Compactification of IIB Theory with Fluxes and Axion-Dilaton String Cosmology

Eiji Konishi\textsuperscript{1} and Jnanadeva Maharana\textsuperscript{2,3,}\textsuperscript{†}

\textsuperscript{1}Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{2}National Laboratory for High Energy Physics (KEK), Tsukuba, Ibaraki 305, Japan
\textsuperscript{3}Institute of Physics, Bhubaneswar-751005, India\textsuperscript{§}

Abstract

Compactification of type IIB theory on torus, in the presence of fluxes, is considered. The reduced effective action is expressed in manifestly S-duality invariant form. Cosmological solutions of the model are discussed in several cases in the Pre-Big Bang scenario.

\textsuperscript{*}E-mail address: konishi.eiji@s04.mbox.media.kyoto-u.ac.jp
\textsuperscript{†}E-mail address: maharana@iopb.res.in
\textsuperscript{§}Present and permanent address
1 Introduction

It is recognized that string theory offers the prospect of unifying the fundamental forces of Nature \cite{1}. The developments in string theory have shed lights in our understanding of the physics of black holes and have addressed important problems in cosmology. Furthermore, there is a lot of progress to establish connections between string theory and the standard model of particle physics which comprehensively explains a vast amount of experimental data. String theory is endowed with a rich symmetry structure. Notable among them are dualities \cite{2, 3, 4}. The strong-weak duality, S-duality, relates strong and weak coupling phases. In some cases these two phases of the same theory may be related and in some other cases strong and weak coupling regimes of two different theories are S-dual to each other. Type IIB string theory is an example of the former whereas, to recall a familiar example, heterotic string with $SO(32)$ gauge group is related by S-duality to type I theory with same gauge group in $D = 10$. The T-duality, which we mention in passing, is tested perturbatively. A simple illustration is to consider compactification of a spatial coordinate on $S^1$ of radius $R$. The perturbative spectrum of this theory matches with the one where the corresponding spatial coordinate is compactified on a circle of reciprocal radius. It is worthwhile to note that the web of dualities have provided us powerful tools to understand string dynamics in diverse dimensions.

The purpose of this article is to envisage toroidal compcatification of type IIB string theory in presence of constant 3-form fluxes along compact direction. Our goal is to investigate consequences of S-duality for reduced string effective action and examine possible mechanism for breaking of S-duality. Moreover, we elaborate a few points and sharpen them which were not addressed in the present perspectives in earlier works in the context of a four dimensional effective action where the symmetry is enhanced.

The vanishingly small value of cosmological constant, $\Lambda_c$, might be understood by invoking the naturalness argument as has been advocated by us \cite{5, 6, 7, 8}. Although there are several proposals to explain smallness of the cosmological constant, this issue has not been completely resolved to every one’s satisfaction. It is to be noted that at the present juncture S-duality does not manifest as an exact symmetry of Nature. In other words, one of the consequences of S-duality would be to discover magnetically charged particles and dyons. In this respect, we may mention that we have no conclusive experimental evidence so far in favor of S-duality as a symmetry in the domain of low energy. Moreover, we have not found evidence for massless dilaton and axion from ongoing experiments. The axion, one often discusses in string theory framework, finds itself in a difficult position to be identified with the axion that one is looking for in various experiments which appears in the standard model phenomenology and GUT phenomenology \cite{9, 10}. Furthermore, it may be argued from the cosmological considerations that the dilaton should acquire its vacuum expectation value rather early in the evolution of the Universe (see section 2). Thus
the dilaton and axion are expected to acquire mass, and S-duality may be a broken symmetry of Nature. As we shall see, for the model at hand, the reduced action does not admit a cosmological constant term when S-duality is preserved. If S-duality is spontaneously broken nonzero (positive) cosmological constant appears in our model. It is of interest to explore how S-duality is broken. Recently, spontaneous breaking of S-duality has been proposed by one of us by invoking the idea of gauging the $SL(2, \mathbb{R})$ group [11]. Our proposal is, if we invoke naturalness argument due to 't Hooft, the cosmological constant should remain small in the theory we are dealing with. We are aware that the scenario we have envisaged is not close to the Universe described by the standard model. However, we have provided a concrete example realizing the arguments of naturalness to qualitatively argue why $\Lambda_c$ could be small. We are also aware that there are subtle issues related with non-compact symmetries and their breaking. We have not addressed to those problems here.

In recent years, compactification of string theories with fluxes has attracted considerable attention [12, 13, 14, 15, 16, 17]. In the context of brane world scenario such compactifications have assumed special significance. There is a lot of optimism that this approach might eventually allow us to build model which will realize the well tested standard model of particle physics although we have not achieved this objective completely. There has been a lot of progress in the string landscape approach to study the rich vacuum structure of string theory and construct models which are close to the phenomenological descriptions of particle physics. It has been shown that the landscape scenario admits a de Sitter phase with appropriate background configurations and a novel mechanism to break supersymmetry [18]. The toroidal compactification and the symmetries of reduced effective action have been studied extensively [2, 3, 19, 20, 21]. Recently, compactification of heterotic string effective action on $d$-dimensional torus with constant fluxes along compact direction has been investigated in detail [22]. It was shown that the reduced action (with constant fluxes) can be cast in $O(d, d)$ invariant form. Moreover, the moduli acquire potentials which would otherwise not appear in standard dimensional reduction of the 10-dimensional action of heterotic theory.

As we shall show in sequel, the toroidal compactification of type IIB theory in the presence of fluxes associated with NS-NS and R-R backgrounds can be expressed in an S-duality invariant form. The type IIB action can be compactified on a torus to obtain manifestly S-duality invariant reduced effective action following the standard procedure [23]. In the present context, due to the coupling of axion-dilaton $\mathcal{M}$-matrix to the fluxes (see later) a potential term appears. We note that when we toroidally compactify the action and allow constant fluxes along the compact direction the resulting reduced action is no longer invariant under supersymmetry. The supersymmetry can be restored by adding appropriate sources [24]. However, our purpose is to investigate S-duality attributes of the reduced action. Moreover, as we shall elaborate in the following section, we argue that there might be a symmetry consideration in order to understand smallness of the cosmological constant. Furthermore, when we
focus our attention on \( D = 4 \), we can dualize the space-time dependent 3-form field strength and express the effective action in yet another S-duality invariant form as has been noted earlier [25].

We study the cosmological solutions of the effective action [26, 27]. Since in the present scenario axion-dilaton potential emerges due to the coupling of the doublet to the fluxes, it provides an opportunity to reexamine some of the well known issues in string cosmology. Notice that the fluxes dictate the form of the potential. Thus, unlike in the past, we have a frame work to introduce a potential in the effective action whose structure follows from the symmetry considerations. We study the graceful exit problem that arises in the context of Pre-Big Bang cosmology [27, 28]. In a simple case, we truncate the action by setting some of the scalars to zero (although we retain the axion and dilaton); however the axion-dilaton potential is kept in tact while addressing the graceful exit issue [29]. We find that the no-go theorem of Kaloper, Madden and Olive is still valid for our model [30]. We consider another form of truncated action to examine whether the Pre-Big Bang solution accompanied by scale factor duality [31] is admissable. Indeed, for this case we obtain a solution which satisfies the scale factor duality. However, the dilaton blows up at an instant when scale factor approaches a certain value [27]. Therefore, there is an epoch where the tree level string effective action is not trustworthy and the loop corrections have to be accounted for.

The paper is organized as follows. The next section is devoted to dimensional reduction of the ten dimensional action when fluxes are present along compact directions. It is shown that, for a simple compactification scheme, when \( D = 4 \), the action can be expressed in manifestly \( SL(3, \mathbb{R}) \) invariant form where moduli parameterize the coset \( \frac{SL(3, \mathbb{R})}{SO(3)} \), although this result was known for a while [25, 32]; we discuss this aspect from another point of view. Section III is devoted to solving the equations of motion. We consider different (truncated) version of the 4-dimensional action to obtain cosmological solutions. It is worthwhile to point out that during the initial developments of string cosmology the potentials were introduced by hand to explore various cosmological solutions. One of our motivations to study string cosmology starting from a nonsupersymmetric model is that it is expected that in the early Universe supersymmetry might not be preserved. The fourth section is devoted to discussions about the no-go theorem. A short appendix contains a detail calculation of axion-dilaton and scale factor evolution in the context of graceful exit problem.

## 2 Effective Action

The massless excitations of type IIB string theory consist of dilaton, \( \hat{\phi} \), axion, \( \hat{\chi} \), graviton, \( \hat{g}_{MN} \), two 2-form potential, \( \hat{B}^{(i)}_{MN}, \) \( i = 1, 2 \) and a 4-form potential \( \hat{C}_{MNPQ} \) with self-dual field strength. Note however, that a covariant 10-dimensional effective action for type IIB theory cannot be written down when we want to incorporate 5-
form self-dual field strength. In what follows, we present the 10-dimensional action in the string frame metric and do not include contribution of the field strength of $\hat{C}_{MNPQ}$. This omission does not affect our study of the symmetry properties of the action. As a notational convention we denote the fields in 10-dimensions with a hat. The action is

$$\hat{S} = \frac{1}{2} \int d^{10}x \sqrt{-\hat{G}} \left\{ e^{-2\hat{\phi}} \left( \hat{R} - \frac{1}{12} \hat{H}_{MNP}^{(1)} \hat{H}^{(1)\ MNP} + 4(\partial \hat{\phi})^2 \right) - \frac{1}{2} (\partial \hat{\chi})^2 \right. $n

$$ - \frac{1}{12} \hat{\chi}^2 \hat{H}_{MNP}^{(1)} \hat{H}^{(1)\ MNP} - \frac{1}{6} \hat{\chi} \hat{H}_{MNP}^{(1)} \hat{H}^{(2)\ MNP} - \frac{1}{12} \hat{H}_{MNP}^{(2)} \hat{H}^{(2)\ MNP} \right\}. \quad (1)$$

It is more convenient to consider string effective action which is expressed in terms of Einstein frame metric $\hat{g}_{MN}$ while discussing S-duality transformations and the invariance properties of the action. The two metrics are related by $\hat{g}_{MN} = e^{-\frac{1}{2}\hat{\phi}} \hat{G}_{MN}$, and the corresponding action is \[33, 34\]

$$\hat{S}_E = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\hat{g}} \left\{ \hat{R}_{\hat{g}} + \frac{1}{4} \text{Tr}(\partial_N \hat{\mathcal{M}} \partial^N \hat{\mathcal{M}}^{-1}) - \frac{1}{12} \hat{\mathcal{H}}_{MNP}^{T} \hat{\mathcal{M}} \hat{\mathcal{H}}^{MNP} \right\} \quad (2)$$

where the axion-dilaton moduli matrix $\hat{\mathcal{M}}$ and the H-fields are defined as follows

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\chi} e^{\hat{\phi}} + e^{-\hat{\phi}} & e^{\hat{\phi}} \\ \hat{\chi} e^{\hat{\phi}} & e^{\hat{\phi}} \end{pmatrix}, \quad \hat{\mathcal{H}}_{MNP} = \begin{pmatrix} \hat{H}^{(1)} \\ \hat{H}^{(2)} \end{pmatrix}_{MNP}. \quad (3)$$

The action Eq.(2) is invariant under the transformations

$$\hat{\mathcal{M}} \to \Lambda \hat{\mathcal{M}} \Lambda^T, \quad \hat{H} \to (\Lambda^T)\ ^{-1} H, \quad \hat{g}_{MN} \to \hat{g}_{MN}, \quad (4)$$

$\Lambda \in SL(2, \mathbb{R})$ and $\Sigma$ is metric of $SL(2, \mathbb{R})$

$$\Sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (5)$$

$$\Lambda \Sigma \Lambda^T = \Sigma, \quad \Sigma \Lambda \Sigma = \Lambda^{-1}, \quad (6)$$

$$\hat{\mathcal{M}} \Sigma \hat{\mathcal{M}} = \Sigma, \quad \Sigma \hat{\mathcal{M}} \Sigma = \hat{\mathcal{M}}^{-1}. \quad (7)$$

In order to facilitate toroidal compactification we choose the following upper triangular form for the vielbein

$$\hat{e}^A_M = \begin{pmatrix} e^r \alpha \beta E^a \\ 0 \end{pmatrix}. \quad (8)$$

Here $M, N \cdots$ denote the global indices and $A, B \cdots$ the local Lorentz indices. The 10-dimensional metric $\hat{g}_{MN} = e^A_M e^B_N \eta_{AB}$, where $\eta_{AB}$ is the 10-dimensional Lorentz metric. Here $\mu, \nu = 0, 1, 2, \cdots, D - 1$ and $\alpha, \beta = D, \cdots, 9$. Note that $r, s, \cdots$ denote
the $D$-dimensional local indices whereas $a, b, \cdots$ are corresponding rest of the local indices, taking values $D, \cdots, 9$.

Thus

$$ g_{\mu\nu} = e^\alpha_\mu e^\delta_\nu \eta_{rs} , \quad G_{\alpha\beta} = E^a_\alpha E^b_\beta \delta_{ab} . \quad (9) $$

$\eta_{rs}$ is flat space Lorentzian metric. With the above form of $E^A_M$ we note that

$$ \sqrt{-\hat{g}} = \sqrt{-g} \sqrt{G} . \quad (10) $$

Let us denote the space-time coordinates as $\{x^\mu, \mu = 0, 1, \cdots, D-1\}$ and the compact coordinate of $T^d$ as $\{Y^\alpha, \alpha = D, \cdots, 9\}$.

If we assume that the backgrounds do not depend on the compact coordinates $\{Y^\alpha\}$, then the 10-dimensional Einstein frame effective action reduces to $[20,21,23,35]$.

$$ S_E = \frac{1}{2} \int d^D x \sqrt{-g} \sqrt{G} \left\{ R + \frac{1}{4} \partial_\mu G_{\alpha\beta} \partial^\mu G^{\alpha\beta} + \partial_\mu \ln G \partial^\mu \ln G - g^{\mu\lambda} g^{\nu\rho} G_{\alpha\beta} F^\alpha_{\mu\nu} F^\beta_{\lambda\rho} - \frac{1}{4} G^{\alpha\beta} g^{\gamma\delta} \partial_\mu B^{(i)\alpha\beta} \partial^\mu B^{(j)\alpha\beta} - \frac{1}{4} G^{\alpha\beta} g^{\gamma\delta} g_{\mu\rho} H^{(i)\gamma\delta} \mathcal{M}_{ij} H^{(j)\mu\rho} \right. $$

$$ \left. + \frac{1}{4} \text{Tr}(\partial_\mu \mathcal{M} \Sigma \partial^\mu \mathcal{M} \Sigma) \right\} \quad (11) $$

by adopting the standard procedure for toroidal compactification of string effective action.

Note that the gauge fields $A^\alpha_\mu$ appear due to the prescription of vielbein keeping in mind that the action will be dimensionally reduced. When dimensional reduction is carried out these Abelian gauge fields are associated with the $d$-isometries. Moreover, the gauge fields $A^{(i)\alpha\mu}, \alpha = D, \cdots, 9, i = 1, 2$ come from the dimensional reduction of $\hat{B}^{(i)\alpha\beta}$ as is well known. Besides scalars $G_{\alpha\beta}$, additional set of scalars $B^{(i)\alpha\beta}$ also appear from the reduction of $\hat{B}^{(i)\alpha\beta}$.

The above action is expressed in the Einstein frame, $G$ being determinant of $G_{\alpha\beta}$. If we demand $SL(2,\mathbb{R})$ invariance of the above action, then the backgrounds are required to satisfy following transformation properties$^\dagger$.

$$ \mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T , \quad H^{(i)\mu\nu} \rightarrow (\Lambda^T)_{ij}^{-1} H^{(j)\mu\nu} , \quad (12) $$

$$ A^{(i)\alpha\mu} \rightarrow (\Lambda^T)_{ij}^{-1} A^{(j)\alpha\mu} , \quad B^{(i)\alpha\beta} \rightarrow (\Lambda^T)_{ij}^{-1} B^{(j)\alpha\beta} . \quad (13) $$

and

$$ g_{\mu\nu} \rightarrow g_{\mu\nu} , \quad A^\alpha_\mu \rightarrow A^\alpha_\mu , \quad G_{\alpha\beta} \rightarrow G_{\alpha\beta} . \quad (14) $$

We focus our attention on the $D = 4$ effective action and truncate it by setting some backgrounds to zero. From now on we set $A^\beta_\mu = 0$. There are two 2-form $^\dagger$If toroidal compactification of action eq.(1) is carried out then reduced action is not expressible in manifestly S-duality invariant form$^{[36]}$.
fields $\tilde{B}^{(i)}_{MN}$ coming from NS-NS and R-R sectors. When we compactify them to 4-dimensions, we keep only $B^{(i)}_{\mu\nu}$. The other field components $B^{(i)}_{\alpha\beta} = 0$, and for the time being we also set $B^{(i)}_{\alpha\beta} = 0$. The 4-dimensional action is

$$S_E^{(4)} = \frac{1}{2} \int d^4x \sqrt{-G} \sqrt{g} \left[ R + \frac{1}{4} \{ \partial_\mu G_{\alpha\beta} \partial^\mu G^{\alpha\beta} + \partial_\mu \ln G \partial^\mu \ln G \} - \frac{1}{12} H^{(i)}_{\mu\nu\rho} M^{ij} H^{(j)}_{\mu\nu\rho} ight. + \left. \frac{1}{4} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma) \right].$$

(15)

According to the compactification procedure adopted above, the field strengths along compact directions are vanishing. The case of constant nonzero fluxes will be considered later.

We choose a simple compactification scheme where only a single modulus, $y(x)$, appears and we adopt the following form of the metric

$$ds^2_{10} = g_{\mu\nu} dx^\mu dx^\nu + e^{\sqrt{3}y} dY^\alpha dY^\beta \delta_{\alpha\beta}.$$

(16)

The resulting action is

$$S_E^{(4)} = \frac{1}{2} \int d^4x \sqrt{-g} e^{\sqrt{3}y} \left[ R + \frac{5}{2} \partial_\mu y \partial^\mu y - \frac{1}{12} H^{(i)}_{\mu\nu\rho} M_{ij} H^{(j)\mu\nu\rho} + \frac{1}{4} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma) \right].$$

(17)

By rescaling the space-time metric, $g_{\mu\nu}$ we remove the overall factor of $e^{\sqrt{3}y}$ and bring the above action to the following form (we still denote the new space-time metric as $g_{\mu\nu}$).

The action is written by

$$S_E^{(4)} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - 2(\nabla y)^2 - \frac{e^{2\sqrt{3}y}}{12} H^{(i)}_{\mu\nu\rho} M_{ij} H^{(j)\mu\nu\rho} + \frac{1}{4} \text{Tr}(\partial_\mu \Sigma \partial^\mu \Sigma) \right].$$

(18)

Note that the 3-form field strengths $H^{(i)}_{\mu\nu\rho}$ can be dualized to trade for two pseudo scalars, $\sigma_i(x)$, $i = 1, 2$ in four dimensions. Moreover, the equations of motion for $H^{(i)}_{\mu\nu\rho}$ are conservation laws. Therefore, we expect that the five moduli $y(x)$, $\sigma_1(x)$, $\sigma_2(x)$, $\chi(x)$ and $\phi(x)$ will also reflect a symmetry. Indeed, they parameterize the coset $SL(3,R)/SO(3)$ [25]. Thus the action can be expressed in the following form

$$S_E^{(4)} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{4} \text{Tr}(\nabla U \nabla U^{-1}) \right]$$

(19)

using $SL(3,R)/SO(3)$ $U$-matrix

$$U = e^{2\phi - \frac{2}{\sqrt{3}y}} \begin{pmatrix}
1 & \chi & \sigma_1 - \chi \sigma_2 \\
\chi & \chi^2 + e^{-2\phi} & \chi(\sigma_1 - \chi \sigma_2) - \sigma_2 e^{-2\phi} \\
\sigma_1 - \chi \sigma_2 & \chi(\sigma_1 - \chi \sigma_2) - \sigma_2 e^{-2\phi} & (\sigma_1 - \chi \sigma_2)^2 + \sigma_2^2 e^{-2\phi} + e^{-4\phi + 2\sqrt{3}y}
\end{pmatrix}$$

(20)
with $\det U = 1$.

Note that the equation of motion for the $U$-matrix is a conservation law,

$$\partial_\mu (\sqrt{-g} U^{-1} \partial^\mu U) = 0$$

representing five equations of motions since there are only five fields parameterizing $U$-matrix. When the action is expressed in terms of the scalars instead of the $U$-matrix, the corresponding equations of motion apparently give the impressions as if there are interaction potentials among the scalars.

Therefore, one is led to believe that there are $SL(2, \mathbb{R})$ invariant potential contradicting the result of [23]. However, when suitable linear combinations of the equations of motion are taken, (derived from an action with component fields not of the form of the action (19)) indeed, one obtains five current conservation equations which match with the same number of conservation laws as one derives from an $SL(3, \mathbb{R})/SO(3)$ symmetric action.

Thus far we have considered the scenario when backgrounds do not depend on compact coordinates. However, it is worthwhile to note that the effective action depends on the field strengths. Therefore, if constant field strengths are added to the effective action they will not affect the equations of motion modulo appearance of a cosmological constant term in certain cases. The appearance of cosmological constant term would be ruled out if additional symmetry constraints are imposed (for example, positive cosmological constant is not admissible if supersymmetry is desired).

Now let us examine how the contribution of the fluxes appears in the effective action. They contribute as the field strengths, 3-forms, coming from the NS-NS and R-R sectors. Thus their additional contribution to equation (17) will be

$$-\frac{1}{12} \sqrt{-g} \sqrt{G} T^\alpha_\gamma \partial_{M} H_{\alpha^\prime}^\beta \partial_{\gamma^\prime} G^{\alpha^\prime} G^{\beta^\prime} G^{\gamma^\prime}.$$  (22)

When we consider our simple compactification scheme, (after space-time metric has been rescaled to arrive at (18)) this term is

$$-\frac{e^{-2\sqrt{3}y}}{12} \sqrt{-g} H_{\alpha^\prime}^{(i)} M_{ij} H_{\gamma^\prime}^{(j)} \partial_{\alpha^\prime^\prime} \partial_{\gamma^\prime^\prime}.$$  (23)

with the appearance of this term the four dimensional effective action, $S^D_4$, (when $H_{\mu\nu\lambda}^{(i)}$ have been dualized) takes the form

$$S^D_4 = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ R + \frac{1}{4} \text{Tr}(\partial_\mu M^{-1} \partial^\mu M) - 2 \partial_\mu y \partial^\mu y - \frac{e^{2\sqrt{3}y}}{2} \partial_\mu \sigma T M \partial^\mu \sigma 
- \frac{e^{-2\sqrt{3}y}}{12} H_{\alpha^\prime}^{(i)} M H_{\gamma^\prime}^{(j)} \right]$$  (24)
where $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ transforms as $SL(2, \mathbb{R})$ doublet like $\sigma \rightarrow (\Lambda^T)^{-1}\sigma$ and $\mathcal{H}_{\alpha\beta\gamma} = \begin{pmatrix} \mathcal{H}^{(1)}_{\alpha\beta\gamma} \\ \mathcal{H}^{(2)}_{\alpha\beta\gamma} \end{pmatrix}$ are the 3-form fluxes along compact directions.

The above action merits some discussions. We recall that in absence of $\mathcal{H}_{\alpha\beta\gamma}$ the action (19) is expressed in manifestly $SL(3, \mathbb{R})$ invariant form. The presence of the constant flux breaks $SL(3, \mathbb{R})$ symmetry. However, the above action is manifestly $SL(2, \mathbb{R})$ invariant. Moreover, notice the coupling of $\mathcal{M}$-matrix to the fluxes. Thus we have coupling of the $SL(2, \mathbb{R})$ multiplet to gravity and to a constant source term. It is well known that due to the coupling to the fluxes, the tadpoles make their appearances. Moreover, supersymmetry is not maintained in this scenario. As alluded earlier our goal is to explore various aspects of S-duality and we are aware that supersymmetries are not preserved. We would like to dwell on some other aspects due to the presence of fluxes in what follows.

The model under considerations is reminiscent of the works on chiral dynamics. It is worthwhile to note the issues addressed in the context of $\sigma$-model approach to pion physics. We might have a scenario where $SL(2, \mathbb{R})$ symmetry could be broken spontaneously. Alternatively, the S-duality could be broken explicitly if we assign specific configurations of the fluxes. Although the last term in the above action is $SL(2, \mathbb{R})$ invariant, once we assign a specific configuration to the fluxes the symmetry is explicitly broken and still one has an axion-dilaton potential depending on the choice of the flux that breaks the symmetry.

We mention in passing that dilaton is expected to settle down to its ground state value in early epochs of the universe. It is well known that dilaton has two roles. It belongs to the massless spectrum of string theory. More importantly, the vacuum expectation value of dilaton determines gauge couplings, Yukawa couplings (hence fermionic masses) and a lot of other important parameters. To rephrase arguments to Damour and Polyakov [37], dilaton must acquire its ground state value (say $\phi_0$) much before the era of nucleosynthesis. Otherwise, the so well tested results of big bang model in that era will be affected since delicate nuclear reactions depend crucially on masses of nucleons (and light nuclei) and other fermions which in turn are determined by fermionic Yukawa couplings which can be computed in principle from underlying string theory (all these are controlled by vacuum expectation value of dilaton).

Furthermore, at the present epoch of the universe there is no conclusive experimental evidence for a weak (as weak as gravity) long range repulsive universal interaction. Should dilaton remain massless, today, we expect such a repulsive force. Moreover, there are limits on dilaton mass to be $m_{\text{dil}} \geq 10^{-3}$ MeV.

If we consider the scenario of spontaneous symmetry breaking, we shall get a term like the cosmological constant as is obvious from eq. (21). Suppose the $\mathcal{M}$-matrix assumes a nonzero vacuum expectation value. Then, we may write $\mathcal{M} = \mathcal{M}_0 + \tilde{\mathcal{M}}$, $\mathcal{M}_0$ is the constant vacuum expectation value and $\tilde{\mathcal{M}}$ is being the fluctuation, $<\tilde{\mathcal{M}}>_0 = 0$. 
Then
\[-\frac{1}{12}\sqrt{-g}H^\alpha_{\beta\gamma}M_0H^{\alpha'\beta'\gamma'}\delta^{\alpha\alpha'}\delta^{\beta\beta'}\delta^{\gamma\gamma'}\] (25)
is the cosmological constant term when the ten dimensional action is compactified on a torus of constant moduli, i.e. the radius of compactification is chosen to be a constant in eq. (16) to start with (see discussion before eq. (34)).

3 Equations of Motion

In this section we intend to present equations of motion associated with the actions considered in the previous section and look for solutions. We shall envisage cosmological scenario. It will be shown that we can obtain exact cosmological solutions for the Friedmann-Robertson-Walker (FRW) metric in certain cases. We note that some of the interesting aspects of type IIB string cosmology have been studied by several authors and a detail references can be found in the review of article of Copeland, Lidsey and Wands.\[25\] However, as mentioned above, in some of those investigations either axion-dilaton potential were not incorporated or the potentials were introduced on ad hoc basis. Our approach differs from earlier works in this sense. In what follows we shall consider the \( k = 0 \) FRW metric, the spatially flat metric.

Let us first consider the equations of motion corresponding to action (19). The matter field equation is that of the \( SL(3,\mathbb{R}) \) nonlinear \( \sigma \)-model coupled to gravity
\[\partial_\mu(\sqrt{-g}U^{-1}\partial^\mu U) = 0 .\] (26)
The Einstein equation is
\[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{8}\sqrt{-g}\partial_\mu U^{-1}\partial_\nu U .\] (27)

In the cosmological scenario the FRW metric is
\[ds^2 = -dt^2 + a^2(t)dr^2 + a^2(t)d\Omega^2 .\] (28)
The equations of motion (using the 0-0 component of Einstein equation and matter equation)
\[16\dot{h} + 24h^2 - \text{Tr}(U^{-1}\dot{U}) = 0 ,\] (29)
with \( h = \dot{a}/a \) as the Hubble parameter, and
\[\partial_t(a^3U^{-1}\dot{U}) = 0 .\] (30)
Thus
\[a^3U^{-1}\dot{U} = A ,\] (31)
and \( A \) is a constant \( 3 \times 3 \) matrix.
It follows from the Hamiltonian constraint that

\[ 6h^2 + \frac{1}{4} \text{Tr}(U^{-1} \dot{U}) = 0. \]  

(32)

Thus combining these two equations, we solve for the scale factor and the \(U\)-matrix

\[ a(t) \sim t^{1/3}, \quad U = e^{A \text{int}}. \]  

(33)

Note that in our approach we are able to present a cosmological solution for the FRW metric. This is analogous to the solution obtained in [38] while dealing with \(O(d, d)\) symmetric action. Here we solve for the scale factor and the \(U\)-matrix parameterized in terms of the five fields, \(\phi, \chi, y, \sigma_1\) and \(\sigma_2\). The solution involves the integration constant \(A\) (now a \(3 \times 3\) matrix with five independent components). We may recall here, in their approach to type IIB string cosmology, Copleland, Lidsey and Wands [25] identified various \(SL(2, R)\) subgroups of \(SL(3, R)\) parameterized in terms of the different combinations of the above five fields besides the space-time metric. In that formulation, interactions among these fields (always involving derivatives or currents) were identified as potentials. Thus in the cosmological scenario, one encountered potentials. These authors found several types of solutions for homogeneous cosmologies for various \(SL(2, R)\) subgroups of the duality group \(SL(3, R)\). We would like to point out that from a purely group theoretic perspective, in the present approach, we are able to arrive the cosmological solution. This is an elegant and powerful method.

Let us turn our attention to the effective action \(S_D^D\), eq. (24), where we have already dualized the three forms \(H^{(i)}_{\mu \nu \rho}\) to \(\sigma_i\), and we have taken into account the presence of fluxes. In the cosmological scenario under consideration, the equations of motion cannot be solved exactly. The presence of fluxes breaks \(SL(3, R)\) to S-duality group \(SL(2, R)\). If we examine for equations of motion, the variation of modulus, \(y\), couples to \(\mathcal{M}\)-matrix and to \(\sqrt{-g} \partial_\mu \sigma^T \mathcal{M} \partial^\mu \sigma\). The equations of motion for \(\sigma_i\) are current conservations.

The following clarifying remark is in order at this stage. Let us consider a case when the 10-dimensional theory has been compactified to the 4-dimensional theory with a constant radius of compactification. In other words, \(y\), appearing in eq. (16) carries no space-time dependence. In such a case, the scalar field \(y(x)\) is constant and not dynamical to start with; however, \(B^{(i)}_{\alpha \beta}(x)\) can still appear as (scalar) dynamical degree of freedom. This situation is different from the case where the volume modulus stabilizes to some constant value due to a mechanism built into the theory. Moreover, if one retains the modulus, \(y(x)\), as dynamical in the reduced action and sets it to a constant value at the level of equation of motion, then the equation of motion for \(y\) (although it is set to a constant at that stage) imposes a constraint equation for other fields. This situation is to be contrasted with the former situation where \(y\) is already taken to be a constant right from the beginning when we compactify the 10-dimensional action. To illustrate further, if we considered \(G_{\alpha \beta}\) to be a constant then their corresponding kinetic energy term is eq. (11) will be absent whereas all
other terms will be present. In this paper when we use the phrase “frozen modulus” or freezing of modulus it is to be understood that $y(x)$ is set to be a constant when $D = 10$ action is compactified. Let us consider the following simple form of the action where we set $\sigma_i = 0$ and $y$ is a constant radius of compactification. The resulting action is

$$S_4 = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ R + \frac{1}{4} \text{Tr}(\partial_\mu M \Sigma \partial^\mu M \Sigma) - \frac{1}{12} \mathcal{H}_{\alpha\beta\gamma}^T M \mathcal{H}^{\alpha\beta\gamma} \right]. \quad (34)$$

We define the last term as $-\text{Tr} SM$ to simplify the notations where

$$S = \frac{1}{12} \begin{pmatrix} \mathcal{H}_{\alpha\beta\gamma}^{(1)} & \mathcal{H}_{\alpha\beta\gamma}^{(1)} & \mathcal{H}_{\alpha\beta\gamma}^{(1)} \\ \mathcal{H}_{\alpha\beta\gamma}^{(1)} & \mathcal{H}_{\alpha\beta\gamma}^{(2)} & \mathcal{H}_{\alpha\beta\gamma}^{(2)} \\ \mathcal{H}_{\alpha\beta\gamma}^{(1)} & \mathcal{H}_{\alpha\beta\gamma}^{(2)} & \mathcal{H}_{\alpha\beta\gamma}^{(2)} \end{pmatrix}. \quad (35)$$

The equation of motion associated with the $M$-matrix is

$$\partial_\mu (\sqrt{-g} \partial^\mu \mathcal{M} \mathcal{M}^{-1}) + \sqrt{-g} \mathcal{M} S - \sqrt{-g} \Sigma \mathcal{M} \Sigma = 0 \quad (36)$$

in the matrix notations. The above equation is derived from the action (34) keeping in mind that variation of $\mathcal{M}$ is constrained since $\mathcal{M} \in SL(2, \mathbb{R})$. Furthermore, if $\mathcal{M}$ were an ordinary matrix (not constrained) then its variation of the last term in action (34) will result in $S_{ij}$ in the equation of motion rather than combination of terms like $\mathcal{M} S$ and $S \mathcal{M}$. If we consider a situation where the $SL(2, \mathbb{R})$ symmetry is broken spontaneously, then in this phase, we may express $\mathcal{M} = \mathcal{M}_0 + \hat{\mathcal{M}}$ where $\mathcal{M}_0$ is the constant $SL(2, \mathbb{R})$ matrix and $\hat{\mathcal{M}}$ is its fluctuation. Now let us turn our attention to Einstein equation. In spontaneously symmetry broken phase the constant flux does couple to both $\mathcal{M}_0$ and $\hat{\mathcal{M}}$. Notice that the former is like a cosmological constant in the Einstein equation whereas the latter contributes to the stress energy momentum tensor in the usual fashion.

Now let us discuss the cosmological scenario when the metric and all other fields depend only on the cosmic time. In the presence of the flux term, we shall get a complete set of equations if we include $y(t)$, the modulus, and the other two sets of pseudo scalars, $\sigma_i(t)$, in the action. As noted earlier, the presence of flux introduces axion-dilaton potential, call it $\Lambda(\phi, \chi)$, which is $SL(2, \mathbb{R})$ invariant. We truncate the action with (and still the S-duality invariance is maintained) when moduli of torus is constant and $\sigma_i = 0$. Notice that the presence of the S-duality invariant potential offers the prospect of exhibiting some of the interesting aspects of string cosmology.

We recall that, in the Pre-Big Bang proposal of Gasperini and Veneziano, one encountered the problem of graceful exit. It was observed that graviti-dilaton cosmology encountered the no-go theorem [29] such that classically the two regimes could not be smoothly connected under certain situations. It was hoped that axion-dilaton cosmology might overcome the difficulty. We shall examine the graceful exit issues in the following.
The 4-dimensional effective action now we consider (including the scalars $B^{(i)}_{\alpha\beta}$) is

$$S_E^{(4)} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{4} \text{Tr}(\partial_{\mu} \mathcal{M}^{-1} \partial^{\mu} \mathcal{M}) - \frac{1}{4} \partial_{\mu} B^{(i)}_{\alpha\beta} \mathcal{M}_{ij} \partial^{\mu} B^{(j)\alpha\beta} ight. $$

$$\left. - \frac{1}{2} \partial_{\mu} \sigma^T \mathcal{M} \partial^{\mu} \sigma - \frac{1}{12} \mathcal{H}^{(i)}_{\alpha\beta\gamma} \mathcal{M}_{ij} \mathcal{H}^{(j)\alpha\beta\gamma} - 2(\partial y)^2 \right]. \quad (37)$$

In a cosmological scenario where modulus is frozen, the above action goes over to

$$S_E^{(4)} = \frac{1}{2} \int dt \sqrt{-g} \left[ R - \frac{1}{4} \text{Tr} \mathcal{M}^{-1} \mathcal{M} + \frac{1}{4} \dot{B}^{(i)}_{\alpha\beta} \mathcal{M}_{ij} \dot{B}^{(j)\alpha\beta} + \frac{1}{2} \dot{\sigma}^T \mathcal{M} \dot{\sigma} ight. $$

$$\left. - \frac{1}{12} \mathcal{H}^{(i)}_{\alpha\beta\gamma} \mathcal{M}_{ij} \mathcal{H}^{(j)\alpha\beta\gamma} \right]. \quad (38)$$

Equations of motion are

$$\frac{d}{dt} (\sqrt{-g} \mathcal{M}_{ij} \dot{B}^{(j)}_{\alpha\beta}) = 0 , \quad (39)$$

$$\frac{d}{dt} (\sqrt{-g} \mathcal{M}_{ij} \dot{\sigma}_j) = 0 , \quad (40)$$

$$\frac{d}{dt} (\sqrt{-g} (\dot{\mathcal{M}} \mathcal{M}^{-1})) + \sqrt{-g} \mathcal{M} \tilde{S} - \sqrt{-g} \Sigma \tilde{S} \mathcal{M} \Sigma = 0 \quad (41)$$

where $\tilde{S}$ also contains terms involving

$$\tilde{S} = \frac{1}{12} \mathcal{H}^{(i)}_{\alpha\beta\gamma} \mathcal{H}^{(j)\alpha\beta\gamma} - \frac{1}{2} \dot{\sigma}^i \dot{\sigma}^j - \frac{1}{4} \dot{B}^{(i)}_{\alpha\beta} \dot{B}^{(j)\alpha\beta} . \quad (42)$$

The first two equations (39) and (40) imply that the time integrations are

$$\mathcal{M}_{ij} \dot{B}^{(j)}_{\alpha\beta} = C^i_{\alpha\beta} , \quad \sqrt{-g} \mathcal{M}_{ij} \dot{\sigma}_j = D_i \quad (43)$$

$C$ and $D$ being time independent.

Thus when we consider equations of motion (41) associated with the $\mathcal{M}$-matrix and utilize the relation (43) in them in the definition of $\tilde{S}$, then we notice that elimination of the time derivatives of $B^{(i)}_{\alpha\beta}$ and $\sigma^i$ from $\tilde{S}$ leads to additional potential terms that depend on $\mathcal{M}$-matrix.

Let us discuss the presence of the axion-dilaton potential (the last term) in the action. One might hope that with the presence of a potential $\Lambda(\phi, \chi)$ it might be possible to circumvent the no-go theorem of Kaloper, Maddern and Olive for graceful exit [30] in the context of Pre-Big Bang scenario. In a simple setting where we consider axion-dilaton and the potential due to the fluxes (maintaining S-duality invariance), we find that the no-go theorem still holds (see the appendix for details).

Now we proceed to present an illustrative example for a cosmological solution. We compactify action (1) to $D = 4$ on $T^6$ of constant moduli. Then set $B^{(i)}_{\mu\nu} = 0,$
\[ B^{(1)}_{\mu\alpha} = 0, \quad B^{(2)}_{\mu\alpha} = 0, \quad A^{\alpha}_\mu = 0 \] and the flux \( \mathcal{H}^{(1)}_{\alpha\beta\gamma} = 0 \). The resulting action is
\[
\tilde{S} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ e^{-2\phi} \left\{ R + 4(\partial\phi)^2 \right\} - \left( \frac{1}{2} \partial^\mu B^{(2)}_{\alpha\beta} \partial^\nu B^{(2)\alpha\beta} + \frac{1}{12} \partial^\mu \mathcal{H}^{(2)}_{\alpha\beta\gamma} \mathcal{H}^{(2)\alpha\beta\gamma} \right) \right].
\] (44)

Where \( H^{(2)}_{\mu\nu\lambda} \) has been dualized to yield \( \sigma^{(2)} \). We have expressed the action (44) in such a way that the two terms in the curly bracket, multiplied by \( e^{-2\phi} \), are the contributions from the NS-NS states with 3-form set to zero. The rest of the terms come from the R-R sector and these are not multiplied by over all factor of \( e^{-2\phi} \).

Following Gasperini and Veneziano [27], we denote the second piece as \( S_m \), such that \( \delta S_m / \delta \phi = 0 \). When we go to the cosmological setting all the fields depend on cosmic time \( t \) and we can define \( 2\varphi = 2\phi - 3\ln \tilde{a} \), with \( ds^2 = -dt^2 + \tilde{a}^2(t)dx_idx_j\delta_{ij} \). \( \tilde{a}(t) \) is the scale factor in this string frame metric. Thus scalars coming from R-R sector, including \( \sigma_2 \), do not couple to dilaton. Furthermore, if we define density and pressure as
\[
T^0_0 = \varrho, \quad T^i_j = p\delta^i_j
\] (45)
with \( 2\delta S_m / \delta g^{\mu\nu} = T_{\mu\nu} \)

the massless scalars are pressureless fluids. However, the flux contributes a negative pressure ((\( \mathcal{H}^{(2)} \))^2). In order to establish correspondence with [27] we redefine the shifted dilaton \( \tilde{\phi} = 2\varphi \) the relevant quantities are defined as follows: \( \tilde{\varrho} = \varrho \tilde{a}^3 \), \( \tilde{p} = p\tilde{a}^3 \). The resulting equations are
\[
\dot{\tilde{\varrho}}^2 - 3\tilde{h}^2 = e^{\tilde{\varrho}} \tilde{\varrho},
\] (46)
\[
\dot{\tilde{h}} - \tilde{\varrho} \dot{\tilde{\varrho}} = \frac{1}{2} e^{\tilde{\varrho}} \tilde{p},
\] (47)
\[
2\dot{\tilde{\varrho}} - \dot{\tilde{\varrho}}^2 - 3\tilde{h}^2 = 0
\] (48)

where \( \tilde{h} \equiv \frac{\dot{\tilde{a}}}{\tilde{a}} \).

The solution to shifted dilaton is
\[
\tilde{\varphi} = \ln \frac{12}{k} \frac{\tilde{a}^\sqrt{3}}{(1 - \tilde{a}^\sqrt{3})^2}
\] (49)

\( k \) being the constant of integration. The scale factor is related to cosmic time through the following relation
\[
\frac{t}{t_0} = \frac{1}{\sqrt{3}} (\tilde{a}^\sqrt{3} - \tilde{a}^{-\sqrt{3}}) + 2\ln \tilde{a}
\] (50)
and \( t_0 \) is another integration constant. The scaled density and pressure (denoted by \( \tilde{\varrho} \) and \( \tilde{p} \) above) satisfy the relation
\[
\tilde{\varrho} = k\tilde{h}^2 \quad \text{and} \quad \tilde{p} = \frac{2}{\sqrt{3} \left(1 + \frac{\tilde{a}^\sqrt{3}}{\tilde{a}^{-\sqrt{3}}} \tilde{\varrho}\right)}.
\] (51)
We remind the reader that the equations of motion for R-R scalars lead to conservation law (corresponding charge conservation) like the previous case. We note that \( \{\tilde{a}, \tilde{\phi}\} \) satisfy scale factor duality during the entire epoch \( \tilde{a}(t) = \tilde{a}^{-1}(-t) \), \( \tilde{\phi}(\tilde{a}) = \tilde{\phi}(\tilde{a}^{-1}) \) and \( \tilde{\phi}(\tilde{a}) = \tilde{\phi}(\tilde{a}^{-1}) \). However as \( \tilde{a}(t) \rightarrow 1 \) dilaton assumes large value and it blows up at \( \tilde{a} = 1 \). Therefore, during this era we cannot trust the tree level string effective action and higher order corrections to the effective action have to be taken into account. Furthermore, the truncated action Eq. (44) is no longer manifestly S-duality invariant since we have removed some of the fields from the action associated with \( SL(2, R) \) transformation. This is a scenario where S-duality is broken explicitly which specific choice of flux. It is not obvious at this stage whether one can address the graceful exit issue in this frame work if all the scalars appearing in action (38) are included. It requires exhaustive study of that effective action.

As we have discussed in the appendix, if we take another truncated version of the action, keeping different set of fields and the full potential, it is not possible to circumvent the no-go theorem. In view of the above discussions the present work offers new opportunities to address the graceful exit problem. Furthermore, other compactification schemes might be adopted to study cosmology from some of the points of view presented here.

4 Summary and Discussion

In this section we summarize our results and discuss some future directions of investigations. We have adopted toroidal compactification of the type IIB string effective action and included contribution of 3-form fluxes along compact directions. We have imposed a requirement that the reduced action remains S-duality invariant. Thus the axion-dilaton potential generated due to the presence of fluxes is \( SL(2, R) \) invariant. Of course such a compactification scheme inherently contains tadpoles and supersymmetries are not preserved. We are aware of this aspect. However, our goal is to study the conventional string cosmology in this approach and address some of the issues. One of the interesting scenario, which is a special feature of graviton-axion-dilaton (string) cosmology is to understand accelerated expansion of the Universe according to Pre-Big Bang hypothesis. One of the issues encountered in the Pre-Big Bang approach is the graceful exit problem. In other words how does the Universe transit from its rapid accelerated expansion regime to the present FRW phase. It has been argued that such a transition is forbidden under certain circumstances due to no-go theorems. There were attempts to introduce phenomenological potentials to circumvent this difficulty. Moreover, it was very difficult to construct S-duality invariant axion-dilaton potential those days. However, with the present compactification scheme the potential is manifestly S-duality invariant. Therefore, we thought it to be appropriate to examine the graceful exit problem. We considered a truncated version of the reduced effective action (setting certain scalar fields to zero) where axion
and dilaton which parameterize $\frac{SL(2,\mathbb{R})}{SO(2)}$ were retained in the action and the $SL(2,\mathbb{R})$ invariant potential was also retained. We find that the no-go theorem of Kaloper, Madden and Olive is still valid.

The four dimensional action contains the axion-dilaton potential. The action is that of a non-linear sigma model. It is possible that the global $SL(2,\mathbb{R})$ symmetry might be spontaneously broken. Moreover, as alluded to in the introduction S-duality is not realized in Nature as an exact symmetry. However, it is believed that it is an exact symmetry of string theory. In this context, we may recall that when the M-theory is compactified on $T^2$ to type IIB theory in 9-dimensions, the resulting action can be cast in S-duality invariant form [39]. Moreover, ”the coupling constants” (the expectation values of dilaton and axion are the coupling constants) belong to $T^2$ on which M-theory is compactified. Thus the two coupling constants have geometric origin as has been argued by Schwarz [39]. Similarly, from the F-theory point of view axion-dilaton doublet of 10-dimensional type IIB theory has also geometrical interpretations [40].

This symmetry must be broken below the string scale. There are several reasons why dilaton, which appears as a massless excitation along with the graviton, is not expected to remain massless at the present epoch. There are arguments from the cosmological standpoint that dilaton must settle its ground state much before the era of nucleosynthesis. If we accept S-duality to be an exact stringy symmetry and that the dilaton does not remain massless at lower scales then the symmetry must be broken. Moreover, from the perspective of the standard model of particle physics axion is expected to be a light weakly interacting pseudoscalar particle. In a qualitative framework string theory contains many axions. However, if we identify the S-duality partner of dilaton to be the axion that cosmologist search for and standard particle physics model introduces then it is important to understand how S-duality is broken. If S-duality, realized non-linearly, is broken spontaneously then, in our model, it introduces a cosmological constant. We are unable to determine the symmetry breaking scale. We remind the reader that here one is discussing breaking of the non-compact $SL(2,\mathbb{R})$ symmetry. We are aware that in some cases one might encounter technical difficulties due to the noncompact nature of $SL(2,\mathbb{R})$. In the context of cosmological constant problem we would like to invoke the naturalness argument advanced by ’t Hooft to argue that $\Lambda_c$ is small [41]. Naturalness argument, to put qualitatively, says that if an exact symmetry dictates that a parameter in a theory is to vanish then that parameter will remain small when the symmetry is broken. In other words when a symmetry is restored if a parameter is forced to be zero the parameter will assume small value in the broken phase. It has been pointed out by ’t Hooft that if cosmological constant, $\Lambda_c$, is set equal to zero in the Einstein-Hilbert action then there is no enhancement of any symmetry of the action. However, contrast this case with a theory which contains fermions. If fermion mass is set to zero, the action is invariant under chiral symmetry. In a more general setting we encounter situations such that exact symmetry require certain parameter have vanishing values. If some parameters
assume small values, then the symmetries could be approximate.

We argue that cosmological constant arising out of spontaneous breaking of S-duality will remain small since in the unbroken phase it vanishes. However, our argument does not prevent from appearance of cosmological constant when other symmetries are broken. At every stage of symmetry breaking there is a contribution to the vacuum energy. The cosmological observations imply presence of a vanishingly small cosmological constant. It is worth pursuing our proposal to seek for a symmetry of the effective field theories that describe phenomena today such that the naturalness argument may be invoked to understand why \( \Lambda_c \) is so small. In our study of cosmological scenarios, we have dealt with four dimensional effective actions. In such cases we arrived at the reduced action by compactifying on torus of constant modulus. Thus the volume modulus is set to constant by hand. In other words we have no way to stabilize the volume modulus.

It will be interesting to solve the Wheeler-De Witt equation for the cosmological case in the minisuperspace framework. It is well known that the graceful exit forbidden in the classical string cosmology, due to the no-go theorems, could be achieved if one adopts the quantum version and solves the corresponding WDW equation \[42\]. The axion-dilaton WDW equation has some novel features \[43, 44\] in that the S-duality symmetry may be exploited to obtain the wave function for axion-dilaton purely from the group theoretic consideration. In fact the dynamics of the two scalars in similar to motion of a particle on the surface of a pseudosphere \[44\] in the background FRW metric. We may hope that similar considerations could apply when we include the presence of fluxes. However, it is obvious that the cosmological version of eq.(34) cannot be solved when we consider the corresponding WDW equation. Intuitively, we can argue that, although the interaction term is invariant under \( SL(2, \mathbb{R}) \) rotations, the axion-dilaton wave function of the corresponding Hamiltonian cannot be obtained as representations of \( SL(2, \mathbb{R}) \) as was the case in the absence of fluxes. If we denote the generators of \( SL(2, \mathbb{R}) \) as \( \Sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \Sigma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), then the \( \mathcal{M} \)-matrix can be expanded using unit matrix and \( \Sigma^i \) as a basis: \( \mathcal{M} = v_0 \mathbf{1} + v_1 \Sigma^1 + v_2 \Sigma^2 + v_3 \Sigma^3 \). It turns out from structure of \( \mathcal{M} \) is such that \( v_3 = 0 \) and the three (time dependent) coefficients satisfy the condition \( v_0^2 - v_1^2 - v_2^2 = 1 \), defining surface of the pseudosphere as mentioned. The axion-dilaton kinetic energy term expressed in terms of \( \{v_i; i = 0, 1, 2\} \) can be written as the Laplace-Beltrami operator. When we go over to the polar coordinates of the pseudosphere it becomes quite transparent that this is the Casimir of \( SL(2, \mathbb{R}) \) \[44\]. In the presence of the fluxes, in general, the potential will not commute with any of the generators of \( SL(2, \mathbb{R}) \) although it commutes with the Casimir. Therefore, if the flux is along a special direction (say the term commutes with the compact generator of \( SL(2, \mathbb{R}) \)) then we can obtain wave functions (for the "angular part") which are characterized by the Casimir and the eigenvalues of the compact generator. These
solutions will be infinite dimensional as is the case with unitary representations of \( SL(2, \mathbb{R}) \). But it is not possible to solve the full WDW equation in a closed form analytically for the scale factor part. However, it is worthwhile to resort to some approximation like WKB and examine various solutions to address the graceful exit problem.

We have all along discussed \( SL(2, \mathbb{R}) \) duality symmetry in the context of string cosmology which has been our main focus. It is well known that, in string theory, the S-duality symmetry corresponds to the discrete \( SL(2, \mathbb{Z}) \) subgroup of \( SL(2, \mathbb{R}) \). The axion-dilaton potential will be further constrained under \( SL(2, \mathbb{Z}) \). It is argued that \( SL(2, \mathbb{Z}) \) is a robust symmetry of string theory. We have addressed the issue of S-duality symmetry breaking in this article. However, it is beyond the scope of present investigation to determine the scale of the symmetry breaking. In other words, although we argue that in the present epoch axion and dilaton are expected to acquire mass, we are unable to identify the scale of the symmetry breaking. We hope to take up some of the issues in future.

**Acknowledgements**

One of us (EK) would like to thank Professor Tohru Eguchi for valuable discussions and advices. JM would like to acknowledge fruitful discussions with Professors Ashok Das, Hikaru Kawai, Romesh Kaul, and P. K. Tripathy. The gracious hospitality of Professor Yoshihisa Kitazawa and the Theoretical High Energy Physics Division of KEK is gracefully acknowledged by JM.

5 **APPENDIX: Graceful Exit Problem with Axion-Dilaton Potential**

In this short appendix we discuss the graceful exit problem following Kaloper, Madden and Olive. Note that our axion comes from R-R sector and therefore the relevant equations are modified appropriately. Our starting point is the 4-dimensional action eq. (34) written in terms of component fields in string frame metric. We have pulled out an over all factor of \( e^{-2\phi} \) and therefore the R-R field \( \chi \) and the flux term get multiplied accordingly. The last term (potential is defined to be 2\( \Lambda(\phi, \chi) \)). The action is

\[
S^{(4)} = \frac{1}{2} \int d^4x \sqrt{-g} e^{-2\phi} \left[ R + 4(\partial\phi)^2 - \frac{1}{2} e^{2\phi}(\partial\chi)^2 - 2\Lambda(\phi, \chi) \right].
\]  
(52)

In the cosmological case, all the fields are dependent on cosmic time (we consider the FRW metric with \( k = 0 \)). We denote the scale factor in the string frame as \( \tilde{a}(t) \) and corresponding Hubble parameter as \( \tilde{h} = \frac{\dot{\tilde{a}}}{\tilde{a}} \). Then the axion equation of motion is

\[
\ddot{\chi} + \frac{3}{\tilde{h}} \dot{\chi} + 2 e^{-2\phi} \frac{\partial \Lambda(\phi, \chi)}{\partial \chi} = 0.
\]  
(53)
The other equations are

\[-3\ddot{\tilde{h}}^2 + 6\dot{\phi}\ddot{\tilde{h}} - 2\dot{\phi}^2 + \frac{1}{4} e^{2\phi}\dot{\chi}^2 + \Lambda(\phi, \chi) = 0, \tag{54}\]

\[4(\ddot{\phi} - \dot{\phi}^2 + 3\ddot{\tilde{h}}\dot{\phi}) - 6\dot{\tilde{h}} - 12\ddot{\tilde{h}}^2 + 2\Lambda(\phi, \chi) - \frac{\partial\Lambda(\phi, \chi)}{\partial\phi} = 0, \tag{55}\]

\[4(\ddot{\phi} - \dot{\phi}^2 + 2\ddot{\tilde{h}}\dot{\phi}) - 4\dot{\tilde{h}} - 6\ddot{\tilde{h}}^2 + 2\Lambda(\phi, \chi) - \frac{1}{2} e^{2\phi}\dot{\chi}^2 = 0. \tag{56}\]

Eq. (54) is the Hamiltonian constraint which follows from variation of the lapse function, \(N(t)\) and then setting \(N(t) = 1\) as is the standard practice. The other two equations are associated with variation of dilaton and the scale factor.

We recall that the Hamiltonian constraint is quadratic in \(\dot{\phi}\) and it leads to

\[\dot{\phi} = \frac{3\ddot{h} \pm \sqrt{3\ddot{h}^2 + 2\Lambda + \rho/2}}{2}, \tag{57}\]

with \(\rho = e^{2\phi}\dot{\chi}^2\). Moreover, as is well known the other two equations are utilized to eliminate \(\ddot{\phi}\) and \(\dot{\phi}^2\), which miraculously led to the Pre-Big Bang scenario, and we are left with an equation for \(\dot{\tilde{h}}\)

\[\dot{\tilde{h}} = \pm\tilde{h}\sqrt{3\ddot{h}^2 + 2\Lambda + \rho/2} - \frac{1}{2} \frac{\partial\Lambda}{\partial\phi} + \frac{\rho}{4}. \tag{58}\]

Where the sign is appropriately chosen from Eq. (57). The potential we recall is

\[\text{Tr}(\mathcal{H}\mathcal{H}^T\mathcal{M}) \tag{59}\]

where \(\mathcal{H}\mathcal{H}^T\) is to be understood as a \(2 \times 2\) matrix with the (internal) tensor indices of \(\mathcal{H}^{(1)}\) and \(\mathcal{H}^{(2)}\) are appropriately contracted. The case where the \(SL(2,\mathbb{R})\) symmetry remains unbroken, the structure of the potential is such that it is positive everywhere and therefore, the no-go theorem of Kaloper, Madden and Olive remains valid.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, Vol I and Vol II, Cambridge University Press, 1987.

J. Polchinski, String Theory, Vol I and Vol II, Cambridge University Press, 1998.

K. Becker, M. Becker and J. H. Schwarz, String Theory and M-Theory: A Modern Introduction, Cambridge University Press, 2007.

B. Zwiebach, A First Course in String Theory, Cambridge University Press, 2004.

[2] A. Sen, Int. J. Mod. Phys. A 9, 3707, (1994) [arXiv:hep-th/9402002].

[3] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. C 244, 77 (1994) [arXiv:hep-th/9401139].
[4] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Nucl. Phys. Suppl. 41, 1 (1995) [arXiv:hep-th/9410237].

[5] J. Maharana and H. Singh, Phys. Lett. B 368, 64 (1996) [arXiv:hep-th/9506213]; S. Kar, J. Maharana and H. Singh, Phys. Lett. B 374, 43 (1996) [arXiv:hep-th/9507063].

[6] J. Maharana, Int. J. Mod. Phys. D 14, 2245 (2005).

[7] J. E. Lidsey, On Cosmology and Symmetry of Dilaton-Axion Cosmology, [arXiv:gr-qc/9609063].

[8] E. Konishi, Axion-Dilaton Gauged S-Duality and Its Symmetry Breaking, [arXiv:0710.1228 [hep-th]].

[9] P. Svrcek and E. Witten, JHEP 0606, 051, (2006), P. Svrcek, Cosmological Constant and Axions in String Theory, [arXiv:hep-th/0607086].

[10] K. E. Kim, A Review on Axions and the Strong CP Problem, [arXiv:0909.2595 [hep-th]].

[11] E. Konishi, Prog. Theor. Phys. 121, 1125 (2009) [arXiv:0902.2565 [hep-th]].

[12] K. Dasgupta, G. Rajesh and S. Sethi, JHEP 9908, 023 (1999) [arXiv:hep-th/9908088].

[13] R. Bousso and J. Polchinski, JHEP, 0006, 006 (2000) [arXiv:hep-th/0004134].

[14] S. B. Giddings, S. Kachru and J. Polchinski, Phys. Rev. D 66, 106006 (2002) [arXiv:hep-th/0105097].

[15] M. R. Douglas and S. Kachru, Rev. Mod. Phys. 79, 733 (2007) [arXiv:hep-th/0610102].

[16] M. Grana, Phys. Rep. C 423, 91 (2006) [arXiv:hep-th/0509003].

[17] F. Denef, Constructing String Vacua (Les Houches Lecture), arXiv:08003.1194 [hep-th].

[18] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. D 68, 046005 (2003) [arXiv:hep-th/0301240].

[19] J. Scherk and J. H. Schwarz, Nucl. Phys. B 153, 61 (1979).

[20] J. Maharana and J. H. Schwarz, Nucl. Phys. B 390, 3 (1993) [arXiv:hep-th/9207016].

[21] S. F. Hassan and A. Sen, Nucl. Phys. B 375, 103 (1992) [arXiv:hep-th/9109038].
[22] N. Kaloper and R. C. Myers, JHEP 9905, 010 (1999) [arXiv:hep-th/9901045].

[23] J. Maharana, Phys. Lett. B 402, 64 (1997) [arXiv:hep-th/9703009].

[24] P. K. Tripathy and S. P. Trivedi, JHEP 030, 028 (2003) [arXiv:hep-th/0301139]; S. Kachru, M. Schultz and S. P. Trivedi, JHEP 0310, 007 (2003) [arXiv:hep-th/0201028]; J. Kumar and J. D. Wells, JHEP 0509, 067 (2005) [arXiv:hep-th/0506252].

[25] E. J. Copeland, J. E. Lidsey and D. Wands, Phys. Rev. D 58, 043503 (1998) [arXiv:hep-th/9708153].

[26] E. J. Copeland, J. E. Lidsey and D. Wands, Phys. Rep. C 337, 343 (2000) [arXiv:hep-th/9909061].

[27] M. Gasperini and G. Veneziano, Phys. Rep. C 373, 1 (2003) [arXiv:hep-th/0207130].

[28] M. Gasperini and G. Veneziano, Astropart. Phys. 1, 317 (1993) [arXiv:hep-th/9211021].

[29] R. Brustein and G. Veneziano, Phys. Lett. B 329, 429 (1994) [arXiv:hep-th/9403060]; N. Kaloper, R. Madden and K. A. Olive, Nucl. Phys. B 452, 677 (1995) [arXiv:hep-th/9506027].

[30] N. Kaloper, R. Madden and K. A. Olive, Phys. Lett. B 371, 34 (1996) [arXiv:hep-th/9510117].

[31] G. Veneziano, Phys. Lett. B 265, 287(1991).

[32] J. Maharana, Phys. Lett. B 372, 53 (1996) [arXiv:hep-th/9511159]; J. Maharana (unpublished work) 1996.

[33] C. M. Hull, Phys. Lett. B 357, 545 (1995) [arXiv:hep-th/9506194].

[34] J. H. Schwarz, Phys. Lett. B 360, 13 (1995) [arXiv:hep-th/9508143].

[35] S. Ferrara, C. Kounnas and M. Porrati, Phys. Lett. B 181, 263 (1986).

[36] S. Roy, Int. J. Mod. Phys. A 13, 4445 (1998) [arXiv:hep-th/9705016].

[37] T. Damour and A. M. Polyakov, Nucl. Phys. B 423, 532 (1994) [arXiv:hep-th/9401069]; also see T. Damour and A. M. Polyakov, Gen. Rel. Grav. 26, 1171 (1994) [arXiv:gr-qc/9411069].

[38] K. Meissner and G. Veneziano, Mod. Phys. Lett. A 6, 3397 (1991); K. Meissner and G. Veneziano, Phys. Lett. B 267, 33 (1991).
[39] J. H. Schwarz, Phys. Lett. B 367, 97 (1996) arXiv:hep-th/9510086.

[40] C. Vafa, Nucl. Phys. B 469, 403 (1996) arXiv:hep-th/9602022.

[41] G. ’t Hooft, Under the Spell of the Gauge Principle, World Scientific, 1994.

[42] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B 472, 349 (1996) arXiv:hep-th/9602087.

[43] J. Maharana, S. Mukherji and S. Panda, Mod. Phys. Lett. A 12, 447 (1996) arXiv:hep-th/9701115.

[44] J. Maharana, Int. J. Mod. Phys. A 20, 1441 (2005) arXiv:hep-th/0405039.