ON THE $K$-THEORY OF TORIC STACK BUNDLES

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ABSTRACT. Simplicial toric stack bundles are smooth Deligne-Mumford stacks over smooth varieties with fibre a toric Deligne-Mumford stack. We compute the Grothendieck $K$-theory of simplicial toric stack bundles and study the Chern character homomorphism.

1. Introduction

Simplicial toric stack bundles, as defined in [10], are bundles over a smooth base variety $B$ with fibers toric Deligne-Mumford stacks in the sense of [5]. In this paper we compute the Grothendieck $K$-theory of simplicial toric stack bundles.

In [10], the construction of toric Deligne-Mumford stacks was slightly generalized by extending the notion of stacky fans. A stacky fan $\Sigma$ is a triple $\Sigma := (N, \Sigma, \beta)$, where $N$ is a finitely generated abelian group of rank $d$, $\Sigma$ is a simplicial fan in the lattice $N = \mathbb{Q}/\mathbb{Z} \subseteq \mathbb{Q}$, and $\beta : \mathbb{Z}^m \to N$ is a map determined by integral vectors $b_1, \ldots, b_n, b_{n+1}, \ldots, b_m \in \mathbb{N}$ ($m \geq n$) satisfying the condition that for $1 \leq i \leq n$ the image $\overline{b}_i \in \overline{N}$ under the projection $N \to \overline{N}$ generates the ray $\rho_i \in \Sigma$. We call $\{b_{n+1}, \ldots, b_m\}$ the extra data in $\Sigma$. The stacky fan $\Sigma$ yields an exact sequence,

$$1 \rightarrow \mu \rightarrow G \overset{\alpha}{\rightarrow} (\mathbb{C}^*)^m \rightarrow T \rightarrow 1$$

where $T = (\mathbb{C}^*)^d$. We associated to $\Sigma$ a toric Deligne-Mumford stack $\mathcal{X}(\Sigma) := [Z/G]$, where $Z = (\mathbb{C}^n \backslash \mathcal{V}(J_\Sigma)) \times (\mathbb{C}^*)^{m-n}$, the ideal $J_\Sigma$ is the irrelevant ideal of the fan $\Sigma$, and $G$ acts on $Z$ via the homomorphism $\alpha : G \to (\mathbb{C}^*)^m$ above.

Removing the extra data $\{b_{n+1}, \ldots, b_m\}$ from the map $\beta$ yields $\beta_{\text{min}} : \mathbb{Z}^n \to N$ given by the integral vectors $\{b_1, \ldots, b_n\}$. The triple $\Sigma_{\text{min}} := (N, \Sigma, \beta_{\text{min}})$ is the stacky fan in the sense of [5]. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{\text{min}})$ is isomorphic to $\mathcal{X}(\Sigma)$, see [10]. The stacky fan $\Sigma_{\text{min}}$ may be interpreted as the minimal representation of the associated toric Deligne-Mumford stack.

Let $P \to B$ be a principal $(\mathbb{C}^*)^m$-bundle, let $^P\mathcal{X}(\Sigma)$ be the quotient stack $[(P \times (\mathbb{C}^*)^m, Z)/G]$, where $G$ acts on $B$ trivially and on $(\mathbb{C}^*)^m$ via the map $\alpha$ above. Then $^P\mathcal{X}(\Sigma)$ is a toric stack bundle over $B$ with fibre the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$. The extra data $\{b_{n+1}, \ldots, b_m\}$ in $\Sigma$ can be put into the $\text{Box}(\Sigma)$ which do not influence the structure of the toric stack bundle $^P\mathcal{X}(\Sigma)$. The choice of torsion and nontorsion extra data does affect the structure of $^P\mathcal{X}(\Sigma)$, but not the Chen-Ruan (orbifold) cohomology, see [10].

Let $\rho_i \in \Sigma$ be a ray. There is a corresponding line bundle $L_i$ over $^P\mathcal{X}(\Sigma)$, which is the trivial line bundle $\mathcal{C}$ over $P \times (\mathbb{C}^*)^m, Z$ with the $G$-action given by the $i$-th component of the map $\alpha$. The
ray $\rho_i$ also defines a line bundle $L_i$ over $X(\Sigma)$ via the $i$-th component of $\alpha$. The line bundle $L_i$ can be taken as the twist $P(L_i)$ of $L_i$ by the principle $(\mathbb{C}^*)^n$-bundle $P$.

Let $R$ denote the character ring of the group $G_{\text{min}}$, which is isomorphic to $DG(\beta_{\text{min}})$ in the Gale dual map $\beta_{\text{min}}^\vee : \mathbb{Z}^n \to DG(\beta_{\text{min}})$. Every character $\chi \in R$ gives a line bundle $L_\chi$ over $P\times X(\Sigma)$. The line bundle $L_i$ is given by the standard character $\chi_i$ induced by the standard generator $x_i$ on $\mathbb{Z}^n$. We let $x_i$ represent the class $[L_i]$ in the $K$-theory. Let $M = N^*$ be the dual of $N$. For $\theta \in M$, let $\xi_\theta \to B$ be the line bundle coming from the principal $T$ bundle $E \to B$ by “extending” the structure group via $\chi^\theta : T \to \mathbb{C}^*$, where $E \to B$ is induced from the $(\mathbb{C}^*)^m$-bundle $P$ via the map $(\mathbb{C}^*)^m \to T$ in (1). Let $\{v_1, \cdots, v_d\}$ be a basis of $\mathbb{N} = \mathbb{Z}^d$, we choose a basis $\{u_1, \cdots, u_d\}$ of $M$, which is dual to $\{v_1, \cdots, v_d\}$. Write $\xi_i = \xi_{u_i}$.

Let $K(B)$ be the $K$-theory ring of the smooth variety $B$. Let $C(P\Sigma)$ be the ideal in the ring $K(B) \otimes R$ generated by the elements

$$(2) \quad \sum_{1 \leq j \leq n} x_j^{(\theta, b_j)} - \prod_{1 \leq i \leq d} (\xi_i^\vee)^{(\theta, v_i)}_{\theta \in M},$$

where $\xi_i^\vee$ is the dual of the line bundle $\xi_i$. Let $I_{\Sigma}$ be the ideal generated by

$$(3) \quad \prod_{i \in I} (1 - x_i)$$

where $I \subseteq [1, \cdots, n]$ such that $\{\rho_i|i \in I\}$ do not form a cone in $\Sigma$.

**Theorem 1.1.** Let $K_0(P\times X(\Sigma))$ be the Grothendieck $K$-theory ring of the toric stack bundle $P\times X(\Sigma)$. Then the morphism

$$\phi : \frac{K(B) \otimes R}{I_{\Sigma} + C(P\Sigma)} \to K_0(P\times X(\Sigma)),$$

which send $\chi$ to $[L_\chi]$, is an isomorphism.

In the reduced case, i.e. the abelian group $N$ is torison-free, the toric Deligne-Mumford stack $X(\Sigma)$ is an orbifold. Then every character of $G$ can be lifted to a character of $(\mathbb{C}^*)^n$. We have the corollary:

**Corollary 1.2.** Let $K_0(P\times X(\Sigma))$ be the Grothendieck $K$-theory ring of the toric stack bundle $P\times X(\Sigma)$ with $X(\Sigma)$ a reduced toric Deligne-Mumford stack. Then the morphism

$$\phi : \frac{K(B)[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]}{I_{\Sigma} + C(P\Sigma)} \to K_0(P\times X(\Sigma)),$$

which send $x_i$ to $[L_i]$, is an isomorphism.

Our proof of the main theorem is based on computations of the $K$-theory rings of toric Deligne-Mumford stacks [6], and of toric bundles [16].

This paper is organized as follows. The basic construction of toric stack bundles defined in [10] is reviewed in Section 2. Chen-Ruan orbifold cohomology ring of toric stack bundles is discussed in Section 3. In Section 4 we compute the $K$-theory ring of toric stack bundles, and in Section 5 we show that there is a Chern character isomorphism from the $K$-theory of the toric stack bundle to the Chen-Ruan cohomology ring. In Section 6 we give an interesting example, where we compute
the $K$-theory ring of finite abelian gerbes over smooth varieties and compare with the Chen-Ruan cohomology calculated in [10].

**Conventions.** In this paper we work algebraically over the field of complex numbers. We use the rational numbers $\mathbb{Q}$ as coefficients of (orbifold) Chow ring and (orbifold) cohomology ring. By an orbifold we mean a smooth Deligne-Mumford stack with trivial generic stabilizer. We refer to [5] for the construction of Gale dual $(\beta)^{\vee} : \mathbb{Z}^{m} \to DG(\beta)$ from $\beta : \mathbb{Z}^{m} \to N$. We write $\mathbb{C}^{\ast} = \mathbb{C} \setminus \{0\}$. $N^{\ast}$ denotes the dual of $N$ and $N \to \overline{N}$ is the natural map modulo torsion.

For the cones in $\Sigma$, we assume that the rays $\rho_1, \ldots, \rho_n$ span a top dimensional cone $\sigma \in \Sigma$, and $\rho_{d+1}, \ldots, \rho_n$ are the other rays. Let $v_i \in \rho_i$ be such that $\{v_1, \ldots, v_d\}$ is a basis of $\overline{N} = \mathbb{Z}^d$. Let $\{u_1, \ldots, u_d\}$ be the dual basis in $M = N^{\ast}$.

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2. Toric Stack Bundles

In this section we review the basic construction of toric stack bundles, see [10] for details.

2.1. Toric Deligne-Mumford Stacks. Let $N$ be a finitely generated abelian group of rank $d$ and $\overline{N} = N/\mathbb{N}_{tor}$ the lattice generated by $N$ in the $d$-dimensional vector space $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$. Write $b$ for the image of $b$ under the natural map $N \to \overline{N}$. Let $\Sigma$ be a rational simplicial fan in $N_{\mathbb{Q}}$. Suppose $\rho_1, \ldots, \rho_n$ are the rays in $\Sigma$. We fix $b_i \in N$ for $1 \leq i \leq n$ such that $b_i$ generates the ray $\rho_i$. Let $\{b_{n+1}, \ldots, b_n\} \subset N$. We consider the homomorphism $\beta : \mathbb{Z}^{m} \to N$ determined by the elements $\{b_1, \ldots, b_m\}$. We require that $\beta$ has finite cokernel.

**Definition 2.1.** The triple $\Sigma := (N, \Sigma, \beta)$ is called a stacky fan.

**Remark 2.2.** If $m = n$, then $\Sigma$ is the stacky fan in the sense of Borisov-Chen-Smith [5].

The stacky fan $\Sigma$ determines two exact sequences:

$$
0 \to DG(\beta)^{\ast} \to \mathbb{Z}^{m} \xrightarrow{\beta} N \to \text{Coker}(\beta) \to 0,
$$

$$
0 \to N^{\ast} \to \mathbb{Z}^{m} \xrightarrow{\beta^{\vee}} DG(\beta) \to \text{Coker}(\beta^{\vee}) \to 0,
$$

where $\beta^{\vee}$ is the Gale dual of $\beta$. As a $\mathbb{Z}$-module, $\mathbb{C}^{\ast}$ is divisible, so it is an injective $\mathbb{Z}$-module, and hence the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^{\ast})$ is exact (see e.g [14]). This yields an exact sequence:

$$
1 \to \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta^{\vee}), \mathbb{C}^{\ast}) \to \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^{\ast}) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{m}, \mathbb{C}^{\ast}) \to \text{Hom}_{\mathbb{Z}}(N^{\ast}, \mathbb{C}^{\ast}) \to 1.
$$

Write $\mu := \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta^{\vee}), \mathbb{C}^{\ast})$, $G := \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^{\ast})$, $T := \text{Hom}_{\mathbb{Z}}(N^{\ast}, \mathbb{C}^{\ast})$, then the above sequence reads

$$
1 \to \mu \to G \xrightarrow{\alpha} (\mathbb{C}^{\ast})^{m} \to T \to 1,
$$

which is the same as (1). Define $Z = (\mathbb{C}^{n} \setminus \{0\}) \times (\mathbb{C}^{\ast})^{m-n}$, where $J_{Z}$ is the irrelevant ideal of the fan $\Sigma$. There exists a natural action of $(\mathbb{C}^{\ast})^{m}$ on $Z$. The group $G$ acts on $Z$ through the map $\alpha$ in (4). The quotient stack $[Z/G]$ is associated to the groupoid $G \rightrightarrows Z$. The morphism $\varphi : Z \times G \to Z \times Z$ to be $\varphi(x, g) = (x, g \cdot x)$ is finite, hence $[Z/G]$ is a Deligne-Mumford stack.
Definition 2.3. For a stacky fan $\Sigma = (N, \Sigma, \beta)$, define $X(\Sigma) := [Z/G]$.

Let $\Sigma$ be a stacky fan. Let $\beta_{\text{min}} : \mathbb{Z}^n \to N$ be the map given by the first $n$ integral vectors $\{b_1, \ldots, b_n\}$ in the map $\beta$. Then $\Sigma_{\text{min}} = (N, \Sigma, \beta_{\text{min}})$ is a stacky fan, which we call the minimal stacky fan. From the definitions, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & 0 \\
\downarrow{\beta_{\text{min}}} & & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \\
0 & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
$$

From the definition of Gale dual, we compute that $DG(\tilde{\beta}) = \mathbb{Z}^{m-n}$ and $\tilde{\beta}'$ is an isomorphism. So by Lemma 2.3 in [5], applying the Gale dual yields

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^n & \longrightarrow & 0 \\
\downarrow{\tilde{\beta}'} & & \downarrow{\tilde{\beta}'} & & \downarrow{\tilde{\beta}'} & & \downarrow{\tilde{\beta}'} & & \\
0 & \longrightarrow & DG(\beta) & \longrightarrow & DG(\beta_{\text{min}}) & \longrightarrow & 0.
\end{array}
$$

Taking $\text{Hom}_Z(-, \mathbb{C}^*)$ functor, we get

$$
\begin{array}{ccccccc}
1 & \longrightarrow & G_{\text{min}} & \xrightarrow{\varphi_1} & G & \longrightarrow & (\mathbb{C}^*)^{m-n} & \longrightarrow & 1 \\
\downarrow{\alpha_{\text{min}}} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \\
1 & \longrightarrow & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^{m-n} & \longrightarrow & 1.
\end{array}
$$

Let $\varphi_0 : \mathbb{C}^n \setminus \mathcal{V}(J_{\Sigma}) \to Z$ be the inclusion defined by $z \mapsto (z, 1)$. So

$$(\varphi_0 \times \varphi_1, \varphi_0) : (((\mathbb{C}^n \setminus \mathcal{V}(J_{\Sigma})) \times G_{\text{min}} \cong \mathbb{C}^n \setminus \mathcal{V}(J_{\Sigma})) \to (Z \times G \cong Z)$$

defines a morphism between groupoids. Let $\varphi : [(\mathbb{C}^n \setminus \mathcal{V}(J_{\Sigma})) / G_{\text{min}}] \to [Z/G]$ be the morphism of stacks induced from $(\varphi_0 \times \varphi_1, \varphi_0)$.

Proposition 2.4 ([10]). The morphism $\varphi : X(\Sigma_{\text{min}}) \to X(\Sigma)$ is an isomorphism.

2.2. Toric Stack Bundles. In this section we introduce the toric stack bundle $P X(\Sigma)$. Let $P \to B$ be a principal $(\mathbb{C}^*)^m$-bundle over a smooth variety $B$. Let $G$ act on the fibre product $P \times_{(\mathbb{C}^*)^m} Z$ via $\alpha$ in (4).

Definition 2.5. Define the toric stack bundle $P X(\Sigma) \to B$ to be the quotient stack

$$
P X(\Sigma) := [(P \times_{(\mathbb{C}^*)^m} Z) / G].
$$

Let $\Sigma$ be a stacky fan. For a cone $\sigma \in \Sigma$, define $\text{link}(\sigma) := \{ \tau : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0 \}$. Let $\{\tilde{r}_1, \ldots, \tilde{r}_t\}$ be the rays in $\text{link}(\sigma)$. Then $\Sigma/\sigma = (N(\sigma) = N/N_{\sigma}, \Sigma/\sigma, \beta(\sigma))$ is a stacky fan, where $\beta(\sigma) : \mathbb{Z}^{l+m-n} \to N(\sigma)$ is given by the images of $b_1, \ldots, b_l, b_{l+1}, \ldots, b_m$ under $N \to N(\sigma)$. From the construction of toric Deligne-Mumford stacks, we have $X(\Sigma/\sigma) := [Z(\sigma)/G(\sigma)]$, where $Z(\sigma) = (A^l \setminus \mathcal{V}(J_{\Sigma/\sigma})) \times (\mathbb{C}^*)^{m-n}$, $G(\sigma) = \text{Hom}_Z(DG(\beta(\sigma)), \mathbb{C}^*)$. We have an action of $(\mathbb{C}^*)^m$ on $Z(\sigma)$ induced by the natural action of $(\mathbb{C}^*)^{l+m-n}$ on $Z(\sigma)$ and the projection $(\mathbb{C}^*)^m \to (\mathbb{C}^*)^{l+m-n}$. As in [10], let

$$
P X(\Sigma/\sigma) = [(P \times_{(\mathbb{C}^*)^m} (\mathbb{C}^*)^{l+m-n} \times (\mathbb{C}^*)^{l+m-n} Z(\sigma)) / G(\sigma)]
$$

$$
= [(P \times_{(\mathbb{C}^*)^m} Z(\sigma)) / G(\sigma)].
$$
Proposition 2.6 ([10]). Let σ be a cone in the stacky fan Σ, then $P\mathcal{X}(\Sigma/\sigma)$ defines a closed substack of $P\mathcal{X}(\Sigma)$.

For each top dimensional cone σ in Σ, denote by $\text{Box}(\sigma)$ the set of elements $v \in N$ such that $v = \sum_{\rho_i \subseteq \sigma} a_i \bar{b}_i$ for some $0 \leq a_i < 1$. Elements in $\text{Box}(\sigma)$ are in one-to-one correspondence with elements in the finite group $N(\sigma) = N/N_\sigma$, where $N(\sigma)$ is a local group of the stack $\mathcal{X}(\Sigma)$. If $\tau \subseteq \sigma$ is a subcone, we define $\text{Box}(\tau)$ to be the set of elements in $v \in N$ such that $v = \sum_{\rho_i \subseteq \tau} a_i \bar{b}_i$, where $0 \leq a_i < 1$. Clearly $\text{Box}(\tau) \subset \text{Box}(\sigma)$. In fact the elements in $\text{Box}(\tau)$ generate a subgroup of the local group $N(\sigma)$. Let $\text{Box}(\Sigma)$ be the union of $\text{Box}(\sigma)$ for all $d$-dimensional cones $\sigma \in \Sigma$. For $v_1, \ldots, v_n \in N$, let $\sigma(\overline{v}_1, \ldots, \overline{v}_n)$ be the unique minimal cone in Σ containing $\overline{v}_1, \ldots, \overline{v}_n$.

The following description for the inertia stack of $P\mathcal{X}(\Sigma)$ is found in [10].

Proposition 2.7. Let $P\mathcal{X}(\Sigma) \to B$ be a toric stack bundle over a smooth variety $B$ with fibre $\mathcal{X}(\Sigma)$, the toric Deligne-Mumford stack associated to the stacky fan Σ. Then its $r$-th inertia stack is

$$I_r(P\mathcal{X}(\Sigma)) = \bigcap_{(v_1, \ldots, v_r) \in \text{Box}(\Sigma)^r} P\mathcal{X}(\Sigma/\sigma(v_1, \ldots, v_r)).$$

3. The Chen-Ruan Orbifold Cohomology of Toric Stack Bundles.

In this section we describe the ring structure of the orbifold cohomology of toric stack bundles.

3.1. Orbifold Cohomology. The Chen-Ruan Chow ring of projective toric Deligne-Mumford stacks was computed in [5], and generalized to semi-projective case in [12]. The calculation for Chen-Ruan orbifold cohomology ring is the same. In this section we assume that the toric Deligne-Mumford stacks are semi-projective.

For $\theta \in M = N^*$, let $\chi^\theta : (\mathbb{C}^\times)^m \to \mathbb{C}^\times$ be the map induced by $\theta \circ \beta : \mathbb{Z}^m \to \mathbb{Z}$. Let $\xi_\theta \to B$ be the line bundle $P \times_{\mathcal{X}^\theta} \mathbb{C}$. We introduce the deformed ring $H^*(B)[N]^\Sigma = H^*(B) \otimes \mathbb{Q}[N]^\Sigma$, where $\mathbb{Q}[N]^\Sigma := \bigoplus_{\xi \in N} \mathbb{Q} \cdot y^\xi$, $y$ is a formal variable, and $H^*(B)$ is the cohomology ring of $B$. The multiplication of $\mathbb{Q}[N]^\Sigma$ is given by

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{v}_1 \in \sigma, \overline{v}_2 \in \sigma, \\ 0 & \text{otherwise}. \end{cases}$$

Let $I(P\Sigma)$ be the ideal in $H^*(B)[N]^\Sigma$ generated by the following elements:

$$c_1(\xi_\theta) + \sum_{i=1}^n \theta(b_i)y^{b_i},$$

and $H^*_{CR}(P\mathcal{X}(\Sigma))$ the Chen-Ruan cohomology ring of the toric stack bundle $P\mathcal{X}(\Sigma)$.

Theorem 3.1 ([10]). Let $P\mathcal{X}(\Sigma) \to B$ be a toric stack bundle over a smooth variety $B$ as above. We have an isomorphism of $\mathbb{Q}$-graded rings:

$$H^*_{CR}(P\mathcal{X}(\Sigma)) \cong \frac{H^*(B)[N]^\Sigma}{I(P\Sigma)}.$$
From the definition of Chen-Ruan cohomology ring, we have
\[(9) \quad H^*_{CR}(P\mathcal{X}(\Sigma)) = \bigoplus_{v \in \text{Box}(\Sigma)} H^*(P\mathcal{X}(\Sigma/\sigma(\mathcal{V})))\]
The closed substack \(P\mathcal{X}(\Sigma/\sigma(\mathcal{V}))\) is also a toric stack bundle over \(B\) with fibre being the toric Deligne-Mumford stack \(\mathcal{X}(\Sigma/\sigma(\mathcal{V}))\) associated to the quotient stacky fan \(\Sigma/\sigma(\mathcal{V})\). Let
\[\text{link}(\sigma(\mathcal{V})) = \{\rho_1, \ldots, \rho_t\}.\]
Let \(I_{\Sigma/\sigma(\mathcal{V})}\) be the ideal of \(H^*(B)[y^{\tilde{b}_1}, \ldots, y^{\tilde{b}_r}]\) generated by
\[\{y^{\tilde{b}_{i_1}} \cdots y^{\tilde{b}_{i_k}} | \rho_{i_1}, \ldots, \rho_{i_k} \text{ do not span a cone in } \Sigma/\sigma(\mathcal{V})\}\].
Then the cohomology ring of \(P\mathcal{X}(\Sigma/\sigma(\mathcal{V}))\) is isomorphic to the Stanley-Reisner ring of the quotient fan over the cohomology ring \(H^*(B)\) of the base \(B\):
\[(10) \quad H^*(P\mathcal{X}(\Sigma/\sigma(\mathcal{V}))) \cong \frac{H^*(B)[y^{\tilde{b}_1}, \ldots, y^{\tilde{b}_r}]}{I_{\Sigma/\sigma(\mathcal{V})} + \mathcal{I}(P\Sigma/\sigma(\mathcal{V}))}.
\]
**Remark 3.2.** As pointed out in [6], the Chen-Ruan cohomology ring \(H^*_{CR}(P\mathcal{X}(\Sigma))\) is not Artinian in general if \(N\) has torsion, since it has degree zero elements. If \(N\) is free, i.e. the toric Deligne-Mumford stack is reduced, then \(H^*_{CR}(P\mathcal{X}(\Sigma))\) is an Artinian module over the cohomology ring \(H^*(B)\) of the base.

### 3.2. Obstruction Bundle

The key gradient of Chen-Ruan orbifold cup product is the orbifold obstruction bundle defined over the double inertia stacks. We review it here for the latter use.

The stack \(P\mathcal{X}(\Sigma)\) is an abelian Deligne-Mumford stack, i.e. the local groups are all abelian groups. The 3-twisted sector sectors of \(P\mathcal{X}(\Sigma)\) are given by triples \((v_1, v_2, v_3)\) for \(v_1, v_2, v_3 \in \text{Box}(\Sigma)\) such that \(v_1 + v_2 + v_3\) belongs to \(N\).

For any 3-twisted sector \(P\mathcal{X}(\Sigma/(v_1, v_2, v_3))\), the normal bundle \(N(P\mathcal{X}(\Sigma/(v_1, v_2, v_3))/P\mathcal{X}(\Sigma))\) splits into the direct sum of line bundles under the group action. It follows from the definition that if \(\mathcal{V} = \sum_{\rho_i \subseteq \mathcal{V}} \alpha_i \mathcal{V}^h\), then the action of \(v\) on the normal bundle \(N(P\mathcal{X}(\Sigma/(v_1, v_2, v_3))/P\mathcal{X}(\Sigma))\) is given by the diagonal matrix \(\text{diag}(\alpha_i)\). Let \(e : P\mathcal{X}(\Sigma/(v_1, v_2, v_3)) \to P\mathcal{X}(\Sigma)\) be the embedding. According to [8] the obstruction bundle \(Ob_{(v_1, v_2, v_3)}\) over \(\mathcal{X}(\Sigma/(v_1, v_2, v_3))\) is defined as
\[Ob_{(v_1, v_2, v_3)} := (H^1(\mathcal{C}, \mathcal{O}_\mathcal{C}) \otimes e^* T_{P\mathcal{X}(\Sigma)})|_{(v_1, v_2, v_3)},\]
where \((v_1, v_2, v_3)\) is the subgroup generated by \(v_1, v_2, v_3\) and \(\mathcal{C}\) is the \((v_1, v_2, v_3)\)-cover over the Riemann sphere \(\mathbb{P}^1\). Details can be found in [8]. Let \(v_1 + v_2 + v_3 = \sum_{\rho_i \subseteq \mathcal{V}} \alpha_i \mathcal{V}^h\). We will use the following description of the Euler class of the obstruction bundle:

**Proposition 3.3 (see [7], [9]).** Let \(P\mathcal{X}(\Sigma/(v_1, v_2, v_3))\) be a 3-twisted sector of the stack \(P\mathcal{X}(\Sigma)\) such that \(v_1, v_2, v_3 \neq 0\). Then the Euler class of the obstruction bundle \(Ob_{(v_1, v_2, v_3)}\) is
\[(11) \quad Ob_{(v_1, v_2, v_3)} = \prod_{\alpha_i = 2} c_1(L_i)|_{\mathcal{X}(\Sigma/(v_1, v_2, v_3))},\]
where \(L_i\) is the line bundle over \(P\mathcal{X}(\Sigma)\) determined by the ray \(\rho_i\).
4. The $K$-Theory of Toric Stack Bundles

In this section we study the Grothendieck ring of toric stack bundles and prove the main theorem.

4.1. The $K$-Theory of Toric Deligne-Mumford Stacks. We recall the result of [6]. Let $\Sigma$ be a stacky fan and $\mathcal{X}(\Sigma)$ the corresponding toric Deligne-Mumford stack. For each ray $\rho_i$ in the fan $\Sigma$, define the line bundle $L_i$ over $\mathcal{X}(\Sigma)$ to be the quotient of the trivial line bundle $\mathbb{Z} \times \mathbb{C}$ over $\mathbb{Z}$ under the action of $G$ on $\mathbb{C}$ through $i$-th component of $\alpha$ in (4). Let $x_i$ represent the class $[L_i]$ in the Grothendieck $K$-theory ring.

Let $R$ be the character ring of the group $G_{\text{min}}$. Let $\text{Cir}(\Sigma)$ be the ideal in $K(B) \otimes R$ generated by the elements

$$\left( \prod_{1 \leq j \leq n} x_j^{(\theta, v_j)} - 1 \right)_{\theta \in M}. \tag{12}$$

Let $I_{\Sigma}$ be the ideal generated by

$$\prod_{i \in I} (1 - x_i) = 0, \tag{13}$$

where $I \subseteq [1, \cdots, n]$ such that $\{\rho_i | i \in I\}$ do not form a cone in $\Sigma$. According to [6], the Grothendieck $K$-theory ring $K_0(\mathcal{X}(\Sigma))$ of $\mathcal{X}(\Sigma)$ can be described as follows.

**Theorem 4.1 ([6]).** For a toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$, the morphism

$$\phi : \frac{R}{I_{\Sigma} + \text{Cir}(\Sigma)} \to K_0(\mathcal{X}(\Sigma)),$$

which send $\chi$ to $[L_{\chi}]$, is an isomorphism.

Let $\Sigma_{\text{min}}$ be the minimal stacky fan associated to $\Sigma$. There is an underlying reduced stacky fan $\Sigma_{\text{red}} = \langle N, \Sigma, \bar{\beta} \rangle$, where $\bar{N} = N/N_{\text{tor}}, \bar{\beta} : \mathbb{Z}^n \to \bar{N}$ is the natural projection given by the vectors $\{\bar{b}_1, \cdots, \bar{b}_n\} \subseteq \bar{N}$. Consider the following diagram

$$\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{\beta} & \bar{N} \\
\downarrow{id} & \downarrow{\bar{\beta}} & \downarrow{N} \\
\mathbb{Z}^n & \xrightarrow{\beta} & \bar{N}.
\end{array}$$

Taking Gale duals yields

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{N}^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\bar{\beta}^\vee} & DG(\bar{\beta}) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & N^* & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta^\vee} & DG(\beta) & \longrightarrow & \text{coker}(\beta^\vee) & \longrightarrow & 0.
\end{array} \tag{14}$$
Applying $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to (14) yields
\begin{align*}
1 & \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 1 \\
1 & \longrightarrow 1 \longrightarrow \overline{G} \xrightarrow{\overline{\alpha}} (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 1,
\end{align*}
(15)

The stack $\mathcal{X}(\Sigma_{\text{red}})$ is a toric orbifold. By construction $\mathcal{X}(\Sigma_{\text{red}}) = [\mathbb{Z}/G]$, where $G = \text{Hom}_\mathbb{Z}(DG(\beta), \mathbb{C}^*)$ and $DG(\beta)$ is the Gale dual $\beta^\vee : \mathbb{Z}^n \to \mathbb{N}^\vee$ of the map $\beta$. We can see from (15) that every character of $G$ can be represented as a character of $(\mathbb{C}^*)^n$. So we have:

**Theorem 4.2.** For the reduced toric Deligne-Mumford stack $\mathcal{X}(\Sigma_{\text{red}})$ the morphism
\[ \phi : \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]_{I_\Sigma + C(\Sigma)} \longrightarrow K_0(\mathcal{X}(\Sigma_{\text{red}})), \]
which send $x_i$ to $[L_i]$, is an isomorphism.

### 4.2. Proof of Theorem 1.1
Let $\Sigma$ be a stacky fan, and $\mathcal{X}(\Sigma)$ the associated toric Deligne-Mumford stack. Let $P \to B$ be a principle $(\mathbb{C}^*)^m$-bundle over the smooth variety $B$. Then we have the toric stack bundle $\pi : P\mathcal{X}(\Sigma) \to B$. For each ray $\rho_i$ in the fan $\Sigma$, we have a line bundle $L_i$ over $\mathcal{X}(\Sigma)$. Twist it by the principal $(\mathbb{C}^*)^m$-bundle $P$, we get the line bundle $L_i$ over the toric stack bundle $P\mathcal{X}(\Sigma)$.

As in [5] and [10] we have a codimension one closed substack $\mathcal{X}(\Sigma/\rho_j) \subset \mathcal{X}(\Sigma)$. There is a canonical section $s_j$ of the line bundle $L_j$ whose zero locus is $\mathcal{X}(\Sigma/\rho_j)$.

Suppose that $\rho_{j_1}, \ldots, \rho_{j_r}$ do not span a cone in $\Sigma$. The section $s = (s_{j_1}, \ldots, s_{j_r})$ of $L_{j_1} \oplus \cdots \oplus L_{j_r}$ is nowhere vanishing and extends to a nowhere vanishing section
\[ P(s) : P\mathcal{X}(\Sigma) \longrightarrow L_{j_1} \oplus \cdots \oplus L_{j_r}, \]
after twisting by the principle $(\mathbb{C}^*)^m$-bundle $P$. Hence by Remark 4.4 in [16],
\begin{equation}
\prod_{1 \leq p \leq r} (1 - L_{j_p}) = 0.
\end{equation}

(16)

For any $\theta \in M$, the $P$-equivariant isomorphism of bundles over $\mathcal{X}(\Sigma)$
\[ \prod_{1 \leq j \leq n} L_j^{(\theta, b_j)} \cong L_\theta \]

yields an isomorphism of bundles over $P\mathcal{X}(\Sigma)$,
\[ \prod_{1 \leq j \leq n} L_j^{(\theta, b_j)} \cong L_\theta. \]

Since $L_\theta = \prod_{1 \leq i \leq d} \xi_i^{-(\theta, v_i)}$, we obtain
\begin{equation}
\prod_{1 \leq j \leq n} L_j^{(\theta, b_j)} \cong \xi_\theta^{\nu_j}, \quad \text{where} \quad \xi_\theta = \prod_{1 \leq i \leq d} \xi_i^{(\theta, v_i)}.
\end{equation}
(17)
Consider the following map
\[
\varphi : \frac{K(B) \otimes R}{I_\Sigma + C(P\Sigma)} \rightarrow K_0(P\mathcal{X}(\Sigma)), \quad b \otimes \chi \mapsto \left[\pi^*b \otimes L\chi\right], \quad b \in K(B), \chi \in R.
\]
We prove that \(\varphi\) is surjective by induction on the dimension of \(B\). It is obvious when \(B\) is a point.

Let \(U \subset B\) be a Zariski open subset and \(Z = B \setminus U\). Consider the following diagram with exact rows (see [18], Section 3.1 for the exactness of the bottom row):
\[
\begin{array}{cccc}
K_0(Z) \otimes K(\mathcal{X}(\Sigma)) & \longrightarrow & K_0(B) \otimes K(\mathcal{X}(\Sigma)) & \longrightarrow \quad K_0(U) \otimes K(\mathcal{X}(\Sigma)) \\
\downarrow & & \downarrow & \downarrow \\
K_0(\pi^{-1}Z) & \longrightarrow & K_0(P\mathcal{X}(\Sigma)) & \longrightarrow \quad K_0(\pi^{-1}U),
\end{array}
\]
where \(\pi : P\mathcal{X}(\Sigma) \rightarrow B\) is the structure map. By Lemma 4.3 below, the vertical map on the right of (18) is surjective. Then by induction the map \(\varphi\) is surjective since \(\text{dim}(Z) < \text{dim}(B)\).

Now we prove that \(\varphi\) is injective. Let \(\sum_{i=1}^m b_i[F_i] \in K(B) \otimes R\) such that
\[
\varphi\left(\sum_{i=1}^m b_i[F_i]\right) = \sum_{i=1}^m \pi^*b_i \otimes [F_i] = 0,
\]
where \(F_i\) is the twist of \(F_i\) by the \((\mathbb{C}^*)^m\)-bundle \(P\). The sheaf \(F_i\) is generated by \(L_j\)'s corresponding to rays and the torsion line bundles corresponding to torsion subgroup in \(G_{\text{min}}\). From the relations in (16) and (17), it is easy to see that if one of \(b_i \neq 0\), then \(\sum_{i=1}^m \pi^*b_i \otimes [F_i] \neq 0\). So \(\varphi\) is injective, hence is an isomorphism. The concludes the proof of Theorem 1.1.

**Lemma 4.3.** Let \(U\) be a smooth scheme. Let \([M/G]\) be a quotient stack, where \(M\) admits a cellular decomposition (in the sense of [17]) which is \(G\)-equivariant. Then the map
\[
K_0(U) \otimes K_G(M) \rightarrow K_G(U \times M)
\]
is surjective.

**Proof.** This is an \(G\)-equivariant version of [17], Exposé 0, Proposition 2.13. This may be proven by adopting the arguments in [17], together with the following claims. \(\square\)

**Claim 1.** Let \(X\) be a smooth scheme with trivial \(G\)-action, and \(G\) acts on \(\mathbb{A}^1\). Let \(p : X \times \mathbb{A}^1 \rightarrow X\) be the projection. Then the pull-back \(p^* : K_G(X) \rightarrow K_G(X \times \mathbb{A}^1)\) is surjective.

**Proof of Claim 1.** Let \(V\) be a \(G\)-equivariant vector bundle over \(X \times \mathbb{A}^1\). Then by the non-equivariant version of Claim 1 (see [17], Exposé 0, Proposition 2.9), there is a vector bundle \(V'\) over \(X\) such that \(V = p^*(V')\). Since \(G\) acts trivially on \(X\), it is easy to see that the \(G\)-action on \(V\) naturally yields a \(G\)-action on \(V'\), making \(p^*\) \(G\)-equivariant. \(\square\)

**Claim 2.** Let \(X\) be a smooth \(G\)-scheme and \(Y \subset X\) a smooth closed subscheme preserved by \(G\)-action. Suppose that the quotient \([X/G]\) is a noetherian Deligne-Mumford stack. Set \(U := X \setminus Y\). Then the natural sequence
\[
K_G(Y) \rightarrow K_G(X) \rightarrow K_G(U) \rightarrow 0
\]
is exact.
Proof of Claim 2. The exactness in the middle is a general fact, see e.g. [18], Section 3.1. The surjectivity of the restriction map \( K_G(X) \to K_G(U) \) follows from Claim 3 below (we interpret \( G \)-equivariant sheaves as sheaves on the quotient stacks).

Claim 3. Let \( X \) and \( U \) be as in Claim 2. Let \( \mathcal{F} \) be a coherent sheaf on \([U/G]\). Then there exists a coherent sheaf \( \mathcal{F}' \) on \([X/G]\) such that \( \mathcal{F}'|_{[U/G]} = \mathcal{F} \).

Proof of Claim 3. Define a quasi-coherent sheaf \( \tilde{\mathcal{F}} \) on \([X/G]\) as follows. For an open subset \( V \subset [X/G] \) define \( \tilde{\mathcal{F}}(V) := \mathcal{F}(V \cap [U/G]) \). By construction \( \tilde{\mathcal{F}}|_{[U/G]} = \mathcal{F} \), which is coherent. The Claim then follows from [15], Corollaire 15.5.

5. Combinatorial Chern Character

In this section we study the Chern character homomorphism from the \( K \)-theory to Chen-Ruan cohomology. For simplicity, we assume that the toric Deligne-Mumford stack \( X(\Sigma) \) is reduced.

In Section 5.1 we generalize two results in [6], which give the module isomorphism of the Chern character. In Section 5.2 we use the Chern character homomorphism in [9] to show that the Chern character is an ring isomorphism.

5.1. The Module Chern Character. By Theorem 1.1

\[
K_0(PX(\Sigma), \mathbb{C}) := K_0(PX(\Sigma)) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \frac{K(B) \otimes R}{I_{\Sigma} + C(P_{\Sigma})} \otimes \mathbb{C},
\]

where \( R \cong \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \). Let \( \widetilde{R} \) denote the right-hand side of (19). Again let \( [\xi_i] \in K(B, \mathbb{C}) := K(B) \otimes_{\mathbb{Z}} \mathbb{C} \) represent the class of \( \xi_i \) in the \( K \)-theory of \( B \). The following Lemma generalizes [6], Lemma 5.1.

Lemma 5.1. The maximum ideals of \( \widetilde{R} \) as \( K(B, \mathbb{C}) \)-algebras are in bijective correspondence with elements of \( \text{Box}(\Sigma) \). A box element \( v = \sum_{\rho_i \subset \sigma} a_i \bar{b}_i \) corresponds to the \( n \)-tuple \( (y_1, \ldots, y_n) \in K(B, \mathbb{C})^n \) such that

\[
y_i = \begin{cases} 
e^{2\pi i a_i} \sqrt{\xi_i} & \text{if } \rho_i \subset \sigma, \\ 1 & \text{otherwise}, \end{cases}
\]

where \( \xi_i \in K(B, \mathbb{C}) \) and \( r_i \) is the order of \( e^{2\pi i a_i} \).

Proof. The maximal ideals of \( \widetilde{R} \) viewed as \( K(B, \mathbb{C}) \)-algebras correspond to points \( (y_1, \ldots, y_n) \) in \( K(B, \mathbb{C})^n \) such that

\[
\prod_{1 \leq j \leq n} y_j^{(\theta b_i)} - \prod_{1 \leq i \leq d} (\ne^{2\pi i a_i}) = 0
\]

and

\[
\prod_{i \in I} (1 - x_i) = 0
\]

for \( \theta \) and \( I \) in (2) and (3).

Suppose that the \( K(B, \mathbb{C}) \)-point \( (y_1, \ldots, y_n) \) satisfies the above condition. Since \( \prod_{i \in I} (1 - x_i) = 0 \), there is some cone \( \sigma \in \Sigma \) such that \( y_i = 1 \) for \( \rho_i \) outside the cone \( \sigma \). Assume that \( \sigma \) is generated by rays \( \rho_1, \ldots, \rho_k \).
Consider the relation (20). Since this relation holds for any \( \theta \in M \), and \( y_i = 1 \) for \( \rho_i \) outside the cone \( \sigma \), we can take \( \theta : N_{\sigma} \to \mathbb{Z} \), where \( N_{\sigma} \) is the intersection of \( N \) with the rational span of \( \rho_1, \cdots, \rho_k \). Then we can choose \( \theta \) such that \( \theta(v_i) = 1 \), and \( \theta(v_j) = 0 \) for \( j \neq i \). The value \( y_i \) is a \( r_i \)-th root of \( \xi_i \) for some integer \( r_i \). So \( y_i = e^{2\pi i a_i \sqrt{\xi_i}} \). The relation now reads \( \prod_{1 \leq i \leq k} e^{2\pi i a_i (\theta, b_i)} = 1 \), and then \( \sum_i (\theta, b_i) a_i \in \mathbb{Z} \) for all \( \theta \). This is equivalent to \( v = \sum_{\rho_i \in \sigma} a_i \tilde{b}_i \in N \). So the maximal ideals are in one-to-one correspondence to the box elements \( \Box(\Sigma) \).

In the reduced case the ring \( \tilde{R} \) is an Artinian module over \( K(B, \mathbb{C}) \). The localization \( \tilde{R}_v \) can be taken as a submodule of \( \tilde{R} \), which is simple. According to (21), we have

\[
\tilde{R} := \frac{K(B) \otimes R}{I_{\Sigma} + C(\Sigma)} \otimes \mathbb{C} = \bigoplus_{v \in \Box(\Sigma)} \tilde{R}_v.
\]

**Proposition 5.2.** Let \( v \in \Box(\Sigma) \) and \( \sigma(\overline{v}) \) the minimal cone in \( \Sigma \) containing \( \overline{v} \). Then the \( K(B, \mathbb{C}) \)-

algebra \( \tilde{R}_v \) is isomorphic to the cohomology of the closed substack \( \overline{P^*X(\Sigma/\sigma(\overline{v}))} \) of the toric stack bundle \( \overline{P^*X(\Sigma)} \).

**Proof.** Let \( \sigma(\overline{v}) \) be generated by the rays \( \rho_1, \cdots, \rho_k \), and let \( \overline{v} = \sum_{1 \leq i \leq k} a_i \overline{v}_i \) with \( a_i \in (0, 1) \). For the rest of rays \( \rho_{k+1}, \cdots, \rho_n \), we may assume that \( \rho_{k+1}, \cdots, \rho_l \) are contained in some cone \( \sigma' \) containing \( \sigma \), and \( \rho_{l+1}, \cdots, \rho_n \) are not.

Now localizing gives the \( K(B, \mathbb{C}) \)-algebra \( \tilde{R}_v \). Then \( x_i - 1 \) is nilpotent for \( i > k \), and \( x_i - e^{2\pi i a_i \sqrt{\xi_i}} \) is nilpotent for \( 1 \leq i \leq k \). Similar to Lemma 5.2 of [6], let

\[
z_i = \begin{cases} 
\log(x_i), & i > k; \\
\log(x_i e^{-2\pi i a_i (\sqrt{\xi_i})^{-1}}), & 1 \leq i \leq k.
\end{cases}
\]

Now we work over the quotient ring \( \tilde{R}_1 \) of \( \tilde{R} \) by a sufficiently high power of the maximal ideal. Using the same method as in [6], we see that \( z_j = 0 \) in \( \tilde{R}_v \) for \( j > l \). And the relations

\[
\prod_{i \in I}(x_i - 1) = 0
\]

are translated to

\[
\prod_{i \in I_{\Sigma/\sigma}} z_i = 0,
\]

where \( I_{\Sigma/\sigma} \) represents the subset of \( \{k + 1, \cdots, l\} \) such that \( \{\rho_i \mid i \in I_{\Sigma/\sigma}\} \) are not contained in any cone of \( \Sigma/\sigma \). (Note that \( \{\rho_{k+1}, \cdots, \rho_l\} \) are the link set of \( \sigma \)). So the relations \( \prod_{i \in I}(x_i - 1) = 0 \) determine the relations \( \prod_{i \in I_{\Sigma/\sigma}} z_i = 0 \) in the quotient fan \( \Sigma/\sigma \).

Let \( ch : K(B, \mathbb{C}) \to H^*(B, \mathbb{C}) \) be the Chern character isomorphism from the \( K \)-theory of \( B \) to the cohomology. Then \( ch(\xi_i) = e^{c_1(\xi_i)} \).

Consider the linear relations

\[
\prod_{1 \leq j \leq n} x_j^{(\theta, b_j)} - \prod_{1 \leq i \leq d} (\xi_i^{\nu})^{(\theta, v_i)} = 0
\]
for $\theta \in M$. Replacing the relations by $z_i$ we get

\begin{equation}
\prod_{i=1}^{k} e^{2\pi i a_{i} \langle \theta, b_i \rangle} \xi_i^{\langle \theta, v_i \rangle} \prod_{i=1}^{k+l} e^{z_i \langle \theta, b_j \rangle} - \prod_{1 \leq i \leq d} (\xi_i^{\vee})^{\langle \theta, v_i \rangle} = 0.
\end{equation}

Let $N_{\sigma(v)}$ be the sublattice generated by $\sigma(v)$, and $N(\sigma(v)) = N/N_{\sigma(v)}$. Let $\overline{N}(\sigma(v))$ be the free part of $N(\sigma(v))$, and $M(\sigma(v)) := N(\sigma(v))^*$. Consider the following diagram:

\begin{equation}
N \xrightarrow{\pi} N(\sigma(v)) \xrightarrow{\theta} \overline{N}(\sigma(v)) \xrightarrow{\widetilde{\theta}} \mathbb{Z}
\end{equation}

where $\pi$ is the natural morphism. For any $\widetilde{\theta} \in M(\sigma(v))$, there is an element $\theta \in M$ induced from diagram (23). Since $\xi_{\theta} = \prod_{1 \leq i \leq d} \xi_i^{\langle \theta, v_i \rangle}$ and $e^{c_1(\xi_{\theta})} = ch(\xi_{\theta})$, passing to the quotient fan $\Sigma/\sigma(v)$ in the lattice $\overline{N}(\sigma(v))$ the equation (22) becomes

\[ e^{\sum_{i=k+1}^{k+l} z_i \langle \theta, b_i \rangle} - e^{c_1(\xi_{\theta}^{\vee})} = 0. \]

So these relations yield

\[ \sum_{i=k+1}^{k+l} z_i \langle \theta, b_i \rangle + c_1(\xi_{\theta}) = 0 \]

which are exactly the linear relations in the cohomology ring of toric stack bundles. Since $x_1, \cdots, x_k$ can be represented as linear combinations of $z_{k+1}, \cdots, z_l$, the algebra $\widetilde{R}_v$ is isomorphic to the ring $H^*(B)[z_{k+1}, \cdots, z_l]$ with relations

\[ \prod_{i \in I_{\overline{\Sigma}/\sigma}} z_i = 0, \quad \text{and} \quad \sum_{i=k+1}^{k+l} z_i \langle \theta, b_i \rangle + c_1(\xi_{\theta}) = 0. \]

So compared to the result in (10), $\widetilde{R}_v$ is isomorphic to $H^*(P \mathcal{X}(\Sigma)/\sigma, \mathbb{C})$.

The decomposition (9) then yields the following.

**Theorem 5.3.** Assume that the toric Deligne-Mumford stack $\mathcal{X}(\Sigma)$ is semi-projective. There is a Chern character map from $K_0^p(\mathcal{X}(\Sigma), \mathbb{C})$ to the Chen-Ruan cohomology $H_{CR}^*(P \mathcal{X}(\Sigma), \mathbb{C})$ which is a module isomorphism.

**Proof.** By Theorem 1.1, Lemma 5.1 and Proposition 5.2 the Chern character map

\[ ch : K_0^p(\mathcal{X}(\Sigma), \mathbb{C}) \longrightarrow H_{CR}^*(P \mathcal{X}(\Sigma), \mathbb{C}) \]

defined by $\mathcal{L} \mapsto ch(\mathcal{L})$ is a module isomorphism.
5.2. **Ring Homomorphism.** In this section we use the stringy $K$-theory product defined in [9] to study the ring homomorphism of Chern character.

Let $P\mathcal{X}(\Sigma) = [(P \times_{(C^*)^m} Z)/G]$ be the toric stack bundle associated to the stacky fan $\Sigma$ and the smooth variety $B$. Its $K$-theory admits the following decomposition (see e.g. [4], [2], [20]):

\[
K(P\mathcal{X}(\Sigma)) = K_G(P \times_{(C^*)^m} Z) = (K(I_G(P \times_{(C^*)^m} Z)))^G = \sum_{g \in G} (K(P \times_{(C^*)^m} Z)^g)^G.
\]

(24)

By Proposition [2,7] and [10], the twisted sectors $P\mathcal{X}(\Sigma/\sigma(\mathfrak{v}))$ of $P\mathcal{X}(\Sigma)$ are indexed by the box elements $\mathfrak{v} \in Box(\Sigma)$. For each $\mathfrak{v}$ in the box, there exists a unique $g \in G$ such that

\[
P\mathcal{X}(\Sigma/\sigma(\mathfrak{v})) \cong ([P \times_{(C^*)^m} Z]^g/G].
\]

Let $\mathcal{F}_{\mathfrak{v}1}, \mathcal{F}_{\mathfrak{v}2} \in K_0(P\mathcal{X}(\Sigma))$. The stringy $K$-theory product of [9] is defined by

\[
\mathcal{F}_{\mathfrak{v}1} \ast \mathcal{F}_{\mathfrak{v}2} = (I \circ e_3)_*(e_1^*\mathcal{F}_{\mathfrak{v}1} \otimes e_2^*\mathcal{F}_{\mathfrak{v}2} \otimes \lambda^{-1}(Ob_{\mathfrak{v}1,\mathfrak{v}2,\mathfrak{v}3}))
\]

where

\[
e_i : P\mathcal{X}(\Sigma/\sigma(v_1, v_2, v_3)) \longrightarrow P\mathcal{X}(\Sigma/\sigma(v_i))
\]

is the evaluation map, and $I : P\mathcal{X}(\Sigma/\sigma(v)) \to P\mathcal{X}(\Sigma/\sigma(v^{-1}))$ is the involution map. Needless to say, the stringy $K$-theory product is defined in a way very similar to that of Chen-Ruan cup product.

Let $P\mathcal{X}(\Sigma/\sigma(v))$ be a twisted sector and $W_v = T_{P\mathcal{X}(\Sigma)|P\mathcal{X}(\Sigma/\sigma(v))}$. Define $W_{v,k}$ to be the eigenbundle of $W_v$, where $v$ acts by multiplication by $\zeta^k = e^{2\pi ik/r}$. Following [9], we define

\[
\mathcal{T}_v := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{v,k}.
\]

For $v = \sum_{\rho_i \in \sigma(\mathfrak{v})} \alpha_i b_i$, this reads

\[
\mathcal{T}_v = \bigoplus_{\rho_i \in \sigma(\mathfrak{v})} \alpha_i \mathcal{L}_i.
\]

**Theorem 5.4.** The “stringy” Chern character morphism

\[
ch_{orb} : K_0(P\mathcal{X}(\Sigma), \mathbb{C}) \longrightarrow H^*_CR(P\mathcal{X}(\Sigma), \mathbb{C}), \quad ch_{orb}(\mathcal{F}_v) := ch(\mathcal{F}_v)td^{-1}\mathcal{T}_v
\]

is an isomorphism as rings under the stringy $K$-theory product.

**Proof.** This is a special case of [9], Theorem 9.5. By Theorem [5,3], the morphism is a module isomorphism. It remains to check the product. Let $\mathcal{L}_i$ and $\mathcal{L}_j$ be line bundles over $P\mathcal{X}(\Sigma)$ as defined before, then we have

\[
ch_{orb}(\mathcal{L}_i \ast \mathcal{L}_j) = ch(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \lambda^{-1}(\oplus_{\alpha_i=2} \mathcal{L}_i)^*) \cdot td^{-1}\mathcal{T}_{v_3^{-1}},
\]

\[
ch_{orb}(\mathcal{L}_i) \cup_{CR} ch_{orb}(\mathcal{L}_j) = ch(\mathcal{L}_i) \cdot td^{-1}\mathcal{T}_{v_1} ch(\mathcal{L}_j) \cdot td^{-1}\mathcal{T}_{v_2} \cdot e(\oplus_{\alpha_i=2} \mathcal{L}_i).
\]

Since

\[
td(\oplus_{\alpha_i=2} \mathcal{L}_i) \cdot ch(\lambda^{-1}(\oplus_{\alpha_i=2} \mathcal{L}_i)^*) = e(\oplus_{\alpha_i=2} \mathcal{L}_i)
\]

\[
td^{-1}\mathcal{T}_{v_1} \cdot td^{-1}\mathcal{T}_{v_2} \cdot td(\oplus_{\alpha_i=2} \mathcal{L}_i) = td^{-1}\mathcal{T}_{v_3^{-1}};
\]

\[
ch_{orb}(\mathcal{L}_i \ast \mathcal{L}_j) = ch(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \lambda^{-1}(\oplus_{\alpha_i=2} \mathcal{L}_i)^*) \cdot td^{-1}\mathcal{T}_{v_3^{-1}}.
\]
we conclude \( \text{ch}_{\text{orb}}(L_i \ast L_j) = \text{ch}_{\text{orb}}(L_i) \cup_{\text{CR}} \text{ch}_{\text{orb}}(L_j). \)

6. Example: Finite Abelian Gerbes

In [10], the degenerate case of toric stack bundles, namely finite abelian gerbes over smooth varieties, were studied. In this section we compute their \( K \)-theory. We first recall the construction of finite abelian gerbes.

Let \( N = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_s} \) be a finite abelian group, where \( p_1, \ldots, p_s \) are prime numbers and \( n_1, \ldots, n_s > 1 \). Let \( \beta : \mathbb{Z} \to N \) be given by the vector \( (1, 1, \cdots, 1) \). \( N_{\mathbb{Q}} = 0 \) implies that \( \Sigma = 0 \), then \( \Sigma = (N, \Sigma, \beta) \) is a stacky fan. Let \( n = \text{lcm}(p_1^{n_1}, \cdots, p_s^{n_s}) \), then \( n = p_1^{n_1} \cdots p_s^{n_s} \), where \( p_{i_1}, \ldots, p_{i_t} \) are the distinct prime numbers which have the highest powers \( n_{i_1}, \ldots, n_{i_t} \). Note that the vector \( (1, 1, \cdots, 1) \) generates an order \( n \) cyclic subgroup of \( N \). We calculate the Gale dual \( \beta^\vee : \mathbb{Z} \to \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \), where \( \text{DG}(\beta) = \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \). We have the following exact sequence:

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \stackrel{\beta}{\longrightarrow} N \longrightarrow \bigoplus_{i \in \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0,
\]

\[
0 \longrightarrow 0 \longrightarrow \mathbb{Z} \stackrel{(\beta)^\vee}{\longrightarrow} \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow \mathbb{Z} \oplus \bigoplus_{i \in \{i_1, \cdots, i_t\}} \mathbb{Z}_{p_i}^{n_i} \longrightarrow 0.
\]

So we obtain

\[
1 \longrightarrow \mu \longrightarrow \mathbb{C}^* \times \prod_{i \in \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \stackrel{\alpha}{\longrightarrow} \mathbb{C}^* \longrightarrow 1,
\]

where the map \( \alpha \) in (26) is given by the matrix \([n, 0, \cdots, 0]^t\) and \( \mu = \mu_n \times \prod_{i \in \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \cong N \).

The toric Deligne-Mumford stack associated with the data is

\[
\mathcal{X}(\Sigma) = [\mathbb{C}^*/\mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}] = \mathcal{B} \mu,
\]

i.e. the classifying stack of the group \( \mu \).

Let \( L \) be a line bundle over a smooth variety \( B \) and \( L^* \) the principal \( \mathbb{C}^* \)-bundle induced from \( L \) removing the zero section. From our twist we have

\[
L^* \mathcal{X}(\Sigma) = L^* \times_{\mathbb{C}^*} [\mathbb{C}^*/\mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}] = [L^*/\mathbb{C}^* \times \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i}],
\]

which is a \( \mu \)-gerbe \( \mathcal{X} \) over \( B \).

**Remark 6.1.** The structure of this gerbe is a \( \mu_n \)-gerbe coming from the line bundle \( L \) plus a trivial \( \prod_{i \notin \{i_1, \cdots, i_t\}} \mu_{p_i}^{n_i} \)-gerbe over \( B \).

For this toric stack bundle, \( \text{Box}(\Sigma) = N \), the Chen-Ruan cohomology was computed in [10].

**Proposition 6.2 ([10]).** The Chen-Ruan cohomology ring of the finite abelian \( \mu \)-gerbe \( \mathcal{X} \) is:

\[
H^*_{CR}(\mathcal{X}, \mathbb{Q}) \cong H^*(B, \mathbb{Q}) \otimes H^*(\mathcal{B} \mu, \mathbb{Q}),
\]

where \( H^*_{CR}(\mathcal{B} \mu; \mathbb{Q}) = \mathbb{Q}[t_1, \cdots, t_s]/(t_1^{p_1^{n_1}} - 1, \cdots, t_s^{p_s^{n_s}} - 1). \)
For the stacky fan $\Sigma = (N, 0, \beta)$, the minimal stacky fan is given by $\Sigma_{\text{min}} = (N, 0, \beta_{\text{min}})$, where $\beta_{\text{min}} = 0 : 0 \to N$ is the zero map. So the Gale dual map is still the map $\beta_{\text{min}}$, and 

$$G_{\text{min}} = \text{Hom}_\mathbb{Z}(N, \mathbb{C}^*) \cong \mu.$$ 

The characters of $\mu \simeq N$ are given by all the maps $\chi : \mu \to \mathbb{C}^*$. Since $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$, let $\chi_1, \ldots, \chi_s$ be the base generators of the characters of $N$ such that $\chi_1^{p_1^{n_1}}, \ldots, \chi_s^{p_s^{n_s}}$ are trivial. Every character $\chi_i$ determines a line bundle $L_i$ over $X$ such that $L_1^{p_1^{n_1}}$ is trivial. Then Theorem 1.1 implies

**Theorem 6.3.** The $K$-theory ring of the finite abelian gerbe $X$ is:

$$K_0(X) \simeq \frac{K(B)[L_1, \ldots, L_s]}{(L_1^{p_1^{n_1}}, \ldots, L_s^{p_s^{n_s}})}.$$ 

**Remark 6.4.** It is easy to see from Theorem 6.3 the $K$-theory ring of the finite abelian gerbes is independent to the triviality and non triviality of the gerbes.

By Theorem 6.2 and 6.3 we have:

**Theorem 6.5.** There exists a Chern character morphism from the $K$-theory ring $K_0(X, \mathbb{C})$ of the finite abelian $\mu$-gerbe $X$ to the Chen-Ruan cohomology $H_{CR}^*(X, \mathbb{C})$, which is a ring isomorphism.

**Remark 6.6.** Suppose that we have two finite abelian $\mu$-gerbes over $B$, one is trivial and the other is non trivial. We see that the $K$-theory ring and the Chen-Ruan cohomology ring cannot distinguish these two different stacks. However quantum cohomology rings of different gerbes are different in general [3].

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