^{-2q(\frac{1}{2} - \kappa) s} \leq \frac{K_{1,0}^{2q-1}}{2q} e^{-2q(\frac{1}{2} - \kappa) s}
\]
if \(K_{0,1}\) has been chosen large enough independently of the other constants in the bootstrap.

**Step 4 Proof for higher order derivatives.** The proof for higher order derivatives works the very same way and we safely leave it to the reader.

\[\square\]

We can now end the proof of Proposition 13.
Proof of Proposition 13. We reason by contradiction. Let $v$ be a solution to (3.5) with initial value $v(s_0)$ at time $s_0$ that satisfies (3.23) when decomposed according to (3.18). Let $s_1$ denote the supremum of times $\tilde{s} \geq s_0$ such that $v$ is well defined and that the bounds (3.24) and (3.25) hold on $[s_0, \tilde{s}]$. From the initial bounds (3.23) and a continuity argument, one has that $s_1 > s_0$ is well defined. We then prove Proposition 13 by contradiction and assume that $s_1$ is finite. If it is the case, then the bounds (3.24) and (3.25) are strict at time $s_1$ from (3.42), (3.44) and (3.45). Therefore, by a continuity argument there exists $\delta > 0$ such that $v$ is well defined and satisfies (3.24) and (3.25) on $[s_1, s_1 + \delta]$, contradicting the definition of $s_1$.

4. Analysis of the vertical axis

This section is devoted to the proof of Theorem 1 and Proposition 10. The proof of Theorem 1 follows and refines the works [2, 16, 22], and differs in particular in the way we deal with the problem outside the origin, see Lemma 28. For more comparisons with these works, see the comments after Theorem 1. The proof of Proposition 10 then uses a very similar analytical framework.

4.1. Flat blow-up for the semi-linear heat equation

We prove in this subsection the result in Theorem 1 concerning the solution $\xi$ to (1.7). The strategy is the following. We construct an approximate blow-up profile in self-similar variables and show the existence of a true solution staying in its neighbourhood via a bootstrap argument. This existence result relies on the control of the difference of the two functions via a spectral decomposition at the origin and energy estimates far away, showing the existence of a finite number of instabilities only allowing for the use of a topological argument to control them.

The unstable blow-ups are related to unstable analytic backward self-similar solutions of the quadratic equation

$$\xi_t - \xi^2 = 0. \quad (4.1)$$

Their properties are the following.

Proposition 21 (unstable self-similar blow-ups for the quadratic equation). For $k \in \mathbb{N}$, $k \geq 2$, the functions $F_k(Z) := (1 + Z^{2k})^{-1}$ are such that

$$\xi(t, y) = \frac{1}{T - t} F_k \left( \frac{a y}{(T - t)^{\frac{1}{2k}}} \right)$$

is a solution of (4.1) for any $T \in \mathbb{R}$ and $a > 0$. For any $a > 0$, $Z \mapsto F_k(aZ)$ is a solution of the stationary self-similar equation

$$F_k + \frac{1}{2k} Z \partial_Z F_k - F_k^2 = 0. \quad (4.2)$$

The linearised transport operator $H_Z := 1 + \frac{1}{2k} Z \partial_Z - 2F_k(aZ)$ acting on $C^\infty(\mathbb{R})$ has the point spectrum

$$\Upsilon(H_Z) = \left\{ \frac{\ell - 2k}{2k}, \ell \in \mathbb{N} \right\}.$$

The associated eigenfunctions are

$$H_Z \phi_{Z, \ell} = \frac{\ell - 2k}{2k} \phi_{Z, \ell}, \quad \phi_{Z, \ell} = \frac{Z^\ell}{(1 + (aZ)^{2k})^2}. \quad (4.3)$$
Two of them are linked to the invariances of the flow:

\[ \phi_0 = F_k(aZ) + \frac{1}{2k} Z \partial_Z F_k(aZ) = \frac{\partial}{\partial \lambda} \left( \lambda F_k(\lambda^{1/2} bZ) \right) \big|_{\lambda = 1}, \quad \phi_{2k} = Z \partial_Z F_k(aZ) = \frac{\partial}{\partial a} (F_k(\lambda^{1/2} bZ)) \big|_{\lambda = 1}. \]

Proof. The proof is made of direct computations that we safely leave to the reader.

We now introduce for a solution to (1.7) for \( a > 0 \) and \( T > 0 \) the self-similar variables following [13]

\[ Y := \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := aY e^{1/2k - 1/2k}, \quad f(s, Y) = (T-t)\xi(t,y), \quad (4.4) \]

to zoom at the blow-up location, and \( f \) solves the first equation in (3.9). The function that we want to construct here, from (1), should converge to 1 in compact sets of the variable \( Y \). Therefore close to the origin the linearised operator is

\[ H_\rho := -1 + \frac{Y}{2} \partial_Y - \partial_{YY}. \]

Its spectral structure is well-known on the weighted \( L^2 \)-based Sobolev spaces

\[ H^k_\rho := \left\{ f \in H^k_{\text{loc}}(\mathbb{R}), \quad \sum_{k'=0}^k \int_{\mathbb{R}} |\partial_{YY}^k f|^2 e^{-\frac{Y^2}{4}} dY < +\infty \right\} \]

with norm and scalar product

\[ \| f \|^2_{H^k_\rho} := \sum_{k'=0}^k \int_{\mathbb{R}} |\partial_{YY}^k f|^2 e^{-\frac{Y^2}{4}} dY, \quad \langle f, g \rangle_\rho := \int_{\mathbb{R}} f g e^{-\frac{Y^2}{4}} dY, \quad \rho(Y) := e^{-\frac{Y^2}{4}}. \quad (4.5) \]

Proposition 22 (Linear structure at the origin (see e.g. [22])). The operator \( H_\rho \) is essentially self-adjoint on \( C^2_0(\mathbb{R}) \subset L^2(\rho) \) with compact resolvant. The space \( H^0_\rho \) is included in the domain of its unique self-adjoint extension. Its spectrum is

\[ \Upsilon(H_\rho) = \left\{ \frac{\ell - 2}{2}, \quad \ell \in \mathbb{N} \right\}. \]

The eigenvalues are all simple and the associated orthonormal basis of eigenfunctions is given by Hermite polynomials:

\[ h_\ell(Y) := c_\ell \sum_{n=0}^{\lfloor \ell/2 \rfloor} \frac{\ell!}{n!(\ell-2n)!} (-1)^n Y^{\ell-2n}. \quad (4.6) \]

\( h_\ell \) is orthogonal to any polynomial of degree lower than \( \ell - 1 \) for the \( L^2_\rho \) scalar product.

To construct the blow-up solution of Theorem 1, we will use an approximate solution to (3.9) close to \( F_k(bZ) \) that is adapted to the linearised dynamics both at the origin and far away.

Proposition 23 (Approximate blow-up profile). Let \( k \in \mathbb{N} \) with \( k \geq 2 \) and \( s_0 < s_1 \). There exists universal constants \( \bar{c}_{2\ell} \)\( \) with:

\[ \bar{c}_{2k-2} := -2k(2k-1), \quad \bar{c}_{2\ell} := \frac{(2\ell + 2)(2\ell + 1)}{k - \ell} \bar{c}_{2\ell+2} \quad \text{for } \ell = 0, \ldots, k-2, \]

\[ \bar{c}_{2k-2} \]

In fact there is a third one due to translation invariance which is absent here due to even symmetry.
and $0 < \epsilon \ll 1$ small enough such that for $a \in C^1([s_0, s_1], (1 - \epsilon, 1 + \epsilon))$, the profile

$$F[a](s, Y) := F_k(Z) + \sum_{\ell=0}^{k-1} \hat{c}_{2\ell} (a e^{-\frac{k-1-2\ell}{k} s}) 2k - 2\ell \phi_{2\ell}(Z)$$

(4.7)  \text{\bf def:Fb}

satisfies the following identity:

$$\partial_s F[a] + F[a] + \frac{Y}{2} \partial_Y F[a] - F^2[a] - \partial_Y Y F[a] = a_s \partial_a F[a] + \Psi,$$

with the error $\Psi$ satisfying for any $j \geq 0$:

$$\|\partial^j_Z \Psi\|_{L^2_p} \lesssim e^{-\frac{k-1}{k} s} \frac{Z}{|1 + |Z||}^{4k - 2} (1 + |Z|)^{-6k}$$

(4.8)  \text{\bf bd:psirho}

and for $|Y| \geq 1$:

$$|(Z \partial_Z)^j \Psi| \lesssim e^{-\frac{k-1}{k} s} |Z|^{4k - 2 - j} (1 + |Z|)^{-6k}$$

(4.9)  \text{\bf bd:Psi2}

and

$$|\partial^j_Z \Psi| \lesssim e^{-\frac{k-1}{k} s} |Z|^{4k - 2 - j} (1 + |Z|)^{-6k} \text{ for } j = 0, \ldots, 2k, \quad |\partial^j_Z \Psi| \lesssim e^{-\frac{k-1}{k} s} (1 + |Z|)^{-2k - 2 - j} \text{ for } j \geq 2k + 1.$$  (4.10)  \text{\bf bd:Psi3}

The variation with respect to $b$ enjoys the following properties:

$$\langle \partial_a F[a], h_{2k} \rangle_p = c b^{2k-1} e^{-(k-1)s} \left(1 + O(e^{-(k-1)s})\right), \quad c \neq 0,$$

(4.11)  \text{\bf id:pabFproject}

and for $j \in \mathbb{N}$:

$$\|\partial^j_Z \partial_a F[a]\|_{L^2_p} \lesssim e^{-\frac{(k-1)}{k} s} \text{ for } j = 0, \ldots, 2k, \quad \|\partial^j_Z \partial_a F[a]\|_{L^2_p} \lesssim 1 \text{ for } j \geq 2k + 1.$$  (4.12)  \text{\bf bd:pabFrho}

and for $|Y| \geq 1$:

$$|(Z \partial_Z)^j \partial_a F[a]| \lesssim |Z|^{2k} (1 + |Z|)^{-4k}.$$  (4.13)  \text{\bf bd:pabF2}

and

$$|\partial^j_Z \partial_a F[a]| \lesssim |Z|^{2k - j} (1 + |Z|)^{-4k} \text{ for } j = 0, \ldots, 2k, \quad |\partial^j_Z \partial_a F[a]| \lesssim (1 + |Z|)^{-2k - j} \text{ for } j \geq 2k + 1.$$  (4.14)  \text{\bf bd:pabF3}

**Proof.** This is a brute force computation.
Step 1: Estimates for $\Psi$. We first decompose from (4.2) and (4.3):

$$
\Psi = \partial_s F[b] + F[a] + \frac{Y}{2} \partial_Y F[a] - F^2[b] - \partial_Y F[a] - a_s \partial_s F[a]
$$

$$
= F_k(Z) + \frac{1}{2k} Z \partial_Z F_k(Z) - F_k^2(Z) - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z)
$$

$$
+ \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} (H_Z \phi_{2\ell})(Z) - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell+2} (\partial_Z \phi_{2\ell})(Z) - (ae^{-\frac{k-1}{2k}s})^2 \partial_Z F_k(Z)
$$

$$
- \left( \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z) \right)^2
$$

$$
= - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(k - \ell)(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z) - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell+2} (\partial_Z \phi_{2\ell})(Z) - (ae^{-\frac{k-1}{2k}s})^2 \partial_Z F_k(Z)
$$

$$
- \left( \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z) \right)^2
$$

$$
= - \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} (\partial_Z \phi_{2\ell+2}(Z)) - \bar{c}_0(ae^{-\frac{k-1}{2k}s})^{2k+2} \partial_Z \phi_0(Z) - \left( \sum_{\ell=0}^{k-1} \bar{c}_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z) \right)^2
$$

As

$$
\partial_Z \phi_{2\ell} = \frac{2(2\ell - 1)Z^{2\ell-2}}{(1 + Z^{2k})^2} - \frac{4k(4\ell + 2k - 1)Z^{2\ell+2k-2}}{(1 + Z^{2k})^3} + \frac{24k^2 Z^{2\ell+4k-2}}{(1 + Z^{2k})^4}
$$

one deduces that for $\ell = 0, \ldots, k - 2$:

$$
\left| (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} (\partial_Z \phi_{2\ell+2}(Z) - (2\ell + 2)(2\ell + 1) \phi_{2\ell}(Z)) \right|
$$

$$
= \left| (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left( - \frac{4k(4\ell + 2k + 3)kZ^{2\ell+2k}}{(1 + Z^{2k})^3} + \frac{24k^2 Z^{2\ell+4k}}{(1 + Z^{2k})^4} \right) \right|
$$

$$
\lesssim (ae^{-\frac{k-1}{2k}s})^{2k-2\ell} Z^{2\ell+2k}(1 + |Z|)^{-6k}.
$$

Similarly, as

$$
\partial_Z F_k = - \frac{2k(2k - 1)Z^{2k-2}}{(1 + Z^{2k})^2} + \frac{8k^2 Z^{4k-2}}{(1 + Z^{2k})^3}
$$
one deduces that
\[ \left| (ae^{-\frac{k-1}{2k}s})^2 (2k(2k-1)\phi_{2k-2} + \partial_{ZZ} F_k) \right| = \left| (ae^{-\frac{k-1}{2k}s})^2 \frac{8k^2 Z^{4k-2}}{(1 + Z^{2k})^3} \right| \lesssim (e^{-\frac{k-1}{2k}s})^2 Z^{4k-2}(1 + |Z|)^{-6k}. \]

Eventually,
\[ \left| -(ae^{-\frac{k-1}{2k}s})^{2k+2} \partial_{ZZ} \phi_0(Z) \right| = \left| (ae^{-\frac{k-1}{2k}s})^{2k+2} \left( \frac{4k(2k-1)Z^{2k-2}}{(1 + Z^{2k})^3} + \frac{24k^2 Z^{4k-2}}{(1 + Z^{2k})^4} \right) \right| \lesssim (e^{-\frac{k-1}{2k}s})^{2k+2} Z^{2k-2}(1 + |Z|)^{-6k} \]
and
\[ \left( \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \phi_{2\ell}(Z) \right)^2 \lesssim \sum_{\ell=0}^{k-1} (e^{-\frac{k-1}{2k}s})^{4k-4\ell} Z^{4\ell}(1 + |Z|)^{-8k}. \]

From the above identities one deduces that:
\[ |\Psi| \lesssim \sum_{\ell=1}^{k+1} (e^{-\frac{k-1}{2k}s})^{2\ell} |Z|^{4k-2\ell}(1 + |Z|)^{-6k} + \sum_{\ell=0}^{k-1} (e^{-\frac{k-1}{2k}s})^{4k-4\ell} |Z|^{4\ell}(1 + |Z|)^{-8k}. \quad (4.15) \]

For \( \ell = 1, \ldots, k+1 \) one computes that
\[ \int \left( (e^{-\frac{k-1}{2k}s})^{2\ell} Z^{4k-2\ell}(1 + |Z|)^{-6k} \right)^2 e^{-\frac{Y^2}{4}} dY \lesssim \int (e^{-\frac{k-1}{2k}s})^{4\ell} (e^{-\frac{k-1}{2k}s})^{8k-4\ell} e^{-\frac{Y^2}{4}} dY \lesssim e^{-4(k-1)s} \]
and similarly for \( \ell = 0, \ldots, k-1 \):
\[ \int \left( (e^{-\frac{k-1}{2k}s})^{4k-4\ell} Z^{4\ell}(1 + |Z|)^{-8k} \right)^2 e^{-\frac{Y^2}{4}} dY \lesssim e^{-4(k-1)s}. \]

The above two bounds imply (4.8) for \( j = 0 \). For \( |Y| \geq 1 \) one has that for \( \ell = 1, \ldots, k+1 \):
\[ (e^{-\frac{k-1}{2k}s})^{2\ell} Z^{4k-2\ell}(1 + |Z|)^{-6k} \lesssim e^{-\frac{k-1}{k-s}(aY e^{-\frac{k-1}{2k}s})^{2\ell-2} Z^{4k-2\ell}(1 + |Z|)^{-6k} = e^{-\frac{k-1}{k-s}Z^{4k-2}(1 + |Z|)^{-6k}} \]
and similarly for \( \ell = 0, \ldots, k-1 \):
\[ (e^{-\frac{k-1}{2k}s})^{4k-4\ell} Z^{4\ell}(1 + |Z|)^{-8k} \lesssim e^{-\frac{k-1}{k-s}Z^{4k-2}(1 + |Z|)^{-8k}}. \]

The above two bounds yield (4.9) for \( j = 0 \). We claim the the other bounds for \( |\Psi| \) can be proved the same way as (4.15) naturally extends to derivatives.

**Step 2 Estimate for \( \partial_a F \).** We compute:
\[
\begin{align*}
b \partial_a F[a] &= -2k \phi_{2k}(Z) + \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} ((2k - 2\ell)\phi_{2\ell}(Z) + Z\partial_Z \phi_{2\ell}(Z)) \\
&= -2k Z^{2k} \left( \frac{1}{1 + Z^{2k}} \right)^2 + \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left( \frac{2k Z^{2\ell}}{(1 + Z^{2k})^2} - \frac{4k Z^{2\ell+2k}}{(1 + Z^{2k})^3} \right) \\
&= -2k Z^{2k} + \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} 2k Z^{2\ell} \\
&\quad + 2k Z^{4k} - Z^{6k} \left( \frac{1}{1 + Z^{2k}} \right)^2 - \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k}s})^{2k-2\ell} \left( \frac{2k Z^{2\ell+2k}}{(1 + Z^{2k})^2} + \frac{4k Z^{2\ell+4k}}{(1 + Z^{2k})^3} \right)
\end{align*}
\]
One has that:
\[ | -2k \frac{1}{b} \phi_{2k}(Z) | \lesssim Z^{2k}(1 + Z^{2k})^{-2} \]
and for \( \ell = 0, \ldots, k - 1 \):
\[ \left| c_{2\ell}(ae^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} ((2k - 2\ell)\phi_{2\ell}(Z) + Z \partial_Z \phi_{2\ell}(Z)) \right| \lesssim (e^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} Z^{2\ell}(1 + Z^{2k})^{-2}. \]
Therefore:
\[ |\partial_a F[a]| \lesssim \sum_{\ell=0}^{k} (e^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} Z^{2\ell}(1 + Z^{2k})^{-2}. \]
(4.16)

This implies that
\[ |\partial_a F[a]| \lesssim e^{-(k-1)s} \sum_{\ell=0}^{k} Y^{2\ell} \]
which yields (4.12) for \( j = 0 \). For \( |Y| \geq 1 \) one therefore estimates:
\[ |\partial_a F[a]| \lesssim \sum_{\ell=0}^{k} (bYe^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} Z^{2\ell}(1 + Z^{2k})^{-2} \lesssim Z^{2k}(1 + Z^{2k})^{-2} \]
which proves (4.13) for \( j = 0 \). Again, we claim that the other bounds concerning \( \partial_a F[a] \) can be proved along the same lines since (4.16) naturally extends to derivatives. We now compute since \( h_{2k}(Y) \) is orthogonal to any polynomial of degree less or equal than \( 2k - 1 \) and \( Z = e^{-(k-1)s/(2k)} \):
\[ \langle -2kZ^{2k} + \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} 2kZ^{2\ell}, h_{2k} \rangle = \langle -2kZ^{2k}, h_{2k} \rangle = b^{2k}ce^{-(k-1)s}, \ c \neq 0. \]

We then get the desired nondegeneracy (4.11) since
\[ \| 2^{2k} Z^{4k} - Z^{6k} - \sum_{\ell=0}^{k-1} c_{2\ell}(ae^{-\frac{k-1}{2k} s})^{2^{k-2\ell}} 2^{2k}Z^{2\ell} \left( \frac{2kZ^{2\ell+2k}}{1 + Z^{2k}} \right) \|_{L^2} \lesssim e^{-2(k-1)s}. \]

We now show Theorem 1 by showing that there exists a global solution to (3.9) converging to \( F[bZ] \) as \( s \to +\infty \). To this end, we perform a bootstrap argument near the approximate profile \( F[a] \). We decompose the solution in self-similar variables according to (using (4.14)):
\[ f = F[a] + \varepsilon, \quad \varepsilon = \sum_{\ell=0}^{2k-1} c_{\ell} h_{\ell}(Y) + \bar{\varepsilon}, \quad \bar{\varepsilon} \perp \rho \ h_{\ell} \text{ for } \ell = 0, \ldots, 2k, \]
(4.17)
where the \( \perp \rho \) is the orthogonality with respect to the \( L^2_\rho \) scalar product.

**Proposition 24.** There exists \( M \gg 1, K_{J+1} \gg K_J \gg \ldots \gg K_0 \gg 1 \) and \( \bar{K}, s_0 \gg 1 \) large enough and \( 0 < \varepsilon \ll 1 \) small enough such that for any \( \varepsilon_0 \) satisfying the orthogonality (4.17) and
\[ \sum_{j=0}^{J} \int_{|Y| \geq 1} \frac{|(Y\partial_Y)^j \varepsilon_0|^2 dY}{\phi_{2k+1}(Z)} |Y| \leq e^{-\frac{s_0}{4}}, \quad \sum_{j=0}^{J} \int_{|Y| \geq 1} \frac{|(Y\partial_Y)^j \varepsilon_0|^2 dY}{\phi_{2k+1/2}(Z)} |Y| \leq e^{-\frac{s_0}{4}}, \]
(4.18)
\[ \sum_{j=2k+1}^{J} \int_{|Y| \geq 1} \frac{|(Y\partial_Y)^j \varepsilon_0|^2 dY}{\phi_0(Z)} |Y| \leq e^{-\frac{s_0}{4}}, \]
(4.19)
\[ \| \partial_Z^j \varepsilon \|_{L^2_\rho} \leq e^{-(k-\frac{1}{2} + \frac{1}{2k} - \frac{j}{2})s_0} \text{ for } j = 0, \ldots, 2k, \quad \| \partial_Z^j \varepsilon \|_{L^2_\rho} \leq e^{-\frac{s_0}{4}} \text{ for } j \geq 2k + 1, \]
(4.20)
and \(|a(s_0) - 1| \leq \epsilon\), then there exist \(c_0(s_0), ..., c_{2k-2}(s_0)\) with

\[
\left( \sum_{\ell=0}^{2k-1} |c_\ell(s_0)|^2 \right) ^{\frac{1}{2}} \leq \tilde{K} e^{-\frac{1}{2} + \frac{1}{4\pi}s_0}
\]  

(4.21)  

such that the solution \(f\) to (3.9) with initial datum

\[ f(s_0) = F[a](s_0) + \chi_M(Y) \sum_{\ell=0}^{2k-1} c_\ell(s_0) h_\ell(Y) + \tilde{\varepsilon}_0, \]

where \(\chi_M(Y) := \chi(Y/M)\), is a global solution to (3.9), and for all \(s \geq s_0\) one has for \(j = 0, ..., J\):

\[
\int_{|Y| \geq 1} |(Y \partial_Y^j) \tilde{\varepsilon}|^2 \, dY \leq K_j e^{-\frac{1}{2} s},
\]

(4.22)  

\[
\int_{|Y| \geq 1} |(Y \partial_Y^j) \tilde{\varepsilon}|^2 \, dY \leq K_{j+1} e^{-\frac{1}{2} s},
\]

(4.23)  

\[
\int_{|Y| \geq 1} \left| \frac{\partial^j_Z \tilde{\varepsilon}}{\partial^j (Z)} \right|^2 \, dY \leq K_j e^{-\frac{1}{2} s} \text{ for } j \geq 2k + 1,
\]

(4.24)  

\[
\|\partial^j_Z \tilde{\varepsilon}\|_{L^2} \leq \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} \text{ for } j = 0, ..., 2k, \quad \|\partial^j_Z \tilde{\varepsilon}\|_{L^2} \leq \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} \text{ for } j \geq 2k + 1,
\]

(4.25)  

\[
\left( \sum_{\ell=0}^{2k-1} |c_\ell|^2 \right)^{\frac{1}{2}} \leq \tilde{K} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s}
\]

(4.26)  

and there exists an asymptotic limit for \(a\), \(|a^* - 1| \leq 2\epsilon\) such that

\[
|a - a^*| \lesssim (K_1^2 + K_2^2) e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s}.
\]

(4.27)  

We prove Proposition 13 with a classical bootstrap reasoning. In what remains in this subsection we assume that \(f\) is a solution to (3.9) defined on \([s_0, s_1]\) such that the decomposition (4.17) satisfies (4.18), (4.19), (4.20), (4.21), (4.22), (4.23), (4.24), (4.25) and (4.26). The results below will specify the dynamics in this regime and allow to prove Proposition 24 at the end of the subsection.  

**Lemma 25.** There holds for \(j = 0, ..., J - 1\), for any \(Z \in \mathbb{R}\):

\[
|\partial^j_Z \tilde{\varepsilon}| \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} (1 + |Z|)^{\frac{1}{2} - 2k - j},
\]

(4.28)  

and for any \(C > 0\), for all \(|Y| \leq C\):

\[
|\partial^j_Z \tilde{\varepsilon}| \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} \text{ for } j = 0, ..., 2k, \quad |\partial^j_Z \tilde{\varepsilon}| \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} \text{ for } j \geq 2k + 1.
\]

(4.29)  

**Proof. Step 1:** Proof of (4.29). From (4.25) and Sobolev embedding one deduces that for \(|Y| \leq C\):

\[
|\partial^j_Z \tilde{\varepsilon}| \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s}.
\]

From (4.26) and (6.1) for \(j = 0, ..., 2k - 1\) and \(|Y| \leq C\) one has:

\[
|\partial^j_Z (c_\ell h_\ell(Y))| = |e^{\frac{1}{4\pi} j s} \partial^j_Z (c_\ell h_\ell(Y))| \lesssim e^{\frac{1}{4\pi} j s} |c_\ell| \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi}) s} \lesssim \sqrt{K_j} e^{-(k - \frac{1}{2} + \frac{1}{4\pi})}
\]

for \(s_0\) large enough. For \(j \geq 2k\) and \(\ell \leq 2k - 1\) one notices that \(\partial^j_Z h_\ell = 0\). Therefore, one obtains (4.29) from the decomposition (4.17) and the two above bounds.
We now estimate the right hand side. First, since
\[ |\phi_{2k+1/2}| \lesssim |Z\partial Z\phi_{2k+1/2}| \leq |Z\partial Z| \lesssim |Z\partial Z\phi_{2k+1/2}|, \]
we infer that
\[ \frac{|Zj\partial Z\xi|^2}{\phi^2_0(Z)} \lesssim \frac{\|Z\partial Z(Z)\|^2}{\phi_0^2(Z) |Y|} + \frac{\|Z\partial Z(Z)\|^2}{\phi_0^2(Z) |Y|} \lesssim K_je^{-\frac{3}{4}ks}. \]
Since \(|\phi_{2k+1/2}| \lesssim |Z|^{2k+1/2}(1 + |Z|)^{-4k}\) one obtains that for \(|Y| \geq 1\)
\[ |\partial Z\xi| \lesssim \sqrt{K_je^{-\frac{3}{4}ks}} |Z|^{2k+1/2-j}(1 + |Z|)^{-4k}. \]
For \(j \geq 2k + 1\), the fact that \(|\phi_{2k+1/2}(Z) = |Z|^{2k+1}|\phi_0(Z)|\) yields the inequality
\[ |Z|/|\phi_0(Z)| \lesssim 1/|\phi_0(Z)| + |Z|^{j+1}/|\phi_0(Z)| + |Z|^{j+1} \]
from which we infer that
\[ \int_{|Y| \geq 1} \frac{|Z\partial Z(Z)(\partial Z\xi)|^2}{\phi^2_0(Z)} dY \lesssim \int_{|Y| \geq 1} \frac{|\partial Z\xi|^2}{\phi_0^2(Z) |Y|} + \int_{|Y| \geq 1} \frac{|Z\partial Z(Z)|^2}{\phi_0^2(Z) |Y|} \lesssim K_je^{-\frac{1}{4}ks}. \]
The very same reasoning using the above bound and (4.24) gives
\[ |\partial Z\xi| \lesssim \sqrt{K_je^{-\frac{1}{4}ks}} (1 + |Z|)^{-4k}. \]
From the two above bounds one infers that for \(|Y| \geq 1\):
\[ |\partial Z\xi| \lesssim \sqrt{K_je^{-\frac{1}{4}ks}} (1 + |Z|)^{-\frac{1}{2} - 2k - j}. \]
Combining (4.30), (4.31) and (4.29) yields (4.28).

The evolution of \(\varepsilon\) is given by:
\[ \varepsilon_s + \frac{Y}{2} \partial_Y \varepsilon + \varepsilon - 2F_k(Z)\varepsilon + 2(F_k(Z) - F[a]\varepsilon - \partial_Y Y\varepsilon + a_s \partial a F[a] + \Psi - \varepsilon^2 = 0 \]
and that of \(\bar{\varepsilon}\) by:
\[ \bar{\varepsilon}_s + \mathcal{L}_\rho \bar{\varepsilon} + \sum_{\ell=0}^{2k-1} (c_{\ell,s} - \frac{\ell - 2}{2} c_{\ell}) h_{2\ell} + 2(1 - F[a])\varepsilon + a_s \partial a F[a] + \Psi - \varepsilon^2 = 0 \]

**Lemma 26 (Modulation equations).** The following identities hold:
\[ |a_s| \lesssim K_je^{-(\frac{3}{2} + \frac{1}{4}k)s} \quad \text{and} \quad \left| c_{\ell,s} - \frac{\ell - 2}{2} c_{\ell} \right| \lesssim K_je^{-(k-\frac{1}{2} + \frac{1}{4}k)s}. \]

**Proof.** **Step 1 Law for \(a\).** We take the scalar product between (4.33) and \(h_{2k}\), yielding, using (4.11) and (4.17):
\[ a_s c_b 2^{k-1} e^{-(k-1)s} (1 + O(e^{-(k-1)s})) \approx \langle -2(1 - F[a]\varepsilon - \Psi + \varepsilon^2, h_{2k}\rangle. \]
We now estimate the right hand side. First, since \(|F[a] - 1| \lesssim e^{-(k-1)s} \sum_{\ell=0}^{k} Y^{2\ell}\), using (4.25):
\[ |\langle -2(1 - F[a]\varepsilon, h_{2k}\rangle| \lesssim ||e||_{L^2_{\rho}}((1 - F[a])h_{2k}||_{L^2_{\rho}} \lesssim e^{-(k-\frac{1}{2})s} e^{-(k-1)s} \lesssim \sqrt{K_je^{-(2k-\frac{1}{2})s}}. \]
Using the bound on the error (4.8):
\[ |\langle \Psi, h_{2k}\rangle| \lesssim e^{-2(k-1)s}. \]
Finally, using the bounds (4.25), (4.26) and (4.28) for the nonlinear term:
\[ |\langle \epsilon^2, h_{2k} \rangle | \lesssim \| \epsilon \|_{L^2} \| \epsilon \|_{L^\infty} \lesssim \sqrt{K_j} e^{-(k-\frac{1}{2})s} \sqrt{K_j} e^{-\frac{4}{1+\sqrt{2}}s} \lesssim K_j e^{-(k-\frac{1}{2}+\frac{4}{1+\sqrt{2}})s}. \]

Summing the above identities yields (4.34) for \( a_s \).

**Step 2** Law for \( c_\ell \). We take the scalar product between (4.33) and \( h_\ell \) for \( \ell = 0,\ldots,2k-1 \), yielding, using (4.11) and (4.17):
\[
(c_{\ell, s} + \frac{\ell - 2}{2} c_\ell) \| h_{2\ell} \|_{L^2}^2 = -\langle 2(1 - F[a]) \epsilon + a_s \frac{\partial}{\partial a} F[a] + \Psi - \epsilon^2, h_\ell \rangle
\]
Performing the same computations as in Step 1 gives:
\[
|\langle 2(1 - F[a]) \epsilon + \Psi - \epsilon^2, h_\ell \rangle | \lesssim K_j e^{-(k-\frac{1}{2}+\frac{4}{1+\sqrt{2}})s}.
\]
Using the bound for \( a_s \) (4.34) obtained in Step 1 and (4.12) one obtains:
\[
|\langle a_s \frac{\partial}{\partial a} F[a], h_\ell \rangle | \lesssim K_j e^{-(\frac{1}{2} + \frac{4}{1+\sqrt{2}})s} e^{-(k-1)s} = K_j^2 e^{-(k-\frac{1}{2}+\frac{4}{1+\sqrt{2}})s}.
\]
The three previous identities then yield (4.34) for \( c_\ell \).

\[ \text{Lemma 27. At time } s_1 \text{ there holds:} \]
\[
\| \partial_Z^j \tilde{\epsilon}(s_1) \|_{L^2} \leq \sqrt{K_j} \alpha^{-\frac{1}{2}} e^{-(k-\frac{1}{2} + \frac{4}{1+\sqrt{2}})s_1} \text{ for } j = 0,\ldots,2k, \quad \| \partial_Z^j \tilde{\epsilon}(s_1) \|_{L^2} \leq \sqrt{K_j} e^{-\frac{4}{1+\sqrt{2}}s_1} \text{ for } j \geq 2k+1.
\]
\[ (4.35) \]

**Proof.** Set \( w = \partial_Z \tilde{\epsilon} \). Then \( w \) solves from (4.33):
\[
w_s + \frac{j}{2k} w + H_\rho w + \sum_{\ell=0}^{2k-1} (c_{\ell, s} + \frac{\ell - 2}{2} c_\ell) \partial_Z^j (h_{2\ell}) + 2 \partial_Z^j ((1 - F[a]) \epsilon) + a_s \partial_Z \partial_a F[a] + \partial_Z \Psi - \partial_Z^j \epsilon^2 = 0
\]
which yields the following expression for the energy identity:
\[
\frac{d}{ds} \left( \frac{1}{2} \int w^2 e^{-\frac{\chi^2}{4}} dY \right) = -\frac{j}{2k} \int w^2 e^{-\frac{\chi^2}{4}} dY - \langle w, H_\rho w \rangle - \sum_{\ell=0}^{2k-1} (c_{\ell, s} + \frac{\ell - 2}{2} c_\ell) \langle w, \partial_Z^j (h_{2\ell}) \rangle - 2 \langle w, \partial_Z^j ((1 - F[a]) \epsilon) \rangle + \langle w, a_s \partial_Z \partial_a F[a] + \partial_Z \Psi - \partial_Z^j \epsilon^2 \rangle.
\]
For \( j = 0,\ldots,2k \), by integrating by parts one obtains from (4.17) that \( w \) is orthogonal to \( h_\ell \) for \( \ell = 0,\ldots,2k-j \) for the \( L^2 \) scalar product and therefore to any polynomial of degree less or equal to \( 2k - j \). Therefore, from Proposition 22 there holds:
\[
-\langle w, H_\rho w \rangle \lesssim \begin{cases} -\left( k - \frac{j}{2} - \frac{1}{2} \right) \int w^2 e^{-\frac{\chi^2}{4}} dY & \text{if } j = 0,\ldots,2k, \\
\int w^2 e^{-\frac{\chi^2}{4}} dY & \text{if } j = 2k+1,\ldots, J. 
\end{cases}
\]
Let \( 0 < \nu \ll 1 \) be a small constant to be fixed later on. Integrating by parts yields:
\[
- \int w H_\rho w e^{-\frac{\chi^2}{4}} dY \leq -c \int Y^2 w^2 e^{-\frac{\chi^2}{4}} dY + c' \int w^2 e^{-\frac{\chi^2}{4}} dY.
\]
for some constants $c, c' > 0$. Combined with the above estimate one writes:

\[- \int w H \rho e^{-\frac{\nu^2}{4}} dY = -(1 - e^{-2s}) \int w H \rho e^{-\frac{\nu^2}{4}} dY - e^{-2s} \int w H \rho e^{-\frac{\nu^2}{4}} dY \]

\[ \leq -(1 - e^{-2s}) \int w^2 e^{-\frac{\nu^2}{4}} dY + e^{-2s} \left( -c \int Y^2 w^2 e^{-\frac{\nu^2}{4}} dY + c' \int w^2 e^{-\frac{\nu^2}{4}} dY \right) \]

\[ \leq e^{-2s} \int Y^2 w^2 e^{-\frac{\nu^2}{4}} dY + O(e^{-2s} \int w^2 e^{-\frac{\nu^2}{4}} dY) \]

\[ + \left\{ -\left( k - \frac{1}{2} - \frac{1}{4} \right) \int w^2 e^{-\frac{\nu^2}{4}} dY \right\} \text{ if } j = 0, \ldots, 2k, \]

\[ + \left\{ \int w^2 e^{-\frac{\nu^2}{4}} dY \right\} \text{ if } j = 2k + 1, \ldots, J. \]

As we said earlier, $w$ is orthogonal to any polynomial of degree less or equal to $2k - j$ for $j = 0, \ldots, 2k$. For $j \geq 2k + 1$ one notices that $\partial^j_Z h_\ell = 0$ for any $\ell = 0, \ldots, 2k - 1$. Hence the cancellation for $j = 0, \ldots, J$:

\[-\langle w, H \rho \rangle - \sum_{\ell=0}^{2k-1} (c_{\ell,s} + \ell - 2c_{\ell}) \langle w, \partial^j_Z (h_{2\ell}) \rangle = 0 \]

One has from (4.7) that the lower order linear potential satisfies:

\[ |(1 - F[a])| \lesssim |Z|^2 (1 + |Z|)^{-2-2k} \]

which adapts to derivatives. Therefore, applying Leibniz rule and Cauchy-Schwarz one gets that for $j = 0, \ldots, J$:

\[ \left| \langle w, \partial^j_Z ((1 - F[a]) \epsilon) \rangle \right| \leq C \left( \int w^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{j}{2}} \sum_{i=0}^{j-1} \left( \int |\partial^i_Z \epsilon|^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{1}{2}} + Ce^{-\frac{k-1}{2}} \int Y^2 w^2 e^{-\frac{\nu^2}{4}} dY \]

since $Z = Ye^{-\frac{\nu^2}{4k}}$. From (4.34), (4.12), (4.8), (4.25) and Cauchy-Schwarz we estimate for $\nu$ small enough:

\[ \left| \int w \partial^j_Z (a_s \partial_a F[a] + \Psi) e^{-\frac{\nu^2}{4}} dY \right| \lesssim e^{-2s} \int w^2 e^{-\frac{\nu^2}{4}} dY + C(K_j) \left\{ e^{-2(k - \frac{1}{2} - \frac{1}{4} + \frac{4k}{\nu} + s)} \right\} \text{ for } j = 0, \ldots, 2k, \]

\[ e^{-2(\frac{1}{4k} + s)} \text{ for } j = 2k + 1, \ldots, J. \]

Using Leibnitz rule and Cauchy Schwarz, from (4.28), (4.17), (4.25) and (4.26) we infer for the nonlinear term:

\[ \left| \int w \partial^j_Z (e^2 e^{-\frac{\nu^2}{4}} dY) \right| \lesssim \left( \sum_{i=0}^{j-1} \| \partial^i_Z \epsilon \|_{L^\infty} \right) \left( \sum_{i=0}^{j} \int |\partial^i_Z \epsilon|^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{1}{2}} \left( \int w^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{1}{2}} \]

\[ \lesssim e^{-\frac{1}{4k}} \left( e^{-2(k - \frac{1}{2} + \frac{1}{4} + \frac{4k}{\nu})} + \left( \sum_{i=0}^{j} \int |\partial^i_Z \epsilon|^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{1}{2}} \right) \left( \int w^2 e^{-\frac{\nu^2}{4}} dY \right)^{\frac{1}{2}} \]

\[ \lesssim C(K_j) \left\{ e^{-2(k - \frac{1}{2} + \frac{1}{4} + \frac{4k}{\nu} + s)} \right\} \text{ for } j = 0, \ldots, 2k, \]

\[ e^{-2(\frac{1}{4k} + s)} \text{ for } j = 2k + 1, \ldots, J. \]
for $0 < \nu \ll 1$ small enough. Putting all the previous estimates together one obtains for $\nu$ small enough and then for $s_0$ large enough, after some signs inspection:

$$
\frac{d}{ds} \left( \int w^2 e^{-\frac{Y^2}{4}} dY \right) \leq \begin{cases} 
-2 \left( k - \frac{j}{2} + \frac{j}{2} \right) \int w^2 e^{-\frac{Y^2}{4}} dY + C(K_j) e^{-2(\frac{j}{4} - \frac{1}{2} + \frac{\nu}{\pi})} & \text{for } j = 0, \ldots, 2k \\
-\frac{1}{k} \int w^2 e^{-\frac{Y^2}{4}} dY + C(K_j) e^{-2(\frac{j}{\pi} + \nu)} & \text{for } j = 2k + 1 \\
-\frac{j-2k}{k} \int w^2 e^{-\frac{Y^2}{4}} dY + C(K_j) e^{-2(\frac{j}{\pi} + \nu)} + C(K_{j-1}) e^{-\frac{j}{k}} & \text{for } j = 2k + 2, \ldots, J 
\end{cases}
$$

Integrated over time this shows (4.35). We only show that this is the case for $j = 2k + 2, \ldots, J$, the proof being the same for $0 \leq j \leq 2k + 1$. For $j = 2k + 2, \ldots, J$, from (4.20) and the above inequality one deduces that at time $s_1$:

$$
\int w^2 e^{-\frac{Y^2}{4}} dY \leq e^{-\frac{j}{k} s_1} \left( \int w(s_0)^2 e^{-\frac{Y^2}{4}} dY + e^{-\frac{j}{k} s_1} \int_{s_0}^{s_1} \left( C(K_j) e^{\frac{1}{k} - 2\nu} + C(K_{j-1}) e^{\frac{1}{k}} \right) d\tilde{s} \right) \leq C e^{-\frac{j}{k} s_1} + C(K_j) e^{-\frac{j}{k} s_1} e^{-2\nu s_1} + C(K_{j-1}) e^{-\frac{j}{k} s_1} \leq \frac{K_j}{2} e^{-\frac{j}{k} s_1}
$$

for $K_j$ large enough depending on $K_{j-1}$ and $s_0$ large enough. We safely leave the other time integrations to the reader. 

\[\square\]

**Lemma 28.** At time $s_1$ there holds for $j = 0, \ldots, J$:

$$
\int_{|Y| \geq 1} \frac{|(Y \partial_Y)^i \varepsilon(s_1)|^2}{\phi_{2k+1}(Z)} \frac{dY}{|Y|} \leq \frac{K_j}{2} e^{-\frac{j}{4\pi} s_1}, \quad \int_{|Y| \geq 1} \frac{|(Y \partial_Y)^i \varepsilon(s_1)|^2}{\phi_{2k+1/2}(Z)} \frac{dY}{|Y|} \leq \frac{K_j+1}{2} e^{-\frac{j}{4\pi} s_1}, \quad (4.36)
$$

$$
\int_{|Y| \geq 1} \frac{|\partial_Z^j \varepsilon(s_1)|^2}{\phi_0(Z)} \frac{dY}{|Y|} \leq \frac{K_j}{2} e^{-\frac{j}{2k} s_1} \quad \text{for } j \geq 2k + 1. \quad (4.37)
$$

**Proof.** We perform the estimates only for $Y \geq 1$, those for $Y \leq -1$ being exactly the same, thus writing $Y = |Y|$ in the following.

**Step 1:** Proof of (4.36). Let $\chi$ be a smooth cut-off function, with $\chi = 1$ for $Y \geq 2$ and $\chi = 0$ for $Y \leq 1$. Let $\ell = 2k + 1$ or $\ell = 2k + 1/2$. Set $w = (Y \partial_Y)^i \varepsilon$. Then $w$ solves from (4.32) since $[\partial_Y, Y \partial_Y] = 2\partial_Y Y$:

$$
0 = w_s + \frac{Y}{2} \partial_Y w + w - 2F_k(Z)w - \partial_Y Y w + 2(F_k w - (Y \partial_Y)^i (F_k \varepsilon)) + 2(Y \partial_Y)^i ((F_k(Z) - F[a]) \varepsilon)
$$

$$
+ 2 \sum_{n=0}^{j-1} (Y \partial_Y)^i (Y \partial_Y)^n \varepsilon + a_{s}(Y \partial_Y)^i \partial_a F[a] + (Y \partial_Y)^i \Psi - (Y \partial_Y)^i \varepsilon^2
$$
From the above identity we compute, performing integrations by parts, that

\[
\frac{d}{ds} \left( \frac{1}{2} \int \chi \frac{w^2}{\phi_t(Z)} \frac{dY}{Y} \right) = \int \chi \frac{w}{\phi_t(Z)} \frac{1 - w + 2F_k w - \frac{Y}{r} \partial_Y w + \partial_Y w - 2(F_k w - (Y \partial_Y)^2 (F_k v)) + (Y \partial_Y)^2 v}{\phi_t(Z)} dY \\
+ \int \chi \frac{w}{\phi_t(Z)} \frac{1}{Y} \frac{2(Y \partial_Y)^2 ((F_k(Z) - F[a]s) - 2 \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} \partial_Y Y (Y \partial_Y)^n e - (Y \partial_Y)^j (a_s \partial_a F[a] + \Psi)}{\phi_t(Z)} dY \\
- \int \chi \frac{w^2}{\phi_t(Z)} \frac{1}{Y} \left( \frac{a_s (Z \partial_Z \phi_t(Z))}{\phi_t(Z)} - \frac{k - 1 (Z \partial_Z \phi_t(Z))}{2k} \phi_t(Z) \right) dY \\
= \int \chi \frac{w^2}{\phi_{2k+1}^j(Z)} \frac{1}{\phi_t(Z)} - \phi_t(Z) + 2F_k \phi_t(Z) - \frac{Z}{2k} \partial_Z \phi_t(Z) dY - \int \chi \frac{|\partial_Y w|^2}{\phi_t^2(Z)} dY \\
+ \frac{1}{4} \int \partial_Y \chi \frac{w^2}{\phi_t^2(Z)} dY + \frac{1}{2} \int w^2 \partial_Y \left( \frac{\chi}{\phi_t^2(Z)} \frac{1}{Y} \right) dY \\
- 2 \int w(F_k(Z)w - (Y \partial_Y)^2 (F[a]s)) dY - 2 \int \chi \frac{w(Y \partial_Y)^2 ((F_k(Z) - F[a]s) e}{\phi_t^2(Z)} dY \\
- 2 \int \chi \frac{w \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} \partial_Y Y (Y \partial_Y)^n e dY}{Y} - \frac{a_s}{a} \int \frac{w^2 (Z \partial_Z \phi_t(Z)) dY}{\phi_t(Z)} Y \\
- \int \chi \frac{w}{\phi_t(Z)} \left( (a_s \partial_a F[a] + \Psi) + \int \chi \frac{w(Y \partial_Y)^2 (e^2)}{\phi_t^2(Z)} dY + \int \chi \frac{w}{\phi_t^2(Z)} dY \right)
\]

where in the last equality, on the first line one has the main order linear effects, on the second their associated boundary terms, one the third and fourth the lower order linear effects, and on the last line the influence of the forcing and of the nonlinear effects. We now estimate all terms.

For the first term from (4.3):

\[
\int \chi \frac{w^2}{\phi_{2k+1}^j(Z)} \frac{1}{\phi_t(Z)} - \phi_t(Z) + 2F_k \phi_t(Z) - \frac{Z}{2k} \partial_Z \phi_t(Z) dY = - \frac{\ell - 2k}{2k} \int \chi \frac{w^2}{\phi_t^2(Z)} dY.
\]

The second term is dissipative and has a negative sign since \( \chi \) is positive. For the third term, using (4.17), (4.25) and (4.26):

\[
\left| \frac{1}{2} \int \partial_Y \chi \frac{w^2}{\phi_t^2(Z)} dY \right| \lesssim \frac{1}{\phi_t^2(Z)} \left\| w \right\|_{L^2(1 \leq Y \leq 2)}^2
\]

\[
\lesssim \left\| Z \right\|_{L^\infty(1 \leq Y \leq 2)}^{-2\ell} \left( \sum_{n=0}^{j-1} \left\| Z^j \partial_Z^n \right\|_{L^2(1 \leq Y \leq 2)}^2 \right) + \sum_{n=0}^{2k-1} \left| c_t \right|^2
\]

\[
\lesssim e^{-\frac{k-1}{4} s Y^{-2\ell}} \left( \sum_{n=0}^{j-1} \left\| Z^j \partial_Z^n \right\|_{L^2(1 \leq Y \leq 2)}^2 + \sum_{n=0}^{2k-1} \left| c_t \right|^2 \right)
\]

\[
\lesssim e^{-\frac{k-1}{4} Y^{-2\ell}} \lesssim e^{-\frac{k}{4} s}. \quad (4.38)
\]

For the fourth term, we first decompose:

\[
\partial_Y \left( \frac{\chi}{\phi_t(Z) Y} \right) = \partial_Y \chi \frac{1}{\phi_t^2(Z) Y} + 2 \partial_Y \chi \partial_Y \left( \frac{1}{\phi_t(Z) Y} \right) + \chi \partial_Y Y \left( \frac{1}{\phi_t^2(Z) Y} \right).
\]
Since one has
\[
\left| \partial_Y \chi \left( \frac{1}{\phi_k^2(Z)Y} \right) + 2\partial_Y \chi_Y \left( \frac{1}{\phi_k^2(Z)Y} \right) \right| \lesssim |Z|^{-2\ell} 1_{1 \leq Y \leq 2} \lesssim e^{-\frac{k-1}{2}\ell s} 1_{1 \leq Y \leq 2}
\]
we claim that one can perform the very same estimate for the first two terms as (4.38), giving:
\[
\left| \int w^2 \left( \partial_Y \chi \left( \frac{1}{\phi_k^2(Z)Y} \right) + 2\partial_Y \chi_Y \left( \frac{1}{\phi_k^2(Z)Y} \right) \right) dY \right| \lesssim e^{-\frac{1}{2}s}.
\]
For the last term, from a direct computation, for \(|Y| \geq 1\), one has that:
\[
\left| \partial_Y \chi \left( \frac{1}{\phi_k^2(Z)Y} \right) \right| \leq \frac{1}{\phi_k^2(Z)Y^3}.
\]
Therefore, if \(\ell = 2k + 1\), we take some \(0 < \kappa \ll 1\) small enough and split the integral using some \(Y^* \gg 1\) large enough, and use (4.17), (4.25) and (4.26):
\[
\left| \int w^2 \partial_Y \chi \left( \frac{1}{\phi_{2k+1}^2(Z)Y} \right) dY \right| \lesssim \int_{Y^*}^{+\infty} w^2 \frac{1}{\phi_{2k+1}^2(Z)Y^2} dY + \int_{1}^{Y^*} w^2 \frac{1}{\phi_{2k+1}^2(Z)Y^2} dY
\]
\[
\lesssim \|w\|_{L^2(1 \leq Y \leq Y^*)} \|w\|_{L^2(1 \leq Y \leq Y^*)} + \kappa \int \chi \frac{\varepsilon^2}{\phi_{2k+1}^2(Z)Y^2} dY
\]
\[
\lesssim e^{-\frac{k-1}{2}(2k+1)s} e^{-2(k-\frac{1}{2})s} + \kappa \int \chi \frac{\varepsilon^2}{\phi_{2k+1}^2(Z)Y^2} dY \lesssim e^{-\frac{1}{2}s} + \kappa \int \chi \frac{\varepsilon^2}{\phi_{2k+1}^2(Z)Y^2} dY.
\]
If \(\ell = 2k + 1/2\), one uses the fact that \(\phi_{2k+1}(Z) = Z^{1/2} \phi_{2k+1/2}\), so that
\[
\frac{1}{\phi_{2k+1/2}^2(Z)Y^3} = \frac{1}{\phi_{2k+1}^2(Z)Y^2} e^{-\frac{k-1}{2}s},
\]
(4.39) to estimate using (4.22):
\[
\left| \int w^2 \partial_Y \chi \left( \frac{1}{\phi_{2k+1/2}^2(Z)Y} \right) dY \right| \lesssim e^{-\frac{k-1}{2}s} \int_{Y \geq 1} w^2 \frac{1}{\phi_{2k+1}^2(Z)Y^2} dY \lesssim e^{-\left(\frac{1}{2} + \frac{1}{2k}\right)s} \lesssim e^{-\frac{k-1}{2}s}.
\]

Collecting the above bounds one has proven that:
\[
\left| \int w^2 \partial_Y \chi \left( \frac{1}{\phi_k^2(Z)Y} \right) dY \right| \lesssim \left\{ \begin{array}{ll}
\frac{1}{2k+1}s + \kappa \int \chi \frac{w^2}{\phi_{2k+1}^2(Z)Y^2} dY & \text{for } \ell = 2k + 1, \\
e^{-\frac{1}{2}s} & \text{for } \ell = 2k + 1/2.
\end{array} \right.
\]

We turn to the fifth term. We first estimate using Leibniz rule:
\[
|F_k(Z)w - (Y \partial_Y)^j (F[a] \varepsilon)| \lesssim \sum_{n=0}^{j-1} |(Z \partial Z)^{j-n} F[a]| |(Y \partial_Y)^n \varepsilon| \lesssim |Z|^{2k} (1 + |Z|)^{-4k} \sum_{n=0}^{j-1} |(Y \partial_Y)^n \varepsilon|.
\]
If \(\ell = 2k + 1\) we then use Cauchy-Schwarz and (4.22) to obtain:
\[
\left| \int \chi \frac{w(F_k(Z)w - (Y \partial_Y)^j (F[a] \varepsilon)) dY}{\phi_{2k+1}^2(Z)} \right| \leq C \left( \int_{Y \geq 1} \frac{w^2}{\phi_{2k+1}^2(Z)Y} dY \right)^{1/2} \sum_{n=0}^{j-1} \left( \int_{Y \geq 1} \frac{(Y \partial_Y)^n \varepsilon)^2 dY}{\phi_{2k+1}^2(Z)Y} \right)^{1/2}
\]
\[
\leq C(K_{j-1}) \sqrt{K_j} e^{-\frac{k-1}{2}s}.
\]
If \(\ell = 2k + 1/2\) we use the fact that
\[
\frac{|Z|^{2k}}{\phi_{2k+1/2}^2(Z)(1 + |Z|)^{4k}} \leq \frac{1}{\phi_{2k+1}^2(Z)},
\]
which, combined with Cauchy-Schwarz and (4.22), gives:

\[
\left| \int \frac{w(F_k(Z)w - (Y \partial_Y)^j(F[a]\varepsilon))}{\phi^2_{2k+1/2}(Z)} dY \right| \lesssim \left( \int_{Y \geq 1} \frac{w^2}{\phi^2_{2k+1}(Z) Y} dY \right)^{1/2} \sum_{n=0}^{j-1} \left( \int_{Y \geq 1} \frac{(Y \partial_Y)^n \varepsilon}{\phi^2_{2k+1}(Z) Y} dY \right)^{1/2} \lesssim e^{-\frac{k-1}{4} s}.
\]

One has then proven that:

\[
\left| \int \frac{w(F_k(Z)w - (Y \partial_Y)^j(F[a]\varepsilon))}{\phi^2_{2k+1/2}(Z)} dY \right| \lesssim \begin{cases} C(K_{j-1}) \sqrt{K_j} e^{-\frac{k-1}{4} s} & \text{for } \ell = 2k + 1, \\ e^{-\frac{k-1}{4} s} & \text{for } \ell = 2k + 1/2. \end{cases}
\]

We turn to the sixth term. Let \( 0 < \nu \ll 1 \) be small enough. Since from (4.7) for any \( j \in \mathbb{N} \):

\[
|(Z \partial_Z)^j(F_k(Z) - F[a])| \lesssim e^{-\frac{k-1}{4} s},
\]

one has using Cauchy-Schwarz that

\[
\left| \int \frac{w(Y \partial_Y)^j((F_k(Z) - F[a])\varepsilon)}{\phi^2_{\ell}(Z)} dY \right| \lesssim e^{-\frac{k-1}{4} s} \sum_{n=0}^{j} \int_{Y \geq 1} \frac{|(Y \partial_Y)^n \varepsilon|^2}{\phi^2_{\ell}(Z) Y} dY
\]

which using (4.22) implies that

\[
\left| \int \frac{w(Y \partial_Y)^j((F_k(Z) - F[a])\varepsilon)}{\phi^2_{\ell}(Z)} dY \right| \lesssim \begin{cases} e^{-\left(\frac{k-1}{4} + \nu\right)s} & \text{for } \ell = 2k + 1, \\ e^{-\left(\frac{k-1}{4} + \nu\right)s} & \text{for } \ell = 2k + 1/2. \end{cases}
\]

For the seventh term, we use the fact that \( \partial_Y Y = ((Y \partial_Y)^2 - Y \partial_Y)/Y^2 \) to decompose:

\[
\sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} \partial_Y Y (Y \partial_Y)^n \varepsilon = Y^{-1} \partial_Y w - \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+1} \varepsilon
\]

\[
+ \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+2} \varepsilon - Y^{-2} (Y \partial_Y)^{j+1} \varepsilon.
\]

For the first term, we integrate by parts and find:

\[
\int \frac{w Y^{-1} \partial_Y w}{\phi^2_{\ell}(Z)} dY = -\frac{1}{2} \int w^2 \partial_Y \left( \frac{1}{\phi^2_{\ell}(Z) Y^2} \right) dY - \frac{1}{2} \int \partial_Y \chi \frac{w^2}{\phi^2_{\ell}(Z) Y^2} dY \lesssim \int_{Y \geq 1} \frac{w^2}{\phi^2_{\ell}(Z) Y^3} dY.
\]

If \( \ell = 2k + 1 \), we take \( 0 < \kappa \ll 1 \) small enough and \( Y^* \) large enough so that, using (4.17), (4.25) and (4.26):

\[
\int_{Y \geq 1} \frac{w^2}{\phi^2_{\ell}(Z) Y^3} dY \lesssim \int_1^{Y^*} \frac{w^2}{\phi^2_{\ell}(Z) Y^3} dY + \int_{Y^*}^{+\infty} \frac{w^2}{\phi^2_{\ell}(Z) Y^3} dY
\]

\[
\lesssim \| \frac{1}{\phi^2_{\ell}(Z)} \|_{L^{\infty}(1 \leq Y \leq Y^*)} \| w \|^2_{L^2} + \kappa \int \frac{w^2}{\phi^2_{\ell}(Z)} \frac{dY}{Y} \lesssim e^{-\frac{k}{4} s} + \kappa \int \frac{w^2}{\phi^2_{\ell}(Z)} \frac{dY}{Y}.
\]

If \( \ell = 2k + 1/2 \), we use (4.39), (4.22) and obtain

\[
\int_{Y \geq 1} \frac{w^2}{\phi^2_{\ell}(Z) Y^3} dY \lesssim e^{-\frac{k-1}{4k} s} \int_{Y \geq 1} \frac{w^2}{\phi^2_{2k+1}(Z)} dY \lesssim e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} \lesssim e^{-\frac{k}{4} s}.
\]
If $\ell = 2k + 1$, one estimates using (4.22) that
\[
\int \chi \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+1} \varepsilon + \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+2} \varepsilon - Y^{-2} (Y \partial_Y)^{j+1} \varepsilon \, dY \phi_{2k+1}^2
\]
\[
\sim \sum_{n=0}^{j-1} \int_{Y \geq 1} \frac{(Y \partial_Y)^{j} \varepsilon^2 \, dY}{Y^2 \phi_{2k+1}^2} \lesssim C(K_{j-1}) e^{-\frac{a}{2\pi s}}.
\]

If $\ell = 2k + 1/2$, one estimates using (4.22) and (4.39) that
\[
\int \chi \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+1} \varepsilon + \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y^{-2} (Y \partial_Y)^{n+2} \varepsilon - Y^{-2} (Y \partial_Y)^{j+1} \varepsilon \, dY \phi_{2k+1}^2
\]
\[
\lesssim e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} \sum_{n=0}^{j-1} \int_{Y \geq 1} \frac{(Y \partial_Y)^{j} \varepsilon^2 \, dY}{\phi_{2k+1}^2} \lesssim e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s}.
\]

One has therefore proven that:
\[
\left| \int \chi \frac{w \sum_{n=0}^{j-1} (Y \partial_Y)^{j-1-n} Y (Y \partial_Y)^{n} \varepsilon \, dY}{\phi_{2}^2(Z) \phi_{1}(Z)} \right| \\lesssim \begin{cases} 
C(K_{j-1}) \sqrt{K_{j}} e^{-\left(\frac{3}{4k}\right)s} + \kappa \int \chi \frac{w^2}{\phi_{2k+1}^2(Z)} \, dY & \text{for } \ell = 2k + 1, \\
- e^{-\frac{1}{k}s} & \text{for } \ell = 2k + 1/2.
\end{cases}
\]

Since $|Z \partial_Z \phi_{\ell}/\phi_{\ell}|$ is bounded, one infers for the eighth term from (4.34), (4.22) and (4.23):
\[
\left| \frac{a_{\varepsilon}}{a} \int \chi \frac{w^2}{\phi_{1}^2(Z)} (Z \partial_Z \phi_{\ell})(Z) \, dY \right| \lesssim \begin{cases} 
e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} & \text{for } \ell = 2k + 1, \\
e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} & \text{for } \ell = 2k + 1/2.
\end{cases}
\]

For the ninth term, from (4.9):
\[
\int_{Y \geq 1} \frac{(Z \partial_Z \Psi)^2 \, dY}{\phi_{2}^2} \lesssim e^{-\frac{k-1}{4k}2s} \int_{Y \geq 1} \frac{|Z|^{8k-4} (1 + |Z|)^{-12k} \, dY}{|Z|^{2l(1 + |Z|)^{-8k}}}
\]
\[
\lesssim e^{-\frac{k-1}{4k}2s} \int_{e^{-\frac{k-1}{4k}2s}}^{+\infty} \frac{|Z|^{8k-4-2l} (1 + |Z|)^{-4k} \, dZ}{Z} \lesssim e^{-\frac{k-1}{4k}2s} \lesssim e^{-\frac{3}{2\pi s}}
\]

and from (4.34) and (4.13):
\[
\int_{Y \geq 1} \frac{|a_{\varepsilon}(Z \partial_Z)^{j} \partial_{\varepsilon} F[a](Z)|^2 \, dY}{\phi_{1}^2(Z)} \lesssim \left| \frac{a_{\varepsilon}}{\phi_{1}^2(Z)} \right| \int_{Y \geq 1} \chi \frac{|Z|^{4k} (1 + |Z|)^{-8k} \, dY}{|Z|^{2l(1 + |Z|)^{-8k}}}
\]
\[
\leq \left| \frac{a_{\varepsilon}}{\phi_{1}^2(Z)} \right| \int_{e^{-\frac{k-1}{4k}2s}}^{+\infty} \frac{|Z|^{4k-2l} \, dZ}{Z} \lesssim e^{-\frac{3}{2\pi s}}.
\]

Therefore, using Cauchy Schwarz, (4.22) and (4.23), for $\nu$ small enough
\[
\left| \int \chi \frac{w}{\phi_{1}(Z)} (Y \partial_Y)^{j} (a_{\varepsilon} \partial_{\varepsilon} F[a] + \Psi) \, dY \right| \lesssim \begin{cases} 
e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} & \text{for } \ell = 2k + 1, \\
e^{-\left(\frac{1}{2} + \frac{1}{4k}\right)s} & \text{for } \ell = 2k + 1/2.
\end{cases}
\]
Finally, for the last term, using (4.28), (4.22) and (4.23) for \( \nu \) small enough.

\[
\left| \int \frac{w(Y \partial_Y)^j(\varepsilon^2) dY}{\phi^2(Z)} \right| \lesssim \left( \int \frac{w \cdot dY}{Y} \right)^\frac{1}{2} \left( \sum_{n=0}^{j-1} \| (Y \partial_Y)^n \varepsilon \|_{L^\infty} \right) \left( \sum_{n=0}^{j} \left( \int \frac{(Y \partial_Y)^{n-1} \varepsilon^2 dY}{\phi^2} \right)^\frac{1}{2} \right)
\]

\[
\lesssim \begin{cases} e^{-\left(\frac{1}{2k+\nu}\right)s} & \text{for } \ell = 2k + 1, \\ e^{-\left(\frac{1}{2k+\nu}\right)s} & \text{for } \ell = 2k + 1/2. \end{cases}
\]

Combining all the above estimates, for \( \nu \) small enough and for \( s_0 \) large enough, then gives the two identities:

\[
\frac{d}{ds} \left( \int \frac{\chi \left( (Y \partial_Y)^j(\varepsilon^2) dY \right)}{\phi^2_{2k+1}(Z)} \right) \leq - \left( \frac{1}{k} - \kappa \right) \int \frac{\chi (Y \partial_Y)^j(\varepsilon^2) dY}{Y} + C(K_{j-1}) \sqrt{K_j} e^{-\frac{1}{2k}s}
\]

\[
\frac{d}{ds} \left( \int \frac{\chi \left( (Y \partial_Y)^j(\varepsilon^2) dY \right)}{\phi^2_{2k+1/2}(Z)} \right) \leq -\frac{1}{2k} \int \frac{\chi (Y \partial_Y)^j(\varepsilon^2) dY}{Y} + C(K_j) e^{-\left(\frac{1}{2k+\nu}\right)s}.
\]

with the convention \( K_{-1} = 1 \). Choosing the constants \( K_j \) one after another then yields (4.36) after integration in time using (4.18) (and (4.25) for the zone \( 1 \leq s \leq 2 \)).

**Step 2: Proof of (4.37).** Let \( j \geq 2k + 1 \). We claim that this bound can be proved the very same way as in step 1. The main argument is the following. \( w := \partial_Z^j \varepsilon \) solves from (4.32):

\[
w_s + j + 2k w + \frac{Y}{2} \partial_Y w - 2F_k(Z)w - \partial_{YY}w = 2(\partial_Z^j(F_k(Z)\varepsilon) - F_k(Z)w) - 2\partial_Z^j((F_k(Z) - F[a])\varepsilon) - \partial_Z^j(a_s \partial_s F[a] + \Psi)
\]

The function \( \phi_0(Z) \) is a stable eigenvalue of the operator without dissipation in the left hand side:

\[
\left( j + 2k + \frac{Y}{2} \partial_Y - 2F_k(Z) \right) \phi_0(Z) = \frac{j - 2k}{2k} \phi_0
\]

and in particular \((j - 2k)/2k \geq 1/(2k) \geq 1/2k\) since \( j \geq 2k + 1 \). As \( 1/(2k) \geq 1/(4k) \), one can then prove (4.37) as in Step 1, checking that all the terms in the right hand side of the equation for \( w \) are lower order, and that the boundary terms at the origin are controlled by (4.25).

\[\square\]

We can now end the proof of Proposition 24.

**Proof of Proposition 24.** We reason by contradiction and assume that for any initial value of the unstable parameters \((c_\ell(s_0))_{0 \leq \ell \leq 2k-1}\) satisfying (4.21) the corresponding solution ceases to satisfy the bounds of the Proposition at some time \( s^* \). Define the mapping

\[
\Phi : B_{R_2k-1}(0, \tilde{K} e^{-\left(k - \frac{1}{2} + \frac{1}{4k}\right)s_0}) \rightarrow B_{R_2k-1}(0, \tilde{K} e^{-\left(k - \frac{1}{2} + \frac{1}{4k}\right)s_0})
\]

\[
(c_\ell(s_0))_{0 \leq \ell \leq 2k-1} \rightarrow (c_\ell(s^*))_{0 \leq \ell \leq 2k-1}
\]

where \( B \) denotes the Euclidean ball. From (4.35), (4.36) and (4.37) and a continuity argument, we deduce that necessarily the bound (4.26) has to fail after \( s^* \), meaning that it is saturated at time \( s^* \), implying that \( \Phi \) maps \( B_{R_2k-1}(0, \tilde{K} e^{-\left(k - \frac{1}{2} + \frac{1}{4k}\right)s_0}) \) to its boundary \( S_{R_2k-1}(0, \tilde{K} e^{-\left(k - \frac{1}{2} + \frac{1}{4k}\right)s_0}) \). From standard arguments \( \Phi \) is a continuous mapping. If initially

\[
\sum_{\ell=0}^{2k-1} |c_\ell(s_0)|^2 = \tilde{K}^2 e^{-2(k - \frac{1}{2} + \frac{1}{4k})s_0}
\]
then from (4.34) the solution leaves the bootstrap due to the outgoing condition
\[
\partial_s \left( \frac{\sum_{\ell=0}^{2k-1} |c_\ell(s)|^2}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s}} \right) (s_0) = 2 \left( k - \frac{1}{2} + \frac{1}{4k} \right) \frac{\sum_{\ell=0}^{2k-1} |c_\ell(s_0)|^2}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s_0}} + 2 \frac{\sum_{\ell=0}^{2k-1} c_\ell(s_0) \partial_s c_\ell(s_0)}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s_0}}
\]
\[
= 2 \left( k - \frac{1}{2} + \frac{1}{4k} \right) \frac{\sum_{\ell=0}^{2k-1} |c_\ell(s_0)|^2}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s_0}} + 2 \frac{\sum_{\ell=0}^{2k-1} c_\ell(s_0) \left( -\frac{\ell}{2} - 2 \right) + O(K f e^{-2k - \frac{1}{2} + \frac{\ell}{4})s_0})}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s_0}}
\]
\[
= \frac{2}{K^2 e^{-2(k-\frac{1}{2} + \frac{\ell}{4})s_0}} \sum_{\ell=0}^{2k-1} \left( k - \frac{1}{2} + \frac{1}{4k} - \frac{\ell}{2} - \frac{2}{2} \right) |c_\ell(s_0)|^2 + O \left( \frac{K_j}{K} \right)
\]
\[
\geq 1 + \frac{1}{4k} + O \left( \frac{K_j}{K} \right) > 0
\]
for $\tilde{K}$ large enough depending on $K_j$. Therefore in that case $s^* = s_0$ and $\Phi ((c_\ell(s_0))_{0 \leq \ell \leq 2k-1}) = (c_\ell(s_0))_{0 \leq \ell \leq 2k-1}$, meaning that $\Phi$ is the identity on the sphere $S_{R^{2k-1}}(0, \tilde{K} e^{-2k - \frac{1}{2} + \frac{\ell}{4})s_0})$. This is a contradiction to Brouwer’s theorem. Therefore there exists at least one initial condition $(c_\ell(s_0))_{0 \leq \ell \leq 2k-1}$ such that the solution satisfies the bounds of the Proposition 24 for all times, ending the proof.

\[\square\]

\section{The coupled linear heat equation}

We now turn to the proof of Proposition 10. We keep the notations of the previous section. The proof is similar and simpler to the one of the related Theorem 1 concerning $\zeta$. Indeed, there are no nonlinear effects and instabilities. For a solution $\zeta$ to $(LFH)$ we start by going again to self-similar variables
\[
Y = \frac{y}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad g(s, y) = (T - t)^4 \zeta(x, t), \quad Z := \frac{a^s Y}{e^{\frac{t}{2k-1}}},
\]
Then $g$ solves the second equation in (3.9). Throughout this section, we assume that $f$ is the solution to the first equation in (3.9) satisfying the properties of Proposition 24. In particular since $f = F_k(a^s e^{-(k-1)/(2k)} Y)$ at leading order, and since the dissipation is lower order, the main order equation reads in $Z$ variable:
\[
g_s + M_Z g = 0, \quad M_Z := 4 - 4F_k(a Z) + \frac{Z}{2k} \partial_Z.
\]

\begin{proposition}[Spectral structure for $M_Z$]

The operator $M_Z$ acting on $C^\infty(\mathbb{R})$ has point spectrum $\sigma(M_Z) = \{\ell/(2k), \ \ell \in \mathbb{N}\}$ and the associated eigenfunctions are
\[
\psi_\ell := \frac{Z^\ell}{(1 + (a Z)^{2k})^{\frac{1}{4}}}, \quad M_Z \psi_\ell = \psi_\ell.
\]

\end{proposition}

\textbf{Proof.} The result comes from a direct computation. \[\square\]

$M_Z$ having a nontrivial kernel and non-negative spectrum, we expect formally the solution to approach an element of its kernel as $s \to +\infty$. Near the origin, as $f = F_k(a^s e^{-(k-1)/(2k)} Y)$ at leading order and since $F_k = 1$ near the origin, the main order equation for $g$ in the zone $|Y| \lesssim 1$ is:
\[
g_s + M_{\rho} g = 0, \quad M_{\rho} := \frac{Y}{2} \partial_Y - \partial_Y Y.
\]
Proposition 31. Let \( J \in \mathbb{N} \). There exist \( K_{J+1} \gg K_{j} \gg \ldots \gg K_{0} \), \( s_{0} \gg 1 \) large enough and \( 0 < \epsilon \ll 1 \) small enough such that for \( \epsilon(s_{0}) = \epsilon_{0} \) satisfying the orthogonality condition (4.41) and

\[
\sum_{j=0}^{J} \int_{|Y| \geq 1} \frac{|(Z\partial_{Z})^{j}\varepsilon_{0}|^{2}}{\psi_{1}^{2}(Z)} \, dY \leq e^{-\frac{3}{4}s_{0}}, \quad \sum_{j=0}^{J} \int_{|Y| \geq 1} \frac{|(Z\partial_{Z})^{j}\varepsilon_{0}|^{2}}{\psi_{1/2}^{2}(Z)} \, dY \leq e^{-\frac{1}{4}s_{0}},
\]

(4.42)

\[
\sum_{j=1}^{J} \int_{|Y| \geq 1} \frac{|\partial_{Z}^{j}\varepsilon_{0}|^{2}}{\psi_{0}^{2}(Z)} \, dY \leq e^{-\frac{1}{4}s_{0}},
\]

(4.43)

\[
\|\varepsilon_{0}\|_{L_{2}^{0}}^{2} \leq e^{-\frac{1}{4}s_{0}}, \quad \|\partial_{Z}^{j}\varepsilon_{0}\|_{L_{2}^{0}}^{2} \leq e^{-\frac{1}{4}s_{0}}, \quad \text{for } j = 1, ..., J,
\]

(4.44)

and an initial parameter \( |b(s_{0}) - 1| \leq \epsilon \) the solution \( g \) to the second equation in (3.9) then satisfies for all \( s \geq s_{0} \), for \( j = 0, ..., J \):

\[
\int_{|Y| \geq 1} \frac{|(Z\partial_{Z})^{j}\varepsilon|^{2}}{\psi_{1}^{2}(Z)} \, dY \leq K_{j}e^{-\frac{1}{4}s}, \quad \int_{|Y| \geq 1} \frac{|(Z\partial_{Z})^{j}\varepsilon|^{2}}{\psi_{1/2}^{2}(Z)} \, dY \leq K_{j+1}e^{-\frac{1}{4}s},
\]

(4.45)

\[
\int_{|Y| \geq 1} \frac{|\partial_{Z}^{j}\varepsilon|^{2}}{\psi_{0}^{2}(Z)} \, dY \leq K_{j}e^{-\frac{1}{4}s},
\]

(4.46)

\[
\|\varepsilon_{0}\|_{L_{2}^{0}}^{2} \leq K_{0}e^{-\frac{1}{4}s}, \quad \|\partial_{Z}^{j}\varepsilon_{0}\|_{L_{2}^{0}}^{2} \leq K_{j}e^{-\frac{1}{4}s}, \quad \text{for } j = 1, ..., J,
\]

(4.47)

and there exists an asymptotic parameter \( |b^{*} - 1| \leq 2\epsilon \) such that \( |b - b^{*}| \lesssim e^{-k(s - 1)} \).

The rest of the subsection is devoted to the proof of Proposition 31. In what follows we assume that \( g \) solves (3.9) and satisfies the bounds of Proposition 31 on some time interval \([s_{0}, s_{1}]\), and perform modulation and energy estimates to improve those bounds. Proposition 31 is then proved at the end of the subsection. First, from the Sobolev bootstrap bounds one deduces pointwise bounds.

Lemma 32. There holds on \([s_{0}, s_{1}]\) for \( j = 0, ..., J - 1 \):

\[
|\partial_{Z}^{j}\varepsilon| \lesssim e^{-\frac{1}{4}s}(1 + |Z|)^{\frac{1}{2} - 8k}.
\]

(4.48)

Proof. First, since \( \partial_{Z} = e^{\frac{k}{2k}}\partial_{Y} \), one deduces from (4.47) that \( \|\varepsilon\|_{H_{\rho}^{s}} \lesssim e^{-s/(2k)} \). Therefore, from Sobolev,

\[
|\partial_{Z}^{j}\varepsilon| \lesssim e^{-\frac{1}{4}s} \quad \text{for } |Y| \leq 1.
\]

(4.49)
Then, applying (A.2), as \(|Z\partial_Z\psi_{1/2}| \lesssim |\psi_{1/2}| \lesssim |Z\partial_Z\psi_{1/2}|:
\[
\|Z^j\partial_Z^j\varepsilon\|_{\psi_{1/2}(Z)}^2 \lesssim \|Z^j\partial_Z^j\varepsilon\|_{\psi_{1/2}(Z)}^2 \lesssim \|Z\partial_Z(Z^j\partial_Z^j\varepsilon)\|_{L^2(|Y| \geq 1, dY)}^2 \leq \sum_{k=0}^{j+1} \|Z^j\partial_Z^j\varepsilon\|_{\psi_{1/2}(Z)}^2 \lesssim e^{-\frac{1}{4k}s}
\]
from (4.45). From the definition of \(\psi\) this implies that
\[
|\partial_Z^j\varepsilon| \lesssim e^{-\frac{1}{4k}s}|Z|^\frac{j}{2} - (1 + |Z|)^{-sk} \text{ for } |Y| \geq 1.
\] (4.50)

Finally, since \(\psi_{1/2}(Z) = |Z|^{1/2}\psi_0(Z)\), for \(j \geq 1\) one has the inequality
\[
\frac{|Z|}{\psi_0(Z)} \lesssim \frac{1}{\psi_{1/2}(Z)} + \frac{|Z|^{j+1}}{\psi_{1/2}(Z)}.
\]
This implies from (4.45) and (4.46) the estimate for \(j \geq 1:\)
\[
\int_{|Y| \geq 1} \frac{|Z\partial_Z(Z^j\varepsilon)|^2 dY}{|Y|} \lesssim \int_{|Y| \geq 1} \frac{|\partial_Z^{j+1}\varepsilon|^2 dY}{\psi_0(Z)} + \int_{|Y| \geq 1} \frac{|Z^{j+1}\partial_Z^{j+1}\varepsilon|^2 dY}{\psi_0(Z)} \lesssim e^{-\frac{1}{4k}s}.
\]
Thus, from (A.2) one obtains for \(j \geq 1:\)
\[
|\partial_Z^j\varepsilon| \lesssim e^{-\frac{1}{4k}s}(1 + |Z|)^{-sk} \text{ for } |Y| \geq 1.
\] (4.51)

The bounds (4.49), (4.50) and (4.51) then imply the desired result (4.48).

From (3.9) and (4.40), the evolution of \(\varepsilon\) is given by the following equation:
\[
b_sF_k^4(Z) + \varepsilon_s + \mathcal{M}\varepsilon + \tilde{\mathcal{M}}\varepsilon + \Psi = 0,
\] (4.52)
where
\[
\mathcal{M} := 4 - 4F_k(Z) + \frac{Y}{2}\partial_Y - \partial_Y, \quad \tilde{\mathcal{M}} := -4(f - F_k(Z)),
\]
and
\[
\Psi := -4bF_k^4(Z)(f - F_k(Z)) - b(e^{-\frac{k-1}{4k}s})^2\partial_Z(Z^4)(F_k^4)(Z)
\]
From the various bounds of Proposition 24 and Lemma 25 we infer the following estimates for the above objects.

**Lemma 33.** One has the following bounds:
\[
|\partial_Z^j(f - F_k(Z))| \lesssim e^{-\frac{1}{4k}s}(1 + |Z|)^{\frac{j}{2} - 2k - j},
\] (4.53)
\[
|f - F_k(Z)|^2 \lesssim e^{-\frac{1}{4k}s},
\] (4.54)
\[
|(Z\partial_Z)^j(\Psi)| \lesssim e^{-\frac{1}{4k}s}(1 + |Z|)^{\frac{j}{2} - 10k}, \quad j = 0, 1, 2
\]
\[
\|\Psi\|^2_{L^2} \lesssim e^{-\frac{1}{4k}s}, \quad ||\partial_Z\Psi||^2_{L^2} \lesssim e^{-\frac{1}{4k}s} \quad \text{and} \quad ||\partial_Z^j\Psi||^2_{L^2} \lesssim e^{-\frac{1}{4k}s} \quad \text{for } j \geq 2,
\] (4.55)
\[
\int_{|Y| \geq 1} \frac{|(Y\partial_Y)^j\Psi|^2 dY}{\psi_{1/2}(Z)} \lesssim \int_{|Y| \geq 1} \frac{|(Y\partial_Y)^j\Psi|^2 dY}{\psi_{1/2}(Z)} \lesssim e^{-\frac{1}{4k}s}, \quad j = 0, \ldots, J,
\] (4.56)
\[
\int_{|Y| \geq 1} \frac{|\partial_Z^j\Psi|^2 dY}{\psi_{1/2}(Z)} \lesssim e^{-\frac{1}{4k}s}, \quad j = 1, \ldots, J.
\] (4.57)
Proof. These are direct bounds implied by the estimates of Proposition 24, and the behaviour of the corresponding eigenfunctions given by Propositions 21 and 29.

We start by computing the evolution of the modulation parameter.

Lemma 34. There holds on \([s_0, s_1]\):

\[ |b_s| \lesssim e^{-(k-1)s}. \]  (4.58)

Proof. One takes the scalar product between (4.52) and \(h_0 = 1\) in \(L^2_\rho\), yielding using (4.41):

\[ b_s\langle F^4(Z^*), h_0\rangle_\rho = \langle -\mathcal{M}\varepsilon - \tilde{\mathcal{M}}\varepsilon - \Psi, h_0\rangle_\rho. \]

First, since \(|F_k(Z) - 1| \lesssim Z^{2k}(1 + |Z|)^{-2k}\) one has the projection in the left hand side is non-degenerate:

\[ \langle F^4(Z^*), h_0\rangle_\rho = 1 + O(e^{-(k-1)s}) \]

and that, since \(\mathcal{M} = \mathcal{M}_\rho + 4(1 - F(Z^*))\):

\[ \langle \mathcal{M}\varepsilon, h_0\rangle_\rho = 0 + \langle \varepsilon, 4(1 - F(Z^*))h_0\rangle_\rho = O(e^{-(k-1)\frac{1}{2}s}). \]

using (4.41), (4.47) and the fact that \(|Z^{2k}| \lesssim e^{-(k-1)s}|Y|^{2k}\). Then, from (4.54) one computes:

\[ \langle \tilde{\mathcal{M}}\varepsilon, h_0\rangle_\rho = -4\langle \varepsilon, (f - F(Z^*))h_0\rangle_\rho = O(e^{-(k-1)\frac{1}{2}s}). \]

Finally, using (4.55):

\[ \langle \Psi, h_0\rangle_\rho = O(e^{-(k-1)s}). \]

From the above identities one gets the desired result (4.58).

We then perform an energy estimate in the zone \(|Y| \lesssim 1\).

Lemma 35. There holds at time \(s_1\):

\[ \|\varepsilon\|_{L^2_\rho} \leq \frac{\sqrt{K_0}}{2} e^{-\frac{1}{2} s_1}, \quad \|\partial_x^j\varepsilon\|_{L^2_\rho} \leq \frac{\sqrt{K_j}}{2} e^{-\frac{1}{2} s_1}, \quad \text{for } j = 1, \ldots, J. \]  (4.59)

Proof. Step 1 Estimate for \(\varepsilon\). Let \(0 < \kappa \ll 1\) be an arbitrarily small constant. Then from (4.52) we infer that:

\[ \frac{d}{ds} \frac{1}{2} \int \varepsilon^2 e^{-\frac{\varepsilon^2}{4}} \, dY = \int \varepsilon \left( -b_s F^4(Z^*) - \mathcal{M}\varepsilon - \tilde{\mathcal{M}}\varepsilon - \Psi \right) e^{-\frac{\varepsilon^2}{4}} \, dY. \]

For the first term, from (4.41) and (4.58), using Cauchy-Schwarz:

\[ b_s \int \varepsilon F^4(Z^*) e^{-\frac{\varepsilon^2}{4}} \, dY = b_s \int \varepsilon (1 - F^4(Z^*)) e^{-\frac{\varepsilon^2}{4}} \, dY = O(e^{-(2(k-1)+\frac{1}{4})s}). \]

For the second, as \(\mathcal{M} = \mathcal{M}_\rho + 4(1 - F_k(Z))\), using (4.48):

\[ - \int \varepsilon \mathcal{M}\varepsilon e^{-\frac{\varepsilon^2}{4}} \, dY = - \int \varepsilon \mathcal{M}_\rho e^{-\frac{\varepsilon^2}{4}} \, dY + 4 \int (F_k(Z) - 1) e^{\varepsilon} e^{-\frac{\varepsilon^2}{4}} \, dY \]

\[ \leq - \frac{1}{2} \int \varepsilon^2 e^{-\frac{\varepsilon^2}{4}} \, dY + O(\|\varepsilon\|_{L^2_\rho} \|\varepsilon\|_{L^\infty} \|F_k(Z) - 1\|_{L^2_\rho}) \leq - \frac{1}{2} \int \varepsilon^2 e^{-\frac{\varepsilon^2}{4}} \, dY + O(e^{-(k-\frac{1}{2} + \frac{1}{4})s}). \]

For the third, from (4.53) and (4.47):

\[ \left| \int \varepsilon \tilde{\mathcal{M}}\varepsilon e^{-\frac{\varepsilon^2}{4}} \, dY \right| \lesssim \|\varepsilon\|^2_{L^2_\rho} \|f - F(Z^*)\|_{L^\infty} \lesssim e^{-(1+\frac{1}{4})s}. \]
Finally, for the fourth, from (4.55):
\[
\left| \int \varepsilon \Psi e^{-\frac{\varepsilon^2}{4}} dY \right| \lesssim \|\varepsilon\|_{L^2_\rho} \|\Psi\|_{L^2_\rho} \lesssim e^{-(k-\frac{1}{2})s}.
\]

Combining the above expressions one obtains:
\[
\frac{d}{ds} \left( \int e^{\frac{\varepsilon^2}{4}} dY \right) \leq -\int e^{\frac{\varepsilon^2}{4}} dY + C(K)e^{-(1+\frac{1}{k})s}.
\]

For \(K\) and \(s_0\) large enough, when reintegrated in time, using (4.44) this gives:
\[
\int e^{\frac{\varepsilon^2}{4}} dY \leq \kappa Ke^{-s} \leq Ce^{-s} + C(K)e^{-\frac{1}{4}s_0}e^{-s} \leq \frac{K^2}{4}
\]
if \(s_0\) has been taken large enough.

**Step 2:** Higher order derivatives. Let \(1 \leq j \leq J\) and define \(w := \partial^j_Z \varepsilon\). Then from (4.52) the evolution of \(w\) is:
\[
w_s + \frac{j}{2K} + \mathcal{M}_\rho w + 4(1-F_k(Z))w + 4(F_k(Z)w - \partial^j_Z(F_k(Z)\varepsilon)) - \partial^j_Z((F - F_k(Z))\varepsilon) + \partial^j_Z((\Psi + b_kF_k(Z)) = 0.
\]

From the above equation, we infer that:
\[
\frac{d}{ds} \left( \int w^2_{L^2_\rho} \right) = -\frac{j}{2K} \int w^2_{L^2_\rho} - \|\partial^j w\|^2_{L^2_\rho} + 4((F_k(Z) - 1)w, w)_{\rho} + 4(\partial^j_Z(F_k(Z)\varepsilon) - F_k(Z)w, w)_{\rho}
\]
\[
+ (\partial^j_Z((F - F_k(Z))\varepsilon), w)_{\rho} - (\partial^j_Z((\Psi + b_kF_k(Z)), w)_{\rho}.
\]

Let \(0 < \nu \ll 1\) be a small constant to be fixed later on. We estimate all terms in the right hand side. First, from (A.1) one has:
\[
-\|\partial^j w\|^2_{L^2_\rho} \leq -c e^{-\nu s} |Y|^2_{L^2_\rho} + e^{-\nu s} \int w^2_{L^2_\rho} \leq -c e^{-\nu s} |Y|^2_{L^2_\rho} + C e^{-\frac{1}{k} + \nu s}, \quad c > 0.
\]

Next, since \(|1 - F_k(Z)| \leq |Z^{2k}|(1 + |Z|)^{-2k} \lesssim e^{-(k-1)s/k}|Y|^2\) we infer that:
\[
|\langle (F_k(Z) - 1)w, w\rangle_{\rho} | \lesssim e^{-\frac{1}{k} + \nu s} |Y|^2_{L^2_\rho}.
\]

From (4.47), as \(\partial^j_Z F_k\) is bounded, we infer using Cauchy-Schwarz that:
\[
\left| \langle \partial^j_Z(F_k(Z)\varepsilon) - \partial^j_Z(F_k(Z)w, w)_{\rho} \right| \lesssim \begin{cases} e^{-\frac{1}{k} + \nu s} & \text{for } j = 1, \\ \sqrt{K_{j-1}} \sqrt{K_j} e^{-\frac{1}{k} + \nu s} & \text{for } j \geq 2. \end{cases}
\]

Using (4.47) and (4.53) we infer:
\[
\left| \langle \partial^j_Z((F - F_k(Z))\varepsilon), w\rangle_{\rho} \right| \lesssim \left( \sum_{i=0}^{J} \|\partial^i_Z \varepsilon\|^2_{L^2_\rho} \right) \left( \sum_{i=0}^{J} \|\partial^i_Z(F - F_k(Z))\|_{L^\infty} \right) \lesssim e^{-\frac{1}{k} + \nu s}.
\]

Finally, from (4.47) and (4.58), (4.55) and Cauchy-Schwarz:
\[
\left| \langle \partial^j_Z((\Psi + b_kF_k(Z)), w\rangle_{\rho} \right| \lesssim \begin{cases} e^{-\frac{1}{k} + \nu s} & \text{for } j = 1, \\ \sqrt{K_{j-1}} \sqrt{K_j} e^{-\frac{1}{k} + \nu s} & \text{for } j \geq 2. \end{cases}
\]

Collecting the above estimates, one finds finally that for \(\nu\) small enough and \(s_0\) large enough:
\[
\frac{d}{ds} \left( \int w^2_{L^2_\rho} \right) \leq \begin{cases} -\frac{1}{k} \int w^2_{L^2_\rho} + O(e^{-\frac{1}{k} + \nu s}) & \text{for } j = 1, \\ -\frac{1}{k} \int w^2_{L^2_\rho} + \sqrt{K_{j-1}} \sqrt{K_j} e^{-\frac{1}{k} + \nu s} & \text{for } j \geq 2. \end{cases}
\]

Reintegrated in time using (4.44), this yields the desired bound (4.47) for \(j \geq 1\).
Lemma 36. There holds at time $s_1$, for $j = 0, \ldots, J$:
\[
\int_{|Y| \geq 1} \frac{|(Z\partial_Z)^j \varepsilon|^2}{\psi_1^2(Z)} \frac{dY}{|Y|} \leq K_j e^{-\frac{4\kappa^s}{4\kappa^s}}.
\]
\[
\int_{|Y| \geq 1} \frac{|(Z\partial_Z)^j \varepsilon(s_1)|^2}{\psi_1^{2/4}} \frac{dY}{|Y|} \leq \frac{K_{j+1}}{2} e^{-\frac{4\kappa^s}{4\kappa^s}},
\]
We treat the boundary terms using (4.47):
\[
\int_{|Y| \geq 1} \frac{\partial_j^2 \varepsilon^2}{\psi_0^2(Z)} \frac{dY}{|Y|} \leq \frac{K_j}{2} e^{-\frac{4\kappa^s}{4\kappa^s}} \quad \text{for} \quad j \geq 1.
\]

Proof. We only perform the analysis for $Y \geq 1$ since it is exactly the same for $Y \leq -1$, thus writing $|Y| = Y$.

**Step 1: Bounds for $\varepsilon$.** Let $0 < \kappa \ll 1$ be an arbitrarily small constant. Let $\chi$ be a smooth and positive cut-off function, $\chi = 1$ for $Y \geq 2$ and $\chi = 0$ for $Y \leq 1$. Let $\ell = 1$ or $\ell = 1/2$. We compute first the identity by integrating by parts and using Proposition 29:
\[
\frac{d}{ds} \frac{1}{2} \left( \int \chi \frac{\varepsilon^2}{\psi_1^2(Z)} \frac{dY}{|Y|} \right) = \frac{\ell}{2k} \int \chi \frac{\varepsilon^2}{\psi_1^2(Z)} \frac{dY}{|Y|} - \int \chi \frac{\partial_Y \varepsilon^2}{\psi_1^2(Z)} \frac{dY}{|Y|} + \frac{1}{4} \int \partial_Y \varepsilon \frac{\varepsilon^2}{\psi_1^2(Z)} \frac{dY}{|Y|}
\]
\[
+ \frac{1}{2} \int \varepsilon^2 \left( \frac{\partial_Y \chi}{\psi_1^2(Z)} + 2\partial_Y \chi \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) + \chi \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) \right) \frac{dY}{|Y|}
\]
\[
+ \int \varepsilon^2 \left( \frac{h^4}{\psi_1^2(Z)} \right) \frac{dY}{|Y|} - \int \chi \left( \frac{\varepsilon}{\psi_1^2(Z)} \left( b_s F_k(Z) + \Psi \right) \frac{dY}{|Y|} \right).
\]
We treat the boundary terms using (4.47):
\[
\left| \frac{1}{4} \int \partial_Y \chi \frac{\varepsilon^2}{\psi_1^2(Z)} \frac{dY}{|Y|} + \frac{1}{2} \int \varepsilon^2 \left( \frac{\partial_Y \chi}{\psi_1^2(Z)} + 2\partial_Y \chi \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) \right) \frac{dY}{|Y|} \right|
\]
\[
\lesssim \left\| \varepsilon \right\|_{L^2_p} \left( \left\| \frac{1}{\psi_1^2(Z)} \right\|_{L^\infty(1 \leq Y \leq 2)} + \left\| \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) \right\|_{L^\infty(1 \leq Y \leq 2)} \right) \lesssim e^{-s + \frac{1}{2} \kappa^s}
\]
\[
\lesssim \begin{cases} 
  e^{-\left(\frac{3}{4}\kappa^s + 1\right)} & \text{if } \ell = 1, \\
  e^{-\left(\frac{3}{4}\kappa^s + 1\right)} & \text{if } \ell = \frac{1}{2}.
\end{cases}
\]
Next, notice that for $Y \geq 1$,
\[
\left| \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) \right| \lesssim \frac{1}{Y^3 \psi_1^2(Z)}.
\]
If $\ell = 1$, we then decompose for $Y^* \geq 1$ large enough depending on $\kappa$:
\[
\left| \int \varepsilon^2 \partial_Y \left( \frac{1}{\psi_1^2(Z)} \right) \frac{dY}{|Y|} \right| \leq C \int \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|}
\]
\[
\leq C \int_{Y \leq Y^*} \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|} + C \int_{Y \geq Y^*} \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|}
\]
\[
\leq C \left\| \varepsilon \right\|_{L^2_p} \left\| \frac{1}{\psi_1^2(Z)} \right\|_{L^\infty(1 \leq Y \leq Y^*)} \kappa + \kappa \int_{Y \geq Y^*} \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|}
\]
\[
\leq C(K_0) e^{-\left(\frac{3}{4}\kappa^s + 1\right)} + \kappa \int \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|}.
\]
If $\ell = 1/2$, we use the fact that $1/(Y^2 \psi_1^2(Z)) = e^{-(k-1)/(2k)s}/\psi_1^2(Z)$ to obtain from (4.45):
\[
\left| \int \varepsilon^2 \partial_Y \left( \frac{1}{\psi_1^2(Z) Y^2} \right) \frac{dY}{|Y|} \right| \lesssim \int \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|} \lesssim e^{-\left(\frac{3}{4}\kappa^s \right) + \kappa} \int \chi \frac{\varepsilon^2}{\psi_1^2(Z) Y^2} \frac{dY}{|Y|} \lesssim e^{-\left(\frac{3}{4}\kappa^s + 1\right)}.
\]
The lower order linear term is estimated via (4.53) and (4.45): 

$$\left| \int \frac{\varepsilon^2}{\psi^2_\ell(Z)} (f - F_k(Z)) \frac{dY}{|Y|} \right| \lesssim \| f - F_k(Z) \|_{L^\infty} \int \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \lesssim \begin{cases} e^{-\left(\frac{1}{2k} + \frac{1}{k}\ell\right)s} & \text{if } \ell = 1, \\ e^{-\left(\frac{1}{2k} + \frac{1}{k}\ell\right)s} & \text{if } \ell = \frac{1}{2}, \end{cases}$$

The error terms are estimated via (4.58), (4.56) and (4.45):

$$\left| \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} b_s F^4_k(Z) \frac{dY}{|Y|} \right| \lesssim |b_s| \left| \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \left| \int \chi \frac{F_k^8(Z)}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}}$$

$$\lesssim e^{-(k-1)s} \left( \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right)^{\frac{1}{2}} \int_\epsilon e^{-\frac{k}{1-\ell}} \frac{1}{|Z|^{2\ell}} \frac{dZ}{Z} \lesssim \begin{cases} e^{-\left(\frac{1}{2k} + \frac{1}{2k} - \frac{1}{2k}\ell\right)s} & \text{if } \ell = 1, \\ e^{-\left(\frac{1}{2k} + \frac{1}{2k} - \frac{1}{2k}\ell\right)s} & \text{if } \ell = \frac{1}{2}, \end{cases}$$

and via Cauchy-Schwarz using (4.56):

$$\left| \int \chi \frac{\varepsilon}{\psi^2_\ell(Z)} \Psi \frac{dY}{|Y|} \right| \lesssim \left| \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \left| \int \chi \frac{|\Psi|^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \lesssim \begin{cases} C\sqrt{K_0}e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = 1, \\ e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = \frac{1}{2}. \end{cases}$$

We now collect all the previous estimates and obtain:

$$\frac{d}{ds} \left( \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right) \leq \begin{cases} -\left(\frac{1}{2k} - \kappa\right) \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} + C\sqrt{K_0}e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = 1, \\ -\frac{1}{2k} \int \chi \frac{\varepsilon^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} + C(K_0)e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = \frac{1}{2}, \end{cases}$$

if $\kappa$ has been chosen small enough, and $s_0$ large enough. The two above differential inequalities yield the desired results (4.60) when reintegrated in time using (4.42) and (4.47), if $K_0$ has been chosen large enough independently of the other constants in the bootstrap.

**Step 2:** *Proof of (4.60) for $Z\partial_Z \varepsilon$. Let $\ell = 1$ or $\ell = 1/2$ and define $w := Z\partial_Z \varepsilon$. It solves from (4.52):*

$$w_s + M_Z w - \partial Y Y w - 4Z\partial_Z (F_k(Z))\varepsilon + 2\partial Y Y\varepsilon - Z\partial_Z ((f - F_k(Z))\varepsilon) + Z\partial_Z (w_s F^4_k(Z) + \Psi) = 0.$$ 

Therefore, one infers that

$$\frac{d}{ds} \left( \int \chi \frac{w^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right) = \begin{cases} -\ell \frac{1}{2k} \int \chi \frac{w^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} - \int \chi \frac{|\partial Y w|^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} + \frac{1}{4} \int \partial Y \chi \frac{w^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \\ + \frac{1}{2} \int \frac{w}{\psi^2_\ell(Z)} \left( \frac{\partial Y \chi}{\psi^2_\ell(Z)} + 2\partial Y \chi \partial Y \left( \frac{1}{\psi^2_\ell(Z)} \right) + \chi \partial Y \left( \frac{1}{\psi^2_\ell(Z)} \right) \right) \frac{dY}{|Y|} \\ + 4 \int \chi \frac{wZ\partial Z (F_k(Z))\varepsilon}{\psi^2_\ell(Z)} |Y| \frac{dY}{|Y|} - \int \frac{1}{4} w \left( \frac{\partial Y \chi}{\psi^2_\ell(Z)} + \chi \partial Y \left( \frac{1}{\psi^2_\ell(Z)} \right) \right) \frac{dY}{|Y|} \\ + 4 \int \frac{w}{\psi^2_\ell(Z)} Z\partial_Z ((f - F_k(Z))\varepsilon) \frac{dY}{|Y|} - \int \chi \frac{w}{\psi^2_\ell(Z)} Z\partial_Z (w_s F^4_k(Z) + \Psi) \frac{dY}{|Y|}. \end{cases}$$

We treat the boundary terms using (4.47) and the fact that $|w| = |Z\partial_Z \varepsilon| \leq e^{-\left(\frac{1}{2k}\ell\right)s} |\partial Z \varepsilon|$ for $1 \leq Y \leq 2$:

$$\left| \frac{1}{4} \int \partial Y \chi \frac{w^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right| = \left| \frac{1}{2} \int \frac{w^2}{\psi^2_\ell(Z)} \left( \frac{\partial Y \chi}{\psi^2_\ell(Z)} + 2\partial Y \chi \partial Y \left( \frac{1}{\psi^2_\ell(Z)} \right) \right) \frac{dY}{|Y|} - \int \frac{w^2}{\psi^2_\ell(Z)} \frac{dY}{|Y|} \right|$$

$$\lesssim e^{-\frac{k-1}{k}s} \left\| \partial Z \varepsilon \right\|_{L^p}^2 \left( \left\| \frac{1}{\psi^2_\ell(Z)} \right\|_{L^\infty(1 \leq Y \leq 2)} + \left\| \partial Y \left( \frac{1}{\psi^2_\ell(Z)} \right) \right\|_{L^\infty(1 \leq Y \leq 2)} \right)$$

$$\lesssim e^{-\frac{1}{2k}s + \frac{k-1}{k}(\ell-1)s} \leq \begin{cases} e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = 1, \\ e^{-\left(\frac{1}{2k}\ell\right)s} & \text{if } \ell = \frac{1}{2}. \end{cases}$$
As in Step 1, since for \( Y \geq 1, |\partial_{\psi}^1(1/(\psi_1^2(Z)))| \leq 1/(\psi_1^2(Z)|Y|) \) one deduces that if \( \ell = 1 \), for some \( Y^* \gg 1 \) large enough:

\[
\left| \frac{1}{2} \int w^2 \chi \partial_{\psi}^2 \left( \frac{1}{\psi_1^2(Z)(\psi_1^2(Z))} \right) dY - \int w^2 \frac{1}{Y} \chi \partial_Y \left( \frac{1}{\psi_1^2(Z)(\psi_1^2(Z))} \right) \frac{dY}{|Y|} \right| \leq C \int \chi \frac{w^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|}
\]

\[
\leq C \int_{Y \leq Y^*} \chi \frac{w^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|} + C \int_{Y \geq Y^*} \chi \frac{w^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|}
\]

\[
\leq C \|w\|_{L^2} \|\frac{1}{\psi_1^2(Z)}\|_{L^\infty} \kappa \int_{Y \geq Y^*} \chi \frac{w^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|}
\]

\[
\leq C(K_1)e^{-(\frac{3}{4k} + \frac{1}{4k})s} + C \int \chi \frac{\varepsilon^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|}
\]

using (4.47). If \( \ell = 1/2 \), we use the fact that \( 1/(\psi_{1/2}^2(Z)) = e^{-(k-1)/(2k)s}/\psi_1^2(Z) \) to obtain from (4.45):

\[
\left| \frac{1}{2} \int w^2 \chi \partial_{\psi}^2 \left( \frac{1}{\psi_{1/2}^2(Z)(\psi_{1/2}^2(Z))} \right) dY - \int w^2 \frac{1}{Y} \chi \partial_Y \left( \frac{1}{\psi_{1/2}^2(Z)} \right) \frac{dY}{|Y|} \right| \leq C \int \chi \frac{w^2}{\psi_{1/2}^2(Z)|Y|^2} \frac{dY}{|Y|}
\]

\[
\leq C \int \chi \frac{w^2}{\psi_{1/2}^2(Z)|Y|^2} \frac{dY}{|Y|} \leq C \sqrt{K_1K_0} e^{-\frac{4}{3k}s}
\]

as \( |Z \partial_Z F_k(Z)| \leq |Z|^{2k(1 + |Z|)^{-4k}} \leq 1 \). For \( \ell = 1/2 \), since

\[
\frac{|Z|^{2k(1 + |Z|)^{-4k}}}{\psi_{1/2}^2(Z)} \leq \frac{1}{\psi_1^2(Z)}
\]

as \( |\psi_1(Z)| = |Z|^{1/2}|\psi_{1/2}(Z)| \), one uses Cauchy-Schwarz, (4.45):

\[
\left| \int \chi \frac{w Z \partial_Z (F_k(Z)) \varepsilon}{\psi_{1/2}^2(Z)} dY \right| \leq \int \chi \frac{w^2}{\psi_{1/2}^2(Z)|Y|^2} \frac{dY}{|Y|} \leq C \sqrt{K_1K_0} e^{-\frac{4}{3k}s}
\]

The lower order linear term is estimated via (4.53), (4.45):

\[
\left| \int \frac{w}{\psi_1^2(Z)} Z \partial_Z ((f - F_k(Z)) \varepsilon) dY \right| \leq (\|f - F_k(Z)\|_{L^\infty} + \|Z \partial_Z (f - F_k(Z))\|_{L^\infty}) \left( \int \chi \frac{\varepsilon^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|} + \int \frac{w^2}{\psi_1^2(Z)|Y|^2} \frac{dY}{|Y|} \right)
\]

\[
\leq \begin{cases} 
  e^{-\left(\frac{1}{4k} + \frac{1}{4k}\right)s} & \text{if } \ell = 1, \\
  e^{-\left(\frac{1}{2k} + \frac{1}{4k}\right)s} & \text{if } \ell = \frac{1}{2}.
\end{cases}
\]
The error terms are estimated via (4.58), (4.56), (4.45):

\[
\left| \int \chi \frac{w}{\psi_t^2(Z)} b_s Z \partial_Z (F_k^i(Z)) \frac{dY}{|Y|} \right| \lesssim |b_s| \left| \int \chi \frac{w^2}{\psi_t^2(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \left| \int \chi \frac{|Z|^{4k}(1+|Z|)^{-20k}}{\psi_t^2(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \\
\lesssim e^{-2(k-1)s} \left| \int \chi \frac{\varepsilon^2}{\psi_t^2(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \int_{e^{-k-1}}^{+\infty} |Z|^{4k-\ell}(1+|Z|)^{-4k} \frac{dZ}{Z}
\]

and via Cauchy-Schwarz using (4.56):

\[
\left| \int \chi \frac{w}{\psi_t^2(Z)} Z \partial_Z \Psi \frac{dY}{|Y|} \right| \leq \left| \int \chi \frac{\varepsilon^2}{\psi_t^2(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \left| \int \chi \frac{|Z\partial_Z \Psi|^2}{\psi_t^2(Z)} \frac{dY}{|Y|} \right|^{\frac{1}{2}} \leq \begin{cases} 
C \sqrt{K_0} e^{-\left(\frac{m}{s}\right)s} & \text{if } \ell = 1, \\
e^{-\left(\frac{1}{s} + \frac{k-1}{ks}\right)s} & \text{if } \ell = \frac{1}{2}.
\end{cases}
\]

We now collect all the previous estimates and obtain:

\[
\frac{d}{ds} \left( \int \chi \frac{w^2}{\psi_t^2(Z)} \frac{dY}{|Y|} \right) \leq \begin{cases} 
- \left(\frac{1}{k} - \kappa\right) \int \chi \frac{w^2}{\psi_t^2(Z)} \frac{dY}{|Y|} + C \sqrt{K_0} \sqrt{K_1} e^{-\left(\frac{m}{s}\right)s} & \text{if } \ell = 1, \\
- \frac{1}{2k} \int \chi \frac{\varepsilon^2}{\psi_t^2(Z)} \frac{dY}{|Y|} + C(K_1) e^{-\left(\frac{1}{s} + \frac{k-1}{ks}\right)s} & \text{if } \ell = \frac{1}{2},
\end{cases}
\]

if \( \kappa \) has been chosen small enough, and \( s_0 \) large enough. The two above differential inequalities yield the desired results (4.60) when reintegrated in time using (4.42) and (4.47), if \( K_1 \) has been chosen large enough depending on \( K_0 \).

**Step 3: End of the proof.** We claim that the bounds (4.60) for higher order derivatives, as well as the bound (4.61), can be proved with verbatim the same argument that were used in Step 1 and Step 2. We safely leave the proof to the reader.

\[\square\]

**Proof of Proposition 31.** We use a bootstrap argument. Let \( s_1 \geq s_0 \) be the supremum of times \( \bar{s} \geq s_0 \) such that all the bounds of Proposition 31 hold on some time interval \([s_0, \bar{s}]\). Then (4.42), (4.42), (4.45) and (4.44) imply \( s_1 > s_0 \) by a continuity argument. Assume by contradiction that \( s_1 < +\infty \). Then the bounds (4.45), (4.45), (4.46) and (4.47) are strict at time \( s_1 \) from (4.47), (4.60), (4.60) and (4.61). From a continuity argument there exists \( \delta > 0 \) such that (4.45), (4.45), (4.46) and (4.47) hold on \([s_1, s_1 + \delta]\), contradicting the definition of \( s_1 \). Thus \( s_1 = +\infty \) and Proposition 31 is proved.

\[\square\]

5. Proof of Theorem 2

In this section we prove Theorem 2. The proof is the same as the one of Theorem 3, only few details change. Namely, the analysis is now based on the stable blow-up of the self-similar heat equation \( \xi_t - \xi^2 - \xi_{yy} = 0 \) whose properties are classical [2, 3, 16, 22]. We therefore just sketch the proof, with an emphasis on the differences between this proof and that of Theorem 3. We consider only the case \( i = 1 \) with the profile \( \Psi_1 \) for Burgers equation, the proof being the same for \( i \geq 2 \). We define the self-similar variables

\[
X := \sqrt{\frac{b}{6(T-t)^2}}, \quad Y := \frac{y}{\sqrt{T-t}}, \quad s := -\log(T-t), \quad Z := \frac{Y}{8\sqrt{s}}.
\]
and
\[ u(t, x, y) = \sqrt{\frac{6}{5}} (T-t)^{\frac{1}{5}} v(s, X, Y), \]

The first step is to obtain precise information for the behaviour of the derivatives of the solution on the transverse axis.

5.1. Analysis on the transverse axis \{x = 0\}

We start by showing the first part of Theorem 1 and the analogue of Proposition 10. Define for a solution \( u \) to (1.1):
\[ \xi(t, y) = -u_x(t, 0, y), \quad \zeta(t, y) = (T-t)^{-1} f(s, Y), \quad s(t) = \partial_x^3 u(t, 0, y), \quad \zeta(t, y) = (T-t)^{-4} g(s, Y). \]
Then \((f, g)\) solve the system (3.9).

**Claim:** For \(0 < T \ll 1\) small enough, for any \( b > 0 \), there exists a solution to (3.9) such that
\[ f(s, Y) = \frac{1}{1 + Z^2} + \tilde{f}, \quad |\partial_x^j \tilde{f}| \lesssim s^{\frac{1}{2}} (1 + |Z|)^{-\frac{3}{2} - j}, \]
\[ g(s, Y) = \frac{b}{(1 + Z^2)^{\frac{1}{4}}} + \tilde{g}, \quad |\partial_x^j \tilde{g}| \lesssim s^{\frac{1}{2}} (1 + |Z|)^{-8 + \frac{1}{2} - j}. \]

These estimates are very similar to (3.7) and (3.8), but the smallness of the error is in powers of \(s^{-1}\) and not of \(e^{-s}\) anymore. This loss will however not be a problem for the sequel.

**Sketch of proof of the claim:** We adapt the strategy of Section 4. We first construct a solution \( f \) to \((NLH)\) satisfying (5.1). We take as an approximate solution to \((NLH)\) the profile
\[ F[a](s, Y) := \frac{1}{1 + \left( \frac{1}{8s} + a \right) Y^2} + \left( \frac{1}{4s} + 2a \right) \frac{1}{1 + \left( \frac{1}{8s} + a \right) Y^2}. \]

Note that here the corrective parameter \(a\) will satisfy \(|a| \lesssim |\log(s) s^{-2}|\). The main difference between the stable blow-up and the flat blow-ups for \((NLH)\) is then the following. The scaling parameter in the flat case (4.4) corresponds to leading order to that of the inviscid case and is not affected by the dissipation (4.27), whereas in the stable blow-up case the dissipation has a modulation effect on this parameter, and forces it to tend to 0 through a logarithmic correction.

Following the proof of Proposition 23, the approximate profile satisfies the following identity:
\[ \partial_s F[a] + F[a] + \frac{Y}{2} \partial_Y F[a] - F^2[a] - \partial_Y F[a] = -\left( \frac{1}{4s^2} + \frac{4a}{s} \right) h_2 + \Psi = R \]
where \(h_2\) is defined by (4.6), and where for a corrective modulation parameter \(a\) satisfying the a priori bound \(|a| \lesssim |\log(s) s^{-2}|\) and \(|as| \lesssim |\log(s)| s^{-3}\) the errors \(\Psi\) and \(R\) satisfy:
\[ \|\Psi\|_{L^2_p} \lesssim s^{-3}, \quad \|\partial_x^j R\|_{L^2_p} \lesssim s^{-3+\frac{j}{2}} \text{ for } j = 0, 1, 2 \text{ and } \|\partial_x^j R\|_{L^2_p} \lesssim s^{-1} \text{ for } j \geq 3, \]
\[ \int_{|Y| \geq 1} \frac{|Z\partial_x^j R|^2}{|Y|} dY \lesssim s^{-1} \text{ and } \int_{|Y| \geq 1} \frac{|\partial_x^j R|^2}{|\phi_0(Z)|^2 |Y|} dY \lesssim s^{-1} \text{ for } j \geq 3, \]
where \(\phi_j\) denotes the eigenfunction
\[ \phi_j(Z) = \frac{Z^j}{(1 + Z^2)^2}, \quad H_Z \phi_j = \frac{j - 2}{2} \phi_j, \quad H_Z = 1 - \frac{2}{1 + Z^2} + \frac{Z}{2} \partial_Z. \]
We then show the existence of a global solution to \((NLH)\) close to \(F[a]\) by a bootstrap argument following Proposition 24. We decompose \(f\) as
\[
f = F[a] + \varepsilon = F[a] + c_1 + \varepsilon, \quad \langle \varepsilon, h_2 \rangle_\rho = 0, \quad \langle \varepsilon, 1 \rangle_\rho = \langle \varepsilon, h_2 \rangle_\rho = 0
\]
where the orthogonality conditions fix the value of \(a\) and \(c_1\) in a unique way. We claim that there exists a global solution to the first equation in (3.9) satisfying:
\[
|a(s)| \lesssim s^{-2}, \quad |c_1(s)| \lesssim s^{-2}, \quad \|\varepsilon\|_{L^2_\rho} \lesssim s^{-3}, \quad \|\partial^j_{\rho} \varepsilon\|_{L^2_\rho} \lesssim s^{-3+\frac{1}{j}} \quad j = 1, 2, \quad \|\partial^j_{\rho} \varepsilon\|_{L^2_\rho} \lesssim s^{-1-j} \quad j \geq 3,
\]
\[
\int_{|Y| \geq 1} \frac{|(Z\partial Z)^j \varepsilon|^2}{|\phi_j(Z)|^2} \frac{dY}{|Y|} \lesssim s^{-\frac{1}{2}} \quad \text{and} \quad \int_{|Y| \geq 1} \frac{|\partial^j_{\rho} \varepsilon|^2}{|\phi_j(Z)|^2} \frac{dY}{|Y|} \lesssim s^{-\frac{1}{2}} \quad \text{for } j \geq 3.
\]
To prove this fact, one first performs modulation estimates, then energy estimates at the origin with the \(\rho\) weight, and then energy estimates outside the origin as in Lemmas 26, 27 and 28.

The evolution equation near the origin reads from (5.3)
\[
c_{1,s} - c_1 - \left(\frac{1}{4s^2} + \frac{4a}{s}\right) - \left(a_s + \frac{2}{s}a\right) h_2 + \tilde{c}_s + \mathcal{L} \tilde{c} = 2(F[a] - 1) \varepsilon - \varepsilon^2 + \Psi = 0.
\]
The modulation estimates are therefore a consequence of the spectral structure of \(\mathcal{L}\) in Proposition 22, giving in the bootstrap regime when projecting the above equation on 1 and \(h_2\):
\[
\left|a_s + \frac{2}{s}a\right| \lesssim s^{-3}, \quad |c_{1,s} - c_1| \lesssim s^{-2}.
\]
The first inequality, when reintegrated in time, gives \(|a| \lesssim |\log(s)| s^{-2}\. The second inequality shows an instability, and the use of Brouwer’s fixed point theorem then implies the existence of a trajectory such that \(|c_1(s)| \lesssim s^{-2}\. The orthogonality conditions for \(\tilde{c}\) imply the spectral damping \(\langle \tilde{c}, \mathcal{L} \tilde{c} \rangle \geq \|\tilde{c}\|_{L^2_\rho}^2\) since \(\tilde{c}\) is even, implying the energy identity
\[
\frac{d}{ds} \left(\frac{1}{2} \|\varepsilon\|_{L^2_\rho}^2\right) \leq -(1 - Cs^{-1} - C\|\varepsilon\|_{L^\infty}) \|\varepsilon\|_{L^2_\rho}^2 + C^{-6}.
\]
This yields the desired estimates for \(\varepsilon\) when reintegrated with time, and the same technique applies to control its derivatives. In the far field, the analysis is the same as in Lemma 28, the equation for \(\varepsilon\) reads
\[
\varepsilon_s + H_Z \varepsilon - \partial_Y \varepsilon - 2(F[a] - \frac{1}{1 + Z^2}) \varepsilon - \varepsilon^2 + R = 0, \quad H_Z = 1 - \frac{2}{1 + Z^2} + \frac{Y}{2} \partial_Y.
\]
Let \(\chi\) be a non-negative smooth cut-off function with \(\chi = 0\) for \(|Y| \leq 1\) and \(\chi = 1\) for \(|Y| \geq 2\). One obtains from this equation the following energy estimate:
\[
\frac{d}{ds} \left(\frac{1}{2} \int \chi \frac{\varepsilon^2}{|\phi_{5/2}(Z)|^2} \frac{dY}{|Y|}\right) \leq -\left(\frac{1}{4} - \kappa - \|\varepsilon\|_{L^\infty}\right) \int \chi \frac{\varepsilon^2}{|\phi_{5/2}(Z)|^2} \frac{dY}{|Y|} - \int \chi \frac{(\partial_Y \varepsilon)^2}{|\phi_{5/2}(Z)|^2} \frac{dY}{|Y|}
+ O\left(s^{-\frac{5}{2}} \|\varepsilon\|_{L^2_\rho}^2\right) + O(s^{-1})
\]
where \(0 < \kappa \ll 1\) is an arbitrary small positive number, since \(|\phi_{5/2}| \sim |Z|^{5/2} \sim |Y|^{5/2} s^{-\frac{5}{2}}\) on compact sets. Thanks to the damping, this estimate is reintegrated in time and shows the weighted decay outside the origin for \(\varepsilon\). The analogous estimates for the derivatives are showed similarly. The strategy that we just explained allows one to close the bootstrap estimates. Using the Sobolev embedding (A.2), this concludes the proof of the existence of a solution \(f\) to \((NLH)\) satisfying (5.1).
Once the properties of $f$ are known, the analysis of $(LFH)$ follows very closely the one performed in Subsection 4.2. We decompose $g$ solution to the second equation in (3.9) according to

$$g(s, Y) = b(s) f^4(s, Y) + \bar{e}, \quad \langle \bar{e}, 1 \rangle = 0$$

where $f$ is the solution $(NLH)$ we just constructed. The evolution equation then reads

$$b_s f^4 + \bar{e}_s + 4 \bar{e} - 4 f \bar{e} + \frac{Y}{2} \partial_Y \bar{e} - \partial_Y Y \bar{e} + \bar{R} = 0, \quad \bar{R} = -12 b (\partial_Y f)^2 f^2.$$  

For $|b| \lesssim 1$ the error satisfies from the properties of $f$ already showed:

$$\|\bar{R}\|_{L^2_t} \lesssim s^{-2}, \quad \|\partial^j \bar{R}\|_{L^2_t} \lesssim s^{-1} \text{ for } j \geq 1,$$

and

$$\int_{|Y| \geq 1} \frac{|(Z \partial_Y Z^j \bar{R}| dY}{|Y|} \lesssim s^{-1}, \quad \int_{|Y| \geq 1} \frac{|(Z \partial_Y Z^j \bar{R}| dY}{|Y|} \lesssim s^{-1} \text{ for } j \geq 1,$$

where $\psi_j$ denotes the eigenfunction

$$\psi_j(Z) = \frac{Z^j}{(1 + Z^2)^{\frac{1}{2}}}, \quad \mathcal{M} \psi_j = j \psi_j, \quad \mathcal{M}_Z = 4 - \frac{4}{1 + Z^2} + \frac{Z}{2} \partial_Y Z.$$

We claim that there exists a solution satisfying the estimates

$$|b_s| \lesssim s^{-2}, \quad \|\bar{e}\|_{L^2_t} \lesssim s^{-2}, \quad \|\partial^j \bar{e}\|_{L^2_t} \lesssim s^{-1} \text{ for } j \geq 1,$$

and

$$\int_{|Y| \geq 1} \frac{|(Z \partial_Y Z^j \bar{e}| dY}{|Y|} \lesssim s^{-\frac{1}{2}}, \quad \int_{|Y| \geq 1} \frac{|(Z \partial_Y Z^j \bar{e}| dY}{|Y|} \lesssim s^{-\frac{1}{2}} \text{ for } j \geq 1.$$  

Similarly, we prove this property by a bootstrap analysis, following closely the analysis of Lemmas 34, 35 and 36. The equation close to the origin reads

$$b_s f^4 + \bar{e}_s + \frac{Y}{2} \partial_Y \bar{e} - \partial_Y Y \bar{e} + 4(1 - f) \bar{e} + \bar{R} = 0.$$  

Taking the $L^2_t$ scalar product against the constant 1 then yields indeed the modulation equation

$$|b_s| \lesssim s^{-2}.$$  

Similarly, from the spectral gap $\|\partial_Y \bar{e}\|_{L^2_t} \geq \|\bar{e}\|_{L^2_t}$ one deduces the energy identity

$$\frac{d}{ds} \left( \frac{1}{2} \|\bar{e}\|_{L^2_t}^2 \right) \leq -(1 - Cs^{-\frac{1}{2}}) \|\bar{e}\|_{L^2_t}^2 + Cs^{-4}$$

which yields the corresponding estimate $\|\bar{e}\|_{L^2_t} \lesssim s^{-2}$ when re-integrated with time. The corresponding estimates for higher order derivatives are showed the same way. In the far field the evolution equation reads

$$b_s f^4 + \bar{e}_s + 4 \bar{e} - \frac{4}{1 + Z^2} \bar{e} + \frac{Y}{2} \partial_Y \bar{e} - \partial_Y Y \bar{e} + 4 \bar{f} \bar{e} + \bar{R} = 0.$$  

This equation enjoys the following energy estimate for an arbitrary constant $0 < \kappa \ll 1$:

$$\frac{d}{ds} \left( \frac{1}{2} \int \frac{\bar{e}^2}{|\psi_1(Z)|^2} dY \right) \leq - \left( \frac{1}{4} - \kappa \right) \int \frac{\bar{e}^2}{|\psi_1(Z)|^2} dY - \int \frac{\chi_1 (\partial_Y \bar{e})^2}{|\psi_1(Z)|^2} dY + O(s^{\frac{3}{2}} \|\bar{e}\|_{L^2_t}^2) + O(s^{-1})$$

This estimate is re-integrated in time and shows the expected weighted decay outside the origin for $\bar{e}$. The estimates for the derivatives are showed the same way. Using the Sobolev embedding (A.2), one then obtained the existence of a solution $g$ to $(LFH)$ satisfying (5.2).
5.2. Analysis of the full 2-d problem

We now follow the analysis of Section 3. Let \( f \) and \( g \) be the solutions to \((NLH)\) and \((LFH)\) satisfying (5.1) and (5.2). For simplicity we fix \( b = 6 \), so that \( X = x/(T-t)^{3/2} \). We take the same blow-up profile as in the proof of Theorem 3, adjusting the cut-off between the inner and outer zones. We set for \( 0 < d \ll 1 \) a cut-off function \( \chi_d(s,Y) := \chi(Y/(d s)) \) where \( \chi \) is a smooth nonnegative function with \( \chi(Y) = 1 \) for \( |Y| \leq 1 \) and \( \chi(Y) = 0 \) for \( |Y| \geq 2 \). We decompose our solution to (5.5) according to:

\[
v(s,X,Y) = Q + \varepsilon, \quad Q = \chi_d(s,Y)\tilde{\Theta} + (1-\chi_d(s,Y))\Theta_e
\]

where (for \( d \) small enough \( f \) and \( g \) do not vanish from (5.1) and (5.2))

\[
\tilde{\Theta}(s,X,Y) = \sqrt{6}g^{-\frac{3}{2}}f^3 \Psi_1 \left( \frac{g^\frac{1}{2}f^{-\frac{1}{2}}}{\sqrt{6}} X \right)
\]

and where \( \Theta_e \) is the exterior profile (recall that \( \tilde{X} = X/(1+Z^2)^{3/2} \)):

\[
\Theta_e(s,X,Y) = \left( -X f(s,Y) + X^3 g(s,Y) \right) e^{-\tilde{X}^2}.
\]

We adjust our initial datum \( v(s_0) \) such that \( -\partial_X v(s_0,0,Y) = f(s_0,Y) \) and \( \partial_Y^2 v(s_0,0,Y) = g(s_0,Y) \). This way, since \( v \) odd in \( x \) and even \( y \), one has that \( \partial_Y^2 \varepsilon(s,0,Y) = 0 \) on the transverse axis for \( j = 0,1,2,3,4 \) for all times \( s \geq s_0 \). The time evolution of the remainder \( \varepsilon \) is:

\[
\varepsilon_s + \mathcal{L} \varepsilon + \tilde{\mathcal{L}} \varepsilon + R + \varepsilon \partial_X \varepsilon = 0
\]

where

\[
\mathcal{L} = -\frac{1}{2} + \partial_X \Theta + \left( \frac{3}{2} X + \Theta \right) \partial_X + \frac{1}{2} Y \partial_Y - \partial_{YY},
\]

\[
\Theta(s,X,Y) = (1 + Z^2)^{\frac{3}{2}} \Psi_1 \left( \frac{X}{1 + Z^2} \right), \quad \tilde{\mathcal{L}} \varepsilon = (Q - \Theta) \partial_X \varepsilon + (\partial_X Q - \partial_X \Theta) \varepsilon,
\]

and

\[
R = Q_s - \frac{1}{2} Q + \frac{3}{2} X \partial_X Q + \frac{1}{2} Y \partial_Y Q + Q \partial_X Q - \partial_{YY} Q.
\]

From Proposition 12 the inviscid linearised operator has eigenvalues of the form \((j + \ell - 3)/2\) for \((j,\ell) \in \mathbb{N}\) with associated eigenfunction

\[
\varphi_{j,\ell}(X,Z) = Z^\ell F_k^{1/2}(Z) \times \frac{(-1)^j \Psi_1^j \left( F_k^{3/2}(Z) X \right)}{1 + 3 \Psi_1^2 \left( F_k^{3/2}(Z) X \right)}.
\]

The sizes of the important objects are

\[
|\Theta(X,Z)| \approx |X| \left( (1 + |Z|)^3 + |X| \right)^{\frac{1}{2} - 1} \approx (1 + |Z|) |\tilde{X}| (1 + |\tilde{X}|)^{\frac{1}{2} - 1}
\]

\[
|\varphi_{4,0}(X,Z)| \approx |X|^4 \left( (1 + |Z|)^3 + |X| \right)^{\frac{1}{2} - 4} \approx (1 + |Z|)^4 |\tilde{X}|^4 (1 + |\tilde{X}|)^{\frac{1}{2} - 4}.
\]

We claim that thanks to (5.1) and (5.2) one has the following estimates for the error, which can be proved as in the proof of Lemma 17:

\[
\sum_{0 \leq j_1 + j_2 \leq 1} \int_{\mathbb{R}^2} \frac{(\partial_{Z}^j A \tilde{Z}^2 R)^{2q} + ((Y \partial_Y)^{j_1} A^j R)^{2q}}{\varphi_{4,0}(X,Z) |X| |Y|} dX dY \lesssim s^{-q}
\]
where $A$ is given by (3.22). From (5.1) and (5.2) one also deduces the following estimates for the lower order linear term
\[ |\partial^j_Z (Q - \Theta)| \lesssim s^{-\frac{j}{2}} |X| (1 + |Z|)^{-j}, \]
\[ |\partial^j_Z \partial^i_X (Q - \Theta)| \lesssim s^{-\frac{j}{2}} (1 + |X|)^{-j} (1 + |Z|)^{-j}. \]
which can be proved as in the proof of Lemma 18. We can therefore perform the same energy estimates as in Lemmas 19 and 20. Indeed, the leading order linear estimate (3.36) holds also true in that case, and with the above control on the lower order linear term and of the error $R$ one obtains:
\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY \right) 
\leq - \left( \frac{1}{2} - \frac{C}{q} - C \|\partial_X \varphi\|_{L^\infty} \right) \int_{\mathbb{R}^2} \frac{\varepsilon^{2q}}{\varphi_{4,0}^2(X, Y)} dX dY - \frac{2q - 1}{q^2} \int |\partial_Y (\varepsilon^q)|^2 dX dY + C s^{-q}.
\]

The same type of energy estimates hold when applying $A, Z$ or $Z \partial_Z$, up to terms involving lower order derivatives. For example, one can derive the following estimate:
\[
\frac{d}{ds} \left( \frac{1}{2q} \int_{\mathbb{R}^2} \frac{(A \varepsilon)^{2q}}{\varphi_{4,0}^2(X, Z)} dX dY \right) 
\leq - \left( \frac{1}{2} - \frac{C}{q} - C \|\partial_X \varphi\|_{L^\infty} - C \varepsilon \|\varphi\|_{L^\infty} \right) \int_{\mathbb{R}^2} \frac{(A \varepsilon)^{2q}}{\varphi_{4,0}^2(X, Y)} dX dY - \frac{2q - 1}{q^2} \int |\partial_Y (\varepsilon^q)|^2 dX dY + C \varepsilon^q + C \int \frac{\varepsilon^{2q}}{\varphi_{4,0}^2(X, Y)} dX dY.
\]
This implies that the analogue of Proposition 13 holds, i.e. that we can bootstrap the following estimates for the remainder $\varepsilon$:
\[
\sum_{0 \leq j_1 + j_2 \leq 2} \left( \int_{\mathbb{R}^2} \frac{|(\partial^j_Z A^j_Z \varepsilon)^{2q}}{\varphi_{4,0}^2(X, Y)} dX dY \right)^{\frac{1}{2q}} + \left( \int_{\mathbb{R}^2} \frac{|(Y \partial_Y)^{j_1} A^j_Z \varepsilon)^{2q}}{\varphi_{4,0}^2(X, Y)} dX dY \right)^{\frac{1}{2q}} \lesssim \frac{1}{\sqrt{s}}.
\]

This estimate, together with the weighted Sobolev embedding (B.2), gives the following pointwise estimates on $\varepsilon$ as in Lemma 14:
\[
|\varepsilon| \lesssim s^{-\frac{1}{2}} (1 + |Z|)^{\frac{3}{2}} |X|^4 (1 + |X|)^{\frac{5}{2}} \lesssim s^{-\frac{1}{2}} |X|,
|\partial_X \varepsilon| \lesssim s^{-\frac{1}{2}} (1 + |Z|)^{-1} |X|^3 (1 + |X|)^{\frac{5}{2}} \lesssim s^{-\frac{1}{2}},
|\partial_Z \varepsilon| \lesssim s^{-\frac{1}{2}} (1 + |Z|) |X|^4 (1 + |X|)^{\frac{5}{2}} \lesssim s^{-\frac{1}{2}}.
\]

By using the above estimate in the decomposition 5.4, combined with (5.1) and (5.2), we get that on compact sets in the variables $X$ and $Z$ there holds the estimate:
\[
v = \Theta + O_{C_1} (s^{-\frac{1}{2}}).
\]
This then ends the proof of Theorem 2.

A. One-dimensional functional analysis results

Lemma 37 (Poincaré inequality in $L^2_\rho$). For any $f \in H^1_\rho$ defined by (4.5) one has that $Yf \in L^2_\rho$ with
\[
\|Yf\|_{L^2_\rho} \lesssim \|f\|_{H^1_\rho}, \quad \text{(A.1) eq:Poincare}
\]
Proof. We prove (A.1) for smooth and compactly supported functions, and its extension to \(H^1_\rho\) follows by a density argument. Performing an integration by parts one first finds
\[
\int Y\varepsilon\partial_Y e^{-\frac{Y^2}{4}}dY = \frac{1}{4} \int Y^2\varepsilon^2 e^{-\frac{Y^2}{4}}dY - \frac{1}{2} \int \varepsilon^2 e^{-\frac{Y^2}{4}}dY.
\]
Therefore, using Cauchy-Schwarz and Young’s inequalities one obtains:
\[
\int Y^2\varepsilon^2 e^{-\frac{Y^2}{4}}dY \leq 2\varepsilon \int Y^2\varepsilon e^{-\frac{Y^2}{4}}dY + \frac{2}{\varepsilon} \int |\partial_Y\varepsilon|^2 e^{-\frac{Y^2}{4}}dY + 2 \int \varepsilon^2 e^{-\frac{Y^2}{4}}dY.
\]
Taking \(0 < \varepsilon < 1/2\) yields the desired result.

\[\square\]

**Lemma 38.** If \(\varepsilon \in H^1_{\text{loc}}(\{|Y| \geq 1\})\) is such that \(\int_{|Y| \geq 1}(\varepsilon^2 + (Y\partial_Y\varepsilon)^2)|Y|^{-1}dY\) is finite, then \(\varepsilon\) is bounded with:
\[
\|\varepsilon\|_{L^\infty(|Y| \geq 1)} \lesssim \|\varepsilon\|_{L^2(|Y| \geq 1, \frac{dY}{Y})} + \|Y\partial_Y\varepsilon\|_{L^2(|Y| \geq 1, \frac{dY}{Y})} \tag{A.2}\]

**Proof.** Assume that the right hand side of (A.2) is finite. Let \(A \geq 1\) and \(v(Z) = \varepsilon(AZ)\). Then, changing variables and using Sobolev embedding gives for some \(C\) independent on \(A\):
\[
\|\varepsilon\|_{L^\infty(A, 2A))} = \|v\|_{L^\infty(1, 2)} \leq C \left( \int A^2 v^2(Z) dZ + \int A |\partial_Z v|^2(Z)dZ \right) \\
\leq C \left( \int_A^{2A} v^2(Y) dY + \int_A^2 A|\partial_Y v|^2(Y)dY \right) \\
\leq C \left( \int_{|Y| \geq 1} v^2(Y) \frac{dY}{|Y|} + \int_{|Y| \geq 1} |Y\partial_Y v|^2(Y) \frac{dY}{|Y|} \right)
\]
Taking the supremum with respect to \(A\) in the above estimate yields (A.2).

\[\square\]

**B. Two-dimensional functional analysis results**

**Lemma 39.** Let \(q \in \mathbb{N}^+\). Then for any \(u \in W^{1,2q}_{\text{loc}}(\mathbb{R}^2)\) one has:
\[
\|u\|_{L^\infty(\mathbb{R}^2)}^{2q} \leq C(q) \left( \int_{\mathbb{R}^2} u^{2q} \frac{dX dY}{|X| \langle Y \rangle} + \int_{\mathbb{R}^2} (X\partial_X u)^{2q} \frac{dX dY}{|X| \langle Y \rangle} + \int_{\mathbb{R}^2} (Y\partial_Y u)^{2q} \frac{dX dY}{|X| \langle Y \rangle} \right) \tag{B.1}
\]

**Proof.** The result follows from the classical Sobolev embedding and a scaling argument. Assume that the right hand side of (B.1) is finite. Let \(A \in \mathbb{R}\) and change variables \(\hat{X} = X/A\) and \(u(X,Y) = v(\hat{X}, Y)\). From Sobolev embedding one has that
\[
\|u\|_{L^\infty(|A| \leq |X| \leq 2A, |Y| \leq 1)} = \|v\|_{L^\infty(|1| \leq |\hat{X}| \leq 2, |Y| \leq 1)} \\
\leq C(q) \int_{1 \leq |\hat{X}| \leq 2, |Y| \leq 1} (u^{2q} + (\partial_{\hat{X}} v)^{2q} + (\partial_Y v)^{2q}) d\hat{X} dY \\
\leq C(q) \int_{A \leq |X| \leq 2A, |Y| \leq 1} (u^{2q} + (A\partial_X u)^{2q} + (\partial_Y u)^{2q}) \frac{dX dY}{A} \\
\leq C(q) \left( \int_{\mathbb{R}^2} u^{2q} \frac{dX dY}{|X| \langle Y \rangle} + \int_{\mathbb{R}^2} (X\partial_X u)^{2q} \frac{dX dY}{|X| \langle Y \rangle} + \int_{\mathbb{R}^2} (Y\partial_Y u)^{2q} \frac{dX dY}{|X| \langle Y \rangle} \right).
\]
Corollary 40. Let \( q \in \mathbb{N}^\ast \). Then for any \( u \in W^{1,2q}_{loc}(\mathbb{R}^2) \) one has:

\[
\left\| \frac{u}{\phi_{j,0}(X,Z)} \right\|_{L^\infty(\mathbb{R}^2)}^{2q} \leq C(q) \left( \int_{\mathbb{R}^2} \frac{u^{2q}}{\phi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} + \int_{\mathbb{R}^2} \frac{(X \partial_X u)^{2q}}{\phi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} + \int_{\mathbb{R}^2} \frac{(\langle Y \rangle \partial_Y u)^{2q}}{\phi_{j,0}^{2q}(X,Z)} \frac{dX dY}{|X| |Y|} \right)
\]  

(B.2)  

Proof. Assume that the right hand side of (B.2) is finite. First, notice from (3.16) that

\[ |X \partial_X \phi_{X,j}| \sim |\phi_{X,j}| \]

From this and (3.13) one deduces that:

\[
|X \partial_X \phi_{j,0}(X,Z)| = \left| X \partial_X \left( \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \phi_{X,j} \left( F_{k}^{\frac{3}{2}}(Z)X \right) \right) \right|
\]

\[
= \left| \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} (X \partial_X \phi_{X,j}) \left( F_{k}^{\frac{3}{2}}(Z)X \right) \right| \sim \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \left| \phi_{X,j} \right| \left( F_{k}^{\frac{3}{2}}(Z)X \right) = \left| \phi_{j,0}(X,Z) \right|.
\]

This implies that

\[
\left| X \partial_X \left( \frac{1}{\phi_{j,0}(X,Z)} \right) \right| = \left| \frac{X \partial_X \phi_{j,0}(X,Z)}{\phi_{j,0}^2(X,Z)} \right| \sim \left| \frac{1}{\phi_{j,0}(X,Z)} \right|.
\]

Assume now that \( |Y| \leq 1 \). Since \( \partial_Y = e^{-(k-1)/(2k)s} \partial_Z \) then

\[
|\partial_Y \phi_{j,0}(X,Z)| = \left| e^{-\frac{1}{2k}s} \partial_Z \left( \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \phi_{X,j} \left( F_{k}^{\frac{3}{2}}(Z)X \right) \right) \right|
\]

\[
\leq \left| \partial_Z \left( \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \right) \phi_{X,j} \left( F_{k}^{\frac{3}{2}}(Z)X \right) \right| + \frac{3}{2} \left| \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \partial_Z F_{k}(Z) \right| \left( X \phi_{X,j} \right) \left( F_{k}^{\frac{3}{2}}(Z)X \right)
\]

\[
\leq C \left( 1 + Z^{2k} \right)^{\frac{4}{2}-1} \left| \phi_{X,j} \left( F_{k}^{\frac{3}{2}}(Z)X \right) \right| = C |\phi_{j,0}(X,Z)|
\]
Lemma 41. Let $j_1, j_2 \leq \mathbb{N}$. Then there exists a constant $C > 0$ such that for any function $u \in C^{j_1+j_2}(\mathbb{R}^2)$, for $A$ defined by (3.22) there holds:

$$\frac{1}{C} \sum_{j_1' = 0}^{j_1} \sum_{j_2' = 0}^{j_2} |Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} \partial_X^{j_2'} u| \leq \sum_{j_1' = 0}^{j_1} \sum_{j_2' = 0}^{j_2} |(Z \partial_Z)^{j_1} (X \partial_X)^{j_2} u| \leq C \sum_{j_1' = 0}^{j_1} \sum_{j_2' = 0}^{j_2} |Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} \partial_X^{j_2'} u|,$$

and for $j_2 \geq 1$:

$$|\partial_Z^{j_2} A^{j_2} u| \leq C \sum_{j_1' = 0}^{j_1} (1 + |Z|)^{-(j_1 - j_1')} |X|^{j_2'} |\partial_Z^{j_1'} \partial_X^{j_2'} u|,$$

$$|(Z \partial_Z)^{j_1} A^{j_2} u| \leq C \sum_{j_1' = 0}^{j_1} |Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} \partial_X^{j_2'} u|,$$

$$|X|^{j_2'} |\partial_Z^{j_2'} \partial_X^{j_2'} u| \leq C \sum_{j_1' = 0}^{j_1} (1 + |Z|)^{-(j_1 - j_1')} |\partial_Z^{j_1'} A^{j_2} u|,$$

$$|Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} \partial_X^{j_2'} u| \leq C \sum_{j_1' = 0}^{j_1} \sum_{j_2' = 1}^{j_2} |(Z \partial_Z)^{j_1'} A^{j_2} u|.$$

Proof. (B.3) follows from an easy induction argument that we leave to the reader.
**Step 1** Proof of (B.4). We first claim that there exists a family of profiles \((f_{j_2,j_2'})_{j_2,j_2' \leq j_2}\) such that

\[ A^{j_2}u = \sum_{j_2'=1}^{j_2} f_{j_2,j_2'} \partial_X^{j_2'} u, \tag{B.8} \]

and satisfying for any \(k_1,k_2 \in \mathbb{N}\):

\[
|\partial_Z^{k_1} \partial_X^{k_2} f_{j_2,j_2'}| \lesssim (1 + |Z|)^{-k_1} (1 + |X|)^{\min(-(k_2-j_2'),0)} |X|^{|\max(j_2'-k_2,0)|}. \tag{B.9}
\]

We prove this fact by induction on \(j_2 \in \mathbb{N}^*\). From (3.22), (B.8) holds for \(j_2 = 1\) with \(f_{1,1} = 3X/2 + F_k^{-3/2}(Z)\Psi_1(F_k^{3/2}(Z)X)\). Since from Proposition 5, \(\partial_X \Psi_1 \leq 0\) and is minimal at the origin we infer that

\[
\frac{1}{2} |X| \leq \left| \frac{3}{2} X + F_k^{-3/2}(Z)\Psi_1(F_k^{3/2}(Z)X) \right| \lesssim \frac{3}{2} |X| \tag{B.10}
\]

and from (3.14) we infer that for \(k_1,k_2 \in \mathbb{N}\):

\[
\left| \partial_Z^{k_1} \partial_X^{k_2} \left( F_k^{-3/2}(Z)\Psi_1(F_k^{3/2}(Z)X) \right) \right| \lesssim \begin{cases} (1 + |Z|)^{-k_1} |X| & \text{if } k_2 = 0, \\ (1 + |Z|)^{-k_1} (1 + |X|)^{-k_2} & \text{if } k_2 \geq 1. \end{cases}
\]

which proves (B.9) for \(j_2 = 1\), and thus the claim is true for \(j_2 = 1\). Assume now that the claim is true for some \(j_2 \in \mathbb{N}^*\). Then, using (B.8) for the integers \(j_2\) and 1:

\[
A^{j_2+1}u = f_{1,1} \partial_X \left( \sum_{j_2'=1}^{j_2} f_{j_2,j_2'} \partial_X^{j_2'} u \right) = \sum_{j_2'=1}^{j_2} f_{1,j_2,j_2'} \partial_X^{j_2'} u + f_{1,j_2,j_2} \partial_X^{j_2+1} u
\]

and the claim is true from the bounds (B.9) for the integers \(j_2\) and 1. Hence it is true for all \(j_2 \in \mathbb{N}^*\). We now apply Leibniz formula and obtain from (B.8):

\[
\partial_Z^{j_1} A^{j_2} u = \sum_{j_2'=1}^{j_2} \sum_{j_1'=0}^{j_1} C_{j_1'}(\partial_Z^{j_1-j_1'} f_{j_2,j_2'}) \partial_Z^{j_1'} \partial_X^{j_2'} u, \quad |\partial_Z^{j_1-j_1'} f_{j_2,j_2'}| \lesssim (1 + |Z|)^{-(j_1-j_1')} |X|^{j_2'}
\]

where the second estimate comes from (B.9), and (B.4) is proven.

**Step 2** Proof of (B.5). This is a direct consequence of (B.3) and (B.4):

\[
|Z \partial_Z)^{j_1} A^{j_2} u| \lesssim \sum_{j_1'=0}^{j_1} |Z|^{j_1'} |\partial_Z^{j_1'} A^{j_2} u| \lesssim \sum_{j_1'=0}^{j_1} \sum_{j_2'=1}^{j_2} |Z|^{j_1'} (1 + |Z|)^{-(j_1'-j_1)} |X|^{j_2'} |\partial_Z^{j_1'} (X \partial_X)^{j_2'} u|,
\]

\[
\lesssim \sum_{j_1'=0}^{j_1} \sum_{j_2'=1}^{j_2} \sum_{j_1'=0}^{j_1} \sum_{j_2'=1}^{j_2} |Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} (X \partial_X)^{j_2'} u| \lesssim \sum_{j_1'=0}^{j_1} \sum_{j_2'=1}^{j_2} |Z|^{j_1'} |X|^{j_2'} |\partial_Z^{j_1'} (X \partial_X)^{j_2'} u|.
\]

**Step 3** Proof of (B.6). First, from (B.3) one has:

\[
|X|^{j_2} |\partial_Z^{j_1} \partial_X^{j_2'} u| \lesssim \sum_{j_1'=1}^{j_1} |\partial_Z^{j_1'} (X \partial_X)^{j_2'} u|. \tag{B.11}
\]

\[\text{an:idAj2}\]

\[\text{an:bdj2j2p}\]
We then claim that there exists a family of profiles \((g_{j_2,j'_2})_{j_2 \leq j_2'}\) such that
\[
(X\partial X)^{j_2} u = \sum_{j'_2=1}^{j_2} g_{j_2,j'_2} A^{j'_2} u, \tag{B.12}
\]
and satisfying for any \(k_1, k_2 \in \mathbb{N}\):
\[
|\partial_Z^{k_1} \partial_X^{k_2} g_{j_2,j'_2}| \lesssim (1 + |Z|)^{-k_1} (1 + |X|)^{-k_2}. \tag{B.13}
\]
From (3.22), (B.12) holds for \(j_2 = 1\) with
\[
g_{1,1} = \frac{X}{\frac{3}{2}X + F_k^{-\frac{3}{2}}(Z)\Psi_1(F_k^\frac{3}{2}(Z)X)} = \left(\frac{\tilde{X}}{\frac{3}{2}\tilde{X} + \Psi_1(\tilde{X})}\right)(F_k^{\frac{3}{2}}(Z)X).
\]
From (3.14) and (B.10) one has that for any \(k_2 \in \mathbb{N}\),
\[
|\partial_X \left(\frac{\tilde{X}}{\frac{3}{2}\tilde{X} + \Psi_1(\tilde{X})}\right)| \lesssim (1 + |\tilde{X}|)^{-k_2}
\]
and hence from (3.14) the estimate (B.13) holds for \(g_{1,1}\). The claim can then be proven by induction on \(j_2 \in \mathbb{N}^*\) by the same techniques as in Step 1, and we safely omit its proof here. Using (B.12), (B.13) and Leibniz formula then yields that
\[
|\partial_Z^{j_1} (X\partial X)^{j_2} u| \lesssim \sum_{j'_2=0}^{j_2} \sum_{j'_1=0}^{j_1} |\partial_Z^{j_1-j'_1} g_{j_2,j'_2} \partial_Z^{j'_2} A^{j'_2} u| \lesssim \sum_{j'_2=0}^{j_2} \sum_{j'_1=0}^{j_1} (1 + |Z|)^{-(j_1-j'_1)} |\partial_Z^{j'_2} A^{j'_2} u|.
\]
This implies (B.6) using (B.11).

**Step 4 Proof of (B.7).** This is a direct consequence of (B.3) and (B.6):
\[
|Z|^{j_1} |X|^{j_2} |\partial_Z^{j_1} \partial_X^{j_2} u| \lesssim \sum_{j'_2=0}^{j_2} \sum_{j'_1=0}^{j_1} |Z|^{j_1} (1 + |Z|)^{-(j_1-j'_1)} |\partial_Z^{j'_2} A^{j'_2} u| \lesssim \sum_{j'_2=0}^{j_2} \sum_{j'_1=0}^{j_1} |Z|^{j'_2} |\partial_Z^{j'_2} A^{j'_2} u|,
\]
\[
\lesssim \sum_{j'_2=0}^{j_2} \sum_{j'_1=0}^{j_1} |(Z\partial_Z)^{j'_1} A^{j'_2} u|.
\]

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