Disorder-Induced Critical Phenomena in Hysteresis: A Numerical Scaling Analysis

Olga Perković, Karin A. Dahmen, and James P. Sethna

Laboratory of Atomic and Solid-State Physics, Cornell University, Ithaca, NY 14853–2501.

Experimental systems with a first order phase transition will often exhibit hysteresis when out of equilibrium. If defects are present, the hysteresis loop can have different shapes: with small disorder, the hysteresis loop has a macroscopic jump, while for large disorder the hysteresis loop is smooth. The transition between these two shapes is critical, with diverging length scales and power laws. We simulate such a system with the zero temperature random field Ising model, in 2, 3, 4, 5, 7, and 9 dimensions, with systems of up to 1000³ spins, and find the critical exponents from scaling collapses of several measurements. The numerical results agree well with the analytical predictions from a renormalization group calculation [13].

05.70.Jk,75.10.Nr,75.40.Mg,75.60.Ej

I. INTRODUCTION

The increased interest in real materials in condensed matter physics has brought disordered systems into the spotlight. Dirt changes the free energy landscape of a system, and can introduce metastable states with large energy barriers [1]. This can lead to extremely slow relaxation towards the equilibrium state. On long length scales and practical time scales, a system driven by an external field will move from one metastable local free-energy minimum to the next. The equilibrium, global free energy minimum and the thermal fluctuations that drive the system toward it, are in this case irrelevant. The state of the system will instead depend on its history.

The motion from one local minima to the next is a collective process involving many local (magnetic) domains in a local region - an avalanche. In magnetic materials, as the external magnetic field H is changed continuously, these avalanches lead to the magnetic noise: the Barkhausen effect [2,3]. This effect can be picked up as voltage pulses in a coil surrounding the magnet. The distribution of pulse (avalanche) sizes follows a power law. Internal field. At the critical temperature and field, the correlation length diverges, and the distribution of pulse (avalanche) sizes follows a power law.

We have argued earlier [12] that the Barkhausen noise experiments can be quantitatively explained by a model with two tunable parameters (external field and disorder), which exhibits universal, non-equilibrium collective behavior. The model is athermal and incorporates collective behavior through nearest neighbor interactions. The role of dirt or disorder, as we call it, is played by random fields. This paper presents the results and conclusions of a large scale simulation of that model: the non-equilibrium zero-temperature Random Field Ising Model (RFIM), with a deterministic dynamics. The results compare very well to our expansion [14,15], and to experiments in Barkhausen noise [12]. A more detailed comparison to experimental systems is in process [16].

The paper is divided as follows. Section II quickly reviews the model. Section III explains the simulation method that we use. Section IV explains the data analysis and shows results for the simulation in 2, 3, 4, and 5 dimensions, as well as in mean field. Section V gives a comparison between the simulation and the expansion exponents, and a comparison between the shape of the magnetization curves in 5, 7, and 9 dimensions, and the predicted shape from the expansion. Section VI summarizes the results. This is followed by three appendices that cover derivations that were omitted in the text for continuity.

II. THE MODEL

To model the long-range, far from equilibrium, collective behavior mentioned in the previous section, we define spins $s_i$ on a hypercubic lattice, which can take two values: $s_i = \pm 1$. The spins interact ferromagnetically with their nearest neighbors with a strength $J_{ij}$, and are sitting in a uniform magnetic field $H$ (which is directed
along the spins). Dirt is simulated by a random field \( h_i \), associated with each site of the lattice, which is given by a gaussian distribution function \( \rho(h_i) \):

\[
\rho(h_i) = \frac{1}{\sqrt{2\pi R}} e^{-\frac{h_i^2}{2R}} \quad (1)
\]

of width proportional to \( R \) which we call the disorder parameter, or just disorder. The hamiltonian is then

\[
\mathcal{H} = - \sum_{<i,j>} J_{ij} s_i s_j - \sum_i (H + h_i) s_i \quad (2)
\]

For the analytic calculation, as well as the simulation, we have set the interaction between the spins to be independent of the spins and equal to one for nearest neighbors, \( J_{ij} = J = 1 \), and zero otherwise.

The dynamics is deterministic, and is defined such that a spin \( s_i \) will flip only when its local effective field \( h_i^{eff} \):

\[
h_i^{eff} = J \sum_j s_j + H + h_i \quad (3)
\]

changes sign. All the spins start pointing down (\( s_i = -1 \) for all \( i \)). As the field is adiabatically increased, a spin will flip. Due to the nearest neighbor interaction, a flipped spin will push a neighbor to flip, which in turn might push another neighbor, and so on, thereby generating an avalanche of spin flips. During each avalanche, the external field is kept constant. For large disorders, the distribution of random fields is wide, and spins will tend to flip independently of each other. Only small avalanches will exist, and the magnetization curve will be smooth. On the other hand, a small disorder implies a narrow random field distribution which allows larger avalanches to occur. As the disorder is lowered, at the disorder \( R = R_c \) and field \( H = H_c \), an infinite avalanche in the thermodynamic system will occur for the first time, and the magnetization curve will show a discontinuity. Near \( R_c \) and \( H_c \), we find critical scaling behavior and avalanches of all sizes. Therefore, the system has two tunable parameters: the external field \( H \) and the disorder \( R \). We found from the mean field calculation \([4,13]\) and the simulation that a discontinuity in the magnetization exists for disorders \( R \leq R_c \), at the field \( H_c(R) \geq H_c(R_c) \), but that only at \( (R_c, H_c) \), do we have critical behavior. For finite size systems of length \( L \), the transition occurs at the disorder \( R_c^{eff}(L) \) near which avalanches first begin to span the system in one of the \( d \) dimensions (spanning avalanches). The effective critical disorder \( R_c^{eff}(L) \) is larger than \( R_c \), and \( R_c^{eff}(L) \to R_c \) as \( L \to \infty \).

### III. Algorithm

There are several methods that can be used to simulate the above model. The simplest but most time and space (memory) consuming method starts by assigning a random field to each spin on the hypercubic lattice. At the beginning of the simulation, all the spins are pointing down. The external field \( H \) is then increased by small increments, starting from a large negative value. After each increase of the field, all the spins are checked to find if one of them should flip (a spin flips when its effective field changes sign). If a spin flips, its neighbors are checked, and so on until no spins are left that can flip. Then, the external field is further increased, and the process repeated. Since the external magnetic field is increased in equal increments, a large amount of time is spent searching the lattice for spins that can flip. The increments have to be big enough to avoid searching the lattice when there are no spins that can flip, but small enough so that two or more spins far apart don’t flip at the same field. This is the method used experimentally, but it is suited only for “that kind of” massively parallel computing.

A variation on the above method, removes the searching through the lattice that is done even if there are no spins that can flip. It involves looking at all the spins, finding the next one that will flip and then increasing the external field so that it does. The average searching time for a flip is decreased, but is still very large. Far from the critical point, where spins will tend to flip independently of each other, the time for searching scales like \( N^2 \) where \( N \) is the number of spins in the system.

The search time can be further decreased if the random fields are initially ordered in a list. The first spin that will flip is the one on “top” of the list. The external field is increased until the effective field of the top spin become zero, and the spin flips. We then check its nearest neighbors, and so on, while keeping the external field constant. When no spins are left to flip, the external field needs to be increased again. The change in the external field \( \Delta H \), necessary to flip the next spin, is found by looking for the spin whose random field \( h_i \) satisfies:

\[
h_i \geq -(H_{old} + \Delta H) - (2n_\uparrow - z)J \quad (4)
\]

where \( H_{old} \) is the field at which the previous spins have flipped, \( z \) is the coordination number, and \( n_\uparrow \) is the number of nearest neighbors pointing up (\( s_j = +1 \)) for spin \( s_i \). In general, there will be a minimum of \( z + 1 \) spins to check from the list, since \( n_\uparrow \) can have the integer value between zero and \( z \). The spin for which equation (4) is satisfied for the smallest \( \Delta H \), and for which the number of up neighbors is \( n_\uparrow \), will flip. In general, more than \( z + 1 \) spins will need to be checked because a spin can satisfy equation (4) for some value of \( n_\uparrow \) but might not have that number of up neighbors, or the spin might have already flipped. This algorithm decreases the searching time since not all the spins need to be checked to find the next spin that will flip. Our early simulation work \([3,13]\) used this method. In practice, about half of the time was spent for the \( N \log_2 N \) initial sorting of the list of random field numbers, where \( N \) is the total number
of spins in the system. The big drawback of this method (as for the ones mentioned above) is the huge amount of storage space needed to store the random fields, the positions of each spin, and the values of the spins. This becomes particularly important when larger size systems are simulated.

The results in this paper use a more sophisticated algorithm which removes the need for a large storage space. It revolves around the idea that the change $\Delta H$ in the external field, between two avalanches, follows a probability distribution since the random fields $h_i$ are given by a Gaussian distribution. The increments $\Delta H$ in the external field should be chosen according to that distribution. The probability distribution itself is not known explicitly, but its integral from 0 to some finite $\Delta H$ is $P_{\text{noflip}}(\Delta H)$, that no $\uparrow$ spin will flip in the whole system during a field change less than $\Delta H$. It is given by:

$$P_{\text{noflip}}(\Delta H) = \Pi_{n_{\uparrow}} P_{n_{\uparrow}}(\Delta H)$$

(5)

where the product is over $n_{\uparrow} = 0, 1, ..., z$, and $P_{n_{\uparrow}}(\Delta H)$ is the probability for a down spin with $n_{\uparrow}$ up nearest neighbors not to flip when the external field changes by less than $\Delta H$:

$$P_{n_{\uparrow}}(\Delta H) = \left(1 - \frac{\int_{0}^{H_{\text{local}}(n_{\uparrow})} \rho(f) df}{\int_{0}^{H_{\text{local}}(n_{\uparrow})} \rho(f) df} \right)^{N_{n_{\uparrow}}}$$

(6)

The function $\rho(f)$ is the random field distribution function, and $H_{\text{local}}(n_{\uparrow})$ and $H_{\text{local}}^{\text{new}}(n_{\uparrow})$ are defined respectively as:

$$H_{\text{local}}(n_{\uparrow}) = -H - (2n_{\uparrow} - z)J$$

(7)

and

$$H_{\text{local}}^{\text{new}}(n_{\uparrow}) = -(H + \Delta H) - (2n_{\uparrow} - z)J.$$  

(8)

$P_{n_{\uparrow}}(\Delta H)$ gives the probability that a spin with $n_{\uparrow}$ up nearest neighbors has not flipped before the field has reached the external magnetic field value $H$:

$$P_{n_{\uparrow}}(H_{\text{local}}(n_{\uparrow})) = \frac{1}{2} + \int_{0}^{H_{\text{local}}(n_{\uparrow})} \rho(f) df$$

(9)

and $N_{n_{\uparrow}}$ is the number of down spins that have $n_{\uparrow}$ up neighbors.

A field increment $\Delta H$ that has the required probability distribution is found by choosing a uniform random number between zero and one and solving for $\Delta H$ from equation (5), by setting the probability $P_{\text{noflip}}(\Delta H)$ equal to the value of the random number. Once the increment $\Delta H$ is known, we can find the next spin that will flip. We first calculate $P_{n_{\uparrow}}(\Delta H)$ for a down spin with $n_{\uparrow}$ up neighbors to flip at the new field $H + \Delta H$:

$$P_{\text{flip}}(n_{\uparrow}) = \frac{R_{n_{\uparrow}}}{R_{\text{tot}}}$$

(10)

where

$$R_{n_{\uparrow}} = \frac{N_{n_{\uparrow}} \rho(H_{\text{local}}^{\text{new}}(n_{\uparrow}))}{P_{n_{\uparrow}}(H_{\text{local}}^{\text{new}}(n_{\uparrow}))}$$

(11)

is the rate at which down spins with $n_{\uparrow}$ up neighbors would flip, and $R_{\text{tot}}$ is the sum of the rates $R_{n_{\uparrow}}$ for all $n_{\uparrow}$. The spin that flips will have $k$ up neighbors, which is found by satisfying the following inequality:

$$\sum_{n_{\uparrow}=0}^{k} P_{\text{flip}}(n_{\uparrow}) > C > \sum_{n_{\uparrow}=0}^{k-1} P_{\text{flip}}(n_{\uparrow})$$

(12)

where the cutoff $C$ is a random number between 0 and 1. Once $k$ is known, a spin is then randomly picked from the list of down spins with $k$ up neighbors.

After the first spin has flipped, its neighbors are checked. The probability for one of the neighbors, with $(n_{\uparrow} + 1)$ up nearest neighbors, to flip at $H + \Delta H$, given that it has not yet flipped, is:

$$P_{\text{next}}(n_{\uparrow}, H + \Delta H) = 1 - \frac{1}{2} + \int_{0}^{H_{\text{local}}^{\text{new}}(n_{\uparrow} + 1)} \rho(f) df$$

(13)

When all the neighbors have been checked, the size of the avalanche is stored, as well as all the other measurements. The external magnetic field $H$ is then incremented again by finding the next $\Delta H$, starting back with equation (4).

The important characteristic of this method is that the random fields are not assigned to the spins at the beginning of the simulation, which for large system sizes decreases memory requirements tremendously (asymptotically, we use one bit per spin). This method has allowed us to simulate system sizes of up to $30000^2$, $1000^3$, $80^4$, and $50^5$ spins. The majority of the data analysis was performed on systems of sizes $7000^2$, $320^3$, $80^4$, and $50^5$. The SP1 and SP2 supercomputers at the Cornell Theory Center, and IBM RS6000 model 560 and J30 workstations were used for the simulation. Using this new algorithm, close to the critical disorder, one run (for a particular random field configuration) for a $320^3$ system took more than 1 CPU hour on a SP1 node at the Cornell Theory Center, while it took close to 37 CPU hours for a $800^3$ system on an IBM RS6000 model 560 workstation. Far above the critical disorder $R_c$, the simulation time increases substantially: 40% above the critical disorder, for the $320^3$ system, the simulation time was six times longer than for the simulation at 10% above $R_c$.

IV. THE SIMULATION RESULTS

The following measurements were obtained from the simulation as a function of disorder $R$:
• the magnetization $M(H, R)$ as a function of the external field $H$.
• the avalanche size distribution integrated over the field $H$: $D_{int}(S, R)$.
• the avalanche correlation function integrated over the field $H$: $G_{int}(x, R)$.
• the number of spanning avalanches $N(L, R)$ as a function of the system length $L$, integrated over the field $H$.
• the discontinuity in the magnetization $\Delta M(L, R)$ as a function of the system length $L$.
• the second $(S^2)_{int}(L, R)$, third $(S^3)_{int}(L, R)$, and fourth $(S^4)_{int}(L, R)$ moments of the avalanche size distribution as a function of the system length $L$, integrated over the field $H$.

In addition, we have measured:
• the avalanche size distribution $D(S, H, R)$ as a function of the field $H$ and disorder $R$.
• the distribution of avalanche times $D(t_{int})(S, t)$ as a function of the avalanche size $S$, at $R = R_c$, integrated over the field $H$.

The data obtained from the simulation was used to find and describe the critical transition. It was analyzed using scaling collapses. The mean field calculation for our model shows that near the critical point, the magnetization curve has the scaling form

$$M(H, R) - M_c(H_c, R_c) \sim |r|^{\beta} \mathcal{M}_{\pm}(h/|r|^{\delta})$$

where $M_c$ is the critical magnetization (the magnetization at $H_c$, for $R = R_c$), $r = (R_c - R)/R$ and $h = (H - H_c)$ are the reduced disorder and reduced field respectively, and $\mathcal{M}_{\pm}$ is a universal scaling function ($\pm$ refers to the sign of $r$). Both $r$ and $h$ are small. The critical exponent $\beta$ gives the scaling for the magnetization at the critical field $H_c$ ($h = 0$). Its mean field value is $1/2$, and the mean field value of $\delta$ is $3/2$. (Appendix A gives a short review on why scaling and scaling functions occur near a critical point, and why they have the form they do).

The significance of scaling for experimental and numerical data is as follows. If the magnetization data, for example, is plotted against the field $H$, there would be one data curve for each disorder $R$ (fig. 3a). While if we plot $|r|^{-\beta}M(H, R)$ against $h/|r|^{\delta}$, all the curves close to $R_c$ and $H_c$ will collapse (fig. 3b) onto either one of two curves: one for $r < 0$ ($\mathcal{M}_-$), and one for $r > 0$ ($\mathcal{M}_+$). The functions $\mathcal{M}_{\pm}$ depend only on the combination $h/|r|^{\delta}$ and not on the field $H$ and disorder $R$ separately, and are therefore universal. Usually, the exponents are unknown and scaling or data collapses are used to obtain them: the exponents are varied until all the curves lie on top of each other. This method is useful for analyzing numerical as well as experimental data, and is often preferred to “data fitting”, as we will show.

Numerical simulations and experiments are done on finite size systems. Often the properties of the system will depend on the linear size $L$. Functions that depend on the system’s length are analyzed using finite size collapses. An example is the number $N$ of spanning avalanches: $N(L, R) \sim L^\theta N(L^{1/\nu}|r|)$ (to be explained later). If $N$ is plotted against $R$, there would be one data curve for each length $L$. The exponents $\theta$ and $\nu$ are obtained by plotting $L^{-\theta}N(L, R)$ against $L^{1/\nu}|r|$ onto one curve (the collapse), and extracting the exponents.

The scaling forms we use for the collapses do not include corrections that are present when the data is not taken in the limit $R \to R_c$ and $L \to \infty$ (see appendix A for corrections that exist in those limits). On the other hand, finite size effects close to $R_c$ become important. It is thus necessary to extrapolate to $R \to R_c$ and $L \to \infty$ to obtain the correct exponents. We have done a mean field simulation to test our extrapolation method. The mean field exponents can be calculated analytically, but it is useful to check that the numerical results from the mean field simulation, for disorders away from $R_c$ and for finite sizes, extrapolate to the analytical values at $R = R_c$ and $1/L = 0$. We will see that this indeed occurs, and we will use the same extrapolation method in 3, 4, and 5 dimensions.

The mean field simulation was done with the same code, but with some changes. In mean field, the interactions between spins are infinite in range (each spin interacts when every spin in the system with the same interaction). This means that distances and positions are not relevant, and therefore we don’t need to keep track of the spins and their neighbors; we just need to know the total number of flipped spins, and the value of the external field $H$. The following section will show the results of the mean field simulation and explain the extrapolation method. Then, we will turn to results in 3, 4, and 5 dimensions. And finally, we will cover the more subtle scaling behavior in two dimensions.

### A. Mean Field Simulation

The mean field simulation shows how well the results for the critical exponents, obtained close to $R_c$ and for finite size systems (finite number of spins), extrapolate to the calculated values for a system in the thermodynamic limit, at the critical disorder. Thus, we will omit in this section some details that are only relevant for understanding the non-mean field simulation results. We start with the curves for the magnetization as a function of the field for different values of the disorder, which we find are not useful for extracting critical exponents. We then go on to measurements of spin avalanche sizes and their moments. Avalanches that span the system from one “side” to another will also be mentioned although since in mean field there are no “sides”, we will define what we mean by a mean field spanning avalanche. Since distances are irrelevant in mean field, we do not have any correlation measurements, but we can still apply what we learn from other collapses in mean field to the correlation
Figure 1 shows the magnetization curves, and figure 2a shows a scaling collapse for a $10^6$ mean field spin system and $r < 0$ ($R > R_c$). As mentioned earlier, near the critical point ($R_c = \sqrt{2/\pi}$ for $J = 1$, in mean field), the magnetization scales like \[ M(H, R) - M_c(H_c, R) \sim |r|^\beta \mathcal{M}_\pm(h/|r|^{\beta \delta}) \] (15) where $\pm$ refers to the sign of the reduced disorder $r = (R_c - R)/R_c$ and $h = (H - H_c)$. The mean field critical exponents are $\beta = 1/2$ and $\beta \delta = 3/2$. Notice in figure 2a that the scaling region around $M_c = 0$ and $H_c = 0$ is very small; figure 2b shows that a substantially different set of critical exponents leads to a similar looking collapse. In general, the critical field $H_c$ and the critical magnetization $M_c$ are not zero as in mean field, and $M_c$ is not well determined numerically. In dimensions that we simulate (2 through 5), the critical region is not only small but it is also poorly defined, which does not sufficiently constrain the values of the exponents. This makes the magnetization function $M(H, R)$ a poor choice for extracting critical exponents.

The critical magnetization $M_c$ can be removed from the scaling form if we look at the first derivative of the magnetization with respect to the field instead. $dM/dH$ scales like:

\[
\frac{dM}{dH}(H, R) \sim |r|^{\beta - \beta \delta} \mathcal{M}_\pm(h/|r|^{\beta \delta})
\] (16)

where $\mathcal{M}_\pm$ denotes the derivative of the scaling function $\mathcal{M}_\pm$ with respect to its argument $h/|r|^{\beta \delta}$. The $dM/dH$ mean field curves and collapses are shown in figure 2 and figures 3(a–b). Notice that the incorrect exponents $\beta = 0.4$ and $\beta \delta = 1.65$ give a better collapse (fig. 3). Figure 3 shows a close up of figure 2(a), alongside with three (thin) curves for disorders: 0.80, 0.81, and 0.82. These are not measured in the simulation (the finite number of mean field spins we use give rise to finite size effects near $R_c$ as we will see soon); instead they are numerically calculated from the mean field implicit equation for the magnetization \[ M(H) = 1 - 2 \int_{-\infty}^{-J^* M(H) - H} \rho(f) df \] (17)

where $J^*$ denotes the coupling of one spin to all the other spins in the system, and $\rho(f)$ is the random field distribution function.

The scaling collapse converges to the expected scaling function (dashed thick line) as we get closer to the critical disorder. The expected scaling form is also obtained from an analytic expression derived in mean field \[ g^3 + \frac{12}{\pi} g - \frac{12 \sqrt{2}}{\pi^{3/2} R_c} y = 0. \] (18)
We again find that the critical exponents and $R_c$, obtained from the $dM/dH$ curves, are ill-determined. In finite dimensions, that is even more true since we have another parameter to fit: $H_c$. For dimensions 3, 4, and 5, we extract $\beta$, $\beta\delta$, $H_c$, and $R_c$ by other means and simply show the resulting collapse of the $M(H)$ and $dM/dH$ curves as a check.

As mentioned earlier, the spins flip in avalanches of varying sizes. The distribution of all the avalanches that occur at a disorders $R$ while the external field $H$ is raised adiabatically from $-\infty$ to $+\infty$ is plotted in figure 3. The curves in this plot are normalized by the number of spins in the system, and therefore represent the probability per spin for an avalanche of size $S$ to occur in the hysteresis loop, at disorder $R$. The curves can be normalized to one if they are divided by the total number of avalanches in the loop, and multiplied by the number of spins in the system.

Often in experiments, the binned avalanche size distribution, which contains only avalanches that occur in a small range of fields around a particular value of the field $H$, is measured instead. The scaling form for this distribution [20] is [314]:

$$D(S, R, H) \sim S^{-\tau} \mathcal{D}_\pm(S^\sigma|r|, h/|r|^{\beta \delta})$$

where $S$ is the size of the avalanche and is large, and $r$ and $h$ are small. In mean field, $\sigma = 1/2$ and $\tau = 3/2$. The scaling form for the integrated avalanche size distribution is obtained by integrating the above form over all fields:

$$D_{\text{int}}(S, R) \sim \int S^{-\tau} \mathcal{D}_\pm(S^\sigma|r|, h/|r|^{\beta \delta}) \, dh$$

With the change of variable $u = h/|r|^{\beta \delta}$, equation (20) becomes:

$$D_{\text{int}}(S, R) \sim S^{-\tau} |r|^{-\beta \delta} \int \mathcal{D}_\pm(S^\sigma|r|, u) \, du$$

The integral in equation (21) is a function of $S^\sigma|r|$ only, so we can write it as:

$$(S^\sigma|r|)^{-\beta \delta} \mathcal{D}_\pm^{(\text{int})}(S^\sigma|r|)$$

(22)

to obtain the scaling form for the integrated avalanche size distribution:

$$D_{\text{int}}(S, R) \sim S^{-(\tau+\sigma \beta \delta)} \mathcal{D}_\pm^{(\text{int})}(S^\sigma|r|)$$

(23)

to obtain equation (22), we have assumed that the integral in (21) converges. This is usually safe to do since the distribution curves near the critical point drop off exponentially for large arguments. The same kind of argument can be made for the integrals of other measurements as well.
Close-up of the mean field \(dM/dH\) curves collapse in figure 4a. Also plotted are three curves (thin lines) calculated using the mean field analytic solution to \(M(H)\) (see text). These are for \(R = 0.80, 0.81,\) and \(0.82.\) We see that the scaling collapse, at the mean field exponents, of the \(dM/dH\) curves converges to the expected mean field scaling function (thick dashed line), as \(R \to R_c.\)

Mean field integrated avalanche size distribution curves for \(10^6\) spins and disorders \(R = 0.912, 0.974, 1.069, 1.197,\) and \(1.460\) (from right to left). The straight line is the slope of the power law behavior in mean field: \(\tau + \sigma \beta \delta = 9/4.\)

Figures 6a and 6b show two collapses with different critical exponents of the curves from figure 5, using the scaling form in equation (23). Notice that the collapse with the incorrect exponents \(\tau + \sigma \beta \delta = 2.4\) and \(\sigma = 0.44\) is better than the collapse with the mean field exponents \(\tau + \sigma \beta \delta = 9/4\) and \(\sigma = 1/2.\) Although the distribution curves in figures 6a and 6b have disorders that are far from the critical disorder \(R_c = 0.79788456,\) the curves collapse but with the wrong exponents.

Scaling collapse of three integrated avalanche size distribution curves in mean field, for disorders: \(1.069, 1.197,\) and \(1.460.\) The curves are smoothed over 5 data points before they are collapsed. The collapse is done using the mean field values of the exponents \(\sigma\) and \(\tau + \sigma \beta \delta\) (1/2 and 9/4 respectively), and \(r = (R_c - R)/R.\) (b) Same curves and scaling form as in (a), but with the exponents \(\sigma = 0.44\) and \(\tau + \sigma \beta \delta = 2.4.\) The collapse is better for the incorrect exponents! We use this “best” collapse to extract exponents for figures 8a and 8b, and then extrapolate to \(R = R_c\) to obtain the correct mean field exponents.

It is surprising that these curves collapse at all since the scaling form is correct only for \(R\) close to \(R_c.\) Corrections to scaling become important away from the critical point, but it seems that the scaling form has enough “freedom” that collapses are possible even far from \(R_c.\) In the limit of \(R \to R_c,\) we expect that the exponents obtained from such collapses will converge to the mean field value, and that the extrapolation will remove the question of scaling corrections. To test this, we have collapsed three curves at a time, and plotted the values of the exponents extracted from such collapses against the average of the reduced disorder \(|r|\) for the three curves, which we call \(|r|_{\text{avg}}\) (figures 8a and 8b). In these figures, notice two things. First, the linear extrapolation to \(|r|_{\text{avg}} = 0\) agrees quite well with the mean field exponent values, and second, the points obtained by doing collapses using \(r = (R_c - R)/R\) either converge faster to the mean field exponents or do as well as the points ob-
tained from collapses done with \( r = (R_c - R)/R_c \). This is true for all the extrapolations that we have done in mean field. Other models (see for example \([21]\)) exhibit this behavior, and experimentalists seem to have known about this for a while \([22]\).

In dimensions 2 to 5, we obtain the exponents \( \tau + \sigma \beta \delta \) and \( \sigma \) in the limit \( R \to R_c \), using the above linear extrapolation method. For other collapses, if the two extrapolation results differ substantially, we “bias” our result towards the \( r = (R_c - R)/R_c \) extrapolated value of the exponent.

Notice in figures \(8(a)\) and \(8(b)\) that the scaling function \( \bar{D}^{(int)}(S,R) \) has a “bump” (the \(-\) sign indicates that the collapse is for curves with \( R > R_c \)). Although we will come back to this point when we talk about the results in 5 and lower dimensions, it is interesting to know what the shape of the scaling function \( \bar{D}^{(int)}(S,R) \) is. In appendix B, we calculate the mean field scaling function for \( r < 0 \) (equations \([B14]\) and \([B15]\)):

\[
\bar{D}^{(int)}(-rS^\sigma) = \frac{e^{-rS^\sigma / 2}}{\pi \sqrt{2}} \times \int_0^\infty e \left( -(-r)S^\sigma u - \frac{u^2}{2} \right) \left( -rS^\sigma + u \right) \frac{du}{\sqrt{u}}
\]

where \( \sigma = 1/2 \).

A closed analytic form can not be obtained, but we can find the behavior of this function for small and large arguments \(-rS^\sigma\). For small arguments \( X = -rS^\sigma \), the scaling function is a polynomial in \( X \) \([B17]\), while for large arguments, the scaling function is given by the product of an exponential decay in \( X^2 \) and the square root of \( X \) \([B19]\). We can then try to fit our data (the scaling collapse) with a function that will incorporate a polynomial and an exponential decay (as an approximation to the real function). We obtain:

\[
e^{-\frac{X^2}{2}} (0.204 + 0.482X - 0.391X^2 + 0.204X^3 - 0.048X^4)
\]

This form has the expected exponential behavior at large \( X \), but the wrong pre-factor. On the other hand, for small \( X \), the above function is analytic. A better approach might be to use a parametric representation \([23]\), which we have not yet tried.

Equation \([23]\) can be compared with the curve obtained by numerically integrating the scaling function \( \bar{D}^{(int)}(S,R) \) in equation \([24]\). Figure \(8\) shows the fit in black (equation \([23]\)) to the collapsed data, for curves (dashed lines) of different disorder, and system size \( S = 10^6 \) and \( S = 10^7 \) spins. The grey curve is the “real” scaling function obtained from the numerical integration of equation...
Notice that the scaling collapse (done with the mean field values of the exponents $\tau + \beta \delta$ and $\sigma$) of even a system of $10^7$ spins and within 7% of $R_c$ (i.e., $R = 0.854$) (this is the curve with the smallest peak in the graph) is not close to the “real” scaling function (the thick grey curve). The error is within 5% for this curve (within 10% for the fit). However, as $R \rightarrow R_c$, the avalanche size distribution curves seem to be approaching the “real” scaling function (grey curve). It is important to keep in mind when analyzing experimental or numerical data as we will in 5 and lower dimensions, that the scaling collapse most likely does not give the limiting curve one would obtain for $1/L \rightarrow 0$ and $R \rightarrow R_c$, even for what seems like a large size, and close to the critical disorder.

The avalanches in the avalanche size distribution are finite, by which we mean that they don’t span the system. We have mentioned earlier that due to the finite size of a system, close to the critical disorder $R_c$, the largest avalanche or avalanches will span the system from one side to another. We will talk about spanning avalanches in more details later, but for now we just need to know that the number $N$ of spanning avalanches scales as $N(L, R) \sim L^{\theta} N_\pm (L^{1/\nu} |r|)$ where $N_\pm$ is a scaling function ($\pm$ indicates the sign of $r$), $L$ is the linear size of the system, $\theta$ is the exponent that arises from the existence of more than one spanning avalanche, and $\nu$ is the correlation length ($\xi$) exponent: $\xi \sim |r|^{-\nu}$.

As was mentioned earlier, in mean field there is no meaning to distance or lattice, and thus there are no “sides”. Purely for the purpose of testing our extrapolation method for finite size scaling collapses in the mean field simulation, we have defined a mean field “spanning avalanche” to be one with more than $\sqrt{S_{mf}}$ spins flipping at a field $H$, where $S_{mf}$ is the total number of spins in the system. (Note that the mean field exponents are valid for dimensions 6 and above, but that in those dimensions distances do have a meaning.) Using the above definition of a mean field spanning avalanche, it can be shown (see appendix C) that the scaling form for their number is:

$$N_{mf}(S_{mf}, R) \sim S_{mf}^{\frac{\theta}{1/\nu}} N_{\pm}^{\frac{\nu}{1/\nu}} (S_{mf}^{1/\nu} |r|)$$

and that the values of the exponents $\theta$ and $1/\nu$ are $3/8$ and $1/4$ respectively. $N_{mf}$ is the number of mean field spanning avalanches, while $N_{\pm}^{mf}$ is a universal scaling function. The exponents $\theta$ and $1/\nu$ are defined by the arbitrary definition for a spanning avalanche. Because of how they are defined, their values are different from the mean field values of $1/\nu = 2$ and $\theta = 1$, obtained from the renormalization group [14,15] and the exponent...
scaling relation $1/\sigma = (d - \theta)\nu - \beta$ \cite{14,15,6}.

Figure 10 shows the number of mean field spanning avalanches as a function of disorder, for several sizes, as well as the scaling collapse of the data. Note that the number of spanning avalanches close to the critical disorder $R_c = \sqrt{2/\pi}$ increases with the size $S_{mf}$ of the system, and that the peaks are getting narrower. The scaling collapse in the inset, shows only the three largest curves. For smaller sizes, the peaks do not collapse well with the larger size systems presumably due to finite size effects. The extrapolation plots for $\theta$ and $1/\tilde{\nu}$ are shown in figures 11a and 11b. On the horizontal axis of these two plots is the geometric mean of $1/S_{mf}$ for the three curves that are collapsed together, analogous to the extrapolation method used for the integrated avalanche size distribution. Note that the extrapolation to $1/S_{mf} \rightarrow 0$ for $\tilde{\theta}$ does not seem to be linear, and that the value of $1/\tilde{\nu}$ from the linear extrapolation of the $r = (R_c - R)/R$ data agrees better with the mean field value than the value obtained from the linear extrapolation of the $r = (R_c - R)/R_c$ data.

Note that we measure the avalanche size distribution only for disorders at which there are no “mean field spanning avalanches” (for a 10$^8$ system, that is for $R \geq 0.912$), since that’s what we do in dimensions 2 through 5 (finite dimensions) to avoid large finite size effects. For the second moments of the avalanche size distribution measurements (see below), the spanning avalanches were removed (same as in finite dimensions).

We have also measured the change in the magnetization $\Delta M$ due to all the spanning avalanches, as a function of the disorder $R$ (figure 12a). This gives us an independent measurement of the exponent $\beta$. In the thermodynamic limit above the critical disorder, there are no spanning avalanches so the change in the magnetization $\Delta M$ will be zero, while for small disorders the change in the magnetization will converge to one. Close and below the critical disorder $R_c$, at the critical field, the scaling form for the change in the magnetization due to the spanning avalanches will be (from equation (13)):

$$\Delta M(H = H_c, R) \sim |r|^\beta.$$  \hspace{1cm} (27)

For finite size systems, as shown in the figure, the change in the magnetization is not zero above the critical disorder: the data has to be analyzed using finite size scaling.

![Figure 12](image-url)

**FIG. 12.** (a) Change in the magnetization due to spanning avalanches as a function of disorder $R$. The data is for several mean field system sizes. The critical disorder is $R_c = 0.79788456$. The statistical errors are not larger than 0.005 (in units of $\Delta M$). (b) Mean field scaling collapse of the change in the magnetization curves for sizes $S_{mf} = 1000, 8000, 64000, 512000$. The exponents are $1/\tilde{\nu} = 0.25$ and $\beta/\tilde{\nu} = 0.125$ and $r = (R_c - R)/R$. The part of the curve that is collapsed is for $R > R_c$.

The dependence on the system size $S_{mf}$ can be brought in through a scaling function (see references \cite{18,19}) that we call $\Delta M_{\pm}$:

$$\Delta M(S_{mf}, R) \sim |r|^{\beta} \Delta M_\pm(S_{mf}^{1/\tilde{\nu}} |r|)$$  \hspace{1cm} (28)

where $\tilde{\nu}$ is defined above, and $\pm$ refers to the sign of $r$. We are free to define the scaling function $\Delta M_{\pm}$ as:

$$\Delta M_{\pm}(S_{mf}^{1/\tilde{\nu}} |r|) \equiv (S_{mf}^{1/\tilde{\nu}} |r|)^{-\beta} \Delta \bar{M}_\pm(S_{mf}^{1/\tilde{\nu}} |r|),$$  \hspace{1cm} (29)

where $\Delta \bar{M}_\pm$ is now a different scaling function. The scaling form for the change of the magnetization $\Delta M$ then becomes:

$$\Delta M(S_{mf}, R) \sim S_{mf}^{-\beta/\tilde{\nu}} \Delta \bar{M}_\pm(S_{mf}^{1/\tilde{\nu}} |r|).$$  \hspace{1cm} (30)

Figure 12b shows a collapse of the data using this scaling form. The collapse is done for disorders close to and
above the critical disorder, that is, for \( r < 0 \). The scaling function in figure 14b, in the range of the collapse, is therefore \( \Delta M \).

![Figure 13](image1.png)

**Figure 13.** (a) and (b) **Mean field exponents** \( 1/\nu \) and \( \beta/\nu \) respectively, from collapses of the magnetization change due to spanning avalanches (see text). The extrapolation to \( (1/S_{mf})_{gm} = 0 \) agrees with the calculated values.

Values for the exponents \( 1/\nu \) and \( \beta/\nu \) extracted from such collapses at several geometric average reciprocal sizes are shown in figures 13a and 13b. (These plots are done the same way as for the spanning avalanches exponents.) The linear extrapolation to \( (1/S_{mf})_{gm} = 0 \) is in very good agreement with the calculated values. Note that the extrapolation for \( 1/\nu \) of the \( r = (R_c - R)/R \) data gives again a better agreement with the calculated value than the extrapolation using the \( r = (R_c - R)/R_c \) data. The exponent \( \beta \) in 3, 4, and 5 dimensions is calculated from \( \beta/\nu \), which is extracted from the above kind of collapse. The obtained value is used to check the collapse of the \( M(H) \) and \( dM/dH \) data curves.

![Figure 14](image2.png)

**Figure 14.** **Mean field second moments** of the avalanche size distribution integrated over the field \( H \), for several different sizes. More than 20 points are used for each curve; each point being an average of a few to several hundred random field configurations. The error bars for the \( S_{mf} = 1,000 \) curve are too small to be shown. Curves at \( S_{mf} = 125 \) and 343 are not shown. The inset shows the collapse of these four curves at \( \epsilon = -(r + \sigma \beta \delta - 3)/\sigma \nu = 3/8 \) and \( 1/\nu = 1/4 \), which are the mean field calculated values.

Another quantity that is related to the avalanches is the moment of the size distribution. We have measured the second, third, and fourth moment, and we will show how the second moment scales and collapses in mean field. The second moment is defined as:

\[
\langle S^2 \rangle = \int S^2 D(S, R, H, S_{mf}) \, dS
\]

where \( D(S, R, H, S_{mf}) \) is the avalanche size distribution mentioned above, but with the system size \( S_{mf} \) included as a variable since we are looking for the finite size scaling form, as is clear from the data in figure 14. Recall that only non-spanning avalanches are included in the distribution function \( D(S, R, H, S_{mf}) \). Equation (31) can be written in terms of the scaling form for large sizes \( S \) of the avalanche size distribution \( D \):

\[
\langle S^2 \rangle \sim \int S^2 \, S^{-\tau} \, \tilde{D}_\pm(S^\sigma |r|, h/|r|^{\beta \delta}, S_{mf}^{1/\nu} |r|) \, dS
\]

As we have seen before, the dependence on the system size in the scaling function \( \tilde{D}_\pm \) is given by \( S_{mf}^{1/\nu} |r| \) where \( \nu \) is defined above through the definition of a mean field spanning avalanche. If we define

\[
\tilde{D}_\pm(S^\sigma |r|, h/|r|^{\beta \delta}, S_{mf}^{1/\nu} |r|) = (S^\sigma |r|)^{-2/\sigma - 1} \, \tilde{D}_\pm(S^\sigma |r|, h/|r|^{\beta \delta}, S_{mf}^{1/\nu} |r|)
\]

where \( \tilde{D}_\pm \) is a different scaling function, and let \( u = S|r|^{1/\sigma} \), we obtain:

\[
\langle S^2 \rangle \sim |r|^{(r - 3)/\sigma} \int \tilde{D}_\pm(u^\sigma, h/|r|^{\beta \delta}, S_{mf}^{1/\nu} |r|) \, du
\]

The integral in equation (34) is a function of \( h/|r|^{\beta \delta} \) and \( S_{mf}^{1/\nu} |r| \) only, so we can write:
 whose scaling form can be obtained by integrating the
Hermite of the distribution integrated over the field
to large sizes agrees with the calculated values for these
universal scaling function.

\[ \langle S^2 \rangle \sim |r|^{(r-3)/\sigma} \frac{1}{\sigma} \int S^{(2)}_x (h/|r|^\beta \delta, S^{1/\nu}_{m_f} |r|) \, dh \]  
(35)

which is the second moment scaling form, and \( S^{(2)}_x \) is a universal scaling function.

\( u = h/|r|^\beta \delta \), and call the remaining integral:

\[ \int S^{(2)}_x (h/|r|^\beta \delta, S^{1/\nu}_{m_f} |r|) \, dh = \]

\( (S^{1/\nu}_{m_f} |r|)^{(r+\sigma \delta -3)/\sigma} \bar{S}^{(2)}_x \)  
(36)

to obtain the second moment of the avalanche size distribution integrated over the magnetic field \( H \):

\[ \langle S^2 \rangle_{int} \sim S^{-1/2}_{m_f} S^{1/\nu}_{m_f} |r| \]  
(38)

where \( S^{(2)}_x \) is a universal scaling function (± indicates the sign of \( r \)). The mean field value for \(- (r + \sigma \beta \delta - 3)/\sigma \nu \)
is 3/8.

Figure 14 shows the integrated second moments of non-spanning mean field avalanches for several system sizes, and a collapse using the scaling form in equation (35). Figures 15a and 15b show the values for 1/\( \nu \) and \(- (r + \sigma \beta \delta - 3)/\sigma \nu \) respectively, for several geometric average reciprocal sizes, and show how well they linearly extrapolate to 1/\( S_{mf} \) → 0. These plots are done the same way as for the mean field spanning avalanches. Notice that for 1/\( \nu \), the linear extrapolation of the data using \( r = (R_c - R)/R \) gives a much better agreement with the calculated value than the linear extrapolation of the data obtained using \( r = (R_c - R)/R_c \).

To summarize this section, we have shown that the values of the critical exponents extracted from our mean field simulation by scaling collapses, extrapolate to the expected (calculated) values for \( R \rightarrow R_c \) and 1/\( S_{mf} \) → 0. Thus corrections to scaling due to finite sizes as well as finite size effects near the critical point seem to be avoided by extrapolation. The same extrapolation method is therefore used for extracting exponents in 3, 4, and 5 dimensions, which we will see next. The results in 2 dimensions will be shown last.

B. Simulation Results in 3, 4, and 5 Dimensions

1. Magnetization Curves

The magnetization as a function of the external field \( H \) is measured for different values of the disorder \( R \). Initially all the spins are pointing down (\( s_i = -1 \) for all \( i \)). The field is then slowly raised from a large negative value, until a spin flips. When the first spin has flipped, the external field is held constant while all the spins in the avalanche are flipping. The change in the magnetization due to this avalanche is just twice the size of the avalanche.

Figure 16a shows the magnetization curves obtained from our simulation in 3 dimensions for several values of the disorder \( R \). Similar plots can be obtained in 4 and 5 dimensions. \( R_c \) is the value of the disorder at which the discontinuity ("jump") in the magnetization curve appears. The critical disorder \( R_c \) is the value of the disorder at which this discontinuity appears for the first time as the amount of disorder is decreased, for a system in the thermodynamic limit. For finite size systems, like the ones we use in our simulation, the "jump" will occur earlier. The effective critical disorder for a system of size \( L \) is larger than the critical disorder \( R_c \) (1/L = 0). The critical disorder \( R_c \) is found from finite size scaling collapses of the spanning avalanches and second moments of the avalanche size distribution which will be covered later.

The values are listed in Table I.
We have seen in mean field that the magnetization curves near the transition scale as

\[ m(H, R) \sim |r|^\beta \mathcal{M}_\pm \left( \frac{h}{|r|} \right)^{\beta \delta} \]  
(39)

where \( m = M(H, R) - M_c(H_c, R_c), \) \( h = H - H_c, \) and \( \mathcal{M}_\pm \) is the corresponding scaling function. The critical magnetization \( M_c \) and critical field \( H_c \) are not universal quantities; in our mean field simulation and the hard-spin mean field model for our system \[14], both are zero; however they are non-zero quantities in a soft-spin model \[4].

In general, the scaling variables in \[39\] need not be \( r \) and \( h \), but can instead be some “rotated” variables \( r' \) and \( h' \) \[28\] which to first approximation can be written as:

\[ r' = r \pm ah \]  
(40)

and:

\[ h' = h \pm br \]  
(41)

(See appendix A for these and other corrections.) The constants \( a \) and \( b \) are not universal and the critical exponents do not depend on them (for the mean field data \( a = b = 0 \)). In equation \[39\], the scaling variables \( r \) and \( h \) should be replaced by the “rotated” variables \( r' \) and \( h' \), but since the measurements in our simulation are in terms of \( r \) and \( h \), we rewrite the scaling form in terms of those. We find that in the leading order of scaling behavior, the magnetization scales like:

\[ M(H, R) - M_c \sim |r|^{\beta} \widetilde{\mathcal{M}}_\pm \left( \frac{(h + br)}{|r|} \right)^{\beta \delta}. \]  
(42)

The correction \( br \) is dominant for \( R \to R_c \), and can not be ignored. The opposite is true for \( ah \) (see appendix A).

From the previous equation, the parameters that need to be fitted are \( M_c, H_c, \beta, \beta \delta, \) and the “tilting” constant \( b \). These should be found by collapsing the magnetization curves onto each other. As in mean field, we find that collapses of magnetization curves in 3, 4, and 5 dimensions do not define well the value of the critical magnetization \( M_c \). Furthermore, we observe strong correlations between the parameters, which lead to weak constraints on their values.

To remove the dependence on the critical magnetization \( M_c \), we can look at the collapse of \( dM/dH \) which scales like:

\[ \frac{dM}{dH}(H, R) \sim |r|^{\beta - \beta \delta} \widetilde{\mathcal{M}}_\pm \left( \frac{(h + br)}{|r|} \right)^{\beta \delta} \]  
(43)

Although \( M_c \) does not appear in the above form, the other parameters are still not uniquely defined by the collapse. We find that we need to extract \( \beta \) from the magnetization discontinuity \( \Delta M \) collapses, and \( \beta \delta \) and \( H_c \) from the binned avalanche size distribution collapses rather than from the magnetization curves themselves.

Using the values obtained from these collapses, and the value of \( R_c \), the “tilting” constant \( b \) is then found from magnetization curve collapses (figure 16b).
The collapsed curves show the scaling collapse using the same exponent and disorder $R_{4a})$. This is because the curves are still far from the critical curve, neither did they for the mean field theory curves (figure $R_{4b}$). The curves are smoothed by 10 data points before they are collapsed. (b) Scaling collapse of the data in (a) with $\beta = 0.036$, $\beta\delta = 1.81$, $b = 0.39$, $H_c = 1.435$, and $R_c = 2.16$. While the curves are not collapsing onto a single curve, neither did they for the mean field theory curves (figure $R_{4b}$). This is because the curves are still far from the critical disorder $R_c$.

Figure 17a shows the curves for the derivative of the magnetization $\Delta M$ with respect to the field $H$ for disorders $R = 2.35, 2.4, 2.45, 2.5, 2.6, 2.7, 2.85, 3.0$, and $3.2$ (highest to lowest peak), in 3 dimensions. The curves are smoothed by 10 data points before they are collapsed. (b) Scaling collapse of the data in (a) with $\beta = 0.036$, $\beta\delta = 1.81$, $b = 0.39$, $H_c = 1.435$, and $R_c = 2.16$. While the curves are not collapsing onto a single curve, neither did they for the mean field theory curves (figure $R_{4b}$). This is because the curves are still far from the critical disorder $R_c$.

Figure 17 shows the curves for the derivative of the magnetization with respect to the field $H$, and figure 17b shows the scaling collapse using the same exponent and parameter values as in figure 16b. The collapsed curves have disorders larger than the critical disorder: below $R_c$, the fluctuations are larger and the collapses are less reliable.

Since we found that $b \neq 0$ ($b = 0.39$ in 3d), the scaling variables are indeed some $r'$ and $h'$, and not the variables we measure: $r$ and $h$. Therefore, the scaling functions will in general be functions of a different combination of scaling variables from the ones we used in mean field, where the scaling variables are $r$ and $h$. However, we find in appendix A that the measurements that are integrated over the external field $H$ remove the “tilt” parameter $b$ (other analytic corrections might still be important though). This is true for the integrated avalanche size distribution, the avalanche correlation (integrated over the field), the number of spanning avalanches, the moment of the avalanche size distribution, and the time distribution of avalanche sizes. In the sections that treat these measurements, we will ignore the “rotation” of axis to simplify the presentation. Note that the change in the magnetization $\Delta M$ due to the spanning avalanches is integrated over only a small range of external fields (wherever there are spanning avalanches). On the other hand, the binned avalanche size distribution is not integrated over the field $H$, and we therefore examine this measurement more carefully.

2. Avalanche Size Distribution

a. Integrated Avalanche Size Distribution In our model the spins often flip in avalanches, which are collective flips of neighboring spins at a constant external field $H$. These avalanches come in different sizes. The integrated avalanche size distribution is the size distribution of all the avalanches that occur in one branch of the hysteresis loop (for $H$ from $-\infty$ to $\infty$). Figure 18 shows some of the raw data (thick lines) in 3 dimensions. Note that the curves follow a power law behavior over several decades. Even 50% away from criticality (at $R = 3.2$), there are still two decades of scaling, which implies that the critical region is large. In experiments, a few decades of scaling could be interpreted in terms of self-organized criticality (SOC). However, our model and simulation suggest that several decades of power law scaling can still be present rather far from the critical point (note that the size of the critical region is non-universal). In the figure, the cutoff in the power law which diverges as the critical disorder $R_c$ is approached ($R_c = 2.16$ in 3 dimensions), is a signature that the system is away from criticality, and that a parameter can be tuned (here $R$) to bring it to the transition. This cutoff scales as $S \sim |r|^{-1/\nu}$, where $S$ is the avalanche size and $r = (R_c - R)/R$ is the reduced disorder.

The power law for the curves of figure 18 can be obtained through scaling collapses. A plot is shown in the inset of figure 18. The scaling form is (see mean field section)

$$D_{int}(S, R) \sim S^{-(\tau + \sigma\beta\delta)} \bar{D}_{int}(S^{\sigma}|r|)$$

where $\bar{D}_{int}$ is the scaling function (the $-$ sign indicates that the collapsed curves are for $R > R_c$). The critical exponents $\tau + \sigma\beta\delta = 2.03$ and $\sigma = 0.24$ are obtained from collapses and linear extrapolation of the extracted values to $R = R_c$ (figures 18a and 19a), as was done in mean field. (Although the “real” scaling variables are $r'$ and $h'$, when integrating over the field $H$ we recover the same form as in mean field; see appendix A.) Table 1 lists all the exponents extracted from scaling collapses, and extrapolated to $R \to R_c$ and $1/L \to 0$. 

![Figure 17](image-url)
FIG. 18. Avalanche size distribution integrated over the field \( H \) in 3 dimensions, for \( \text{320}^3 \) spins and disorders \( R = 4.0, 3.2, 2.6 \). The last curve is at \( R = 2.25 \), for a \( 1000^3 \) spin system. The \( \text{320}^3 \) curves are averages over up to 16 initial random field configurations. All curves are smoothed by 10 data points before they are collapsed. The inset shows the scaling collapse of the integrated avalanche size distribution curves in 3 dimensions, using \( \tau = (R_c - R)/R \), \( \tau > 0.5 \beta \delta \beta = 0.23 \) and \( \sigma = 0.24 \), for sizes \( 160^3, 320^3, 800^3 \), and \( 1000^3 \), and disorders ranging from \( R = 2.25 \) to \( R = 3.2 \) \((R_c = 2.16)\). The two top curves in the collapse, at \( R = 3.2 \), show noticeable corrections to scaling. The thick dark curve through the collapse is the fit to the data (see text). In the main figure, the distribution curves obtained from the fit to the collapsed data are plotted (thin lines) alongside the raw data (thick lines). The straight dashed line is the expected asymptotic power law behavior: \( S^{-2.03} \), which does not agree with the measured slope of the raw data due to the shape of the scaling function (see text).

We have mentioned earlier that the mean field scaling function \( D^{(\text{int})}(X) \), with \( X = S^\beta |r| \) and \( r < 0 \), is a polynomial for small \( X \) and gives an exponential in \( X^{1/\sigma} \) (\( 1/\sigma = 2 \) in mean field) multiplied by \( X^\beta \) (\( \beta = 1/2 \) in mean field) for large \( X \) (see mean field section and appendix B). As we have done in mean field, we can try to fit the scaling function \( \bar{D}^{(\text{int})}(X) \) in dimensions 5 and below with a product of a polynomial and an exponential function. This is done in 3 dimensions in the inset of figure 18 (thick black line through the data). The phenomenological fit is:

\[
D^{(\text{int})}(X) = e^{-0.789 X^{1/\sigma}} \times (0.021 + 0.002 X + 0.531 X^2 - 0.266 X^3 + 0.261 X^4)
\]

with \( 1/\sigma = 4.20 \) which is obtained from scaling collapses. The distribution curves obtained using the above fit are plotted (thin lines in figure 18) alongside the raw data (thick lines). They agree remarkably well even far above \( R_c \). We should recall though, from the mean field discussion (see figure 1), that the fitted curve to the collapsed data can differ from the “real” scaling function even for large sizes and close to the critical disorder (in mean field the error was about 10%). We expect a similar behavior in finite dimensions.

FIG. 19. (a) and (b) \( \tau + \sigma \beta \delta \) and \( \sigma \) respectively, from collapses of the integrated avalanche size distribution curves for a \( \text{320}^3 \) spin system. The data is plotted as in mean field. The two closest points to \( |r|_{\text{avg}} = 0 \) are for a \( 800^3 \) system, for a collapse using curves with disorder: 2.26, 2.28, 2.30, 2.32, 2.34, and 2.36. The extrapolation to \( |r|_{\text{avg}} = 0 \) gives: \( \tau + \sigma \beta \delta = 2.03 \) and \( \sigma = 0.24 \).

The scaling function in the inset of figure 18 has a peculiar shape: it grows by a factor of ten before cutting off. The consequence of this shape is that in the simulations, it takes many decades in the size distribution for the slope to converge to the asymptotic power law. This can be seen from the comparison between a straight line fit through the \( R = 2.25 \) \((1000^3)\) curve in figure 18 and the asymptotic power law \( S^{-2.03} \) obtained from scaling collapses and the extrapolation (thick dashed straight line in the same figure). A similar “bump” exists in other dimensions and mean field as well. Figure 20 shows the scaling functions in different dimensions and in mean field. In this graph, the scaling functions are normalized to one and the peaks are aligned (the scaling forms allow this). The curves plotted in figure 20 are not raw data but fits to the scaling collapse in each dimensions, as was done in the inset of figure 18. The mean field and 3 dimensions curves are given by equations (23) and (24) respectively. For 5, 4, and 2 dimensions, we have respectively:

\[
\bar{D}^{(\text{int})}(X) = e^{-0.518 X^{1/\sigma}} \times
\]
(0.112 + 0.459X - 0.260X^2 + 0.201X^3 - 0.050X^4) \quad (46)

\[ D_\pm^\text{(int)}(X) = e^{-0.954X^{1/\sigma}} \times (0.058 + 0.396X + 0.248X^2 - 0.140X^3 + 0.026X^4) \quad (47) \]

with $1/\sigma = 2.35, 3.20, \text{and} 10.0$. The errors in the fits are in the same range as for the mean field simulation data (see figure 9). The 2 dimensional fit plotted in grey will be covered further in the next section.

\[ D_\pm^\text{(int)}(X) = e^{-1.076X^{1/\sigma}} \times (0.492 - 4.472X + 14.702X^2 - 20.936X^3 + 11.303X^4) \quad (48) \]

From figure 20 we can conclude that in each dimension (and in mean field!), a straight line fit to the integrated avalanche size distribution data is going to give the *wrong* critical exponent, and that only by doing scaling collapses and an extrapolation the asymptotic value can be found. This is shown for 3 dimensions in figure 18, and was found to be true in other dimensions as well. We will next see that this is different for the *binned* avalanche size distribution. The value for the slope obtained from a linear fit to the data agrees very well with the value obtained from the scaling collapses.

**FIG. 20. Integrated avalanche size distribution scaling functions in 2, 3, 4, and 5 dimensions, and mean field.** The curves are fits (see text) to the scaling collapses done with exponents from Table II and VII. The peaks are aligned to fall on (1,1). Due to the “bump” in the scaling function the power law exponent can not be extracted from a linear fit to the raw data.

**FIG. 21.** (a) *Binned in H* avalanche size distribution in 4 dimensions for a system of $80^4$ spins at $R = 4.09$ ($R_c = 4.10$). The critical field is $H_c = 1.265$. The curves are averages over close to 60 random field configuration. Only a few curves are shown. (b) Scaling collapse of the binned avalanche size distribution for $H < H_c$ (upper collapse) and $H > H_c$ (lower collapse). The critical exponents are $\tau = 1.53$ and $\sigma \beta = 0.54$, and the critical field is $H_c = 1.265$. The bins are at fields: 1.162, 1.185, 1.204, 1.220, 1.234, 1.245, 1.254, 1.276, 1.285, 1.296, 1.310, 1.326, 1.345, and 1.368. Notice that the two scaling functions do not have a “bump” (see text).

**b. Binned in H Avalanche Size Distribution** The avalanche size distribution can also be measured at a field $H$ or in a small range of fields centered around $H$. We have measured this *binned* in $H$ avalanche size distribution for systems at the critical disorder $R_c$, ($r = 0$). To obtain the scaling form, we start from the distribution of avalanches at field $H$ and disorder $R$ (eqn. [19]):

\[ D(S, R, H) \sim S^{-\tau} D_{\pm}(S^\sigma |r|, |h|/|r|^{\beta \delta}) \quad (49) \]

where as before $D_{\pm}$ is the scaling function and $\pm$ indicates the sign of $r$. (For most of our data, we can ignore the corrections due to the “rotation” of axis as explained in appendix A.) The scaling function can be rewritten as $D_{\pm}(S^\sigma |r|, (S^\sigma |r|)^{\beta \delta}|h|/|r|^{\beta \delta})$, where $D_{\pm}$ is a new scaling function. Letting $R \rightarrow R_c$, the scaling for the avalanche size distribution at the field $H$, measured at the critical disorder $R_c$ is:
$D(S, H) \sim S^{-\tau} \mathcal{D}_z(|h|S^{\sigma\beta\delta})$  \hspace{1cm} (50)

![Figure 22](image1.png)

**FIG. 22.** Values for the exponent $\tau$ extracted from the binned in $H$ avalanche size distribution curves in 4 dimensions, for a $80^4$ spin system at $R = 4.09$ ($R_c = 4.10$). The critical field is $H_c = 1.265$. The exponent $\tau$ is found from this linear extrapolation to $\Delta H_{avg} = 0$. The exponent $\sigma\beta\delta$ is calculated from the value of $\tau + \sigma\beta\delta$, extracted from the integrated avalanche size distribution, and the value of $\tau$ from this plot.

Figure 22 shows the binned in $H$ avalanche size distribution curves in 4 dimensions, for values of $H$ below the critical field $H_c$. The simulation was done at the best estimate of the critical disorder $R_c$ (4.1 in 4 dimensions). The binning in $H$ is logarithmic and started from an approximate critical field $H_c$ obtained from the magnetization curves; better estimates of $H_c$ are obtained from the binned distribution data curves and their collapses. Our best estimate for the critical field $H_c$ in 4 dimensions is $1.265 \pm 0.007$. The scaling form for the logarithmically binned data is the same as in equation (44), if the log-binned data is normalized by the size of the bin. Figure 23 shows the scaling collapse for our data, both below and above the critical field $H_c$. The “top” collapse gives the shape of the $\mathcal{D}_- (H < H_c)$ function, while the “bottom” collapse gives the $\mathcal{D}_+ (H > H_c)$ function. Above the critical field $H_c$, there are spanning avalanches in the system. These are not included in the binned avalanche size distribution collapse shown in figure 23.

The exponent $\tau$ which gives the power law behavior of the binned avalanche size distribution is obtained from an extrapolation similar to previous ones (figure 22), but with the field $H$ ($\Delta H_{avg}$ in figure 22) is the algebraic average of $H - H_c$ for three curves collapsed together) as the variable instead of the disorder $R$. The exponent $\sigma\beta\delta$ is found to be very sensitive to $H_c$, while $\tau$ is not. We have therefore used the values of $\tau + \sigma\beta\delta$ and $\sigma$ from the integrated avalanche size distribution collapses, and $\tau$ from the binned avalanche size distribution collapses to further constrain $H_c$ (by constraining $\sigma\beta\delta$), and to calculate $\beta\delta$. The latter is then used to obtain collapses of the magnetization curves. We should mention here that $H_c$ in all the dimensions is difficult to find and that it is influenced by finite sizes. The values listed in Table I are the best estimates obtained from the largest system sizes we have. Nevertheless, systematic errors for $H_c$ could be larger than the errors given in Table I. This implies possible systematic errors for $\sigma\beta\delta$ which depends on $H_c$, and for $\beta\delta$ which is calculated from $\sigma\beta\delta$. These could also be larger than the errors listed in Table I.

From figure 22, we see that the two binned avalanche size distribution scaling function do not have a “bump” as does the scaling function for the integrated avalanche size distribution (inset in figure 21). Therefore, we expect that the exponent $\tau$ which gives the slope of the distribution in figure 22 can also be obtained by a linear fit through the data curve closest to the critical field. Figure 23 shows the curve for the $H = 1.265$ bin (dashed curve) as well as the linear fit. The slope from the linear fit is 1.55 while the value of $\tau$ obtained from the collapses and the extrapolation in figure 22 is $1.53 \pm 0.08$.

The avalanche correlation function $G(x, R, H)$ measures the probability that a flipping spin will trigger, through an avalanche of spins, another spin a distance $x$ away. From the renormalization group description \[14\][15], close to the critical point and for large distances

3. **Avalanche Correlation**

The avalanche correlation function $G(x, R, H)$ measures the probability that a flipping spin will trigger, through an avalanche of spins, another spin a distance $x$ away. From the renormalization group description \[14\][15], close to the critical point and for large distances...
where \( r \) and \( h \) are respectively the reduced disorder and field, \( G_\pm (\pm \text{indicates the sign of } r) \) is the scaling function, \( d \) is the dimension, \( \xi \) is the correlation length, and \( \eta \) is called the “anomalous dimension”. The correlation length \( \xi(r, h) \) is a macroscopic length scale in the system which is roughly on the order of the mean linear extent of the avalanches for a system away from the critical point.

At the critical field \( H_c \) (\( h=0 \)) and near \( R_c \), the correlation length scales like \( \xi \sim |r|^{-\nu} \), while for small field \( h \) it is given by \( \xi \sim |r|^{-\nu} \gamma_\pm (h/|r|^{\beta \delta}) \) where \( \gamma_\pm \) is a universal scaling function. The avalanche correlation function should not be confused with the cluster or “spin-spin” correlation which measures the probability that two spins a distance \( x \) away have the same value. (The algebraic decay for this other, spin-spin correlation function at the critical point \((r = 0 \text{ and } h = 0)\), is \( 1/x^{d-4+\tilde{\eta}} \) [14].)

We have measured the avalanche correlation function integrated over the field \( H \), for \( R > R_c \). For every avalanche that occurs between \( H = -\infty \) and \( H = +\infty \), we keep a count on the number of times a distance \( x \) occurs in the avalanche. To decrease the computational time not every pair of spins is selected; instead we do a statistical average for \( S \) pairs where \( S \) is the size of the avalanche. Our simulation seems to indicate that the difference between this statistical average and the exact measurement is less than the fluctuations obtained from measurements of the correlation function for different realizations of the random field distribution. The data is saved in “distance” bins separated by 0.5 and starting at a distance of 1.0 (the self correlation is not included), and is normalized by the number of neighbors at each distance. The spanning avalanches are not included in our correlation measurement. Figure 24 shows several avalanche correlation curves in 3 dimensions for \( L = 320 \). The scaling form for the avalanche correlation function integrated over the field \( H \), close to the critical point and for large distances \( x \), is obtained by integrating equation (51):
\[ G_{int}(x, R) \sim \int \frac{1}{x^{d-2+\eta}} G_\pm \left( \frac{x}{\xi(r, h)} \right) dh \] (52)

Near the critical point \( \xi(r, h) \sim |r|^{-\nu} \gamma_\pm (h/|r|^{\beta}) \). Defining \( u = h/|r|^{\beta} \), equation (52) becomes:

\[ G_{int}(x, R) \sim |r|^{\beta} x^{-(d-2+\eta)} \int G_\pm \left( x/|r|^{-\nu} \gamma_\pm (u) \right) du \] (53)

The integral \( \mathcal{I} \) in equation (53) is a function of \( x/|r|^\nu \) and can be written as:

\[ \mathcal{I} = (x/|r|^\nu)^{-\beta \nu/\nu} \tilde{G}_\pm (x/|r|^\nu) \] (54)

to obtain the scaling form:

\[ G_{int}(x, R) \sim \frac{1}{x^{d-\beta \nu}} \tilde{G}_\pm (x/|r|^\nu) \] (55)

where we have used the scaling relation \((2-\eta)\nu = \beta \delta - \beta\) (see [14][16] for the derivation).

![Avalanche Correlation vs Distance](image)

**FIG. 26. Anisotropies in the avalanche correlation function.** The curves are for a system of 320\(^3\) spins at \( R = 2.35 \). Four curves are shown on the graph: one is the avalanche correlation function integrated over the field \( H \) (as in figure 24), while the other three are measurements of the correlation along the three axes, the six face diagonals, and the four body diagonals. Avalanches involving more than four spins show no noticeable anisotropy: the critical point appears to have spherical symmetry. The same result is found in 2 dimensions.

Figure 26 shows the integrated avalanche correlation curves collapse in 3 dimensions for \( L = 320 \) and \( R > R_c \). The exponent \( \nu \) is obtained from such collapses by extrapolating to \( R = R_c \) (figure 25) as was done for other collapses. The exponent \( \beta/\nu \) can be obtained from these collapses too, but it is much better estimated from the magnetization discontinuity covered below. The value of \( \beta/\nu \), listed in Table 1 alongside all the other exponents, is derived from the magnetization discontinuity collapses only.

We have also looked for possible anisotropies in the integrated avalanche correlation function in 2 and 3 dimensions. The anisotropic integrated avalanche correlation functions are measured along “generalized diagonals”: one along the three axis, the second along the six face diagonals, and the third along the four body diagonals. We compare the integrated avalanche correlation function and the anisotropic integrated avalanche correlation functions to each other, and find no anisotropies in the correlation, as can be seen from figure 21.

4. Spanning Avalanches

The critical disorder \( R_c \) was defined earlier as the disorder \( R \) at which an infinite avalanche first appears in the system, in the thermodynamic limit, as the disorder is lowered. At that point, the magnetization curve will show a discontinuity at the magnetization \( M_c(R_c) \) and field \( H_c(R_c) \). For each disorder \( R \) below the critical disorder, there is one infinite avalanche that occurs at a critical field \( H_c(R) > H_c(R_c) \) [14][15], while above \( R_c \) there are only finite avalanches. This is the behavior for an infinite size system. In a finite size system far below and above \( R_c \) the above picture is still true, but close to the critical disorder, as we approach the transition, the avalanches get larger and larger, and we expect that one of them will be on the order of the system size and span the system from one “side” to another in at least one direction. This avalanche is not the infinite avalanche; it is only the largest avalanche that occurs close to the critical point. If the system was larger, this avalanche would be non-system spanning. Such an avalanche (which spans the system) we call a spanning avalanche.

In our numerical simulation, we find that for finite sizes \( L \), there are not one but many such avalanches in 4 and 5 dimensions (and maybe 3), and that their number increases as the system size increases. Figures 27(a-c) show the number of spanning avalanches as a function of disorder \( R \), for different sizes and dimensions. In 4 and 5 dimensions, the spanning avalanche curves become more narrow as the system size is increased. Also, the peaks shift toward the critical value of the disorder (4.1 and 5.96 respectively), and the number of spanning avalanches at \( R_c \) increases. This suggests that in 4 and 5 dimensions, for \( L \rightarrow \infty \), there will be one infinite avalanche below \( R_c \), none above, and an infinite number of spanning avalanches at the critical disorder \( R_c \). (These spanning avalanches are infinite avalanches for \( L \rightarrow \infty \).) In 3 dimensions, the results are not conclusive, which can be noticed from figure 27a, but also from the value of the spanning avalanche exponent \( \theta = 0.15 \pm 0.15 \) defined below (a value of 0 implies only one infinite or spanning avalanche at \( R_c \) as \( L \rightarrow \infty \).)
FIG. 27. (a) **Number of spanning avalanches** *N* in 3 dimensions, occurring in the system between *H* = −∞ to *H* = ∞, as a function of the disorder *R*, for linear sizes *L*: 20 (dot-dashed), 40 (long dashed), 80 (dashed), 160 (dotted), and 320 (solid). The critical disorder *R*\(_c\) is at 2.16. The error bars for each curve tend to be smaller than the peak error bar for disorders above the peak and larger for disorders below the peak. They are not given here for clarity. Note that the number of avalanches increases only slightly as the size is increased. (b) **Number of spanning avalanches in 4 dimensions.** The critical disorder is 4.1. (c) **Number of spanning avalanches in 5 dimensions.** The critical disorder is 5.96. Both in 4 and 5 dimensions, the peaks grow and shift towards *R*\(_c\) as the size of the system is increased. (d) Collapse of the spanning avalanche curves in 4 dimensions for linear sizes *L* = 20, 40, and 80. The exponents are \(\theta = 0.32\) and \(\nu = 0.89\), and the critical disorder is *R*\(_c\) = 4.10. The collapse is done using \(r = (R_c - R)/R\).

In percolation, a similar multiplicity of infinite clusters (as the system size is increased) is found for dimensions above 6 which is the upper critical dimension (UCD). The UCD is the dimension at and above which the mean field exponents are valid. Below 6 dimensions, there is only one such infinite cluster. The existence of a diverging number of infinite clusters in percolation is associated with the breakdown of the hyperscaling relation above 6 dimensions. Since a hyperscaling relation is a relation between critical exponents that includes the dimension *d* of the system, it is always only satisfied up to and including the upper critical dimension. In our system, the upper critical dimension is also 6, but we find spanning avalanches in dimensions even below that. In a comment by Maritan *et al.* [29], it was suggested that our system should satisfy the hyperscaling relation: \(d\nu - \beta = 1/\sigma\) which is also the one found in percolation [23]. But since our system has spanning avalanches below the upper critical dimension, this hyperscaling relation breaks down below 6 dimensions. Due to the existence of many spanning avalanches near *R*\(_c\), the new “violation of hyperscaling” relation for dimensions 3 and above becomes [14,16]:

\[
(d - \theta)\nu - \beta = 1/\sigma
\]  

(56)

where \(\theta\) is the “breakdown of hyperscaling” or spanning avalanches exponent defined below. One can check that our exponents in 3, 4, and 5 dimensions and mean field satisfy this equation (see Tables II and III).

For the simulation, we define a spanning avalanche to be an avalanche that spans the system in one direction. We average over all the directions to obtain better statistics. Depending on the size and dimension of the system and the distance from the critical disorder, the number of spanning avalanches for a particular value of disorder *R* is obtained by averaging over as few as 5 to as many as 2000 different random field configurations. We define the exponent \(\theta\) such that the number *N* of spanning avalanches, at the critical disorder *R*\(_c\), increases with the linear system size as: \(N \sim L^\theta\) (\(\theta > 0\)). The finite size scaling
form \[18,19\] for the number of spanning avalanches close to the critical disorder is:

\[
N(L, R) \sim L^\theta N_\pm(L^{1/\nu}|r|) \tag{57}
\]

where \(\nu\) is the correlation length exponent and \(N_\pm\) is the corresponding scaling function (\(\pm\) indicates the sign of \(r\)). The corrections to scaling are subdominant as explained in appendix A. The collapse is shown in figure 27d. The values for \(\theta\) and \(\nu\) from collapses of curves of sizes \(L = 20, 30, 40,\) and 80 in 4 dimensions, are shown in Table IV. (We show the results and collapses in 4 dimensions here since the existence of spanning avalanches in 3 dimensions is not conclusive.) These values are used along with the results from other collapses to obtain Table II. In the analysis of the avalanche size distribution, magnetization, and correlation functions for \(R > R_c\), how close we chose to come to the critical disorder \(R_c\) was determined by the spanning avalanches: we include no values \(R\) below the first value which exhibited a spanning avalanche.

5. Magnetization Discontinuity

We have mentioned earlier that in the thermodynamic limit, at and below the critical disorder \(R_c\), there is a critical field \(H_c(R) > H_c(R_c)\) at which the infinite avalanche occurs. Close to the critical transition, for \(r\) small and \(H = H_c(R)\), the change in the magnetization due to the infinite avalanche scales as (equation (39)):

\[
\Delta M(R) \sim r^\beta \tag{58}
\]

where \(r = (R_c - R)/R\), while above the transition, there is no infinite avalanche.

In finite size systems, the transition is not as sharp: we have spanning avalanches above the critical disorder. If we measure the change in the magnetization due to all the spanning avalanches (the infinite avalanche is included too), the scaling form for that quantity is going to depend on the system size \(L\) analogous to the scaling of the number of spanning avalanches:

\[
\Delta M(L, R) \sim |r|^{\beta} \Delta M_\pm(L^{1/\nu}|r|) \tag{59}
\]

where \(\Delta M_\pm\) is a universal scaling function. (Since \(\Delta M(L, R)\) is measured at \(h' = 0\), corrections to scaling are subdominant; see also appendix A.) Defining a new universal scaling function \(\tilde{\Delta} M_\pm:\n\]

\[
\Delta M_\pm(L^{1/\nu}|r|) \equiv (L^{1/\nu}|r|)^{-\beta} \tilde{\Delta} M_\pm(L^{1/\nu}|r|) \tag{60}
\]

we obtain the scaling form:

\[
\Delta M(L, R) \sim L^{-\beta/\nu} \tilde{\Delta} M_\pm(L^{1/\nu}|r|) \tag{61}
\]

Figures 28a and 28b show the change in the magnetization due to the spanning avalanches in 4 dimensions, and a scaling collapse of that data (similar results exist.
in 3 and 5 dimensions). Notice that as the system size increases, the curves approach the \(|r|^{\beta/\nu}\) behavior. The exponents \(1/\nu\) and \(\beta/\nu\) are extracted from scaling collapses (figure 28b) and are listed in table \(V\). The value of \(\beta\) is calculated from \(\beta/\nu\) and the knowledge of \(\nu\), and is the value used for collapses of the magnetization curves (see earlier).

The second moment of the avalanche size distribution was defined earlier (see the mean field simulation section). We found that the scaling form of the integrated second moment \(\langle S^2 \rangle_{\text{int}}\) is

\[
\langle S^2 \rangle_{\text{int}} \sim L^{-\frac{\tau + \sigma \beta \delta - 3}{\sigma \nu}} S^{(2)}(L^{1/\nu}|r|)
\]

where \(L\) is the linear size of the system, \(r\) is the reduced disorder, \(S^{(2)}\) is the scaling function, and \(\nu\) is the correlation length exponent. The corrections are subdominant (appendix A). We can similarly define the third and fourth moment, with the exponent \(\frac{1}{\nu} = \frac{\tau + \sigma \beta \delta}{\sigma \nu}\) replaced by \(-\frac{\tau + \sigma \beta \delta - 4}{\sigma \nu}\) and \(-\frac{\tau + \sigma \beta \delta - 5}{\sigma \nu}\) respectively. Figures 29a and 29b show the second moments data in 5 dimensions for sizes \(L = 5, 10, 20,\) and 30, and a collapse (again, results in 3 and 4 dimensions are similar and we have chosen to show the curves in 5 dimensions for variety). The curves are normalized by the average avalanche size integrated over all fields \(H: \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S \, dS \, dH.\) The spanning avalanches and the infinite avalanche are not included in the calculation of the moments. The collapse does not include the \(L = 5\) curve because, due to finite size effects, this curve does not collapse well with the larger size curves. Table \(VI\) shows the values of the exponents and \(R_c\) from the collapses. The exponents for the third and fourth moment can be calculated from this table, and we find that they agree with the values obtained from their respective collapses.

7. Avalanche Time Measurement

The exponents we have measured so far are static scaling exponents: they do not depend on the dynamics of the model. If we measure the time an avalanche takes to occur, we are making a dynamical measurement. The time measurement in the numerical simulation is done by increasing the time “meter” by one for each shell of spins in the avalanche; it corresponds to a synchronous dynamics, where, when all unstable spins are flipped, time is incremented by one, and the new list of unstable spins is generated. The scaling relation between the time \(t\) it takes an avalanche to occur and the size \(S\) of that avalanche for small disorder \(R\) can be found by noting that the characteristic duration of an avalanche is proportional to the correlation length \(\xi\) to the power \(z\):

\[
t \sim \xi^z
\]

(63)

The exponent \(z\) is known as the dynamical critical exponent. Equation (63) gives the scaling for the time it takes for a spin to “feel” the effect of another distance \(\xi\) away. Since the correlation length \(\xi\) scales like \(r^{-\nu}\) close to the critical disorder, and the characteristic size \(S\) as \(r^{-1/\sigma}\), the time \(t\) then scales with large sizes as:

\[
t \sim S^{\nu z}
\]

(64)
(a) Avalanche time distribution curves in 3 dimensions, for avalanche size bins from about 2000 to 40000 spins (from upper left to lower right corner). The system size is $800^3$ at $R = 2.26$. The curves are from only one random field configuration. (b) Scaling collapse of curves in (a). The values of the exponents are $\sigma \nu z = 0.57$ and $(\tau + \sigma \beta \delta + \sigma \nu z)/\sigma \nu z = 4.0$.

In our simulation, we measure the distribution of times for each avalanche size $S$. The distribution of times $D_t(S, R, H, t)$ for an avalanche of size $S$ close to the critical field $H_c$ and critical disorder $R_c$ is

$$D_t(S, R, H, t) \sim S^{-q} \mathcal{D}^{(t)}_\pm (S^\sigma |r|, h/|r|^\beta \delta, t/S^\nu z)$$

(65)

where $q = \tau + \sigma \nu z$, and is defined such that

$$\int_{-\infty}^{+\infty} \int_1^{\infty} D_t(S, R, H, t) \, dH \, dt = S^{-(\tau + \sigma \beta \delta)} \mathcal{D}^{(int)}_\pm (S^\sigma |r|)$$

(66)

where $\mathcal{D}^{(int)}_\pm$ was defined in the integrated avalanche size distribution section. The avalanche time distribution integrated over the field $H$, at the critical disorder ($r = 0$) is:

$$D^{(int)}_t(S, t) \sim t^{-(\tau + \sigma \beta \delta + \sigma \nu z)/\sigma \nu z} \mathcal{D}^{(int)}_\pm (t/S^\nu z)$$

(67)

which is obtained from equation (65) in a derivation analogous to the one for the integrated avalanche size distribution scaling form.

Figures 30a and 30b show the avalanche time distribution integrated over the field $H$ for different avalanche sizes, and a collapse of these curves using the above scaling form, for a $800^3$ system at $R = 2.260$ (just above the range where spanning avalanches occur). The data is saved in logarithmic size bins, each about 1.2 times larger than the previous one. The time is also measured logarithmically (next bin is 1.1 times larger than the previous one). The extracted value for $z$ in 3 dimensions is $1.68 \pm 0.07$. The results for other dimensions are listed in Table II.

C. Simulation Results in 2 Dimensions

The critical transition in the shape of the hysteresis loop is observed in the simulation, and expected from the renormalization group [14,15] in 3, 4, and 5 dimensions. We also found that the upper critical dimension, at and above which the mean field exponents become correct, is six. Furthermore, in one dimension, we expect that in a thermodynamic system, with an unbounded distribution of random fields, there will be no infinite avalanche for $R > 0$. That will be so because if there is any randomness, there will be a spin in the linear chain that will have the “right” value for its random field to stop the first avalanche. For a bounded distribution of random fields, the scaling behavior near the transition will not be universal [14]; instead, it will depend on the exact shape of the tails of the distribution of random fields. Then, the question that remains is: what happens in two dimensions?

From the simulation and a few arguments that we are about to show, we conjecture that the two dimensional exponents will have the values: $\tau + \sigma \beta \delta = 2$, $\tau = 3/2$, $1/\nu = 0$, and $\sigma \nu = 1/2$. (The other exponents (except $z$) can be found from exponent relations [44] using these values.) The “arguments” are as follows.

It is quite possible that two is the lower critical dimension (LCD) for our system. At the lower critical dimension, the critical exponents are often ratios of small integers, and it is often possible to derive exact solutions. Since the geometry in 2 dimensions allows for at most one system spanning avalanche, the “breakdown of hyperscaling” exponent $\theta$ (see section IV B) must be zero, and the hyperscaling relation [14,15] is restored:

$$\frac{1}{\sigma \nu} = d - \frac{\beta}{\nu}$$

(68)

We know that this relation is violated in 4 and 5 dimensions, and is probably violated in 3 dimensions. In dimensions above two, the hyperscaling relation is modified by the exponent $\theta$ which gives a measure of the number of spanning avalanches near the critical transition, as a function of the system size. Figure 31 shows the number of spanning avalanches in 2 dimensions for several system sizes. Notice that, as assumed, there is not more than one spanning avalanche in the system.
FIG. 31. Number of spanning avalanches in 2 dimensions as a function of disorder $R$, for several system sizes. The data points are averages between as little as 10 to as many as 2200 random field configurations. Some typical error bars near the center of the curves are shown; error bars are smaller toward the ends. Note that there is no more than one spanning avalanche.

We use two more arguments to derive the critical exponents. In 2 dimensions, we find that the avalanches “look” compact (figure 32). (The avalanches in 3 dimensions are not compact (figure 33).) This implies that $1/\sigma \nu = 2$, which leads to $\beta/\nu = 0$ from equation (68). Furthermore, it is often the case that in the lower critical dimension, the Harris criterion \cite{14} $\nu \beta \delta \geq 2d$ becomes saturated (an equality); so in 2 dimensions we expect $\beta \delta / \nu = 1$. From this and the previous result, the exponent which gives the decay in space of the avalanche correlation function

$$\eta = 2 + \frac{\beta}{\nu} - \frac{\beta \delta}{\nu}$$

(see references \cite{14,16} for the derivation of all the exponent relations) becomes equal to $\eta = 1$.

Since at the LCD the correlation length typically diverges exponentially as the critical point is approached, we expect $\nu \rightarrow \infty$, and $\beta$ can be finite. Using the exponent relation \cite{14,16}:

$$\tau - 2 = \sigma \beta (1 - \delta),$$

we further find that $\tau = 3/2$ and $\tau + \sigma \beta \delta = 2$.

We must mention that our firm conjectures about the exponents in two dimensions must be contrasted with our lack of knowledge about the proper scaling forms. As mentioned above, at the LCD the correlation exponent $\nu$ typically diverges, although some combinations of critical exponents stay finite (hence $\sigma \nu = 1/2$). Those which diverge and those which go to zero usually must be replaced by exponents and logs, respectively. We have used three different RG-scaling ansätze to model the data in two dimensions. (1) We used the traditional scaling form $\xi \sim |R_c - R|^\nu$, deriving $\nu = 5.3 \pm 1.4$ and $R_c = 0.54 \pm 0.04$. These collapses worked as well as any, but the large value for $\nu$ (and larger value still for $1/\sigma = 10 \pm 2$) makes one suspicious. (2) We used a scaling form suggested by Bray and Moore \cite{32} in the context of the equilibrium thermal random field Ising model at the LCD, where $R_c = 0$: if they assume that $R$ is a marginal direction, then by symmetry the flows must start with $R^3$, leading to $\xi \sim e^{(\hat{g}/R_c - R)^2} \equiv e^{(\hat{g}/R^2)}$. This form has the fewest free parameters, and most of the collapses were about as good as the others (except notably for the finite-size scaling of the moments of the avalanche size distribution, which did not collapse well once spanning avalanches became common). (3) We developed another possible scaling form, based on a finite $R_c$ and $R$ marginal, which generically has a quadratic flow under coarse-graining: here $\xi \sim e^{(\hat{g}/R_c - R)}$. We find $R_c = 0.42 \pm 0.04$. The rational behind these three forms is shown in appendix A.

The results from data collapses in two dimensions were obtained from measurements of the spanning avalanches, the second moments of the avalanche size distribu-
Measurements that require the knowledge of the critical randomness are the binned avalanche size distribution from which we extract the exponents $\tau$ and $\beta\delta$, the critical magnetic field $H_c$, and the avalanche time measurement which gives the exponent $z$. These measurements were not obtained at the critical disorder because $R_c$ is not well defined as was mentioned above, and because for low disorders (less than 0.71 for a 7000$^2$ system), the system flips in one infinite avalanche, and such measurements are therefore not possible. We have nevertheless estimated the values of some of these exponents and of $H_c$, from data obtained at a larger disorder (where there is no spanning avalanche). From the avalanche size distribution binned in $H$ at $R = 0.71$ and $L = 7000$, and the magnetization curves, we find that the critical field $H_c$ is around 1.32. A straight line fit through the data agrees with a possible value of $\tau = 3/2$ (the conjectured value). From the time distribution of avalanche sizes for a system of 30000$^2$ spins, at $R = 0.65$, we measured (from a straight linear fit) the exponent $\sigma\nu z$ to be 0.64. The other exponents were obtained from scaling collapses as follows.

Figure 34a shows the second moments of the avalanche size distribution for several system sizes. The collapses using the three different scaling forms are shown in figures 34(b-d). The first one (figure 34b) is:

$$\langle S^2 \rangle_{\text{int}} \sim L^{-\left(\tau + \sigma\beta\delta - 3\right)/\sigma\nu} \tilde{S}^{(2)}_{\text{int}}(L \mid |r|^\nu)$$

(72)

which is the kind of scaling form used in 3, 4, and 5 dimensions. This form assumes $\xi \sim |r|^{-\nu}$. The exponents are $(\tau + \sigma\beta\delta - 3)/\sigma\nu = -1.9$ and $\nu = 5.25$, and $r = (R_c - R)/R$ with $R_c = 0.54$. The second scaling form (figure 34c) is:

$$\langle S^2 \rangle_{\text{int}} \sim L^{-\left(\tau + \sigma\beta\delta - 3\right)/\sigma\nu} \tilde{S}^{(2)}_{\text{int}}(L e^{-\tilde{a}/|R_c - R|^2})$$

(73)

which is obtained from $\xi \sim e^{a(|R_c - R|)}$. The values of the exponents and parameters are: $(\tau + \sigma\beta\delta - 3)/\sigma\nu = -1.9$, $\tilde{a} = 3.4$ ($\tilde{a}$ is not universal), and $R_c = 0$ (by assumption; see previous paragraph). Notice that this collapse is not as good as the other two: a better collapse is obtained with $R = 0.15$ and $\tilde{a} = 2.0$. If this is the correct scaling form and $R_c = 0$, this discrepancy can be due to finite size effects. The third scaling form is (figure 34d):

$$\langle S^2 \rangle_{\text{int}} \sim L^{-\left(\tau + \sigma\beta\delta - 3\right)/\sigma\nu} \tilde{S}^{(2)}_{\text{int}}(L e^{-\tilde{b}/|R_c - R|})$$

(74)

which is obtained from $\xi \sim e^{\tilde{b}(|R_c - R|)}$. The values of the exponents and parameters are: $(\tau + \sigma\beta\delta - 3)/\sigma\nu = -1.9$, $\tilde{b} = 2.05$ ($\tilde{b}$ is also non-universal), and $R_c = 0.42$. As it is clear from the last three figures, collapses with these different scaling forms are comparable. Notice that the exponent $(\tau + \sigma\beta\delta - 3)/\sigma\nu$ is the same for the three collapses, but that $1/\nu$ is zero for the last two (by assumption) while it is 0.19 for the first collapse. Let’s now look at the collapses of the integrated avalanche size distribution curves, which are not finite size scaling measurements.

FIG. 33. Largest avalanche occurring in the hysteresis loop in a 40$^3$ spins system near the critical point. The avalanche is not compact.
FIG. 34. (a) **Second moments of the avalanche size distribution in 2 dimensions**, integrated over the external field $H$, for several system sizes. The data points are averages over up to 2200 random field configurations. Error bars are smaller than shown for larger disorders. (b), (c), and (d) Scaling collapses of the second moments of the avalanche size distribution in 2 dimensions, integrated over the field $H$. The curves that are collapsed are of size: $50^2$, $100^2$, $300^2$, $500^2$, $1000^2$, $3000^2$, $5000^2$, $7000^2$, and $30000^2$. See text for the scaling forms, and the values of the exponents and parameters.

Figure 34 shows the integrated avalanche size distribution curves for a $7000^2$ spin system, at several values of the disorder $R$. Earlier, in figure 20, we saw the fit to the scaling collapse of such curves, done using the same scaling form as in 3, 4, and 5 dimensions:

$$D_{\text{int}}(S, R) \sim S^{-(\tau + \sigma \beta \delta)} \bar{D}_{\text{int}}^\sigma(|r|)$$  \hspace{1cm} (75)

(The $-$ sign indicates that the collapsed curves are for $r < 0$, i.e. $R > R_c$.) However, $S^\sigma |r|$ might not be the appropriate scaling argument in 2 dimensions. First, from figure 20, the scaling curve in 2 dimensions differs dramatically from the scaling curves in higher dimensions for small arguments $X = S^\sigma |r|$. The mean field scaling function $\bar{D}_{\text{int}}^\sigma(X)$ is a polynomial for small $X$, and we expected (and found) a similar behavior in 5, 4 and 3 dimensions (but notice that the scaling function in 3 dimensions is starting to look like the curve in 2 dimensions for small $X$). In 2 dimensions, if we collapse our data (figure 34b) using the scaling form:

$$D_{\text{int}}(S, R) \sim S^{-(\tau + \sigma \beta \delta)} D_{\text{int}}^\sigma(|r|^{1/\sigma})$$  \hspace{1cm} (76)

with $\tau + \sigma \beta \delta = 2.04$, $1/\sigma = 10$, and $r = (R_c - R)/R$, we find that the scaling function for small $X = S^\sigma |r|^{1/\sigma}$ looks linear with power one! This might imply that the scaling function $D_{\text{int}}^\sigma(|r|^{1/\sigma})$ (eqn. (76)) is the one that is analytic for small arguments in 2 dimensions.
Second, we conjectured above that the values for $\sigma$ and $1/\nu$ are probably zero in 2 dimensions, and that only the combination $\sigma \nu$ is finite ($\sigma \nu = 1/2$). It follows that, for the other two scaling forms we use, the arguments of the scaling function should be $S e^{-a/(\sigma \nu |R - R_c|^2}$ and $S e^{-b/(\sigma \nu |R - R_c|^2}$, and not $S \sigma e^{-a/|R - R_c|^2}$ and $S \sigma e^{-b/|R - R_c|^2}$ respectively. This is analogous to using $S |r|^{1/\sigma}$ in the scaling form (76). We should mention here that both equation (75) and equation (76) give the same scaling exponents $\tau + \sigma \beta \delta$ and $\sigma$, and that in all our scaling collapses we have assumed that the same scaling argument is valid for small and large $X$ (and in between). This in general, does not have to be true.

Equation (76) is therefore one of the three scaling forms we use. The second scaling form is:

**FIG. 35.** (a) Integrated avalanche size distribution curves for several disorders in 2 dimensions, at the system size $L = 7000$. The curves are averages over 10 to 20 random field configurations, and have been smoothed. (b), (c), and (d) Scaling collapses of the data from (a) using the three scaling forms and the exponents from the text. The collapsed curves have disorders: 0.72, 0.74, 0.77, and 0.80. The straight grey line in each of the plots has a slope of one.

**FIG. 36.** Integrated avalanche size distribution curve in 2 dimensions, for a system of 30000$^2$ spins, at $R = 0.650$. Shown are two linear fits to the data: one for small sizes and the other for large sizes. The slope for the fit at small $S$ is 0.90. The fit was done for sizes in the range $[10, 250]$. The slope differs by less than 5% when the range is changed ($S$ is never larger than 400 though). The slope for the fit at large $S$ is 1.78. The slope differs by less than 2% when the range is changed ($S$ is never smaller than 10000). The conjectured value for $\tau + \sigma \beta \delta$ is 2 which is different from 1.78. This is similar to the behavior we saw in 3, 4, and 5 dimensions. On the other hand, for small sizes we expect the exponent $\tau + \sigma \beta \delta - 1 = 1$ (see text). Again, the two measurements don’t completely agree, but the slope from our data does seem to indicate such a behavior.
shown in figure 35d, with \(\tau + \sigma \beta \delta = 2.04, \hat{b} / \sigma \nu = 4.0\) (which makes \(\sigma \nu = 0.51\)), and \(r = R_c - R\) with \(R_c = 0.42\). Again, not only are all three collapses comparable, but the exponents extracted from them are as well. The exponent for the slope of the distribution is \(\tau + \sigma \beta \delta = 2.04\) for the three collapses, and the exponent combination \(\sigma \nu\) is around 0.51 (for the first collapse \(\sigma = 0.10\), while \(\nu = 5.25\) from the equivalent second moment collapse). Figures 35(b-d) show that the scaling function \(D^{(\mathrm{int})}\) seems to be linear with slope one for small arguments (the grey lines have slope one) and that the constant term in the polynomial expansion is zero (or close to zero). This leads to a singular scaling function correction to the avalanche size distribution exponent \(\tau + \sigma \beta \delta\) for small non-zero \(X\):

\[
D_{\mathrm{int}}(S, R) \sim S^{-(\tau + \sigma \beta \delta)} D^{(\mathrm{int})(1)}(S) \sim S^{-(\tau + \sigma \beta \delta) + 1}
\]

with slope one for small distances, as is the integrated avalanche size distribution for small sizes.

The spanning avalanches data are also analyzed using three scaling forms similar to those used for the second moments of the avalanche size distribution collapses. The exponent \(\theta\) is poorly defined from these collapses (and is therefore not listed in Table VII), although the data does collapse for the exact value: \(\theta = 0\).

The three collapses for all the measurements we have done are very similar. This is not a surprise: it is always hard to distinguish large power laws (\(\nu\) and \(1/\sigma\) are large in the “linear argument” scaling form (eqns. (72) and (76))) from exponentials. Although some of the exponents have very different values in the three collapses, the average of the exponents from the three methods agree within the error bars with each method (see figure 38) and our conjectures. In conclusion, although we do not know the correct scaling form for the data in 2 dimensions, the possible three scaling forms we mention give exponent values that are compatible with each other and with our conjectures (see Table VII). (Table VIII gives the conjectured values for the exponents that have not been measured in the collapses.) Much larger system sizes might be necessary to obtain more conclusive results.
FIG. 37. (a) Avalanche correlation function in 2 dimensions, integrated over the external field \( H \), for several disorders \( R \) and the system size \( L = 30000 \). Only the curve with the smallest disorder is an average over several random field configuration. (b) Scaling collapse of the avalanche correlation curves in 2 dimensions, for a system of 30000² spins. The exponent values are: \( \nu = 5.25 \) and \( \beta/\nu = 0 \). The critical disorder is \( R_c = 0.54 \), and \( r = (R - R_c)/R \). Notice that for small \( x|r|^{\nu} \), the scaling function looks singular with a power close to one (the straight line has a slope of one).

V. COMPARISON WITH THE ANALYTICAL RESULTS

We have compared the simulation results with the renormalization group analysis of the same system [14,15]. According to the renormalization group the upper critical dimension (UCD), at and above which the critical exponents are equal to the mean field values, is six. Close to the UCD, it is possible to do a \( 6 - \epsilon \) expansion (\( \epsilon \) is small and greater than 0), and obtain estimates for the critical exponents and the magnetization scaling function, which can then be compared with our numerical results. Furthermore, at dimension eight there is a prediction for another transition. Below eight dimensions, there is a discontinuity in the slope of the magnetization curve as it approaches the “jump” in the magnetization (\( R < R_c \)), while above eight dimensions the approach is smooth.

![Graph showing numerical and analytical results for five of the critical exponents obtained in dimensions two to six (in six dimensions, the values are the mean field ones). The other exponents can be obtained from scaling relations [14,16]. The exponent values in figure 38 are obtained by extrapolating the results of scaling collapses to either \( R \to R_c \) or \( 1/L \to 0 \) (see section on simulation results). In two dimensions, which is possibly the lower critical dimension, the plotted values are averages from the three different scaling forms used to collapse the data and extract the exponents. The error bars shown span all three \( \text{ansätze} \), and are compatible with our conjectures from the previous section. The long-dashed lines are the \( \epsilon \) expansions to first order for the exponents \( \tau + \sigma \beta \delta \), \( \tau \), \( \sigma \nu z \), and \( \sigma \nu \). The three short-dashed lines [14] are Borel sums [33] for \( 1/\nu \). The lowest is the variable-pole Borel sum from LeGuillou et al. [33], the middle uses the method of Vladimirov et al. to fifth order, and the upper uses the method of LeGuillou et al., but without the pole and with the correct fifth order term. Notice that the numerical values converge nicely to the mean field predictions, as the dimension approaches six, and that the agreement between the numerical values and the \( \epsilon \) expansion is quite impressive.]

FIG. 38. Numerical values (filled symbols) of the exponents \( \tau + \sigma \beta \delta \), \( \tau \), \( 1/\nu \), \( \sigma \nu z \), and \( \sigma \nu \) (circles, diamond, triangles up, squares, and triangle left) in 2, 3, 4, and 5 dimensions. The empty symbols are values for these exponents in mean field (dimension 6). Note that the value of \( \tau \) in 2d is the conjectured value: we have not extracted \( \tau \) from scaling collapses (see text). We have simulated sizes up to 30000², 1000³, 80⁴, and 50⁵, where for 320³ for example, more than 700 different random field configurations were measured. The long-dashed lines are the \( \epsilon \) expansions to first order for the exponents \( \tau + \sigma \beta \delta \), \( \tau \), \( \sigma \nu z \), and \( \sigma \nu \). The short-dashed lines are Borel sums [33] for \( 1/\nu \). The lowest is the variable-pole Borel sum from LeGuillou et al. [33], the middle uses the method of Vladimirov et al. to fifth order, and the upper uses the method of LeGuillou et al., but without the pole and with the correct fifth order term. The error bars denote systematic errors in finding the exponents from extrapolation of the values obtained from collapses of curves at different disorders \( R \). Statistical errors are smaller.

Figure 38 shows the numerical and analytical results for five of the critical exponents obtained in dimensions two to six (in six dimensions, the values are the mean field ones). The other exponents can be obtained from scaling relations [14,16]. The exponent values in figure 38 are obtained by extrapolating the results of scaling collapses to either \( R \to R_c \) or \( 1/L \to 0 \) (see section on simulation results). In two dimensions, which is possibly the lower critical dimension, the plotted values are averages from the three different scaling forms used to collapse the data and extract the exponents. The error bars shown span all three \( \text{ansätze} \), and are compatible with our conjectures from the previous section. The long-dashed lines are the \( \epsilon \) expansions to first order for the exponents \( \tau + \sigma \beta \delta \), \( \tau \), \( \sigma \nu z \), and \( \sigma \nu \). The three short-dashed lines [14] are Borel sums [33] for \( 1/\nu \). The lowest is the variable-pole Borel sum from LeGuillou et al. [33], the middle uses the method of Vladimirov et al. to fifth order, and the upper uses the method of LeGuillou et al., but without the pole and with the correct fifth order term [14]. Notice that the numerical values converge nicely to the mean field predictions, as the dimension approaches six, and that the agreement between the numerical values and the \( \epsilon \) expansion is quite impressive.
FIG. 39. Comparison between six simulation curves (thin lines) and the $dM/dH$ curve (thick dashed line) obtained from a parametric form to third order in $\epsilon$. The six curves are for a system of 30$^5$ spins at disorders: 7.0, 7.3, and 7.5 ($R_c = 5.96$ in 5 dimensions), and for a system of 50$^5$ spins at disorders: 6.3, 6.4, and 6.5 (for larger fields, these are closer to the dashed line in the figure). All the curves have been stretched/shrunk in the horizontal and vertical direction to lie on each other, and shifted horizontally.

The $\epsilon$ expansion can be an even more powerful tool if it can predict the scaling functions. This has been done for the magnetization scaling function of the pure Ising model in $4-\epsilon$ dimensions [34,35]. Since the $\epsilon$ expansion for our model is the same as the one for the equilibrium RFIM [14], and the latter has been mapped to all orders in $\epsilon$ to the corresponding expansion of the regular Ising model in two lower dimensions [14,36,37], we can use the results obtained in [34,35]. This is done in figure 39, which shows the comparison between the $dM/dH$ curves obtained in 5 dimensions at $R = 6.3, 6.4, 6.5, 7.0, 7.3, 7.5$ ($R_c = 5.96$) (the curves have been stretched/shrunk to lie on top of each other, and shifted horizontally so that the peaks align), and the parametric form (thick dashed line) for the scaling function of $dM/dH$, to third order in $\epsilon$, where $\epsilon = 1$ in 5 dimensions (see [35]). As we see, the agreement is very good in the scaling region (close to the peaks). This brings up the possibility of using the $\epsilon$ expansion for the scaling function to extract the critical exponents from simulation or experimental data. So far though, only the scaling function for the magnetization has been obtained.

FIG. 40. Magnetization curves in 5, 7, and 9 dimensions. The disorders for these curves are $R = 3.3, 4.7,$ and 6.0 for $30^5$, $10^7$, and $5^9$ size systems respectively. The dashed lines represent the “jump” in the magnetization. Notice that in 9 dimensions the approach to the “jump” seems to be continuous.

As another check between the simulation and the renormalization group, we have looked for the predicted transition in eight dimensions. Figure 40 shows the magnetization curves in 5, 7, and 9 dimensions (system sizes: $30^5$, $10^7$, and $5^9$) for values of the disorder equal to $2^d$, where $d$ is the dimension. These values of disorder are below the critical disorder in dimensions below six, and are expected to be below for dimensions 7 and 9 as well. For 5d and 7d, the approach to the “jump” in the magnetization is discontinuous. Above the eight dimension, the approach is continuous (see close ups in figure 41). This is as expected from the renormalization group [14]. We have also looked at $dM/dH$, which appears clearly to diverge in $d = 9$ and not in $d = 7$ (figure 42).

VI. CONCLUSION

We have used the zero temperature random field Ising model, with a Gaussian distribution of random fields, to model a random system that exhibits hysteresis. We found that the model has a transition in the shape of the hysteresis loop, and that the transition is critical. The tunable parameters are the amount of disorder $R$ and the external magnetic field $H$. The transition is marked by the appearance of an infinite avalanche in the thermodynamic system. Near the critical point, $(R_C, H_C)$, the scaling region is quite large: the system can exhibit power law behavior for several decades, and still not be near the critical transition. This is important to keep in mind whenever experimental data are analyzed. If a tunable parameter can be found, a system that appears to be SOC, might in reality have a disorder induced critical point.
FIG. 41. (a) and (b) Closeup of the magnetization curves in 7 and 9 dimensions respectively from figure 40. In 8 dimensions, there is a prediction from the renormalization group \[14\] that there is a transition in the way the jump is approached (see text).

We have extracted critical exponents for the magnetization, the avalanche size distribution (integrated over the field and binned in the field), the moments of the avalanche size distribution, the avalanche correlation, the number of spanning avalanches, and the distribution of times for different avalanche sizes. These values are listed in Table I and Table VII and were obtained as an average of the extrapolation results (to \(R \to R_c\) or \(L \to \infty\)) from several measurements. For example, the correlation length exponent \(\nu\) is the average value from three different collapses: the correlation function, the spanning avalanches, and the second moments of the avalanche size distribution. As shown earlier, the numerical results compare well with the \(\epsilon\) expansion \[14,15\]. Furthermore, the renormalization group work predicts another transition in eight dimensions, which we find in the simulation as well. Comparisons to experimental Barkhausen noise measurements \[12\] are very encouraging, and a more comprehensive review of possible experiments that exhibit disorder–driven critical phenomena similar to our model is under way \[16\].

Finally, we should mention that there are other models for avalanches in disordered magnets. There is a large body of work on depinning transitions and the motion of the single interface \[38,39\]. In these models, avalanches occur only at the growing interface. Our model though, deals with many interacting interfaces: avalanches can grow anywhere in the system. Similar models exist with random bonds \[40,41\] and random anisotropies. In the random bonds model, the interaction \(J_{ij}\) between neighboring spins \(i\) and \(j\) is random. The zero temperature random bond Ising model \[40,41\] also exhibits a critical transition in the shape of the hysteresis loop, where the mean bond strength is analogous to our disorder \(R\). It has been argued numerically \[11\] and analytically \[14\], that as long as there are no long-range forces \[9\] and correlated disorder, the random bond and the random field Ising model are in the same universality class. A comparison between our simulation and the results in reference \[41\] show that the 3 dimensional results agree quite nicely. However, in 2 dimensions, there are large differences, which we believe occur because of the small system sizes used by the authors for their simulation (only up to \(L = 100\)). We have seen that our results (see section on the 2 dimensional simulation) are very size dependent. Looking back for example at figure 31, we find that for a system of \(L = 100\) spins, a “good” estimate for the critical disorder \(R_c\) would indeed be 0.75 as was found in \[11\]. However, we find after increasing the system size that the critical disorder \(R_c\) is 0.54 or lower.
We acknowledge the support of DOE Grant #DE-FG02-88-ER45364 and NSF Grant #DMR-9419506. We would like to thank Sivan Kartha and Bruce W. Roberts for their initial ideas on the “probabilities” algorithm. Furthermore, we would like to thank M. E. J. Newman, J. A. Krumhansl, J. Souletie, and M. O. Robbins for helpful conversations. This work was conducted on the SP1 and SP2 at the Cornell National Supercomputing Facility (CNSF), funded in part by the National Science Foundation, by New York State, and by IBM, and on IBM 560 workstations and the IBM J30 SMP system (both donated by IBM). We would like to thank CNSF and IBM for their support. Further pedagogical information using Mosaic is available at http://www.lassp.cornell.edu/sethmAHysteresis.

APPENDIX A: DERIVATION OF THE VARIOUS SCALING FORMS AND CORRECTIONS

In this paper we make extensive use of scaling collapses. Many variations are important to us: Widom scaling, finite-size scaling, singular corrections to scaling, analytic corrections to scaling, rotating axes, and exponentially diverging correlation length scaling. The underlying theoretical framework for scaling is given by the renormalization group, developed by Wilson and Fisher in the context of critical phenomena and by now well explicated in a variety of texts [18, 31, 43].

We have discovered that we can derive all the scaling forms and corrections that have been important to us from two simple hypotheses (found in critical regions): universality and invariance under reparameterizations. Universality is the statement that two completely different systems will behave the same near their critical point [2] (for example, they can have exactly the same kinds of correlations). Reparameterization invariance is the statement that smooth changes in the units or methods of measurement should not affect the critical properties. We use these properties to develop the scaling forms and corrections we use in this paper. Each example we cover will build on the previous ones while developing a new idea.

For our first example, consider some property $F$ of a system at its critical point, as a function of a scale $x$. $F$ might be the spin-spin correlation function as a function of distance $x$ (or it might be the avalanche probability distribution as a function of size $x$, etc.) If two different experimental systems are at the same critical point, their $F$’s must agree. It would seem clear that they cannot be expected to be equal to one another: the overall scale of $F$ and the scale of $x$ will depend on the microscopic structure of the materials. The best one could imagine would be that

$$F_1(x_1) = AF_2(Bx_2) \quad (A1)$$

gives the ratio of the domain sizes.

Now, consider comparing a system with itself, but with a different measuring apparatus. Universality in this self-referential sense would imply $F(x) = AF(Bx)$, for suitable $A$ and $B$. If instead of using finite constant $A$ and $B$, we arrange for an infinitesimal change in the measurement of length scales, we find:

$$F(x) = (1 - \alpha \epsilon) F\left(1 - \epsilon x\right) \quad (A2)$$

where $\epsilon$ is small and $\alpha$ is some constant. Taking the derivative of both sides with respect to $\epsilon$ and evaluating it at $\epsilon = 0$, we find

$$F(x) \sim x^{-\alpha}. \quad (A3)$$

The function $F$ is a power–law! The underlying reason why power–laws are seen at critical points is that power laws look the same at different scales.

Now consider a new measurement with a distorted measuring apparatus. Now $F(x) \sim A\left[ F\left( B(x) \right) \right]$ where $A$ and $B$ are some nonlinear functions. For example, one might measure the number of microscopic domains $x$ flipped in an avalanche, or one might measure the total acoustic power $B(x)$ emitted during the avalanche; these two “sizes” should roughly scale with one another, but nonlinear amplifications will occur while the spatial extent of the avalanche is small compared to the wavelength of sound emitted: we expand $B(x) = Bx + b_0 + b_1/x + \ldots$ Similarly, our microphone may be nonlinear at large sound amplitudes, or the absorption of sound in the medium may be nonlinear: $A(F) = AF + a_2 F^2 + \ldots$ So,

$$A\left[ F\left( B(x) \right) \right] \approx \ A\left( F(Bx) + F'(Bx)(b_0 + b_1/x + \ldots) + F''(Bx)(\ldots) + a_2 F^2(Bx) + \ldots \right) \quad (A4)$$

We can certainly see that our assumption of universality cannot hold everywhere: for large $F$ or small $x$ the assumption of reparameterization invariance prevents any simple universal form. Where is universality possible? We can take the power-law form $F(x) \sim x^{-\alpha} = x^{\log A/\log B}$ which is the only form allowed by linear reparameterizations and plug it into (A4), and we see that all these nonlinear corrections are subdominant (i.e., small) for large $x$ and small $F$ (presuming $\alpha > 0$). If $\alpha > 1$, the leading correction is due to $b_0$ and we expect $x^{-\alpha-1}$ corrections to the universal power law at small distances; if $0 < \alpha < 1$ the dominant correction is due to $a_2$, and we expect corrections of order $x^{-2\alpha}$. Thus our assumptions of universality and reparameterization invariance both lead us to the power-law scaling forms and inform us as to some expected deviations from these forms. Notice that the simple rescaling led to the universal power-law predictions, and that the more complicated nonlinear
rescalings taught us about the dominant corrections: this will keep happening with our other examples.

For our second example, let us consider a property $K$ of a system, as a function of some external parameter $R$, as we vary $R$ through the critical point $R_c$ for the material (so $r = R - R_c$ is small). $K$ might represent the second moment of the avalanche size distribution, where $R$ would represent the value of the randomness; alternatively $K$ might represent the fractional change in magnetization $\Delta M$ at the infinite avalanche. If two different experimental systems are both near their critical point, then the dependence of $K_1$ and $K_2$ on “temperature” $R$ must agree, up to overall changes in scale. Thus, using a simple linear rescaling $K(r) = (1 - \mu r)K((1 - \epsilon)r)$ leads as above to the prediction

$$K(r) = r^{-\mu}. \quad \text{(A5)}$$

Now let us consider nonlinear rescalings, somewhat different than the one discussed above. In particular, the nonlinearity of our measurement of $K$ can be dependent on $r$. So,

$$A_r\left(K(r)\right) = a_0 + a_1 r + a_2 r^2 + \ldots + a_0 K(r) + \ldots \quad \text{(A6)}$$

If $\mu > 0$, these analytic corrections don’t change the dominant power law near $r = 0$. However, if $\mu < 0$, all the terms $a_n$ for $n < -\mu$ will be more important than the singular term! Only after fitting them to the data and subtracting them will the residual singularity be measurable.

For the fractional change in magnetization: $\Delta M \sim r^\beta$ has $0 < \beta < 1$ (at least above three dimensions), so we might think we need to subtract off a constant term $a_0$, but $\Delta M = 0$ for $R \geq R_c$, so $a_0$ is zero. On the other hand, in a previous paper [1], we discussed the singularity in the area of the hysteresis loop: Area $\sim r^{2-\alpha}$, where $2 - \alpha = \beta + \beta\delta$ is an analogue to the specific heat in thermal systems. Since $\alpha$ is near zero (slightly positive from our estimates of $\beta$ and $\delta$ in 3, 4, and 5 dimensions), measuring it would necessitate our fitting and subtracting three terms (constant, linear, and quadratic in $r$); we did not measure the area for that reason.

For our third example, let’s consider a function $F(x, r)$, depending on both a scale $x$ and an external parameter $r$. For example, $F$ might be the probability $D_{\text{init}}$ that an avalanche of size $x$ will occur during a hysteresis loop at disorder $r = R - R_c$. Universality implies that two different systems must have the same $F$ up to changes in scale, and therefore that $F$ measured at one $r$ must have the same form as if measured at a different $r$. To start with, we consider a simple linear rescaling:

$$F(x, r) = (1 - \alpha r)F((1 - \epsilon)x, (1 + \zeta\epsilon)r). \quad \text{(A7)}$$

Taking the derivative of both sides with respect to $\epsilon$ gives a partial differential equation that can be manipulated to show $F$ has a scaling form. Instead, we change variables to a new variable $y = x^\delta r$ (which satisfies $y' = y$ to order $\epsilon$). If $\tilde{F}(x, y) \equiv F(x, r)$ is our function measured in the new variables, then

$$F(x, r) = \tilde{F}(x, y) = (1 - \alpha \epsilon)\tilde{F}((1 - \epsilon)x, y) \quad \text{ (A8)}$$

and $-\alpha \tilde{F} = x \partial \tilde{F}/\partial x$ shows that at fixed $y$, $F \sim x^{-\alpha}$, with a coefficient $\tilde{F}(y)$ which can depend on $y$. Hence we get the scaling form

$$F(x, r) \sim x^{-\alpha}F(x^\delta r). \quad \text{(A9)}$$

This is just Widom scaling. The critical exponents $\alpha$ and $\zeta$ and the scaling function $F(x^\delta r)$ are universal (two different systems near their critical point will have the same critical exponents and scaling functions). We don’t need to discuss corrections to scaling for this case, as they are similar to those discussed above and below (and because none were dominant in our cases).

Notice that if we sit at the critical point $r = 0$, the above result reduces to equation (A3) so long as $F(0)$ is not zero or infinity. If, on the other hand, $F(y) \sim y^n$ as $y \to 0$, the two-variable scaling function gives a singular correction to the power-law near the critical point: $F(x, r) \sim x^{-\alpha}F(x^\delta r) \sim x^{-\alpha+n\gamma}$ for $x \ll r^{-1/\zeta}$; only when $x \sim r^{-1/\zeta}$ will the power-law $x^{-\alpha}$ be observed. This is what happened in two dimensions to the integrated avalanche size distribution (figures 37 and 38) and the avalanche correlation functions (figure 37b).

For the fourth example, we address finite-size scaling of a property $K$ of the system, as we vary a parameter $r$. If we measure $K(r, L)$ for a variety of sizes $L$ (say, all with periodic boundary conditions), we expect (in complete analogy to (A4))

$$K(r, L) \sim r^{-\mu}K(rL^{1/\nu}). \quad \text{(A10)}$$

Now, suppose our “thermometer” measuring $r$ is weakly size-dependent, so the measured variable is $C(r) = r + c/L + c_2/L^2 + \ldots$. The effects on the scaling function is

$$K\left(C(r), L\right) \sim r^{-\mu} \times \left(K(rL^{1/\nu}) + (cL^{1/\nu - 1} + c_2L^{1/\nu - 2})K'(rL^{1/\nu}) + \ldots \right). \quad \text{(A11)}$$

In two and three dimensions, $\nu > 1$ and these correction terms are subdominant. In four and five dimensions, we find $1/2 < \nu < 1$, so we should include the term multiplied by $c$ in equation (A11). However, we believe this first term is zero for our problem. For a fixed boundary problem (all spins “up” at the boundary) with a first-order transition, there is indeed a term like $c/L$ in $r(L)$ [5]. At a critical transition, the leading correction to $r(L)$ can be $c/L$ or a higher power of $L$ ($1/L^2$ and so on). This seems to depend on the model studied, the geometry of
the system, and the boundary conditions (free, periodic, ferromagnetic, ...). Furthermore, for the same kind of model, the coefficient \( c \) itself depends on the geometry and boundary conditions, and it can even vanish, which leaves only higher order corrections. In a periodic boundary conditions problem like ours, we expect that the correction is smaller than \( c/L \). Our finite-size scaling collapses for spanning avalanches \( N \), the second moments \( \langle S^2 \rangle \), and the magnetization jump \( \Delta M \), were successfully done by letting \( c = 0 \).

For the fifth example, consider a property \( K \) depending on two external parameters: \( r \) (the disorder for example) and \( h \) (could be the external magnetic field \( H - H_c \)). Analogous to (A9), \( K \) should then scale as

\[
K(r, h) \sim r^{-\mu} \mathcal{K}(h/r^{\beta \delta}).
\]  
(A12)

Consider now the likely dependence of the field \( h \) on the disorder \( r \). A typical system will have a measured field which depends on the randomness: \( \tilde{h}(h) = h + br + b_2 r^2 + \ldots \) (Corresponding nonlinearities in the effective value of \( r \) are subdominant.) This system will have

\[
K(r, \tilde{h}(h)) = r^{-\mu} \times \left( \mathcal{K}(h/r^{\beta \delta}) + (br + b_2 r^2) r^{-\beta \delta} \mathcal{K}'(h/r^{\beta \delta}) \right).
\]  
(A13)

Now, for our system \( 1 < \beta \delta < 2 \) for dimensions three and above. This means that the term multiplied by \( b \) is dominant over the critical scaling singularity: unless one shifts the measured \( h \) to the appropriate \( h' = h + br \), the curves will not collapse (e.g., the peaks will not line up horizontally). We measure this (non-universal) constant for our system (Table I), using the derivative of the magnetization with field \( dM/dH(r, h) \). The magnetization \( M(r, h) \) and the correlation length \( \xi(r, h) \) should also collapse according to equation (A13) (but with \( h + br \) instead of \( h \)); we don’t directly measure the correlation length, and the collapse of \( M(r, h) \) in figure 10b includes the effects of the tilt \( b \). In two dimensions, \( \beta \delta \) is large (probably infinite), so in principle we should need an infinite number of correction terms: in practise, we tried lining up the peaks in the curves (with no correction terms); because we did not know \( \beta \) (which we usually obtained from \( \Delta M \), which gives \( \beta/\nu = 0 \) in two dimensions), we failed to extract reliable exponents in two dimensions from \( dM/dH \).

For the sixth example, suppose \( F \) depends on \( r, h, \) and a size \( x \). Then from the previous analysis, we expect

\[
F(x, r, h) \sim x^{-\alpha} \mathcal{F}(x^{\epsilon}r, h/r^{\beta \delta}h).
\]  
(A14)

Notice that universality only removes one variable from the scaling form. One could in practice do two–variable scaling collapses (and we believe someone has probably done it), but for our purposes these more general scaling forms are used by fixing one of the variables. For example, we measure the avalanche size distribution at various values of \( h \) (binned in small ranges), at the critical disorder \( r = 0 \). We can make sense of equation (A14) by changing variables from \( h/r^{\beta \delta} \) to \( x^{\epsilon} \).

Before we can set \( r = 0 \), we must see what are the possible corrections to scaling in this case. If the disorder \( r \) depends on the field, then instead of the variable \( r \), we must use \( r + ah \) (the analysis is analogous to the one in example five; other corrections are subdominant). Setting \( r = 0 \) now, leaves \( F \) dependent on its first variable, as well as the second:

\[
F(x, r, h) \sim x^{-\alpha} \mathcal{F}(x^{\epsilon}r, h/r^{\beta \delta}h) \approx x^{-\alpha} \times \mathcal{F}(0, x^{\epsilon}h) + ahx^{\epsilon} \mathcal{F}(1, 0)(0, x^{\epsilon}h),
\]  
(A16)

where \( \mathcal{F}(1, 0) \) is the derivative of \( \mathcal{F} \) with respect to the first variable (keeping the second fixed).

For the binned avalanche size distribution, \( x^{\epsilon} \) is \( S^\tau \), where \( 0 \leq \sigma < 1/2 \) as we move from two dimensions to five and above. Thus, the correction term will only be important for rather large avalanches, \( S > h^{-1/\sigma} \), so long as we are close to the critical point. Expressed in terms of the scaling variable, important corrections to scaling occur if the scaling variable \( X = S^\tau h > h^{-1+\beta} \). For us, \( \beta \delta > 3/2 \), and we only use fields near the critical field \( h < 0.08 \), so the corrections will become of order one when \( X = 4 \) for the largest \( h \) we use. In 3 and 4 dimensions, this correction does not affect our scaling collapses, while in 5 dimensions some of the data needs this correction. We have tried to avoid this problem (since we don’t measure our data such that it can be used in a two–variable scaling collapse) by concentrating on collapsing the regions in our data curves where this correction is negligible.

A similar analysis can be done for the avalanche time distribution, which has two “sizes” \( S \) and \( t \) and one parameter \( r \) which is set to zero; because we integrate over the field \( h \) the correction in (A16) does not occur, and other scaling corrections are small.

Finally, we discuss the unusual exponential scaling forms we developed to collapse our data in two dimensions. If we assume that the critical disorder \( R_e \) is zero and that the linear term in the rescaling of \( r \) vanishes (\( \zeta \nu r \) in equation (A7) vanishes), then from symmetry the correction has to be cubic, and equation (A7) becomes

\[
F(x, r) = (1 - \alpha \epsilon) F((1 - \epsilon)x, (1 + k\epsilon r^2)r).
\]  
(A17)

with \( k \) (which is not universal) and \( \alpha \), \( \epsilon \) constants, and \( \epsilon \) small.

Taking the derivative of both sides with respect to \( \epsilon \) and setting it equal to zero gives a partial differential equation for the function \( F \). To solve for \( F \), we do a change of variable: \((x, r) \rightarrow (x, y) \) with \( y = x e^{-\alpha \epsilon}/r^2 \). The constant \( \alpha^* \) is determined by requiring that \( y \) rescales onto itself to order \( \epsilon \): we find \( \alpha^* = 1/2k \). We then have:
\[ 0 = -\alpha \hat{F}(x, y) - \frac{\partial \hat{F}}{\partial x} x \]  

(A18)

which gives

\[ F(x, r) = x^{-\alpha} \hat{F}\left(x e^{-1/2k r^2}\right). \]  

(A19)

This is one of the forms we use in 2 dimensions for the scaling collapse of the second moments \((\langle S^2 \rangle_{\text{int}})\), the avalanche size distribution \(D_{\text{int}}\) integrated over the field \(H\), the avalanche correlation \(G_{\text{int}}\), and the spanning avalanches \(N\). We use another form too which is obtained by assuming that the critical disorder \(R_c\) is not zero but that the linear term in the rescaling of \(r\) still vanishes. Instead of equation (A17), we have:

\[ F(x, r) = (1 - \alpha \epsilon) F\left((1 - \epsilon) x, (1 + \epsilon r) r\right). \]  

(A20)

The function \(F\) becomes:

\[ F(x, r) = x^{-\alpha} \hat{F}\left(x e^{-1/l r}\right). \]  

(A21)

The corrections to scaling for the last two forms (equations (A19) and (A21)) are similar to the ones discussed above. They are all subdominant.

**APPENDIX B: FULL DERIVATION OF THE MEAN FIELD SCALING FORM FOR THE INTEGRATED AVALANCHE SIZE DISTRIBUTION**

The mean field scaling form for the integrated avalanche size distribution \(D_{\text{int}}(S, R)\) was obtained in section IV A using the scaling form of the avalanche size distribution \(D(S, R, H)\). The scaling form for \(D_{\text{int}}(S, R)\) can also be obtained by integrating the avalanche probability distribution \(D(S, t)\) (derived originally in [13]) directly:

\[ D_{\text{int}}(S, R) = \int_{-\infty}^{+\infty} \rho(-JM - H) D(S, t) dH \]  

(B1)

where \(\rho(-JM - H)\) is the probability distribution for the random fields, and \(\rho(-JM - H) dH\) is the probability for a spin to flip between fields \(-JM(H) - H\) and \(-JM(H + dH) - (H + dH)\). \(D(S, t)\) is the probability of having an avalanche of size \(S\), a small “distance” \(t \equiv 2J \rho(-JM - H) - 1\) from the infinite avalanche at \(\rho(-JM - H) = 1/2J\), given that a spin has flipped at \(-JM - H\) [13]. (The scaling form for the non-integrated avalanche size distribution \(D(S, R, H)\) (eq. 19) is obtained from \(D(S, t)\) by expressing \(t\) as a function of \(R\) and \(H\) [13]). \(J\) is the coupling of a spin to all others in the system, \(H\) is the external magnetic field, and \(R\) is the disorder. The advantage of this procedure is that we can find out something about the scaling function \(D_{\text{int}}\).

The average mean field magnetization \(M\) and the avalanche probability distribution \(D(S, t)\) are given by [13,14]:

\[ M(H, R) = 1 - 2 \int_{-\infty}^{-JM(H) - H} \rho(f) df, \]  

(B2)

and

\[ D(S, t) = \frac{SS^{-2}}{(S - 1)!} (t + 1)^{S - 1} e^{-S(t + 1)} \]  

(B3)

To solve equation (B4), let’s define the variable \(y = (-JM - H)/(\sqrt{2} R)\) and rewrite the integral as:

\[ D_{\text{int}}(S, R) = \sqrt{2} R \times \int_{-\infty}^{+\infty} \rho(\sqrt{2} Ry) D(S, 2J \rho(\sqrt{2} Ry) - 1) \times \left(1 - 2J \rho(\sqrt{2} Ry)\right) dy, \]  

(B4)

where we have used:

\[ \frac{dy}{dH} = \frac{1}{\sqrt{2} R} \left(-J - \frac{2}{1 - 2J \rho(-JM - H)} - 1\right) \]  

(B5)

Since we are interested in the behavior of the integrated avalanche distribution for large sizes, the factorial in equation (B3) can be expanded using Stirling’s formula. To first order, we have:

\[ (S - 1)! \approx \frac{SS\sqrt{2\pi}}{eS \sqrt{S}} \]  

(B6)

Substituting this and the random field distribution function \(\rho\),

\[ \rho(\sqrt{2} Ry) = \frac{1}{\sqrt{2\pi R}} e^{-y^2}, \]  

(B7)

in equation (B4), we obtain:

\[ D_{\text{int}}(S, R) \approx C \left(\frac{R_c}{R}\right)^S \times \int_{-\infty}^{+\infty} e^{-S \left(y^2 + \frac{R_c}{R} e^{-y^2}\right)} \left(1 - \frac{R_c}{R} e^{-y^2}\right) dy, \]  

(B8)

where \(C = S^{-\frac{3}{2}} e^{S R_c/(\pi R \sqrt{2})}\), and \(S\) is large.

For disorders above but close to the critical disorder \(R_c\), we have:

\[ \left(\frac{R_c}{R}\right)^S = e^{S \log\left(\frac{R_c}{R}\right)} \approx e^S \left(1 - \left(\frac{R_c}{R}\right) - \left(\frac{R_c}{R}\right)^2 - \left(\frac{R_c}{R}\right)^3 - \ldots\right) \]  

(B9)
If we assume that only terms up to $S(1 - R_c/R)^2$ are important (terms like $S(1 - R_c/R)^3$ and $S(1 - R_c/R)^4$ go to zero as $R \rightarrow R_c$), and we note that the integrand in equation (B8) is an even function of $y$, equation (B8) becomes:

$$D_{int}(S, R) \approx 2 \frac{C}{\pi \sqrt{2}} \left[ \int_0^\infty e^{-S \left( \frac{R_c}{R} \right)^2 + \frac{u^2}{2}} \left( 1 - \frac{R_c}{R} e^{-y^2} \right) dy \right]$$

(B10)

The asymptotic behavior of the above integral, as $S \rightarrow \infty$, is obtained using Laplace’s method [47]. The idea is as follows. The asymptotic behavior as $S \rightarrow \infty$ of the integral:

$$I(S) = \int_a^b f(x) e^{S\phi(x)} \, dx$$

(B11)

can be found by integrating over a small region $[c-\epsilon, c+\epsilon]$ (instead of the interval $[a, b]$) around the maximum of the function $\phi(x)$ at $x = c$, since in the asymptotic expansion, the largest contribution to the integral will be from this region. The corrections will be exponentially small. The maximum of $\phi$ must be in the interval $[a, b]$, $f(x)$ and $\phi(x)$ are assumed to be real continuous functions, and $f(c) \neq 0$. $f(x)$ and $\phi(x)$ can now both be expanded around $x = c$, and the integral solved. Often the integral is easier to handle if the limit of integration is extended to infinity. This will add only exponentially small corrections in the asymptotic limit of $S \rightarrow \infty$.

Let’s apply this method to equation (B10). The function in the exponential has a maximum at $y = 0$. The function $\left( 1 - \frac{R_c}{R} e^{-y^2} \right)$ is not zero there if $R \neq R_c$. We can thus expand both functions in the integral of equation (B10) around $y = 0$. Defining $u = y \sqrt{S}$, we obtain:

$$D_{int}(S, R) \approx C_1 \times \left[ \int_0^\infty e^{-\sqrt{S} \left( \frac{R_c}{R} \right)^2 + \frac{u^2}{2}} \left( 1 - \frac{R_c}{R} + \frac{R_c}{R} \frac{u}{\sqrt{S}} - \frac{R_c}{2R \sqrt{S}} u^2 + ... \right) du \right]$$

(B12)

where

$$C_1 = \frac{1}{\pi \sqrt{2}} \frac{R_c}{R} S^{-\frac{3}{2}} e^{-\frac{1}{2} \left( \frac{R_c}{R} \right)^2},$$

(B13)

$S$ is large, $R$ is close to but not equal to $R_c$, and only terms up to $S(1 - R_c/R)^2$ are non-vanishing. In the asymptotic limit of $S \rightarrow \infty$ we can ignore terms with powers of $S$ in the denominator, and look at the distribution for $R$ close to $R_c$. To first order in $r = (R_c - R)/R$, $R_c/R \approx 1$ and $1 - R_c/R \approx -r$, which gives:

$$D_{int}(S, R) \approx \frac{1}{\pi \sqrt{2}} S^{-\frac{3}{2}} e^{-\frac{1}{2} (-r)^2} \times \int_0^\infty e^{-(-r)\sqrt{S} u - \frac{u^2}{2}} (-r \sqrt{S} + u) \frac{du}{\sqrt{u}}$$

(B14)

where we have expanded the integration to infinity. As mentioned above, this will only add exponentially small corrections in the asymptotic limit of $S \rightarrow \infty$. Equation (B14) is the integrated avalanche size distribution in mean field for large sizes $S$, and finite $S/r^2$. We see right away that it gives the correct scaling form:

$$D_{int}(S, R) \sim S^{-\frac{3}{2}} \bar{D}_{\pm}^{(int)} \left( \sqrt{S} |r| \right)$$

(B15)

where $\pm$ indicates the sign of $r$, the exponent $\tau + \sigma \beta$ and $\sigma$ are 9/4 and 1/2 respectively, and the scaling function $\bar{D}_{\pm}^{(int)}$ is:

$$\bar{D}_{\pm}^{(int)} \left( \sqrt{S} |r| \right) = e^{-\frac{\sqrt{\pi} |r|^2}{2}} \bar{F}_\pm \left( \sqrt{S} |r| \right).$$

(B16)

The function $\bar{F}_\pm \left( \sqrt{S} |r| \right)$ is proportional to the integral in equation (B11). Note that the above result is equivalent to the one obtained (eqn. 23) by integrating the scaling form of $D(S, R, H)$ over the field $H$.

What is the behavior of the scaling function $\bar{D}_{\pm}^{(int)}(X)$ for small and large positive arguments $X = \sqrt{S} (-r) > 0$ ($R > R_c$)? From equations (B14) and (B16), for small arguments we have a polynomial in $X$:

$$\bar{D}_{\pm}^{(int)}(X) \approx A + BX + CX^2 + O(X^3)$$

(B17)

These parameters can be calculated numerically. We obtain in mean field:

$$\bar{D}_{\pm}^{(int)} \approx 0.232 + 0.243X - 0.174X^2 - 0.101X^3 + 0.051X^4$$

(B18)

On the other hand, for large arguments we find:

$$\bar{D}_{\pm}^{(int)}(X) \approx \pi^{1/2} e^{-\frac{\sqrt{\pi} X}{2}} \sqrt{X} \left( 1 + O(X^{-2}) \right)$$

(B19)

In general, for all dimensions, in equation (B19) the exponential is of $X^{1/\sigma}$ (1/\sigma = 2 in mean field), since the exponent $\sigma$ gives the exponential cutoff to the power law distribution for large $X$, and the power of $X$ is $\beta$ ($\beta = 1/2$ in mean field). One can see the latter by expanding the distribution function $D_{int}(S, R)$ in terms of $1/S$ ($S$ is large), analogous to [43]:

$$D_{int}(S, R) = \sum_{n=1}^\infty f_n(r) S^{-n}$$

(B20)

Since $X = S^\sigma (-r)$, then we can write $S = X^{1/\sigma} (-r)^{-1/\sigma}$ and obtain:
\[ D_{\text{int}}(S, R) = \sum_{n=1}^{\infty} f_n(r) X^{-n/\sigma} (-r)^{n/\sigma} \] (B21)

The scaling function \( \mathcal{D}_{\text{int}}(X) \) scales like \( S^{(\tau+\sigma \beta \delta)} \times D_{\text{int}}(S, R) \):

\[
\mathcal{D}_{\text{int}}(X) \sim \sum_{n=1}^{\infty} f_n(r) X^{-n/\sigma} X^{(\tau+\sigma \beta \delta)/\sigma} \times (-r)^{n/\sigma} (-r)^{-(\tau+\sigma \beta \delta)/\sigma} \] (B22)

and since it is only a function of \( X \), it must satisfy:

\[
\mathcal{D}_{\text{int}}(X) \sim \sum_{n=1}^{\infty} g_n X^{-n/\sigma} X^{(\tau+\sigma \beta \delta)/\sigma} \] (B23)

where \( g_n \) is independent of \( r \).

The exponent combination \( (\tau+\sigma \beta \delta)/\sigma \) can be rewritten as:

\[
\frac{\tau + \sigma \beta \delta}{\sigma} = \frac{2 \tau + 2 \sigma \beta \delta - 2}{\sigma} = \frac{2 \tau}{\sigma} + \beta \] (B24)

where we have used the scaling relation \[14,16\]: \( \beta - \beta \delta = (\tau - 2)/\sigma \). Thus we have for the scaling function \( \mathcal{D}_{\text{int}}(X) \):

\[
\mathcal{D}_{\text{int}}(X) \sim X^\beta \sum_{n=1}^{\infty} g_n X^{-n/\sigma} = X^\beta \mathcal{K}(X^{1/\sigma}) \] (B25)

which shows (compare to equation \[B14\]) that the power of \( X \) is indeed the exponent \( \beta \).

We have used the results of the expansion of the mean field scaling function \( \mathcal{D}_{\text{int}}(X) \) for small and large parameters (equations \[B17\] and \[B19\]), to build a fitting function to the integrated avalanche size distribution scaling functions in 2, 3, 4, and 5 dimensions, described in section IV B.

Finally, note from equation \[B17\] that the scaling function:

\[
\mathcal{D}_{\text{int}}(S_{\tau}^2) = e^{-\langle S_{\tau}^2 \rangle} F_{-}(S_{\tau}^2) \] (B26)

used earlier in reference \[14\], is not analytic for small arguments \( S_{\tau}^2 \), from which we conclude that the appropriate scaling variable should be \( \sqrt{S} (r) \) and not \( S_{\tau}^2 \). (Notice that this no longer seems true in two dimensions; see section on 2 dimensional results.)

**APPENDIX C: DERIVATION OF THE MEAN FIELD SCALING FORM FOR THE SPANNING AVALANCHES**

We have defined earlier a mean field spanning avalanche to be an avalanche larger than \( \sqrt{S_{mf}} \), where \( S_{mf} \) is the *total* size of the system. We want to derive the scaling form for the number of such avalanches in half of the hysteresis loop (for \( H \) from \(-\infty \) to \(+\infty \)) as a function of the system size \( S_{mf} \) and the disorder \( R \). The number of mean field spanning avalanches is proportional to the probability of having avalanches of size larger than \( \sqrt{S_{mf}} \). Since we want the number of spanning avalanches, we need to multiply this probability by the total number of avalanches. For large system sizes, these scales with the system size \( S_{mf} \) (corrections are subdominant). We thus obtain by integrating over equation \[B13\] (which gives the scaling form for the probability distribution of avalanches of size \( S \) in the hysteresis loop):

\[
N_{mf}(S_{mf}, R) \sim S_{mf} \times \int_{\sqrt{S_{mf}}}^{\infty} S^{-\frac{\tau}{4}} e^{-\frac{\langle S_{\tau}^2 \rangle}{2}} \bar{F}_{\pm}\left(\sqrt{S}_{\tau}|r|\right) dS \] (C1)

Let’s define \( u = \sqrt{S}_{\tau}|r| \), then equation \(C1\) can be written as:

\[
N_{mf}(S_{mf}, R) \sim 2 S_{mf} |r|^\frac{\tau}{4} \times \int_{|r|\sqrt{S_{mf}}}^{\infty} u^{-\frac{\tau}{4}} e^{-\frac{\langle S_{\tau}^2 \rangle}{2}} \bar{F}_{\pm}(u) du \] (C2)

The integral \( \mathcal{I} \) is a function of \( S_{mf}^\frac{1}{2} |r| \) only, and we can write it as:

\[
\mathcal{I} = \left( S_{mf}^\frac{1}{2} |r| \right)^{-\frac{\tau}{4}} N_{mf}^r \left( S_{mf}^\frac{1}{2} |r| \right) \] (C3)

to obtain the scaling form for the number \( N_{mf} \) of mean field spanning avalanches:

\[
N_{mf}(S_{mf}, R) \sim S_{mf}^\frac{3}{2} N_{mf}^r \left( S_{mf}^\frac{1}{2} |r| \right) \] (C4)

From this scaling form, we can extract the exponents \( \bar{\theta} = 3/8 \), and \( 1/\nu = 1/4 \). This form is used for collapses of the spanning avalanche curves in mean field (see mean field section).

[1] J. P. Sethna, J. D. Shore, and M. Huang, Phys. Rev. B 44, 4943 (1991), and references therein; J. D. Shore, Ph.D. Thesis, Cornell University (1992), and references therein; T. Riste and D. Sherrington, *Phase Transitions and Relaxation in Systems with Competing Energy Scales* (Proc. NATO Adv. Study Inst., Geilo, Norway, April 13-23, 1993), and references therein; M. Mézard, G. Parisi, M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987), and references therein; K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1993), and references therein; G. Grinstein and J. F. Fernandez, Phys. Rev. B 29, 6389 (1984); J. Villain, Phys. Rev. Lett. 52, 1543 (1984); D. S. Fisher, Phys. Rev. Lett. 56, 416 (1986).
For a good introduction on static scaling and finite size scaling, see: N. Goldenfeld, *Lectures on Phase Transition and the Renormalization Group* (Addison Wesley, 1992).

In previous papers [2] and [3], the scaling function for the avalanche size distribution was defined as $D_\pm(S|r|^{1/\nu}, h/|r|^{\beta\delta})$. Since this function is not analytic in mean field for small $S|r|^{1/\nu}$ (see appendix B), we use in this paper the scaling function $D_\pm(S|r|^{1/\nu}, h/|r|^{\beta\delta})$, which in mean field is analytic for small $S|r|^{1/\nu}$. Note that the scaling function $D_\pm^{(int)}(S|r|)$ in reference [12] has been mistakenly written as $D_{int}(S|r|)$.

Hong Ji and Mark O. Robbins, Phys. Rev. B 46, 14519 (1992).

Jean Souletie, personal communication.

P. Schofield, Phys. Rev. Lett. 22, 606 (1969); D. S. Gaunt and C. Domb, J. Phys. C 3, 1442 (1970).

Plots for all the dimensions can be obtained from the raw data at the Web site: http://www.lassp.cornell.edu/LASSP_Science.html.

M. E. Fisher in *Critical Phenomena*, Lecture Notes in Physics, Vol. 186, Ed. F. J. W. Hahne (Springer–Verlag, Berlin 1983).

In 3 and 4 dimensions, below the critical field $H_c$, we have not found any spanning avalanches. In 5 dimensions, spanning avalanches appeared below the critical field $H_c = 1.175$. This is due to the small system size ($30^3$) used for the 5 dimensional simulation. We however, do not expect substantial corrections to the exponents derived from that data.

L. de Arcangelis, J. Phys. A 20, 3057 (1987); A. Coniglio, Springer Proc. Phys., 5, 84 (1985).

D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, revised second edition (Taylor & Francis, London, Bristol, PA 1994).

A. Maritan, M. Cieplak, M. R. Swift and J. Banavar, Phys. Rev. Lett. 72, 946 (1994); J. P. Sethna, K. A. Dahmen, S. Kartha, J. A. Krumhansl, O. Perković, B. W. Roberts, and J. D. Shore, Phys. Rev. Lett. 72, 947 (1994).

P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, No. 3 (1977).

J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *The Theory of Critical Phenomena* (Clarendon Press, Oxford, 1992). S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Massachusetts, 1976).

A. J. Bray and M. A. Moore, J. Phys. C 18, L927 (1985);

H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S. A. Larin (Phys. Lett. B 272, 39 (1991) and erratum in Phys. Lett. B 319, 545 (1993)) provide the expansion for the pure, equilibrium Ising exponents to fifth order in $\epsilon = 4 - d$. A. A. Vladimirov, D. I. Kazakov, and O. V. Tarasov, (Sov. Phys. JETP 50 (3), 521 (1979) and references therein) introduce a Borel resummation method with one parameter, which is varied to accelerate convergence. J.C. LeGuillou and J. Zinn-Justin (Phys. Rev. B 21, 3976 (1980)) do a coordinate transformation with a pole at $\epsilon = 3$, and later (J. Physique Lett. 46, L137 (1985) and J. Physique 48, 19 (1987)) make the placement of the pole a variable parameter (leading to a total of four real acceleration parameters for a fifth order expansion!). Unfortunately, LeGuillou et al. used a form for the fifth order term which turned out to be incorrect (Kleinert, above). The $\epsilon$ expansion is an asymptotic series, which need not determine a unique underlying function (J. Zinn-Justin “Quantum Field Theory and Critical Phenomena”, 2nd edition, Clarendon Press, Oxford (1993)). Our model likely has non-perturbative correct-
tions (as did the equilibrium, thermal random-field Ising model; G. Parisi, lectures given at the 1982 Les Houches summer school XXXIX “Recent advances in field theory and statistical mechanics” (North Holland), and references therein).

[34] E. Brézin, D. J. Wallace, and K. G. Wilson, Phys. Rev. Lett. 29, 591 (1972), and Phys. Rev. B 7, 232 (1973); G. M. Avdeeva and A. A. Midgal, J.E.T.P. Letters 16, 178 (1972); D. J. Wallace and R. P. K. Zia, Phys. Lett. A 46, 261 (1973); D. J. Wallace and R. P. K. Zia, J. Phys. C 7, 3480 (1974); C. Domb and M. S. Green, Phase Transitions and Critical Phenomena, Vol. 6, Section 5, (Academic Press, NY, 1976), and references therein.

[35] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 2nd edition (Clarendon Press, Oxford 1993).

[36] A. Aharony, Y. Imry, and S. K. Ma, Phys. Rev. Lett. 37, 1364 (1976); G. Parisi and N. Sourlas, Phys. Rev. Lett. 43, 744 (1979).

[37] G. Parisi, lectures given at the 1982 Les Houches summer school XXXIX “Recent advances in field theory and statistical mechanics” (North Holland), and references therein.

[38] H. Ji and M. O. Robbins, Phys. Rev. B 46, 14519 (1992) and references therein; O. Narayanan and D. S. Fisher, Phys. Rev. Lett. 68, 3615 (1992) and Phys. Rev. B 46, 11520 (1992); D. Ertaş and M. Kardar, Phys. Rev. E 49, R2532 (1994); C. Myers and J.P. Sethna, Phys. Rev. B 47 11171 (1993) and Phys. Rev. B 47 11194 (1993).

[39] T. Nattermann, S. Stepanow, L. H. Tang and H. Leschhorn, J. Phys. II France 2, 1483 (1992); O. Narayanan and D. S. Fisher, Phys. Rev. B 48, 7030 (1993).

[40] C. M. Coram, A. E. Jacobs, N. Heinig, and K. B. Winterbon, Phys. Rev. B 40, 6992 (1989); E. Vives and A. Planes, Phys. Rev. B 50, 3839 (1994).

[41] E. Vives, J. Goicoechea, J. Ortín, and A. Planes, Phys. Rev. E 52, R5 (1995).

[42] K. G. Wilson, Phys. Rev. B 4, 3174, 3181 (1971); K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972); K. G. Wilson and J. Kogut, Phys. Rep. C 12, 75 (1974).

[43] J. M. Yeomans, Statistical Mechanics of Phase Transitions (Clarendon Press, Oxford 1992); M. Plischke and B. Berghersen, Equilibrium Statistical Physics (Prentice Hall, NJ 1989); S. Ma, Modern Theory of Critical Phenomena (Benjamin, Massachusetts, 1976).

[44] For systems with short-range interactions, the critical behavior of a system depends only on the dimensionality $d$ and the symmetry group of the hamiltonian $\mathcal{H}$, and not on the microscopic details of the system. Two systems are in the same universality “class” if they have the same dimensionality and their hamiltonian is in the same symmetry group.

[45] M. E. Fisher and A. N. Berker, Phys. Rev. B 26, 2507 (1982), and references therein.

[46] M. E. Fisher and A. E. Ferdinand, Phys. Rev. Lett. 19, 169 (1967); M. E. Fisher in Critical Phenomena, Proceedings of the 51st “Enrico Fermi” Summer School, Ed. M. S. Green (Academic Press, Varenna, Italy, 1973) (pages 73-97); M. E. Fisher, J. Vac. Sci. Technol. 10, 665 (1973); see also [34] pages 153-157.

---

### TABLE I. Numerical values for the critical disorders and fields, and the “tilt” parameter $b$ (see section on magnetization curves collapses) in 3, 4, and 5 dimensions extracted from scaling collapses. The critical disorder is obtained from collapses of the spanning avalanches and the second moments of the avalanche size distribution. The critical field is obtained from the binned avalanche size distribution and the magnetization curves. $H_c$ is affected by finite sizes, and systematic errors could be larger than the ones listed here. The mean field values are calculated analytically $[13,14]$, $R_c$, $H_c$, and $b$ are not universal characteristics of the system.

| Dimension | $R_c$ | $H_c$ | $b$ |
|-----------|-------|-------|-----|
| 3d        | 2.16±0.03 | 4.10±0.02 | 0.39±0.08 |
| 4d        | 5.96±0.02 | 5.06±0.07 | 0.46±0.05 |
| 5d        | 0.7978±0.06 | 1.175±0.004 | 0.23±0.08 |

TABLE I. Numerical values for the critical disorders and fields, and the “tilt” parameter $b$ (see section on magnetization curves collapses) in 3, 4, and 5 dimensions extracted from scaling collapses. The critical disorder is obtained from collapses of the spanning avalanches and the second moments of the avalanche size distribution. The critical field is obtained from the binned avalanche size distribution and the magnetization curves. $H_c$ is affected by finite sizes, and systematic errors could be larger than the ones listed here. The mean field values are calculated analytically $[13,14]$. The “tilt” $b$ is obtained from the $dM/dH$ collapses using the values for the critical disorder and field from this table and the values for the exponents from Table II. Only the parameter $b$ is allowed to vary. The values in 2 dimensions (which are not listed) are less accurate. Depending on the scaling form we obtain a critical disorder of 0.45, or 0.54. The critical field is around 1.32 and is estimated from the binned in $H$ avalanche size distribution and magnetization curves (see text). The “tilt” $b$ was not measured. $R_c$, $H_c$, and $b$ are not universal characteristics of the system.
**TABLE II.** Values for the exponents extracted from scaling collapses in 3, 4, and 5 dimensions. The mean field values are calculated analytically [13,14]. $\nu$ is the correlation length exponent and is found from collapses of avalanche correlations, number of spanning avalanches, and moments of the avalanche size distribution data. The exponent $\theta$ is a measure of the number of spanning avalanches and is obtained from collapses of that data. $(\tau + \sigma \beta \delta - 3)/\sigma \nu$ is obtained from the second moments of the avalanche size distribution collapses. $1/\sigma$ is associated with the cutoff in the power law distribution of avalanche sizes integrated over the field $H$, while $\tau + \sigma \beta \delta$ gives the slope of that distribution. $\tau$ is obtained from the binned avalanche size distribution collapses. $d + \beta/\nu$ is obtained from avalanche correlation collapses and $\beta/\nu$ from magnetization discontinuity collapses. $\sigma \nu z$ is the exponent combination for the time distribution of avalanche sizes and is extracted from that data.

| measured exponents | 3d       | 4d       | 5d       | mean field |
|--------------------|----------|----------|----------|------------|
| $1/\nu$            | 0.71 ± 0.09 | 1.12 ± 0.11 | 1.47 ± 0.15 | 2          |
| $\theta$           | 0.015 ± 0.015 | 0.32 ± 0.06 | 1.03 ± 0.10 | 1          |
| $(\tau + \sigma \beta \delta - 3)/\sigma \nu$ | -2.90 ± 0.16 | -3.20 ± 0.24 | -2.95 ± 0.13 | -3         |
| $1/\sigma$         | 4.2 ± 0.3   | 3.20 ± 0.25 | 2.35 ± 0.25 | 2          |
| $\tau + \sigma \beta \delta$ | 2.03 ± 0.03 | 2.07 ± 0.03 | 2.15 ± 0.04 | 9/4        |
| $\tau$             | 1.60 ± 0.06 | 1.53 ± 0.08 | 1.48 ± 0.10 | 3/2        |
| $d + \beta/\nu$    | 3.07 ± 0.30 | 4.15 ± 0.20 | 5.1 ± 0.4   | 7 (at $d_c = 6$) |
| $\beta/\nu$        | 0.025 ± 0.020 | 0.19 ± 0.05 | 0.37 ± 0.08 | 1          |
| $\sigma \nu z$     | 0.57 ± 0.03 | 0.56 ± 0.03 | 0.545 ± 0.025 | 1/2        |
| calculated exponents | 3d     | 4d     | 5d     | mean field |
|----------------------|--------|--------|--------|------------|
| $\sigma\beta\delta$ | 0.43 ± 0.07 | 0.54 ± 0.08 | 0.67 ± 0.11 | 3/4        |
| $\beta\delta$       | 1.81 ± 0.32 | 1.73 ± 0.29 | 1.57 ± 0.31 | 3/2        |
| $\beta$              | 0.035 ± 0.028 | 0.169 ± 0.048 | 0.252 ± 0.060 | 1/2        |
| $\sigma\nu$         | 0.34 ± 0.05 | 0.28 ± 0.04 | 0.29 ± 0.04 | 1/4        |
| $\eta = 2 + (\beta - \beta\delta)/\nu$ | 0.73 ± 0.28 | 0.25 ± 0.38 | 0.06 ± 0.51 | 0          |

TABLE III. Values for exponents in 3, 4, and 5 dimensions that are not extracted directly from scaling collapses, but instead are derived from Table II and the exponent relations (see [14,16]). The mean field values are obtained analytically [13,14]. Both $\sigma\beta\delta$ and $\beta\delta$ could have larger systematic errors than the errors listed here. See the binned avalanche size distribution section for details.
TABLE IV. Exponent values and critical disorder $R_c$ from collapses of spanning avalanche curves in 4 dimensions. Three curves (different linear size $L$) are collapsed together, with $r = (R_c - R)/R$ and $r = (R_c - R)/R_c$. Tables IV, V, and VI give information equivalent to that given in for example figures 8a and 8b. Graphs showing two points with an extrapolation to $1/L \to 0$ seemed unnecessary.

| Exponents and $R_c$ | $L=10,20,40$ | $L=20,40,80$ |
|---------------------|--------------|--------------|
| $1/\nu$             | 0.96 ± 0.07  | 1.05 ± 0.10  | 1.12 ± 0.06 |
| $\theta$            | 0.35 ± 0.10  | 0.32 ± 0.04  | 0.32 ± 0.04 |
| $R_c$               | 4.09 ± 0.02  | 4.095 ± 0.015| 4.10 ± 0.01 |
| Exponents | $L=10,20,40$ | $L=20,40,80$ |
|-----------|-------------|-------------|
| $r = (R_c - R)/R$ | $r = (R_c - R)/R$ | $r = (R_c - R)/R$ | $r = (R_c - R)/R$ |
| $1/\nu$ | $1.10 \pm 0.04$ | $1.24 \pm 0.08$ | $1.10 \pm 0.05$ | $1.11 \pm 0.05$ |
| $\beta/\nu$ | $0.195 \pm 0.035$ | $0.19 \pm 0.05$ | $0.18 \pm 0.05$ | $0.20 \pm 0.06$ |

TABLE V. Exponent values for $1/\nu$ and $\beta/\nu$, obtained from scaling collapses of the change of the magnetization $\Delta M$ due to the spanning avalanches. Three curves of different size $L$ are collapsed together with $r = (R_c - R)/R$ and $r = (R_c - R)/R_c$, where $R_c = 4.10 \pm 0.02$. See also comment in Table IV.
TABLE VI. Exponent values from the collapses of second moments of the avalanche size distribution curves in 5 dimensions. Three curves of different size L are collapsed together with $r = (R_c - R)/R$ and $r = (R_c - R)/R_c$, where $R_c = 5.96 \pm 0.02$. See also comment in Table IV.
TABLE VII. Conjectured and measured values for some exponents in 2 dimensions. We don’t have a conjectured value for the exponent combination $\sigma \nu z$, but the measured value is $0.64 \pm 0.02$. (*) Note that the distribution of avalanche sizes at $R_c$ in two dimensions, integrated over the loop, will scale as $S^{-(\tau + \sigma \beta \delta) + \omega}$, where the correction $\omega \sim 1$ is due to the singularity in the scaling function $D^{(\text{int})}(X) \sim X^\omega$ as $X \to 0$. See text section IV, figure 35, and appendix A for details. A similar argument can be made for the avalanche correlation measurement (integrated over the field $H$), where due to the singularity of the scaling function $\tilde{G}_-$, the scaling for small $x |r|$ is $x^{-(d+\beta/\nu) + \tilde{\omega}}$, with $\tilde{\omega} \sim 1$ (see text and figure 37b).

| 2d | $1/\nu$ | $(\tau + \sigma \beta \delta - 3)/\sigma \nu$ | $\sigma$ | $\tau + \sigma \beta \delta^*$ | $\tau$ | $\sigma \nu$ | $\beta/\nu^*$ |
|----|--------|---------------------------------|-------|---------------------------------|------|--------|--------|
| conj. | 0      | -2                              | 0     | 2                               | 3/2  | 1/2    | 0      |
| meas. | $0.13 \pm 0.13$ | $-1.9 \pm 0.1$ | $0.10 \pm 0.02$ | $2.04 \pm 0.04$ | $0.0 \pm 0.0$ | $0.51 \pm 0.08$ | $0.03 \pm 0.06$ |

TABLE VIII. Conjectured values for some exponents in 2 dimensions. These exponents were not extracted from collapses (see text).