Necessary criterion for approximate recoverability

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Abstract

A tripartite state \( \rho_{ABC} \) forms a Markov chain if there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \) acting only on the \( B \)-part that perfectly reconstructs \( \rho_{ABC} \) from \( \rho_{AB} \). To achieve an approximate reconstruction, it suffices that the conditional mutual information \( I(A:C|B)_\rho \) is small, as shown recently. Here we ask what conditions are necessary for approximate state reconstruction. This is answered by a lower bound on the relative entropy between \( \rho_{ABC} \) and the recovered state \( \mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \). The bound consists of the conditional mutual information and an entropic correction term that quantifies the disturbance of the \( B \)-part by the recovery map.

1 Introduction

A recovery map is a trace-preserving completely positive map that reconstructs parts of a composite system. More precisely, for a tripartite state \( \rho_{ABC} \) on \( A \otimes B \otimes C \) we can consider a recovery map \( \mathcal{R}_{B \rightarrow BC} \) from \( B \) to \( B \otimes C \) that reconstructs the \( C \)-part from the \( B \)-part only. If such a reconstruction is perfectly possible, i.e., if

\[
\rho_{ABC} = \mathcal{R}_{B \rightarrow BC}(\rho_{AB})
\]  

we call \( \rho_{ABC} \) a (quantum) Markov chain in order \( A \leftrightarrow B \leftrightarrow C \).\(^1\)

The structure of Markov chains is well understood. A state \( \rho_{ABC} \) is a Markov chain if and only if there exists a decomposition of the \( B \) system as \( B = \bigoplus_j (b_j^L \otimes b_j^R) \) such that

\[
\rho_{ABC} = \bigoplus_j P(j) \rho_{Ab_j^L} \otimes \rho_{b_j^R C} \, ,
\]  

with states \( \rho_{Ab_j^L} \) on \( A \otimes b_j^L \), \( \rho_{b_j^R C} \) on \( b_j^R \otimes C \), and a probability distribution \( P \).\(^2\) Alternatively, a state \( \rho_{ABC} \) is a Markov chain if and only if its conditional mutual information \( I(A:C|B)_\rho \) vanishes\(^2\), where \( I(A:C|B)_\rho := H(AB)_\rho + H(BC)_\rho - H(B)_\rho - H(ABC)_\rho \) and \( H(A)_\rho := -\text{tr} \rho_A \log \rho_A \) is the von Neumann entropy of the marginal state on \( A \). Furthermore, in case \( I(A:C|B)_\rho = 0 \) the Petz recovery map (also known as transpose map)

\[
\mathcal{T}_{B \rightarrow BC} : X_B \mapsto \rho_B^{\frac{1}{2}} X_B \rho_B^{\frac{1}{2}} \otimes \text{id}_C \rho_B^{\frac{1}{2}}
\]  

reverses perfectly, i.e., (1) holds with \( \mathcal{R}_{B \rightarrow BC} = \mathcal{T}_{B \rightarrow BC} \).

Tripartite states \( \rho_{ABC} \) that have a small conditional mutual information are called approximate Markov chains. The justification for this terminology is a recent result\(^1\) proving that for any state \( \rho_{ABC} \) there exists a recovery map \( \mathcal{R}_{B \rightarrow BC} \) such that

\[
I(A:C|B)_\rho \geq -\log F(\rho_{ABC}, \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \, ,
\]  

\(^1\)We usually omit the identity map \( \mathcal{I} \) in our notation when its use is clear from the context. For example, we write \( \mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \) instead of \( (\mathcal{I}_A \otimes \mathcal{R}_{B \rightarrow BC})(\rho_{AB}) \). We will drop the order of the Markov chain if it is \( A \leftrightarrow B \leftrightarrow C \).
where \( F(\tau, \omega) := \|\sqrt{\tau} \sqrt{\omega}\|_1^2 \) denotes the \emph{fidelity} between \( \tau \) and \( \omega \).\footnote{Recall that for any two states \( \tau \) and \( \omega \) we have \( F(\tau, \omega) \in [0,1] \) and that \( F(\tau, \omega) = 1 \) if and only if \( \tau = \omega \) (see, e.g., \cite{24}).} Inequality (4) shows that the Markov property (1) approximately holds whenever the conditional mutual information is small. However, there exist tripartite states with a small conditional mutual information whose distance to any Markov chain is nevertheless large \cite{10, 17}. As a consequence, approximate quantum Markov chains are not necessarily close to quantum Markov chains. We refer to Appendix A for a more detailed explanation of this phenomenon.

Inequality (4) has been refined in a series of works \cite{8, 5, 31, 36, 32, 18, 30}. The most recent (and strongest) result \cite[Theorem 4.1]{30} shows that for any state \( \rho_{ABC} \) we have
\[
I(A : C | B)_{\rho} \geq D_{\mathcal{M}}(\rho_{ABC} \| \mathcal{P}_{B \rightarrow BC}(\rho_{AB})) \tag{5}
\]
for the explicit recovery map
\[
\mathcal{P}_{B \rightarrow BC}(\cdot) := \int_{-\infty}^{\infty} \beta_0(dt) \mathcal{P}_B^{[t]}(\cdot) \quad \text{with} \quad \mathcal{P}_B^{[t]}(\cdot) = \rho_{BC}(\rho_B^{-1+i t X_B \rho_B^{-1+i t} \otimes \text{id}_C}) \tag{6}
\]
and the probability measure
\[
\beta_0(dt) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1} dt \tag{7}
\]
on \( \mathbb{R} \). \( D_{\mathcal{M}} \) denotes the \emph{measured relative entropy}, which is defined as
\[
D_{\mathcal{M}}(\rho \| \sigma) := \sup_{M \in \mathcal{M}} D(M(\rho) \| M(\sigma)) \tag{8}
\]
where \( \mathcal{M} \) is the set of all classical-quantum channels \( M(\omega) = \sum_x \langle x | M_x \omega | x \rangle | x \rangle \langle x | \) with \( \{ M_x \} \) an orthonormal basis. A simple property of the measured relative entropy ensures that \( D_{\mathcal{M}}(\tau \| \omega) \geq -\log F(\tau, \omega) \) for all states \( \tau, \omega \) \cite{31}. Inequality (16) which shows that (5) implies (4). We further note that the recovery map \( \mathcal{P}_{B \rightarrow BC} \) given in (6) is \emph{universal} in the sense that it only depends on \( \rho_{BC} \) and it satisfies \( \mathcal{P}_{B \rightarrow BC}(\rho_B) = \rho_{BC} \).

Inequality (4) shows that there always exists a recovery map whose recovery quality (measured in terms of the logarithm of the fidelity) is of the order of the conditional mutual information. This shows that a small conditional mutual information is a sufficient condition for a state to be approximately recoverable. In other words, (4) gives an entropic characterization for the set of tripartite states that can be approximately recovered.

In this work, we are interested in an opposite statement. This corresponds to an inequality that bounds the distance between \( \rho_{ABC} \) and any reconstructed state \( \mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \) from below with an entropic functional of \( \rho_{ABC} \) and the recovery map \( \mathcal{R}_{B \rightarrow BC} \) that involves the conditional mutual information. Such an inequality is the converse to (4), and gives a necessary condition for approximate recoverability.

### 1.1 Main result

For any trace-preserving completely positive map \( \mathcal{E} \) on a system \( S \) we denote by \( \text{Inv}(\mathcal{E}) \) the set of density operators \( \tau \) on \( S \) which are left invariant under the action of \( \mathcal{E} \), i.e.,
\[
\text{Inv}(\mathcal{E}) := \{ \tau : \mathcal{E}(\tau) = \tau \} \tag{9}
\]
We may now quantify the deviation of any state \( \rho \) from the set \( \text{Inv}(\mathcal{E}) \) by
\[
\Lambda_{\mathcal{E}}(\rho) := \inf_{\tau \in \text{Inv}(\mathcal{E})} D_{\text{max}}(\rho \| \tau) \tag{10}
\]
where $D_{\text{max}}(\omega | \sigma) := \inf \{ \lambda \in \mathbb{R} : \omega \leq 2^\lambda \sigma \}$ denotes the max-relative entropy. The $\Lambda_{\text{max}}$-quantity has the property that it is zero if and only if $\mathcal{E}$ leaves $\rho$ invariant, i.e.,

$$\Lambda_{\text{max}}(\rho \| \mathcal{E}) = 0 \iff \mathcal{E}(\rho) = \rho .$$

**Main result.** We prove that for any state $\rho_{ABC}$ on $A \otimes B \otimes C$ and any recovery map $\mathcal{R}_{B \rightarrow BC}$ from the $B$ system to the $B \otimes C$ system we have

$$D(\rho_{ABC} \| \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \geq I(A : C | B)_\rho - \Lambda_{\text{max}}(\rho_{AB} \| \mathcal{R}_{B \rightarrow B}) ,$$

where $D(\tau | \sigma) := \text{tr} \tau \log \tau - \text{tr} \tau \log \sigma$ if $\text{supp}(\tau) \subseteq \text{supp}(\sigma)$ and $+\infty$ otherwise denotes the relative entropy, and $\mathcal{R}_{B \rightarrow B} := \mathcal{R}_{C \leftarrow B} \circ \mathcal{R}_{B \rightarrow BC}$ is the action of the recovery map $\mathcal{R}_{B \rightarrow BC}$ on $B$. We refer to Theorem 3.1 for a more precise statement.

**Cases where the $\Lambda_{\text{max}}$-term vanishes.** To interpret the term $\Lambda_{\text{max}}$ in (12), note that the recovery map $\mathcal{R}_{B \rightarrow BC}$ generally not only reads the content of system $B$ in order to generate $C$, but also disturbs it. $\Lambda_{\text{max}}$ quantifies the amount of this disturbance of $B$, taking system $A$ as a reference. In particular, $\Lambda_{\text{max}}(\rho_{AB} \| \mathcal{R}_{B \rightarrow B}) = 0$ if $\mathcal{R}_{B \rightarrow BC}$ is “read only” on $B$, i.e., if $\rho_{AB} = \mathcal{R}_{B \rightarrow B}(\rho_{AB})$. Inequality (12) then simplifies to

$$D(\rho_{ABC} \| \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \geq I(A : C | B)_\rho .$$

We further note that in case $\mathcal{R}_{B \rightarrow BC}$ is a recovery map that is “read only” on $B$ its output state $\sigma_{ABC} := \mathcal{R}_{B \rightarrow BC}(\rho_{AB})$ is a Markov chain since

$$H(A | B)_\rho \leq H(A | BC)_\sigma \leq H(A | B)_\sigma = H(A | B)_\rho ,$$

where the two inequality steps follow from the data-processing inequality [19, 20] applied for $\mathcal{R}_{B \rightarrow BC}$ and $\text{tr}_C$, respectively and hence $I(A : C | B)_\sigma = H(A | B)_\sigma - H(A | BC)_\sigma = 0$.

### 1.2 Tightness of the main result

We next discuss several aspects concerning the tightness of (12). This will also give a better understanding about the role of the $\Lambda_{\text{max}}$-term. We first note that by combining (5) with (12) we obtain

$$D_{\text{max}}(\rho_{ABC} | \mathcal{P}_{B \rightarrow BC}(\rho_{AB})) \leq I(A : C | B)_\rho \leq \inf_{\mathcal{R}_{B \rightarrow BC}} \left\{ D(\rho_{ABC} \| \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) + \Lambda_{\text{max}}(\rho_{AB} \| \mathcal{R}_{B \rightarrow B}) \right\} ,$$

where the recovery map $\mathcal{P}_{B \rightarrow BC}$ on the left-hand side is given by (6) and the infimum is over all recovery maps $\mathcal{R}_{B \rightarrow BC}$ that map $B$ to $B \otimes C$. The main difference between the lower and upper bound for the conditional mutual information given by (15) and (16), respectively, is the $\Lambda_{\text{max}}$-term. In the following we will discuss the necessity and optimality of this term.

**Classical case.** Inequalities (15) and (16) hold with equality in case $\rho_{ABC}$ is a classical state, i.e.,

$$\rho_{ABC} = \sum_{a, b, c} P_{ABC}(a, b, c) |a\rangle \langle a|_A \otimes |b\rangle \langle b|_B \otimes |c\rangle \langle c|_C ,$$

for some probability distribution $P_{ABC}$. To see this, we first note that if $\rho_{ABC}$ is classical (in which case $\rho_{ABC}$ and all its marginals commute pairwise) a straightforward calculation gives

$$I(A : C | B)_\rho = D(\rho_{ABC} \| \mathcal{T}_{B \rightarrow BC}(\rho_{AB})) ,$$

\footnote{Note that the max-relative entropy has a definiteness property which ensures that for a sequence $(\omega_k)_{k \in \mathbb{N}}$ of states such that $\lim_{k \to \infty} D_{\text{max}}(\tau \| \omega_k) = 0$ we have $\lim_{k \to \infty} \omega_k = \tau$. This follows from the fact that $-\log F(\tau, \omega) \leq D_{\text{max}}(\tau \| \omega)$ [2, 3] and the definiteness property of the fidelity $[34, 1]$, i.e., $\lim_{k \to \infty} F(\tau, \omega_k) = 1$ implies $\lim_{k \to \infty} \omega_k = \tau$.}
for the Petz recovery map $T_{B \rightarrow BC}$ defined in (3). Furthermore, if $\rho_{ABC}$ is classical $T_{B \rightarrow BC}(\rho_{AB}) = \rho_{BC} \rho_B^{-1} \rho_{AB}$. We further see that $\text{tr} C T_{B \rightarrow BC}(\rho_{AB}) = T_B(\rho_{AB}) = \rho_{AB}$ and hence

$$\Lambda_{\max}(\rho_{AB} \| T_B \| B) = 0. \quad (19)$$

This shows that in the classical case (16) is an equality and that the Petz recovery map $T_{B \rightarrow BC}$ minimizes the right-hand side of (16).

We further note that in the classical case the measured relative entropy coincides with the relative entropy and the rotated Petz recovery map $\rho_{B \rightarrow BC}$ that satisfies (15) simplifies to the Petz recovery map $T_{B \rightarrow BC}$. This together with (18) then shows that (15) holds with equality in the classical case.

**Necessity of the $\Lambda_{\max}$-term.** A natural question regarding (12) is whether the $\Lambda_{\max}$-term is necessary. Here we show that this is indeed the case by constructing an example proving that a large conditional mutual information does not imply that all recovery maps are bad and hence the $\Lambda_{\max}$-term is indispensable.

More precisely, in Section 4.1 we construct a generic example showing that for any constant $\kappa < \infty$ there exists a classical state $\rho_{ABC}$ (i.e., a state of the form (17)) such that

$$\kappa D_{\max}(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB})) < I(A : C | B)_{\rho}, \quad (20)$$

for some recovery map $R_{B \rightarrow BC}$ that satisfies $R_{B \rightarrow BC}(\rho_B) = \rho_{BC}$. A similar construction (also given in Section 4.1) shows that there exists another classical state $\rho_{ABC}$ such that

$$\kappa D_{\max}(R_{B \rightarrow BC}(\rho_{AB}) \| \rho_{ABC}) < I(A : C | B)_{\rho}, \quad (21)$$

for some recovery map $R_{B \rightarrow BC}$ that satisfies $R_{B \rightarrow BC}(\rho_B) = \rho_{BC}$.

These constructions therefore show that the term $\Lambda_{\max}(\rho_{AB} \| R_{B \rightarrow B})$, which measures the deviation from a “read only” map on $B$, is necessary to obtain a lower bound on the relative entropy between a state and its reconstruction version. The example has an even stronger implication. It shows that the $\Lambda_{\max}$-term is necessary even if one tries to bound the max-relative entropy between a state and its reconstruction version, i.e., $D_{\max}(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB}))$, which cannot be smaller than $D(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB}))$, from below. The two strict inequalities (20) and (21) show that the $\Lambda_{\max}$-term is also necessary if one would allow for swapping the two arguments of the relative (or even max-relative) entropy. Furthermore, restricting the set of recovery maps such that they satisfy $R_{B \rightarrow BC}(\rho_B) = \rho_{BC}$ still requires the $\Lambda_{\max}$-term.

Since for classical states (18) holds, these examples also show that for the task of minimizing the relative entropy between $\rho_{ABC}$ and its reconstructed state $R_{B \rightarrow BC}(\rho_{AB})$ the Petz recovery map can be far from being optimal — even in the classical case. The examples further show that considering recovery maps that leave the $B$ system invariant (i.e., they only “read” the $B$-part) is a considerable restriction. We refer to Section 4.1 for more information about these examples.

**Optimality of the $\Lambda_{\max}$-term.** Even in the case where $\rho_{ABC}$ is not classical, (12) is still close to optimal. We present two arguments why this is the case. First, we show that the $\Lambda_{\max}$-term cannot be replaced by a relative entropy measure that is smaller than the max-relative entropy. More precisely, (12) is violated if the max-relative entropy in the definition of $\Lambda_{\max}(\rho_{AB} \| R_{B \rightarrow B})$ is replaced with any $\alpha$-Rényi relative entropy for any $\alpha \in \left(\frac{1}{2}, \infty\right]$. We refer to Section 4.2 for more information.

Second, we show that the $\Lambda_{\max}$-term cannot be defined as a distance between $\rho_{AB}$ and $R_{B \rightarrow B}(\rho_{AB})$.

The $\Lambda_{\max}$-term in (12) quantifies the (max-relative entropy) distance between $\rho_{AB}$ and its closest state that is invariant under $R_{B \rightarrow B}$. A natural question is if (12) remains valid if the $\Lambda_{\max}$-term is replaced by the (max-relative entropy) distance between $\rho_{AB}$ and $R_{B \rightarrow B}(\rho_{AB})$, i.e., $D_{\max}(\rho_{AB} \| R_{B \rightarrow B}(\rho_{AB}))$.

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4 The max-relative entropy and its properties are discussed in more detail in Section 2. It is the largest sensible relative entropy measure.
This however is ruled out. To see this we recall that by the example presented above in (20) there exists a tripartite state $\rho_{ABC}$ and a recovery map $R_{B \rightarrow BC}$ such that
\begin{equation}
2D_{\text{max}}(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB})) < I(A : C|B)_{\rho}.
\end{equation}
The data-processing inequality for the max-relative entropy [13, 33] and the fact that the max-relative entropy cannot be smaller than the relative entropy then imply
\begin{equation}
D(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB})) < I(A : C|B)_{\rho} - D_{\text{max}}(\rho_{AB} \| R_{B \rightarrow B}(\rho_{AB})) ,
\end{equation}
which shows that (12) is no longer valid for the modified $\Lambda_{\text{max}}$-term described above.

1.3 Related results

Using the continuity of the conditional entropy, it is possible to derive an upper bound for the conditional mutual information of a state $\rho_{ABC}$ in terms of its distance to any reconstructed state $\sigma_{ABC} := R_{B \rightarrow BC}(\rho_{AB})$, where $R_{B \rightarrow BC}$ denotes an arbitrary recovery map [4, 15]. This leads to a lower bound on the relative entropy between $\rho_{ABC}$ and $R_{B \rightarrow BC}(\rho_{AB})$ that however depends on the dimension of the $A$ system. To see this, let $[0,1] \ni x \mapsto h(x) := -x \log x - (1-x) \log(1-x)$ denote the binary entropy function and let $\Delta(\tau,\omega) := \frac{1}{2} \| \tau - \omega \|_1$ be the trace distance between $\tau$ and $\omega$. The data-processing inequality [19, 20] implies that
\begin{equation}
I(A : C|B)_{\rho} = H(A|B)_{\rho} - H(A|BC)_{\rho} \leq H(A|BC)_{\sigma} - H(A|BC)_{\rho} .
\end{equation}
By the improved Alicki-Fannes inequality [38, Lemma 2] we find
\begin{equation}
I(A : C|B)_{\rho} \leq 2\Delta(\rho,\sigma) \log(\text{dim } A) + (1 + \Delta(\rho,\sigma)) h\left(\frac{\Delta(\rho,\sigma)}{1 + \Delta(\rho,\sigma)}\right)
\end{equation}
\begin{equation}
\leq 2\sqrt{\Delta(\rho,\sigma)} \left( \log(\text{dim } A) + 1 \right) ,
\end{equation}
where we used that $(1 + x)h\left(\frac{1}{1+x}\right) \leq 2\sqrt{x}$ for all $x \in [0,1]$ and $\Delta(\rho,\sigma) \in [0,1]$. Together with Pinsker’s inequality [28, 12] this gives
\begin{equation}
D(\rho_{ABC} \| R_{B \rightarrow BC}(\rho_{AB})) \geq \frac{2}{\ln 2} \Delta(\rho_{ABC}, R_{B \rightarrow BC}(\rho_{AB})) \geq \frac{I(A : C|B)_{\rho}^2}{8 \ln 2 (\log(\text{dim } A) + 1)} .
\end{equation}
The fact that this bound explicitly depends on the dimension of the system $A$ is unsatisfactory. Furthermore, the example discussed above in (20) shows that such a dependence on the dimension is unavoidable.

A different approach to derive an upper bound for the conditional mutual information of a state $\rho_{ABC}$ in terms of its distance to a reconstructed state $R_{B \rightarrow BC}(\rho_{AB})$ was taken in [14, Theorem 11 and Remark 12] (see also [30, Proposition F.1]). It was shown that for any state $\rho_{ABC}$
\begin{equation}
\int_{-\infty}^{\infty} \beta_0(dt) D_{\tau}(\rho_{ABC} \| P_{B \rightarrow BC}^{[t]}(\rho_{AB})) \geq I(A : C|B)_{\rho} ,
\end{equation}
where $\beta_0$ and $P_{B \rightarrow BC}^{[t]}$ are given in (7) and (6), respectively and $D_{\tau}(\tau\|\omega) := \log \text{tr} \tau^2 \omega^{-1}$ denotes Petz’ Rényi relative entropy of order 2 [25]. The examples discussed above imply that the left-hand side of (28) can be much larger than the relative entropy between $\rho_{ABC}$ and $R_{B \rightarrow BC}(\rho_{AB})$ for the optimal recovery map $R_{B \rightarrow BC}$. In other words, rotated Petz recovery maps are generally far from optimal recovery maps.

2 One-shot relative entropies

The goal of this section is to derive a triangle-like inequality for the relative entropy (see Lemma 2.1) which will be used in the proof of our main result, i.e., Theorem 3.1. To understand Lemma 2.1 we need to review a few properties of one-shot relative entropy measures.
2.1 Preliminaries

Let $S(A)$ and $P(A)$ denote the set of density and nonnegative operators on $A$, respectively. For any linear operator $L$ on $A$, the trace norm is given by $\|L\|_1 := \text{tr}|L|$ with $|L| := \sqrt{L^*L}$. For $\rho, \sigma \in P(A)$ we write $\rho \ll \sigma$ if the support of $\rho$ is contained in the support of $\sigma$. Within this document our Hilbert spaces are assumed to be separable. We define the min-relative entropy \cite{29} as

$$D_{\min}(\rho\|\sigma) := -\log \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = -\log F(\rho, \sigma)$$

(29)

and the max-relative entropy \cite{13, 29} as

$$D_{\max}(\rho\|\sigma) := \inf\{\lambda \in \mathbb{R} : \rho \leq 2^\lambda \sigma\}.$$  

(30)

As the names suggest, the min-relative entropy cannot be larger than the max-relative entropy, or more precisely we have

$$D_{\min}(\rho\|\sigma) \leq D(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma),$$

(31)

with strict inequalities in the generic case \cite{23, 33}. The max-relative entropy turns out to be the largest relative entropy measure that satisfies the data-processing inequality and is additive under tensor products \cite[Section 4.2.4]{33}.

The min- and max-relative entropy can be seen as the extreme points of a family of relative entropies called minimal quantum Rényi relative entropy (also known as sandwiched Rényi relative entropy) \cite{23, 37}. For $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty)$ and $\rho, \sigma \in P(A)$, this family is defined as

$$D_\alpha(\rho\|\sigma) := \left\{ \begin{array}{ll} \frac{1}{\alpha-1} \log \frac{1}{\text{tr}_\rho} \text{tr} \left( \sigma^{\frac{1}{1-\alpha}} \rho \sigma^{\frac{1}{\alpha}} \right)^\alpha & \text{if } \rho \ll \sigma \land \alpha < 1 \\ \infty & \text{otherwise}. \end{array} \right.$$  

(32)

It can be shown \cite{23} that

$$D_{\frac{1}{2}}(\rho\|\sigma) = D_{\min}(\rho\|\sigma), \quad \lim_{\alpha \rightarrow 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \quad \text{and} \lim_{\alpha \rightarrow \infty} D_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma).$$

(33)

Furthermore the minimal quantum Rényi relative entropy is monotone in $\alpha \in [\frac{1}{2}, \infty)$ \cite[Theorem 7]{23}, i.e.,

$$D_\alpha(\rho\|\sigma) \leq D_{\alpha'}(\rho\|\sigma) \quad \text{for } \alpha \leq \alpha'.$$

(34)

2.2 Triangle-like inequality for relative entropy

It is well-known that the relative entropy does not satisfy the triangle inequality. For the three (classical) qubit states $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}\text{id}_2$, $\sigma = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}\text{id}_2$, and $\omega = \frac{1}{2}\text{id}_2$ we have $D(\rho\|\sigma) > D(\rho\|\omega) + D(\omega\|\sigma)$. The following lemma proves a triangle-like inequality for the minimal quantum Rényi relative entropy.

Lemma 2.1. Let $\rho \in S(A)$, $\sigma, \omega \in P(A)$ and let $\alpha \in [\frac{1}{2}, \infty)$. Then

$$D_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\omega) + D_{\max}(\omega\|\sigma).$$

(35)

Proof. For $\alpha \in [\frac{1}{2}, 1)$, the function $t \mapsto t^\frac{1}{1-\alpha}$ is operator monotone on $[0, \infty)$ \cite[Theorem V.1.9]{6}. Furthermore, the function $X \mapsto \text{tr}X^\alpha$ is monotone on the set of Hermitian operators \cite[Theorem 2.10]{9}. By definition of the max-relative entropy we find

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \log \text{tr} \left( \rho^{\frac{1}{1-\alpha}} \sigma^{\frac{1}{1-\alpha}} \rho^{\frac{1}{1-\alpha}} \right)^\alpha \leq D_\alpha(\rho\|\omega) + D_{\max}(\omega\|\sigma).$$

(36)

For $\alpha \in (1, \infty)$ the argument is exactly the same, where we note that $t \mapsto t^\frac{1}{-\alpha}$ is operator antimonotone \cite{33}. The case $\alpha = 1$ then follows by continuity. \qed
The following remarks show that Lemma 2.1 is optimal and that there is not much flexibility to prove triangle-like inequalities for the relative entropy different than (35).

**Remark 2.2.** Lemma 2.1 is optimal in the sense that (35) is no longer valid if $D_{\max}$ is replaced with $D_\alpha$ for any $\alpha \in [\frac{3}{2}, \infty)$. To see this, let $p \in (0, 1)$ and consider three classical distributions on $\{0, 1\} \times \{0, 1\}$ defined by

$$
P_{XY}(x, y) := \begin{cases} \frac{p}{1-p} & \text{if } x = y = 0 \\ \frac{1-p}{3} & \text{otherwise} \end{cases}, \quad Q_{XY}(x, y) := \begin{cases} \frac{p}{1-p} & \text{if } x = y = 1 \\ \frac{1-p}{3} & \text{otherwise} \end{cases}, \quad S_{XY}(x, y) = \frac{1}{4}.$$  

(37)

A simple calculation shows that

$$D(P||Q) = \frac{4p-1}{3} \log \frac{3p}{1-p}$$

(38)

$$D(P||S) = p \log 4p + (1-p) \log \frac{4(1-p)}{3}$$

(39)

$$D_\alpha(S||Q) = \frac{1}{\alpha-1} \log \left( \frac{3^\alpha}{4^\alpha(1-p)^\alpha} + \frac{1}{4^\alpha p^{\alpha-1}} \right).$$

(40)

Choosing $p = 1 - 2^{-\alpha}$ reveals that

$$D(P||Q) > D(P||S) + D_\alpha(S||Q) \quad \text{for all } \alpha \in \left[ \frac{3}{2}, \infty \right).$$

(41)

In the limit $\alpha \to \infty$ the strict inequality (41) becomes an equality.

**Remark 2.3.** The statement of Lemma 2.1 is no longer true if the max-relative entropy and the relative entropy on the right-hand side of (35) are exchanged. To see this consider the three classical binary probability distributions

$$P(x) := \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{otherwise} \end{cases}, \quad Q(x) := \begin{cases} 1-\varepsilon & \text{if } x = 0 \\ \varepsilon & \text{otherwise} \end{cases}, \quad S(x) := \begin{cases} 1-p & \text{if } x = 0 \\ \frac{p}{2} & \text{otherwise} \end{cases},$$

(42)

with $p, \varepsilon \in (0, 1)$. This gives

$$D(P||Q) = (1-p) \log \frac{1-p}{1-\varepsilon} + p \log \frac{p}{\varepsilon}$$

(43)

$$D(S||Q) = \frac{2-p}{2} \log \frac{2-p}{2-2\varepsilon} + \frac{1-p}{2} \log \frac{p}{2\varepsilon}$$

(44)

$$D_{\max}(P||S) = \max \left\{ \log \frac{2(1-p)}{2-p}, \log 2 \right\} = 1.$$  

(45)

For $p = \frac{7}{8}$ and $\varepsilon = \frac{1}{8}$ we find that

$$D(P||Q) > D_{\max}(P||S) + D(S||Q).$$

(46)

This shows that it is crucial which term in Lemma 2.1 carries a max-relative entropy.

**Remark 2.4.** The relative entropy satisfies a triangle-like inequality different from Lemma 2.1. For the log-Euclidean $\alpha$-Rényi divergence $D^\alpha_\alpha(\omega||\sigma) := \frac{1}{1-\alpha} \log \text{tr} \rho^{\alpha} \log^{1-\alpha} \rho + \log^{\alpha} \sigma$ it is known [22] that

$$D(\rho||\sigma) \leq \frac{\alpha}{\alpha-1} D(\rho||\omega) + D^\alpha_\alpha(\omega||\sigma) \quad \text{for } \alpha \in (1, \infty).$$

(47)

We also note that $D^\infty_\alpha(\omega||\sigma) \leq D_{\max}(\omega||\sigma)$ which shows that in the limit $\alpha \to \infty$ we obtain Lemma 2.1 for the case $\alpha = 1$. 

7
3 Main result and proof

**Theorem 3.1.** Let $A$, $B$, and $C$ be separable Hilbert spaces, let $\rho_{ABC} \in S(A \otimes B \otimes C)$, and let $\mathcal{R}_{B \rightarrow BC}$ be a trace-preserving completely positive map from $B$ to $B \otimes C$. Then

\[
D(\rho_{ABC} \Vert \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \geq I(A : C|B)_{\rho} - \Lambda_{\text{max}}(\rho_{AB} \Vert \mathcal{R}_{B \rightarrow BC}).
\]  

(48)

The quantity $\Lambda_{\text{max}}(\rho_{AB} \Vert \mathcal{R}_{B \rightarrow B})$ is defined in (10) and $\mathcal{R}_{B} := \text{tr}_{C} \circ \mathcal{R}_{B \rightarrow BC}$. To prove the assertion of Theorem 3.1 we make use of a known lemma stating that the conditional mutual information of a tripartite density operator is bounded from above by the smallest relative entropy distance to Markov chains. Let $\text{MC}(A \otimes B \otimes C)$ denote the set of Markov chains on $A \otimes B \otimes C$.

**Lemma 3.2** ([17, Theorem 4]). Let $\rho_{ABC} \in S(A \otimes B \otimes C)$. Then

\[
I(A : C|B)_{\rho} \leq \inf_{\mu \in \text{MC}} D(\rho_{ABC} \Vert \mu_{ABC}).
\]  

(49)

**Proof of Lemma 3.2.** We provide a proof for the known statement of Lemma 3.2 since the original proof given in [17, Theorem 4] has been simplified considerably (see the short note after the acknowledgements in [17]). The resulting proof is elegant and leads to a good understanding of the assertion of the lemma.

By definition of the relative entropy and the conditional mutual information we find for all $\rho_{ABC} \in S(A \otimes B \otimes C)$ and all $\mu_{ABC} \in \text{MC}(A \otimes B \otimes C)$

\[
D(\rho_{ABC} \Vert \mu_{ABC}) + D(\rho_{B} \Vert \mu_{B}) - D(\rho_{AB} \Vert \mu_{AB}) - D(\rho_{BC} \Vert \mu_{BC}) = I(A : C|B)_{\rho} + \nu,
\]  

(50)

where

\[
\nu := \text{tr} \rho_{ABC} \log \mu_{ABC} + \text{tr} \rho_{B} \log \mu_{B} - \text{tr} \rho_{AB} \log \mu_{AB} - \text{tr} \rho_{BC} \log \mu_{BC}.
\]  

(51)

Using the structure of Markov chains given by (2) we have that

\[
\mu_{ABC} = \bigoplus_{j} P(j) \mu_{AB}^{j} \otimes \mu_{b_{j} C} \quad \text{for} \quad B = \bigoplus_{j} b_{j}^{R} \otimes b_{j}^{R},
\]  

(52)

with $\mu_{AB}^{j} \in S(A \otimes b_{j}^{R})$ and $\mu_{b_{j} C} \in S(b_{j}^{R} \otimes C)$. A straightforward calculation then shows that $\nu = 0$. The data-processing inequality [21, 35] ensures that $D(\rho_{B} \Vert \mu_{B}) \leq D(\rho_{AB} \Vert \mu_{AB})$ and the nonnegativity of the relative entropy (follows by Klein’s inequality) guarantees that $D(\rho_{BC} \Vert \mu_{BC}) \geq 0$. This then completes the proof. \( \square \)

In order to prove Theorem 3.1 we need one more lemma that relates the distance to Markov chains with the $\Lambda_{\text{max}}$-quantity defined in (10).

**Lemma 3.3.** Let $\rho_{AB} \in S(A \otimes B)$ and $\mathcal{R}_{B \rightarrow BC}$ be a trace-preserving completely positive map. Then

\[
\inf_{\mu \in \text{MC}} D_{\text{max}}(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \Vert \mu_{ABC}) \leq \Lambda_{\text{max}}(\rho_{AB} \Vert \mathcal{R}_{B \rightarrow B}),
\]  

(53)

where $\mathcal{R}_{B \rightarrow B} = \text{tr}_{C} \circ \mathcal{R}_{B \rightarrow BC}$ is the reduction of $\mathcal{R}_{B \rightarrow BC}$ to the output space $B$.

**Proof of Lemma 3.3.** The data processing inequality for the max-relative entropy [13, 33] implies that

\[
\inf_{\rho_{ABC}} \{ D_{\text{max}}(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \Vert \mu_{ABC}) : \mu_{ABC} \in \text{MC} \}
\]

\[
\leq \inf_{\rho_{ABC}, \tau_{AB}} \{ D_{\text{max}}(\mathcal{R}_{B \rightarrow BC}(\rho_{AB}) \Vert \mathcal{R}_{B \rightarrow BC}(\tau_{AB})) : \mathcal{R}_{B \rightarrow BC}(\tau_{AB}) \in \text{MC}, \tau_{AB} \in S(A \otimes B) \}
\]

\[
\leq \inf_{\tau_{AB}} \{ D_{\text{max}}(\rho_{AB} \Vert \tau_{AB}) : \mathcal{R}_{B \rightarrow BC}(\tau_{AB}) \in \text{MC}, \tau_{AB} \in S(A \otimes B) \}.
\]  

(54)

(55)
Furthermore, because the data processing inequality for the conditional entropy [19, 20] implies that 
\[ H(A|BC)_{R_{B→BC}(τ_{AB})} \geq H(A|B)_{τ_{AB}} \] 
for any \( τ_{AB} \in S(A \otimes B) \), we also have 
\[ τ_{AB} \in \text{Inv}(R_{B→B}) \implies H(A|BC)_μ \geq H(A|B)_μ \] 
for \( μ_{ABC} = R_{B→BC}(τ_{AB}) \). \( \text{(56)} \)

Note that the inequality on the right hand side of the implication must, again by the data processing 
inequality, be an equality, which means that \( I(A : C|B)_μ = 0 \) and, hence, that \( μ = MC \). This proves 
the general implication 
\[ τ_{AB} \in \text{Inv}(R_{B→B}) \implies R_{B→BC}(τ_{AB}) \in MC. \] \( \text{(57)} \)

We now use it to obtain 
\[ Λ_{max}(ρ_{AB}||R_{B→B}) = \inf_{τ_{AB}} \{ D_{max}(ρ_{AB}||τ_{AB}) : τ_{AB} \in \text{Inv}(R_{B→B}) \} \] 
\[ \geq \inf_{τ_{AB}} \{ D_{max}(ρ_{AB}||τ_{AB}) : R_{B→BC}(τ_{AB}) \in MC, τ_{AB} \in S(A \otimes B) \}. \] \( \text{(58)} \)

Combining this with (55) completes the proof. \( \square \)

**Proof of Theorem 3.1.** Let \( μ_{ABC} \) be a Markov chain. Combining Lemma 3.2 with Lemma 2.1 applied 
for \( α = 1, ρ = μ_{ABC}, σ = μ_{ABC} \) and \( ω = R_{B→BC}(ρ_{AB}) \) gives 
\[ D(ρ_{ABC}||R_{B→BC}(ρ_{AB})) \geq I(A : C|B)_ρ - \inf_{μ \in MC} D_{max}(R_{B→BC}(ρ_{AB})||μ_{ABC}). \] \( \text{(60)} \)

Lemma 3.3 then proves the assertion of Theorem 3.1.5

### 4 On the tightness of the main result

In this section we construct examples that show two things. First, there exist classical tripartite states 
with a large conditional mutual information that, however, can be recovered well. This shows the 
necessity of the \( Λ_{max} \)-term in the main bound (48) — even if the relative entropy was replaced by the 
largest possible relative entropy measure, i.e., the max-relative entropy. Furthermore, the violation 
of such a bound without the \( Λ_{max} \)-term can be made arbitrarily large. Second, our example shows 
that (48) is no longer valid if the max-relative entropy in the definition of \( Λ_{max}(ρ_{AB}||R_{B→B}) \) is replaced 
with any \( α \)-Rényi relative entropy for any \( α ∈ [\frac{1}{2}, ∞) \).

Both examples will be classical, i.e., we consider tripartite states of the form (17). Such states are 
special as the corresponding density operators of the states and all its marginals are simultaneously 
diagonalizable. As a result, we can use the classical notion of a distribution to describe such states.

#### 4.1 A large conditional mutual information does not imply bad recovery

Let \( X = \{1, 2, \ldots, 2^n\} \) for \( n ∈ N, p, q ∈ [0, 1] \) such that \( p + q ≤ 1 \), and consider two independent random 
variables \( E_Z \) and \( E_Y \) on \( \{0, 1\} \) and \( \{0, 1, 2\} \), respectively, such that \( P(E_Z = 0) = p + q, P(E_Y = 0) = p, \) 
and \( P(E_Y = 1) = q \). Let \( X \sim U(X) \), where \( U(X) \) denotes the uniform distribution on \( X \) and define 
two random variables by 
\[ Z := \begin{cases} X & \text{if } E_Z = 0 \\
U_Z & \text{otherwise} \end{cases} \quad \text{and} \quad Y := \begin{cases} X & \text{if } E_Y = 0 \\
Z & \text{if } E_Y = 1 \\
U_Y & \text{otherwise} \end{cases}. \] \( \text{(61)} \)

where \( U_Y \sim U(X) \) and \( U_Z \sim U(X) \) are independent. This defines a tripartite distribution \( P_{XYZ} \). A 
simple calculation reveals that 
\[ H(X|YEYZ) = pH(X|XEZ) + qH(X|ZEZ) + (1 - p - q)H(X|Y E Z) \] \( \text{(62)} \)

5We note that (60) is stronger than (48) and therefore may be of independent interest.
Similarly we find
\[
H(X|YZE_Y E_Z) = q(1 - p - q)H(X|U_Z) + (1 - p - q)H(X|U_Y) + (1 - p - q)(1 - p).
\]  

We thus obtain
\[
I(X : Z|Y)_P = H(X|Y) - H(X|YZ) \geq H(X|YE_Y E_Z) - I(X : E_Y E_Z|Y Z) - I(X : E_Y E_Z|Y Z) \geq n(1 - p - q)(p + q) - \log 6. \tag{69}
\]

We next define a recovery map \(R_{Y \rightarrow Y'}Z\) that creates a tuple of random variables \((Y', Z')\) out of \(Y\). Let the recovery map be such that
\[
(Y', Z') := (p^2 + q + pq)(Y, Y) + \frac{1}{2}(1 - p^2 - q - pq)(Y, U) + \frac{1}{2}(1 - p^2 - q - pq)(U', Y),
\]

where \(U, U'\) are independent uniformly distributed on \(X\). Let
\[
Q_{XY'Z'} := R_{Y \rightarrow Y'}Z(P_{XY})
\]
denote the distribution that is generated when applying the recovery map (described above) to \(P_{XY}\). In the following we will assume that \(n\) is sufficiently large. It can be verified easily that \(Q_{Y'Z'} = P_{YZ}\). Since \(P_{XYZ}\) and \(Q_{XY'Z'}\) are classical distributions we have
\[
D_{\text{max}}(P_{XYZ}(Q_{XY'Z'})) = \max_{x, y, z} \log \frac{P_{XYZ}(x, y, z)}{Q_{XY'Z'}(x, y, z)}.
\]

We note that \(P(X = Y) = p + pq + q^2\) according to the distribution \(P_{XY}\) and hence
\[
D_{\text{max}}(P_{XYZ}(Q_{XY'Z'}))
= \max \left\{ \log \frac{(p + q)^2}{P(X = Y)(p^2 + q + pq)}, \log \frac{(1 - p - q)q}{P(X \neq Y)(p^2 + q + pq)}, \log \frac{(p + q)(1 - p - q)}{P(X = Y)(1 - p^2 - q - pq)} \right\},
\]  

and
\[
D_{\text{max}}(Q_{XY'Z'}(P_{XYZ}))
= \max \left\{ \log \frac{P(X = Y)(p^2 + q + pq)}{(p + q)^2}, \log \frac{P(X \neq Y)(p^2 + q + pq)}{(1 - p - q)q}, \log \frac{P(X = Y)\frac{1}{2}(1 - p^2 - q - pq)}{(p + q)(1 - p - q)}, \log \frac{P(X \neq Y)(1 - p^2 - q - pq)}{(1 - p - q)^2} \right\}. \tag{73}
\]

We are now ready to state the conclusion of this example. For \(\kappa < \infty\), \(p = \frac{1}{2}\), \(q = 0\), and \(n\) sufficiently large we find by combining (69) with (72)
\[
\kappa D_{\text{max}}(P_{XYZ}(R_{Y \rightarrow Y}Z(P_{XY}))) = \kappa < \frac{n}{4} - \log 6 \leq I(X : Z|Y)_P. \tag{74}
\]

For \(\kappa < \infty\), \(p = q = \frac{1}{4}\), and \(n\) sufficiently large (69) and (73) imply
\[
\kappa D_{\text{max}}(R_{Y \rightarrow Y}Z(P_{XY})) = \kappa \log \frac{15}{8} < \frac{n}{4} - \log 6 \leq I(X : Z|Y)_P. \tag{75}
\]

This shows that there exist classical tripartite distributions \(P_{XYZ}\) with a large conditional mutual information \(I(X : Y|Z)_P\) and a recovery map \(R_{Y \rightarrow Y}Z\) such that \(R_{Y \rightarrow Y}Z(P_{XY})\) is close to \(P_{XYZ}\) and \(R_{Y \rightarrow Y}Z(P_Y) = P_{YZ}\). The closeness is measured with respect to the max-relative entropy.
4.2 Tightness of the $\Lambda_{\text{max}}$-term

In this section we construct a classical example showing that our main result, i.e., (48) is essentially tight in the sense that it is no longer valid if the max-relative entropy in the definition of $\Lambda_{\text{max}}(\rho_{AB}\|R_{B\rightarrow B})$, given in (10), is replaced with an $\alpha$-Rényi relative entropy for any $\alpha < \infty$. More precisely, for $\alpha \in [1, \infty]$ we define

$$\Lambda_{\alpha}(\rho\|\mathcal{E}) := \inf_{\tau \in \text{inv}(\mathcal{E})} D_{\alpha}(\rho\|\tau).$$

(76)

For $\alpha = \infty$ we have $\Lambda_{\infty}(\rho\|\mathcal{E}) = \Lambda_{\text{max}}(\rho\|\mathcal{E})$. In this section we show that for all $\alpha < \infty$ there exits a (classical) tripartite state $\rho_{ABC}$ and a recovery map $R_{B\rightarrow BC}$ that satisfies $R_{B\rightarrow BC}(\rho_{B}) = \rho_{BC}$ such that

$$D(\rho_{ABC}\|R_{B\rightarrow BC}(\rho_{AB})) < I(A : C|B)_{\rho} - \Lambda_{\alpha}(\rho_{AB}\|R_{B\rightarrow B}).$$

(77)

To see this consider the following classical example (where we switch to the classical notation). Let $\mathcal{S} = \{0, \ldots, 2^n - 1\}$ and consider a tripartite distribution $Q_{XYZ}$ defined via the random variables $X \sim U(\mathcal{S})$ and $Y = Z$. Let $Q'_{XYZ}$ be the distribution defined via the random variables $X \sim U(\mathcal{S})$, $Y \sim U(\mathcal{S})$ where $X$ and $Y$ are independent and $Z = (X + Y) \mod 2^n$. For $p \in [0,1]$ we define a binary random variable $E$ such that $P(E = 0) = p$. Consider the distribution

$$P_{XYZ} = \begin{cases} Q_{XYZ} & \text{if } E = 0 \\ Q'_{XYZ} & \text{if } E = 1. \end{cases}$$

(78)

We next define two recovery maps $\bar{R}_{Y\rightarrow Y',Z'}$ and $\tilde{R}_{Y\rightarrow Y',Z'}$ that create the tuples $(Y',Z')$ out of $Y$ such that

$$(Y',Z') = (Y,Y) \quad \text{and} \quad (Y',Z') = (U,(Y - U) \mod 2^n),$$

(79)

where $U \sim U(\mathcal{S})$, respectively. We then define another recovery map as

$$R_{Y\rightarrow Y',Z'} := p\bar{R}_{Y\rightarrow Y',Z'} + (1-p)\tilde{R}_{Y\rightarrow Y',Z'}.$$  

(80)

We note that the recovery map satisfies $R_{Y\rightarrow Y',Z'}(P_Y) = P_{YZ}$. A simple calculation shows that

$$H(X|YE)_{\rho} = pH(X|Y)_{Q} + (1-p)H(X|Y)_{Q'} = (1-p)n$$

and

$$H(X|YZE)_{\rho} = pH(X|YZ)_{Q} + (1-p)H(X|YZ)_{Q'} = 0.$$  

(81)

(82)

We thus find

$$I(X : Z|Y)_{\rho} = H(X|Y) - H(X|YZ) \geq H(X|YE) - H(X|YZ) - I(X : E|YZ)$$

(83)

$$\geq (1-p)n - h(p).$$

(84)

(85)

The distribution $R_{Y\rightarrow Y',Z'}(P_{XY})$ generated by applying the recovery map to $P_{XY}$ can be decomposed as

$$R_{Y\rightarrow Y',Z'}(P_{XY}) = p\left(p\bar{S}_{XYZ} + (1-p)\tilde{S}_{XYZ}\right) + (1-p)\left(p\bar{S}'_{XYZ} + (1-p)\tilde{S}'_{XYZ}\right),$$

(86)

where $\bar{S}_{XYZ} = \bar{R}_{Y\rightarrow Y',Z'}(Q_{XY})$, $\tilde{S}_{XYZ} = \bar{R}_{Y\rightarrow Y',Z'}(Q_{XY})$, $\bar{S}'_{XYZ} = \tilde{R}_{Y\rightarrow Y',Z'}(Q_{XY})$, and $\tilde{S}'_{XYZ} = \tilde{R}_{Y\rightarrow Y',Z'}(Q_{XY})$. The joint convexity of the relative entropy [11, Theorem 2.7.2] then implies

$$D(P_{XYZ}\|R_{Y\rightarrow Y',Z'}(P_{XY}))$$
\[
\leq pD(Q_{XYZ} \parallel p\tilde{S}_{XYZ} + (1-p)\tilde{S}_{XYZ}) + (1-p)D(Q'_{XYZ} \parallel p\tilde{S}'_{XYZ} + (1-p)\tilde{S}'_{XYZ})
\]  
(87)

A simple calculation shows that

\[
D(Q_{XYZ} \parallel p\tilde{S}_{XYZ} + (1-p)\tilde{S}_{XYZ}) = \sum_{x=y=z} Q_{XYZ}(x, y, z) \log \frac{Q_{XYZ}(x, y, z)}{p\tilde{S}_{XYZ}(x, y, z) + (1-p)\tilde{S}_{XYZ}(x, y, z)} 
\leq \frac{2^{-n}}{p2^{-n}} = \log \frac{1}{p}
\]  
(88)

and

\[
D(Q'_{XYZ} \parallel p\tilde{S}'_{XYZ} + (1-p)\tilde{S}'_{XYZ}) = \sum_{x,y,z=x+y \mod 2^n} Q'_{XYZ}(x, y, z) \log \frac{Q'_{XYZ}(x, y, z)}{p\tilde{S}'_{XYZ}(x, y, z) + (1-p)\tilde{S}'_{XYZ}(x, y, z)} 
\leq \frac{2^{-2n}}{p2^{-2n}} = \log \frac{1}{p}
\]  
(89)

We thus have

\[
D(P_{XYZ} \parallel R_{Y \rightarrow Y'; Y}(P_{XY})) \leq \log \frac{1}{p}
\]  
(90)

We note that the recovery map \( R_{Y \rightarrow Y'} = \text{tr}_{Z'} \circ R_{Y \rightarrow Y'; Z} \) leaves the uniform distribution \( Q'_{XY} \) invariant, i.e., \( R_{Y \rightarrow Y'}(Q'_{XY}) = Q'_{XY} \). As a result we find

\[
\Lambda_\alpha(P_{XY} \parallel R_{Y \rightarrow Y'}) \leq D_\alpha(P_{XY} \parallel Q_{XY}) = \frac{1}{\alpha - 1} \log (2^{-n}(1-p)^\alpha(2^n - 1) + 2^{-n}(1-p + p2^n)^\alpha)
\]  
(91)

where the final step follows by definition of the \( \alpha \)-Rényi relative entropy and a straightforward calculation.

Recall that we need to prove (77), which in the classical notation reads as

\[
D(P_{XYZ} \parallel R_{Y \rightarrow Y'; Y}(P_{XY})) + \Lambda_\alpha(P_{XY} \parallel R_{Y \rightarrow Y'}) < I(X : Z | Y)p,
\]  
(92)

for all \( \alpha < \infty \). As mentioned in (34), the \( \alpha \)-Rényi relative entropy is monotone in \( \alpha \) which shows that it suffices to prove (93) for all \( \alpha \in (0, \infty) \), where \( \alpha_0 \geq 0 \) can be arbitrarily large.

Combining (91) and (92) shows that for any \( \alpha \in (0, \infty) \) where \( \alpha_0 \) is sufficiently large, \( p = \alpha^{-2} \), and \( n = \alpha \)

\[
D(P_{XYZ} \parallel R_{Y \rightarrow Y'; Y}(P_{XY})) + \Lambda_\alpha(P_{XY} \parallel R_{Y \rightarrow Y'}) \leq 2 \log \alpha + \frac{1}{\alpha - 1} \log \left(1 + 2^{-\alpha}(1 + \alpha^{-2}2^{\alpha})\right)
\]  
(93)

where we used that \((1 - \alpha^{-2})^2(2^n - 1) \leq 2^n\) for \( \alpha \geq 1 \). Using the simple inequality \( \log(1+x) \leq \log x + \frac{2}{x} \) for \( x \geq 1 \) gives

\[
D(P_{XYZ} \parallel R_{Y \rightarrow Y'; Y}(P_{XY})) + \Lambda_\alpha(P_{XY} \parallel R_{Y \rightarrow Y'})
\leq 2 \log \alpha - \frac{\alpha}{\alpha - 1} + \frac{\alpha}{\alpha - 1} \log \left(1 + \frac{2^\alpha}{\alpha^2}\right) + \frac{2 - \alpha}{\alpha - 1} \left(1 + \frac{2^\alpha}{\alpha^2}\right)^{-\alpha}
\]  
(94)

where the final step is valid since \( \alpha \) is assumed to be sufficiently large. Using once more \( \log(1+x) \leq \log x + \frac{2}{x} \) for \( x \geq 1 \) gives

\[
D(P_{XYZ} \parallel R_{Y \rightarrow Y'; Y}(P_{XY})) + \Lambda_\alpha(P_{XY} \parallel R_{Y \rightarrow Y'})
\]  
(95)
\[\leq 2 \log \alpha + \frac{\alpha}{\alpha - 1} \left( \alpha - 2 \log \alpha - 1 + \frac{2\alpha^2}{2^\alpha} \right) + 2^{-\alpha}\]  
(97)

\[= \alpha - \frac{2}{\alpha - 1} \log \alpha + 2^{-\alpha} \text{poly}(\alpha),\]  
(98)

where \(\text{poly}(\alpha)\) denotes an arbitrary polynomial in \(\alpha\). As a result, we obtain for a sufficiently large \(\alpha\)

\[D(P_{XYZ} \| R_{Y \rightarrow Y'} | (P_{XY})) + \Lambda_\alpha(P_{XY} \| R_{Y \rightarrow Y'}) < \alpha - \frac{2}{\alpha} \log \alpha - 1 + 2^{-\alpha}\]  
(99)

\[\leq \alpha - \alpha^{-1} - h(\alpha^{-2})\]  
(100)

\[\leq I(X : Z|Y)_P.\]  
(101)

The two steps (99) and (100) are both valid because \(\alpha\) is sufficiently large. The final step uses (85).

This example shows that (48) is no longer valid if the \(\Lambda_{\text{max}}\)-term is replaced with a \(\Lambda_\alpha\)-term for any \(\alpha \in [\frac{1}{2}, \infty)\).\footnote{The example does not work in the limit \(\alpha \to \infty\).} Note also that this example implies Remark 2.2 on the tightness of the triangle-like inequality for the relative entropy.

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**A Approximate Markov chains can be far from Markov chains**

As mentioned in the introduction, it is known \([10, 17]\) that there exist tripartite states with a small conditional mutual information whose distance to any Markov chain is nevertheless large. For example, consider a state \(\rho_{S_1, \ldots, S_d} = |\psi\rangle\langle\psi|_{S_1, \ldots, S_d}\) on \(S_1 \otimes \cdots \otimes S_d\) with \(\dim S_k = d > 1\) for all \(k = 1, \ldots, d\), where

\[|\psi\rangle_{S_1, \ldots, S_d} := \sqrt{\frac{1}{d!}} \sum_{\pi \in S_d} \text{sign}(\pi) |\pi(1)\rangle \otimes \cdots \otimes |\pi(d)\rangle\]  
(102)

is the Slater determinant, \(S_d\) denotes the group of permutations of \(d\) objects, and \(\text{sign}(\pi) := (-1)^L\), where \(L\) is the number of transpositions in a decomposition of the permutation \(\pi\). The chain rule and the trivial upper bound for the mutual information show that we have

\[I(S_1 : S_2 \ldots S_d)_\rho = \sum_{k=2}^d I(S_1 : S_k|S_2 \ldots S_{k-1})_\rho \leq 2 \log d.\]  
(103)

Because the mutual information is nonnegative, there exists \(k \in \{2, \ldots, d\}\) such that

\[I(S_1 : S_k|S_2 \ldots S_{k-1})_\rho \leq \frac{2}{d-1} \log d,\]  
(104)

which can be arbitrarily small (as \(d\) gets large). By definition, the reduced state \(\rho_{S_1, S_k}\) is the antisymmetric state on \(S_1 \otimes S_k\) that is far from separable \([7, \text{p. 53}]\). More precisely, for any separable state \(\sigma_{S_1, S_k}\) on \(S_1 \otimes S_k\) we have \(\Delta(\rho_{S_1, S_k}, \sigma_{S_1, S_k}) \geq \frac{1}{2}\), where \(\Delta(\tau, \omega) := \frac{1}{2} \| \tau - \omega \|_1\) denotes the trace.
distance between $\tau$ and $\omega$. For any state $\mu_{S_1 \cdots S_k}$ on $S_1 \otimes \cdots \otimes S_k$ that forms a Markov chain in order $S_1 \leftrightarrow S_2 \ldots S_{k-1} \leftrightarrow S_k$, it follows by (2) that its reduced state $\mu_{S_1 S_k}$ on $S_1 \otimes S_k$ is separable. Using the monotonicity of the trace distance under trace-preserving completely positive maps [24, Theorem 9.2] we thus find

$$\Delta(\rho_{S_1 \cdots S_k}, \mu_{S_1 \cdots S_k}) \geq \Delta(\rho_{S_1 S_k}, \mu_{S_1 S_k}) \geq \frac{1}{2},$$

(105)

showing that the state $\rho_{S_1 \cdots S_k}$ despite having a conditional mutual information that is arbitrarily small (see (104)) is far from any Markov chain.

As discussed in the introduction, states with a small conditional mutual information are called approximate Markov chains (which is justified by (4)). The example in this appendix shows that approximate quantum Markov chains are not necessarily close to quantum Markov chains.

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