On the Energy of Stringy Black Holes

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Abstract

It is well-known that one of the most interesting and challenging problems of General Relativity is the energy and momentum localization. There are many attempts to evaluate the energy distribution in a general relativistic system. One of the methods used for the energy and momentum localization is the one which used the energy-momentum complexes. After the Einstein work, a large number of definitions for the energy distribution was given. We mention the expressions proposed by Landau and Lifshitz, Papapetrou, Bergmann, Weinberg and Møller. The Einstein, Landau and Lifshitz, Papapetrou, Bergmann and Weinberg energy-momentum complexes are restricted to calculate the energy distribution in quasi-Cartesian coordinates. The energy-momentum complex of Møller gives the possibility to make the calculations in any coordinate system.

In this paper we calculate the energy distribution of three stringy black hole solutions in the Møller prescription. The Møller energy-momentum complex gives us a consistent result for these three situations.

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1 INTRODUCTION

It is well-known that one of the most interesting and challenging problems of general relativity is the energy and momentum localization. Numerous attempts have been made in the past for a solution, and this problem still attracts considerable attention in the literature, and remains an important issue to be settled. The different attempts at constructing an energy-momentum density don’t give a generally accepted expression.

Related on the method that used the energy-momentum complexes we can say that there are various energy-momentum complexes including those of Einstein [1]-[2], Landau and Lifshitz [3], Papapetrou [4], Bergmann [5], Weinberg [6] and Møller [7]. Also, there are doubts that these prescriptions could give acceptable results for a given space-time. The problem is that with different energy-momentum complexes we can obtain different expressions for the energy associated with a given space-time. This is because most of the energy-momentum complexes are restricted to the use of particular coordinates. But the results obtained by several authors [8]-[9] demonstrated that the energy-momentum complexes are good tools for evaluating the energy and momentum in general relativity. Although, the energy-momentum complexes of Einstein, Landau and Lifshitz, Papapetrou, Bergmann and Weinberg are coordinate dependent they can give a reasonable result if calculations are carried out in quasi-Cartesian coordinates. Some interesting results sustain this conclusion.

On the other hand, Møller [7] constructed an expression which enables one to calculate the energy distribution in any coordinate system not only in quasi-Cartesian coordinates. Also, Lessner in his important work [10] pointed out that the problem lies with the interpretation of the result from a special relativistic point of view instead of a general relativistic one. In conclusion, the Møller prescription can be used with success to evaluate the energy distribution of a given space-time. Many results recently obtained [11] support the conclusion given by Lessner in his recent paper that the Møller definition is a powerful concept of energy and momentum in general relativity.
In this paper we calculate the energy distribution of some stringy black hole solutions in the Møller prescription. We study the energy associated with these solutions because we think they can furnishes us interesting results. We have three cases. The first case is that of the dual solution known as the magnetic black hole solution. The metric is obtained by multiplying the electric metric in the Einstein frame by a factor $e^{-2\Phi}$. In the second case we consider the metric which describes a non-asymptotically black hole solution in dilaton-Maxwell gravity and was given by Chan, Mann and Horne [13]. First, we perform the calculations for the string metric (see Kar [13]). In the last case we study the string metric for the magnetic black hole (Kar [13]). The Møller energy-momentum complex gives us consistent results for these three situations. Through the paper we use geometrized units ($G = 1, c = 1$) and follow the convention that Latin indices run from 0 to 3.

2 ENERGY IN THE MØLLER PRESCRIPTION

The low energy effective theory largely resembles general relativity with some new ”matter” fields as the dilaton, axion etc [13]-[14]. A main property of the low-energy theory is that there are two different frames in which the features of the space-time may look very different. These two frames are the Einstein frame and the string frame and they are related to each other by a conformal transformation ($g^E_{\mu\nu} = e^{-2\Phi}g^S_{\mu\nu}$) which involves the massless dilaton field as the conformal factor. The string ”sees” the string metric. Many of the important symmetries of string theory also rely of the string frame or the Einstein frame [15].

The action for the Einstein-dilaton-Maxwell theory is given by

$$S_{EDM} = \int d^4x \sqrt{-g}e^{-2\Phi}[R + 4g_{\mu\nu}\nabla^\mu\Phi\nabla^\nu\Phi - \frac{1}{2}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}].$$

(1)

Varying with respect to the metric, dilaton and Maxwell fields we get the field equations for the theory given as

$$R_{\mu\nu} = -2\nabla_\mu\Phi\nabla_\nu\Phi + 2F_{\mu\lambda}F^{\nu\lambda},$$

(2)

$$\nabla^\nu(e^{-2\Phi}F_{\mu\nu}) = 0,$$

(3)
\[4 \nabla^2 \Phi - 4(\nabla \Phi)^2 + R - F^2 = 0. \tag{4}\]

The metric (in the string frame) which solve the Einstein-dilaton-Maxwell field equations to yield the electric black hole is given by

\[ds^2 = A(1 + \frac{2M \sinh^2 \alpha}{r})^{-2} dt^2 - \frac{1}{A} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \tag{5}\]

where \(A = 1 - \frac{2M}{r}\).

In the string frame the dual solution known as the magnetic black hole is obtained by multiplying the electric metric in the Einstein frame by a factor \(e^{-2\Phi}\). Therefore, the magnetic black hole metric is given by

\[ds^2 = \frac{A}{B} dt^2 - \frac{1}{AB} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \tag{6}\]

with \(B = 1 - \frac{Q^2}{Mr}\).

Recently, some non-asymptotically flat black hole solutions in dilaton-Maxwell gravity are due to Chan, Mann and Horne [12]. The metric of the solution in the Einstein frame is given by

\[ds^2 = \frac{C}{\gamma^2} dt^2 - \frac{1}{C} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \tag{7}\]

where \(C = r^2 - 4 \gamma M\).

The string metric is obtained by performing the usual conformal transformation on the metric. This turns out to be

\[ds^2 = \frac{r^2 D}{\gamma^2} dt^2 - \frac{1}{D} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \tag{8}\]

with \(D = 1 - \frac{2\sqrt{2} r^2 M}{Qr}\).

The string metric for the magnetic black hole is given by

\[ds^2 = \frac{2Q^2 E}{\gamma^4} dt^2 - \frac{2Q^2}{r^2 E} dr^2 - 2Q^2 r^2 d\theta^2 - 2Q^2 \gamma^2 \sin^2 \theta d\varphi^2, \tag{9}\]

where \(E = 1 - \frac{4M}{r}\).

We think that another important and challenging problems related to these black hole solutions is that of their energy distributions. We consider
that a good manner to evaluate the energy associated with a black hole solution in string theory in the one which used the Møller energy-momentum complex [7]. This is because of the above mentioned important results obtained in the Møller prescription and, also, of the Lessner opinion [10] and Cooperstock very important hypothesis [16].

In the following we present the definition of the Møller energy-momentum complex. Its good properties indicates it as a good tool for energy-momentum localization in general relativity.

As we pointed out above, Møller gave an expression which allows us to perform the calculations in any coordinate system. Also, Møller argued that his expression enables one to obtain the same values for the total energy and momentum as the Einstein energy-momentum complex for a closed system.

The Møller energy-momentum complex $M^k_i$ [14] is given by

$$M^k_i = \frac{1}{8 \pi} \chi^{kl}_i,$$  \quad (10)

where

$$\chi^{kl}_i = \sqrt{-g} \left( \frac{\partial g_{in}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^n} \right) g^{km} g^{nl}. \quad (11)$$

The quantity $\chi^{kl}_i$ is the Møller superpotential and satisfies the antisymmetric property

$$\chi^{kl}_i = -\chi^{lk}_i. \quad (12)$$

$M_{0}^{0}$ is the energy density and $M_{0}^{\alpha}$ are the momentum density components.

Also, $M^k_i$ satisfies the local conservations laws

$$\frac{\partial M^k_i}{\partial x^k} = 0. \quad (13)$$

The energy of the physical system in a four-dimensional background is given by

$$E = \oint M_{0}^{0} \, dx^1 \, dx^2 \, dx^3 =$$

$$= \frac{1}{8 \pi} \oint \frac{\partial \chi_{0}^{0}}{\partial x^1} \, dx^1 \, dx^2 \, dx^3. \quad (14)$$
Using the Gauss theorem we get the energy contained in a sphere of radius $r$. We have

$$E = \frac{1}{8\pi} \oint \chi_0^{0l} \, d\theta \, d\varphi. \quad (15)$$

First, we consider the dual solution in the string frame which is the magnetic black hole described by the metric given by the $(6)$. For this solution the non-zero $\chi_i^{kl}$ components of the Møller energy-momentum complex are given by

$$\chi_0^{01} = \frac{2 M^2 - Q^2}{M r - Q^2} r \sin \theta. \quad (16)$$

Now, we use (15) and (16) and we obtain the for the energy distribution the expression

$$E(r) = \frac{1}{2} \frac{2 M^2 - Q^2}{M r - Q^2} r. \quad (17)$$

In the Fig.1 we give the graphic representation of the energy. $E$ is plotted against $r$ on X-axis and $Q$ on Y-axis. For the mass $M$ we get the value $M = 1$. 

Figure 1:
From (17) it results that the energy depends on the mass \( M \) and charge \( Q \).

For the solution described by the metric given by (8) we get the non-zero \( \chi^{kl}_i \) components of the Møller energy-momentum complex

\[
\chi^{01}_0 = \left( \frac{2 r^2}{\gamma^2} - \frac{2 \sqrt{2} M r}{Q} \right) \sin \theta. \tag{18}
\]

With (15) and (18) we have for the energy

\[
E(r) = \left( \frac{r^2}{\gamma^2} - \frac{\sqrt{2} M r}{Q} \right). \tag{19}
\]

In the Fig.2 \( E \) is plotted against \( r \) on X-axis and \( Q \) on Y-axis. For the mass \( M \) and \( \gamma \) we get the values \( M = 1 \) and \( \gamma = 1 \).

Figure 2:

We observe from (19) that the energy distribution depends on the mass \( M \), charge \( Q \) and \( \gamma \).
In the case of the magnetic black hole given by (9) we obtain the non-vanishing $\chi_{i^k l}$ components of the Møller energy-momentum complex

$$\chi^{01}_0 = \frac{8 M Q^2}{\gamma^2} r \sin \theta. \quad (20)$$

Using (15) and (20) we obtain for the energy distribution

$$E(r) = \frac{4 M Q^2 r}{\gamma^2}. \quad (21)$$

In the Fig.3 we have the graphic representation of the energy. $E$ is plotted against $r$ on X-axis and $Q$ on Y-axis. For the mass $M$ and $\gamma$ we get the values $M = 1$ and $\gamma = 1$.

![Figure 3](image)

From (21) it results that the energy depends on the mass $M$, charge $Q$ and $\gamma$. 

8
3 DISCUSSION

The subject of the localization of energy continues to be an open one since Einstein has given his important result of the special theory of relativity that mass is equivalent to energy. Misner et al [17] sustained that to look for a local energy-momentum means that is looking for the right answer to the wrong question. Also, they concluded that the energy is localizable only for spherical systems. On the other hand, Cooperstock and Sarracino [18] demonstrated that if the energy is localizable in spherical systems then it is also localizable in any space-times. Bondi [19] gave his opinion that "a non-localizable form of energy is not admissible in general relativity, because any form of energy contributes to gravitation and so its location can in principle be found". Also, Chang, Nester and Chen [20] showed that the energy-momentum complexes are actually quasilocal and legitimate expression for the energy-momentum. They concluded that there exist a direct relationship between energy-momentum complexes and quasilocal expressions because every energy-momentum complexes is associated with a legitimate Hamiltonian boundary term.

Even the method of localization of energy with several energy-momentum complexes has many adepts there was, also, many criticism related to the use of the energy-momentum complexes in energy and momentum localization. The main lack of the energy-momentum complexes is that most of these restrict one to calculate in quasi-Cartesian coordinates.

As we pointed out above, in Introduction, the most energy-momentum complexes are dependent on the coordinate system. Only the Møller energy-momentum complex allows us to make the calculations in any coordinate system. Another argument that sustains the use of the Møller prescription is the Lessner [10] conclusion that the Møller energy-momentum complex is a powerful representation of energy and momentum in general relativity. The viewpoint of Lessner [10] is that "The energy-momentum four-vector can only transform according to special relativity only if it is transformed to a reference system with an everywhere constant velocity. This cannot be achieved by a global Lorentz transformation".

In this paper we choose to study the energy associated with some stringy black hole solution [13] because those solutions are interesting to study. we obtain results for three stringy black hole solutions in the string frame. The energy of the magnetic black hole, which is the dual solution in the string frame, in eq. (17) is \( \lim_{r \to \infty} E(r) = M - \frac{Q^2}{2M} \).
Our result show that the total energy is dependent on charge $Q$ and differs to previous investigations. It shows us that the stringy black holes in string frame are interesting for study. The total energy of the other two non-asymptotically flat black hole solutions, in eqs. (8) and (9), become infinite. From eqs. (19) and (21) we found out that the Møller energy-momentum complex furnishes us consistent results of non-asymptotically flat black hole solutions.

4 REFERENCES

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