THE BETTI NUMBERS OF A DETERMINANTAL VARIETY

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Abstract. We determine the Poincaré polynomial of the determinantal variety \{\det = 0\} in the projective space associated with the monoid of \(n \times n\) matrices.

Keywords: Determinantal variety, Betti numbers, Chow groups, Borel-Moore homology

MSC: 14M12, 20M32, 14C15

1. Introduction

In this note, we look closely at the homology groups of a classical variety. Let \(Y^0\) denote the semigroup defined by the vanishing of the determinant polynomial in \(n \times n\) matrices. More precisely, we set \(Y^0 := M \setminus G\), where \(M\) is the monoid of all linear operators on an \(n\) dimensional complex vector space \(V\), and \(G = GL(V)\). The purpose of this note is to describe the Poincaré polynomial of the quotient,

\[
Y := \mathbb{P}(Y^0) = \frac{(Y^0 \setminus \{0\})}{\mathbb{C}^*},
\]

where \(\mathbb{C}^*\) is the center of \(GL(V)\).

It is not difficult to see that if \(\dim V = 2\), then \(Y\) is isomorphic to the quadric surface \(\mathbb{P}^1 \times \mathbb{P}^1\) in \(\mathbb{P}^3\). In particular, the Poincaré polynomial of \(Y\) is given by \(1 + 2t^2 + t^4\). However, in general, \(Y\) has a large singular locus, which is given by the projectivization of a \(G \times G\)-orbit closure, \((GeG \setminus \{0\})/\mathbb{C}^*\), where \(e\) is an idempotent of rank \(n - 2\) in \(M\). It is natural question to ask for a description of the Poincaré polynomial of \(Y\) for \(n := \dim V > 2\). It turns out that the degrees as well as the coefficients of monomials in \(P_Y(t)\) have interesting patterns, although \(P_Y(t)\) is neither symmetric nor unimodal.

Our main result and its corollary are the following statements.

Theorem 1.2. Let \(Y\) denote the determinantal variety defined as in (1.1). If \(V\) is \(n\) dimensional, then the homology groups of \(Y\) satisfy the following isomorphisms:

\[
H_k(Y) \cong \begin{cases} 
0 & \text{if } k \text{ is odd and } k < n^2 - 1; \\
\mathbb{Z} & \text{if } k \text{ is even and } k < n^2 - 1; \\
H_{k+1-(n^2-1)}(\text{PSU}_n) & \text{if } k \text{ is odd and } k \geq n^2 - 1.
\end{cases}
\]

Finally, if \(k\) is even and \(k \geq n^2 - 1\), then we have \(H_k(Y)/H_{k+1-(n^2-1)}(\text{PSU}_n) \cong \mathbb{Z}\). Here, \(\text{PSU}_n\) denotes the projective special unitary group.

Let us denote by \(P_{\text{PSU}_n}(t)\) the polynomial \(\prod_{i=1}^{n-1}(1 + t^{2i+1})\). In other words, \(P_{\text{PSU}_n}(t)\) is the Poincaré polynomial of \(\text{PSU}_n\). It is easy to check that, starting from \(n = 5\) the polynomial \(P_{\text{PSU}_n}(t)\) is no longer unimodal. On the other hand, as a product of palindromic polynomials, \(P_{\text{PSU}_n}(t)\) is palindromic for all \(n\).

Let us write \(P_{\text{PSU}_n}(t)\) in the form \(P_{\text{PSU}_n}(t) = \sum_{i=0}^{n^2-1} b_i t^i\) with \(b_i \in \mathbb{Z}_{\geq 0}\).

Corollary 1.3. Let \(Y\) denote the determinantal variety defined as in (1.1). If \(V\) is \(n\) dimensional, then the Poincaré polynomial of \(Y\) is expressible as a sum of two polynomials,

\[
P_Y(t) = A(t) + \tilde{B}(t),
\]

Date: March 15, 2019.
where $A(t) = 1 + t^2 + \cdots + t^{2\left\lfloor \frac{n-1}{2} \right\rfloor}$, and $\overline{B}(t)$ is the polynomial that is obtained from

$$B(t) := t^{n^2-1}P_{PSU_n}(t) = \sum_{i=n^2-1}^{2(n^2-1)} b_{i-(n^2-1)} t^i$$

by adding 1 to the coefficients of the terms $b_{i-(n^2-1)} t^i$ with $i$ odd.

Note that a complete description of the (torsion in the) cohomology ring of $PSU_n$ has recently been given by Haibao Duan in [3].

2. Preliminaries

We start with reviewing some well known facts about the Chow groups and Borel-Moore homology groups. We follow the presentation in [5, Chapter 19]; if $X$ is a topological space, then $H_*(X)$ stands for the Borel-Moore homology group with integer coefficients.

2.1. Let $k$ be a nonnegative integer, and let $X$ be a scheme. The free abelian group generated by all $k$ dimensional subvarieties of $X$ is denoted by $Z_k X$. The elements of $Z_k X$ are called $k$-cycles. A $k$-cycle $\alpha$ is said to be rationally equivalent to 0, and written $\alpha \sim 0$ if there exist an integer $n$ and rational functions $f_i \in \mathcal{O}(Y_i)$ ($i = 1, \ldots, s$) in such a way that $\alpha = \sum[\text{div}(f_i)]$. The set of $k$-cycles which are rationally equivalent to 0 is a subgroup of $Z_k(X)$, denoted by $\text{Rat}_k(X)$. The quotient group $A_k(X) := Z_k(X)/\text{Rat}_k(X)$ is called the group of $k$-cycles modulo rational equivalence, or the $k$-th Chow group. The total Chow group $A_*(X) := \bigoplus_{k=0}^\infty A_k(X)$ is a graded abelian group; if $X$ is equidimensional, then $A_{\dim X}(X)$ is freely generated by the classes of irreducible components of $X$.

If $X$ is an equidimensional scheme, by replacing $Z_k(X)$ with $Z^k(X)$, that is the group of $k$-codimensional cycles, we have the Chow group

$$A^k(X) := Z^k(X)/\text{Rat}_{\dim X-k}(X) = A_{\dim X-k}(X).$$

We set $A^*(X) := \oplus A^i(X)$. If $X$ is smooth, then there is an intersection pairing on $A^*(X)$, and hence, $A^*(X)$ becomes a ring.

Let $s$ be an element of $\mathbb{Z}_{\geq 0} \cup \{ \ast \}$. We will denote the vector spaces $A^s(X) \otimes \mathbb{Q}$ and $A_s(X) \otimes \mathbb{Q}$ by $A^s(X)_{\mathbb{Q}}$ and $A_s(X)_{\mathbb{Q}}$, respectively.

Chow groups behave nicely with respect to certain classes of morphisms.

1. If $f : X \to Y$ is a proper morphism, then there is a (covariant) homomorphism

$$f_* : A_k(X) \to A_k(Y).$$

2. If $f : X \to Y$ is a flat morphism of relative dimension $n$, then there is a (contravariant) homomorphism

$$f^* : A_k(Y) \to A_{k+n}(X).$$

Let $i : Y \hookrightarrow X$ be an inclusion of a closed subscheme $Y$ into a scheme $X$. Let $U$ denote the complement $X - Y$ and let $j : U \to X$ denote the inclusion. Then there is an exact sequence

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

for all $k$. To understand the image of $i_*$ in $A_k(X)$ we need to consider Edidin and Graham’s version of Bloch’s higher Chow groups.

Let $X$ be a quasi-projective scheme, and let $\Delta^k$ denote the algebraic version of the regular $k$-simplex:

$$\Delta^k = \text{Spec}(\mathbb{Z}[t_1, \ldots, t_k]/(t_1 + \cdots + t_k - 1)).$$

A face of $X \times \Delta^k$ is the subscheme of the form $X \times \Delta^m$, where the second factor $\Delta^m$ is the image of an injective canonical morphism $\rho : \Delta^m \to \Delta^k$. We denote by $Z^i(X, \bullet)$ the complex whose $k$-th term is the group of cycles of codimension $i$ in $X \times \Delta^k$ which intersect properly all of the faces in $X \times \Delta^k$. In [1], Bloch considered the following higher Chow groups:

$$\text{CH}^i(X, m) := H_m(Z^i(X, \bullet)).$$

Let $Z_p(X, \bullet)$ denote the complex whose $k$-th term is the group of cycles of dimension $p + k$ in $X \times \Delta^k$ intersecting the faces properly.
Lemma 2.4. Let $Y$ be a closed, not necessarily equidimensional subscheme of an equidimensional scheme $X$. Then there is a long exact sequence of higher Chow groups:

\begin{align}
&\cdots \to A_p(Y, 1) \to A_p(X) \to A_p(X - Y) \to 0.
\end{align}

Proof. See [4, Lemma 4]. \qed

Remark 1. It is not clear if the localization long exact sequence terminates for an arbitrary scheme.

2.2. The Borel-Moore homology groups of a space are defined by using ordinary cohomology groups as follows. Let $Y$ be a topological space that is embedded as a closed subspace of $\mathbb{R}^n$ for some positive integer $n$. Then the $q$th Borel-Moore homology of $Y$, denoted by $\bar{H}_q(Y)$ is defined by

$$\bar{H}_q(Y) = H^{n-q}(\mathbb{R}^n, \mathbb{R}^n \setminus Y).$$

(1) If $f : Y \to X$ is a proper morphism of complex schemes, then there are covariant homomorphisms $f_* : \bar{H}_i(Y) \to \bar{H}_i(X)$.

(2) If $j : U \to Y$ is an open imbedding, then there are contravariant restriction homomorphisms $j^* : \bar{H}_i(Y) \to \bar{H}_i(U)$.

(3) If $Y$ is the complement of $U$ in $X$ and $i : Y \to X$ is the closed imbedding, then there is a long exact sequence

\begin{align}
&\cdots \to \bar{H}_{i+1}(U) \to \bar{H}_i(Y) \xrightarrow{i_*} \bar{H}_i(X) \xrightarrow{j^*} \bar{H}_i(U) \to \bar{H}_{i-1}(Y) \to \cdots
\end{align}

(4) If $X$ is a disjoint union of a finite number of spaces, $X = X_1 \cup \cdots \cup X_n$, then $\bar{H}_i(X) = \oplus \bar{H}_i(X_j)$.

(5) There is a K"unneth formula for Borel-Moore homology.

(6) If $X$ is an $n$-dimensional complex scheme, then $\bar{H}_i(X) = 0$ for all $i > 2n$, and $\bar{H}_{2n}(X)$ is a free abelian group with one generator for each irreducible component $X_i$ of $X$. The generator corresponding to $X_i$ will be denoted by $cl(X_i)$. More generally, we will use the following notation: If $V$ is a $k$-dimensional closed subscheme of $X$, and $i : V \to X$ is the closed imbedding, then $cl_X(V)$ stands for $i_*cl(V)$, which lives in $\bar{H}_{2k}(X)$. If confusion is unlikely, we will omit the subscript $X$ from the notation.

(7) If $f : V \to W$ is a proper, surjective morphism of varieties, then $f_*cl(V) = deg(V/W) \cdot cl(W)$. Since we do not need it for our purposes, we will not define $deg(V/W)$ here; see [5, Section 1.4] for its definition.

(8) For any complex scheme $X$, there is a homomorphism from algebraic $k$-cycles on $X$ to the $k$-th Borel-Moore homology, $cl : Z_k(X) \to \bar{H}_k(X)$, defined by $cl(\sum n_i[V_i]) = \sum n_i cl_X(V_i)$. This homomorphism factors through the “algebraic equivalence” (which we didn’t introduce), hence, by composition, it induces a homomorphism from the $k$-th Chow group of $X$ onto the $2k$-th Borel-Moore homology. We will denote the resulting homomorphism by $cl$ also, and call it the cycle map.

(9) If a complex scheme $X$ has a cellular decomposition, then the cycle map $cl_X : A_k(X) \to \bar{H}_{2k}(X)$ is an isomorphism (see [5, Section 19.1.11]).

(10) Finally, let us mention that if $X$ is an $n$-dimensional oriented manifold, then $\bar{H}_k(X) \cong H^{n-k}(X)$.

For further details of this useful homology theory, see [2].
3. Proof

We will use the following notation in the sequel:

\( M \) : the monoid of \( n \times n \) matrices defined over \( \mathbb{C} \);

\( G \) : the general linear group of \( n \times n \) matrices defined over \( \mathbb{C} \);

\( T \) : the maximal torus consisting of diagonal matrices in \( G \);

\( Z \) : the center of \( G \);

\( X \) : the projectivization of \( M \), \( X := (M \setminus \{0\})/Z \);

\( Y_{0} \) : the vanishing locus of the determinant in \( M \);

\( Y \) : the projectivization of \( Y \), \( Y := Y_{0}/Z \);

\( U \) : the projectivization of \( G \), \( U := G/Z = \text{PGL}_{n} \).

Since \( Y \) is a projective variety, we have \( \bar{H}_{q}(Y) = H_{q}(Y) \) for \( q \in \{0, \ldots, \dim Y\} \). Of course, similar equalities hold true for \( X \cong \mathbb{P}^{n^{2} - 1} \) as well. Both of the spaces \( X \) and \( Y \) are path connected, therefore, we have \( H_{0}(X) = H_{0}(Y) = \mathbb{Z} \). The complement of \( Y \) in \( X \) is given by the group \( U \). Since \( U \) is open in \( X \), there is a long exact sequence for their Borel-Moore homology,

\[ \cdots \rightarrow \bar{H}_{q}(Y) \rightarrow \bar{H}_{q}(X) \rightarrow \bar{H}_{q}(U) \rightarrow \bar{H}_{q-1}(Y) \rightarrow \cdots, \]

As complex projective spaces have zero odd homology, the long exact sequence in (3.1) breaks up into short exact sequences. More precisely, for \( q = 1, \ldots, n^{2} - 1 \), we have

\[ 0 \rightarrow \bar{H}_{2q+1}(U) \rightarrow H_{2q}(Y) \rightarrow H_{2q}(X) \rightarrow \bar{H}_{2q}(U) \rightarrow H_{2q-1}(Y) \rightarrow 0, \]

We will identify \( U = \text{PGL}_{n} \) with the (complex) projective special linear group, \( \text{PSL}_{n} \). In turn, as a real manifold, \( \text{PSL}_{n} \) has the (Cartan-Malcev-Iwasawa) decomposition \( \text{PSL}_{n} \cong \text{PSU}_{n} \times \mathbb{R}^{s} \), where \( \text{PSU}_{n} \) is the projective special unitary group, and \( s = n^{2} - 1 \). Note that, as a (real) Lie group, \( \text{PGL}_{n} \) is an oriented 2\((n^{2}-1)\)-dimensional manifold, therefore, its Borel-Moore homology groups are actually cohomology groups,

\[ \bar{H}_{q}(U) = H^{2(n^{2}-1)-q}(U) = H^{2(n^{2}-1)-q}(\text{PSL}_{n}) = H^{2(n^{2}-1)-q}(\text{PSU}_{n}). \]

The unitary groups are compact. Since \( \text{PSU}_{n} \) is a \((n^{2}-1)\)-manifold, by Poincaré duality, we see the following fact.

**Lemma 3.3.** The homology groups of \( U = \text{PGL}_{n} \) are given by

\[ \bar{H}_{q}(U) = \begin{cases} 0 & \text{if } q < n^{2} - 1, \\ H_{q-(n^{2}-1)}(\text{PSU}_{n}) & \text{if } q \geq n^{2} - 1. \end{cases} \]

By using (3.4) and the short exact sequence in (3.2), we determine the homology groups of \( Y \) in lower degrees.

**Lemma 3.5.** The homology groups \( H_{q}(Y) \) for \( q < n^{2} - 1 \) are given by

\[ H_{q}(Y) = \begin{cases} 0 & \text{if } q \text{ is odd and } q < n^{2} - 1, \\ \mathbb{Z} & \text{if } q \text{ is even and } q < n^{2} - 1. \end{cases} \]

**Remark 2.** Since \( Y \) is an irreducible hypersurface in \( X \), the knowledge of the lower degree homology groups as in (3.6) could also be obtained by using the Lefschetz hyperplane theorem, see [7, Corollary 1.24].

We are now ready to state and prove our main result that is stated in the introduction.

**Proof of Theorem 1.2.** For \( q \in \{1, \ldots, n^{2} - 2\} \), we have the commuting diagram of Chow groups and Betti numbers as in Figure 3.

We have two remarks in order:

1. Since \( X \) has a cellular decomposition, the vertical map \( c_{l_{X}} \) is an isomorphism.

2. Secondly, as a result of a deep result Totaro, we know that the Chow groups of \( U \) are almost always zero, except at the degree \( n^{2}-1 \), where it is \( \mathbb{Z} \). Indeed, by [6, Theorem 16.6], we know that the Chow ring \( A^{*}(\text{GL}_{n} / \mathbb{C}^{*}) \), which is Poincaré dual to \( A_{*}(U) \), is isomorphic to \( \mathbb{Z} \). In particular, \( A_{q}(U) = 0 \) for \( q \in \{0, \ldots, n^{2} - 3\} \), and \( A_{0}(U) \cong \mathbb{Z} \).
As a consequence of these two remarks, we see that, for $q \in \{1, \ldots, n^2 - 3\}$, the map $i^*$ in the top row of diagram in (3) is zero, hence, the top $j^*$ is surjective. It follows that the bottom $j^*$ is surjective as well. But then, by the exactness of the bottom row, the kernel of the bottom $i^*$ is equal to $\bar{H}_{2q}(X)$, hence it is the zero map. In other words, we have $\bar{H}_{2q}(Y)/\bar{H}_{2q+1}(U) \cong \mathbb{Z}$ and $\bar{H}_{2q-1}(Y) \cong \bar{H}_{2q}(U)$. Thus, combining these isomorphisms with Lemma 3.5, we finish the proof of our theorem.

It is now easy to verify that the Poincaré polynomial of $Y$ is as given in Corollary 1.3.

REFERENCES

[1] Spencer Bloch. Algebraic cycles and higher $K$-theory. Adv. in Math., 61(3):267–304, 1986.
[2] A. Borel and J. C. Moore. Homology theory for locally compact spaces. Michigan Math. J., 7:137–159, 1960.
[3] Haibao Duan. The cohomology of the projective unitary groups. arXiv e-prints, page arXiv:1710.09222, Oct 2017.
[4] Dan Edidin and William Graham. Equivariant intersection theory. Invent. Math., 131(3):595–634, 1998.
[5] William Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]: Springer-Verlag, Berlin, second edition, 1998.
[6] Burt Totaro. Group cohomology and algebraic cycles, volume 204 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2014.
[7] Claire Voisin. Hodge theory and complex algebraic geometry. II, volume 77 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.